EMA 3702 Lab.

Manual Handbook

Strength of materials Laboratory Brief Technical and Lecture Notes

Stress Terms

Stress is defined as force per unit area. It has the same units as pressure, and in fact pressure is one special variety of stress. However, stress is a much more complex quantity than pressure because it varies both with direction and with the surface it acts on.

Compression
> Stress that acts to shorten an object.

Tension
> Stress that acts to lengthen an object.

Normal Stress
> Stress that acts perpendicular to a surface. Can be either compressional or tensional.

Shear
> Stress that acts parallel to a surface. It can cause one object to slide over another. It also tends to deform originally rectangular objects into parallelograms. The most general definition is that shear acts to change the angles in an object.

Hydrostatic
> Stress (usually compressional) that is uniform in all directions. A scuba diver experiences hydrostatic stress. Stress in the earth is nearly hydrostatic.

Directed Stress
> Stress that varies with direction. Stress under a stone slab is directed; there is a force in one direction but no counteracting forces perpendicular to it. This is why a person under a thick slab gets squashed but a scuba diver under the same pressure doesn't. The scuba diver feels the same force in all directions.

We only see the results of stress as it deforms materials. Even if we were to use a strain gauge to measure in-situ stress in the materials, we would not measure the stress itself. We would measure the deformation of the strain gauge (that's why it's called a "*strain* gauge") and use that to infer the stress.

Strain Terms

Strain is defined as the amount of deformation an object experiences compared to its original size and shape. For example, if a block 10 cm on a side is deformed so that it becomes 9 cm long, the strain is (10-9)/10 or 0.1 (sometimes expressed in percent, in this case 10 percent.) Note that strain is dimensionless.

Theoritical Back Grounds

$\varepsilon = (\delta L)/L$

Longitudinal or Linear Strain
> Strain that changes the length of a line without changing its direction. Can be either compressional or tensional.

Compression
> Longitudinal strain that shortens an object.

Tension
> Longitudinal strain that lengthens an object.

Shear
> Strain that changes the angles of an object. Shear causes lines to rotate.

Infinitesimal Strain
> Strain that is tiny, a few percent or less. Allows a number of useful mathematical simplifications and approximations.

Finite Strain
> Strain larger than a few percent. Requires a more complicated mathematical treatment than infinitesimal strain.

Homogeneous Strain
> Uniform strain. Straight lines in the original object remain straight. Parallel lines remain parallel. Circles deform to ellipses. Note that this definition rules out folding, since an originally straight layer has to remain straight.

Inhomogeneous Strain
> How real geology behaves. Deformation varies from place to place. Lines may bend and do not necessarily remain parallel.

Terms for Behavior of Materials

Elastic
> Material deforms under stress but returns to its original size and shape when the stress is released. There is no permanent deformation. Some elastic strain, like in a rubber band, can be large, but in rocks it is usually small enough to be considered infinitesimal.

Brittle
> Material deforms by fracturing. Glass is brittle. Rocks are typically brittle at low temperatures and pressures.

Ductile
> Material deforms without breaking. Metals are ductile. Many materials show both types of behavior. They may deform in a ductile manner if deformed slowly, but fracture if deformed too quickly or too much. Rocks are typically ductile at high temperatures or pressures.

Viscous
> Materials that deform steadily under stress. Purely viscous materials like liquids deform under even the smallest stress. Rocks may behave like viscous materials under high temperature and pressure.

Plastic
> Material does not flow until a threshold stress has been exceeded.

Viscoelastic
> Combines elastic and viscous behavior.

Beams

A **beam** is a structural member which carries loads. These loads are most often perpendicular to its longitudinal axis, but they can be of any geometry. A beam supporting any load develops internal stresses to resist applied loads. These internal stresses are bending stresses, shearing stresses, and normal stresses.

Beam types are determined by method of support, not by method of loading. Below are three types of beams that will be investigated in this course:

1. Simple Support Beam: 2. Cantilever Beam:
3. Indeterminate Statically Beam Support
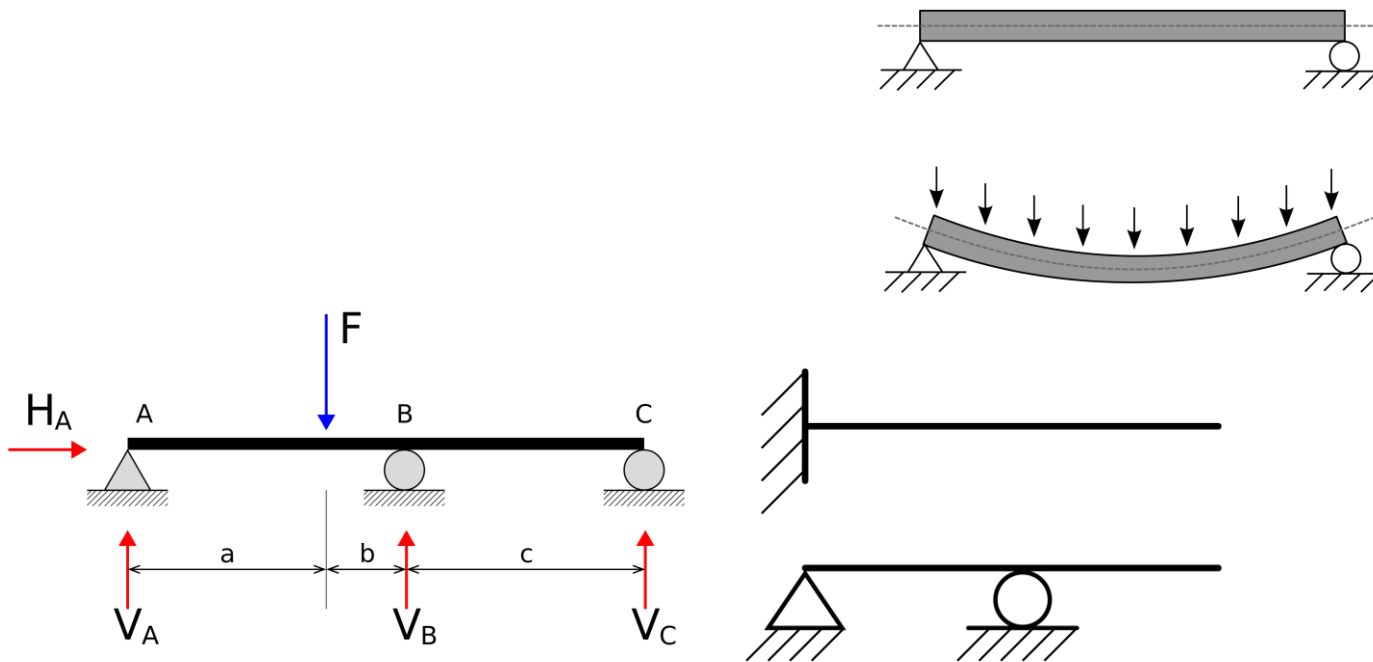
The first two types are statically determinate,
meaning that the reactions, shears and moments can be found by the laws of statics alone. Continuous beams are statically indeterminate. The internal forces of these beams cannot be found using the laws of statics alone. Early structures were designed to be statically determinate because simple analytical methods for the accurate structural analysis of indeterminate structures were not developed until the first part of this century. A number of formulas have been derived to simplify analysis of indeterminate beams.

Beam Loading Conditions:

The two beam loading conditions that either occur separately, or in some combination, are:

A. Concentrated Load
B. Distributed Laod

CONCENTRATED

Either a force or a moment can be applied as a concentrated load. Both are applied at a single point along the axis of a beam. These loads are shown as a "jump" in the shear or moment diagrams. The point of application for such a load is indicated in the diagram above. Note that this is NOT a hinge! It is a point of application. This could be point at which a railing is attached to a bridge, or a lamppost on the same.

DISTRIBUTED

Distributed loads can be uniformly or non-uniformly distributed. Both types are commonly found on all kinds of structures. Distributed loads are shown as an angle or curve in the shear or moment diagram. A uniformly distributed load can evolve into a one with unevenly uniformly distributed load (snow melting to ice at the edge of a roof), but are normally assumed to act as given. These loads are often replaced by a singular resultant force in order to simplify the structural analysis.

Introduction Beam Design:

Normally a beam is analyzed to obtain the maximum stress and this is compared to the material strength to determine the design safety margin. It is also normally required to calculate the deflection on the beam under the maximum expected load. The determination of the maximum stress results from producing the shear and bending moment diagrams. To facilitate this work the first stage is normally to determine all of the external loads.

Nomenclature

e = strain
σ = stress (N/m^2)
E = Young's Modulus = σ /e (N/m^2)
y = distance of surface from neutral surface (m).
R = Radius of neutral axis (m).
I = Moment of Inertia (m^4 - more normally cm^4)
Z = section modulus = I/y_{max} (m^3 - more normally cm^3)
M = Moment (Nm)
w = Distributed load on beam (kg/m) or (N/m as force units)
W = total load on beam (kg) or (N as force units)
F= Concentrated force on beam (N)
S= Shear Force on Section (N)
L = length of beam (m)
x = distance along beam (m)

Calculation of external forces

To allow determination of all of the external loads a free-body diagram is construction with all of the loads and supports replaced by their equivalent forces. A typical free-body diagram is shown below.
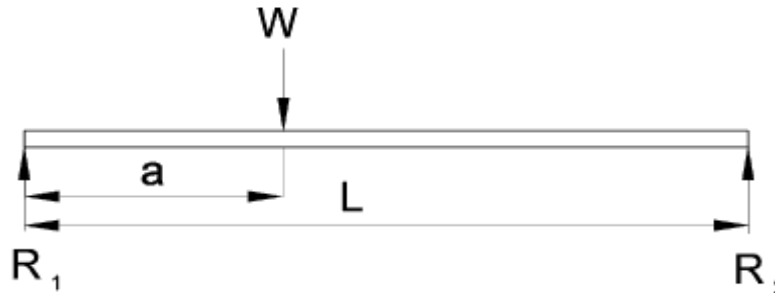
The unknown forces (generally the support reactions) are then determined using the equations for plane static equilibrium.

Equations for plane static equilibrium (in two dimension)

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = 0$$

For example considering the simple beam above the reaction R_2 is determined by Summing the moments about R_1 to zero

$$R_2 . L - W.a = 0 \text{ Therefore } R_2 = W.a / L$$

R_1 is determined by summing the vertical forces to 0

$$W - R_1 - R_2 = 0 \text{ Therefore } R_1 = W - R_2$$

Shear and Bending Moment Diagram

The shear force diagram indicates the shear force withstood by the beam section along the length of the beam.
The bending moment diagram indicates the bending moment withstood by the beam section along the length of the beam.
It is normal practice to produce a free body diagram with the shear diagram and the bending moment diagram position below

For simply supported beams the reactions are generally simple forces. When the beam is built-in the free body diagram will show the relevant support point as a reaction force and a reaction moment....

Sign Convention

The sign convention used for shear force diagrams and bending moments is only important in that it should be used consistently throughout a project. The sign convention used on this page is as below.
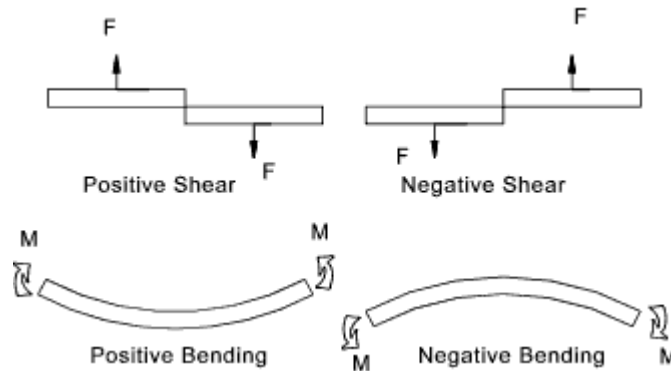

Typical Diagrams

A shear force diagram is simply constructed by moving a section along the beam from (say) the left origin and summing the forces to the left of the section. The equilibrium condition states that the forces on either side of a section balance and therefore the resisting shear force of the section is obtained by this simple operation

The bending moment diagram is obtained in the same way except that the moment is the sum of the product of each force and its distance(x) from the section. Distributed loads are calculated buy summing the product of the total force (to the left of the section) and the distance(x) of the centroid of the distributed load.

The sketches below show simply supported beams with on concentrated force.
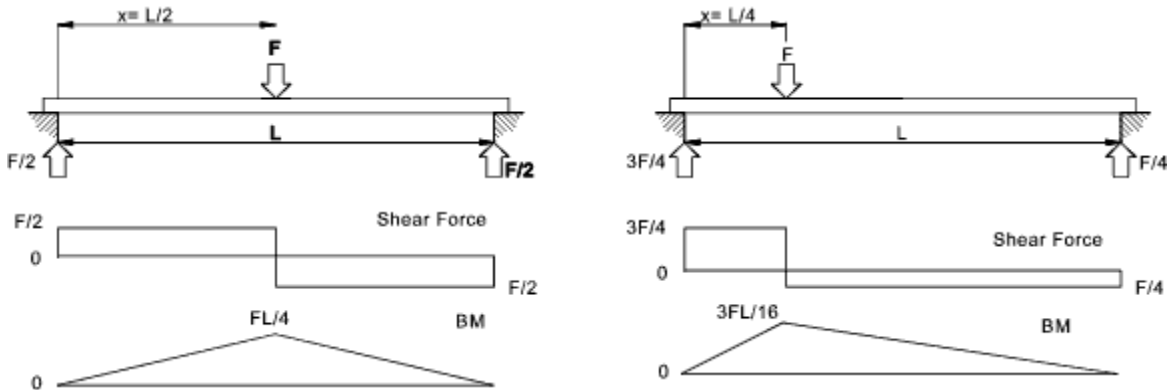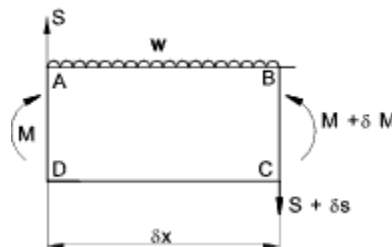

The sketches below show Cantilever beams with three different load combinations.
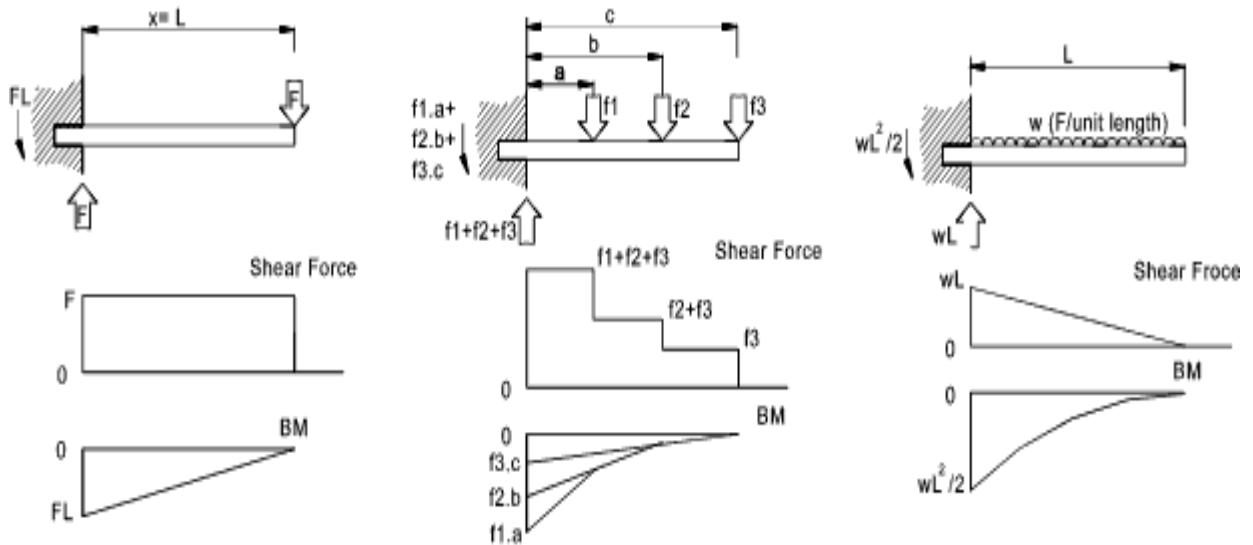
Note: The force shown if based on loads (weights) would need to be converted to force units i.e. 50kg = 50x9,81(g) = 490 N.

Shear Force Moment Relationship

Consider a short length of a beam under a distributed load separated by a distance δx.

The bending moment at section AD is M and the shear force is S. The bending moment at BC = M + δM and the shear force is S + δS.

The equations for equilibrium in 2 dimensions results in the equations.. Forces.

$$S - w.\delta x = S + \delta S$$
Therefore making δx infinitely small then.. $dS/dx = -w$

Moments.. Taking moments about C

$$M + S\delta x - M - \delta M - w(\delta x)^2/2 = 0$$
Therefore making δx infinitely small then.. $dM/dx = S$

Therefore putting the relationships into integral form.

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$$-\int w.dx = \int dS = S \quad ..and.. \quad \int S.dx = \int dM = M$$

The integral (Area) of the shear diagram between any limits results in the change of the shearing force between these limits and the integral of the Shear Force diagram between limits results in the change in bending moment...

Torsion
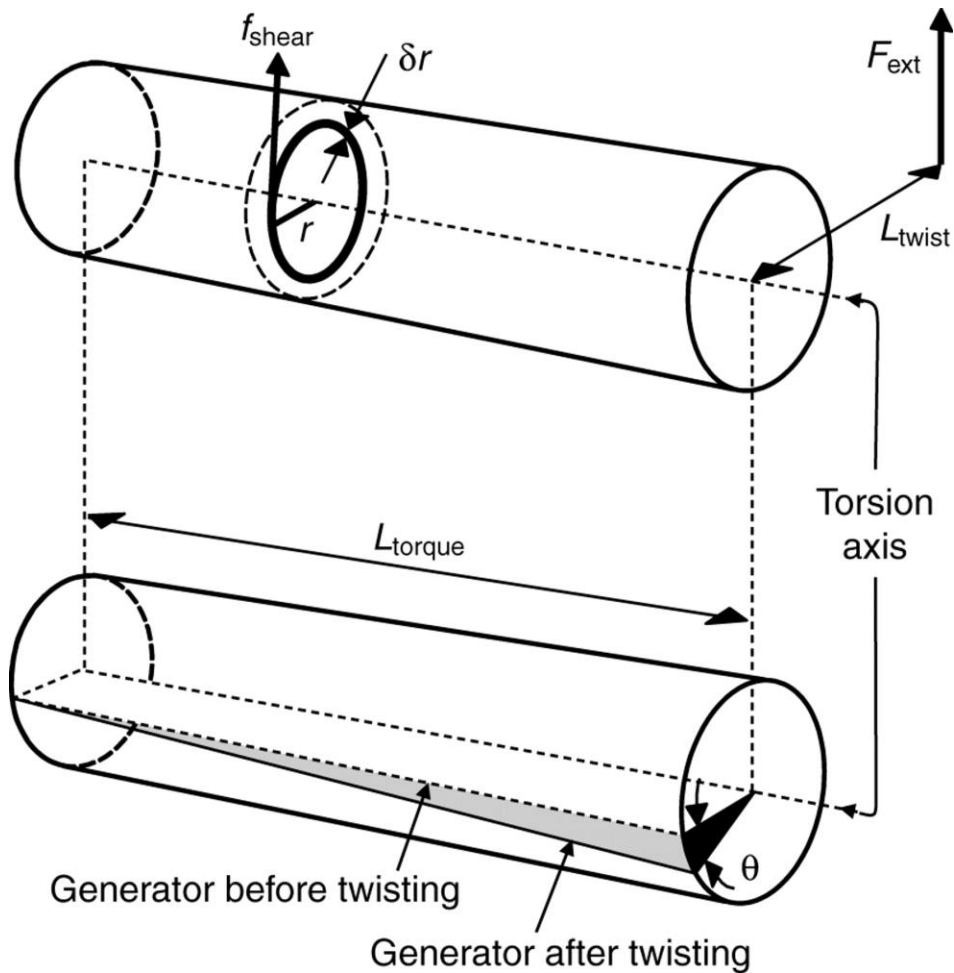
In solid mechanics, **torsion** is the twisting of an object due to an applied torque. In circular sections, the resultant shearing stress is perpendicular to the radius.

For solid or hollow shafts of uniform circular cross-section and constant wall thickness, the torsion relations are:

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\varphi}{\ell}$$

where:

- R is the outer radius of the shaft.
- τ is the maximum shear stress at the outer surface.
- φ is the angle of twist in radians.
- T is the torque (N·m or ft·lbf).

- ℓ is the length of the object the torque is being applied to or over.
- G is the shear modulus or more commonly the modulus of rigidity and is usually given in gigapascals (GPa), lbf/in^2 (psi), or lbf/ft^2.
- J is the torsion constant for the section . It is identical to the polar moment of inertia for a round shaft or concentric tube only. For other shapes J must be determined by other means. For solid shafts the membrane analogy is useful, and for thin walled tubes of arbitrary shape the shear flow approximation is fairly good, if the section is not re-entrant. For thick walled tubes of arbitrary shape there is no simple solution, and FEA may be the best method.
- the product GJ is called the torsional rigidity.

The shear stress at a point within a shaft is:

$$\tau_{\varphi z} = \frac{Tr}{J}$$

where:

- r is the distance from the center of rotation

Note that the highest shear stress is at the point where the radius is maximum, the surface of the shaft. High stresses at the surface may be compounded by stress concentrations such as rough spots. Thus, shafts for use in high torsion are polished to a fine surface finish to reduce the maximum stress in the shaft and increase its service life.

The angle of twist can be found by using:

$$\varphi = \frac{T\ell}{JG}.$$

Polar moment of inertia

The polar moment of inertia for a solid shaft is:

$$J = \frac{\pi}{2} r^4$$

where r is the radius of the object.

The polar moment of inertia for a pipe is:

$$J = \frac{\pi}{2}\left(r_o^4 - r_i^4\right)$$

where the o and i subscripts stand for the outer and inner radius of the pipe.

For a thin cylinder

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$$J = 2\pi R^3 t$$

where R is the average of the outer and inner radius and t is the wall thickness.

Failure mode

The shear stress in the shaft may be resolved into principal stresses via Mohr's circle. If the shaft is loaded only in torsion then one of the principal stresses will be in tension and the other in compression. These stresses are oriented at a 45 degree helical angle around the shaft. If the shaft is made of brittle material then the shaft will fail by a crack initiating at the surface and propagating through to the core of the shaft fracturing in a 45 degree angle helical shape. This is often demonstrated by twisting a piece of blackboard chalk between one's fingers.

Deflection of Beams

The deformation of a beam is usually expressed in terms of its deflection from its original unloaded position. The deflection is measured from the original neutral surface of the beam to the neutral surface of the deformed beam. The configuration assumed by the deformed neutral surface is known as the elastic curve of the beam.
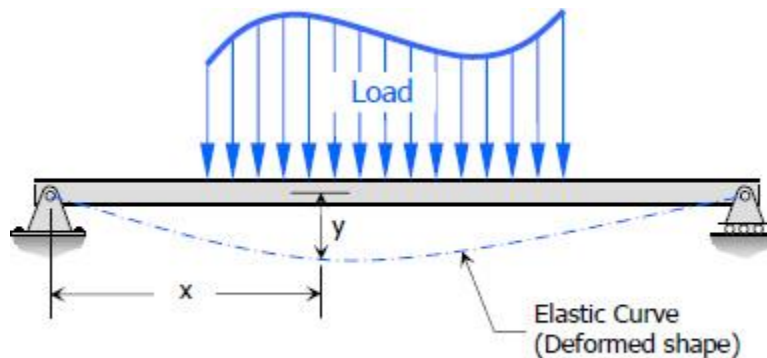

Figure: Elastic curve

Methods of Determining Beam Deflections

Numerous methods are available for the determination of beam deflections. These methods include:

1. Double-integration method
2. Area-moment method
3. Strain-energy method (Castigliano's Theorem)
4. Three-moment equation
5. Conjugate-beam method

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6. Method of superposition
7. Virtual work method

Of these methods, the first two are the ones that are commonly used.

Introduction

> The stress, strain, dimension, curvature, elasticity, are all related, under certain assumption, by the theory of simple bending. This theory relates to beam flexure resulting from couples applied to the beam without consideration of the shearing forces.

Superposition Principle

> The superposition principle is one of the most important tools for solving beam loading problems allowing simplification of very complicated design problems..

> For beams subjected to several loads of different types the resulting shear force, bending moment, slope and deflection can be found at any location by summing the effects due to each load acting separately to the other loads.

Nomenclature

> e = strain
> E = Young's Modulus = σ /e (N/m^2)
> y = distance of surface from neutral surface (m).
> R = Radius of neutral axis (m).
> I = Moment of Inertia (m^4 - more normally cm^4)
> Z = section modulus = I/y_{max}(m^3 - more normally cm^3)
> F = Force (N)
> x = Distance along beam
> δ = deflection (m)
> θ = Slope (radians)
> σ = stress (N/m^2)

Simple Bending

> A straight bar of homogeneous material is subject to only a moment at one end and an equal and opposite moment at the other end...
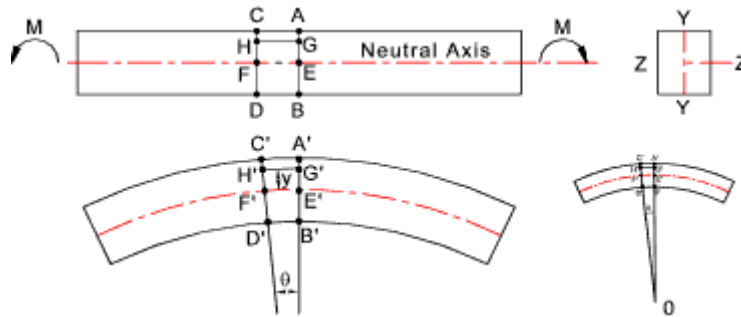
Assumptions

The beam is symmetrical about Y-Y
The traverse plane sections remain plane and normal to the longitudinal fibres after bending (Beroulli's assumption)
The fixed relationship between stress and strain (Young's Modulus)for the beam material is the same for tension and compression ($\sigma = E.e$)

Consider two section very close together (AB and CD).
After bending the sections will be at A'B' and C'D' and are no longer parallel.   AC will have extended to A'C' and BD will have compressed to B'D'
The line EF will be located such that it will not change in length.   This surface is called neutral surface and its intersection with Z_Z is called the neutral axis
The development lines of A'B' and C'D' intersect at a point 0 at an angle of θ radians and the radius of E'F' = R
Let y be the distance(E'G') of any layer H'G' originally parallel to EF..Then

$$H'G'/E'F' = (R+y)\theta /R \theta = (R+y)/R$$

And the strain e at layer H'G' =

$$e = (H'G'- HG) / HG = (H'G'- HG) / EF = [(R+y)\theta - R \theta] /R \theta = y /R$$

The accepted relationship between stress and strain is $\sigma = E.e$ Therefore

$$\sigma = E.e = E. \, y /R$$
$$\sigma / E = y / R$$

Therefore, for the illustrated example, the tensile stress is directly related to the distance above the neutral axis.   The compressive stress is also directly related to the distance below the neutral axis.   Assuming E is the same for compression and tension the relationship is the same.

As the beam is in static equilibrium and is only subject to moments (no vertical shear forces) the forces across the section (AB) are entirely longitudinal and the total compressive forces must balance the total tensile forces. The internal couple resulting from the sum of ($\sigma.dA \, .y$) over the whole section must equal the externally applied moment.
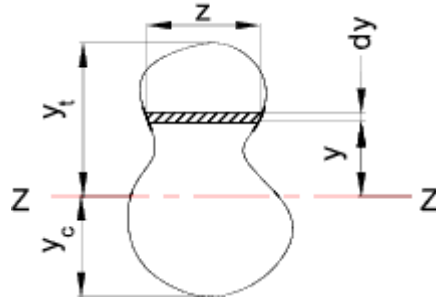
$$\sum(\sigma.\delta A) = 0 \quad \text{therefore} \quad \sum(\sigma.z\ \delta y) = 0$$

As $\sigma = \frac{yE}{R}$ therefore $\frac{E}{R}\sum(y.\ \delta A) = 0$ and $\frac{E}{R}\sum(y.\ z\delta y) = 0$

This can only be correct if $\Sigma(y\delta a)$ or $\Sigma(y.z.\delta y)$ is the moment of area of the section about the neutral axis. This can only be zero if the axis passes through the centre of gravity (centroid) of the section.

The internal couple resulting from the sum of (σ.dA .y) over the whole section must equal the externally applied moment. Therefore the couple of the force resulting from the stress on each area when totalled over the whole area will equal the applied moment

The force on each area element = $\sigma.\ \delta A = \sigma.\ z.\ \delta y$

The resulting moment = $y.\sigma.\ \delta A = \sigma.\ z.\ y.\ \delta y$

The total moment $\quad M = \sum(y.\sigma.\delta A)$ and $\sum(\sigma.z.y\ \delta y)$

Using $\frac{E}{R}y = \sigma$

$M = \frac{E}{R}\sum(y^2.\ \delta A)$ and $\frac{E}{R}\sum(z.y^2\delta y)$

$\sum(y^2\ \delta A)$ is the Moment of Inertia of the section(I)

From the above the following important simple beam bending relationship results

$$\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}$$

It is clear from above that a simple beam subject to bending generates a maximum stress at the surface furthest away from the neutral axis. For sections symmetrical about Z-Z the maximum compressive and tensile stress is equal.
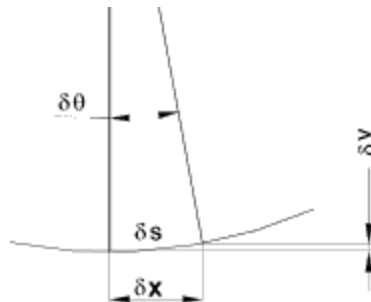
$$\sigma_{max} = y_{max}.\ M\ /\ I$$

The factor $I\ /y_{max}$ is given the name section Modulus (Z) and therefore

$$\sigma_{max} = M / Z$$

Values of Z are provided in the tables showing the properties of standard steel sections

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Deflection of Beams

Below is shown the arc of the neutral axis of a beam subject to bending.

For small angle $dy/dx = \tan \theta = \theta$
The curvature of a beam is identified as $d\theta /ds = 1/R$
In the figure $\delta\theta$ is small and δx; is practically $= \delta s$; i.e $ds /dx = 1$

$$\frac{1}{R} = \frac{d\theta}{ds} = \frac{d\theta}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

From this simple approximation the following relationships are derived.

$$\frac{1}{R} = \frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$Slope = \theta = \frac{dy}{dx} = \int\left(\frac{d^2y}{dx^2}\right)dx = \int \frac{M}{EI} dx$$

Integrating between selected limits.

The deflection between limits is obtained by further integration.

$$Deflection = x = \int \theta \, dx = \int\left(\frac{dy}{dx}\right)dx = \int\int \frac{M}{EI} dx$$

It has been proved ref <u>Shear - Bending</u> that $dM/dx = S$ and $dS/dx = -w = d^2M /dx$
Where S = the shear force M is the moment and w is the distributed load /unit length of beam.   therefore

$$S = \frac{dy}{dx}\left(EI\frac{d^2y}{dx^2}\right) = EI\frac{d^3y}{dx^3} \text{ and } -w = EI\frac{d^4y}{dx^4}$$

If w is constant or a integratatable function of x then this relationship can be used to arrive at general expressions for S, M, dy/dx, or y by progressive integrations with a constant of integration being added at each stage. The properties of the supports or fixings may be used to determine the constants. (x= 0 - simply supported, dx/dy = 0 fixed end etc)

In a similar manner if an expression for the bending moment is known then the slope and deflection can be obtained at any point x by single and double integration of the relationship and applying suitable constants of integration.

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

Singularity functions can be used for determining the values when the loading a not simple ref Singularity Functions

Example - Cantilever beam

Consider a cantilever beam (uniform section) with a single concentrated load at the end. At the fixed end x = 0, dy = 0 , dy/dx = 0
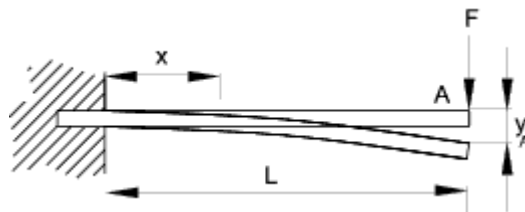
From the equilibrium balance ..At the support there is a resisting moment -FL and a vertical upward force F.
At any point x along the beam there is a moment $F(x - L) = M_x = EI \, d^2y/dx^2$

$$EI\frac{d^2y}{dx^2} = -F(L-x) \quad \text{Integrating}$$

$$EI\frac{dy}{dx} = -F\left(Lx - \frac{x^2}{2}\right) + C_1 \quad \ldots..(C_1=0 \text{ because dy/dx = 0 at x = 0 })$$
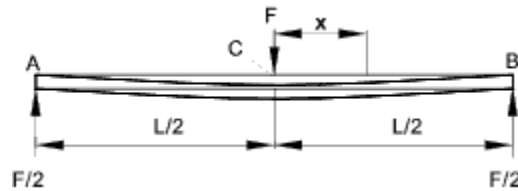
Integrating again

$$EI\, y = -F\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2 \quad \ldots..(C =0 \text{ because y = 0 at x = 0 })$$

At end A $\left(\frac{dy}{dx}\right)_A = -\frac{F}{EI}\left(L^2 - \frac{L^2}{2}\right) = -\frac{FL^2}{2EI}$ and $y_A = -\frac{F}{EI}\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = -\frac{FL^3}{3EI}$

Example - Simply supported beam

Consider a simply supported uniform section beam with a single load F at the centre. The beam will be deflect symmetrically about the centre line with 0 slope (dy/dx) at the centre line. It is convenient to select the origin at the centre line.

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{F}{2}\left(\frac{L}{2}+x\right) - Fx\right] = \frac{F}{2EI}\left(\frac{L}{2}-x\right) \quad \text{Integrating}$$

$$\frac{dy}{dx} = \frac{F}{2EI}\left(\frac{Lx}{2} - \frac{x^2}{2}\right) + C_1 \dots (C_1 = 0 \text{ because } dy/dx = 0 \text{ at } x = 0)$$

Integrating again $y = \frac{F}{2EI}\left(\frac{Lx^2}{4} - \frac{x^3}{6}\right) + C_2$

$y = 0$ when $x = L/2$ therefore $\frac{F}{2EI}\left(\frac{L^3}{8} - \frac{L^3}{12}\right) + C_2 = 0$

and thus $C_2 = -\frac{FL^3}{48EI}$

At end B $\left(\frac{dy}{dx}\right)_b = -\frac{F}{2EI}\left(\frac{L^2}{4} - \frac{L^2}{8}\right) = \frac{FL^2}{16EI}$ and $y_B = \frac{F}{2EI}\left(\frac{L^3}{8} - \frac{L^3}{12}\right) - \frac{FL^3}{48EI} = 0$

At centre C $y_c = -\frac{Fl^3}{48EI}$ (slope $\frac{dy}{dx} = 0$ by symmetry)
$x = 0$

Moment Area Method

This is a method of determining the change in slope or the deflection between two points on a beam. It is expressed as two theorems...

Theorem 1
If A and B are two points on a beam the change in angle (radians) between the tangent at A and the tangent at B is equal to the area of the bending moment diagram between the points divided by the relevant value of EI (the flexural rigidity constant).

Theorem 2
If A and B are two points on a beam the displacement of B relative to the tangent of the beam at A is equal to the moment of the area of the bending moment diagram between A and B about the ordinate through B divided by the relevant value of EI (the flexural rigidity constant).

Examples ..Two simple examples are provide below to illustrate these theorems

Example 1) Determine the deflection and slope of a cantilever as shown..
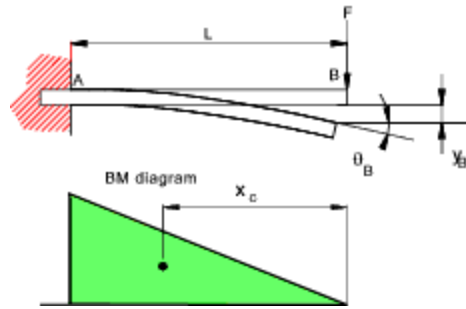
BM diagram

The bending moment at $A = M_A = FL$
The area of the bending moment diagram $A_M = F.L^2/2$
The distance to the centroid of the BM diagram from $B = x_c = 2L/3$
The deflection of $B = y_b = A_M.x_c/EI = F.L^3/3EI$
The slope at B relative to the tan at $A = \theta_b = A_M/EI = FL^2/2EI$

Example 2) Determine the central deflection and end slopes of the simply supported beam as shown..

$$E = 210 \text{ GPa} ......I = 834 \text{ cm}^4...... EI = 1,7514. 10^6 \text{Nm}^2$$
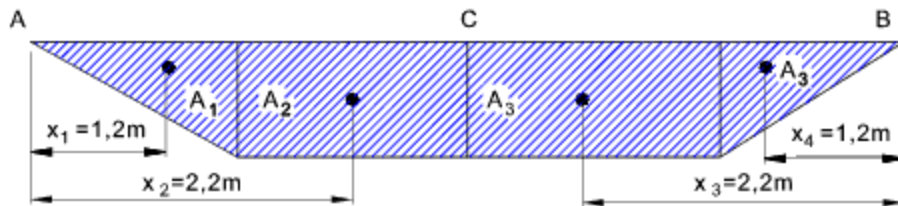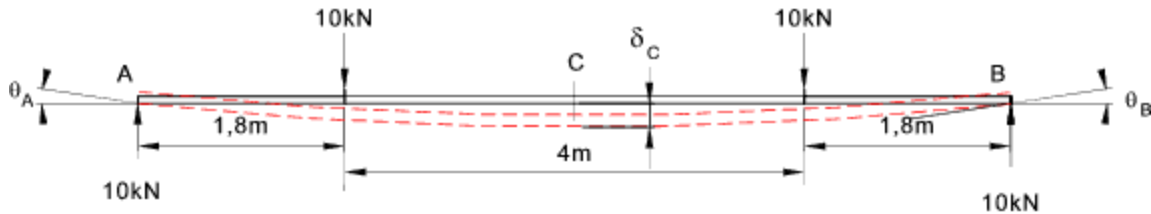
Bending Moment Diagram

$A_1 = 10.1,8.1,8/2 = 16,2\text{kNm}$
$A_2 = 10.1,8.2 = 36\text{kNm}$
$A_2 = 10.1,8.2 = 36\text{kNm}$
$A_1 = 10.1,8.1,8/2 = 16,2\text{kNm}$
$x_1 = \text{Centroid of } A_1 = (2/3).1,8 = 1,2$
$x_2 = \text{Centroid of } A_2 = 1,8 + 1 = 2,8$
$x_3 = \text{Centroid of } A_3 = 1,8 + 1 = 2,8$
$x_4 = \text{Centroid of } A_4 = (2/3).1,8 = 1,2$

The slope at A is given by the area of the moment diagram between A and C divided by EI.

$$\theta_A = (A_1 + A_2)/EI = (16{,}2+36){\cdot}10^3 / (1{,}7514{\cdot}10^6)$$
$$= 0{,}029 \text{rads} = 1{,}7 \text{ degrees}$$

The deflection at the centre (C) is equal to the deviation of the point A above a line that is tangent to C.
Moments must therefore be taken about the deviation line at A.

$$\delta_C = (A_M{\cdot}x_M)/EI = (A_1 x_1 + A_2 x_2)/EI = 120{,}24{\cdot}10^3/(1{,}7514{\cdot}10^6)$$
$$= 0{,}0686m = 68{,}6mm$$

Mohr's Circle

Introduced by Otto Mohr in 1882, Mohr's Circle illustrates principal stresses and stress transformations via a graphical format,
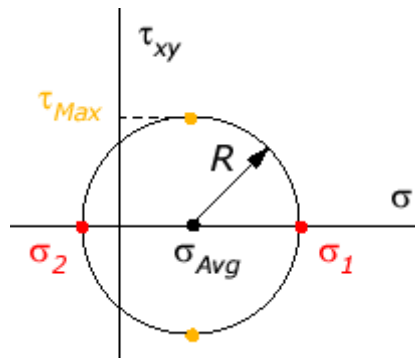
The two principal stresses are shown in **red**, and the maximum shear stress is shown in **orange**. Recall that the normal stresses equal the principal stresses when the stress element is aligned with the principal directions, and the shear stress equals the maximum shear stress when the stress element is rotated 45° away from the principal directions.

As the stress element is rotated away from the principal (or maximum shear) directions, the normal and shear stress components will always lie on Mohr's Circle.

Mohr's Circle was the leading tool used to visualize relationships between normal and shear stresses, and to estimate the maximum stresses, before hand-held calculators became popular. Even today, Mohr's Circle is still widely used by engineers all over the world.

Derivation of Mohr's Circle

To establish Mohr's Circle, we first recall the stress transformation formulas for plane stress at a given location,

$$\begin{cases} \sigma_{x'} - \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\[4mm] \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{cases}$$

Using a basic trigonometric relation ($\cos^2 2\theta + \sin^2 2\theta = 1$) to combine the two above equations we have,

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^{\,2} = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}$$

This is the equation of a circle, plotted on a graph where the abscissa is the normal stress and the ordinate is the shear stress. This is easier to see if we interpret σ_x and σ_y as being the two principal stresses, and σ_{xy} as being the maximum shear stress. Then we can define the average stress, σ_{avg}, and a "radius" R (which is just equal to the maximum shear stress),

$$\sigma_{\text{Avg}} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}}$$

The circle equation above now takes on a more familiar form,

$$\left(\sigma_{x'} - \sigma_{\text{Avg}}\right)^2 + \tau_{x'y'}^{\,2} = R^2$$

The circle is centered at the average stress value, and has a radius R equal to the maximum shear stress, as shown in the figure below,
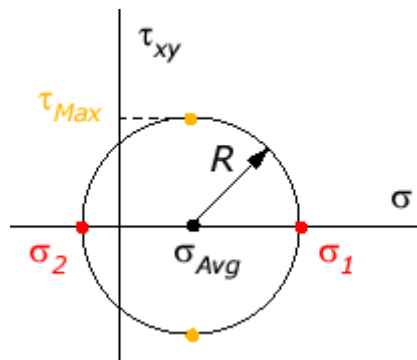
STRAIN GAUGES
 Introduction
Stress is not directly measurable with
current technology. We can measure
the *load* that we apply to a specimen,
but we cannot easily measure the *load
per unit area*, stress, at any point on
the specimen. Instead, the
experimental analysis of engineering stresses must be based on strains, which can be used to
calculate stresses. Using measured values of strain and knowledge of the mechanical properties
of the material such as the modulus of elasticity and Poisson's ratio, stresses can be calculated
from the appropriate stress-strain relationship of the material.

Cantilever Support   Strain Gage   Fine Wire   Aluminum Test Beam
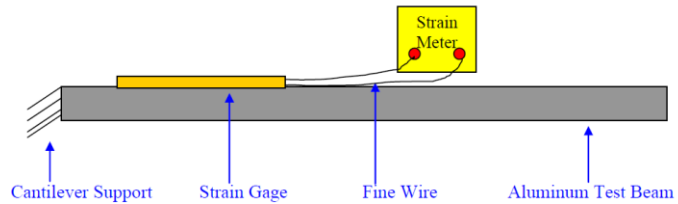
There are many types of commercially available strain gauges used in experimental mechanics.
Examples are acoustical, capacitance, inductance, mechanical, optical, piezioresistive, resistance,
and semiconductor. The optical strain gauges come in two types, diffraction and interferometric.
Perhaps the most versatile and widely used gauge is the bonded electrical resistance strain gauge
which is the type of gauge attached to the test specimens in Experiments 1 and 2 in this course.
Experiment 5 uses a Berry Strain Gauge, which is a particular type of mechanical strain gauge.

 Bonded Electrical Resistance Strain Gauge
The bonded electrical resistance strain gauge consists of a metallic strain-sensing element
encapsulated by a thin polyimide film that acts as an insulator attached to tabs for leadwire
connections. The sensing element is a grid of very thin metal alloy. The entire assembly can then
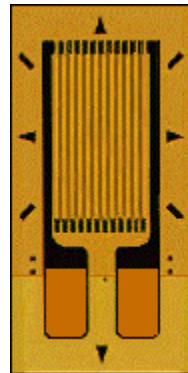be bonded to the specimen so that the gauge moves in unison with the specimen. Figure

Figure F1: **A strain gauge with a uniaxial pattern
for measuring strain in the direction of the gridlines.**

F.1 shows a graphic of a uniaxial strain gauge. As the gauge is stretched, the thin wires elongate,
increasing the electrical resistance of the gauge in direct proportion to the strain. The
measurement of the change in electrical resistance, which is measured as a change in voltage in a
bridge type circuit, is converted to strain is by dividing by a gauge factor. Equation F.1 illustrates
this simple relationship.

$$\Box R/R = (\text{Gauge Factor})(\Box L/L) \qquad (\text{Eq. F-1})$$

Where: R = resistance
$\Box$R = change in resistance

Dr. S. E. Beladi, PE   EMA 3702 Lab   P a g e | **21**
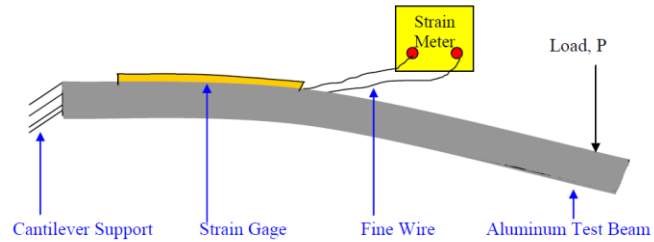   **Theoritical Back Grounds**

L = original length
ΔL = change in length (ΔL/L = strain)
The gauge factor is a property of the particular gauge being used and is usually between 2.00 and 2.20

The Figure above shows the arrangements of installation of strain gauges to a cantilever beam.

There is a wide selection of grid configurations, sizes, and alloy compositions to accommodate test conditions, including temperature extremes, dynamic or cyclic loading, etc. "Rosette" configuration strain gauges have two or three sensing elements and are used to determine principal strains and principal directions for general two dimensional plane stress problems.

Three element strain gauge rosettes will be used in Experiments 5 as well as Mechanical Measurements Labs.
 Strain gauge resistance changes are very small, on the order of 10^{-3} Ω to 10^{-6} Ω, due to micro strains occurring in engineering materials when stressed. Conventional ohmmeters are not capable of measuring resistances with enough precision to detect these small changes. Instead, potentiometer circuits and Wheatstone bridge circuits are used to convert these very small resistance changes to voltage signals that can be observed and recorded. The Wheatstone bridge circuits used in the laboratory are shown below. The dummy gauge in Figure F.2b is used to compensate for strain changes due to temperature only. The instruments in the laboratory are calibrated periodically and include adjustments for the strain gauge sensitivity (resistance), bridge balance (zero) and gauge factor. The instruments' readings are in proportion to strain.
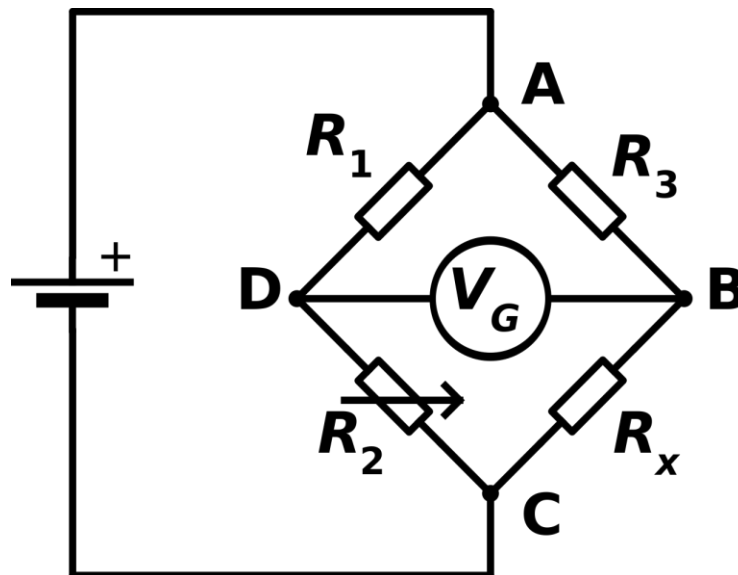

Figure F2: **Different types of variable resistance bridges.**
A short explanation of how the instruments in the laboratory work is as follows. In Figure F.2 the bridge is initially "balanced" so that R1$\square$R3 = R2$\square$R4 and the signal voltage V_G is zero. The

Active leg of the bridge (R1) is the strain gauge and the resistance of R1 varies as the structure is loaded. If the resistance of R1 increases (tension), the signal voltage (V_G) increases in proportion to the strain. The signal voltage is amplified and displayed.

Principal Stresses and Strains from Strain Gauge Rosettes

Principal stresses cannot be directly determined from a strain rosette. Principal strains can be calculated from the measurements taken from the strain rosette and then Hooke's Law can be used to convert principal strains to principal stresses. This section refers to homogeneous, isotropic materials in the linear-elastic range: Rectangular rosette (0□□- 45□□- 90□)

A rectangular rosette has three gauges so that measurements of normal strain can be taken along three axes the rosettes used in this course are Rectangular and have three gauges oriented 45□□from each other. In the example that follows, the three axes of the rosette are numbered 1, 2, 3 counterclockwise and measure the strains □**1,** □**2,** and □**3** respectively. The whole rosette is oriented at an arbitrary angle, □p, measured from gauge number 1 with respect to the principal axes of the strain on the specimen.

For 3 dimensional Rosette with 0,-60 and -120 Degrees:

(1) $\varepsilon_{p,q} = (\varepsilon_1 + \varepsilon_1 + \varepsilon_1)/3 \pm (\sqrt{2}/3)\{[\,(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2]\}^{1/2}$

And

(2) $\sigma_{p,q} = E/3\,\{(\varepsilon_1 + \varepsilon_1 + \varepsilon_1)(1\text{-}v) \pm (\sqrt{2}/(1+v))\{[\,(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2]\}^{1/2}\}.$

And angle of principal stress are calculated from:

(3) $\Phi_{p,q} = \frac{1}{2}\,.\mathbf{Tan^{-1}}\,(\sqrt{3}).\{[\,(\varepsilon_2 - \varepsilon_1)]/[\,(\varepsilon_1 - \varepsilon_2) + (\varepsilon_1 - \varepsilon_3)]\}.$
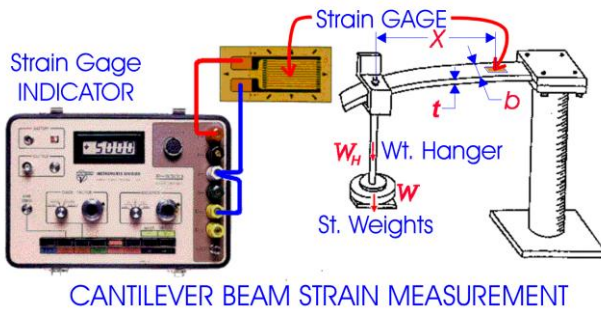
These equations could be used to obtain the principal stresses and strains.
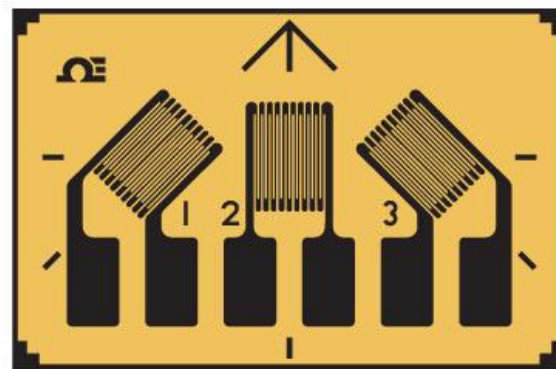
SGD-1/120-RY21

**Methods of Measurements of Data Using Wheatstone Bridge and Strain gauge indicators.**

The strain gauge subjected to deformation, ie. For a cantilever beam is attached to input channel of a Wheatstone Bridge, as shown below.

CANTILEVER BEAM STRAIN MEASUREMENT

The bridge is then balanced by appropriate balancing resistance, such that the measured voltage in the bridge is Zero.

The output of Bridge is connected directly to input of Balanced Strain Gauge meter, and the direct strains, for the material gauge factor is obtained in (μ-length/μ-length) units.

Connections for Full, Half and quarter Bridges are shown below.

Theoritical Back Grounds
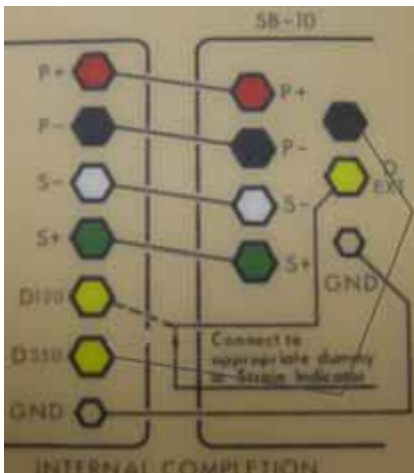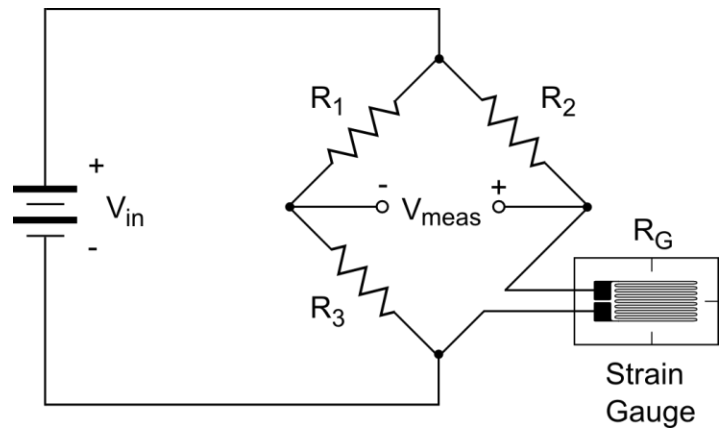
Connections:
1. the connections between strain indicator and balance unit are as the above photo
2. Strain gage #1 (on top of the beam), single wire goes to RED terminal at CH#9, two ground
   wires go to the WHITE & YELLOW terminals.
3. Strain gage #2 (on bottom of the beam), single wire goes to RED terminal at CH#10, two
   ground wires go to the WHITE & YELLOW terminals.

1. Press "**AMP ZERO** ±strain value to "0" for strain gage #2.

2 2. Press "**GAGE FACTOR**", and set it to "value listed on Beam"
3. Place the "**WEIGHT HANGER**"
 4. Press "**RUN**", switch to the **CHANNEL #10,** and use the "**BALANCE KNOB at CH#10**" to
balance the strain value to "0" for strain gage #1.
5. The strain is shown in the strain gauge indicator panel.

Hardness

Hardness is one of the most basic mechanical properties of engineering materials.
Hardness test is practical and provide a quick assessment and the result can be used as a good indicator for material selections. This is for example, the selection of materials suitable for metal forming dies or cutting tools. Hardness test is also employed for quality assurance in parts which require high wear resistance such as gears.
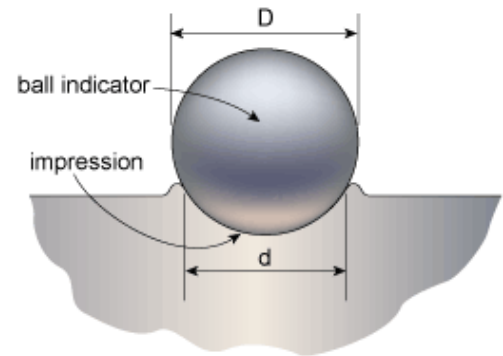
The nomenclature of hardness comes in various terms depending on the techniques used for hardness testing and also depends on the hardness levels of various types of materials. A scratch hardness test is generally used for minerals, giving a wide range of hardness values in a Moh.s scale at minimum and maximum values of 1 and 10 respectively. For example, talcum provides the lowest value of 1 while diamond gives the highest of 10. The basic principle is that the harder material will leave a scratch on a softer material. Hardness values of metals generally fall in a range of 4-8 in Moh.s scale, which is not practical to differentiate hardness properties for engineering applications. Therefore, indentation hardness measurement is conveniently used for metallic materials. A deeper or wider indentation indicates a less resistance to plastic deformation of the material being tested, resulting in a lower hardness value.

The indentation techniques involve Brinell, Rockwell, Vickers and Knoop. Different types of indenters are applied for each type. The standard test methods according to the American Society Testing and Materials (ASTM) available are, for instance, ASTM E10-07a (Standard test method for Brinell hardness of metallic materials), ASTM E18-08 (Standard test method for Rockwell hardness of metallic materials) and ASTM E92-41 (Standard test method for Vickers hardness of metallic materials) These hardness testing techniques are selected in relation to specimen dimensions, type of materials and the required hardness information. Their principles and testing methods are mentioned as follow.
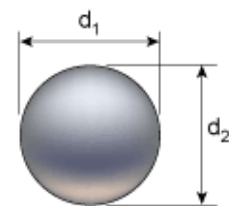
1. *Brinell Hardness Test*

(a) Brinell indentation

(b) measurement of impression diameter

Brinell hardness test was invented by J.A. Brinell in 1900 using a steel ball indenter with a 10 mm diameter. The steel ball is pressed on a metal surface to provide an impression as demonstrated in figure 1. This impression should not be distorted and must not be too deep since this might cause too

Theoritical Back Grounds

much of plastic deformation, leading to errors of the hardness values.

Different levels of material hardness result in impression of various diameters and depths. Therefore different loads are used for hardness testing of different materials as listed in table 1. Hard metals such as steels require a 3,000 kgf load while brass and aluminum involve the loads of 2,000 and 1,000 or 500 kgf respectively. For materials with very high hardness, a tungsten carbide ball is utilized to avoid the distortion of the ball.

In practice, pressing of the steel ball on to the metal surface is carried out for 30 second, followed by measuring two values of impression diameters normal to each other using a low magnification macroscope. An average value is used for the calculation according to equation 1

$$\mathbf{BHN = P/\{(\pi D/2).(D - \sqrt{D^2 - d^2})\} = P/(\pi Dt)}$$

Where:

P is the applied load, kg
D is the diameter of the steel ball, mm
d is the diameter of the indentation, mm
t is the depth of impression, mm

Note: This BHN values has a unit of kgf.mm-2 (1 kgf.mm-2 = 9.8 MPa) which cannot be compared to the average mean pressure on the impression. Generally, the metal surface should be flat without oxide scales or debris because these will significantly affect the hardness values obtained. A good sampling size due to a large steel ball diameter is advantageous for materials with highly different microstructures or microstructural heterogeneity. Scratches or surface roughness have very small effects on the hardness values measured. However, there are some disadvantages of Brinell hardness test. These are errors arising from the operator themselves (from diameter measurement) and the limitation in measuring of too small samples.
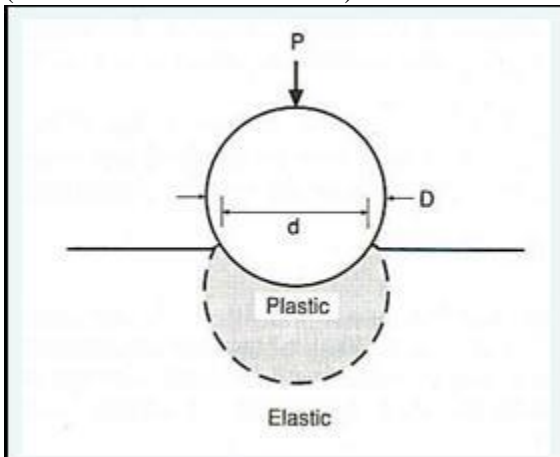
If we considered the plastic zone beneath the Brinell indenter, this plastic region is surrounded by elastic material which obstructs the plastic flow. This condition is said to be plane strain compressive where plastic deformation is limited. If the metal is very rigid, the metal flow upwards surrounding the indenter is possible as illustrated in figure
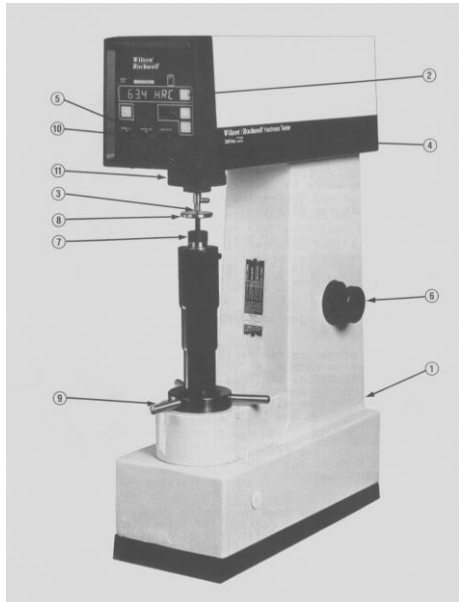
1. Power switch
2. Test scale scroll key
3. Indenter
4. Indenter display
5. Major load (kg) display
6. Weight selector dial
7. Anvil
8. Specimen
9. Capstan handwheel
10. Minor load (kg) display

Figure 2 – A Wilson Rockwell Model

However this situation is rarely seen because the metal displaced by the indenter is accounted for by the reduced volume of elastic material.

Rockwell Hardness Test

Rockwell hardness test is commonly used among industrial practices because the Rockwell testing machine offers a quick and practical operation and can also minimize errors arising from the operator. The depth of an indentation determines the hardness values. There are two types of indenters, Brale and steel ball indenters. The former is a round-tip cone with an included angle of 120o whereas the latter is a hardened steel ball with their sizes ranging from 1.6-12.7 mm. Therefore different combinations of indenters and loads selected are suitable for hardness testing of various materials. This is for example; the R scale is employed for soft materials such as polymers while the A scale is suitable for hardness testing of hard materials such as tool materials according to table 1.

The testing procedure starts with indenting a flatly ground metal surface with a diamond or hardened steel ball with a minor load of 10 kgf to position the metal surface as shown in figure above.

The depth of the impression caused by the minor load will be recorded as *H1* onto the machine before applying a major load level according to a standard as shown in table 2 and is recorded as *H2*. The difference of the depths ($\Delta H = H1-H2$) when applying the minor and the major loads indicates the hardness value of the material. If the depth difference is small, the deformation resistance of the metal is high, resulting in a high Rockwell hardness value. The hardness value will be displayed on a dial or a screen, having 100 divisions and each division represents a depth of 0.002 mm. Therefore the hardness value can be determined from a relationship as follows

$$HRX = M - \Delta H/2$$

Where ΔH is *H1-H2* and *M* is the maximum scale which equals 100 in general for testing with the diamond indenter (scale A, C and D). The M value equals 130 when testing with a steel ball for Rockwell scales B, E, M, and R.

The Rockwell hardness units are in RA, RB and RC (or HRA, HRB, HRC), depending on material's hardness. Tables 1 and 2 summarize loads and types of an indenter utilized for each scale. There are two types of indenters used, Brale indenter and steel ball indenters as mentioned previously. The applied major loads vary from 60, 100 and 150 kgf, also depending on the Rockwell hardness scale utilized. For instance, hardened steel is tested on a Rockwell scale C using a Brale indenter and at a major load of 150 kgf. On the Rockwell scale C, the obtained hardness values range from RC 20 F RC 70. Metals with lower hardness are tested on a Rockwell scale B using a 1.6 mm diameter steel ball at a 100 kgf major load, providing RB 0 F RB 100 hardness values. Rockwell scale A offers a wider range of hardness values which can be used to test materials ranging from annealed brass to cemented carbide. Due to high accuracy, the Rockwell hardness test is commonly conducted for measuring hardness of heat-treated steels. Furthermore, the smaller indenter (in comparison to that of Brinell hardness test) facilitates hardness measurement in small areas. However, this technique requires good surface preparation since the hardness values obtained is significantly affected by rough and scratched surfaces. There are several considerations for Rockwell hardness test:

- Require clean and well positioned indenter and anvil
- The test sample should be clean, dry, smooth and oxide-free surface
- The surface should be flat and perpendicular to the indenter
- Low reading of hardness value might be expected in cylindrical surfaces
- Specimen thickness should be 10 times higher than the depth of the indenter
- The spacing between the indentations should be 3 to 5 times of the indentation diameter
Loading speed should be standardized.

Vickers Hardness Test

Vickers hardness test requires a diamond pyramid indenter with an included angle of 136°. This technique is also called a diamond pyramid hardness test (DPH) according to the shape of the indenter. To carry on the test, the diamond indenter is pressed on to a prepared metal surface to cause a square-based pyramid indentation as illustrated in figure 4.
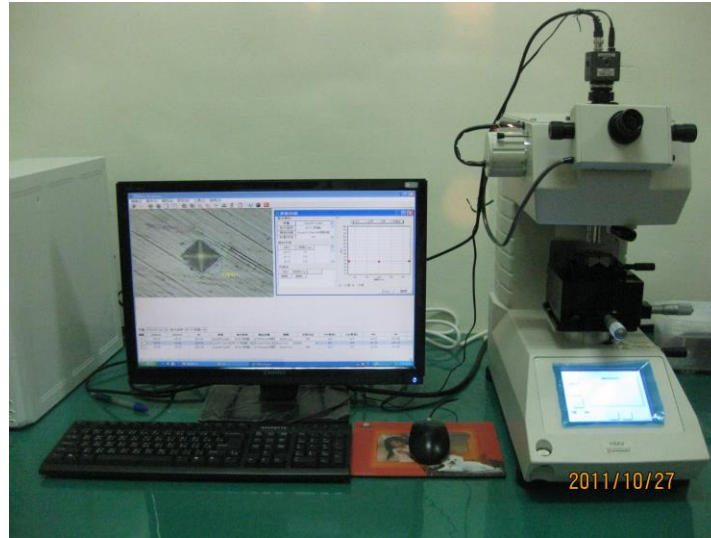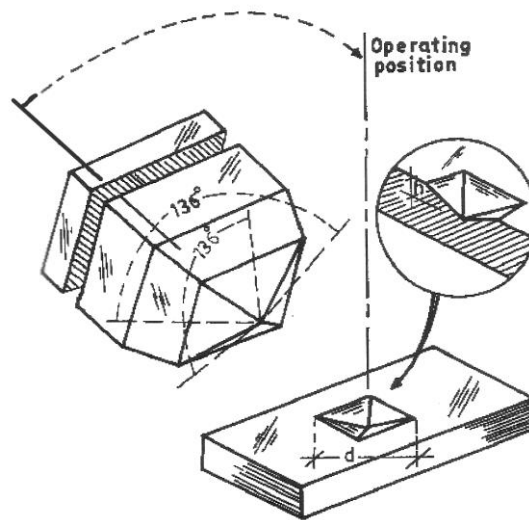

Figure: Vicker Micro hardness Equipment and Data acquisition.

The Vickers hardness value (VHN) can be calculated from the applied load divided by areas of indentation, at which the latter is derived from the diagonals of the pyramid as expressed in the equation below

$$VHN = 2PSin(\theta/2) /d^2 = 1854.4P/d^2$$

Theoritical Back Grounds
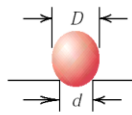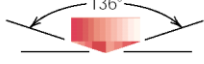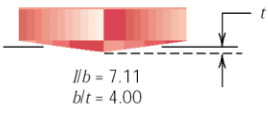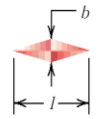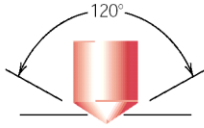
Where P is the applied load in grams (*g*), ϕ is the indenter face angle of 136°, and d is the mean diagonal length in μm. The constant 1854.4 incorporates the value of *sin*(ϕ/2) and other conversion factors to give VHN a unit of *kg/mm²*.

Procedure for microhardness test:

1. Turn on the tester.
2. Select and install the indenter (Vickers or Knoop), if not already installed.
3. Place the weights selected on the loading pan.
4. Place the specimen in the tester and turn the 40$\times$ objective lens into place. Focus on the specimen surface with the focusing control until surface features can be seen.
5. Gently turn the loading handle clockwise to raise the weights and the indenter, and turn the indenter into place. Slowly release the loading handle counter-clockwise to apply the load. Leave the indenter on the specimen for 10 to 15 s.
6. Raise the indenter by turning the loading handle clockwise gently, and turn the objective lens back into place.
7. Focus on the specimen surface to view the indentation. Measure length of the long diagonal (Knoop) or both diagonals (Vickers) of the indentation with the scale in the microscope. The numbers on the scale are length measured in 0.001 mm. alternatively, the diagonal lengths can be determined by moving a point on the scale from a corner to the opposite corner of the impression under microscope and noting the difference in micrometer readings (numbers on the fine scale are in 0.01 mm).
8. Calculate microhardness number using the appropriate formula.

Various Hardness Techniques.

Test	Indenter	Shape of Indentation		Load	Formula for Hardness Number[a]
		Side View	*Top View*		
Brinell	10-mm sphere of steel or tungsten carbide	D ; d	d	P	$HB = \dfrac{2P}{\pi D[D - \sqrt{D^2 - d^2}]}$
Vickers microhardness	Diamond pyramid	136°	d_1 ; d_1	P	$HV = 1.854 P / d_1^2$
Knoop microhardness	Diamond pyramid	t ; $l/b = 7.11$; $b/t = 4.00$	b ; l	P	$HK = 14.2 P / l^2$
Rockwell and Superficial Rockwell	Diamond cone ; $\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$ in. diameter steel spheres	120°		60 kg, 100 kg, 150 kg } Rockwell ; 15 kg, 30 kg, 45 kg } Superficial Rockwell	