1D UNSTEADY OPEN CHANNEL FLOWS

Qiantang River Tidal Bore

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Examples of Wave Generating Mechanisms

- Sudden Change at control structures:
  - sudden opening or closing of hydraulic gates;
  - dam break;
  - sudden increase/decrease in demand at a power station;
  - ship-lock operation.
- Surges
- Tides
- Large Runoff: Flood wave in rivers; sewer surcharging.

Movies

Flooding in Greece

https://www.youtube.com/watch?v=b7OPQlzxrGo

Qiantang River Tide Wave, China

https://www.youtube.com/watch?v=ILi0_p1xt_Y

Various dam breaks

https://www.youtube.com/watch?v=LZDJ6zPHYAM

Saint-Venant equations

- 1D Saint-Venant equations

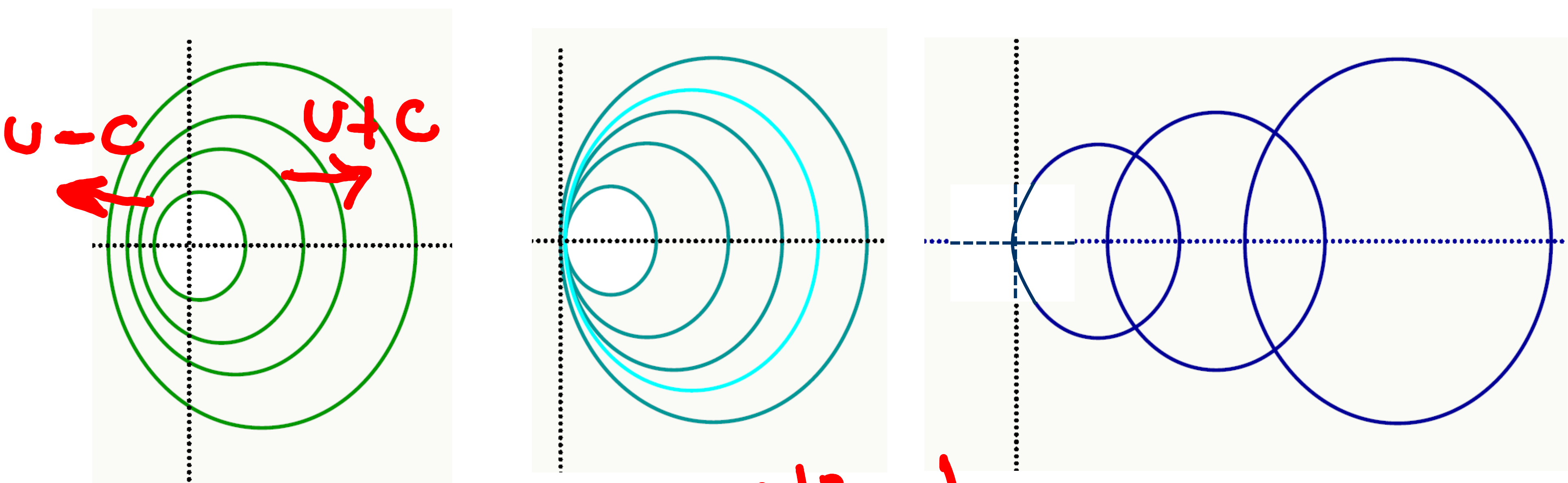
$$\frac{\partial y}{\partial t} + D\frac{\partial V}{\partial x} + V\frac{\partial y}{\partial x} = 0$$ **(1) Continuity**

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x} = g(S_o - S_f)$$ **(2) Momentum**

- x = distance along the channel, t = time, V = cross-sectional averaged velocity, y = depth of flow, D = A/T, A = cross sectional area, T = top width, S_o = bed slope, S_f = friction slope

Propagation of a disturbance in subcritical, critical and supercritical flows

$\longrightarrow$ Flow



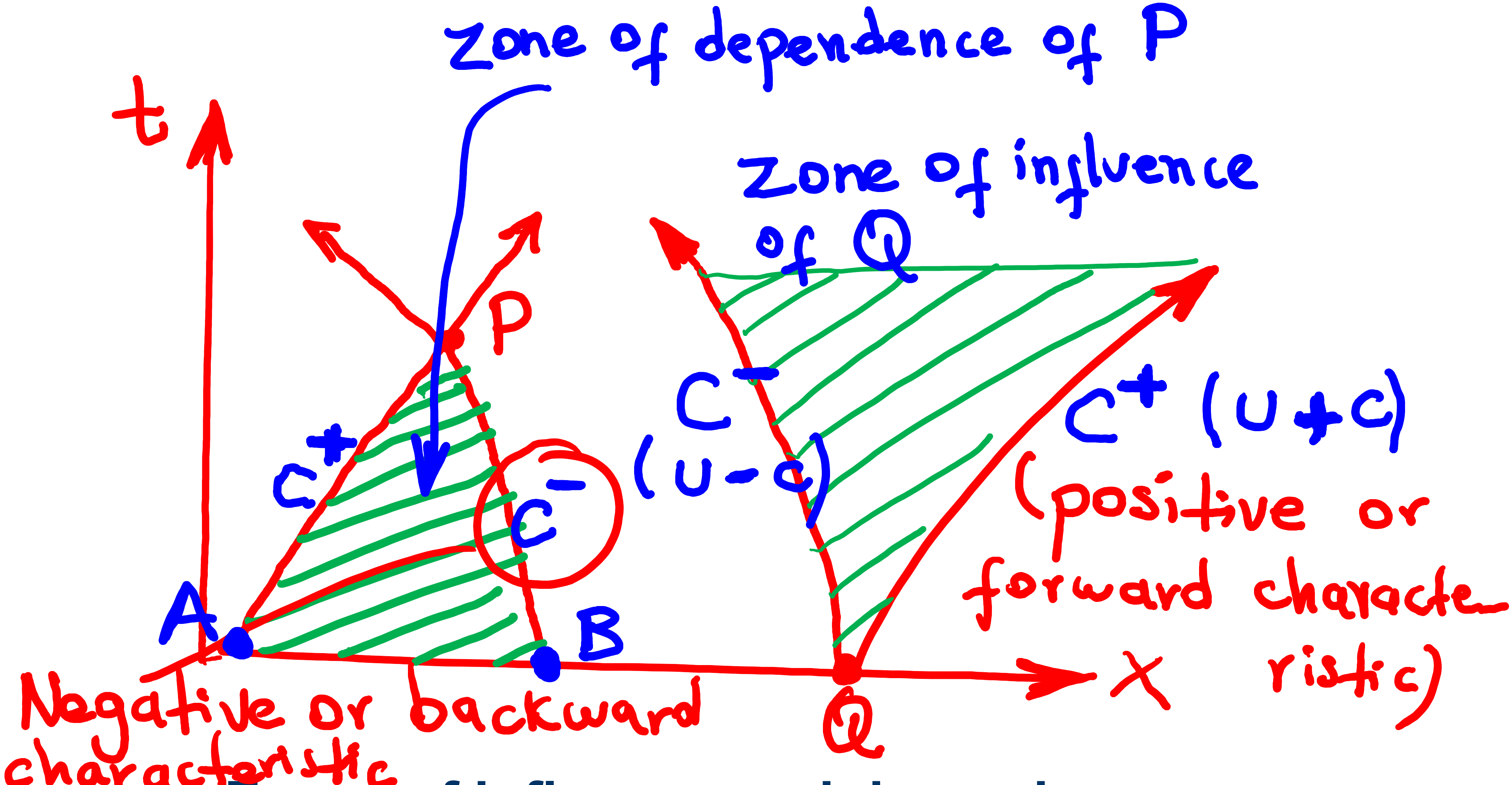
Subcritical
$\left(V < \sqrt{gA/T}\right)$

critical
$V = \sqrt{gA/T}$

supercritical
$\left(V > \sqrt{gA/T}\right)$

Flow direction

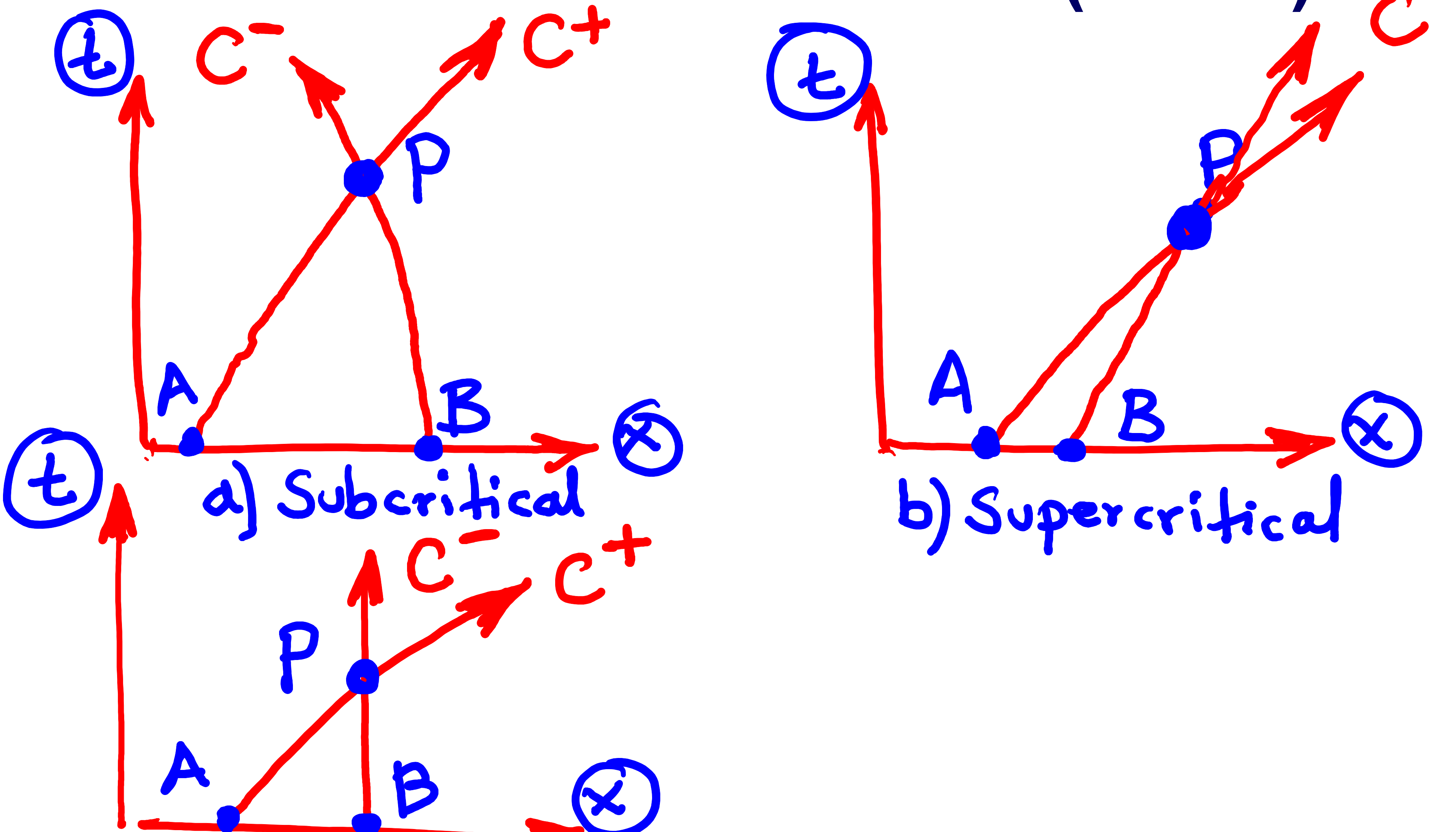
Method of Characteristics



Zones of influence and dependence

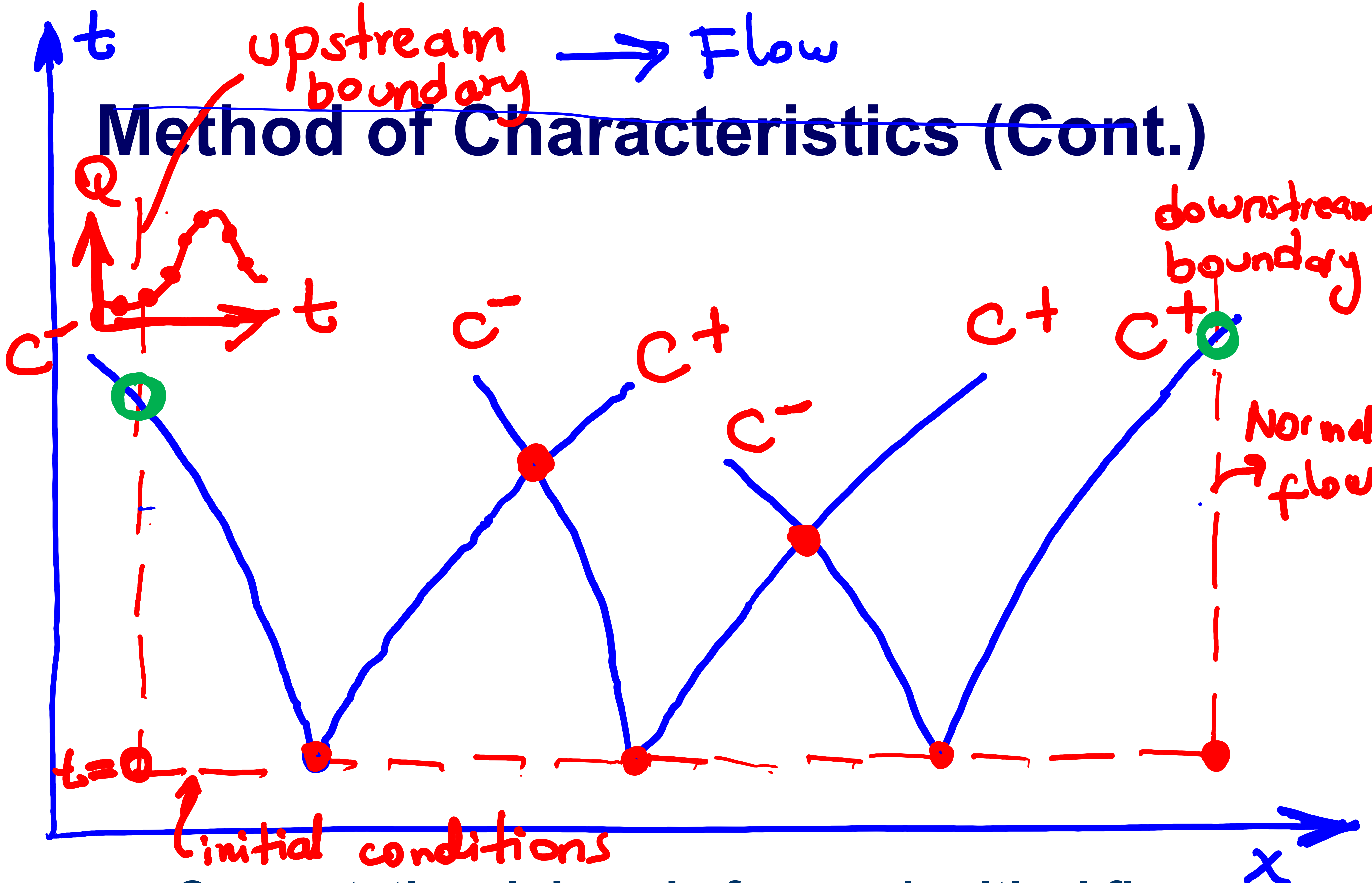
(Subcritical flow)

Method of Characteristics (Cont.)



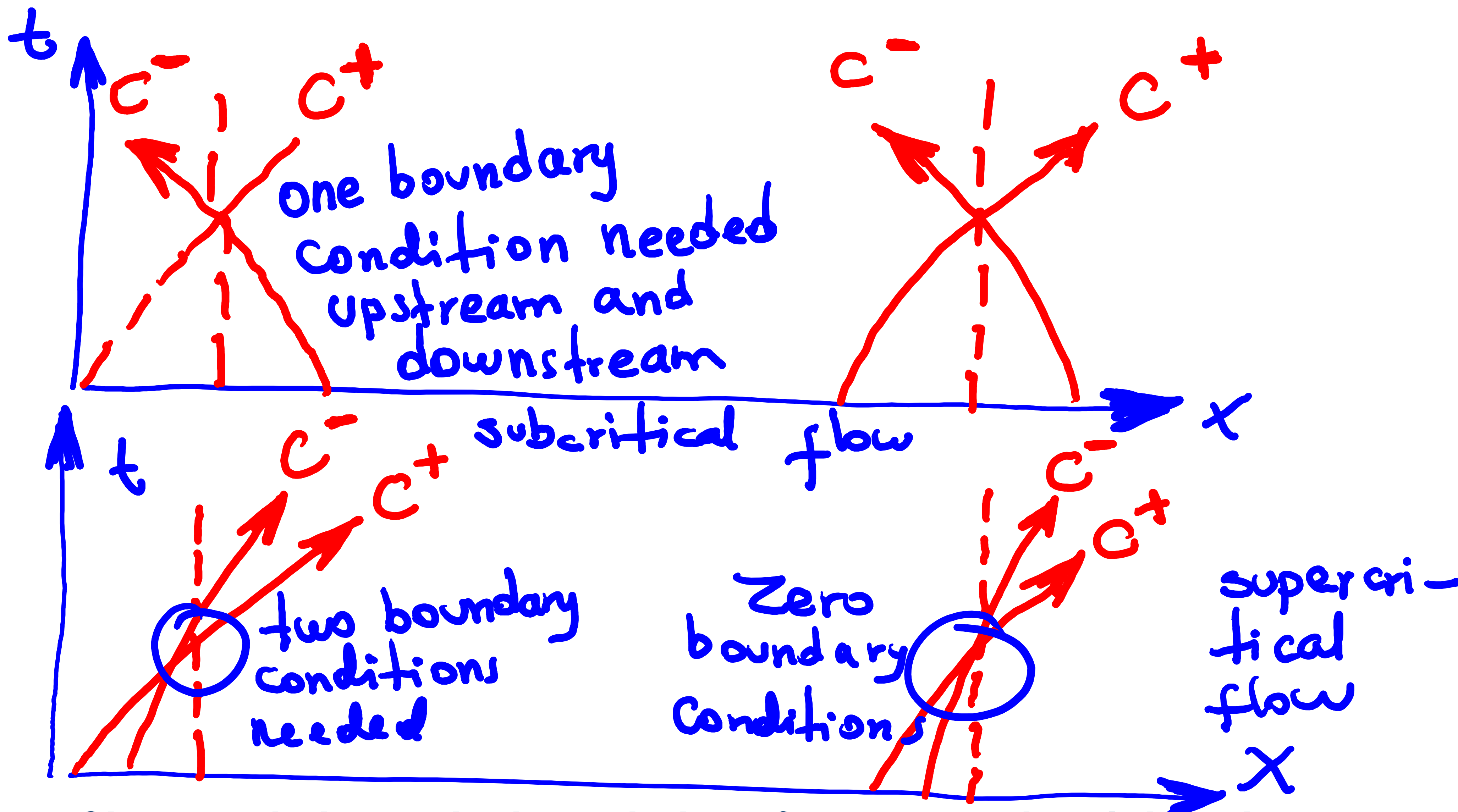
Characteristics for various types of flows

Method of Characteristics (Cont.)



Computational domain for a subcritical flow

Method of Characteristics (Cont.)



Characteristics at the boundaries of a computational domain

Method of Characteristics (Cont.)

- Multiplying the Continuity equation (Eq. 1) by $\lambda = \pm\, g/c$ and adding it to the Momentum equation (Eq. 2) yields

$$\left[\frac{\partial}{\partial t} + (v \pm c)\frac{\partial}{\partial x}\right] v \pm \frac{g}{c}\left[\frac{\partial}{\partial t} + (v \pm c)\frac{\partial}{\partial x}\right] y = g(S_o - S_f) \quad (3)$$

- The above equation is a pair of equations along characteristics given by

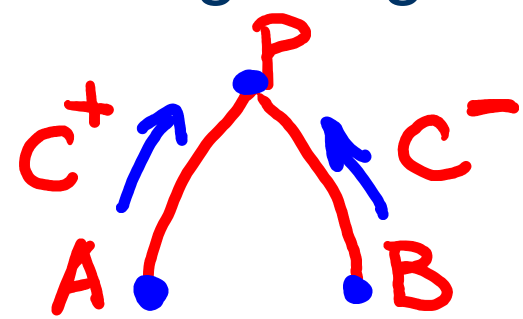
$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\frac{dx}{dt}$$

$$\frac{dv}{dt} \pm \frac{g}{c}\frac{dy}{dt} = g(S_o - S_f), \text{ along } \frac{dx}{dt} = v \pm c \quad (4)$$

$u-c$ (C^-) $u+c$ (C^+)

Method of Characteristics (Cont.)

- By multiplying equation (4) by *dt* and integrating along AP and BP (see slide 6), we obtain



Along AP:

$$V_P - V_A + \left(\frac{g}{c}\right)_A (y_P - y_A) = g(S_0 - S_f)_A (t_P - t_A)$$

Along BP:

$$V_P - V_B - \left(\frac{g}{c}\right)_B (y_P - y_B) = g(S_0 - S_f)_B (t_P - t_B)$$

$c = \sqrt{gy}$

Method of Characteristics (Cont.)

For rectangular or wide channels:

- Equation (4) can be written as

C^+

$$\frac{d(V+2c)}{dt} = g(S_o - S_f) \quad \left[\frac{dx}{dt} = V + c \right]$$

Along AP

$$\frac{d(V-2c)}{dt} = g(S_o - S_f) \quad \left[\frac{dx}{dt} = V - c \right]$$

C^-

Along BP

Method of Characteristics (Cont.)

- For a horizontal and frictionless channel, the characteristic system of equation becomes:

$$V+2c = J^{+} \left[\frac{dx}{dt} = V+c \right]$$

$$V-2c = J^{-} \left[\frac{dx}{dt} = V-c \right]$$

constants

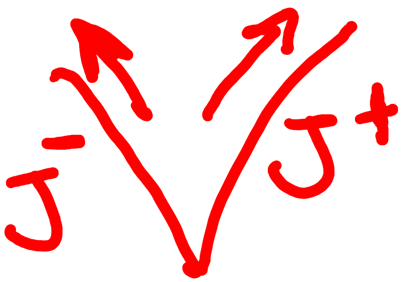
Riemann invariants

The **constants** *V+2c* and *V-2c* are called **Riemann invariants**

Method of Characteristics (Cont.)

Riemann invariants (For Circular channels):

(See León, A., Ghidaoui, M., Schmidt, A., and García, M. (2006). “Godunov-Type Solutions for Transient Flows in Sewers.” J. Hydraul. Eng., 132(8), 800–813.):



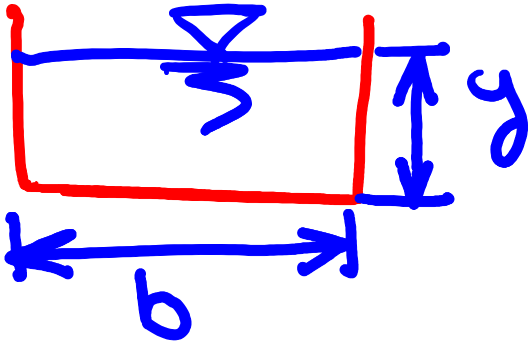
$$u_L + \phi_L = u_\star + \phi_\star = J^+ \left[\frac{dx}{dt} = V + c \right]$$

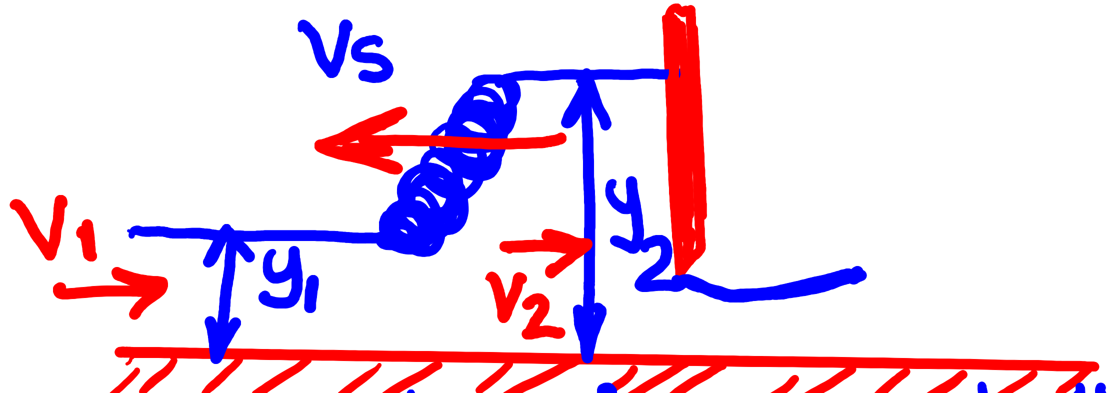
$$u_\star - \phi_\star = u_R - \phi_R = J^- \left[\frac{dx}{dt} = V - c \right]$$

$$\phi = \sqrt{g\frac{d}{8}} \int_0^\theta \frac{1 - \cos\theta}{\sqrt{(\theta - \sin\theta)\sin(\frac{\theta}{2})}}\, d\theta$$

$$\phi = \sqrt{g\frac{d}{8}} \left[\sqrt{3}\,\theta - \frac{\sqrt{3}}{80}\,\theta^3 + \frac{19\sqrt{3}}{448000}\,\theta^5 + \frac{\sqrt{3}}{10035200}\,\theta^7 + \frac{491\sqrt{3}}{27 \times 7064780800}\,\theta^9 + \dots \right]$$

Positive Surges in rectangular channels





Surge or bore is moving to the left with speed Vs

* choose control volume moving with the surge

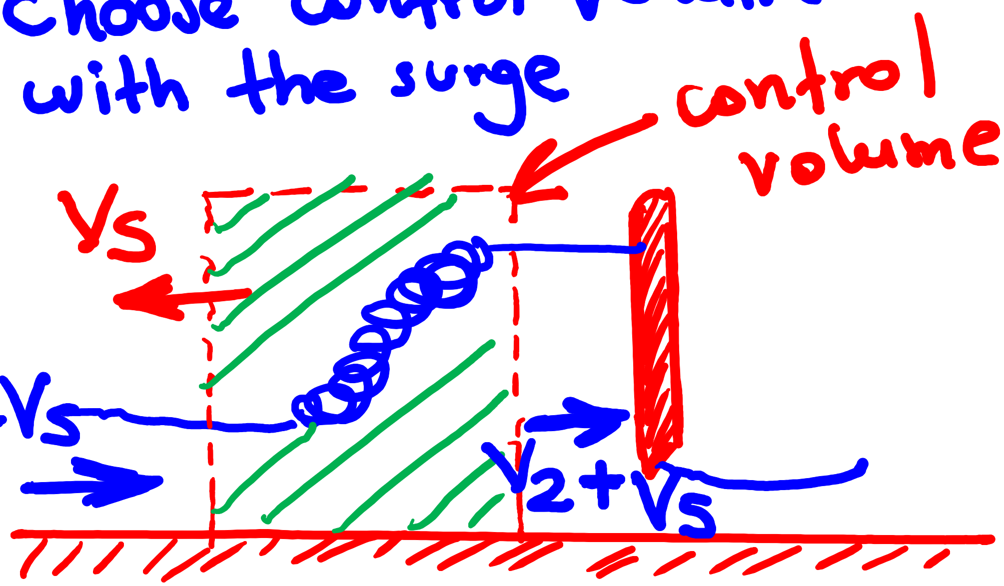
Continuity: $Q_1 = Q_2$

$$(V_1 + V_s)\cancel{b}\, y_1 = (V_2 + V_s)\cancel{b}\, y_2$$

$$V_1 y_1 + V_s y_1 = V_2 y_2 + V_s y_2$$

$$-V_s (y_2 - y_1) = V_2 y_2 - V_1 y_1$$

$$\boxed{V_2 y_2 = V_1 y_1 - V_s (y_2 - y_1)} \quad \ldots (1)$$



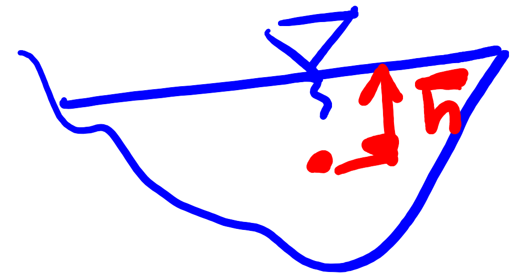
Momentum : $\Sigma F = \dot{m}(V_2 - V_1)$ $\qquad \dot{m} = \rho A V$

$$\Sigma F = \rho A_1 (V_1 + V_s)\left[V_2 + \cancel{V_s} - (V_1 + \cancel{V_s})\right]$$

$$\Sigma F = \rho b y_1 (V_1 + V_s)(V_2 - V_1)$$

$$F_{pressure} = \gamma \bar{h} A$$

$$F_1 = \rho g \frac{y_1}{2} b y_1$$



$$F_2 = \rho g \frac{y_2}{2} b y_2$$

For rectangular channels

$$\cancel{\rho} g \cancel{b} \frac{y_1^2}{2} - \cancel{\rho} g \cancel{b} \frac{y_2^2}{2} = \cancel{\rho} \cancel{b} y_1 (V_1 + V_s)(V_2 - V_1) \qquad \bar{h} = \frac{y}{2}$$

$$g\left(\frac{y_1^2 - y_2^2}{2}\right) = (V_1 + V_s)\, y_1 (V_2 - V_1)$$

$$= (V_1 + V_s)\left(V_2 y_2 \left[\frac{y_1}{y_2}\right] - V_1 y_1\right)$$

From ①

$$= (V_1+V_s)\left(\left[V_1 y_1 - V_s(y_2-y_1)\right]\frac{y_1}{y_2} - V_1 y_1\right)$$

$$= (V_1+V_s)\left[V_1\frac{y_1^2}{y_2} - V_s y_1 + V_s\frac{y_1^2}{y_2} - V_1 y_1\right]$$

$$= (V_1+V_s)\left(V_1\left[\frac{y_1^2}{y_2} - y_1\right] + V_s\left[\frac{y_1^2}{y_2} - y_1\right]\right)$$

$$= (V_1+V_s)\left(\frac{y_1^2 - y_1 y_2}{y_2}\right)(V_1+V_s)$$

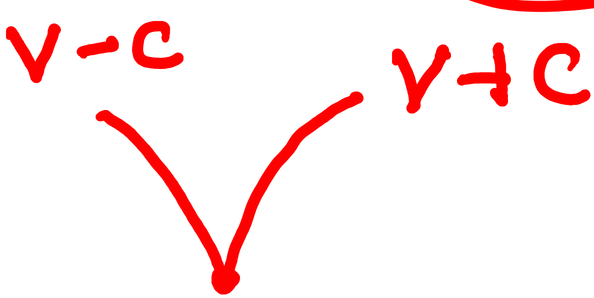
$$= (V_1+V_s)^2\frac{y_1}{y_2}(y_1-y_2)$$

$$g\frac{(y_1^2 - y_2^2)}{2} = (V_1 + V_s)^2 \frac{y_1}{y_2}(y_1 - y_2)$$

$$y_1^2 - y_2^2 = (y_1 + y_2)(y_1 - y_2)$$

$$g\left(\frac{y_1 + y_2}{2}\right) = (V_1 + V_s)^2 \frac{y_1}{y_2}$$

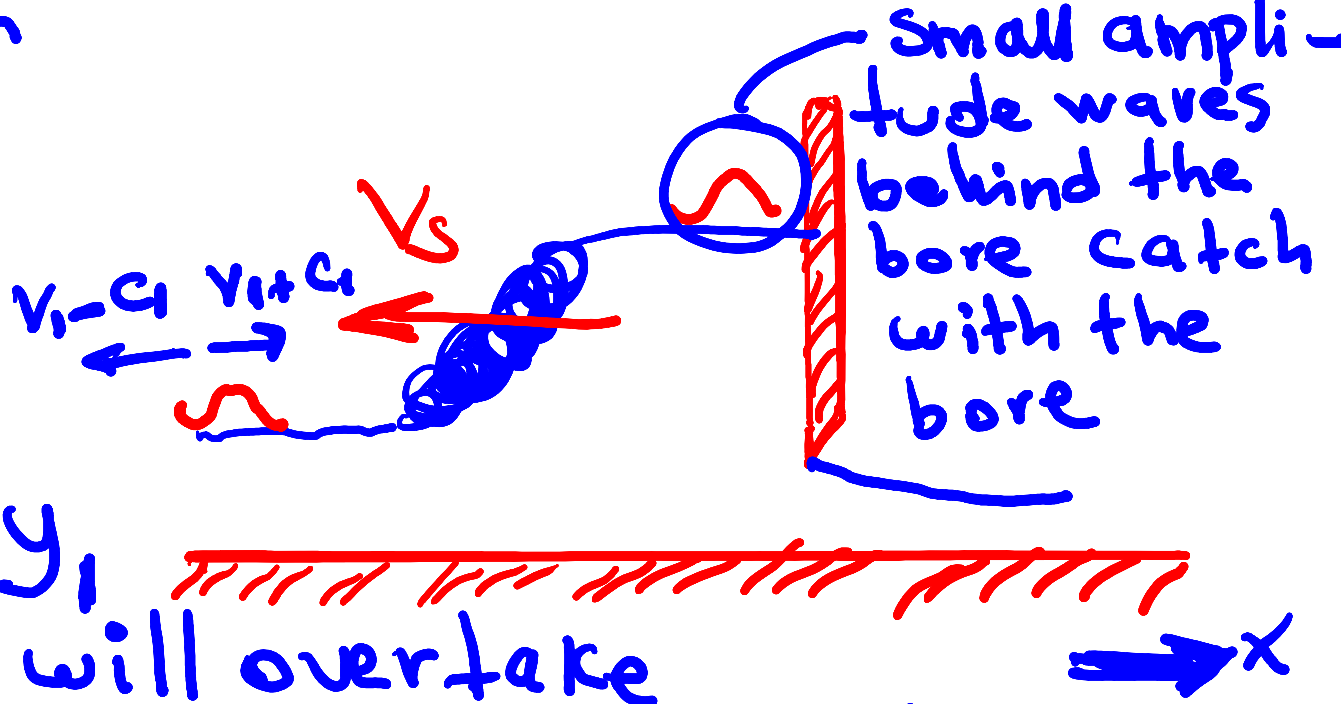
$$\boxed{V_s = -V_1 + \sqrt{g\left(\frac{y_1 + y_2}{2}\right)\frac{y_2}{y_1}}}$$



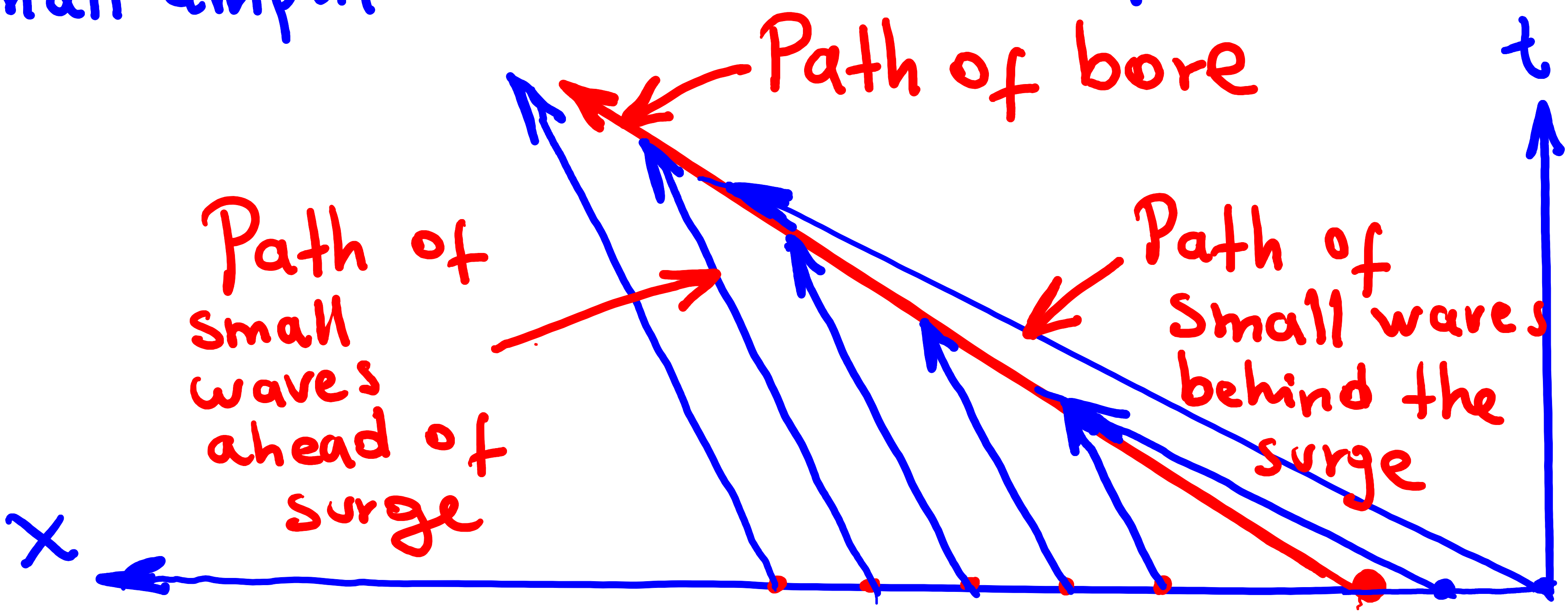
V_s : absolute velocity of the bore or surge

$$V_1 - C_1 = V_1 - \sqrt{g y_1}$$

$$\|V_1 - C_1\| = \sqrt{g y_1} - V_1$$



Because $y_2 > y_1$ surge or bore will overtake small amplitude waves ahead of it



what if $V_s = 0$ (steady hydraulic jump)

$$0 = -V_1 + \sqrt{g\frac{(y_1+y_2)}{2}\frac{y_2}{y_1}}$$

$$\frac{V_1^2}{gy_1} = \frac{\cancel{g}}{\cancel{g}y_1}\frac{(y_1+y_2)}{2}\frac{y_2}{y_1}$$

$$Fr_1 = \frac{V_1}{\sqrt{gy_1}}$$

$$Fr_1^2 = \frac{V_1^2}{gy_1}$$

$$Fr_1^2 = \frac{1}{2}\left[\frac{y_1}{y_1} + \frac{y_2}{y_1}\right]\left[\frac{y_2}{y_1}\right]$$

1

$$2Fr_1^2 = \left(1+\frac{y_2}{y_1}\right)\left(\frac{y_2}{y_1}\right) \rightarrow \left(\frac{y_2}{y_1}\right)^2 + \frac{y_2}{y_1} - 2Fr_1^2 = 0$$

$$ax^2+bx+c=0$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\frac{y_2}{y_1} = \frac{-1 \pm \sqrt{1^2-4(1)(-2Fr_1^2)}}{2(1)}$$

Not possible

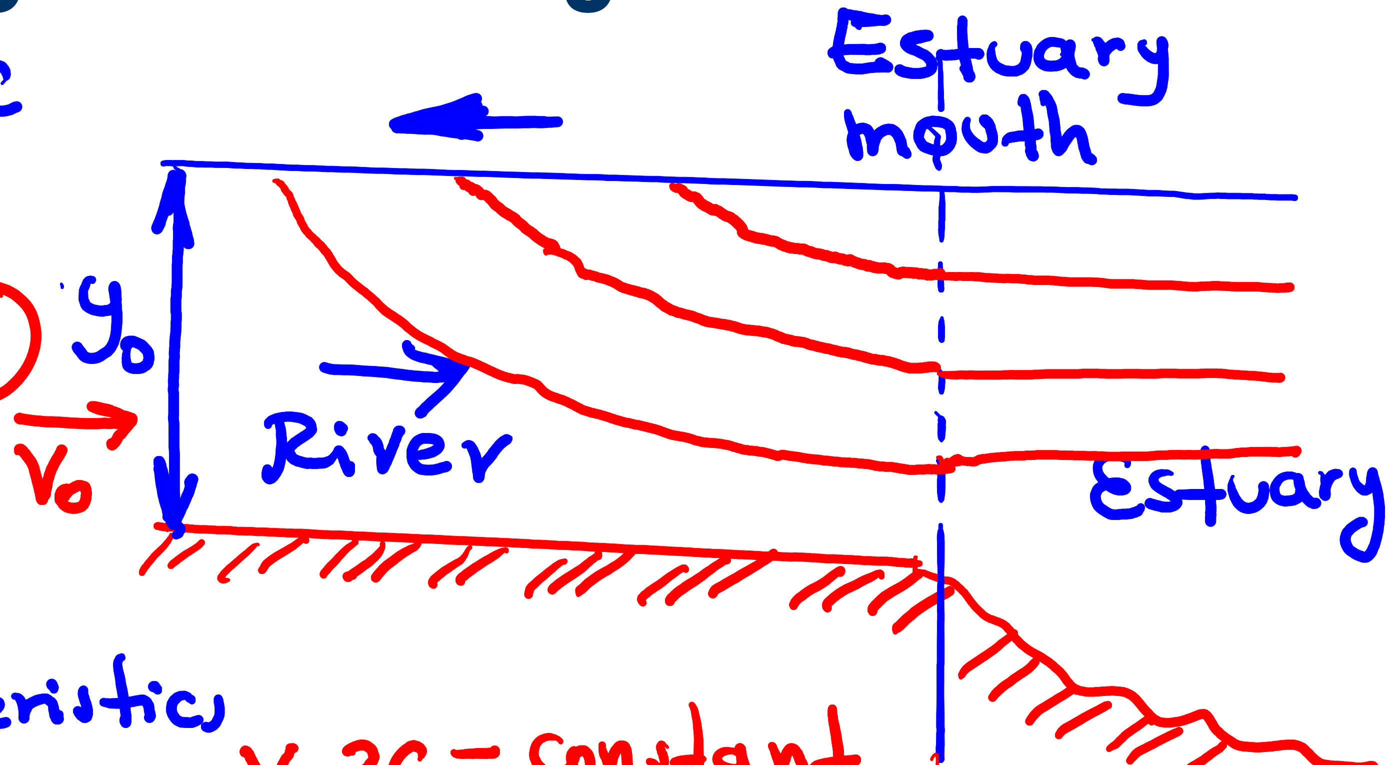
$$\boxed{\frac{y_2}{y_1} = \frac{-1 + \sqrt{1 + 8 Fr_1^2}}{2}}$$

Classical equation for steady hydraulic jumps.

Negative Surges in rectangular channels

$$V_0 + 2C_0 = V + 2C$$

$$\boxed{V = V_0 + 2C_0 - 2C} \quad (1)$$



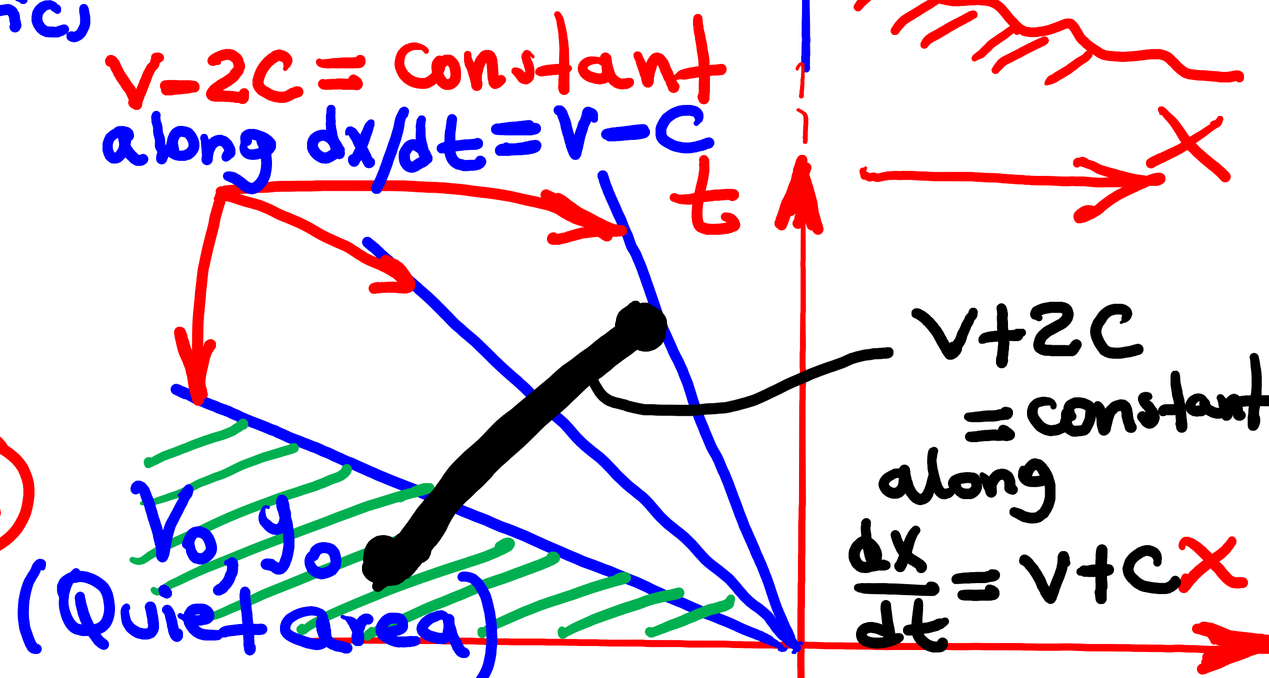
∴ * Wave motion

$$\frac{dx}{dt} = V - C$$

because characteristics are straight lines

$$\frac{x}{t} = V - C$$

$$\boxed{V = \frac{x}{t} + C} \quad (2)$$



Combining 1 and 2

$$\frac{x}{t} + c = V_0 + 2c_0 - 2c$$

$$3c = -\frac{x}{t} + V_0 + 2\sqrt{gy_0}$$

$$\boxed{c = \frac{-\frac{x}{t} + V_0 + 2\sqrt{gy_0}}{3}}$$

$$c = \sqrt{gy} \rightarrow y = \frac{c^2}{g}$$

$$\boxed{y = \frac{\left(-\frac{x}{t} + V_0 + 2\sqrt{gy_0}\right)^2}{9g}}$$

In 2

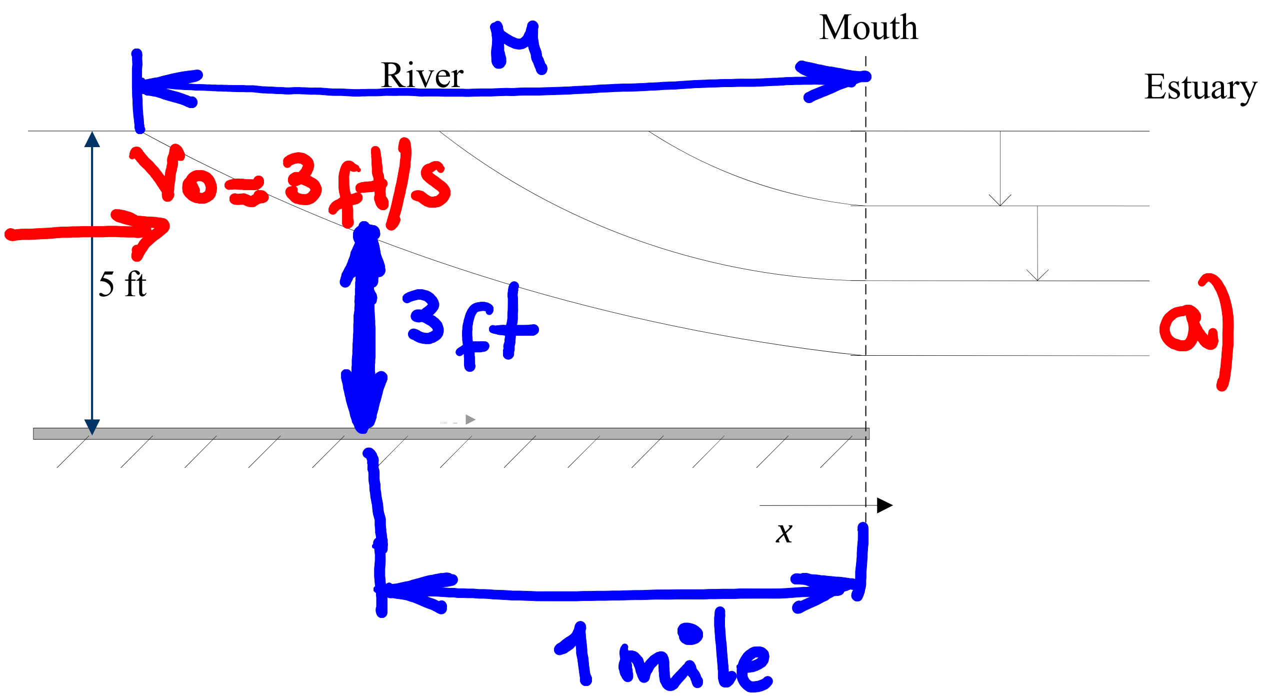
$$V = \frac{x}{t} + C$$

$$V = \frac{x}{t} + \frac{\left(-\frac{x}{t} + V_0 + 2\sqrt{gy_0}\right)}{3}$$

$$\boxed{V = \frac{1}{3}\left(\frac{2x}{t} + V_0 + 2\sqrt{gy_0}\right)}$$

Example of Application:

Water flows at a uniform depth of 5ft and V_o = 3ft/s in a rectangular channel. The outlet consists of a large estuary. The initial level of the estuary is equal to that in the channel. The level in the estuary commences to fall at a rate of 1ft/hr for 3hrs. How long does it take for the river level to fall by 2ft at a section 1 mile upstream of the mouth? At this time, how far upstream will the river level just begin to start falling? Note: 1 mile = 5280 ft and 1 ft = 0.3048 m.



a) time ?

b) M ?

a) $x = -1$ mile

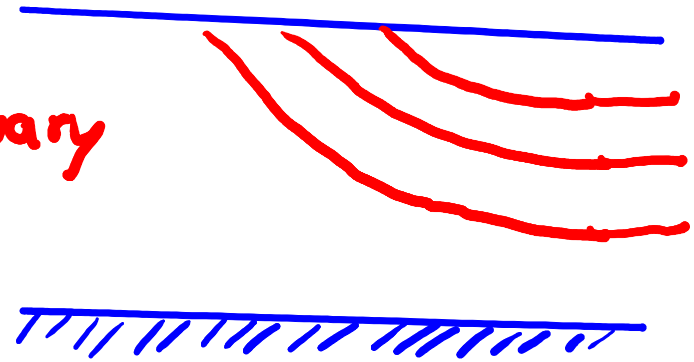
$= -5280$ ft

$y = 3$ ft

$y_o = 5$ ft

$V_o = 3$ ft/s

time = time 1 + time 2



time 1 = time to propagate from estuary mouth to 1 mile upstream

time 2 = time for water level at estuary mouth to drop 2 ft

time 2 = 2 hr

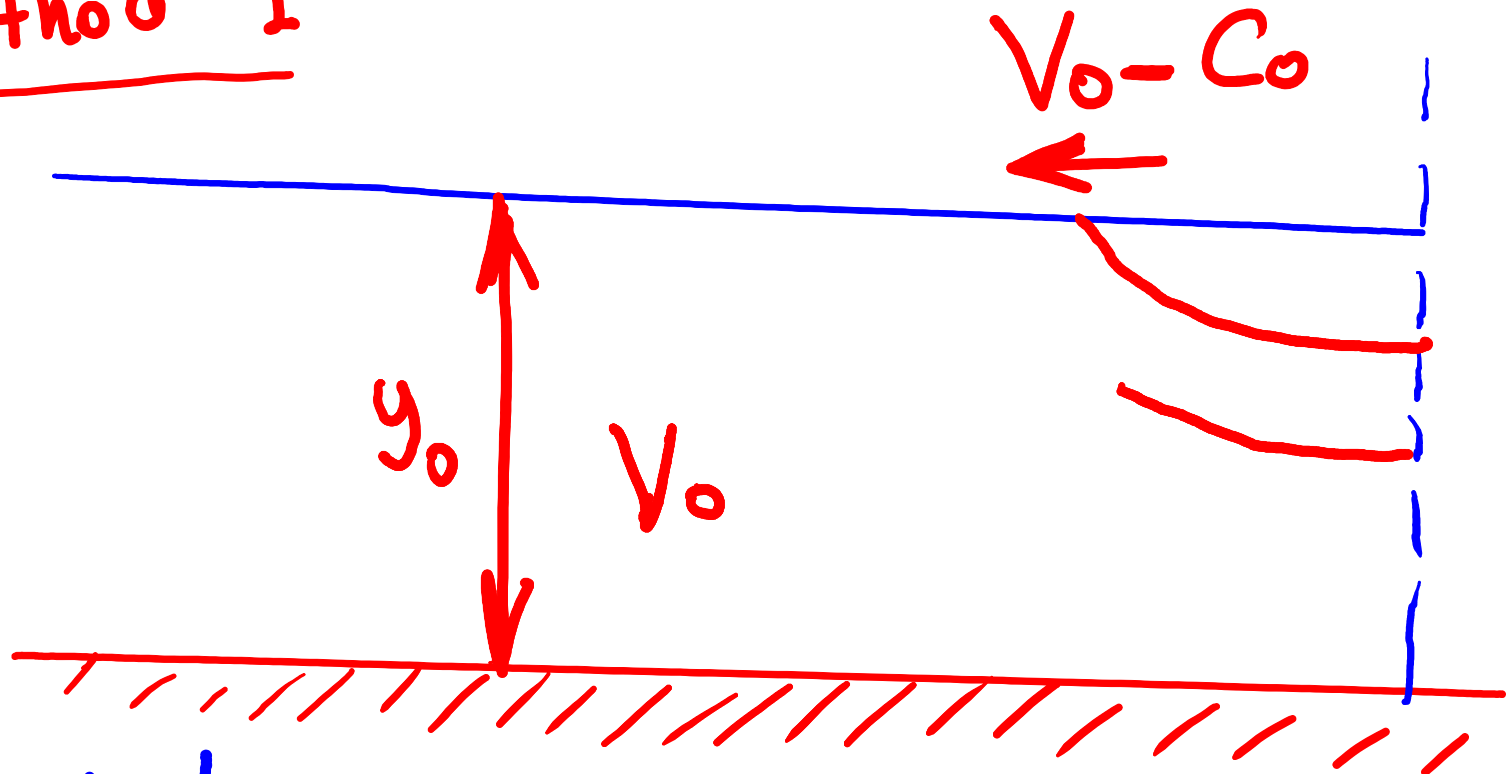
$$* \quad \frac{x}{t} = V_0 + 2\sqrt{g y_0} - 3\sqrt{g y}$$

$$\frac{-5280}{t} = 3 + 2\sqrt{32.2 \times 5} - 3\sqrt{32.2 \times 3}$$

$$t_1 = 4766\ s = 79.4\ min$$

$$\therefore \quad time = 120 + 79.4 = \underline{199.4\ min}$$

b) Method 1



$$e = v \cdot t$$

$$M = \left(3 - \sqrt{32.2 \times 5}\right) 199.4 \times 60\ s$$

$$\boxed{M = -21.95 \text{ miles}}$$

* Method 2:

$$\frac{x}{t} = V_0 + 2\sqrt{g y_0} - 3\sqrt{g y}$$

Note: initial water depth at estuary mouth is 5 ft. ["t" will be total time]

$y = 5\,ft$

$t = 199.4\ min$

$= 11{,}964\ s$

$$\frac{x}{11964} = 3 + 2\sqrt{32.2 \times 5} - 3\sqrt{32.2 \times 5}$$

$x = -115914\ ft$

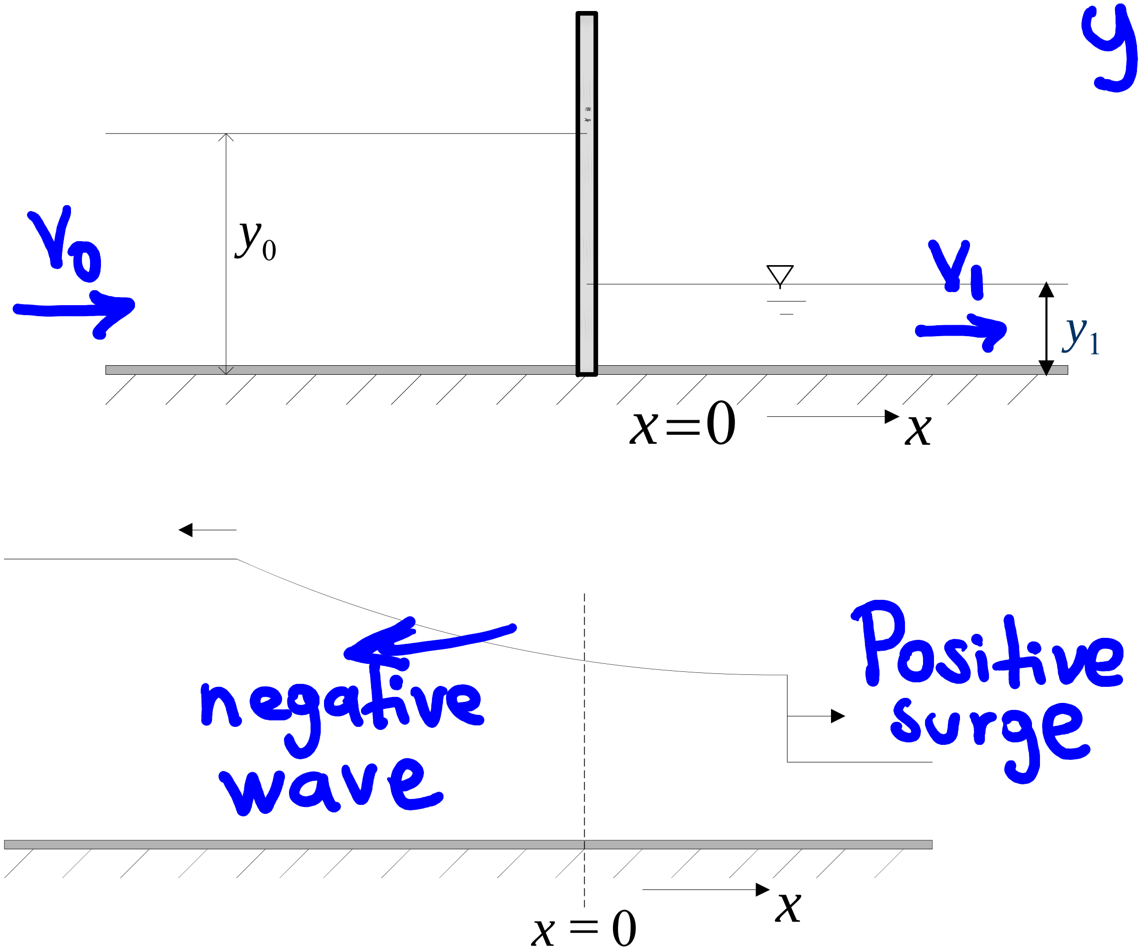
$x = -21.95\ miles$

Dam break problem and its solution

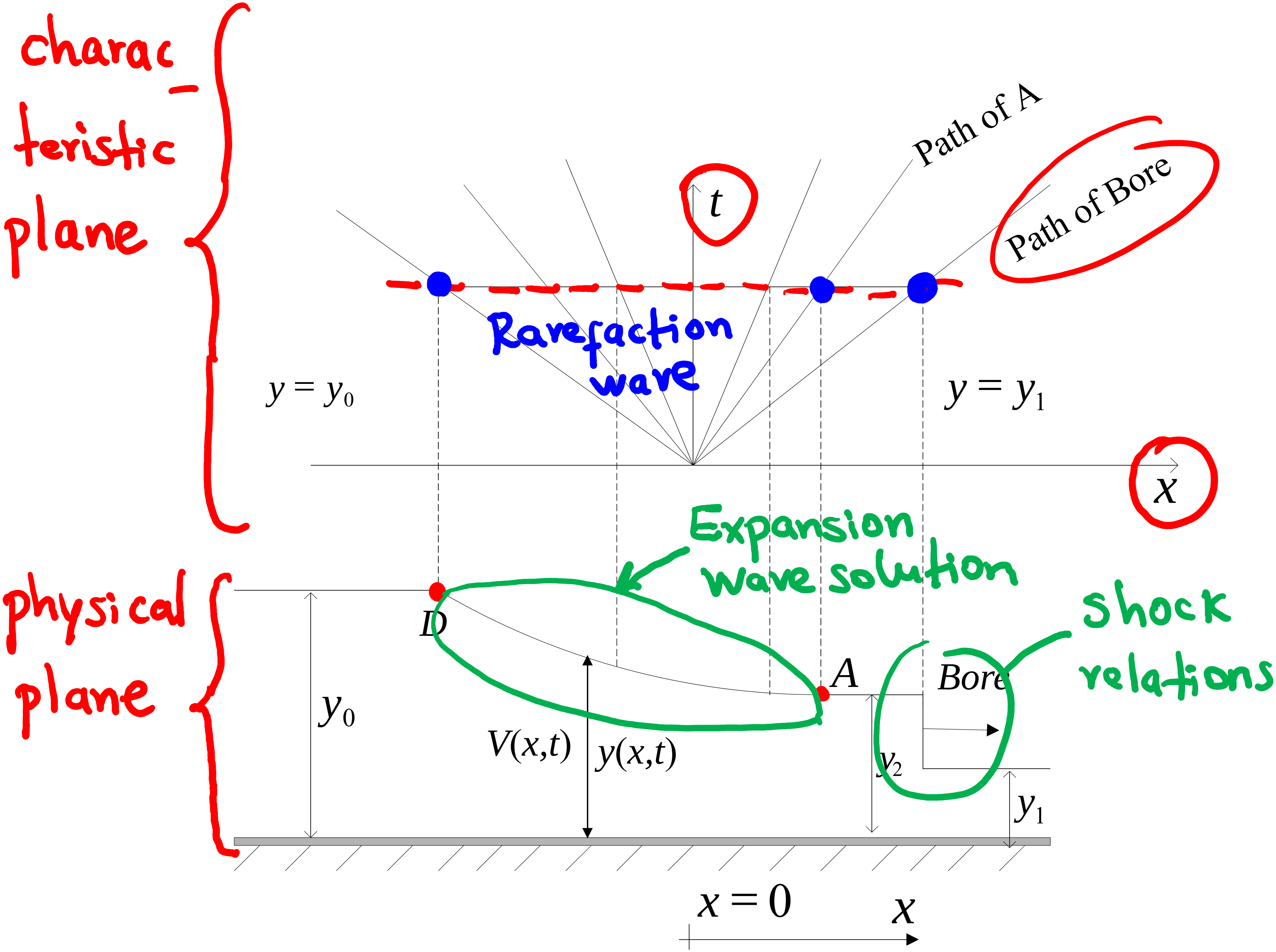
A sudden dam collapse results in a large flow at dam location, which in turn leads to a positive surge downstream of dam and a negative surge upstream of dam.

Given (known) quantities are:

y_1, y_0, V_1, V_0



Physical Plane



charac-
teristic
plane
physical
plane
t
x
Path of A
Path of Bore
Rarefaction
wave
$y = y_0$
$y = y_1$
Expansion
wave solution
Shock
relations
D
A
Bore
y_0
$V(x,t)$
$y(x,t)$
y_2
y_1
$x = 0$
x

Solution in the rarefaction or expansion wave region $X_D \leq X \leq X_A$

$$y_{(x,t)} = \frac{\left(-\frac{x}{t} + V_0 + 2\sqrt{g y_0}\right)^2}{9g}$$

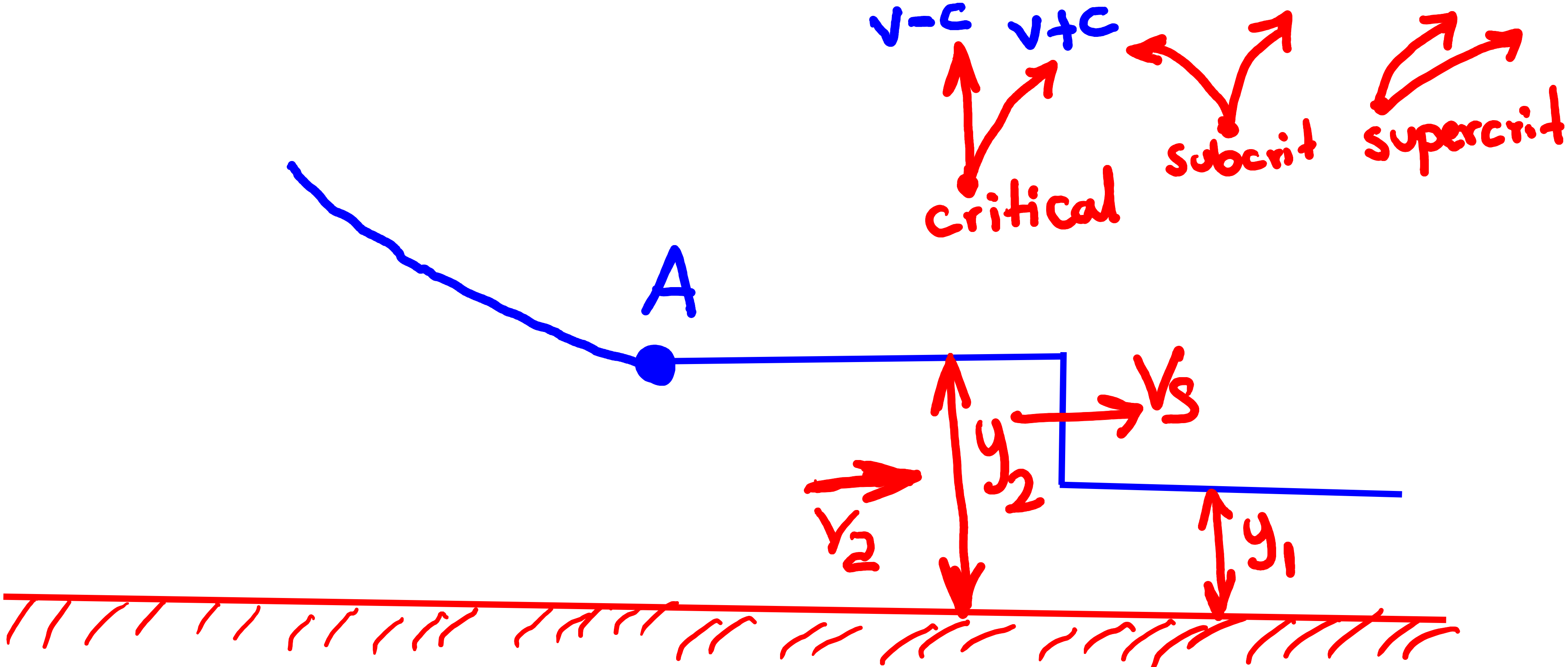
Solution upstream of rarefaction wave $(x < X_D)$

$y = y_0$

if $V_0 = 0$, $X_D = t(-\sqrt{g y_0})$

*Path of point A

what if $V_2 = C_2$, then poin A remains fixed



$$\frac{dx}{dt} = V - C \rightarrow \frac{x}{t} = V - C \quad (S_o = S_f)$$

$$x_A = t(V_2 - c_2)$$

$$c_2 = \sqrt{g y_2}$$

$$x_A = t(V_2 - \sqrt{g y_2})$$

if $V_2 > \sqrt{g y_2}$, A moves downstream

if $V_2 < \sqrt{g y_2}$, A moves upstream

* With some algebra:

Point A moves downstream if $\frac{y_1}{y_0} < 0.14$

" " " upstream if $\frac{y_1}{y_0} > 0.14$

* If point "A" moves downstream, dam break location falls within rarefaction wave.

Thus, at $x=0$ (dam break location)

$$y_{x=0,t} = \frac{\left(0+0+2\sqrt{gy_0}\right)^2}{9g} = \frac{4\cancel{g}\,y_0}{9\cancel{g}}$$

$$\boxed{y_{x=0,t} = \frac{4}{9}y_0}$$

$$V = \frac{1}{3}\left(\cancel{\frac{2x}{t}}^{\,0} + \cancel{V_0}_{\,0} + 2\sqrt{g y_0}\right) = \frac{2}{3}\sqrt{g y_0}$$

$$C = \frac{2}{3}\sqrt{g y_0}$$ [Left for you as exercise]

$V = C$ (critical flow)

$$\underset{\text{flow rate}}{Q} = V \cdot A = \frac{2}{3}\sqrt{g y_0}\; b\left(\frac{4}{9} y_0\right) = \frac{8b}{27}\sqrt{g}\, y_0^{3/2}$$

Example : $y_0 = 50\,m$, $b = 10\,m$

$V_0 = 0$

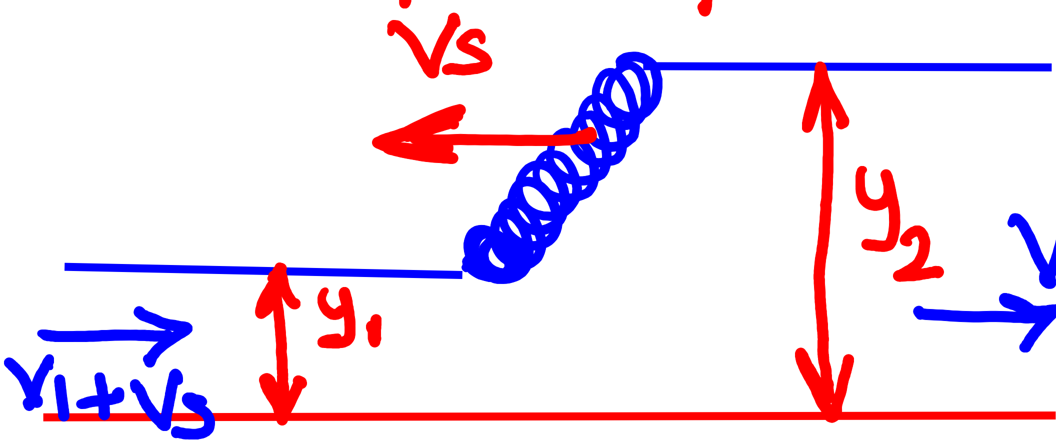
$$V = \frac{2}{3}\sqrt{9.8 \times 50} = 14.7\,m/s$$

$$Q = \frac{8}{27} \times 10 \times \sqrt{9.8}\,(50)^{3/2} = 3279\ m^3/s$$

* Solution inside the bore region

$$x_A < x < x_{bore}$$

For our previous problem

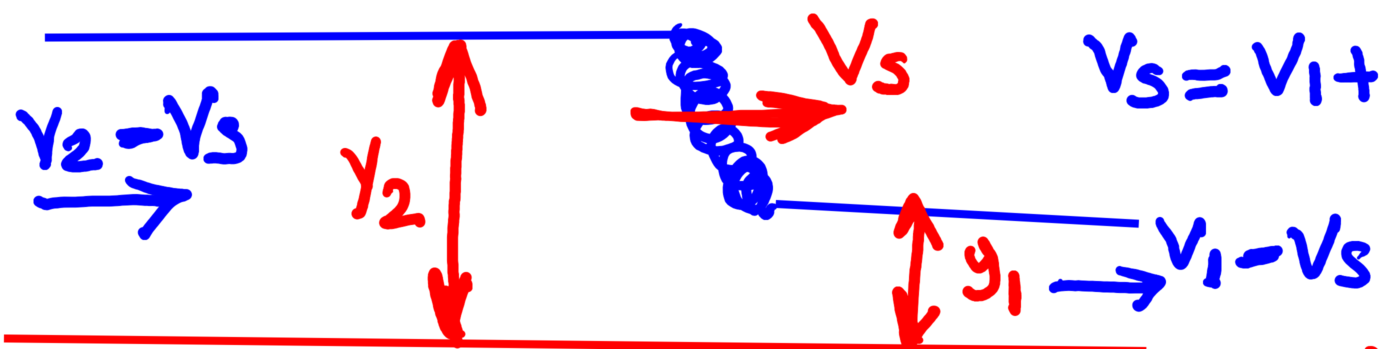


[1] $-V_s(y_2-y_1)=V_2y_2-V_1y_1$

[2] $V_s=-V_1+\sqrt{g\frac{(y_1+y_2)}{2}\frac{y_2}{y_1}}$

For our dam break problem

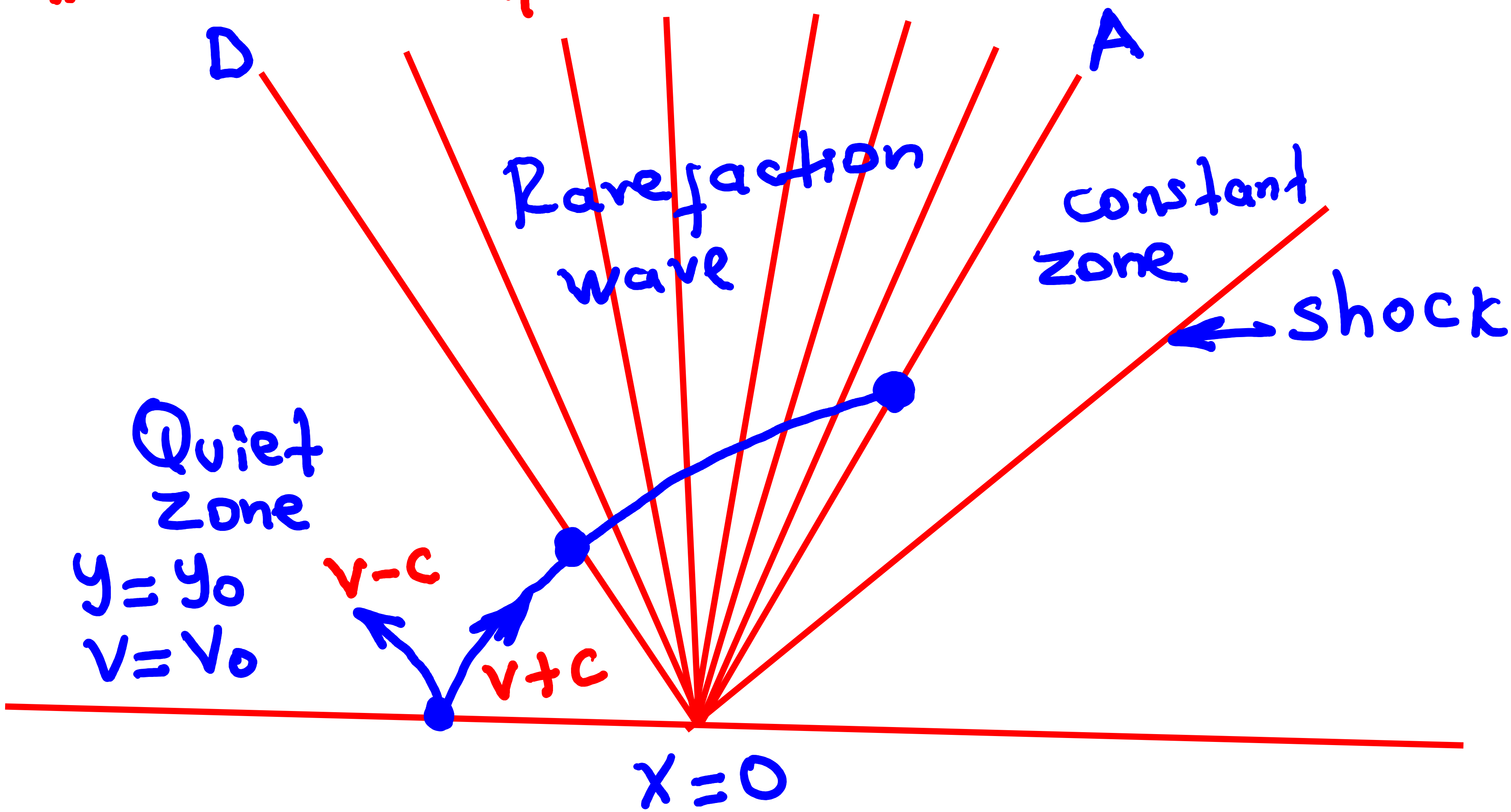
$$V_s(y_2-y_1)=V_2y_2-V_1y_1 \quad (1)$$



$$V_s=V_1+\sqrt{g\frac{(y_1+y_2)}{2}\frac{y_2}{y_1}} \quad (2)$$

Unknowns:
V_s, V_2, y_2

* One more equation is needed.



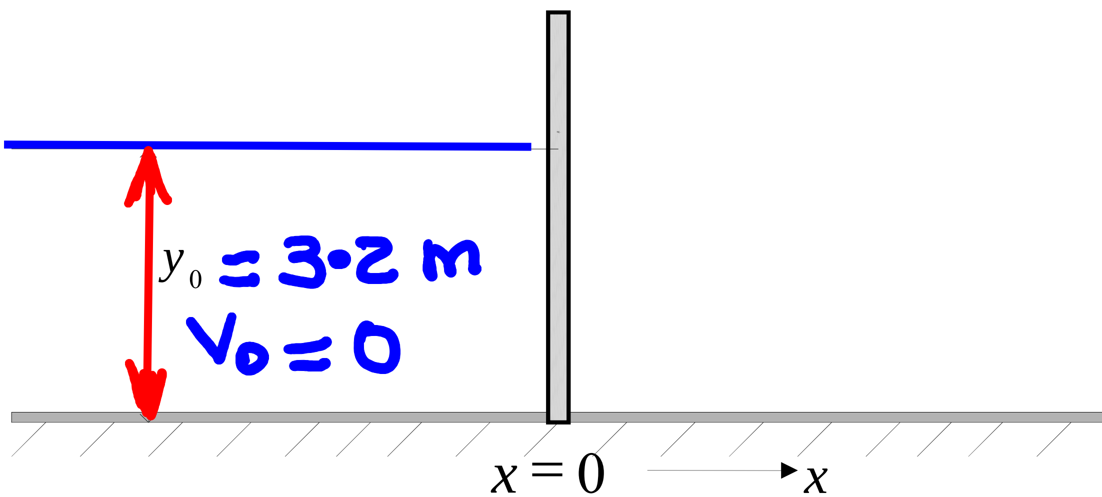
Along $\frac{dx}{dt} = v + c$, $V_0 + 2C_0 = V_2 + 2C_2$

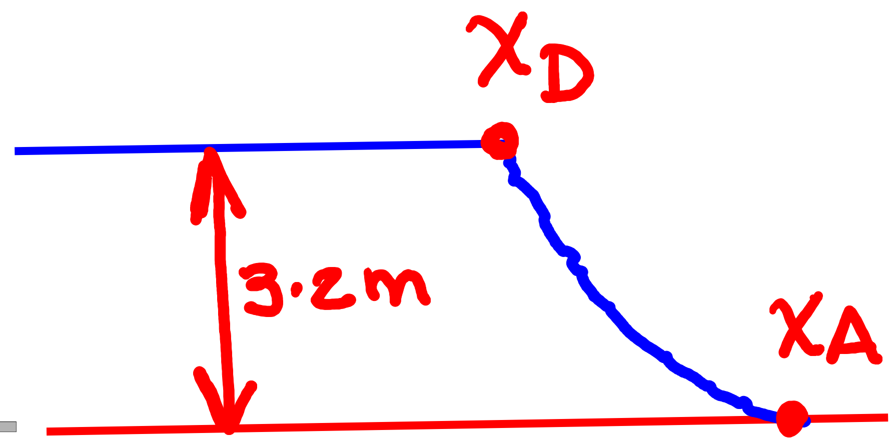
$$V_0 + 2\sqrt{g y_0} = V_2 + 2\sqrt{g y_2} \quad \cdots (3)$$

$\therefore$ V_s, V_2 and y_2 can be now determined.

Example of Application:

Stagnant water of depth 3.2 m is stored in a long rectangular irrigation channel. A maneuver error resulted in the sudden opening of the gate. Determine the solution of the water surface profile after 5, 30 and 60 minutes of gate opening.





* If $x < X_D$, $y = 3.2$ m

* Inside rarefaction wave

$$y_{x,t} = \frac{\left(-\frac{x}{t} + V_0 + 2\sqrt{g y_0}\right)^2}{9g}$$

* How fast x_D is moving

$$3.2 = \frac{\left(-\frac{x_D}{t} + 0 + 2\sqrt{9.8 \times 3.2}\right)^2}{9 \times 9.8}$$

$$x_D = -5.6t$$

after 5 min (300 seconds)

$$x_D = -1680\ m$$

* How fast x_A is moving

$$0 = \frac{\left(-\frac{x_A}{t} + 0 + 2\sqrt{9.8 \times 3.2}\right)^2}{9 \times 9.8}$$

$$x_A = 11.2t$$

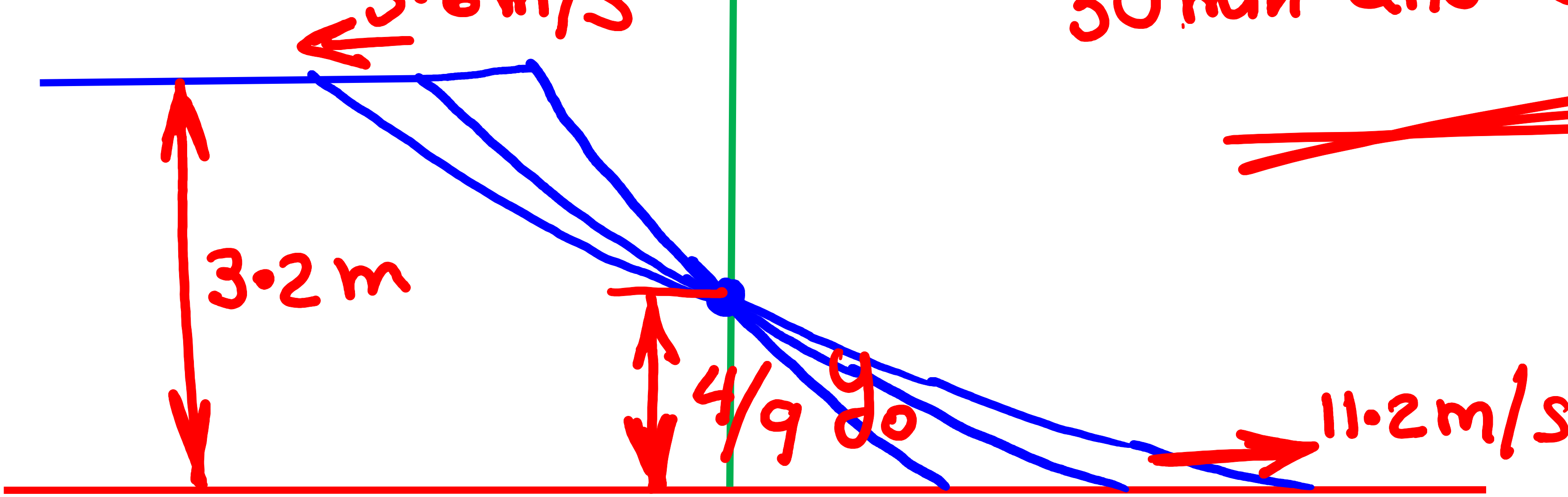
* Full solution

$y = 3.2$ if $x < -5.6t$

$$y = \frac{\left(-\frac{x}{t} + v_0 + 2\sqrt{g y_0}\right)^2}{9g} \qquad -5.6t \le x \le 11.2t$$

$y = 0 \quad x > 11.2t$

Plot this solution for $t = 5$ min (300 sec), 30 min and 60 min



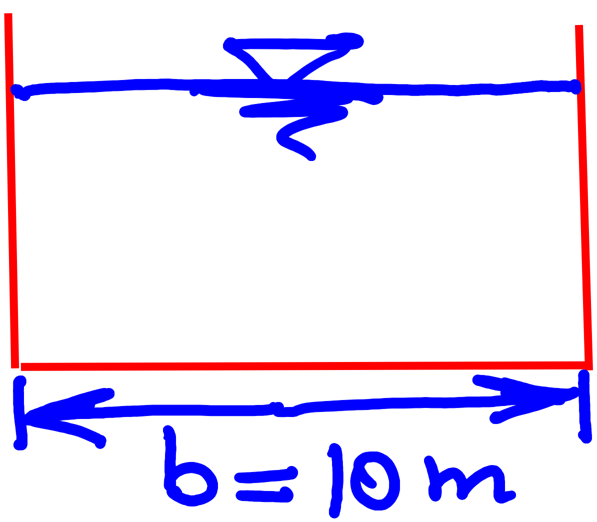
Example of Application:

The headrace channel in a hydroelectric power plant has a rectangular cross section with a width of b =10.0 m. This channel has a bed slope of S_o= 0.002 and the coefficient of Manning is n = 0.02. During the normal turbine operation the flow is uniform, having a discharge of Q = 40 m^3/s. Due to a sudden load rejection, the gate in the headrace channel is partially, but rapidly closed, thus decreasing the discharge to Q_f = 0.5 m^3/s.

(i) Determine the hydraulics of the waves traveling downstream and upstream from the gate;

(ii) Determine the type of front for these waves; (bore, negative wave)

(iii) How long will it take for the wave to arrive at a station located at a distance of 0.5 km upstream from the gate?

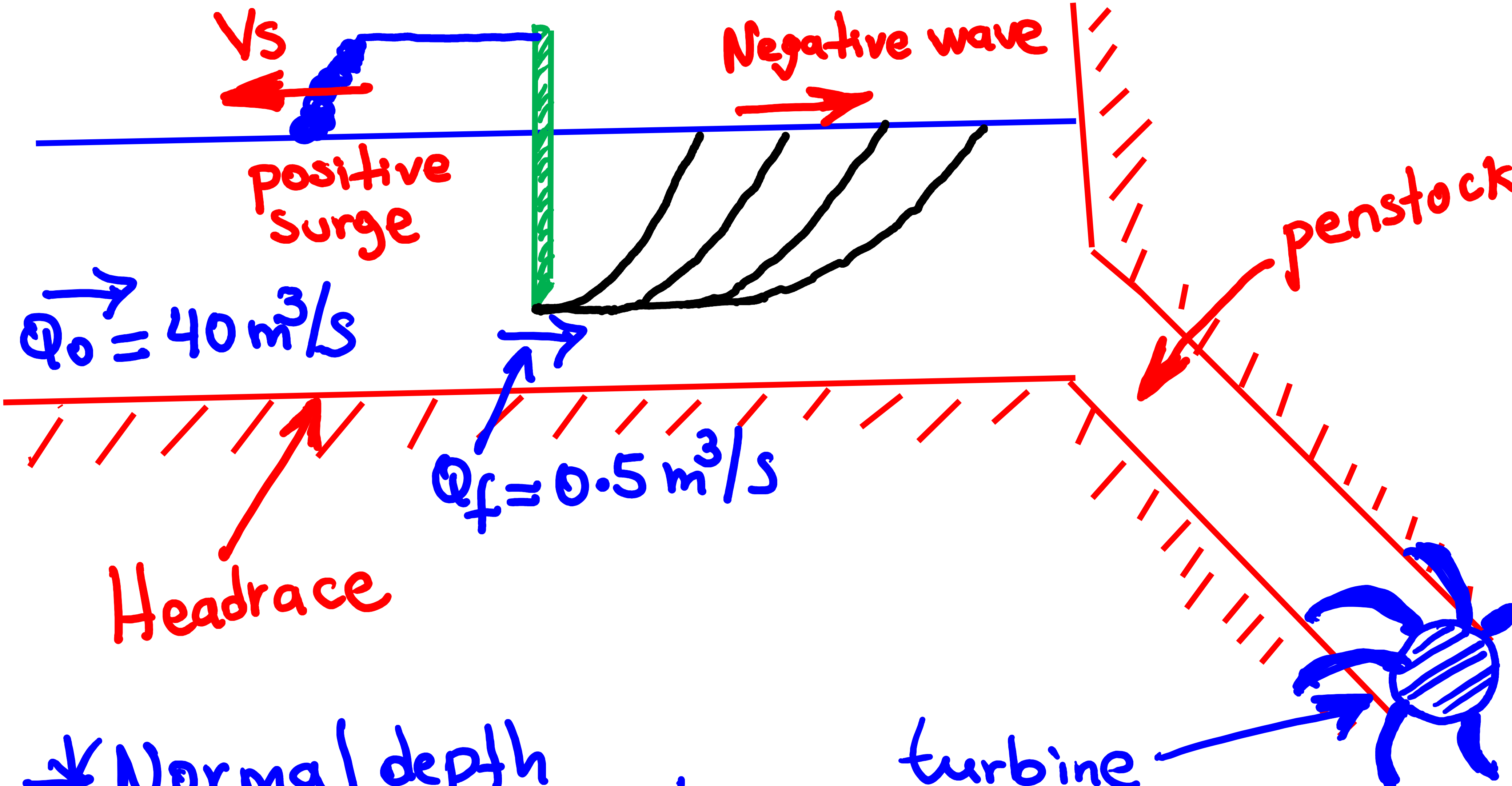


$S_o = 0.002$

$n = 0.02$

$Q_0 = 40\ m^3/s$

$Q_f = 0.5\ m^3/s$ (after partial closure)



* Normal depth

$$Q_0 = \frac{1}{n} A R^{2/3} S_o^{1/2}$$

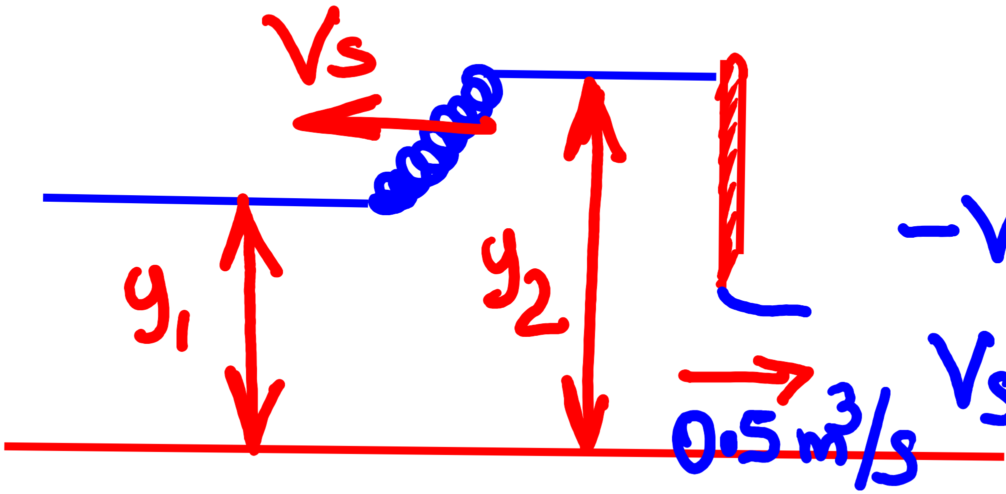
$$40 = \frac{1}{0.02} \frac{(10y)^{5/3}}{(10+2y)^{2/3}} \times 0.002^{1/2}$$

$y_0 = 1.58\ m$, $V_0 = \frac{Q}{A} = \frac{40}{10 \times y_0}$, $V_0 = 2.53\ \frac{m}{s}$

$$F_0 = \frac{V_0}{\sqrt{g y_0}} = \frac{2.53}{\sqrt{9.8(1.58)}} = 0.64$$

(subcritical flow)

(i) Positive surge



$y_1 = 1.58\ m$

$V_1 = 2.53\ m/s$

$$-V_s(y_2 - y_1) = V_2 y_2 - V_1 y_1 \quad (1)$$

$$V_s = -V_1 + \sqrt{g\frac{(y_2+y_1)}{2}\frac{y_2}{y_1}} \quad (2)$$

$$0.5 = V_2(10\, y_2)$$

$$V_2 = 0.5/(10\, y_2) \cdots (3)$$

3 in 1

$$-V_s(y_2-1.58)=\frac{0.5\,\cancel{y_2}}{10\,\cancel{y_2}}-2.53\times 1.58$$

$$V_s=-\frac{\left(\frac{0.5}{10}-2.53\times 1.58\right)}{y_2-1.58}\quad\cdots\ 4$$

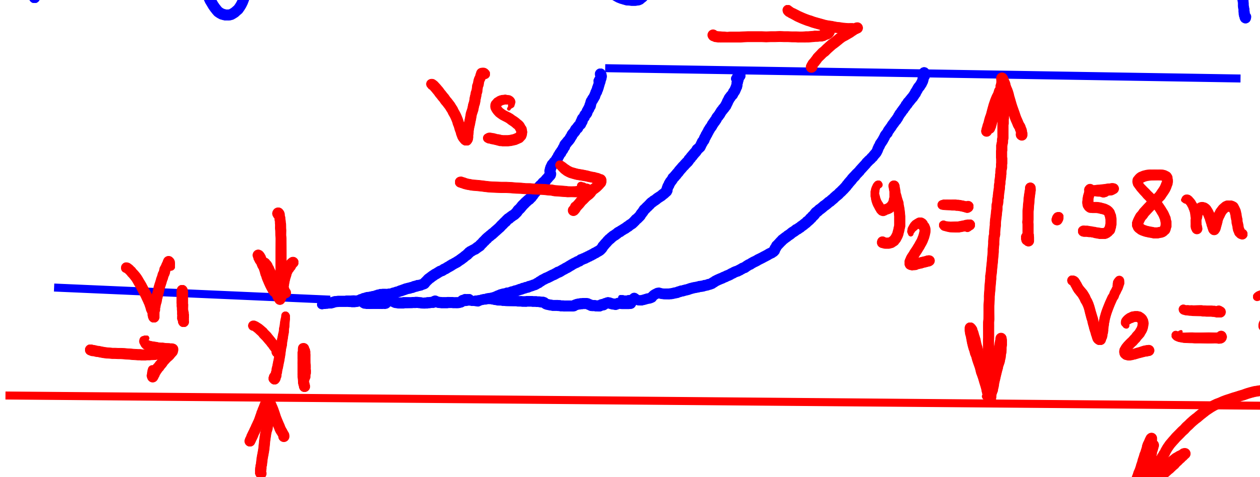
2 = 4

$$-2.53+\sqrt{\frac{9.8(y_2+1.58)}{2}\,\frac{y_2}{1.58}}=\frac{-\left(\frac{0.5}{10}-2.53\times 1.58\right)}{y_2-1.58}$$

$\boxed{y_2=2.71\,m}$, $\boxed{V_2=0.018\ m/s}$

$\boxed{V_s=-3.49\ m/s}$

* Negative surge [Wave front is unstable and tends to flatten out]



Unknowns
V_1, y_1, V_S

* Continuity : $V_S = \frac{V_2 y_2 - V_1 y_1}{y_2 - y_1}$ ①

$V_1 y_1 (10) = 0.5$

$V_1 = \frac{0.05}{y_1}$ ③

$V_S = V_1 + \sqrt{\frac{g(y_1+y_2)}{2} \frac{y_2}{y_1}}$ ②

③ in ① and then ① = ②

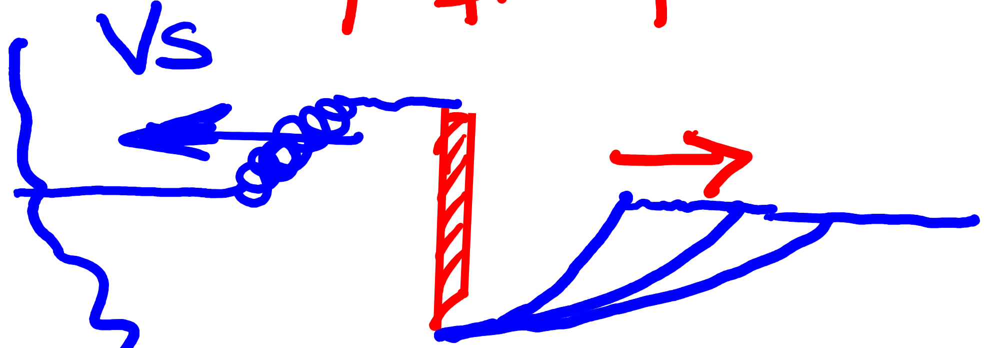
Solve for y_1,

$y_1 = 0.776 m$

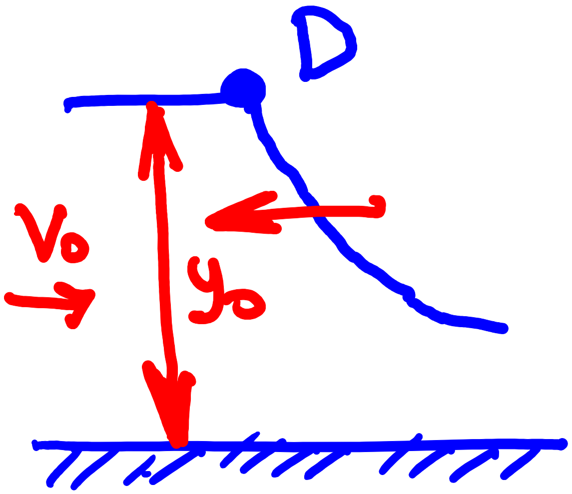
$V_S = 4.91 m/s$

$V_1 = 0.064 m/s$

Now, let's calculate the water profile for the negative surge.

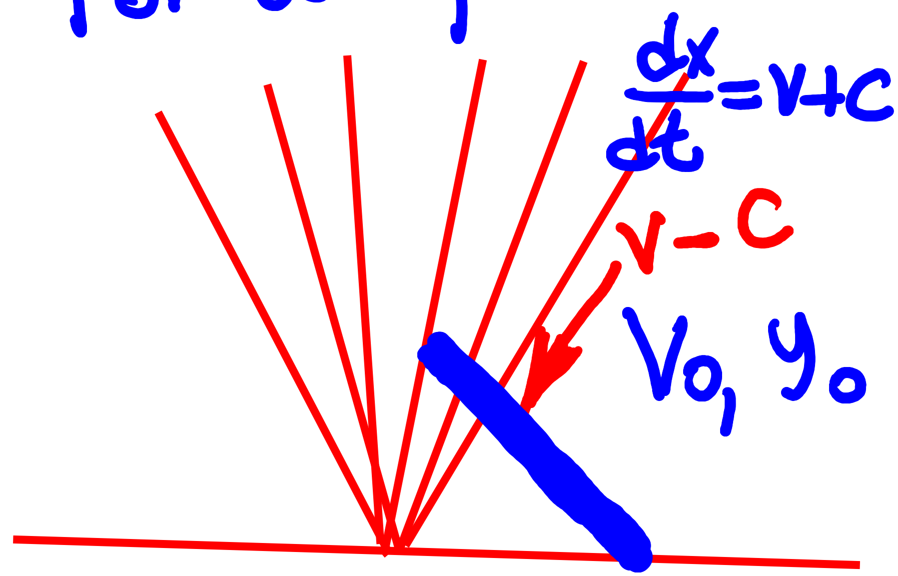


* For our dam bream problem



$$\frac{x}{t} = V_o + 2\sqrt{g y_o} - 3\sqrt{g y}$$

For our problem



$$V_o - 2C_o = V - 2C \quad (1)$$

$$\frac{dx}{dt} = V + C \quad (2)$$

* Because characteristic are straight lines for our case $\frac{x}{t} = v + c \quad \ldots (3)$

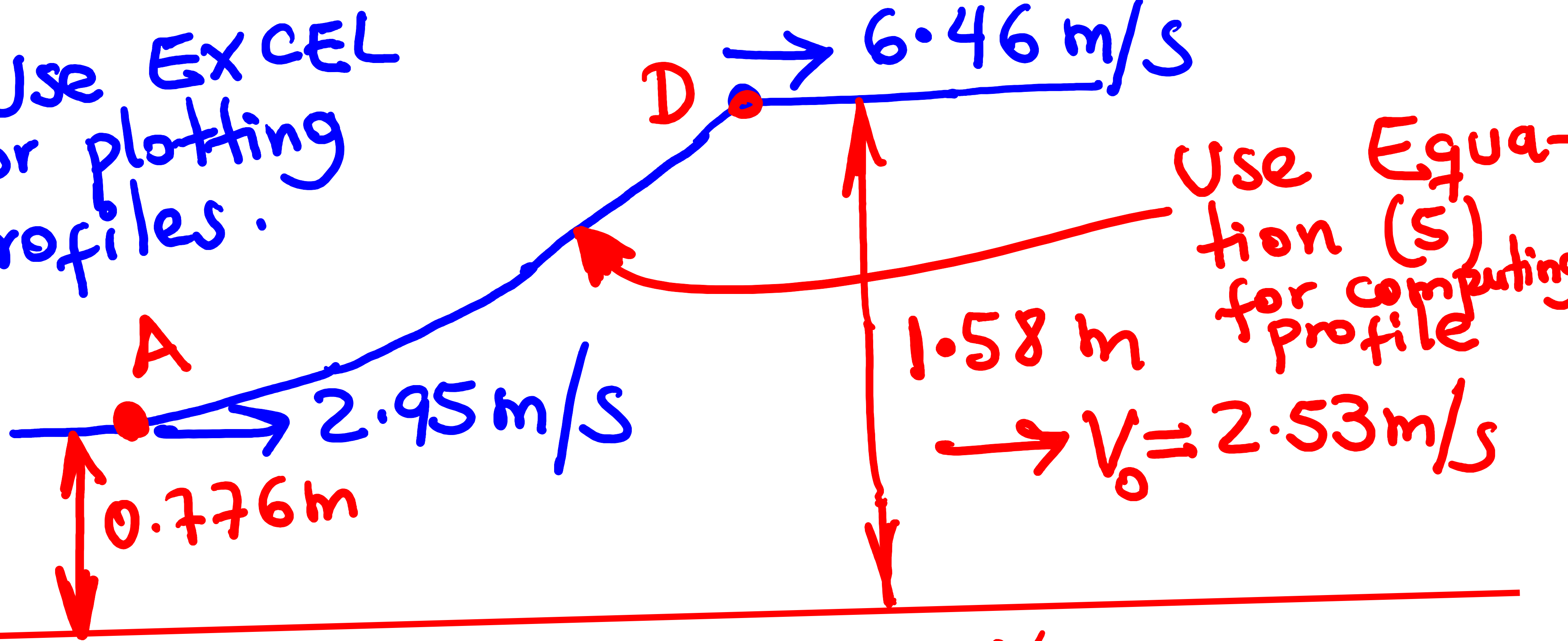
From (1) $\quad v = V_0 - 2C_0 + 2C \quad \ldots (4)$

(4) in (3)

$$\frac{x}{t} = V_0 - 2C_0 + 2C + C$$

$$\frac{x}{t} = V_0 - 2C_0 + 3\sqrt{gy} \quad (5)$$

Use EXCEL for plotting profiles.



* Locations of x_D and x_A as a function of time

* $$\frac{x_D}{t} = 2.53 - 2\sqrt{9.8 \times 1.58} + 3\sqrt{9.8 \times 1.58}$$

$$x_D = 6.46\,t$$

Likewise,

$$x_A = 2.95\,t$$

ii)

* Shock (steep front)
* rarefaction wave (initial negative wave is step and flattens out with time

iii) $e = v \cdot t$

$500 = V_s \cdot t$

3·49 m/s

$t = \frac{500}{3 \cdot 49} = 144\,s$