1

Frequency analysis (Cont.)

Lecture 16, 05/23/2013

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Adapted from textbook and notes of Philip B. Bedient

Binomial Distribution

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The probability of getting x successes followed by n-x failures is the product of prob of n independent events: $p^x (1-p)^{n-x}$

This represents only one possible outcome. The number of ways of choosing x successes out of n events is the binomial coeff. The resulting distribution is the Binomial or B(n,p).

$$\boxed{\begin{array}{l} \mathrm{P(x)} = \dfrac{n!}{x!(n-x)!}\, p^x (1-p)^{n-x} \\ \mathrm{x} = 0,\ 1,\ 2,\ 3,\ \ldots,\ \mathrm{n} \end{array}}$$

P(occurrence of event)

1-P(No occurrence of event)

3

Example: Pilot of commercial aircraft

If the probability of an aircraft accident is 1/10,000, what is the probability of an aircraft accident for someone that travels 100 times in his/her lifetime?

Solution

$$P_{\text{accident at any time}} = \frac{1}{10,000} = 0.0001$$

$$P_{\text{no accident in 100 times}} = (1-P)^{100} = (1-0.0001)^{100} = 0.9900$$

$$P_{\text{accident in 100 times}} = 1-0.9900 = 0.01\ (1\%)$$

	O	O	O	..... O
	100 times			
Accident	P	P	P	..... P
No Accid.	1-P	1-P	1-P	1-P

Risk and Reliability 4

The probability of at least one success in "n" years, where the probability of success in any year is 1/T, is called the RISK.

Prob. success = $p = 1/T$ *and* Prob. failure = $1-p$

$$RISK = 1 - P(0)$$
$$= 1 - \text{Probab. of failure in "n" consecutive years}$$
$$= 1 - (1-p)^n$$

RISK $= 1 - (1 - 1/T)^n$ Danger

$Reliability = (1 - 1/T)^n$ Safety

Risk Example

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What is the probability of occurrence of at least one 50 yr flood in a 30 year mortgage period.

The probability of success in any year is $1/T = 1/50 = 0.02$

$$\text{RISK} = 1 - (1 - 1/T)^n = 1 - (1 - 0.02)^{30}$$

$$= 1 - (0.98)^{30} = 0.455 \text{ or } 46\%$$

This risk is very high. If the risk is too high, you may want to increase the design level. For instance for a 100 year flood, $p = 0.01$

$$\text{RISK} = 1 - (0.99)^{30} = 0.26 \text{ or } 26\%$$

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Return Periods for Various Degrees of Risk and Expected Design Life

Table 3–3 Return Periods for Various Degrees of Risk and Expected Design Life [Eq. (3–51)]

Risk (%)	Reliability (%)	Expected Design Life, *n* (yr)							
		2	5	10	15	20	25	50	100
75	25	2.0	4.1	7.7	11.3	14.9	18.5	36.6	72.6
63	37	2.6	5.5	10.6	15.6	20.6	25.6	50.8	101.1
50	50	3.4	7.7	14.9	22.1	29.4	36.6	72.6	144.8
40	60	4.4	10.3	20.1	29.9	39.7	49.4	98.4	196.3
30	70	6.1	14.5	28.5	42.6	56.6	70.6	140.7	280.9
25	75	7.5	17.9	35.3	52.6	70.0	87.4	174.3	348.1
20	80	9.5	22.9	45.3	67.7	90.1	112.5	224.6	448.6
15	85	12.8	31.3	62.0	92.8	123.6	154.3	308.2	615.8
10	90	19.5	48.0	95.4	142.9	190.3	237.8	475.1	949.6
5	95	39.5	98.0	195.5	292.9	390.4	487.9	975.3	1950.1
2	98	99.5	248.0	495.5	743.0	990.5	1238.0	2475.4	4950.3
1	99	199.5	498.0	995.5	1493.0	1990.5	2488.0	4975.5	9950.4
0.5	99.5	399.5	998.0	1995.5	2993.0	3990.5	4988.0	9975.5	19950.5

For a risk of 1% (Reliability 99%), and Design life of 50 years, the return period is 4975.5 (~5000 years)

Example: Problem 3.3 Textbook

7 1

A temporary cofferdam is being designed to protect a 5-yr construction project from the 25-yr flood. What is the probability that the cofferdam will be overtopped:

(a) at least once during the 5-yr project, (this is risk)

(b) not at all during the project, (this is reliability)

(c) in the first year only,

(d) in the fourth year and fifth year exactly?

(e) Two times in the next 5 years.

(f) Three times in the next five years

(g) Five times in the next five years

Solution

$T = 25$ yrs

$\text{Risk} = 1 - \left(1 - \frac{1}{T}\right)^n$

$\text{Reliability} = 1 - \text{Risk}$

a) $n = 5$, $\text{Risk} = 1 - \left(1 - \frac{1}{25}\right)^5 = 1 - 0.96^5 = \underline{0.185}$

7 2

b) $\left(1-\frac{1}{25}\right)^5 = 0.815$

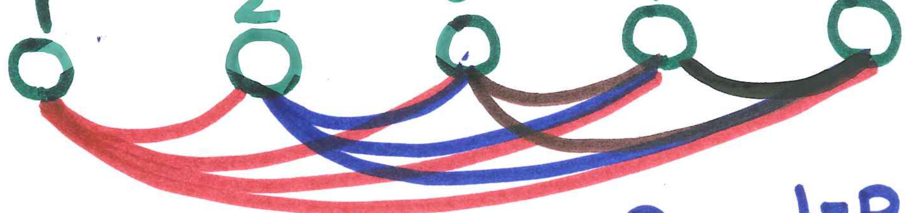
c) $P(1-P)^4 = 0.04 \times 0.96^4 = 0.034$

d) 1 2 3 4 5

$1-P \quad 1-P \quad 1-P \quad P \quad P$

$(1-P)^3 P^2 = 0.00142 = 0.14\%$

e) 1 2 3 4 5 } combination C^n_x



$P^2(1-P)^3$

$1-P \quad 1-P \quad P \quad P \quad 1-P$

$$P = C^n_x P^2(1-P)^3 = \frac{n!}{(n-x)!\,x!} P^2(1-P)^3 = \frac{5\times4\times\cancel{3!}\times0.04^2\times0.96^3}{\cancel{3!}\,2!}$$

$$P = 10 \times 0.04^2 \times 0.96^3 = 0.0142 = 1.4\%$$

7 3

f

O O O O O

$1-p \quad p \quad p \quad p \quad 1-p$

$$P = C^5_3 \, p^3(1-p)^2 = \frac{5\times 4\times \cancel{3!}}{2!\cancel{3!}} \times 0.04^3 \times 0.96^2$$

$$P = 0.000589 = \underline{0.058\%}$$

g

$$P = C^5_5 \, p^5(1-p)^0 = \frac{\cancel{5!}}{0!\cancel{5!}} \times 0.04^5 \times 0.96^0$$

$$P = 1 \times 0.04^5 \times 1 = \underline{1.02\times 10^{-7}}$$

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Normal distribution

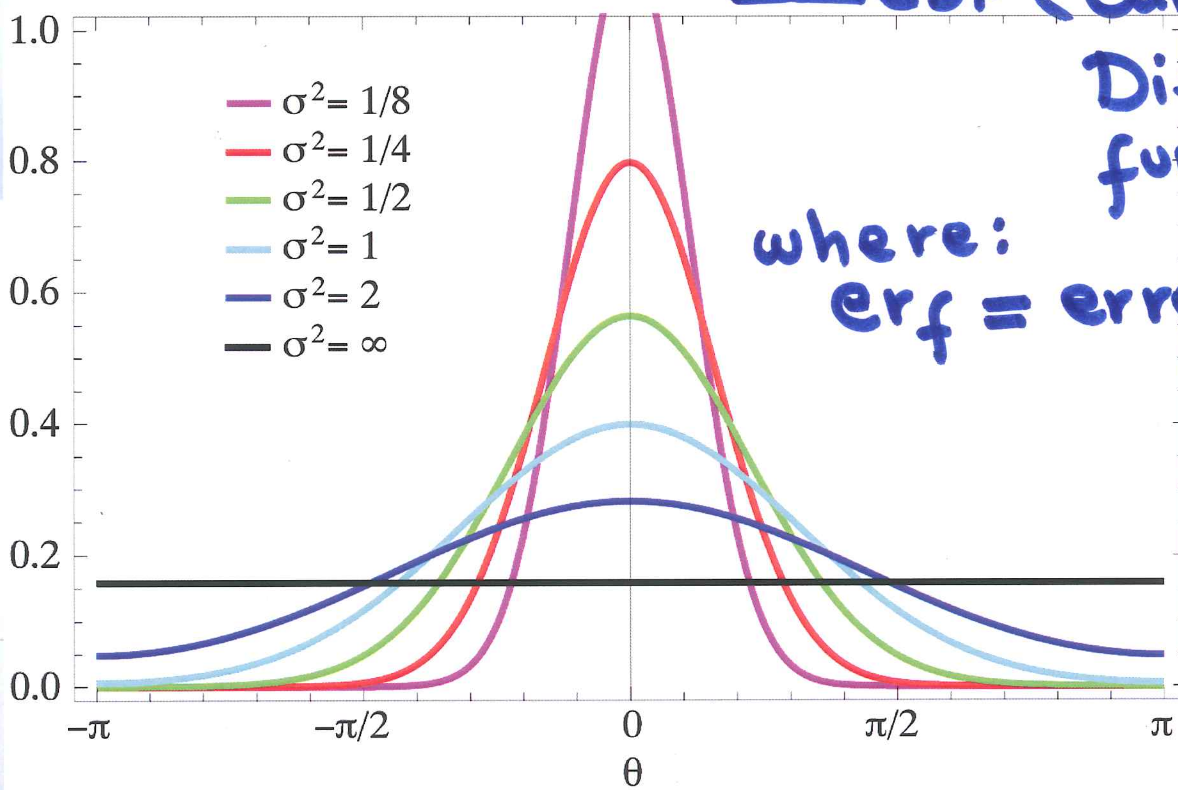
PDF (probability distribution function)

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}$$

$$F(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2\sigma^2}}\right)\right]$$

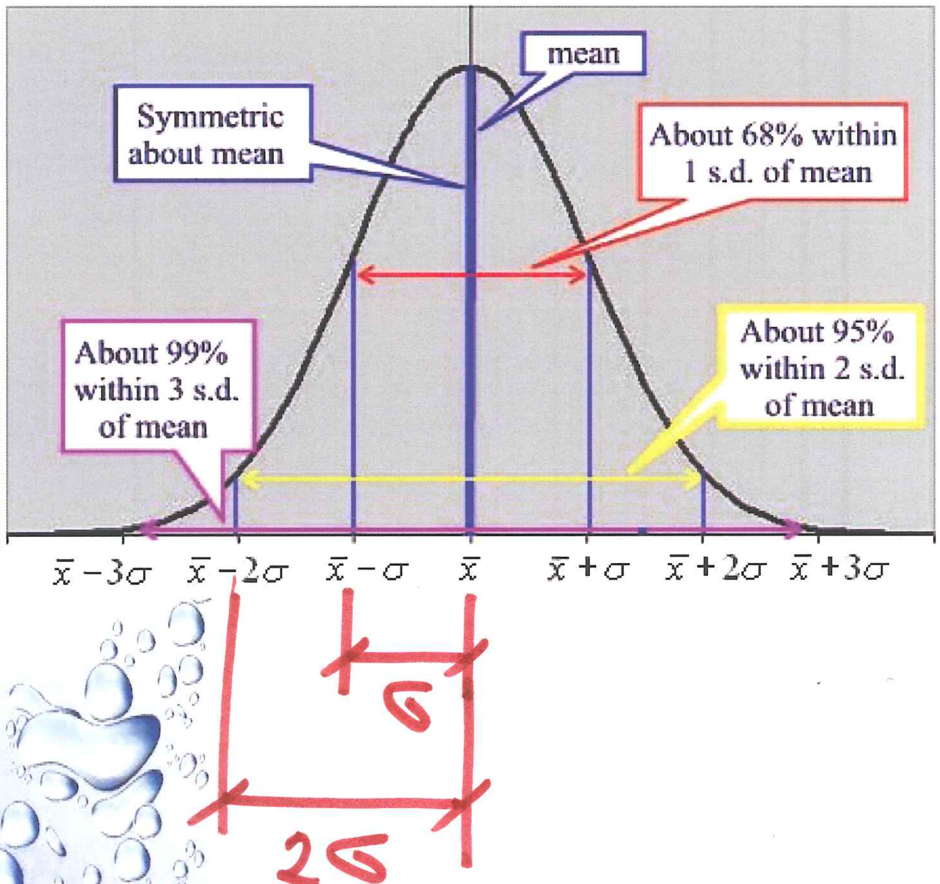
CDF (Cumulative Distribution function)

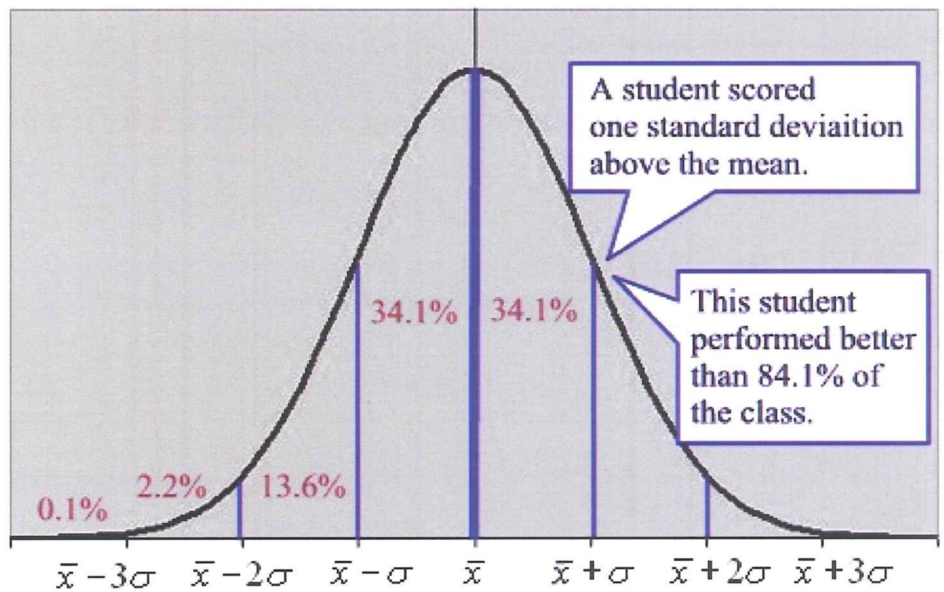
where: erf = error function



Source: http://en.wikipedia.org/wiki/File:WrappedNormalPDF.png

Normal distribution





Source: http://www.regentsprep.org/Regents/math/algtrig/ATS2/NormalLesson.htm

Log-Normal distribution

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$X = e^{\mu+\sigma Z}$

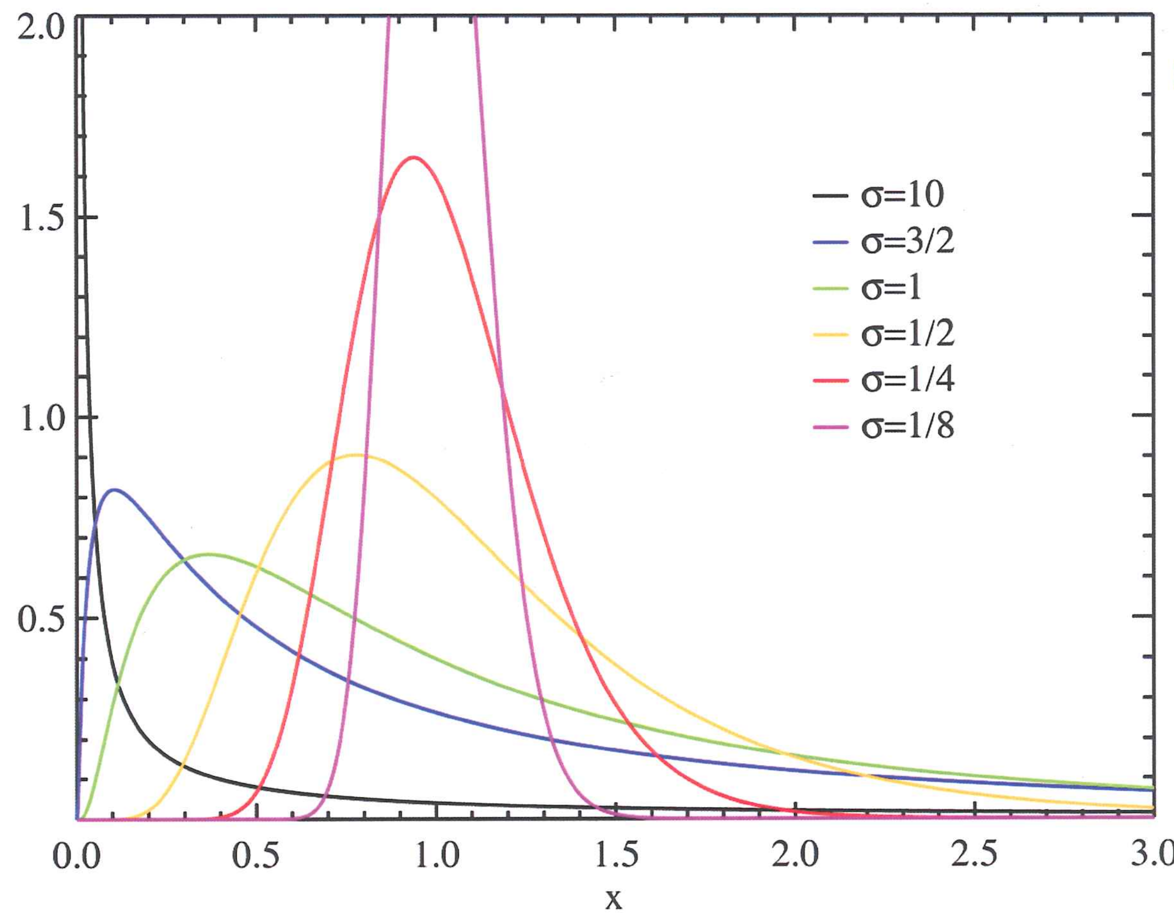
PDF

$$f_X(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}, \quad x > 0$$

CDF

$$F_X(x;\mu,\sigma) = \frac{1}{2}\operatorname{erfc}\left[-\frac{\ln x - \mu}{\sigma\sqrt{2}}\right] = \Phi\left(\frac{\ln x - \mu}{\sigma}\right),$$

where: erfc = complementary error function



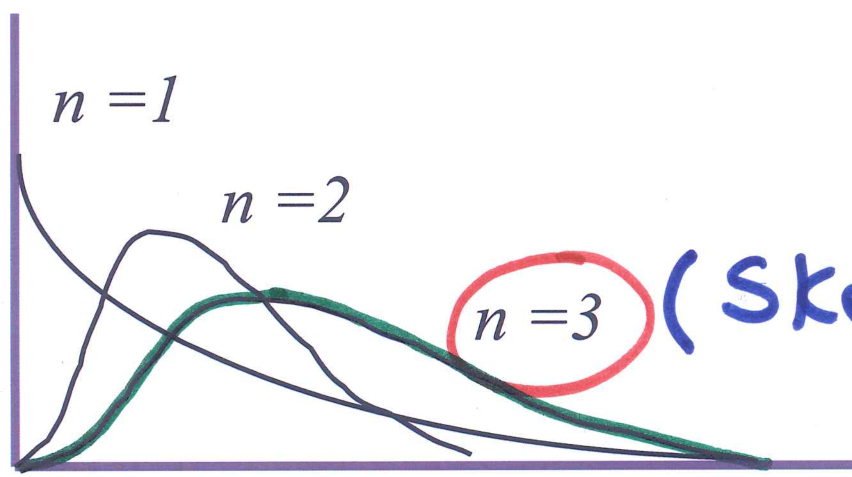
http://en.wikipedia.org/wiki/Log-normal#cite_note-JKB-1

Gamma Distribution (Pearson type 3) 11

$$Q_n = \frac{1}{K\Gamma(n)}\left(\frac{t}{k}\right)^{n-1} e^{-t/K}$$

$Mean$ or $E(t) = nK$

$Var = nK^2$ where $\Gamma(n) = (n-1)!$



Log [Pearson type 3] ⇒ Log pearson → Most recommended for hydrological analysis.

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Summary of Distributions

Distribution	Probability density function	Range	Parameters in terms of sample moments
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$	$-\infty < x < \infty$	$\mu = \bar{x}$ $\sigma = S_x$
Log-normal	$f(x) = \frac{1}{x\sigma_y\sqrt{2\pi}} \exp\left[-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right]$ where $y = \ln x$	$x > 0$	$\mu_y = \bar{y}$ $\sigma_y = S_y$
Exponential	$f(x) = \frac{1}{\beta} e^{-x/\beta}$	$x \geq 0$	$\beta = \bar{x}$
Gamma (two-parameter)	$f(x) = \frac{1}{\beta\Gamma(\alpha)} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta}$	$x \geq 0$	$\alpha = \frac{\bar{x}^2}{S_x^2} = \frac{1}{C_V^2}$; $\beta = \frac{S_x^2}{\bar{x}}$
Pearson type III (three-parameter gamma)	$f(x) = \frac{1}{\beta\Gamma(\alpha)} \left(\frac{x-c}{\beta}\right)^{\alpha-1} e^{-(x-c)/\beta}$	$x \geq c$	$\alpha = \frac{4}{C_s^2}$; $\beta = \frac{S_x C_s}{2}$ $c = \bar{x} - \alpha\beta$
Log-Pearson type III	$f(x) = \frac{1}{x\beta\Gamma(\alpha)} \left(\frac{y-c}{\beta}\right)^{\alpha-1} e^{-(y-c)/\beta}$ where $y = \ln x$	$y = \ln x \geq c$	$\alpha = \frac{4}{C_{sy}^2}$; $\beta = \frac{S_y C_{sy}}{2}$ $c = \bar{y} - \alpha\beta$
Gumbel (EV I)	$f(x) = \frac{1}{\alpha} \exp\left[-\frac{x-u}{\alpha} - \exp\left(\frac{x-u}{\alpha}\right)\right]$	$-\infty < x < \infty$	$\alpha = 0.78 S_x$ $u = \bar{x} - 0.45 S_x$

(Extreme Values)

http://daad.wb.tu-harburg.de/?id=516

Parameters of Dist'n

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Distribution	Normal x	LogN $Y = log x$	Gamma x	Exp t
Mean	μ_x	μ_y	nk	1/k
Variance	σ_x^2	σ_y^2	nk^2	$1/k^2$
Skewness	zero	zero	$2/n^{0.5}$	2

y is normally distributed, where $y = \log x$ (x = original data)

Normal, LogN, LPIII

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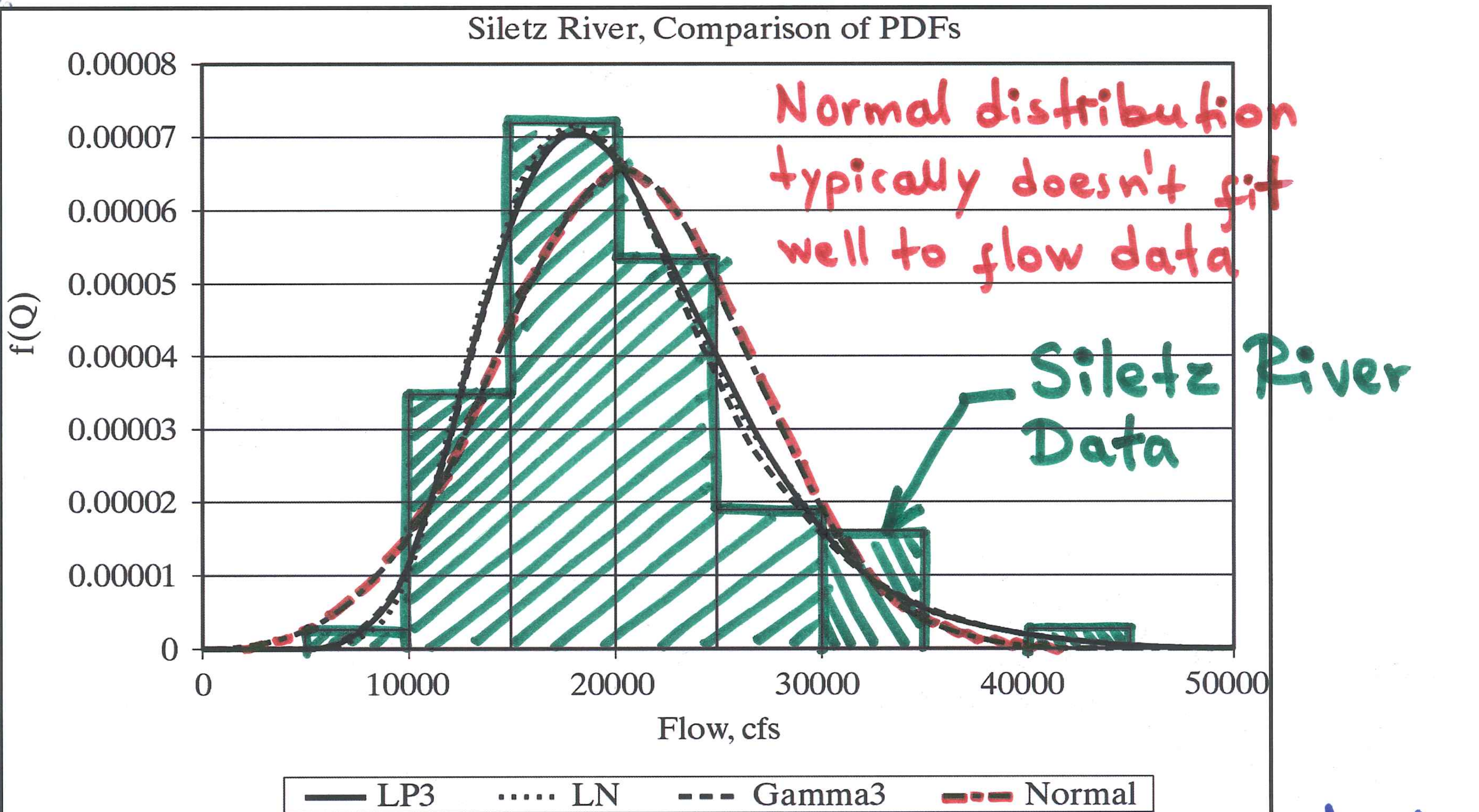

LP3 (Log Pearson type 3 most recommended for hydrological studies)

Figure 3.15

Four PDFs fit to data for the Siletz River. Fit is by the method of moments, as shown in the text, with moments given in Example 3.3.