Chapter 1

Problem Solving with Mathematical Models

Mathematical Model

– A mathematical model is the collection of variables and relationships needed to describe pertinent features of such a problem.

Operations Research (OR) [1.1]

– Is the study of how to form mathematical models of complex engineering and management problems and how to analyze them to gain insight about possible solutions.

EXAMPLE 1.1: Mortimer Middleman

Mortimer Middleman--friends call him MM--operates a modest wholesale diamond business. Several times each year MM travels to Antwerp, Belgium, to replenish his diamond supply on the international market. The wholesale price there averages approximately $700 per carat, but Antwerp market rules require him to buy at least 100 carats each trip. Mortimer and his staff then resell the diamonds to jewelers at a profit of $200 per carat. Each of the Antwerp trips requires 1 week, including the time for Mortimer to get ready, and costs approximately $2000.

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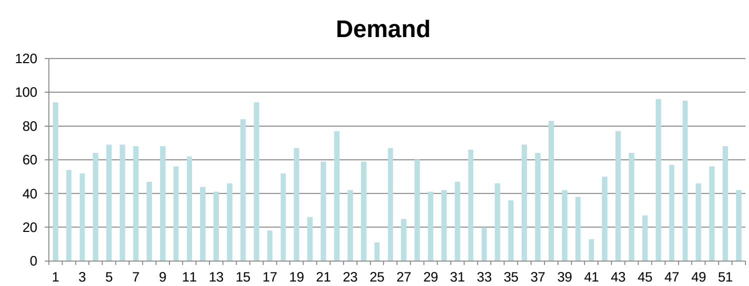
EXAMPLE 1.1: Mortimer Middleman

Customer demand values in Figure 1.1(a) show that business has been good. Over the past year, customers have come in to order an average of 55 carats per week.



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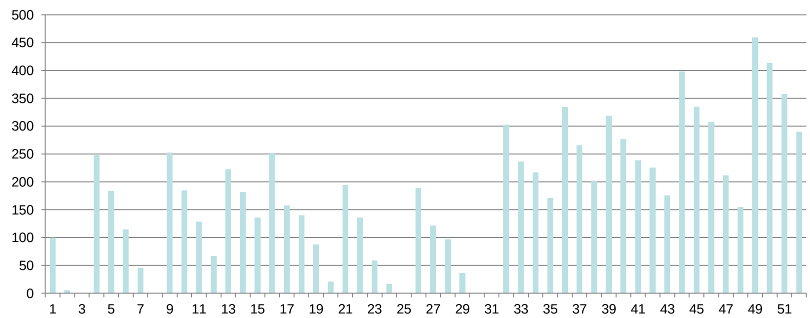
EXAMPLE 1.1: Mortimer Middleman

Part (c) of Figure 1.1 illustrates Mortimer's problem. Weekly levels of on-hand diamond inventory have varied widely, [Figure 1.1(c)]



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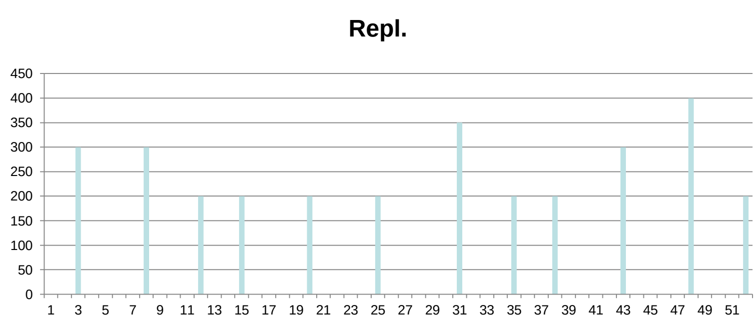
EXAMPLE 1.1: Mortimer Middleman

depending on the ups and downs in sales and the pattern of MM's replenishment trips [Figure 1.1(b)].



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EXAMPLE 1.1: Mortimer Middleman

Sometimes Mortimer believes that he is holding too much inventory. The hundreds of carats of diamonds on hand during some weeks add to his insurance costs and tie up capital that he could otherwise invest. MM has estimated that these holding costs total 0.5 % of wholesale value per week (i.e., 0.005 x $700 = $3.50 per carat per week).

At other times, diamond sales—and Mortimer's $200 per carat profit—have been lost because customer demand exceeded available stock [see Figure 1.1(d)]. When a customer calls, MM must either fill the order on the spot or lose the sale.

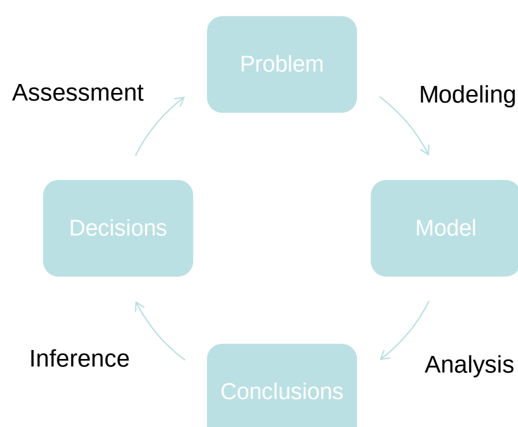
EXAMPLE 1.1: Mortimer Middleman

[Figure 1.1(d)].

EXAMPLE 1.1: Mortimer Middleman

Adding this all up for the past year, MM estimates holding costs of $38,409, unrealized profits from lost sales of $31,600, and resupply travel costs of $24,000, making the annual total $94,009. Can he do better?

1.2 Optimization and the Operations Research Process



3 Dimension of the Problem

- The **decisions** open to the decision makers
- The **constraints** limiting decision choices
- The **objectives** making some decisions preferred to others [1.2]

EXAMPLE 1.1: Mortimer Middleman

- **Decisions:** reorder point, order quantity
- **Constraints:** non-negative, order quantity > 100
- **Objectives:** to minimize total cost (holding + replenishment + lost-sale)

Optimization and Mathematical Programming

- Optimization models (also called mathematical programs) represent problem choices as decision variables and seek values that maximize or minimize objective functions of the decision variables subject to constraints on variable values expressing the limits on possible decision choices. [1.3]

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EXAMPLE 1.1: Mortimer Middleman

- **Decision Variables:**
  - $q \triangleq$ reorder quantity
  - $r \triangleq$ reorder point
  - ($\triangleq$ means "is defined to be")
- **Constraints:**
  - $q \geq 100$
  - $r \geq 0$
- **Objective function:**
  - $c(q, r) \triangleq$ total cost using a reorder quantity of q and a reorder point r

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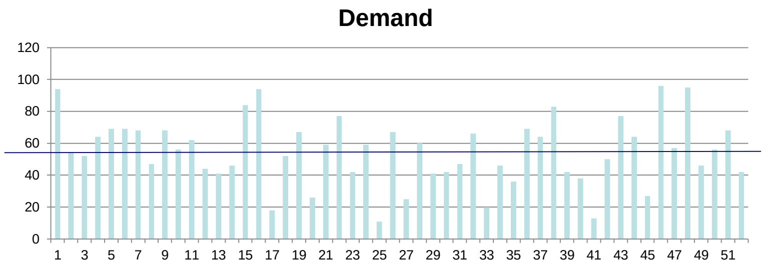
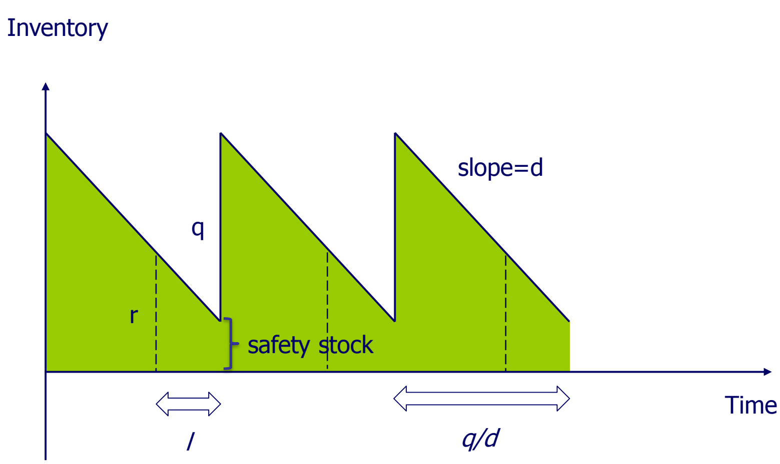
Constant-Rate Demand Assumption

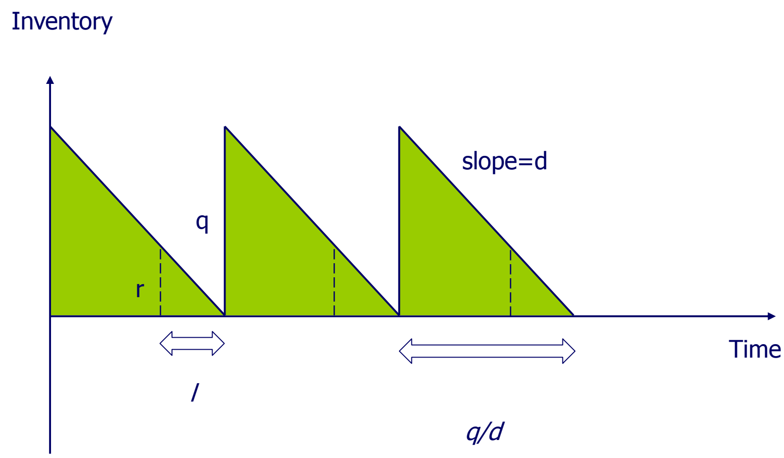


- Assume constant-rate demand of 55 carats/wk

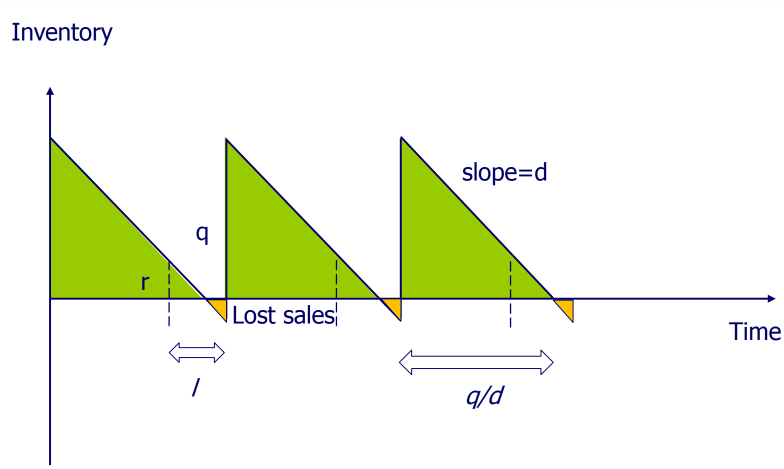
Inventory with safety stock



Inventory without safety stock or lost sales



Inventory with lost sales



EXAMPLE 1.1: Constant-Rate Demand Model

Minimize $c = 3.50\left[(r-55)+\frac{q}{2}\right]+\frac{2000}{q/55}$ **(1.1)**

Subject to

q ≥ 100

r ≥ 55

Feasible and Optimal Solutions

- A feasible solution is a choice of values for the decision variables that satisfies all constraints. (e.g. q=200, r=90)
- Optimal solutions are feasible solutions that achieve objective function value(s) as good as those of any other feasible solutions. [1.4]

EXAMPLE 1.1: Constant-Rate Demand Model

r* = 55

$$c(q,r) \triangleq 3.50\left[\frac{q}{2}\right] + 2000\frac{q}{55} \qquad (1.2)$$

$$q = \pm\sqrt{\frac{2(2000)(55)}{3.50}} \approx 250.7$$

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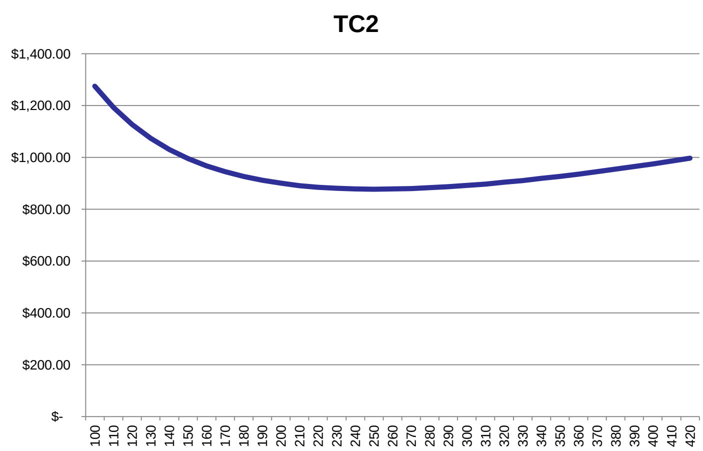
EXAMPLE 1.1: Constant-Rate Demand Model



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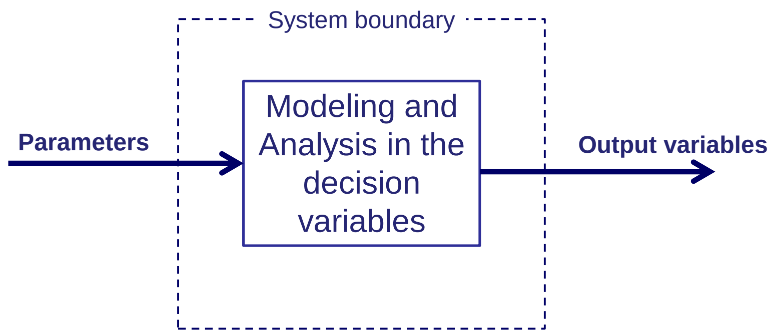
1.3 System Boundaries, Sensitivity Analysis, Tractability, and Validity



EOQ Under Constant-Rate Demand

- **Parameters:**
  - $d \triangleq$ weekly demand (55 carats)
  - $f \triangleq$ fixed cost of replenishment ($2000)
  - $h \triangleq$ cost per carat per wk for holding inventory ($3.50)
  - $s \triangleq$ cost per carat of lost sales ($200)
  - $l \triangleq$ lead time between reaching the reorder point and receiving a new supply (1 week)
  - $m \triangleq$ minimum order size (100 carats)

EOQ Under Constant-Rate Demand

- **Decision variables (if lost sales are not allowed):** [1.5]

  optimal order quantity $q^* = \sqrt{\frac{2fd}{h}}$

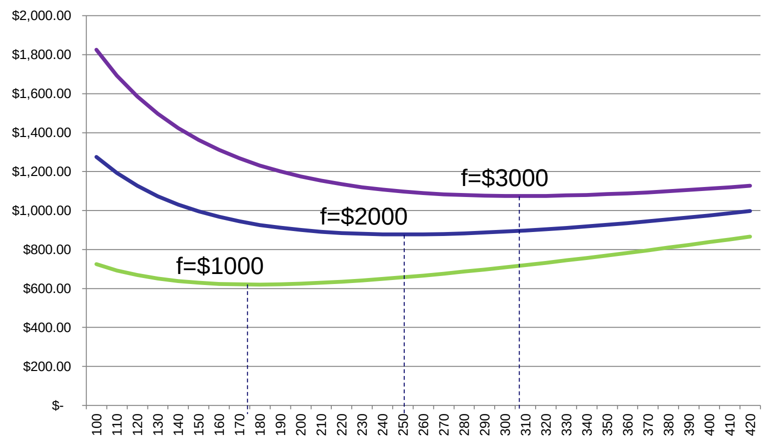
  optimal reorder point $r^* = ld$

  - *Provided* $q^* \geq m$
  - $q^* \geq ld$

Sensitivity Analysis

- Sensitivity analysis is an exploration of results from mathematical models to evaluate how they depend on the values chosen for parameters. [1.6]

Sensitivity Analysis



Closed-Form Solutions

- Closed-form solutions represent the ultimate in analysis of mathematical models because they provide both immediate results and rich sensibility analysis. [1.7]

Tractability versus Validity

- Tractability in modeling means the degree to which the model admits convenient analysis – how much analysis is practical. [1.8]
- Validity of a model is the degree to which inference drawn from the model hold for the real system. [1.9]
- OR analysts almost always confront a tradeoff between validity of models and their tractability to analysis. [1.10]

1.4 Descriptive Models and Simulation

- A simulation model is a computer program that simply steps through the behavior of a system of interest and reports experience.

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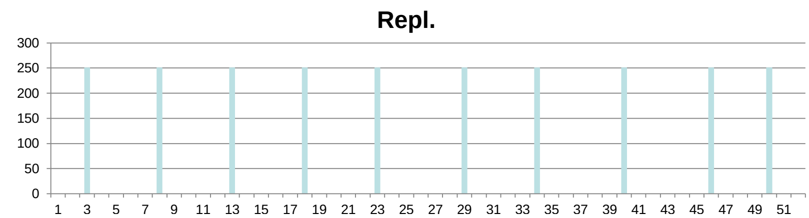
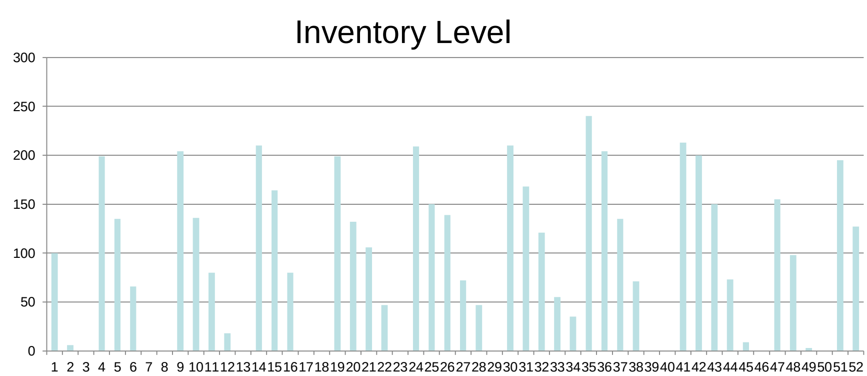
Simulation over MM’s History



Simulation over MM’s History



Total Cost: $108,729.75

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Simulation Model Validity

- Simulation models often possess high validity because they track true system behavior fairly accurately. [1.11]

Descriptive versus Prescriptive Models

- Descriptive Models evaluate fixed decision alternatives rather than indicating good choices.
- Descriptive Models yield fewer analytical inferences than prescriptive, optimization models because they take both input parameters and decisions as fixed. [1.12]

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1.5 Numerical Search and Exact versus Heuristic Solutions

$c(q,r)$ ≜ total cost computed by simulation with reorder point fixed at r and reorder quantity at q

MM's problem than reduces to the math model:

$$\begin{aligned} &\text{minimize} && c(q,r) && (1.3) \\ &\text{subject to} && q \geq 100,\ r \geq 0 \end{aligned}$$

Numerical Search

- Numerical search is the process of systematically trying different choices for the decision variables, keeping track of the feasible one with the best objective function value found so far.

Numerical Search

$q^{(0)} = 251$, $r^{(0)} = 55$, $c(q^{(0)}, r^{(0)}) = \$108,621$
$q^{(1)} = 251$, $r^{(1)} = 65$, $c(q^{(1)}, r^{(1)}) = \$108,421$ ↓
$q^{(2)} = 251$, $r^{(2)} = 75$, $c(q^{(2)}, r^{(2)}) = \$63,254$ ↓
$q^{(3)} = 251$, $r^{(3)} = 85$, $c(q^{(3)}, r^{(3)}) = \$63,054$ ↓
$q^{(4)} = 251$, $r^{(4)} = 95$, $c(q^{(4)}, r^{(4)}) = \$64,242$ ↑
increasing q
$q^{(5)} = 261$, $r^{(5)} = 85$, $c(q^{(5)}, r^{(5)}) = \$95,193$ ↑
decreading q
$q^{(6)} = 241$, $r^{(6)} = 85$, $c(q^{(6)}, r^{(6)}) = \$72,781$ ↑
STOP
$q^{(3)} = 251$, $r^{(3)} = 85$, $c(q^{(3)}, r^{(3)}) = \$63,054$

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Numerical Search – A Different Start

$q^{(0)} = 251$, $r^{(0)} = 145$, $c(q^{(0)}, r^{(0)}) = \$56,904$
$q^{(1)} = 251$, $r^{(1)} = 155$, $c(q^{(1)}, r^{(1)}) = \$59,539$ ↑
decreading r
$q^{(2)} = 251$, $r^{(2)} = 135$, $c(q^{(2)}, r^{(2)}) = \$56,900$ ↓
$q^{(3)} = 251$, $r^{(3)} = 125$, $c(q^{(3)}, r^{(3)}) = \$59,732$ ↑
increasing q
$q^{(4)} = 261$, $r^{(4)} = 135$, $c(q^{(4)}, r^{(4)}) = \$54,193$ ↓
$q^{(5)} = 271$, $r^{(5)} = 135$, $c(q^{(5)}, r^{(5)}) = \$58,467$ ↑
STOP
$q^{(4)} = 261$, $r^{(4)} = 135$, $c(q^{(4)}, r^{(4)}) = \$54,193$

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Numerical Search

- Inferences from numerical search are limited to specific points explored unless mathematical structure in the model supports further deduction. [1.13]

Exact versus Heuristic Optimization

- An exact optimal solution is a feasible solution to an optimization model that is probably as good as any other in objective function value.
- A heuristic or approximate optimum is a feasible solution derived from perspective analysis that is not guaranteed to yield an exact optimum. [1.14]
- Losses from settling for heuristic instead of exact optimal solutions are often dwarfed by variations associated with questionable model assumptions and doubtful data. [1.15]
- The appeal of exact optimal solutions is that they provide both good feasible solutions and certainty about what can be achieved under a fixed set of model assumptions. [1.16]

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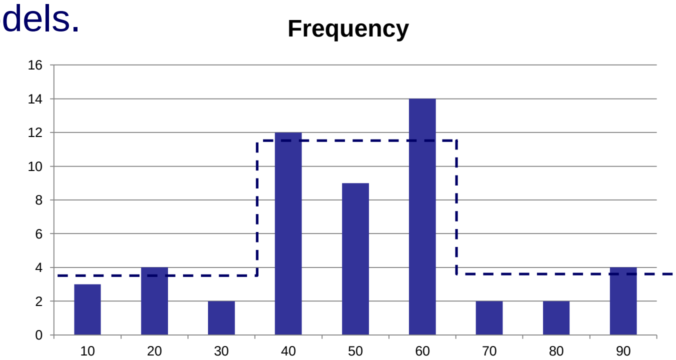
1.6 Deterministic versus Stochastic Models

- A mathematical model is termed deterministic if all parameter values are assumed to be known with certainty, and probabilistic or stochastic if it involves quantities known only in probability. [1.17]

Random Variables and Realizations

- Random variables (in Caps) represent quantities known only in terms of a probability in stochastic models.



Stochastic Simulation (Monte Carlo Analysis)

1. Randomly generating a sequence of realizations for input parameters.
2. Simulating each realization against chosen values for the decision variables.

- Besides providing only descriptive analysis, stochastic simulation models impose the extra analytic burden of having to estimate results statistically from a sample of system realizations. [1.18]

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Tradeoffs between Deterministic and Stochastic Models

- The power and generality of available mathematical tools for analysis of stochastic models does not nearly match that available for deterministic models. [1.19]
- Most optimization models are deterministic – not because OR analysts really believe that all problem parameters are known with certainty, but because useful prescriptive results can often be obtained only if stochastic variation is ignored. [1.20]

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1.7 Perspectives

- Informal <> Formal models
- Validity <> Tractability
- Other considerations: understood, time frame, computer power, data collection…
- The model-based OR approach to problem solving works best on problems important enough to warrant the time and resources for a careful study. [1.21]

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