For Cartesian tensor analysis:

"Introduction to Continuum Mechanics" by Lai, Rubin, and Krempl, Pergamon Press

Introduction to the Mechanics of a Continuous Medium, by Malvern, Prentice Hall

For Curvilinear Tensor Analysis:

"Tensor Analysis Theory and Application to Geometry and Mechanics of Continua" by Sokolnikoff, Wiley. (This one is out of print I think)

"Foundations of Solid Mechanics" by Fung, Prentice Hall

The Equations of Elasticity in Curvilinear Coordinates

1. Equations of Motion (Eqns. of Equilibrium)

Cartesian: $\dfrac{\partial \sigma_{ij}^{©}}{\partial x_j} + F_i^{©} = \rho a_i^{©}$

$$\boxed{\sigma_{ij}^{©}|_j + F_i^{©} = \rho a_i^{©}}$$

NOTE:

Moment Equilibrium in cartesian coordinates gave us $\sigma_{ij}^{©} = \sigma_{ji}^{©}$

or

$\sigma_{ij}(x) = \sigma_{ji}(x)$

$\sigma^{ij}(x) = \sigma^{ji}(x)$

etc...

$$\boxed{\begin{aligned}\sigma_{ij}(\theta) &= \sigma_{ji}(\theta)\\ \sigma^{ij}(\theta) &= \sigma^{ji}(\theta)\end{aligned}}$$

$$\boxed{\hat{\sigma}_{ij} = \hat{\sigma}_{ji}}$$

$$\boxed{\begin{aligned}
\sigma^{ij}|_j(x) + F^i(x) &= \rho a^i(x)\\
\sigma_i{}^j|_j(x) + F_i(x) &= \rho a_i(x)\\
\sigma_{ij}|^j(x) + F_i(x) &= \rho a_i(x)\\
\sigma^i{}_j|^j(x) + F^i(x) &= \rho a^i(x)
\end{aligned}}$$

$$\left.\begin{aligned}
\sigma^{ij}\big|_j(\theta) + F^i(\theta) &= \rho a^i(\theta) \\
\sigma_i{}^j\big|_j(\theta) + F_i(\theta) &= \rho a_i(\theta)
\end{aligned}\right\}$$

Either of these can be used

$$\left.\begin{aligned}
\sigma_{ij}\big|^j(\theta) + F_i(\theta) &= \rho a_i(\theta) \\
\sigma^i{}_j\big|^j(\theta) + F^i(\theta) &= \rho a^i(\theta)
\end{aligned}\right\}$$

Involve "contravariant" derivatives so not useful.

$$\boxed{\sigma^{ij}\big|_j + F^i = \rho a^i}$$

Form most often used

$$\boxed{\sigma_i{}^j\big|_j + F_i = \rho a_i}$$

most general

$$\boxed{\underbrace{\frac{\partial \sigma^{ij}}{\partial \theta^j}}_{\text{3 terms}} + \underbrace{\Gamma^i_{kj}\,\sigma^{kj}}_{\text{9 terms}} + \underbrace{\Gamma^j_{kj}\,\sigma^{ik}}_{\text{9 terms}} + F^i = \rho a^i}$$

$\boxed{i=1}$

$$\left(\frac{\partial\sigma^{11}}{\partial\theta^1} + \frac{\partial\sigma^{12}}{\partial\theta^2} + \frac{\partial\sigma^{13}}{\partial\theta^3}\right) +$$

$$\Big(\Gamma^1_{11}\sigma^{11} + \Gamma^1_{12}\sigma^{12} + \Gamma^1_{13}\sigma^{13} + \Gamma^1_{21}\sigma^{21} + \Gamma^1_{22}\sigma^{22} + \Gamma^1_{23}\sigma^{23}$$
$$+ \Gamma^1_{31}\sigma^{31} + \Gamma^1_{32}\sigma^{32} + \Gamma^1_{33}\sigma^{33}\Big) +$$

$$\Big(\left[\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right]\sigma^{11} + \left[\Gamma^1_{21} + \Gamma^2_{22} + \Gamma^3_{23}\right]\sigma^{12}$$
$$+ \left[\Gamma^1_{31} + \Gamma^2_{32} + \Gamma^3_{33}\right]\sigma^{13}\Big) + F^1 = \rho a^1$$

etc... for $i = 2, 3$

Example – Cylindrical Coordinates

$\theta^1 = r$ $F^1 = F^r$ $a^1 = a^r$ $\sigma^{11} = \sigma^{rr}$

$\theta^2 = \theta$ $F^2 = F^\theta$ $a^2 = a^\theta$ $\sigma^{12} = \sigma^{r\theta}$

$\theta^3 = z$ $F^3 = F^z$ $a^3 = a^z$ etc...

$\Gamma^1_{22} = -r$ $h_1 = 1$

$\Gamma^2_{12} = 1/r$ $h_2 = r$

$\Gamma^2_{21} = 1/r$ $h_3 = 1$

all other Γ's $= 0$

$\boxed{i=1}$

$$\boxed{\frac{\partial \sigma^{rr}}{\partial r} + \frac{\partial \sigma^{r\theta}}{\partial \theta} + \frac{\partial \sigma^{rz}}{\partial z} + \left(-r\sigma^{\theta\theta}\right) + \left(\frac{1}{r}\sigma^{rr}\right) + F^{r} = \rho a^{r}}$$

If we now introduce Physical Components

$$\left.\begin{aligned} \hat{F}_{i} &= h_i F^{i} \quad \text{(no sum)} \\ \hat{a}_{i} &= h_i a^{i} \quad \text{(no sum)} \\ \hat{\sigma}_{ij} &= h_i h_j \sigma^{ij} \quad \text{(no sums)} \end{aligned}\right\}$$

These relations true for any curvilinear coordinate system (not just orthogonal ones)

$$\longrightarrow \quad \hat{F}_r = F^r$$

$$\hat{a}_r = a^r$$

$$\hat{\sigma}_{rr} = \sigma^{rr}$$

$$\hat{\sigma}_{r\theta} = r\,\sigma^{r\theta} \quad \longrightarrow \quad \sigma^{r\theta} = \frac{\hat{\sigma}_{r\theta}}{r}$$

$$\hat{\sigma}_{rz} = \sigma^{rz}$$

$$\hat{\sigma}_{\theta\theta} = r^2\sigma^{\theta\theta} \quad \longrightarrow \quad \sigma^{\theta\theta} = \frac{\hat{\sigma}_{\theta\theta}}{r^2}$$

$$\longrightarrow \quad \frac{\partial \hat{\sigma}_{rr}}{\partial r} + \frac{\partial}{\partial \theta}\left(\frac{\hat{\sigma}_{r\theta}}{r}\right) + \frac{\partial \hat{\sigma}_{rz}}{\partial z} - r\left(\frac{\hat{\sigma}_{\theta\theta}}{r^2}\right) + \frac{\hat{\sigma}_{rr}}{r} + \hat{F}_r = \rho \hat{a}_r$$

$$\boxed{\frac{\partial \hat{\sigma}_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \hat{\sigma}_{r\theta}}{\partial \theta} + \frac{\partial \hat{\sigma}_{rz}}{\partial z} + \frac{\hat{\sigma}_{rr} - \hat{\sigma}_{\theta\theta}}{r} + \hat{F}_r = \rho \hat{a}_r}$$

With similar calculations, the other two equations ($i = 2,3$) become the following (in physical components):

$$\boxed{\begin{gathered}\frac{\partial \hat{\sigma}_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \hat{\sigma}_{\theta\theta}}{\partial \theta} + \frac{\partial \hat{\sigma}_{\theta z}}{\partial z} + \frac{2}{r}\hat{\sigma}_{\theta r} + \hat{F}_\theta = \rho \hat{a}_\theta \\ \frac{\partial \hat{\sigma}_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \hat{\sigma}_{\theta z}}{\partial \theta} + \frac{\partial \hat{\sigma}_{zz}}{\partial z} + \frac{\hat{\sigma}_{rz}}{r} + \hat{F}_z = \rho \hat{a}_z\end{gathered}}$$

<u>NOTE</u>: At this point, most books drop the "hats" on the physical components, and just forget about covariant, contravariant, etc... components.

just call them stresses at this point

$\boxed{2.}$ Strain - Displacement Relations → Small disp form

<u>Cartesian</u>: $\epsilon_{ij}^{©} = \frac{1}{2}\left(\frac{\partial u_i^{©}}{\partial x_j} + \frac{\partial u_j^{©}}{\partial x_i}\right)$

$$\boxed{\epsilon_{ij}^{©} = \frac{1}{2}\left(u_{i|j}^{©} + u_{j|i}^{©} \right)}$$

Since $\epsilon_{ij}^{©} = \epsilon_{ji}^{©}$
we can show

$$\boxed{\begin{aligned} \epsilon_{ij} &= \epsilon_{ji} \\ \epsilon^{ij} &= \epsilon^{ji} \end{aligned}}$$

$$\begin{aligned}
\epsilon_{ij}(x) &= \frac{1}{2}\left(u_i|_j(x) + u_j|_i(x) \right) \\
\epsilon^{ij}(x) &= \frac{1}{2}\left(u^{i|j}(x) + u^{j|i}(x) \right) \\
\epsilon^{i}{}_{j}(x) &= \frac{1}{2}\left(u^i|_j(x) + u_j|^i(x) \right) \\
\epsilon_{i}{}^{j}(x) &= \frac{1}{2}\left(u_i|^j(x) + u^j|_i(x) \right)
\end{aligned}$$

$$\begin{aligned}
\epsilon_{ij}(\theta) &= \frac{1}{2}\left(u_i|_j(\theta) + u_j|_i(\theta) \right) \\
\epsilon^{ij}(\theta) &= \frac{1}{2}\left(u^{i|j}(\theta) + u^{j|i}(\theta) \right) \\
\epsilon^{i}{}_{j}(\theta) &= \frac{1}{2}\left(u^i|_j(\theta) + u_j|^i(\theta) \right) \\
\epsilon_{i}{}^{j}(\theta) &= \frac{1}{2}\left(u_i|^j(\theta) + u^j|_i(\theta) \right)
\end{aligned}$$

[Only equation which doesn't involve a <u>contravariant derivative</u>. Thus, we use this one.]

$$\boxed{\epsilon_{ij} = \frac{1}{2}\left[u_{i|j} + u_{j|i} \right]}$$

not necessarily cartesian

$$\epsilon_{ij} = \frac{1}{2}\left[\left(\frac{\partial u_i}{\partial \theta^j} - \Gamma_{ij}^{k} u_k\right) + \left(\frac{\partial u_j}{\partial \theta^i} - \Gamma_{ji}^{k} u_k\right)\right]$$

<u>but</u> $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$

$$\rightarrow \boxed{\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial \theta^j} + \frac{\partial u_j}{\partial \theta^i}\right] - \Gamma_{ij}^{k} u_k}$$

<u>9 equations, but only 6 are unique since $\epsilon_{ij} = \epsilon_{ji}$</u>

$\boxed{i=1,\ j=1}$

$$\epsilon_{11} = \frac{\partial u_1}{\partial \theta^1} - \left(\Gamma_{11}^{1} u_1 + \Gamma_{11}^{2} u_2 + \Gamma_{11}^{3} u_3\right)$$

$\boxed{i=1,\ j=2}$

$$\epsilon_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial \theta^2} + \frac{\partial u_2}{\partial \theta^1}\right] - \left(\Gamma_{12}^{1} u_1 + \Gamma_{12}^{2} u_2 + \Gamma_{12}^{3} u_3\right)$$

<u>etc...</u>

Example - Cylindrical Coordinates

$$\theta^1 = r \qquad u_1 = u_r \qquad \epsilon_{11} = \epsilon_{rr}$$
$$\theta^2 = \theta \qquad u_2 = u_\theta \qquad \epsilon_{12} = \epsilon_{r\theta}$$
$$\theta^3 = z \qquad u_3 = u_z \qquad \text{etc...}$$

$$\Gamma^1_{22} = -r \qquad h_1 = 1$$
$$\Gamma^2_{12} = 1/r \qquad h_2 = r$$
$$\Gamma^2_{21} = 1/r \qquad h_3 = 1$$

all other Γ's $= 0$

$\boxed{i = j = 1}$

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} - (0 + 0 + 0)$$

$$\boxed{\epsilon_{rr} = \frac{\partial u_r}{\partial r}}$$

$\boxed{i = 1,\ j = 2}$

$$\boxed{\epsilon_{r\theta} = \frac{1}{2}\left[\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r}\right] - \left(\frac{1}{r} u_\theta\right)}$$

etc...

If we now introduce Physical Components:

$$\left.\begin{aligned} \hat{u}_i &= \frac{u_i}{h_i} \quad \text{(no sum)} \\ \hat{\epsilon}_{ij} &= \frac{\epsilon_{ij}}{h_i h_j} \quad \text{(no sums)} \end{aligned}\right\} \text{True only for orthogonal curvilinear coordinates}$$

$$\rightarrow \quad \hat{u}_r = u_r$$

$$\hat{u}_\theta = \frac{u_\theta}{r} \rightarrow u_\theta = r\hat{u}_\theta$$

$$\hat{\epsilon}_{rr} = \epsilon_{rr}$$

$$\hat{\epsilon}_{r\theta} = \frac{\epsilon_{r\theta}}{r} \rightarrow \epsilon_{r\theta} = r\hat{\epsilon}_{r\theta}$$

$\boxed{i=j=1}$

$$\boxed{\hat{\epsilon}_{rr} = \frac{\partial \hat{u}_r}{\partial r}}$$

$\boxed{i=1,\ j=2}$

$$r\hat{\epsilon}_{r\theta} = \frac{1}{2}\left[\frac{\partial \hat{u}_r}{\partial \theta} + \frac{\partial}{\partial r}(r\hat{u}_\theta)\right] - \frac{1}{r}(r\hat{u}_\theta)$$

$$r\hat{\epsilon}_{r\theta} = \frac{1}{2}\left[\frac{\partial \hat{u}_r}{\partial \theta} + \hat{u}_\theta + r\frac{\partial \hat{u}_\theta}{\partial r}\right] - \hat{u}_\theta$$

$$\boxed{\hat{\epsilon}_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial \hat{u}_r}{\partial \theta} + \frac{\partial \hat{u}_\theta}{\partial r} - \frac{\hat{u}_\theta}{r}\right]}$$

$$\gamma_{r\theta} = 2\hat{\epsilon}_{r\theta} = \frac{1}{r}\frac{\partial \hat{u}_r}{\partial \theta} + \frac{\partial \hat{u}_\theta}{\partial r} - \frac{\hat{u}_\theta}{r}$$

Christoffel

With similar calculations, the other four unique equations become the following (in physical components):

$\boxed{i=j=2}$
$$\hat{\epsilon}_{\theta\theta} = \frac{1}{r}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{\hat{u}_r}{r}$$

$\boxed{i=j=3}$
$$\hat{\epsilon}_{zz} = \frac{\partial \hat{u}_z}{\partial z}$$

$\boxed{i=2,\ j=3}$
$$\hat{\epsilon}_{\theta z} = \frac{1}{2}\left(\frac{\partial \hat{u}_\theta}{\partial z} + \frac{1}{r}\frac{\partial \hat{u}_z}{\partial \theta}\right)$$

$\boxed{i=1,\ j=3}$
$$\hat{\epsilon}_{rz} = \frac{1}{2}\left(\frac{\partial \hat{u}_r}{\partial z} + \frac{\partial \hat{u}_z}{\partial r}\right)$$

$\boxed{3.}$ Stress-Strain Relations (Hooke's Law)

Cartesian:

$$\boxed{\epsilon^{\copyright}_{ij} = \left(\frac{1+\nu}{E}\right)\sigma^{\copyright}_{ij} - \frac{\nu}{E}\,\sigma^{\copyright}_{kk}\,\cancelto{g^{\copyright}_{ij}}{\delta_{ij}}}$$

$$\epsilon_{ij}(x) = \left(\frac{1+\nu}{E}\right)\sigma_{ij}(x) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(x)\right) g_{ij}(x)$$

$$\epsilon^{ij}(x) = \left(\frac{1+\nu}{E}\right)\sigma^{ij}(x) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(x)\right) g^{ij}(x)$$

$$\epsilon^{i}_{\cdot j}(x) = \left(\frac{1+\nu}{E}\right)\sigma^{i}_{\cdot j}(x) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(x)\right) g^{i}_{\cdot j}(x)$$

$$\epsilon_{i\cdot}^{\ \cdot j}(x) = \left(\frac{1+\nu}{E}\right)\sigma_{i\cdot}^{\ \cdot j}(x) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(x)\right) g_{i\cdot}^{\ \cdot j}(x)$$

[All of these are acceptable for evaluation purposes]

$$\epsilon_{ij}(\theta) = \left(\frac{1+\nu}{E}\right)\sigma_{ij}(\theta) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(\theta)\right) g_{ij}(\theta)$$

$$\epsilon^{ij}(\theta) = \left(\frac{1+\nu}{E}\right)\sigma^{ij}(\theta) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(\theta)\right) g^{ij}(\theta)$$

$$\epsilon^{i}_{\cdot j}(\theta) = \left(\frac{1+\nu}{E}\right)\sigma^{i}_{\cdot j}(\theta) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(\theta)\right) \cancelto{\delta^{i}_{j}}{g^{i}_{\cdot j}(\theta)}$$

$$\epsilon_{i\cdot}^{\ \cdot j}(\theta) = \left(\frac{1+\nu}{E}\right)\sigma_{i\cdot}^{\ \cdot j}(\theta) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(\theta)\right) \cancelto{\delta_{i}^{j}}{g_{i\cdot}^{\ \cdot j}}$$

Lets choose

$$\boxed{\epsilon^{i}_{\cdot j} = \left(\frac{1+\nu}{E}\right)\sigma^{i}_{\cdot j} - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}\right)\delta^{i}_{j}}$$

$\boxed{i = j = 1}$

$$\epsilon^{1}_{\cdot 1} = \left(\frac{1+\nu}{E}\right)\sigma^{1}_{\cdot 1} - \frac{\nu}{E}\left(\sigma^{1}_{\cdot 1} + \sigma^{2}_{\cdot 2} + \sigma^{3}_{\cdot 3}\right)$$

$$\boxed{\epsilon^{1}_{\cdot 1} = \frac{1}{E}\left(\sigma^{1}_{\cdot 1} - \nu\,\sigma^{2}_{\cdot 2} - \nu\,\sigma^{3}_{\cdot 3}\right)}$$

$\boxed{i = 1,\ j = 2}$

$$\epsilon^{1}_{\cdot 2} = \frac{1+\nu}{E}\,\sigma^{1}_{\cdot 2}$$

$$\boxed{\epsilon^{1}_{\cdot 2} = \frac{1}{2G}\,\sigma^{1}_{\cdot 2}}$$

etc...

Example – Cylindrical Coordinates

$\theta^1 = r$	$\sigma^{1}_{\cdot 1} = \sigma^{r}_{\cdot r}$	$\epsilon^{1}_{\cdot 1} = \epsilon^{r}_{\cdot r}$	$h_1 = 1$
$\theta^2 = \theta$	$\sigma^{1}_{\cdot 2} = \sigma^{r}_{\cdot \theta}$	$\epsilon^{1}_{\cdot 2} = \epsilon^{r}_{\cdot \theta}$	$h_2 = r$
$\theta^3 = z$	etc...	etc...	$h_3 = 1$

$\boxed{i = j = 1}$

$$\boxed{\epsilon^{r}_{\cdot r} = \frac{1}{E}\left(\sigma^{r}_{\cdot r} - \nu\,\sigma^{\theta}_{\cdot \theta} - \nu\,\sigma^{z}_{\cdot z}\right)}$$

$\boxed{i = 1,\ j = 2}$

$$\boxed{\epsilon^{r}_{\cdot \theta} = \frac{1}{2G}\,\sigma^{r}_{\cdot \theta}}$$

etc...

Physical Components

$$\hat{\sigma}_{ij} = \frac{h_i}{h_j}\,\sigma^{i}_{\cdot j} \qquad \hat{\epsilon}_{ij} = \frac{h_i}{h_j}\,\epsilon^{i}_{\cdot j}$$

Valid for Orthogonal Curvilinear Systems Only

$$\rightarrow \quad \hat{\sigma}_{rr} = \sigma^{r}_{\cdot r} \qquad \hat{\epsilon}_{rr} = \epsilon^{r}_{\cdot r}$$
$$\hat{\sigma}_{\theta\theta} = \sigma^{\theta}_{\cdot \theta} \qquad \hat{\epsilon}_{\theta\theta} = \epsilon^{\theta}_{\cdot \theta}$$
$$\hat{\sigma}_{zz} = \sigma^{z}_{\cdot z} \qquad \hat{\epsilon}_{zz} = \epsilon^{z}_{\cdot z}$$

$$\hat{\sigma}_{r\theta} = \frac{1}{r}\sigma^{r}_{\cdot \theta} \qquad \hat{\epsilon}_{r\theta} = \frac{1}{r}\epsilon^{r}_{\cdot \theta}$$

$\boxed{i = j = 1}$

$$\boxed{\hat{\epsilon}_{rr} = \frac{1}{E}\left(\hat{\sigma}_{rr} - \nu\hat{\sigma}_{\theta\theta} - \nu\hat{\sigma}_{zz}\right)}$$

$\boxed{i = 1,\ j = 2}$

$$\cancel{r}\,\hat{\epsilon}_{r\theta} = \frac{1}{2G}\,\cancel{r}\,\hat{\sigma}_{r\theta}$$

$$\boxed{\hat{\epsilon}_{r\theta} = \frac{1}{2G}\hat{\sigma}_{r\theta}}$$

Similarly:

$$\boxed{\begin{aligned}
\hat{\epsilon}_{\theta\theta} &= \frac{1}{E}\left(\hat{\sigma}_{\theta\theta} - \nu\hat{\sigma}_{rr} - \nu\hat{\sigma}_{zz}\right)\\
\hat{\epsilon}_{zz} &= \frac{1}{E}\left(\hat{\sigma}_{zz} - \nu\hat{\sigma}_{rr} - \nu\hat{\sigma}_{\theta\theta}\right)\\
\hat{\epsilon}_{\theta z} &= \frac{1}{2G}\hat{\sigma}_{\theta z}\\
\hat{\epsilon}_{rz} &= \frac{1}{2G}\hat{\sigma}_{rz}
\end{aligned}}$$

The Equations of Elasticity in Curvilinear Coordinates

$\boxed{1.}$ Equations of Motion (Eqns. of Equilibrium)

Cartesian: $\dfrac{\partial \sigma_{ij}^{©}}{\partial x_j} + F_i^{©} = \rho a_i^{©}$

NOTE:

Moment Equilibrium in cartesian coordinates gave us $\sigma_{ij}^{©} = \sigma_{ji}^{©}$

or

$\sigma_{ij}(x) = \sigma_{ji}(x)$

$\sigma^{ij}(x) = \sigma^{ji}(x)$

etc...

$$\boxed{\begin{array}{l} \sigma_{ij}(\theta) = \sigma_{ji}(\theta) \\ \sigma^{ij}(\theta) = \sigma^{ji}(\theta) \end{array}}$$

$$\boxed{\hat{\sigma}_{ij} = \hat{\sigma}_{ji}}$$

$$\boxed{\sigma_{ij}^{©}|_j + F_i^{©} = \rho a_i^{©}}$$

$$\boxed{\begin{aligned} \sigma^{ij}|_j(x) + F^i(x) &= \rho a^i(x) \\ \sigma_i{}^j|_j(x) + F_i(x) &= \rho a_i(x) \\ \sigma_{ij}|^j(x) + F_i(x) &= \rho a_i(x) \\ \sigma^i{}_j|^j(x) + F^i(x) &= \rho a^i(x) \end{aligned}}$$

$$\sigma^{ij}\big|_j(\theta) + F^i(\theta) = \rho a^i(\theta)$$

$$\sigma_{i\cdot}^{\cdot j}\big|_j(\theta) + F_i(\theta) = \rho a_i(\theta)$$

Either of these can be used

$$\sigma_{ij}\big|^j(\theta) + F_i(\theta) = \rho a_i(\theta)$$

$$\sigma^{i}_{\cdot j}\big|^j(\theta) + F^i(\theta) = \rho a^i(\theta)$$

Involve "contravariant" derivatives so not useful.

$$\boxed{\sigma^{ij}\big|_j + F^i = \rho a^i}$$

Form most often used

$$\boxed{\sigma_{i\cdot}^{\cdot j}\big|_j + F_i = \rho a_i}$$

$$\boxed{\underbrace{\frac{\partial \sigma^{ij}}{\partial \theta^j}}_{\text{3 terms}} + \underbrace{\Gamma^i_{kj}\,\sigma^{kj}}_{\text{9 terms}} + \underbrace{\Gamma^j_{kj}\,\sigma^{ik}}_{\text{9 terms}} + F^i = \rho a^i}$$

$\boxed{i=1}$

$$\left(\frac{\partial\sigma^{11}}{\partial\theta^1} + \frac{\partial\sigma^{12}}{\partial\theta^2} + \frac{\partial\sigma^{13}}{\partial\theta^3}\right) +$$

$$\left(\Gamma^1_{11}\sigma^{11} + \Gamma^1_{12}\sigma^{12} + \Gamma^1_{13}\sigma^{13} + \Gamma^1_{21}\sigma^{21} + \underline{\Gamma^1_{22}\sigma^{22}} + \Gamma^1_{23}\sigma^{23} + \Gamma^1_{31}\sigma^{31} + \Gamma^1_{32}\sigma^{32} + \Gamma^1_{33}\sigma^{33}\right) +$$

$$\left(\left[\Gamma^1_{11} + \underline{\Gamma^2_{12}} + \Gamma^3_{13}\right]\sigma^{11} + \left[\Gamma^1_{21} + \Gamma^2_{22} + \Gamma^3_{23}\right]\sigma^{12} + \left[\Gamma^1_{31} + \Gamma^2_{32} + \Gamma^3_{33}\right]\sigma^{13}\right) + F^1 = \rho a^1$$

etc... for $i = 2,3$

Example – Cylindrical Coordinates

$\theta^1 = r$	$F^1 = F^r$	$a^1 = a^r$	$\sigma^{11} = \sigma^{rr}$
$\theta^2 = \theta$	$F^2 = F^\theta$	$a^2 = a^\theta$	$\sigma^{12} = \sigma^{r\theta}$
$\theta^3 = z$	$F^3 = F^z$	$a^3 = a^z$	etc...

$\Gamma^1_{22} = -r$ $\qquad h_1 = 1$

$\Gamma^2_{12} = 1/r$ $\qquad h_2 = r$

$\Gamma^2_{21} = 1/r$ $\qquad h_3 = 1$

all other Γ's $= 0$

$\boxed{i=1}$

$$\boxed{\begin{aligned}&\frac{\partial\sigma^{rr}}{\partial r}+\frac{\partial\sigma^{r\theta}}{\partial\theta}+\frac{\partial\sigma^{rz}}{\partial z}+\left(-r\sigma^{\theta\theta}\right)\\&\quad+\left(\frac{1}{r}\sigma^{rr}\right)+F^{r}=\rho a^{r}\end{aligned}}$$

<u>If we now introduce Physical Components</u>

$$\left.\begin{aligned}\hat{F}_{i}&=h_{i}F^{i}\quad(\text{no sum})\\\hat{a}_{i}&=h_{i}a^{i}\quad(\text{no sum})\\\hat{\sigma}_{ij}&=h_{i}h_{j}\,\sigma^{ij}\quad(\text{no sums})\end{aligned}\right\}$$ These relations true for any curvilinear coordinate system (not just orthogonal ones)

$$\longrightarrow\quad\hat{F}_r=F^r$$

$$\hat{a}_r=a^r$$

$$\hat{\sigma}_{rr}=\sigma^{rr}$$

$$\hat{\sigma}_{r\theta}=r\,\sigma^{r\theta}\quad\longrightarrow\quad\sigma^{r\theta}=\frac{\hat{\sigma}_{r\theta}}{r}$$

$$\hat{\sigma}_{rz}=\sigma^{rz}$$

$$\hat{\sigma}_{\theta\theta}=r^2\sigma^{\theta\theta}\quad\longrightarrow\quad\sigma^{\theta\theta}=\frac{\hat{\sigma}_{\theta\theta}}{r^2}$$

$$\longrightarrow\quad\frac{\partial\hat{\sigma}_{rr}}{\partial r}+\frac{\partial}{\partial\theta}\left(\frac{\hat{\sigma}_{r\theta}}{r}\right)+\frac{\partial\hat{\sigma}_{rz}}{\partial z}-r\left(\frac{\hat{\sigma}_{\theta\theta}}{r^2}\right)$$

$$+\frac{\hat{\sigma}_{rr}}{r}+\hat{F}_r=\rho\hat{a}_r$$

$$\boxed{\frac{\partial \hat{\sigma}_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \hat{\sigma}_{r\theta}}{\partial \theta} + \frac{\partial \hat{\sigma}_{rz}}{\partial z} + \frac{\hat{\sigma}_{rr} - \hat{\sigma}_{\theta\theta}}{r} + \hat{F}_r = \rho \hat{a}_r}$$

With similar calculations, the other two equations ($i = 2,3$) become the following (in physical components):

$$\boxed{\begin{aligned} &\frac{\partial \hat{\sigma}_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \hat{\sigma}_{\theta\theta}}{\partial \theta} + \frac{\partial \hat{\sigma}_{\theta z}}{\partial z} + \frac{2}{r}\hat{\sigma}_{\theta r} + \hat{F}_\theta = \rho \hat{a}_\theta \\ &\frac{\partial \hat{\sigma}_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \hat{\sigma}_{\theta z}}{\partial \theta} + \frac{\partial \hat{\sigma}_{zz}}{\partial z} + \frac{\hat{\sigma}_{rz}}{r} + \hat{F}_z = \rho \hat{a}_z \end{aligned}}$$

NOTE: At this point, most books drop the "hats" on the physical components, and just forget about covariant, contravariant, etc... components.

$\boxed{2.}$ **Strain - Displacement Relations**

Cartesian :
$$\epsilon_{ij}^{©} = \frac{1}{2}\left(\frac{\partial u_i^{©}}{\partial x_j} + \frac{\partial u_j^{©}}{\partial x_i}\right)$$

$$\boxed{\epsilon_{ij}^{©} = \frac{1}{2}\left(u_{i|j}^{©} + u_{j|i}^{©} \right)}$$

Since $\epsilon_{ij}^{©} = \epsilon_{ji}^{©}$

we can show

$$\boxed{\begin{aligned} \epsilon_{ij} &= \epsilon_{ji} \\ \epsilon^{ij} &= \epsilon^{ji} \end{aligned}}$$

$$\boxed{\begin{aligned}
\epsilon_{ij}(x) &= \frac{1}{2}\left(u_i|_j(x) + u_j|_i(x) \right) \\
\epsilon^{ij}(x) &= \frac{1}{2}\left(u^i|^j(x) + u^j|^i(x) \right) \\
\epsilon^{i}{}_{j}(x) &= \frac{1}{2}\left(u^i|_j(x) + u_j|^i(x) \right) \\
\epsilon_{i}{}^{j}(x) &= \frac{1}{2}\left(u_i|^j(x) + u^j|_i(x) \right)
\end{aligned}}$$

$$\boxed{\begin{aligned}
\epsilon_{ij}(\theta) &= \frac{1}{2}\left(u_i|_j(\theta) + u_j|_i(\theta) \right) \\
\epsilon^{ij}(\theta) &= \frac{1}{2}\left(u^i|^j(\theta) + u^j|^i(\theta) \right) \\
\epsilon^{i}{}_{j}(\theta) &= \frac{1}{2}\left(u^i|_j(\theta) + u_j|^i(\theta) \right) \\
\epsilon_{i}{}^{j}(\theta) &= \frac{1}{2}\left(u_i|^j(\theta) + u^j|_i(\theta) \right)
\end{aligned}}$$

[Only equation which doesn't involve a contravariant derivative. Thus, we use this one.]

$$\boxed{\epsilon_{ij} = \frac{1}{2}\left[u_{i}|_{j} + u_{j}|_{i} \right]}$$

$$\epsilon_{ij} = \frac{1}{2}\left[\left(\frac{\partial u_i}{\partial \theta^j} - \Gamma^k_{ij} u_k\right) + \left(\frac{\partial u_j}{\partial \theta^i} - \Gamma^k_{ji} u_k\right)\right]$$

<u>but</u> $\Gamma^k_{ij} = \Gamma^k_{ji}$

$$\rightarrow \quad \boxed{\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial \theta^j} + \frac{\partial u_j}{\partial \theta^i}\right] - \Gamma^k_{ij} u_k}$$

<u>9 equations, but only 6 are unique since $\epsilon_{ij} = \epsilon_{ji}$</u>

$\boxed{i=1,\ j=1}$

$$\epsilon_{11} = \frac{\partial u_1}{\partial \theta^1} - \left(\Gamma^1_{11} u_1 + \Gamma^2_{11} u_2 + \Gamma^3_{11} u_3\right)$$

$\boxed{i=1,\ j=2}$

$$\epsilon_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial \theta^2} + \frac{\partial u_2}{\partial \theta^1}\right] - \left(\Gamma^1_{12} u_1 + \underline{\Gamma^2_{12}} u_2 + \Gamma^3_{12} u_3\right)$$

<u>etc...</u>

Example - Cylindrical Coordinates

$$\theta^1 = r \qquad u_1 = u_r \qquad \epsilon_{11} = \epsilon_{rr}$$
$$\theta^2 = \theta \qquad u_2 = u_\theta \qquad \epsilon_{12} = \epsilon_{r\theta}$$
$$\theta^3 = z \qquad u_3 = u_z \qquad \text{etc...}$$

$$\Gamma_{22}^{1} = -r \qquad h_1 = 1$$
$$\Gamma_{12}^{2} = 1/r \qquad h_2 = r$$
$$\Gamma_{21}^{2} = 1/r \qquad h_3 = 1$$

<u>all other Γ's = 0</u>

$\boxed{i = j = 1}$

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} - (0 + 0 + 0)$$

$$\boxed{\epsilon_{rr} = \frac{\partial u_r}{\partial r}}$$

$\boxed{i = 1,\ j = 2}$

$$\boxed{\epsilon_{r\theta} = \frac{1}{2}\left[\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r}\right] - \left(\frac{1}{r} u_\theta\right)}$$

etc...

If we now introduce Physical Components:

$$\left.\begin{aligned} \hat{u}_i &= \frac{u_i}{h_i} \quad \text{(no sum)} \\ \hat{\epsilon}_{ij} &= \frac{\epsilon_{ij}}{h_i h_j} \quad \text{(no sums)} \end{aligned}\right\} \text{True only for orthogonal curvilinear coordinates}$$

$$\longrightarrow \quad \hat{u}_r = u_r$$

$$\hat{u}_\theta = \frac{u_\theta}{r} \longrightarrow u_\theta = r\hat{u}_\theta$$

$$\hat{\epsilon}_{rr} = \epsilon_{rr}$$

$$\hat{\epsilon}_{r\theta} = \frac{\epsilon_{r\theta}}{r} \longrightarrow \epsilon_{r\theta} = r\hat{\epsilon}_{r\theta}$$

$\boxed{i=j=1}$

$$\boxed{\hat{\epsilon}_{rr} = \frac{\partial \hat{u}_r}{\partial r}}$$

$\boxed{i=1,\ j=2}$

$$r\hat{\epsilon}_{r\theta} = \frac{1}{2}\left[\frac{\partial \hat{u}_r}{\partial \theta} + \frac{\partial}{\partial r}(r\hat{u}_\theta)\right] - \frac{1}{r}(r\hat{u}_\theta)$$

$$r\hat{\epsilon}_{r\theta} = \frac{1}{2}\left[\frac{\partial \hat{u}_r}{\partial \theta} + \hat{u}_\theta + r\frac{\partial \hat{u}_\theta}{\partial r}\right] - \hat{u}_\theta$$

$$\boxed{\hat{\epsilon}_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial \hat{u}_r}{\partial \theta} + \frac{\partial \hat{u}_\theta}{\partial r} - \frac{\hat{u}_\theta}{r}\right]}$$

$$\gamma_{r\theta} = 2\hat{\epsilon}_{r\theta} = \frac{1}{r}\frac{\partial \hat{u}_r}{\partial \theta} + \frac{\partial \hat{u}_\theta}{\partial r} - \frac{\hat{u}_\theta}{r}$$

With similar calculations, the other four unique equations become the following (in physical components):

$\boxed{i=j=2}$

$\boxed{i=j=3}$

$\boxed{i=2,\ j=3}$

$\boxed{i=1,\ j=3}$

$$\boxed{\begin{aligned}
\hat{\epsilon}_{\theta\theta} &= \frac{1}{r}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{\hat{u}_r}{r} \\
\hat{\epsilon}_{zz} &= \frac{\partial \hat{u}_z}{\partial z} \\
\hat{\epsilon}_{\theta z} &= \frac{1}{2}\left(\frac{\partial \hat{u}_\theta}{\partial z} + \frac{1}{r}\frac{\partial \hat{u}_z}{\partial \theta}\right) \\
\hat{\epsilon}_{rz} &= \frac{1}{2}\left(\frac{\partial \hat{u}_r}{\partial z} + \frac{\partial \hat{u}_z}{\partial r}\right)
\end{aligned}}$$

$\boxed{3.}$ Stress-Strain Relations (Hooke's Law)

Cartesian:

$$\boxed{\epsilon^{c}_{ij} = \left(\frac{1+\nu}{E}\right)\sigma^{c}_{ij} - \frac{\nu}{E}\,\sigma^{c}_{kk}\,\cancel{\delta_{ij}}\; g^{c}_{ij}}$$

$$\epsilon_{ij}(x) = \left(\frac{1+\nu}{E}\right)\sigma_{ij}(x) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(x)\right)g_{ij}(x)$$

$$\epsilon^{ij}(x) = \left(\frac{1+\nu}{E}\right)\sigma^{ij}(x) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(x)\right)g^{ij}(x)$$

$$\epsilon^{i}_{\cdot j}(x) = \left(\frac{1+\nu}{E}\right)\sigma^{i}_{\cdot j}(x) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(x)\right)g^{i}_{\cdot j}(x)$$

$$\epsilon_{i\cdot}^{\ \cdot j}(x) = \left(\frac{1+\nu}{E}\right)\sigma_{i\cdot}^{\ \cdot j}(x) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(x)\right)g_{i\cdot}^{\ \cdot j}(x)$$

[All of these are acceptable for evaluation purposes]

$$\epsilon_{ij}(\theta) = \left(\frac{1+\nu}{E}\right)\sigma_{ij}(\theta) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(\theta)\right)g_{ij}(\theta)$$

$$\epsilon^{ij}(\theta) = \left(\frac{1+\nu}{E}\right)\sigma^{ij}(\theta) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(\theta)\right)g^{ij}(\theta)$$

$$\epsilon^{i}_{\cdot j}(\theta) = \left(\frac{1+\nu}{E}\right)\sigma^{i}_{\cdot j}(\theta) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(\theta)\right)\cancel{g^{i}_{\cdot j}(\theta)}\;\delta^{i}_{j}$$

$$\epsilon_{i\cdot}^{\ \cdot j}(\theta) = \left(\frac{1+\nu}{E}\right)\sigma_{i\cdot}^{\ \cdot j}(\theta) - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}(\theta)\right)\cancel{g_{i\cdot}^{\ \cdot j}}\;\delta_{i}^{j}$$

Lets choose

$$\boxed{\epsilon^{i}_{\cdot j} = \left(\frac{1+\nu}{E}\right) \sigma^{i}_{\cdot j} - \frac{\nu}{E}\left(\sigma^{k}_{\cdot k}\right) \delta^{i}_{j}}$$

$\boxed{i = j = 1}$

$$\epsilon^{1}_{\cdot 1} = \left(\frac{1+\nu}{E}\right) \sigma^{1}_{\cdot 1} - \frac{\nu}{E}\left(\sigma^{1}_{\cdot 1} + \sigma^{2}_{\cdot 2} + \sigma^{3}_{\cdot 3}\right)$$

$$\boxed{\epsilon^{1}_{\cdot 1} = \frac{1}{E}\left(\sigma^{1}_{\cdot 1} - \nu\,\sigma^{2}_{\cdot 2} - \nu\,\sigma^{3}_{\cdot 3}\right)}$$

$\boxed{i = 1,\ j = 2}$

$$\epsilon^{1}_{\cdot 2} = \frac{1+\nu}{E}\,\sigma^{1}_{\cdot 2}$$

$$\boxed{\epsilon^{1}_{\cdot 2} = \frac{1}{2G}\,\sigma^{1}_{\cdot 2}}$$

etc...

Example – Cylindrical Coordinates

$\theta^1 = r$	$\sigma^{1}_{\cdot 1} = \sigma^{r}_{\cdot r}$	$\epsilon^{1}_{\cdot 1} = \epsilon^{r}_{\cdot r}$	$h_1 = 1$
$\theta^2 = \theta$	$\sigma^{1}_{\cdot 2} = \sigma^{r}_{\cdot \theta}$	$\epsilon^{1}_{\cdot 2} = \epsilon^{r}_{\cdot \theta}$	$h_2 = r$
$\theta^3 = z$	etc...	etc...	$h_3 = 1$

$\boxed{i = j = 1}$

$$\boxed{\epsilon^{r}_{\cdot r} = \frac{1}{E}\left(\sigma^{r}_{\cdot r} - \nu\,\sigma^{\theta}_{\cdot \theta} - \nu\,\sigma^{z}_{\cdot z}\right)}$$

$\boxed{i = 1,\ j = 2}$

$$\boxed{\epsilon^{r}_{\cdot \theta} = \frac{1}{2G}\,\sigma^{r}_{\cdot \theta}}$$

etc...

Physical Components

$$\hat{\sigma}_{ij} = \frac{h_i}{h_j} \sigma^{i}_{\cdot j}$$
$$\hat{\epsilon}_{ij} = \frac{h_i}{h_j} \epsilon^{i}_{\cdot j}$$

Valid for Orthogonal Curvilinear Systems Only

$$\hat{\sigma}_{rr} = \sigma^{r}_{\cdot r} \qquad \hat{\epsilon}_{rr} = \epsilon^{r}_{\cdot r}$$
$$\hat{\sigma}_{\theta\theta} = \sigma^{\theta}_{\cdot\theta} \qquad \hat{\epsilon}_{\theta\theta} = \epsilon^{\theta}_{\cdot\theta}$$
$$\hat{\sigma}_{zz} = \sigma^{z}_{\cdot z} \qquad \hat{\epsilon}_{zz} = \epsilon^{z}_{\cdot z}$$

$$\hat{\sigma}_{r\theta} = \frac{1}{r}\sigma^{r}_{\cdot\theta} \qquad \hat{\epsilon}_{r\theta} = \frac{1}{r}\epsilon^{r}_{\cdot\theta}$$

$\boxed{i=j=1}$

$$\boxed{\hat{\epsilon}_{rr} = \frac{1}{E}\left(\hat{\sigma}_{rr} - \nu\hat{\sigma}_{\theta\theta} - \nu\hat{\sigma}_{zz}\right)}$$

$\boxed{i=1,\ j=2}$

$$\cancel{r}\,\hat{\epsilon}_{r\theta} = \frac{1}{2G}\cancel{r}\,\hat{\sigma}_{r\theta}$$

$$\boxed{\hat{\epsilon}_{r\theta} = \frac{1}{2G}\hat{\sigma}_{r\theta}}$$

Similarly:

$$\boxed{\begin{aligned}
\hat{\epsilon}_{\theta\theta} &= \frac{1}{E}\left(\hat{\sigma}_{\theta\theta} - \nu\hat{\sigma}_{rr} - \nu\hat{\sigma}_{zz}\right)\\
\hat{\epsilon}_{zz} &= \frac{1}{E}\left(\hat{\sigma}_{zz} - \nu\hat{\sigma}_{rr} - \nu\hat{\sigma}_{\theta\theta}\right)\\
\hat{\epsilon}_{\theta z} &= \frac{1}{2G}\hat{\sigma}_{\theta z}\\
\hat{\epsilon}_{rz} &= \frac{1}{2G}\hat{\sigma}_{rz}
\end{aligned}}$$

2nd Order Tensor Components

(Extension of Vector Rules)

(Definitions) A 2nd Order tensor $\underline{\underline{T}}$ has:

Contravariant components T^{ij}

Covariant components T_{ij}

Dot determines what is first

Mixed Components $T^{i}_{\cdot j}$ and $T_{i}^{\cdot j}$

resolved → physical components $\hat{T}_{ij}$ ← unit vector

according to unit vector → cartesian components $T_{ij}^{©}$ ←

which transform using the following tensor component transformation rules:

contravariant → $$T^{ij\prime} \equiv \frac{\partial \theta^{i\prime}}{\partial \theta^{k}} \frac{\partial \theta^{j\prime}}{\partial \theta^{l}} T^{kl}$$

covariant → $$T_{ij}{}' \equiv \frac{\partial \theta^{k}}{\partial \theta^{i\prime}} \frac{\partial \theta^{l}}{\partial \theta^{j\prime}} T_{kl}$$

Mixed → $$T^{i}_{\cdot j}{}' \equiv \frac{\partial \theta^{i\prime}}{\partial \theta^{k}} \frac{\partial \theta^{l}}{\partial \theta^{j\prime}} T^{k}_{\cdot l}$$

$$T_{i}^{\cdot j}{}' \equiv \frac{\partial \theta^{k}}{\partial \theta^{i\prime}} \frac{\partial \theta^{j\prime}}{\partial \theta^{l}} T_{k}^{\cdot l}$$

ignore

$$T^{ij} = \frac{\partial \theta^{i}}{\partial \theta^{k\prime}} \frac{\partial \theta^{j}}{\partial \theta^{l\prime}} T^{kl\prime}$$

etc.…

↓

Cartesian

$T_{ij}^{©} = a_{ik} a_{jl} T_{kl}^{©}$

$\frac{dx_i'}{dx_k}$ $\frac{dx_j'}{dx_l}$

in cartesian coordinates

$\hat{V}_i = h_i V^i$

Invariant Notation

dyads

$$\underline{\underline{T}} = T^{ij}\,\vec{g}_i\,\vec{g}_j$$
$$= T_{ij}\,\vec{g}^i\,\vec{g}^j$$
$$= T^{i}_{\cdot j}\,\vec{g}_i\,\vec{g}^j$$
$$= T_{i\cdot}^{\ j}\,\vec{g}^i\,\vec{g}_j$$
$$= \hat{T}_{ij}\,\vec{\ell}_i\,\vec{\ell}_j$$
$$= T^{(c)}_{ij}\,\vec{e}_i\,\vec{e}_j$$

invariant quanties formed using repeated subscript/superscript pairs

cartesian (simplification)

Note that for $\theta^i = x_i$

$$\boxed{T^{(c)}_{ij} = T_{ij} = T^{ij} = T^{i}_{\cdot j} = T_{i\cdot}^{\ j} = \hat{T}_{ij}}$$

since $\vec{g}_i = \vec{g}^i = \vec{\ell}_i = e_i$

$$\frac{\partial\theta^k}{\partial\theta^{i'}} = \frac{\partial\theta^{i'}}{\partial\theta^k} = \frac{\partial x_k}{\partial x_i'} = \frac{\partial x_i'}{\partial x_k} = a_{ik}$$

all transformations are the same!

NOTE: Special Cases $\left.\begin{array}{l}\theta^i = x_i \\ \theta^{i'} = x_i'\end{array}\right\}$ Cartesian Systems

(1.) $T'_{ij} = \dfrac{\partial\theta^k}{\partial\theta^{i'}}\dfrac{\partial\theta^l}{\partial\theta^{j'}} T_{kl}$ < covariant

$$\overset{\copyright}{T}{}'_{ij} = \frac{\partial x_k}{\partial x_i'}\frac{\partial x_l}{\partial x_j'} \overset{\copyright}{T}_{kl}$$

$$\boxed{\overset{\copyright}{T}{}'_{ij} = a_{ik}\, a_{jl}\, \overset{\copyright}{T}_{kl}}$$

$$T'^{ij} = \frac{\partial\theta^{i'}}{\partial\theta^k}\frac{\partial\theta^{j'}}{\partial\theta^l} T^{kl}$$

$$\overset{\copyright}{T}{}'^{ij} = \frac{\partial x_i'}{\partial x_k}\frac{\partial x_j}{\partial x_l} \overset{\copyright}{T}{}^{kl}$$

$$\boxed{\overset{\copyright}{T}{}'^{ij} = a_{ik}\, a_{jl}\, \overset{\copyright}{T}{}^{kl}}$$

$\overset{\copyright}{T}{}'_{ij}$ $\qquad$ $\overset{\copyright}{T}_{kl}$

Really the same equation!

(2.) $\underline{\theta^{i'} \ne \theta^i}$, $\underline{\theta^i = x_i}$

$$T^{ij} = \frac{\partial\theta^i}{\partial x_k}\frac{\partial\theta^j}{\partial x_l} \overset{\copyright}{T}_{kl}$$

etc...

See next page

(2.) Continued

In general,

If we establish that a set of components $T^{©}_{ij}$ are cartesian tensor components, [redacted] we can easily extend these components to be curvilinear tensor components.

The covariant, contravariant, & mixed components of the curvilinear tensor are defined from the cartesian components by:

In cartesian coordinates, all components are the same:

$$\left(T^{ij} = T_{ij} = T^{i}{}_{j} = T_{i}{}^{j}\right) \equiv T^{©}_{ij}$$

In a curvilinear, the various coefficients are defined by:

σ_{YY}

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \vdots \\ \sigma_{yz} \end{pmatrix}$$

$$\boxed{\begin{aligned}
T^{ij}(\theta) &\equiv \frac{\partial \theta^{i}}{\partial x_{k}} \frac{\partial \theta^{j}}{\partial x_{l}} T^{kl}(x) = \frac{\partial \theta^{i}}{\partial x_{k}} \frac{\partial \theta^{j}}{\partial x_{l}} T^{©}_{kl} \\
T_{ij}(\theta) &\equiv \frac{\partial x_{k}}{\partial \theta^{i}} \frac{\partial x_{l}}{\partial \theta^{j}} T_{kl}(x) = \frac{\partial x_{k}}{\partial \theta^{i}} \frac{\partial x_{l}}{\partial \theta^{j}} T^{©}_{kl} \\
T^{i}{}_{j}(\theta) &\equiv \frac{\partial \theta^{i}}{\partial x_{k}} \frac{\partial x_{l}}{\partial \theta^{j}} T^{k}{}_{l}(x) = \frac{\partial \theta^{i}}{\partial x_{k}} \frac{\partial x_{l}}{\partial \theta^{j}} T^{©}_{kl} \\
T_{i}{}^{j}(\theta) &\equiv \frac{\partial x_{k}}{\partial \theta^{i}} \frac{\partial \theta^{j}}{\partial x_{l}} T_{k}{}^{l}(x) = \frac{\partial x_{k}}{\partial \theta^{i}} \frac{\partial \theta^{j}}{\partial x_{l}} T^{©}_{kl}
\end{aligned}}$$

The various types of tensor components (in a particular coordinate system) are related to each other through the metric components:

$$T^{ij} = g^{ik} g^{jl} T_{kl}$$

$$T_{ij} = g_{ik} g_{jl} T^{kl}$$

$$T^{i}{}_{j} = g^{ik} T_{kj} = g_{jk} T^{ik}$$

etc...

the metric components "raise" and "lower" the indices of tensor components

Proof

look at def for g^{ij}

$$T^{ij} = \frac{\partial\theta^i}{\partial x_k}\frac{\partial\theta^j}{\partial x_l}\frac{\partial\theta^m}{\partial x_k}\frac{\partial\theta^n}{\partial x_l} T_{mn}$$

$$T^{ij} = \left(\frac{\partial\theta^i}{\partial x_k}\frac{\partial\theta^m}{\partial x_k}\right)\left(\frac{\partial\theta^j}{\partial x_l}\frac{\partial\theta^n}{\partial x_l}\right) T_{mn}$$

$$T^{i'j'} = \frac{\partial\theta^{i'}}{\partial\theta^k}\frac{\partial\theta^{j'}}{\partial\theta^l} T^{kl}$$

let $\theta^{i'} \to \theta^i$

$\theta^i \Rightarrow x_i$

Combine

$$T^{ij} = \frac{\partial\theta^i}{\partial x_k}\frac{\partial\theta^j}{\partial x_l} T_{kl} \quad \Rightarrow \quad T_{kl} = \frac{\partial\theta^m}{\partial x_k}\frac{\partial\theta^n}{\partial x_l}$$

(★) See bottom of page – FIRST

Proof $\left(\text{of } T^{ij} = g^{ik} g^{jl} T_{kl} \right)$

$$T^{ij\,\prime} = \frac{\partial \theta^{i\,\prime}}{\partial \theta^{k}} \frac{\partial \theta^{j\,\prime}}{\partial \theta^{l}} T^{kl} \qquad\qquad T_{ij}^{\prime} = \frac{\partial \theta^{k}}{\partial \theta^{i\,\prime}} \frac{\partial \theta^{l}}{\partial \theta^{j\,\prime}} T_{kl}$$

Let $\theta^{i\,\prime} \rightarrow \theta^{i}$, $\theta^{i} \rightarrow x_i$ Let $\theta^{i\,\prime} = x_i$, $\theta^{i} = \theta^{i}$

$$\boxed{T^{ij} = \frac{\partial \theta^{i}}{\partial x_k} \frac{\partial \theta^{j}}{\partial x_l} T_{kl}^{©}} \qquad\qquad T_{ij}^{©} = \frac{\partial \theta^{k}}{\partial x_i} \frac{\partial \theta^{l}}{\partial x_j} T_{kl}$$

$$\rightarrow \boxed{T_{kl}^{©} = \frac{\partial \theta^{m}}{\partial x_k} \frac{\partial \theta^{n}}{\partial x_l} T_{mn}}$$

Combine

$$T^{ij} = \frac{\partial \theta^{i}}{\partial x_k} \frac{\partial \theta^{j}}{\partial x_l} \left(\frac{\partial \theta^{m}}{\partial x_k} \frac{\partial \theta^{n}}{\partial x_l} T_{mn} \right)$$

$$T^{ij} = \left(\frac{\partial \theta^{i}}{\partial x_k} \frac{\partial \theta^{m}}{\partial x_k} \right) \left(\frac{\partial \theta^{j}}{\partial x_l} \frac{\partial \theta^{n}}{\partial x_l} \right) T_{mn}$$

$$\boxed{T^{ij} = g^{im} g^{jn} T_{mn}} \checkmark$$

$\vec{g}_i = \vec{g}^{i} = \vec{e}_i$ (★) NOTE THAT FOR CARTESIAN COORDINATES

$$\frac{\partial \theta^{i}}{\partial \theta^{k}} = \frac{\partial x_i'}{\partial x_k} = \frac{\partial x_k}{\partial x_i'} = a_{ik} \qquad T^{ij} = T_{ij} = T^{i}_{\cdot j} = T_{i}^{\cdot j} = \hat{T}_{ij} = T_{ij}^{©}$$

base vectors and

N^{th} Order Tensor Components

An N^{th} Order tensor $\underline{\underline{T}}$ has:

→ Contravariant components $T^{ijkl\ldots}$
→ Covariant Components $T_{ijkl\ldots}$

Mixed Components $T^{ij}_{\cdot\cdot kl\ldots}$, $T^{i\cdot k}_{\cdot j \cdot l\ldots}$, etc....

Which transform using:

$$T^{ijkl\ldots'} = \left(\frac{\partial\theta^{i'}}{\partial\theta^m}\frac{\partial\theta^{j'}}{\partial\theta^n}\frac{\partial\theta^{k'}}{\partial\theta^o}\frac{\partial\theta^{l'}}{\partial\theta^p}\ldots\right)T^{mnop\ldots}$$

$$T'_{ijkl\ldots} = \left(\frac{\partial\theta^m}{\partial\theta^{i'}}\frac{\partial\theta^n}{\partial\theta^{j'}}\frac{\partial\theta^o}{\partial\theta^{k'}}\frac{\partial\theta^p}{\partial\theta^{l'}}\ldots\right)T_{mnop\ldots}$$

etc....

And are related by:

$$T^{ijkl\ldots} = \left(g^{im}g^{jn}g^{ko}g^{lp}\ldots\right)T_{mnop\ldots}$$

$$T_{ijkl\ldots} = \left(g_{im}\,g_{jn}\,g_{ko}\,g_{lp}\ldots\right)T^{mnop\ldots}$$

etc....

In Cartesian Coordinates, All components are the same :

$$T^{c}_{ijkl\ldots} = T_{ijkl\ldots} = T^{ijkl\ldots} = T^{i\,\cdot\,k}_{\cdot\,j\,\cdot\,l\ldots} \quad \text{etc.\ldots}$$

Invariant Notation :

$$\begin{aligned} \mathbf{T} &= T_{ijkl\ldots}\, \vec{g}^{\,i}\vec{g}^{\,j}\vec{g}^{\,k}\vec{g}^{\,l}\ldots \\ &= T^{ijkl\ldots}\, \vec{g}_i\vec{g}_j\vec{g}_k\vec{g}_l\ldots \end{aligned}$$

etc. ...

Unit Natural Base Vectors (subscript)

$$h_i \equiv |\vec{g}_i|$$ "Coordinate System Scale Factors"

$$\vec{e}_i \equiv \frac{\vec{g}_i}{|\vec{g}_i|} = \frac{\vec{g}_i}{h_i} \qquad \text{(NO SUM)}$$

cursive e

Metric of the Coordinate System

$$|\vec{g}_i|^2 = \overbrace{\vec{g}_i \cdot \vec{g}_i = g_{ii}}^{\text{NO SUM}}$$

$$\boxed{g_{ij} \equiv \vec{g}_i \cdot \vec{g}_j}$$

$$\boxed{h_i = \sqrt{g_{ii}} \quad \text{NO SUM}}$$

NOTE: $g_{ij} = g_{ji}$

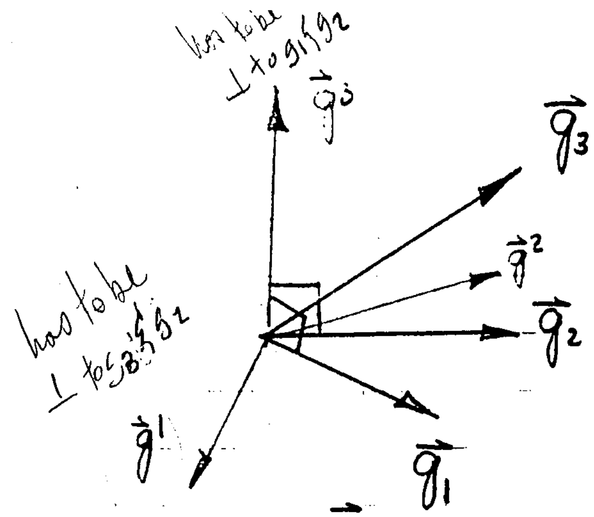
Dual Base Vectors $\vec{g}^i$ (super script)
(Reciprocal Base Vectors)

$$\boxed{\vec{g}^i \cdot \vec{g}_j \equiv \delta^i_j} \equiv \delta_{ij} \quad \text{Kronecker Delta} \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}$$

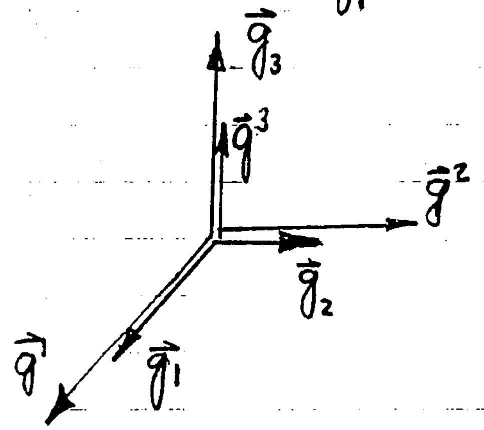
$\cancel{g_i = g^i}$

$\cancel{h_{ij} g^j = u_i}$

we will try to adopt the convention that the number of subscript & superscript free indices should be the same on each side of an equation.



non-orthogonal curvilinear system



Orthogonal curvilinear system
(spherical and cylindrical case)

Reciprocal Metric

$$\boxed{g^{ij} \equiv \vec{g}^i \cdot \vec{g}^j}$$

NOTE $g^{ij} = g^{ji}$

Relate the $\vec{g}_i$'s to the $\vec{g}^j$'s

$$\vec{g}^i = A_{ij}\,\vec{g}_j \qquad \text{(what are the } A_{ij}\text{?)}$$

$$\vec{g}^i \cdot \vec{g}^k = (A_{ij}\,\vec{g}_j)\cdot \vec{g}^k$$

$$g^{ik} = A_{ij}\,\delta_j^k$$

$$g^{ik} = A_{ik}$$

$$\boxed{\vec{g}^i = g^{ij}\,\vec{g}_j}$$

NOTE THAT THIS DOESNT HELP US TOO MUCH. WE MAY KNOW THE $\vec{g}_i$ but we won't know the g^{ij} until we get the $\vec{g}^j$.

to get the $\vec{g}^i$

Similarly

$$\boxed{\vec{g}_i = g_{ij}\,\vec{g}^j}$$

$$\underline{I} = \delta_{ij}\,\vec{e}_i\vec{e}_j$$

curvi linear $\quad \underline{I} = g_{ij}\,\vec{g}^i\vec{g}^j$

These equations are an example of a common trend we will see in the future:

$V^i = g^{ij} V_j$ "raises the index"

$V_i = g_{ij} V^j$ "lowers the index"

Analogous to δ_{ij} changing the index:

$V_i = \delta_{ij} V_j$

$$g_{ij} = \vec{g}_i \cdot \vec{g}_j$$

$$g_{ij} = \left(\frac{\partial x_k}{\partial \theta^i} \vec{e}_k \right) \cdot \left(\frac{\partial x_l}{\partial \theta^j} \vec{e}_l \right)$$

$$g_{ij} = \frac{\partial x_k}{\partial \theta^i} \frac{\partial x_l}{\partial \theta^j} \underbrace{\vec{e}_k \cdot \vec{e}_l}_{\delta_{kl}}$$

$$\boxed{g_{ij} = \frac{\partial x_k}{\partial \theta^i} \frac{\partial x_l}{\partial \theta^j} \delta_{kl} = \frac{\partial x_k}{\partial \theta^i} \frac{\partial x_k}{\partial \theta^j}}$$

good eq.

Assume

$$\boxed{\vec{g}^i = A_{ij} \vec{e}_j}$$

(Find the A_{ij})

By Definition:

$$\vec{g}^i \cdot \vec{g}_k = A_{ij} \vec{e}_j \cdot \vec{g}_k = \delta^i_k$$

$$A_{ij} \vec{e}_j \cdot \left(\frac{\partial x_l}{\partial \theta^k} \vec{e}_l \right) = \delta^i_k$$

$$A_{ij} \frac{\partial x_l}{\partial \theta^k} \underbrace{(\vec{e}_j \cdot \vec{e}_l)}_{\delta_{jl}} = \delta^i_k$$

$$A_{ij} \frac{\partial x_j}{\partial \theta^k} = \delta^i_k$$

$A_{ij} = \frac{\partial \theta^i}{\partial x_j}$ works in general

something

Since then

$$A_{ij}\frac{\partial x_j}{\partial \theta^k} = \frac{\partial \theta^i}{\partial x_j}\frac{\partial x_j}{\partial \theta^k} = \frac{\partial \theta^i}{\partial \theta^k} = \delta^i_k$$

So

$$\boxed{\vec{g}^{\,i} = \frac{\partial \theta^i}{\partial x_j}\,\vec{e}_j}$$

[Reciprocals — Thus the name Reciprocal Base Vectors]

$$\vec{g}_i = \frac{\partial x_j}{\partial \theta^i}\,\vec{e}_j$$

$$g^{ij} = \vec{g}^{\,i}\cdot\vec{g}^{\,j}$$

$$g^{ij} = \left(\frac{\partial \theta^i}{\partial x_k}\vec{e}_k\right)\cdot\left(\frac{\partial \theta^j}{\partial x_l}\vec{e}_l\right)$$

$$\boxed{g^{ij} = \frac{\partial \theta^i}{\partial x_k}\frac{\partial \theta^j}{\partial x_l}\delta_{kl} = \frac{\partial \theta^i}{\partial x_k}\frac{\partial \theta^j}{\partial x_k}}$$

$$\vec{e}_i = \frac{\partial \theta^j}{\partial x_i} \vec{g}_j$$

$$\vec{e}_i = \frac{\partial \theta^j}{\partial x_i} \left(g_{jk} \vec{g}^k \right)$$

$$\vec{e}_i = \frac{\partial \theta^j}{\partial x_i} \left(\frac{\partial x_l}{\partial \theta^j} \frac{\partial x_l}{\partial \theta^k} \right) \vec{g}^k$$

$$\vec{e}_i = \cancelto{\delta_{il}}{\frac{\partial x_l}{\partial x_i}} \frac{\partial x_l}{\partial \theta^k} \vec{g}^k$$

$$\vec{e}_i = \frac{\partial x_i}{\partial \theta^k} \vec{g}^k$$

$$\boxed{\vec{e}_i = \frac{\partial x_i}{\partial \theta^j} \vec{g}^j}$$

$$g^{ij}\, g_{jk} = \left(\frac{\partial x_l}{\partial \theta^i} \frac{\partial x_l}{\partial \theta^j} \right) \left(\frac{\partial \theta^j}{\partial x_m} \frac{\partial \theta^k}{\partial x_m} \right)$$

$$= \frac{\partial x_l}{\partial \theta^i} \frac{\partial \theta^k}{\partial x_m} \left(\frac{\partial x_l}{\partial \theta^j} \frac{\partial \theta^j}{\partial x_m} \right)$$

$$= \frac{\partial x_l}{\partial \theta^i} \frac{\partial \theta^k}{\partial x_m} \cancelto{\delta_{lm}}{\frac{\partial x_l}{\partial x_m}}$$

$$= \frac{\partial x_l}{\partial \theta^i} \frac{\partial \theta^k}{\partial x_l}$$

$$= \frac{\partial \theta^k}{\partial \theta^i}$$

$$\boxed{g^{ij}\, g_{jk} = \delta^i_k} \qquad [g^{ij}] = [g_{ij}]^{-1}$$

$$\rightarrow \quad [g^{ij}][g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similary $\boxed{g_{ij}\, g^{jk} = \delta_i^k}$

Example – Cartesian Coordinates

$$\theta^i = x_i \qquad\qquad h_i = |g_i|$$

$$\vec{r} = x_i \vec{e}_i \qquad\qquad h_1 = h_2 = h_3 = 1$$

$$\vec{g}_i = \frac{\partial \vec{r}}{\partial \theta^i} = \frac{\partial \vec{r}}{\partial x_i} = \frac{\partial (x_j \vec{e}_j)}{\partial x_i} = \frac{\partial x_j}{\partial x_i} \vec{e}_j = \delta_{ij} \vec{e}_j = \vec{e}_i$$

$$\boxed{\vec{g}_i = \vec{e}_i}$$

$$g_{ij} = \vec{g}_i \cdot \vec{g}_j = \vec{e}_i \cdot \vec{e}_j$$

$$\boxed{g_{ij} = \delta_{ij}}$$

$$\vec{g}^i = \frac{\partial \theta^i}{\partial x_j} \vec{e}_j = \frac{\partial x_i}{\partial x_j} \vec{e}_j = \delta_{ij} \vec{e}_j$$

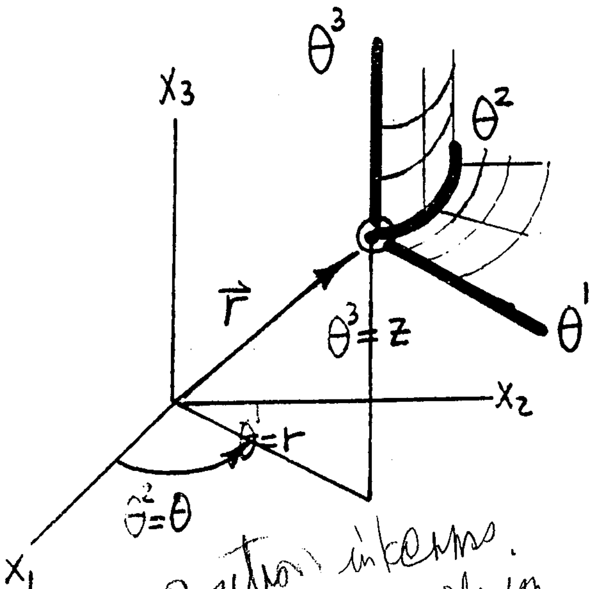
$$\boxed{\vec{g}^i = \vec{e}_i} \qquad \boxed{g^{ij} = \delta^{ij}} \leftarrow \text{only to keep convention}$$

in cartesian there is

$$\delta_{ij} = \vec{g}^i \cdot \vec{g}_j = \vec{g}^i \cdot \vec{e}_j$$

$$\vec{g}^i = e_i$$

Example - Cylindrical Coordinates



$$\boxed{\begin{aligned}\theta^1 &\equiv r\\ \theta^2 &\equiv \theta\\ \theta^3 &\equiv z\end{aligned}}$$

$$\boxed{\begin{aligned}x_1 &= r\cos\theta\\ x_2 &= r\sin\theta\\ x_3 &= z\end{aligned}}\quad\begin{aligned}&= \theta^1\cos\theta^2\\ &= \theta^1\sin\theta^2\\ &= \theta^3\end{aligned}$$

Position in terms of cartesian

$$\vec{r} = x_i\,\vec{e}_i$$

$$\boxed{\vec{r} = (r\cos\theta)\vec{e}_1 + (r\sin\theta)\vec{e}_2 + z\vec{e}_3}$$

θ^1 θ^2 θ^3

$$r = \sqrt{x_1^2 + x_2^2}$$

$$\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

$$z = x_3$$

Rewrite so that $x_1 = x$, $x_2 = y$, $x_3 = z$!

covariant

$$\vec{g}_i = \frac{\partial \vec{r}}{\partial \theta^i} = \overbrace{\frac{\partial x_j}{\partial \theta^i}}^{a_{ij}}\,\vec{e}_j$$

$$\vec{g}_1 = \frac{\partial \vec{r}}{\partial \theta^1} = \frac{\partial \vec{r}}{\partial r} = \cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2 \Leftarrow$$ from $\vec{r}$ in cartesian

or

$$\vec{g}_1 = \frac{\partial x_1}{\partial \theta^1}\vec{e}_1 + \frac{\partial x_2}{\partial \theta^1}\vec{e}_2 + \underbrace{\frac{\partial x_3}{\partial \theta^1}}_{0}\vec{e}_3 = \frac{\partial x_k}{\partial \theta^1}\vec{e}_k$$

$$\vec{g}_1 = \cancelto{\cos\theta}{\frac{\partial x_1}{\partial r}}\vec{e}_1 + \cancelto{\sin\theta}{\frac{\partial x_2}{\partial r}}\vec{e}_2 + \cancelto{0}{\frac{\partial x_3}{\partial r}}\vec{e}_3 = \cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2 \checkmark$$

$\theta^1 = r$ $\theta^2 = \theta$ $\vec{r} = (r\cos\theta)\hat{e}_1 + (r\sin\theta)\hat{e}_2 + z\hat{e}_3$

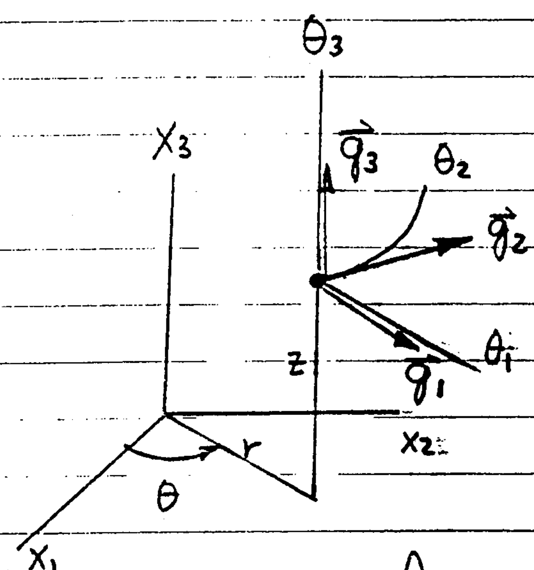
$\theta^3 = z$

$$\vec{g_2} = \frac{\partial \vec{r}}{\partial \theta^2} = \frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\, \vec{e_1} + r\cos\theta\, \vec{e_2}$$

$$\vec{g_3} = \frac{\partial \vec{r}}{\partial \theta^3} = \frac{\partial \vec{r}}{\partial z} = \vec{e_3}$$

Orthogonal set

$$\boxed{\begin{aligned} \vec{g_1} &= \cos\theta\, \vec{e_1} + \sin\theta\, \vec{e_2} \\ \vec{g_2} &= -r\sin\theta\, \vec{e_1} + r\cos\theta\, \vec{e_2} \\ \vec{g_3} &= \vec{e_3} \end{aligned}}$$



$$g_{ij} = \vec{g_i} \cdot \vec{g_j}$$

Cylindrical Coordinates are

important → Orthogonal Curvilinear Coordinates

$$g_{11} = \cos^2\theta + \sin^2\theta = 1$$

$$g_{22} = r^2\sin^2\theta + r^2\cos^2\theta = r^2$$

$$g_{33} = 1$$

$$g_{12} = g_{21} = g_{13} = g_{31} = g_{23} = g_{32} = 0$$

$$\boxed{[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Scale Factors

$$h_i \equiv |\vec{g}_i| = \sqrt{g_{ii}} \quad \text{(NO SUM)}$$

$$\boxed{\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= 1 \end{aligned}}$$

$$\vec{\ell}_i = \frac{\vec{g}_i}{|\vec{g}_i|} \quad \text{(NO SUM)}$$

Script ℓ

$$\boxed{\begin{aligned} \vec{\ell}_1 &= \vec{g}_1 = \cos\theta\, \vec{e}_1 + \sin\theta\, \vec{e}_2 \\ \vec{\ell}_2 &= \frac{\vec{g}_2}{r} = -\sin\theta\, \vec{e}_1 + \cos\theta\, \vec{e}_2 \\ \vec{\ell}_3 &= \vec{g}_3 = \vec{e}_3 \end{aligned}}$$

(we often call these $\vec{e}_r, \vec{e}_\theta, \vec{e}_z$ in Dynamics etc...)

$$\vec{g}^i = \frac{\partial \theta^i}{\partial x_j} \vec{e}_j \qquad \text{def } g^i \cdot g_j = \delta^i_j$$

$$\vec{g}^1 = \frac{\partial \theta^1}{\partial x_j} \vec{e}_j = \frac{\partial r}{\partial x_1} \vec{e}_1 + \frac{\partial r}{\partial x_2} \vec{e}_2 + \frac{\partial r}{\partial x_3} \vec{e}_3$$

$$\frac{\partial r}{\partial x_1} = \frac{\partial \sqrt{x_1^2 + x_2^2}}{\partial x_1} = \frac{1}{2}(x_1^2 + x_2^2)^{-1/2}\, 2x_1 = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$\frac{\partial r}{\partial x_1} = \frac{r\cos\theta}{r} = \cos\theta$$

Bringing it Back to Cylindrical

[illegible] $\frac{\partial r}{\partial x_3} = 0$

$$\frac{\partial r}{\partial x_2} = \frac{\partial}{\partial x_2}\left(\sqrt{x_1^2+x_2^2}\right) = \frac{1}{2}\left(x_1^2+x_2^2\right)^{-1/2} 2x_2 = \frac{x_2}{\sqrt{x_1^2+x_2^2}}$$

$$\frac{\partial r}{\partial x_2} = \frac{r\sin\theta}{r} = \sin\theta$$

$$\frac{\partial r}{\partial x_3} = 0$$

$$\longrightarrow \quad \underline{\vec{g}^{\,1} = \cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2}$$

Known from before

OR USE DEFINITION: $\vec{g}^{\,i}\cdot\vec{g}_j = \delta^i_j$ let $\boxed{\vec{g}^{\,1} = A_i\,\vec{e}_i}$

$$\vec{g}^{\,1}\cdot\vec{g}_1 = A_1\cos\theta + A_2\sin\theta = 1$$

$$\vec{g}^{\,1}\cdot\vec{g}_2 = -A_1 r\sin\theta + A_2 r\cos\theta = 0$$

$$\vec{g}^{\,1}\cdot\vec{g}_3 = A_3(1) = 0$$

$$\longrightarrow A_3 = 0$$

$$A_1 = \cos\theta$$

$$A_2 = \sin\theta \qquad \longrightarrow \quad \underline{\vec{g}^{\,1} = \cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2} \checkmark$$

$$\boxed{\begin{aligned} \vec{g}^1 &= \cos\theta\, \vec{e}_1 + \sin\theta\, \vec{e}_2 \\ \vec{g}^2 &= \left(-\frac{\sin\theta}{r}\right)\vec{e}_1 + \left(\frac{\cos\theta}{r}\right)\vec{e}_2 \\ \vec{g}^3 &= \vec{e}_3 \end{aligned}}$$

$$\left(\vec{g}^1 = \vec{g}_1\right)$$
$$\left(\vec{g}^2 = \frac{\vec{g}_2}{r^2}\right)$$
$$\left(\vec{g}^3 = \vec{g}_3\right)$$

$$g^{ij} = \vec{g}^i \cdot \vec{g}^j$$

$$\boxed{[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\cancel{[g_{ij}]}$

orthogonal

NOTE THAT $[g_{ij}]$ & $[g^{ij}]$ are matrix inverses!

The other eqns. we established can be checked

$$\left\{ \begin{aligned} \vec{g}^i &= g^{ij}\vec{g}_j \\ \vec{g}_i &= g_{ij}\vec{g}^j \\ \vec{e}_i &= \frac{\partial \theta^j}{\partial x_i}\vec{g}_j \\ \vec{e}_i &= \partial x_i \, \vec{g}^j \end{aligned} \right\} \checkmark$$

Superscript / Subscript
pair for invariant

Vector Components

important

$\vec{V} = V^i \vec{g}_i$
$\vec{V} = V_i \vec{g}^i$
$\vec{V} = \hat{V}_i \vec{e}_i$

$\rightarrow$ $\boxed{V^i}$ are the <u>contravariant components</u> of vector $\vec{V}$

$\rightarrow$ $\boxed{V_i}$ are the <u>covariant components</u> of vector $\vec{V}$

$\rightarrow$ $\boxed{\hat{V}_i}$ are the <u>physical components</u> of vector $\vec{V}$

$\vec{V} = V_i^{©} \vec{e}_i$

Cartesian Components
New notation

[The reasons for the positioning on the indices will become clear later when we consider coordinate system transformations.]

Contravariant × Covariant become invariant

Ex

$$\vec{V} = V^i \vec{g}_i = V^i \left(g_{ij}\, \vec{g}^j \right) = \left(g_{ij} V^i \right) \vec{g}^j$$

$$\vec{V} = V_j \vec{g}^j$$

$$\rightarrow \quad V_j = g_{ij} V^i$$

$$\boxed{V_i = g_{ij} V^j}$$

(since $g_{ij} = g_{ji}$)

Similarly

$$\boxed{V^i = g^{ij} V_j}$$

the metric components "raise" and "lower" indices of vector components

<u>Covariant $\leftrightarrow$ contravariant components transformations</u>

$$\vec{V} \cdot \vec{g}_i = (V_j \vec{g}^j) \cdot \vec{g}_i$$

$$= V_j (\vec{g}^j \cdot \vec{g}_i)$$

$$= V_j \, \delta^j_i$$

$$= V_i$$

$$\vec{V} \cdot e_i = V_i$$

$$\boxed{V_i = \vec{V} \cdot \vec{g}_i} \qquad (\vec{V} = V_i \vec{g}^i \quad \& \quad \vec{V} = V^i \vec{g}_i)$$

Similarly

$$\boxed{V^i = \vec{V} \cdot \vec{g}^i}$$

NOTE – NUMBER OF ~~#~~ free (SUB/SUPER)SCRIPTS ARE THE SAME ON EACH SIDE.

for a proper indicial notation question

$$\vec{V} = (V^i \vec{g}_i) \cdot g_j$$

$$\vec{V} = g_{ij} V^j$$

$$\vec{V} = V_i$$

$$\hat{V} \quad \vec{e}_i = \sum_{i=1}^{3} \hat{V}_i \frac{g_i}{h_i}$$

Physical Components

$$\vec{V} = \hat{V}_i \vec{e}_i = \sum_{i=1}^{3} \hat{V}_i \frac{\vec{g}_i}{\underbrace{h_i}_{\sqrt{g_{ii}}}} = \sum_{i=1}^{3} \left(\frac{\hat{V}_i}{h_i}\right) \vec{g}_i$$

$$\vec{V} = V^i \vec{g}_i$$

$$\longrightarrow \boxed{\begin{array}{ll} V^i = \dfrac{\hat{V}_i}{h_i} & \underline{\text{NO SUM}} \\ \hline \hat{V}_i = h_i V^i & \underline{\text{NO SUM}} \end{array}}$$

$\sqrt{g_{ii}}$

$$\hat{V}_i = h_i V^i \quad (\text{NO SUM})$$

$$\boxed{\hat{V}_i = h_i g^{ij} V_j \qquad (\text{NO SUM } i)}$$

NOTE: For Cartesian Coordinates

$$\vec{g}_i = \vec{g}^i = \vec{e}_i = \vec{e}_i$$

$$\longrightarrow \boxed{V^i = V_i = \hat{V}_i} \equiv V_i^{c}$$

$\vec{V} = V_i \vec{e}_i$ is still appropriate!

Coordinate System Transformations

covariant - transform from one system to another

θ^i & $\theta^{i'}$

$$\boxed{\begin{aligned} \theta^{i'} &= \theta^{i'}(\theta^1, \theta^2, \theta^3) \\ \theta^i &= \theta^i(\theta^{1'}, \theta^{2'}, \theta^{3'}) \end{aligned}}$$

$\vec{V} = V_i \vec{g}^i = V^i \vec{g}_i$
$\vec{V} = V_i' \vec{g}^{i'} = V^{i'} \vec{g}_i'$

BASE VECTOR TRANSFORMATIONS: (definitions of base)

$$\vec{g}_i' = \frac{\partial \vec{r}}{\partial \theta^{i'}} \qquad \vec{g}_i = \frac{\partial \vec{r}}{\partial \theta^i}$$

$$\vec{g}_i' = \frac{\partial \vec{r}}{\partial \theta^j} \frac{\partial \theta^j}{\partial \theta^{i'}} \quad \text{(chain rule)} \qquad \vec{g}_i = \frac{\partial \vec{r}}{\partial \theta^{j'}} \frac{\partial \theta^{j'}}{\partial \theta^i}$$

$$\boxed{\vec{g}_i' = \frac{\partial \theta^j}{\partial \theta^{i'}} \vec{g}_j}$$

notice position of primes

covariant

$$\boxed{\vec{g}_i = \frac{\partial \theta^{j'}}{\partial \theta^i} g_j'}$$

e_i ...

covariant transformation

covariant ~~contravariant~~ transformation

cart equiv

$$e_i' = a_{ij} e_j$$

$$\frac{dx_j}{dx_i'}$$

set $\frac{dx_j}{dx_i} \frac{\theta^j}{\theta^{i'}}$

g_j & g_i' are covariant components

$$\vec{g}^{\,i'} = \frac{\partial \theta^{i'}}{\partial x_j}\,\vec{e}_j \qquad\qquad \vec{g}^{\,i} = \frac{\partial \theta^{i}}{\partial x_j}\,\vec{e}_j$$

$$\vec{g}^{\,i'} = \frac{\partial \theta^{i'}}{\partial \theta^k}\,\underbrace{\frac{\partial \theta^{k}}{\partial x_j}\,\vec{e}_j}_{\vec{g}^{\,k}} \qquad \text{(chain rule)}$$

$$\vec{g}^{\,i'} = \frac{\partial \theta^{i'}}{\partial \theta^k}\,\vec{g}^{\,k}$$

$$\boxed{\vec{g}^{\,i'} = \frac{\partial \theta^{i'}}{\partial \theta^j}\,\vec{g}^{\,j}} \qquad\qquad \boxed{\vec{g}^{\,i} = \frac{\partial \theta^{i}}{\partial \theta^{j'}}\,\vec{g}^{\,j'}}$$

contravariant

notice position of primed component

<u>NOTE</u>: <u>If the primed system is a Cartesian System</u>:

$\Rightarrow \theta^{i'} = x_i$

$$\longrightarrow \vec{g}_i = \frac{\partial x_j}{\partial \theta^i}\,\vec{e}_j$$

$$\longrightarrow \vec{g}^{\,i} = \frac{\partial \theta^i}{\partial x_j}\,\vec{e}_j$$

<u>Same as before</u>!

<u>NOTE</u>: <u>If both primed & unprimed systems are Cartesian Systems</u>:

$$\theta^i = x_i\ ,\ \theta^{i'} = x_i'$$

$$\Rightarrow \vec{e}_i' = \frac{\partial x^j}{\partial x_i'}\,\vec{e}_j = a_{ij}\,\vec{e}_j$$

$$\Rightarrow \vec{e}_i = \frac{\partial x_j'}{\partial x_i}\,\vec{e}_j' = a_{ji}\,\vec{e}_j'$$

<u>Same as before</u>!

Component Transformations

$\vec{V} = V_i \vec{g}^i$ controvariant

$\vec{V} = V_i \left(\frac{\partial \theta^i}{\partial \theta^{j'}} \vec{g}^{j'} \right)$

$\vec{V} = \left(\frac{\partial \theta^i}{\partial \theta^{j'}} V_i \right) \vec{g}^{j'}$ — $\vec{V} = V'_j \vec{g}^{j'}$

$\rightarrow \left[V'_j = \frac{\partial \theta^i}{\partial \theta^{j'}} V_i \right]$ → conclusion relation between covariant

$$\boxed{V'_i = \frac{\partial \theta^j}{\partial \theta^{i'}} V_j}$$ ← Basis of covariant transformations

Similarly

$$\boxed{V_i = \frac{\partial \theta^{j'}}{\partial \theta^i} V'_j}$$

$$\boxed{V^{i'} = \frac{\partial \theta^{i'}}{\partial \theta^j} V^j}$$

Basis of Contravariant transformations

$$\boxed{V^i = \frac{\partial \theta^i}{\partial \theta^{j'}} V^{j'}}$$

Again, if $\theta^i = x_i$, $\theta^{i'} = X_i$

then $\boxed{V_i = a_{ij} V_j}$, $V^{i'} = a_{ij} V^j$

Note that:

$$\vec{g}_{i'} = \frac{\partial \theta^j}{\partial \theta^{i'}} \vec{g}_j$$ ← covariant (sub script

$$\vec{g}^{i'} = \frac{\partial \theta^{i'}}{\partial \theta^j} \vec{g}^j$$ ← contravariant (sup script

→ $\vec{g}_i$ transforms like the covariant components v_i

→ they are often called **covariant base vectors**

→ $\vec{g}^i$ transform like the contravariant components v^i

→ they are often called **contravariant base vectors**

Covariant Derivatives

& Useless "Contravariant Derivatives"

Goal – Find out how cartesian tensor components such as $\frac{\partial \phi}{\partial x_i}$, $\frac{\partial v_i^{©}}{\partial x_j}$, $\frac{\partial T_{ij}^{©}}{\partial x_k}$, etc... transform to curvilinear components.

Cartesian Tensor Components

Remember

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x_i}\vec{e}_i \qquad \frac{\partial \phi}{\partial x_i} = \phi|_i^{©}$$

$$\vec{\nabla}\vec{v} = \frac{\partial v_i^{©}}{\partial x_j}\vec{e}_i\vec{e}_j \qquad \frac{\partial v_i^{©}}{\partial x_j} = v_i|_j$$

$$\vec{\nabla}\underline{\underline{T}} = \frac{\partial T_{ij}^{©}}{\partial x_k}\vec{e}_i\vec{e}_j\vec{e}_k \qquad \frac{\partial T_{ij}^{©}}{\partial x_k} = T_{ij}^{©}|_k$$

etc...

All components (cov., contra., mixed) are the same in a cartesian system:

$$\frac{\partial \phi}{\partial x_i} \equiv \phi|_i^{©} = \phi|_i(x) = \phi|^i(x)$$

$$\frac{\partial v_i^{©}}{\partial x_j} \equiv v_i^{©}|_j = v_i|_j(x) = v^i|_j(x)$$
$$= v_i|^j(x) = v^i|^j(x)$$

$$\frac{\partial T_{ij}^{©}}{\partial x_j} \equiv T_{ij}^{©}|_k = T_{ij}|_k(x) = T^{ij}|_k(x) = T^i{}_{\cdot j}|_k(x) = T_{i\cdot}{}^{j}|_k(x)$$
$$= T_{ij}|^k(x) = T^{ij}|^k(x) = T^i{}_{\cdot j}|^k(x) = T_{i\cdot}{}^{j}|^k(x)$$

NOTE: A curvilinear tensor has 2^N diff. types of components

The so-called covariant derivatives – since the derivative indice appears as a subscript. The covariant derivatives will end up being the logical ones to use.

Scalar Field

Cartesian

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x_i}\vec{e}_i \equiv \phi|_i(x)\,\vec{e}_i \qquad \vec{g}^{\,i}(x)$$

$$\equiv \phi|^{i}(x)\,\vec{e}_i \qquad \vec{g}_i(x)$$

Curvilinear

$$\boxed{\begin{aligned}\vec{\nabla}\phi &= \phi|_i(\theta)\,\vec{g}^{\,i}(\theta)\\ \vec{\nabla}\phi &= \phi|^{i}(\theta)\,\vec{g}_i(\theta)\end{aligned}}$$

Covariant & contravariant Transformation to extend the cartesian tensor components to curvilinear coordinate components!

$$\phi|_i(\theta) = \frac{\partial x_j}{\partial \theta^i}\,\phi|_j(x)$$

$$\phi|_i(\theta) = \frac{\partial x_j}{\partial \theta^i}\,\frac{\partial \phi}{\partial x_j} \quad \text{(chain Rule)}$$

$$\boxed{\phi|_i(\theta) = \frac{\partial \phi}{\partial \theta^i}}$$

Nice and Short!

$$\phi|^{i}(\theta) = \frac{\partial \theta^i}{\partial x_j}\,\phi|^{j}(x)$$

$$\phi|^{i}(\theta) = \frac{\partial \theta^i}{\partial x_j}\,\frac{\partial \phi}{\partial x_j}$$

(No further reduction possible ...)

Better to get $\phi|^i$ from $\phi|_i$ using

$$\boxed{\phi|^{i} = g^{ij}\,\phi|_j}$$

We will always use :

$$\boxed{\vec{\nabla}\phi = \phi|_i\,\vec{g}^{\,i} = \frac{\partial \phi}{\partial \theta^i}\,\vec{g}^{\,i}}$$

Vector Field

Cartesian

$$\vec{\nabla}\vec{v} = \frac{\partial v_i^{©}}{\partial x_j}\vec{e}_i\vec{e}_j \equiv v_i^{©}|_j \, \vec{e}_i\vec{e}_j$$

$\vec{g}_i\vec{g}_j \ (x)$

$$\vec{\nabla}\vec{v} = v_i|_j\,(x)\,\vec{e}_i\vec{e}_j = v^i|_j\,(x)\,\vec{e}_i\vec{e}_j$$

$$= v_i|^j\,(x)\,\vec{e}_i\vec{e}_j = v^i|^j\,(x)\,\vec{e}_i\vec{e}_j$$

Curvilinear

$$\vec{\nabla}\vec{v} = v_i|_j\,(\theta)\,\vec{g}^i\vec{g}^j = v^i|_j\,(\theta)\,\vec{g}_i\vec{g}^j$$
$$= v_i|^j(\theta)\,\vec{g}^i\vec{g}_j = v^i|^j(\theta)\,\vec{g}_i\vec{g}_j$$

(cov. derivative - cov. component) $\quad v_i|_j\,(\theta) = \dfrac{\partial x_k}{\partial \theta^i}\dfrac{\partial x_l}{\partial \theta^j}\, v_k|_l\,(x) = \dfrac{\partial x_k}{\partial \theta^i}\dfrac{\partial x_l}{\partial \theta^j}\, \underline{v_k|_l^{©}}$

(cov. derivative of contra. comp.) $\quad v^i|_j\,(\theta) = \dfrac{\partial \theta^i}{\partial x_k}\dfrac{\partial x_l}{\partial \theta^j}\, v^k|_l\,(x) = \dfrac{\partial \theta^i}{\partial x_k}\dfrac{\partial x_l}{\partial \theta^j}\, \underline{v_k|_l^{©}}$

(contra. der. of cov. comp.) $\quad v_i|^j\,(\theta) = \dfrac{\partial x_k}{\partial \theta^i}\dfrac{\partial \theta^j}{\partial x_l}\, v_k|^l\,(x) = \dfrac{\partial x_k}{\partial \theta^i}\dfrac{\partial \theta^j}{\partial x_l}\, \underline{v_k|_l^{©}}$

(contra. der. of contra. comp.) $\quad v^i|^j\,(\theta) = \dfrac{\partial \theta^i}{\partial x_k}\dfrac{\partial \theta^j}{\partial x_l}\, v^k|^l\,(x) = \dfrac{\partial \theta^i}{\partial x_k}\dfrac{\partial \theta^j}{\partial x_l}\, \underline{v_k|_l^{©}}$

$$\boxed{\frac{\partial v_k^{©}}{\partial x_l}}$$

Similar to the scalar field case, we will be able to show that the "covariant" derivatives $v_i|_j$ & $v^i|_j$ are related to the real derivatives $\frac{\partial v_i(\theta)}{\partial \theta^j}$ & $\frac{\partial v^i(\theta)}{\partial \theta^j}$. The "contravariant" derivatives $v^i|^j$ and $v_i|^j$ will show no simple reductions.

$$V_i|_j(\theta) = \frac{\partial x_k}{\partial \theta^i}\frac{\partial x_l}{\partial \theta^j} V_{k|l}^{©}$$

$$= \frac{\partial x_k}{\partial \theta^i}\frac{\partial x_l}{\partial \theta^j}\frac{\partial V_k^{©}}{\partial x_l}$$

<u>Introduce:</u> $V_k^{©} = V_k(x) = \dfrac{\partial \theta^m}{\partial x_k} V_m(\theta)$

$$V_i|_j(\theta) = \frac{\partial x_k}{\partial \theta^i}\frac{\partial x_l}{\partial \theta^j}\left[\frac{\partial}{\partial x_l}\left(\frac{\partial \theta^m}{\partial x_k} V_m\right)\right]$$

$$= \frac{\partial x_k}{\partial \theta^i}\frac{\partial x_l}{\partial \theta^j}\left[\frac{\partial^2 \theta^m}{\partial x_l \partial x_k} V_m + \frac{\partial \theta^m}{\partial x_k}\frac{\partial V_m}{\partial x_l}\right]$$

$$= \underbrace{\frac{\partial x_k}{\partial \theta^i}\frac{\partial \theta^m}{\partial x_k}}_{\frac{\partial \theta^m}{\partial \theta^i} = \delta_{mi}}\underbrace{\frac{\partial x_l}{\partial \theta^j}\frac{\partial V_m}{\partial x_l}}_{\frac{\partial V_m}{\partial \theta^j}} + \left(\frac{\partial x_k}{\partial \theta^i}\frac{\partial x_l}{\partial \theta^j}\frac{\partial^2 \theta^m}{\partial x_k \partial x_l}\right) V_m$$

$$V_i|_j(\theta) = \frac{\partial V_i}{\partial \theta^j} + \underbrace{\left(\frac{\partial x_k}{\partial \theta^i}\frac{\partial x_l}{\partial \theta^j}\frac{\partial^2 \theta^m}{\partial x_k \partial x_l}\right)}_{\equiv -\Gamma_{ij}^{\ \ m}} V_m$$

$$\boxed{V_i|_j(\theta) = \frac{\partial V_i}{\partial \theta^j} - \Gamma_{ij}^{\ \ m} V_m}$$

$$\boxed{\Gamma_{ij}^{\ \ m} = -\frac{\partial x_k}{\partial \theta^i}\frac{\partial x_l}{\partial \theta^j}\frac{\partial^2 \theta^m}{\partial x_k \partial x_l}}$$

<u>Christoffel Symbols of the 2nd Kind</u>

$$V^i\big|_j(\theta) = \frac{\partial \theta^i}{\partial x_k}\frac{\partial x_\ell}{\partial \theta^j} V_{k|\ell}^{©}$$

$$= \frac{\partial \theta^i}{\partial x_k}\frac{\partial x_\ell}{\partial \theta^j}\frac{\partial V_k^{©}}{\partial x_\ell}$$

<u>Introduce:</u>

$$V_k^{©} = V^k(x) = \frac{\partial x_k}{\partial \theta^m} V^m(\theta)$$

$$= \frac{\partial \theta^i}{\partial x_k}\frac{\partial x_\ell}{\partial \theta^j}\left[\frac{\partial}{\partial x_\ell}\left(\frac{\partial x_k}{\partial \theta^m} V^m\right)\right]$$

$$= \frac{\partial \theta^i}{\partial x_k}\frac{\partial x_\ell}{\partial \theta^j}\left[\frac{\partial}{\partial x_\ell}\left(\frac{\partial x_k}{\partial \theta^m}\right) V^m + \frac{\partial x_k}{\partial \theta^m}\frac{\partial V^m}{\partial x_\ell}\right]$$

$$= \underbrace{\frac{\partial \theta^i}{\partial x_k}\frac{\partial x_k}{\partial \theta^m}}_{\frac{\partial \theta^i}{\partial \theta^m} = \delta_{im}}\underbrace{\frac{\partial x_\ell}{\partial \theta^j}\frac{\partial V^m}{\partial x_\ell}}_{\frac{\partial V^m}{\partial \theta^j}} + \underbrace{\frac{\partial \theta^i}{\partial x_k}\frac{\partial x_\ell}{\partial \theta^j}\frac{\partial}{\partial x_\ell}\left(\frac{\partial x_k}{\partial \theta^m}\right)}_{\equiv \Omega^i_{mj}} V^m$$

$$\boxed{V^i\big|_j = \frac{\partial V^i}{\partial \theta^j} + \Omega^i_{mj} V^m}$$

$$\boxed{\Omega^i_{mj} \equiv \frac{\partial \theta^i}{\partial x_k}\frac{\partial x_\ell}{\partial \theta^j}\frac{\partial}{\partial x_\ell}\left(\frac{\partial x_k}{\partial \theta^m}\right)}$$

Later we will show that $\boxed{\Omega^i_{mj} = \Gamma^i_{mj}}$

(<u>not obvious at this point</u>)

$$X = F(x)G(x)H(x)$$

$$\frac{dX}{dx} = \frac{dF(x)}{dx}G(x)H(x) + F(x)\left(\frac{dG(x)}{dx}H(x) + G(x)\frac{dH(x)}{dx}\right)$$

$$\frac{dX}{dx} = \frac{dF(x)}{dx}G(x)H(x) + F(x)\frac{dG(x)}{dx}H(x) + F(x)G(x)\frac{dH(x)}{dx}$$

Similar calculations reveal that:

$v_i|^j(\theta)$ not related to $\dfrac{\partial v_i}{\partial \theta^j}$ & v_m

$v^i|^j(\theta)$ not related to $\dfrac{\partial v^i}{\partial \theta^j}$ & v^m

We typically do not "use" the contravariant derivatives, since they are difficult (more difficult) to evaluate.

We can find alternative formulas for the Γ_{ij}^{k} & Ω_{ij}^{k}, and actually show they are equal:

Consider a different approach:

$$\frac{\partial v_i(\theta)}{\partial \theta^j} = \frac{\partial v_i(\theta)}{\partial x_k} \frac{\partial x_k}{\partial \theta^j} \qquad \text{(chain rule)}$$

$$v_i(\theta) = \frac{\partial x_l}{\partial \theta^i} v_l(x)$$

$$= \frac{\partial}{\partial x_k}\left[\frac{\partial x_l}{\partial \theta^i} v_l(x)\right] \frac{\partial x_k}{\partial \theta^j}$$

$$\frac{\partial v_i(\theta)}{\partial \theta^j} = \frac{\partial}{\partial x_k}\left(\frac{\partial x_l}{\partial \theta^i}\right) v_l(x) \frac{\partial x_k}{\partial \theta^j} + \frac{\partial x_l}{\partial \theta^i}\frac{\partial x_k}{\partial \theta^j} \cancel{\frac{\partial v_l(x)}{\partial x_k}}$$

© $v_l|k = v_l|_k(x)$

$$\frac{\partial V_i(\theta)}{\partial \theta^j} = \underbrace{\frac{\partial\left(\frac{\partial x_\ell}{\partial \theta^i}\right)}{\partial x_k}\frac{\partial x_k}{\partial \theta^j}}_{\frac{\partial\left(\frac{\partial x_\ell}{\partial \theta^i}\right)}{\partial \theta^j} = \frac{\partial^2 x_\ell}{\partial \theta^i \partial \theta^j}} V_\ell(x) + \underbrace{\frac{\partial x_\ell}{\partial \theta^i}\frac{\partial x_k}{\partial \theta^j} V_{\ell|k}(x)}_{V_{i|j}(\theta)}$$

$$\frac{\partial V_i(\theta)}{\partial \theta^j} = \frac{\partial^2 x_\ell}{\partial \theta^i \partial \theta^j} V_\ell(x) + V_{i|j}(\theta)$$

$$= \frac{\partial \theta^m}{\partial x_\ell} V_m(\theta)$$

$$\rightarrow \boxed{V_{i|j}(\theta) = \frac{\partial V_i}{\partial \theta^j} - \left(\frac{\partial^2 x_\ell}{\partial \theta^i \partial \theta^j}\frac{\partial \theta^m}{\partial x_\ell}\right) V_m}$$

<u>In our previous calculation</u>, <u>we had</u> :

$$V_{i|j}(\theta) = \frac{\partial V_i}{\partial \theta^j} + \underbrace{\left(\frac{\partial x_k}{\partial \theta^i}\frac{\partial x_\ell}{\partial \theta^j}\frac{\partial^2 \theta^m}{\partial x_k \partial x_\ell}\right)}_{\left(\text{we called this } -\Gamma^m_{ij}\right)} V_m$$

Comparison of the two formulas gives:

$$\boxed{V_i|_j = \frac{\partial V_i}{\partial \theta^j} - \Gamma_{ij}^{\ m} V_m}$$ for both

if

$$\boxed{\Gamma_{ij}^{\ m} = -\frac{\partial x_k}{\partial \theta^i}\frac{\partial x_\ell}{\partial \theta^j}\frac{\partial^2 \theta^m}{\partial x_k \partial x_\ell} = +\frac{\partial^2 x_\ell}{\partial \theta^i \partial \theta^j}\frac{\partial \theta^m}{\partial x_\ell}}$$

(NOT OBVIOUS)

Version never seen

Version found in all books

(Easier to Use)

Either formula can be used to evaluate the Christoffel Symbols of the 2nd Kind.

Note that this formula shows us that:

$$\boxed{\Gamma_{ij}^{\ m} = \Gamma_{ji}^{\ m}}$$

Note that the "book version" is easier to use for two reasons:

Called symbols because they are NOT tensor components

1. Only 3 terms (compared to 9) in the sum.
2. It is typically easier to evaluate derivatives of the cartesian coordinates with respect to the curvilinear coordinates.

If we use the alternate approach with $\dfrac{\partial v^i(\theta)}{\partial \theta^j}$ we get:

$$\frac{\partial v^i(\theta)}{\partial \theta^j} = \frac{\partial v^i(\theta)}{\partial x_k}\frac{\partial x_k}{\partial \theta^j}$$

$$= \frac{\partial}{\partial x_k}\left(\frac{\partial \theta^i}{\partial x_\ell} v^\ell(x)\right)\frac{\partial x_k}{\partial \theta^j}$$

$$= \frac{\partial^2 \theta^i}{\partial x_k \partial x_\ell}\frac{\partial x_k}{\partial \theta^j} v^\ell(x) + \overbrace{\frac{\partial \theta^i}{\partial x_\ell}\frac{\partial x_k}{\partial \theta^j}\frac{\partial v^\ell(x)}{\partial x_k}}^{v^i|_j(\theta)}$$

$$= \frac{\partial x_\ell}{\partial \theta^m} v^m(\theta) \qquad V_\ell|_k^{(c)} = v^\ell|_k(x)$$

$$\frac{\partial v^i(\theta)}{\partial \theta^j} = \left(\frac{\partial x_k}{\partial \theta^j}\frac{\partial x_\ell}{\partial \theta^m}\frac{\partial^2 \theta^i}{\partial x_k \partial x_\ell}\right) v^m(\theta) + v^i|_j(\theta)$$

or

$$v^i|_j(\theta) = \frac{\partial v^i(\theta)}{\partial \theta^j} + \underbrace{\left(-\frac{\partial x_k}{\partial \theta^j}\frac{\partial x_\ell}{\partial \theta^m}\frac{\partial^2 \theta^i}{\partial x_k \partial x_\ell}\right)}_{\Gamma^i_{jm} = \Gamma^i_{mj}} v^m(\theta)$$

(from last page)

$$\boxed{v^i|_j(\theta) = \frac{\partial v^i}{\partial \theta^j} + \Gamma^i_{mj} v^m}$$

Remember that in our previous calculation we had:

$$V^i|_j(\theta) = \frac{\partial V^i}{\partial \theta^j} + \Omega^i_{mj} V^m$$

where: $$\Omega^i_{mj} = \frac{\partial \theta^i}{\partial x_k} \frac{\partial x_\ell}{\partial \theta^j} \frac{\partial}{\partial x_\ell}\left(\frac{\partial x_k}{\partial \theta^m}\right)$$

Comparison of the two formulas gives:

$$\boxed{\Omega^i_{mj} = \Gamma^i_{mj} = \frac{\partial^2 x_\ell}{\partial \theta^m \partial \theta^j} \frac{\partial \theta^i}{\partial x_\ell} = -\frac{\partial x_k}{\partial \theta^m} \frac{\partial x_\ell}{\partial \theta^j} \frac{\partial^2 \theta^i}{\partial x_k \partial x_\ell} = \frac{\partial \theta^i}{\partial x_k} \frac{\partial x_\ell}{\partial \theta^j} \frac{\partial}{\partial x_\ell}\left(\frac{\partial x_k}{\partial \theta^m}\right)}$$

★ Easiest to Evaluate

SUMMARY for Vector Field

$$\boxed{\begin{aligned} V_i|_j(\theta) &= \frac{\partial V_i}{\partial \theta^j} - \Gamma_{ij}^{\;m} V_m \\ V^i|_j(\theta) &= \frac{\partial V^i}{\partial \theta^j} + \Gamma^i_{mj} V^m \end{aligned}}$$

$V_i|^j(\theta)$ & $V^i|^j(\theta)$ are <u>inconvenient</u>

$$\boxed{\Gamma_{ij}^{\;k} = \underbrace{\frac{\partial^2 x_\ell}{\partial\theta^i \partial\theta^j}\frac{\partial\theta^k}{\partial x_\ell}}_{\text{This formula is easiest to evaluate!}} = -\frac{\partial x_\ell}{\partial\theta^i}\frac{\partial x_m}{\partial\theta^j}\frac{\partial^2\theta^k}{\partial x_\ell \partial x_m} = \frac{\partial\theta^k}{\partial x_\ell}\frac{\partial x_m}{\partial\theta^j}\frac{\partial}{\partial x_m}\left(\frac{\partial x_\ell}{\partial\theta^i}\right)}$$

This formula is <u>easiest</u> <u>to evaluate</u>!

(Its the one you will find in other books)

Tensor Field

Cartesian:

$$\vec{\nabla}\,\mathbb{T} = \frac{\partial T_{ij}^{\copyright}}{\partial x_k}\,\vec{e}_i\,\vec{e}_j\,\vec{e}_k \equiv T_{ij}|_k^{\copyright}\,\vec{e}_i\,\vec{e}_j\,\vec{e}_k$$

$$\vec{\nabla}\,\mathbb{T} = \boxed{T_{ij}|_k\,(x)}\;\vec{e}_i\,\vec{e}_j\,\vec{e}_k$$

$$= T^{ij}|_k\,(x)\;\vec{e}_i\,\vec{e}_j\,\vec{e}_k$$

etc... (8 forms)

Curvilinear

$$\vec{\nabla}\,\mathbb{T} = T_{ij}|_k\,(\theta)\;\vec{g}^i\,\vec{g}^j\,\vec{g}^k$$

$$= T^{ij}|_k\,(\theta)\;\vec{g}_i\,\vec{g}_j\,\vec{g}^k$$

etc... (8 forms)

Covariant Derivatives

$$T_{ij}|_k\,(\theta) = \frac{\partial x_l}{\partial \theta^i}\frac{\partial x_m}{\partial \theta^j}\frac{\partial x_n}{\partial \theta^k}\,T_{lm}|_n\,(x) = \frac{\partial x_l}{\partial \theta^i}\frac{\partial x_m}{\partial \theta^j}\frac{\partial x_n}{\partial \theta^k}\,T_{lm|n}^{\copyright}$$

$$T^{ij}|_k\,(\theta) = \frac{\partial \theta^i}{\partial x_l}\frac{\partial \theta^j}{\partial x_m}\frac{\partial x_n}{\partial \theta^k}\,T^{lm}|_n\,(x) = \frac{\partial \theta^i}{\partial x_l}\frac{\partial \theta^j}{\partial x_m}\frac{\partial x_n}{\partial \theta^k}\,T_{lm|n}^{\copyright}$$

$T^{i}_{\cdot j}|_k\,(\theta)$

$T_{i}^{\cdot j}|_k\,(\theta)$

Contravariant Derivatives

$T_{ij}|^k\,(\theta)$

$T^{ij}|^k\,(\theta)$

$T^{i}_{\cdot j}|^k\,(\theta)$

$T_{i}^{\cdot j}|^k\,(\theta)$

etc...

Using procedures similar to those used with the vector field, we can show that:

1. The contravariant derivatives do not evaluate to any convenient form — we won't use.

2. The covariant derivatives are:

$$T_{ij}|_k(\theta) = \frac{\partial T_{ij}}{\partial \theta^k} - \Gamma^l_{ik} T_{lj} - \Gamma^l_{jk} T_{il}$$

$$T^{ij}|_k(\theta) = \frac{\partial T^{ij}}{\partial \theta^k} + \Gamma^i_{lk} T^{lj} + \Gamma^j_{lk} T^{il}$$

$$T^{i}{}_{\cdot j}|_k(\theta) = \frac{\partial T^{i}{}_{\cdot j}}{\partial \theta^k} + \Gamma^i_{lk} T^{l}{}_{\cdot j} - \Gamma^l_{jk} T^{i}{}_{\cdot l}$$

$$T_{i\cdot}{}^{j}|_k(\theta) = \frac{\partial T_{i\cdot}{}^{j}}{\partial \theta^k} - \Gamma^l_{ik} T_{l\cdot}{}^{j} + \Gamma^j_{lk} T_{i\cdot}{}^{l}$$

Christoffel Symbols of the First Kind

$$\boxed{\Gamma_{ijk} \equiv g_{\ell k} \Gamma_{ij}^{\ \ \ell}}$$

$$\boxed{\Gamma_{ij}^{\ \ k} = g^{\ell k} \Gamma_{ij\ell}}$$

Think of $\Gamma_{ij}^{\ \ k}$ as $\Gamma_{ij\cdot}^{\ \ \ k}$

since $\Gamma_{ij}^{\ \ \ell} = \Gamma_{ji}^{\ \ \ell}$

we have:

$$\boxed{\Gamma_{ijk} = \Gamma_{jik}}$$

Some Other Formulas which can be Established:

$$\frac{\partial \vec{g}_i}{\partial \theta^j} = \Gamma_{ij}^{\ k} \vec{g}_k = \Gamma_{ijk} \vec{g}^k$$

$$\frac{\partial \vec{g}^i}{\partial \theta^j} = -\Gamma_{jk}^{\ i} \vec{g}^k$$

$$\frac{\partial \vec{V}}{\partial \theta^i} = V_j|_i \, \vec{g}^j = V^j|_i \, \vec{g}_j$$

$$\Gamma_{ij}^{\ k} = \frac{\partial \vec{g}_i}{\partial \theta^j} \cdot \vec{g}^k$$

$$\Gamma_{ijk} = \frac{\partial \vec{g}_i}{\partial \theta^j} \cdot \vec{g}_k$$

$$\frac{\partial g_{ij}}{\partial \theta^k} = \Gamma_{ikj} + \Gamma_{jki}$$

$$\Gamma_{ijk} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \theta^j} + \frac{\partial g_{jk}}{\partial \theta^i} - \frac{\partial g_{ij}}{\partial \theta^k}\right]$$

$$\Gamma_{ij}^{\ i} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial \theta^j} = \frac{1}{2g} \frac{\partial g}{\partial \theta^j}$$

Evaluation of the Christoffel Symbols

Use $$\boxed{\Gamma_{ij}^{\ k} = \frac{\partial^2 x_\ell}{\partial\theta^i\partial\theta^j}\frac{\partial\theta^k}{\partial x_\ell}}$$ (one sum-on ℓ)

or $$\boxed{\Gamma_{ij}^{\ k} = g^{\ell k}\,\Gamma_{ij\ell} = \frac{g^{\ell k}}{2}\left[\frac{\partial g_{i\ell}}{\partial\theta^j} + \frac{\partial g_{j\ell}}{\partial\theta^i} - \frac{\partial g_{ij}}{\partial\theta^\ell}\right]}$$ (one sum - on ℓ)

for any given coordinate there are $3^3 = 27$ of these

[Since $\Gamma_{ij}^{\ k} = \Gamma_{ji}^{\ k}$ it ends up only ~~18~~ **16** are unique]

Cartesian

$$\theta^i = x_i \longrightarrow \frac{\partial x_i}{\partial\theta^j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

$$g_{ij} = g^{ij} = \delta_{ij} = (0 \text{ or } 1)$$

1st or 2nd form above $\longrightarrow$ $\boxed{\Gamma_{ij}^{\ k} = 0}$ $\boxed{\Gamma_{ijk} = 0}$

Cylindrical

$\theta^1 = r$	$x_1 = x$	$x = r\cos\theta$	$r = \sqrt{x^2+y^2}$
$\theta^2 = \theta$	$x_2 = y$	$y = r\sin\theta$	$\theta = \tan^{-1}(y/x)$
$\theta^3 = z$	$x_3 = z$	$z = z$	$z = z$

$g_{11} = g_{33} = 1$

$g_{22} = r$

$g_{ij} = 0 \ (i \neq j)$

$$\Gamma_{22}^{\ 1} = \frac{\partial^2 x_\ell}{\partial\theta^2\partial\theta^2}\frac{\partial\theta^1}{\partial x_\ell} = \frac{\partial^2 x_\ell}{\partial\theta^{\,2}}\frac{\partial r}{\partial x_\ell}$$

$$= \frac{\partial^2 x}{\partial\theta^2}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial\theta^2}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial\theta^2}\frac{\partial r}{\partial z}$$

$$\Gamma^1_{22} = (-r\cos\theta)\left(\frac{2x}{2\sqrt{x^2+y^2}}\right) + (-r\sin\theta)\left(\frac{2y}{2\sqrt{x^2+y^2}}\right) + 0$$

$$\Gamma^1_{22} = (-r\cos\theta)\left(\frac{r\cos\theta}{r}\right) + (-r\sin\theta)\left(\frac{r\sin\theta}{r}\right)$$

$$\Gamma^1_{22} = -r\left(\cos^2\theta + \sin^2\theta\right)$$

$$\boxed{\Gamma^1_{22} = -r}$$

Similarly $\boxed{\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}}$

and all other $\boxed{\Gamma^k_{ij} = 0}$