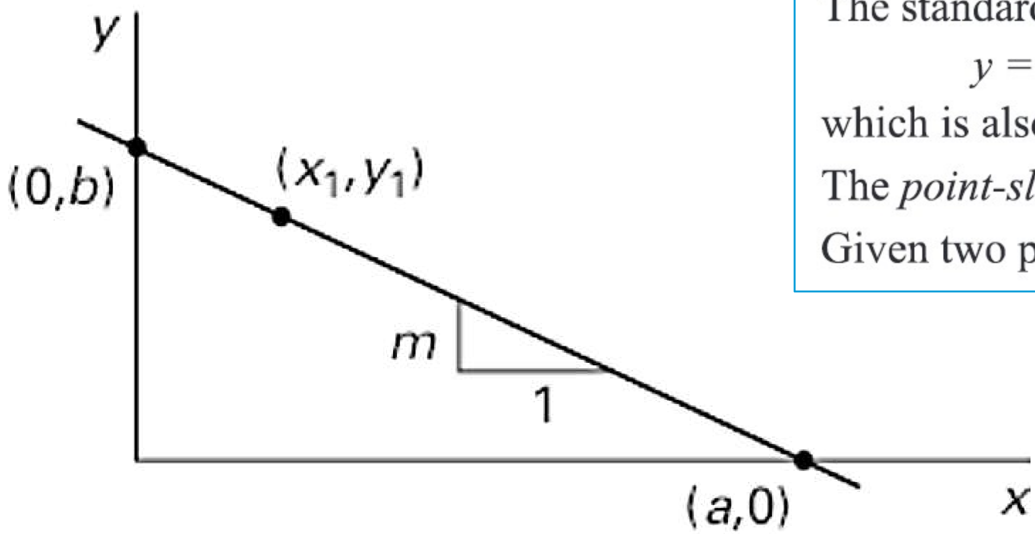
Analytic Geometry and Trigonometry

CHAPTER 2. FE CIVIL REVIEW AND PRACTICE PROBLEMS, PPI.

1. STRAIGHT LINES

Figure 2.1 Straight Line



The general form of the equation is

$$Ax + By + C = 0$$

The standard form of the equation is

$$y = mx + b,$$

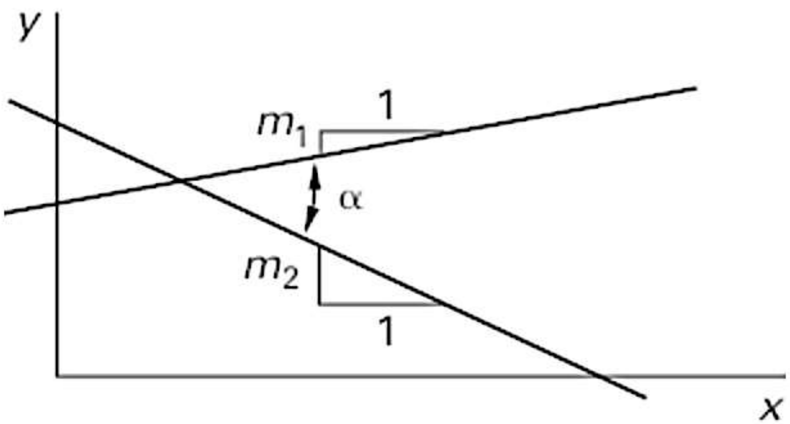
which is also known as the *slope-intercept* form.

The *point-slope* form is $y - y_1 = m(x - x_1)$

Given two points: slope, $m = (y_2 - y_1)/(x_2 - x_1)$

Angle Between Two Lines

Figure 2.2 *Two Lines Intersecting in Two-Dimensional Space*



The angle between lines with slopes m_1 and m_2 is

$$\alpha = \arctan\,[(m_2 - m_1)/(1 + m_2 \cdot m_1)]$$

Two lines are perpendicular if $m_1 = -1/m_2$

Two lines are parallel if $m_1 = m_2$, $\alpha = 0$.

Example 1

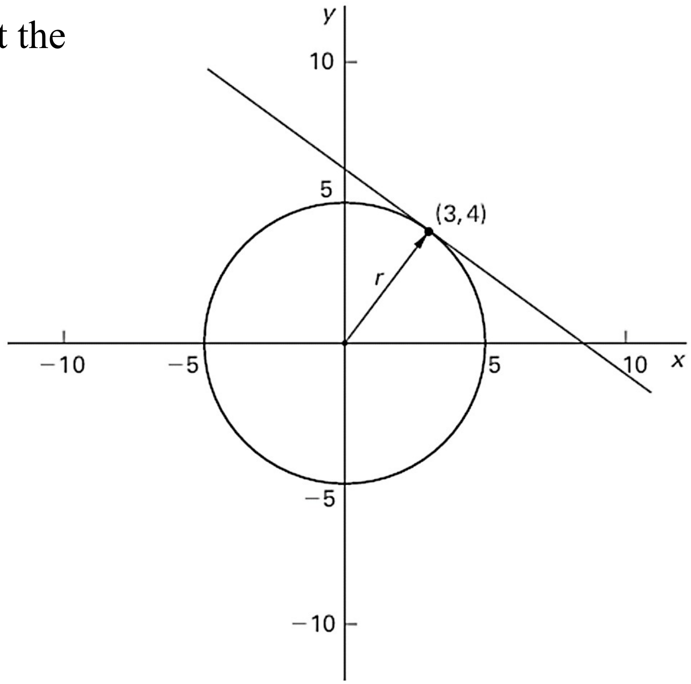
A circle with a radius of 5 is centered at the origin. What is the standard form of the equation of the line tangent to this circle at the point (3,4)?

(A) $x = \frac{-4}{3}y - \frac{25}{4}$

(B) $y = \frac{3}{4}x + \frac{25}{4}$

(C) $y = \frac{-3}{4}x + \frac{9}{4}$

(D) $y = \frac{-3}{4}x + \frac{25}{4}$



Example 2

The angle between the line $y = -7x + 12$ and the line $y = 3x$ is most nearly

(A) 22°

(B) 27°

(C) 33°

(D) 37°

2. QUADRATIC EQUATION

$$ax^2 + bx + c = 0$$

$$x = \text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac > 0$, the roots are real and unequal.
- If $b^2 - 4ac = 0$, the roots are real and equal. This is known as a *double root.*
- If $b^2 - 4ac < 0$, the roots are complex and unequal.

Example 3

What is a real solution of the equation $50x^2 + 5(x-2)^2$ = -1?

(A) -6.12 and -3.88

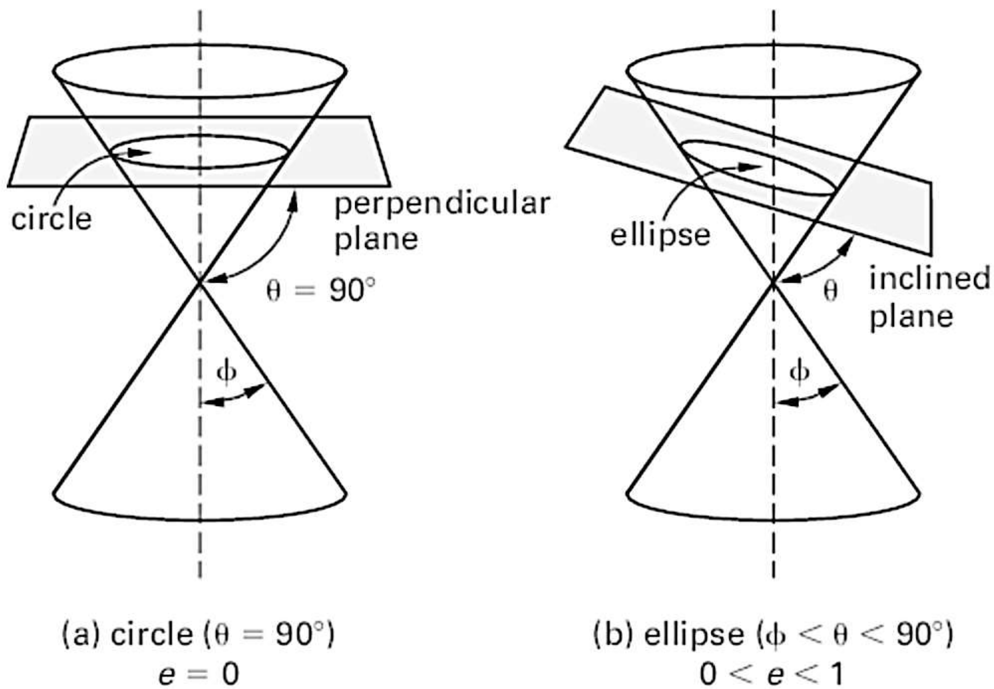
(B) -0.52 and 0.7

(C) 7.55

(D) No real solution

3. CONIC SECTIONS

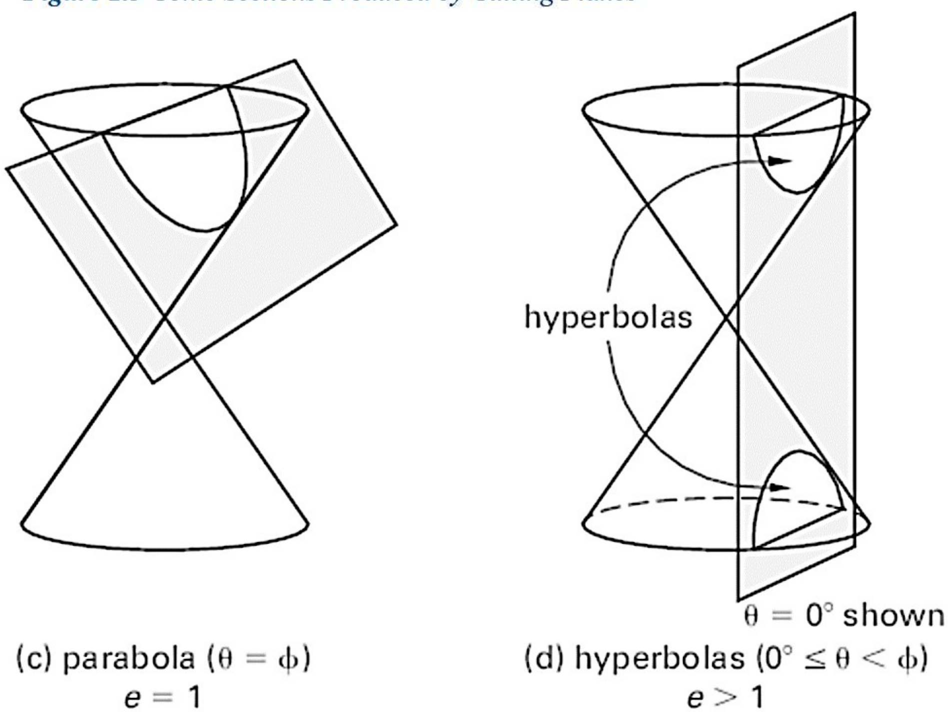
Figure 2.3 *Conic Sections Produced by Cutting Planes*



e = eccentricity = $\cos\theta / (\cos\phi)$

3. CONIC SECTIONS

Figure 2.3 *Conic Sections Produced by Cutting Planes*



Conic Section Equation

The general form of the conic section equation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where not both A and C are zero.

If $B^2 - 4AC < 0$, an *ellipse* is defined.

If $B^2 - 4AC > 0$, a *hyperbola* is defined.

If $B^2 - 4AC = 0$, the conic is a *parabola*.

If $A = C$ and $B = 0$, a *circle* is defined.

If $A = B = C = 0$, a *straight line* is defined.

Example 4

What kind of conic section is described by the following equation?

$$4x^2 - y^2 + 8x + 4y = 15$$

(A) circle

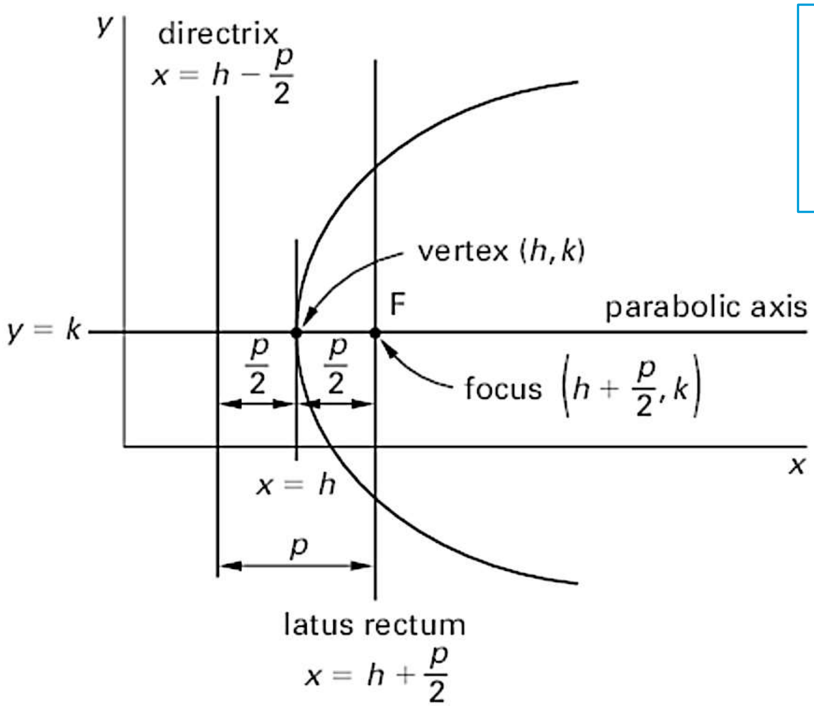
(B) ellipse

(C) parabola

(D) hyperbola

Standard Form Eqn. Parabola

Figure 2.4 *Parabola*



$(y - k)^2 = 2p(x - h)$; Center at (h, k)
is the standard form of the equation. When $h = k = 0$,
Focus: $(p/2, 0)$; Directrix: $x = -p/2$

Equation for a vertical parabola:

$$(x - h)^2 = 2p(y - k)$$

Example 5

What is the equation of a parabola with a vertex at $(4, 8)$ and a directrix at $y = 5$?

(A) $(x-8)^2 = 12(y-4)$

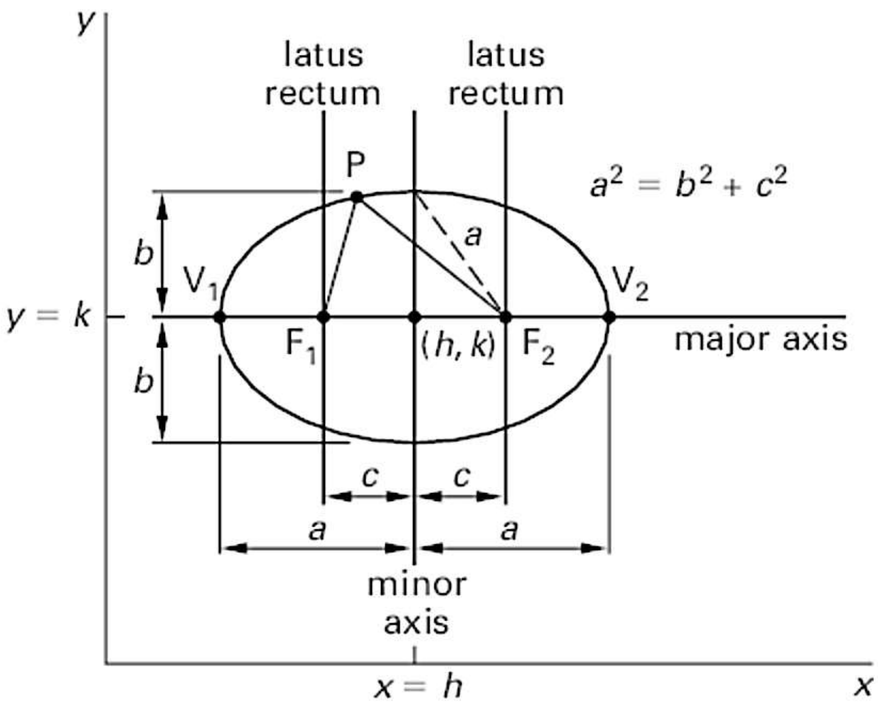
(B) $(x-4)^2 = 12(y-8)$

(C) $(x-4)^2 = 6(y-8)$

(D) $(y-8)^2 = 12(x-4)$

Standard Form Eqn. Ellipse

Figure 2.5 *Ellipse (with horizontal major axis)*



$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1;$$ Center at (h, k)

is the standard form of the equation. When $h = k = 0$,

Eccentricity: $e = \sqrt{1 - (b^2/a^2)} = c/a$

$b = a\sqrt{1 - e^2}$;

Focus: $(\pm ae, 0)$; Directrix: $x = \pm a/e$

Area $= \pi ab$

Example 6

What is the area of the ellipse which equation is $4x^2 + y^2 - 24x + y - 2 = 0$?

(A) 540

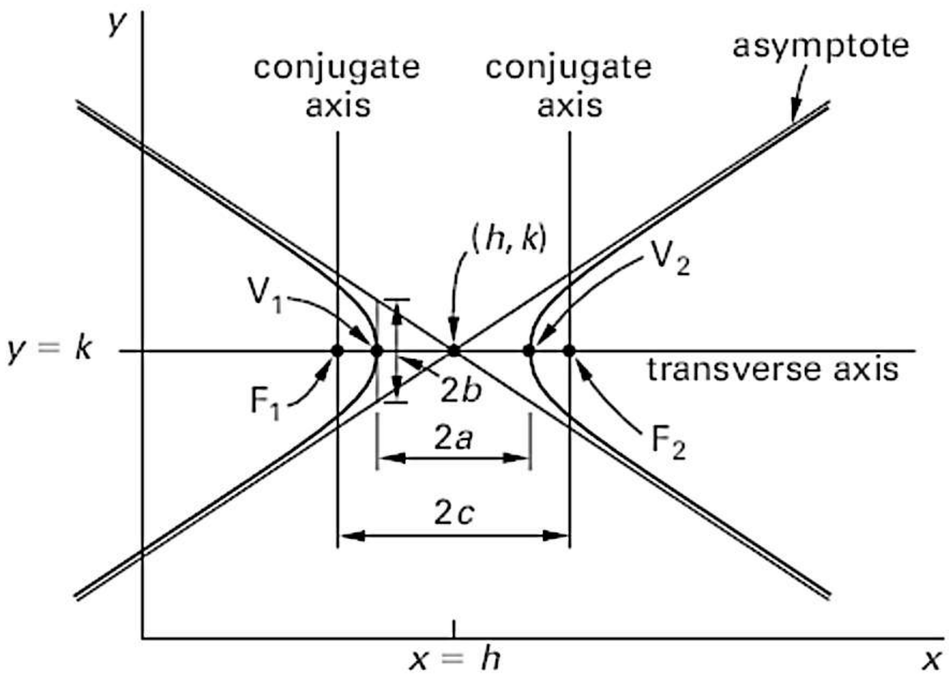
(A) 172

(B) 60

(C) 24

Standard Form Eqn. Hyperbola

Figure 2.6 *Hyperbola*



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1;\ \text{Center at } (h, k)$$

is the standard form of the equation. When $h = k = 0$,

Eccentricity: $e = \sqrt{1 + (b^2/a^2)} = c/a$

$b = a\sqrt{e^2 - 1};$

Focus: $(\pm\, ae, 0)$; Directrix: $x = \pm\, a/e$

Example 7

What is the angle between the asymptotes of the hyperbola which equation is
$9(x-2)^2-16(y-5)^2=144$?

(A) 37

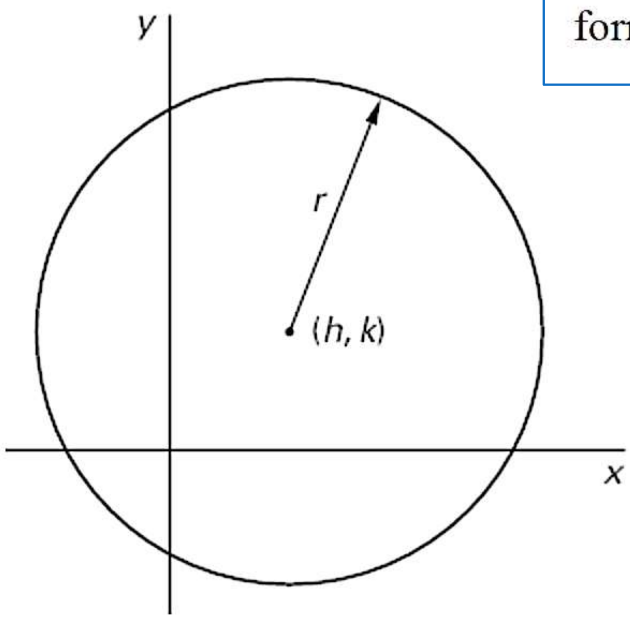
(A) 74

(B) 126

(C) 143

Standard Form Eqn. Circle

Figure 2.7 Circle



$(x-h)^2 + (y-k)^2 = r^2$; Center at (h, k) is the standard form of the equation with radius $r = \sqrt{(x-h)^2 + (y-k)^2}$

$$x^2 + y^2 + 2ax + 2by + c = 0$$

is the normal form of the conic section equation, if that conic section has a principal axis parallel to a coordinate axis.

$h = -a$; $k = -b$

$r = \sqrt{a^2 + b^2 - c}$

If $a^2 + b^2 - c$ is positive, a *circle*, center $(-a, -b)$.

If $a^2 + b^2 - c$ equals zero, a *point* at $(-a, -b)$.

If $a^2 + b^2 - c$ is negative, locus is *imaginary*.

Example 8

What is the radius of the circle which equation is $x^2 + y^2 - 4x + 8y = 7$?

(A) $\sqrt{3}$

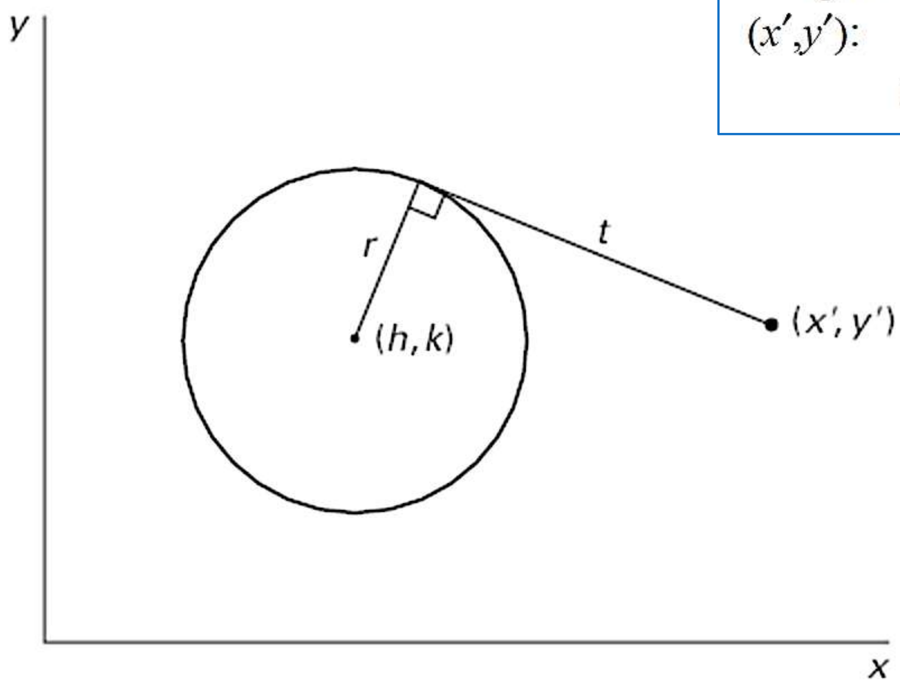
(A) $2\sqrt{5}$

(B) $3\sqrt{3}$

(C) $4\sqrt{3}$

Length of a Circle Tangent Line

Figure 2.8 Tangent to a Circle from a Point



Length of the tangent line from a point on a circle to a point (x', y'):

$$t^2 = (x' - h)^2 + (y' - k)^2 - r^2$$

The distance between two points is

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Example 9

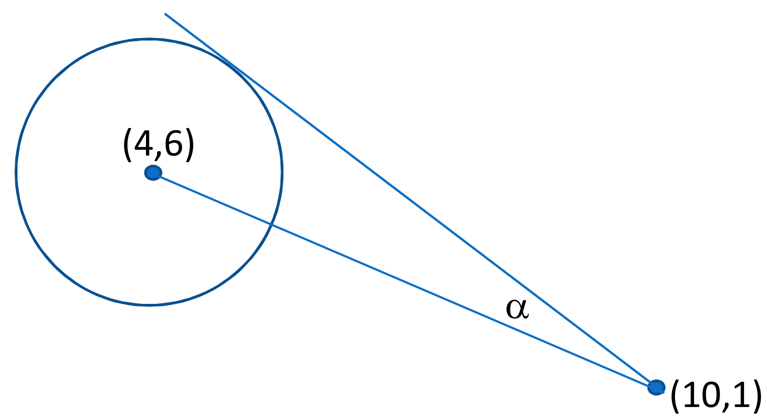
A tangent is drawn from the point (10,1) to the circle with center at (4,6) and radius equal to 3. What is the angle α between the tangent and the radial line shown?

(A) 22.8 deg

(A) 24.6 deg

(B) 36.8 deg

(C) 65.4 deg



4. QUADRIC SURFACE SPHERE

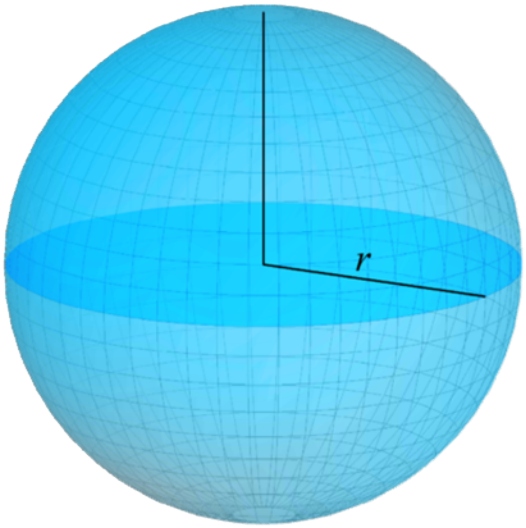
The standard form of the equation is

$$(x-h)^2+(y-k)^2+(z-m)^2=r^2$$

with center at (h, k, m).

In a three-dimensional space, the distance between two points is

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$



Example 10

What is the radius of the sphere with center at the origin and passing through the point (8,1,6)?

(A) 10

(A) 65

(B) $\sqrt{101}$

(C) 100

5. TRIGONOMETRY

A triangle is a three-sided closed polygon with three angles whose sum is 180º (π rad).

A right triangle is a triangle in which one of the angles is 90º (π/2 rad).

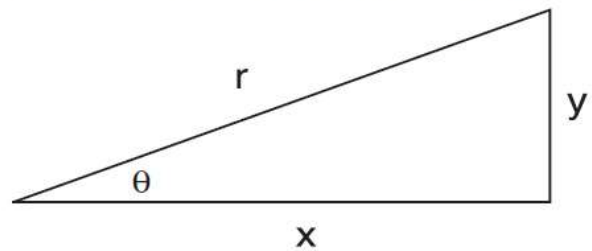
Choosing one acute angle as a reference, the sides of the triangle are called adjacent side, x, opposite side, y, and the hypotenuse, r.

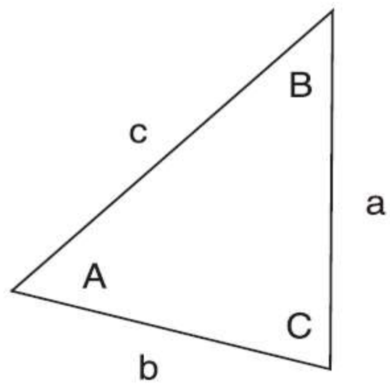
Trigonometric functions are defined using a right triangle.

$\sin\theta = y/r$, $\cos\theta = x/r$

$\tan\theta = y/x$, $\cot\theta = x/y$

$\csc\theta = r/y$, $\sec\theta = r/x$





Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Example 11

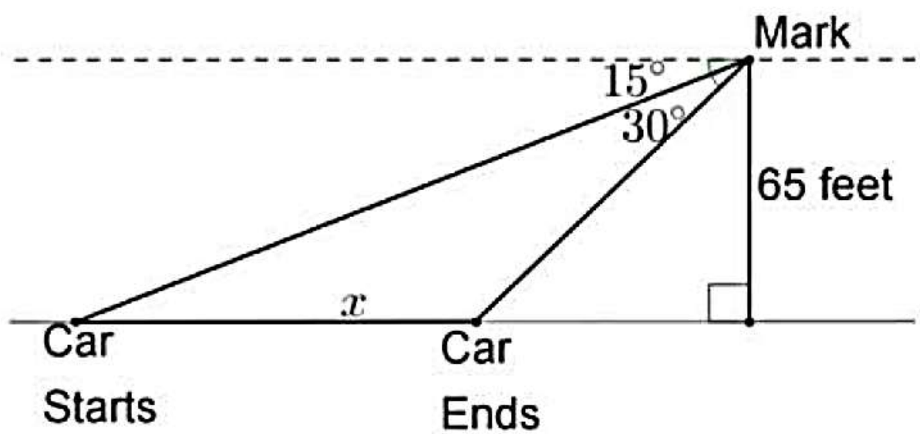
From the fourth story of a building (65 feet) Mark observes a car moving towards the building driving on the street below. If the angle of depression of the car changes from 15o to 45o while he watches, how far did the car travel?

(A) 65 ft

(A) 124 ft

(B) 178 ft

(C) 243 ft



Example 12

A surveyor measures a traverse segment AB to be 120 m long. Next he measures angles $\angle BAC = 55^o$and $\angle ABC = 45^o$ to a benchmark C. The distance AC (m) is more likely

(A) 167.18

(A) 144.3

(B) 99.81

(C) 86.16

Example 13

What is an equivalent expression for $\sin 2\alpha$?

(A) $-2\sin\alpha\cos\alpha$

(B) $\frac{1}{2}\sin\alpha\cos\alpha$

(C) $\dfrac{2\sin\alpha}{\sec\alpha}$

(D) $2\sin\alpha\cos\dfrac{\alpha}{2}$

To determine whether two trig expressions are equivalent

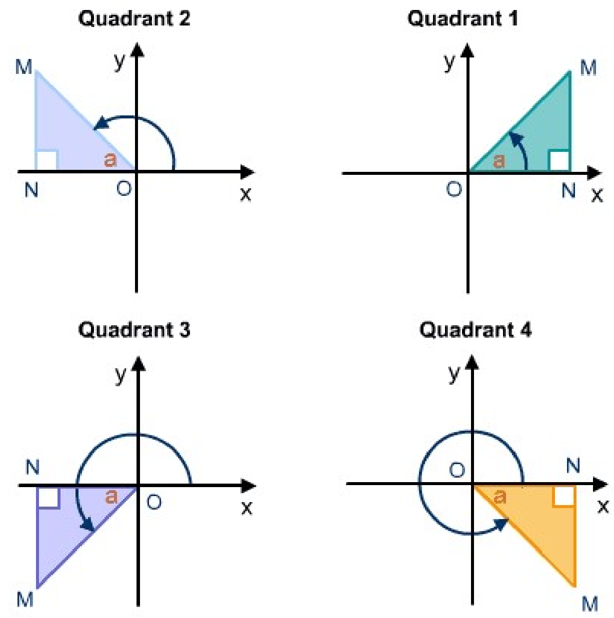
a) Use trigonometric identities on handbook, or

b) Just substitute an angle into both and see if they come out the same. DO NOT substitute special angles like 0, 30, 45,60, 90, 180, etc. - use something like 17 deg. instead.

Example 14

Which is true regarding the signs of the natural functions for angles between 90° and 180°? *(figure is not given)*

(A) The tangent is positive
(B) The cotangent is positive
(C) The cosine is negative
(D) The sine is negative



Example 15

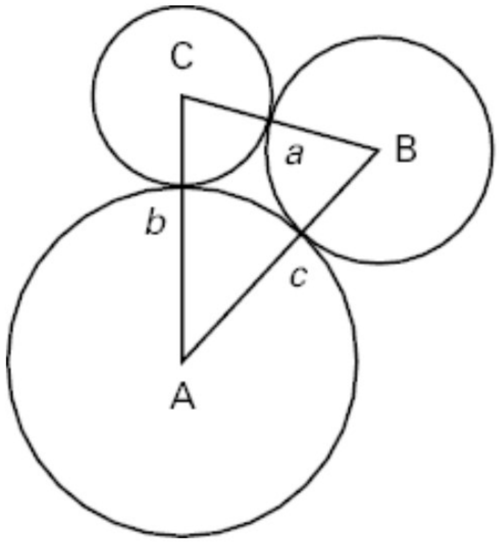
Three circles of radii 110 m, 140 m, and 220 m are tangent to one another. What are the interior angles of the triangle formed by joining the centers of the circles? *(figure is not given)*

(A) 34.2°, 69.2°, and 76.6°

(B) 36.6°, 69.1°, and 74.3°

(C) 42.2°, 62.5°, and 75.3°

(D) 47.9°, 63.1°, and 69.0°



Example 16

For some angle θ, csc θ = -8/5. What is cos 2θ?

(A) 7/32
(B) 1/4
(C) 3/8
(D) 5/8

a) Use trigonometric identities on handbook, or
b) Use calculator:

Find $\sin\theta = 1/\csc\theta = -5/8$
Find $\theta = \sin^{-1}(-5/8)$
Then find $\cos 2\theta$

6. MEASUREMENT OF AREAS

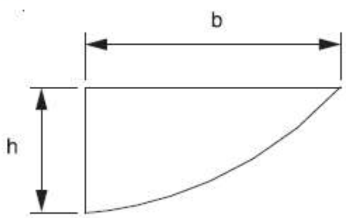
Nomenclature

A = total surface area

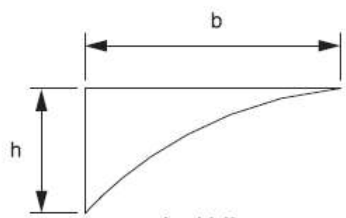
P = perimeter

V = volume

Parabola

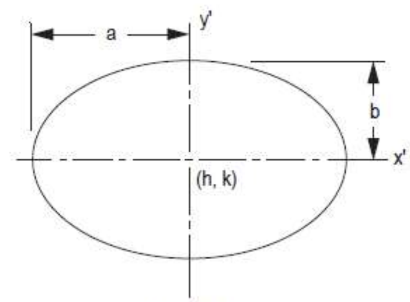


A = 2bh/3



A = bh/3

Ellipse



A = πab

$$P_{approx} = 2\pi\sqrt{(a^2 + b^2)/2}$$

$$P = \pi(a+b)\left[\begin{array}{l} 1 + (1/2)^2\lambda^2 + (1/2 \times 1/4)^2\lambda^4 \\ +(1/2 \times 1/4 \times 3/6)^2\lambda^6 + (1/2 \times 1/4 \times 3/6 \times 5/8)^2\lambda^8 \\ +(1/2 \times 1/4 \times 3/6 \times 5/8 \times 7/10)^2\lambda^{10} + \ldots \end{array}\right],$$

where

$$\lambda = (a-b)/(a+b)$$

Example 17

Find the area of the region defined by the following expressions:

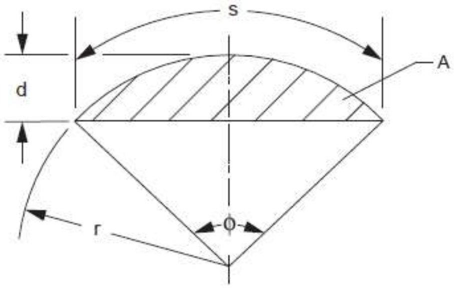
$y > 3x^2 - 2$

$y < 5$ and $x > 0$

(A) 7.1
(B) 5.6
(C) 4.8
(D) 3.6

Circular Segment

♦

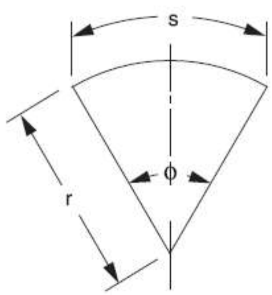


$$A = \left[r^2(\phi - \sin\phi)\right]/2$$

$$\phi = s/r = 2\left\{\arccos\left[(r - d)/r\right]\right\}$$

Circular Sector

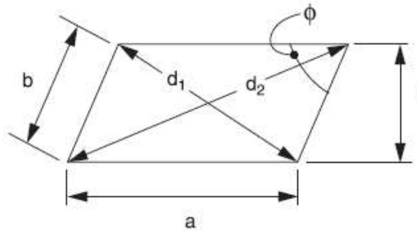
♦



$$A = \phi r^2/2 = sr/2$$

$$\phi = s/r$$

Parallelogram



$$P = 2(a + b)$$

$$d_1 = \sqrt{a^2 + b^2 - 2ab(\cos\phi)}$$

$$d_2 = \sqrt{a^2 + b^2 + 2ab(\cos\phi)}$$

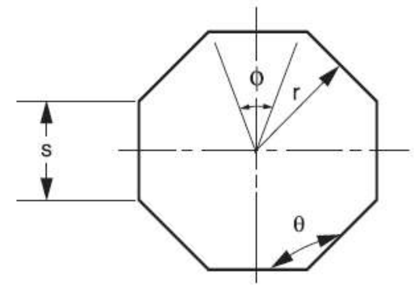
$$d_1^2 + d_2^2 = 2\left(a^2 + b^2\right)$$

$$A = ah = ab(\sin\phi)$$

If $a = b$, the parallelogram is a rhombus.

Regular Polygon (n equal sides)

♦



$$\phi = 2\pi/n$$

$$\theta = \left[\frac{\pi(n - 2)}{n}\right] = \pi\left(1 - \frac{2}{n}\right)$$

$$P = ns$$

$$s = 2r\left[\tan(\phi/2)\right]$$

$$A = (nsr)/2$$

Example 19

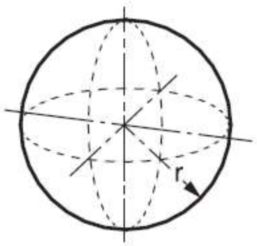
Sewer pipelines are designed to run partially full at maximum discharge. Find the area of flow when a 8 ft-diameter pipeline runs water at a depth of 6 ft.

(A) 22.6 ft^2
(B) 40.4 ft^2
(C) 60.8 ft^2
(D) 72.5 ft^2

7. VOLUMES

Sphere

♦

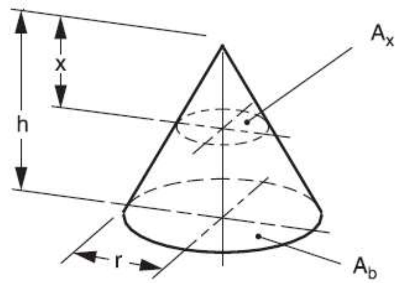


$$V = 4\pi r^3/3 = \pi d^3/6$$

$$A = 4\pi r^2 = \pi d^2$$

Right Circular Cone

♦



$$V = (\pi r^2 h)/3$$

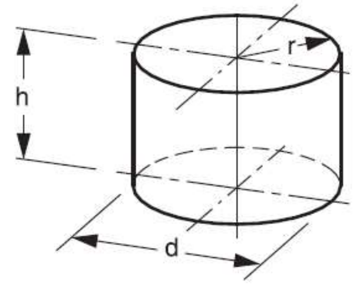
$$A = \text{side area} + \text{base area}$$

$$= \pi r\left(r + \sqrt{r^2 + h^2}\right)$$

$$A_x : A_b = x^2 : h^2$$

Right Circular Cylinder

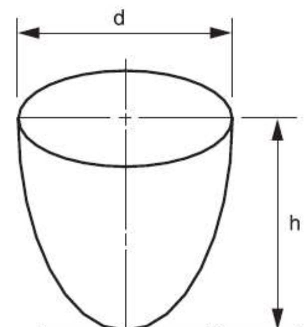
♦



$$V = \pi r^2 h = \frac{\pi d^2 h}{4}$$

$$A = \text{side area} + \text{end areas} = 2\pi r(h + r)$$

Paraboloid of Revolution



$$V = \frac{\pi d^2 h}{8}$$

Example 20

A line $y = 2x$ is rotated about its y – axis. The volume of the revolved solid between $y = 0$ and $y = 10$ is:

(A) 50 $\pi/3$
(B) 50 π
(C) 250 $\pi/3$
(D) 100 π