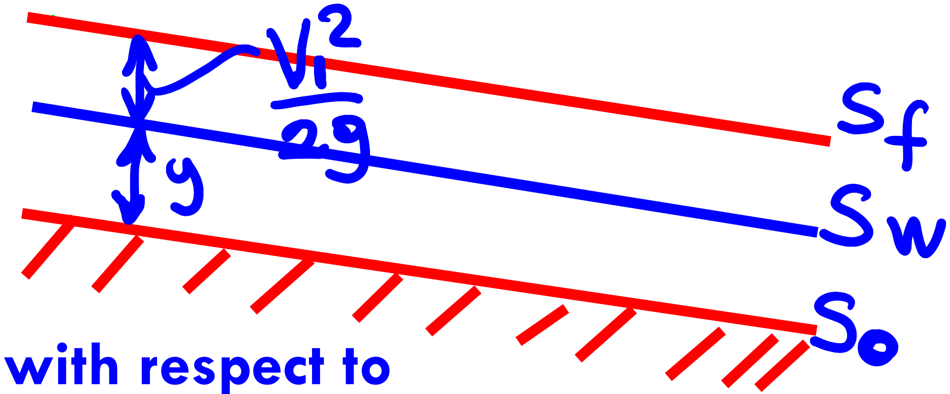
Uniform Flow $\frac{dy}{dx} = 0, \frac{dv}{dx} = 0$

Arturo S. Leon, PhD, PE, D.WRE

Uniform Flow



$S_0 = S_w = S_f$

- Flow properties remain **constant with respect to distance.**
- Uniform flow in open channels is understood to mean **steady uniform flow**
- Uniform flow is possible only in **prismatic channels.**
- **Slope** of the energy line S_f, slope of the water surface S_w and bottom slope S_0 will all be **equal to each other.**

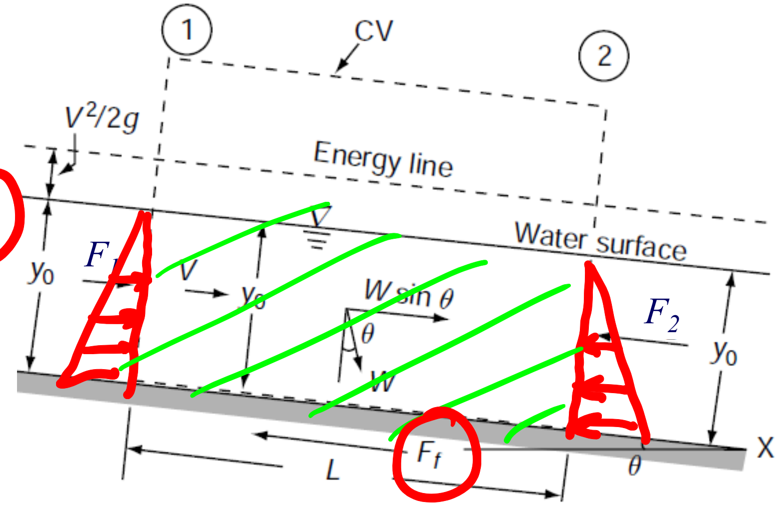
$\sin\theta \approx \tan\theta$

$S_o = \tan\theta$

Momentum Analysis

Momentum equation on control volume

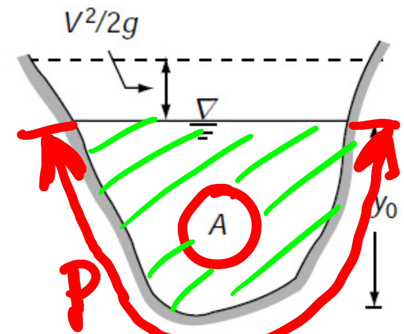
$$\cancel{F_1} - W\sin\theta - F_f - \cancel{F_2} = \cancel{M_2} - \cancel{M_1} \quad ..(1)$$



where F_1 and F_2 are the pressure forces and M_1 and M_2 are the momentum fluxes at Sections 1 and 2, respectively, W = fluid weight and F_f = shear force at the boundary.

Since the flow is uniform: $F_1 = F_2$ and $M_1 = M_2$

$\underline{W = \gamma AL}$ and $F_f = \tau_o PL$



Cross-section

where τ_0 = average shear stress on the wetted perimeter of length P. Replacing sin θ by S_0 (= bottom slope),

P: wetted perimeter

In (1) $\quad \underline{\gamma ALS_o} = \tau_o PL$

$$\tau_o = \gamma \frac{A}{P} S_o = \gamma R S_o$$

where $R = A/P$ is the hydraulic radius.

$$\boxed{\tau_o = \gamma R S_o}$$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

θ small

$\cos\theta \approx 1$

Chezy Formula

$R = \frac{A}{P}$

Expressing the average shear stress τ_0 as $k\rho V^2$, where k is a coefficient which depends on the nature of the surface and flow parameters

$$k\rho V^2 = \gamma R S_o$$

$$\boxed{V = C\sqrt{R S_o}}$$

Used in most of Europe.

More accurate than Manning's equation

where $C = \sqrt{\frac{\gamma}{\rho}\frac{1}{k}}$

C is the Chezy coefficient which depends on the nature of the surface and the flow

Dimensions of C are $[L^{1/2}\ T^{-1}]$ and it can be made dimensionless by dividing it by $g^{1/2}$.

Darcy–Weisbach Friction Factor (f)

- Pipe Flow: Surface can be termed **hydraulically smooth**, **rough** or in **transition** depending on the relative thickness of the **roughness magnitude** to the thickness of the **laminar sub-layer.** Classification is as follows:

[Can also be used for open channel flow]

$$\frac{\varepsilon_s u_*}{\nu} < 4 \text{ (hydraulically-smooth wall)}$$

$$4 < \frac{\varepsilon_s u_*}{\nu} < 60 \text{ (transitional regime)}$$

$$\frac{\varepsilon_s u_*}{\nu} < 60 \text{ (full rough flow)}$$

Where ε_s = equivalent sand grain roughness,
u_* = shear velocity ($u_* = \sqrt{gRS_o}$)
ν = kinematic viscosity

Darcy–Weisbach Friction Factor f

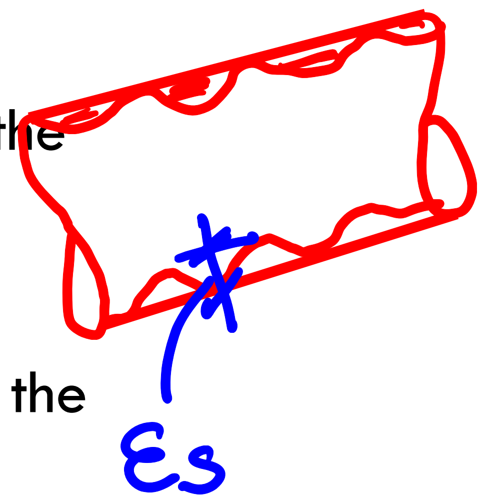
- For **pipe flow**, the Darcy–Weisbach equation is

$$h_f = f\frac{L}{D}\frac{V^2}{2g}$$

where h_f = head loss due to friction in a pipe of diameter D and length L; ***f* = Darcy–Weisbach friction factor.**

- For **smooth pipes**, ***f*** is found to be a function of the **Reynolds number** ($\mathrm{Re} = VD/\nu$) only.

- For **rough turbulent flows**, ***f*** is a function of the **relative roughness** (ε_s/D) and is independent of the Reynolds number.



- In the **transition regime**, **both** the Reynolds number and relative roughness **play important roles**.

Darcy–Weisbach Friction Factor f

- For **open channels**, the Darcy–Weisbach equation is

Replace D with 4R

$$h_f = f\frac{L}{4R}\frac{V^2}{2g}$$

$R = \frac{A}{P}$ (pipe)

$R = \frac{\cancel{\pi} D^{\cancel{2}}}{4\cancel{\pi D}} = \frac{D}{4}$

$\boxed{D = 4R}$

which on rearranging gives $V = \sqrt{\frac{8g}{f}}\sqrt{R}\sqrt{\frac{h_f}{L}}$

- Noting that for uniform flow in an open channel h_f/L = slope of the energy line = $\mathbf{S_f = S_0}$,

$$\boxed{V = C\sqrt{RS_o}} \qquad C = \sqrt{\frac{8g}{f}}$$

Manning's Formula [Mostly used in the USA]

- Proposed by **Robert Manning**, an Irish engineer, for uniform flow in open channels,

Semi-empirical formula

$$V = \frac{k}{n} R^{2/3} S_o^{1/2}$$

$k = 1$ (SI units)
$k = 1.49$ (English units)
n = a roughness coefficient known as Manning's n $[L^{-1/3}T]$.

- Comparing with the Chezy formula, $V = C\sqrt{RS_o}$

$$C = \frac{1}{n} R^{1/6}$$

- From previous slide $C = \sqrt{\frac{8g}{f}} = \frac{1}{n} R^{1/6}$

$$f = \left(\frac{n^2}{R^{1/3}}\right)(8g)$$

Velocity Distribution

Wide Channels: channels with **large aspect ratio** B/y_0 , as for example in rivers, the flow can be considered to be **two dimensional.**

1. **Velocity-defect Law:** Fully developed velocity distributions are similar to the **logarithmic form of velocity-defect law** found in turbulent flow in pipes.

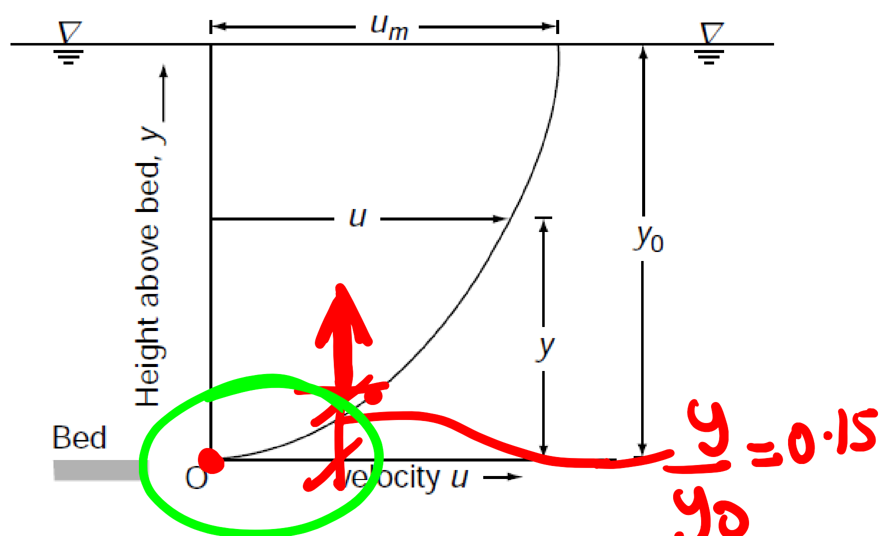
- **Maximum velocity** u_m occurs essentially at the **water surface**
- **The velocity** u **at a height** y above the bed in a channel having uniform flow at a depth y_0 is given by the velocity defect law for $y/y_0 > 0.15$ as

$$\frac{u_m - u}{u_*} = -\frac{1}{k} \ln \frac{y}{y_0}$$

Applicable to both rough and smooth boundaries

$$= -\frac{2.3}{k} \log_{10} (y/y_0)$$

where u_* = shear velocity = $\sqrt{\tau_0/\rho} = \sqrt{gRS_0}$
R = hydraulic radius, S_0 = longitudinal slope, and k = Karman constant = 0.41 for open channel flows



Velocity Distribution (Cont.)

2. Law of the Wall (inner wall region):

- For **smooth boundaries**, the law of the wall is found applicable in the inner wall region (**$y/y_0 < 0.20$**) as

$$\frac{u}{u_*} = \frac{1}{k} \ln \frac{y u_*}{\nu} + A_s$$

Constants $k = 0.41$ and $A_s = 5.29$ regardless of the Froude number and Reynolds number of the flow

- For **completely rough turbulent flows**, the velocity distribution in the wall region (**$y/y_0 < 0.20$**) is given by

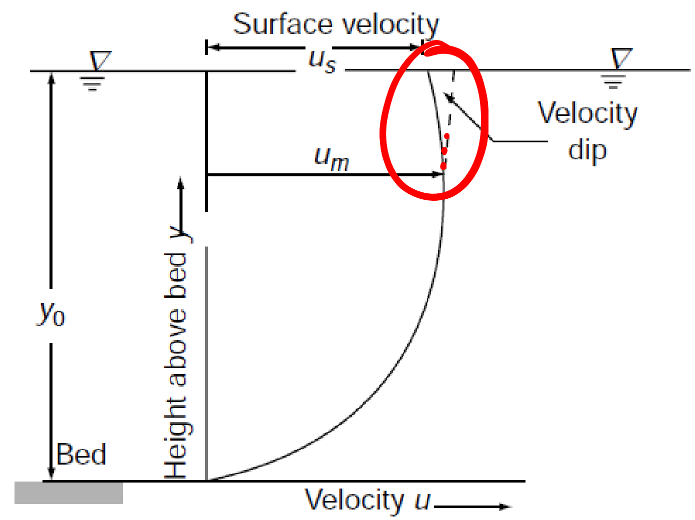
$$\frac{u}{u_*} = \frac{1}{k} \ln \frac{y}{\varepsilon_s} + A_r$$

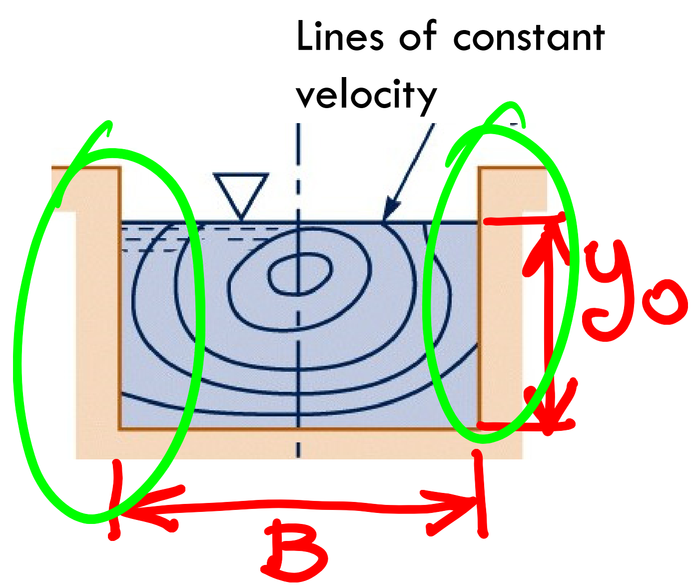
where ε_s = equivalent sand grain roughness. It has been found that k is a universal constant irrespective of the roughness size. Values of k = 0.41 and A_r = 8.5 are appropriate.

Velocity Distribution (Cont.)

$\frac{B}{y_0} \leq 5$ (Narrow)

- **Channels with Small Aspect Ratio:** channels which are **not wide enough to have two dimensional flow**, the resistance of the sides will be significant to alter the two-dimensional nature of the velocity distribution
  - Occurrence of **velocity-dip**, where the maximum velocity occurs not at the free surface but rather some distance below it because of secondary currents
  - The **critical ratio of B/y_0** above which the velocity-dip becomes insignificant has been found to be about 5.0.
  - Channels with **$B/y_0 \leq 5$ can be classified as narrow channels**





Manning's Roughness Coefficient *n*

- Selection of a value for **"*n*" is subjective**, based on one's own experience and engineering judgement
- **Cowan** has developed a procedure to estimate the value of roughness factor n of natural channels in a systematic way

$$n = (n_b + n_1 + n_2 + n_3 + n_4)m$$

Where n_b = a base value of *n* for a straight uniform smooth channel in natural material

n_1 = correction for surface irregularities

n_2 = correction for variation in shape and size of the cross section

n_3 = correction for obstructions

n_4 = correction for vegetation and flow conditions

m = correction for meandering of the channel

Manning's Roughness Coefficient *n*

- Empirical Formulae for *n*

Strickler formula: $n = \frac{d_{50}^{1/6}}{21.1}$

Where d_{50} **is in meters** and represents the particle size in which 50% of the bed material is finer.

Meyer et al. for coarse grained beds: $n = \frac{d_{90}^{1/6}}{26}$

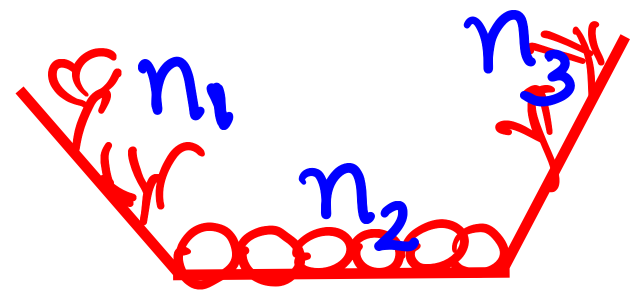
useful in predicting *n* in mountain streams paved with coarse gravel and cobbles

Where d_{90} **is in meters** in which 90% of the particles are finer than d_{90}.

Equivalent or Composite Roughness *n*

$n_1 \neq n_2$

- Different parts of the channel perimeter may have **different roughness**
  - Canals in which **only the sides are lined**,
  - laboratory flumes with glass side walls and rough bed, natural rivers with **sandy bed and sides with vegetation.**



- Necessary to determine an equivalent roughness coefficient that can be **applied to the entire cross-sectional perimeter** for use in Manning's formula

Manning's Roughness Coefficient *n*

Horton's Method of Equivalent Roughness Estimation

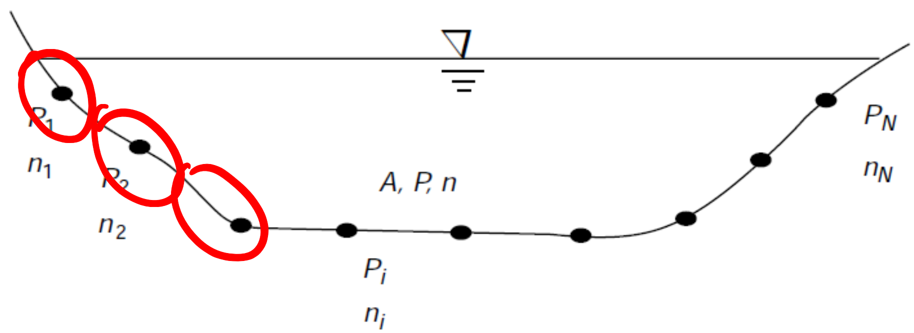
Consider a channel having its perimeter composed of ***N* types of roughness**, P_1, P_2, …P_N are the lengths of these *N* parts and n_1, n_2, … n_N are the respective roughness coefficients

Let each part P_i be associated with a **partial area A_i** such that

$$\sum_{i=1}^{N} A_i = A_1 + A_2 + \dots + A_i + \dots + A_N = A$$

It is assumed that the **mean velocity in each partial area** is the same as the mean velocity *V* for the entire cross-section,

$$V_1 = V_2 = \dots = V_i = \dots = V_N = V$$



By the Manning's formula

$$S_o^{1/2} = \frac{V_1 n_1}{R_1^{2/3}} = \frac{V_2 n_2}{R_2^{2/3}} = \dots = \frac{V_i n_i}{R_i^{2/3}} = \frac{V n}{R^{2/3}}$$

Where *n* = equivalent roughness

$$\left(\frac{A_i}{A}\right)^{2/3} = \frac{n_i P_i^{2/3}}{n P^{2/3}} \qquad A_i = A\frac{n_i^{3/2} P_i}{n^{3/2} P}$$

$$\sum_{i=1}^{N} A_i = A = A\frac{\sum_{i=1}^{N}\left(n_i^{3/2} P_i\right)}{n^{3/2} P} \qquad n = \left[\frac{\sum_{i=1}^{N} P_i n_i^{3/2}}{\sum_{i=1}^{N} P_i}\right]^{2/3}$$

If the **Darcy–Weisbach friction formula** is used under the same assumption of (i) Velocity being equal in all the partial areas, and (ii) slope S_0 is common to all partial areas, then

$$\frac{h_f}{L} = S_o = \frac{fV^2}{8gR} = \frac{fV^2 P}{8gA} \quad \text{Hence} \quad \frac{V^2}{8gS_o} = \frac{A}{Pf} = \frac{A_i}{P_i f_i}$$

Thus $\frac{A_i}{A} = \frac{P_i f_i}{Pf}$ and on summation $\sum_{i=1}^{N} \frac{A_i}{A} = \frac{\sum_{i=1}^{N} P_i f_i}{Pf} = 1$

$$\sum_{i=1}^{N} P_i f_i = Pf \qquad f = \frac{\sum_{i=1}^{N} P_i f_i}{P}$$

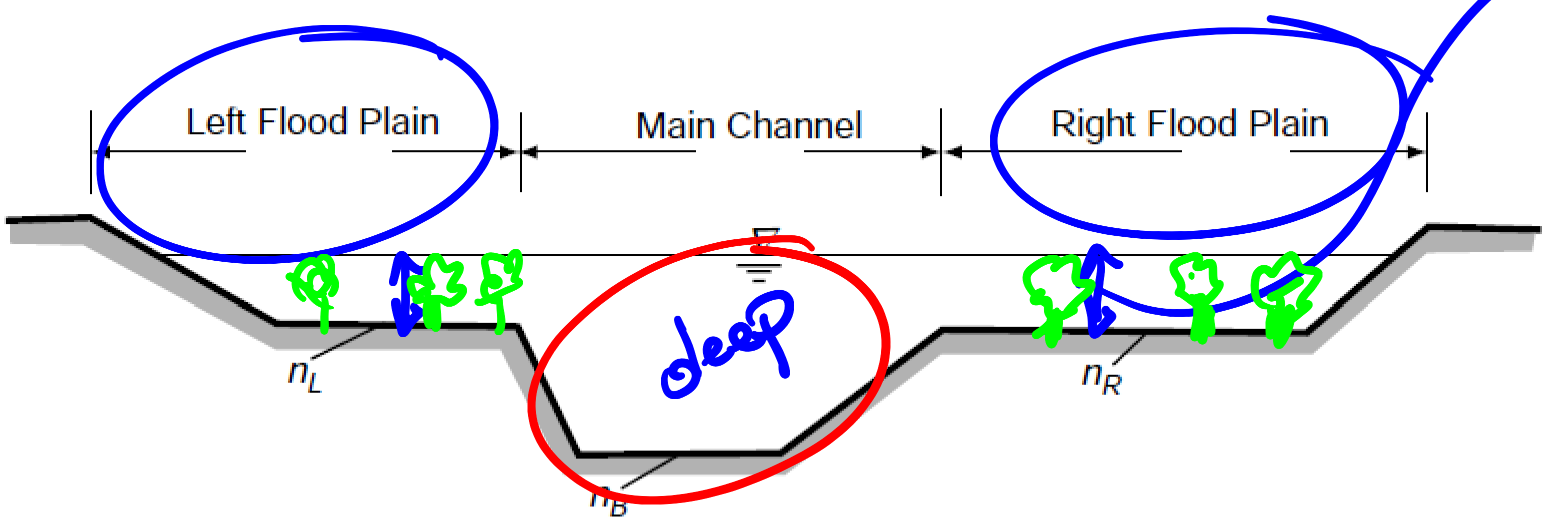
Equivalent or Composite Roughness (n_c)

Formula	Assumption	Reference
$n_c = \left[\frac{\sum_{i=1}^{N} P_i n_i^{3/2}}{\sum_{i=1}^{N} P_i}\right]^{2/3}$	The **velocities** corresponding to the different **sub-areas are equal to one another** as well as equal to the mean velocity of the whole cross-section.	Horton, 1933
$n_c = \left[\frac{\sum_{i=1}^{N} P_i n_i^{2}}{\sum_{i=1}^{N} P_i}\right]^{1/2}$	**Total resisting force** is equal to the **sum of the resisting forces** in each sub-section and the hydraulic radius of each sub-section is equal to the hydraulic radius of the whole cross-section.	Einstein and Banks, 1951
BEST!! $n_c = \frac{PR^{5/3}}{\sum_{i=1}^{N} \frac{P_i R_i^{5/3}}{n_i}}$	Writes **total discharge** as the **sum of the discharges** in the sub-sections. Was found to be the **most accurate based on measurements in 36 US streams**.	Lotter, 1933

Compound Channels

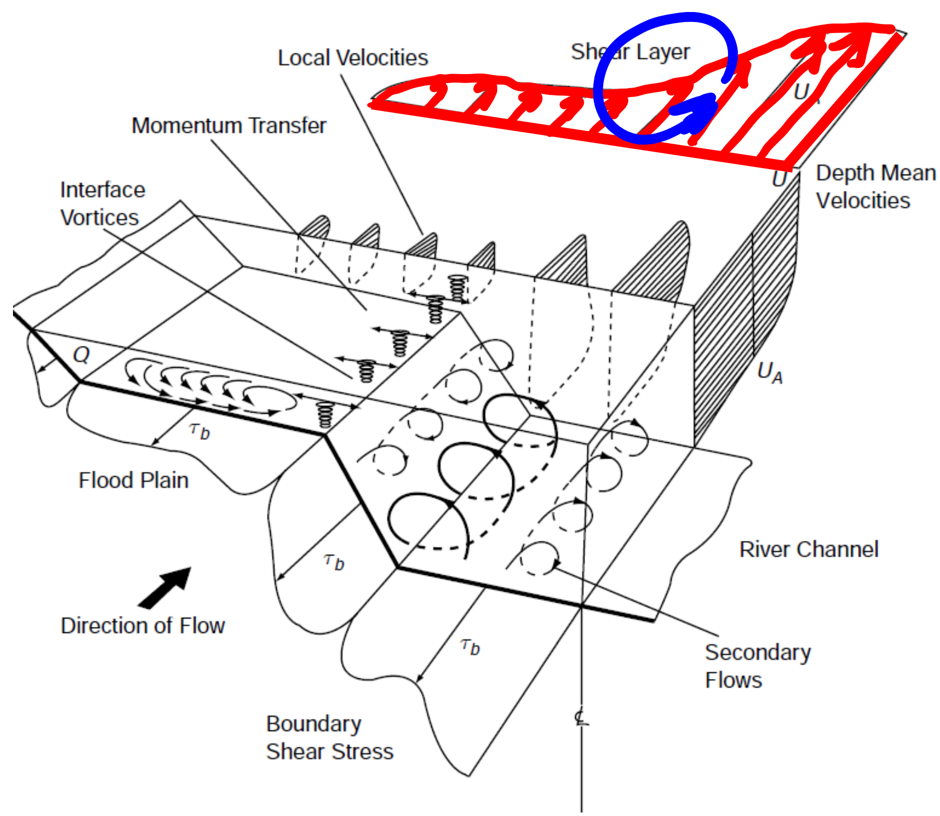
- Channel section composed of a **main deep portion** and one or **two flood plains** that carry high-water flows.
- **Main channel carries the dry weather** flow and during wet season, the flow may spillover the banks of the main channel to adjacent flood plains.

shallow



Compound Channels

- **Velocity of flow in the flood plain is smaller than in the main channel** due to relative smaller water depth and higher bed roughness.
- At the junction of the main channel with the flood plain a set of **vortex structures** having vertical axis extending up to the water surface exist.
  - This vortex set is believed to be responsible for **momentum exchange** between the main and shallow water flows.

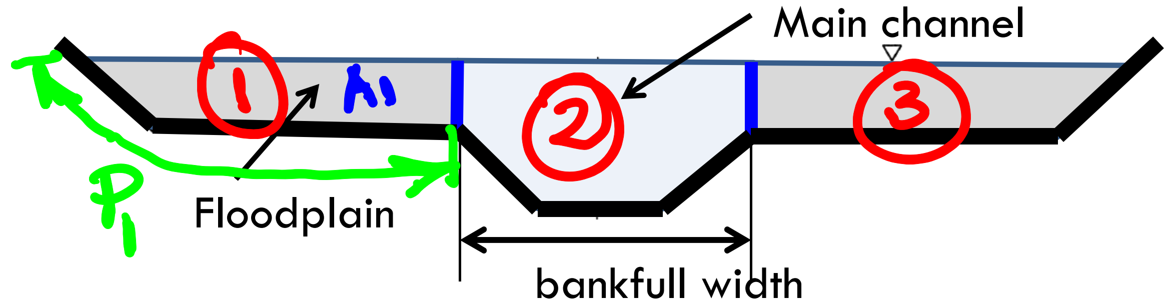


Compound Channels

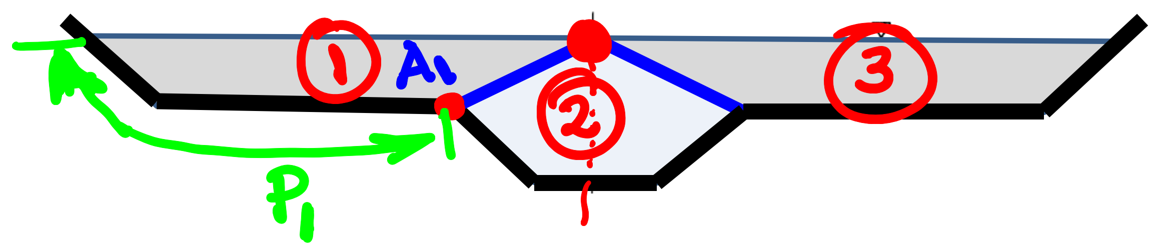
- Manning's formula is applied to the compound channel by considering a **common conveyance *K* and a common energy slope S_f** for the entire section to obtain the discharge as $Q = K\sqrt{S_f}$
- To account for the **different hydraulic conditions of the main and flood plain sections**, the channel is considered to be divided into subsections with **each subsection having its own conveyance**, K_i.
  - Sum of conveyances gives the total channel conveyance ($\Sigma Ki = K$) for use in discharge computation.
  - known as **Divided Channel method (DCM)**

Divided Channel Method (DCM)

1. Vertical Interface Method:



2. Diagonal Interface Method:



- Length of the **vertical and diagonal interface** is **not included in the calculation of the wetted perimeter** of either the over bank flow or the main channel flow

- It is known the **DCM over estimates the discharge** to some extent.

The best hydraulic cross section

The best hydraulic cross section is defined as the section of **maximum flow rate (Q) for a constant hydraulic area** (*A*), slope (S_o), and roughness coefficient (*n*).

$$Q = \frac{k}{n} A R^{2/3} S_o^{1/2}$$

A is constant

Q maximum

$$Q = \text{constant} * \left(\frac{A}{P}\right)^{2/3}$$

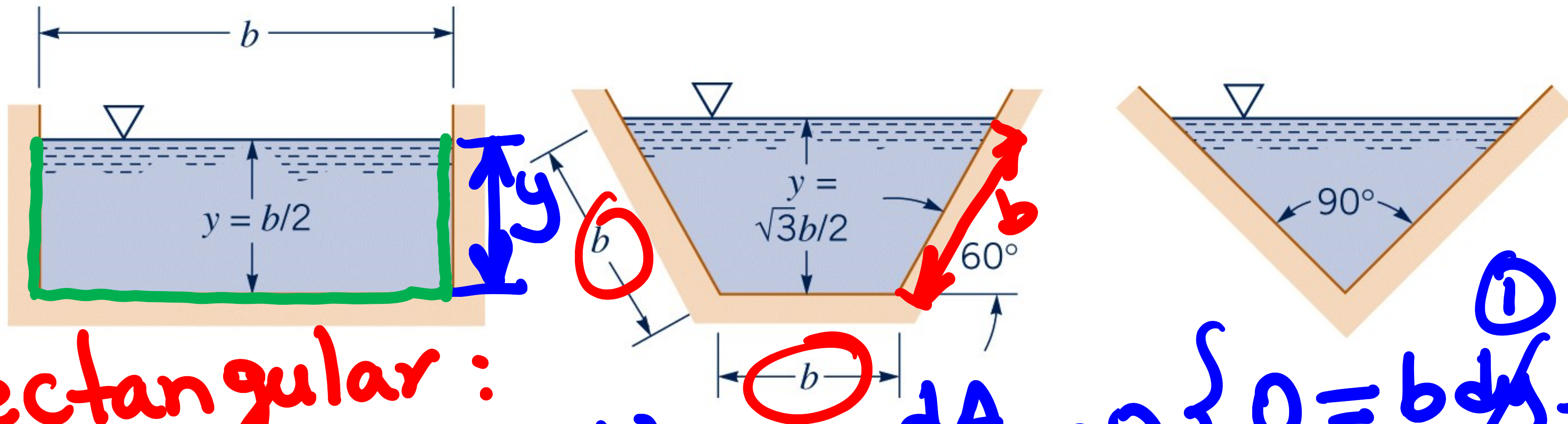
$$Q \sim \frac{\text{constant}}{P^{2/3}}$$

Q will be maximum when P is minimum

$$Q = \frac{m}{P^{2/3}}$$

$$\frac{dP}{dy} = 0$$

The best hydraulic cross-section for various shapes



Rectangular:

A = by (constant)

P = b + 2y (minimum)

$\frac{dP}{dy} = 0 \rightarrow 0 = \frac{db}{dy} + 2 \rightarrow \frac{db}{dy} = -2$

$\frac{dA}{dy} = 0 \left\{ 0 = b\frac{\cancel{dy}}{\cancel{dy}} + y\frac{db}{dy} \right.$ ①

$\frac{db}{dy} = -\frac{b}{y}$

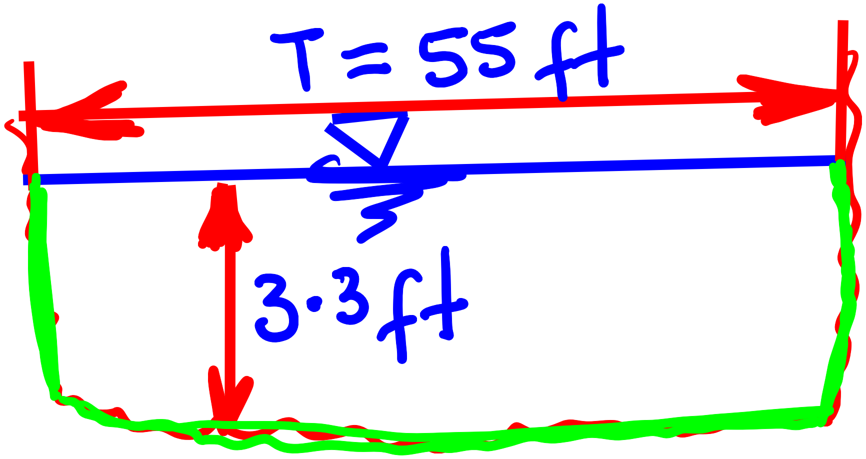
$-2 = -\frac{b}{y}$

$\boxed{b = 2y}$

Example:

The following data are obtained for a particular reach of the Provo River in Utah: A = 183 ft^2, free-surface width = 55 ft, average depth = 3.3 ft, R_h = 3.32 ft, V = 6.56 ft/s, length of reach = 116 ft, and elevation drop of reach = 1.04 ft. Determine (a) the average shear stress on the wetted perimeter, (b) the Manning coefficient, n, and (c) the Froude number of the flow.

$R_h = 3.32\ ft$



$A = 183\ ft^2$

$T = 55\ ft$

$V = 6.56\ ft/s$

$L = 116\ ft$

$\Delta h = 1.04\ ft$

$$S_o = \frac{\Delta h}{L} = \frac{1.04}{116} = 0.00897$$

a) Average shear stress:

$$\tau_0 = \gamma R S_0 = 62.4 \frac{lb}{ft^3} \times 3.32\, ft \times 0.00897$$

$$\boxed{\tau_0 = 1.8 \frac{lb}{ft^2}}$$

b) $n = ??$

$$V = \frac{k}{n} R^{2/3} S_0^{1/2}$$

$$n = \frac{k R^{2/3} S_0^{1/2}}{V}$$

$$n = \frac{1.49 \times 3.32^{2/3} \times 0.00897^{1/2}}{6.56}$$

$$n = 0.0469$$

c) $Fr = \dfrac{V}{\sqrt{g\dfrac{A}{T}}}$

For any cross-section

$$Fr = \frac{6.56}{\sqrt{32.2 \times \dfrac{183}{55}}} = 0.63$$

subcritical flow

Example:

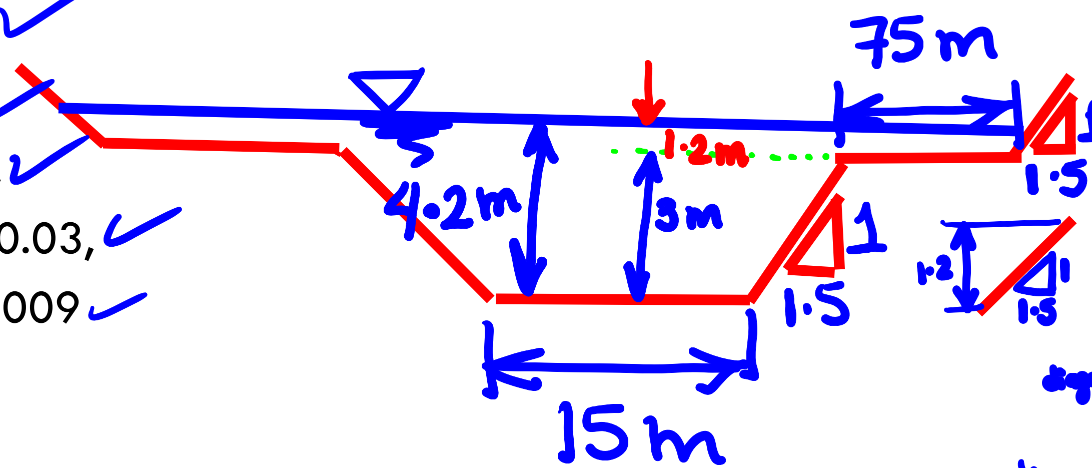
$n_M = 0.03$

$n_F = 0.05$

$S = 0.0009$

A compound channel is symmetrical in cross section and has the following geometric properties.

- Main channel:
  - Trapezoidal cross section,
  - Bottom width = 15.0 m,
  - Side slopes = 1.5H:1V,
  - Bank full depth = 3.0 m,
  - Manning's coefficient = 0.03,
  - Longitudinal slope = 0.0009
- Flood plains:
  - Width = 75 m,
  - Side slope = 1.5H:1V,
  - Manning's coefficient = 0.05,
  - Longitudinal slope = 0.0009.
- Compute the uniform flow discharge for a flow with total depth of 4.2 m by using DCM with (i) diagonal interface, and (ii) vertical interface procedures.



$\frac{1.2}{1} = \frac{x}{1.5}$

$x = 1.8$

$\text{diagonal} = \sqrt{1.2^2 + 1.8^2}$

$\text{diagonal} = 2.16$

$Q = ??$

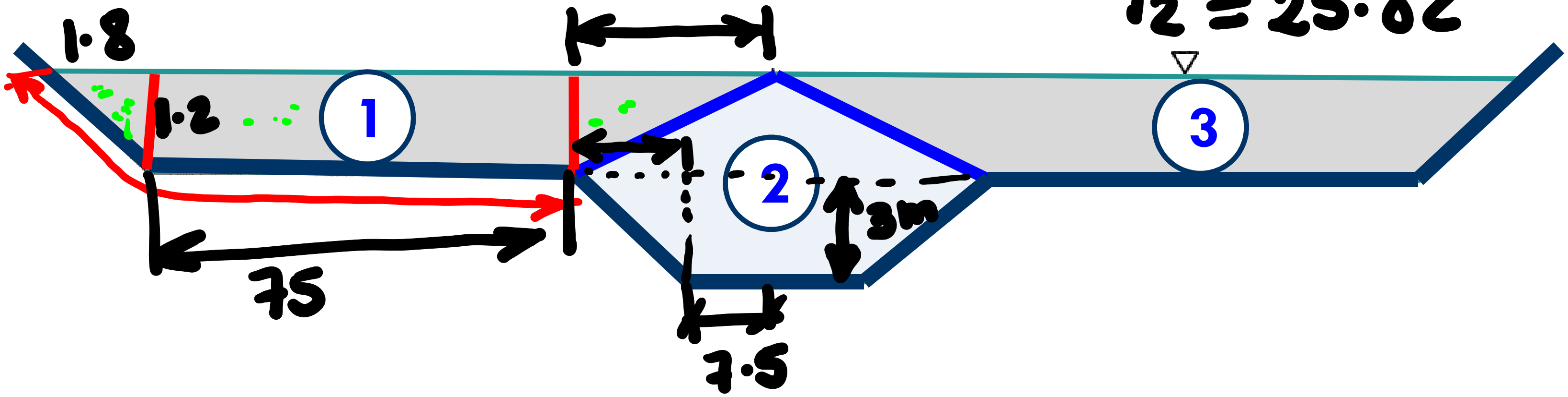
a) Diagonal

$P_1 = 75 + 2.16 = 77.16\,m$, $P_2 = b + 2y\sqrt{1+z^2}$

$P_3 = 77.16\,m$

$P_2 = 15 + 2\times3\sqrt{1+1.5^2}$

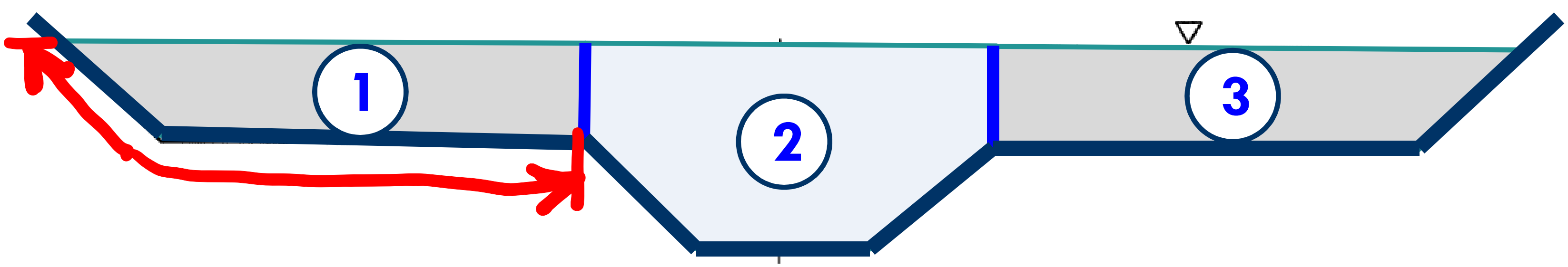
$P_2 = 25.82$



b) Vertical

$P_1 = 77.16\,m$, $P_2 = 25.82$

$P_3 = 77.16\,m$



* Diagonal :

$$A_1 = \frac{1.8 \times 1.2}{2} + 75 \times 1.2 + \frac{1.2 \times 12}{2}$$

$A_1 = 98.28$, $A_3 = 98.28\ m^2$

$$A_2 = 1.2 \times 12 + \left(\frac{15+24}{2}\right) \times 3 = 72.9\ m^2$$

$R_1 = A_1/P_1$, $R_1 = 1.274\ m$

$R_2 = A_2/P_2$

$R_2 = 2.824\ m$

* flow discharges.

Manning's formula $Q = \frac{k}{n} A R^{2/3} S^{1/2}$

$k = 1.0$ (SI)

$Q_1 = \frac{1}{0.05} * 98.28 * 1.274^{2/3} * 0.0009^{0.5}$

$= 69.29 \; m^3/s$

$Q_2 = \frac{1}{0.03} * 72.90 * 2.824^{2/3} * 0.0009^{0.5} = 145.64 \; m^3/s$

$Q_3 = 69.29 \; m^3/s$.

$Q = Q_1 + Q_2 + Q_3 = 284.21 \; m^3/s$

Diagonal interface Method.

Vertical interface

$$A_1 = \frac{1.8\times1.2}{2} + 1.2\times 75 = 91.08\ m^2$$

$$A_2 = 24\times1.2 + \left(\frac{15+24}{2}\right)\times3 = 87.3\ m^2$$

$$A_3 = 91.08\ m^2.$$

$$R_1 = \frac{91.08}{77.16} = 1.18\ m\ ,\qquad R_3 = 1.18 m$$

$$R_2 = 3.382\ m$$

* **Flow discharges.**

$$Q_1 = \frac{1}{0.05} * 91.08 * 1.18^{2/3} * 0.0009^{0.5} = 61.04\ m^3/s$$

$$Q_2 = \frac{1}{0.03} \times 87.3 \times 3.382^{2/3} * 0.0009^{0.5} = 196.70\ m^3/s$$

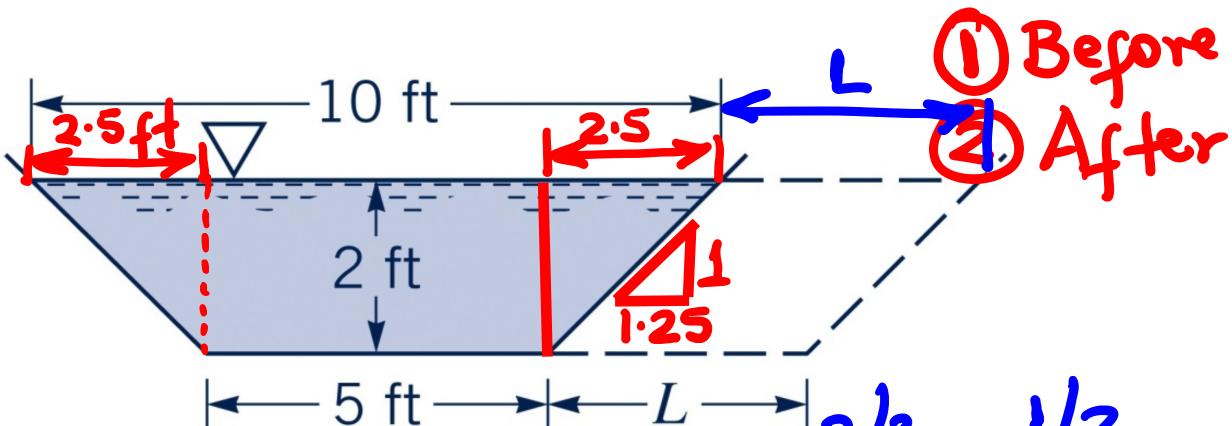
$$Q_3 = 61.04\ m^3/s$$

$$\therefore \quad Q_{total} = Q_1 + Q_2 + Q_3 = 318.77 \ m^3/s$$

Vertical interface method.

Example:

The canal shown in the figure below is to be widened so that it can carry twice the amount of water. Determine the additional width, ***L***, required if all other parameters (i.e., flow depth, bottom slope, surface material, side slope) are to remain the same.



$$\frac{Q_1}{Q_2} = \frac{1}{2} \quad \ldots \quad (1)$$

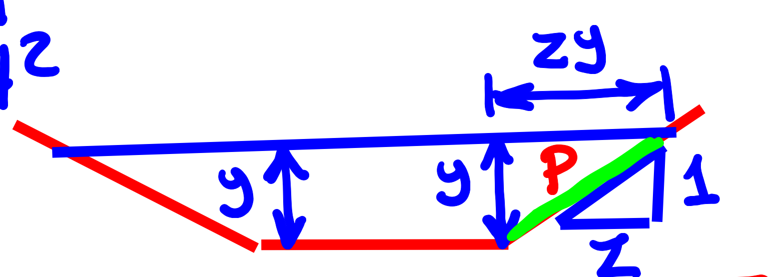
$L = ??$

* Manning's: $Q = \frac{k}{n} A R^{2/3} S_o^{1/2}$

$$A_1 = \frac{(5+10)}{\cancel{2}} \times \cancel{2} = 15\ ft^2$$

$$P_1 = 5 + 2\times 2\sqrt{1+1.25^2}$$

$$P_1 = 11.4\ ft$$



$$P = b + 2\sqrt{y^2 + z^2y^2}$$

$$R_1 = \frac{A_1}{P_1} = 1.316 \text{ ft}$$

$$P = b + 2y\sqrt{1+z^2}$$

$$A_2 = \left(\frac{b+B}{2}\right)h = \frac{(5+L+10+L)}{\cancel{2}} * \cancel{2}$$

$$A_2 = 15 + 2L$$

$$P_2 = P_1 + L = 11.4 + L$$

$$R = \frac{15+2L}{11.4+L}$$

In 1

$$\frac{\cancel{\frac{k}{n}} A_1 R_1^{2/3} \cancel{S_o^{1/2}}}{\cancel{\frac{k}{n}} A_2 R_2^{2/3} \cancel{S_o^{1/2}}} = \frac{1}{2}$$

$$\frac{15 \times 1.316^{2/3}}{(15+2L)\left(\frac{15+2L}{11.4+L}\right)^{2/3}} = \frac{1}{2} \rightarrow L = 5.94 \text{ ft}$$

Example:

A trapezoidal channel is to be designed to carry a discharge of 75 m^3/s with a velocity of 1.75 m/s. The side slopes of the channel are 2:1 (2 Horizontal and 1 Vertical) and the Manning's roughness *n* is 0.030. If the channel is designed for **maximum hydraulic efficiency** conditions, what should be bottom width and height of the channel?

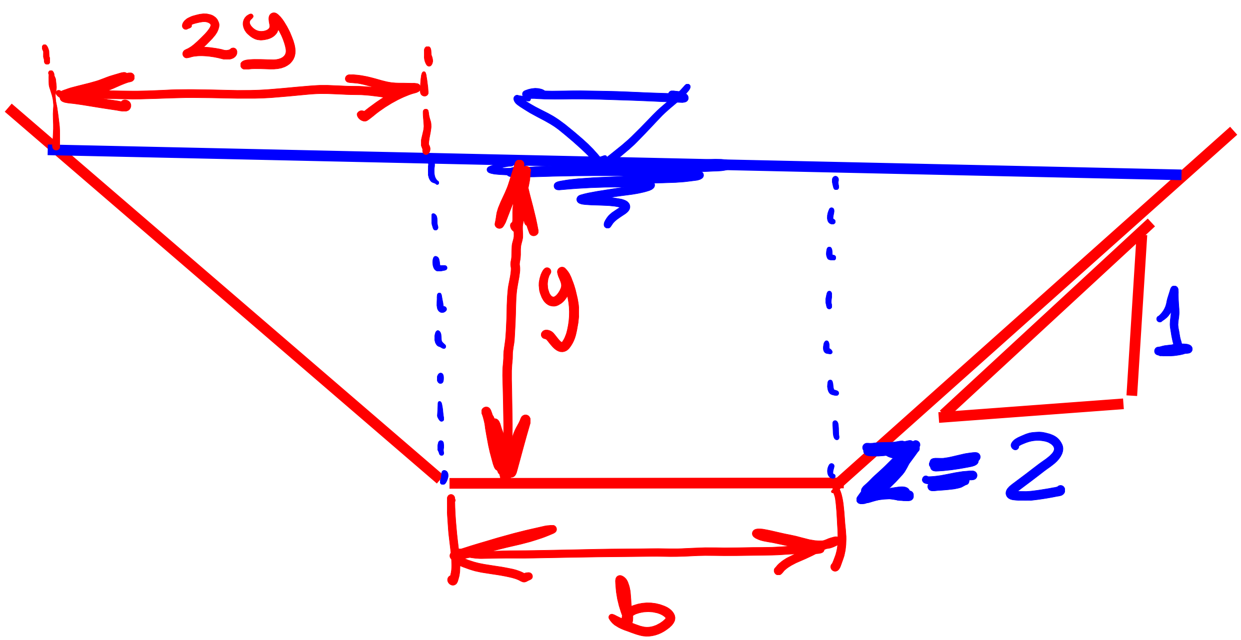
$Q = 75\ m^3/s$

$V = 1.75\ m/s$

$n = 0.03$

$b = ??$

$y = ??$



* $Q = V \cdot A \rightarrow \boxed{A = \frac{75}{1.75} = 42.86\,m^2}$

* Manning's

$$Q = \frac{k}{n} A R^{2/3} S_o^{1/2}$$

* Maximum hydraulic efficiency:

P is minimum $\left(\frac{dP}{dy} = 0\right)$

$$P = b + 2y\sqrt{1+z^2}$$

$$P = b + 2y\sqrt{1+2^2}$$

$$P = b + 2\sqrt{5}\,y$$

$$\frac{dP}{dy} = 0 \qquad 0 = \frac{db}{dy} + 2\sqrt{5} \longrightarrow \frac{db}{dy} = -2\sqrt{5} \quad \ldots\ (1)$$

* Area is contant

$$\frac{dA}{dy} = 0 \qquad A = by + 2y \cdot y$$

$$A = by + 2y^2$$

$$0 = b\,\cancel{\frac{dy}{dy}}^{\,1} + y\frac{db}{dy} + 4y$$

$$0 = b + y\frac{db}{dy} + 4y \quad \dots \text{②}$$

① in ②

$$0 = b + y(-2\sqrt{5}) + 4y$$

$$b = y(2\sqrt{5} - 4)$$

$$\boxed{b = 0.472y}$$

$$\text{Area} = 42.86 \text{ m}^2$$

$$\text{Area} = by + 2y^2 = 42.86$$

$$0.472y^2 + 2y^2 = 42.86$$

$$2.472\,y^2 = 42.86$$

$$\boxed{y = 4.16\,m}$$

$$\boxed{b = 1.96\,m}$$