Gradually varied flow

$\frac{dy}{dx} \ll 1$

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$\frac{dV}{dx} \ll 1$

Gradually varied flow (GVF)

- Steady non-uniform flow in a prismatic channel with **gradual changes in its water surface elevation**
- For example,
  - **backwater** produced by a dam or weir across a river
  - **drawdown** produced at a sudden drop in a channel
- In GVF
  - **velocity varies** along the channel
  - bed slope, water surface slope, and energy **slope will all differ from each other**

Gradually varied flow (GVF)

- Two basic assumptions in GVF analysis
  - **Pressure distribution at any section is assumed to be hydrostatic**
    - Gradual changes in the surface curvature give rise to **negligible normal accelerations**
  - **Resistance to flow** at any depth is assumed to be given by the corresponding **uniform flow equation**, such as the Manning's formula
    - with the condition that the **slope term to be used** in the equation is the **energy slope (S_e)** and not the bed slope

Hence, S_e is often replaced by S_f $\quad S_f = \frac{n^2 V^2}{R^{4/3}}$ (SI units)

Differential equation of GVF

$\alpha \approx 1.0$

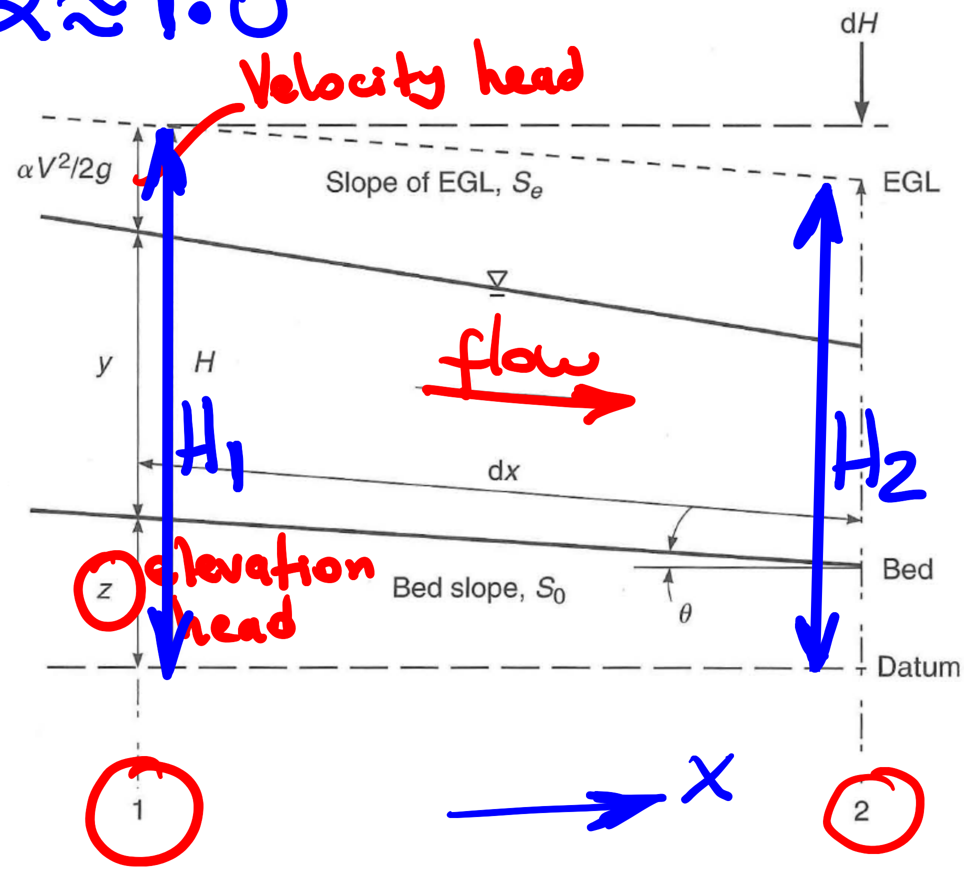
$$H = z + y + \frac{\alpha V^2}{2g}$$

$$S_o = \frac{z_1 - z_2}{x_2 - x_1} = -\frac{dz}{dx}$$

$$S_e = \frac{H_1 - H_2}{x_2 - x_1} = -\frac{dH}{dx}$$

$$S_f = \frac{\tau_o}{\gamma R}$$

$$S_e \approx S_f$$



Where:

S_e = energy grade line slope

S_f = friction slope

S_o = bed slope

$$H = z + y + \frac{V^2}{2g} = z + y + \frac{Q^2}{2gA^2} \qquad V = \frac{Q}{A}$$

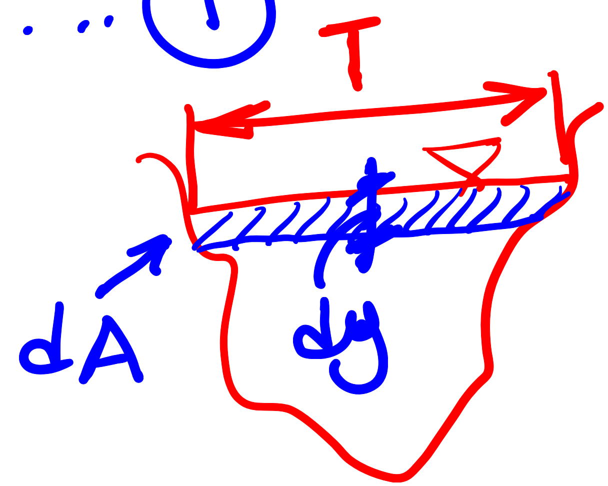
$$d(x^n) = nx^{n-1}$$

$$\frac{dH}{dx} = -S_e = -S_f$$

$$-S_f = \underbrace{\frac{dz}{dx}}_{-S_o} + \frac{dy}{dx} + \frac{Q^2}{2g}\frac{d}{dx}\left(A^{-2}\right)$$

$$-S_f = -S_o + \frac{dy}{dx} + \frac{Q^2}{\cancel{2g}}(-\cancel{2})A^{-3}\frac{dA}{dx}$$

$$S_o - S_f = \underline{\frac{dy}{dx}} - \frac{Q^2}{gA^3}\underbrace{\frac{dA}{dy}}_{\approx T}\cdot\underline{\frac{dy}{dx}} \quad \ldots \text{(1)}$$



$$dA \approx T\,dy$$

$$\frac{dA}{dy} \approx T$$

$$F = \frac{V \cdot A}{A\sqrt{g\frac{A}{T}}} = \frac{Q}{A\sqrt{gA/T}}$$

In ①

$$S_o - S_f = \frac{dy}{dx} - \frac{Q^2}{gA^3} T \frac{dy}{dx} = \frac{dy}{dx}\left[1 - \frac{Q^2 T}{gA^3}\right]$$

$$\frac{dy}{dx} = \frac{S_o - S_f}{1 - F^2}$$

$$\boxed{\frac{dy}{dx} = \frac{S_o - S_f}{1 - F^2} \qquad \frac{dy}{dx} = \frac{S_o - S_f}{1 - \frac{\alpha Q^2 T}{gA^3}}}$$

Classification of Water Surface Profiles

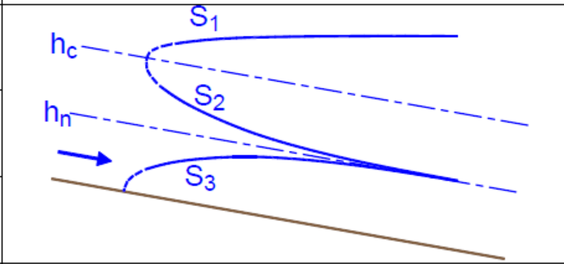
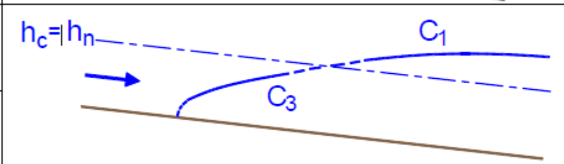
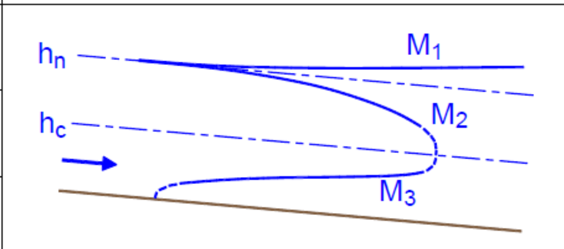
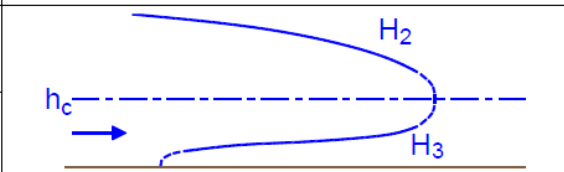
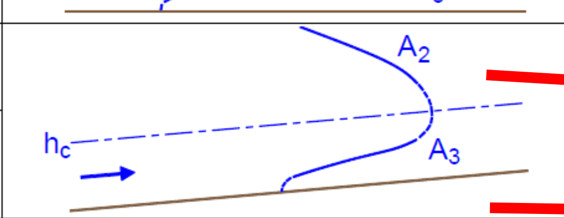
bed slope

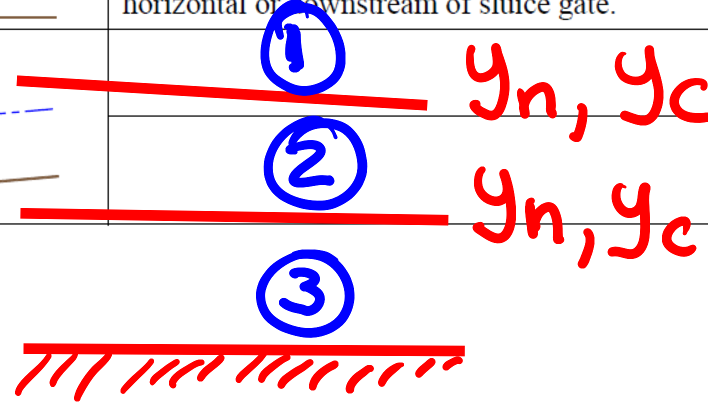
friction slope

$$\frac{dy}{dx} = \frac{S_o - S_f}{1 - F^2}$$

- Process of identification of possible flow profiles as a **prelude to quantitative computations**
- As $y \to y_0$, $dy/dx \to 0$, i.e. the water surface approaches the **normal depth line asymptotically**.
- As $y \to y_c$, $dy/dx \to \infty$, i.e. the water surface meets the **critical depth line vertically**.
  - **high curvatures at critical depth zones violate the assumption** of gradually-varied nature of the flow
    - Hence, **GVF computations have to end or commence a short distance away from the critical-depth location.**

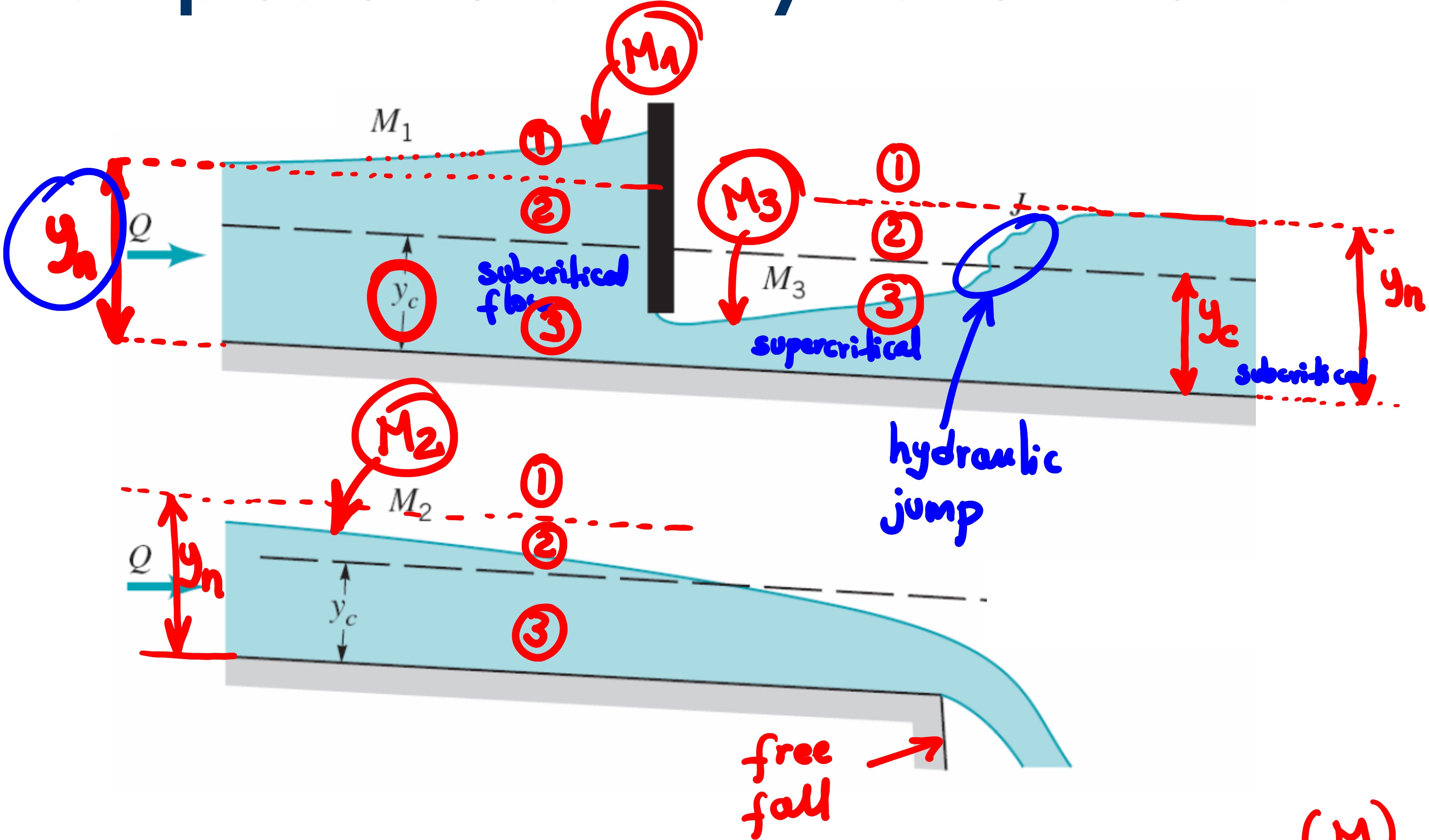
Classification of Water Surface Profiles

Type	Symbol	Definition	Sketches	Examples
STEEP (normal flow is supercritical)	S1	$h > h_c > h_n$		Hydraulic jump upstream with obstruction or reservoir controlling water level downstream.
	S2	$h_c > h > h_n$		Change to steeper slope.
	S3	$h_c > h_n > h$		Change to less steep slope.
CRITICAL (undesirable; undular unsteady flow)	C1	$h > h_c = h_n$		
	C3	$h_c = h_n > h$		
MILD (normal flow is subcritical)	M1	$h > h_n > h_c$		Obstruction or reservoir controlling water level downstream.
	M2	$h_n > h > h_c$		Approach to free overfall.
	M3	$h_n > h_c > h$		Hydraulic jump downstream; change from steep to mild slope or downstream of sluice gate.
HORIZONTAL (limiting mild slope; $h_n \rightarrow \infty$)	H2	$h > h_c$		Approach to free overfall.
	H3	$h_c > h$		Hydraulic jump downstream; change from steep to horizontal or downstream of sluice gate.
ADVERSE (upslope)	A2	$h > h_c$		
	A3	$h_c > h$		



Source: Hydraulic notes, David Apsley

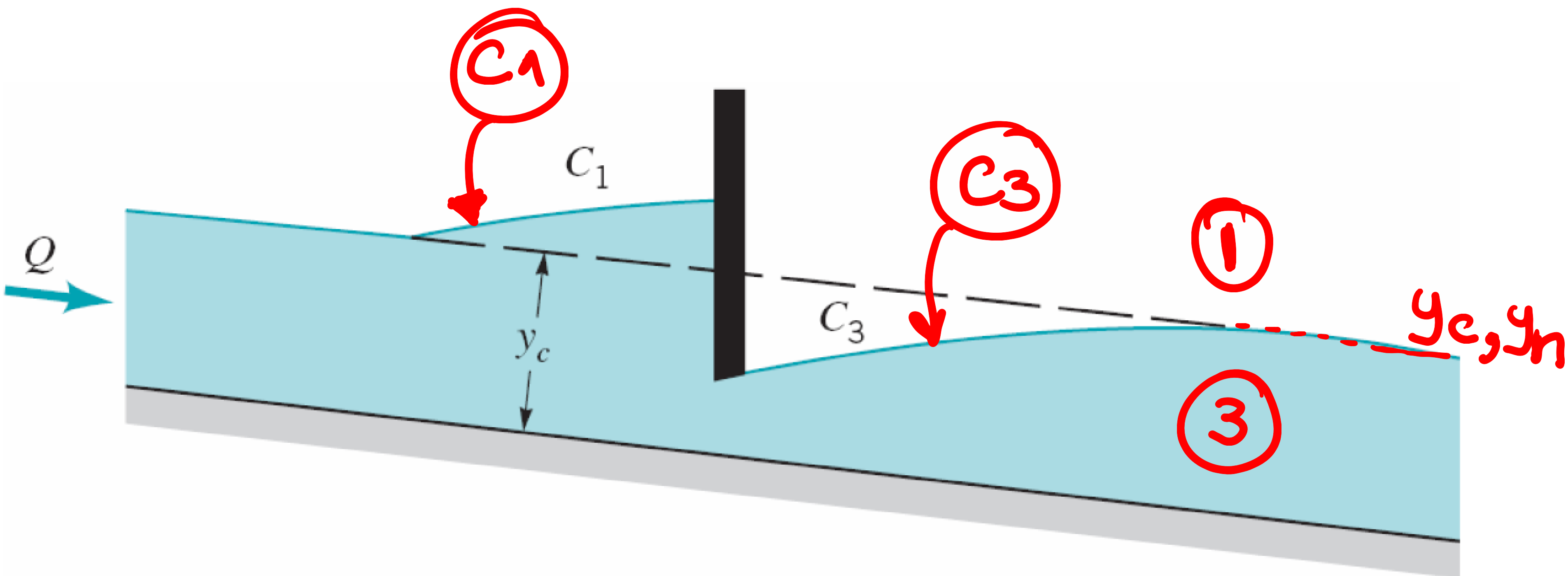
Examples of Gradually Varied Flows



(M)

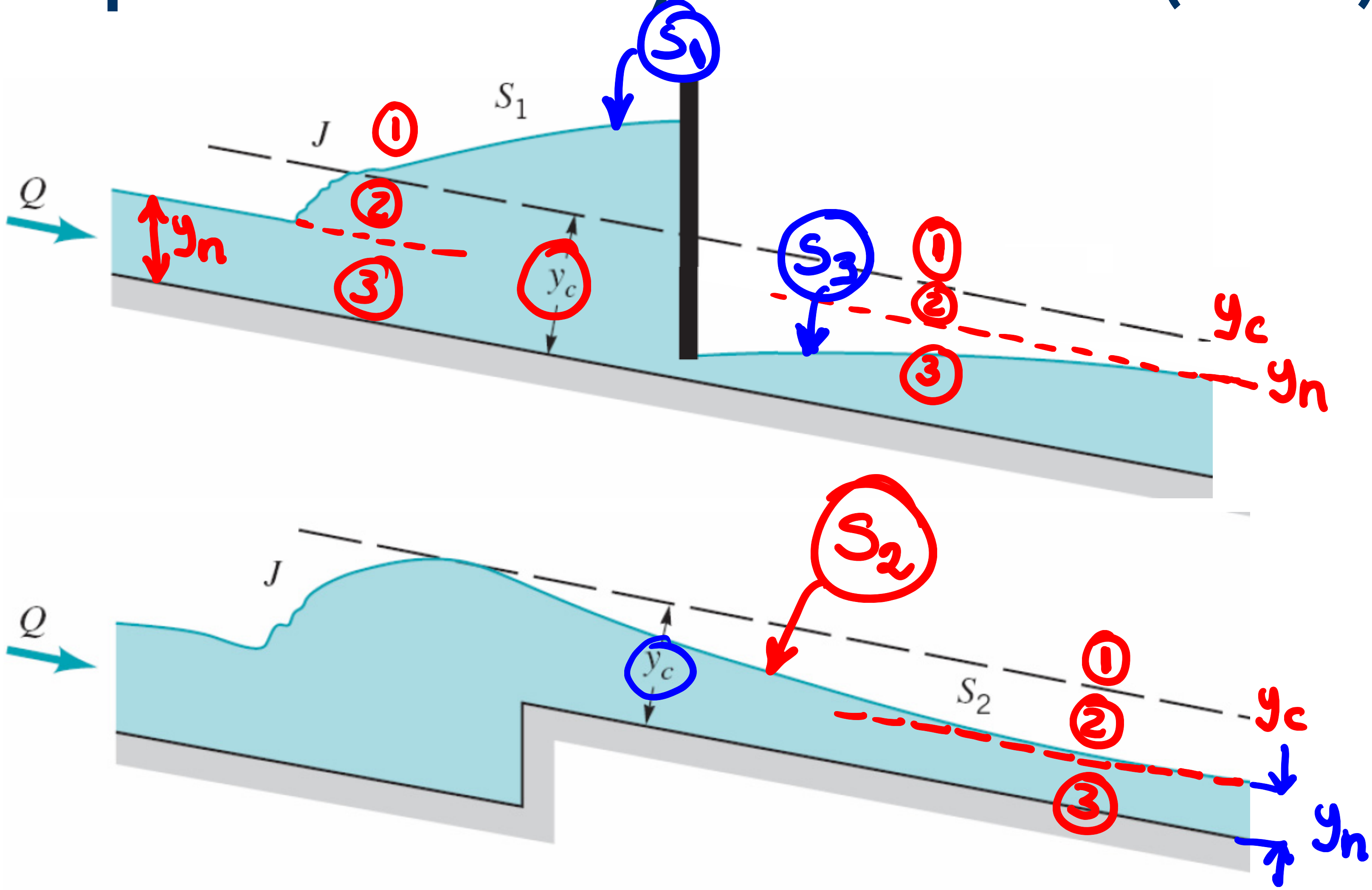
Typical surface configurations for nonuniform depth flow with a **mild slope**

Examples of Gradually Varied Flows (Cont.)



Typical surface configurations for nonuniform depth flow with a **critical slope** (c) $y_n = y_c$

Examples of Gradually Varied Flows (Cont.)



Typical surface configurations for nonuniform depth flow with a **steep slope** (S)

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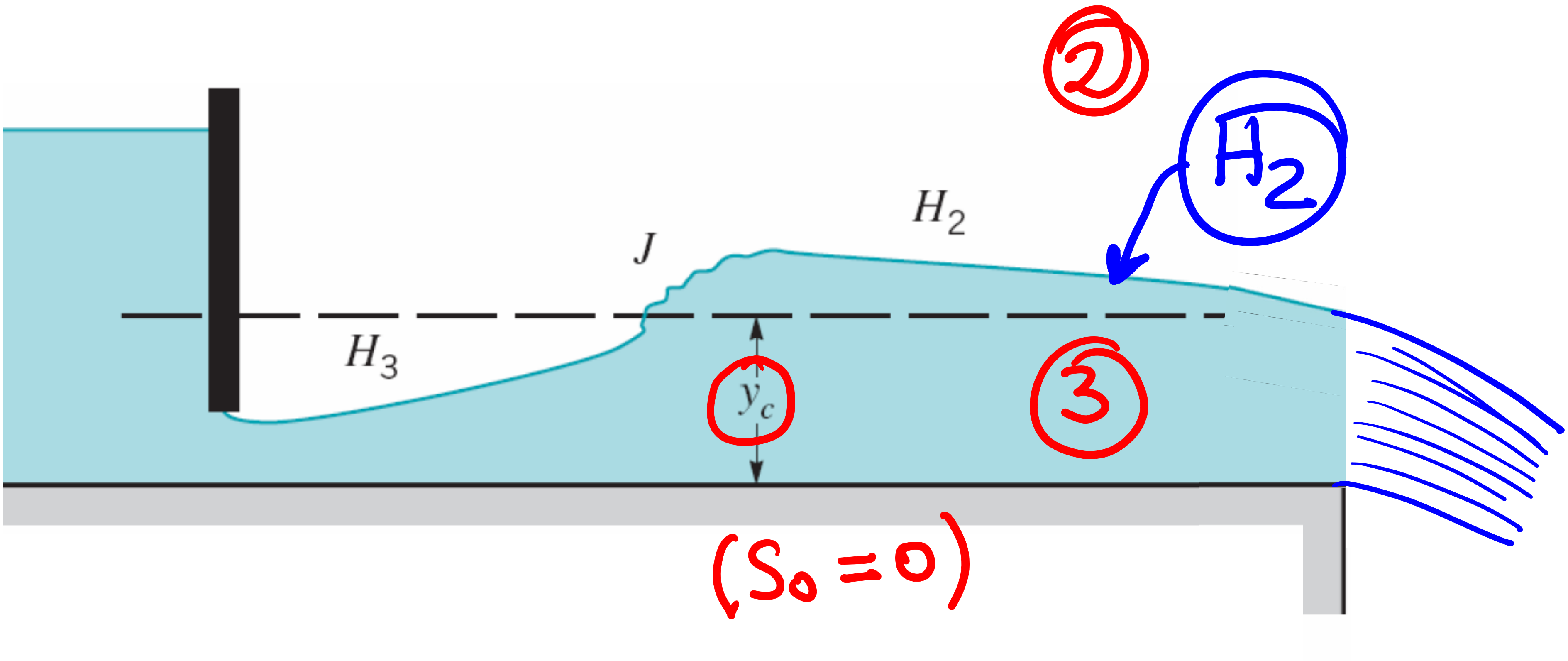
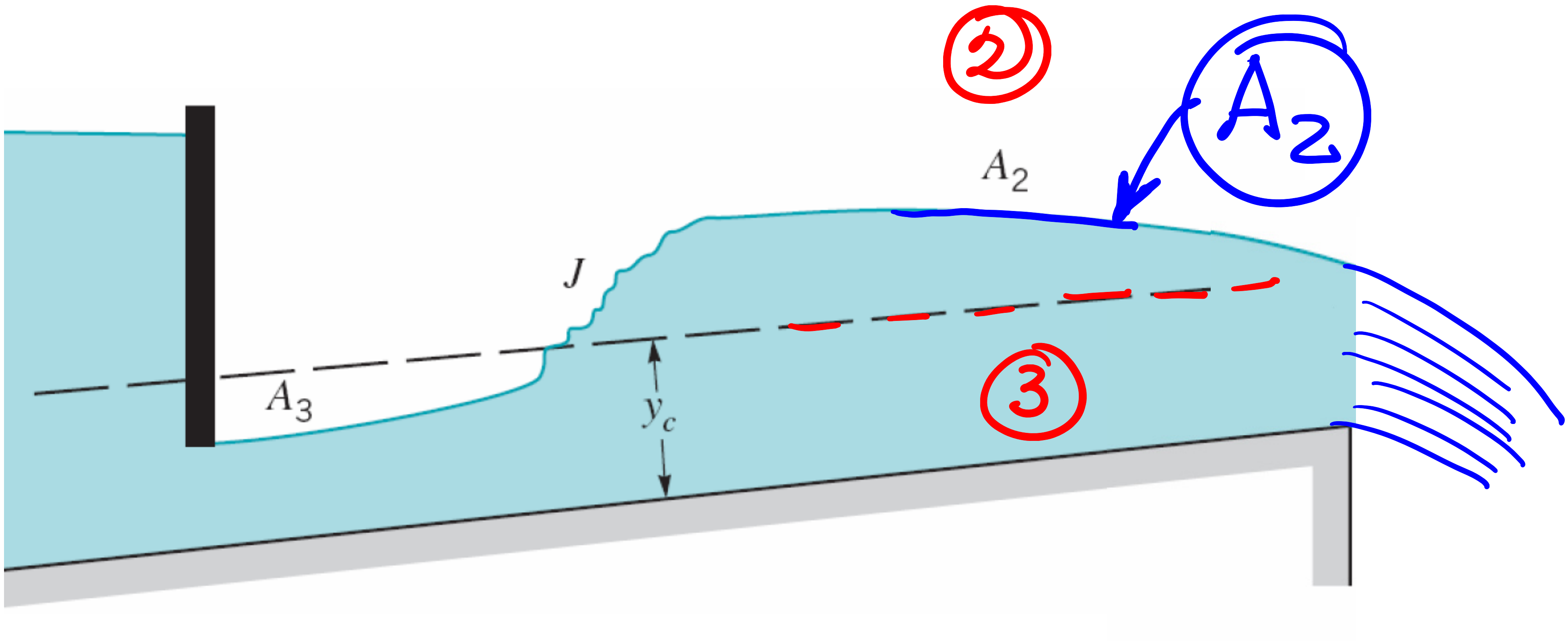
Examples of Gradually Varied Flows (Cont.)



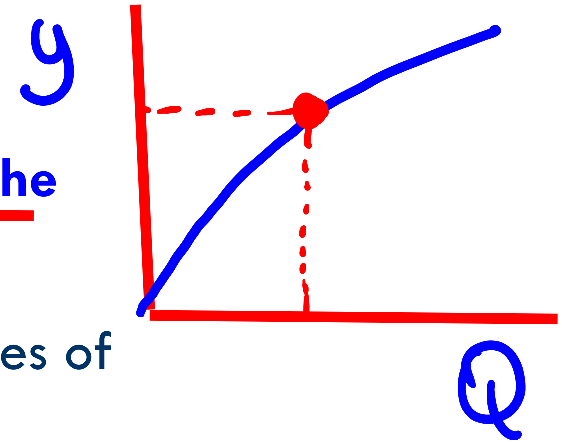
Typical surface configurations for nonuniform depth flow with a **horizontal slope** (H)

Examples of Gradually Varied Flows (Cont.)

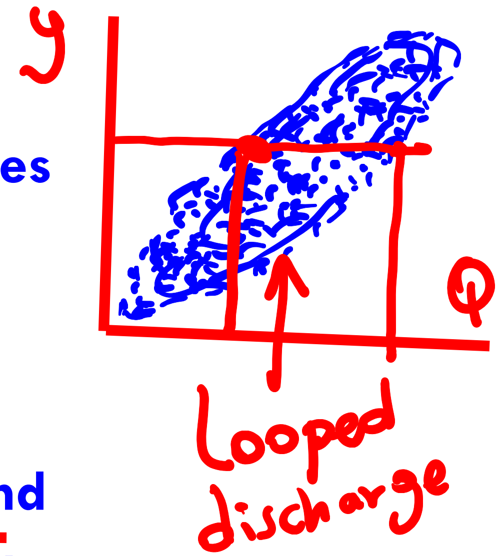


Typical surface configurations for nonuniform depth flow with **adverse slope** (A)

Control Sections

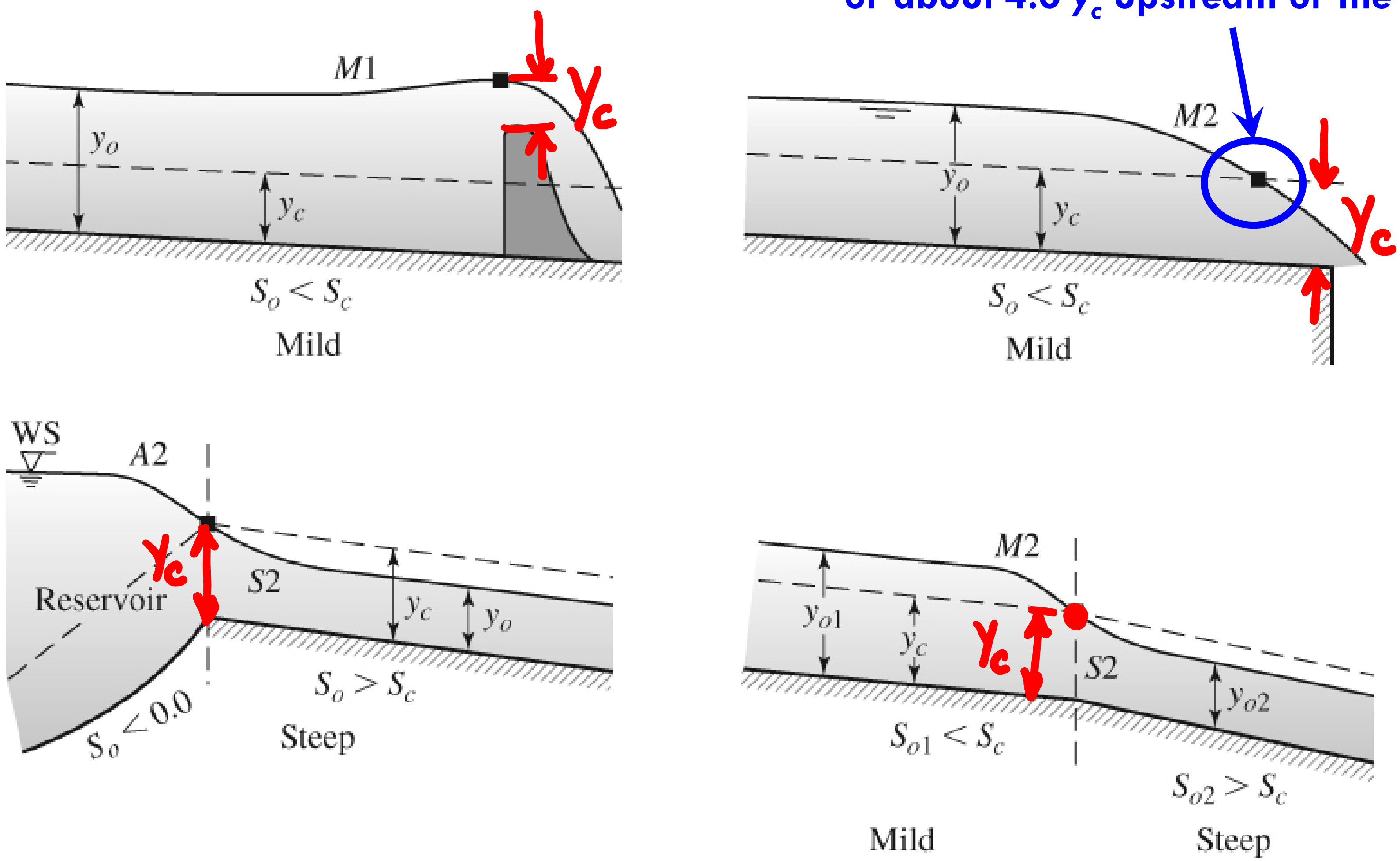


- Section in which a **fixed relationship exists between the discharge and depth of flow.**
- Weirs, spillways sluice gates are some typical examples of structures which give rise to control sections.
- Critical depth is also a control point.
  - However, it is **effective in a flow profile which changes from subcritical to supercritical flow.**
- Control sections provide a key to the identification of proper profile shapes.
  - **Subcritical flows have controls in the downstream end**
  - **Supercritical flows have controls in the upstream end**
  - Hence, the direction of computation of subcritical profiles is upstream, and for supercritical, it is downstream.



Control Sections

Due to curvature of the streamlines, critical depth actually occurs at a distance of about 4.0 y_c upstream of the drop

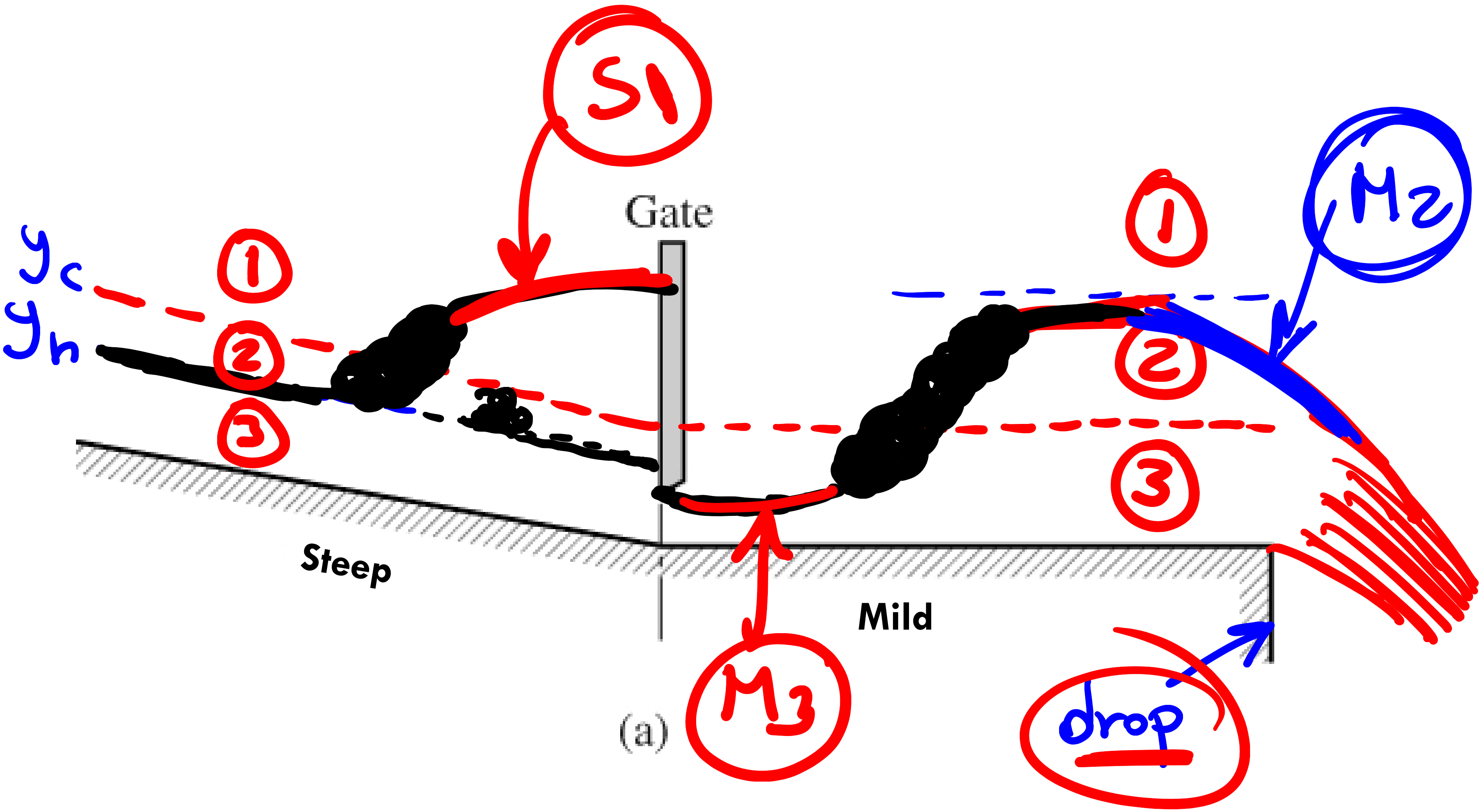


Bold squares show the control sections

Normal depth is also an option [Very long and is not affected by boundaries]

Example

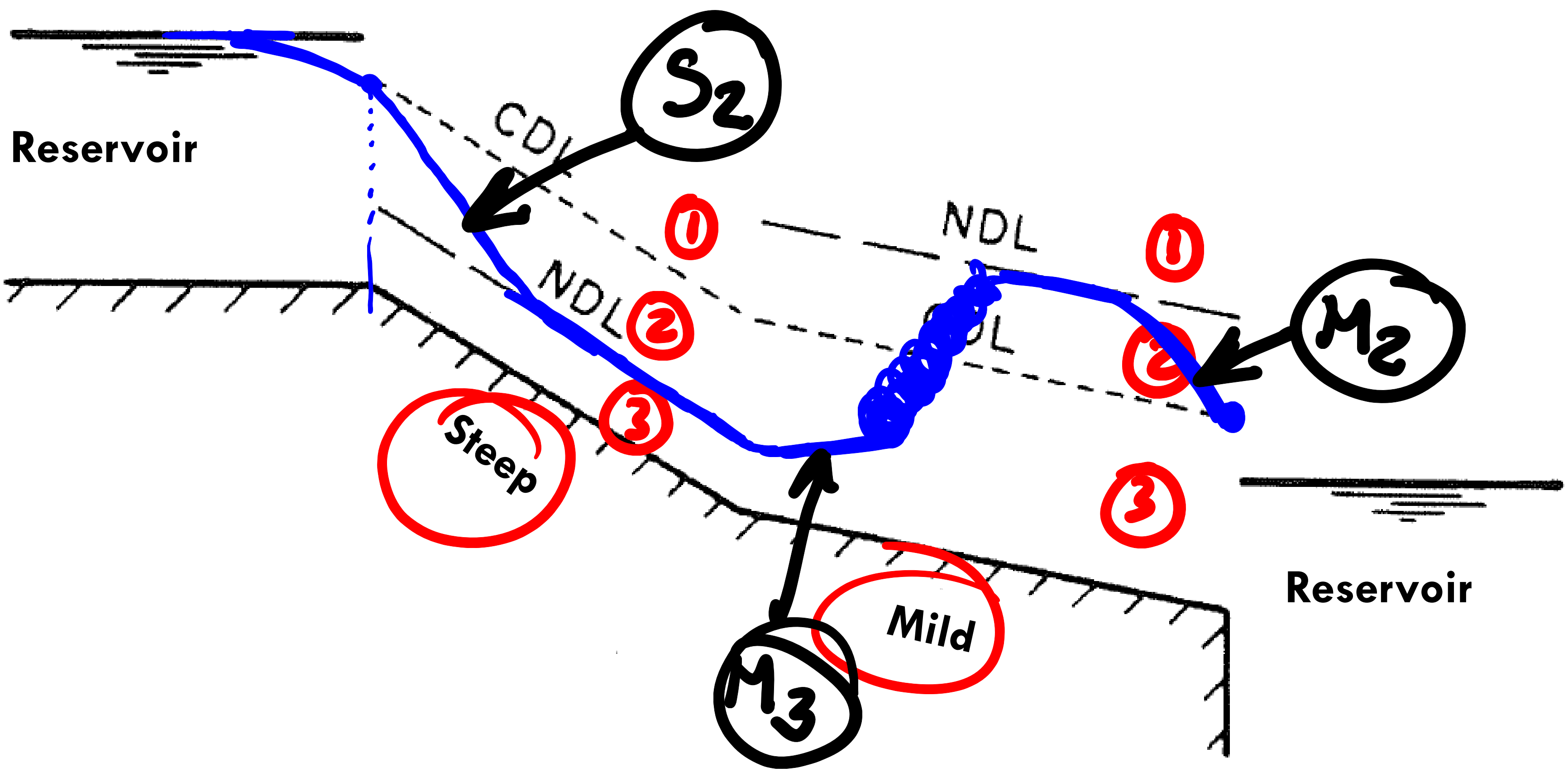
Sketch the water surface profile for the two-reach open-channel system below. A gate is located between the two reaches and the second reach ends with a sudden fall.



(a)

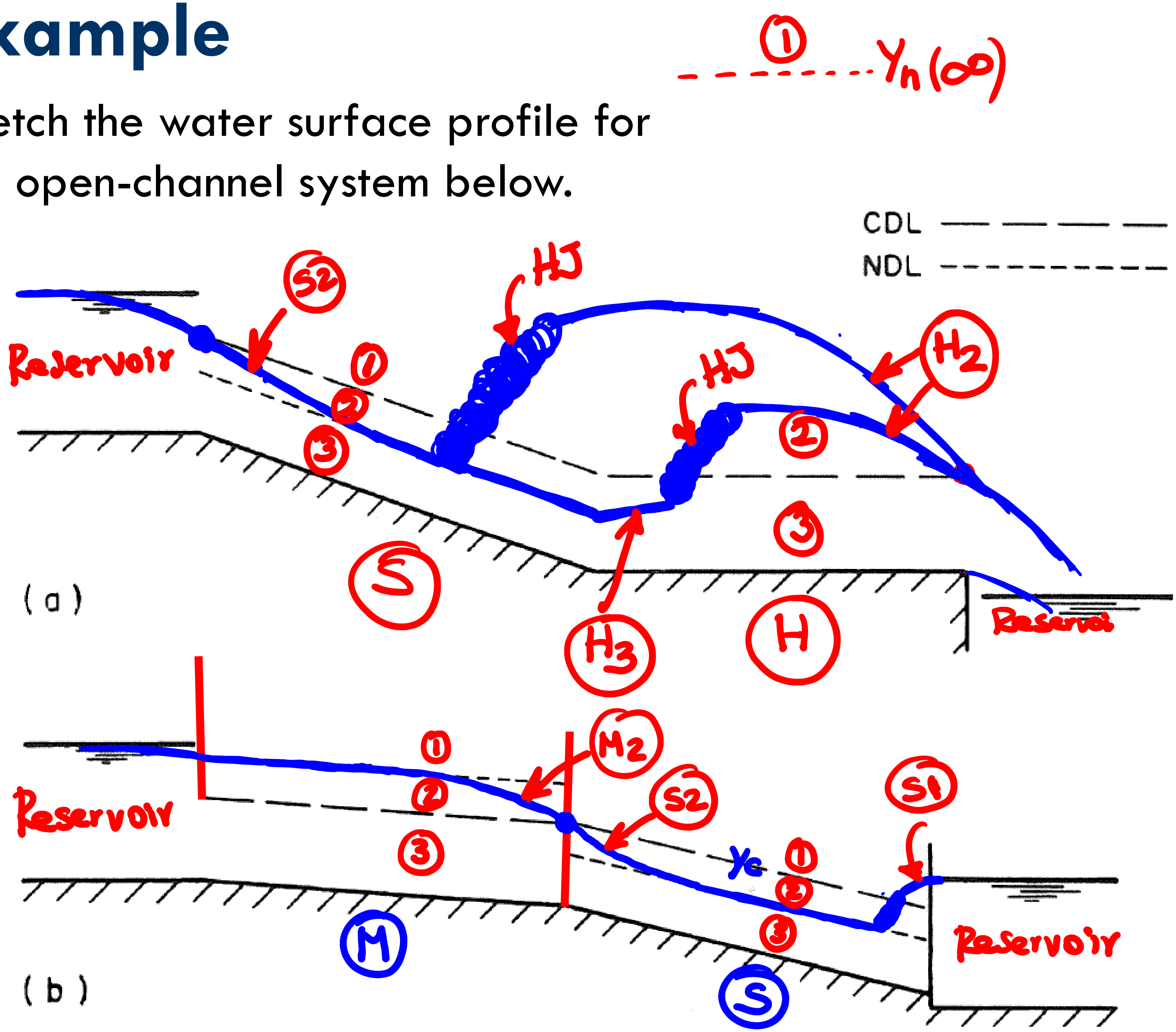
Example

Sketch the water surface profile for the open-channel system below.



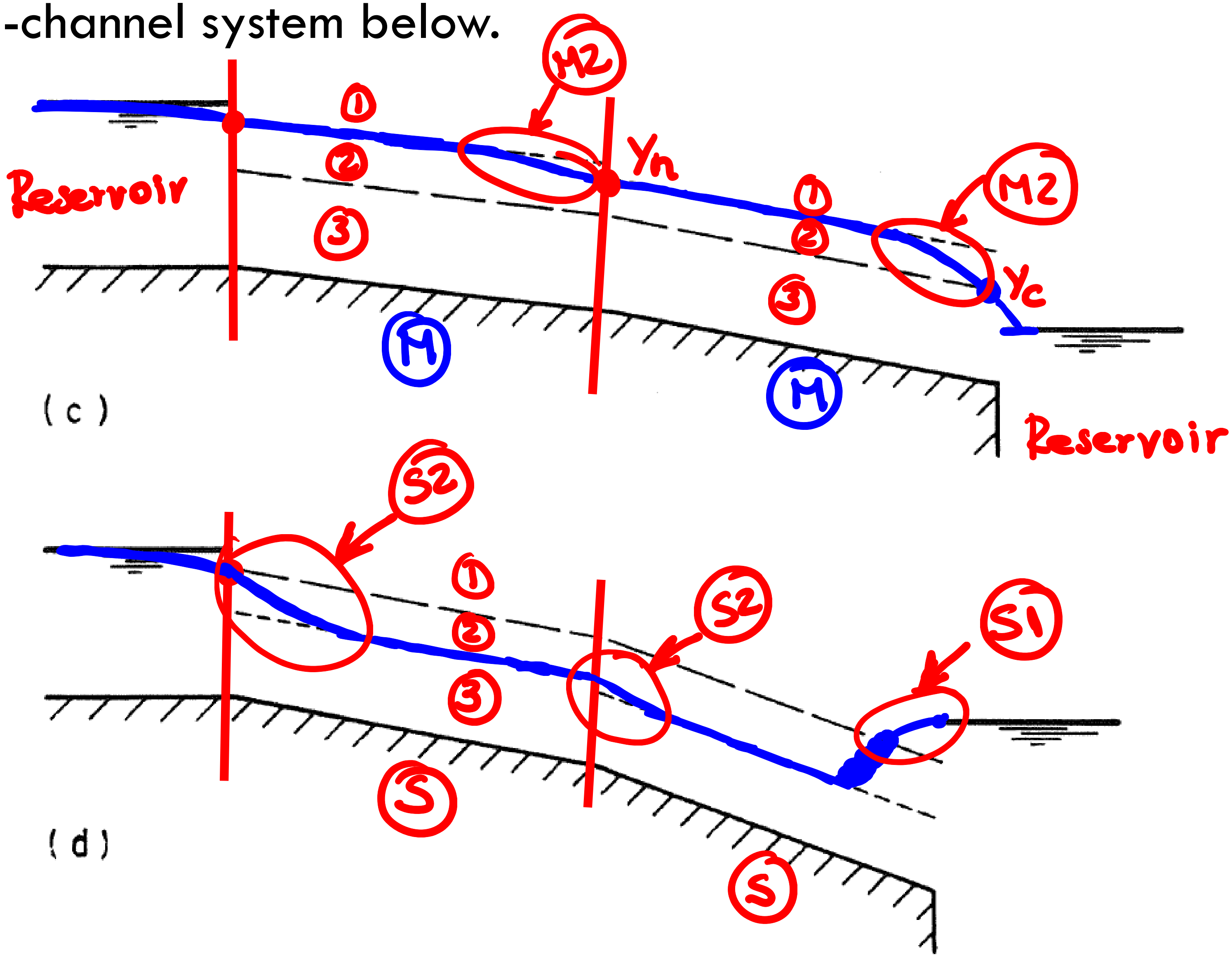
Example

Sketch the water surface profile for the open-channel system below.



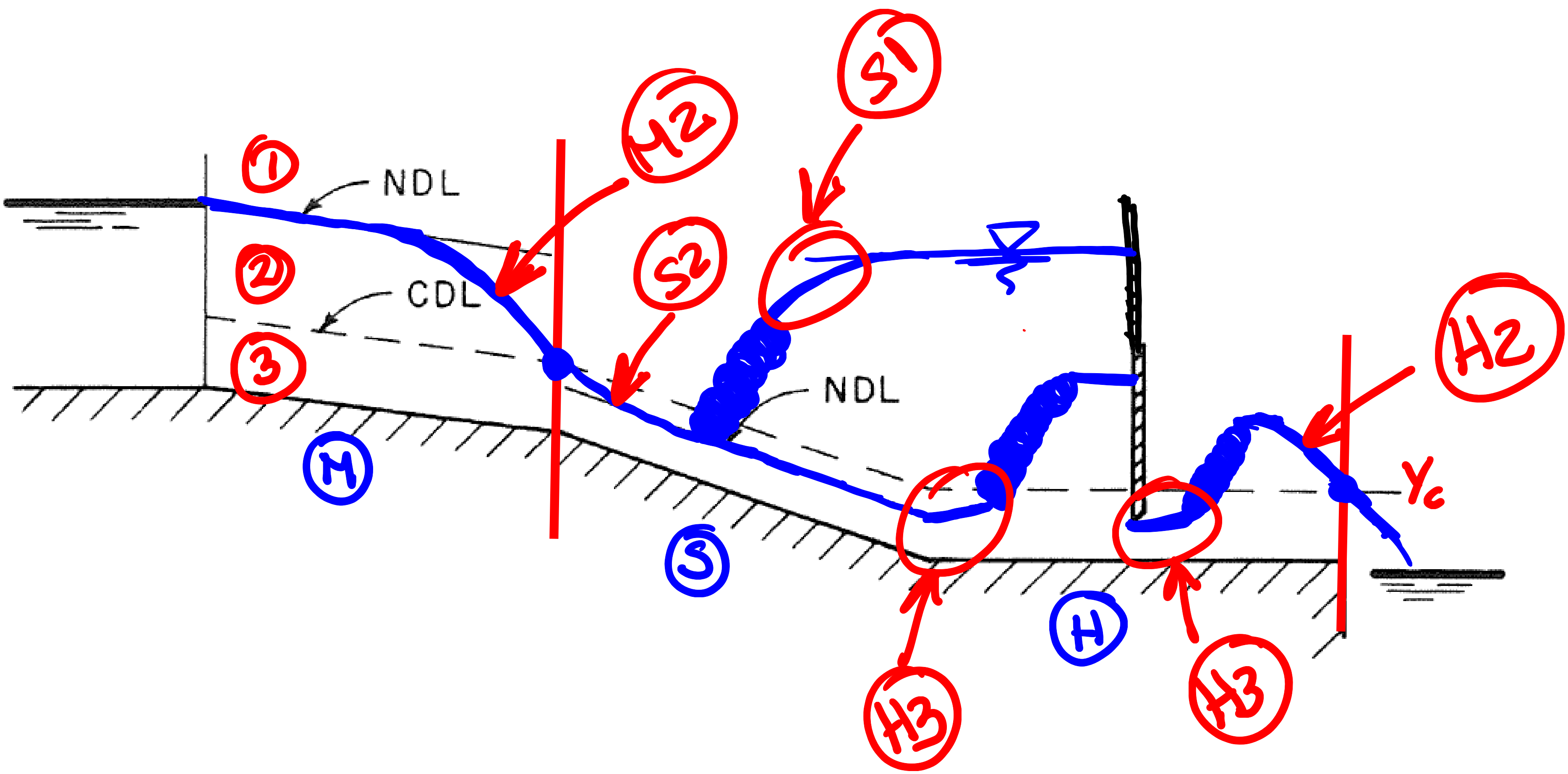
Example

Sketch the water surface profile for the open-channel system below.



Example

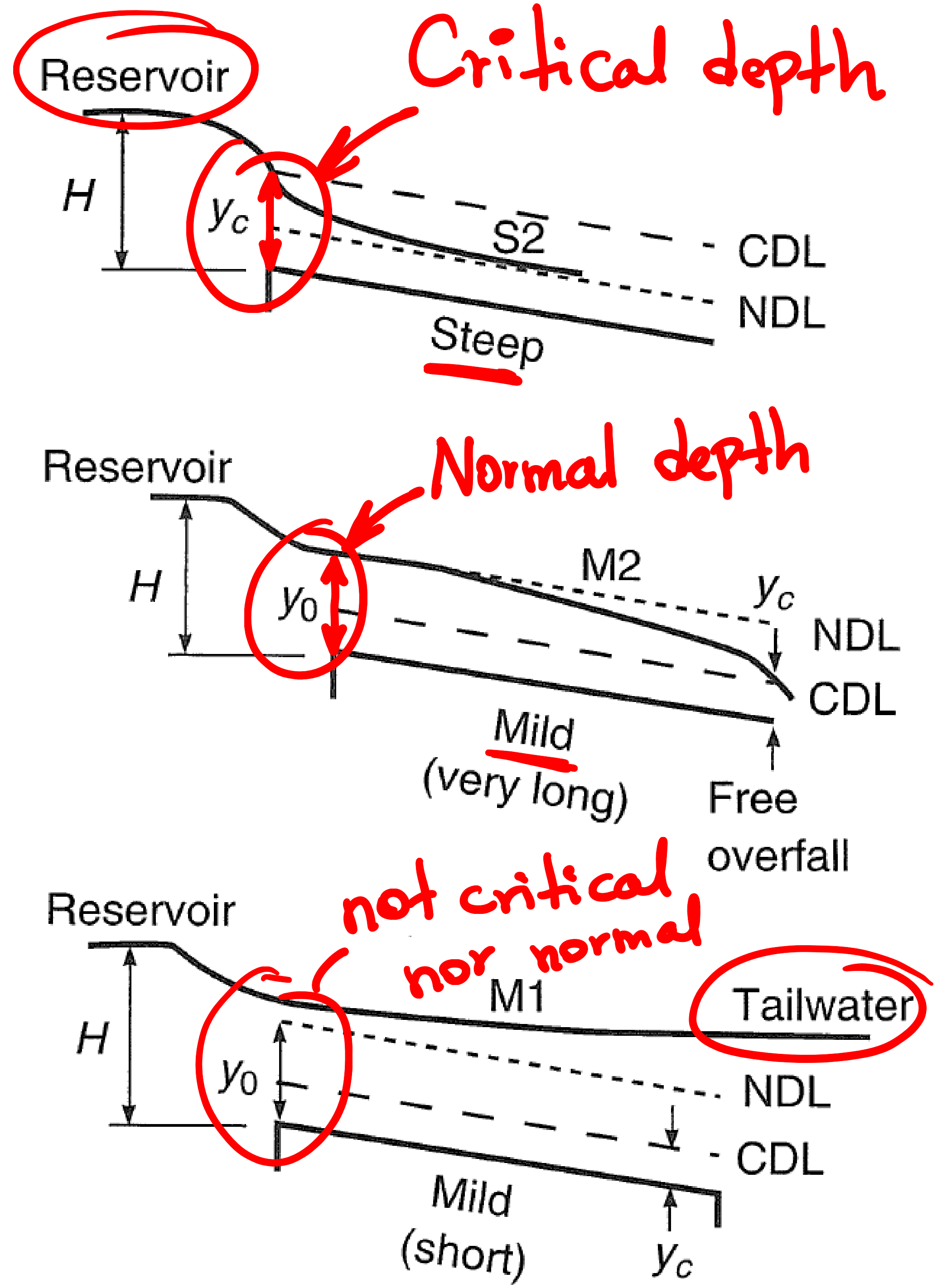
Sketch the water surface profile for the open-channel system below.



Lake Discharge Problem

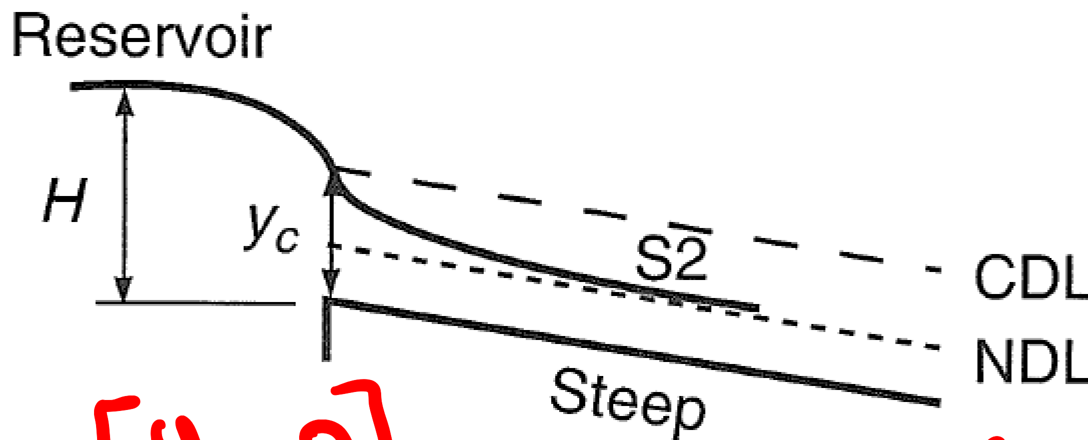
Difficult to know discharge, because it is **unclear whether slope is mild or steep**

*we will use the trial-error approach



Lake Discharge Problem (Cont.)

* Assume slope is steep



$$H = y_c + \frac{V_c^2}{2g}$$

$$H = y_c + \frac{Q^2}{2gA_c^2} \quad \ldots\ (1) \qquad [y_c, Q]$$

$$T_c = f(y_c)$$

* $Fr = 1$ [critical flow]

$$\frac{A_c^3}{T_c} = \frac{Q^2}{g} \quad \ldots\ (2) \qquad [y_c, Q]$$

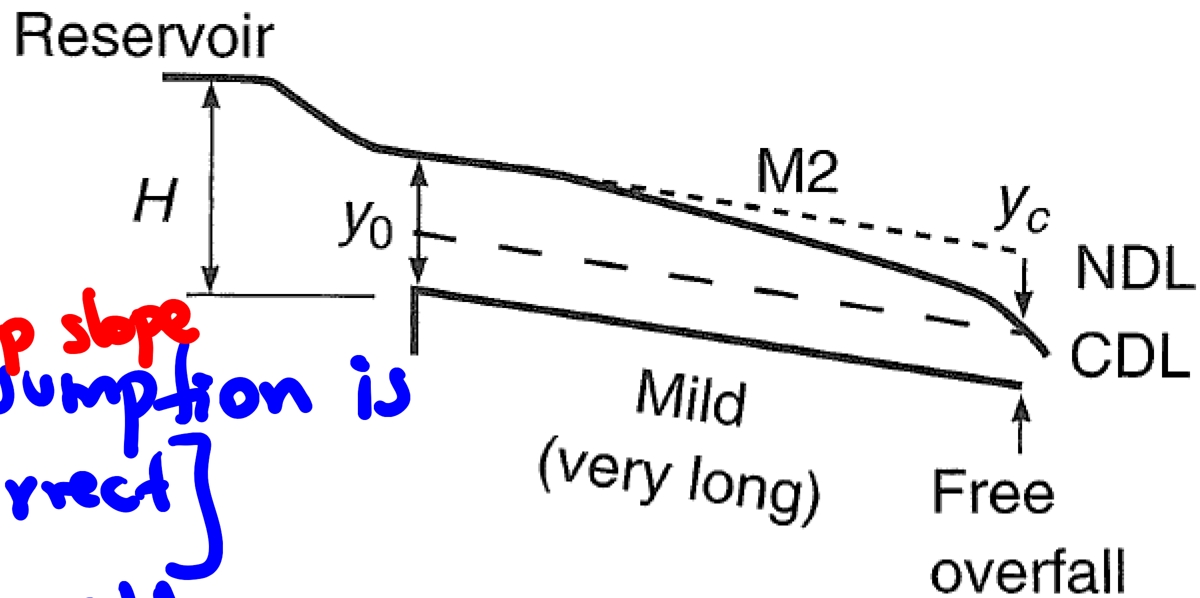
Two equations and two unknowns.

* How to verify assumption is correct??

Manning's eq: $Q = \frac{k}{n} A R^{2/3} S_0^{1/2}$

Lake Discharge Problem (Cont.)

$$S_c = \frac{n^2 Q^2}{k^2 A_c^2 R_c^{4/3}}$$



if $S_0 > S_c$ [steep slope assumption is correct]

else, slope is mild

y_0 [normal depth]

* slope is mild

$$H = y_0 + \frac{Q^2}{2g A_0^2} \quad \cdots (1) \quad [y_0, Q]$$

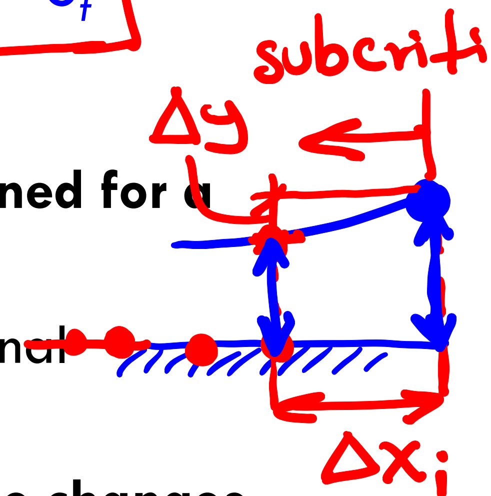
$A_0, R_0 = f(y_0)$

$$Q = \frac{k}{n} A_0 R_0^{2/3} S_0^{1/2} \quad [y_0, Q]$$

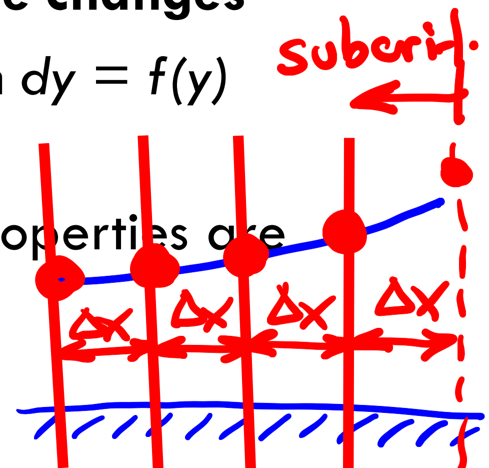
Two equations and two unknowns [✓]

Water surface profile computation

$$\boxed{\frac{dy}{dx} = \frac{S_o - S_e}{1 - F^2}} \qquad \boxed{S_e = S_f}$$



- Two types of methods
  1. **Explicit or direct step method: distance is determined for a specified depth change**
     - Mostly for prismatic channels where cross-sectional properties don't change with distance *x*.
  2. **Implicit methods: depth is computed from distance changes**
     - Unknown appears on both sides of the equation $dy = f(y)$ dx
     - For natural channels for which cross-sectional properties are determined beforehand at particular locations.



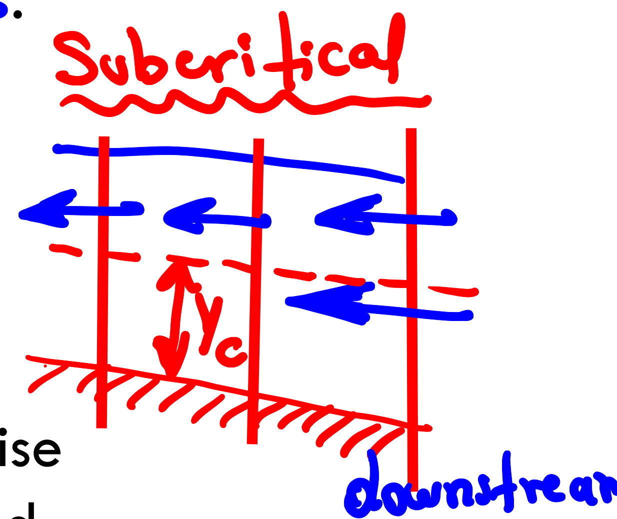
- Assumptions
  – **slope of the energy** grade line, S_e, can be evaluated from Manning's or Chezy's equation using the **local value of depth.**

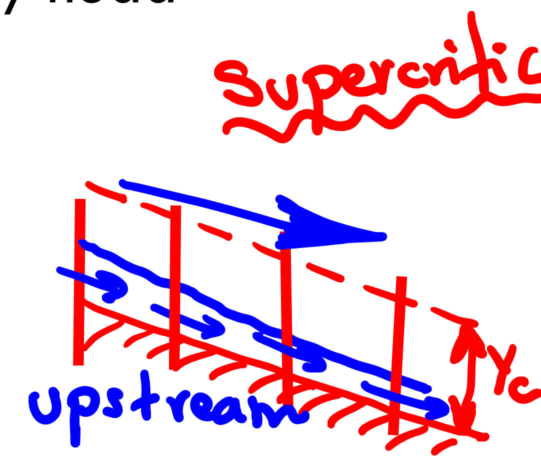
Standard Step Method

Subcritical Δx 100–1000m

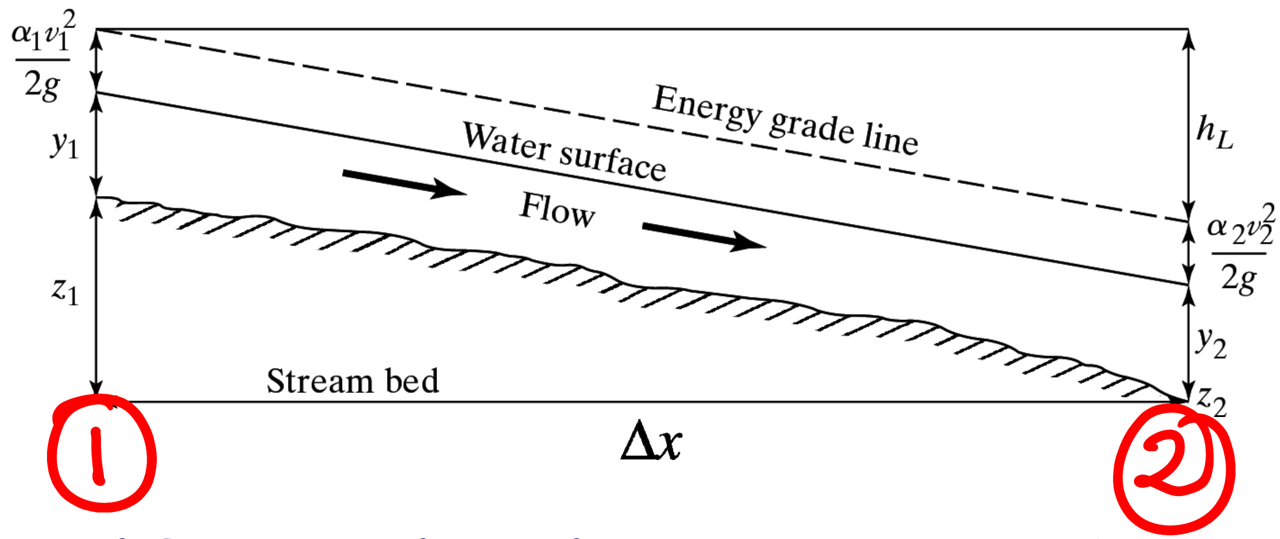
Supercritical Δx 1–20 m

- This method is used in **most** practical **GVF solvers**.
- This method solves sequentially for $y_1, y_2, y_3, \ldots$ **starting at the control section** (upstream or downstream end) with known water depth y_0.
- **Step size** (Δx) **must be small** enough so that changes in water depth aren't very large. Otherwise estimates of the friction slope and the velocity head are inaccurate
  1. For **subcritical flows**, calculations **start downstream.**
  2. For **supercritical flows**, calculations **start upstream.**





Standard Step Method (cont.)



- Equation of GVF in the form of the energy equation (with $\alpha = 1$) :

$$z_1 + y_1 + \frac{V_1^2}{2g} = z_2 + y_2 + \frac{V_2^2}{2g} + \bar{S}_e \Delta x$$

head losses

- Solve for **y_1 (Subcritical flow)** or **y_2 (supercritical flow)**

Where:

$\bar{S}_e$ = mean slope of the energy grade line

Δx = reach length

Mean slope of the energy grade line:

- Average conveyance: $\bar{S}_e = \dfrac{Q^2}{\left[\frac{K_1+K_2}{2}\right]^2}$ Default in HEC-2 and HEC-RAS
- Average EGL slope: $\bar{S}_e = \dfrac{s_{e_1}+s_{e_2}}{2}$ Most accurate for M1 profiles
- Geometric mean slope: $\bar{S}_e = \dfrac{Q^2}{K_1 K_2}$ Default in WSPRO
- Harmonic mean slope: $\bar{S}_e = \dfrac{2 s_{e_1} s_{e_2}}{s_{e_1}+s_{e_2}}$ Most accurate for M2 profiles

Where: $S_{e\,j} = \left(\dfrac{Q n_j}{k A_j R_j^{2/3}}\right)^2$

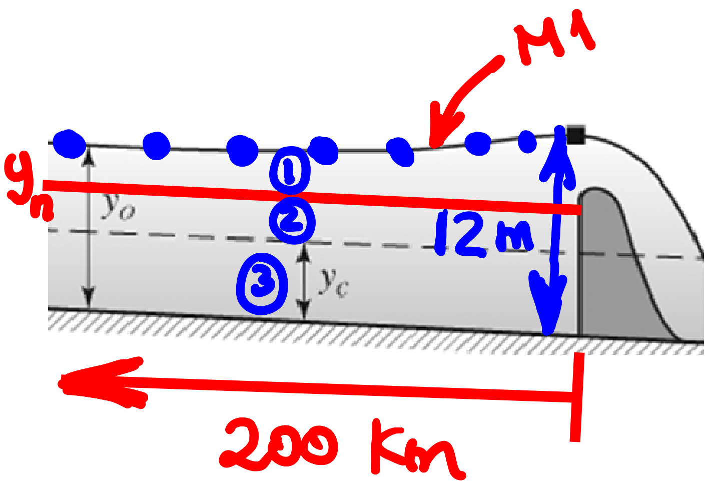
k = 1 (SI units)
k = 1.49 (English units)
n = Manning's roughness

Mixed-flow regime:

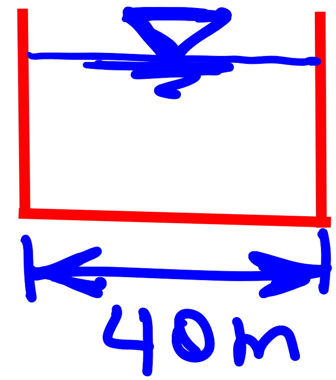
- When there is **occurrence of both supercritical and subcritical** depths in a river reach
  - For example, a hydraulic jump in a reach
- Intersection of the momentum function for upstream supercritical and downstream subcritical profile determines the **location of hydraulic jump**.
- **Several programs** are available for modeling mixed flow regimes
  - **Annel2** (Arturo Leon)
  - **HEC-RAS** (USACE)
  - **WSPRO** (USGS)

Example

A rectangular concrete-lined channel (n = 0.015) has a constant bed slope of 0.0001 and a bottom width of 40 m. A control gate at the dam increased the depth at the dam to **12 m** when the discharge is 300 m^3/s. Compute the water surface profile from the dam up to 200 km upstream of the dam. (**See Excel spreadsheet in Canvas**).



$Q = 300 \ m^3/s$
$n = 0.015$
$S = 0.0001$
$b = 40 \ m$



* Normal depth

Manning's eq.

$$Q = \frac{k}{n} A R^{2/3} S_o^{1/2}$$

$k = 1.0$ (SI)

$$\boxed{y_n = 4.6448\ m}$$

* Critical depth

$$F = 1$$

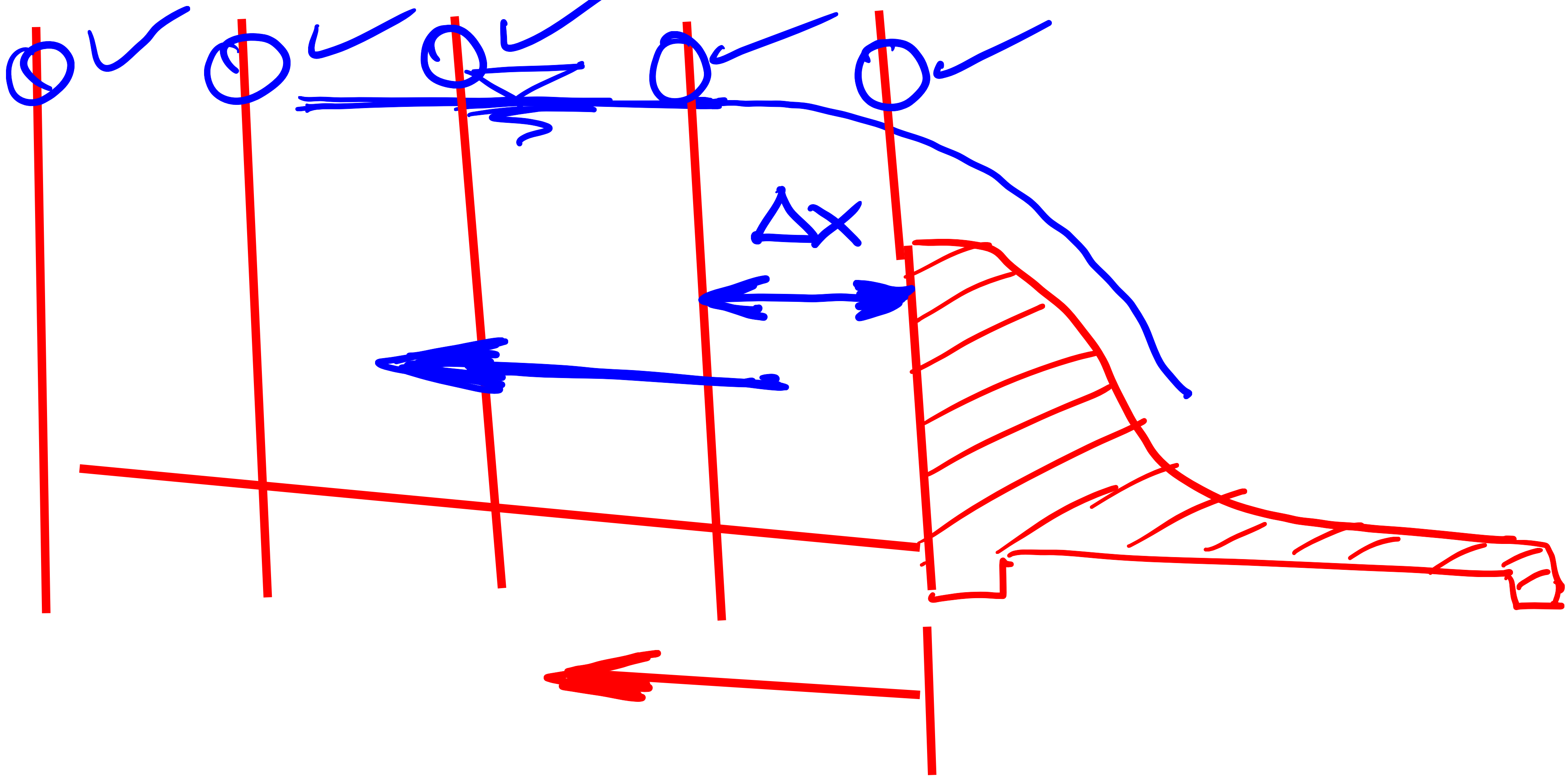
$$\frac{Q^2 T}{g A^3} = 1 \rightsquigarrow \boxed{Y_c = 1.7899\ m}$$

~ 12

$$y > Y_n > Y_c$$

[Subcritical flow]

M1 profile



Calculation direction

Solution with Excel Spreadsheet

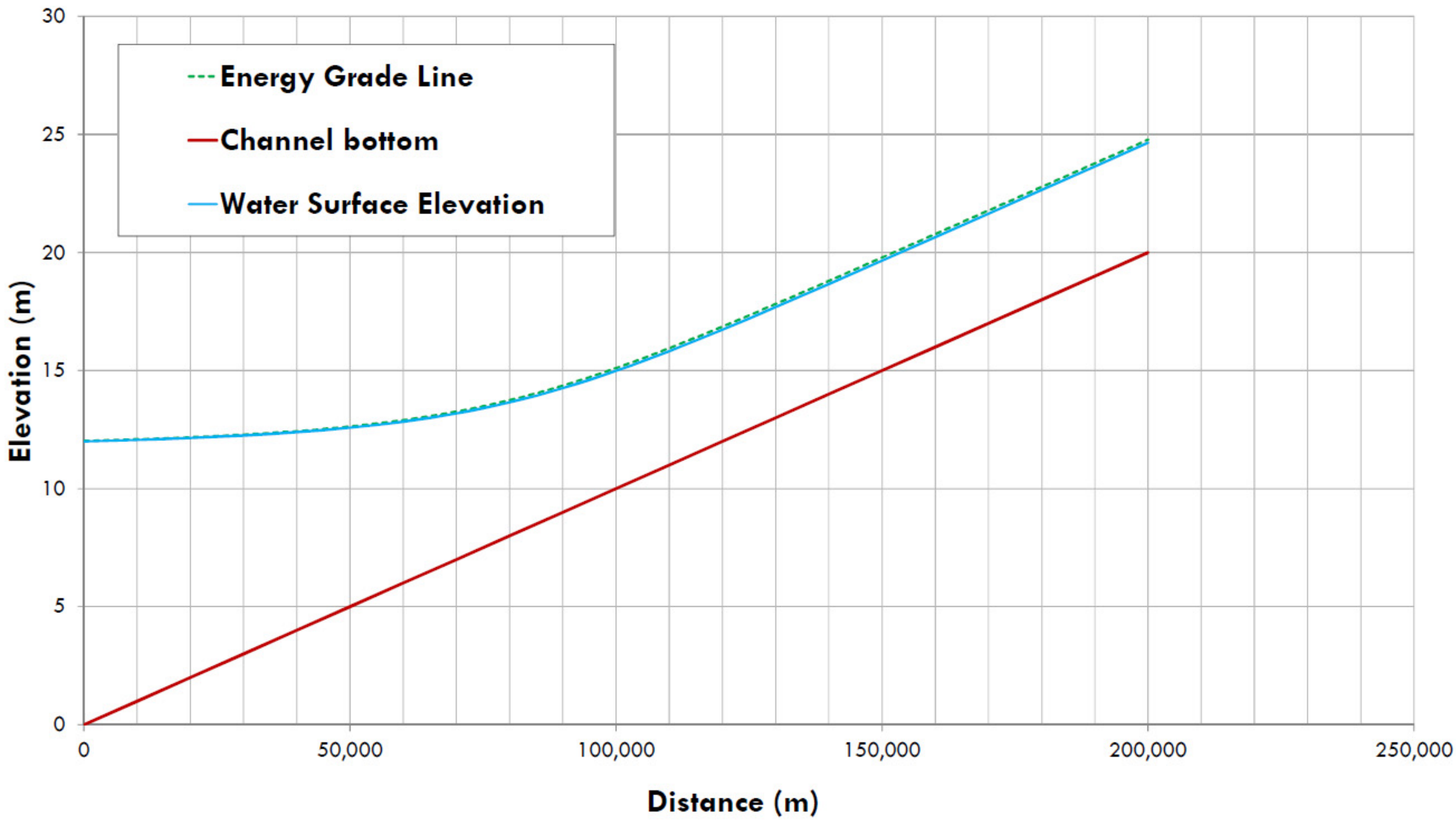
Gradually Varied Flow for rectangular channels, Arturo S. Leon

Q (m3/s) =	300
So (Slope)	0.0001
n Manning =	0.015
b (m) =	40
Initial depth (m) :	12
delta X (m) =	10000
tolerance =	0.0000001

Write energy equation in flow direction

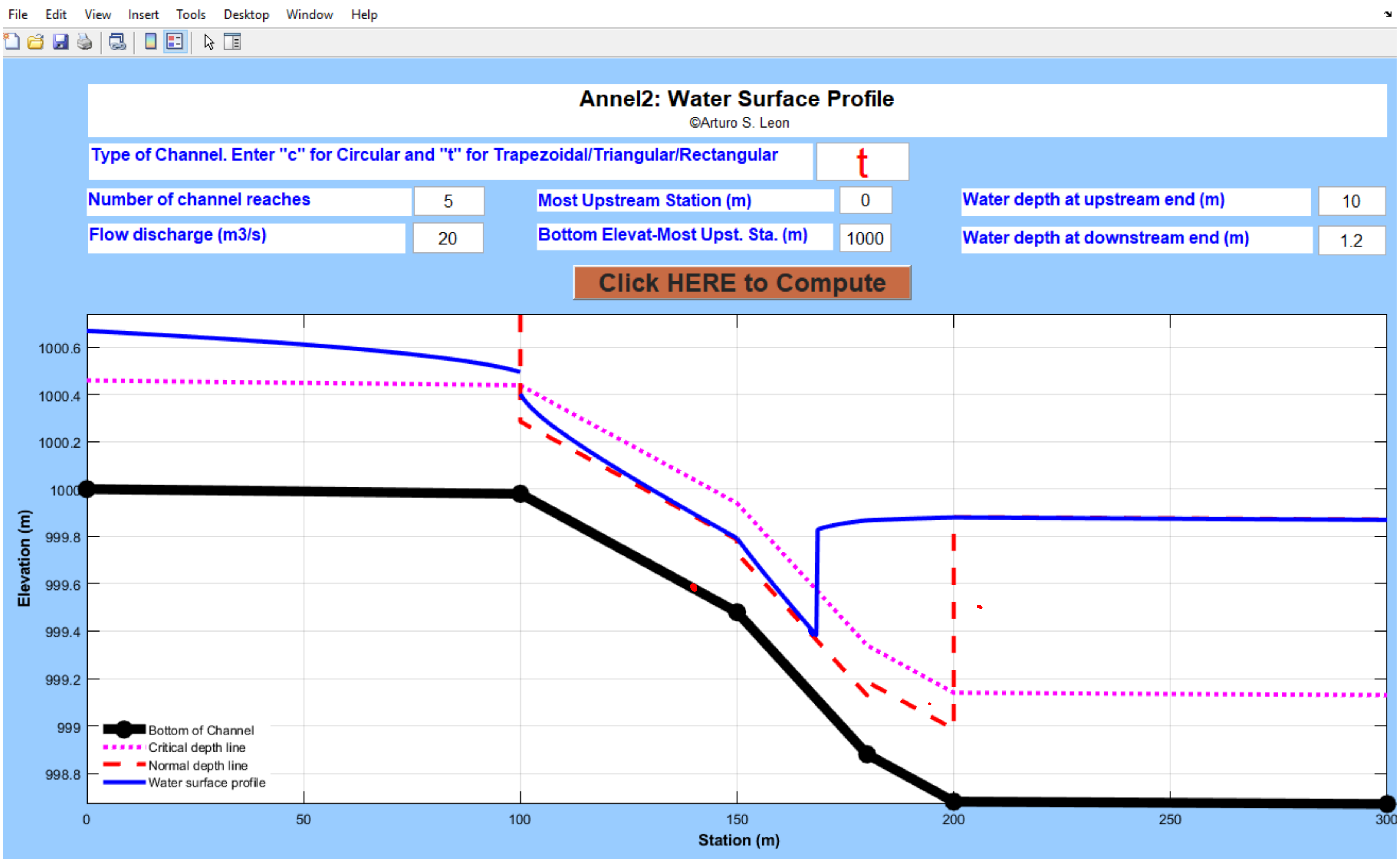
x	depth y	Z (m)	A (m^2)	P (m)	R (m)	V (m/s)	WSE (z + y)	H = z + y + v^2/(2g)	Sf	average Sf	F(y) = 0
0	12.000	0.000	480.000	64.000	7.500	0.625	12.000	12.020	5.98679E-06		0.000000
10000	11.064	1.000	442.565	62.128	7.123	0.678	12.064	12.088	7.54314E-06	6.76497E-06	0.000000
20000	10.146	2.000	405.831	60.292	6.731	0.739	12.146	12.174	9.6742E-06	8.60867E-06	0.000000
30000	9.252	3.000	370.068	58.503	6.326	0.811	12.252	12.285	1.26394E-05	1.11568E-05	0.000000
40000	8.392	4.000	335.670	56.783	5.911	0.894	12.392	12.433	1.68143E-05	1.47269E-05	0.000000
50000	7.580	5.000	303.206	55.160	5.497	0.989	12.580	12.630	2.27056E-05	1.976E-05	0.000000
60000	6.837	6.000	273.466	53.673	5.095	1.097	12.837	12.898	3.08857E-05	2.67956E-05	0.000000
70000	6.186	7.000	247.443	52.372	4.725	1.212	13.186	13.261	4.17165E-05	3.63011E-05	0.000000
80000	5.654	8.000	226.152	51.308	4.408	1.327	13.654	13.744	5.47855E-05	4.8251E-05	0.000001
90000	5.256	9.000	210.232	50.512	4.162	1.427	14.256	14.360	6.8436E-05	6.16107E-05	0.000002
100000	4.988	10.000	199.524	49.976	3.992	1.504	14.988	15.103	8.03141E-05	7.4375E-05	0.000003
110000	4.826	11.000	193.048	49.652	3.888	1.554	15.826	15.949	8.88782E-05	8.45962E-05	0.000002
120000	4.737	12.000	189.465	49.473	3.830	1.583	16.737	16.865	9.41499E-05	9.15141E-05	0.000001
130000	4.690	13.000	187.604	49.380	3.799	1.599	17.690	17.821	9.70553E-05	9.56026E-05	0.000000
140000	4.667	14.000	186.674	49.334	3.784	1.607	18.667	18.799	9.85532E-05	9.78042E-05	0.000000
150000	4.655	15.000	186.218	49.311	3.776	1.611	19.655	19.788	9.9298E-05	9.89256E-05	0.000000
160000	4.650	16.000	185.998	49.300	3.773	1.613	20.650	20.783	9.96615E-05	9.94798E-05	0.000000
170000	4.647	17.000	185.891	49.295	3.771	1.614	21.647	21.780	9.98373E-05	9.97494E-05	0.000000
180000	4.646	18.000	185.840	49.292	3.770	1.614	22.646	22.779	9.99219E-05	9.98796E-05	0.000000
190000	4.645	19.000	185.816	49.291	3.770	1.615	23.645	23.778	9.99625E-05	9.99422E-05	0.000000
200000	4.645	20.000	185.804	49.290	3.770	1.615	24.645	24.778	9.9982E-05	9.99723E-05	0.000000
										SUM	0.00001

Solution with Excel Spreadsheet (Cont.)



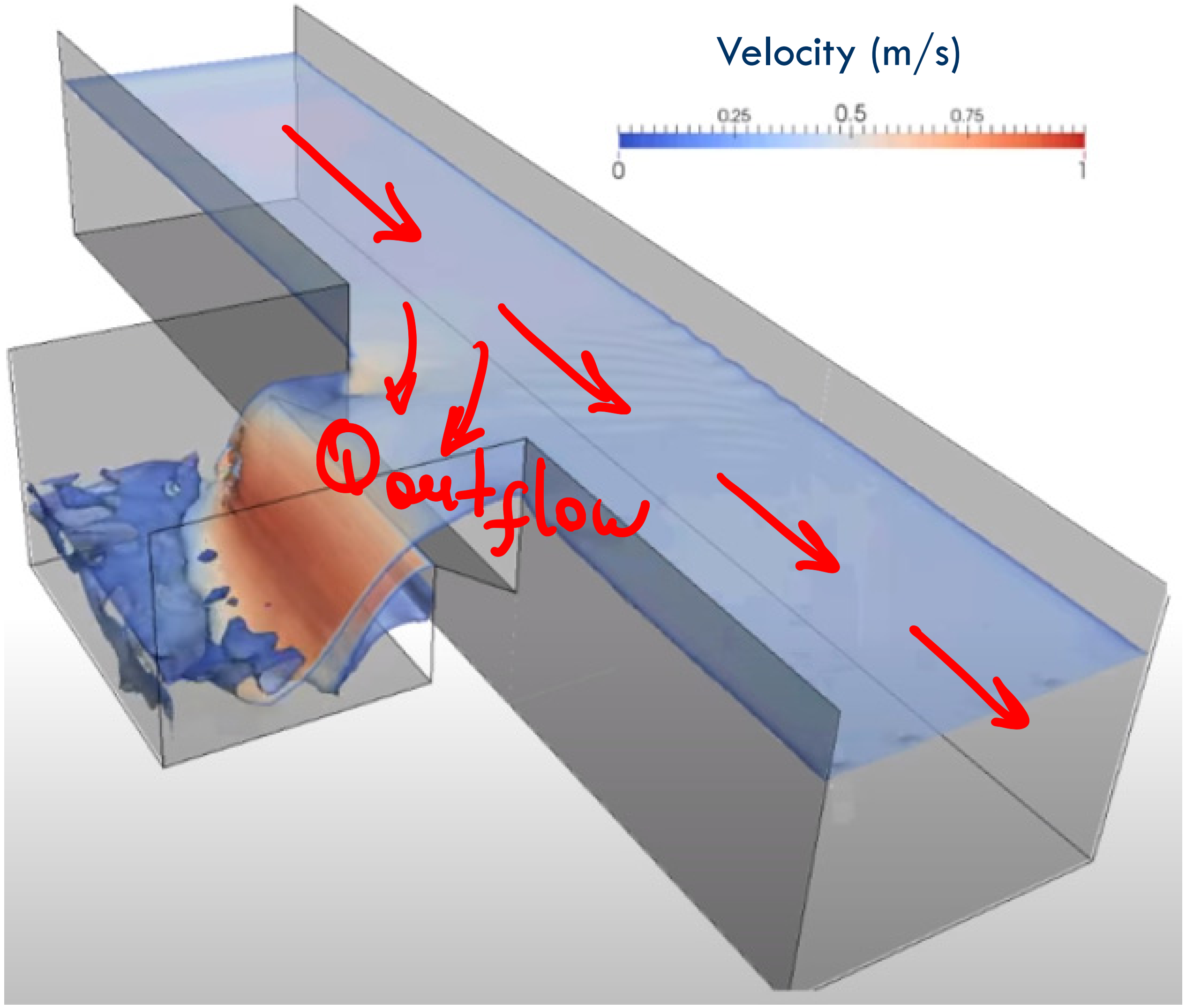
Show Annel2 demo

Download software from: https://web.eng.fiu.edu/arleon/Annel2.html



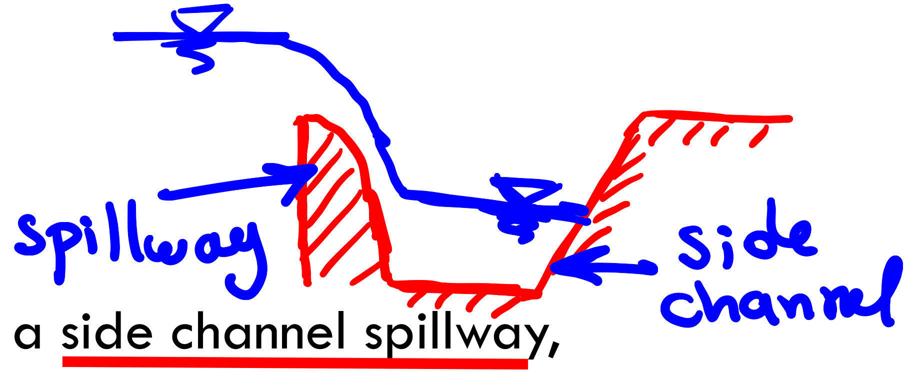
Spatially Varied Flow (Lateral Weir)

Lateral weirs



Source: https://www.youtube.com/watch?v=WayG2RgOwT8

Lateral inflow

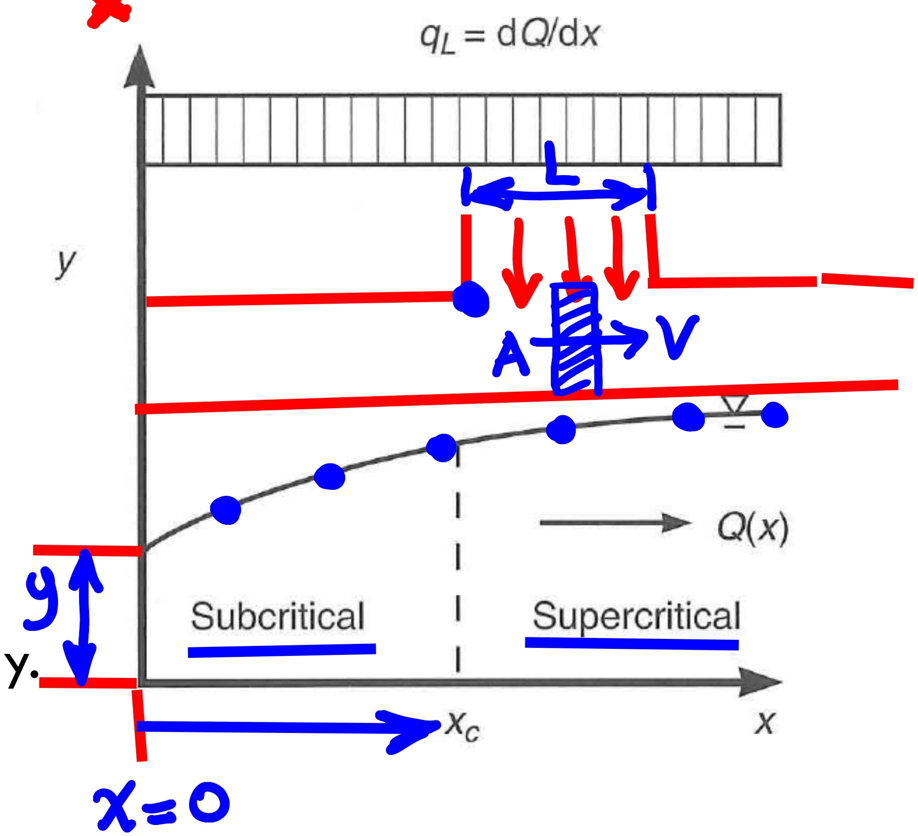


- For the case of **lateral inflow**, such as a side channel spillway, the general unsteady momentum equation can be simplified (for perpendicular entry of lateral flow) to

$Q_x = q_L x$

$$\boxed{\frac{dy}{dx} = \frac{S_o - S_f - \frac{2q_L V}{gA}}{1 - F^2}}$$

where S_f = friction slope and q_L = lateral inflow rate per unit of channel length. In the case of the side channel spillway, q_L is a constant, such that the channel discharge Q(x) = $q_L x$, where x = 0 at the upstream end of the channel.



- The equation above is solved numerically.

- **Critical conditions** can occur anywhere, with **subcritical upstream** and **supercritical downstream** of critical point.

Lateral inflow (cont.)

- **Location of the critical section** can be shown to be given by (Henderson, 1966)

$$\boxed{x_c = \frac{8q_L^2}{gB^2\left[S_o - \frac{gP}{C^2B}\right]^3}}$$

in which x_c = location of critical section; q_L = lateral inflow per unit channel length; B = channel top width; S_0 = bed slope; P = wetted perimeter; and C = Chezy resistance coefficient

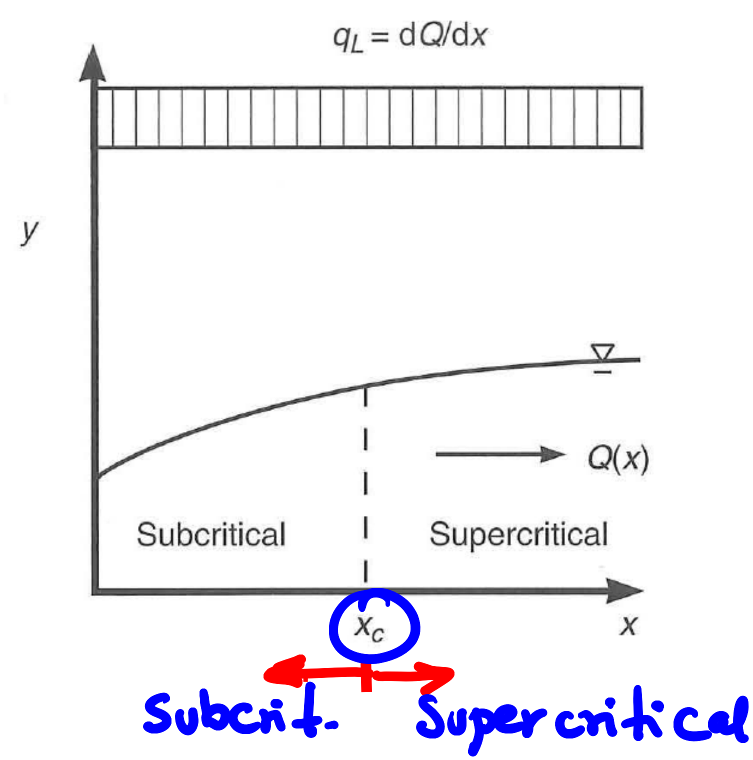
- Froude number is equal to unity at the critical section:

$$F^2 = \frac{Q^2(x)B_c}{gA_c^3} = 1$$

where $Q(x) = q_L x$.

- If $x_c > L$, the channel side length, the control is at the downstream end of the channel with **subcritical flow in the entire channel.**

- **Otherwise**, the flow is **subcritical upstream** of x_c and **supercritical downstream.**

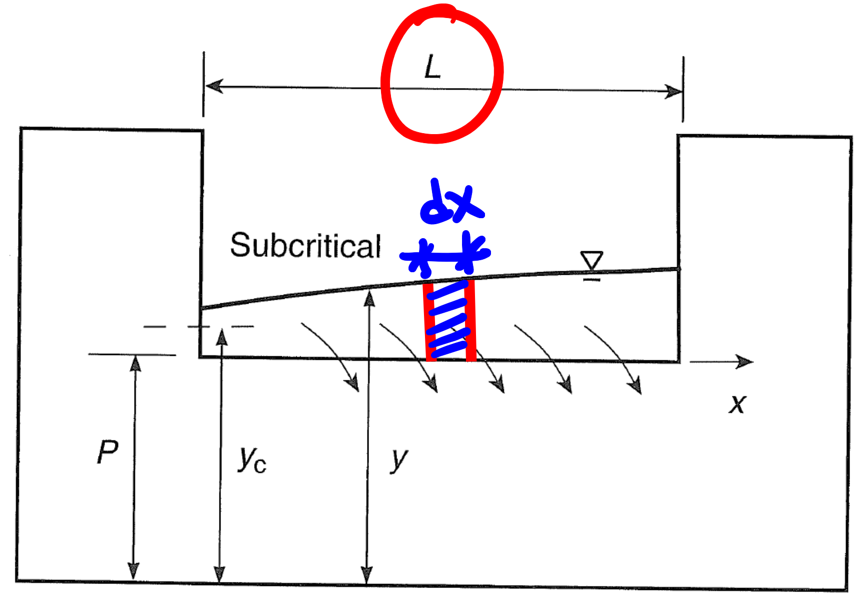


Lateral Outflow

- In the case of **lateral outflow** the direction of the lateral momentum flux is unknown
- Since the weir is a local disturbance, **energy losses along the weir are relatively small**
  - the energy approach is used more often
  - if **we assume that dE/dx = 0**, then for a rectangular channel of width *b*

$$\frac{dy}{dx} = \frac{Q(x)y\left(-\frac{dQ}{dx}\right)}{gb^2y^3 - Q^2}$$

$$\frac{dy}{dx} = \frac{\frac{q_L V}{gA}}{1 - F^2}$$



- Where $q_L = -dQ/dx$ can be obtained from the discharge equation for a sharp-crested weir:

$$q_L = -\frac{dQ}{dx} = C_1\sqrt{2g}(y-P)^{3/2}$$

$$C_1 = \frac{2}{3}C_d$$

Expression for C_d given by Hager (1999) that accounts for the lateral outflow angle and approach velocity

$$C_d = 0.636\sqrt{\frac{1-P/E}{3-2y/E-P/E}}$$

Lateral Outflow (Cont.)

- Because we assume the **energy grade line to be horizontal**, the energy equation gives the discharge at any section as

$$Q = by\sqrt{2g(E-y)}$$

where b = width of the channel and E = known constant specific energy.

- Integrating equation for dy/dx, gives the result obtained by De Marchi (Benefield, Judkins, and Parr 1984)

$$\boxed{\frac{xC_1}{b} = \frac{2E-3P}{E-P}\sqrt{\frac{E-y}{y-P}} - 3\sin^{-1}\sqrt{\frac{E-y}{E-P}} + \text{constant}}$$

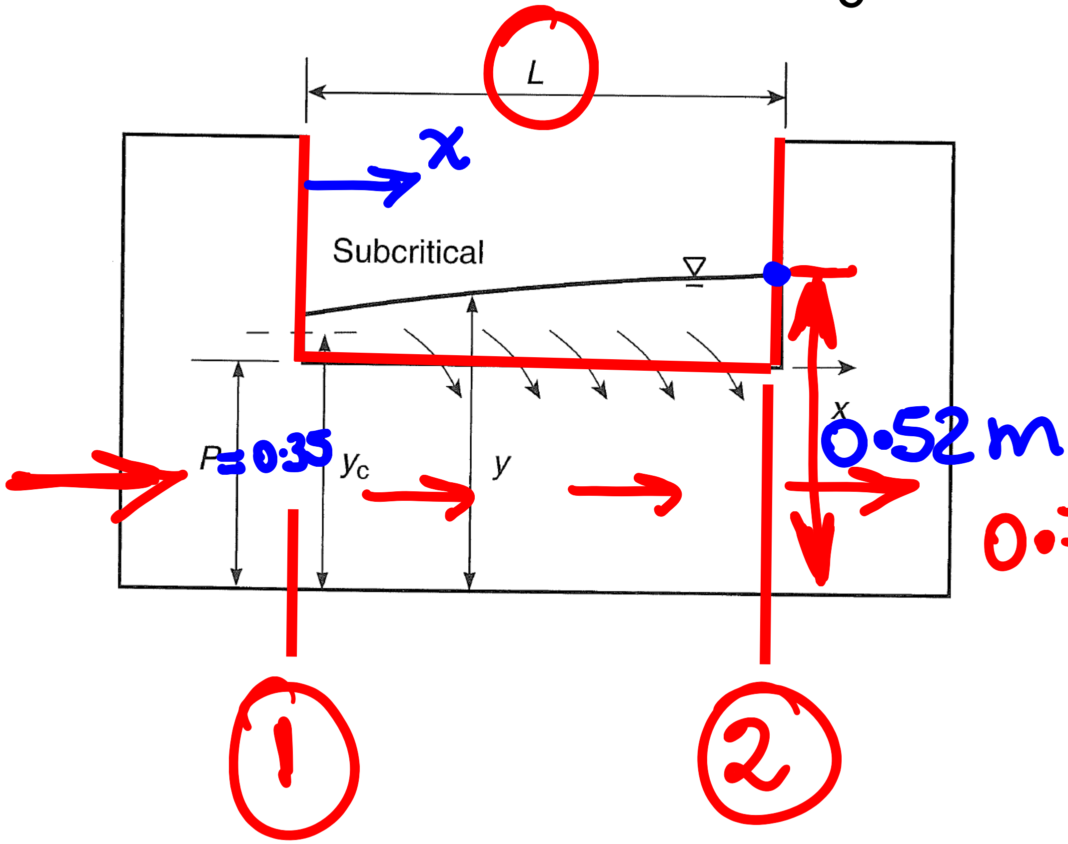
C_1 = weir discharge coefficient; P = height of weir crest above channel bottom;

- Hager (1987) showed that the outflow equation used by de **Marchi is exact only for small Froude numbers**.

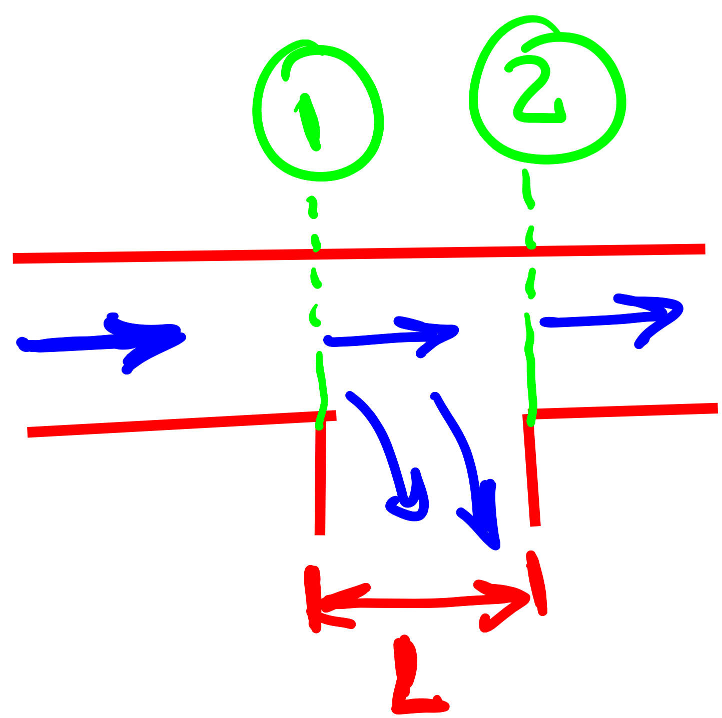
Example

A rectangular side discharge weir has a height of 0.35 m. It is located in a rectangular channel having a width of 0.7 m. If the downstream depth is 0.52 m for a discharge of 0.27 m^3/s, how long should the weir be for a lateral discharge of 0.21 m^3/s?

$P = 0.35$ m



Plan view



$y_2 = 0.52\,m$, $Q_2 = 0.27\,m^3/s$

$L = ??$ $Q_L = 0.21\,m^3/s$

$Q_1 = Q_2 + Q_L = 0.21 + 0.27$

$Q_1 = 0.48\,m^3/s$.

* $E = ??$ E = specific energy = constant $= y + \frac{V^2}{2g}$

$$E = y + \frac{Q^2}{2gA^2}$$

At section 2, we know y_2 and Q

$$E = y_2 + \frac{Q_2^2}{2g(by_2)^2} = 0.52 + \frac{0.27^2}{2 \times 9.8\,(0.7 \times 0.52)^2}$$

$$\boxed{E = 0.548\ m}$$

$$V_2 = \frac{Q_2}{A_2} = \frac{0.27}{0.7 \times 0.52} = 0.74\ m/s.$$

$$F_2 = \frac{V_2}{\sqrt{g y_2}} = \frac{0.74}{\sqrt{9.8 \times 0.52}} = 0.33$$ [subcritical flow]

* $y_1 = ?$ How to get it?

E is constant $\quad E_1 = E_2$

$$y_1 + \frac{Q_1^2}{2g(0.7 y_1)^2} = 0.548$$

$$y_1 + \frac{0.48^2}{2 \times 9.8 (0.7 y_1)^2} = 0.548$$

$y_1 = 0.392\ m$ [subcritical flow]

(highest positive root)

* Using De Marchi Equation

$$\frac{L C_1}{b} = \frac{2E-3P}{E-P}\sqrt{\frac{E-y}{y-P}} - 3\sin^{-1}\sqrt{\frac{E-y}{E-P}}\Bigg|_{y_1}^{y_2}$$

$$\frac{2E-3P}{E-P} = 0.232$$

M

$$L = \frac{b}{C_1}\left[\left(0.232\sqrt{\frac{0.548-0.52}{0.52-0.35}} - 3\sin^{-1}\sqrt{\frac{0.548-0.52}{0.548-0.35}}\right) - \left(0.232\sqrt{\frac{0.548-0.392}{0.392-0.35}} - 3\sin^{-1}\sqrt{\frac{0.548-0.392}{0.548-0.35}}\right)\right]$$

. . . . 1

$b = 0.7\,m$

$C_1 = \frac{2}{3} C_d$

$$Cd = 0.636\sqrt{\frac{1 - P/E}{3 - \frac{2y}{E} - \frac{P}{E}}}$$

Average Cd

$$\overline{Cd} = \frac{Cd(y_1) + Cd(y_2)}{2}$$

For y_1:

$$Cd = 0.636\sqrt{\frac{1 - \frac{0.35}{0.548}}{3 - 2 \times \frac{0.392}{0.548} - \frac{0.35}{0.548}}}$$

$$Cd = 0.39$$

For y_2:

$$Cd = 0.636\sqrt{\frac{1 - \frac{0.35}{0.548}}{3 - 2 \times \frac{0.52}{0.548} - \frac{0.35}{0.548}}}$$

$$Cd = 0.56$$

$$\overline{C_d} = \frac{0.39 + 0.56}{2} = 0.475$$

In 1

$$L = \frac{0.7}{\frac{2}{3}(0.475)} \times M = \boxed{3.9\,m}$$