Department of Mechanical and Materials Engineering
Florida International University
FRACTURE MECHANICS - EGM 6570
Spring 2016

Instructor:
Dr. Cesar Levy
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Class Schedule:
Initial meeting will be 1230-145pm Jan 12
Room: please be at the first meeting EC1116

TEXTBOOK:

Anderson, T.L., *Fracture Mechanics: Fundamentals and Applications*, 3nd Ed., CRC Press (2005).
Notes handed out in class

RECOMMENDED LITERATURE:

Broek, D., *Elementary Engineering Fracture Mechanics,* Kluwer Academic Publishers (1987).
Timoshenko, S.P., and Goodier, J.N., *Theory of Elasticity*, McGraw Hill (1970).
Hellan, K., *Introduction to Fracture Mechanics*, McGraw-Hill (1985).
Cherepanov, G.P. *Methods of brittle Fracture*, McGraw-Hill, (1979).

EXAMS AND GRADES:

2 exams	(25% each)
Writing Assignment	(25%)
Final Exam	(25%)

GRADING POLICY:

95-100	A	75-80	B-	55-60	D
90-95	A-	70-75	C+	55 & below	F
85-90	B+	65-70	C		
80-85	B	60-65	C-		

COURSE CONTENTS:

1. Introduction to Fracture Mechanics, Conventional Design Criteria, Structural Failure in the Past
2. Theoretical Fracture Strength, Crack Modes, Fracture at Stresses Below Theoretical Fracture Strength, Griffith Contribution
3. Energy Principles, Elastic Crack Tip Study, Review of Theory of Elasticity, 2-D Elasticity,
4. Mode III solution, Definition of Tractions, Solution of Laplace Equation using Complex Variables
5. Cauchy-Riemann Equations and Mode III continued, near and far field solutions, energy of deformation
6. Energy of Deformation for Mode III continued, Energy of Deformation for a finite body using Superposition, Crack Extension Force, Stress Intensity Factor
7. Crack Extension Force, Stress Intensity Factor for Mode III (continued), Plane Stress, Plane Strain Problems, Mode I problem, Mode II problem,
8. Crack Extension Force, SIFs for Different Configurations, Compliance Change due to Crack, Constant SIF samples
9. LEFM, Griffith-Irwin Analysis, Crack Tip Plasticity, Plastic Zone Shapes, Stress Redistribution,
10. Fracture Toughness Testing (FTT), Relation Between Crack Extension Resistance Curves and Fracture Surfaces, Pop-in
11. Practical Aspects of FTT, Plate Thickness, Instrumentation for Fracture, Crack Opening Displacement
12. Sample Shapes, Introduction to the J Integral
13. J Integral Continued
14. J. Integral (Mode III), Eshelby Derivation of J Integral, Theoretical Basis for Measurement of J Integral
15. Experimental Determination of J Integral, J Integral for Elastic-Plastic Materials, J Integral as Fracture Criterion

16. J Integral to Describe Crack Tip Singularities (linear elastic materials, power law hardening materials), Elastic Viscous Analogy, Nonlinear Viscous Materials, Plastic Fracture Mechanics -- Dugdale Mushkelishvili Model
17. Dugdale Mushkelishvili Model (continued), Mechanisms of Crack Nucleation
18. Crack Nucleation and Slip Bands, Inclusions, Cottrell Theory of Brittle Fracture and Ductile Brittle Transition in Steels
19. Cottrell Theory of Brittle Fracture and Ductile Brittle Transition in Steels (continued), Ductile Brittle Transition Temperature (DBTT)
20. Grain Size Dependence of DBTT, Fatigue and Fatigue Design
21. Plastic Flow and Fatigue, cyclic effect, Mechanics of Fatigue Crack Growth,
22. Random Loading on Fatigue Crack Growth (FCG) Rate, Microscopic Aspects of FCG
23. FCG in Ductile Materials, Wertman Theory
24. Wertman Theory (Continued), Comparison to FCG rate
25. Hydrogen Embrittlement,
26. Thermodynamics of Hydrogen Embrittlement

Final Exam is will be announced

Note: This syllabus may be changed during the course of the semester. All changes will be announced in class.

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Fracture of Solids

Class Notes

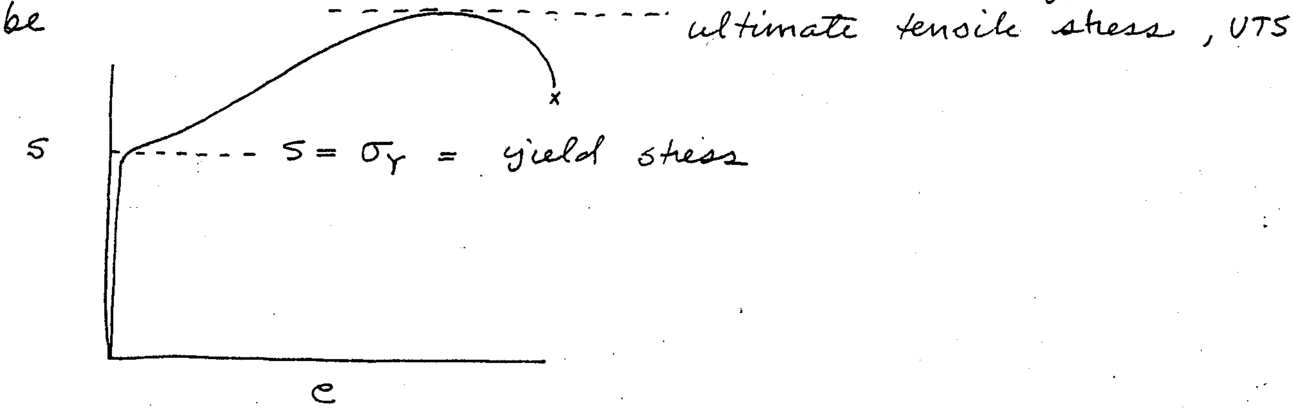
I Introduction to Fracture

Conventional Design Criteria and the Occurance of Fracture

As a first guess it would seem that the most direct way to assess the fracture properties of a given structural material would be to perform a tension test to failure and evaluate the results.

a) Ductile Metal

the engineering stress – strain curve would typically be



we typically record and tabulate:

σ_Y = yield stress ~ measure of shear resistance not directly related to fracture or breaking

UTS = ultimate tensile strength ~ measure of plastic instability (strain hardening competes with geometrical softening)

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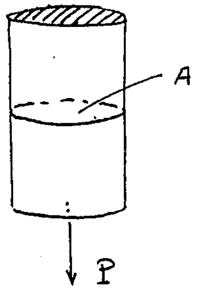


for a ductile metal

$$P = \sigma A$$

$$dP = \underset{\text{(strain hardening)}}{A\,d\sigma} + \underset{\text{(geometrical softening)}}{\sigma\,dA}$$

$d\sigma$ +ive $\qquad$ dA − ive

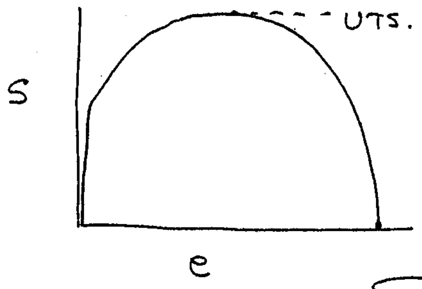
between yield stress and UTS, $|A\,d\sigma| > |\sigma\,dA|$

at UTS $\quad dP = A\,d\sigma + \sigma\,dA = 0$

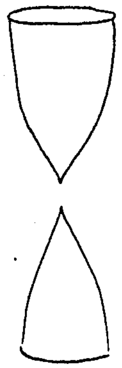
consequently, UTS has <u>nothing</u> to do with breaking in a ductile metal.

But breaking does occur eventually. How?

1) For metal with <u>unlimited</u> ductility : <u>shear failure</u>, sample would draw to a point at which time the load would have dropped to zero.



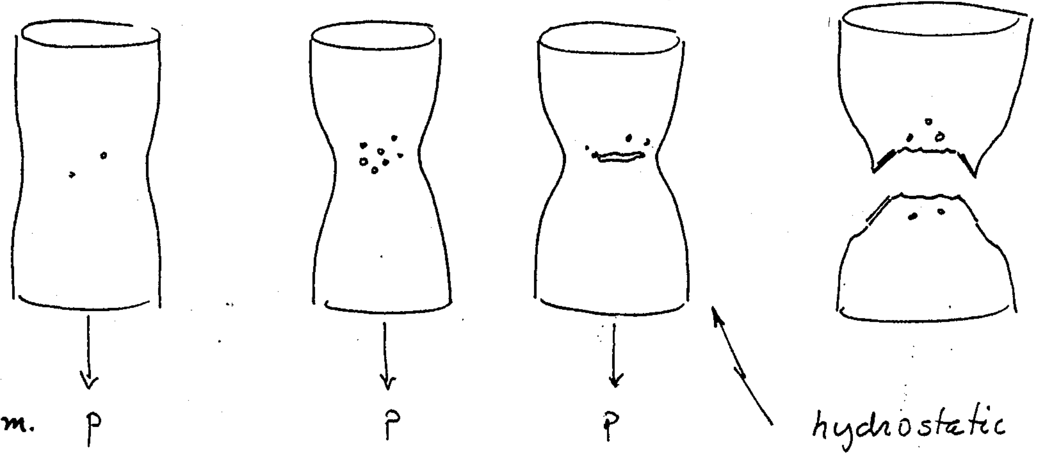
In this idealized case failure occurs entirely by plastic instability – called <u>shear failure</u>



sample failed by shear failure.

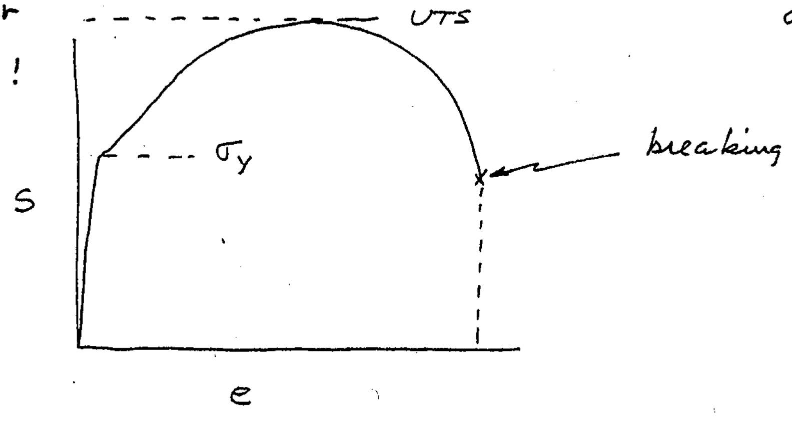
2) For Metal with limited ductility: <u>cup and cone failure</u>

failure begins to occur in the necked region by the formation and coalesence of voids. These voids grow under the hydrostatic tension in the neck, ultimately forming a crack. The crack grows until the remaining crossection cannot support the load.



hydrostatic stresses arise in the neck here because of the shape of the <u>neck</u>.

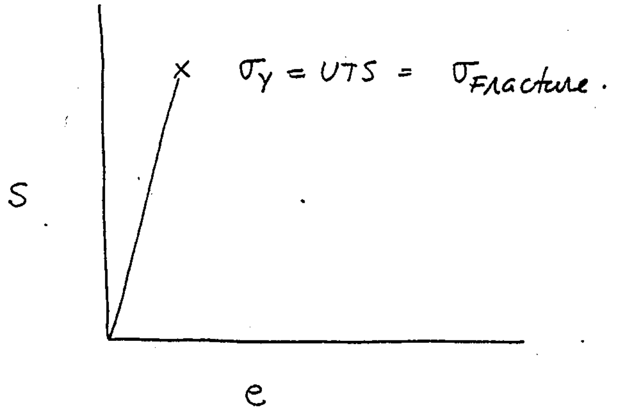
void form. could occur before UTS!



Conclusion: for ductile metal σy, UTS unrelated to fracture, fracture is complex in tensile test. (We will also see that the tensile test is poor test for evaluating fracture properties because the the tensile bar is smooth and free to deform. Most applications involve notches or cracks where constraints will limit the extent of plastic flow possible.).

(b) Brittle Material

Here the tensile test has a simple result. Essentially, $\sigma_Y = UTS = \sigma_{Fracture}$.



In this case the measured value of UTS has little practical value as it depends sensitively on surface preparation etc.

The usual practice of designing on the Yield Stress or UTS (with appropriate safety factors) is ordinarily sucessful because:

1) for metals with some ductility, design stresses are usually very very low so that fracture does not occur

2) Very Brittle materials are usually not subjected to tension.

But, as we continue to push the strengths of structural materials towards their theoretical strengths (through structural control), catastrophic failures will become more common.

also, structural failures often become human tragedies of major proportions.

Structural Failures of The Past

The societal impact of some of the more important structural failures of the past are summarized by E.R. Parker, Brittle Behavior of Engineering Structures, Wiley (1957)

1) Early observation of Brittle Fracture of Steel.

paper by Nathaniel Barnaby, "The use of Steel in Naval Construction" J. Brit. Iron and Steel Inst. (1879)

> "Recent cases have occurred of fracture in Bessemer bars ... from some trifling blow or strain – they nearly all took place during the late severe weather at Chatham"

Discussion by Kirk ... Steel plates, when cold, on being thrown down, split right up. Pieces cut from each side of the split stood all the Admiralty tests"

2) Molasses Tank Explosion — Boston, Mass. 1919

12 persons drowned in Molasses (some horses too)

40 injured

here calc. stress ~ 26 ksi, actual stresses around rivit holes ~ 50 ksi UTS = 55 ksi, stress too high

3) Methane Storage Tank Explosion — Cleveland 1944

one of most disasterous brittle fractures in history

128 persons killed

$ 7×10^6 damage

plate not properly heat treated

4) World War II Merchant Ships

5000 built — 1000 cracked by 1946

200 serious damage

9 T-2 Tankers, 7 Liberty ships

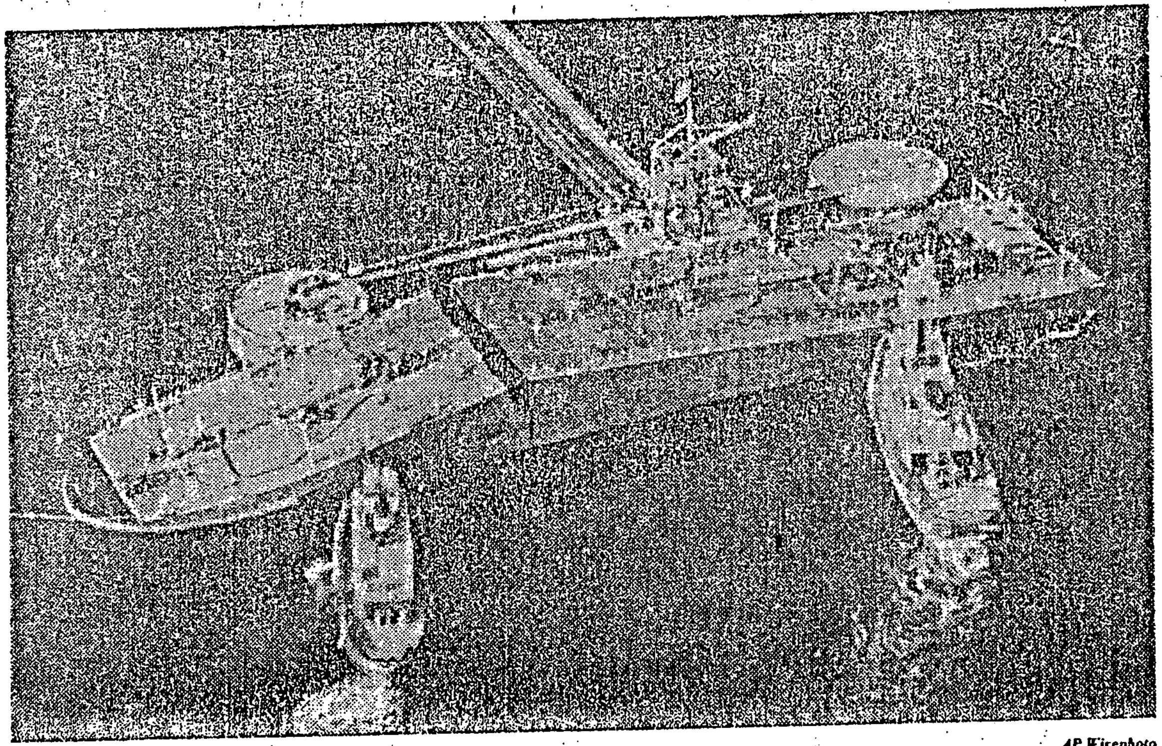
sustained complete fracture!

cargo ships continued breaking thru 50's and 60's even though profession sensitive to problem in 1947.

Tank Barge I.O.S. 3301 — built in 1971 584 ft.

fractured in calm seas in 1972!

A Slippery Split



AP Wirephoto

Coast Guard oil spill experts yesterday were standing by a 340-foot oil barge that split in half Monday night as it was being loaded at the Atlantic Terminal Corp. docks in Newington, N.H. The barge was loaded with about one million gallons of oil when it broke in half. Officials said that little of the oil had leaked out of the barge, the Bouchard 105.

FINAL S.F. SCHOOL PLAN

From Page 1

result of declining school enrollments.

The revised plan includes a grade school to a sixth-through-eighth grade "middle" school.

Under that plan, the elementary children would have then been eight will be bused to the Pelton Middle School along with some students from the Bayview/Hunters Point area," Alioto said at a press briefing prior to the board

SPIES

From Page 1

bond. They are to be formally arraigned today in Alexandria.

The indictment charges Humphrey and Truong with having conspired with a number of Vietnamese officials to deliver to Hanoi clasified State Department communications "relating to the national defense of the United States."

At the time of his arrest Humphrey was a program evaluator for the information agency, the Justice Department said, and held the rank of Foreign Sercie Information Officer 4. He served in South Vietnam from 1969 and 1971 and did two tours of duty in West Germany.

Truong, generally known as David Truong, was admitted to the United States in 1964 to study economics at Stanford University, where he graduated in 1968.)

He majored in economics and political science, the Justice Department said, and had applied for admission to the United States as a permanent resident alien.

His employer at the Animal Health Institute, which is a few blocks from Humphrey's office, said Truong was paid about $9000 a year and was also a part-time graduate student in economics at George Washington University. The institute represents the manufacturers of medicines for animals.

Exact information about Truong's parents was unavailable, but Fritz Kessinger, his employer, said that according to Truong they still live in the southern part of Vietnam and he had hoped to visit them in Vietnam next Christmas.

The indictment said that Hum-

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5) Bridge Failures

1962 Kings Bridge, Melbourne, Aust.

Brittle Fracture, 40°F.

1967 Dec 15 Point Pleasant Bridge, West. Va.

46 dead

6) 1973 Molasses Tank Explosion, Bellview N.J.

March 22 1973! What a mess.

So we still have not got the problem licked.

7) Many examples of fatigue failure

Helicopter Crash LAX 1967 etc.

8) Most Serious - yet to happen - Supertanker Fracture!

These examples indicate why it is important to develop a better understanding of Mechanics and Mechanisms of Fracture. To learn how to design for fracture resistance in materials.

Reading Assignment:

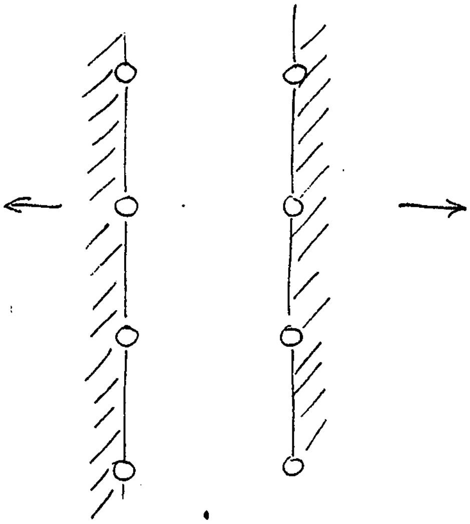
E.R. Parker, Brittle Behavior of Engineering Structures, Wiley (1957) p. 253 - 272

Hertzberg, p 229 - 232

Rolfe and Barsom, p 1-7

Theoretical Fracture Strength (Frenkel ~ 1920)

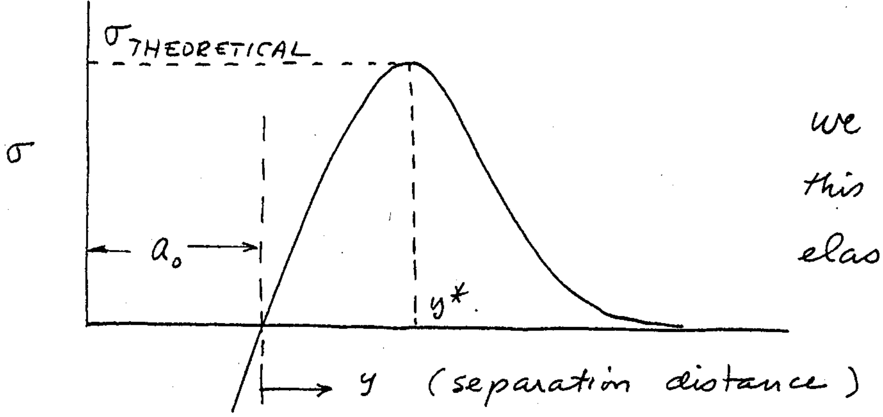
We wish to estimate the upper limit of tension stress which can be supported by a crystal.



Imagine that a crystal is to be pulled apart along a crystal plane.

the stress builds up then falls off exponentially: assume

$$\sigma = A\, y \exp(-By)$$



we wish to match this relation to the elastic modulus.

$$E = \frac{\Delta\sigma}{\Delta\epsilon} = \frac{\Delta\sigma}{\Delta l/l} = \frac{l\,\Delta\sigma}{\Delta l} = a_o \frac{\Delta\sigma}{\Delta y}$$

$$\left.\frac{d\sigma}{dy}\right)_{y=0} = A\exp(-By) - BAy\exp(-By)\Big]_{y=0} = A$$

$$E = a_o \left.\frac{d\sigma}{dy}\right)_{y=0} = a_o A \quad \text{hence} \quad \boxed{A = \frac{E}{a_o}}$$

now we match the force law to the surface energy:

$$2\gamma_s = \int_0^\infty \sigma\, dy = A\int_0^\infty y\exp(-By)\,dy$$

can integrate by parts and get

$$2\gamma_s = \frac{A}{B^2} \quad ; \quad \boxed{B = \sqrt{\frac{E}{2\gamma_s a_o}}}$$

We want $\sigma_{Theoretical}$ which is σ evaluated at $\frac{d\sigma}{dy} = 0$.

$$\frac{d\sigma}{dy} = A \exp(-By) - BAy\exp(-By) = 0.$$

$$(1 - By^*) A \exp(-By) = 0 \quad ; \quad y^* = \frac{1}{B}$$

hence

$$\sigma_{Theoret} = \frac{E}{a_o} y^* \exp(-1) = \frac{E}{a_o B e}$$

but $\frac{1}{B} = \sqrt{\frac{2\gamma_s a_o}{E}}$ so

$$\sigma_{Theoret} = \frac{1}{e}\frac{E}{a_o}\sqrt{\frac{2\gamma_s a_o}{E}} = \frac{1}{e}\sqrt{\frac{2\gamma_s E}{a_o}}$$

$$\boxed{\sigma_{Theoret} \simeq 0.5\sqrt{\frac{\gamma_s E}{a_o}}}$$

Typical values:

$E \approx 10^{12}$ dynes/cm^2

$\gamma_s \approx 10^3$ ergs/cm^2

$a_o \approx 3 \times 10^{-8}$ cm

$\sigma_{Theoret} \approx 0.9 \times 10^{11}$ dynes/cm

$$\sigma_{Theoret} \approx \frac{E}{10}.$$

or $\frac{E}{\sigma_{theoretical}} \approx 10$ – compare this with table (some whiskers indeed show theoretical strength).

Table 2.2
σ_f = Maximum
Observed Strength [11]

Material	(psi × 10^{-6})	E(psi × 10^{-6})	E/σ_f
Music wire	0.4	20	72
Silica fibers	3.5	14	4
Silica rods	1.9	14	7
Iron whiskers	1.9	43	23
Al_2O_3 whiskers	2.2	72	33
NaCl whiskers	0.16	6.3	40
BeO whiskers	2.8	49	17
Silicon whiskers	0.94	24	26
Silicon (bulk)	0.75	24	32
TiC (bulk)	0.80	70	87
Boron	0.35	51	145
Ausformed steel	0.45	29	64

Most structural materials fail at stresses well below $\sigma_{Theoretical}$, because ƒ stress concentrations associated with notches and cracks.

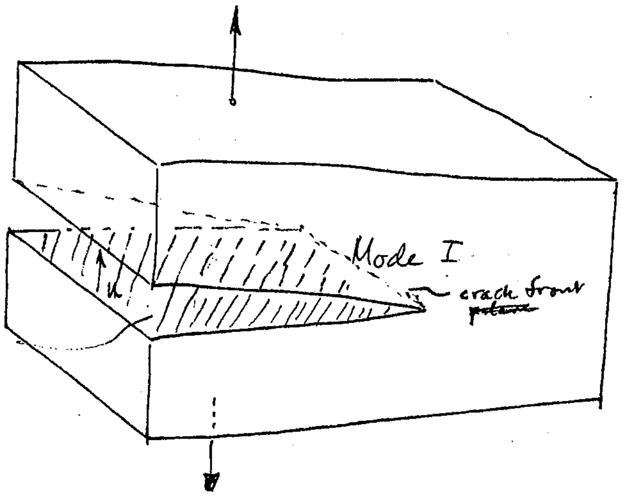
Crack Modes

crack geometry as related to applied stress.

Mode I Crack Opening Mode

$u \perp$ crack plane.

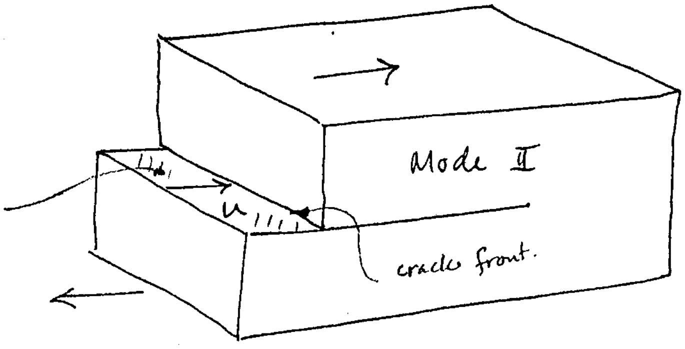
$u \perp$ crack front



<ins>Mode II</ins> Shear Mode

u ⊥ crack front

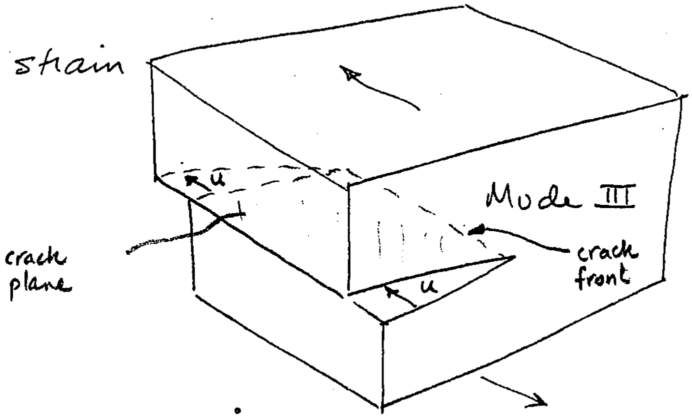
u ∥ crack plane



<ins>Mode III</ins> Anti-plane strain

u ∥ crack front

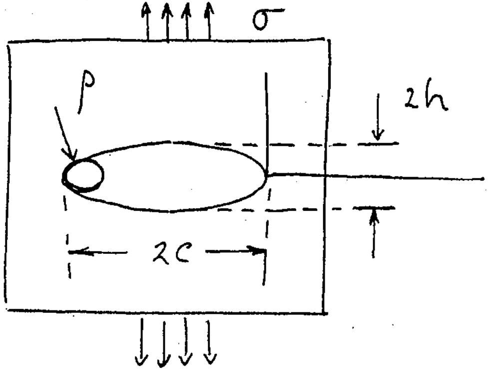
u ∥ crack plane

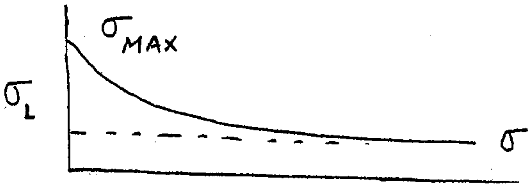


Fracture at Stresses Below $\sigma_{Theoretical}$

Fracture occurs at $\sigma < \sigma_{Theoretical}$ because cracks concentrate the stress so that $\sigma_{Local} = \sigma_{Theoret.}$ even though the nominal stress remains well below $\sigma_{Theoretical}$.

Inglis, Trans. Inst. Nav. Arch. <ins>55</ins> pt. I 219 (1913) stress concentration for elliptical hole in a thin plate (plane stress - to be discussed later).





Inglis solution : $\sigma_M = \sigma\left(1 + \frac{2c}{h}\right)$

For an ellipse : radius of curvature at tip

$$\rho = \frac{h^2}{c} \qquad h = \sqrt{c\rho}$$

hence

$$\sigma_M = \sigma\left(1 + 2\sqrt{\frac{c}{\rho}}\right)$$

and when $c >> \rho$ (sharp crack)

$$\sigma_M \rightarrow 2\sigma\sqrt{\frac{c}{\rho}} .$$

Now we may assume that fracture can occur only if (valid only for perfectly elastic matl)

$$\sigma_M \geq \sigma_{Theoretical} = 0.5\sqrt{\frac{\gamma_s E}{a_0}}$$

or when

$$2\sigma\sqrt{\frac{c}{\rho}} \geqslant 0.5\sqrt{\frac{\gamma_s E}{a_0}}$$

$$\sigma = \frac{1}{2}\sigma_M\sqrt{\frac{\rho}{c}} \geqq 0.25\sqrt{\frac{\gamma_s E}{a_0}}\sqrt{\frac{\rho}{c}}$$

now if $\rho \rightarrow 0$, $\sigma \rightarrow 0$. But this is unrealistic because

$$\rho \rightarrow a_0 \quad \text{(as a lower limit)}$$

so that for sharp cracks

$$\sigma = 0.25\sqrt{\frac{\gamma_s E}{c}}$$ (really only approximate).

This condition is necessary for cleavage (perfectly brittle) fracture but not sufficient. We must require that when crack grows, the applied stress is sufficient to provide the necessary work to create the two surfaces. – analysis above implies no limit in principle; ie as $\rho \to 0$ $\sigma \to 0$, but this ignores energetics.

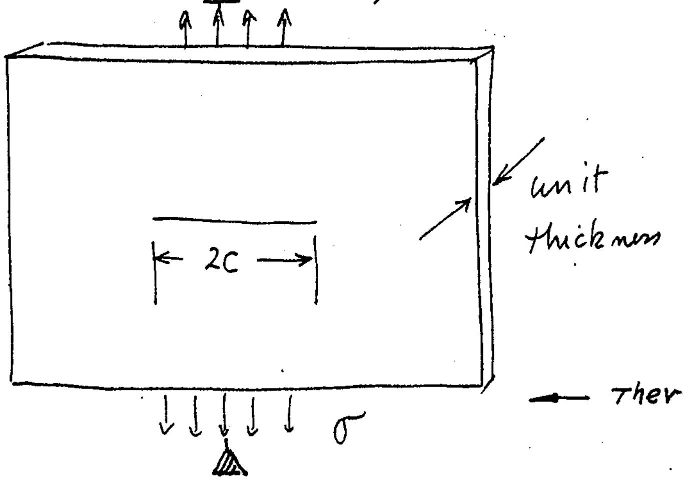
<u>Great Contribution of A. A. Griffith</u>,

"The Phenomena of Rupture and Flow in Solids"
Phil. Trans. of Roy. Soc. 221 A (1920) p 163

Journal Ref. 15.

Reprinted in Trans ASM, <u>61</u>, 861 (1968)
(Metallurgical Classics).

Following Treatment innovated by Griffith. (errors in Griffiths original work (corrected by Gilman) will be discussed later when we deal with crack elasticity.



thin plate (plane stress).
containing through slit of length 2c.

(Mode I)

Let U_o be potential energy of uncracked plate under stress.

when crack of length 2c is formed the potential energy of the system becomes

$$U_{Total} = U_o + \Delta U_s + \Delta U_{el} + \Delta U_{wk}$$

ΔU_s potential energy associated with two surfaces created

ΔU_{el} increase in elastic energy of the body (work is evidently done by the stress σ when the crack forms (and opens))

ΔU_{wk} increase in potential energy of work reservoir (actually U_{wk} is -ive)

For plane stress $\boxed{\Delta U_{el} = \frac{\pi c^2 \sigma^2}{E}}$ (we will derive the Mode III equivalent of this later)

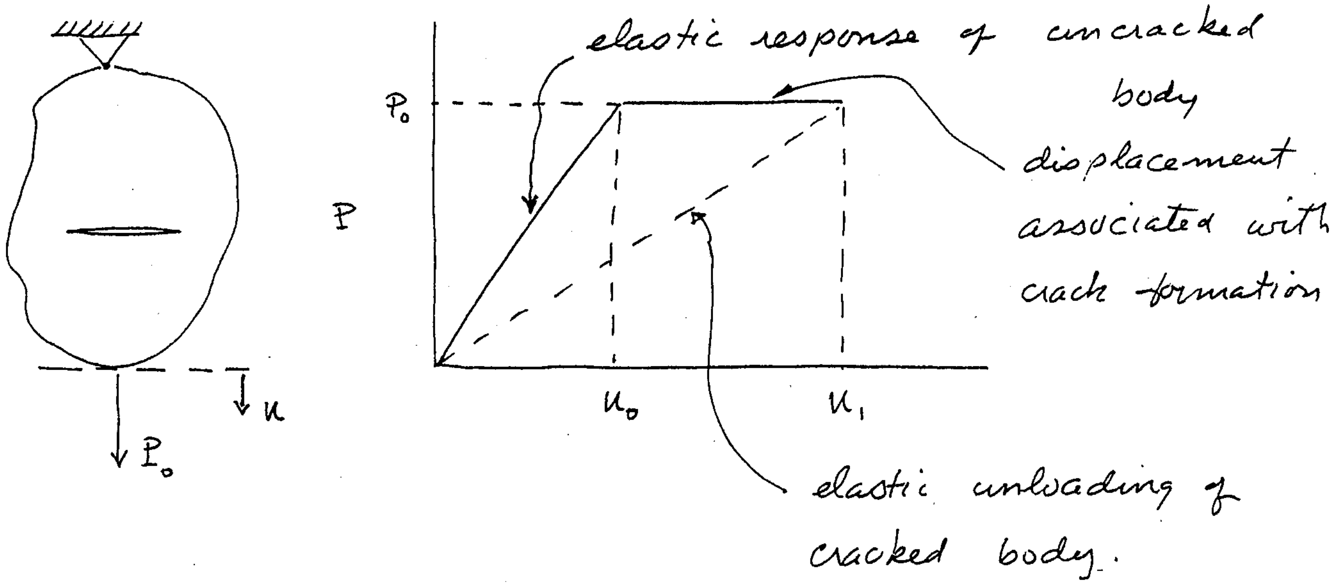
$$\boxed{\Delta U_s = 2 \gamma_s 2c = 4 \gamma_s c}$$

What is ΔU_{wk}?

we will make a simple argument here to show that

$$\Delta U_{wk} = -2\,\Delta U_{el}$$

Consider a <u>linear</u> elastic body:



Now $\Delta U_{wk} = -P_o(u_1 - u_o) =$ − work absorbed by sample when crack grows.

But

$$\Delta U_{el} = \tfrac{1}{2} P_o u_1 - \tfrac{1}{2} P_o u_o$$ (change in strain energy)

So

$$\Delta U_{el} = \tfrac{1}{2} P_o (u_1 - u_o).$$

Consequently

$$\boxed{\Delta U_{wk} = -2\,\Delta U_{el}}$$

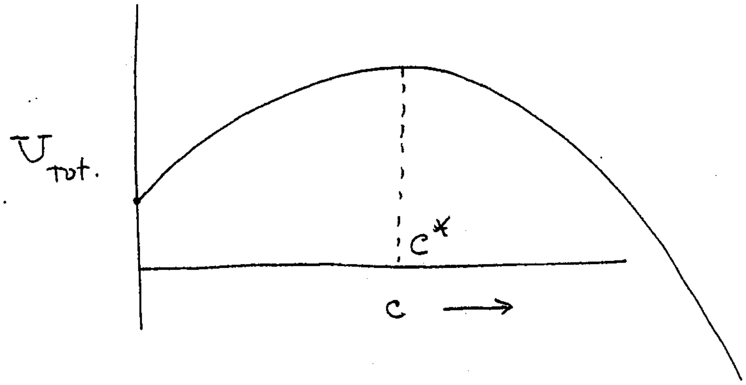
Now the total potential energy of the system is

$$U_{total} = U_0 + \underbrace{4\gamma_s c}_{\text{related to } R \cdot 2c} + \overbrace{\Delta U_{el} \quad - 2\,\Delta U_{el}}^{\text{related to } G_I \cdot 2c}$$

$$= U_0 + 4\gamma_s c - \frac{\pi c^2 \sigma^2}{E}$$

G_I crack extension force
energy release rate
R crack resistance force

This potential varies in the following way with c



For $c > c^*$, unstable crack growth can occur.

$$\frac{\pi c^2 \sigma^2}{E} = \frac{\pi \cancel{c^2}}{\cancel{E}} \cdot \frac{2\gamma_s \cancel{E}}{\cancel{\pi c}} = 2\gamma_s c$$

$$\frac{\partial U_{tot}}{\partial c} = 4\gamma_s - \frac{2\pi\sigma^2 c}{E} = 0$$

hence $\boxed{\sigma = \sqrt{\frac{2\gamma_s E}{\pi c}}}$ the Famous Griffith Eqnn.

Generally, for sharp cracks $\rho \approx a_0$, the local stress at the crack tip reaches the theoretical stress $\sigma_{theoret}$ before the Griffith condition is satisfied. Hence the Griffith condition is both necessary and sufficient for unstable crack growth.

2. $B \geq 2.5 \left(K_{Ic}/\sigma_y \right)^2$: This requirement arises from the consideration that we want only MODE I type fracture

Proof:

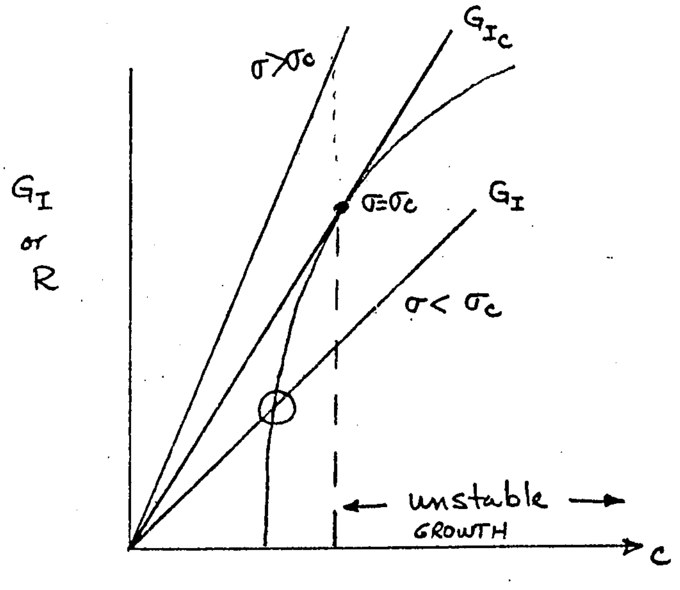
- As has been shown to you in class in order for cracks to propagate for perfectly brittle materials, the crack extension force $G_I = 2\gamma_s$, where γ_s is the surface energy. However for materials that deform plastically, then crack extension will only occur when $G_I = 2\gamma_s + p$ where p is the plastic work of crack extension. p is not a constant and depends on the size of the plastic zone, σ_y, the work hardening rate, etc AND they all in turn depend on the crack length.
- if we define $R \equiv$ crack extension resistance $= 2\gamma_s + p$, then for unstable growth we must have that

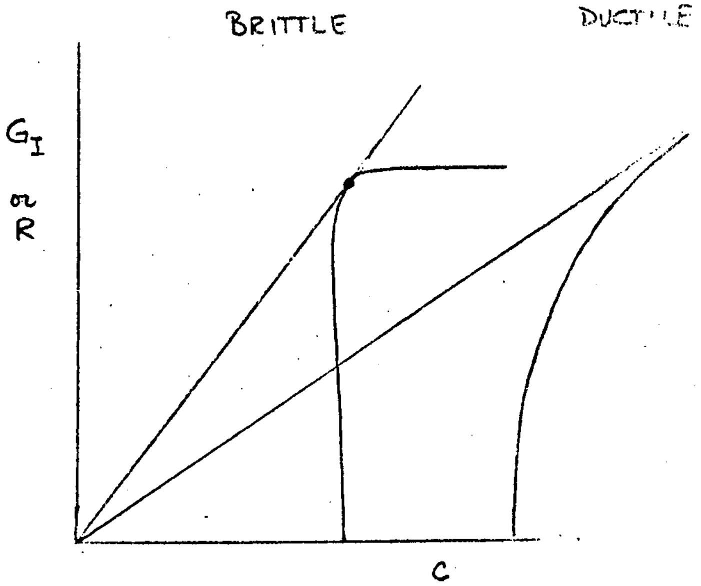
$$G_I \geq R$$

and also

$$\partial G_I/\partial c \geq \frac{\partial R}{\partial c}$$

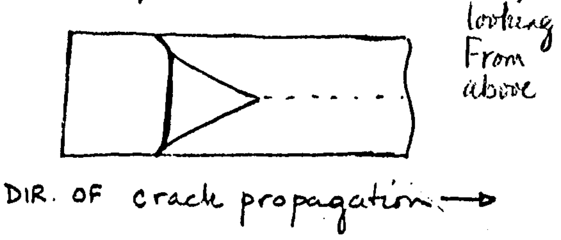
Thus if we remember that $G_I = \dfrac{\sigma^2 \pi c}{2\mu}(1-\nu)$ and look at a typical G_I versus c curve,

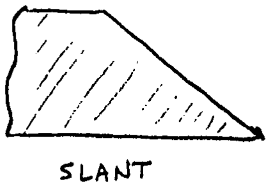




Note that the brittle material shows little plastic deformation and has a well defined $G_I = R$ point of intersection and occurs below the ductile material point of intersection

What has been found is that as the plate is made thinner the R curve will vary and will no longer have a distinct intersection point. The reason for this is the growth of "shear lips" from the free surface and the thinness of the plate (plane stress conditions).





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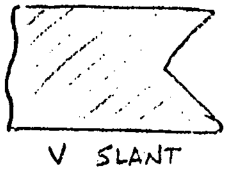
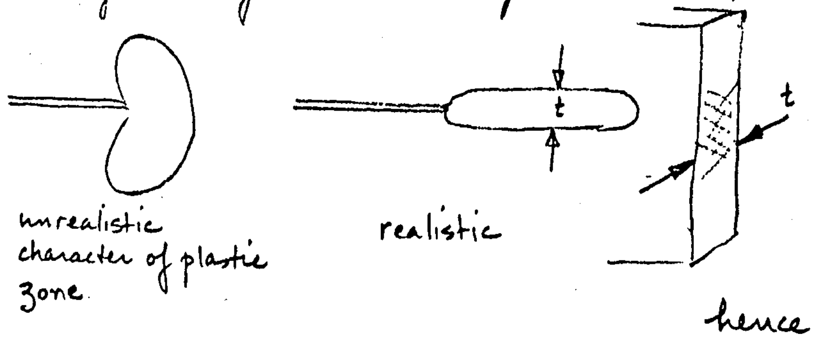
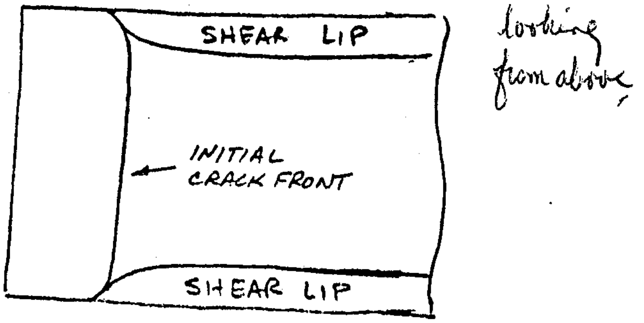
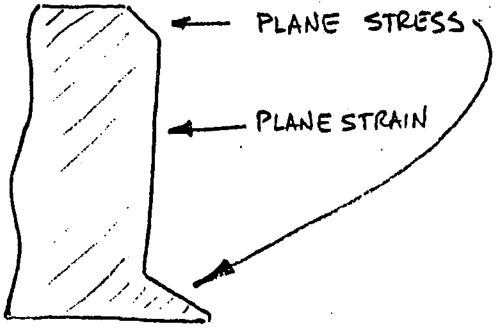


The growth of the shear lips is due to the plastic zone being constrained in the thickness direction. So it will spread in front of the crack tip. The mechanism that will cause crack extension will be due to failure in <u>shear</u> (mode III); hence we see the slant formation.

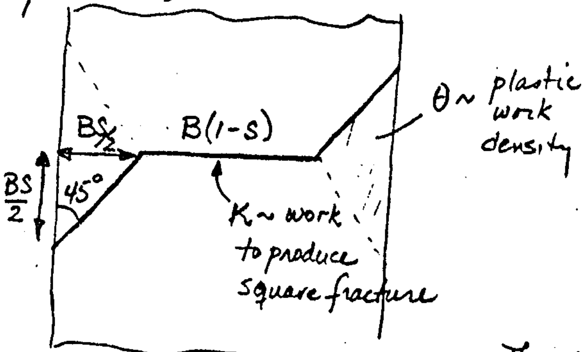


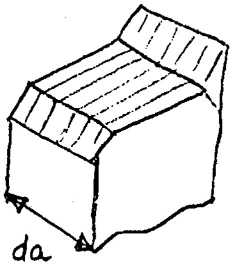
As the plate width is increased, the formation of the shear lips is reduced due to the plane strain effect and the cross-section will look like this





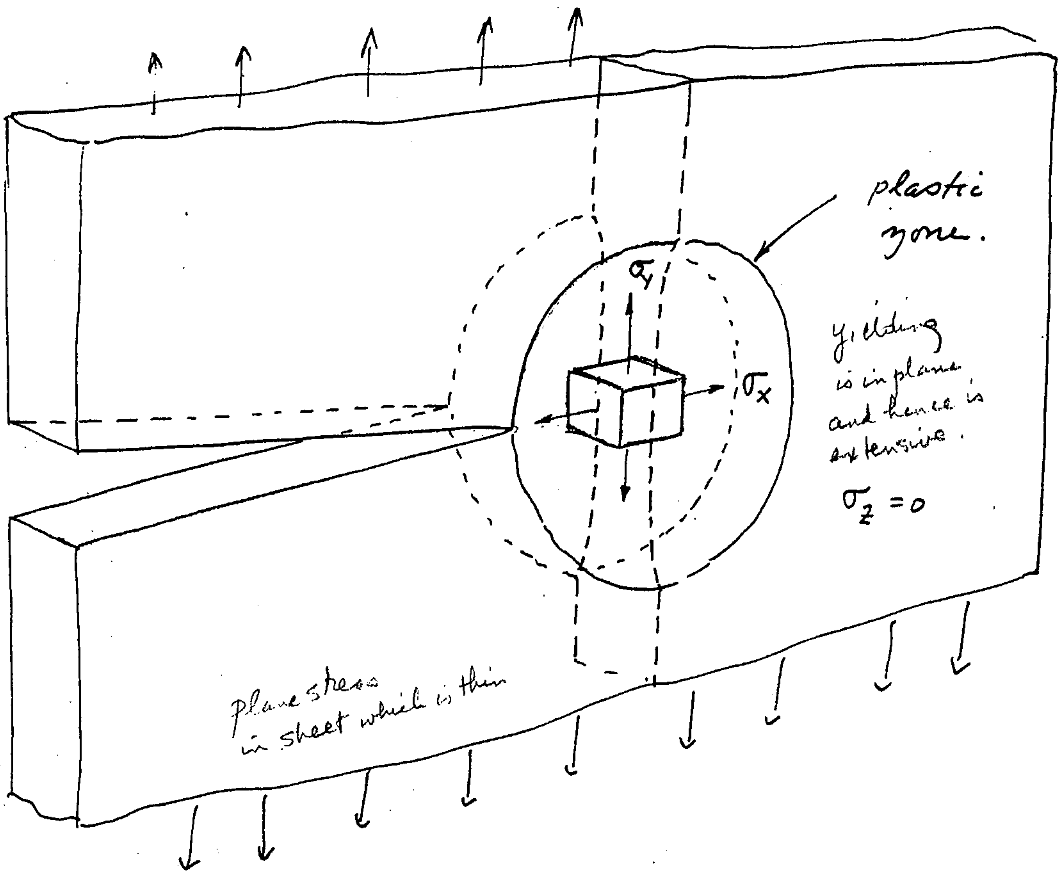
Many have proposed models to describe what occurs here. One such model is that of Krafft, Sullivan and Boyle (1961) modified by Bluhm





1. square fracture $\neq f(c_0)$
2. shear lips are assumed to occur at 45°
3. flat fracture is a surface phenomenon
4. Shear lip is volumetric

The work done to create the crack surface da is:

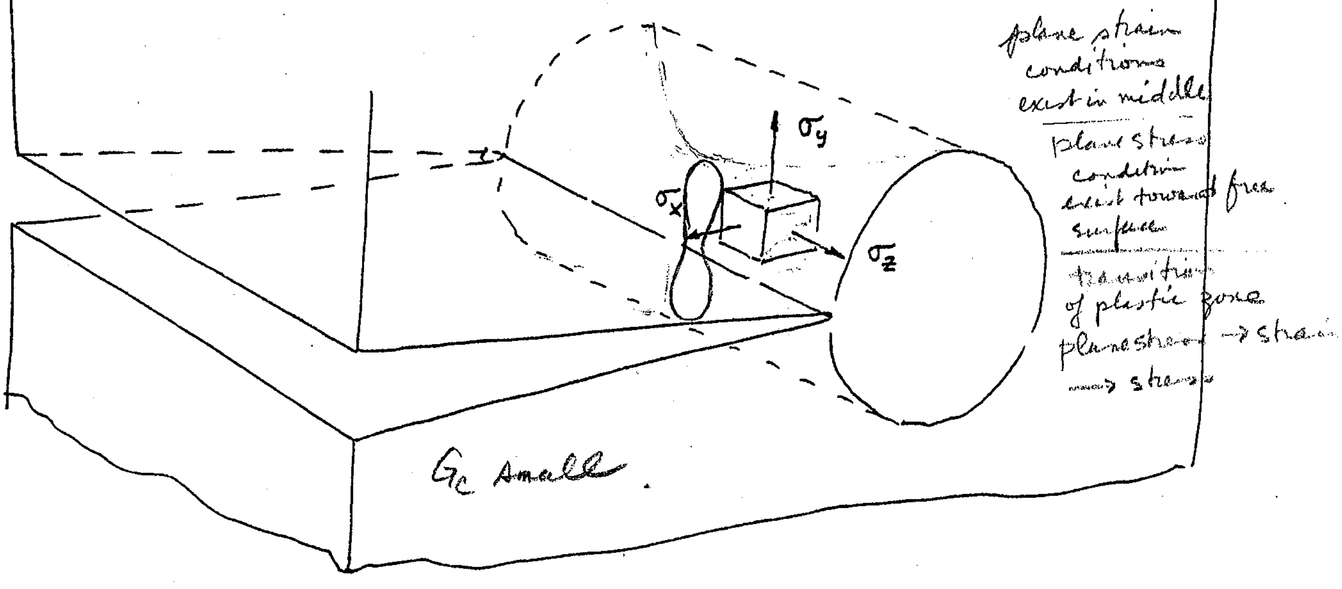


yielding is extensive in this case because shear stresses can be large

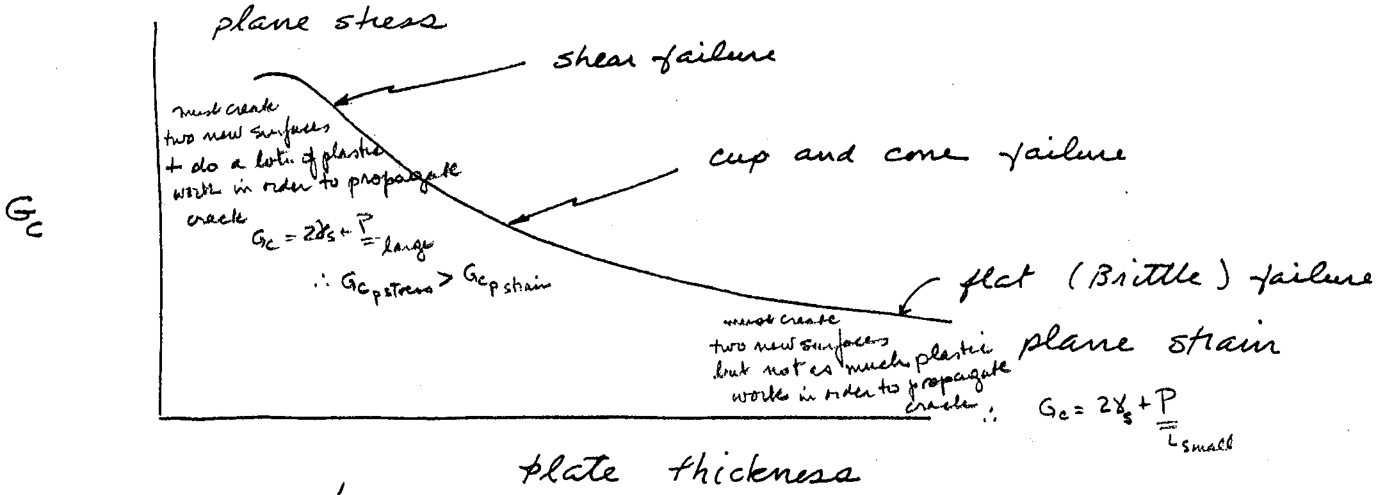
$$\frac{\sigma_y - \sigma_z}{2} = \frac{\sigma_y - 0}{2} = \frac{\sigma_y}{2}$$

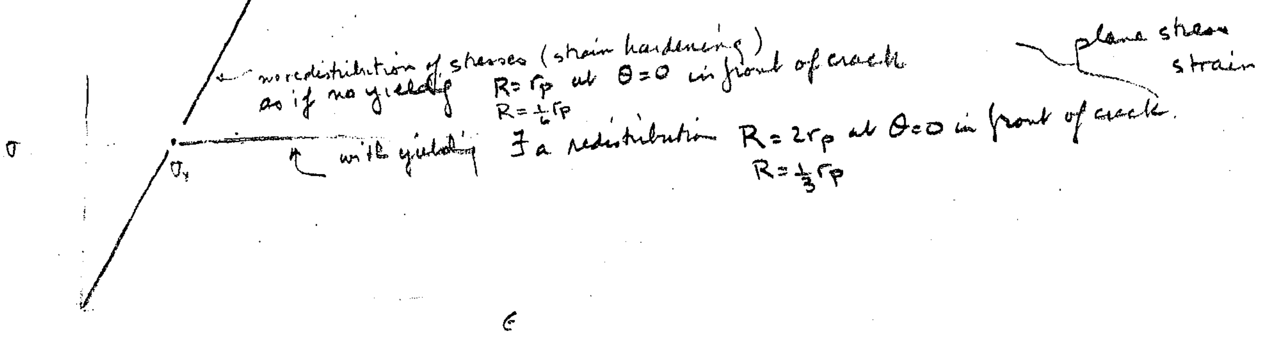
materials generally tough in plane stress.

<u>Plane Strain</u> : Consider crack in a <u>thick plate</u>, then the plastic zone at the crack tip is under <u>plane strain</u> : $\epsilon_z = 0$



Relation to Fracture Mode (Qualitative)

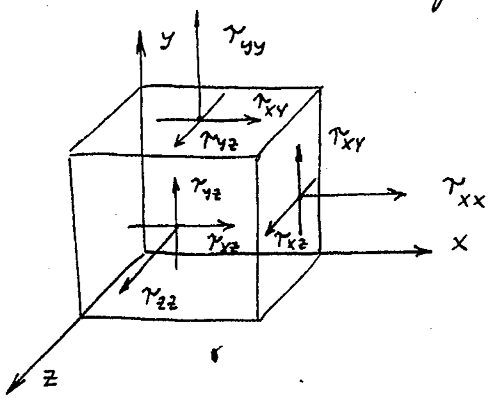




II Elastic Properties of Cracks

We use the theory of elasticity to determine the properties of cracks. The following quantities and relations are fundamental to the theory of elasticity. Any elementary book on elasticity would give these relations

Stress (cartesian system)



stress tensor $\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$

Equilibrium

$\Sigma F = 0$ on any element leads to following set of equilibrium equations

$$\frac{\partial \tau_{ij}}{\partial x_j} = 0 \qquad i = x, y, z$$

where repeated j means sum over $j = x, y, z$.

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$\vdots$ etc.

Strain

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad e_{ij} = \begin{pmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{pmatrix}$$

where u_i $i = 1, 2, 3$ are displacements.
x, y, z (used interchangably).

<u>Hookes Law</u> (Linear Elasticity (Isotropic))

$$e_{xx} = \frac{1}{E}\left[\tau_{xx} - \nu(\tau_{yy} + \tau_{zz}) \right]$$

E = Youngs Modulus
ν = Poisson's Ratio.

$\vdots$ etc.

$$e_{xy} = \frac{1}{2\mu} \tau_{xy}$$

μ = shear modulus.

$\vdots$ etc.

<u>Two Dimensional Elasticity</u>. (for solving Modes I, II, III cracks).
(τ_{ij}, e_{ij} $\neq f(z)$) ie $\frac{\partial}{\partial z}$(anything) = 0

a) Modes I, II: Plane problems of elasticity, principal plane remains planar. e_{xz}, $e_{yz} = 0$
hence τ_{xz}, $\tau_{yz} = 0$

$$\frac{\partial \tau_{ij}}{\partial x_j} = 0$$

Equilibrium equations become

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} = 0$$

with Hookes Law and the strain definitions these equations may be reduced to

$$\nabla^2(\tau_{xx} + \tau_{yy}) = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\tau_{xx} + \tau_{yy}) = 0$$

This can be obtained by starting with the compatibility relation

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} - 2\frac{\partial^2 e_{xy}}{\partial x \partial y} = 0$$

which is obviously satisfied considering the strains

$$e_{xx} = \frac{\partial u_x}{\partial x} \qquad e_{yy} = \frac{\partial u_y}{\partial y} \qquad e_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

With Hookes law (plane stress)

$$\frac{\partial^2}{\partial y^2}\left\{\frac{1}{E}\left[\tau_{xx} - \nu\,\tau_{yy}\right]\right\} + \frac{\partial^2}{\partial x^2}\left\{\frac{1}{E}\left[\tau_{yy} - \nu\,\tau_{xx}\right]\right\} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}$$

From the equilibrium relations:

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \tau_{xx}}{\partial x^2}$$

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \tau_{yy}}{\partial y^2}$$

so

$$\frac{1}{E}\left\{\frac{\partial^2 \tau_{xx}}{\partial y^2} - \nu\frac{\partial^2 \tau_{yy}}{\partial y^2}\right\} + \frac{1}{E}\left\{\frac{\partial^2 \tau_{yy}}{\partial x^2} - \nu\frac{\partial^2 \tau_{xx}}{\partial x^2}\right\} = \frac{1}{2\mu}\left(-\frac{\partial^2 \tau_{xx}}{\partial x^2} - \frac{\partial^2 \tau_{yy}}{\partial y^2}\right)$$

$$\frac{\partial^2 \tau_{xx}}{\partial y^2} - \nu\frac{\partial^2 \tau_{yy}}{\partial y^2} + \frac{\partial^2 \tau_{yy}}{\partial x^2} - \nu\frac{\partial^2 \tau_{xx}}{\partial x^2} + (1+\nu)\frac{\partial^2 \tau_{xx}}{\partial x^2} + (1+\nu)\frac{\partial^2 \tau_{yy}}{\partial y^2} = 0$$

hence

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\tau_{xx} + \tau_{yy}) = 0 \qquad \text{QED.}$$

1) Plane stress :

$$\tau_{zz} = 0 \quad ; \quad e_{zz} = \frac{1}{E}\left\{ -\nu (\tau_{xx} + \tau_{yy}) \right\}$$

2) Plane Strain

$$e_{zz} = 0 \quad ; \quad \tau_{zz} = \nu (\tau_{xx} + \tau_{yy})$$

For either plane stress or plane strain, the equilibrium equations above are satisfied by The general Airy stress function method. Pick ϕ so that

$$\tau_{xx} = \frac{\partial^2 \phi}{\partial y^2} \quad ; \quad \tau_{yy} = \frac{\partial^2 \phi}{\partial x^2} \quad ; \quad \tau_{xy} = - \frac{\partial^2 \phi}{\partial x \partial y}$$

Then the compatibility equations are satisfied if ϕ is a solution to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0 \quad ; \quad \nabla^4 \phi = 0$$

$\nabla^4 \phi = 0$ is bi-harmonic equation for stress function ϕ.

For problems of Plane stress, plane strain, one finds ϕ which also satisfies boundary conditions to problem.

This method is used by P.C. Paris and G. C. Sih, "Stress Analysis of Cracks", ASTM STP 381 (1964) p 30 symposia ref. # 5.

b) Mode III Anti-Plane Strain - first $\frac{\partial}{\partial z}(\text{anything}) = 0$ (2-D)

$u_1 = u_2 = 0$ $u_3 = u_3(x,y)$ only $\tau_{zx}, \tau_{zy} \neq 0$

$u_x = u_y = 0$ $u_z = u_z(x,y)$

Equilibrium equation is $\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0$

$\left(\frac{\partial \tau_{ij}}{\partial x_j} = 0\right)$

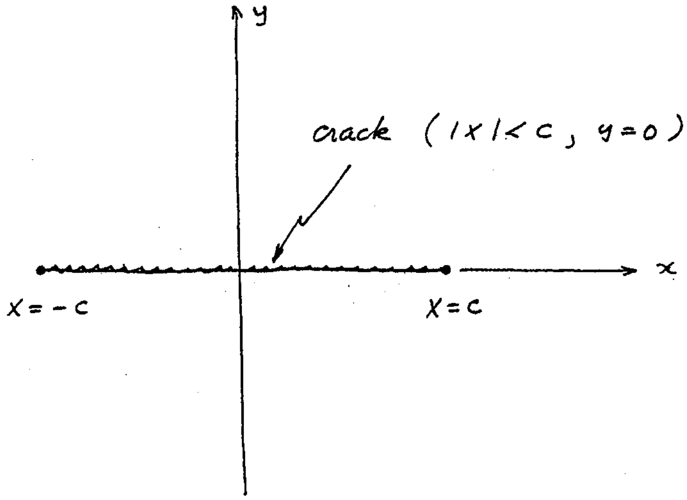
since $\tau_{zx} = 2\mu\, e_{zx} = \mu\left(\cancel{\frac{\partial u_x}{\partial z}} + \frac{\partial u_z}{\partial x}\right) = \mu \frac{\partial u_z}{\partial x}$

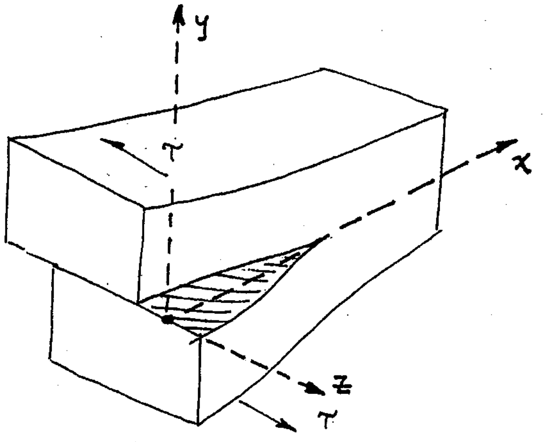
$$\tau_{zy} = 2\mu\, e_{zy} = \mu\left(\frac{\partial u_z}{\partial y} + \cancel{\frac{\partial u_y}{\partial z}}\right) = \mu \frac{\partial u_z}{\partial y}$$

then equilibrium equation becomes

$$\mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2}\right) = \mu\left(\nabla^2 u_z\right) = 0 \qquad \boxed{\nabla^2 u_z = 0}$$ La Places Eqn.

Stress Analysis of Mode III Crack (Barnett Analysis).





Boundary Conditions :

(i) crack surface traction free $\tau_{ij} \cdot n_j = 0 \Rightarrow \tau_{zy} = 0$ ($|x| < c$, $y \to 0$)

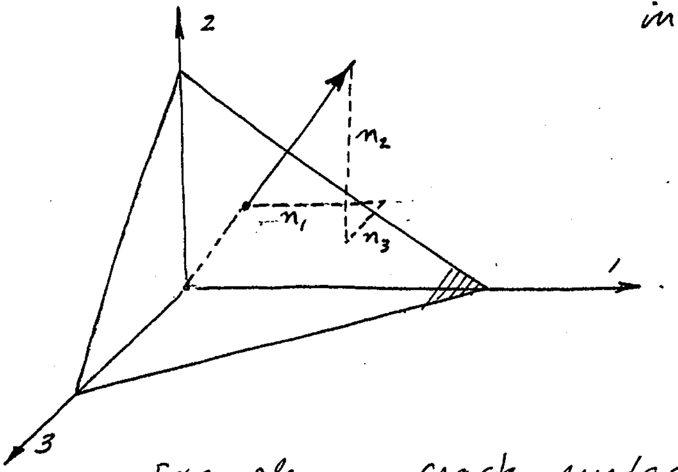
(ii) $\tau_{zx} \to 0$ as $x^2 + y^2 \to \infty$ $\tau_{zy} \to -\tau$ as $x^2 + y^2 \to \infty$

a note on traction stress boundary conditions

Free surface boundary conditions are expressed as

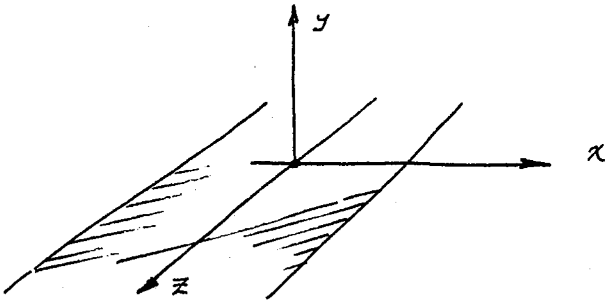
$$\tau_{ij}\, n_j = 0$$

where τ_{ij} is the stress tensor and n_j is the unit normal to the surface. The repeated index j implies



$$\tau_{11}\, n_1 + \tau_{12}\, n_2 + \tau_{13}\, n_3 = 0$$
$$\tau_{21}\, n_1 + \tau_{22}\, n_2 + \tau_{23}\, n_3 = 0$$
$$\tau_{31}\, n_1 + \tau_{32}\, n_2 + \tau_{33}\, n_3 = 0$$

Example: crack surface on xz plane



$$n_j = (0, 1, 0)$$

hence $\tau_{12} = \tau_{22} = \tau_{32} = 0$

Now for a mode III crack (anti-plane strain) the only non-zero stresses are τ_{32}, τ_{31}.

hence $\tau_{ij}\, n_j = 0$ means $\tau_{32} = 0$ on crack surface.

Solution to La Places equation

any function $u_3(\xi)$ of a complex variable $\xi = x + iy$ solves
$\nabla^2 u_3 = 0 = \frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} = 0$. Now using chain rule

$$\frac{\partial u_3}{\partial x} = \frac{du_3}{d\xi}\frac{\partial \xi}{\partial x} = \frac{du_3}{d\xi} \; ; \quad \frac{\partial^2 u_3}{\partial x^2} = \frac{d}{d\xi}\left(\frac{du_3}{d\xi}\right)\frac{\partial \xi}{\partial x} = \frac{d^2 u_3}{d\xi^2}$$

$$\frac{\partial u_3}{\partial y} = \frac{du_3}{d\xi}\frac{\partial \xi}{\partial y} = i\,\frac{du_3}{d\xi} \; ; \quad \frac{\partial^2 u_3}{\partial y^2} = \frac{d}{d\xi}\left(i\,\frac{du_3}{d\xi}\right)\frac{\partial \xi}{\partial y} = i^2\,\frac{d^2 u_3}{d\xi^2} = -\frac{d^2 u_3}{d\xi^2}$$

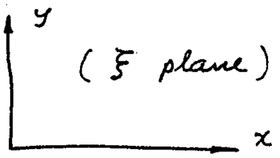
hence

$$\nabla^2 u_3 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u_3 = \frac{d^2 u_3}{d\xi^2} + (-)\,\frac{d^2 u_3}{d\xi^2} = 0$$

thus $\nabla^2 u_3 = 0$ is satisfied.

Now we need some properties of Complex variables

consider a function



$$Z_{III} = Z_{III}(\xi) = Re\, Z_{III} + i\, Im\, Z_{III}$$

$$= \alpha(x,y) + i\,\beta(x,y)$$

consider the derivative

$$\frac{dZ_{III}}{d\xi} = \lim_{\Delta\xi = \Delta x + i\Delta y \to 0} \left\{ \frac{\alpha(x+\Delta x, y+\Delta y) - \alpha(x,y)}{\Delta x + i\Delta y} + i\,\frac{\beta(x+\Delta x, y+\Delta y) - \beta(x,y)}{\Delta x + i\Delta y} \right\}$$

path for $\Delta y = 0$

$$\frac{d Z_{III}}{d \xi} = \frac{\partial \alpha}{\partial x} + i \frac{\partial \beta}{\partial x}$$

path for $\Delta x = 0$

$$\frac{d Z_{III}}{\partial \xi} = \frac{1}{i} \frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial y}$$

if Z_{III} is a unique function of the complex variable ξ then

$$\left. \frac{d Z_{III}}{d \xi} \right|_{\Delta y = 0} = \left. \frac{d Z_{III}}{d \xi} \right|_{\Delta x = 0}$$

and

$$\frac{\partial \alpha}{\partial x} + i \frac{\partial \beta}{\partial x} = \frac{1}{i} \frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial y}$$

$$i \frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial x} = \frac{\partial \alpha}{\partial y} + i \frac{\partial \beta}{\partial y}$$

hence

$$\boxed{\begin{aligned} \frac{\partial \alpha}{\partial x} &= \frac{\partial \beta}{\partial y} \\ \frac{\partial \alpha}{\partial y} &= - \frac{\partial \beta}{\partial x} \end{aligned}}$$

Cauchy - Riemann Relations.
(C-R)

From the C-R Relations $\nabla^2 \alpha = \nabla^2 \beta = 0$
$(\alpha = Re\, Z_{III} \quad , \quad \beta = Im\, Z_{III})$

so Z_{III}, $Re\, Z_{III}$, $Im\, Z_{III}$ all satisfy LaPlace's eqn.

Mode III Crack Solution

Now we let:

$$u_3 = \frac{1}{\mu} Im\, Z_{III} = \frac{\beta}{\mu}$$

we know this satisfies $\nabla^2 u_3 = 0$ - we will see this solution will work. satisfy boundary conditions etc.

now let us look for stresses.: $\tau_{zx} = \mu \frac{\partial u_3}{\partial x}$ $\tau_{zy} = \mu \frac{\partial u_3}{\partial y}$

$$\left.\frac{dZ_{III}}{d\xi}\right)_{\Delta Y=0} = \underbrace{\frac{\partial \alpha}{\partial x} + i \frac{\partial \beta}{\partial x} = \frac{\partial \beta}{\partial y}}_{\text{C-R Relation}} + i \frac{\partial \beta}{\partial x}$$

but $\frac{\partial \beta}{\partial x} = \mu \frac{\partial u_3}{\partial x} = \tau_{zx}$

recognize that

$$\frac{\partial \beta}{\partial y} = \mu \frac{\partial u_3}{\partial y} = \tau_{zy}$$

so

$$\frac{dZ_{III}}{d\xi} = \tau_{zy} + i\,\tau_{zx}$$

or $$\tau_{zy} = Re\left\{\frac{dZ_{III}}{d\xi}\right\}$$

$$\tau_{zx} = Im\left\{\frac{dZ_{III}}{d\xi}\right\}$$

Now let us consider the following function, What type of stress field would be produced?

$$Z_{III} = A\,(\xi^2 - c^2)^{1/2}$$

since Z_{III} is directly related to the displacement

$$u_3 = \frac{1}{\mu}\, Im\, Z_{III}$$

we pick a function which goes to zero at $\xi = \pm c$

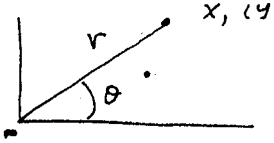
and which varies as $\xi^{1/2}$ near the crack tips

$$Z_{III} = A\left\{(\xi - c)(\xi + c)\right\}^{1/2} = A(\xi^2 - c^2)^{1/2}$$

Let us find the stress field (stress boundary condition).

$$\frac{dZ_{III}}{d\xi} = A\,\tfrac{1}{2}\,(\xi^2 - c^2)^{-1/2}\,2\xi = \frac{A\xi}{(\xi^2 - c^2)^{1/2}} = \frac{A\xi}{\left\{(\xi - c)(\xi + c)\right\}^{1/2}}$$

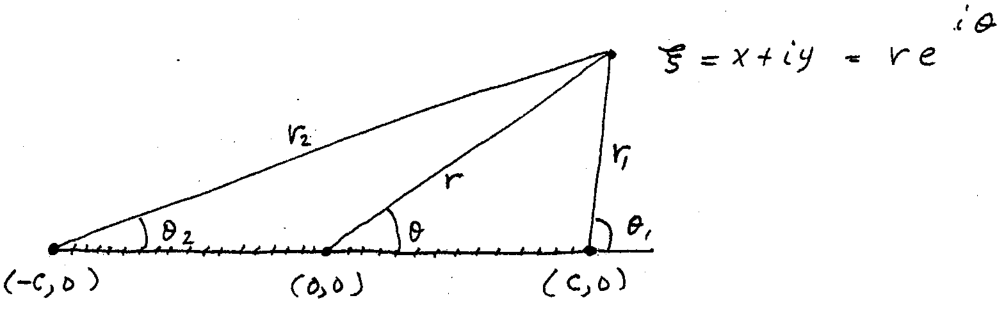
Now it is convenient to express this in polar coordinates in the ξ plane



Since $\xi = x + iy$

$$\xi = r\cos\theta + i r \sin\theta$$

$$\xi = r(\cos\theta + i\sin\theta) = re^{i\theta}$$



$$\xi = re^{i\theta}$$

$$\xi + c = r_2 e^{i\theta_2}$$

$$\xi - c = r_1 e^{i\theta_1}$$

so that

$$\frac{dZ_{III}}{d\xi} = \frac{A r e^{i\theta}}{(r_1 e^{i\theta_1})^{1/2}(r_2 e^{i\theta_2})^{1/2}}$$

$$= A\frac{r}{\sqrt{r_1 r_2}}\, e^{i\left(\theta - \frac{\theta_1 + \theta_2}{2}\right)}$$

or

$$\frac{dZ_{III}}{d\xi} = A \frac{r}{\sqrt{r_1 r_2}} \cos\left(\theta - \frac{\theta_1+\theta_2}{2}\right) + i A \frac{r}{\sqrt{r_1 r_2}} \sin\left(\theta - \frac{\theta_1+\theta_2}{2}\right)$$

Therefore

$$\tau_{ZY} = Re \frac{dZ_{III}}{d\xi} = \frac{Ar}{\sqrt{r_1 r_2}} \cos\left(\theta - \frac{\theta_1+\theta_2}{2}\right)$$

$$\tau_{ZX} = Im \frac{dZ_{III}}{d\xi} = \frac{Ar}{\sqrt{r_1 r_2}} \sin\left(\theta - \frac{\theta_1+\theta_2}{2}\right)$$

Evaluation of constant, A.

as $r \to \infty$ $r_1 \to r_2 \to \infty$ and $\theta_1 \to \theta_2 \approx \theta$
(ie, from far away, the points $(-c,0)$, $(0,0)$, $(c,0)$ look like the same points)

hence
$$\lim_{r\to\infty} \cos\left(\theta - \frac{\theta_1+\theta_2}{2}\right) = 1$$

$$\lim_{r\to\infty} \sin\left(\theta - \frac{\theta_1+\theta_2}{2}\right) = 0$$

So

$$\lim_{r\to\infty} \tau_{ZY} = A \qquad \lim_{r\to\infty} \tau_{ZX} = 0$$

so take

$$\underline{\underline{A = -\tau}}$$

Now we have the solution

$$Z_{III} = -\tau\left(\xi^2 - c^2\right)^{1/2} .$$

For this to be correct we need to make sure that

$$\tau_{zy} = 0 \quad \text{for} \quad y = 0 \quad |x| < c \quad (\text{on crack plane}).$$

The stresses are

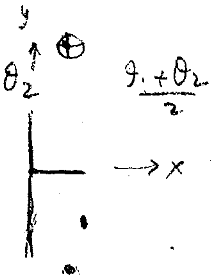
$$\tau_{zy} = \frac{A r}{\sqrt{r_1 r_2}} \cos\left(\theta - \frac{\theta_1 + \theta_2}{2}\right)$$

$$\tau_{zx} = \frac{A r}{\sqrt{r_1 r_2}} \sin\left(\theta - \frac{\theta_1 + \theta_2}{2}\right)$$

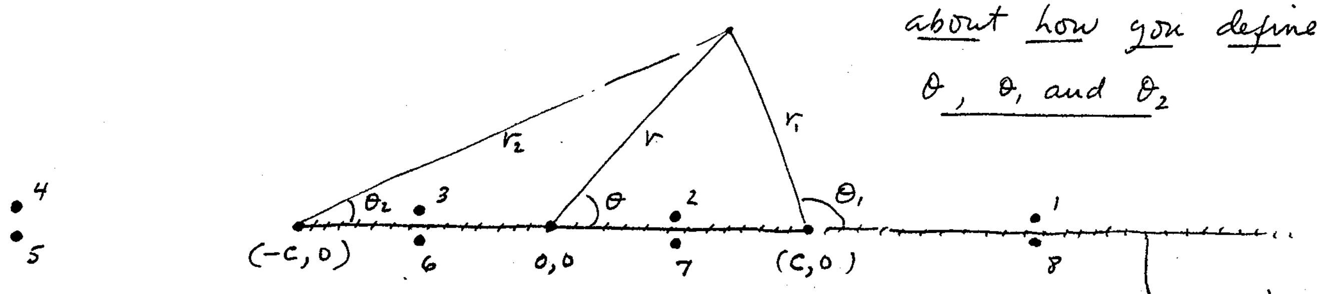
$\theta = \pi/2$ $\quad \theta_1 = \pi - \theta_2 \quad \frac{\theta_1 + \theta_2}{2} = \pi/2$

$\tau_{zy} = \frac{A r}{r_1}$

$\tau_{zx} = 0$



must be careful about how you define θ, θ_1 and θ_2



Branch cuts to deal with singular points.

The θ coordinates are

	1	2	3	4	5	6	7	8
θ	0	0	π	π	π	π	2π	2π
θ_1	0	π	π	π	π	π	π	2π
θ_2	0	0	0	π	π	2π	2π	2π
τ_{zy}	$\frac{Ar}{\sqrt{r_1 r_2}}$	0	0	$\frac{Ar}{\sqrt{r_1 r_2}}$	$\frac{Ar}{\sqrt{r_1 r_2}}$	0	0	$\frac{Ar}{\sqrt{r_1 r_2}}$

consequently the solution

$Z_{III} = -\tau(\xi^2 - c^2)^{1/2}$ satisfies the boundary conditions.

Recall that

$$Z_{III} = -\tau\sqrt{\xi - c}\sqrt{\xi + c} = -\tau\sqrt{r_1 e^{i\theta_1}}\sqrt{r_2 e^{i\theta_2}}$$

$$Z_{III} = -\tau\sqrt{r_1 r_2}\; e^{i\frac{\theta_1+\theta_2}{2}}$$

$$= -\tau\sqrt{r_1 r_2}\left\{\cos\frac{\theta_1+\theta_2}{2} + i\sin\frac{\theta_1+\theta_2}{2}\right\}$$

$\pi + \delta = \theta_1$

δ

$\theta_2 = 2\pi - \delta$

$\frac{\theta_1+\theta_2}{2} = \frac{3\pi}{2}$

Now since we started with

$$u_3 = \frac{1}{\mu}\, Im\, Z_{III}$$

Then

$$\boxed{u_3 = -\frac{\tau}{\mu}\sqrt{r_1 r_2}\,\sin\left(\frac{\theta_1+\theta_2}{2}\right)}$$

u_3 on free surface

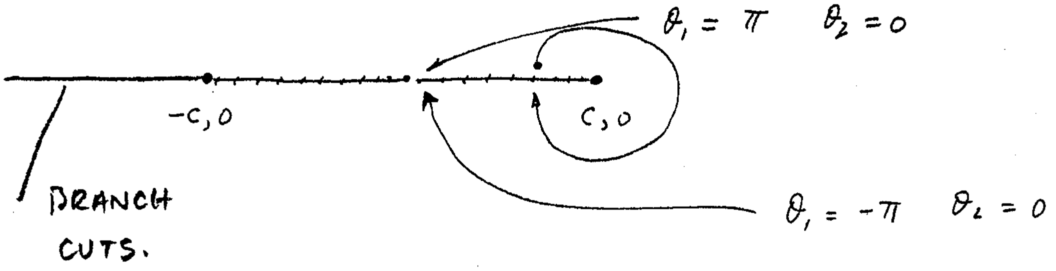
$-\frac{\tau}{\mu} r_1$ above

$\frac{\tau}{\mu} r_1$ below

$\Delta u = \frac{2\tau}{\mu} r_1$

$r_1 = \sqrt{c^2 + y^2}$

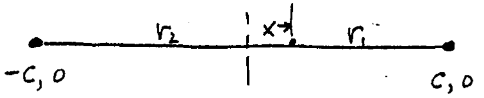
We can now look at relative displacements of the crack faces, taking care to properly write $\theta, \theta_1, \theta_2$.



Top crack face: $u_3^T = -\frac{\tau}{\mu}\sqrt{r_1 r_2}\,\sin\left(\frac{\pi}{2}\right) = -\frac{\tau}{\mu}\sqrt{r_1 r_2}$

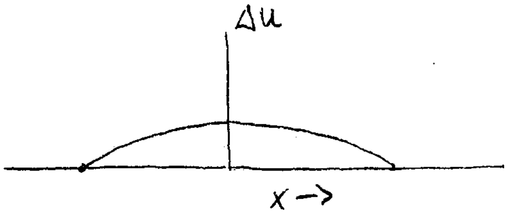
bottom crack face $u_3^b = -\frac{\tau}{\mu}\sqrt{r_1 r_2}\,\sin\left(-\frac{\pi}{2}\right) = \frac{\tau}{\mu}\sqrt{r_1 r_2}$

hence $\Delta u_3 = u_3^b - u_3^T = \frac{2\tau}{\mu}\sqrt{r_1 r_2}$



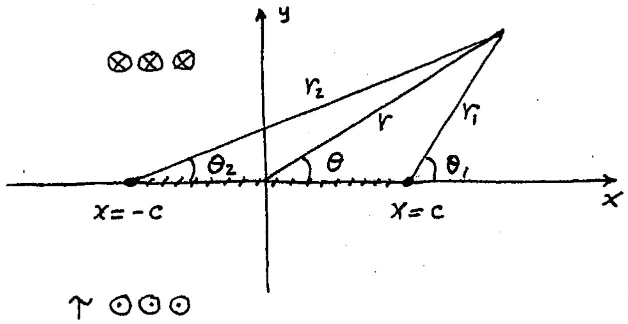
$r_1 = c - x \qquad r_2 = c + x \qquad r_1 r_2 = (c^2 - x^2)$

$$\boxed{\Delta u = \frac{2\tau}{\mu}\sqrt{c^2 - x^2}}$$



Note in passing that the Mode III crack can be composed of screw dislocations: the dislocations would be distributed in such a way as to produce the above displacement discontinuity.

Summary for Mode III Crack



general

$$\tau_{zy} = -\frac{\tau r}{\sqrt{r_1 r_2}} \cos\left(\theta - \frac{\theta_1+\theta_2}{2}\right)$$

$$\tau_{zx} = -\frac{\tau r}{\sqrt{r_1 r_2}} \sin\left(\theta - \frac{\theta_1+\theta_2}{2}\right)$$

$$u_3 = -\frac{\tau}{\mu}\sqrt{r_1 r_2}\, \sin\left(\frac{\theta_1+\theta_2}{2}\right)$$

near crack tip ($x = c \quad y = 0 \quad \theta_2 = \theta = 0 \quad r_2 = 2c \quad r = c$)

$$\tau_{zy} = -\tau \frac{c}{\sqrt{2cr_1}} \cos\left(0 - \frac{\theta_1+0}{2}\right) = -\tau\sqrt{\frac{c}{2r_1}}\cos\frac{\theta_1}{2}$$

$$\tau_{zx} = -\tau \frac{c}{\sqrt{2cr_1}} \sin\left(0 - \frac{\theta_1+0}{2}\right) = \tau\sqrt{\frac{c}{2r_1}}\sin\frac{\theta_1}{2}$$

$$u_3 = -\frac{\tau}{\mu}\sqrt{2cr_1}\,\sin\frac{\theta_1}{2}$$

$x=-c \quad y=0 \quad \theta_2 / 0 \quad \theta=\pi \quad \theta_1=\pi$

$\theta - \frac{\theta_1+\theta_2}{2} = \pi - \theta_1/2 - \theta_2/2 = \frac{\pi}{2} - \frac{\theta}{2}$

τ_{zy} , $\sin\frac{\theta}{2}$

τ_{zx} $\cos\frac{\theta}{2}$

u_3 $\cos\theta/2$

we will see later that the stress intensity factor K_{III} is usually defined as (for this particular Problem).

$$K_{III} = \tau\sqrt{\pi c}$$

so the stresses are: and the displacements

$$\tau_{ZY} = -\frac{K_{III}}{\sqrt{2\pi r_1}}\cos\frac{\theta_1}{2} \qquad u_3 = -\frac{K_{III}}{\mu}\sqrt{\frac{2r_1}{\pi}}\sin\frac{\theta_1}{2}$$

$$\tau_{ZX} = \frac{K_{III}}{\sqrt{2\pi r_1}}\sin\frac{\theta_1}{2}$$

← Stresses always written in this form, regardless of loading. K will depend on loading situation

far away from the crack $x^2+y^2 \longrightarrow \infty$ $\theta_1 = \theta_2 = \theta$

$r_1 = r_2 = r$

$$\left.\begin{aligned}\tau_{ZY} &= -\tau\\ \tau_{ZX} &= 0\end{aligned}\right\}$$

we shall see later that these will have to be more carefully specified to get the energy of deformation.

$$u_3 = -\frac{\tau}{\mu}r\sin\theta = -\frac{\tau}{\mu}y$$

Note: K_{III} completely characterizes the stress and strain in the vicinity of the crack tip.

$K_{III} = \tau\sqrt{\pi c}$ STRESS INTENSITY FACTOR.

For linear elastic fracture mechanics (LEFM), fracture occurs at a critical value of K_{III}.

delete

Energy of Deformation for Mode III Crack

The energy of deformation (strain energy change associated with crack formation) was key element of Giffith equation derivation

We will attempt to get strain energy for an infinite body with a crack. We will get the wrong answer (as did Giffith). Then we will revise the treatment for finite body.

We will let $\mathcal{E}_\ell^F$ be the false energy of deformation (for infinite body)
For linear elasticity we can write

$$\mathcal{E}_{el}^F = \frac{1}{2}\iiint_V \overset{\sigma : \nabla u}{\tau_{ij}\, e_{ij}}\, dV = \frac{1}{2}\iiint_V \tau_{ij}\, \frac{1}{2}\left\langle \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\rangle dV$$

but since $\tau_{ij} = \tau_{ji}$ the two terms here (when summed over i and j) are the same so that

$$\mathcal{E}_{el}^F = \frac{1}{2}\iiint_V \tau_{ij}\, \frac{\partial u_i}{\partial x_j}\, dV = \frac{1}{2}\iiint_V \frac{\partial}{\partial x_j}\left(\tau_{ij}\, u_i\right) dV$$

since $\frac{\partial \tau_{ij}}{\partial x_j} = 0$ equil.

Now we may use the Divergence Theorem which is

$$\iiint_V \frac{\partial}{\partial x_j}(A_j)\, dV = \oiint_S A_j\, \overset{n_j\, ds}{ds_j}$$

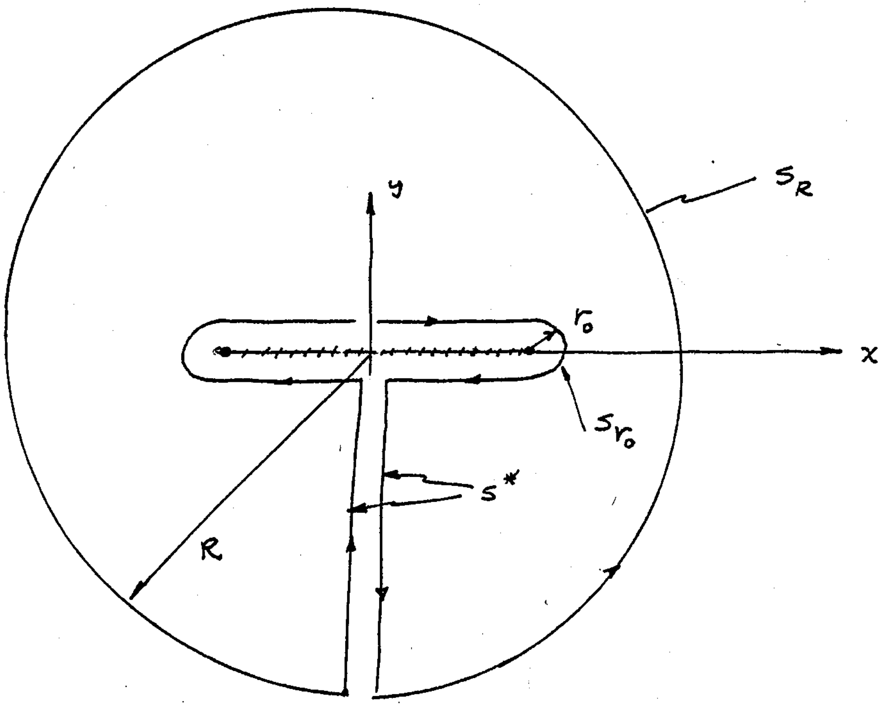
if A_j is continuous and single valued.

delete

so that

$$\mathcal{E}_{el}^{F} = \frac{1}{2} \oiint_{S} \tau_{ij}\, u_i \; dS_j$$

Since the Mode III Crack Problem is a two dimensional problem we can do everythings on a "per unit length" basis and the $\oiint_{S}$ is path which surrounds the material where both τ_{ij} and u_i are single valued and continuous.



The path shown surrounds the material completely and does not include any singular points.

$$\mathcal{E}_{el}^{F} = \frac{1}{2} \oiint_{S_R + S_{r_0} + S^*} \tau_{ij}\, u_i \, dS_j$$

(a) $\oiint_{S^*} = 0$ since $\tau_{ij}\, u_i$ continuous on S^* and we integrate in opposite directions over S^* (dS_j points away)

Start here

(b) on flat part of S_{r_0} $\tau_{ij}\, dS_j = 0$ (the surface is traction free)

so

$$\underset{\text{flat part of } S_{r_0}}{\oiint} = 0$$

and on circular part of S_{r_0}

$\tau_{ij} \propto r_0^{-1/2}$ $u_i \propto r_0^{1/2}$ $dS_j \propto r_0$

as $r_0 \to 0$ $\underset{S_{r_0}}{\oiint} \to \propto r_0 \to 0$

therefore

$$\mathcal{E}_{el}^{F} = \frac{1}{2} \underset{S_R}{\oiint} \tau_{ij}\, u_i\, dS_j = \frac{R}{2} \int_0^{2\pi} d\theta \left[\tau_{ij}\, u_i \right]_{R = \text{constant}}$$

$i, j = z, r$

per unit length.

(see next page).

which is

$$\mathcal{E}_{el}^{F} = \frac{R}{2} \int_0^{2\pi} d\theta \left[\tau_{zr}\, u_z \right]_{r = R \to \infty}$$

$$\left[\mathcal{E}_{el}^{F} = \frac{1}{2} \int_0^{2\pi} \tau_{zr}\, u_z\, R\, d\theta \right]$$

have to do this $R \to \infty$ because soln valid for $R \to \infty$

Now we need to evaluate u_z and τ_{zr} on the boundary $r = R \to \infty$.

<u>The displacement</u>

$$u_z = -\frac{\tau}{\mu} \operatorname{Im} \sqrt{\xi^2 - c^2}$$

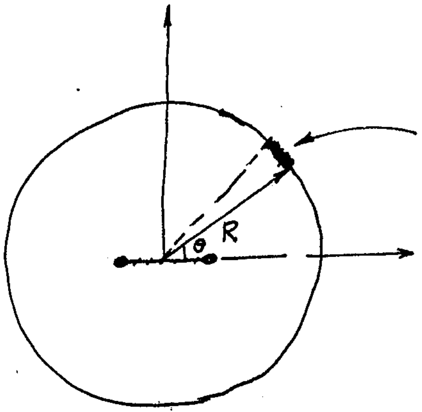
as $|\xi|^2 = x^2 + y^2 \to \infty$ $\sqrt{\xi^2 - c^2} = \xi \left\{ 1 - \left(\frac{c}{\xi} \right)^2 \right\}^{1/2}$

$$= \xi \left\{ 1 - \frac{c^2}{2\xi^2} + O\left(\frac{1}{\xi^4} \right) \cdots \right\}$$

$$= \xi - \frac{c^2}{2\xi} + O\left(\frac{1}{\xi^3} \right)$$

we have argued that

$$\mathcal{E}_{el}^{F} = \frac{1}{2} \oiint_{S_R} \tau_{ij}\, u_i\, dS_j$$



consider the area element.
$dS = R\,d\theta$

Let us figure out what

$$\tau_{ij}\, u_i\, dS_j$$

dS_j = j^{th} component of area vector.

means for the above element we use the r, θ, z coordinates.

$$\begin{array}{ll|lll}
 & j = & r & \theta & z \\
 & & x & y & z \\
i & & & & \\
\| & & & & \\
r & x & \tau_{rr}\, u_r\, dS_r & \tau_{r\theta}\, u_r\, dS_\theta & \tau_{rz}\, u_r\, dS_z \\
\theta & y & \tau_{\theta r}\, u_\theta\, dS_r & \tau_{\theta\theta}\, u_\theta\, dS_\theta & \tau_{\theta z}\, u_\theta\, dS_z \\
z & z & \boxed{\tau_{zr}\, u_z\, dS_r} & \tau_{z\theta}\, u_z\, dS_\theta & \tau_{zz}\, u_z\, dS_z \\
 & & \tau_{zx}\, u_z\, dS_x & \tau_{zy}\, u_z\, dS_y &
\end{array}$$

now

$\tau_{ij}\, u_i\, dS_j$ is the sum of these terms;

since $\tau_{zr}\, u_z\, dS_r$ is the only non-zero term we have

$$\mathcal{E}_{el}^{F} = \frac{1}{2} \oiint_{S_R} \tau_{rz}\, u_z\, dS_r = \frac{1}{2} \int_0^{2\pi} \tau_{rz}\, u_3\, R\, d\theta\,.$$

so as $|\xi| \to \infty$ $\quad u_3 = -\frac{\tau}{\mu} \, Im \, (\xi - \frac{c^2}{2\xi})$

on S_R $\quad \xi = R e^{i\theta}$

$$u_3 \approx -\frac{\tau}{\mu} \, Im \left(R e^{i\theta} - \frac{c^2}{2R} e^{-i\theta} \right)$$

$$R e^{i\theta} = R (\cos\theta + i \sin\theta)$$

$$e^{-i\theta} = (\cos\theta - i \sin\theta)$$

$$u_3 = -\frac{\tau}{\mu} R \left\{ \sin\theta + \frac{1}{2} \left(\frac{c}{R} \right)^2 \sin\theta \right\}$$

$$\boxed{u_3 = -\frac{\tau R}{\mu} \sin\theta \left\{ 1 + \frac{1}{2} \left(\frac{c}{R} \right)^2 \right\}}$$

The Tractions

now we need $\tau_{Zr} = \tau_{ZX} \cos\theta + \tau_{ZY} \sin\theta$ on $\xi = R e^{i\theta}$

recall that

$$\frac{d Z_{III}}{d\xi} = \tau_{ZY} + i \tau_{ZX} = \frac{d}{d\xi} \left(- \tau (\xi^2 - c^2)^{1/2} \right)$$

so

$$\tau_{ZY} + i \tau_{ZX} = - \tau \frac{\xi}{\sqrt{\xi^2 - c^2}} = - \tau \frac{\xi}{\xi \sqrt{1 - (\frac{c}{\xi})^2}} = - \tau \left\{ 1 - \left(\frac{c}{\xi} \right)^2 \right\}^{-1/2}$$

expanding for $\xi >> c$

$$\tau_{ZY} + i \tau_{ZX} = - \tau \left\{ 1 + \frac{c^2}{2\xi^2} \right\} = - \tau \left\{ 1 + \frac{c^2}{2R^2} e^{-i2\theta} \right\}$$

$$\tau_{zy} + i\,\tau_{zx} = -\tau\left\{ 1 + \frac{c^2}{2R^2}\left[\cos 2\theta - i \sin 2\theta\right]\right\}$$

so that

$$\tau_{zy} = -\tau\left\{ 1 + \frac{c^2}{2R^2} \cos 2\theta\right\}$$

$$\tau_{zx} = +\,\tau \frac{c^2}{2R^2} \sin 2\theta$$

so

$$\tau_{zr} = \tau_{zx} \cos\theta + \tau_{zy} \sin\theta$$

$$= \tau \frac{c^2}{2R^2} \sin 2\theta \cos\theta - \tau \sin\theta - \tau \frac{c^2}{2R^2} \cos 2\theta \sin\theta$$

$$= -\tau\left\{(-\sin 2\theta \cos\theta + \cos 2\theta \sin\theta)\frac{c^2}{2R^2} + \sin\theta\right\}$$

$$(-2\sin\theta\cos^2\theta + (2\cos^2\theta - 1)\sin\theta)$$

$$(-\sin\theta)$$

$$\boxed{\tau_{zr} = -\tau \sin\theta\left\{1 - \frac{c^2}{2R^2}\right\}}$$

Now finally

$$\mathcal{E}_{el}^{F} = \frac{R}{2}\int_0^{2\pi} d\theta \left[\tau_{zr}\, u_3\right]_{r=R\to\infty}$$

$$\mathcal{E}_{el}^{F} = \frac{R}{2}\int_0^{2\pi}\left[-\tau \sin\theta\left\{1 - \frac{c^2}{2R^2}\right\}\right]\left[-\frac{\tau R}{\mu}\sin\theta\left\{1 + \frac{c^2}{2R^2}\right\}\right] d\theta$$

$$= \frac{\tau^2 R^2}{2\mu}\int_0^{2\pi}\left\{1 - \frac{c^4}{4R^4}\right\} d\theta \sin^2\theta = \frac{\tau^2}{2\mu}\pi R^2\left\{1 - \frac{c^4}{4R^4}\right\}$$

now as $R \to \infty$

$$\mathcal{E}_{el}^{F} = \frac{1}{2}\int_0^{2\pi} \left[\tau_{zr} u_z\right]_{R\to\infty} R\, d\theta$$

$$\mathcal{E}_{el}^{F} = \frac{\tau^2}{2\mu}\pi R^2$$

(this is the strain energy of the uncracked body).

independent of c. Help! this can't be correct. the problem is that we have tried to deal with an <u>infinite body</u>.

Let us see why the answer we have here is incorrect.

originally we demanded that at $\xi \to \infty$ $\tau_{ij} = \tau_{ij}^{APPLIED}$

$$\tau_{zy} = -\tau$$

$$\tau_{zx} = 0$$

or equivalently

$$\left.\begin{aligned} \tau_{zr} &= -\tau \sin\theta \\ \tau_{z\theta} &= -\tau\cos\theta \end{aligned}\right\} \text{ at } \xi \to \infty$$

But our solution was

$$\tau_{zr} = -\tau \sin\theta \left\{ 1 - \frac{c^2}{2R^2} \right\}.$$

hence the stress we used contains an extra positive term

$$\Delta\tau_{zr} = \tau \sin\theta \frac{c^2}{2R^2}$$

which goes to zero only when $R \to \infty$, not when $R >> c$.

Consequently there is an extra term in our calculation of the energy of deformation which is

$$\Delta \mathcal{E}_{el} \sim \Delta \tau_{zr} \, u_3 \, ds$$

but $\Delta \tau_{zr} \sim \frac{1}{R^2}$

$u_3 \sim R$

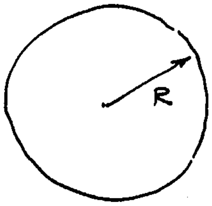
$ds = R\,d\theta \sim R$

hence $\Delta \mathcal{E}_{el} \rightarrow$ finite as $R \rightarrow \infty$. so we got the wrong answer.

Energy of Deformation for a Finite Body

We need to require that on S_R

$$\tau_{zr} = -\tau \sin\theta$$



We do this by adding another solution to satisfy this condition:
(Because of linear elastic assumption we can superimpose such solutions).

We want a solution which will make $\Delta \tau_{rz} = 0$ on S_R.
This is easy

pick $u_3^* = \alpha y$ (satisfies $\nabla^2 u_3^* = 0$).

so that $\tau^*_{zy} = \mu\alpha$ and $\tau^*_{zr} = \mu\alpha \sin\theta$

now on S_R

$$\tau_{zr}\Big)_{Total} = -\tau \sin\theta\left(1 - \frac{c^2}{2R^2}\right) + \mu\alpha \sin\theta \equiv -\tau \sin\theta .$$

(this makes $\Delta\tau_{zr} = 0$)

thus we pick $\alpha = -\dfrac{\tau c^2}{2R^2\mu}$

Now we can do the strain energy problem correctly.

$$\mathcal{E}_{el}^{True} = \mathcal{E}_{el}^{T} = \frac{1}{2}\iint_{S_R} \tau_{zr}\, u_3^{TOTAL}\, dS_R = \frac{1}{2}\int_0^{2\pi} \tau_{zr}\, u_z^{TOTAL}\, R\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} R\, d\theta\, [-\tau \sin\theta][u_3 + u_3^*] \quad \text{on } S_R \cdot \; y = R\sin\theta$$

(per unit length basis).

$$= \frac{1}{2}\int_0^{2\pi} R\, d\theta\, [-\tau\sin\theta]\left\{-\frac{\tau R}{\mu}\sin\theta\left[1+\frac{1}{2}\left(\frac{c}{R}\right)^2\right] - \frac{\tau c^2}{2R^2\mu}\cdot R\sin\theta\right\}$$

$$= \frac{\tau R}{2}\int_0^{2\pi} d\theta\, \sin^2\theta \left(\frac{\tau R}{\mu}\right)\left(1 + \frac{c^2}{2R^2} + \frac{c^2}{2R^2}\right)$$

note: $\underset{=\pi}{\int_0^{2\pi} \sin^2\theta\, d\theta} = \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}\Big|_0^{2\pi}$

$$= \frac{\tau^2 R^2}{2\mu}\left\{1 + \frac{c^2}{R^2}\right\}\cdot \pi$$

$$\mathcal{E}_{el}^{T} = \underset{\substack{\uparrow \\ \text{energy of uniformly loaded plate} \neq f(c)}}{\frac{\tau^2}{2\mu}(\pi R^2)} + \underset{\substack{\uparrow \\ \text{increase in strain energy due to crack.}}}{\frac{\pi\tau^2 c^2}{2\mu}}$$

$\dfrac{\partial U}{\partial c}$ $\dfrac{\pi\tau^2 c^2}{2\mu}$

increase in strain energy due to crack.

energy of uniformly loaded plate $\neq f(c)$

So that

$$\boxed{\mathcal{E}_{el}^{T} = \mathcal{E}_{0} + \frac{\pi c^{2} \tau^{2}}{2\mu}}$$

form used in Griffith Analysis.

Recall that the change in mechanical potential energy on crack formation is

$$\Delta U_{mechanical} = \Delta U_{el} + \Delta U_{WK}$$

but that $\Delta U_{WK} = -2\,\Delta U_{el}$ always

so

$$\Delta U_{mechanical} = -\,\Delta U_{el} = -\,\frac{\pi c^{2} \tau^{2}}{2\mu}$$ (For Mode III).

Thus the crack extension force is (for each end of the crack).

$$G_{III} = -\frac{1}{2}\frac{\partial\,\Delta U_{mechanical}}{\partial c} = \frac{1}{2}\frac{\pi c \tau^{2}}{\mu} = \frac{1}{2}\frac{K_{III}^{2}}{\mu}$$

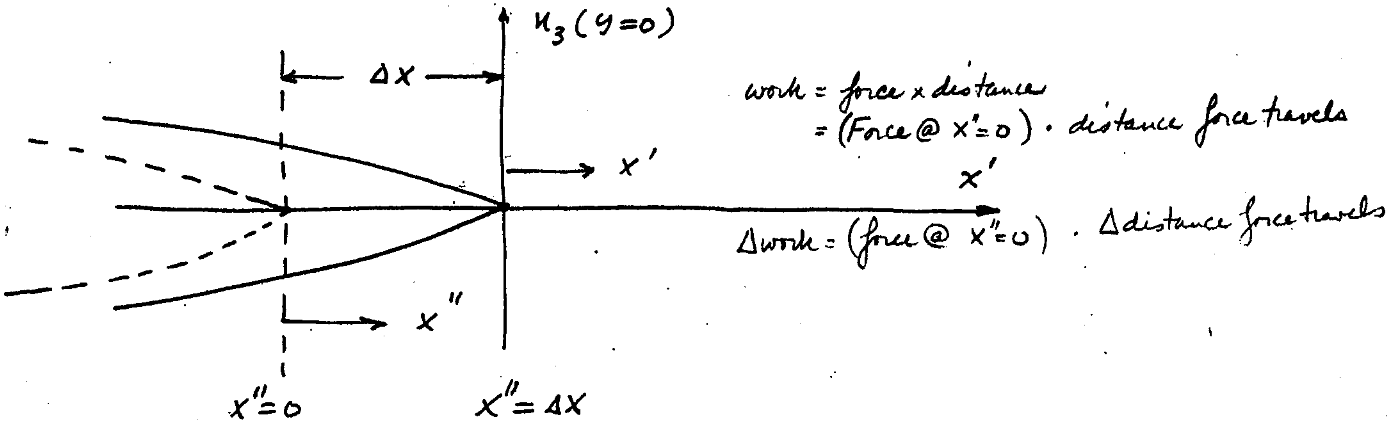
where $\boxed{K_{III} \equiv \tau\sqrt{\pi c}}$ Stress Intensity Factor

$$\boxed{G_{III} = \frac{K_{III}^{2}}{2\mu}}$$

The factor ½ enters because we want the crack extension force for one end of the crack only.

Before moving to the problems of Mode I and Mode II cracks let us determine G_{III} in a different way. This will be useful later when we try to get the crack extension forces for the other modes.

Consider the tip of a Mode III crack.
The displacement along the crack plane is.



Now suppose we exert forces on the crack face near the crack tip and close up a portion of the crack. This would move the crack front to $x' = -\Delta x$

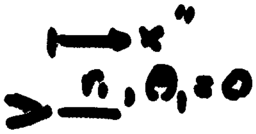
The work required for this process is evidently

$$(WK) = \underbrace{2}_{\text{two crack faces.}} \int_{x''=0}^{x''=\Delta x} \frac{1}{2}\, dx''\; \tau_{yz}\Big)_{\text{crack tip at } x''=0} \; u_z\Big)_{\text{crack tip at } x'=0}$$

(one has to be very careful not to get the wrong sign here).

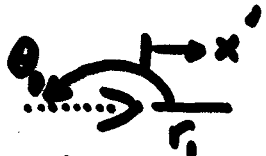
the stresses are

$$|\tau_{yz}| = \frac{K_{III}}{\sqrt{2\pi r_1}} \cos\frac{\theta_1}{2} = \frac{K_{III}}{\sqrt{2\pi x''}} \qquad K_{III} = \tau\sqrt{\pi c}$$



the displacements are

$$|u_3| = \frac{K_{III}}{\mu}\sqrt{\frac{2r_1}{\pi}} \sin\frac{\theta_1}{2} = \frac{K_{III}}{\mu}\sqrt{\frac{2}{\pi}(\Delta x - x'')}$$



$\theta_1 = \pi$

$x'' = x' + \Delta x$

$r_1 = x'$

So the work to close up the crack tip is

$$(\Delta) WK = \int_{\alpha''=0}^{\alpha''=\Delta x} \frac{K_{III}}{\sqrt{2\pi x''}} \frac{K_{III}}{\mu}\sqrt{\frac{2}{\pi}(\Delta x - x'')} \quad dx''$$

$$= \frac{K_{III}^2}{\mu}\frac{1}{\pi}\int_0^{\Delta x}\left(\frac{\Delta x - x''}{x''}\right)^{1/2} dx''$$

Now let $x'' = \Delta x \cos^2\theta$

$dx'' = -2\Delta x \cos\theta \sin\theta \, d\theta$

$$\int_0^{\Delta x}\left(\frac{\Delta x - x''}{x''}\right)^{1/2} dx'' = \int_{\theta=\frac{\pi}{2}}^{\theta=0}\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)^{1/2}(-)2\,\Delta x \cos\theta \sin\theta \, d\theta$$

$$= \int_{\frac{\pi}{2}}^{0}\left(\frac{\sin^2\theta}{\cos^2\theta}\right)^{1/2}(-)\,2\,\Delta x\cos\theta\sin\theta\,d\theta$$

$$\int_{\frac{\pi}{2}}^{0} \left(\frac{\sin^2\theta}{\cos^2\theta} \right)^{1/2} (-)\ 2\Delta X \cos\theta \sin\theta \, d\theta = \int_{0}^{\frac{\pi}{2}} 2\Delta X \sin^2\theta \, d\theta$$

$$= 2\Delta X \cdot \frac{\pi}{4}$$

so

$$(\Delta) WK = \frac{K_{III}^2}{\mu} \frac{1}{\pi} \, 2\Delta X \cdot \frac{\pi}{4} = \frac{K_{III}^2}{2\mu} \Delta X$$

Now by definition

$$G_{III} = \lim_{\Delta X \to 0} \left(\frac{\Delta(WK)}{\Delta X} \right) = \frac{K_{III}^2}{2\mu}$$

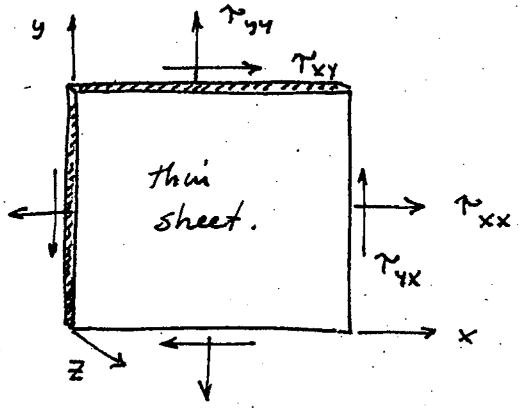
$$\boxed{G_{III} = \frac{K_{III}^2}{2\mu}}$$

hence we can get the crack extension force from the stresses and displacements directly.

<u>Plane Stress and Plane Strain</u> (for planar problems, all planes remain planar during deformation) ($\tau_{zx} = \tau_{zy} = 0$).

2D elasticity problems are idealized in the sense that there are no 2D solids. Thus all real crack problems are 3D problems, plane stress and plane strain are limiting conditions.

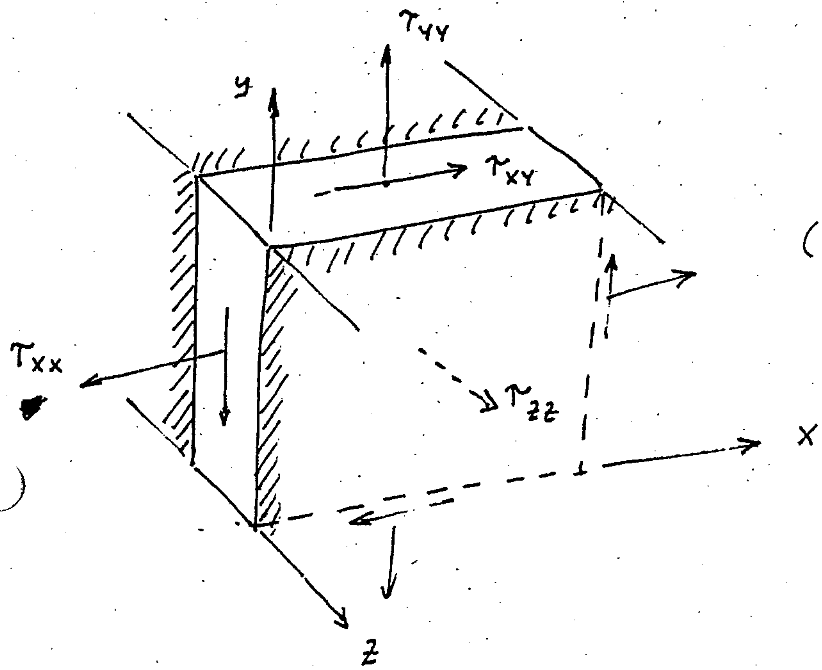
<u>Plane Stress</u> (defined by $\tau_{zz} = 0$)



$\tau_{zz} = 0$

($\tau_{zx} = \tau_{zy} = 0$ for planar prob.).

<u>Plane Strain</u> (defined by $u_z = 0$)



$u_z = 0 \Rightarrow e_{zz} = 0$

$e_{zx} = e_{zy}$

($\tau_{zx} = \tau_{zy} = 0$ for planar prob.).

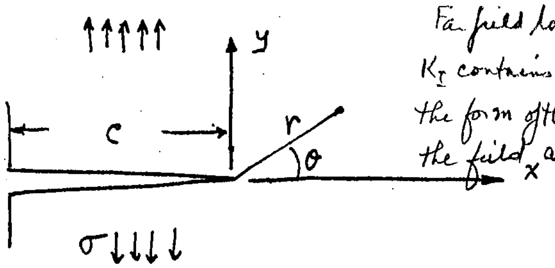
walls attached to planar surface require

$u_z = 0 \quad \tau_{zz} \neq 0$

Mode I Crack Stress Field (Paris and Sih).

(crack tip stresses and displacements only)

Plane Strain ($u_z = 0$)



Far field loading is σ
K_I contains geometry & loading of crack
the form of the solution is dependent only on the field around the notch.

general equations for crack tip stress / displacement fields — valid for all kinds of loading. K_I depends on geometry & loading.

$$\tau_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\tau_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\boxed{K_I = \sigma\sqrt{\pi c}}$$

imbedded crack in a plate loaded at ∞

(Stress Intensity Factor for biaxial stress state) (to be discussed later).

Planestrain implies $\tau_{zz} = \nu(\tau_{xx} + \tau_{yy})$

$$u_1 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}} \cos\frac{\theta}{2}\left[1 - 2\nu + \sin^2\frac{\theta}{2}\right]$$

$$u_2 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}} \sin\frac{\theta}{2}\left[2 - 2\nu - \cos^2\frac{\theta}{2}\right]$$

$$u_3 = 0$$

Plane Stress $\tau_{zz} = 0$

use above relations except:

$$\boxed{K_I = \sigma\sqrt{\pi c}}$$

1) $\tau_{zz} = 0$ 2) replace $\nu \Rightarrow \frac{\nu}{1+\nu}$

transformations to plane stress from plane strain $E \Rightarrow E\left(\frac{1+2\nu}{(1+\nu)^2}\right)$

these are general transformations which can be used in going from plane strain to plane stress or vice versa.

Suppose stresses, strains, displacements given for a problem of plane strain
to convert to plane stress

$$\nu \Rightarrow \frac{\nu}{1+\nu}$$

$$\Rightarrow \text{ replace by}$$

$$E \Rightarrow E\left(\frac{1+2\nu}{(1+\nu)^2}\right)$$

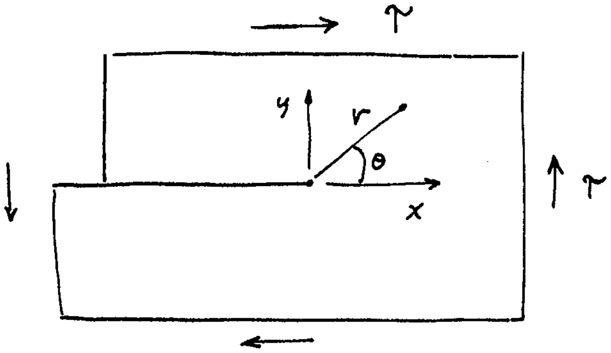
Suppose stresses, strains, displacements given for a problem of plane stress
to convert to plane strain

$$\nu \Rightarrow \frac{\nu}{1-\nu}$$

$$E \Rightarrow \frac{E}{1-\nu^2}$$

Mode II Crack Stress Field

Plane Strain



form of near field solution is the same but if you change loading & geometry K_{II} changes

$$\tau_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left[2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right]$$

$$\tau_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos\frac{3\theta}{2}$$

$$\tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\tau_{zz} = \nu(\tau_{xx} + \tau_{yy}) \qquad \boxed{K_{II} = \tau\sqrt{\pi c}}$$

$$u_1 = \frac{K_{II}}{\mu}\sqrt{\frac{r}{2\pi}} \sin\frac{\theta}{2}\left[2 - 2\nu + \cos^2\frac{\theta}{2}\right]$$

$$u_2 = \frac{K_{II}}{\mu}\sqrt{\frac{r}{2\pi}} \cos\frac{\theta}{2}\left[-1 + 2\nu + \sin^2\frac{\theta}{2}\right]$$

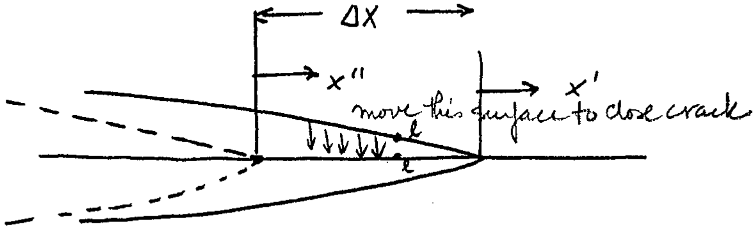
$$u_3 = 0$$

Plane Stress (same as above except)

$$\tau_{zz} = 0 \qquad \nu \Rightarrow \frac{\nu}{1+\nu} \qquad E \Rightarrow E\,\frac{1+2\nu}{(1+\nu)^2} \qquad \boxed{K_{II} = \tau\sqrt{\pi c}}$$

Crack Extension Forces

Let us determine the crack extension force for Mode I, plane strain using the crack tip closing technique.



∴ must move crack distance from when ~~crack was at~~ tip was at X' to when it is X''

∴ must apply a force to surface on pt ℓ .3. traction goes from 0 when crack tip is at X' to τ_{yy} when it is at X''

$$\Delta\,(WK) = 2 \int_{x''=0}^{x''=\Delta X} \tfrac{1}{2}\, dx''\, \tau_{yy} \Big)_{\substack{\text{crack tip}\\ \text{at } x''=0}} \cdot\; u_y \Big)_{\substack{\text{crack tip}\\ \text{at } x'=0}}$$

∴ no contribution from shear since $\tau_{xy} = 0$ along crack plane

but

$$\tau_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] = \frac{K_I}{\sqrt{2\pi x''}}$$

$$u_2 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\,\sin\frac{\theta}{2}\left[2 - 2\nu - \cos^2\frac{\theta}{2}\right] =$$

$$= \frac{K_I}{\mu}\, 2(1-\nu)\sqrt{\frac{\Delta X - x''}{2\pi}}$$

so

$$\Delta\,(WK) = \int_{x''=0}^{x''=\Delta X} \frac{K_I}{\sqrt{2\pi x''}}\; \frac{K_I}{\mu}\, 2(1-\nu)\sqrt{\frac{\Delta X - x''}{2\pi}}\; dx''$$

$$= \frac{K_I^2(1-\nu)}{\pi\mu}\int_{x''=0}^{x''=\Delta X}\sqrt{\frac{\Delta X - x''}{x''}}\; dx'' = \frac{K_I^2(1-\nu)}{\pi\mu}\, 2\Delta X \cdot \frac{\pi}{4}$$

$$\Delta(WK) = \frac{K_I^2(1-\nu)}{2\mu}\,\Delta X$$

now

$$G_I = \lim_{\Delta X \to 0} \frac{\Delta(WK)}{\Delta X} = \frac{K_I^2(1-\nu)}{2\mu} \quad (\text{plane strain})$$

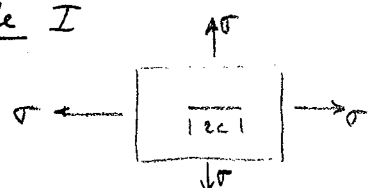
and for plane stress

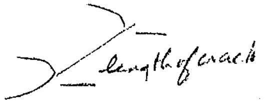
$$\nu \Longrightarrow \frac{\nu}{1+\nu} \qquad G_I = \frac{K_I^2}{2\mu}\left(1-\frac{\nu}{1+\nu}\right) = \frac{K_I^2}{2\mu(1+\nu)}$$

$$\mu = \frac{E}{2(1+\nu)} \qquad \frac{P}{in^2}\cdot in \quad \frac{P}{in^2} = \text{Pounds/in} \ (\text{plane stress}).$$

Table of Results (uniformly stressed plates)

Mode I

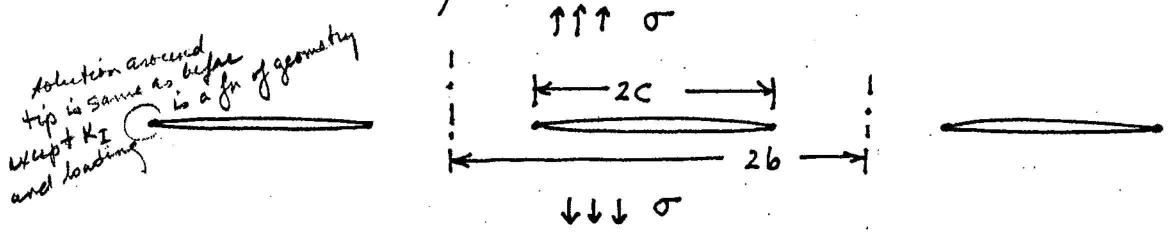




		Stress Intensity Factor	Crack Extension Force
Mode I	Plane Strain	$\sigma\sqrt{\pi c}$ $\frac{\text{force}}{(\text{length})^{3/2}}$	$\frac{K_I^2(1-\nu)}{2\mu}$ units force/unit length of crack
	plane stress	$\sigma\sqrt{\pi c}$	$\frac{K_I^2}{2\mu(1+\nu)} = \frac{K_I^2}{E}$
Mode II	Plane strain	$\tau\sqrt{\pi c}$	$\frac{K_{II}^2(1-\nu)}{2\mu}$
	plane stress	$\tau\sqrt{\pi c}$	$\frac{K_{II}^2}{2\mu(1+\nu)} = \frac{K_{II}^2}{E}$
Mode III		$\tau\sqrt{\pi c}$	$\frac{K_{III}^2}{2\mu}$

A Summary of Stress Intensity Factors
(Paris and Sih)

1. Infinite array of cracks, loaded at ∞



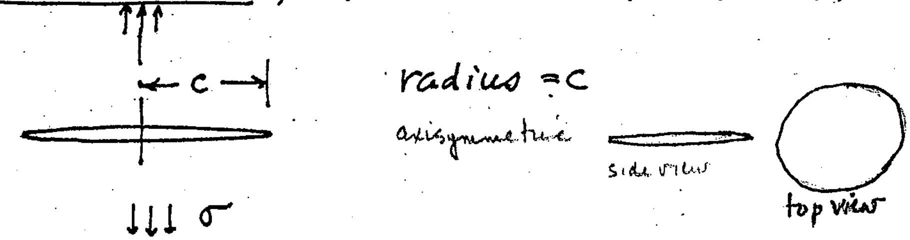
solution around tip is same as before except K_I is a fn of geometry and loading

$$K_I = \sigma\sqrt{\pi c}\left(\frac{2b}{\pi c}\tan\frac{\pi c}{2b}\right)^{1/2}$$

as $b \to \infty$: let $\frac{\pi c}{2b} = \xi$, $b \to \infty$ $\xi \to 0$

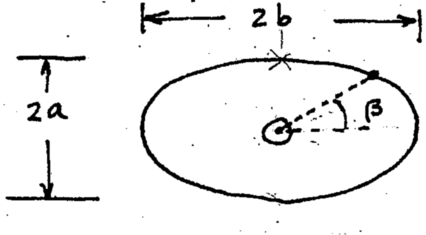
$\left(\frac{1}{\xi}\tan\xi\right)^{1/2} \to 1$

2. Imbedded Circular Crack, loaded in tension at ∞



$$K_I = \frac{2}{\pi}\sigma\sqrt{\pi c}$$

3. Imbedded Elliptical Crack, loaded in tension at ∞

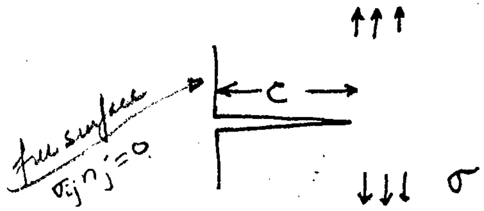


stress applied normal to crack plane

$$K_I = \frac{\sigma\sqrt{\pi a}}{\Phi_0}\left(\sin^2\beta + \left(\frac{a}{b}\right)^2\cos^2\beta\right)^{1/4}$$

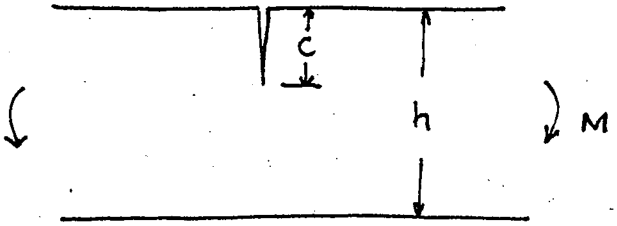
$$\Phi_0 = \int_0^{\pi/2}\left[1 - \left(\frac{b^2 - a^2}{b^2}\right)\sin^2\theta\right]^{1/2} d\theta$$

4. Edge Crack, loaded in tension at ∞



$$K_I = 1.12\,\sigma\sqrt{\pi c}$$

5. Edge Crack, subjected to in-plane bending



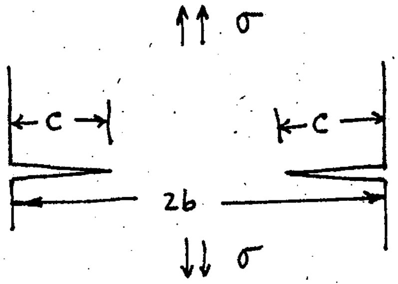
polynomial

$$K_I = \frac{6M}{(h-c)^{3/2}}\, g\left(\frac{c}{h}\right)$$

$g\left(\frac{c}{h}\right)$ given on p. 42

Paris and Sih

6. Double-Symmetric Edge Cracks, loaded in tension

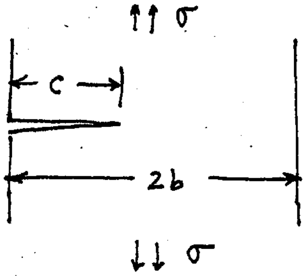


$$K_I = \sigma\sqrt{\pi c}\left(\frac{2b}{\pi c}\tan\frac{\pi c}{2b}\right)^{1/2} h\left(\frac{c}{b}\right)$$

$h\left(\frac{c}{b}\right)$ given on p. 43

Paris and Sih

7. Single Edge Crack, loaded in Tension at ∞



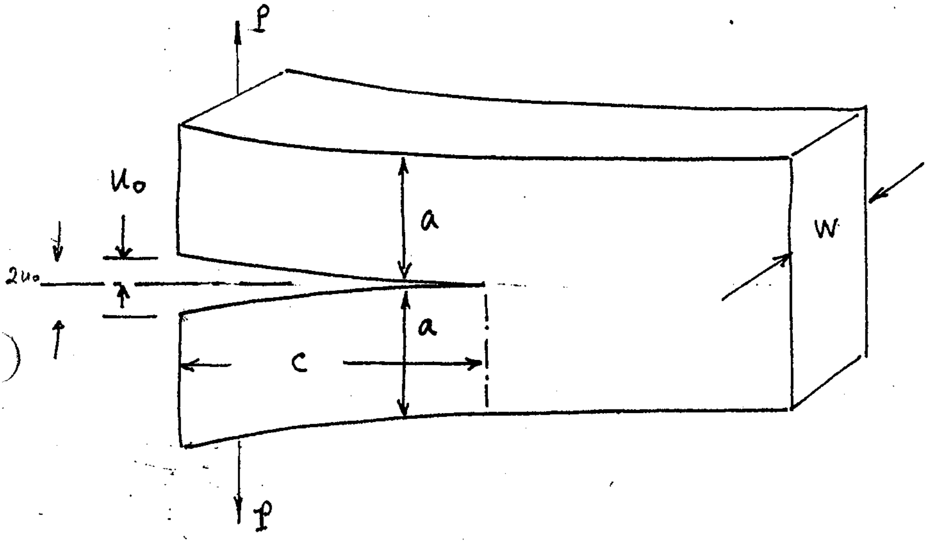
$$K_I = \sigma\sqrt{\pi c}\; k\left(\frac{c}{b}\right)$$

$k\left(\frac{c}{b}\right)$ given on p. 44 Paris and Sih

$$k(c/b) = \frac{1}{\sqrt{\pi}}\left[1.99 - 0.41\left(\frac{c}{2b}\right) + 18.7\left(c/2b\right)^2 - 38.48\left(c/2b\right)^3 + 53.85\left(c/2b\right)^4\right]$$

The Double Cantilever Beam Specimen (DCB Specimen)

We consider now a case in which the loading differs from the cases previously discussed and the stress intensity factor and crack extension force take a different form. Consider the double - cantilever beam to be deformed in plane stress.



used to control crack growth

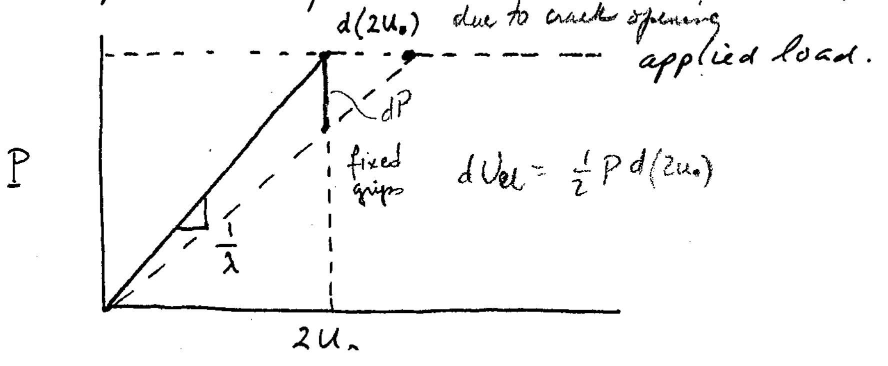
normally when $K_I \rightarrow K_{I_c}$ crack will start to grow & propagate uncontrollably

$c >> a$

For a crack of length c as shown the compliance of the sample loaded as shown is

$$\lambda = \frac{2u_0}{P} = f(\text{thickness of beam \& length of crack})$$

If one were to measure the opening displacement as a function of load we would have



If the crack is permitted to extend by dc then the opening displacement will increase by $d(2u_0)$ under the constant load.

The change in total mechanical energy when the crack advances dc is

$$dU_{TOT} = \underset{(+ive)}{dU_{el}} + \underset{(-ive)}{dU_{POT\,E.}}$$

you will recall for linear elastic bodies we can write

$$dU_{el} = \frac{1}{2} P\, d(2u_0)$$

$$dU_{POT\,E} = -2\,dU_{el} = -P\,d(2u_0)$$

so that

$$dU_{TOT} = -\frac{1}{2} P\, d(2u_0) = -\frac{P^2}{2}\,d\lambda$$

from the definition of the compliance.

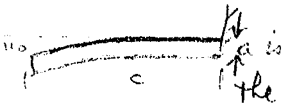
Now the crack extension force is evidently (total crack extension force)

$$G^{TOTAL} = -\frac{dU_{tot}}{dc} = \frac{P^2}{2}\frac{d\lambda}{dc}$$

Therefore, to find the crack extension force we have to find the compliance, λ, and how it depends on c.

Now the compliance for the double-cantilever beam sample can be easily calculated with beam theory. (Timoshenko & Goodier p38) We will treat these as two independent beams stuck into a wall.

$$u_o = \frac{Pc^3}{3EI}(1-\nu^2)$$



elementary beam theory is basically plane stress. The E here is for plane strain.

for plane strain

where I is the moment of Inertia $= \frac{wa^3}{12}$

so

$$u_o = \frac{4P}{wE}\left(\frac{c}{a}\right)^3 (1-\nu^2)$$

Now it is evident that

$$\lambda = \frac{2u_o}{P} = \frac{8}{wE}\left(\frac{c}{a}\right)^3 (1-\nu^2)$$

and thus

$$\frac{d\lambda}{dc} = \frac{24}{wE}\frac{c^2}{a^3}(1-\nu^2)$$

so that

$$G^T = \frac{P^2}{2}\frac{d\lambda}{dc} = \frac{12P^2}{wE}\cdot\frac{c^2}{a^3}(1-\nu^2)$$

* note: for uniformly stressed plate K^2 is proportional to crack extension force, which is force per unit length

Now for Mode I in plane strain we know

* (see note above)

$$\frac{G^T}{w} = \frac{K_I^2(1-\nu)}{2\mu} \quad \frac{\#^2}{in^3} \Big/ \#/in \qquad \text{so} = \frac{\#}{in}$$

$$K_I^{\,2} = \frac{2\mu\, G^T}{w\,(1-\nu)}$$ for plane strain

but $$\mu = \frac{E}{2(1+\nu)}$$

$$K_I^{\,2} = \frac{E\,G^T}{w\,(1-\nu^2)} = \frac{E}{w\,(1-\nu^2)} \cdot \frac{12\,P^2}{w\,E}\,\frac{c^2}{a^3}\cancel{(1-\nu^2)}$$

plane strain G^T

$$K_I^{\,2} = \frac{12\,P^2c^2}{\cancel{(1-\nu^2)}\,w^2\,a^3}\cancel{(1-\nu^2)}$$

$$K_I = \frac{2\sqrt{3}}{\cancel{\sqrt{(1-\nu^2)}}}\;\frac{P\,c}{w\,a^{3/2}}\cancel{\sqrt{(1-\nu^2)}}$$

$$K_I = 2\sqrt{3}\,\frac{PC}{w\,a^{3/2}}$$

under constant load P, The stress intensity factor increases with c

Suppose the displacement $2u_0$ is fixed. Then the load P varies with compliance as

$$P = \frac{2u_0}{\lambda} = 2u_0\,\frac{wE}{8}\left(\frac{a}{c}\right)^3 \frac{1}{(1-\nu^2)}$$ as $c\uparrow$ $P\downarrow$

$$\frac{\delta P}{\delta c} = 2u_0\,\frac{wE}{8}\left(-3\,\frac{a^3}{c^4}\right) = -\frac{3P}{c}$$

$$K_I = \frac{2\sqrt{3}}{\cancel{\sqrt{1-\nu^2}}} \cdot \frac{2u_0\,\frac{wE}{8}\left(\frac{a}{c}\right)^3 c}{w\,a^{3/2}}\cancel{\sqrt{1-\nu^2}} = \frac{\sqrt{3}}{4\cancel{\sqrt{1-\nu^2}}}\;\frac{2u_0\,E}{c^2}\,a^{3/2}\cancel{\sqrt{1-\nu^2}}$$

Thus as $c\uparrow$ $K_I\downarrow$

$$K_I = \frac{\sqrt{3}}{4}\,\frac{E}{c^2}\,a^{3/2}\cdot 2u_0$$

under constant displacement $2u_0$: K_I decreases with crack length. so stable crack growth possible.

Constant Stress Intensity Factor Samples.

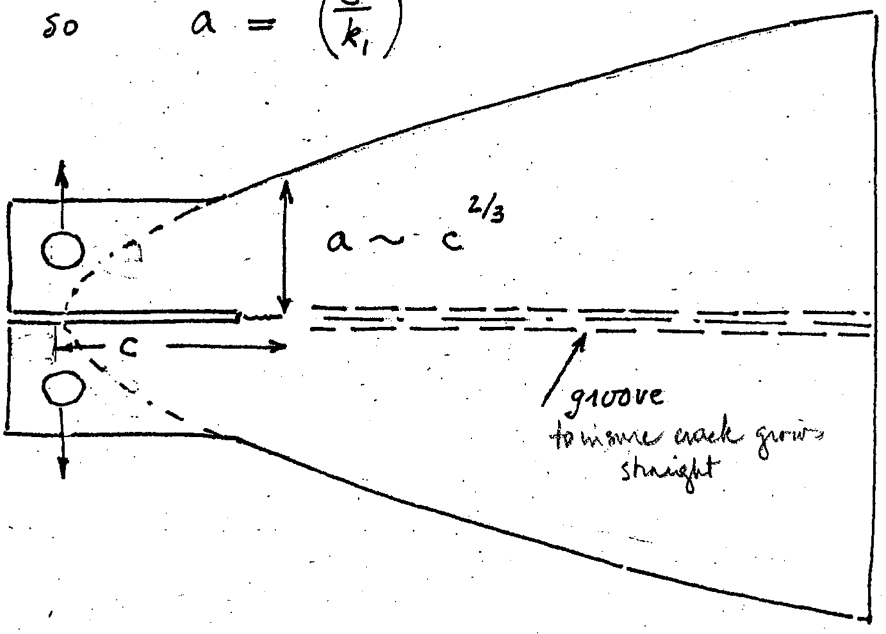
Based on double cantilever beam analysis

$K_I = \frac{2\sqrt{3}}{\cancel{\sqrt{1-\nu^2}}} \frac{Pc}{w\,a^{3/2}} \sqrt{1-\nu^2}$ to maintain constant K_I for constant P, we hold $c/a^{3/2}$ constant.

$= 2\sqrt{3}\,\frac{Pc}{W a^{3/2}}$

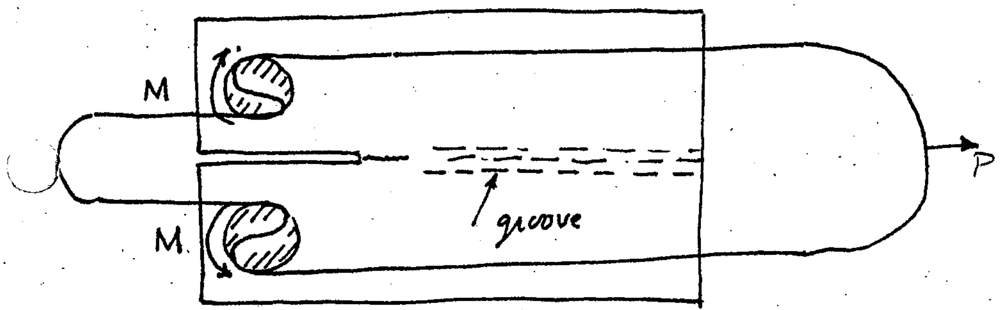
let $\frac{c}{a^{3/2}} = k_1$

so $a = \left(\frac{c}{k_1}\right)^{2/3}$



$K_I \neq f(c)$ for constant P

For standard double cantilever beam sample $Pc = M$ moment applied to each arm.



$K_I = \frac{2\sqrt{3}}{\sqrt{1-\nu^2}} \frac{M}{w\,a^{3/2}} \cancel{\sqrt{1-\nu^2}}$

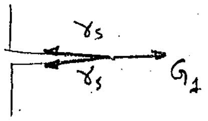
$K_I \neq f(c)$ for constant M.

Linear Elastic Fracture Mechanics

(consider Mode I, plane strain)

Griffith Analysis - perfectly elastic and brittle solids, eg glass, ceramics

* Fracture occurs when crack extension force G_I is equal to or greater than total surface tension force acting at crack tip.



$$G_I \geq 2\gamma_s \qquad \text{for fracture.}$$

$$G_I = \frac{K_I^2(1-\nu)}{2\mu} \geq 2\gamma_s$$

$$K_I \geq \left(\frac{4\mu\gamma_s}{1-\nu}\right)^{1/2}$$

Stress Intensity Factor.

but $K_I = \sigma\sqrt{\pi c}$ for loading @ ∞

so

$$\sigma \geq \left(\frac{4\mu\gamma_s}{\pi(1-\nu)c}\right)^{1/2}$$

Griffith Eqn.

Condition of Fracture may be expressed as

$$G_I \geq G_{I_c} = 2\gamma_s$$

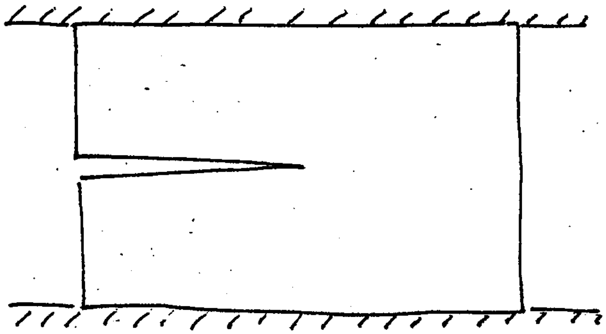
$$K_I \geq K_{I_c} = \left(\frac{4\mu\gamma_s}{1-\nu}\right)^{1/2}$$

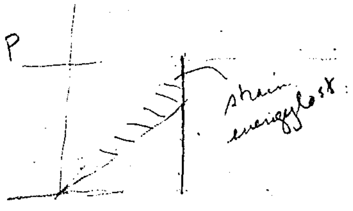
$$\sigma \geq \sigma_F = \left(\frac{4\mu\gamma_s}{\pi(1-\nu)c}\right)^{1/2}$$

All equivalent expressions for condition of fracture.

Terminology:

G_I – <u>crack extension force</u> (dynes / cm)
<u>strain energy release rate</u> (ergs / cm^2)





fixed grip

$\frac{1}{2}\int \sigma\varepsilon\, dV$

$\frac{1}{2} u \delta P$

for fixed grip condition, only driving force for crack extension is reduction of strain energy, so G_I is the strain energy release rate

G_{I_c} - <u>work of fracture</u> (ergs / cm^2)

K_I = <u>stress intensity factor</u> fn of geometry & load only

K_{I_c} = <u>fracture toughness</u>,
<u>crack tougness</u>.

LHS is a function of geom/loads

RHS fns of material property

$G_I \geq G_{I_c}$

$K_I \geq K_{I_c}$

~~$\lambda P = 2u_0$~~

~~$d\lambda P + dP\lambda = d(2u_0)$~~

~~if $d(2u_0) = 0 \Rightarrow \lambda\, dP = -P d\lambda$~~

~~$dU_{el} = \frac{1}{2}(2u_0)\delta P = \frac{1}{2}(\lambda P \delta P)$~~

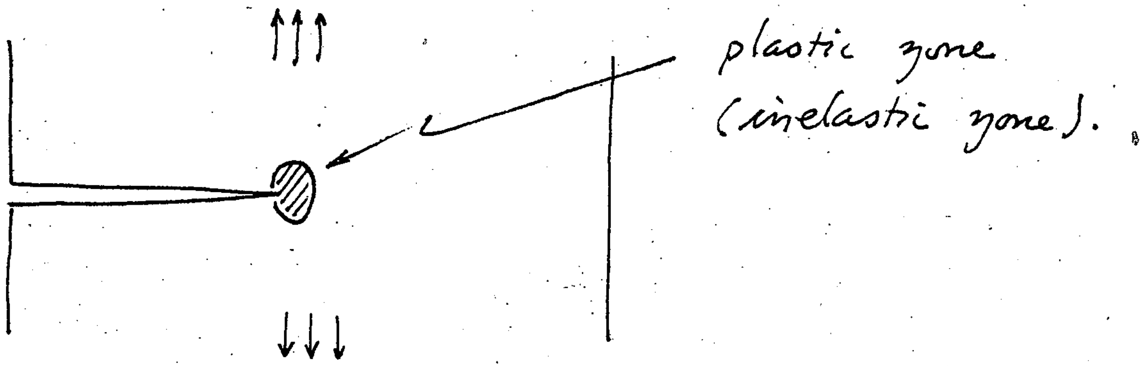
~~$=$~~

~~$G = -\frac{dU_{TOT}}{dt} + \frac{1}{2}(2u_0)\delta P = +\frac{1}{2}(\lambda P \delta P)$~~

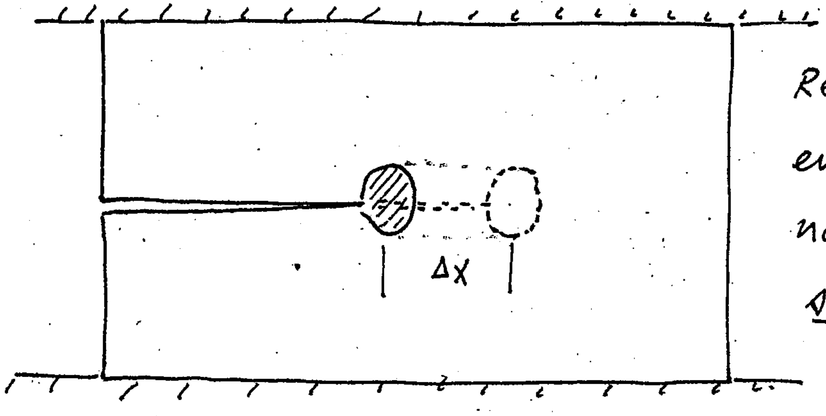
~~$= \frac{3}{2c}\lambda P^2$~~

Griffith - Irwin Analysis

For metals and even some ceramics, the high stresses at the crack tip cause inelastic deformation to occur. (plastic flow in the case of metals, microcracking in the case of ceramics). Thus stresses at the crack tip are not those given by the elastic theory.



But! if plastic zone is small compared to crack length c, then body as a whole behaves elastically. Thus, the thermodynamic extension force is not affected by localized inelastic processes. — This is basis of Griffith - Irwin Analysis



Reduction of strain energy per Δx not affected by small plastic zone.

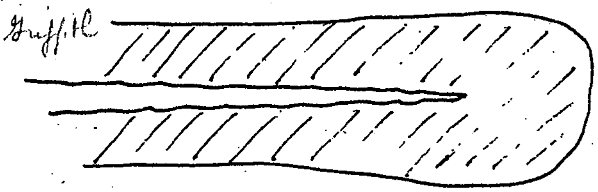
Assumption of LEFM is that G_I and K_I are exactly the same as those for perfectly brittle solids.

So fracture occurs when

$G_I \geq G_{I_c}$ but G_{I_c} (work of fracture) includes not only surface creation $2\gamma_s$, but also plastic (inelastic) work of fracture, P.

$$G_{I_c} = 2\gamma_s + P$$

Irwin modif of Griffith anal



P = plastic work per unit crack extension.

or, in terms of stress intensity factor,

$$K_I \geq K_{I_c} = \left(\frac{2\mu\,(2\gamma_s + P)}{(1-\nu)} \right)^{1/2}$$

for metals

$P \gg 2\gamma_s$

This is provided we have localized yielding and assuming that plastic zone is small. Plastic zone affect G_{I_c} but we calculated G_I from body as a whole.

Crack Tip Plasticity

We now wish to estimate the effects of crack tip plasticity on the stability of cracks under stress. First we want to estimate the size and shape of the plastic zone.

Let us assume that the material in question work hardens sufficiently rapidly that the elastic stress fields can be used without modification. We wish to find the boundary of the region about the crack tip, outside of which all deformation remains elastic.

We will assume that the material yields according to the von Mises yield criterion: That is

$$\Longrightarrow \quad \frac{(\tau_{xx}+p)^2}{2} + \frac{(\tau_{yy}+p)^2}{2} + \frac{(\tau_{zz}+p)^2}{2} + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2 \geqslant \frac{\sigma_Y^2}{3}$$

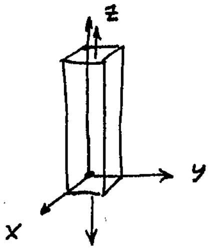
uniaxial yield stress

where p is the hydrostatic pressure

$$p = -\frac{(\tau_{xx} + \tau_{yy} + \tau_{zz})}{3}$$

and where σ_y is the uniaxial yield stress.

Consider the case of simple tension:



$$\tau_{zz} = \sigma \qquad \tau_{xx} = \tau_{yy} = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$

$$p = -\frac{\sigma}{3}$$

so yielding when

$$\frac{1}{2}\left(\frac{4}{9}+\frac{1}{9}+\frac{1}{9}\right)\sigma^2 \geq \frac{\sigma_Y}{3}^2$$

$$(1-\tfrac{1}{3})^2+(-\tfrac{1}{3})^2+(-\tfrac{1}{3})^2$$

$$\frac{\sigma^2}{3} \geq \frac{\sigma_Y^2}{3} \qquad \boxed{\sigma \geq \sigma_Y}$$

Now let us consider the yielding at the tip of the crack: (Mode I).

$$\tau_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\tau_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

Plane Stress	Plane Strain
$\tau_{zz} = 0$	$\tau_{zz} = \nu(\tau_{xx}+\tau_{yy})$
$\tau_{xz} = \tau_{yz} = 0$	$\tau_{zz} = \dfrac{2\nu K_I}{\sqrt{2\pi r}}\cos\dfrac{\theta}{2}$
	$\tau_{xz} = \tau_{yz} = 0$

Plane Stress

Now investigate yielding via von mises criterion

$$\tau_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\theta(1-\ldots);\ \tau_{zz} =$$

$$\tau_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\theta(1+\ldots)$$

$$-\left(\frac{\tau_{yy}}{3}+\frac{\tau_{xx}}{3}\right) = p \doteq -\frac{2K_I}{3\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$$\tau_{xx}+p = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[\frac{1}{3}-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\tau_{yy} + p = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[\frac{1}{3} + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\tau_{zz} + p = p = -\frac{2K_I}{3\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

so

$$\left[\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\right]^2\left\{\frac{1}{2}\left[\frac{1}{3} - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]^2 + \frac{1}{2}\left[\frac{1}{3} + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]^2\right.$$

$$\left. + \frac{1}{2}\left[-\frac{2}{3}\right]^2 + \left(\sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)^2\right\} \geq \frac{\sigma_y^2}{3}$$

$$\frac{K_I^2}{2\pi r}\cos^2\frac{\theta}{2}\left\{\frac{1}{2}\left[\frac{1}{9} - \frac{2}{3}\sin\frac{\theta}{2}\sin\frac{3\theta}{2} + \sin^2\frac{\theta}{2}\sin^2\frac{3\theta}{2}\right]\right.$$

$$+ \frac{1}{2}\left[\frac{1}{9} + \frac{2}{3}\sin\frac{\theta}{2}\sin\frac{3\theta}{2} + \sin^2\frac{\theta}{2}\sin^2\frac{3\theta}{2}\right]$$

$$\left. + \frac{1}{2}\left[\frac{4}{9}\right] + \sin^2\frac{\theta}{2}\cos^2\frac{3\theta}{2}\right\} \geq \frac{\sigma_y^2}{3}$$

$$\frac{K_I^2}{2\pi r}\cos^2\frac{\theta}{2}\left\{\frac{1}{3} + \sin^2\frac{\theta}{2}\sin^2\frac{3\theta}{2} + \sin^2\frac{\theta}{2}\cos^2\frac{3\theta}{2}\right\} \geq \frac{\sigma_y^2}{3}$$

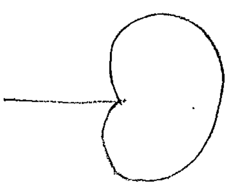
$$\frac{K_I^2}{2\pi r}\cos^2\frac{\theta}{2}\left\{\frac{1}{3} + \sin^2\frac{\theta}{2}\right\} \geq \frac{\sigma_y^2}{3} \longleftarrow$$

result.
plane stress.

Solve for r

so $$r = \frac{K_I^2}{2\pi\sigma_y^2}\cos^2\frac{\theta}{2}\left\{1 + 3\sin^2\frac{\theta}{2}\right\}$$

r_p



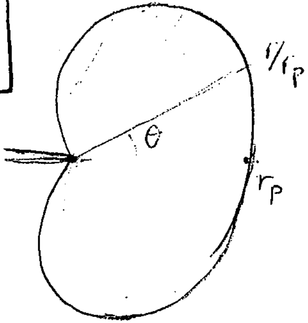
Note that when $\theta = 0$ $\qquad K_I = \sigma\sqrt{\pi c} \Rightarrow K_I^2/\pi = \sigma^2 c$

$$\underset{\theta=0}{r} = \frac{K_I^{\,2}}{2\pi\,\sigma_y^{\,2}} = r_p = \frac{c}{2}\left(\frac{\sigma}{\sigma_y}\right)^2$$

for uniformly stressed plate.

or

$$\boxed{\frac{r}{r_p} = \cos^2\frac{\theta}{2}\cdot\left\{1 + 3\sin^2\frac{\theta}{2}\right\}}$$



Plane Strain

$$p = -\frac{K_I}{\sqrt{2\pi r}}\left\{\frac{2}{3}\cos\frac{\theta}{2} + \frac{2\nu}{3}\cos\frac{\theta}{2}\right\}$$

$$= -\frac{2}{3}\frac{K_I}{\sqrt{2\pi r}}(1+\nu)\cos\frac{\theta}{2}$$

$$\tau_{xx} + p = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1 - \frac{2}{3}(1+\nu) - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\tau_{yy} + p = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1 - \frac{2}{3}(1+\nu) + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\tau_{zz} + p = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[2\nu - \frac{2}{3}(1+\nu)\right]$$

so

$$\left[\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\right]^2\left\{\frac{1}{2}\left[\frac{1-2\nu}{3} - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]^2 + \frac{1}{2}\left[\frac{1-2\nu}{3} + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]^2\right.$$

$$\left.\frac{1}{2}\left[-\left(2\,\frac{(1-2\nu)}{3}\right)\right]^2 + \sin^2\frac{\theta}{2}\cos^2\frac{3\theta}{2}\right\} \geq \frac{\sigma_y^{\,2}}{3}$$

$$\begin{aligned}\frac{K_I^2}{2\pi r}\cos^2\frac{\theta}{2}\Bigg\{ &\frac{1}{2}\left[\frac{(1-2\nu)^2}{9} - 2\frac{(1-2\nu)}{3}\sin\frac{\theta}{2}\sin\frac{3\theta}{2} + \sin^2\frac{\theta}{2}\sin^2\frac{3\theta}{2}\right] \\ &\frac{1}{2}\left[\frac{(1-2\nu)^2}{9} + 2\frac{(1-2\nu)}{3}\sin\frac{\theta}{2}\sin\frac{3\theta}{2} + \sin^2\frac{\theta}{2}\sin^2\frac{3\theta}{2}\right] \\ &\frac{1}{2}\ \frac{4}{9}(1-2\nu)^2 + \sin^2\frac{\theta}{2}\cos^2\frac{3\theta}{2}\Bigg\} \geq \frac{\sigma_Y^2}{3}\end{aligned}$$

$$\frac{K_I^2}{2\pi r}\cos^2\frac{\theta}{2}\left\{\frac{(1-2\nu)^2}{9} + \frac{2}{9}(1-2\nu)^2 + \sin^2\frac{\theta}{2}\sin^2\frac{3\theta}{2} + \sin^2\frac{\theta}{2}\cos^2\frac{3\theta}{2}\right\} \geq \frac{\sigma_Y^2}{3}$$

$$\frac{K_I^2}{2\pi r}\cos^2\frac{\theta}{2}\left\{\frac{1}{3}(1-2\nu)^2 + \sin^2\frac{\theta}{2}\right\} \geq \frac{\sigma_Y^2}{3}$$

result. plane strain.

so solve for r

$$r = \underbrace{\frac{K_I^2}{2\pi\sigma_Y^2}}_{r_p}\cos^2\frac{\theta}{2}\left\{(1-2\nu)^2 + 3\sin^2\frac{\theta}{2}\right\}$$

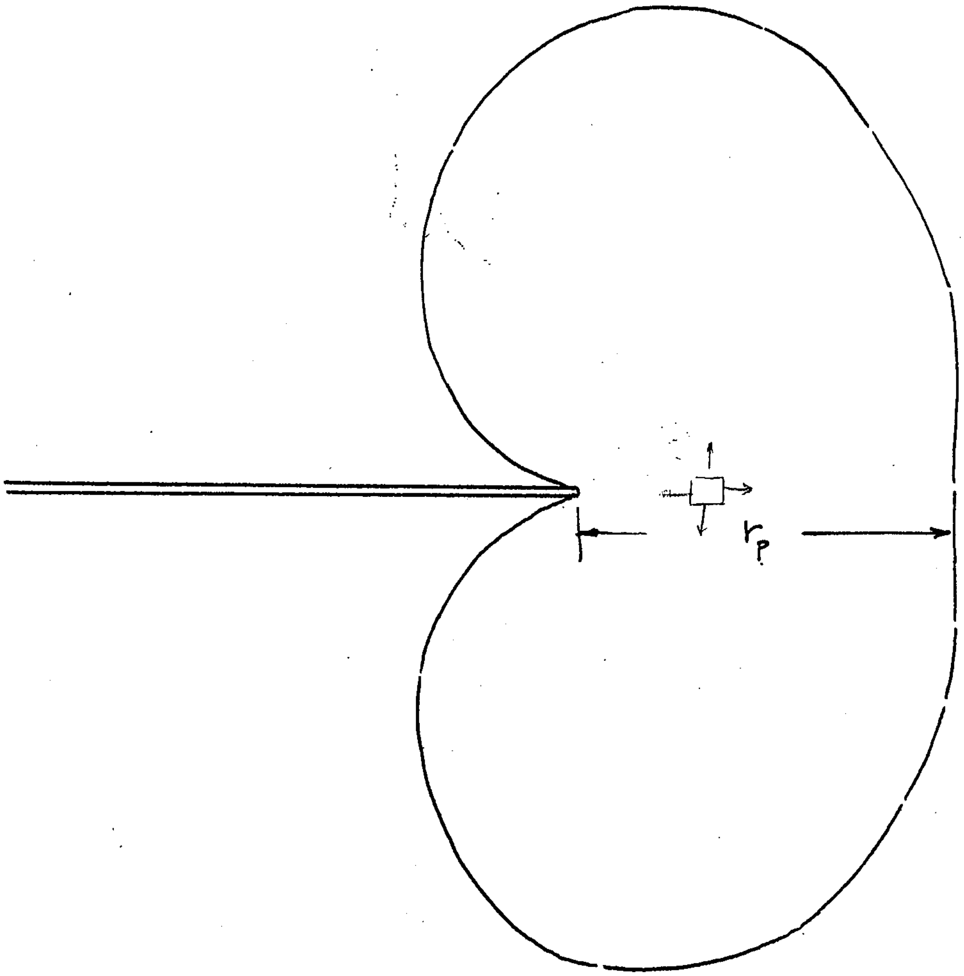
so

$$\boxed{\frac{r}{r_p} = \cos^2\frac{\theta}{2}\left\{(1-2\nu)^2 + 3\sin^2\frac{\theta}{2}\right\}}$$

$$r_p = \frac{c}{2}\left(\frac{\sigma}{\sigma_Y}\right)^2$$

as before

Plastic Zone Shapes

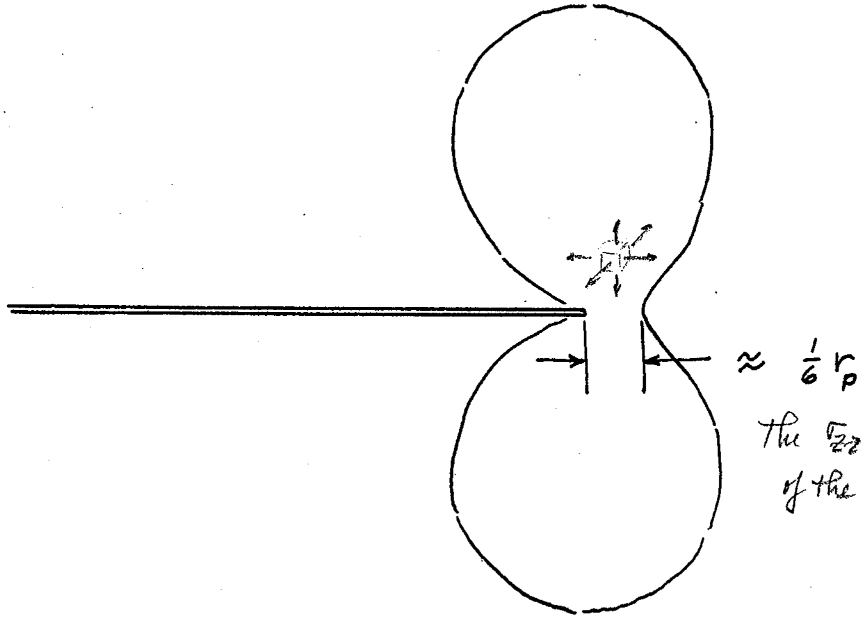


plane stress

$$\frac{r}{r_p} = \cos^2\frac{\theta}{2}\left[1 + 3\sin^2\frac{\theta}{2}\right]$$

$$r_p = \frac{K_I^2}{2\pi\sigma_y^2}$$

$$= \frac{c}{2}\left(\frac{\sigma}{\sigma_y}\right)^2$$



plane strain

$$\frac{r}{r_p} = \cos^2\frac{\theta}{2}\left[(1-2\nu)^2 + 3\sin^2\frac{\theta}{2}\right]$$

$\nu = 0.3$

The σ_{zz} stresses limit the size of the plastic zone

For a thick plate we have plane stress conditions at the surface of the plate and plane strain conditions at the center of the plate. The plastic zone then takes the following shape.

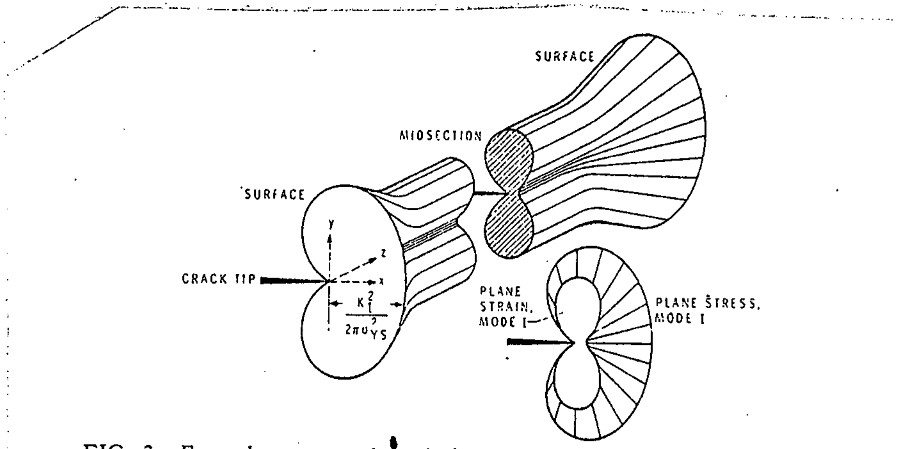


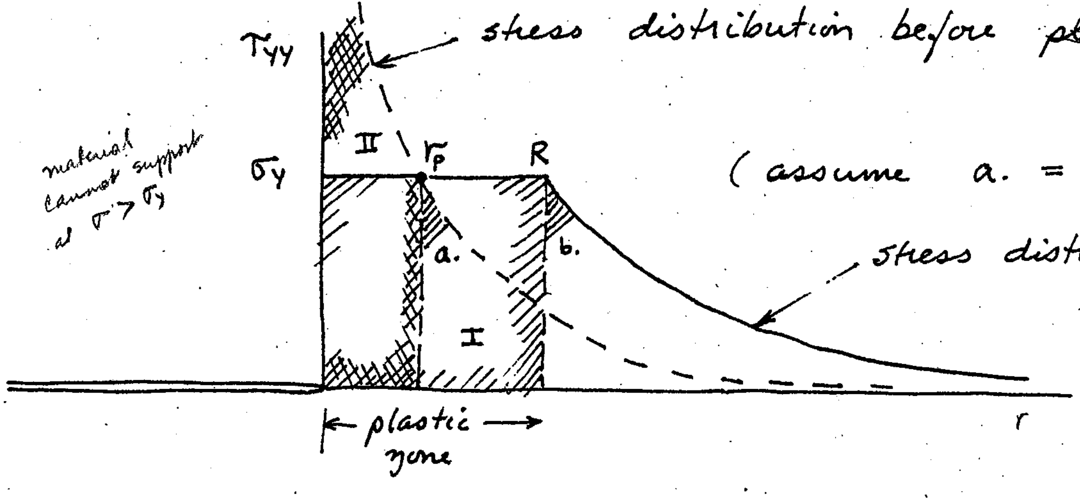
FIG. 3—*Formal representation of plastic zone at the front of a through-thickness crack in a plate.*

Redistribution of the Stresses at The Crack Tip due to Yielding

An estimate of the redistribution of stresses can be made by requiring that the load bearing capacity of the $x=0$ plane in front of the crack remain unchanged when yielding occurs. Thus r_p is too small for the plastic zone depth.

stress distribution before ~~plastic zone~~ yielding

material cannot support $\sigma > \sigma_y$



(assume a. = b.)

stress distribution after ~~plastic zone~~ yielding.

Assuming elastic tail is simply shifted to right, and that it supports the same load as before it follows that

$$II = I$$

or

$$\int_0^{r_p} \tau_{yy}\, dx = \sigma_y R$$

$$\int_0^{r_p} \frac{K_I}{\sqrt{2\pi x}}\, dx = \frac{K_I}{\sqrt{2\pi}}\, 2 r_p^{1/2} = \sigma_y R$$

but $r_p = \dfrac{K_I^2}{2\pi \sigma_y^2}$ or $K_I = \sqrt{2\pi}\, \sigma_y\, r_p^{1/2}$ so

$$= \sqrt{2\pi r_p}\, \sigma_y$$

$$= \sqrt{\pi c}\, \sigma_y \quad \therefore \; c = 2r_p$$ at yield $\sigma = \sigma_y$

$$\boxed{2 r_p = R}$$

Thus plastic zone is about twice as deep as we had calculated previously

So we use

$$R = 2 r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_y}\right)^2$$ <u>plane stress</u> mode I

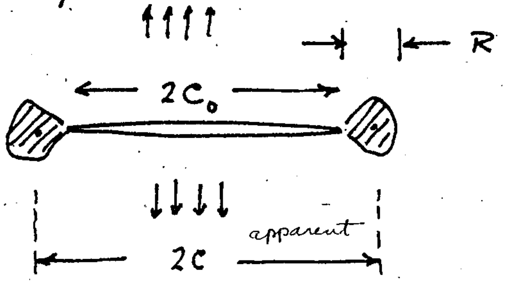
and extend this to plane strain as

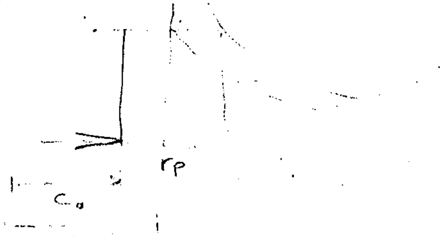
$$R = 2\, \frac{1}{6}\, r_p = \frac{1}{6\pi}\left(\frac{K_I}{\sigma_y}\right)^2$$ <u>plane strain</u> mode I

Fundamentals of Fracture Toughness Testing

We want minimum plastic flow. we want $K_I \rightarrow K_{IC}$ before $\sigma \rightarrow \sigma_y$

Consider a plate loaded in tension:





$\therefore c = c_o + r_p \therefore 2c = 2c_o + \underbrace{2r_p}_{R}$

The stress intensity factor is $\sigma\sqrt{\pi C_0}$ before the plastic zones form and $\sigma\sqrt{\pi C}$ afterward. The effective length of the crack is

$$2C \approx 2C_0 + R$$

where for a plane strain, mode I crack

$$R = \frac{1}{6}\,\frac{1}{\pi}\,\frac{K_I^2}{\sigma_y^2}$$

so

$$C = C_0 + \frac{1}{12\pi}\,\frac{K_I^2}{\sigma_y^2}$$

During loading the stress intensity factor is

$$K_I = \sigma\sqrt{\pi C_0 + \frac{1}{12}\,\frac{K_I^2}{\sigma_y^2}} = \sigma\sqrt{\pi C}$$

As the stress σ increases the stress intensity factor increases not only directly (σ), but also indirectly due to an increase in the effective length of the crack. Solving for the stress intensity factor gives

$$K_I = \sigma\sqrt{\pi c_0}\ \frac{1}{\left\{1 - \frac{1}{12}\left(\frac{\sigma}{\sigma_Y}\right)^2\right\}^{1/2}}$$

As applied stress increases during a test

$$\sigma \longrightarrow \sigma_Y$$

and

$$K_I \rightarrow K_{Ic}$$

If $\sigma = \sigma_Y$ before $K_I = K_{Ic}$ then general yielding occurs and the plastic zones extend to ∞ and hence the small plastic zone restiction does not hold and hence LEFM does not apply. We must make $K_I = K_{Ic}$ before $\sigma = \sigma_Y$ to have an adequate fracture toughness test. For a given crack length $2C_0$, the condition for fracture is

$$K_I = K_{Ic} = \sigma\sqrt{\pi C_0}\ \frac{1}{\left\{1 - \frac{1}{12}\left(\frac{\sigma}{\sigma_Y}\right)^2\right\}^{1/2}}$$

Solving for $2C_0$ we get

$$2C_0 = \frac{2}{\pi}\left(\frac{K_{Ic}}{\sigma}\right)^2\left\{1 - \frac{1}{12}\left(\frac{\sigma}{\sigma_Y}\right)^2\right\} \qquad (K_I = K_{Ic})$$

crack length to produce unstable crack growth.

or

$$2c_0 = \frac{2}{\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2 \left\{ \left(\frac{\sigma_y}{\sigma}\right)^2 - \frac{1}{12} \right\} \qquad K_I = K_{Ic}$$

The crack length that will produce yielding $\sigma = \sigma_y$ when $K_I = K_{Ic}$ is then (letting $\sigma = \sigma_y$)

$$\boxed{2c_0 = \frac{11}{12}\,\frac{2}{\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2} \qquad \underline{\underline{\sigma = \sigma_y \quad \text{when} \quad K_I = K_{Ic}}}$$

if $2c_0 >> \frac{22}{12\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2$ then $K_I \rightarrow K_{IC}$ when $\sigma < \sigma_y$. cracking

if $2c_0 << \frac{22}{12\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2$ than $\sigma \rightarrow \sigma_y$ when $K_I < K_{Ic}$ general yield

$\approx \frac{7}{12}$

When

$$2c_0 \overset{\sim}{>} \frac{1}{2}\left(\frac{K_{Ic}}{\sigma_y}\right)^2 \qquad K_I = K_{Ic} \qquad \sigma < \sigma_y$$

or when

$$2c_0 < \frac{1}{2}\left(\frac{K_{Ic}}{\sigma_y}\right)^2 \qquad K_I = K_{Ic} \qquad \sigma > \sigma_y$$

clearly this is an unacceptable condition for it indicates that general yielding will have occurred prior to fracture ($K_I = K_{Ic}$).

To make an adequate measurement of K_{Ic} we must require that

$$2c_0 >> \frac{1}{2}\left(\frac{K_{Ic}}{\sigma_y}\right)^2$$

$$2c_0 >> \frac{1}{2}\left(\frac{K_{Ic}}{\sigma_y}\right)^2$$

(how much greater can only be determined experimentally.)

Srawley and Brown suggest

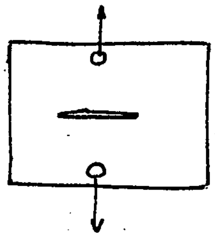
$$2C_0 > (10)\ \frac{1}{2}\left(\frac{K_{Ic}}{\sigma_y}\right)^2$$

which seems reasonable.

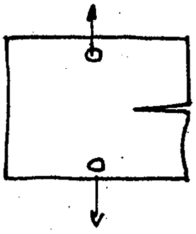
$$\boxed{C_0 > 2.5\left(\frac{K_{Ic}}{\sigma_y}\right)^2}$$ for cracking to occur

Crack Extension Resistance

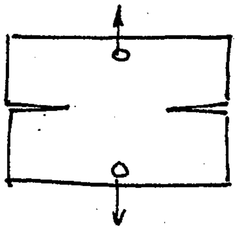
All practical fracture toughness tests involve loading pre-cracked samples, measuring crack extension force G_I or stress intensity factor K_I at which crack runs unstably.



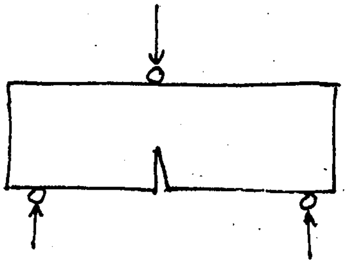
center
cracked
panel



single
edge
crack
(tension)



double
edge
crack



single
edge
crack
(bending)

For perfectly brittle materials G_I increases with loading until $G_I = 2\gamma_s$, at which time unstable crack growth occurs. No crack extension occurs until the instability is reached.

But for materials that deform inelastically (plastically) we expect that unstable crack growth will not occur until $G_I = 2\gamma_s + p$ where p is the plastic work of crack extension. But p is not a constant! It depends on the size of the plastic zone (as well as the yield strength and work hardening rate etc) which in turn depends on the crack length. To illustrate the effects of plastic flow on crack extension we define

$$R = 2\gamma_s + p = \underline{\text{crack extension resistance}}.$$

R = (plastic flow, strain hardening, [illegible] of σ, crack length, geometry, plate thi surface energy)

We let R depend on the plastic properties, σ_Y & θ_{WH}, the crack length, c, and the plate thickness, also depends on applied stress.

For unstable crack growth we must have not only

$$G_I \geqslant R$$ crack extension force must be $\geqslant$ crack extension resis

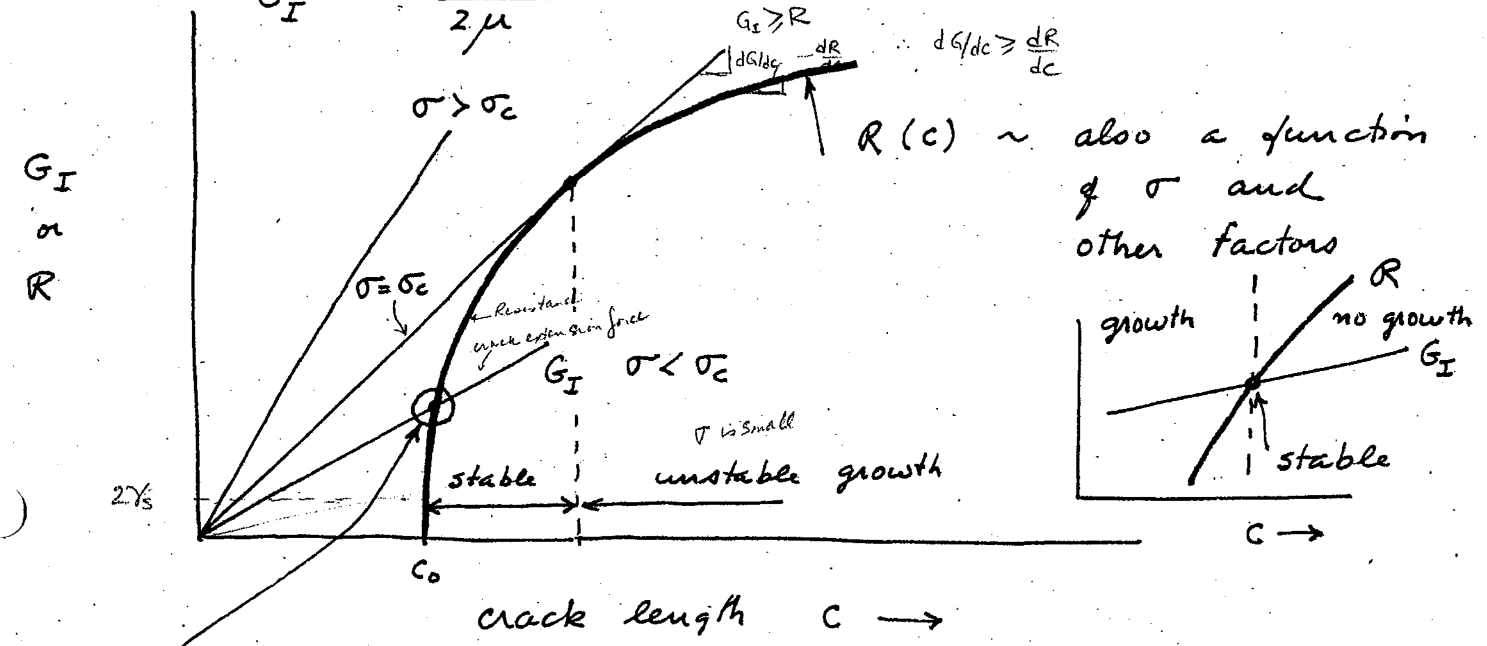
but also

$$\frac{dG_I}{dc} \geqslant \frac{dR}{dc}$$

Proof of 2nd statement

<u>Consider the Following Hypothetical R Curve</u>

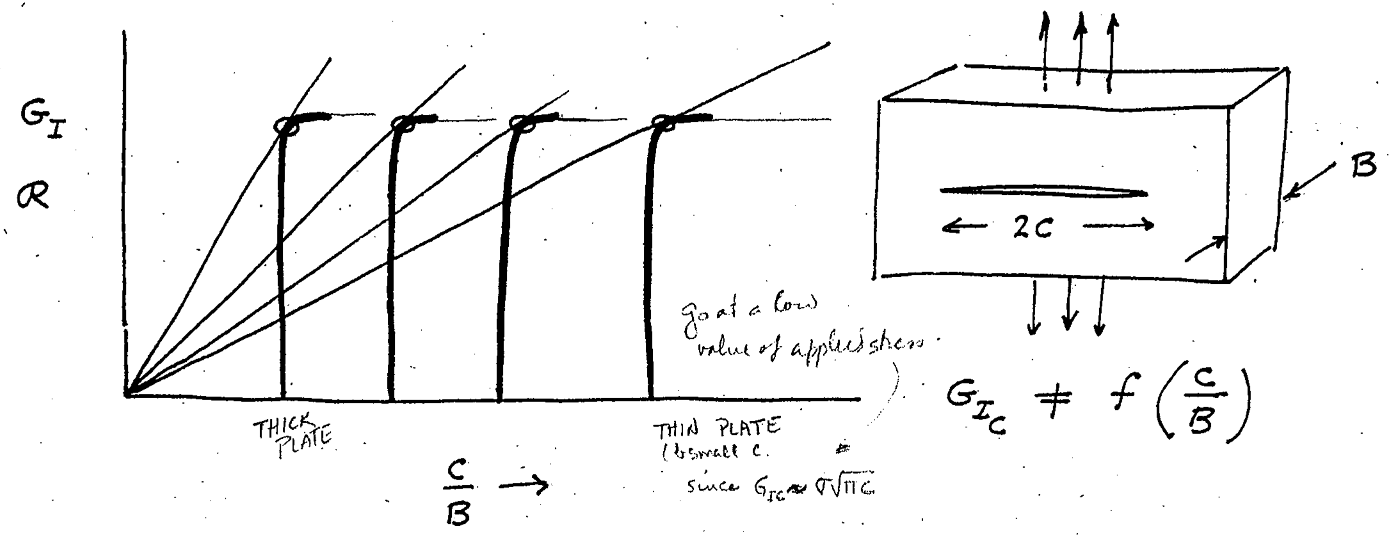
$$G_I = \frac{\sigma^2 \pi c}{2\mu}(1-\nu)$$



R here depends on $\sigma_Y, \theta_{WH}, c, \sigma$ etc.

Hypothetical R curves

Brittle Material – sharply defined instability

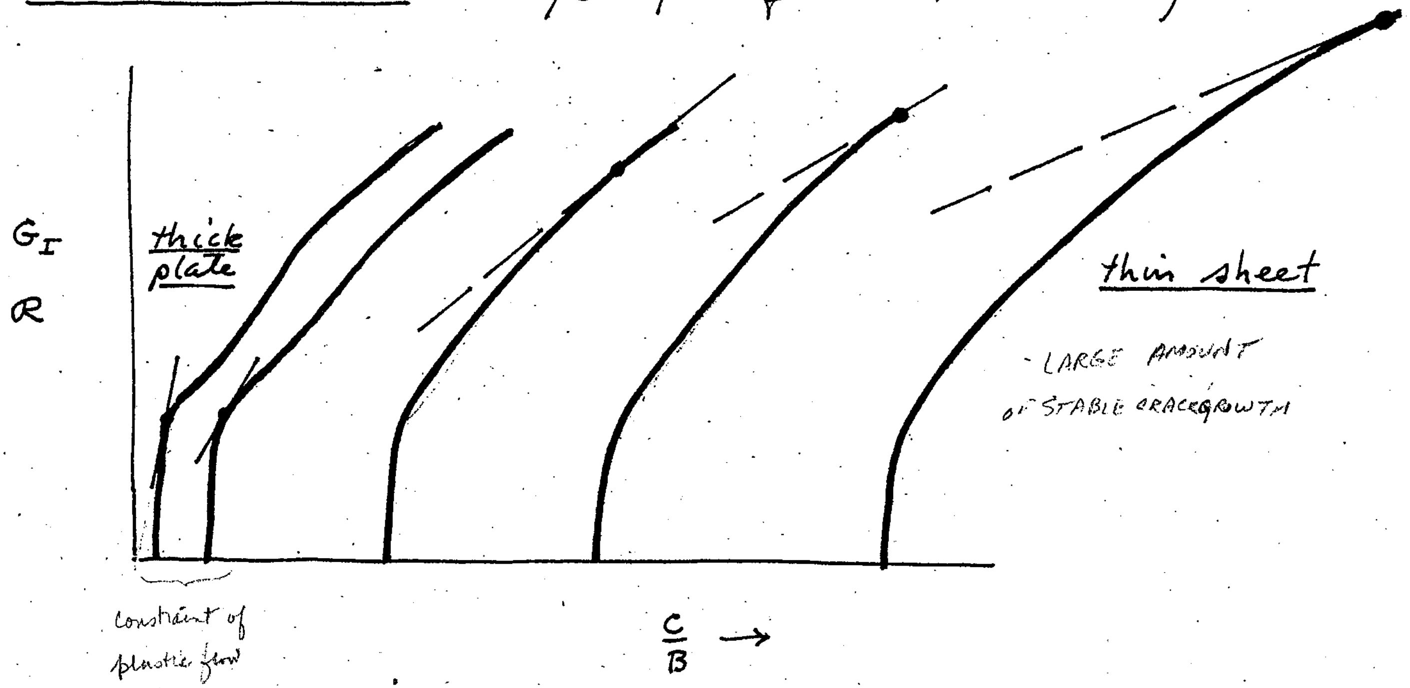


Go at a low value of applied stress

since $G_{Ic} \propto \sigma\sqrt{\pi c}$

$$G_{I_C} \neq f\left(\frac{c}{B}\right)$$

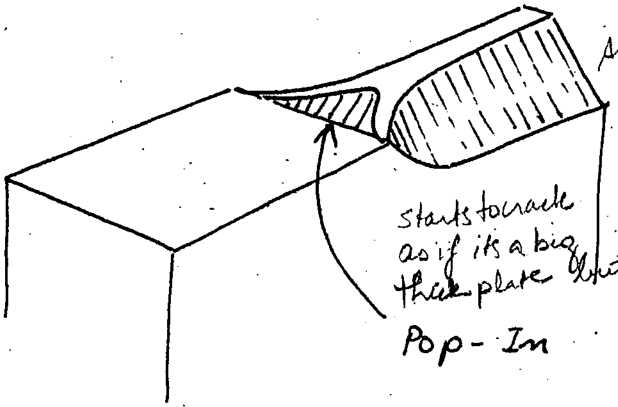
Ductile Material – poorly defined instability



- LARGE AMOUNT OF STABLE CRACKGROWTH

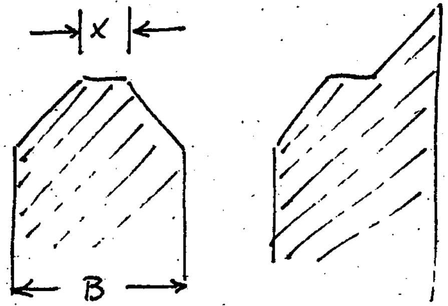
Constraint of plastic flow

Pop-In (meta - Instability)



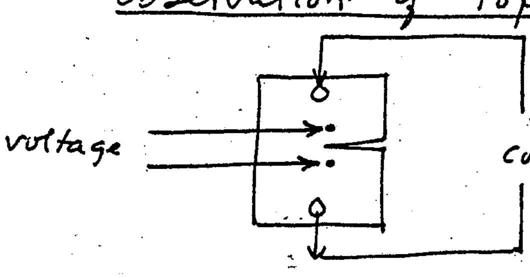
Pop-In meta-instability - in middle of plate.

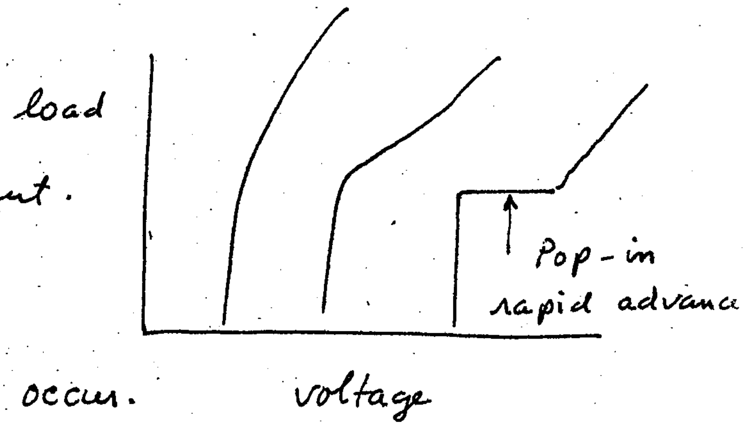
mixed mode



$$\% \text{ square} = 100 \frac{X}{B}$$

Observation of Pop-In



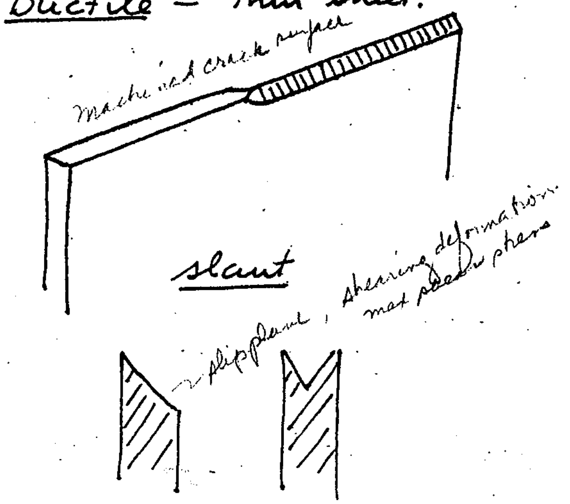


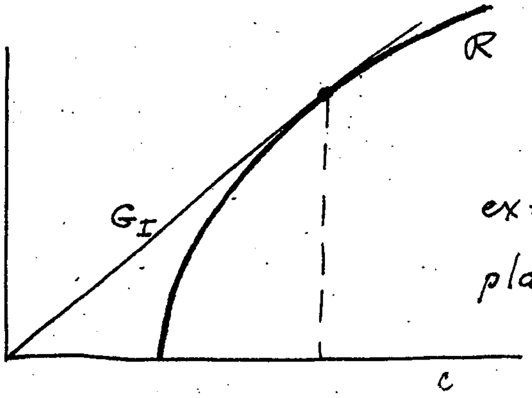
Can actually hear Pop-In occur.
- acoustic signal when part of crack advances. - but shear lips prevent unstable growth.

Relation Between Fracture Surfaces & R Curves

Fracture Surface Type: (macroscopic)

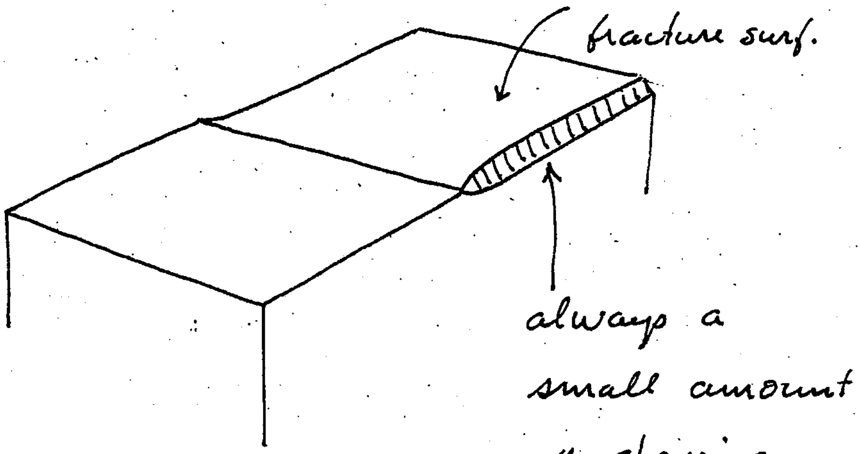
Ductile – Thin sheet.

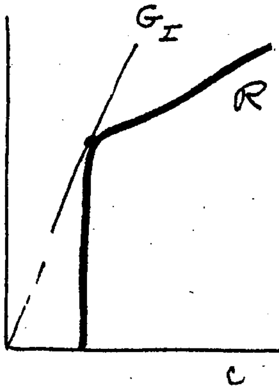




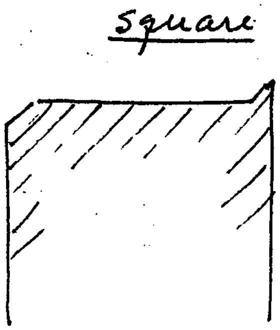
extensive plastic def.

Brittle – thick plate

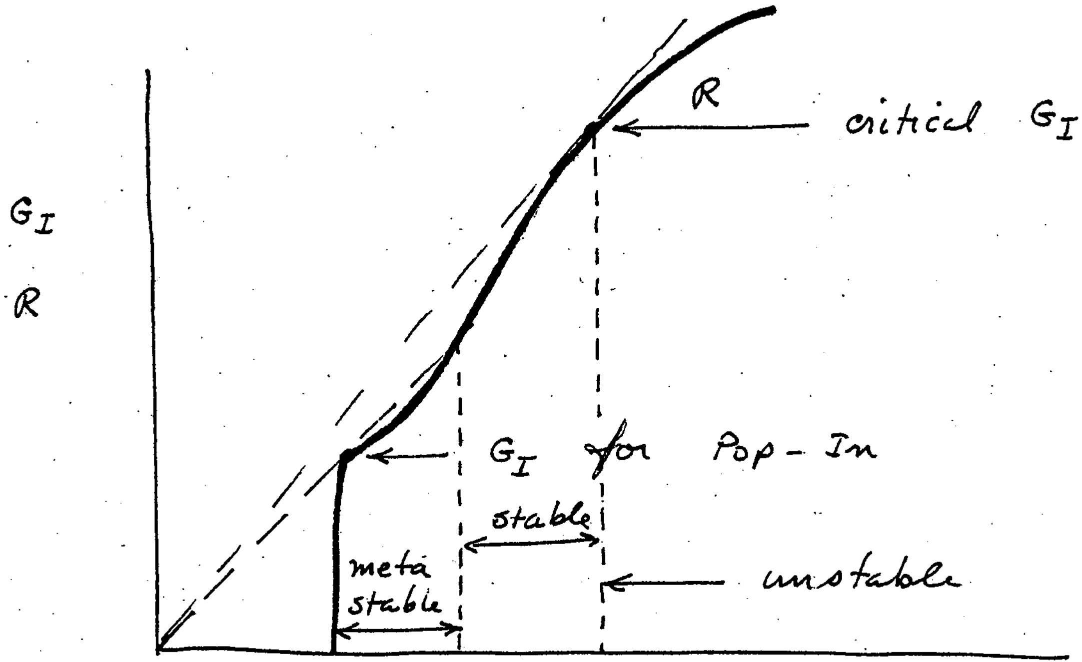




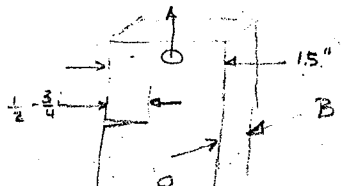
virtually no plastic flow.



R Curve Representation of Pop-In



Summary of G_c vs Plate Thickness



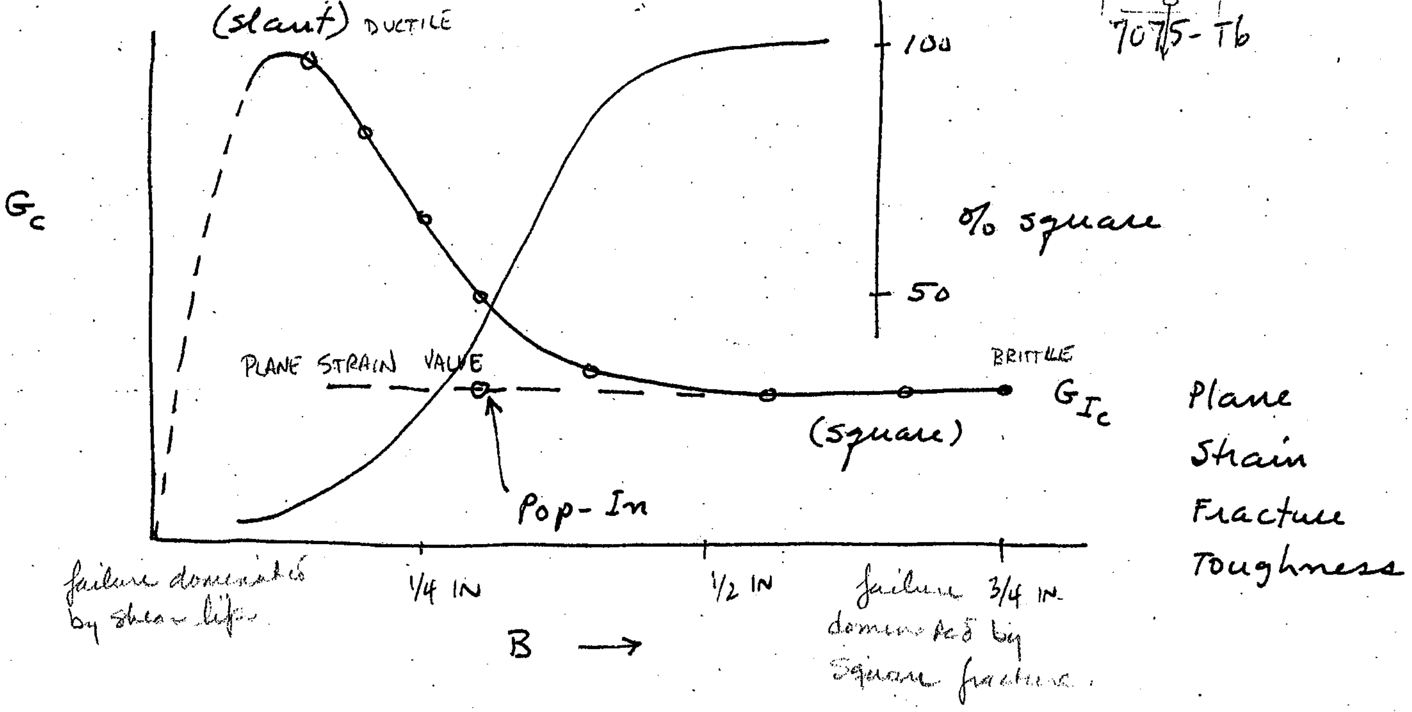
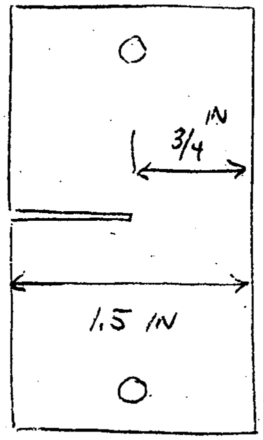
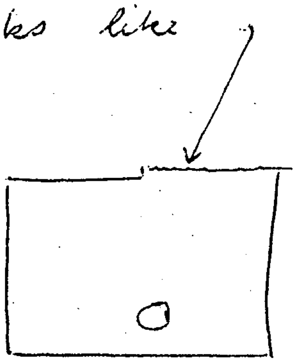


Plate Size Effects – A Speculation

The Present Results were obtained for samples of 7075-T6 shaped as follows

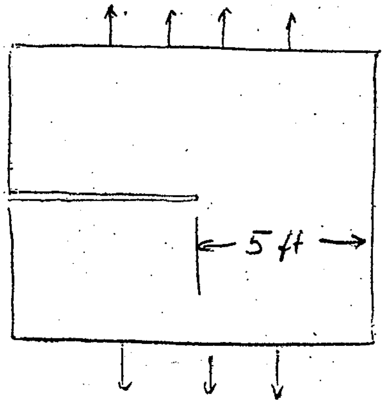


For 3/4 in thick plate the fractured sample looks like



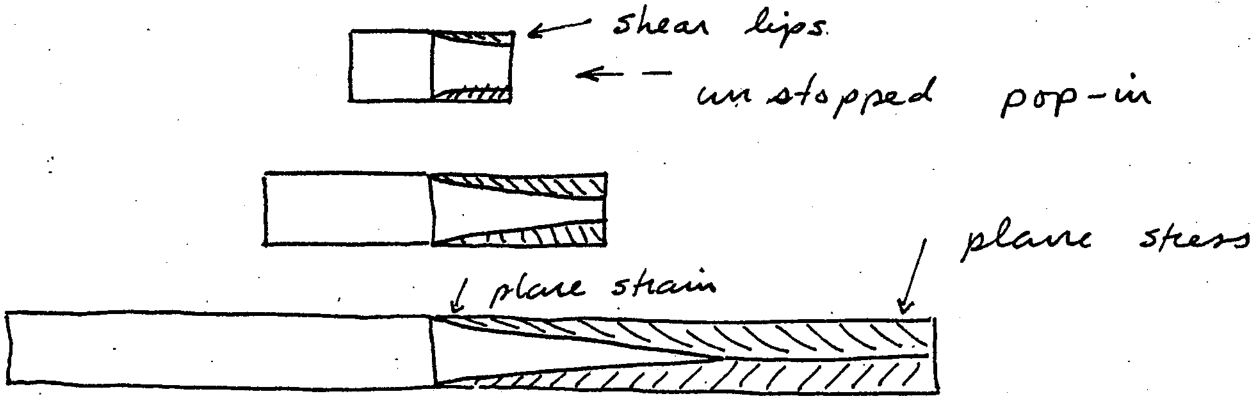
square fracture (plane strain).

To understand the cracking of a very big plate (still 3/4 in thick) say 10 ft x 10 ft., we consider that the above square fracture is simply an un-stopped pop-in!



If the crack has farther to run, then eventually the pop-in will stop and the crack will ultimately develop <u>plane stress</u> conditions. The fracture surface will consist of giant shear lips. The first 3/4 IN of travel will be nearly the same as in the small sample. What is pop-in in the big plate is plane strain fracture in the small one

Consider fracture surface appearance as a function of the dimensions of the plate (3/4^{IN} thick)



For small sample, pop-in produces complete failure. For large plates, the pop-in phenomenon starts in the same way but does not produce indefinite crack extension. The giant shear lips eventually stop the crack extension.

Note of caution: In very big plates pop-in alone may constitute failure. In a gas tank, for example, pop-in might permit leakage. Thus one cannot necessarily take advantage of the <u>plane stress</u> nature of crack extension in very big plates.

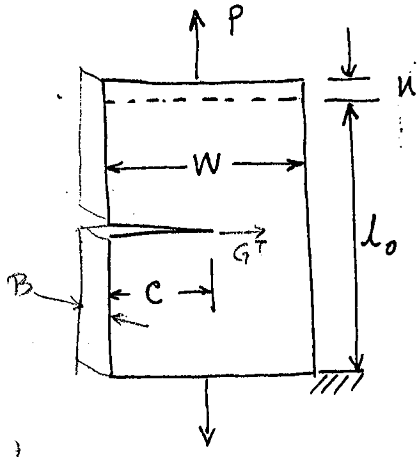
Practical Aspects of Fracture Toughness Testing

Because the plane strain fracture toughness represents a minimum crack resistance (maximum constraint), we focus our attention on that quantity: K_{I_c}.

Calibration of Stress Intensity factors for Samples with Finite Dimensions (K Calibration)

Although we have given rigorous solutions to certain idealized problems (infinite plate, etc), in practice one cannot rely on these expressions. The reason is that it would be impractical to test a sample sufficiently large to closely approximate the infinite plate conditions. Consequently K must be determined experimentally for each sample type.

Consider the single edge crack sample:



The compliance is defined as

$$\lambda = \frac{u}{P}$$

u measured by displacement gage

P measured by load cell.

One measures the compliance as a function of crack length c

Recall on page 54 that the crack extension force is

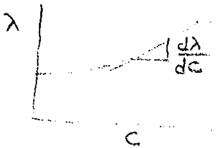
$$G^T = \frac{P^2}{2} \frac{d\lambda}{dC}$$

(good for linear elastic body, not just double cantilever beam configuration)

The stress intensity factor then becomes

$$\boxed{K_I = \sqrt{\frac{2\mu G^T}{(1-\nu)B}} = \sqrt{\frac{\mu}{(1-\nu)B}}\, P \sqrt{\frac{d\lambda}{dc}}}$$

So measured value of the crack length dependence of the compliance gives K.



The results of such experiments may be expressed as

$$K_I = Y \cdot \frac{P\sqrt{c}}{BW} = Y\sigma\sqrt{c}$$

nominal stress

B = plate thickness

W = plate width

where Y is a polynomial in $\left(\frac{c}{W}\right)$

$$Y = 1.99 - 0.41\left(\frac{c}{W}\right) + 18.70\left(\frac{c}{W}\right)^2 - 38.48\left(\frac{c}{W}\right)^3 + 53.85\left(\frac{c}{W}\right)^4$$

for infinite plate $K_I = \sqrt{\pi}\,\sigma\sqrt{c} \sim 1.77\,\sigma\sqrt{c}$

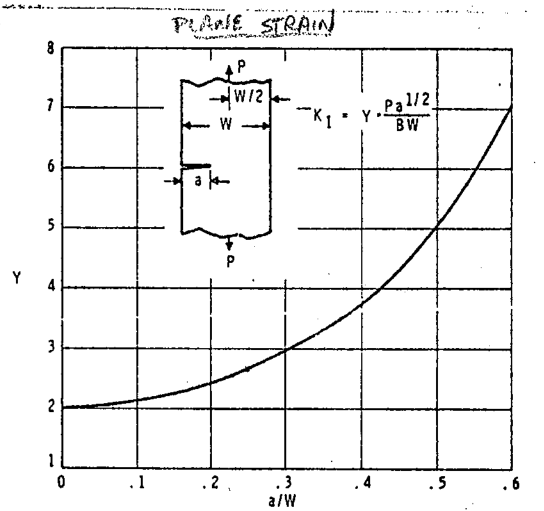


FIG. 6—*K calibration for single-edge-crack tension specimen.*

Other K Calibrations   Ref. Brown + Srawley STP 410

Center - Cracked Plate

$$K_I = \frac{P\sqrt{c}}{BW}\cdot Y \qquad Y = 1{,}77 + 0{,}227\left(\frac{2c}{W}\right) - 0{,}510\left(\frac{2c}{W}\right)^2 + 2{,}7\left(\frac{2c}{W}\right)^3$$

Double - Edge Cracked Plate

$$K_I = \frac{P\sqrt{c}}{BW}\cdot Y \qquad Y = 1.98 + 0.36\left(\frac{2c}{W}\right) - 2.12\left(\frac{2c}{W}\right)^2 + 3.42\left(\frac{2c}{W}\right)^3$$

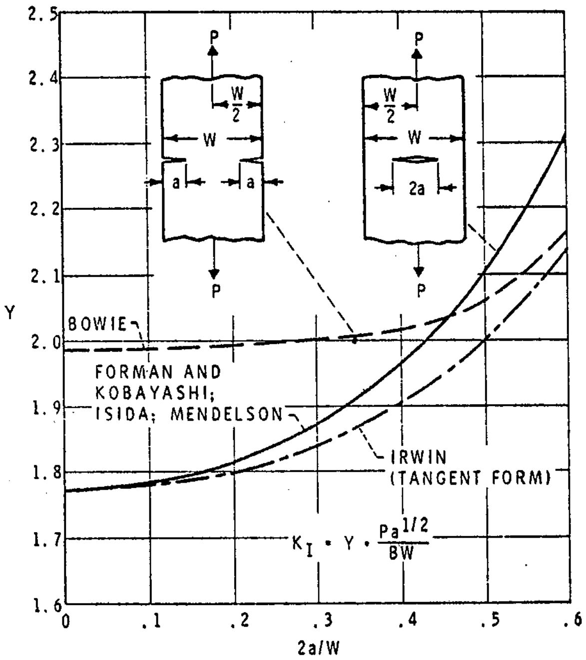


FIG. 5—*K calibrations for the center-cracked and double-edge-cracked specimens.*

Single - Edge Cracked Bend Specimen

$$K_I = \frac{6M\sqrt{c}}{BW^2} \cdot Y \qquad Y = A_0 + A_1\left(\frac{c}{W}\right) + A_2\left(\frac{c}{W}\right)^2 + A_3\left(\frac{c}{W}\right)^3 + A_4\left(\frac{c}{W}\right)^4$$

$$\sigma = \frac{M}{I}$$

	A_0	A_1	A_2	A_3	A_4
Pure Bending (4 point)	+1.99	−2.47	+12.97	−23.17	+24.80
3 Point Bending					
S/W = 8	+1.96	−2.75	+13.66	−23.98	+25.22
S/W = 4	+1.93	−3.05	+14.53	−25.11	+25.80

where M = bending moment.

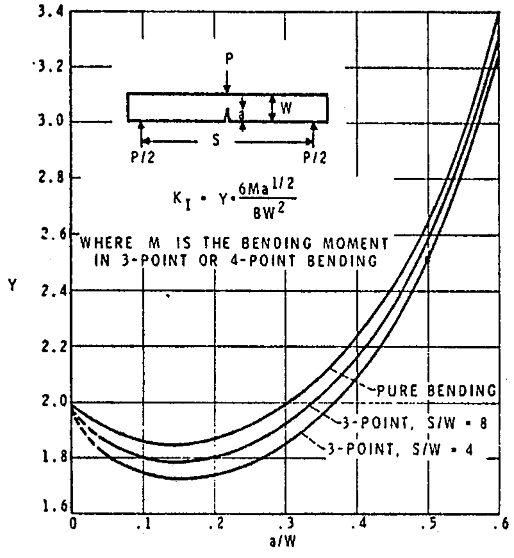


FIG. 7—*K calibrations for bend specimens.*

Specimen Size Requirements

We have argued that to limit yielding we must make large samples with long cracks. Thus $K_I \rightarrow K_{I_c}$ before $\sigma \rightarrow \sigma_y$. From our analysis we expect

$$C_o \geqslant 2.5 \left(\frac{K_{I_c}}{\sigma_y} \right)^2$$

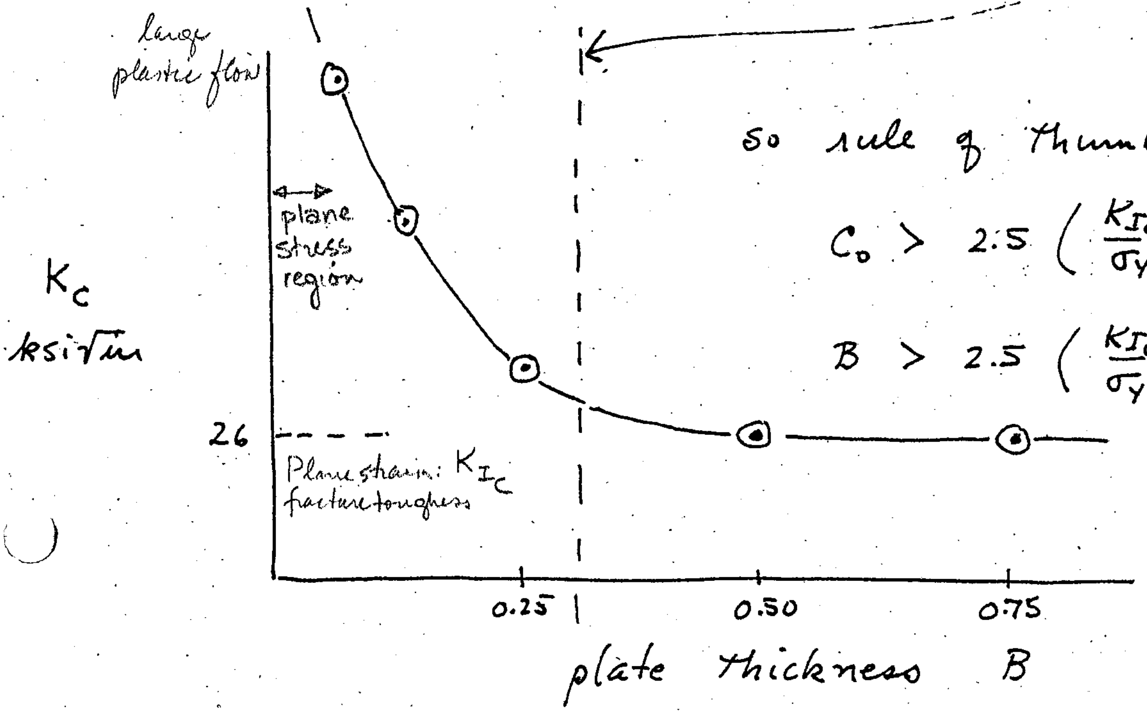
this needs to be checked. Also, How thick must sample be for plane strain conditions?

Consider 7075-T6 (MS+E 202C experiment).

$\sigma_y = 75$ ksi

$K_{I_c} = 26$ ksi $\sqrt{in}$

$$2.5 \left(\frac{K_{I_c}}{\sigma_y} \right)^2 = 0.3 \text{ in}$$



so rule of thumb:

$$C_o > 2.5 \left(\frac{K_{I_c}}{\sigma_y} \right)^2$$

$$B > 2.5 \left(\frac{K_{I_c}}{\sigma_y} \right)^2 \longleftarrow$$

usually more difficult to achieve due to how the specimen is machine

Specimen size Requirements

we have already discussed the fact that there is a critical thickness to achieve plane strain conditions and to measure K_{I_c}. Also, there is a critical crack length. Generally speaking, there are no reliable techniques for predicting the critical crack length or critical thickness. One simply measures K_c to determine if it is independent of crack length or thickness.

Crack Length Effects

Example: Maraging steel. $\sigma_y = 242$ KSI

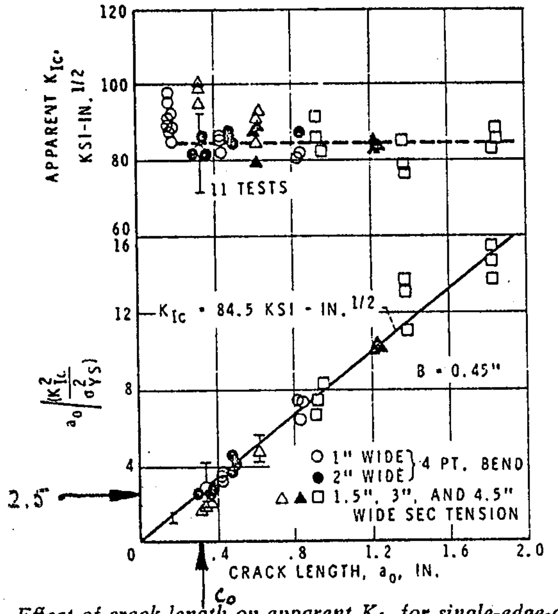


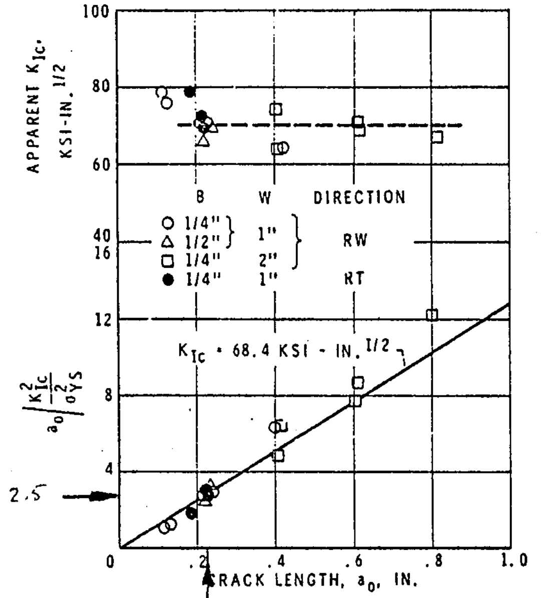
FIG. 10—*Effect of crack length on apparent K_{Ic} for single-edge-cracked tension and bend tests on 242 yield strength maraging steel.*

פלדת מריג׳ינג 242

Conclusion:

$$\frac{C_o}{\left(K_{Ic}^2/\sigma_Y^2\right)} \geqslant 2.5 \qquad \text{hence} \qquad C_o \geqslant 2.5 \left(\frac{K_{Ic}}{\sigma_Y}\right)^2$$

Example: Maraging Steel $\sigma_y = 259$ ksi



min C_o = arrow

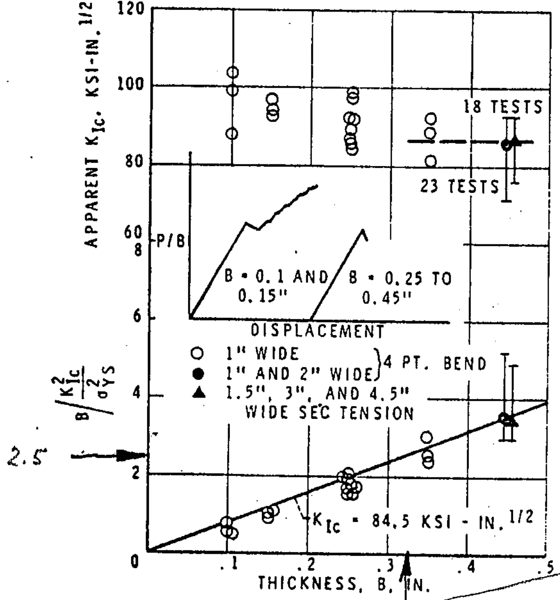
FIG. 11—*Effect of crack length on apparent K_{Ic} for 4-point bend tests on 259 ksi yield strength maraging steel.*

פלדת מריג׳ינג

conclusion $C_o \geqslant 2.5 \left(\frac{K_{Ic}}{\sigma_y}\right)^2$

Plate Thickness Effects

Example: Maraging Steel $\sigma_y = 242$ ksi



min B

conclusion:

$B \geqslant 2.5 \left(\frac{K_{Ic}}{\sigma_y}\right)^2$

FIG. 13—*Effect of thickness on apparent K_{Ic} for 242 ksi yield strength maraging steel tested using bend and single-edge-crack tension specimens.*

Example: Maraging Steel $\sigma_Y = 259$ Ksi

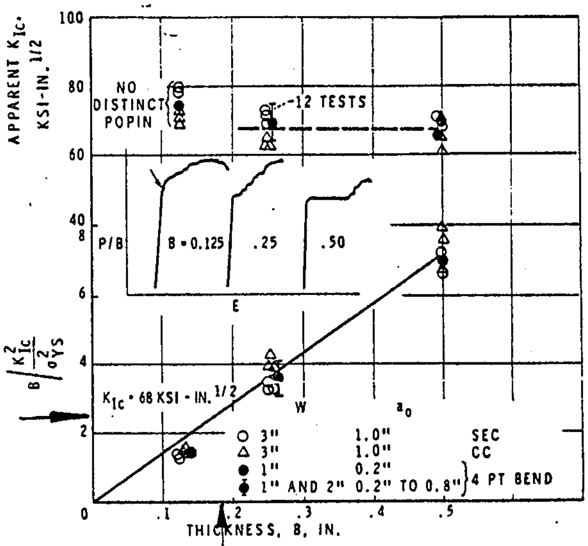


FIG. 14—*Effect of thickness on popin behavior and apparent K_{Ic} for 259 ksi yield strength maraging steel tested using several specimen types.*

conclusion :

$$B \geq 2.5 \left(\frac{K_{Jc}}{\sigma_Y}\right)^2$$

Instrumentation

measurement of crack face displacements.

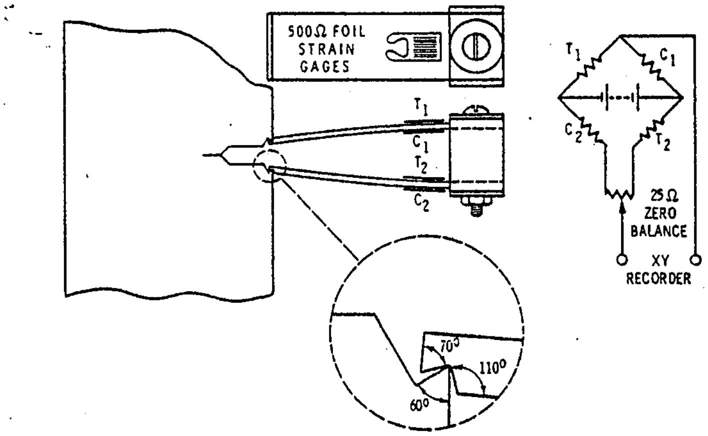


FIG. 19—*Double cantilever beam gage and method of mounting on crack-notched specimen for displacement measurement (designed by J. E. Srawley).*

It is possible to compute the crack length from the measured displacements across the crack faces.

$\frac{u}{P}$ not exactly the compliance

$\frac{EuB}{P}$

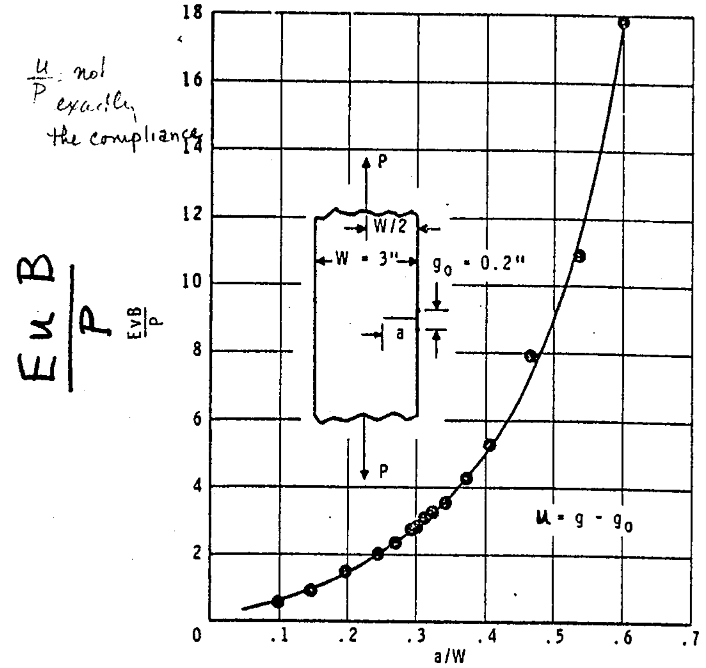
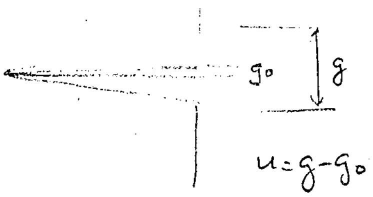


FIG. 22—*Calibration curve for converting displacement measurements to crack length single-edge-crack tension specimens.*



$u = g - g_0$

gives a more sensitive measurement of crack running; however u & P are not actually related by $\underline{u = \lambda P}$ since u is not measured where P is applied.

<u>Sample shapes</u>

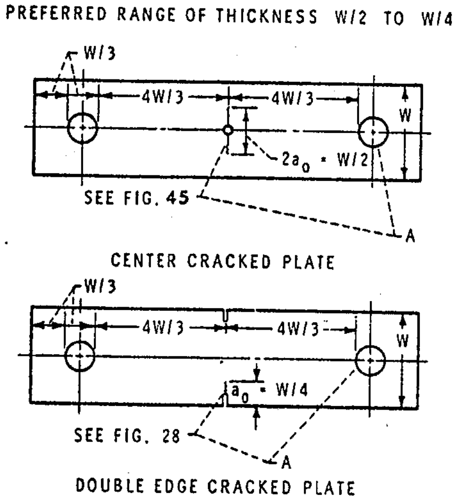


FIG. 42—*Proportions for center- and double-edge-cracked plate specimens. A-surfaces must be symmetric to specimen centerline within W/1000.*

✓

W = 1.5 IN

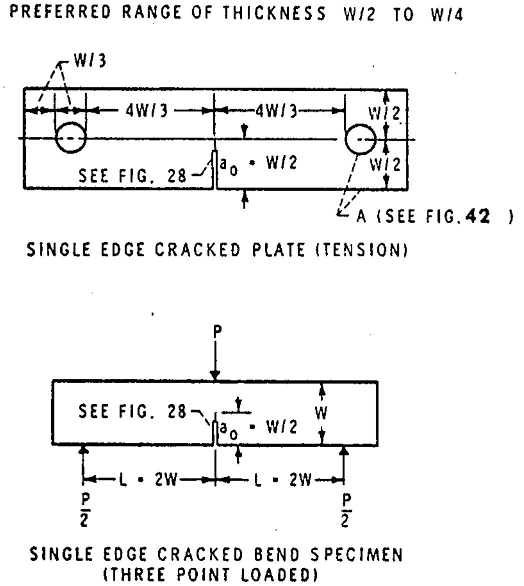


FIG. 43—*Proportions for single-edge-cracked tension and bend specimens.*

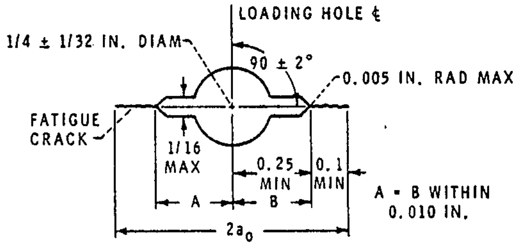


FIG. 45—*Fatigue crack starter for center-cracked plate specimens.*

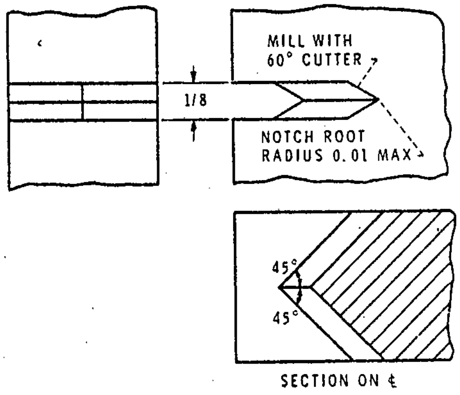


FIG. 28—*Chevron notch for edge-crack plate specimens.*

The J Integral

As we have discussed before, Griffith's Thermodynamic treatment of unstable crack growth applies strictly only in the case of perfectly elastic, brittle materials. Thus far no rigorous treatment for the case of elasti-plastic materials has been presented. Irwin's modification of the Griffith theory to include plasticity is useful in a qualitative way and often provides a basis for solving engineering problems. However the Irwin modification ($2\gamma_s + p$) does not realistically account for the redistribution of stresses at the crack tip when plastic flow occurs.

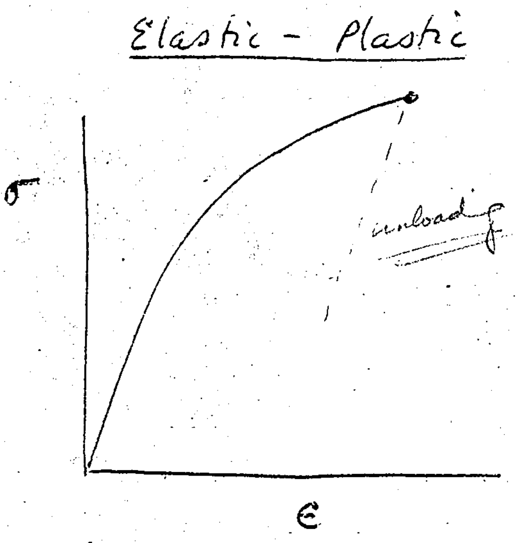
About 50 years ago J.R. Rice developed a method of analysis, called The J Integral Metho That permits an exact treatment of certain aspects of The crack tip stresses and strains. (restricted to 2 dimensional fields).

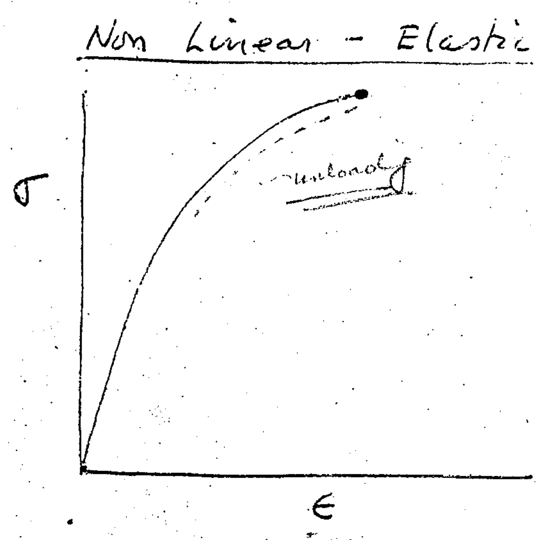
Ref. J.R. Rice Journal Applied Mech. June 1968, 379.

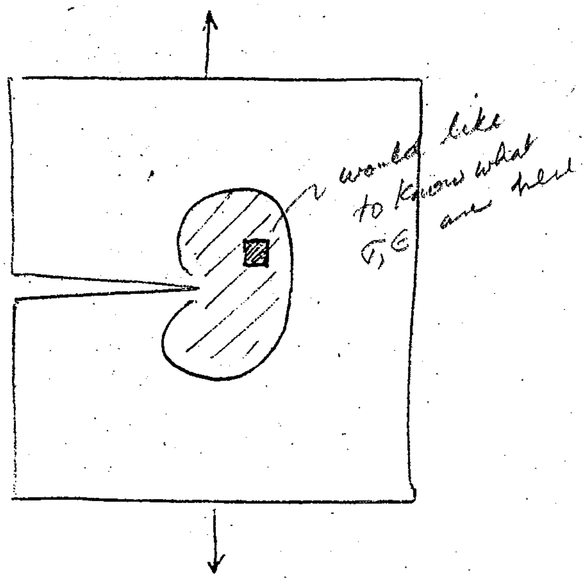
The key idea of the J Integral Method is That the stresses and strains at the

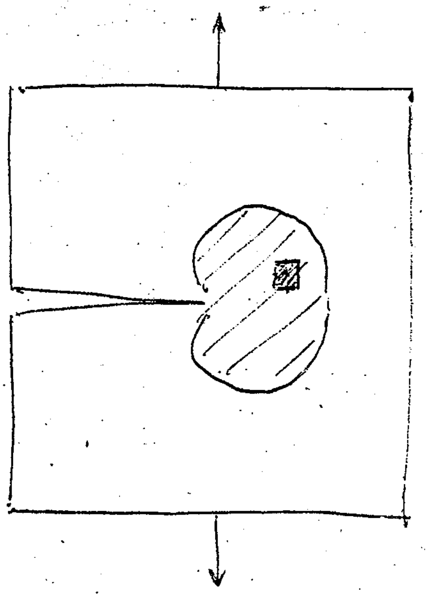
ONLY GOOD FOR SMALL STRAINS

tip of a crack in an elastic - plastic body are exactly the same as the stresses and strains at the tip of a crack in a non-linear elastic body provided that the stress - strain relationships are the same.


Elastic - Plastic


Non Linear - Elastic





does not work for crack unloading

For monotonic loading (no unloading) — elastic-plastic problem indistinguishable from non-linear elastic problem.

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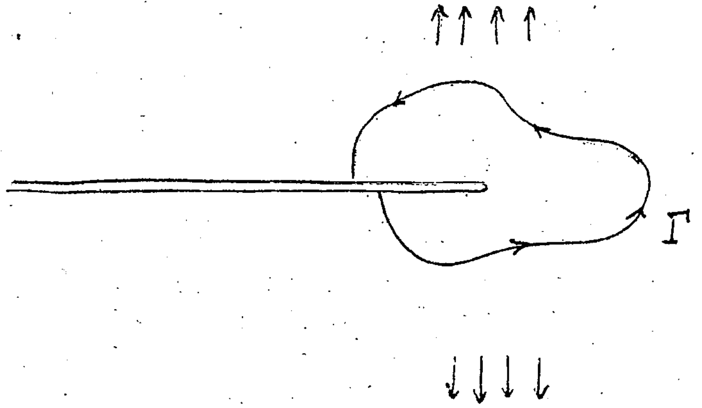
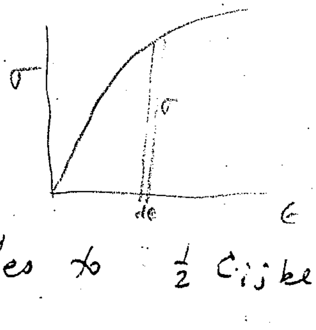
If we know something about the stresses and strains for the non-linear elastic problem, then the results apply to the elastic-plastic case as well.

With the above relationships in mind, we consider a sharp crack in a non-linear elastic solid.



Consider any path Γ which starts at the bottom crack face and travels in a contraclockwise sense about the crack tip to the top crack face. Define (in the usual way) the strain energy density for this non-linear elastic material as

$$W = \int_0^{\epsilon} \sigma_{ij} \, d\epsilon_{ij}$$



(for linear elasticity this integrates to $\frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl}$ or $\frac{1}{2} \sigma_{ij} \epsilon_{ij}$)

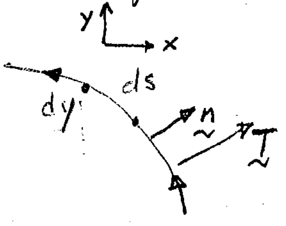
The strain energy density is a state property from which the stress may be obtained as

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$$

The J Integral – A Path Independent Integral

Define

$$J = \int_{\Gamma} \left(W\,dy - T_i \frac{\partial u_i}{\partial x}\, ds \right)$$



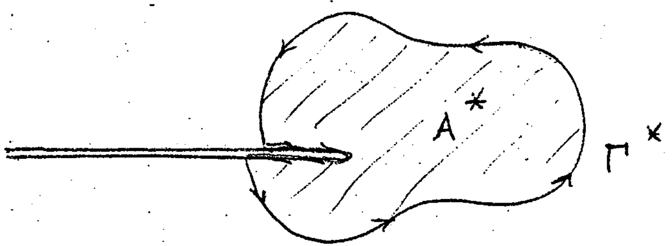
Where T_i is the traction vector defined according to the outward pointing normal along Γ

u_i is the displacement vector

ds is an element of arc along Γ

dy is the y component of ds.

For <u>any</u> non-linear elastic material, J is independent of the path Γ. Consider any <u>closed</u> path Γ^* that encloses an area A^*:



The J Integral becomes

$$\int_{\Gamma^*}\left(W\,dy - T_i \frac{\partial u_i}{\partial x}\,ds\right) = \iint_{A^*}\left[\frac{\partial W}{\partial x} - \frac{\partial}{\partial x_j}\left(\sigma_{ij}\frac{\partial u_i}{\partial x}\right)\right]dx\,dy$$

by the Green-Gauss Theorem. Any good book on Advanced Calculus shows

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

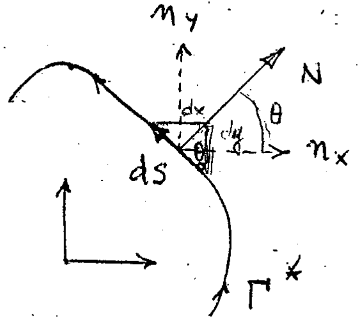
"Greens Theorem"
Adv. Calc.
Kaplan
p. 239.

To show this we write

$$T_i = \sigma_{ij}\, n_j .$$

or

$$T_i = \sigma_{ix}\, n_x + \sigma_{iy}\, n_y$$



$n_x\,ds = dy$

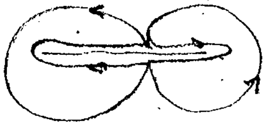
$n_y\,ds = -dx$

[redacted]

$$\int_{\Gamma^*}\left(W\,dy - (\sigma_{ix}\, n_x + \sigma_{iy}\, n_y)\frac{\partial u_i}{\partial x}\,ds\right)$$

$$\int_{\Gamma^*} \underset{\underset{Q}{\Uparrow}}{\left(W - \sigma_{ix}\frac{\partial u_i}{\partial x}\right)} dy + \underset{\underset{P}{\Uparrow}}{\left(\sigma_{iy}\frac{\partial u_i}{\partial x}\right)} dx$$

So using Green-Gauss Theorem



$$\int_{\Gamma^*} \left(W\,dy - T_i \frac{\partial u_i}{\partial x}\,ds \right)$$

$$= \iint_{A^*} \left[\frac{\partial}{\partial x}\left(W - \sigma_{ix} \frac{\partial u_i}{\partial x} \right) - \frac{\partial}{\partial y}\left(\sigma_{iy} \frac{\partial u_i}{\partial x} \right) \right] dx\,dy$$

$$= \iint_{A^*} \left[\frac{\partial W}{\partial x} - \frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial x} \right) \right] dx\,dy$$

Now we can show that $\iint_{A^*} [\quad]\,dx\,dy = 0$

$$\frac{\partial W}{\partial x} = \frac{\partial W}{\partial \epsilon_{ij}} \frac{\partial \epsilon_{ij}}{\partial x} = \sigma_{ij} \frac{\partial \epsilon_{ij}}{\partial x}$$

$W = \int \sigma_{ij}\,d\epsilon_{ij}$
$\partial W = \sigma_{ij}\,\partial\epsilon_{ij}$

But since

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\frac{\partial W}{\partial x} = \frac{1}{2} \sigma_{ij} \left[\frac{\partial}{\partial x} \frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x} \frac{\partial u_j}{\partial x_i} \right]$$

since we are double summing & $\sigma_{ij} = \sigma_{ji}$
$\sigma_{ij} \frac{\partial u_i}{\partial x_j} = \sigma_{ji} \frac{\partial u_j}{\partial x_i}$

$$= \sigma_{ij} \frac{\partial}{\partial x} \frac{\partial u_i}{\partial x_j}$$ since $\sigma_{ij} = \sigma_{ji}$

$$= \sigma_{ij} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x}$$ if quantities are continuous & if body is isotropic

Finally since $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$ (equilibrium)

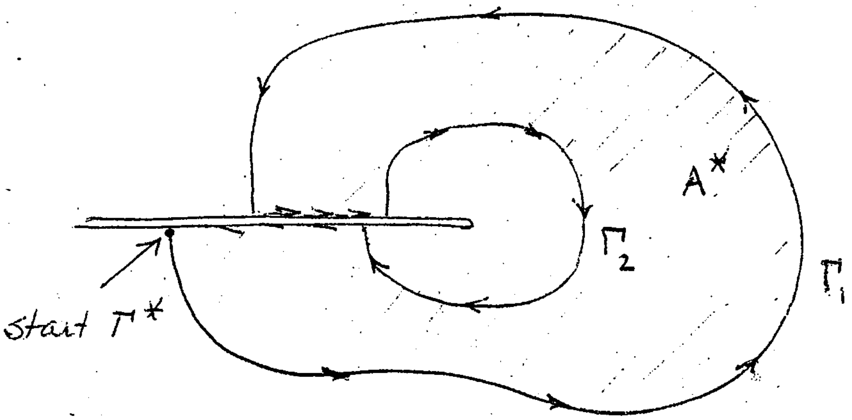
we can write

$$\frac{\partial W}{\partial x} = \sigma_{ij} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x} = \frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial x} \right) = \cancelto{0}{\frac{\partial \sigma_{ij}}{\partial x_j}} \frac{\partial u_i}{\partial x} + \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x}\right)$$

Thus the integral $\iint_{A^*} [\quad] \, dx\, dy = 0$
and

$$\int_{\Gamma^*} \left(W dy - T_i \frac{\partial u_i}{\partial x} ds \right) = 0$$ for any closed path Γ^*

Now consider any two paths Γ_1 and Γ_2.



along crack faces $dy = 0$ and $T_i = 0$ so

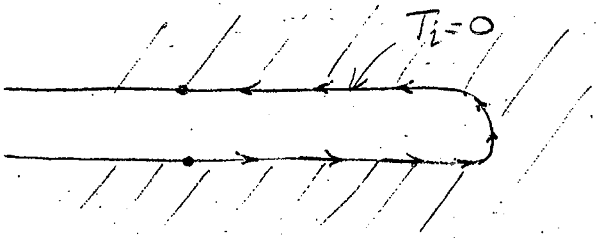
$$\int_{\Gamma^*} = \int_{\Gamma_1} + \int_{\Gamma_2} = 0$$

Thus $$\int_{\Gamma_1} = -\int_{\Gamma_2}$$

Thus the integral

$$J = \int_{\Gamma} \left(W dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$ is <u>path independent</u>

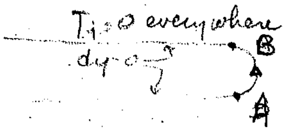
Since J is path independent, it may be evaluated along any path, even one that goes though the crack tip field. Suppose Γ_t is taken along crack surface (for a smooth crack tip)



can be aribitrarily sharp!

Since no tractions on surfaces,

$$J = \int_{\Gamma_t} W dy$$



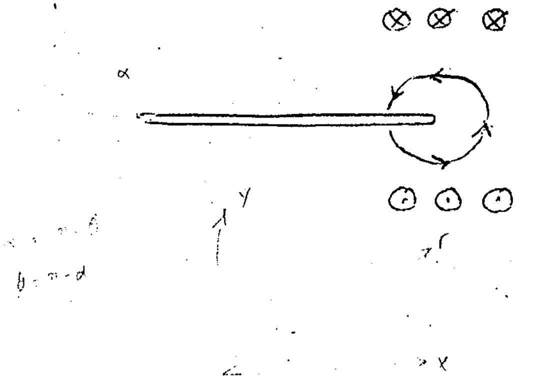
$$\therefore J = \int_A^B W dy$$

<u>so J depends only on the strain energy density at the crack tip!</u>

J Integral – Mode III Linear Elastic

$$J = \int_{\Gamma} \left(W\,dy - T_i \frac{\partial u_i}{\partial x} \right) ds$$

To illustrate the meaning of the J integral consider the linear elastic Mode III crack tip field.



$$\tau_{zy} = -\frac{K_{III}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$$

$$\tau_{zx} = \frac{K_{III}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}$$

$$u_z = -\frac{K_{III}}{\mu}\sqrt{\frac{2r}{\pi}} \sin\frac{\theta}{2}$$

$$\tau_{zr} = \tau_{zx}\cos\theta + \tau_{zy}\sin\theta \quad = \frac{K_{III}}{\sqrt{2\pi r}} \; -\sin\left(\frac{\theta}{2}\right) \qquad \sin\theta/2\cos\theta - \cos\theta/2\sin\theta \quad \theta/2 - \theta$$

$$\tau_{z\theta} = -\tau_{zx}\sin\theta + \tau_{zy}\cos\theta \qquad \overset{\cos\theta/2}{\cos\theta/2\sin\theta - \cos\theta\sin\theta/2 = \sin\theta/2}$$

Using Trig identities

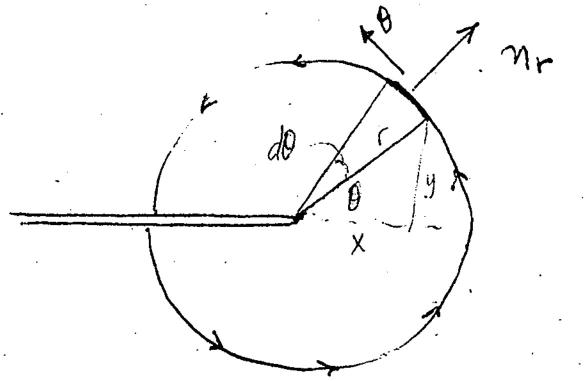
$$\tau_{zr} = -\frac{K_{III}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \qquad e_{zr} = -\frac{K_{III}}{2\mu\sqrt{2\pi r}} \sin\frac{\theta}{2}$$

$$\tau_{z\theta} = -\frac{K_{III}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \qquad e_{z\theta} = -\frac{K_{III}}{2\mu\sqrt{2\pi r}} \cos\frac{\theta}{2}$$

Now the strain energy density is

$$W = \frac{1}{2}\tau_{ij}e_{ij} = \frac{1}{2}\left(\tau_{zr}e_{zr} + \tau_{z\theta}e_{z\theta}\right) = \frac{K_{III}^2}{8\pi r\mu}$$

The traction vector is



$$T_i = \tau_{ij}\, n_j = \tau_{ir}\, n_r + \tau_{i\theta}\, n_\theta$$

$$T_z = \tau_{zr}\, n_r = \tau_{zr}$$

$$u_z = -\frac{K_{III}}{\mu}\sqrt{\frac{2r}{\pi}}\,\sin\frac{\theta}{2}$$

but $r\sin\theta = y$ $r\cos\theta = x$ $x^2+y^2 = r^2$

$\tan\theta = \frac{y}{x}$

$$u_z = -\sqrt{\frac{2}{\pi}}\,\frac{K_{III}}{\mu}\,(x^2+y^2)^{1/4}\sin\left(\frac{1}{2}\tan^{-1}\frac{y}{x}\right)$$

$$\frac{\partial u_z}{\partial x} = -\sqrt{\frac{2}{\pi}}\,\frac{K_{III}}{\mu}\left[\frac{1}{4}(x^2+y^2)^{-3/4}\,2x\,\overset{\cos}{\sin}\left(\frac{1}{2}\tan^{-1}\frac{y}{x}\right)\right.$$

$$\left. + (x^2+y^2)^{1/4}\cos\left(\frac{1}{2}\tan^{-1}\frac{y}{x}\right)\frac{1}{2}\,\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)\right]$$

$-\sin(\quad)$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{K_{III}}{\mu}\left[\frac{1}{2}r^{-3/2}\,r\cos\theta\,\overset{\cos}{\sin}\frac{\theta}{2} + r^{1/2}\cos\frac{\theta}{2}\,\frac{1}{2}\,\frac{-y}{x^2+y^2}\right]$$

$\theta/2$ $\sin(\quad)$

$$= -\frac{2K_{III}}{\sqrt{2\pi r}\,\mu}\left[\frac{1}{2}\overset{\cos}{\sin}\frac{\theta}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) - \frac{1}{2}\,2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right]$$

$+ \sin\theta/2 \quad 2\sin\theta/2 \cos\theta/2$

$\cos\theta/2 (\quad) + \sin^2\theta/2 \cos\theta/2$

$$= + \frac{K_{III}}{\sqrt{2\pi r}\,\mu}\left[\sin\frac{\theta}{2}\right]$$

$\frac{1}{2}\cos\theta/2\,(c^2 - s^2 + 2s^2)$

Now the J integral is

$$J = \int_{-\pi}^{\pi} \left[W\,dy - T_i \frac{\partial u_i}{\partial x}\,ds \right].$$

integrates to zero

$$= \int_{-\pi}^{\pi} \left[\frac{K_{III}^2}{8\pi r \mu} r\cos\theta\,d\theta \mp \overbrace{\left(-\frac{K_{III}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\right)}^{+\tau_{zr}} \overbrace{\frac{K_{III}}{\sqrt{2\pi r}\,\mu} \sin\frac{\theta}{2}}^{\frac{\partial u_z}{\partial x}}\, r\,d\theta \right]$$

$$= \frac{K_{III}^2}{2\pi\mu} \int_{-\pi}^{\pi} \sin^2\frac{\theta}{2}\,d\theta$$

Let $\alpha = \frac{\theta}{2}$ $\quad d\theta = 2\,d\alpha$.

$$J = \frac{K_{III}^2}{\pi\mu} \int_{-\pi/2}^{\pi/2} \sin^2\alpha\,d\alpha = \frac{K_{III}^2}{2\mu} = G_{III}$$

So for linear elastic crack, the J integral is the crack extension force G_{III}. only!

For Mode I

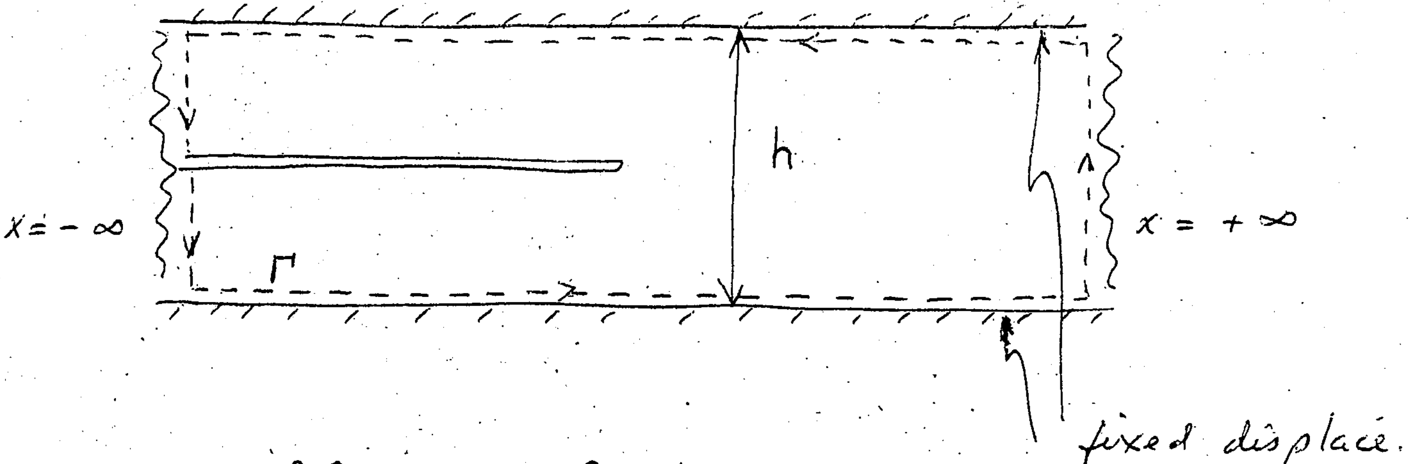
$$J = G_I = \frac{K_I^2(1-\nu)}{2\mu}$$

For Mode II

$$J = G_{II} = \frac{K_{II}^2(1-\nu)}{2\mu}$$

Evaluation of The J Integral by Convenient Path Selection

Infinite strip with crack, subjected to constant displacement.



$$J = \int_\Gamma \left[W\,dy - T_i \frac{\partial u_i}{\partial x}\,ds \right]$$

along clamped boundaries $dy = 0$ $\frac{\partial u_i}{\partial x} = 0$

since displacements are uniform ∴ $\frac{\partial u_i}{\partial x} = 0$.

at $x = -\infty$ $W = 0$ $\frac{\partial u_i}{\partial x} = 0$

since body is far away from crack tip would be like 2 cantilevers one displacing relative to the other

since body rigidly displaces at ∞

so J determined by $x = +\infty$ where $\frac{\partial u_i}{\partial x} = 0$

since body displaces rigidly ∴ $u_i(x+dx, y) = u_i(x,y)$

$$J = W_{x=\infty}\, h$$

uniform at ∞ since it is so far away from tip

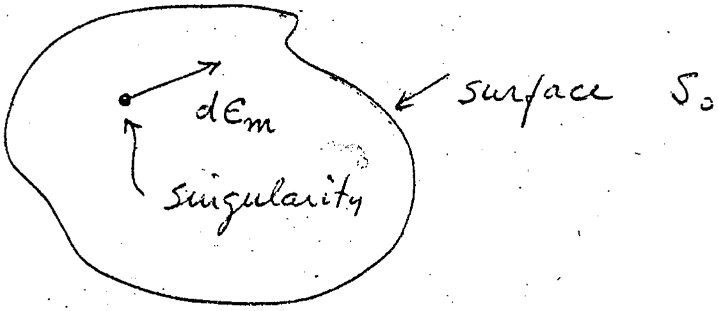
Illustrates That J Integral valid for crack tip can be evaluated at $x = \infty$.

Eshelby - Barnett Derivation of J Integral

The J integral was discovered by Rice in connection with a study of the crack tip strain singularity in an elastic - perfectly plastic material. The same integral had been discovered by Eshelby about 10 years earlier.

J. D. Eshelby Solid State Physics Vol 3 1956 Academic Press.

Eshelby derived a general expression for the force on an elastic singularity (dislocation, crack tip)



$$F_m = -\frac{\delta E^{TOTAL}}{\delta \epsilon_m} = \oiint_{S_0} \left[\overbrace{W \delta_{jm}}^{\text{external potential}} - \overbrace{\tau_{ij} \frac{\partial u_i}{\partial x_m}}^{\text{internal potential}} \right] dS_j$$

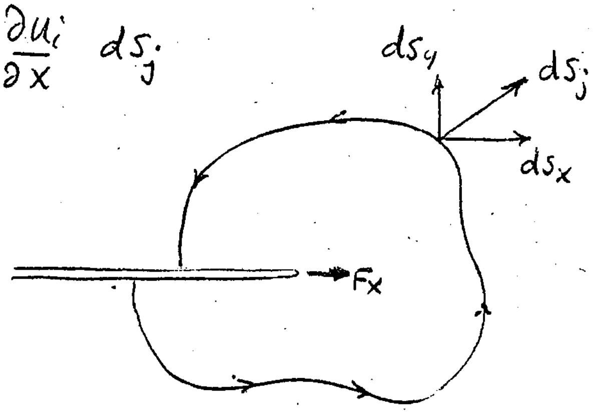
Castigliano's theorem?

$$= \oiint_{S_0} P_{jm} \, dS_j$$

$$P_{jm} = W \delta_{jm} - \tau_{ij} \frac{\partial u_i}{\partial x_m} = \text{elastic energy-momentum tensor}$$

Consider 2D crack, compute force in x direction

$$F_x = \int_\Gamma W \underset{\overset{\uparrow}{\text{kronecker } \delta}}{\delta_{jx}}\, ds_j - \tau_{ij} \frac{\partial u_i}{\partial x}\, ds_j$$

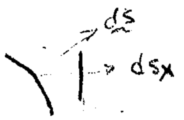


on per unit length basis.

$$\delta_{ij} = 0 \qquad i \neq j$$

$$\delta_{ij} = 1 \qquad i = j$$

$$ds_j = n_j\, ds$$



$$\delta_{jx}\, ds_j = \underset{\substack{\parallel \\ 1}}{\delta_{xx}}\, ds_x + \underset{\substack{\parallel \\ 0}}{\delta_{yx}}\, ds_y = dy$$

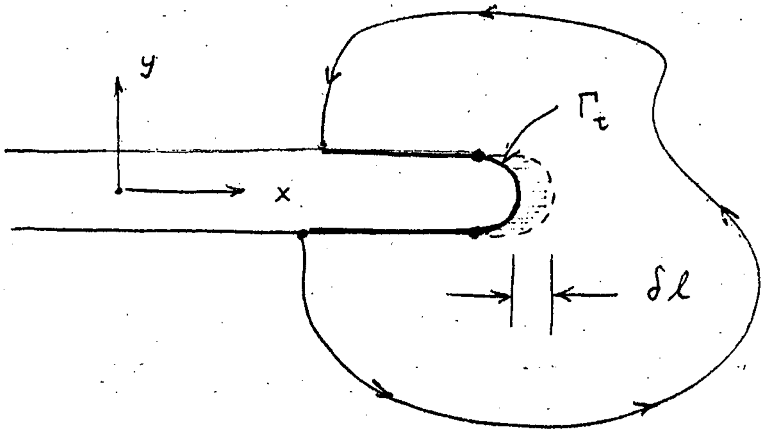
$$F_x = \int_\Gamma W\, dy - \underbrace{\tau_{ij}\, n_j}_{\downarrow}\, \frac{\partial u_i}{\partial x}\, ds$$

$$J = F_x = \int_\Gamma \left[W\, dy - T_i \frac{\partial u_i}{\partial x}\, ds \right]$$

The J. Integral.

Theoretical Basis for Measurement of J Integral

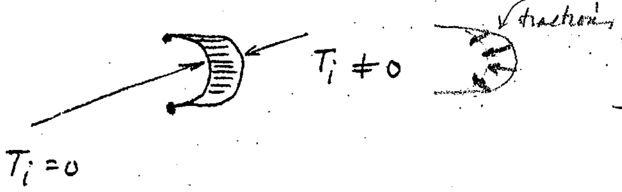
Consider identical non-linear elastic bodies containing smooth notches of lengths l and $l + \delta l$

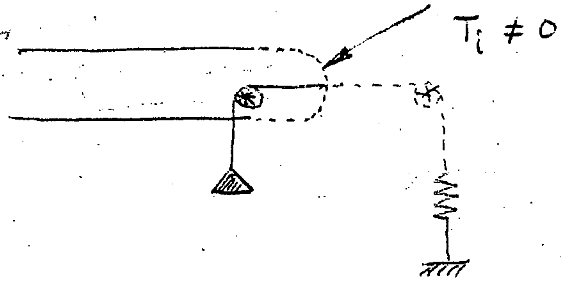


The J integral, which is path independent, may be evaluated along the crack tip surface. Since $dy = 0$ and $T_i = 0$ along the flat crack surfaces and $T_i = 0$ along the curved surface the J integral may be written as

$$J = \int_{\Gamma_t} W \, dy$$

Now consider extending the notch by cutting out the shaded region of thickness δl





we imagine that the tractions at the crack tip are removed – the decrease in total potential energy E in the body is exactly the increase in potential energy of the external work reservoir. So relaxation produces no net change in potential energy.

The total energy of the removed material causes the potential energy of the system to change by

$$\delta E = -\delta l \int_{\Gamma_t} W\,dy$$

Thus it follows that

$$J = \int_{\Gamma_t} W\,dy = -\frac{\delta E}{\delta l}$$

where E is the total potential (mechanical) energy of the system.

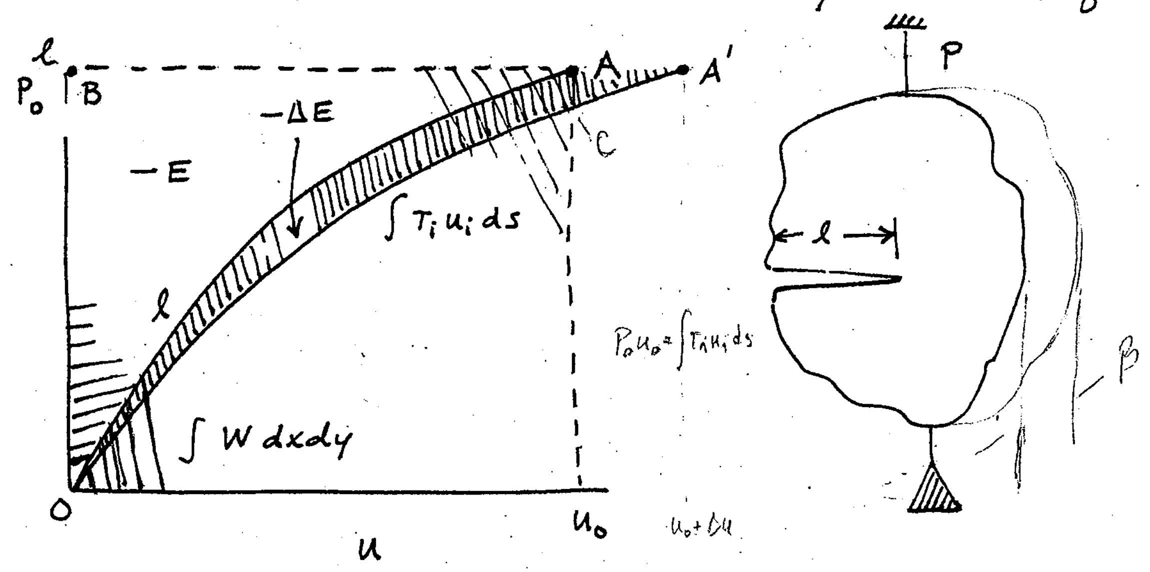
$$\overset{\text{Gibbs free energy}}{E} = \overset{\text{Helmholtz free energy}}{\int_{Area} W\,dx\,dy} - \overset{\text{external work (PV work)}}{\int_{Line} T_i\,u_i\,ds}$$

Experimental Determination of J Integral

Ref. J. A. Begley and J. D. Landes, ASTM STP 514 (1972)

As noted above, the J integral can be found by measuring the total mechanical energy E for two samples having slightly different crack lengths.

Consider load-displacement record for a non-linear elastic material containing a crack of length l.



For Fixed Load

The total mechanical energy is

$$E = \int_{Area} W \, dx\,dy - \int_{Line} T_i u_i \, ds$$

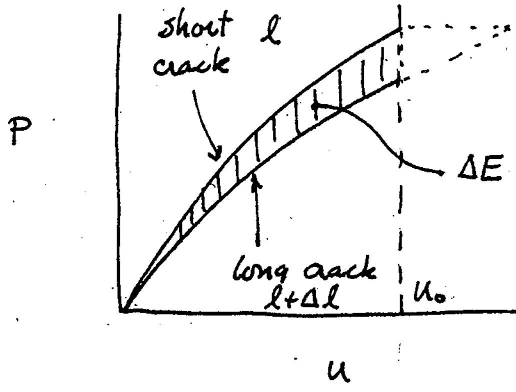
and is represented by the negative of the area OABO. Now if the crack length were $l + \Delta l$ the load-displacement relation would be as shown. The mechanical energy would be $-$ OA'BO. Now J is simply

$$J = -\frac{\Delta E}{\Delta l}\frac{1}{B}$$

ΔE is total change in mechanical energy.

$\frac{\Delta E}{B}$ on a per unit thickness basis.

J Integral for Fixed Displacement



(like strain energy release)

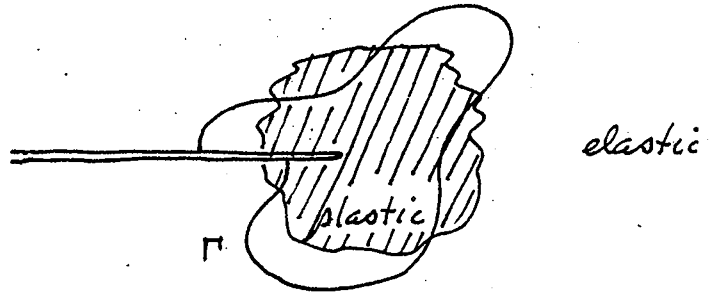
$$J = -\frac{\Delta E}{\Delta l}\frac{1}{B}$$

this is the difference between fixed force / fixed grips

thus if for small changes in crack length this area is of higher order ∴ force/disp fixed conditions are almost same.

J Integral for Elastic - Plastic Material

Now consider a crack in an elastic-plastic material



Provided the loading is monotonic (no unloading) and no crack growth occurs, the stresses and strains in an elastic-plastic material are exactly the same as those for a non-linear elastic material. Further, the stresses and strains are characterized by the J integral

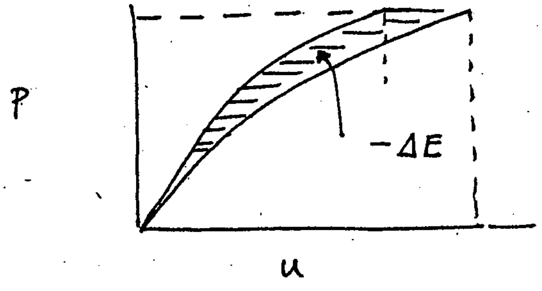
$$J = \int_{\Gamma} W dy - T_i \frac{\partial u_i}{\partial x} ds$$

where W is not the strain energy density, but is

$$W = \int_0^{\epsilon} \sigma_{ij} \, d\epsilon_{ij}$$

The area under the stress strain curve. Since the elastic-plastic and non-linear elastic materials are indistinguishable, the crack tip stress and strain field is characterized by J.

We can measure J by measuring the P - u relation for two samples having crack lengths l and $l + \Delta l$



Note: J Integral is <u>not</u> the strain energy release rate for elastic - plastic!

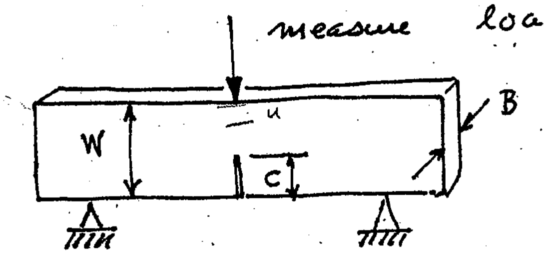
then

$$J = - \frac{\Delta E}{\Delta l} \frac{1}{B}$$

No matter how much plastic flow occurs (or how non-linear the material, the crack tip stress field (or the stress field along any path) is characterized by J.

J Integral as Fracture Criterion

Ni-Cr-Mo-V Rotor Steel -

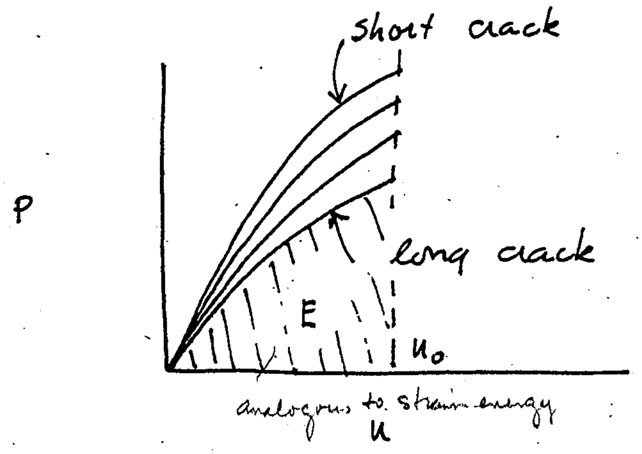


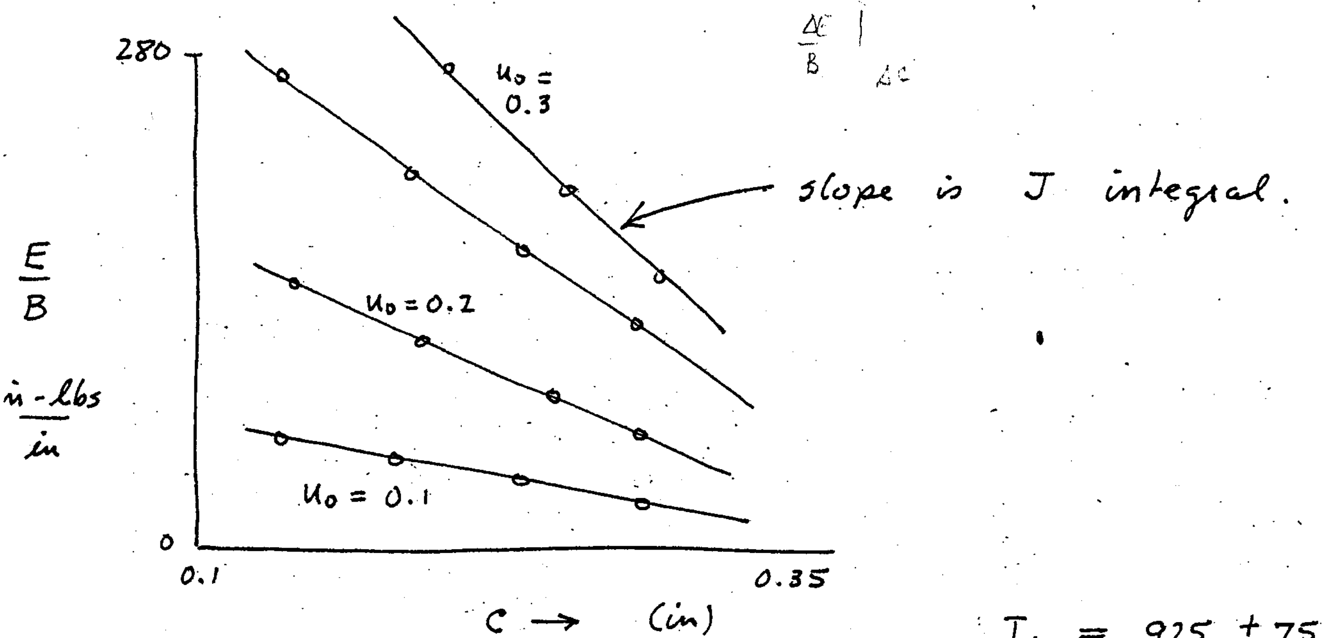
load P and displacement u.

$W = 0.474$ in

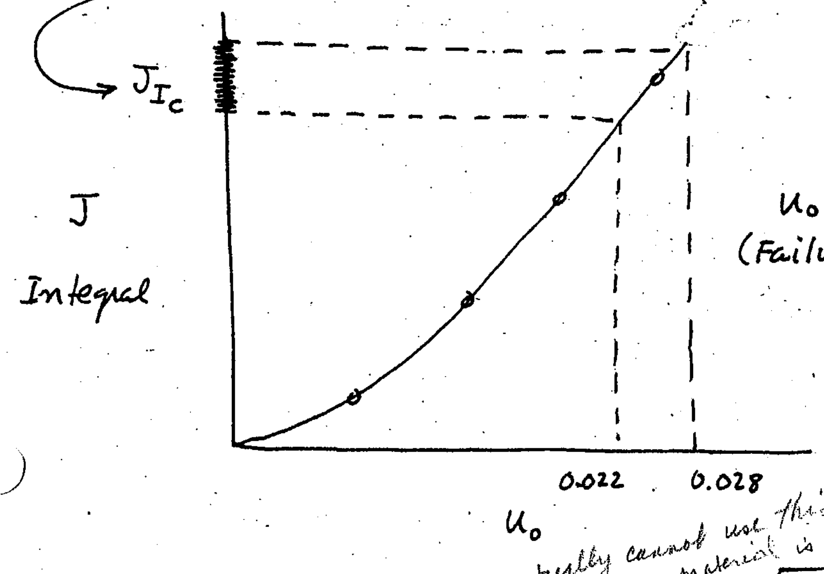
$B = 0.394$ in

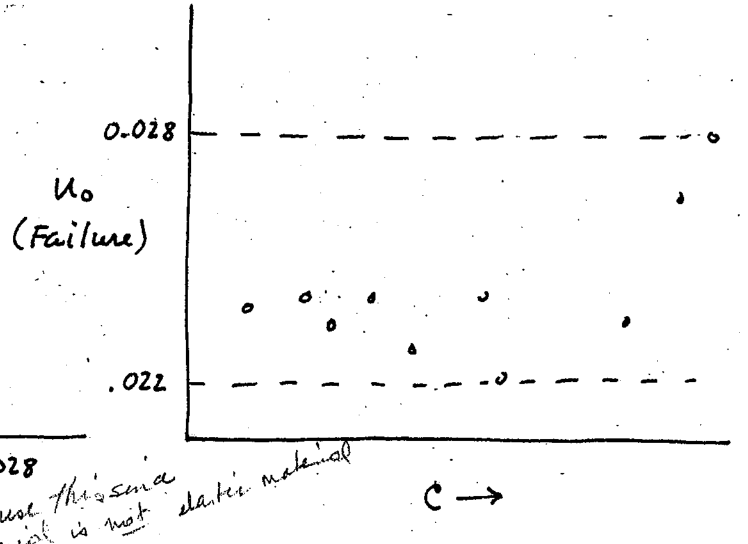
C variable





$J_{I_c} = 925 \pm 75$ $\frac{\text{in-lbs}}{\text{in}^2}$





really cannot use this since material is not elastic material

This result gives $K_{I_c} = \sqrt{\frac{J_{I_c} E}{1-\nu^2}} = 170$ ksi$\sqrt{\text{in}}$ very close to measured value. *

* using conventional techniques.

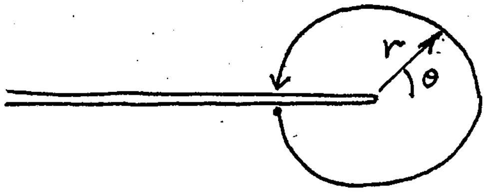
Use of J Integral to Describe Crack Tip Singularities

$$J = \int_{\Gamma} [W\,dy - T_i \frac{\partial u_i}{\partial x}\,ds]$$

$y = r \sin\theta$

$dy = r \cos\theta \, d\theta$

$ds = r\,d\theta$



$$\frac{J}{r} = \int_{-\pi}^{\pi} \left\{ W \cos\theta - T_i \frac{\partial u_i}{\partial x} \right\} d\theta$$

Since J is path independent ($\neq r$), the integrand { } must be inversely proportional to r. Since the bracketed terms are of order $\sigma_{ij}\,\epsilon_{ij}$ it must follow that

$$\sigma_{ij}\,\epsilon_{ij} \sim \frac{f(\theta)}{r} \qquad \text{as } r \to 0$$

Linear Elastic Materials — $\sigma_{ij} \sim r^{-1/2}$ $\quad \epsilon_{ij} \sim r^{-1/2}$

§ 3.2.3 Anderson

Power Law Hardening Materials - Non linear materials

J. R. Rice and G. F. Rosengren, J. Mech. Phys. Sol. 16, 1 (1968)

N. L. Goldman and J. W. Hutchinson, Int. J. Solid. Struct. 11, 575 (1974)

Suppose:

$$\sigma = A\,\epsilon^{N}$$

$N = 1/2$ for parabolic hardening

for non linear elastic materials

assume stress singularity

$$\sigma \sim r^{-m}$$

then

$$\epsilon \sim \sigma^{1/N} \sim r^{-m/N}$$

Now by the J integral

$$\sigma \epsilon = r^{-m} \, r^{-m/N} \sim r^{-1}$$

Thus $$m + \frac{m}{N} = 1$$

$$m \left(\frac{N+1}{N} \right) = 1 \qquad m = \frac{N}{N+1}$$

Thus for power law hardening material

$$\sigma \sim r^{-\left(\frac{N}{N+1} \right)} \qquad \epsilon \sim r^{-\left(\frac{1}{N+1} \right)}$$

For parabolic hardening $N = 1/2$

$$\sigma \sim r^{-1/3} \qquad \epsilon \sim r^{-2/3}$$

Elastic – Viscous Analogy.

The formalism of linear viscosity $\S$ 4.3.2.3

$$\tau_{ij} \sim \dot{e}_{ij}$$

is exactly the same as that for linear elasticity

$$\tau_{ij} \sim e_{ij}$$

Just replace e_{ij} by $\dot{e}_{ij}$ and change constants. Thus for a crack in a linear viscous material crack tip fields are:

$$\tau_{ij} \sim r^{-1/2} \qquad \dot{e}_{ij} \sim r^{-1/2}$$

The formalism of non-linear viscosity

$$\tau_{ij} \sim \dot{e}_{ij}^{N} \qquad N \neq 1$$

is also exactly the same as that for non-linear elasticity

$$\tau_{ij} \sim e_{ij}^{N} \qquad N \neq 1$$

Thus for non linear viscous material the crack tip fields are

$$\tau_{ij} \sim r^{-\frac{N}{N+1}} \qquad \dot{e}_{ij} \sim r^{-\frac{1}{N+1}}$$

In terms of power law creep

$$\dot{e}_{ij} \sim \tau_{ij}^{\,n} \qquad N = 1/n$$

So

$$\tau_{ij} \sim r^{-\frac{1/n}{1/n+1}} \sim r^{-\frac{1}{n+1}}$$

$$\dot{e}_{ij} \sim r^{-\frac{1}{\frac{1}{n}+1}} \sim r^{-\frac{n}{n+1}}$$

For $n = 5$ $\tau_{ij} \sim r^{-1/6}$ $\dot{e}_{ij} \sim r^{-5/6}$

$\dot{J}$ Integral Rate

For non-linear viscous material (elastically rigid)

$$\dot{J} = \int_{\Gamma} \dot{W}\, dy - T_i \frac{\partial \dot{u}_i}{\partial x}\, ds$$

$$W = \int_0^{e} \sigma_{ij}\, d\dot{e}_{ij} \qquad \dot{u}_i = \text{displacement rate.}$$

Stresses and strain rates in creeping materials characterized by $\dot{J}$.

Plastic Fracture Mechanics (Microscopic Approach)

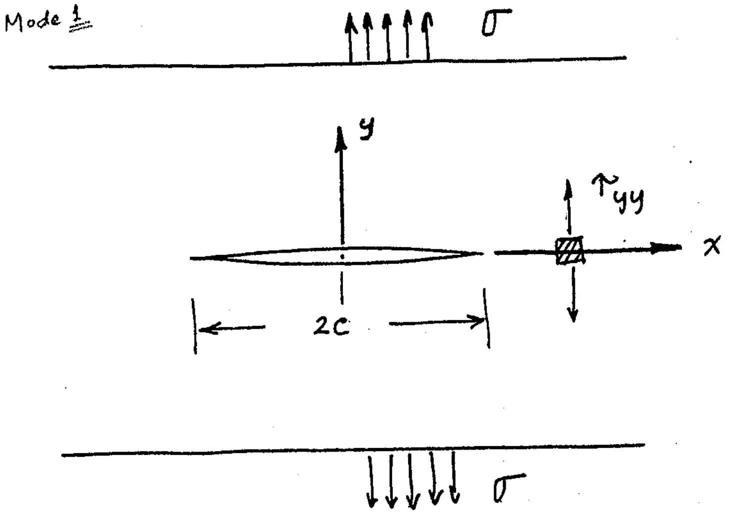
Plane Stress Fracture Mechanics.

Although we have discussed the *qualitative* aspects of plastic zones and their effect on G_c, we have not attempted to compute G_c from the plastic properties.

References : Ref. 18, 19, 20, 21, 22
Read: G.T. Hahn and A. R. Rosenfield Acta Met. 13 293 (1965)

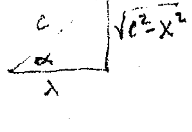
Dugdale - Muskhelishvili (DM) Model

This model of plastic zone is based on Muskhelishvili solution for stresses on the plane of the crack.



From page 2 of answer sheet to problem sets 1 and 2, the tension stress on the crack plane ($\theta = \theta_1 = \theta_2 = 0$ $r = x$ $r_1 = x - c$ $r_2 = x + c$), hence :
From Westergaard soln

$$\left.\tau_{yy}\right)_{\substack{y=0\\x>c}} = \frac{\sigma x}{\sqrt{x^2-c^2}}$$



Muskhelishvili gives this result in a different form:

$$\left.\tau_{yy}\right)_{\substack{y=0\\x>c}} = \sigma \coth \alpha \qquad \text{where } \cosh\alpha = \left(\frac{x}{c}\right)$$

These are evidently different ways to express the same relation:

if $\cosh\alpha = \left(\frac{x}{c}\right)$, then since $\cosh^2\alpha - \sinh^2\alpha = 1$

$$\sinh\alpha = \sqrt{\left(\frac{x}{c}\right)^2 - 1}$$

so that

$$\coth\alpha = \frac{\cosh\alpha}{\sinh\alpha} = \frac{\frac{x}{c}}{\sqrt{\left(\frac{x}{c}\right)^2-1}} = \frac{x}{\sqrt{x^2-c^2}}$$

Hence according to Muskhelishvili

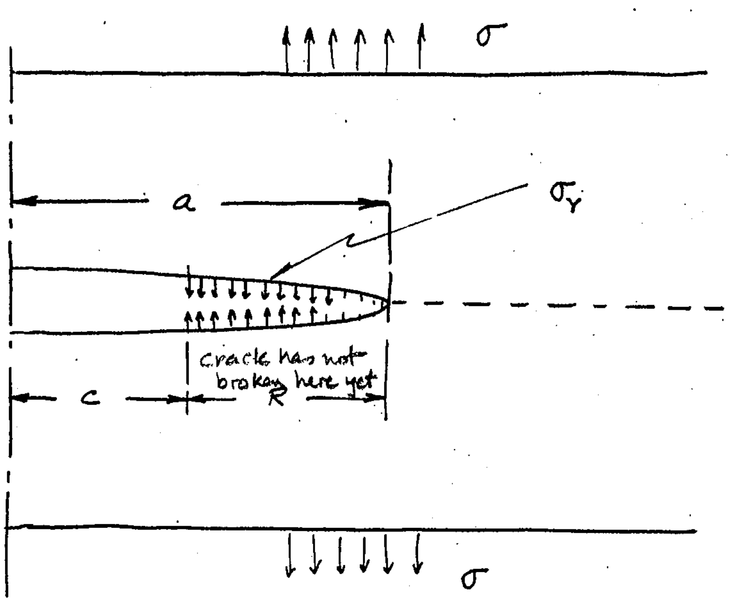
$$\left.\tau_{yy}\right)_{\substack{y=0\\x>c}} = \sigma\coth\alpha = \frac{\sigma x}{\sqrt{x^2-c^2}}$$

as before with the complete Westergaard analysis.

Dugdale – Muskhelishvili (DM) Model.

We simulate plastic zones by considering "an extended crack" w/ partial loading on the crack face

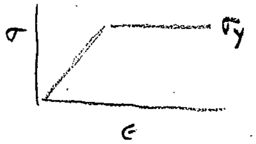
We think of the yield stress as supplying a "cohesive" stress binding the crack faces together.



plastic zone is where crack faces are bound together with stress σ_Y.

we assume plastic zones are confined to region near $y=0$.

ie plane stress problems only.



Muskhelishvili solution for Partially Loaded Faces.
(reported in Hahn and Rosenfield)

$$\tau_{yy}\begin{pmatrix} y=0 \\ x>a \end{pmatrix} = \left(\sigma - \frac{2\beta\sigma_Y}{\pi}\right)\coth\alpha + \sigma\left\{1 - \frac{1}{\beta}\tan^{-1}\left\{\frac{\sin 2\beta}{\cos 2\beta - e^{2\alpha}}\right\} - Q\frac{e^{\alpha}}{4\beta}\right\}$$

where $\cosh\alpha = \frac{x}{a}$ $\qquad$ Q = complex function of α and β.

$\cos\beta = \frac{c}{a}$

Hahn and Rosenfield report $Q\frac{e^{\alpha}}{4\beta} << 1$ (neglect)

now we examine the <u>elastic - plastic boundary</u> $x = a$

here $\alpha \to 0$ so that $\coth\alpha = (\tanh\alpha)^{-1} \approx \frac{1}{\alpha} \to \infty$

If we require the stresses to remain FINITE at the elastic-plastic boundary then the coefficient of coth α must be zero.

$$\left(\sigma - \frac{2\beta\sigma_Y}{\pi}\right) = 0$$

hence $\beta = \frac{\pi}{2}\left(\frac{\sigma}{\sigma_Y}\right)$

so

$$\cos\beta = \frac{c}{a} = \cos\left(\frac{\pi}{2}\frac{\sigma}{\sigma_Y}\right) = \frac{a-R}{a} = \frac{c}{a}$$

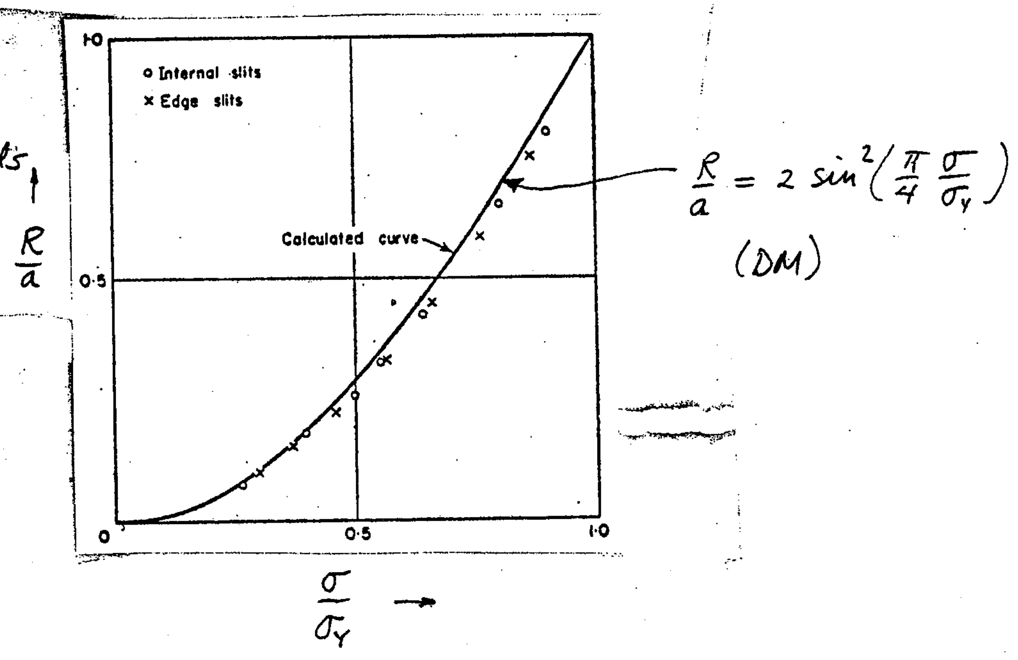
or $$\frac{a-R}{a} = 1 - \frac{R}{a} = \cos\left(\frac{\pi}{2}\frac{\sigma}{\sigma_Y}\right)$$

$$\frac{R}{a} = 1 - \cos\left(\frac{\pi}{2}\frac{\sigma}{\sigma_Y}\right) = 2\sin^2\left(\frac{\pi}{4}\frac{\sigma}{\sigma_Y}\right)$$

under plane stress condition

Comparison between DM Model and Experiment

D. S. Dugdale
J. Mech Phys. Solids,
1960
(experimental)



For small plastic zones

$$\frac{R}{a} \approx \frac{R}{c} \simeq 2\left(\frac{\pi}{4}\frac{\sigma}{\sigma_y}\right)^2 = \frac{\pi^2}{8}\left(\frac{\sigma}{\sigma_y}\right)^2$$

hence

$$R \cong c \ \frac{\pi^2}{8}\left(\frac{\sigma}{\sigma_y}\right)^2$$ (about the same as result on p 67 of notes).

Now the stresses on the crack plane are

$$\left.\tau_{yy}\right)_{\substack{y=0\\ x>a}} = \sigma + \frac{\sigma}{\beta}\tan^{-1}\left\{\frac{\sin 2\beta}{e^{2\alpha} - \cos 2\beta}\right\}$$

where $\beta = \frac{\pi}{2}\left(\frac{\sigma}{\sigma_y}\right)$

or

$$\frac{\tau_{yy}}{\sigma_y} = \frac{\sigma}{\sigma_y} + \frac{2}{\pi}\tan^{-1}\left\{\frac{\sin 2\beta}{e^{2\alpha} - \cos 2\beta}\right\}$$

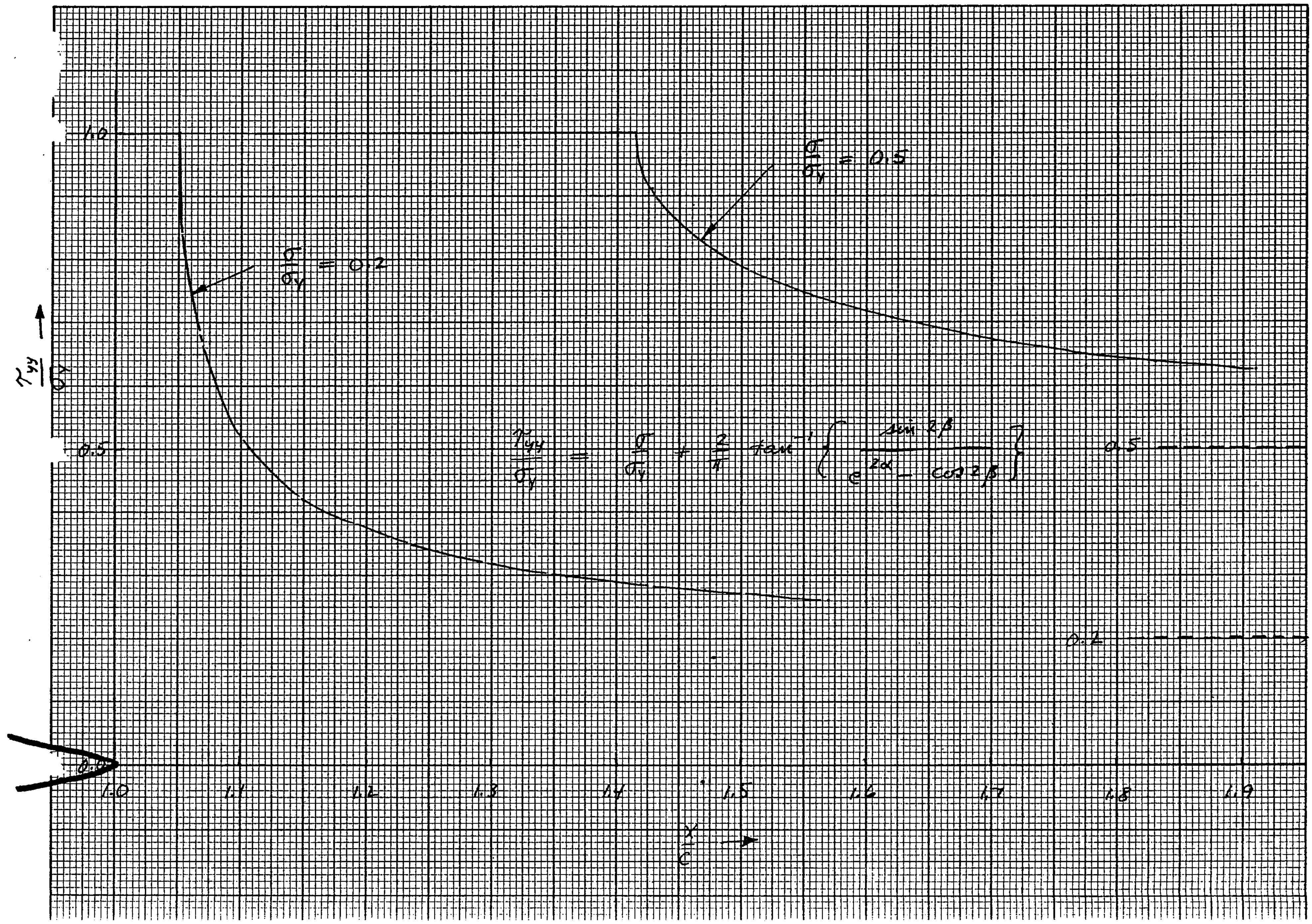
where

$$\alpha = \cosh^{-1}\left(\frac{x}{a}\right) = \cosh^{-1}\left\{\frac{x}{c}\cos\beta\right\}$$

to solve this equation it is useful to have

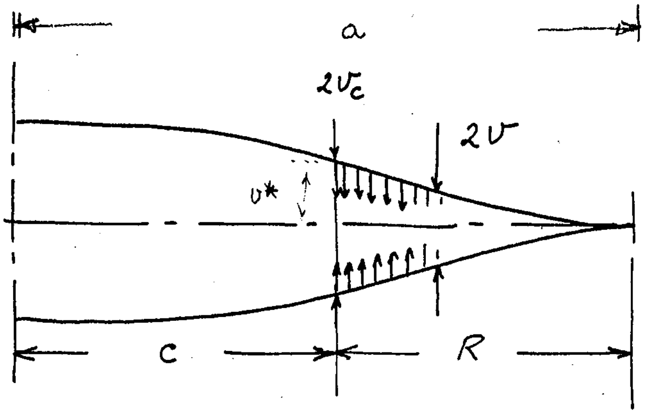
$$\cosh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\left(\frac{x}{a}\right) + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right\}$$

The stress distributions are shown on the next page.


$\frac{\tau_{yy}}{\sigma_y} = \frac{\sigma}{\sigma_y} + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\sin 2\beta}{e^{2\alpha} - \cos 2\beta} \right\}$
$\frac{\sigma}{\sigma_y} = 0.2$
$\frac{\sigma}{\sigma_y} = 0.5$
$\frac{\tau_{yy}}{\sigma_y}$
1.0
0.5
0.5
0.2
0.0
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
$\frac{x}{c}$

Prediction of Displacements at the Crack Surface with DM Model

(from Goodier and Field)



v = normal displacement of crack surface

v_c = critical displacement of crack surface.

v^* – displacement at x = c, before critical values of cracking occurs

General Solution

$$v = \frac{a\sigma_Y}{\pi E}\left(\cos\theta \ln\left\{\frac{\sin^2(\beta-\theta)}{\sin^2(\beta+\theta)}\right\} + \cos\beta \ln\left\{\frac{(\sin\beta+\sin\theta)^2}{(\sin\beta-\sin\theta)^2}\right\}\right)$$

where $\theta = \cos^{-1}\frac{x}{a}$

$$\cos\beta = \frac{c}{a} = \cos\left(\frac{\pi}{2}\frac{\sigma}{\sigma_Y}\right)$$

Now we want the crack opening displacement, v_c. So we take

$$\lim_{x\to c} v^* = v_c = \lim_{\theta\to\beta} v(\theta,\beta)$$

$$v^* = \lim_{\theta\to\beta}\frac{a\sigma_Y}{\pi E}\left(\cos\theta \ln\left\{\frac{\sin^2(\beta-\theta)}{\sin^2(\beta+\theta)}\right\} + \cos\beta \ln\left\{\frac{(\sin\beta+\sin\theta)^2}{(\sin\beta-\sin\theta)^2}\right\}\right)$$

$$v^* = \frac{a\,\sigma_Y}{\pi E} \cos\beta \lim_{\theta \to \beta} 2 \ln \left\{ \frac{\sin(\beta-\theta)}{\sin(\beta+\theta)} \frac{(\sin\beta + \sin\theta)}{(\sin\beta - \sin\theta)} \right\}$$

now let $\beta = x - y \quad \sin\beta = \sin x \cos y - \cos x \sin y$

$\theta = x + y \quad \sin\theta = \sin x \cos y + \cos x \sin y$

$$v^* = \frac{2a\,\sigma_Y}{\pi E} \cos\beta \lim_{y \to 0} \ln \left\{ \frac{\sin(-2y)\ 2 \sin x \cos y}{\sin(2x)\,(-)\,2 \cos x \sin y} \right\}$$

$$v^* = \frac{2a\,\sigma_Y}{\pi E} \cos\beta \lim_{y \to 0} \ln \left\{ \frac{\cancel{2}\,\cancel{\sin y} \cos y\ \cancel{2}\,\cancel{\sin x} \cos y}{\cancel{2}\,\cancel{\sin x} \cos x\ \cancel{2} \cos x\,\cancel{\sin y}} \right\}$$

$$v^* = \frac{2a\,\sigma_Y}{\pi E} \cos\beta \lim_{\substack{y \to 0 \\ x \to \beta}} \ln \left\{ \frac{\cos^2 y}{\cos^2 x} \right\}$$

$$v^* = \frac{4a\,\sigma_Y}{\pi E} \cos\beta \ln(\sec\beta)$$

$$\cos\beta = \frac{c}{a}$$

$$\beta = \frac{\pi}{2} \frac{\sigma}{\sigma_Y}$$

$$\boxed{v^* = \frac{4c\,\sigma_Y}{\pi E} \ln\left(\sec\left(\frac{\pi}{2} \frac{\sigma}{\sigma_Y}\right)\right)}$$

Now consider a <u>low</u> stress limit: for very thin sheets

$$\underline{\frac{\sigma}{\sigma_y} << 1}$$

$$\sec\beta = \frac{1}{\cos\beta} = \frac{1}{\cos 2\cdot\beta/2}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\sec\beta = \frac{1}{1-2\sin^2\frac{\beta}{2}}$$

$\ln\sec\beta = -\ln\left(1-2\sin^2\frac{\beta}{2}\right)$; now since $\beta = \frac{\pi}{2}\sigma/\sigma_y$ & $\sigma/\sigma_y << 1 \Rightarrow \beta << 1$

$$\ln(1-x) \approx -x \qquad x << 1$$

$$\ln\sec\beta \approx -\ln\left(1-2\sin^2\frac{\beta}{2}\right) \approx 2\sin^2\frac{\beta}{2} \approx \frac{\beta^2}{2}.$$

so that

$$\ln\sec\beta \approx \frac{\pi^2}{8}\left(\frac{\sigma}{\sigma_y}\right)^2$$

Thus

$$v_4^* \approx \frac{4\sigma_y c}{\pi E}\,\frac{\pi^2}{8}\left(\frac{\sigma}{\sigma_y}\right)^2 = \underline{\frac{\pi c\,\sigma^2}{2E\sigma_y}} = \cancel{\frac{K_I^2}{2E\sigma_y}}$$

So that from DM Model if $v^* = v_{crit.}$

$$\sigma_c = \sqrt{\frac{2E\sigma_y v_c}{\pi c}} \qquad \text{but} \qquad \sigma_c = \frac{K_c}{\sqrt{\pi c}}$$

So

$$K_c = \sqrt{E\sigma_y 2v_c} = \sqrt{E G_c} \qquad p\ 52. \qquad G_c = K_c^2/E$$

↑ plane stress

So we conclude:

$$\boxed{G_c = \sigma_y \, 2V_c}$$

where surface energy is put into σ_y

makes sense - work of crack extension.

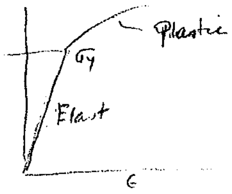
we call $2V_c$ the critical crack opening displacement.
or simply:

$$2V_c = C.O.D.$$

so

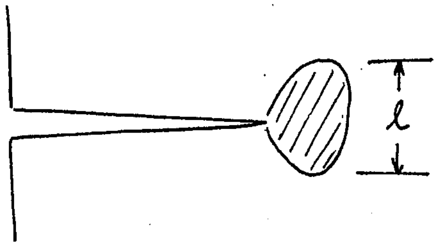
$$G_c \text{ (work of fracture) } = \sigma_y \cdot (C.O.D).$$

This means we could estimate G_c or K_c if σ_y and C.O.D. were known.



estimate $\sigma_y \equiv$ macroscopic yield stress.

estimate C.O.D. from fracture strain.



l = distance over which plastic zone extends.

then $C.O.D \approx l \, \bar{\epsilon}$ $\quad$ $\bar{\epsilon}$ = average fracture strain.

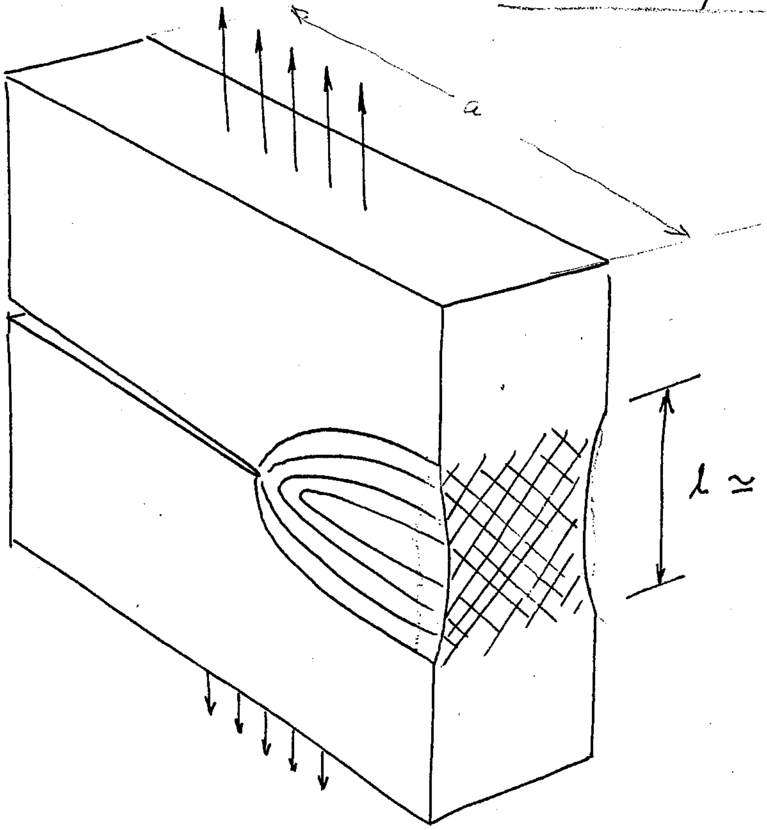
Because of distribution of strain in plastic zone might expect

$$\bar{\epsilon} \approx \frac{\epsilon_f}{2} \qquad \epsilon_f = \text{true fracture strain.}$$

so

$$C.O.D. = l \cdot \frac{\epsilon_f}{2}$$

But! what is l? For plane stress it is reasonable to expect $l \approx t$ (plate thickness).



hence

$$C.O.D \simeq \frac{\epsilon_f t}{2}$$

so that

$$\boxed{G_c \simeq \sigma_y \frac{\epsilon_f t}{2}}$$

Prediction by Hahn and Rosenfield.

Plane Stress - Plastic Zone.

Comparison of DM Model with Measured K_c (Plane Stress – Thin Sheet) (Hahn and Rosenfield)

$$K_c = \sqrt{G_c E} \quad \text{(plane stress)}$$

$$K_c = \sqrt{\sigma_Y \frac{\epsilon_f}{2} t E}$$

1) 4330 Steel :

$\sigma_Y = 189,000$ psi
$\epsilon_f = 0.45$
$t = 0.140$ in
$E = 30 \times 10^6$ psi

K_c (calculated) $= 423$ ksi $\cdot\sqrt{in}$

K_c (measured) $= 300$ ksi $\cdot\sqrt{in}$

2) 2219-T87 Aluminum :

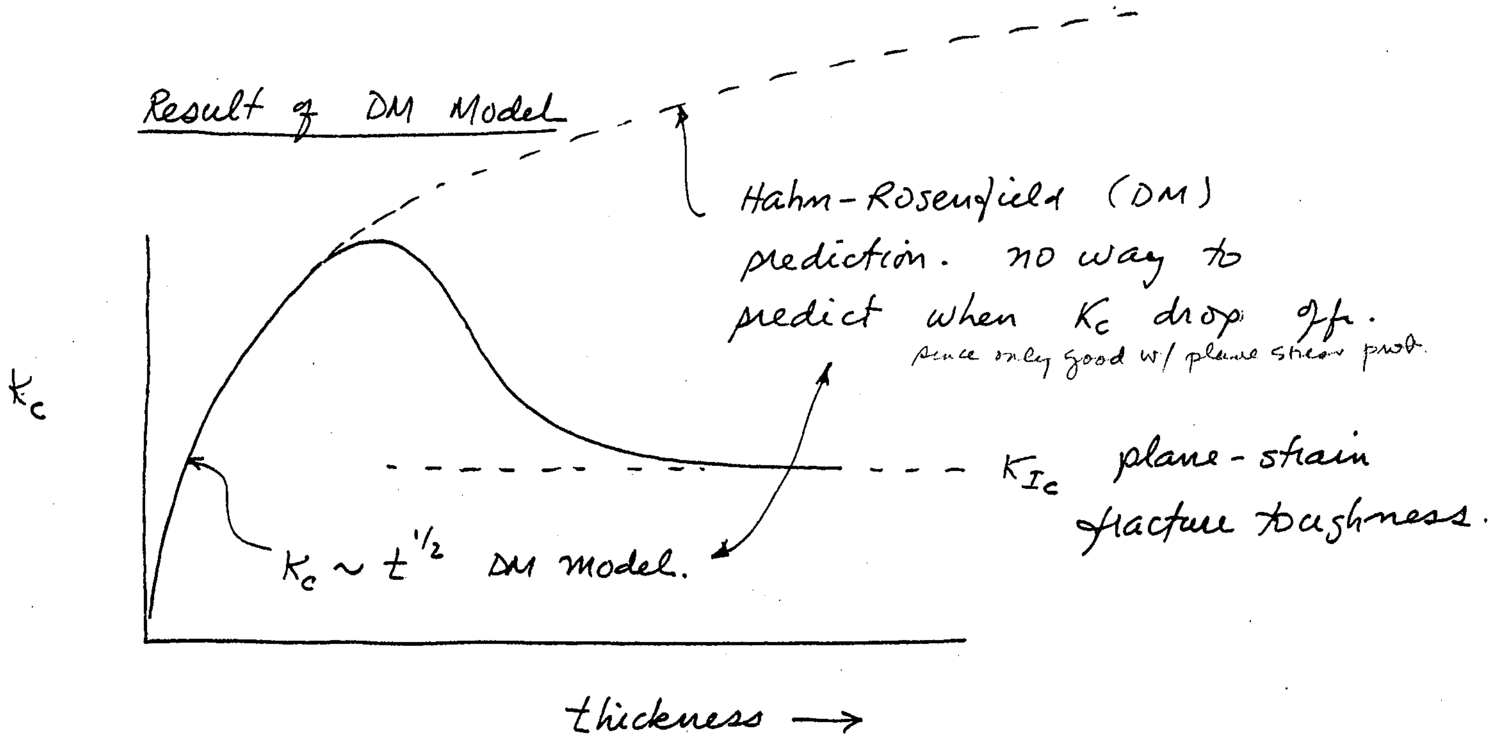
$\sigma_Y = 59,000$ psi
$\epsilon_f = 0.30$
$t = 0.10$ in
$E = 11 \times 10^6$ psi

K_c (calculated) $= 99$ ksi $\cdot\sqrt{in}$

K_c (measured) $= 110$ ksi $\cdot\sqrt{in}$

Note: this analysis valid only for very thin sheet - plane stress. We cannot predict when K_c or G_c would begin to fall off with increasing thickness.

Result of DM Model



Existing cracks:
Casting - shrinkages
Wrought
└ weak phase sulfides
stringers
powder metallurgy
porosity - no bonding (dirty surfaces)

← even if no existing

Mechanisms of Crack Nucleation

Except for perfectly brittle materials such as glass, cracks are almost always produced by inhomogeneous plastic deformation. The requirement of plastic flow for cracks which grow by microvoid coalescence is obvious. The need for plastic flow for cleavage needs some justification.

Low (1959), showed that σ_y in compression, same as σ_F in tension for Fe at -196°C.

Zackay et.al.
p. 404
Fracture
Vol I

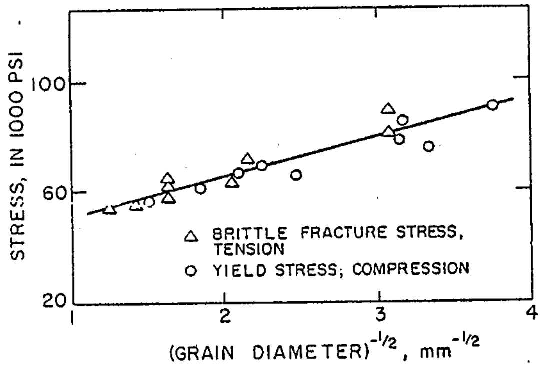
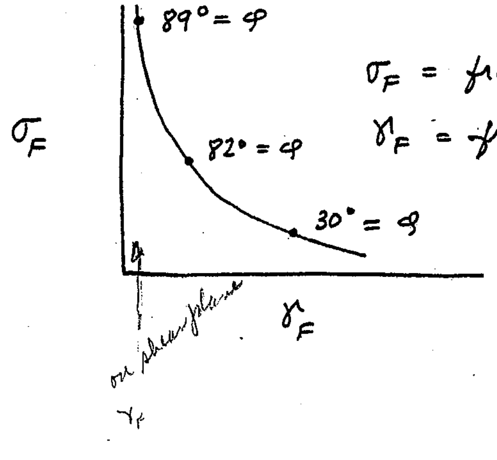


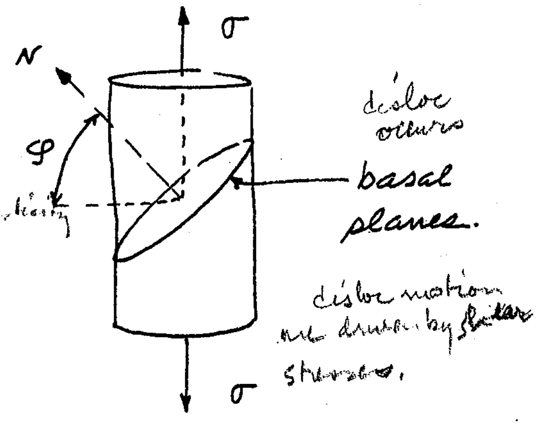
FIG. 8. Effect of grain size on yield stress in compression and cleavage stress in tension, medium carbon structural steel tested at −196° C (Low, 1959).

Fracture of HCP Zn Crystals



σ_F = fracture stress
γ_F = fracture strain. measure of degree of plasticity

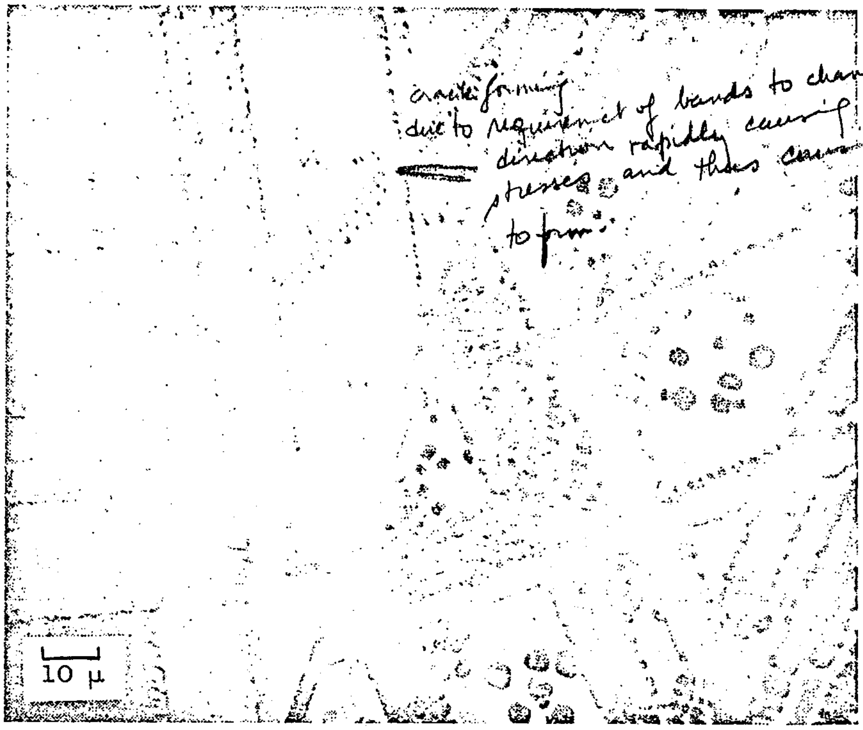
hence - shear on basal plane makes fracture easier.



Direct evidence of nucleation of cracks by inhomogeneous plastic flow.

plastic flow is confined to bands causing large stress concent

Johnson,
Stokes and Li
(Petch, p372)
Fracture Vol I
MgO.



cracks forming due to requirement of bands to change direction rapidly causing large stresses and thus causing cracks to form.

FIG. 13. Cracks at end of a slip band where stopped by a grain boundary, in MgO (Johnston, Stokes, and Li, 1962).

Gilbert et. al.
(Tetelman, p 242)

Mo

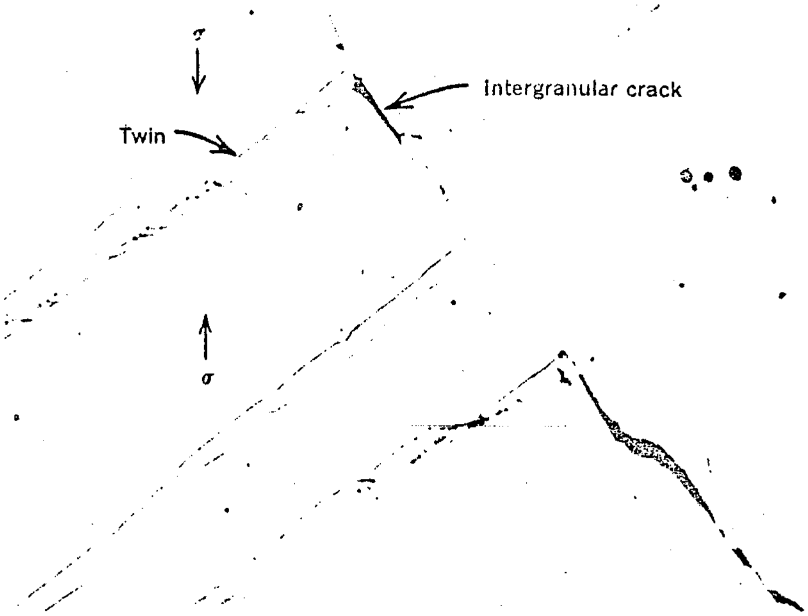
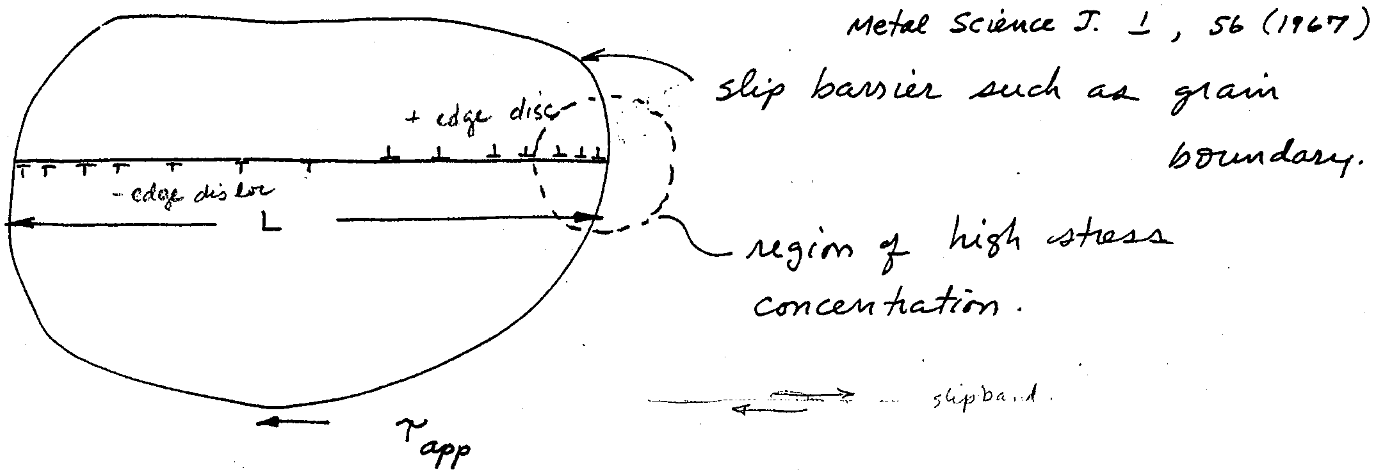


Fig. 6.6. Intergranular microcrack formed by twinning under compressive loading in Mo. 750×. *Courtesy A. Gilbert et al [7] and Acta Met.*

Stroh Model for Crack Nucleation

(Basic idea due to Zener).

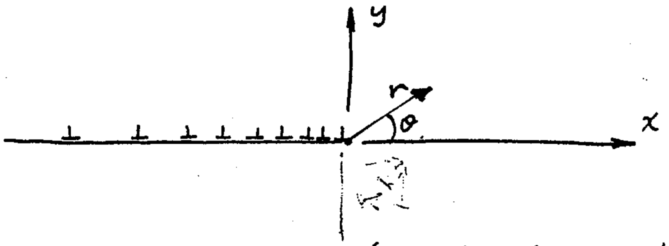
Recent review:
E. Smith & J. T. Barnby
Metal Science J. 1, 56 (1967)



Treat slip band as freely slipping Mode II Crack.

The effective stress is $(\tau_{app} - \tau_i) = \tau_e$ where τ_i is the friction stress for plastic flow in the band.

at the tip of the slip band



From p 50 (Mode II crack)

Look at problem in plane strain

$$\tau_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left[2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right]$$

$$\tau_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos\frac{3\theta}{2}$$

$$K_{II} = (\tau_{app} - \tau_i)\sqrt{\pi c}$$

$$\tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

In any mechanics book you can find a general expression for the maximum normal stress for a two dimensional stress state:

$$\sigma_n^{max} = \frac{\tau_{xx}+\tau_{yy}}{2} + \sqrt{\left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

Plugging in the expressions for the Mode II crack we have (after much algebra)

$$\sigma_n^{max} = \frac{K_{II}}{\sqrt{2\pi r}}\left\{\sqrt{1-\underbrace{3\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}_{\frac{3}{4}\sin^2\theta}} - \sin\theta\right\}$$

$\frac{\sqrt{2\pi r}}{K_{II}}\frac{\partial\sigma_n}{\partial\theta}$ =

This function is maximum at $\theta = 270°$; $\frac{3}{2}$.

So

$$\sigma_n^{max} = \frac{\tau_e\sqrt{\pi c}}{\sqrt{2\pi r}}\cdot\frac{3}{2}$$

but $c = \frac{L}{2}$

hence

$$\sigma_n^{max} = \frac{3}{4}\tau_e\sqrt{\frac{L}{r}} = \frac{3}{4}(\tau_{app}-\tau_i)\sqrt{\frac{L}{r}}$$

We suppose that a crack is nucleated when

see pg 8

$$\sigma_n^{max} \geq \sigma_{Theoretical} = 0.5\sqrt{\frac{\overset{\text{surface energy}}{\gamma_s} E}{a_0}}$$

a_0 ← lattice param

$$\frac{3}{4}(\tau_{app}-\tau_i)\sqrt{\frac{L}{r}} \geq 0.5\sqrt{\frac{\gamma_s E}{a_0}}$$

so

$$\tau_{app} - \tau_i \geqslant \frac{2}{3}\sqrt{\frac{\gamma_s E r}{a_o L}}$$

The highest stress is at $r \approx a_o$, so

$$\tau_{app} - \tau_i \geqslant \frac{2}{3}\sqrt{\frac{\gamma_s E}{L}}$$

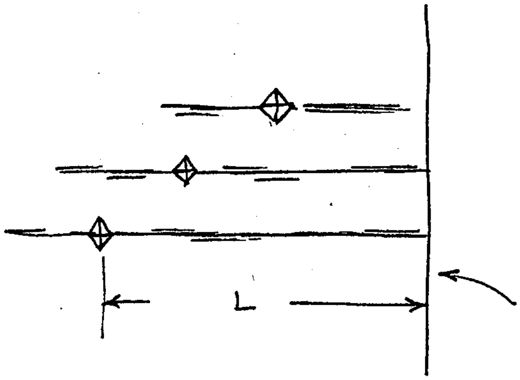
would get a slightly different answer if the crack plane is required to go through the tip of the pile-up.

note: this predicted grain size dependance compares favorably with experiment of Low (Pg 107).

<u>A direct Experiment</u>:

Ku and Johnston

MgO crystals. (heat treat it to pin dislocations, indent to produce fresh dislocations a given distance from grain boundary (using simple nail indentation).



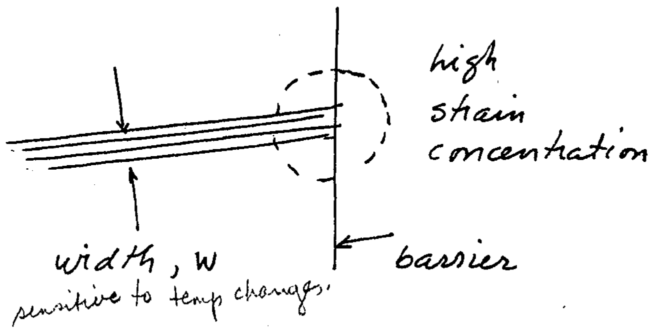
measured σ_F : $\sigma_F \propto \frac{1}{L^{1/2}}$.

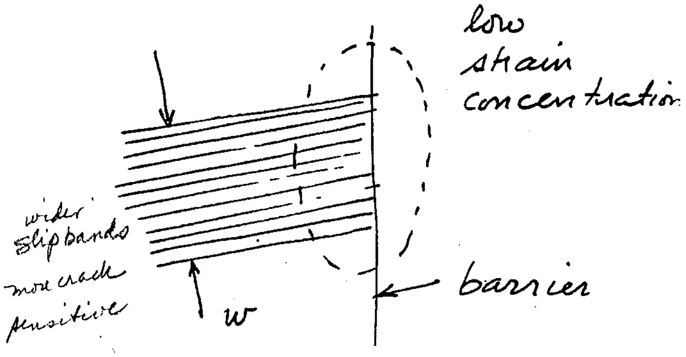
as predicted by theory.

Microstructural Aspects of Crack Nucleation by Blocked Slip Bands

Slip Band Width

intensity of local strains or stresses depends on displacement gradient. Hence depends on both slip band displacement and width





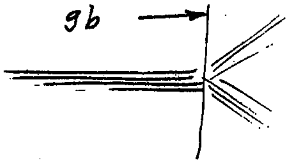
Low temperature - cross-slip (dislocation widening) more difficult
narrow slip band.

High Temperature - cross-sli
easy - wide slip band.

Conclusion: Lower temperature ~ more crack sensitive because of narrow slip bands.
- this is special conclusion for the idealized situation described. does not always hold.

Effect of Grain Boundary Microstructure

1) Clean Grain Boundaries (no second phase).



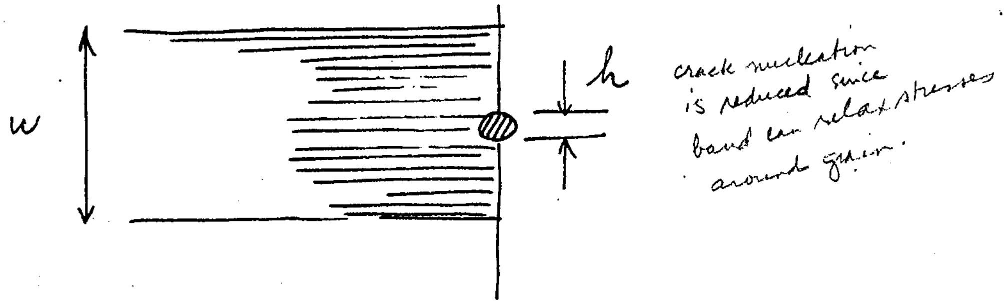
here plastic relaxation

plastic relaxation. easy except for brittle matl

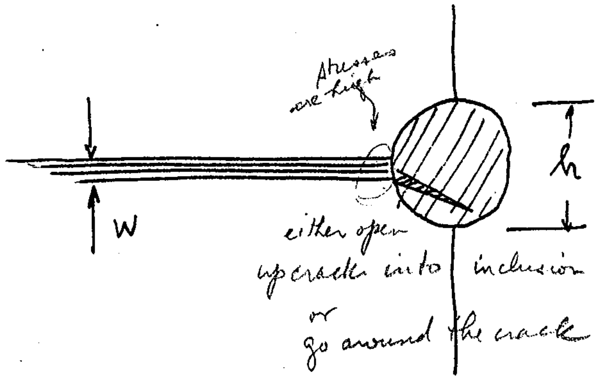
some of localized stresses can be relieved into next

2) Grain Boundaries with Second Phase Inclusions.
(carbides, oxides etc).
generally inclusions brittle and strong.

i) when $w >> h$, plastic relaxation can occur.



ii) when $w << h$, no plastic relaxation hence, crack nucleation



Microstructural Conclusions

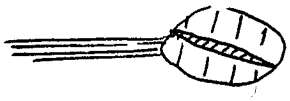
1) keep slip band lengths short
fine grain size, high twin density,
fine dispersion of second phase

2) avoid large brittle (or strong? inclusions at grain boundaries.

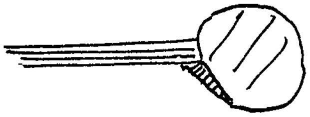
Void Nucleation at Inclusions

stresses built up when a slip band is blocked by an inclusion lead to:

i) a crack in the inclusion



ii) a crack in the phase boundary



In either case a void nucleus will have formed such that after more deformation we will have well formed voids adjecent to the inclusion.

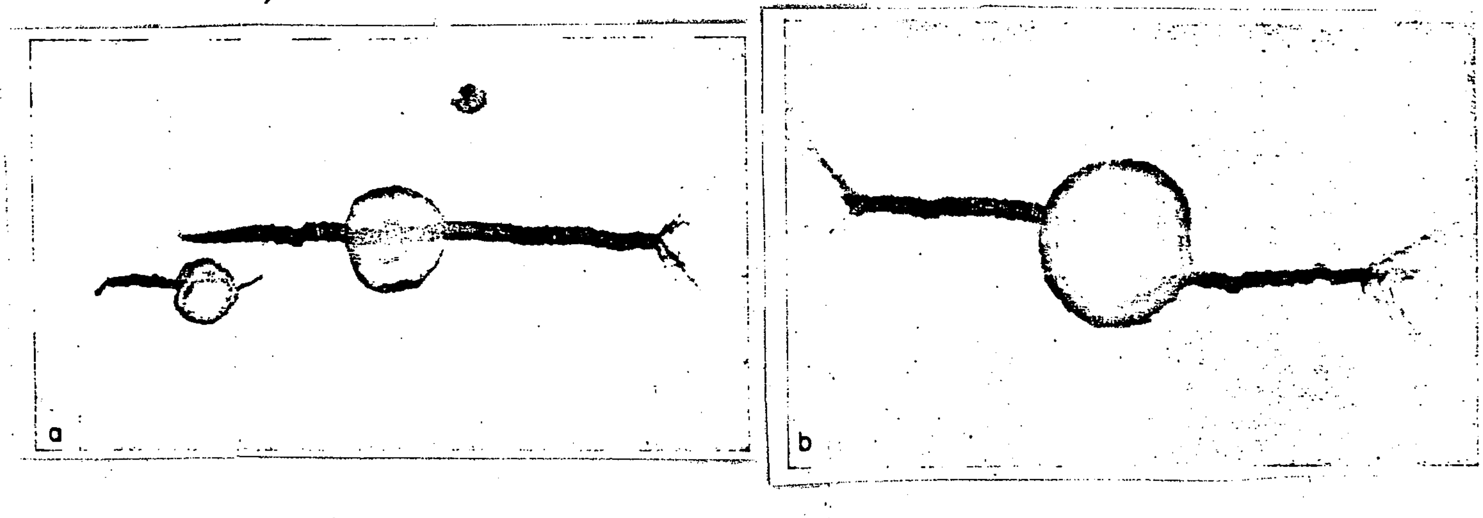
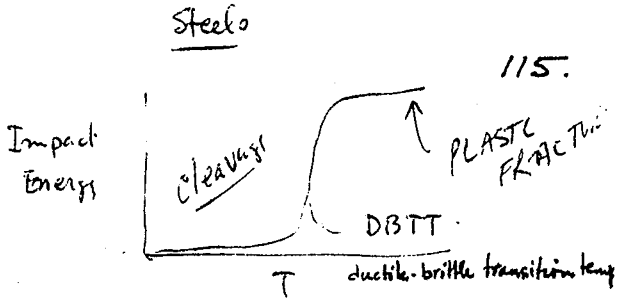


FIG. 12. Crack nucleation at second-phase particles in ultra-high strength steels: (a) D6aC steel, (625×), (b) D6aC steel, (750×), (c) maraging steel (1900×).

Zackay, Gerberich, Parker p 410

Fracture Vol I

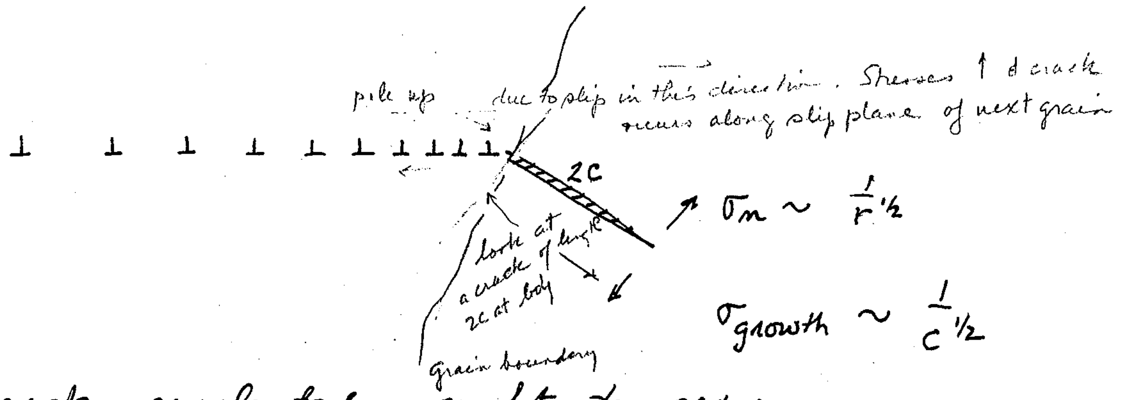


Cottrell Theory of Brittle Fracture

Reference: A. A. Cottrell 1958 Institute of Metals Div. Lecture
"Theory of Brittle Fracture in Steels and
Similar Metals"
Trans. AIME Vol 212 p 192 (1958)

Stroh theory for fracture essentially a "nucleation" theory. The idea is that if a crack can be nucleated, then it will always grow:

The reason for this is that the singularity for a blocked slip band, $r^{-1/2}$, is the same as the crack length dependence $c^{-1/2}$ for crack growth.



$$\sigma_m \sim \frac{1}{r^{1/2}}$$

$$\sigma_{growth} \sim \frac{1}{c^{1/2}}$$

so once crack nucleated, ought to grow.

$$\tau_{nucleation} = \underset{\substack{\text{friction}\\ \text{stress required to}\\ \text{move dislocation initially}}}{\tau_i} + \frac{2}{3}\sqrt{\frac{\gamma_s E}{L}}$$

But there may be some "flaws" in this approach.

1) cracking by this model not affected by hydrostatic tension, only shear stresses.

Reason not as much plastic work done ∴ $G_c \to G_{Ic}$.

yet we know thick sections are more brittle, hydrostatic pressure makes more ductile.

Cottrell concludes that <u>Growth</u> of the microcrack may be a more difficult step in fracture process.

2) Petch has also concluded there must be a flaw in the Stroh argument. He notes that

$$\tau_y = \tau_i + K_y d^{-1/2} \qquad \text{Hall - Petch.}$$

where K_y independent of impurities, temperature etc.

and that $\tau_{nucl} < \tau_y$ for fracture to occur.

$$\tau_{nucleation} = \tau_i + \frac{2}{3}\sqrt{\gamma_s E}\ d^{-1/2} \qquad L = d$$

grain size

by Stroh. Now if $\tau_{nucleation} = \tau_{fracture}$ then changing the value of τ_i should not change brittleness because τ_y and τ_F would be affected similarly. ie by adding new elements

But. a) adding Si to Fe increases τ_i <u>and</u> brittleness

Cottrell's Theory of Ductile - Brittle Transition in Steels

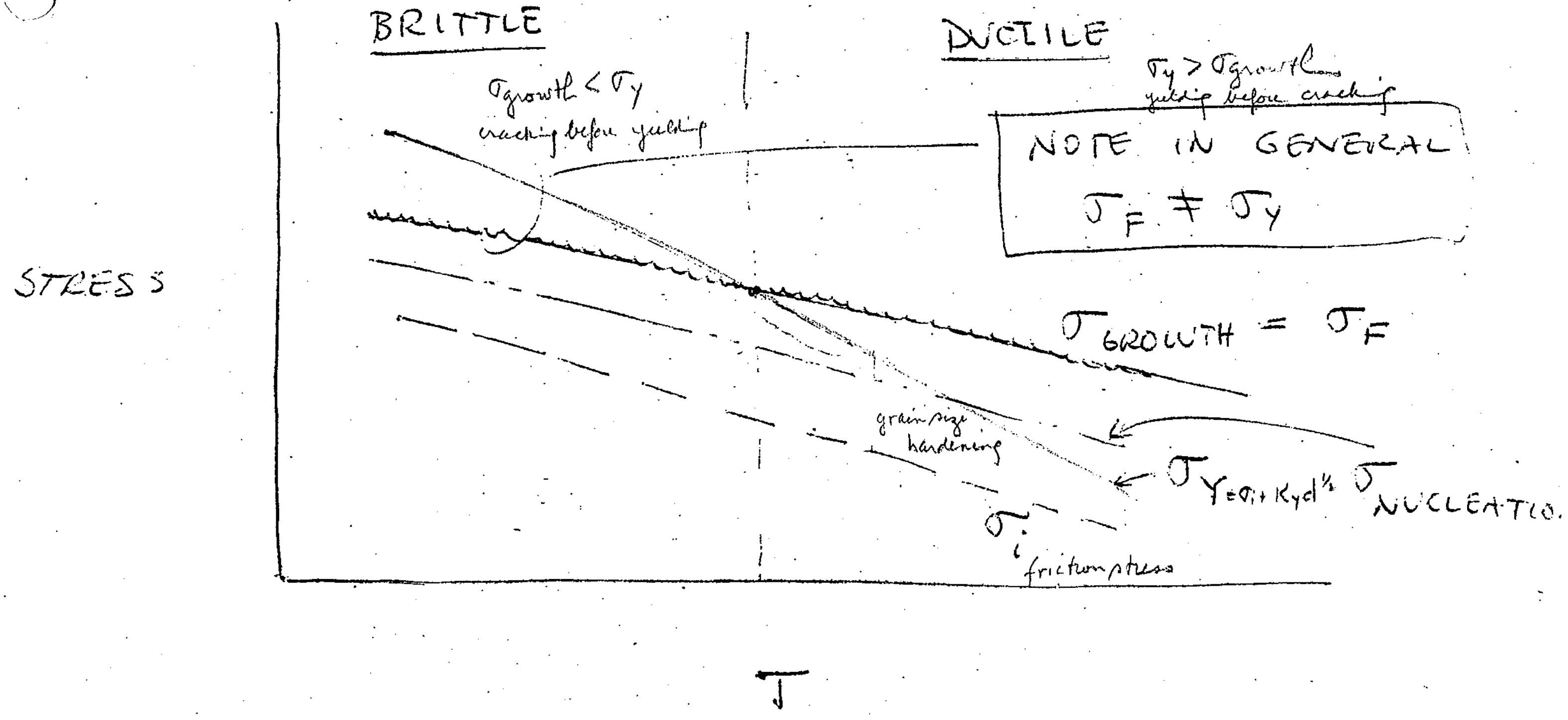
Cottrell assumes that when brittle fracture occurs $\sigma_F = \sigma_{GROWTH} > \sigma_{NUCLEATION} > \sigma_i$

$\uparrow$ friction stress.

The ductile-brittle transition is defined by

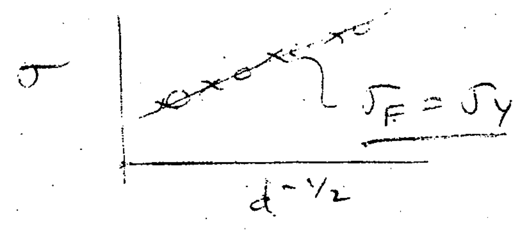
$$\sigma_F = \sigma_Y$$

The transition is represented by



Note: This theory does not predict the low result exactly.

UNLESS VERY CLOSE TO DBTT



b) irradiation hardening Fe increases τ_i
and brittleness.

So Petch also concluded Growth of the microcrack might be the rate limiting.

Cottrell envisioned three separate processes:

1) yield stress to start slip bands
2) crack nucleation stress:

$$\tau_{nucleation} = \tau_i + \frac{2}{3}\sqrt{\frac{\gamma_s E}{L}}$$

3) crack growth stress — ASSUMED to be the most difficult step in process.

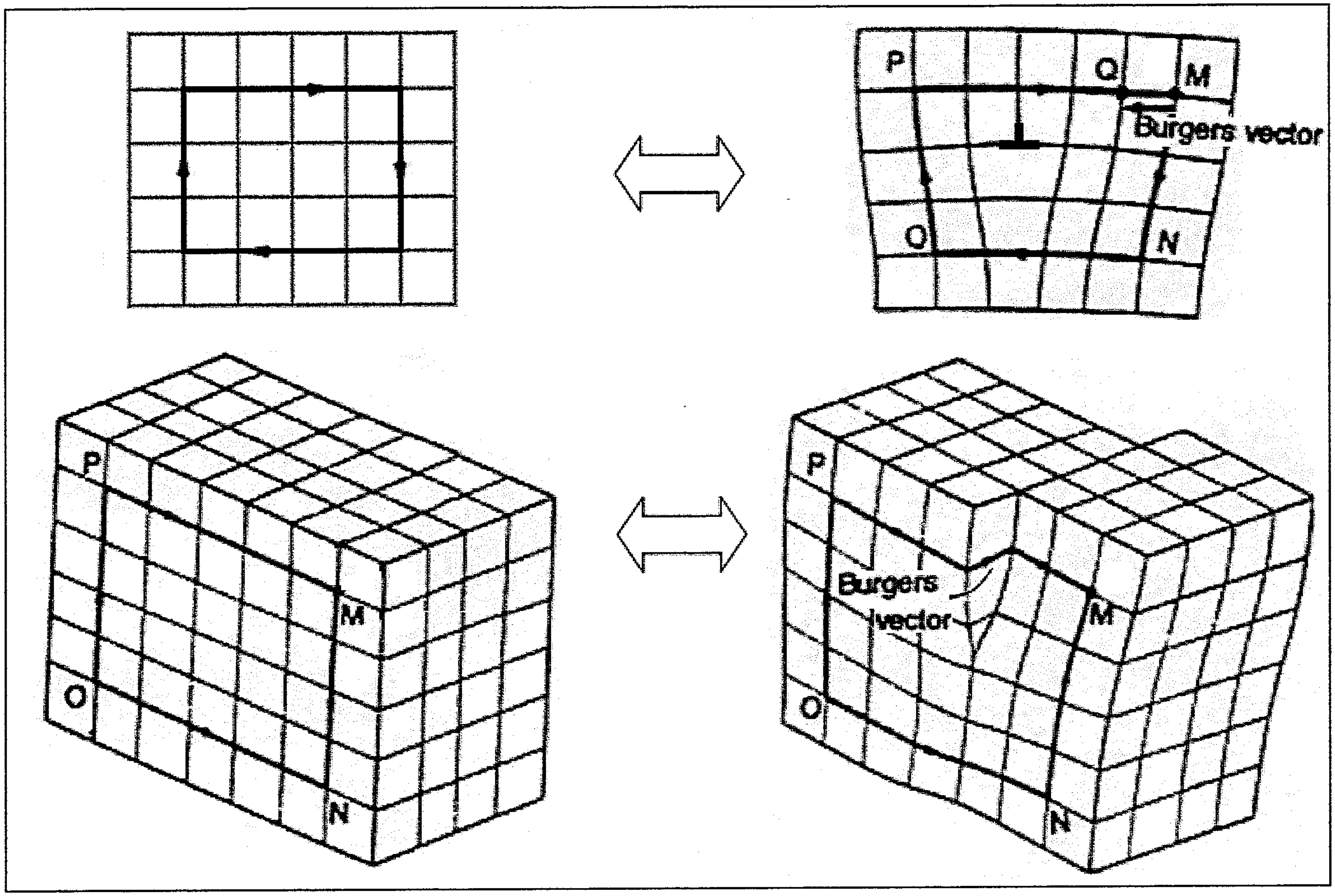
For brittleness we will compare σ_y (Tensile yield stress with σ_{growth} (tensile stress for crack growth).
In general:

Brittle: $\sigma_{growth} < \sigma_y$

Ductile: $\sigma_{growth} > \sigma_y$

Cottrell's Fracture Mechanism for BCC Metals

1) pile-ups at grain boundaries in pure metals ought to be relaxed by plastic flow in adjacent grains. (grain boundary acting as source of dislocations, etc).



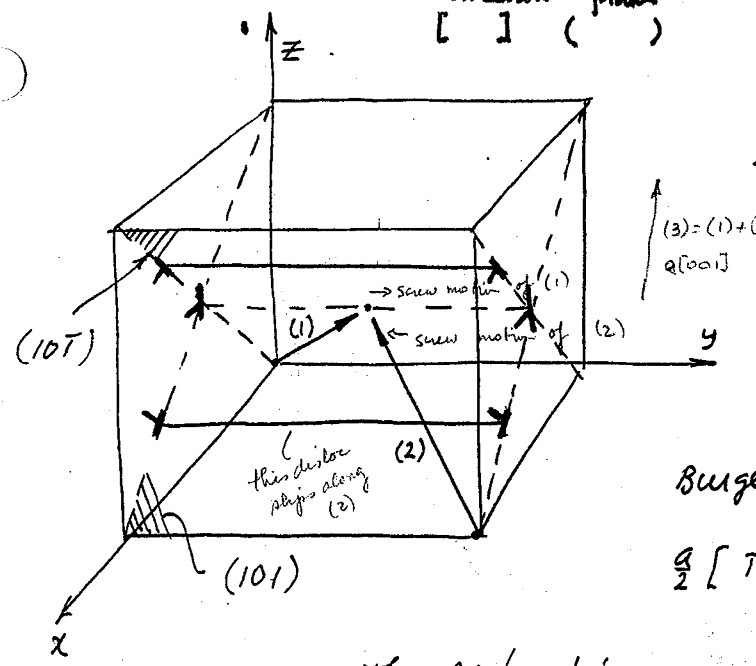
Edge

Screw

2) Brittle Fracture occurs in single crystals where grain boundaries are not present.

So Cottrell considered a dislocation mechanism. Actually the Cottrell model is more general than it looks. It includes inhomogeneous plastic flow with provisions when the stress can assist crack growth.

Consider intersecting slip dislocations in BCC



$\vec{b}$

1) $\frac{a}{2}[111]\ (10\bar{1})$ screw + edge disloc

2) $\frac{a}{2}[\bar{1}\bar{1}1]\ (101)$ screw + edge disloc

} will give an edge disloc

Following reaction is energetically feasible:

Burgers Vector Reaction

$$\frac{a}{2}[\bar{1}\bar{1}1] + \frac{a}{2}[111] \longrightarrow a[001]$$

The dislocation energy goes as μb^2, where $b = |\vec{b}|$

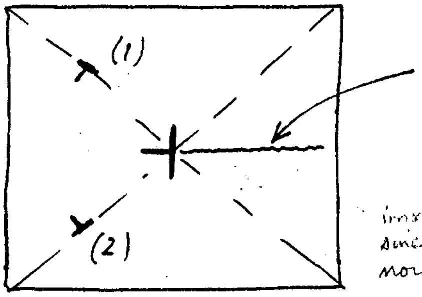
before after

so $$\frac{a^2}{4}(\sqrt{3})^2 + \frac{a^2}{4}(\sqrt{3})^2 \gtrless a^2$$

$$1.5\,a^2 > a^2$$

hence the reaction is energetically favorable.

a side View of Reaction



Reacting dislocations (1) and (2) produce fracture dislocation.

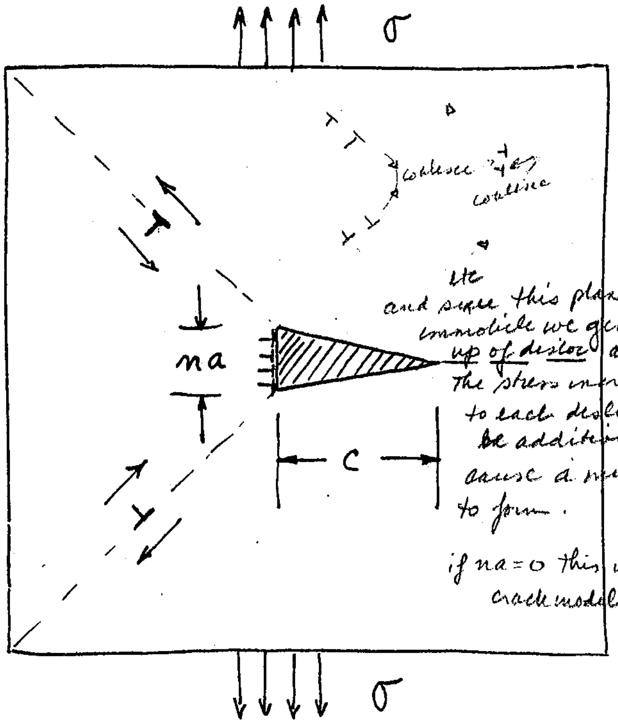
immobile disloc. since this is not a normal slip plane

as dislocations ~~co~~coaleoce and produce Fracture dislocations with Burgers vectors $b = a(001)$ a tiny microcrack will form.

NOTE: ONLY FIRST TWO DISLOCATIONS COME TOGETHER SPONTANEOUSLY - AFTER THAT MUST BE FORCED!

<u>Complete Model</u>

When n pairs of dislocations come together and form n Fracture dislocations, we have a <u>wedge</u> tending to open the crack.



etc
and since this plane is immobile we get pile up of disloc and hence the stress increase due to each disloc tend to be additive and cause a micro crack to form.

if $na = 0$ this is the griffith crack models

Note:

1) inhomogeneous plastic flow produces crack nucleation

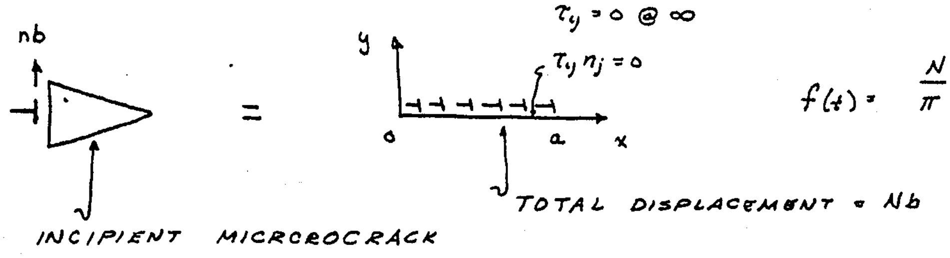
2) plastic flow <u>and</u> tensile stress makes crack grow.

GO TO Pg 120

MS 238 - AM 216 B

(HOMEWORK)

1.



$$f(t) = \frac{N}{\pi} \frac{1}{\sqrt{t(a-t)}}$$

(a) USING THE CONTINUOUSLY DISTRIBUTED DISLOCATION APPROACH, FIND AN EXPRESSION FOR THE SHAPE OF THE SURFACE OF THE "INCIPIENT MICROCRACK" & ILLUSTRATE YOUR RESULT GRAPHICALLY (i.e. - HOW GOOD IS THE "TRIANGULAR" MICROCRACK APPROXIMATION?)

(b) SHOW THAT $\mathcal{E}_{el} = \frac{\mu N^2 b^2}{4\pi(1-\nu)} \ln \frac{4R}{e^2 a} + \mathcal{E}_0$

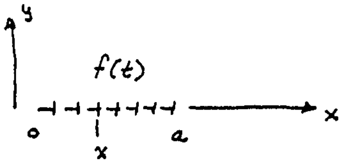
$e = 2.71828$

$R \sim$ CRYSTAL SIZE

$\mathcal{E}_0$ = CONSTANT INDEPENDENT OF a.

MS 238 - AM 216 B

(ANSWER SHEET)



1. $u_y^{TOP} - u_y^{BOTTOM} = \Delta u_y = b \int_x^a f(t)\,dt$

∴ IF v = DISPLACEMENT OF TOP HALF OR CRACK:

$$v = \frac{\Delta u_y}{2} = \frac{b}{2}\int_x^a \frac{N}{\pi} \frac{dt}{\sqrt{t(a-t)}} = \frac{Nb}{2\pi}\int_x^a \frac{dt}{\sqrt{t(a-t)}}$$

$$\frac{v}{Nb/2} = \frac{1}{\pi}\int_x^a \frac{dt}{\sqrt{t(a-t)}}$$

LE $\omega = \frac{a}{2} - t$

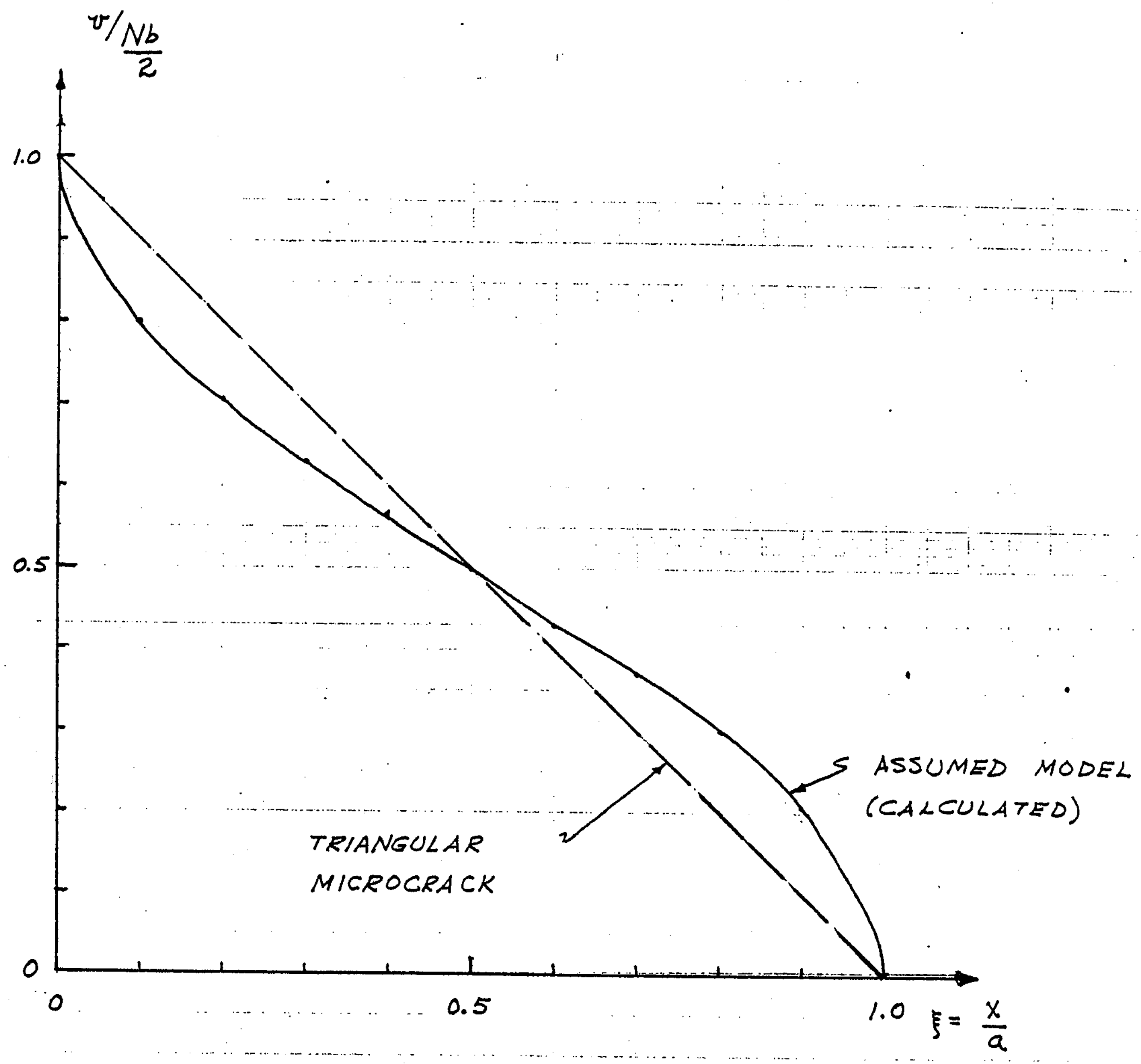
$$\frac{v}{Nb/2} = \frac{1}{\pi}\int_{-a/2}^{a/2-x} \frac{d\omega}{\sqrt{\left(\frac{a}{2}\right)^2 - \omega^2}} = \frac{1}{\pi}\sin^{-1}\frac{\omega}{a/2}\Bigg|_{-a/2}^{a/2-x}$$

$$= \frac{1}{\pi}\left\{\sin^{-1}\left(1 - \frac{x}{a/2}\right) + \frac{\pi}{2}\right\}$$

OR CALLING $\xi = x/a$

$$\underline{\underline{\frac{v}{Nb/2} = \left\{\frac{1}{2} + \frac{1}{\pi}\sin^{-1}(1 - 2\xi)\right\}}} \quad ; \quad \frac{v(0)}{Nb/2} = 1 \quad ; \quad \frac{v(1)}{Nb/2} = 0$$

(SEE GRAPH)



STROH-COTTRELL INCIPIENT MICROCRACK
DISPLACEMENT OF CRACK SURFACE

area under the arrow is the energy related to crack propagation

LET $x = a\omega$

$$\int_0^a f(x) \ln|x-t|\,dt = \frac{Nb}{\pi}\int_0^1 \frac{\ln a + \ln\omega}{\sqrt{\omega(1-\omega)}}\,d\omega$$ INDEPENDENT OF t

$$\therefore \int_0^a f(t)\,dt \int_0^a f(x)\ln|x-t|\,dt = \frac{Nb}{\pi}\int_0^1 \frac{\ln a + \ln\omega}{\sqrt{\omega(1-\omega)}}\,d\omega \int_0^a f(t)\,dt$$

$$= \frac{(Nb)^2}{\pi}\int_0^1 \frac{\ln a + \ln\omega}{\sqrt{\omega(1-\omega)}}\,d\omega$$

$$\mathcal{E}^1_{el} = \mathcal{E}_0 + \frac{\mu}{4\pi(1-\nu)}(Nb)^2\left\{(\ln R - 1) - \frac{\ln a}{\pi}\int_0^1\frac{d\omega}{\sqrt{\omega(1-\omega)}} - \frac{1}{\pi}\int_0^1\frac{\ln\omega}{\sqrt{\omega(1-\omega)}}\,d\omega\right\}$$

NOW $$\int_0^1 \frac{d\omega}{\sqrt{\omega(1-\omega)}} = \pi$$

$$\int_0^1 \frac{\ln\omega}{\sqrt{\omega(1-\omega)}}\,d\omega = \int_{-1/2}^{1/2}\frac{\ln(1-2v) - \ln 2}{\sqrt{(\frac{1}{2})^2 - v^2}}\,dv = \pi\ln\frac{1+\sqrt{1-(\frac{1}{2})^2(2^2)}}{2} - \pi\ln 2$$

$\omega = \frac{1}{2} - v$

$$= -2\pi\ln 2$$

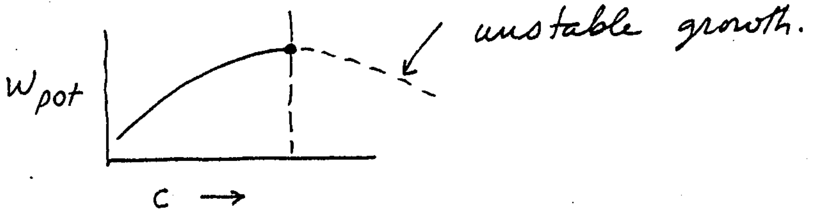
$$\therefore \mathcal{E}^1_{el} = \mathcal{E}_0 + \frac{\mu N^2b^2}{4\pi(1-\nu)}\left\{\ln R - 1 - \ln a + 2\ln 2\right\}$$

$$\boxed{\mathcal{E}^1_{el} = \mathcal{E}_0 + \frac{\mu N^2 b^2}{4\pi(1-\nu)}\ln\frac{4R}{ea}}$$

MISTAKE IN NOTES — HAD e^2. DOESN'T AFFECT $\frac{\partial E_T}{\partial a}$.

We want to find condition of instability. Following Griffith approach we write $W_{pot.}$ and find

$$\frac{\partial W_{pot}}{\partial c} = 0$$



Now what is W_{pot}?

1) <u>elastic energy of n crack dislocations</u> even if no applied stress.

$$\frac{\mu (na)^2}{4\pi(1-\nu)} \ln\left(\frac{4R}{c}\right)$$

Basic equation for dislocation energy.

here we have Burgers Vector na
core radius $\sim c$
since we take derivative, $\frac{\partial}{\partial c}$, numerical constants in ln term unimportant.

2) <u>surface energy of crack</u> in griffith analysis length of crack = 2c

$2\gamma c$

we treat a perfectly brittle crack even though we later have to put $\gamma = 10^3 \gamma_s$ to use the theory. A defect in the present state of this model.

work terms assoc. w/ applied stress

3) increase in strain energy of body due to presence of the crack MINUS work done by external stress when crack is formed. elast. strain energy + pot energy due to loading.

$$-\frac{\pi(1-\nu)\sigma^2 c^2}{8\mu}$$ Griffith term.

These terms are taken together as usual

note: This is plane strain result. with extra factor of 4 in denominator because the total length of our crack is c rather than 2c as usual.

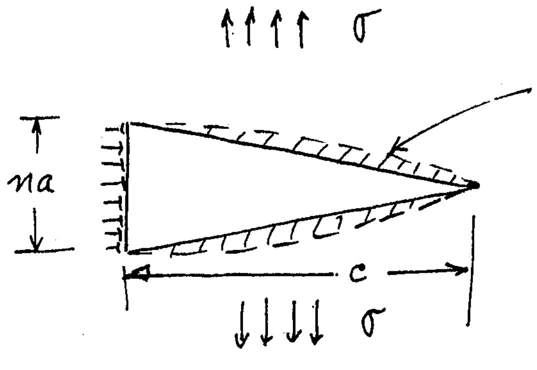
4) work done by external stress when wedge shape void forms by dislocation coalescence.

$$-\frac{\sigma n a c}{2}$$ (negative because reduction of potential energy of external work reservoir.).

$= \sigma \cdot \underline{\text{area}}$ ≈ pdV term.

This term not included in any of the above.

note:



these displacements are included in item 3 above.

So that

$$W_{pot} = \frac{\mu (na)^2}{4\pi(1-\nu)} \ln\left(\frac{4R}{c}\right) + 2\gamma c - \frac{\pi(1-\nu)\sigma^2 c^2}{8\mu} - \frac{\sigma n a c}{2}$$

Let us set

$$c_1 = \frac{\mu (na)^2}{8\pi(1-\nu)\gamma}$$

$$c_2 = \frac{8\mu\gamma}{\pi(1-\nu)\sigma^2} \qquad \text{so that} \quad \left(\frac{c_1}{c_2}\right)^{1/2} = \frac{\sigma n a}{8\gamma}$$

with these constants,

$$W_{pot}(c) = 2\gamma\left[c_1 \ln\left(\frac{4R}{c}\right) + c - \frac{c^2}{2c_2} - 2\left(\frac{c_1}{c_2}\right)^{1/2} c \right]$$

To find critical crack length,

$$\frac{\partial W_{pot}(c)}{\partial c} = 0 = 2\gamma\left[c_1 \frac{c}{4R}\left(\frac{-4R}{c^2}\right) + 1 - \frac{c}{c_2} - 2\left(\frac{c_1}{c_2}\right)^{1/2} \right]$$

$$0 = 2\gamma\left[-\frac{c_1}{c} + 1 - \frac{c}{c_2} - 2\left(\frac{c_1}{c_2}\right)^{1/2} \right]$$

$$0 = \frac{2\gamma}{c c_2}\left[-c_1 c_2 + c c_2 - c^2 - 2 c c_2 \left(\frac{c_1}{c_2}\right)^{1/2} \right]$$

Finally:

$$\boxed{c^{*2} - \left[1 - 2\left(\frac{c_1}{c_2}\right)^{1/2}\right] c_2 c^* + c_1 c_2 = 0}$$

$$\frac{\partial^2 W_{pot}}{\partial c^2} = 2\gamma\left[\frac{c_1}{c^2} - \frac{1}{c_2}\right] < 0$$

hence $\frac{1}{c_2} > \frac{c_1}{c^2}$ or $\boxed{c^{*2} > c_1 c_2}$

for instability ?
(check this).

Let us study the solution:

1) <u>suppose $\sigma \to 0$</u> then $c_2 = \frac{8\mu\gamma}{\pi(1-\nu)\sigma^2} \to \infty$

hence

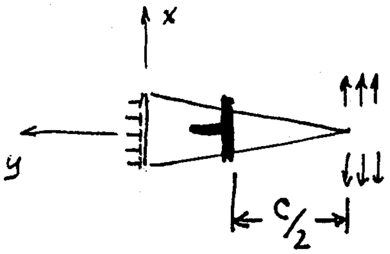
$$c^{*2} - c^* c_2 + c_1 c_2 = 0$$

$$c^{*2} = (c^* - c_1)\, c_2$$

since $c_2 \to \infty$ the only solution is $c^* \to c_1$

Hence $\sigma \to 0$ $\boxed{c^* \to c_1} = \frac{\mu (na)^2}{8\pi(1-\nu)\gamma}$

Physical meaning:



σ_{xx} (from dislocation)

$$\sigma_{xx} = \frac{\mu\, n a}{2\pi(1-\nu)\, c/2}$$

now let

$$\sigma_{xx} = \sigma_{griffith} = \sqrt{\frac{8\gamma\mu}{\pi(1-\nu]\, c}}$$

$$\frac{\mu \ n a}{2\pi(1-\nu)\, c/2} = \sqrt{\frac{8\gamma\mu}{\pi(1-\nu)c}}$$

$$\frac{\mu^2 (na)^2}{\pi^2 (1-\nu)^2 c^2} = \frac{8\gamma\mu}{\pi(1-\nu)c}$$

$$c = \frac{\mu(na)^2}{8\pi(1-\nu)\gamma} = c_1$$

So when $\sigma \to 0$ the critical crack length $c^* \to c_1$ is <u>approximately</u> the unstable crack length for the wedging dislocations <u>alone</u>.

2) <u>suppose $n = 0$</u> then $c_1 = \frac{\mu(na)^2}{8\pi(1-\nu)\gamma} = 0$

hence

$$c^{*2} - c^* c_2 = 0$$

solution $\boxed{c^* = c_2} = \frac{8\mu\gamma}{\pi(1-\nu)\sigma^2}$

<u>Physical Meaning</u>: The Griffith crack length as expected.

General solution for $\sigma \neq 0 \quad n \neq 0$

$$c^2 - \left[1 - 2\left(\frac{c_1}{c_2}\right)^{1/2}\right] c_2\, c + c_1 c_2 = 0 \qquad \text{(we drop the } * \text{)}.$$

$$c = \frac{c_2\left[1 - 2\left(\frac{c_1}{c_2}\right)^{1/2}\right] \pm \sqrt{c_2^2\left[1 - 2\left(\frac{c_1}{c_2}\right)^{1/2}\right]^2 - 4c_1c_2}}{2}$$

$$= \frac{c_2}{2}\left\{\left[1 - 2\left(c_1/c_2\right)^{1/2}\right] \pm \sqrt{\left[1 - 2\left(c_1/c_2\right)^{1/2}\right]^2 - 4c_1/c_2}\right\} = \frac{c_2}{2}\left\{\left[1 - 2\left(c_1/c_2\right)^{1/2}\right] \pm \sqrt{1 - 4\left(c_1/c_2\right)^{1/2}}\right\}$$

Consider the special value of $\left(\frac{c_1}{c_2}\right)^{1/2} = \frac{1}{4}$

Under $\sqrt{}$ we have

$$\left[c_2 - 2\left(c_1c_2\right)^{1/2}\right]^2 - 4c_1c_2 = c_2^2 - 4c_2\left(c_1c_2\right)^{1/2} + 4\cancel{c_1}c_2 - 4\cancel{c_1}c_2$$

$$= c_2^2\left[1 - 4\left(\frac{c_1}{c_2}\right)^{1/2}\right]$$

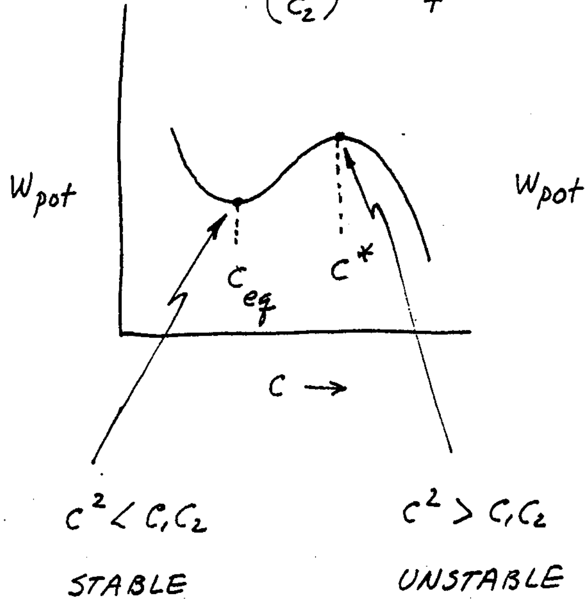
When $\left(\frac{c_1}{c_2}\right)^{1/2} = \frac{1}{4}$ two real roots coincide

When $\left(\frac{c_1}{c_2}\right)^{1/2} < \frac{1}{4}$ two real roots exist, the smaller value of c gives the stable crack length.

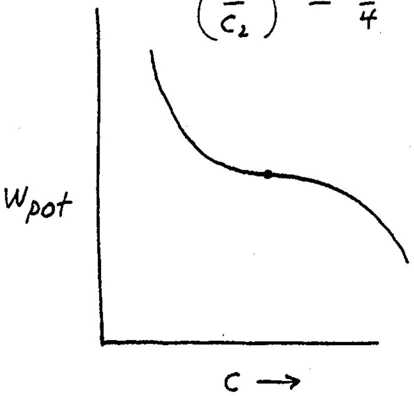
When $\left(\frac{c_1}{c_2}\right)^{1/2} > \frac{1}{4}$ no real roots present, no values of c can satisfy eqns., unstable crack growth.

or

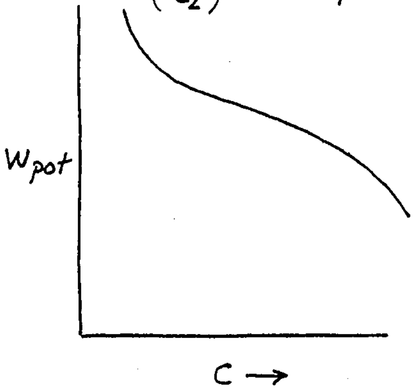
$\left(\frac{c_1}{c_2}\right)^{1/2} < \frac{1}{4}$



$\left(\frac{c_1}{c_2}\right)^{1/2} = \frac{1}{4}$



$\left(\frac{c_1}{c_2}\right)^{1/2} > \frac{1}{4}$



no stable crack growth - once it goes it doesn't stop.

- when $\left(\frac{c_1}{c_2}\right)^{1/2} < \frac{1}{4}$ cracks nucleated but won't grow

$\left(\frac{c_1}{c_2}\right)^{1/2} > \frac{1}{4}$ cracks nucleated and growth.

Hence, unstable fracture occurs when

$$\left(\frac{c_1}{c_2}\right)^{1/2} \geqslant \frac{1}{4}$$

$$\frac{\sigma n a}{8\gamma} \geqslant \frac{1}{4}$$

$$\boxed{\sigma n a \geqslant 2\gamma}$$

Cottrell Fracture Criterion.

The Fracture criterion can be expressed in convenient form by replacing .

shear yield stress by Tresca theory.

at DBTT $$\sigma_F \rightarrow \sigma_Y = 2\tau_Y = 2\left(\tau_i + k_y d^{-1/2}\right)$$ Hall-Petch

represent by

$$na \rightarrow \frac{(\tau_Y - \tau_i)\,d}{\mu} = \frac{k_Y\, d^{1/2}}{\mu}$$ Pile-up Theory.

Hence

$$\sigma \cdot na = 2\left(\tau_i + k_y d^{-1/2}\right)\frac{k_y\, d^{1/2}}{\mu} \gtreqless 2\gamma$$ at DBTT for fracture growth

brittle

$$\boxed{\left(\tau_i\, d^{1/2} + k_y\right) k_y \gtreqless \gamma\mu}$$ at DBTT

ductile

= at DBTT

k_y independent of Temperature.

τ_i strongly temperature dependent

So according to Cottrell Theory:

1) large grain ⟶ more brittle (brittle at H. Temp).
2) large k_y ⟶ more brittle
3) large τ_i ⟶ more brittle.

Results:

For Fe at 78°K (liquid nitrogen).
find from experimental data (σ_y, d, k_y μ)
That $\gamma \simeq 18000$ erg/cm^2. about 10 times γ_s.

For Mo at 200°K
find from experimental data (σ_y, d, k_y, μ)
that $\gamma \simeq 8000$ erg/cm^2

Grain size dependence of DBTT

at Fracture $(\tau_i d^{1/2} + k_y) k_y = \tau_y k_y d^{-1/2} \geqslant \gamma\mu$

if we assume $\gamma\mu \simeq$ constant. (Big assumption in view of values of γ above).

Then

$$\frac{\frac{\partial}{\partial d^{1/2}}\left(\tau_y k_y d^{-1/2}\right)}{\frac{\partial}{\partial T}\left(\tau_y k_y d^{-1/2}\right)} = -\frac{\delta T}{\delta d^{1/2}}$$

gives grain size dependence of Transition Temperature

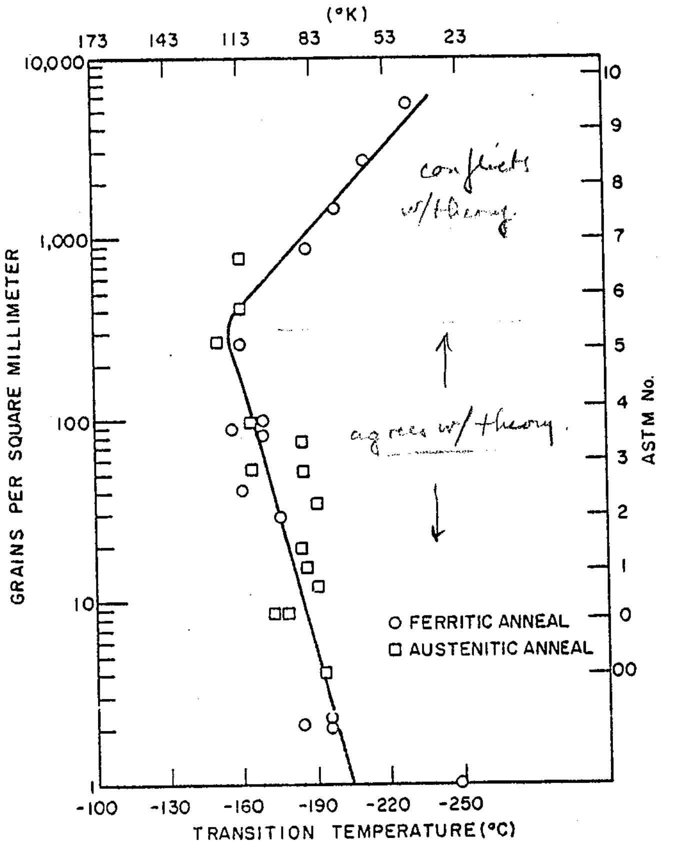
comes from: $d\left(\tau_y k_y d^{-1/2}\right) = \frac{\partial}{\partial d^{1/2}}(\ )\,\delta d^{1/2} + \frac{\partial}{\partial T}(\ )\,\delta T$
$= 0$

prediction agrees with expt. for Fe. (sometimes).

But — The theory does not work well for other BCC metals.

For example:

F.G. Tahmoush, E.P. Abrahamson II and N.J. Grant
Trans AIME 227 505 (1963)



conflicts w/theory

↑

agrees w/theory.

↓

Fig. 2—Grain size vs transition temperature for pure iron.

Also,

Thornley and Wronski Met. Sci J. 6 113 1972.
Cr, W, and Mo all show
wrong grain size dependence of DBTT.

Fatigue

References
Ref. 1, Ch 8
Ref 3, p 339, 383
Ref 4
Ref 8, Ch VI
Ref 9, Ch 5
Ref 11
Ref 28, 29, 30, 31

also
MS 257 (Enroll in Engr 257)
Fatigue of Metal Structures
Prof H.O. Fuchs

Maria Ronay, Fatigue of High Strength Materials, Ch 7.
Fracture Vol III Ed: Liebowitz p 431 (1971)

General Introduction

Fatigue - failure produced by oscillating stresses at stress levels below macroscopic yield stress.

Obviously important for engineering structures
vibrating machinery, rotating shafts,
aircraft structures - many different frequencies (high frequencies in engine - low frequencies in landing gear, 1 cycle per flight)
also, almost all engineering structures subjected to fatigue loading of some kind.
example: periodic pressurization of pressure vessel
cyclic temperature leading to thermal stresses.

low cycle fatigue - starting process up

Huge body of knowledge relating to fatigue: our treatment will focus on <u>physics</u> and <u>mechanics</u> of fatigue crack nucleation and propagation.

Engineering Characteristics of Fatigue

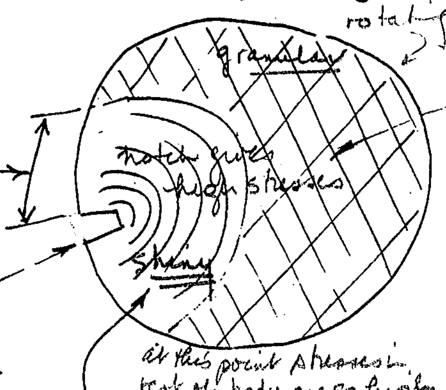
In service structures

1) Fatigue crack nucleation at stress concentration
2) Crack propagation to critical size
3) Unstable failure of remaining ligament. due to overloa

"Typical" Fatigue Failure

fatigue crack growth

nucleation of fatigue crack ⇒ keyway in shaft.



unstable failure of ligament due to overload.

at this point stresses in rest of body are so high that unstable fracture occurs

circular markings "shell" markings - various positions of crack front. (corrosion)

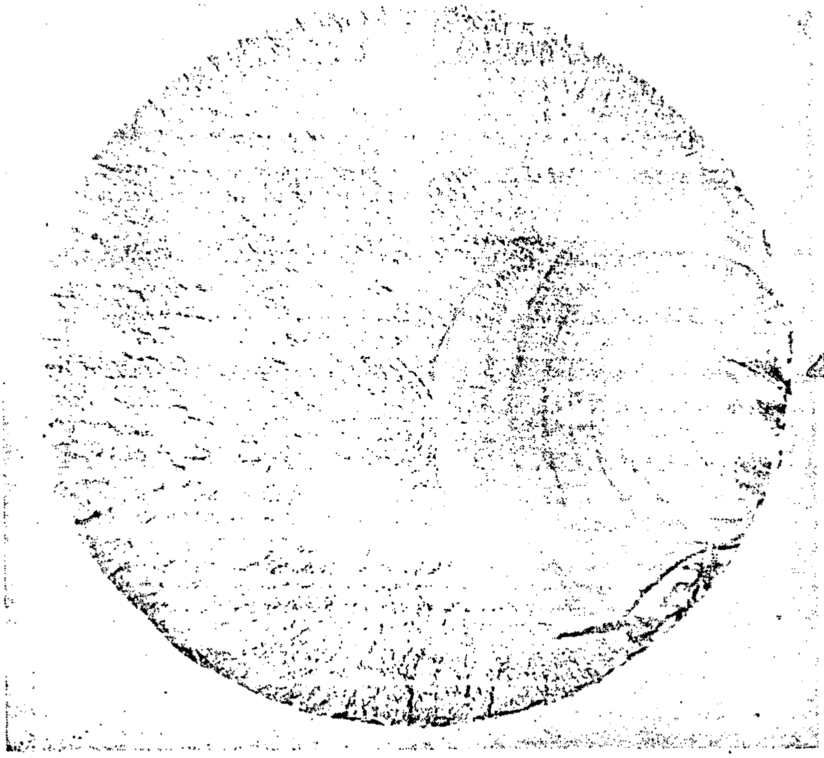


Fig. 5.78. Photograph of a fatigue failure in a 10-in.-diameter piston rod. (Courtesy John M. Lessells, Lessells Associates, Inc.)

crack nucleation

crack $K_I > K_{Ic}$ of material

From Chalmers,

Physical Metallurgy

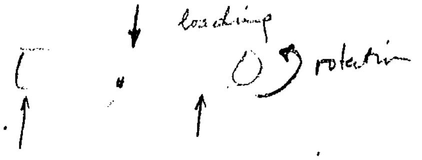
1959

p. 212

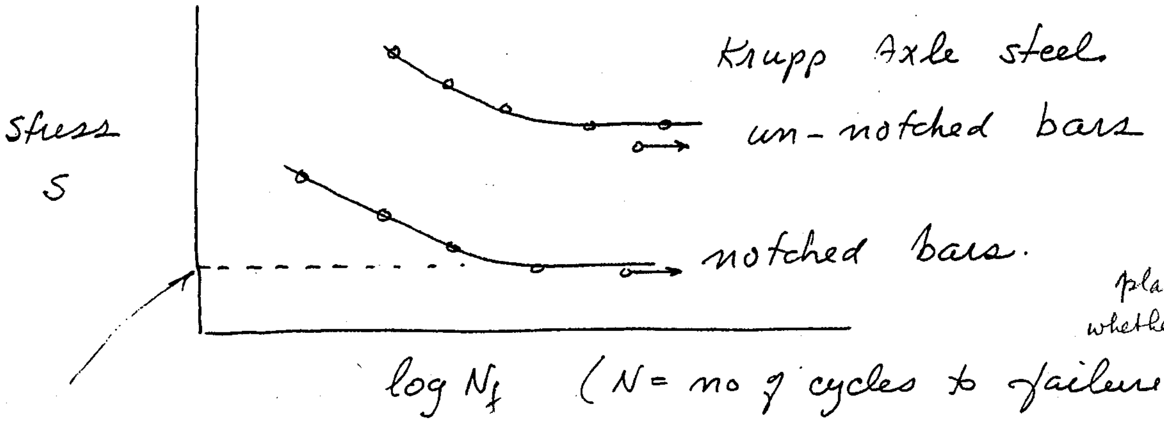
Brief History

Axle failures in Europe in 1850's – led to fatigue simulation of fatigue failures

Wöehler (Germany)
Fairbairn (England) } S-N Curve.



notch will experience compression/tension w/ rotation



~ 1860

plastic flow governs whether an endurance limit exists

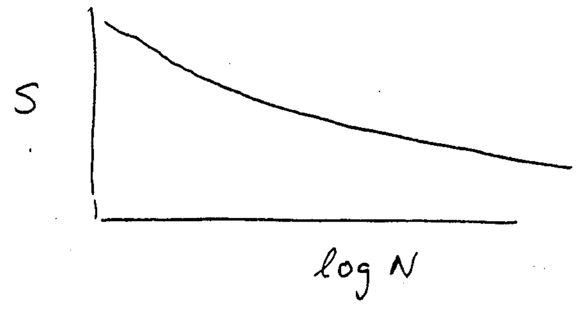
Endurance Limit

note: 1) observed endurance limit
2) observed effect of notches.

Different Approaches to Fatigue Design

1. Safe Life Approach – keep loads low enough so that fatigue cracks do not form.
Conventional approach – based on S-N curve and endurance limit
    a) But!, not all materials show Endurance limit.

Al alloys
or most alloys at
elevated temperatures



b) also, Endurance limit approach assumes we know surface condition – not always true. we know sharp notches or flaws in matl. will change S-N curve or will remove endurance limit.

Ref. 8 cites study by Donaldson and Anderson of 197 aircraft fatigue failures. – of these, 196 were initiated at mechanical defects, such as tool marks, burrs, gouges, fret marks etc.

So in service we have to anticipate existence of cracks and flaws.

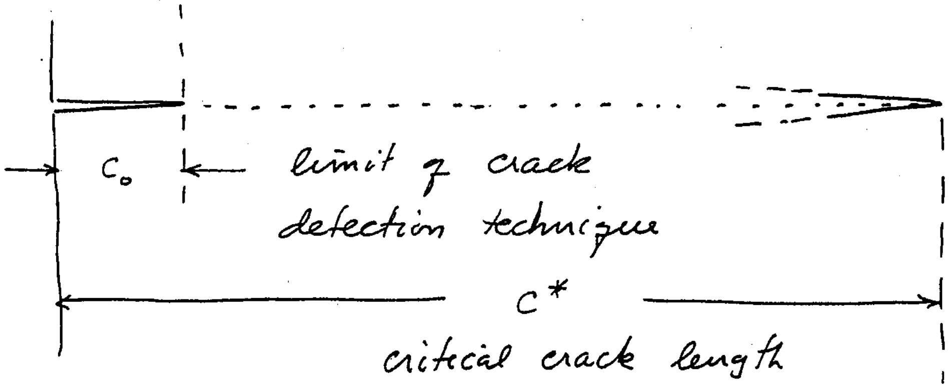
2. Fail Safe Approach – multiple load path structure.
– redundant design such that if failure occurs, other elements in the structure will take up the load. (this design philosophy is not restricted to fatigue design).

a) But, can't always afford the extra weight or expense of fail safe design.

b) Some structures are inherently difficult to fail-safe design – landing gear in aircraft.

3) Safe Inspection Time - based on fracture mechanics and fatigue crack growth rates. Assumes presence of cracks. and always growing.

schematically:



need an equation relating $\frac{dc}{dN} \sim (\Delta K)$

critical crack length based on calculated stress intensity factors and fracture toughness. $K_I = K_{I_c}$

$$\text{Fatigue Crack Growth rate} = \frac{dc}{dt} = f(\Delta K)$$

With this technique must inspect sufficiently often that cracks will not grow from c_o to c^* between inspections:

$$t_I = \text{inspection interval} \leqslant \frac{c^* - c_o}{\frac{dc}{dt}}$$

Currently popular - both for low T and high T structures, (even in blades, vanes, turbine wheels etc.).

Basic Reqmt: growth rate vs Stress Intensity Factor.

Evidence for Plastic Flow in Fatigue

Like brittle fracture, plastic flow must occur for fatigue failure to occur. We should have local inhomogeneous deformation

Fatigue Failure does not occur in perfectly elastic brittle matls. Glass, certain metal single crystals at low temperatures do not fail by fatigue – they either break on first stress cycle or they never break!

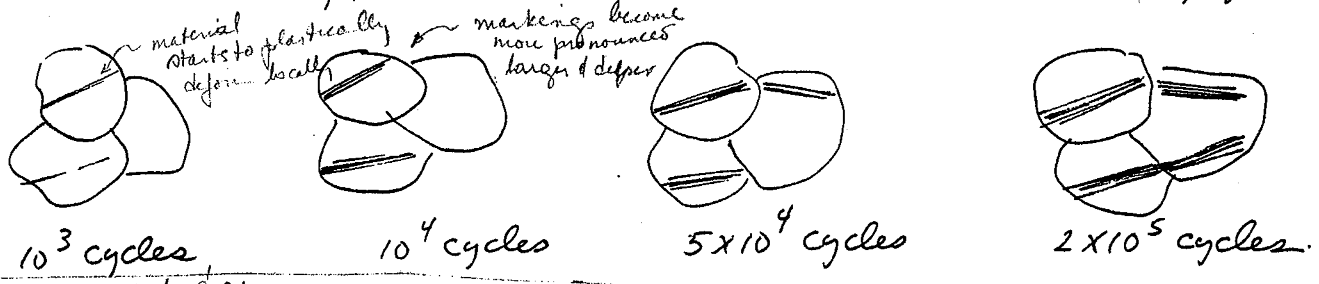
Evidence of plastic flow prior to fatigue failure.

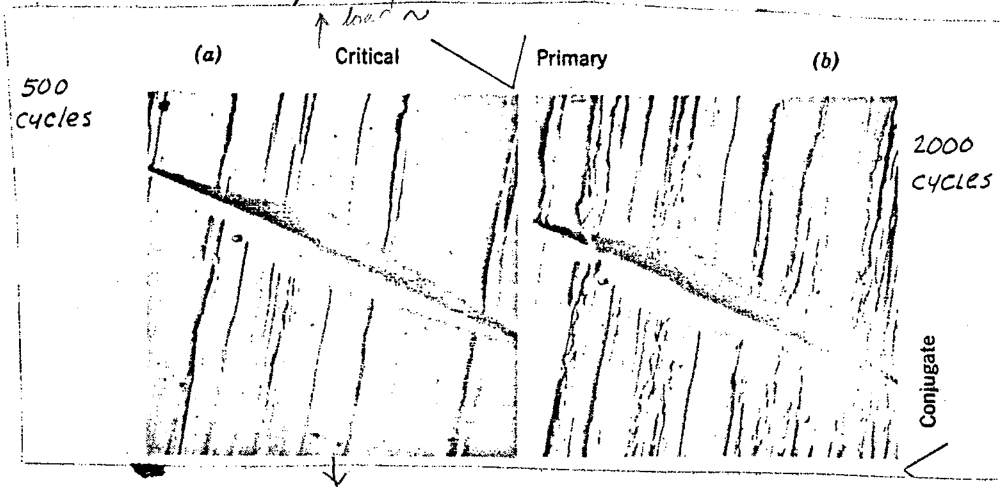
Ewing and Humphrey 1903.

slip lines on the polished and etched surface of wrought Iron. Observed slip bands in some surface grains. – Then fatigue cracks.

$\sigma < \sigma_{yield}$

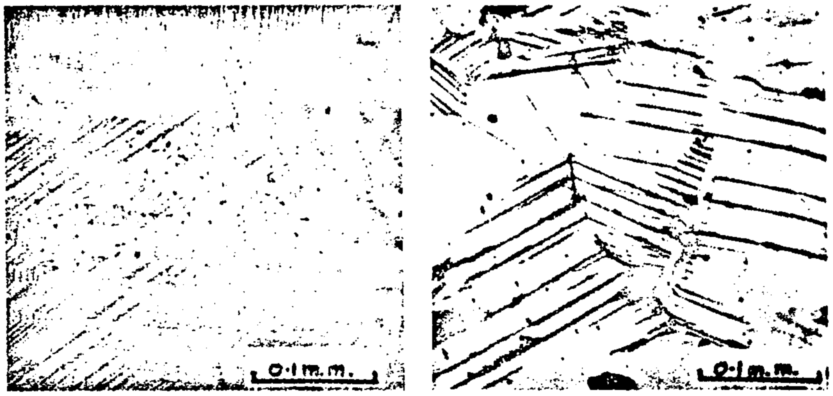
Now know: Deformation Bands concentrate with increasing cycles.





Cu surface:
From: W.A. Backofen
p 442, Ref. 4
(bands intensify with cycles).

For annealed metals, bands in cyclic straining more concentrated than uniaxial straining.



(a) (b)

Fig. 12. Comparison of slip band patterns in polycrystalline copper for unidirectional straining (a) and reversed cyclic straining (b) (Bullen et al.[55]).

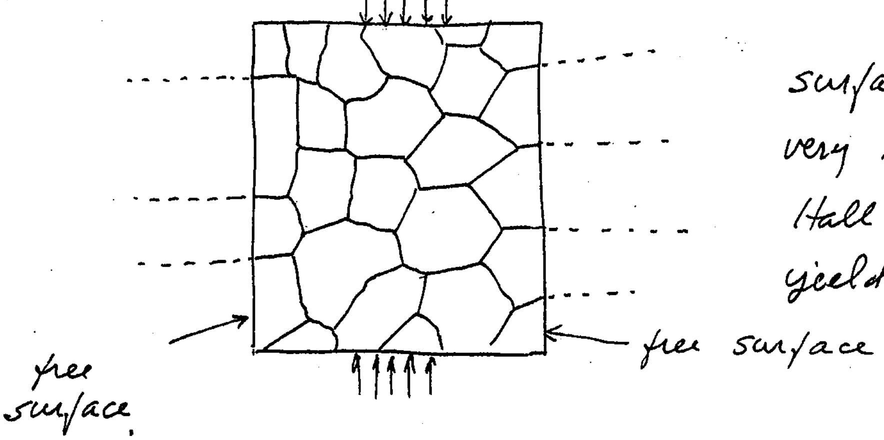
Ref 3
p 214

Role of Surface in Fatigue Damage

one reason

effective size of grain at surface > grain inside. Grain appears to be large, so in hall-petch sense since slip occurs in largest crystal, surface grains would yield fir

Yielding tends to start at surface grains first. Because of lack of constraint of surface grains.



surface grains act like very large grains. — Hall-Petch gives lower yield stress for slip in surface grains.

So, although the whole sample essentially, elastic, the surface grains can be plastically deformed.

1) <u>at low N or low σ</u>

light slip lines at surface, easily removed by electropolishing — indefinite fatigue life.

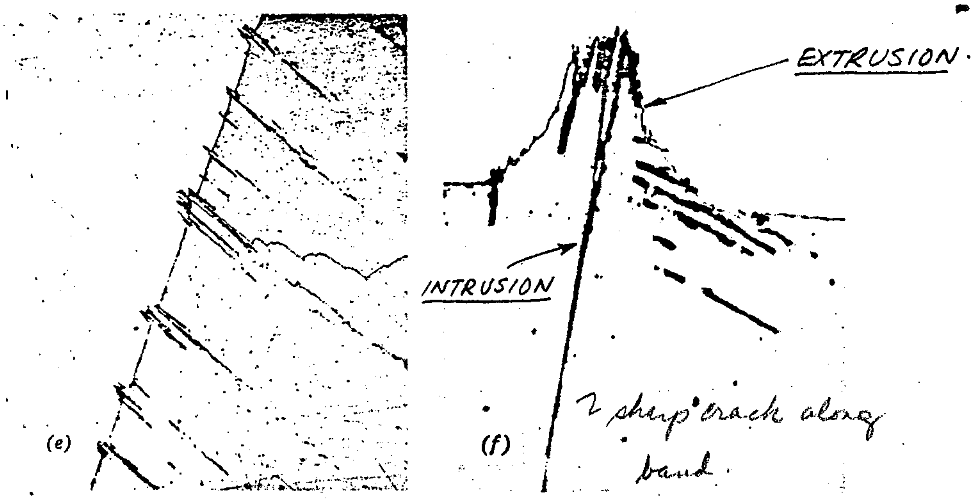
2) <u>med N or med σ</u>

heavy slip lines at surface, must electropolish heavily to remove.

3) <u>high N or high σ</u>

slip band activity leads to surface roughening. <u>Intrusions</u> and <u>Extrusions</u> (discovered by Forsyth 1953)

in Cu:



Tetelman and M^c Evily p 359.

The intrusions and extrusions produce stress concentrations and these in turn lead to <u>crack</u> <u>nucleation</u>.

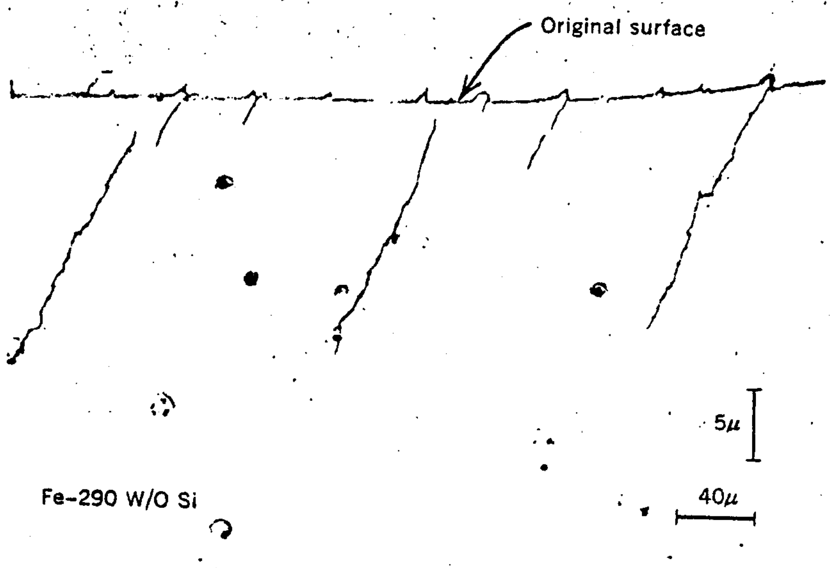
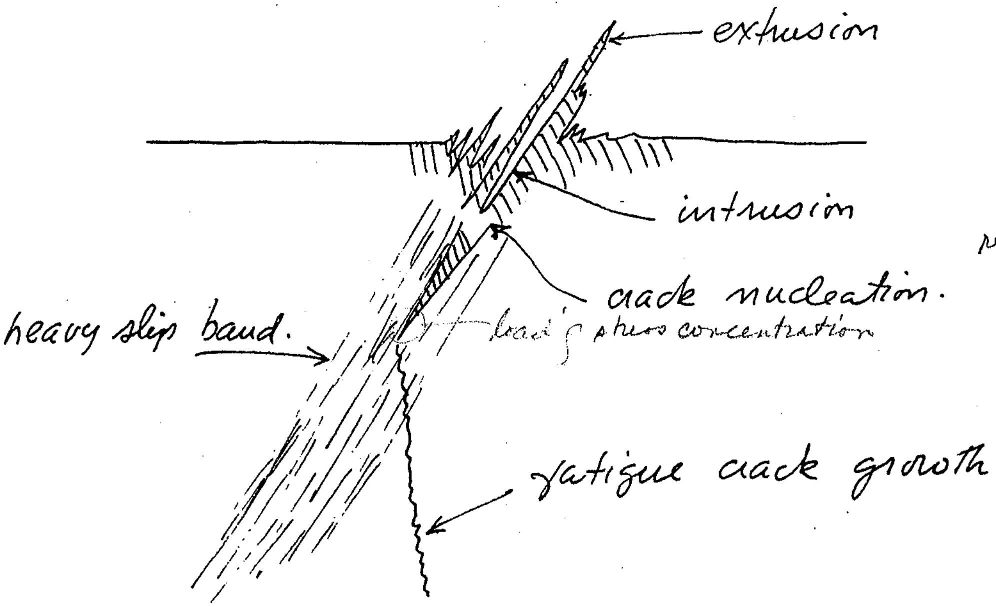


Fig. 8.9. Taper section of a slip bands shown in Fig. 8.7*g*. Crack formation associated with merging of ridges and with reentrant angles of ridges [117].

Tetelman
and
McEvily
p 361

Schematic of Fatigue Crack Initiation



result of irreversible deformation

direction of growth.

(based on stress distribution).
& the way loading is applied.

Engineering Implications

1) Keep surfaces smooth – so surface yielding requires highest possible stresses. (known for a very long time).

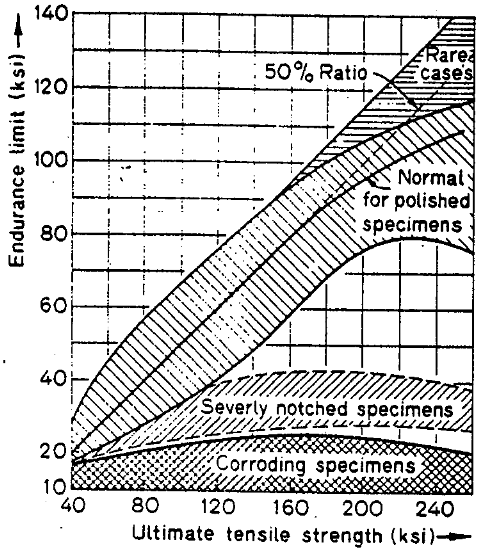


FIG. 13. Relation between fatigue strength and tensile strength for polished, notched, and corroded specimens of steels (Bullens, 1938).

(note date).

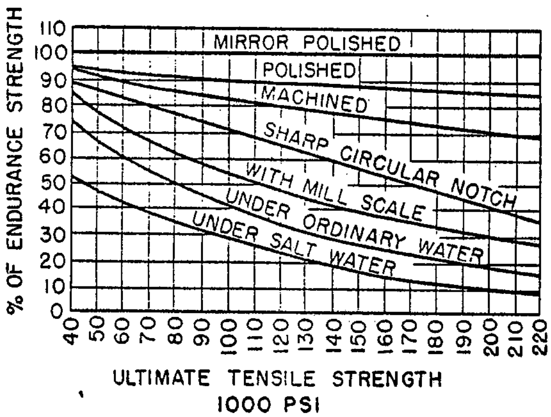


Fig. 3.22. **The effect of surface finish on endurance limit.** The endurance strength of a mirror polished bar is taken as 100 per cent and the endurance limits of other bars with different surface finishes are taken as percentages of the endurance limit of the mirror polished bar. Note that the higher the ultimate strength of the material, the more harmful the effect of surface imperfections as regards fatigue strength. Ordinary water and salt water have damaged the surfaces of the bars used in two of the tests, so that fatigue failure is accelerated. This type of failure is known as *corrosion fatigue*. [After D. Landau, *Fatigue of Metals*. The Nitralloy Corp., 1942.]

(note date)

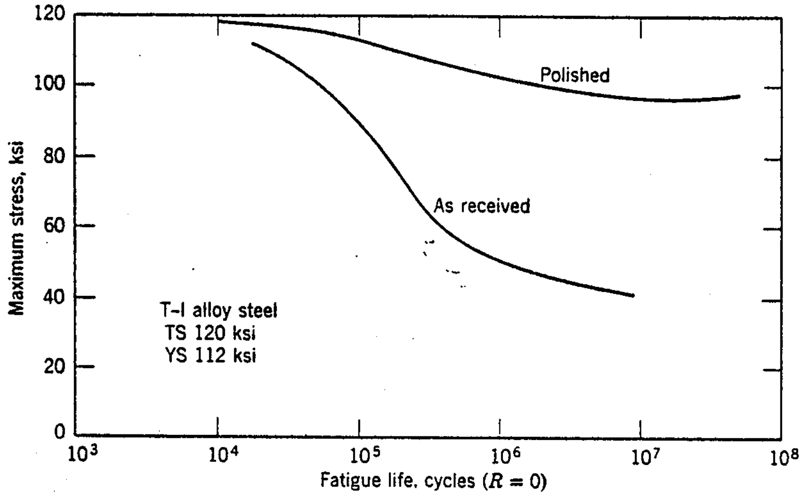


Fig. 8.6. S-N curves for T-1 steel in polished [22] and as-received [21] conditions.

Tetelman & McEvily p 356.

2) Keep surface in Compression (Residual Stresses)

shot peening — Prof. Fuchs is world expert here.

causes plastic deformation on surface crowds surface crystals & puts residual stresses on surface & under crystals

3) Make surface strong

requires high stresses for surface yielding.

shot peening

chrome plating

carburizing, nitriding etc.

~~Microstructural Implications of Fatigue Crack Nucleation~~

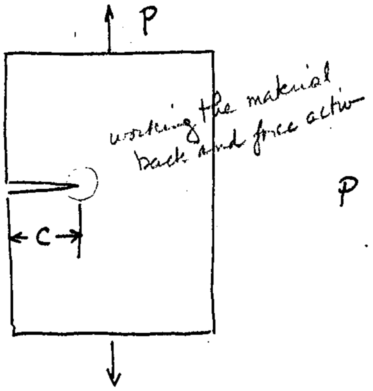
~~Fatigue strength of precipitation strengthened alloys (Al) particularly low - related to plasticity aspects of fatigue crack nucleation.~~

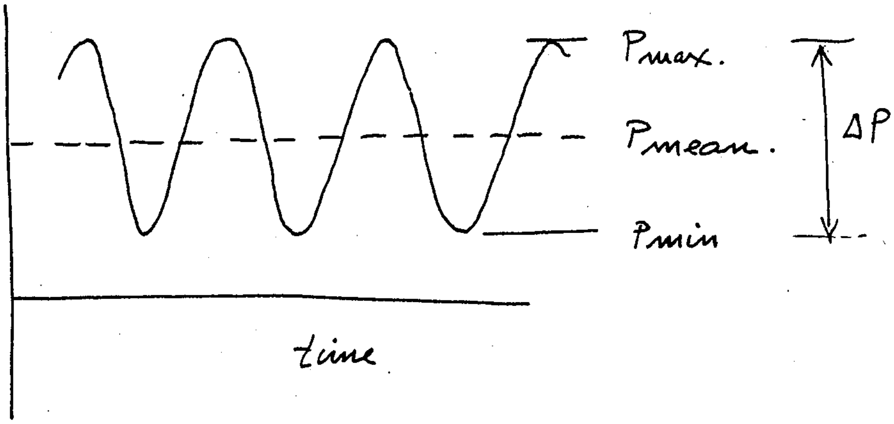
~~McEvily & Boettner 2024-T6 Al-Cu-Mg-Mn~~
~~Cycled 25000 psi, 10% of life, age 150°C for 16 hr.~~
~~renews life!! cyclic loading causes damage.~~

Mechanics of Fatigue Crack Growth

Most important aspect is crack growth rate vs Stress Intensity Factor. This relation forms the basis for engineering control of fatigue failure

Notation





Corresponding to these loads we have

$$K_{max} \sim P_{max} \qquad \text{define} \quad \beta \equiv \frac{P_{max}}{P_{min}} = \frac{K_{max}}{K_{min}}$$

$$K_{min} \sim P_{min}$$

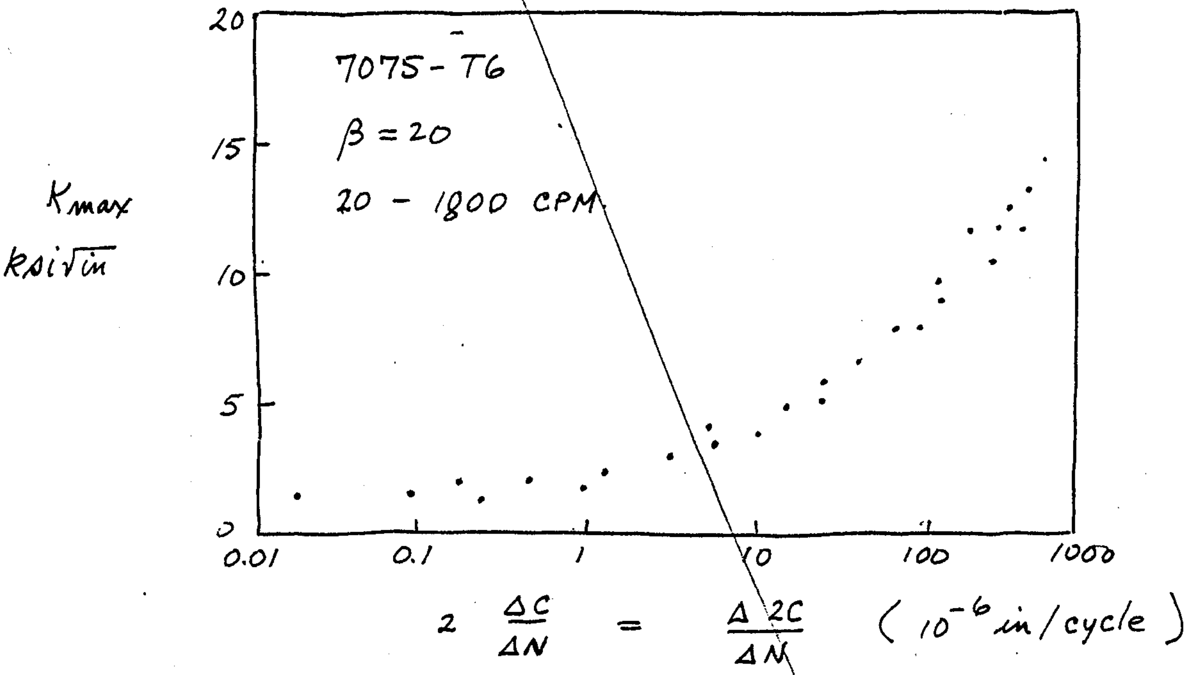
$$K_{mean} \sim P_{mean} \qquad \gamma \equiv \frac{P_{mean}}{\Delta P} = \frac{K_{mean}}{\Delta K}$$

$$K_{max} - K_{min} = \Delta K \sim \Delta P$$

Suppose we measure crack growth rate $\frac{dc}{dN} = \frac{\Delta c}{\Delta N}$

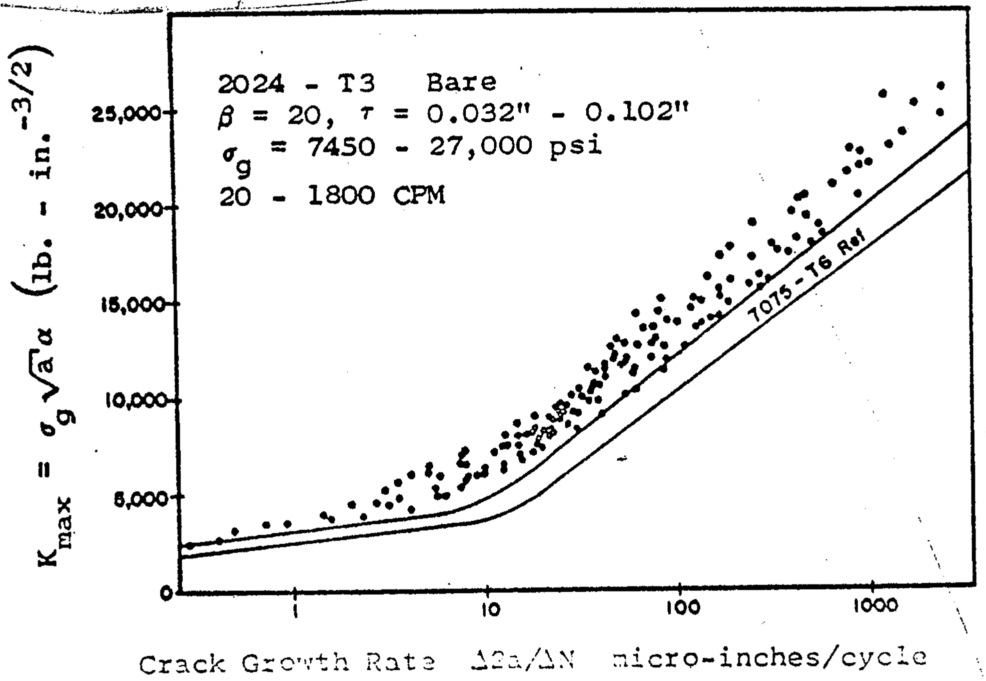
for given alloy (say 7075-T6) for various P_{max} but for constant β. If crack growth uniquely related to ΔK, should get unique relation with K_{max}.

get



from Ref. 8, p VI-8

Similar result for 2024 - T3

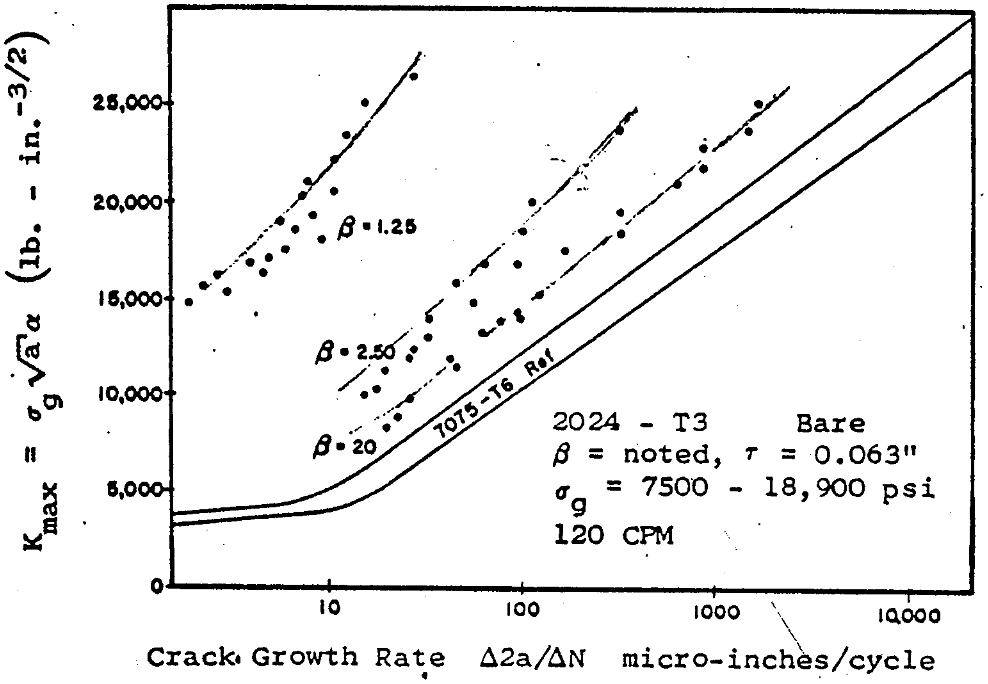


from Ref 8, p VI-8.

(scatter due to strain rate effects on deformation).

* Tests from BAC and NACA TN 4394 and Proceedings, Third Congress of Applied Mechanics, 1958, pp595 - 604.

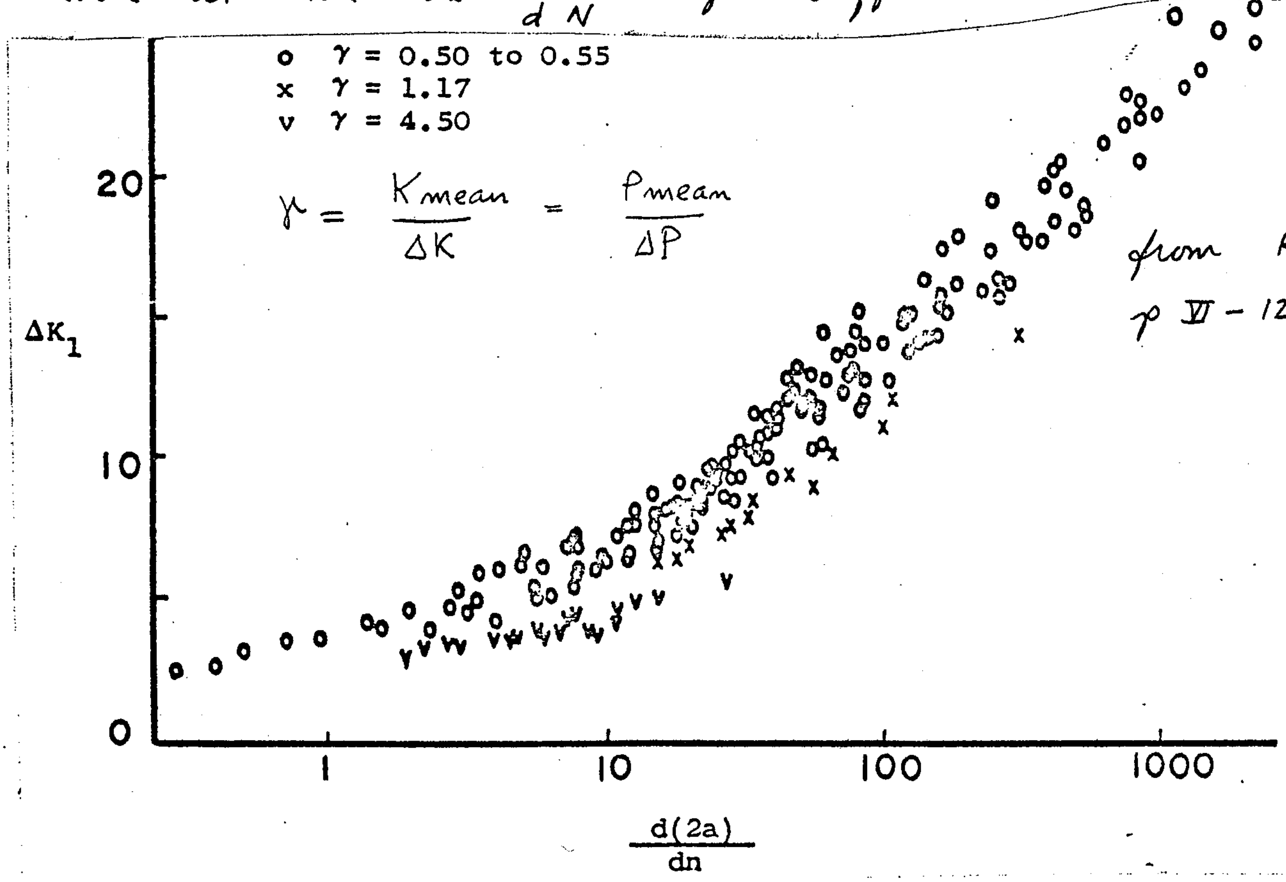
Look at effects of β, important too.



from Ref 8, p VI-10

* BAC Tests

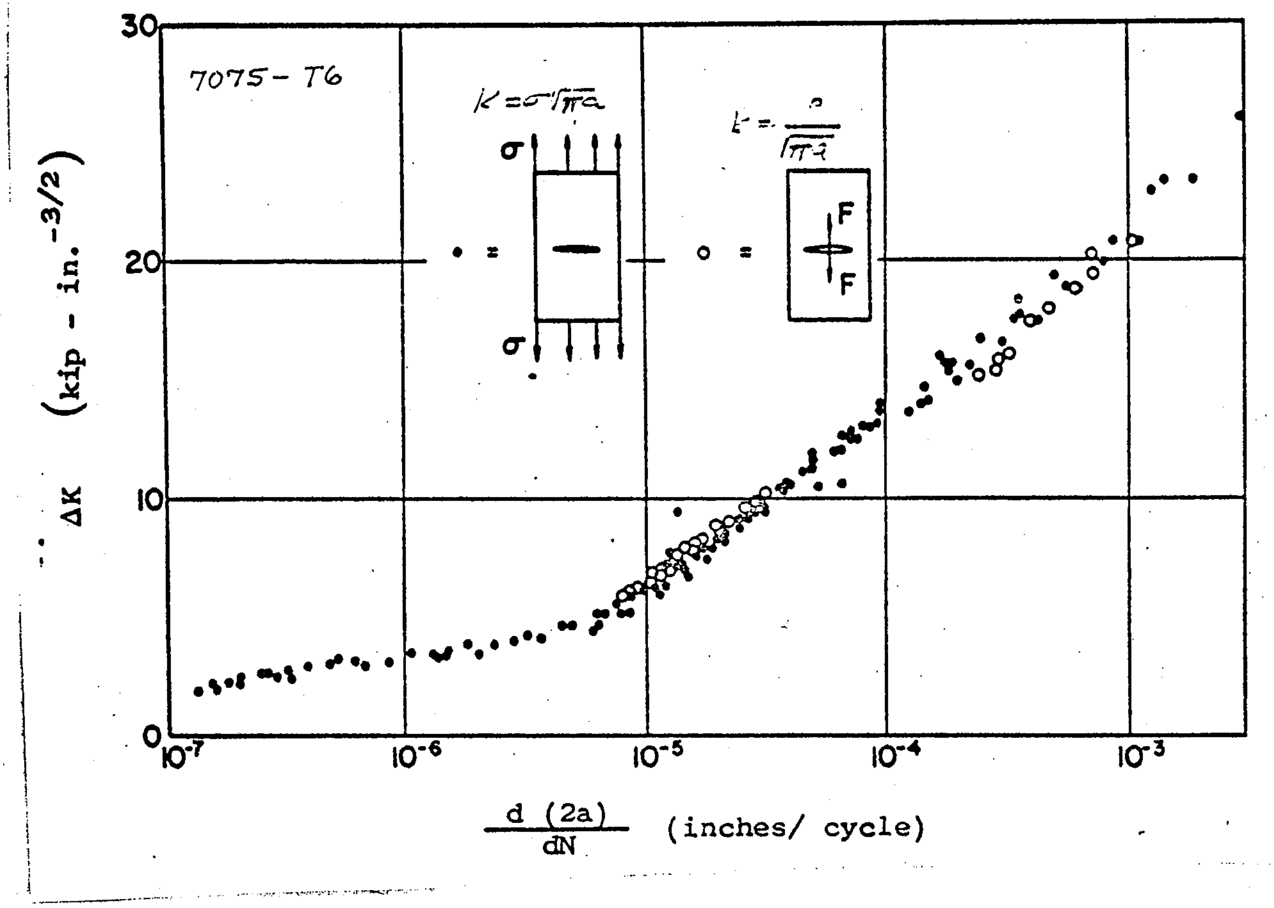
Look at ΔK vs $\frac{d\,2C}{d\,N}$ for different values of γ.



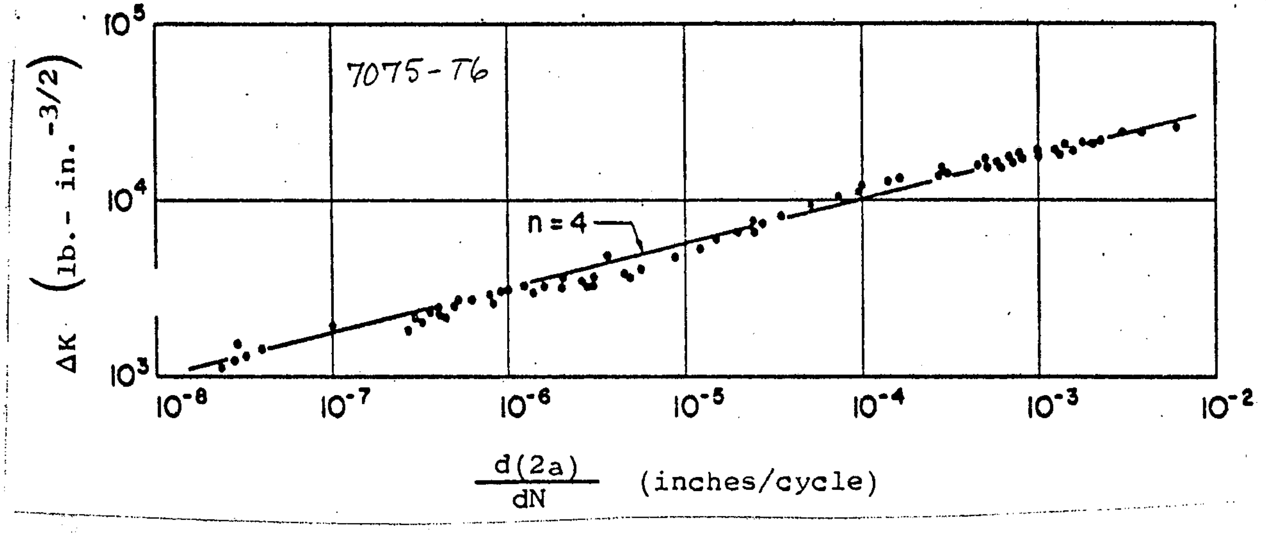
$$\gamma = \frac{K_{mean}}{\Delta K} = \frac{P_{mean}}{\Delta P}$$

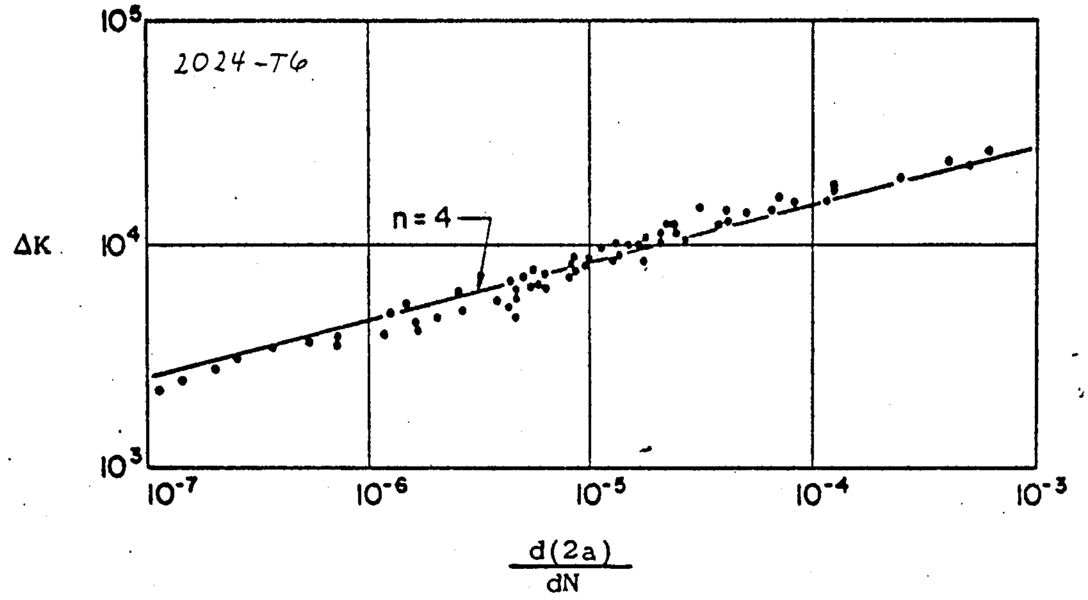
from Ref. 8, p VI-12

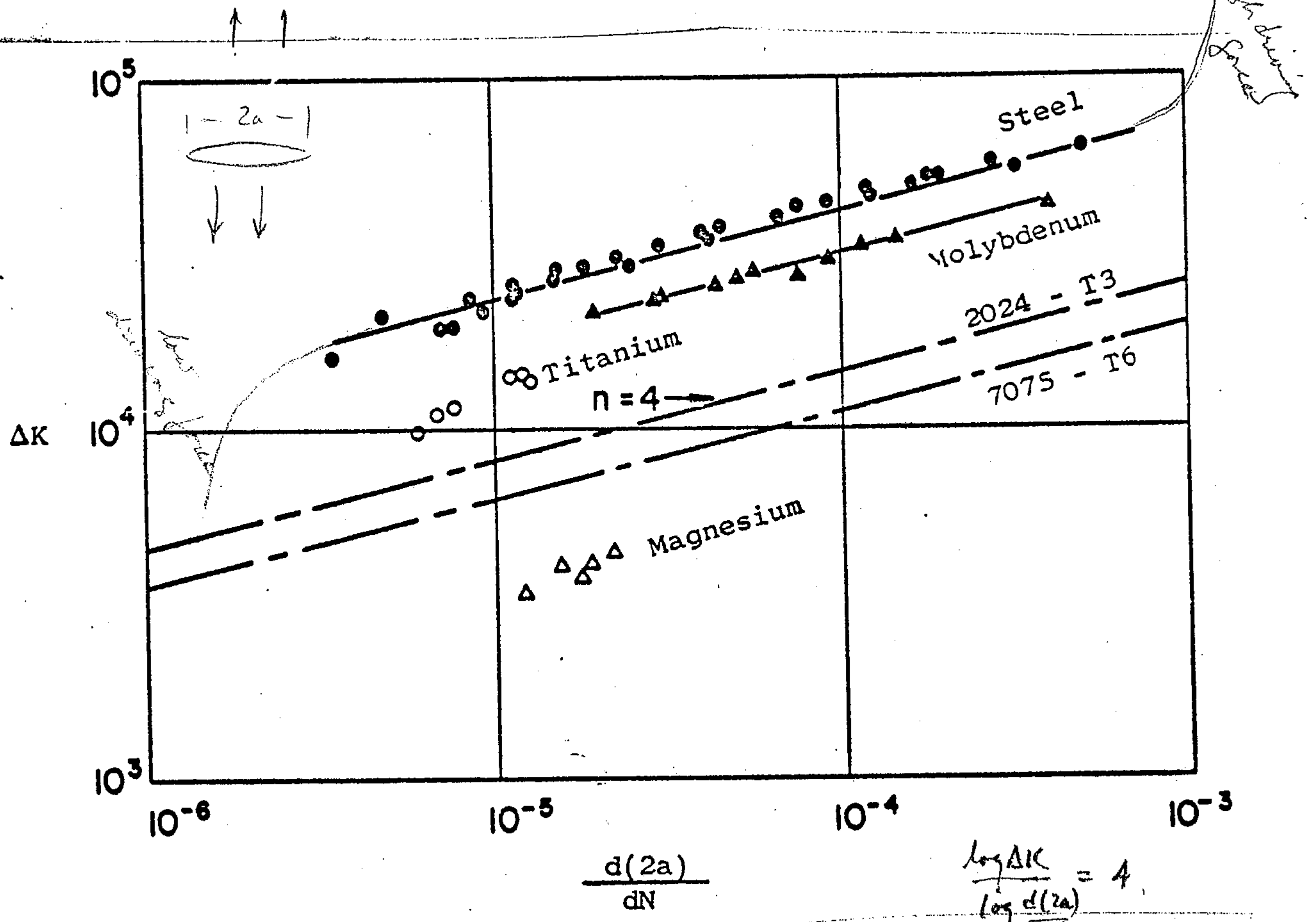
Now consider different configurations – uniformly stressed plate and wedge force load.



Therefore, the general correlation for $\frac{\Delta 2c}{\Delta N}$ is with ΔK.







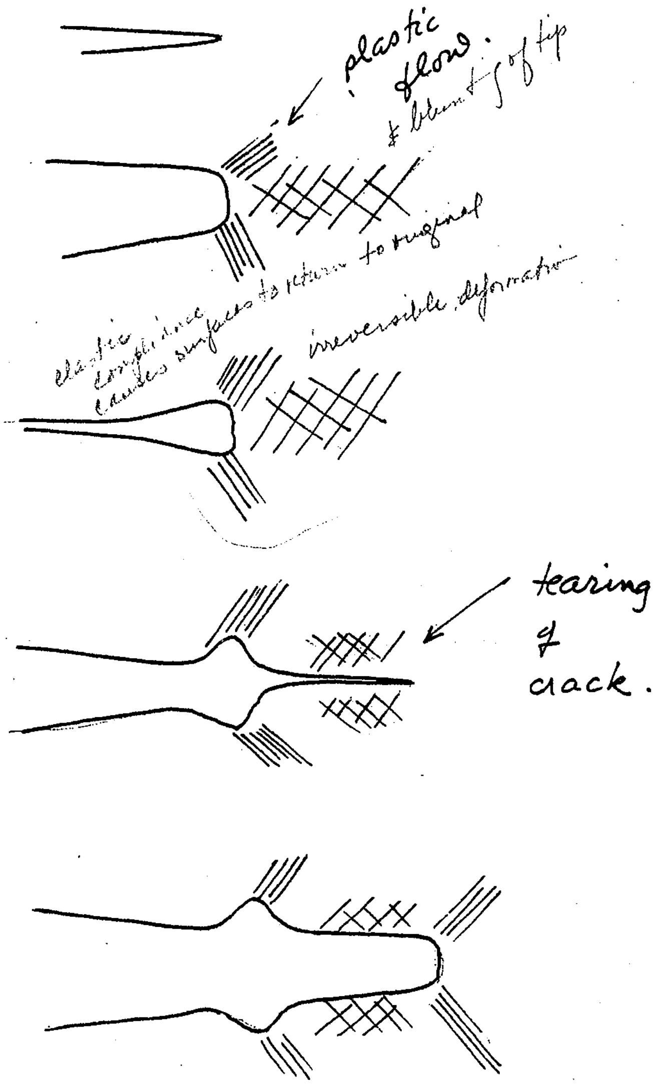
$$\frac{\log \Delta K}{\log \frac{d(2a)}{dN}} = 4.$$

Microscopic Aspects of Fatigue Crack Growth

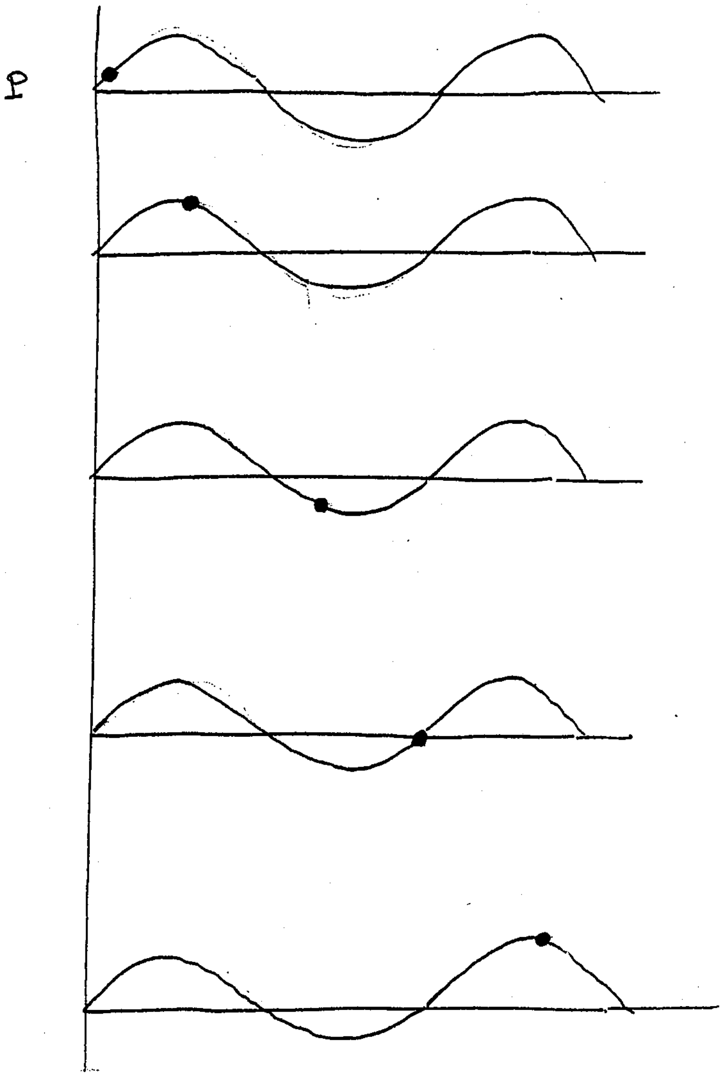
explanation of striations on surface.

Ref. C. Laird and G.C. Smith
Phil Mag. 7 847 (1962)

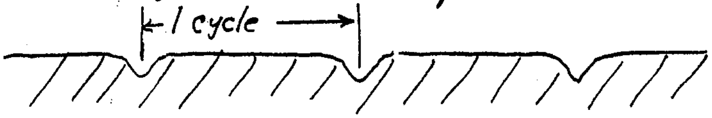
For ductile metals, we have ripples on fracture surface which indicate discontinuous movement of fatigue crack front.



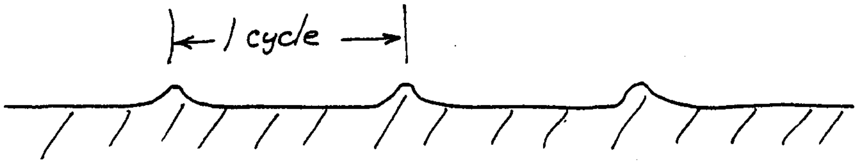
Crack Configuration



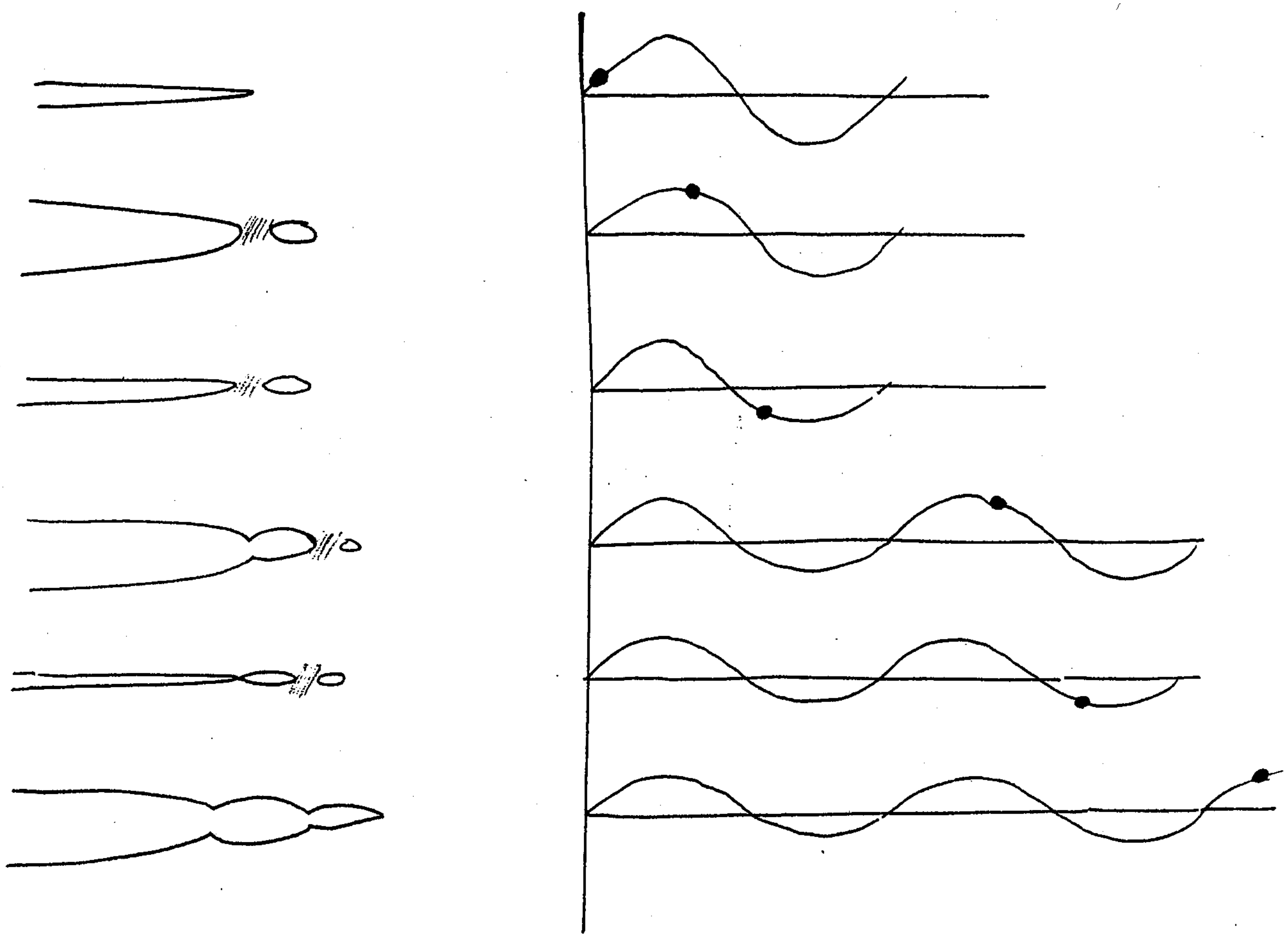
Resulting Fracture Surface



Another model will produce following fracture surface.



This would be produced by void growth ahead of fatigue crack tip. (Forsyth and Ryder).



This structure useful because we can relate the macroscopic growth rate $\left(\frac{\Delta c}{\Delta N}\right)_{macro.}$ to $\left(\frac{\Delta c}{\Delta N}\right)_{micro.}$

$$\left(\frac{\Delta c}{\Delta N}\right)_{macro} = \frac{\text{crack length at failure} - \text{initial crack length}}{N_f \text{ (no cycles to failure).}}$$

$$\left(\frac{\Delta c}{\Delta N}\right)_{micro} = \Delta c \text{ (from replica or scanning microscopy)}$$

find
$$\left(\frac{\Delta c}{\Delta N}\right)_{macro} = \left(\frac{\Delta c}{\Delta N}\right)_{micro}.$$

also, the actual crack growth rate, $\frac{dc}{dt}$, is

$$\frac{dc}{dt} = f\ \frac{dc}{dN}.$$

Fatigue striations

note: macroscopic "shell" markings not striations and do not represent crack front positions. Only the microscopic striations represent crack front positions.

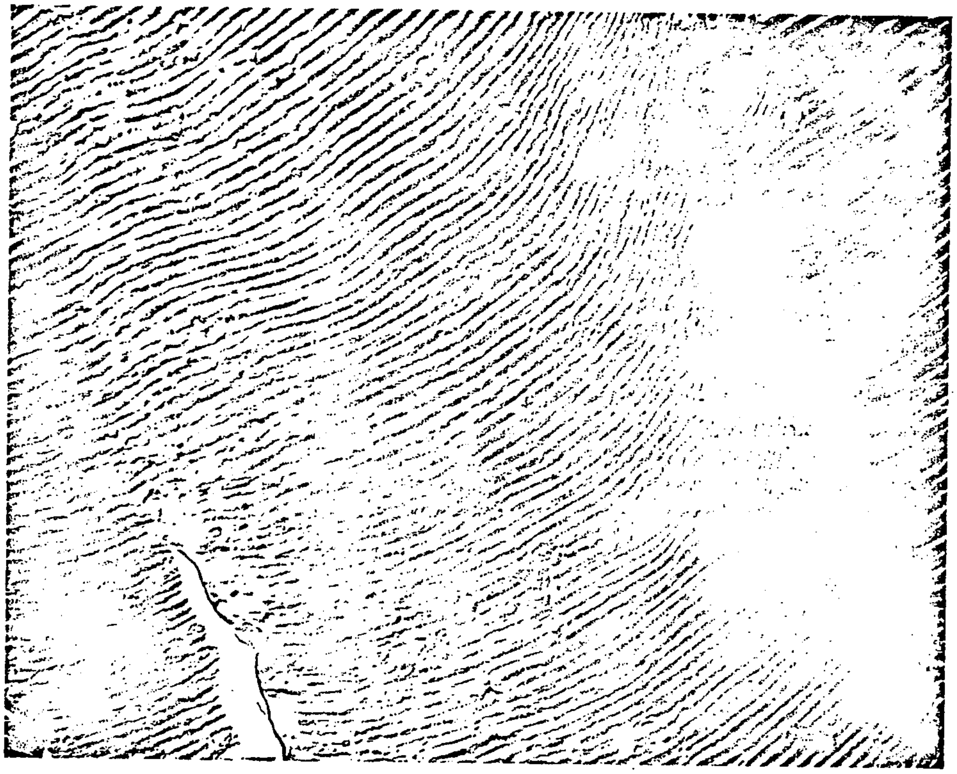


Fig. 5.7

From Forsyth "The Physical Basis of Fatigue" (1969) p. 87.

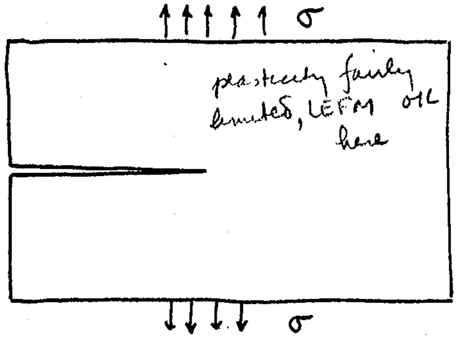
3/4/75

Theory of Fatigue Crack Growth in Ductile Metals

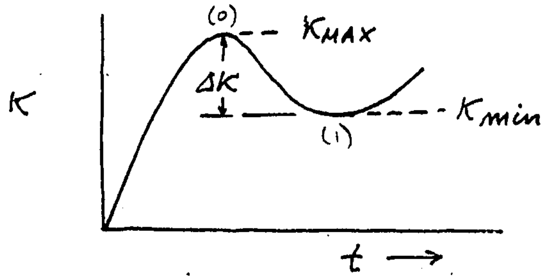
J WERTMAN Int. J Fracture Mech 3, 460 (1966

Qualitative Aspects

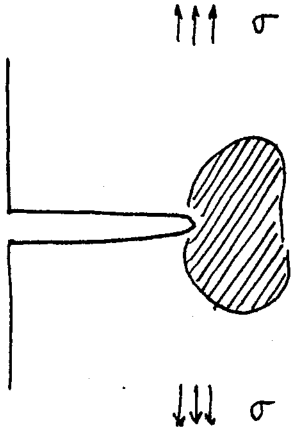
Consider a sharp crack machined into a plate



Consider following variation in Stress Intensity Factor



On loading $\left(\frac{dK}{dt} = +\text{ive}\right)$
and plastic zone is formed.



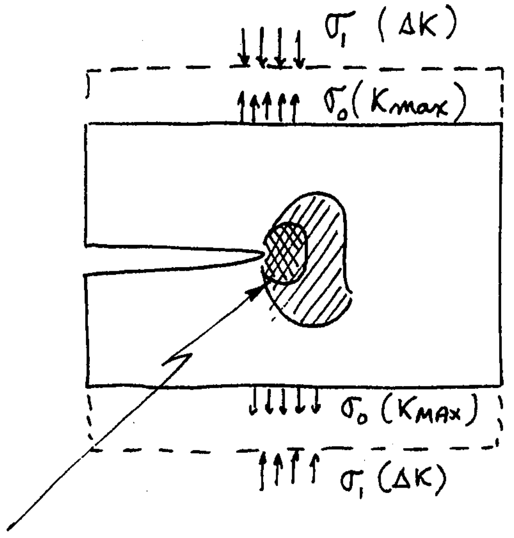
In actual practice the crack tip will be blunted by plastic flow and the singularity will be removed.

Theoretical treatments of this problem will assume that the singularity is not removed.

Now consider the unloading region $\left(\frac{dK}{dt} = -\text{ive}\right)$. The decreasing stress intensity factor can be modeled by superimpos an increasing K of the opposite sign.

whether it yields on back cycle [(0) to (1)] depends on bluntness of tip, and bluntness may not allow any more plastic flow. Material strain hardens. Wertman analysis doesn't take into account bluntness, But always assumes a sharp crack

Based on Bilby-Cottrell-Swinden elastic-plastic treatment Mode III analog of DM



extent of reverse yielding if it occurs.

In a real case (treated qualitatively here) we have to distinguish between two cases

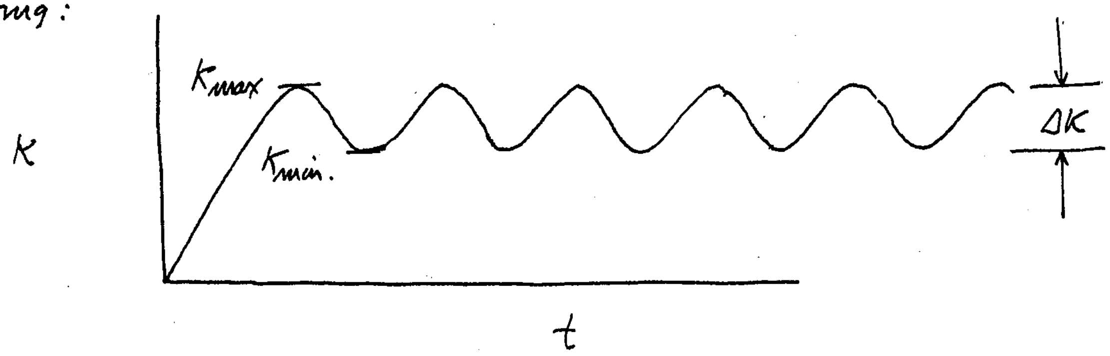
a) ΔK large enough to cause reverse yielding at the crack tip (this depends on how blunt the crack tip became on the first cycle.)

b) ΔK too small to cause reverse yielding.

First consider (b), no reverse yielding. – fatigue crack would never grow because plastic flow would stop after first. loading cycle.

Now consider (a), reverse yielding – here material at crack tip will plastically flow in reverse direction as shown.

Now assume that K continues to cycle according to the following:



Now continuing to reverse the K will cause reverse plastic flow to accumulate in the crack tip. We have two possible results.

1) cracking eventually occurs at the crack tip.
2) work hardening raises the yield strength sufficiently that after some number of cycles, plastic flow no longer occurs.

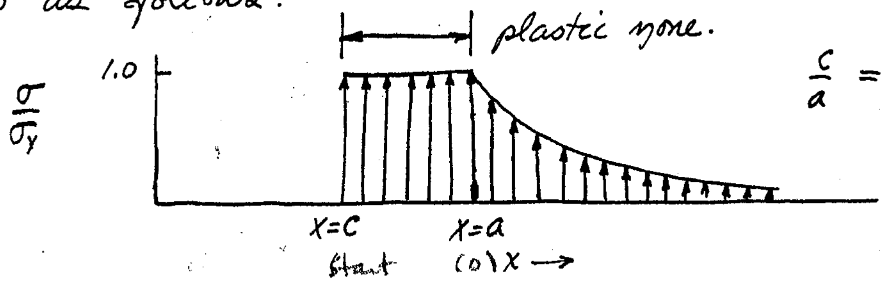
The above *qualitative* analysis indicates that oscillating stress intensity factors will *not* *always* produce fatigue crack growth. The quantitative theory below predicts fatigue crack growth *always* because of the assumption of the stress singularity at the tip.

Weertman's Theory of Fatigue Crack Growth

J. Weertman, International J. Fract. Mech. 2 460 (1966).

Theory based on Bilby, Cottrell and Swinden (BCS).
Proc. Roy. Soc. (London) A 272 304 (1963)
model of Mode III Elastic - Plastic Crack.

The BCS Theory is the Mode III equivalent of the Dugdale - Muskhelishvili (DM) Model (Mode I). discussed earlier (see pp 91 - 97). On loading, the stress distribution is as follows.



$$\frac{c}{a} = \cos\beta = \cos\left(\frac{\pi}{2}\frac{\sigma}{\sigma_y}\right)$$

To be somewhat consistent with the Weertman notation we will let $D(x)$ be the total displacement between the crack faces. (Remember, this displacement is evaluated in the plastic zone and is interpreted as plastic displacement. From Goodier & Field the displacement $D(x)$ is (see p 96)

$$D(x) = \frac{2a\sigma_y}{\pi E}\left[2\cos\theta \ln\left\{\frac{\sin(\beta-\theta)}{\sin(\beta+\theta)}\right\} + 2\cos\beta \ln\left\{\frac{\sin\beta + \sin\theta}{\sin\beta - \sin\theta}\right\}\right]$$

where $\theta = \cos^{-1}\left(\frac{x}{a}\right)$ $\qquad \beta = \cos^{-1}\left(\frac{c}{a}\right) = \frac{\pi}{2}\frac{\sigma}{\sigma_y}$.

Following Weertman we will put this in terms of x explicitly

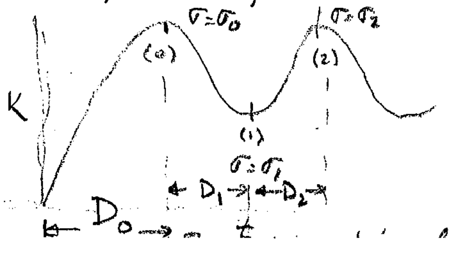
$$D(x) = \frac{4a\sigma_y}{\pi E}\left[\frac{x}{a}\ln\left\{\frac{x\sqrt{a^2-c^2} - c\sqrt{a^2-x^2}}{x\sqrt{a^2-c^2} + c\sqrt{a^2-x^2}}\right\} + \frac{c}{a}\ln\left\{\frac{\sqrt{a^2-c^2} + \sqrt{a^2-x^2}}{\sqrt{a^2-c^2} - \sqrt{a^2-x^2}}\right\}\right]$$

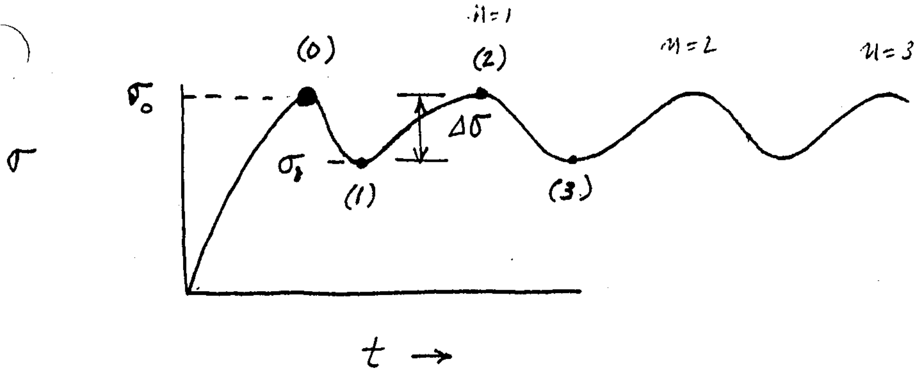
or finally for $\sigma \uparrow$ to σ_0 gives a despl in the plastic zone of

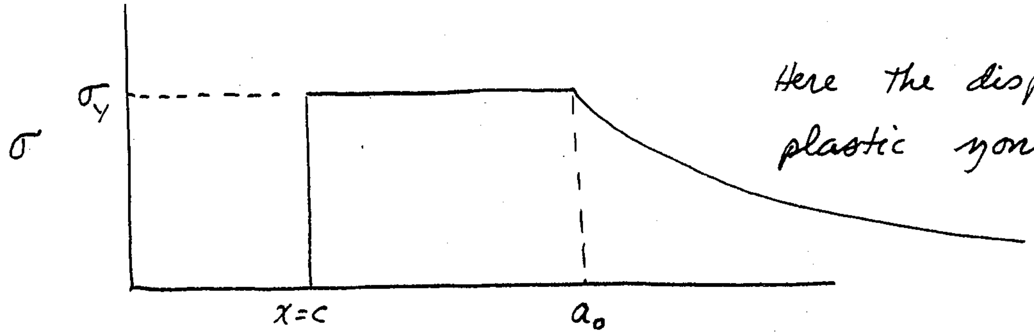
$$D_0(x) = \frac{4\sigma_y}{\pi E}\left[c\ln\left\{\frac{\sqrt{a_0^2-c^2} + \sqrt{a_0^2-x^2}}{\sqrt{a_0^2-c^2} - \sqrt{a_0^2-x^2}}\right\} - x\ln\left\{\frac{x\sqrt{a_0^2-c^2} + c\sqrt{a_0^2-x^2}}{x\sqrt{a_0^2-c^2} - c\sqrt{a_0^2-x^2}}\right\}\right]$$

where c is considered given and a_0 is the plastic zone extension corresponding to the load, σ_0:

$$a_0 = \frac{c}{\cos\left(\frac{\pi}{2}\frac{\sigma_0}{\sigma_y}\right)}$$





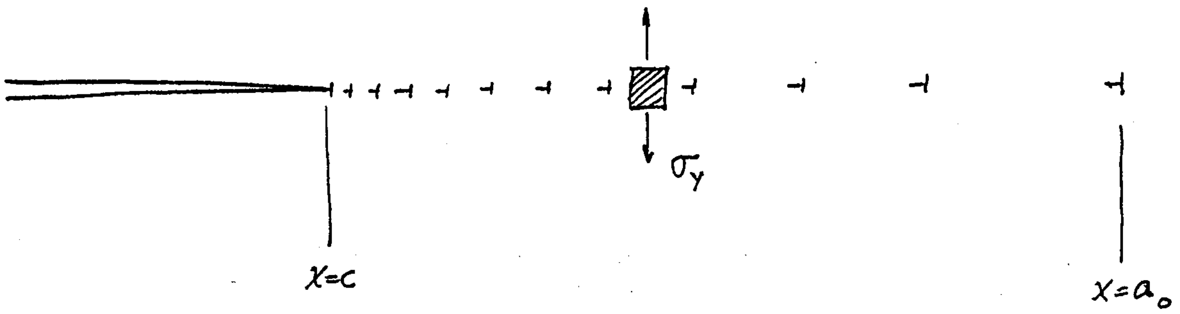


Here the displacements in the plastic zone are $D_0(x)$ (previous page).

Now consider the unloading part of the cycle. For the BCS or DM models we can regard the crack tip (at $x=c$) as remaining infinitely sharp so that <u>any</u> reduction in stress intensity factor will produce reverse yielding. We treat this by superimposing (since this is LEFM) a stress intensity factor ΔK of the reverse sign. If we reapply the BCS or DM models we have to account for the residual stresses in the plastic zone due to the first half cycle of deformation.

This is discussed by J. Weertman
Bull. Seismological Soc. of Amer.
<u>54</u> 1035 (1964).

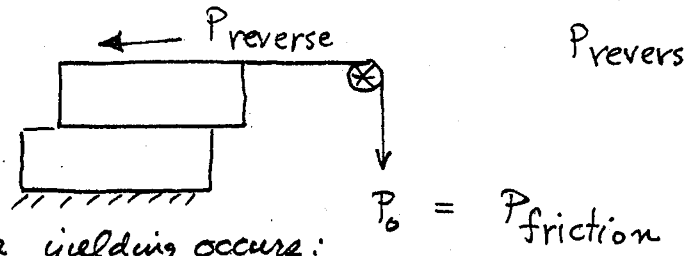
In terms of dislocations, the first half cycle produces the following distribution of inhomogeneous strain



This distribution of dislocations keeps the tension stress equal to σ_y in the plastic zone.

Now for reverse yielding the local compressive stress must be $-2\sigma_y$ because we have to overcome the residual stresses and cause reverse yielding.

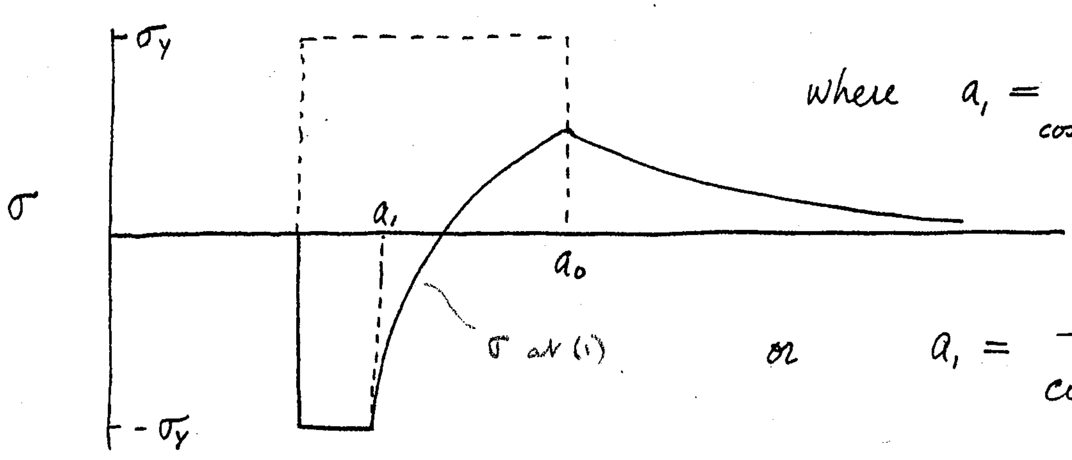
Analogous to:



TO cause reverse sliding.

$P_{reverse} = 2P_{FRICTION}$

So when reverse yielding occurs:



$$\text{where} \quad a_1 = \frac{c}{\cos\left(\frac{\pi}{2} \cdot \frac{\Delta\sigma}{2\sigma_y}\right)}$$

yield strength needed to go from (0) to (1)

$$\text{or} \quad a_1 = \frac{c}{\cos\left(\frac{\pi}{4} \frac{\sigma_0 - \sigma_1}{\sigma_y}\right)}$$

The displacements associated with the unloading, ΔK, are

2× operative yield stress (last time we had only $4\sigma_y$)

$$D_1(x) = -\frac{8\sigma_y}{\pi E}\left[c \ln\left\{\frac{\sqrt{a_1^2-c^2}+\sqrt{a_1^2-x^2}}{\sqrt{a_1^2-c^2}-\sqrt{a_1^2-x^2}}\right\} - x \ln\left\{\frac{x\sqrt{a_1^2-c^2}+c\sqrt{a_1^2-x^2}}{x\sqrt{a_1^2-c^2}-c\sqrt{a_1^2-x^2}}\right\}\right]$$

note this minus

So at point (1) in the loading cycle the displacements are

$$D(x) = D_0(x) + D_1(x).$$

Increasing the load again to (2) introduces new displacements $D_2(x)$ given by

$$D_2(x) = +\frac{8\sigma_y}{\pi E}\left[c \ln\left\{\frac{\sqrt{a_2^2-c^2}+\sqrt{a_2^2-x^2}}{\sqrt{a_2^2-c^2}-\sqrt{a_2^2-x^2}}\right\} - x \ln\left\{\frac{x\sqrt{a_2^2-c^2}+c\sqrt{a_2^2-x^2}}{x\sqrt{a_2^2-c^2}-c\sqrt{a_2^2-x^2}}\right\}\right]$$

$$\text{where} \quad a_2 = \frac{c}{\cos\left(\frac{\pi}{4}\,\frac{\sigma_0-\sigma_1}{\sigma_y}\right)}$$

so that at (2) the total displacements are

$$D(x) = D_0(x) + D_1(x) + D_2(x)$$

we note that $\Delta\sigma_y$ [to go from (1) to (2)] $= 2\sigma_y$

Now if the material were capable of undergoing reverse plastic strains indefinitely, the crack would never advance. (since $D_1 = -D_2$) However, we know there is a limit to the extent of reverse plastic straining that can be imposed. Following Weertman we will assume that

The crack will begin to advance when the ABSOLUTE plastic displacement at the crack tip $x=c$ reaches a critical value D^*

When

$$|D_0(x=c)| + |D_1(x=c)| + |D_2(x=c)| \cdots = D^*,$$

Crack Growth Begins!

$$|D_0(x=c)| = \frac{8c\,\sigma_Y}{\pi E} \ln\left(\frac{a_0}{c}\right)$$

(see p 97 for limiting process)

also

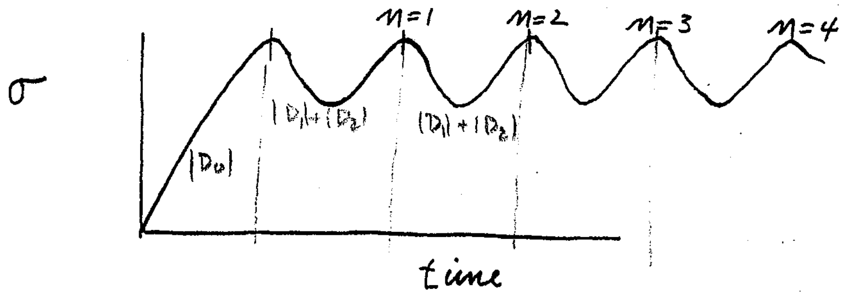
$$|D_1(x=c)| + |D_2(x=c)| = \frac{8c\;2\sigma_Y}{\pi E} \ln\left(\frac{a_1}{c}\right) + \frac{8c\;2\sigma_Y}{\pi E} \ln\left(\frac{a_2}{c}\right)$$

where $$a_1 = a_2 = \frac{c}{\cos\left(\frac{\pi}{4}\frac{(\sigma_0-\sigma_1)}{\sigma_Y}\right)}$$

so

$$|D_1(x=c)| + |D_2(x=c)| = 4\,\frac{8c\,\sigma_Y}{\pi E} \ln\left(\frac{a_1}{c}\right)$$

If we count the number of cycles n as follows:



Then

$$\sum |D(x=c)| = |D_0(x=c)| + n\{|D_1(x=c)| + |D_2(x=c)|\}$$

for initiation & crack growth

$$\sum |D(x=c)| = \frac{8c\sigma_Y}{\pi E}\left\{ \ln\frac{a_0}{c} + 4n \ln\frac{a_1}{c}\right\} = D^*$$

For the case of $n=0$, we have the Hahn and Rosenfield result for the COD.

Suppose we set D^* so large that

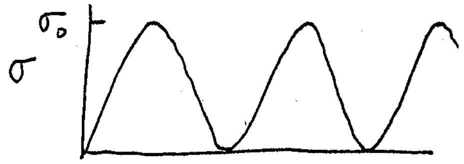
$$\sum |D(x=c)| = D^*$$ only when n is very large.

so that we can neglect the first term in the above. Then the crack would begin to grow when

$$D^* \simeq \frac{8c\,\sigma_Y}{\pi E} 4n \ln\frac{a_1}{c} \qquad \text{where } a_1 = \frac{c}{\cos\left(\frac{\pi}{4}\frac{\sigma_0-\sigma_1}{\sigma_Y}\right)}$$

For the special case of $\sigma_1 = 0$, $\Delta\sigma = \sigma_0$

ie



$\cos\left(\pi/4 \frac{\sigma_0}{\sigma_y}\right) \sim 1 \therefore \frac{a_1}{c} = 1+\epsilon$; $\ln\left(\frac{a_1}{c}\right) = \epsilon$

and for small stresses $\frac{\sigma_0}{\sigma_y} \ll 1$, we have

$$D^* \simeq \frac{8c\,\sigma_y}{\pi E} 4n \cdot \frac{\pi^2}{32}\left(\frac{\sigma_0}{\sigma_y}\right)^2 = \frac{\pi c\,\sigma_0^2\, n}{E\,\sigma_y}$$

Or, cracking begins at

$$n = n^* = \frac{E \sigma_Y D^*}{\pi c \sigma_0^2} .$$

basically C.O.D

since $\sigma_Y D^* = G_c$ not needed inconsistent w/model.

and $E G_c = \cancel{(1-\nu^2)} K_{I_c}^2$

and $\pi c \sigma_0^2 = (\Delta K)^2$

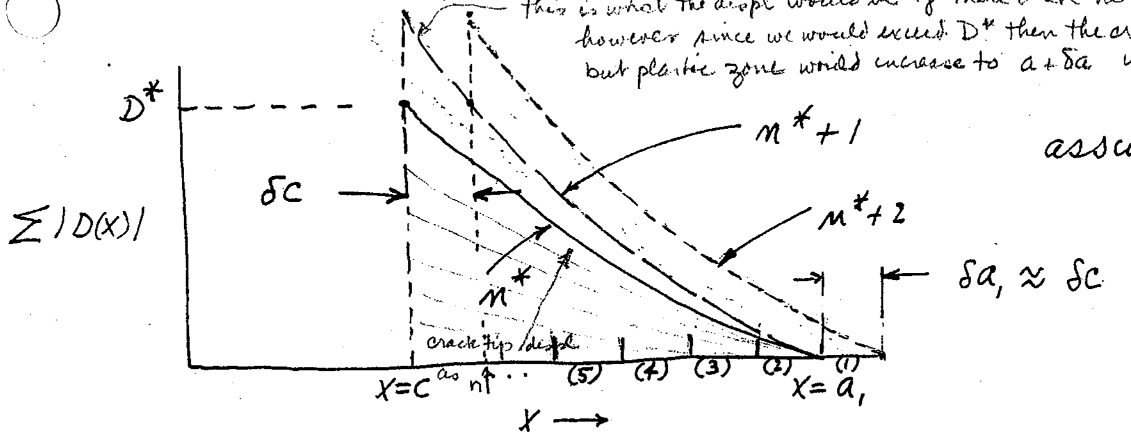
not needed

$$\boxed{n^* = \cancel{(1-\nu^2)} \left(\frac{K_{Ic}}{\Delta K}\right)^2}$$

Now, how far will the crack propagate when the critical displacement is reached? Certainly the crack will not run catastrophically at that point. It will run until the absolute magnitude of the displacements no longer exceed D^*.

Consider the following schematic:

this is what the displ would be if there were no cut off D^* however since we would exceed D^* then the crack would grow to $\delta c + c$ but plastic zone would increase to $a + \delta a$ where $\delta a \approx \delta c$



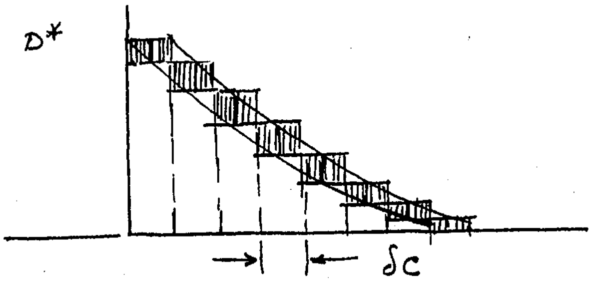
assumes $D_0(x) \approx 0$

Imagine the plastic zone to be divided into elements δc in width (as shown).

Evidently the absolute sum of displacements (for one cycle) over elements 1, 2, 3, 4 .... over the entire plastic zone must be D^* at steady state.

This is represented approximately as

$$D^* \simeq \frac{\int_c^a |D(x)|_{N=1}\, dx}{\delta c}$$ with $D_0(x) = 0$ as before



$$D^* \delta c = \int_c^a \left\{ |D_1(x)| + |D_2(x)| \right\} dx$$

$$= \frac{16\,\sigma_Y}{\pi E} \int_c^{a_1} c \ln\left\{ \frac{\sqrt{a_1^2-c^2} + \sqrt{a_1^2-x^2}}{\sqrt{a_1^2-c^2} - \sqrt{a_1^2-x^2}} \right\} - x \ln\left\{ \frac{x\sqrt{a_1^2-c^2} + c\sqrt{a_1^2-x^2}}{x\sqrt{a_1^2-c^2} - c\sqrt{a_1^2-x^2}} \right\} dx$$

$$D^* \delta c = \frac{16\,\sigma_Y\, c}{\pi E} \left[\sqrt{a_1^2 - c^2} \left(\frac{\pi}{2} - \sin^{-1}\left(\frac{c}{a_1}\right)\right) - 2c \ln\left(\frac{a_1}{c}\right) \right]$$

(Integral given by Weertman.).

Hence the advance of the fatigue crack for each cycle is

$$\frac{\delta c}{\delta N} = \frac{16\,\sigma_Y\, c}{\pi E D^*} \left[\sqrt{a_1^2 - c^2} \left(\frac{\pi}{2} - \sin^{-1}\left(\frac{c}{a_1}\right)\right) - 2c \ln\left(\frac{a_1}{c}\right) \right]$$

now recall that

$$\frac{c}{a_1} = \cos \beta_1 = \cos\left(\frac{\pi}{4} \frac{\sigma_0 - \sigma_1}{\sigma_Y}\right)$$

We focus our attention on small stresses $\frac{\sigma_0 - \sigma_1}{\sigma_Y} << 1$, small β_1 :

then

$$\frac{\delta c}{\delta N} = \frac{16 \sigma_Y c}{\pi E D^*}\left[\sqrt{\frac{c^2}{\cos^2\beta_1} - c^2}\left(\frac{\pi}{2} - \sin^{-1}(\cos\beta_1)\right) + 2c \ln \cos\beta_1\right]$$

$$= \frac{16 \sigma_Y c^2}{\pi E D^*}\left[\tan\beta_1\left(\frac{\pi}{2} - \left\{\frac{\pi}{2} - \beta_1\right\}\right)\right.$$

$$\left. + 2\left\{-\frac{\beta_1^2}{2} - \frac{\beta_1^4}{12} - \frac{\beta_1^6}{45} - \cdots\right\}\right]$$

$$= \frac{16 \sigma_Y c^2}{\pi E D^*}\left[\beta_1\left\{\cancel{\beta_1} + \frac{\beta_1^3}{3} + \frac{2\beta_1^5}{15} \cdots\right\}\right.$$

to cancel other

$$\left. + 2\left\{-\cancel{\frac{\beta_1^2}{2}} - \frac{\beta_1^4}{12} - \frac{\beta_1^6}{45} \cdots\right\}\right]$$

$$\cong \frac{16 \sigma_Y c^2}{\pi E D^*} \frac{\beta_1^4}{6} = \frac{16 \sigma_Y c^2 \pi^4 (\sigma_0 - \sigma_1)^4}{\pi E D^* 6\, 4^4 \sigma_Y^4}$$

$$\frac{\delta c}{\delta N} = \frac{\pi \left\{ (\sigma_0 - \sigma_1)\sqrt{\pi c} \right\}^4}{96\ E\ D^*\ \sigma_y^3} = \frac{\pi (\Delta K)^4}{96\ E\ D^*\ \sigma_y^3}$$

But referring to Hahn and Rosenfield (DM Model) we have

$$D^* \sigma_y = G_c \quad \text{(critical crack extension force).}$$

For thick plate, $G_c = \frac{K_{Ic}^2 (1-\nu)}{2\mu}$

$$\text{and} \quad \mu = \frac{E}{2(1+\nu)}$$

so

$$G_c = \frac{K_{Ic}^2}{E} (1-\nu^2)$$

now in place of $D^*\sigma_y$ we write $\frac{K_{Ic}^2}{E}(1-\nu^2)$

$$\boxed{\delta c = \frac{\pi}{96} \frac{(\Delta K)^4}{\sigma_y^2\ K_{I_c}^2 (1-\nu^2)}}$$

1) 4th power dependence as observed.

2) Estimate order of magnitude for 7075-T6.

$K_{I_c} = 30,000$ psi $\sqrt{in}$

$\sigma_y = 75,000$ psi.

$\nu = 0.35$

7075-T6:

$$\delta c = 8 \times 10^{-21} (\Delta K)^4$$

so that

$$\frac{d(2c)}{dN} = 1.6 \times 10^{-20} (\Delta K)^4$$ Theory.

From experiment (p 146 of notes)

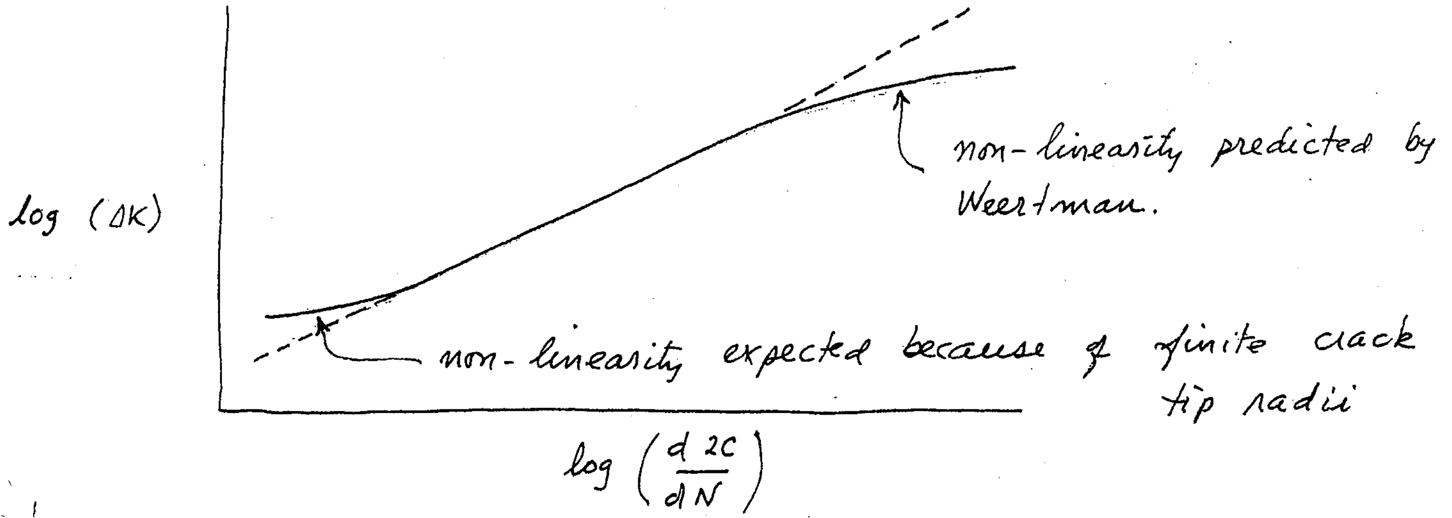
$$\frac{d\,(2c)}{dN} = 10^{-20} (\Delta K)^4$$ Experiment

Fantastic Agreement. !! within a factor of 2 of absolute growth rate.

Non-Linearity at High Stress Intensity Factors.

The crack growth per cycle is

$$\delta c = \frac{16 \sigma_Y c^2}{\pi E D^*} \left\{ \frac{\beta_1^4}{6} + \frac{4}{45} \beta_1^6 + \dots \right\}$$



non-linearity predicted by Weertman.

non-linearity expected because of finite crack tip radii

Further comparison between Weertman theory and fatigue crack growth rates.

For both theory and experiment we have

$$\frac{d2c}{dN} = A\,(\Delta K)^4 \qquad A_{Theoretical} = \frac{2\pi}{96\,\sigma_Y^{\,2}\,K_{Ic}^{\,2}(1-\nu^2)}$$

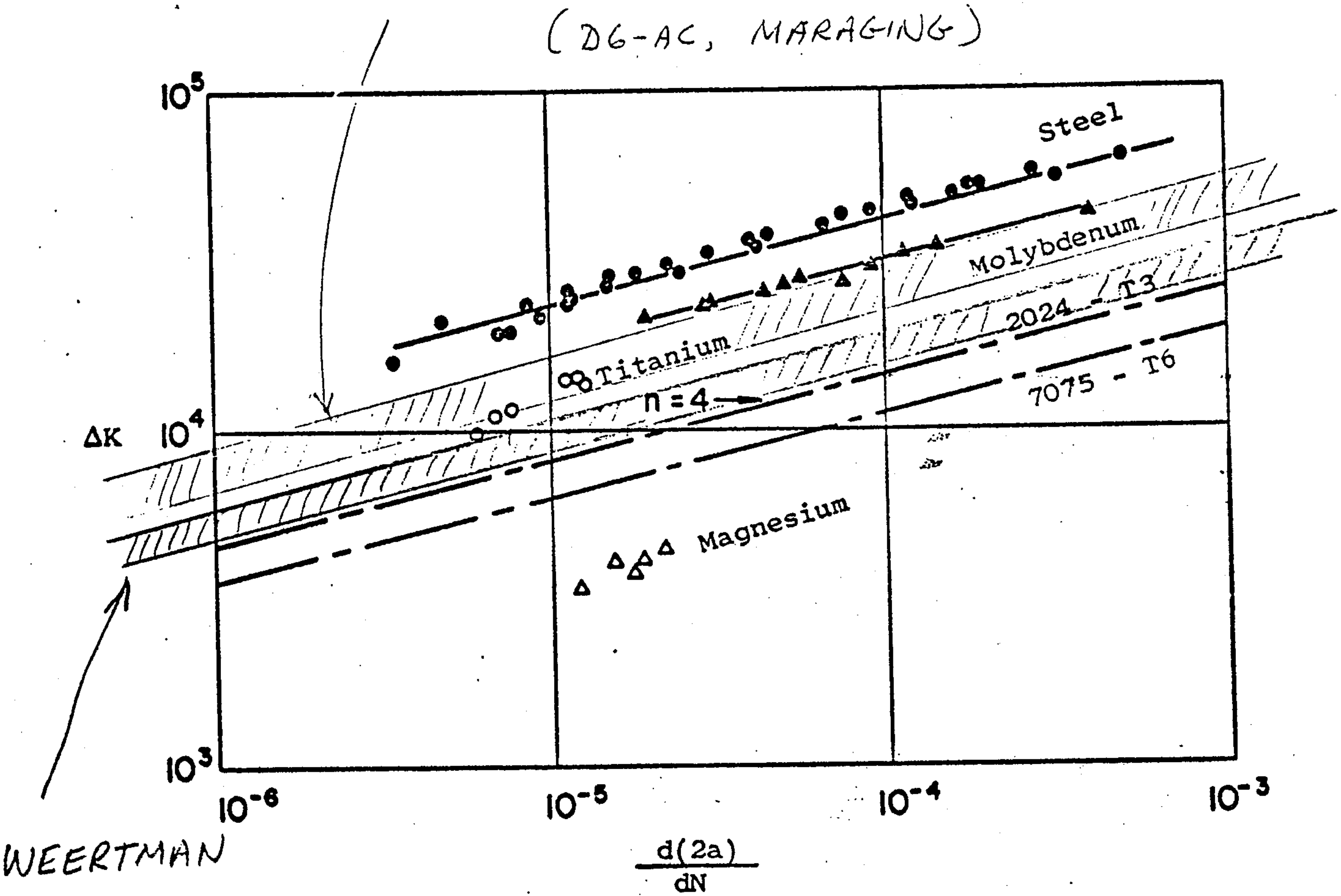
identifies what $\frac{\delta c}{\delta N}$ depends on $(K_{Ic}, \sigma_Y, \Delta K)$

Material	$A_{expt.}$	Material	K_{Ic}	σ_Y	$A_{Theoretical}$
Al - 7075 - T6	10^{-20}	Al 7075-T6	30	75	1.6×10^{-20}
"Titanium"	5×10^{-22}	Ti-6Al-4V	35	160	2.3×10^{-21}
			65	154	7.3×10^{-22}
"Steel"	4×10^{-23}	D6-AC	45	249	5.8×10^{-22}
			90	212	2×10^{-22}
		Maraging	52	285	3.3×10^{-22}
			68	259	2.3×10^{-22}
			84	242	1.7×10^{-22}
Al- 2024-T6	2×10^{-21}	Al-2024-T4	47.5	48	1.4×10^{-20}

theory is only good for $\frac{\delta c}{\delta N} \sim (\Delta K)^4$ but as we note $\left(\begin{array}{l}\text{low driving forces}\\ \text{high driving forces}\end{array}\right.$ the theory will break down.

Figure VI-21 -- Data on Various Materials

WEERTMAN THEORY FOR VARIOUS STEELS
(DG-AC, MARAGING)



WEERTMAN
THEORY
FOR Ti-6Al-4V

Figure VI-19 -- Data on 7075 T6 Aluminum Alloy

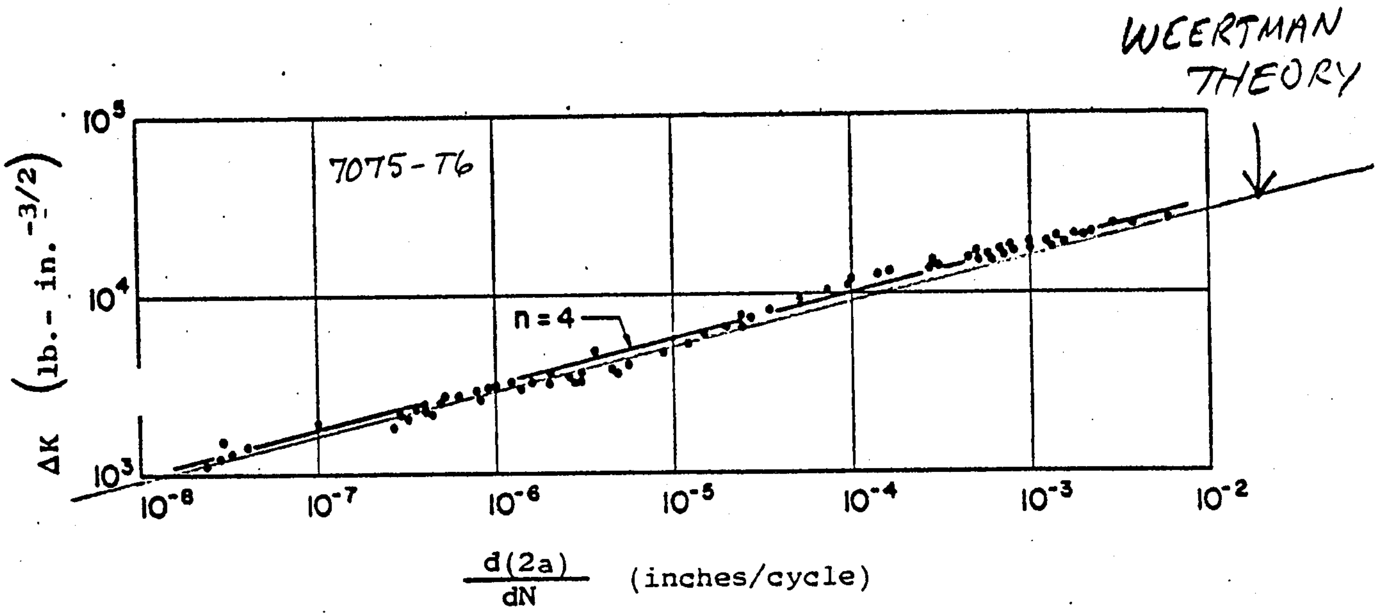
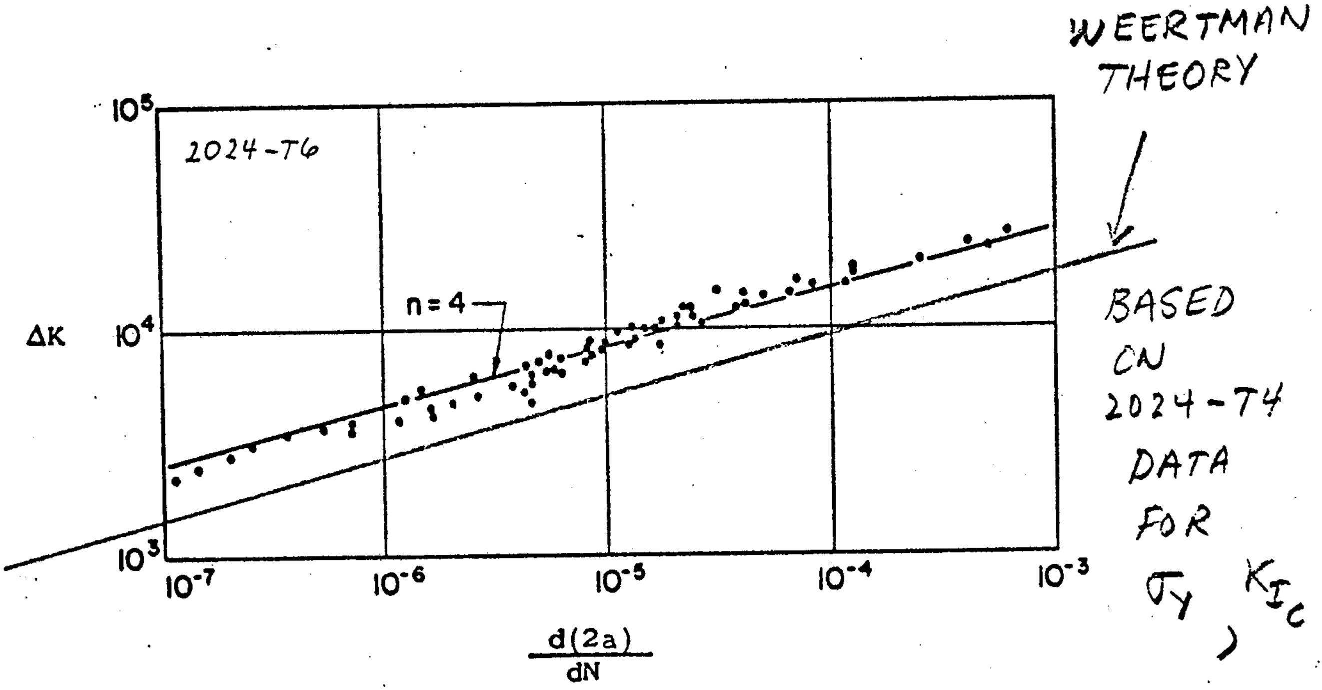


Figure VI-20 -- Data on 2024 - T6 Aluminum Alloy



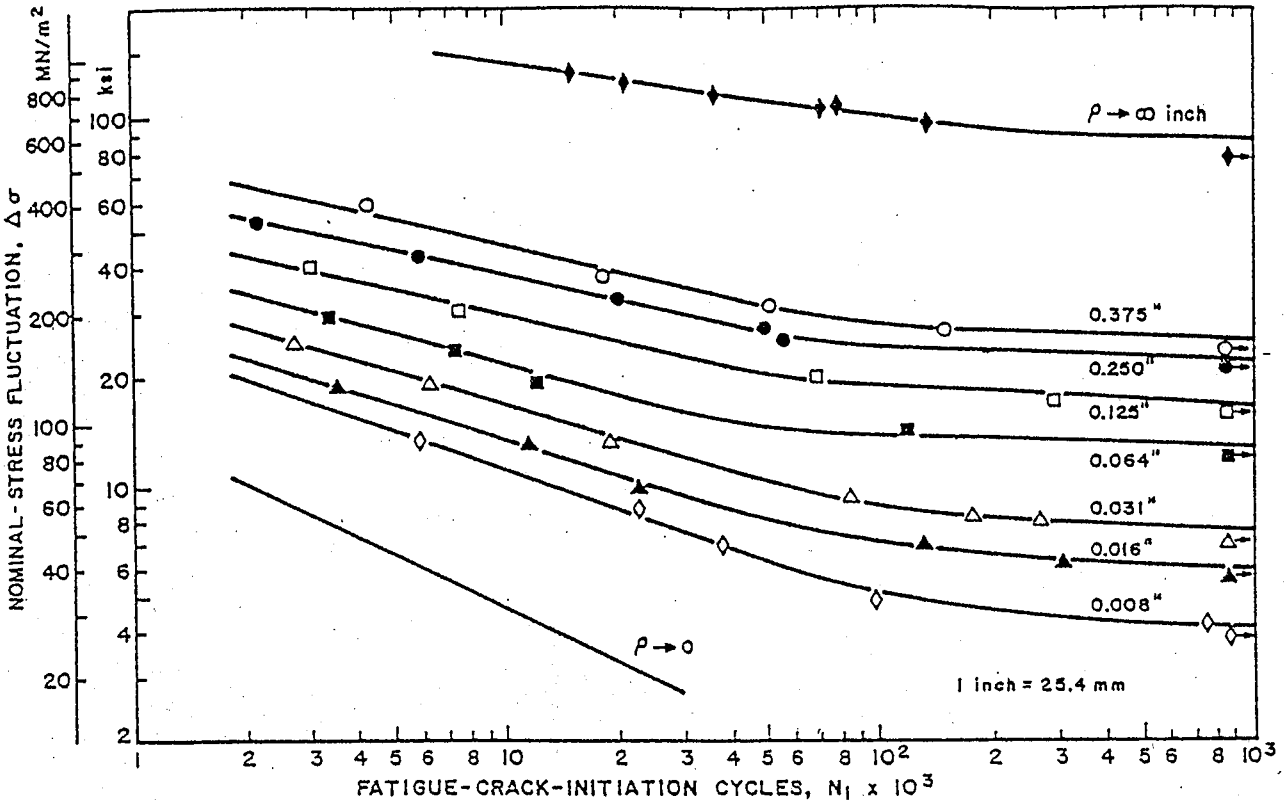


FIG. 7.6. Dependence of fatigue-crack initiation of HY-130 steel on nominal-stress fluctuations for various notch geometries.

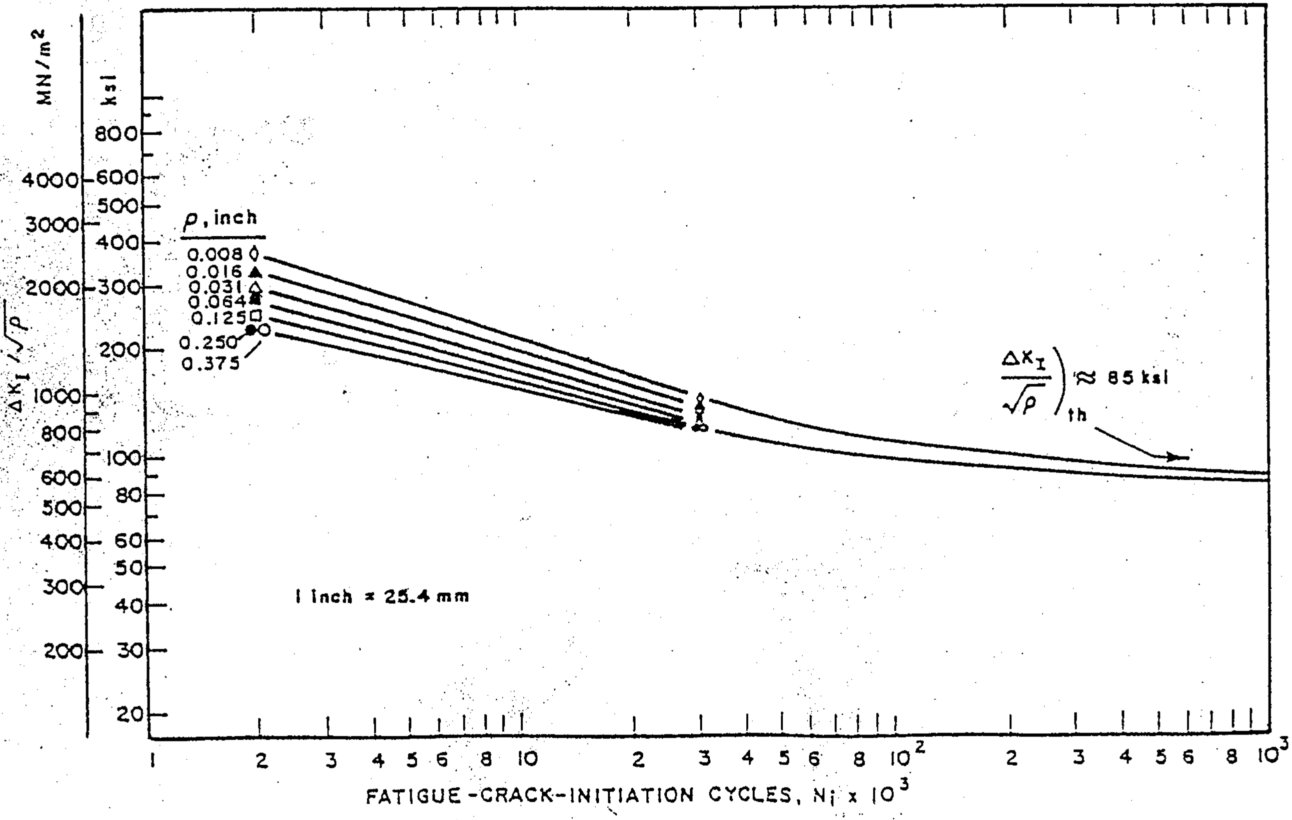


FIG. 7.7. Correlation of fatigue-crack-initiation life with the parameter $\Delta K_I/\sqrt{\rho}$ for HY-130 steel.

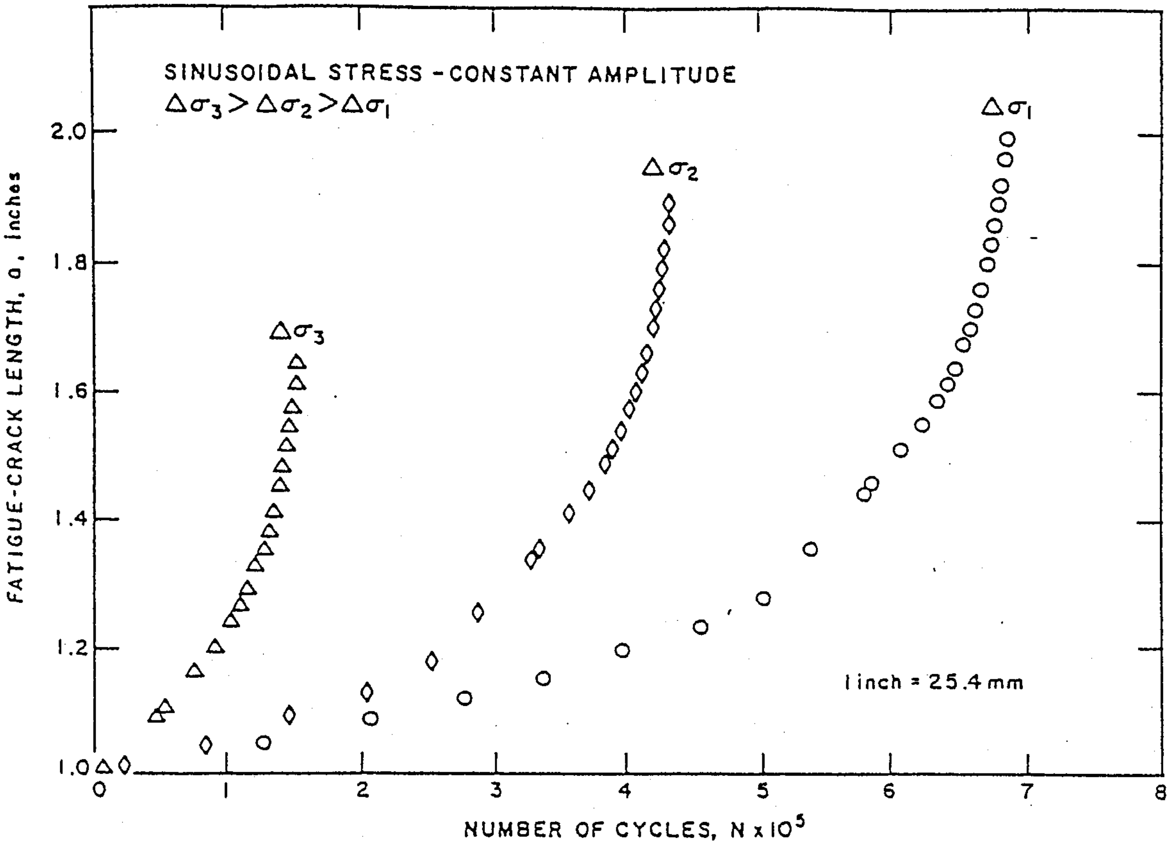


FIG. 8.2. Effect of cyclic-stress range on crack growth.

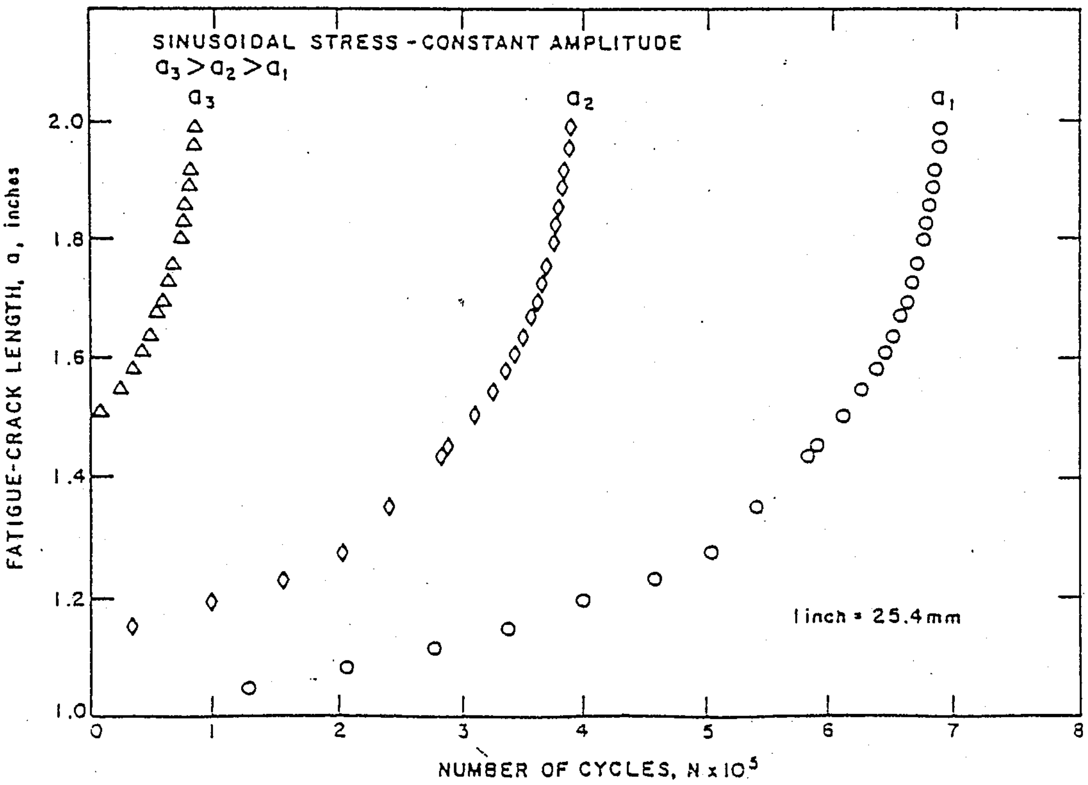


FIG. 8.3. Effect of initial crack length on crack growth.

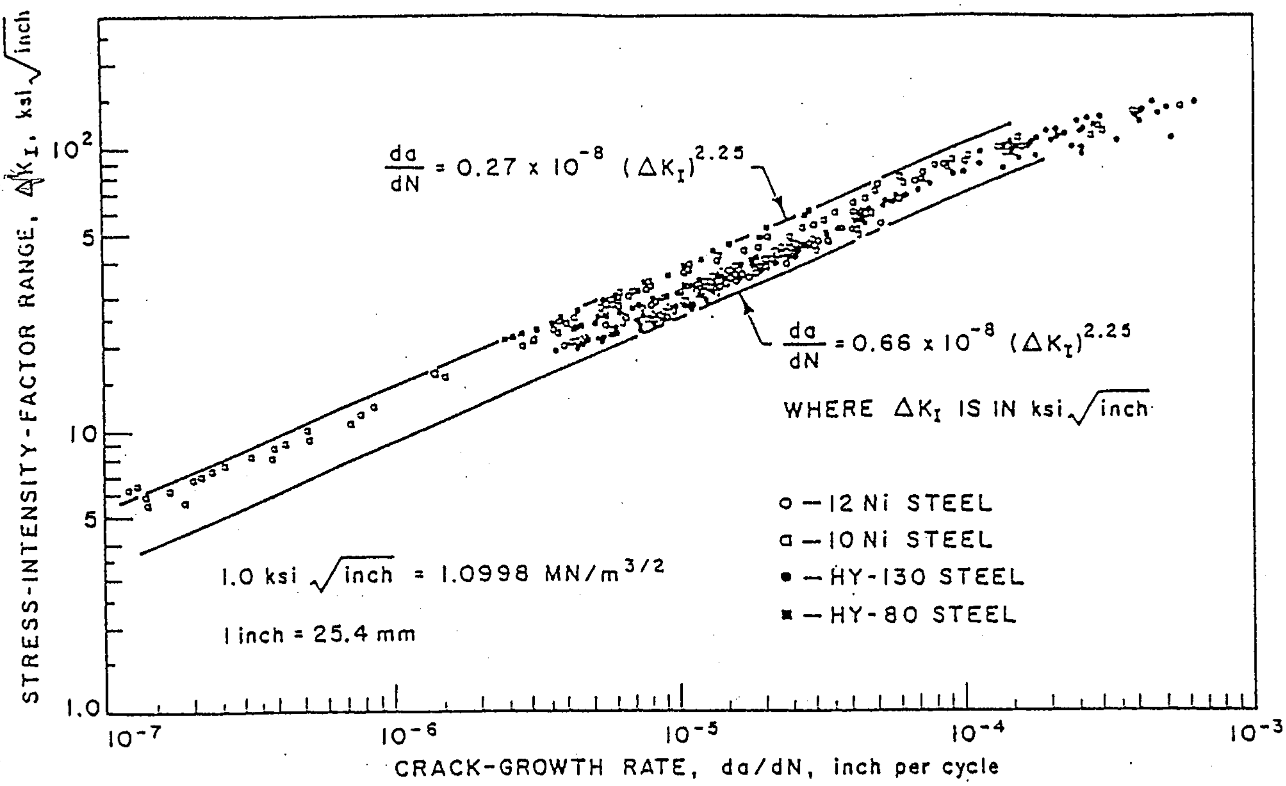
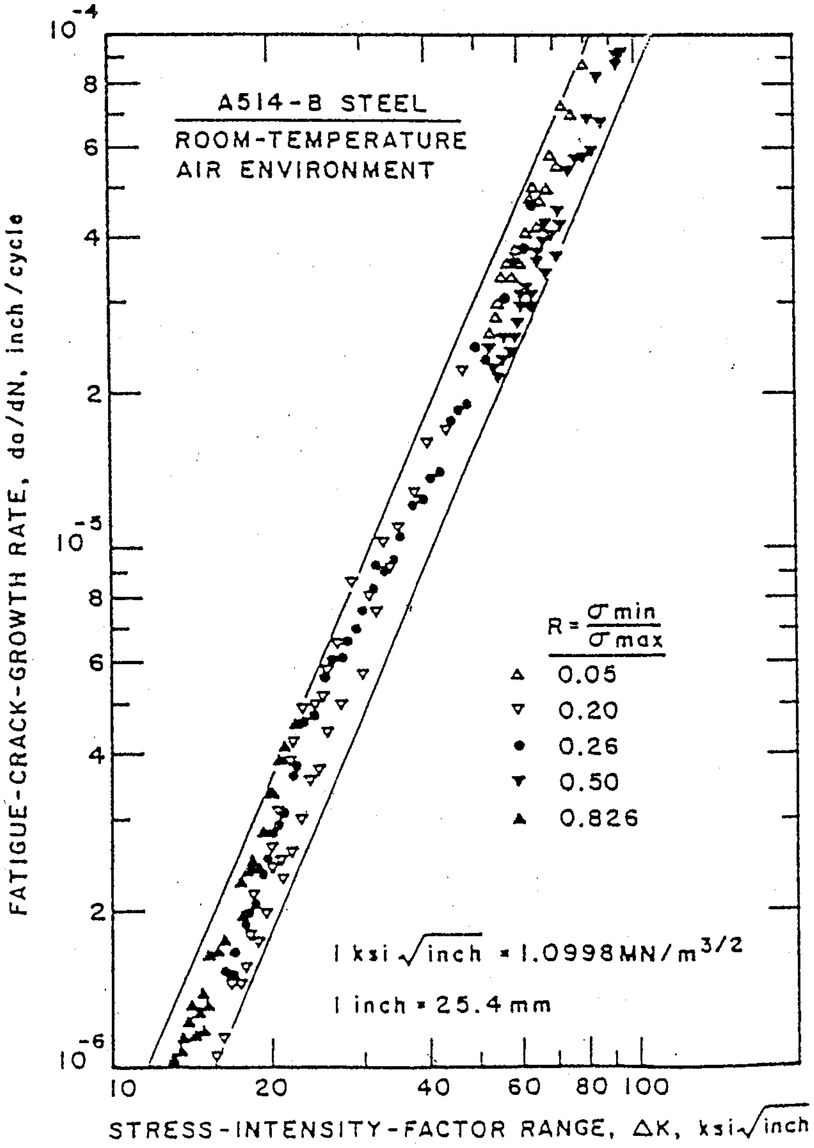


FIG. 8.5. Summary of fatigue-crack-propagation for martensitic steels.



$$\frac{da}{dN} = 0.66 \times 10^{-8}(\Delta K_I)^{2}$$

where: a = in.,
K_I = ksi$\sqrt{\text{in.}}$

Ferrite-Pearlite Steels

$$\frac{da}{dN} = 3.6 \times 10^{-10}(\Delta K_I)^{3.0}$$

where a = in.,
ΔK_I = ksi$\sqrt{\text{in.}}$

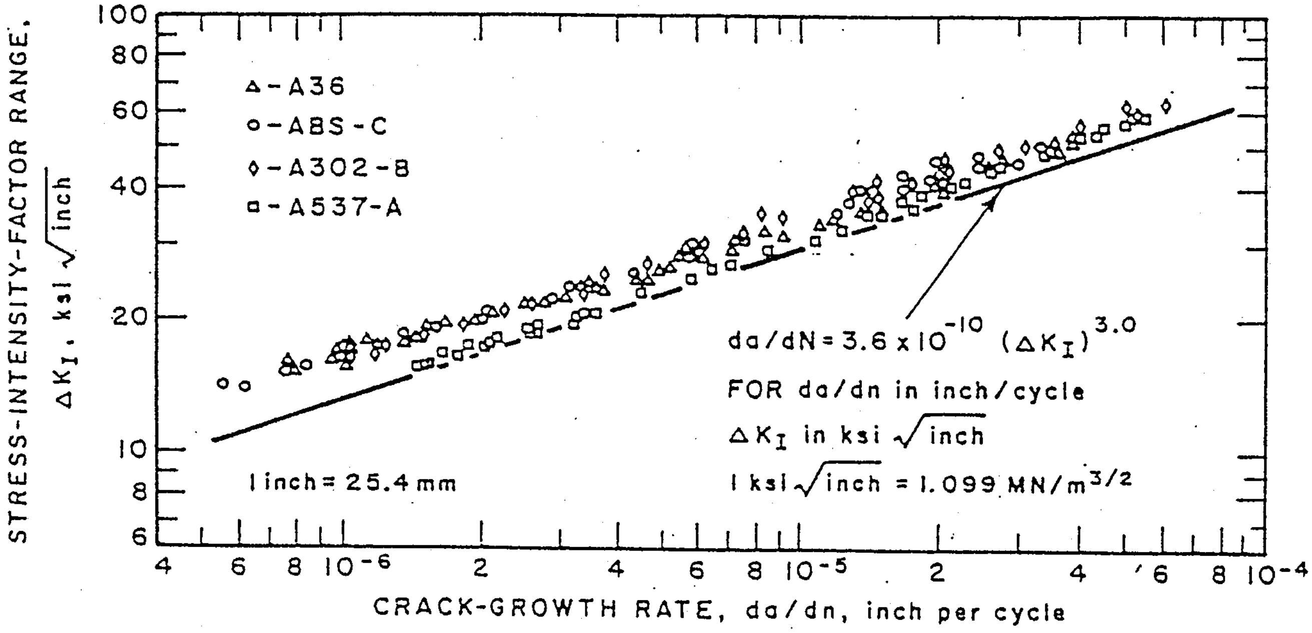


FIG. 8.8. Summary of fatigue-crack-growth data for ferrite-pearlite steels.

$$\frac{da}{dN} = 3.0 \times 10^{-10}(\Delta K_{\mathrm{I}})^{3.25}$$

$a =$ in.,
$K_{\mathrm{I}} =$ ksi$\sqrt{\text{in.}}$

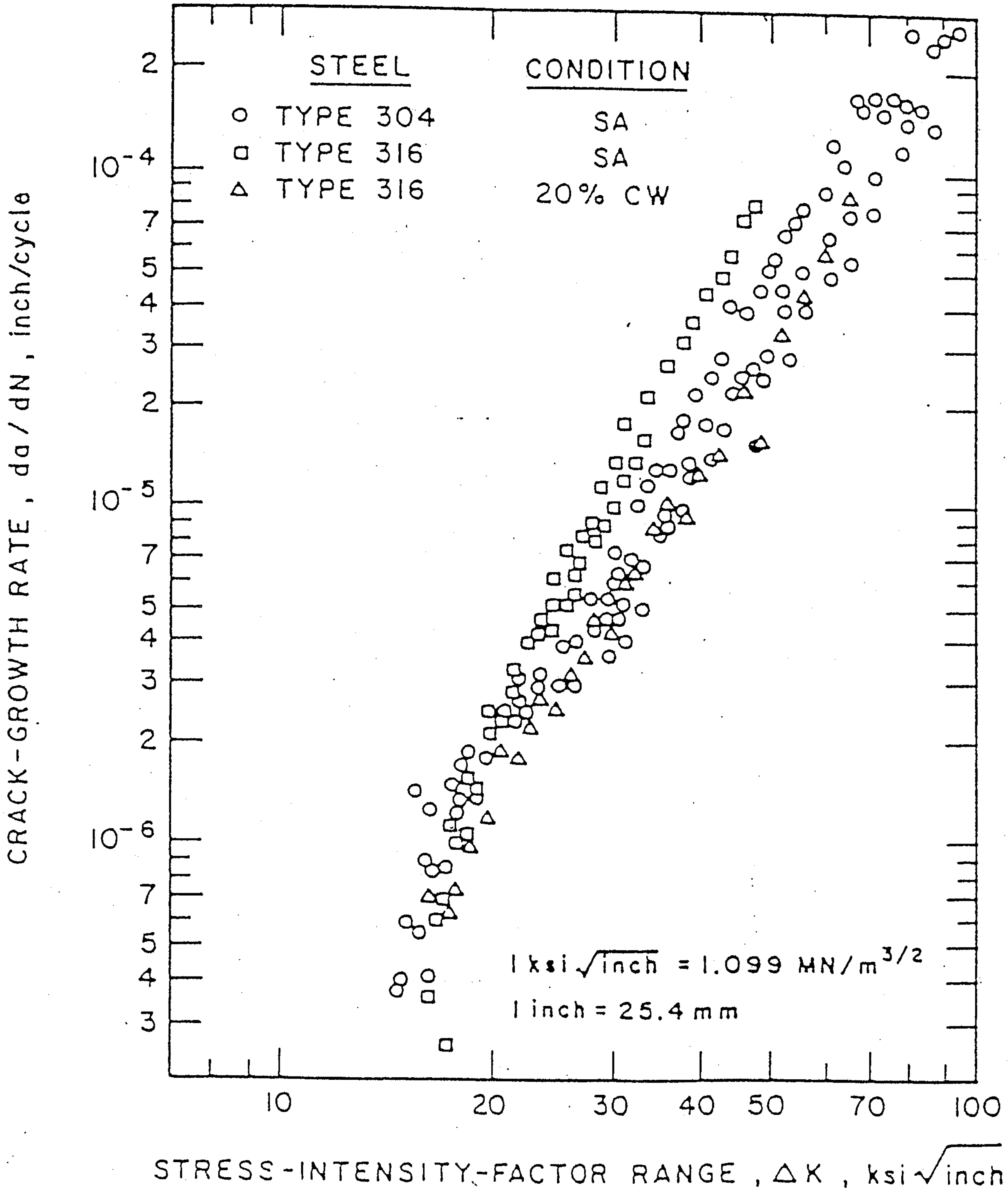
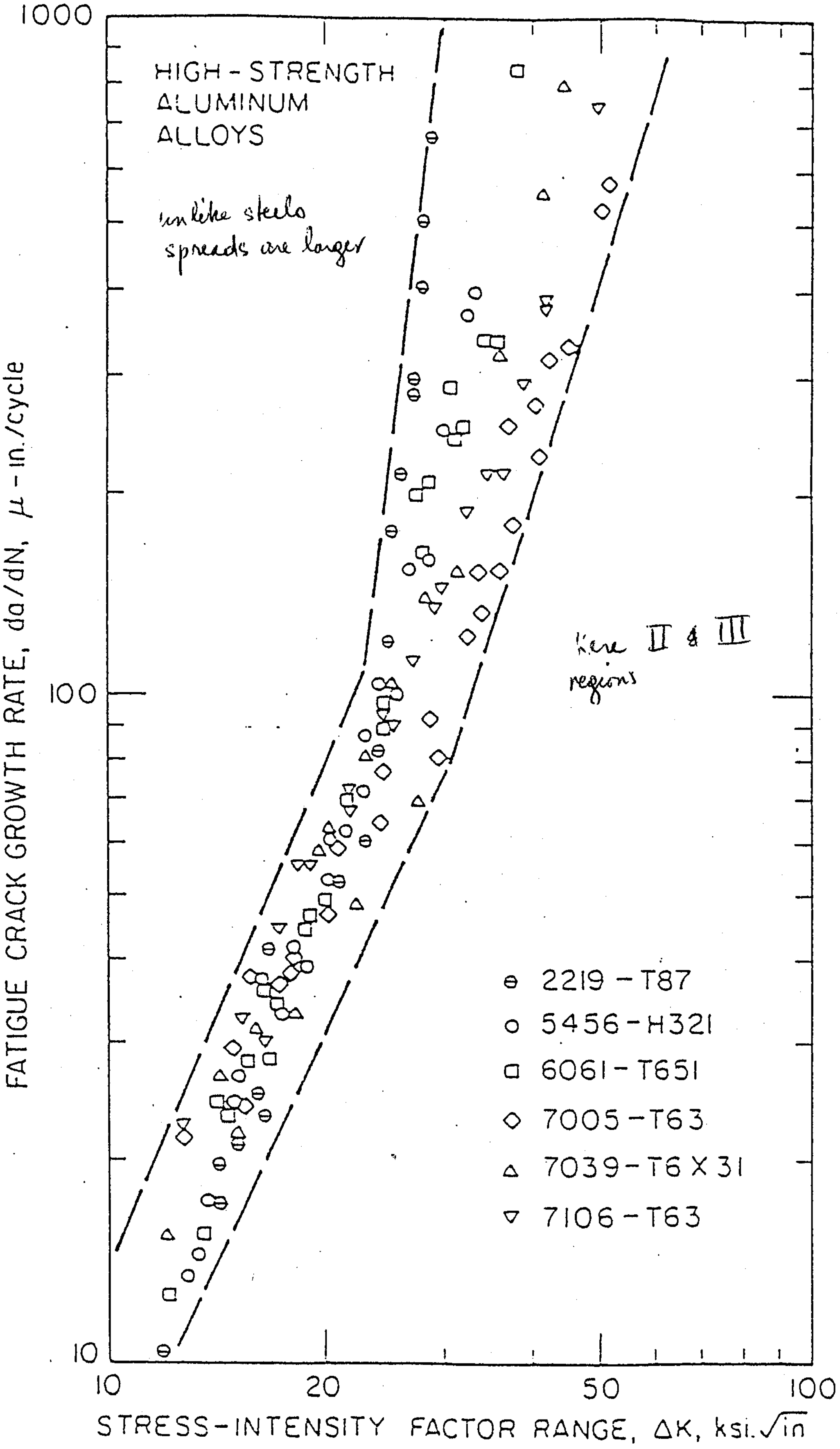


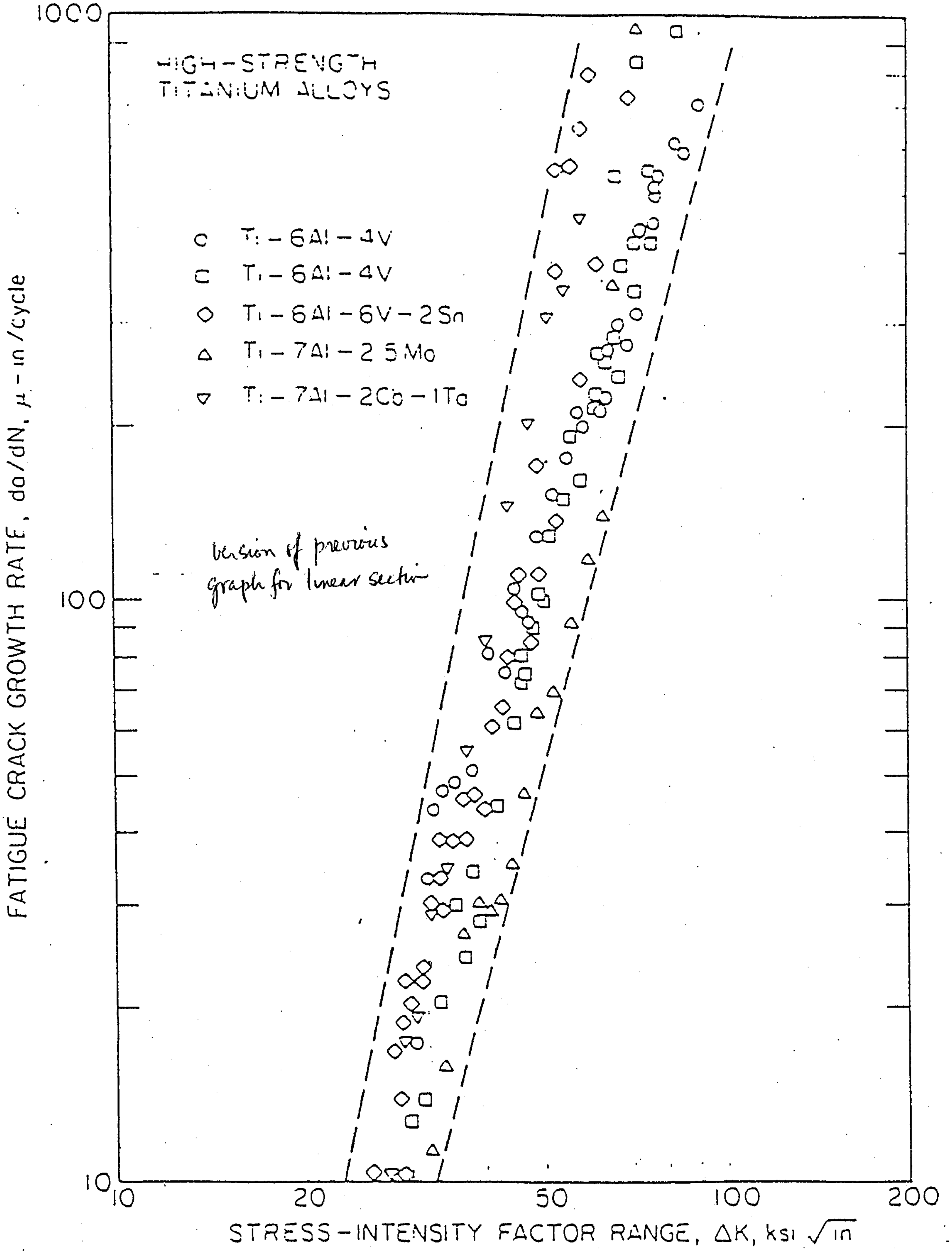
FIG. 8.9. Fatigue-crack-growth rate of solution-annealed type 304 and



unlike steels spreads are larger

here II & III regions

FIG. 8.13. Summary plot of *da/dN* versus ΔK for six aluminum alloys. The yield strengths of these alloys range from 34 to 55 ksi.



version of previous
graph for linear section

FIG. 8.14. Summary plot of *da/dN* versus ΔK data for five titanium alloys ranging in yield strength from 110 to 150 ksi.

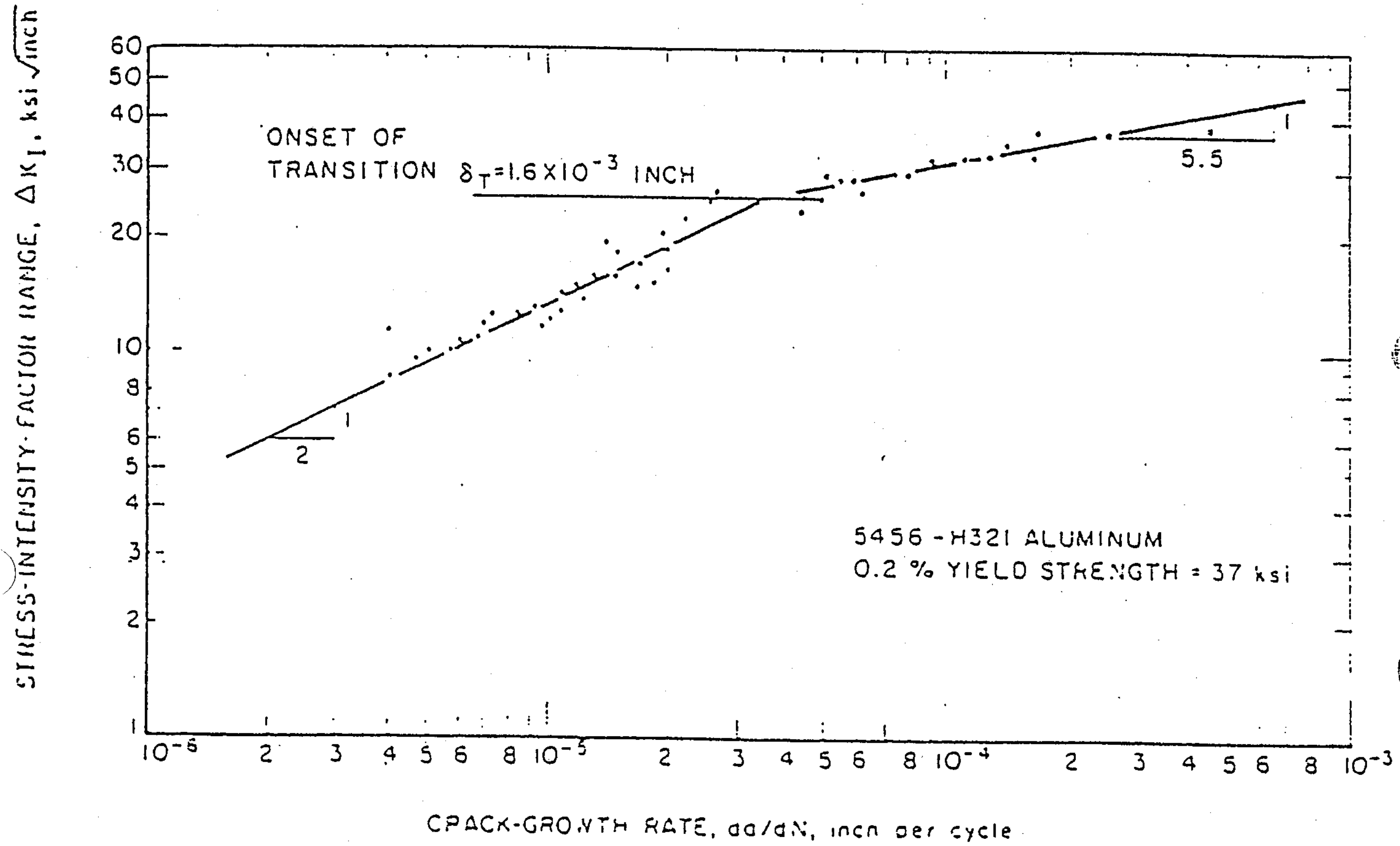


FIG. 8.15. Fatigue-crack-growth rate as a function of stress intensity for 5456-H321 aluminum.

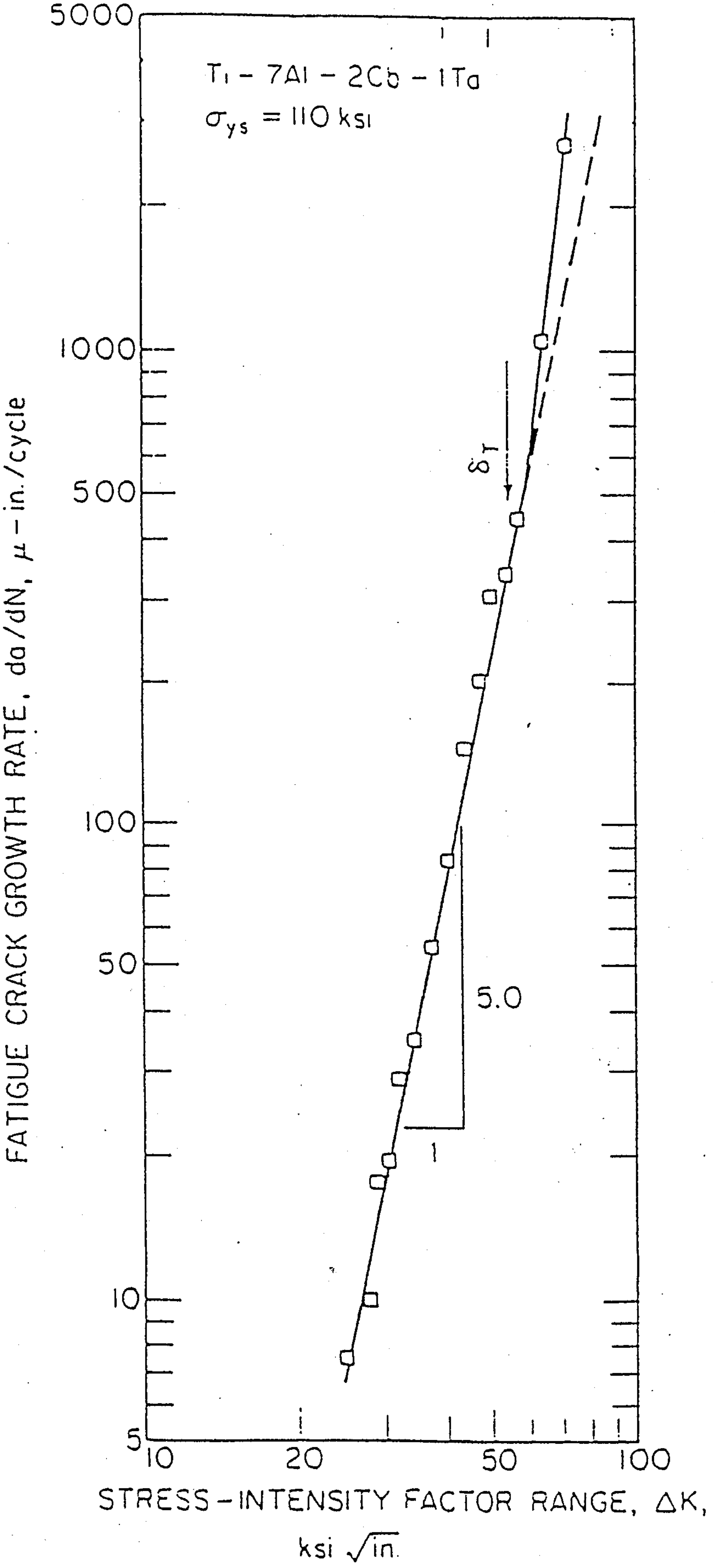


FIG. 8.16. Fatigue-crack-growth rate in a titanium alloy.

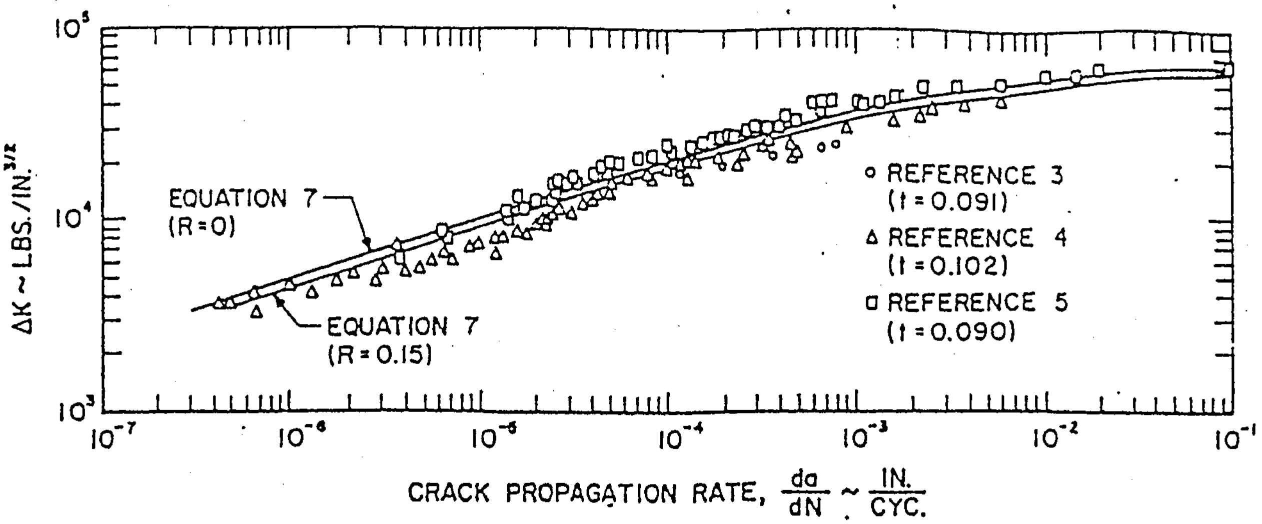


Fig. 1 Comparison of experimental and theoretical crack-propagation rates in 7075-T6 aluminum plate for $R = 0$ to $R = 0.15$

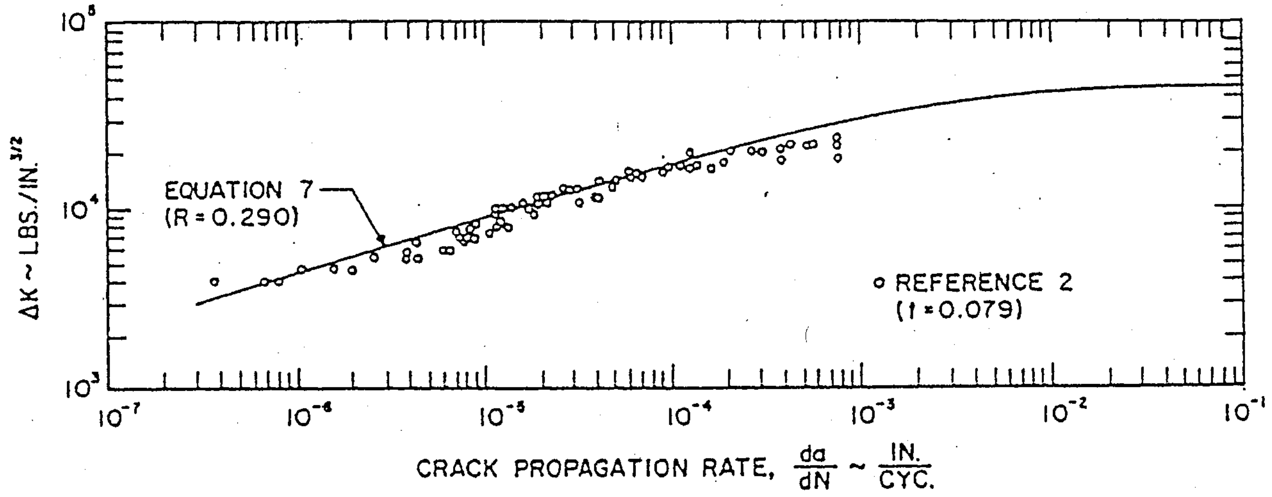


Fig. 2 Comparison of experimental and theoretical crack-propagation rates in 7075-T6 aluminum plate for $R = 0.285$ to $R = 0.297$

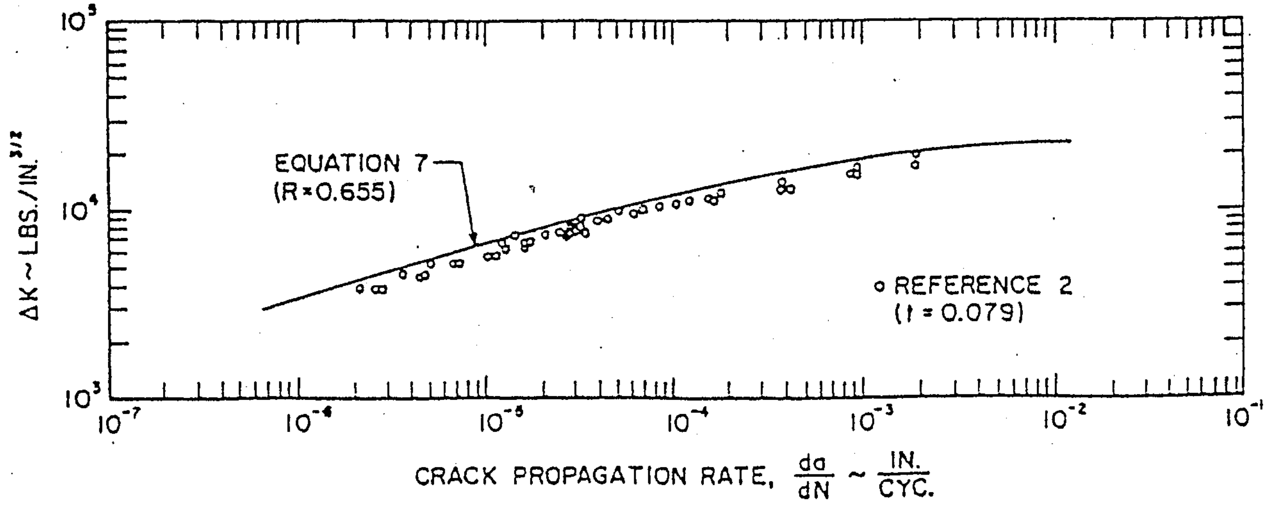
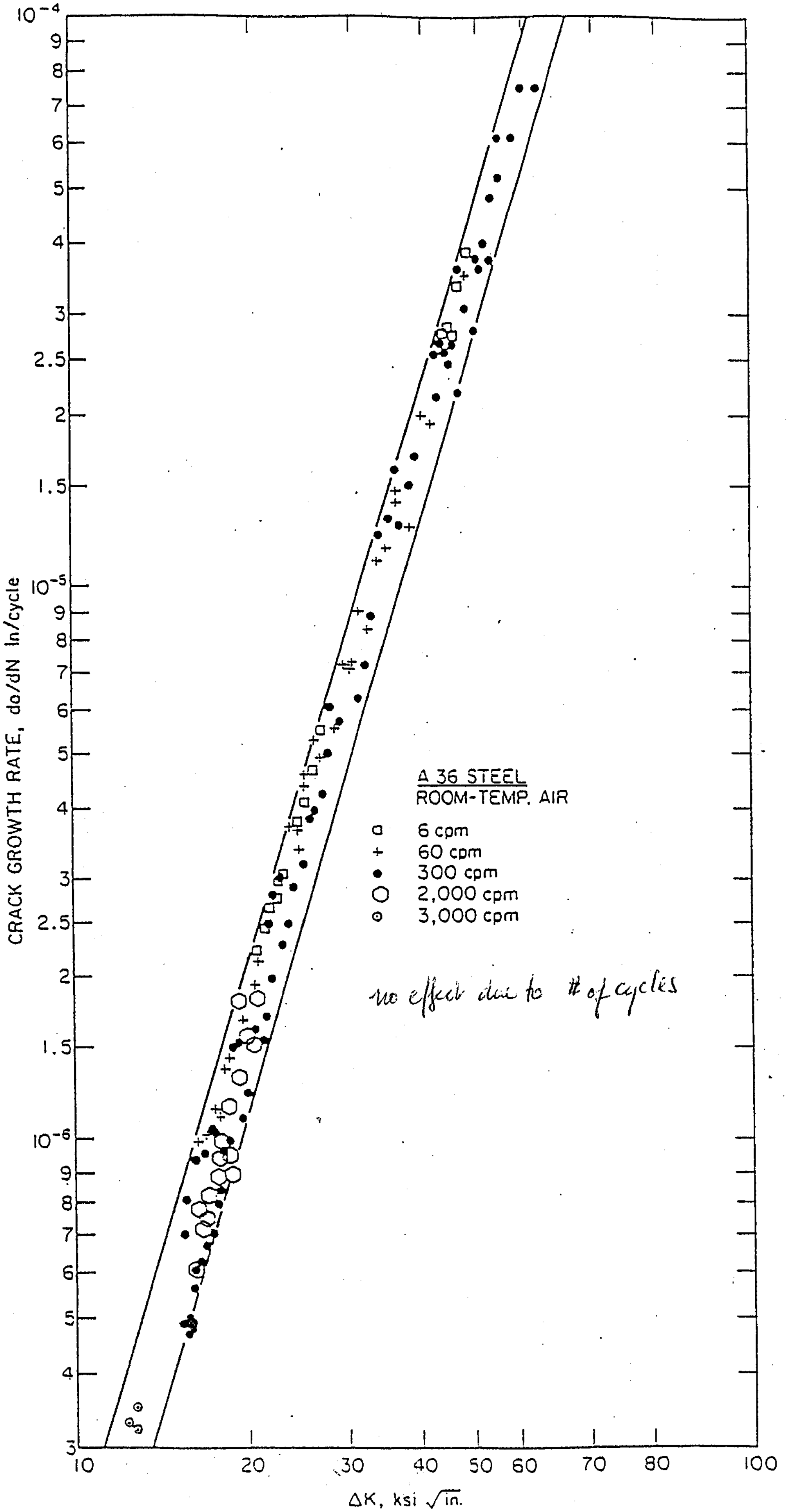
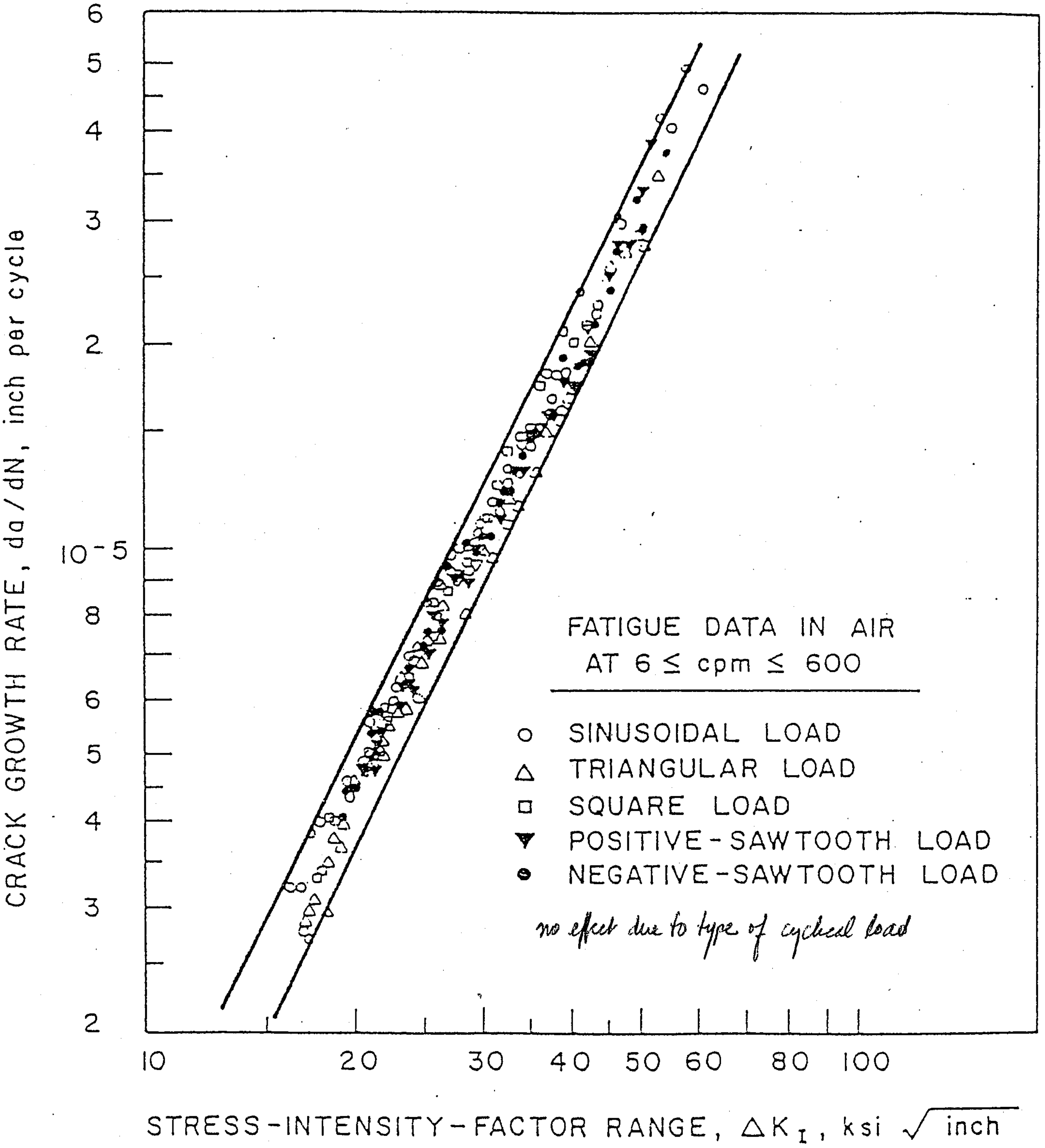


Fig. 3 Comparison of experimental and theoretical crack-propagation rates in 7075-T6 aluminum plate for $R = 0.655$

$$\frac{da}{dN} = \frac{C(\Delta K)^n}{(1 - R)K_c - \Delta K}$$



no effect due to # of cycles



no effect due to type of cycled load

FIG. 8.20. Fatigue-crack-growth rates in 12Ni-5Cr-3Mo steel under various cyclic-stress fluctuations with different stress-time profiles.

FATIGUE - LIFE PREDICTION

The Basic Algorithm

a) Determine the initial crack size a_o by measuring it, or by estimating it based on the NDT resolution.

b) Determine the relation K_I (a) for the given structure and loading conditions.

c) Calculate K_T

d) Determine the crack critical size a_{cr} based on K_{IC} or K_T.

e) Determine experimentally or from handbook the appropriate fatigue-crack growth rate equation e.g.,:

$$\boxed{\frac{da}{dN} = A_p\,(\Delta K_I)^{n_p}}$$

f) Determine ΔK_I from the prescribed loading

g) Integrate from a_o to a_{cr}

$$\boxed{N_f = \int_{a_o}^{a_{cr}} \frac{da}{Ap\,(\Delta K_I)^{n_p}}}$$

להלן נביא דוגמא מספרית:

נתון לוח פלדה A514 עם סדק דק קטן $a/w \ll 1$. כמו כן נתון:

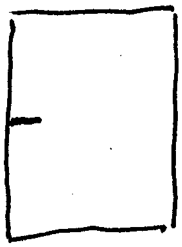
א. $\sigma_{ys} = 100\ Ksi = 689\ MN/m^2$

ב. $K_{IC} = 150\ Ksi\sqrt{in} = 165\ MNm^{-3/2}$

ג. $a_0 = 0.3'' = 7.6\ mm$

ד. $\sigma_{max} = 45\ Ksi = 310\ MNm^{-2}$

$\sigma_{min} = 25\ ksi = 172\ MNm^{-2}$



$\sigma_{max} - \sigma_{min} = \Delta\sigma = 20\ Ksi = 138\ MNm^{-2}$

ה. $K_I \cong 1.12\,\sigma\sqrt{\pi a} = 1.985\,\sigma\sqrt{a}$

פתרון ע"י האלגוריתם

א. $a_0 = 0.3''$

ב. $K_I = 1.985\,\sigma\sqrt{a}$

ג. חשוב a_{cr} עבור $\sigma_{max} = 45 Ksi$

$$a_{cr} = \left(\frac{K_{IC}}{1.985\times\sigma_{max}}\right)^2 = \left(\frac{150}{1.985\times 45}\right)^2 \approx 2.8'' \quad (71.1\ mm)$$

for steel A514

ד. $$\frac{da}{dN} = 0.66\times10^{-8}(\Delta K)^{2.25}$$

ה. $$\Delta K = 1.985\cdot\Delta\sigma\sqrt{a} = 1.985\times 20\sqrt{a}$$

$$\Delta K = 39.7\sqrt{a_{av}}$$

ו. עבור כל אינקרמנט $\Delta a = 0.1''$ נקבל גידול ב N הנתון ע"י

for $\frac{da}{dN} = 0.66\times10^{-8}(\Delta K)^{2.5} \longrightarrow$ $$\Delta N = \frac{\Delta a}{0.66\times10^{-8}(39.7\sqrt{a_{av}})^{2.25}} =$$

TABLE 8.2 Fatigue-Crack-Growth Calculations

$$\Delta N = \frac{\Delta a}{0.66 \times 10^{-8}[1.98(\Delta\sigma)\sqrt{a_{avg}}]^{2.25}}$$

where $\Delta a = 0.10$ in. (2.54 mm)
$\Delta\sigma = 20$ ksi (138 MN/m^2)

a_0 (*in.*)	a_f (*in.*)	a_{avg} (*in.*)	ΔK (*ksi* $\sqrt{in.}$)	ΔN (*cycles*)	$\sum N$ (*cycles*)
0.3	0.4	0.35	23.5	12,500	12,500
0.4	0.5	0.45	26.7	9,750	22,250
0.5	0.6	0.55	29.4	7,550	29,800
0.6	0.7	0.65	32.2	6,150	35,950
0.7	0.8	0.75	34.6	5,200	41,150
0.8	0.9	0.85	36.6	4,600	45,750
0.9	1.0	0.95	38.8	4,100	49,850
1.0	1.1	1.05	40.5	3,700	53,550
1.1	1.2	1.15	42.5	3,300	56,850
1.2	1.3	1.25	44.5	2,950	59,800
1.3	1.4	1.35	46.1	2,700	62,500
1.4	1.5	1.45	47.7	2,550	65,050
1.5	1.6	1.55	49.3	2,350	67,400
1.6	1.7	1.65	51.0	2,200	69,600
1.7	1.8	1.75	52.5	2,050	71,650
1.8	1.9	1.85	54.0	1,900	73,550
1.9	2.0	1.95	55.6	1,800	75,350
2.0	2.1	2.05	56.8	1,700	77,050
2.1	2.2	2.15	58.5	1,600	78,650
2.2	2.3	2.25	59.6	1,500	80,150
2.3	2.4	2.35	60.8	1,450	81,600
2.4	2.5	2.45	62.5	1,400	83,000
2.5	2.6	2.55	63.5	1,350	84,350
2.6	2.7	2.65	64.8	1,200	85,550
2.7	2.8	2.75	66.0	1,150	86,700

1 in. = 25.4 mm
1 ksi$\sqrt{\text{in.}}$ = 1.1 MN/m$^{3/2}$

a_{max}

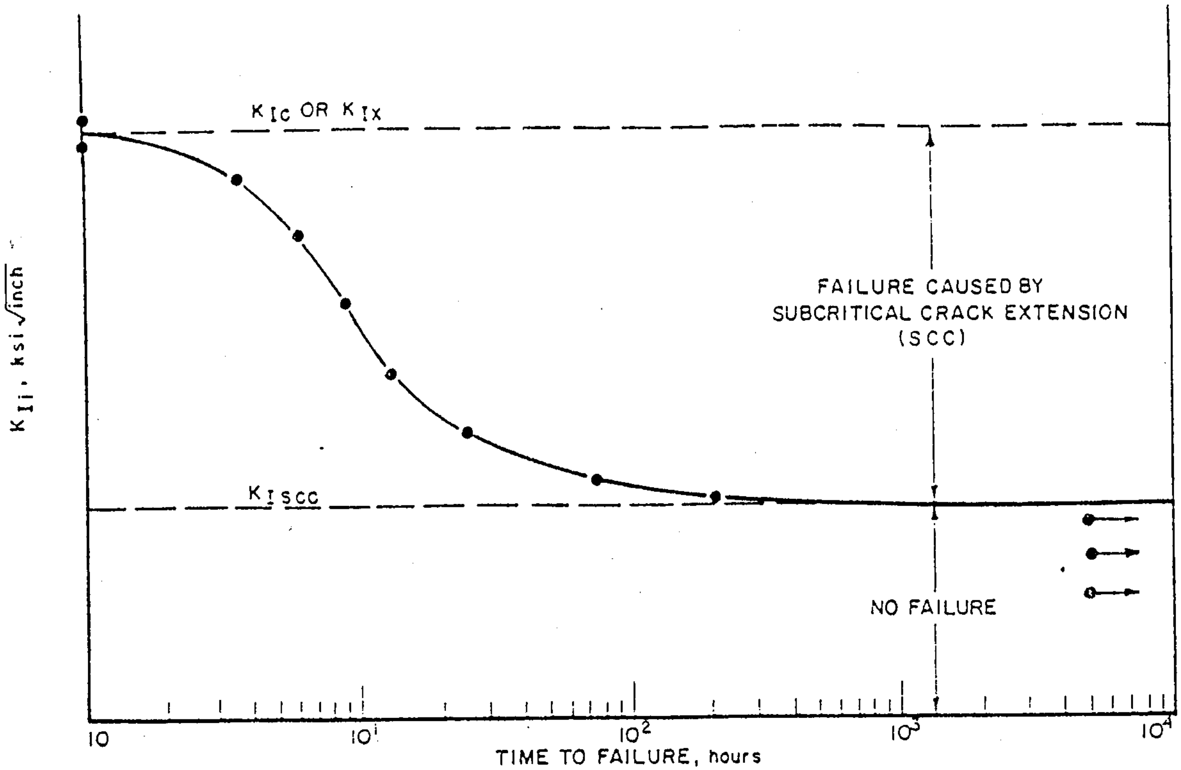


FIG. 10.3. Procedure to obtain K_{Iscc} with precracked cantilever-beam specimens.

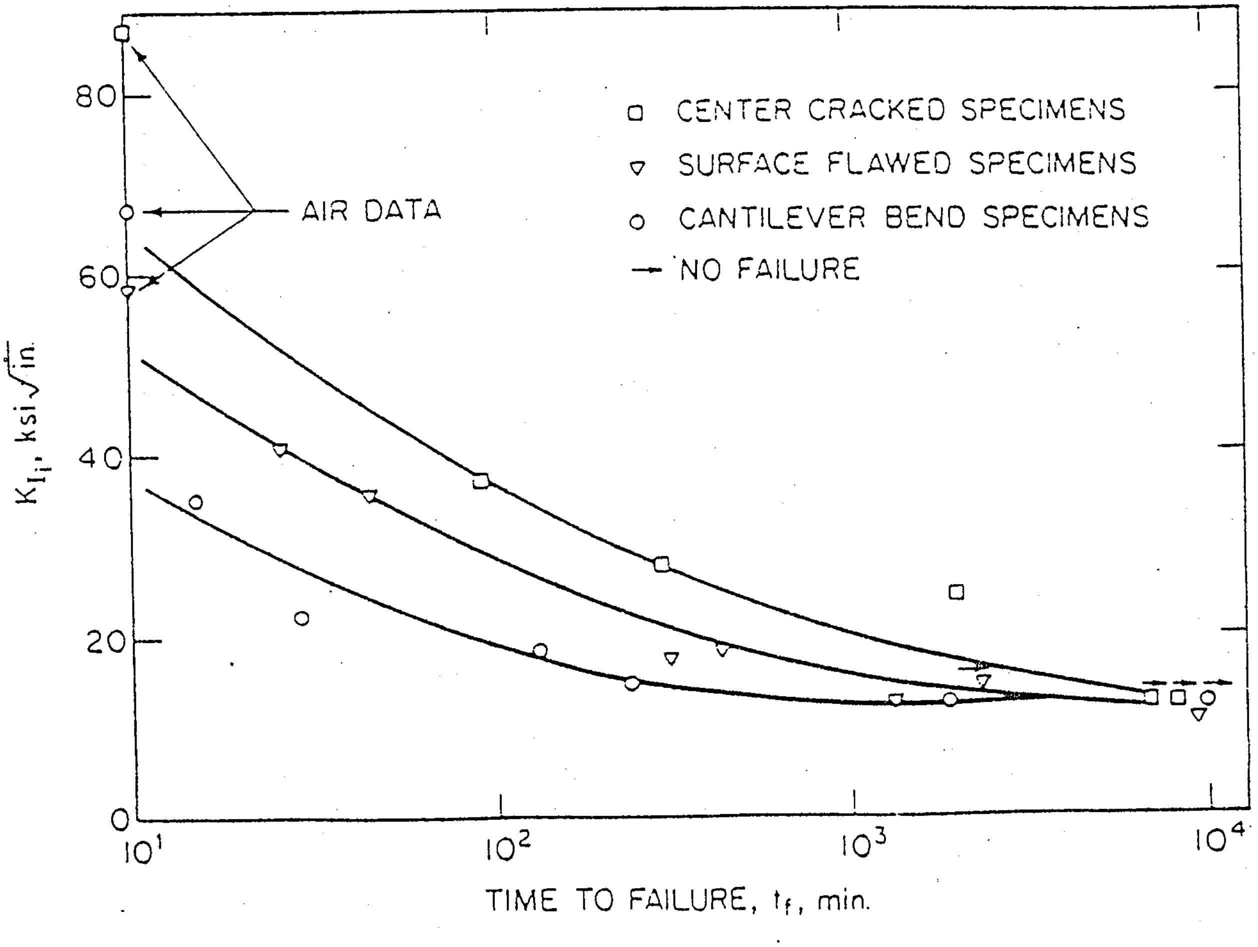


FIG. 10.8. Influence of specimen geometry on the time to failure (AISI

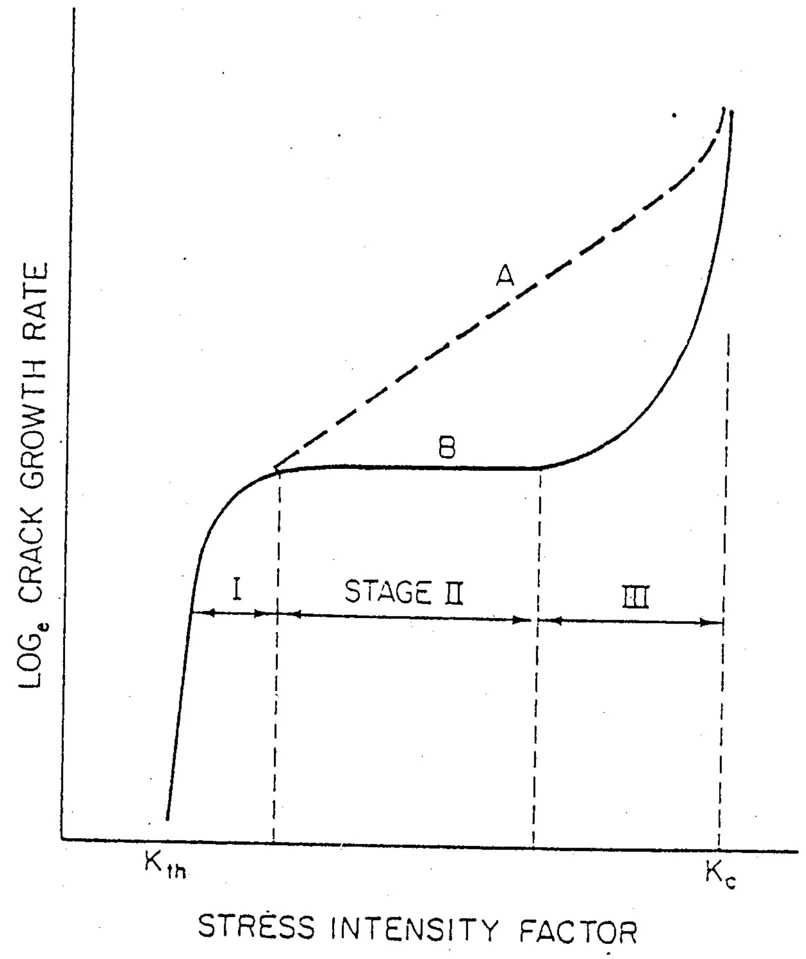
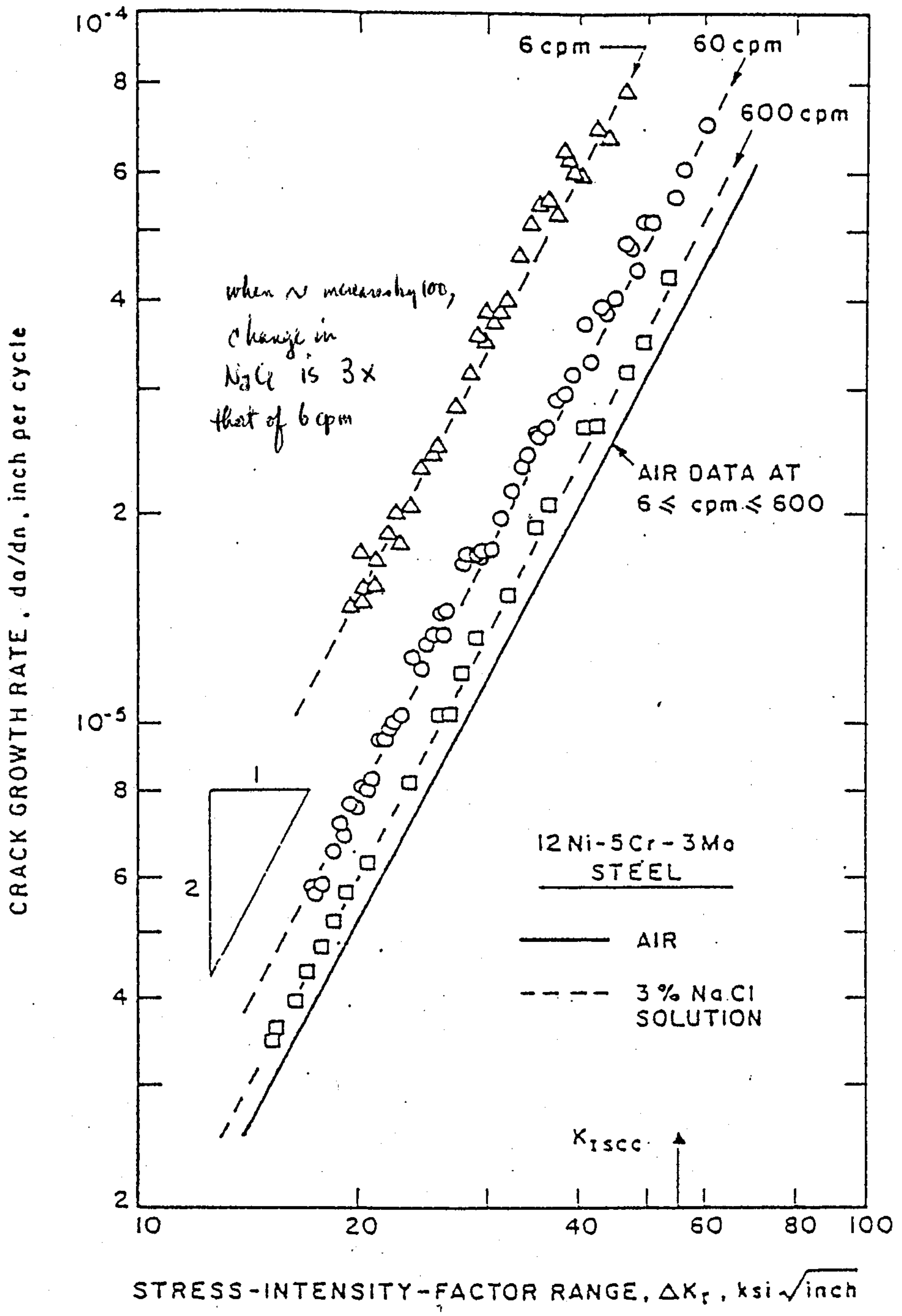


FIG. 10.17. Schematic illustration of the functional relationship between stress-intensity factor (K) and subcritical crack growth rate (da/dt).

K_{ISCC}/K_{IC} for Various Materials - Environment Systems

Material	Temp [°F]	σ_y [ksi]	Environment	K_{ISCC}/K_{IC}
6Al-4V-T_i	RT	160	Methanol	0.24
	RT	160	Freon	0.58
	RT	160	H_2O_2 (30%)	0.74
	RT	160	H_2O_2 (60%)	0.83
	RT	160	NaCl + H_2O	0.82
	RT	160	H_2O	0.80
	RT	160	He	0.90
	RT	160	Aerozine 50	0.82
	90	160	H_2O_2 (30%)	0.71
	90	160	H_2O_2 (60%)	0.75
	105	160	Mono-Hydrazine	0.75
	110	160	Aerozene 50	0.75
2219 T87 Alumina	RT	58	Air	0.96
	370	66	Liquid Hydrogen	0.82
	423	72	Liquid Hydrogen	0.85
4330	RT	205	H_2O	0.24
4340	RT	> 200	NaCl + H_2O	0.20
6Al-4V-T_i Weldments	RT	200	NaCl + H_2O (spray)	> 0.70
	RT	235	NaCl + H_2O	> 0.70
	RT	170	NaCl + H_2O	> 0.70

$$\frac{da}{dN} = D(t)\,\Delta K^2$$



when ν increases by 100,
change in
NaCl is 3x
that of 6 cpm

FIG. 11.2. Corrosion-fatigue-crack growth data as a function of test frequency.

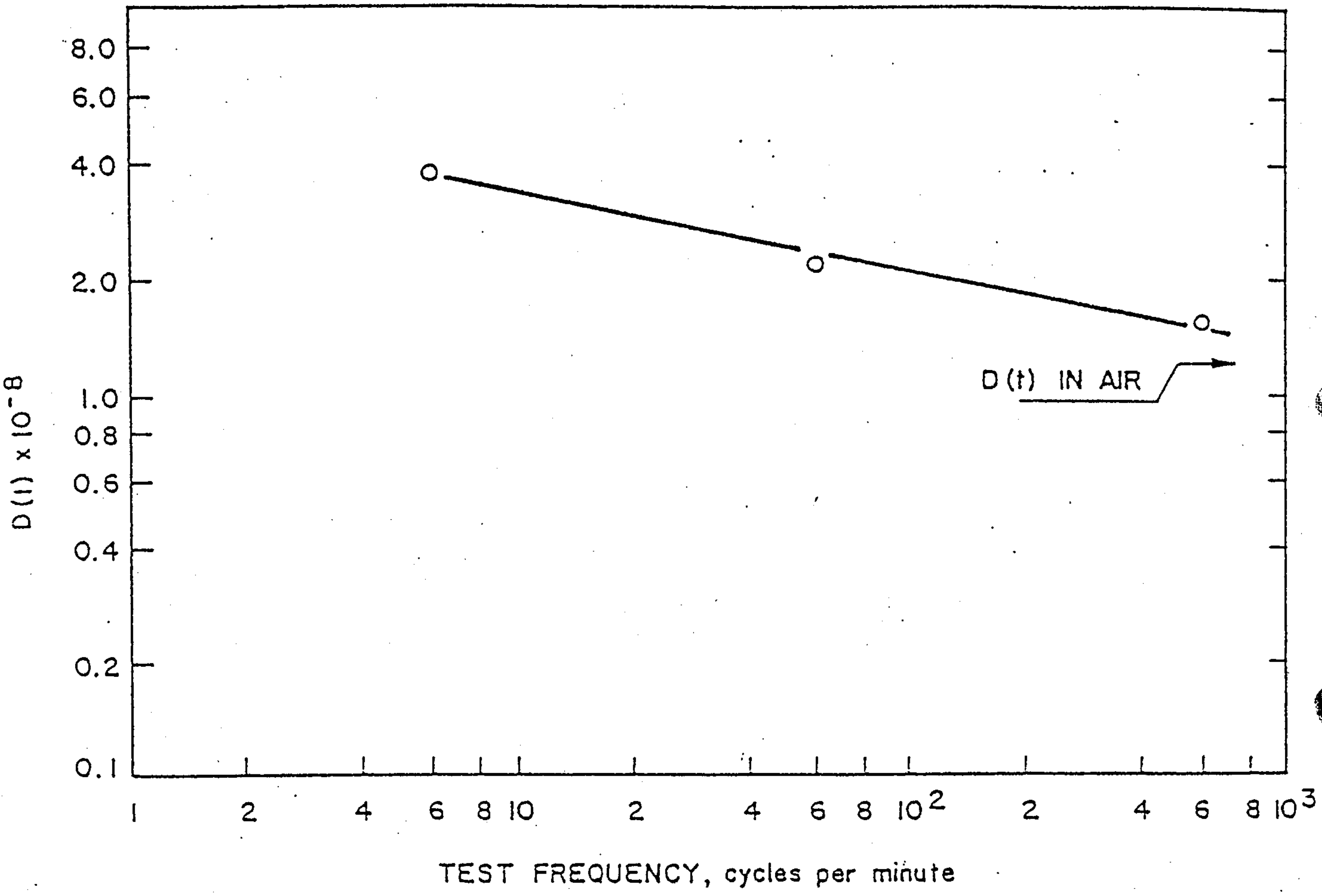


FIG. 11.3. Correlation between time-dependent function $D(t)$ and test frequency for 12Ni-5Cr-3Mo steel tested in sodium chloride solution.

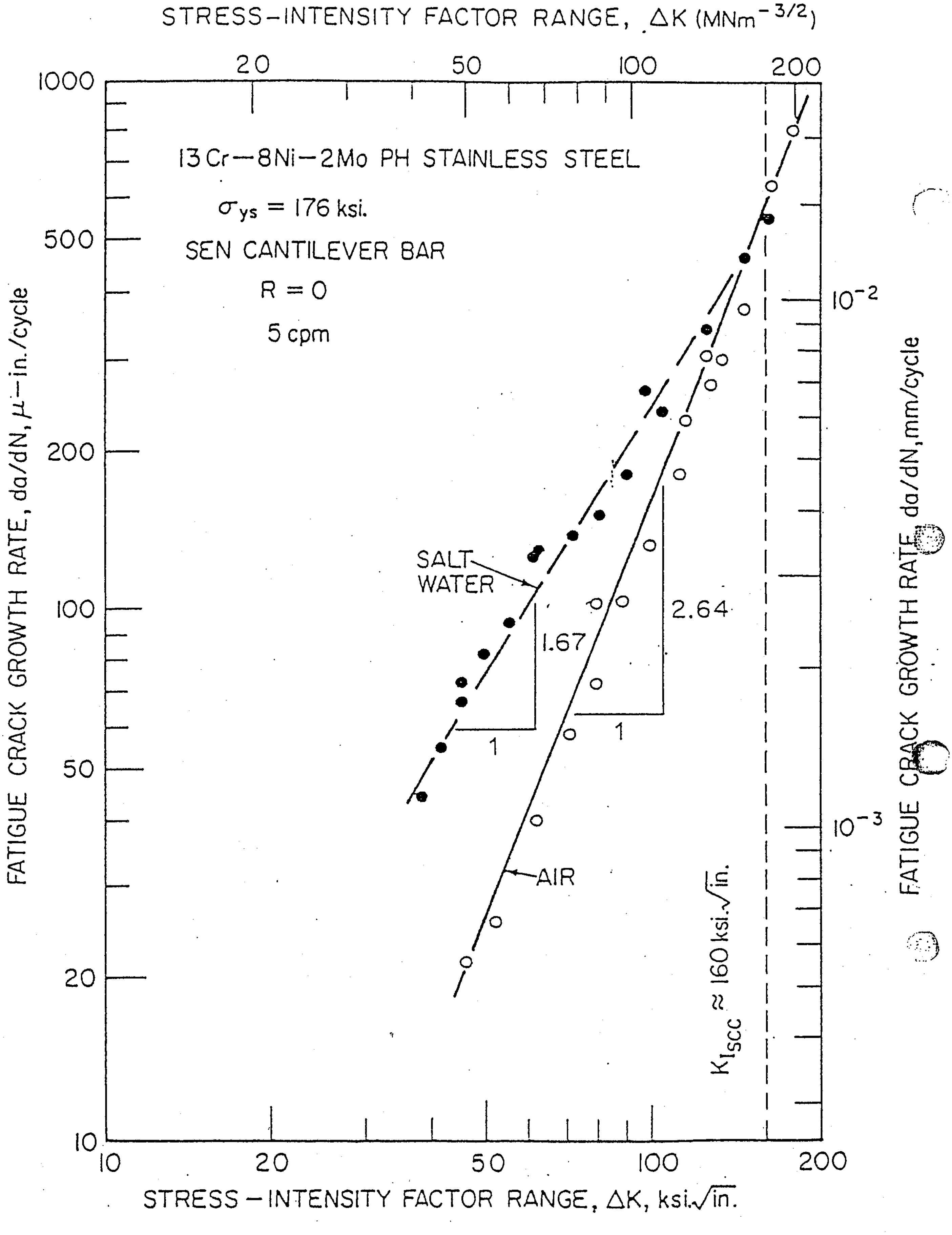


FIG. 11.4. Air and salt water fatigue crack growth rate behavior of 13Cr-8 Ni-2 Mo PH stainless steel.

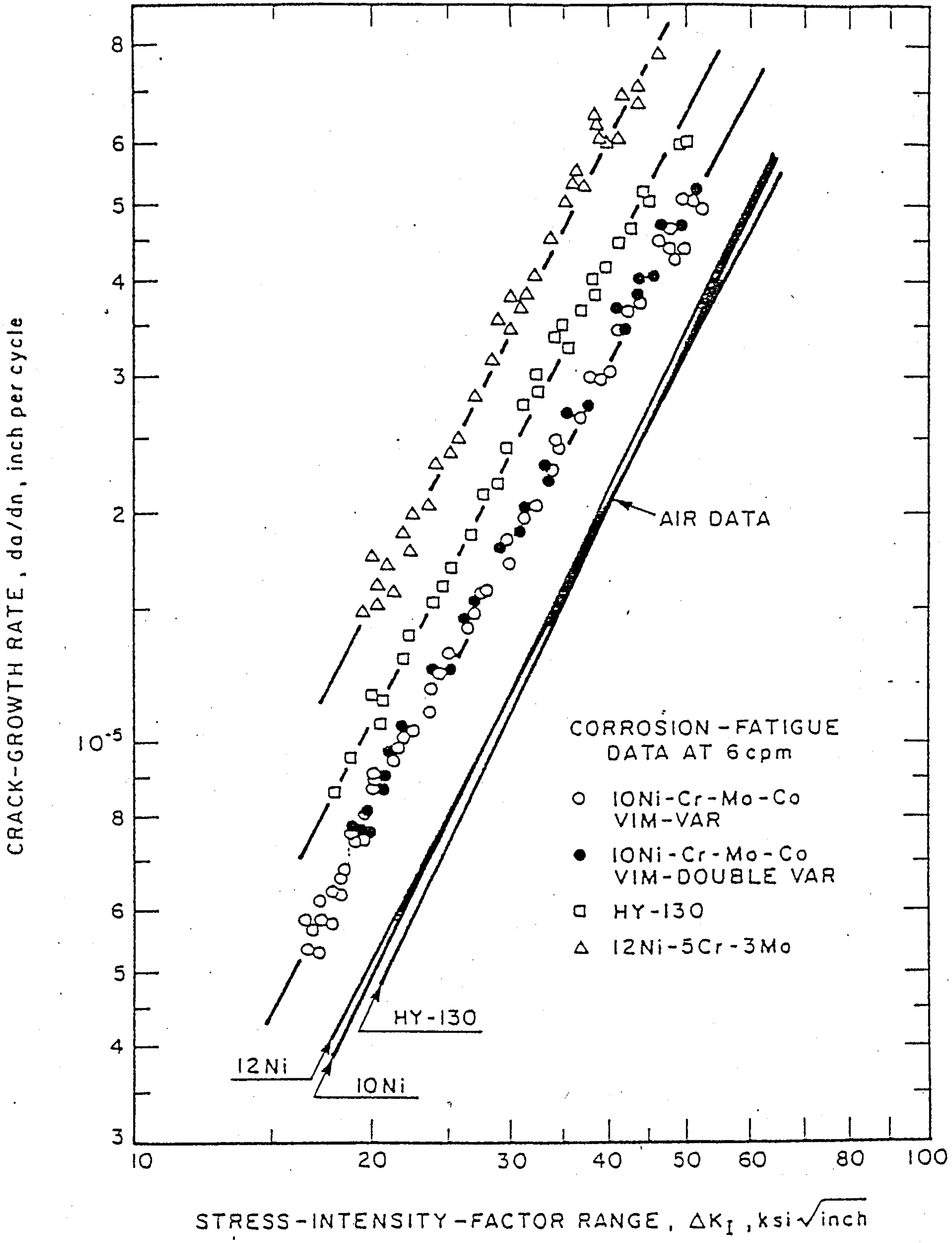


FIG. 11.5. Fatigue-crack-growth rates in air and in 3 % solution of sodium hloride below K_{Iscc} for various high-yield-strength steel.

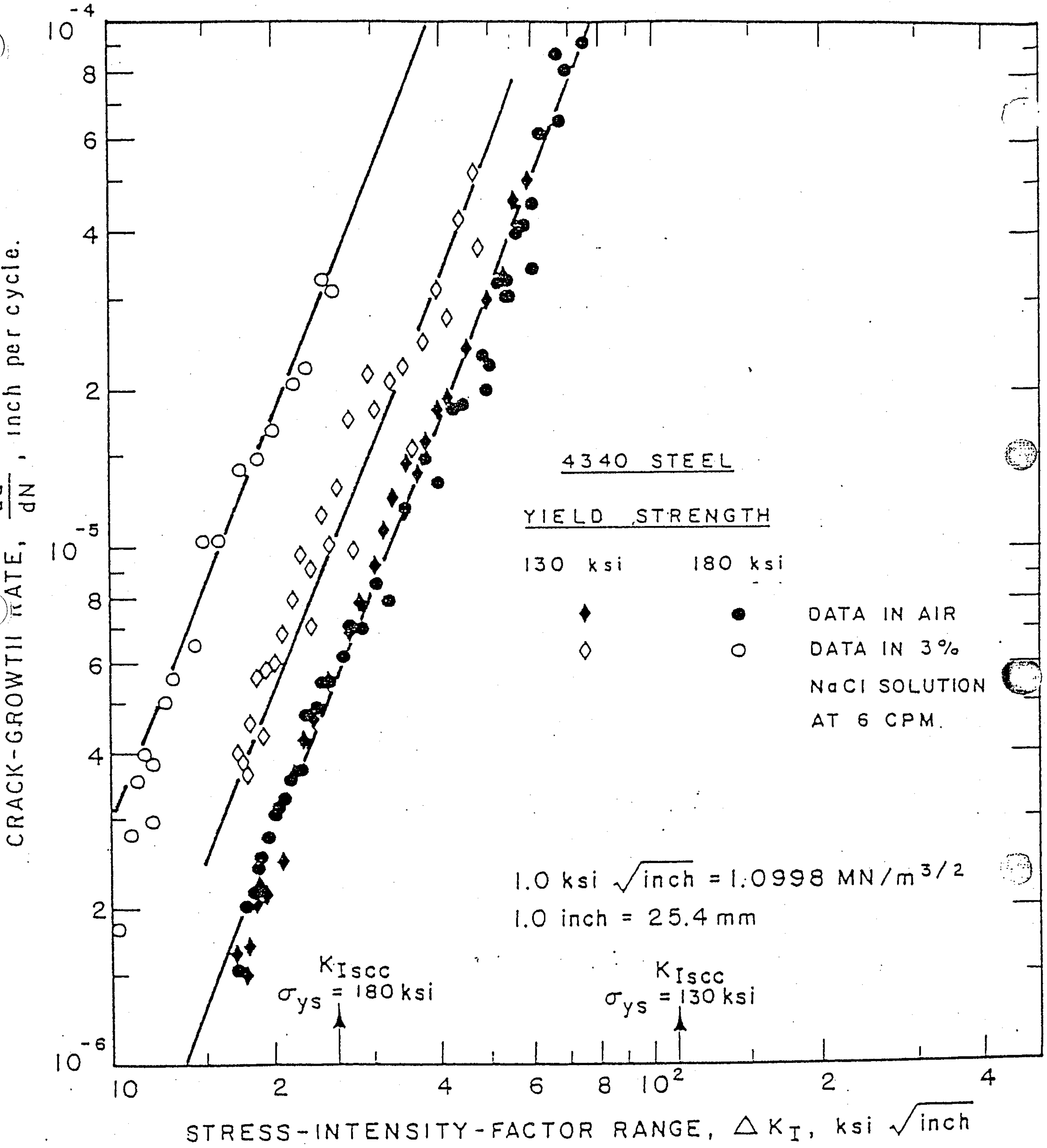


FIG. 11.6. Corrosion-fatigue-crack-growth data.

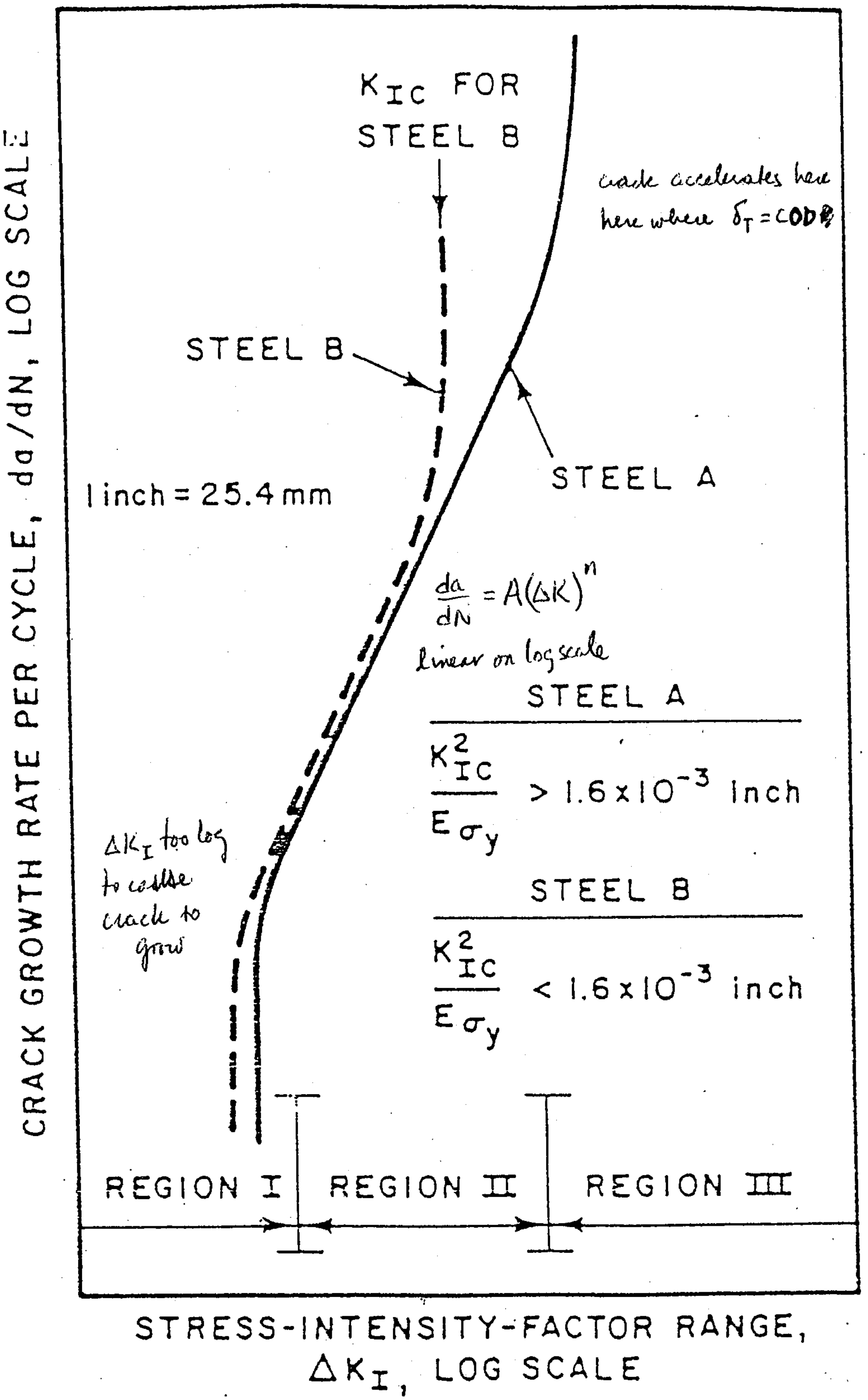


FIG. 8.4. Schematic representation of fatigue-crack growth in steel.

$$\delta_T = \frac{K_T^2}{E\sigma_{ys}} = 1.6 \times 10^{-3} \text{ in.} \quad (0.04 \text{ mm})$$

K_T = stress-intensity-factor-range value corresponding to onset of acceleration in fatigue-crack-growth rates.

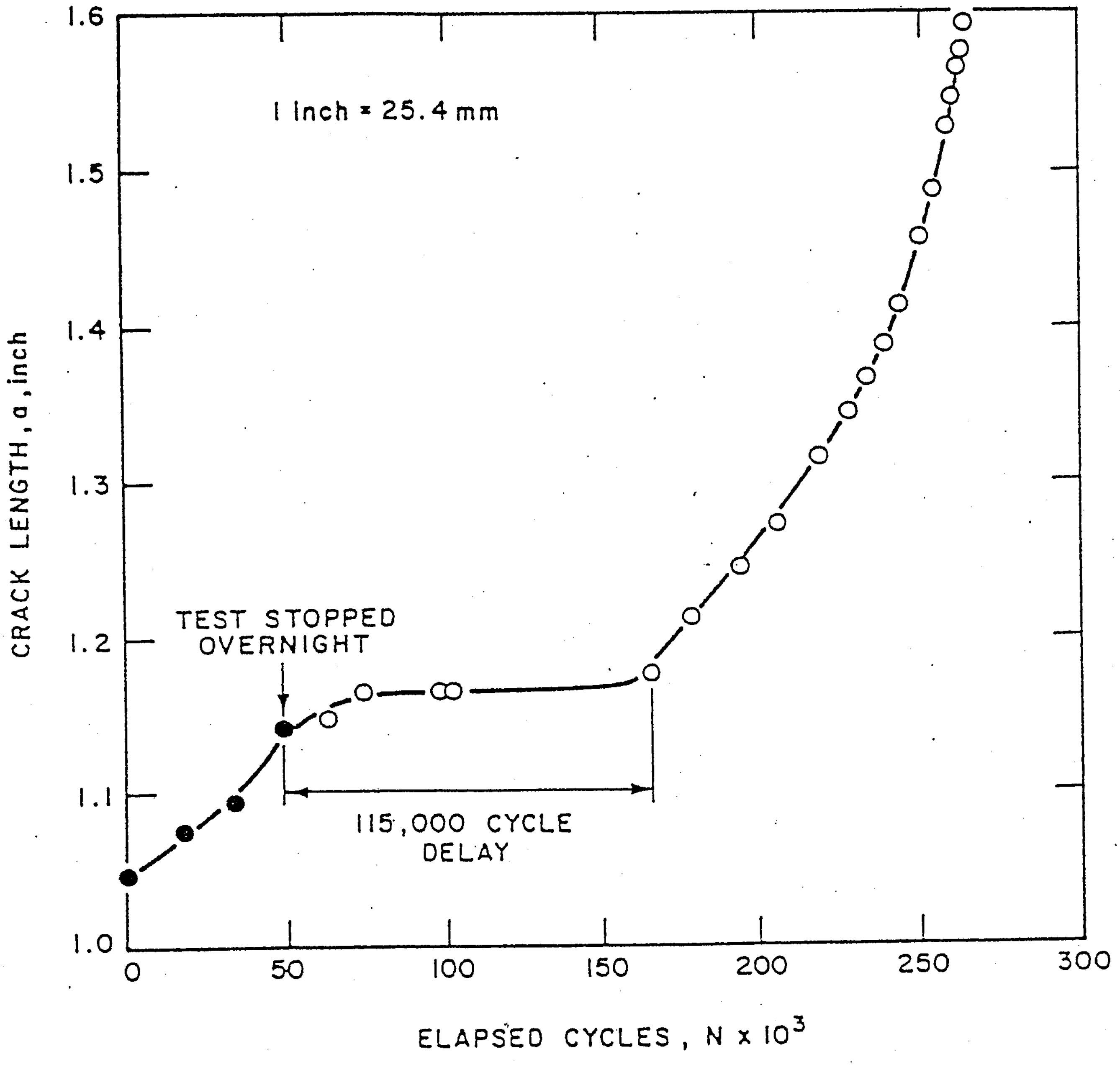


FIG. 11.21. Retardation of corrosion-fatigue-crack-growth rate under wet and dry environmental conditions for A514 Grade F steel.

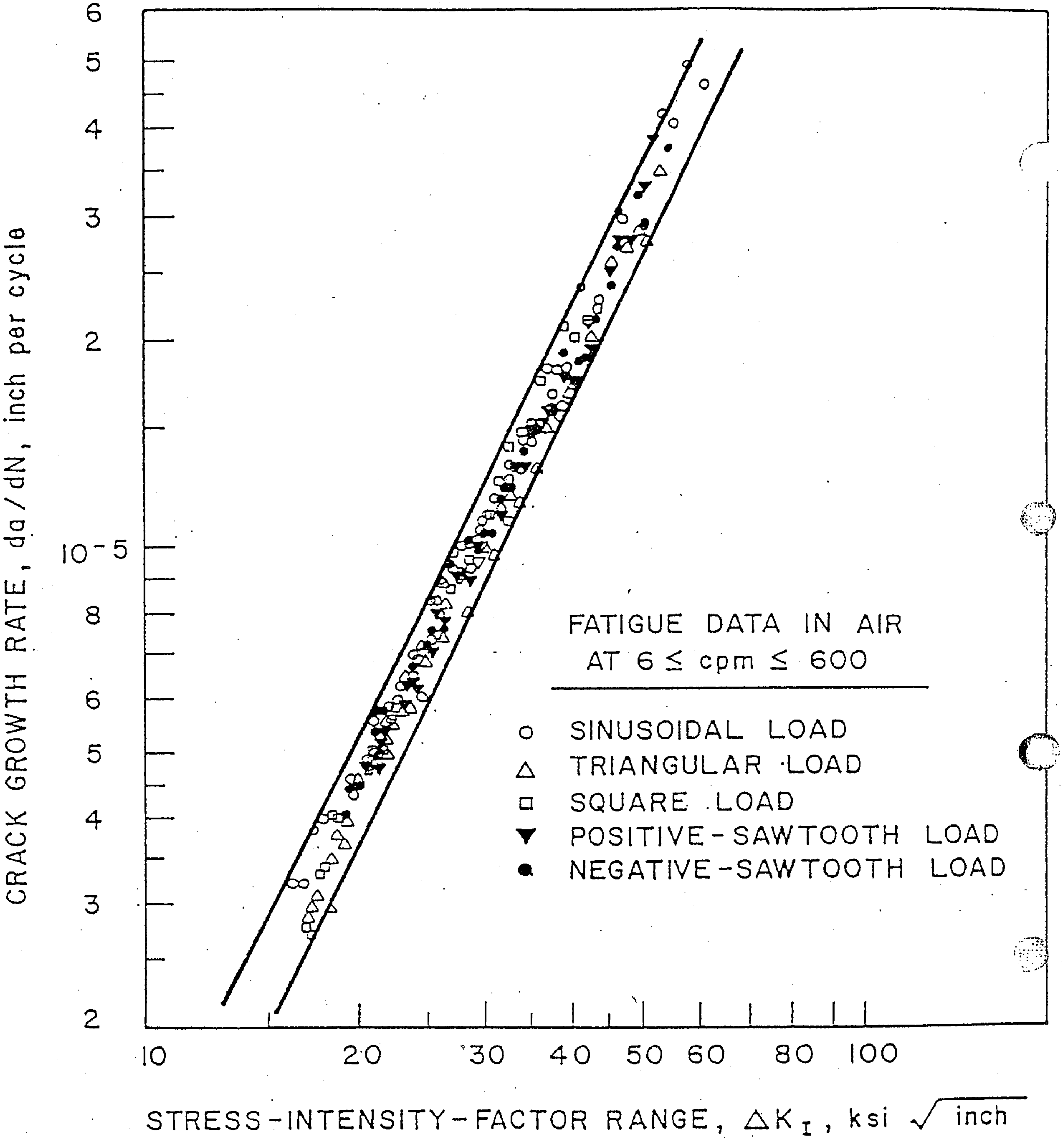


FIG. 11.22. Fatigue-crack-growth rates in 12Ni-5Cr-3Mo steel under various cyclic-stress fluctuations with different stress-time profiles.

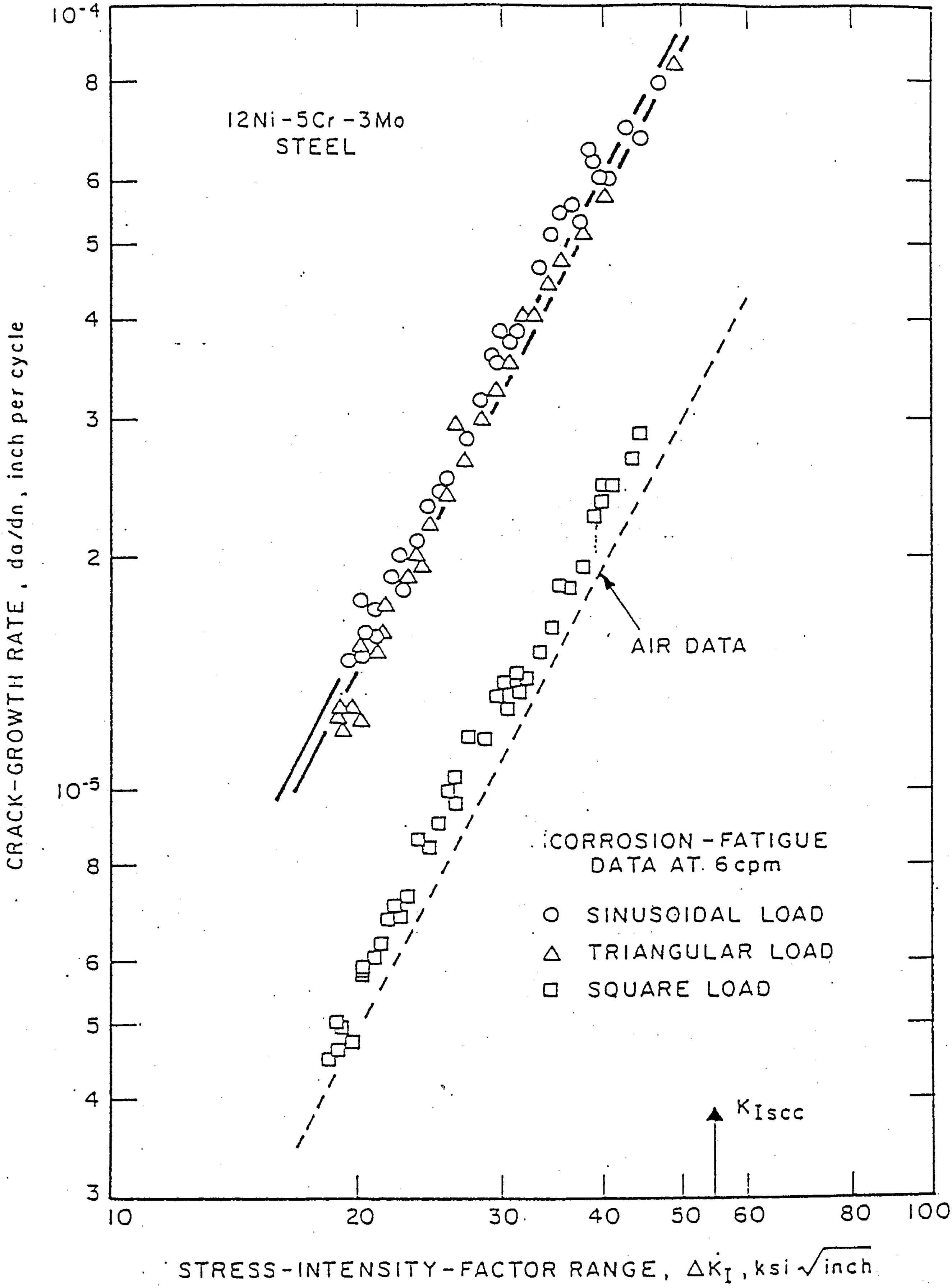


FIG. 11.23. Corrosion-fatigue-crack-growth rates below K_{Iscc} under sinusoidal, triangular, and square loads.

1.

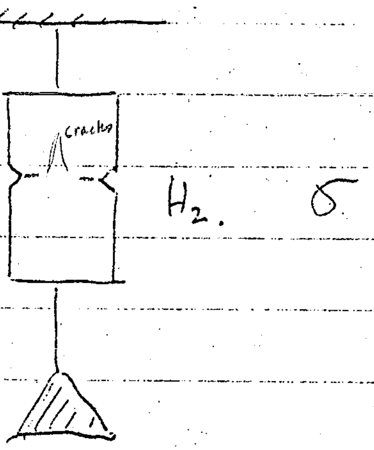
Hydrogen Embrittlement see notes next page

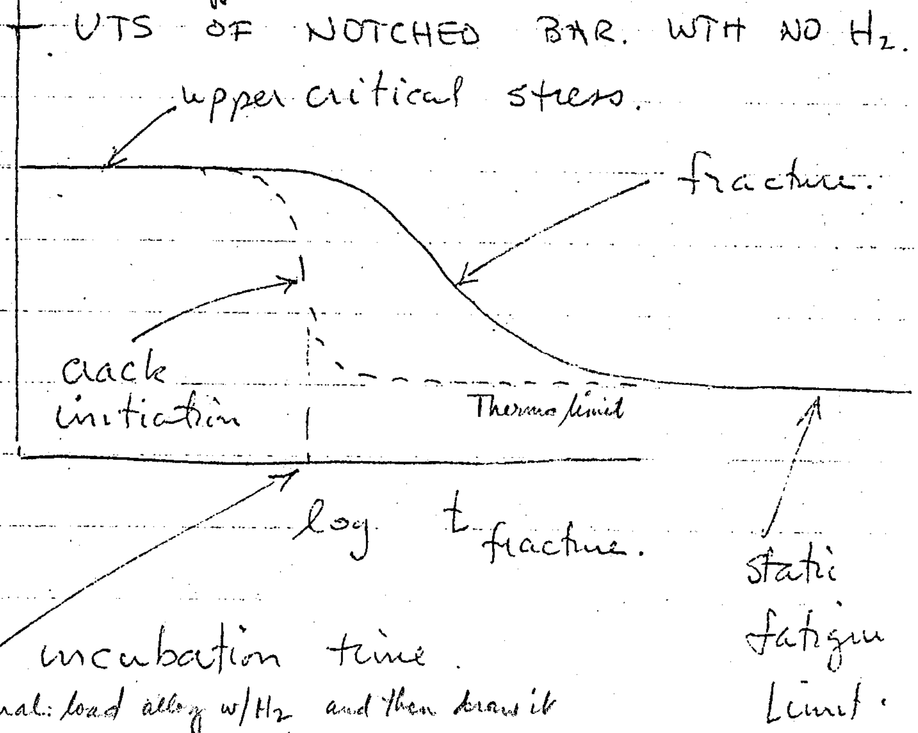
We study since there are pressure vessels containing chemicals gases etc, that fail due to embrittlement.

A. R. Troiano, Trans. ASM 52 (1960)
R. A. Oriani, German Phys. Chem J., 76 (1972)
both reprinted in Hydrogen Damage – ASM.

Hydrogen embrittlement – reduction of ductility in presence of H_2.
- high strength materials, steels, titanium
- slow strain rates or elevated temperature. (paced by H_2 diffusion).
- time dependent due to H_2 diffusion

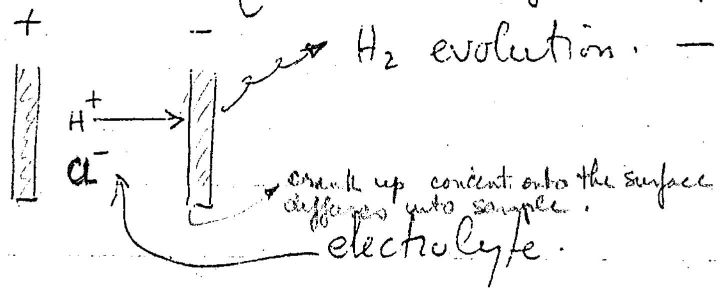
Static Fatigue





Hydrogen Charging

– normal: load alloy w/H_2 and then draw it out as you do the experiment

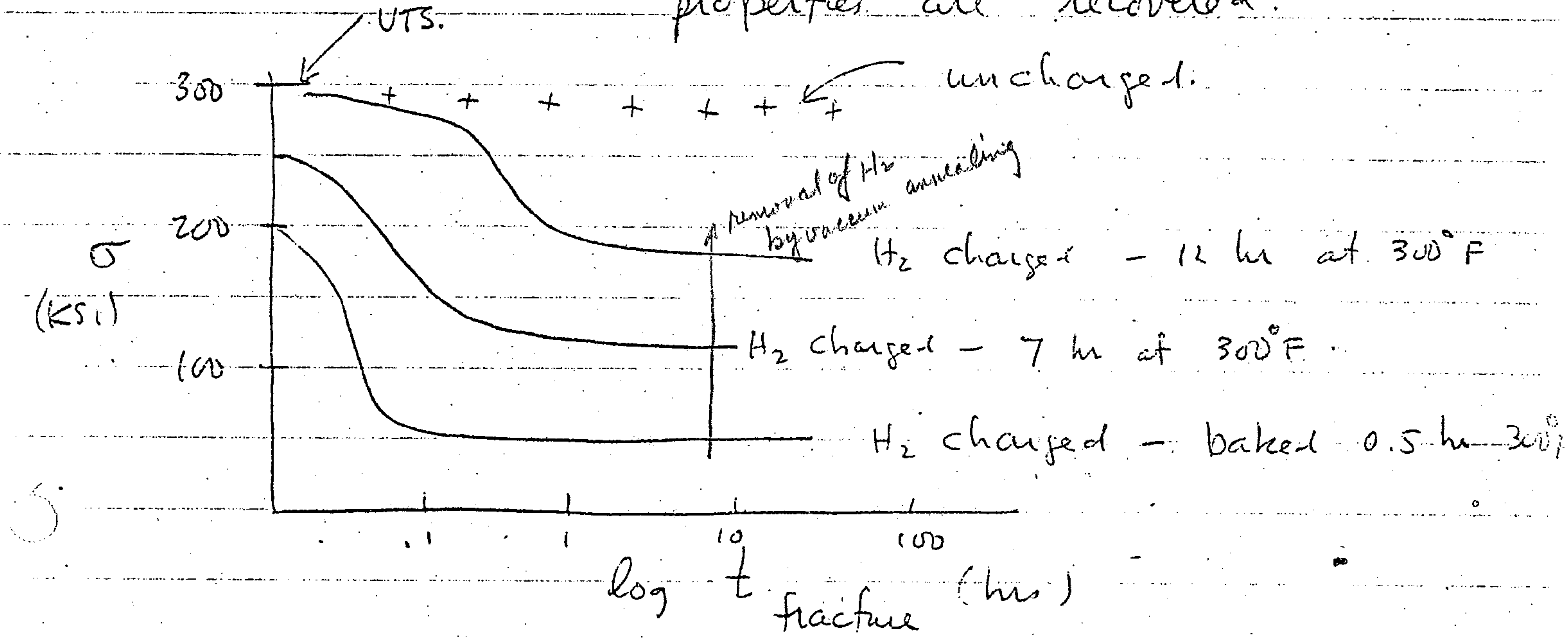


H_2 evolution. – some H dissolves into ~~and~~ lattice. – Hydrogen Charging.

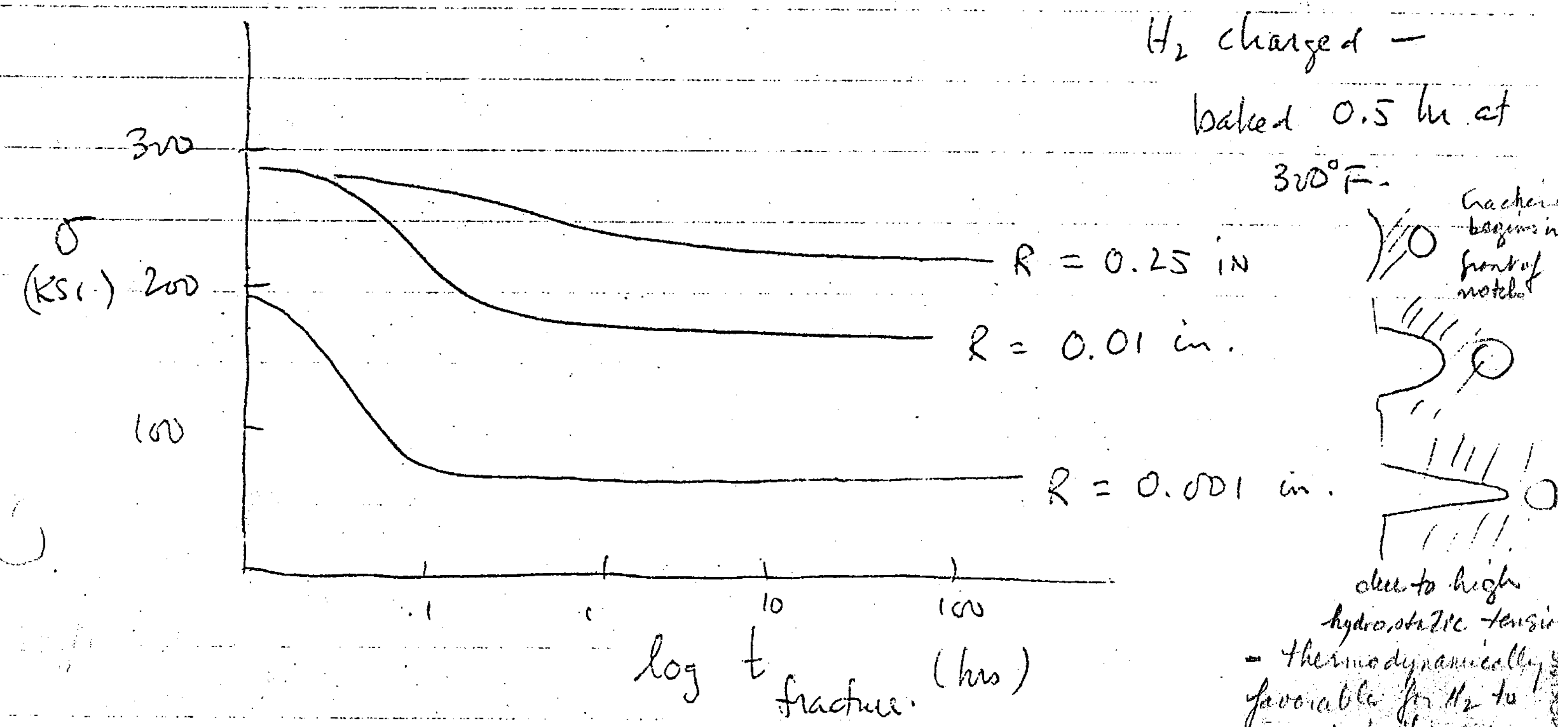
Hydogen Concentration Effects

reduction of ductility in high strength metallic alloys subjected to H_2 at high stresses

H_2 charge – then bake in vaccuum at 300°F. H diffuses to surface and H_2 is evolved. – mechanical properties are recovered.

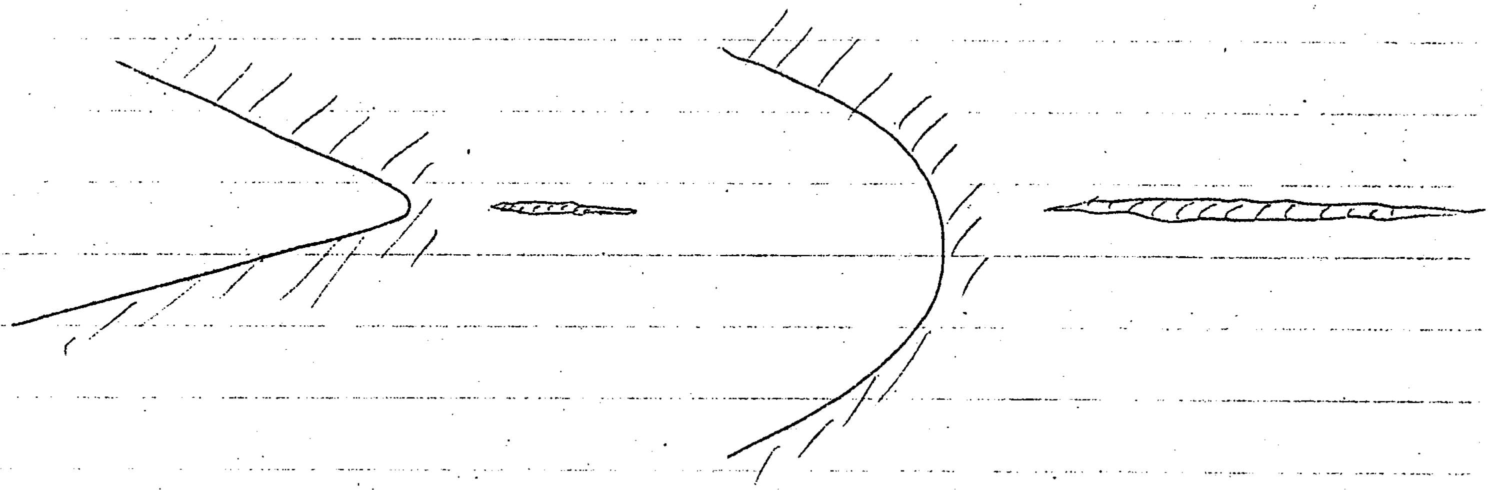


Notch Radius Effects

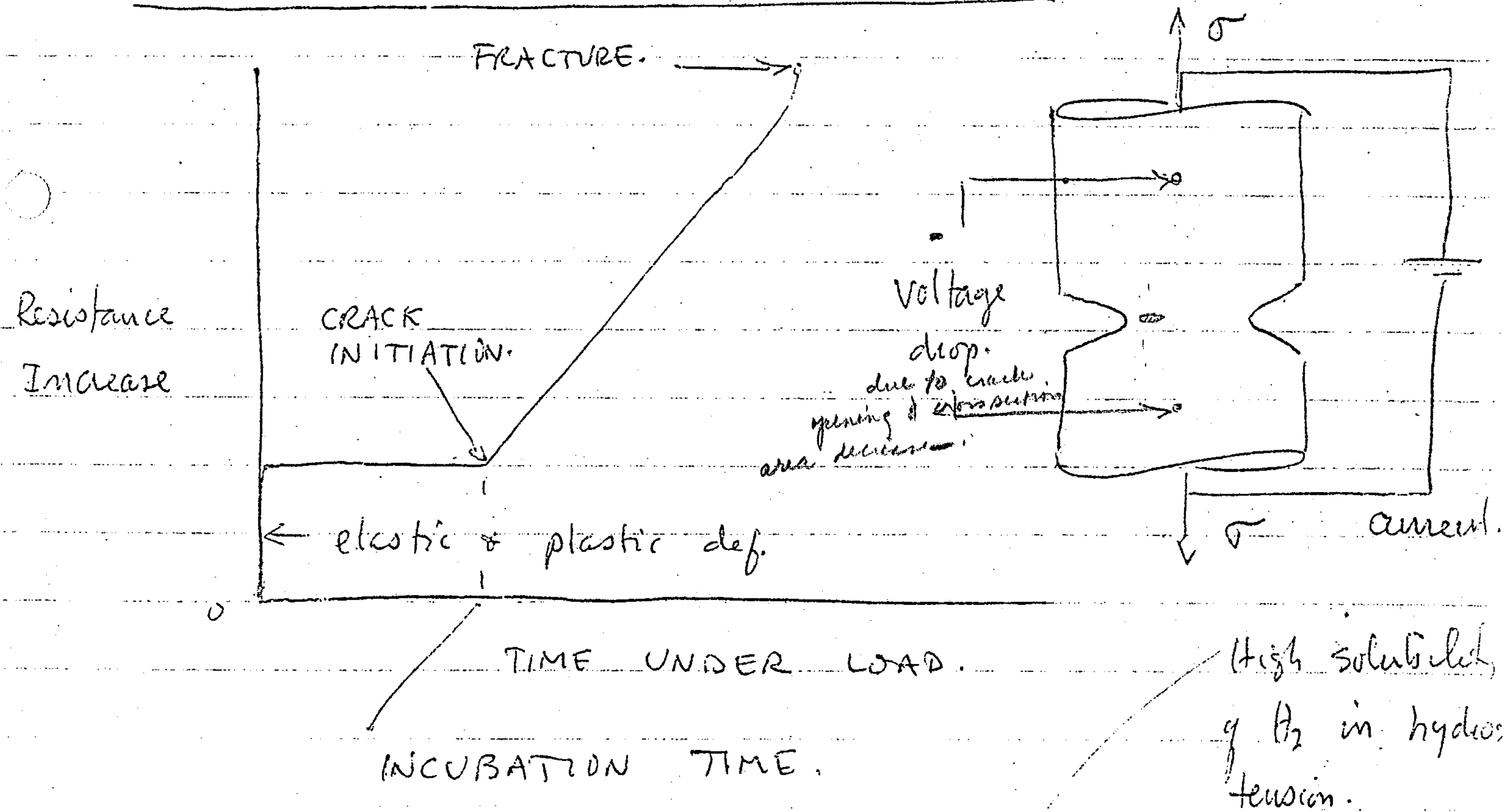


due to high hydrostatic tension
– thermodynamically favorable for H_2 to
– migrate to this areas

Observed Cracking — cracks form at root of notch.



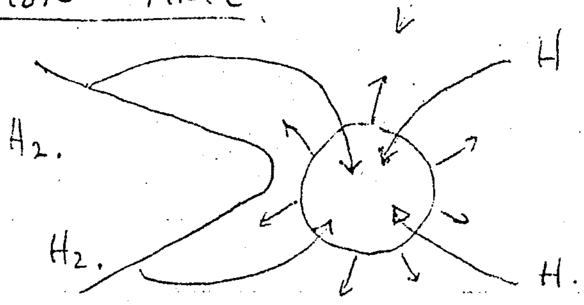
Crack Initiation — Incubation Times.



High solubility of H_2 in hydros tension.

THEORY OF INCUBATION TIME:

– CONTROLLED BY H diffusion in Fe lattice.



crack nucleates when H concentration reaches crit val.

Einstein Mobility:

D for H_2 in Fe.

$$\underset{\text{drift velocity}}{V} = \frac{\overset{\text{diffusivity}}{D}}{kT}\cdot F$$

driving force - related to stress state.

But $D = D_0 \exp\left(-\frac{Q}{kT}\right)$

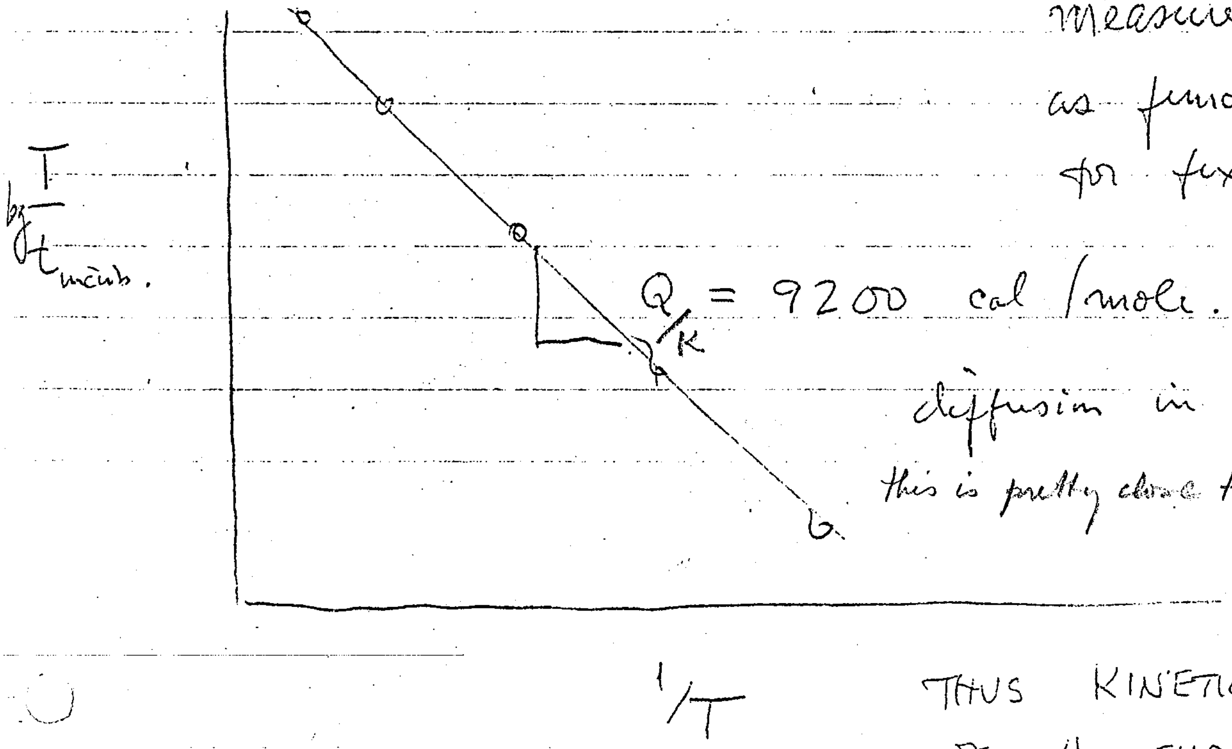
Boltzman Constant

$$\frac{1}{t_{\text{incubation}}} \sim V \sim \frac{D_0 \exp^{-Q/kT}}{kT}\cdot F$$

movement of molecules to area of hydrostatic tension depends on diffusion of H_2 in Fe.

or $$\frac{T}{t_{\text{incubation}}} \sim \exp\left(-Q/kT\right)$$

if we plot $\log T/t$ vs $\frac{1}{T}$ that should give us heat of diffusion Q



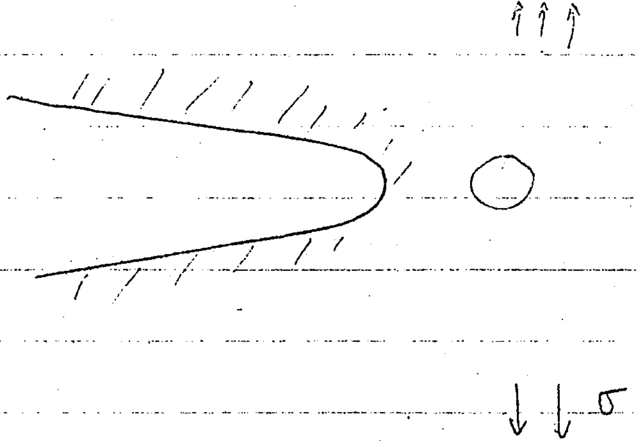
measure $t_{\text{incub.}}$ as funct. of T for fixed σ.

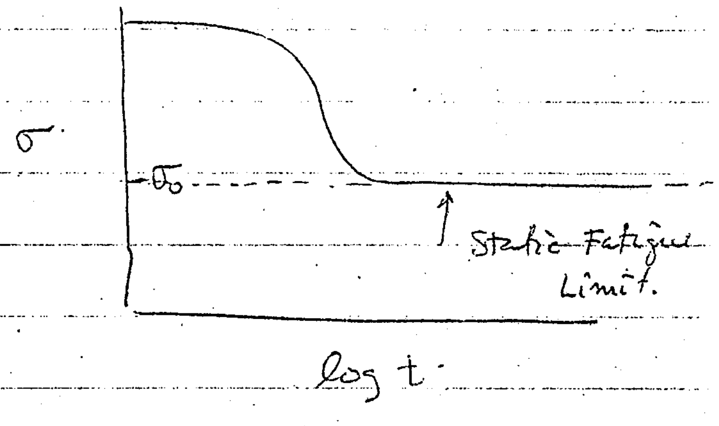
$Q = 9200$ cal/mole. $\simeq Q_{\text{Hydrogen}}$ diffusion in Fe.

this is pretty close to actual Q.

THUS KINETIC ASPECTS OF H_2 EMBRITTLEMENT RELATED TO H DIFFUSION IN Fe.

THERMODYNAMICS OF Hydogen Embrittlement.





Below σ_0, – no failure – no crack nucleation regardless of time elapsed.
suggests that <u>critical conditions</u> are not met.
As first approx, say hydrogen concentration in notch does not reach critical value.

<u>Effect of Stress on H Concentration.</u>

$$C_H(x) = C_0 \exp \frac{-p\Omega}{RT}$$

Ω = vol expansion assoc. with 1 atom of H in lattice.

Since H_2 are interstitial atom & fits into holes of lattice <u>but</u> also volume ↑

Thus $C_H(x)$ high in regions of hydrostatic tension $p<0$

$C_H(x)$ low in regions of hydrostatic compression $p>0$

p = hydostatic pressure

$$p = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

Now assume that a crack nucleates when the $C_H(x)$ reaches a critical value C_{crit}. Then the condition of crack nucleation is

$$\frac{C_{crit}}{C_0} = \exp\left\{+\frac{\Omega}{kT}\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\right\}$$

this doesn't take into account the fact that the applied loads driv H_2 to the location of greatest $+p$. And the fracture depends not on on H_2 but on applied loa too, However this is discussed in Oriani's paper, We assume everything is lumped in C_H

or

$$\frac{3kT}{\Omega}\ln\frac{C_{crit}}{C_0} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

So at a given Temperature, T, and given hydrogen environment, C_0, then failure (nucleation will occur when

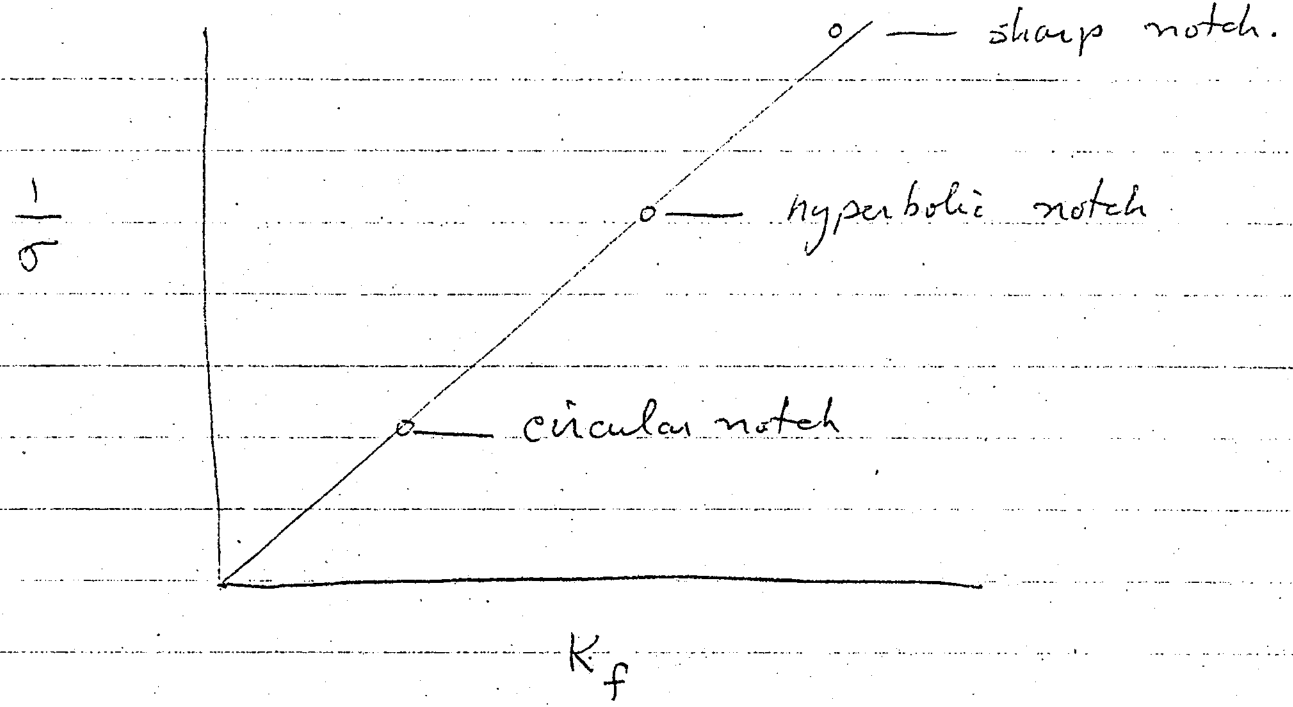
$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \overset{\text{const.}}{\text{critical value}} = \frac{3kT}{\Omega}\ln\frac{C_{crit}}{C_0}$$

But $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = K_f\,\sigma = \text{const.}$

↑ stress concentration factor

So for various notches (at same T and C_0)

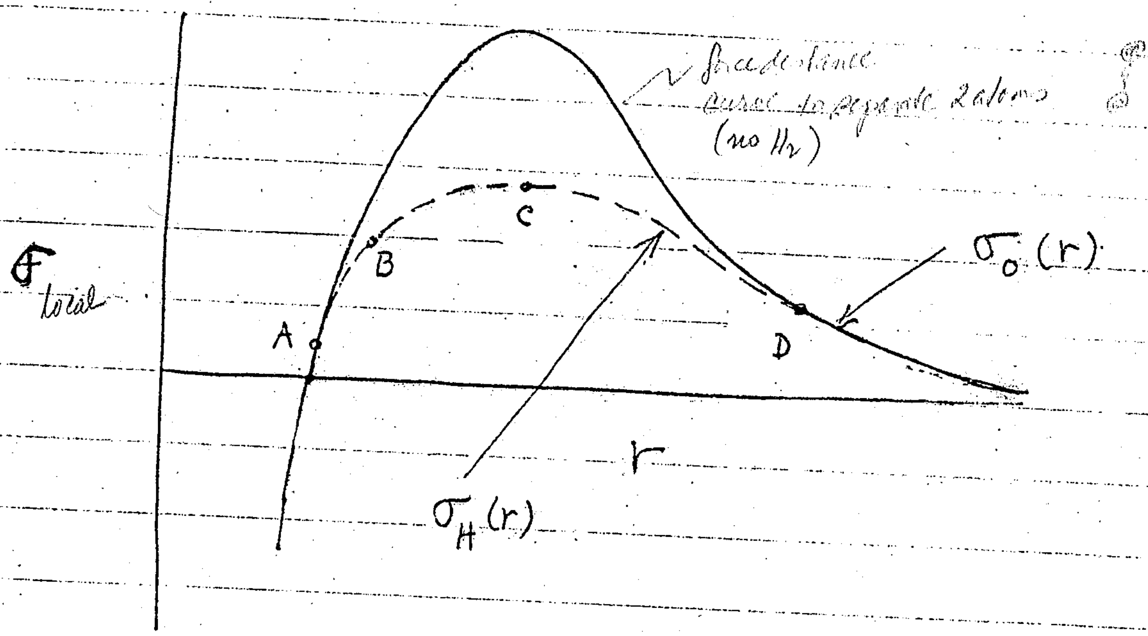
$$\frac{1}{\sigma} = \text{const}\ K_f.$$

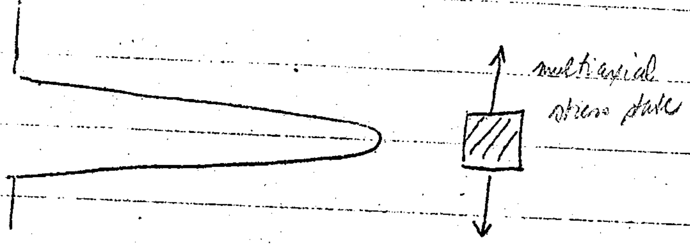


The Troiano treatment is only a first approximation as it assumes that fracture is initiated at a particular H concentration regardless of the stress! That is, the role of the stress is only to concentrate the H_2, not to break the atoms apart.

A more complete statement of this problem was given by Oriani who includes consideration of both the stress it self and its H concentration potential.

Oriani – Based on Effect of H_2 on Cohesive Forces. 1972.





A. Stress is very small – ~~no~~ low concentration of H_2,
F – x curve virtually the same as no H

B. Stress now large enough to concentrate H_2,
so F–x curve is affected

C. Large stresses now concentrate H_2 and F–x
curve is changed significantly – Considering
applied σ have reached point of instability,

D. At large separations, σ is low again so
H conc is low and