# Specific Energy

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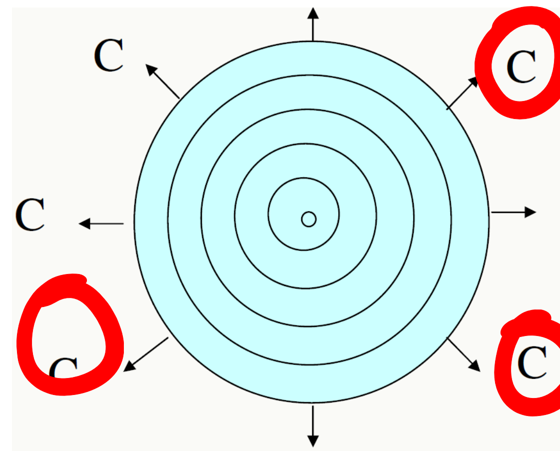
c: gravity wave speed

# Wave speed in open channel flows

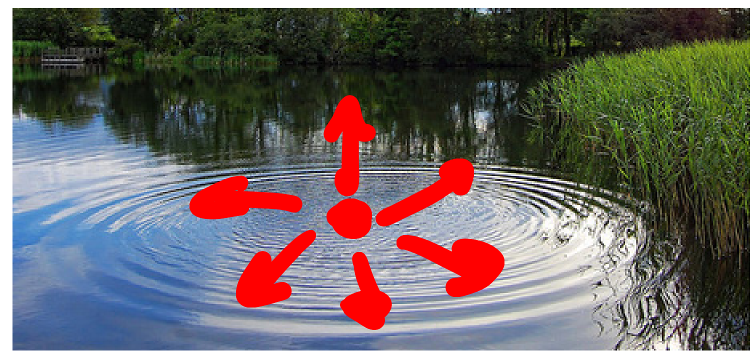
**Any cross-section channel:**

$$c=\sqrt{\frac{gA}{T}}$$



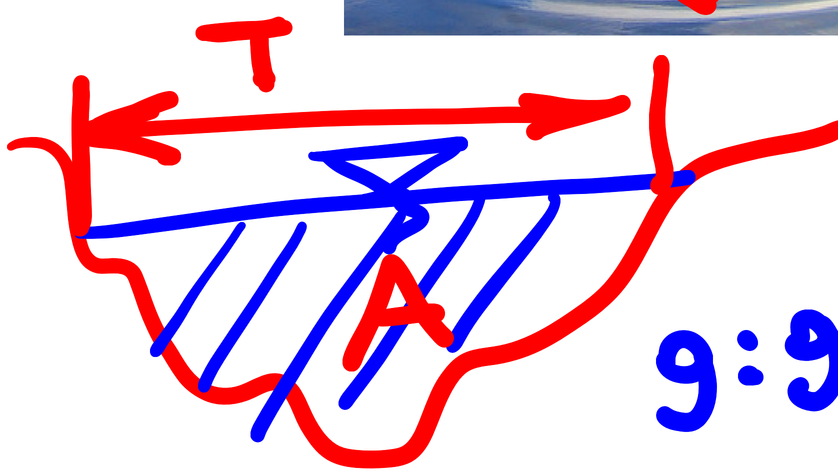

**Rectangular channels:**

$$c=\sqrt{gy}$$



A: hydraulic area

T: Surface width




g: gravity

# Froude Number:

$$Fr \sim \frac{\text{Inertia}}{\text{Gravity}}$$

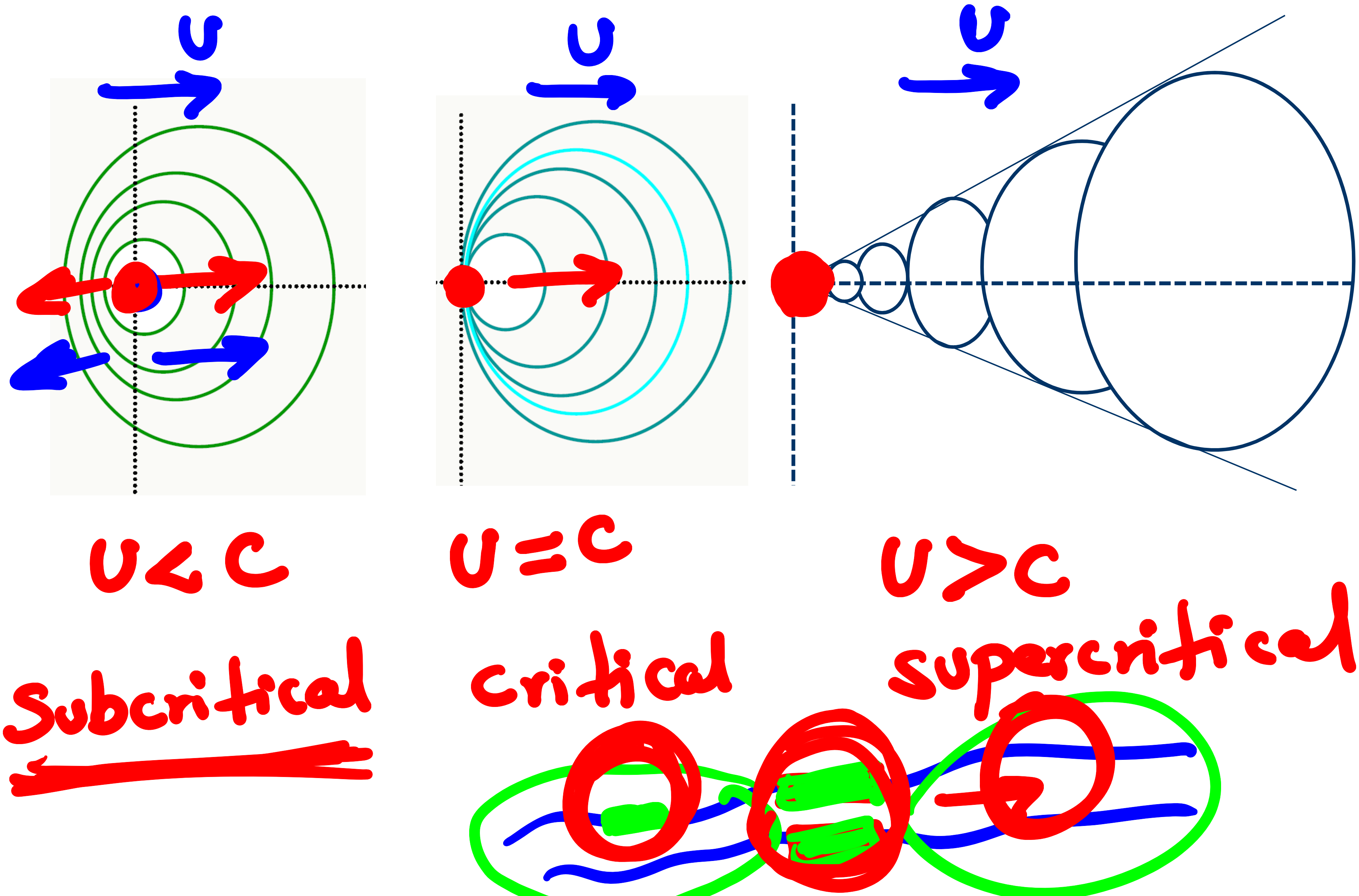
$$\boxed{Fr = \frac{V}{c}}$$

$$Fr = \frac{V}{\sqrt{g\frac{A}{T}}}$$

*V* is the average velocity
c is the gravity wave speed

- Subcritical flow: $Fr < 1$
- Critical flow: $Fr = 1$
- Supercritical flow: $Fr > 1$

# Propagation of a disturbance in subcritical, critical and supercritical flows



# How about other type of forces in open channel flows ?

$$\mathrm{Re} \sim \frac{\text{Inertia}}{\text{Viscous}} \qquad \mathrm{Re} = \frac{VR_h}{\nu}$$

$V$ is the average velocity

$R_h$ is the hydraulic radius

$\nu$ is kinematic viscosity

- Laminar flow: Re < 500
- Transitional flow: $500 < Re < 12{,}500$
- Turbulent flow: Re > 12,500

In Civil Engineering practice

# Geometric elements for different channel cross sections

| | *rectangular* | *trapezoidal* | *triangular* | *circular* | *parabolic* |
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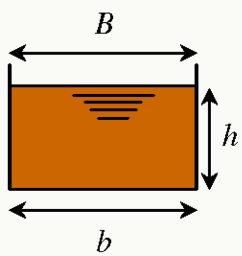
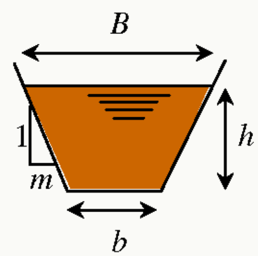
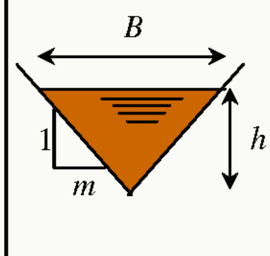
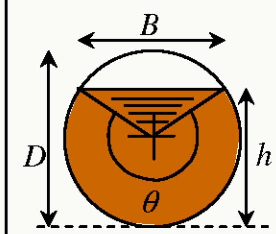
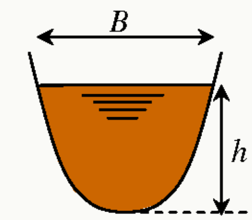
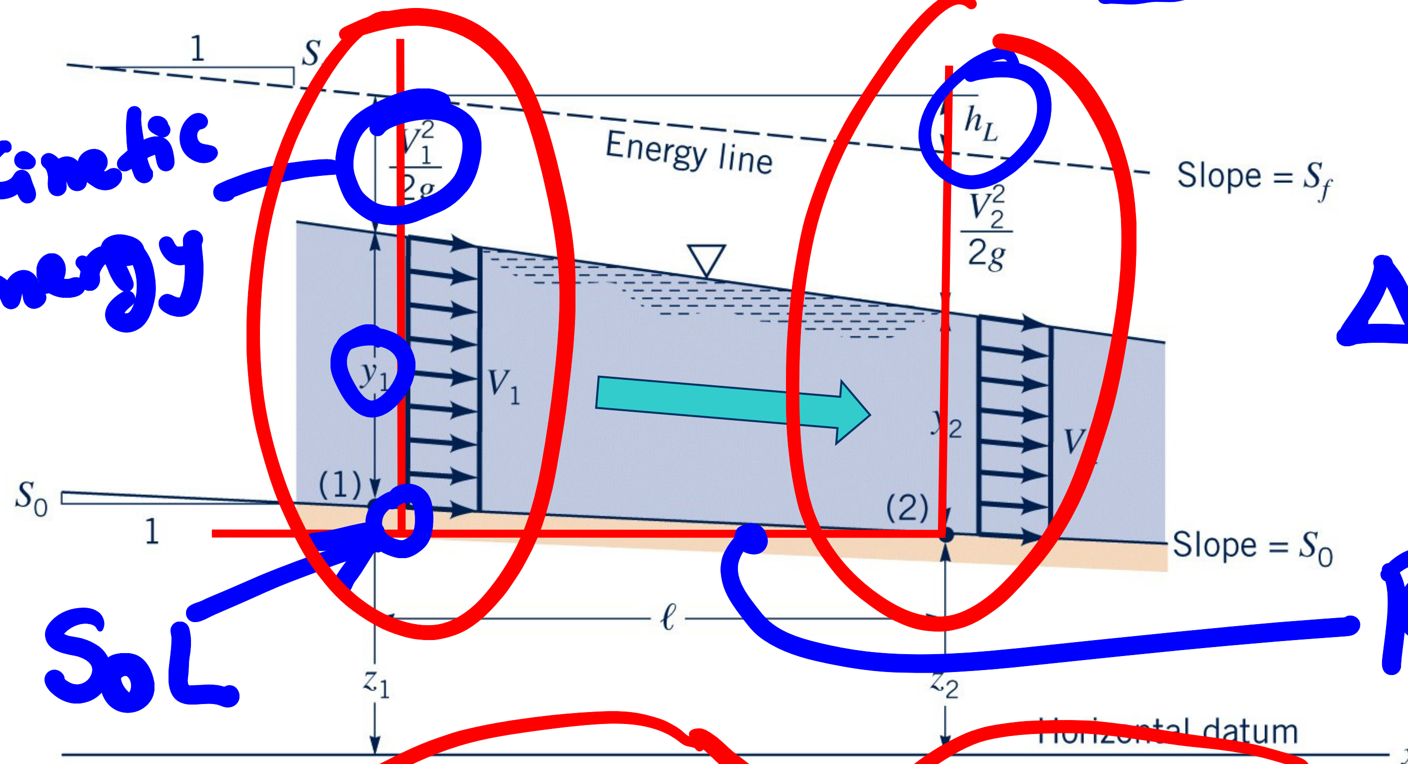
| |  |  |  |  |  |
| *flow area* $A$ | $bh$ | $(b+mh)h$ | $mh^2$ | $\frac{1}{8}(\theta-\sin\theta)D^2$ | $\frac{2}{3}Bh$ |
| *wetted perimeter* $P$ | $b+2h$ | $b+2h\sqrt{1+m^2}$ | $2h\sqrt{1+m^2}$ | $\frac{1}{2}\theta D$ | $B+\frac{8}{3}\frac{h^2}{B}$ * |
| *hydraulic radius* $R_h$ | $\frac{bh}{b+2h}$ | $\frac{(b+mh)h}{b+2h\sqrt{1+m^2}}$ | $\frac{mh}{2\sqrt{1+m^2}}$ | $\frac{1}{4}\left[1-\frac{\sin\theta}{\theta}\right]D$ | $\frac{2B^2h}{3B^2+8h^2}$ * |
| *top width* $B$ | $b$ | $b+2mh$ | $2mh$ | $(\sin\theta/2)D$ *or* $2\sqrt{h(D-h)}$ | $\frac{3}{2}\frac{A}{h}$ |
| *hydraulic depth* $D_h$ | $h$ | $\frac{(b+mh)h}{b+2mh}$ | $\frac{1}{2}h$ | $\left[\frac{\theta-\sin\theta}{\sin\theta/2}\right]\frac{D}{8}$ | $\frac{2}{3}h$ |

* *Valid for* $0<\xi\le 1$ *where* $\xi = 4h/B$

*If* $\xi>1$ *then* $P=(B/2)\left[\sqrt{1+\xi^2}+(1/\xi)\ln\left(\xi+\sqrt{1+\xi^2}\right)\right]$

# Specific Energy

$$E = y + \frac{V^2}{2g}$$




Kinetic energy

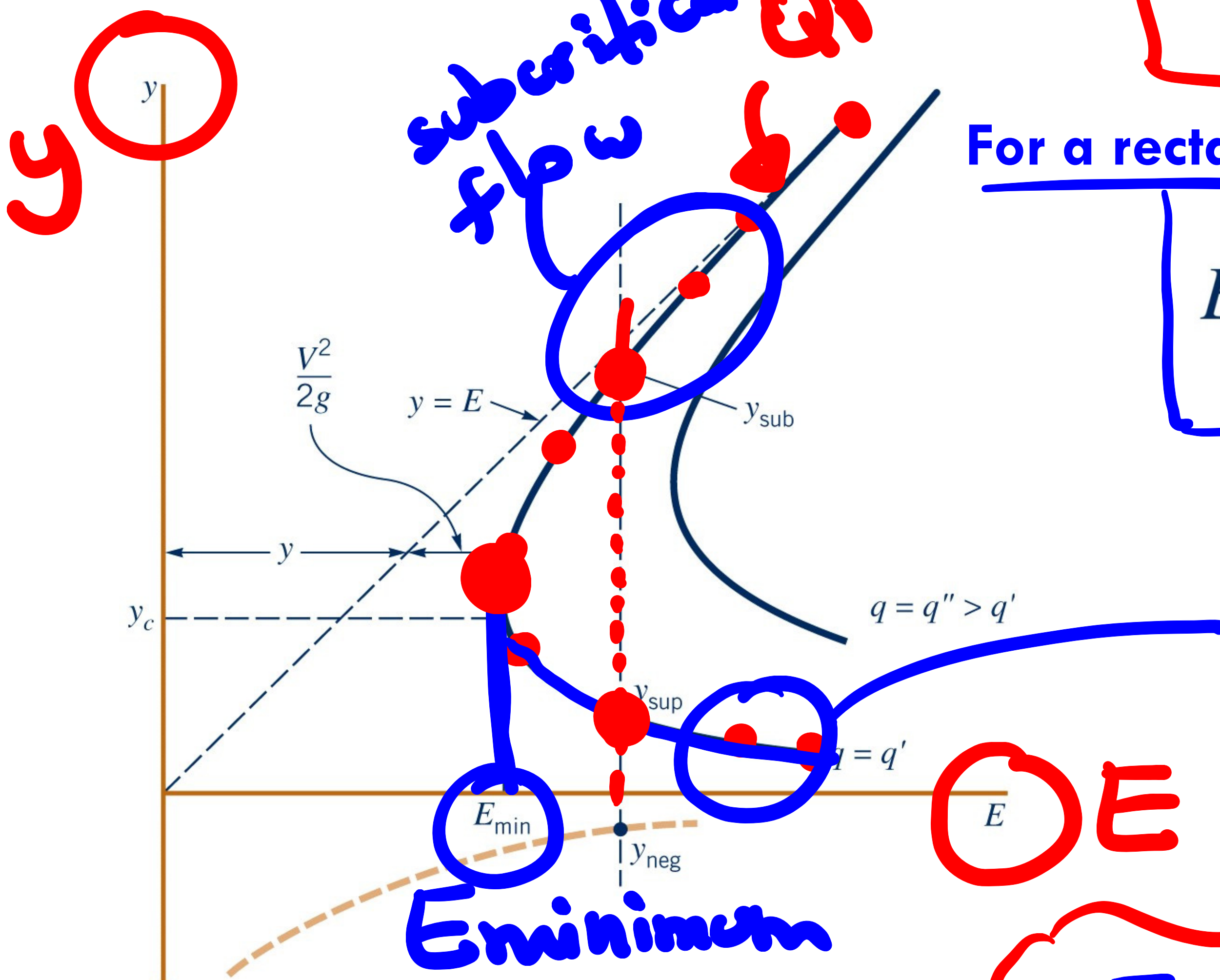
$\Delta z = S_o L$

$S_o L$

Reference

$$S_o L + \underbrace{y_1 + \frac{V_1^2}{2g}}_{E_1} = \underbrace{y_2 + \frac{V_2^2}{2g}}_{E_2} + h_L$$

# Specific Energy




Specific energy diagram

$$\boxed{E = y + \frac{V^2}{2g}}$$

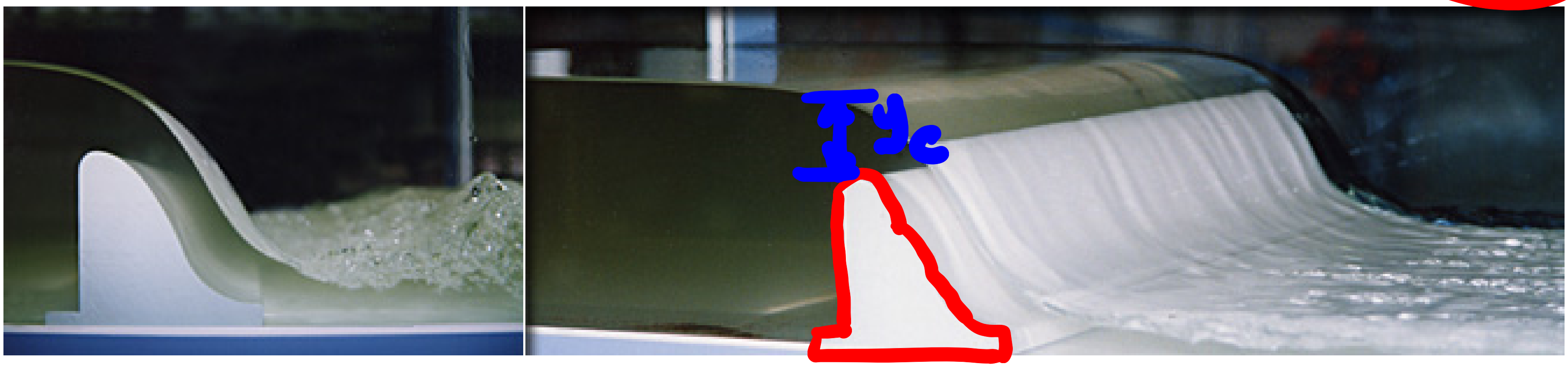
For a rectangular channel, $q = Q/b$

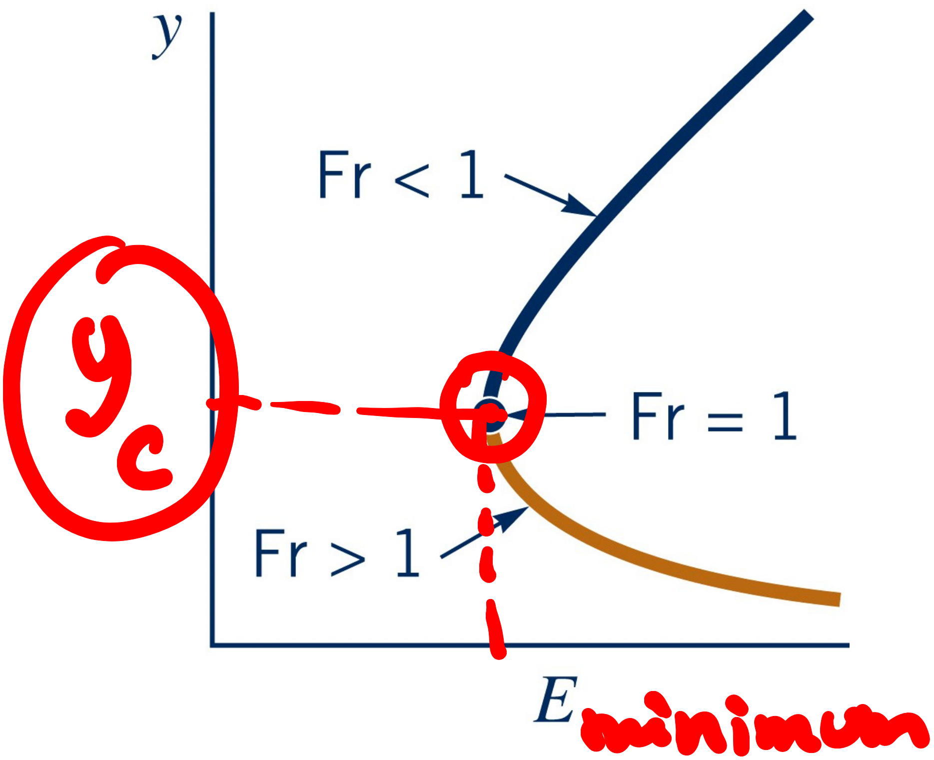
$$\boxed{E = y + \frac{q^2}{2gy^2}}$$

$$E = y + \frac{q^2}{2gy^2}$$

# Critical depth

2 solution +
1 solution −




$$Q = A \cdot V$$

$$Q = (b y_c)(V_c)$$

also is Known

$$dx^n = nx^{n-1}$$

# Critical depth (Cont.)

$$E = y + \frac{V^2}{2g} = y + \frac{Q^2}{2gA^2}$$

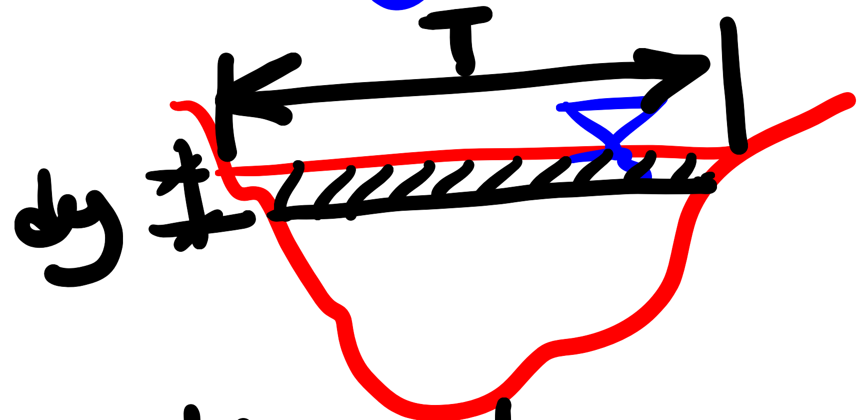
To determine $E_{min}$: $\dfrac{dE}{dy} = 0$

$$0 = 1 + \frac{Q^2}{2g}\frac{d}{dy}\left(A^{-2}\right)$$

$$0 = 1 + \frac{Q^2}{\cancel{2}g}(-\cancel{2})A^{-3}\frac{dA}{dy}$$

$$\frac{Q^2}{gA^3}\underbrace{\frac{dA}{dy}}_{T} = 1$$

$$\boxed{\frac{Q^2 T}{gA^3} = 1}$$




$$dA = T\,dy$$

$$dA/dy = T$$

For a rectangular channel ($q = Q/b$)

$$\boxed{y_c = \left(\frac{q^2}{g}\right)^{1/3}}$$

$$q = \frac{Q}{b}$$

# Flow Choking

$$\boxed{\Delta z_c = E_1 - E_c}$$

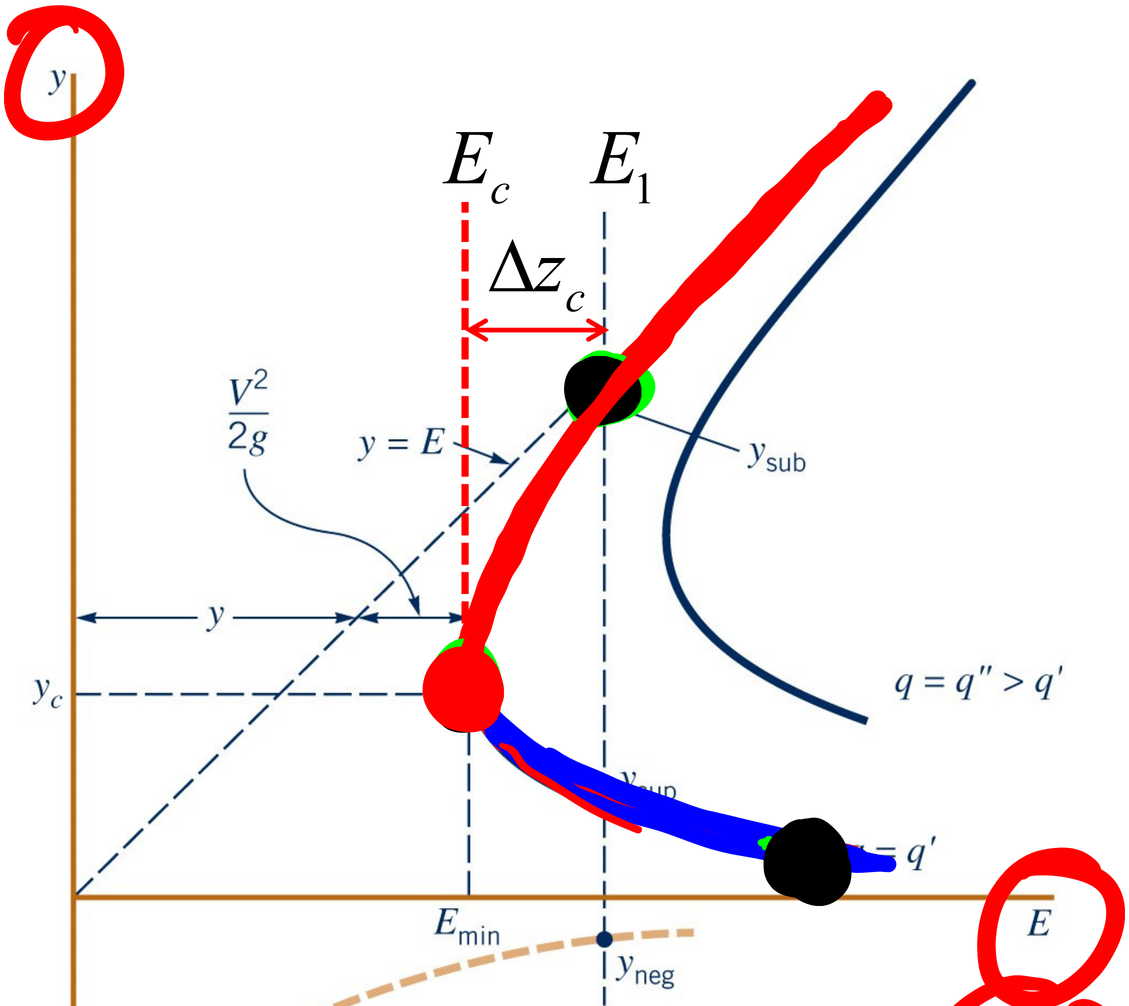
**For a rectangular channel:**

$$\Delta z_c = E_1 - \frac{3}{2}y_c$$

**For a non-rectangular channel:**

$$\Delta z_c = E_1 - \left(y_c + \frac{D_c}{2}\right)$$

$$D_c = \frac{A_c}{T_c}$$




$$E_c = y_c + \frac{V_c^2}{2g}$$

$$E_c = y_c + \frac{y_c}{2} = \frac{3}{2}y_c$$

# Example

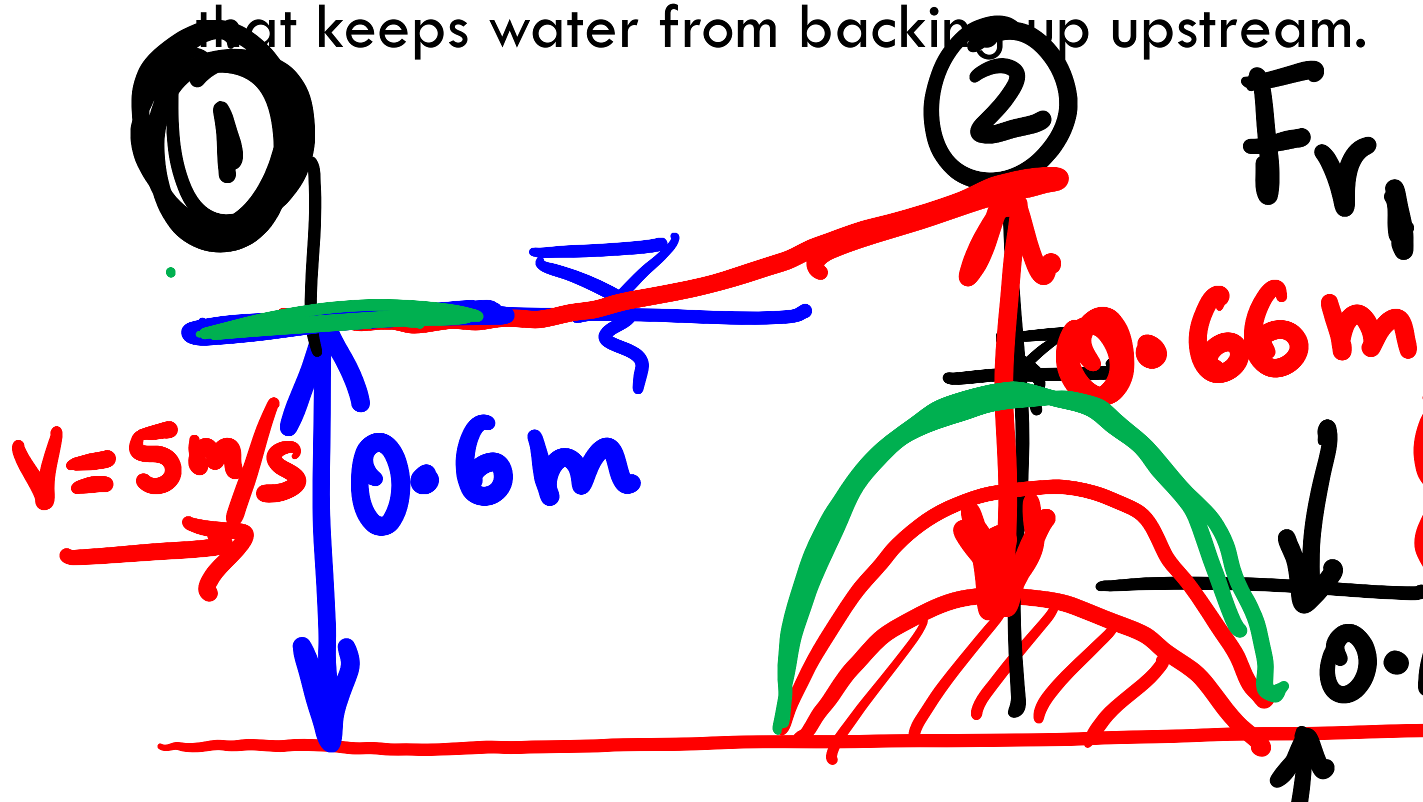
Consider a channel where the upstream velocity is 5.0 m/s and the upstream flow depth is 0.6 m. The flow then passes over a bump 15 cm in height.

(a) Compute the flow depth and velocity on the crest of the bump.

(b) Compute the maximum allowable bump height that keeps water from backing up upstream.

$$Fr = 1 = \frac{V_c}{\sqrt{g y_c}}$$

$$1 = \frac{V_c^2}{g y_c}$$




$$Fr_1 = \frac{V_1}{\sqrt{g y_1}} = \frac{5}{\sqrt{9.8 \times 0.6}}$$

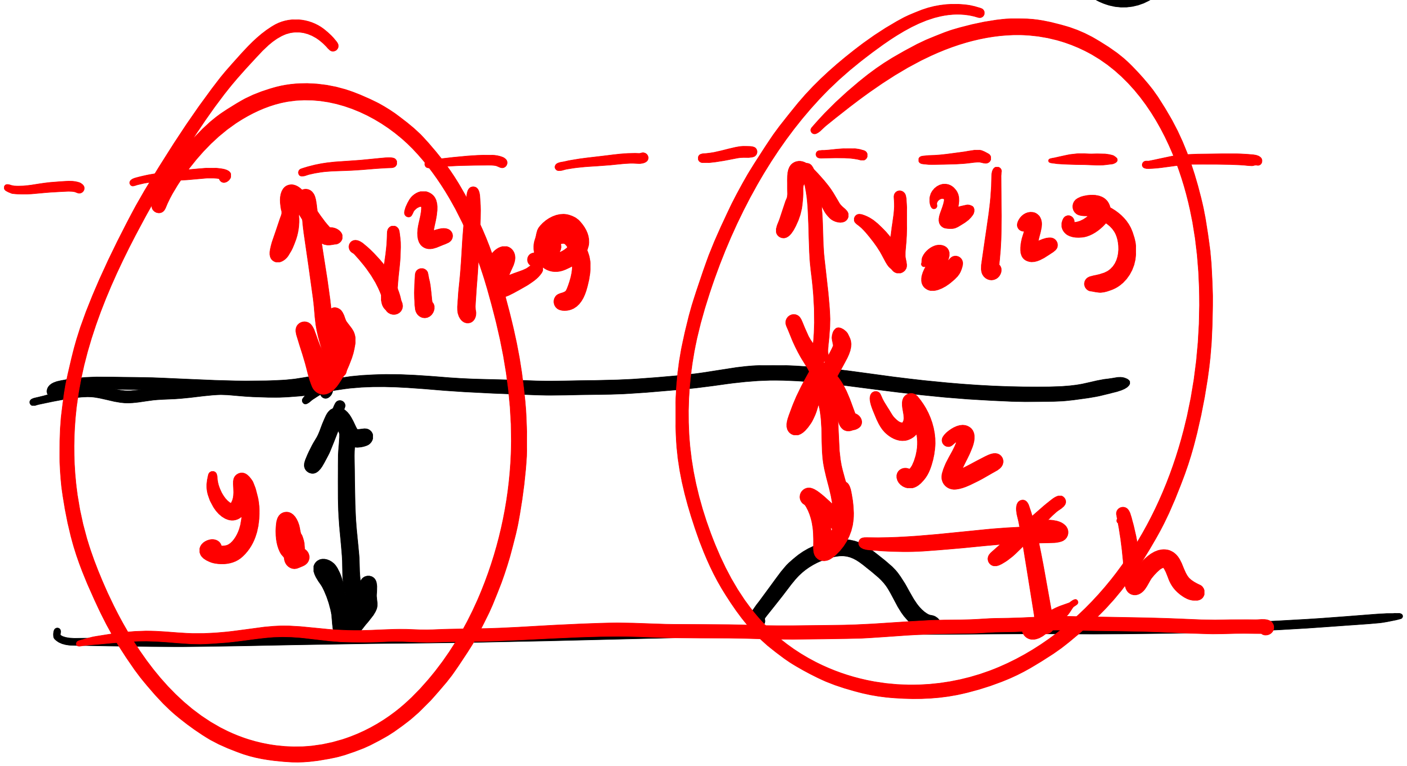
$$\boxed{Fr_1 = 2.06}$$

Supercrit

a) $y_2 = ??$ , $V_2 = ??$

Neglect head losses 1 - 2

$$y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g} + 0.15\,m$$



$$0.6 + \frac{5^2}{2 \times 9.8} = y_2 + \frac{V_2^2}{2g} + 0.15 \quad \ldots \text{①}$$

$$Q_1 = Q_2$$

$$A_1 V_1 = A_2 V_2$$

$$\cancel{b} y_1 V_1 = \cancel{b} y_2 V_2$$

$$0.6 \times 5 = y_2 V_2$$

$$\boxed{V_2 = 3 / y_2}$$

In 1

$$0.6 + \frac{5^2}{2 \times 9.8} = y_2 + \frac{9}{2 \times 9.8\, y_2^2} + 0.15$$

super 1 $y_2 = 0.66\,m$ — supercritical

sub 2 $y_2 = 1.52\,m$

3 $y_2 = -$

$$E_1 = y_1 + \frac{V_1^2}{2g}$$

$$y_c = \sqrt[3]{q^2/g}$$

b) $h_{max} = E_1 - \frac{3}{2} y_c$

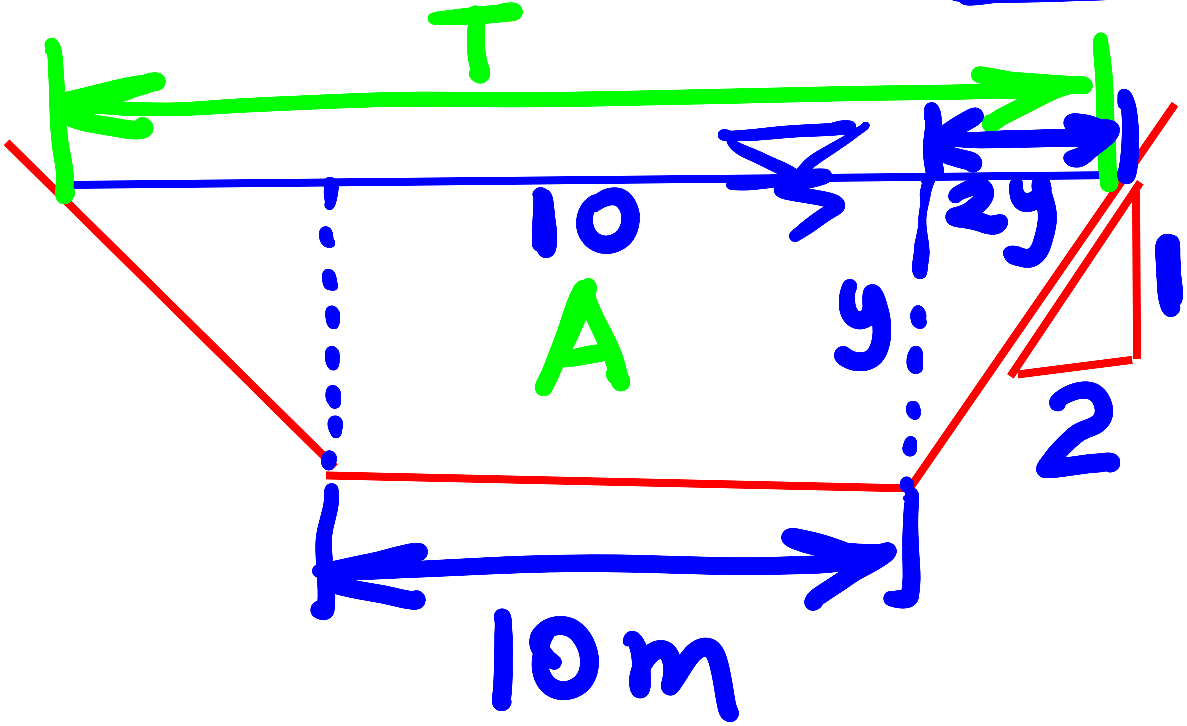
$= 0.41\,m$

# Example

Compute the critical depth in a trapezoidal channel for a flow of 30 m$^3$/s. The channel bottom width is 10 m, side slopes are 2H:1V.

$Q = 30\ m^3/s$

$$\boxed{\frac{Q^2 T}{g A^3} = 1} \quad \dots (1)$$




$$A = \frac{(10 + 4y + 10)}{2} \times y$$

$$A = 10y + 2y^2$$

$$T = 10 + 4y$$

In 1

$$\frac{30^2(10+4y)}{9.8\times(10y+2y^2)} = 1$$

$$\boxed{y_c = 0.91\,m}$$

# Example

Water flows at a depth of 2.15 m and a unit discharge of 5.5 m$^2$/s in a rectangular channel. Energy losses can be neglected.

(a) What is the maximum height *h* of a raised bottom that will permit the flow to pass over it without increasing the upstream depth?

(b) Sketch the water surface and energy grade line.

(c) If the channel bottom is raised greater than *h*, discuss a type of change that may take upstream of the transition.

$q = 5.5\ m^2/s$

$y_1 = 2.15\ m$

$q = \frac{Q}{b} = \frac{\cancel{b} y \cdot V}{\cancel{b}}$

$V = q/y$

$V_1 = 2.56\ m/s$

a) $h_{max} = E_1 - E_c = E_1 - \frac{3}{2} y_c$

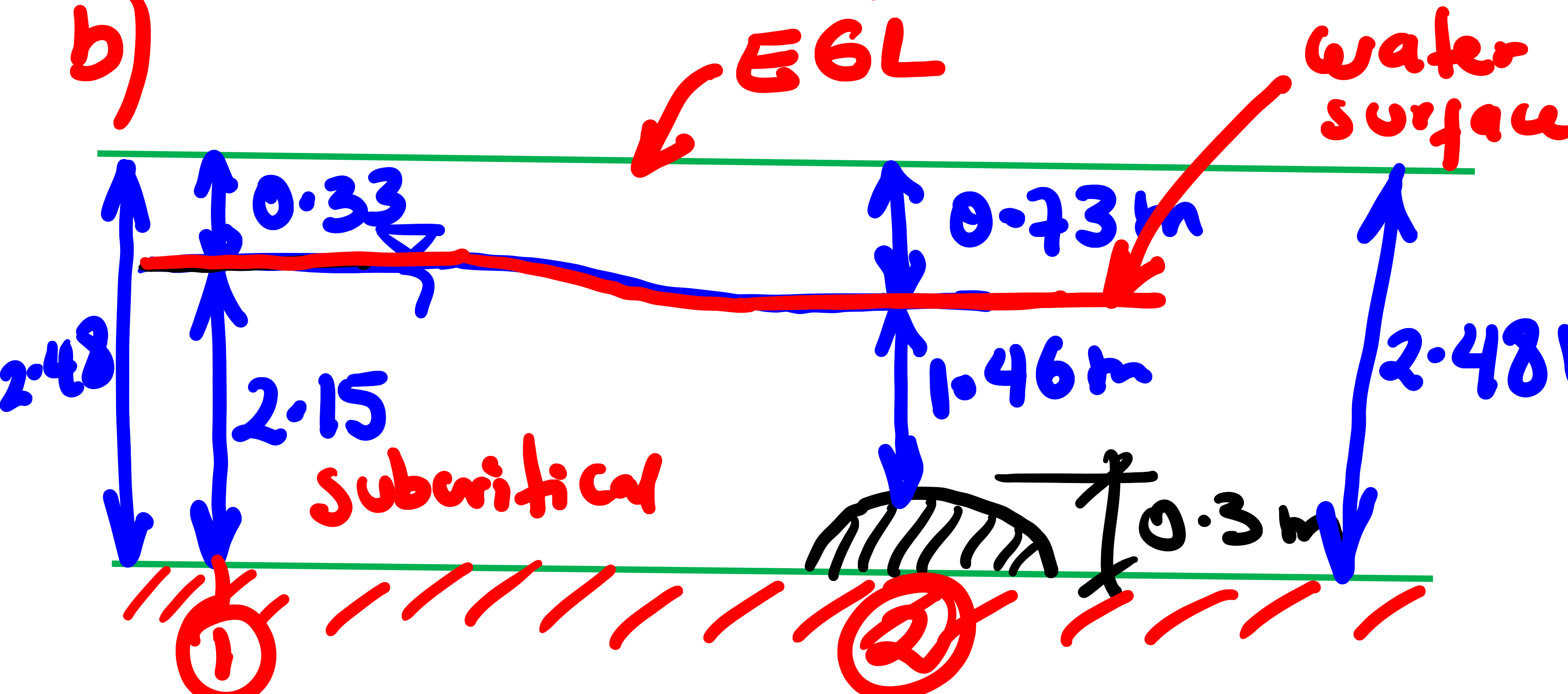
$y_c = \sqrt[3]{\frac{q^2}{g}} = \sqrt[3]{\frac{5.5^2}{9.8}} = 1.46\ m$

$$E_1 = y_1 + \frac{V_1^2}{2g} = 2.15 + \frac{2.56^2}{2\times 9.8} = 2.48\,m$$

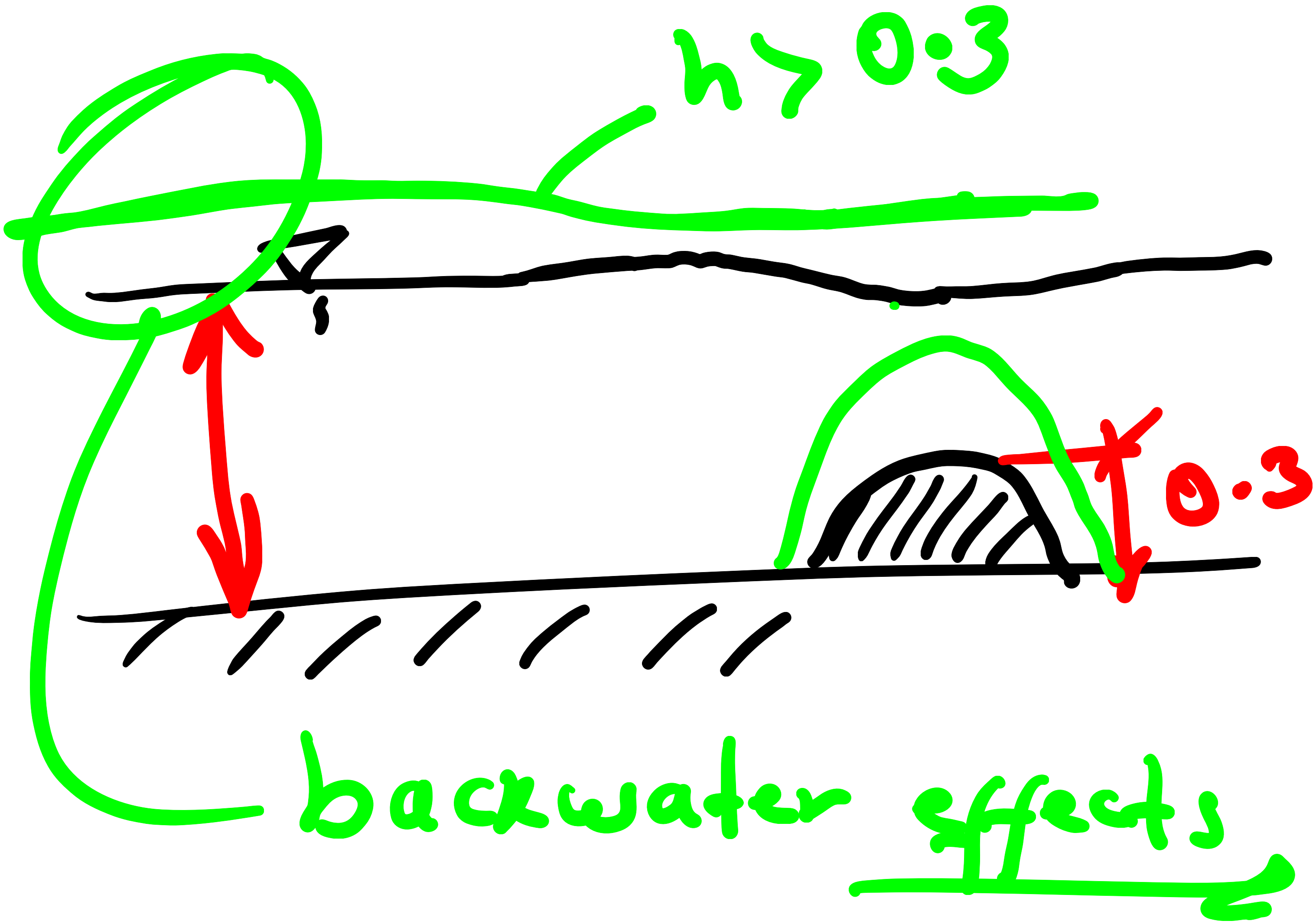
$$h_{max} = 2.48 - \frac{3}{2}\times 1.46$$

$$\boxed{h_{max} = 0.30\,m}$$

b)

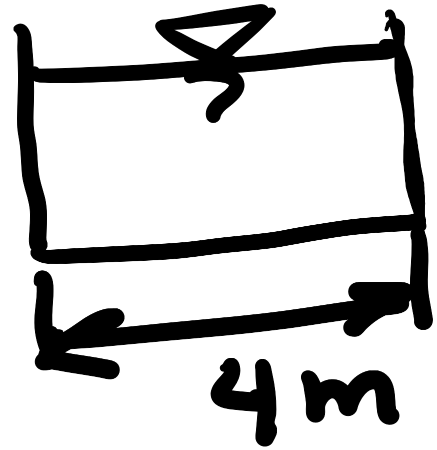


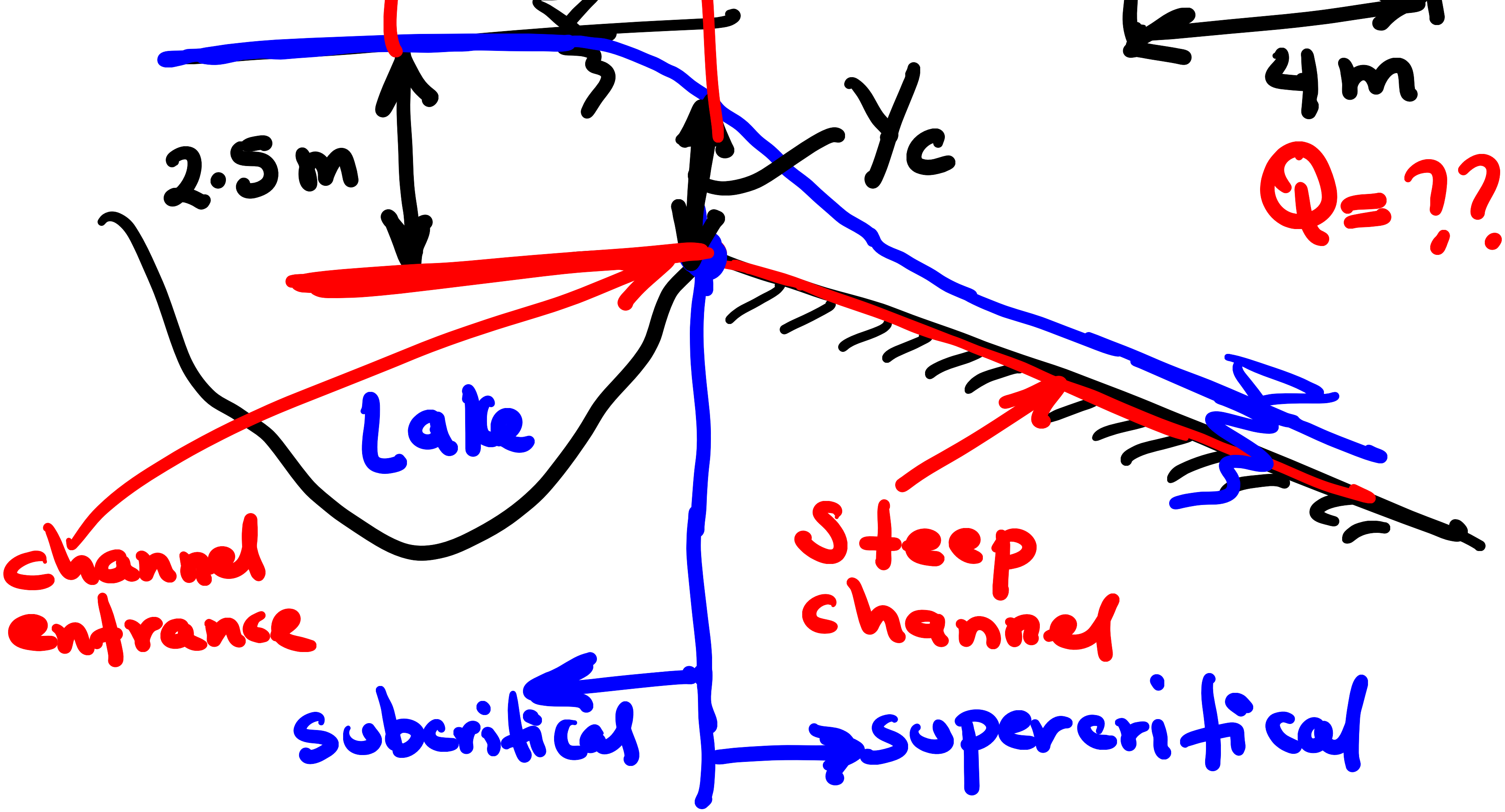
(c) Flow upstream will still be supercritical but there will be backwater effects.



# Example

A lake discharges into a steep channel. At the channel entrance the lake level is 2.5 m above the channel bottom. Neglecting losses, find the discharge if the channel section is rectangular with a width $b = 4$ m.



* Energy ≈ 0

$$y_1 + \cancel{\frac{V_1^2}{2g}} + 0 = y_c + \frac{V_c^2}{2g} + 0$$

$$2.5 + 0 = y_c + \frac{1}{2} y_c$$

$$\frac{3}{2} y_c = 2.5$$

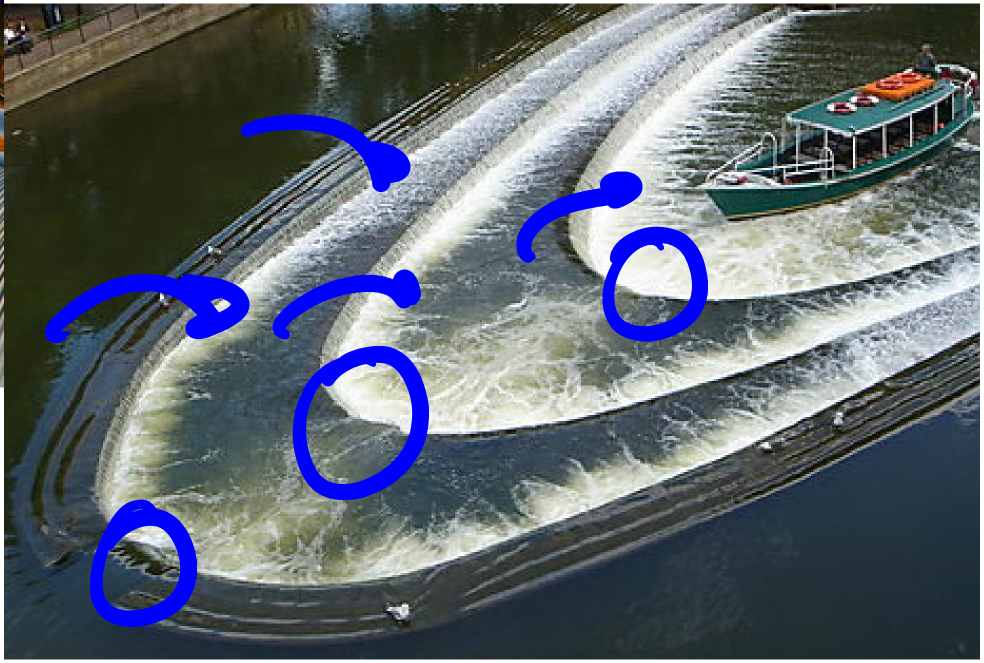
$$\boxed{y_c = 1.67\,m}$$

* $y_c = \sqrt[3]{\frac{q^2}{g}}$

$$(1.67)^3 = \left(\frac{Q}{4}\right)^2 \frac{1}{9.8}$$

$$\boxed{Q = 26.9 \ m^3/s}$$

# Weirs



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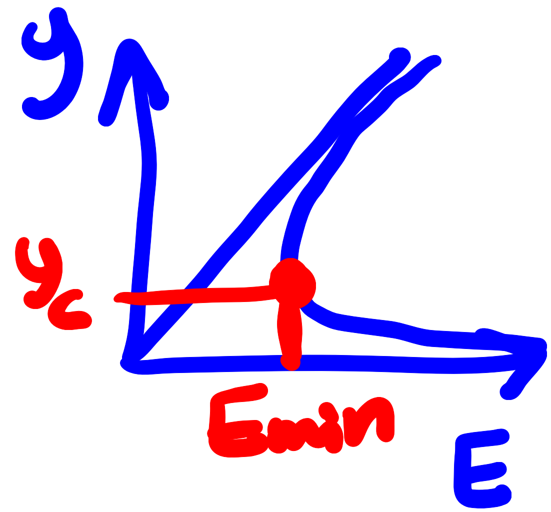
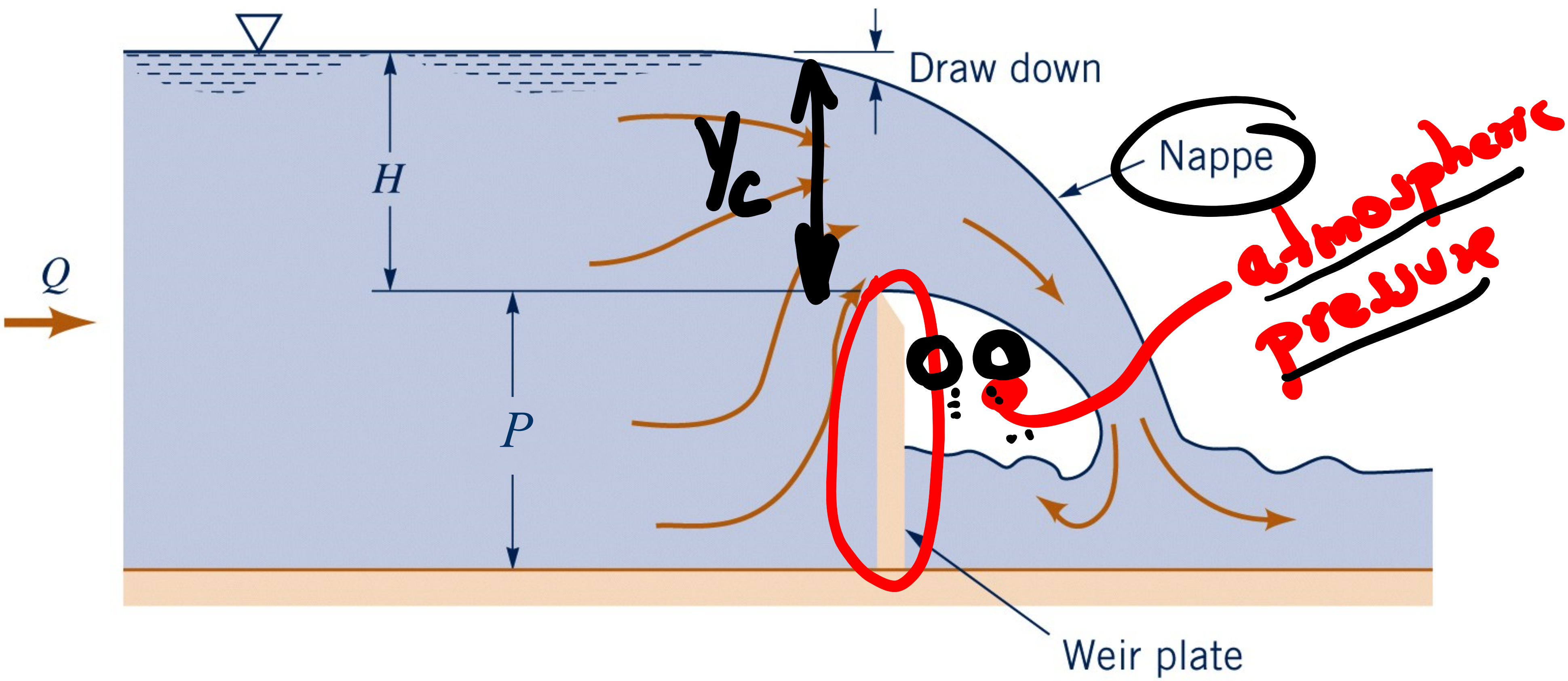
# Weirs

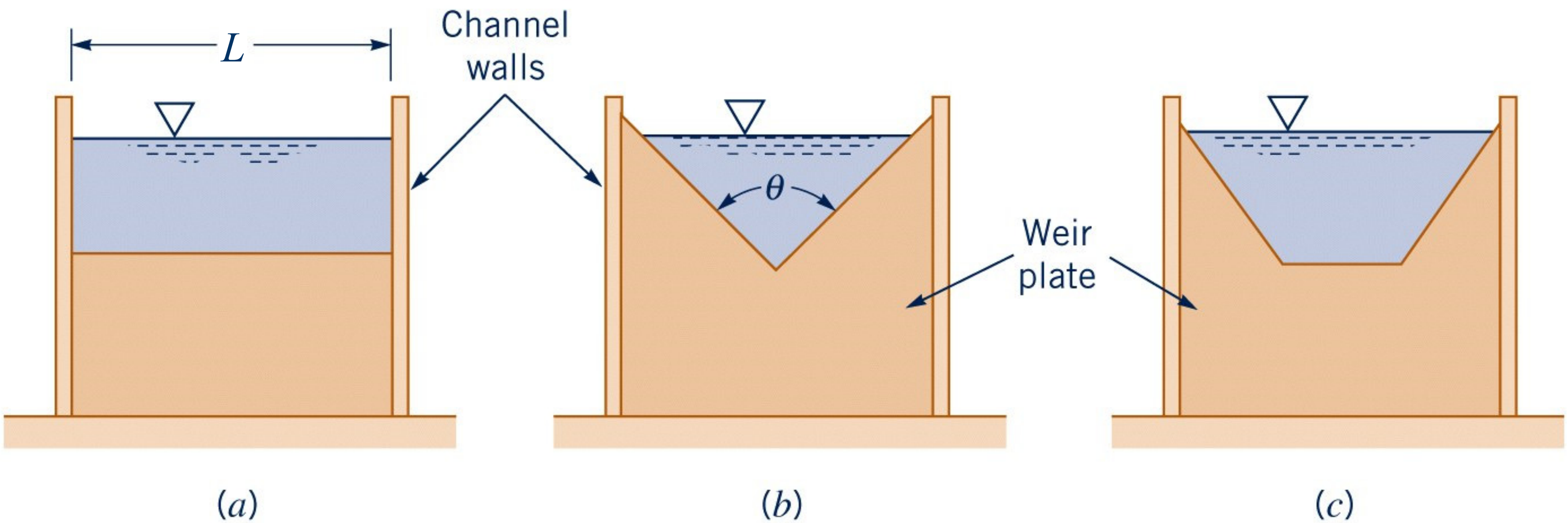
- Weirs are often used to measure discharges in open channel flows.
- By creating an obstruction, **critical depth is forced to occur** and, therefore, the **unique relationship between depth and discharge** can be used to measure flow.
- Three types are presented herein:
  1. Sharp-crested Rectangular Notch Weir
  2. Sharp-crested triangular Notch Weir
  3. Broad-crested Weir



# Sharp-Crested Weir

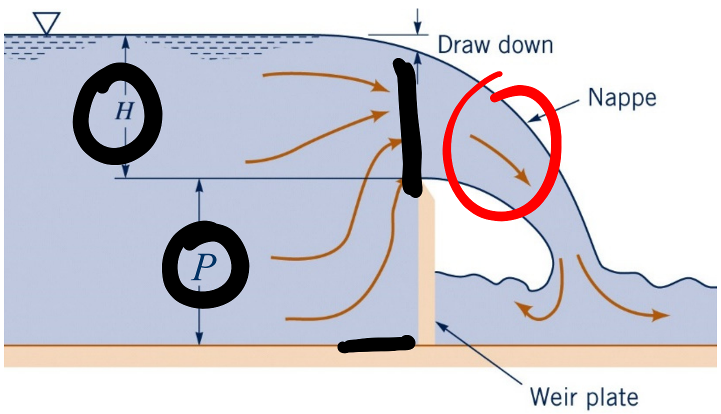


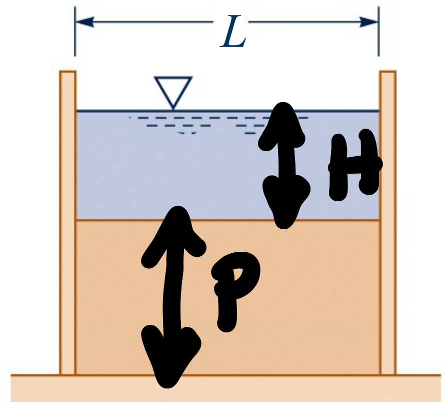
# Sharp-Crested Weir - Geometry




**Rectangular** **Triangular** **Trapezoidal**

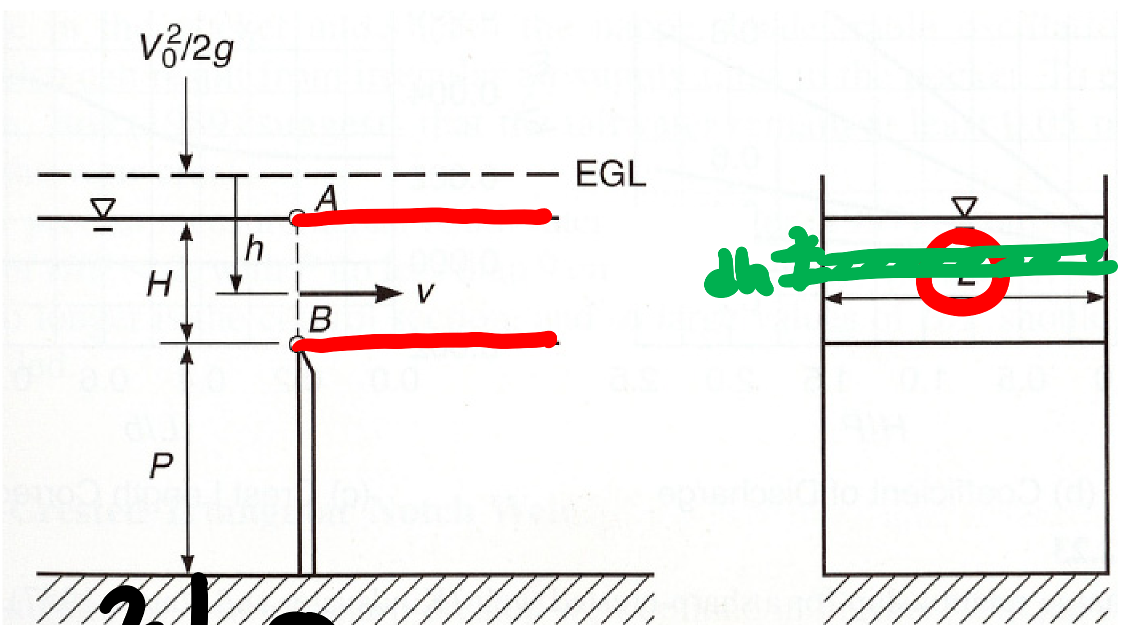
# Ideal discharge in a sharp-crested rectangular weir



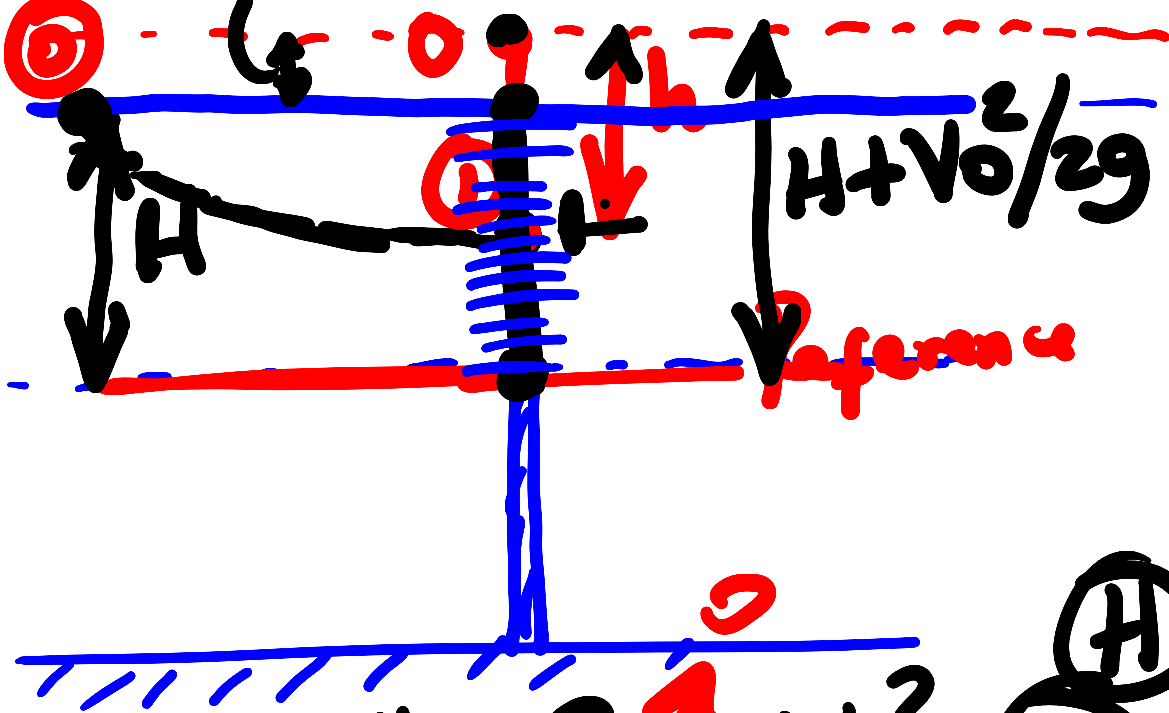




**Assumptions:**

- No head losses
- Atmospheric pressure across section AB
- No vertical contraction of the nappe.




$V_0^2/2g$




$$Q = V \cdot A$$

$$\int dQ = \int V \cdot dA$$

$$dA = L\,dh \quad \therefore \; 1$$

Bernoulli $\frac{\cancel{P_0}}{\gamma} + \frac{V_0^2}{2g} + \overset{H}{Z_0} = \frac{\cancel{P_1}}{\gamma} + \frac{V_1^2}{2g} + z_1$

$$\cancel{H} + \cancel{\frac{V_0^2}{2g}} = \frac{V_1^2}{2g} + \cancel{H} + \cancel{\frac{V_0^2}{2g}} - h$$

$$V_1^2 = 2gh \rightarrow \boxed{V_1 = \sqrt{2gh}}$$

In 1

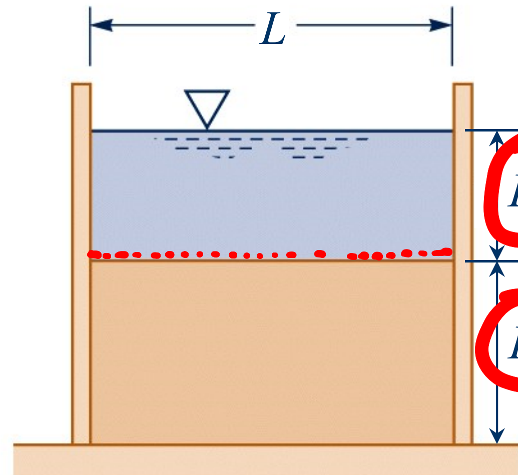
$$Q = \int_{V_0^2/2g}^{H+V_0^2/2g} \sqrt{2gh}\, L\, dh = \sqrt{2g}\, L \int_{V_0^2/2g}^{H+V_0^2/2g} h^{1/2}\, dh$$

$$Q = \sqrt{2g}\, L\, \frac{2}{3} h^{3/2} \Big|_{V_0^2/2g}^{H+V_0^2/2g}$$

$$\boxed{Q_{ideal} = \frac{2}{3}\sqrt{2g}\, L \left[ \left(H+\frac{V_0^2}{2g}\right)^{3/2} - \left(\frac{V_0^2}{2g}\right)^{3/2} \right]}$$

# Rectangular Weir

$$Q = \frac{2}{3}\sqrt{2g}\, C_d L H^{3/2}$$




Where $C_d$ is the rectangular weir coefficient given by the well-known Rehbock formula

$$C_d = 0.611 + 0.08\frac{H}{P}$$

The variation of $C_d$ for rectangular sharp-crested weirs is valid for $H/P \leq 5.0$.

# Triangular Weir

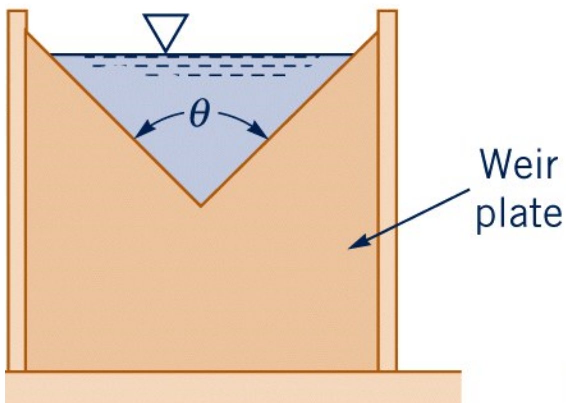
$$Q = C_{de}\frac{8}{15}\tan\left(\frac{\theta}{2}\right)\sqrt{2g}\,(H + k_h)^{5/2}$$

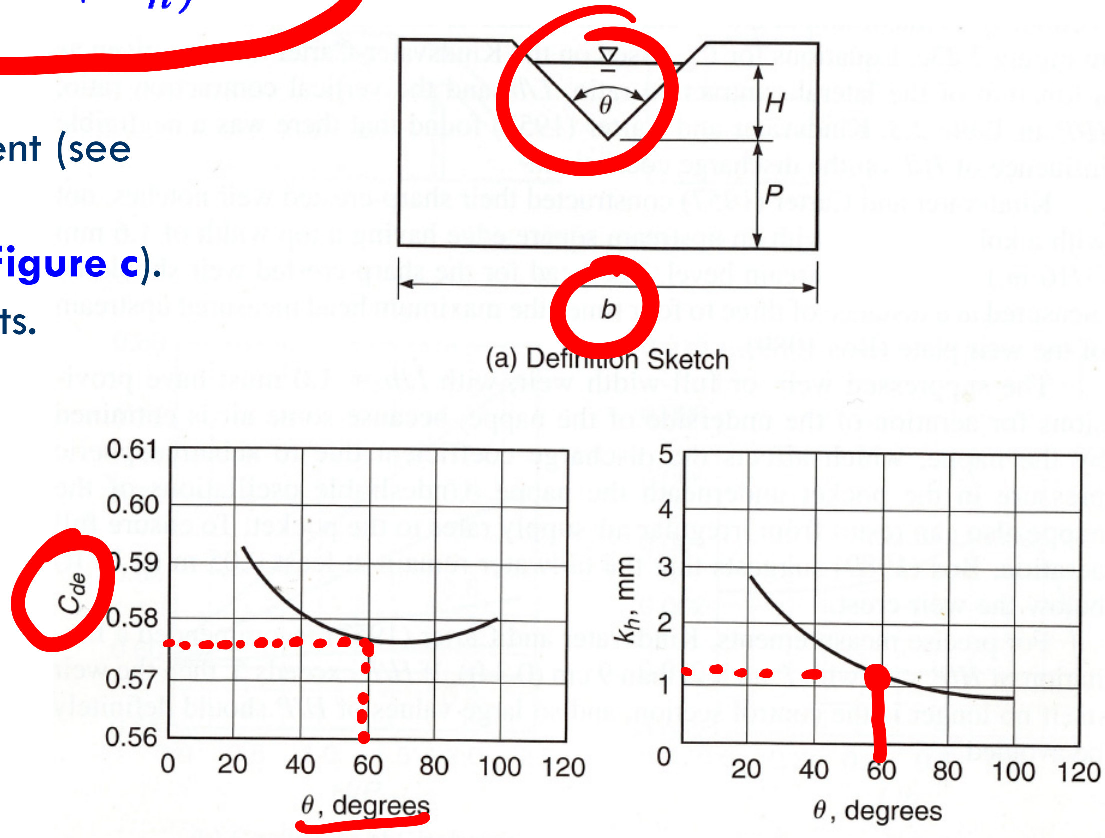
Where:

$C_{de}$ = Effective discharge coefficient (see **Figure b**)

$K_h$ = Head correction in mm (see **Figure c**).

Need to convert mm to proper units.




(a) Definition Sketch

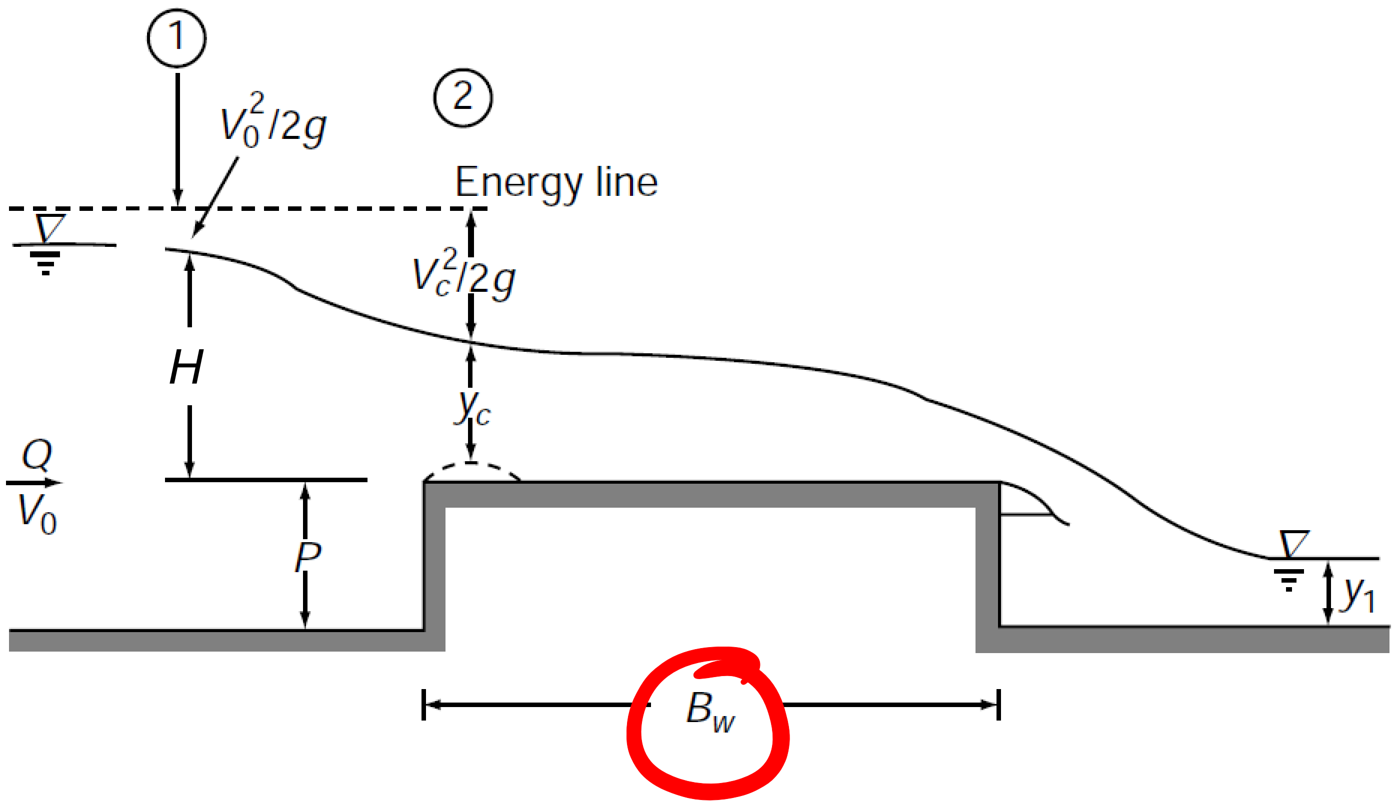
(b) Coefficient of Discharge

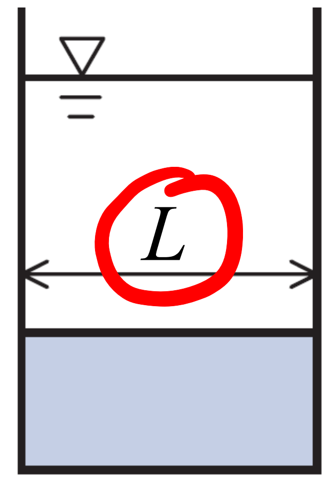
(c) Head Correction

**Effective discharge coefficient** and **Head correction** for triangular weirs. $C_{de}$ values above are valid when $H/P \leq 0.4$ and $P/b \leq 0.2$.

# Broad-Crested Weir in rectangular channels

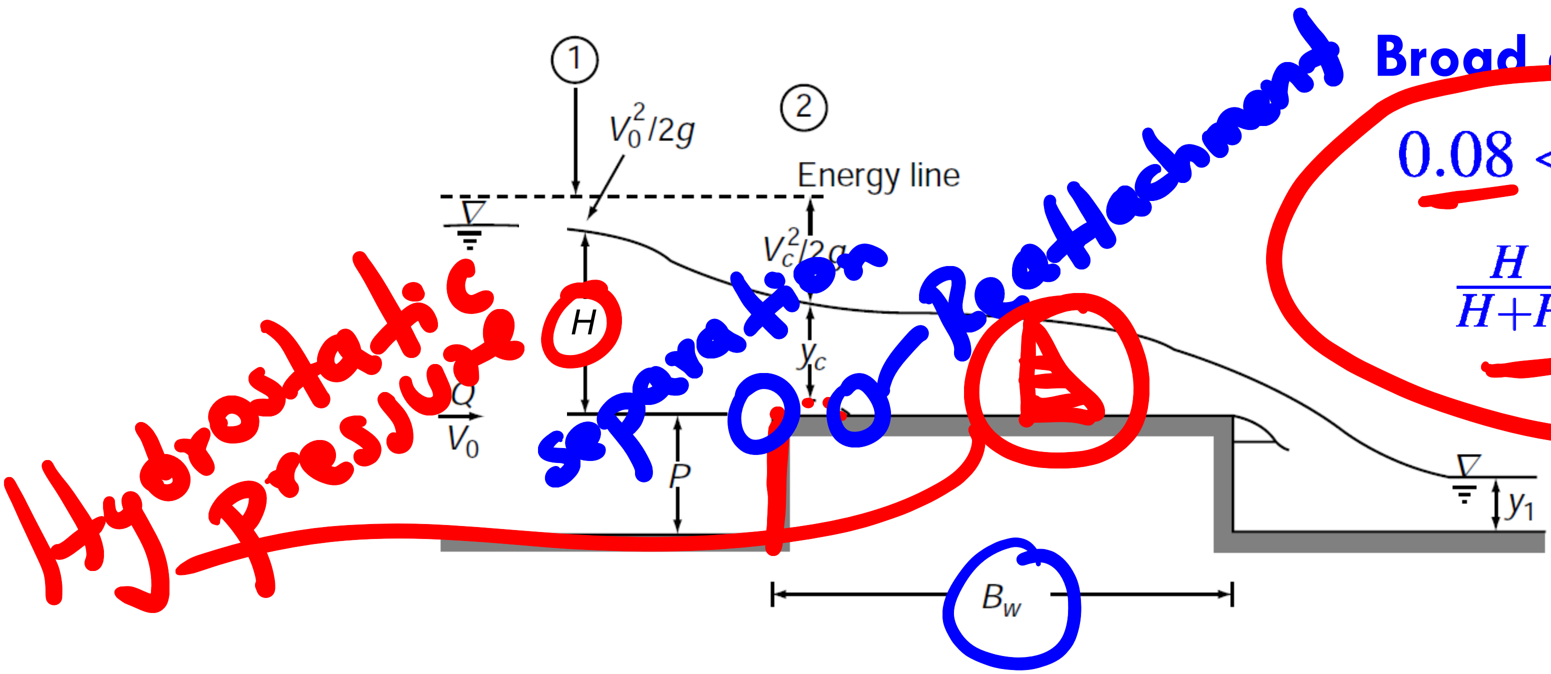
- Weirs with a **finite crest width** in the direction of flow
- Wide variety of crest and cross-sectional shapes of the weir are used in practice.
- Here we present a horizontal broad-crested weir in a **rectangular channel**.



# Broad-Crested Weir

- This weir has a sharp upstream corner which causes the flow to separate and then reattach enclosing a **separation bubble.**
- If the width $B_w$ of the weir is sufficiently long, the curvature of the stream lines will be small and the hydrostatic pressure distribution will prevail over most of its width.
- The weir will act like an inlet with subcritical flow upstream of the weir and supercritical flow over it.
- A critical-depth control section will occur at the upstream end-probably at a location where the bubble thickness is maximum.




**Broad crested weir:**

$$0.08 < \frac{H}{B_w} < 0.33$$

$$\frac{H}{H+P} \leq 0.35$$

# Broad-Crested Weir

$$Q = C_v C_d \frac{2}{3}\left(\frac{2}{3}g\right)^{1/2} L H^{3/2}$$

*Where* $L$ = length of the weir measured in a transverse direction to the flow and $B_w$ = width of the weir measured in the longitudinal direction.

$C_v$ can be related to the variable $C_d A^*/A_1$

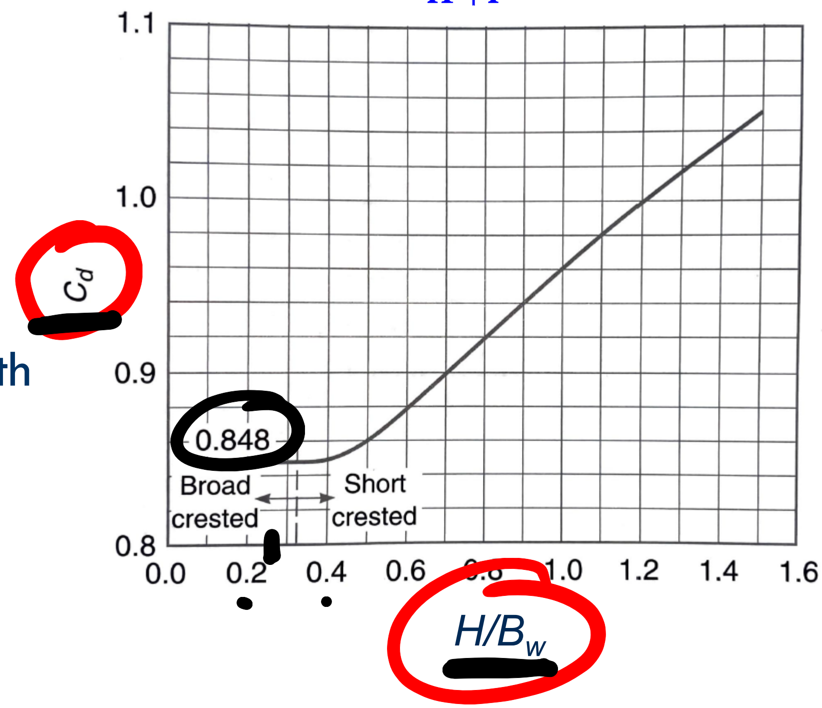
$A^* = LH$, $A_1 = L(H + P)$

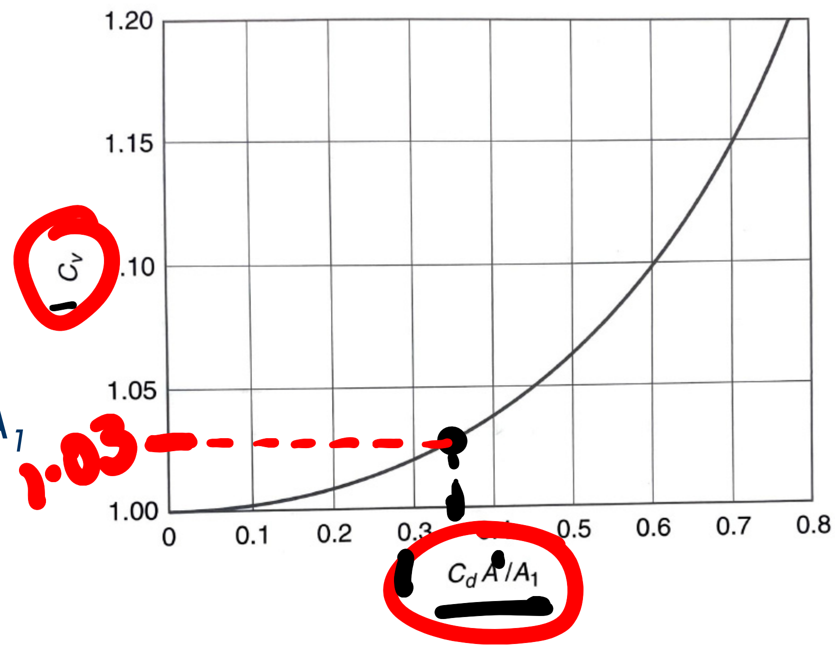
$$C_d \frac{A^*}{A_1} = C_d \frac{H}{H+P}$$

## Steps to find Q:

- Find $C_d$ using provided chart with $H/B_w$
- With $C_d$ calculate $C_d A^*/A_1$
- Find $C_v$ using provided chart with $C_d A^*/A_1$
- Plug in values to compute flow discharge

**$C_d$ for $\frac{H}{H+P} \leq 0.35$**

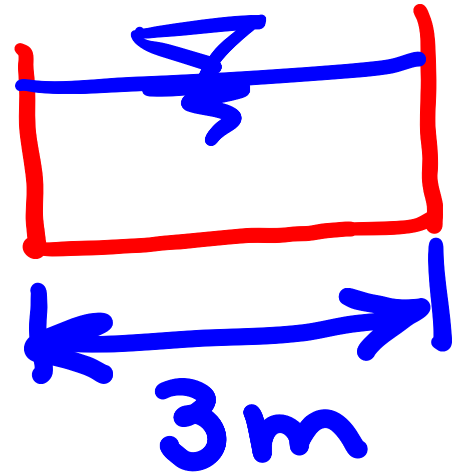


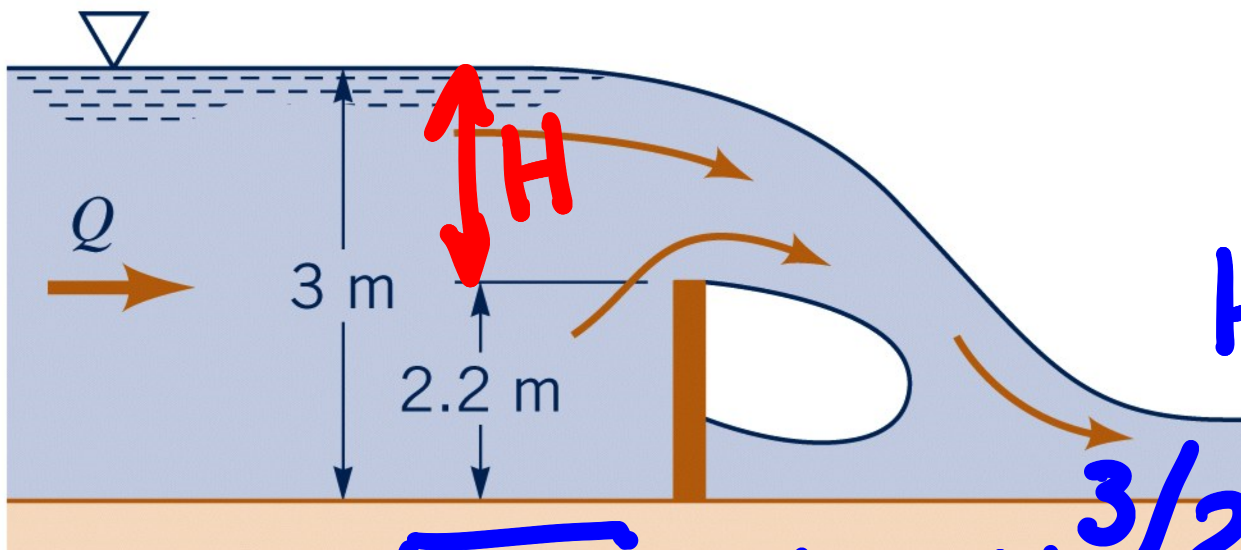



# Example

Water flows over the rectangular sharp-crested weir in a 3m-wide channel as shown in the figure below. Find the flow discharge.

$Q = ?$




$H = 3 - 2.2 = 0.8\,m$

$P = 2.2\,m$

$L = 3\,m$

$$Q = \frac{2}{3}\sqrt{2g}\,C_d\,L\,H^{3/2}$$

$$C_d = 0.611 + 0.08\,\frac{H}{P}$$

$$C_d = 0.611 + 0.08 \times \frac{0.8}{2.2} = 0.64$$

$$Q = \frac{2}{3}\sqrt{2 \times 9.8} \times 0.64 \times 3 \times 0.8^{3/2}$$

$$\boxed{Q = 4.055 \ m^3/s}$$

# Example

A broad-crested weir has a crest length of $B_w = 0.75$ m, crest width of $L = 1.0$ m, and crest height of $P = 0.30$ m. The water surface at the approach section is 0.20 m above the crest – that is, $H = 0.20$ m. Determine the discharge.

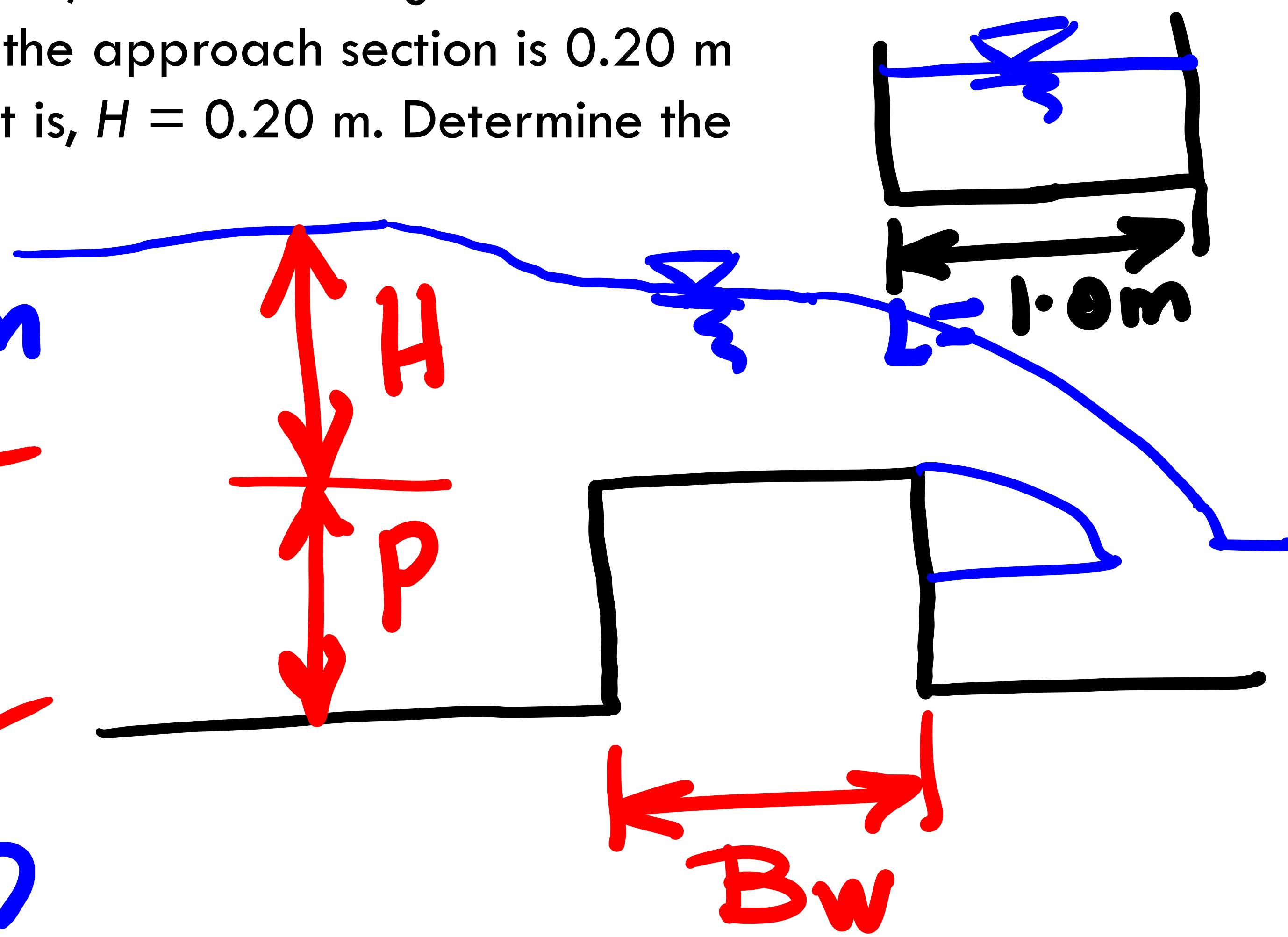
$B_w = 0.75\,m$

$L = 1.0\,m$

$P = 0.3\,m$

$H = 0.2\,m$

$Q = ??$



$$Q = C_v\, C_d\, \frac{2}{3} \sqrt{\frac{2}{3} g}\; L\, H^{3/2}$$

$$* \quad \frac{H}{B_w} = \frac{0.2}{0.75} = 0.27 \xrightarrow[\text{chart}]{} C_d = 0.848$$

$$* \quad C_d \frac{A^*}{A_1} = C_d \frac{H}{H+P} = 0.848 * \frac{0.2}{0.5} = 0.34$$

$$\xrightarrow{\text{chart}} C_v = 1.03$$

$$Q = 1.03 * 0.848 \left(\frac{2}{3}\right) \sqrt{\frac{2}{3} \times 9.8} \times 1.0 \times 0.2^{1.5}$$

$$\boxed{Q = 0.133 \ m^3/s}$$

$$\boxed{Q = 133 \ \text{Liter/s}}$$