Open Channel and Pressurized flow
Governing Equations

Irrawaddy River, Myanmar

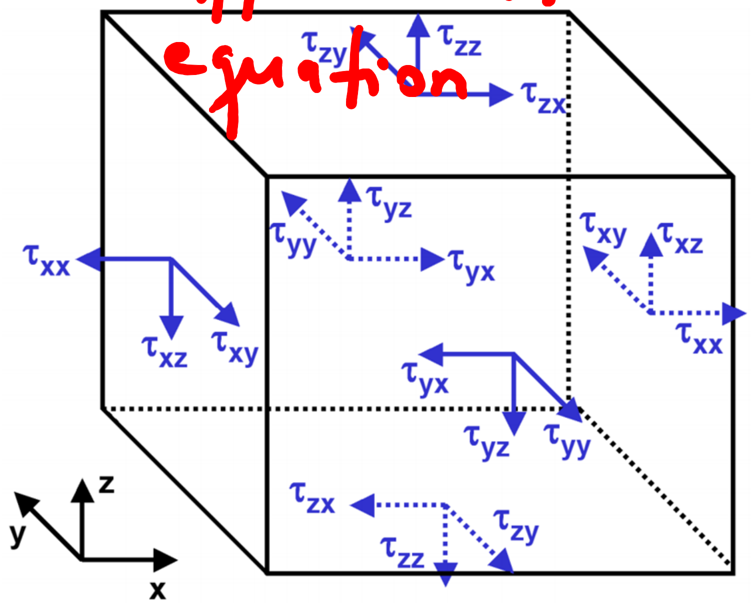
Arturo S. Leon, Ph.D, P.E., D.WRE
Florida International University

Navier-Stokes Equations

Partial differential equation

A system of 4 nonlinear PDE of mixed hyperbolic-parabolic type describing the fluid hydrodynamics in 3D

(**Continuity** and **Momentum**)



Operators

Gradient:

$$\nabla \boldsymbol{u} = \left(\partial \boldsymbol{u} / \partial \mathbf{x}, \partial \boldsymbol{u} / \partial \mathbf{y}, \partial \boldsymbol{u} / \partial \mathbf{z} \right)$$

Divergence:

$$\mathbf{div}\, \vec{\boldsymbol{u}} = \nabla \cdot \vec{\boldsymbol{u}} = \partial \boldsymbol{u}_x / \partial \mathbf{x} + \partial \boldsymbol{u}_y / \partial \mathbf{y} + \partial \boldsymbol{u}_z / \partial \mathbf{z}$$

Laplacian:

$$\nabla^2 \boldsymbol{f} = \partial^2 \boldsymbol{f} / \partial \mathbf{x}^2 + \partial^2 \boldsymbol{f} / \partial \mathbf{y}^2 + \partial^2 \boldsymbol{f} / \partial \mathbf{z}^2$$

Newton's Second Law: $\Sigma\vec{F} = m \cdot \vec{a}$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad (1)$$

$a = \frac{dV}{dt}$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \qquad (2)$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \qquad (3)$$

$$a_i = \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}$$

Local acceleration

convective acceleration

Notation: Einstein

i,i [indicates
j,j   sum]

$$f_x = \overset{\text{Body force}}{g_x} + \frac{1}{\rho}\left[\overset{\text{Pressure force}}{-\frac{\partial p}{\partial x}} + \overbrace{\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}^{\text{Shear force}}\right] \quad (4)$$

$$f_y = g_y + \frac{1}{\rho}\left[-\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right] \quad (5)$$

$$f_z = g_z + \frac{1}{\rho}\left[-\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right] \quad (6)$$

normal stress

The **stresses are related to fluid element displacements** by invoking the **Stokes viscosity law** for an incompressible fluid.

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x},\ \tau_{yy} = 2\mu\frac{\partial v}{\partial x},\ \tau_{zz} = 2\mu\frac{\partial w}{\partial x} \qquad (7)$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (8)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad (9)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad (10)$$

Substituting eqs. 7-10 into eqs. 4-6, we get

$$f_x = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \quad (11)$$

$$f_y = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] \quad (12)$$

$$f_z = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] \quad (13)$$

$$f_i = g_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

Using Einstein notation

The three Navier-Stokes momentum equations can be written in compact form as

Momentum

$$\boxed{\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{-1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + g_i} \quad \text{(A1)}$$

The equation of continuity for an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

expanding

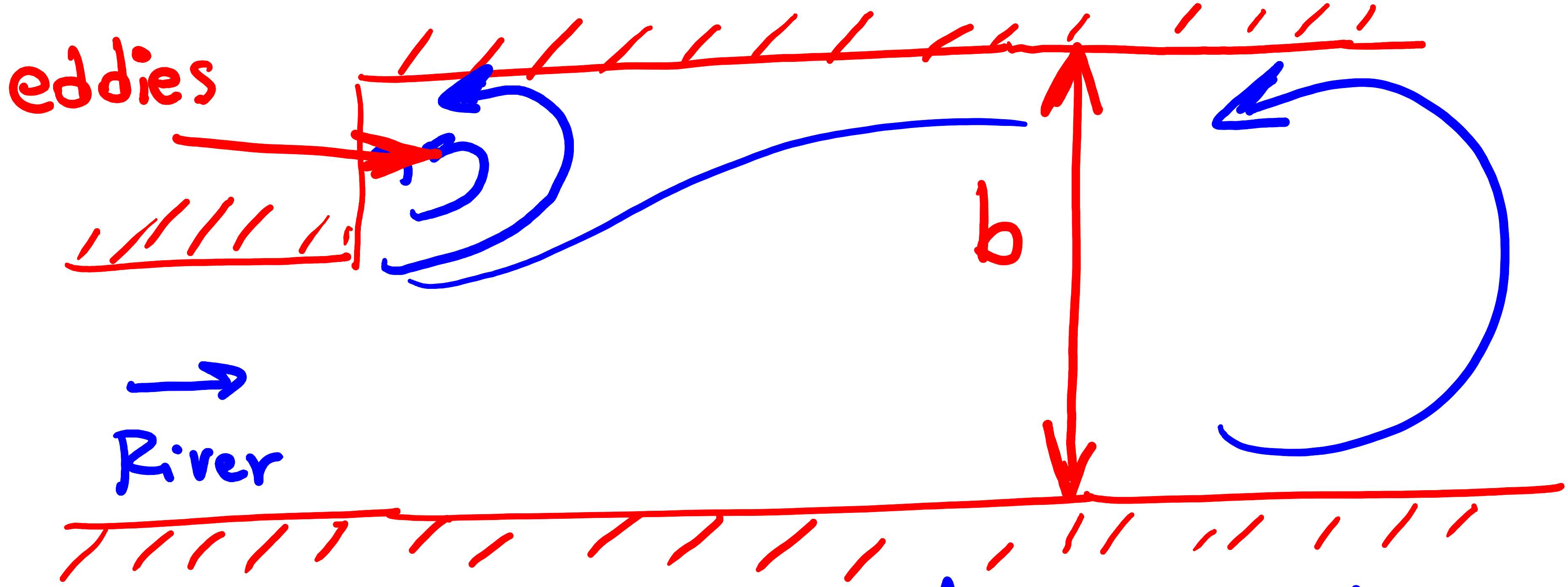
$$\frac{\partial u_i}{\partial x_i} = 0 \qquad \text{(A2)}$$

Continuity

Turbulence

Free surface flows occurring in nature are almost always turbulent. A few characteristic of turbulence are:

1. Irregularity: The **flow consists of a spectrum of different scales** (eddy sizes) where largest eddies are of the order of the flow geometry (i.e. flow depth or width, etc). At the other end of the spectra we have the **smallest eddies which are by viscous forces** (stresses) dissipated into internal energy.



Largest eddy scales with "b"

Smallest eddies are dissipated into internal energy

2. Diffusivity:

The **turbulence increases the exchange of momentum** in flow thereby increasing the resistance (wall friction).

3. Large Reynolds Number:

Turbulent flow occurs at high Reynolds number. For example, the transition to turbulent flow in pipes occurs at R~2000.

$$Re = \frac{V \cdot D}{\nu} \quad [\text{pipes}]$$

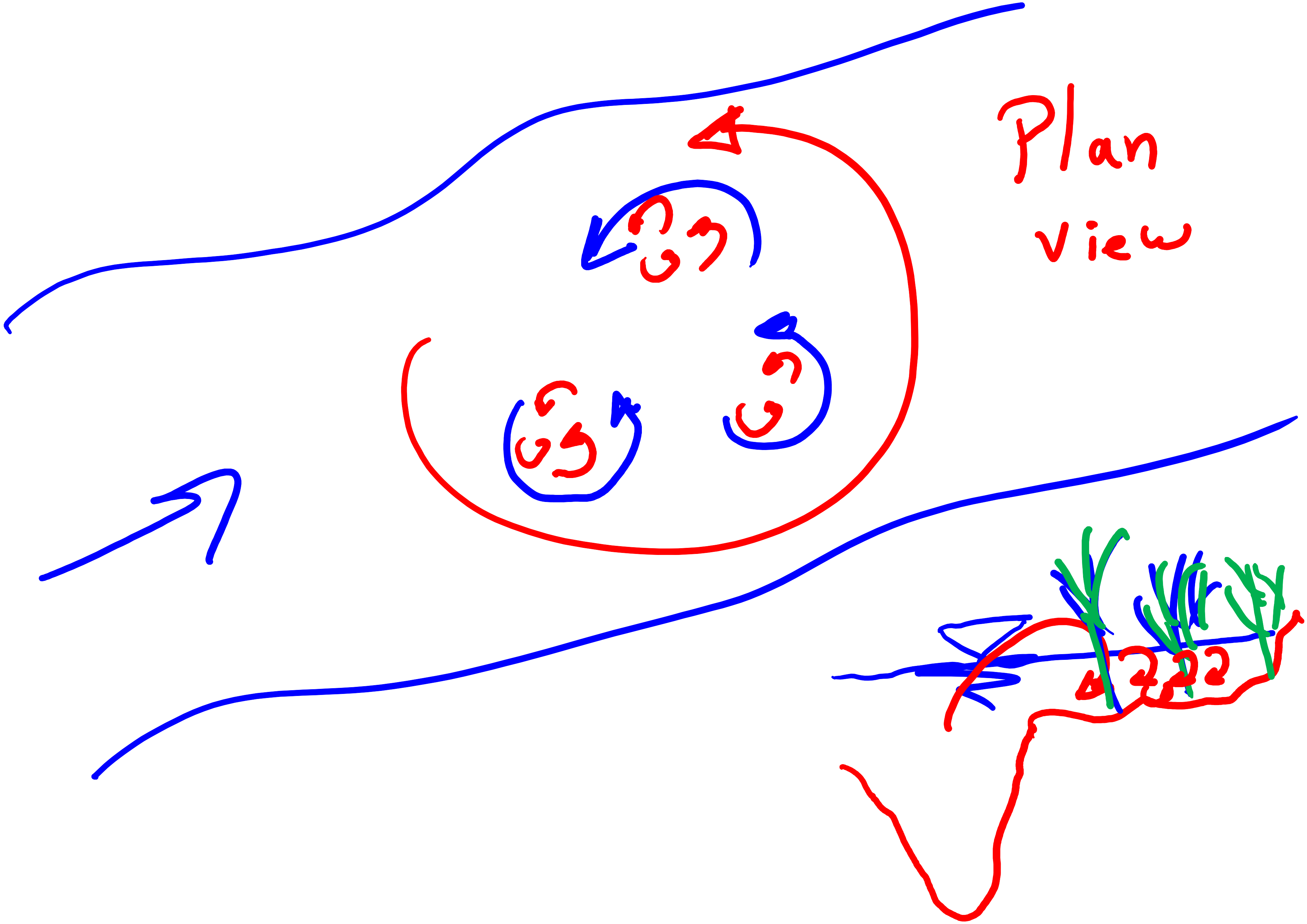
V: velocity

D: Diameter of pipe

ν: Kinematic viscosity

4. Dissipation:

Turbulent flow is dissipative, which means that kinetic energy in the **small eddies are transformed into internal energy**. The small eddies receive the kinetic energy from slightly larger eddies. The **slightly larger eddies receive their energy from even larger eddies** and so on. The **largest eddies extract their energy from the mean flow**. This process of transferred energy from the largest turbulent scales (eddies) to the smallest is called **cascade process**.



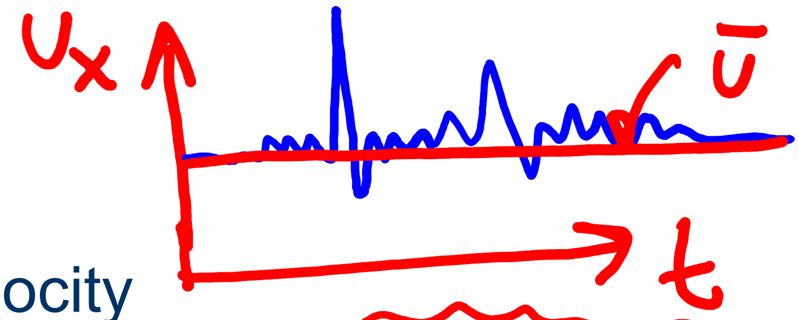
Plan view

Side view

$\bar{U}$: average velocity

u' : velocity fluctuation

Turbulence



u, v, w, p instantaneous values of velocity components and pressure, respectively

u', v', w', p' are fluctuations of velocity components and pressure, respectively

$$u = \bar{u} + u'$$

$$v = \bar{v} + v'$$

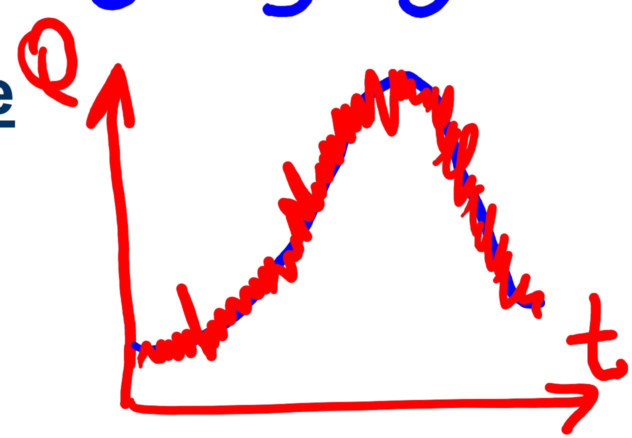
$$w = \bar{w} + w'$$

$$p = \bar{p} + p'$$

Why decompose variables?

- We are usually **interested in the mean values rather than the time histories**.
- When we want to solve the Navier-Stokes equation numerically, **it would require a very fine grid to resolve all turbulent scales** and it would also require a **fine time resolution** since turbulent flow is always unsteady.

$$y = \bar{y} + y'$$



Reynolds Time-averaged Navier-Stokes (RNS) Equations

local acceleration

Convective acceleration

pressure gradient

$$\underbrace{\frac{\partial \bar{u}}{\partial t}}_{\text{local acceleration}} + \underbrace{\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}}_{\text{convective acceleration}} = g_x - \underbrace{\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}}_{\text{pressure gradient}} + \frac{\partial}{\partial x}\left\{-\rho\overline{u'u'} + \mu\frac{\partial \bar{u}}{\partial x}\right\} +$$

$$\frac{\partial}{\partial y}\left\{-\rho\overline{u'v'} + \mu\frac{\partial \bar{u}}{\partial y}\right\} + \frac{\partial}{\partial z}\left\{-\rho\overline{u'w'} + \mu\frac{\partial \bar{u}}{\partial z}\right\}$$

Momentum equation in x-direction

Reynold stresses

viscous stresses

We can write similar equations for the Momentum in the *y*- and *z*-directions.

$\overline{u'} = 0$

$\overline{u'u'} \neq 0$

RNS Equations (Cont.)

Momentum

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \tau_{vij}}{\partial x_j} + \frac{1}{\rho}\frac{\partial \tau_{Rij}}{\partial x_j} + g_i$$

$$\frac{\partial \bar{u}_i}{\partial x} = 0$$

continuity

Molecular viscosity

where $\tau_{vij} = \nu \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)$

Viscous stress tensor

$$\tau_{Rij} = -\rho \overline{u_i' u_j'}$$

Reynold stress tensor

$\tau_{Rij} \gg \tau_{vij}$

for turbulent flows

Closure Problem

- 3 velocity components, one pressure and 6 Reynold stress terms = 10 unknowns
- No. of equations=4 [Mx, My, Mz, continuity]
- We need to close the problem to obtain a solution.
- Turbulence modeling tries to represent the **Reynold stresses** in terms of **the time-averaged velocity components.**

<u>Unknowns :</u> U, V, W, P

$\overline{u'u'}, \overline{u'v'}, \overline{u'w'}, \overline{w'w'}, \overline{v'v'}, \overline{v'w'}$

\# equations available : 4

\# unknowns : 10

Turbulence Models

Boussinesq Model

An algebraic equation is used to compute a turbulent viscosity, often called ***eddy viscosity***. The **Reynolds stress tensor is expressed in terms of the time-averaged velocity gradients and the turbulent viscosity**.

turbulent viscosity

$$-\overline{u_i' u_j'} = \nu_t \left(\frac{\partial \overline{U}_j}{\partial x_i} + \frac{\partial \overline{U}_i}{\partial x_j} \right)$$

Widely used in practice

k-ε Turbulence Model

<u>Two transport equations are solved</u> which describe the transport of the **<u>turbulent kinetic energy, *k*</u>** and its **<u>dissipation, *ε*</u>**. The eddy viscosity is calculated as

k : Kinetic energy

ε : dissipation

k

ε

$$\nu_t = c_\mu \frac{k}{\varepsilon^2}$$

RNS Equations and River Flow Simulation

widely

RNS equations are not used often for river flow simulation because of **computational burden**.

In practice

Method of choice for unsteady flows in rivers, streams and overland flow is typically **1D Saint-Venant equations or 2D shallow-water equations**

2D Saint Venant Equations

- Obtained from RNS equations by depth-averaging.
- Suitable for modeling flow over floodplains, flow over a dyke and flow through a breach.
- Assumptions: **hydrostatic pressure distribution, small channel slope**

$S < \sim 10\%$

2D Saint Venant Equations (cont.)

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}+\frac{\partial (hv)}{\partial y}=0$$

Continuity Equation

$$\frac{\partial (hu)}{\partial t}+\frac{\partial (hu^2)}{\partial x}+\frac{\partial (huv)}{\partial y}+gh\frac{\partial h}{\partial x}=-gh\frac{\partial z_b}{\partial x}-gn^2u\frac{\sqrt{u^2+v^2}}{h^{1/3}}$$

M_x

$$\frac{\partial (hv)}{\partial t}+\frac{\partial (huv)}{\partial x}+\frac{\partial (hv^2)}{\partial y}+gh\frac{\partial h}{\partial y}=-gh\frac{\partial z_b}{\partial y}-gn^2v\frac{\sqrt{u^2+v^2}}{h^{1/3}}$$

M_y

Unknowns: u, v, h

n: Manning's roughness

1D Saint-Venant Equation (Cont.)

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0$$

continuity

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}+g\frac{\partial h}{\partial x}=g(S_0-S_f)$$

Mx

The 1D Saint-Venant equations can be written in conservative form as: $\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}}{\partial x}=\mathbf{S}$

conservative form

S_f, S_e [friction slope or energy slope]

$$\underbrace{\mathbf{U}=\begin{bmatrix}\rho_w A\\ \rho_w Q\end{bmatrix}}_{\text{used in man-made channels}},\quad \mathbf{F}=\begin{bmatrix}\rho_w Q\\ \rho_w\frac{Q^2}{A}+A\bar{p}+\rho_w g A_f H\end{bmatrix}$$

$$and\quad \mathbf{S}=\begin{bmatrix}0\\ (S_o-S_e)\rho_w g A\end{bmatrix}$$

Source: Leon, A. S., Ghidaoui, M. S., Schmidt, A. R. and Garcia, M. H. (2010) “A robust two-equation model for transient mixed flows.” *Journal of Hydraulic Research*, 48(1), 44-56

Simplified Equations of Saint-Venant

(Non-conservative)

Local acceleration

Convective acceleration

pressure gradient

gravity

friction

$$\underbrace{\frac{\partial V}{\partial t} + \underbrace{V\frac{\partial V}{\partial x} + \underbrace{g\frac{\partial h}{\partial x} = \underbrace{g(S_o - S_f)}_{\text{kinematic wave}}}_{\text{diffusive wave}}}_{\text{dynamic quasi-steady wave}}}_{\text{dynamic wave}}$$

Relative Weight of Each Term in SV Equation

Order of magnitude of each term in SV equation for a flood on river Rhone (**Fr** ~ 0.50)

$$\frac{1}{g}\frac{\partial V}{\partial t} \quad O(10^{-5})$$
Local acceleration

$$\frac{V}{g}\frac{\partial V}{\partial x} \quad O(10^{-5})$$
Convective accele-ration

$$g\frac{\partial h}{\partial x} \quad O(10^{-3})$$
Pressure gradient

$$S_o \quad O(10^{-3})$$
gravity

$$S_f \quad O(10^{-3})$$
friction

Source: Henderson, 1966 (Open-Channel Flow)

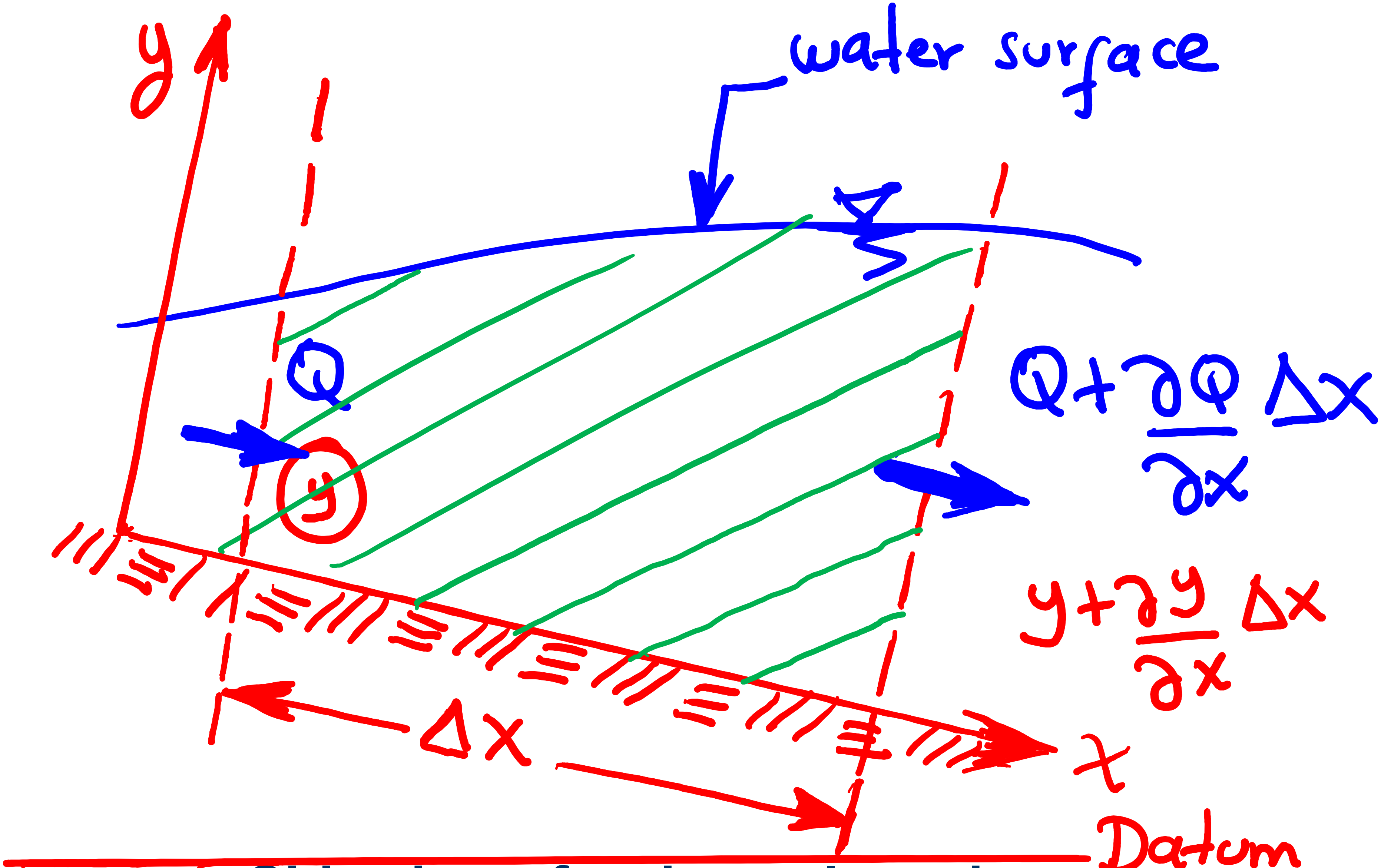
One-Dimensional Modeling:

Most unsteady flows are well represented by one-dimensional theory because, in most cases, the **component of the motion along the channels is far greater than the transverse motion**. One-dimensional models are best suited for in-channel flows and when floodplain flows are minor.

Two-Dimensional Modeling:

Two-dimensional models should be used when **transverse motion is important** such as when inundation involves relatively large floodplain areas.

Derivation of the Saint-Venant equations



Side view of a channel reach

Derivation (Cont.)

$$\Sigma I - \Sigma O = \frac{\Delta S}{\Delta t}$$

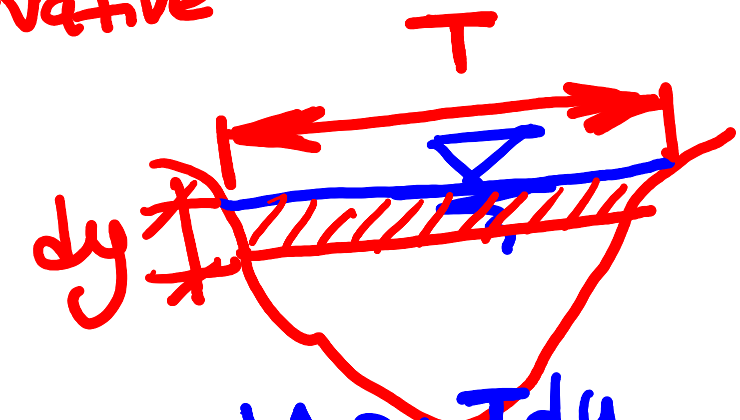
Continuity Equation :

$$\cancel{Q} - \left(\cancel{Q} + \frac{\partial Q}{\partial x}\Delta x\right) = \frac{\partial}{\partial t}\left(A\,\Delta x\right)$$

$$\cancel{\Delta x}\,\frac{\partial A}{\partial t} + \cancel{\Delta x}\,\frac{\partial Q}{\partial x} = 0$$

$$\boxed{\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0} \quad \ldots (1)$$

Conservative form



$$dA \approx T\,dy$$

$$* \quad \frac{\partial A}{\partial t} = \overbrace{\frac{\partial A}{\partial y}}^{\approx T}\frac{\partial y}{\partial t} \approx T\,\frac{\partial y}{\partial t}$$

Derivation (Cont.)

$$* \quad \frac{\partial Q}{\partial x} = \frac{\partial (A \cdot V)}{\partial x} = A\frac{\partial V}{\partial x} + V\frac{\partial A}{\partial x}$$

In (1)

$$\boxed{V\frac{\partial A}{\partial x} + A\frac{\partial V}{\partial x} + T\frac{\partial y}{\partial t} = 0}$$

Non-conservative form

* Momentum equation

$$\sum \vec{F} = m \cdot \vec{a} \quad [\text{Newton's } 2^{nd} \text{ Law}]$$

(m) $\rho = \frac{m}{\forall} \rightarrow \boxed{m = \rho A \Delta x}$

Derivation (Cont.)

$V = U_x$

$$a = \frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x}$$

V

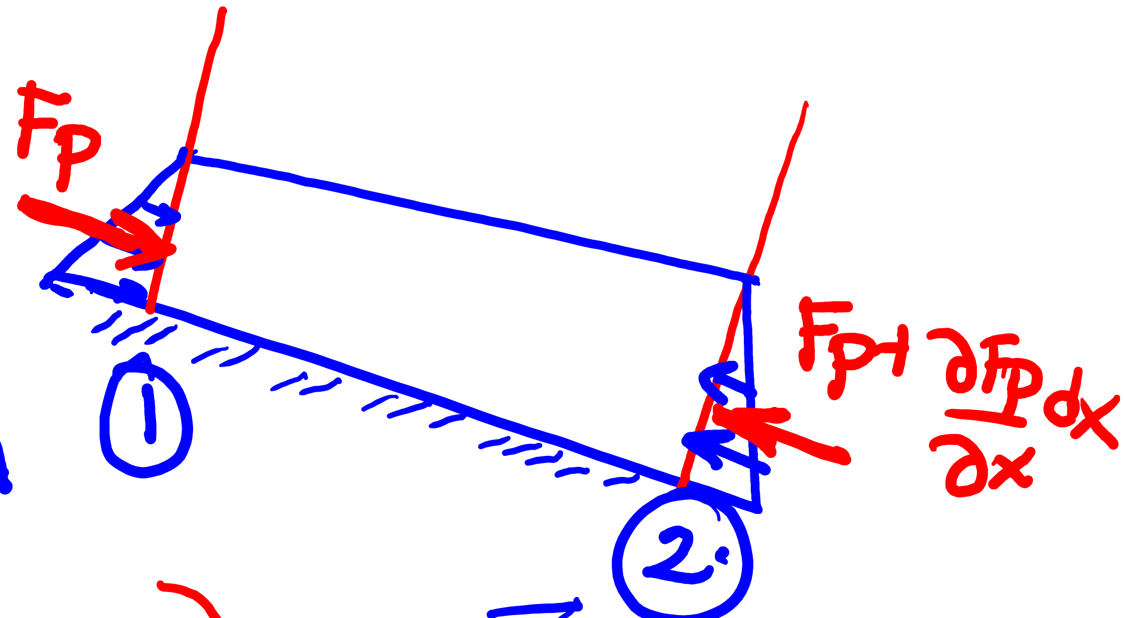
* **Forces**

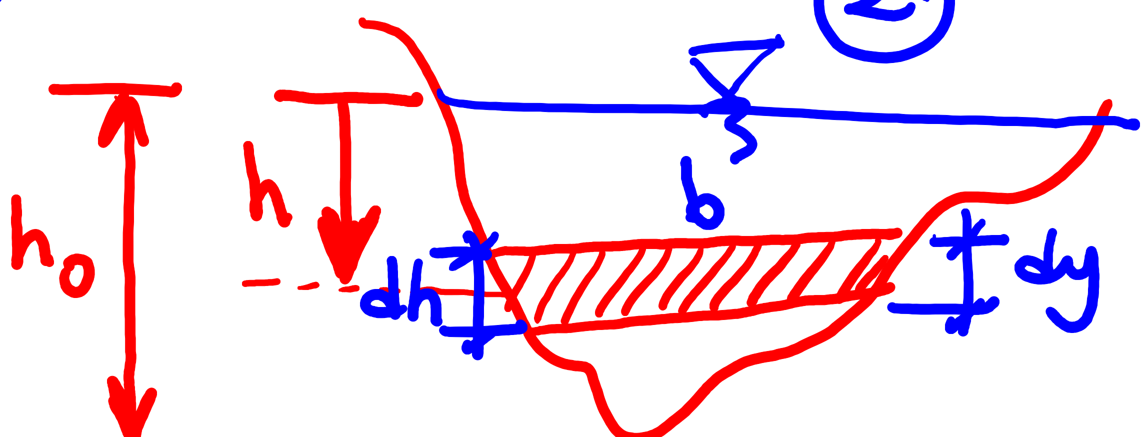
Pressure force

$$F_p = \int_0 \gamma h b\, dh \quad \ldots (2)$$

$$\boxed{\bar{h}\cdot A = \int h b\, dh} = \int h\, dA$$

$$\bar{h} = \int h\, dA / A$$





Derivation (Cont.)

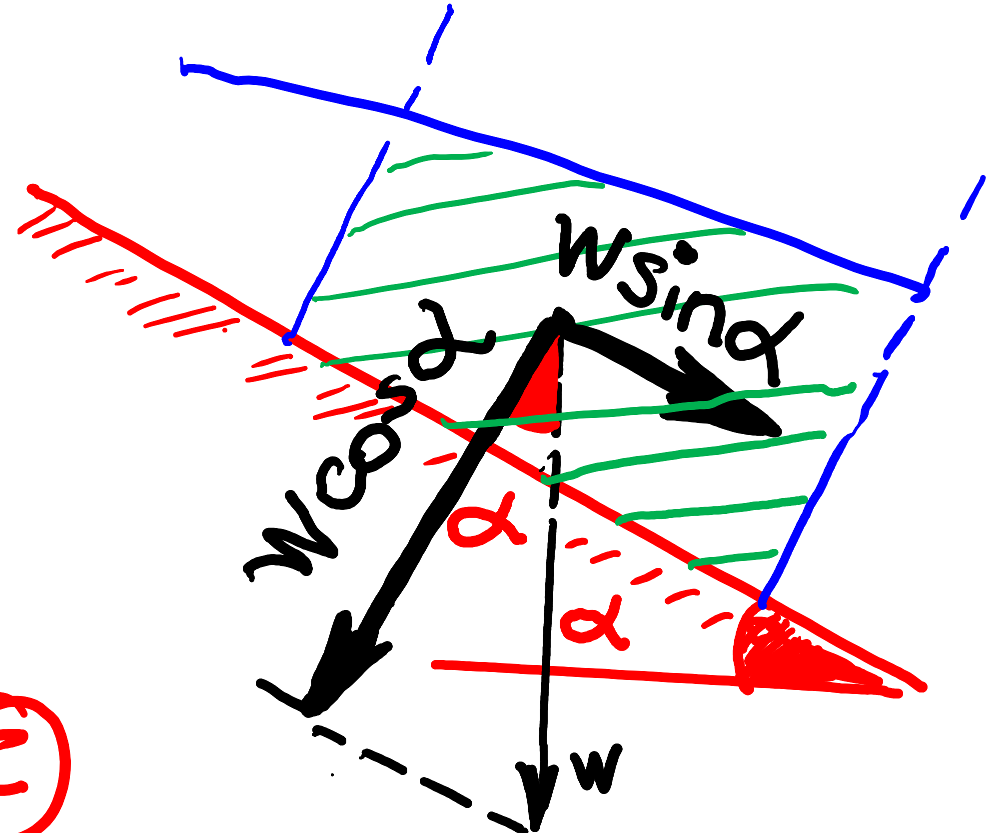
In ② $\boxed{F_p = \gamma \bar{h} A}$

$$F_{p_1} - F_{p_2} = \cancel{F_p} - \left(\cancel{F_p} + \frac{\partial F_p}{\partial x}\Delta x\right)$$

$$= -\frac{\partial(\gamma \bar{h} A)}{\partial x}\,\Delta x$$

$$= -\gamma \Delta x \frac{\partial}{\partial x}(\bar{h}\cdot A)$$

Derivation (Cont.)



* Gravity force

$$F_g = \gamma A \Delta x \sin\alpha$$
$$= \rho g A \Delta x \sin\alpha$$

* Friction force

$$F_F = \tau p \Delta x = \underbrace{\gamma R S_f}_{\tau} \, p \Delta x \qquad R = \frac{A}{P}$$

$$= \gamma \frac{A}{\cancel{P}} S_f \cdot \cancel{P} \Delta x = \underline{\gamma A \Delta x S_f}$$

Negative force because is against the flow

Derivation (Cont.)

Combining forces:

$$\frac{\cancel{(g)}}{(g)}\cancel{\rho} A \cancel{\Delta x}\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -\cancel{\gamma}\cancel{\Delta x}\frac{\partial}{\partial x}(A\bar{h})$$
$$+\cancel{\rho g} A \cancel{\Delta x} \sin\alpha - \cancel{\gamma} A \cancel{\Delta x}\, S_f$$

Dividing by A

$$\frac{1}{g}\left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right] = -\frac{1}{A}\frac{\partial}{\partial x}(A\bar{h}) + \sin\alpha - S_f$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{g}{A}\frac{\partial}{\partial x}(A\bar{h}) - g\sin\alpha + g S_f = 0$$

(*)

For small α, $\cos\alpha \approx 1.0$
$\sin\alpha \approx \tan\alpha$

Derivation (Cont.)

Slope $(S_o = \tan\alpha)$ $\therefore$ $S_o \approx \sin\alpha$

$g\sin\alpha \approx g S_o$

In (*)

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{g}{A}\underline{\frac{\partial}{\partial x}(A\bar{h})} - g(S_o - S_f) = 0$$

$$\frac{\partial}{\partial x}(A\bar{h}) = \frac{\partial}{\partial x}\left(\cancel{A}\cdot\frac{\int h\,dA}{\cancel{A}}\right) = \frac{\partial}{\partial x}\left(\int h\,dA\right)$$

Derivation (Cont.)

$f = h, \quad dy = dA$

$x = x, \quad t = t$

Leibnitz rule:

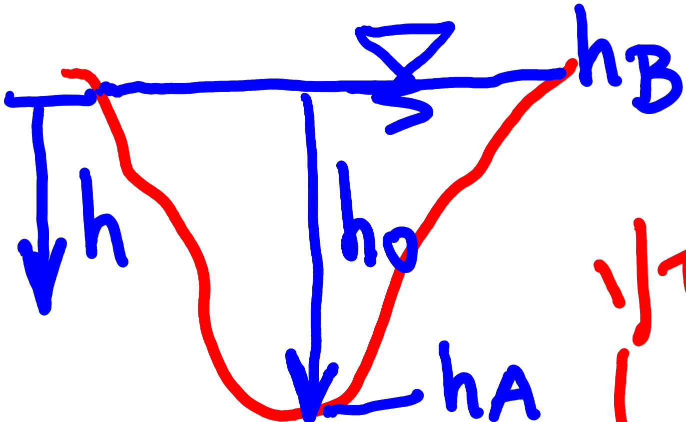
$$\int_{A(x,t)}^{B(x,t)} \frac{\partial f}{\partial x}(x,y,t)\,dy = \frac{\partial}{\partial x}\int_{A(x,t)}^{B(x,t)} f(x,y,t)\,dy - f(B)\frac{\partial B}{\partial x} + f(A)\frac{\partial A}{\partial x}$$

$$\therefore \frac{\partial}{\partial x}\left(\int_0^A h\,dA\right) = \int \frac{\partial h}{\partial x}\,dA + \cancel{h_B}^{\,0}\frac{\partial A}{\partial x} \;\; 0$$

Derivation (Cont.)

$$-h_A \cancel{\frac{\partial o}{\partial x}}$$

0



$1/T$

$$\frac{\partial}{\partial x}\int h\,dA = \int \frac{\partial h}{\partial x}\,dA$$

$$\therefore \int \frac{\partial h}{\partial x}\,dA = \int \frac{\partial h}{\partial A}\frac{\partial A}{\partial x}\,dA = \int \frac{1}{T}\frac{\partial A}{\partial x}\,dA = \frac{1}{T}\frac{\partial A}{\partial x}\underbrace{\int dA}_{A}$$

$$= \cancel{\frac{1}{T}\frac{\partial A}{\partial h}}\frac{\partial h}{\partial x}\cdot A = A\frac{\partial h}{\partial x}$$

$$\therefore \boxed{\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial x} = g(S_o - S_f)}$$

Used in river modeling

Compressible Waterhammer Flow Equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}$$

$$\mathbf{U} = \begin{bmatrix} \rho_f A_f \\ \rho_f Q \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho_f Q \\ \rho_f \dfrac{Q^2}{A_f} + A_f p \end{bmatrix} \quad and$$

$$\mathbf{S} = \begin{bmatrix} 0 \\ (S_o - S_e)\rho_f g A_f \end{bmatrix} \quad p = p_{ref} + a^2(\rho_f - \rho_{ref})$$

Source: Leon, A. S., Ghidaoui, M. S., Schmidt, A. R. and Garcia, M. H. (2010) "A robust two-equation model for transient mixed flows." *Journal of Hydraulic Research*, 48(1), 44-56.

Coupled open-channel and pressured unsteady flows

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}$$

$$\mathbf{U} = \begin{bmatrix} \rho_w A \\ \rho_w Q \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho_w Q \\ \rho_w \dfrac{Q^2}{A} + A\bar{p} + \rho_w g A_f H \end{bmatrix}$$

$$and \quad \mathbf{S} = \begin{bmatrix} 0 \\ (S_o - S_e)\rho_w g A \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} \rho_f A_f \\ \rho_f Q \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho_f Q \\ \rho_f \dfrac{Q^2}{A_f} + A_f p \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 0 \\ (S_o - S_e)\rho_f g A_f \end{bmatrix}$$

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