Momentum Equation

● Open channel flow with complex internal flow patterns can have high energy loss, the nature of which is difficult to estimate.

● In cases of complex internal flow patterns, the energy equation cannot be applied to relate flow parameters. In these cases, the momentum equation with suitable assumptions is recommended.

● For instance, the application of the momentum equation to hydraulic jumps yields meaningful results.
Hydraulic Jump

- Occurs when a **supercritical** flow meets a **subcritical** flow.
- Jump consists of a steep change in the water-surface elevation with a **reverse flow roller** on the major part.
- **Roller entrains** considerable quantity of **air** and the surface has white, frothy and choppy appearance.

See Video: https://www.youtube.com/watch?v=XsYgODmmiAM
Momentum function

For steady flow and horizontal channel

\[ F_{p1} - F_{p2} = \rho Q (V_2 - V_1) \]  
\[ F_{p} = \rho h_c \cdot A \]

\[ Q = A \cdot V \]
\[ V = \frac{Q}{A} \]

\[ h_c \cdot A = \int h dA \]
\[ h_c = \frac{\int h dA}{A} \]

Where: \( h_c \) = centroid of the area \( A \), \( F_p \) = pressure force at the control surface
Momentum function (Cont.)

In (1)

\[ Y \circ h c_1 A_1 - \circ h c_2 A_2 = \circ Q \left( \frac{Q}{A_2} - \frac{Q}{A_1} \right) \]

\[ A_1 h c_1 + \frac{Q^2}{g A_1} = A_2 h c_2 + \frac{Q^2}{g A_2} \]

\[ M = A \cdot h c + \frac{Q^2}{g A} \]

M: Momentum function

\[ M_1 = M_2 \]
Momentum function ($M$) for various channels

\[ M = Ah_c + \frac{Q^2}{gA} \]

\[ M = \frac{by^2}{2} + \frac{Q^2}{gby} \]

\[ M = \frac{by^2}{2} + \frac{Q^2}{gby} \]

\[ M = \frac{by^2}{2} + \frac{Q^2}{gby} \]

\[ \theta = 2 \cos^{-1}[1 - 2(y/d)] \]
Hydraulic Jump in a rectangular channel

Hydraulic jump in a horizontal, rectangular channel

\[ H_1 = H_2 \]

\[ A_1 h_{c1} + \frac{Q^2}{gA_1} = A_2 h_{c2} + \frac{Q^2}{gA_2} \]

\[ \left( \frac{b y_1^2}{2} + \frac{Q^2}{gby_1} = \frac{b y_2^2}{2} + \frac{Q^2}{gby_2} \right) / b \]

\[ q = \frac{Q}{b} \]

\[ \frac{y_1^2}{2} + \frac{Q^2}{b^2 g y_1} - \frac{1}{2} = \frac{y_2^2}{2} + \frac{Q^2}{b^2 g y_2} - \frac{1}{2} \]

\[ \frac{y_1^2}{2} - \frac{y_2^2}{2} = \frac{q^2}{g} \left( \frac{1}{y_2} - \frac{1}{y_1} \right) \]

\[ \frac{y_2^2}{2} \]
Hydraulic Jump in a rectangular channel (Cont.)

\[ 1 - \left( \frac{y_2}{y_1} \right)^2 = \frac{2g^2}{9} \left[ \frac{1}{y_2y_1^2} - \frac{1}{y_1^3} \right] \] .... \( 1 \)

\[ F_1 = \frac{v_1}{\sqrt{gy_1}} = \frac{9}{by_1\sqrt{gy_1}} = \frac{9}{y_1\sqrt{gy_1}} \]

\[ F_1^2 = \frac{q^2}{gy_1^3} \] .... \( 2 \)
Hydraulic Jump in a rectangular channel (Cont.)

In \( \text{I} \) \( 1 - \left( \frac{y_2}{y_1} \right)^2 = \frac{2q^2}{gy_1^3} \left[ \frac{y_1}{y_2} - 1 \right] \)

\[
F_1^2 = \frac{y_2}{y_1}
\]

\[1 - a^2 = 2F^2 \left( \frac{1}{a} - 1 \right)\]

\[1 - a) (1 + a) = 2F^2 (1 - a)\]

\[
a^2 + a - 2F_1^2 = 0
\]

Belanger momentum equation

\[
y_2 = \frac{1}{2} \left( -1 + \sqrt{1 + 8F_1^2} \right)
\]

Upstream flow is known

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
Hydraulic Jump in a rectangular channel (Cont.)

In terms of $F_2$ (Subcritical Froude number)

$$\frac{y_1}{y_2} = \frac{1}{2} \left( -1 + \sqrt{1 + 8F_2^2} \right)$$

Jump Losses:

$$E_L = \frac{(y_2-y_1)^3}{4y_1y_2}$$

$$\frac{E_L}{E_1} = \frac{\left( \frac{y_2}{y_1} - 1 \right)^3}{4 \left( \frac{y_2}{y_1} \right) \left( 1 + \frac{F_1^2}{2} \right)}$$

where: $E_L = E_1 - E_2$ (Head loss)

$E = y + \frac{V^2}{2g}$

$z$: elevation head
## Classification of Hydraulic Jumps

<table>
<thead>
<tr>
<th>Fr&lt;sub&gt;1&lt;/sub&gt;</th>
<th>y&lt;sub&gt;2&lt;/sub&gt;/y&lt;sub&gt;1&lt;/sub&gt;</th>
<th>Classification</th>
<th>Sketch</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 1</td>
<td>1</td>
<td>Jump impossible</td>
<td><img src="image1" alt="Sketch" /></td>
</tr>
<tr>
<td>1 to 1.7</td>
<td>1 to 2.0</td>
<td>Standing wave or undulant jump</td>
<td><img src="image2" alt="Sketch" /></td>
</tr>
<tr>
<td>1.7 to 2.5</td>
<td>2.0 to 3.1</td>
<td>Weak jump</td>
<td><img src="image3" alt="Sketch" /></td>
</tr>
<tr>
<td>2.5 to 4.5</td>
<td>3.1 to 5.9</td>
<td>Oscillating jump</td>
<td><img src="image4" alt="Sketch" /></td>
</tr>
<tr>
<td>4.5 to 9.0</td>
<td>5.9 to 12</td>
<td>Stable, well-balanced steady jump; insensitive to downstream conditions</td>
<td><img src="image5" alt="Sketch" /></td>
</tr>
<tr>
<td>&gt;9.0</td>
<td>&gt;12</td>
<td>Rough, somewhat intermittent strong jump</td>
<td><img src="image6" alt="Sketch" /></td>
</tr>
</tbody>
</table>

**Source:** Munson, Young and Okiishi’s Fundamentals of Fluid Mechanics, 8th Edition
Hydraulic Jump Variations

(a) jump caused by a change in channel slope, (b) submerged jump

(a) free jump

(b) Tailwater submerged jump

Roller
Example:

Under appropriate conditions, water flowing from a faucet, onto a flat plate, and over the edge of the plate can produce a circular hydraulic jump as shown in the figure below. Consider a situation where a jump forms 3.0 in from the center of the plate with depths upstream and downstream of the jump of 0.05 in and 0.20 in, respectively. Determine the flow rate from the faucet.

\[ Q = ? \]
\[ y_1 = 0.05 \text{ in} \]
\[ y_2 = 0.20 \text{ in} \]

(English units)
\[ \frac{y_2}{y_1} = \frac{1}{2} \left( -1 + \sqrt{1 + 8Fr_1^2} \right) \]

\[ \frac{0.2}{0.05} = \frac{1}{2} \left( -1 + \sqrt{1 + 8Fr_1^2} \right) \]

\[ Fr_1 = 3.16 \]

\[ Fr = \frac{V}{\sqrt{9g}} \]

\[ 3.16 = \frac{V_1}{\sqrt{32.2 \times 0.05}} \]

\[ V_1 = \frac{1.16 ft}{s} \]
\[ Q = A_v V_1 = b y_1 v_1 \]

\[ b = 2\pi (\frac{3}{12}) \]

\[ Q = 2\pi (\frac{3}{12}) \times 0.05 \times 1.16 \]

\[ Q = 0.00759 \text{ ft}^3/\text{s} \]
Water enters a reach of a rectangular channel where $y_1 = 0.5 \text{ m}$, $b = 7.5 \text{ m}$, and $Q = 20 \text{ m}^3/\text{s}$. It is desired that a hydraulic jump occur upstream (location 2) of the sill and on the sill critical conditions exist (location 3). Other than across the jump, losses can be neglected. Determine the following:

(a) Depths at locations 2 and 3
(b) Required height of the sill, $h$
(c) Resultant force acting on the sill
(d) Sketch the water surface and energy grade line

Example:
a) \( y_2, y_3 \)

\[
\frac{y_2}{y_1} = \frac{1}{2} \left( -1 + \sqrt{1 + 8F_{r1}^2} \right)
\]

\[
\frac{y_2}{0.5} = \frac{1}{2} \left( -1 + \sqrt{1 + 8 \times 2.41^2} \right)
\]

\( y_2 = 1.47m \)

\[
+ y_3 = y_c
\]

\[
y_c = \sqrt[3]{\frac{92}{9}}
\]

\[
F_{r1} = \frac{V_1}{\sqrt{9y_1}}
\]

\[
V_1 = \frac{20}{(7.5 \times 0.9)}
\]

\[
V_1 = 5.33\%
\]

\[
F_{r1} = \frac{5.33}{\sqrt{9.8 \times 0.5}}
\]

\[
F_{r1} = 2.41
\]

\[
f = \frac{f_0}{9}
\]
\[ y_c = \sqrt{\left(\frac{20}{3.5}\right)^2} = 0.90 \text{ m} \]

\[ y_3 = 0.90 \text{ m} \]

\[ \begin{align*}
V_2 &= \frac{Q}{A_2} \\
V_2 &= 1.81 \text{ m/s} \\
V_3 &= \frac{Q}{A_3} \\
V_3 &= 2.96 \text{ m/s}
\end{align*} \]

\[ \begin{align*}
\text{b) } h &= \? \\
E_2 &= E_3 \\
y_2 + \frac{v_2^2}{2g} + z_2 &= y_3 + \frac{v_3^2}{2g} + z_3
\end{align*} \]

\[ \begin{align*}
1.47 + \frac{1.81^2}{2 \times 9.8} &= 0.9 + \frac{2.96^2}{2 \times 9.8} + h
\end{align*} \]
$h = 0.29 \text{ m}$
\[ \Sigma F = \dot{m}(V_B - V_A) \]

\[ F_2 - F_4 = \rho Q(V_4 - V_2) \quad \text{(*)} \]

\[ F_2 = \gamma A_2 Y_{c2} = 1000 \times 9.8 \times (7.5 + 1.47) \times \frac{1.47}{2} \]

\[ F_4 = \gamma A_4 Y_{c4} \quad \text{\underline{\underline{\gamma_4 \text{ is unknown}}}} \]

\[ E_4 = E_3 = E_2 \quad \text{\underline{\underline{E_2 = E_4}}} \]
In (\(\text{Eq}(\text{t})\)) \(F = 1.26 \times 10^4 \text{ N}\)

\[ \begin{align*}
\text{Given:} & \\
\text{Known:} & \\
\text{Required:} & \\
\text{Solution:} & \\
\end{align*} \]

\[ m = 8 \text{ kg} \]

\[ v = 2 \text{ m/s} \]

\[ y_4 = 0.59 \text{ m} \]

\[ \frac{2x9.8}{2x9.8} = y_4 + 20^2 \]

\[ \frac{1.47 + 181^2}{18} = y_4 + \text{th} \]

\[ y_2^2 = \frac{2g}{2 \text{ g} + 2} \]

\[ 0^2 + \frac{2g}{2 \text{ g} + 2} \]

\[ \text{Choose the positive root} \]

\[ z \neq 1 \text{ or } 2 \]

\[ \text{Superzentury} \]

\[ \text{Smaller positive value} \]

\[ \text{Flow 3 roots} \]

\[ \text{Flow is supercritical} \]
Surges

A tidal bore in Morecambe Bay, the United Kingdom

Video of a tidal bore in China
https://www.youtube.com/watch?v=axAxtsyHreQ
Surges

- Surges belong in a discussion of **unsteady flows**.
- However, they can be **transformed into a steady flow problem** by superimposing a surge velocity to make the surge stationary.
- For an observer moving at the speed of the surge, this becomes a **steady-flow formation of a hydraulic jump**.

Source: https://www.mdpi.com/2073-4441/11/1/28
Surges in rectangular channels

**Continuity**

Stationary jump: \( Q_1 = Q_2 = A \cdot V \)

\[ y_1 \cdot V_1 = y_2 \cdot V_2 \]

Moving jump:

\[ (V_1 + V_s) \cdot y_1 = (V_2 + V_s) \cdot y_2 \]

\[ V_s = \frac{V_1 \cdot y_1 - V_2 \cdot y_2}{y_2 - y_1} \]
Surges (cont.)

**Momentum**

Stationary jump:

\[ \frac{y_1^2}{2} - \frac{y_2^2}{2} = \frac{q^2}{2g} \left( \frac{1}{y_2} - \frac{1}{y_1} \right) = \frac{v_1^2 y_1^2}{2g} \left( \frac{1}{y_2} - \frac{1}{y_1} \right) \]

Moving jump:

\[ \frac{(y_1 + y_2)(y_1 - y_2)}{2} = \frac{(v_1 + v_s)^2 y_1^2}{2g} \left( \frac{y_1 - y_2}{y_2} \right) \]
\[
\frac{(V_t + V_s)^2}{gY_1} = \frac{1}{2} \left[ \frac{y_2}{y_1} + \left(\frac{y_2}{y_1}\right)^2 \right]
\]
Example of Application:
A 2m wide rectangular channel carries a discharge of 1 m³/s at a flow depth of 1m. A sluice gate located in the channel is suddenly lowered and it is desired to produce a 0.1m high surge upstream of the gate. Find the velocity of the surge and the flow velocity at a section after the surge has passed. Assume a frictionless and horizontal channel.

\[ Q = 1 \text{ m}^3/\text{s} \]
\[ V_1 = \frac{Q}{A_1} = \frac{1}{2 \times 1} = 0.5 \text{ m/s} \]

\[ y_1 = 1 \text{ m} \]
\[ y_2 = 1.1 \text{ m} \]

*Momentum:*
\[
\left( \frac{V_1 + V_s}{g y_1} \right)^2 = \frac{1}{2} \frac{y_2}{y_1} \left( 1 + \frac{y_2}{y_1} \right)
\]

\[
\left( \frac{0.5 + V_s}{9.8 \times 1} \right)^2 = \frac{1}{2} \left( \frac{1.1}{1} \right) \left( 1 + \frac{1.1}{1} \right)
\]

\[ V_s = 2.86 \text{ m/s} \]

*Continuity:*
\[
(V_1 + V_s) y_1 = (V_2 + V_s) y_2
\]
\[
(0.5 + 2.86) \times 1.0 = (V_2 + 2.86) \times 1.1
\]

\[ V_2 = 0.19 \text{ m/s} \]
Example of Application:
What is the velocity of the surge and surge height in previous example if the gate is completely closed?

\[ V_s = ?? \]
\[ \Delta h = ?? \]

\[ \Delta h > 0.1 \text{ m} \]

\[ y_1 = 1.0 \text{ m} \]
\[ V_2 = 0 \text{ m/s} \]
\[ V_1 = 0.5 \text{ m/s} \]
* Momentum: \[
\frac{(V_1 + V_s)^2}{9.8 \times 1} = \frac{1}{2} \frac{y_2}{y_1} \left(1 + \frac{y_2}{y_1} \right)
\]

\[\text{In eqn (1):} \frac{(0.5 + V_s)^2}{9.8} = \frac{1}{2} \frac{y_2}{1.0} \left(1 + \frac{y_2}{1.0} \right) \Rightarrow V_s = 3.02 \text{ m/s} \]

* Continuity: \[
(V_1 + V_s) y_1 = (V_2 + V_s) y_2
\]

\[\Rightarrow \text{In eqn (2):} \frac{(0.5 + V_s) \times 1.0}{y_2} = (0 + V_s) y_2 \]

\[\Rightarrow y_2 = \frac{0.5 + V_s}{V_s} \]

\[\Rightarrow y_2 = 0.5 + 3.02 = 3.52 \text{ m} \]

\[\Delta h = y_2 - y_1 = 1.17 - 1.0 \text{ m} = 0.17 \text{ m} \]
Bridge piers

Source: https://commons.wikimedia.org/wiki/File:Bridge_Piers_P2110004_US_27_Central_Ave.JPG
Bridge piers

- Obstruction caused by bridge piers in subcritical flows leads to backwater effects upstream and can even cause choking.

- Two types of flow.

- Type I flow: The depth decreases when passing through the constriction with the flow remaining subcritical.

- Type II flow: Choking occurs with critical depth existing in the constriction.

Source: https://engineeringmaster.in/2017/04/05/how-bridges-are-built-over-water/
Bridge piers

Type I Flow: Subcritical approach flow \textit{without choking}

Type II Flow: Subcritical approach flow \textit{with choking} and supercritical flow downstream of pier
Bridge piers (Cont.)

**Type I Flow**

\[ M_1 = M_4 + \frac{D}{\gamma} \]

\[ M = Ah_c + \frac{Q^2}{gA} \]

\[ D = C_D\rho A_p \frac{V_1^2}{2} \]

- \( C_D \) = Drag coefficient
- \( A_p \) = Frontal area of pier
- \( h_1^* \) = Change in depth or backwater

(b) Profile of Type I Flow
Bridge piers

For a rectangular channel:

\[
\frac{M_1}{S} = \frac{M_4}{S} + \frac{D}{8S}
\]

\[
\frac{y_1^2 + \frac{Q^2}{2g b^2 y_1}}{S} = \frac{y_4^2 + \frac{Q^2}{2g b^2 y_4}}{S} + \frac{C_D (a y_1) V_1^2}{2g S}
\]

\[
\frac{y_1^2 + \frac{q^2}{2g y_1}}{S} = \frac{y_4^2 + \frac{q^2}{2g y_4}}{S} + \frac{C_D a y_1 V_1^2}{2g S}
\]

\[
M_1 = M_4 + \frac{D}{8}
\]

\[
M = \frac{b y_1^2 + \frac{Q^2}{2g b y_1}}{S}
\]
\[ R = \frac{C_D a}{S} \]

\[ h_1^* = y_1 - y_4 \text{ (backwater height)} \]

\[ \lambda = \frac{y_1 - y_4}{y_4} = \frac{y_1}{y_4} - 1 \]

\[ \lambda + 1 = \frac{y_1}{y_4} \quad \lambda + 2 = \frac{y_1}{y_4} + 1 \]

In ①

\[ \frac{y_1^2 - y_4^2}{2} + \frac{q^2 - \frac{q^2}{2}}{g y_1 g y_4} = R g y_1 V_1^2 \]

\[ \frac{y_1^2 - y_4^2}{2} = \frac{y_4^2}{2} \left[ \left( \frac{y_1}{y_4} \right)^2 - 1 \right] = \frac{y_4^2}{2} \left( \frac{y_1}{y_4} - 1 \right) \left( \frac{y_1}{y_4} + 1 \right) \]
\[
\frac{y_1^2 - y_4^2}{2} = \frac{y_4^2}{2} \lambda (\lambda + 2) \tag{*}
\]

\[
\frac{q^2}{g_{q_1}} - \frac{q^2}{g_{q_4}} = \sqrt{\frac{v_4}{g_{q_4}} y_4^2 \left[ \frac{y_4}{y_1} - 1 \right]}
\]

\[
F_{q_4} = \frac{v_4}{\sqrt{g_{q_4}}}
\]

\[
F_{q_4}^2 = \frac{v_4^2}{g_{q_4}}
\]

\[
= F_{q_4}^2 y_4^2 \left( \frac{y_4}{y_1} - 1 \right) = F_{q_4}^2 y_4^2 \left( \frac{1}{\lambda + 1} - 1 \right) = F_{q_4}^2 y_4^2 \left( \frac{-\lambda}{\lambda + 1} \right)
\]
\[ \frac{C_D a g_1 V_1^2}{2gS} = \frac{Rg_1 V_1^2}{2g} = \frac{R y_1 V_4^2 y_4^2}{y_1^2} \]

\[ = \frac{y_4^2 y_4^2}{y_1 y_4} \left( \frac{R}{2g} \right) R_4 = \frac{y_4^2 y_4^2}{y_1 y_4} \left( \frac{R}{2g} \right) R_4 \]

\[ = \frac{F r_4^2 y_4^2}{2} \left( \frac{y_4}{y_1} \right) R_4 = \frac{F r_4^2 y_4^2}{2} \left( \frac{1}{\lambda+1} \right) R_4 \]

* Substituting:

\[ \frac{y_4^2}{2} \lambda (\lambda+2) + F r_4^2 y_4^2 \left( -\frac{\lambda}{\lambda+1} \right) = \frac{F r_4^2 y_4^2}{2} \left( \frac{R}{2(\lambda+1)} \right) \]

\[ \lambda (\lambda+2) + 2 F r_4^2 \left( -\frac{\lambda}{\lambda+1} \right) = \frac{F r_4^2}{2} \frac{R}{\lambda+1} \]
\[ \lambda (\lambda + 2) = \frac{F_{y4}^2}{\lambda + 1} \left[ R + 2\lambda \right] \]

\[ R = \frac{C_D a}{s} \]

\[ F_{y4}^2 = \frac{\lambda (\lambda + 1) (\lambda + 2)}{R + 2\lambda} \]

\[ \lambda = h_i^* \quad \frac{y_4}{y_4} \]

\[ h_i^* = y_1 - y_4 \]

\[ F_4^2 = \frac{\lambda (\lambda + 1) (\lambda + 2)}{C_D a} + 2\lambda \]
Solution for backwater caused by bridge piers in Type I flow

\[ F_4^2 = \frac{\lambda(\lambda+1)(\lambda+2)}{C_Da/s + 2\lambda} \]
Example of application

For a river flow between bridge piers 3 m in diameter with a spacing of 20 m, determine the backwater using the momentum method if the downstream depth is 4.0 m and the downstream velocity is 1.9 m/s. Assume a drag coefficient of 2.0 for the bridge piers.

\[ h_1^* = y_1 - y_4 \]

\[ F_{r4} = \frac{\lambda(\lambda+1)(\lambda+2)}{C_D a} + 2\lambda \]

\[ q = 3 \text{m} \]
\[ S = 20 \text{m} \]
\[ y_4 = 4 \text{m} \]
\[ V_4 = 1.9 \text{m/s} \]
\[ C_D = 2.0 \]
\[ \frac{C_D a}{S} = \frac{2.0 \times 3}{20} = 0.3 \]

* \[ F_{r_4} = \frac{V_4}{\sqrt{g y_4}} = \frac{1.9}{\sqrt{9.8 \times 4}} = 0.303 \]

In Eq. 1

\[ (0.303)^2 = \frac{\lambda (\lambda+1)(\lambda+2)}{0.3+2\lambda} \]

\[ \lambda = 0.0148 \]

\[ \lambda = \frac{h_i^*}{y_4} = 0.0148 \rightarrow h_i^* = 0.0148 \times 4 \]

\[ h_i^* = 0.059 m \approx 6 \text{ cm} \]

(backwater)