Corrections to Second Printing,
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as of January 1979

Page	Line	Correction
221	7	Change "Lagerstrom (1975)" to "... (1976)"
235	12*	Change to "Dorrepaal, O'Neill, and Ranger (1976)"
236	6	Change to "Coutanceau, 1974"
249	1	Put reference to ACRIVOS in correct alphabetical order following ABBOTT.
252	5	Change to "DORREPALL, J. M., O'NEILL, M. E., and RANGER, K. B. (1976). Axisymmetric Stokes flow past a spherical cap. J. Fluid Mech. 75, 273-286. [235]"
255	17*	Change to "Lagerstrom, P. A. (1976). Forms of singular asymptotic expansions in layer-type problems. Rocky Mtn. J. Math. 6, 609-635."
257	20	Change to "... downstream. AGARD Conf. Proc. No. 168 Flow Sep., pp. 4-1 to 4-10"
261	13*	Change to "Stos. 16, No. 3, 601-614; J. Appl. Math. Mech. 28, 90-101. [6, 173, 239]"
261	6*	Change to "... series. In Tenth Symposium, Naval Hydrodynamics, eds. R. D. Cooper and S. W. Doroff, 449-457. U. S. Govt. Printing Office, Washington. [215, 243]"

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PERTURBATION METHODS IN
ENGINEERING MECHANICS

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Notes for Course AA 219

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PREFACE

These notes are intended to supplement the author's book, Perturbation Methods in Fluid Mechanics, which was published by Academic Press in 1964, and in an "Annotated edition" by Parabolic Press in 1975. The changes here represent first, of course, modernization to include advances made during a decade of vigorous activity, particularly in the understanding of singular perturbations.

Second, as indicated by the change of title, the range of problems has been broadened to cover other branches of applied mechanics: elasticity, dynamics, orbital mechanics, electrostatics, and the like. Fluid mechanics is still pre-eminent, however, not only because it is the author's field of research, but also because techniques for treating perturbation problems have been largely developed within fluid dynamics.

In the last fifteen years the study of perturbation methods has evolved from an esoteric speciality to a standard subject for graduate students in engineering. Some knowledge of the subject is essential for carrying out analytical research, and hence to an understanding of the current literature. Consequently a number of other books on the subject have appeared. Some are purely mathematical in outlook (Eckhaus 1973), but several are devoted to physical problems (Cole 1968, Nayfeh 1973).

We here insist on the physical viewpoint throughout. This permits us, as engineers, to bolster mathematical reasoning with physical insight. Another advantage is that we can emphasize throughout the role of scales of space and time. This point of view helps to explain why some perturbations are singular, and clarifies the various techniques used to handle them.

References

COLE, J. D. 1968 Perturbation Methods in Applied Mathematics. Blaisdell, Waltham, Mass.

ECKHAUS, WIKTOR 1973 Matched Asymptotic Expansions and Singular Perturbations. North-Holland, Amsterdam

NAYFEH, Ali Hasan 1973 Perturbation Methods. Wiley-Interscience, New York

VAN DYKE, Milton 1964 Perturbation Methods in Fluid Mechanics. Academic, New York. Annotated edition, 1975, Parabolic Press, Stanford, Calif.

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CONTENTS

Chapter I

THE NATURE OF PERTURBATION THEORY

1.1. Approximation in engineering mechanics

A problem in mechanics can be attacked by any one (or a combination) of three techniques: experiment, analysis, or numerical simulation. Engineers are turning increasingly to numerical solution as computers are improved, because of the limitations of analysis.

Analytic solution is impeded by nonlinearity, by awkward geometry, or by unknown boundaries. Few problems in modern technology can realistically be regarded as linear, so that the whole apparatus of classical mathematical physics, based as it is on superposition, is of little avail. The few exact solutions that can be found are usually either trivial (e.g., a parallel stream of fluid, a uniform state of stress) or are self-similar, and therefore often physically unrealistic.

The remaining possibility for analysis is to approximate. Approximation is an art, and the inventors of successfull approximations are remembered by posterity: Prandtl's boundary-layer theory, Rayleigh-Ritz method, Hertz contact theory, etc.

1.2. Systematic approximations

This course is concerned with the systematic approximation of the solutions of physical problems. A systematic, or rational approximation is based on some known exact solution (possibly a trivial one). The problem to be solved is envisioned as differing only slightly, in some sense, from that basic problem. Consequently one may hope that the two solutions also differ only slightly. One accordingly perturbs the basic solution to find an approximation.

Usually one is satisfied with the first approximation, sometimes the second; only rarely does he proceed to high order. However, it is essential that it be possible in principle to continue a rational approximation indefinitely.

Such an approximation is by construction asymptotic; it becomes more accurate as the difference between the basic and actual problems decreases. The series may have zero radius of convergence, yet a few terms provide a close approximation.

Not all useful approximations are rational in this sense. An example of an irrational approximation of considerable utility is the shock-expansion method for supersonic flow past an airfoil or hypersonic flow past a pointed body of revolution (Flügge 1962, p. 78-2), which has resisted determined efforts to embed it into a systematic scheme of successive approximations. However, approximations that appear irrational when originally proposed are often found to be rational on further analysis. Such has been the history, for example, of simple beam theory in elasticity (Timoshenko & Goodier 1951, sec. 21-22) and of lifting-line theory in aerodynamics (Van Dyke 1964).

1.3. Parameter and coordinate perturbations

To systematize a perturbation scheme, the difference between the basic and actual problems is characterized by a perturbation quantity. It can always be defined so as to be small, and tend to zero when the two problems coalesce. (For example, a very long elastic beam or airplane wing has a very small thickness.) We denote the perturbation quantity generically by ε.

In physical problems the perturbation quantity must be dimensionless: it makes no sense to say that a flow is slow or a plate is thin; one says instead that the Reynolds number is low or the thickness ratio small. This dimensionless quantity is often (as in these two examples) the ratio of two lengths:

$$\varepsilon = \frac{s}{L} \tag{1.1}$$

Here the longer reference length L is almost always a characteristic or parametric length. It is not necessarily a geometric dimension, however. For example, if liquid of kinematic viscosity ν flows with speed U, we may take as characteristic length the ratio ν/U, which has the dimensions of length.

Often the shorter reference length s is also a characteristic length. In this case the ratio ε is a parameter in the problem; and as it tends to zero we speak of a parameter perturbation. The thickness ratio of a plate or shell is in this category, and so is the Reynolds number, because it is the ratio of the geometric to the viscous length.

The other possibility is that the shorter length s is a coordinate. For example, in a problem governed by parabolic or hyperbolic differential equations, so that disturbances spread only in one direction (as for supersonic flow, or a viscous boundary layer), we can approximate the solution for distances small compared with the length L. Then ε is not a parameter but a dimensionless coordinate. We then speak of a coordinate perturbation.

The difference between a parameter and a coordinate is somewhat subtle. A parameter is a "variable" that does not change. That is, to solve a given problem one must consider a range of values of coordinates, but not of parameters. Consequently, the governing equations contain derivatives and integrations with respect to the coordinates, but not the parameters.

The perturbation quantity ε, whether it is a parameter or a coordinate, can in unsteady problems also be the ratio of two times. For example, the problem of a sphere set abruptly into motion through a viscous liquid has been approximated for small values of the dimensionless time $t/(\nu/U^2)$.

Occasionally the perturbation quantity cannot be identified as the ratio of either two lengths or two times. An example is the Mach number, which is small if a gas flow is only slightly compressible. The Mach number can be regarded as the ratio of two speeds, or its square as the ratio of two energies, but scarcely as the ratio of two lengths or times.

An unusual example was introduced by Garabedian (1956) in studying the flow of a free jet of liquid squirted from a circular orifice (Fig. 1.1). Although the flow is governed simply by the Laplace equation, the unknown boundary makes the problem impossible to solve exactly. But the corresponding problem in plane flow is simple, thanks to conformal mapping. Garabedian therefore envisaged the problem in $2+\varepsilon$ space dimensions, and approximated the solution for small ε. He thus calculated for $\varepsilon = 1$ a value of the contraction of the jet that agrees perfectly with subsequent numerical computations. This is evidently a parameter perturbation, but of a rather curious kind that illustrates the ingenuity that can be exercised in the choice of perturbation quantity.

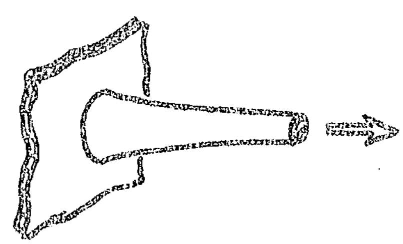


Fig. 1.1. Free jet from circular orifice

1.4. The form of a regular perturbation series

Let the perturbation quantity ε be defined such that in the full problem (governing equations and boundary conditions) it appears only in integral powers including the first.* We then consider first the simple situation of a so-called regular perturbation problem, which means that the approximation is uniformly valid throughout the region of physical interest. Then ε can appear only in integral powers in the solution as well. Thus the perturbation expansion is a formal power series, of the form

$$f(x;\varepsilon) = f_1(x) + \varepsilon f_2(x) + \varepsilon^2 f_3(x) + \ldots \qquad (1.2)$$

* Later, in chapter IV, we shall want to normalize ε in a different way.

More generally, however, the solution might have been normalized differently so as to be multiplied by some power of ε. Then it has the form

$$f(x;\varepsilon) = \varepsilon^{\alpha}\left[f_1(x) + \varepsilon f_2(x) + \varepsilon^2 f_3(x) + \dots\right] \quad (1.3)$$

where α is not a non-negative integer. The factor ε^{α} could evidently be absorbed into f by redefinition, reducing (1.3) to (1.2).

Often the solution is analytic at the origin of ε, in which case (1.2) is a Taylor series with non-zero radius of convergence. For (1.3) the origin may be a regular singular point, in which case the expression in brackets is a Taylor series. In other problems, however, the solution is non-analytic in ε. For example, it might have the form

$$f(x;\varepsilon) = f_1(x) + \varepsilon f_2(x) + \varepsilon^2 f_3(x) + \dots$$
$$+ e^{-1/\varepsilon}\left[g_1(x) + \varepsilon g_2(x) + \dots\right] + \dots \quad (1.4)$$

The term $e^{-1/\varepsilon}$ has an essential singularity at the origin of ε. Nevertheless, it is smaller than any power of ε on the positive real axis*, so that it contributes nothing to the formal power series in the first line of (1.4). Such transcendentally small terms slip through our fingers if we construct the approximation in conventional fashion.

1.5. Examples of regular parameter perturbations

We now examine several examples of regular perturbation series taken from the literature of applied mechanics, and comment briefly on their particularities. These series are typical, except that they have been selected for having more terms than is usual, in order to display their patterns more clearly. We begin with some parameter perturbations.

First, the effects of compressibility of the air flowing past a body were investigated independently by Janzen (1913) and Rayleigh (1916). They perturbed the classical incompressible potential flow past a circle (Fig. 1.2), assuming that the free-stream Mach number M (the ratio of the flow speed U to the undisturbed speed of sound) is small. Higher terms were calculated by Imai (1941) and Simasaki (1956). For the maximum speed, occurring at the sides of the circle, this gives (with the adiabatic exponent γ of the gas taken as 1.405)

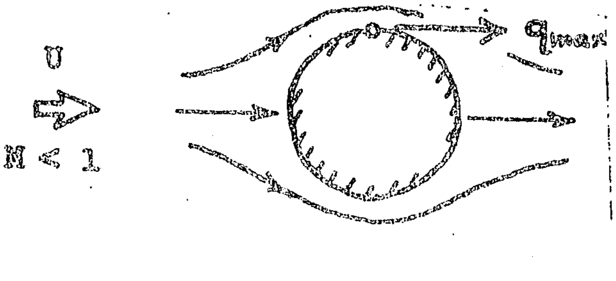


Fig. 1.2. Subsonic potential flow past circle.

* That is, $\lim \varepsilon^n e^{-1/\varepsilon} = 0$ for all n as $\varepsilon \to 0$ through positive real values.

$$\frac{q_{max}}{U} = 2.00000 + 1.16667\, M^2 + 2.58129\, M^4 + 7.53386\, M^6 + 25.69342\, M^8 + 96.79287\, M^{10} + \ldots \qquad (1.5)$$

This is a parameter expansion, of the simple form (1.2). Only even powers of M appear, because the governing equations contain M^2 rather than M. Hoffman (1970) and Reynolds (unpublished) have calculated several more terms by delegating the tedious computations to a digital computer.

The signs are all positive. By analogy with such a simple model function as

$$\frac{1}{1-\varepsilon} = 1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \ldots \qquad (1.6)$$

this pattern of signs suggests a singularity on the positive axis of M. The rapid increase of the coefficients also suggests that the singularity lies considerably closer than $M = 1$. In fact, using techniques discussed in Chapter VIII, Hoffman (1970) shows that, as conjectured by Rayleigh, the series converges only up to the critical Mach number, about 0.40, at which the maximum speed becomes locally sonic.

Second, Stokes (1851) calculated the drag of a sphere in slow viscous flow by neglecting the nonlinear convective terms in the Navier-Stokes equations. Oseen (1910) found a second approximation by linearizing instead about the uniform stream; and Goldstein (1929) added four more terms according to that linearization. This gives

$$C_D = \frac{D}{\pi\rho U^2 a^2} = \frac{6}{R}\left[1 + \frac{3}{4}\left(\frac{R}{2}\right) - \frac{19}{80}\left(\frac{R}{2}\right)^2 + \frac{71}{320}\left(\frac{R}{2}\right)^3 - \frac{30,179}{134,400}\left(\frac{R}{2}\right)^4 + \frac{122,519}{537,600}\left(\frac{R}{2}\right)^5 - \ldots\right] \qquad (1.7)$$

where the last coefficient has been corrected by Shanks (1955). Here $R = Ua/\nu$ is the Reynolds number based on radius a, the expansion being written in powers of $R/2$ to keep the coefficients from decreasing rapidly.

This parameter expansion has the form (1.3) with $\alpha = -1$, simply because the definition of drag coefficient used is inappropriate to small Reynolds numbers. The signs now alternate, a pattern simulated by the model function

$$\frac{1}{1+\varepsilon} = 1 - \varepsilon + \varepsilon^2 - \varepsilon^3 + \ldots \qquad (1.8)$$

This suggests that the nearest singularity lies on the negative axis of R, so that it has no physical significance. The series has been extended to 24 terms by computer (Van Dyke 1970); and analysis of the coefficients, as described in Chapter 8, reveals that convergence is limited by a simple pole at $R = -2.09086$.

1.6. Examples of regular coordinate perturbations

We continue with examples where the perturbation quantity is a dimensionless coordinate rather than a parameter.

Third, Howarth (1938) has calculated the boundary layer on a wall having the inviscid surface speed $U(1-x/8L)$. He perturbs the self-similar solution of Prandtl and Blasius for constant speed U, by expanding for small dimensionless distance x downstream. For the coefficient of skin friction he finds

$$c_f = \frac{\tau}{\rho U^2} = \tfrac{1}{2}(UL/\nu)^{-\frac{1}{2}}(x/L)^{-\frac{1}{2}}\Big[1.328242 - 1.02054\,(x/L) - .06926\,(x/L)^2 - .0560\,(x/L)^3 - .0372\,(x/L)^4 - .0272\,(x/L)^5 - .0212\,(x/L)^6 - .0174\,(x/L)^7 - .0147\,(x/L)^8 - \dots\Big] \quad (1.9)$$

This has the form (1.3) with $\alpha = -\frac{1}{2}$. The signs have the fixed pattern of the model series (1.6), suggesting that, as in our first example (1.5), the nearest singularity lies on the positive real axis. Analysis of the coefficients (Van Dyke 1974) indicates a square-root zero at $x/L = 0.96$, in accord with numerical integrations and local solutions. Fig. 1.3 shows how beyond that value adding successive terms to the sum clearly indicates divergence.

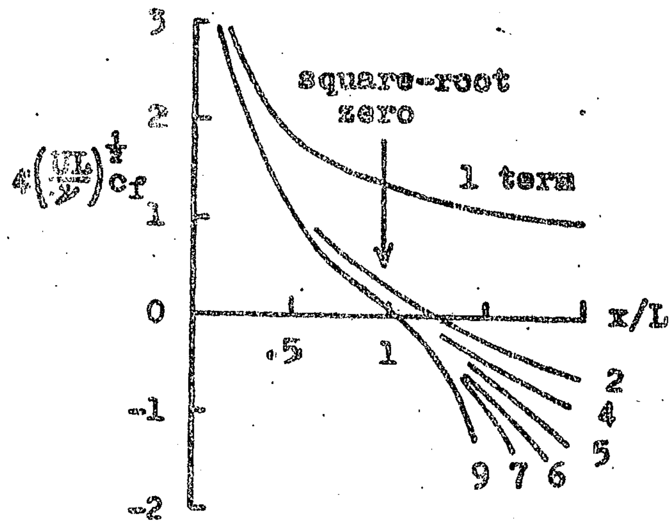


Fig. 1.3. Howarth's series (1.9) for skin friction

The coordinate in a coordinate expansion may also be the time. Our fourth example is the series calculated by Katagiri (1974) for the viscous torque on an infinite disk set impulsively into motion in its own plane with angular velocity Ω. At the first instant the motion at any point is that given by the simple self-similar solution of the Navier-Stokes equations for translation of a plate, and perturbing that result gives for later times the dimensionless torque

$$\tau^* = (\Omega t)^{-\frac{1}{2}}\Big[1.128379 + .188412\,(\Omega t)^2 - .0175867\,(\Omega t)^4 + .00179678\,(\Omega t)^6 - .000181775\,(\Omega t)^8 + .0000181\text{3}4\,(\Omega t)^{10} - .000001789\,(\Omega t)^{12} + .000000175\,(\Omega t)^{14} - .000000017\,(\Omega t)^{16} + \dots\Big] \quad (1.10)$$

The signs alternate, indicating as in our second example (1.7) that the nearest singularity lies on the negative axis. Figure 1.4 suggests that convergence is thereby limited to about $\Omega t = 3$, and later in Chapter 8 we show how the radius of convergence can be estimated accurately as 3.304.

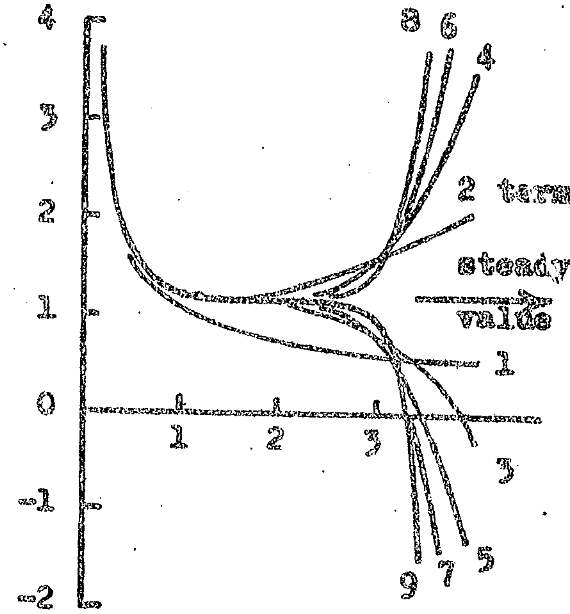


Fig. 1.4. Katagiri's series (1.10) for torque on spun disk

In our first and third examples (1.5, 1.9) the singularity has "physical" significance, though it would be eliminated by adopting more realistic governing equations. On the other hand, in our second and fourth examples (1.7, 1.10) the singularity on the negative axis has no physical meaning. It therefore represents an artificial limitation on the range of utility of the series. We shall see in Chapter 8 how that range can be extended by some form of analytic continuation. Such extension is usually more desirable in a coordinate than a parameter expansion, because one generally wants to proceed to as large values of a coordinate as possible, whereas the practical range of a parameter may be limited. (For example, the thickness ratio of an airfoil, which is assumed small in thin-airfoil theory, seldom exceeds 1/5.)

Another possibility is to supplement the expansion for small values of the coordinate by another expansion for large values. We call this an <u>inverse coordinate expansion</u>. (Of course one sometimes also expands for large as well as small values of a parameter.) An example is the integrated skin friction on a semi-infinite flat plate. The familiar boundary-layer approximation of Prandtl and Blasius is only a first approximation for large distances downstream. Higher approximations (Imai 1957) give, for the drag back to the distance x from the leading edge,

$$C_F = 1.328\, R_x^{-\frac{1}{2}} + 2.326\, R_x^{-1} - 1.102\, R_x^{-\frac{3}{2}} \log R_x + C_1 R_x^{-\frac{3}{2}} + \ldots \qquad (1.11)$$

Here $R_x = Ux/\nu$ is the Reynolds number based on x. The constant C_1 is undetermined. It corresponds to an effective slight shift of the leading edge, and represents a residual effect far downstream of the details of the flow near the leading edge. Additional undetermined constants arise one after another in higher approximations. This appearance of <u>eigensolutions</u> is a characteristic feature of any inverse coordinate expansion, which limits its practicability.

1.7. Singular perturbations: the role of scales

Regular perturbations are relatively rare. The majority of perturbation solutions turn out to be singular, meaning that the straightforward approximation is not uniformly valid, but breaks down somewhere in the region of physical interest.

Such nonuniformity will be disclosed by one or another of several symptoms, as discussed in Chapter 2. In mild cases, higher approximations are found to become increasingly singular in some part of the field. When the nonuniformity is more severe, it may prove impossible to calculate higher approximations, because integrals diverge, boundary conditions cannot be imposed, and so on.

Why this should happen, and how it may be corrected can, in physical problems, be illuminated by considering the role of scales and the formation of dimensionless variables. Suppose that the perturbation quantity is the ratio of two lengths, $\varepsilon = s/L$ (1.1). In solving the problem one naturally introduces dimensionless variables. In particular, the space coordinates $x, y, \ldots$ are made dimensionless by referring them to a characteristic length. But we have two characteristic lengths, s and L, whose ratio is small and regarded as tending to zero. Which shall we choose? Which is the more characteristic?

The surprising answer is that in general neither choice alone is correct; one must make both choices, introducing, say,

$$\bar{x} = \frac{x}{L} \qquad \text{and} \qquad X = \frac{x}{s}. \tag{1.12}$$

This would appear to be a retrograde step, for it doubles the number of independent variables, whereas we aim to reduce their number and so integrate the equations. However the complication is only apparent. The exact roles of these twinned variables depends on which of three categories of singular-perturbation problems is being treated.

The simplest is the category of slow variations. Many practical approximations in mechanics -- for example, simple beam theory or the one-dimensional "hydraulic" approximation for pipe flow (Fig. 1.5) -- exploit the fact that the object under consideration is of much greater extent in one direction than another. As described in Chapter 3, we can construct a systematic scheme of successive approximations by introducing as perturbation parameter the ratio $\varepsilon = s/L$ of some short characteristic transverse dimension s to a characteristic longitudinal dimension L. It turns out that choosing either s or L as reference length leads

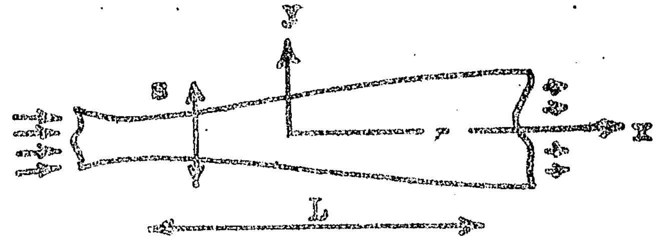


Fig. 1.5. Flow through slowly varying channel

to nonuniformity. That can be avoided, however, by using a mixed system, referring the longitudinal coordinate x to the typical longitudinal length L, and the transverse coordinate y to the transverse dimension s.

The intermediate category comprises what are often called <u>boundary-layer problems</u>. It is exemplified by viscous flow past a body at low as well as high Reynolds number. At low Reynolds number (Fig. 1.6) the nonlinear inertial effects can be neglected inside a region near the body whose extent is of the order of a characteristic body dimension s. In that region $\bar{x} = x/s$, etc. are the appropriate variables. However, the inertial effects reassert themselves far from the body, at distances of the order of the viscous length $L = \nu/U$. A different approximation is required there — that of nearly uniform flow — and the appropriate variables are $X = x/L$, etc.

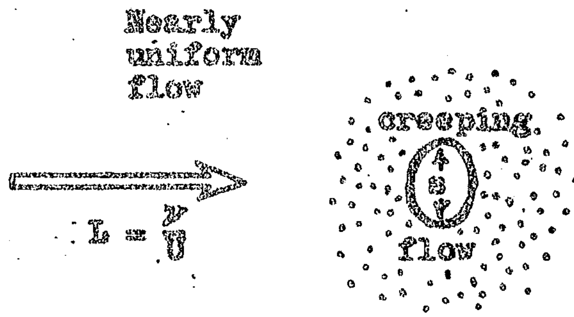


Fig. 1.6. Scales for flow at low Reynolds number

At high Reynolds number the relative magnitudes of the geometric and viscous lengths are reversed. For a simple shape the effects of viscosity are, according to Prandtl's boundary-layer theory, concentrated into a thin layer next to the surface. Thus the flow field can again be divided into two regions, with different length scales. Outside the boundary layer, where the flow is nearly inviscid, the appropriate reference length is the dimension L of the body. The boundary layer itself is an embedded region of slow variation. Its thickness is not the very short viscous length $s = \nu/U$ but $\sqrt{Ls}$, the root-mean square of L and s. This illustrates how two disparate lengths combine to provide an unlimited number of others, of the form $L(s/L)^k$ for any k. Thus distances across the boundary layer are referred to $\sqrt{Ls}$, and distances along it to L.

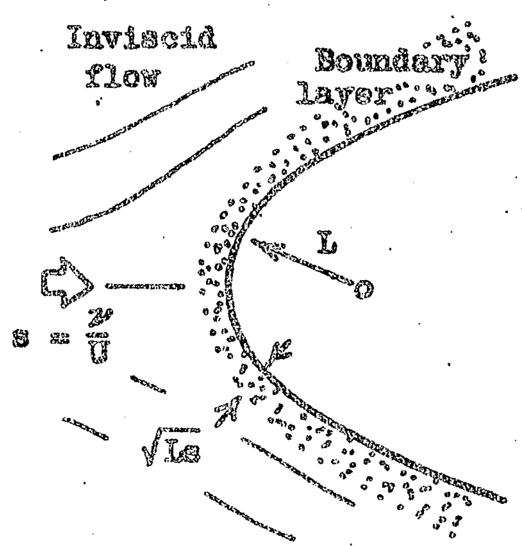


Fig. 1.7. Scales for flow at high Reynolds number

Thus in a boundary-layer problem only one or the other of the twinned variables is used in each region, so that the number of independent variables is nowhere increased. (More complicated problems may involve three or more different regions, with a corresponding number of alternative dimensionless forms of a single coordinate x; but again only one such variable is used in each region.) Both problems lack some boundary conditions, but the solution can be completed by matching the two solutions where they overlap. This is the idea of the method of matched asymptotic expansions, discussed in Chapter 4.

The most complicated of the three categories of singular-perturbation problems is that of slowly modulated oscillations. A simple model is the damping of sound waves by viscosity. Stokes (1845) considered the plane periodic waves produced by a piston oscillating sinusoidally (Fig. 1.8). The obvious scale is the wavelength s, which provides the dimensionless coordinate $X = x/s$. However, the slight viscosity of air damps the waves over a much longer scale L that encompasses many wavelengths, and provides the alternative dimensionless coordinate $\bar{x} = x/L$.

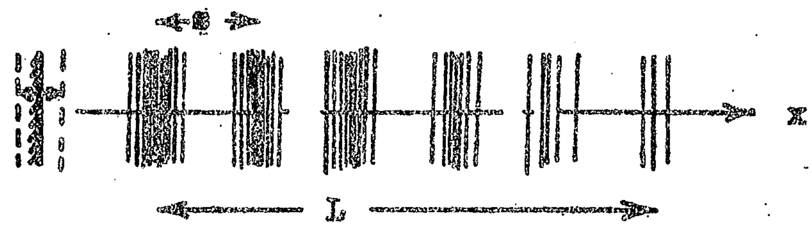


Fig. 1.8. Damping of plane sound waves by viscosity

If the ratio $\varepsilon = s/L$ of these two characteristic lengths is adopted as the small parameter for an approximation, the straightforward solution using X is found to break down on the longer scale: the perturbation is singular. The remedy is again to introduce both X and $\bar{x}$ as independent variables. In this more involved situation, however, the problem obviously cannot be decomposed into regions where only one or the other is the primary variable. The feeble effects of viscosity are no longer concentrated into thin layers, but act throughout the field. (There are boundary layers everywhere!) Both scales are significant at every point, so both variables are important throughout.

Thus the number of independent variables is, in this class of problems, formally increased. However, we shall see that the increase is only apparent, because when integration is being carried out with respect to the short-scale variable X the long-scale variable $\bar{x}$ plays only a parametric role. This is the idea of the method of multiple scales, several versions of which are discussed in Chapter 5.

We have just observed that two disparate scales can combine to form an indefinite number. Hence we will not be surprised to find in Chapter 5 that in some problems the method of multiple scales involves an unlimited number of successively longer scales.

1.8. Common forms of singular-perturbation series

We saw that in a regular perturbation the small quantity ε appears in the solution in the same simple way that it appears in the problem, say in integral powers (aside from a possible multiplicative fractional power). In a singular perturbation this simplicity is often destroyed by the interplay between disparate scales. As a result, the perturbation quantity can appear in the solution in an unlimited variety of complicated ways. Nevertheless, three general categories of solutions can be distinguished. We describe these and give examples for each.

In the first category, the solution involves fractional powers of ε. For example, the unseparated boundary layer on a smooth body has an expansion in inverse half powers of the Reynolds number, although it appears only to the first power in the Navier-Stokes equations. More complicated fractions arise, for instance, in the elastic deflection of a thin-walled toroid under internal pressure (Fig. 1.9). The straightforward approximation is the membrane theory of shells, which neglects the bending rigidity; but this fails in a thin strip about the top and bottom circles where the bending rigidity is essential because the shell lacks compound curvature there. Colbourne & Flugge (1967) treat this elasticity boundary layer by the method of matched asymptotic expansions. In terms of a dimensionless internal pressure ε they find that the angular rotation of the originally topmost element has the expansion

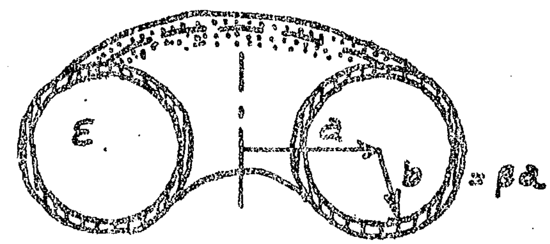


Fig. 1.9. Pressurized thin toroidal shell

$$\chi = \frac{0.7396}{\sqrt{\beta(\beta^2-1)}}\,\varepsilon^{3/4} - \frac{0.3749}{\beta}\,\varepsilon + \text{const.}\ \varepsilon^{5/4} + \dots \qquad (1.13)$$

The 1/16 power of the inverse Reynolds number appears in various problems of boundary-layer interaction, and rather curious 3/11 power is encountered in a theory of Newtonian flow separation. Irrational powers can also occur; Stewartson (1957) and Libby & Fox (1963) show that the expansion for the boundary layer on a semi-infinite flat plate contains "eigensolutions" with the stream function varying as distance to the powers -1.387, -2.314, etc. An unusual sequence of powers appears in Wilson's (1970) analysis of the boundary layer at the entrance to a tube, where the inverse Reynolds number occurs to the powers 1/2, 3/4, 7/8, 15/16, etc., with a condensation at 1.

In the second category, the solution involves a mixture of powers of ε and of its logarithm. For example, whereas in the Oseen linearization the viscous drag of a sphere has an expansion (1.7) in integral powers of Reynolds number, for the full Navier-Stokes equations Proudman & Pearson (1957) and Chester & Breach (1969) show that logarithms appear beginning with the third term:

$$C_D = \frac{6}{R}\left[1 + \frac{3}{8}R + \frac{9}{40}R^2 \log R + \frac{9}{40}\left(\gamma + \frac{5}{3}\log 2 - \frac{323}{360}\right)R^2 + \frac{27}{80}R^3 \log R + \dots\right] \quad (1.14)$$

Here $\gamma = 0.5772$ is Euler's constant, and "log" denotes the natural logarithm.

The detailed pattern varies considerably; the logarithm appears immediately in the first term in slender-body aerodynamics, but is delayed to the fourth term in subsonic thin-airfoil theory. Often the power of the logarithm increases by one every other term, and perhaps that happens in (1.14). Fractional powers may also be mixed with logarithms. And even loglog's have occasionally been encountered in boundary-layer problems.

In the third category, the solution proceeds in integral powers of the reciprocal of the logarithm. For example, Hallen (1928) has calculated the electrostatic capacitance of a slender charged cylindrical conductor (Fig. 1.10) as

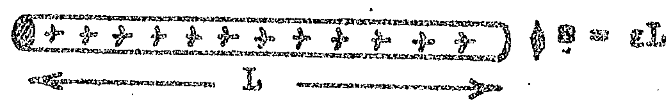


Fig. 1.10. Slender charged cylinder

$$\frac{C}{2\pi\epsilon L} = \frac{1}{\log 2/\varepsilon} + \frac{0.3069}{(\log 2/\varepsilon)^2} + \frac{0.2717}{(\log 2/\varepsilon)^3} + \frac{0.6898}{(\log 2/\varepsilon)^4} + \dots \quad (1.15)$$

Again, Lamb (1911), Proudman & Pearson (1957), and Kaplun (1957) have calculated the drag coefficient of a circular cylinder in a viscous stream as

$$C_D = \frac{D}{\rho U^2 a} = \frac{4\pi}{R}\left[\frac{1}{\log 1/R} - \frac{1.309}{(\log 1/R)^2} + \frac{0.84}{(\log 1/R)^3} + \dots\right] \quad (1.16)$$

where D is the drag per unit length of the cylinder. As in the corresponding axisymmetric case (1.12) the multiplicative factor $1/R$ could be eliminated by a more appropriate definition of the drag coefficient.

This form of expansion is perhaps always associated with circular cylindrical geometry. It ordinarily yields poor accuracy unless ε is unrealistically small. Probably such a series is purely asymptotic, with zero radius of convergence. Proudman & Pearson have suggested that the terms indicated in (1.16) are succeeded by transcendentally small ones, which are smaller by factors of R, R^2, etc. On the scale of $\log 1/R$, these correspond to the $e^{-1/\varepsilon}$, $e^{-2/\varepsilon}$, etc. in (1.4).

1.9. Gauge functions and order symbols

The preceding examples contain a set of simple functions of the perturbation quantity that serve to describe the degree of smallness of the terms in a series. These <u>gauge functions</u> include

$$\varepsilon^N, \quad \varepsilon^\alpha, \quad e^{-c/\varepsilon}, \quad \varepsilon^M(\log \varepsilon)^N, \quad \frac{1}{(\log k/\varepsilon)^M} \qquad (1.17)$$

and these few suffice for the great majority of physical problems.

Of course these forms are not unique. For example, one might for some reason prefer to expand not in powers of ε, but instead in powers of $\sin\varepsilon$, or (as discussed in Chapter 8) in powers of $\varepsilon/(c+\varepsilon)$. These three alternative gauge functions are said to be of the same <u>order</u> as ε tends to zero, because they all vanish at the same rate -- linearly. A multiplicative constant does not affect the order: ε and 2ε are of the same order. In fact, ε and 2000ε are of the same order, which illustrates that order in this mathematical sense is not quite the same thing as physical order of magnitude. (However, we expect that if variables are sensibly normalized in a physical problem, no very large or small constants will appear.)

Euler Transform

The orders of two different gauge functions (or other functions of the perturbation quantity) are compared by using the <u>order symbols</u> o and O (pronounced "little oh" and "big oh"). The symbol o means "of smaller order than." That is, given two functions $g(\varepsilon)$ and $G(\varepsilon)$ we say

$$g(\varepsilon) = o\left[G(\varepsilon)\right] \quad \text{as} \quad \varepsilon \to 0 \quad \text{if} \quad \lim \frac{g(\varepsilon)}{G(\varepsilon)} = 0 \qquad (1.18)$$

The symbol O means "of not greater order than." We say

$$F(\varepsilon) = O\left[G(\varepsilon)\right] \quad \text{as} \quad \varepsilon \to 0 \quad \text{if} \quad \lim \frac{F(\varepsilon)}{G(\varepsilon)} < \infty \qquad (1.19)$$

Thus, for example, $\sin\varepsilon = O(\varepsilon)$. According to the definition, it is also formally true that $\sin\varepsilon = O(\varepsilon^{1/2})$ or $\sin\varepsilon = O(1)$; but these latter statements are less informative. We assume that the sharpest possible estimate is always given. This means using O with the smallest possible gauge function, and using o only when there is insufficient knowledge to use O.

We shall use order symbols mostly as a bookkeeping device to keep track of the error incurred in truncating a perturbation series at a finite number of terms. Thus instead of using the ellipsis (...), we could write three terms of the Janzen-Rayleigh expansion (1.5) as

$$\frac{q_{max}}{U} = 2.00000 + 1.16667\, M^2 + 2.58129\, M^4 + O(M^6) \qquad (1.20)$$

(and we would add "as $M \to 0$" if this were not perfectly clear). However, if we were uncertain of the exact order of the remainder, we would give the less precise estimate

$$\frac{q_{max}}{U} = 2.00000 + 1.16667\,M^2 + 2.58129\,M^4 + o(M^4) \qquad (1.21)$$

For this purpose we need to know how to carry out simple operations with order symbols; but these are mostly obvious. An exception is the fact that although order symbols can be integrated (with respect to either the small quantity or a parameter) they cannot necessarily be differentiated (Erdélyi 1956, sec. 1.1). However, since in engineering research we must ordinarily proceed without rigor, we shall feel free also to differentiate with impunity.

1.10. Asymptotic sequences and asymptotic expansions

Each of our previous examples of a perturbation series (1.5, 1.7, 1.9, 1.10, etc.) is by construction a formal asymptotic expansion. It is formed with the aid of a special sequence of gauge functions appropriate to the particular problem. Each term of this asymptotic sequence, which we denote generically by $\delta_n(\varepsilon)$, is of smaller order than its predecessor:

$$\delta_n(\varepsilon) = o\left[\delta_{n-1}(\varepsilon)\right] \qquad (1.22)$$

We saw (sec. 1.4) that in a regular perturbation the asymptotic sequence may be taken as the integral powers of the small quantity: $\delta_n(\varepsilon) = \varepsilon^n$, though equivalent alternatives may occasionally be preferred (sec. 1.9). In singular perturbations, however, we saw (sec. 1.8) that the asymptotic sequence usually contains fractional powers, logarithms, or exponentials. Clearly the asymptotic sequence for a given problem is neither entirely arbitrary, nor entirely fixed [unless we are content with the standard gauge functions (1.17)].

A finite asymptotic expansion (to N terms) is then a linear combination of the leading terms of the asymptotic sequence:

$$f(\varepsilon) = \sum_{n=1}^{N} c_n \delta_n(\varepsilon) + o(\delta_N) \quad \text{as} \quad \varepsilon \to 0 \qquad (1.23)$$

For either a finite or infinite number of terms this is usually written more simply, and using the special equality sign $\sim$ ("asymptotically equal to") as

$$f(\varepsilon) \sim \sum c_n \delta_n(\varepsilon) \qquad (1.24)$$

and we shall henceforth use this notation. If the function f were known explicitly, the coefficients c_n could be calculated in succession by taking the limit after subtracting the previous terms:

$$c_m = \lim_{\varepsilon \to 0} \left[f(\varepsilon) - \sum_{n=1}^{m-1} c_n \delta_n(\varepsilon)\right] / \delta_m(\varepsilon) \qquad (1.25)$$

Asymptotic series were christened by Poincaré, who restricted attention to the asymptotic power series $\sum c_n \varepsilon^n$. (This term is sometimes also applied to the form $f_0(\varepsilon) \sum c_n \varepsilon^n$.) The generalization to other asymptotic sequences followed naturally. In physical applications the solution, and hence the coefficients c_n, usually depend also upon one or more dimensionless coordinates, for example

$$f(x;\varepsilon) \sim \sum c_n(x)\,\delta_n(\varepsilon) \tag{1.26}$$

Such an expansion to N terms is said to hold uniformly in x if the remainder is $o(\delta_N)$ uniformly in x.

Asymptotic expansions may be added, multiplied, divided, integrated, and differentiated only under mathematical restrictions that are discussed in texts on the subject (Erdélyi 1956, ch. 1). Again, however, since we must ordinarily proceed heuristically in physical problems, we will have to perform these operations frequently without rigorous justification.

An asymptotic expansion may be convergent (with finite or infinite radius of convergence) or divergent. However, we usually find it convenient to reserve the term "asymptotic" for the divergent case. We may also use the expression "purely asymptotic" to emphasize that an expansion has zero radius of convergence.

Stieltjes called such series semi-convergent, an evocative term that has unfortunately fallen into disuse. It underlines the important property that successive terms decrease in magnitude up to a certain point, and only thereafter increase. It is for this reason that purely asymptotic series are useful for computation; for example, for Bessel functions of large argument. The error is of the order of the first term neglected, and in fact frequently less in magnitude than the last term retained. The best approximation is therefore obtained by truncating the series just before the terms begin to increase in magnitude.

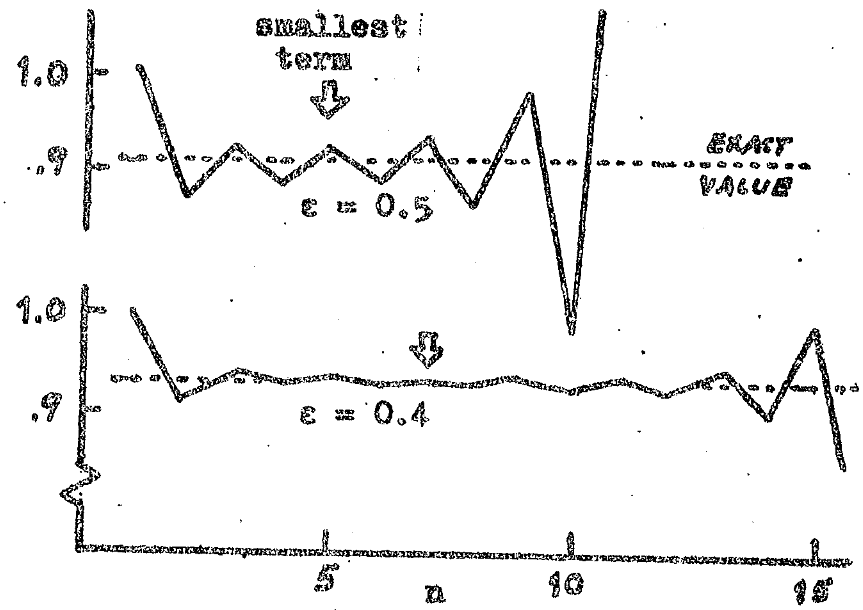


Fig. 1.11. Successive partial sums for bracket in asymptotic series (1.27) for complementary error function

The approximation becomes better, and the number of terms that can profitably be retained increases, as the limit $\varepsilon \to 0$ is approached. These features are illustrated in Fig. 1.11 for the asymptotic expansion of the complementary error function, a function that arises in simple solutions of the heat equation:

$$\operatorname{erfc}\left(\frac{1}{\varepsilon}\right) \sim \frac{1}{\sqrt{\pi}}\,\varepsilon\, e^{1/\varepsilon^2}\left[1 - \frac{1}{2}\varepsilon + \frac{3}{4}\varepsilon^2 - \cdots + (-1)^n\frac{1\cdot 3\cdots(2n-1)}{2^n}\,\varepsilon^{2n}\right] \tag{1.27}$$

Even for such a large value as $\varepsilon = 0.5$, five terms of this divergent series are seen to give the result to within one per cent -- an accuracy that is reached only with twelve terms of the Taylor series for the error function, even though it has infinite radius of convergence.

Successive terms in a purely asymptotic expansion usually alternate in sign, either individually or in groups of two or more. That this is not invariably true, however, is illustrated by the asymptotic expansion for the modified Bessel function

$$I_0\left(\frac{1}{\varepsilon}\right) \sim \sqrt{\frac{\varepsilon}{2\pi}}\, e^{1/\varepsilon}\left[1 + \frac{1^2}{1!8}\varepsilon + \frac{1^2 \cdot 3^2}{2!8^2}\varepsilon^2 + \dots\right] \qquad (1.28)$$

Figure 1.12 shows how in this case the fact that the error is of the order of the first term neglected means that the exact value corresponds to an "inflection point" in the plot of partial sums.

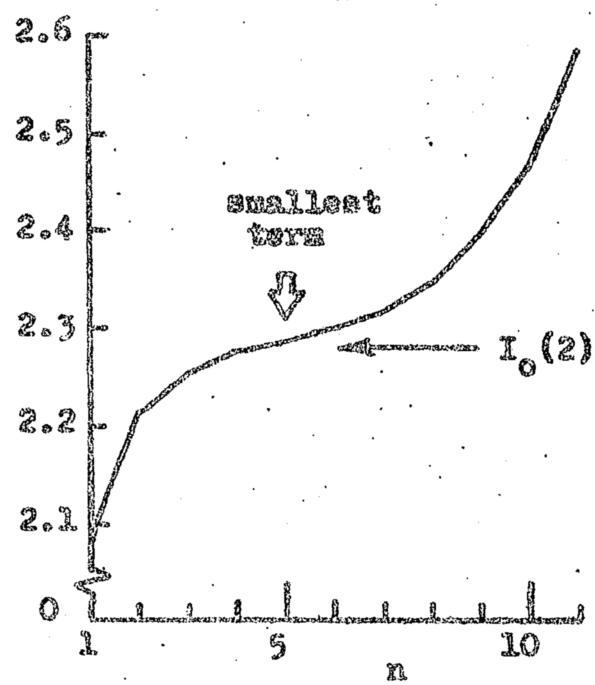


Fig. 1.12. Successive partial sums of asymptotic series (1.28) at $\varepsilon = 0.5$

References

CHESTER, W. & BREACH, D. R. 1969 On the flow past a sphere at low Reynolds number. J. Fluid Mech. 37, 751-760

COLBOURNE, J. R. & FLÜGGE, W. 1967 The membrane theory of the toroidal shell - a singular perturbation problem. Int. J. Non-Linear Mech. 2, 39-53

ERDÉLYI, A. 1956 Asymptotic Expansions. Dover, New York

FLÜGGE, W. 1962 Handbook of Engineering Mechanics. Mc-Graw-Hill, New York

GARABEDIAN, P. 1956 Calculation of axially symmetric cavities and jets. Pacific J. Math. 6, 611-684

GOLDSTEIN, S. 1929 The steady flow of viscous fluid past a fixed spherical obstacle at small Reynolds number. Proc. Roy. Soc. A 123, 225-235

HALLÉN, E. 1929 Lösung zweier Potentialprobleme der Elektrostatik. Arkiv för Mat. Ast. och Fys. 21A, no. 22, 1-44

HOWARTH, L. 1938 On the solution of the laminar boundary-layer equations. Proc. Roy. Soc. A 164, 547-579

IMAI, Isao 1941 On the flow of a compressible fluid past a circular cylinder, II. Proc. Physico-Math. Soc. Japan (3) 23, 180-193

IMAI, Isao 1957 Second approximation to the laminar boundary-layer flow over a flat plate. J. Aeronaut. Sci. 24, 155-156

JANZEN, O. 1913 Beitrag zu einer Theorie der stationaren Strömung kompressibler Flüssigkeiten. Phys. Zeit. 14, 639-643

KAPLUN, S. 1957 Low Reynolds number flow past a circular cylinder. J. Math. & Mech. 6, 595-603

KATAGIRI, M. 1974 Flow due to an impulsive rotation of infinite disk. Phys. Fluids 17, 1463-1464

LAMB, H. 1911 On the uniform motion of a sphere through a viscous fluid. Phil. Mag. (6) 21, 112-121

LIBBY, P. A. & FOX, H. 1963 Some perturbation solutions in laminar boundary-layer theory. Part I. The momentum equation. J. Fluid Mech. 17, 433-449

OSEEN, C. W. 1910 Ueber die Stokes'sche Formel, und über eine verwandte Aufgabe in der Hydrodynamik. Ark. Math. Astronom. Fys. 6, No. 29

PROUDMAN, I. & PEARSON, J. R. A. 1957 Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. J. Fluid Mech. 2, 237-262

RAYLEIGH, Lord 1916 On the flow of compressible fluid past an obstacle. Philos. Mag. (6) 32, 1-6; Scientific Papers 6, 402-406, Cambridge Univ. Press, 1920

SHANKS, D. 1955 Non-linear transformations of divergent and slowly convergent sequences. J. Math. & Phys. 34, 1-42

SIMASAKI, T. 1956 On the flow of a compressible fluid past a circular cylinder. II. Bull. Univ. Osaka Prefecture, Ser. A, 4, 27-35

STOKES, G. G. 1845 On the theories of the internal friction of fluids in motion and of the equilibrium and motion of elastic solids. Trans. Camb. Philos. Soc. 8 287; Math. & Phys. Papers 1, 75-129 (esp. sec. 7), Cambridge Univ. Press, 1880

STOKES, G. G. 1851 On the effect of the internal friction of fluids on the motion of pendulums. Trans. Camb. Philos. Soc. 9, Part II, 8-106; Math. & Phys. Papers 3, 1-141, Cambridge Univ. Press, 1901

TIMOSHENKO, S. & GOODIER, N. 1951 Theory of Elasticity. McGraw-Hill, New York

VAN DYKE, M. 1964 Lifting-line theory as a singular perturbation problem. Arch. Mech. Stos. 16, no. 3; Appl. Math. Mech. (PMM) 28, 90-101

VAN DYKE, M. 1970 Extension of Goldstein's series for the Oseen drag of a sphere. J. Fluid Mech. 44, 365-372

VAN DYKE, M. 1974 Analysis and improvement of perturbation series. Quart. J. Mech. Appl. Math., to appear

WILSON, S. D. R. 1970 Entry flow in a channel. Part 2. J. Fluid Mech. 46, 787-799

EXERCISES

1.1. Perturbation solution of a quadratic equation. Find a systematic approximate solution of the algebraic equation

$$x = 1 - \varepsilon x^2$$

by (a) first neglecting the term εx^2, then iterating on that first approximation to find a second approximation, and repeating to find a third and a fourth; (b) assuming a power-series expansion in ε for x, substituting, expanding, equating like powers, and solving term by term for the first four terms. Contrast these two procedures, and the results they give, from the points of view of simplicity, accuracy, etc. Do you think your results converge for any values of ε? If so, could you determine (or guess) what values? What happened to the other solution of the quadratic equation? How can a corresponding approximation for it be found?

1.2. The relative orders of small quantities. The small parameter or coordinate ε in a perturbation solution typically appears in the forms listed below. It is essential to know how these rank with respect to one another for sufficiently small ε. Arrange them in increasing order (from left to right) for $\varepsilon \to 0$ (e.g., $\varepsilon^2 < \varepsilon < 1$).

$$\varepsilon \log\frac{1}{\varepsilon}, \quad \frac{1}{\log 1/\varepsilon}, \quad \varepsilon^{\frac{1}{2}}, \quad e^{-1/\varepsilon}, \quad \pi^{\varepsilon}, \quad \varepsilon^{\varepsilon}, \quad \varepsilon, \quad \log\frac{1}{\varepsilon}, \quad \log(\log\frac{1}{\varepsilon}).$$

1.3. Perturbation solution of an integral equation. In a study of heat transfer in a boundary layer, Lighthill encountered the integral equation

$$[F(x)]^4 = -\frac{1}{2x^{\frac{1}{2}}} \int_0^x \frac{F'(\xi)\, d\xi}{(x^{3/2} - \xi^{3/2})}, \quad F(0) = 1 .$$

He solved this approximately for small x by substituting the assumed expansion $F(x) = 1 - a_1 x + a_2 x^2 - a_3 x^3 + \ldots$ Carry out this analysis far enough to show that $a_1 = 1.461$. (The definite integrals required are Beta functions; see, for example, Dwight's "Tables of Integrals and Other Mathematical Data," or Abramowitz & Stegun's "Handbook of Mathematical Functions.")

1.4. Reversion of series with logarithmic terms. A standard result (e.g., Dwight's Tables, eq. 50) is that the power series

$$y = ax + bx^2 + cx^3 + dx^4 + \ldots$$

can be reverted to find x as a power series in y:

$$x = \frac{1}{a} y - \frac{b}{a^3} y^2 + \frac{2b^2 - ac}{a^5} y^3 + \ldots .$$

Find three terms in the analogous reversion of the series

$$y = ax + bx^2\log x + cx^2 + dx^3\log^2 x + ex^2\log x + fx^2 + \dots \quad .$$

1.5. Asymptotic series for exponential integral. The exponential integral

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\,dt \qquad (|\arg z| < \pi),$$

has the expansion for large z (e.g., Abramowitz & Stegun, eq. 5.1.51)

$$E_1(z) \sim \frac{e^{-z}}{z}\left(1 - \frac{1}{z} + \frac{1\cdot 2}{z^2} - \frac{1\cdot 2\cdot 3}{z^3} + \dots\right).$$

Verify the first two terms by integrating by parts. Write down the general term by inspection. Then show, using one of the standard tests, that the radius of convergence is zero. By considering $z = 3$, for which $z\,e^z E_1(z) = 0.78625$, demonstrate that a point is reached beyond which additional terms increase the error (cf. Fig. 1.11).

Chapter 2

REGULAR PERTURBATIONS

2.1. Successive approximations

In physical problems we are usually concerned with solving ordinary or partial differential equations (occasionally integral, integro-differential, or difference equations), subject to appropriate auxiliary (initial, boundary, radiation, ...) conditions. We suppose that the full problem is intractible, but that we discover a limiting procedure $\varepsilon \to 0$ for some parameter or coordinate ε in the problem that promises to provide a useful approximation. That is the case if we can solve the basic problem for $\varepsilon = 0$, and if ε is small in realistic applications.

There are then two systematic procedures for calculating successive approximations. (Both were illustrated by the simple mathematical model of Exercise 1.1):

(i) Iteration on the basic approximation

(ii) Substitution of an assumed expansion

Iteration has the advantage that there is no need to guess the asymptotic sequence. It is generated automatically including, for example, unexpected logarithms if they are present. A single iteration step yields a cluster of closely spaced terms, such as $\varepsilon^2 \log^2 \varepsilon$, $\varepsilon^2 \log \varepsilon$, and ε^2.

On the other hand, series expansion is more systematic in a nonlinear problem because it produces only significant results. By contrast, iteration yields one correct new term (or cluster) at each step, together with higher-order terms that are not yet correct and should therefore be disregarded. As a consequence of these relative advantages, iteration is often simpler when only a few terms are required, but series expansion is usually better for high-order approximations like the examples in section 1.4.

Straightforward application of either procedure will break down in a singular-perturbation problem. Then one needs the more elaborate techniques discussed in chapters 4 through 7. However, we shall see that in a realistic physical problem one can seldom decide in advance whether a perturbation is regular or singular. Consequently it is usually most efficient to hope for the best: to suppose that the perturbation is regular, and proceed with a straightforward approximation until some symptom of nonuniformity is encountered.

2.2. The basis of a perturbation solution

The point of departure for any perturbation expansion is a known exact solution, corresponding to $\varepsilon = 0$. It may have one of three successively more complicated forms:

1. A trivial solution. The null condition — quiescent fluid, an unstressed elastic medium, etc. — is always a solution of any governing equations. So is a uniform state, such as a uniform parallel stream, a field of uniform tension, etc. We shall count such a trivial basic solution as the "zeroth approximation," reserving the term "first approximation" for the leading departure from it.

2. A closed-form solution. A solution in terms of elementary or known functions is often found in a linear problem by separating variables in orthogonal coordinates, and occasionally in a nonlinear problem by seeking self-similarity. An example is the simple solution for potential flow past a circle that underlies the Janzen-Rayleigh expansion (1.5). We regard such a basic solution as nontrivial, and accordingly call it the "first approximation."

3. A numerical solution. In nonlinear problems, self-similar solutions usually involve numerical integration of ordinary differential equations. An example is the Prandtl-Blasius solution for the boundary layer on a semi-infinite flat plate, which is the basis of both Howarth's series (1.9) and Imai's series (1.11). It must be admitted, however, that the increasing availability of high-speed computers is tending to erode the distinction between closed-form solution in terms of, say, the error function in Katagiri's series (1.10) and numerical solutions involving, say, the Prandtl-Blasius function $f(\eta)$.

The basic solution is linear or nonlinear according as the governing equations are. However, it is the essence of a perturbation scheme that all subsequent approximations satisfy linear governing equations, together with linear boundary conditions. If the basic solution is known in closed form, one may hope that higher approximations will be too; but if it is known only numerically, higher terms will necessarily also require numerical integration. (In practice, one recomputes the basic solution, rather than using previously tabulated values, at the same time that the perturbation is being calculated.)

2.3. Basic solutions for torsion, Poiseuille flow, or soap film

To introduce the techniques of perturbation, we consider a simple problem and perturb it in a variety of ways. We choose a classical mathematical problem that has familiar interpretations in elasticity, viscous fluid flow, and other branches of continuum mechanics, as well as an analogy in the deflection of a membrane. This is the simple Poisson equation for a finite region in the plane, with a constant forcing term, which can be normalized to unity, together with vanishing boundary values:

$$\nabla^2\phi = \begin{Bmatrix} \phi_{xx} + \phi_{yy} \\ \phi_{rr} + \frac{\phi_r}{r} + \frac{\phi_{\theta\theta}}{r^2} \end{Bmatrix} = -1 , \quad \phi = 0 \text{ on a closed curve.} \quad (2.1)$$

Soap-film analogy. We recall first the soap-film or membrane analogy for this problem (Fig. 2.1). In exploring the phenomenon of surface tension in 1805, Laplace deduced that across a surface with tension σ the pressure jumps from the convex to the concave side by an amount

$$\Delta p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \qquad (2.2)$$

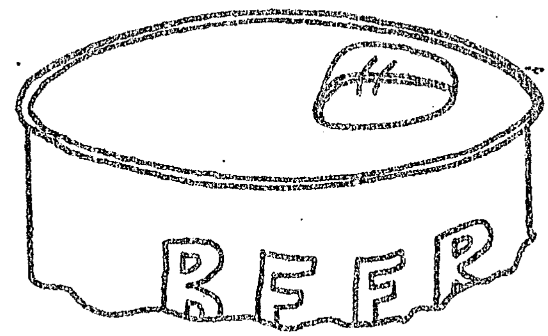


Fig. 2.1. Deflection of liquid film under pressure difference

Here R_1 and R_2 are the principal radii of curvature. If the surface $z = \phi(x,y)$ has small slope (cf. Exercise 2.1), the mean curvature $(1/R_1 + 1/R_2)$ is approximately $-(\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2)$. (The minus sign corresponds to the fact that if the deflection is upward the centers of curvature lie below.) Hence if a soap film or membrane spans a hole of cross-section C in a flat plate and is subject to a slight pressure difference, its deflection is described by

$$\sigma\left(\phi_{xx} + \phi_{yy}\right) = -\Delta p, \quad \phi = 0 \text{ on } C. \qquad (2.3)$$

Because the surface tension σ may be assumed constant as well as Δp, we can choose units such that $\Delta p/\sigma$ is normalized to unity and so obtain (2.1).

Torsion of cylinder. If a long cylindrical shaft is subjected to couples at either end (Fig. 2.2), the details of their application decay exponentially away from the ends, according to St.-Venant's principle (Fung 1965, 300-309).

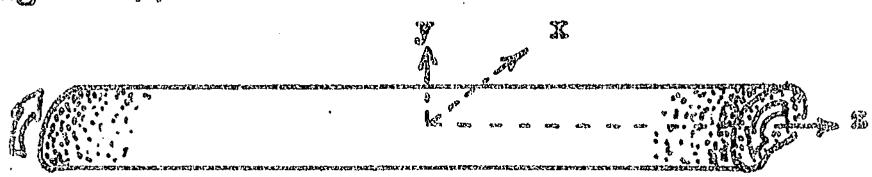


Fig. 2.2. Torsion of long cylinder

These details are negligible beyond a "boundary layer" that extends only a distance from the ends of the order of the maximum width of the cross-section. The problem of linear elasticity away from the ends was solved by St.-Venant in 1855, who verified his assumption that while cross-sections rotate as for a circular section they are warped in the axial direction. The warping function satisfies Laplace's equation; but Prandtl showed that it is easier to work with the related stress function ϕ (Timoshenko & Goodier 1951, p.

261). The equations of equilibrium are satisfied with the shear stresses given by

$$\tau_{xz} = \frac{\partial\phi}{\partial y}, \quad \tau_{yz} = -\frac{\partial\phi}{\partial x} . \tag{2.4}$$

The absence of stresses on the surface implies that ϕ is constant (say zero for a simply-connected cross-section); and ϕ is found to satisfy the differential equation

$$\phi_{xx} + \phi_{yy} = -2G\alpha . \tag{2.5}$$

Here G is the modulus of elasticity in shear and α the angle of twist per unit length. Again we can choose units so that the problem is normalized to (2.1).

Poiseuille flow. If viscous liquid is forced through a long cylindrical tube (Fig. 2.3), the details of the entry and exit conditions again decay exponentially. The length of the exit boundary layer is only a few channel widths, as in the torsion problem; but the entry boundary layer is longer by a factor of the Reynolds number (Van Dyke 1970b).

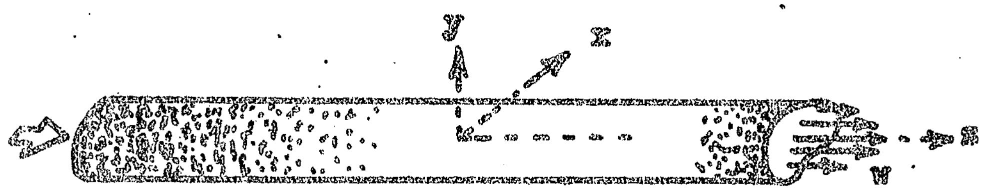


Fig. 2.3. Viscous flow through long cylindrical channel

Elsewhere the velocity has only an axial component, so that continuity is satisfied, and the pressure gradient is a negative constant. The Navier-Stokes equations therefore reduce to a simple requirement on axial momentum:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{dp/dz}{\mu} \tag{2.6}$$

where μ is the viscosity. The no-slip condition requires that $w = 0$ on the boundary, so that again the problem can be normalized to (2.1).

Basic solutions. In the axisymmetric case, the solution of these problems gives the parabolic distribution

$$\phi = \tfrac{1}{4}(1-r^2) . \tag{2.7}$$

For a thin rectangular section, the soap-film analogy (Fig. 2.4) makes clear that the deflection is nearly cylindrical except in boundary layers near the ends that again extend only a distance of the order of the thickness. Elsewhere the solution is again parabolic:

$$\phi = \tfrac{1}{2}(1 - x^2) . \qquad (2.8)$$

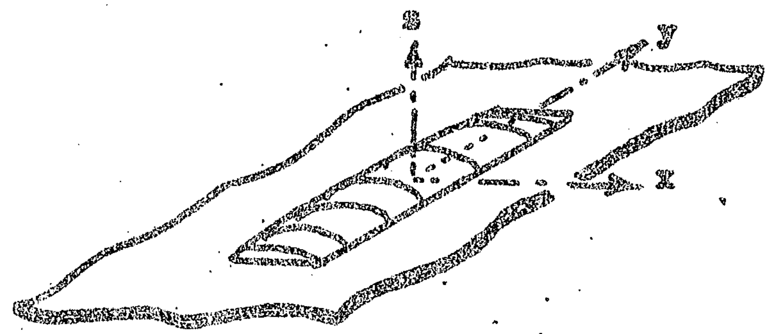


Fig. 2.4. Deflection of soap film over thin rectangular slit

Simple closed-form solutions in terms of elementary functions are also known for the equilateral triangle (Timoshenko & Goodier 1951, p. 266), the ellipse, and other shapes connected with the circle and ellipse (Love 1927, sec. 222).

2.4. Perturbation of governing equations: torsion of aeolotropic prism

The departure from the basic solution represented by a perturbation may make its appearance in the mathematical problem in either the governing equations or the auxiliary conditions (or both). The first possibility is the simplest to handle. We examine it in a very simple problem of torsion (which of course has its counterparts in the other physical interpretations we have discussed).

Love (1927, sec. 226) analyzes the torsion of an "aeolotropic prism" — a non-isotropic cylinder — when the cross-section is a plane of symmetry of structure. A wood rod or a natural crystal has this elastic structure (Fig. 2.5). Love works with St.-Venant's warping function, but we convert to Prandtl's stress function (2.4) in order to exploit our previous results.

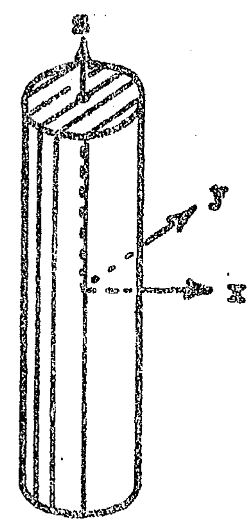


Fig. 2.5. Circular shaft of non-isotropic material

It is reasonable to suppose that the anisotropy is small, represented by a perturbation parameter ε. Then adapting Love's equations, we find that to first order in ε the problem for a circular shaft is

$$(1-\varepsilon)\phi_{xx} + \phi_{yy} = -1 , \qquad (2.9a)$$

$$\phi = 0 \quad \text{at} \quad x^2 + y^2 = 1 . \qquad (2.9b)$$

For $\varepsilon = 0$ we have the basic solution for an isotropic shaft (2.7):

$$\phi_1 = \tfrac{1}{4}(1 - x^2 - y^2) . \qquad (2.10)$$

We may now either iterate or assume an expansion for small ε. The first choice seems simpler, and gives as the problem for the second approximation*

$$\phi_{II_{xx}} + \phi_{II_{yy}} = -1 + \varepsilon\,\phi_{I_{xx}} = -(1+\tfrac{1}{2}\varepsilon),$$
$$\phi_{II} = 0 \quad \text{at } x^2+y^2 = 1 . \tag{2.11}$$

Hence in this simple problem the second approximation is a multiple of the first:

$$\phi_{II} = \tfrac{1}{4}(1+\tfrac{1}{2}\varepsilon)\cdot(1-x^2-y^2) . \tag{2.12}$$

As a matter of fact, this feature is obviously repeated in higher approximations. This suggests that the exact solution is a multiple of the first approximation, and it is then easily found to be

$$\phi = \frac{1}{4(1-\frac{1}{2}\varepsilon)}\,(1-r^2) \tag{2.13}$$

Thus the perturbation solution is not necessary, though without it one might not have anticipated the rather paradoxical result that the stress field is axisymmetric although the problem appears not to be.

In its fluid-mechanical interpretation, the basic axisymmetric solution (2.7) was perturbed by Dean (1927) to calculate the flow in a slightly curved pipe. Similarly, it has been used to find the effects of the rotation of the earth on the flow in a straight pipe.

2.5. Perturbation of boundary: slightly noncircular section

Now suppose that the perturbation appears in the initial or boundary conditions. If only the nature of the condition is varied, the straightforward procedure just described remains applicable. However, complications arise when the position of the boundary varies. This is the situation for free boundaries, whose location depend on the solution, and also for geometry the departs slightly from the basic shape. It is then necessary to transfer the boundary condition.

We illustrate this manipulation for a nearly circular section in torsion (or with laminar flow, or a soap film). Any shape can be described by a Fourier series, and to first order in the deviation each component can be treated separately. We therefore consider only the family of shapes

$$r = 1 + \varepsilon \cos N\theta \tag{2.14}$$

---

* Having used Arabic subscripts for successive terms in a perturbation series, we introduce Roman-numeral subscripts to distinguish the corresponding successive partial sums. Thus, for example, the first two approximations in some problem might be

$$f_I(x;\varepsilon) = f_1(x), \qquad f_{II}(x;\varepsilon) = f_1(x) + \varepsilon f_2(x)$$

This includes first approximations to the ellipse for $N = 2$ and the equilateral triangle for $N = 3$ (Fig. 2.6).

In polar coordinates the problem for $\phi(r, \theta)$ is

$$\phi_{rr} + \frac{\phi_r}{r} + \frac{\phi_{\theta\theta}}{r^2} = -1,$$
$$\phi(1 + \varepsilon \cos N\theta, \theta) = 0. \qquad (2.15)$$

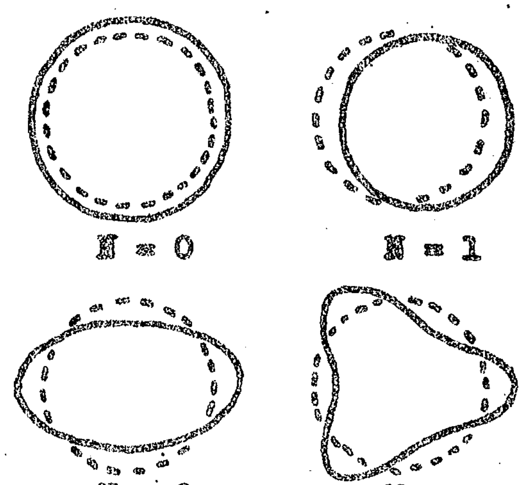


Fig. 2.6. Family of simple shapes close to circle

We now assume a series in powers of ε starting with the basic axisymmetric solution (2.7):

$$\phi = \phi_1 + \varepsilon\phi_2 + \dots$$
$$= \tfrac{1}{4}(1 - r^2) + \varepsilon\phi_2(r,\theta) + \dots \qquad (2.16)$$

However we cannot now simply substitute into the full problem and equate like powers of ε. The difficulty is that ε appears implicitly in the argument of each ϕ_n in the boundary condition. It must be exhibited explicitly by transferring the boundary condition from the perturbed to the basic boundary.

Often, as in this case, the transfer can be effected by expansion in Taylor series (though other expansions may be required in more complicated situations). Thus the boundary condition becomes

$$\begin{aligned}\phi(1+\varepsilon\cos N\theta, \theta) &= \phi(1, \theta) + \varepsilon \cos N\theta\, \phi_r(1, \theta) + O(\varepsilon^2)\\ &= \left[\phi_1(1,\theta) + \varepsilon\phi_2(1,\theta)\right] + \varepsilon\cos N\theta\, \phi_r(1,\theta) + \dots\\ &= \varepsilon\left[\phi_2(1,\theta) - \tfrac{1}{2}\cos N\theta\right] + O(\varepsilon^2) = 0 \qquad (2.17)\end{aligned}$$

Now we are able to equate like powers of ε, which yields for ϕ_2 the problem

$$\phi_{2_{rr}} + \frac{\phi_{2_r}}{r} + \frac{\phi_{2\theta\theta}}{r^2} = 0, \qquad \phi_2(1,\theta) = \tfrac{1}{2}\cos N\theta. \qquad (2.18)$$

The solution is readily found by separation of variables, giving the second approximation

$$\phi = \tfrac{1}{4}(1 - r^2) + \tfrac{1}{2}\varepsilon r^N \cos N\theta + O(\varepsilon^2) \qquad (2.19)$$

Two simple checks can be applied to this result. For $N = 0$ the perturbed shape is simply a circle of radius $1+\varepsilon$ (Fig. 2.6). The axisymmetric solution for a circle of radius a is $\tfrac{1}{4}(a^2 - r^2)$, and setting $a = (1+\varepsilon)$ and expanding reproduces (2.19). For $N = 1$ the perturbed shape is, to $O(\varepsilon)$, just the basic circle shifted to the right a distance ε. The basic solution (2.7) therefore applies with r^2 replaced by $r^2 + \varepsilon^2 - 2r\varepsilon\cos\theta$, and this too reproduces (2.19).

The process of transfer from the basic to the perturbed contour by series expansion is frequently repeated after the solution has been found, in evaluating quantities on the boundary. This time the transfer is not essential, but it produces the simplest form of the result, free of irrelevant higher-order terms.

The first approximation in a perturbation solution is often adequate for all practical purposes; but higher approximations are sometimes needed for greater accuracy or for some special purpose. An example is the fact that here a second approximation is required to show any change in the integral of ϕ over the contour, which gives the torsional rigidity of the shaft (or the rate of viscous flow, or the volume under the soap film). The next step in the process of successive approximations [with care exercised to retain terms of $O(\varepsilon^2)$ in transfering the boundary condition] gives

$$\phi = \tfrac{1}{4}(1-r^2) + \tfrac{1}{4}\varepsilon r^N \cos N\theta - \frac{2N-1}{8}\varepsilon^2(1 + r^{2N}\cos 2N\theta) + \ldots \qquad (2.20)$$

This satisfies our previous check for $N = 0$ [our check for $N = 1$ being inapplicable because (2.14) does not describe a shifted circle to $O(\varepsilon^2)$].

2.6. Shift of perturbation: strained coordinates

It is clear from the last two sections that a problem with a perturbed governing equation is somewhat simpler than one with a perturbed boundary. **We now show how the perturbation can be shifted from the boundary conditions to the equation by slightly straining the coordinates.**

We introduce this idea because coordinates that are slightly strained will play a central role in our treatment of singular perturbations in Chapters 5 and 6. **In simple cases the straining is merely a linear stretching; and that is all that is required in** the present regular-perturbation problem for a nearly circular contour.

We consider an ellipse of major and minor semi-axes $(1+\varepsilon)$ and $(1-\varepsilon)$ (Fig. 2.7). Then the torsion/Poiseuille-flow/soap-film problem is

$$\phi_{xx} + \phi_{yy} = -1,$$

$$\phi = 0 \quad \text{on} \left(\frac{x}{1+\varepsilon}\right)^2 + \left(\frac{y}{1-\varepsilon}\right)^2 = 1. \qquad (2.21)$$

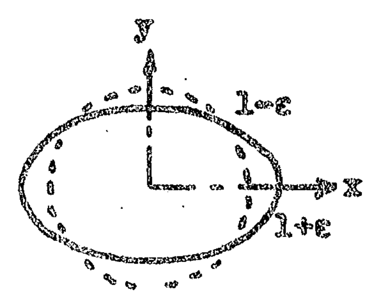


Fig. 2.7. Slightly noncircular ellipse

To first order in ε this is just our previous problem (2.15) with $N = 1$.

It is easy to introduce new slightly stretched coordinates

$$X = \frac{x}{1+\epsilon}, \quad Y = \frac{y}{1-\epsilon} \qquad \text{with} \quad \phi(x,y) = \Phi(X,Y), \tag{2.22}$$

in terms of which the boundary is a true circle. Of course this is achieved at the expense of complicating the differential equation, which becomes

$$(1+\epsilon^2)(\Phi_{XX} + \Phi_{YY}) = -(1-\epsilon^2)^2 + 2\epsilon(\Phi_{XX} - \Phi_{YY}), \tag{2.23}$$

or in polar coordinates R, Θ in the stretched X-Y plane,

$$(1+\epsilon^2)\left(\Phi_{RR} + \frac{\Phi_R}{R} + \frac{\Phi_{\Theta\Theta}}{R^2}\right) = -(1-\epsilon^2)^2 + 2\epsilon\left(\Phi_{RR} - \frac{\Phi_R}{R} - \frac{\Phi_{\Theta\Theta}}{R^2}\right)\cos 2\Theta$$
$$-4\epsilon\left(\frac{\Phi_{R\Theta}}{R} - \frac{\Phi_\Theta}{R^2}\right)\sin 2\Theta. \tag{2.24}$$

Again we assume a regular-perturbation expansion

$$\Phi = \Phi_1 + \epsilon\Phi_2 + \ldots \tag{2.25}$$

where of course

$$\Phi_1 = \tfrac{1}{2}(1 - R^2) \tag{2.26}$$

Then the problem for Φ_2 is found to be simply the homogeneous one

$$\Phi_{2_{RR}} + \frac{\Phi_{2_R}}{R} + \frac{\Phi_{2_{\Theta\Theta}}}{R^2} = 0, \quad \Phi = 0 \quad \text{at} \quad R = 1 \tag{2.27}$$

Consequently Φ_2 is zero. Then reverting to the original unstretched coordinates gives

$$\phi = \Phi_1 + O(\epsilon^2) = \tfrac{1}{2}(1 - X^2 - Y^2) + O(\epsilon^2)$$
$$= \tfrac{1}{2}(1 - r^2) + \tfrac{1}{2}\epsilon\cos 2\theta + \ldots \tag{2.28}$$

and this agrees with the result of the preceding section.

2.7. Thin cross-sections

We now suppose that our section, instead of being nearly circular, is very thin. This will permit us to exploit our second basic solution (2.8). For simplicity we take the section to be symmetric (Fig. 2.8). Then it may be described by

$$y = \pm\epsilon f(x), \tag{2.29}$$

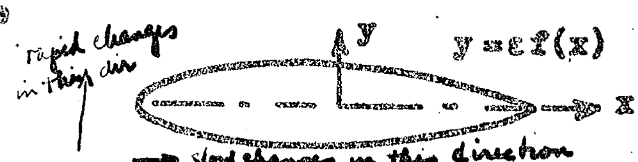


Fig. 2.8. Thin symmetrical section

where f is a smooth function of order unity.

It is clear from the soap-film analogy (cf. Fig. 2.4) that for small thickness ϵ the deflection ϕ will still change rapidly across the section, but relatively slowly along it (except possibly near a blunt end or discontinuity). Then as a first approximation we may neglect ϕ_{xx} compared with ϕ_{yy}, and the differential equation (2.1) becomes simply

$$\phi_{yy} = -1 \tag{2.30}$$

The general solution is $\phi = A + By - \frac{1}{2}y^2$; but of course the dependence on x has not disappeared altogether — it remains in the boundary condition — so that the "constants" of integration A and B may depend upon x. In fact, B vanishes for our symmetric section, and the condition that ϕ vanish at $y = \pm \varepsilon f(x)$ gives $A = \frac{1}{2}\varepsilon^2 f^2(x)$. Thus our first approximation is

$$\phi_I = \frac{1}{2}\left[\varepsilon^2 f^2(x) - y^2\right] \tag{2.31}$$

This is a <u>local</u> <u>cylindrical</u> or <u>quasi-cylindrical</u> approximation. The solution at any station along the section is that for an infinite rectangular section having the local thickness. Other familiar examples of this approximation are, in fluid mechanics, the hydraulic approximation in channels and nozzles and strip theory for wings of high aspect ratio and, in solid mechanics, simple beam theory for slowly varying cross section.

The second approximation is conveniently found by iteration. It satisfies the equation

$$\phi_{II_{yy}} - 1 = -\phi_{I_{xx}} = -\frac{1}{2}\varepsilon^2\left[f^2(x)\right]'' \tag{2.32}$$

This is the same problem as before, with 1 replaced by $1 + \frac{1}{2}\varepsilon^2\left[f^2(x)\right]''$, so the solution is

$$\phi_{II} = \frac{1}{2}\varepsilon^2\left\{1 + \frac{1}{2}\varepsilon^2\left[f^2(x)\right]''\right\}\cdot\left[f^2(x) - y^2/\varepsilon^2\right] \tag{2.32}$$

This has been arranged to show that the expansion proceeds in even powers of ε (and of y/ε). One might then prefer to proceed to subsequent approximations by substituting an assumed series. In any case the third approximation is found to be

$$\phi_{III} = \frac{1}{2}\varepsilon^2\Big\{1 + \frac{1}{2}\varepsilon^2(f^2)'' + \frac{1}{4}\varepsilon^4\Big[(f^2)''(f^2)'' + 2(f^2)'(f^2)''' + (f^2)(f^2)'''' - \frac{1}{6}(f^2)''''(f^2 + y^2/\varepsilon^2)\Big]\Big\}(f^2 - y^2/\varepsilon^2) \tag{2.33}$$

Evidently this procedure can, in principle, be continued indefinitely, at the expense only of a proliferation of terms.

<u>2.8. Prediction of uniformity and nonuniformity</u>

We have advocated (section 2.1) proceeding on the assumption that a perturbation is regular until some nonuniformity is encountered that proves it to be singular. The reason is that it is actually impossible to predict a singular perturbation. We discuss why this is so, and how one can instead recognize nonuniformity in the course of solution.

In the older literature one sometimes reads that a perturbation is singular if the small perturbation quantity multiplies one of the highest derivatives in the differential equation. This criterion was inspired by Prandtl's boundary-layer theory, where the highest derivatives in the Navier-Stokes equations are multiplied by the small viscosity. However, it is relevant only to unbounded regions (or those that grow indefinitely as $\varepsilon \to 0$, as in the last section). Even then it is not foolproof, because we can always manipulate the boundary conditions to suppress the nonuniformity. For example, a parabola in a viscous stream needs a boundary layer to adjust the outer potential flow to the surface speed (Fig. 1.3). However, we could imagine moving the surface (say with many small rollers, as suggested in Fig. 2.9), and if the local surface speed were adjusted to that of the potential flow, no boundary layer would exist.

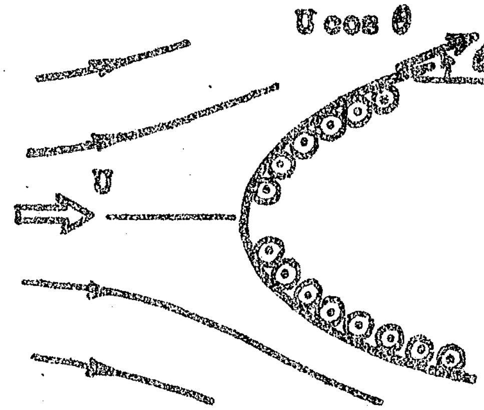


Fig. 2.9. Viscous flow past parabola with skin moving to eliminate boundary layer

In a physical problem one might suppose that a singular perturbation could be predicted on dimensional grounds. We have suggested (Section 1.5) that nonuniformities may arise if the perturbation quantity is the ratio of two characteristic lengths in the problem. This helps to explain why slight deviations from a circular boundary produced a regular perturbation (section 2.4): the two characteristic lengths — the major and minor axes in the case of the ellipse — are nearly equal, so that ε represents their difference rather than their ratio. On the other hand, in boundary-layer theory the Reynolds number is the ratio of two disparate lengths, and our remarks above show that the perturbation may be regular or singular depending on the details of the boundary conditions.

Thus there is no positive test for nonuniformity. However, it appears that this dimensional reasoning does provide instead a positive test for the absence of any nonuniformity:

A perturbation solution of a physical problem will be uniformly valid in the space and time coordinates if the perturbation quantity is not a small parameter that can be interpreted as the ratio of two lengths. (2.34)

Our previous examples all conform to this rule, though in one case the interpretation may appear a bit forced. Thus in the Janzen-Rayleigh expansion (1.5) the perturbation quantity is the Mach number, or its square; and this is the ratio of two speeds, or of two energies, but cannot possibly be regarded as the ratio of two lengths, so the perturbation must be regular. On the other hand, Reynolds number is sometimes interpreted as the ratio of a geometric to a viscous length (as well as the ratio of two forces), so the flow past a sphere at low Reynolds number could be a singular perturbation; this happens for the Navier-Stokes equations (1.14) but not for the Oseen linearization (1.7).

The thickness ratio of a slender rod is nothing else than the ratio of two lengths; and it leads to a singular perturbation for the capacitance (1.15). The elastic deformation of a pressurized thin toroidal shell (1.19) is also a singular perturbation; but it is not immediately obvious that the dimensionless pressure is the ratio of two lengths. However, it is taken as $\varepsilon = pa/Eh$, where p is the pressure, a the internal radius of the toroid, E the modulus of elasticity, and h the thickness of the shell. Hence ε can be interpreted as the ratio of the characteristic length pa/E to the thickness (as well as the ratio of two stresses).

Our rule (2.34) implies that a coordinate perturbation is always regular. The reason is that if x/L is the perturbation quantity, for example, there is no alternative choice of dimensionless coordinate (corresponding to x/ℓ in a parameter perturbation). Thus Howarth's expansion (1.9) for the boundary layer in a retarded flow is necessarily regular (though of course the boundary layer itself is part of a singular perturbation for large Reynolds number).

2.9. Symptoms of nonuniformity: singularities in the solution

If our perturbation quantity is a small parameter formed as the ratio of two characteristic lengths, we should be on the lookout for nonuniformities. A nonuniformity will manifest itself in the solution through singularities within the region of space and time that is of physical significance.* The reason is that a straightforward perturbation expansion has been constructed on the assumption that each term is of smaller order than its predecessor; and this can fail to be so only near a singularity.

We shall therefore, in case of doubt, examine the ratios of successive terms in any supposed regular-perturbation solution. If they fail to remain small, the perturbation is singular; and we may discern the location, size, and shape of the region of nonuniformity.

We reconsider from this point of view our approximation (Section 2.6) for the torsion of a thin shaft. The perturbation parameter ε is clearly the ratio of two characteristic lengths, so the solution (2.33) may be singular. For simplicity, we examine the family of power-law shapes (Fig. 2.10) described by

$$y = \varepsilon f(x) = \varepsilon(1-x^2)^m, \quad -1 \leq x \leq 1, \quad m \geq 0. \qquad (2.35)$$

---

*A singularity outside the field is of no concern. For example, every analytic function except a constant has a singularity somewhere in the complex plane.

The ratio of the second term to the first is

$$\frac{\text{second term}}{\text{first term}} = \tfrac{1}{4}\varepsilon^2\left[f^2(x)\right]''$$

$$= 4m(2m-1)\left[\frac{\varepsilon}{(1-x^2)^{1-m}}\right]^2 +$$

$$+ \text{ less singular terms} \qquad (2.36)$$

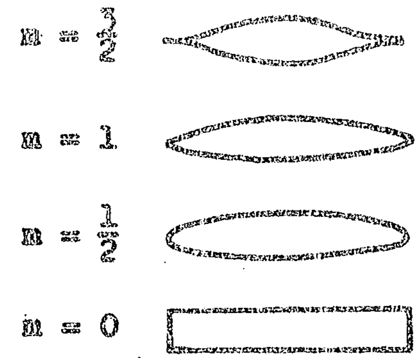


Fig. 2.10. Thin doubly-symmetric power-law sections

This is small, of $O(\varepsilon^2)$ as assumed, throughout the region of physical significance if $m \geqslant 1$ -- that is, if the ends are at least as sharp as a wedge. If they are blunter, this ratio grows without bounds near the ends. The perturbation is then singular, and we see that the region of nonuniformity extends from either end of the section a distance of order $\varepsilon^{1/(1-m)}$.

A curious exception arises for the ellipse. Although the ratio is singular for neighboring power-law shapes on either side, it vanishes for $m = \frac{1}{2}$ thanks to the multiplicative factor $(2m-1)$. This suggests that the perturbation is regular for the ellipse, and this is confirmed by the fact that the exact solution is just $(1+\varepsilon^2)^{-\frac{1}{2}}$ times our first approximation. On the other hand, the factor m makes the ratio vanish also for the rectangle; but the soap-film analogy of Fig. 2.4 suggests that the perturbation is nevertheless singular. The explanation is that the bluntness is so great that we have missed it; the rectangle is not actually described by (2.35) with $m = 0$ (which gives an open-ended strip) but by $H(1-x^2)$, where H is the Heaviside step function, and the ratio is consequently very singular indeed at the ends.

A singularity that is the symptom of nonuniformity is compounded in higher approximations, as for the model function

$$\frac{x}{x+\varepsilon} = 1 - \frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} - \frac{\varepsilon^3}{x^3} + \cdots \qquad (2.37)$$

Otherwise it would represent a true singularity of the solution. Thus the ratio of the third to the second term in (2.33) again behaves like (2.36); and this structure will continue indefinitely.

In problems of indefinite spatial or temporal extent, the nonuniformity -- and hence the singularities -- may arise at infinity. A particular case, which we shall examine in Chapter 5, **is that of so-called <u>secular</u> <u>terms</u>.** A simple example is a linear oscillator with slight damping (Fig. 2.11).

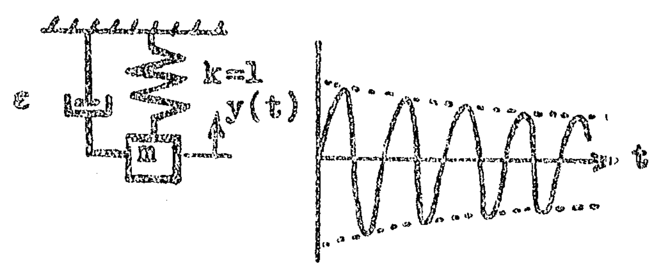


Fig. 2.11. Motion of slightly damped linear oscillator

If disturbed, it will obviously oscillate with slowly decaying amplitude. However, a straightforward perturbation solution has the form

$$y(t) = \sin t - \varepsilon t \sin t + \varepsilon^2\left[\tfrac{1}{2}(t^2+1)\sin t - \tfrac{1}{2} t \cos t\right] + \ldots \qquad (2.38)$$

The terms in $t \sin t$, $t^2 \sin t$, etc. are called secular (from Latin saeculum: a long period of time). **They are clear evidence that this approximation is not uniformly valid for large time, breaking down when t is $O(1/\varepsilon)$.**

2.10. Unsatisfiable boundary conditions: the Stokes paradox

A singularity in the straightforward solution is the usual symptom of nonuniformity. Sometimes, however, the symptom is inability to calculate the next approximation (which may be the first one). The difficulty may be, for example, the appearance of integrals that are divergent and uninterpretable.

Another common difficulty is the inability to satisfy all the boundary conditions. A familiar example is the inability to satisfy the no-slip condition for fluid motion past a body when the viscosity is so small that it is neglected. This difficulty was resolved by Prandtl's boundary-layer theory; and we shall see that in a number of other examples the difficulty has also succumbed recently to treatment as a singular perturbation.

This failure of a straightforward approximation to satisfy all the boundary conditions is sometimes termed a paradox. We mention four paradoxes of this kind:

1. The Stokes paradox. No solution can be found of the approximate equations of slow viscous motion for a cylinder normal to a uniform stream.

2. The Whitehead paradox. No second approximation can be found for slow viscous flow past a finite three-dimensional body (e.g., a sphere) in a uniform stream.

3. The Hertz paradox. No solution with finite displacement can be found for the elastic contact of infinite cylinders (Fig. 2.12a).

4. The surf-board paradox. No solution with finite displacement of the free surface can be found for two-dimensional motion of a plate planing on the surface of deep water at such high speed that gravity is negligible (Fig. 2.12b).

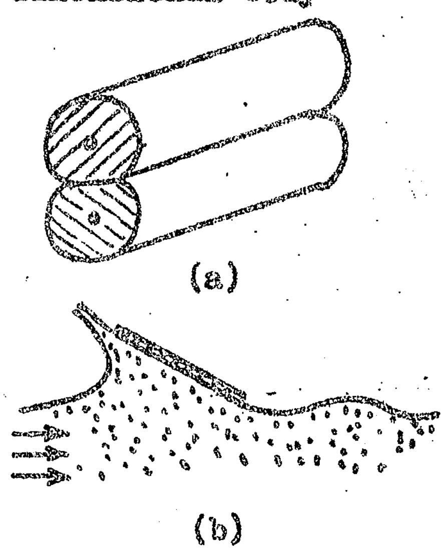


Fig. 2.12. The paradoxes of (a) Hertz for elastic contact, (b) the surf-board.

We show the difficulty in detail for the Stokes paradox, which is typical of all, considering only the circular cylinder. It is convenient to satisfy the continuity equation by working with the stream function

ψ, whose derivatives give the velocity components in Cartesian coordinates according to

$$u = \psi_y , \qquad v = - \psi_x . \tag{2.39}$$

Then eliminating the pressure from the Navier-Stokes equations by cross-differentiation leaves a single nonlinear fourth-order partial differential equation

$$\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y}\right)\nabla^2 \psi = \nu \nabla^4 \psi . \tag{2.40}$$

This expresses a balance of inertial forces on the left and viscous forces on the right. (It is sometimes called the "vorticity equation," because $-\nabla^2\psi$ is the vorticity in plane flow.)

For high viscosity or low speeds -- that is, at low Reynolds number -- this is simplified as was first done by Stokes (1851), who says

> "... the motion is supposed small, on which account it will be allowable to neglect the terms which involve the square of the velocity."

Thus we obtain the linear biharmonic equation:

$$\nabla^4 \psi = 0 . \tag{2.41}$$

In the polar coordinates appropriate to a circle this becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \psi = 0 \tag{2.41b}$$

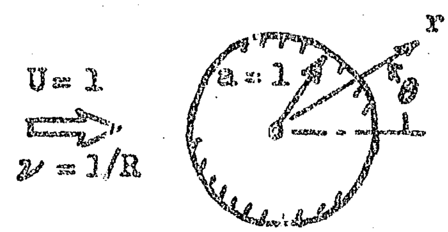


Fig. 2.13. Viscous flow past circle

and if the variables are made dimensionless (Fig. 2.13) by referring lengths to the radius a of the circle and velocities to the free-stream speed U (so that ψ is referred to Ua), the boundary conditions become

$$\text{surface:} \qquad \psi = \psi_r = 0 \qquad \text{at} \quad r = 1, \tag{2.42a}$$

$$\text{upstream:} \qquad \psi \sim y = r \sin\theta \qquad \text{as} \quad r \to \infty \tag{2.42b}$$

Following Stokes, we may assume that, as suggested by the only nonhomogeneous element in the problem, $\psi = \sin\theta \cdot F(r)$. This gives the ordinary differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)^2 F(r) = 0 \tag{2.43}$$

This is an equidimensional equation, so we try $F(r) = r^m$; and substituting yields the indicial equation

$$(m+1)(m-1)^2(m-3) = 0 \tag{2.44}$$

The repeated root indicates the need for a logarithm*, so that

---

* This fact is known to every graduate student nowadays; but Stokes had to multiply the last term in the operator in (2.43) by $(1+\delta)^2$ and finally let δ tend to zero.

$$F(r) = \frac{A}{r} + Br + C\,r\log r + Dr^3 \tag{2.45}$$

(We use "log" always to denote the natural logarithm.) Stokes now imposes the upstream condition, and then observes that he has only the constant A left to satisfy the two surface conditions. We prefer to satisfy the surface conditions instead, leaving

$$\phi = \left[(\tfrac{1}{2}C + D)\frac{1}{r} - (\tfrac{1}{2}C + 2D)\,r + C\,r\log r + D\,r^3\right]\sin\theta \tag{2.46}$$

But now we cannot satisfy the upstream condition. The best we can do is to make the singularity at infinity as weak as possible by setting $D = 0$; but the velocity still grows logarithmically with r rather than becoming uniform.

This is the paradox of Stokes. He concludes erroneously that no steady-state flow exists, the circle continually entraining fluid without limit. However, he does add, without further comment, that "it may not be safe in such an extreme case to neglect the terms depending on the square of the velocity, not that they become unusually large in themselves, but only unusually large compared with the terms retained."

That is, the approximation adopted cannot pass an <u>a posteriori</u> consistency test. Using our solution (2.46) with $D = 0$ to evaluate the ratio of a typical neglected term to one that was retained gives

$$\frac{\text{neglected}}{\text{retained}} = \frac{R\,\frac{\phi_\theta}{r}\,\frac{\partial}{\partial r}\nabla^2\phi}{\frac{\partial^2}{\partial r^2}\nabla^2\phi} = -CR\left(r\log r - \tfrac{1}{2}r + \tfrac{1}{2}\frac{1}{r}\right)\cos\theta \tag{2.47}$$

This is small like R, but not uniformly for large radius r. The neglected terms are as large as those retained when $r = O(1/R)$. The situation is exactly the same in Whitehead's paradox for the sphere; but the difficulty is delayed to the second approximation because the flow is gentler.

Evidently the solution can be completed only by somehow taking into account the nonlinear inertia terms. Oseen (1910) did this approximately by linearizing about the uniform stream rather than a quiescent state. This changes the perturbation from a singular to a regular one, so that the first approximation can be completed for the circle (Lamb 1911), and higher approximations for both the circle and sphere found, presumably in principle to any order (Illingworth 1960). However, the analysis is unnecessarily complicated because the inertial terms are represented even close to the body, where they are negligible.

A simpler and more systematic procedure is to consider the inertial terms only far from the body, where they are important. This means constructing another approximation to supplement that of Stokes. It will fail near the body, just as Stokes's fails far away; but together they describe the entire field. Moreover, they possess an <u>overlap domain</u> of common validity. They can therefore be <u>matched</u> to determine the constants of integration remaining in each. Finally, elements of each can be combined to single <u>composite</u> approximation that is uniformly valid.

This is an outline of the <u>method of matched asymptotic expansions</u>, which we discuss in chapter 4. It was systematically developed by Kaplun (1957) and Proudman & Pearson (1957), who independently applied it to resolve the paradoxes of Stokes and Whitehead and then calculate the next approximation. That technique has more recently been applied also to resolve the paradoxes of Hertz (Schwartz & Harper 1971) and of the surfboard (Rispin 1967).

References

(additional to those cited in the previous chapter)

DEAN, W. R. 1927 Note on the motion of fluid in a curved pipe. Phil. Mag. (7) 4, 208-223

FUNG, Y. C. 1965 Foundations of Solid Mechanics. Prentice-Hall, Englewood Cliffs.

ILLINGWORTH, C. R. 1960 Flow at small Reynolds numbers. Chapter 4 of Laminar Boundary Layers (L. Rosenhead, editor), Clarendon Press, Oxford

KAPLUN, S. 1957 Low Reynolds number flow past a circular cylinder. J. Math. Mech. 6, 595-605

KAPLUN, S. & LAGERSTROM, P. A. 1957 Asymptotic expansions of Navier-Stokes solutions for small Reynolds numbers. J. Math. Mech. 6, 585-593

LAMB, H. 1911 On the uniform motion of a sphere through a viscous fluid. Phil. Mag. (6) 21, 112-119; Hydrodynamics (1932), sec. 343.

LOVE, A. E. H. 1927 A Treatise on the Mathematical Theory of Elasticity. Cambridge & Macmillan; Dover, New York, 1944.

OSEEN, C. W. 1910 Ueber die Stokes'sche Formel, und über eine Verwandte Aufgabe in der Hydrodynamik. Ark. Math. Astron. Fys. 6, No. 29.

PROUDMAN, I. & PEARSON, J. R. A. 1957 Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. J. Fluid Mech. 2, 237-262.

RISPIN, P. P. 1967 A singular perturbation method for non-linear water waves past an obstacle. Ph.D. thesis, Calif. Inst. Tech., 1967

SCHWARTZ, J. & HARPER, E. Y. 1971 On the relative approach of two-dimensional elastic bodies in contact. Int. J. Solids Structures 7, 1613-1626

VAN DYKE, M. 1970b Entry flow in a channel. J. Fluid Mech. 44, 813-823

EXERCISES

<u>2.1. Second approximation for deflection of membrane.</u> Small deflections $\phi(x,y)$ of a soap film or other homogeneous membrane under a pressure difference p are governed by the Poisson equation (2.3). However, if the slope of the membrane is not small, this linearized equation must be replaced by the true nonlinear one (2.2). In Cartesian coordinates this has the rather complicated form

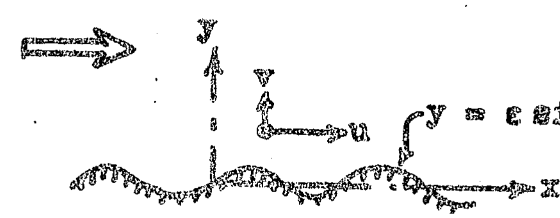
$$\sigma \frac{(1+\phi_x^2)\phi_{yy} + (1+\phi_y^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy}}{(1+\phi_x^2+\phi_y^2)^{3/2}} = -p$$

and in cylindrical polar coordinates with axial symmetry

$$\phi_{rr} + (1+\phi_r^2)\frac{\phi_r}{r} = -(1+\phi_r^2)^{3/2}\,p\,.$$

Consider the case of a membrane spanning a circular hole of radius a in a flat plate. Solve the linearized equation to show that the deflection is parabolic in this approximation. Find a second approximation by iterating on the first. (Expand the radical in Taylor series, use the first approximation to evaluate the nonlinear terms, find a particular integral by trial, add the general solution of the homogeneous linear equation, and impose the boundary conditions.) What is the dimensionless perturbation quantity? Can it be regarded as the ratio of two characteristic lengths? Is this a coordinate perturbation or a parameter perturbation? What would be the form of subsequent terms? Can you imagine calculating them by hand, or with an electronic computer? Verify your second approximation by expanding the exact solution, which obviously (as in a child's bubble pipe) gives spherical deflection.

<u>2.2. Flow past wavy wall.</u> A simple problem that illustrates some of the features of water waves and of thin-ship theory is that of potential flow past an infinite sinusoidal wall with a uniform stream far from the wall. If continuity is satisfied by introducing a stream function according to $u = \psi_y$, $v = -\psi_x$ the dimensionless problem is



$$\psi_{xx} + \psi_{yy} = 0, \qquad \psi = 0 \quad \text{on the wall}$$
$$\psi \sim y \quad \text{far from the wall}$$

No solution of this problem is known in closed form. Show that the first approximation for small ε is

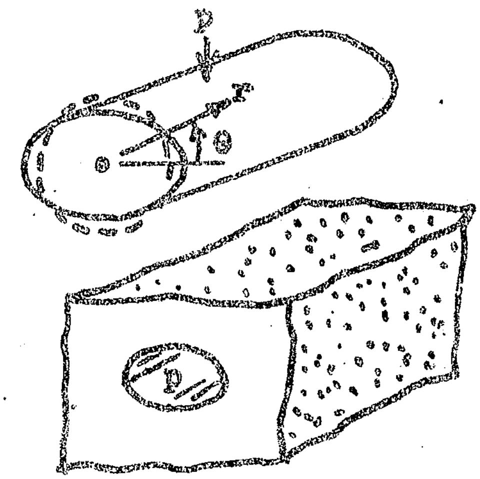
$$\psi \sim y - \varepsilon \sin x \, e^{-y}$$

and carry this to the next approximation. Use your result to calculate the maximum and minimum speed in the field (which occur on the wall at the tops and bottoms of the sinusoidal bumps), putting them in the form $1 + a\varepsilon + b\varepsilon^2$. The necessity for transferring the surface boundary condition can be avoided — at the expense of complicating the differential equation — by using coordinates that conform to the surface. A simple possibility is to replace the ordinate by its value measured from the surface, introducing the new independent variables

$$X = x, \qquad Y = y - \varepsilon \sin x .$$

Carry out the first-order solution on this basis, and show by comparison with the previous result that it actually gives the maximum and minimum speed correct to second order.

<u>2.3. Stresses for slightly noncircular pressurized rod or hole.</u> A long elastic cylinder of nearly circular cross-section, or a long nearly circular hole in a large block of elastic material is subjected to a uniform hydrostatic pressure p. In this situation of plane strain the three non-zero stress components can be given in terms of Airy's stress function ϕ. In polar coordinates the relations are (for example, Timoshenko & Goodier 1951, p. 56)



$$\sigma_r = \frac{\phi_r}{r} + \frac{\phi_{\theta\theta}}{r^2}, \quad \sigma_\theta = \phi_{rr}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\frac{\phi_\theta}{r}$$

and ϕ satisfies the biharmonic equation (2.41b). Consider the cylinder described by $r = R(\theta) = 1 + \varepsilon \cos N\theta$, and calculate ϕ to order ε by perturbing the solutions $\phi = -\frac{1}{2}pr^2$ for a circular rod or $\phi = -a^2 p \log r$ for a circular hole of radius a. (Note that the normal and tangential stresses on the boundary differ from those on the basic circle because the contour has not only shifted radially but also rotated slightly; and the latter is accounted for using Mohr's circle or the formulas on which it is based: e.g., Timoshenko & Goodier 1951, p. 13.) Check your results by using the facts that $N = 0$ represents simply a slight enlargement of the contour, and $N = 1$ corresponds, to order ε, to a sideways translation.

<u>2.4. Plane waves traveling through slowly changing environment.</u> Various problems in mechanics involve the propagation of waves through a medium that is changing its properties with distance, but appreciably only over many wave lengths. Examples are longitudinal elastic waves in a bar of slowly varying diameter, acoustic waves in a trumpet, and acoustic waves traveling through air of slowly varying density. In the approximation of plane waves, these are all governed by the hyperbolic equation

$$\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) - F\frac{\partial^2 u}{\partial t^2} = 0 .$$

Here F is known function of x (the cross-sectional area in the first two examples, and the density in the third) such that the energy in the wave is proportional to the product u^2F. We are interested in periodic waves that in the case of constant F would be described (in dimensionless variables such that the wave speed is unity) by $u = \cos(x-t)$ — say, for definiteness, the solution for waves propagating only in the positive x-direction with $u = \cos t$ at $x = 0$.

Show that even in the simple case $F = 1 + \varepsilon x$ the straightforward approach of assuming a regular perturbation for small ε leads to difficulties at large distances in the second approximation. Carry the third approximation (involving terms in ε^2) at least far enought to show how the difficulty is compounded in higher approximations.

An approximation based on the idea that energy is conserved with distance is

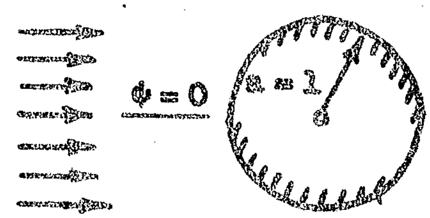
$$u \approx \frac{\cos(x-t)}{\sqrt{F(x)}} .$$

Verify that in the case $F = 1 + \varepsilon x$ this represents an improvement, in that the difficulty is postponed from the second to the third approximation. What does this suggest about the role of scales in this problem? For what $F(x)$ is this approximation exact?

2.5. Circle in symmetric shear flow. Consider the flow past a circular cylinder of unit radius symmetrically placed in a parallel stream of incompressible inviscid fluid having the velocity distribution far upstream

$$u = \cosh(\varepsilon^{\frac{1}{2}} y), \quad v = 0 .$$

For such a rotational flow the stream function ψ satisfies the equation



$$\nabla^2\psi = \begin{Bmatrix} \psi_{xx} + \psi_{yy} \\ \psi_{rr} + \dfrac{\psi_r}{r} + \dfrac{\psi_{\theta\theta}}{r^2} \end{Bmatrix} = -\omega(\psi) ,$$

which expresses the fact that the vorticity ω is constant along each streamline in a plane inviscid flow, and hence a function only of the stream function ψ. Evaluate the function $\omega(\psi)$ by calculating both ω and ψ far upstream and eliminating the coordinates between them. Solve the resulting partial differential equation by perturbing the classical solution $\psi = (r - 1/r)\sin\theta$ for a uniform stream. Thus show that a difficulty arises in the term of order ε because the velocity disturbances grow rather than dying out far upstream.

2.6. Potential flow past nearly circular cylinder. Suppose that a long cylinder having the cross-section of Eq. (2.14) and Fig. 2.6 is placed across a uniform stream of incompressible inviscid fluid. Find an approximation to $O(\varepsilon)$ for the stream function ψ, which is a solution of Laplace's equation that vanishes on the surface and approaches a multiple of y far upstream. (For a circle of radius a in a stream of speed U, $\psi = U(r - a^2/r)\sin\theta$.) Apply the two checks described at the bottom of page 27.

Chapter 3

SLOW VARIATIONS

There are no singular-perturbation problems;
there are only singular-perturbation solutions.

— P. A. Lagerstrom

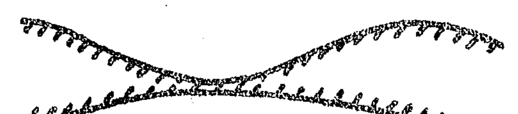
3.1. Slow vs. slight variations

We have seen how a perturbation scheme will frequently, through one symptom or another, reveal itself to be singular. If a uniformly valid solution is required, means must be found for dealing with those difficulties. We will discuss in the following three chapters the three main methods of treating singular-perturbation problems in mechanics.

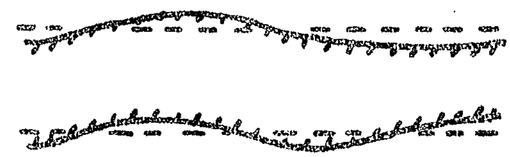
We shall see that each of these methods is based on the idea of manipulating the scales of space and time. That is, a singular perturbation becomes a regular perturbation under appropriate alteration of scales.

As a simple introduction to these ideas we first examine problems of slow variations. This is an approximation familiar in every branch of mechanics, based on the observation that devices of practical utility often have a geometry that varies slowly in one direction compared with the others (Fig. 3.1a). Slow variations in time also fall into this category.

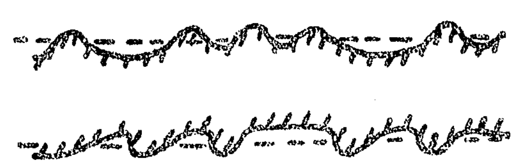
These problems represent a transition between regular and singular perturbations. If they are attacked with a straightforward perturbation procedure the result is found to be nonuniform unless the variation is slight as well as slow (Fig. 3.1b). However, a simple linear contraction of the longitudinal length scale relative to the transverse scale serves to render the perturbation regular. The calculations are actually simplified, yet the result is more general in that it is valid for any slow variation, whether slight or not. (Neither approximation is useful for a "wiggly" shape (Fig. 3.1c), whose variations are slight, but rapid rather than slow.)



(a) Slow



(b) Slight and slow



(c) Slight

Fig. 3.1. Slow and slight variations on a strip.

3.2. Potential flow in slightly varying channel

We begin with the simple problem of plane potential flow through a nearly straight channel. (Of course this problem has its counterparts in electricity, elasticity, and other fields.) For definiteness we consider the hyperbolic channel described in Fig. 3.2, whose double symmetry is a convenience rather than a necessity. (The exact solution could be found using elliptic coordinates; see Morse & Feshbach 1953, p. 1196.)

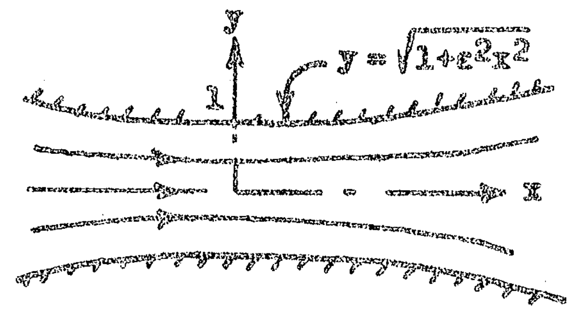


Fig. 3.2. Potential flow through slowly varying hyperbolic channel

It is convenient to work with the stream function ψ. Suppose that we prescribe the flux through the channel, which is proportional to the difference between the values of ψ at the upper and lower walls. Hence the full problem can be normalized to

$$\psi_{xx} + \psi_{yy} = 0, \qquad \psi(x, 0) = 0, \quad \psi(x, \sqrt{1+\varepsilon^2 x^2}) = 1. \tag{3.1}$$

Although the equation is elliptic, no boundary conditions are needed at the "ends" of the channel because they are indefinitely removed (so that the details of entry and exit will have disappeared according to an analog of St.-Venant's principle).

For $\varepsilon = 0$ the solution is simply $\psi = y$, corresponding to the uniform parallel flow in a channel of constant width. If we assume that there is a regular perturbation for small ε, it must proceed in powers of ε^2 with, say,

$$\psi = y + \varepsilon^2 \psi_2(x,y) + \varepsilon^4 \psi_3(x,y) + \ldots \tag{3.2}$$

Substitution shows that each ψ_n must satisfy Laplace's equation and vanish on the axis. The boundary condition on the wall is expanded as

$$\sqrt{1+\varepsilon^2 x^2} + \varepsilon^2 \psi_2(x, \sqrt{1+\varepsilon^2 x^2}) + \ldots = 1 + \tfrac{1}{2}\varepsilon^2 x^2 + \varepsilon^2 \psi_2(x,1) + O(\varepsilon^4) = 1 \tag{3.3}$$

Hence the problem for ψ_2 is

$$\psi_{2xx} + \psi_{2yy} = 0, \qquad \psi_2(x,0) = 0, \quad \psi_2(x,1) = -\tfrac{1}{2} x^2. \tag{3.4}$$

The solution is easily found by trying simple polynomials, or better using complex variable and trying the imaginary parts of simple powers of $(x + iy)$ — which, by symmetry, must be odd powers. Thus we find the second approximation:

$$\psi = y - \frac{1}{6}\varepsilon^2 (3x^2 y - y^3 + y) + \ldots$$
$$u = \psi_y = 1 - \frac{1}{6}\varepsilon^2 (3x^2 - 3y^2 + 1) + \ldots \tag{3.5}$$

Even on the axis of the channel this approximation is evidently not uniformly valid for large x. There is a singularity at $x = \infty$; and the second term in u becomes as large as the first when $x = O(1/\varepsilon)$. This difficulty is compounded in higher approximations. The next approximation is easily found to be

$$\frac{\psi}{y} = 1 - \frac{1}{6}\varepsilon^2(3x^2-y^2-1) + \frac{1}{120}\varepsilon^4(45x^4-90x^2y^2+9y^4+60x^2-20y^2+11) \qquad (3.6)$$

and the indication of nonuniformity where $x = O(1/\varepsilon)$ is reinforced.

It is clear physically why the solution breaks down at such large distances. When $x = O(1/\varepsilon)$ the contour has departed significantly from the basic strip, and the boundary condition has to be transferred over too great a distance to remain valid.

3.3. Quasi-cylindrical approximation: slowly varying channel

The above procedure is not the usual practical way of calculating the flow in a channel, nozzle, or pipeline. Any plumber would apply instead the hydraulic approximation, assuming that the velocity is parallel to the axis and constant across the channel. This gives the familiar quasi-one-dimensional result that the speed u varies inversely as the cross-sectional area A, so that the product uA remains constant to satisfy continuity (or ρuA remains constant in compressible flow with density ρ). In our special case of a hyperbolic channel this gives

$$u = \frac{1}{\sqrt{1+\varepsilon^2x^2}} \qquad (3.7)$$

This hydraulic approximation has the great advantage that it is not restricted to slight variations. It is accurate for even enormous variations of area provided that they take place slowly (Fig. 3.1a), and furthermore quickly recovers its accuracy outside local zones of sudden variation ("St.-Venant's principle").

This is a special simple example of the quasi-cylindrical approximation, which we have already used in analyzing torsion of a thin cross-section (sec. 2.6), and is familiar in many other problems and fields. Some examples are current flow in a slowly varying wire; and bending or torsion of a slowly varying beam or shaft.

On its face the quasi-cylindrical approximation is an ad hoc rather than a systematic one (sec. 1.2). That is, it is not immediately obvious whether it can be embedded in a rational scheme for successively calculating higher approximations. In fact, however, this is easily accomplished by shifting the perturbation from the boundary conditions to the differential equation. As in section 2.5, this is accomplished simply by stretching the coordinates differentially, but now by a very large ratio. That is, we "square up" the geometry by introducing the contracted abscissa

$$X = \varepsilon x \qquad (3.8)$$

(Since only the relative stretching is significant, we could alternatively introduce the magnified ordinate $Y = y/\varepsilon$; and we shall later prefer this choice when the original problem is scaled such that the slowly varying region is "thin" rather than "long" — for example, in Prandtl's viscous boundary layer.)

With this distortion of the coordinates,* the full problem (3.1) becomes

$$\varepsilon^2 \psi_{XX} + \psi_{yy} = 0, \qquad \psi(X,0) = 0; \quad \psi(X, \sqrt{1+X^2}) = 1. \tag{3.9}$$

For $\varepsilon = 0$ the solution is

$$\psi_1 = \frac{y}{\sqrt{1+X^2}} \tag{3.10}$$

and this yields the axial velocity (3.7) of the hydraulic approximation. Furthermore, working with the stream function has the advantage that this provides also a first approximation to the transverse velocity, which is absent from the hydraulic approximation.

We can now systematically calculate higher approximations by assuming a regular expansion

$$\psi = \psi_1 + \varepsilon^2 \psi_2(X,y) + \varepsilon^4 \psi_3(X,y) + \ldots \tag{3.11}$$

The problem for the nth term is (for $n = 2, 3, \ldots$)

$$\psi_{n_{yy}} = -\psi_{(n-1)_{XX}}, \qquad \psi_n(X,0) = \psi_n(X,\sqrt{1+X^2}) = 0 \tag{3.11}$$

Thus instead of seeking a solution of Laplace's equation at each step, as in the previous approximation, we need merely perform two quadratures. Thus the second approximation is easily found as

$$\psi = \frac{y}{\sqrt{1+X^2}}\left[1 - \varepsilon^2 \frac{1-2X^2}{6(1+X^2)}\left(1 - \frac{y^2}{1+X^2}\right) + O(\varepsilon^4)\right] \tag{3.12}$$

There is now no symptom of nonuniformity in this or any higher approximation: the perturbation is regular. For example, far downstream our hyperbolic channel approaches the wedge-shaped channel $y = \pm\varepsilon x$. There our second approximation (3.12) becomes

$$\psi \sim \frac{y}{X}\left[1 + \frac{1}{3}\varepsilon^2\left(1 - \frac{y^2}{X^2}\right) + O(\varepsilon^4, X^{-2})\right] \tag{3.13}$$

and this is just the expansion to $O(\varepsilon^2)$ of the solution for flow in a wedge (Fig. 3.3), produced by a source at the origin:

$$\psi = \frac{\tan^{-1}(y/x)}{\tan^{-1}\varepsilon} = \frac{\tan^{-1}(\varepsilon y/X)}{\tan^{-1}\varepsilon} \tag{3.14}$$

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* A mathematician would also change the notation for ψ to indicate that the stream function is not the same function of X and y as it is of x and y, but we follow the usual engineering practice of using the same symbol when no confusion can result.

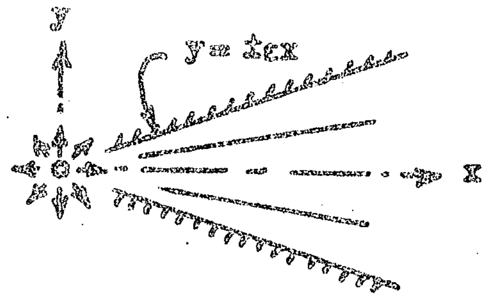


Fig. 3.3. Potential flow through wedge channel from source at vertex

Rewriting our approximation (3.12) in terms of the original abscissa x and expanding for small ϵ (more precisely, for small ϵx) reproduces our earlier result (3.5) based on slight rather than slow variations. This operation reveals that (3.5) converges for $\epsilon x < 1$, in accord with the symptoms of nonuniformity that we detected for $x = O(1/\epsilon)$. **Thus we see that the approximation of slow variations is preferable not only because it is simpler, but also because it is less restrictive than the approximation of slight variations, whose results are included as a special case.**

The calculations are so simple that it is just as easy to treat a channel of general shape. Thus for a symmetric channel described by $y = \pm F(\epsilon x)$ the second approximation is

$$\phi = \frac{y}{F(X)} + \frac{1}{6}\epsilon^2 \left[\frac{1}{F(X)}\right]'' \left[y F^2(X) - y^3\right] + O(\epsilon^4). \qquad (3.15)$$

This is uniformly valid if the function F is smooth and bounded, so that the slope of the channel is everywhere $O(\epsilon)$, the curvature $O(\epsilon^2)$, etc.

Further approximations are found so easily in principle that it would seem that the mounting labor involved could be delegated to a computer. In fact Lucas (1972) has done this for the more complicated problem of laminar flow governed by the Navier-Stokes equations. He has computed, for example, 40 terms for an exponential channel.

3.4. Generalized asymptotic expansions

We have transformed a singular-perturbation problem into a regular perturbation by simply distorting the ratio of longitudinal and transverse scales. The result is an asymptotic expansion (section 1.10), and in fact an asymptotic power series. That is, in each term the dependence on the perturbation quantity ϵ has been neatly extracted in the form of a gauge function $\delta_n(\epsilon) = \epsilon^{2n}$, leaving a multiplicative function of X and y that is independent of ϵ.

This tidy arrangement is destroyed, however, if we revert to the original coordinates. Thus, replacing X by εx in (3.12) gives

$$\phi = \frac{y}{\sqrt{1+\epsilon^2 x^2}} \left[1 - \epsilon^2 \frac{1 - 2\epsilon^2 x^2}{6(1+\epsilon^2 x^2)} \left(1 - \frac{y^2}{1+\epsilon^2 x^2}\right) + O(\epsilon^4)\right] \qquad (3.12b)$$

As indicated by the final order symbol, this can still be regarded as an expansion in powers of ϵ^2. Now, however, the coefficients **also depend on ε. Hence this is no longer an asymptotic expansion in the classical sense. Instead of having the standard form (1.26)**

$$f(x;\ \varepsilon) \sim \sum c_n(x)\,\delta_n(\varepsilon) \qquad (1.26)$$

it has the more complicated form

$$f(x;\ \varepsilon) \sim \sum c_n(x;\ \varepsilon)\,\delta_n(\varepsilon)\ . \qquad (1.26b)$$

Such an expansion has been called a generalized asymptotic expansion. (Erdelyi 1961). The dependence of the coefficients on ε provides flexibility at the expense of additional indeterminacy: with the gauge functions chosen, a given function has only one asymptotic expansion, but an unlimited number of generalized asymptotic expansions. We naturally insist that the gauge function properly represent the smallness of each term; but that still leaves a wide choice for the ε-dependence of each coefficient. Consequently, further rules of procedure must be adopted in each specific application. (In the preceding example, we would require that the coefficients depend not on x and ε separately, but only on their product $X = \varepsilon x$.)

We shall see that, by one technique or another, the solution of any perturbation problem can be found as a single expansion that is uniformly valid throughout the region of interest. In a regular-perturbation problem, that will be an asymptotic expansion in the classical sense (which perhaps has a finite radius of convergence). In a singular-perturbation problem, on the other hand, it is necessarily a generalized asymptotic expansion.

3.5. Closer local approximations

We have seen that the method of slight variations is less general than that of slow variations, being contained in it as a special case. It would seem that these are the first two members of a hierarchy of successively more general methods, each containing all its predecessors. Each is superior to its predecessors because it matches the geometry more closely. Thus the first requires that the fractional change in ordinate be small, the second that changes in the slope be small, and the Nth that changes in a dimensionless form of the (N-1)st derivative be small.

In particular, the third member of this sequence would provide a uniformly valid approximation for slowly varying channels whose fractional change in ordinate and slope may be enormous, but whose curvature is everywhere small. More precisely, the appropriate dimensionless criterion is that the product of the local curvature and ordinate be small. Then the first approximation would not be cylindrical flow at each station, but conical flow (Fig. 3.4).

Fraenkel (1963) has explored this idea for plane viscous flow in a symmetric channel, but the basic conical solution is the Jeffery-Hamel flow, which involves elliptic functions that complicate the analysis. In the simpler case of incompressible potential flow, the basic conical solution is that of Fig. 3.3, due to a source upstream or a sink downstream, according as the local slope is positive or negative. (In the dividing case of zero slope it is uniform parallel flow.)

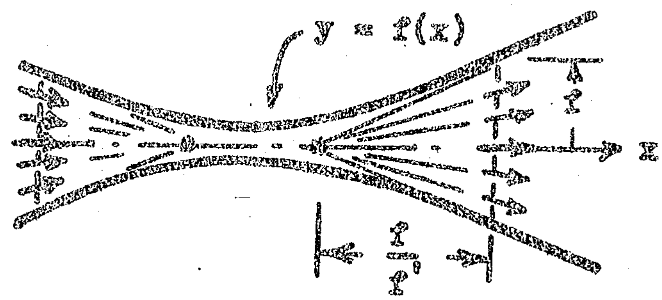


Fig. 3.4. Quasi-conical approximation for potential flow in slowly varying channel

Using (3.14) we see that for a symmetrical channel described by $y = \pm f(x)$ the first quasi-conical approximation is

$$\phi_1(x,y) = \frac{\tan^{-1}[y \cdot f'(x)/f(x)]}{\tan^{-1} f'(x)} . \tag{3.16}$$

As it stands, this is an ad hoc approximation rather than a systematic one. It would be worthwhile to think about how to imbed it into a systematic scheme of successive approximations.

In the same way Massonet (1962) has suggested approximating the stress field in a beam of varying height as that in the tangent wedge at each station. The writer is not aware that this idea has been pursued or systematized.

Evidently it is appropriate to characterize these as quasi-conical approximations, in contrast to the previous quasi-cylindrical one. More generally, one can make piecewise local application of other self-similar solutions. This has been tried in boundary-layer theory using the Falkner-Skan family (Smith 1956), but no one has yet succeeded in embedding it in a systematic scheme so as to calculate a second approximation.

Still more generally, one can imagine piecewise local application of any simple solution. For example, the slightly damped oscillator of Fig. 2.11 on page 28 could be regarded as a sequence of simple harmonic motions (Fig. 3.5). In any such approximation the parameters of the simple solution -- in this case the amplitude, frequency, and phase -- will vary slowly, and a scheme for calculating those variations is required. We shall see in chapter 5 that such a scheme is provided by the method of multiple scales.

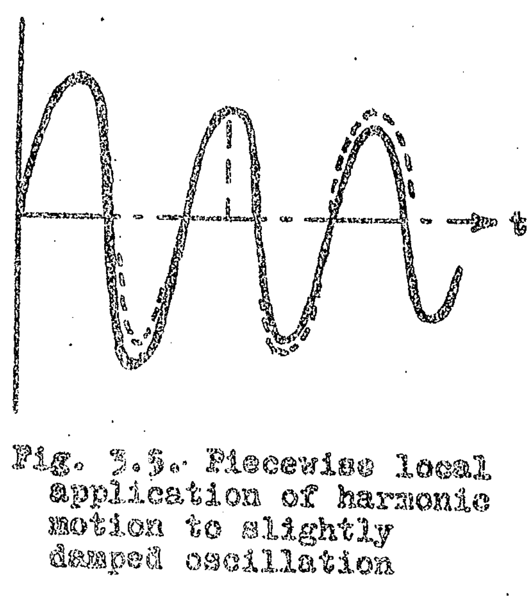


Fig. 3.5. Piecewise local application of harmonic motion to slightly damped oscillation

interior - channel type prob
exterior - airfoil type

3.6. External regions: slender-body theory

Slowly varying geometry is necessary if the approximation of slow variations is to be exploited, but it is not sufficient. In our previous examples the solution as well as the geometry varies slowly along the axis because it is confined to an interior region. For the region exterior to a slowly varying shape the solution will be slowly varying in some cases and to a certain order of approximation, and not in others; and the distinction is not obvious.

We consider the fluid-mechanical problem of flow past a thin or slender body. We use these adjectives in the technical sense that a razor blade is thin, and a needle (even a slightly bent or flattened one) is slender. It turns out that flow over a thin shape, such as an airplane wing, is never slowly varying. On the other hand, flow over a slender body is slowly varying, at least in the first approximation.

For simplicity we consider incompressible flow past a slender body of revolution (Fig. 3.6). Taking its length as unity we describe it in cylindrical polar coordinates by $r = \varepsilon f(x)$, so that ε is the dimensionless thickness ratio, which we will assume to be small. Since this parameter is the ratio of two characteristic lengths, we will be on the lookout for nonuniformity.

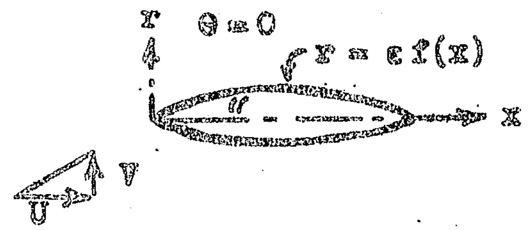


Fig. 3.6. Flow past slender body of revolution

If we work with the velocity potential ϕ the problem is linear, the nonlinearities of the inertia terms in the momentum equations having been isolated in the subsequent calculation of the pressure from the Bernoulli equation. Then the axial and transverse components of the oncoming stream can be considered separately. We start with the crossflow because it is simpler.

We take the crossflow component of velocity V as reference speed. Then the full problem for the velocity potential is

$$\nabla^2\phi = \phi_{xx} + \phi_{rr} + \frac{\phi_r}{r} + \frac{\phi_{\theta\theta}}{r^2} = 0,$$

$$\phi_r = \varepsilon f'(x)\cdot\phi_x \quad \text{at} \quad r = \varepsilon f(x) \quad \text{(TANGENCY)} \tag{3.17}$$

$$\phi \to r\cos\theta \quad \text{as} \quad r \to \infty . \quad \text{(FREE STREAM)}$$

(Here the tangency condition is derived from the fact that at any surface $F(x,y,z) = 0$ the normal component of velocity is grad ϕ · grad F.)

In this problem the small parameter ε appears implicitly in the tangency condition as well as explicitly. It is made to appear only explicitly, and shifted to the differential equation, by again stretching the radial and longitudinal coordinates differentially, so as to "square up" the geometry (Fig. 3.7). In contrast to our channel of

Fig. 3.2, it is the transverse scale that we have chosen to make small, rather than the longitudinal scale large, with our choice of units. We consequently magnify the radial dimensions (as in boundary-layer theory) by introducing $R = r/\varepsilon$, where R is of order unity.

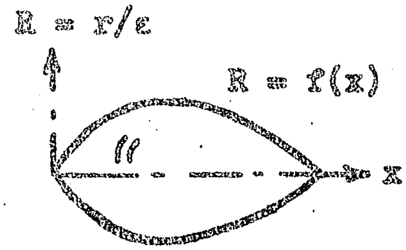


Fig. 3.7. Slender body in magnified radial coordinates

A new feature arises in that the dependent variable also needs magnification. The free-stream condition is the only nonhomogeneous element in the problem, and accordingly sets the scale; and in terms of R it shows that ϕ is also small, of order ε. We therefore set

$$\phi(x,r,\theta) = \varepsilon\Phi(x,R,\theta), \quad R = r/\varepsilon. \tag{3.18}$$

This magnification of ϕ is only a convenience here, because the problem is linear so that it cancels out; but it would be essential in a nonlinear one. Thus our problem (3.17) is transformed to

$$\Phi_{RR} + \frac{\Phi_R}{R} + \frac{\Phi_{\theta\theta}}{R^2} + \varepsilon^2\Phi_{xx} = 0, \quad \Phi_R = \varepsilon^2\Phi_x f'(x) \text{ at } R = f, \quad \Phi \to R\cos\theta \text{ as } R \to \infty. \tag{3.19}$$

Setting $\varepsilon = 0$ now yields the problem for the first approximation. The remarkable simplification that results is the disappearance of the axial coordinate x from the differential equation. It therefore describes two-dimensional flow in the cross-plane, past a circle of radius $f(x)$. It is clear physically (Fig. 3.8) that this simplification corresponds to the fact that fluid flows more easily around than along a slender body. To a first approximation the axial velocity can be neglected.

The solution of this simple problem for the first approximation is just

$$\Phi_1 = \left[R + \frac{f^2(x)}{R}\right]\cos\theta \tag{3.20a}$$

or

$$\phi_1 = \left[r + \varepsilon^2\frac{f^2(x)}{r}\right]\cos\theta. \tag{3.20b}$$

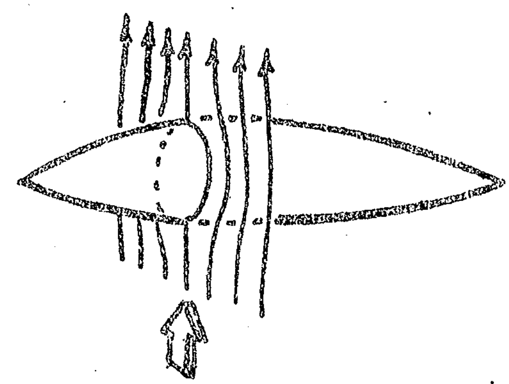


Fig. 3.8. Crossflow past slender body

Of course x has not altogether disappeared from the problem, so it appears here parametrically. Thus we again reap the advantage of working with such an integrated quantity as the stream function or velocity potential, that although the axial component was neglected we find it in the resulting solution. As a consequence the pressure varies along the body, the Bernoulli equation

$$p + \tfrac{1}{2}\rho\,|\text{grad}\,\phi|^2 = \text{const.}, \tag{3.21}$$

giving on the surface

$$p = p_\infty + \tfrac{1}{4}\rho\left[1 - 4\sin^2\theta - 4\varepsilon^2 f'^2(x)\cos^2\theta\right] \qquad (3.21)$$

This pressure distribution produces no net force at any section. However, when a uniform axial velocity is added, the cross-product terms in Bernoulli's equation reveal a lifting force at each section proportional to the local rate of increase of cross-sectional area (Fig. 3.9). This is the airship theory of Munk (1924). It was extended to slender lifting wings by Jones (1946). He noticed that it applies at all subsonic and even supersonic speeds, because in linearized theory the effect of compressibility is merely to add in the differential equation (3.17) a factor $(1-M^2)$ to the term ϕ_{xx} that is neglected.

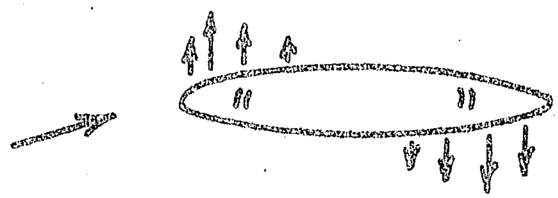


Fig. 3.9. Lift distribution on slender airship

3.7. Symptoms of nonuniformity in slender-body theory

We saw (sec. 3.3) that for the interior of a slowly varying region a differential stretching of coordinates serves to render the perturbation problem regular to any order. For an exterior region, on the other hand, symptoms of nonuniformity will sooner or later appear. The reason is that although the solution is dominated by a layer of slow variation near the boundary, it includes a fringe that extends far from the boundary. (In viscous flow at high Reynolds number, this "fringe" is more important than the boundary layer, so that we cannot begin with the approximation of slow variations.)

For our crossflow problem, this difficulty reveals itself at the next stage. If we assume that (3.19) has a regular-perturbation expansion in powers of ε^2, we find for the second approximation the problem

$$\begin{aligned} \Phi_{2_{RR}} + \frac{\Phi_{2_R}}{R} + \frac{\Phi_{2_{\theta\theta}}}{R^2} &= -\left[f^2(x)\right]'' \frac{\cos\theta}{R}, \\ \Phi_{2_R} &= 2f'^2(x)\cos\theta \quad \text{at } R = f(x), \\ \Phi_2 &\to 0 \quad \text{as } R \to \infty. \end{aligned} \qquad (3.22)$$

A particular integral of this differential equation is

$$\Phi_2 = -R\log R \cdot \tfrac{1}{4}\left[f^2(x)\right]'' \cos\theta, \qquad (3.23)$$

but no complementary solution of the homogeneous equation will cancel the consequent logarithmic singularity in the velocity at infinity.

This is just the difficulty discussed in section 2.9, being the counterpart of the Whitehead paradox because it arises in the second approximation. Our inability to satisfy the distant boundary condition is again a symptom of nonuniformity; and the reason is clear:

at a distance the body presents itself to the flow not as a cylinder but as a finite line of singularities (Fig. 3.10). Consequently an approximation based on the first point of view is not appropriate to a distant boundary condition. These two complementary views, based on different scalings of the same problem, are the basis of the method of matched asymptotic expansions, which we discuss in the next chapter.

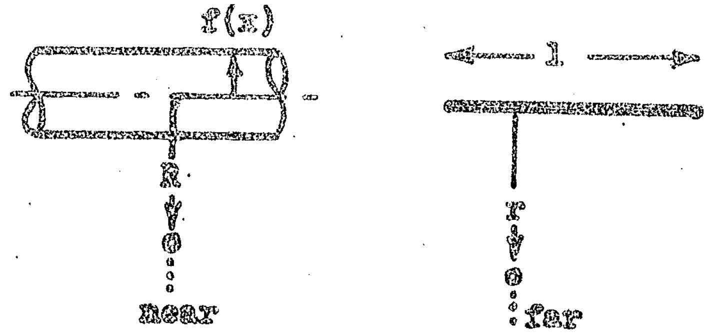


Fig. 3.10. Two complementary views of crossflow past a slender body

The nonuniformity manifests itself already in the first approximation for the axial component of flow (so that this case is the counterpart of the Stokes paradox). Because the problem is axisymmetric, it becomes

$$\nabla^2\phi = \phi_{xx} + \phi_{rr} + \frac{\phi_r}{r} = 0, \quad \phi_r = \varepsilon f'(x)\,\phi_x \ \text{ at } \ r = \varepsilon f(x) \qquad \phi \to x \ \text{ as } \ r \to \infty \tag{3.24}$$

As the body disappears it leaves the uniform stream. We therefore perturb that trivial solution. Magnifying the radial coordinate as before with $R = r/\varepsilon$, we see that the perturbation in ϕ is now of order ε^2. Hence we set

$$\phi = x + \varepsilon^2 \Phi(x,R), \tag{3.25}$$

which transforms the full problem to

$$\Phi_{RR} + \frac{\Phi_R}{R} + \varepsilon^2 \Phi_{xx} = 0, \quad \Phi_R = f'(x)\left[1 + \varepsilon^2 \Phi_x\right] \ \text{ at } \ R = f(x) \qquad \Phi \to 0 \ \text{ as } \ R \to \infty \tag{3.26}$$

In the first approximation (setting $\varepsilon = 0$, or expanding Φ in powers of ε^2) we have again two-dimensional flow in the cross-plane, now axisymmetric and governed by the simple equation

$$\Phi_{1_{RR}} + \frac{\Phi_{1_R}}{R} = 0 \qquad \text{or} \qquad (R\Phi_{1_R}) = 0 . \tag{3.27}$$

As for the slowly varying channel (sec. 3.3) the problem has been reduced to successive quadratures. Integrating once and imposing the tangency condition gives

$$\Phi_{1_R} = \frac{f(x)\, f'(x)}{R} . \tag{3.28}$$

This has the simple interpretation (Fig. 3.11) that to the uniform stream is added a plane radial flow in each section that corresponds to the local rate of increase of the cross-sectional area $S(x) = \pi\varepsilon^2 f^2(x)$.

Integrating again yields

$$\Phi_1 = f(x)f'(x) \log R + g(x) \qquad (3.29)$$

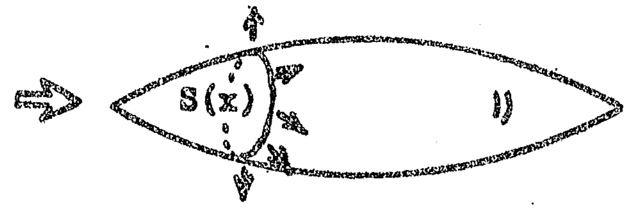


Fig. 3.11. Velocities produced by slender body of revolution in uniform stream

Unfortunately the function of integration g(x) cannot now be found. The distant boundary condition cannot be imposed, again because our approximation is not valid at a distance. Thus the first approximation serves only to predict the radial velocity; it fails to predict the streamwise velocity increment, and hence the pressure on the body.

The remedy that was adopted by aerodynamicists in the 1940's was to introduce the slender-body approximation only after having satisfied the distant boundary condition. This was accomplished in England by applying the Laplace transformation, and in America by representing the body by a distribution of sources and sinks along the axis, so that the distant boundary condition is automatically satisfied, and then approximating for small slope. The same results were later obtained using the method of matched asymptotic expansions.

References

ERDÉLYI, A. 1961 An expansion procedure for singular perturbations. Atti Accad. Sci. Torino 95, 651-672

FRAENKEL, L. E. 1963 Laminar flow in symmetrical channels with slightly curved walls. II. An asymptotic series for the stream function. Proc. Roy. Soc. A 272, 406-428

JONES, R. T. 1946 Properties of low-aspect-ratio pointed wings at speeds below and above the speed of sound. Nat. Adv. Comm. Aero. Rept. 835.

LUCAS, R. D. 1972 A perturbation solution for viscous incompressible flow in channels. Ph.D. dissertation, Stanford Univ.

MASSONET, C. 1962 Elasticity: two-dimensional problems. Chap. 37 of Handbook of Engineering Mechanics (W. Flügge, editor), McGraw-Hill, New York

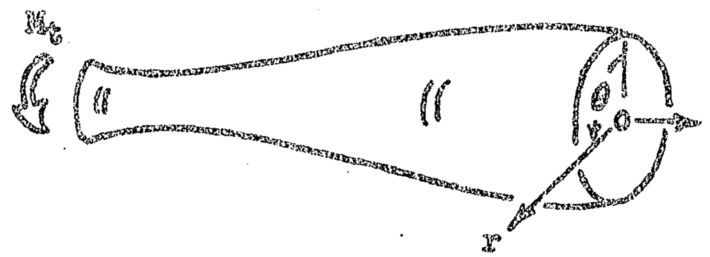
MORSE, P. M. & FESHBACH, H. 1953 Methods of Theoretical Physics. McGraw-Hill, New York

MUNK, M. M. 1924 The aerodynamic forces on airship hulls. Nat. Adv. Comm. Aero. Rept. 184

SMITH, A. M. O. 1956 Rapid laminar boundary-layer calculations by piecewise application of similar solutions. J. Aeronaut. Sci. 23, 901-912

EXERCISES

3.1. <u>Torsion of circular shaft of varying radius.</u> The stresses in a circular shaft of varying radius twisted by couples at the ends are (except for the details near the ends) derivable from Michell's stress function ϕ according to



$$\tau_{r\theta} = -\frac{\phi_z}{r^2}, \quad \tau_{\theta z} = \frac{\phi_r}{r^2}.$$

Then the problem for ϕ is

$$\phi_{rr} - 3\frac{\phi_r}{r} + \phi_{zz} = 0, \qquad \phi = 0 \text{ (say) at } r = 0$$
$$\phi = M_t/2\pi = 1 \text{ (say) on the surface}$$

where M_t is the torque. (a) Find the simple solution for a shaft of constant radius a. (b) Perturb to find the solution to $O(\varepsilon^2)$ for the hyperboloidal shaft described by $r^2 = 1 + \varepsilon^2 z^2$. (c) Find the second approximation for a slowly varying shaft of general form by introducing the contracted abscissa $Z = \varepsilon z$. (d) Compare the results of (c) with those of (b) for the hyperboloidal shaft, and with Föppl's exact solution (Timoshenko & Goodier 1951, p. 309) for a conical shaft of semi-vertex angle α:

$$\phi = \frac{2 - 3z(r^2+z^2)^{-1/2} + z^3(r^2+z^2)^{-3/2}}{2 - 3\cos\alpha + \cos^3\alpha}$$

3.2. <u>Potential flow through axisymmetric duct of varying radius.</u> A problem in fluid mechanics similar to the torsion problem above is that of finding the Stokes stream function ψ for incompressible potential flow through an axisymmetric duct. The streamwise and radial velocity components are given by

$$v_z = \frac{\psi_r}{r}, \quad v_r = -\frac{\psi_z}{r},$$

and the problem for ψ is

$$\psi_{rr} - \frac{\psi_r}{r} + \psi_{zz} = 0, \qquad \psi = 0 \text{ (say) at } r = 0$$
$$\psi = Q/2\pi = 1 \text{ (say) on the wall}$$

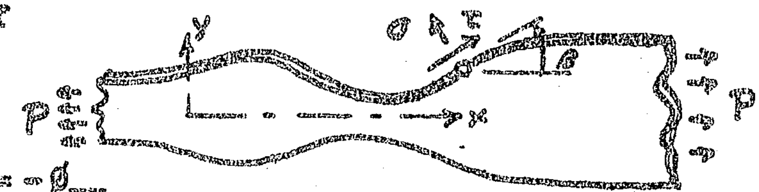
where Q is the volumetric flux through the duct. Carry out the steps corresponding to those in the preceding exercise. That is, find the simple solution for a tube; perturb it to find an approximation for the hyperboloidal duct; find the second approximation for a slowly varying duct using the contracted abscissa; and compare the last result with the approximation for the hyperboloidal duct, and with the exact solution for a conical duct of semi-vertex angle α (which is simply that for a point source.)

3.3. Generalized axisymmetric potential theory for varying contour. Solve the preceding two exercises concurrently by working with the equation of generalized axisymmetric potential theory ("GASPT"):

$$\phi_{rr} - M\frac{\phi_r}{r} + \phi_{zz} = 0$$

where M is here an integer. (Note that cases of physical interest, in addition to the present $M = 3$ for torsion and $M = 1$ for the Stokes stream function, are $M = 0$ for the stream function or velocity potential in plane flow, $M = -1$ for the velocity potential in axisymmetric flow, and others.)

3.4. Stresses in strip under tension. A long symmetrical strip of thin elastic material is held under tension at its distant ends. Because it is thin, the approximation of plane stress is accurate, and the stresses are given in terms of an Airy stress function ϕ by



$$\sigma_x = \phi_{yy}, \quad \sigma_y = \phi_{xx}, \quad \tau_{xy} = -\phi_{xy}$$

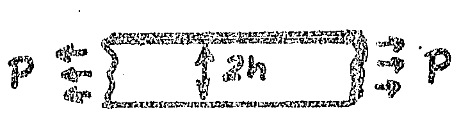
Here ϕ satisfies the biharmonic equation; in Cartesian coordinates

$$\phi_{xxxx} + 2\phi_{xxyy} + \phi_{yyyy} = 0 .$$

The conditions of no normal or tangential stress on the edges of the strip, found by taking components, are, in the notation of the sketch

$$\left.\begin{aligned} \sigma &= \sigma_x \sin^2\beta + \sigma_y \cos^2\beta - 2\tau_{xy}\sin\beta\cos\beta = 0 \\ \tau &= \tau_{xy}(\cos^2\beta - \sin^2\beta) - (\sigma_x - \sigma_y)\sin\beta\cos\beta = 0 \end{aligned}\right\} \text{ at each edge}$$

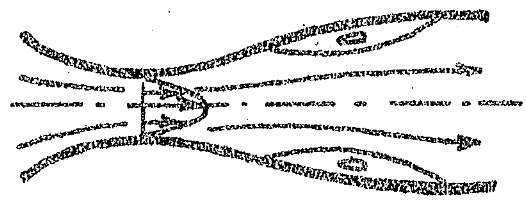
Suppose that the slope $\tan\beta$ of the edge is everywhere small, of order ε. Exploit this to find for any smooth shape a first approximation for ϕ that is correct to within terms of relative order ε^2. You may find it helpful to check with the results indicated below for a uniform strip and a wedge:



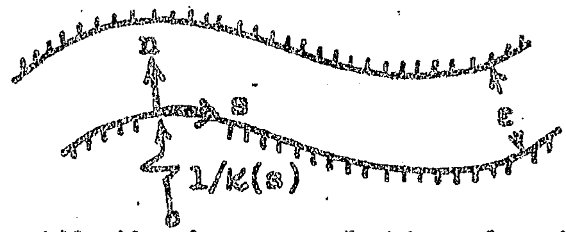
$$\phi = \frac{p}{4h} y^2$$

$$\phi = \frac{p}{2\alpha + \sin 2\alpha} r\,\theta \sin\theta$$

3.5. Plane laminar flow in slowly varying channel. Consider viscous flow through a slowly varying symmetrical channel of arbitrary form. Using the vorticity equation (2.40) for the stream function ψ, show that the first approximation yields a parabolic velocity profile at each station, and skin friction varying inversely as the square of the channel width. Calculate the second approximation. Specialize to the linearly growing channel and find, as a function of Reynolds number, the wall slope for which the skin friction vanishes.



3.6. Plane potential flow in narrow curved duct. A smooth curved duct of constant breadth ε is described by giving the curvature $\kappa(s)$ of its lower wall as a function of the curvilinear distance s along that wall. Set up a systematic scheme for calculating approximately the plane irrotational incompressible flow through the duct, and calculate the second or third approximation for small ε. Check by comparing with the known solution for the annulus between two concentric circles. Suggestions: it is advantageous to use the orthogonal "boundary-layer" coordinates consisting of the distance s along the lower wall and the distance n normal to it. If u and v are the corresponding velocity components, the governing equations of continuity and irrotationality are (Goldstein 1938, Modern Developments in Fluid Mechanics, p. 119)



$$\frac{\partial u}{\partial s} + \frac{\partial}{\partial n}\left[(1+\kappa n)v\right] = 0\,, \qquad \frac{1}{1+\kappa n}\frac{\partial v}{\partial s} - \frac{\partial u}{\partial n} - \frac{\kappa}{1+\kappa n}\,u = 0\,.$$

It is convenient to satisfy continuity by introducing a stream function ψ, and to normalize it to zero on the lower wall and unity on the upper wall.

Chapter 4

THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

Matching, merging;
patching, purging ...

— Sydney Goldstein

4.1. Historical introduction

The most useful procedure for dealing with singular perturbations in mechanics is the generalization of Prandtl's boundary-layer technique that has come to be known as the method of matched asymptotic expansions. Although Prandtl made decisive use of it, he did not claim to have originated the idea, and in fact attributed priority to an 1881 paper of L. Lorenz (Prandtl 1952, p. 414).

Actually, it appears that matched expansions were first used by Laplace (1805). In the appendix to the last volume of his great treatise on Celestial Mechanics,* Laplace calculated the approximate shape of the meniscus in (among many shapes) a large circular cylinder (Fig. 4.1). The equilibrium between the forces of gravity and surface tension is described by a second-order ordinary differential equation, which is so nonlinear as to be inintractable. However, the slope of the meniscus is small almost everywhere in a large vessel, so that the equation can be linearized.

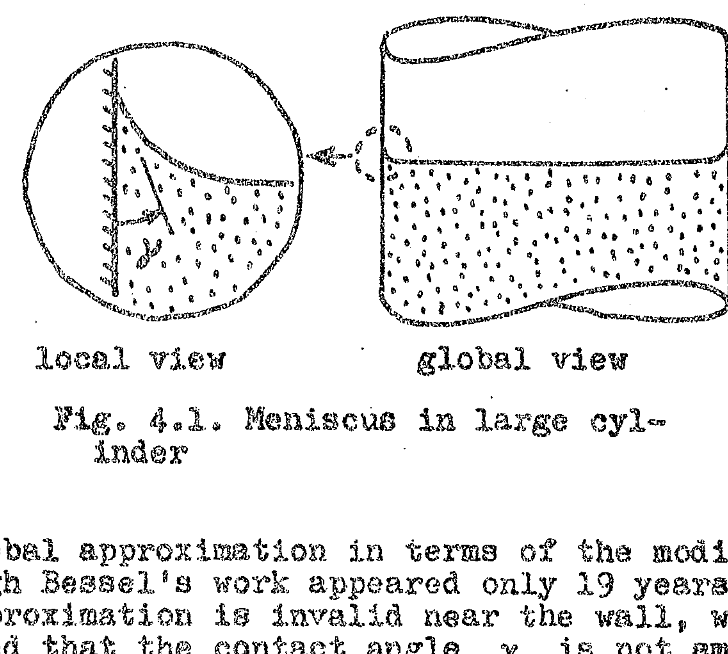


Fig. 4.1. Meniscus in large cylinder

Thus Laplace found a global approximation in terms of the modified Bessel function I_0 (though Bessel's work appeared only 19 years later). Of course this approximation is invalid near the wall, where the slope is large (provided that the contact angle γ is not small); bu that "boundary layer" is so narrow that in it the curvature of the

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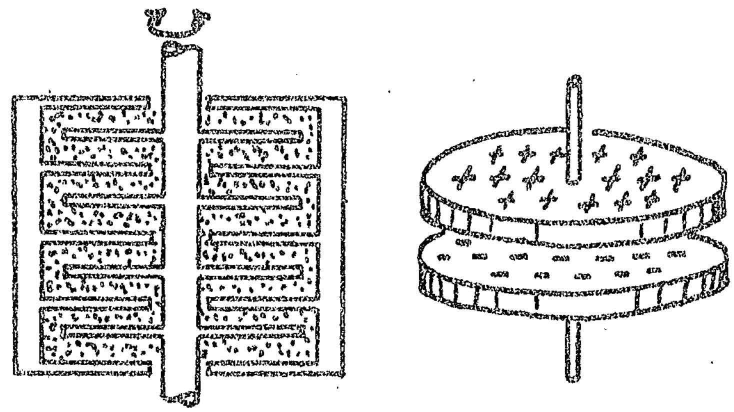
* In this case the original French has gained in translation; in his English version the famous American astronomer and navigator Nathaniel Bowditch has corrected, annotated, and amplified Laplace's text in a running commentary at the bottom of each page.

wall can be neglected. Despite its nonlinearity, this local problem has a solution in terms of trigonometric functions. Laplace joined his global and local approximations in an intuitive but quite correct fashion.

More than a century later the British physicist Rayleigh (1915) independently repeated this calculation. He became aware of Laplace's work just as his paper was in proof, and hastily extended his analysis to the next approximation. He thus became the founder of second-order boundary-layer theory.

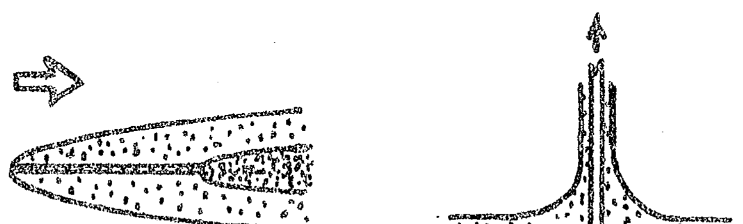
Meanwhile, other nineteenth-century giants of natural philosophy had used the idea of matching in the same natural way. In 1860 Maxwell calculated the slow laminar flow in a circular-disk viscometer (Fig. 4.2a), an assemblage of closely spaced alternating fixed and rotating disks. The motion is one-dimensional except near the rim of each disk, where it is approximately plane. The local problem at the rim requires solving the Laplace equation in two dimensions, which Maxwell did with the aid of his friend William Thomson. Similarly in 1877 Kirchoff calculated the electrostatic capacitance of two thin charged circular disks with small spacing (Fig. 4.2b).



a. Maxwell's viscometer   b. Kirchoff's capacitor

Fig. 4.2. The boundary-layer problems of Maxwell and Kirchoff

The twentieth century began with Prandtl's analysis of the viscous boundary layer -- an idea that incubated quietly in Göttingen for twenty years before spreading rapidly throughout the world. From the point of view of matched expansions one can discern other landmarks in the 1930's and 40's. For example, Goldstein (1930) treated the laminar wake behind a flat plate, matching the Prandtl-Blasius solution to a sub-layer that grows from the trailing edge (Fig. 4.3a). Again, Landau & Levich (1942) calculated the thickness of the liquid film deposited on a sheet that is drawn steadily out of a bath of fluid (Fig. 4.3b). They speak (writing in English) of "chaining" the upper slowly varying



a. wake of flat plate   b. coating of sheet

Fig. 4.3. The boundary-layer problems of Goldstein and of Landau & Levich

solution to the local approximation for a static meniscus below.

These pioneers "chained" or joined their complementary approximations in an intuitive fashion that, bolstered by firm physical insight, sufficed for 150 years. Consequently they might, as suggested by the quotation at the head of this chapter, be humorously contemptuous of more formal matching schemes. However, the insight of even a Prandtl will fail at some stage; and then a systematic procedure is welcome.

Deeper insight into the matching was provided in the 1950's by the work of Kaplun and Lagerstrom (Lagerstrom & Cole 1955, Kaplun 1957, Kaplun & Lagerstrom 1957). They recognized that viscous flow at low as well as high Reynolds number is a singular perturbation, and in that context studied the basis of the matching procedure. Since then, the method of matched asymptotic expansions has been applied to an ever widening range of problems in fluid motion, engineering mechanics, and applied mathematics, and has become a standard technique of asymptotic analysis.

4.2. Deflection of slightly rigid membrane

It is helpful to introduce the method of matched asymptotic expansions in a problem where the source of the nonuniformity and the nature of the boundary layer are physically obvious. Prandtl's boundary layer is unsuitable, because it can only be observed indirectly, and always involves numerical integration. Laplace's meniscus is ideal for qualitative explanation, because any intelligent child comprehends the band of steep slope on the rim of a glass of water; but the mathematical analysis is still complicated.

We choose to begin with a problem in solid mechanics that is moreover just a slight modification of the one we used to introduce regular perturbations in chapter 2. Suppose that the pressurized membrane of section 2.2, instead of being perfectly flexible, has a slight amount of bending rigidity. We may think of a thin metal diaphragm stretched across an orifice under uniform tension σ, and clamped on its periphery (Fig. 4.4). (We must abandon the analogs in Poiseuille flow and in torsion, for lack of a counterpart to rigidity.)

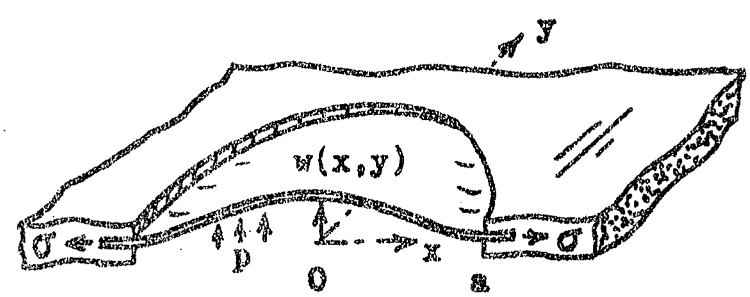


Fig. 4.4. Deflection due to pressure difference across slightly rigid membrane under tension

Small deflections w of the diaphragm are governed by the linear partial-differential equation

$$K\nabla^4 w - \sigma\nabla^2 w = p \qquad (4.1)$$

As in Eq. (2.3), the second term describes the effect of tension (which would become nonlinear, as in Exercise 2.1, if the slopes were not small), and the third term the effect of transverse loading (which we have taken uniform for simplicity). The new first term describes the effect of bending rigidity according to the usual approximations in the theory of thin plates (Way 1962). Here the constant

$$K = \frac{E\,t^3}{12(1-\nu^2)} \tag{4.2}$$

is the flexural rigidity or bending stiffness, where t is the plate thickness and ν Poisson's ratio.

In order to speak of slight rigidity we introduce dimensionless variables, referring the lateral coordinates x, y to a characteristic dimension a of the orifice, and the deflection w to pa^2/σ, which is the maximum deflection in the absence of rigidity, aside from a factor of order unity ($\frac{1}{4}$ for the circle). This reduces the differential equation to

$$\varepsilon \nabla^4 w - \nabla^2 w = 1, \qquad \varepsilon = \frac{K}{\sigma a^2} = \frac{E\,t^3}{12\,\sigma(1-\nu^2)a^2} \tag{4.3}$$

This is our previous equation (2.1) for the membrane, augmented by a stiffness term proportional to ε. This parameter, which measures the relative importance of rigidity and tension, we propose to take small. For clamped edges the boundary conditions are that the deflection and its normal derivative vanish:

$$w = \frac{dw}{dn} = 0 \quad \text{on the contour} \tag{4.4}$$

This problem displays both the warnings of nonuniformity discussed in section 2.7: the small parameter ε multiplies the highest derivatives, and it is the ratio of two characteristic lengths. We would therefore anticipate a singular perturbation even if it were not obvious physically that the membrane approximation fails in a narrow neighborhood of the edge where the rigidity, though small, brings the slope quickly to zero.

Though simpler because it is linear, this problem is analogous to that in Prandtl's boundary-layer theory. The boundary conditions (4.4) are identical if w is identified with the stream function ψ; and the differential equation (4.3) is similar to the vorticity equation (2.40), with ε corresponding to the viscosity ν (or to the inverse Reynolds number in dimensionless terms). The main qualitative difference is that the boundary layer will be seen to be thinner here because Eq. (4.3) drops from fourth to second order when ε disappears, whereas the viscous equation drops only to third order.

4.3. First global approximation for cylindrical deflection

We exhibit the essentials of the method of matched asymptotic expansions, unencumbered by mathematical details, by considering the cylindrical deflection of a diaphragm stretched across a slot (Fig. 4.5). (As in section 2.6 this might be the quasi-cylindrical approximation for a long slowly varying orifice.) Then our problem (Eqs. 4.3, 4.4) reduces to

$$\varepsilon \frac{d^4w}{dx^4} - \frac{d^2w}{dx^2} = 1,$$

$$w = \frac{dw}{dx} = 0 \quad \text{at} \quad x = \pm 1. \tag{4.5}$$

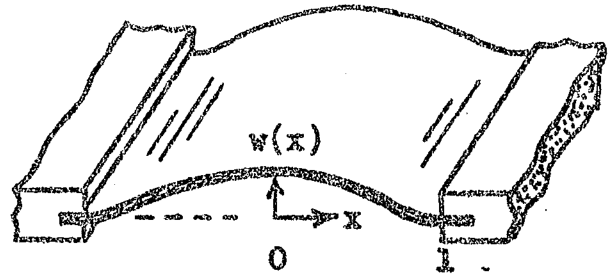


Fig. 4.5. Cylindrical deflection of slightly rigid membrane

The solution is obviously symmetric in x, so that the boundary conditions at the left edge, say, can be replaced with the requirement of symmetry.

This mathematical problem has another slightly different physical interpretation. It describes also the deflection of a beam under tension T and uniform transverse loading P, where now $\varepsilon = EI/Ta^2$, I being the moment of inertia (Fig. 4.6). For a beam of rectangular cross-section, taking the width as unity makes $T = \sigma$ and $P = p$, and then $\varepsilon = Et^3/12\sigma a^2$, the absence of the factor $(1-\nu^2)$ that appears in (4.3) representing the difference between plane stress and plane strain. Cole (1968, pp. 69-76) analyzes this problem in the more general situation when the loading P varies with x; and the reader may want to compare his discussion with the present one. In that case our membrane becomes a string.

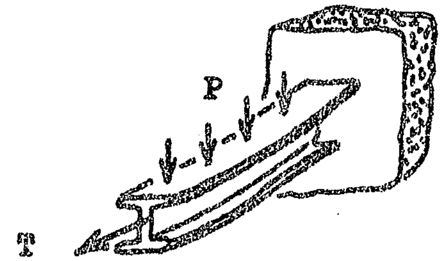


Fig. 4.6. Deflection of uniformly loaded beam in tension

In the membrane approximation we neglect the rigidity, which corresponds to setting $\varepsilon = 0$ in (4.5). Then the general symmetric solution of the differential equation is $w = C - \frac{1}{2}x^2$. However, we cannot impose both boundary conditions at $x = 1$. This is a characteristic dilemma whenever the order of the equation is lowered. There are various ways of answering the question of which condition, if either, is to be retained. That question scarcely arises here, however, because it is impossible to impose the condition of zero slope; and in any case it is clear physically that this condition disappears with the rigidity. Hence the usual membrane solution (2.8) represents a first approximation everywhere except in the immediate vicinity of the edge.

Suppose we try to improve upon this first approximation, for example by substituting

$$w = \tfrac{1}{2}(1 - x^2) + \varepsilon\, w_2(x) + \ldots \tag{4.6}$$

This gives $d^2w_2/dx^2 = 0$, of which the most general symmetric solution is simply a constant. We shall see later that this result is essentially correct. In fact, it is clear physically (Fig. 4.7) that enforcing the rigidity condition of zero slope at the edge reduces the deflection everywhere almost uniformly. However, our solution has reached a dead end, for we have no way of calculating the constant. (The same difficulty arises with the Navier-Stokes equations at high Reynolds number:

dropping the no-slip condition yields as a first approximation an inviscid flow that is valid almost everywhere; but the second approximation cannot be determined.)

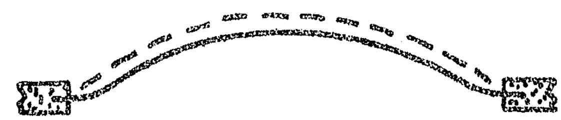


Fig. 4.7. Membrane approximation compared with exact deflection

4.4. First local approximation at edge

There is no singularity here, as in some problems, to indicate the source of difficulty. However, it is abundantly clear physically, as well as from our neglect of boundary conditions, that the difficulty originates at the edges. We therefore follow the general boundary-layer procedure of shifting the origin to the region of nonuniformity, and then magnifying the scale. We consider the left-hand edge, where $x+1 = 0$ (the right-hand edge being treated in exactly the same way by symmetry).

Finding the proper scaling is a crucial step in the method. Here it is simplest to realize that the rigidity, which was previously neglected, must balance the tension near the edge. This corresponds to introducing the magnified abscissa

$$X = \frac{x+1}{\varepsilon^{\frac{1}{2}}} \tag{4.7}$$

so that the problem (4.5) becomes

$$\frac{d^4w}{dX^4} - \frac{d^2w}{dX^2} = \varepsilon, \qquad w = \frac{dw}{dX} = 0 \quad \text{at} \quad X = 0,\ 2/\varepsilon^{\frac{1}{2}} \tag{4.8}$$

The deflection w ought to be correspondingly magnified, for it is obviously smaller near the edge than elsewhere. In a nonlinear problem it is essential to stretch the dependent as well as the independent variables. Here, however, it is not essential because any stretching of w cancels out of the leading terms as a result of linearity.

When the proper scaling is not evident, it is helpful to consider all possible power-law stretchings and single out those that yield nontrivial simplified equations. Thus we would try setting

$$w = \varepsilon^a W(X), \qquad X = \frac{x+1}{\varepsilon^b} \tag{4.9}$$

so that the differential equation (4.5) is transformed to

$$\varepsilon^{1+a-4b}\frac{d^4W}{dX^4} - \varepsilon^{a-2b}\frac{d^2W}{dX^2} = 1 \tag{4.10}$$

Fig. 4.8 shows how the a-b plane is divided into three sectors, within each of which a single one of the three terms in the equation dominates the other two for small ε. Along the dividing rays, pairs of terms balance each other. And at their common intersection all three terms survive.

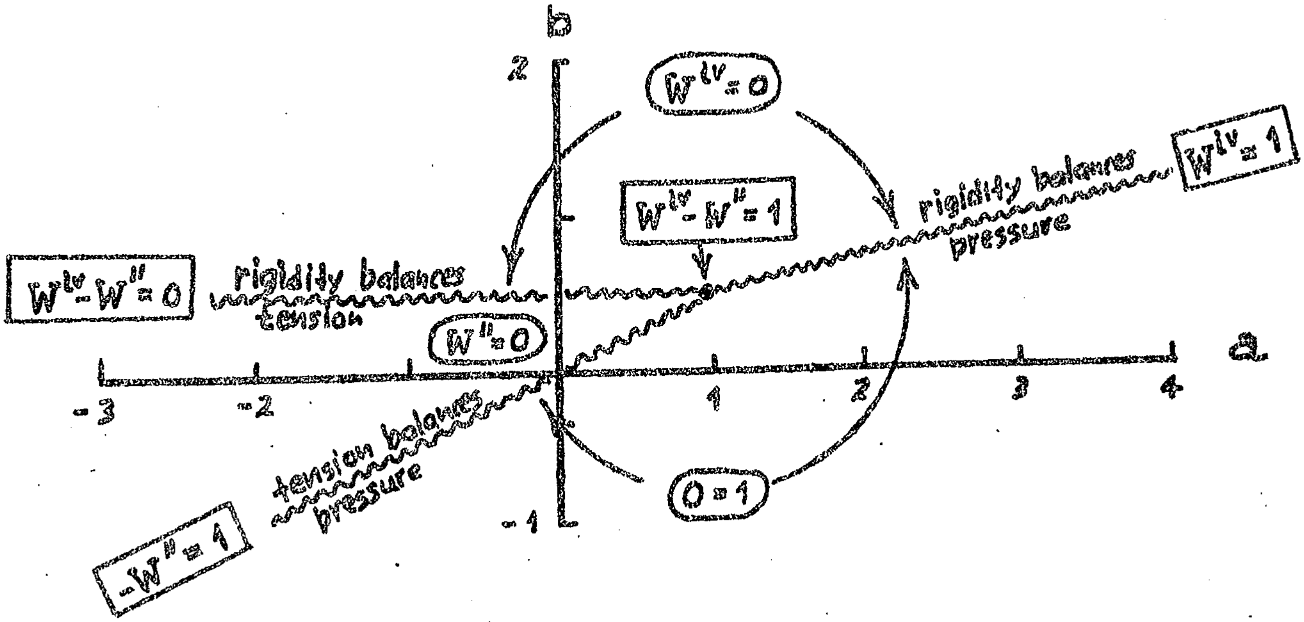


Fig. 4.8. Possible first-order equations for power-law stretchings of x+l and w.

We reject the last possibility since approximation was resorted to because the full equation was supposed to be too difficult to solve. On the other hand, the first case of a single surviving term is too degenerate to be useful. Here it yields the absurdity $0 = 1$ when the pressure is dominant, but always presents the contradiction that a single term is so much larger than the others that it is equal to zero. We are consequently interested in the intermediate situation where pairs of terms balance. **We have already dealt with the balance between tension and pressure in our global approximation. Both the remaining pairs include rigidity.**

If it were not clear physically that rigidity is balanced by tension, both possibilities would have to be considered. If it were balanced instead by pressure, the first local approximation would be a quartic or, with the boundary conditions imposed at $X = 0$, $W_1 = AX^2 + BX^3 + X^4/24$. However, this will not match with the global solution, whereas we shall see that the other choice does.

For $\epsilon = 0$ our equation (4.8) has the general solution

$$w = Ae^{X} + Be^{-X} + CX + D \tag{4.11}$$

We can reject the term in e^X because an exponentially growing term will not match (though if uncertain we would keep it). Then imposing the boundary conditions at the edge in question leaves

$$w = C(X - 1 + e^{-X}) \tag{4.12}$$

The remaining constant is not, of course, to be found by imposing the boundary conditions at the other edge, which lies far outside the range of validity of this local approximation. Instead, it is to be found by matching with the global approximation.

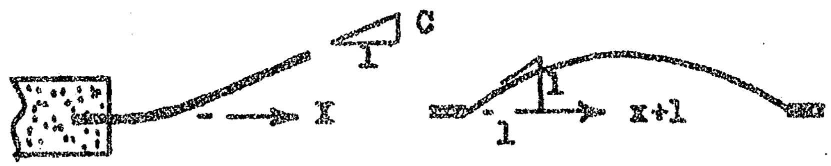
4.5. Matching of global and local approximations

We anticipate that our global and local approximations have a common region of validity — an overlap domain where both apply. If so, we can determine the remaining constant of integration in the local approximation (4.12) by matching there.

Matching is the crucial step in the method of matched asymptotic expansions. Although in some ways intuitively appealing and physically obvious, it is at the same time a subtle and almost magical procedure. We therefore carry out the matching in three different ways, each of which has its proponents — and its advantages and disadvantages — though of course the result is the same in each case.

Intuitive matching. We first emulate the giants of the nineteenth and the first half of the twentieth century, for whom the matching was a natural intuitive process. Our guiding principle is that far away from the edge the local solution should agree with the global solution close to the edge.

For large X the exponential in (4.12) is negligible, so the local approximation grows linearly at its outer edge (Fig. 4.9a), with slope given by



(a) local solution (b) global solution

Fig. 4.9. Matching of slope at edge

$$w \approx \bar{C}X = \frac{\bar{C}}{\varepsilon^{\frac{1}{2}}}(x+1). \qquad (4.13)$$

Conversely, near the edge where $(x+1)$ is small the first global approximation (4.6) also becomes linear (Fib. 4.9b):

$$w \approx (x+1) \qquad (4.14)$$

These match with

$$\bar{C} = \varepsilon^{\frac{1}{2}}. \qquad (4.15)$$

Of course this shows, as anticipated, that w should have been magnified to the same extent as the abscissa in the local solution. It is also now clear that the quartic local solution that would represent a balance of rigidity and pressure cannot match. Neither can the positive exponential.

Matching by intermediate limits. The notion that we have just used of moving far away from the edge in the local solution while moving close to the edge in the global solution can be formalized by introducing intermediate limits associated with a family of intermediate variables. If we restrict attention to powers of ε, the relevant intermediate variables here are

$$\tilde{x} = \frac{x+1}{\varepsilon^{\alpha}}, \qquad 0 < \alpha < \tfrac{1}{2}. \qquad (4.16)$$

The limits on the exponent assure that as $\varepsilon \to 0$ a point lying at fixed x tends toward the edge, but more slowly than in the local solution. It therefore drifts to the outer fringes of the outer solution while approaching the edge in the inner one.

For simplicity we may use only the specific intermediate variable with $\alpha = \frac{1}{4}$, though in general the whole family is used. Then rewriting the local solution (4.12) in terms of the intermediate variable $\tilde{x} = \varepsilon^{\frac{1}{4}} X$ and expanding for small ε gives

$$w = C(\varepsilon^{-\frac{1}{4}}\tilde{x} - 1 + e^{-\tilde{x}/\varepsilon^{1/4}}) \sim \frac{C}{\varepsilon^{\frac{1}{4}}}\tilde{x}\left[1 + O(\varepsilon^{\frac{1}{4}})\right] \qquad (4.17)$$

Performing the same operation on the global approximation gives

$$w = \varepsilon^{\frac{1}{4}}\tilde{x}\left[1 - \tfrac{1}{2}\varepsilon^{\frac{1}{4}}\tilde{x}\right] \qquad (4.18)$$

The leading terms again match with $C = \varepsilon^{\frac{1}{2}}$.

<u>The asymptotic matching principle</u>. In an effort to systematize the matching procedure, the writer (Van Dyke 1964) proposed what he thought to be a variant of Kaplun's scheme of matching by intermediate limits, but which Fraenkel (1969) has shown to be a distinct procedure. It was given in the form*

The m-term local expansion of the n-term global expansion = the n-term global expansion of the m-term local expansion. (4.19)

This is supposed to hold for any m and n, but it is useful only for certain combinations -- often when m is equal to n, or one less.

Occasionally the counting involved in this rule may seem ambiguous; then an alternative version is useful. One may feel uncertain how to count terms, for example, in the following situations: if the asymptotic sequence has gaps (e.g., the series proceeds by integral powers of ε except that the coefficient of ε^3 vanishes identically); if the asymptotic sequences are essentially different for the global and local expansions (e.g., one proceeds by powers of ε,

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* Here we have replaced <u>inner</u> and <u>outer</u> by <u>local</u> and <u>global</u>. The older terms were derived from flows at high and low Reynolds number, where the local region lies inside the global one; but our problem of the deflection of a slightly rigid diaphragm has already provided an example where the positions are reversed. Disregarding the geometrical meaning, Kaplun insisted that the terms be assigned so that the outer solution is, to first order, independent of the inner one: in other words, the outer is the primary and the inner the secondary expansion. However, it does not seem worthwhile to preserve this distinction, which does not exist in every problem, and is somewhat artificial, as exemplified by the fact that the viscous boundary layer is then the inner solution in forced flow but the outer one in free convection. Instead, our terms <u>local</u> and <u>global</u> serve to emphasize once more the important role of scales in perturbation problems. They are, incidentally, close to the terms <u>proximal</u> and <u>distal</u> adopted by our French colleagues.

the other by half powers); if logarithmic terms arise (e.g., $\varepsilon^k \log \varepsilon$ appears as well as ε^k); or if eigensolutions intervene. In such cases, uncertainty is usually avoided by adopting the following unambiguous variant of the rule:

The local expansion to order Δ of the global expansion to order δ = the global expansion to order δ of the local expansion to order Δ. (4.20)

Here $\Delta(\varepsilon)$ and $\delta(\varepsilon)$ are any two gauge functions (not necessarily the same), which may or may not actually appear in the asymptotic sequences for the global and local expansions. Of course this reproduces the original form of the rule (4.19) whenever it is clear how to count terms. In all subsequent general discussion we will, for simplicity, suppose that the counting is unambiguous, so that the original form can be used.

We now apply this rule to (4.6) and (4.12), taking $m = n = 1$. We systematize the procedure by always using the following format:

<u>1-term global expansion:</u>	$\psi = \frac{1}{2}(1-x^2)$	(4.21a)
rewritten in local variables:	$= \varepsilon^{\frac{1}{2}} X(1-\varepsilon^{\frac{1}{2}} X)$	(4.21b)
expanded for small ε:	$= \varepsilon^{\frac{1}{2}} X - \varepsilon X^2$	(4.21c)
1-term local expansion:	$= \varepsilon^{\frac{1}{2}} X$	(4.21d)
<u>1-term local expansion:</u>	$\psi = C(X-1+e^{-X})$	(4.22a)
rewritten in global variables:	$= C\left[\frac{x+1}{\varepsilon^{\frac{1}{2}}} - 1 + e^{-(x+1)/\varepsilon^{\frac{1}{2}}}\right]$	
expanded for small ε:	$= C\left(\frac{x+1}{\varepsilon^{\frac{1}{2}}} - 1 + \text{exp}\right)$	(4.22c)
1-term global expansion:	$= C\,\frac{x+1}{\varepsilon^{\frac{1}{2}}}$	(4.22d)
rewritten in local variables for comparison:	$= CX$	(4.22e)

Here in (4.22c) "exp" stands for exponentially small terms, smaller than any power of ε. The last step is required simply to put the two results into a common notation. Equating them gives once more $C = \varepsilon^{\frac{1}{2}}$.

4.6. Formation of a composite first approximation

We have found a first approximation that is valid everywhere, but it exists in the two pieces shown in Fig. 4.9. This may be adequate in some cases, but often we would like a single uniformly valid approximation. If the local and global solutions cross — as they do in this example — it would be consistent with the approximation simply to switch from one to the other at their intersection. However, the resulting kink makes this an unsatisfactory procedure.

Instead, it is easy to unite the elements of the global and local expansions to form a single uniformly valid composite expansion. The idea is simply to combine the two approximations, and remove the part that they have in common (which was determined in the course of matching).

This combination is usually effected by addition and subtraction. Thus the additive composite is given by

$$\text{composite} = (\text{global}) + (\text{local}) - \begin{pmatrix} \text{local of global} = \\ \text{global of local} \end{pmatrix} \qquad (4.23)$$

Just as in matching, this procedure may be applied to any n terms of the global expansion and m terms of the local expansion (or through orders δ and Δ if the counting is ambiguous); but usually m is taken equal to n, or one less.

For our membrane problem, taking the 1-term elements from Eqs. (4.21a), (4.22b), and (4.22d), with $C = \varepsilon^{\frac{1}{2}}$, gives

$$\begin{aligned} w_{comp} &= \tfrac{1}{2}(1-x^2) + \varepsilon^{\frac{1}{2}}\left(\frac{1+x}{\varepsilon^{\frac{1}{2}}} - 1 + e^{-(x+1)/\varepsilon^{\frac{1}{2}}}\right) - (1+x) \\ &= \tfrac{1}{2}(1-x^2) - \varepsilon^{\frac{1}{2}}\left(1 - e^{-(1+x)/\varepsilon^{\frac{1}{2}}}\right) \qquad (4.24) \end{aligned}$$

Here the last term may be regarded as a correction boundary layer, representing the difference between the true solution and the global approximation near the edge, which is added to the latter to render it uniformly valid. This point of view is common in solid mechanics, whereas in fluid mechanics the full boundary layer is usually considered.

Of course this result has been rendered valid only at the left edge, not at the right one. We could take the attitude that the solution is obviously symmetric, and simply reflect it on the right; but that would leave a kink at the middle, although of magnitude exponentially small in ε. However it is more satisfying to repeat the process, forming a composite from (4.24), regarded as the global approximation, and the right-hand local approximation, which is just the left-hand one with x replaced by $-x$. This gives the composite valid everywhere

$$w_{comp} = \tfrac{1}{2}(1-x^2) - \varepsilon^{\frac{1}{2}}\left(1 - e^{-(1+x)/\varepsilon^{\frac{1}{2}}}\right) - \varepsilon^{\frac{1}{2}}\left(1 - e^{-(1-x)/\varepsilon^{\frac{1}{2}}}\right) \qquad (4.25)$$

In this form this is readily interpreted as the result of adding to the global approximation the correction boundary layers for the left and right edges.

Clearly the composite expansion can be nowhere more accurate than the global and local approximations from which it was formed. Thus it is not a mistake that our composite result (4.25) gives a deflection $w = (-\varepsilon^{\frac{1}{2}} + \exp)$ at either edge instead of zero, for that is the order of the error in the first approximation, which would be reduced in the second. Nevertheless, we might be tempted to remedy this, and slightly simplify the result, by replacing it with the equivalent

$$w_{comp} = \tfrac{1}{2}(1-x^2) - \varepsilon^{\frac{1}{2}}\left[1 - e^{-(1+x)/\varepsilon^{\frac{1}{2}}} - e^{-(1-x)/\varepsilon^{\frac{1}{2}}}\right] \qquad (4.26)$$

At the middle this gives $w = \frac{1}{2} - \varepsilon^{\frac{1}{2}}$, whereas (4.25) gave $\frac{1}{2} - 2\varepsilon^{\frac{1}{2}}$, and no significance can be attached to either term in $\varepsilon^{\frac{1}{2}}$ because it too is a second-order quantity, to be determined in the next approximation. However, the sign seems right according to the physical argument illustrated by Fig. 4.7.

Whereas the constituent global and local approximations are classical asymptotic expansions, of the form (1.24), the composite approximation is not. It has instead the form

$$f(x;\ \varepsilon) \sim \sum c_n(x;\ \varepsilon)\ \delta_n(\varepsilon) \qquad (4.27)$$

which is complicated by the fact that the perturbation quantity ε appears in the coefficients c_n as well as the gauge functions δ_n. Erdelyi (1961) has termed this a generalized asymptotic expansion. Part of the price paid for this generality is that, in contrast to a classical expansion, the generalized asymptotic expansion of a given function is not unique even when the asymptotic sequence $\delta_n(\varepsilon)$ is specified. However, any two alternatives are equivalent, differing only in higher-order terms.

Consequently there are other ways of forming a composite expansion, which yield results that are different but equivalent to a given order. In fact, J. Ellinwood (unpublished) has pointed out that any operation on two functions can be used that has a unique inverse and reasonable properties of monotonicity. Multiplication is such an operation; and in treating the nonuniformity at the leading edge in thin-airfoil and slender-body theory, the author (Van Dyke 1954) preferred the multiplicative composite, formed as

$$\text{composite} = \frac{(\text{global})\cdot(\text{local})}{(\text{local of global}) = (\text{global of local})} \qquad (4.28)$$

However there is always the danger of dividing by zero; and Schneider (1973) has pointed out that this occurs ahead of a round-nosed airfoil. Thus the additive composite (4.23) is to be recommended for general use.

4.7. Second global approximation

To illustrate how the method of matched asymptotic expansions is continued to higher approximations, we proceed to the next stage of the solution for the slightly rigid membrane. We must follow the usual matching order indicated in Fig. 4.10, which involves starting with the first global approximation and then alternating between the global and local expansions. (This order can be by-passed under certain circumstances, of which we shall see an example in the next section.) Thus we turn to the second term of the global expansion.

It is helpful to anticipate the next matching by carrying out the half that can be done now, because this usually serves to indicate the order of the next term to be calculated. Thus following our usual format [cf. Eqs. (4.22)] we have, for the left edge,

1-term local expansion:	$w = \varepsilon^{\frac{1}{2}}(X - 1 + e^{-X})$
rewritten in global variables:	$= 1 + x - \varepsilon^{\frac{1}{2}}\left[1 - e^{-(1+x)/\varepsilon^{\frac{1}{2}}}\right]$
expanded for small ε:	$= 1 + x - \varepsilon^{\frac{1}{2}} + \exp$
2-term global expansion:	$= (1 + x) - \varepsilon^{\frac{1}{2}}$ (4.29)

This suggests that the second term in the global expansion is of order $\varepsilon^{\frac{1}{2}}$, rather than ε [as it would be in the regular perturbation represented by Eq. (4.6)].

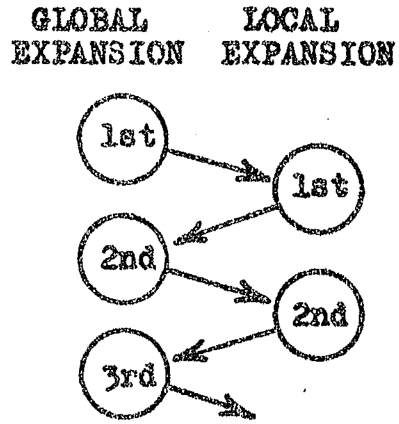


Fig. 4.10. Normal matching order

Substituting

$$w = \tfrac{1}{2}(1 - x^2) + \varepsilon^{\frac{1}{2}} w_2(x) + \ldots \qquad (4.30)$$

into the original differential equation (4.5) gives, as before,

$$\frac{d^2 w_2}{dx^2} = 0 \qquad (4.31)$$

Again the symmetric solution is just a constant, say c_2, which is found by completing the matching with the local solution:

2-term global expansion:	$w = \tfrac{1}{2}(1 - x^2) + c_2 \varepsilon^{\frac{1}{2}}$
rewritten in local variables:	$= \tfrac{1}{2}\varepsilon^{\frac{1}{2}} X(2 - \varepsilon^{\frac{1}{2}}X) + c_2 \varepsilon^{\frac{1}{2}}$
1-term local expansion:	$= \varepsilon^{\frac{1}{2}}(X + c_2)$
rewritten in global variables for comparison:	$= (1 + x) + c_2 \varepsilon^{\frac{1}{2}}$ (4.32)

Then equating (4.29) and (4.32) shows that $c_2 = -1$. This means that our alternative composite (4.26) happens to be correct to second order except near the edges.

In the next step we would, as indicated by Fig. 4.10, return to the local expansion and calculate the second approximation. Instead, we simply observe that any number of terms of either expansion can be extracted from the exact solution of (4.5), which is

$$w = \tfrac{1}{2}(1 - x^2) - \varepsilon^{\frac{1}{2}} \frac{\cosh(1/\varepsilon^{\frac{1}{2}}) - \cosh(x/\varepsilon^{\frac{1}{2}})}{\sinh(1/\varepsilon^{\frac{1}{2}})} \qquad (4.33)$$

4.8. The earth-moon-spaceship problem

The preceding example yielded a series belonging to the first category of singular-perturbation series described in section 1.6 -- those involving fractional powers of the perturbation quantity. We now consider a problem from another branch of mechanics that yields a series in the second category -- involving a mixture of powers and logarithms.

In celestial mechanics the restricted three-body problem deals with one body -- say a "spaceship" -- of such small mass that it does not affect the motion of the other two. If, moreover, one of the two heavy bodies is much the greater -- as a sun and planet, or planet and moon -- it dominates the motion. Thus a spaceship launched from our earth will be only slightly disturbed by the moon (whose mass is 1/80 as great), unless it sooner or later passes close to the moon (Fig. 4.11).

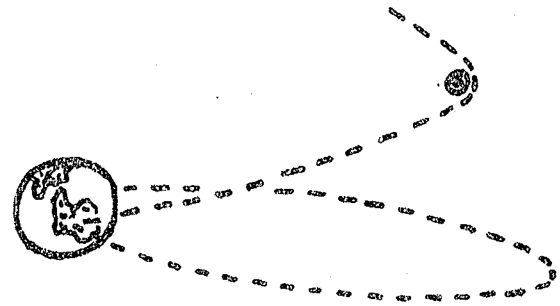


Fig. 4.11. Trajectories of spaceship launched from earth

Laplace associated with the moon a "sphere of influence" at which orbits about the earth and about the moon were to be patched together. Lagerstrom and Kevorkian treated this perturbation problem more systematically using the method of matched asymptotic expansions. We exhibit the essentials with their simplest model of one-dimensional motion for a non-rotating system (Lagerstrom & Kevorkian 1963).

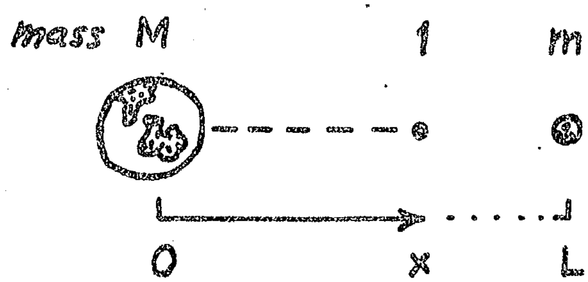


Fig. 4.12. Notation for one-dimensional earth-moon-spaceship problem

In the notation of Fig. 4.12, Newton's law gives for the motion of a spaceship of unit mass

$$\frac{d^2x}{dt^2} = G\left[\frac{m}{(L-x)^2} - \frac{M}{x^2}\right] \quad (4.34)$$

Here G is the universal gravitational constant, and the signs are correct for the spaceship between the earth and moon $(0 < x < L)$. It is simpler to work with the first integral

$$\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2 - G\left(\frac{M}{x} + \frac{m}{L-x}\right) = \text{const.} \qquad (4.35)$$

This is an energy equation, the first term representing the kinetic and the second term the potential energy.

We now make several simplifications. First, we introduce dimensionless variables by referring distance to L, masses to the total mass $(M+m)$ of the system, and time to $[L^3/2G(M+m)]^{1/2}$. Second, we restrict attention to the special case of zero energy (so that the spaceship would reach remote space with zero speed). Third, we measure time from the instant of leaving the (center of the) earth. Fourth, we introduce $\varepsilon = m/(M+m)$ as a small parameter. And fifth we interchange the independent and dependent variables. Thus the problem is reduced to

$$2\left(\frac{dt}{dx}\right)^2 = \frac{x(1-x)}{(1-\varepsilon)-(1-2\varepsilon)x}\ , \qquad t(0) = 0 \tag{4.36}$$

This can be solved in terms of elliptic integrals. However, a perturbation solution for small ε is simpler, more useful, and serves as a model for more intractable problems.

For $\varepsilon = 0$ the moon is absent, and the solution is easily found. Taking that as a first approximation leads to a straightforward expansion in powers of ε:

$$t = \frac{2}{3}x^{3/2} + \varepsilon\left(\frac{2}{3}x^{3/2} + x^{\frac{1}{2}} - \frac{1}{2}\log\frac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}}\right) + \varepsilon^2\left(x^{3/2} + 3x^{\frac{1}{2}} - \frac{27}{16}\log\frac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}} + \frac{3}{8}\frac{x^{\frac{1}{2}}}{1-x}\right) + \ldots \tag{4.37}$$

That this approximation is not valid close to the moon is indicated by a logarithmic singularity at $x = 1$ in the second term and an algebraic one in the third. We have been able to by-pass the usual matching order (Fig. 4.10), and calculate several terms in the global expansion completely, because no boundary conditions are imposed within the local region near the moon.

For the local expansion we must evidently shift the origin of space to $x = 1$, and of time to the approximate time of arrival at the moon, say C_0 [which we will see is in fact given correctly as 2/3 by the first term of the global expansion (4.37)]. However, it is not clear how greatly these variables are to be magnified. The ratio of the first and second terms of the global expansion suggests an exponentially small region of nonuniformity, where $1-x = O(e^{-1/\varepsilon})$; the ratio of the first and third terms suggests $1-x = O(\varepsilon^2)$; and the ratio of the second and third suggests $1-x = O(\varepsilon)$. The gravitational forces balance in (4.34) when $1-x = O(\varepsilon^{\frac{1}{2}})$; but the potential energies of the earth and moon are equal in (4.35) when $1-x = O(\varepsilon)$.

When the proper choice is unclear it is helpful, as before (Sec. 4.4), to consider all possible power-law stretchings. Thus setting

$$X = \frac{1-x}{\varepsilon^a}\ , \qquad T = \frac{t-C_0}{\varepsilon^b} \tag{4.38}$$

transforms the differential equation (4.36) to

$$\left(\frac{dT}{dX}\right)^2 = \varepsilon^{2(a-b)}\frac{X(1-\varepsilon^a X)}{X(1-2\varepsilon)+\varepsilon^{1-a}} \approx \varepsilon^{2(a-b)}\frac{X}{X+\varepsilon^{1-a}}\ . \tag{4.39}$$

(Here we have approximated the right-hand side using $a > 0$, which corresponds to the obvious fact that the region needs to be magnified rather than reduced.) Now this equation degenerates to the contradiction of a single dominant term that vanishes, unless $(a-b) = 0$ and $a \leq 1$. Furthermore, unless $a = 1$ the first approximation to T is linear in X, and will not match. Thus we must choose $a = b = 1$, and the full equation in local variables becomes

$$\left(\frac{dT}{dX}\right)^2 = \frac{X(1-\varepsilon X)}{1+X-2\varepsilon X} \tag{4.40}$$

Let us assume that the local solution has, like the global one, an expansion in integral powers of ε:

$$T(X;\ \varepsilon) = T_1(X) + \varepsilon T_2(X) + \ldots \tag{4.41}$$

Then the first approximation is found to be

$$T_1 = \sinh^{-1}\sqrt{X} - \sqrt{X(1+X)} + C_1 \tag{4.42}$$

Here C_1 is determined by matching; in our usual format:

<u>Global expansion to $O(\varepsilon)$:</u> $\quad t = \frac{2}{3}x^{\frac{3}{2}} + \varepsilon\left(\frac{2}{3}x^{\frac{3}{2}} + x^{\frac{1}{2}} - \frac{1}{2}\log\frac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}}\right)$

rewritten in local variables: $\quad = \frac{2}{3}(1-\varepsilon X)^{\frac{3}{2}} + \varepsilon\left[\frac{2}{3}(1-\varepsilon X)^{\frac{3}{2}} + (1-\varepsilon X)^{\frac{1}{2}} - \frac{1}{2}\log\frac{1+(1-\varepsilon X)^{\frac{1}{2}}}{1-(1-\varepsilon X)^{\frac{1}{2}}}\right]$

expanded to $O(\varepsilon)$: $\quad = \frac{2}{3} + \varepsilon\left(\frac{5}{3} - X - \frac{1}{2}\log\frac{4}{\varepsilon X}\right)$

rewritten in global variables for comparison: $\quad = \frac{2}{3} - (1-x) + \varepsilon\left(\frac{5}{3} - \frac{1}{2}\log\frac{4}{1-x}\right) \quad (4.43)$

<u>Local expansion to $O(\varepsilon)$:</u> $\quad t = C_0 + \varepsilon\left[\sinh^{-1}\sqrt{X} - \sqrt{X(1+X)} + C_1\right]$

rewritten in global vars.: $\quad = C_0 + \varepsilon\left[\sinh^{-1}\sqrt{\frac{1-x}{\varepsilon}} - \sqrt{\frac{(1-x)(1-x+\varepsilon)}{\varepsilon^2}} + C_1\right]$

expanded to $O(\varepsilon)$: $\quad = C_0 - (1-x) + \varepsilon\left[C_1 - \frac{1}{2} + \frac{1}{2}\log\frac{4(1-x)}{\varepsilon}\right] \quad (4.44)$

[The $-\frac{1}{2}\varepsilon$ in the last term comes from the secondary term in the global expansion of the second-order local term, and was overlooked by Lagerstrom and Kevorkian.]

These match if (4.43) and (4.44) are identical, which would require

$$C_0 = \frac{2}{3}, \qquad C_1 = \frac{13}{6} - \frac{1}{2}\log\frac{16}{\varepsilon} \tag{4.45}$$

Strictly, this is not acceptable, because C_1 was supposed to be a pure constant, independent of ε as well as x. However, this slight mismatch by a multiple of $\log(1/\varepsilon)$ is not serious. It simply means that our local expansion (4.41) ought to have started with a term in $\log(1/\varepsilon)$; and this would have been clear if we had anticipated the matching as we did in the previous problem. However, the structure of the expansion is sufficiently simple that starting over on that basis yields just the same result as merely using the above value for C_1.

Thus the local expansion about the moon gives

$$t = \frac{2}{3} - \frac{1}{2}\varepsilon \log\frac{1}{\varepsilon} + \varepsilon\left[\sinh^{-1}\sqrt{X} - \sqrt{X(1+X)} + \frac{13}{6} - 2\log 2\right] + \ldots \quad (4.46)$$

Setting $X = 0$ gives the time of arrival at the (center of the) moon:

$$t_{arr} = \frac{2}{3} - \frac{1}{2}\varepsilon \log\frac{1}{\varepsilon} + \left(\frac{13}{6} - 2\log 2\right)\varepsilon + \ldots \quad (4.47)$$

Higher powers of $\log(1/\varepsilon)$ can be expected in subsequent terms.

It is clear how such a mixture of powers and logarithms of the perturbation quantity cf. Eq. (1.12) is generated by the interplay between the global and local scales whenever either expansion contains a logarithm of the coordinate or any function, such as $\cosh^{-1}$, K_0, etc., that behaves logarithmically for small or large values of the argument. This phenomenon has been termed switchback (cf. Lagerstrom & Carsten 1972).

4.9. Resolution of Stokes Paradox for Circle at Low Reynolds Number

The first triumph of the method of matched asymptotic expansions was its effective resolution of the Stokes paradox (Sec. 2.10) for slow viscous flow past a circle, and the corresponding Whithead paradox for the sphere. We consider the circle. This provides an example of the third category of singular-perturbation series distinguished in Section 1.6 -- those involving integral powers of the reciprocal of the logarithm of the perturbation quantity. Fraenkel (1969) calls this the purely logarithmic case, and shows that it has certain unique properties.

We saw in Section 2.10 that the solution of the biharmonic equation for the stream function that grows as slowly as possible with distance,

$$\psi = C(r\log r - \tfrac{1}{2}r + \tfrac{1}{2}\tfrac{1}{r})\sin\theta \quad (4.48)$$

is invalid where $r = O(1/R)$. We therefore regard this Stokes approximation as the first term of a local expansion, and seek to construct a complementary global approximation valid at large radius.

We introduce the contracted radial variable

$$\rho = Rr \quad (4.49)$$

(not to be confused with the density, there being a shortage of symbols for radial variables when R is pre-empted by the Reynolds number). This corresponds to referring the physical radius to the viscous length ν/U rather than the radius a of the circle. In these terms the circle is described by $\rho = R$, so it shrinks to a point as $R \to 0$. It is therefore clear that the global solution begins with the undisturbed free stream:

$$\psi = y + \ldots = \frac{\rho}{R}\sin\theta + \ldots \quad (4.50)$$

We can now find the constant C by matching these two approximations. In our familiar format we write

<u>1-term global expansion:</u> $\psi = \frac{1}{R}\rho \sin\theta$

rewritten in local vars: $= r\sin\theta$

1-term local expansion, rewritten in global vars: $= \frac{1}{R}\rho\sin\theta$ (4.51)

<u>1-term local expansion:</u> $\psi = C\left[r\log r - \frac{1}{2}r + \frac{1}{2}\frac{1}{r}\right]\sin\theta$

rewritten in global vars: $= C\left[\frac{\rho}{R}\log\frac{1}{R} + \log\rho - \frac{1}{2}\frac{\rho}{R} + \frac{1}{2}\frac{R}{\rho}\right]\sin\theta$

expanded for small R: $= C\left[\frac{1}{R}\log\frac{1}{R}\cdot\rho + \frac{1}{R}(\rho\log\rho - \frac{1}{2}) + \frac{1}{2}R\right]\sin\theta$

1-term global expansion: $= C\,\frac{1}{R}\log\frac{1}{R}\cdot\rho\sin\theta$ (4.52)

These match with $C = \log(1/R)$. As in the earth-moon-spaceship problem we have encountered a switchback phenomenon, showing that the local expansion starts with $\log(1/R)$.

In this purely logarithmic case the matching is marginal, and successive approximations proceed with agonizing slowness. We had to neglect R^{-1} compared with $R^{-1}\log(1/R)$, because otherwise the term in $\rho\log\rho$ would spoil the matching. This means that the expansion proceeds, in higher approximations, in powers of $[\log(1/R)]^{-1}$. The drag coefficient (where D' is drag per unit length) is then

$$C_D \equiv \frac{D'}{\rho U a} \sim 4\pi\frac{1}{R}\left[\frac{1}{\log\frac{1}{R}} + O\left(\frac{1}{\log\frac{1}{R}}\right)^2\right] \qquad (4.53)$$

To proceed to higher approximations, we turn again to the global expansion, in accord with the normal matching order (Fig. 4.10). Thus we substitute

$$\psi = \frac{1}{R}\left[\rho\sin\theta + \frac{1}{\log\frac{1}{R}}\psi_2(\rho,\theta) + \dots\right] \qquad (4.54)$$

in the full Navier-Stokes equations. This gives for ψ_2 the linearized equation for small departures from a uniform stream that was introduced in 1910 by Oseen, and is known as the Oseen approximation. We need the solution for a point disturbance. Using that result, Proudman & Pearson (1957) found a second approximation for the drag:

$$C_D \sim 4\pi\frac{1}{R}\left[\frac{1}{\log\frac{1}{R}} - \frac{\log 4 - \gamma + \frac{1}{2}}{\left(\log\frac{1}{R}\right)^2} + \dots\right] \qquad (4.55)$$

Here $\gamma = 0.5772$ is Euler's constant.

This is in fact just the expansion to two terms of Lamb's (1911) result:

$$C_D \approx 4\pi\frac{1}{R}\frac{1}{\log\frac{4}{R} - \gamma + \frac{1}{2}} \qquad (4.56)$$

which he obtained by solving the linearized Oseen equations approximately for small Reynolds number. As this suggests, the problem is not a singular perturbation if it is solved in the contracted global coordinates. The situation is thus analogous to that of slow variations (and in accord with the quotation at the head of Chapter 3). In the corresponding problem of the sphere, Illingworth (1963) has discussed how higher approximations can be found by iterating on the Navier-Stokes equations starting with the Oseen linearization. However, the method of matched asymptotic expansions has the advantage of producing the solution in the simplest possible way.

The computation becomes so difficult that although Kaplun (1957) claculated the third approximation, no further terms have been found since, nor is it likely that they will be. His result is

$$C_D \sim 4\pi \frac{1}{R}\left[\frac{1}{\log\frac{4}{R}-\gamma+\frac{1}{2}} - \frac{0.87}{\left(\log\frac{4}{R}-\gamma+\frac{1}{2}\right)^3} + O\left(\frac{1}{\log\frac{4}{R}-\gamma+\frac{1}{2}}\right)^4\right] \quad (4.57)$$

The constant 0.87 was found by numerical evaluation of an infinite integral involving products of Bessel functions. Here Proudman & Pearson's two terms have been telescoped into the first by adopting Lamb's form (which comes from expanding the modified Bessel function K_0). Fig. 4.13 shows that this greatly improves the agreement, though the result is still inaccurate except at Reynolds numbers less than unity.

This infinite sequence of powers of $[\log(1/R)]^{-1}$ or $[\log(4/R)-\gamma+\frac{1}{2}]^{-1}$, if it could be found, would still correspond only to the linearized

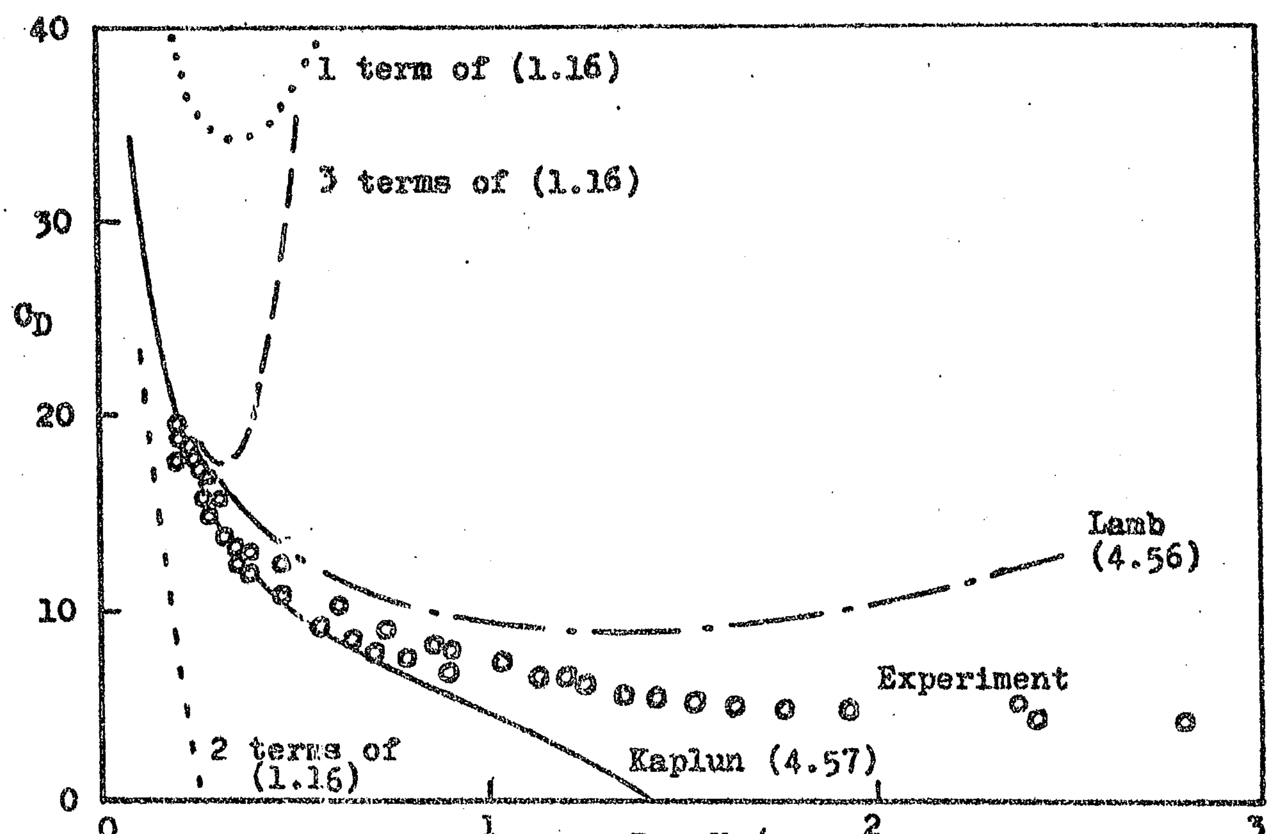


Fig. 4.13. Drag of circle at low Reynolds number

(biharmonic) approximation near the circle. Hence it cannot predict the separation, for example, that begins at about R = 2.2 (Fig. 4.14). The nonlinear terms in the Navier-Stokes equations are essential, but they add transcendentally small terms (sec. 1.4), of relative order R. Proudman & Pearson (1957) discussed briefly the possibility of skipping to those terms. This idea has recently been pursued in other problems. Thus Terrill (1973) finds that including transcendentally small terms significantly improves his solution for viscous flow through a porous pipe. The notion of finding the "complete" asymptotic expansion, including all transcendentally small terms, is dealt with at great length in a new book by Dingle (1973).

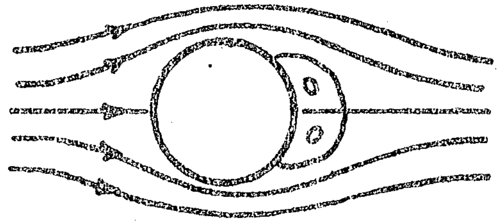


Fig. 4.14. Separated flow past circle at $R \approx 10$

4.10. Multiple boundary layers

Boundary layers are encountered not only in a wide range of problems in mechanics, but in many geometrical configurations, and in a remarkable variety of combinations. However, they can always be treated in principle by appropriate application of the method of matched asymptotic expansions. The practical computational details may become forbidding, but one has the satisfaction of knowing that the method decomposes a problem into its simplest possible elements.

We may distinguish three categories of multiple boundary layers. The simplest to calculate are concentric boundary layers. These arise when a problem involves three or more disparate scales of length. Then a corresponding number of distinct regions can be distinguished, and these are nested one within another like a set of Russian dolls. An example is viscous flow with heat transfer past a body at very low or high Prandtl number, which means that the thermal diffusion of the fluid is much greater than its diffusion of momentum by viscosity. Such problems are solved simply by successive application of the procedure already discussed. Thus for flow at high Prandtl number (Fig. 4.15) the inviscid flow far from the body is first regarded as the global approximation to be matched to the local approximation in the viscous boundary layer, but then the latter is in turn regarded as the global approximation to be matched to the still more local thermal layer. Three different regions are treated in the same way for Kirchoff's capacitor (Fig. 4.2b). As many as five superimposed layers have been dealt with by Bush & Cross (1967) in hypersonic viscous flow.

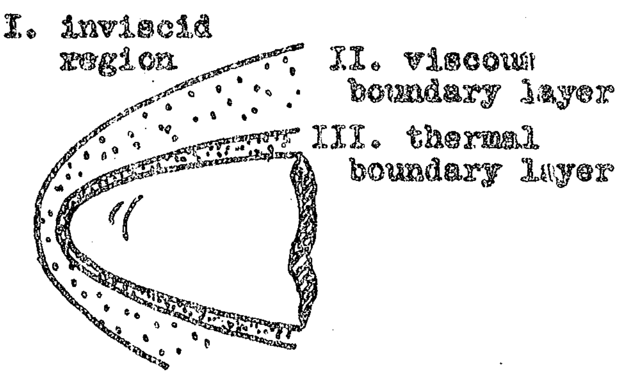


Fig. 4.15. Concentric boundary layers for flow past a body at high Prandtl number

It is more difficult to solve the second category of <u>intersecting</u> (or <u>overlapping</u>) <u>boundary layers</u>. These typically arise at edges and corners. The problem is simplified if the thicknesses of the layers are of different orders. For example, if a circular cylinder filled with slightly viscous fluid is impulsively set into rotation at large Reynolds number $R = \Omega a^2/\nu$ (Fig. 4.16), boundary layers of thicknesses $R^{-1/4}$ and $R^{-1/3}$ grow on the sides, and of thickness $R^{-1/2}$ on the flat ends. (Greenspan & Howard 1963). The three concentric regions on the sides can be treated as described in the preceding paragraph; and the intersection with the layer on the ends is handled directly by matching. However, there remains in the corner a square region of dimension $R^{-1/2}$ that is intractable. This more difficult problem of the intersection of two equal boundary layers arises in pure form, for example, with the meniscus in a corner (Fig. 4.17), or viscous flow along a corner. These intersections have usually required numerical solution.

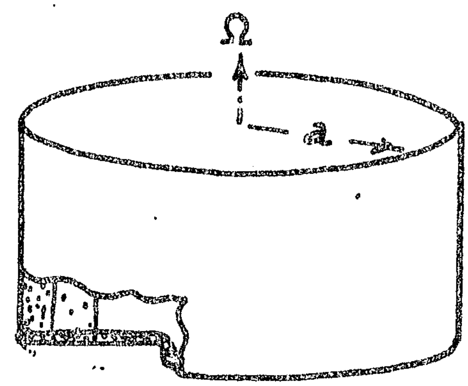


Fig. 4.16. Spin-up of fluid in circular cylinder

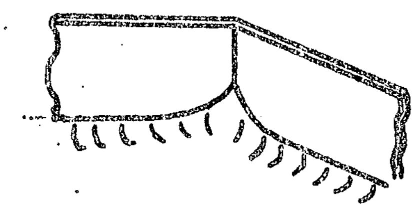


Fig. 4.17. Intersecting meniscuses in a corner

A third category of <u>colliding boundary layers</u> may be distinguished in a moving medium. An example is a sphere spinning steadily in slightly viscous fluid (Fig. 4.18). Fluid is drawn inward at the poles, carried around the surface in a thin boundary layer, and then ejected in an equatorial sheet after the two boundary layers collide at the equator. The term "colliding" is used to emphasize the fact that the boundary layer on either hemisphere could continue well past the equator without separating (Banks 1965) were it not to meet its mirror image head on.

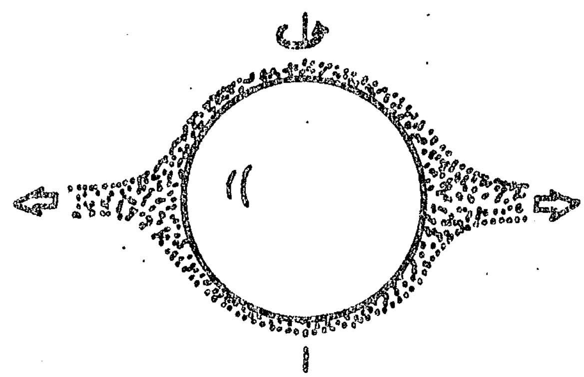


Fig. 4.18. Colliding boundary layers on spinning sphere

<u>4.11. The subtleties of matching</u>

Matching, which is the heart of the method of matched asymptotic expansions, is evidently a delicate and subtle process. The newcomer to the subject may be entitled to regard it as mysterious and even magical.

The process was first systematized by Friedrichs (1955), who called it the "identification technique." Deeper insight into the nature of matching was provided by Kaplun, who introduced such concepts as the overlap domain, intermediate limits and expansions, and the extension principle. The three papers that comprise his published life work (Kaplun 1954, Kaplun 1957, Kaplun & Lagerstrom 1957) have been republished, together with some incomplete notes left unpublished at his death (Kaplun 1967). Further discussion and interpretation has been provided by Lagerstrom and his colleagues in that volume and elsewhere. In particular, Lagerstrom & Casten (1972) give a useful review with the aid of model equations.

The central idea is that of overlap. Two expansions can be matched only if they are both rich enough to cover a common region of validity. This is the case for Prandtl's boundary layer, which in its outer fringes merges into the inviscid flow (Fig. 4.19). By contrast, a more drastic simplification of the Navier-Stokes equations would lead to the Stokes approximation of creeping flow, but that applies only in a thin sheet at the bottom of Prandtl's layer; it is not rich enough to overlap with the inviscid flow.

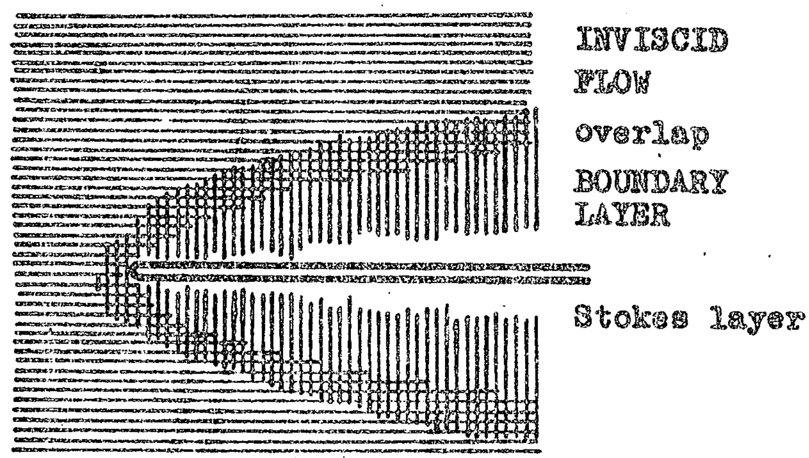


Fig. 4.19. Viscous flow past flat plate, indicating overlap of boundary layer and inviscid flow

If overlap exists, matching rests on the general principle that

> The local expansion of the global expansion should, to appropriate orders, agree with the global expansion of the local expansion. (Lagerstrom 1957) (4.58)

The specific way in which this is carried out in practice is largely a matter of taste. A master in the field may choose to match intuitively, and sniff at any formal scheme. But his intuition will fail at some stage: even Prandtl wavered when he discussed higher-order boundary-layer theory (Prandtl 1934, p. 90). In the book of Cole (1968) matching is always by intermediate limits. The present writer, perhaps for lack of experience, feels insecure with that method, and prefers the "asymptotic matching principle" (4.19) -- or, better, (4.20) -- because it is so automatic it could be programmed for a computer.

Another important study of the mathematical nature of matching has been presented by Fraenkel (1969) in a series of three papers whose significance has been largely overlooked. The first part, in particular, is hard reading for an engineer. But concealed among the new notation, definitions, lemmas, and theorems are several nuggets of practical information. We summarize the three most useful points:

1. The asymptotic matching principle vs. intermediate limits. The asymptotic matching principle (4.19) or (4.20) can be incorrect in some cases, including the two important situations covered by points 2 and 3 below. However it has the advantage of being easier to use than Kaplun's procedure of matching by intermediate limits (sec. 4.5), which depends on the existence of overlap. It succeeds even when applied to local and global expansions that contain too few terms to overlap to the order of the terms being matched.

2. Don't cut between logarithms. Let ε be specified as the ratio of the local to the global scales. Then in matching, all terms must be grouped together that differ by less than some power of ε — that is, one should not separate terms that differ only by a power of the logarithm, such as $\varepsilon^N \log \varepsilon$ and ε^N. This seems plausible, because a change of scale from ε to $c\varepsilon$ will shift powers of $\log c$ among such close terms.

The practical importance of this warning is illustrated by a recent calculation of acoustic diffraction by a thin plate (Crighton & Leppington 1973). The asymptotic sequence for the local expansion is

$$1, \quad \varepsilon, \quad \varepsilon^2 \log \varepsilon, \quad \varepsilon^2, \quad \varepsilon^3 \log \varepsilon, \quad \varepsilon^3, \quad \ldots \tag{4.59}$$

It happens that disregarding Fraenkel's warning and truncating the series at $\varepsilon^2 \log \varepsilon$ gives a correct match; but truncating at $\varepsilon^3 \log \varepsilon$ yields a result that violates a reciprocity principle, and is corrected by retaining also the term in ε^3. This casts doubt on the result (1.12) of Chester and Breach for the drag of a sphere at low Reynolds number, because it has exactly the same asymptotic sequence, and has been truncated at just the point that led to error in the diffraction problem.

3. Forbidden regions in the purely logarithmic case. In the "purely logarithmic case" (sec. 4.9), when both the global and the local expansions proceed in powers of $[\log(1/\varepsilon)]^{-1}$, the asymptotic matching principle (4.19) may fail for certain values of m and n lying within a "forbidden region." This difficulty is actually encountered in the problem of axisymmetric flow past a paraboloid of revolution at low Reynolds number, and in various cylindrical geometries (Exercise 4.2). The remedy seems to be simply to proceed to the next approximation, which ordinarily carries one outside the forbidden region, so that all constants can then be determined by matching.

The following additional remarks about matching may be helpful:

a. Matching is different from patching. The idea of patching two curves is a familiar one, exemplified by Laplace's "sphere of influence" at which orbits about the earth and the moon are joined with continuous tangent, but discontinuities in higher derivatives. By contrast, matching yields a smooth transition, with all derivatives continuous.

b. Matching takes the place of missing boundary conditions. It is used to determine the elements in one or both of the expansions that are unknown for lack of boundary conditions. These elements are usually multiplicative constants, but may also be exponents (Exercise 4.1) and even functions (Exercise 4.3).

c. Matching one quantity matches all. One is free to match any dependent variable that serves the purpose of determining missing elements. Then other dependent variables are also matched, or matching them instead would give the same result. For example, in viscous boundary-layer theory it is customary to match either the stream function ψ or the tangential velocity component $u = \psi_y$, and then the other is automatically matched. (Of course if the dependent variable were a polynomial, matching too high a derivative would yield a correct but useless triviality.) Likewise, it may suffice to match along one line rather than throughout a region. For example, Crighton & Leppington (1973) match on only a single ray from the origin.

d. The asymptotic matching principle holds for any m and n (or in the form (4.20), for any Δ and δ). Ordinarily, when the normal matching order is followed, only the choices $n = m$ (for the right-hand column of Fig. 4.10) and $n = m + 1$ (for the left-hand column) are useful. All other combinations give results that are also correct, but only trivially so.

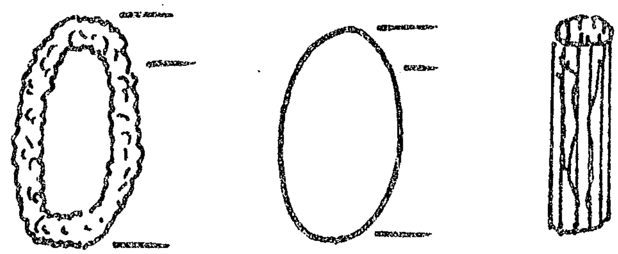
e. Numerical or even experimental results can be matched. It is only necessary that the asymptotic form of the result be known in closed analytic form. For example, in the classical problem of viscous flow past a flat plate(Fig. 4.19) the boundary-layer solution involves the so-called Blasius function $f(\eta)$, which is found by numerical integration of a nonlinear ordinary differential equation. It can be matched using the fact, revealed by a simple asymptotic analysis of the equation, that $f(\eta) \sim \eta - \beta + \ldots$ for large η (that is, in the outer fringes of the boundary layer -- the overlap region).

4.12. Triumphs of the method of matched asymptotic expansions

The literature of engineering mechanics shows that the method of matched asymptotic expansions has by now been applied to several thousand problems. The majority are in the various branches of fluid mechanics, and such allied fields as heat transfer and combustion; but the method is increasingly being applied to diffraction, solid mechanics, celestial mechanics, etc.

Most of these applications are straightforward or even routine. A few stand out, however, for their novelty of technique, or the simplicity and elegance of the results. We sketch a few examples of what are, in the author's opinion, such triumphs of the method.

Its first successes were the definitive resolution of the paradoxes of Stokes and Whitehead in viscous flow theory. We mentioned in sec. 2.9 the resolution of analogous paradoxes -- also involving logarithmic divergence at infinity -- in the theory of elastic contact and of free-surface planing; and in sec. 1.6 the removal of the algebraic singularity in the membrane approximation for a toroidal shell. Another example of a paradox resolved is the motion of a slightly viscous vortex ring (Fig. (4.20). In plane flow, a vortex can be approximated by an inviscid singularity; but in three dimensions a curved concentrated vortex line induces infinite velocity on itself, so that viscosity, though small, is crucial. Tung & Ting (1967) matched the local solution for a two-dimensional viscous vortex to the global solution for an inviscid vortex ring, to find a finite speed for the ring, decreasing logarithmically with time as the core diffuses.



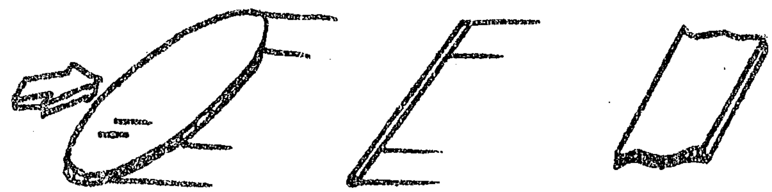
Full problem | Global approximation | Local approximation

Fig. 4.20. Motion of a viscous vortex ring

Another significant role of matched expansions is in systematizing *ad hoc* approximations. A class of free-streamline problems that have been treated is exemplified by the two-dimensional waterfall formed by a uniform inviscid stream flowing over a cliff (Fig. 4.21). It is clear that far from the brink (in the absence of surface tension and instability) the water falls almost freely in a thin sheet along a nearly parabolic arc. Clarke (1965) matched that global approximation to a local approximation departing only slightly from a horizontal stream near the edge. For the aerodynamics of a wing of high aspect ratio (Fig. 4.21), Friedrichs (1953) first showed how Prandtl's lifting-line theory could be embedded in a systematic expansion scheme; and the author (Van Dyke 1964) had the pleasure of working out the details of the third approximation. (At that stage for a smooth planform there appears near the wing tip a still more local region of nonuniformity that has not yet been analyzed.)

Full problem | Global approximation | Local approximation

Fig. 4.21. Plane inviscid waterfall



Full problem | Global approximation | Local approximation

Fig. 4.21. Lifting wing of high aspect ratio

Ogilvie (1970) has recently shown how the method of matched asymptotic expansions is useful in a variety of problems of ship hydrodynamics. In solid mechanics, Messick (1962) has treated boundary layers at the edges of plates and shells, and their intersections at corners.

This brief sampling could be multiplied many times. In fluid mechanics, for example, matching has produced interesting results in the theories of aerodynamic noise, sonic boom, hydrodynamic stability, lubrication, rarefied gases, transonic and hypersonic flow, and water waves. The reader interested in further examples will encounter them in almost every issue of the Journal of Fluid Mechanics, for example, or he can consult the books of Cole (1968), Nayfeh (1973), and Van Dyke (1964) and subsequent references to them in Science Citations.

References

(additional to those cited in previous chapters)

BANKS, W. H. H. 1965 The boundary layer on a rotating sphere. Quart. J. Mech. Appl. Math. 18, 443-454

BUSH, W. B. & CROSS, A. K. 1967 Hypersonic weak-interaction similarity solutions for flow past a flat plate. J. Fluid Mech. 29, 349-359

CLARKE, N. S. 1965 On two-dimensional inviscid flow in a waterfall. J. Fluid Mech. 22, 359-369

CRIGHTON, D. G. & LEPPINGTON, F. G. 1973 Singular perturbation methods in acoustics: diffraction by a plate of finite thickness. Proc. Roy. Soc. A 335, 313-339

DINGLE, R. B. 1973 Asymptotic Expansions: Their Derivation and Interpretation, 1. Academic Press, N.Y.

ERDELYI, A. 1961 An expansion procedure for singular perturbations. Atti Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat. 95, 651-672

FRAENKEL, L. E. 1969 On the method of matched asymptotic expansions. Part 1: A matching principle; Part II: Some applications of the composite series; Part III: Two boundary-value problems. Proc. Camb. Phil. Soc. 5, 209-231; 233-261; 263-284

FRIEDRICHS, K. O. 1953 Special Topics in Fluid Dynamics. N. Y. Univ. Published 1966, Gordon & Breach

FRIEDRICHS, K. O. 1955 Asymptotic phenomena in mathematical physics. Bull. Amer. Math. Soc. 61, 485-504

GREENSPAN, H. P. & HOWARD, L. N. 1963 On a time-dependent motion of a rotating fluid. J. Fluid Mech. 17, 385-404

GOLDSTEIN, S. 1930 Concerning some solutions of the boundary-layer equations in hydrodynamics. Proc. Camb. Phil. Soc. 26, 1-30

ILLINGWORTH, C. R. 1963 Flow at small Reynolds number. Chapter IV, Laminar Boundary Layers (L. Rosenhead, ed.). Clarendon Press, Oxford

KAPLUN, S. 1954 The role of coordinates in boundary-layer theory. Z. Angew. Math. Phys. 5, 111-135; also Chap. 1 of Kaplun 1967

KAPLUN, S. 1957 Low Reynolds number flow past a circular cylinder. J. Math. Mech. 6, 595-603; also Chap. 3 of Kaplun 1967

KAPLUN, S. 1967 Fluid Mechanics and Singular Perturbations (P. A. Lagerstrom, L. N. Howard & C. S. Liu, eds.). Academic Press, N.Y.

KAPLUN, S. & LAGERSTROM, P. A. 1957 Asymptotic expansions of Navier-Stokes solutions for small Reynolds numbers. J. Math. Mech. 6, 585-593; also Chap. 2 of Kaplun 1967

KIRCHOFF, G. 1877 Zur Theorie des Condensators. Monatsb. Acad. Wiss. Berlin, 144-162

LAGERSTROM, P. A. & CASTEN, R. G. 1972 Basic concepts underlying singular perturbation techniques. SIAM Rev. 14, 63-120

LAGERSTROM, P. A. & COLE, J. D. 1955 Examples illustrating expansion procedures for the Navier-Stokes equations. J. Rat. Mech. Anal. 4, 817-882

LAGERSTROM, P. A. & KEVORKIAN, J. 1963 Matched conic approximation to the two fixed force-center problem. Astron. J. 68, 84-92

LAMB, H. 1911 On the uniform motion of a sphere through a viscous fluid. Phil. Mag. (6) 21, 112-119. See also Hydrodynamics, 614-616

LANDAU, L. & LEVICH, B. 1942 Dragging of a liquid by a moving plate. Acta Physicochimica U.R.S.S. 17, 42-54

LAPLACE, P. S. 1805 Mécanique Celeste. English: Celestial Mechanics (N. Bowditch, transl.), Boston, 1839; Chelsea, N.Y., 1966

MAXWELL, J. C. 1866 On the viscosity or internal friction of air and other gases. Phil. Trans. Roy. Soc. 156, 249-268

MESSICK, R. E. 1962 Application of an edge- and corner-layer technique to elastic plates and shells. Ph.D. thesis, Calif. Inst. Tech., Pasadena

MEYER, R. E. & WILSON, L. O. 1973 Stretching and calculus of negligibility. SIAM J. Appl. Math. 25, 388-402

OGILVIE, T. F. 1970 Singular perturbation problems in ship hydrodynamics. Univ. Michigan, Dept. Naval Arch. & Marine Engg., Rept. 096

PRANDTL, L. 1934 The mechanics of viscous fluids. Division G, Aerodynamic Theory (W. F. Durand, editor). Springer, Berlin

PRANDTL, L. 1952 Essentials of Fluid Dynamics. Blackie & Son, London & Glasgow

RAYLEIGH, Lord 1915 On the theory of the capillary tube. Proc. Roy. Soc. A 92, 184-195; Sci. Papers 6, 350-361

SCHNEIDER, W. 1973 A note on the breakdown of the multiplicative composition of inner and outer expansions. J. Fluid Mech. 59, 785-789

TERRILL, R. M. 1973 On some exponentially small terms arising in flow through a porous pipe. Quart. J. Mech. Appl. Math. 26, 347-354

TUNG, C. & TING, L. 1967 Motion and decay of a vortex ring. Phys. Fluids 10, 901-910

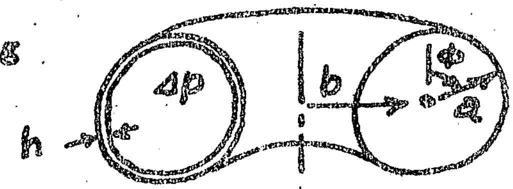
VAN DYKE, M. D. 1954 Subsonic edges in thin-wing and slender-body theory. NACA Tech. Note 3343

VAN DYKE, M. 1964 Lifting-line theory as a singular perturbation problem. Arch. Mech. Stosowanej 16, 601-614; Appl. Math. Mech. 28, 90-101

WAY, S. 1962 Plates. Chapter 39, Handbook of Engineering Mechanics (W. Flügge, ed.). McGraw-Hill, New York

EXERCISES

<u>4.1. Matching for pressurized toroidal shell.</u> In a Stanford Ph.D. thesis the deformation of a slightly pressurized toroidal shell was solved by the method of matched asymptotic expansions. Iterating on the linearized membrane approximation yields for the angular rotation χ of an element of the shell the series (with the abbreviation $s = \sin\phi$)



$\varepsilon = \Delta pa/Eh$, $\beta = b/a$

$$\frac{\chi}{\cos\phi} = \varepsilon\frac{\frac{1}{2}\beta}{s(\beta+s)} + \varepsilon^{1+\gamma} K\frac{2\beta^2+3\beta s}{s^4(\beta+s)} - \varepsilon^2\tfrac{1}{2}(1-2\nu)\frac{\beta(\beta+s)(2\beta-s)}{s^4}$$

$$+\ \varepsilon^2\frac{1}{s^5(\beta+s)^2}\left(3\beta^4+\frac{15}{2}\beta^3 s+\frac{41}{8}\beta^2 s^2-\frac{3}{4}\beta s^3-\frac{1}{8}\beta^2 s^4\right)+o(\varepsilon^2)$$

Here the constant of integration K itself has an expansion $K = K_1 + \varepsilon^\delta K_2 + \ldots$ The positive exponents γ and δ are unknown, as well as the constants K_n. This approximation failes near $\phi = 0$ or π, where the pressure is not resisted by compound curvature of the shell, so that bending rigidity becomes of primary importance. In terms of a boundary-layer variable $\Phi = \phi/\varepsilon^{\frac{1}{4}}\beta^{\frac{1}{2}}$ the approximation in that vicinity is found to be

$$\chi = \frac{\varepsilon^{3/4}}{\beta^{\frac{1}{2}}}\left[-\tfrac{1}{2}\Phi F_0(\Phi) - C_1\{\Phi F_1(\Phi)+1\}\right] + O(\varepsilon)$$

Here F_0 and F_1 are tabulated functions that behave for small values of their argument as

$$F_0(\Phi)\sim\frac{-1}{\Phi^2}-\frac{6}{\Phi^6}+\ldots,\qquad F_1(\Phi)\sim\frac{-1}{\Phi}-\frac{2}{\Phi^5}+\ldots$$

(a) Give a plausible motivation for the choice of $\varepsilon^{\frac{1}{4}}$ in the boundary-layer variable.

(b) Match the two expansions to determine the exponent γ, and the constant K_1 in terms of C_1 (which is itself then found from the symmetry condition of zero rotation at $\phi = \pm\pi/2$).

(c) Form a composite first approximation that is uniformly valid.

<u>4.2. Matching in the purely logarithmic case: sliding rod.</u> An infinitely long circular rod of radius a initially at rest in an unbounded expanse of viscous liquid is impulsively set into motion along its

axis with constant speed U. The problem for the resulting fluid velocity is

$$u_t = \nu(u_{rr} + u_r/r), \qquad \begin{array}{l} u = U \text{ at } r = a \\ u = 0 \text{ at } t = 0,\ r = \infty \end{array}$$

(Alternatively, u is the temperature in a solid of thermal conductivity ν outside a cylinder of radius a whose temperature is suddenly raised by an amount U.) It is convenient to introduce the dimensionless variables $w = u/U$, $R = r/a$, and the similarity variable $\tau = a/2(\nu t)^{\frac{1}{2}}$. Then the problem becomes

$$w_{RR} + w_R/R + 2\tau^3 w_\tau = 0, \qquad \begin{array}{l} w = 1 \text{ at } R = 1 \\ w = 0 \text{ at } \tau = \infty,\ R = \infty \end{array}$$

$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)$

We seek an approximation for large time (hence small τ). Neglecting the term containing τ^3, and imposing the surface boundary condition, gives the approximation $w = 1 + C \log R$. This represents, in fact, the infinite Stokes expansion, of the form

$$w = 1 + \left[\frac{a}{\log 1/\tau} + \frac{b}{(\log 1/\tau)^2} + \frac{c}{(\log 1/\tau)^3} + \ldots\right] \log R$$

because the neglected terms in τ^3 are smaller than any power of $(\log 1/\tau)^{-1}$. However, this cannot satisfy the boundary condition at $R = \infty$. To describe the flow far from the surface we introduce the contracted radius $\rho = \tau R = Ur/\nu$. In these Oseen variables the problem becomes

$$w_{\rho\rho} + (2\rho + 1/\rho)w_\rho + 2\tau w_\tau = 0, \qquad \begin{array}{l} w = 1 \text{ at } \rho = \tau \\ w = 0 \text{ at } \rho = \infty,\ \tau = \infty \end{array}$$

Now neglecting the term containing τ gives as a first approximation a multiple of $E_1(\rho^2)$, where the exponential integral is

$$E_1(z) = \int_z^\infty \frac{e^{-s}}{s}\,ds = -(\log z + \gamma) + O(e^{-z}) \quad \text{for small } z$$
$O(z)$ $\qquad (\gamma = .5772\ldots)$

Matching with the first term of the Stokes expansion shows that the constant is $\frac{1}{2}/(\log 1/\tau)$, with a relative error of the same order. Thus the Oseen expansion has the form

$$w = \frac{1}{\log 1/\tau}\,\tfrac{1}{2}E_1(\rho^2) + O\left(\frac{1}{\log 1/\tau}\right)^2$$

(a) Using the asymptotic matching principle in the special form of (4.20)

> The Stokes expansion to order $(\log 1/\tau)^{-m}$ of the Oseen expansion to order $(\log 1/\tau)^{-n}$ = the Oseen expansion to order $(\log 1/\tau)^{-n}$ of the Stokes expansion to order $(\log 1/\tau)^{-m}$,

try to match for all six combinations with $m = 0, 1, 2$ and $n = 0, 1$. Thus determine the constants a and b in the Stokes expansion, and show that matching is impossible in two of the cases. What is the shape of the forbidden region in the m-n-plane?

(b) Show that the second and third terms of the Stokes expansion can be telescoped into one by rewriting them, for some K, as

$$w = 1 + \left[\frac{a}{\log K/\varepsilon} + \frac{c}{(\log K/\varepsilon)^3} + \cdots\right] \log R$$

(c) What can you say about the utility of a composite expansion in this case?

4.3. Spherically symmetric noise field. The method of matched asymptotic expansions has recently been applied with success to various problems of aerodynamic sound, of which the following is a simple example:

A certain loud source of sound can be idealized as a sphere of radius a through whose surface air is being pumped with radial velocity $V\cos\omega t$. An approximate solution is sought on the basis that the maximum speed V is small compared with the undisturbed speed of sound c. Then entropy changes can be neglected to high order, so that the radial velocity is $\partial\phi/\partial r$, where the velocity potential ϕ satisfies the nonlinear equation of gas dynamics

$$\left[c^2 - (\gamma-1)(\phi_t + \tfrac{1}{2}\phi_r^2)\right](\phi_{rr} + 2\frac{\phi_r}{r}) = \phi_{tt} + 2\phi_r\phi_{rt} + \phi_r^2\phi_{rr} .$$

Here the adiabatic exponent γ is a constant (7/5 for air).

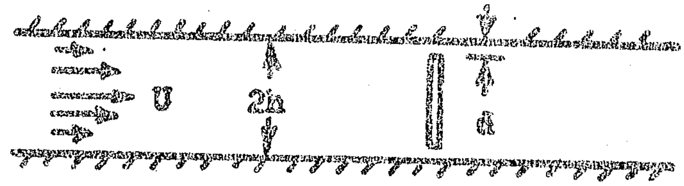
(a) Introduce dimensionless variables appropriate to the flow in the vicinity of the source, referring velocities to V, time to $1/\omega$, and distances to a. Thus show that, provided the dimensionless frequency $\omega a/V$ is of order unity, the nearby flow is incompressible with an error of order M^2, where $M = V/c$ is the Mach number.

(b) Solve the incompressible problem. Then iterate to find the correction of order M^2, carrying the solution just far enough to find evidence of nonuniformity when the radius is as large as the acoustic wavelength c/ω.

(c) Introduce new dimensionless variables appropriate to that distant region, and show that to a first approximation for small M the motion there is governed by the linear wave equation. Match the general solution of the wave equation for outgoing waves, $\phi = f(r-ct)/r$, to the incompressible solution.

4.4. Deflection of slightly rigid triangular membrane. Consider the deflection due to uniform loading of a slightly rigid membrane under

<u>4.6. Qualitative matching for viscous flow past barrier.</u> Oil is being pumped steadily through the passage between two parallel flat plates, and at some point this is nearly blocked by a transverse plate that leaves only small gaps for the oil to squirt through. The motion will certainly obey the Navier-Stokes equations:



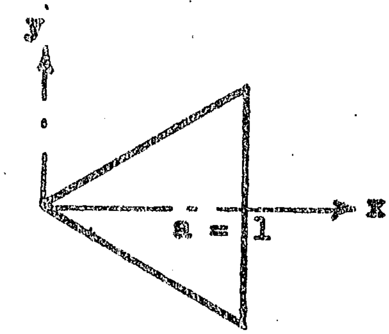
continuity: $u_x + v_y = 0$

x-momentum: $uu_x + vu_y + p_x/\rho = \nu(u_{xx} + u_{yy})$

y-momentum: $uv_x + vv_y + p_y/\rho = \nu(v_{xx} + v_{yy})$

but the problem is obviously too difficult to solve analytically. Suppose you decide to approximate on the basis that the ratio $\varepsilon = d/h$ of gap to channel half-width is small. Explain why this is almost surely a singular perturbation. Describe how it would be treated by the method of matched asymptotic expansions. For both the global and the local problem define the independent variables, sketch the geometry for the first approximation, and indicate what you think the streamlines might look like. Then from overal continuity considerations deduce the order, in terms of ε, of the velocity components in the local problem. Hence deduce from the Navier-Stokes equations the order of the pressure in the local problem. Thus deduce finally that the drag coefficient of the transverse plate (its drag per unit distance into the sketch divided, say, by $\rho U^2 h$) varies as a certain known power of ε.

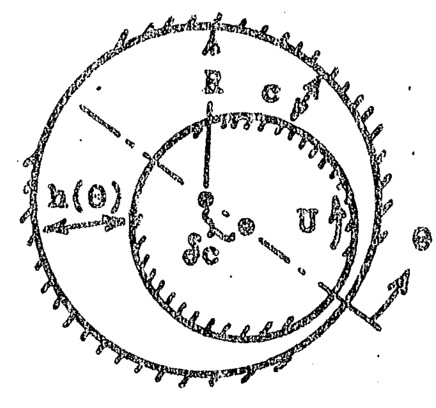
uniform tension when its shape is an equilateral triangle. Find a first approximation to the solution of equations (4.3) and (4.4) for small ε. Use St.-Vénant's result (for the analogous torsion problem) that the solution of $\nabla^2 w = -1$ for an equilateral triangle with $w = 0$ on the boundary is a polynomial of degree three in x and y (and hence simply the product of the three linear functions of x and y that vanish along the edges). Find the boundary-layer correction for the edge $x = 1$. Discuss whether the boundary-layer solution is valid in a corner, and if not, what is required there?



4.5. Heavily loaded journal bearing. Reynolds's theory of lubrication gives the problem for the pressure increment $p(\theta)$ in an infinitely long journal bearing as

$$\frac{d}{d\theta}\left(h^3 \frac{dp}{d\theta}\right) = 6\mu U R \frac{dh}{d\theta} ,$$

$$p(0) = p(2\pi) = 0 .$$



For a small ratio of mean clearance c to radius R, the gap width $h(\theta)$ can be approximated by $c(1 - \delta \cos\theta)$. Then Sommerfeld's solution is

$$p = -\frac{6\mu U R}{c^2} \, \frac{\delta(2 - \delta\cos\theta)\sin\theta}{(2 + \delta^2)(1 - \delta\cos\theta)^2} .$$

Suppose that (perhaps because Sommerfeld's solution was not known) one were to attack the problem by seeking an approximate solution for heavy loading — and hence very small minimum gap — by setting $\delta = (1 - \varepsilon)$ and approximating for small ε. Deduce from Sommerfeld's solution the first two terms of the straightforward (global) expansion for small ε. Verify that the first term satisfies the first-order global problem, obtained by simply setting $\varepsilon = 0$ in the full problem. Deduce the size of the angular region near $\theta = 0$ where the global expansion breaks down, describing it as $\theta = O(\varepsilon^k)$ with an appropriate exponent k. Show that the same estimate is obtained by using the first global approximation to evaluate the ratio of terms neglected to those included in setting $\varepsilon = 0$ in the full problem. Introduce the magnified (local) variable $\Theta = \theta/\varepsilon^k$ into both the problem and Sommerfeld's solution. Verify that it satisfies the local equation to first order. Introduce the intermediate variable $\vartheta = \theta/\varepsilon^{k/2} = \varepsilon^{k/2}\Theta$ into the leading terms of the local and global expansions. Expand both formally to first order for small ε and verify that the two results agree. Rewrite the leading term of the global expansion in terms of the local variable Θ and expand to first order for small ε. Conversely, write the leading term of the local expansion in terms of the global variable θ and expand to first order. Verify that the two results agree.

Chapter 5

THE METHOD OF MULTIPLE SCALES

5.1. Introduction

Most perturbations are singular, and most singular perturbations are of boundary-layer type, so that they can be treated by the method of matched asymptotic expansions. However, there is a large and growing category of problems in mechanics having a more complicated structure. Most of these can be categorized as involving slowly modulated oscillations — vibrations or waves. Such problems cannot be solved by matching, but can be treated by a more general technique that has come to be known as the method of multiple scales.

Of course a boundary-layer problem involves multiple scales, in the sense that in some direction two disparate scales are relevant. However, there is the simplifying feature that at any point only one or the other of the scales is of primary importance. Consequently the field can be subdivided into a global and a local region, in each of which only the single relevant scale need be used. By contrast, in a slowly modulated oscillation both scales are of primary importance everywhere. We might say that it is packed full of boundary layers.

The method of multiple scales consists in working with both scales simultaneously, and regarding them as independent. This seems at first sight a retrograde step, inasmuch as it increases the number of independent variables, and so turns an ordinary differential equation into a partial one, whereas our aim is to solve the problem by reversing the process — for example, by separating variables. However, we shall see that this complication is more apparent than real, because the extra variable appears only as a parameter at each stage. The consequence is that what would be constants of integration in a regular perturbation become slowly varying functions. The crucial step in the method is to determine these functions by examining (but not necessarily solving) the equations for the next approximation, and requiring that the source of nonuniformity there be suppressed.

Evidently multiple scales, being the more general technique, can be used to treat boundary-layer problems as well. That slightly complicates the analysis, because the global and local regions are then treated simultaneously rather than separately. However, the result has the advantage of being uniformly valid, whereas matching yields expansions that have to be combined into a composite series if a single uniform expansion is required. The result is in any case a generalized asymptotic expansion (sec. 3.4, 4.6), and is therefore not unique. This means that there is a certain amount of freedom in the choice of the slowly varying functions, and this can sometimes be exploited for some specific purpose.

Whereas matching is 150 years old, the method of multiple scales has been invented and developed within the last two decades. It appears to have been introduced by Sturrock (1957), who called it the "derivative-expansion method." It has also been called the "two-variable expansion method" (Cole 1968) and, because it was first applied to problems having time as the independent variable, it has facetiously been called "two-timing."

5.2. Near-resonant excitation of a linear oscillator

We are primarily interested in problems of continuum mechanics, which requires the solution of partial differential equations. However, to exhibit the method of multiple scales in its simplest form we borrow from Kevorkian (1966) the problem of an undamped linear oscillator started from rest. As the basis for a perturbation scheme, we suppose that the driving frequency α differs only slightly from the natural frequency $\omega = (k/m)^{\frac{1}{2}}$.

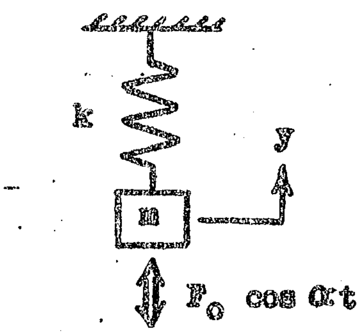


Fig. 5.1. Linear oscillator under sinusoidal excitation

We refer the displacement y to the static deflection F_o/k, and the time to the reciprocal of the natural frequency. Then with $\varepsilon = (\omega - \alpha)/\omega$, the full problem is

$$\ddot{y} + y = \cos(1-\varepsilon)t, \qquad y(0) = \dot{y}(0) = 0. \tag{5.1}$$

A straightforward perturbation expansion gives

$$y = \tfrac{1}{2}\Big[t \sin t - \tfrac{1}{2}\varepsilon(t^2 \cos t - t \sin t) - \tfrac{1}{12}\varepsilon^2(2\,t^3 \sin t + 3\,t^2 \cos t - 3\,t \sin t) + \dots\Big] \tag{5.2}$$

This is clearly invalid for large time, when $t = O(1/\varepsilon)$. The first term increases indefinitely in amplitude, corresponding to exact resonance; but the second and third terms grow to the same order as the first when $t = O(1/\varepsilon)$, and then y is also $O(1/\varepsilon)$. Furthermore, the nonuniformity is evident in even the first term, because the secular terms (sec. 2.9) $t \sin t$, $t \cos t$, etc., fail to display the periodicity that we know the exact solution must possess.*

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* It is strictly periodic if ε is a rational fraction, but in any case globally periodic, repeating itself after $t = 2\pi/\varepsilon$.

Of course in this simple problem the exact solution is well known:

$$y = \frac{1}{\varepsilon(2-\varepsilon)}\Big[(\cos\varepsilon t - 1)\cos t + \sin\varepsilon t \sin t\Big] \tag{5.3}$$

Expanding this formally for small ε (and also small εt) reproduces the perturbation series (5.2), and shows how its successive terms deviate from the true solution (Fig. 5.2).

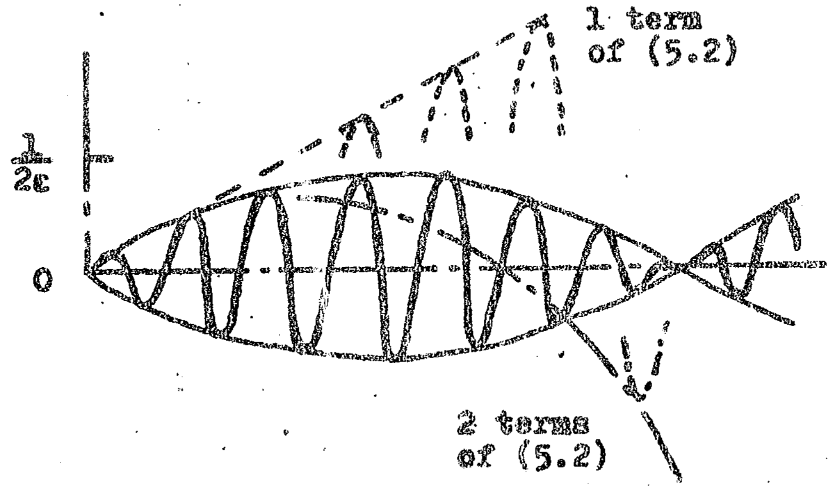


Fig. 5.2. Response of undamped linear oscillator to near-resonant excitation

Our previous experience might suggest that the nonuniformity can be removed by using the method of matched asymptotic expansions, introducing the contracted local variables Y and T, valid near the point at $T = \infty$, according to

$$y = \frac{1}{\varepsilon} Y(T;\varepsilon), \qquad T = \varepsilon t \tag{5.4}$$

However, this transforms the full differential equation to

$$\varepsilon^2 Y'' + Y = \varepsilon \cos\left(\frac{T}{\varepsilon} - T\right) \tag{5.5}$$

$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \cdots = 1 - \frac{T^2}{2}(\frac{1}{\varepsilon} - 1)^2$

which gives as a first approximation nothing more than the average value $Y = 0$. Matched expansions are inapplicable because this is not a boundary-layer problem.

Instead, we recognize that both the short time scale t and the long scale $T = \varepsilon t$ are simultaneously operative by setting

$$y \sim \frac{1}{\varepsilon} y_1(\tau,T) + y_2(\tau,T) + \varepsilon y_3(\tau,T) + \dots, \qquad \begin{matrix}\tau = t\\ T = \varepsilon t\end{matrix} \tag{5.6}$$

(We start the expansion with a term in $1/\varepsilon$ because, as implied by the straightforward expansion, y reaches that order of magnitude as a result of the near resonance. Failing to include this term would again bring secular terms into the following analysis.) Substituting into the differential equation and boundary conditions (5.1), calculating derivatives by the chain rule, and equating like powers of ε yields the sequence of problems

$$\frac{\partial^2 y_1}{\partial \tau^2} + y_1 = 0, \qquad y_1 = \frac{\partial y_1}{\partial \tau} = 0 \quad \text{at } \tau = T = 0$$

$$\frac{\partial^2 y_2}{\partial \tau^2} + y_2 = \cos(\tau - T) - 2\frac{\partial^2 y_1}{\partial \tau \partial T}, \qquad y_2 = \frac{\partial y_2}{\partial \tau} + \frac{\partial y_1}{\partial T} = 0 \quad \text{at } \tau = T = 0$$

$$\frac{\partial^2 y_n}{\partial \tau^2} + y_n = -2\frac{\partial^2 y_{n-1}}{\partial \tau \partial T} - \frac{\partial^2 y_{n-2}}{\partial T^2}, \quad y_n = \frac{\partial y_n}{\partial \tau} + \frac{\partial y_{n-1}}{\partial T} = 0 \quad \text{at } \tau = T = 0$$

$$(n = 3, 4, \ldots) \qquad (5.7)$$

We see that at each stage the slower variable appears only parametrically, so that these are effectively ordinary differential equations. The first has the general solution

$$y_1 = A(T) \cos \tau + B(T) \sin \tau, \qquad A(0) = B(0) = 0 \qquad (5.8)$$

It is typical of the method that the solution can be completed at any stage only by examining the problem for the next approximation (though it need not be completely solved). Thus we inspect the second-order equation, which becomes

$$\frac{\partial^2 y_2}{\partial \tau^2} + y_2 = \left[\cos T - 2B'(T)\right] \cos \tau + \left[\sin T + 2A'(T)\right] \sin \tau \qquad (5.9)$$

The non-homogeneous terms in $\sin \tau$ and $\cos \tau$ will in general produce secular terms in $\tau \cos \tau$ and $\tau \sin \tau$. The simplest way of suppressing that secularity is to annihilate the right-hand side by choosing

$$A'(T) = -\tfrac{1}{2}\sin T, \quad B'(T) = \tfrac{1}{2}\cos T . \qquad (5.10)$$

Then integrating, using the initial conditions on A and B found in the first approximation (5.8), gives

$$A(T) = \tfrac{1}{2}(\cos T - 1), \quad B(T) = \tfrac{1}{2}\sin T . \qquad (5.11)$$

Thus we have found the uniformly valid first approximation

$$y \sim \frac{1}{2\varepsilon}\left[(\cos T - 1) \cos \tau + \sin T \sin \tau\right] + O(1) \qquad (5.12)$$

or finally, restoring the original variable t,

$$y \sim \frac{1}{2\varepsilon}\left[(\cos \varepsilon t - 1) \cos t + \sin \varepsilon t \sin t\right]\left[1 + O(\varepsilon)\right] \qquad (5.13)$$

This is in obvious accord with the exact solution (5.3). In this simple example the process can be continued indefinitely to find successive terms in the expansion (5.6) which will, according to the exact solution, all be simple multiples of the first, giving

$$y \sim \frac{1}{2\varepsilon}\left[(\cos \varepsilon t - 1) \cos t + \sin \varepsilon t \sin t\right](1 + \tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon^2 + \ldots) \qquad (5.14)$$

5.3. The elements of the method of multiple scales

This simple example has illustrated all the essential features of the method of multiple scales. In contrast to a regular-perturbation problem, whose solution is an asymptotic expansion of the classical Poincare form (1.26), or a boundary-layer problem, whose solution can be represented by two such expansions in overlapping regions with correspondingly different scales, the solution here (like a composite expansion) is a <u>generalized</u> <u>asymptotic</u> <u>expansion</u> (sec. 3.4), of the form

$$y \sim \sum_{n=1} \varepsilon^{n-2} y_n(t; \varepsilon) \tag{5.15}$$

We have remarked that such an expansion is far from being unique, so that further rules of procedure are required. Here the basic rule is that the dependence of the y_n on the parameter ε is only through the slower scale εt, so that the expansion specializes to

$$y \sim \sum_{n=1} \varepsilon^{n-2} y_n(t, \varepsilon t) \tag{5.16}$$

Further rules are required to specify the procedure. In particular, one faces the decision whether to write coefficients in terms of the short scale, the long scale, or both. That is, the forcing term in (5.1) can be written variously as

$$\cos(1-\varepsilon)t = \cos(\tau-\varepsilon\tau),\quad \cos(\tau-T),\quad \cos\left(\frac{T}{\varepsilon}-\varepsilon\tau\right),\quad \cos\left(\frac{T}{\varepsilon}-T\right) \tag{5.17}$$

We made without comment the simplest choice — the second of these — and the reader can verify that it alone yields a uniform result.

This might suggest that the coefficients are to be rewritten in terms of the two scales in such a way as to eliminate all explicit dependence on ε. However, in some problems (cf. Exercise 5.3) this would make the first approximation the full solution, so that no approximation was achieved. At present, we can only propose the somewhat unsatisfactory rule that coefficients may be left unchanged except where that leads to nonuniformity.

Thus we may summarize the method of multiple scales, in its simplest form, as follows:

1. **Introduce, in addition to the original dimensionless coordinate, a second slower (or faster) independent variable equal to the first multiplied (or divided) by ε.**

2. **Thereafter (until the approximation is complete), regard these two coordinates as entirely independent, calculating derivatives by the chain rule.**

3. **Leave coefficients in terms of the original coordinate except where that is found to lead to nonuniformity.**

4. **Expand the solution in powers of ε, substitute into the problem, equate like powers of ε, and solve the resulting simpler problems in succession.**

5. **At each stage determine the functions of integration, which depend on the slower coordinate, by requiring that the next term in the expansion be uniformly smaller — hence no more singular — than its predecessor.**

The variety of physical problems that have been treated in the literature using the method of multiple scales show many variations on this basic scheme. It can be applied to a space as well as a time variable (in which case it is natural to speak of short and long, rather than fast and slow, variables). The procedure applies to partial as well as ordinary differential equations -- of elliptic, parabolic, and hyperbolic type. Then the nonuniformity usually arises with respect to only one of the coordinates, and it alone is subdivided into two disparate scales. The ratio of the two scales may be an integral or fractional power of ε, or some more general gauge function; and likewise the perturbation series may proceed in other than integral powers of ε. It may also happen that more than two scales are required, just as for concentric boundary layers (sec. 4.10).

More important, although the simple scheme outlined above usually suffices for a first approximation, it often fails in higher approximations. It must be replaced by a more elaborate version of the method of multiple scales, of which several have been devised. We discuss these in their order of increasing complexity.

5.4. An unlimited number of simple scales

Only rarely in practice do two simple scales suffice to eliminate nonuniformity to arbitrarily high order. Either more scales are needed, or they are not both simple. Cole (1968) and Nayfeh (1973) illustrate this with the simple example of a slightly damped linear oscillator (Fig. 5.3). In dimensionless variables, the equation of motion may be written (with the factor 2 inserted for convenience)

$$\ddot{y} + 2\varepsilon\dot{y} + y = 0 \qquad (5.18)$$

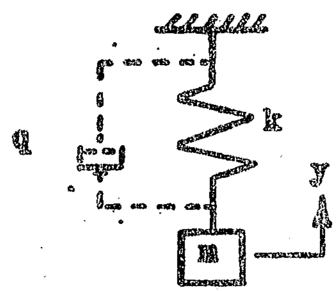


Fig. 5.3. Slightly damped linear oscillator

which has the general solution

$$y = c\,e^{-\varepsilon t}\cos\left(\sqrt{1-\varepsilon^2}\,t + p\right) \qquad (5.19)$$

The straightforward perturbation solution for small ε is

$$y = c\left[\cos(t+p) - \varepsilon t\cos(t+p) + \tfrac{1}{2}\varepsilon^2\left\{t^2\cos(t+p) + t\sin(t+p)\right\} + \dots\right] \qquad (5.20)$$

Again this is clearly invalid when $t = O(1/\varepsilon)$, with a pattern of secular terms similar to (5.2). This would suggest using the two simple scales $\tau = t$ and $T = \varepsilon t$ as before; but that leads, in the third approximation, to

$$y = c\,e^{-T}\left[\cos(\tau+p) + \tfrac{1}{2}\varepsilon^2\tau\sin(\tau+p)\right] \qquad (5.21)$$

The residual secularity here suggests introducing a third, still slower, time scale $T_3 = \varepsilon^2 t$. Then two approximations later a fourth scale is required, and so on indefinitely.

Thus we introduce, in general, an unlimited number of successively slower (or longer) scales,

$$T_1 = t, \quad T_2 = \varepsilon t, \quad T_3 = \varepsilon^2 t, \quad T_4 = \varepsilon^3 t, \ldots \tag{5.22}$$

Because derivatives are then found from

$$\frac{d}{dt} = \frac{\partial}{\partial T_1} + \varepsilon \frac{\partial}{\partial T_2} + \varepsilon^2 \frac{\partial}{\partial T_3} + \ldots \tag{5.23}$$

Sturrock (1957), who invented this procedure, called it the <u>derivative-expansion method</u>. That name is retained by Nayfeh (1973), who gives the most comprehensive treatment of the method of multiple scales, with three variants, of which this is the first.

5.5. One simple, one slightly stretched scale

Although the preceding treatment of the slightly damped oscillator is formally adequate, it is conceptually unsatisfactory because we recognize that the motion does not actually depend on an unending sequence of successively slower scales, but only two. The difficulty is that one of those two scales, $(1-\varepsilon^2)^{\frac{1}{2}} t$, depends on ε in a way sufficiently complicated that it cannot be discerned in advance. Hence another possibility is to find that scale step-by step in the course of solution, by setting

$$\begin{aligned} \tau &= t(1 + a\varepsilon^2 + b\varepsilon^3 + c\varepsilon^4 + \ldots) \\ T &= \varepsilon t \end{aligned} \tag{5.24}$$

and determining the free constants a, b, c, ... so as to suppress secularity.

This procedure, invented by Cole & Kevorkian (1963), is the only version of multiple scales used in the book of Cole (1963), and is the second of the three variants described in the book of Nayfeh (1973). Note that no linear term in ε is required when the faster scale τ is stretched, because it appears in the slower scale.* In some problems, on the other hand, it is the slower scale that requires a slight stretching, in which case we set

$$\begin{aligned} \tau &= t \\ T &= \varepsilon t(1 + A\varepsilon + B\varepsilon^2 + C\varepsilon^3 + \ldots) \end{aligned} \tag{5.25}$$

---

* In our problem of the slightly damped oscillator, it is clear from the exact solution (5.19) that all the other odd powers of ε will also disappear.

Of course the coordinate may have originally been scaled differently so that, for example, in stretching the faster scale we must set

$$\tau = \frac{t}{\varepsilon}(1 + a\varepsilon^2 + b\varepsilon^3 + c\varepsilon^4 + \ldots)$$
$$T = t \qquad (5.26)$$

5.6. Two slightly stretched scales

Neither Cole nor Nayfeh mentions that in some problems (e.g. Peyret 1970) both scales need to be stretched slightly. A simple example involving a partial differential equation (taken from the 1969 final examination) is the damping of plane sound waves by viscosity. Stokes studied this problem in 1845 using a linearized form of his newly developed equations of viscous motion:

$$u_{tt} - c^2 u_{xx} = \frac{4}{3}\nu u_{xxt} . \qquad (5.27)$$

Suppose that waves are generated at the origin of x by a plane piston oscillating sinusoidally (Fig. 5.4). Then it is consistent with the linearization to transfer the boundary condition to the origin (section 2.5), giving

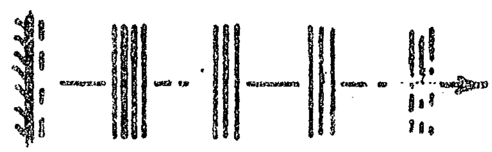


Fig. 5.4. Plane waves from oscillating piston

$$u = U \cos \omega t \quad \text{at} \quad x = 0 \qquad (5.28)$$

Referring the velocity to the speed U of the piston, distance to c/ω (rather than the viscous length ν/c, which is very small), and time to $1/\omega$ yields the dimensionless problem

$$u_{tt} - u_{xx} = \varepsilon u_{xxt} , \qquad \varepsilon \equiv \frac{4}{3}\frac{\nu\omega}{c^2}$$
$$u(0,t) = \cos t \qquad (5.29)$$

Since we will approximate by first neglecting the dissipation, we will also need the Sommerfeld radiation condition of waves moving in only the positive x-direction.

A straightforward perturbation solution for small ε gives

$$u = \cos(t-x) - \tfrac{1}{2}\varepsilon x \cos(t-x) + \tfrac{1}{8}\varepsilon^2\left[x^2\cos(t-x) - 3x \sin(t-x)\right] + \ldots \qquad (5.30)$$

and this shows nonuniformity at large distances, where $x = O(1/\varepsilon)$. Applying our primitive version of the method, using the two simple scales $\xi = x$, $X = \varepsilon x$, yields the uniform first approximation

$$u = e^{-\frac{1}{2}\varepsilon x} \cos(t-x) + O(\varepsilon) \qquad (5.31)$$

This result inspired Rayleigh's observation (Theory of Sound, vol. 2, p. 316) that

> The mellowing of sounds by distance, as observed in mountainous countries, is perhaps to be attributed to friction, by the operation of which the higher and harsher components are gradually eliminated.

In higher approximations, further nonuniformities arise. They can be eliminated either by introducing additional scales (sec. 5.4) or by slightly stretching the two used above. That both the short and the long abscissa must be stretched is clear from the exact solution, which is easily found to be

$$u = e^{-\alpha x}\cos(t-\beta x) \tag{5.32a}$$

where

$$\left.\begin{aligned}\beta^2-\alpha^2 &= \frac{1}{1+\varepsilon^2}\\ 2\alpha\beta &= \frac{\varepsilon}{1+\varepsilon^2}\end{aligned}\right\}\qquad \begin{aligned}\alpha &= \tfrac{1}{2}\varepsilon - \tfrac{5}{16}\varepsilon^3 + \tfrac{63}{128}\varepsilon^5 + \dots\\ \beta &= 1-\tfrac{3}{8}\varepsilon^2 + \tfrac{35}{128}\varepsilon^4 - \dots\end{aligned} \tag{5.32b}$$

Thus we would in general set

$$\begin{aligned}\xi &= x(1+a\varepsilon^2+b\varepsilon^3+c\varepsilon^4+\dots)\\ X &= \varepsilon x(1+A\varepsilon+B\varepsilon^2+C\varepsilon^3+\dots)\end{aligned} \tag{5.33}$$

and proceed as before. In this particular example, of course, it is clear from the exact solution that only the even and odd powers of ε are needed in ξ and X respectively.

5.7. A posteriori stretching

Benney & Newell (1967), following Pritulo (1962), point out that a slight stretching of one or more scales can be imposed *a posteriori* on a perturbation expansion if it is found to be nonuniform. The coefficients in the stretching are determined by requiring that the solution, rewritten in terms of the stretched coordinate, be uniformly valid.

As an example, we reconsider the slightly damped linear oscillator (sec. 5.4). Using the two simple scales t and $T = \varepsilon t$ gave the nonuniform third approximation (5.21)

$$y = c\,e^{-T}\left[\cos(t+p) + \tfrac{1}{2}\varepsilon^2 t\sin(t+p)\right] \tag{5.34}$$

Now instead of introducing the slightly strained fast variable (5.24)

$$\tau = t(1+a\varepsilon^2+\dots) \tag{5.35}$$

into the problem, we may avoid duplication of effort by simply introducing it into this form of the solution. Reverting the series to find $t = \tau(1-a\varepsilon^2+\dots)$, substituting into (5.34), and expanding systematically for small ε gives

$$y = ce^{-\tau}\left[\cos(\tau + p) + (\tfrac{1}{8} + a)\,\varepsilon^2\tau\sin(\tau + p)\right] + O(\varepsilon^4) \qquad (5.36)$$

The secular term is eliminated by choosing $a = -\frac{1}{8}$. Thus, restoring the original variable, we have the uniformly valid third approximation

$$y = ce^{-\varepsilon t}\cos(t + p - \tfrac{1}{8}\varepsilon^2 t) \qquad (5.37)$$

which is just the result of the method of section 5.5.

References

(additional to those cited in previous chapters)

BENNEY, D. J. & NEWELL, A. C. 1967 Sequential time closure for interacting random waves. J. Math. Phys. 46, 363-393

COLE, J. D. & KEVORKIAN, J. 1963 Uniformly valid asymptotic approximations for certain nonlinear differential equations. Nonlinear Differential Equations and Nonlinear Mechanics (J. P. LaSalle & S. Lefschetz, eds.), Academic, N.Y., 113-120

KEVORKIAN, J. 1966 The two variable expansion procedure for the approximate solution of certain nonlinear differential equations. Lectures in Applied Mathematics 7; Space Mathematics, part III, 206-275, Amer. Math. Soc.

PEYRET, R. 1970 Etude de l'écoulement d'un fluide conducteur dans un canal par la méthode des echelles multiples. J. Mécanique 9, 61-97

PRITULO, M. F. 1962 On the determination of uniformly accurate solutions of differential equations by the method of perturbation of coordinates. Prikl. Mat. Mekh. 26, 661-667; English transl. 444-448

STURROCK, P. A. 1957 Non-linear effects in electron plasmas. Proc. Roy. Soc. A 242, 277-299

WINGATE, R. T. & DAVIS, R. T. 1970 Perturbation solution of a hyperbolic equation governing longitudinal wave propagation in certain nonuniform bars. J. Acoust. Soc. Amer. 47, 1334-1337

EXERCISES

5.1. Plane waves in slowly changing environment by multiple scales. Using the method of multiple scales, reconsider the problem of Exercise 2.4:

$$\frac{\partial}{\partial x} F(\epsilon x)\frac{\partial u}{\partial x} - F(\epsilon x)\frac{\partial^2 u}{\partial t^2} = 0\,,$$ $u = \cos t$ at $x = 0$, waves propagating in positive x-direction.

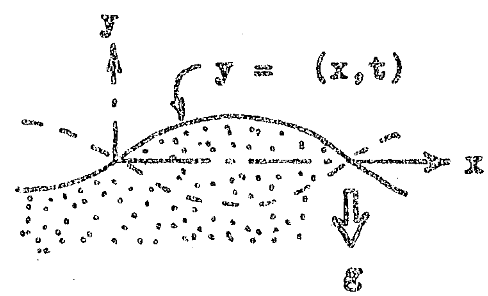
Using two simple space scales, show that the uniformly valid first approximation reproduces the previous result based on the assumption that energy is conserved. Proceed to the second approximation, confirming the statement of Wingate & Davis (1970) that "higher approximations can be obtained by continuing in a similar manner." Show that using Nayfeh's scheme of keeping the simple long scale, but slightly straining the short scale in a nonuniform way according to

$$\xi = x + \epsilon g(\epsilon x) + \dots$$

yields the second approximation as a single term, so that for $F(\epsilon x) = 1 + \epsilon x$ the solution is

$$u = \frac{1}{\sqrt{1+\epsilon x}} \cos\left[x + \frac{\epsilon^2 x}{8(1+\epsilon x)} - t \quad + O(\epsilon^2)\right]$$

5.2. Effect of viscosity on standing deep-water waves. For small kinematic viscosity ν the motion of water waves is approximately irrotational. Hence the velocity may be taken as the gradient of a velocity potential ϕ that satisfies the Laplace equation in the space coordinates. For plane motion, with the upward displacement of the free surface from its mean position given by $y = \eta(x,t)$, the first-order boundary conditions at the surface are



$$\left.\begin{array}{ll} \text{(surface normal stress)} & g\eta + \phi_t = 2\nu\phi_{xx} \\ \text{(surface flux)} & \eta_t - \phi_y = 0 \end{array}\right\} \text{ at } y = 0$$

In deep water the velocity must also decay with depth. (a) Neglecting viscosity, consider standing waves in deep water that are periodic in both space and time, so that $\eta = a \sin kx \cos ct$, where a is the (small) maximum amplitude. Relate c to k, and calculate $\phi(x,y,t)$. (b) Show that including the small effects of viscosity by means of a straightforward perturbation scheme leads to nonuniformity in time. (c) Apply the method of multiple scales to suppress the nonuniformity and so obtain a uniformly valid first approximation.

5.3. The earth-moon-spaceship problem revisited. The one-dimensional nonrotating earth-moon-spaceship problem (sec. 4.8) was

$$2\left(\frac{dt}{dx}\right)^2 = \frac{1}{\frac{1-\varepsilon}{x}+\frac{\varepsilon}{1-x}}, \qquad t(0) = 0$$

This was solved by matching with an inner solution on the magnified scale $X = (1-x)/\varepsilon$. Find a uniformly valid first approximation instead by using both scales simultaneously. (This example illustrates that it is not always possible, in suppressing the nonuniformity, simply to annihilate the nonhomogeneous terms in the higher-order equations.)