Winter 1978
J. P. Johnston

ME 251B

Advanced Fluids Engineering

FINAL EXAM

Notes:

1) This exam should consume not more than 6 hours. Note time spent on cover of blue book. Correct solution of two out of the three problems is worth 80% of total.

2) Use of notes and books permitted.

3) Write up solutions in blue book.

4) Local students (including Honors Coop) turn in exam to my secretary, Ann Ibaraki, Room 501C, by 11:30 a.m. on Thursday, March 22.

   Remote location T.V. students, be sure exam is in my hands no later than March 30th.

5) Course grades: Problem Sets - 60%
   Exam - 40%

Problem 1

Consider two-dimensional, steady, incompressible, laminar flow about a circular cylinder at high Reynolds number, $Re = U_0 a/\nu$. U_0 is the free-stream speed far from the cylinder and a is the radius of the cylinder.

Assume that the velocity distribution at the cylinder surface (outside the boundary layer) is given by the ideal, potential flow distribution,

$$U_e = 2U_0 \sin\alpha ,$$

where α is the angle measured from the forward stagnation point.

Part (a): Obtain an expression for the ratio (θ/a) in terms of Re and α . θ is boundary layer momentum thickness.

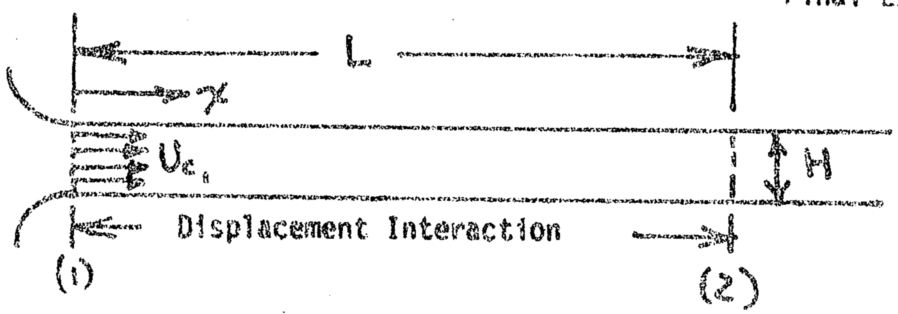
Part (b): Determine the angular location of laminar separation, α_{sep}(degrees).

Part (c): Estimate the maximum Re below which instabilities of all wave length will decay in the boundary layer on the forward half of the cylinder $(\alpha \leq \pm 90°)$.

Part (d): Experiments show that laminar separation can occur at α's less than $\pm 90°$. How can this be explained?

Problem 2

Two-dimensional, incompressible flow enters a long, smooth, parallel walled duct at uniform core speed U_{c1} , and with very thin turbulent boundary layers. The Reynolds number is high and the boundary layers remain turbulent. Assume that the duct Reynolds number, based on duct width H , is $(U_{c1}H/\nu) = 10^5$.



Estimate, by approximate means, the values of the following parameters at station (2), the end of the displacement interaction zone.

$$; \quad \frac{\bar{p}_1 - \bar{p}_2}{\frac{\rho}{2} U_{c_1}^2} \quad ; \quad \frac{F}{\frac{\rho}{2} U_{c_1}^2 L} \quad ; \quad \frac{L}{H}$$

where $\bar{p}$ is time-mean static pressure, and

$$F = \int_0^L \tau_\omega dx$$

is the total drag force (per unit width) on one wall of the duct. Obtain F by two different methods.

You may assume simple power law velocity profiles at all streamwise stations, i.e.,

$$\frac{U}{U_c} = \left(\frac{y}{\delta}\right)^{1/7}$$

and, in part of your solution at least, treat the boundary layer on the duct wall as though it grew at zero pressure gradient, i.e.,

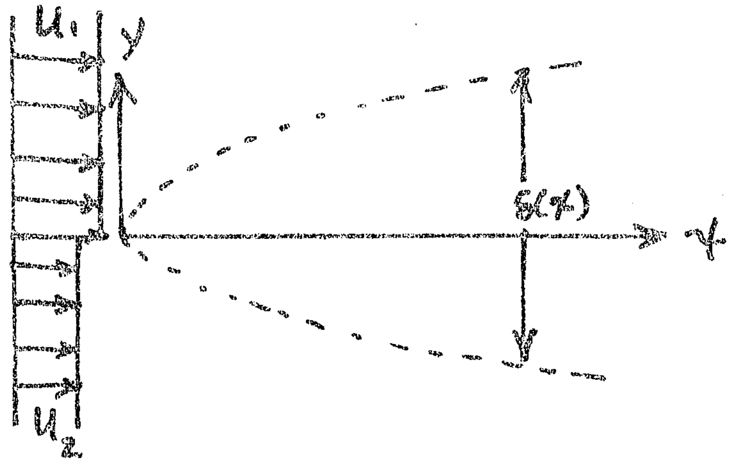
$$C_f = 0.058 \, Re_x^{-0.2}$$

where $C_f = \tau_\omega / \frac{1}{2}\rho U_c^2$ and $Re_x = U_c x / \nu$.

Discuss your results and indicate how you believe they would have to be changed if they were to more accurately represent real flow conditions. That is, are the estimates of the sizes of the parameters large or small. Why?

Problem 3

Obtain a solution for the velocity profile of a steady, incompressible, two-dimensional, laminar free mixing layer in the limiting cases where $(u_1 - u_2) << u_1$.



Discuss the growth, $\delta(x)$ characteristics of the layer's thicknesses and compare the result for the shear stress coefficient on the layer $y = 0$ to results obtained for flow over a flat wall.

ME 251B Prof Johnston

Outline

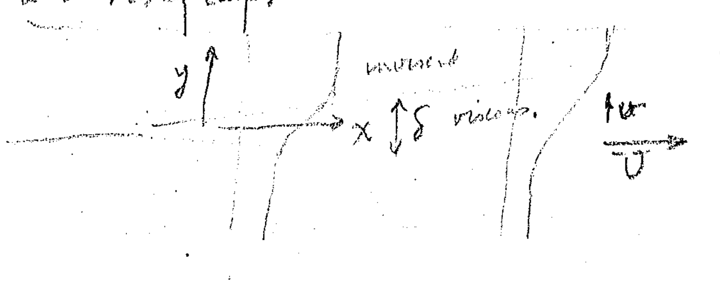
Momentum Transfer in Shear (Boundary) Layers — Mom. Transfer in Boundary Layers Cebeci/Bradshaw

1. Eqns of Viscous flow
   - Cons. of mass (continuity)
   - Navier Stokes
   - Energy
   - Turbulent (mean)
   - Vorticity

- Reading assignments are required
- Books for course / reference / in library on reserve.
  - 3 problems 20% each
  - 1 final 40% run in same manner as before due 22 March 11:30 PM

2. exact & approx sol's on Navier Stokes
3. Thin Shear Layers (BL eqs)
   - Differ form
   - Integral form
   - BC.'s
4. Laminar Shear Layers (similarity solns)
5. Turbulent " "
6. Internal flows
   - Displacement Interactions
   - Shear "
   - Separation & Reattachment
   - Diffusers

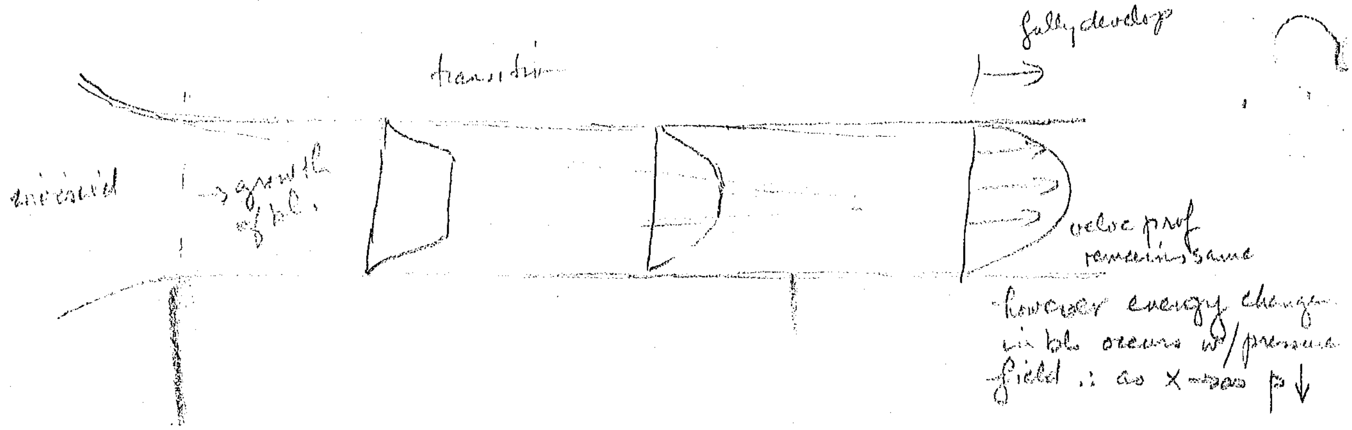
2-D Mixing layer



$\delta(x)$ only $\propto x$

$\frac{\delta}{x} \lesssim \frac{1}{10}$ for both turb & laminar

$\frac{v}{U} \lesssim \frac{1}{10}$ $\qquad \frac{\partial U}{\partial y} \gg \frac{\partial U}{\partial x}$



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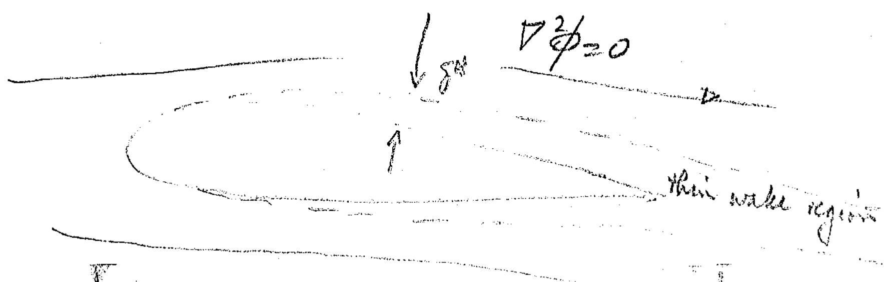
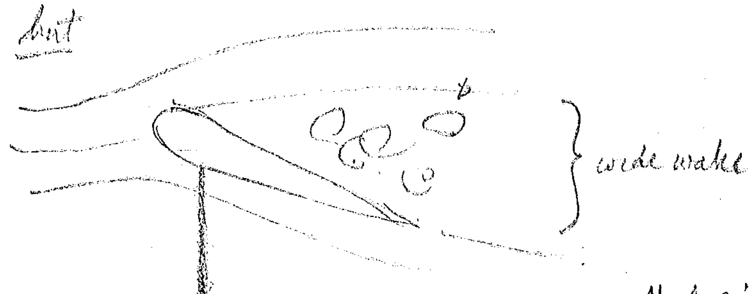


(1) Solve $\nabla^2\phi=0$ for airfoil w/ Kutta condition & No B. Layer (get p on airfoil surface)

(2) with $p_w(s)$ along surface obtain $\delta^*(s)$.

(3) add δ^* to surface & solve $\nabla^2\phi=0$ again

but



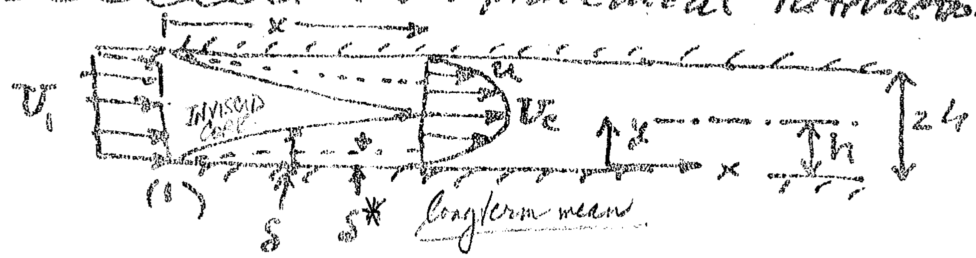
wake affects foil "shape" so $\nabla^2\phi=0$ for foil alone is not sufficient to start step/step s.l.

Need simultaneous solution ie Navier Stokes

Solve for thin viscous layer for separation pt.

Viscous / Inviscid Interaction

External Flow (Text Fig 11.4)
Internal Flow (Displacement interaction)



δ^* = Displacement thickness $\left\{ \begin{array}{l} \rho = \text{const case} \\ \text{2-D, steady in longterm mean} \end{array} \right\}$

$\dot{m}_{in} = \dot{m}_{out}$ continuity

$$\rho\, 2h\, U_1 = 2\rho \int_0^h u\, dy$$ Assume symmetric profile

Subtract $U_e h$ from both sides and change sign

$$U_e h - U_1 h = \int_0^h (U_e - u)\,dy = \int_0^\delta (U_e - u)\,dy$$ since $\int_\delta^h (U_e - u)\,dy = \int_\delta^h 0 \cdot dy = 0$

$$U_e h - U_1 h = U_e \int_0^\delta \left(1 - \frac{u}{U_e}\right) dy = U_e\, \delta^*$$ ρ = const, 2-D plane

$$\int_0^\delta \left(1 - \frac{u}{U_e}\right) dy = \delta^* \text{ is } \underline{\text{displacement thickness}}$$ δ^*/h is the measure of distortion of velocity problem.

so $$U_1 h = U_e (h - \delta^*) = \text{const.}$$ as $\delta^* \uparrow$ $h - \delta^* \downarrow \Rightarrow U_e \uparrow$

$\left\{ \begin{array}{l} \text{increases} \\ \text{with } x \end{array} \right\}$ as "Blocked" area (δ^*) grows

differentiate w/o x so:

$$0 = dU_e (h - \delta^*) - U_e\, d\delta^*$$

$$\therefore \frac{dU_e}{U_e} = \frac{d\delta^*}{(h - \delta^*)}$$

in inviscid core $p + \frac{\rho}{2}V_e^2 =$ const

from Bernoulli Eq. which is valid along center line where viscous effects are nil. ($p(x)$ only assumed)

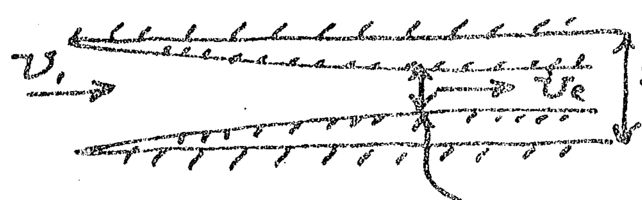
since $\delta/x \ll 1$ ∴ streamlines remain almost parallel and hence by Euler's N eq. $\frac{dp}{dN} \approx 0$ hence $p \neq p(y)$)

$$\frac{dp}{dx} = -\rho V_e \frac{dV_e}{dx}$$

$$\therefore \quad \frac{dp}{dx} = -\frac{\rho V_e}{(h-\delta^*)}\frac{d\delta^*}{dx}$$

viscous/inviscid interaction eqn.

Now as we shall see, the growth of δ^* (or δ) depends on $p(x)$ and from above $p(x)$ depends on δ^* so there is viscous/inviscid interaction!



REAL AREA

$2(h-\delta^*)$ Effective Area

$2\delta^*$ Blocked Area

We need to compute both at same time in many real problems.

Lecture No 2

II. Basic Equations

A.) Mass Conservation → Continuity
Momentum Theorem → Navier-Stokes

B.) Others:
Energy < Mechanical / Thermal
Second Law Thermo → Dissipative Function
Turbulent Eqs of Motion
Vorticity Eq's
Constitutive Eqns for σ_{ij} ; q_i
Eq's of State for Thermo. Pure Substance.

(a.) Some Introductory Review and Nomenclature
(see Mini Course Notes Pages 11, 12, 13, 14 ref.)

Kinematics (subscript notation)

Velocity of a fluid Particle: $\vec{V}(\vec{x}, t)$

$$\vec{V} = \vec{i}\,u + \vec{j}\,v + \vec{k}\,w$$

or $\vec{V} = \vec{i}_i u_i \Longrightarrow u_i \qquad (i = 1, 2, 3)$

just in subscript form

Elementary Cartesian Coordinates

Acceleration $\vec{a}(\vec{x}, t) = \dfrac{D\vec{V}}{Dt} = \vec{i}_i\, a_i$

$$a_i = \frac{Du_i}{Dt} = \underbrace{\frac{\partial u_i}{\partial t}}_{\text{temporal}} + \underbrace{u_j \frac{\partial u_i}{\partial x_j}}_{\text{convective}}$$

this is only Cartesian representation

for $i = 1$, $u_i = u_1 = u$; $u_2 = v$, $u_3 = w$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

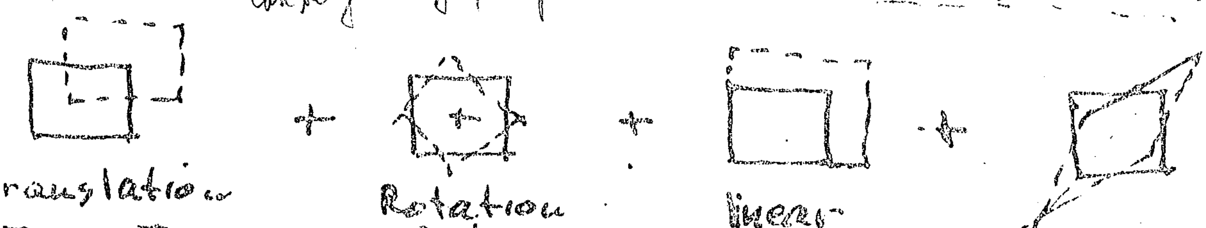
Rate of change of any scalar Property, .. Q following a fluid particle.

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \underbrace{u_j \frac{\partial Q}{\partial x_j}}_{\text{again } \Sigma \text{ convention}}$$

Strain Rates for a small fluid element

Can be given by 4 components

STRAIN RATES



Translation — obtained from $\vec{V}$, $D\vec{V}/Dt$

Rotation Rate $\vec{\Omega} = \frac{1}{2}\vec{\omega}$, $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{V}$

linear extension — Sum of 3 coord gives the dilatation $\Theta = \frac{\partial u_i}{\partial x_i}$

Shear (Angular distortion)

GENERAL FLUID MOTION

translation is created by forces on the mass center ($\underset{\sim}{F} = m\underset{\sim}{a}$)

Rotation rate $\vec{\Omega}$ ≡ average rate of rotation of .. line elements

created by torques (ie shear stress)

Vorticity $2\vec{\Omega} = \vec{\omega} = \text{CURL}\,\vec{V} = \vec{\nabla} \times \vec{V}$

In cart. subscript notation: $\omega_i = \epsilon_{ijk}\left(\frac{\partial u_k}{\partial x_j}\right)$ — Double Contraction on j and k.

cyclic unit tensor

$\epsilon_{ijk} = (0, +1 \text{ or } -1)$ — +1: cyclic, −1: anticyclic

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note that if $\underset{\sim}{\omega} = 0 \Rightarrow$ no shear stress. Irrot flow $\nabla \times \underline{v} = 0$

<u>Extra</u>

<u>Note</u>: <u>general gradient tensor</u>

$\dfrac{\partial u_i}{\partial x_j}$ (9 components <u>not</u> symmetric)

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}$$

ε_{ij} ω_{ij}

$\frac{1}{2}(\nabla u + u\nabla) + \frac{1}{2}(\nabla u - u\nabla)$

$= \dfrac{\partial u_j}{\partial x_i} + \dfrac{\partial u_i}{\partial x_j}$ $\left(\dfrac{\partial u_j}{\partial x_i} - \dfrac{\partial u_i}{\partial x_j}\right)$

any component of rate of strain (ϵ_{ij}) tensor

any related vorticity component ω_{ij}

$\phi_{ij} = \varepsilon_{ij} + \omega_{ij} +$ <u>2nd order terms</u>

stretch tensor.

– symmetric

– nonsymmetric

– zero diagonal

– all fluid element distortion, gives stresses

– fluid rotation gives no stresses.

$e = \frac{dq}{dt} \cdot \frac{1}{q}$ strainrate

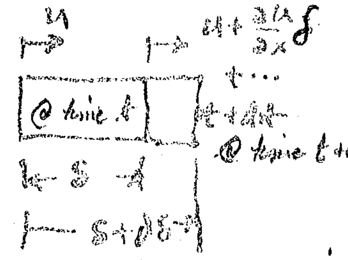
Strain Rate: $\left[\dfrac{\text{Rate of Change of Quant. (e.g. length)}}{\text{Original Quantity}}\right]$ units $\left(\dfrac{6.}{(s^{-1})}\right)$

Extensional Strain Rates (ordinary Cartesian coordinates x, y, z) Quantity: line element

x-dir: $e_{xx} = \partial u / \partial x$

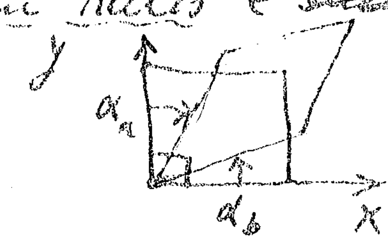
y-dir: $e_{yy} = \partial v / \partial y$

z-dir: $e_{zz} = \partial w / \partial z$



Shear (Angular) strain rates

Quantity:



$(\alpha_a + \alpha_b)$ is change from $90°$

along z-axis (x-y plane): $e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$

along x-axis: $e_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$

along y-axis: $e_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$

Strain Rate tensor (matrix)

$$e_{ij} = \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix}$$

$e_{xz} = e_{zx}$ (example)

← Symmetric about diagonal

Representation in compact notation

$$\boxed{e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

Note e_{ij} sometimes called S_{ij} (Kundu-Cohen notation)

Volumetric strain Rate – Dilatation

$$\Theta = \frac{\text{Rate of change of particle volume}}{\text{Original volume}}$$

original vol $= dx\,dy\,dz = 1$ (arbitrary) $dx = 1$, etc.

Change of vol. = final vol − initial vol.

" $= \left(1 + \frac{\partial u}{\partial x}\cdot 1 \cdot dt\right)\left(1 + \frac{\partial v}{\partial y}\cdot 1 \cdot dt\right)\left(1 + \frac{\partial w}{\partial z}\cdot 1 \cdot dt\right) - 1$

" $= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dt + O(dt^2)$

$$\frac{\text{Rate of Change}}{\text{Original}} = \Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\boxed{\Theta = \frac{\partial u_i}{\partial x_i}}$$

Gradients and Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{" } i = j \end{cases}$$

in matrix form $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

gradient of a scalar, Q:

$$\vec{\nabla} Q = \vec{i}\,\frac{\partial Q}{\partial x} + \vec{j}\,\frac{\partial Q}{\partial y} + \vec{k}\,\frac{\partial Q}{\partial z}$$

In last subs. notation $\frac{\partial Q}{\partial x_i}$ so $\delta_{ij}\frac{\partial Q}{\partial x_i} = \frac{\partial Q}{\partial x_j}$

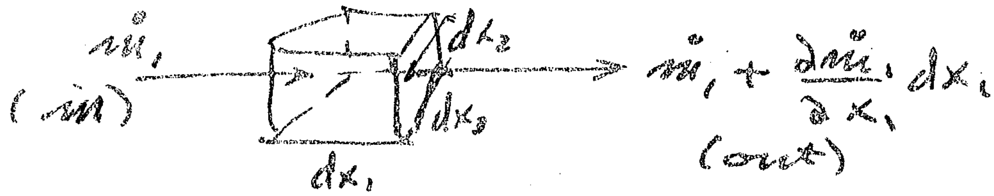
Sometimes call substitution tensor → ← [illegible]

Lecture No. 3

(B.) Mass Conservation: $\overset{\text{rate of creation}}{ROC(m)} = 0$

storage + flow rates = 0

for an elementary c.v. Mass Flow:



$\dot{m}_1 = \rho u_1 \, dx_2 dx_3$

$\dot{m}_2 = \rho u_2 \, dx_3 dx_1$

$\dot{m}_3 = \rho u_3 \, dx_1 dx_2$

what flows in + what's stored = what goes out

$\frac{\partial}{\partial t}\int \rho dV + \int \rho \underset{\sim}{q} \cdot \underset{\sim}{n} \, dA = 0$

Storage: $\frac{\partial}{\partial t}(\rho)\, dx_1 dx_2 dx_3 =$

So: $\frac{\partial \rho}{\partial t} dv + \sum_{i=1,2,3} (\dot{m}_i)_{out} - \sum_{i=1,2,3} (\dot{m}_i)_{in} = 0$

On a per unit vol. basis:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0} \quad (8-1)$$

Split convection term:

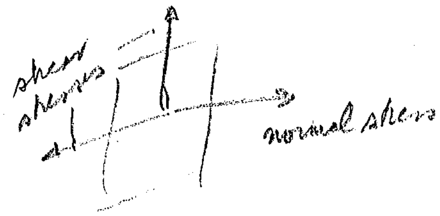
$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial x_i} + u_i \frac{\partial \rho}{\partial x_i} = 0$$

$$\boxed{\frac{D\rho}{Dt} + \rho\, \Theta} \quad (8-2) \quad \text{or} \quad \frac{D\rho}{Dt} + \rho \nabla \cdot q = 0$$

Constant ρ case:

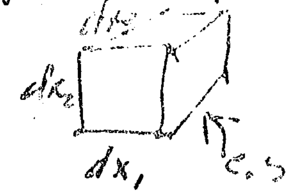
$\nabla \cdot \underset{\sim}{q} = 0$

$$\boxed{\Theta = \frac{\partial u_i}{\partial x_i} = \operatorname{div} \vec{V} = 0}$$


shear stress
normal stress

mom in + mom stored = moment + Σ forces on the body

(C) <u>Momentum theorem</u> (linear) <u>per particle</u>



$$\Sigma F_i = \text{ROC}(\text{Mom}\ i)$$

mass·vel = density·velo·vol.

<u>Storage</u> in 1-direction $\dfrac{\partial(\rho u_1)}{\partial t}\,dx_1 dx_2 dx_3$

in i-dir: $\dfrac{\partial(\rho u_i)}{\partial t}\,dV$

<u>Net Outflow</u>

$\int \frac{\partial}{\partial t}(\rho \underset{\sim}{q})\,dV + \int_A \rho \underset{\sim}{q}(\underset{\sim}{q}\cdot n)\,dA = \Sigma F$ in 1 dir.: $\dfrac{\partial}{\partial x_j}(\underbrace{\rho u_j}_{\text{3 flows}}\ \underbrace{u_1}_{\text{mom/unit mass}})$

or $\int \left(\frac{\partial}{\partial t}(\rho \underset{\sim}{q}) + \rho \underset{\sim}{q}(\nabla\cdot \underset{\sim}{q})\right) dV = \Sigma F$ in i-dir: $\dfrac{\partial}{\partial x_j}(\rho u_j u_i)$

So on a <u>unit</u> volume basis:

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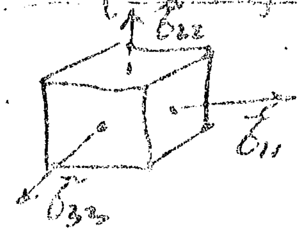
$$\boxed{\underbrace{\rho f_{\text{body } i}}_{\text{body force/unit vol.}}^{\text{or per unit mass}} + f_{\text{surface } i} = \frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_j u_i)}{\partial x_j}} \quad (A\text{-}1)$$

<u>Body forces</u> (gravity, etc.) ~~give~~ give as $\boxed{\rho f_i}$

<u>Surface forces</u>:



$$\left.\begin{aligned} -\bar{p} + \underbrace{\sigma_{11}}_{\text{viscous}} &= \tilde{\sigma}_{11} \\ -\bar{p} + \sigma_{22} &= \tilde{\sigma}_{22} \\ -\bar{p} + \sigma_{33} &= \tilde{\sigma}_{33} \end{aligned}\right\} \text{Normal Stresses}$$

$$-3\bar{p} + (\sigma_{11} + \sigma_{22} + \sigma_{33}) = \tilde{\sigma}_{11} + \tilde{\sigma}_{22} + \tilde{\sigma}_{33}$$

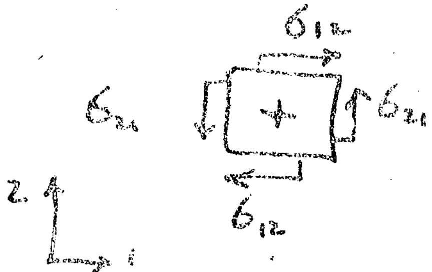
pressure: defined as (− mean normal ~~stress~~ pressure) we use in incompressible cases. Reduces to p at rest (thermo. pressure) total normal

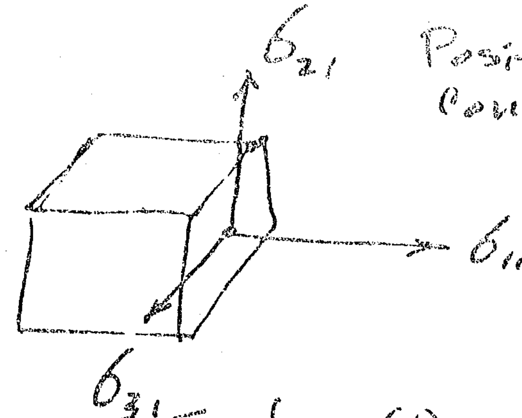
$$-\bar{P} = \frac{1}{3}\left(\bar{\sigma}_{ii} - \sigma_{ii}\right)$$

viscous effects.

<u>handouts</u> <u>Friday</u> 12 Jan. Tutorial list/ HW#1

Shear Stresses





Positive convention

$\sigma_{31} \leftarrow$ face (1)
$\uparrow$ direction (3)

$\sigma_{12} = \sigma_{21}$
due to balance of moments. σ_{11} and σ_{22}

Shear stresses symmetry

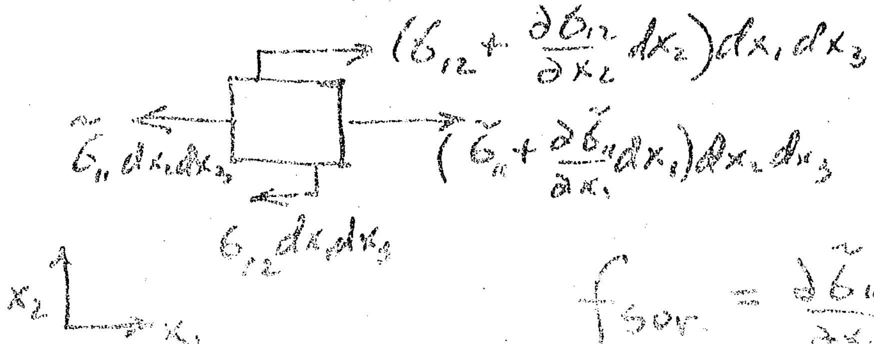
$$\sigma_{12} = \sigma_{21}$$
$$\sigma_{23} = \sigma_{32}$$
$$\sigma_{31} = \sigma_{13}$$

Stress Tensor

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \leftarrow \text{shear forces 1-dir}$$

$\uparrow$ gives force components (explicit $\bar{F}$) on 1 face (see sketch above)

Net Surface Forces (take 1-dir)



note

$$(\tilde{\sigma}_{11} = -\bar{p} + \sigma_{11})$$

$$f_{sur_1} = \frac{\partial \tilde{\sigma}_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}$$

$$\underbrace{\quad\quad\quad\quad}_{}(\partial \sigma_{ij}/\partial x_j)$$

or $$f_{sur_1} = -\frac{\partial \bar{p}}{\partial x_1} + \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}$$

now for i-dir substitute i for 1 and plug in momentum theorem Eq (9-1) and Σ

$$\boxed{\frac{\partial \rho u_i}{\partial t} + \frac{\partial(\rho u_j u_i)}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i} \quad (11-1)$$

Expand second term on r.h.s. + apply (8-1) (mass conserved)

$$\underbrace{\frac{\partial \rho u_i}{\partial t}}_{\rho\frac{\partial u_i}{\partial t} + \cancel{u_i\frac{\partial \rho}{\partial t}}} + \rho u_j \frac{\partial u_i}{\partial x_j} + \underbrace{\cancel{u_i\frac{\partial(\rho u_j)}{\partial x_j}}}_{u_i\left(\cancel{-\frac{\partial \rho}{\partial t}}\right)} = \text{〃} + \text{〃} + \text{〃}$$

$$\rho\frac{Du_i}{Dt} = \rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \text{〃} + \text{〃} + \text{〃}$$

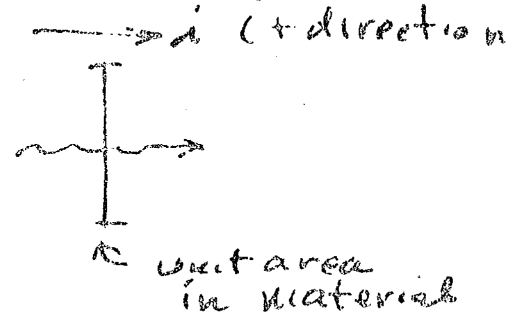
divide by ρ:

$$\boxed{\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} + f_i} \quad (11-2)$$

these are the <u>EQUATIONS OF MOTION</u>

Constitutive Equations – in an isotropic material (usual for fluid)

1. Heat flux vector q_i



Fourier's law

$$\boxed{\begin{aligned} q_i &= -k \frac{\partial T}{\partial x_i} \\ \text{or} \quad \vec{q} &= -k \vec{\nabla} T \end{aligned}} \qquad (12-1)$$

k is scalar property
discuss physical basis!

2. Stress tensor: σ_{ij} for Newtonian fluid

Assume:

1. fluid continuous (mean free path << character length of problem) σ_{ij} related linearly to ϵ_{ij} (no higher powers) since fluids deform continuously until load is removed. [Unlike Hooke's law]

   $\sigma_{ij} = \sigma_{ij}\left(\frac{\partial u_i}{\partial x_j}\right)$ hypothesis

2. fluid isotropic – results independent of coordinate type and direction of orientation to some arbitrary axis

3. When fluid at rest, $e_{ij} = 0$ result reduces to Thermo. pressure p

   $\tilde{\sigma}_{ij} = -p\delta_{ij} + \sigma_{ij}$

the p we used before is thus p, the thermo pressure

i.e. $$\boxed{\tilde{\sigma}_{ij} = -p\,\delta_{ij} + \sigma_{ij}} \qquad (13\text{-}1)$$

reduces to $\tilde{\sigma}_{11} = \tilde{\sigma}_{22} = \tilde{\sigma}_{33} = -p$

when $i = j$ and fluid at rest!

<u>Condition 1 means</u>; for example

$$\sigma_{11} = c_1 \epsilon_{11} + c_2 \epsilon_{22} + c_3 \epsilon_{33} + c_4 \epsilon_{12} + c_5 \epsilon_{23} + c_6 \epsilon_{31}$$

$$\sigma_{22} = c_7 \epsilon_{11} + c_8 \epsilon_{22} + \cdots$$

$$\vdots$$

$$\sigma_{31} = \cdots$$

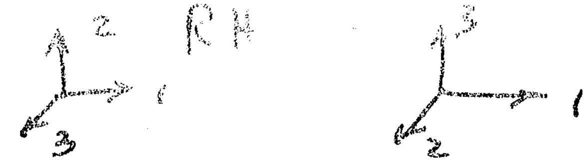
Matrix of coefficients is $6 \times 6 = 36$ in number! These are material properties

<u>Condition 2 (isotropy)</u> saves the day! since one may show that <u>principal axes</u> of σ_{ij} and ϵ_{ij} **coincide** dumps all but 9 coefs, <u>i.e. working</u> <u>in principal axes system</u> (12, 23, 31 = 0) one obtains 9 possible coefs

Note (2)

$$\begin{array}{lll} & \text{ALONG} & \text{ACROSS} \\ \tilde{\sigma}_{11} + p = \sigma_{11} = & C_1 \epsilon_{11} + & C_2 \epsilon_{22} + C_3 \epsilon_{33} \\ \tilde{\sigma}_{22} + p = \sigma_{22} = & C_4 \epsilon_{22} + & C_5 \epsilon_{33} + C_6 \epsilon_{11} \\ \tilde{\sigma}_{33} + p = \sigma_{33} = & C_7 \epsilon_{33} + & C_8 \epsilon_{11} + C_9 \epsilon_{22} \end{array}$$

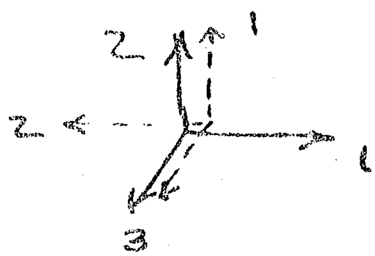
RH LH



Note ① Isotropy says we can have same result with either LH or RH coordinates so

$$C_2 = C_3 \Rightarrow B_1$$
$$C_5 = C_6 \Rightarrow B_2$$
$$C_8 = C_9 \Rightarrow B_3$$

Note ② We could choose any axis as 1, 2 or 3 by rotation of coordinates, and this shouldn't change values of constants in rows



so $C_1 = C_4 = C_7 = B_1$

__all other__ C's $= B_2$

Now: __let__ $B_2 = B$

$B_1 = A + B$

__so__ (still in PA system):

$$\tilde{\sigma}_{11} + p = \sigma_{11} = (A+B)\epsilon_{11} + B(\epsilon_{22} + \epsilon_{33})$$

$$\tilde{\sigma}_{22} + p = \sigma_{22} = (A+B)\epsilon_{22} + B(\epsilon_{33} + \epsilon_{11})$$

$$\tilde{\sigma}_{33} + p = \sigma_{33} = (A+B)\epsilon_{33} + B(\epsilon_{11} + \epsilon_{22})$$

__or__ (in PA system) ($j = i$ only)

$$\sigma_{ij} = + A\epsilon_{ij} + B(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})$$

Pick up notes from prof Reynolds

fluid $\sigma_{ij} = A\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij} B\left(\frac{\partial u_m}{\partial x_m}\right)$

$A = 2\mu \,;\; B = \lambda = \left(\beta - \frac{2}{3}\mu\right)$

for $\rho = \text{const} \rightarrow \frac{\partial u_m}{\partial x_m} = 0$ (from continuity) $\therefore$ we don't need to know B, λ, β

for $\rho \neq \text{const}$

Now $(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \epsilon_{ii} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i}\right)$

$(\quad " \quad) = \operatorname{div} \vec{V} = \Theta$

$$\therefore \quad \sigma_{ij} = A\epsilon_{ij} + B\theta \qquad (15\text{-}1)$$

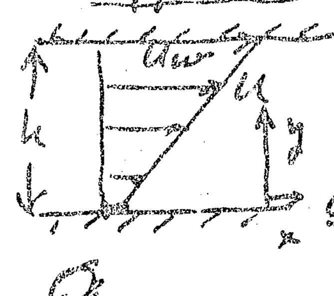
(in PA system)

Now Rotate coordinates from PA → any axis system

$$\boxed{\sigma_{ij} = A\epsilon_{ij} + B\Theta\delta_{ij}} \qquad (15\text{-}2)$$

δ_{ij} must be added because Θ arise ~~those~~ terms can only arise in normal stresses (consider spherical expansions)

Consider experiment which defines μ



$\sigma_{xy} = \mu \frac{du}{dy} = \mu \frac{U_w}{h}$ v comp = 0 since wall travels too

$\sigma_{xy} = A\,\frac{1}{2}\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right) \Rightarrow \frac{A}{2}\frac{du}{dy}$

$\therefore \boxed{A = 2\mu}$ $\frac{MT}{L}$ units

Consider the mean normal (from eq (13-1))

$$\frac{1}{3}(\tilde{\sigma}_{11} + \tilde{\sigma}_{22} + \tilde{\sigma}_{33}) = -p + \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

$$\frac{1}{3}\tilde{\sigma}_{ii} = -p + \frac{1}{3}\sigma_{ii}$$

$$\frac{1}{3}\tilde{\sigma}_{ii} = -p + \frac{2\mu}{3}\epsilon_{ii} + \frac{3B\Theta}{3}$$

Stokes hypothesis $\lambda = -\frac{2}{3}\mu$ or $\beta = 0$

$$\bar{p} = -\frac{1}{3}\tilde{\sigma}_{ii} = -p$$

but $\sigma_{ii} = 2\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2\mu}{3}\left(\frac{\partial u_m}{\partial x_m}\right)$

<u>normal stresses</u> have viscous effect.

Moment eq $\rho \frac{Du_i}{Dt} = \frac{\partial \tilde{\sigma}_{ij}}{\partial x_j} + \rho f_i$

$$\tilde{\sigma}_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda\left(\frac{\partial u_m}{\partial x_m}\right)$$

now $\epsilon_{ii} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i}\right) = \frac{\partial u_i}{\partial x_i} = \theta$

$$\therefore \boxed{\frac{\tilde{\sigma}_{ii}}{3} = -p + \underbrace{\left(B + \frac{2\mu}{3}\right)\theta}_{\text{Deviation of mean normal stress from thermo pressure}}}$$

$\boxed{B \equiv \lambda}$ is called second Coeff of viscosity.

By $\boxed{\text{"Stokes hypothesis" } \lambda = -\tfrac{2}{3}\mu}$

so $\frac{1}{3}\sigma_{ii} = -p$

note: "Bulk viscosity" in Book

$$\beta = \lambda + \frac{2\mu}{3} \left(\begin{matrix}\text{which}\\ =0\end{matrix}\right) \text{ in}$$

Stokes Hypothesis

As a result from eq (15-2)

$$\boxed{\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\theta\delta_{ij}} \quad (16-1)$$

Noting that $\bar{p} = p$ eq (11-2) becomes

press force, viscous shear stress, dilatation

$$\boxed{\frac{Du_i}{Dt} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \frac{1}{\rho}\frac{\partial}{\partial x_i}(\lambda\theta)}$$

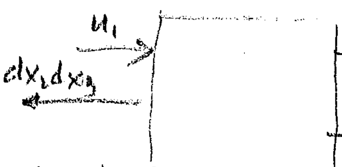
NAVIER-STOKES EQUATIONS (16-2)

Total energy = mech. energy + thermal energy
can be split (if ρ = const) completely.

second law of thermo.

$\vec{\hat{f}} \cdot \vec{r}$ = work of body force

$\frac{\text{work done}}{\text{time}}$ = power $\quad \frac{D}{Dt}\left[\vec{\hat{f}} \cdot \vec{r}\right] = \vec{\hat{f}} \cdot \frac{D\vec{r}}{Dt} = \vec{\hat{f}} \cdot \vec{u} = \hat{f}_i u_i$



u_1

$f = \tilde{\sigma}_{11}\, dx_2 dx_3$

$u_1 + \frac{\partial u_1}{\partial x_1} dx_1$

$\left(\tilde{\sigma}_{11} + \frac{\partial \tilde{\sigma}_{11}}{\partial x_1} dx_1\right) dx_2 dx_3$

$f \cdot \frac{dr}{dt} = |f||u_1| \cos(u,f)$

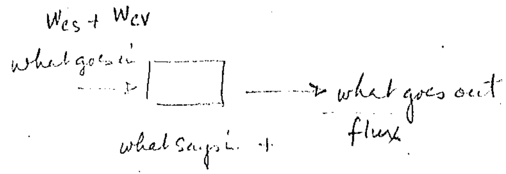
$= -\tilde{\sigma}_{11} u_1\, dx_2 dx_3$ $\quad \cos\pi$

work = $f \cdot dr = |u||f| \cos(u,f)$

$\tilde{\sigma}_{11} u_1 + \frac{\partial}{\partial x_1}(\tilde{\sigma}_{11} u_1)\, dx_1 dx_2 dx_3$ work

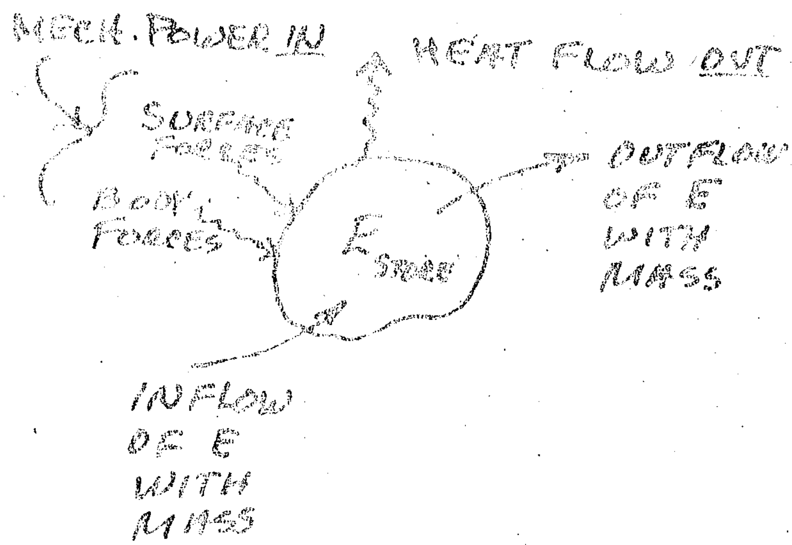
net outflow = + $= \frac{\partial}{\partial x_1}(\tilde{\sigma}_{11} u_1)\, dx_1 dx_2 dx_3$

net inflow = − net outflow $= -\frac{\partial}{\partial x_1}(\tilde{\sigma}_{11} u_1)\, dx_1 dx_2 dx_3$



THE ENERGY EQ's

Conservation of Energy: $\boxed{ROC(E) = 0}$



WHERE

$$E_{STOR} = \int_{CV} e\,\rho\,dV$$

Specific energy:

$$e = \tilde{u} + \frac{V^2}{2} +$$

↑ internal thermal

$$\tilde{u} = \left(h - \frac{1}{\rho}p\right)$$

TOTAL ENERGY EQ for fixed c.s. $dV = 1$:

$$0 = \begin{pmatrix}\text{STORAGE}\\ \text{RATE}\end{pmatrix} + \begin{pmatrix}\text{NET E}\\ \text{OUTFLOW}\end{pmatrix} + \begin{pmatrix}\text{NET HEAT}\\ \text{OUTFLOW}\end{pmatrix} - \begin{pmatrix}\text{NET INFLOW}\\ \text{OF MECH. POWER}\end{pmatrix}$$

mass flow + energy

$$\boxed{0 = \frac{\partial(\rho e)}{\partial t} + \underbrace{\frac{\partial(\rho u_j e)}{\partial x_j}}_{\text{energy flux}} + \frac{\partial q_j}{\partial x_j} - \left[\underbrace{\frac{\partial(u_i \tilde{\sigma}_{ij})}{\partial x_j}}_{\text{total surface forces}} + \underbrace{f_i u_i}_{\text{Body Forces}}\right]} \quad (17\text{-}1)$$

e = energy/mass

E = energy/vol.

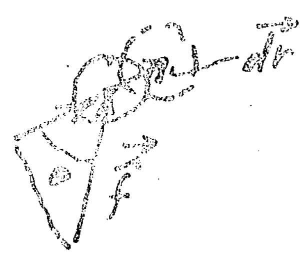
work by fluid is +

total surface forces: $\tilde{\sigma}_{ij} = -\delta_{ij}p + \sigma_{ij}$

Body Forces: $f_i = \rho g_i$

(1.) Look at Body Force Work $\hat{f}_i u_i$ term:

displace mass particle $d\vec{r}$ with $\vec{f}_i$ work done is:



$$f\cos\theta\, dr = \vec{f}\cdot d\vec{r}$$

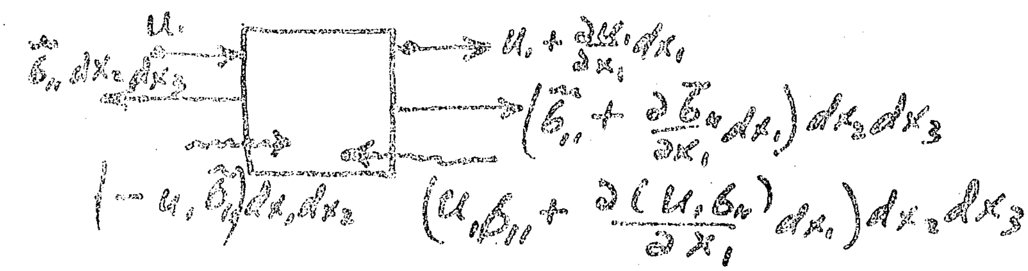
Power applied (inflow)

$$\vec{f}\cdot\frac{d\vec{r}}{dt} = \vec{f}\cdot\vec{V} = f_i u_i$$

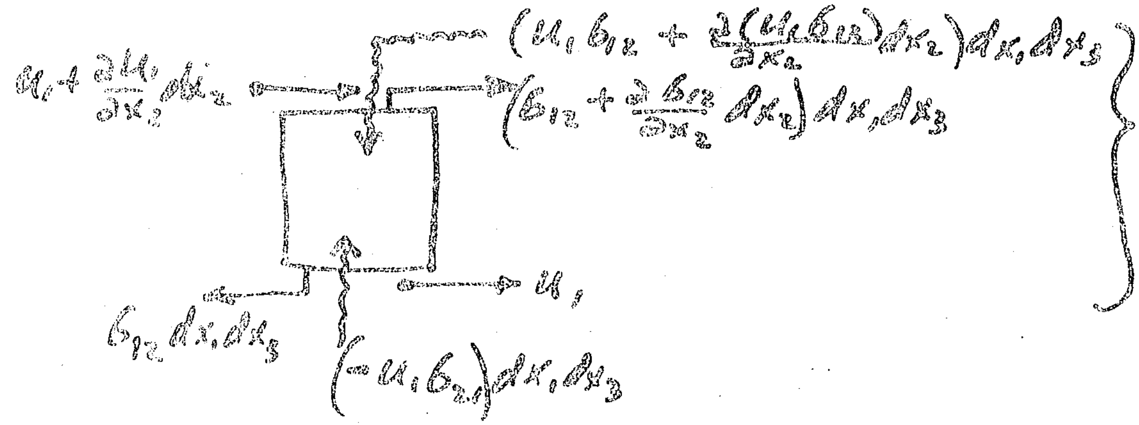
so per unit volume: $\boxed{\hat{f}_i u_i}$

(2.) Look at Rate at which work done on a C.S.

Take forces in 1-dir ($\longrightarrow$ work on element) ($\longrightarrow$ displacement rate)



NET/$d\tau$: $\dfrac{\partial(u_1\tilde{\sigma}_{11})}{\partial x_1}$



$\dfrac{\partial(u_1\hat{\sigma}_{12})}{\partial x_2}$

AND SIMILARLY FOR σ_{13} $\quad\Big\}\ \dfrac{\partial(u_1\tilde{\sigma}_{13})}{\partial x_3}$

MECHANICAL ENERGY EQ

An equation for calculation of $(V^2/2)$

$$\boxed{K.E. = \tfrac{1}{2}V^2 = \tfrac{1}{2}u_i u_i}$$

Obtain by multiplying eq. (11-2) (see p. 2f), where $\tilde{\sigma}_{ij} = -\delta_{ij}p + \sigma_{ij}$, by ρu_i and Σing:

$$\rho u_i \frac{Du_i}{Dt} = u_i \frac{\partial \tilde{\sigma}_{ij}}{\partial x_j} + u_i f_i \rho$$

Note: $u_i du_i = d(u_i^2/2) = d(V^2/2)$

so:
$$\boxed{\rho \frac{D}{Dt}\left(V^2/2\right) = u_i \frac{\partial \tilde{\sigma}_{ij}}{\partial x_j} + u_i \rho f_i} \quad (19-2)$$

Also using continuity one can obtain $\frac{D\rho}{Dt} = 0$ from cont. $\Rightarrow V^2 D\rho/Dt = 0$ hence $V^2 D\rho/Dt + \rho \frac{D}{Dt}(V^2/2) = \frac{D}{Dt}(\rho V^2/2)$

$$\boxed{\frac{\partial(\rho V^2/2)}{\partial t} + \frac{\partial\left(\rho u_j \frac{V^2}{2}\right)}{\partial x_j} = u_i \frac{\partial \tilde{\sigma}_{ij}}{\partial x_j} + u_i f_i} \quad (19-3)$$

$$\begin{bmatrix}\text{Rate of}\\ \text{Storage}\\ \text{of K.E.}\end{bmatrix} + \begin{bmatrix}\text{Net Rate}\\ \text{of out-}\\ \text{flow of K.E.}\end{bmatrix} = \underbrace{\begin{bmatrix}\text{Rate of}\\ \text{work by}\\ \text{surface}\\ \text{stresses}\end{bmatrix}}_{} + \begin{bmatrix}\text{Body}\\ \text{force}\\ \text{work}\\ \text{Rate}\end{bmatrix}$$

These are, potentially, the reversible parts of total work, $\frac{\partial(u_i \tilde{\sigma}_{ij})}{\partial x_j}$.

the other part is "Dissipation" $\Phi = \tilde{\sigma}_{ij}\frac{\partial u_i}{\partial x_j}$

(see p. 22).

Total energy $\left(e = \tilde{u} + V^2/2\right)$ internal (thermal energy)

heat transfer — work transfer — transfer due to viscous — pot energy due to body force

$$\rho \frac{De}{Dt} = -\frac{\partial q_j}{\partial x_j} - \frac{\partial (u_j p)}{\partial x_j} + \frac{\partial (u_i \sigma_{ij})}{\partial x_j} + \rho f_i u_i \quad (1)$$

KE (mechanical) $\frac{d}{dt}$ KE of flowing fluid

$$\rho \frac{D}{Dt}\left(\frac{V^2}{2}\right) = -u_j \frac{\partial p}{\partial x_j} + u_i \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i u_i \quad (2)$$

flow work — reversible

Thermal (internal E $= \tilde{u}$)

$$\rho \frac{D\tilde{u}}{Dt} = -\frac{\partial q_j}{\partial x_j} - p\left(\frac{\partial u_j}{\partial x_j}\right) + \sigma_{ij}\frac{\partial u_i}{\partial x_j} \quad (3) = (1)-(2)$$

pdV work — reversible — dissipation fn. Φ

THERMAL ENERGY EQ

(Total Eq.) − (Mechanical Eq.)

(17-1) − (19-2)

WHERE : $e = \tilde{u} + V^2/2$

gravity field is in the work term.

$$\frac{\partial \rho e}{\partial t} = \frac{\partial \rho \tilde{u}}{\partial t} + \frac{\partial \rho V^2/2}{\partial t}, \quad \frac{\partial}{\partial x_j}(\rho u_j e) = \frac{\partial}{\partial x_j}(\rho u_j \tilde{u}) + \frac{\partial}{\partial x_j}\left(\rho u_j \frac{V^2}{2}\right)$$

but $\frac{D}{Dt}\left(\rho V^2/2\right) = u_i \frac{\partial \sigma_{ij}}{\partial x_j} + f_i u_i$ giving

$$\underbrace{\frac{\partial(\rho \tilde{u})}{\partial t} + \frac{\partial(\rho u_j \tilde{u})}{\partial x_j}}_{\frac{D}{Dt}\rho\tilde{u} = \rho\frac{D\tilde{u}}{Dt} + \tilde{u}\frac{D\rho}{Dt}^{\,0}} = -\frac{\partial q_j}{\partial x_j} + \sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

$$\rightarrow \rho \frac{D\tilde{u}}{Dt} = -\frac{\partial q_j}{\partial x_j} + \delta_{ij}(-p)\frac{\partial u_i}{\partial x_j} + \tau_{ij}\frac{\partial u_i}{\partial x_j}$$

So:

$$\boxed{\rho \frac{D\tilde{u}}{Dt} = \left(-\frac{\partial q_j}{\partial x_j}\right) + \left(-p\Theta\right) + \tau_{ij}\frac{\partial u_i}{\partial x_j}} \quad (20\text{-}1)$$

[Rate of increase of internal thermal energy] = [Rate of heat transfer into the particle] + [Work of Compression (−pdV)] + ["Dissipation" by Viscous stress]

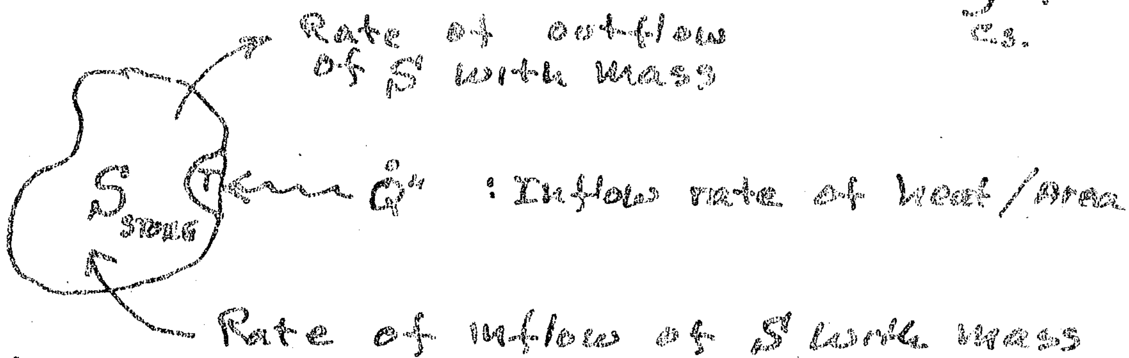
Work of compression: = 0 for ρ = const.

another form ($h = \tilde{u} + p/\rho$):

$$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} - \frac{\partial q_j}{\partial x_j} + \tau_{ij}\frac{\partial u_i}{\partial x_j} \quad (20\text{-}2)$$

also see eq (59), p. 21 of minicourse notes

SECOND LAW OF THERMO: $\mathrm{ROC}(S) \geq \int_{c.s.} \frac{\dot{Q}''}{T} dA$



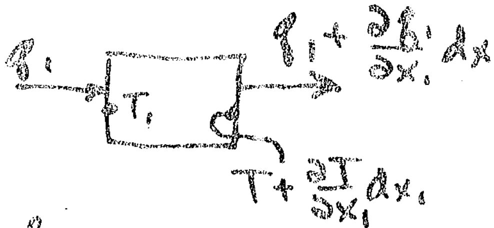
$\boxed{S = \rho s}$

For a control vol. of unit volume, $dV = 1$:

$$\boxed{\underbrace{\frac{\partial(\rho s)}{\partial t}}_{\substack{\text{Storage}\\\text{rate of}\\ S}} + \underbrace{\frac{\partial(\rho u_j s)}{\partial x_j}}_{\substack{\text{Net out-}\\\text{flow rate}\\\text{of } S}} = \underbrace{-\frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right)}_{\substack{\text{Reversible}\\\text{Creation}\\\text{Rate}}} + \underbrace{\mathcal{P}}_{\substack{\text{Irreversible}\\\text{Production}}}} \quad (2.1-1)$$

$\boxed{\mathcal{P} \geq 0}$

Look at Reversible Creation term:



$$\left[\frac{q_1}{T} - \frac{q_1 + \frac{\partial q_1}{\partial x_1} dx_1}{T + \frac{\partial T}{\partial x_1} dx_1}\right] dx_2 dx_3$$

for side pair 1 in the limit becomes

$$-\frac{\partial}{\partial x_1}\left(\frac{q_1}{T}\right) dV$$

$$\rho \frac{Ds}{Dt} = -\frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right) + \mathcal{P}$$

suppose $\rho = \text{const}$; from cont $\frac{\partial u_j}{\partial x_j} = 0 \Rightarrow$ ① $= 0$

or $\quad -\frac{q_j}{T^2}\frac{\partial T}{\partial x_j} \geq 0 \quad \Rightarrow$ if $-q_j \geq 0 \Rightarrow \frac{\partial T}{\partial x_j} \geq 0$

if isothermal $T = \text{const} \Rightarrow$ ② $= 0 \quad \Rightarrow \quad \frac{\sigma_{ij}}{T}\frac{\partial u_i}{\partial x_j}$

for $\Phi \geq 0 \Rightarrow \mu \geq 0$ and $(3\lambda + 2\mu) \geq 0$ $\qquad$ (remember stokes hypothesis $3\lambda + 2\mu = 0$)

USE (21-1) TO EVALUATE $\wp$

$$\wp = \rho \frac{Ds}{Dt} + s\left(\cancel{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j}}\right) + \frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right)$$

USE GIBBS' EQ: $\boxed{T\,ds = dh - \frac{1}{\rho}dp}$ (22-1)

this assumes $s = s(h, p)$

Simple compressible substance

in the form:

$$T\frac{Ds}{Dt} = \left(\frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}\right)$$

and Thermal Energy Eq (20-2) to obtain

$$\boxed{\wp = \left[\frac{\sigma_{ij}}{T}\frac{\partial u_i}{\partial x_j} + \frac{-q_j}{T^2}\frac{\partial T}{\partial x_j}\right] \geq 0} \quad (22-2)$$

① ②

This equation provides constraints on constitutive Equations and visa-versa

this is general result

Term ① is viscous dissipation term

It should be positive: For Newtonian fluid: $\sigma_{ij} = 2\mu e_{ij} + \delta_{ij}\lambda\theta$

$$\boxed{\Phi = \sigma_{ij}\frac{\partial u_i}{\partial x_j} = 2\mu\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right]\left(\frac{\partial u_i}{\partial x_j}\right) + \lambda\theta^2}$$ is positive (22-3)

Term ② is due to heat transfer and if Fourier's Law holds, i.e. $q_j = -k\frac{\partial T}{\partial x_j}$

$$\left(-q_j\frac{\partial T}{\partial x_j}\right) = +k\left(\frac{\partial T}{\partial x_j}\right)^2$$ is Positive, Heat transfer also gives Irreversibility!

Navier Stokes 7 unknown u_i (3) , $p, \rho, \underbrace{\mu, \lambda}_{\text{Stokes theor reduces to 6}}$

(3 eq)

Added Eqns

Cont (1 eq) $\frac{D\rho}{Dt} + \rho \nabla \cdot \underset{\sim}{u} = 0$

Total Energy $\rho \, De/Dt = \ldots$

adds two unknown $\tilde{u}$ and k (thermal conductivi

Total 9 - unknowns 5 eqns.

need eqns of state (2) $p = p(\tilde{u}, S)$ $\rho = \rho(\tilde{u}, S)$

Stokes hypot (1)

empirical info of μ, λ variation wrt each other

$\mu = \mu(T, p)$ data

ME 251 B 77/78
EQUATION SUMMARY

(1.) KINEMATICS

$$\vec{V} = u_i$$

$$\vec{a} = a_i = \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}$$

Q any scalar property:

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + u_j \frac{\partial Q}{\partial x_j}$$

Rotation and Vorticity:

$$2\vec{\Omega} = \vec{\omega} = \text{curl } \vec{V} = \vec{\nabla} \times \vec{V}$$

$$\omega_i = \epsilon_{ijk} \left(\frac{\partial u_k}{\partial x_j} \right) \quad \left\{ \begin{array}{l} \epsilon_{ijk} = 0 \\ \text{unless } i \neq j \neq k, \\ +1 \text{ if cyclic} \\ -1 \text{ if anticyclic} \end{array} \right.$$

Strain rates:

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Volumetric strain rate (Dilatation):

$$\theta = \frac{\partial u_i}{\partial x_i} = \text{div } \vec{V}$$

Kronecker delta: $\delta_{ij} = \left\{ \begin{array}{l} 0 \text{ when } i \neq j \\ 1 \text{ when } i = j \end{array} \right\}$

so $\delta_{ij} u_i = u_j$

$$\delta_{ij} \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_i}$$

(2.) MASS CONSERVATION :

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0 \qquad (8-1)$$

$$\frac{D\rho}{Dt} = -\rho\Theta \qquad (8-2)$$

(3.) Momentum theorem :

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_j u_i)}{\partial x_j} = \rho f_{body_i} + f_{surface_i}$$

$$\rho \frac{Du_i}{Dt} = \rho f_{body_i} + f_{surface_i} \qquad (9-1)$$

Stress tensor :

$$\tilde{\sigma}_{ij} = -\delta_{ij} p + \underbrace{\sigma_{ij}}_{\text{viscous parts}} \qquad (13-1)$$

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} + \underbrace{f_i}_{\text{(BODY FORCE)}} \qquad (11-2)$$

(4.) CONSTITUTIVE EQ'S:

Conduction heat transfer: $q_i = -k \dfrac{\partial T}{\partial x_i}$ (12-1)

Newtonian viscous fluid:

$$\sigma_{ij} = 2\mu e_{ij} + \lambda \theta \delta_{ij} \qquad (16\text{-}1)$$

STOKES HYPOTHESIS: $\lambda = -2\mu/3$

(5.) NAVIER-STOKES EQUATIONS

$$\frac{Du_i}{Dt} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \frac{1}{\rho}\frac{\partial(\lambda\theta)}{\partial x_i} \qquad (16\text{-}2)$$

special case: ρ = const ($\theta = 0$) and μ = const.

— Constant properties — $\boxed{\mu/\rho \equiv \nu}$

$$\boxed{\frac{Du_i}{Dt} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}} \qquad (25\text{-}1)$$

Simple Compressible Subst: $\vec{u} = (p, \rho)$, etc

$$T\frac{Ds}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} \qquad (22\text{-}1)$$

(6.) ENERGY EQUATIONS:

Total Energy is ($e = \hat{u} + V^2/2$)

$$\rho \frac{De}{Dt} = -\frac{\partial q_j}{\partial x_j} \overset{-}{\cancel{+}} \frac{\partial (u_j p)}{\partial x_j} + \frac{\partial (u_i \sigma_{ij})}{\partial x_j} + \rho f_i u_i \quad (17\text{-}1)$$

or for: ($h_0 = h + V^2/2$), stagnation enthalpy

$$\boxed{\rho \frac{Dh_0}{Dt} = \frac{\partial p}{\partial t} - \frac{\partial q_j}{\partial x_j} \overset{-}{\cancel{+}} \frac{\partial (u_j p)}{\partial x_j} + \frac{\partial (u_i \sigma_{ij})}{\partial x_j} + \rho f_i u_i} \quad (26\text{-}1)$$

(7.) MECHANICAL ENERGY: ($V^2/2$)

$$\rho \frac{D}{Dt}\left(\frac{V^2}{2}\right) = -u_j \frac{\partial p}{\partial x_j} + u_i \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i u_i \quad (19\text{-}2)$$

(8.) THERMAL ENERGY: ($\hat{u}$)

$$\rho \frac{D\hat{u}}{Dt} = -\frac{\partial q_j}{\partial x_j} - p\Theta + \sigma_{ij} \frac{\partial u_i}{\partial x_j} \quad (20\text{-}1)$$

or $$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} - \frac{\partial q_j}{\partial x_j} + \sigma_{ij} \frac{\partial u_i}{\partial x_j} \quad (20\text{-}2$$

"Dissipation" Function (Newtonian fluid):

$$I = \sigma_{ij} \frac{\partial u_i}{\partial x_j} = 2\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right) + \lambda \Theta^2$$

(22-3)

no unique solutions guaranteed for this set of eqs & conditions

Nature shows this too:

see bottom of page

NAVIER-STOKES EQ'S - EXACT + APPROX SOLUTIONS.

START WITH GENERAL EQ: (16-2)

$$\frac{Du_i}{Dt} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(\lambda \frac{\partial u_j}{\partial x_j}\right)$$

, AND CONTINUITY (8-2)

$$\frac{D\rho}{Dt} = -\rho\left(\frac{\partial u_j}{\partial x_j}\right)$$

, AND TOTAL ENERGY (17-1)

, AND EQUATIONS OF STATE FOR VARIOUS THERMO PROPERTIES AND TRANSPORT PROPERTIES; μ, λ, k.

ADD BOUNDARY CONDITIONS:

(1) NO SLIP AT WALL

(2) GIVEN T_{wall} OR $(q_i)_{wall}$

(3) CONDITIONS ON ALL PARAMETERS AND VARIABLES AT INFLOW AND OUT

AND INITIAL CONDITIONS

THESE ARE NON-LINEAR, ILL-CONDITIONED, HIGH ORDER EQ'S AND CONDITIONS. NO GENERAL UNIQUE SOLUTION GUARANTEED.

SYSTEM (IN NATURE) SHOWS:

1. INSTABILITIES, BREAKDOWNS
2. NON LINEAR EFFECTS, TURBULENCE
3. HIGHLY PARAMETER DEPENDENT R_N and M

EXACT SOLUTIONS ARE ALL "PARTICULAR" AND ONLY ABOUT 70 EXIST (TOP ESTIMATE) take $-\frac{\partial}{\partial y}\frac{Du_1}{Dt} + \frac{\partial}{\partial x}\frac{Du_2}{Dt}$

and defining $\omega = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}$ then we obtain

$\frac{D\omega}{Dt} = \nu \nabla^2 \omega$ for no body forces, constant density, constant viscosity, 2-D flow

NAVIER-STOKES EQ'S — SPECIAL CASE OF CONSTANT PROPERTIES ($\rho, \mu, k, C_v, \cdots$)

$$\frac{Du_i}{Dt} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad (25\text{-}1)$$

AND

$$\frac{\partial u_j}{\partial x_j} = 0 \qquad (25\text{-}2)$$

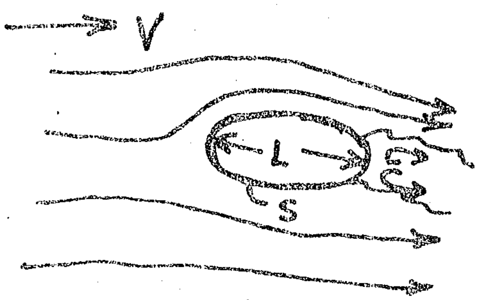
ARE 4 EQ'S IN 4 UNKNOWNS (u_i AND p)

THIS CASE IS UNCOUPLED FROM ENERGY EQ. WHICH FOR $\hat{u} = C_v T$ AND $q_i = -k\,(\partial T/\partial x_i)$ BECOMES AN EQ FOR TEMPERATURE, SEE EQ (20-1):

$$\frac{DT}{Dt} = \frac{k}{\rho C_v}\left(\frac{\partial T}{\partial x_i \partial x_i}\right) + \frac{2\nu}{C_v}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right) \qquad (25\text{-}3)$$

TO BE SOLVED GIVEN u_i FIELD

FOR THIS SPECIAL CASE CONSIDER A GENERAL PROBLEM OF STREAMING FLOW OVER A BODY



s : surface

Let: $f_i = 0$

PARAMETERS : ρ, ν, V, L

NORMALIZE VARIABLES

$\hat{x}_i = \frac{x_i}{L}$; $\hat{t} = \frac{tV}{L}$ (independent)

$\hat{u}_i = \frac{u_i}{V}$; $\hat{p} = \frac{p}{\rho V^2}$ (dependent)

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j}\right)$$

$$\frac{V \cdot V}{L}\frac{D\hat{u}_i}{D\hat{t}} = -\frac{1}{\rho}\cdot\frac{\rho V^2}{L}\frac{\partial \hat{p}}{\partial \hat{x}_i} + \frac{\nu V}{L^2}\frac{\partial \hat{u}_i}{\partial \hat{x}_j \partial \hat{x}_j}$$

$$\frac{V^2}{L}(\quad) = \frac{V^2}{L}(\,) + \frac{V^2}{L}\cdot\frac{\nu}{LV}(\quad)$$

or $$\frac{D\hat{u}_i}{D\hat{t}} = -\frac{\partial \hat{p}}{\partial \hat{x}_i} + \frac{1}{R}\frac{\partial^2 \hat{u}_i}{\partial \hat{x}_j \partial x_j}$$

AFTER "NORMALIZATION" EQ'S ARE:

$$\left[\frac{\partial \hat{u}_i}{\partial \hat{t}} + \hat{u}_j \frac{\partial \hat{u}_i}{\partial \hat{x}_j}\right] = -\frac{\partial \hat{p}}{\partial \hat{x}_i} + \frac{1}{R}\left(\frac{\partial^2 \hat{u}_i}{\partial \hat{x}_j \partial \hat{x}_j}\right)$$

$$\frac{\partial \hat{u}_j}{\partial \hat{x}_j} = 0$$

THE B.C.'S ARE (STEADY CASE)

$\hat{u}_i(S) = $ (NO SLIP)

$\hat{u}_i(\infty) = 1$

$\hat{p}(\infty) = p_\infty/\rho V^2 \Rightarrow 0$ {arbitrary in ρ = const flow.}

PARAMETER: $R = \frac{VL}{\nu} = \frac{\rho V L}{\mu}$

$R = \frac{\text{Inertia Forces (ma)}}{\text{Viscous Forces}}$

EQ. SYSTEM IS: (1) NON-LINEAR
(2) PARAMETER DEPENDENT
(3) ELLIPTIC (BOUNDARY VALUE PROBLEM)

SPECIAL LIMITS:

(1.) $R \Rightarrow \infty$; VISCOUS FORCES $\Rightarrow 0$ potential flow solution

HIGEST ORDER TERM: $\nabla^2 u_i \Rightarrow 0$
CANNOT SATISFY (NO SLIP) B.C. AT R=0
TROUBLES - NEED B.L. MODEL

(2.) $R \Rightarrow 0$; inertia Forces $\Rightarrow 0$
EQ'S BECOME LINEAR AND PARAMETER FREE WITH PROPER NORMALIZATION. CAN STILL SATISFY (NO SLIP) AT R=0.
"CREEPING FLOW" LIMIT

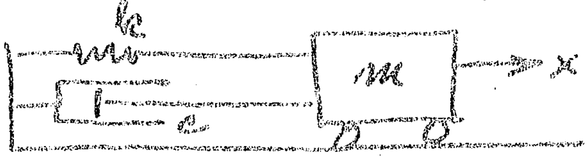
$0 = \frac{\partial p}{\partial x_1} - \frac{\mu}{\rho}\left(\frac{\partial u_i}{\partial x_j^2}\right.$

vorticity equation

$\nabla^4 \psi = 0$

Illustration of Problem at high Re No. Limit

Spring - Mass - Dashpot System



$\to x$; dx/dt ; d^2x/dt^2
(V) (a)

Basic Eq. $\Sigma F = ma$

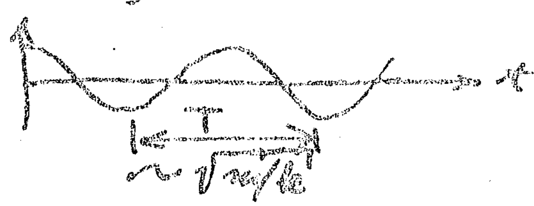
$$-kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

Normalize w/o parameters

$$\boxed{\frac{d^2x}{dt^2} + \left(\frac{c}{m}\right)\frac{dx}{dt} + \left(\frac{k}{m}\right)x = 0}$$

Solution #1 (no damping) as $c/m = 0$

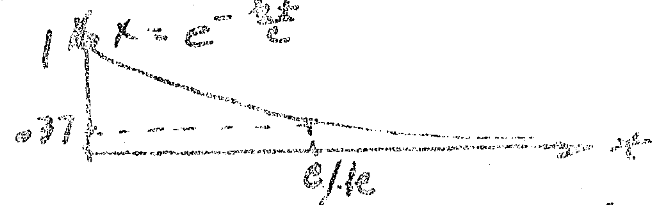
$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$$



oscillatory or periodic

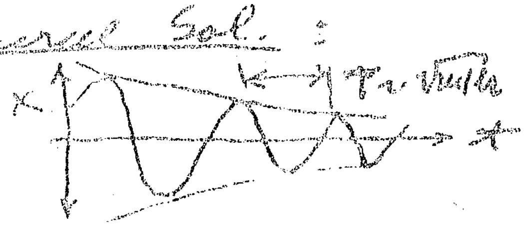
Solution #2 (no mass) as $m\, d^2x/dt^2 = 0$

$$\frac{dx}{dt} + \left(\frac{k}{c}\right)x = 0$$

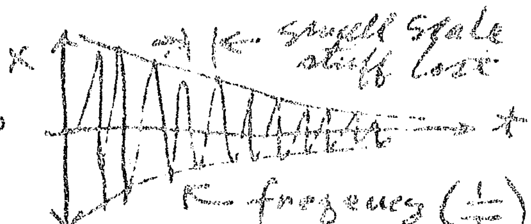


damped, exponential decay

General Sol.:



as $m \to 0$



Solution #2 gives only "envelope" says nothing about $f \to \infty$. We throw out d^2x/dt^2, the highest order term! Very dangerous.

SPECIAL CASE — $\boxed{\text{TWO-DIM PLANE FLOW WITH CONSTANT } \rho, \nu}$

CONTINUITY:
$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \quad (31-1)$$

NAVIER-STOKES (MOMENTUM) EQ (25-1)

$$\frac{Du}{Dt} = f_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \quad (31-2a)$$

$$\frac{Dv}{Dt} = f_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \quad (31-2b)$$

WHERE: $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

VORTICITY EQ FORM OF (31-2a) AND 31-2b) FOR CONSERVATIVE BODY FORCES

$$\boxed{\frac{D\omega}{Dt} = \nu\nabla^2\omega} \quad \text{WHERE: } \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (31-3)$$

INTRODUCE STREAMFUNCTION (SATIS. (31-1))

$$\boxed{u = \frac{\partial\psi}{\partial y} \;;\; v = -\frac{\partial\psi}{\partial x} \;;\; \omega = -\nabla^2\psi} \quad (31-4)$$

SO EQ OF MOTION BECOMES, FROM (31-3)

$$\boxed{\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) = \nu\nabla^2(\nabla^2\psi)} \quad (31-5)$$

NOTE: ALL IRROTATIONAL FLOWS $\omega = -\nabla^2\psi = 0$ SATISFY (31-5), BUT THEY CANNOT SATISFY NO SLIP BOUNDARY CONDITIONS!

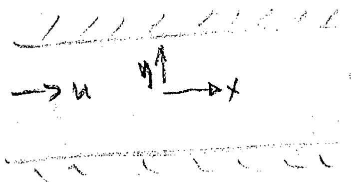
Parallel flow – assume 1 veloc comp only $u \neq 0$, $v = w = 0$

Continuity $\Rightarrow \frac{\partial u}{\partial x} = 0$ $\quad \rho = \text{const}$

Mom eq $\Rightarrow \frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

$$u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}} = 0 \text{ since } \frac{\partial u}{\partial x} = 0 \qquad \text{if } \frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$$

For steady flow $\frac{\partial u}{\partial t} \to 0$ $\quad \frac{\partial u}{\partial z} = 0$ for 2-D problem



$$\therefore \; -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right) = 0$$

$\Rightarrow u = f(y)$ only $\quad$ developed flow

or $+\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu f''(y) \Rightarrow \frac{\partial p}{\partial x} = \mu f''(y)$

from 2-other eqs of motion since $w, v = 0 \Rightarrow \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$ $\quad \therefore p = p(x)$ only

$\therefore \frac{\partial p}{\partial x} = \text{const.}$

let $\frac{\partial p}{\partial x} = k$ $\quad \therefore p = kx + C_2$

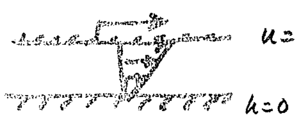
$$f'' = \frac{k}{\mu}\frac{y^2}{2} + C_1 y + C_2$$

$u = 0$ @ ~~c, y=c~~ $y = \pm\frac{h}{2}$ $\Rightarrow C_1, C_2$ as fn of k, μ

CLASSES OF EXACT (PARTICULAR) SOLUTIONS TO N-S EQ'S WITH NO-SLIP

I. CONVECTIVE ACCEL. VANISHES, $u_j \frac{\partial u_i}{\partial x_j} = 0$ (LAMINAR)

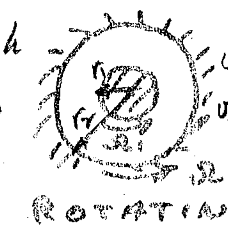
A. COUETTE SHEAR FLOWS (STEADY & UNSTEADY)



$y = h$

$y = 0$

PLANE



$v_\theta \neq 0$ Ω_1

$v_z = v_r = 0$

Ω_2

BC: $v_\theta = \omega_1 r_1,\ r = r_1$; $v_\theta = \omega_2 r_2,\ r = r_2$

ROTATING

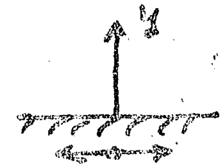
mom r-dir / θ dir

$\frac{\rho v_\theta^2}{r} = \frac{dp}{dr}$ / $\frac{d^2 v_\theta}{dr^2} + \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = 0$

B. STEADY FULLY-DEVELOPED DUCT FLOWS

C. UNSTEADY " " "

D. UNSTEADY SEMI-INFINITE FLOW



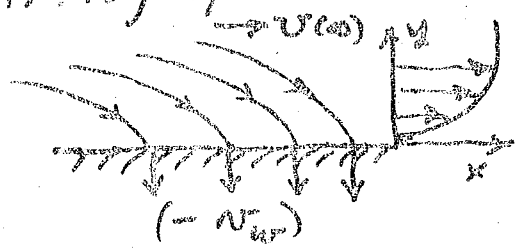
suddenly accel. flat plate
STOKES' 1st PROB

$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$ $u = V_0 @ y = 0$ $t > 0$; $u = 0 @ y = \infty$ $t < 0$

$U_w = U_0 \cos \omega t$
Oscillating Plate
STOKES' 2nd PROB

II. CONVECTIVE ACCEL. LINEARIZED; $U_j \frac{\partial u_i}{\partial x_j}$

A. Asymptotic Suction Flows



$u = U(1 - \exp(y v_w / \nu))$

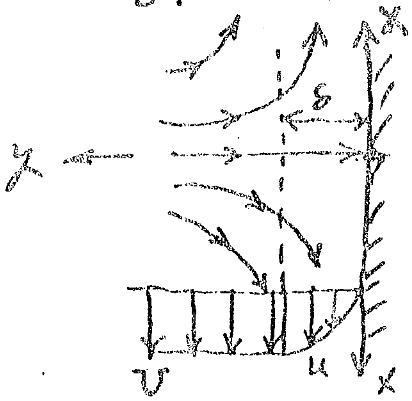
$v = v_w$

III. SPECIAL SIMILARITY SOLUTIONS:

e.g. $u(x,y) = u(\eta)$
BY TRANSFORMATION OF COORD'S.

A. ROTATING INFINITE DISK

B. STAGNATION POINT FLOW:



OUTSIDE δ to CLOSE APPROX.

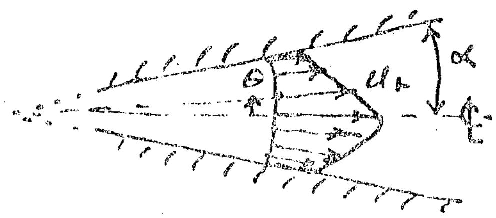
$$\left.\begin{array}{l} u \to U = ax \\ v \to V = -ay \end{array}\right\} \text{Potential Sol.}$$

$$\delta \approx 2.4\sqrt{\nu/a}$$

$$\eta = \sqrt{\frac{a}{\nu}}\, y$$

$$u = U\,(\text{function of } \eta)$$

C. CONVERGENT-DIVERGENT CHANNEL (SOURCE-SINK IN A WEDGE REGION)



$$u_r = (u_r)_{max}\, f(\eta)$$

$$\eta = \theta/\alpha$$

IV. SOLUTIONS AT $Re_D = 0$ - CREEPING FLOW LIMIT (viscous effects >> inertial effects)

A. STOKES SPHERE FLOW (3-D)

B. ~~CYLINDER~~ Sphere (2-D) WITH OSEEN'S IMPROVEMENT.

Consider constant density & viscosity

CREEPING FLOW RE. No. = 0

CONSIDER DIFF. EQ'S AND "NORMALIZE"

$$\rho \frac{D u_i}{D t} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

PARAMETERS : ρ, μ, V, L

let: $\hat{x}_i = x_i / L$; $\hat{t} = tV/L$

$\hat{u}_i = u_i / V$; $\hat{p} = \dfrac{pL}{\mu V}$ ← so that $\hat{p}$ will be bounded - if defined as $P/\rho V^2$, as $V \to 0$ $\hat{p} \to \infty$. different from P. 28 in this way

SO EQ BECOMES

$$\rho \frac{V^2}{L}\left(\frac{D \hat{u}_i}{D \hat{t}}\right) = \frac{\mu V}{L^2}\left(-\frac{\partial \hat{p}}{\partial \hat{x}_i} + \frac{\partial^2 \hat{u}_i}{\partial \hat{x}_j \partial \hat{x}_j}\right)$$

$\mu \to \infty$ $V \to 0$
$\Rightarrow \mu V \to$ const.

and so

$$\cancel{\left(\frac{\rho V L}{\mu}\right)} \frac{D \hat{u}_i}{D \hat{t}} = -\frac{\partial \hat{p}}{\partial \hat{x}_i} + \frac{\partial^2 \hat{u}_i}{\partial \hat{x}_j \partial \hat{x}_j}$$

$\to 0$ as $R \to 0$

∴ Creeping flow Eqs are (dimensional form)

$$\left[\begin{array}{ll} \text{momentum} & \dfrac{\partial p}{\partial x_i} = \mu \dfrac{\partial^2 u_i}{\partial x_j \partial x_j} \\ \text{continuity } \rho = \text{const} & \dfrac{\partial u_j}{\partial x_j} = 0 \end{array}\right] \qquad (34\text{-}1)$$

EQ'S ARE LINEAR, HIGESTORDER TERM FROM N-S REMAINS, ~~NO-SLIP~~ B.C. STILL CAN BE APPLIED. Parameter free in the non-dimensional case

Special Properties:

(I) Take $\frac{\partial}{\partial x_i}\left[\frac{\partial p}{\partial x_i}\right] = \mu \frac{\partial}{\partial x_i}\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j}\right)$

$$\frac{\partial^2 p}{\partial x_i \partial x_i} = \mu \frac{\partial^2}{\partial x_j \partial x_j}\left(\cancel{\frac{\partial u_i}{\partial x_i}}\right)^{=0 \text{ By Continuity}}$$

$$\therefore \quad \boxed{\nabla^2 p = 0} \qquad (35\text{-}1)$$

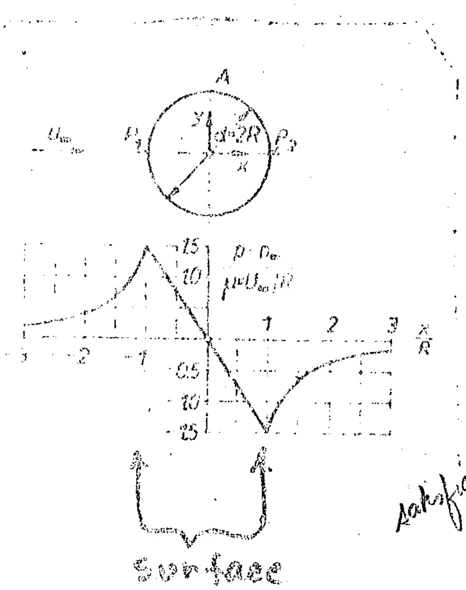
So Pressure field obeys Laplace's Eq.

(II) For 2-D Flow go to equation for streamfunction and set inertia terms to zero; From (31-5)

$$\boxed{\nabla^4 \psi = 0} \qquad (35\text{-}2)$$

DRAG OF A SPHERE (STOKES' SOLUTION)

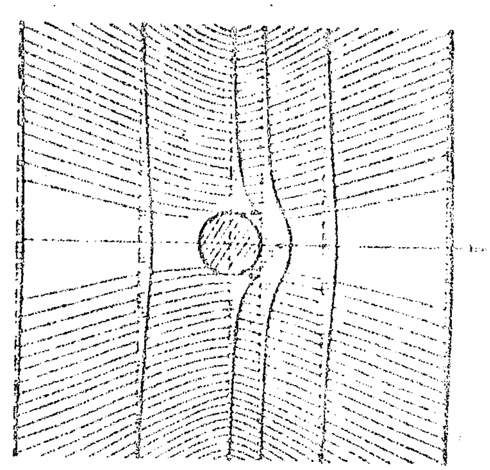
– see Schlichting, Ref. 1 –



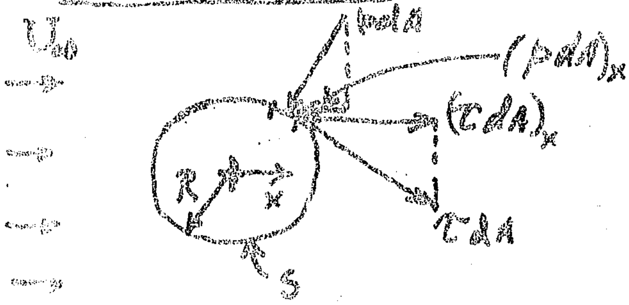
Note
⟷
Symmetry

satisfies no slip

Streamlines ($U_\infty = 0$ but Sphere Moving)
no wake which is not correct



STOKES' DRAG:



SPHERE

$$F_D = \int_S (\tau\,dA)_x - \int_S (p\,dA)_x$$

$$\boxed{F_D = 6\pi\mu R U_\infty}$$ for $R \neq 0$

Drag Coef:

$$C_D = \frac{F_D}{\pi R^2 \left(\frac{\rho U_\infty^2}{2}\right)} = \frac{24}{R_N}$$

Where: $R_N = \frac{\rho U_\infty (2R)}{\mu}$

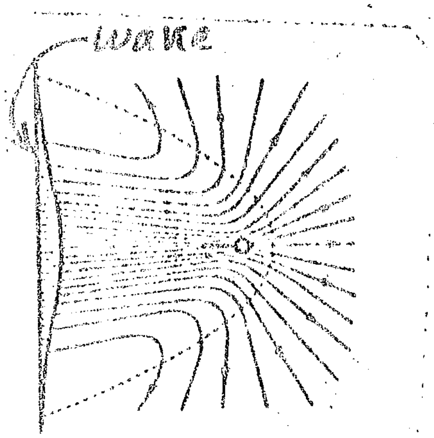
OSEEN'S "IMPROVEMENT"? (linearized inertia)

$$\left.\begin{aligned} u_1 = u &= U_\infty + u' \\ u_2 = v &= v' = u_2' \\ u_3 = w &= w' = u_3' \end{aligned}\right\} \; u', v', w' \text{ small except close to sphere}$$

valid far away from sphere.

When put into full N.S. Eqs and non-linear terms omitted one obtains

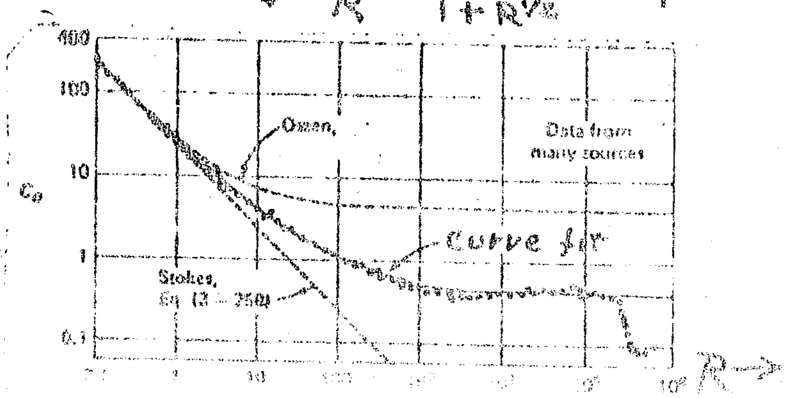
$$\rho U_\infty \frac{\partial u_i'}{\partial x_1} + \frac{\partial p}{\partial x_i} = \mu \frac{\partial^2 u_i'}{\partial x_j \partial x_j} \; ; \quad \frac{\partial u_j}{\partial x_j} = 0$$



Streamlines (sphere moving → at U_∞)

Oseen Drag: $C_D = \frac{24}{R}\left[1 + \frac{3R}{16} + \frac{9R^2 \ln R}{160} + \dots\right]$ GBDP (19.57)

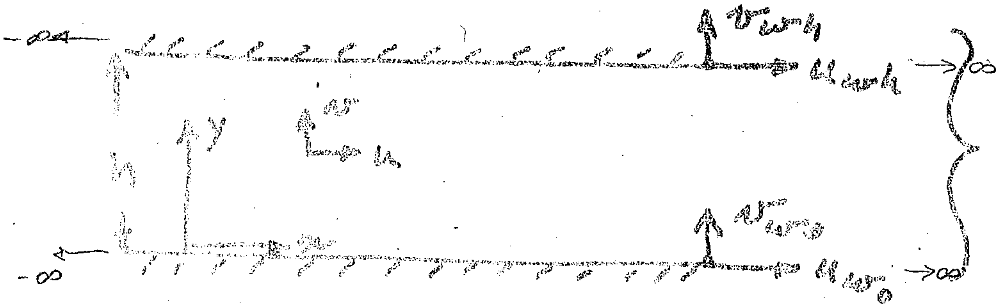
Curve fit $C_D = \frac{24}{R} + \frac{6}{1+R^{1/2}} + 0.4$



Slider-bearing problems – Check [illegible] note.

EXACT SOLUTION OF CONST. PROPERTY NAVIER-STOKES — "DUCT" FLOWS

• CASE OF ZERO OR LINEAR CONVECTIVE DERIVATIVE FOR 2-D, PLANE FLOW •



B.C.'s could have suction, blowing, moving walls.

Assume: $w = 0$, $\rho = \text{const}$, $\mu = \text{const}$, LAMINAR FLOW ONLY

fully-developed flow (f-d) assumptions

No variation of u, v w/o x.

$\frac{\partial u}{\partial x} = 0$, $\frac{\partial v}{\partial x} = 0$; fluid motion is not fn of x

Continuity: $\left[\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}\right] = 0$ from (f-d) $\Rightarrow v = \text{const}$

Therefore $v = \text{const}$ or funct. of t
velocity through walls

$$v = v_{w_0} = v_{w_h} = v_w(t)$$

UNDER THESE CONDITIONS N-S BECOMES

non steady conditions

$$\frac{\partial u}{\partial t} + \overset{\frac{\partial u}{\partial x}=0}{} v_w \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad \overset{\frac{\partial^2 u}{\partial x^2}=0}{} \qquad (37\text{-}1)$$

$$\underbrace{\frac{d v_w}{d t} + 0}_{\text{linear convective derivative}} \overset{\frac{\partial v}{\partial y}=0}{} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \underset{\frac{\partial^2 v}{\partial y^2}}{0} \qquad \frac{\partial^2 v}{\partial x^2} = 0 \qquad (37\text{-}2)$$

look at special case steady flow, $u(0)=0$ $u(h)=U$, $v_w=0$

then $-G = \nu \frac{\partial^2 u}{\partial y^2}$

$\therefore -\frac{G}{\nu}y + K_1 = \frac{\partial u}{\partial y}$

$-\frac{G y^2}{2\nu} + K_1 y + K_2 = u$

INTEGRATE EQ (37-2) ON y

$$-\frac{p}{\rho} = \left(\frac{dv_w}{dt}\right) y + \mathbb{C}(x,t)$$

$f(t,y)$

DIFF. W/O x AND EQUATE TO (37-1)

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial \mathbb{C}}{\partial x} = \underbrace{\left[\frac{\partial u}{\partial t} + v_w \frac{\partial u}{\partial y} - \nu \frac{\partial^2 u}{\partial y^2}\right]}_{\text{function of } y,t \text{ only}}$$

by f-d assumpt

thus $\frac{\partial \mathbb{C}}{\partial x} = G(t)$ only

IN GENERAL:

$$\mathbb{C}(x,t) = G(t)x + H(t)$$

SATISFIES ABOVE CONDITION SO PRESSURE SATISFIES GENERAL SOLUTION

$$\boxed{-\frac{p}{\rho} = \left(\frac{dv_w}{dt}\right) y + G(t)x + \underbrace{\cancel{H}(t)}_{\text{ARBITRARY FLUCTUATING PRESSURE LEVEL FOR WHOLE SYSTEM}}^{0}} \quad (38\text{-}1)$$

AND $\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x} = G(t)}$ (38-2)

G(t) IS RELATED TO ΔP ALONG CHANNEL

$$\Delta P = p_{IN} - p_{out} \quad \underline{AT} \quad y = \text{const}$$

$$L = x_{out} - x_{IN}$$

$$\boxed{\begin{array}{l} G(t) = \Delta p / \rho L \\ \text{IS PRESCRIBED (HERE)} \end{array}} \quad (38\text{-}3)$$

$$\boxed{v_w(t) \text{ IS ALSO PRESCRIBED}}$$

$$\therefore \quad -p/\rho = \frac{dv_w}{dt} y + G(t)x$$

$$\therefore \quad \frac{\partial u}{\partial t} + v_w \frac{\partial u}{\partial y} = G(t) + \nu^2 \frac{\partial^2 u}{\partial y^2}$$

BOUNDARY CONDITIONS u_{w0}, u_{wh}

IN THIS CASE OF NO CONVECTIVE ACCELERATION ALONG X-AXIS OBSERVER CAN MOVE AT ARBITRARY VELOCITY W/O X SO IT IS PERFECTLY GENERAL TO SET $\boxed{u_{w0} = 0 \text{ AND TO PRESCRIBE } u_{wh} = U(t).}$

NOW SOLVE EQ. (37-1) FOR GIVEN VALUES OF $v_w(t)$, $U_w(t)$, $G(t)$ AND OBTAIN VELOCITY PROFILES $u(y, t)$.

EXAMPLES:

(I.) STEADY FLOWS W/O SUCTION ($v_w = 0$)

$$v_w = 0 \;, \quad \partial/\partial t = 0$$

B.C.'s: $u(h) = U$; $u(0) = 0$

D.Eq. (37-1) BECOMES ($G = \Delta P/\rho L$):

$$0 = G + \nu \frac{d^2u}{dy^2}$$

INTEGRATE TWICE since $u \neq f(x,t)$ $u = u(y)$ only

$$\frac{du}{dy} = -\frac{G}{\nu} y + K_1$$

$$u = -\frac{1}{2}\frac{G}{\nu} y^2 + K_1 y + \cancel{K_2} \qquad \cancel{K_2} = 0 \;; \; u(0) = 0 \text{ B.C.}$$

SECOND B.C. $u(h) = U$ GIVES K_1 AND SO SOL. IS

$$\boxed{u = \frac{G}{2\nu}(hy - y^2) + \frac{U}{h} y} \qquad (39\text{-}1)$$

"NORMALIZE" SOL. ON CHARACTERISTIC SPEED, u_c, AND DIMENSION h

$$\boxed{\eta = y/h}$$

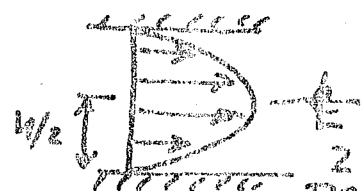
SO
$$\boxed{\frac{u}{u_c} = \frac{1}{2}\,\frac{G h^2}{\nu u_c}(\eta - \eta^2) + \frac{U}{u_c}(\eta)} \quad (40\text{-}1)$$

CHOICE OF u_c DEPENDS ON CASE!

(A) Wall at rest $(U = u(1) = 0)$

$u_c = \dfrac{G h^2}{\nu}$ MAKES EQ (40-1) PARAMETER FREE, i.e.

$$\frac{u}{u_c} = \frac{1}{2}(\eta - \eta^2)$$

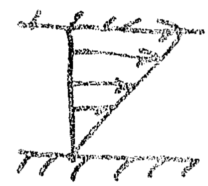


2-D POUSEUILLE FLOW

(B) PRESSURE GRAD = 0, $(G = 0)$

$u_c = U$ SO

$$\frac{u}{u_c} = \eta$$



COUETTE FLOW

(C) GENERAL CASE $(U \neq 0,\ G \neq 0)$

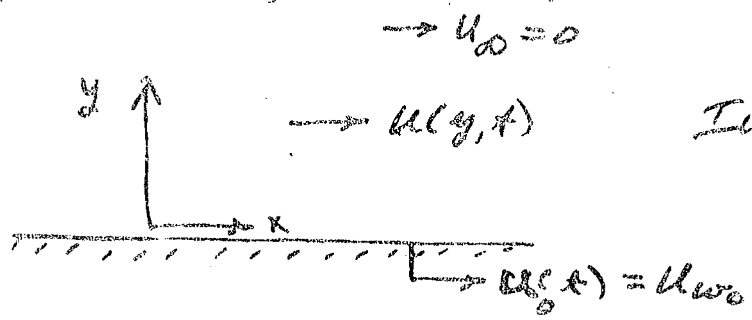
CHOOSE U_c THAT DOESN'T $\to 0$ IN EITHER LIMIT $U \to 0$ OR $G \to 0$. FOR EXAMPLE USE VOL. FLOW RATE:

$$u_c = \frac{Q}{h} = \frac{1}{h}\int_0^h u\,dy = \int_0^1 u\,d\eta$$

So: $$u_c = \frac{G h^2}{12\nu} + \frac{U}{2}$$

Stokes' 1st Problem

II) Unsteady Flow, Moving Wall



Initially at rest

constant after some time t_0

Look at previous, Set: $G(t) = -\frac{1}{\rho}\frac{\partial p}{\partial x} = 0$

$\frac{\partial p}{\partial x} = 0$ since domain is ∞ in extent.

$v_w = 0$

Let: $h \rightarrow \infty$

$U_{w_0} = U_0(t)$

$U_{w_h} = U_\infty = 0$

Eq. to be solved:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

Parameters: ν, U_0 boundary Param.

Normalize: No characteristic scale length so use the following (NO EXTERNAL CHAR SCALE only internal scale δ)

U_0 : characteristic speed

ν / U_0 : " length dimension

ν / U_0^2 : " time dimension

$t = \frac{\nu}{U_0^2}\hat{t}$; $y = \frac{\nu}{U_0}\hat{y}$; $u = U_0 \hat{u}$

∴ D.E. is parameter free:

$$\cancel{\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}} \qquad \frac{\partial \hat{u}}{\partial \hat{t}} = \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} \quad \text{parabolic eq.} \quad (41\text{-}1)$$

so are I.C.'s: $\hat{u}(\hat{y}, \hat{t} \leq 0) = 0$ fluid at rest

and B.C.'s $\hat{u}(\hat{y} = 0, \hat{t}) = 1$ start up condition (41-2)

Every ba...

Method of Similarity transformation

Seek transformation: $\eta = \eta(\hat{y}, \hat{t})$

So that $\hat{u} = f(\eta)$ — let η be complex in the deriv of least order, simple in most order

which is compatible with B.C.'s & I.C.'s

$$\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{df}{d\eta}\frac{\partial \eta}{\partial \hat{t}} = f'\left(\frac{\partial \eta}{\partial \hat{t}}\right)$$

$$\frac{\partial \hat{u}}{\partial \hat{y}} = f' \frac{\partial \eta}{\partial \hat{y}}$$

$$\frac{\partial^2 \hat{u}}{\partial \hat{y}^2} = f''\left(\frac{\partial \eta}{\partial \hat{y}}\right)^2 + f'\left(\frac{\partial^2 \eta}{\partial \hat{y}^2}\right)$$

So plug into eq. (41-1) and obtain

$$\boxed{f'' + g f' = 0} \qquad (42\text{-}1)$$

$g = g(\eta, \hat{y}, \hat{t}, \eta_{,\hat{y}\hat{y}}, \eta_{,\hat{t}}, \eta_{,\hat{y}})$

where
$$g = \frac{(\partial^2\eta/\partial \hat{y}^2) - \partial\eta/\partial \hat{t}}{(\partial\eta/\partial \hat{y})^2} \qquad (42\text{-}2)$$

Now if we are to have a similarity transformation $g(\eta)$ only. Solution of (42-2) is a problem in functional analysis to find $g(\eta)$, if it exists.

For our purposes choose: $\eta = c\,\hat{y}\,\hat{t}^{m}$

so $\frac{\partial^2 \eta}{\partial \hat{y}^2} = 0$ and (42-2) assume the highest deriv in $g = 0$ $\Rightarrow$ $\eta_{,\hat{y}\hat{y}} = 0$

1. make parameter free
2. go to similarity transformation

becomes $$-\frac{\partial \bar{\eta}}{\partial \bar{x}} = g(\eta)\left(\frac{\partial \eta}{\partial \bar{y}}\right)^2$$

$$-c\bar{y}\,n\,\bar{x}^{(n-1)} = g(\eta)\,c^2\,\bar{x}^{2n}$$

by trying $g(\eta) = \eta = c\,\bar{y}\,\bar{x}^{n}$

above becomes

$$-\cancel{c\bar{y}}\,n\,\bar{x}^{(n-1)} = \cancel{c\bar{y}}\,\bar{x}^{n}\,c^2\,\bar{x}^{2n}$$

$$-n\,\bar{x}^{(n-1)} = c^2\,\bar{x}^{(3n)}$$

powers on $\bar{x}$ must be the same so

$$(n-1) = (3n)$$

$$\boxed{n = -\tfrac{1}{2}}$$

also $-n = c^2$

so $\boxed{c = \sqrt{\tfrac{1}{2}}}$

<u>now</u> $\boxed{\eta = \frac{\bar{y}}{\sqrt{2\bar{x}}} = \frac{y}{\sqrt{2\nu x}} \quad \text{and } g = \eta}$

<u>D.E.</u> $f'' + \eta f' = 0 \Rightarrow \frac{f''}{f'} = -\eta \Rightarrow \frac{df'}{f'} = -\eta\, d\eta$

with $f(\infty) = 0$, $f(0) = 1$

and $\frac{u}{u_0} = f$

Change variable as $f' = h(\zeta)$

D.E. becomes $h' + \zeta h = 0$

whose general solution using $\int$ factor is

$$\frac{df}{d\zeta} = h = c_1 e^{-\zeta^2/2}$$

$$\therefore \quad f = c_1 \int_0^{\zeta} e^{-\varepsilon^2/2}\, d\varepsilon + c_2$$

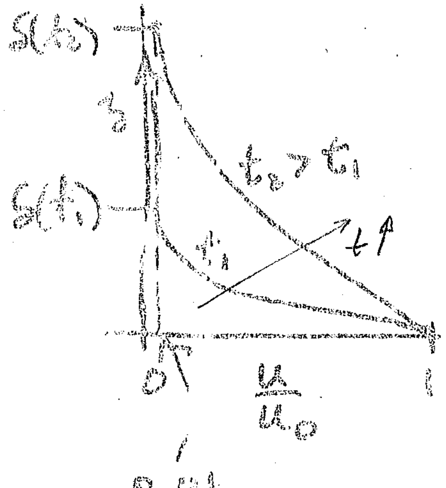
Using conditions $f(\infty) = 0$, $f(0) = 1$
(see Sabersky & Acosta)

$$f = 1 - \sqrt{\frac{2}{\pi}} \int_0^{\zeta} e^{\varepsilon^2/2}\, d\varepsilon$$

Change of variable to $\boxed{\beta = \frac{\zeta}{\sqrt{2}} = \frac{y}{2\sqrt{\nu t}}}$

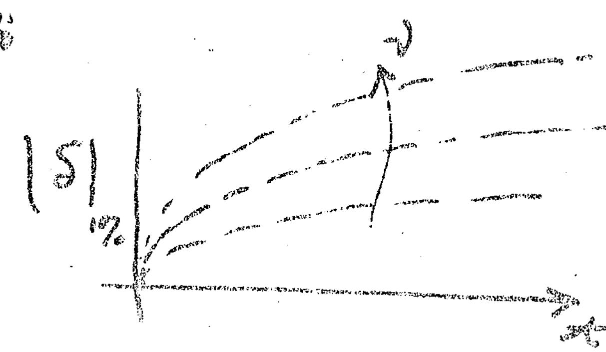
gives $\boxed{\frac{u}{u_0} = 1 - \mathrm{erf}(\beta) = \mathrm{erfc}(\beta)}$

where $\mathrm{erf}(\beta) = \frac{2}{\sqrt{\pi}} \int_0^{\beta} e^{-x^2}\, dx$



diffusion layer thickness

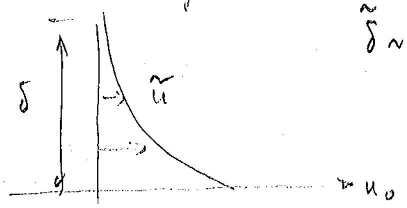
$$|\delta|_{1\%} \approx 3.64 \sqrt{\nu t}$$



1/26

1. Introduce b.l or thin shear layer (TSL)
2. Eqs & BC's of TSL in differential / integral

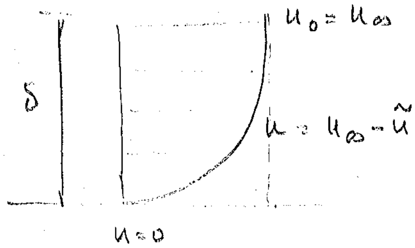
Stokes 1st problem



$\tilde{\delta} \sim \sqrt{\nu t}$

fluid at rest, ~~body~~ wall is jerked to right

by transformation



$\delta = \tilde{\delta} \sim \sqrt{\nu t}$

fluid jerked to right / wall at rest

If $d\delta/dx$ slow ~~this~~, this section looks like Stokes 1st prob

where $U_\infty = u_0$, $l \approx U_\infty t$

$\delta_{LAMIN} \sim \sqrt{\nu l / U_\infty}$ $\qquad \dfrac{\delta}{l} \sim \dfrac{1}{\sqrt{Re}}$ where $Re = \dfrac{U_\infty l}{\nu}$

$\sqrt{\nu t}$

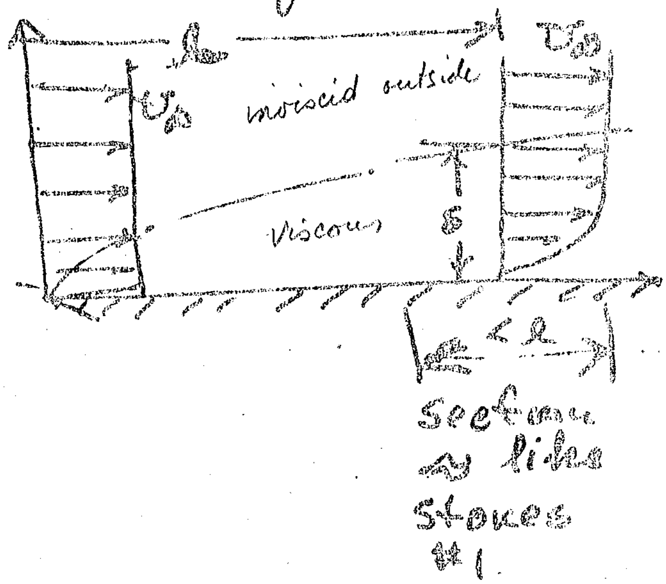
rapid changes in velocity in bl

for $\delta_{TURBULENT} \sim (Re)^{-1/5}$

$\delta_{LAMIN} \sim Re^{-1/2}$

P. 45

Imagine steady B.L. on flat plate at U_∞ = const. Here flow not quite parallel, i.e $\frac{\partial u}{\partial x} \neq 0$



but a each section is roughly like the previous problem where $\ell \approx U_\infty t$

So reasoning by analogy

$$\delta \approx K\sqrt{\nu \ell / U_\infty}$$

or

$$\frac{\delta}{\ell} \approx K \frac{1}{\sqrt{\frac{U_\infty \ell}{\nu}}} = \frac{K}{\sqrt{Re_\ell}}$$

In practice this is exact if laminar and $Re_\ell \gg 1$, value of $K \approx 5.0$ for flat plate. However shape of profile not exactly erfc(β).

① non dimensionalize ~~and~~ x, y, u, v and p

② plug into PDE $<$ cont / 2 mom.

③ keep ~~most~~ highest order term and use Euler Eq to obtain order

of $\frac{\partial p}{\partial x}$, $\Rightarrow$

y mom will give $\frac{\partial p}{\partial y} = 0 \Rightarrow \frac{\partial p}{\partial x} = \frac{dp}{dx}$

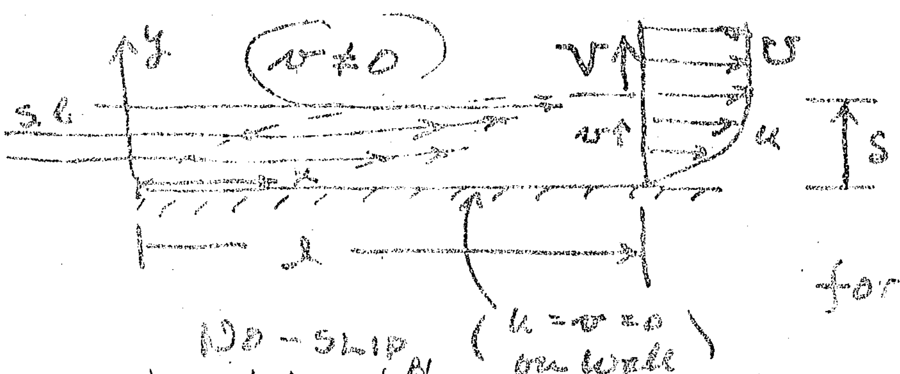
IV.)

THE THIN SHEAR LAYER (TSL) at High Reynolds Number.

A.) THE TSL APPROX. AND B.L. EQ'S. $\delta/l \ll 1$ & $Re \gg 1$

Basic Assumption $\delta/l \ll 1$ at High Reynolds number. This is emperical

How to Estimate terms!



"FLAT" WALL

for $y \geq \delta$
$u \to U$
$v \to V$

Do estimates of changes in BL

Estimates of Gradients if NO SUDDEN CHANGES INSIDE B.L. (O.K. IN LAM. B.L. AND TURB. B.L. (outside of sublayer))

$$\frac{\partial u}{\partial y} \sim \frac{u(y=\delta) - u(y=0)}{(y=\delta) - (y=0)} \sim \frac{U-0}{\delta - 0} \sim \frac{U}{\delta}$$

$$\frac{\partial u}{\partial x} \sim \frac{u(x=l) - u(x=0)}{(x=l) - (x=0)} \sim \frac{0-U}{l-0} \sim \frac{U}{l}$$

$$\frac{\partial^2 u}{\partial x^2} \sim \frac{U}{l^2}$$ (order of magnitude) "of order mag. w/o ref. to sign"

don't know the relationship of V

$$\frac{\partial v}{\partial y} \sim \frac{V}{\delta} \; ; \; \frac{\partial v}{\partial x} \sim \frac{V}{l} \; ; \; \frac{\partial^2 v}{\partial y^2} \sim \frac{V}{\delta^2}$$

P. 47 Assume 2-D, plane steady flow & Flat wall

Look at Continuity Eq:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\sim \frac{U}{\ell} \qquad \sim \frac{V}{\delta} = 0$$

if $\frac{U}{\ell}$ is same order (order 1) $\Rightarrow \frac{V}{\delta}$ is order 1

if $\frac{\partial v}{\partial y} \neq O\left(\frac{\partial U}{\partial x}\right) \Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow V = 0$ everywhere (for 1)

(this is fully developed flow assumption)

So that both terms remain; this case is not fully-developed flow but has growth so $(v \neq 0)$ we obtain

$$\boxed{\frac{V}{U} = \frac{\delta}{\ell}} \ll (\sim 1) \text{ at high Re}$$

"Normalize" Momentum Eqs so terms in D/Dt side are (~ 1) or smaller

Choose: $\tilde{u} = u/U$; $\tilde{v} = v/V = v\ell/U\delta$

$\tilde{x} = x/\ell$; $\tilde{y} = y/\delta$ $\quad \tilde{p} = p/\rho U^2 \quad \tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{\rho U^2}$

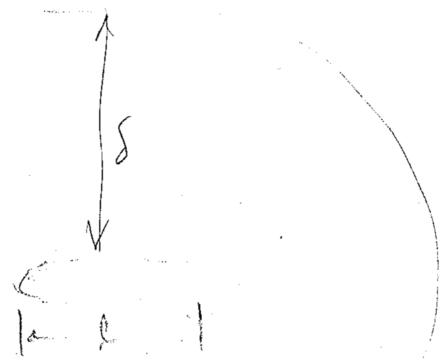
Continuity $\quad \underset{\sim 1}{\frac{\partial \tilde{u}}{\partial \tilde{x}}} + \underset{\sim 1}{\frac{\partial \tilde{v}}{\partial \tilde{y}}} = 0 \qquad$ Steady mean 2-D

x-eq $\quad \underset{\sim 1}{\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}} + \underset{\sim 1}{\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}} = + \underset{\sim 1}{\frac{1}{\rho U^2}}\left[-\frac{\partial p}{\partial \tilde{x}} + \frac{\partial \sigma_{xx}}{\partial \tilde{x}} + \left(\frac{\ell}{\delta}\right)\frac{\partial \sigma_{xy}}{\partial \tilde{y}}\right]$

y-eq $\quad \underbrace{\left(\frac{\delta}{\ell}\right)\left[\underset{\sim 1}{\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}}} + \underset{\sim 1}{\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}}}\right]}_{\sim\left(\frac{\delta}{\ell}\right)} = \underbrace{\frac{1}{\rho U^2}\left[\left(\frac{\ell}{\delta}\right)\frac{\partial p}{\partial \tilde{y}}\right.}_{\sim\left(\frac{\delta}{\ell}\right)} \left. + \left(\frac{\ell}{\delta}\right)\frac{\partial \sigma_{yy}}{\partial \tilde{y}} + \frac{\partial \sigma_{xy}}{\partial \tilde{x}}\right]$

↑

(case 1)


δ

Case 1. Inviscid flow (neglect σ's) and obtain Euler's Eq's
we see that pressure forces balance inertia and for terms in each eq. to balance

$$\frac{1}{\rho U^2}\frac{\partial p}{\partial x} \sim 1 \quad (x\text{-eq})$$

Re is not a parameter

$$\frac{1}{\rho U^2}\frac{\partial p}{\partial y} \sim \left(\frac{\delta}{\ell}\right)^2 \quad (y\text{-eq})$$ this term

is really same order as other terms (~ 1) since $\delta \doteq \ell$ in "outer" inviscid flow.

Case 2. Turbulent S.L.'s Here all σ's of about same size

e.g. turbulent B.L. $0.4\,\sigma_{xx} \approx \sigma_{xy}$ ...

Maximum values, all else $(\sigma_{xy})_{B.L.} \approx 0.001\rho U^2$

But, in a turbulent wake or jet (free S.L.)
$$(\sigma_{xy})_{free} \sim 0.01\rho U^2$$

Here if σ's are to interact all in eq's they must balance inertia terms. Since $d\tilde{x} \sim d\tilde{y}$ and $\sigma_{xx} \sim \sigma_{xy}$ the terms are small re to other terms and remaining terms are at least

$$\frac{1}{\rho U^2}\frac{\partial \sigma_{xy}}{\partial \tilde{y}} \sim \frac{\delta}{\ell} \quad \text{and} \quad \frac{1}{\rho U^2}\frac{\partial \sigma_{yy}}{\partial \tilde{y}} \sim \left(\frac{\delta}{\ell}\right)^2$$

or $\frac{(\sigma_{xy})_{max}}{\rho U^2} \sim \begin{Bmatrix} 0.001 \\ 0.01 \end{Bmatrix} \sim \frac{\delta}{\ell}$ gives estimate of max ...

p. 49

Consequently, for turbulent flow the dimensional eqs are

$$\boxed{\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \sigma_{xy}}{\partial y} \\ O\left(\frac{\delta}{\ell}\right) &= \cancel{-\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial \sigma_{yy}}{\partial y}} \end{aligned}} \quad (49\text{-}1)$$

when terms on l.h.s. of equal order are retained. In practice the second eq. is small $(\delta/\ell)^2$ w/o first so it can be dropped in most cases. Hence

$$\frac{\partial p}{\partial x} \Longrightarrow \frac{dp}{dx} \quad \text{and must}$$

be part of "given" data.

Case 3 Laminar flow S.L.

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x} = \frac{2\mu U}{\ell}\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)$$

$$\sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\mu U}{\ell}\left[\left(\frac{\ell}{\delta}\right)\frac{\partial \bar{u}}{\partial \bar{y}} + \left(\frac{\delta}{\ell}\right)\frac{\partial \bar{v}}{\partial \bar{x}}\right]$$

$$\sigma_{yy} = 2\mu\frac{\partial v}{\partial y} = \frac{2\mu U\left(\frac{\delta}{\ell}\right)}{\delta}\left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)$$

Rewriting the eqs $(\nu = \mu/\rho)$ $\left(Re \equiv \frac{U\ell}{\nu}\right)$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{1}{\rho U^2}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\nu}{U\ell}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}\right) + \frac{\nu}{U\ell}\left(\frac{\ell}{\delta}\right)^2\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$\left(\frac{\delta}{\ell}\right)\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \left(\frac{\delta}{\ell}\right)\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} = -\frac{1}{\rho U^2}\left(\frac{\ell}{\delta}\right)\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{\nu}{U\ell}\left(\frac{\ell}{\delta}\right)\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} + \frac{\nu}{U\ell}\left(\frac{\ell}{\delta}\right)\frac{\partial^2 \tilde{u}}{\partial \tilde{x}\partial \tilde{y}}$$

Now as $Re \to \infty$

it is "observed" $\frac{\delta}{\ell} \Rightarrow \frac{1}{\sqrt{Re}}$ or $\frac{(\ell/\delta)^2}{Re} \to 1$

As a consequence we see second equation is small compared to first and again

$$\left(\partial p/\partial y\right) \lesssim \rho U^2\left(\frac{\delta}{\ell}\right)^2 \approx 0 \quad w/o$$

the term $\partial p/\partial x = dp/dx$ (given)

In dimensional form the B.L. equations for a flat plate are for

LAMINAR STEADY B.L.

$$\boxed{\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}} \qquad (50\text{-}1)$$

with B.C.'s on u (at $y = 0$ and as for example) and v (at the wall).

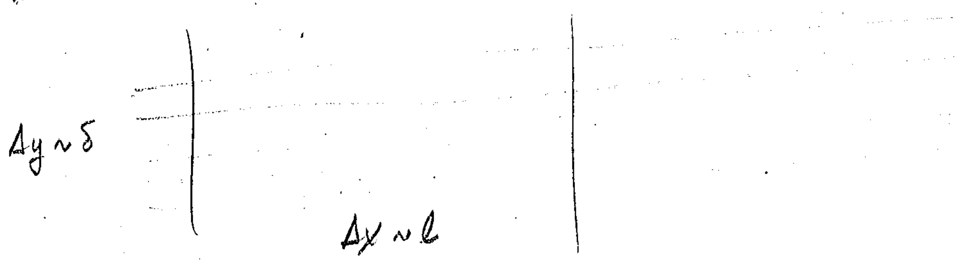
Stream: $p + \frac{\rho}{2}(u_e^2 + v_e^2) = \text{const}$ for $y \geq \delta$

$$\frac{dp}{dx} + \rho\left(u_e \frac{du_e}{\partial x}\right) + \underbrace{v_e \frac{\partial v_e}{\partial x}}_{\text{small}} = 0 \text{ at } y = \delta$$

then $\frac{1}{\rho}\frac{dp}{dx} = -u_e \frac{du_e}{\partial x} \leftarrow$ must be given

must also be given b c at wall.

numerical



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IV. B.)) INTEGRAL EQ'S FOR B.L. (TSL)

IF B.L. EXISTS THEN ONLY IMPORT. STRESSES

$$\boxed{\tau = \sigma_{xy} = \mu\frac{\partial u}{\partial y} + (\sigma_{xy})_{turb}}$$ turbulent definition where at wall $\sigma_{xy}|_{turb} = 0$ but outside (but $\leq\delta$) $\mu\frac{\partial u}{\partial y} = 0$ (51-1)

ALSO $\boxed{\frac{\partial p}{\partial y} = 0}$ SO $\boxed{\frac{\partial p}{\partial x} = \frac{dp}{dx} = -\rho u_e \frac{du_e}{dx}}$

(51-2)

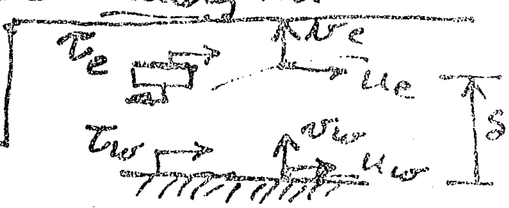
BASIC FACTS WHEN $\boxed{\frac{\delta}{\ell} \ll \sim 1}$ TSL

EQ'S

$$\boxed{\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (\rho = \text{const.}) \\ &u_e \equiv u(y \geq \delta) \text{ free-stream} \end{aligned}}$$ turbulent definition (51-3)

Now Momentum Integral Eq. (steady flow)

Note: in turb. flow u and v are long term average velocities so flow may be steady in the mean



from (51-3)

integrate cont. $$v_e = -\int_0^\delta \frac{\partial u}{\partial x}dy + v_w \qquad (51-4)$$

$2u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} = (\underline{v}\cdot\nabla)u + u(\nabla\cdot\underline{v})_{=0}$ $-u\frac{\partial u}{\partial x}$ fn of x only

$$\int_0^\delta \frac{\partial u^2}{\partial x}dy + \int_0^\delta \frac{\partial(uv)}{\partial y}dy = u_e\frac{du_e}{dx}\int_0^\delta dy + \frac{1}{\rho}(\tau_e - \tau_w)$$

(51-5)

EQ (51-5) BECOMES:

$$\int_0^\delta \left(\frac{\partial u^2}{\partial x} - u_e \frac{du_e}{dx} \right) dy + u_e \underbrace{v_e}_{\int_0^\delta \frac{\partial u}{\partial x} dy + v_w} - u_w v_w = \frac{\tau_e - \tau_w}{\rho}$$

EQ (51-4) GIVES v_e SO ABOVE EQ. IS:

$$\int_0^\delta \left(\frac{\partial u^2}{\partial x} - u_e \frac{du_e}{dx} - u_e \frac{\partial u}{\partial x} \right) dy = \frac{\tau_e - \tau_w}{\rho} - v_w (u_e - u_w)$$

NOW L.H.S. BECOMES $\left[-\frac{d}{dx}(u_e^2 \theta) - \delta^* u_e \frac{du_e}{dx} \right]$ BY FOLLOWING METHOD ON P. 51 OF TEXT AND USING LEIBNITZ'S RULE, i.e.

$$\frac{d}{dx} \int_0^{\delta(x)} f\, dy = \int_0^{\delta(x)} \frac{\partial f}{\partial x} dy + f_e \frac{d\delta}{dx}$$

AND WE OBTAIN THE M.I. EQ:

$$\boxed{\begin{aligned} &\frac{d}{dx}(u_e^2 \theta) + \delta^* u_e \frac{du_e}{dx} = \frac{(\tau_w - \cancelto{0}{\tau_e})}{\rho} + v_w (u_e - u_w) \qquad (52\text{-}1) \\ &\text{OR} \\ &\frac{d\theta}{dx} = \frac{(\tau_w - \cancelto{=0\ (\text{NORMALLY})}{\tau_e})}{\rho u_e^2} + \frac{v_w}{u_e}\left(1 - \frac{u_w}{u_e}\right) - (H+2)\frac{\theta}{u_e}\frac{\partial u_e}{\partial x} \qquad (52\text{-}2) \end{aligned}}$$

WHERE: DISPLACEMENT THICKNESS: $\delta^* = \int_0^\delta \left(1 - \frac{u}{u_e}\right) dy$

for $y > \delta$ $1 - \frac{u}{u_e} = 0$ $\therefore \int_0^\delta = \int_0^y$

MOMENTUM THICKNESS: $\theta = \int_0^\delta \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy$

for $y > \delta$ $1 - \frac{u}{u_e} = 0$ $\therefore \int_0^\delta = \int_0^y$

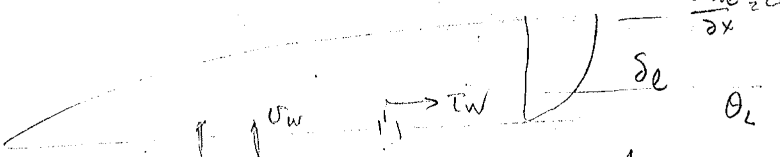
SHAPE FACTOR $H \equiv \delta^*/\theta$

SEE TEXT FOR OTHER CASES ($\rho \neq$ const, etc)

$$\theta = fn\left(\tau_w, H = \delta^*/\theta\right)$$

Drag on Flat plate with v_w

$u_e = \text{const}$ $\quad \frac{\partial u_e}{\partial x} = 0 \qquad u_w = 0$



$$\int_0^l \frac{d\theta}{dx}\,dx = \int_0^l \frac{\tau_w}{\rho u_e^2}\,dx + \frac{v_w}{u_e}\int_0^l dx$$

$$\theta\Big|_0^l \Rightarrow \frac{2\theta_l}{l} = 2C_D + 2C_Q \qquad C_Q = \frac{1}{l}\int$$

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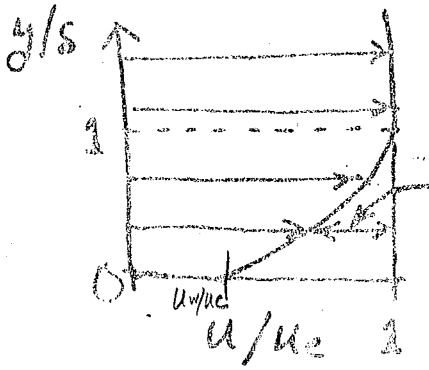
INTERPRETATION AND USE OF M.I. EQ



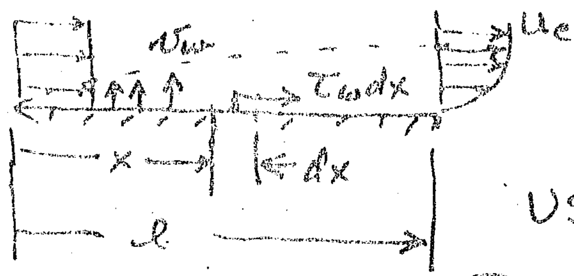
$\left(1-\frac{u}{u_e}\right)$: MASS FLOW RATE DEFICIT / AREA : $\dot{m}_d$

$\dot{m}_d u_e$: MOMENTUM flow RATE DEFICIT / AREA

EXAMPLE DRAG ON PLATE WITH v_w (SUCTION / BLOWING) BUT WITH $u_w = 0$

IN THIS CASE X-MOM TRANSFER TO/FROM PLATE $= 0$ ($u_w = 0$) SO DRAG FORCE IS ALL DUE TO τ_w, i.e. DRAG PER UNIT WIDTH OF 2-D PLATE IS FOR B.L. ON ONE SIDE, ($u_e = $ CONST).

$\frac{\partial u_e}{\partial x} = 0$



$$F_D = \int_0^{\ell} \tau_w \, dx$$

USING (52-1) GIVES

$$F_D = \rho u_e^2 \theta \Big|_0^{\ell} - \rho u_e \int_0^{\ell} v_w \, dx$$

Define Drag Coef:

since $\theta_{x=0} = 0$

$$C_D = \frac{F_D}{\frac{1}{2}\rho u_e^2 \ell} = 2\frac{\theta_\ell}{\ell} - 2C_Q \qquad (53\text{-}1)$$

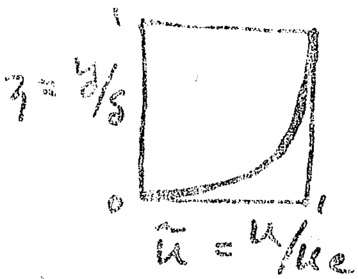
Where: $C_Q = \frac{1}{\ell}\int_0^{\ell}\left(\frac{v_w}{u_e}\right)dx = \frac{Q}{\ell u_e}$ (53-2) average flowrate coeff.

$$Q = \int_0^{\ell} v_w \, dx$$

DRAG LAWS – SIMILARITY PROFILES

BASED ON ASSUMPTIONS ABOVE HOW DO WE RELATE C_D (IN 53-1) TO INDEPENDENT PARAMETER, THE PLATE RE NO, $R \equiv (U_e \ell / \nu)$?

ASSUME IN THIS EXAMPLE THAT ALL VELOCITY PROFILES AT ALL x STATIONS ARE IDENTICAL IN SHAPE, I.E. SIMILAR ie collapse all velo profiles



$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_w = \frac{\mu U_e}{\delta} \left(\frac{\partial \tilde{u}}{\partial \eta} \right)_{\eta=0}$$

$$\beta \equiv \left(\frac{\partial \tilde{u}}{\partial \eta} \right)_{\eta=0} = \text{Const. for similar profiles}$$

$$\therefore \boxed{\tau_w = \frac{\mu U_e}{\delta} \beta} \qquad (54\text{-}1)$$

ALSO:

$$\boxed{\theta = \delta \int_0^1 \tilde{u} (1 - \tilde{u}) d\eta = \delta \alpha} \qquad (54\text{-}2)$$

$$\alpha = \int_0^1 \tilde{u} (1 - \tilde{u}) d\eta = \text{Constant for similar profiles}$$

M.I. Eq at any x is (U_e = Const.) $u_w = 0$ $\frac{dp}{dx} = 0$

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho U_e^2} + \frac{v_w}{U_e}$$

$$\alpha \frac{d\delta}{dx} = \frac{\nu}{\delta U_e} \beta + \frac{v_w}{U_e}$$

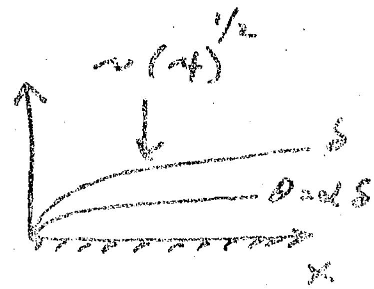
Similarity demands $\alpha, \beta \neq f(x)$.

for α and β = Const. this may be $\int_0^x$ if we assume

$$\boxed{\frac{v_w}{U_e} = \frac{K}{\delta}}$$

v_w decreases in same manner as $\tau_w \downarrow$ as $\delta \uparrow$

$$\alpha \int_0^\delta \delta\, d\delta = \left[\frac{\nu\beta}{u_e} + K\right]\int_0^x dx$$



$$\alpha \frac{\delta^2}{2} = \left[\frac{\nu\beta}{u_e} + K\right] x$$

$$\theta^2 = 2\alpha\left[\frac{\nu\beta l}{u_e l} + \frac{lK}{l}\right] x$$

So: $\theta = \alpha\delta = \left(2\alpha\left[\frac{\nu\beta l}{u_e} + \frac{K l}{l}\right]\right)^{1/2} (x)^{1/2} = c^{\frac{1}{2}} x^{\frac{1}{2}} l^{\frac{1}{2}}$

Define

So: $\boxed{C = 2\alpha\left[\frac{\beta}{Re} + \frac{K}{l}\right]}$ (55-1) $\therefore \theta = C^{1/2} x^{1/2}$

$$\boxed{\left(\frac{\theta}{l}\right) = \alpha\left(\frac{\delta}{l}\right) = C^{1/2}\left(\frac{x}{l}\right)^{1/2}} \qquad (55-2)$$

So at $x = l$, END OF PLATE

$$\theta_l = C^{1/2}\, l$$

$$\delta = c^{\frac{1}{2}} x^{\frac{1}{2}} l^{\frac{1}{2}}$$

From (53-1) where $C_Q = \frac{1}{l}\int_0^l \frac{K}{\delta}\, dx = \frac{K\alpha}{l^2 c^{\frac{1}{2}} l^{\frac{1}{2}}}\int_0^l \frac{dx}{x^{\frac{1}{2}}}$

$$2x^{\frac{1}{2}}\Big|_0^l$$

$$C_Q = \frac{K}{l^2}\frac{\alpha l^{1/2}}{C^{1/2}}\int_0^l \frac{dx}{x^{1/2}} = \frac{2K\alpha}{l\, C^{1/2}} \leftarrow \frac{2K\alpha}{l c^{\frac{1}{2}}}$$

We see:

$$C_D = 2C^{1/2} - 4\frac{K\alpha}{l C^{1/2}}$$

OR

$$C_D = 2\sqrt{2}\,\alpha^{1/2}\left\{\left[\frac{\beta}{Re} + \frac{K}{l}\right]^{1/2} - \frac{K}{l}\left[\frac{\beta}{Re} + \frac{K}{l}\right]^{-1/2}\right\} \qquad (55-3)$$

CASE OF VERY WEAK SUCTION OR BLOWING:

$$\left(\frac{v_w}{u_e}\right)_{x=l} = \left(\frac{K}{\delta}\right) \ll 1$$

$$K \ll \underbrace{\delta \ll l}_{\text{B.L. APPROXIMATION}}$$

$$\therefore \boxed{\frac{K}{l} \lll 1}$$

NOW FROM (55-2) $\quad \dfrac{\delta_l}{l} = \dfrac{C^{1/2}}{\alpha}$

$$\text{So} \quad \frac{K}{l} = \left(\frac{K}{\delta_l}\right)\left(\frac{\delta_l}{l}\right) = \left(\frac{K}{\delta_l}\right)\frac{C^{1/2}}{\alpha}$$

Plug into C_Q term of (55-3)

$$\boxed{C_D = 2C^{1/2} - 4\left(\frac{K}{\delta_l}\right) = 2C^{1/2} - 4\left(\frac{v_w}{u_e}\right)} \quad (55\text{-}4)$$

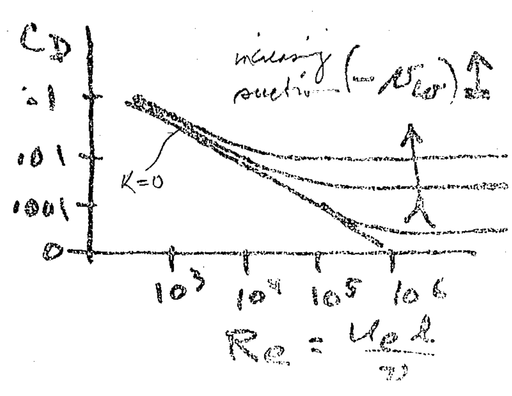
AND SO WITH ($K/l \lll 1$) WE OBTAIN

$$C_D = 2\sqrt{2\alpha}\left[\frac{\beta}{Re} + \cancel{\frac{K}{l}}\right]^{1/2} - 4\left(\frac{v_w}{u_e}\right)_{x=l}$$

0 — if small ($\ll 1$)

$$\boxed{C_D \approx \frac{2\sqrt{2\alpha\beta}}{\sqrt{Re}} - 4\left(\frac{v_w}{u_e}\right)_{x=l}}$$

when SUCTION small



if suction $K<0 \Rightarrow v_w<0$

BOUNDARY LAYERS WITH PRESSURE GRADIENTS

Consider steady, 2-D, laminar b.l. Eq's (51-3) where parameter ν absorbed by rescaling v and y; i.e. v and y are stretched by factor $1/\sqrt{\nu}$ $\quad \nu \sim \frac{1}{Re}$

namely:

$$\tilde{v} = \frac{(v)_{physical}}{\sqrt{\nu}} \; ; \; \tilde{y} = \frac{(y)_{physical}}{\sqrt{\nu}} \Rightarrow$$

DE rescales to $u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} = u_e\frac{du_e}{dx} + \frac{\partial^2 u}{\partial \tilde{y}^2}$

Using streamfunction Eq's (51-3) become

$$\left[\begin{array}{c} \frac{\partial\psi}{\partial y}\left(\frac{\partial^2\psi}{\partial x\,\partial y}\right) - \frac{\partial\psi}{\partial x}\left(\frac{\partial^2\psi}{\partial y^2}\right) = u_e u_e' + \frac{\partial^3\psi}{\partial y^3} \\ \text{where: } u = \frac{\partial\psi}{\partial y} \; ; \; v = -\frac{\partial\psi}{\partial x} \end{array}\right] \quad (57\text{-}1)$$

B.C.'s are:

no flow through wall

$$\frac{\partial\psi}{\partial y} = 0 \quad \text{at } y=0 \; ; \; \frac{\partial\psi}{\partial y} \to u_e \text{ as } y \to \infty$$

note still inside bl since we rescale.

$$\frac{\partial\psi}{\partial x} = 0 \quad \text{at } y = 0$$

Since laminar flow are parabolic $\Rightarrow$ need initial condition in order to start solution

Similarity Transformation ASSUME

A transformation where $g(x)$ is a scale thickness of the local B.L., define a new similarity coordinate,

$$\boxed{Y = \frac{y}{g}} \quad (58\text{-}1) \quad \text{For similarity} \quad \boxed{\frac{u}{u_e} = F'(Y)} \quad (58\text{-}2)$$

Let: $$\boxed{\psi = u_e(x)\, g(x)\, F(Y)} \quad (58\text{-}3)$$

$$u = \frac{\partial\psi}{\partial y} = u_e g F' \frac{\partial Y}{\partial y} = u_e g F'\left(\frac{1}{g}\right) = u_e F'$$

so we see that separable form for ψ looks OK and $F(Y)$ is dimensionless stream function. We can obtain v from

$$-v = \frac{\partial\psi}{\partial x} = u_e g F'\left(\frac{\partial Y}{\partial x}\right) + (u_e g)' F$$

$$\boxed{\left(\frac{\partial Y}{\partial x}\right) = -\frac{y}{g^2} = -\frac{Y}{g}}$$

so $$\boxed{\frac{v}{u_e} = F'Y - \frac{(u_e g)'}{u_e} F} \quad (58\text{-}4)$$

From these we obtain B.C.'s

At wall $(Y=0)$: $F'=0$, $F=0$ (see v eq above)

At Free stream $(Y \to \infty)$: $F' \to 1$

And with considerable manipulation the D.E. becomes if $F(Y, x)$ generally

$$\left[F''' + \alpha F F'' + \beta (1 - F'^2) = \gamma \right] \quad (59\text{-}1)$$

where α and β are given by the ODE's

$$\left[\alpha = g (u_e g)' \quad \text{and} \quad \beta = g^2 u_e' \right] \quad (59\text{-}2)$$

and the R.H.S. of (59-1) is γ

$$\left[\gamma = u_e g^2 \left(F' \frac{\partial F'}{\partial x} - F'' \frac{\partial F}{\partial x} \right) \right] \quad (59\text{-}3)$$

NOW FOR SIMILARITY SOLUTIONS, $F(Y$ only$)$

$\gamma = 0$ and α and β must be constants. It is shown in Ref 1, pp 139-141 THAT IF $2\alpha - \beta \neq 0$ THEN WE MAY LET $\alpha = 1$ AND DEFINE A NEW CONSTANT m SUCH THAT

$$\left[m = \frac{\beta}{2 - \beta} \quad \text{AND} \quad \beta = \frac{2m}{m+1} \right] \quad (59\text{-}4)$$

ALSO THE SCALE THICKNESS

$$\left[g = \left(\frac{2}{m+1} \right)^{1/2} \left(\frac{x}{u_e} \right)^{1/2} \right] \quad (59\text{-}5)$$

AND THE SIMILARITY coordinate is

FOUND TO BE (WHERE y IS THE PHYSICAL COORDINATE, NOT $y/\sqrt{\nu}$)

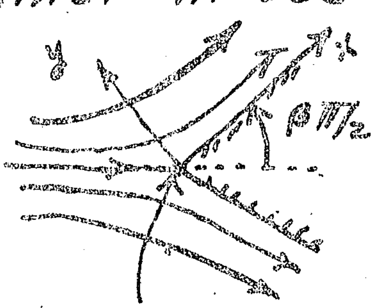
$$\left[\, Y = \left(\frac{m+1}{2}\right)^{1/2} \eta \quad \text{AND} \quad \eta = \left(\frac{u_e}{\nu x}\right)^{1/2} y \,\right] \quad (60\text{-}1)$$

USED IN "TEXT"

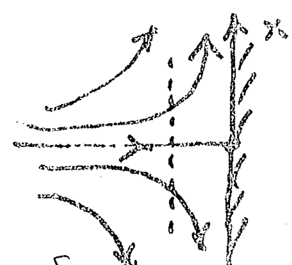
THE FREE STREAM VELOCITY ~~XXX~~ must VARY AS x^m, i.e.

$$\left[\, u_e = C x^m \,\right] \qquad \frac{du_e}{dx} = C m x^{m-1} = \frac{m u_e}{x} \qquad \therefore\ m = \frac{x}{u_e}\frac{du_e}{dx} \quad (60\text{-}2)$$

WE NOTE THAT FOR $0 \le m < \infty$ OR $0 \le \beta < 2$ THIS CORRESPONDS TO POTENTIAL FLOW OVER AN INFINITE WEDGE OF HALF ANGLE $(\beta\pi/2)$. THESE ARE CALLED "WEDGE FLOWS"



STAG. POINT

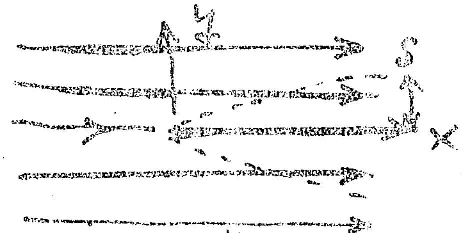


$\delta = \text{const}$ $u_e \sim x$

$\beta = 1$

$m = 1$

BLUNT STAG. PT.



$\delta \sim x^{1/2}$ $u_e = \text{const}$

$\beta = 0$

$m = 0$

BLASIUS FLAT PLATE

SHAPE OF VELOCITY PROFILES GIVEN BY SOLUTION OF

$$\boxed{F''' + FF'' + \beta(1 - F'^2) = 0} \quad (60\text{-}3)$$

SUBJECT TO $\left\{ \begin{array}{ll} F' = 0,\ F = 0 & \text{AT } Y = 0 \\ F' = 1 & \text{AT } Y = \infty \end{array} \right\}$

EXPRESS SOLUTION IN TERMS OF "TEXT" SIMILARITY COORDINATES

$$\eta = \left(\frac{u_e}{\nu x}\right)^{1/2} y \quad , \text{ where } \eta = \left(\frac{2}{m+1}\right)^{1/2} Y$$

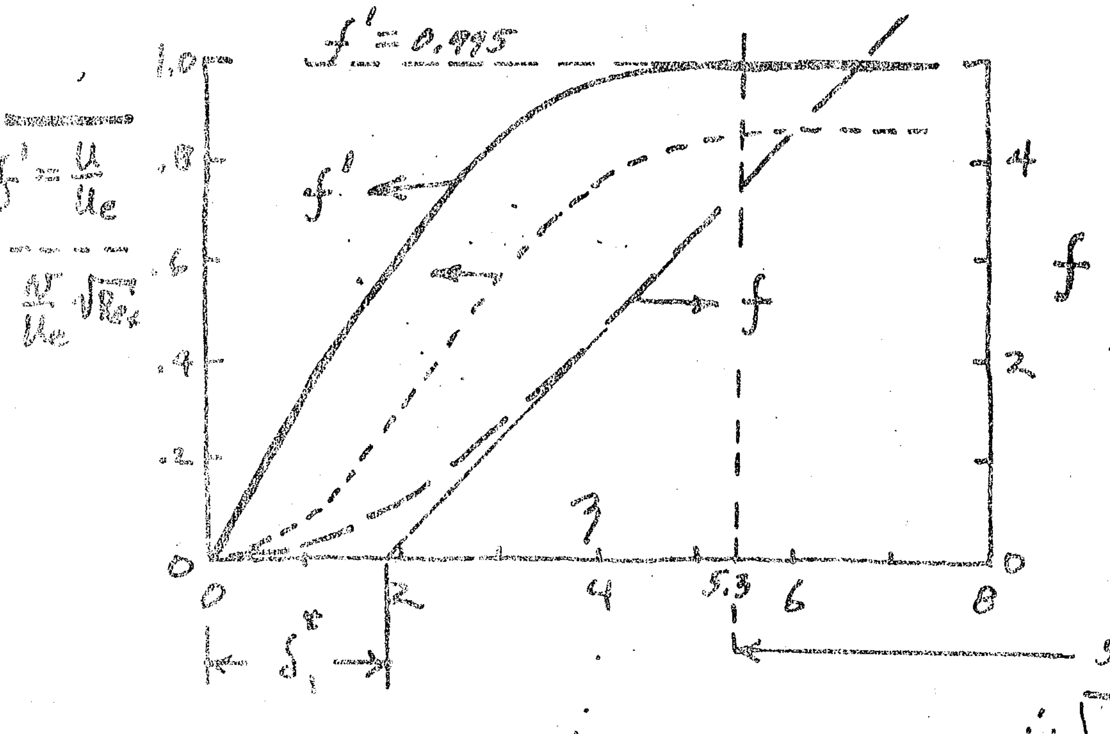
$$\left(\frac{u}{u_e}\right) = f'(\eta) \quad , \text{ where } f = \left(\frac{2}{m+1}\right)^{1/2} F$$

$$\left(\frac{v}{u_e}\right)\sqrt{Re_x} = \frac{\eta}{2} f' - \left(\frac{m+1}{2}\right) f \quad , \text{ where } Re_x = \frac{u_e x}{\nu}$$

CASE OF "BLASIUS" FLAT PLATE $(\beta = m = 0)$

D. EQ. IS: $\boxed{F''' + FF'' = 0}$

SOLUTION IS (FIG'S 5.2, 5.3 IN "TEXT")



$$\delta^* = \delta\int_0^\delta (1 - f')\,d\eta$$
$$\theta^* = \delta\int_0^\delta f'(1 - f')\,d\eta$$

$$5.3 = \sqrt{\frac{u_e}{\nu x}}\,\delta_{.995}$$

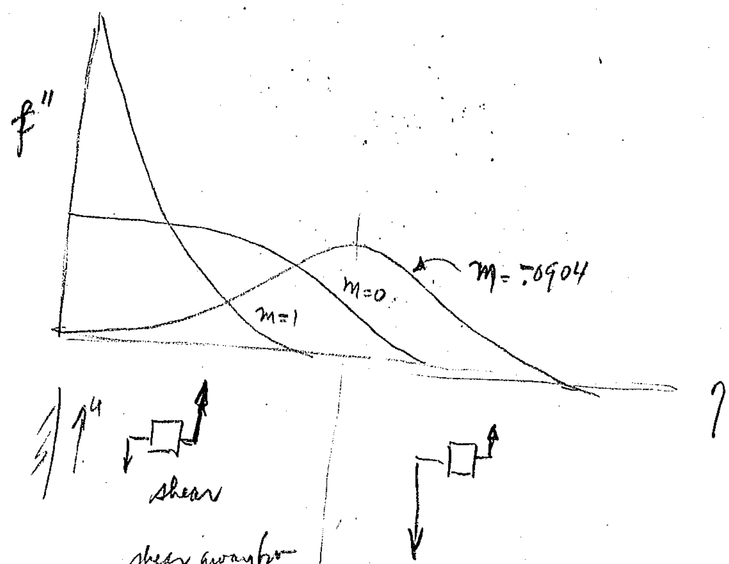
$$\therefore \boxed{\frac{\delta}{x} = \frac{5.3}{\sqrt{Re_x}}}$$

$$\frac{\tau_w}{\frac{1}{2}\rho u_e^2} = C_f = \frac{f''(\eta = 0)}{\sqrt{Re_x}} = \frac{0.664}{\sqrt{Re_x}}$$

$$\left.\begin{aligned} \delta^* &= \delta_1^* / \sqrt{u_e/\nu x} = 1.721 / \sqrt{u_e/\nu x} \\ \theta^* &= \theta_1 / \sqrt{u_e/\nu x} = 0.664 / \sqrt{u_e/\nu x} \end{aligned}\right\} H = \frac{\delta^*}{\theta} = 2.591$$

$$\frac{\tau_w}{\rho u_e^2}\sqrt{Re_x} = f''(0)$$

for $m < 0$ $\frac{dp}{dx} \uparrow$



shear away from wall is higher & allows fluid to move against pressure grad

2/5/79

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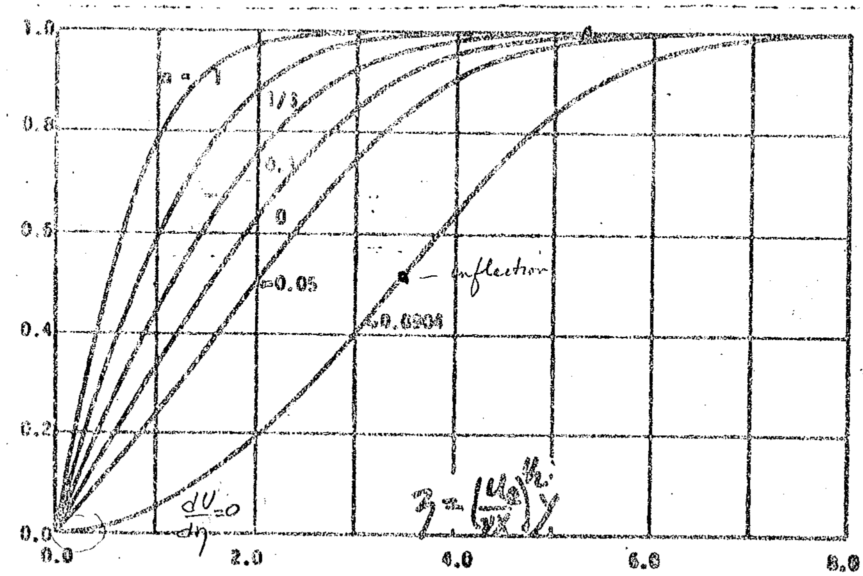
CASES OF "WEDGE" FLOW ($0 \le m \le \infty$) AND ADVERSE PRESSURE GRADIENT UP TO SEPARATION (HARTREE SOLUTIONS TO FALKNER + SKAN PROBLEM

$f' = \frac{u}{u_e}$



Free Stream

$u_e = C x^m$

C = const.

$u_e' = Cm x^{m-1} = \frac{m u_e}{x}$

$\frac{dp}{dx} = -\rho u_e u_e' = -\frac{\rho m u_e^2}{x}$

if $m = 0$ $\frac{dp}{dx} = 0$ or P = const

Table 5.1 Solutions of the Falkner-Skan Equation for Positive Wall Shear

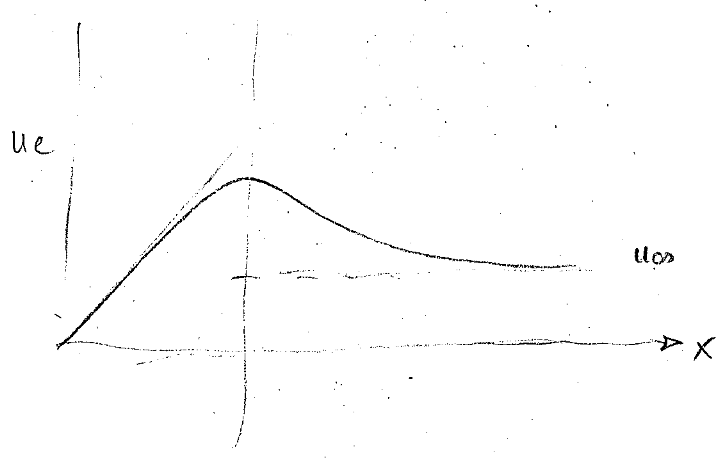
m	$f''(0) \equiv \frac{1}{2} c_f \sqrt{Re_x}$	$\delta_1^* \equiv \delta^* \sqrt{u_0/\nu x}$	$\theta_1 \equiv \theta \sqrt{u_0/\nu x}$	H	
1	1.23259	0.64791	0.29234	2.216	BLUNT STAG. POINT
1/3	0.75745	0.98536	0.42900	2.297	
0.1	0.49657	1.34782	0.55660	2.422	
0	0.33206	1.72074	0.66412	2.591	FLAT PLATE
−0.01	0.31148	1.78000	0.67892	2.622	
−0.05	0.21351	2.1174	0.75148	2.818	
−0.0904	0.0	3.4277	0.86797	3.949	SEPARATION

STAG. Point Case $y = \delta$ at $\eta = 3.0$

$$\therefore \delta \left(\frac{u_e}{\nu x}\right)^{1/2} = 3.0$$

$$\delta \left(\frac{C x}{\nu x}\right)^{1/2} = 3.0$$

$$\therefore \delta = 3.0 \left(\frac{\nu}{C}\right)^{1/2} = \text{const.}$$

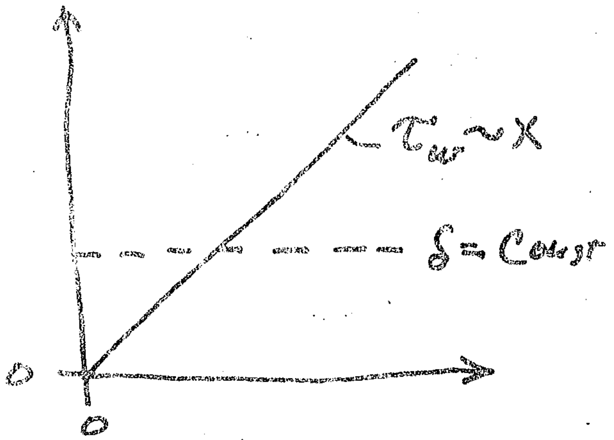


ue
uoo
x

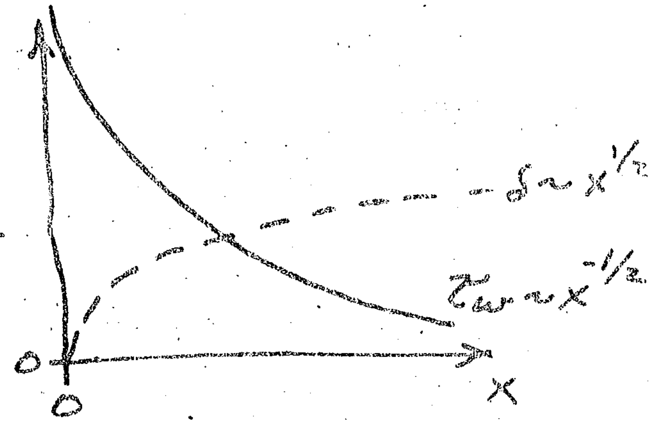
Stag. Point shear stress at wall:

$$\frac{\tau_w}{\rho u_e^2}\sqrt{\frac{u_e x}{\nu}} = 1.23259$$

$$\tau_w = 1.23259\left(\frac{\nu\rho^2 u_e^3}{x}\right)^{1/2} = 1.233\left(\mu\rho C^3 x^2\right)^{1/2} = Kx$$



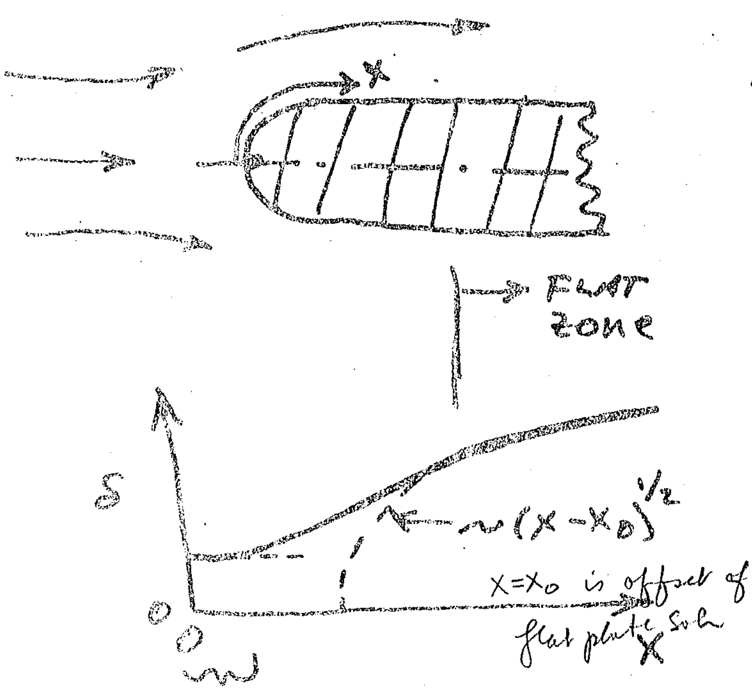
BLUNT STAG. PT.

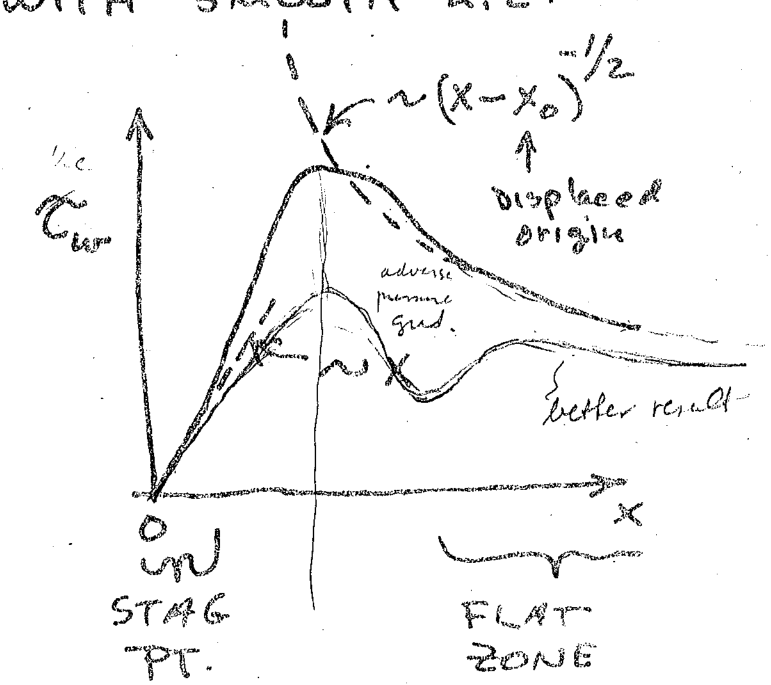


FLAT PLATE

A REAL EXAMPLE!

CONSIDER PLATE WITH SMOOTH L.E.





for non similar flows

① Solve DE for laminar flow, $\tau = \mu \frac{\partial u}{\partial y}$ fortran prog pp 256-280

② Integral EQ methods using

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho u_e^2} - (2+H)\frac{\theta}{u_e} u_e'$$

veloc profile

$$\frac{u}{u_e} = f\left(\left(\eta = \frac{y}{\delta}\right), \Pi_1(x), \Pi_2(x), \ldots\right)$$

for similar flows only.

no. parameters

normally take in one param

$m = \frac{x}{u_e}\frac{du_e}{dx}$ is location dependent

Laminar B.L.'s

Solution for Non-similar flows

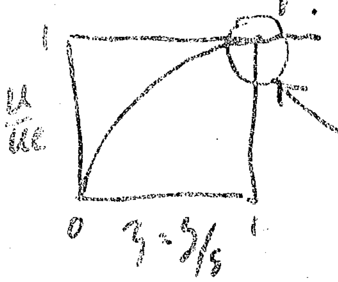
Represent vel. profiles ($\eta = y/\delta$)

$$\frac{u}{U_e} = f\left(\frac{y}{\delta}\right) \quad \text{for "Similar" flows of } M = \text{const.}$$

But in general cases other parameters may be important too

$$\frac{u}{U_e} = f\left(\eta, \Pi_1(x), \Pi_2(x), \cdots\right)$$

Establishment of profiles by use of Conditions on velocity profiles for case of solid walls and edges



Wall : $\left(u/U_e\right)_w = 0$ no slip

Edge : $\left(u/U_e\right)_e = 1$ matching

But also $\left(\frac{\partial u}{\partial y}\right)_e = \left(\frac{\partial^2 u}{\partial y^2}\right)_e = \left(\frac{\partial^n u}{\partial y^n}\right)_e = 0$

From b.l. eq. at wall we also obtain

$$\cancel{u}^{\,0 \text{ no slip bc}} \frac{\partial u}{\partial x} + \cancel{v}^{\,0 \text{ no flow thru wall}} \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$\therefore \boxed{\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_w = \frac{1}{\rho}\frac{dp}{dx} = -U_e U_e'} \quad (64\text{-}1)$$

Called wall compatibility condition

A possible parameter that controls velocity profiles based on (64-1) is

$$\frac{\nu U_e}{\delta^2}\left(\frac{\partial^2 (u/U_e)}{\partial \eta^2}\right)_{\eta=0} = -U_e U_e'$$

Possible Π = Const for F,-S "Similarity"

2nd Par. $P = \Lambda = \frac{\delta^2 u_e'}{\nu}$ for 4th order poly Pohlhausen

$P = \lambda = \frac{\theta^2 u_e'}{\nu}$ Thwaites

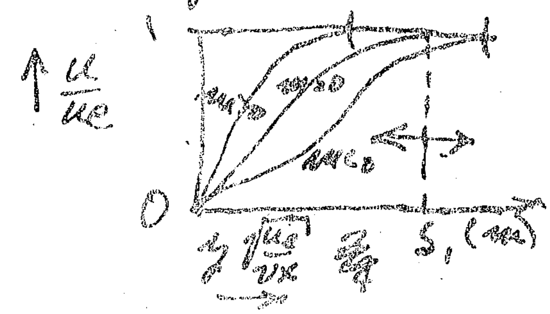
or
$$\boxed{\left(\frac{\partial^2 (u/u_e)}{\partial \eta^2}\right)_w = -\left(\frac{\delta^2}{\nu}\frac{du_e}{dx}\right) = -\Lambda} \qquad (65\text{-}1)$$

defines a parameter Λ which controls curvature of dimensionless profile at the wall.

As a next order of approximation above simple similarity, one could assume a one-parameter family

$$\frac{u}{u_e} = f(\eta, \Lambda) \quad \text{where } \Lambda \neq \text{const. with } x$$

Example, we could use similar profiles
i.e, Value of Λ for Falkner-Skan similarity profiles



$$\delta = \delta_1 \sqrt{\frac{\nu x}{u_e}} \; ; \quad \delta_1 = \delta_1(m)$$

$$u_e = Cx^m$$

$$u_e' = Cmx^{m-1} = \frac{mCx^m}{x} = \frac{mu_e}{x}$$

$$\therefore \left(\Lambda\right)_{sim} \equiv \frac{\delta^2}{\nu} u_e' = \frac{\delta_1^2}{\nu}\left(\frac{\nu x}{u_e}\right) \frac{m u_e}{x}$$

$$\boxed{\left(\Lambda\right)_{similar} = m\,\delta_1^2} = \text{function of } (m) \quad (65\text{-}1)$$

$\therefore$ given value of Λ we could identify a similarity profile from the F.-S. family. We call this a <u>one-parameter family</u>

Other one-parameter families can be constructed. The most common is the 4th order polynomial.

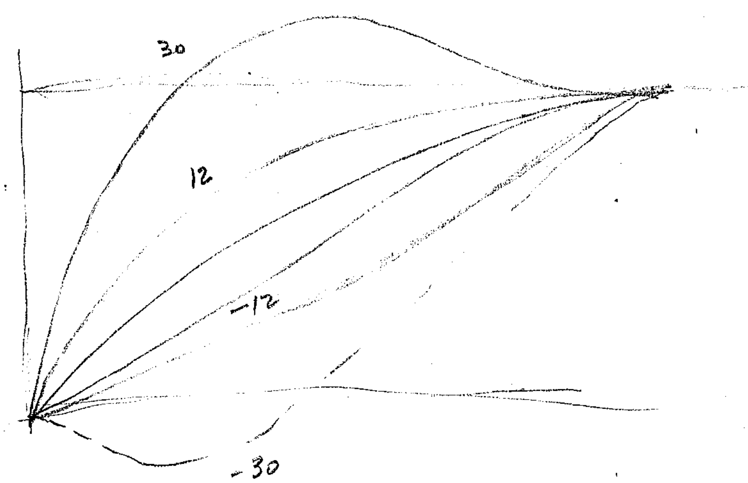
Must use $U = f(\eta)$

to satisfy $U=0 \quad \eta=0$ no slip $\Rightarrow a_0 = 0$

$\left.\frac{dU}{d\eta}\right|_{\eta=1} = 0$ veloc profile slope at outer edge

$\left.\frac{d^2U}{d\eta^2}\right|_{\eta=1} = 0$

$\left.\frac{d^2U}{d\eta^2}\right|_{\eta=0} = -\Lambda$ pressure based on what goes on in flow near wall.



$\Lambda < 0$ are adverse pressure gradients

$\therefore \; |\Lambda| < 12$

Pohlhausen (4th order Polynomial)

Assume: $$\frac{u}{u_e} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 \quad (1)$$

$$\frac{\partial\left(\frac{u}{u_e}\right)}{\partial\eta} = 0 + a_1 + 2a_2\eta + 3a_3\eta^2 + 4a_4\eta^3 \quad (2)$$

$$\frac{\partial^2\left(\frac{u}{u_e}\right)}{\partial\eta^2} = 0 + 0 + 2a_2 + 6a_3\eta + 12a_4\eta^2 \quad (3)$$

Apply $\left(\frac{u}{u_e}\right)_{\eta=0} = 0$ to (1) and get $a_0 = 0$

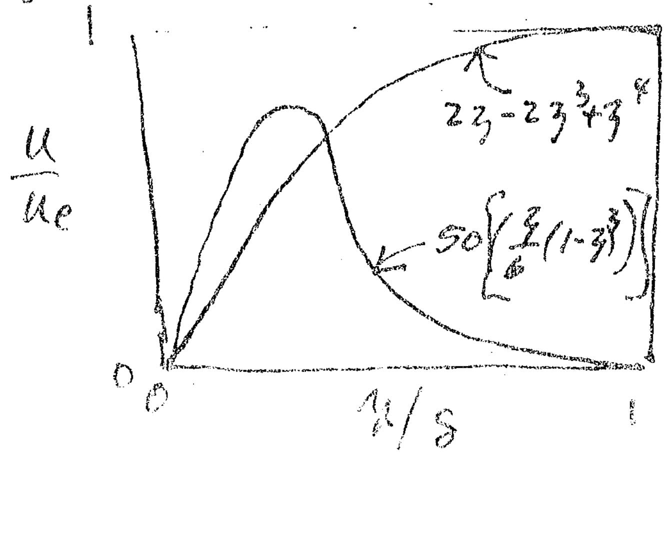
$\left(u/u_e\right)_{\eta=1} = 1$ and

then $\left(\frac{\partial\left(\frac{u}{u_e}\right)}{\partial\eta}\right)_{\eta=1} = 0$; $\left(\frac{\partial^2\left(\frac{u}{u_e}\right)}{\partial\eta^2}\right)_{\eta=1} = 0$;

and $\left(\frac{\partial^2(u/u_e)}{\partial\eta^2}\right)_{\eta=0} = -\Lambda$

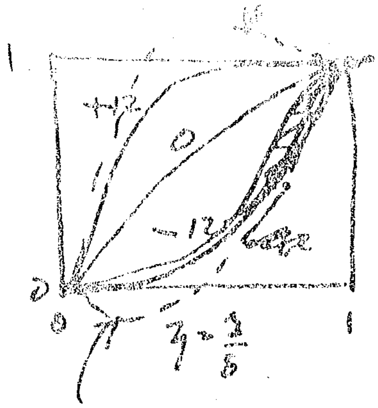
gives equation in 4 unknowns to obtain a_1, a_2, a_3, a_4. As a result of this

$$\boxed{\frac{u}{u_e} = \left[2\eta - 2\eta^3 + \eta^4\right] + \Lambda\left[\frac{\eta(1-\eta)^3}{6}\right]} \quad (66\text{-}1)$$



Profiles for various values of Λ shown in Fig 5.5 of the text.

$\Lambda > +12$ Polynomial no good



$\Lambda < -12$
Polynomial
no good

Useful range for Λ

adverse pressure grad $\quad -12 \leq \Lambda \leq 12 \quad$ favorable pressure grad

stress = 0 as wall

Wall shear stress

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu u_e}{\delta}\left(\frac{\partial(u/u_e)}{\partial \eta}\right)_{\eta=0}$$

from the polynomial

$$\boxed{\tau_w = \frac{\mu u_e}{\delta}\left[2 + \frac{\Lambda}{6}\right]}$$

let $\frac{\mu u_e}{\delta}$ be the dimensionless value (67-1) for τ_w

$\therefore$ $\tau_w/\mu u_e$ is a dimensionless quantity

and we see that $\Lambda = -12$ corresponds to flow <u>separation</u>: for 4th order Polynomial.

For <u>Falkner-Skan similarity</u> profiles shown in fig 5.4 of Text

$$\delta_1 \approx 8.0, \quad m = -0.0904 \text{ at}$$

separation, $\therefore$ From eq (65-1) we obtain

$$\boxed{(\Lambda)_{\substack{F\text{-}S \\ sep}} \approx -0.0904 \times (8)^2 \approx -5.8}$$

Compared to $\Lambda = -12$ for Pohlhausen $\Lambda = -5.8$ would ~~sep~~ not be separation for Pohlhausen. In fact Pohlhausen method (see text pp 107) doesn't give good laminar separation flow solution

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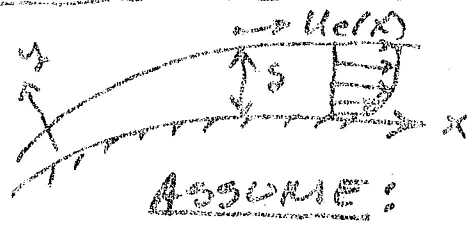
LAMINAR B.L., NON SIMILAR APPROX METHOD.

MOMENTUM INTEGRAL EQ.



GIVEN: $U_e(x)$

$$\frac{d U_e}{dx} = U_e'$$

$$\sigma_w = 0$$

ASSUME:

ONE PARAMETER PROFILE MODEL

$$\frac{u}{U_e} = f\left(\eta = \frac{y}{\delta}, \Pi\right)$$

PARAMETER: (1) 4'th order Polynomial

$$\Lambda = \frac{\delta^2}{\nu} U_e'$$ use when we have exact soln.

(2) Other definition used when working w/experimental data

$$\lambda = \frac{\theta^2 U_e'}{\nu} = \left(\frac{\theta}{\delta}\right)^2 \Lambda$$

FROM EQ. (52-2)

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho U_e^2} - (2+H)\frac{\theta}{U_e} U_e' \qquad (52-2)$$

MULTIPLY by $(\theta U_e/\nu)$

$$\frac{U_e}{\nu}\frac{d\theta^2}{dx} = 2\left[\frac{\tau_w \theta}{\mu U_e} - (2+H)\frac{\theta^2 U_e'}{\nu}\right]$$

$$U_e \frac{d}{dx}\left(\frac{\lambda}{U_e'}\right) = 2\left[\left(\frac{\tau_w \theta}{\mu U_e}\right) - (2+H)\lambda\right] \qquad (58-1)$$

for adverse pressure grads. you cannot represent the veloc. profile in terms of one param only

$$\cancel{v = f(\eta, \Lambda)}$$

$$v = f(\eta, \Lambda_1, \Lambda_2)$$

NOW ALL TERMS ARE DIMENSIONLESS AND IT IS REASONABLE TO ASSUME THAT THE TWO PARAMETER, $\left(\frac{\tau_w \theta}{\mu U_e}\right)$, AND, $(H = \delta^*/\theta)$, ARE FUNCTIONS OF THE PROFILE PARAMETER, λ, i.e.

$$\frac{\tau_w \theta}{\mu U_e} = \ell(\lambda) \quad \text{AND} \quad H(\lambda) \qquad (69\text{-}1)$$

<u>THWAITES</u> METHOD: THWAITES DID NOT ASSUME SOME PARTICULAR PROFILE, e.g. 4'th order, BUT CORRELATED ALL AVAILABLE (1949) EXACT AND EXPERIMENTAL RESULTS AND OBTAINED RESULTS BELOW

SHEAR AND SHAPE FUNCTIONS CORRELATED BY THWAITES (1949)

λ	$H(\lambda)$	$\ell(\lambda)$	λ	$H(\lambda)$	$\ell(\lambda)$
+0.25	2.00	0.500	−0.056	2.94	0.122
0.20	2.07	0.463	−0.060	2.99	0.113
0.14	2.18	0.404	−0.064	3.04	0.104
0.12	2.23	0.382	−0.068	3.09	0.095
0.10	2.28	0.359	0.072	3.15	0.085
+0.080	2.34	0.333	−0.076	3.22	0.072
0.064	2.39	0.313	−0.080	3.30	0.056
0.048	2.44	0.291	−0.084	3.39	0.038
0.032	2.49	0.268	−0.086	3.44	0.027
0.016	2.55	0.244	−0.088	3.49	0.015
0.0	2.61	0.220	−0.090	3.55 (Separation)	0.000
−0.016	2.67	0.195			
−0.032	2.75	0.168			
−0.040	2.81	0.153			
−0.048	2.87	0.138			
−0.052	2.90	0.130			

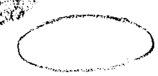
SEE EQ'S (5.6.20), P. 110, IN TEXT

Thwaites's methods doesn't depend on derivatives as Pohlhausen is. It only depends on u_e

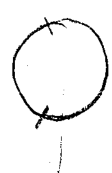
Similarity Sol. $\frac{u}{u_e} = \left(\frac{x}{c}\right)^m$ Hartree (1939)

Non Sim. $\frac{u}{u_e} = 1 - \frac{x}{c}$ Howarth (1938)
Assumes series solution x, y

$\left\{ \begin{array}{l} u_e = \text{const} \\ v_N = \pm \text{const} \end{array} \right\}$ blow/suction Iglish (1949) NACA TM 1205 1949



Schubauer's steady flow over ellipse NACA Rept 652 1939

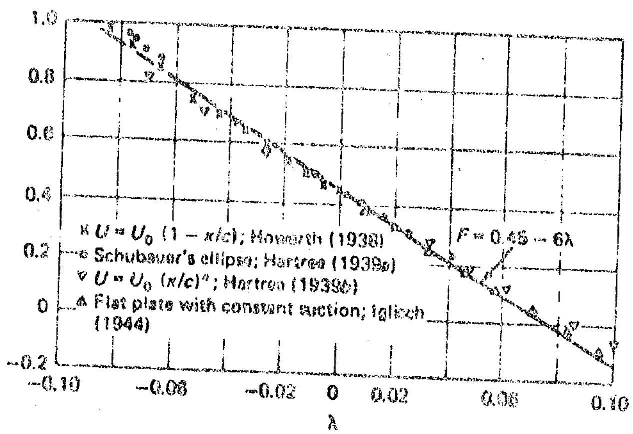


Görtler ~1957

PUT THIS INTO M.I. EQ, (68-1):

$$\frac{u_e}{\nu}\frac{d\theta^2}{dx} = 2\left[\ell(\lambda) - (2 + H(\lambda))\lambda\right] \equiv F(\lambda)$$

THAT IS THE R.H.S. MUST BE A FUNCTION OF λ. FROM THWAITES WE SEE



$F = 0.45 - 6\lambda$ CORRELATION

SUBSTITUTE $F(\lambda)$ INTO M.I. EQ AND MULTIPLY BY u_e^5

$$\frac{u_e^6}{\nu}\frac{d(\theta^2)}{dx} = u_e^5\left(0.45 - 6\,\frac{\theta^2}{\nu}\frac{du_e}{dx}\right)$$

SO WE OBTAIN

$$\frac{1}{\nu}\frac{d}{dx}\left(\theta^2 u_e^6\right) = 0.45\,u_e^5$$

UPON INTEGRATION

$$\boxed{\frac{\theta^2 u_e^6}{\nu} = 0.45\int_0^x u_e^5\,dx + \left(\frac{\theta^2 u_e^6}{\nu}\right)_{x=0}} \qquad (70\text{-}1)$$

NOTE: R.H. Term = 0 at leading edge of flat plate and at a stagnation point.

2/16/79 ÷ problem set #2

Outline of Sect VI
Turbulent shear flows
1. Intro to transition to turb.
2. Turb. BL on flatplate, u_e = const crude approx

3. Elements of Stability Theory and transition in wall BL

4. Turbulent flow
- Reynolds stresses
- Eqn of motion
- thin shear layers ϵ and l models
- far field wakes/jets | s.l.

5. Turbulent BL
- description of mean veloc profile law of wall
- C_f laws
- eddy visc. + mixing length
- Integral eq analysis (entrainment methods)

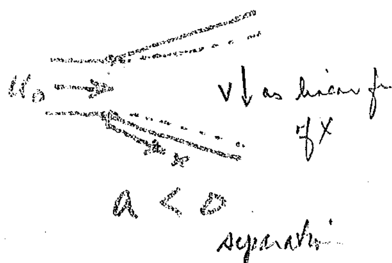
Outline in Section VII
Internal flows
- zones of interaction
- blockage
- applic to diffuser flows
- real flow effects
- viscid/inviscid interactions

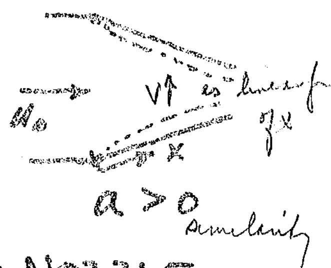
accelerating flows tend to similarity solution

EXAMPLE OF USE OF THWAITES METHOD:

CASE: $\boxed{U_e = U_0 + aX}$

WHERE a = CONST, $U_0 = U_e$ at $X = 0$





$\delta = 0$ at $X = 0$

$\boxed{U_e' = a}$

2-D DIFFUSER 2-D NOZZLE

(SMALL INCLUDED ANGLES)

{NO "DISPLACEMENT" INTERACTION}

APPLY M.I. EQ (70-1) Howarth

$$\frac{\theta^2}{\nu}(U_0 + aX)^6 = 0.45 \int_0^X (U_0 + aX)^5 \, dX$$

$$\frac{\theta^2}{\nu}(U_0 + aX)^6 = 0.45 \left(\frac{(U_0 + aX)^6 - U_0^6}{6a} \right)$$

$$\boxed{\lambda = \frac{\theta^2 a}{\nu} = \frac{0.45}{6}\left[1 - \left(\frac{1}{1 + \frac{aX}{U_0}}\right)^6\right]} \qquad (71-1)$$

NOZZLE $(a > 0)$ let $\frac{aX}{U_0} \gg 1$, i.e. $X \to \infty$

(71-1) REDUCES TO $\frac{\theta^2 a}{\nu} = 0.075$ (71-2)

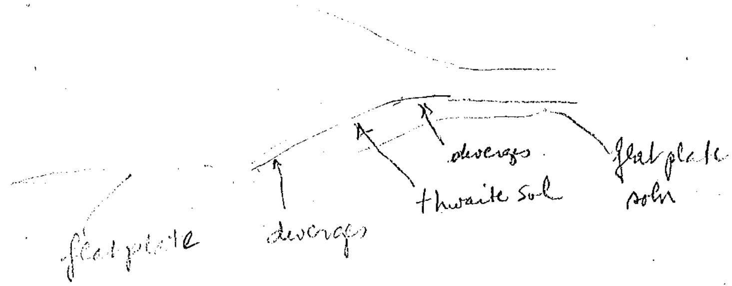
THIS IS LIKE approaching A SIMILARITY SOLUTION

$$U_e = aX; \quad a = U_e/X$$

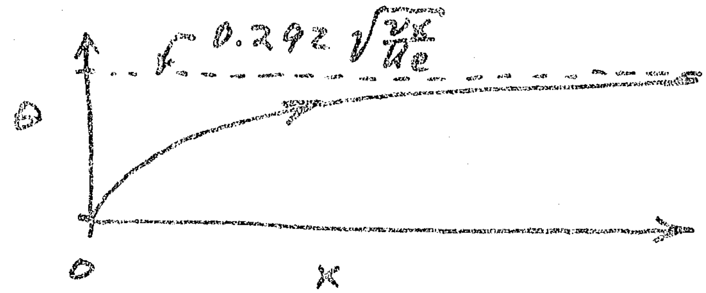
FOR $m = 1$, TABLE 5.1 GIVES $\theta_{ES}\sqrt{\frac{U_e}{\nu X}} = .29234$

FROM (71-2) $\theta\sqrt{\frac{U_e}{\nu X}} = 0.274$

$\lambda = \theta^2 a$


diverges
thwaite sol
flat plate
soln
flat plate
diverges

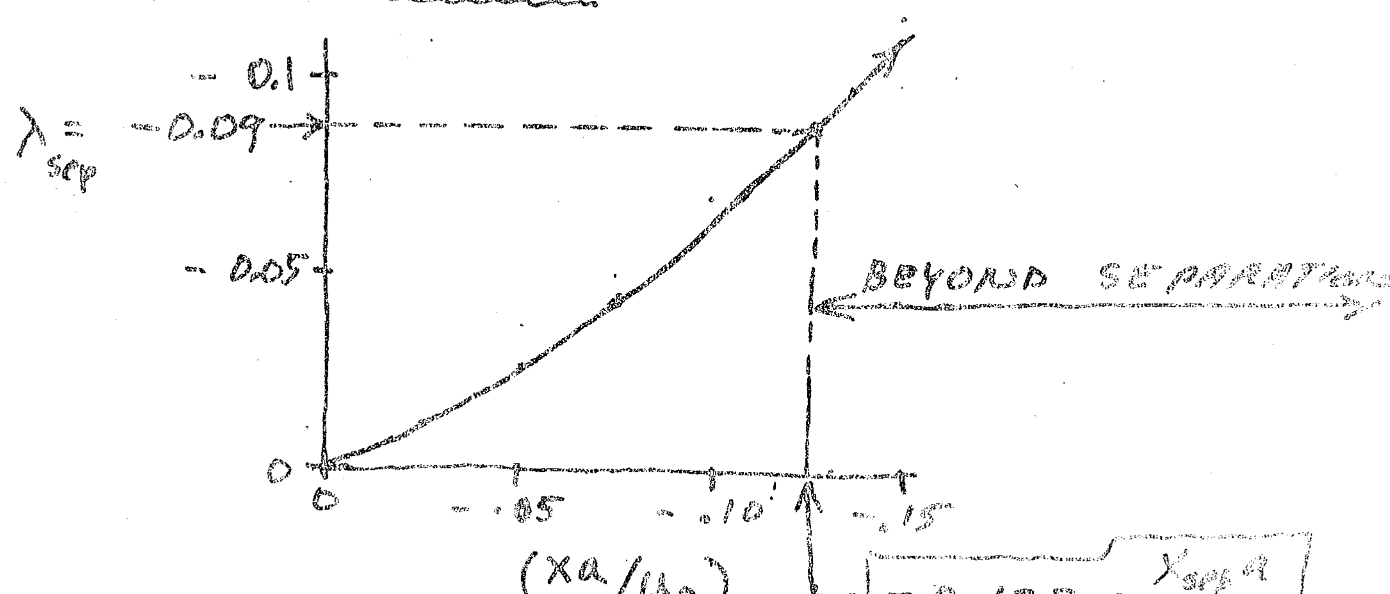
B.L. GROWTH ALONG A NOZZLE OF THIS TYPE TENDS TO SIMILARITY SOL.



? what if $u_e = u_0 + a x^m$

Same

DIFFUSER ($a < 0$)



$$\boxed{-0.123 = \frac{x_{sep}\, a}{u_0}}$$

PRESSURE COEF AT SE

$$(u_e)_{sep} = u_0\left(1 - \frac{x_{sep} a}{u_0}\right) = u_0(0.877)$$

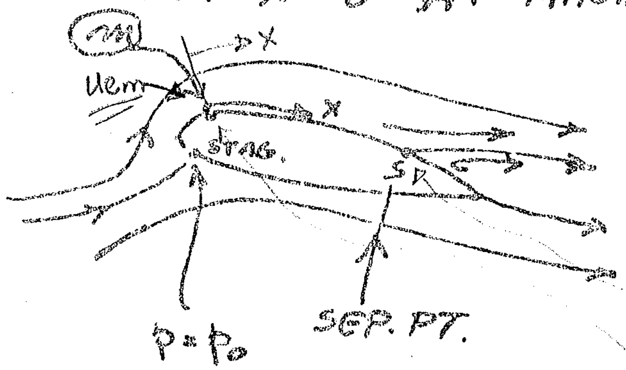
$$\therefore \boxed{\frac{u_{e\,sep}}{u_0} = 0.877} \quad \text{rate of sep } \frac{d\lambda}{dx} \qquad (72\text{-}1)$$

$$\boxed{(C_p)_{sep} = \frac{p_{sep} - p_0}{\frac{\rho}{2} u_0^2}\bigg|_{\substack{\text{bl@} \\ \text{0 thickness}}} = 1 - \left(\frac{u_{e\,sep}}{u_0}\right)^2 = 0.231} \qquad (72\text{-}2)$$

if flow is laminar then we get only 23% recovery

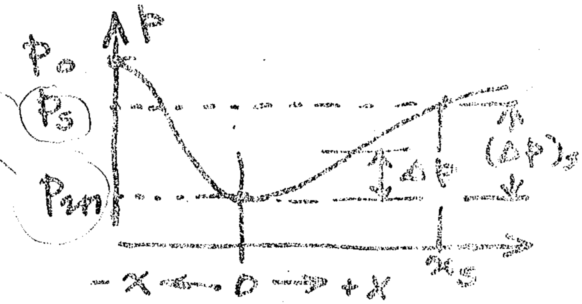
CRITERIA FOR 2-D, LAMINAR SEPARATION

SET X = 0 AT MINIMUM PRESSURE POINT



Pressure + U_e distributions

M IS POINT WHERE P IS MIN. AND U_e MAX

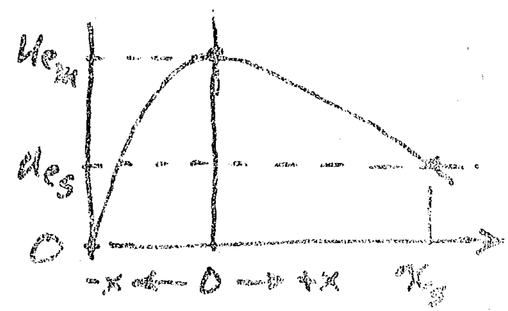


PRESS. COEF TO SEPARATION:

$$(C_p)_s = \frac{(\Delta P)_s}{\frac{\rho}{2} U_{em}^2} = 1 - \left(\frac{U_{es}}{U_{em}}\right)^2$$

VELOCITY RATIO TO SEPARATION:

$$\left(\frac{U_{es}}{U_{em}}\right)$$



VALUES BASED ON MANY SOLUTIONS AND EXPERIMENTS!

U_{es}/U_{em} = 0.88 TO 0.97

$(C_p)_s$ = 0.226 TO 0.056

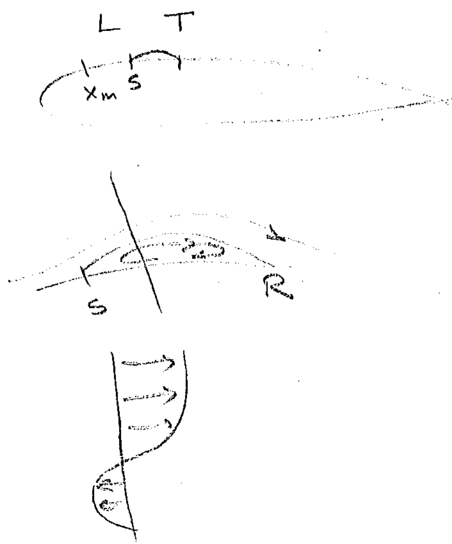
} be $\neq 0$ @ $X = X_m$ ∴ since $\delta \neq 0$ at $X = X_m$ there is a sensitivity to δ i.e. $\frac{ax}{U_0} = f\left(\frac{\delta_0 a}{\nu U_0}\right)$

VALUES FROM EXAMPLE USING "THWAITES METHOD WHERE $\delta = 0$ AT $X = 0$; $\frac{U_e}{U_{em}} = 1 - \frac{ax}{U_{em}}$

$U_{es}/U_{em} = 0.887$

$(C_p)_s = 0.231$

LOWER VALUES OF $(C_p)_s$ OBTAIN WHEN $\delta \neq 0$ AT $X = 0$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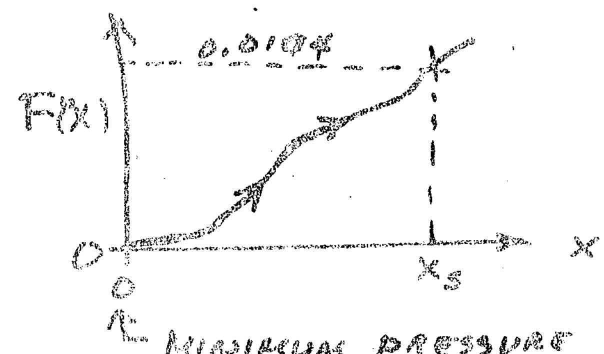
CURL + SKAN (Aeronautical Q., vol. 8, 1957 pp 257-268)

GIVES METHOD FOR ESTIMATION OF X_S, SEPARATION POINT

$$\underset{\text{distance from } x_m}{X^2}\, C_p \left(\frac{dC_p}{dx}\right)^2 \equiv F(x)$$

DETERMINE $F(x)$ AND OBTAIN X_S WHEN $F(x) \geq 0.0104$

must make a correction for initial boundary thickness



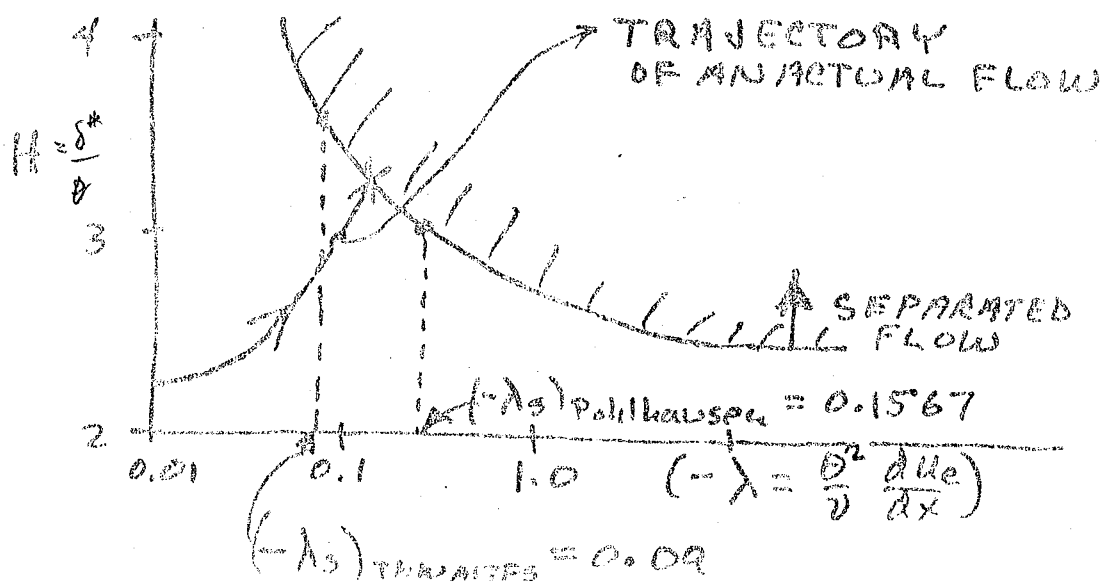
NOTE:

$$C_p = \frac{p - p_m}{\frac{\rho}{2} U_{e_m}^2}$$

LIU + SANDBORN (Aero. Q., vol. 19, 1968, pp 235-242)

CORRELATE VELOCITY PROFILES AT SEPARATION AND SHOW THAT ONE-PARAMETER NOT SUFFICIENT AT SEP. THEY USE:

$$\left(\frac{u}{U_e}\right)_s = f\left(\frac{y}{\delta^*}, \lambda, H\right)$$



2/12/79

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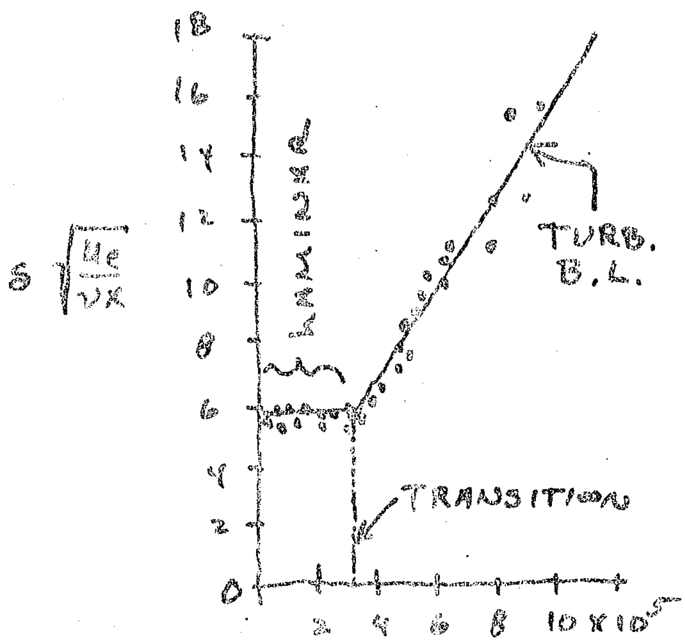
VI TURBULENT SHEAR LAYERS

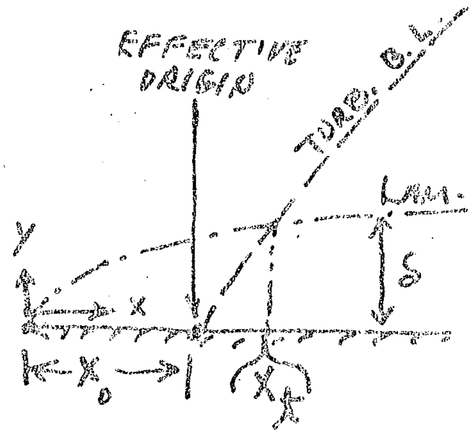
(A) AN INTRODUCTION TO TURBULENCE AND TRANSITION TO TURBULENCE

CONSIDER FLOW OVER FLAT PLATE (2-D)



$Re_x = U_e x/\nu$

(REF. 1, P. 38)



LAMINAR: $\delta \sim \sqrt{x}$

TURBULENT: $\delta \sim (x-x_0)^{4/5}$

(low Re_x) $10^5 \sim 10^7$

- IN THIS CASE, $(Re_x)_t = 3.2 \times 10^5$
- IN SOME CASES VALUES AS HIGH AS $(Re_x)_t = 5 \times 10^6$ ARE OBTAINED. DEPENDING ON MANY FACTORS: AMONG THEM ARE
  - FREE STREAM DISTURBANCES (TURB.)
  - WALL DISTURBANCES (DIRT, BUMPS, TRIPS)
  - PRESSURE GRADIENTS $\partial p/\partial x > 0$ early
  - high Re will bring transitions naturally

VALUES of $(Re_{\delta^*})_t$ AND $(Re_\theta)_t$ AT $(Re_x)_t$ FOR FLAT PLATE.

$$\delta^* = 1.72\sqrt{\frac{\nu x}{u_e}} \quad ; \quad Re_{\delta^*} = \frac{u_e \delta^*}{\nu} = 1.72\sqrt{Re_x}$$

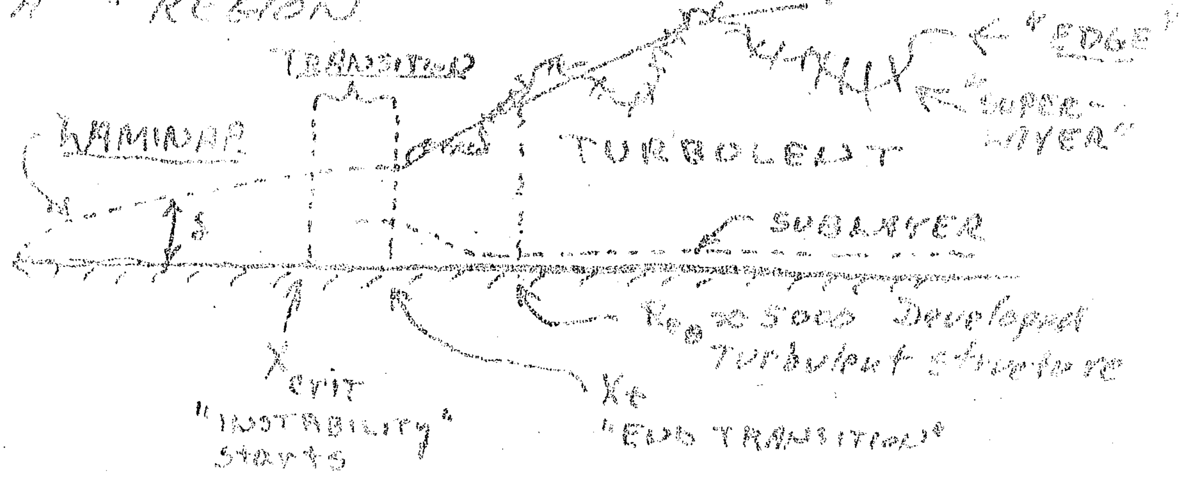
$$\theta = 0.664\sqrt{\frac{\nu x}{u_e}} \quad ; \quad Re_\theta = \frac{u_e \theta}{\nu} = 0.664\sqrt{Re_x}$$

$$\delta \approx 5.3\sqrt{\frac{\nu x}{u_e}} \quad ; \quad Re_\delta = \frac{u_e \delta}{\nu} = 5.3\sqrt{Re_x}$$

TRANSITION VALUES

	LOW (HIGH DISTURBANCE) – INTERNAL FLOW – TURBOMACHINES	HIGH (DISTURBANCE FREE) – FREE FLIGHT – CLEAN SUBSONIC aircraft	
$(Re_x)_t$	3×10^5	5×10^6	
$(Re_{\delta^*})_t$	940	3800	based on $\frac{\partial p}{\partial x} = 0$
$(Re_\theta)_t$	360	1500	
$(Re_\delta)_t$	2900	12000	

TRANSITION DOES NOT OCCUR AT A POINT
IT IS A "REGION"



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TURB. B.L. ON A FLAT PLATE

— CRUDE RESULTS —

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho u_e^2} \qquad \theta(x) = ? \quad \tau_w(x) = ?$$

VELOCITY PROFILE APPROX.

OUTSIDE THE SUBLAYER:

$$\frac{u}{u_e} = \left(\frac{y}{\delta}\right)^{1/n}$$ based on turbulent pipe flows.

THIS $\to 1$ AS $y \to \delta$

$\to 0$ AS $y \to 0$

BUT WON'T WORK AT WALL SINCE

$$\frac{\partial u}{\partial y} = \frac{u_e}{\delta^{1/n}} \frac{1}{n} y^{\left(\frac{1}{n} - 1\right)}$$

GOES to ∞ at $y = 0$ FOR $n > 1$

(SEE P. 77)

INTEGRAL PARAMETERS:

$$\frac{\delta^*}{\delta} = \int_0^{\delta} \left(1 - \frac{u}{u_e}\right) d\left(\frac{y}{\delta}\right)$$

u is a mean velocity

$$\frac{\delta^*}{\delta} = \frac{1}{1+n} \quad ; \quad \frac{\theta}{\delta} = \frac{n}{(1+n)(2+n)}$$

FOR SMOOTH PLATES WHERE $R_{e_x} \lesssim 6 \times 10^7$

$n \simeq 7$

$\delta^* = \delta/8 \quad ; \quad \theta = 7\delta/72$

COMPARE

$H = 1.29$ (ACTUAL EXP. VALUES UP TO $H = 1.4$)

$(H)_{LAM} \simeq 2.59$

FOR ROUGH PLATES VALUES OF $n \simeq 4$ to 6 ARE BETTER

COMPARISON OF BOUNDARY LAYERS
VELOCITY PROFILES

y = Distance from surface u = Velocity

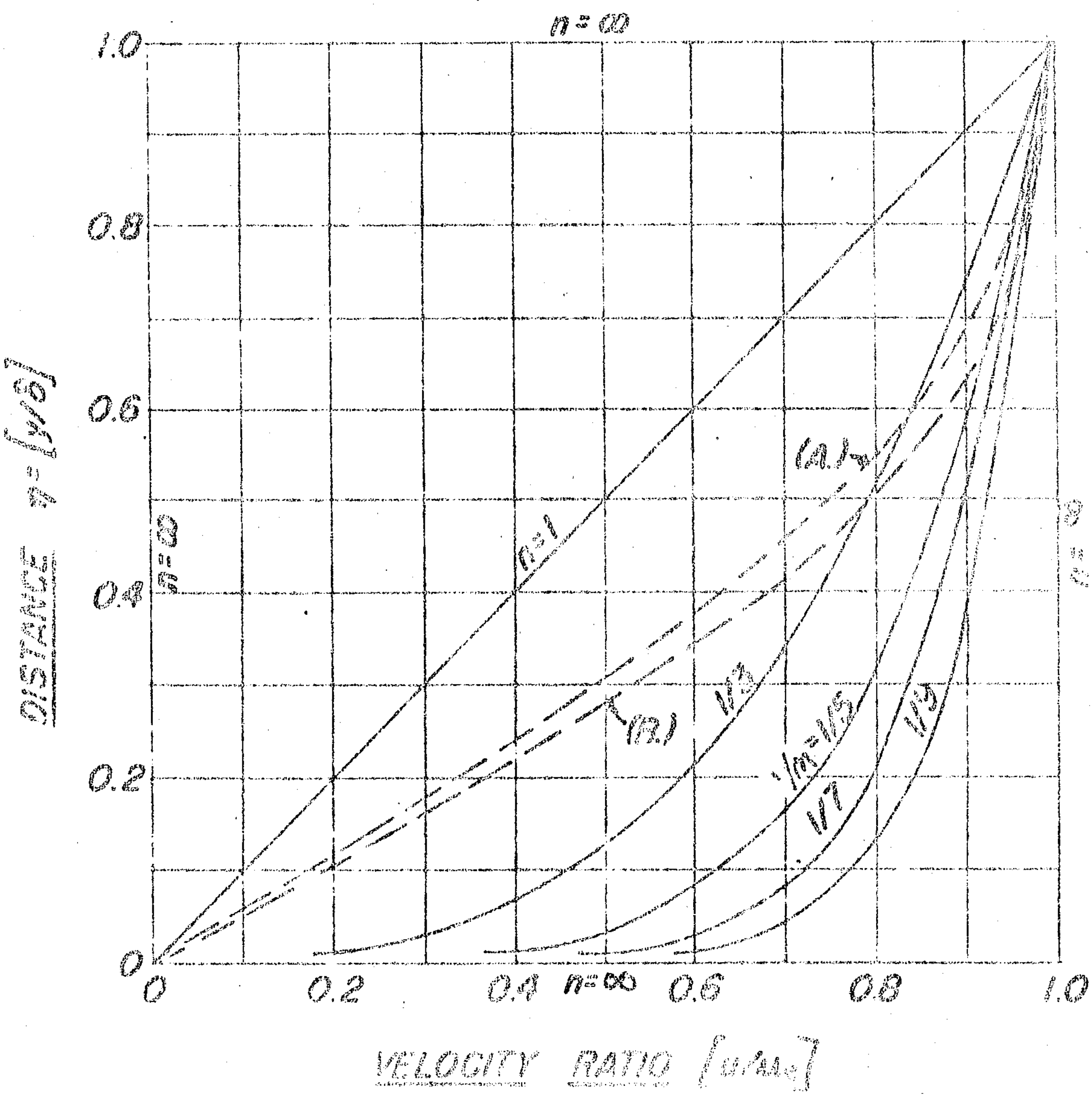
δ = Boundary layer thickness u_e = Velocity where $y = \delta$

– – – – Laminar Profiles:

(A.) Blasius Solution if δ taken where $u/u_e = 0.990$

(B.) Pohlhausen Approximation for $\lambda = 0$, (i.e. $dp/dx = 0$)

———— Power Law Profiles, $u/u_e = (y/\delta)^{1/n}$



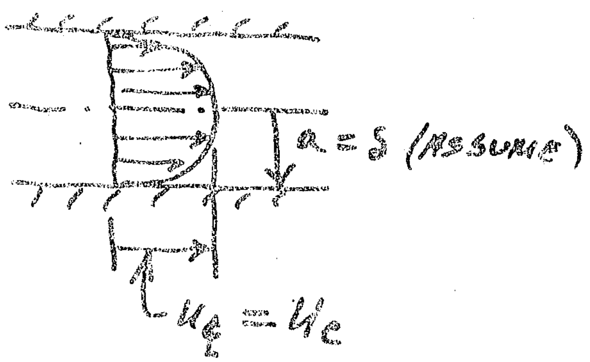
turb wakes in ~~[illegible]~~ are 10 times as high as wall flows since the wall
isn't there to provides the damping effect

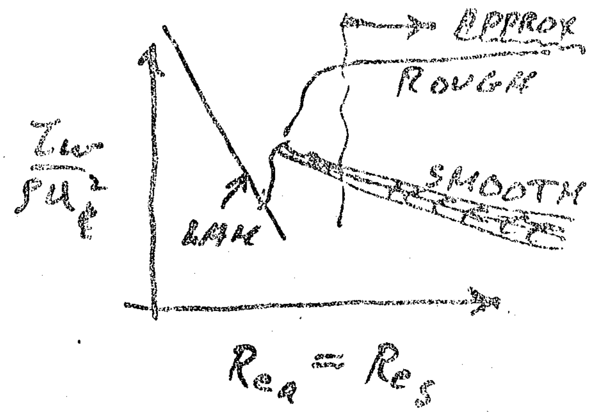
NOW SOLVE MOM. INTEG. EQ

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho u_e^2}$$

HOW TO OBTAIN τ_w SINCE POWER LAW GIVES $\tau_w = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0} \to \infty$?

ATTEMPT TO USE OTHER "DATA" FOR EXAMPLE, TURB. PIPE FLOW





log - log scales

OBTAIN:

$$\tau_w = \rho u_e^2 A\left(\frac{\nu}{u_e \delta}\right)^{1/m}$$

M.I. EQ BECOMES: $\left[\frac{n}{(1+n)(n+2)}\right]\frac{d\delta}{dx} = A\left(\frac{\nu}{u_e}\right)^{1/m}\frac{1}{(\delta)^{1/m}}$

UPON INTEGRATION:

$$\left(\frac{m}{m+1}\right)\delta^{\left(\frac{m+1}{m}\right)} = \frac{A(1+n)(2+n)}{n}\left(\frac{\nu}{u_e}\right)^{1/m}(x - x_0)$$

FOR $n = 7$, $m = 4$, $A = 0.0225$:

$$\left(\frac{\delta}{x}\right) = 0.37\left(R_{e(x-x_0)}\right)^{-1/5} \qquad \delta \sim (x - x_0)^{4/5}$$

$$c_f = \frac{\tau_w}{\frac{\rho}{2} u_e^2} = 0.058\left(R_{e(x-x_0)}\right)^{-1/5}$$

good for $10^7 < Re$

2/14/79 Discuss process of transition
Basic instability of laminar flow

1. Small disturbance eqns.
2. Search for Critical cond.
3. Neutral Stability curves

B. TRANSITION TO TURBULENCE

THE TRANSITION PROCESS (SIMPLIFIED):

LAMINAR STABLE FLOW

↓

(1.) INSTABILITY — (Re_{crit})

↓

(2.) RAPID GROWTH OF SOME MODES ω, λ (α)

↓

(3.) BREAKDOWN TO "SPOTS" OF TURBULENCE

↓

(4.) GROWTH AND SPREAD OF "SPOTS"

↓

(5) TURBULENT FLOW ($Re_{transition}$)

FACTORS KNOWN TO AFFECT ALL/SOME OF THE PHASES OF THE PROCESS

- SOUND (ACOUSTIC LEVEL DISTURBANCES)
- TURBULENCE IN FREE STREAM From outside
- WALL ROUGHNESS
- PRESSURE GRADIENT
- DENSITY VARIATIONS WITH y
- HEAT TRANSFER AT WALL
- WALL CURVATURE
- SYSTEM ROTATION creating coriolis forces.
- POLYMERS IN VERY LOW H_2O CONCENTRATIONS
- - - - - - ETC. - - - -

$$\cancel{\frac{\partial \bar u}{\partial t}} + \frac{\partial u'}{\partial t} + \cancel{\bar u \frac{\partial \bar u}{\partial x}} + \bar u \frac{\partial u'}{\partial x} + u' \frac{\partial \bar u}{\partial x} + \cancel{u' \frac{\partial u'}{\partial x}} + \cancel{\bar v \frac{\partial \bar u}{\partial y}} + \bar v \frac{\partial u'}{\partial y} + v' \frac{\partial \bar u}{\partial y} + \cancel{v' \frac{\partial u'}{\partial y}} = \cancel{-\frac{1}{\rho}\frac{\partial}{\partial x}\bar p} - \frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu \cancel{\nabla^2 \bar u} + \nu \nabla^2 u'$$

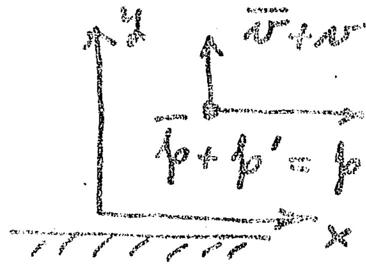
drop 2nd & higher order

drops out since $u'\frac{\partial \bar u}{\partial x} \ll v'\frac{\partial \bar u}{\partial y}$ for thin shear layer

~~mixing parallel flow ie $\frac{\partial \bar u}{\partial x} = 0$~~

ELEMENTS OF STABILITY THEORY
(SEE TEXT, PP 281–301)

CONSIDER 2-D FLOW (sufficient here) upon which are imposed disturbances which fluctuate about the mean:



In "stationary" flow:

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{v}}{\partial t} = \frac{\partial \bar{p}}{\partial t} = 0$$

$$\bar{u} = \frac{1}{T}\int_{t_0}^{t_0+T} u\, dt \text{ , etc.}$$

NAVIER–STOKES EQS. Laminar constant viscosity & density

$$\frac{\partial u'}{\partial t} + (\bar{u}+u')\frac{\partial(\bar{u}+u')}{\partial x} + (\bar{v}+v')\frac{\partial(\bar{u}+u')}{\partial y} = -\frac{1}{\rho}\frac{\partial}{\partial x}(\bar{p}+p') + \nu\nabla^2(\bar{u}+u') \qquad (80\text{-}1)$$

Keep only 1st order terms only

$$\frac{\partial v'}{\partial t} + (\bar{u}+u')\frac{\partial(\bar{v}+v')}{\partial x} + (\bar{v}+v')\frac{\partial(\bar{v}+v')}{\partial y} = -\frac{1}{\rho}\frac{\partial}{\partial y}(\bar{p}+p') + \nu\nabla^2(\bar{v}+v') \qquad (80\text{-}2)$$

and

$$\frac{\partial}{\partial x}(\bar{u}+u') + \frac{\partial}{\partial y}(\bar{v}+v') = 0 \quad \text{Continuity} \qquad (80\text{-}3)$$

ASSUME: DISTURBANCES ARE SMALL AND DISCARD NON-LINEAR TERMS SUCH AS $u'\frac{\partial u'}{\partial x}$, ETC.

ALSO TSL APPROX

so $u'\frac{\partial \bar{u}}{\partial x} \ll v'\frac{\partial \bar{u}}{\partial y}$

since $\frac{\partial \bar{u}}{\partial y} \gg \frac{\partial \bar{u}}{\partial x}$

Disturbance + mean eqns = navier stokes w/ thin shear layer approx.
in the limit of perturbation problem we decouple disturb & mean eqns.
$\therefore$ mean eqns $= 0 \Rightarrow$ disturbance eqns $= 0$ (thus we've linearized everything so far)

IF DISTURBANCES ARE $=0$ EQ'S (80-1, -2, -3) ARE THE "MEAN" FLOW EQUATIONS OF STEADY LAMINAR FLOW.

SUBTRACT "MEAN" FLOW EQ'S FROM LINEARIZED DISTURBANCE EQUATIONS AND OBTAIN — 2-D DISTURBANCE EQ'S

$$\left.\begin{aligned}
\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \bar{v}\frac{\partial u'}{\partial y} + v'\frac{\partial \bar{u}}{\partial y} &= -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu\nabla^2 u' \\
\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} + \bar{v}\frac{\partial v'}{\partial y} + v'\frac{\partial \bar{v}}{\partial y} &= -\frac{1}{\rho}\frac{\partial p'}{\partial y} + \nu\nabla^2 v' \\
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= 0
\end{aligned}\right\} (81-1)$$

PARALLEL-FLOW APPROXIMATION IN TSL'S WHERE DISTURBANCE AMPLIFICATION RATES SMALL IN X-DIRECTION

$$\left.\begin{aligned}
\bar{v} = 0 \quad &; \quad \bar{u}(y) \text{ ONLY} \\
\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} = \nu\frac{\partial^2 \bar{u}}{\partial y^2} \quad &; \quad \frac{\partial \bar{p}}{\partial y} = 0
\end{aligned}\right\} (81-2)$$

- EXACT IN FULLY-DEVELOPED FLOW
- GOOD APPROX IN BOUNDARY LAYER
- $\bar{v} \sim .1\bar{u}$, $\delta/L < 1$ but not too small — FAIR " IN WAKES, JETS, SHEAR LAYERS

USE (81-2) TO OBTAIN 2-D PARALLEL FLOW DISTURBANCE EQUATIONS

$$\left\{\begin{aligned}
\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + v'\frac{\partial \bar{u}}{\partial y} &= -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu\nabla^2 u' \\
\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} &= -\frac{1}{\rho}\frac{\partial p'}{\partial y} + \nu\nabla^2 v' \\
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= 0
\end{aligned}\right\} (81-3)$$

that $\frac{\partial}{\partial y}(1) - \frac{\partial}{\partial x}(2)$ and let ψ

let $u' = \frac{\partial \psi}{\partial y}$ $v' = -\frac{\partial \psi}{\partial x}$ so that continuity eqn.

$$\psi = \phi(y) e^{i(\alpha x - \omega t)}$$

$$\frac{\partial \psi}{\partial x} = \phi \, i\alpha e^{i(\alpha x - \omega t)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\phi \alpha^2 e^{i(\alpha x - \omega t)}$$

$$\frac{\partial \psi}{\partial y} = \phi' e^{i(\quad)}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \phi'' e^{i}$$

$$\Delta \psi = (-\alpha^2 \phi + \phi'') e^{i(\quad)}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

ELIMINATE p' FROM EQ'S (B1-3) BY CROSS-DIFFERENTIATION AND SUBTRACTION. INTRODUCE A DISTURBANCE STREAM FUNCTION, ψ, TO SATISFY CONTINUITY, SO EQ'S (B1-3.) REDUCE TO:

shitty notation

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \bar{u}\frac{\partial}{\partial x}(\nabla^2\psi) - \bar{u}''\frac{\partial\psi}{\partial x} = \nu\nabla^2(\nabla^2\psi) \qquad (B2-1)$$

WHERE:

$$u' = \frac{\partial\psi}{\partial y};\ v' = -\frac{\partial\psi}{\partial x};\ \bar{u}'' = \frac{d^2\bar{u}}{dy^2} \qquad \Delta^2\psi \neq 0$$

B.C.'S FOR A B.L. WITH DISTURBANCE FREE CONDITIONS OUTSIDE B.L.

at $y=0$ (wall) ; $u' = v' = 0$

at $y \to \infty$; $u' = v' \to 0$

SMALL DISTURBANCE (ONE COMPONENT OF A SPECTRUM). LETS ASSUME:

$$\psi = \phi(y)\, e^{i[\alpha x - \omega t]} = \phi(y)\, e^{i[\alpha x - \omega_r t]}\, e^{\omega_i t} \qquad (B2-2)$$

$\phi_r \cos - \phi_i \sin$

COMPLEX AMPLITUDE: $\phi = \phi_r + i\,\phi_i$

" FREQUENCY: $\omega = \omega_r + i\,\omega_i$

WAVE NUMBER: $\alpha = 2\pi/\lambda$

WAVE LENGTH: λ

TAKE REAL PART FOR PHYSICAL SOL.

$$\left[\ \psi_r = \phi_r\left[\cos(\alpha x - \omega_r t) - \left(\frac{\phi_i}{\phi_r}\right)(\sin(\alpha x - \omega_r t)\right] e^{\omega_i t}\ \right]$$

EQ. OF A TRAVELLING WAVE (B2-3)

CHARACTERISTICS :

$\phi_r(y)$: AMPLITUDE ; ω_r : WAVE FREQ.

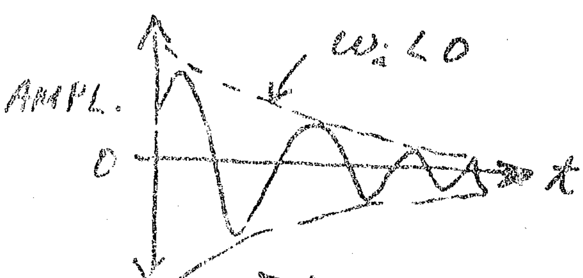
$\left(\frac{\phi_i}{\phi_r}\right)$: Phase shift ; ω_i : DAMPING FACTOR

$$c \equiv \frac{\omega}{\alpha} = \underbrace{\frac{\omega_r}{\alpha}}_{\text{REAL WAVE SPEED}} + i\,\frac{\omega_i}{\alpha}$$: COMPLEX TRAVELLING WAVE SPEED

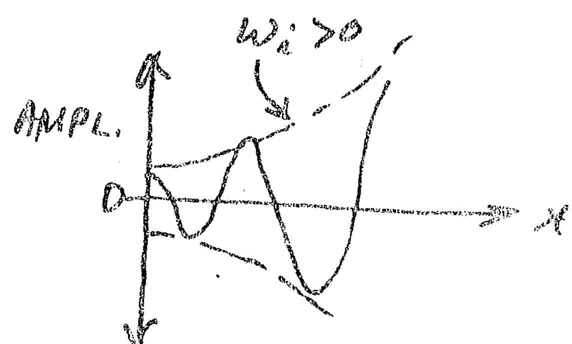
DISTURBANCE OBTAINED FROM DEF'S.

$$\left.\begin{aligned} u' &= \frac{\partial\psi}{\partial y} = \phi'(y)\,e^{i(\alpha x - \omega t)} \\ v' &= -\frac{\partial\psi}{\partial x} = -i\alpha\,\phi(y)\,e^{i(\alpha x - \omega t)} \end{aligned}\right\} \quad (83.1)$$

WE SEE THAT ω_i CONTROLS GROWTH OF ψ, u' and v'



DAMPED



EXPONENTIAL GROWTH

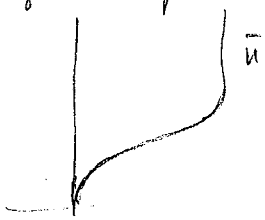
CRITICAL CONDITIONS OCCUR WHEN

$$\boxed{\omega_i = 0}$$

Lord Rayleigh solves

$$(\bar{u}-c)(\phi''-\alpha^2\phi)-\bar{u}''\phi=0$$ totally stable.

If an inflection point in gradients are unstable.



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ORR - SOMMERFIELD EQUATION: (TEMPORAL GROWTH OR DECAY) [( )' = ∂/∂y, etc.)]

$$\boxed{(\bar{u} - c)(\phi'' - \alpha^2\phi) - \bar{u}''\phi = -\frac{i\nu}{\alpha}\left[\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi\right]} \quad (84-1)$$

OBTAINED BY USE OF (82-2) IN (82-1) FOR B.L. SUBJECT TO B.C.'S ON ϕ;

$$\begin{bmatrix} y = 0 \; ; \; \phi = \phi' = 0 \\ y \to \infty \; ; \; \phi = \phi' = 0 \end{bmatrix} \begin{matrix} \text{WALL} \\ \text{DISTURBANCE FREE FREE STREAM} \end{matrix}$$

THIS IS AN EIGEN-VALUE PROBLEM TO DETERMINE THE COMPLEX EIGEN FUNCTION: ϕ AND EIGEN-VALUE c GIVEN: $\bar{u}(y)$, A WAVE NUMBER $\alpha = 2\pi/\lambda$ AND THE KINEMATIC VISCOSITY, ν, (OR $Re = u_e\delta/\nu$).

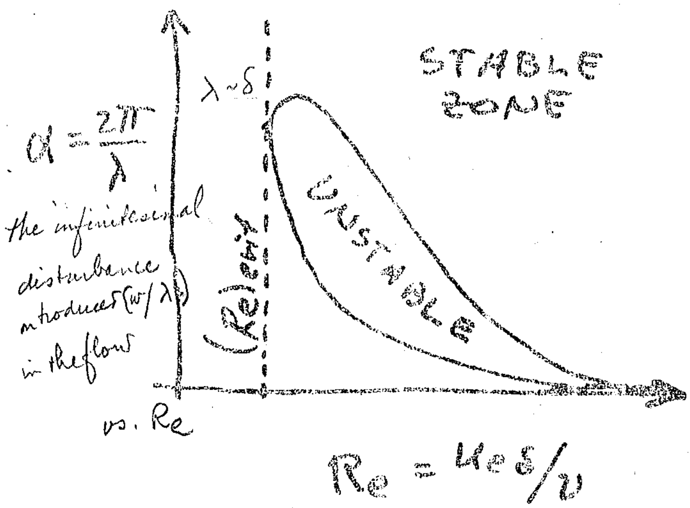
NEUTRAL STABILITY (Re_{crit}) OBTAINS WHEN $c = c_r + i c_i$ = REAL NUMBER, i.e. WHEN $\boxed{\omega_i = 0, \text{OR } c_i = \omega_i/\alpha = 0}$

UNSTABLE SOLUTIONS OBTAIN WHEN

$$\boxed{\omega_i > 0, \text{OR } c_i > 0}$$

if $\frac{dp}{dx} \leq 0$ then $\rho u \frac{du}{dx} + \frac{dp}{dx} = 0 \Rightarrow \frac{du}{dx} \geqslant 0$

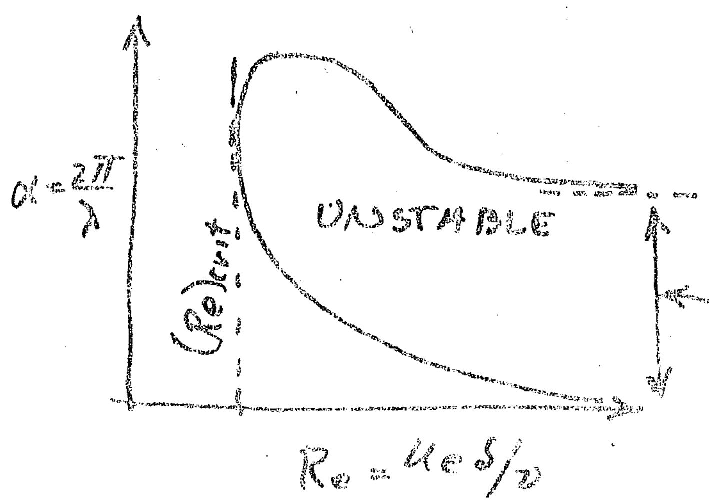
NEUTRAL STABILITY CURVES



CURVES FOR CASES WHERE

$$\frac{dP}{dx} \leq 0$$

(FAVORABLE AND ZERO) – VISCOUS INSTABILITY



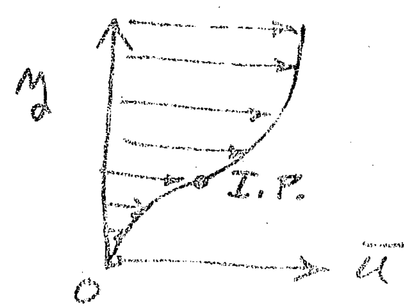
CURVES FOR UNFAVORABLE PRESSURE GRAD'S

$$dP/dx > 0$$

UNSTABLE TO "LONG" WAVES AT $Re \to \infty$

INVISCID INSTABILITY AS A RESULT OF INFLECTED MEAN VELOCITY PROFILES

- B.L. in ADVERSE PRESSURE GRADS.
- WAKES
- JETS
- SHEAR LAYERS



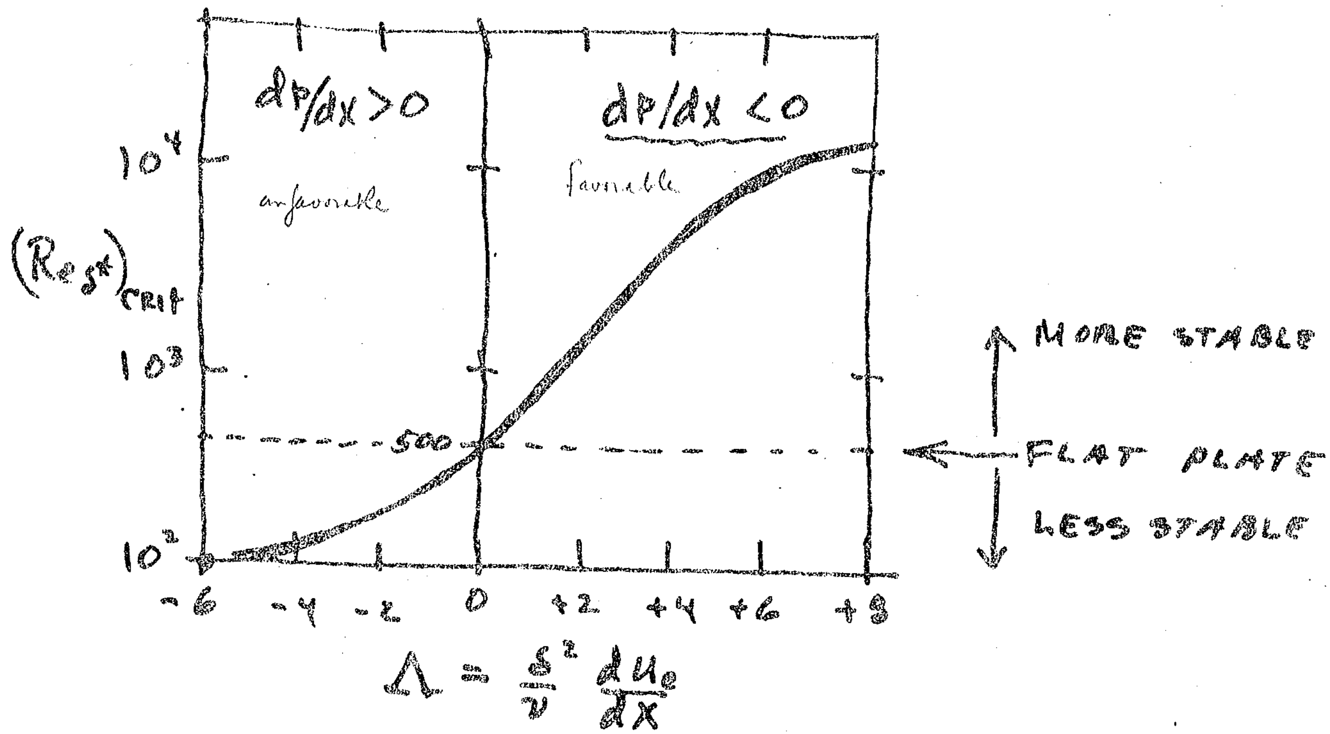
VALUES OF LOWEST CRITICAL Re

$(Re_{\delta^*})_{CRIT} \cong 500$ FOR FLAT PLATE LAM. B.L.

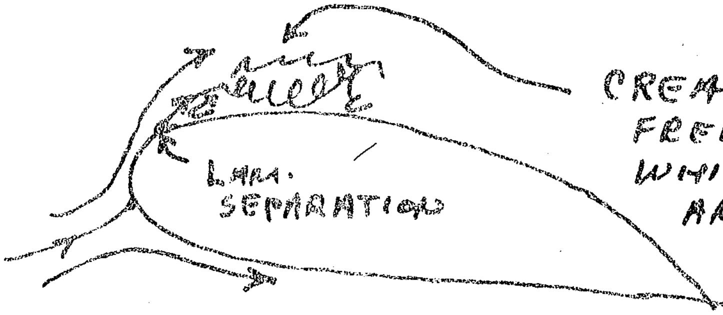
$(Re_\theta)_{crit} \approx 200$

FOR B.L.'S IN PRESSURE GRADIENTS:

USING SIMILARITY SOLUTIONS FOR $\bar{U}(y)$ – SEE FIG. 9.10 IN TEXT



IN ADVERSE PRESSURE GRADIENTS SEPARATION MAY OCCUR BEFORE TRANS.



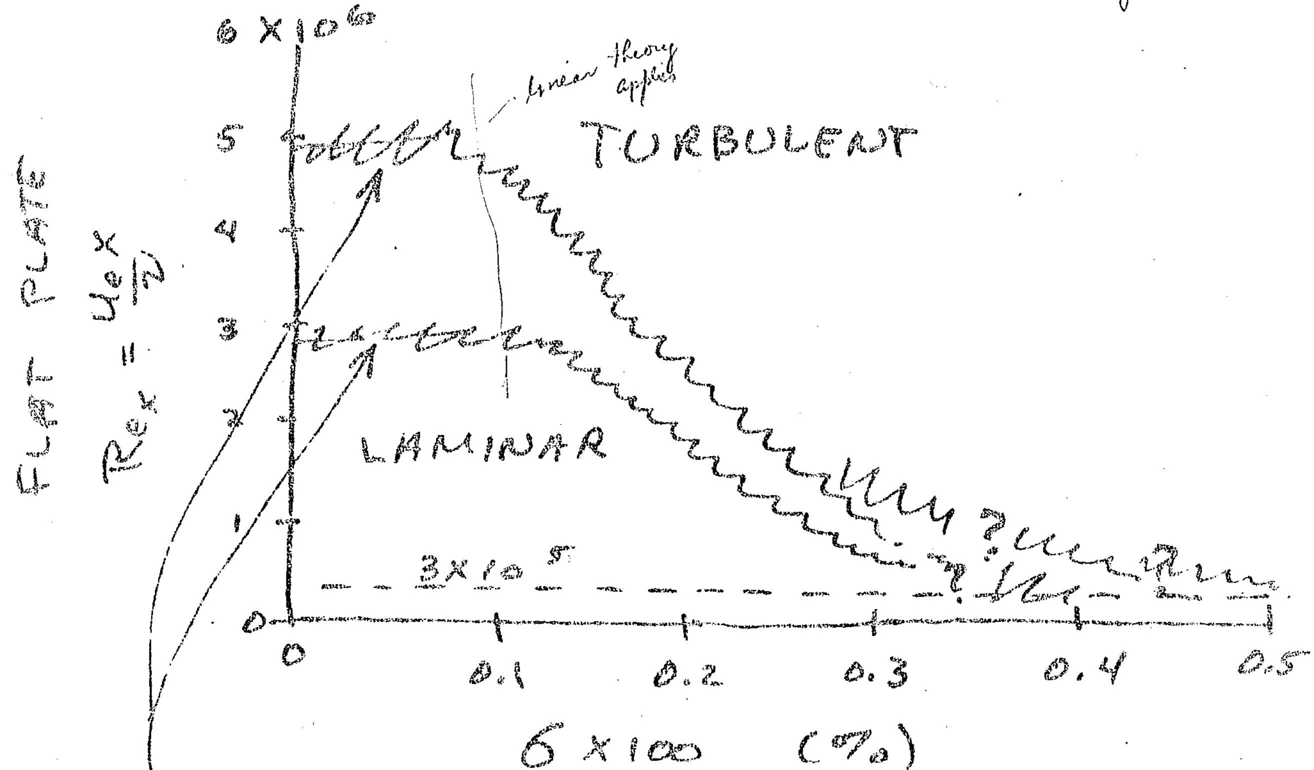
CREATES VERY UNSTABLE FREE-SHEAR LAYER WHICH GOES TURBULENT AND "REATTACHES".

TRANSITION EFFECTS:

(1.) FREE-STREAM TURBULENCE LEVEL

DEFINE INTENSITY OF TURBULENCE (DISTURBANCES): $\sigma = \dfrac{\sqrt{(\overline{u_e'^2}+\overline{v_e'^2}+\overline{w_e'^2})\,1/3}}{u_e}$

disturbance fluctuation

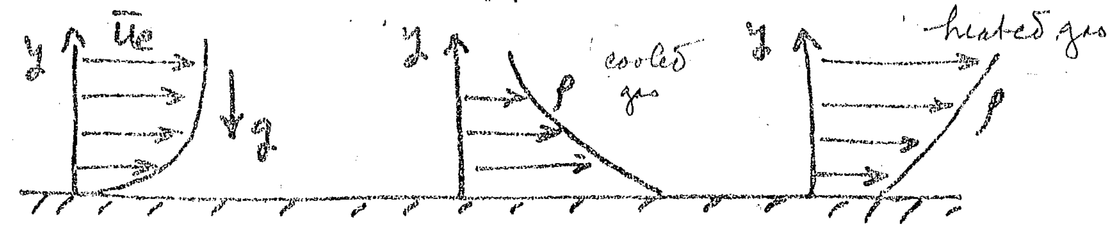


LOW DISTURBANCE LEVEL LIMITS FOR FLAT PLATE FLOW

NOTE: $(Re_x)_{crit} = \cancel{1.45 \times 10^5}$ 8.5×10^4

USING $\dfrac{\delta^*}{x} = \dfrac{1.721}{\sqrt{Re_x}}$ AND $\left(\dfrac{u_e \delta^*}{\nu}\right)_{crit} = 500$

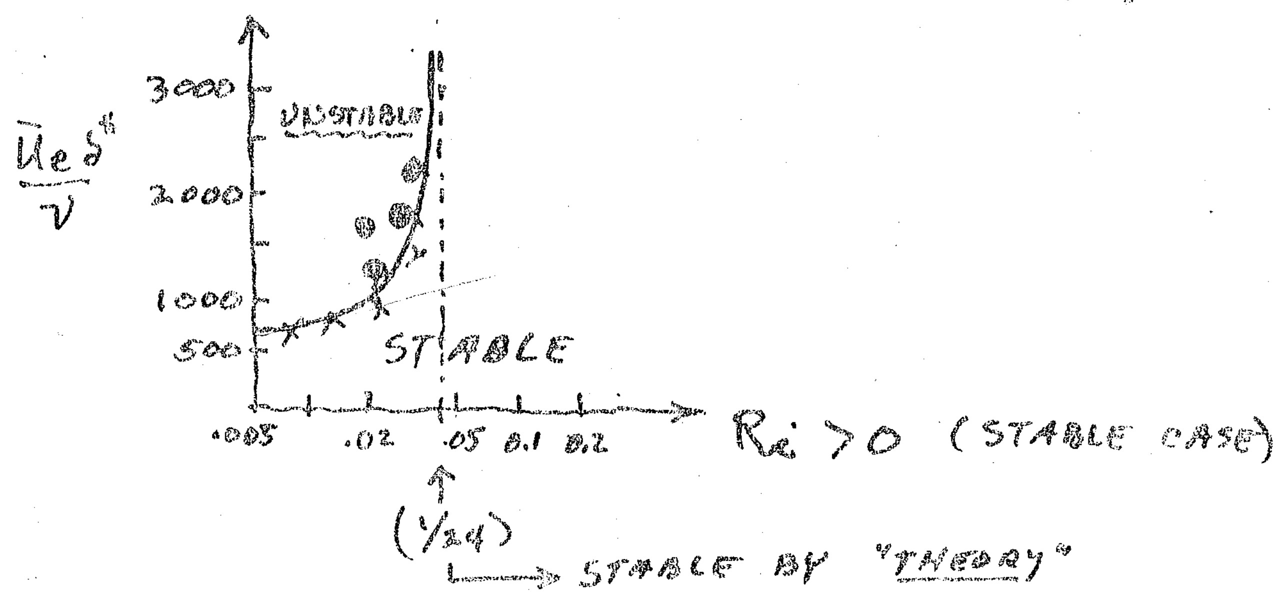
(2.) DENSITY GRADIENTS:



Velocity Profile

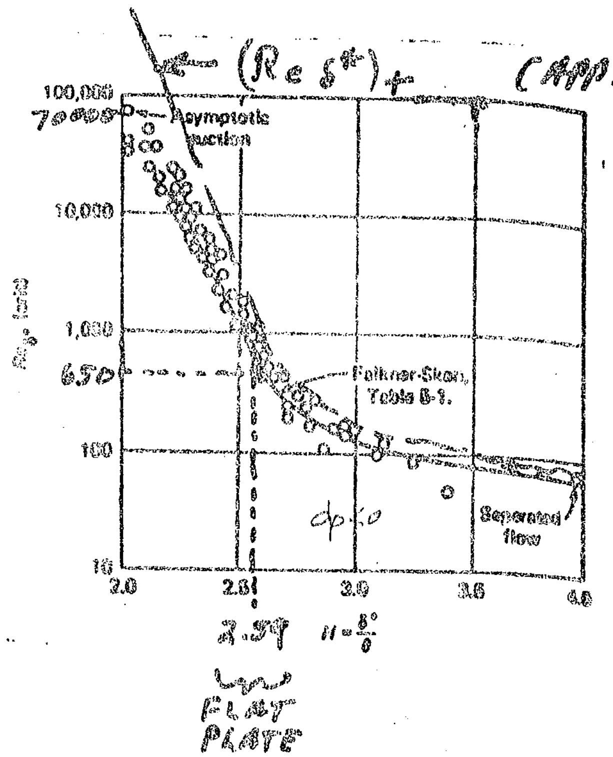
	CASE A	CASE B
	STABLE	UNSTABLE
	$\frac{\partial \rho}{\partial y} < 0$	$\frac{\partial \rho}{\partial y} > 0$

Richardson Number: $Ri = \dfrac{-(g/\rho)\, \partial\rho/\partial y}{\left(\frac{\partial u}{\partial y}\right)_{y=0}}$



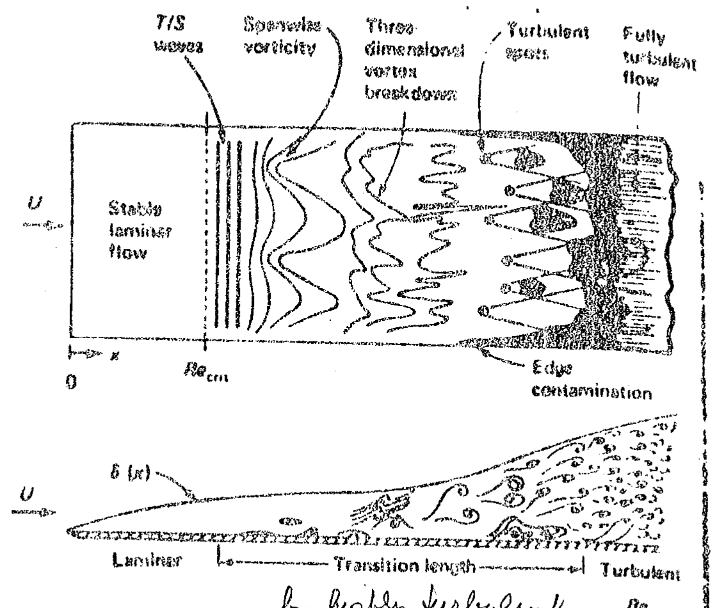
SEE REF. 1, PP 491-493

(3.) PRESSURE GRADIENTS AND SUCTION AT WALL { o – various calculations non-similar; ―― similar laminar profiles



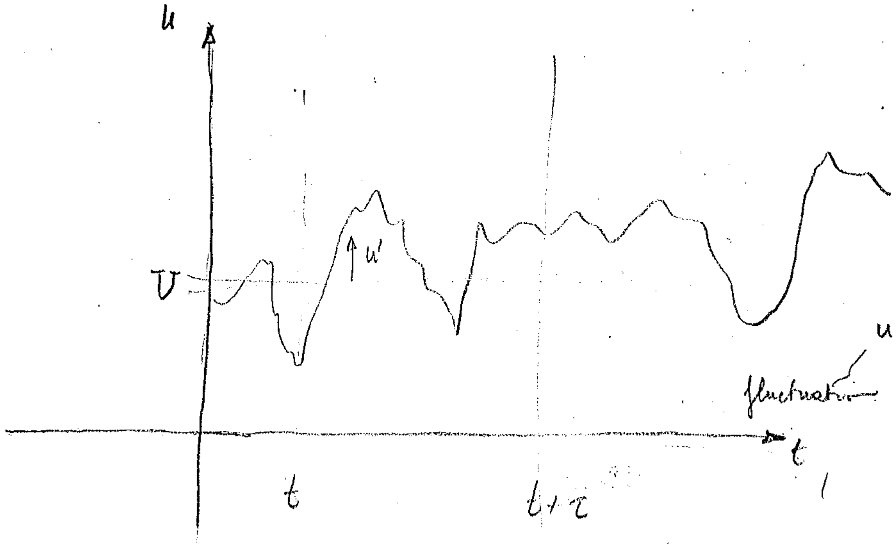
NOTE: EXACT VALUES OF $(R_e)_{transition}$ CANNOT BE CORREL. EXACTLY ON H. IN ADVERSE PRESSURE GRADIENTS TRANSITION IS COMPLETED RAPIDLY.

DISCUSSION OF PROCESS OF TRANSITION



$Re_\theta > 1000$

for highly turbulent flows this transition region collapses and all these effects can occur ~~within~~ but not as



$u' = \tilde{u} - U$ or mean

fluctuation

total value

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ME 251 B 2/2/79 p. 90

TURBULENT FLOW

e.) TURBULENC EQ'S

Statistical defn.

1.) REYNOLDS STRESSES + EQ'S OF MOTION

AVERAGES: TIME: $\bar{f} = \lim_{T \to \infty} \left[\frac{1}{T} \int_{t_0}^{t_0+T} f \, dt \right]$

ENSEMBLE: $\bar{f}(t) = \lim_{N \to \infty} \left[\frac{1}{N} \sum_{1}^{N} f_N(t) \right]$

- STATIONARY PROCESS: $\frac{\partial U}{\partial t} = 0$ (like a steady flow)

  TIME AVERAGE INDEPENDENT OF t_0

- "ERGODIC" HYPOTHESIS: FOR STATIONARY PROCESS, $(\bar{f})_{TIME} = (\bar{f})_{ENSEMBLE}$

MEAN VALUES: (CHANGE OF NOTATION)

Velocities: $\bar{u}_i \equiv U_i$

Pressure, density, etc: $\bar{p}, \bar{\rho}$;

FLUCTUATIONS: $u_i'(t) \equiv u_i(t)$

$p'(t)$; $\rho'(t)$;

TOTAL VEL: $(U_i + u_i) = \tilde{u}_i$

MEAN SQUARES: (Components of turbulence intensity) $\rightarrow$ $\overline{u_i^2} = \frac{1}{T \to \infty} \int_0^T u_i^2 \, dt$

RMS

$\overline{(p')^2} = \frac{1}{T \to \infty} \int_0^T (p')^2 \, dt$

variance in stats

AUTO CORRELATION $\overline{u_i u_j} = \overline{u_i^2}$

CROSS CORRELATIONS: $\overline{u_i u_j}$; $\overline{u_i p'}$

normally this should be zero. However in turbulence this is no the case.

CORRELATION COEF'S: $\dfrac{\overline{u_i u_j}}{\left(\overline{u_i^2}\,\overline{u_j^2}\right)^{1/2}}$; $\dfrac{\overline{u_i p'}}{\sqrt{\overline{u_i^2}}\sqrt{\overline{p'^2}}}$

0 < correl coff < 1 in turbulence analysis normally correlcoff .2 to .5

LEVELS OF FLUCTUATION:

$$\frac{\overline{u_i^2}}{U_i^2} \; ; \; \frac{\overline{u_i^2}}{U_{ref}^2} \; ; \; \frac{\overline{u_i u_j}}{U_{ref}^2} \; ; \text{ etc.}$$

REYNOLDS STRESSES EQ's of Motion

STARTING WITH (9-1) and (11-2) where σ_{ij} IS VISCOUS COMPONENT OF STRESS.

$$\frac{\partial(\rho \tilde{u}_i)}{\partial t} + \frac{\partial(\rho \tilde{u}_i \tilde{u}_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} + \cancel{X_i}^{\text{OMIT}} \qquad (91\text{-}1)$$

Introduce $\tilde{u}_i = U_i + u_i$ and average. Let $\rho' = 0$ so $\rho = \bar{\rho}$ (Neglect density fluctuations):

even for $M_\infty \sim 2$ to 5 — Now time average everything

$$\rho \frac{\partial(U_i + \cancel{\overline{u_i}}^{\,0})}{\partial t} + \rho \frac{\partial(U_i U_j + \cancel{\overline{u_i}}^{\,0} U_j + \cancel{\overline{u_j}}^{\,0} U_i + \overline{u_i u_j})}{\partial x_j} = -\frac{\partial(p + \cancel{\overline{p'}}^{\,0})}{\partial x_i} + \frac{\partial(\overline{\sigma}_{ij} + \cancel{\overline{\sigma'_{ij}}}^{\,0})}{\partial x_j}$$

average of fluct.

Now: $$\boxed{\overline{\sigma}_{ij} = 2\mu \bar{e}_{ij} = \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)} \qquad (91\text{-}2)$$

IS THE VISCOUS STRESS DUE TO MEAN MOTION

AND $\boxed{-\rho\, \overline{u_i u_j}}$ (91-3) IS THE "REYNOLDS" STRESS TENSOR

EQ'S OF MOTION FOR MEAN FLOW

FROM ABOVE: NOTE!

$$\frac{DU_i}{Dt} = \frac{\partial U_i}{\partial t} + \frac{\partial (U_i U_j)}{\partial x_j} = \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}$$

$U_i \frac{\partial U_j}{\partial x_j} + U_j \frac{\partial U_i}{\partial x_j}$ obequent

WE OBTAIN EQ'S OF MOTION

momentum exchange flux.

$$\boxed{\frac{DU_i}{Dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\bar{\sigma}_{ij} - \rho\,\overline{u_i u_j}\right)} \quad (92\text{-}1)$$

CONTINUITY IS (ρ = const.)

$$\boxed{\frac{\partial U_i}{\partial x_i} = 0} \quad (92\text{-}2)$$

BECAUSE

$$0 = \overline{\frac{\partial \tilde{u}_i}{\partial x_i}} = \frac{\partial U_i}{\partial x_i} + \cancel{\frac{\partial \overline{u_i}}{\partial x_i}}^{=0}$$

ALSO NOTE WE CAN OBTAIN A CONTINUITY EQ. FOR FLUCTUATIONS (ρ = CONST):

$$0 = \frac{\partial \tilde{u}_i}{\partial x_i} = \cancel{\frac{\partial U_i}{\partial x_i}}^{=0} + \frac{\partial u_i}{\partial x_i}$$

$$\therefore \quad \boxed{\frac{\partial u_i}{\partial x_i} = 0} \quad (92\text{-}3)$$

FOR THE 2-D TBL APPROX., EQ'S FOR MEAN ARE IF FLOW STEADY IN MEAN

$$\boxed{\begin{aligned} U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y} - \overline{uv}\right) \\ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} &= 0 \\ \frac{\partial \bar{p}}{\partial y} &= 0 \end{aligned}} \quad (92\text{-}4)$$

"Reynolds" shear stress $-\rho\overline{uv}$

TURBULENT SHEAR FLOWS (TSL CASE)

- THE PROBLEM - HOW TO MODEL THE REYNOLDS SHEAR STRESS *ad hoc methods*

$$\left[-\rho \overline{uv} \right] = ?$$

THE "EDDY-VISCOSITY" METHOD

$$\left[-\rho \overline{uv} = \rho \epsilon_M \frac{\partial U}{\partial y} \right] \qquad (93\text{-}1)$$

THE "MIXING LENGTH" DEFINED

$$\left[\ell = \frac{\sqrt{-\overline{uv}}}{\partial U / \partial y} \frac{\frac{ft}{sec}}{\frac{1}{sec}} \right] \qquad (93\text{-}2)$$

HENCE WE MUST "MODEL" EITHER ϵ_M OR ℓ. THEY ARE RELATED BY

$$\left[\epsilon_M = \ell^2 \left| \frac{\partial U}{\partial y} \right| \right] \qquad (93\text{-}4)$$

DIFFERENT ASSUMPTIONS USED FOR:

- BOUNDARY LAYERS (WALL LAYER EFFECT)
- FREE SHEAR LAYERS *$\overline{uv}$, $\overline{vw}$, $\overline{uw}$ are normally different*
  - SHEAR LAYERS
  - WAKES
  - JETS

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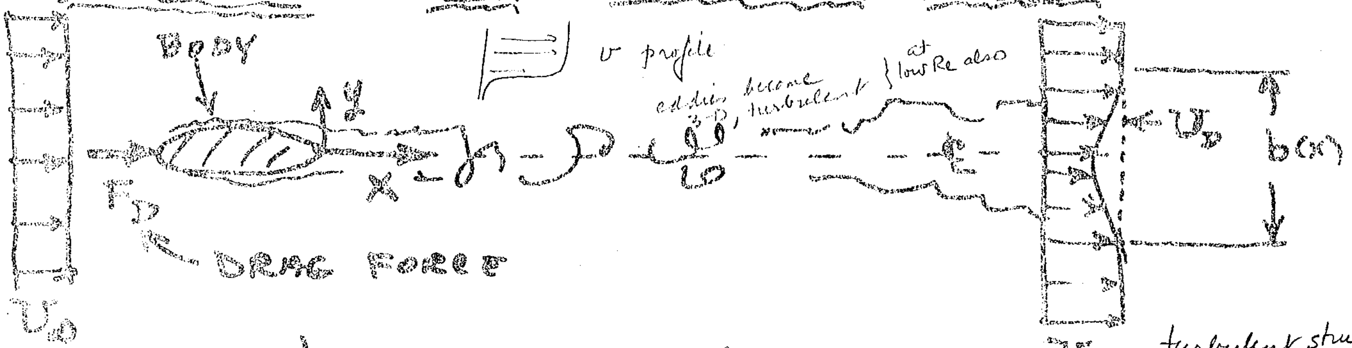
Free Shear Layers

Turb / Lam Low Re

Near & Far Field problem.

FREE TURBULENT SHEAR LAYERS:

EXAMPLE: THE FAR FIELD WAKE



NEAR FIELD

TURBULENCE STRUCTURE DEPENDS ON BODY SHAPE.

FAR FIELD

turbulent structure in local equilib. there is no history effect of previous location.

$$\boxed{U_D = (U_\infty - U) \ll U_\infty}$$

defect veloc.

TURBULENCE STRUCTURE NEARLY "UNIVERSAL"

$U_D(x, y)$

$U_{D_{max}}(x)$ (MAX. AT ℄)

$\boxed{b(x) - \text{WAKE WIDTH}}$

BASIC EQ'S (92-4) far-field results steady state solution

ASSUME: • $\bar{p} = p_\infty = \text{CONST.}$ due to thin shear layer approx = boundary layer

• $\nu \dfrac{\partial U}{\partial y} \lll -\overline{uv}$ (FULLY TURBULENT)

$$\left.\begin{aligned} U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} &= \frac{\partial}{\partial y}\left(-\overline{uv}\right) \\ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} &= 0 \end{aligned}\right\} (94\text{-}1)$$

B.C.'s:
$$\left.\begin{aligned} &U \to U_\infty \text{ as } y \to \infty \\ &\frac{\partial U}{\partial y} = 0 \text{ at } y = 0 \text{ (℄ symmetry)} \end{aligned}\right\} (94\text{-}2)$$

LINEARIZATION IN FAR WAKE, EQ'S (94-1) ARE

$$(U_\infty - U_D)\left(-\frac{\partial U_D}{\partial x}\right) + V\left(-\frac{\partial U_D}{\partial y}\right) = \frac{\partial(-\overline{uv})}{\partial y} \quad (95\text{-}1)$$

$$-\frac{\partial U_D}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad (95\text{-}2)$$

USING: $\partial y \hat{=} b$; $\partial x \hat{=} x$; $\partial U_D \hat{=} U_{Dm}$; $\partial V \hat{=} V$ into continuity eqn.

SO: $\left[V \simeq \frac{b}{x} U_{Dm}\right]$ IS USEFUL FOR

ESTIMATE ORDER OF TERMS IN (95-1):

$$-U_\infty \frac{\partial U_D}{\partial x} + U_D \frac{\partial U_D}{\partial x} - V \frac{\partial U_D}{\partial y} = \frac{\partial(-\overline{uv})}{\partial y}$$

ORDER: $\left[\frac{U_\infty U_{Dm}}{x}\right] + \left[\cancel{\frac{U_{Dm}^2}{x}}\right] - \left[\cancel{\frac{b}{x} U_{Dm} \frac{U_{Dm}}{b}}\right] = [\sim]$

SMALL WHEN $U_D \ll U_\infty$

So EQ TO SOLVE IN FAR WAKE IS: with continuity eq

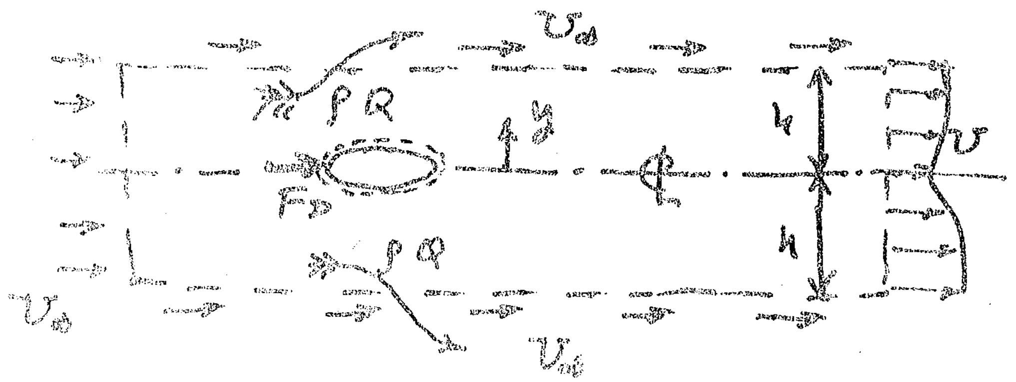
$$\boxed{-U_\infty \frac{\partial U_D}{\partial x} = \frac{\partial(-\overline{uv})}{\partial y}} \text{ TURB.} \quad (95\text{-}3)$$

FOR LAMINAR CASE THIS WOULD BE:

$$\boxed{U_\infty \frac{\partial U_D}{\partial x} = \nu \frac{\partial^2 U_D}{\partial y^2}} \text{ LAM.} \quad (95\text{-}4)$$

SAME B.C.'S, SHOWN IN EQ'S (94-2).

CONDITION RELATING WAKE TO DRAG FORCE



$h > \delta/2$

CONS. OF MASS: $\left[\dot{m}_{IN} = \dot{m}_{OUT}\right]$

$$2\rho\int_0^h U_\infty\,dy = 2\rho Q + 2\rho\int_0^h U\,dy$$

$$\therefore \quad Q = \int_0^h (U_\infty - U)\,dy = \int_0^\infty U_D\,dy \qquad (96\text{-}1)$$

MOMENTUM IN X-DIR: $\left[\sum \vec{F}_x = \overrightarrow{\text{Mom}}_{x\,OUT} - \overrightarrow{\text{Mom}}_{x\,IN}\right]$

$$-F_D = \left[2\rho Q U_\infty + 2\rho\int_0^h U^2\,dy\right] - 2\rho\int_0^h U_\infty^2\,dy$$

WITH USE OF (96-1)

$$-\frac{F_D}{2\rho} = \int_0^h \left(U_\infty(U_\infty - U) + U^2 - U_\infty^2\right)dy$$

$$\frac{F_D}{2\rho} = \int_0^h U(U_\infty - U)\,dy \quad \text{— momentum thickness}$$

$$\boxed{\frac{F_D{}^{/\text{unit width}}}{\rho} = 2\int_0^\infty U\,U_D\,dy = \int_{-\infty}^{+\infty} U\,U_D\,dy} \quad (96\text{-}2)$$

NOW "LINEARIZED" FORM OF DRAG LAW

$$\frac{F_D}{\rho} = \int_{-\infty}^{+\infty} (U_\infty - U_D) U_D \, dy = \int_{-\infty}^{+\infty} (U_\infty U_D - \underbrace{\cancel{U_D^2}}_{\text{small}}) \, dy$$

$$\boxed{\therefore \frac{F_D}{\rho U_\infty} = \int_{-\infty}^{+\infty} U_D \, dy}$$

FOR $U_D \lll U_\infty$ LINEARIZED FORM (97-1)

NOW SOLUTION USING "PRANDTL" EDDY VISCOSITY MODEL

RESUME:

$$\boxed{\epsilon_m = k\, b(x)\, U_{Dm}(x)}$$

this came about because of the microscopic factor of momentum transfer

FROM (93-1) $\therefore$

$$\overline{uv} = k\, b(x)\, U_{Dm}(x) \left(-\frac{\partial U_D}{\partial y}\right)$$

NOTE: stress increases as b (local wake width $\leftrightarrow$ eddy size) and U_{Dm} (maximum vel. difference) increase.

EQUATION (95-3) BECOMES:

$$\left[\frac{\partial U_D}{\partial x} = \left(\frac{\overset{\text{vary from flow to flow}}{k\, b U_{Dm}}}{U_\infty}\right) \frac{\partial^2 U_D}{\partial y^2}\right] \qquad (97\text{-}1)$$

NOTE similarity to Laminar Eq. (95-4)

$$\left[\frac{\partial U_D}{\partial x} = \left(\frac{\nu}{U_\infty}\right) \frac{\partial^2 U_D}{\partial y^2}\right] \qquad (95\text{-}4)$$

ASSUME "SIMILARITY" SOLUTION TO (97-1) WHERE

$$\left[\eta = \frac{y}{b(x)} \quad , \text{AND} \quad \frac{U_D}{U_{DM}(x)} = c\, f(\eta)\right]$$

USING DRAG CONDITION - EQ (97-1) integral constant

$$\left[\frac{F_D}{\rho U_\infty} = c\, U_{DM}\, b \int_{-\infty}^{+\infty} f\, d\eta\right] = \int_{-\infty}^{\infty} U_D\, dy \qquad (98-1)$$

SINCE R.H.S = CONST. THEN $C\, U_{DM}\, b = \text{CONST.}$
ALSO OR

$$\left[U_{DM}\, b = A\right] \quad \text{IS REQUIRED } (A = \text{Constant}) \qquad (98-2)$$

NOW SUBSTITUTE (98-1) INTO (97-1)
WHERE:

$$\frac{\partial^2 U_D}{\partial y^2} = c\, U_{DM}\, f''\left(\frac{1}{b^2}\right)$$

$$\frac{\partial U_D}{\partial x} = c\, f\, U'_{DM} - c\, U_{DM}\, f'\, \eta\, \frac{b'}{b}$$

INSERT INTO (97-1) USING (98-2)

$$c\, f\, U'_{DM} - c\, U_{DM}\, f'\, \eta\, \frac{b'}{b} = \frac{k A}{U_\infty}\, \frac{c\, U_{DM}}{b^2}\, f''$$

$$f'' + \frac{U_\infty}{kA}\left[b\, b'(\eta f') - b^2 \frac{U'_{DM}}{U_{DM}}(f)\right] = 0 \qquad (98-3)$$

NOW FROM (98-2) $\dfrac{U'_{DM}}{U_{DM}} = -\dfrac{b'}{b}$

SO EQ (98-3) BECOMES:

$$f'' + \underbrace{\frac{U_\infty}{kA}(bb')}\left(\eta f' + f\right) = 0 \qquad (99\text{-}1)$$

THIS MUST BE A CONSTANT.
IT IS ARBITRARY AND SET = 1/2

$$\text{So} \quad bb' = \frac{1}{2}\frac{db^2}{dx} = \frac{1}{2}\frac{kA}{U_\infty}$$

$$\therefore \quad b^2 = \frac{kA}{U_\infty}x + \cancel{\text{Const}}^{\,=0}$$

set an undefined origin of coords ∋ $b(0)=0$.

So:

$$\left.\begin{aligned} b &= \sqrt{\frac{kA}{U_\infty}}\sqrt{x} \;;\quad \eta = \sqrt{\frac{U_\infty}{kA}}\frac{y}{\sqrt{x}} \\ \frac{U_{DM}}{U_\infty} &= \frac{A}{bU_\infty} = \sqrt{\frac{A}{kU_\infty}}\sqrt{\frac{1}{x}} \end{aligned}\right\} (99\text{-}2)$$

AND VELOCITY PROFILE FROM

$$f'' + \frac{1}{2}\eta f' + \frac{1}{2}f = 0 \qquad (99\text{-}2)$$

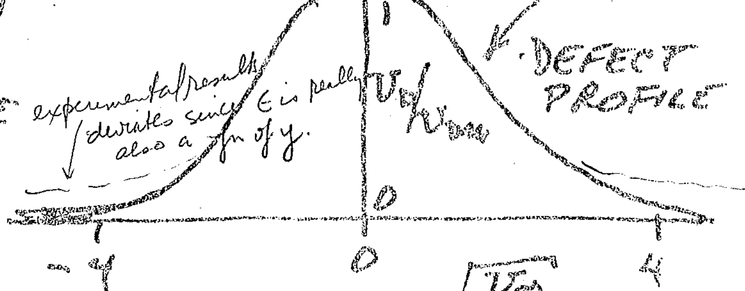
With B.C.s $f(\infty) = 0 \quad (y \to \infty,\ U_D \to 0)$

$f'(0) = 0 \quad (y = 0,\ \text{symmetry})$

SOLUTION GAUSS ERROR-DISTRIBUTION

$$\boxed{f = \exp(-\eta^2/4)} \quad (99\text{-}3)$$

NOTE: ALSO VALID FOR LAMINAR CASE WHERE $kA = \nu$
SEE: (95-4)



$$\eta = y\sqrt{\frac{U_\infty}{kAx}}$$

by finding & fitting data we define kA & X_0 (origin of coordinates) X_0 is normally near trailing edge. Still A is still unknown – Hence we must get it by experiment

OBTAIN CONSTANT C FROM EQ. FOR DRAG, EQ (98-1) — {C_D IS DRAG COEFF BASED ON AREA $L \times 1$}

$$\frac{1}{2}C_D = \frac{F_D}{\rho U_\infty^2 L} = C\left(\frac{U_{DM}}{U_\infty}\frac{b}{L}\right)\int_{-\infty}^{+\infty} f\, d\eta$$

BUT:
$$\int_{-\infty}^{+\infty} f\, d\eta = \int_{-\infty}^{+\infty} e^{-\eta^2/4}\, d\eta = 2\sqrt{\pi}$$

length of body

$$\therefore\quad C = \frac{C_D}{4\sqrt{\pi}}\left(\frac{U_\infty}{U_{DM}}\right)\left(\frac{L}{b}\right)$$

SO:
$$\frac{U_D}{U_\infty} = \frac{U}{U_{DM}}\cdot\left(\frac{U_{DM}}{U_\infty}\right) = C f(\eta)\left(\frac{U_{DM}}{U_\infty}\right) = \frac{C_D}{4\sqrt{\pi}}\left(\frac{L}{b}\right) f(\eta)$$

USING (99-2)

$$\left\{ \frac{U_D}{U_\infty} = \frac{C_D}{4\sqrt{\pi}}\sqrt{\frac{U_\infty L}{kA}}\left(\frac{x}{L}\right)^{-1/2} \exp\left[\frac{-y^2 U_\infty}{4 kA x}\right] \right\} \quad (100\text{-}1)$$

NOTE FOR LAMINAR FLOW USE $\nu \Rightarrow kA$

ASSUME LAMINAR FLAT PLATE DRAG FOR EXAMPLE:

$$\frac{U_D}{U_\infty} = \frac{0.664}{\sqrt{\pi}}\underbrace{\sqrt{\frac{\nu}{kA}}}_{\text{USUALLY} \ll 1 \text{ SINCE } kA \gg \nu}\left(\frac{x}{L}\right)^{-1/2}\exp\left[\frac{-y^2 U_\infty}{4 kA x}\right]$$

VALID BEYOND $X \gtrsim 3L$ IN FAR FIELD

free shear flows –
for field / near field sol.

2. Turb bound layers – intro
to mean profile $U(y)$
shear stress $\tau(y)$
where $\tau = \mu \frac{\partial U}{\partial y} - \rho \overline{uv}$

near field of jet can appear w/ or w/o free shear layer whose velo profile

~~Coeff~~ velocity = $c x^0$
for free shear layer, $c \sim 1/10$ for turb
$\sim 1/1000$ for laminar

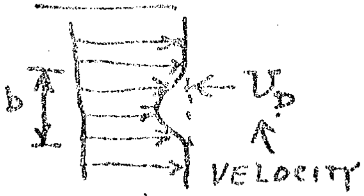
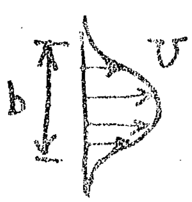
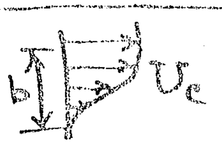
SIMILARITY FUNCTIONS – FAR FIELD FREE SHEAR LAYERS

ALL CASES FOR $\partial p/\partial x = 0$

b = CHARACTISTIC WIDTH

U_D = WAKE VELOCITY DEFECT

$U_c = U_1 - U_2$ = SHEAR LAYER VEL. INCREMENT

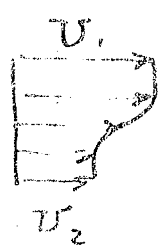
		b		VELOCITY	
		LAM	TURB	LAM	TURB
WAKE 	PLANE, 2-D	$x^{1/2}$	$x^{1/2}$	$x^{-1/2}$	$x^{-1/2}$
	AXISYMMETRIC	$x^{1/2}$	$x^{1/3}$	x^{-1}	$x^{-2/3}$
JET 	PLANE, 2-D	$x^{2/3}$	x^{1}	$x^{-1/3}$	$x^{-1/2}$
	AXISYMMETRIC	x^{1}	x^{1}	x^{-1}	x^{-1}
	FREE SHEAR LAYER	$x^{1/2}$	x	x^{0}	x^{0}

differences are due to geometrical & dimensional reasons.

NOTE ON FREE SHEAR LAYER IN TURB. FLOW

$\xi = \sigma \dfrac{y}{x}$ or at EDGE (ξ = CONST, $y \approx b$)

$b = \text{Const.} \dfrac{x}{\sigma}$; $(1/\sigma)$ is spread rate parameter and depends on $r = U_2/U_1$.



Approximately ($\sigma_0 \approx 11$ to 13):

$$\left(\frac{1}{\sigma}\right) = \left(\frac{1}{\sigma_0}\right)\left[\frac{1-r}{1+r}\right]$$

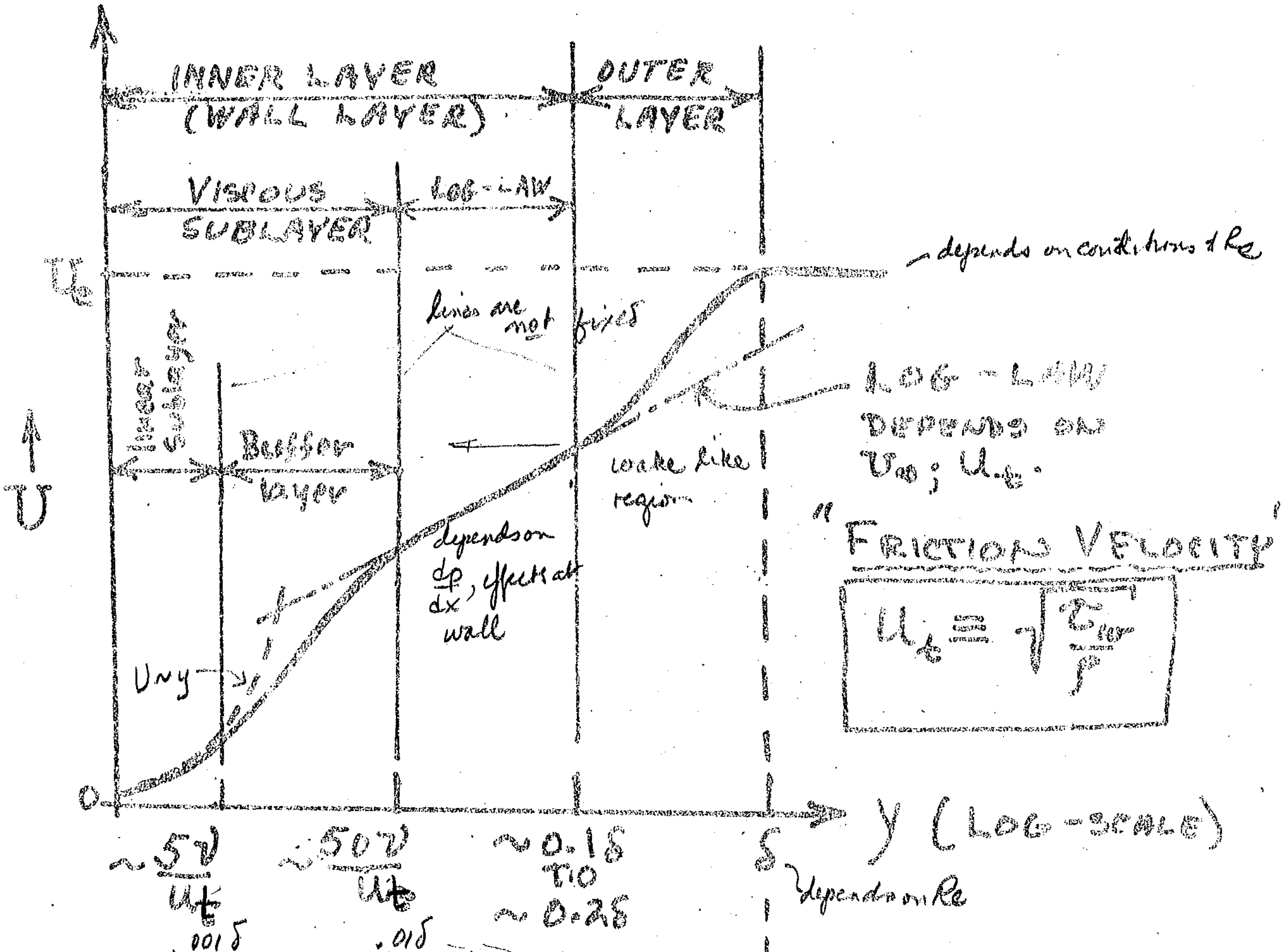
is max. FOR $U_2 = 0$

low FOR $U_2 \approx U_1$

D.) TURBULENT BOUNDARY LAYERS

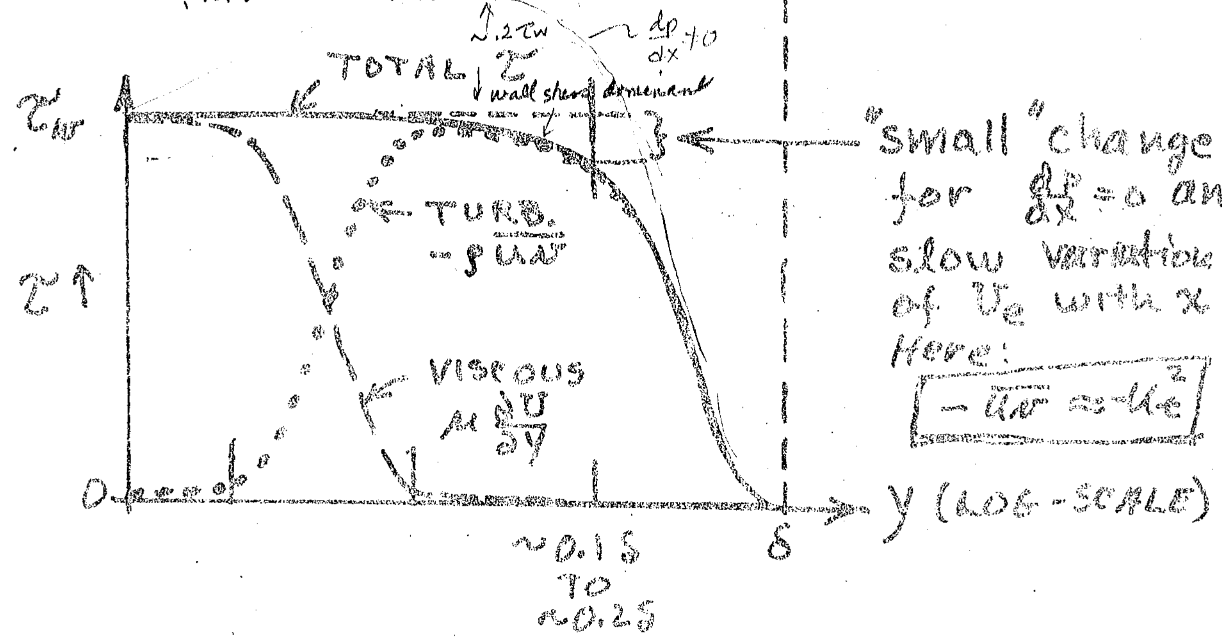
(1.) MEAN VELOCITY PROFILE ON SMOOTH PLATE

(2.) SHEAR STRESS PROFILE (zero or small $\frac{dp}{dx}$'s)



"FRICTION VELOCITY"

$$\boxed{U_\tau = \sqrt{\frac{\tau_w}{\rho}}}$$



"Small" change for $\frac{dp}{dx} = 0$ and slow variation of U_e with x

Here:

$$\boxed{-\overline{u'v'} \approx U_\tau^2}$$

Inner Sublayer ~ .001δ

Buffer layer ~ .01δ

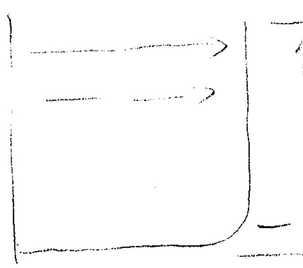
log-law region ~ .1δ

Outer layer

Defect relation $(U_e - U)$

Equil cond in pressure gradient flows (β)

overlap → gives law of wall for β=const



outer what happens here depends on what happened upstream due to convection

inner turbulent eddies must die at wall but very slowly. Turbulent stress bearing eddies are greatest near wall

rate of decay ~ rate of creating

difference is diffused into outer

controlled by local condition: eddies decay & convert to heat

in outer region no balance of production & decay

LOCAL EQULIBRIUM – CONDITIONS DEPEND ON LOCAL PARAMETERS – $U_\infty, \delta, \nu, \rho, (\tau_w \text{ or } u_t)$

INNER LAYER – ALMOST ALWAYS CLOSE TO LOCAL EQULIBRIUM

STRESS CARRING EDDIES ARE SMALL AND HAVE SHORT "LIFE TIMES"

OUTER LAYER – IN LOCAL EQULIBRIUM ONLY IN SPECIAL CASES

STRESS CARRING EDDIES ARE LARGE (RANGE IN SIZE FROM 0.1δ TO δ) AND HAVE "LIFE TIMES" THAT ARE LONG ($T_L \sim (10 \text{ TO } 50)\frac{\delta}{U_\infty}$)

LAW OF THE WALL (OUT TO $y \sim 0.1\delta$)

(1) LINEAR SUBLAYER (Viscous stress dominates)

$$y < (5\nu/u_t)$$

$$\tau = \tau_w = \mu\left(\frac{\partial U}{\partial y}\right)_0 = \rho u_t^2$$

Taylor series: in velocity

$$U = 0 + y\left(\frac{\partial U}{\partial y}\right)_0 + \frac{1}{2}y^2\left(\frac{\partial^2 U}{\partial y^2}\right)_0 + O(y^3)$$

Velocity profile

$$U = y\left(\frac{\tau_w}{\mu}\right) + \frac{1}{2}y^2\left(\frac{1}{\mu}\frac{dp}{dx}\right) + O(y^3)$$

or

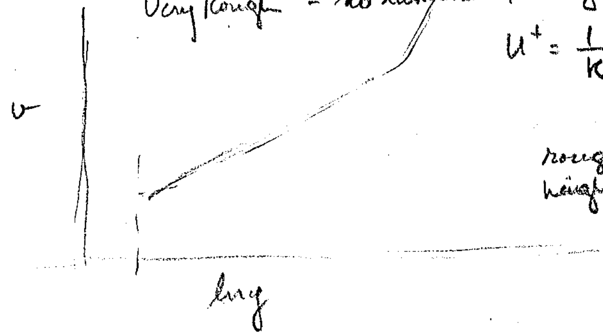
$$\frac{U}{u_t} = \left(\frac{y\,u_t}{\nu}\right) + \frac{1}{2}\left(\frac{y u_t}{\nu}\right)^2\left[\frac{\nu}{u_t^3\rho}\frac{dp}{dx}\right] + O(y^3)$$

or DEFINING

$$\boxed{u^+ = \frac{U}{u_\tau} \quad \text{and} \quad y^+ = \frac{y\,u_t}{\nu}}$$

$\frac{\tau_w}{\mu u_t} = \frac{u_t}{\nu}$ $\quad \tau_w = \rho u_t^2$

Very Rough - no laminar subregion



$$u^+ = \frac{1}{k}\ln\left(\frac{y}{k}\right) + B$$

roughness height

$\left.\begin{array}{l}\text{shape}\\ \text{size}\\ \text{distrib}\end{array}\right\}$ of roughness elements

So: $$\boxed{u^+ = y^+} + \frac{1}{2} y^{+2}\left[\frac{\nu}{u_\tau^3}\frac{dp}{dx}\right] + \cdots \quad (104\text{-}1)$$

only valid at wall $\boxed{\text{as } y^+ \to 0}$

$P^+ = \frac{\nu}{u_\tau^3}\frac{1}{\rho}\frac{dp}{dx}$

(2.) LOG-LAW REGION $\left(\frac{50\nu}{u_\tau} < y < 0.1 \text{ to } 0.2\delta\right)$

where turbulence stress dominates

HERE IT IS OBSERVED THAT

$$U = (\text{Const.})\ln y + (\text{Const.})$$

or $$\frac{dU}{dy} = \frac{\text{Const.}}{y}$$

The Const. does not depend on μ since this is turbulent region. It must be related to $\tau = -\rho\overline{uv}$ which is approximately equal to τ_w for $y \leq 0.1\,\delta$. Therefore, for dimensional reasons.

$$\boxed{\frac{dU}{dy} = \frac{u_\tau}{K y}} \quad (104\text{-}2)$$

$K = 0.41$ is the Kármán Constant and the log-law becomes

$$\boxed{u^+ = \frac{1}{K}\ln y^+ + C} \quad (104\text{-}3)$$

$C \approx 5.0$ for smooth surfaces, $\frac{dp}{dx} = 0$

A GENERAL FORM OF THE "LAW OF THE WALL" VALID FOR WHOLE INNER LAYER COMES FROM IDEA OF LOCAL EQUILIBRIUM WHERE τ_w, ν, ρ ARE ONLY PARAMETERS THAT CONTROL MEAN VELOCITY PROFILE, i.e.

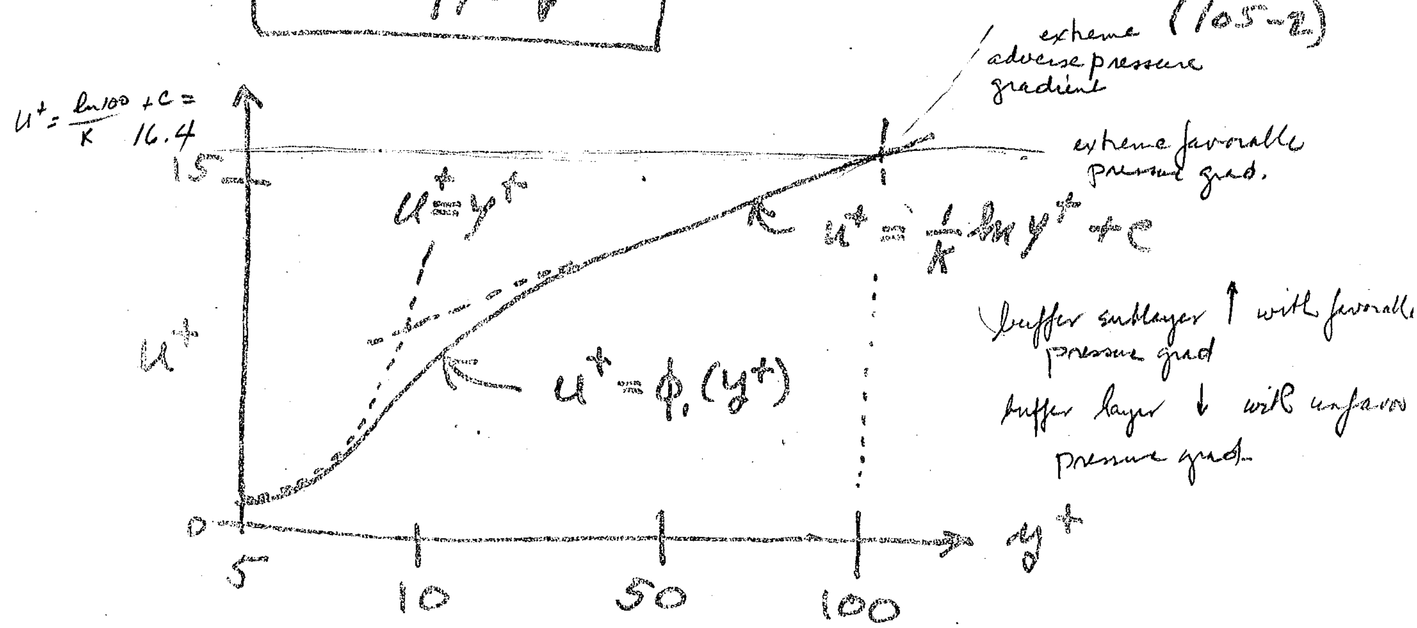
not independent since $-\overline{uv} = \frac{\tau_w}{\rho}$

$$\boxed{U = f(y;\ \tau_w, \nu, \rho)} \qquad (105\text{-}1)$$

DIMENSIONAL ANALYSIS LEADS TO THE FORM:

$$\frac{U}{u_\tau} = \phi_1\left(\frac{y u_\tau}{\nu}\right)$$

or $\boxed{u^+ = \phi_1(y^+)}$ General form of the law of the wall (105-2)



for nonzero pressure grad & suction at wall $\frac{V_w}{U_e}$

$$u^+ = \phi_1(y^+, P^+, V_0^+)$$

~ 30% decay i veloc in outer layer
∴ use $U_e - U$ in perturbation analysis
assume $U_e - U = f$(whatever is happening in outer layer)

$$U_e - U = f\left(y; \delta, -\overline{uv}, \frac{dp}{dx}\right)$$
↳ inner edge of layer.

Turb BL
Outer Layer Veloc Prof
(1) Equil Defect "Law"
(2) Coles "Law of wake"
(3) More general empirical laws $\quad C_f = A\, Re_\theta^{1/m}$ for flat plat flow

Eddy viscosity & mixing length models

wake in a free shear layer is characterized by $\frac{U_e - U}{U_\tau}$ not $\frac{U_e}{U_\tau}$

$$\beta = \frac{\delta^*}{\tau_w}\frac{d\bar{p}}{dx} = -\frac{\delta^* \rho U_e}{\tau_w}\left(\frac{dU_e}{dx}\right) = -\frac{\delta^*}{u_\tau^2} U_e \frac{dU_e}{dx}$$

$$\lambda = \frac{\theta^2}{\nu}\frac{dU_e}{dx} \qquad H = \delta^* \theta$$

$$\beta = \frac{2H\lambda}{C_f(\quad)\,\theta\sqrt{\quad}}$$

(2) THE OUTER LAYER HERE FLOW FEELS NO DIRECT IMPACT OF VISCOSITY AND "FEELS" RETARDING EFFECT OF WALL SHEAR STRESS. IT ALSO RESPONDS TO THE PRESSURE GRADIENT. A DEFECT (WAKE LIKE) RELATIONSHIP IS APPROPRIATE

$$\boxed{(U_e - U) = f\left(y;\ \tau_w, \rho, \delta, \frac{d\bar{p}}{dx}\right)} \qquad (106\text{-}1)$$

IN DIMENSIONLESS FORM ($u_\tau^2 = \tau_w/\rho$)

$$\boxed{\frac{U_e - U}{u_\tau} = g\left(\frac{y}{\delta};\ \frac{\delta}{\tau_w}\frac{d\bar{p}}{dx}\right)} \qquad (106\text{-}2)$$

validity demonstrated by data.

EQUILIBRIUM - DEFECT FLOWS: CLAUSER FOUND FROM EXPERIMENTS THAT DEFECT PROFILES FORMED UNIVERSAL SHAPES FOR CONSTANT VALUES OF β WITH x

$$\frac{U_e - U}{u_\tau} = g_{equil}\left(\frac{y}{\Delta},\ \beta = \frac{\delta^*}{\tau_w}\frac{d\bar{p}}{dx}\right) \qquad (106\text{-}3)$$

actually $\left(\frac{y}{\Delta},\ \frac{\Delta}{\tau_w}\frac{d\bar{p}}{dx}\right)$

REMEMBER IN LAMINAR FLOW β IS CONST. WHEN $\lambda = (\theta^2/\nu)(dU_e/dx)$ IS CONSTANT OR WHEN $U_e \sim x^m$ WHICH IS ALSO APPROXATELY THE EQUILIBRIUM-DEFECT FREE STREAM VELOCITY DISTRIBUTION FOR TURBULENT FLOWS

$$\Delta = \int_0^\infty \left(\frac{U_e - U}{u_\tau}\right) dy = \frac{\delta^*}{\sqrt{\frac{c_f}{2}}} = \frac{\delta^* U_e}{u_\tau} \qquad (106\text{-}4)$$

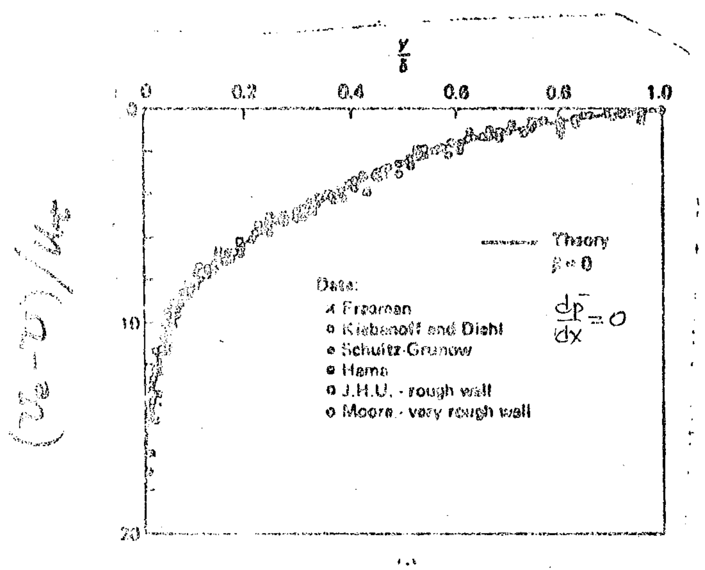
IS THE "DEFECT THICKNESS".

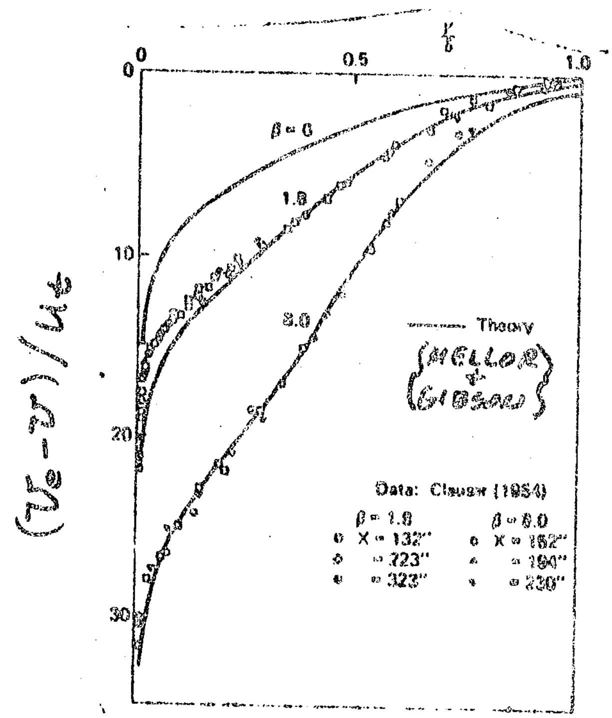
$\Delta u_\tau = \delta^* U_e$

$\Delta \sqrt{\frac{c_f}{2}} = \delta^*$

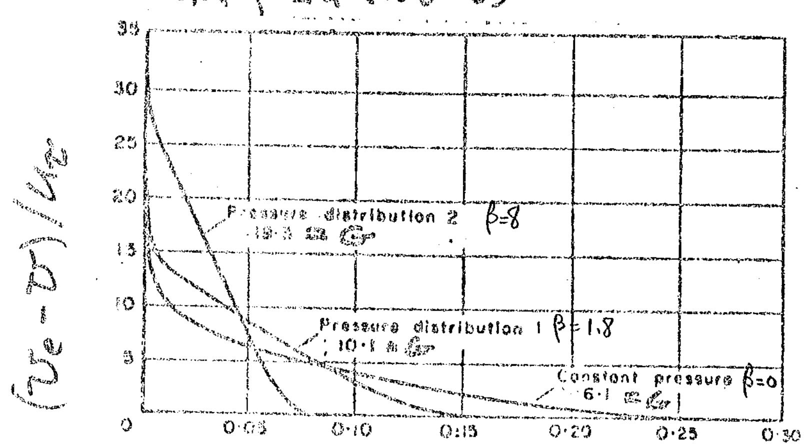
if $\beta, G \neq$ const then everything so ~~far~~ far we've said is not true & no similarity soln exists
however if $u_e \propto x^m$ β, G are not fn's of x but are constant.

FOR EQUILIBRIUM ADVERSE AND ZERO PRESSURE GRADIENTS IT IS SUFFICIENT TO PLOT VERSUS y/δ, SO





WHEREAS IN PROPER FORM, EQ (106-3)



$$\frac{y}{\Delta} = \frac{y}{\delta^*}\frac{U_e}{u_\tau}$$

G is the CLAUSER (1954) SHAPE FACTOR:

$$G = \int_0^\infty \left(\frac{U_e - U}{u_\tau}\right)^2 d\left(\frac{y}{\Delta}\right)$$

WHICH IS CONST. FOR EACH $\beta = \frac{\delta^*}{\tau_w}\frac{dp}{dx}$

FOR ALL X-STATIONS ALONG FLOW

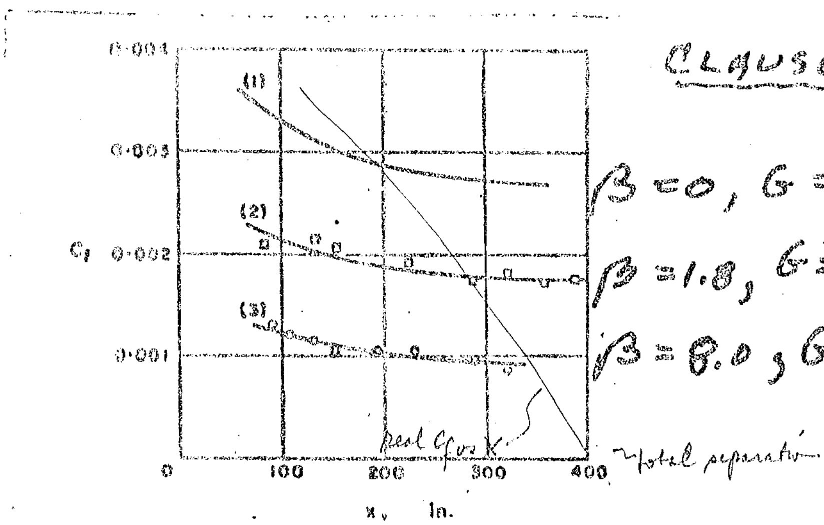
THE OUTER LAYERS ARE "SELF-PRESERVING" IN THESE CASES.

SKIN FRICTION COEF. AND SHAPE FACTOR H FOR EQUILIBRIUM-DEFECT CASES.

$$\beta = \text{Const} \quad \text{and} \quad G = \text{Const}.$$

Approximately: $G \simeq 6.1\sqrt{\beta + 1.81} - 1.7$

But $C_f(x)$ and $H(x)$ are not const.



CLAUSER DATA

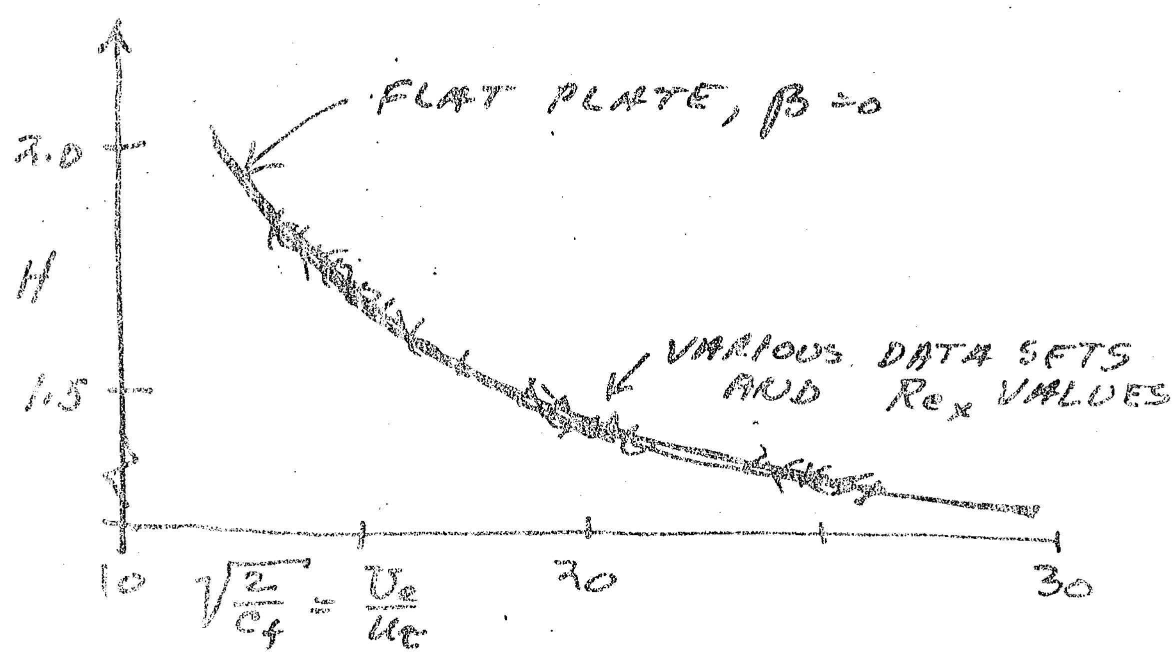
$\beta = 0,\ G = 6.1$

$\beta = 1.8,\ G = 10.1$

$\beta = 8.0,\ G = 19.3$

ALSO FROM DEFINITIONS: $H = \frac{\delta^*}{\theta} = \left(1 - G\sqrt{\frac{C_f}{2}}\right)^{-1}$

SO $H(x) \neq$ Const



for $\beta = 0$

inner layer $u^+ = \phi_1\left(\frac{y u_\tau}{\nu}\right) = u/u_\tau$

$$u^+ = u_e^+ + \phi_2(y/\delta) = u_e^+ + \phi_2\left(\frac{y u_\tau}{\nu} \cdot \frac{\nu}{u_\tau \delta}\right)$$

equate the two at some point let $b = \frac{\nu}{u_\tau \delta}$ $y^+ = y\frac{u_\tau}{\nu}$

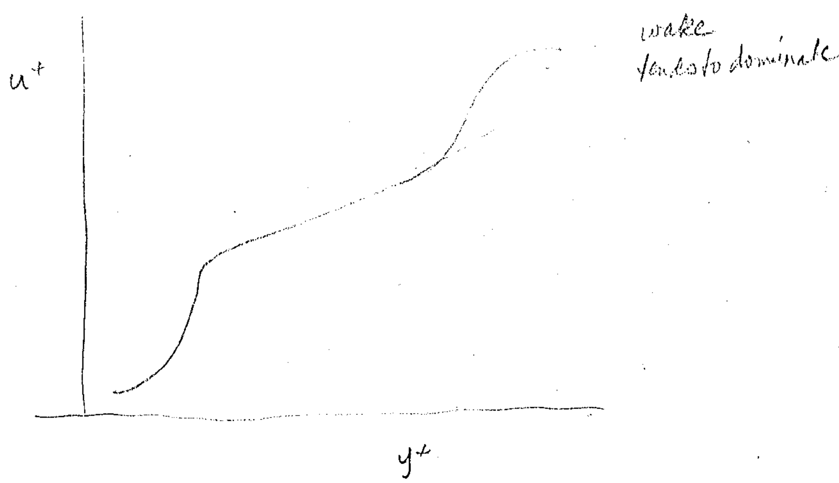
$$\phi_1(y^+) = u_e^+ + \phi_2(y^+ \cdot b)$$

assume $\frac{1}{K_1}\ln y^+ + \tilde{C} = u_e^+ + \frac{1}{K_2}\ln(y^+) + \frac{1}{K_2}\log b$

$$\therefore \Rightarrow K_1 = K_2 \quad \underbrace{\tilde{C} = \frac{u_e^+}{u_\tau} + \frac{1}{K}\log b}_{} = \frac{C}{u_\tau}$$

$\therefore$ if indep of pressure gradient
$\Rightarrow$ we can determine $\tilde{C}$ (which depends on local wall conditions, but it is fixed by overall outer flow).

Coles



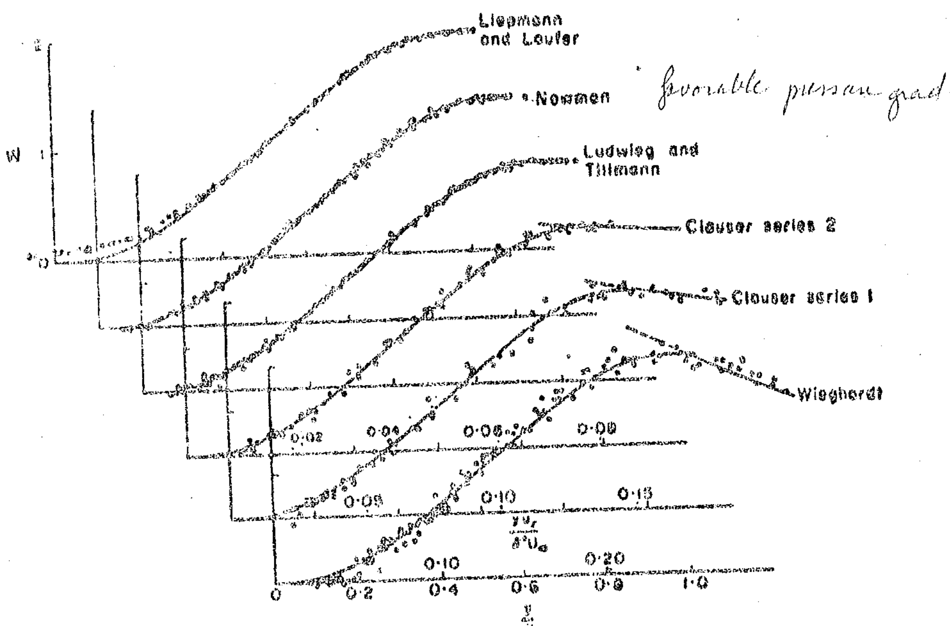
GENERAL OUTER LAYER — "LAW OF THE WAKE" — PROPOSED BY COLES TO FIT NON-EQUILIBRIUM CASES BUT TO UTILIZE THE WAKE-LIKE CHARACTERISTIC. HE GAVE

$$\boxed{\frac{U}{u_\tau} = \phi_1\left(\frac{y u_\tau}{\nu}\right) + \frac{\Pi}{K}\, w\left(\frac{y}{\delta}\right)} \qquad (108\text{-}1)$$

$K = 0.41$, THE KÁRMÁN CONSTANT

$\Pi(x)$ IS THE COLES WAKE PARAMETER

FOR THE DEVIATION FUNCTION $w(y/\delta)$ HE FIT FROM MANY DATA



MOST WORKERS TAKE: $\boxed{w \approx 1 - \cos(\pi y/\delta)}$ (108-2)

A BETTER APPROXIMATION THAT GIVES $(\partial U/\partial y)_{y=\delta} = 0$ IS GIVEN BY ($\zeta = y/\delta$):

$$(\Pi w) = \zeta^2(1-\zeta) + 2\Pi\,\zeta^2(3 - 2\zeta)$$

IS OCCASIONALLY USED.

EVALUATION OF Π (OR δ) CAN BE OBTAINED IN THE SMOOTH WALL CASE FROM (108-1) AT $y=\delta$, $U=U_e$

$$\frac{U_e}{u_\tau}=\frac{1}{0.41}\ln\left(\frac{\delta u_\tau}{\nu}\right)+5.0+\frac{2\Pi}{0.41} \qquad (110\text{-}1)$$

(FOR RELATIONS TO δ^*, θ, etc. SEE TEXT, p175). WHAT WE SEE IS THAT PARAMETERS ARE FUNCTIONALLY DEPENDENT.

$$C_f=2\left(\frac{u_\tau}{U_e}\right)^2=F_{C_f}(Re_\delta,\Pi)$$
$$H=F_{H_1}(Re_\delta,C_f) \quad ,\text{etc.}$$

THESE ARE TWO-PARAMETER RATHER THAN SIMPLE ONE-PARAMETER GIVEN ON PP 75-78 OF THESE NOTES WHERE

$U/U_e=(y/\delta)^n$ SO: $H=\frac{2+n}{n}$

and CORRESPONDING $C_f=\text{const.}\,(Re_x)^{1/\ldots}$

MORE GENERAL EMPERICAL C_f LAWS

(1) WHEN $\beta=0$, (flat plate) one-parameter WILL DO ($\Pi=0.55$). AN IMPLICIT FORMULA FOR WIDE RANGE OF Re_x

$$\frac{1}{\sqrt{C_f}}=1.7+4.15\log_{10}(C_f Re_x) \qquad (110\text{-}2)$$

pressure grad effect here implicit

FOR $5\times10^5<Re_x<10^7$ A SIMPLER FORM

$$C_f=\frac{0.059}{Re_x^{1/5}} \qquad (110\text{-}3$$

no pressure grad here.

(2.) EQUILIBRIUM-DEFECT FLOWS

WHERE $\beta = \frac{\delta^*}{\tau_w}\frac{dP}{dx} = \text{CONST.}$ OR

$U_e \sim x^m$.

HERE (110-1) MAY BE USED

$$\sqrt{\frac{2}{C_f}} = \frac{1}{0.41}\ln\left[\left(\frac{\delta U_e}{\nu}\right)\sqrt{\frac{C_f}{2}}\right] + 5.0 + \frac{2\Pi}{0.41} \qquad (111\text{-}1)$$

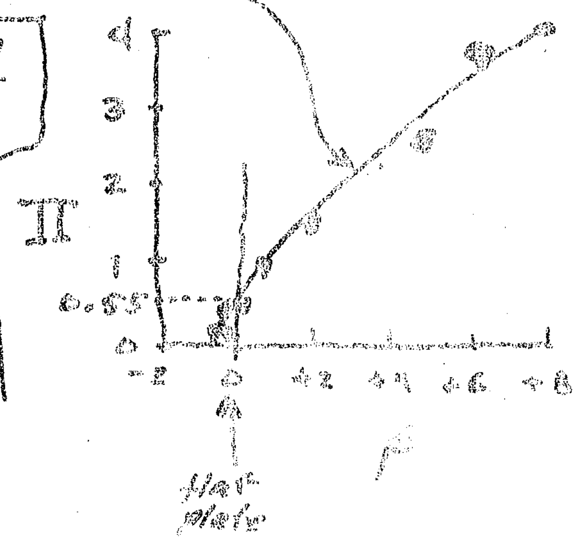
WHERE β and Π ARE APPROX. RELATED BY

$$\Pi \simeq 0.8(\beta + 0.5)^{0.75}$$

and $\frac{\delta^*}{\delta} \simeq \frac{1+\Pi}{0.41}\sqrt{\frac{C_f}{2}}$

etc.

THIS METHOD SUGGESTED BY WHITE FOR NON-EQUILIBRIUM FLOWS TOO.



(3.) NON-EQUILIBRIUM ADVERSE PRESSURE FLOWS NOT AT SEPARATION THE LUDWIEG-TILLMANN (1949) FORMULA IS OFTEN USED FOR ±10% ACCURACY

more widely used

$$C_f = 0.246\, Re_\theta^{-0.268}\, 10^{-0.678H} \qquad (111\text{-}2)$$

WHERE $Re_\theta = \frac{U_e\theta}{\nu}$

Exam given out 16^{th} in class and return 22^{nd} March at 11 AM
will be like in class final

Review

Turbulent, Smooth Wall
2^{nd}D B.L.

outer layer — large eddies decay slowly — non local equil

inner layer $\sim .1\delta$ — greatly energetic dissipate rapidly — local equil

wall layers

Law of the wall: $u^{+} = \phi_w (y^{+}, p^{+}, v_0^{+})$

where $u^{+} = U/u_\tau$ $\quad p^{+} = \dfrac{\nu}{u_\tau^3 \rho}\dfrac{d\bar{p}}{dx}$
$y^{+} = y u_\tau/\nu$
$v_0^{+} = v_w/u_\tau$ if suction or blowing exists

exact: Outer layer defect veloc profile

$$\frac{u_e - U}{u_\tau} = g\left(\frac{y}{\Delta}, \frac{\Delta}{\tau_w}\, d\bar{p}/dx\right) \qquad \Delta = \int_0^\infty \left(\frac{u_e - u}{u_\tau}\right) dy$$

$$G \text{ profile shape param} = \int_0^\infty \left(\frac{u_e - u}{u_\tau}\right)^2 d\left(y/\Delta\right) = \text{const.}$$

defect equil layer: Approx $\quad \beta = \dfrac{\delta^*}{\tau_w}\dfrac{d\bar{p}}{dx} = \text{const}$ if equil outer flow and $u_e \propto x^m$

$\Delta u_\tau = \delta^* u_e$; $\quad \dfrac{u_\tau}{u_e} = \delta^*/\Delta$ varies slowly w/ $x = \sqrt{\dfrac{c_f}{2}}$

General Profile Descrip
Law of the wall + wake

$$u/u_\tau = \phi_w (y^{+}, p^{+}, v_0^{+}) + \frac{\Pi}{K} w\left(y/\delta\right)$$

w/outer layer equil $\Pi = .8(\beta + .5)^{.75}$ empirical data fit

K = Karman const = .41
Π coles "wake" param
$w = 1 - \cos(\pi y/\delta)$
Π related to δ^*/δ, c_f, θ/δ & H by eqs 2.34B,a in extra notes e.g.

$$\Pi = \frac{\delta^*}{\delta} K \sqrt{2/c_f} - 1$$

E.) EDDY VISCOSITY AND MIXING LENGTH ON SMOOTH WALLS:

$$\epsilon_M = -\overline{uv}/\left(\frac{\partial u}{\partial y}\right), \quad \ell = \sqrt{-\overline{uv}}/(\partial u/\partial y) \Big\} \; e_m = \ell^2\left|\frac{\partial u}{\partial y}\right|$$

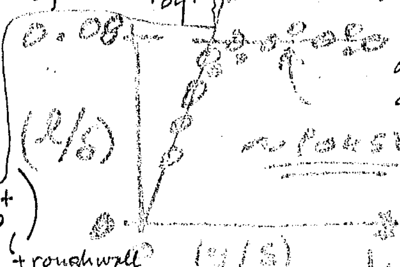
OUTER LAYERS (WAKE LIKE):

VARIOUS MODELS USED BY VARIOUS WORKERS

$\epsilon_M \propto U_e\delta^*$; $\epsilon_m \propto u_\tau \delta$; $\ell \propto \delta$ $\Big|\; e_m = \ell^2\left|\frac{\partial U}{\partial y}\right| = b^2\delta^2\left|\frac{\partial U}{\partial y}\right|$

or

$e_m = \begin{vmatrix} .016 \\ .020 \end{vmatrix} U_e \delta^*$ o.k. for equil defect



INNER LAYERS

$$\frac{\epsilon_m}{\nu} = f_1\left(\frac{y u_\tau}{\nu}, P^+, V_0^+\right)$$

(+ roughwall param.)

NOW IN LOG-LAW ZONE (fully turbulent)

$$\ell \approx Ky \qquad \text{OFTEN ASSUMED}$$

$$\text{BUT} \quad \frac{\partial U}{\partial y} = \frac{\sqrt{-\overline{uv}}}{\ell} \approx \frac{u_\tau}{Ky} \quad \left\{ \begin{array}{l} \text{SINCE } u_\tau^2 = \frac{\tau_w}{\rho} \\ \text{AND } \tau_w = -\rho\overline{uv} \end{array} \right\}$$

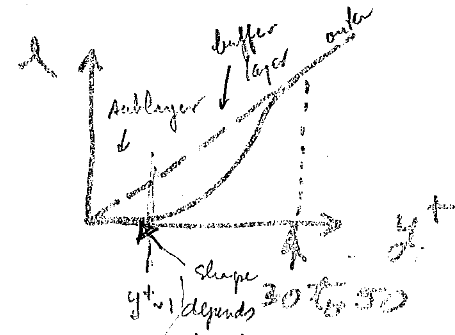
GIVES LOG-LAW OF THE WALL

$$\frac{U}{u_\tau} = \frac{1}{K}\ln\left(\frac{y u_\tau}{\nu}\right) + C \quad \Rightarrow e_m = K^3 y$$

cannot be used near wall but

BUT, BECAUSE VISCOUS STRESS IMPORTANT CLOSE TO THE WALL WE USE VAN DRIEST DAMPING ON ℓ, i.e.

$$\ell = Ky\left[1 - \exp\frac{-y^+}{A^+}\right]$$

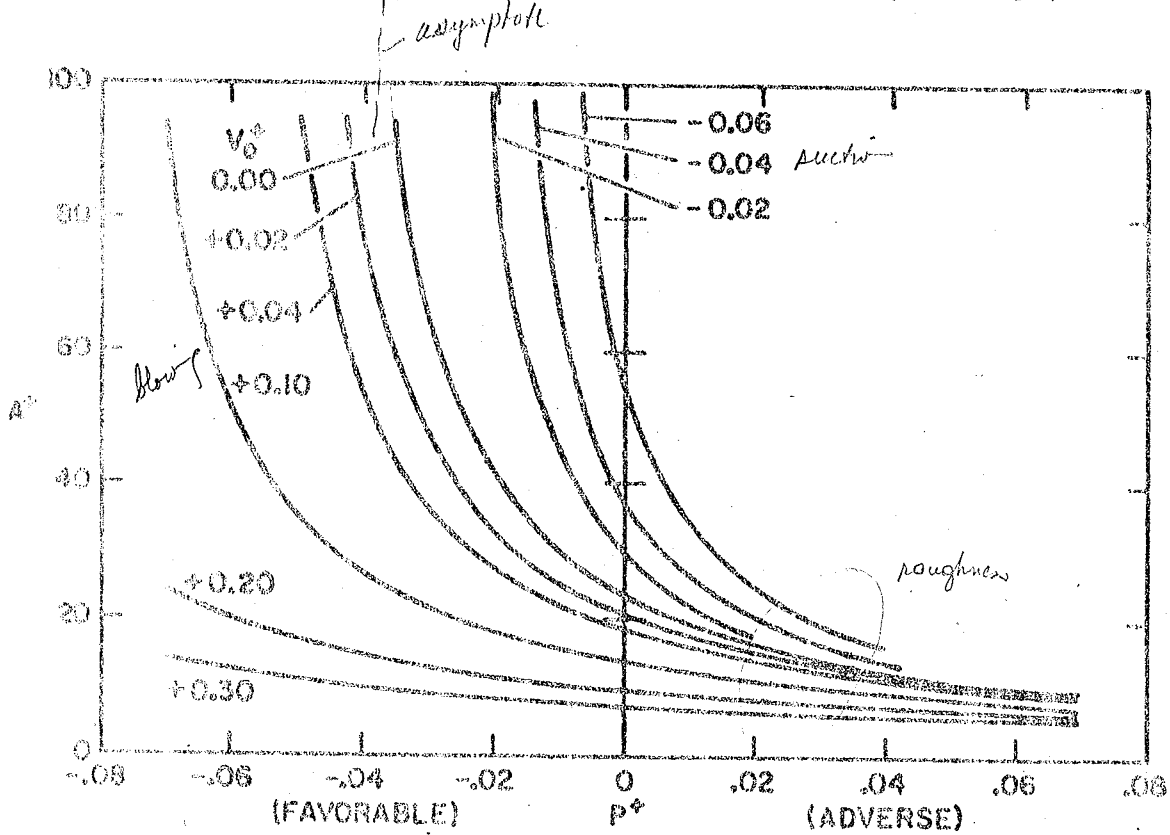


FOR $\beta = 0$ (flat plate)

$$\left(\frac{A u_\tau}{\nu}\right) \equiv A^+ \simeq 26$$

A^+ DEPENDS ON OTHER WALL PARAMETERS SUCH AS $P^+ = \frac{\nu}{U_e^3}\frac{dP}{dX} = \left(\frac{\nu}{U_\tau\delta^*}\right)\beta$ AND $V_0^+ = \frac{v_w}{u_\tau}$

FOR FLOWS WITH SLOW CHANGES OF P^+ AND V_0^+ IN X-DIRECTION CRAWFORD + KAYS (STAN-5) RECOMMEND (SEE REPORT HMT-23, 1976)



WE SEE THAT FAVORABLE PRESS. GRADIENTS AND SUCTION ($V_0^+ < 0$) MAY MAKE VISCOUS REGION VERY THICK. RETRANSITION IS POSSIBLE to laminar since sublayer & buffer layer moves further out

Methods of Solution of Turb. BL

$c_f(x)$, $\delta(x)$, $\delta^*(x)$ etc

(1) Algebraic eq $c_f = .058\, Re^{-1/5}$ $\left(\frac{dp}{dx} = 0\right)$

(2) O.D.E. or Integral

See 1968 Stanford Conference

(3) PDE or Diff eq.

(a) Zero equ models $l(y)$

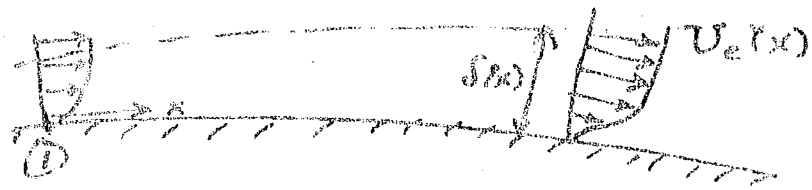
turbulent kinetic energy

(b) 1 Eq. models, T.K.E. $\left(-\overline{uv} = a(k)\right)$

(c) 2 Eq " , T.K.E. + dissipation

(d) Multi eqn models.

INTEGRAL ANALYSIS OF 2-D TURB. B.L.



GIVEN: CONDITIONS AT ①, FLUID PROPERTIES (ρ, ν) AND $\bar{P}(x)$ OR $U_e(x)$

FIND: $\delta, \delta^*, \theta, H$ etc. AS FUNCTIONS OF x
ALSO PREDICT SEPARATION.

MOST METHODS USE:

(1.) M.I. EQ SUCH AS (52-1) OR (52-2):

$$\frac{d\theta}{dx} = \frac{c_f}{2} + \cancel{\frac{V_w}{U_e}}^{\text{(IGNORE HERE)}} - (H+2)\frac{\theta}{U_e}\frac{dU_e}{dx} \qquad (52-2)$$

(2.) AN EMPIRICAL C_f EQ SUCH AS THE "LUDWIEG-TILLMAN" EQUATION:

$$c_f = 0.246\, R_{e_\theta}^{-0.268}\, 10^{-0.678H} \qquad (111-2)$$

(3.) AN EMPIRICAL DIFF. EQ FOR H OR OTHER RELATED PARAMETER.

(SEE DISCUSSION IN: WHITE, F.M., "VISCOUS FLUID FLOW", McGRAW-HILL, P 512-530 AND PROC. "COMPUTATION OF TURBULENT BOUNDARY LAYERS - 1968 AFOSR-IFP-STANFORD CONFERENCE" VOL. I S.J. KLINE, ET AL., TSD ME. DEPT, STANFORD.)

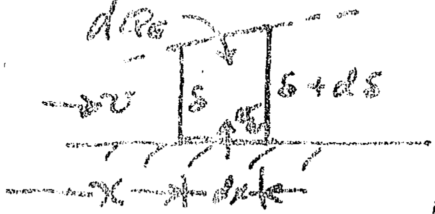
WHITE OFFERS A SIMPLE METHOD based on the law of the wall-wake - WHERE (2.) AND (3.) ARE SIMPLE ALGEBRAIC EQUATIONS. RESULTS WILL BE GOOD AS LONG AS FLOW DOESN'T DEPART TOO MUCH FROM "EQUILIBRIUM"

Eqs 108-2, 110-~~1~~, and add a lag eqn on Π $\quad \frac{d\Pi}{dx} = \frac{\lambda}{\theta}(\Pi_{equil} - \Pi)$
since transition of eddies as they move downstream shows that they don't accept equil until further on down

developed so as not to use lag eqn.

ENTRAINMENT METHODS

CONSIDER ENTRAINMENT INTO THE B.L.



Mass Cons. for C.V. (ρ=const)

$$\dot{m}_{in} = \dot{m}_{out}$$

$$\cancel{\int_0^\delta v\,dy} + v_w\,dx + dQ_E = \cancel{\int_0^\delta v\,dy} + d\int_0^\delta v\,dy$$

$$v_E \equiv \frac{dQ_E}{dx}$$

IS ENTRAINMENT VELOCITY (RATE)

So:

$$v_E = \frac{d}{dx}\int_0^\delta v\,dy - \cancel{v_w} \qquad (115\text{-}1)$$

neglect here

now

$$\int_0^\delta v\,dy = V_e\delta - \int_0^\delta (V_e - v)\,dy = V_e(\delta - \delta^*)$$

SO (115-1)

$$\boxed{v_E = \frac{d}{dx}\left[V_e(\delta - \delta^*)\right]} \qquad (115\text{-}2)$$

HEAD'S MODEL FOR v_E

DEFINE A NEW SHAPE PARAMETER:

$$H_1 \equiv \frac{\delta - \delta^*}{\theta}$$

HEAD SAID, BASED ON CORRELATION OF VARIOUS DATA SETS

$$v_E = V_e F(H_1) \quad \text{AND} \quad H_1 = G(H)$$

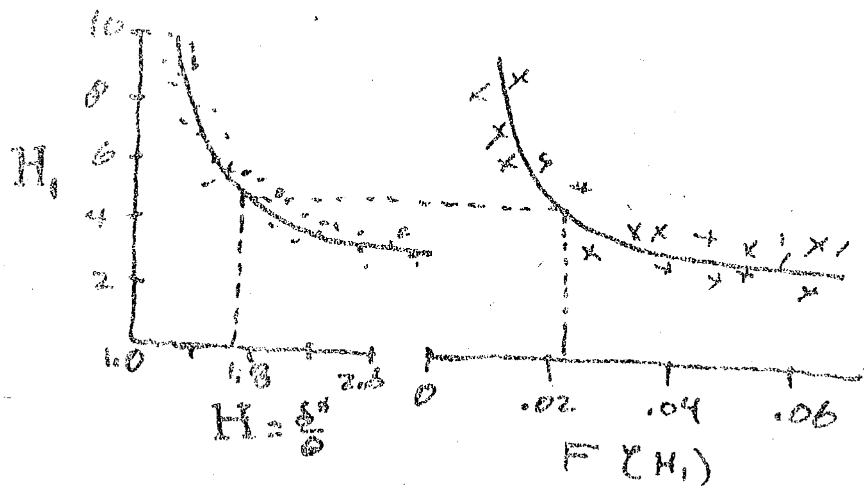
auxiliary eq w/ 2nd mom eqn.

AS A RESULT (115-2) GIVES A D.E. AS AN AUXILIARY EQ TO SOLVE

$$\frac{d}{dx}\left(V_e\,\theta\,H_1\right) = V_e\,F(H_1) \qquad (116\text{-}1)$$

$$\text{WHERE } H_1 = G(H) \qquad (116\text{-}2)$$

EQ'S FOR F AND G GIVEN IN TEXT, P. 173. HEADS CORRELATIONS SHOWN BELOW.



OTHER MODELS FOR v_E USE A VELOCITY SCALE FOR v_E BASED ON A TURBULENCE VELOCITY SCALE, $\tilde{Q}$, i.e. "FIRST REYNOLDS" METHOD (K = UNIVERSAL CONST.)

$$v_E = K\,\tilde{Q} \qquad \tilde{Q} = \sqrt{\text{turb K.E.}}$$

THEIR RESULT IS A DIFF. EQ. SET ($K_3 \approx 0.014$)

$$\frac{d}{dx}\left(\frac{1}{2}v_E^2\,I\right) = K_3\,u_\tau\,v_E \qquad (116\text{-}3)$$

AND FROM (115-1) WHERE $I = \int_0^\delta \tau\,dy$

$$v_E = \frac{dI}{dX} \qquad (116\text{-}4)$$

Dr. W. J. McCroskey

NASA Ames 965-5835 M.S. Helicopter Aircraft Fluid Dynamics Rotor

shear layer – no region of potential flow avail.

VII INTERNAL FLOWS

- DISPLACEMENT INTERACTIONS
  - (1.) DEVELOPING FLOWS IN PIPES AND DUCTS
  - (2.) DIFFUSER SEPARATION — FLOW REGIMES —
- SHEAR LAYER INTERACTIONS
- SEPARATION + REATTACHMENT

ZONES OF INTERACTION — NOZZLE/DUCT AS AN EXAMPLE —

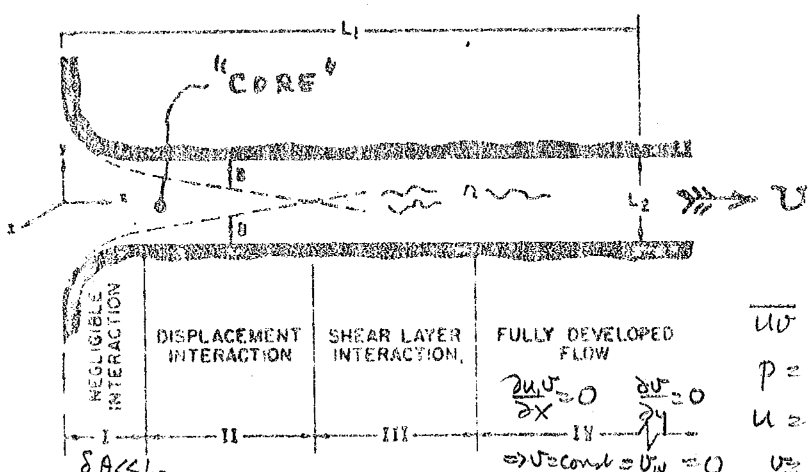


Fig. 3.1. Zones of interaction in duct flow. Note: zone boundaries are in reality poorly defined

$\overline{uv} = f(y)$ only
$p = p(y)$ only if it is
$u = u(y)$ only
$v = v(y)$ only

DUCT DIMENS.

L_1 – AXIAL LENGTH

L_2 – TRANSVERSE LENGTH

ZONE I.

- $L_1 \ll L_2$
- INVISCID CORE WIDTH $\approx L_2$
- $\delta \ll L_2$
- EXAMPLES: (i) THIN AIRFOIL AT HIGH Re No. AND LOW ANGLE OF ATTACK; (ii) SHORT NOZZLE AT HIGH Re No.

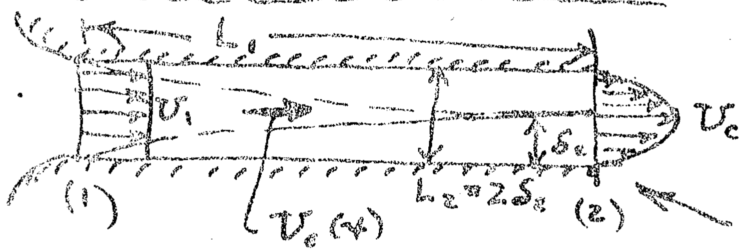
ZONE II – DISPLACEMENT INTERACTION *Nozzle Duct*

- TURB. B.L. – $L_1 \stackrel{<}{\approx} (10 \text{ to } 20)\, L_2$ *depends on initial cond. & initial geom.*
- LAMINAR B.L. – DEPENDENT ON

$$Re_2 = \frac{\bar{U} L_2}{\nu} \quad ; \quad \bar{U} \equiv \frac{Q}{A} = \frac{\dot{m}}{\rho A}$$

LAMINAR B.L. THIN IN SOME CASES FOR VERY LARGE x.

CONSIDER LAMINAR CASE:



$U_{c_2} \approx 2U_1 = 2\bar{U}$

B.L.'s MERGE AT $x = L_1$

Neglect Press. Grad. effects so

$$\delta \approx \frac{5x}{Re_x^{1/2}} = \frac{5\nu^{1/2} x^{1/2}}{U_c^{1/2}}$$

USE AVERAGE VALUE OF $U_c = \frac{1}{2}(U_{c_1} + U_{c_2}) = \frac{3\bar{U}}{2}$

SO:

$$L_2 = 2\delta_2 \approx \frac{10}{\sqrt{3/2}} \left(\frac{\nu}{\bar{U}}\right)^{1/2} L_1^{1/2}$$ *do get strong pressure gradients*

$$\frac{L_2}{L_1} \approx 8.16 \left(\frac{\nu}{\bar{U}}\right)^{1/2} \frac{1}{L_1^{1/2}}$$

$$\left(\frac{L_2}{L_1}\right)^2 \approx 67 \frac{\nu}{\bar{U}} \frac{1}{L_1} \cdot \left(\frac{L_2}{L_2}\right)$$

GIVES $\boxed{\frac{L_1}{L_2} \approx \frac{Re_{L_2}}{67}} \boxed{Re_{L_2} = \frac{\bar{U} L_2}{\nu}}$ (118-1)

APPROX. RESULT

EXACT SOLUTIONS TO DEVELOPMENT IN 2-D DUCTS INDICATE

$$\frac{L_1}{L_2} \approx \frac{Re_{L_2}}{25} + 0.5$$

↑ VERY LOW Re EFFECT

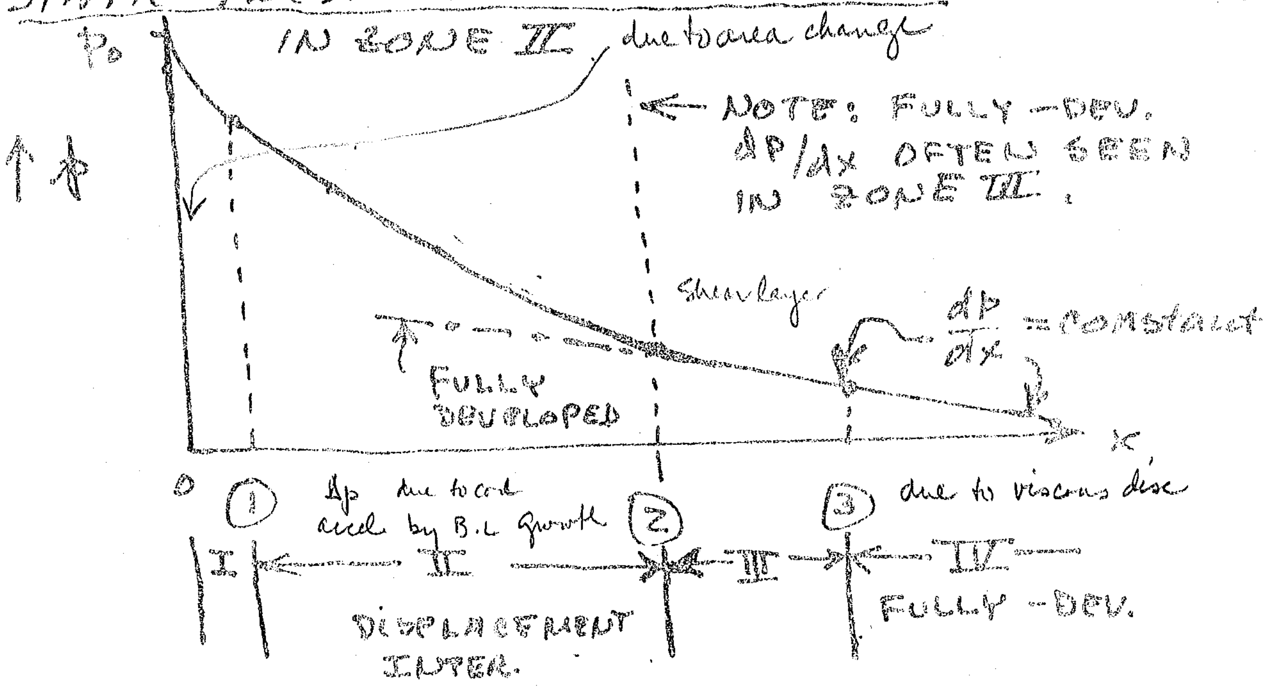
SO: FOR

Re_{L_2}	L_1/L_2
10^3	15 TO 40
10^4	150 TO 400

150 ↑ FROM (118-1); 400 ↑ FROM EXACT

TRANSITION OCCURES AT $Re_{L_2} \approx 3 \times 10^3$ SO L_1/L_2 DROPS TO ≈ 20 BETWEEN Re_{L_2} OF 10^3 AND 10^4

STATIC PRESSURE IN ALL ZONES — TRANSITION



ZONE III - SHEAR LAYER INTERACTION

- POTENTIAL "CORE" DISSAPEARS
- VELOCITY PROFILE ADJUST (CHANGE SHAPE)
- B.L; FLOW $\xrightarrow{\text{transition}}$ FULLY-DEV. FLOW

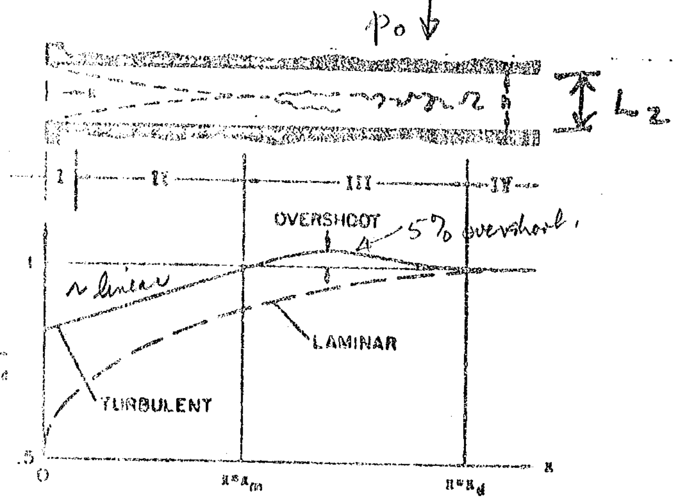


Fig. 3.6. Characteristic core region (centerline) flow speed distributions for inlet regions of constant area ducts: not to scale

NOTE:
TURB. CASE U_c OVERSHOOT

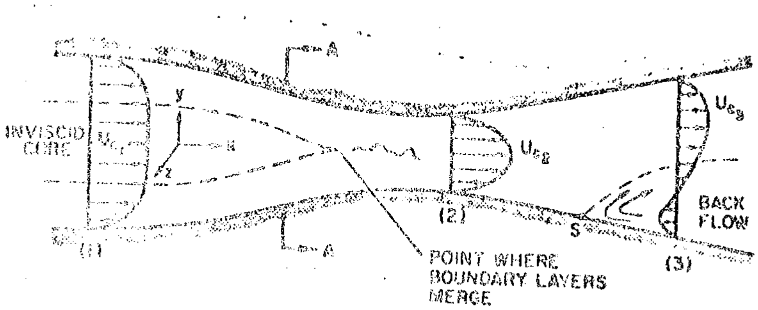
ZONE IV - FULLY-DEVELOPED

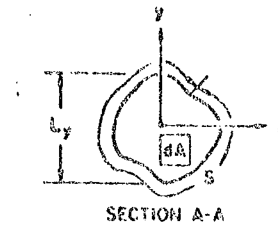
- WITH LAMINAR ZONE III ($Re_{L_2} \lesssim 3\times10^3$) STARTS AT x_d
- TURB. B.L. IN ZONE III THEN MEAN VELOCITY PROFILES, $U(y)$, ESTABLISHED AT $L_1/L_2 \approx 40$

  BUT TURBULENT F.-D. STRUCTURE MAY NOT BE SEEN UPSTREAM OF $L_1/L_2 \approx 100$.
- SELDOM ENCOUNTERED -

INTERNAL FLOW

- STEADY FLOW IN A STRAIGHT, SLENDER DUCT – BLOCKAGE





A : SECTION AREA
S : PERIMETER

SLENDER FLOW CONDITIONS: – SAME AS TSL

- $\frac{\partial p}{\partial x} = \frac{dp}{dx}$; $\frac{\partial p}{\partial y} \approx 0$ $\frac{\partial p}{\partial z} \approx 0$
- $U \gg v, w$ • $\frac{\partial U}{\partial y} ; \frac{\partial U}{\partial z} \gg \frac{\partial U}{\partial x}$

$v, w \lesssim .01U$ can be neglected

STAGNATION PRESSURE (ρ = const.):

$$P = p + \frac{\rho}{2}(U^2 + \cancel{v^2} + \cancel{w^2})$$

small

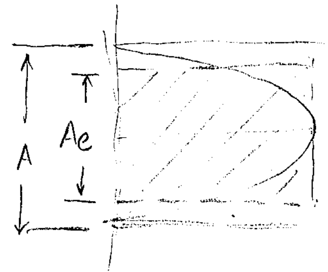
CENTRAL CONDITIONS (ALSO "CORE" IN REGION AHEAD OF B.L. MERGER):

$$P_c = p + \frac{\rho}{2} U_c^2$$

P_c = CONSTANT IN POTENTIAL "CORE" (Bernoulli Eqn along center streamline)

VARIATION OF STATIC PRESSURE:

$$\boxed{dp = dP_c - \rho U_c \, dU_c} \qquad (121-1)$$


A
Ae

BLOCKAGE CONCEPT:

A_E : EFFECTIVE AREA — AREA THAT WOULD PASS FULL FLOW ($Q = \dot{m}/\rho$) AT MAXIMUM LOCAL SPEED, V_e. SO

$$\boxed{A_E = Q/V_e} \quad \text{AND} \quad \boxed{A_E < A}$$

A_B : BLOCKED AREA

$$\boxed{A_B = A - A_E}$$

So $A_B = A - \frac{Q}{V_e} = \int_A dA - \int_A \frac{V}{V_e} dA$

$$\boxed{A_B = \int_A \left(1 - \frac{V}{V_e}\right) dA}$$

∴ DISPLACEMENT THICKNESS: $\boxed{\delta^* = \frac{A_B}{S}}$ ← perimeter of duct

(REDUCES TO CORRECT 2-D FORM)

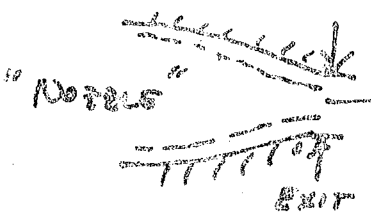
EFFECT OF dA_B AND dA ON dV_e:

$$Q = V_e A_E = V_e (A - A_B)$$

Log. Differentiation gives

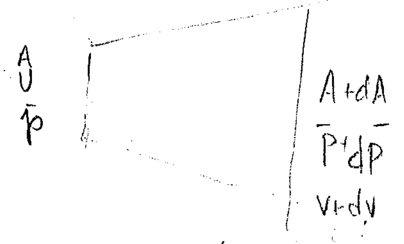
$$\boxed{\frac{dV_e}{V_e} = \frac{dA_B}{(A - A_B)} - \frac{dA}{(A - A_B)}} = \frac{dA_B - dA}{A_e} \quad (122-1)$$

So AS $dA_B \uparrow$ AND $dA \downarrow$ IN x-direction THEN $dV_e \uparrow$ AND VISE VERSA.



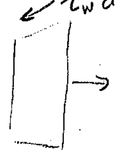
$\longrightarrow V_e > \frac{Q}{A}$ BECAUSE OF "BLOCKAGE"

Mom theorem.



$$\sum_{body} F_x + \sum_{sur} F_x = \frac{d}{dx}\left[\int^{A} \rho U^2 dA\right] dx$$

we assume pg body force = 0



surface forces w/ ρ = const

$$-A\frac{d\bar{p}}{dx} - \int^{A} \tau_w ds = \rho \frac{d}{dx}\int^{A} U^2 dx$$

define momentum deficit area $\quad A_\theta = \int^{A}\left(1-\frac{U}{U_c}\right)\left(\frac{U}{U_c}\right) dA$

MI eq becomes for ρ=const straight slender duct flows.

$$\rho\frac{d}{dx}\left(U_c^2 A_\theta\right) = A_B\frac{d\bar{p}}{dx} + \int \tau_w ds + A_F\frac{dp_c}{dx}$$

pressure grad + or − I — wall friction + or − II — loss in core total press. − III always smooths profile

if both positive will cause the veloc profile to distort

if fully developed flow $dU/dx = 0$ $\quad d\bar{p} = dp_c$ so for MI eq

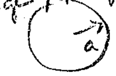
$$\frac{d\bar{p}}{dx} = \frac{1}{A}\int_s \tau_w ds$$

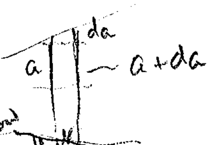
for a pipe $A = \pi a^2$, $S = 2\pi a$ $\quad \frac{d\bar{p}}{dx} = \frac{2}{a}\tau_w$ for uniform shear $\qquad$ let $A_\theta = \theta$, $A_B = \delta^*$, $U_c = U_e$ in mom integ eq

2-D BL. $\quad A_\theta = \theta_a S \quad A_B = \delta_a^* S \quad S = 1$

$$\frac{dA_\theta}{dx} = \frac{C_f}{2} - \left(2 + A_B/A_\theta\right)\frac{A_\theta}{U_c}\frac{dU_c}{dx}$$

Axially sym $\quad S = 2\pi a$





rate of change of periphery as a fn of x

$$\frac{d\theta_a}{dx} = \frac{C_f}{2} - \left(2 + \frac{\delta_a^*}{\theta_a}\right)\frac{\theta_a}{U_c}\frac{dU_c}{dx} - \underbrace{\frac{\theta_a}{a}\frac{da}{dx}}_{\text{accounts for convergence or divergence of flow geometrical effect}}$$

$$\theta_a = A_\theta/2\pi a\,; \quad \delta_a^* = A_B/2\pi a$$

EFFECT OF "BLOCKAGE" ON STATIC PRESSURE CHANGE

USE EQ (122-1) IN (121-1) AND OBTAIN

$$dp = \frac{\rho U_e^2}{(A - A_B)} \left[dA - dA_B \right] + d P_c \qquad (123\text{-}1)$$

(1) (2) (3)

Term (1): Geometric Effect can cause increase (diffuser) or decrease (nozzle) OF dp.

Term (2): Blockage Effect Generally tends to reduce dp since dA_B nearly always positive

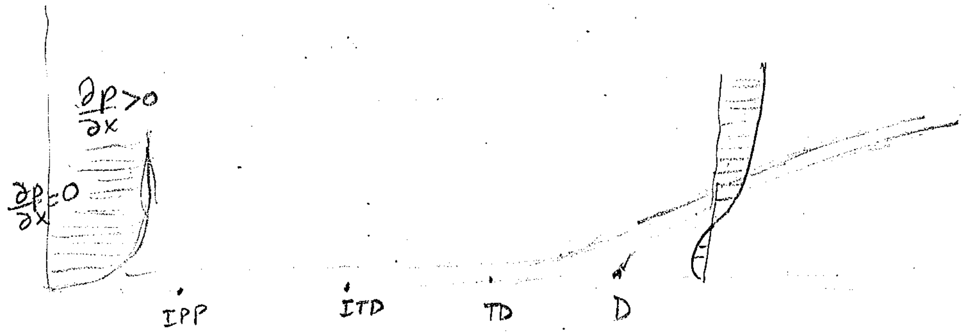
Term (3): Core total pressure Effect

(a) Zero up to "SHEAR LAYER" INTERACTION ZONE.

(b) Negative DOWNSTREAM OF "SHEAR LAYER" INTERACTION — Δ Loss from 2nd law of thermo.

CASES: • SHORT NOZZLES ($dA/dx < 0$) NEGLECT dA_B EFFECT AT FIRST ITERATION. $dP_c = 0$. USE FIRST dp/dx TO OBTAIN B.L. SOLUTION.

• A = CONST. (PIPES & DUCTS) • NEAR INLET (dA_B) WHOLE EFFECT WHERE $dP_c = 0$ • FULLY-DEV. REGION, $dA_B = 0$ AND $dP_c = dp$.

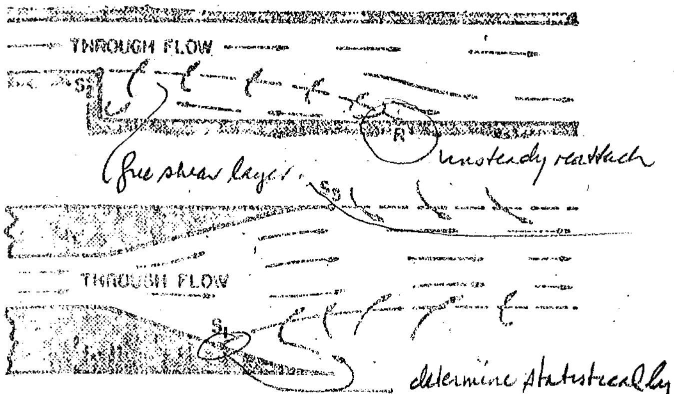


incepient pressure point ~ 1% backflow (instantaneous)
intermittent transitory detachment ~ 20% backflow (instantaneous)
transitory detatchment ~ 50% "

detachment D $\tau_w = 0$

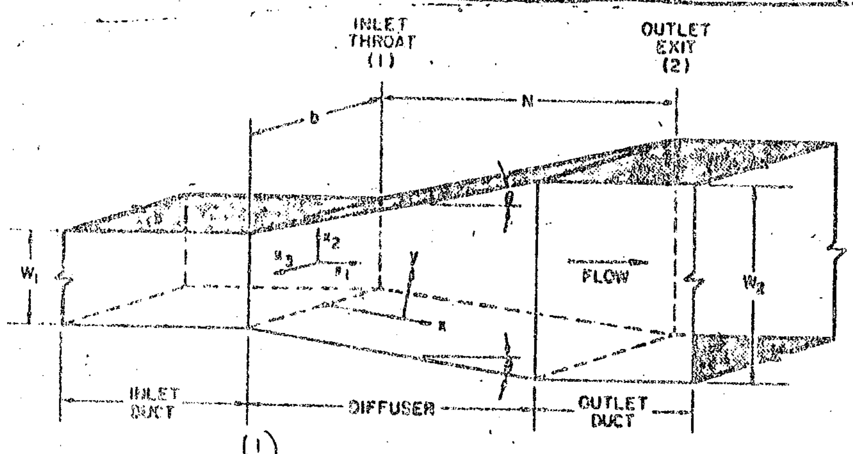
FLOW IN DIFFUSERS

- PHENOMINA OF SEPARATION AND REATTACHMENT DOMINANT



- S_1 – STEP EDGE
- S_2 – TRAILING EDGE
- S_3 – FAIRED SURFACE
- R – REATTACHMENT

DIFFUSER GEOMETRY (2-D) AND PARAMETERS



endwall b.l. don't play important ~~role~~ part

AS (ASPECT RATIO) : $b/W_1 \gg 1$

2θ (INCLUDED ANGLE)

N/W_1 (LENGTH TO WIDTH RATIO)

AR (AREA RATIO) : W_2/W_1

NOTE FOR SYMMETRIC CASE:

$$\boxed{AR = 1 + \frac{N}{W_1} 2 \tan\theta}$$

Assume steady flow @ inlet
incomp
irrot core at (1)
neglect tail pipe
thin turb bl at (1)

PERFORMANCE PARAMETERS:

- PRESSURE RECOVERY
- EFFECTIVENESS
- OUTLET VELOCITY PROFILE SHAPES
- OUTLET TURBULENCE "STATE"

PRESS. RECOVERY COEFF:

$$C_p \equiv \frac{(p_2 - p_1)}{\frac{1}{2}\rho U_{c_1}^2} = \frac{(p_2 - p_1)}{(P_{c_1} - p_1)}$$ core veloc at 1

OR $$\bar{C}_p \equiv \frac{(p_2 - p_1)}{\frac{1}{2}\rho \bar{U}_1^2} \quad \text{where } \bar{U}_1 = \frac{Q}{A_1} = \frac{\dot{m}}{\rho_1 A_1}$$

FOR "IDEAL, ρ = CONST., 1-D FLOW" W/O LOSS

bernoulli's $$C_{p_i} = 1 - \left(\frac{\bar{U}_2}{\bar{U}_1}\right)^2 = 1 - \frac{1}{AR^2}$$

EFFECTIVENESS: $$\xi \equiv \frac{p_2 - p_1}{(p_2 - p_1)_{1\text{-D, ideal, w/o loss}}}$$

WHERE ρ = CONST.

$$\xi = \frac{\bar{C}_p}{C_{p_{ideal}}} \quad \left\{\text{since } (\bar{U}_1)_{1\text{-D}} = \bar{U}_1\right\}$$

also $$\xi = \left(\frac{U_{c1}}{\bar{U}_1}\right)^2 \frac{C_p}{C_{p_i}} = \left(\frac{A_1}{A_{E1}}\right)^2 \frac{C_p}{C_{p_i}}$$

Loss Coefficient: (NEED DETAILS OF $\bar{U}_1(y)$, $\bar{U}_2(y)$)

PRACTICAL DEFINITION:

FOR SLENDER CASES WHERE ($p_1(x)$ and $p_2(x)$):

$$\Delta \bar{P} = (p_1 - p_2) + \frac{1}{2}\rho(\bar{U}_1^2 - \bar{U}_2^2)$$

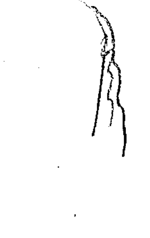
$$\bar{\omega} = \left(\Delta \bar{P} / \tfrac{1}{2}\rho \bar{U}_1^2\right) = \left[C_{p_i} - \bar{C}_p\right] = \bar{H}_L$$

head loss coeff

wakes are less detrimental that jets as far as blockage

tail pipes act as a fully developed flow ?

wall curvature : need 2-D potential effect to be included



PARAMETERS THAT AFFECT DIFFUSER PERFORMANCE

- MACH NO. : (U_1/a_1) not an important until you get to $M \geq 1.2$
- REY. NO. : $(U_1 W_1/\nu)$
- ASPECT RATIO : $(b/W_1) \leftarrow$ (SHAPE)
- INLET VELOCITY PROFILES :
- INLET TURBULENCE :
  - INTENSITY
  - STRUCTURE (EDDY SIZE AND ORIENTATION)

SHAPE OF CROSS SECTION SUCH AS CONICAL, ANNULAR, 2-D, OTHER

AT SUBSONIC MACH NO. AND WHEN

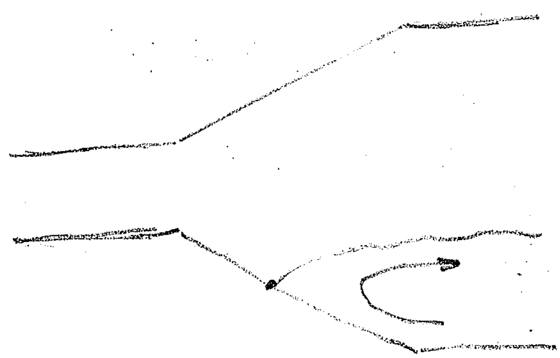
$$AS = b/W_1 > 4 \quad \text{(SMALL END WALL EFFECTS)}$$

$$Re_{W_1} = \frac{U_1 W_1}{\nu} > 10^4 \quad \text{(TURB. B.L.)}$$

THE MOST IMPORTANT PARAMETER, OTHER THAN GEOMETRY, IS

BLOCKAGE FACTOR : $\boxed{B_1 = A_{B_1}/A_1}$ reduces performance of diffuser

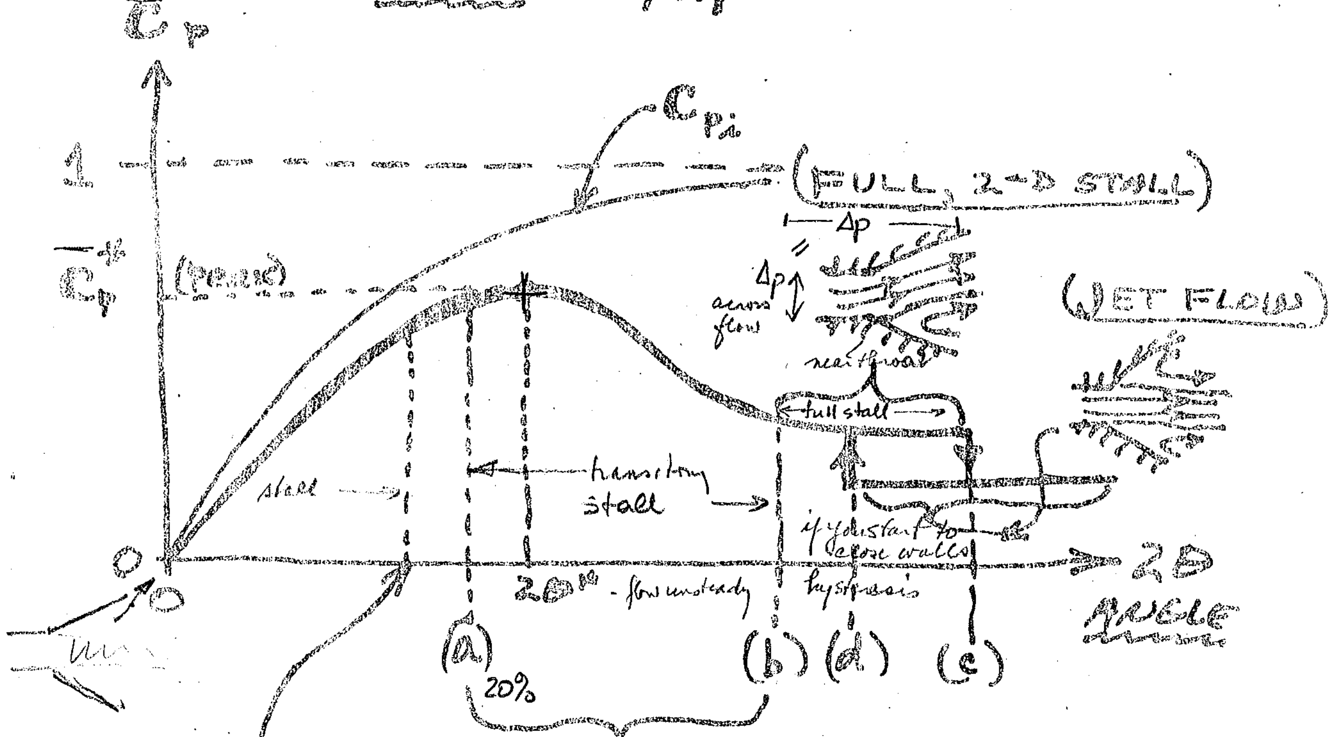
WHEN INLET PROFILES ARE B.L. TYPE WITH NEARLY UNIFORM "CORE" AT (1) AND MODEST TO LOW TURBULENCE INTENSITY, δ (p.87) $\leq$ (1 to 2)%.



near $2\theta^*$

TYPICAL PERFORMANCE (FLOW REGIMES)

2-D, AS > 4, $6 < 0.02$, $B_1 < 0.05$
$Re_1 > 10^4$, $U_1/a_1 < 0.8$
FIXED: N/W_1



(TRANSITORY STALL)
STARTS WHEN ~20% OF EXIT AREA IN BACK FLOW INSTANTANEOUSLY

FIRST STALL AT EXIT WHEN $(2\theta) \approx 0.8\,(2\theta)_{line\,(aa)}$

HERE DISPLACEMENT INTERACTION VERY STRONG, SO PREDICTION OF SEPARATION REQUIRES "SIMULTANEOUS" SOLUTION OF CORE AND B.L.'S

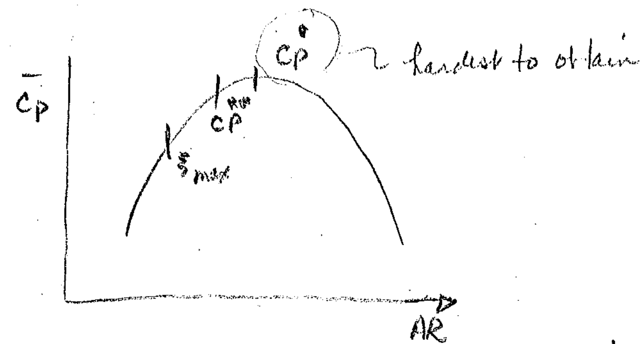
Recovery Optima

Pressure recovery C_p^* (const N/W_1)

at constant AR (

to max ξ (const N/W_1)

$(\bar{\bar{H}}_2)_{min}$ (Const N/W_1)



go to pg 127 for blockage

FLOW REGIME MAP (2-D CASE FROM WORK OF FOX + KLINE)

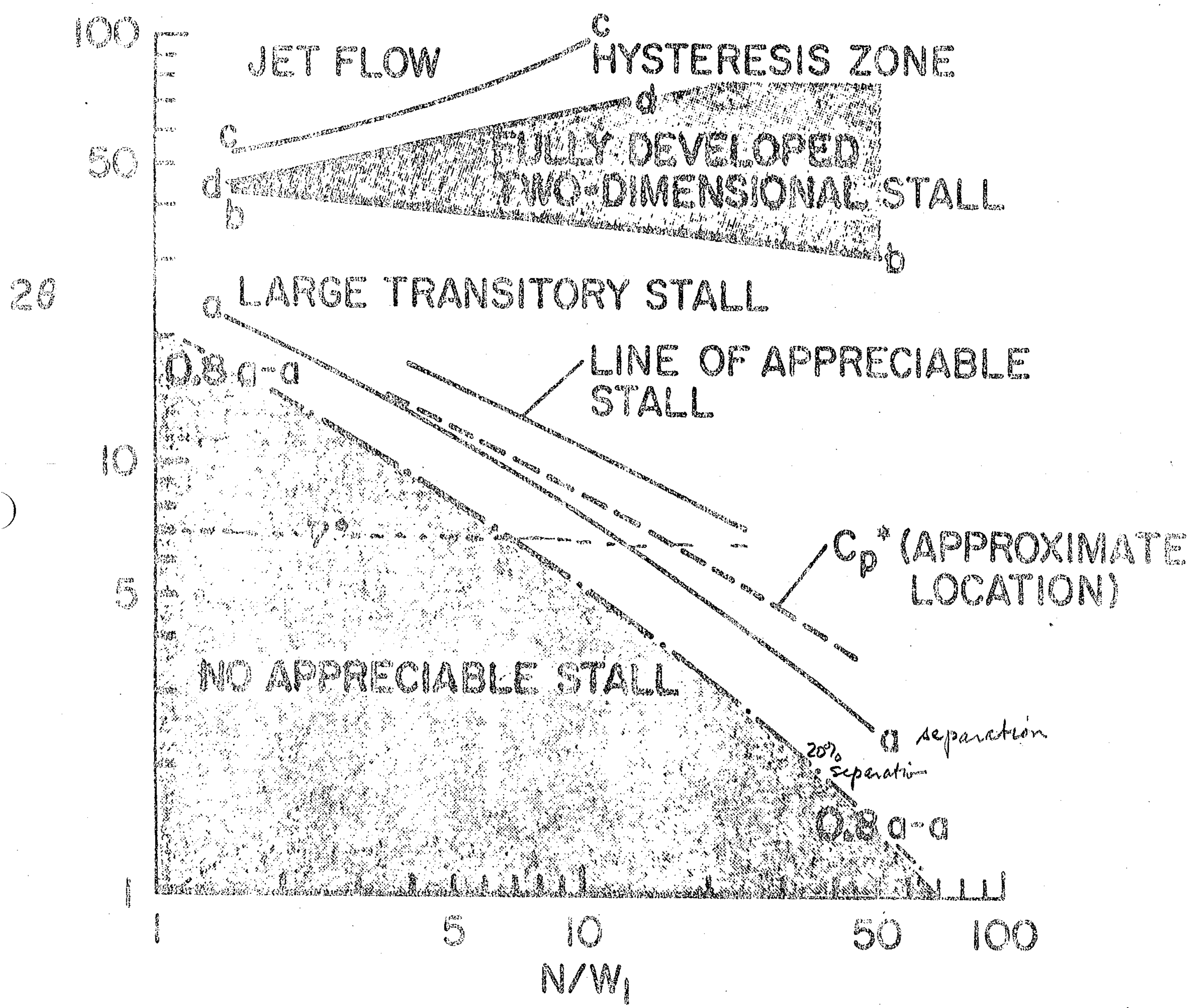


Fig. 1. Straight-walled diffuser flow-regime chart of Fox and Kline [1].

DISCUSS SOME DESIGN IMPLICATIONS

E.1.1.2 A Simple Method

Sovran + Klomp (Ref. 3) have correlated many types of straight diffuser data (Fig. 18) in the chart shown. And have worked out a simple method for estimating C_p^*

Using $P_c = p + \frac{\rho}{2} U_c^2$ we obtain

$$\boxed{\overline{C}_p = \frac{1}{E_1^2} - \frac{1}{E_2^2 AR^2} - w_c}$$

$E_1 = 1 - B_1$ (given) $B = A_B/A$

$E_2 = 1 - B_2$ (to be found)

$w_c = \frac{\overline{P}_{c1} - \overline{P}_{c2}}{\frac{\rho}{2} U_1^2}$ (zero if B_1 small and little b.l. mixing.

Sovran + Klomp correlated much data and did many calculations using Momentum Integral b.l. theory and found that

$(E_2)_{theory}$ correlates well w.

$AR(100B_1)^{1/4}$

Data also collapses on these coordinates at least near peak recovery geometry.

This result is shown in Fig. 24 of of Ref. 3 - Show method. It is of very limited value in region above RR (100 B.)$^{1/4}$ of 5.0

Show how this result has been used to predict AS ratio effect data of Powers. (Fig. 96)