Overview of Finite Volume Methods for Solution of the Shallow Water Equations in 1D and 2D

Prof. Arturo S. Leon, Ph.D., P.E., D.WRE
Florida International University

Evolving from Finite Difference (FD) to Finite Volume (FV)

- Over the last several decades, the shallow water equations in 1D and 2D were solved mostly using Finite Difference (FD) techniques.
- Since about a decade ago (~2005), there is more emphasis on using Finite-Volume (FV) methods for the solution of the shallow water equations in 1D and 2D
- A FV solution approach, similar to what was added for 2D modeling will be available for 1D modeling in HEC-RAS version 5.1

1D HEC-RAS (< V. 5.1)

Preissmann Scheme (Finite Difference)

- This method has been widely used (e.g., ***HEC-RAS)***
- The advantage of this method is that variable spatial grid may be used
- Steep wave fronts may be properly simulated by varying the weighting coefficient

Preissmann Scheme cont…

$$\frac{\partial f}{\partial t}=\frac{(f_i^{k+1}+f_{i+1}^{k+1})-(f_i^{k}+f_{i+1}^{k})}{2\Delta t}$$

$$\frac{\partial f}{\partial x}=\frac{\alpha(f_{i+1}^{k+1}-f_i^{k+1})}{\Delta x}+\frac{(1-\alpha)(f_{i+1}^{k}-f_i^{k})}{\Delta x}$$

$$f=\frac{1}{2}\alpha(f_{i+1}^{k+1}+f_i^{k+1})+\frac{1}{2}(1-\alpha)(f_{i+1}^{k}+f_i^{k})$$

- Where α is a weighting coefficient
- By selecting a suitable value for α, the scheme may be made totally explicit (α=0) or implicit (α=1)
- Usually, the scheme is stable if $0.6< \alpha \leq 1$.

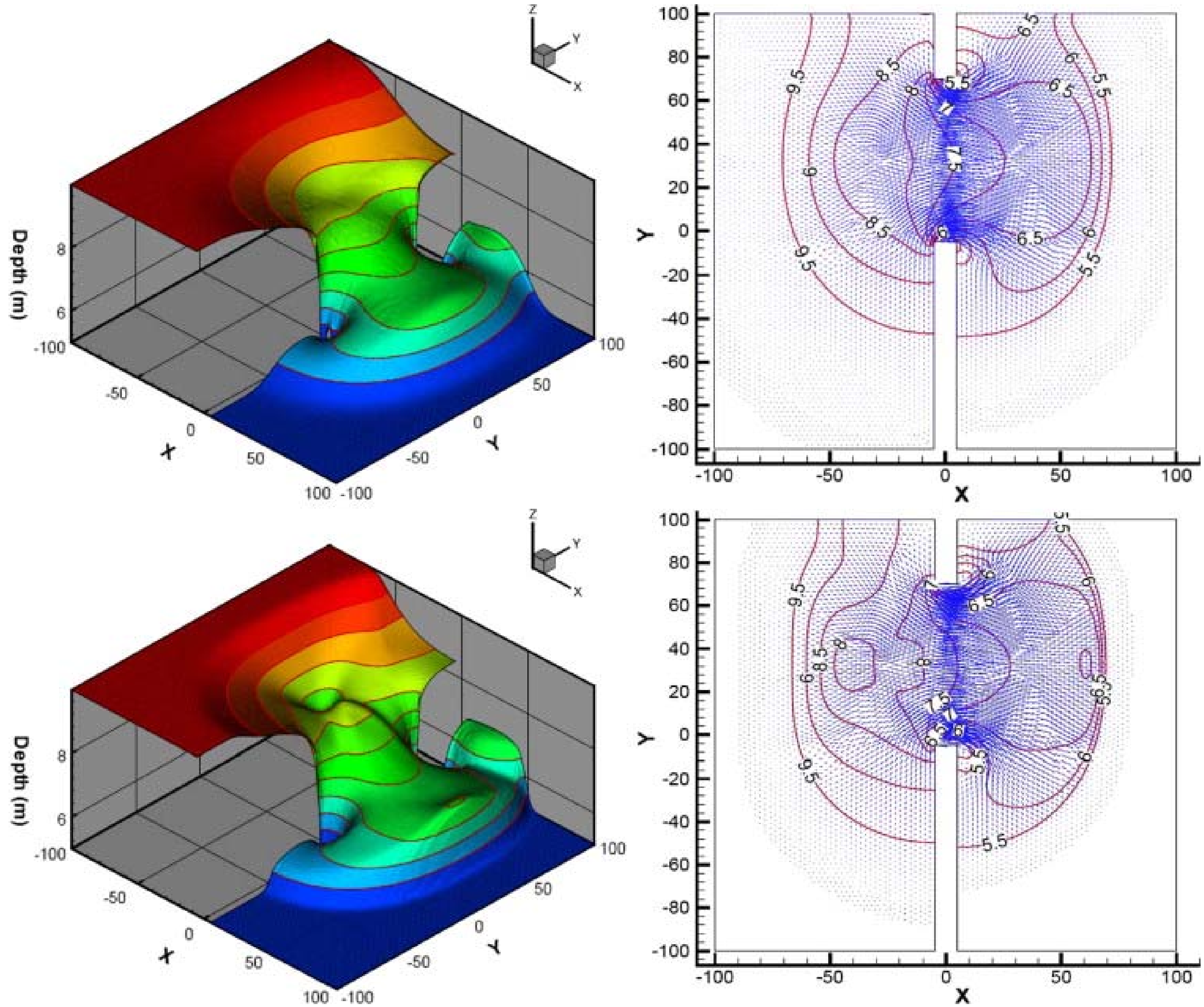
Finite Volume Methods (1D-2D)

Adapted from Lecture Notes on Shallow-Water equations by Andrew Sleigh, Toro (1999,2001) and Leon et al. (2006, 2010)

Finite Volume Shock-Capturing Methods

- Ability to handle extreme flows
- Transitions between subcritical / supercritical flows are easily handled
  - Other techniques have problems with trans-critical flows
- Steep wave fronts can be accurately simulated

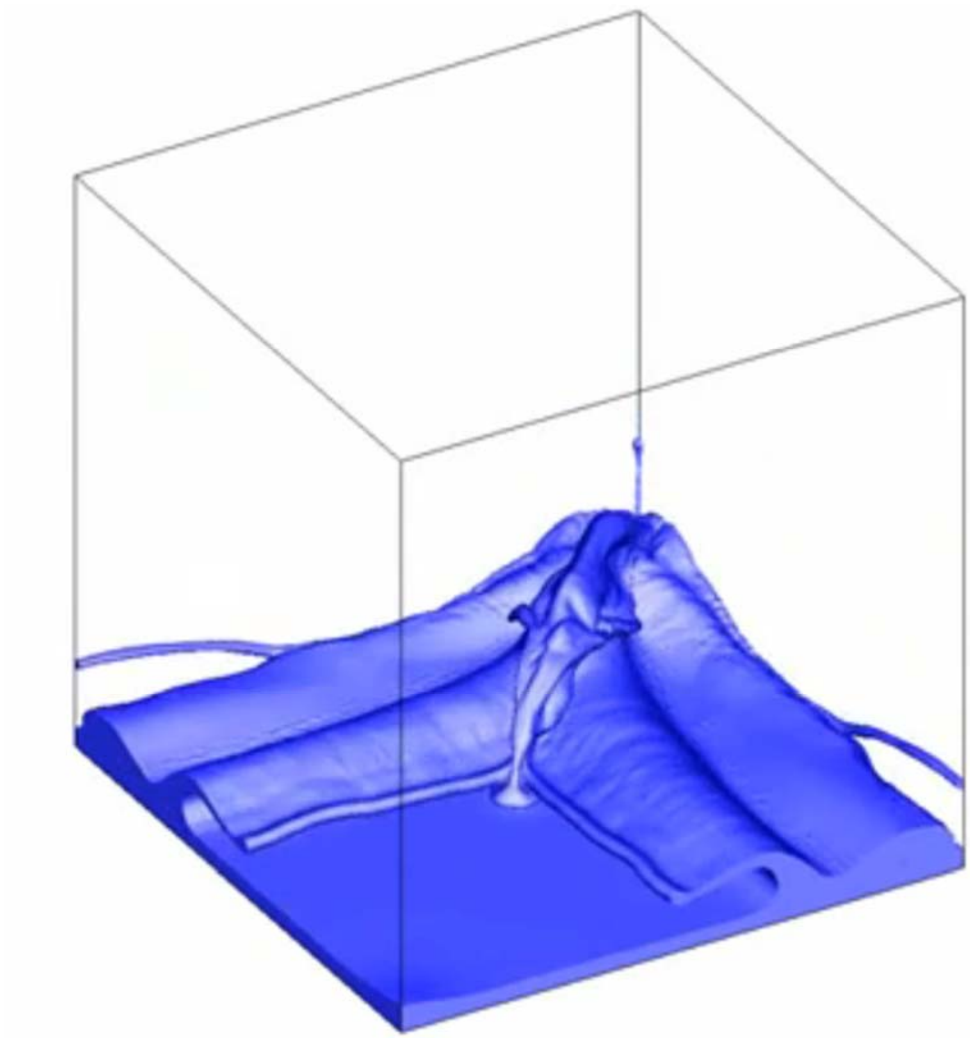
Dam break



Source: An unstructured node-centered finite volume scheme for shallow water flows with wet/dry fronts over complex topography, Nikolos and Delis

Dam break (animation)

http://www.youtube.com/watch?v=-QXUViTi_b0



Shallow-water equations in 1D

Governing equations in conservative form

$$U_t + F(U)_x = S(U)$$

$$U = \begin{bmatrix} A \\ Q \end{bmatrix}$$

$$F(U) = \begin{bmatrix} Q \\ \dfrac{Q^2}{A} + gI_1 \end{bmatrix}$$

$$S(U) = \begin{pmatrix} 0 \\ gI_2 + gA(S_o - S_f) \end{pmatrix}$$

I_1 and I_2

- **Trapezoidal channel**
  - Base width *B*, Side slope $S_L = Y/Z$

$$I_1 = h^2\left(\frac{B}{2} + h\frac{S_L}{3}\right)$$

$$I_2 = h^2\left(\frac{1}{2}\frac{dB}{dx} + \frac{h}{3}\frac{dS_L}{dx}\right)$$

- **Rectangular, $S_L = 0$**

$$I_1 = \frac{h^2 B}{2} = \frac{A^2}{2B} \quad I_2 = \frac{A^2}{2B^2}\frac{dB}{dx}$$

- **Source term** $S_f = \dfrac{Q|Q|n^2}{A^2 R^{(4/3)}}$

Rectangular Prismatic

$$U = \begin{bmatrix} h \\ hu \end{bmatrix}$$

$$F(U) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2} gh^2 \end{bmatrix}$$

$$S(U) = \begin{pmatrix} 0 \\ gh\left(S_o - S_f\right) \end{pmatrix}$$

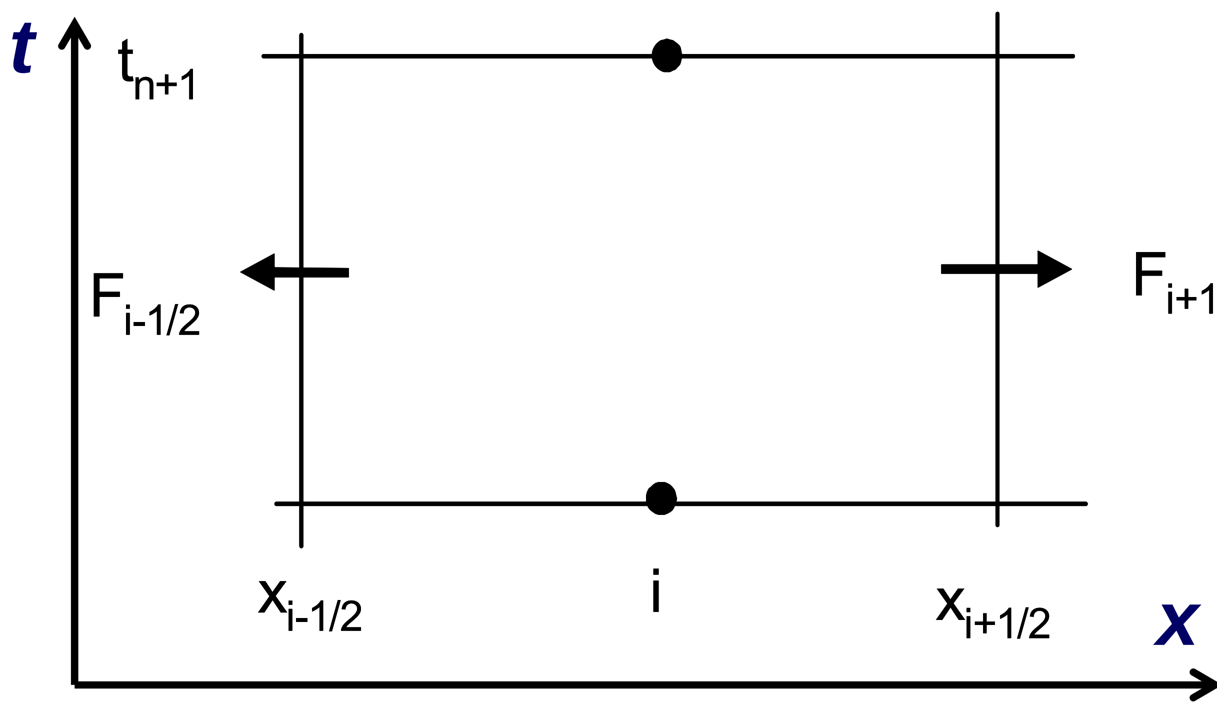
$$\oint_V \left[U\,dx + F(U)dt\right] = \int_V S(U)dU$$

Finite volume formulation

- For homogeneous form
  - i.e. without source terms

$$\oint_V \left[U\,dx + F(U)\,dt\right] = 0$$

rectangular control volume in *x-t* space



Finite Volume Formulation (Cont.)

Defining as integral averages

$$U_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U\left(x, t_n\right)\, dx \qquad U_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U\left(x, t_{n+1}\right)\, dx$$

$$F_{i-1/2} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F\left(U\left(x_{i-1/2}, t\right)\right)\, dt \quad F_{i+1/2} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F\left(U\left(x_{i+1/2}, t\right)\right)\, dt$$

Finite volume formulation becomes

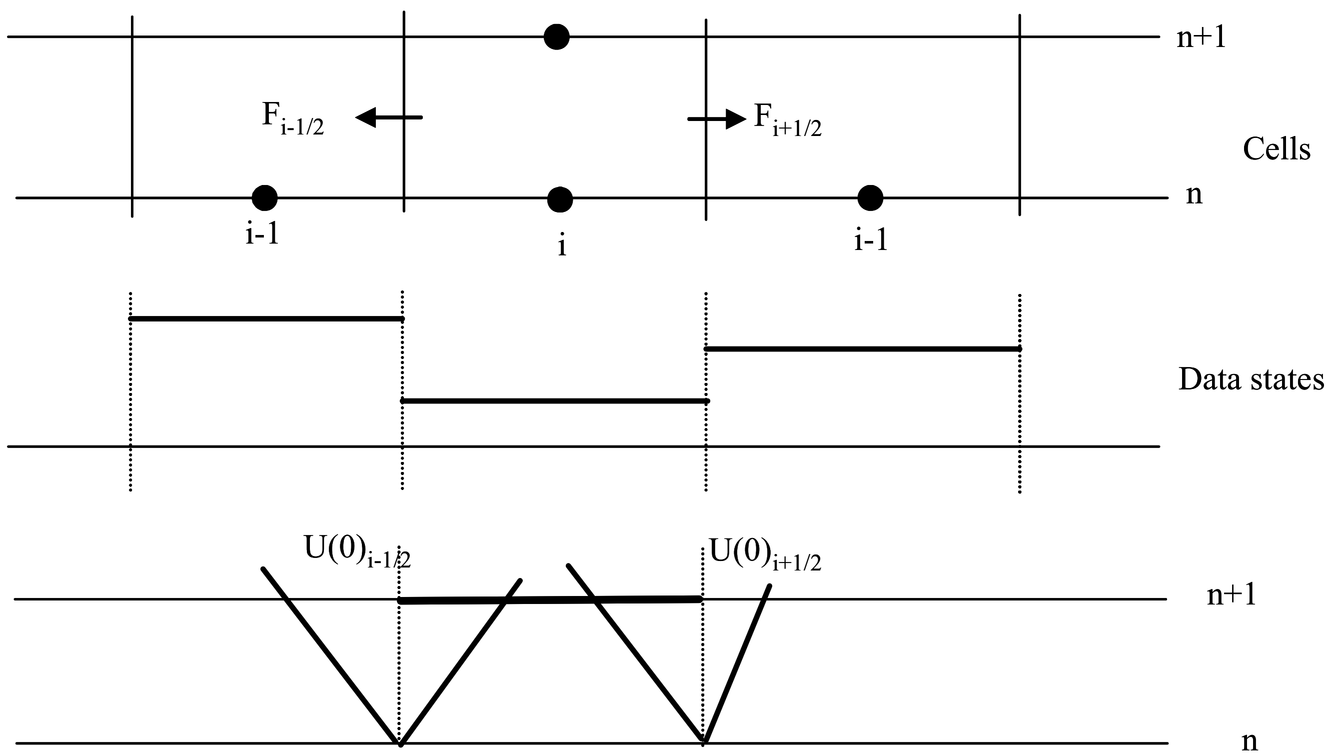
$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right]$$

Finite Volume Formulation (Cont.)

- So far no approximation was made
- The solution now depends on how the integral averages are estimated
- In particular, the inter-cell fluxes $F_{i+1/2}$ and $F_{1-1/2}$ need to be estimated.

Godunov method for flux comput.

- Uses information from the wave structure
- Assume piecewise linear data states



- Flux calculation is solution of local Riemann problem

Riemann Problem

- The Riemann problem is an **initial value problem** defined by

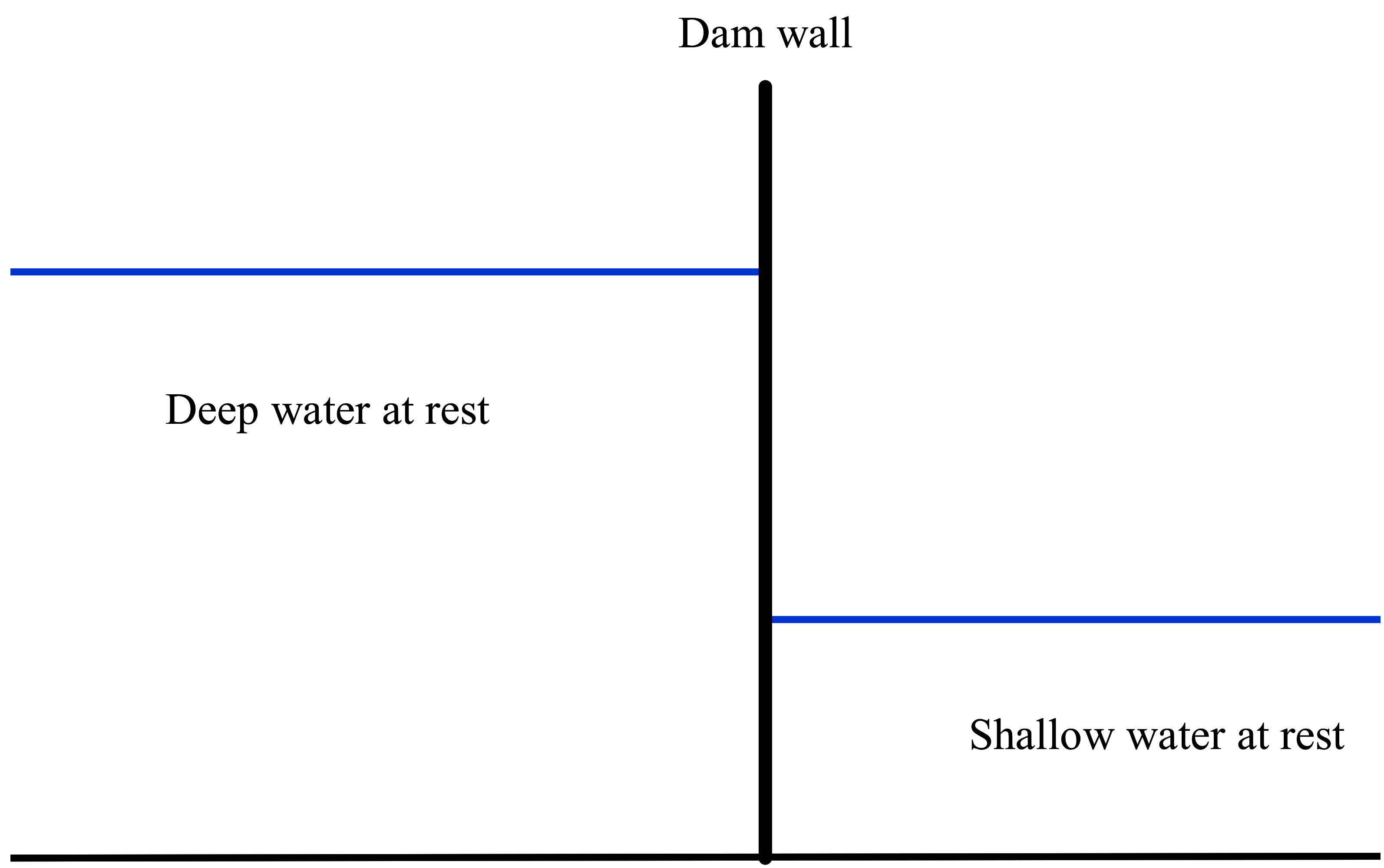
$$U_t + F(U)_x = 0$$

$$U(x, t_n) = \begin{cases} U_i^n \ \ if \ \ x < x_{i+1/2} \\ U_{i+1}^n \ \ if \ \ x > x_{i+1/2} \end{cases}$$

- The solution of this problem can be used to estimate the flux at $x_{i+1/2}$

Riemann Problem (Cont.)

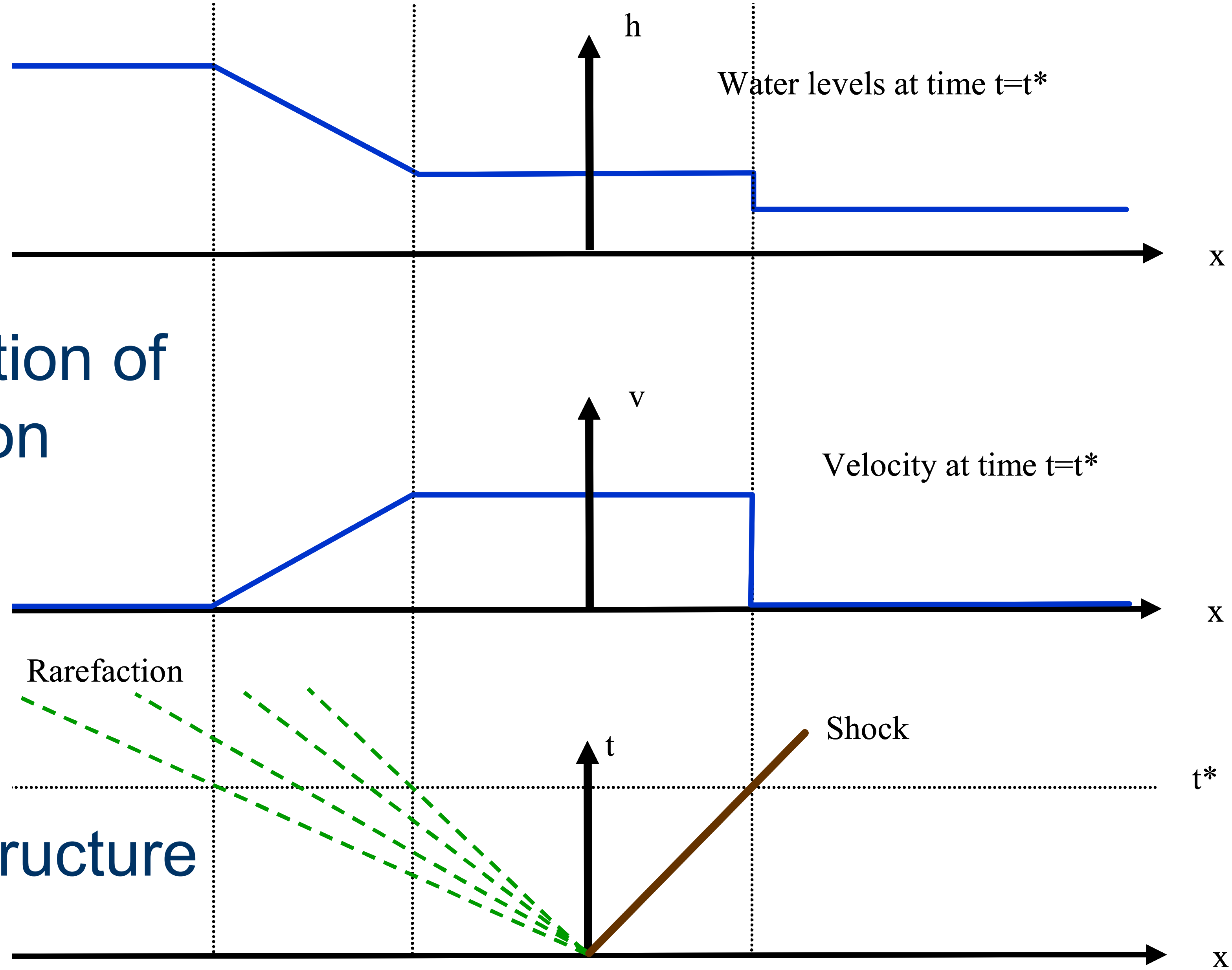
The Riemann problem is a **generalisation of the dam break problem**



Dam Break Solution

Evolution of solution

Wave structure



Exact Solution

- Toro (1992) demonstrated an exact solution
- Considering all possible wave structures a single non-linear algebraic equation gives solution.

Exact Solution

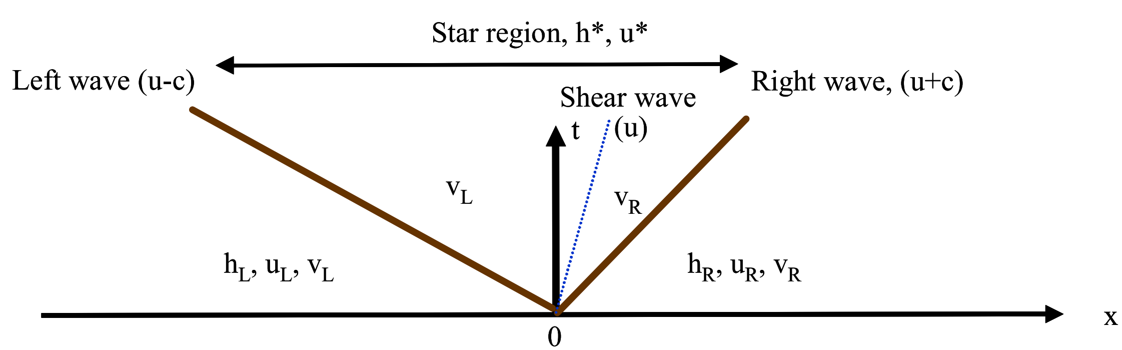
- Consider the local Riemann problem

$$U_t + F(U)_x = 0$$

$$U = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix} \qquad F(U) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix}$$

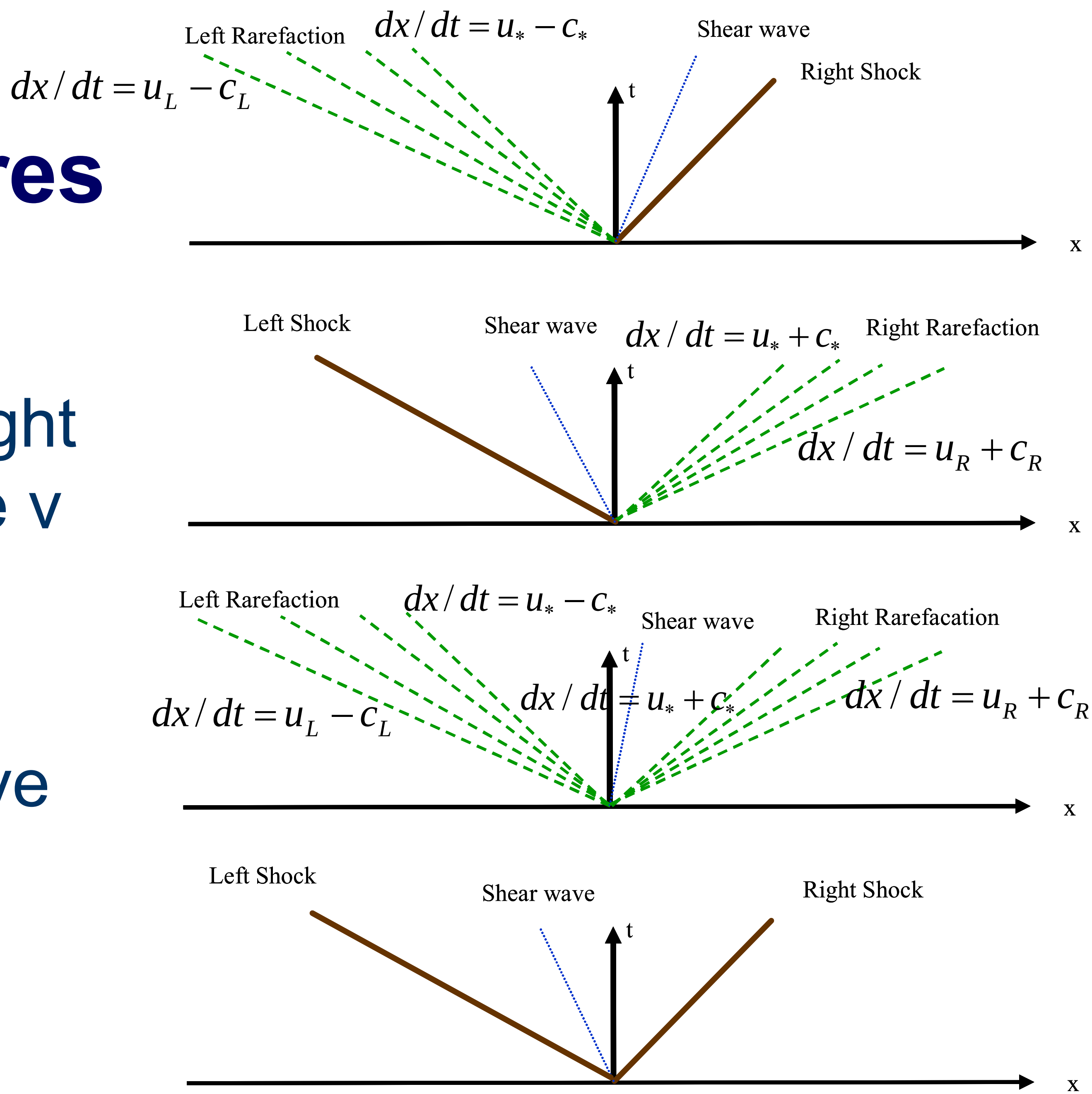
$$U(x,0) = \begin{cases} U_L \ \ if \ \ x < 0 \\ U_R \ \ if \ \ x > 0 \end{cases}$$

- Wave structure



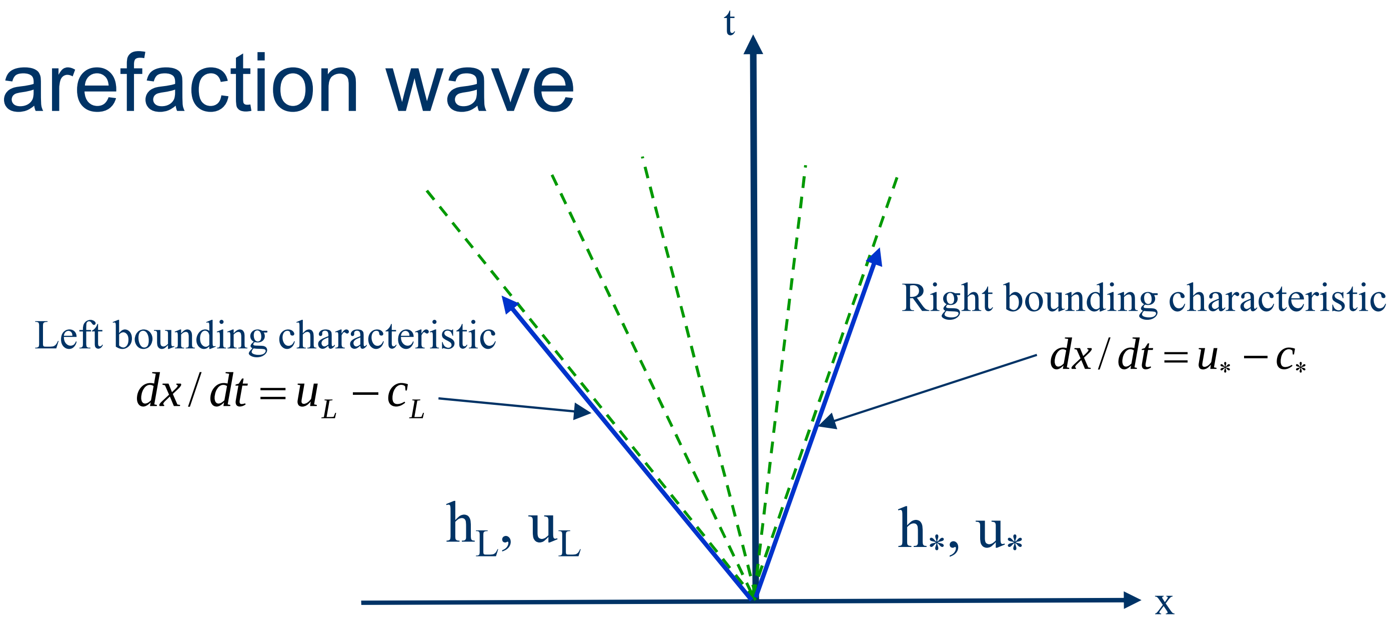
Possible Wave structures

- Across left and right wave h, u change v is constant
- Across shear wave v changes, h, u constant



Conditions across each wave

- Left Rarefaction wave



- Smooth change as move in x-direction
- Bounded by two (backward) characteristics

Crossing the rarefaction

- We cross on a forward characteristic

$$u + 2c = \text{constant}$$

- States are linked by:

$$u_L + 2c_L = u_* + 2c_*$$

- or $$u_* = u_L + 2(c_L - c_*)$$

Solution inside the left rarefaction

- The backward characteristic equation is $\frac{dx}{dt} = u - c$
- For any line in the direction of the rarefaction

- Crossing this the following applies: $u_L + 2c_L = u + 2c$

- Solving gives $c = \frac{1}{3}\left(u_L + 2c_L - \frac{dx}{dt}\right)$ $u = \frac{1}{3}\left(u_L + 2c_L + 2\frac{dx}{dt}\right)$

- On the *t* axis dx/dt = 0

$$c = \frac{1}{3}\left(u_L + 2c_L\right) \qquad u = \frac{1}{3}\left(u_L + 2c_L\right)$$

Right rarefaction

- Bounded by forward characteristics $\frac{dx}{dt} = u + c$
- Cross it on a backward characteristic

$$u_R - 2c_R = u_* - 2c_* \qquad u_* = u_R + 2(c_* - c_R)$$

- In rarefaction $c = \frac{1}{3}\left(-u_R + 2c_R + \frac{dx}{dt}\right) \qquad u = \frac{1}{3}\left(u_R - 2c_R + 2\frac{dx}{dt}\right)$

- On the *t* axis dx/dt = 0

$$c = \frac{1}{3}(-u_R + 2c_R) \qquad u = \frac{1}{3}(u_R - 2c_R)$$

Shock waves

- Two constant data states are separated by a discontinuity or jump
- Shock moving at speed S_i
- Using Conservative flux for left shock

$$U_L = \begin{bmatrix} h_L \\ h_L u_L \end{bmatrix} \qquad U_* = \begin{bmatrix} h_* \\ h_* u_* \end{bmatrix}$$

Conditions across shock

- Rankine-Hugoniot condition

$$F(U_*) - F(U_L) = S_i (U_* - U_L)$$

- Entropy condition

$$\lambda_i(U_L) > S_i > \lambda_i(U_*)$$

- $\lambda_{1,2}$ are equivalent to characteristics.
- They tend towards being parallel at shock

Shock analysis

- Change frame of reference, add S_L

$$\hat{u}_* = u_* - S_L \qquad \hat{u}_L = u_L - S_L$$

$$\hat{U}_L = \begin{bmatrix} h_L \\ h_L \hat{u}_L \end{bmatrix} \qquad \hat{U}_* = \begin{bmatrix} h_* \\ h_* \hat{u}_* \end{bmatrix}$$

- Rankine-Hugoniot (mass and momentum) gives

$$h_* \hat{u}_* = h_L \hat{u}_L$$

$$h_* \hat{u}_*^2 + \frac{1}{2} g h_*^2 = h_L \hat{u}_L^2 + \frac{1}{2} g h_L^2$$

Shock analysis

- Mass flux conserved $$M_L \equiv h_* \hat{u}_* = h_L \hat{u}_L$$
- From momentum eqn. $$M_L = -\frac{1}{2} g \left(\frac{h_*^2 - h_L^2}{\hat{u}_* - \hat{u}_L} \right)$$
- Using $\hat{u}_* = M_L / h_*$ $\quad \hat{u}_L = M_L / h_L$

$$M_L = \sqrt{\frac{1}{2} g \left(h_* + h_L\right) h_* h_L}$$

- also

$$\hat{u}_* - \hat{u}_L = u_* - u_L \qquad M_L = -\frac{1}{2} g \left(\frac{h_*^2 - h_L^2}{u_* - u_L} \right)$$

Left Shock Equation

- Equating gives

$$u_* = u_L - f_L(h_*, h_L) \qquad f_L(h_*, h_L) = (h_* - h_L)\sqrt{\frac{1}{2}g\left(\frac{h_* + h_L}{h_* h_L}\right)}$$

- Also

$$S_L = u_L - c_L q_L \qquad q_L = \sqrt{\frac{1}{2}\left(\frac{(h_* + h_L)h_*}{h_L^2}\right)}$$

Right Shock Equation

- Similar analysis gives

$$u_* = u_R + f_R(h_*, h_R) \qquad f_R(h_*, h_R) = (h_* - h_R)\sqrt{\frac{1}{2}g\left(\frac{h_* + h_R}{h_* h_R}\right)}$$

- Also

$$S_R = u_R + c_R q_R \qquad q_R = \sqrt{\frac{1}{2}\left(\frac{(h_* + h_R)h_*}{h_R^2}\right)}$$

Complete equation

- Equating the left and right equations for u_*

$$u_* = u_L - f_L(h_*, h_L) \qquad u_* = u_R + f_R(h_*, h_R)$$

$$u_R - u_L + f_L(h_*, h_L) + f_R(h_*, h_R) = 0$$

- Which is the iterative of the function of Toro (2001)

$$f(h_*) \equiv f_L(h_*, h_L) + f_R(h_*, h_L) + \Delta u = 0$$

Steps to determine exact solution

Determine which wave

- Which wave is present is determined by the change in data states thus:

  - $h^* > h_L$ left wave is a shock
  - $h^* \leq h_L$ left wave is a rarefaction

  - $h^* > h_R$ right wave is a shock
  - $h^* \leq h_R$ right wave is a rarefaction

Solution Procedure

- Construct this equation

$$f(h) = f_L(h, h_L) + f_R(h, h_R) + \Delta u$$

- And solve iteratively for h (=h*).
  - The functions may change in each iteration

f(h)

- The function f(h) is defined as

$$f(h) = f_L(h, h_L) + f_R(h, h_R) + \Delta u \qquad \Delta u = u_R - u_L$$

$$f_L = \begin{cases} 2\left(\sqrt{gh} - \sqrt{gh_L}\right) & if \qquad h \le h_L \quad (rarefaction) \\ (h - h_L)\sqrt{\frac{1}{2}g\left(\frac{h + h_L}{hh_L}\right)} & if \qquad h > h_L \quad (shock) \end{cases}$$

$$f_R = \begin{cases} 2\left(\sqrt{gh} - \sqrt{gh_R}\right) & if \qquad h \le h_R \quad (rarefaction) \\ (h - h_R)\sqrt{\frac{1}{2}g\left(\frac{h + h_R}{hh_R}\right)} & if \qquad h > h_R \quad (shock) \end{cases}$$

- And u_* $$u_* = \frac{1}{2}(u_L + u_R) + \frac{1}{2}\left[f_R(h_*, h_R) - f_L(h_*, h_L) \right]$$

Iterative solution

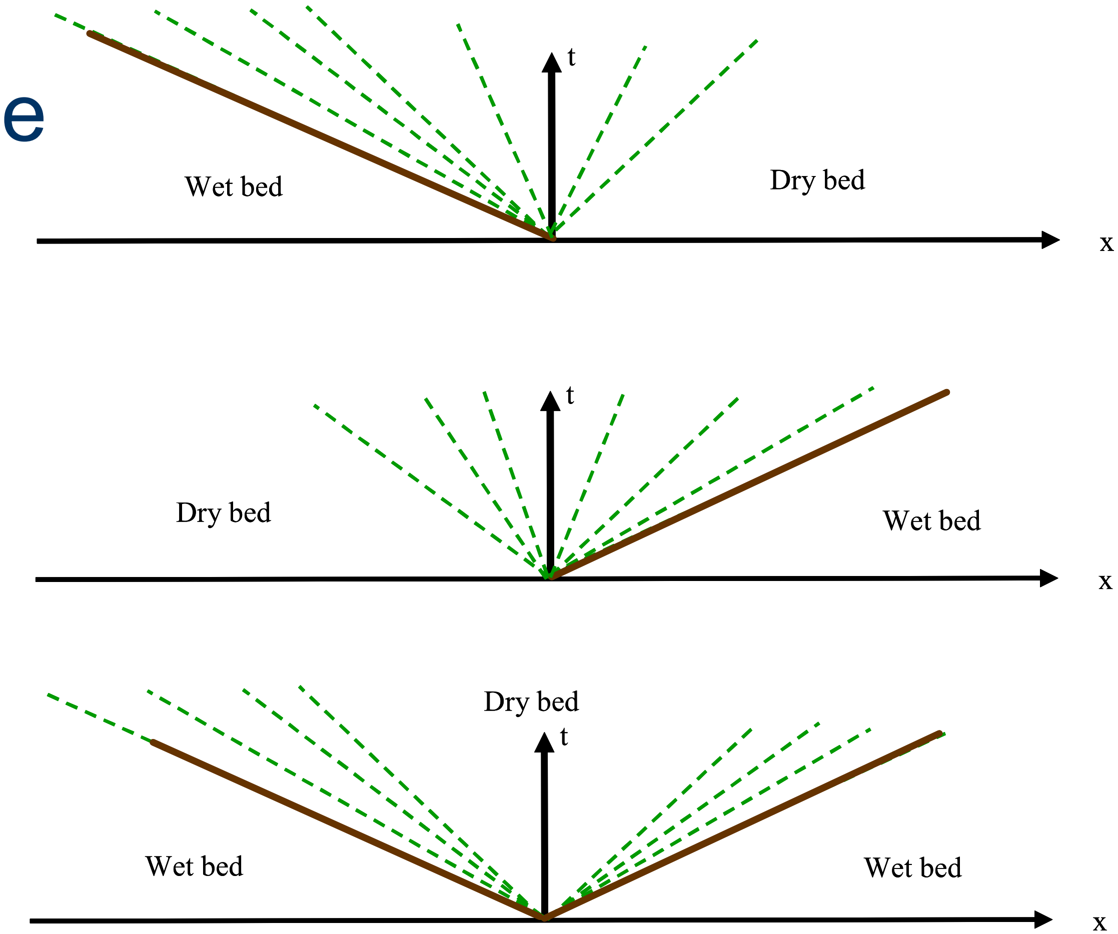
- The function is well behaved and solution by Newton-Raphson is fast
  - (2 or 3 iterations)

- One problem – if negative depth calculated!
- This is a dry-bed problem.
- Check with depth positivity condition:

$$\Delta u \equiv u_R - u_L < 2\left(c_L + c_R\right)$$

Dry–Bed solution

- Dry bed on one side of the Riemann problem
- Dry bed evolves
- Wave structure is different.



Dry-Bed Solution (Cont.)

- Solutions are explicit
  - Need to identify which applies – (simple to do)
- Dry bed to right

$$u_* = \frac{1}{3}\left(u_L + 2c_L\right) \qquad c_* = \frac{1}{3}\left(u_L + 2c_L\right) \qquad h_* = c_*^2 / g$$

- Dry bed to left

$$u_* = \frac{1}{3}\left(u_R - 2c_R\right) \qquad c_* = \frac{1}{3}\left(-u_R + 2c_R\right)$$

- Dry bed evolves h* = 0 and u* = 0
  - Fails depth positivity test

Shear wave (discontinuities that arise from eigenmodal analysis)

- The solution for the shear wave is straight forward.
  - If $v_L > 0$ $v^* = v_L$
  - Else $v^* = v_R$
- Can now calculate inter-cell flux from h^*, u^* and v^*
  - For *any* initial conditions

Approximate Riemann Solvers – 1D Model

- No need to use exact solution

  Exact solutions require iterations and are computationally expensive
- For some problems, exact solutions may not exist
- Many Riemann solvers are available (Roe’s and HLL are most popular)

Toro Two-Rarefaction Solver

- Assume two rarefactions
- Take the left and right equations

$$u_* = u_L + 2(c_L - c_*) \qquad u_* = u_R + 2(c_* - c_R)$$

- Solving gives

$$c_* = \frac{u_L - u_R}{4} + \frac{c_L + c_R}{2}$$

- For critical rarefaction use solution earlier

Toro Two-Shock Solver

- Assuming the two waves are shocks

$$h_* = \frac{q_L h_L + q_R h_R + u_L - u_R}{q_L + q_R}$$

$$u_* = \frac{1}{2}(u_L - u_R) - \frac{1}{2}\left[(h_* - h_R)q_R - (h_* - h_L)q_L\right]$$

$$q_L = \sqrt{\frac{g(h_o + h_L)}{2h_o h_L}} \qquad q_R = \sqrt{\frac{g(h_o + h_R)}{2h_o h_R}}$$

- Use two rarefaction solver to give h_0

Approximate Riemann Solvers – 1D

Roe's Solver

- Governing equations are approximated as:

$$U_t + F(U)_x \equiv U_t + AU_x \approx U_t + \tilde{A}U_x$$

- Where $\tilde{A}$ is obtained by *Roe averaging*

$$c_L = \sqrt{gh_L}\,;\; c_R = \sqrt{gh_R}$$

$$\tilde{h} = \sqrt{h_L h_R}$$

$$\tilde{u} = \frac{u_L\sqrt{h_L} + u_R\sqrt{h_R}}{\sqrt{h_L} + \sqrt{h_R}}$$

$$\tilde{c} = \sqrt{\frac{1}{2}\left(c_L^2 + c_R^2\right)}$$

Roe's Solver (Cont.)

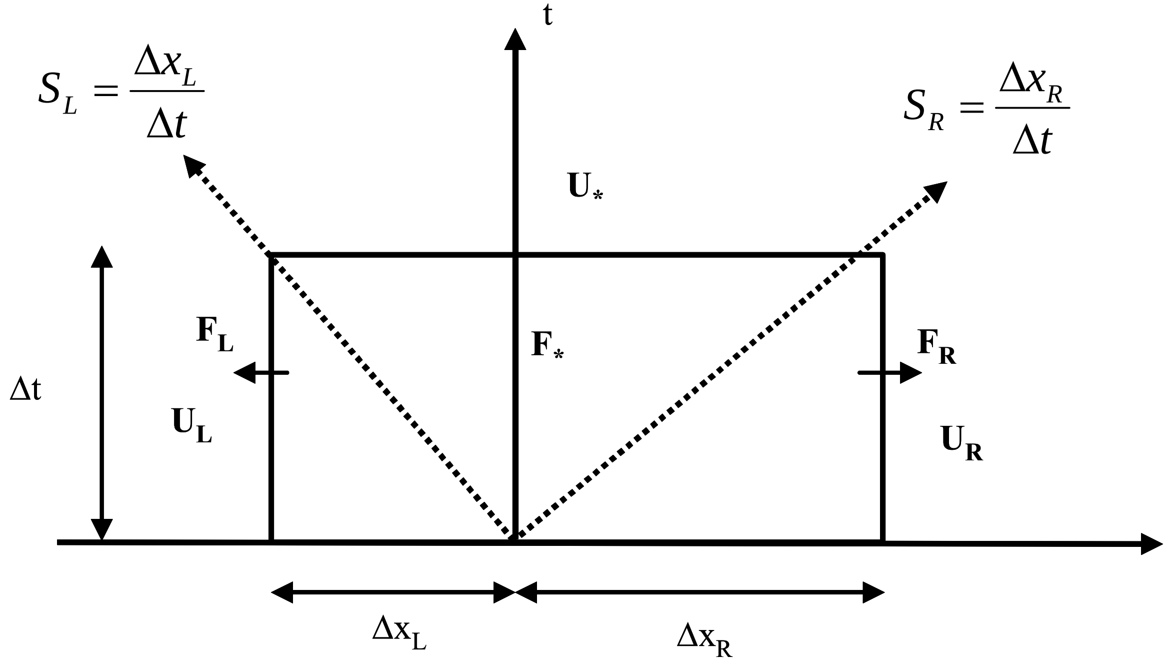
- Properties of matrix
  - Eigen values $\tilde{\lambda}_1 = \tilde{u} - \tilde{c}$ $\quad \tilde{\lambda}_2 = \tilde{u} + \tilde{c}$
  - Right Eigen vectors $\tilde{R}^{(1)} = \begin{bmatrix} 1 \\ \tilde{u} - \tilde{c} \end{bmatrix}$ $\quad \tilde{R}^{(2)} = \begin{bmatrix} 1 \\ \tilde{u} + \tilde{c} \end{bmatrix}$
  - Wave strengths $\tilde{\alpha}_1 = \frac{1}{2}\left(\Delta h - \frac{\tilde{h}}{\tilde{c}}\Delta u\right)$ $\quad \tilde{\alpha}_2 = \frac{1}{2}\left(\Delta h + \frac{\tilde{h}}{\tilde{c}}\Delta u\right)$

$$\Delta h = h_R - h_L; \quad \Delta u = u_R - u_L$$

- Flux is given by $F_{i+1/2} = \frac{1}{2}\left(F_i^n + F_{i+1}^n\right) - \frac{1}{2}\sum_{j=1}^{2} \tilde{\alpha}_j \left|\tilde{\lambda}_j\right| \tilde{\mathbf{R}}^{(j)}$

HLL Solver

- Harten, Lax, Van Leer
- Assume wave speed
- Construct volume
- Integrate



$$\Delta x_L U_L + \Delta x_R U_R - \left(\Delta x_L + \Delta x_R\right) U_* - \Delta t U_R + \Delta t U_L = 0$$

$$U_* = \left(S_R U_R - S_L U_L + F_L - F_R\right) / \left(S_R - S_L\right)$$

Or,

$$F_* = \left(S_R F_L - S_L F_R + S_L S_R \left(U_R - U_L\right)\right) / \left(S_R - S_L\right)$$

HLL Solver

- What wave speeds to use?
  - One option:

$$S_L = \min\left(u_L - c_L, u_{TR} - c_{TR}\right)$$

$$S_R = \min\left(u_R + c_R, u_{TR} + c_{TR}\right)$$

- For dry bed (right)

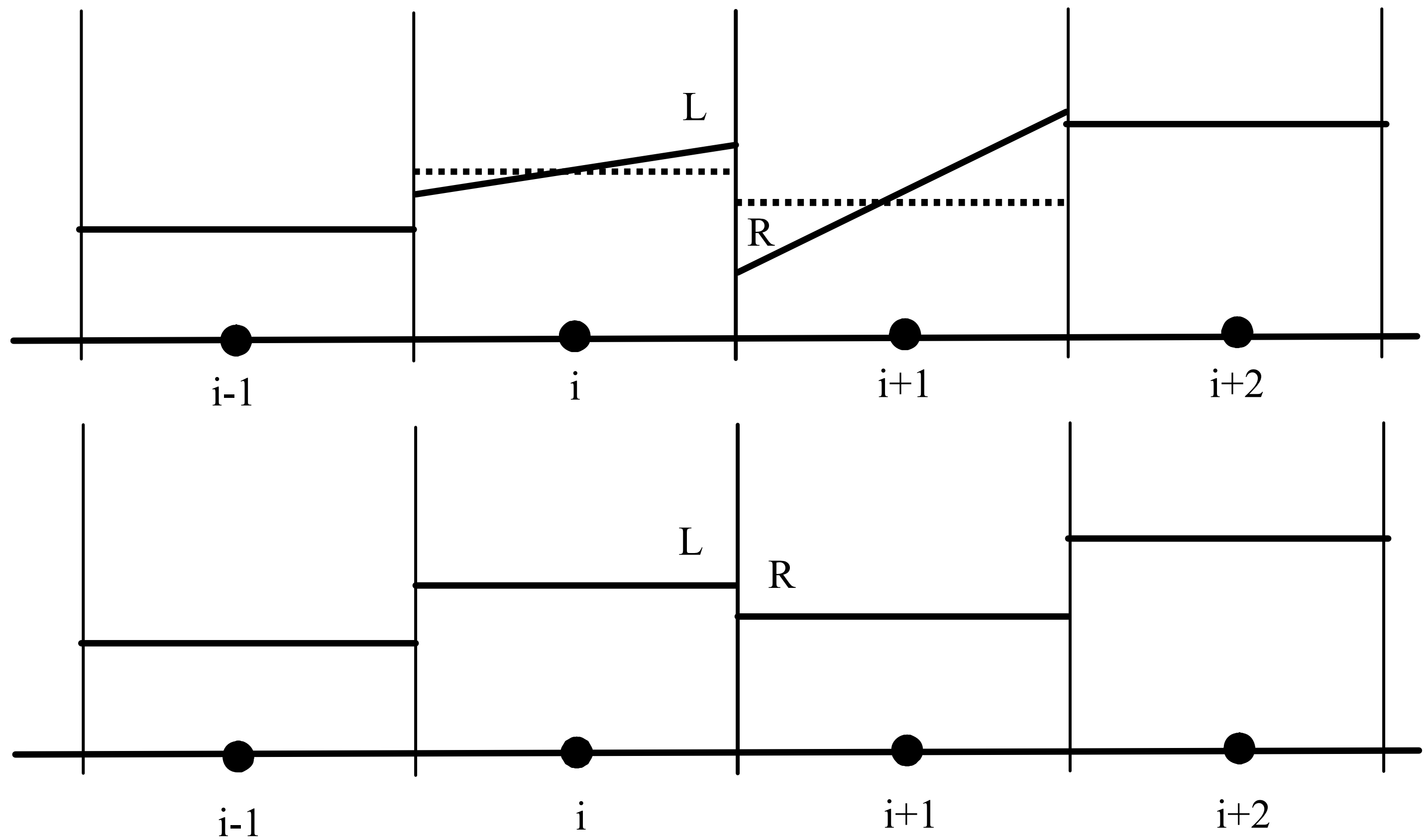
$$S_L = u_L - c_L$$

$$S_R = u_R + 2c_R$$

- Simple, but robust

Higher-Order in Space

- Construct Riemann problem using surrounding cells
- May create oscillations
- Piecewise reconstruction



- Need to use limiters

Limiters

- Obtain a gradient for variable in cell *i*, Δ*i*

$$U_L = U_i + \frac{1}{2}\Delta x \Delta_i$$

$$U_R = U_{i+1} - \frac{1}{2}\Delta x \Delta_{i+1}$$

- Gradient obtained from *Limiter* functions
- Provide gradients at cell faces

$$\Delta_{i+1/2} = a = \frac{u_{i+1} - u_i}{x_{i+1/2} - x_i}$$

$$\Delta_{i-1/2} = b = \frac{u_i - u_{i-1}}{x_i - x_{i-1}}$$

- Limiter Δ*i* =G(a,b)

Limiters (Cont.)

- A general limiter

$$G(a,b)=\begin{cases}\max\left[0,\min(\beta a,b),\min(a,\beta b)\right] & for\ a>0\\ \min\left[0,\max(\beta a,b),\max(a,\beta b)\right] & for\ a<0\end{cases}$$

  - β=1 give MINMOD,
  - β=2 give SUPERBEE

- Van Leer

$$G(a,b)=\frac{a|b|+|a|b}{|a|+|b|}$$

Higher order in time

- Needs to advance half time step
- **MUSCL-Hancock**
  - Primitive variable
  - Limit variable
  - Evolve the cell face values 1/2Δt:

$$\mathbf{W}_t + \mathbf{A}(\mathbf{W})\mathbf{W}_x = 0$$

$$\overline{\mathbf{W}}_i^{L,R} = \mathbf{W}_i^{L,R} + \frac{1}{2}\frac{\Delta t}{\Delta x}\mathbf{A}\left(\mathbf{W}_i^n\right)\left[\mathbf{W}_i^L - \mathbf{W}_i^R\right]$$

- Update as normal solving the Riemann problem using evolved W_L, W_R

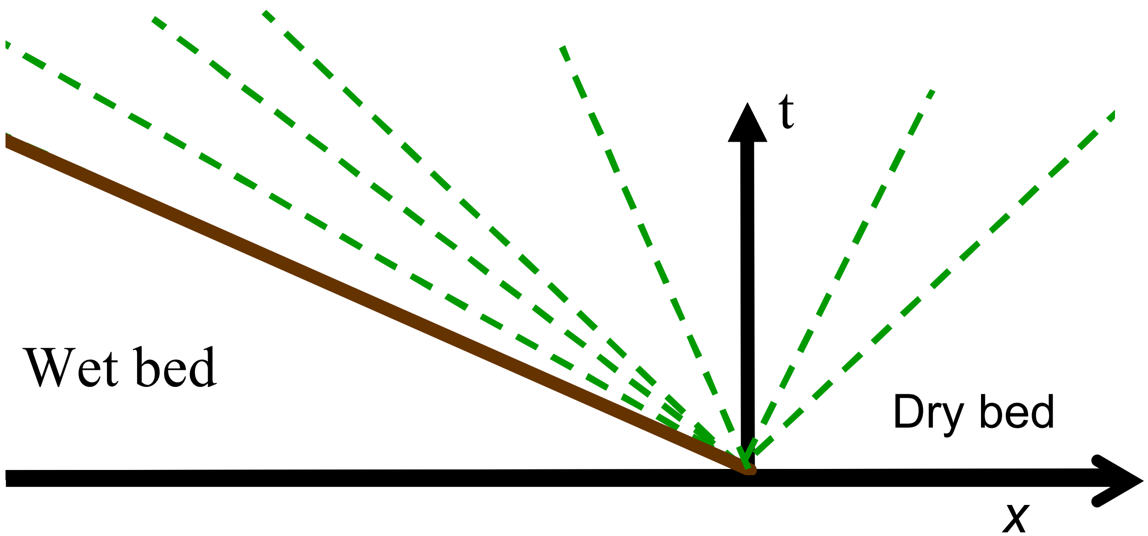
$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right]$$

Wet / Dry Fronts

- Wet / Dry fronts are difficult
  - Source of error
  - Source of instability

- Found in:
  - Filling of storm-water and combined sewer systems
  - Flooding - inundation

Dry front speed



$$S_{*L} = u + c$$

$$u_L + 2c_L = u + 2c$$

$$c = 0$$

$$S_{*L} = u_L + 2c_L$$

- Dry front is fast
- Can cause problem with time-step / Courant number

Solutions for wet/dry fronts

- Most popular way is to artificially wet bed
- Provide a small water depth with zero velocity
- Can drastically affect front speed
- Need to be very careful about dividing by zero

Boundary Conditions

- Set flux on boundary
  - Directly
  - Ghost cell
- Wall u, v = 0. Ghost cell $u_{n+1} = -u_n$
- Transmissive Ghost cell $h_{n+1} = h_n$

$$u_{n+1} = u_n$$

Source Terms

- “Lumped” in to one term and integrated
- Attempts at “upwinding source”
- Current time-step
  - Could use the half step value
- E.g.

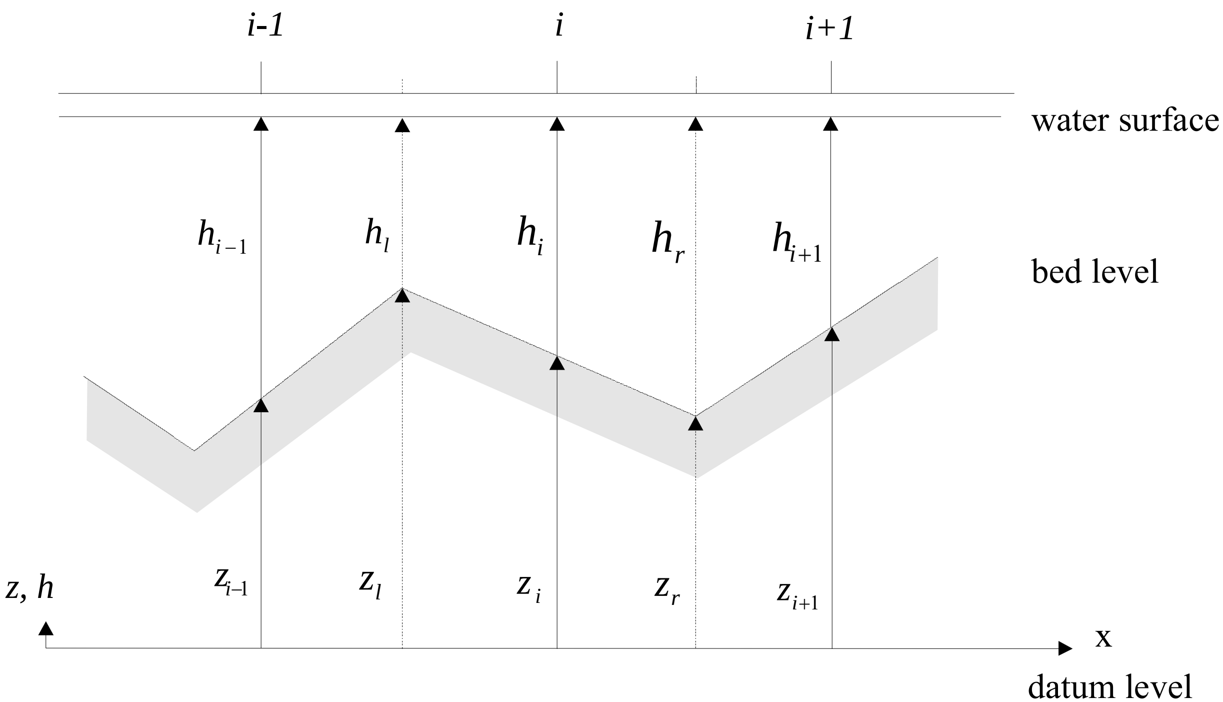
$$S = \begin{bmatrix} 0 \\ gh\left(\frac{\partial z}{\partial x} - \frac{u|u|}{K^2}\right) + \frac{\phi^2}{2B}\frac{\partial B}{\partial x} \end{bmatrix}$$

$$K = CR^{1/2}$$

$$K = \frac{R^{2/3}}{n}$$

Main Problem is Slope Term

- Flat still water over uneven bed starts to move.
- Problem with discretisation of $$gh\frac{\partial z}{\partial x}$$



$$gh_i\frac{(z_r - z_l)}{\Delta x}$$

Discretisation

- Discretised momentum eqn

$$\left(hu\right)_i^{n+1} = \left(hu\right)_i + \frac{\Delta t}{\Delta x}\left(\left(hu^2\right)_l - \left(hu^2\right)_r + \frac{gh^2{}_l}{2} - \frac{gh^2{}_r}{2}\right) - \frac{\Delta t}{\Delta x} gh_i\left(z_r - z_l\right)$$

- For flat, still water

$$\left(hu\right)_i^{n+1} = g\frac{\Delta t}{\Delta x}\left(\left(\frac{h_l^2}{2} + h_i z_l\right) - \left(\frac{h_r^2}{2} + h_i z_r\right)\right) = 0$$

- Require

$$\frac{h_l^2}{2} + h_i z_l = \frac{h_r^2}{2} + h_i z_r$$

A solution

- Assume a “datum” depth, measure down.

For horizontal water surface: $\frac{\partial z}{\partial x} = -\frac{\partial h_i}{\partial x}$

$$g\ h_i \frac{\partial z}{\partial x} = -g\ h_i \frac{\partial h_i}{\partial x} \approx g \frac{1}{2} \frac{\partial h_i^2}{\partial x}$$

$$g \frac{1}{2} \frac{(h_i + z_i - z_r)^2 - (h_i + z_i - z_l)^2}{\Delta x}$$

- Momentum eqn:
  - Flat surface $h_l = h_i + z_i - z_l$ $h_r = h_i + z_i - z_r$

$$(hu)_i' = \frac{g}{2} \frac{\Delta t}{\Delta x} \left(h_l^2 - h_r^2 + (h_i + z_i - z_r)^2 - (h_i + z_i - z_l)^2 \right)$$

Shallow-water in two-dimensions

- In 2-d we have an extra term:

$$U_t + F(U)_x + G(U)_y = S(U)$$

$$F(U) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix}$$

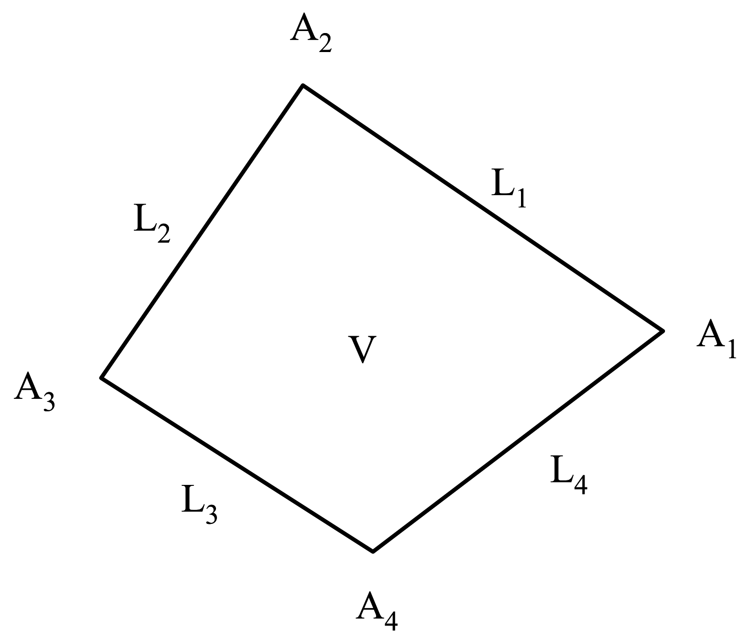
$$U = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix} \quad S(U) = \begin{pmatrix} 0 \\ gh(S_{o_x} - S_{f_x}) \\ gh(S_{o_y} - S_{f_y}) \end{pmatrix}$$

$$G(U) = \begin{bmatrix} hu \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{bmatrix}$$

- Friction $S_{f_x} = \frac{n^2}{h^{(1/3)}} u\sqrt{u^2 + v^2}$

Finite Volume in 2-D

If nodes and sides are labelled as :

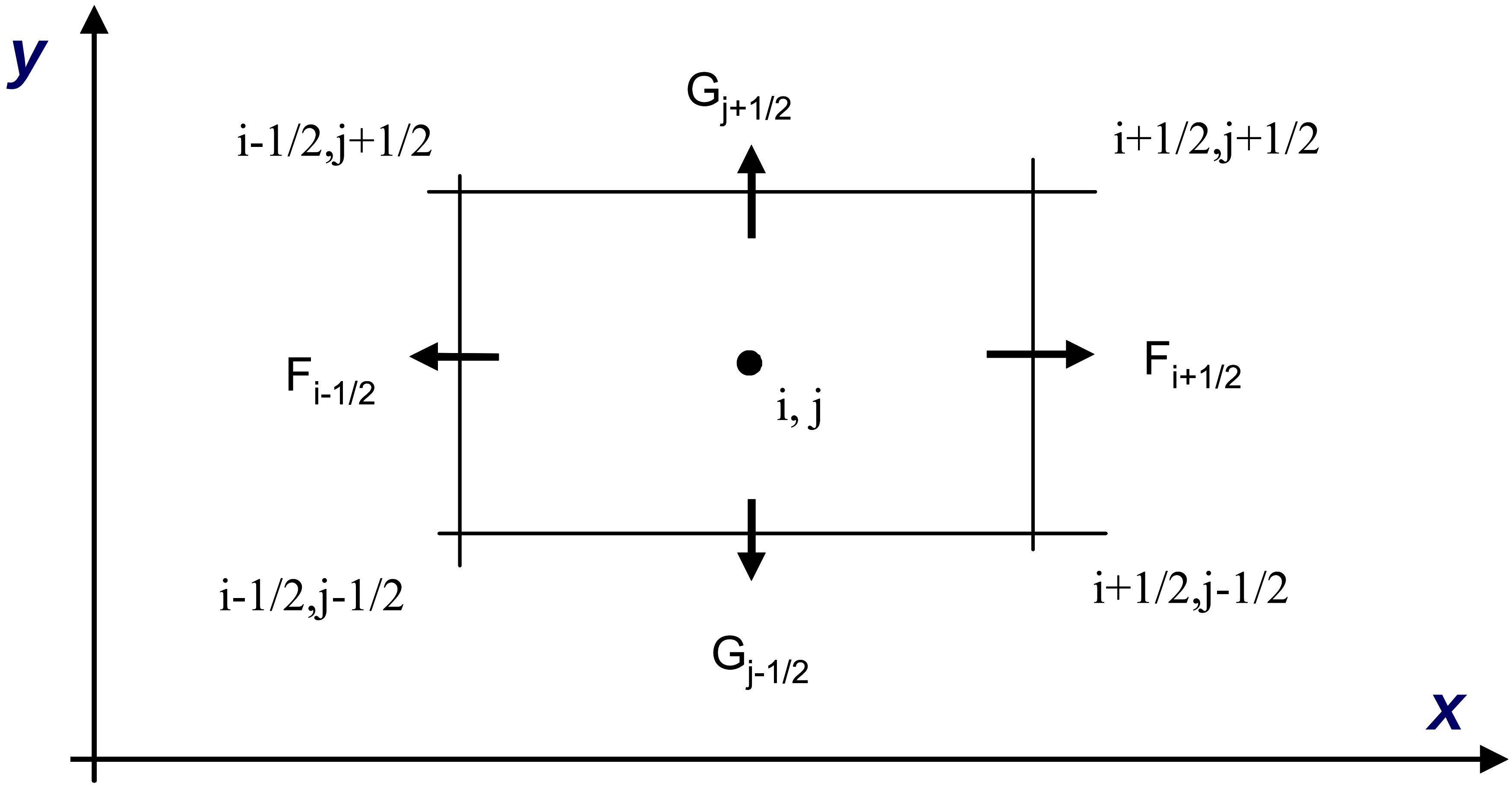


Solution is

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{|V|}\left(Fn_{s1}L_1 + Fn_{s2}L_2 + Fn_{s3}L_3 + Fn_{s4}L_4 \right)$$

Where Fn_{s1} is normal flux for side 1 etc.

FV 2-D Rectangular Grid



Solution is

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2,j} - F_{i-1/2,j}\right] - \frac{\Delta t}{\Delta y}\left[G_{i,j+1/2} - G_{i,j-1/2}\right]$$

Approximate Riemann Solvers – 2D

Roe's Solver is simple and one of the most popular. This solver will be used in this class.

Roe's Solver:

Eigen values

$$\tilde{\lambda}_1 = \tilde{u} - \tilde{c} \qquad \tilde{\lambda}_2 = \tilde{u} \qquad \tilde{\lambda}_3 = \tilde{u} + \tilde{c}$$

Right Eigen vectors

$$\tilde{R}^{(1)} = \begin{bmatrix} 1 \\ \tilde{u} - \tilde{c} \\ \tilde{v} \end{bmatrix} \qquad \tilde{R}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \tilde{R}^{(3)} = \begin{bmatrix} 1 \\ \tilde{u} + \tilde{c} \\ \tilde{v} \end{bmatrix}$$

Roe's Solver (Cont.)

Where: $c_L = \sqrt{gh_L};\ c_R = \sqrt{gh_R}$

$$\tilde{h} = \sqrt{h_L h_R}$$

$$\tilde{c} = \sqrt{\frac{1}{2}\left(c_L^2 + c_R^2\right)}$$

$$\tilde{u} = \frac{u_L\sqrt{h_L} + u_R\sqrt{h_R}}{\sqrt{h_L} + \sqrt{h_R}}$$

$$\tilde{v} = \frac{v_L\sqrt{h_L} + v_R\sqrt{h_R}}{\sqrt{h_L} + \sqrt{h_R}}$$

Roe's Solver (Cont.)

Wave strengths

$$\tilde{\alpha}_1 = \frac{\Delta u_1(\tilde{u}+\tilde{c}) - \Delta u_2}{2\tilde{c}}; \quad \tilde{\alpha}_2 = \Delta u_3 - \tilde{v}\Delta u_1;$$

$$\tilde{\alpha}_3 = \frac{-\Delta u_1(\tilde{u}-\tilde{c}) + \Delta u_2}{2\tilde{c}}$$

Where:

$$\Delta u_1 = h_R - h_L; \quad \Delta u_2 = u_R h_R - u_L h_L \ \text{ or } q_R - q_L;$$

$$\Delta u_3 = h_R v_R - h_R v_L$$

Roe's Solver (Cont.)

Numerical Flux is given by (Toro 2001)

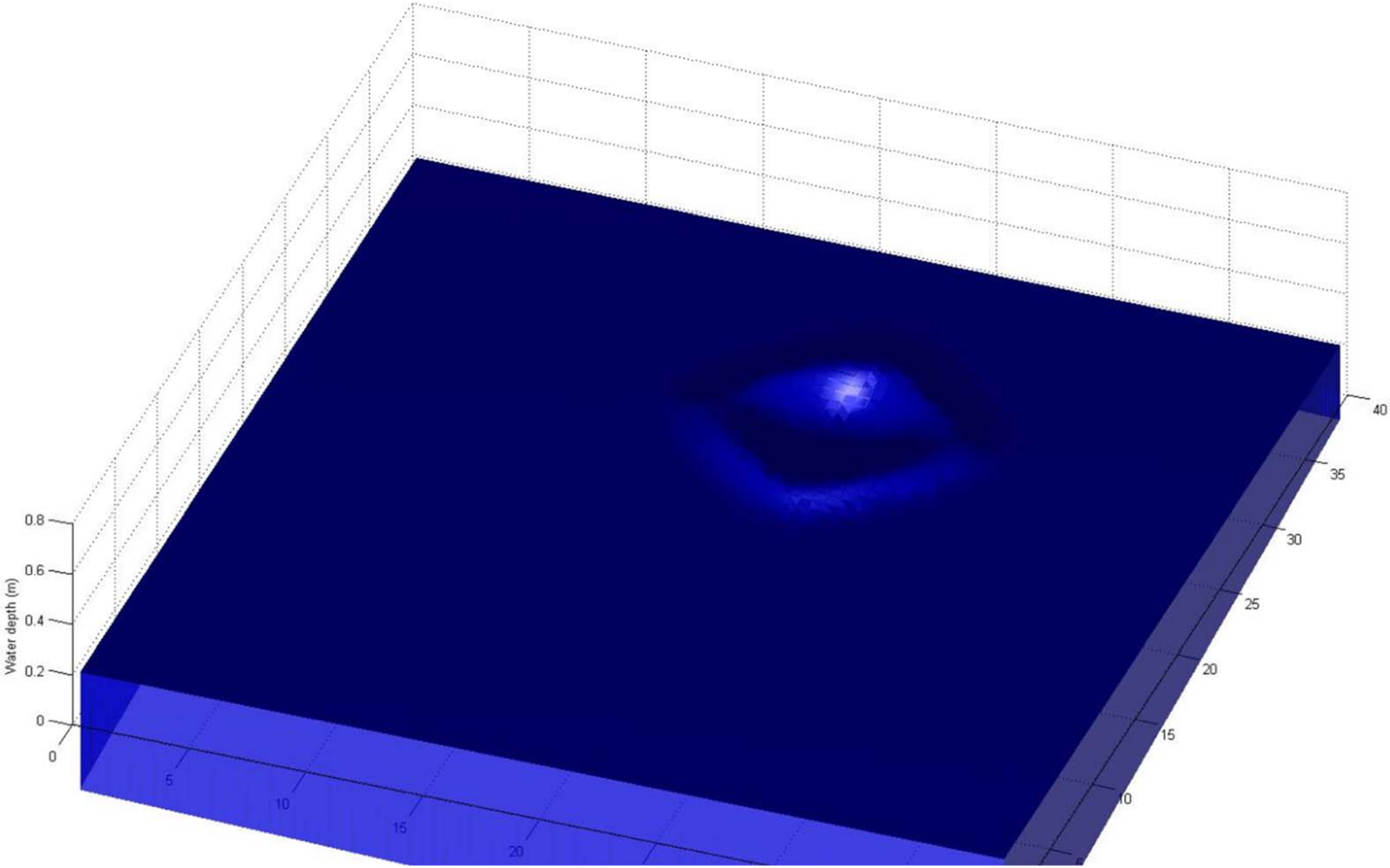
$$F_{i+1/2} = \frac{1}{2}\left(F_i^n + F_{i+1}^n\right) - \frac{1}{2}\sum_{j=1}^{3} \tilde{\alpha}_j \left|\tilde{\lambda}_j\right| \tilde{\mathbf{R}}^{(j)}$$

Update of solution:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2,j} - F_{i-1/2,j}\right] - \frac{\Delta t}{\Delta y}\left[G_{i,j+1/2} - G_{i,j-1/2}\right]$$

Show Demo in MATLAB for Solution of 2D Shallow Water Equations using Roe's solver (Droplets and Dam break problem)

Download Matlab files from Canvas.



References

- E.F. Toro. *Riemann Solvers and Numerical Methods for Fluid Dynamics*. Springer Verlag (2nd Ed.) 1999.
- E.F. Toro. *Shock-Capturing Methods for Free-Surface Flows*. Wiley (2001)
- Lecture notes on Shallow-Water equations by Andrew Sleigh
- **Leon, A. S**., Ghidaoui, M. S., Schmidt, A. R. and Garcia, M. H. (2010) “A robust two-equation model for transient mixed flows.” *Journal of Hydraulic Research*, 48(1), 44-56.
- **Leon, A. S**., Ghidaoui, M. S., Schmidt, A. R. and Garcia, M. H. (2006) “Godunov-type solutions for transient flows in sewers”. *Journal of Hydraulic Engineering,* 132(8), 800-813.