Florida International University, Department of Civil and Environmental Engineering

CWR 3201 Fluid Mechanics, Fall 2018

Dimensional Analysis and Similitude

Source: https://www.theglobeandmail.com/politics/article-canada-invests-another-54-million-into-development-of-f-35-stealth/

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Isabella Lake Dam Hydraulic model, CA

https://www.youtube.com/watch?v=aDhd88lWtbc

6.1 Introduction

- **Dimensional analysis** is used to keep the required experimental studies to a minimum.
  - Based off **dimensional homogeneity** [**all terms in an equation should have the same dimension**.]

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

$/z_1$

Bernoulli's equation: Dimension of each term is length

$$\frac{P_1}{\gamma z_1} + \frac{V_1^2}{2g z_1} + 1 = \left(\frac{P_2}{\gamma z_2} + \frac{V_2^2}{2g z_2} + 1\right)\frac{z_2}{z_1}$$

Bernoulli's equation in this form: Each term is dimensionless

6.1 Introduction (Cont.)

- **Similitude** is the study of predicting prototype conditions from model observations.
  - Uses dimensionless parameters obtained in dimensional analysis.

- Two approaches can be used in dimensional analysis:
  - Buckingham π-theorem: Theorem that organizes steps to ensure dimensional homogeneity.
  - Extract dimensionless parameters from the differential equations and boundary conditions.

6.2 Dimensional Analysis

6.2.1 Motivation

In the interest of saving time and money in the study of fluid flows, the fewest possible combinations of parameters should be utilized.

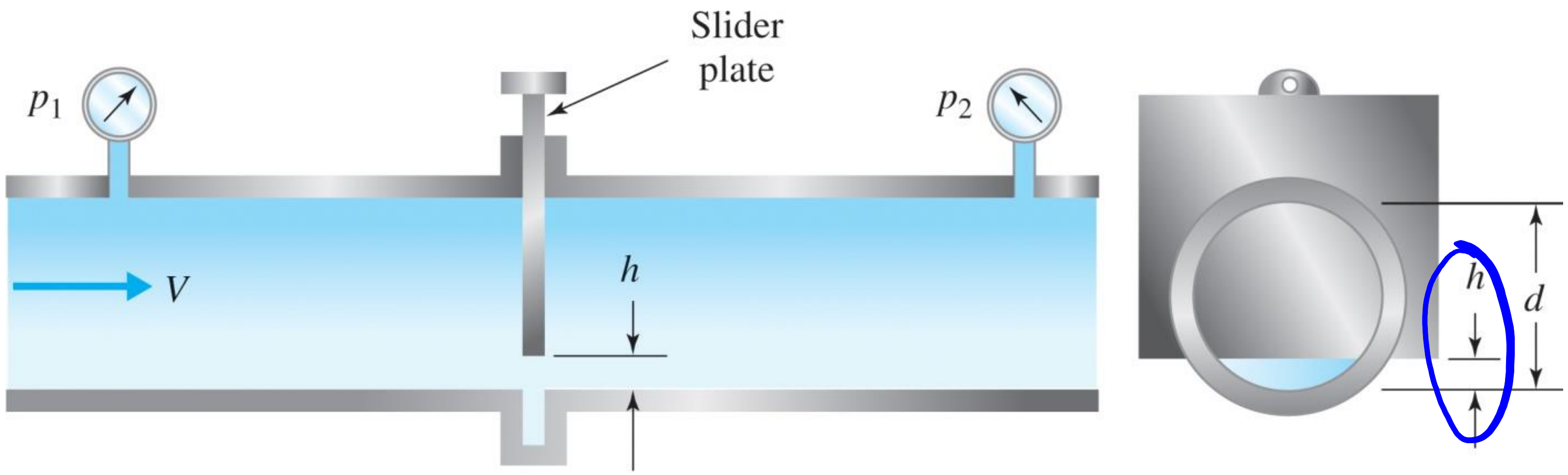


Fig. 6.2 Flow around a slider valve.

- For pressure drop across a slider valve above:
  - We can assume that it depends on pipe mean velocity *V*, fluid density *ρ*, fluid viscosity *μ*, pipe diameter *d*, and gap height *h*

$$\Delta p = f(V, \rho, \mu, d, h)$$

6.2.1 Motivation (Cont.)

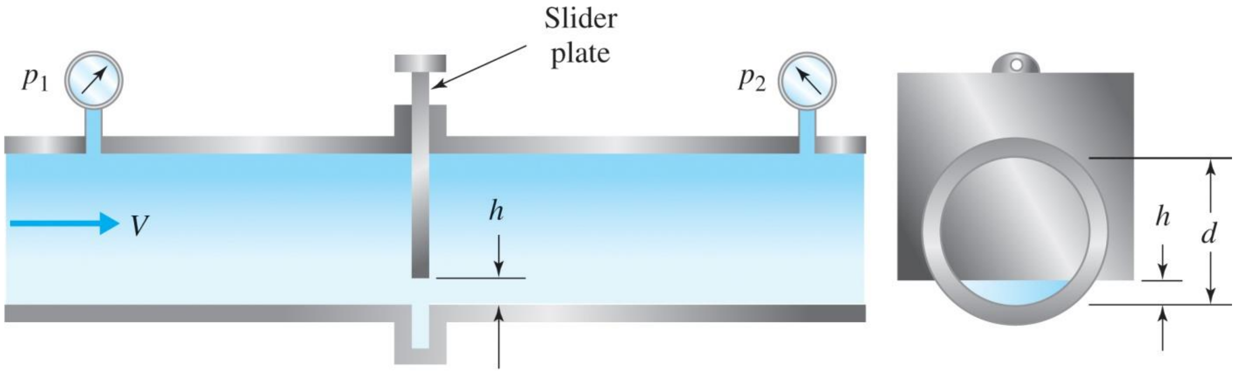


Fig. 6.2 Flow around a slider valve.

- Could fix all parameters except velocity and find pressure dependence on average velocity.
- Repeat with changing diameter, etc.

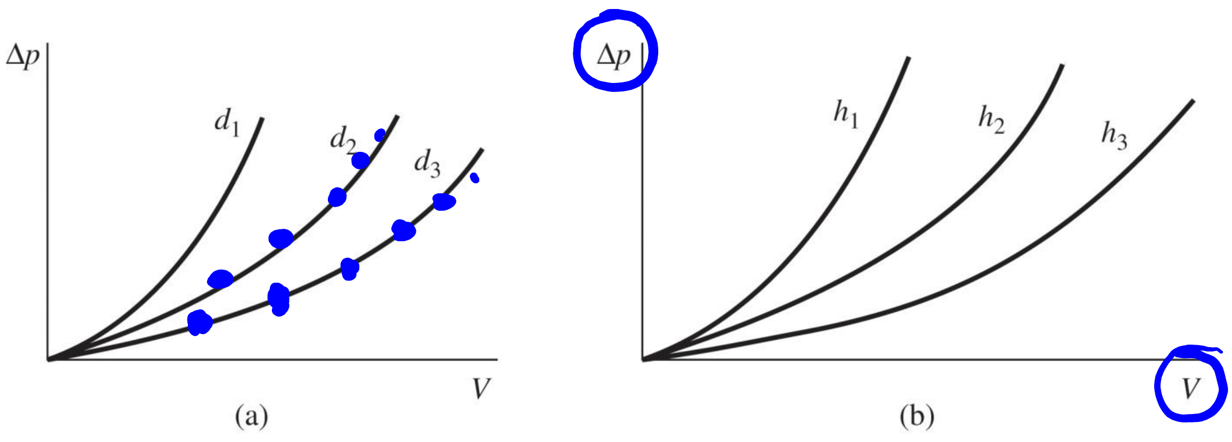


Fig. 6.3 Pressure drop versus velocity curves: (a) ρ, μ, h fixed; (b) ρ, μ, d fixed.

6.2.1 Motivation (Cont.)

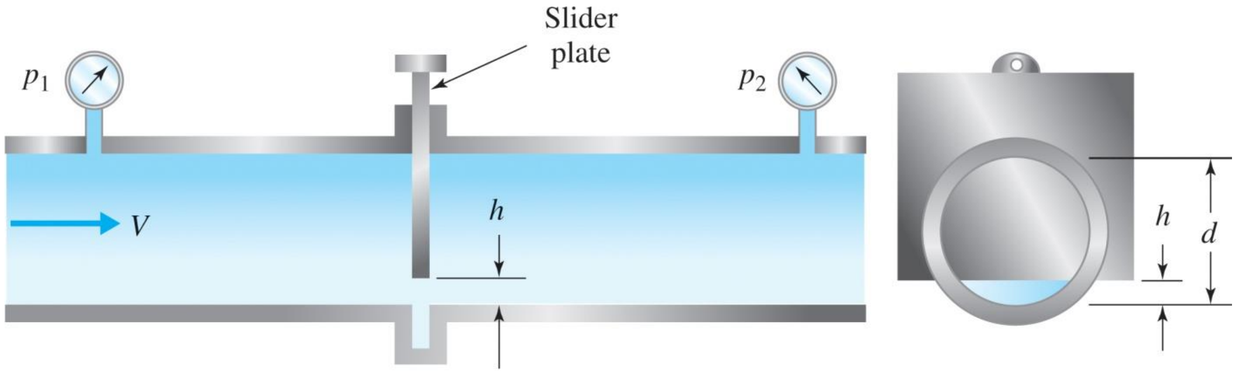


Fig. 6.2 Flow around a slider valve.

$$[\Delta p = f(V, \rho, \mu, d, h)]$$

- The equation could be rewritten in terms of dimensionless parameters as:

$$\frac{\Delta p}{\rho V^2} = f\left(\frac{V\rho d}{\mu}, \frac{h}{d}\right)$$

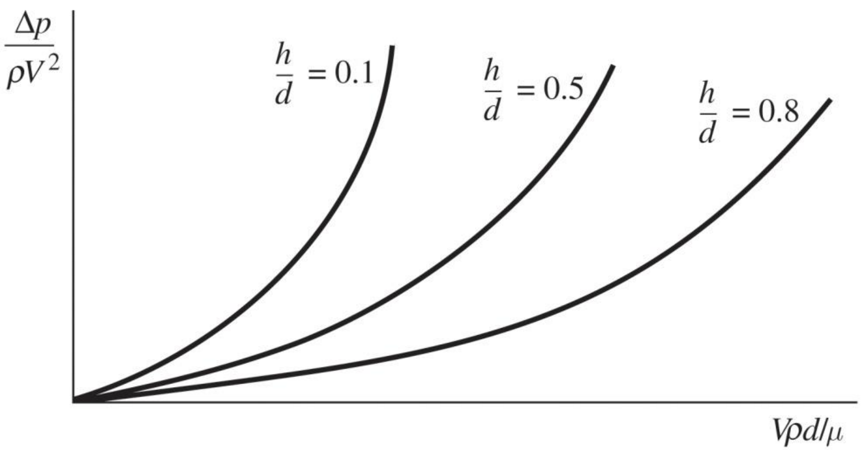


Fig. 6.4 Dimensionless pressure drop versus dimensionless velocity.

6.2.2 Review of Dimensions

- All quantities have a combination of dimensions of length, time, mass, and force by Newton's Second Law:

Table 6.1 Symbols and Dimensions of Quantities Used in Fluid Mechanics

Quantity	*Symbol*	*Dimensions*
Length	l	L
Time	t	T
Mass	m	M
Force	F	ML/T^2
Velocity	V	L/T
Acceleration	a	L/T^2
Frequency	ω	T^{-1}
Gravity	g	L/T^2
Area	A	L^2
Flow rate	Q	L^3/T
Mass flux	$\dot{m}$	M/T
Pressure	p	M/LT^2
Stress	τ	M/LT^2
Density	ρ	M/L^3
Specific weight	γ	M/L^2T^2
Viscosity	μ	M/LT
Kinematic viscosity	ν	L^2/T
Work	W	ML^2/T^2
Power, heat flux	$\dot{W}, \dot{Q}$	ML^2/T^3
Surface tension	σ	M/T^2
Bulk modulus	B	M/LT^2

6.2.3 Buckingham π-Theorem

- In any problem, a dependent variable x_1 is expressed in terms of independent variables, i.e., $x_1 = f(x_2, x_3, x_4, \ldots, x_n)$ *[n: Number of variables]*

π-Terms

- The Buckingham *π* –theorem states that (*n*-*m*) dimensionless groups of variables, called *π* –terms, can be related by

$$\Pi_L = f_1(\Pi_1, \Pi_2, \ldots \Pi_{n-m})$$

  - ***m*: Number of basic dimensions included in the variables.** $\begin{pmatrix} M & L & T \\ F & L & T \end{pmatrix}$
  - **π_1: includes the dependent variable; remaining *π*-terms include only independent variables.**

- For a successful dimensional analysis, a dimension must occur at least twice or not at all.

Example: The flow rate Q in an open channel depends on the hydraulic radius R, the cross-sectional area A, the wall roughness height e, gravity g, and the slope S. Relate Q to the other variables using

(a) the M-L-T system

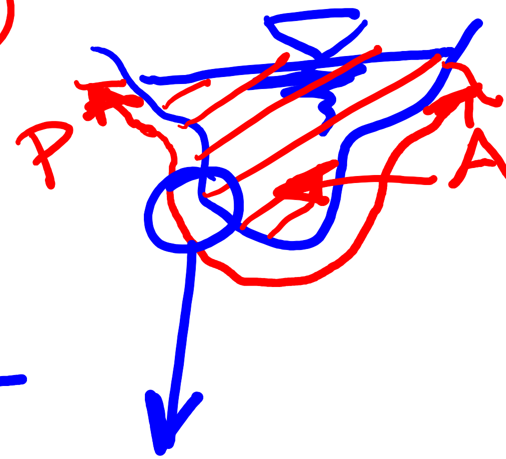
(b) the F-L-T system.

$n = 6$

$$R = \frac{A}{P}$$

P: Wetted perimeter

A: hydraulic area



$$Q = f(R, A, e, g, S)$$

$$[Q] = \frac{L^3}{T}, \quad [R] = \frac{L^2}{L} = L$$

$$[A] = L^2, \quad [e] = L, \quad [g] = \frac{L}{T^2}$$

$$[S] = 1$$

basic dimensions = 2 (L, T) [m=2]

$$\Pi_1 = f_1(\Pi_2, \Pi_3, \dots \Pi_{\underset{n-m}{4}})$$

$\Pi_1 = Q R^{a_1} g^{b_1}$

$\Pi_2 = A R^{a_2} g^{b_2}$

$\Pi_3 = e R^{a_3} g^{b_3}$

$\Pi_4 = S R^{a_4} g^{b_4}$

Repeating variables

$[g], [R]$

Π_1 $L^3 T^{-1} L^{a_1} (L T^{-2})^{b_1} = 0$

$3 + a_1 + b_1 = 0$

$-1 - 2b_1 = 0 \rightarrow b_1 = \frac{-1}{2}$

$a_1 = -5/2$

Π_2 $L^2 L^{a_2} (L T^{-2})^{b_2} = 0$

$2 + a_2 + b_2 = 0$

$-2b_2 = 0 \rightarrow \boxed{b_2 = 0}$

$a_2 = -2$

Π_3

$L L^{a_3} (L T^{-2})^{b_3} = 0$

$\boxed{b_3 = 0}$

$a_3 = -1$

Π_4

$$L^{a_4} + (LT^{-2})^{b_4} = 0$$

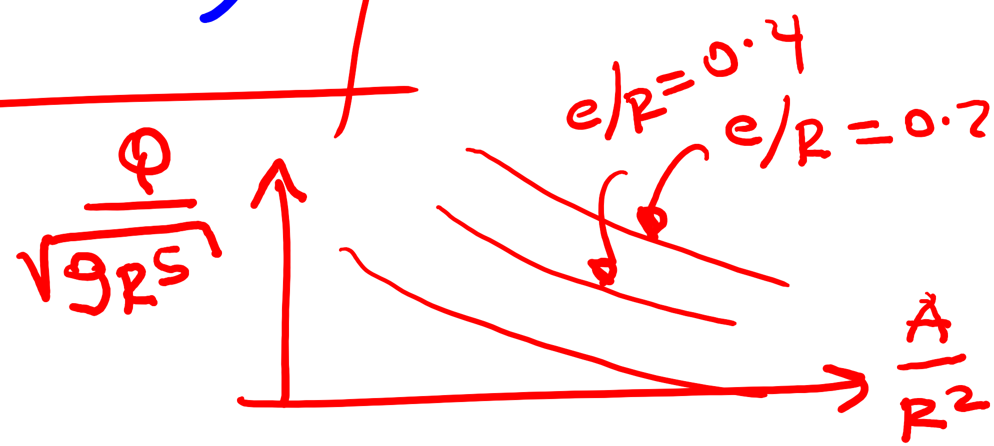
$$b_4 = 0$$

$$a_4 = 0$$

*

$$\Pi_1 = \frac{Q}{\sqrt{R^5 g}}, \quad \Pi_2 = \frac{A}{R^2}, \quad \Pi_3 = \frac{e}{R}, \quad \Pi_4 = S$$

$$\boxed{\frac{Q}{\sqrt{gR^5}} = f\left(\frac{A}{R^2}, \frac{e}{R}, S\right)}$$



6.2.4 Common Dimensionless Parameters

- Each dimensionless number can be written as a ratio of two forces.

$\rho = \frac{m}{\forall} \rightarrow m = \rho L^3$

$$F_P = \text{pressure force} = \Delta p A \sim \Delta p l^2$$

$$F_I = \text{inertial force} = mV\frac{dV}{ds} \sim \rho l^3 V \frac{V}{l} = \rho l^2 V^2$$

$$F_\mu = \text{viscous force} = \tau A = \mu \frac{du}{dy} A \sim \mu \frac{V}{l} l^2 = \mu l V$$

$$F_g = \text{gravity force} = mg \sim \rho l^3 g$$

$$F_B = \text{compressibility force} = BA \sim \rho \frac{dp}{d\rho} l^2 = \rho c^2 l^2$$

$$F_\omega = \text{centrifugal force} = mr\omega^2 \sim \rho l^3 l \omega^2 = \rho l^4 \omega^2$$

$$F_\sigma = \text{surface tension force} = \sigma l$$

Euler number

$$\mathrm{Eu} \propto \frac{\text{pressure force}}{\text{inertial force}}$$

Reynolds number

$$\mathrm{Re} \propto \frac{\text{inertial force}}{\text{viscous force}}$$

Froude number

$$\mathrm{Fr} \propto \frac{\text{inertial force}}{\text{gravity force}}$$

Mach

$$\mathrm{M} \propto \frac{\text{inertial force}}{\text{compressibility force}}$$

Strouhal

$$\mathrm{St} \propto \frac{\text{centrifugal force}}{\text{inertial force}}$$

Weber number

$$\mathrm{We} \propto \frac{\text{inertial force}}{\text{surface tension force}}$$

6.2.4 Common Dimensionless Parameters (Cont.)

Table 6.2 Common Dimensionless Parameters in Fluid Mechanics

Parameter	*Expression*	*Flow situations where parameter is important*
Euler number	$\dfrac{\Delta p}{\rho V^2}$	Flows in which pressure drop is significant: most flow situations
Reynolds number (pipe flows)	$\dfrac{\rho l V}{\mu}$	Flows that are influenced by viscous effects: internal flows, boundary layer flows
Froude number (open channel)	$\dfrac{V}{\sqrt{lg}}$	Flows that are influenced by gravity: primarily free surface flows
Mach number	$\dfrac{V}{c}$	Compressibility is important in these flows, usually if $V > 0.3\ c$
Strouhal number	$\dfrac{l\omega}{V}$	Flow with an unsteady component that repeats itself periodically
Weber number (bubble flows) (multiphase flows)	$\dfrac{V^2 l\rho}{\sigma}$	Surface tension influences the flow; flow with an interface may be such a flow

6.3 Similitude

6.3.1 General Information

- Study of predicting prototype conditions from model observations.

- If a model study has to be performed:
  - Need a quantity measured on the model (subscript *m*) to predict an associated quantity on the prototype (subscript *p*).
  - This needs **dynamic similarity between the model and prototype.**
  - Forces which act on corresponding masses in the model flow and prototype flow are in the same ratio throughout the entire flow field.

6.3.1 General Information (Cont.)

- If inertial forces, pressure forces, viscous forces, and gravity forces are present:

$$\boxed{\frac{(F_I)_m}{(F_I)_p} = \frac{(F_p)_m}{(F_p)_p}} = \frac{(F_\mu)_m}{(F_\mu)_p} = \frac{(F_g)_m}{(F_g)_p}$$

Due to dynamic similarity at corresponding points in the flow fields.

These can be rearranged as

$$\frac{(F_I)_m}{(F_p)_m} = \frac{(F_I)_p}{(F_p)_p} \qquad \frac{(F_I)_m}{(F_\mu)_m} = \frac{(F_I)_p}{(F_\mu)_p} \qquad \frac{(F_I)_m}{F_{g_m}} = \frac{(F_I)_p}{(F_g)_p}$$

$$\boxed{Eu_m = Eu_p} \qquad \boxed{Re_m = Re_p} \qquad \boxed{Fr_m = Fr_p}$$

Euler number Reynolds

Froude

6.3.1 General Information (Cont.)

- **Kinematic Similarity**: Velocity ratio is a constant between all corresponding points in the flow fields.
  - **Streamline pattern around the model is the same as that around the prototype except for a scale factor.**

$$\frac{V_m}{V_p} = \text{constant}$$

- **Geometric Similarity**: Length ratio is a constant between all corresponding points in the flow fields.
  - **Model has the same shape as the prototype.**

$$\frac{L_m}{L_p} = \text{constant}$$

6.3.1 General Information (Cont.)

To ensure complete similarity between model and prototype:

- Geometric similarity must be satisfied. $L_m/L_p = \text{constant}$
- Mass ratio of corresponding fluid elements is a constant.
- Dimensionless numbers in model and prototype should be equal.

Open-channel flows

$$\text{Euler number, Eu} = \frac{\Delta p}{\rho V^2}$$

{Pipe flows}

$$\text{Reynolds number, Re} = \frac{V\rho l}{\mu}$$

$$\text{Froude number}^2\text{, Fr} = \frac{V}{\sqrt{lg}}$$

$$\text{Mach number, M} = \frac{V}{c}$$

$$\text{Strouhal number}^2\text{, St} = \frac{l\omega}{V}$$

$$\text{Weber number}^2\text{, We} = \frac{V^2 l\rho}{\sigma}$$

6.3.2 Confined Flows

- A confined flow is a flow that has no free surface (liquid-gas surface) or interface (two different liquids).
- Can only move within a specific region (e.g., internal flows in pipes).

- Isn't influenced by gravity or surface tension.
- Dominant effect is viscosity in incompressible confined flows.
- Relevant forces are pressure, inertial, and viscous forces.
  - Dynamic similarity is obtained if the ratios between the model and the prototype are the same.
- **Hence, only the Reynolds number is the dominant dimensionless parameter.**
  - **If compressibility effects are significant, Mach number would become important.**

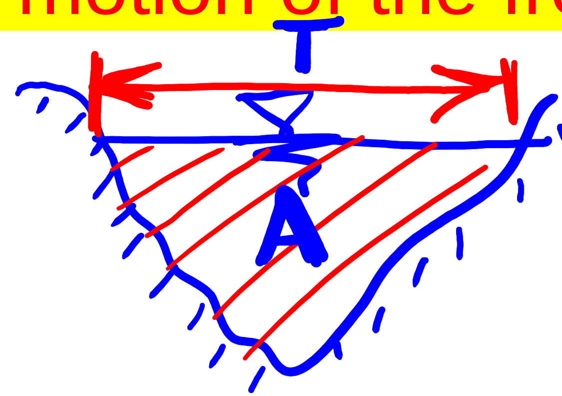
$Re = \frac{V \cdot D}{\nu}$

$\boxed{Re_m = Re_p}$

6.3.3 Free-Surface Flows (Open-channel flows)

- A free-surface flow is a flow where part of the boundary involves a pressure boundary condition.
    - E.g., Flows over weirs and dams, flows in channels, flows with two fluids separated by an interface, etc.

- Gravity controls the location and motion of the free surface.
- Viscous effects are significant



- **Requires the Froude number similitude**

T: Length of free surface width

$$\boxed{Fr_m = Fr_p}$$

$$Fr = \frac{V}{\sqrt{gA/T}}$$

Example: A 1:5 scale model of a large pump is used to test a proposed change. The prototype pump produces a pressure rise of 600 kPa at a mass flux of 800 kg/s. Determine the mass flux to be used in the model and the expected pressure rise.

(a) Water at the same temperature is used in both model and prototype.

(b) The water in the model study is at 30°C and the water in the prototype is at 15°C.

$\frac{L_p}{L_m} = 5$

$\Delta p_p = 600$ kPa

$\dot{m}_p = 800$ kg/s

$\dot{m}_m = ?$

$\Delta P_m = ?$

a) $Temp_m = Temp_p$ $(\nu_m = \nu_p)$

Pipe flow: $Re_m = Re_p$

$$\frac{V_m \cdot L_m}{\nu_m} = \frac{V_p \cdot L_p}{\nu_p} \rightarrow \frac{V_m}{V_p} = \cancel{\left(\frac{\nu_m}{\nu_p}\right)} \frac{L_p}{L_m}$$

1

$$\frac{V_m}{V_p} = 5 \quad \dots (1)$$

Mass flux $= \rho L^2 \cdot V$

$$\frac{\dot{m}_m}{\dot{m}_p} = \frac{\rho_m L_m^2 V_m}{\rho_p L_p^2 V_p}$$

(1)

$$\frac{\dot{m}_m}{800\ \text{kg/s}} = \left(\frac{1}{5}\right)^2 \cdot 5 \Rightarrow \boxed{\dot{m}_m = 160\ \text{kg/s}}$$

$$* \ \Delta P_m \sim \rho_m V_m^2 \qquad \left[F \sim \rho L^2 V^2 \right.$$

$$\Delta P_p \sim \rho_p V_p^2 \qquad \left. P \sim \rho V^2 \right]$$

(1)

$$\frac{\Delta P_m}{\Delta P_p} = \cancel{\frac{\rho_m}{\rho_p}} \left(\frac{V_m}{V_p}\right)^2 \rightarrow \Delta P_m = \Delta P_p \left[5^2\right]$$

$$\Delta P_m = 600 \times 25$$

$$= \underline{15{,}000\ \text{kpa}}$$

b) $\nu_m = 0.8 \times 10^{-6}\ m^2/s$

$\nu_p = 1.14 \times 10^{-6}\ m^2/s$

$$Re_m = Re_p$$

$$\frac{V_m \cdot L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$$

$$\frac{V_m}{V_p} = \frac{\nu_m}{\nu_p} \cdot \frac{L_p}{L_m}$$

$$\frac{V_m}{V_p} = \frac{\nu_m}{\nu_p} \cdot \left(\frac{L_p}{L_m}\right)$$

$$\frac{V_m}{V_p} = \frac{0.8}{1.14} \times 5 = 3.51$$

$$\frac{\dot{m}_m}{\dot{m}_p} = \frac{\rho_m L_m^2 V_m}{\rho_p L_p^2 V_p} = \left(\frac{1}{5}\right)^2 \cdot 3.51$$

1

$$\dot{m}_m = 800 \times \frac{1}{25} \times 3.51 = 112.3\ kg/s$$

$$* \quad \frac{\Delta P_m}{\Delta P_p} = \frac{\rho_m V_m^2}{\rho_p V_p^2} = 3.51^2$$

1

$$\Delta P_m = 600\ kPa \times 3.51^2 = \underline{7392\ kPa}$$