Autumn 1978
J. P. Johnston

ME 251A

Advanced Fluids Engineering

James P. Johnston, Office: 501F, Phone: 497-4024

Text: Fluid Flow by Sabersky, Acosta and Hauptmann Got

Course Assistants: Jay Gillis, John Eaton, Jalal Ashjaee
Office - 501H

COURSE OUTLINE

I. Introduction (Study Chapter 1, Sections 2.1, 2.2, 2.3, 2.4, 2.8; Read: Sections 2.5, 2.6, 2.7)

- A. Observations on fluid mechanics and engineering
- B. Fluid continuum and its properties
- C. Kinematics of flowing fluids -- the field approach and introduction of compact notation
- D. Application -- development of continuity equations
- E. Application -- development of equations of motion for an inviscid fluid

II. Global Equations (Study: Chapter 4; Read: mini-course notes)

- A. Control volume forms of Basic Laws
    1. Conservation of Mass
    2. Conservation of Energy (1st Law of Thermodynamics)
    3. Linear Momentum Theorem (Newton's laws of motion)
    4. Moment of Momentum Theorem
    5. Second Law of Thermodynamics
- B. Applications to systems of finite size

III. Applications of equations of motion for an inviscid flow (Study: Chapter 3)

- A. Euler's equations in various coordinate systems
    1. rectangular, cylindrical, spherical
    2. local streamline coordinates (S,N)
- B. Bernoulli's equations--steady and unsteady flow of an incompressible fluid, steady Barotropic flow, steady incompressible flow, gravity as a body force and conservative body forces.
- C. Applications in systems of finite size -- use and misuse of Bernoulli in conjunction with use of Global Equations.

IV. Steady Inviscid Flow (Study: Chapter 6) 4wks

- A. Discussion of applicability
- B. Rotation, vorticity and circulation

C. Applications to inviscid, irrotational flows
D. Velocity potential and streamfunction, Laplace equation and boundary conditions -- inviscid irrotational flows
E. Some approximate methods of solution
F. Comparison of real flows to results from potential theory

V. Introduction to equations of motion of a Newtonian Viscous Fluid (Study: Sections 2.5, 2.6, 2.7; Read: Chapter 7)

Note: It is unlikely that we will get into the material of section V. In any case, it will be covered in the first few weeks of M.E. 251B, Winter Quarter.

VI. Similitude and Dimensional Analysis (Read and Review: Chapter 5, Study if not familiar to you)

Note: We will not lecture on the material in section VI. You are, however, responsible for it at final exam time -- be prepared!

Examinations: There will be one midterm and a 3-hour final examination.

Problems: You should attempt to solve a number of the exercises from the text for your own benefit. A list of suggested exercises is given below. Answers to these exercises are provided in the back of the book.

Several problem sets to be written up for a grade will be assigned during the quarter. These must be presented in a neat, readable format consistent with professional standards and must include:

(1) Problem statement with appropriate diagrams and definition of nomenclature.
(2) Basic principles, laws and equations employed.
(3) Simplifying assumptions stated with reasons.
(4) Summary of results and conclusions.
(5) Very brief discussion (if required).
(6) Proper citation of reference material other than class notes.

Suggested Exercises (not to be passed in for grade). Textbook problem numbers:

(Weeks 1-2): 1.5, 2.2, 2.4, 2.5, 2.8, 2.10, 2.13, 2.14, 2.20, 2.23, 2.24, 2.26, 2.30.

(Weeks 3-5): 3.3, 3.4, 3.5, 3.8, 3.10, 3.13, 3.15, 3.21, 3.31, 3.34, 4.4, 4.7, 4.10, 4.15, 4.17, 4.19, 4.26, 4.31.

(Weeks 6-10): 6.1, 6.4, 6.5, 6.6, 6.10, 6.11, 6.18, 6.21, 6.39, 6.42.

Grades: will be determined approximately as follows:

Problem sets:	30% final grade	
Midterm exam:	10% final grade	midterm (open book) take home
Final exam:	60% final grade	~~Open book~~ 3 hr exam

Tutorial Hours: Each class member will be assigned to a tutoral group which will meet with a course assistant for one hour each week. The purpose of these meetings is to provide for open and general discussion of current problems, exercises, lecture material, etc. It is not intended to be a working session for group solution of assigned problem sets.

We will separate the class into three groups and pick three separate hours for the weekly meetings of each group. Please indicate on blank back of blue card four possible choices in order of preference from the following list of available hours:

Tuesday	11:00-12:00 noon
Thursday	11:00-12:00 noon
Monday	3:15-4:05 p.m.
Wednesday	3:15-4:05 p.m.
Friday	3:15-4:05 p.m.
Tuesday	12:00-1:00 p.m.
Wednesday	12:00-1:00 p.m.
Thursday	12:00-1:00 p.m.

Monday through Thursday evenings: 7:00-8:00 p.m.

Course Ass't
12-2 W,F
1:30-3:30 Th.

Course Assistants: Besides leading the tutorials, the course assistants will be available for private consultations during a limited number of hours each week. Office hours will be posted and enforced. Take advantage of this opportunity, but please do not abuse the privilege. These persons are also busy graduate students with deadlines and pressures of their own.

Motion Picture Films on Fluid Mechanics will be shown almost every Monday for a 30 minute period starting at 12:15 p.m. A schedule is attached. Although attendance at the films is not required, it is highly recommended. In particular, I urge you to see the films scheduled on Oct. 9, Oct. 23, Oct. 30, and Nov. 20.

ME 251A
Autumn 1978
J. P. Johnston

REFERENCE AND TEXT BOOKS IN FLUID MECHANICS -- A VERY SHORT SELECTION

1) General Introduction and First Course Books:

Sabersky, R. H., A. S. Acousta and E. G. Hauptmann, Fluid Flow - A First Course in Fluid Mechanics, Macmillan: New York.

Fox, R. W. and A. T. McDonald, Introduction to Fluid Mechanics, John Wiley & Sons, Inc.: New York.

2) Intermediate Books for Special Purposes:

Shapiro, A. H., The Dynamics and Thermodynamics of Compressible Fluid Flow, Vols. 1 and 2, The Ronald Press: New York.

Karamcheti, K., Principles of Ideal-Fluid Aerodynamics, John Wiley & Sons, Inc.: New York.

Tennekes, H. and J. L. Lumley, A First Course in Turbulence, The MIT Press: Cambridge, Massachusetts.

White, F. M., Viscous Fluid Flow, McGraw-Hill Book Co.: New York.

Wallis, G. B., One-Dimensional Two-Phase Flow, McGraw-Hill Book Co.: New York.

Dixon, S. L., Fluid Mechanics and Thermodynamics of Turbomachinery, Second Edition, Pergamon Press: Oxford, New York.

3) Advanced Texts and Referance Works:

Batchelor, G. K., An Introduction to Fluid Dynamics, Cambridge University Press: Cambridge, United Kingdom.

Schlichting, H., Boundary Layer Theory, (Latest Edition), McGraw-Hill Book Co.: New York.

Monin, A. S. and A. M. Yaglom, Statistical Fluid Mechanics, The MIT Press: Cambridge, Massachusetts.

Bradshaw, P. (Editor), "Turbulence," Topics in Applied Physics, Vol. 12, Springer-Verlag: Berlin, Heidelberg, New York.

Streeter, V. L. (Editor), Handbook of Fluid Dynamics, McGraw-Hill Book Co.: New York.

Kline, S. J., Similitude and Approximation Theory, McGraw-Hill Book Co.: New York (Out of Print).

4) Description of NCFMF Films on Fluid Mechanics:

Shapiro, A. H. (Editor), Illustrated Experiments in Fluid Mechanics, The MIT Press: Cambridge, Massachusetts.

ME 251A/255
Autumn 1978
J. P. Johnston
S. J. Kline

ME 251A/255

SCHEDULE OF FLUID MECHANICS FILMS

Time: Films start at 12:15 p.m. and run, on the average, for 30 minutes.

Place: Room 300

Oct. 9	Eulerian and Lagrangian Description in Fluid Mechanics
Oct. 16	Surface Tension in Fluid Mechanics
Oct. 23	Pressure Fields and Fluid Acceleration
Oct. 30	Flow Visualization
Nov. 6	Waves in Fluids
Nov. 13	Channel Flow of a Compressible Fluid
Nov. 20	Vorticity, Parts 1 and 2 (longer than average film)
Nov. 27	Deformation of Continuous Media
Dec. 4	Rheological Behavior of Fluids

NOTE: A paperback book which discusses all these, and other films of the NCFMF series is available. It is entitled, "Illustrated Experiments in Fluid Mechanics," A. Shapiro, Editor; M.I.T. Press, (Price $7.50).

If there is enough interest, we could place a group order later in the quarter.

Advanced Fluids JP Johnston RM501F Hr: M~~T~~Th 1-2

251B - Intro to Viscous Shear Flows
Basic EQ's n-s EQ
Energy, Second Law, Turbulent EQ
Exact & Approx. Sol. of n-s Eq
Thin shear layers (boundary layers)
Laminar BL
Turbulent " stable models
Applic to Internal flows
3D flow (if time)

Fluid Mechanics - study of fluids in motion & forces on fluids due to motion.

Fluids Engineering - orderly applications of technikes and basic knowledge to analysis and design of fluid flows and related machines.

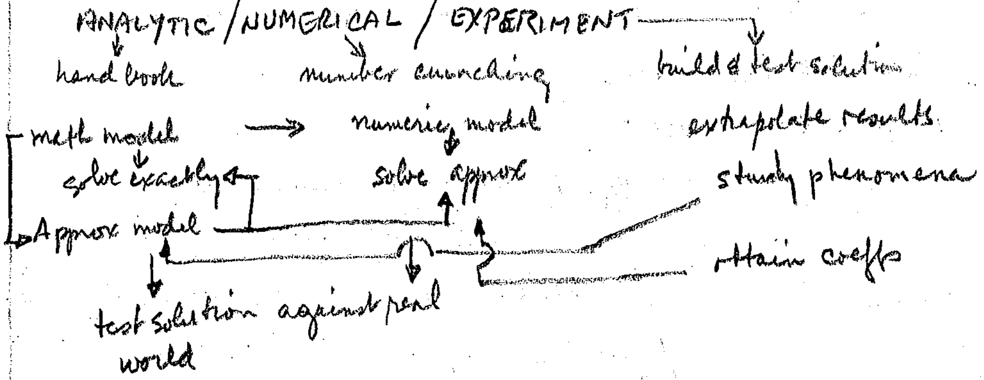
Classes of Systems

1. Propulsion production
2. Fluids transport tanks, ducts
3. Measurement (of flow, velocity, pressure, temp, etc.)
4. Energy & power exchange (pumps, fans, turbines etc.)
    - solid - fluid energy exchange operate close to 100%
    - fluid - fluid energy exchange operate well below 50%
5. Vehicle dynamics C_M, C_D, lift, drag
6. Flow mixing - (combustion process)

9/28

Factors in Engineering Problems
Approaches to Problems - Factors for Solution to Real vs. Academic Problems
Time (sec vs. years)
Money (0 → million of $)
People (technician, BS/MS/PhD, consultant)
Equip & Other Resources (pan + pen → computers)
(glue + wood → 40x80 Ames tunnels)

Approaches to Problems



Fluid Continuum & Properties

Molecular - fluids at low pressures deal with each molecules

<u>Physical idea</u>

a. Volume $\delta V = \delta x \, \delta y \, \delta z \approx (\delta x)^3$

mean free path length λ

b. continuum exists if $\delta x \gg \lambda$

for air (1 atm, 20°C) δV holds 3×10^{10} molecules $\therefore \lambda << \delta x$

Continuum properties

$\rho = \frac{m}{V}$, $P = \frac{F}{A}$, Temp, stress (normal/tangential), μ, ν, velocity

Continuum 'SCALAR PROPERTY as a 'field function

$$q = q(t; x, y, z)$$

function must be single valued continuous and piecewise differentiable

Jump conditions for shock wave allowed at Boundary.

— Because of the above chain rule diff is applicable

$$dq = \underbrace{\frac{\partial q}{\partial t} dt}_{\text{local deriv}} + \underbrace{\frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy + \frac{\partial q}{\partial z} dz}_{\text{convective deriv}}$$

10/2/78

Wednesday evening tutorial 7-8 PM

$$dq = \frac{\partial q}{\partial t} dt + \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy$$ let i represent $(1,2,3) = (x,y,z)$

$$= \frac{\partial q}{\partial t} dt + \frac{\partial q}{\partial x_i} dx_i$$ use summation convention if repeated index

VECTOR APPROACH

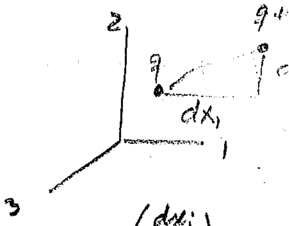
$$\nabla q = \left[\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right] q$$ q being a scalar function he

$$d\underset{\sim}{r} = \underset{\sim}{i}\,dx + \underset{\sim}{j}\,dy + \underset{\sim}{k}\,dz = \underset{\sim}{n}_i\,dx_i$$

$$dq = \nabla q \cdot d\underset{\sim}{r} = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dx + \frac{\partial q}{\partial z} dz = \frac{\partial q}{\partial x_i} dx_i$$

$$dq = \underbrace{\frac{\partial q}{\partial t} dt}_{\text{temporal local effec}} + \underbrace{\frac{\partial q}{\partial x_i} dx_i}_{\text{SPATIAL CONVECTIVE}}$$

$$\left[\frac{dq}{dt}\right]_{\text{particle}} = \frac{Dq}{Dt}$$ substantial deriv



$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \left(\frac{dx_i}{dt}\right)_{\text{part}} \frac{\partial q}{\partial x_i}$$

change in property even if not moving

$$\left(\frac{dx_i}{dt}\right)_{\text{part}} = \text{velocity} = \frac{Dx_i}{Dt} = u_i$$ let $\underset{\sim}{q} = \underset{\sim}{n}_i u_i$

$$\text{accel} = \underset{\sim}{a} = \frac{D\underset{\sim}{q}}{Dt} = \underset{\sim}{n}_i a_i$$

particle accel a_i $(i = 1,2,3)$ vector (rank 1) $\rightarrow$ $$a_i = \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \rightarrow \sum_j$$

Scalar $\longrightarrow$ rank 0 (no free indices) $$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + u_j \frac{\partial q}{\partial x_j}$$

rank 2 means there are 2 free indices

$\frac{\partial u_i}{\partial x_j}$ is a tenson of rank 2 $\left\{ \begin{array}{l} \text{vorticity (rank 1 since its a vector)} \\ \text{rate of strain} \end{array} \right.$

Contraction

vector → scalar : contracting from higher order to lower order

tensor → vector $u_j \frac{\partial u_i}{\partial x_j}$

Contraction is a dot product $\underset{\sim}{A} \cdot \underset{\sim}{B} = \text{scalar } C$

in general
vector form $\underset{\sim}{a} = \frac{\partial \underset{\sim}{q}}{\partial t} + (\underset{\sim}{q} \cdot \nabla) \underset{\sim}{q}$ is a dyadic product.

$$\underset{\sim}{a} = \frac{\partial \underset{\sim}{q}}{\partial t} + \nabla\left(\frac{\underset{\sim}{q}^2}{2}\right) - \underset{\sim}{q} \times (\nabla \times \underset{\sim}{q})$$

$$\nabla \times \underset{\sim}{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ u_1 & u_2 & u_3 \end{vmatrix} \quad \text{curl } \underset{\sim}{q}$$

$$= \omega_i \quad (\text{vorticity}) = \underset{\substack{\wedge \\ \text{unit tensor (rank 3)}}}{\epsilon_{ijk}} \left(\frac{\partial u_k}{\partial x_j} \right)$$

cyclic +1
anticycle −1
if i=j, j=k, k=i 0
or i=j=k

10/4/78

Lecture notes will be made up (Goto Eng Library & see tape tomorrow)

Today's lecture:

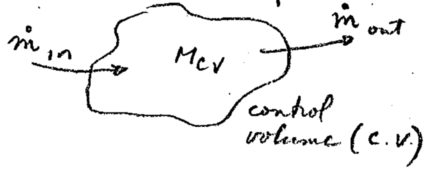
Kinematics Eulerian vs. Lagrangian

Pathline

Streamline

Steady vs. Unsteady Flow.

Conservation of Mass



R.O.C. of Mass in volume = 0

$$(\dot{m}_{out} - \dot{m}_{in}) + \frac{dM_{cv}}{dt} = 0$$

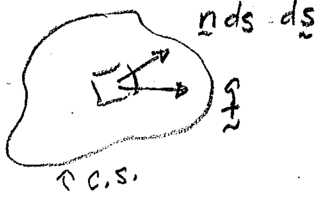
no mass can be created (rate of creation = 0

Continuity Eq

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{q}) \quad \text{or} \quad \frac{1}{\rho}\frac{D\rho}{Dt} = \frac{\partial u_i}{\partial x_i} = \nabla\cdot\underset{\sim}{q}$$

$$\text{if } \rho = \text{const} \implies \nabla\cdot\underset{\sim}{q} = 0$$

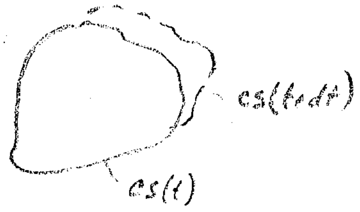
Integral form



$$0 = \int_{cv} \frac{\partial \rho}{\partial t}\, d\forall + \int_{cs} \rho \underset{\sim}{q}\cdot\underset{\sim}{n}\, ds$$ fixed cs

mass storage net efflux

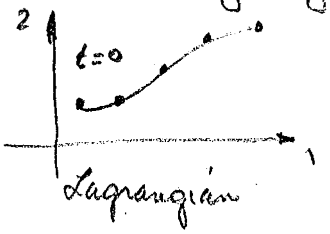
if c.s. is moving then



$$0 = \frac{d}{dt}\int_{cv} \rho\, d\forall + \int_{cs} \rho\, \underset{\sim}{q}_{rel}\cdot\underset{\sim}{n}\, ds$$

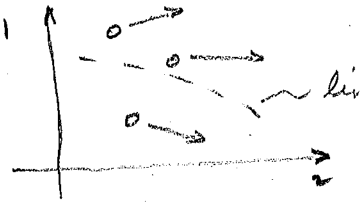
$$\underset{\sim}{q}_{rel} = \underset{\sim}{q} - \underset{\sim}{q}_{surface}$$

PATHLINES - Trajectory of a fluid particle



Stream line

A continuous line everywhere tangent to the velocity vectors at a given instant of time



Stream tube

Connected set of streamlines through which mass flows through at any given instant of time. no mass flows across stream tube.

To show unsteady flow $\frac{\partial}{\partial t}\underline{q} \neq 0$ use two dyed beads: if they mark 2 different path lines @ t & $t+\delta t$ flow is unsteady

Streakline

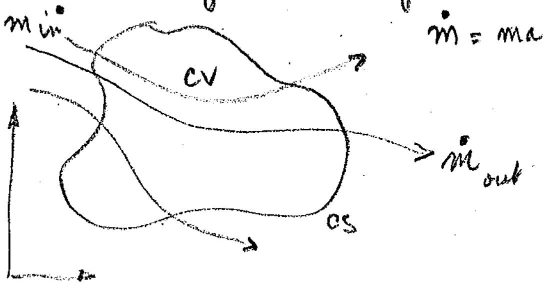
line marking particles passing through a point

In steady flow all three (path, stream, streak) lines are identical

CONSERV of MASS

from Eulerian CV, CS are general concepts

$\dot{m}$ = mass flow rate



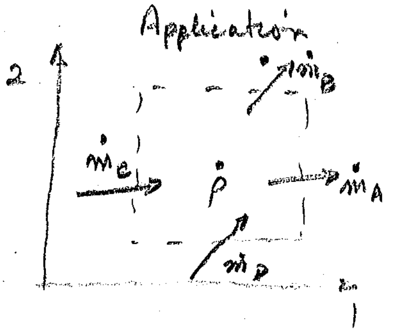
for infinitesimal use fixed

$$d\forall = \prod_{i=1}^{3} dx_i$$

LAGRANGIAN $M = \text{const}$ $\frac{dM}{dt} = 0$

$$\text{R.O.C.}(M) = 0 \qquad (\dot{m}_{out} - \dot{m}_{in}) + \frac{dM_{cv}}{dt} = 0$$

Application



2-DIM $d\forall = dx_1 dx_2 \cdot 1$

$M_{cv} = \rho dx_1 dx_2 + h.o.t.$

$\frac{\partial M_{cv}}{\partial t} = \frac{\partial \rho}{\partial t} dx_1 dx_2$ since $d\forall$ is fixed

storage term

$\dot m_A = \rho u_1 dx_2 + \frac{\partial}{\partial x_1}(\rho u_1 dx_2)\frac{dx_1}{2}$ in & out flow terms.

$\dot m_c = \rho u_1 dx_2 - \frac{\partial}{\partial x_1}(\rho u_1 dx_2)\frac{dx_1}{2}$

$\dot m_A - \dot m_c = \frac{\partial}{\partial x_1}(\rho u_1) dx_2 dx_1$ since dx_2 is fixed

same $\dot m_{out} - \dot m_{in} = (\dot m_A - \dot m_c) + (\dot m_B - \dot m_D) = d\forall\left[\frac{\partial}{\partial x_1}(\rho u_1) + \frac{\partial}{\partial x_2}(\rho u_2)\right]$

thus ROC(M): $\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i)\right] d\forall = 0$

In differential derivations you can either look at each indiv property or a property which is relevant to what you are looking at

ie $(\rho + \frac{\partial \rho}{\partial x})(u + \frac{\partial u}{\partial x}) = \rho u + \frac{\partial}{\partial x}(\rho u) + h.o.t.$

or $(\rho u) + \frac{\partial}{\partial x}(\rho u)$

10/6/78

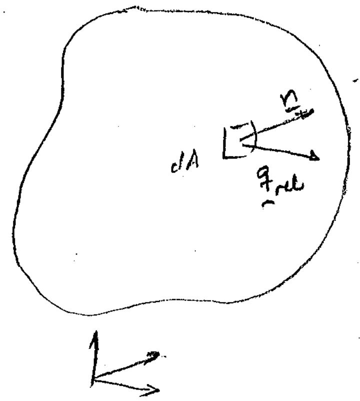
Pick up notes from Karen Olds 500F

TODAY

Conservation Eq: $\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{q} = 0$

Incompress $\nabla \cdot \underline{q} = 0$

Integral Form of ROC (M) = 0 fixed & general C.S. Cases



$\underset{\sim}{n} \cdot \underset{\sim}{q}_{rel} = q_{rel} \cos\theta$

$q = \underset{\sim}{q}_{surf} + \underset{\sim}{q}_{rel}$

fluid flow vel rel. to coords

surface vel rel to coords

fluid vel relative to surface

rate of mass flow across surface $\rho\, \underset{\sim}{q}_{rel} \cdot d\underset{\sim}{A} = \rho\, q_{rel} \cos\theta\, dA \underset{out}{>0}$

From ROC(M) = 0

$$\frac{dM_{cv}}{dt} + (\dot{m}_{out} - \dot{m}_{in}) = 0$$

$$\frac{d}{dt}\int_{cv} \rho\, d\forall + \int_{cs} \rho\, \underset{\sim}{q}_{rel} \cdot d\underset{\sim}{A} = 0$$

For ~~fixed~~ CS $\underset{\sim}{q}_s = 0$ & $\frac{d}{dt}$ becomes $\frac{\partial}{\partial t}$ since $\frac{d}{dt}\, dV = 0$

$$\int_{cv} \frac{\partial\rho}{\partial t}\, d\forall + \int_{cs} \rho\, \underset{\sim}{q}_{rel} \cdot d\underset{\sim}{A} = 0$$

Continuity : $\rho_{,t} + \nabla\cdot(\rho \underset{\sim}{q}) = 0$

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = \underbrace{\frac{\partial\rho}{\partial t} + u_j \frac{\partial}{\partial x_j}\rho} + \rho \frac{\partial u_j}{\partial x_j} = 0$$

$$\frac{D\rho}{Dt} + \rho\, \nabla\cdot \underset{\sim}{q} = 0$$

— Assume ρ = const (liquids for which $\frac{\Delta\rho}{\rho}$ small)(gases for which $\frac{\Delta\rho}{\rho}$ smal

Continuity becomes $\nabla\cdot \underset{\sim}{q} = 0$ or $\frac{\partial u_j}{\partial x_j} = 0$

for a fixed CS $\int \frac{\partial\rho}{\partial t}\, d\forall$ drops out & $\int_{cs} \rho\, \underset{\sim}{q} \cdot d\underset{\sim}{A} = 0 \Rightarrow \int_{cs} \underset{\sim}{q} \cdot d\underset{\sim}{A} = 0$

for a general CS you can't drop the CV since $\frac{d}{dt}\int d\forall$ is changi

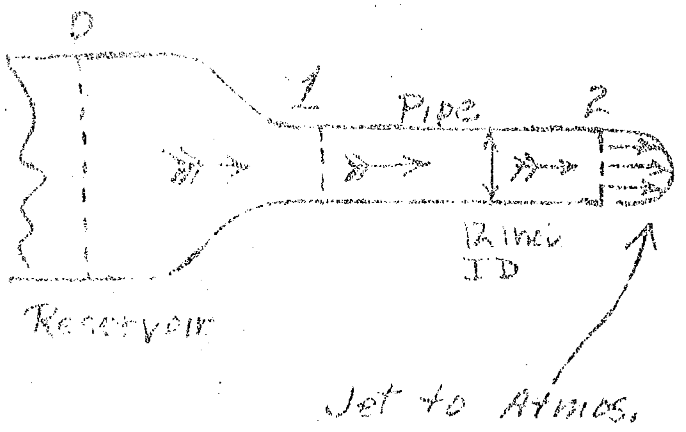
Autumn 1978
J.P. Johnston

ME 251A

Problem Set No. 2

(Due Monday, October 23)

Problem 1



Water flows steadily through the system shown with a volume flow of 39.3 ft^3/sec. At section 1, following the contraction from the pressurized reservoir, the pressure is 10 psi gage and the velocity is uniform over the cross-section. At section 2, where the pipe discharges into the atmosphere, the velocity varies over the cross-section according to the one-seventh power law for turbulent flow:

$$\frac{u}{u_c} = \left(\frac{R-r}{R}\right)^{1/7}$$

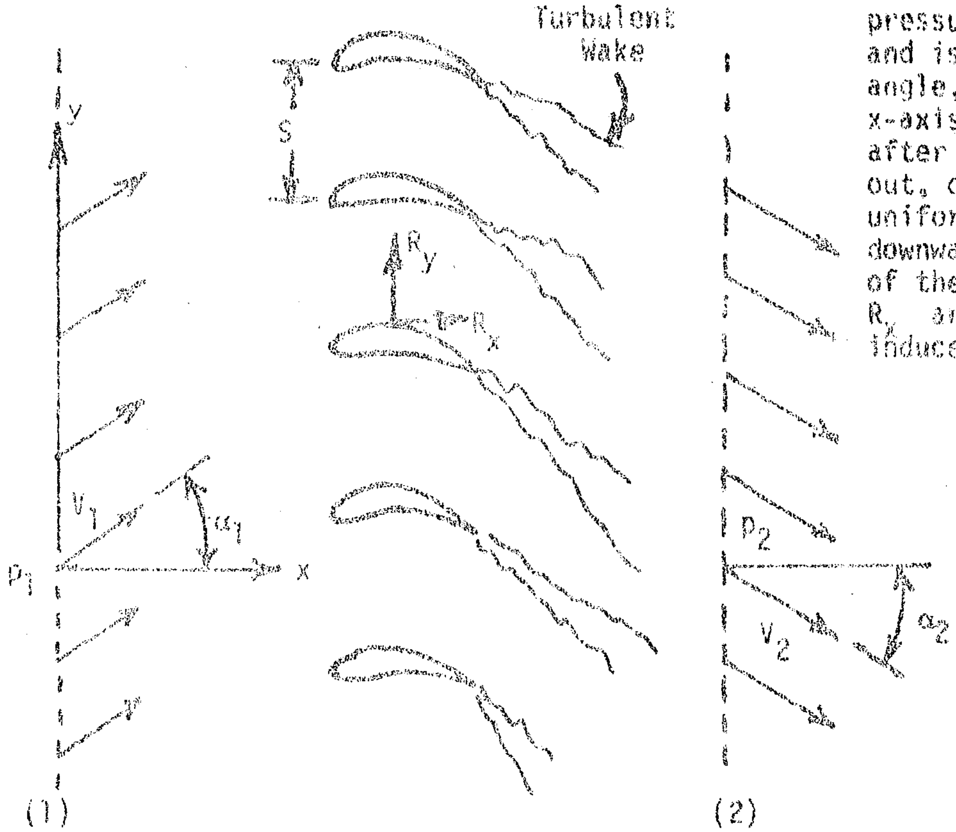
where u is the velocity at the radius r; u_c is the velocity at the centerline; and R is the pipe radius (6").

(a) Calculate the pressure (psi gage) in the reservoir, assuming no friction in the contraction piece and that the diameter of the reservoir is three times as great as the diameter of the pipe.

(b) Calculate the total tensile force (lbf) in the pipe wall acting in the axial direction at section 1. Assume that the pipe has a very thin wall.

Problem 2

Sketched below is an infinite cascade of equally spaced, identical airfoils. The fluid flows without significant density change, and, except for the turbulence in the wakes of blades, the flow is steady. Far upstream at (1) the flow is uniform at static pressure, p_1, velocity, V_1, and is directed upward at an angle, α_1, with respect to the x-axis. Far downstream at (2), after the turbulent wakes mix out, conditions are again uniform but the flow angle is downward at α_2, the magnitude of the inlet angle. R_y and R_x are the respective total flow induced forces on one blade.



Derive expressions for the forces R_y and R_x in terms of α_1, α_2, p_1, p_2, V_1, ρ, and S, the blade spacing.

For a given blade force ratio (R_x/R_y) and when $\alpha_1 = \alpha_2 = \alpha$, at what angle α is the total pressure loss coefficient,

$$\varepsilon = \frac{p_{o_1} - p_{o_2}}{\frac{\rho}{2} V_1^2} ,$$

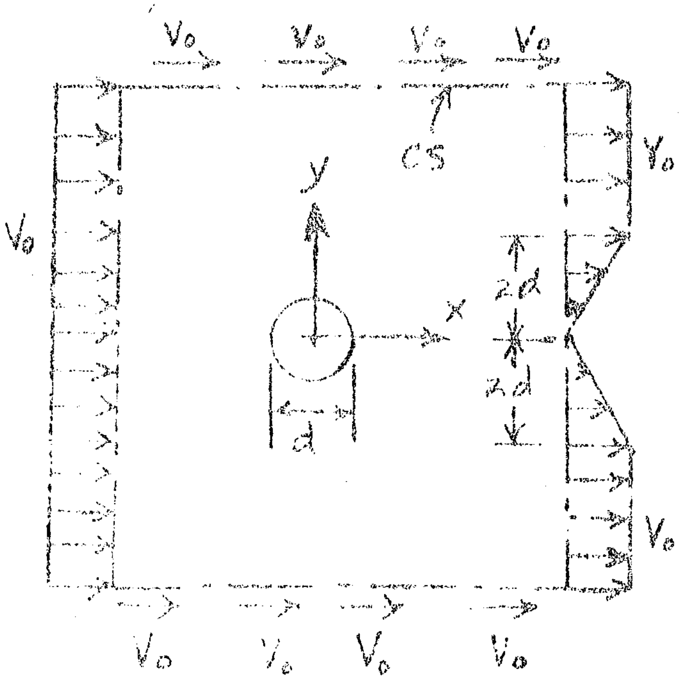
a maximum? Note that total pressure (elevation changes neglected) in an incompressible fluid is defined as

$$p_o = p + \frac{\rho}{2} V^2$$

at any point in the flow.

ME 251A
Problem Set No. 2
Autumn 1978

PROBLEM 3



In an experiment to determine drag, a circular cylinder of diameter d was immersed in a steady, two-dimensional incompressible flow. Measurements of velocity and pressure were made at the boundaries of the control surface shown. The pressure was found to be uniform over the entire control surface. The x-component of velocity at the control surface boundary was approximately as indicated by the sketch.

From the measured data, show that the drag coefficient of the cylinder, based on the projected area and on the free stream dynamic head,

$$C_d \equiv \frac{\text{Drag force per unit length}}{1/2\ \rho V_0^2 d} = \frac{4}{3}$$

(4/3 is the correct answer!)

initial cond.

$$p(t=0, x_i) \;;\; \rho(t=0, x_i) \;;\; u_i(t=0, x_i)$$

Boundary condition

(1) Normal velocity must match

$$(u_n)_{fluid} = (u_n)_{wall}$$

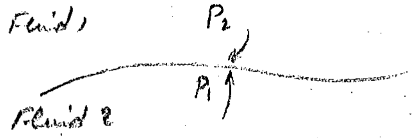
(2) No slip (viscous fluid only – all real fluids)

$$(u_t)_{fluid} = (u_t)_{wall} \quad \text{for fluid continuum}$$

(3) pressure

solid surface p, ρ are dependent

fluid surface interface



$$(p_1 - p_2) = \frac{\sigma}{R}$$

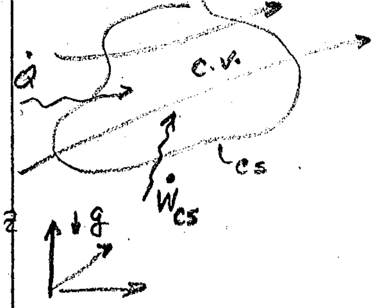
σ = surface tension

R = local radius of curvature.

for small droplets this Δp is significant

For inviscid fluid no slip conditions are no longer independent BC that can be specified but becomes dependent

Global Eqs



$$\frac{d}{dt}\int_{cv} \rho \, d\forall + \int_{cs} \rho \, \underline{q}_{rel} \cdot d\underline{A} = 0 \quad \text{(mass)}$$

let e : total energy/mass (energy)

$$\frac{d}{dt}\int_{cv} e\rho \, d\forall \quad \int_{cs} e(\rho \underline{q}_{rel} \cdot d\underline{A}) - \dot{Q}$$

(heat flows in – conduction & radiation only)

$$+ (-\dot{W}_{cs}) \quad + (-\dot{W}_{cv}) = 0$$

(workdone at CS on the material inside)

(work done in CV but no interaction at CS)

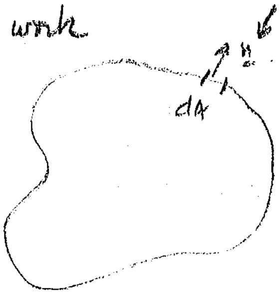
$$e = \tilde{u} + \frac{V^2}{2} + gz + \ldots$$ (Chemical / Electric)

(internal thermal energy) KE/unit mass PE/unit mass

$h = \tilde{u} + P/\rho$ P/ρ is a work term & not really an e term

look at $\dot{W}_{cs} = \dot{W}_{\substack{flow\\work}}$ (few) $+ \dot{W}_{cs,other}$

flow work



$F_p'' =$ force/unit area (pressure force)
$= p(-n)$

$$\int_{cs} F_p'' \cdot (q_{rel} + q_{surf})\, dA = \dot{W}_p$$

$$= -\int p(q_{rel} + q_{surf}) \cdot dA = \dot{W}_{cs}$$

$$= \underbrace{-\int_{cs} p\, q_{rel} \cdot dA}_{\text{flow work}} \underbrace{- \int_{cs} p\, q_{surf} \cdot dA}_{\text{piston work}}$$

$$= -\int_{cs} \frac{p}{\rho} (\rho q_{rel}\, dA) - \int p\, q_{surf} \cdot dA$$

⇒ go to energy if $\dot{W}_{cs}$ is broken down into flow work + other

then $$\frac{\partial}{\partial t} \int \rho e\, d\forall + \int \left(\tilde{u} + \frac{p}{\rho} + gz + \frac{q^2}{2}\right)(\rho q_{rel} \cdot dA) - \dot{Q} - \dot{W}_{\substack{cs\\other}} - \dot{W}_{cv} = 0$$

Linear Momentum

ROC$(\rho q) = \Sigma F$ on CV

$$\frac{d}{dt} \int \rho q\, d\forall + \int_{cs} q\, (\rho q_{rel}) \cdot dA = \int_{cs} F_s''\, dA + \int_{cv} F_B'''\, d\forall$$

Angular Momentum

$$\frac{d}{dt} \int (r \times q) \rho\, d\forall + \int_{cs} (r \times q)(\rho q_{rel} \cdot dA) = \int_{cs} r \times F_s''\, dA$$
$$+ \int_{cv} r \times F_B'''\, d\forall$$

10/12/78

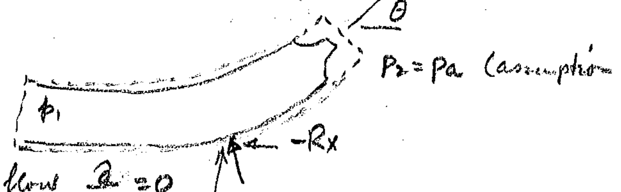
Global Equations -

1. PLM - forces on elbow
2. PAM - Torque & power on reaction turbine (Lawn sprinkler)
3. Drag on flat wall with mass injection.

pg 25 of Johnston's notes.

<u>Find</u> forces on elbow at support

<u>Given</u>



Assume steady flow $\frac{\partial}{\partial t} = 0$

incomp " ρ = const

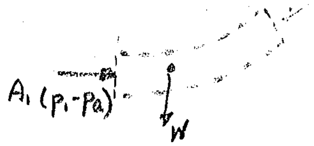
1-D flow at jet, inlet

force applied only at one point.

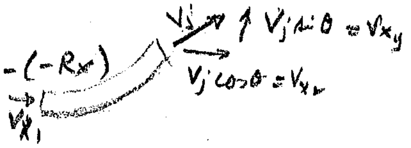
define control surface

Basic principles

$$ROC\ (mom_{x/y}) = \sum F_{x/y}$$



$$\sum F_x = A(p_1 - p_a) \text{ ~~[illegible]~~ } - (-R_x)$$
$$\sum F_y = -W + R_x$$



$$\frac{\partial}{\partial t}\int_{cv} V_x \rho\, dV + \int_{out} V_x\, d\dot{m}_{out} - \int_{in} V_x \left(\overset{\rho \underline{V}\cdot dA}{d\dot{m}}\right) = \sum F_x$$

fixed control surface $\frac{\partial}{\partial t}\int \;=\; \int \cancelto{0}{\frac{\partial}{\partial t}(V_x \rho)}\, dV$

since flow is 1-D $\Rightarrow \int V_x(\ ) = V_x \int(\ )$ since V_x is assumed = const

Pg 28 lawn sprinkler problem

Find torque & Power = F×V = Torque × N

10/15/78

look at notes pg 35

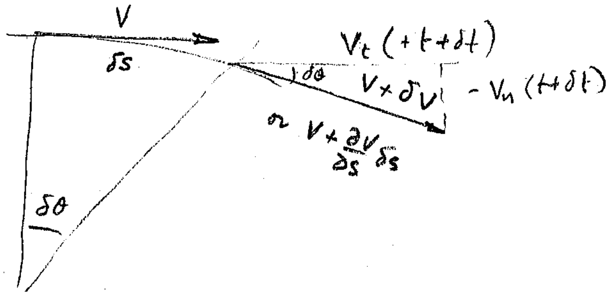
$$\frac{\partial}{\partial t}u_i + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + f_{b_i}$$

Euler's equation - cartesian

Euler's equations - streamline (or natural) will be developed.

δs is measured along streamline

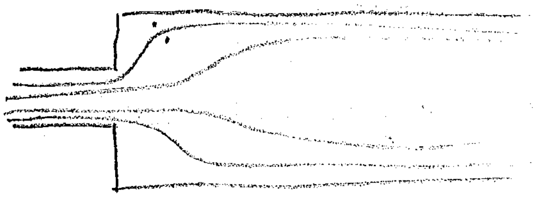
streamline tangential to veloc vector



s equation $$\frac{\partial V_s}{\partial t} + V\frac{\partial V}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + f_s$$

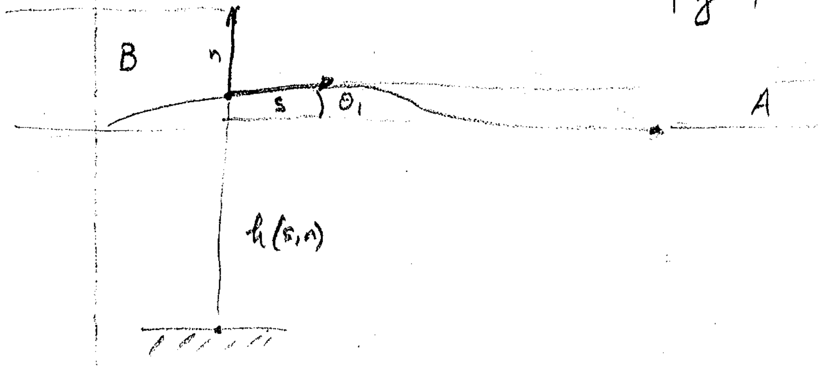
n equation $$\frac{\partial V_n}{\partial t} + \left(\frac{-V^2}{R}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial n} + f_n$$

if steady flow, no body forces: n eq becomes $-\frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n}$



if flow is quasi straight $\frac{\partial p}{\partial n} \to 0$ $R \to \infty$

pg 40



$$A+B = \frac{\partial h}{\partial s}ds + \frac{\partial h}{\partial n}dn$$

10/20/78

Bernoulli's <u>N-eq</u> special case: steady, circular streamlines, $\rho = \text{const}$

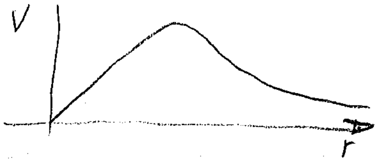
$\frac{\partial V_n}{\partial t} = 0$ $V(r)$ $n = r = R$

$$\cancel{\frac{\partial V_n}{\partial t}} - \frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} - \frac{\partial (gh)}{\partial n}$$

$$\text{so } \frac{V^2}{r} = \frac{1}{\rho}\frac{d(p+\rho g h)}{dr} \qquad p^* = (p+\rho g h)$$ piezometric pressure

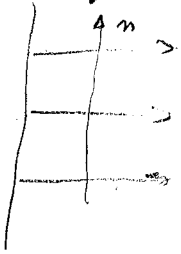
$$\int \frac{V^2 dr}{r} = \frac{1}{\rho}\left(p_2^* - p_1^*\right) \quad \text{if } \rho = \text{const}$$

<u>Case of Vortex</u> $V(0)=0$ $V(\infty)=0$



as $r\uparrow\infty$ $p^*\uparrow$ $\therefore$ $p_{\text{℄}} < p^*$

Case of parallel flow $r \to \infty$



$P_2^* = P_1^*$ $p^* = \text{const}$ <u>across stream</u>

P_1^*

basic assump: for inviscid flow

We can show the n-eq is also true for viscous flow.

Energy equation, steady($\frac{\partial V}{\partial t} = 0$), no heat term, no work terms ($\dot W_{cs}, \dot W_{cv}$) will be the same as the inviscid bernoulli eq along a streamline (steady, baratropi

ME251A

Final Exam

Open Book / Take Home

Pick up Dec 8   Due: Noon Dec 12 (Locals)

ME251B   Winter   Momentum transfer in BL

Bradshaw and & Cebeci

Autumn 1978
J. P. Johnston
October 4, 1978

ME 251A

Problem Set No. 1

(Due Friday, Oct. 13)

Problem 1

Use the law of conservation of mass to derive a differential equation for the velocity components v and w in an "inviscid, thin-film" of an incompressible fluid on a circular, cylindrical surface. The radial thickness of the fluid film, b , and the instantaneous radius of the cylinder's surface, a , will appear in the resulting equation.

In cylindrical coordinates (r,θ,z) the velocity components are u , v , and w respectively where, in general, each component depends on r , θ , z and time, t . See sketch below.

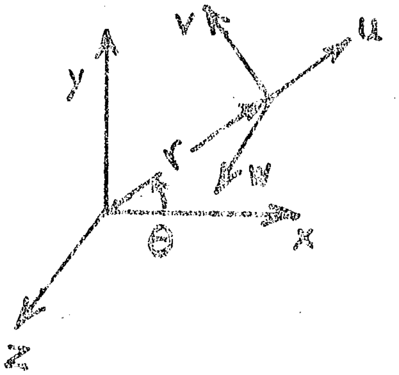
For the "thin-film" approximation $b \ll a$ and u is very small compared to v and w . If, in addition, the no-slip boundary condition is neglected, one may assume "inviscid" flow and thereby neglect dependence of v and w on the radial coordinate, r .

Assume that film thickness, b , is a known function of surface coordinates and time, i.e.,

$$b(\theta,z,t) = \text{given} ,$$

and that the cylinder's surface radius, a , is constant in θ and z , but is a known function of time, i.e.,

$$a(t) = \text{given} .$$



CYLINDRICAL COORDINATES

Problem 2

For steady, two-dimensional flow of a compressible fluid, the continuity relation is given by:

$$u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + \rho [\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}] = 0$$

Demonstrate the validity of this relation by applying the principle of conservation of mass to a system of fixed mass rather than by use of a small control surface (fixed in space) as was done in class. Hint: Work with a small volume of mass all parts of which move with the flow.

Problem 3

Consider the two-dimensional, steady motion of a viscous, compressible fluid in the x, y plane.

Let σ_x and σ_y denote the normal stresses (positive when tensile) acting in the x and y directions; τ_{xy} the shear stress in the x-direction acting on a y-plane, positive when fluid above exerts a rightward force on fluid below; etc.; u and v the x and y components of the velocity $\vec{V}$; ρ the density.

(a) Neglect body forces, and derive x and y equations of motion similar to the Euler ecuations for an inviscid fluid.

(b) Demonstrate from the theorem of moment of momentum that $\tau_{xy} = \tau_{yx}$.

(c) In the case of inviscid flow $\tau_{xy} = \tau_{yx} = 0$. Demonstrate that $\sigma_x = \sigma_y = -p$ where p, the pressure, is a compressive stress that is a scaler function of position and time, but is independent of the direction of any elementary area on which it can create a force.

(d) Express your results in subscript Tensor notation if possible, and see if they may be generalized to fully three-dimensional fluid motion. Discuss some of the implications of these results. Be concise!

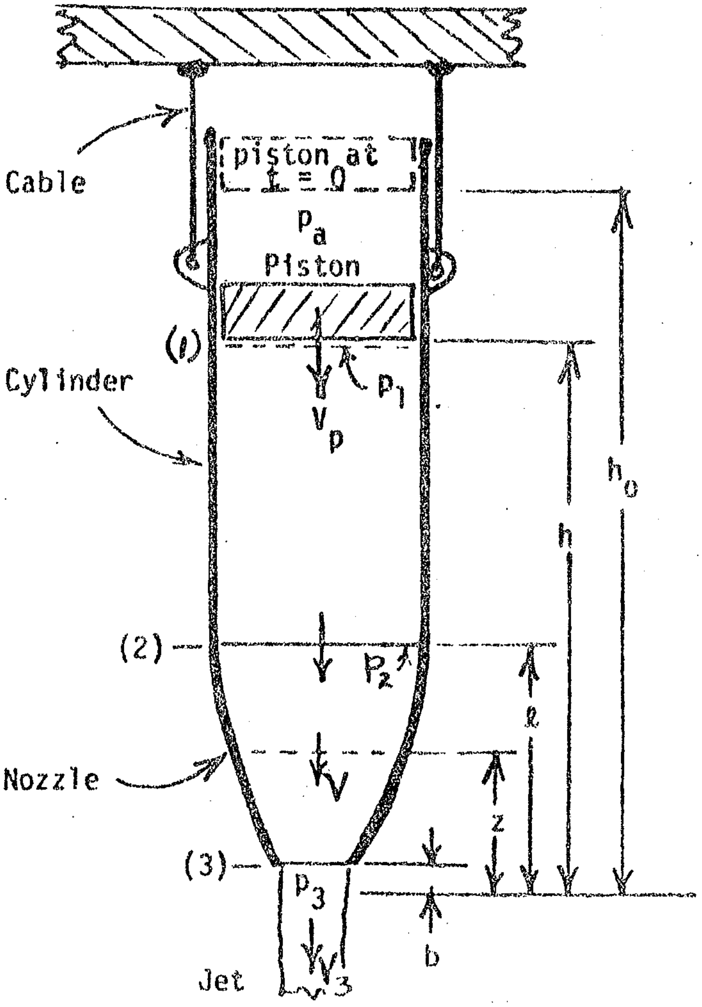
Autumn 1978
J. P. Johnston
October 23, 1978

ME 251A

Problem Set No. 3
(Due Wednesday, November 1)

Problem 1

Consider the system sketched below. It comprises a piston that fits into a cylinder-nozzle assembly full of liquid of density ρ. The piston contacts the liquid and may be assumed to slide freely without friction or leakage at the cylinder walls. The nozzle shape is approximately as sketched, and it has a linear variation of cross-sectional area A with respect to elevation, i.e., $A/A_c = z/\ell$, where A_c is the area of the cylinder z, ℓ, and all vertical dimensions are measured from a datum plane at a distance b below the nozzle outlet, station-3. The whole assembly is suspended from an inertially stationary platform by cables.



Definitions:	Units:
W_c = dry weight of cylinder-nozzle	N
W_p = piston weight	N
A_c = cylinder area	m^2
A = nozzle area	m^2
h = instantaneous piston height at time t	m
t = time	s
ρ = liquid density	$Kg\ m^{-3}$
p = pressure	Nm^{-2}
V_p = piston velocity	ms^{-1}
g = acceleration of gravity	ms^{-2}

At some instant of time earlier than that shown above the downward flow was started by unblocking the nozzle outlet, station-3, where the liquid issues as a jet into the atmosphere at $p_3 = p_a$. Atmospheric pressure always acts on the top of the piston which started its downward motion at h_o .

In the following we are considering only the part of the process from $t = 0$, $h = h_o$, $V_p = 0$ until the piston is just about to touch the nozzle, $h = \ell$.

Part (A): Obtain a simple expression for the instantaneous downward fluid velocity, V , in the nozzle as a function of local nozzle area A , and piston velocity V_p . Also, show how the instantaneous jet velocity V_3 is related to A_3 and V_p .

Part (B): Obtain from basic principles an expression for the instantaneous fluid pressure p_1, just under the piston in terms of p_a , W_p , g , A_c , and the rate of change of piston speed (dV_p/dt) .

Part (C): Obtain the instantaneous pressure p_2 (at station-2) in terms of p_1 , ρ , g , (dV_p/dt) , h and ℓ . State your assumptions clearly.

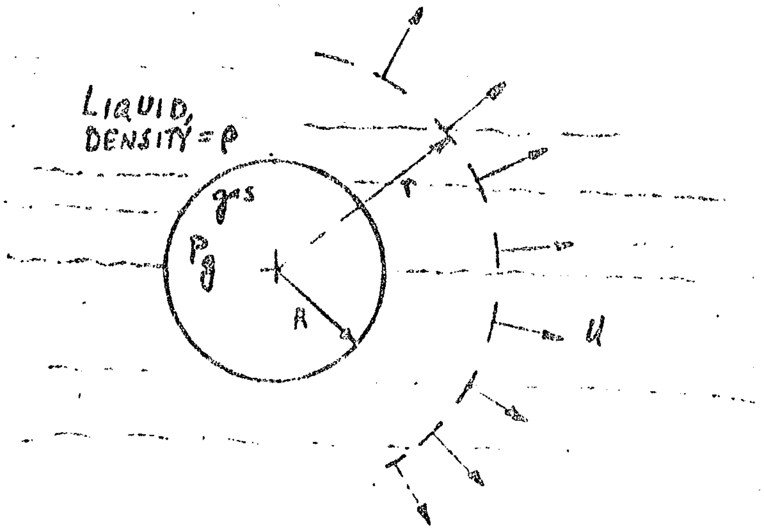
Part (D): Derive a differential equation that would permit one to solve for h , the instantaneous piston height, as a function of time. Express your result for this part in terms of ℓ , b , A_c , A_3 , ρ , g , p_a and p_1 . The pressure, p_1 , under the piston should be considered to be a "known" function of time which could be obtained by simultaneous use of the result derived in Part (B). Again, state assumptions clearly.

Part (E): Derive an expression for the instantaneous total tensile force F_T in suspension cables in terms of W_p , W_c , ρ , g , A_c , A_3 , h , ℓ , b , V_p and (dV_p/dt) . Show all forces and flows on clearly labeled, neatly drawn control surfaces.

Part (F): Discuss the procedure you might use to solve for the time period $t = T$, for the piston to reach the nozzle $h = \ell$.

Part (G): (Extra Credit) Solve for T , $h(t)$, and $[F_t(t) - W_c]$ by any appropriate method for the special case where fluid is liquid water, piston mass is 1kg , cylinder diameter is 500 mm , nozzle diameter is 250 mm , $h_o = 4$ m and $\ell = 2$ m.

Problem 2



Consider a bubble of high-pressure gas expanding in an incompressible liquid in a spherically-symmetrical fashion. The gas is not soluble in the liquid, and the liquid does not evaporate into the gas. At any instant R is the radius of the bubble, dR/dt is the velocity of the interface, p_g is the gas pressure (assumed uniform in the bubble), p_∞ is the liquid pressure at a great distance from the bubble. Gravity is to be neglected. The following questions pertain to the formulation of an analysis which will lead to the details of the pressure and velocity distributions and to the rate of bubble growth:

(a) Show that at any instant

$$u = \frac{R^2}{r^2}\frac{dR}{dt}$$

where u is liquid velocity at radius r .

(b) Show that the rate of growth of the bubble is described by the equation

$$R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2 + \frac{2\sigma}{\rho R} = \frac{p_g - p_\infty}{\rho}$$

where σ is the surface tension at the gas-liquid interface.

(c) What additional information or assumptions would be necessary in order to establish the bubble radius R as a function of time? Explain how you would use this information.

Autumn 1978
J. P. Johnston

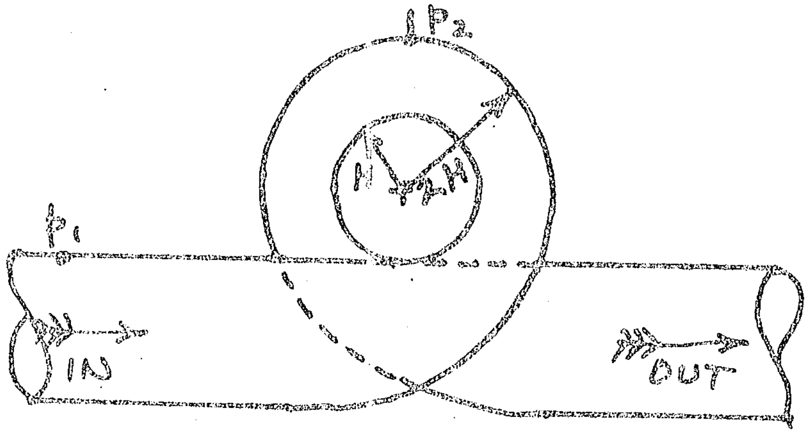
ME 251A

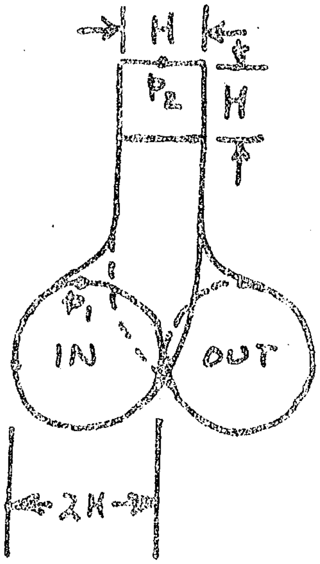
Problem Set No. 4

(Due: Friday, November 17)

Note: Problems 1 and 2 in this set are to be treated as a "take home quiz". That is, you are to solve them without consultation or discussion with any other person or any reference to material other than your notes and the course text book (Honor Code applies).

PROBLEM 1 An inventor comes to you with a new type flow rate meter based on a venturi that makes a 360 degree loop. He claims that his device can measure the volumn flow rate, Q (m^3/s) with a sensitivity coefficient, K, twice as large as the K for the same device if it were stretched out straight, i.e., if it were uncurled. The device has the dimensions shown below in two views.





The sensitivity coefficient is defined in the equation:

$$KQ = A\sqrt{\frac{2(P_1-P_2)}{\rho}}$$

Where: A = cross sectional area at the throat (minimum area). (m^3)

ρ = density (assume constant), (kg/m^3)

P_1 = measured upstream wall static pressure, (N/m^2)

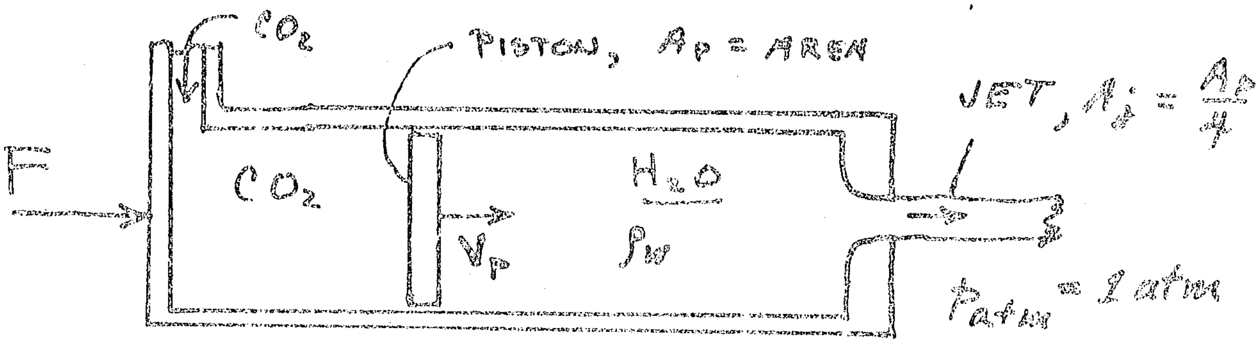
P_2 = measured wall static pressure at the throat on the outer curved wall, (N/m^2)

H = BASIC dimensions of the throat (arbitrary size), (m)

Part a: Do you believe the inventor's claim? Prove your answer by analysis.

Part b: Could you make any suggestions for simple changes (such as change of pressure tap locations) that would increase K ? If so, by approximately how much? Back up your claims by analysis.

PROBLEM 2: A simple jet propulsor for a novel toy is constructed by separating a cylindrical chamber with a piston. Compressed CO_2, admitted at the end, pushes the piston down the cylinder. The piston, in turn, pushes water on the other side of the chamber out through a nozzle having a cross sectional area 1/4 of the piston area.



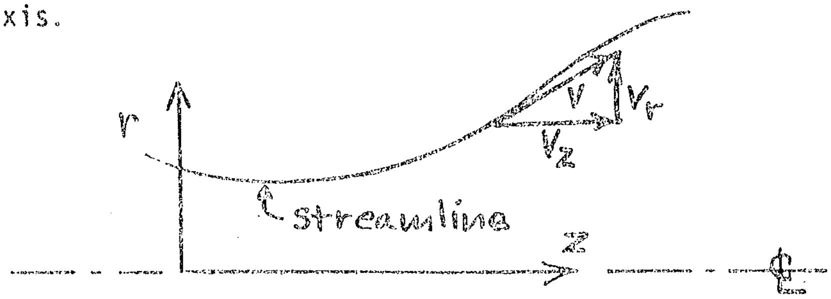
Pressure in the jet from the nozzle is the same as that of the surrounding atmosphere. The apparatus is horizontal with respect to g .

Part a: Estimate the force axial (thrust) F necessary to restrain the propulsor during a static test for the part of the cycle of operations when the piston speed, V_p , is constant. Express your results in terms of ρ_w , A_p , and V_p . Neglect the effects of the CO_2 mass relative to the mass of the water.

Part b: After the starting transient, the absolute pressure in the cylinder is not to exceed 1.1 atmospheres. In addition, it is desired to obtain a steady thrust of 5 lbf for a period of one second. Estimate the minimum length and the corresponding internal diameter of the cylinder.

Neglect any friction between cylinder wall and piston. Also, neglect the starting and stopping transients.

PROBLEM 3: This problem concerns axisymmetric flow; flow where the fluid motion is identical in every flat plane that passes through a straight line called the axis. We shall use cylindrical coordinates (r, θ, z) along which directions the respective velocity components are v_r, v_θ, and v_z. The axis of symmetry is the z-axis. The swirl (tangential) velocity component is zero, $v_\theta = 0$, and all gradients with respect to θ are zero, $\partial/\partial\theta = 0$. Streamlines may be drawn on the r-z plane. These lines are also traces of "stream surfaces" which are cylindrical surfaces of revolution about the z-axis.



For incompressible flow $(\rho = \text{const.})$ the equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0$$

The vorticity vector $\vec{\omega}$ can have only one component, namely

$$\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$$

ME 251A
Autumn 1978
Problem Set No. 4

In the following parts you should assume ρ = constant and steady flow .

Part (A): Obtain (by any means) the stream function, $\psi(r,z)$, and derive a differential equation for ψ which allows the flow field to satisfy the condition of irrotationality.

Part (B): Is it still true for your ψ , as in the case of plane two-dimensional flow, that the volumetric rate of flow between any two "stream surfaces" is proportional to the difference $(\psi_2 - \psi_1)$? Prove your answer and if the answer is "yes it is true" give the constant of proportionality. Note: First you must show that streamlines correspond to lines of ψ = constant.

Part (C): Define a velocity potential $\phi(r,z)$, and develop a differential equation for ϕ in terms of the independent variables r and z .

Part (D): Are lines of constant ϕ everywhere perpendicular to lines of constant ψ ? Hint: Can you show that $(\vec{\nabla}\phi) \cdot (\vec{\nabla}\psi) = 0$?

PROBLEM 4: Solve textbook problems 6.2 (page 186) and 6.7 (page 194).

Autumn 1978
J. P. Johnston

ME 251A

Advaned Fluids Engineering

FINAL EXAM

Notes:

(1) This take home exam should not consume more than 4 hours. Note the time spent on the front of the blue book.

(2) Use of books and notes is permitted.

(3) Write solutions in blue book, tape or staple in the solution sheets for problem 3.

(4) Local students turn in exam to my secretary, Ann Ibaraki, Room 501C, by 11:30 a.m., Tuesday, December 12th.

Remote location TV students, be sure the exam is in my hands no later than December 19th.

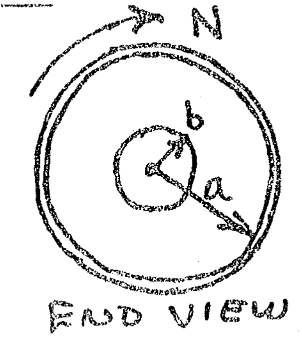
(5) Course grading:

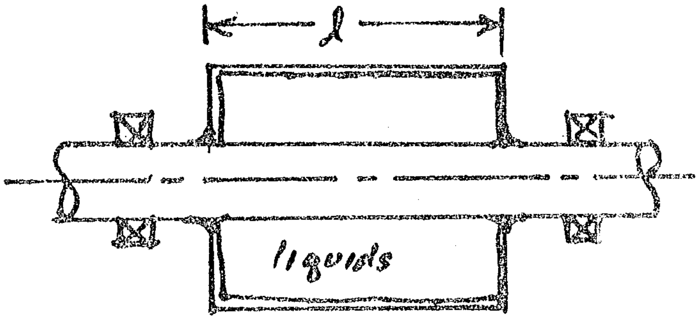
- Problem Sets - 40%
- Home Quiz (Problem Set #4, Problems 1 & 2) - 10%
- Final Exam (4 Problems of Equal Weight) - 50%

100%

ME 251A
Final Exam

PROBLEM 1



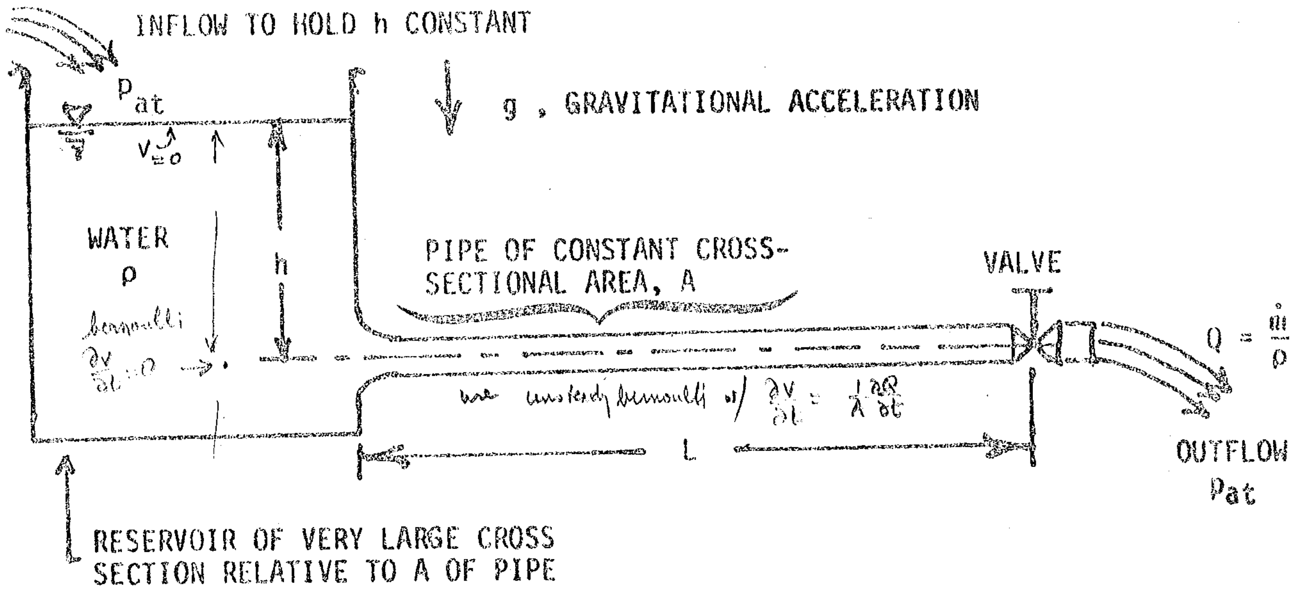


Consider a cylindrical drum of inside radius a and axial length ℓ. The drum is mounted on a central shaft of radius b, through its axis of symmetry, about which it is rotated at constant speed N (radians/sec). An immiscible mixture of liquids of unknown densities fills the drum. After a long period of rotation, the fluid is in a state of solid body rotation (e.g., $v_r = v_z = 0$, $v_\theta = Nr$ from $r = b$ to $r = a$). All the liquids have undergone centrifugal separation and are located in separate annular rings.

It is claimed that the final, steady state difference in fluid static pressure between the fluid at the cylinder wall ($r = a$) and at the shaft ($r = b$) can be obtained knowing only ℓ, N and M, the total mass of the mixture in the drum. Is this true? You may assume, for purposes of simplicity that there are only two unknown liquids in the mixture. Gravity is to be neglected also.

use radial Bernoulli
w/ constant ~~[illegible]~~ mass in drum
assume no surface tension between
fluids.

PROBLEM 2



Water flows steadily from a very large reservoir through a very long, horizontal pipe under constant h . A valve at the end of the pipe is closed in a controlled way so that the valve opening is a known function of time. Flow enters the reservoir and leaves the pipe at atmospheric pressure.

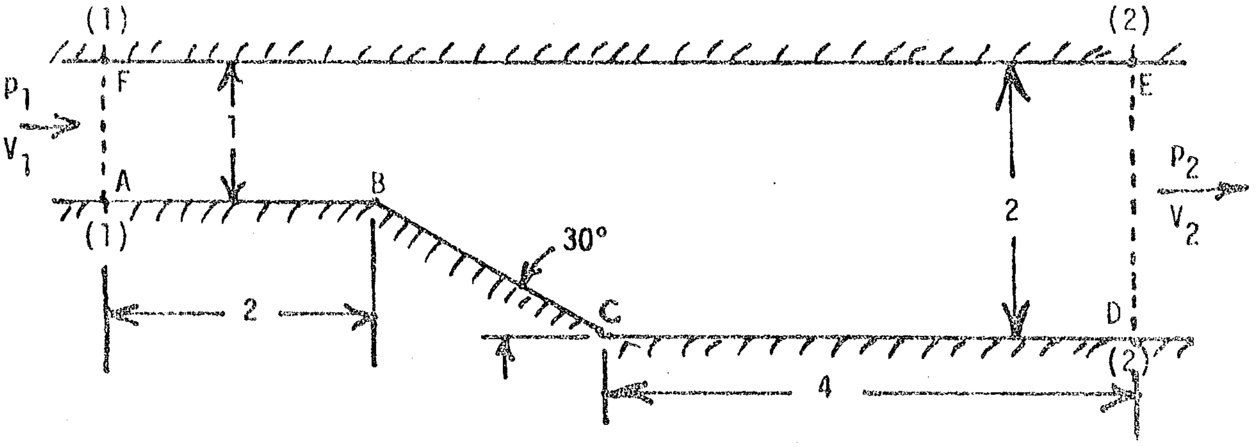
The characteristic of the valve may be described in terms of the equation:

$$Q = KA\sqrt{\frac{\Delta p}{\rho}}$$

where Q is the volumetric flow rate, and Δp is the drop in static across the valve, from just ahead of to just downstream of the valve. K , the dimensionless resistance coefficient, depends on the geometric shape of the valve's internal parts and its degree of opening. K decreases in a known way with time as the valve is closed.

For this system, during the period of valve closure, derive an approximate expression for the rate of change of flow rate, dQ/dt , in terms of the following parameters: ρ , g , Q , L , h , A and K . Neglect the viscosity and compressibility of water.

PROBLEM 3



Consider the two-dimensional, plane, incompressible inviscid flow in the duct shown above. Flow velocity is uniform and parallel at inlet, section (1) and outlet, section (2). Neglect gravity.

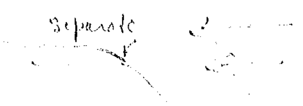
Part a: Determine the field of ϕ-lines and ψ-lines by an approximate method. Carefully trace your final results on the diagram of the duct at the top of the solution sheet. (Two identical sheets are attached, one for scratch work and one for final results.)

Part b: Sketch the shapes of the distributions of static pressure (in dimensionless form, $C_p = (p-p_1)/\frac{1}{2}\rho V_1^2$) along the two walls: ABCD, the bottom wall, and EF, the top wall. Place your final results on the graph in the center of the attached solution sheet.

Part c: Provide analytical expressions, each valid to an arbitrary multiplicative constant, which will describe the flow in regions very close to the sharp corners, B and C. Discuss the differences in the flows at B and C.

$p_B = -\infty$ $V_B = \infty$ $V_C = 0$ $p = p_{stag}$ @ b ∴ $\alpha = \frac{7\pi}{6}$ ∴ choose $z^{\pi/\alpha}$ @ c $\alpha = \frac{5\pi}{6}$

Part d: Discuss some of the effects viscosity might have on the flow for the cases of normal flow and for reverse flow. Make sketches on the ducts drawn at the bottom of the solution sheet to illustrate your discussion.



ME 251A
Final Exam

PROBLEM 4

Consider the complex potential given by

$$\Phi = U_\infty \left\{ z + ae^{i2\pi z/\lambda} \right\}$$

where U_∞, a and λ are real constants.

Part a: Find the streamfunction, ψ, for the flow represented by this Φ. Express your result as a function of the rectangular coordinates, x and y, and the system constants. $\Phi = \phi + i\psi$

Part b: Show that this Φ represents streaming flow over a wavy wall whose approximate shape is that of a sine wave when the ratio of wave amplitude to wave length is very small. Hint: the series expansion of $e^x = 1 + \sum_{M=1}^{\infty} \frac{x^M}{M!}$.

$V = \frac{\partial\psi}{\partial y} = U_\infty$ for large y

$\psi = 0$ is the value of stream fn at wall; assume $y/\lambda << 1$

Part c: Determine the points of maximum and minimum static pressure on the wavy wall and show that the maximum pressure is related to p_∞, the free stream pressure far from the wall, by the approximation

use bernoulli;
since @ V_{min} $p = p_{max}$ where $V^2 = u^2 + v^2$
V_{max} $p = p_{min}$

$$p_{max} = p_\infty + \left(\frac{4\pi a}{\lambda}\right) \tfrac{1}{2} \rho U_\infty^2$$

How smooth must the wall be if the indicated value of free stream speed, U_∞, is not 1% smaller or larger than its true value? The true value is based on stagnation pressure p_0 and p_∞ whereas the indicated value is based on a wall static pressure and p_0. Note that one cannot generally determine the location of a wall pressure tap with respect to the crests on a wavy wall of very low ratio $\overset{a/\lambda}{\wedge}$ amplitude to wave length. Neglect gravity in your analysis.

$p_0 = p_\infty + \rho \frac{V_\infty^2}{2}$

$p_s + \frac{1}{2}\rho V_w^2 = p$

$\therefore \frac{p_s - p_\infty}{\frac{1}{2}\rho V_\infty^2} = \frac{V_w^2}{U_\infty^2}$

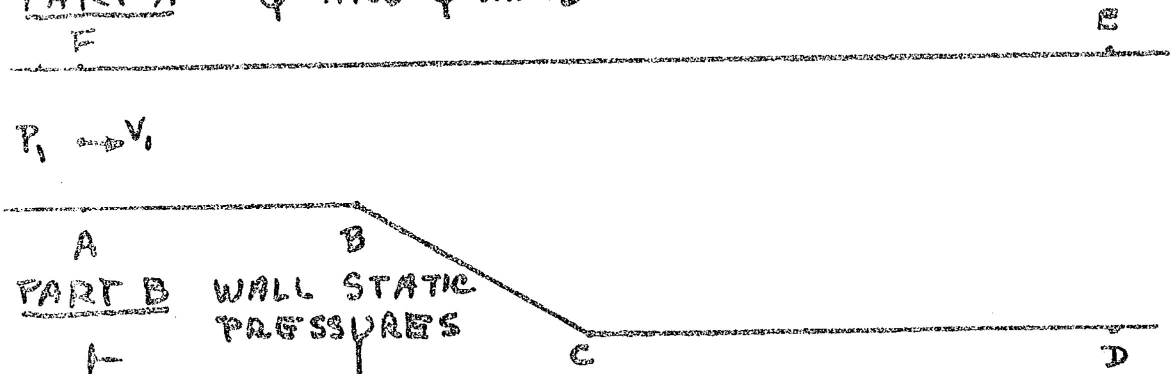
∴ we want

$1 - \left(\frac{V_w}{V_\infty}\right)^2 =$

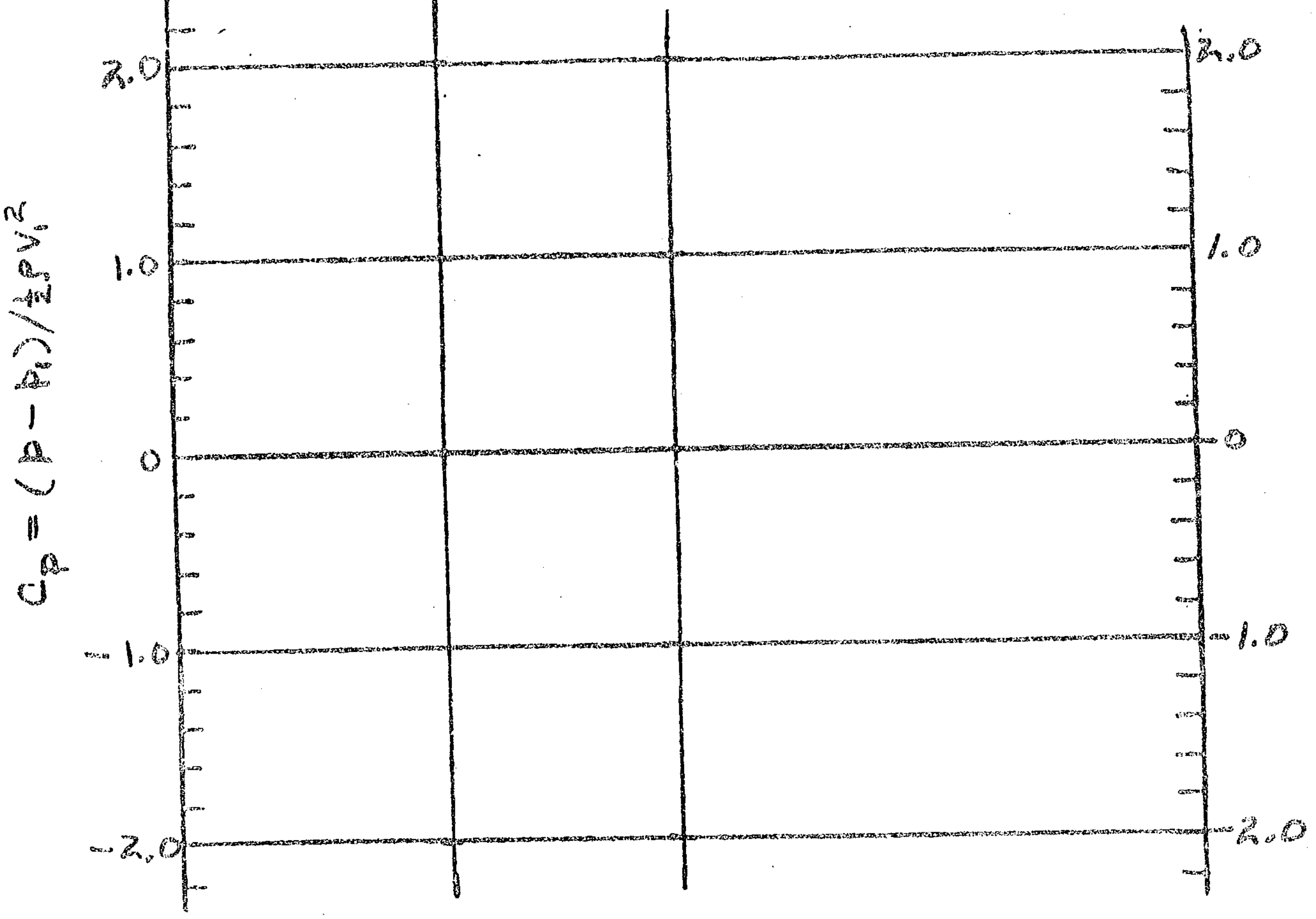
$V_w = 1.01 V_\infty$

ME 251A FINAL EXAM SOLUTION SHEET FOR PROB. 3

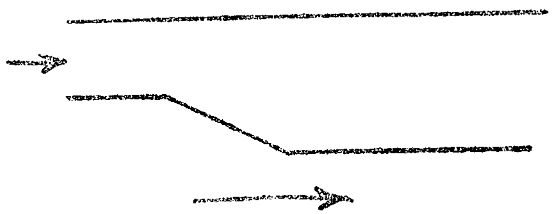
PART A φ AND ψ LINES



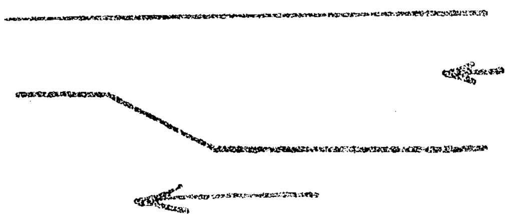
PART B WALL STATIC PRESSURES



PART D VISCOUS FLOW PATTERNS



NORMAL FLOW

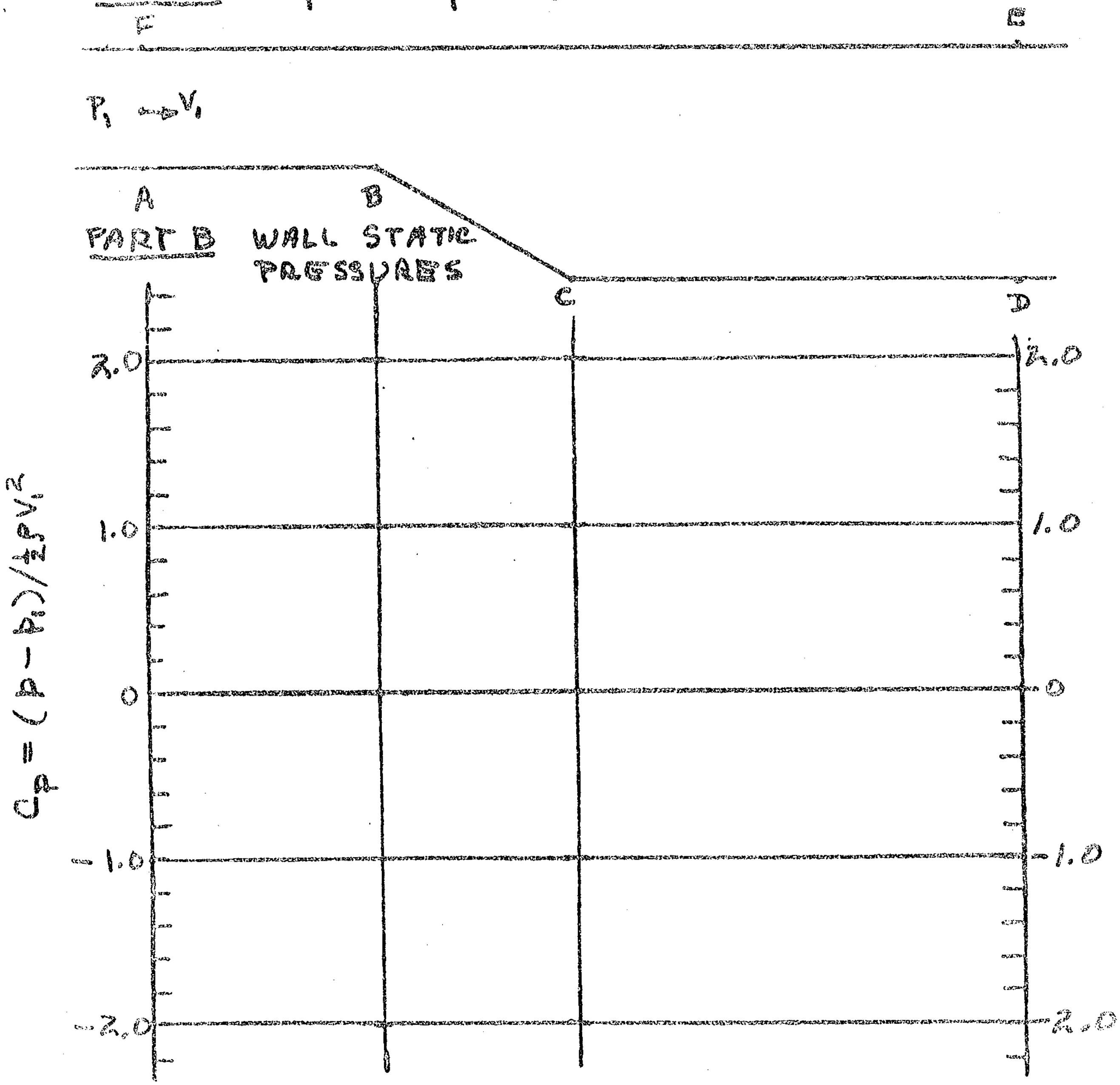


REVERSE FLOW

PRINT NAME:

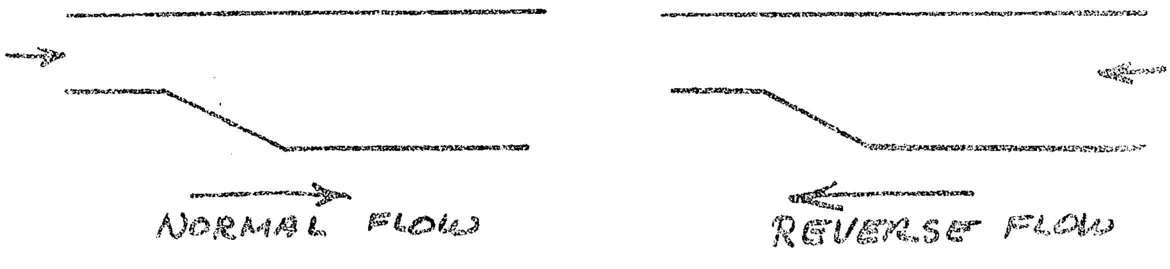
ME 251A FINAL EXAM SOLUTION SHEET FOR PROB. 3

PART A ϕ AND ψ LINES



PART B WALL STATIC PRESSURES

PART D VISCOUS FLOW PATTERNS



PRINT NAME:

1.5

since flow is incompressible cont. $\frac{D\rho}{Dt} + \rho \nabla \cdot \underset{\sim}{q} = 0 \Rightarrow \nabla \cdot \underset{\sim}{q} = 0$

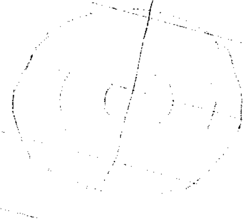
∴ if $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ then u, v are components of velocity

2.8 $\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta} v_\theta = 0$ if $v_\theta = 0 \Rightarrow (r v_r)_{,r} = 0$ or $v_r = \frac{\text{const}}{r}$

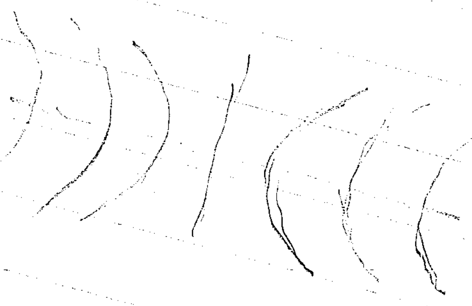
this is like a flow splashing into a plate

if $v_r = 0 \Rightarrow v_\theta =$ const.

2.10



$\psi = x^2 + y^2 =$ const



$\psi = x^2 - y^2 =$ const

2.13

$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$ $v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$

Cont. equation for incomp in 3D w/ $\partial/\partial\theta = 0$ is $\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial}{\partial z} v_z = 0$

Subst. gives $-\frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial z} + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = 0$ if $\partial z \partial r = \partial r \partial z$ which

is true for UC cont. functions.

for steady compress. $\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$ or if $\rho u = \rho_0 \frac{\partial \psi}{\partial y}$, $\rho v = \rho_0 \frac{\partial \psi}{\partial x}$

ere $\rho_0 =$ const. then

$$\frac{\partial}{\partial x}\left(\rho_0 \frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\rho_0 \frac{\partial \psi}{\partial x}\right) = \rho_0\left[\psi_{xy} - \psi_{yx}\right] = 0$$

d $d\psi = -\frac{\rho v}{\rho_0}dx + \frac{\rho u}{\rho_0}dy$

$+ \rho \nabla \cdot \underset{\sim}{q} = 0$ (continuity)

$+ \rho(x, y, z, t)$ since density since fluid

$\nabla \cdot q = 0$

2.23 full eqs are

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v}{r}$$

since flow is steady $\frac{\partial}{\partial t} = 0$; $v_r = 0$

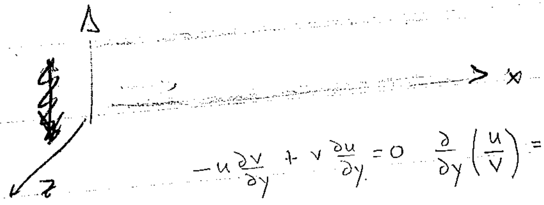
and f_r, f_θ are 0 and. p = const in

$$-\frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

2.22 steady $\frac{\partial}{\partial t} = 0$ p = const $f_y = -g$ & since range we can assume $p \neq p(x)$

$P \neq P(z)$

$$(q\cdot\nabla) q = -\frac{1}{\rho}\frac{\partial p}{\partial y} j - g j$$



and from cont $\nabla\cdot q = 0 \Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$

$-u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} = 0$ $\frac{\partial}{\partial y}\left(\frac{u}{v}\right) = 0$

and $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0$; $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g$

$\frac{u}{v} = f(x)$; $u^2\frac{\partial}{\partial x}\left(\frac{v}{u}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g$; $-u^2 \cdot \frac{f'(x)}{f^2(x)} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g \Rightarrow \frac{u^2}{f^2}f' = v^2 f' = +\frac{1}{\rho}\frac{\partial p}{\partial y} + g =$

since $\underset{f(x) \neq y}{}$ $\underset{f(y)}{}$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{matrix} 0 \\ -1/\rho \end{matrix}$$

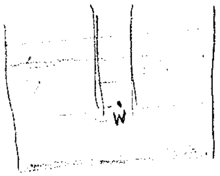
$u = v f(x)$

$= \frac{\partial v}{\partial x} f + v f' = -\frac{\partial v}{\partial y}$

2.24 if $v_\theta = \Omega r \Rightarrow r\,dr = \frac{1}{\rho\Omega^2}dp$ or $\frac{r^2}{2} = \frac{p}{\rho\Omega^2} + C$

if $v_\theta = \frac{c}{r} \Rightarrow \frac{c^2}{r^3} = \frac{1}{\rho}\frac{dp}{dr}$ or $\frac{dr}{r^3} = \frac{1}{\rho c^2}dp \Rightarrow -\frac{r^{-2}}{2} = \frac{1}{\rho c^2}p +$

2.26



$p = -\rho g y$ caused by a volume of water of $\pi(a^2 - b^2)y$

$p + \frac{\delta p}{\delta t} = -\rho g(y + \delta y) = -\rho g\frac{\delta y}{\delta t}$ caused by $\pi(a^2 - b^2)\frac{\delta y}{\delta t}$ amount w

$\dot{w} = \frac{a^2 - b^2}{a^2}$?

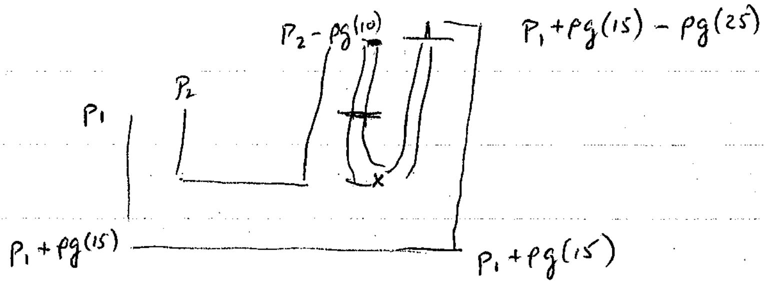
Acct L. LEVY

Password
~~Protection~~ Cesar ME 200.A (6 hrs)
ME 250a (6)

get allocation for faculty sponsorship of lots account ME 291

fill out back of intro to lots handout return to Queenette Baur @ Lots office

2-30

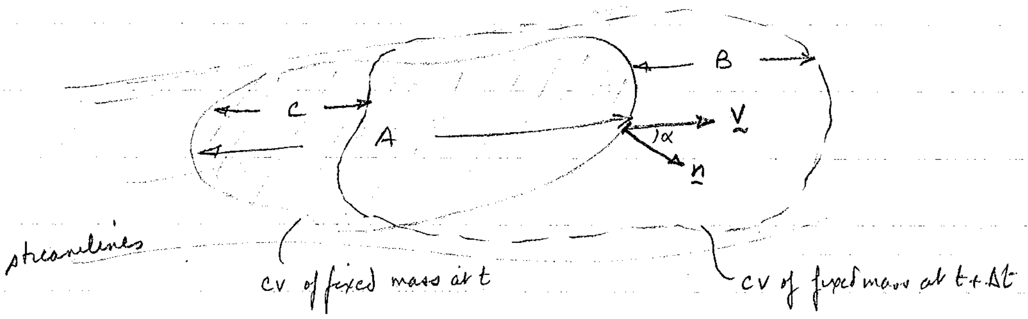


$$P_2 - \rho g(10') + \rho g(2.5) + \cancel{13.6\rho g x} = P_1 + \rho g(15) - \rho g(25) + \rho g\underset{\substack{27.2\\6.8}}{(13.6)}(2.5 + \cancel{x})$$

$$P_2 - P_1 + \overset{\rho g(15)}{\cancel{\rho g(25)}} - \rho g(7.5') = -\rho g(10) + \rho g(34)$$

$$= \rho g\,(41.5 - 10) = \rho g(31.5)$$

$$\frac{(62.4)(31.5)}{144} = 13.65$$



$$m_A(t) = m_A(t+\Delta t) - m_c(t+\Delta t) + m_B(t+\Delta t)$$ since this is fixed mass

where $m_j(p)$ is the mass of region j at time p

$$\therefore \quad m_A(t+\Delta t) - m_A(t) = m_c(t+\Delta t) - m_B(t+\Delta t)$$

$$\therefore \text{ look at } \lim_{\Delta t\to 0}\left[\frac{m_A(t+\Delta t) - m_A(t)}{\Delta t}\right] = \lim_{\Delta t\to 0}\left[\frac{m_c(t+\Delta t)}{\Delta t} - \frac{m_B(t+\Delta t)}{\Delta t}\right]$$

$$\downarrow$$

$$\frac{\partial}{\partial t}(m)_{cv} = \frac{\partial}{\partial t}\int_{cv} \rho \, d\mathcal{V}$$

$$\lim_{\Delta t\to 0} \frac{m_c(t+\Delta t)}{\Delta t} = \dot{m}_{in}$$ mass flow rate into the CV through the CS

$$\lim_{\Delta t\to 0} \frac{m_B(t+\Delta t)}{\Delta t} = \dot{m}_{out}$$ mass flow rate out of the CV through the CS

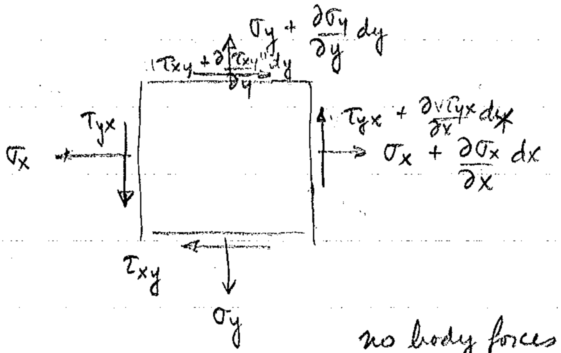
$$\dot{m}_{in} - \dot{m}_{out} = \int_{A_{in}} \rho \, \underset{\sim}{V}\cdot\underset{\sim}{n}\, dA - \int_{A_{out}} \rho\, \underset{\sim}{V}\cdot\underset{\sim}{n}\, dA = -\oint_{CS} \rho\, \underset{\sim}{V}\cdot\underset{\sim}{n}\, dA = -\int_{CV} \nabla\cdot(\rho \underset{\sim}{V})\, d\mathcal{V}$$

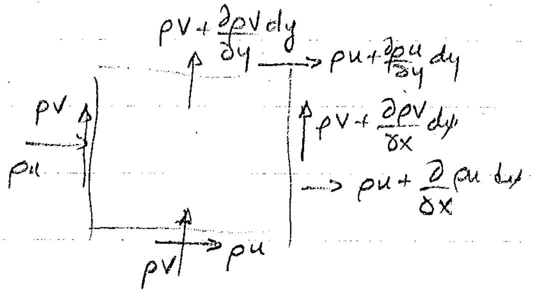
or $$\frac{\partial}{\partial t}\int_{CV} \rho\, d\mathcal{V} + \int_{CV} \nabla\cdot(\rho \underset{\sim}{V})\, d\mathcal{V} = 0$$

for steady flow $\frac{\partial}{\partial t}(\ ) = 0$ $\quad \therefore \int_{CV} \nabla\cdot(\rho \underset{\sim}{V})\, d\mathcal{V} = 0$ since true for any $d\mathcal{V} \Rightarrow$

$$\nabla\cdot(\rho \underset{\sim}{V}) = 0$$ and in 2dim $\Rightarrow \rho\,\nabla\cdot\underset{\sim}{V} + \underset{\sim}{V}\cdot\nabla\rho = 0$

Look at





no body forces look at control volume

Σ forces on the fluid is the time rate of change of momentum thera volume.

$$\Sigma \text{Forces} = \frac{\partial \sigma_x}{\partial x} dx\, dy\, \underline{i} + \frac{\partial \sigma_y}{\partial y} dy\, dx\, \underline{j} + \frac{\partial \tau_{yx}}{\partial x} dy\, dx\, \underline{j} + \frac{\partial \tau_{xy}}{\partial y} dx\, dy\, \underline{i}$$

~~time rate of change~~ of momentum $\frac{\partial \rho u}{\partial y} dy\, dx\, \underline{i} + \frac{\partial \rho u}{\partial x} dy\, dx\, \underline{i} + \frac{\partial}{\partial y} \rho v\, dx\, dy\, \underline{j} + \frac{\partial}{\partial x} \rho v\, dx\, dy\, \underline{j}$

$\therefore \quad \cancel{\frac{\partial \rho u}{\partial x}} + \cancel{\frac{\partial}{\partial y} \rho u}$ using newton's law then $F = \frac{D}{Dt}(m\underline{v}) = \rho\, dx\, dy\, \frac{D\underline{v}}{Dt}$

or $\rho\, dx\, dy \frac{Du}{Dt} = \frac{\partial \sigma_x}{\partial x} dx\, dy + \frac{\partial \tau_{xy}}{\partial y} dx\, dy$

$\therefore \quad \rho \frac{Du}{Dt} = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}$ for steady flow $\frac{Du}{Dt} = (\underline{v}\cdot\nabla)u$

$\rho \frac{Dv}{Dt} = \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y}$ $\frac{Dv}{Dt} = (\underline{v}\cdot\nabla v)$

$$\frac{D(\rho\underline{v})}{Dt} = \frac{\partial}{\partial t}(\rho\underline{v}) + \underline{v}\cdot\nabla(\rho\underline{v}) = \frac{\partial}{\partial t}(\rho\underline{v}) + \rho(\underline{v}\cdot\nabla)\underline{v} + \underline{v}\cdot(\underline{v}\cdot\cancelto{0}{\nabla\rho})\ \text{cont.}$$

$$\left| \begin{aligned} (\underline{v}\cdot\nabla)u &= \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \\ (\underline{v}\cdot\nabla)u &= \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \end{aligned} \right|$$

note that $\sigma_x = -p + \sigma_x'$
$\sigma_y = -p + \sigma_y'$
gives same type of result

$$\sum \underline{r}\times\underline{F} = \cancel{\frac{\partial}{\partial t}}\int \underline{r}\times\rho\underline{V}\,d\mathcal{V}^{\nearrow 0} + \int \underline{r}\times\underline{V}\,(\rho\underline{V}\cdot\underline{n})\,dA$$

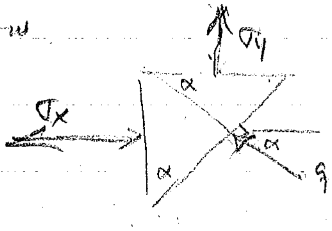
steady

$$\sum m = \frac{\partial\sigma_x}{\partial x}dx\,dy\cdot\frac{dy}{2} - \left(\tau_{yx} + \frac{\partial\tau_{yx}}{\partial x}dx\right)dy\,dx - \frac{\partial\sigma_y}{\partial x}dy\,dx\cdot\frac{dx}{2}$$

$$+ \left(\tau_{xy} + \frac{\partial\tau_{xy}}{\partial y}dy\right)dx\,dy$$

$\tau_{xy} = \tau_{yx}$ otherwise when shrunk to a point the volume would have to rotate w/ infinite angular velocity

1. 1st derive inviscid equation
2. show



$$\sigma_x\,dy = q\cos\alpha\,ds = q\,dy \Rightarrow \sigma_x = q$$

$$\sigma_y\,dx = q\sin\alpha\,ds = q\,dx \Rightarrow -q = \sigma_y$$

$\therefore\ \sigma_x = \sigma_y = p$ or

$\therefore$ since for inviscid equation $\sigma_x, \sigma_y = -p$

$$\Rightarrow (\underline{V}\cdot\nabla)\underline{V} = -\nabla p$$

$$u_j\frac{\partial}{\partial x_j}u_i = -\frac{\partial p}{\partial x_i}$$

NOTES FOR
A WARM-UP FOR THE THERMOSCIENCES

A MINI-COURSE FOR NEW
GRADUATE STUDENTS IN
MECHANICAL ENGINEERING
AT STANFORD

W. C. REYNOLDS

WCR
Fall, 1979

Table of Contents

I. A BRIEF GUIDE TO PROBLEM SOLVING IN THE THERMOSCIENCES

In our graduate courses in the Thermosciences at Stanford we emphasize a professional approach to problem solution. For the most part, homework problems will be analyses of the type that you might encounter in your professional work, and we expect you to prepare these in a professional manner. If you learn to work systematically, you can do most of these problems in a single pass. If you tend to think randomly and work sloppily, you will have only yourself to blame for long hours spent doing and re-doing and writing and re-writing homework problems.

A. General Methodology

The following methodology will usually get you through a problem efficiently and correctly:

1. Define the system. This is important to aid your thinking and to communicate your analysis to the reader. The following points should be considered:
   a. Use dots on a sketch to define the boundaries of the system being analyzed. Place the boundaries where you know something or where you want to know something; often the analysis varies significantly with slight differences in boundary location, so be very clear in your definition.
   b. A control mass is a fixed piece of matter; a control volume is a defined region in space through which matter flow. Clearly identify which you have enclosed by the dots.
   c. Coordinate systems are often important. If they are, define them on the sketch. If they move, or if the system moves with respect to them, indicate the appropriate velocities or displacements.
2. Define the forces or flows
   a. If you are doing momentum analysis, indicate the significant forces on the sketch, and define their sign convention by an arrow. Show the forces exerted on the system defined.
   b. If you are doing an energy analysis, show all the significant energy flows (as heat, as work, and convected), on the sketch, and define the direction of positive energy flow by an arrow.
3. Define the model. This is done by listing the simplifying assumptions (steady-state, one-dimensional flow, adiabatic device, frictionless device, perfect mixing, negligible kinetic energy, etc.). The reader should be able to ascertain the model that you are using completely by looking at your system sketch and reading your idealization list.

4. Indicate the time basis. Your analyses will either be made over the time for the process to occur, or on a "rate basis." Indicate which.

5. Apply fundamental principles. Express the pertinent basic principle (conservation of mass, conservation of energy, momentum principle, etc.) as it applies to your defined system in terms of the symbols defined on your system sketch. There should be a one-to-one correspondence between the terms on your sketch and the terms in your basic equation.

   Most mistakes in analysis come at this point, usually from failure of the analyst to properly and consistently define the system, forces, or flows. So check your equation and your sketch at this point.

6. Bring in auxiliary information. You may need to use equations of state, rate equations, or other information. It is a good idea to count your unknowns and count your equations, and be sure that these numbers are the same before you start doing algebra. If you don't have enough equations to cover all the unknowns, ask yourself what basic principle or what information has not yet been used; often this helps you find an additional equation.

7. Solve the problem algebraically if this is easy or desirable, or numerically if algebraic solutions are tedious or impractical. Use a hand calculator to avoid numerical errors. Check units. If an answer has units, be sure to give them (the number will be of no use to anyone without the units).

8. Non-dimensionalization. If your analysis is of general utility, you may want to express the results non-dimensionally by defining appropriate dimensionless variables and parameters (e.g., P/P_o vs x/d).

9. Discuss the significance of your result, limitations on it because of modeling simplifications, and, if appropriate, economic or environmental factors what should be considered in subsequent work on the project.

Make your analysis "read." Number equations, refer to them by number, insert a few words here and there so that the reader can follow your analysis without having to guess what you are doing. An analysis which is unclear wastes the time of the reader (your boss), and is unprofessional.

Use straight-edges and french curves in drawing sketches and graphs. Your work should make the person reading it feel that you care about the work and his reaction to it; if you don't care, perhaps you should find another job, or another profession.

B. Basic Principles for a Control Mass

The basic principles used most often in the Thermosciences are:

1. Conservation of mass
2. Conservation of energy
3. Second law of thermodynamics
4. Momentum principles (linear and angular)

We have developed a very appealing way of formulating these basic principles through the use of the general idea of "production." In general, for energy, mass, entropy, momentum, or applies,

$$\mathcal{P} \equiv \text{production} = \text{outflow} - \text{inflow} + \text{increase in storage}$$

If this is not obvious to you, perhaps this is:

$$\text{Inflow} + \text{production} = \text{outflow} + \text{increase in storage}$$

or this:

$$\text{Increase in storage} = \text{inflow} - \text{outflow} + \text{production}$$

Inflows are like deposits, outflows like withdrawals, production like interest, and the increase in storage like the increase in the balance in your checking account. If none of the (identical) expressions above makes senst to you, your bank account is probably in serious trouble.

On a rate basis, we have

$$\begin{aligned}\dot{\mathcal{P}} \equiv \text{rate of production} &= \text{rate of outflow} - \text{rate of inflow} \\ &\quad + \text{rate of increase in storage}\end{aligned}$$

It is this form that we use in most analysis. Find a form that you understand and can remember.

Once you have your bookkeeping straight, the basic principles are simple:

1. Energy and mass are not produced.
2. Entropy can be produced, but can never be destroyed: a reversible process is one that produces no entropy.
3. Momentum production rate is proportional to the applied force; angular momentum production rate is proportional to the applied torque.

With proper bookkeeping, these basic principles apply equally well to a control mass or a control volume.

We assume that these basic principles are all familiar to you in some form. If they are not, you have remedial work to do.

For a control mass CM, which by definition has no mass flows across the control surface (CS), the basic principles may be expressed by the following simple mathematics:

Conservation of mass

$$\dot{\mathcal{P}}_{\text{mass}} = \frac{dM}{dt} = 0 \tag{1}$$

M = mass of CM
$\dot{\mathcal{P}}$ = "rate of creation of"

Conservation of energy

$$\dot{\mathcal{P}}_{\text{energy}} = \frac{dE}{dt} + \sum (\dot{Q} + \dot{W})_{\text{ouw}} - \sum (\dot{Q} + \dot{W})_{\text{in}} = 0 \tag{2}$$

E = energy of CM
$\dot{Q}$ = rate of energy transfer across CS as heat
$\dot{W}$ = rate of energy transfer across CS as work

Second law of thermodynamics

$$\dot{\mathcal{P}}_{\text{entropy}} = \frac{dS}{dt} + \sum (\dot{Q}/T)_{\text{out}} - \sum (\dot{Q}/T)_{\text{in}} \geq 0 \tag{3}$$

S = entropy of CM
T = absolute temperature of CS where $\dot{Q}$ crosses
We interpret Q/T as a "rate of entropy flow with heat." See Reynolds and Perkins, Engineering Thermodynamics, McGraw-Hill

Linear momentum

$$\dot{\mathcal{P}}_{\overrightarrow{\text{momentum}}} = \frac{d\vec{m}}{dt} = g_c \vec{F}_{\text{net}} \tag{4}$$

$\vec{m}$ = momentum of CM in $\overrightarrow{\text{direction}}$ $(M\vec{V})$
$\vec{F}_{\text{net}}$ = net force on CM in $\overrightarrow{\text{direction}}$
g_c = constant in Newton's law
  g_c = 1 in SI system
  g_c = 32.17 ft-lbm/lbf-sec^2 in English system

Angular momentum

$$\dot{\mathcal{P}}_{\overrightarrow{\text{angular momentum}}} = \frac{d\vec{a}}{dt} = g_c \vec{\tau}_{\text{net}} \tag{5}$$

$\vec{a}$ = angular momentum of CM in $\overrightarrow{\text{direction}}$ $(\vec{r} \times \vec{m})$
$\vec{\tau}_{\text{net}}$ = net torque on CM in $\overrightarrow{\text{direction}}$ $(r \times \vec{F})$

Note that the expressions for the $\dot{\mathcal{P}}$ terms in Eqns. (1)-(5) are *bookkeeping* statements. The physics is expressed by

$$\dot{P}_{\text{mass}} = 0 \;, \quad \dot{P}_{\text{energy}} = 0 \;, \quad \dot{P}_{\text{entropy}} \geq 0$$

$$\dot{P}_{\text{momentum}} = g_c\vec{F} \;, \quad \dot{P}_{\substack{\text{angular}\\ \text{momentum}}} = g_c\vec{T} \tag{6}$$

If you are impressed by mathematical elegance (be sure you understand the physics first!), the integral formulations of the basic principles, again for a control mass, may appeal to you. In the expressions below an overdot denotes a rate of flow, a " denotes a quantity per unit area, and a "' denotes a quantity per unit volume. Thus, for example, $\dot{Q}''$ is a heat transfer rate per unit of surface area, e.g., J/s-m^2. Also,

ρ = mass per unit volume
e = energy per unit mass $\quad u + \dfrac{V^2}{2g_c}$
F_s = surface force acting on CM
F_b = body force acting on CM
dV = elemental volume of CM
dA = elemental area of CS

$$\dot{P}_{\text{mass}} = \frac{d}{dt}\int_{CM} \rho dV = 0 \tag{7}$$

$$\dot{P}_{\text{energy}} = \frac{d}{dt}\int_{CM} \rho e dV - \underbrace{\int_{CS} \dot{Q}''dA}_{\substack{\text{net inflow}\\ \text{of energy}\\ \text{as heat}}} - \underbrace{\int_{CS} \vec{F}_s''\cdot\vec{V}dA - \int_{CM} F_b''' \, VdV}_{\substack{\text{net inflow of energy}\\ \text{as work}}} = 0 \tag{8}$$

$$\dot{P}_{\text{entropy}} = \frac{d}{dt}\int_{CM} \rho s dV - \underbrace{\int_{CS} (\dot{Q}''/T)\, dA}_{\substack{\text{net inflow}\\ \text{of entropy}\\ \text{with heat}}} \geq 0 \tag{9}$$

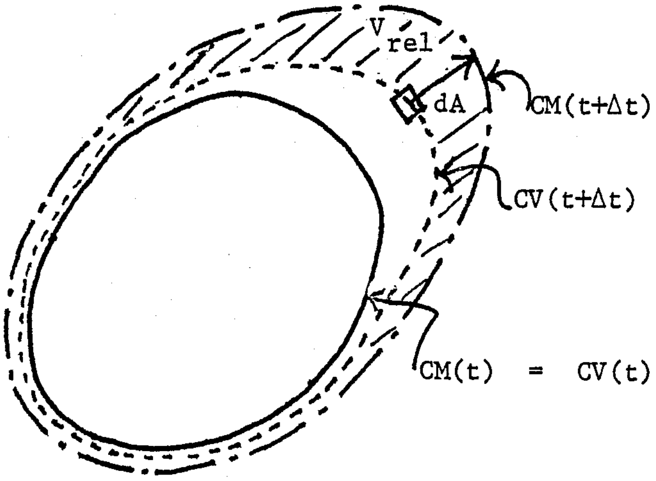
$$\dot{P}_{\overrightarrow{\text{momentum}}} = \frac{d}{dt}\int_{CM} \rho\vec{V}dV = g_c\left\{\underbrace{\int_{CS} \vec{F}_s''dA + \int_{CM} \vec{F}_b'''dV}_{\text{net force}}\right\} \tag{10}$$

$$\dot{P}_{\substack{\overrightarrow{\text{angular}}\\ \text{momentum}}} = \frac{d}{dt}\int \rho\vec{r}x\vec{V}dV = g_c\left\{\underbrace{\int_{CS} \vec{r}x\vec{F}_s'''dA + \int_{CM} \vec{r}x\vec{F}_b'''dV}_{\text{net torque}}\right\} \tag{11}$$

C. Basic principles for a control volume

You undoubtedly have used control volumes in your undergraduate work. However, most likely this has been limited to control volumes which are not changing shape or accelerating, i.e., fixed volumes fixed in space (or translating at a constant velocity). If the control volume is changing shape or accelerating, things get more complicated mathematically, although the physical ideas are quite straightforward.

To develop the general control volume equations, let's carry out a completely general (except for relativistic effects!) control volume transformation. To do this we consider a control mass CM(t) which, at time t, exactly coincides with the control volume at time t, CV(t). At a slightly later time $t+\Delta t$ the CM and CV will occupy different regions in space, $CM(t+\Delta t)$ and $CV(t+\Delta t)$. The sketch below describes this with sufficient generality.



Now, we note that each of the basic principles involves a term of the form

$$\frac{d}{dt}\int_{CM} g\,dV$$

The control volume transformation simply amounts to expressing this property of the <u>CM</u> in terms of properties of the <u>CV</u>. To do this we need to know how to differentiate a three-dimensional integral over a time-dependent domain. Chances are you have never seen this in your math courses, so we will have to work it out from basics (remember, we emphasize basics!). From the definition of a derivative,

$$\frac{df}{dt} \equiv \lim_{\Delta t\to 0}\left[\frac{f(t+\Delta t) - f(t)}{\Delta t}\right] \qquad (12)$$

we find

$$\frac{d}{dt}\int_{CM(t)} g(\vec{x},t)\,dv = \lim_{\Delta t\to 0}\left[\frac{\int_{CM(t+\Delta t)} g(\vec{x},t+\Delta t)\,dV - \int_{CM(t)} g(\vec{x},t)\,dV}{\Delta t}\right] \qquad (13)$$

Next we break the first integral on the right into two parts (two connected regions in space),

$$\int_{CM(t+\Delta t)} g(\vec{x},t+\Delta t)dV = \int_{CV(t+\Delta t)} g(\vec{x},t+\Delta t)dV + \int_{SP} g(\vec{x},t+\Delta t)dV \tag{14}$$

where SP denotes the shaded region of space shown in the sketch (the part of the CM that has passed out of the CV). Since CM(t) = CV(t), the right hand side of (13) is

$$\lim_{\Delta t\to 0}\left\{\left[\frac{\int_{CV(t+\Delta t)} g(\vec{x},t+\Delta t)dV - \int_{CV(t)} g(\vec{x},t)dV}{\Delta t}\right] + \frac{1}{\Delta t}\int_{SP(t+\Delta t)} g(\vec{x},t+\Delta t)dV\right\}$$

Ah ha! The limit of the term in the square brackets is, by definition, the time-derivative of the g-integral over the control volume,

$$\frac{d}{dt}\int_{CV(t)} g dV$$

This represents the rate of change of the storage of g-stuff in the control volume. The remaining term in the limit can be evaluated by noting that the element of volume of the shade portion can be written as

$$dV = \vec{V}_{rel}\,\Delta t\cdot d\vec{A} \tag{15}$$

where $\vec{V}_{rel}$ is the velocity of the material relative to the control volume boundary, and $d\vec{A}$ is an outward-normal vector representing an element of area of the control volume surface. Note that this can be interpreted as an element of volume which has passed out of the control volume (at points where matter flows into the control volume, $\vec{V}_{rel}\cdot d\vec{A}$ would be negative, so this includes inflows as well). The area integration that covers SP is just the surface of the control volume (we now denote) the CV surface by CS); so, when we take the limit we find

$$\frac{d}{dt}\int_{CM(t)} g dV = \frac{d}{dt}\int_{CV(t)} g dV + \int_{CS(t)} g\,\vec{V}_{rel}\cdot d\vec{A} \tag{16}$$

Again, the first term on the right is the rate of change of g-stuff in the CV, and the second term is the net outflow of g-stuff from the CV.

The other surface and volume integrals appearing in the CM expressions of the basic principles may be replaced by exactly the same CV integrals, since the CM and CV coincide at time t . This is all we need to do to get the basic principles in control volume form. They are summarized below, in rate of production form.

$$\dot{P}_{\text{mass}} = \frac{d}{dt}\int_{CV}\rho\, d\mathcal{V} + \int_{CS}(\rho\vec{V}_{rel}\cdot d\vec{A}) = 0 \tag{17}$$

$$\begin{aligned}\dot{P}_{\text{energy}} &= \frac{d}{dt}\int_{CV}\rho e\, d\mathcal{V} + \int_{CS} e(\rho\vec{V}_{rel}\cdot d\vec{A}) - \int_{CS}\dot{Q}''dA \\ &\quad - \int_{CS}\vec{F}''_s\cdot\vec{V}dA - \int_{CV}\vec{F}'''_b\cdot\vec{V}d\mathcal{V} = 0\end{aligned} \tag{18}$$

$$\begin{aligned}\dot{P}_{\text{entropy}} &= \frac{d}{dt}\int_{CV}\rho s\, d\mathcal{V} + \int_{CS} s(\rho\vec{V}_{rel}\cdot d\vec{A}) \\ &\quad - \int_{CS}(\dot{Q}''/T)\ dA \geq 0\end{aligned} \tag{19}$$

$$\begin{aligned}\dot{P}_{\text{momentum}} &= \frac{d}{dt}\int_{CV}\rho\vec{V}d\mathcal{V} + \int_{CS}\vec{V}(\rho\vec{V}_{rel}\cdot d\vec{A}) \\ &= g_c\left\{\int_{CS}\vec{F}''_s dA + \int_{CV}\vec{F}'''_b d\mathcal{V}\right\}\end{aligned} \tag{20}$$

$$\begin{aligned}\dot{P}_{\substack{\text{angular}\\ \text{momentum}}} &= \frac{d}{dt}\int_{CV}\rho\vec{r}\times\vec{V}d\mathcal{V} + \int_{CS}\vec{r}\times\vec{V}(\rho\vec{V}_{rel}\cdot dA) \\ &\quad + g_c\left\{\int_{CS}\vec{r}\times\vec{F}''_s dA + \int_{CM}\vec{r}\times\vec{F}'''_b d\mathcal{V}\right\}\end{aligned} \tag{21}$$

Note that the integral terms involving $\rho\vec{V}_{rel}\cdot d\vec{A}$ represent <u>convective</u> outflows from the control volume; Professor Moffat likes to call these "mass-associated transports," since $\rho\vec{V}_{rel}\cdot d\vec{A}$ is the mass flow rate across area dA . Note also that the <u>relative</u> velocity $\vec{V}_{rel}$ is used in computing the flow rate, while the <u>absolute</u> velocity $\vec{V}$ must be used for the momentum and angular momentum terms. Remember that the momentum equations must be written in an inertial reference frame, as must the energy equation if kinetic energy is important (kinetic energy is only $MV^2/2g_c$ in an inertial frame!)

If you are wondering where the enthalpy is in the energy equation, the u-part of h is contained in the convected e term. The Pv-part of h (flow work) is contained in the surface force integral. If we split the surface stress into a pressure part and a residual part (which might be viscous stresses),

$$F''_s = -P + f''_s \tag{22}$$

then we can combine the P part with the convected flow to bring in the enthalpy.

The previous equations are elegant, but the ideas are extremely simple. We recommend that you start each problem from the basic physical ideas, not from the fancy integral formulations. Simple equations expressing the same basic ideas are given below:

$$\dot{\mathcal{P}}_{mass} = \frac{dM}{dt} + \sum \dot{M}_{out} - \sum \dot{M}_{in} = 0 \tag{23}$$

M = mass of CV

$\dot{M}$ = mass flow rate = $A\rho V_{rel}$

$$\dot{\mathcal{P}}_{energy} = \frac{dE}{dt} + \sum(\dot{Q}+\dot{W}+\dot{M}h^{o})_{out} - \sum(\dot{Q}+\dot{W}+\dot{M}h^{o})_{in} = 0 \tag{24}$$

E = energy of CV

h^{o} = $h + V^2/2g_c$ (total enthalpy)

$\dot{W}$ = work of viscous and body forces (the flow work is in h).

$$\dot{\mathcal{P}}_{entropy} = \frac{dS}{dt} + \sum(\dot{M}s)_{out} - \sum(\dot{M}s)_{in} + (\dot{Q}/T)_{out} - (\dot{Q}/T)_{in} \geq 0 \tag{25}$$

S = entropy of CV

s = entropy per unit mass

$$\dot{\mathcal{P}}_{momentum} = \frac{d\vec{m}}{dt} + \sum(\dot{M}\vec{V})_{out} - \sum(\dot{M}V)_{in} = g_c\vec{F}_{net} \tag{26}$$

$$\dot{\mathcal{P}}_{\substack{angular\\momentum}} = \frac{d\vec{a}}{dt} + \sum(\dot{M}\vec{r} \times \vec{V})_{out} - \sum(\dot{M}\vec{r} \times \vec{V})_{in} = g_c \vec{T}_{net} \tag{27}$$

To summarize, the basic principles are all very simple:

1. Energy and mass cannot be produced.
2. Entropy production must be positive, or zero in the limit of a reversible process.
3. The rate of momentum production is proportional to the force applied to the control volume.
4. The rate of angular momentum production is proportional to the torque applied to the control volume.

The rate of production is

$$\dot{P} = \text{rate of increase in storage} + \text{outflow rate} - \text{inflow rate} \quad .$$

When energy flows as heat across the boundaries of a system, entropy flows with the heat at the rate $\dot{Q}/T$.

These are the essential ideas to remember. Then, with the help of a good system sketch and careful thinking, you can work out specific equations in specific cases, and thereby handle <u>any</u> problem that you might encounter. Master these ideas to be a Master of Mechanical Engineering.

II. NOTATIONS AND PROCEDURES FOR DERIVATION OF PDE'S

Many of the graduate engineering courses at Stanford contain a substantial amount of engineering science. Usually this takes the form of solutions to the partial differential equations (PDE's) governing the physical phenomena. There are easy and hard ways to derive and write PDE's. The purpose of this section is to provide you with some basic notations that simplify the writing of PDE's and to give you some tips on how to derive them easily.

A. Vector Notation

We presume you are familiar with vector algebra and vector fields. For example, the velocity of a particle is described by a vector $\vec{V}$ with three components; the velocity in a fluid is described by the velocity vector field $\vec{V}(\vec{x})$. Thus, while one could write the fluid velocity at a point by the set of three symbols (u,v,w) which are the velocity components in the $x,y,$ and z directions, it is shorter to write the same information as $\vec{V}$, which is then understood to have the three components $u,v,$ and w. If your vector algebra is rusty, you should do some review; remind yourself of the meaning of $\vec{A}\cdot\vec{B}$ and $\vec{A}\times\vec{B}$.

You may not have encountered vector calculus, but this is not a complicated subject. To differentiate a vector with respect to a scalar, for example $d\vec{V}/dt$, you simply differentiate each component separately to obtain the component of the derivative.

The central differential vector operator is the gradient or grad. This operator can operate either on a scalar or a vector. The gradient represents the rate of change of the operand in the three spatial directions. In cartesian coordinates,

$$\text{grad}(\ ) = \vec{\nabla}(\ ) = \vec{i}\,\frac{\partial(\ )}{\partial x} + \vec{j}\,\frac{\partial(\ )}{\partial y} + \vec{k}\,\frac{\partial(\ )}{\partial z} \qquad (1)$$

Here $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors in the x, y, and z directions, respectively. For example, Fourier's law of heat conduction (the conduction rate equation) can be written as

$$\vec{q} = -\kappa \text{ grad } T = -\kappa\vec{\nabla}T \qquad (2)$$

where $\vec{q}$ is the heat flux vector (W/m^2), T is temperature, and κ is the thermal conductivity.

The gradient operator, dot-product operating on a vector, is called the <u>divergence</u> of the vector,

$$\text{div } \vec{V} = \vec{\nabla}\cdot\vec{V} \qquad (3)$$

In cartesian coordinates, if $\vec{V} = (u,v,w)$,

$$\text{div } \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad (4)$$

For example, the equation expressing conservation of mass for an incompressible fluid is div $\vec{V} = 0$, where $\vec{V}$ is the fluid velocity.

The gradient operator, cross-product operating on a vector, is called the curl,

$$\text{curl}\ \vec{V} = \vec{\nabla} \times \vec{V} \tag{5}$$

For example, in cartesian coordinates, with $\vec{V} = (u,v,w)$,

$$\vec{\nabla} \times \vec{V} = \vec{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \vec{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{6}$$

In fluid mechanics, the curl of the velocity is called the vorticity.

The gradient operator, applied twice, becomes an important scalar operator, the Laplacian; in cartesian coordinates,

$$\vec{\nabla} \cdot \vec{\nabla}(\ ) = \nabla^2(\ ) = \frac{\partial^2(\ )}{\partial x^2} + \frac{\partial^2(\ )}{\partial y^2} + \frac{\partial^2(\ )}{\partial z^2} \tag{7}$$

Two important identities that follow from the definitions are (S is a scalar and $\vec{V}$ is a vector):

$$\text{curl}[\text{grad}(S)] = 0 \tag{8}$$

$$\text{div}[\text{curl}(\vec{V})] = 0 \tag{9}$$

Thus, for example, the vorticity is "divergence-free."

As we shall see, the vector notations described above allow one to represent the equations describing physical problems in a very compact form. The terms $\vec{\nabla}S$, $\vec{\nabla} \cdot \vec{V}$, and $\vec{\nabla} \times \vec{V}$ have basic meanings which are independent of the coordinate system, but they look different in various coordinate systems. For example, in cylindrical coordinates,

$$\nabla^2 S = \frac{\partial^2 S}{\partial r^2} + \frac{1}{r}\frac{\partial S}{\partial r} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2} + \frac{\partial^2 S}{\partial z^2} \tag{10}$$

For the forms of the other operators in other coordinate systems, consult a book on vector calculus (or, better yet, derive them youself from transformations from the cartesian forms).

B. Cartesian Tensor Notation

The cartesian tensor notation is another way that we use to simplify the derivation and writing of PDE's in cartesian coordinates. It involves the use of subscripts. Thus, x_i is used to represent "the ith x", with $x_1 = x$, $x_2 = y$, and $x_3 = z$. Similarly, V_i is used to represent the ith V; $V_1 = u$, $V_2 = v$, $V_3 = w$.

It is the subscript summation convention that simplifies the writing. If an index is repeated, it is to be understood that a summation is to be carried out over all possible values of the repeated subscript. Thus, for example,

$$V_i V_i \quad \text{means} \quad \sum_{i=1}^{3} V_i V_i = V_1^2 + V_2^2 + V_3^2 \tag{11}$$

$$\frac{\partial V_i}{\partial x_i} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3} \tag{12}$$

$$V_j \frac{\partial V_i}{\partial x_j} = V_1 \frac{\partial V_i}{\partial x_1} + V_2 \frac{\partial V_i}{\partial x_2} + V_3 \frac{\partial V_i}{\partial x_3} \tag{13}$$

In properly derived forms, a subscript is never repeated more than once, and this fact can be used to check derivations. Also, the summed indices must be "summed out" in all terms. Thus, for example

$$V_j \frac{\partial V_i}{\partial x_j} + V_i V_j \tag{14}$$

is an improper form, because there is no summation over the j index in the second term.

From (12) it is evident that

$$\text{div } \vec{V} = \frac{\partial V_i}{\partial x_i} \tag{15}$$

Also,

$$\nabla^2 S = \frac{\partial^2 S}{\partial x_j \partial x_j} \tag{16}$$

This subscript summation convention is also useful in discussing problems in linear algebra. For example, consider

$$A_{ij} x_j = y_i \tag{17}$$

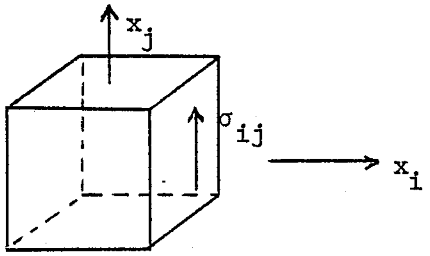
Since j is summed, but i is not, this represents the system of n algebraic equations.

$$\begin{array}{c} A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n = y_1 \\ \vdots \qquad\qquad\qquad \vdots \\ A_{n1}x_1 + A_{n2}x_2 + \dots + A_{nn}x_n = y_n \end{array} \tag{18}$$

In deriving PDE's involving deformation, the symbol S_{ij} is used to denote the rate of strain tensor,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \tag{19}$$

and the symbol σ_{ij} is used to denote the stress on a plane perpendicular to the i axis in the direction of the j axis.



The delta function, δ_{ij}, is defined to be 1 if i and j are equal, and zero if not. Thus $\delta_{11} = 1$, $\delta_{12} = 0$. By the rules of the summation convention, $\delta_{ii} = 3$.

For example, the constitutive equation of a Newtonian fluid is

$$\sigma_{ij} = - P\delta_{ij} + 2\mu S_{ij} \tag{20}$$

where P is the pressure. As an exercise, you should write out the nine components of the stress tensor using (20).

The curl operation can be expressed with the introduction of a third-order alternating tensor, ε_{ijk}. This is defined to be zero if any two of the three indices i, j, or k are equal, to be +1 if they are in ascending order (123, 231, or 312), and -1 if they are in descending order (321, 213, 132). Then

$$\text{curl } \vec{V} = \varepsilon_{ijk} \frac{\partial V_k}{\partial x_j} \tag{21}$$

Note that both k and j are repeated, and hence must be summed. You should write out the components of the curl vector in detail to verify that they are indeed as given by (6).

Tensors (S_{ij} and σ_{ij} are tensors) have other important mathematical properties and facilitate working in generalized coordinate systems. However, to start it is important only that you become comfortable with the cartesian tensor notation described above.

C. Derivative Notations

You are familiar with the symbols d and ∂. In convective problems the combination of terms

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y} + w\frac{\partial(\ )}{\partial z} \tag{22}$$

always arises. This represents the rate of change following a fluid particle that moves with velocity components u, v, w. This is called the <u>substantial derivative</u>. Note that

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u_j\frac{\partial(\ )}{\partial x_j} \tag{23}$$

or, in cartesian coordinates,

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \vec{V}\cdot\vec{\nabla}(\ ) \tag{24}$$

Some books, papers, and professors use other shorthand notations for partial derivatives. For example, a subscript often denotes partial differentiation with respect to the subscript

$$u_x = \partial u/\partial x \quad , \quad u_{yy} = \partial^2 u/\partial y^2$$

However, some people (including some professors) use u_x to denote u at x, so you had better be sure that you understand what is meant in each case.

Sometimes the symbol ∂_x is used to mean $\partial/\partial x$. Or the symbol ∂_i is used to mean $\partial/\partial x_i$. Thus, some will write

$$\text{div}\ \vec{V} = \partial_i V_i \tag{25}$$

The real chalk savers use an even more streamlined notation, in which subscripts after commas imply partial differentiation; thus

$$V_{i,i} = \frac{\partial V_i}{\partial x_i} = \text{div}\ \vec{V} \tag{26}$$

$$S_{,ii} = \frac{\partial^2 S}{\partial x_i \partial x_i} = \nabla^2 S \tag{27}$$

$$V_j V_{i,j} = V_j\frac{\partial V_i}{\partial x_j} = (\vec{V}\cdot\vec{\nabla})\vec{V} \tag{28}$$

They also use an overdot to represent time differentiation,

$$\dot{V}_i = \partial V_i/\partial t \tag{29}$$

However, the overdot also is used to represent a rate of flow (stuff/sec), so again you must know what is going on to keep things straight.

In linear problems operator notation is frequently used to symbolize an equation. For example, if we define the operator L by

$$L(\ ) = \frac{\partial^2(\ )}{\partial x_i \partial x_i} + \lambda^2(\ ) \tag{30}$$

then $Lu = 0$ means

$$\frac{\partial^2 u}{\partial x_i \partial x_i} + \lambda^2 u = 0 \tag{31}$$

The following equations are the Navier-Stokes equations for incompressible Newtonian fluid flow, written in a variety of notations. You can see for yourself the value of the simplification.

a) <u>Longhand</u>

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{32a}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{32b}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{32c}$$

b) <u>Vector notation</u> (cartesian coordinates)

$$\rho\left[\frac{\partial V}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\text{grad } P + \mu\nabla^2\vec{V} \tag{33}$$

c) <u>Cartesian tensor</u>

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{34}$$

or, in chalksaver form,

$$\rho(\dot{u}_i + u_j u_{i,j}) = -P_{,i} + \mu u_{i,jj} \tag{35}$$

d) MSMO Notation*

$$N = 0 \tag{36}$$

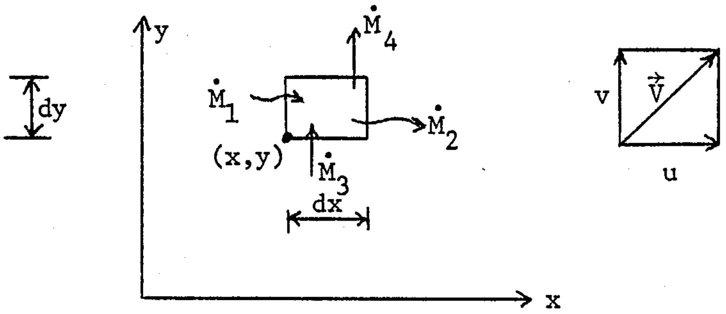
where N represents the above equations.

D. Tips for Deriving PDE's

PDE's are derived by applying the basic principles to elemental control volumes. The choice of control volume depends upon the geometry of interest and the degree of symmetry in the problem under study. For example, one might use cartesian or cylindrical or spherical coordinate systems. Or one might take advantage of radial symmetry to reduce a cylindrical coordinate problem to two dimensions (r and z).

The terms appearing in the basic equations are evaluated to $O(\Delta)$, where Δ is the size of the infinitesimal control volume. To do this requires expansion of the field variable in a Taylor series about some point. This point is usually taken as one corner of the control volume, but it makes absolutely no difference which point is used; the same PDE is derived by any (proper) approach.

To illustrate the ideas, let's derive the equation expressing the conservation of mass for two-dimensional flow in an incompressible fluid. The control volume to be used is shown below:



Since the density is fixed, mass cannot be accumulating inside. So the mass balance yields

$$\dot{M}_1 + \dot{M}_3 = \dot{M}_2 + \dot{M}_4 \tag{37}$$

where the $\dot{M}$ terms represent mass flows across the surfaces indicated. The velocity varies across face 1; to O(dy)** we can express M_1 in terms of the velocity at the center of this face (the point x, y+dy/2). By Taylor's series expansion,

$$u(x,y+dy/2) = u + \frac{\partial u}{\partial y}\frac{dy}{2} + O(dy^2) \tag{38}$$

---

* Maximum Simplicity Maximum Obscurity

** See page 32, Section D for meaning of O( ) .

where it is understood that u and $\partial u/\partial y$ are evaluated at point (x,y). So

$$\dot{M}_1 = \left[u + \frac{\partial u}{\partial y}\frac{dy}{2} + O(d^2)\right]\rho\ dy\cdot 1 \tag{39}$$

The 1 represents the depth of the control volume into the paper. Now, for face 2 we approximate the velocity by its value at the middle of face 2, i.e., at the point (x+dx,y+dy/2),

$$\dot{M}_2 = \left[u + \frac{\partial u}{\partial y}\frac{dy}{2} + \frac{\partial u}{\partial x}\,dx + O(d^2)\right]\rho\ dy\cdot 1 \tag{40}$$

Now, when we subtract (39) from (40), the u term and the term involving dy/2 will cancel and we obtain

$$\dot{M}_2 - \dot{M}_1 = \rho\,\frac{\partial u}{\partial x}\,dxdy\cdot 1 + O(d^3) \tag{41}$$

If we had neglected to properly expand in the y direction (as most books and professors do), we would have gotten the same result, since there is no difference in y between faces 1 and 2 of the control volume.

Similarly, by expanding the vertical velocity field v, we can express

$$\dot{M}_3 = \rho\left[v + \frac{\partial v}{\partial x}\frac{dx}{2} + O(d^2)\right]dx\cdot 1 \tag{42}$$

$$\dot{M}_4 = \rho\left[v + \frac{\partial v}{\partial x}\frac{dx}{2} + \frac{\partial v}{\partial y}\,dy + O(d^2)\right]dy\cdot 1 \tag{43}$$

So

$$\dot{M}_4 - \dot{M}_3 = \rho\,\frac{\partial v}{\partial y}\,dxdy\cdot 1 \tag{44}$$

Note that both terms in the basic equation, i.e., (41) and (44), contain the same factor dxdy. This will always be the case in a proper derivation of a PDE. Now, if we substitute (41) and (44) into the basic mass balance (37) and divide by dxdy, we find

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{45}$$

which is the desired PDE. Note that the PDE comes from the O(d) terms; the O(1) terms will always cancel, and the $O(d^2)$ terms will be negligible for infinitesimal control volumes, in any proper derivation.

If instead we had placed the control volume such that the point (x,y) was <u>centered</u> in face 1, then we would have had

$$\dot{M}_1 = \rho u\ dy\cdot 1 \tag{46a}$$

$$\dot{M}_2 = \rho \left(u + \frac{\partial u}{\partial x} dx \right) dy \cdot 1 \tag{46b}$$

Or

$$\dot{M}_2 - \dot{M}_1 = \rho \frac{\partial u}{\partial x} dxdy \tag{47}$$

as before. If we place the point (x,y) at the <u>center</u> of the box, then

$$\dot{M}_1 = \rho \left(u - \frac{\partial u}{\partial x} \frac{dx}{2} \right) dy \cdot 1 \qquad \dot{M}_2 = \rho \left(u + \frac{\partial u}{\partial x} \frac{dx}{2} \right) dy \cdot 1 \tag{48}$$

or

$$\dot{M}_2 - \dot{M}_1 = \rho \frac{\partial u}{\partial x} dxdy \tag{49}$$

as before. So it does not matter where the expansion point (x,y) is placed in the control volume.

If the fluid were not incompressible, we would need to account for density variations. This requires expansion of the density field. For example,

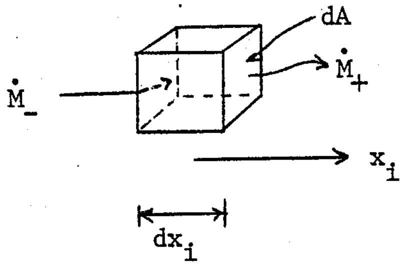
$$\dot{M}_1 = \left[p + \frac{\partial \rho}{\partial y} \frac{dy}{2} + 0(d) \right] \left[u + \frac{\partial u}{\partial y} \frac{dy}{2} + 0(d) \right] \tag{50}$$

which you could multiply out. However, you can save yourself a lot of unnecessary work by instead expanding the product ρu directly,

$$\dot{M}_1 = \rho u + \frac{\partial (\rho u)}{\partial y} \frac{dy}{2} + 0(d^2) \tag{51}$$

You can carry out the multiplications to convince yourself that (46) and (47) are identical. In problems involving product terms such as ρuh, it is always easier to expand the entire term in a Taylor's series rather than expanding term-by-term and multiplying. With a little experience, you will be able to get the <u>difference</u> term directly, e.g., $\partial(\rho uh)/\partial x$.

If we had to do it in three cartesian dimensions, we might find it helpful to get the difference terms in the ith direction, then sum over $i = 1, 2,$ and 3. Thus:



$$\left(\dot{M}_+ - \dot{M}_- \right)_i = \rho \frac{\partial u_i}{\partial x_i} \underbrace{dx_i dA}_{dV} \quad \text{(no summation)} \tag{52}$$

Summing and dividing by ρdV,

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{53}$$

You will have plenty of practice deriving governing ODE's and PDE's in your Thermosciences courses. Save yourself work and trouble by using the simplest notations and procedures applicable to the problem.

E. Basic Equations for Continuity, Momentum, Energy, and Entropy for Fluid Flow

You should apply the methods and definitions given above to derive the following equations for compressible fluid flow. In these equations, u_i and f_i are the velocity and body force per unit of volume in the ith direction, σ_{ij} is the stress in the fluid acting on a plane perpendicular to the i axis in the direction of the j axis, q_i is the heat flux (W/m^2) in the ith direction, e is the energy per unit mass of the fluid (internal energy plus kinetic energy), and p is the entropy production rate per unit volume.

1. Continuity (conservation of mass)

$$\underset{\text{storage rate}}{\frac{\partial \rho}{\partial t}} + \underset{\text{net outflow rate}}{\frac{\partial}{\partial x_j}(\rho u_j)} = 0 \tag{54}$$

2. Momentum (ith direction)

$$\underset{\text{storage rate}}{\frac{\partial}{\partial t}(\rho u_i)} + \underset{\text{net outflow rate}}{\frac{\partial}{\partial x_j}(\rho u_i u_j)} = \underset{\text{net force}}{\frac{\partial}{\partial x_j}(\sigma_{ji}) + f_i} \tag{55}$$

3. Energy (rate per unit)

$$\underset{\text{storage rate}}{\frac{\partial}{\partial t}(\rho e)} + \underset{\substack{\text{net outflow}\\ \text{rate with}\\ \text{mass}}}{\frac{\partial}{\partial x_j}(\rho u_j e)} - \underset{\substack{\text{net inflow}\\ \text{power of}\\ \text{surface}\\ \text{forces}}}{\frac{\partial}{\partial x_j}(u_i \sigma_{ji})} - \underset{\substack{\text{power}\\ \text{input}\\ \text{of body}\\ \text{forces}}}{f_i u_i} + \underset{\substack{\text{net out-}\\ \text{flow power}\\ \text{heat}}}{\frac{\partial}{\partial x_j} q_j} = 0 \tag{56}$$

III. MATHEMATICS REVIEW

The purpose of this section is to remind you of the main mathematical tools that will be used in your Thermosciences courses. The review is necessarily superficial, and covers only those aspects common to several courses. You should make a serious review of complex variables, ODE's, the calculus of functions of several variables, and expansion techniques, to be sure that you are on top of these matters as you begin to Master the Thermosciences.

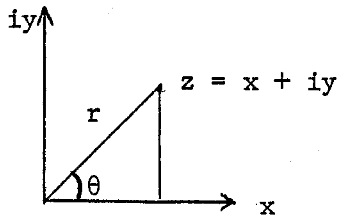
A. Complex Variables

Complex variables are used in the solution of many kinds of physical problems. We expect that you have had some contact with complex variables, for example in a course on electric circuits or control theory; at the graduate level you will also encounter complex variables in the solution of PDE's. This brief review will cover the basic ideas needed in your Thermosciences courses.

1. Definitions and manipulations

A complex variable is a number pair with certain special properties. The pair might be the two coordinates of a two-dimensional cartesian coordinate system (x,y), or two physical variables related to one another in some particular way (velocity potential and stream function, or temperature and heat flux). The two quantities are distinguished from one another by regarding one as a coordinate on the real axis and the other as the coordinate on the y axis,

$$z = x + iy \qquad (1)$$



Consider the Taylor's series for $e^{i\theta}$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^2}{4!} - \ldots + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right]$$

We see that the two series are exactly $\cos\theta$ and $\sin\theta$. So

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Defining $r = \sqrt{x^2+y^2} = |z|$, it is evident that $x = r\cos\theta$, $y = r\sin\theta$, so that an alternative way to represent the complex variable z is

$$z = re^{i\theta} \qquad (2)$$

The polar form (2) is most useful for manipulations involving products of complex variables. The cartesian form (1) is most useful for operations involving sums of complex variables. Thus, for the product of two complex numbers,

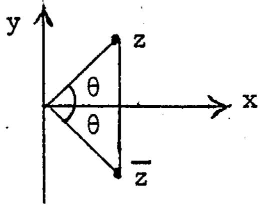
$$z_1 z_2 = r_1 e^{i\theta_1} \quad r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} \tag{3}$$

and for addition

$$z_1 + z_2 = (x_1+iy_1) + (x_2+iy_2) = (x_1+x_2) + i(y_1+y_2) \tag{4}$$

Note that addition of complex numbers corresponds to vector addition in the complex plane.

The complex conjugate $\bar{z}$ of a complex variable $z = x+iy$ is defined by



$$\bar{z} = x - iy = re^{-i\theta} \tag{5}$$

Note that

$$z\bar{z} = r^2 = |z|^2 \tag{6}$$

2. Functions of a complex variable

Functions of complex variables also have real and imaginary parts. For example,

$$f(z) = z^2 = (x^2-y^2) + i2xy = G(x,y) + iH(x,y) \tag{7}$$

Note that the real functions G and H are not, in general, themselves expressible in terms of z .

A function of a complex variable is called analytic if f(z) and df/dz exist (are uniquely definable and finite) and single valued except possibly at selected singular points. For analyticity, df/dz must be independent of the direction of the vector dz . In other words, if $dz = \alpha dx + i\beta dy$, df/dz must have the same value for any α and β . This will be true if G and H satisfy certain conditions. First let $\beta = 0$, $\alpha = 1$. Then, if $f = G + iH$,

$$\frac{df}{dz} = \frac{df}{dx} = \frac{\partial G}{\partial x} + i\frac{\partial H}{\partial x} \tag{8}$$

Now, let $\beta = 1$ and $\alpha = 0$. Then,

$$\frac{df}{dz} = \frac{df}{idy} = -i\frac{df}{dy} = -i\frac{\partial G}{\partial y} + \frac{\partial H}{\partial y} \tag{9}$$

Equating the real and imaginary parts of (8) and (9),

$$\frac{\partial G}{\partial x} = \frac{\partial H}{\partial y} \qquad \frac{\partial H}{\partial x} = -\frac{\partial G}{\partial y} \tag{10a,b}$$

These are the Cauchy-Riemann equations; they must be satisfied (except at singular points) by any analytic function of a complex variable.

Differentiating (10a) with respect to x, and using (10b),

$$\frac{\partial^2 G}{\partial x^2} = \frac{\partial^2 H}{\partial x \partial y} = -\frac{\partial^2 G}{\partial y^2}$$

So,

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0 \qquad (\nabla^2 G = 0) \tag{11}$$

Similarily, you should prove that

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0 \qquad (\nabla^2 H = 0) \tag{12}$$

We see that both the real and imaginary parts of a function of a complex variable satisfy the Laplace equation. Analytic functions of complex variables therefore are used to construct solutions to physical problems described by Laplace equations.

For example, in two-dimensional conduction heat transfer, the temperature field is described by

$$\nabla^2 T = 0 \tag{13}$$

So, the real or imaginary part of any analytic function of a complex variable is the solution to some temperature problem.

In fluid flow idealized as incompressible and irrotational, one deals with the complex potential $\Phi(z)$

$$\Phi(z) = \phi + i\psi \tag{14}$$

The real part ϕ is the velocity potential, and the imaginary part ψ is the stream function. Both ϕ and ψ satisfy Laplace's equation. Solution to ideal fluid flow problems are constructed by combining appropriate functions of a complex variable to produce a solution that satisfies the boundary conditions.

You will use these ideas in several Thermosciences courses. If you plan a Ph.D. in the Thermosciences, a course in complex variables, from the mathematics department is recommended.

B. Ordinary Differential Equations

Ordinary differential equations arise in the solution of one-dimensional physical problems. They also arise during the course of the solution of partial differential equations. You must be ready to deal comfortably, quickly, and precisely with ODE's if you are to Master in the Thermosciences.

1. First order equations

These don't arise very often. Those that do are generally solvable by separating the variables and integrating. Linear equations are solved by the general methods for linear equations of any order.

2. Linear ODE's

We can express an arbitrary linear ODE of nth order by

$$L_n y = f \tag{15}$$

The differential operator L_n is a linear, of the form

$$L_n = a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \dots + a_o \tag{16}$$

where the coefficients a_i are functions of x ; the right hand side, $f(x)$, makes the equation inhomogeneous. An equation is homogeneous (in the dependent variable) if the dependent variable y can be replaced by Cy without change in the equation.

You should recall that the general solution of an nth order linear ODE consists of the sum of a particular solution, which takes care of the inhomogeneous term f , and the general solution of the associated homogeneous equation $L_n y = 0$. The general solution of the homogeneous equation will be the sum of n linearly independent solutions y_i . Hence, the complete solution of the inhomogeneous equation is (subscript summation convention used)

$$y = C_i y_i + y_p \tag{17}$$

where the homogeneous solutions satisfy $L_n y_i = 0$ and the particular solution satisfies $L_n y_p = f$. The boundary conditions only are used to determine the n constants C_i .

In solving linear ODE's encountered in real problems, the homogeneous and particular solutions may be generated analytically or numerically. The general structure of the solution, as outlined above, is particularly helpful in constructing a numerical solution with minimum effort.

If the coefficients a_i are all constant, the solutions to the homogeneous equations will be of exponential form. Assuming $y = e^{px}$, $L_n y = 0$ gives

$$\sum_{i=0}^{n} a_n p^n = 0 \tag{18}$$

The n roots of this equation define the n linearly independent homogeneous solutions. If one root is repeated, then the solution corresponding to the repeated root will be of the form xe^{px} .

Second order equations are most common. Two special cases which arise very frequently are

$$y'' + \lambda^2 y = 0 \tag{19}$$

$$y'' - \lambda^2 y = 0 \tag{20}$$

Equation (19) gives $p^2 + \lambda^2 = 0$, so $p = \pm i$, and $y = e^{\pm i\lambda x}$. Thus, the general solution of (19) may be written in a variety of ways, each of which is useful for particular type of boundary conditions:

$$y = C_1 e^{i\lambda x} + C_2 e^{-i\lambda x} \tag{21a}$$

$$y = C_3 \sin(\lambda x) + C_4 \cos(\lambda x) \tag{21b}$$

$$y = C_5 \sin[\lambda(x-L)] + C_6 \cos[\lambda(x-L)] \tag{21c}$$

Similiarly, for (20) $p^2 - \lambda^2 = 0$, so $p = \pm\lambda$, and the solution can be written as

$$y = c_1 e^{\lambda x} + C_2 e^{-\lambda x} \tag{22a}$$

$$y = C_3 \sinh(\lambda x) + C_4 \cosh(\lambda x) \tag{22b}$$

$$y = C_5 \sinh[\lambda(x-L)] + C_6 \cosh[\lambda(x-L)] \tag{22c}$$

Second order homogeneous equations with non-constant coefficients have solutions in terms of "special functions; for example, Bessel functions, Legendre polynomials, spherical harmonics, hypergeometric functions, etc. One good source for information on these functions is:

> Abromowitz and Stegun, Handbook of Mathematical Functions, Publication 55 of the Amer. Math. Society (Dover Reprint).

If you have not yet encountered these functions, don't fear them. Just regard them as you would the special functions with which you are familiar, such as $\sin x$, e^x , and $\cosh x$. The special functions have series expansions, which are developed from the equations. It will take two linearly independent solutions to complete the solution to a homogeneous second-order differential equation; all the information you need to solve problems is in the handbooks. For example, the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 \qquad (23)$$

has the general solution

$$y = C_1 J_o(x) + C_2 Y_o(x) \qquad (24)$$

where $J_o(x)$ and $Y_o(x)$ are Bessel functions. To apply boundary conditions at $x = 1$ and $x = 2$ you would simply use this solution and look up $J_o(1)$, $J_o(2)$, $Y_o(1)$, $Y_o(2)$, just as you would if the solution were $C_1 \sin x + C_2 \cos x$. If you have not encountered special functions, you should take a graduate course on PDE.

Particular solutions of linear ODE's can be obtained by a variety of methods. If the right-hand side f is a simple polynomial, the particular solution will be a polynomial and you can assume $y_p = \sum b_m x^m$ and determine the b_n by matching coefficients of powers of x . Similar games can be played if the right-hand side of a constant-coefficients equation is of exponential form. Often it is helpful to split the right-hand side up into several parts, $f = f_1 + f_2 + \ldots$, and then construct separate solutions for each part of f . You should see that the complete particular solution is simply the sum of these partial particular solutions.

If you cannot construct a particular solution by direct methods like this, there is a technique that always will work (but it is to messy for simple problems). This is the method of variation of parameters. The idea is to assume a particular solution of the form

$$y_p = A_i(x) y_i(x) \quad \text{(summation convention)} \qquad (25)$$

where the functions $y_i(x)$ are the n linearly independent solutions of the homogeneous equation $L_n y = 0$. Differentiating,

$$y_p' = A_i y_i' + A_i' y_i$$

We cleverly choose the A_i such that

$$A_i' y_i = 0 \tag{26a}$$

Then,

$$y_p'' = A_i y_i'' + A_i' y_i'$$

Again (if n >2) we choose

$$A_i' y_i' = 0 \tag{26b}$$

This process is repeated up through $y^{(n-1)}$, giving us n-1 equations constraining the A_i'. Finally, we substitute into the homogeneous ODE. Since

$$y_p^{(n)} = A_i y_i^{(n)} + A_i' y_i^{(n-1)}$$

and since $L_n y_i = 0$, the ODE gives

$$A_i' \sum_{k=0}^{n} a_k y_i^{(k-1)} = f \tag{27}$$

Equation (27) and the n-1 equations (26) form a set of n <u>linear algebraic</u> equations for the A_i'. These can be solved by linear algebra, to give $A_i' = g_i(x)$. One then integrates these to get A_i (any integral will give a particular solution, so the limits of integration are arbitrary). So, the method permits construction of a particular solution for any linear ODE. If special functions are involved, the particular solutions will be messy integrals of these functions, and it is not likely that a solution can be dealt with analytically. However, numerical particular solutions can always be used, with analytical or numerical homogeneous solutions.

Systems of linear ODE's are treated in a similar manner. In particular, if the coefficients are constant the homogeneous solutions will be of exponential form, with roots given by the roots of a system of algebraic equations.

3. Systems of first order equations

These arise directly in the solution of lumped-parameter transient heat transfer problems, and elsewhere. The general form may be written as

$$y_i' = f_i(x,\underline{y}) \qquad i = 1,2,\ldots,n \tag{28}$$

Note that f could be a linear or a nonlinear function of the dependent variable vector $\underline{y}$.

Any ODE may be reduced to a system of first-order ODE's by defining the variables appropriately. For example, consider

$$u''' + 2uu' = 0 \tag{29}$$

let

$$\begin{aligned} y_1 &= u \\ y_2 &= u' \\ y_3 &= u'' \end{aligned} \tag{30}$$

Then we have the three equations

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= -2y_1y_2 \end{aligned} \tag{31}$$

Computer programs exist for solution of systems of first-order ODE initial value problems in the form of (28). These are very simple to use. The user has to write a subroutine that describes the functions on the right-hand side, a driver program to call the integrating subroutine, and provide initial values for the dependent variables y_i . The computer does the rest. In sophisticated routines the accuracy is automatically adjusted to keep the solution within a preset error. This type of computer program is used to solve the complex planetary orbital calculations for satellite dynamics. It can also be used to solve any ODE arising in the Thermosciences (it is not the best way if an analytical solution is possible), such as the Blasius boundary layer equation (29). A variety of routines of this type are available at the Stanford Computation Center.

For example, suppose one such routine is FODES(N,Y,DSUB,XO,XMAX). This routine solves a system of N first-order equations, constructing the solution in Y . DSUB is a subroutine, written by the user, that specifies the right hand sides $f_i(x,\underline{y})$. XO and XMAX are the initial and final values of X . The initial values for Y(I) must be loaded before the routine FODES is called. To solve the problem outlined above, we would write a simple program as follows:

```
SUBROUTINE DSUB (X,Y,F)
DIMENSION Y(3),F(3)
F(1)=Y(2)
F(2)=Y(3)
F(3)=-2Y(1)*Y(2)
RETURN
END

DIMENSION Y(3)
EXTERNAL DSUB
Y(1)=0
Y(2)=0
Y(3)=-1.0
CALL FODES(3,Y,DSUB,0.,10.)
STOP
END
```

FODES would carry out the calculation as an initial value problem, printing out the values of X and the Y(I) from X=0 to X=10.

Programs like this solve initial value problems. If a boundary value problem is involved, and the problem is linear, one can construct the appropriate number of linearly independent homogeneous solutions using arbitrary initial conditions, a particular solution, and then combine them as in (17) to get the general solution. The constants would then be determined by the actual boundary conditions.

C. The Calculus of Functions of Several Variables

PDE solutions involve the calculus of functions of more than one variable. We presume that you are acquainted with the concept of a partial derivative,

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \left\{ \frac{f(x+\Delta x,y) - f(x,y)}{\Delta x} \right\}$$

Note that the other independent variable(s) are treated as constants in taking a partial derivative with respect to one of the independent variables.

The total differential of a function of several variables is

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots = \frac{\partial f}{\partial x_i} dx_i \qquad (33)$$

Changes of variables are often necessary. This is accomplished through the chain rule. For example, if

$$\xi = \xi(x,y) \qquad (34a)$$

$$\eta = \eta(x,y) \qquad (34b)$$

Then to transform an equation for $f(x,y)$ to one for $f(\xi,\eta)$ one uses

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} \tag{35a}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} \tag{35b}$$

As an exercise, you should practice by transforming Laplace's equation from Cartesian coordinates to cylindrical coordinates. This is messy, but not impossible, if you are careful while doing the differentiations.

D. Epsilontics

Epsilontics refers to the analysis of small terms. It is a technique for dealing with indeterminant forms that is superior to L'Hopital's rule, and it is very important in approximation analysis.

The "order symbol" $0(x)$ is used to keep tract of the size of terms. If we say that $f(\varepsilon) = 0(g(\varepsilon))$, we mean that

$$\frac{f(\varepsilon)}{g(\varepsilon)} \rightarrow \text{constant as} \quad \varepsilon \rightarrow 0 \tag{36}$$

For example, $\sin\varepsilon = 0(\varepsilon)$; $\cos \varepsilon = 0(1)$; $1 - e^{\varepsilon^2} = 0(\varepsilon^2)$.

The 0 symbol is used in expansions to tell the size of the remainder:

$$e^x = 1 + x + x^2 + 0(x^3) \tag{37a}$$

$$\sin x = x - \frac{x^3}{6} + 0(x^5) \tag{37b}$$

If you need to evaluate an indeterminate form, don't mess around with L'Hopital's rule. Just expand the numerator and denominator in Taylor's series about the point in question, and examine the ratio in the neighborhood of the point. For example,

$$\frac{\sin x}{x} = \frac{x + 0(x^3)}{x} = 1 + 0(x^2) \tag{38}$$

so, $(\sin x)/x \rightarrow 1$ as $x \rightarrow 0$.

There is a powerful method of analytical solution called perturbation analysis. The general idea is to find an expansion of the solution about some special point where the solution is known or easily found. for example, consider the equation

$$y'' + \varepsilon y^2 + y = 0 \tag{39}$$

4. Entropy

$$\mathcal{P} = \frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x_j}(\rho u_j s) + \frac{\partial}{\partial x_j}(q_j/T) \qquad (57)$$

entropy production rate	storate rate	net outflow rate with mass	net outflow rate with heat

5. Mechanical energy

An equation describing the <u>mechanical</u> energy flows may be derived by multiplying the momentum equation by u_i (and summing), using continuity. (In some textbooks this is stated as a fundamental energy balance, but it is a derived result.)

$$\frac{\partial}{\partial t}(\rho V^2/2) + \frac{\partial}{\partial x_j}(u_j \rho V^2/2) = f_i u_i + u_i \frac{\partial}{\partial x_j}(\sigma_{ji}) \qquad (58)$$

net K.E. storage rate	net K.E. outflow rate	power input by body forces	power input by surface forces

6. Thermal energy

An equation governing the thermal energy may be obtained by subtracting this equation from the basic energy equation. This is sometimes stated in books as a fundamental equation, but it is a derived result. In terms of the enthalpy h,

$$e = h - P/\rho + V^2/2$$

this equation, and its interpretation, is

$$\underbrace{\frac{\partial(\rho h)}{\partial t} - \frac{\partial P}{\partial t}} + \frac{\partial}{\partial x_j}(u_j \rho h) = \frac{\partial}{\partial x_j}(u_j P) - \frac{\partial q_j}{\partial x_j} + \sigma_{ji}\frac{\partial u_i}{\partial x_j} \qquad (59)$$

net thermal storage rate	net thermal outflow rate	power input by reversible expansion	net heat outflow rate	power input from viscous dissipation

This equation is often used as the starting point for development of equations governing the temperature field in fluid flows.

7. Irreversibility

Using the thermodynamic relation (the Gibbs equation)

$$Tds = dh - dP/\rho$$

(57) and (59) combine to yield

$$\underset{\substack{\text{rate of}\\ \text{entropy}\\ \text{production}\\ \text{per cent}\\ \text{volume}}}{p} = \underset{\substack{\text{irrever-}\\ \text{sibility}\\ \text{rate due}\\ \text{to vis-}\\ \text{cous}\\ \text{stresses}}}{\frac{1}{T}\sigma'_{ji}\frac{\partial u_i}{\partial x_j}} - \underset{\substack{\text{irrever-}\\ \text{sibility}\\ \text{rate due}\\ \text{to heat}\\ \text{transfer}}}{\frac{q_j}{T^2}\frac{\partial T}{\partial x_j}} \geq 0 \qquad (60)$$

Note that the convective terms do not contribute to the irreversibility; irreversibility arises only as a result of viscous stresses or heat transfer. σ'_{ji} is the viscous part of the total stress tensor.

This equation is used as the basis for setting constraints on the constitutive relationships between stress and strain-rate, and between heat flux and temperature gradient; these constitutive relationships must be such as to keep (60) satisfied for all possible fields.

For $\varepsilon > 0$ the problem is non-linear and an exact solution is impossible. However, for small ε (light damping) we can obtain a solution, at least for x not too large, in a Taylor's series about $\varepsilon = 0$.

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + 0(\varepsilon^3) \tag{40}$$

Substituting (39) into (40), and demanding that the equation be satisfied to each order of ε , one obtains a hierarchy of equations

$$y_0'' + y_0 = 0 \tag{41}$$

$$y_1'' + y_1 = -y_0^2 \tag{42}$$

$$y_2'' + y_2 = -2y_0 y_1 \tag{43}$$

Equation (41) is easily solved; the higher-order equations (42), (43), etc. are all linear, so they can be solved. Thus, the solution is obtained by purely analytical methods.

This particular solution breaks down after some time (or x) , because the higher-order solutions become vary large. It is possible to construct a solution method that is valid for all times (uniformly valid). One has to be particularly clever in the epsilontics. Professor Van Dyke gives an excellent course, Perturbation Methods in Fluid Mechanics, which is highly recommended to students who have a good background in PDE's and who are interested in analysis of this type. Professor Keller's courses, Methods of Mathematical Physics, also deal with these topics.

The Complex Potential Function for Application to the Solution of Incompressible, Irrotational, Two-Dimensional Flow

J. P. Johnston

October 1968

Introduction

Some of the most powerful methods for solution of problems in inviscid, incompressible, two-dimensional flow involve the use of complex functions. The purpose of these notes is to: (i) review some basic theorems on the use of complex numbers, variables, and functions; (ii) introduce the concept of an analytic function of a complex variable; (iii) show how the potential function and stream function for two-dimensional, irrotational, incompressible flow are combined to form the complex potential function; and, (iv) review the characteristics of a number of the basic functions which may be linearly superposed to give various desired flow solutions.

Review of Complex Numbers*

One may describe the position of a point, P , in Cartesian Coordinates (x, y-plane) by the use of the complex number z . The x, y-plane may be referred to as the z-plane or physical plane.

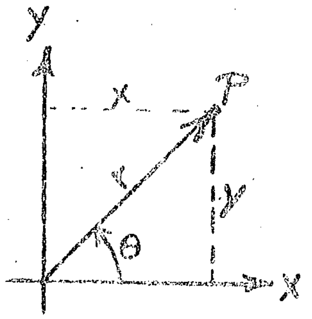


Fig. 1 The z-plane

By definition:

$x \triangleq$ real axis

$y \triangleq$ imaginary axis

and $z \triangleq x + iy$ (1)

where i . is the imaginary number.

$i \quad \sqrt{-1}$ so that $i^2 = -1$, $i^3 = -\sqrt{-1}$, $1/i = -i$, etc.
The absolute value of z is,

$$|z| = |x + iy| = \left| \sqrt{x^2 + y^2} \right| = r \qquad (2)$$

---

[*] See Churchill, Complex Variables and Applications, McGraw-Hill.

From the geometry of Fig. 1 it is seen that

$$r \cos \theta = x$$

$$r \sin \theta = y$$

In polar (r, θ) coordinates z is given by

$$z = x + iy = r(\cos \theta + i \sin \theta)$$

One may show by expansion in power series of $\cos \theta$, $\sin \theta$, and $e^{i\theta}$ that

$$\left.\begin{aligned} e^{i\theta} &= \cos \theta + i \sin \theta \\ e^{-i\theta} &= \cos \theta - i \sin \theta \end{aligned}\right\} \qquad (3)$$

Thus one obtains the useful form

$$z = re^{i\theta} \qquad (4)$$

The complex number, z, is said to have a real part and an imaginary part:

$x = R(z)$, the real part of z and

$y = I(z)$, the imaginary part of z.

Two complex numbers are equal if their real and imaginary parts are equal. Hence if

$$z_1 = x_1 + iy_1 = r_1 \; e^{i\theta_1}$$

$$z_2 = x_2 + iy_2 = r_2 \; e^{i\theta_2}$$

then $z_1 = z_2$ if $x_1 = x_2$ and $y_1 = y_2$ or equivalently if $r_1 = r_2$ and $\theta_1 = \theta_2$.

Addition and subtraction of complex numbers corresponds to vector addition in the z-plane, see Fig. 2.

$$z_3 = z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$

so

$$z_3 = (x_1 + x_2) + i(y_1 + y_2) \qquad (5)$$

Multiplication of complex numbers, see Fig. 3, is given by

$$z_3 = (z_1)(z_2) = (x_1 + iy_1)(x_2 + iy_2) \ .$$

$$z_3 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_2) \ .$$

and in the case of r , θ coordinates by,

$$z_3 = (z_1)(z_2) = (r_1e^{i\theta_1})(r_2e^{i\theta_2})$$

$$z_3 = r_1r_2e^{i(\theta_1 + \theta_2)} \tag{7}$$

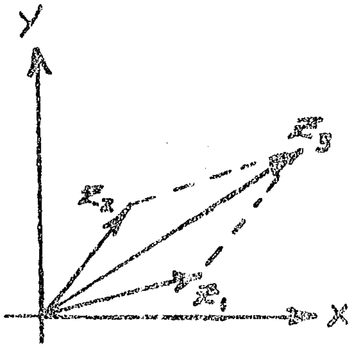


Fig. 2 Addition

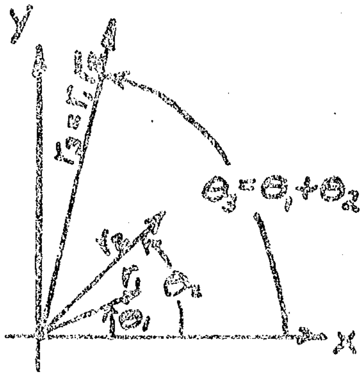


Fig. 3 Multiplication

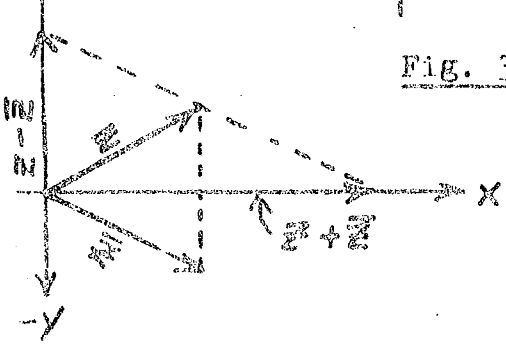


Fig. 4 The complex number z and its conjugate $\bar{z}$

The conjugate, $\bar{z}$, of a complex number is defined as

$$\bar{z} \triangleq x - iy = re^{-i\theta} \tag{8}$$

From this definition it is seen that (see Fig. 4),

$$z + \bar{z} = 2x$$

$$z - \bar{z} = i(2y)$$

$$(z)(\bar{z}) = x^2 + y^2 = r^2$$

From the latter expression and eq. (2) it is seen that the absolute value of a complex number is

$$|z| = \left|\sqrt{(z)(\bar{z})}\right| = r \qquad (9)$$

One may construct functions of complex numbers (or variables). Such functions of z have real and imaginary parts, i.e.,

$$f(z) = R[f(z)] + i\, I[f(z)]$$

The real and imaginary parts of f(z) are each real, algegraic functions of x and y (or r and θ). The rules already outlined for manipulation of the complex number (or variable), z , also hold for the complex function. For example, the conjugate of f(z) is written as

$$\overline{f(z)} = R[f(z)] - i\, I[f(z)]$$

Analytic Function of a Complex Variable

A function f(z) is said to be analytic if f(z) and (df/dz) exist (i.e., are finite) and single valued except possibly at a finite number of singular points. To require that (df/dz) be finite and single valued is equivalent to requiring that the limit

$$\frac{df}{dz} \triangleq \lim_{\Delta z \to 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z}\right] = \lim_{\Delta z \to 0} \left[\frac{\Delta f}{\Delta z}\right]$$

exist uniquely as $\Delta z \to 0$ from any direction in the complex (z) plane. An analytic function thus has this property.

For analytic function f(z) = G(x, y) + i H(x, y) where

G(x, y) is real part of f(x)

H(x, y) is imaginary part of f(x)

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \left(\frac{\Delta f}{\Delta z}\right) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{\Delta G + i\Delta H}{\Delta x + i\Delta y}\right)$$

Since Δz may approach zero from any direction if f is analytic, let $\Delta z \to 0$ along a line parallel to the x (real) axis. Then y = constant and $\Delta y = 0$

$$\left.\frac{df}{dz}\right|_{y=const} = \lim_{\Delta x \to 0} \left(\frac{\Delta G + i\Delta H}{\Delta x}\right) = \lim_{\Delta x \to 0} \left(\frac{\Delta G}{\Delta x} + i\,\frac{\Delta H}{\Delta x}\right) = \frac{\partial G}{\partial x} + i\,\frac{\partial H}{\partial x}$$

also
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\,(G + i\,H) = \frac{\partial G}{\partial x} + i\,\frac{\partial H}{\partial x}$$

so that
$$\left.\frac{df}{dz}\right|_{y=const} = \frac{\partial f}{\partial x} \qquad (10)$$

Let $\Delta z \to 0$ approach zero along the imaginary, y, axis. In this case x = constant and $\Delta x = 0$.

$$\left.\frac{df}{dz}\right|_{x=const} = \lim_{\Delta y \to 0} \left[\frac{\Delta G + i\Delta H}{i\Delta y}\right] = \lim_{\Delta y \to 0} \left[\frac{\Delta G}{i\Delta y} + \frac{i\Delta H}{i\Delta y}\right] = -i\,\frac{\partial G}{\partial y} + \frac{\partial H}{\partial y}$$

but
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\,(G + iH) = \frac{\partial G}{\partial y} + i\,\frac{\partial H}{\partial y}$$

Multiplying this equation by $-i$ and comparing it to $\left.\frac{df}{dz}\right|_{x=const}$ gives as a result:

$$\left.\frac{df}{dz}\right|_{x=const} = -i\,\frac{\partial f}{\partial y} \qquad (11)$$

Since f is assumed to be analytic, df/dz must be single valued or

$$\left.\frac{df}{dz}\right|_{y=const} = \left.\frac{df}{dz}\right|_{x=const}$$

or from eqs. (10) and (11) one obtains

$$\frac{\partial f}{\partial x} = -i\,\frac{\partial f}{\partial y}$$

Expanding the above in terms of the functions F(x, y) and H(x, y) one obtains

$$\frac{\partial G}{\partial x} + i\,\frac{\partial H}{\partial x} = -i\,\frac{\partial G}{\partial y} + \frac{\partial H}{\partial y}$$

Equating the real and imaginary parts of this equation yields the Cauchy-Riemann Equations:

$$\boxed{\frac{\partial G}{\partial x} = \frac{\partial H}{\partial y} \quad \text{and} \quad \frac{\partial G}{\partial y} = -\frac{\partial H}{\partial x}} \qquad (12)$$

which limit the types of functions G and H which form f(z). G and H are scaler point functions of x and y. Hence

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 G}{\partial x \partial y} = \frac{\partial^2 G}{\partial y \partial x}$$

Combining these requirements with eqs. (12) gives

$$\left.\begin{aligned} \nabla^2 H &= \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0 \\ \nabla^2 G &= \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0 \end{aligned}\right\} \qquad (13)$$

As a result it is seen that both the real and imaginary parts of an analytic function of a complex variable are solutions of Laplace's Equation.

The Complex Potential Function, $\Phi(z)$

For incompressible flow one can define two scaler point functions $\phi(x, y)$ and $\psi(x, y)$ if the flow is two-dimensional and irrotational.

$\phi(x, y) \triangleq$ Velocity potential

$\psi(x, y) \triangleq$ Stream function

By definition, in terms of the x and y fluid velocity components

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \triangleq u \quad \text{and} \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \triangleq v$$

The continuity (conservation of mass) condition and the irrotationality condition require that

$$\nabla^2\phi = 0 \quad ,\text{and} \quad \nabla^2\psi = 0$$

It is therefore seen that one can construct a <u>complex potential function</u>

$$\Phi(z) = \phi(x, y) + i\ \psi(x, y) \tag{14}$$

and $\Phi(x)$ <u>will be an analytic function of z</u> since by construction ϕ and ψ satisfy eqs. (12) and are solutions of Laplace's Equation.

From eqs. (10) and (11) it is seen that

$$\frac{d\Phi}{dz} = \frac{\partial\Phi}{\partial x} = -i\ \frac{\partial\Phi}{\partial y}$$

and thus

$$\frac{d\Phi}{dz} = \frac{\partial\phi}{\partial x} + i\ \frac{\partial\psi}{\partial x} = -i\ \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial y} = -i\ v + u$$

New functions $\bar{w}$ and w are defined so that

$$\frac{d\Phi}{dz} = u - iv \triangleq \bar{w}$$

and

$$w \triangleq u + iv \tag{15}$$

$\bar{w}$ is the conjugate of the complex velocity, w. w and $\bar{w}$ may be treated like any other complex numbers in the <u>Hodograph Plane</u>, Fig. 5.

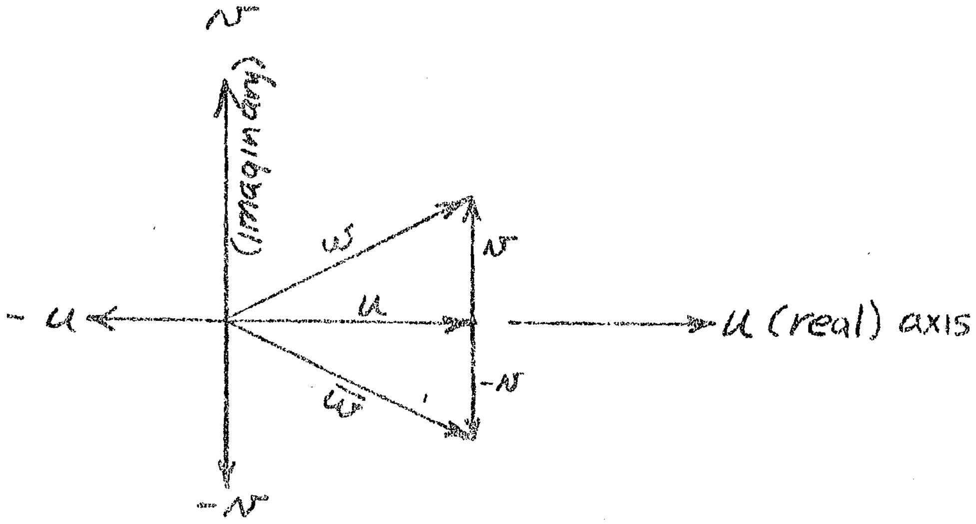


Fig. 5 Hodograph Plane

Elementary Complex Potential Functions (Basic Flows)

Here we will describe a few commonly used analytic functions which form the basic flows. These elementary functions may be linearly superposed since the real and imaginary parts of the functions are solutions to Laplace's Equation. Even with a short catalogue of basic flows, many different flow solutions may be formed.

1. Uniform Parallel Flow may be described by

$$\boxed{\Phi(z) = (a + ib)z} \qquad (16)$$

This expands to (a and b are constants):

$$\Phi(z) = (a + ib)(x + iy) = (ax - by) + i(ay + bx)$$

Hence following eq. (14)

$$\phi = ax - by \quad \text{and}$$

$$\psi = ay + bx$$

A streamline is a line of constant ψ, so $\psi = ay + bx = \text{const}$ on a streamline, and $d\psi = 0 = ady + bdx$ along a streamline. Thus the slope of a streamline is;

$$\left[\frac{dy}{dx}\right]_{\text{streamline}} = -\frac{b}{a}$$

Also it is seen that from

$$\frac{d\Phi}{dz} = a + ib = \bar{w}$$

and $\bar{w} = u - iv$

thus $a = u$

$b = -v$

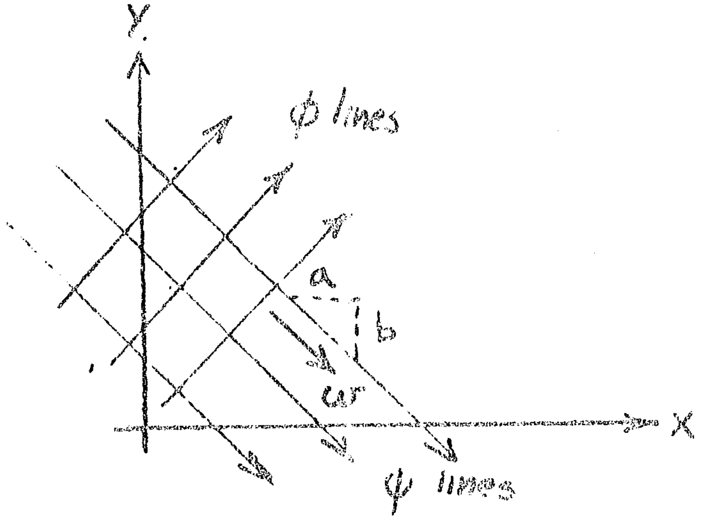


Fig. 6 ϕ and ψ for uniform parallel flow (a and b positive constants)

For flow uniform and parallel to x axis at velocity w one can set $b = 0$ and $a = w$ so

$$\Phi(z) = w\,z \qquad (17)$$

If we want to tilt this flow at any angle α to the x axis

$$\Phi(z) = w\,e^{-i\alpha}z \quad (18)$$

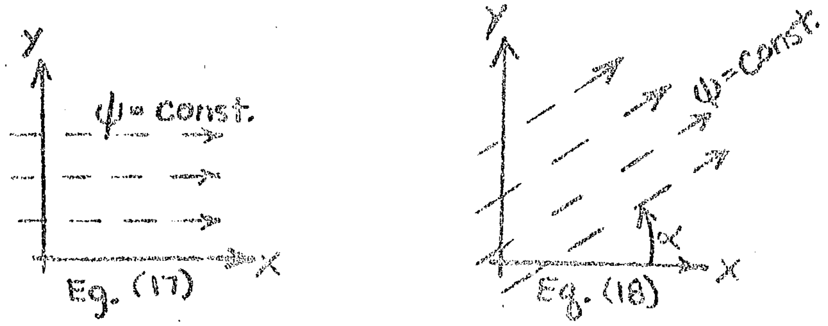


Fig. 7

2. Source and Sink Flow is described by the function

$$\boxed{\Phi(z) = \frac{Q}{2\pi}\ \ell n\ z} \quad (19)$$

where the constant Q is called the source or sink "strength" ($Q > 0$ if source; $Q < 0$ if sink).

Since $z = r\,e^{i\theta}$

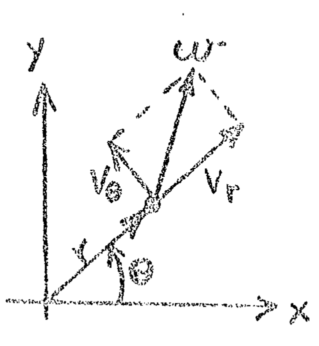


Fig. 8 Polar Components of velocity

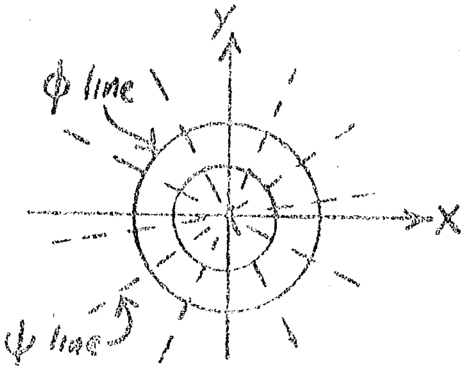


Fig. 9 ϕ and ψ for source and sink one may expand $\Phi(z)$ to read

$$\Phi(z) = \frac{Q}{2\pi}\,[\ell n\ r + i\theta] = \phi + i\psi$$

and thus $\phi = \frac{Q}{2\pi}\,\ell n\ r$ and $\psi = \frac{Q}{2\pi}\,\theta$

It may be shown that the polar velocity components, Fig. 8, may be obtained from

$$\left.\begin{aligned} V_r &= \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \cancel{\frac{\partial\phi}{\partial\phi}} = \frac{\partial\phi}{\partial r} \\ V_\theta &= -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \end{aligned}\right\} \qquad (20)$$

For source and sink flow it is seen that

$$V_\theta = 0$$

$$V_r = \frac{Q}{2\pi r}$$

All flow is radial at volume flow rate Q .

3. <u>The Point Vortex Flow</u> is described by the function

$$\Phi(z) = -i\,\frac{\Gamma}{2\pi}\,\ell n\ z = -i\,\frac{\Gamma}{2\pi}\,[\ell n\ r + i\theta] \qquad (21)$$

where Γ is the circulation for a circuit surrounding $z = 0$. The potential and streamfunctions are

$\phi = \frac{\Gamma}{2\pi}\,\theta$ and

$\psi = -\frac{\Gamma}{2\pi}\,\ell n\ r$

Eqs. (20) give

$V_r = 0$

$V_\theta = \frac{\Gamma}{2\pi r}$

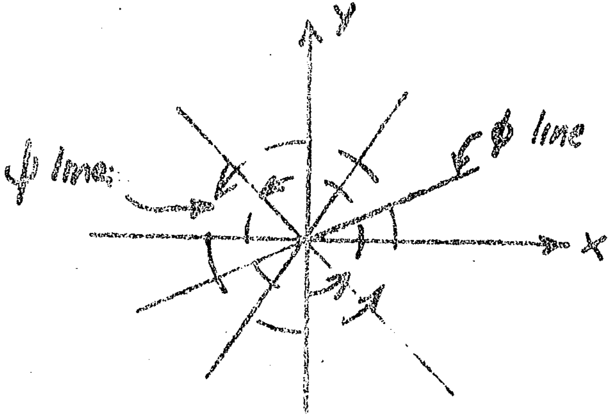


Fig. 10 ϕ and ψ lines for point vortex, arrows on ψ lines give flow direction for Γ positive

Taking a ψ (streamline) line as a circuit to calculate the circulation gives

$$\Gamma = 2\pi r\ V_\theta$$

so Γ is the circulation.

Note that $\Phi(z)$ is not analytic at $z = 0$ since

$$\frac{d\Phi}{dz} = \bar{w} = -i\ \frac{\Gamma}{2\pi}\ \frac{1}{z}$$

For finite Γ, $\bar{w}$ and w are infinite at $z = 0$. Since the whole flow is around the origin, $z = 0$, is irrotational all vorticity associated with Γ is, concentrated at the singular point $z = 0$.

4. The Doublet (dipole) may be constructed by the linear superposition of a sink of strength $-Q$ at $x = a$ and source of strength Q at $x = -a$ (Q and a are real positive constants). The strength of a doublet is defined as $S = Qa/\pi$. S is held constant in a limiting process as the constant a is caused to approach zero, i.e., source and sink are brought together so that they produce a new $\Phi(z)$.

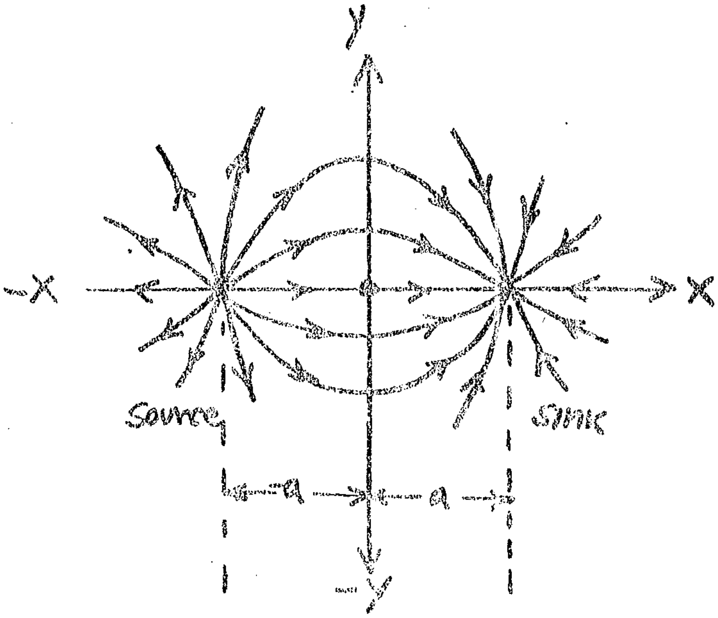


Fig. 11 Superposed source and sink (ψ lines)

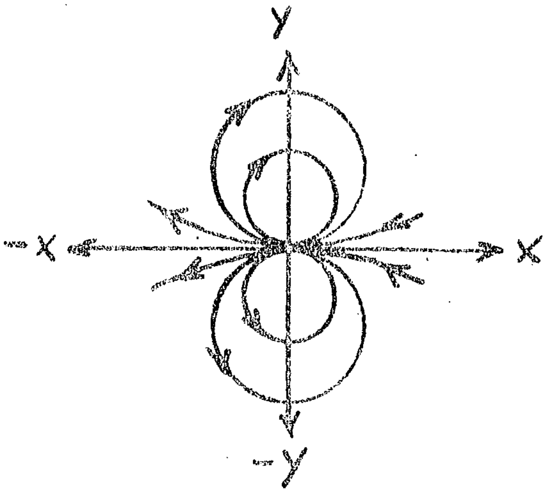


Fig. 12 Doublet (ψ lines)

The complex potential for the source and sink, Fig. 11, is

$$\Phi_S(z) = \frac{Q}{2\pi}\ln(z + a) - \frac{Q}{2\pi}\ln(z - a) = \frac{S}{2a}\ln\frac{(z + a)}{(z - a)}$$

The complex potential of the doublet is

$$\Phi(z) = \lim_{\substack{a \to 0 \\ S=\text{const}}} \Phi_S(z) = S \lim_{a \to 0}\left[\frac{\ln(z + a) - \ln(z - a)}{2a}\right] = \frac{0}{0}$$

which is indeterminate. Apply L'Hospital's Rule

$$\Phi(z) = S \lim_{a \to 0}\left[\frac{\frac{1}{(z+a)} - \frac{(-1)}{(z-a)}}{2}\right] = S \lim_{a \to 0}\left[\frac{(z-a) + (z+a)}{2(z+a)(z-a)}\right] = S\frac{2z}{2z^2}$$

Hence the complex potential of the doublet is

$$\boxed{\Phi(z) = \frac{S}{z} = \frac{S}{re^{i\theta}} = \frac{S}{r}e^{-i\theta}} \qquad (22)$$

also (see eq. (3))

$$\Phi(z) = \frac{S}{r}(\cos\theta - i\sin\theta) = \phi + i\psi$$

so that $\phi = \frac{S}{r}\cos\theta$ and $\psi = -\frac{S}{r}\sin\theta$. The shape of a streamline (ψ = constant) is a circle as can be seen from Fig. 13.

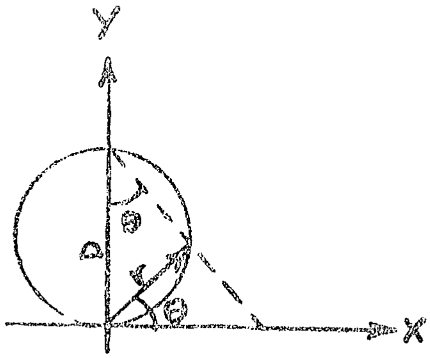


Fig. 13

On a circle of diameter D

$$r = D\sin\theta$$

so that

$$\frac{\sin\theta}{r} = \frac{1}{D} = \text{const} = \frac{-\psi}{S}$$

5. The Class of Corner or Wedge Flows may be described by a potential (A and n are real constants)

$$\boxed{\Phi(z) = \frac{A}{n} z^n} \tag{23}$$

In expanded notation

$$\Phi(z) = A \frac{(re^{i\theta})^n}{n} = A \frac{r^n e^{in\theta}}{n} = A \frac{r^n}{n} (\cos n\theta + i \sin n\theta)$$

and thus

$$\phi = A \frac{r^n}{n} \cos n\theta \quad \text{and} \quad \psi = A \frac{r^n}{n} \sin n\theta$$

The complex conjugate velocity $\bar{w}$ is

$$\bar{w} = \frac{d\Phi}{dz} = Az^{n-1} = Ar^{n-1} e^{i(n-1)\theta} = u - iv$$

and the absolute value of the velocity is

$$|w| = Ar^{n-1} = V \tag{24}$$

at $r = 0$ one can conclude from eq. (24) that:

when $n > 1$ then $V = 0$ (concave corners + wedges)

when $n < 1$ then $V = \infty$ (convex corners)

Points where $V = 0$ or $V = \infty$ may be singular points in $\Phi(z)$ and are thus points where streamline may have discontinuities in slope. Streamlines are line of constant

$$\psi = A \frac{r^n}{n} \sin n\theta$$

ψ is zero along straight radial lines where

$$\sin n\theta = 0 \text{ , or } \quad n\theta = 0, \pi, 2\pi, 3\pi, \ldots$$

$$n\theta = k\pi \ (k = 0, 1, 2, 3, \ldots)$$

or when

$$\theta = \frac{k\pi}{n} \quad \text{where} \quad k = 0, 1, 2, 3, \ldots \tag{25}$$

Examples where $n > 1$ (Concave corners and wedges)

Here it is useful to "solidify" streamlines of $\psi = 0$ in order to examine physical flows for various values of n.

(1) Case of $n = 3/2$, $\psi = A \frac{r^{3/2}}{3/2} \sin 3/2\ \theta$,

From eq. (25) $\psi = 0$ when $\theta = 0,\ 2\pi/3,\ 4\pi/3,\ 6\pi/3 = 2\pi = 0$ which represents flows shown in Figs. 14 and 15.

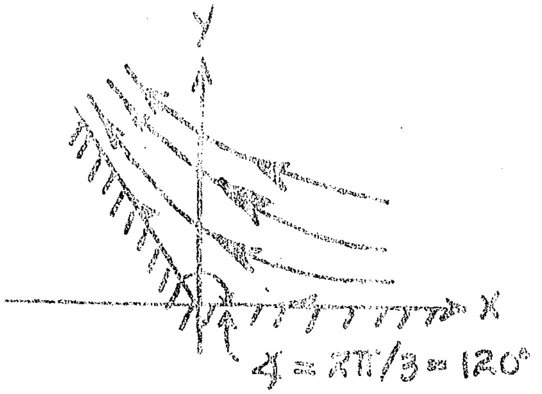


Fig. 14 Concave Corner (120°)

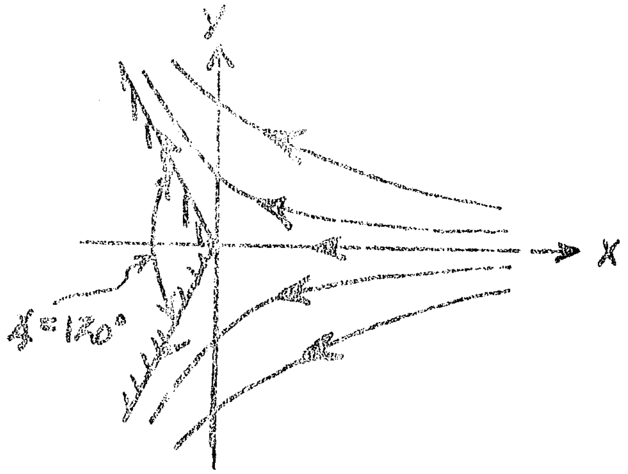


Fig. 15 Wedge (120°)

(2) Case of $n = 2$, $\psi = A \frac{r^2}{2} \sin 2\theta$,

Here $\psi = 0$ along lines for which $\theta = 0,\ \pi/2,\ \pi,\ 3\pi/2,\ 2\pi = 0$ that represents flow in a 90° corner or stagnation point flow, Figs. 16 and 17.

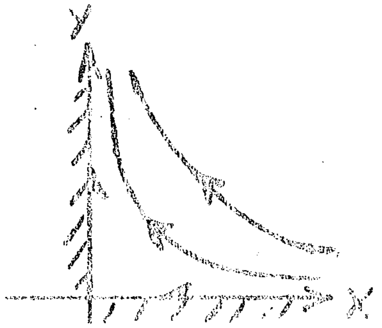


Fig. 16 Concave Corner (90°)

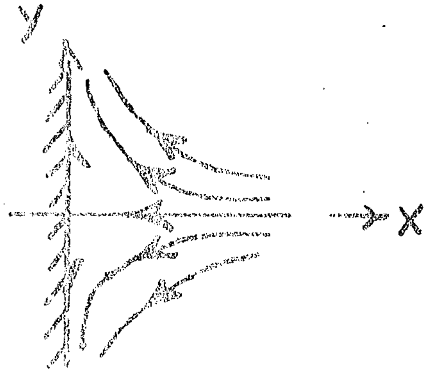


Fig. 17 Wedge (180°) Stagnation point flow on blunt body

For the cases shown in Figs. 16 and 17,

$$\bar{w} = \frac{\partial \Phi}{\partial z} = Az^{2-1} = Az = Ax + iAy = u - iv$$

Equating real and imaginary parts gives

$$\begin{aligned} u &= Ax \\ v &= -Ay \end{aligned} \quad ; \quad |w| = |A|\, r$$

In order to give the streamlines shown in Figs. 16 and 17, A must be a negative constant.

(3) Case of $n = 3$, $\psi = A \frac{r^3}{3} \sin 3\theta$,

Here $\psi = 0$ along $\theta = 0$, $\pi/3$, $2\pi/3$, π , $4\pi/3$, $5\pi/3$, $2\pi = 0$ to give the flows shown in Fig. 18.

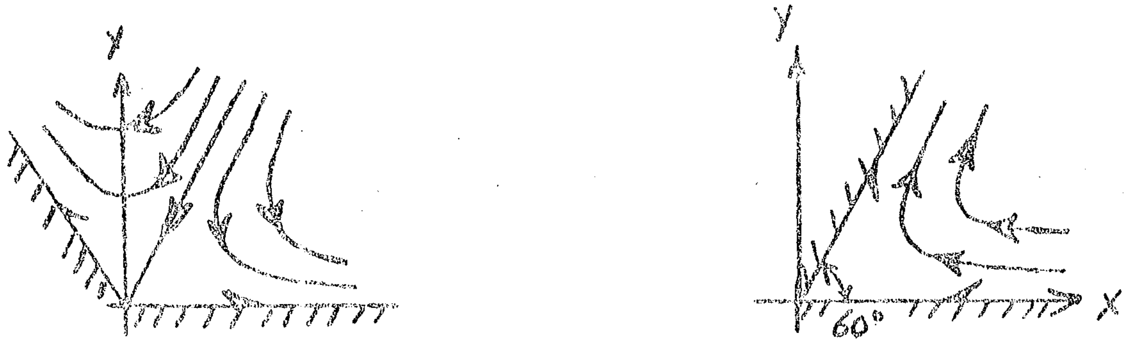


Fig. 18 Corners for $n = 3$

Examples where $n < 1$ (Convex Corners)

(4) Case of $n = 2/3$, $\psi = A \frac{r^{2/3}}{2/3} \sin \frac{2\theta}{3}$,

Here $\psi = 0$ along $\theta = 0$, $3\pi/2$, 3π , ... which represents flow over a 90° corner, Fig. 19.

(5) Case of $n = 1/2$, $\psi = A \frac{r^{1/2}}{1/2} \sin \frac{\theta}{2}$,

Here $\psi = 0$ along $\theta = 0$, 2π , 4π , ... Fig. 19 also illustrates this case.

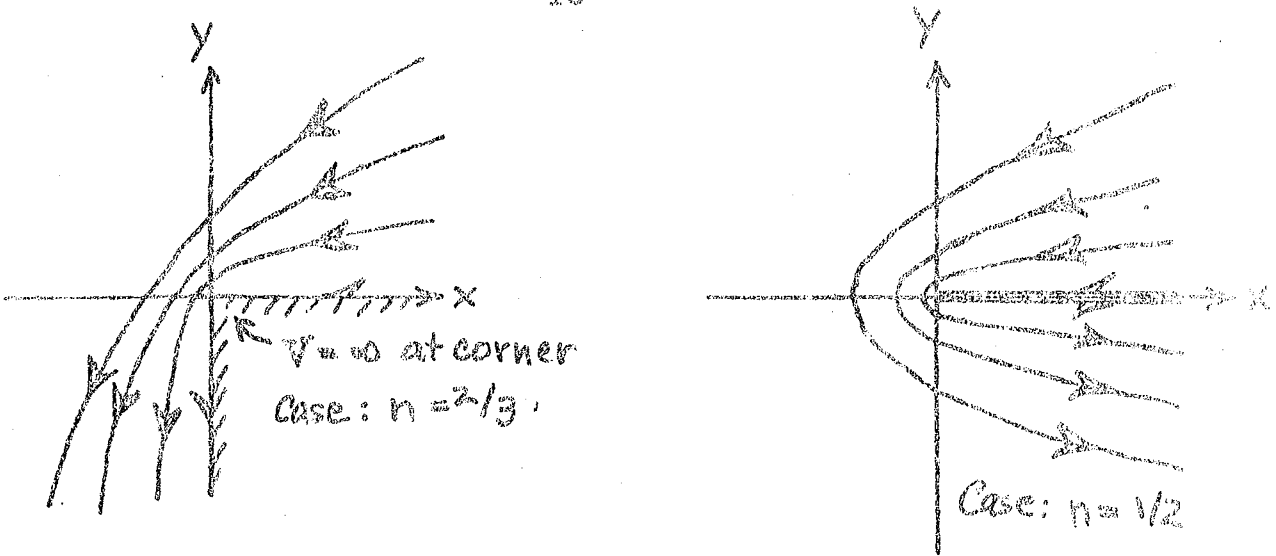


Fig. 19 Convex Corner Flows ($n < 1$)

In conclusion, one should note that when $n = 1$ the potential function reduces to that of uniform parallel flow, eq. (17), where the constant $A = w$. The class of wedge flows is restricted to

$$1 < n \leq 2 \quad (\text{wedge} \measuredangle \leq 180^\circ)$$

and they all exhibit $V = 0$ (stagnation point) at apex of wedge.

When $n = 1/2$, the convex corner flows achieve their physical limit of 180° of turn. Thus the class of convex corner flows is limited to

$$1/2 \leq n < 1$$

All these cases exhibit $V = \infty$ at the corner.

ME 251A

Advanced Fluids Engineering

Class Notes

by: J. P. Johnston

Forward:

These unedited notes are intended to ease your job of note taking. I hope you find them of some value. Be warned that they contain some errors so please check all important results for yourself. Note also that pages 71-78 do not exist.

My thanks to J. Ashjaee who prepared lectures 16, 17, and 18 and J. Eaton who did 22, 23, and 24 during the Autumn of 1977.

ME 251 A - LECTURE 1 77/78

SOME DEFINITIONS

FLUID MECHANICS - BASIC STUDY OF FLUIDS, FORCES IN FLUIDS AND MOTION OF FLUIDS — (A SCIENCE, BRANCH OF PHYSICS)

FLUIDS ENGINEERING - THE ORDERLY APPLICATION OF SCIENTIFIC KNOWLEDGE TO THE ANALYSIS AND DESIGN OF SYSTEMS THAT DEAL WITH FLUIDS AT REST AND IN MOTION.

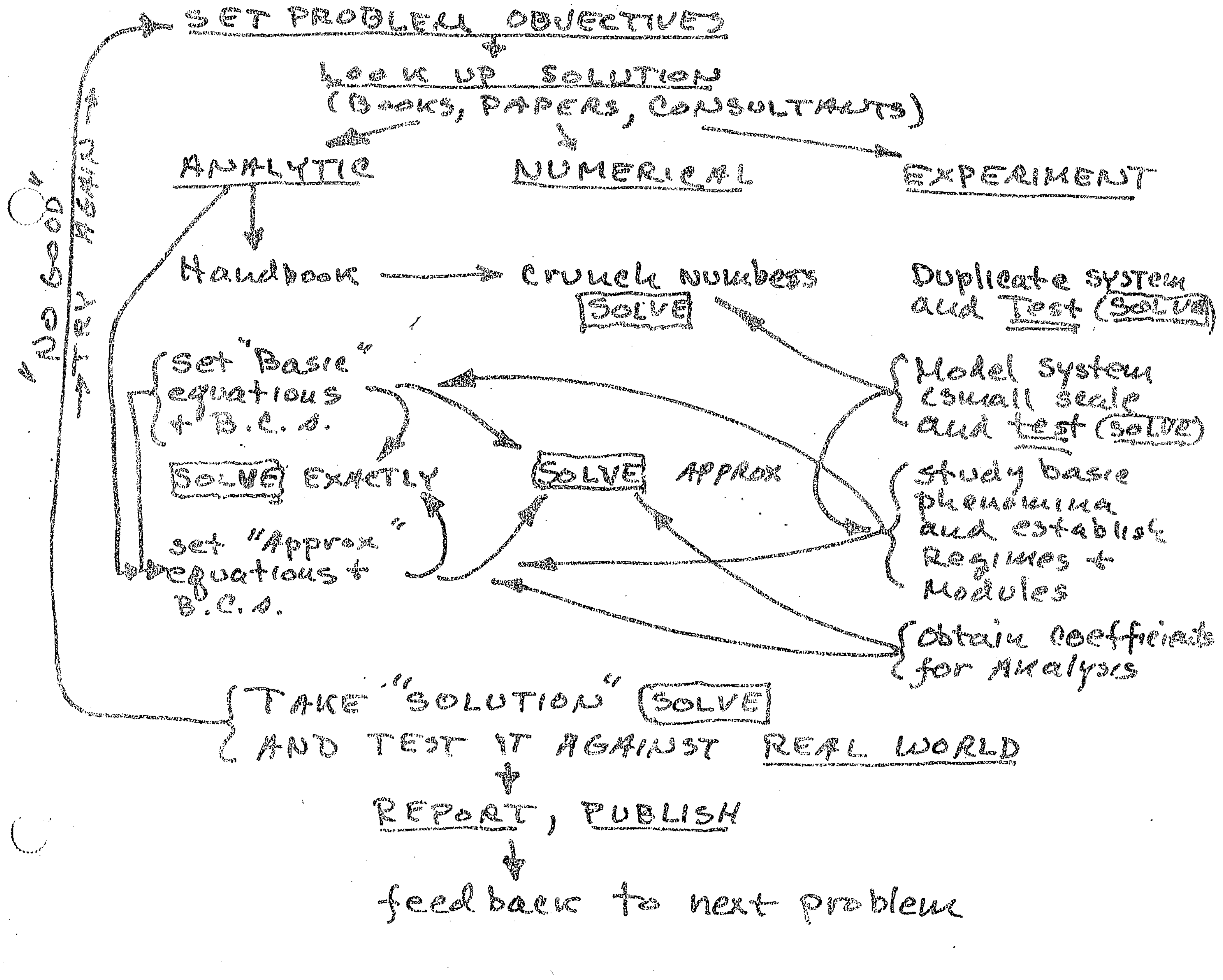
CLASSES OF SYSTEMS

- PROPULSION SYSTEMS
  - ROCKETS, TURBOJETS
- FLUID TRANSPORT SYSTEMS
  - PIPE LINES, TANK CARS & TRUCKS
- MEASUREMENT SYSTEMS
  - FLOW RATE, PRESSURE, VELOCITY
- ENERGY, POWER EXCHANGE SYSTEMS
  - FLUID/SOLID (PUMPS, TURBINES)
  - FLUID/FLUID (EJECTOR)
  - FLUID/ELECTRIC (MHD GENERATOR)
- VEHICLES IN MOTION
  - DRAG, LIFT
- FLOW MIXING
  - POLLUTANT DISPERSAL
  - TURBULENT COMBUSTION

APPROACH TO REAL ENGINEERING PROBLEMS

FACTORS.

- TIME TO SOLUTION (SECS → YEARS)
- $ AVAILABLE (NONE → M$)
- PEOPLE AVAILABLE (TECHNICIAN → B.S. ENGR → M.S. ENGR → Ph.D. (CONSULTANT))
- EQUIPMENT AVAILABLE (PAD + PEN → IBM 7600, STAR, ILLIAC) (GLUE + WOOD → 40X80 WINDTUNNEL)



FLUID $\rightarrow$ COLLECTION OF MOLECULES

FLUID $\rightarrow$ CONTINUUM DESCRIPTION

PHYSICAL IDEA

VOLUME: $\delta\forall = \delta x\,\delta y\,\delta z \approx (\delta x)^3$

MEAN FREE PATH: λ

CONTINUUM OK. IF $\delta x \gg \lambda$

MOST PRACTICAL FLOWS $\delta x > 10^{-3}$ CM
(NOTE EXCEPTIONS) $\delta\forall > 10^{-9}$ CM^3

FOR AIR (N_2, O_2) AT N.P.T. (20°C, 1 atm) THIS $\delta\forall$ CONTAINS $\sim 3\times10^{10}$ MOLECULES SO CONTINUUM PROPERTIES CAN BE DEFINED

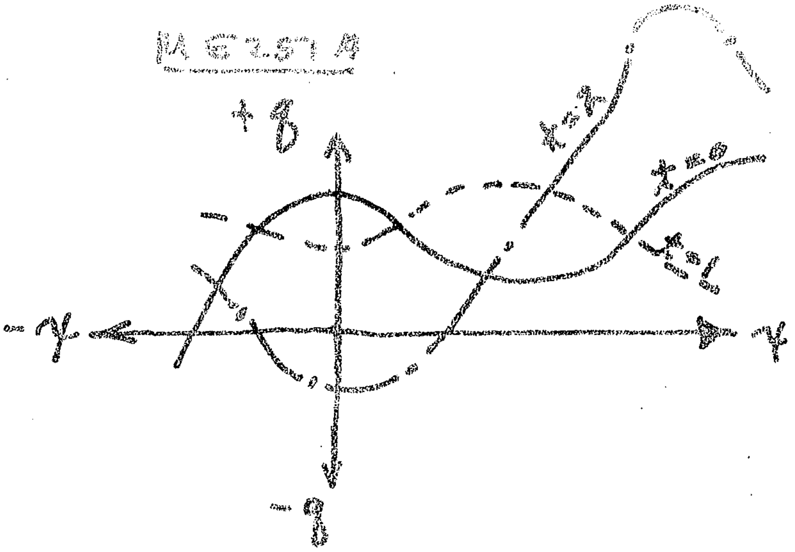
CONTINUUM PROPERTIES (EXAMPLES)

ρ Density $-(M/\forall)$; $v = 1/\rho$ Specific Vol. $-(\forall/M)$
T Temperature $-(\Theta)$
p Pressure $-(F/A)$
$\hat{u}$ Specific internal energy $-(E/\forall)$
h Enthalpy $(\hat{u} + pv) - (E/\forall)$
σ normal stress $-(F_\perp/A)$
τ Shear stress $-(F_\parallel/A)$
μ Viscosity $-(Ft/AL)$
ν Kinematic Viscosity $(\mu/\rho) - (L^2/t)$
c_p Specific heat at p=const $(\partial h/\partial T)_p - (E/\forall\Theta)$
k Ratio of Specific Heats $(c_p/c_v) - (0)$
$V(u_i)$ Speed (Velocity) $-(L/t)$

SYMBOL FOR ANY "SCALER" PROPERTY AT A GIVEN POINT IN TIME (t) AND SPACE

$$q = q(t;\ \underbrace{x, y, z}_{\text{rectangular coordinates}})$$

FIELD FUNCTION (MAY BE + OR −)



$$q = q(t; x, y = \text{const}, z = \text{const.})$$

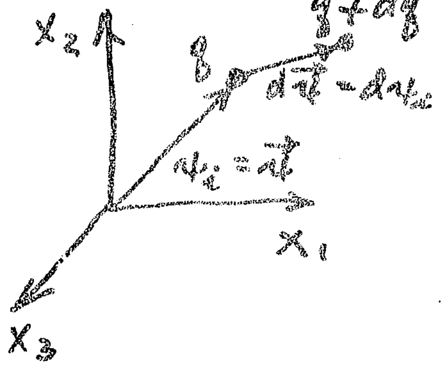
IN DOMAIN q IS SINGLE VALUED, PIECEWISE DIFFERENTIABLE, CONTINUOUS. HENCE

$$dq = \frac{\partial q}{\partial t}dt + \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy + \frac{\partial q}{\partial z}dz$$

IN <u>SUBSCRIPT NOTATION</u> (x_i, dx_i WHERE $i = 1, 2, 3$)

$$dq = \frac{\partial q}{\partial t}dt + \underbrace{\frac{\partial q}{\partial x_i}dx_i}$$

SUMMATION CONVENTION



DOT PRODUCT OF 2 VECTORS: $(\vec{\nabla} q)\cdot d\vec{x}$

$$\vec{\nabla} q = \vec{i}\,\frac{\partial q}{\partial x} + \vec{j}\,\frac{\partial q}{\partial y} + \vec{k}\,\frac{\partial q}{\partial z}$$

$$d\vec{x} = \vec{i}dx + \vec{j}dy + \vec{k}\partial z$$

$$dq = \frac{\partial q}{\partial t}dt + \vec{\nabla} q \cdot d\vec{x}$$

Total change = Temporal change + Spatial change

$\underbrace{\text{Temporal change}}$ "LOCAL" EFFECT

$\underbrace{\text{Spatial change}}$ "CONVECTIVE" EFFECT IF $d\vec{x}$ REPRESENTS FLUID PARTICLE DISPLACEMENT, $\vec{V}dt$

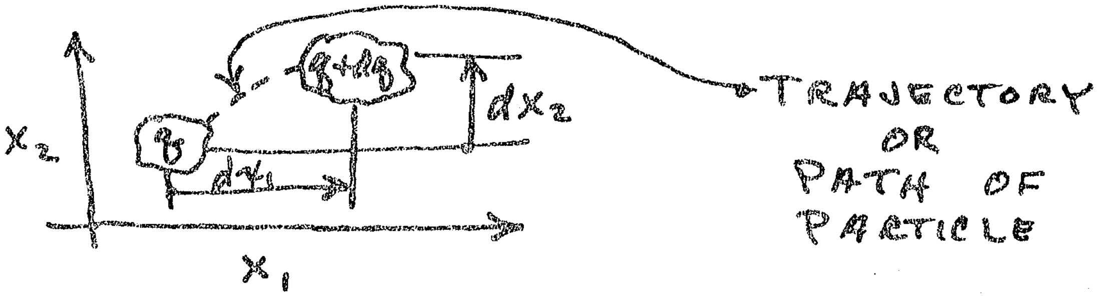
FIELD KINEMATICS

CALCULATE RATE OF CHANGE OF A q FOR AT MATERIAL POINT PARTICLE,

$$\left[\frac{dq}{dt}\right]_{\text{PARTICLE}}$$

DEFINE NEW SYMBOL – <u>SUBSTANTIAL OR MATERIAL DERIVATIVE</u>

$$\frac{Dq}{Dt} = \left[\frac{dq}{dt}\right]$$ AS WE FOLLOW PARTICLE AT AN INSTANT



$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \left\{ \left[\frac{dx_i}{dt}\right]_{\substack{\text{Particle}\\\text{Path}}} \times \left[\frac{\partial q}{\partial x_i}\right] \right\}$$

$$\left[dx_i\right]_{\substack{\text{Particle}\\\text{path}}} \equiv \underline{\text{DISPLACEMENT}}$$

$$\left[\frac{dx_i}{dt}\right]_{\substack{\text{Particle}\\\text{path}}} \equiv \frac{Dx_i}{Dt} \equiv u_i \equiv \underline{\text{PARTICLE VELOCITY}}$$

(Symbol $\vec{V}$ often used)

$$\boxed{\begin{array}{l} \dfrac{Dq}{Dt} = \dfrac{\partial q}{\partial t} + u_i \dfrac{\partial q}{\partial x_i} \\[2ex] \dfrac{Dq}{Dt} = \dfrac{\partial q}{\partial t} + \vec{V} \cdot \vec{\nabla}(q) \end{array}}$$

<u>or</u>

FIELD KINEMATICS (CONT.) - FLUID PARTICLE

DISPLACEMENT:

$$\vec{dx} = \vec{i}_1 dx_1 + \vec{i}_2 dx_2 + \vec{i}_3 dx_3 = \vec{i}_i dx_i$$

VELOCITY:

$$\vec{V} = \left[\frac{\vec{dx}}{dt}\right]_{\text{Particle}} = \vec{i}_1 u_1 + \vec{i}_2 u_2 + \vec{i}_3 u_3 = \vec{i}_i u_i$$

ACCELERATION:

$$\vec{a} = \left[\frac{d\vec{V}}{dt}\right]_{\text{Particle}} = \frac{D\vec{V}}{Dt} = \vec{i}_i a_i$$

EACH COMPONENT of $\vec{a}$ in rect. coords

$$a_i = \frac{Du_i}{Dt} \qquad (i = 1, 2, 3)$$

Now:

$$\boxed{\frac{Du_i}{Dt}} = \frac{\partial u_i}{\partial t} + \left[\frac{dx_j}{dt}\right]_{\text{Particle}} \times \left[\frac{\partial u_i}{\partial x_j}\right] \boxed{= \underbrace{\frac{\partial u_i}{\partial t}}_{\substack{\text{local} \\ \text{accel.}}} + \underbrace{u_j\left(\frac{\partial u_i}{\partial x_j}\right)}_{\substack{\text{convective} \\ \text{accel.}}}}$$

EXAMINE CONVECTIVE ACCELERATION TERM

$$u_j\left(\frac{\partial u_i}{\partial x_j}\right) = u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3} \qquad (i = 1, 2, 3)$$

$\left(\frac{\partial u_i}{\partial x_j}\right)$ IS A TENSOR OF RANK 2
(9 - components)

$u_j\left(\frac{\partial u_i}{\partial x_j}\right)$ IS A contraction: dot product of a vector and tensor of rank 2 in this case

INVARIANT VECTOR FORM (UNIT vectors ≠ CONST.)

$$\vec{a} = \frac{D\vec{V}}{Dt} = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local accel.}} + \underbrace{\vec{\nabla}\left(\frac{V^2}{2}\right) - \vec{V} \times (\text{curl } \vec{V})}_{\text{Convective accel.}}$$

Sometimes written improperly as

$$(\vec{V} \cdot \nabla \vec{V})$$

Vorticity FIELD

$$\vec{\omega} = \text{curl } \vec{V} = \vec{\nabla} \times \vec{V}$$

In rectangular coords

$$\vec{\omega} = \begin{vmatrix} \vec{i}_1 & \vec{i}_2 & \vec{i}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

in subscript notation: $\vec{\omega} = \vec{i}_i \, \omega_i$ and

$$\omega_i = \varepsilon_{ijk} \left(\frac{\partial u_k}{\partial x_j}\right) \qquad \left[\sum_{j=1}^{3} \sum_{k=1}^{3}\right]$$

where is a unit tensor

$\varepsilon_{ijk} = 0$ if any indices are equal, e.g. ε_{122}

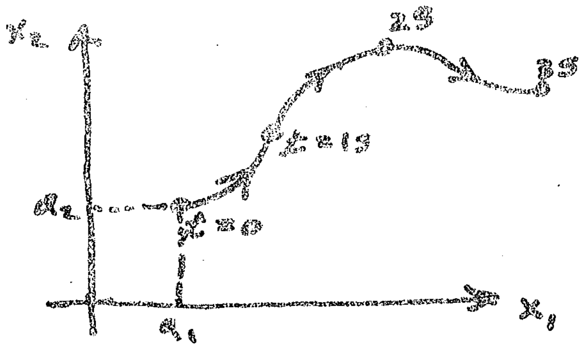
$\varepsilon_{ijk} = 1$ if indices are cyclic, e.g. ε_{123}

$\varepsilon_{ijk} = -1$ if indices are anticyclic, e.g. ε_{132}

KINEMATICS EULERIAN VERSUS LAGRANGIAN

IN EULERIAN FLUID MECHANICS WE ARE GENERALLY NOT INTERESTED IN DISPLACEMENT BUT SOLVE UP TO VELOCITY FIELD ONLY

PATHLINE, THE TRAJECTORY OF A FLUID PARTICLE IS A LAGRANGIAN IDEA (since you are following the particle)

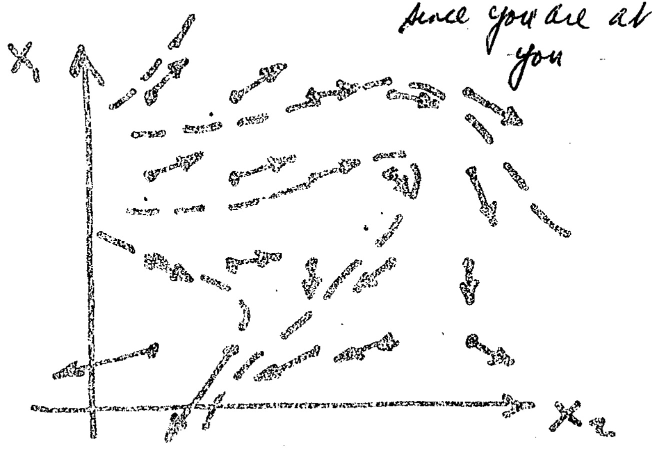


THE COORDINATES OF A PATHLINE ARE OBTAINED FROM TIME HISTORY OF VELOCITY FIELD

$$x_i = a_i + \int_0^t u_i\,dt$$

WHERE a_i ($i=1,2,3$) specified and $u_i = u_i(x_i, t)$ KNOWN.

STREAMLINE, A CONTINUOUS LINE EVERYWHERE TANGENT TO VELOCITY VECTORS AT AN INSTANT OF TIME (AN EULERIAN IDEA) since you are at a fixed point in space & ~~watch~~ what goes by you



FIELD OF STREAMLINES AT AN INSTANT

STREAMLINES ARE IDENTICAL TO PATHLINES ONLY IN STEADY FLOW

STEADY FLOW IS WHEN

$$\frac{\partial u_i}{\partial t} = \frac{\partial \vec{V}}{\partial t} = 0$$

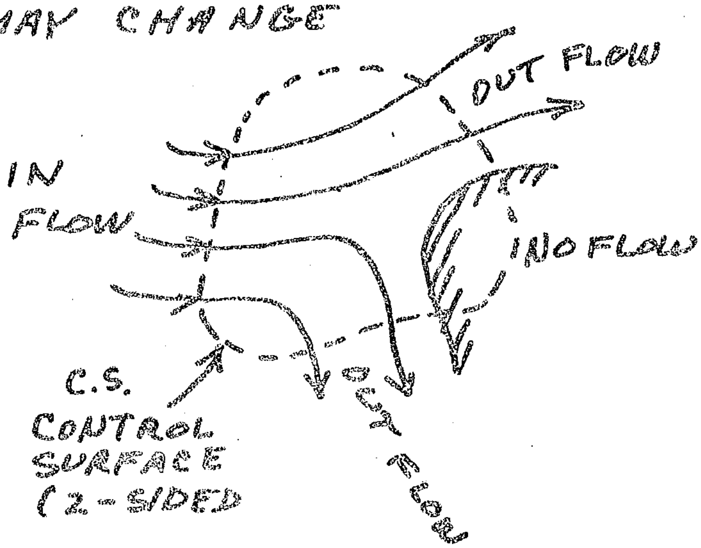
CONSERVATION OF MASS (2 points of view) IN RATE FORM

LAGRANGIAN - MASS OF FIXED IDENTITY

$$\boxed{\frac{DM}{Dt} = 0} \quad \text{OR} \quad M = \text{CONST.}$$

EULERIAN - LOOK AT MASS FLOW THRU A CONTROL SURFACE WHICH SURROUNDS A CONTROL VOLUME IN WHICH MASS CONTAINED MAY CHANGE

$$\boxed{R.O.C.(M) = 0}$$



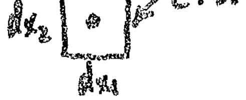
$$\left\{\begin{matrix}\text{Net rate of} \\ \text{out flow of} \\ \text{Mass thru C.S.}\end{matrix}\right\} + \left\{\begin{matrix}\text{Rate of increase} \\ \text{of Mass inside C.S.} \\ \text{(Storage rate)}\end{matrix}\right\} = 0$$

Define: $\overset{\circ}{M}$ = MAGNITUDE OF MASS FLOW RATE THRU A SECTION OF C.S.

∴ R.O.C. (M) = 0

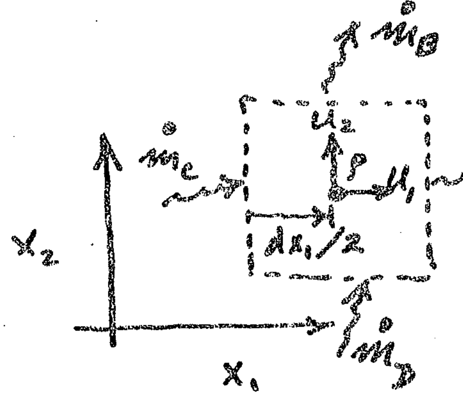
$$\boxed{(\dot{m}_{out} - \dot{m}_{in}) + \frac{dM_{cv}}{dt} = 0}$$

Application: C.V. of elemental size whose sides are fixed with respect to (w/o) coordinates for fluid velocity



Assume for simplicity: 2-D situation

$$d\forall = dx_1 \cdot dx_2 \cdot (1)$$



$$\dot{m}_A = \rho u_1 dx_2 + \left(\frac{\partial(\rho u_1)}{\partial x_1}\frac{dx_1}{2}\right)dx_2$$

Taylor's series up to order (dx_2)

$$M_{cv} = \rho\, dx_1 dx_2$$

$$\frac{dM_{cv}}{dt} = \frac{\partial \rho}{\partial t} dx_1 dx_2 \quad \leftarrow \left\{\text{at fixed point, the centroid of c.v.}\right\}$$

NOW IN THE X_1 DIRECTION

$$\dot{m}_A - \dot{m}_C = \Big(\underset{O(dX)^0}{\cancel{\rho u_1 dx_2}} + \underset{O(dx)^1}{\frac{\partial(\rho u_1)}{\partial x_1}\frac{dx_1}{2}dx_2}\Big) - \Big(\underset{O(dx)^0}{\cancel{\rho u_1 dx_2}} - \underset{O(dx)^1}{\frac{\partial(\rho u_1)}{\partial x_1}\frac{dx_1}{2}dx_2}\Big)$$

<u>SO:</u>

$$\dot{m}_A - \dot{m}_C = \frac{\partial(\rho u_1)}{\partial x_1} dx_1 dx_2$$

FROM C.S. SHOWN:

$$(\dot{m}_{out} - \dot{m} = (\dot{m}_A + \dot{m}_B) - (\dot{m}_C + \dot{m}_D)$$

$$'' = (\dot{m}_A - \dot{m}_C) + (\dot{m}_B - \dot{m}_D)$$

$$'' = \left(\frac{\partial(\rho u_1)}{\partial x_1} dx_1 dx_2\right) + \left(\frac{\partial(\rho u_2)}{\partial x_2} dx_2 dx_1\right)$$

SO FINALLY R.O.C.(M) = 0 IS:

$$\frac{\partial \rho}{\partial t}\cancel{dx_1 dx_2} + \frac{\partial(\rho u_1)}{\partial x_1}\cancel{dx_1 dx_2} + \frac{\partial(\rho u_2)}{\partial x_2}\cancel{dx_2 dx_1} = 0$$

NOW RELAX 2-D ASSUMPTION SO FULL CONTINUITY EQ IS OBTAINED:

$$0 = \frac{\partial \rho}{\partial t} + \underbrace{\frac{\partial(\rho u_1)}{\partial x_1} + \frac{\partial(\rho u_2)}{\partial x_2} + \frac{\partial(\rho u_3)}{\partial x_3}}_{\underbrace{\text{OPERATOR } \partial/\partial x_i \text{ on}}_{\frac{\partial(\rho u_i)}{\partial x_i}}\ \underbrace{(\rho u_i)}_{\text{MASS FLUX}}}$$

So various forms are:

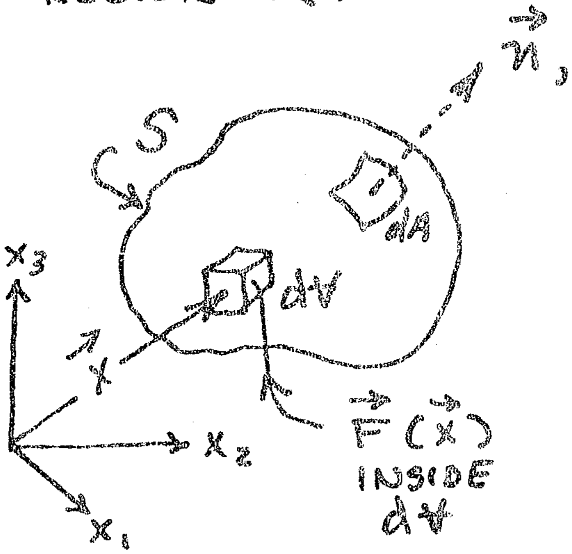
$$\boxed{\begin{aligned} 0 &= \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} \\ 0 &= \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) \\ 0 &= \frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho\vec{V}) \end{aligned}} \quad (11-1)$$

↑ EQ NO's

(PAGE − #)

ANOTHER APPROACH - DIVERGENCE THEOREM (GAUSS' THEOREM)

$\vec{F}$ IS ANY VECTOR FUNCTION OF $\vec{x}$ INSIDE A SMOOTH, CLOSED SURFACE S ABOUT A REGION R.



$\vec{n}$, UNIT OUTWARD NORMAL:

$$|\vec{n}| = \sqrt{\vec{n}\cdot\vec{n}} = 1$$

AREA VECTOR (ON S):

$$\vec{dA} = \vec{n}\, dA$$

$$\boxed{\int_R \vec{\nabla}\cdot\vec{F}\, dV = \int_S (\vec{n}\cdot\vec{F})\, dA}$$

APPLY TO CONTINUITY, FIRST $\int$ CONT. EQ OVER A FIXED CV. since $\frac{\partial}{\partial t}\int_{CV} \rho\, dV = \int_{CV} \cancelto{0}{\rho \frac{\partial}{\partial t}(dV)} + \int \frac{\partial \rho}{\partial t} dV$

$$0 = \int_{CV} \frac{\partial \rho}{\partial t} dV + \int_{CV} \vec{\nabla}\cdot(\rho\vec{V})\, dV$$

APPLY DIVERGENCE THEOREM TO RIGHT HAND (R.H.) TERM. $R \rightarrow CV$, $S \rightarrow CS$, $\vec{F} \rightarrow (\rho\vec{V})$

$$\boxed{0 = \int_{CV} \frac{\partial \rho}{\partial t} dV + \int_{CS} \rho\vec{V}\cdot\vec{dA}}$$

FIXED CV

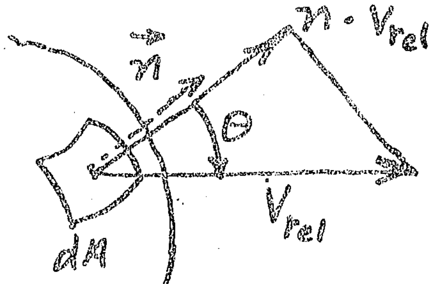
FIXED CS

$\vec{V}$ = absolute vel

MORE GENERAL FORM FOR CV CONS. OF MASS: MOVING IN COORD. SPACE AND NOT OF CONST. VOL.

$$\boxed{0 = \frac{d}{dt}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}_{rel} \cdot d\vec{A}}$$

$\vec{n} \cdot \vec{V}_{rel} = V_{rel} \cos\theta$



→ CS
surface
can move
at $\vec{V}_s(t, \vec{r})$

$\vec{V} = \vec{V}_s + \vec{V}_{rel}$

fluid velocity
w/o origin
of coordinates
(on CS only)

RATE OF MASS FLOW ACROSS SURFACE:

$(\rho \vec{V}_{rel} \cdot d\vec{A}) = \rho(\vec{n} \cdot \vec{V}_{rel}) dA$

$(\quad " \quad) = \rho V_{rel} \cos\theta \, dA$

$(\quad " \quad) > 0$ (OUTFLOW)

$(\quad " \quad) < 0$ (INFLOW)

$(\quad " \quad) = 0$ (NO FLOW)

SO EXPRESSION ABOVE EQUIVALENT TO

$$\boxed{0 = \frac{dM_{cv}}{dt} + (\dot{m}_{out} - \dot{m}_{IN})}$$

OTHER CONCLUSIONS ON CONTINUITY

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i}$$

$$0 = \underbrace{\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i}} + \rho \frac{\partial u_i}{\partial x_i}$$

$$0 = \frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i}$$

OR

$$\boxed{0 = \frac{1}{\rho}\frac{D\rho}{Dt} + \underbrace{\frac{\partial u_i}{\partial x_i}}_{\vec{\nabla}\cdot\vec{V} \overset{\text{OR}}{=} \text{div}\,\vec{V}}} \qquad (14-1)$$

SLIGHTY COMPRESSIBLE FLUID:

$$\frac{d\rho}{\rho} = d(\ln \rho) \text{ VERY SMALL}$$

EXAMPLE: 1. GAS WITH SMALL P AND T CHANGE

2. LIQUIDS (MOST)

3. ANY FLUID WHERE $|\vec{V}| \ll$ LOCAL SPEED OF SOUND

IN THE LIMIT, AN INCOMPRESSIBLE FLUID: ρ = CONST. IS ASSUMED, SO:

$$\boxed{(\partial u_i / \partial x_i) = 0} \qquad (14-2)$$

IN GENERAL, IF $\rho \approx$ CONST ABOVE APPLIES IN UNSTEAGY FLOW TOO, I.E. WHERE $\partial u_i/\partial t \neq 0$.

ME 251A LECTURE 5 77/78

EQUATIONS OF MOTION - INVISCID FLUID

NEWTON'S SECOND LAW OF PARTICLE MOTION WHERE $\vec{V}$ AND $\vec{a}$ IN "INERTIAL" COORDS

$$\vec{F} \propto \frac{D}{Dt}(m\vec{V})$$

$$\vec{F} = \frac{1}{g_c}\frac{D(m\vec{V})}{Dt} = \frac{m}{g_c}\frac{D\vec{V}}{Dt} + \frac{\vec{V}}{g_c}\frac{Dm}{Dt}$$

FOR SINGLE FIXED MASS PARTICLE

$$\boxed{\vec{F} = \frac{m}{g_c}\frac{D\vec{V}}{Dt} = \frac{m}{g_c}\vec{a}}$$

DIMENSIONS AND (UNITS)

SYSTEM	PRIMARY	SECONDARY
Mks (SI)	M (kg)	F ($kg\, m/s^2 \equiv N$) (Newton
	L (m)	$\vec{V}$ (m/s)
	t (s)	$\vec{a}$ (m/s^2)
	g_c (0) = 1	p ($\frac{N}{m^2} \equiv Pa$) pascal
English Engrg (absolute)	F (lbf)	M ($lbf\, s^2/ft \equiv slug$)
	L (ft)	$\vec{V}$ (ft/s)
	t (s)	$\vec{a}$ (ft/s^2)
	g_c (0) = 1	p ($lbf/ft^2 \equiv psf$)
English Engrg	F (lbf)	$g_c = 32.17\left(\frac{ft\ lb}{s^2\ lbf}\right)$
	L (ft)	
	t (s)	
	M (lbM)	

A FEW USEFUL CONVERSIONS OF UNITS

1 (slug) = 32.17 (lbm)

1 (kg) = 2.2046 (lbm)

1 (Bar) = 0.98692 (atm) = 10^5 (N/m^2)

1 (kPa) = 10^3 (N/m^2) = 0.145038 (lbf/in^2)
(psi)

1 (ft) = 0.3048 (m/s)

DROP g_c IN ALL DERIVATIONS

SET $g_c = 1$ SO: $\boxed{\vec{F} = m\frac{D\vec{V}}{Dt}}$

FORCES: $\vec{F} = \Sigma$ ALL APPLIED FORCES

CLASSIFICATION: SURFACE FORCES, F_S
BODY FORCES, F_B

BODY FORCES: GRAVITY, $\boxed{F_g}$
(ACTION AT A DISTANCE)
ELECTROSTATIC
ELECTROMAGNETIC
⋮

SURFACE FORCES: SURFACE TENSION
(CONTACT)
(AT PHASE INTERFACES)
NORMAL, $F_\perp$
SHEAR, $F_\parallel$

BODY FORCES PER UNIT MASS:

$$\vec{f}_B = \vec{F}_B / (\rho\, d\forall)$$

NEWTON'S LAW FOR PARTICLE OF MASS $(\rho\, d\forall)$

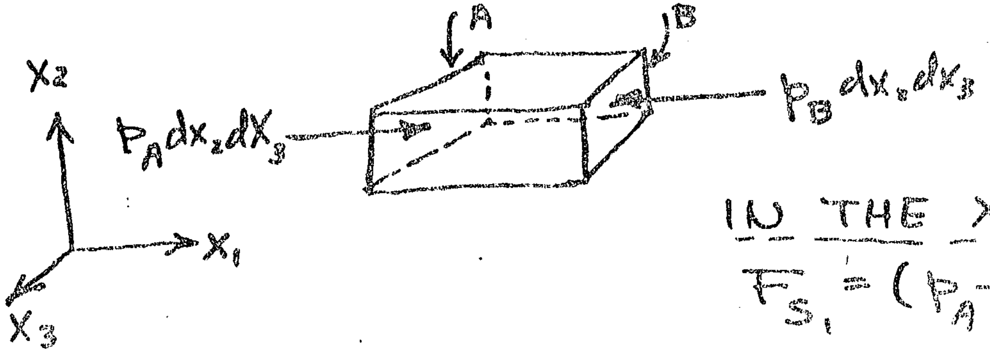
$$\boxed{F_{S_i} + (\rho\, d\forall) f_{B_i} = (\rho\, d\forall)\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right]}$$

$\boxed{i = 1, 2, 3}$

INVISCID FLUID: $F_{\parallel} = 0$, $F_{\perp} = p\, dA$

COMPRESSIVE

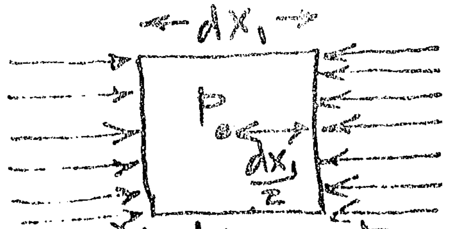
SURFACE FORCES ON A PARTICLE



IN THE X_1 DIRECTION:

$F_{S_1} = (p_A - p_B) dx_2 dx_3$

$'' = \left(-\frac{\partial p}{\partial x_1}\right)(dx_1 dx_2 dx_3)$



$p_A = p - \frac{\partial p}{\partial x_1}\frac{dx_1}{2}$ $p_B = p + \frac{\partial p}{\partial x_1}\frac{dx_1}{2}$

SIDE VIEW
(PRESSURES)

IN THE X_i DIRECTION:

$$\boxed{F_{S_i} = -\frac{\partial p}{\partial x_i}(d\forall)}$$

NEWTON'S LAW FOR INVISCID FLOW

(per unit mass by ÷ with $(\rho\, d\forall)$)

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x_i} + f_{B_i} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}} \qquad (18-1)$$

$$\boxed{i = 1, 2, 3}$$

THIS GIVES 3 EQ'S IN 5 UNKNOWNS

$(p, \rho, u_i \; ; \; i = 1, 2, 3)$

CONTINUITY, EQ (11-1) or (14-1) INTRO'S. ANOTHER EQ BUT NO MORE UNKNOWNS.

TO COMPLETE SYSTEM OF EQUATIONS WE NEED MORE INFO. FOR EXAMPLE, ASSUME ISENTROPIC PERFECT GAS SO: $p\rho^{-k} = \text{const.}$ IF INCOMPRESSIBLE FLUID THEN $\rho = \text{CONST}$ SO WE HAVE

$\rho = \text{CONST}$ - GIVEN

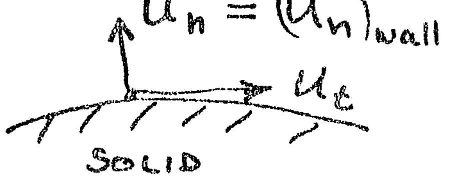
Momentum EQ (18-1) - IN 4 UNKNOWNS

Continuity EQ (14-2) - $\partial u_i / \partial x_i = 0$

THESE ARE SUFFICIENT WITH <u>BOUNDARY AND INITIAL CONDITIONS</u>

INITIAL COND'S $p(t=0, x_i)$,
$u_i(t=0, x_i)$,
ETC.

BOUNDARY COND'S



SOLID WALL:

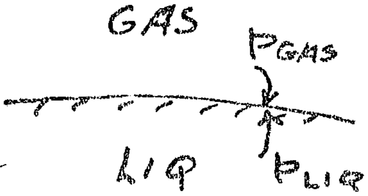
NO SLIP (VISCOUS EFFECT)

$$u_t = (u_t)_{wall}$$

INVISCID FLUID

u_t NOT SPECIFIED.

FREE SURFACE:



$$p_{gas} - p_{liq} = \frac{\sigma}{R}$$

σ ~ surface tension

R ~ instantaneous radius of curvature

INVISCID, NEGLECT SURFACE TENSION:

$$p_{GAS} = p_{LIQ}$$

EULER'S EQUATIONS (GENERAL $\vec{F}_B$)

BESIDES EQ (18-1) GENERAL VECTOR FORMS

$$\boxed{-\frac{1}{\rho}\vec{\nabla} p + \vec{f}_b = \frac{\partial \vec{V}}{\partial t} + \vec{\nabla}\frac{V^2}{2} - \vec{V} \times (\vec{\nabla} \times \vec{V})} \qquad (19-1)$$

AND CONTINUITY ARE A SET

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0} \qquad (19-2)$$

IF ρ = CONST. (19-2) BECOMES

$$\boxed{\vec{\nabla} \cdot \vec{V} = 0} \qquad (19-3$$

GLOBAL EQ'S

BASIC LAWS FOR FINITE CONTROL VOLUME

1. MASS CONSERVATION [R.O.C. (m) = 0]

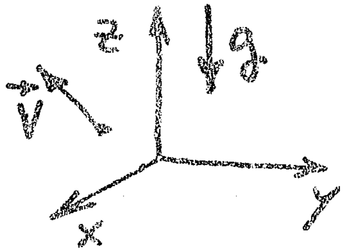
$$\boxed{\frac{d}{dt}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{V}_{rel} \cdot d\vec{A} = 0} \qquad (20\text{-}1)$$

2. ENERGY CONSERVATION [R.O.C. (E) = 0]

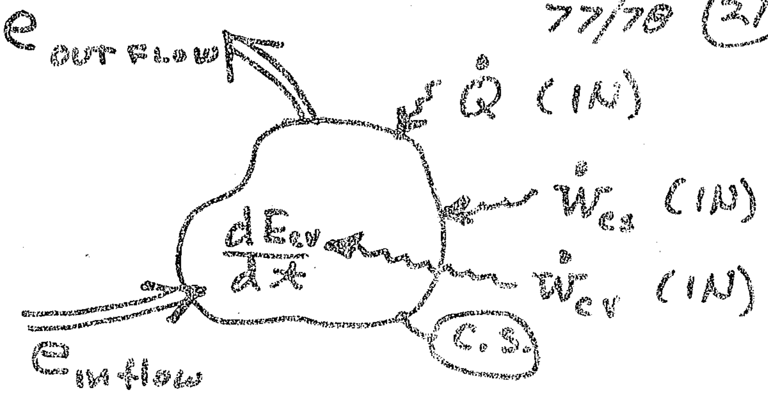
– INERTIAL COORDS. WHERE $mV^2/2$ IS K.E.

$$\underbrace{\frac{d}{dt}\int_{cv} e \rho \, d\forall}_{\text{NET RATE OF INCREASE OF } \boxed{E_{cv}}.} + \underbrace{\int_{cs} e (\rho \vec{V}_{rel} \cdot d\vec{A})}_{\text{NET OUT FLOW RATE OF } e \text{ WITH MASS.}} - \underbrace{\int_{cs} \dot{Q}'' dA}_{\text{NET IN FLOW RATE OF HEAT, } \boxed{\dot{Q}}.}$$

$$- \underbrace{\int_{cs} (\vec{F}_s'' \cdot \vec{V}) dA}_{\text{NET INFLOW OF WORK AT CS } \boxed{\dot{W}_{cs}}.} - \underbrace{\int_{cv} (\vec{F}_B''' \cdot \vec{V}) d\forall}_{\text{NET INFLOW OF WORK INSIDE CV } \boxed{\dot{W}_{cv}}} = 0$$



INERTIAL
COORD'S WITH
GRAVITY ACCEL.
IN $-z$



SYMBOLIC ENERGY CONSERVATION

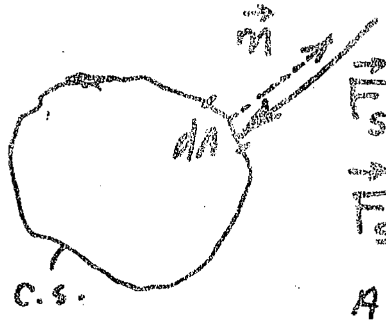
SPECIFIC ENERGY

$$\boxed{e = \tilde{u} + \frac{V^2}{2} + gz + \cdots\cdots\cdots}$$

WORK AT C.S. [INTO CV]

$$\dot{W}_{cs} = \dot{W}_{fw} + \dot{W}_{cs\,(\text{OTHER})}$$

FLOW WORK – WORK DUE TO PRESSURE FORCES THAT ARE ASSOCIATED WITH FLUID FLOW ACROSS CS. (FORCE ON C.S.)



$$\vec{F}'_{s_p} = p(-\vec{n}) = -p\vec{n}$$

$$\vec{F}''_{s_p} \cdot \vec{V} = -p\,\vec{V}\cdot\vec{n}$$

ALSO: $\vec{V} = \vec{V}_{rel} + \vec{V}_s$

$$\therefore \int_{CS} (\vec{F}''_{s_p} \cdot \vec{V})\, d\vec{A} = \underbrace{-\int_{C.S.} \frac{p}{\rho}(\rho \vec{V}_{rel} \cdot d\vec{A})}_{\text{FLOW WORK}} + \int_{C.S.} (-p)\vec{V}_s \cdot d\vec{A}$$

SO:

$$\dot{W}_{cs} = -\underbrace{\int_{cs}\frac{p}{\rho}(\rho\vec{V}_{rel}\cdot\vec{dA})}_{\text{FLOW WORK}} + \underbrace{\underbrace{\int_{c.s.}(-p\vec{V}_s\cdot\vec{dA})}_{\substack{\text{WORK DUE}\\ \text{EXPANSION}\\ \text{OR CONTRACT-}\\ \text{ION OF CS}\\ \text{(PISTON}\\ \text{EFFECT)}}} + \underbrace{\dot{W}_{\tau}}_{\substack{\text{SHEAR}\\ \text{STRESS}\\ \text{WORK}\\ \text{(includes}\\ \text{Shaft Work)}}} + \underbrace{\dot{W}_{st}}_{\substack{\text{SURFACE}\\ \text{TENSION}\\ \text{WORK}}}}_{\dot{W}_{cs(OTHER)}}$$

NOW REWRITE ENERGY CONS. EQ. AND COMBINE FLOW WORK WITH ENERGY FLOW WITH MASS TERM. AS A RESULT

$$\boxed{\begin{aligned}&\frac{d}{dt}\int_{CV} e\rho\, d\forall + \int_{CS}(\underbrace{\tilde{u}+\frac{p}{\rho}}_{h\ (\text{ENTHALPY})}+\frac{V^2}{2}+gz+\cdots)(\rho\vec{V}_{rel}\cdot\vec{dA})\\ &-\dot{Q} - \dot{W}_{CS(OTHER)} - \dot{W}_{CV} = 0\end{aligned}} \tag{22-1}$$

3. LINEAR MOMENTUM THEOREM

$$R.O.C.\,(\overrightarrow{MOM}) = \sum \overrightarrow{(\text{APPLIED FORCES})}$$

$$\boxed{\begin{array}{l} \dfrac{d}{dt}\displaystyle\int_{CV} \vec{V}\rho\, d\forall + \int_{CS} \vec{V}(\rho \vec{V}_{rel}\cdot d\vec{A}) = \int_{CS} \vec{F}_S''\, dA + \int_{CV} \vec{F}_B'''\, d\forall \\ \underline{\text{OR}} \\ \dfrac{d}{dt}\displaystyle\int_{CV} u_i\, \rho\, d\forall + \int_{CS} u_i(\rho \vec{V}_{rel}\cdot d\vec{A}) = \int_{CS} F_{S_i}''\, dA + \int_{CV} F_{B_i}'''\, dA \end{array}}$$

(23-1)

$$\underbrace{\hphantom{\frac{d}{dt}\int u_i \rho\, d\forall}}_{\substack{\text{RATE OF}\\ \text{INCREASE}\\ \text{OF MOM.}\\ \text{INSIDE C.V.}}} \quad \underbrace{\hphantom{+\int u_i(\rho \vec{V}_{rel}\cdot d\vec{A})}}_{\substack{\text{NET RATE}\\ \text{OF OUT FLOW}\\ \text{OF MOMENTUM}}} \quad \underbrace{\hphantom{\int F_{S_i}'' dA + \int F_{B_i}''' dA}}_{\substack{\text{SURFACE}\\ +\\ \text{BODY FORCES}}}$$

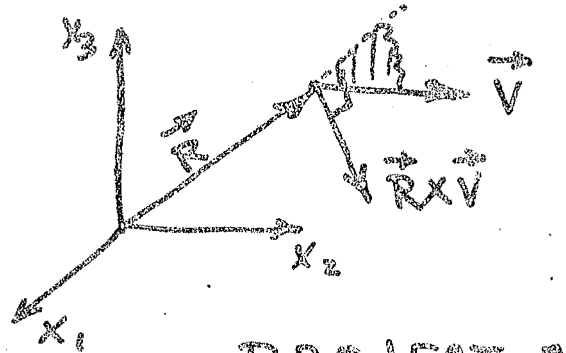
$$\overbrace{\frac{d}{dt}\int u_i\, dm} + \overbrace{\int_{\text{Outflow}} u_i\, d\dot{m} - \int_{\text{Inflow}} u_i\, d\dot{m}} = \overbrace{F_{S_i} + F_{B_i}}$$

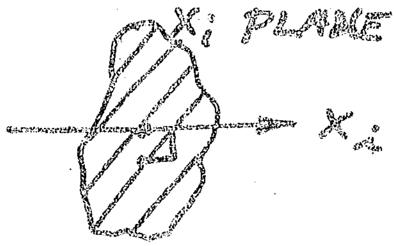
4. MOMENT OF MOMENTUM

R.O.C. $(\overrightarrow{\text{ANGULAR MOM.}}) = \sum(\overrightarrow{\text{APPLIED TORQUE}})$

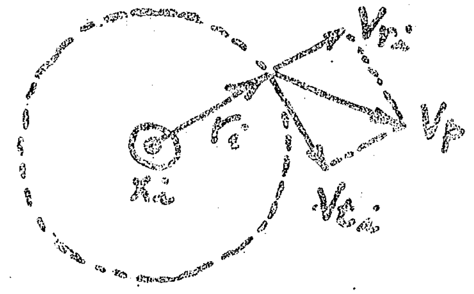
$$\frac{d}{dt}\int_{CV}(\vec{R}\times\vec{V})\rho\, dV + \int_{CS}\vec{R}\times\vec{V}(\rho\vec{V}_{rel}\cdot\vec{dA}) = \int_{CS}\vec{R}\times\vec{F}_s''\, dA + \int_{CV}(\vec{R}\times\vec{F}_B''')dA$$

INTERPRETATION:





PROJECT ABOVE ON X_i PLANE
$\vec{V}_p$ IS PROJECTED $\vec{V}$, r_i IS PROJ $\vec{R}$

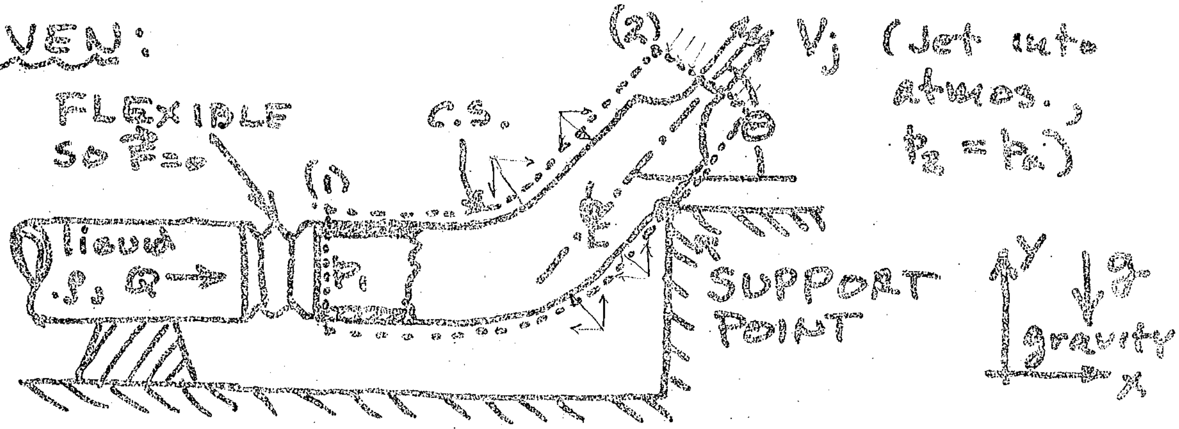


$$\boxed{\frac{d}{dt}\int_{CV}(r V_t)_i\,\rho\, dV + \int_{CS}(r V_t)_i\,\rho\vec{V}_{rel}\cdot\vec{dA} = \int_{CS}(r F_{s_t}'')_i\, dA + \int_{CV}(r F_{B_t}''')dV} \quad (24-1)$$

GLOBAL EQ'S — APPLICATIONS

EXAMPLE – 1 FIND THE FORCE TO RESTRAIN AN ELBOW-NOZZLE

GIVEN:



ρ = liquid density (kg/m³)

Q = Volume rate of flow (m³/s)

A_1, A_2 : inside ⊥ areas at (1) and (2)

ASSUME:

1. STEADY FLOW; $\partial \vec{V} / \partial t = 0$

2. INCOMPRESSIBLE; ρ = const

3. ONE-DIM. FLOW ACROSS (1) AND (2)

4. ALL SUPPORT FORCE, $\vec{R}$, APPLIED AT ONE POINT

$$\Sigma F = p_1 A_1 \underset{\sim}{i} - p_2 A_1 (\cos\theta \underset{\sim}{i} + \sin\theta \underset{\sim}{j}) + R_y \underset{\sim}{j} - R_x \underset{\sim}{i} - W_t \underset{\sim}{j}$$

$$V_j = \frac{V_1 A_1}{A_2}$$

$$-V_1 (\rho V_1 A_1) \underset{\sim}{i} + V_j (\rho V_1 A_1)(\cos\theta \underset{\sim}{i} + \sin\theta j)$$

force on fluid = − force on elbow

force on elbow = − force on support

force on fluid = force on support

$$P_1 A_1 - P_a A_1 \cancel{\cos\theta} + R_x = V_j \cos\theta (\rho V_1 A_1) - V_1 \rho V_1 A_1)$$

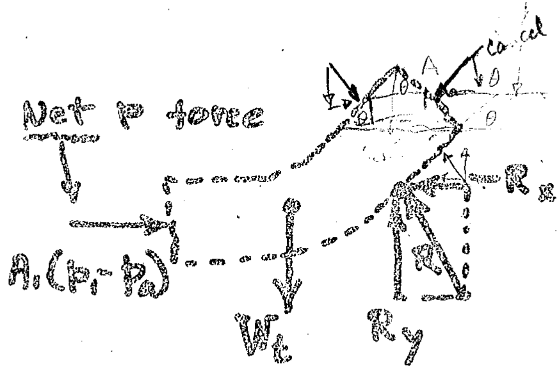
$$-\cancel{P_a A_1} \sin\theta + R_y - W_t = V_j (\rho V_1 A_1) \sin\theta$$

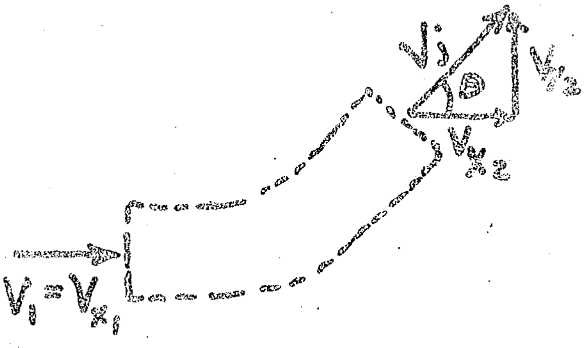
force on Support

force on elbow by supp

ME251A 7/7/70 ㉖

CONTROL VOLUMES FOR FORCES AND FLOWS





FORCES

$$\vec{R} = \vec{i} R_x + \vec{j} R_y$$

MOMENTUM FLUXES

$$V_{x_2} = V_j \cos\theta$$

$$V_{y_2} = V_j \sin\theta$$

momentum / unit area

(1.) R.O.C. (MASS) = 0

$$\cancel{\frac{\partial M_{cv}}{\partial t}} + \dot{m}_{out} - \dot{m}_{in} = 0$$

$$0;\ \frac{\partial \rho}{\partial t} = 0 \quad + \rho A_2 V_j - \rho A_1 V_1 = 0$$

and fixed c.s.

So:

$$\boxed{\begin{array}{l} V_j = \dfrac{A_1 V_1}{A_2} = \dfrac{Q}{A_2} \\ V_1 = \dfrac{Q}{A_1} \end{array}}$$

(2.) LINEAR MOMENTUM (FIXED C.S.)

$$R.O.C\ \overrightarrow{(\text{MOMENTUM})} = \sum \vec{F}_s + \sum \vec{F}_B$$

EQ'S FOR X AND Y ARE:

$$\int_{c} \underbrace{\cancel{\frac{\partial(\rho V_x)}{\partial t}}}_{=0,\ \textcircled{1}+\textcircled{2}} + \int_{out} V_x\, d\dot{m} - \int_{in} V_x\, d\dot{m} = \sum F_{S_x} + \sum F_{B_x}$$

$$\int_{CV} \cancel{\frac{\partial(\rho V_y)}{\partial t}} + \int_{out} V_y\, d\dot{m} - \int_{in} V_y\, d\dot{m} = \sum F_{S_y} + \sum F_{B_y}$$

WITH 1-D FLOW AT (1) AND (2), ASSUMPTION ③

AND $\int_{in} d\dot{m} = \int_{out} d\dot{m} = \rho Q$ FROM R.O.C.(M) = 0

$$(V_j \cos\theta)\rho Q - (V_1)\rho Q = A_1(p_1 - p_a) - (-R_x)$$

$$(V_j \sin\theta)\rho Q - (0)\rho Q = (-W_t) + (R_y)$$

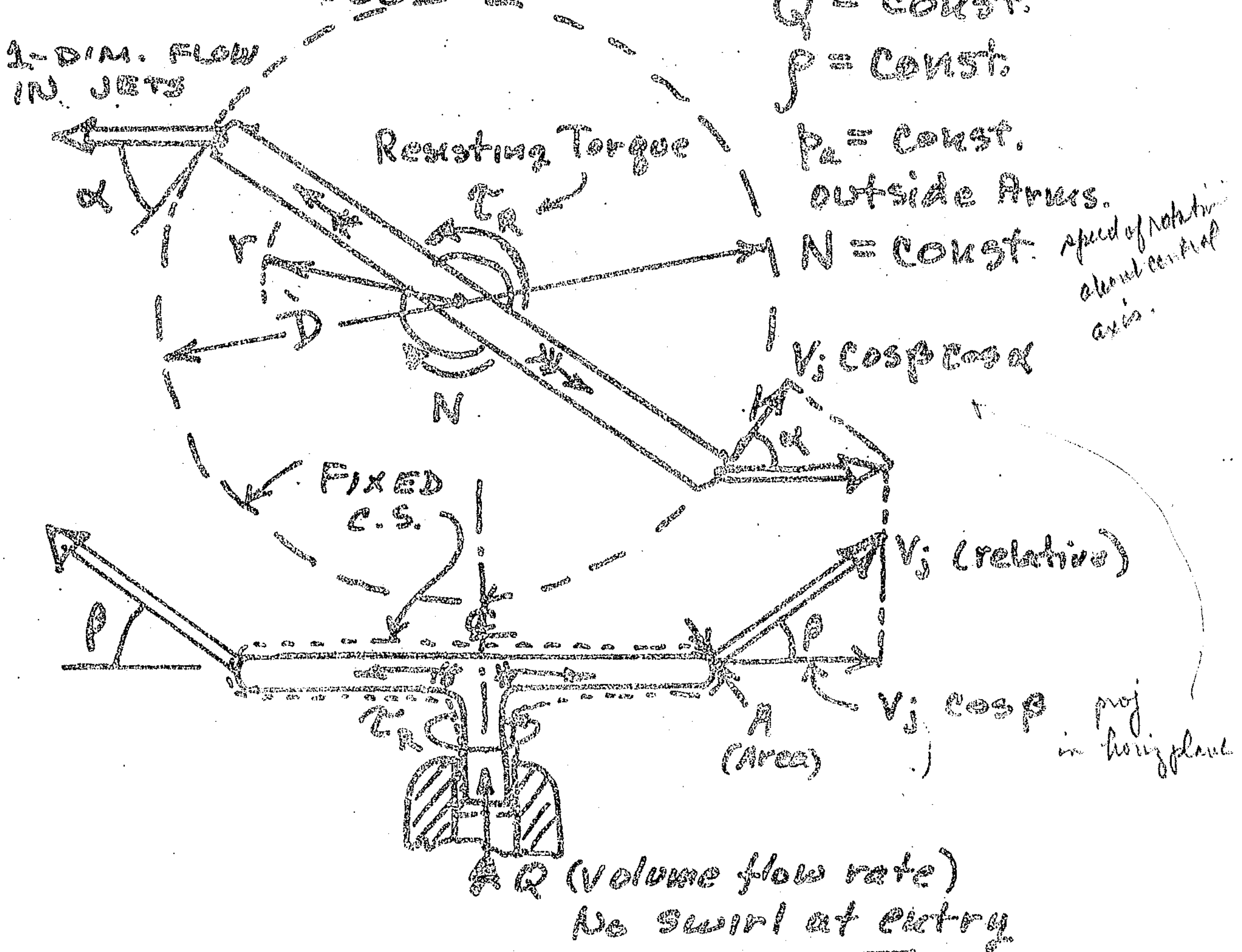
REDUCED TO USEFUL WORKING FORM:

$$\left[\begin{aligned} R_x &= A_1(p_a - p_1) + \left(\frac{\cos\theta}{A_2} - \frac{1}{A_1}\right)\rho Q^2 \\ R_y &= W_t + \frac{1}{A_2}(\sin\theta)\rho Q^2 \end{aligned}\right]$$

EXAMPLE - 2 THE "LAWN SPRINKLER" OR "REACTION WATERTURBINE"

FIND: STEADY RATE OF ROTATION, N, OF TWO ARMED SPRINKLER SHOWN AND GENERALIZE TO REACTION TURBINE. GIVEN:

Q = CONST.
ρ = CONST.
p_a = CONST. outside Arms.
N = CONST speed of rotation about central axis.



(1.) R.O.C. (MASS) = 0 GIVES $\boxed{V_j = \frac{1}{2}\frac{Q}{A}}$ (20-1)

ALSO: V_{rel} = CONST w/o $*$ AT ALL POINTS INSIDE SYSTEM

mass flow $\rho Q_{in} = (\rho Q_{out})$ $\therefore A V_j = \frac{1}{2} Q_{in}$ since coming out of 2 jets

$V_{tangent} = r \times N$ (N is really ω) for steady rotation N = const ⇒ V_t = const

(2.) MOMENT OF MOMENTUM ABOUT AXIS

DEFINE τ_R AS + DIRECTION

$$\sum \tau_{t} = \underbrace{\int_{CV} r \frac{\partial (V_t \rho)}{\partial t} dV}_{= 0 \text{ steady state}} + \int_{CS} r V_t (\rho \vec{V}_{rel} \cdot d\vec{A})$$

BECAUSE $\dfrac{\partial V_t}{\partial t} = \dfrac{\partial}{\partial t}\Big[\overset{\text{absolute velocity}}{V_{rel_t} + (-N) r}\Big] = 0$

AND $\dfrac{\partial \rho}{\partial t} = 0$

NOW SURFACE FLUX OF MOM. OF MOM.

$r V_t = r\left(V_{j_t} + \frac{1}{2} D(-N)\right)$: AT OUTLET

ROTOR SPEED DN/2

$$\boxed{V_{j_t} = V_j \cos\beta \cos\alpha} \quad (29-1)$$

relative to system

$\dot{m} = \rho Q$

BOTH ARMS

SO!

$$\boxed{\tau_R = \left(V_{j_t} - \frac{ND}{2}\right) \int_{out} d\dot{m} = \left(V_{j_t} - \frac{ND}{2}\right) \rho Q} \quad (29-2)$$

THIS RESULT IS INDEPENDENT OF NO. OF ARMS.

now $\int_{in} r \underline{V} (\rho \underline{V}_{rel}) d\underline{A} = 0$

since $\underline{V}_{in} = \frac{Q}{A} \underset{\sim}{k}$

$d\underline{A} = ds\, \hat{i}_r$

$\underset{\sim}{k} \cdot \hat{i}_r = 0$

since $\underset{\sim}{k} \perp i_r$

IF n ARMS EQ (28-1) BECOMES

$$\boxed{V_j = \frac{1}{n} \frac{Q}{A}} \quad (29-3)$$

COMBINE (29-1), (29-3) WITH (29-2) SO:

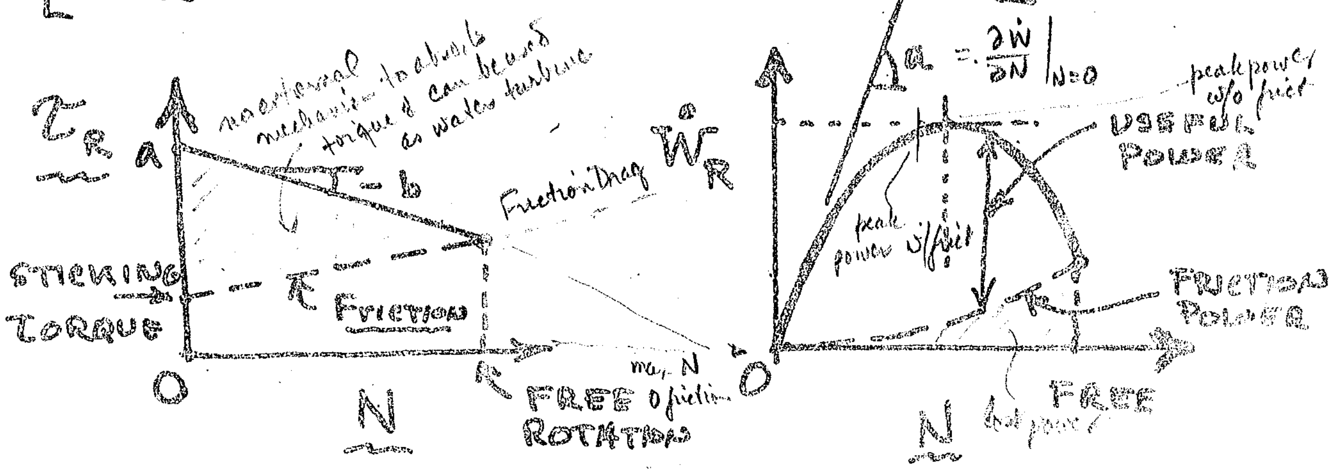
$$\tau_R = \left[\frac{D}{2\pi}(\rho Q^2)\frac{1}{A}\overset{V_{jt}}{\cos\beta\cos\alpha}\right] - \left[\frac{D^2}{4}\rho Q\right]N$$

$$\tau_R = [a] - [b]N$$

$$N = \frac{[a]}{[b]} - \frac{\tau_R}{[b]} = \left(\frac{2}{\pi}\right)\frac{Q}{DA}\cos\beta\cos\alpha - \frac{4\tau_R}{\rho Q D^2}$$

MECHANICAL POWER DELIVERED TO ENVIORMENT (USEFUL + FRICTION LOSSES)

$$\left[\dot{W}_R = \tau_R N = [a]N - [b]N^2\right]$$

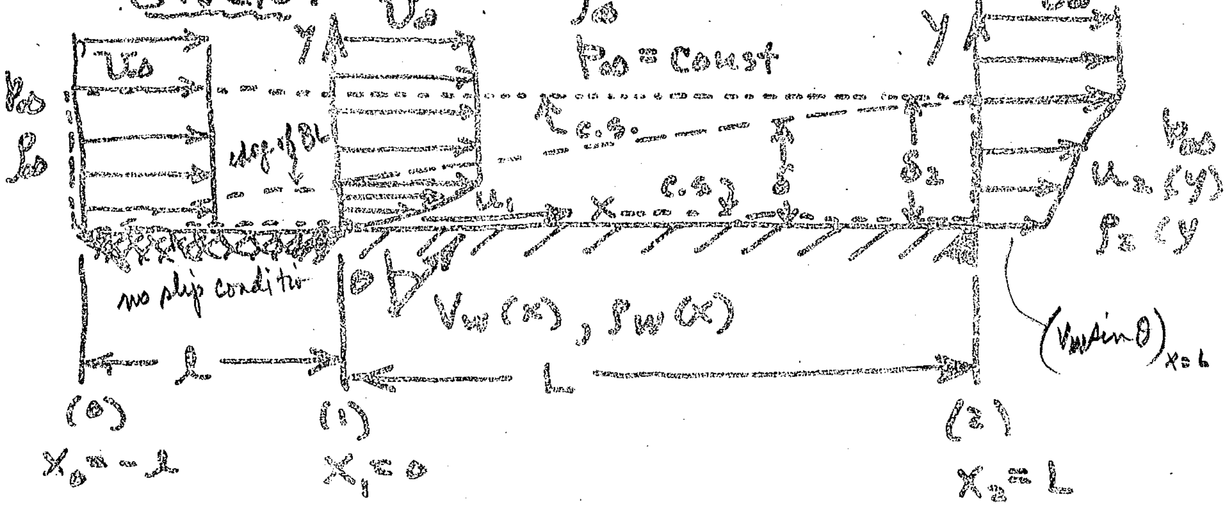


DESIGN PARAMETERS IN [a] AND [b]

EXAMPLE - 3 DRAG OF FLAT WALL WITH MASS INJECTION - (2-D FLOW)

FIND: TOTAL F_D PER UNIT WIDTH

GIVEN:

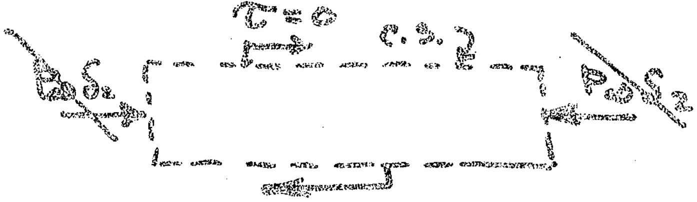


KNOWN: ALL DIMENSIONS, P_∞, ρ_∞, U_∞, $u_2(y)$, $\rho_2(y)$, $\theta(x)$, $V_w(x)$, $\rho_w(x)$.

ASSUME:
1. 2-D FLOW, UNIT WIDTH
2. STEADY FLOW ($\partial\vec{V}/\partial t$) = 0
3. $\partial p/\partial x = 0$
4. C.S. AS RECTANGLE SHOWN; $(l + L)$ LONG, δ_2 HIGH.
5. F_D = TOTAL DRAG (ON PLATE IN + X DIRECTION)
6. STREAMLINES OUTSIDE $(y > \delta)$ BOUNDARY LAYER HAVE P_∞ = CONST., U_∞ = CONST., NO SHEAR STRESS.
7. $P(y) = P_\infty$ = CONST ACROSS δ_2

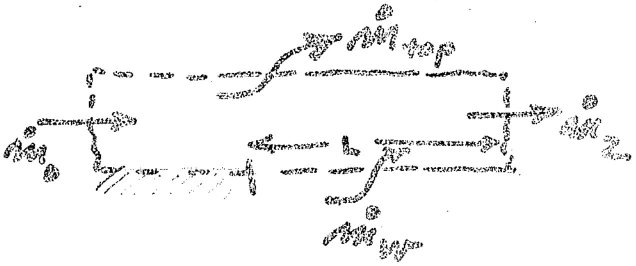
ok for turbulent flow if we use long time average } 2.

from prandtl BL dimension analysis

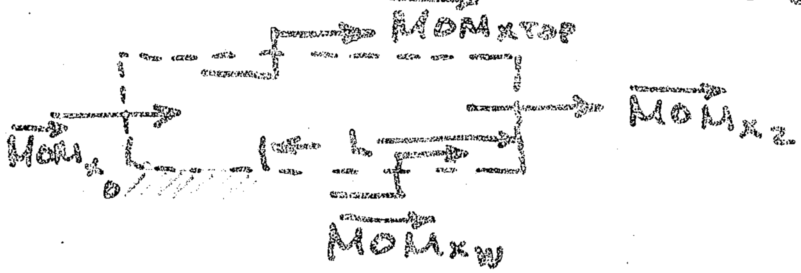


free of plate on fluid $F_D = \sum F_x$

FORCES IN X - DIRECTION



MASS FLOWS ACROSS CS



MOMENTUM FLOWS IN X - DIRECTION

(1.) R.O.C. (MASS) $\dot m_{out} - \dot m_{IN} = 0$

So $\boxed{\dot m_{top} = \dot m_0 + \dot m_W - \dot m_2}$

$\underline{V}\cdot\underline{n}\, dA$ $\quad \underline{n} = -\underline{j}$

$$\boxed{\dot m_{top} = \delta_2 \rho_\infty V_\infty + \int_0^L \rho_W V_W \cos\theta \, dx - \int_0^{\delta_2} \rho_2 u_2 \, dy} \quad (32-1)$$

$\underline{V} = |V| \underline{e}_1$

$\therefore \underline{V}\cdot\underline{n} = |V| \cos(\underline{n}, \underline{e})$

ALL QUANTITIES ON R.H.S. KNOWN

(2.) LINEAR MOM. IN +x DIRECTION

$$\int_{CV} \cancel{\frac{\partial(\rho u)}{\partial t}}^{\,0} dV + \int_{OUT} u\, d\dot{m} - \int_{IN} u\, d\dot{m} = \sum F_x$$

$$\boxed{-F_D = \Big[\underbrace{\overset{\dot{m}}{\int_0^{\delta_2}} \rho_2 u_2^2\, dy}_{u_2\, d\dot{m}_2} + U_\infty \underbrace{\dot{m}_{top}}_{\substack{\text{OBTAIN} \\ \text{FROM} \\ (32\text{-}1)}} \Big] - \Big[\underbrace{\rho_\infty U_\infty^2 \delta_2}_{U_\infty \dot{m}_0} + \underbrace{\int_0^L \rho_w V_w^2 \cos\theta \sin\theta\, dx}_{(\dot{M}OM\, x_w)} \Big]}$$

$\underline{V} = V_w \sin\theta$

$\underline{V}_{\perp} \cdot dA = V_w \cos\theta\, dx$

(33-1)

SPECIAL CASE:

ASSUME: $\rho_w = \rho_\infty = \rho_2 = \boxed{\text{const.} \equiv \rho}$

$V_w \cos\theta \equiv \boxed{v_w = \text{const.}}$

$V_w \sin\theta \equiv \boxed{u_w = \text{const.}}$

SO: Karman/Polhausen method $\boxed{\eta_2 \equiv y/\delta_2}$

$$+F_D = \rho U_\infty^2 \delta_2 + \rho v_w u_w L - \rho U_\infty^2 \delta_2 \int_0^1 \left(\frac{u_2}{U_\infty}\right)^2 d\eta_2 - U_\infty \dot{m}_{top}$$

$$\dot{m}_{top} = \delta_2 \rho U_\infty + \rho v_w L - \rho U_\infty \delta_2 \int_0^1 \left(\frac{u_2}{U_\infty}\right) d\eta_2$$

COMBINE:

$$\boxed{\frac{F_D}{\delta_2 \rho U_\infty^2} = \int_0^1 \left(\frac{u_2}{U_\infty}\right)\left(1 - \frac{u_2}{U_\infty}\right) d\eta_2 + \left(\frac{v_w}{U_\infty}\right)\left(\frac{u_w - U_\infty}{U_\infty}\right)\frac{L}{\delta_2}}$$

(33-2)

$$C_D = \frac{\text{drag force}}{\text{dynamic press} \times \text{area of plate}}$$

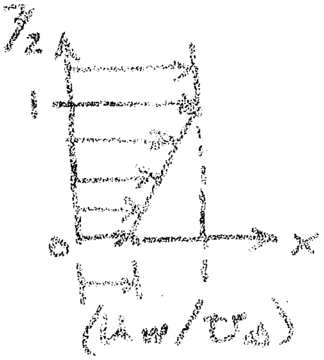
effect due to injection

$$C_D = \frac{F_D}{\frac{1}{2}\rho V_\infty^2 (L+l)} = \frac{2\delta_2}{L+l}\int_0^1 \left(\frac{u_2}{V_\infty}\right)\left(1 - \frac{u_2}{V_\infty}\right) d\eta_2 + \frac{2L}{L+l}\left(\frac{v_w}{V_\infty}\right)\left(\frac{u_w - V_\infty}{V_\infty}\right)$$

depend on $R_N = \frac{V_\infty (L+l)}{\nu_\infty}$

DISCUSSION

(B) SPECIAL CASE WHERE



$$\frac{u_2}{U_\infty} = \left(1-\frac{u_w}{U_\infty}\right)\eta + \frac{u_w}{U_\infty}$$

SO $$\int_0^1 \left(\frac{u_2}{U_\infty}\right)\left(1-\frac{u_2}{U_\infty}\right)d\eta = \frac{1}{6} - \frac{5}{6}\left(\frac{u_w}{U_\infty}\right) + \frac{2}{3}\left(\frac{u_w}{U_\infty}\right)^2$$

$$C_D = \frac{F_D}{\text{PLATE AREA}\,(\rho U_\infty^2/2)} = \frac{2F_D}{(L+\ell)\rho U_\infty^2}$$

$$\boxed{C_D = \frac{1}{3}\frac{\delta_2}{L+\ell}\left[1-5\left(\frac{u_w}{U_\infty}\right)+4\left(\frac{u_w}{U_\infty}\right)^2\right] + 2\left(\frac{v_w}{U_\infty}\right)\left[1-\frac{u_w}{U_\infty}\right]\left(\frac{L}{L+\ell}\right)}$$

NOTE: SPECIAL CASE $\frac{u_w}{U_\infty} = \frac{v_w}{U_\infty} = 0$

$$C_D = \underbrace{\frac{1}{3}}\left(\frac{\delta_2}{L+\ell}\right)$$ NO SLIP AT WALL

TURBULENT B.L.
WHERE $u_2/U_\infty = (\eta)^{1/7}$
HAS (14/72)

APPLICATIONS OF EQ'S OF AN INVISCID FLOW (EQ'S 18-1), (19-1)

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + f_{bi} \qquad \begin{matrix}(i=1,2,3)\\ \left(\sum_{j=1}^{3}\right)\end{matrix}$$

AND

$$\frac{\partial \vec{V}}{\partial t} + \vec{\nabla}\left(\frac{V^2}{2}\right) - \vec{V}\times(\vec{\nabla}\times\vec{V}) = -\frac{1}{\rho}\vec{\nabla}p + \vec{f}_b$$

THESE ARE EULER'S EQ'S IN rect. coordinates AND general vector form

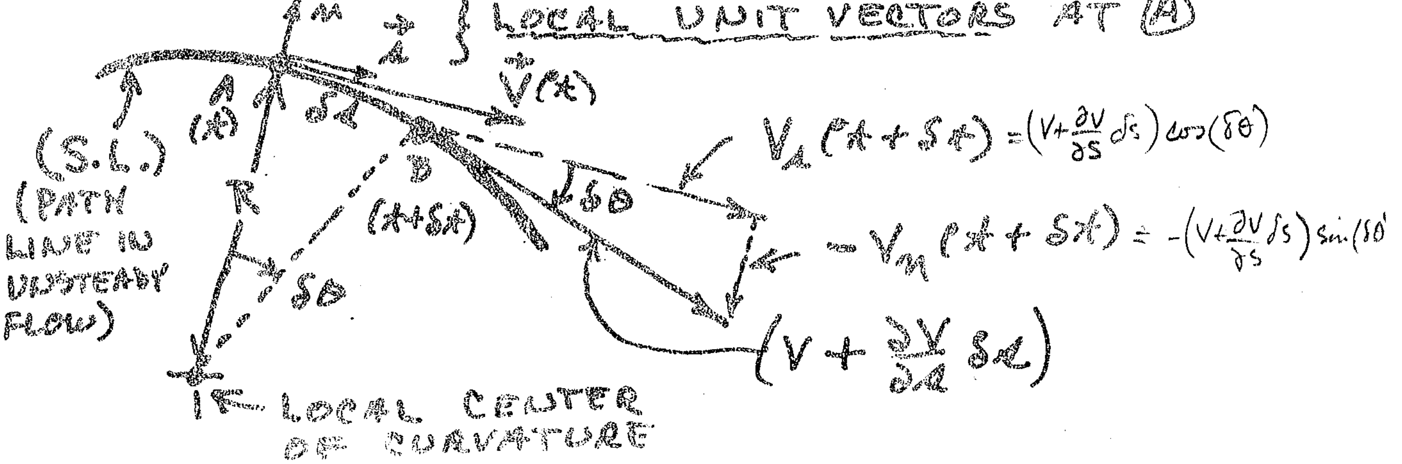
EULER'S EQ'S IN LOCAL STREAMLINE (NATURAL) COORDINATES

Starting Assumptions (useful but not necessary):

(1.) 2-D, plane flow

(2.) Steady Flow, $\partial\vec{V}/\partial t = 0$

KINEMATICS OF POINT PARTICLE:



VELOCITY AND ACCELERATION AT Ⓐ

$$\vec{V} = \vec{s}\,V_s(t) + \vec{n}\,V_n(t)$$

But:

$$\boxed{\vec{V} = \vec{s}\,V(t) + \vec{n}\,(0)}$$

$$\boxed{\vec{a} = \vec{s}\,a_s(t) + \vec{n}\,a_n(t)}$$

CALCULATION OF a_s AND a_n

$$a_s = \lim_{\delta t \to 0}\left[\frac{V_s(t+\delta t) - V_s(t)}{\delta t}\right]$$

$$a_s = \lim_{\delta t \to 0}\left[\frac{\left(V + \frac{\partial V}{\partial s}\delta s\right)\cos(\delta\theta) - V}{\delta t}\right]$$

BUT IN limit $\delta t \to 0$

$\cos(\delta\theta) \to 1$, $\sin(\delta\theta) \to \delta\theta$

$\delta s \to V\,\delta t$. SO:

this a_s is for Steady flow.

$a_s = V\dfrac{\partial V}{\partial s}$, THE CONVECTIVE ACCELERATION

IN UNSTEADY FLOW ADD LOCAL, SO:

$$\boxed{a_s = \frac{\partial V_s}{\partial t} + V\frac{\partial V}{\partial s}} \quad (36\text{-}1)$$

$$a_n = \lim_{\delta t \to 0} \left[\frac{V_n(t+\delta t) - V_n(t)}{\delta t} \right]$$

$$a_n = \lim_{\delta t \to 0} \left[\frac{-\left(V + \frac{\partial V}{\partial s}\delta s\right)\sin(\delta\theta) - 0}{\delta t} \right]$$

$$a_n = \lim_{\delta t \to 0} \left[-V \underbrace{\frac{\delta\theta}{\delta t}}_{R\delta\theta = \delta s} - \frac{\partial V}{\partial s} \underset{\substack{\downarrow \\ V}}{\frac{\delta s}{\delta t}} \underset{\substack{\downarrow \\ 0}}{\delta\theta} \right]$$

$$\therefore \; a_n = \lim_{\delta t \to 0} \left[- \frac{\delta s}{\delta t} \frac{V}{R} \right] = - \frac{V^2}{R}$$

CENTRIFUGAL ACCELERATION — A CONVECTIVE EFFECT.

IN UNSTEADY FLOW ADD LOCAL ACCELERATION SO:

$$\boxed{a_n = \frac{\partial V_n}{\partial t} - \frac{V^2}{R}} \quad (37\text{-}1)$$

NOTE ALL QUANTITIES FUNCTIONS OF t AND s (DISTANCE ALONG PATHLINE) IN EQ'S (36-1), (37-1)

FORCES ON PARTICLE IN s-n COORDS

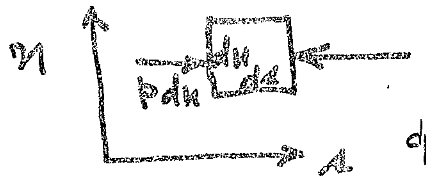
INVISCID FLOW: ① PRESSURE FORCES, f_p

② BODY FORCES, f_b

(a) Conservative

(b) Gravity (Special case), f_g

PRESSURE FORCES



$(p + \frac{\partial p}{\partial s} ds) dn$ (s-direction)

$dp = -\frac{\partial p}{\partial s} ds\, dn$ now $\rho \frac{dp}{ds\, dn} = \frac{\text{pressure}}{\text{unit mass}}$

PER UNIT MASS:

$$\vec{f}_p = \vec{s} f_{p_s} + \vec{n} f_{p_n}$$

$$\boxed{\begin{aligned} f_{p_s} &= -\frac{1}{\rho}\frac{\partial p}{\partial s} \\ f_{p_n} &= -\frac{1}{\rho}\frac{\partial p}{\partial n} \end{aligned}} \quad (38\text{-}1)$$

CONSERVATIVE BODY FORCES (PER UNIT MASS)

$$\vec{f}_b = \vec{s} f_{b_s} + \vec{n} f_{b_n}$$

IF (U) IS A SCALER POINT FUNCTION THEN:

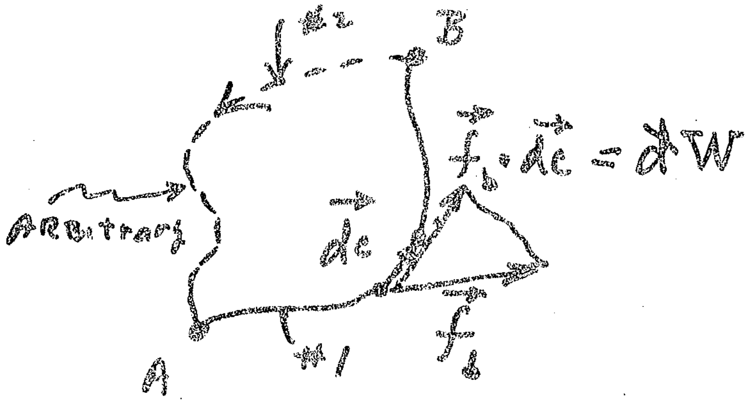
$$\boxed{(\vec{f}_b)_{\text{conservative}} = \vec{\nabla} U = \frac{\partial U}{\partial x_i}} \quad (38\text{-}2)$$

why is this a conservative force field – check the work done

NOTE: $U(x,y)$; $U(X_i)$ $i=1,2,3$;
$U(r,\theta,z)$; etc. MUST BE
SINGLE VALUED, SMOOTH, ETC.

WHY CALL $\vec{\nabla} U$ CONSERVATIVE?

CALCULATE WORK DONE BY $\vec{f}_b$ AS A PARTICLE TRACES A PATH IN SPACE



if $\vec{f}_b = \vec{\nabla} U$

$$\text{đ}W = \vec{\nabla} U \cdot d\vec{l}$$

$$\text{đ}W = dU$$

$$\therefore \int_A^B \text{đ}W = \int_A^B dU = U_B - U_A$$

ALSO: $$\int_B^A \text{đ}W = U_A - U_B$$ depends on end points only (not on path.)

$$\therefore \oint \text{đ}W = \underset{\substack{\text{PATH} \\ \#1}}{\int_A^B} \text{đ}W + \underset{\substack{\text{PATH} \\ \#2}}{\int_B^A} \text{đ}W = (U_B - U_A) - (U_A - U_B) = 0$$

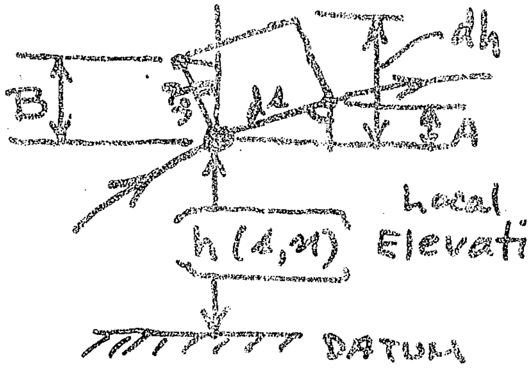
NET EFFECT ON SYSTEM $= 0$

GRAVITY BODY FORCES

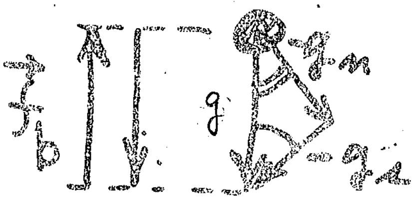
$\vec{g}$: GRAVITATIONAL ACCEL. OR FORCE ON MATTER PER UNIT MASS

$$\vec{f}_g = \vec{g} \quad ; \quad f_g = g$$

$$\boxed{\vec{f}_g = \vec{s}\, g_s + \vec{n}\, g_n}$$



$$dh = A + B = \frac{\partial h}{\partial s} ds + \frac{\partial h}{\partial n} dn$$



USING CORRESPONDING LEGS AND ANGLES ONE OBTAINS

$$\cos \measuredangle = -\frac{g_s}{g} = \frac{A}{ds} = \frac{\partial h}{\partial s}$$

$$\cos \measuredangle = -\frac{g_n}{g} = \frac{B}{dn} = \frac{\partial h}{\partial n}$$

$$\therefore \quad \boxed{\vec{f}_g = \vec{s}\left(g\frac{\partial h}{\partial s}\right) + \vec{n}\left(g\frac{\partial h}{\partial n}\right)} \quad (40-1)$$

FOR $g = \text{Const}$; $\boxed{U = (-g h)}$ (40-2)

$$\vec{f}_g = \vec{\nabla}(-gh)$$

FINALLY:

EULER'S S + n EQ'S

ALONG A STREAMLINE (s+ in $\vec{V}$ dir)

$$\boxed{\frac{\partial V_s}{\partial t} + V\frac{\partial V}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + \frac{\partial U}{\partial s}} \quad (41\text{-}1)$$

NORMAL TO A STREAMLINE (n+ in direction out from local center of curvature)

$$\boxed{\frac{\partial V_n}{\partial t} - \frac{V^2}{\underset{\text{instantaneous radius of curvature}}{R}} = -\frac{1}{\rho}\frac{\partial p}{\partial n} + \frac{\partial U}{\partial n}} \quad (41\text{-}2)$$

FOR CONSTANT g FIELD AT A POINT h ABOVE DATUM

$$\boxed{U = -gh} \quad (40\text{-}2)$$

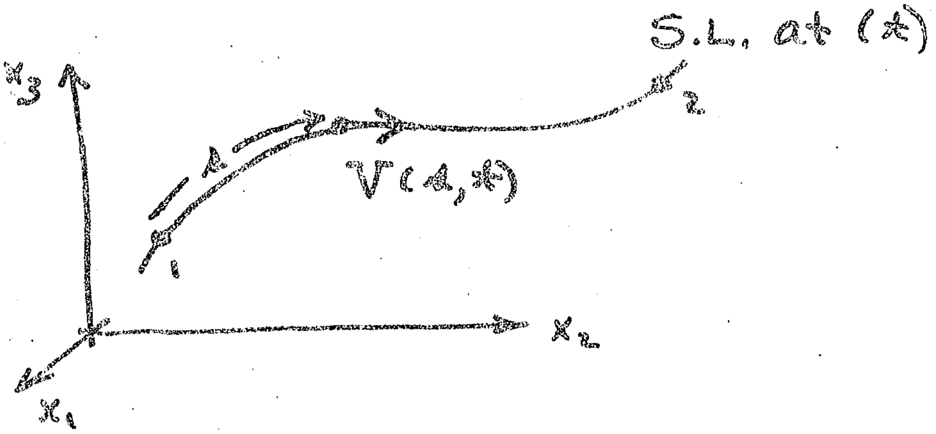
ASSUMPTIONS: (1) INVISCID FLOW
(2.1) g FIELD CONST
or
(2.2) CONSERVATIVE BODY FORCE

INTEGRATION OF EULER'S S-EQ

INTEGRATE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial V^2}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial s} - \frac{\partial U}{\partial s} = 0$$

ALONG A STREAMLINE AT AN INSTANT OF TIME (t)



this is from the fact that streamline varies location w/time

since integration is wrt space coordinates only

$$\underbrace{\int \frac{\partial V}{\partial t}\,ds}_{\text{TERM Ⓐ}} + \frac{V^2}{2} + \underbrace{\int \frac{dp}{\rho}}_{\text{TERM Ⓑ}} - U = B(t) \qquad (42-1)$$

or

$$\int_1^2 \frac{\partial V}{\partial t}\,ds + \frac{1}{2}\left(V_2^2 - V_1^2\right) + \int_1^2 \frac{dp}{\rho} - (U_2 - U_1) = 0$$

VALID WHEN ASSUMPTIONS ARE:

(1) INVISCID FLOW no shear term

(2) CONSERVATIVE BODY FORCE

(3) ALONG A STREAMLINE

ME251A 7/7/78 ㊸

EXTENSIONS OF THIS INTEGRAL

ASSUMPTION (A) STEADY FLOW

$$\boxed{\frac{\partial V}{\partial t} = 0} \quad \text{SO} \quad \boxed{\text{TERM } Ⓐ = 0}$$

BUT INTEGRATION CONSTANT B(t) MAY STILL BE TIME DEPENDENT

ASSUMPTION (B) BAROTROPIC FLUID

$$\boxed{p(\rho) \ \underline{\text{or}} \ \rho(p)}$$

THUS IN TERM Ⓑ $\boxed{\frac{dp}{\rho} = \frac{dp}{\rho(p)} = di}$

WHERE PROPERTY IS TERM Ⓑ $\boxed{i = \int \frac{dp}{\rho} + \text{const.}}$

EXAMPLE: ISENTROPIC FLOW OF A SIMPLE COMPRESSIBLE SUBST.

GIBBS EQ OF STATE:

$$T \underset{=0}{\cancel{ds}} = dh - \frac{1}{\rho} dp$$

$$\therefore \underline{dh} = \frac{1}{\rho} dp = \underline{di}$$

$$\underline{i = h + \text{const}}$$

ME 251A 7/7/78 (44)

EXAMPLE ISOTHERMAL FLOW OF A PERFECT GAS

$$\rho = \frac{p}{RT} \quad ; \quad T = \text{const.}$$

$$i = RT \int \frac{dp}{p} + \text{const}$$

$$\underline{i = RT \ln p + \text{const}}$$

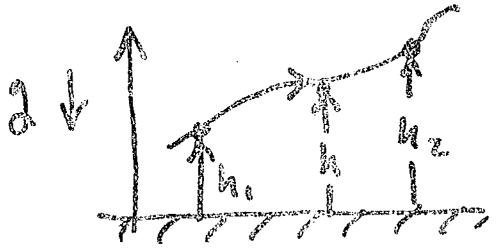
ASSUMPTION (5A) INCOMPRESSIBLE

$$\boxed{\rho = \text{const}}$$

SO TERM B IS $\boxed{\frac{p}{\rho} + \text{const}}$

CLASSICAL BERNOULLI THEOREM

TAKE EQ (42-1) WITH ASSUMPTIONS (4), (5A) AND GRAVITY BODY FORCE



indefinite form

$$\frac{V^2}{2} + \frac{p}{\rho} + gh = B \qquad (44-1)$$

or

$$\frac{V_1^2}{2} + \frac{p_1}{\rho} + gh_1 = \frac{V_2^2}{2} + \frac{p_2}{\rho} + gh_2$$

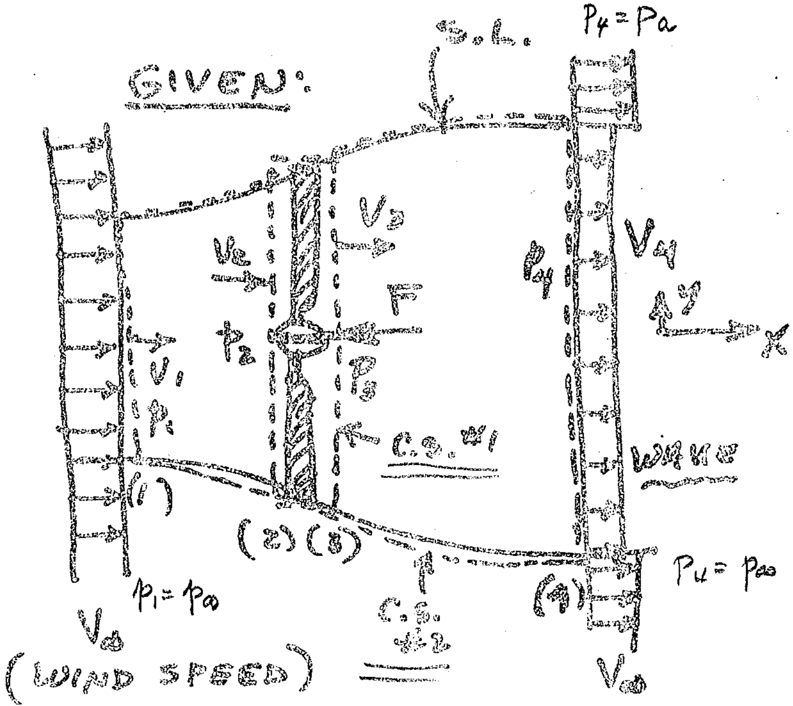
$$U = -gh$$

ME 251 A   LECTURE 8   77/78 ⑩

APPLICATIONS OF BERNOULLI

EXAMPLE-1 FIND: EXPRESSIONS FOR MAX THEORETICAL POWER AND THRUST FORCE ON AN AXIAL WIND TURBINE IN A STEADY (GUST FREE) WIND OF UNIFORM SPEED.

GIVEN:



ASSUME

1. UNIFORM, SWIRL FREE FLOW AT 1, 2, 3 AND 4.
2. $p_1 = p_4 = p_\infty$ AND ALL p's CONST ACROSS SECTIONS (1-D flow)
3. ρ = CONST.
4. STEADY FLOW
5. $A_2 = A_3$
6. SHEAR STRESSES = 0 FROM 1 TO 2 AND FROM 3 TO 4

CONTROL SURFACE #1

CONS. OF MASS:

$$\cancel{\rho_2 A_2} V_2 = \cancel{\rho_3 A_3} V_3$$

Using assump #3, 5

$$\therefore \boxed{V_2 = V_3 = V_D}$$

V_D = MEAN VELOCITY THRU ROTOR DISK

WHOSE AREA = $\boxed{A_D = A_2 = A_3}$

LINEAR MOMENTUM (+ X DIRECTION):

$$\Sigma F_x = \cancel{\frac{\partial}{\partial t}\int_{CV} V_x \rho\, dV} + \int_{OUT} V_x\, d\dot{m} - \int_{IN} V_x\, d\dot{m}$$

C.S. #1

$$-F + p_2 A_2 - p_3 A_3 = \dot{m}\cancel{V_3} - \dot{m}\cancel{V_2}$$

since conserv of mass eq ⇒ $V_2 = V_3$

$$\therefore \boxed{F = (p_2 - p_3)A_D} \quad (46-1)$$

C.S. #2 NOTE: $p_1 = p_4 = p_\infty$

ASSUME: $p = p_\infty$ ALONG S.L. FROM (1) → (4) TOO

$p_\infty \rightarrow$

$$\Sigma F_x = -F + p_1 A_1 - \underbrace{p_4 A_4}_{p_\infty(A_1 - A_4)}$$

$+ p_\infty$ [surface area component in x direct] → $A_4 - A_1$

$\therefore p_\infty[A_4 - A_1]$

$$-F = \dot{m}V_4 - \dot{m}V_1 = \rho A_D V_D (V_4 - V_1)$$

and $-F + p_\infty(A_1 - A_4) + p_\infty(A_4 - A_1)$

hence only pressure forces give you $-F$ on CS#2

$$\boxed{F = \rho A_D V_D (V_1 - V_4)} \quad (46-2)$$

AS A RESULT OF TWO EQ'S ABOVE

$$\boxed{(p_3 - p_2) = \rho V_D (V_4 - V_1)} \quad (46-3)$$

NOW APPLY BERNOULLI FROM (1) → (2) AND FROM (3) → (4). SO:

$$\boxed{\begin{aligned} p_1 + \tfrac{1}{2}\rho V_1^2 &= p_2 + \tfrac{1}{2}\rho V_2^2 \\ p_3 + \tfrac{1}{2}\rho V_3^2 &= p_4 + \tfrac{1}{2}\rho V_4^2 \end{aligned}} \quad (46-4)$$

WHY CAN'T WE APPLY BERNOULLI FROM (1) → (4), (2) → (3) OR (2) → (4)?

$$(P_2 - P_3) = \cancel{(P_1 - P_4)} + \frac{\rho}{2}(V_1^2 \cancel{- V_2^2} \cancel{+ V_3^2} - V_4^2)$$

since $V_2 = V_3$

since $P_1 = P_4 = P_\infty$

→ because there's energy (to stream tube) transfered

NOW NOTE THAT $\boxed{p_1 = p_4 = p_\infty}$

SOLVE EQ'S (46-4): $\boxed{V_2 = V_3 = V_D}$

$$\boxed{p_3 - p_2 = \frac{1}{2}\rho\left(V_4^2 - V_1^2\right)} \quad (47\text{-}1)$$

COMPARE THIS TO RESULT (46-3) AND WE SEE THAT

$$\boxed{V_D = \frac{1}{2}\left(V_1 + V_4\right)} \quad (47\text{-}2)$$

NOW USE C.S. #1 TO OBTAIN THE POWER FROM THE TURBINE USING CONSERVATION OF ENERGY:

$$\cancel{\frac{d}{dt}\int_{CV} e\rho\, dV}^{\,\text{steady},\ 0} + \int_{OUT}\left(\tilde{u} + \frac{p}{\rho} + \frac{V^2}{2} + gz\right)d\dot{m} - \int_{IN}\left(\tilde{u} + \frac{p}{\rho} + \frac{V^2}{2} + gz\right)d\dot{m}$$

$$\left(\text{LOSS FREE}\right)$$

$$-\cancel{\dot{Q}} - \dot{W}_{CS(OTHER)} - \cancel{\dot{W}_{CV}}^{\,0} = 0$$

in REVERSIBLE, INCOMP. FLOW (LOSS FREE)

$$\dot{Q} = \int_{out} \tilde{u}\, d\dot{m} - \int_{in} \tilde{u}\, d\dot{m}$$

$$\Rightarrow \left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_3 - \left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_2 = \frac{\dot{W}_{cs}}{\dot{m}}$$

Ie the difference in bernoulli term must be = work done to the CV.
normally bernoulli should give $\dot{W}_{cs} = 0$

ME251A 77/78 ㊽

$$-\dot{W}_{\substack{C.S.\\(OTHER)}} = \mathbb{P}_{Turbine} = \mathbb{P}_T$$

SHAFT POWER TO ELECTRIC GENERATOR GRINDING WHEEL, PUMP, ETC.

THUS ENERGY EQ. BECOMES FOR LOSS FREE CONDITIONS AT C.S. #1 $\Rightarrow V_2 = V_3$ and $z_3 = z_2$

$$\frac{p_3}{\rho}\dot{m} - \frac{p_2}{\rho}\dot{m} + \mathbb{P}_{T_{MAX}} = 0$$

Note: NO SWIRL AT (3) IS DESIGN POINT REQUIREMENT ON TURBINE BLADES, SHAFT SPEED, ETC.

SINCE: $\dot{m} = \rho A_D V_D$ not correct unitwise

$$\mathbb{P}_{T_{MAX}} = A_D V_D (p_2 - p_3) \text{X}$$

$$-w_s = \frac{P_3 - P_2}{\rho} + \frac{1}{2}\cancelto{0}{(V_3^2 - V_2^2)} + g\cancelto{0}{(z_3 - z}$$

$$-w_s = \frac{1}{2}(V_4^2 - V_1^2) = \frac{1}{2}V_1^2(\eta^2 - 1)$$

$$= \frac{1}{2}(V_4 + V_1)(V_4 - V_1)$$

$w_s = V_D(V_1 - V_4)$
per unit mass

USE EQ (47-1) AND (47-2)

$$\mathbb{P}_{T_{MAX}} = A_D \frac{1}{2}(V_4 + V_1)\frac{\rho}{2}(V_4^2 - V_1^2)$$

$$\mathbb{P}_{T_{MAX}} = \underbrace{(\rho A_D \cdot V_1)}_{\dot{m}_\infty}\underbrace{\left(\frac{V_1^2}{2}\right)}_{(K.E.)_\infty}\frac{1}{2}(\eta + 1)(1 - \eta^2)$$

WHERE: $\boxed{\eta = \frac{V_4}{V_1}}$

$\frac{lb\text{-}ft}{\frac{lb\text{-}sec^2}{ft}} = \frac{ft^2}{sec^2}$ //

POWER COEFFICIENT:

$$C_{\mathcal{P}} = \frac{\mathcal{P}_T}{(\rho A_D V_\infty)\left(\frac{V_\infty^2}{2}\right)}$$

MAX THEORETICAL POWER COEFF:

$$C_{\mathcal{P}_{MAX}} = \frac{1}{2}\left(\zeta - \zeta^3 + 1 - \zeta^2\right)$$

FIND VALUE OF ζ AT OPTIMUM

$$\frac{d C_{\mathcal{P}_{MAX}}}{d\zeta} = 0 = \frac{1}{2}\left(1 - 3\zeta^2 - 2\zeta\right)$$

WHICH IS SATISFIED FOR $\boxed{\zeta = \frac{1}{3}}$

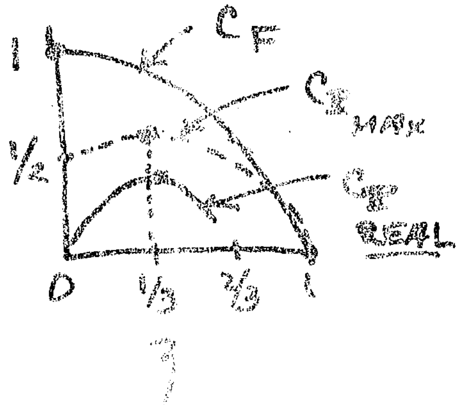
SO $\boxed{C_{\mathcal{P}_{MAX}} \leq \frac{16}{27} \leq 0.593}$

THRUST (FROM 46-2) THRUST COEF.

$$\frac{F}{\rho A_D V_\infty^2/2} = C_F = (1+\zeta)(1-\zeta) = 1 - \zeta^2$$

AT $\zeta = 1/3$; $C_F = 8/9$

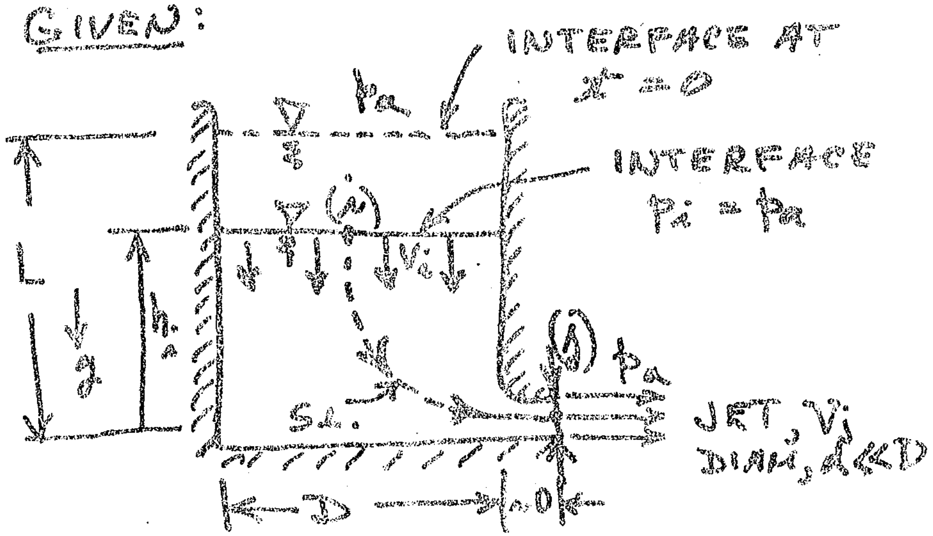
NOTE: $C_{\mathcal{P}}$ AND C_F CURVES ARE MAX AT DIFFERENT POINTS



ME 251 A 11/28

EXAMPLE - 2 FIND: TIME NECESSARY TO DRAIN A LARGE TANK THROUGH A SMALL NOZZLE

GIVEN:



ASSUME:

1. ρ = CONST.
2. INVISCID FLOW
3. V_j AND V_i ARE ~~UNIFORM~~ unifor.
4. $\vec{f}_B = -g$
5. NOZZLE VOLUME SMALL RE TANK VOLUME

CONSERVATION OF MASS:

C.S. INCLUDES ALL FLUID INSIDE TANK AT TIME t

$$\frac{d}{dt}\int_{CV}\rho\, d\forall + \int_{OUT}\rho V_{rel}\cdot dA - \int_{IN}\rho V_{rel}\cdot dA = 0$$

$$\cancel{\rho}\frac{d}{dt}(\forall)_{CV} + \cancel{\rho}\frac{\pi}{4}d^2 V_j - 0 = 0$$

BUT $(\forall)_{CV} = \frac{\pi}{4}D^2 h_i$ $\therefore$ $\boxed{\frac{dh_i}{dt} = -\frac{d^2}{D^2} V_j}$

Since nozzle volume is small.

([illegible])

ME 251A 77/78 (51)

APPLY $\int$ EULER'S S-EQ. ALONG S.L.

instantaneous heights

$$0 = \underbrace{\int_i^j \frac{\partial V}{\partial t}\,ds}_{} + \frac{1}{2}\underbrace{(V_j^2 - V_i^2)}_{\boxed{V_i = \frac{d^2}{D^2} V_j}\ (51-1)} + \frac{1}{\rho}\underbrace{(p_j - p_i)}_{p_j = p_i = p_a} + g\underbrace{(h_j - h_i)}_{h_j \approx 0}$$

mass conservation

$$0 \approx \underbrace{h_i \frac{dV_i}{dt}}_{} + \underbrace{\frac{V_j^2}{2}\left(1 + \frac{d^4}{D^4}\right)}_{\text{CONS. OF MASS}} + 0 - g h_i$$

ALONG S.L. $V \approx V_i$ EXCEPT IN AND VERY NEAR NOZZLE ESPECIALLY WHEN $h_i \gg D$ SO APPROXIMATE TERM IS

$$\approx \frac{dV_i}{dt}\int_i^j ds \approx \frac{dV_i}{dt} h_i$$

assume that $\int \frac{dV}{dt} ds$ near i is negligible wrt rest of tank

RESULTING APPROX. EQ IS (USING 51-1)

$$\boxed{0 = \frac{d^2}{D^2} h_i \frac{dV_j}{dt} + \frac{V_j^2}{2}\left(1 - \frac{d^4}{D^4}\right) - g h_i} \quad (51-2)$$

GIVEN INITIAL COND'S EQ.(51-1) AND (50-1) ARE SOLVED FOR $h_i(t)$, $V_j(t)$

I.C.'s: $h_i = L$ AT $t = 0$

$h_i \approx 0$ AT $t = T$, TIME TO EMPTY.

LIMITING CASE $\left(\frac{d}{D} \lll 1\right)$ SO FROM EQ (51-2) WE OBTAIN "QUASI-STATIC" RESULT:

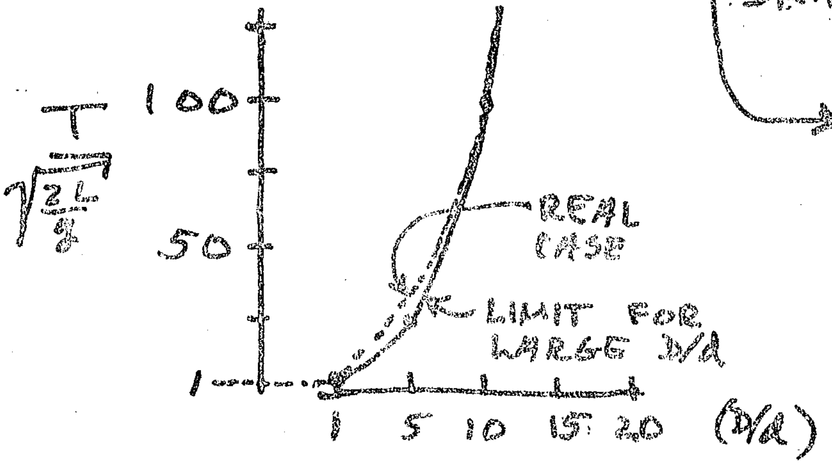
drop $\frac{dh_i}{dt} \cdot \frac{d^2}{D^2}$ & $V_j^2 \cdot \frac{d^4}{D^4}$ terms small w.r.t

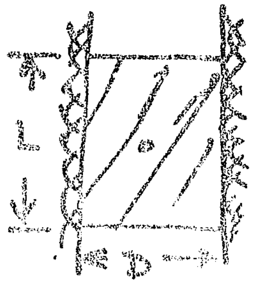
$$V_j^2 = 2 g h_i$$

PUT THIS INTO EQ (50-1) AND INTEGRATE

$$\int_0^L \frac{dh_i}{\sqrt{h_i}} = -\sqrt{2g}\left(\frac{d^2}{D^2}\right)\int_0^T dx$$

SO:
$$T = \underbrace{\sqrt{\frac{2L}{g}}\left(\frac{D}{d}\right)^2}_{\text{TIME FOR DRAINAGE OF SLUG}}$$





NUMBERS: LET: L = 1 ft, g = 32.17 ft^2/sec

D/d = 10

SO: T = 25 sec

STEADY INVISCID FLOW

HOW DO WE FIND THE $\vec{V}(\vec{x})$ AND STREAMLINES?

RESTRICT ATTENTION TO:

1. STEADY FLOW or $\frac{\partial v}{\partial t} = 0$
2. INVISCID " (mixing, boundary layer, shock waves problems)
3. ρ = CONSTANT
4. Conservative $\underline{f_b}$

INTEGRAL OF EULER EQ'S ALONG $\underline{\underline{A}}$ S.L. which satisfies the 4 items above.

$$\boxed{\frac{V^2}{2} + \frac{p}{\rho} - U = B} \quad (53\text{-}1)$$

B IS CONSTANT FOR $\underline{\underline{A}}$ S.L.

WHEN IS B = CONST. ON $\underline{ALL}$ $\underline{S.L.'s}$?

$\underline{ONE}$ POSSIBLE CONDITION:

$$\vec{\nabla} B = 0 \quad \underline{\underline{or}} \quad \frac{\partial B}{\partial x_i} = 0$$

because $\nabla\left(\frac{v^2}{2} + P/\rho - U\right) = 0$ with $\underline{f}_B = \nabla U$. Using the euler eq & seeing where it comes from momentum gives

$$\text{Euler eq:} \; -\frac{\partial v}{\partial t} + \underline{V} \times (\nabla \times \underline{V}) = \nabla\left(\frac{V^2}{2} + \frac{P}{\rho} - U\right) = 0$$

$\nabla B = 0 \Rightarrow \frac{\partial v}{\partial t}$, $\nabla \times \underline{V} = \underline{\omega}$ as kinematic conditions

but $\frac{\partial v}{\partial t} = 0$ because of steady condition $\therefore \longrightarrow \underline{V} \times \underline{\omega} = 0$

3 solutions $\left\{\begin{matrix}\underline{V} = 0 \\ \text{fluid at rest}\end{matrix}\right\}$, $\left\{\begin{matrix}\underline{V} \times \underline{\omega} = 0 \text{ if } \underline{V} \parallel \underline{\omega} \\ \text{Beltrami flow}\end{matrix}\right\}$, $\underline{\omega} = 0$ (irrotational flow)

when can we employ $\nabla \times \underline{V} = 0$ (from momentum)

for steady flow $\nabla \cdot \underline{V} = 0$ (from Conservation)

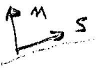
$\Rightarrow \nabla \times \underline{V}, \nabla \cdot V \Rightarrow \Delta u = \Delta v = 0$ from $\underline{V} = (u, v)$

2-D FLOW AS A SPECIAL CASE

FROM EULER'S n-EQ. THE CONDITION IS:

$$\boxed{\frac{\partial B}{\partial n}=0 \quad \text{EVERYWHERE}}$$

OPERATE ON EQ (53-1) SO:



$$\frac{\partial}{\partial n}\left(\frac{V^2}{2}+\frac{p}{\rho}-U\right)=\frac{\partial B}{\partial n}$$

$$V\frac{\partial V}{\partial n}+\frac{1}{\rho}\frac{\partial p}{\partial n}-\frac{\partial U}{\partial n}=\frac{\partial B}{\partial n} \qquad (54-1)$$

BUT EULER'S n-EQ IS (41-2 w steady flow)

$$-\frac{V^2}{R}+\frac{1}{\rho}\frac{\partial p}{\partial n}-\frac{\partial U}{\partial n}=0 \qquad (54-2)$$

SUBTRACT (54-2) FROM (54-1) SO:

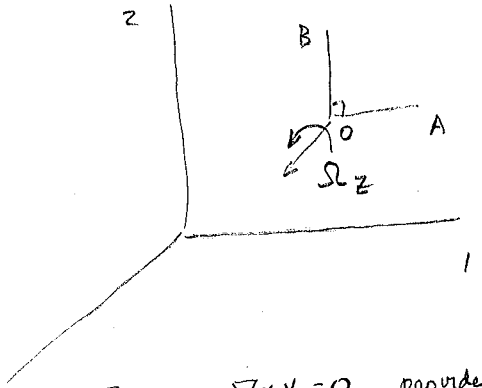
$$\boxed{V\left(\frac{\partial V}{\partial n}+\frac{V}{R}\right)=\frac{\partial B}{\partial n}} \qquad (54-3)$$

CONDITIONS FOR B = CONST. EVERYWHERE

(1.) FLUID AT REST : $V=0$

OR (2.) $\left(\frac{\partial V}{\partial n}+\frac{V}{R}\right)=0$; A KINEMATIC CONDITION

for irrotational flow



$$\Omega_z = \frac{\Omega_{OA} + \Omega_{OB}}{2}$$

where OA & OB are ⊥ line elements in the 1,2 plane

3 $\nabla \cdot \underline{V} = 0$, $\nabla \times \underline{V} = 0$ provide enough equations to solve for velocity field if <u>we</u> <u>relax</u> no slip condition (i.e potential flow)

positive rotation is right hand rule

vorticity $= \nabla \times \underline{V} = \underline{\omega} = 2\underline{\Omega}$ (2 × rate of rotation = vorticity)

WHAT IS CONDITION (2.)?

WE SHALL SHOW:

$$\frac{\partial V}{\partial n} + \frac{V}{R} = -2\Omega_z$$

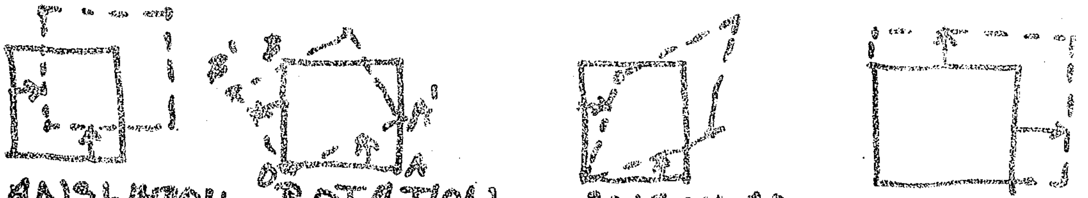
WHERE Ω_z IS LOCAL RATE OF FLUID ROTATION. THUS CONDITION (2.) IS THE STATEMENT

$$\boxed{\Omega_z = 0}$$

THE FLUID IS IRROTATIONAL

FLUID ROTATION

ELEMENTS OF KINEMATICS OF A SMALL, FINITE FLUID ELEMENT



TRANSLATION ROTATION ANGULAR DISTORTION (SHEAR STRAIN) LINEAR STRAIN

ROTATION RATE:

AVERAGE RATE OF ROTATION OF TWO PERPENDICULAR, INFINITESMAL LINES IN THE FLUID (E.G. OA AND OB ABOVE).

- for s, n coordinate system which is a local coordinate system. $\omega_z' = -\left(\frac{\partial V}{\partial n} + \frac{V}{R}\right)$ where z′ is the binormal axis which changes unless n, s are in a plane ⊥ to x, y plane.

$\Omega_{z(\;)}$ = rate of rotation about z axis of the ( ) element.

definition of rotation arbitrarily picks the angles as being mutually ⊥.

IN GENERAL RATE OF ROTATION IS A VECTOR:

$$\vec{\Omega} = \vec{i}_1\Omega_1 + \vec{i}_2\Omega_2 + \vec{i}_3\Omega_3$$

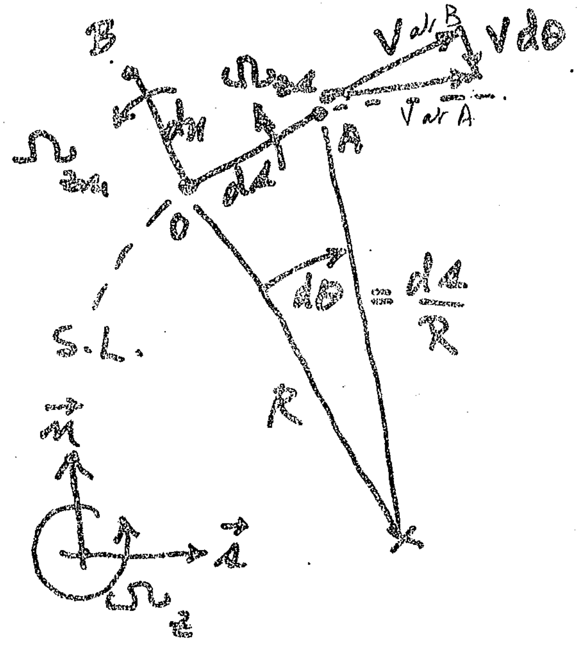
IN S-N COORDINATE WHERE Z-AXIS PERPENDICULAR TO PLANE OF FLOW

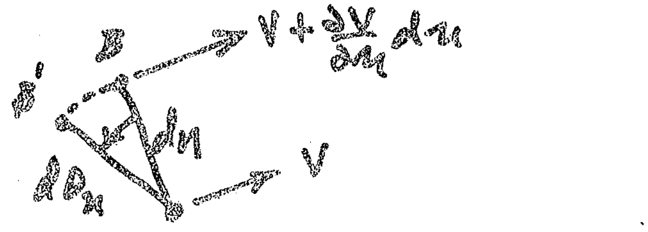
min curvature
of the line
lies in the plane,

$$\Omega = \vec{s}\,\Omega_s + \vec{n}\,\Omega_n + \underline{\underline{k\,\Omega_z}}$$

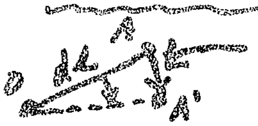
Ω_{zn} AND Ω_{zs} ARE RATES OF ROTATION OF LINE ELEMENTS dn AND ds.

$$\Omega_z = \frac{1}{2}\left(\Omega_{zs} + \Omega_{zn}\right)$$





$$\Omega_{zn} = \frac{d\theta_n}{dt} = \frac{-\frac{\partial V}{\partial n}dn}{dn} = -\frac{\partial V}{\partial n}$$

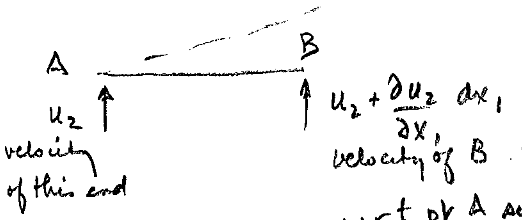


$V\,d\theta\,dt = AA'$

$-d\theta_s = d\theta$

$$\Omega_{zs} = \frac{d\theta_s}{dt} = -\frac{d\theta}{dt} = -\frac{\frac{Vd\theta dt}{ds}}{dt}$$

$$\Omega_{zs} = -\frac{Vd\theta}{ds} = -\frac{V}{R}$$



velocity of B where $\frac{\partial u_2}{\partial x_1} dx_1$ is relative velocity wrt pt A since u_2 will vary as one moves along dx_1

CONSEQUENTLY:

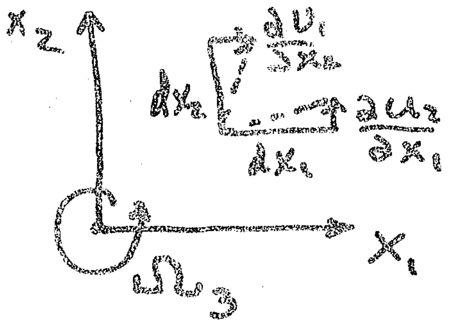
$$\Omega_z = \frac{1}{2}\left(-\frac{\partial V}{\partial x} - \frac{V}{R}\right)$$

$$\boxed{2\Omega_z = -\left(\frac{\partial V}{\partial x} + \frac{V}{R}\right)}$$

VORTICITY ALONG z-AXIS (= 0 FOR IRROTATIONAL FLOW)

RECTANGULAR COORDINATES

VORTICITY: $\vec{\omega} = \vec{i}_i \omega_i$



$$2\Omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = \omega_3$$

GENERAL

$$2\Omega_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \omega_i$$

EXAMPLE: SET $i = 3$

$$2\Omega_3 = \omega_3 = \varepsilon_{3jk}\frac{\partial u_k}{\partial x_j}$$

$$\boxed{=0}$$

$$\omega_3 = \varepsilon_{31k}\frac{\partial u_k}{\partial x_1} + \varepsilon_{32k}\frac{\partial u_k}{\partial x_2} + \varepsilon_{\boxed{33}k}\frac{\partial u_k}{\partial x_3}$$

Expand again on k

$$\omega_3 = \cancel{\varepsilon_{3\boxed{11}}}\frac{\partial u_1}{\partial x_1} + \varepsilon_{312}\frac{\partial u_2}{\partial x_1} + \varepsilon_{\boxed{3}1\boxed{3}}\frac{\partial u_3}{\partial x_1}$$

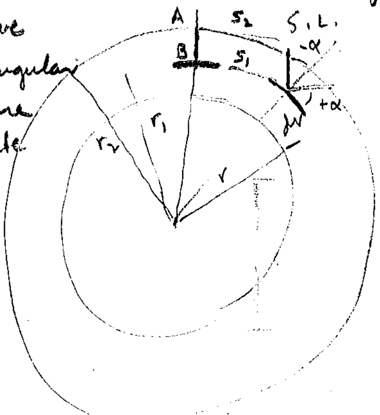
$$+ \varepsilon_{321}\frac{\partial u_1}{\partial x} + \varepsilon_{3\boxed{22}}\frac{\partial u_2}{\partial x_2} + \varepsilon_{\boxed{3}2\boxed{3}}\frac{\partial u_3}{\partial x_1}$$

for flow in a circular pattern $v_r = 0 \Rightarrow \omega_z = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\cancel{\partial v_r}}{\cancel{\partial \theta}}$

this looks like the n equation i.e. $r d\theta = ds$ and $dr = dn$

look at special case - flow along circular streamline
r = local radius of curvature $dr = dn$

the two lines have gone through angular rotation which are equal but opposite in sign ∴
The average rotation = 0.



$r = R$

$dn = dr$

$$-\omega = \frac{\partial v}{\partial n} + \frac{v}{R} = \frac{\partial v}{\partial r} + \frac{v}{r}$$

for irrotational flow $\omega = 0$ i.e. $\nabla \times \underline{v} = 0 \Rightarrow$

$$\therefore \frac{dv}{v} = \frac{-dr}{r}$$

$$\ln v + \ln r = C$$

Since vr = const
and $r_2 > r_1$ $V_2 < V_1$
then in time t
A has moved $V_2 t = S_2$
B has moved $V_1 t = S_1 > S_2$

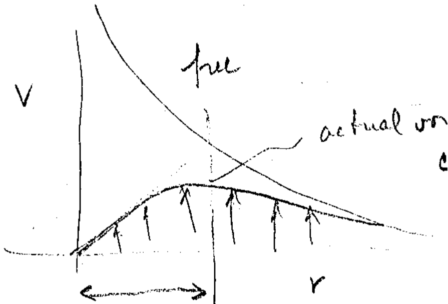
∴ rV = const

not that @ $V = \infty$ at $r = 0$ singularity

@ $r = \infty$ $V = 0$

Vr = const is a free vortex

V/r = const is a forced vortex (in order for the velocity to be finite at $r = 0$)



actual vortex is a combination of free and forced vortex

r_{core} = core radius

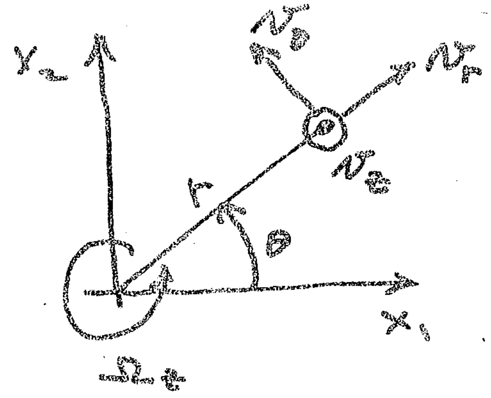
AS A RESULT, SINCE

$$\epsilon_{321} = -1$$

$$\epsilon_{312} = +1$$

$$\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

CYLINDRICAL COORDINATES:



$$2\Omega_z = \omega_z = \left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\right) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

$$2\Omega_r = \omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}$$

$$2\Omega_\theta = \omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$$

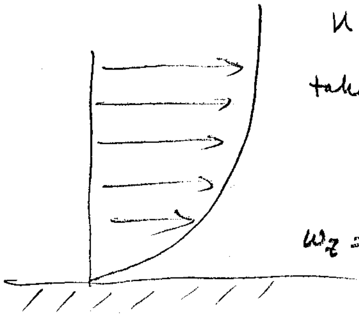
GENERAL VECTOR FORM

$\vec{\Omega}$: FLUID ROTATION

$\vec{\omega}$: VORTICITY

$$2\vec{\Omega} = \vec{\omega} = \vec{\nabla} \times \vec{V} = \text{CURL } \vec{V}$$

IRROTATIONAL FLOW IS <u>CURL FREE</u>



$u_0 = \text{const}$

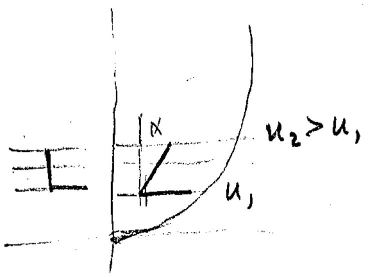
taking $\frac{u(y)}{U_0} = 1 - \left(1 - \frac{y}{\delta}\right)^2$

for $y \le \delta$ $v(y)$ is assumed to be 0.

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} = -\frac{2U_0}{\delta}\left(1 - \frac{y}{\delta}\right)$$

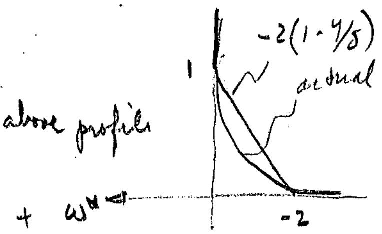
define $\frac{\omega_z \delta}{U_0} = \omega^* = -2\,(1 - y/\delta)$

this shows that vorticity exists in b l ! how ? since for $y < \delta \Rightarrow \omega^* \neq 0$



$\therefore$ angle $\alpha = \frac{\Delta u}{\Delta y} \neq 0$

for the above profile



MHD nonconservative forces acting on particle

$$\frac{D\bar{\omega}}{Dt} = \overset{\text{dilatation}}{\bar{\omega}\,(\nabla \cdot \underline{V})} + \underbrace{\overbrace{\nabla p \times \nabla\left(\tfrac{1}{\rho}\right)}^{p \parallel \text{ to density grad}}}_{=0 \text{ for } \rho = \text{const}} + \overset{\text{non conserv}}{\nabla \times [f_{NC} + f_\mu]} + \underbrace{\bar{\omega} \cdot \overbrace{(\nabla \bar{V})}^{\substack{\text{3-D effects}\\ \text{dyad}}}}_{=0 \text{ for 2 d flow}}$$

for stratified fluid $\frac{\partial \rho}{\partial y} \neq 0$

$\bar{\omega} \cdot \nabla \bar{V} = \omega_j \frac{\partial u_i}{\partial x_j}$: for 2-D flow ω_1, ω_2 are 0 and $\frac{\partial u_i}{\partial x_3} = 0$
plane since for plane 2-D flow vorticity only exists along ω_3 since only 2-dim flow

PHYSICAL CONDITIONS FOR $\vec{\omega} = 0$

FACTORS WHICH CREATE VORTICITY

1. NON-CONSERVATIVE FORCES

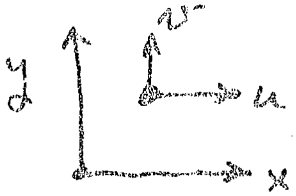
$\frac{D\vec{\omega}}{Dt} \neq 0$

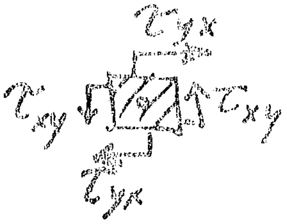
   a. BODY-FORCE TORQUES Dilatation $\left(\frac{1}{\rho}\frac{D\rho}{Dt} = \nabla \cdot \underline{v}\right)$
   b. PRESSURE-FORCE " not // to density gradients
   c. VISCOUS-FORCE " $\nabla \times$ nonconservative forces $\neq 0$

2. DIFFERENTIAL HEATING
3. THREE-DIMENSIONAL FLOW (TRANSFORMS VORTICITY COMPONENTS)

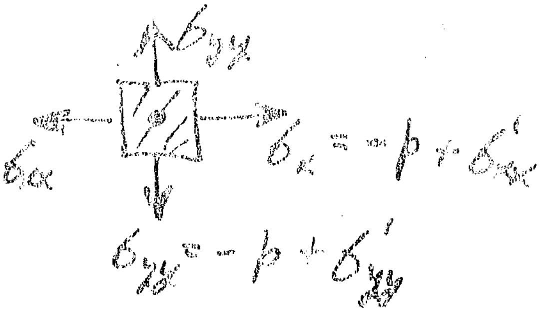
(TO BE DISCUSSED AT GREATER DEPTH IN ME 251B) take $\nabla \times$ eqs of motion

EXAMPLE: 2-D FLOW WITH SIMPLE NEWTONIAN VISCOSITY (μ = CONST.); ρ = CONST., CONSERVATIVE BODY FORCES:





SHEAR STRESSES



Normal Stresses

Constitutive eqs for μ = const.

angular or shear strain

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

linear strain

$$\sigma'_{xx} = \mu\left(2\frac{\partial u}{\partial x}\right)$$

$$\sigma'_{yy} = \mu\left(2\frac{\partial v}{\partial y}\right)$$

EQ'S OF MOTION: $(\nu = \mu/\rho)$ ν = Constant

for non Constant ν $\frac{\partial}{\partial x}\nu\left(\frac{\partial u}{\partial x}\right)$

$$(x\text{-EQ}):\quad \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \overset{\text{Conservative}}{\frac{\partial U}{\partial x}} + \nu\nabla^2 u$$

$$(y\text{-EQ}):\quad \underbrace{\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial U}{\partial y}}_{\text{EULER'S EQ'S}} + \underbrace{\nu\nabla^2 v}_{\text{ADDED VISCOUS FORCES}}$$

WHERE:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

CROSS-DIFFERENTIATE AND SUBTRACT TO ELIMINATE p AND U, $\nabla\times\frac{D\bar{V}}{Dt} = \frac{D}{Dt}\underbrace{(\nabla\times\bar{V})}_{\omega_z}$

$$\frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) - \frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) = \nu\left[\frac{\partial}{\partial x}(\nabla^2 v) - \frac{\partial}{\partial y}(\nabla^2 u)\right]$$

BY DEFINITION: $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

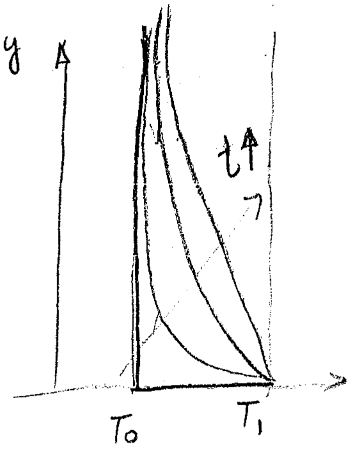
SO EQ ABOVE BECOMES:

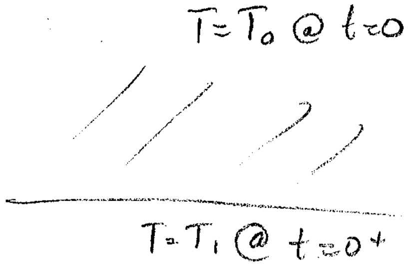
$$\boxed{\frac{D}{Dt}(\omega_z) = \nu\nabla^2(\omega_z)} \quad (60\text{-}1)$$

$$\left\{\begin{array}{l}\text{RATE OF INCREASE}\\ \text{OF VORTICITY OF}\\ \text{A FLUID PARTICLE}\end{array}\right\} = \nu \times \left[\begin{array}{l}\text{RATE OF DIFF-}\\ \text{USION BY GRADIENTS}\\ \text{OF FLUX OF } \omega_z\end{array}\right]$$

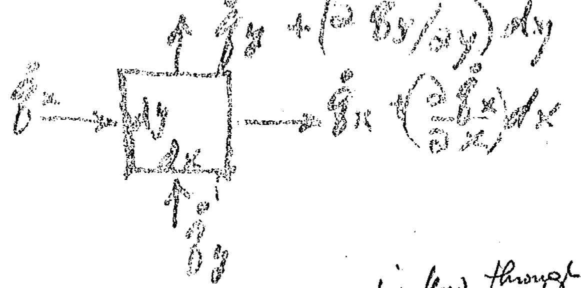
if w is linear in y no vorticity in fluid

w is non linear in y vorticity in fluid


y
t
To
T₁


T=T₀ @ t=0
T=T₁ @ t=0+

COMPARE TO 2-D THERMAL ENERGY EQ. (CONDUCTION HEAT TRANSFER IN A SLOWLY MOVING LIQUID OR GAS)



fourier's law of heat conductn

$$\left\{ \begin{aligned} \dot{q}_x &= \left(-k \frac{\partial T}{\partial x}\right) dy \\ \dot{q}_y &= \left(-k \frac{\partial T}{\partial y}\right) dx \end{aligned} \right\}$$

heat diffuses due to temp changes

net inflow through x surface

So
$$\frac{D\tilde{u}}{Dt} dx dy = -\left(\frac{\partial \dot{q}_x}{\partial x}\right) dx - \left(\frac{\partial \dot{q}_y}{\partial y}\right) dy$$

WHERE $\tilde{u} = \rho c_p T$

$\alpha = (k/\rho c_p)$ THERMAL DIFFUSIVITY

So:
$$\boxed{\frac{D}{Dt}(T) = \alpha \nabla^2 (T)} \quad (61-1)$$

$$\left[\begin{array}{l}\text{RATE OF INCREASE}\\ \text{OF THERMAL}\\ \text{ENERGY OF A}\\ \text{PARTICLE}\end{array}\right] = \alpha \times \left[\begin{array}{l}\text{RATE OF DIFFUSION}\\ \text{BY HEAT}\\ \text{CONDUCTION}\end{array}\right]$$

ANALOGOUS EFFECTS AND QUANTITIES:

VORTICITY, ω_z $\Longleftrightarrow$ TEMPERATURE, T

KINEMATIC VISCOSITY, ν $\Longleftrightarrow$ THERMAL DIFFUSIVITY, α

$$\left\{\begin{array}{l}\text{DIFFUSION}\\ \text{OF VORTICITY}\\ \text{DRIVEN BY}\\ \text{VORTICITY}\\ \text{GRADIENTS}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\text{DIFFUSION OF}\\ \text{HEAT (TEMP.)}\\ \text{BY TEMPERATURE}\\ \text{GRADIENTS}\end{array}\right\}$$

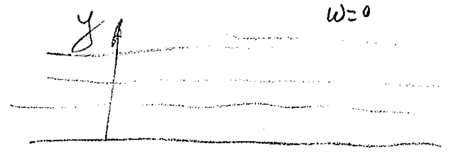
Unsteady start of a flat plate

$u = v = 0$ at $t = 0$

@ $t = 0^+$ u exists where at some δ $u = U_\infty$

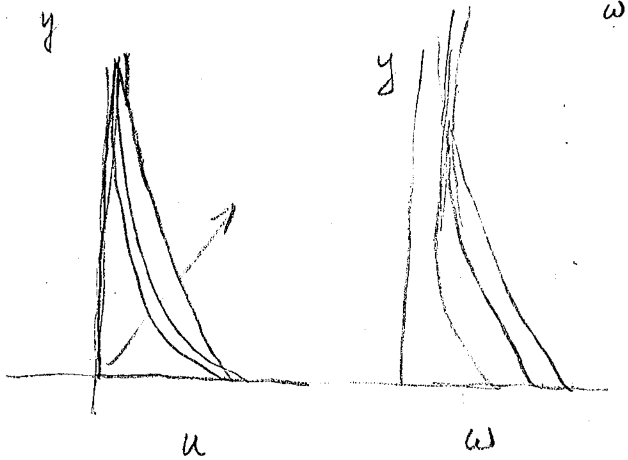
$v = 0$ and $\frac{\partial}{\partial x} = 0$ since infinite flat plate

$\omega = 0$



↳ U = const will cause vorticity at wall

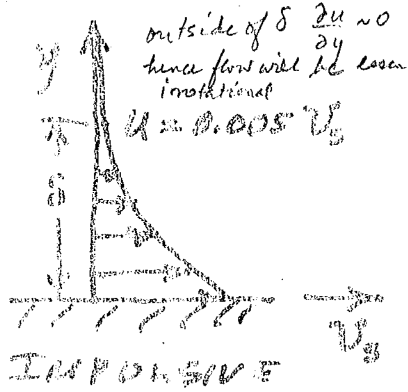
@ $t = 0$



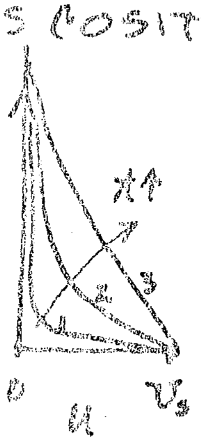
$\omega = -\frac{\partial u}{\partial y}$ as time went on the vorticity will decay as $t \uparrow$ by diffusion

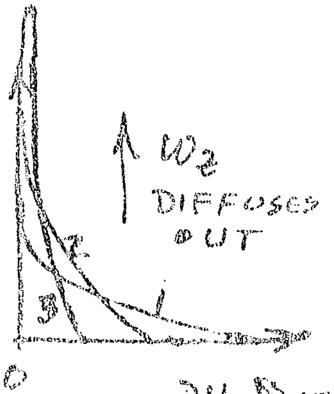
TEMP. DIFFERENCES (GRADIENTS)
VORTICITY DIFFERENCES (GRADIENTS)
ARE CAUSES FOR DIFFUSION

<u>NO SLIP B.C.</u> INTRODUCES VORTICITY DIFFERENCES WHICH ARE DIFFUSED OUT FROM A SURFACE BY VISCOSITY



IMPULSIVE MOTION AFTER TIME t





$$-\omega_z = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$

CASE WHERE $v = 0$, $\frac{\partial}{\partial x} = 0$,
EQ (60-1) BECOMES

$$\frac{\partial \omega_z}{\partial t} + \overset{0}{\cancel{u\frac{\partial \omega_z}{\partial x}}} + \overset{0}{\cancel{v\frac{\partial \omega_z}{\partial y}}} = \nu\left(\overset{0}{\cancel{\frac{\partial^2 \omega_z}{\partial x^2}}} + \frac{\partial^2 \omega_z}{\partial y^2}\right)$$

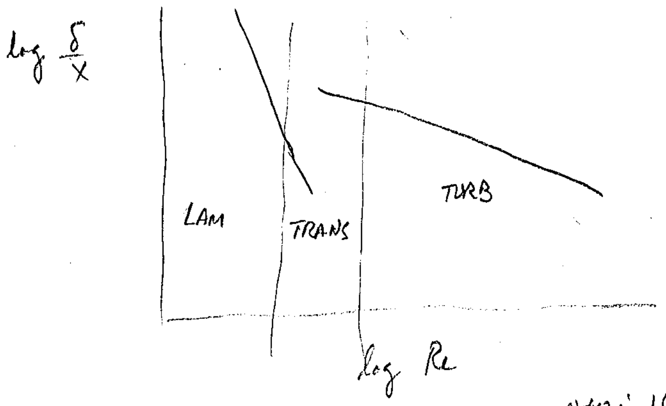
$$\frac{\partial \omega_z}{\partial t} = \nu \frac{\partial^2 \omega_z}{\partial y^2}$$

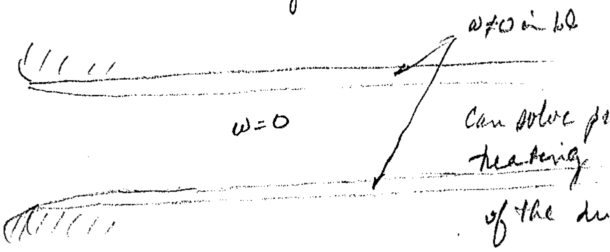
<u>also</u> $$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

SEE TEXT EXAMPLE 2.5 (PP 58-60)

$$\boxed{\delta \sim \sqrt{\nu t}}$$

for $Re < 10^3$ $\frac{\delta}{x} \sim o(1)$ and viscosity effects diffuse rapidly into the flow. Thus we really should *not* use irrotational flow theory.



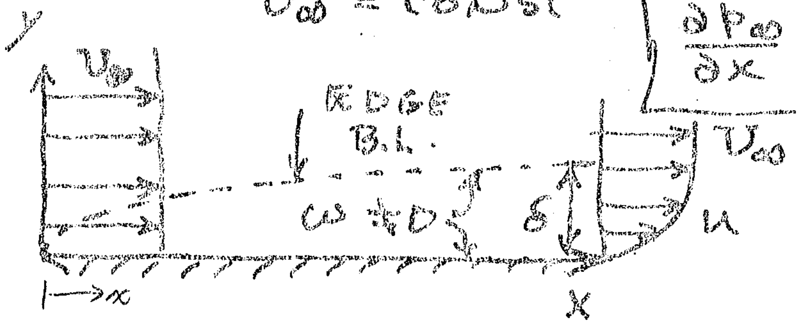


Can solve problem in here by treating the bl as boundary of the duct

PHYSICAL CONDITIONS FOR IRROTATIONAL FLOW (CONT.)

STEADY FLOW – SOME FACTS (W/O PROOF) CONCERNING BOUNDARY LAYERS AND OTHER REGIONS WHERE $\vec{\omega} \neq 0$.

CASE I FLAT PLATE B.L. UNDER U_∞ = CONST $\boxed{\frac{\partial p_\infty}{\partial x} = 0}$ ZERO PRESSURE GRADIENT



$\frac{\partial p}{\partial x} = 0$

DEFINE REYNOLDS NO: $\boxed{Re_x = \frac{U_\infty X}{\nu}}$

FOR AIR (60°F, 1 atm): $\nu = 160 \times 10^{-6}$ (ft²/s) more visc

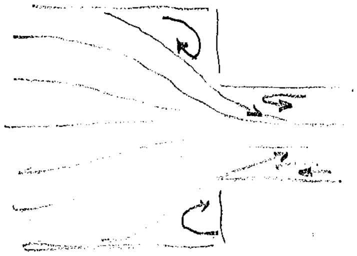
FOR WATER (" , "): $\nu = 11 \times 10^{-6}$ (ft²/s)

FLOW REGIME	Re_x RANGE	$\left(\frac{\delta}{x}\right)$ RANGE	FORMULA $\left(\frac{\delta}{x}\right) \approx$
LAMINAR*	$10^3 - 10^5$	0.16 – 0.016	$5\, Re_x^{-0.5}$
TRANSITION	$10^5 - 10^6$	0.016 – 0.023	- - - - - - -
TURBULENT	$10^6 - 10^8$	0.023 – 0.0093	$0.37\, Re_x^{-0.2}$

* B.L. THEORY ? FOR $Re_x < (10^2 - 10^3)$

$Re_x = 10^5 - 10^6$ — FLOW OVER COMPRESSOR & TURBINE BLADES

$Re_x = 10^8 - 10^9$ – FLOW OVER LARGE SHIP HULL



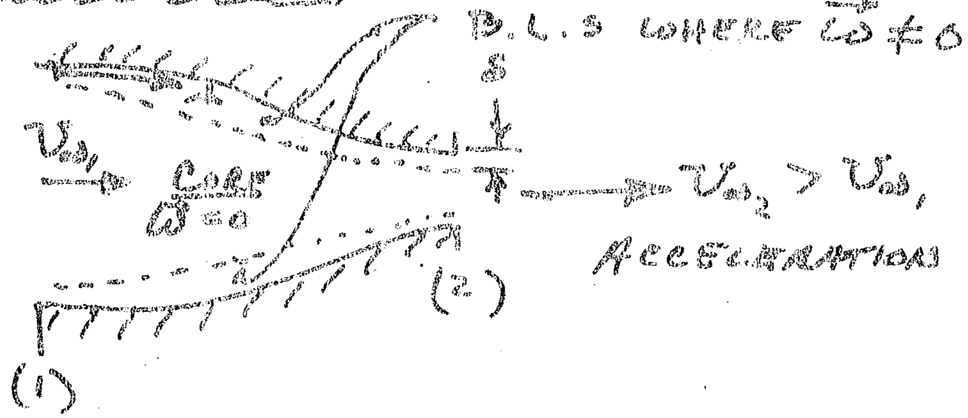
in some regions $\frac{\partial p}{\partial x} > 0$ $\Rightarrow$ separation of high b.l.

ME 251 A 7/7/78 (67)

CASE II FAVORABLE PRESSURE GRADIENT

$$\boxed{\frac{\partial p_\infty}{\partial x} < 0}$$

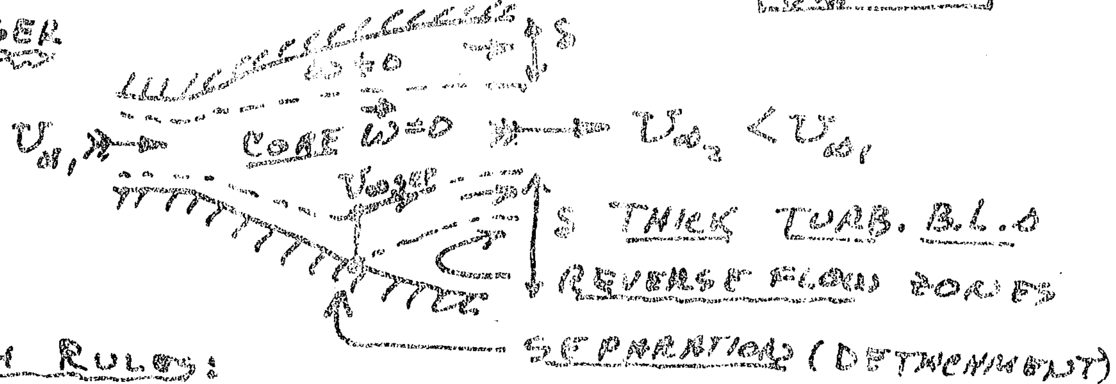
NOZZLE (SUBSONIC)



- WALL B.L.s THIN
- LIKELY TO BE LAMINAR EVEN AT HIGH Re No.

CASE III ADVERSE PRESS. GRAD. $\boxed{\frac{\partial p_\infty}{\partial x} > 0}$

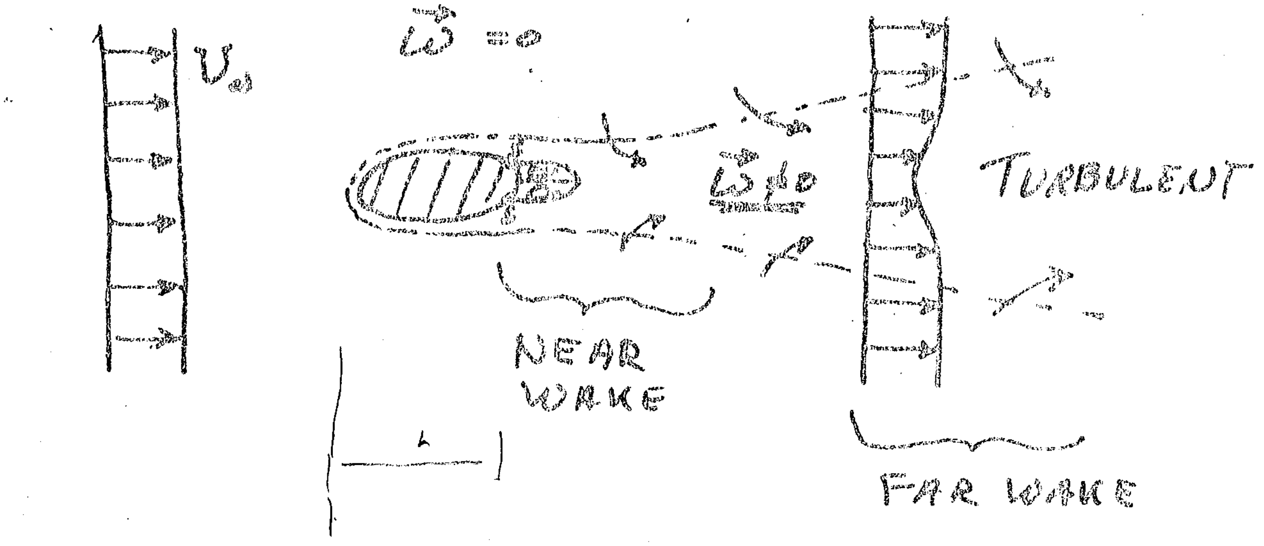
DIFFUSER

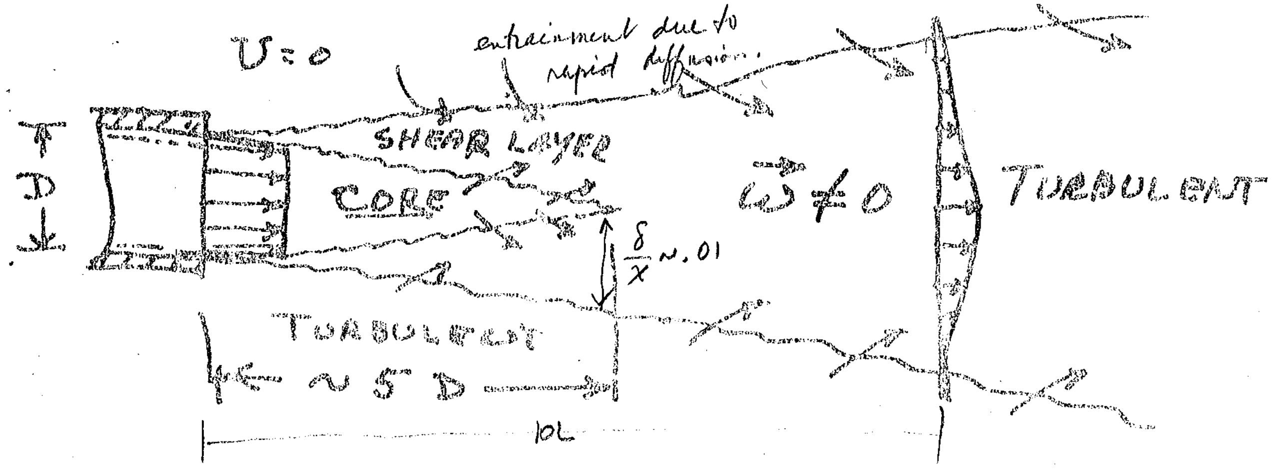


ROUGH RULES:

- VERY LOW Re WHERE LAMINAR B.L.'s
  $U_{\infty_{SEP}} \gtrsim 0.9 U_{\infty_1}$ to $0.99 U_{\infty}$
- HIGHER Re WHERE TURBULENT B.L.'s
  $U_{\infty_{SEP}} \approx (0.6 \text{ to } 0.7) U_{\infty_1}$

CASE IV WAKES AND JETS





- TRANSITION AT VERY LOW Re
- RAPID DIFFUSION ($\nu_{EFF} > 1000\nu$)
- GROWTH BY ENTRAINMENT

turbulent is dominant characteristic of free shear layers

for an incompressibility we have 2 unknowns need 2 eqs for u, v
$\therefore$ use cont & vorticity

for compress we have 3 unk (ρ, u, v) need 3 eq
$\therefore$ use cont + vorticity + equation of state (for $s = \text{const}$ $P/\rho^k = \text{const}$)

III for free surfaces where surface tension (σ) effects ≈ 0
then $p_{gas} = p_{liq}$ at surface

- Assume uniform // flow in long ducts: If we assume // walls
$v = 0 \Longrightarrow \omega = -\frac{\partial u}{\partial y} = 0$ also from continuity $\frac{\partial u}{\partial x} = 0 \Rightarrow u = \text{const}$
or uniform flow in long ducts
- We assume influence of elbows in ducts on inlets & outlets negligible
- we assume also body is still & fluid moves over it

SOLUTION OF STEADY INVISCID FLOWS

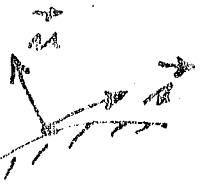
- **BASICS** $\frac{\partial \rho}{\partial t} = 0$

EQUATIONS
(RECT. COORD'S.)

	GENERAL	2-D PLANE
I. CONS. MASS		
gas (M<.3) liquid ρ = CONST.	$\frac{\partial u_i}{\partial x_i} = 0$	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$
gas (M>.3) $\rho \neq$ CONST.	$\frac{\partial (\rho u_i)}{\partial x_i} = 0$	$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0$
II. VORTICITY (EQ'S MOTION) $\vec{\omega} = 0$ (IRROTATIONAL)	$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$ $(i = 1, 2, 3)$	$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ $\frac{\partial V}{\partial n} + \frac{V}{R} = 0$ n,s syste

III. BOUNDARY CONDITIONS

$$\vec{V}_{WALL} = \vec{s}\, V_s + \vec{n}\, V_n$$



we neglect no slip condition ∴
SOLID WALL'S ARE STREAMLINES

IV. PRESSURES FROM BERNOULLI EQ:

$$\frac{p}{\rho} - U + \frac{1}{2}(u_i u_i) = B = \text{CONST.}$$

$$\frac{p}{\rho} - U + \frac{1}{2}(u^2 + v^2) = B = \text{CONST}$$

by using cont + $(\bar{\omega} = 0)$ + B.C. gives

1. for an incompressible flow p is really a pressure difference wrt some ref pressure
2. for compressible flow p is an absolute pressure.

STREAM FUNCTION $\psi(x,y)$

(TEXT · pp 22-26)

2-D PLANE FLOW

BY DEFINITION:

$$\rho u = \frac{\partial \psi}{\partial y} \; ; \; \rho v = -\frac{\partial \psi}{\partial x}$$

PLUG IN CONTINUITY

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$\nabla\psi = \left(+\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right)$

$\nabla \times (\nabla\psi) =$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

SO ψ IDENTICALLY SATISFIES CONTINUITY.

(IF ρ = CONST. USE: $u = \frac{\partial \psi}{\partial y}$; $v = -\frac{\partial \psi}{\partial x}$)

for well behaved smooth fn.

PROPERTIES OF ψ:

1. ψ - LINES ARE IDENTICAL TO STREAMLINES. SOLID WALLS ARE LINES OF CONSTANT ψ-VALUE.
   ψ-lines are pathlines
2. FLOW RATE PER UNIT DEPTH (z-DIRECTION) BETWEEN TWO ψ-LINES IS PROPORTIONAL TO DIFFERENCE IN ψ-VALUES FOR THE TWO LINES

Proof of Property (1.) for ρ = const.

$$\vec{\nabla}\psi = \vec{i}\left(\frac{\partial\psi}{\partial x}\right) + \vec{j}\left(\frac{\partial\psi}{\partial y}\right) = \vec{i}(-v) + \vec{j}(u)$$

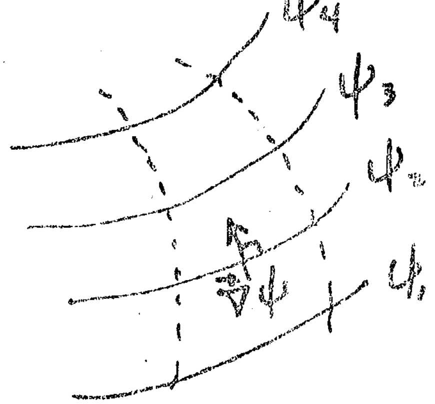
BUT $\vec{V} = \vec{i}(u) + \vec{j}(v)$

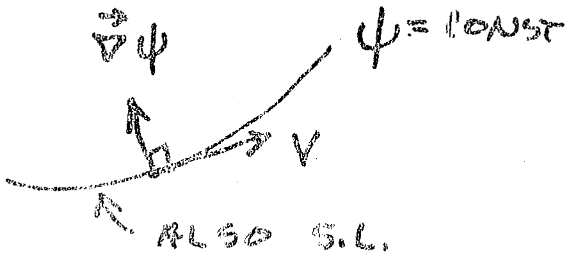
SO: $\vec{V}\cdot(\vec{\nabla}\psi) = -vu + vu = 0$

THEREFORE: $\vec{V} \perp \vec{\nabla}\psi$

BUT: $\vec{V} \parallel$ STREAMLINES

WE CONCLUDE: ψ-LINES SAME AS STREAMLINES, since ψ-lines are $\perp \nabla\psi$





ψ = CONST. $\perp \vec{\psi}$

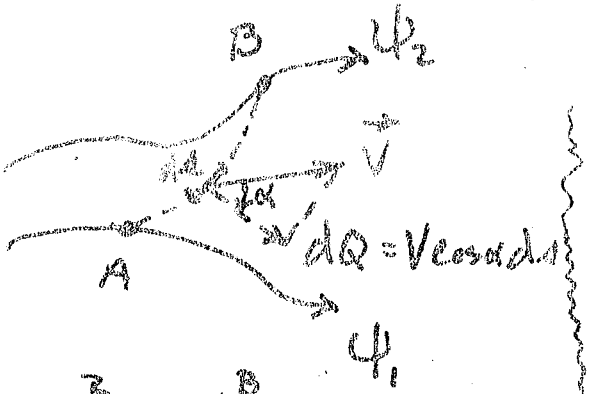
$$\boxed{\psi\text{-LINES ARE STREAMLINES}}$$

ME 251A 7-/70 (19)

Proof of Property (2.)

ASSUME: ρ = CONST (NOT NECESSARY)

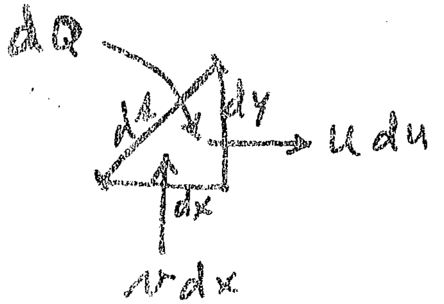
Q = Volume flow rate per unit depth between streamlines 1 and 2



$dQ = V\cos\alpha\, ds$

$$Q = \int_A^B dQ = \int_A^B V\cos\alpha\, ds$$

CONS. MASS FOR SMALL C.S.



$$\boxed{dQ + v\,dx = u\,dy}$$

USING GIVES:

$$Q = \int_A^B (u\,dy - v\,dx) = \int_A^B \left(\frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial x}dx\right) = \int_A^B d\psi$$

$$\therefore\ Q = \psi_B - \psi_A$$

$$\boxed{Q = \psi_1 - \psi_2}$$ Q.E.D.

SOLUTION EQ'S USING ψ

(1.) ψ SATISFIES CONTINUITY

(2.) IT MUST ALSO SATISFY IRROTATIONAL FLOW

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

FOR ρ = CONST. CASE

$$\omega_z = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$$

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0}$$

THE 2-D LAPLACE EQ MUST BE SATISFIED! WITH ψ SPECIFIED AT REGIME BOUNDARIES.

SOLUTION $\psi(x,y)$ GIVES u, v AND $p(x,y)$ FROM BERNOULLI:

$$p = \rho B + \rho U - \frac{\rho}{2}\left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2\right]$$

STREAM FUNCTION LIMITED TO 2-D FLOW SO WE RESORT TO

POTENTIAL FLOW (CHAPT. 6)

POTENTIAL FUNCTION (SCALAR POINT FUNCTION) $\phi(x_i, t)$

t OUT FOR STEADY FLOW
will be shown related to $\underline{V}$

IS DEFINED SO THAT:

$$\text{CURL}\,(\vec{\nabla}\phi) = 0$$

$$\vec{\nabla} \times (\vec{\nabla}\phi) = 0$$

if $\underline{\omega} = 0$ $\nabla \times \underline{V} = 0$
$\Rightarrow \underline{V} = \nabla\phi$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \begin{aligned} &\vec{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) \\ &+ \vec{j}\left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right) \\ &+ \vec{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) \end{aligned} = 0$$

IF: $\vec{V} = \vec{\nabla}\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$

$$\vec{V} = \vec{i}\,u + \vec{j}\,v + \vec{k}\,w$$

THEN

$$\vec{\nabla} \times (\vec{\nabla}\phi) = \vec{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \vec{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \vec{k}\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)$$

$$'' \quad = \vec{i}\,\omega_x + \vec{j}\,\omega_y + \vec{k}\,\omega_z = \vec{\omega}$$

BOTH SIDES OF EQ

$$\vec{\nabla} \times (\vec{\nabla}\phi) = \vec{\omega}$$

ARE ZERO IF ϕ IS SCALER POINT FUNCTION WHICH GIVES VELOCITY FIELD BY

$$\vec{V} = \vec{\nabla}\phi$$

AND IF FLOW IRROTATIONAL,

$$\vec{\omega} = 0$$

NO other RESTRICTIONS ON DENSITY OR TEMPORAL VARIATIONS, ie $\frac{\partial}{\partial t}(\rho, \text{etc})$ WE CALL FLOW WHERE THESE CONDITIONS SATISFIED A "POTENTIAL FLOW" WHERE $\phi(\vec{x}, t)$ IS THE "VELOCITY POTENTIAL"

EXAMPLE:

$$\phi = Ax^2 + Bxy + Cy^2$$

$$\frac{\partial \phi}{\partial z} = w = 0 ; \quad u = \frac{\partial \phi}{\partial x} = 2Ax + By ; \quad v = \frac{\partial \phi}{\partial y} = Bx + 2Cy$$

(2-D, PLANE FLOW)

$$\vec{\omega} = \vec{i}(0) + \vec{j}(0) + \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\vec{\omega} = \vec{k}(B - B) \Rightarrow 0 \quad \boxed{\text{FOR ANY } A, B, C}$$

in order to find ϕ we must satisfy conservation of mass and for constant ρ $\nabla \cdot \underline{V} = 0 \Rightarrow \nabla \cdot \nabla\phi = \Delta\phi = 0$

SATISFY CONTINUITY ($\rho = $ CONST.)

$$\boxed{\vec{\nabla} \cdot \vec{V} = 0}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) = 0$$

$$\boxed{\begin{gathered}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \\ \nabla^2 \phi = 0\end{gathered}} \quad (81\text{-}1)$$

LAPLACE'S EQ. MUST BE SATISFIED BY ϕ

FOR THE EXAMPLE (P. 80):

$\phi = Ax^2 + Bxy + Cy^2$

SATISFIES $\vec{\omega} = 0$ FOR ANY A, B AND C

$; \frac{\partial \phi}{\partial x} = 2Ax + By ; \frac{\partial^2 \phi}{\partial x^2} = 2A$

$; \frac{\partial \phi}{\partial y} = Bx + 2Cy ; \frac{\partial^2 \phi}{\partial y^2} = 2C$

$; \frac{\partial \phi}{\partial z} = 0 ; \frac{\partial^2 \phi}{\partial z^2} = 0$

IF: $\nabla^2 \phi = 0$

THEN: $2A + 2C = 0$

$\therefore \boxed{A = -C}$ REQUIRED TO SATISFY CONTINUITY note ψ satisfies continuity ; to satisfy irrot $\Delta\psi = 0$ restriction is that ψ is for 2D plane flow

note ϕ satisfies irrot ; to satisfy cont $\Delta\phi = 0$ no restrictions we only require irrot

PROPERTIES OF ϕ (SATISFIES $\nabla^2\phi=0$)

1. ϕ - LINES ARE PERPENDICULAR TO STREAMLINES

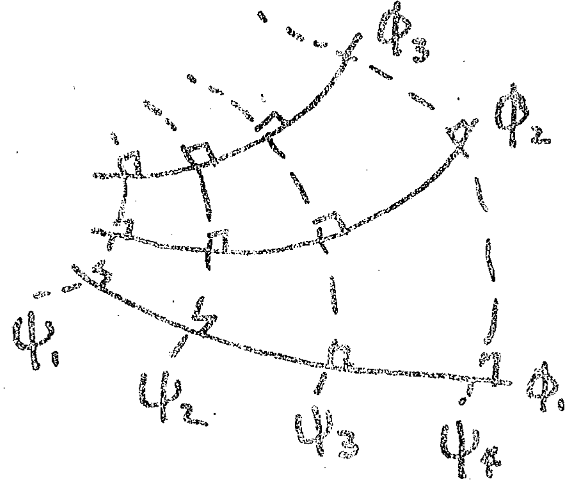
2. IN 2-D, PLANE FLOW ϕ-LINES ARE PERPENDICULAR TO ψ-LINES

PROOF: SINCE $\vec{V} = \vec{\nabla}\phi$

AND $\vec{\nabla}\phi \perp$ ϕ-LINES (LINE OF ϕ = CONST.)

SO: $\vec{V}$ IS $\perp$ TO ϕ-LINES

IN 2-D PLANE FLOW (ρ = CONST.)



BOTH ϕ AND ψ SATISFY LAPLACE (2-D)

$$\boxed{\begin{array}{c}\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0\\[2ex]\dfrac{\partial^2\psi}{\partial x^2}+\dfrac{\partial^2\psi}{\partial y^2}=0\end{array}}$$

$\vec{V} = \vec{i}u + \vec{j}v$

AND $u = \dfrac{\partial\phi}{\partial x} = \dfrac{\partial\psi}{\partial y}$

$v = \dfrac{\partial\phi}{\partial y} = -\dfrac{\partial\psi}{\partial x}$

3 types of BC

DIRICHLET	ϕ, ψ is given on ∂R
NEUMANN	$\frac{\partial \phi}{\partial n}, \frac{\partial \psi}{\partial n}$ on ∂R
MIXED	ϕ / ψ, or $\frac{\partial \phi}{\partial n} / \frac{\partial \psi}{\partial n}$ given

max, min of ψ, ϕ will be on boundary.

ME 251A   LECTURE 14   7/78 (83)

LAPLACE EQ. ; $\boxed{\nabla^2 \phi = 0}$; $\underset{\text{2-D PLANE}}{\nabla^2} \psi = 0$

CHARACTERISTICS:

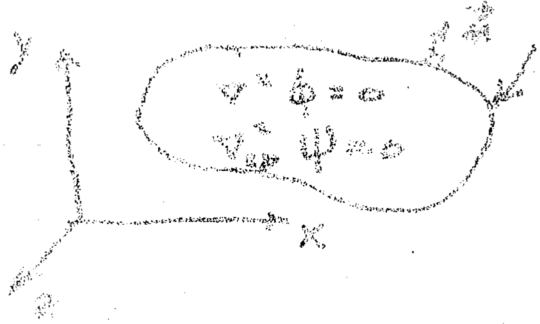
1. LINEAR IN ϕ AND $\frac{\partial \phi}{\partial x_i}$'s   Superposition is greatest asset

   not product of deriv or deriv squared

2. HOMOGENEOUS:
   - (COMPARE TO $\nabla^2 \phi = f(x_i)$ WHERE $f \neq 0$ "POISSON EQ."

3. PARAMETER FREE:
   - SOLUTION INDEPENDENT OF SCALE LENGTHS AND CHARACTERISTIC FLOW SPEEDS.

SOLUTION AND BOUNDARY CONDITIONS



REGION, R, SURROUNDED BY SURFACE, S.

SOLUTION INSIDE S DEPENDS ON B.C.'s ON S
relates ϕ to spatial coord.

(1.) DIRICHLET B.C.'s

$\boxed{\phi \text{ (or } \psi) \text{ SPECIFIED ON } S}$

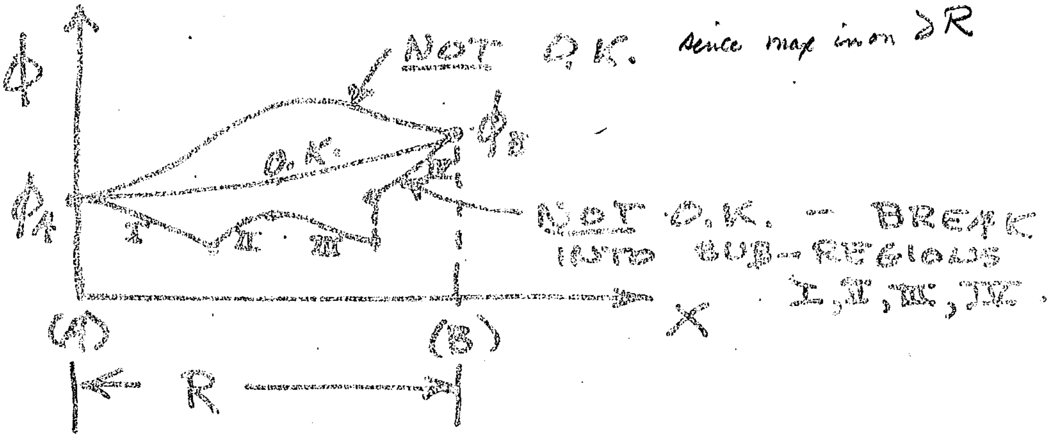
SOLUTION (ϕ or ψ):

(A) IS UNIQUE AND SINGLE VALUED

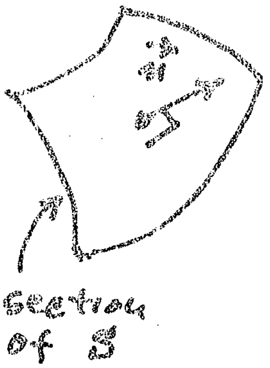
(B) NEVER DISPLAYS MAX OR MIN VALUE INSIDE R.

(C) IS SMOOTH AND CONTINUOUS INSIDE R.

CONSIDER $\phi(x, y = \text{const.}, z = \text{const.})$



(2.) NEUMANN B.C.'S



normal velocity to surface

$$\boxed{\begin{gathered}\frac{\partial \phi}{\partial n} \quad \left(\text{OR } \frac{\partial \psi}{\partial n}\right) \\ \text{SPECIFIED ON } S\end{gathered}}$$

SOLUTION (ϕ OR ψ) SAME AS DIRICHLET B.C.'S EXCEPT SOLUTION ONLY UNIQUE TO ARBITRARY CONSTANT, K., i.e.

solution is unique up to arbitrary function.

$$\phi \quad \underline{\text{OR}} \quad \phi \pm K$$

SATISFY B.C.'S AND $\nabla^2 \phi = 0$.

NO PROBLEM IN FLOW PROBLEMS SINCE WE WANT $\vec{V} = \vec{\nabla}\phi = \vec{\nabla}(\phi \pm K)$.

ME 251A

7/7/78

SUPERPOSITION OF SOLUTIONS:

LINEAR SUPERPOSITION OF ϕ'S (OR ψ'S):

LET: $\phi_1, \phi_2, \phi_3 \cdots$ ALL SATISFY $\nabla^2 \phi_i = 0$ $(i = 1, 2, \cdots m)$

NEW SOLUTION

$$\phi_N = A_1 \phi_1 + A_2 \phi_2 + \cdots$$

$$\phi_N = A_i \phi_i \qquad (\textstyle\sum_i \text{ over } i = 1, 2, \cdots m)$$

ALSO SATISFIES $\nabla^2 \phi_N = 0$

SINCE: $\nabla^2 \phi_N = \nabla^2 (A_i \phi_i) = A_i \underbrace{\nabla^2 \phi_i}_{=0} = 0$

SUPERPOSITION OF VELOCITY FIELDS:

$$\vec{V}_N = A_1 \vec{V}_1 + A_2 \vec{V}_2 + \cdots$$

OR $\vec{\nabla} \phi_N = A_1 \vec{\nabla} \phi_1 + A_2 \vec{\nabla} \phi_2 + \cdots$

OPERATE ON ABOVE WITH $\vec{\nabla} \cdot (\;)$

NOTE THAT $\vec{\nabla} \cdot \vec{\nabla} = \nabla^2$ (PROVE FOR YOURSELF) SO

$$\nabla^2 \phi_N = A_1 \nabla^2 \phi_1 + A_2 \nabla^2 \phi_2 + \cdots$$

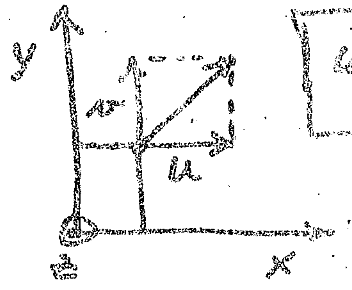
$$0 = 0 + 0 + \cdots$$

SUPERPOSITION OF PRESSURE FIELDS:

NOT POSSIBLE since $p = f(V^2)$ not V

POTENTIAL FLOW - REVIEW OF EQUATIONS FOR SPECIAL CASE:

2-D, PLANE FLOW
STEADY FLOW
ρ = CONST $\nabla \cdot \underline{V} = 0$
IRROTATIONAL, $\omega = 0$ $\nabla \times \underline{V} = 0$

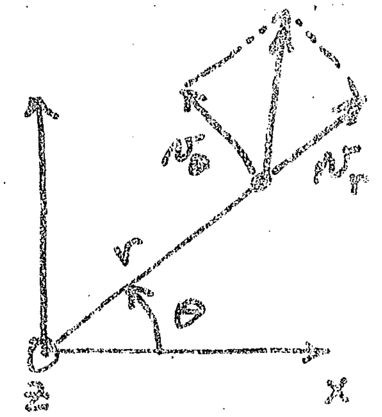


$$\boxed{\omega_z = \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} = 0} \quad (86-1)$$

AND

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \quad (86-2)$$

CONTINUITY



$$\boxed{\omega_z = \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right] = 0} \quad (86-3)$$

$$\boxed{\frac{1}{r}\left[\frac{\partial (v_r r)}{\partial r} + \frac{\partial v_\theta}{\partial \theta}\right] = 0} \quad (86-4)$$

CONTINUITY

BERNOULLI $\left[V^2 = u^2 + v^2 = v_r^2 + v_\theta^2\right]$

$$\boxed{\frac{p}{\rho} - U + \frac{1}{2}V^2 = \text{CONSTANT}}$$

ME251A 7/7/78 (87)

STREAM FUNCTION:

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0} \quad (87\text{-}1)$$

WHERE: $u = \frac{\partial \psi}{\partial y}$

$v = -\frac{\partial \psi}{\partial x}$

$$\boxed{\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0} \quad (87\text{-}2)$$

WHERE: $v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$

$v_\theta = -\frac{\partial \psi}{\partial r}$

VELOCITY POTENTIAL:

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0} \quad (87\text{-}3)$$

WHERE: $u = \frac{\partial \phi}{\partial x}$

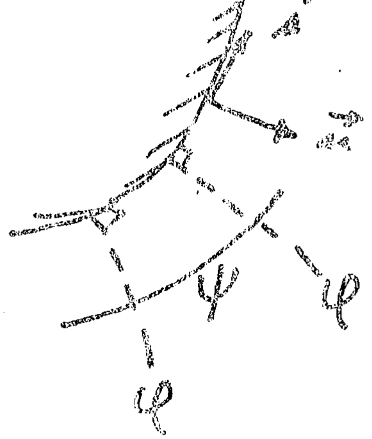
$v = \frac{\partial \phi}{\partial y}$

$$\boxed{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0} \quad (87\text{-}4)$$

WHERE: $v_r = \frac{\partial \phi}{\partial r}$

$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$

BOUNDARY CONDITIONS ON WALL OR S.L.



$\psi \parallel \partial R$

$\varphi \perp \partial R$

$$V = \frac{\partial \phi}{\partial s} = \frac{\partial \psi}{\partial n} \quad \left\{ \begin{matrix} \text{TO BE} \\ \text{OBTAINED} \end{matrix} \right\}$$

$$0 = \frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial s} \quad \left\{ \begin{matrix} \text{TO BE} \\ \text{GIVEN} \end{matrix} \right\}$$

OR $\psi = \psi_{WALL}$

A. Analytic Exact

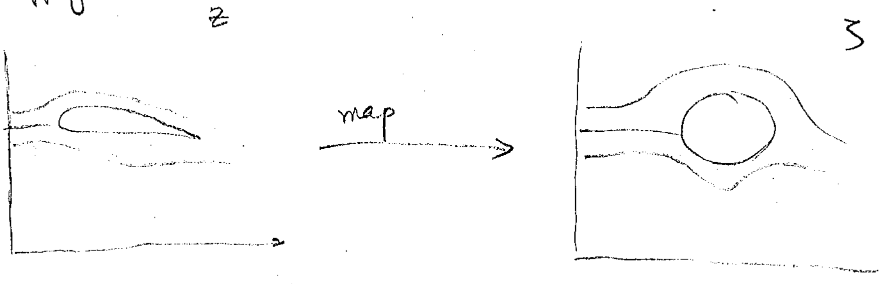
(1) efficient for only simple boundary shapes

(2) 2-D flows use complex potential function

$\Phi(z) = \phi + i\psi$ when $\phi(x,y)$, $\psi(x,y)$ are the stream fn ψ, velocity potential ϕ

$$\frac{d\Phi}{dz} = \bar{w} = u - iv$$

(a) Conformal mapping



Solve problem in ζ plane

inverse map soln back to ~~z~~ plane

(b) Singularities superposition

Source sink

$V_r r = \text{const}$

$V_\theta = 0$

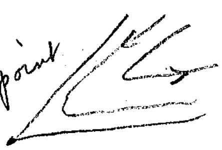
Point Vortex

$V_\theta r = \text{const}$

$V_r = 0$

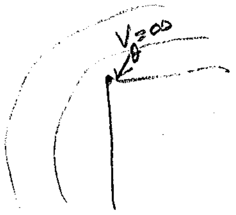
dipole (doublet)

(c) stag point



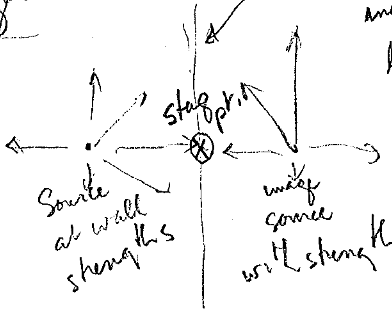
V=0

not singular



Method of images

at this line, sources cancel and line is a reflection line



Source at wall strength s

image source with strength -s

ME251A 77/78 (80)

METHODS OF SOLUTION (NOT ALL INCLUSIVE)

(A.) ANALYTIC - EXACT

- SEPARATION OF VARIABLES

  $\phi = F(x) \cdot G(y)$

  AND SERIES OF HARMONIC FUNCTIONS

- ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE
  - (a.) SUPERPOSITION AND SERIES
  - (b.) CONFORMAL MAPPING
  - (c.) METHOD OF SINGULARITIES (SOURCES, SINKS, POINT VORTICES, DOUBLETS, ETC.)
  - (d.) METHOD OF IMAGES

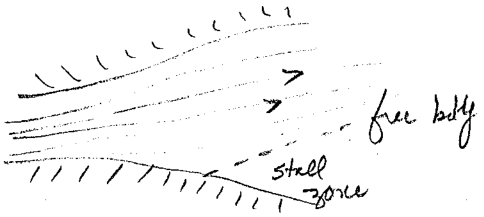
(B.) ANALYTIC - APPROXIMATE

- ORTHOGONAL SQUARES (PENCIL AND ERASER) —— (FAST, CHEAP, POOR ACCURACY) only (2-D)
- STREAMLINE CURVATURE —— (SEMI-ANALYTIC, LIMITED SCOPE, FAIR ACCURACY)
- ANALOGUE —————— (MORE ACCURATE, NEEDS EQUIPMENT, FAST, GOOD FOR INTERNAL FLOWS)
  - $\nabla^2 T = 0$ • THERMAL CONDUCTION
  - $\nabla^2 E = 0$ • ELECTRIC "

better used for internal flows where streamlines ~~ca~~ can be approx

(C) NUMERICAL — APPROXIMATE

- FINITE DIFFERENCE — (GOOD FOR INTERNAL FLOW, COMPUTER NEEDED, CODES AVAILABLE, ACCURATE, FAST)
  MESH — RELAXATION OR MATRIX INVERSION
  - popular in fluids
- FINITE ELEMENTS — (SAME AS ABOVE)
  popular in elasticity (structural mechanics community)
- CAUCHY-INTEGRAL METHOD (DEVELOPED HERE PAST 8 YEARS) — (SEMI-ANALYTIC, MATRIX INVERSION, GOOD FOR "FREE" B.C.'S AND EXTERNAL FLOWS)
  iterative work with a "free" boundary
- OTHERS (MANY)



$Q = \Delta\psi$

$Q \sim b\bar{V}$ or $\bar{V} \approx \frac{\Delta\psi}{b}$

ME 251A LECTURE 15 77/78

SOME APPROXIMATE METHODS

ORTHOGONAL SQUARES:

EQUIPMENT: PAD, PENCIL, ERASER,
COMPASS, STRAIGH EDGE, ETC.

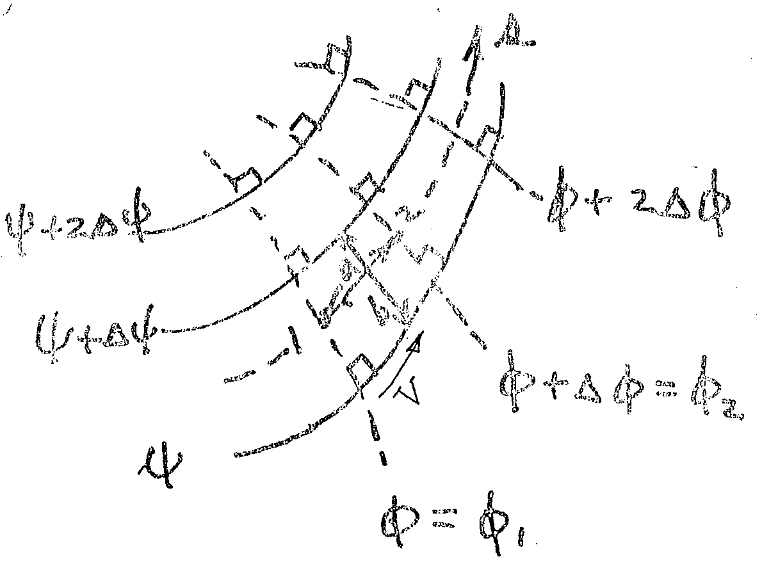
BASIS: ORTHOGONALITY OF ϕ AND ψ

LIMITS: 2-D, PLANE FLOWS, INCOMPRESSIBLE
LOW ACCURACY

ADVANTAGES: FAST, CHEAP, GOOD FOR ESTIMATES

"SQUARE" MESH:

LET $\Delta\phi$ AND $\Delta\psi$ BE CONSTANTS



TAYLOR SERIES FROM
1 → 2 ALONG s

$$\phi_2=\phi_1+a\left(\frac{\partial\phi}{\partial s}\right)_1+\frac{a^2}{2}\left(\frac{\partial^2\phi}{\partial s^2}\right)_1+\ldots$$

SO

$$\Delta\phi\approx a\frac{\partial\phi}{\partial s}+\ldots$$

SIMILARLY ALONG n

$$\Delta\psi\approx b\frac{\partial\psi}{\partial n}+\ldots$$

BUT IN ADDITION: $V = \frac{\partial\phi}{\partial s} \approx \frac{\Delta\phi}{a}$

AND: $Q \approx Vb \approx \Delta\psi$

SO $V \approx \frac{\Delta\psi}{b} = \frac{\partial\psi}{\partial n}$

THUS:

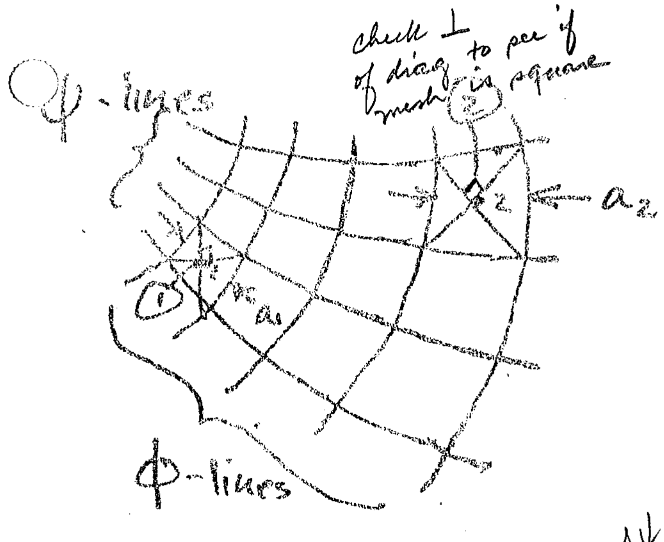
$$V = \frac{\Delta\phi}{a} = \frac{\Delta\psi}{b}$$

$$\boxed{V = \frac{\text{Const}}{a} = \frac{\text{Const}}{b}}$$

ALSO

$$\boxed{\frac{\Delta\phi}{\Delta\psi} = \frac{a}{b} = \text{Const}}$$

hold anywhere in mesh



IF $\Delta\phi \Leftrightarrow \Delta\psi$

$a = b$

AND MESH MUST BE "SQUARE" EVERYWHERE

AND VELOCITY RATIO

$$\frac{\frac{\Delta\psi}{b_2}}{\frac{\Delta\psi}{b_1}} = \frac{V_2}{V_1} = \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

PRESS. COEF.

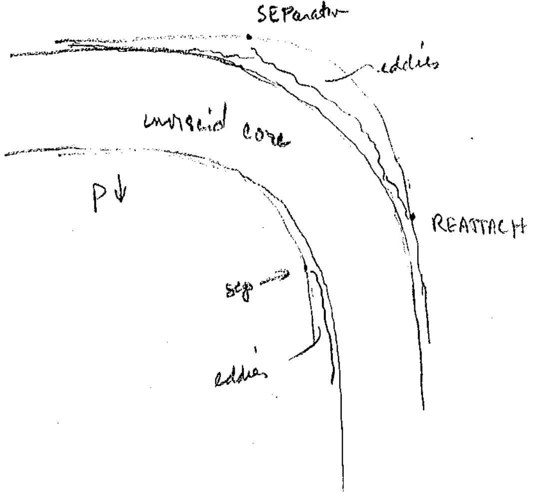
$$\frac{p_2 - p_1}{\frac{1}{2}\rho V_1^2} = 1 - \left(\frac{V_2}{V_1}\right)^2 = 1 - \left(\frac{a_1}{a_2}\right)^2$$

FOR TWO POINTS ① AND ② IN FLOW FIELD.

accuracy of method ±10%

potential flow is inviscid flow ⇒ totally reversible

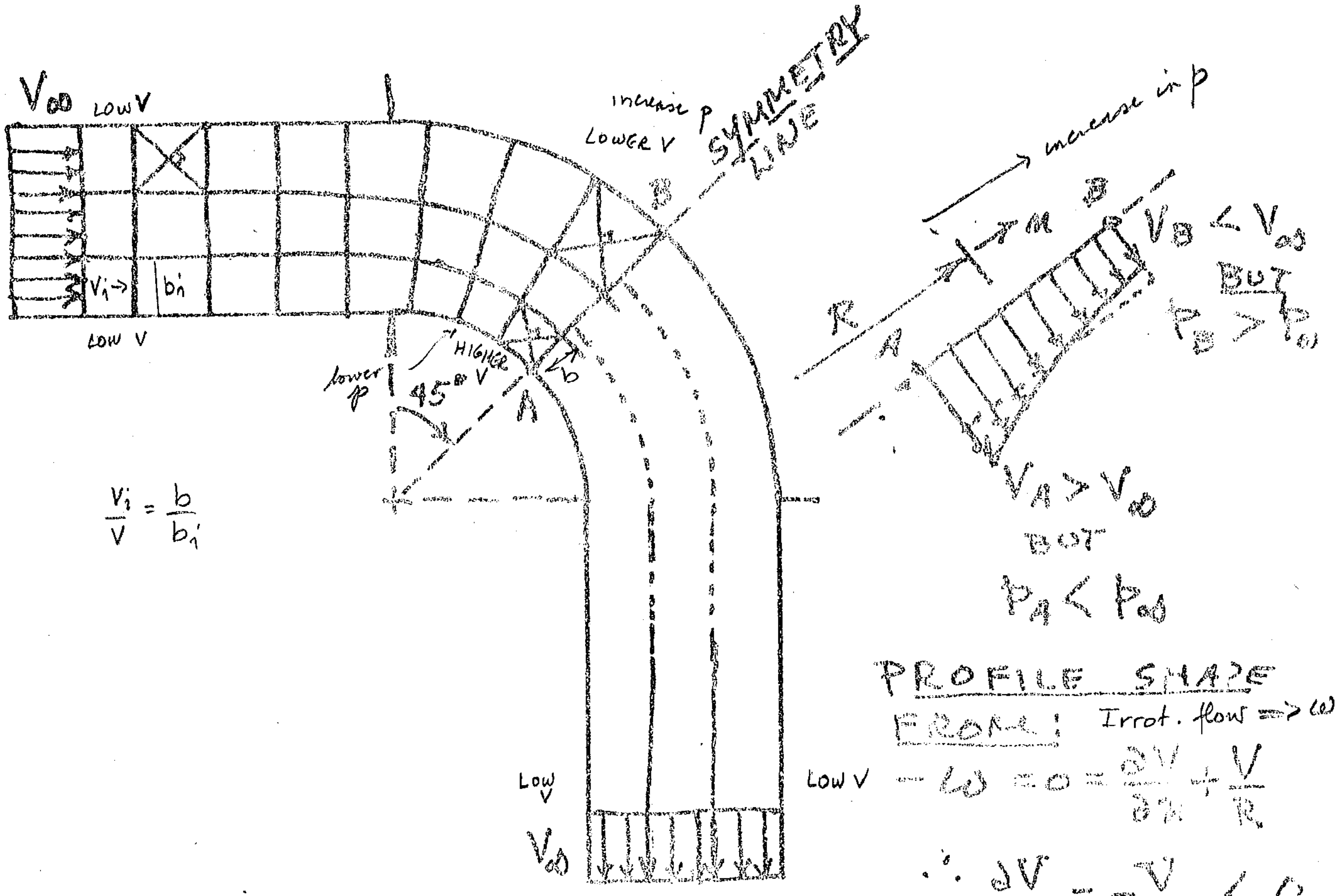
general comments hold however method of orthog squares only applies to 2-D flows



potential flow is a guide since actual flows behave differently.
We can get primary Δp's over the bend

ME 251 A 77/78 (92)

EXAMPLE: UNIFORM PARALLEL FLOW IN A 90° ELBOW; $R_{OUT} = 2R_{IN}$



$$\frac{V_1}{V} = \frac{b}{b_1}$$

PROFILE SHAPE FROM: Irrot. flow => $\omega = 0$

$$-\omega = 0 = \frac{\partial V}{\partial n} + \frac{V}{R}$$

$$\therefore \frac{\partial V}{\partial n} = -\frac{V}{R} < 0$$

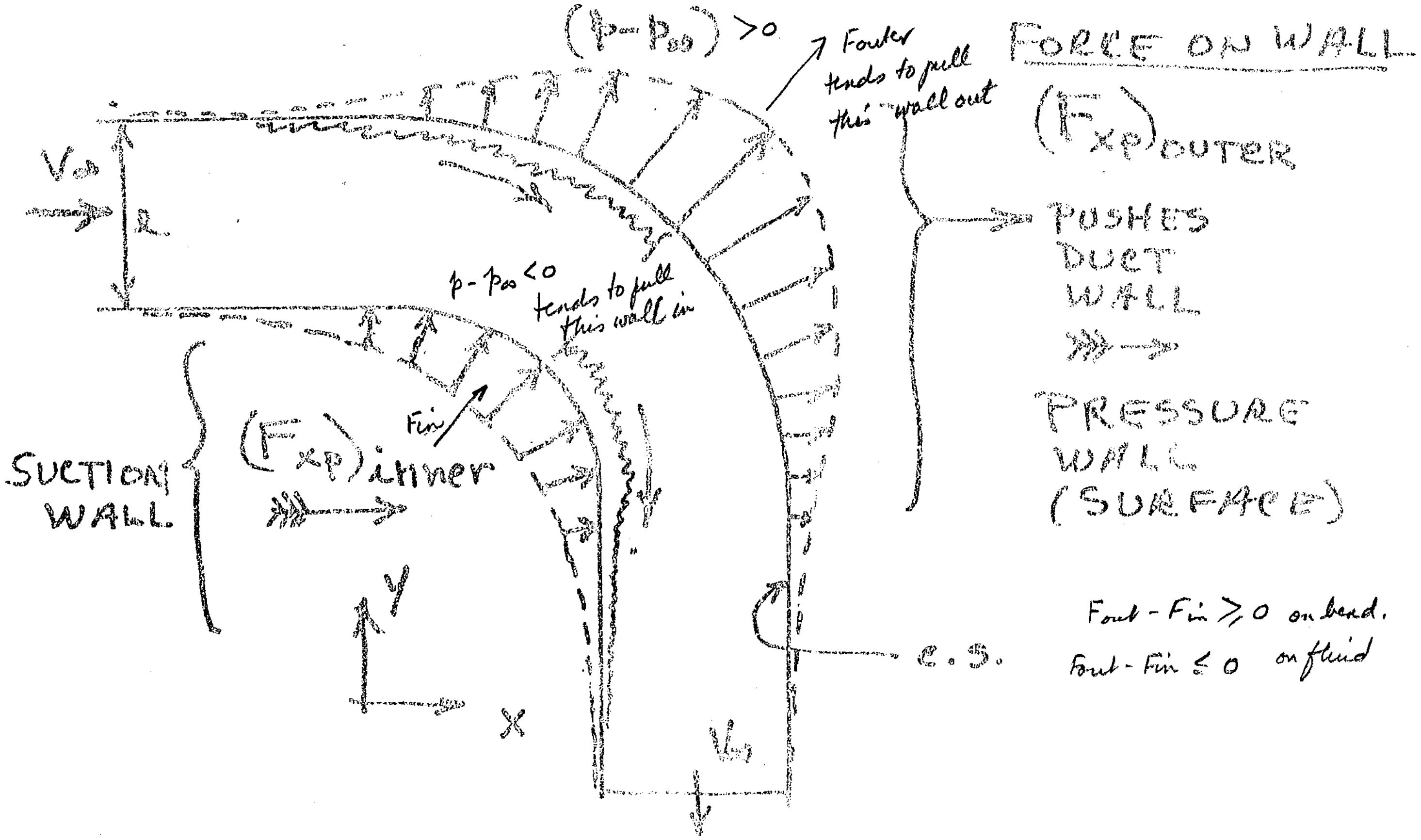
SO $V(\downarrow)$ AS $R(\uparrow)$

EULER'S N-EQ

$$\frac{\partial p}{\partial n} = \rho \frac{V^2}{R} > 0$$

$$\therefore \quad p(\uparrow) \text{ AS } R(\uparrow)$$

REGIONS OF ADVERSE PRESSURE GRADIENT; $(\partial p/\partial s)_{WALL} > 0$



Fout − Fin ≥ 0 on bend.

Fout − Fin ≤ 0 on fluid

MOMENTUM IN + X DIRECTION ON FLUID INSIDE C.S.

$$\Sigma F_x = \dot{m}\,(0) - \dot{m}\,V_\infty$$

$$\Sigma F_x = -\rho\,V_\infty^2\,l$$

ALSO

$$\Sigma F_x = -(F_{xp})_{INNER} - (F_{xp})_{OUTER}$$

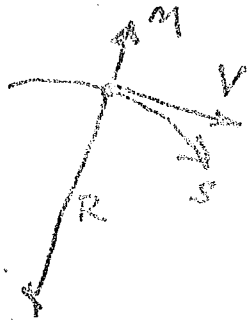
POTENTIAL SOLUTION SHOWS SOURCE OF NET FORCE ON WALLS.

ME 251A 7 7/78 (79)

STREAMLINE CURVATURE

EQUIPMENT: ANALYTIC

BASIS: $\omega = 0$ IN S-N COOR.



$$\frac{\partial V}{\partial n} + \frac{V}{R} = 0$$

we assume changes of V in S direction << change of V in n direction.

IF $\partial V/\partial s \ll \partial V/\partial n$

SO $\boxed{\frac{dV}{V} \approx -\frac{dn}{R}}$

Integrate along n

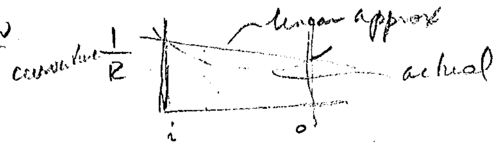
$$\ln V = -\int \frac{dn}{R} + \text{Const.}$$

$$\boxed{V = \exp\left(-\int \frac{dn}{R}\right) + \text{Const.}}$$

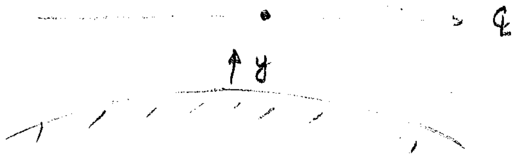
LIMITS: 2-D FLOW, QUASI-ONE DIM.

ADVANTAGES: FAST, FAIR ACCURACY
NOT LIMITED TO $\rho = \text{CONST.}$,
USEFUL IN DESIGN
PROCESS (INVERSE METHOD)

for elbow



R is radius of streamlines not pipe.



$$\bar{V} = \frac{Q}{2bH}$$

$$\left.\frac{1}{R}\right|_{\text{throat}} = \frac{1}{a} \qquad \bar{V} = \frac{1}{b}\int_0^b V\,dy$$

$$\left.\frac{1}{R_c}\right|_{\text{center}} = 0$$

$$\frac{1}{\bar{R}} = \frac{1}{2a}$$

$$\bar{v} = \frac{1}{b}\int_0^b v\,dy = -\frac{1}{b}\cdot R\int_t^c dV$$
$$= -\frac{2a}{b}(V_c - V_t)$$
$$\therefore \quad V_c = V_t - \frac{b}{2a}\bar{v}$$

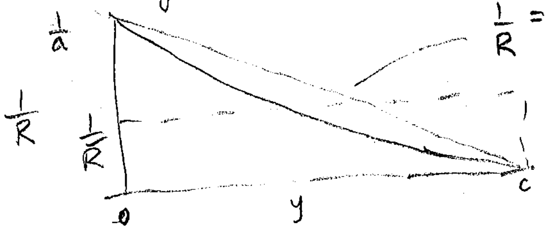
$$\frac{dV}{V} = -\frac{dy}{R}$$

use $V \approx \bar{V}$ and $\frac{1}{R} \approx \frac{1}{R} = \frac{\frac{1}{a} + \frac{1}{\infty}}{2} = \frac{1}{2a}$

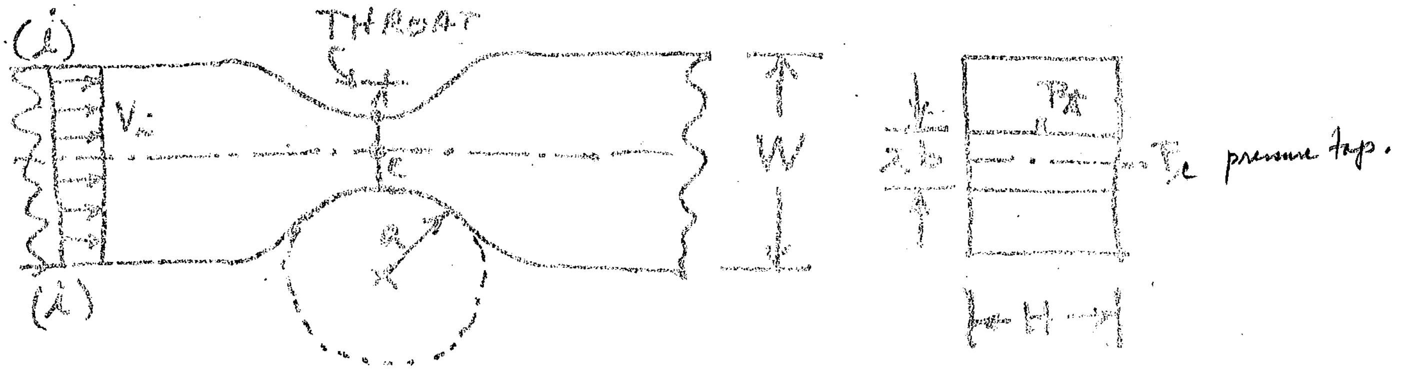
Gives $V_c = V_t - \bar{V}\left(\frac{b}{2a}\right)$

we know that for a better approx

$$\frac{1}{R} = \frac{1}{a} + \frac{1}{a}\left(\frac{y}{b}\right)$$



EXAMPLE FIND ΔP ACROSS THROAT OF 2-D VENTURI AND ESTIMATE OF OF FLOW SENSITIVITY COEFF, K.



$$\Delta P = (p_i - p_t)$$

$$\text{sensitivity coeff } K\,Q = (2bH)\sqrt{2(p_i - p_t)/\rho} = VA$$

$$K = f\left(\frac{A_i}{A_c}, \text{flowshape}\right)$$

we want K large in order to measure the sensitivity in Δp

ASSUME
1. steady flow
2. 2-D, plane flow
3. ρ = const.
4. $\mu \approx 0$ (SLIP AT WALLS)
5. Uniform, parallel flow at (i)

BECAUSE OF (2) → (5) FLOW IS irrotational AT THROAT

$$\frac{\partial V}{\partial n} + \frac{V}{R} = 0$$

BECAUSE OF SYMMETRY ABOUT THROAT

$$\frac{\partial V}{\partial s} = 0 \text{ ACROSS } t\ c$$

SO $$\boxed{\frac{dV}{V} = -\frac{dn}{R}}$$ AT THROAT ONLY if $\frac{dV}{V} = -\frac{dy}{R}$

$$\frac{dV}{V} = -\left[\frac{1}{a} - \frac{1}{a}\left(\frac{y}{b}\right)\right] dy$$

integrate from $y=0$ to $y=b$

$$\ln\left(\frac{V_c}{V_t}\right) = -\frac{1}{2}\left(\frac{b}{a}\right)$$

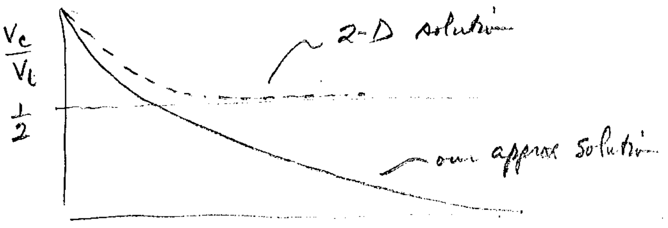
$$\frac{V_c}{V_t} = e^{-\frac{1}{2}\left(\frac{b}{a}\right)} \cong 1 - \frac{1}{2}\left(\frac{b}{a}\right) + \left(\frac{1}{8}\frac{b^2}{a^2}\right) - \dots$$

for small $\frac{b}{a}$ $\Rightarrow$

$$\frac{V_c}{V_t} \cong 1 - \frac{1}{2}\frac{b}{a}$$

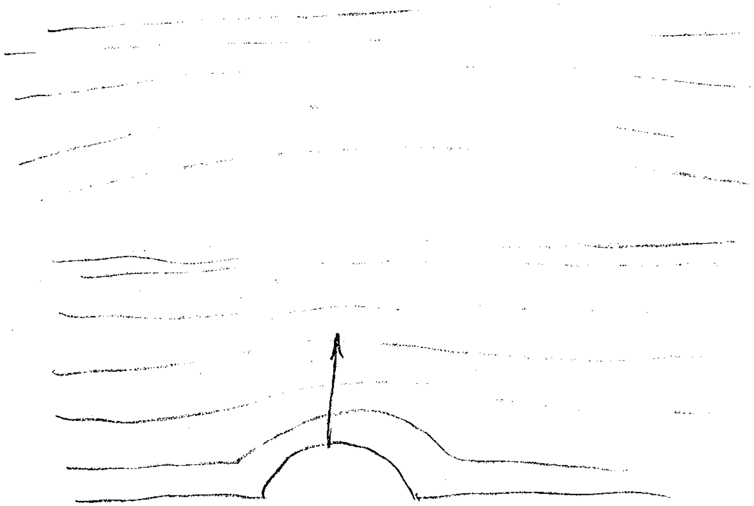
$$\therefore V_c \cong V_t - \frac{1}{2}\frac{b}{a}V_t$$

for large $\frac{b}{a}$ $\frac{V_c}{V_t} \to 0$ which is incorrect. $\frac{V_c}{V_t} \to \frac{1}{2}$ for 2-D problems.

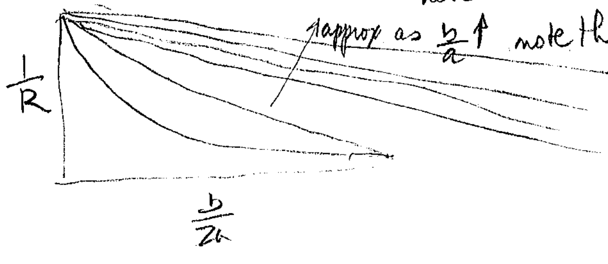


for the case where $\frac{b}{a}$ is small

note $\frac{1}{R}$ is slow smooth variation



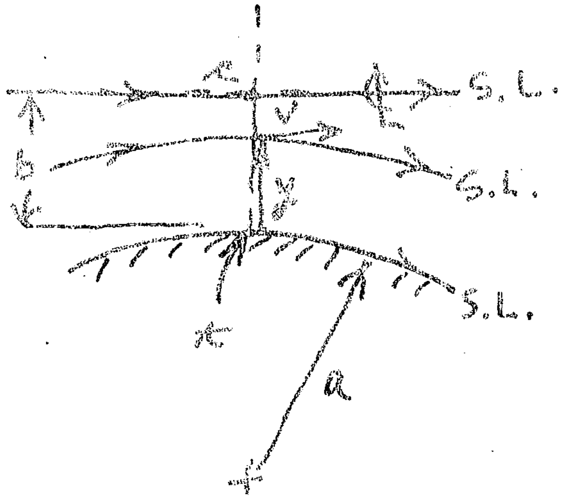
note that as $y \uparrow$ $\frac{1}{R} \to 0$.



note that straight line approx becomes worse as $\frac{b}{2a}$ becomes larger & larger $\therefore$ streamline method is better for small $\frac{b}{}$.

ME 251A

77/78 ⑨⑥



MEAN SPEED

$$\bar{V} = \frac{1}{b}\int_0^b V\,dy = \frac{Q}{(2bH)}$$

MEAN (AVERAGE) CURVATURE

$$\frac{1}{R} = \frac{1}{2}\left(\frac{1}{R_x} + \frac{1}{R_z}\right) = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{\infty}\right)$$

$$\frac{1}{R} = \frac{1}{2a}$$

So EQ (95-1) IS APPROXIMATELY $(dn = dy)$

$$\int_{V_t}^{V} dV = -\frac{\bar{V}}{2a}\int_0^y dy \qquad dV = -V\frac{dn}{R}$$

$$\boxed{\begin{aligned} V &\approx V_t - \bar{V}\left(\frac{y}{2a}\right) \\ V_e &= V_t - \bar{V}\left(\frac{b}{2a}\right) \end{aligned}} \quad (96\text{-}1)$$

(note: $V_t = V_{max}$)
since $V \downarrow$ as $y \uparrow$

Euler's n-EQ.

$$\frac{\partial p}{\partial n} = \frac{dp}{dy} = \rho\frac{V^2}{R} \approx \rho\frac{\bar{V}^2}{2a} + \cdots$$

$$\text{So } \boxed{\Delta p = (p_e - p_i) \approx \rho\frac{\bar{V}^2}{2}\left(\frac{b}{a}\right) + \cdots} \quad (96\text{-}2)$$

(1/2 ANSWER)

$$\bar{V} = \frac{1}{b}\int_0^b \left(V_t - \bar{V}\frac{y}{2a}\right)dy = V_t - \bar{V}\,\frac{1}{4}\left(\frac{b}{a}\right)$$

$$\boxed{V_t = \bar{V}\left(1 + \frac{1}{4}\frac{b}{a}\right)} \quad (96\text{-}3)$$

NOW BERNOULLI FROM (i) → (t)

$$\frac{p_i}{\rho} + \frac{V_i^2}{2} = \frac{p_t}{\rho} + \frac{V_t^2}{2}$$

$$V_t^2\left(1 - \frac{V_i^2}{V_t^2}\right) = \frac{2(p_i - p_t)}{\rho}$$

$$\bar{V}^2\left(1 + \frac{1}{4}\frac{b}{a}\right)^2\left(1 - \left(\frac{V_i}{\bar{V}}\right)^2 \frac{1}{\left(1 + \frac{1}{4}\frac{b}{a}\right)^2}\right) = \frac{2(p_i - p_t)}{\rho}$$

BUT $V_i \cancel{H} W = \bar{V}\, 2b \cancel{H}$ continuity

$$\bar{V} = Q/(2bH)$$

SO! $$\bar{V} = \frac{Q}{2bH} = \underbrace{\frac{1}{\sqrt{\underbrace{\left(1 + \frac{1}{4}\frac{b}{a}\right)^2}_{\approx \text{ CORRECTION FOR THROAT CURVATURE}} - \left(\frac{2b}{W}\right)^2}}}_{K\text{ : SENSITIVITY COEFFICIENT}} \sqrt{\frac{2(p_i - p_t)}{\rho}}$$

NOTE: CURVATURE INCREASES SENSITIVITY

ELECTRIC ANALOGUE

EQUIPMENT: SHALLOW PAN OF WATER OR "TELEDELTOS" PAPER, PROBES, WIRES, VOLTAGE SOURCE, VOLTAGE DIVIDER, GALVANOMETER, ETC.

BASIS: ELECTRIC POTENTIAL IN THIN CONDUCTING SHEET (BELOW)

LIMITS: 2-D, PLANE, ρ = CONST.

ADVANTAGES: EASY WHEN SETUP AVAILABLE, SOLVES MANY COMPLEX BOUNDARY SHAPES, FAIR ACCURACY ($\sim \pm 5\%\ V_{max}$)

ELECTRIC PROBLEM

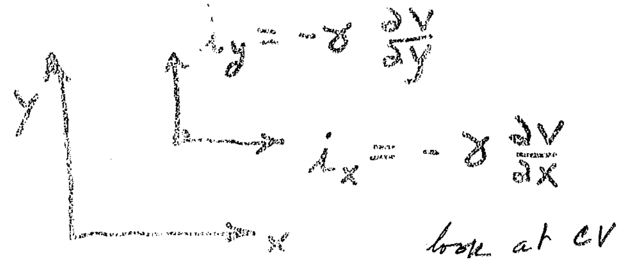
$\vec{i}$ = current density; $I = \int_A \vec{i} \cdot d\vec{A}$ is current through area A

$\vec{\nabla} V$ = gradient of electric potential or, $-\vec{E} = \vec{\nabla} V$, electric field strength

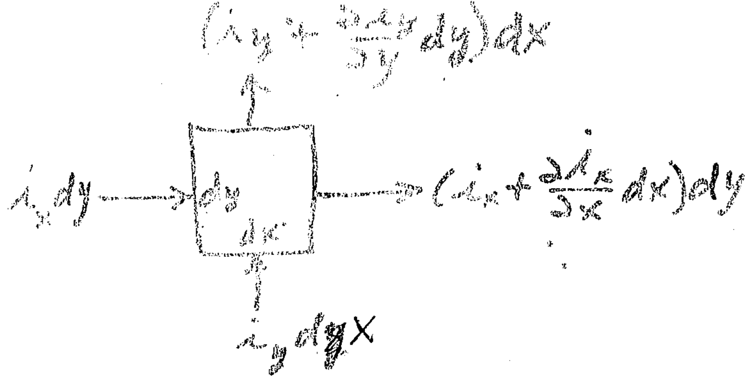
$\vec{i} = +\gamma \vec{E} = -\gamma \vec{\nabla} V$, generalized Ohm's law where

$\gamma = \frac{1}{r}$ electric conductivity (1/resistivity) of material

LOOK AT 2-D, PLANE SITUATION (UNIT DEPTH)



LAW CONS. OF CHARGE (NO SOURCES AND STEADY CASE): CURRENT (OUT − IN) = 0.



CURRENT FLOWS

BOUNDARY COND'S.

$$\boxed{(V = \text{Const}) \Leftrightarrow \psi = (\text{Const})}$$

$$\frac{\partial V}{\partial n} = 0 \Longleftrightarrow \frac{\partial \phi}{\partial n} = 0$$

SO:

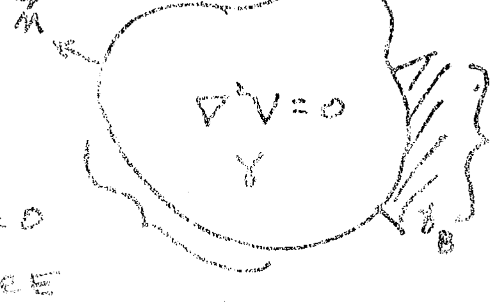
$$\left(\frac{\partial i_x}{\partial x}\right)\cancel{dx\,dy} + \left(\frac{\partial i_y}{\partial y}\right)\cancel{dy\,dx} =$$

$$\frac{\partial}{\partial x}\left(-\gamma\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\gamma\frac{\partial V}{\partial y}\right) = 0$$

IF γ = CONST.

$$\boxed{\nabla^2 V = 0}$$

ANALOGOUS to ϕ OR ψ



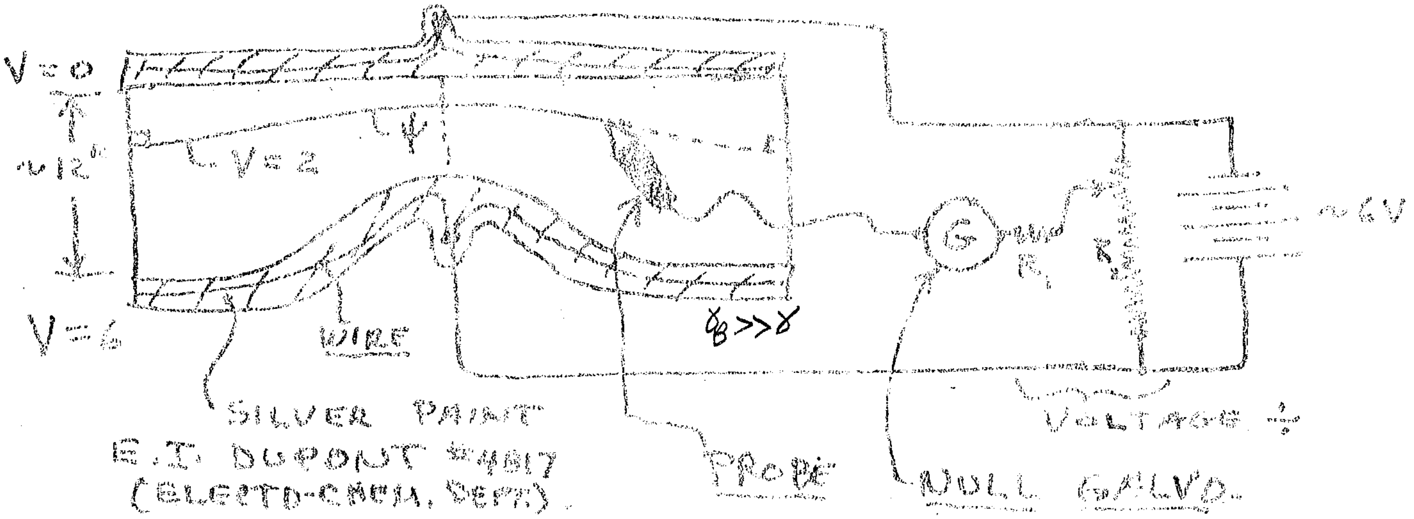
V = Const. if Boundary a SUBSTANCE of $\gamma_B \gg\gg \gamma$

$\frac{\partial V}{\partial n} = 0$ WHERE $\gamma_B \ll\ll \gamma$ (INSULATOR)

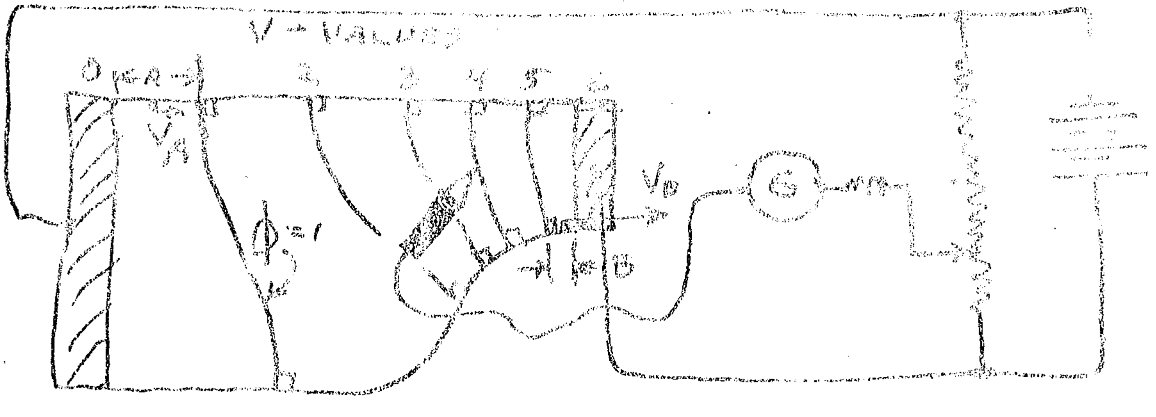
ME 251 A 7/7/73 (P00)

APPLICATION ("TELODELTOS" PAPER FROM WESTERN ELECTRIC)

γ = const ± 5%



SET UP FOR EXAMPLE OF VENTURI TO OBTAIN STREAMLINES



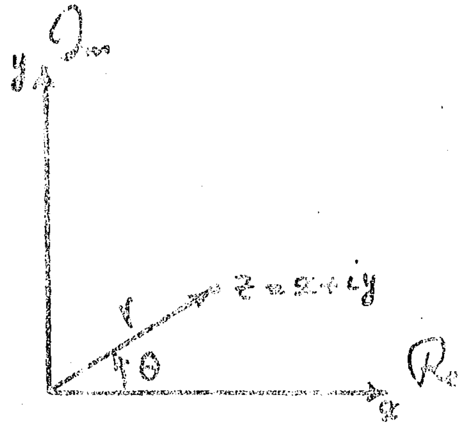
SET UP FOR φ-LINES AND WALL SPEEDS

$$\frac{V_B}{V_A} = \frac{A}{B}$$

$\rho = \text{const}$ $\vec{\omega} = 0$ 2-D plane flow

we will look at flows.

Review of Complex Numbers



$$z = x + iy$$
$$= r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

Addition and Multiplication

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$
$$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = r_1r_2\, e^{i(\theta_1+\theta_2)}$$
$$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

The Complex Conjugate of $z = x+iy$ is defined as

$$\bar{z} \text{ or } z^* \triangleq x - iy = re^{-i\theta}$$

Differentiation

A function $f(z)$ is said to have a derivative at z_0 if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and has the same value for any mode of approach of z to z_0.

Analytic Functions

If $f(z)$ has a derivative at z_0 and also at each point in the neighborhood of z_0 then $f(z)$ is said to be analytic at z_0

Tests of Analyticity

1. Cauchy - Riemann Equations

The real and imaginary parts of an analytic function, $f(z) = g(x,y) + i\,h(x,y)$, satisfy Cauchy-Riemann equations

$$\boxed{\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} \quad \text{and} \quad \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}}$$

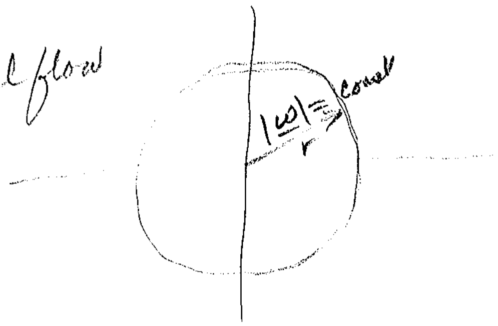
2. Laplace's Equation

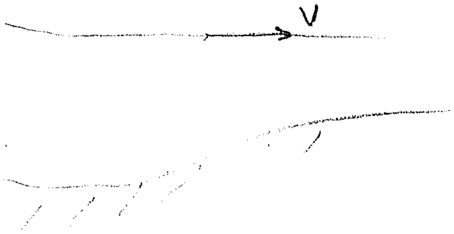
The real and imaginary parts of an analytic function are <u>harmonic</u>

$$\boxed{\nabla^2 g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0 \;, \quad \nabla^2 h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0}$$

3. Others (direct method, geometrical method).

shallow channel flow





The Complex Potential Function

For 2-D, incompressible potential flow,

$$\nabla^2 \phi = 0 \quad \text{and} \quad \nabla^2 \psi = 0$$

where ϕ is the velocity potential and ψ is the stream function.

Define the complex potential function as

$$\Phi(z) = \phi(x,y) + i\,\psi(x,y)$$

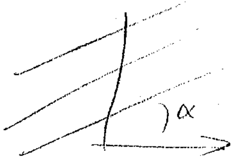
then $\Phi(z)$ is analytic.

$$\Phi'(z) = \frac{d\Phi}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv = \overline{w}$$

where $w = u + iv$ is the complex velocity. The magnitude of velocity is the modulus of w

$$V = |w| = |\overline{w}|$$

all stagnation points $w = 0$ on hodograph plane



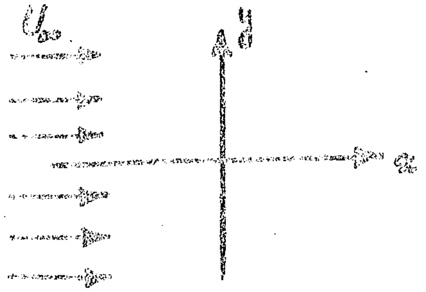
$$\Phi = we^{-i\alpha}z$$

$$= wre^{i\underbrace{(\theta-\alpha)}_{\text{rotation of the axis}}}$$

$$\therefore \Phi = wz' \qquad z' = re^{i(\theta-\alpha)}$$

Elementry Flows

1. UNIFORM PARALLEL



$$\boxed{\Phi(z) = U_\infty z}$$

$$\Phi'(z) = u - iv = U_\infty$$

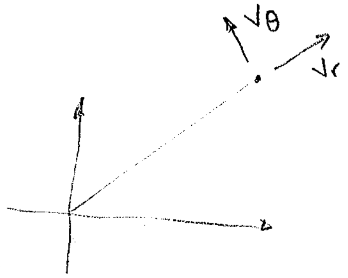
$\therefore\quad u = U_\infty\ ,\ v = 0$ This is a parallel flow in +x direction

$$\Phi(z) = U_\infty z = U_\infty x + i\, U_\infty y$$

$$\therefore\quad \phi = U_\infty x\ ,\ \psi = U_\infty y$$

In general $\Phi(z) = \bar{w}\, z = (u - iv)\, z$ is a uniform parallel flow.

$$\phi = \frac{Q}{2\pi} \ln r \qquad \psi = \frac{Q}{2\pi}\theta$$

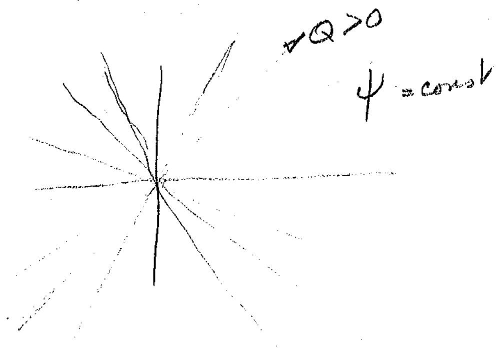


$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi r}$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = 0$$

as $r \uparrow$ $v_r \longrightarrow 0$

Q is source/sink strength



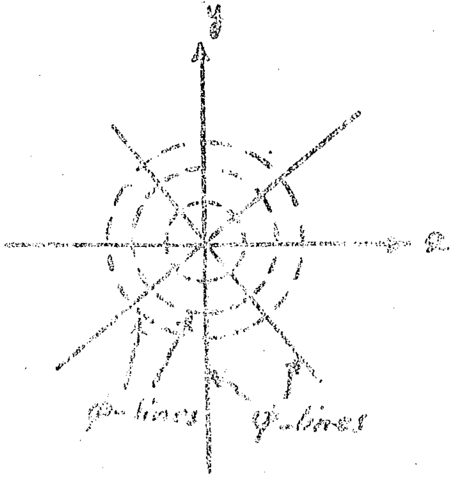
let source pt be z_0 then $z - z_0 = w$ translates z_0 to orig

$$\Phi(z) = \Phi(w) = \frac{Q}{2\pi}\ln w = \frac{Q}{2\pi}\ln(z - z_0)$$

$$\omega_z = \frac{\partial v_r}{r\,\partial\theta} - \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} = 0 \therefore \text{irrot}$$

2. SOURCE AND SINK

$$\boxed{\Phi(z) = \frac{Q}{2\pi} \ln z}$$



$Q > 0$: Source

$Q < 0$: Sink

Q is the source or sink strength.

$$\Phi(z) = \frac{Q}{2\pi} \ln (re^{i\theta}) = \frac{Q}{2\pi} (\ln r + i\theta)$$

$$\therefore \quad \phi = \frac{Q}{2\pi} \ln r \; ; \quad \psi = \frac{Q}{2\pi} \theta$$

$$v_r = \frac{\partial \phi}{\partial r} = \frac{Q}{2\pi r} \; , \quad v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0$$

More generally, $\Phi(z) = \frac{Q}{2\pi} \ln(z - z_0)$ for a source or sink located at z_0

Γ = circulation
(circulation) strength of vortex

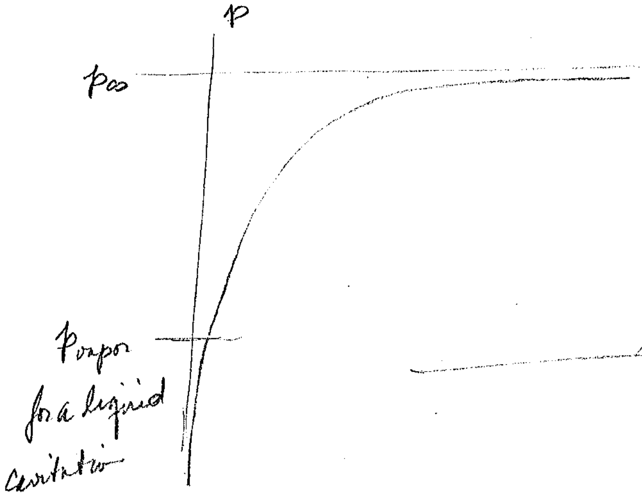
$v_\theta \longrightarrow 0$ as $r \longrightarrow \infty$

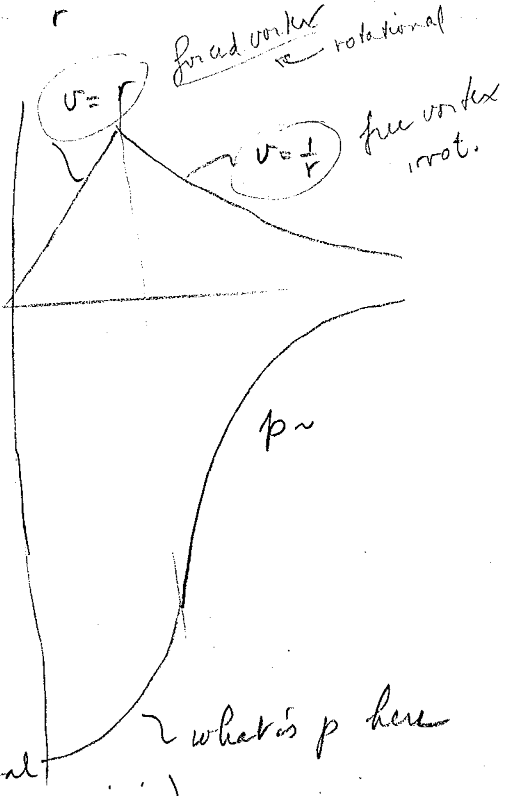
Bernoulli's eq to get p

or Euler's n equation $\frac{dp}{dr} = \rho\frac{v_\theta^2}{r} = \rho\frac{\Gamma^2}{r^3\pi^2}$ $\quad \frac{dp}{dr} \sim \frac{1}{r^3}$

$$p_\infty = p + \rho\frac{v_\theta^2}{2}$$

$$p_\infty = p + \rho\frac{\Gamma^2}{8\pi^2 r^2} \quad \text{or} \quad p = p_\infty - \frac{\rho}{2}\left(\frac{\Gamma}{2\pi}\right)^2\frac{1}{r^2}$$





Now $\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{\partial v_r}{r\partial\theta}$

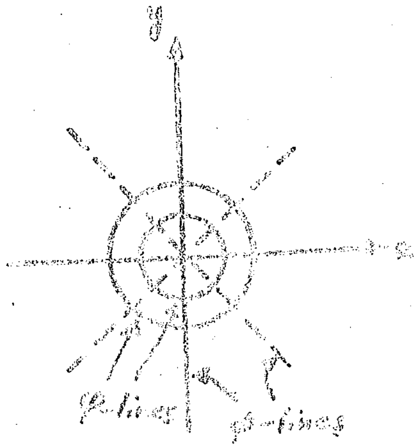
$$-\frac{\Gamma}{2\pi r^2} + \frac{\Gamma}{2\pi r^2} = 0 = \omega_z$$

contradicts $\oint \underline{v}\cdot d\underline{s} = \Gamma$

reason is that ϕ can be multivalued
in a region where flow is rotational
ie where a singularity exists (here at origin).

3. POINT VORTEX

$$\boxed{\Phi(z) = -\frac{i\Gamma}{2\pi} \ln z}$$



$\Gamma > 0$ Counter-clockwise

$\Gamma < 0$ Clockwise

Γ is called the vortex strength.

$$\Phi(z) = -\frac{i\Gamma}{2\pi} \ln re^{i\theta} = -\frac{i\Gamma}{2\pi}(\ln r + i\theta)$$

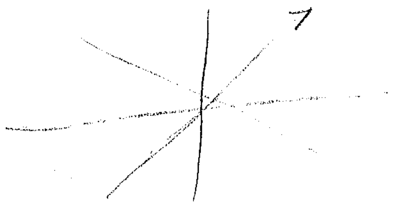
$$\Phi(z) = \frac{\Gamma}{2\pi}\theta - \frac{i\Gamma}{2\pi}\ln r$$

$$\therefore \quad \phi = \frac{\Gamma}{2\pi}\theta \quad , \quad \psi = -\frac{\Gamma}{2\pi}\ln r$$

$$v_r = \frac{\partial\phi}{\partial r} = 0$$
$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r}$$

Again $\Phi(z) = -\frac{i\Gamma}{2\pi}\ln(z - z_0)$ for a point vortex located at z_0

for a source



monopole

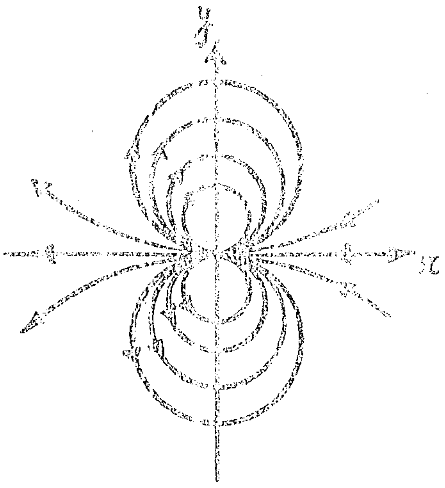
$$v_r = \frac{Q}{2\pi} \frac{1}{r}$$

$v_r \sim \frac{1}{r}$ 2-D

$v_r \sim \frac{1}{r^2}$ 3-D

4. DOUBLET (DIPOLE)

$$\boxed{\Phi(z) = \frac{S}{z}}$$



ψ-lines when S>0

S is the strength of the dipole.

$\phi = \frac{S}{r}\cos\theta$, $\psi = -\frac{S}{r}\sin\theta$

For a doublet at $z=z_0$: $\Phi(z) = \frac{S}{z-z_0}$

Note: A doublet is defined as the limit, as $a \to 0$, of a source and a sink of strength Q and $-Q$ located at $z=a$ and $z=-a$, respectively, the limit being taken in such a way that the doublet strength, defined as $S = \frac{Qa}{\pi}$, remains constant.

Extension: A quadrupole (2^2 pole) is defined in a manner analogous to the dipole (2^1 pole), as the limit of two equal and opposite dipoles. In general the complex potential due to a 2^n pole at $z=z_0$ is

$$\Phi_n(z) = \frac{(-1)^n k}{(z-z_0)^n}$$

where k is the strength of 2^n-pole.

5. CORNER AND WEDGE FLOWS

$$\boxed{\Phi(z) = \frac{A}{n} z^{n}}$$

where $n = \frac{\pi}{\alpha}$ and α is the angle of the Corner.

$$\phi = A \frac{r^{n}}{n} \cos n\theta$$

$$\psi = A \frac{r^{n}}{n} \sin n\theta$$

$$\Phi'(z) = A z^{n-1} = A r^{n-1} e^{i(n-1)\theta} = \bar{w}$$

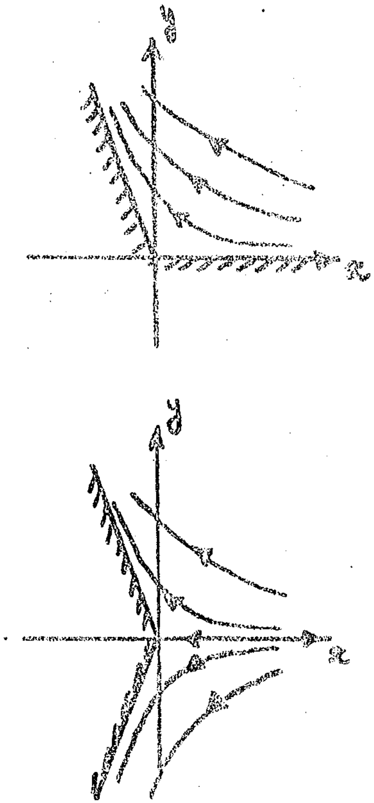
$$V = |\bar{w}| = A r^{n-1}$$

$$v_r = \frac{\partial \phi}{\partial r} = A r^{n-1} \cos n\theta$$

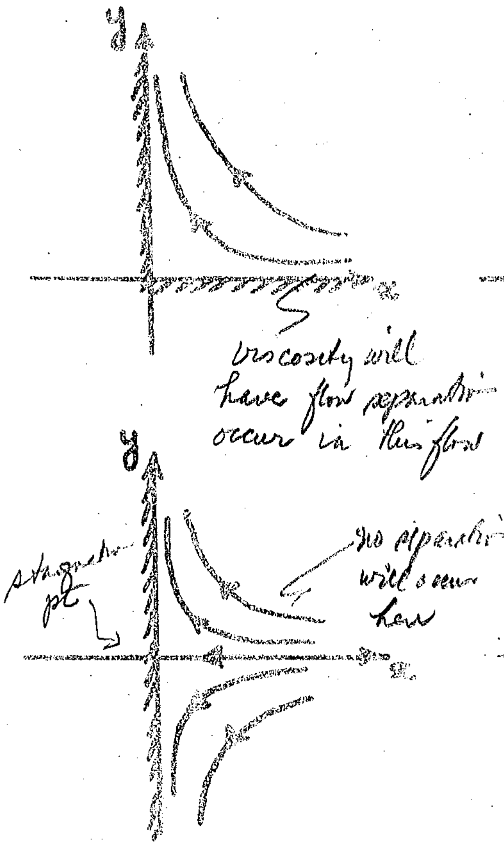
$$v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -A r^{n-1} \sin n\theta$$

at the Corner vertex $(r=0)$ $\begin{cases} V = 0 & \text{if } n > 1 \quad \text{Concave} \\ V = \infty & \text{if } n < 1 \quad \text{Convex} \end{cases}$

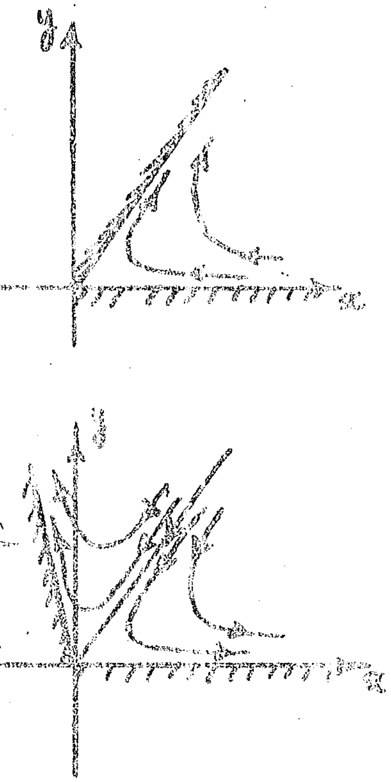
Examples where $n > 1$



$n = 3/2$

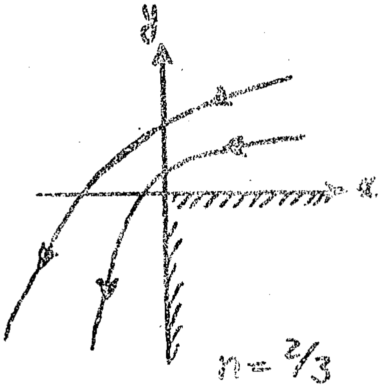


$n = 2$

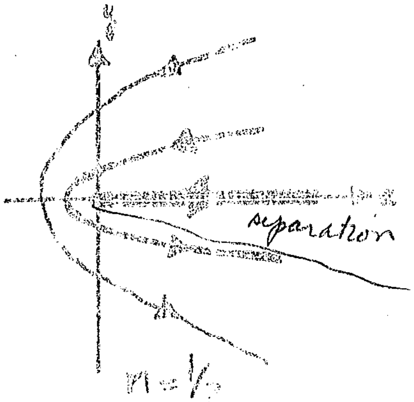


$n = 3$

Examples where $n < 1$



$n = 2/3$



$n = 1/2$

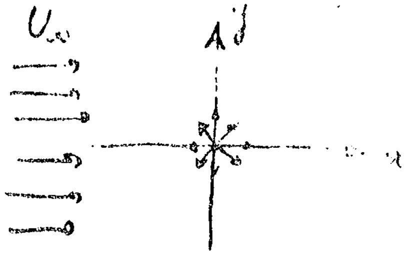
We will Solidify streamlines and look for Stagnation points.

Superposition of Basic Flows

If $f_1(z), f_2(z), \cdots, f_n(z)$ are analytic functions at $z = z_0$ so is their linear combination. This gives the idea of superposition of known flows in order to construct new flows or to satisfy the boundary conditions.

Example: Examine the flow field due to a parallel flow and source flow. Specifically determine the stagnation point and the body shape resulted from "solidification of the streamline" through the stagnation point. Also check the possibility of laminar separation.

Parallel + Source



$$\Phi(z) = U_\infty z + \frac{Q}{2\pi} \ln z$$

$$= \underbrace{U_\infty x}_{\varphi_p} + i \underbrace{U_\infty y}_{\psi_p} + \underbrace{\frac{Q}{2\pi} \ln r}_{\varphi_Q} + i \underbrace{\frac{Q}{2\pi} \theta}_{\psi_Q}$$

Now we can graphically construct $\psi = \psi_p + \psi_Q$

(See next page)

$$\phi = U_\infty x + \frac{Q}{2\pi} \ln(\sqrt{x^2+y^2})$$

$$\psi = U_\infty y + \frac{Q}{2\pi} \tan^{-1}(y/x)$$

Velocity Field $\quad \overline{w} = \frac{d\Phi}{dz} = u - iv = U_\infty + \frac{Q}{2\pi} \frac{1}{z} = V_\infty + \frac{Q}{2\pi} \frac{\bar{z}}{z\bar{z}}$

$$\overline{W} = u - iv = \Phi'(z) = U_\infty + \frac{Q}{2\pi} \frac{1}{z}$$

$$\overline{W} = U_\infty + \frac{Q}{2\pi} \frac{x - iy}{r^2} = \left(U_\infty + \frac{Q}{2\pi} \frac{x}{r^2}\right) - i\left(\frac{Q}{2\pi} \frac{y}{r^2}\right)$$

Stagnation points where $u = 0$ and $v = 0$,

$\therefore \quad y_s = 0$

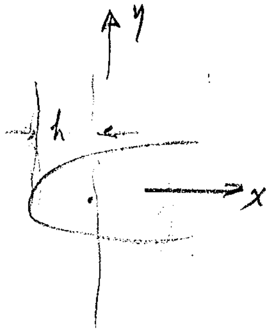
$x_s = -\frac{Q}{2\pi U_\infty} = -h \quad$ (See fig)

also $\theta_s = \pi$

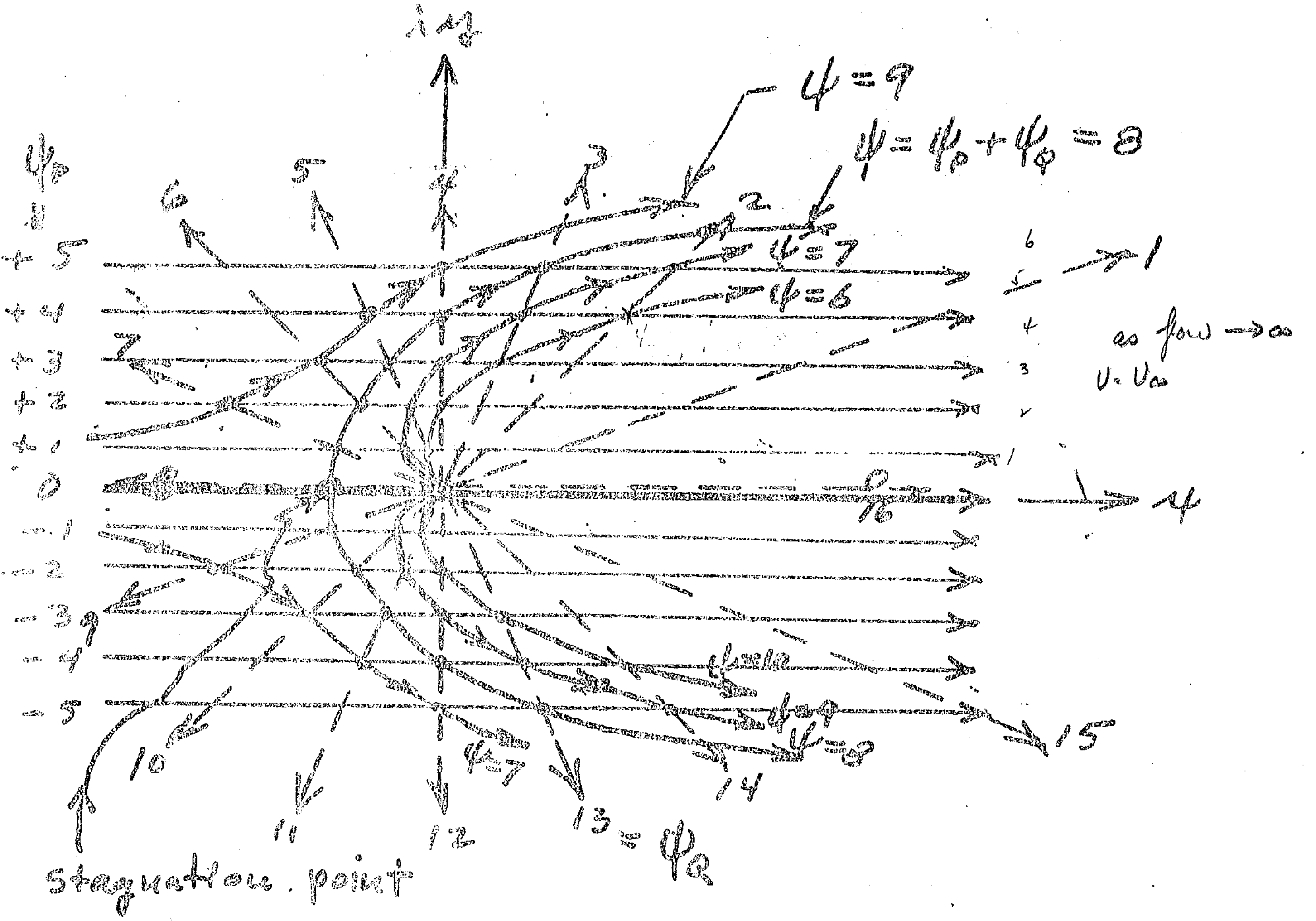
$\bar{w} = 0$ Stag pt $y = 0$

$$U_\infty + \left(\frac{Q}{2\pi}\right)\frac{h}{h^2+0} = 0$$

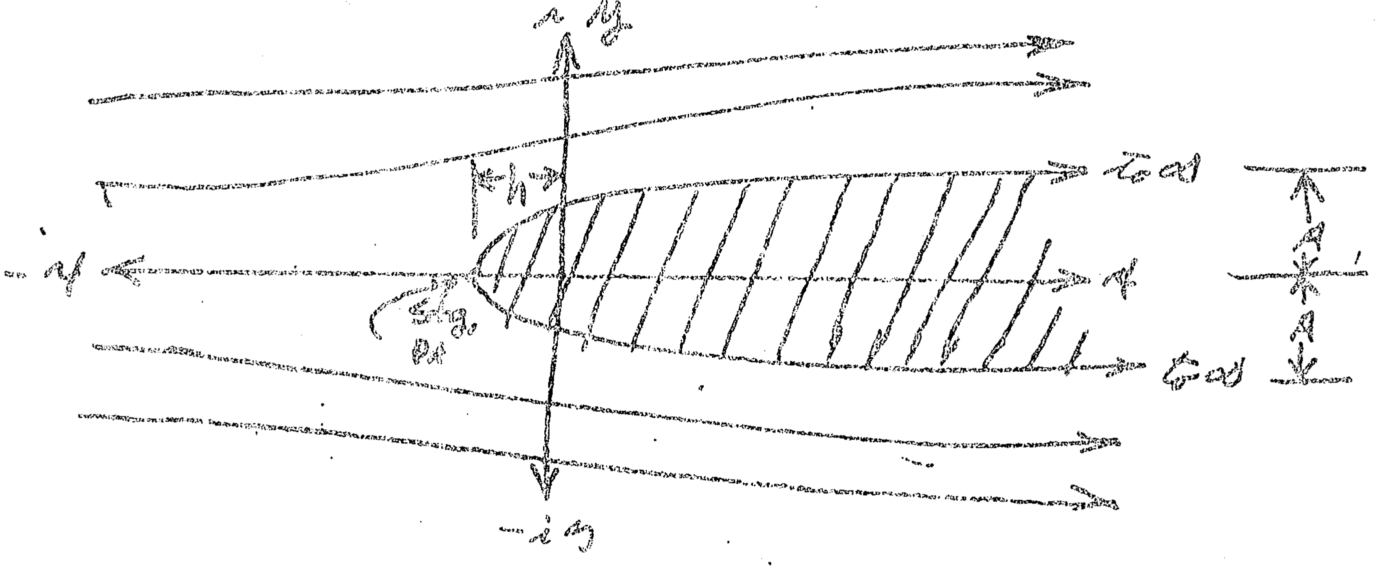
$$\boxed{h = -\frac{Q}{2\pi U_\infty}}$$ where



$$\frac{u}{U_\infty} = 1 + \frac{\hat{x}}{\hat{r}^2} \qquad \frac{v}{U_\infty} = \frac{\hat{y}}{\hat{r}^2}$$ where $\hat{x}, \hat{y}, \hat{r} = \frac{x}{h}, \frac{y}{h}, \frac{r}{h}$



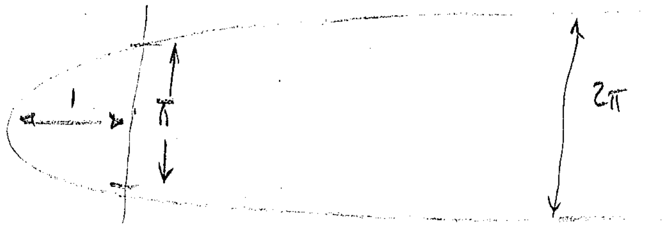
We see that a stag. pt. lies on the $\psi = 8$ stream line. If we solidify this line we would solve the problem of streaming flow over a symmetrical half-body shaped line:



since $\psi = V_\infty y + \frac{Q}{2\pi}\theta$, on the streamline is $y = 0$ and $\theta = \pi$

$\therefore \quad \psi_s = \frac{Q}{2}$ or $\psi = V_\infty y_{surf} + \frac{Q}{2\pi}\underset{\left(\tan^{-1}\left(\frac{y_s}{x_s}\right)\right)}{\theta_{surf}} = \frac{Q}{2}$

at $x_s = +\infty$ $\quad \theta_{surface} = \pm 0 \quad \hat{y}_s = \frac{y_s}{h} = \pm\pi$ see last page
& next page on back



$p + \frac{\rho}{2}V_\infty^2 = p_0$ pressure in the flow

From $\bar{w} = u - iv$,

$$\boxed{\frac{u}{U_\infty} = 1 + \frac{\hat{x}}{\hat{r}^2} \, , \quad \frac{v}{U_\infty} = \frac{\hat{y}}{\hat{r}^2}}$$

where $\hat{r}^2 = \hat{x}^2 + \hat{y}^2$, $\hat{x} = \frac{x}{h}$, $\hat{y} = \frac{y}{h}$

Character of Solution for $r^2 = x^2 + y^2 \rightarrow \infty$

$$\boxed{\frac{u_\infty}{U_\infty} = 1} \, , \quad \boxed{\frac{v_\infty}{U_\infty} = 0} \quad \text{so} \quad \boxed{V_\infty = U_\infty}$$

which are proper boundary conditions for free flight in an infinite atmosphere.

Shape of the body - "Solidification of stream-line" through the stagnation point,

$$\psi_S = U_\infty y_S + \frac{Q}{2\pi} \theta_S = \frac{Q}{2\pi}(\pi) = \frac{Q}{2}$$

Stagnation point is on the body surface, therefore the equation for the body shape would be,

$$\boxed{\frac{Q}{2\pi} \theta + U_\infty y = \frac{Q}{2}}$$

$$\therefore \quad \boxed{y = \frac{Q}{2U_\infty}\left(1 - \frac{\theta}{\pi}\right) \quad \text{or} \quad \hat{y} = \frac{y}{h} = \pi - \theta}$$

$$\frac{y}{h} = \frac{Q}{2U_\infty h \pi}(\pi - \theta)$$

$$h = -\frac{Q}{2\pi U_\infty} \quad \therefore \quad \hat{y} = (\theta - \pi)$$

Since $\theta = \tan^{-1}\left(\frac{\hat{y}}{\hat{x}}\right)$,

$$\boxed{\hat{y} = \pi - \tan^{-1}\frac{\hat{y}}{\hat{x}}}$$

@ $x = +\infty$, $\theta = 0$ so $\boxed{(\hat{y})_{x=+\infty} = \pi}$

@ $x = 0$, $\theta = \pi/2$ so $\boxed{(\hat{y})_{x=0} = \pi/2}$

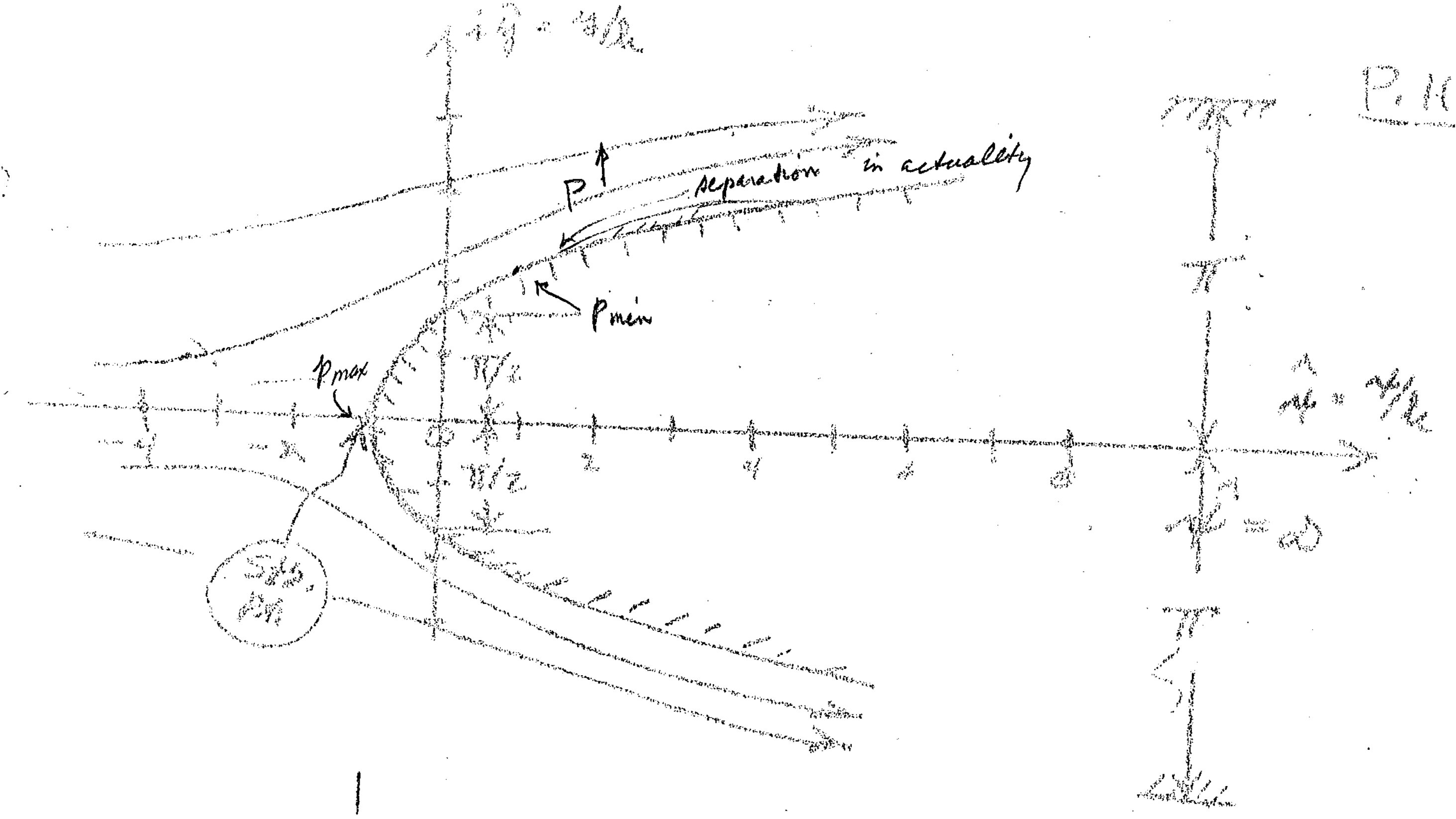
Pressure distribution

Using Bernoulli equation,

$$\boxed{\frac{p_0 - p}{\frac{1}{2}\rho U_\infty^2} = \left(1 + \frac{\hat{x}}{\hat{\rho}^2}\right)^2 + \left(\frac{\hat{y}}{\hat{\rho}^2}\right)^2}$$ (See next page)

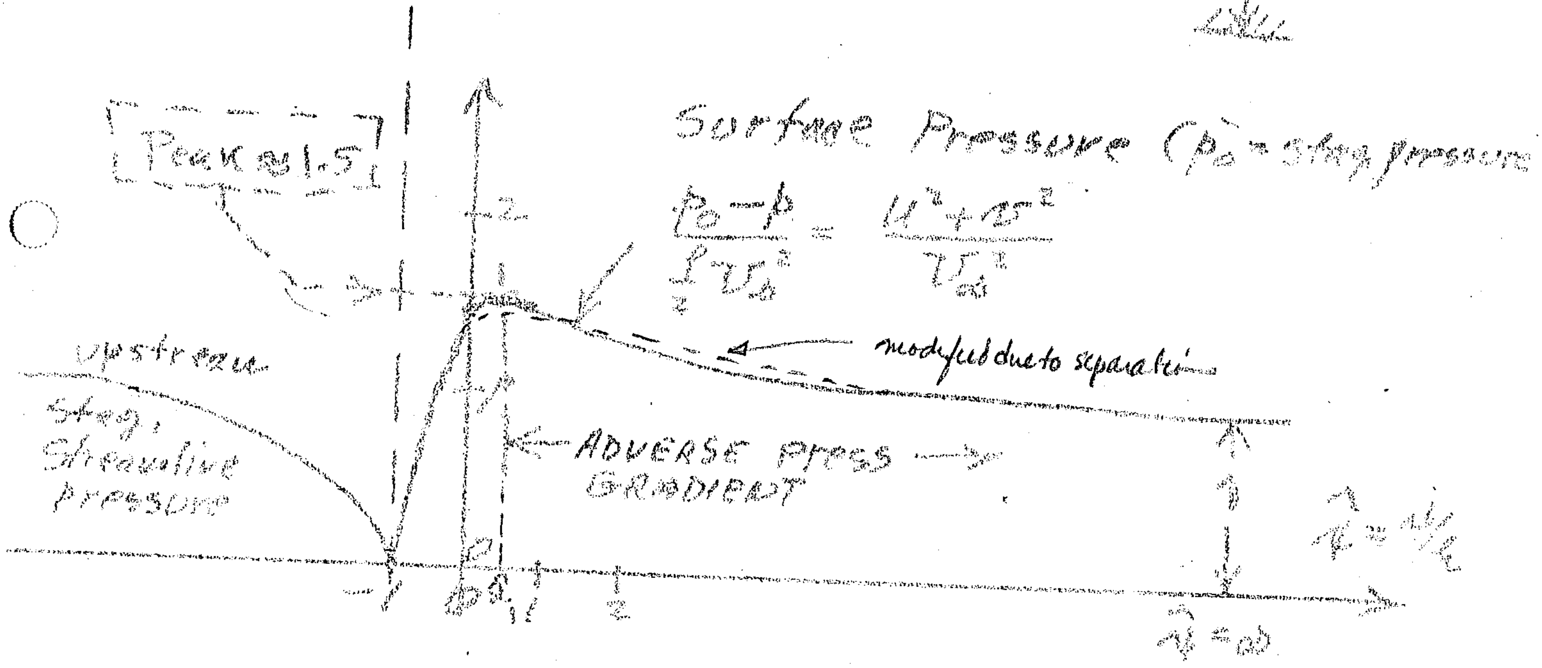
Practical Implications

1. Nose of a pitot-static probe – position of static holes should be at least 1 to 2 width back of nose, at $\hat{x} > 1$ to 2 (2π).
2. Possibility of laminar flow separation at $\hat{x} > \hat{x}_1 \simeq 1/2$.



Surface Pressure (p_0 = stag pressure)

$$\frac{p_0 - p}{\frac{\rho}{2} U_0^2} = \frac{u^2 + v^2}{U_\infty^2}$$

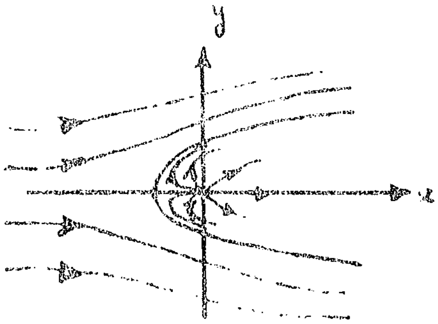


You can create any body by using sources & sinks at
some given distances & given strength

ME 251A LECTURE 18 77/78 (11a)

Classical Examples of Superposition

1. PARALLEL + SOURCE



$$\Phi(z) = U_\infty z + \frac{Q}{2\pi} \ln z$$

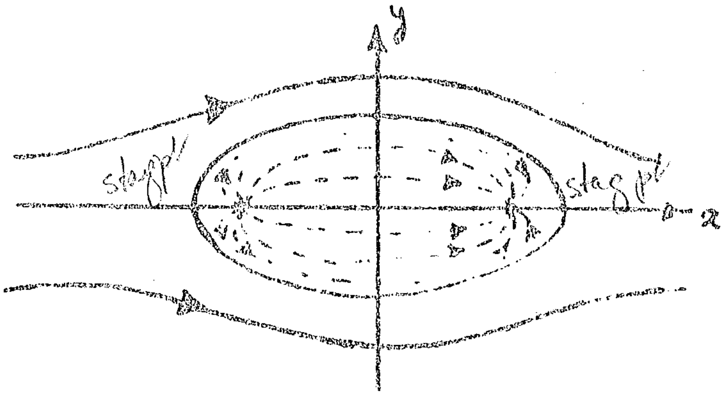
Flow over the half-body examined in last lecture.

2. PARALLEL + SOURCE + SINK (Equal strength)

$$\Phi(z) = U_\infty z + \frac{Q}{2\pi} \ln (z + x_0) - \frac{Q}{2\pi} \ln (z - x_0)$$

Flow about a Rankine oval

See text p. 193



3. PARALLEL + DOUBLET (flow over the cylinder)

$$\Phi(z) = U_\infty z + \frac{S}{z}$$

$$= U_\infty r(\cos\theta + i\sin\theta) + \frac{S}{r}(\cos\theta - i\sin\theta)$$

$$= \underbrace{U_\infty r\cos\theta\left(1+\frac{S}{U_\infty r^2}\right)}_{\phi} + i\underbrace{U_\infty r\sin\theta\left(1-\frac{S}{U_\infty r^2}\right)}_{\psi}$$

$$\Phi'(z) = U_\infty - \frac{S}{z^2} = \underbrace{U_\infty - \frac{S}{r^2}\cos 2\theta}_{u} + i\underbrace{\frac{S}{r^2}\sin 2\theta}_{-v}$$

Stagnation points $\begin{cases} v=0 : & \sin 2\theta = 0 \\ u=0 : & U_\infty - \frac{S}{r^2}\cos 2\theta = 0 \end{cases}$

$$\therefore \begin{cases} \theta_s = 0, \pi \\ r_s = \sqrt{\frac{S}{U_\infty}} \equiv a \end{cases} \quad \text{and } \psi_s = 0$$

Solidify the streamline through the stagnation point:

$$U_\infty r\sin\theta\left(1-\frac{S}{U_\infty r^2}\right) = 0 \quad \text{since } \theta \text{ is arbitrary} \Rightarrow r = \sqrt{\frac{S}{U_\infty}} = a$$

$$\therefore \quad \theta = 0, \pi \quad \text{and} \quad \boxed{r = \sqrt{\frac{S}{U_\infty}} = a}$$ as eqn. for body shape

Thus flow over a circular cylinder.

Velocity Field

$$\begin{cases} v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U_\infty \cos\theta\left(1 - \frac{a^2}{r^2}\right) \\ v_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty \sin\theta\left(1 + \frac{a^2}{r^2}\right) \end{cases}$$

$$V = \left(v_r^2 + v_\theta^2\right)^{1/2} = \left(1 + \frac{a^4}{r^4} - \frac{2a^2}{r^2}\cos 2\theta\right)^{1/2}$$

On the cylinder $(r=a)$,

$$\boxed{\begin{cases} v_{r_c} = 0 \\ v_{\theta_c} = -2U_\infty \sin\theta \end{cases} \qquad V_c = 2U_\infty \sin\theta}$$

Pressure distribution

From Bernoulli Eqn.

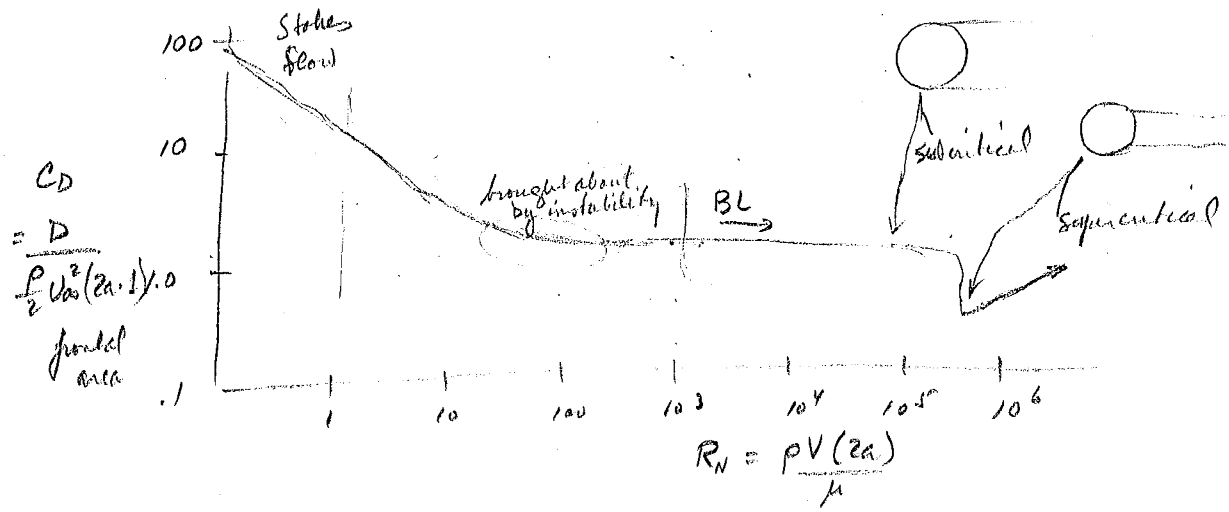
$p' = p + \rho g h \qquad p_\infty' = p_\infty + \rho g h$

$$\frac{p' - p_\infty'}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{V^2}{U_\infty^2} = 2\frac{a^2}{r^2}\cos 2\theta - \frac{a^4}{r^4}$$

On the cylinder $(r=a)$: $\boxed{\frac{p_c - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 2\cos 2\theta - 1}$

laminar flow separation possible for $0 < \theta < \pi/2$ since $dp > 0$

when you make flow turbulent separation pt moves back on the body & flow stays on the body a lot longer.



From Schlichting

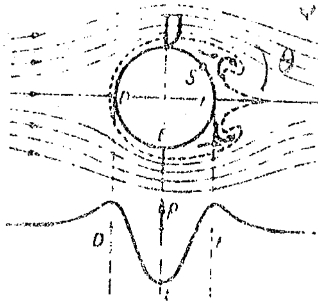
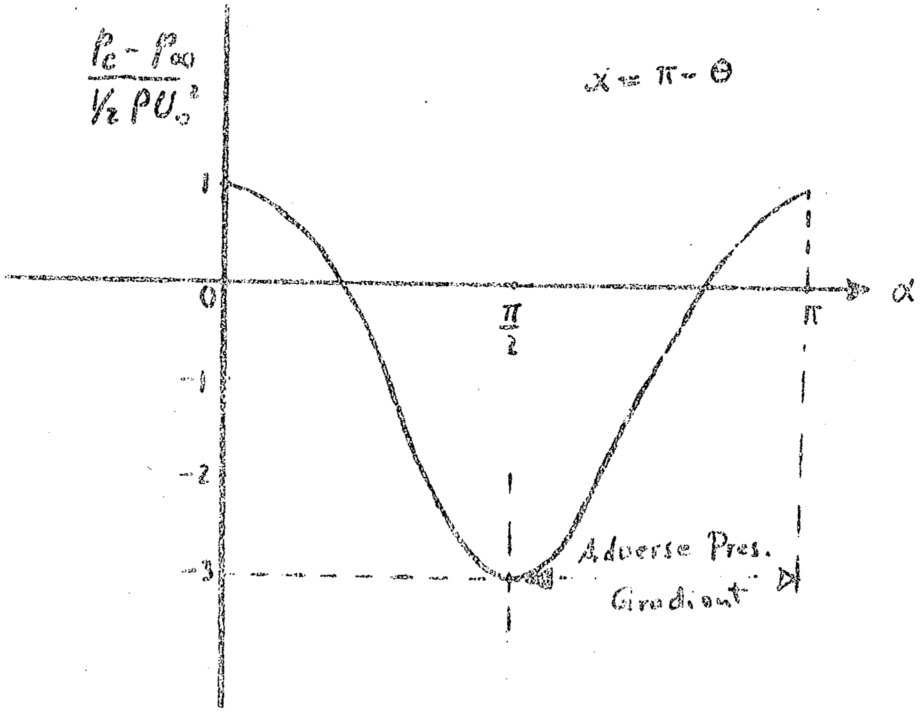


Fig. 2.4. Boundary-layer separation and vortex formation on a circular cylinder (diagrammatic)
S = point of separation



Drag and Lift

$$D = 2\int_0^{\pi} P_c \cos\theta \; a\,d\theta \quad \Rightarrow \quad D = 0 \text{ (D'Alembert Paradox)}$$

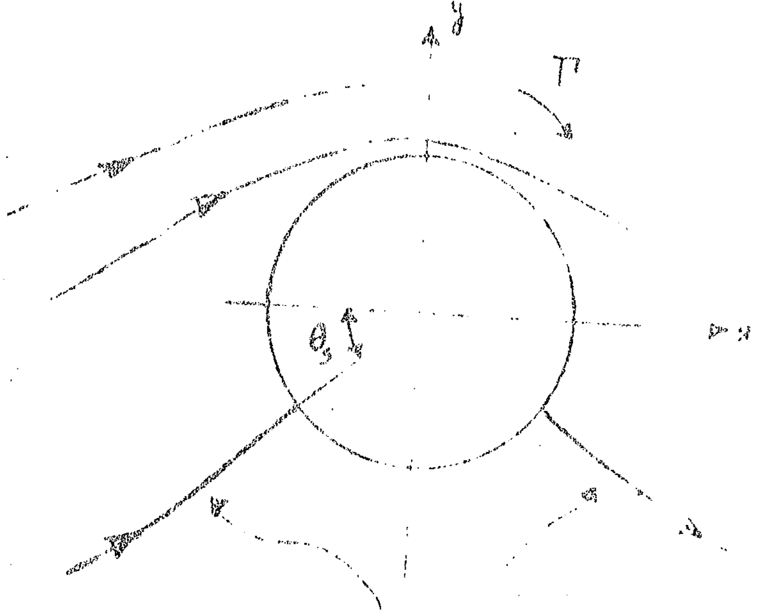
$$L = 2\int_0^{\pi} P_c \sin\theta \; a\,d\theta \quad \Rightarrow \quad L = 0$$

4. PARALLEL + DOUBLET + VORTEX

(flow over cylinder w/ lift)

$$\Phi(z) = U_\infty z + \frac{S}{z} + \frac{i\Gamma}{2\pi} \ln z$$

This combination represents flow over a circular cylinder with circulation. Do this problem as an exercise and check:

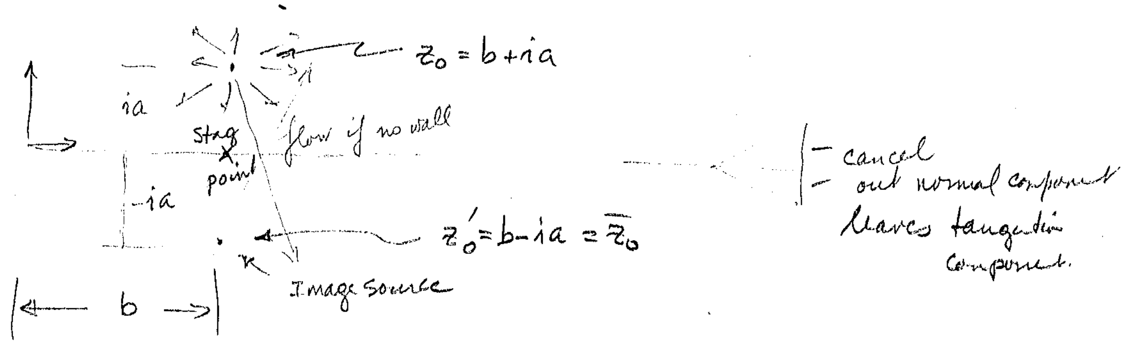


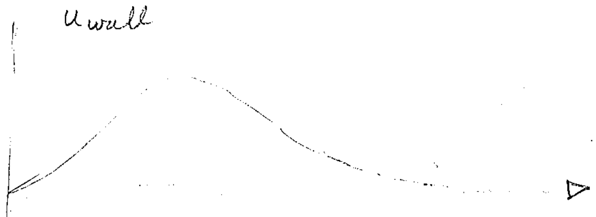
$$\begin{cases} \theta_s = \sin^{-1} \dfrac{\Gamma}{4\pi a U_\infty} \\ r_s = a = \sqrt{\dfrac{S}{U_\infty}} \end{cases}$$

Also calculate drag and lift

$D = 0$

$L = \rho U_\infty \Gamma$: An example of "Kutta-Joukowsky" theorem which will be discussed later.





$$\Phi_{\text{source}} = \frac{Q}{2\pi} \ln (z - z_0)$$

$$\Phi_{\text{image}} = \frac{Q}{2\pi} \ln (z - \bar{z}_0)$$

$$\Phi = \Phi_{\text{source}} + \Phi_{\text{image}} = \frac{Q}{2\pi} \ln (z - z_0) + \frac{Q}{2\pi} \ln (z - \bar{z}_0)$$

$$= \frac{Q}{2\pi} \ln (z - z_0)(z - \bar{z}_0)$$

METHOD OF IMAGES

This is a special type of superposition to satisfy the boundary conditions.

Example 1. Source over flat wall.

$\psi = \frac{Q}{2\pi}\tan^{-1}\left(\frac{y-y_0}{x-x_0}\right)$

on $y=0$

$\psi = \frac{Q}{2\pi}\tan^{-1}\left(\frac{-y_0}{x-x_0}\right)$

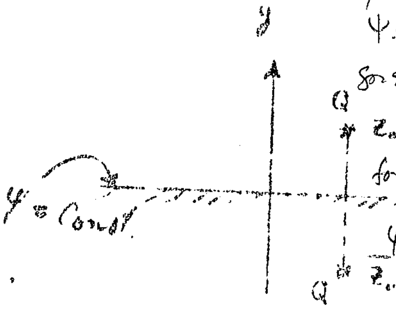
for sink at $z=\bar{z}_0$

$-\frac{Q}{2\pi}\tan^{-1}\left(\frac{-(-y}{x-}\right.$

for source at $z=\bar{z}_0$

$\psi = \frac{Q}{2\pi}\tan^{-1}\left(\frac{-(-y_0)}{x-x_0}\right)$

$\therefore$ image = source at $z=\bar{z}_0$



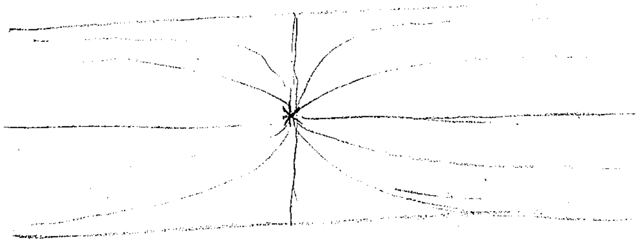
A source is located at z_0 and the b.c is, $\psi = 0$ on $y=0$.

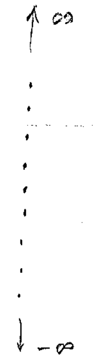
Put a source of the same strength at the mirror image of z_0, i.e. $\bar{z}_0$, then, due to symmetry, the x-axis would be a streamline. The complex potential for this problem then can be written as,

$$\Phi(z) = \frac{Q}{2\pi}\ln(z-z_0) + \frac{Q}{2\pi}\ln(z-\bar{z}_0)$$

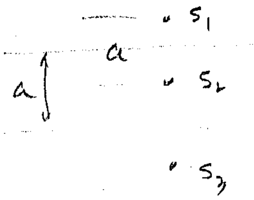
$$\boxed{\Phi(z) = \frac{Q}{2\pi}\ln(z-z_0)(z-\bar{z}_0)}$$

Suppose we have a source with a pipe





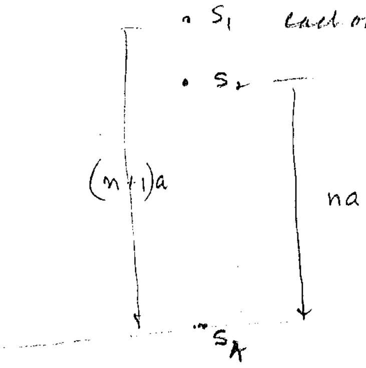
Only 2 sources on either side will cause walls to bow since S_1 effect on wall betw S_2, S_3 will bow it.



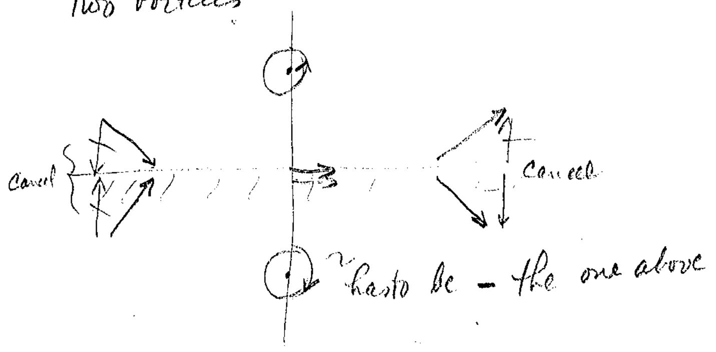
∴ to decrease the bow we can add more sources which are further away from S_2 etc

The effect of S_1, S_2 on S_n is such that they about equal each other as $n \to \infty$

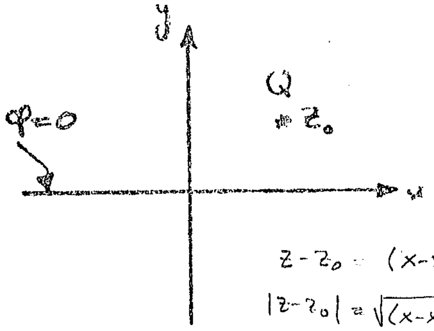
$$\Phi = \sum_{n=-\infty}^{\infty} \frac{Q}{2\pi} \ln(z - ian)$$



Two vortices



Example 2. A source of strength Q is located at $z = z_0$. Find the complex potential for this flow in the upper half-plane subject to the condition: $\varphi = 0$ on $y = 0$



$z - z_0 = (x-x_0) + iy_0$ (y

$|z-z_0| = \sqrt{(x-x_0)^2+y_0^2}$

$\theta = \tan^{-1}\left(\frac{y-y_0}{x-y_0}\right)$

along $y=0$

$|z-z_0| = x-x_0$

$\varphi_{so} = \frac{Q}{2\pi} \ln \sqrt{(x-y_0)^2+y_0^2}$

$\varphi_{si} = -\frac{Q}{2\pi} \sqrt{(x-x_0)^2+y_0^2}$

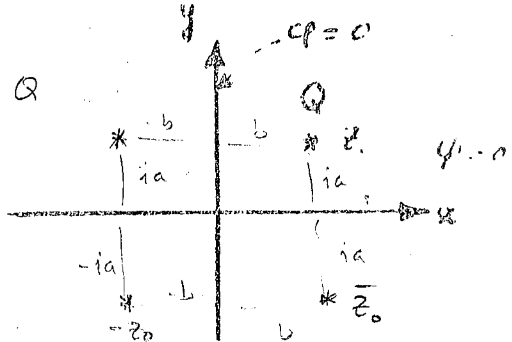
This problem can be solved by putting a sink of strength $-Q$ at $z = \bar{z}_0$.

$$\boxed{\Phi(z) = \frac{Q}{2\pi} \ln \frac{z-z_0}{z-\bar{z}_0}}$$

$-b+ia = -(b-ia)$
$= -\bar{z}_0$

Exercise 1.

Write the complex potential for the flow shown in the region $x>0$ and $y>0$



$$\Phi(z) = \frac{Q}{2\pi} \ln\left(\frac{z-z_0}{z-\bar{z}_0}\right) + \frac{Q}{2\pi} \ln\left(\frac{z+\bar{z}_0}{z+z_0}\right)$$

Exercise 2.

Do problem 6.15 in text

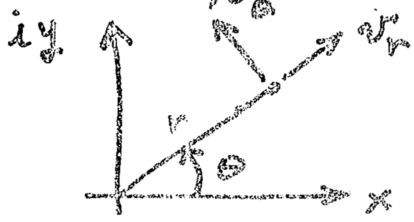
CIRCULATION AND VORTICITY

(1.) THE POINT (FREE) VORTEX:

$$\Phi = -i\frac{\Gamma}{2\pi}\ln(z) \qquad (\Gamma \text{ is real})$$

$$\Phi = -i\frac{\Gamma}{2\pi}(\ln r + i\theta) = \underbrace{\frac{\Gamma\theta}{2\pi}}_{\varphi}\underbrace{-i\frac{\Gamma}{2\pi}\ln r}_{\psi = -\frac{\Gamma}{2\pi}\ln r}$$

POLAR COORDS:

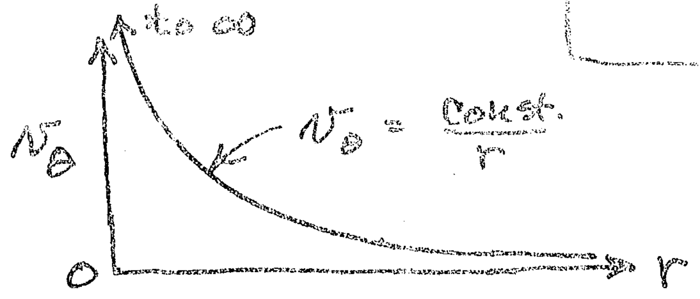


$$v_r = \frac{\partial\varphi}{\partial r} = 0$$

$$v_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \frac{1}{r}\frac{\Gamma}{2\pi}$$

$$\boxed{v_\theta r = V r = \frac{\Gamma}{2\pi} \qquad (123\text{-}1)}$$

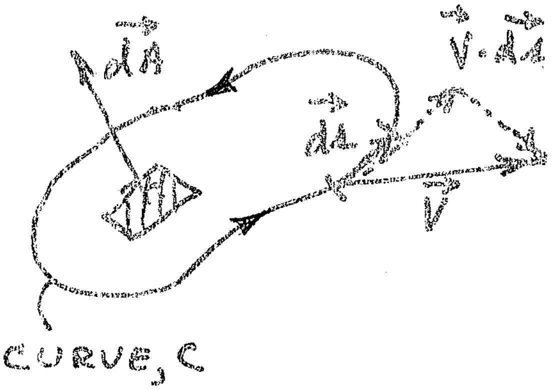
$$V^2 = v_\theta^2 + \cancel{v_r^2}$$



Γ IS A QUANTITY WE WILL SEE IS CALLED THE CIRCULATION

CIRCULATION DEFINED

CONSIDER "ANY" CLOSED CURVE IN THE FLUID



$$\boxed{\Gamma = \oint_C \vec{V}\cdot d\vec{s}}$$

+ IF COUNTER-CLOCKWISE W/O $d\vec{A}$

(124-1)

STOKES THEOREM ($\vec{F}$ ARBITRARY)

F arbitrary vector field smooth and belonging to C,

$$\oint_C \vec{F}\cdot d\vec{s} = \iint_{A} (\vec{\nabla}\times\vec{F})\cdot d\vec{A}$$

A ENCLOSED BY C

IF $\vec{F} = \vec{V}$ THEN $\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{V} = \vec{\omega}$

AND

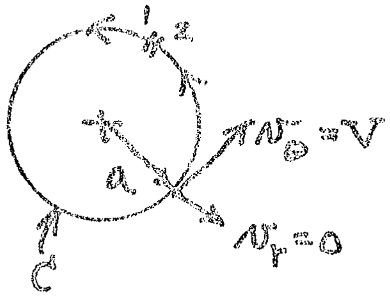
$$\boxed{\Gamma = \iint \vec{\omega}\cdot d\vec{A}}$$

(124-2)

if $\nabla\times\underline{V} = 0 \Rightarrow \Gamma = 0$

IRROTATIONAL FLOW INSIDE C MEANS CIRCULATION = 0

APPLY TO A POINT VORTEX WITH CIRULAR CURVE ABOUT ORIGIN



$$\Gamma = \oint_C \vec{V} \cdot d\vec{s} = \int_1^2 v_\theta \, a \, d\theta$$

$$v_\theta = \frac{1}{a} \frac{\Gamma}{2\pi} \quad \underline{\underline{\text{ON}}} \quad C$$

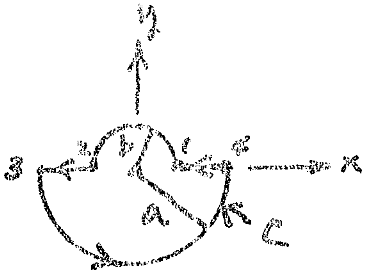
$$\therefore \quad \Gamma = \frac{\Gamma}{2\pi} \int_1^2 \frac{1}{\cancel{a}} \cancel{a} \, d\theta = \frac{\Gamma}{\cancel{2\pi}} (\cancel{2\pi})$$

$$\text{SO} \quad \Gamma = \Gamma \quad \text{Q.E.D.}$$

HOWEVER THIS RESULT IS MORE GENERAL

Γ = CONST. FOR "ANY" C

EXAMPLE:

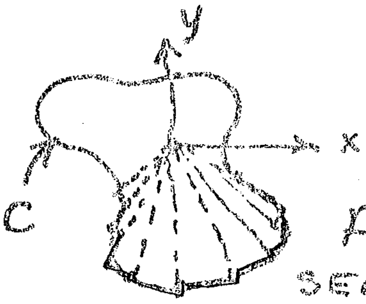


← for pt vortex →

$$\Gamma = \int_1^2 v_\theta \, b \, d\theta + \int_2^3 \cancel{v_r}^{\,0} \, dr + \int_3^4 v_\theta \, a \, d\theta - \int_4^1 \cancel{v_r}^{\,0} \, dr$$

$$\Gamma = \frac{\Gamma}{2\pi} \int_0^\pi \frac{1}{b} \, b \, d\theta + 0 + \frac{\Gamma}{2\pi} \int_\pi^{2\pi} \frac{1}{a} \, a \, d\theta - 0$$

$$\Gamma = \frac{\Gamma}{2\pi} (\pi + \pi) = \Gamma \quad \text{Q.E.D.}$$



ARBITRARY C, $\Gamma = \Gamma$ for the point vortex
SEGMENT C ALONG RADIUS AND ARC

$L = -\rho U_\infty \Gamma$ Kutta-Joukowsky theorm for Aerodynamics Lift

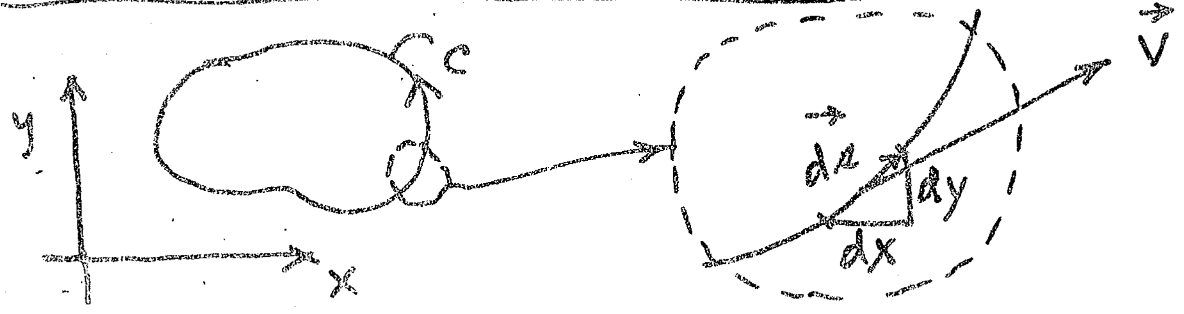
if we assume $\rho = \text{const}$, $\mu = 0$, $\omega_z = 0$ →
then we get

$\Gamma = \iint \omega_z \, dx \, dy = 0$ since $\omega_z = 0$ but $\Gamma = \oint d\Gamma$

note that φ, Γ are not equal in general since

φ can be multivalued if we integrate $\int d\varphi$ around a singularity

CIRCULATION IN 2-DIMENSIONS



$$\vec{ds} = \vec{i}\,dx + \vec{j}\,dy$$
$$\vec{V} = \vec{i}\,u + \vec{j}\,v$$

$$\Gamma = \oint_c \vec{V}\cdot\vec{ds} = \oint_c (u\,dx + v\,dy) = \oint_c d\Gamma$$

General in derivation

$d\Gamma = u\,dx + v\,dy$ (IS NOT AN EXACT DIFFERENTAL SINCE $\Gamma \neq 0$ IN GENERAL)

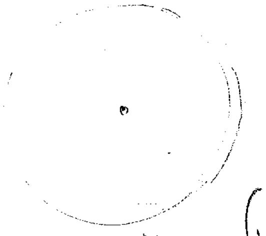
LET: $\begin{cases} u = \dfrac{\partial \varphi}{\partial x} \\ v = \dfrac{\partial \varphi}{\partial y} \end{cases}$

$$\therefore \quad d\Gamma \overset{?}{=} \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = d\varphi$$

HOWEVER $\Gamma \neq \varphi \neq \iint \vec{\omega}_z \cdot \vec{dA}$

SINCE φ NOT DEFINED FOR ROTATIONAL FLOW. φ DEFINED INSIDE A REGION WHERE $\vec{\omega} = 0$, $\Gamma = 0$ FOR C ABOUT REGION

φ NOT SINGLE VALUED FOR REGION WHERE $\Gamma \neq 0$ (BOOK PP 188-191)



since we can decrease a or increase b to 1

$\Gamma_C = 0 \implies$ for

$$\Gamma_c = \int \underline{V} \cdot \underline{ds} = \int_A^B \frac{\Gamma}{2\pi r} r d\theta = \frac{\Gamma}{2\pi} \theta_a^B$$

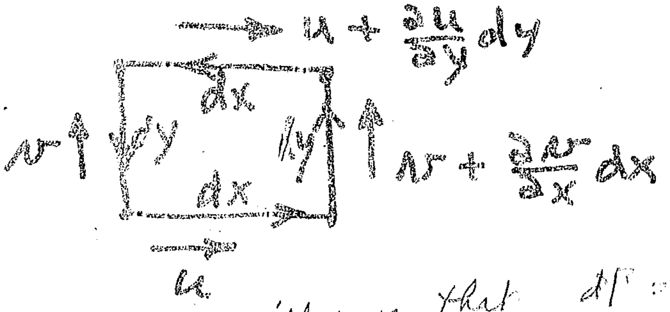
for full circle around origin $\theta = 2\pi \quad \therefore \Gamma \neq 0$

as we keep on making more & more ~~[illegible]~~ circuits about O

$\Gamma_c = \Gamma k$ — # of circuits

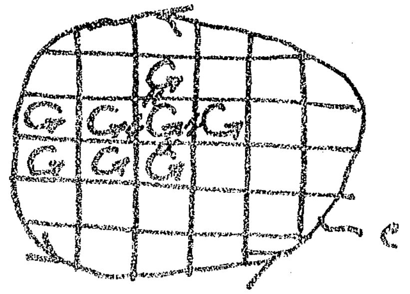
STOKES THEOREM IN 2-D FLOW

ELEMENTARY CIRCUIT :



we will prove that $d\Gamma = \omega_z dxdy$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$



$$\oint_c d\Gamma = \Gamma$$

interior circuits cancel

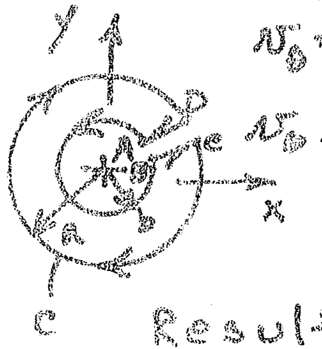
$$d\Gamma = u\,dx + \left(v + \frac{\partial v}{\partial x}dx\right)dy - \left(u + \frac{\partial u}{\partial y}dy\right)dx - v\,dy$$

$d\vec{s} = -\vec{i}dx$ $d\vec{s} = -\vec{j}dy$

$$d\Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy = \omega_z dA$$

$$\therefore \boxed{\Gamma_{2D} = \iint \omega_z dA} \quad (127\text{-}1)$$

Apply to Point Vortex Flow FOR CIRCUIT that does not include ORIGIN



$v_\theta = \frac{\Gamma}{a 2\pi}$ OUTER

$v_\theta = \frac{\Gamma}{b 2\pi}$ INNER

$$\Gamma_c = \Gamma_{A\to B} + \underbrace{\overset{v_r=0}{\cancel{\Gamma_{B\to C}}}} + \Gamma_{C\to D} + \underbrace{\overset{v_r=0}{\cancel{\Gamma_{D\to A}}}}$$

cancel

$$\Gamma_c = \underset{\text{inner}}{v_\theta}(2\pi b) - \underset{\text{outer}}{v_\theta}(2\pi a)$$

Result indep. of a & b $\boxed{\Gamma_c = \Gamma - \Gamma = 0}$

$\Gamma_c = 0$ FOR WHOLE FLOW FIELD EXCEPT $r = 0$.
ORIGIN IS SINGULAR POINT WHERE ω_z IS CONCENTRATED (p.125, $\Gamma \neq 0$ FOR C AROUND ORIGIN)

bound vortex   associated w/ body in fluids
free vortex – associated w/ fluid

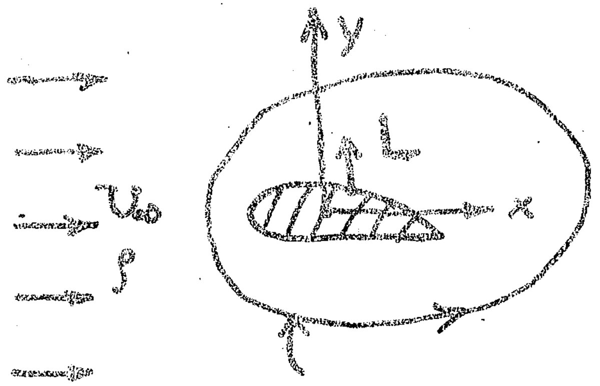
*

change in direction of this flow
is due to forces exerted on the
blade.

AERODYNAMIC LIFT

KUTTA-JOUKOWSKY THEOREM

CONSIDER FLOW OF A UNIFORM STREAM OF VELOCITY U_∞ AND DENSITY ρ.



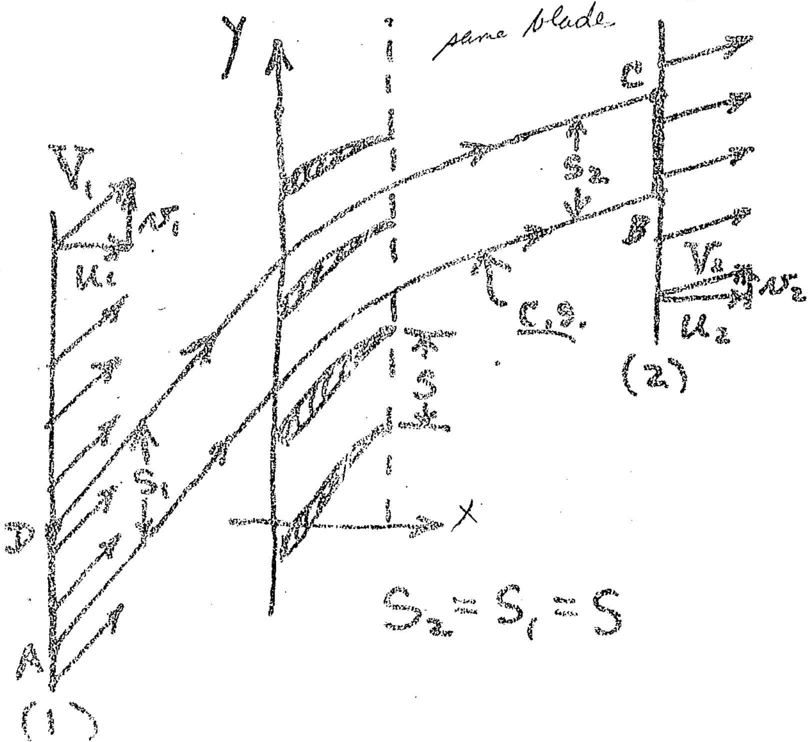
L = NET LIFT FORCE ON BODY (IN +Y DIRECTION)

$$\boxed{L = -\rho U_\infty \Gamma_c}$$

EVALUATE Γ_c ON ANY CIRCUIT ABOUT BODY

PROOF: 1. SEE PAGES 206-208 IN TEXT.

2. CASCADE METHOD, BELOW.



$S_2 = S_1 = S$

ASSUME:

1. 2-D FLOW
2. STEADY "
3. ρ = CONST.
4. INVISCID
5. (1) AND (2) AT $X = \pm\infty$
6. UNIFORM PARALLEL FLOW AT (1) AND (2)
7. C.S. ON REPEATING S.L.'S A → B AND D → C

CONS. OF MASS FOR C.S. ABCD

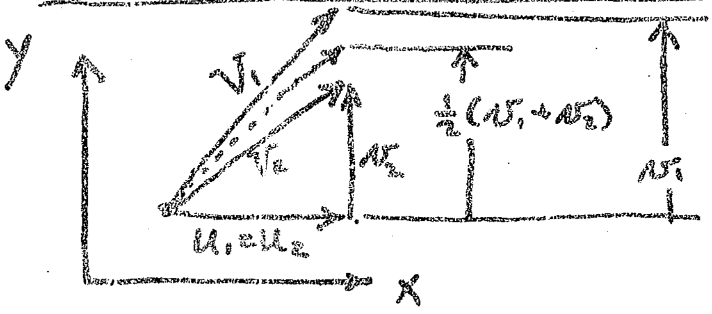
$$\dot{m}_2 = \dot{m}_1$$

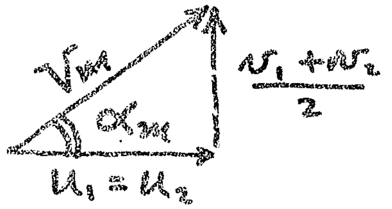
$$\cancel{\rho_2}(\vec{V}_2 \cdot \vec{dA}_2) = \cancel{\rho_1}(\vec{V}_1 \cdot \vec{dA}_1)$$

$$u_2 \cancel{S\rho_2} = u_1 \cancel{S\rho_1}$$

$$\therefore \quad \boxed{u_2 = u_1} \quad (129\text{-}1)$$

VELOCITY TRIANGLES AT (1) AND (2)





DEFINE MEAN
VELOCITY AND
ANGLE

BERNOULLI FROM (1) → (2)

$$p_2 - p_1 = \frac{\rho}{2}\left(V_1^2 - V_2^2\right)$$

disappeared $u_1 = u_2$

$$p_2 - p_1 = \frac{\rho}{2}\left(\cancel{u_1^2} + v_1^2 - \cancel{u_2^2} - v_2^2\right) = \frac{\rho}{2}\left(v_1^2 - v_2^2\right)$$

$$\boxed{p_2 - p_1 = \frac{\rho}{2}(v_1 + v_2)(v_1 - v_2)} \quad (129\text{-}2)$$

start

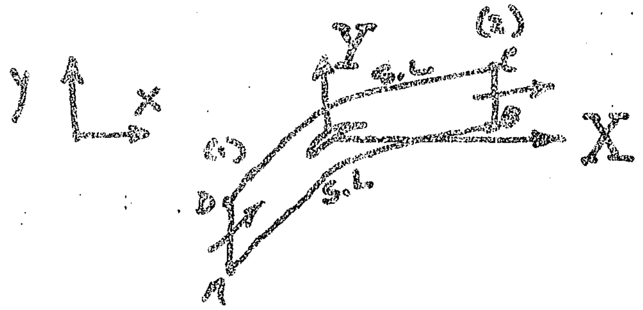
MOMENTUM IN X AND Y DIRECTIONS

NOTE: SIDES A-B AND D-C ARE IDENTICAL (REPEATING) S.L.S

(1) NO MOMENTUM CROSSES S.L.'S

(2) FORCES ON A-B CANCEL FORCES ON D-C

THEREFORE, WE NEED ONLY FORCES AND MOMENTUM FLOWS ON ENDS B-C AND A-D TO OBTAIN NET FORCES X AND Y BY BLADE ON FLOW



$$\sum F_x = (p_1 - p_2)s + X$$

$$\sum F_y = Y$$

LINEAR MOMENTUM (PER UNIT DEPTH)

(x-dir) ⟶ $(p_1 - p_2)S + X = \dot{m}(u_2 - u_1)$

(y-dir) ⟶ $\boxed{Y = \dot{m}(v_2 - v_1)}$

SINCE $u_2 = u_1$ ⟶ $\boxed{X = S(p_2 - p_1)}$

(130-1)

FORCES ON BLADE

$$X_B = -X \quad \text{AND} \quad Y_B = -Y$$

RESULTANT FORCE MAGNITUDE:

$$R_B = \sqrt{X_B^2 + Y_B^2} = \sqrt{\underbrace{\left(s\,(p_2 - p_1)\right)^2}_{\text{USE } (129-2)} + \left(\rho s u_1 (v_2 - v_1)\right)^2}$$

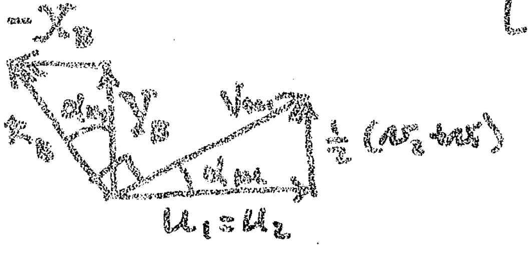
$$R_B = \rho s\,(v_1 - v_2)\underbrace{\sqrt{\left(\tfrac{1}{2}(v_1 + v_2)\right)^2 + u_1^2}}_{V_m \text{ from Vel. } \Delta}$$

$$\boxed{R_B = \rho s\,(v_1 - v_2)\,V_m} \quad (131-1)$$

FORCE RATIO

$$\frac{X_B}{Y_B} = \frac{X}{Y} = \frac{-\frac{1}{2}(v_2 + v_1)}{u_1}$$

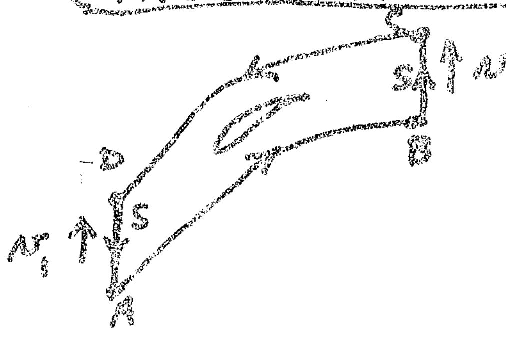
$$\text{OR} \quad \boxed{\frac{-X_B}{Y_B} = \tan \alpha_m}$$



WE SEE THAT

$$\boxed{\vec{R}_B \perp \vec{V}_m} \quad (131-2)$$

CIRCULATION ON ABCD



$$\Gamma = \cancel{\Gamma_{AB}} + \Gamma_{BC} + \cancel{\Gamma_{CD}} + \Gamma_{DA}$$

CANCEL
(SAME LINES
IN ± DIRECTION)

$$\Gamma = S v_2 - S v_1$$

$$\boxed{\Gamma = -S(v_1 - v_2)} \quad (132\text{-}1)$$

COMBINE WITH (131-1)

$$R_B = \rho S (v_1 - v_2) V_m$$

TO OBTAIN

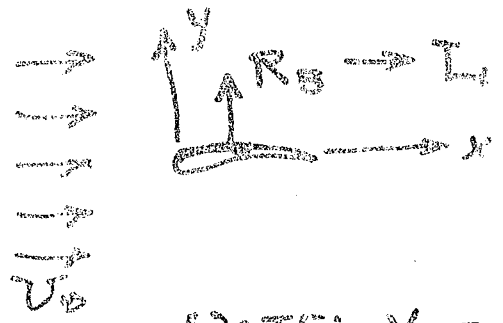
$$\boxed{R_B = -\rho \Gamma V_m} \quad (131\text{-}1)$$

TAKE LIMIT AS $S \to \infty$ WHILE $\Gamma = \text{CONST.}$

FROM (132-1): $(v_1 - v_2) \to 0$

FROM (129-2): $(p_2 - p_1) \to 0$

FLOW BECOMES UNIFORM WHERE $V_m \to U_0$
IF COORDINATES ROTATED



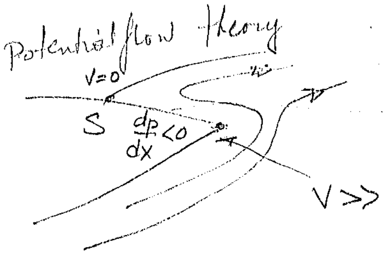
AND AS A RESULT
OF (131-1)

$$R_B \to L = \rho \Gamma U_0$$

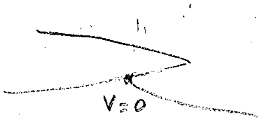
NOTE: $Y_B = R_B$

$X_B = 0$ SINCE $N_1 = N_2 = 0$
IN ROTATED COORDINATES

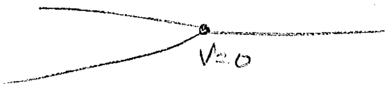
Kutta Condition



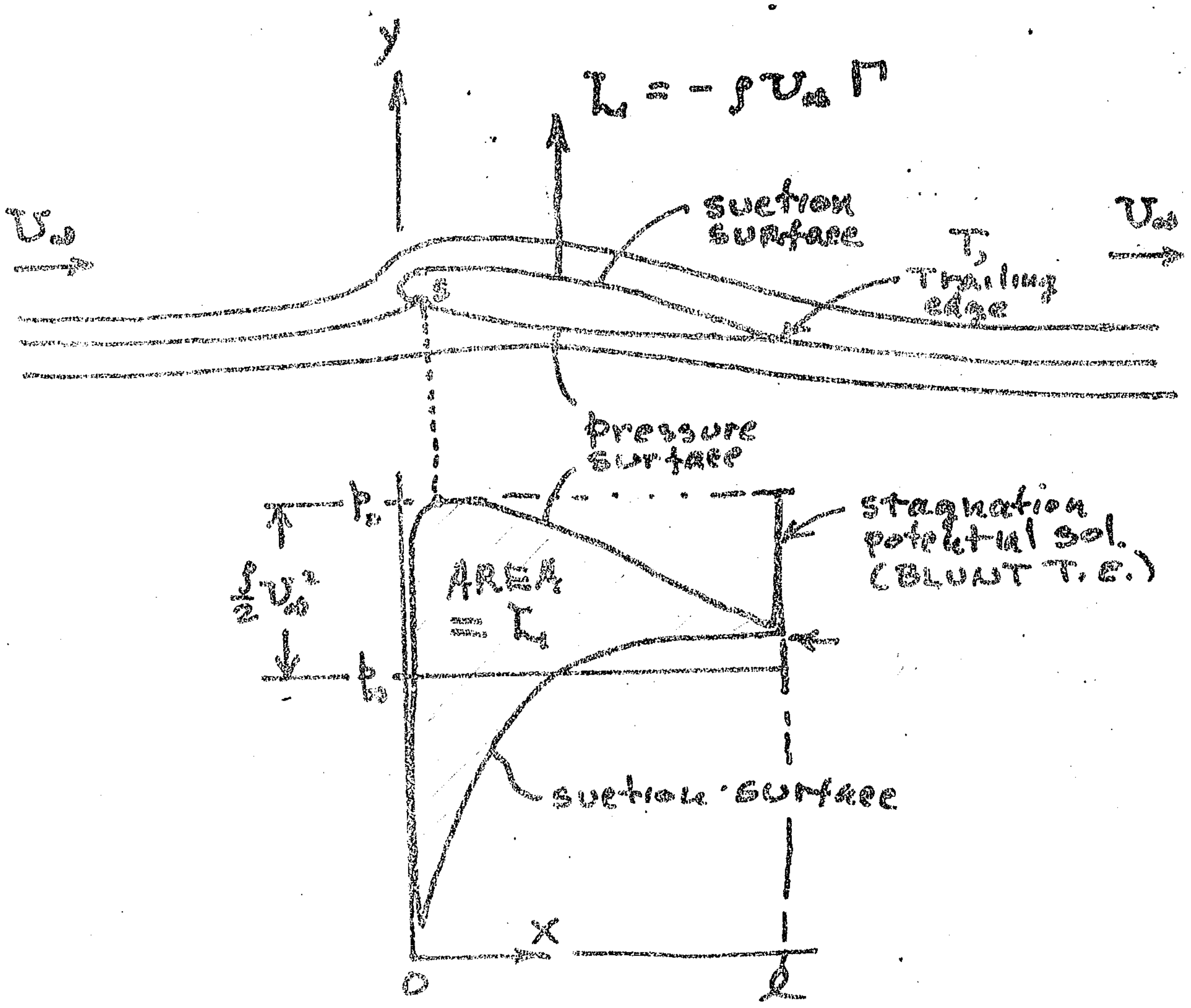
this will cause flow separation due to adverse pressure grad.



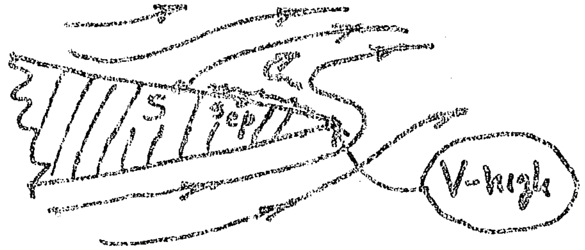
this will also cause separation. Only unseparated solution is



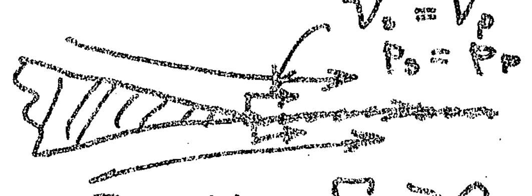
LIFT ON 2-D AIRFOILS



KUTTA CONDITION CONTROLS Γ



START UP, $\Gamma = 0$
sep. pt. moves to
trailing edge

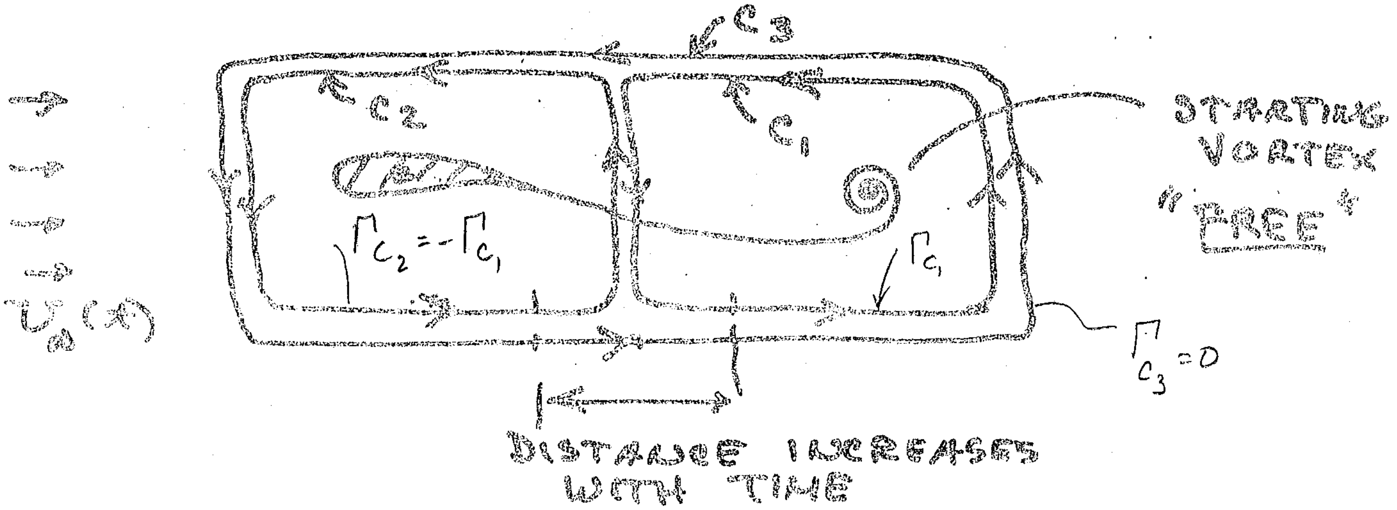


FINAL, $\Gamma > 0$

BOUND VORTEX
"INSIDE" OR ON
SURFACE OF
AIRFOIL

there's some momentum ff in fluid due to lift force applied to fluid for finite time.

STARTING VORTEX: AIRFOIL PUT INTO MOTION FROM REST ($U_\infty = 0$ at $t = 0$)



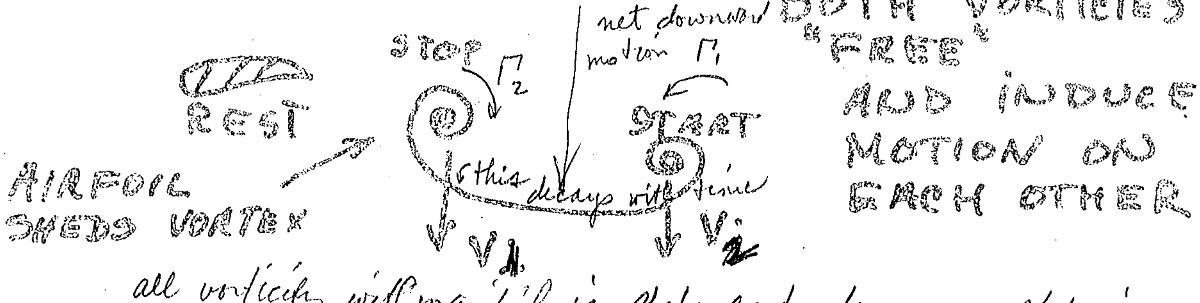
$$\Gamma_{C_2} = -\Gamma_{C_1}$$

$\Gamma_{C_3} = 0$ FOR ALL TIME

- "BOUND VORTICITY" MOVES WITH AIRFOIL
- STARTING VORTEX HAS ω_z AT CENTER AND IS "FREE" TO MOVE WITH FLUID

$$\boxed{T_L = -\rho U_\infty \Gamma_{C_2}}$$

STOPPING VORTEX: FLOW BROUGHT TO REST



all vorticity with airfoil is shed and becomes a stopping vortex.

vortexline is a line everywhere tangent to vorticity field.

effects of finite span
1. reduce lift w/ respect to 2-D value
2. induce drag.

ME 251A 77/78 (135)

TRAILING VORTICIES!

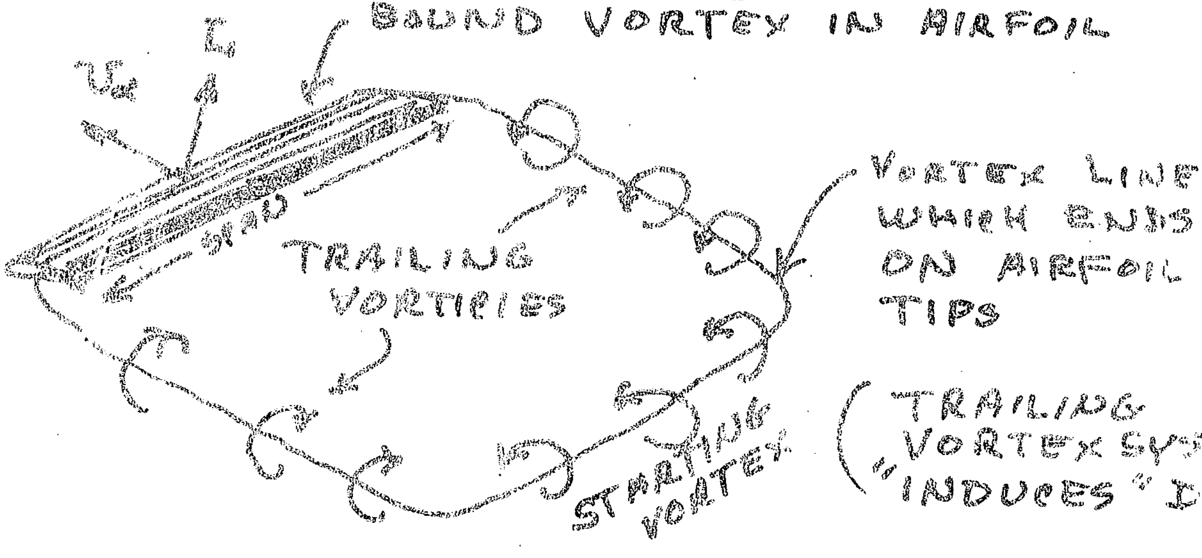
IN 3-D FLOW VORTICIES
ARE MARKED BY VORTEX LINES

BASIC THEOREM

"VORTEX LINES
CANNOT END/BEGIN
INSIDE FLUID"

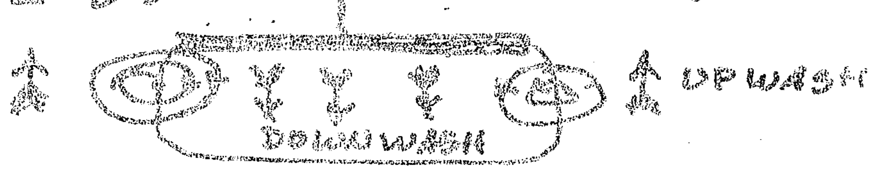
→ ME 251 B ←

FINITE AIRFOIL (3-D FLOW)



(TRAILING
"VORTEX SYSTEM
"INDUCES" DRAG)

(LIFT IS LESS THAN 2-D) ⟶ $\uparrow L < \left[-(\rho U_\infty \Gamma_{2D}) \times (\text{SPAN})\right]$ due to effect of trailing vortex syst.

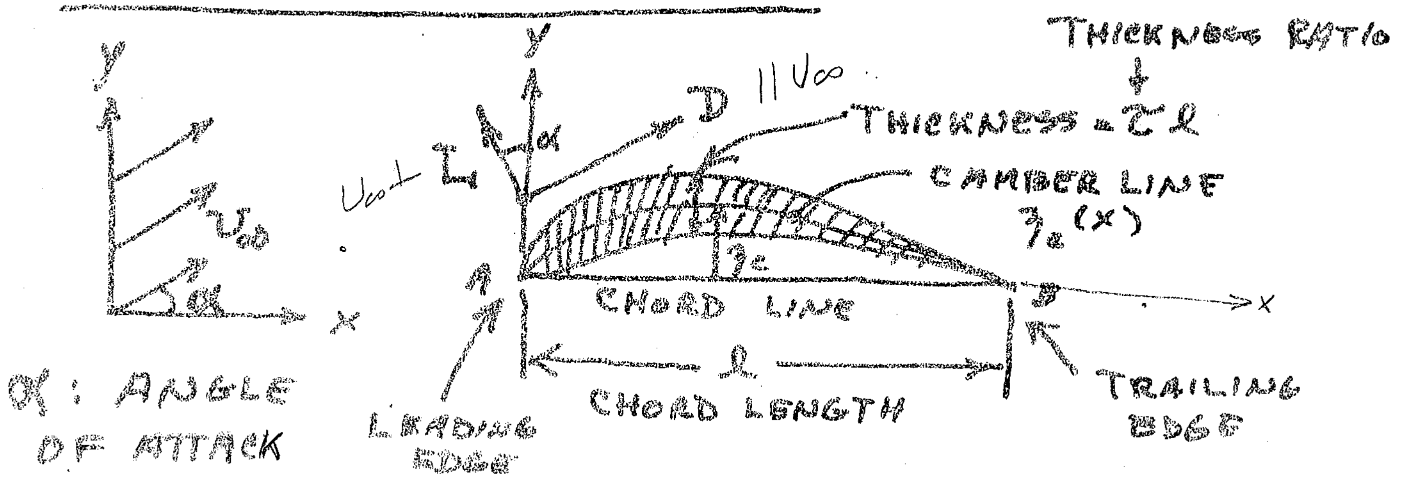


VIEW FROM BEHIND

Final Exam: will have 4 problems

Exam will be avail Dec 8
" " due Dec 12 by 11^{30}AM

AIRFOIL NOMENCLATURE



2-D AIRFOIL A_p: PLANFORM AREA = chord length × span

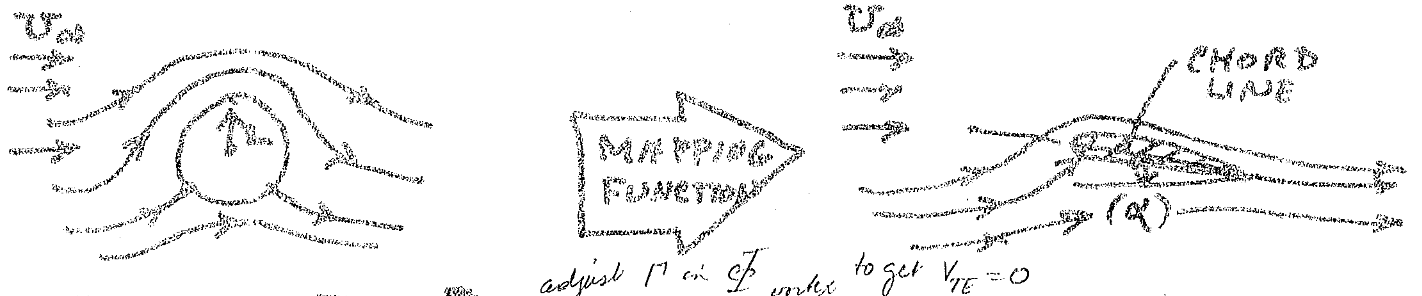
$A_p = l$ (PER UNIT DEPTH)

LIFT COEF: $C_L = \frac{L}{\frac{\rho}{2} U_\infty^2 A_p} = \frac{2L}{\rho U_\infty^2 l} = \frac{-2\Gamma}{U_\infty l}$

DRAG COEF. (STREAMLINED BODIES) $C_D = \frac{D}{\frac{\rho}{2} U_\infty^2 A_p} = \frac{2D}{\rho U_\infty^2 l}$

Start.

GENERAL SOLUTION BY CONFORMAL MAPPING



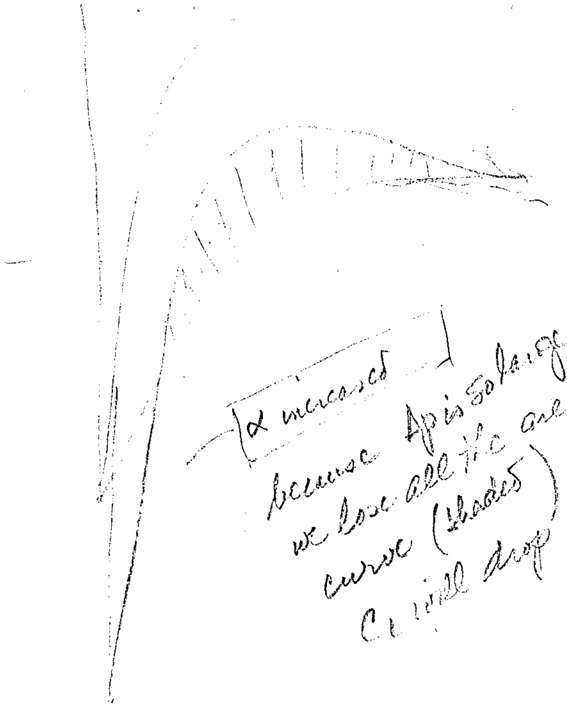
adjust Γ in Φ vortex to get $V_{TE} = 0$

$\Phi = U_\infty x + \Phi + \Phi$
DOUBLET + VORTEX

SO: RESULTS GIVE

$$\Gamma = -4\pi a U_\infty \sin(\alpha + \beta)$$

PARAMETERS a AND β DEPEND ON AIRFOIL SHAPE AND SIZE (l).



α increased

because Δp is so large we lose all the area under curve (shaded) and hence C_L will drop causing stall

OR IN GENERAL

$$\boxed{C_L = 8\pi\left(\frac{a}{\ell}\right)\sin(\alpha+\beta)} \quad (137\text{-}1)$$

THIN, SLIGHTLY C~~H~~AMBERED FOILS

$\frac{t}{\ell}_{MAX} < 1/10$ thin

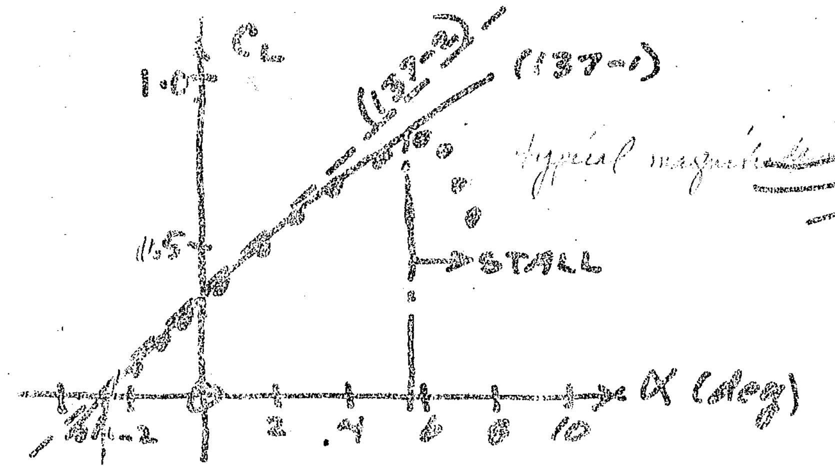
$\left|\left(\frac{y_c}{\ell}\right)_{MAX}\right| < 1/10$ cambered slightly

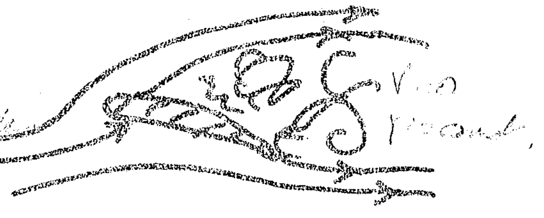
AT SMALL ANGLES OF ATTACK

$\alpha \simeq \sin\alpha$

THE EQ (137-1) IS:

$$\boxed{C_L = 8\pi\left(\frac{a}{\ell}\right)(\alpha+\beta)} \quad (137\text{-}2)$$





STALLED CASE

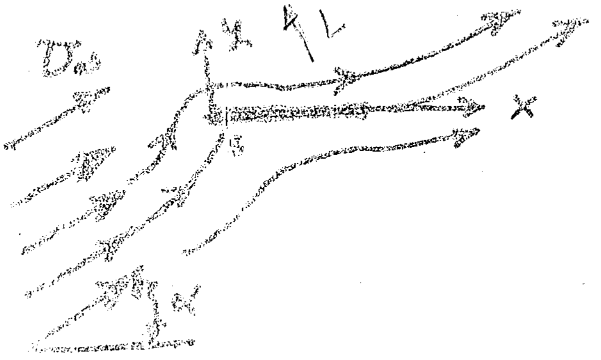
$\alpha_0 = -\beta$: NO LIFT ∡ OF ATTACK

DEPENDS ON CHAMBER SHAPE a/ℓ

$\therefore C_L = 8\pi\frac{a}{\ell}(\alpha - \alpha_0)$

EXAMPLES (EXACT SOLUTIONS):

- THIN ($\tau = 0$) FLAT PLATE (THEORY FOLLOWS)



NOTE:

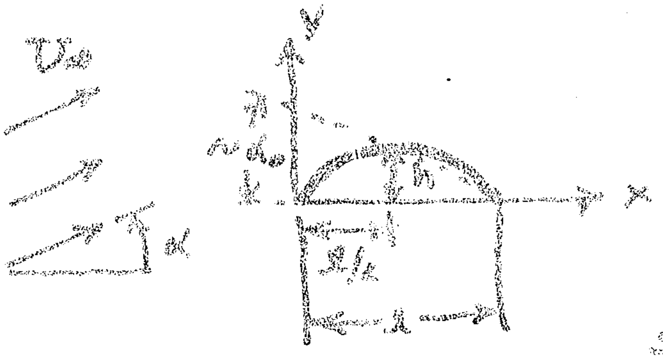
SUCTION PEAK SINGULARITY AT LEADING EDGE

$a = l/4$ no camber

$\beta = -\alpha_0 = 0$

So: $\boxed{C_L = 2\pi \sin\alpha}$ (138-1)

- THIN CIRCULAR ARC ($\tau = 0$)



$a = \frac{l}{4}\left[1+\left(\frac{2h}{l}\right)^2\right]^{-1/2}$

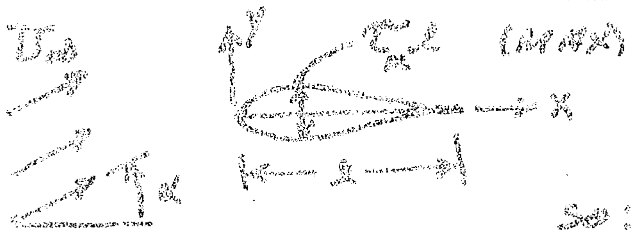
$-\alpha_0 = \beta = \arctan\left(\frac{2h}{l}\right)$

For $2h/l \ll 1$

$-\alpha_0 = \beta \approx \frac{2h}{l}$

So: $\boxed{C_L = 2\pi\left[1+\left(\frac{2h}{l}\right)^2\right]^{-1/2}\sin(\alpha+\beta)}$ (138-2)

- SYMMETRIC JOUKOWSKY FOIL (τ_m small)



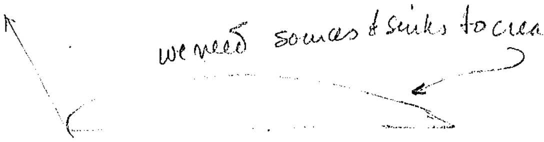
$a \approx \frac{l}{4}(1 + 0.77\tau_m)$

$\beta = -\alpha_0 = 0$

So: $\boxed{C_L \approx 2\pi(1+0.77\tau_m)\sin\alpha}$ (138-3)

COMPLETE SUMMARY SEE BOOKS:
"FLUID-DYNAMIC LIFT" BY S. F. HOERNER
"FLUID-DYNAMIC DRAG"
ORDER DIRECT FROM: HOERNER FLUID DYNAMICS, BOX 342, BRICK TOWN, N.J. 08723

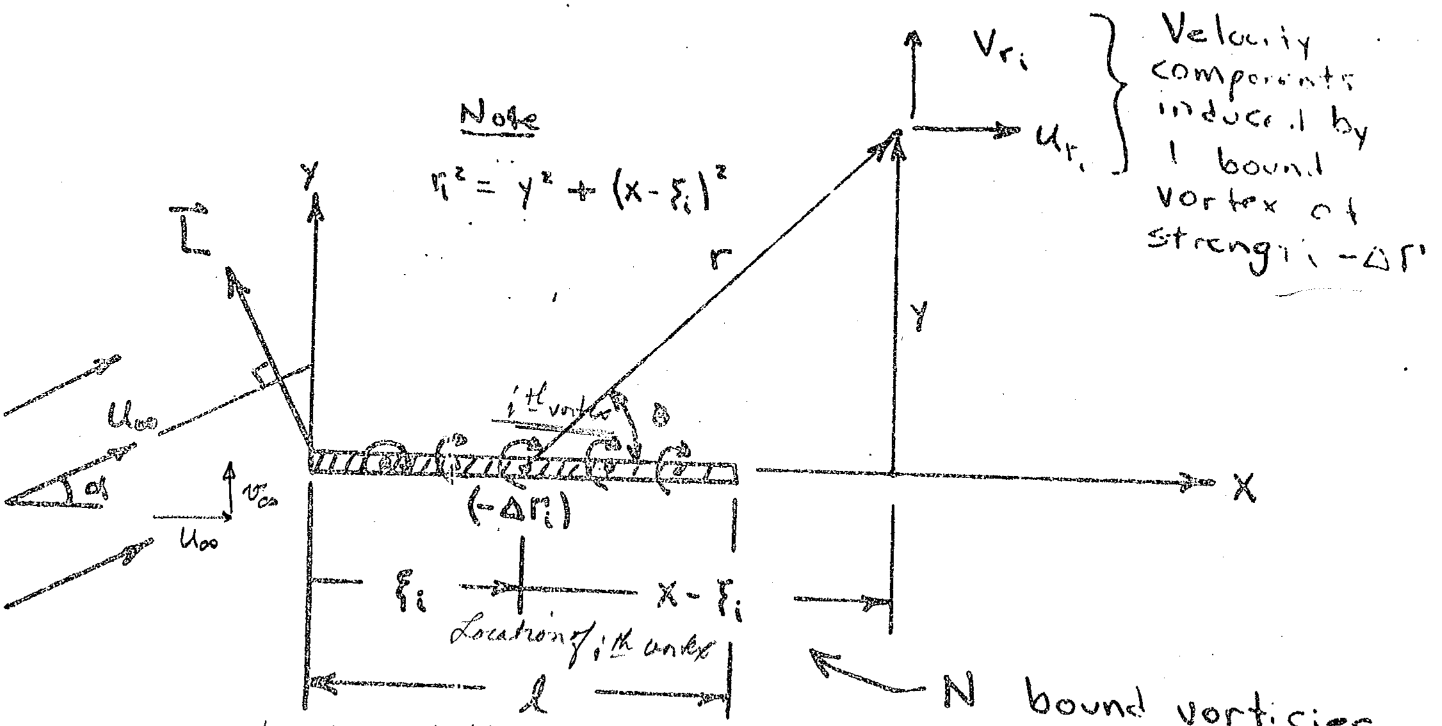
if an air foil existed in flow.



we need sources & sinks to create this streamline

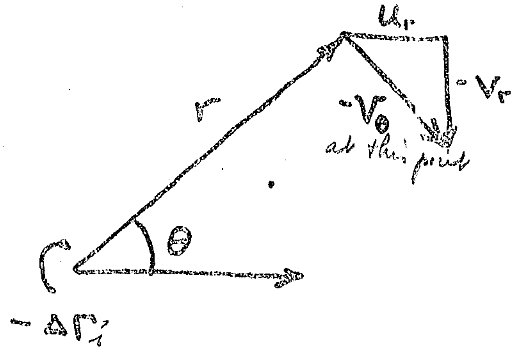
we will pick N pts with $\Delta\Gamma_i$ variable. We will then pick $\Delta\Gamma_i$ $\ni$ we satisfy B.C.

LIFT ON A THIN PLATE



to get correct b/y conditions we need an ∞ no. of bound vortices since N<∞ we can only satisfy b.c. for N discrete points

VELOCITY FIELD INDUCED BY SINGLE VORTEX at any general pt.



$$-V_\theta = \frac{-\Delta\Gamma_i}{2\pi}\frac{1}{r}$$

$$u_r = -V_\theta \frac{y}{r} = \frac{-\Delta\Gamma_i}{2\pi}\frac{y}{r^2}$$

$$-V_r = -V_\theta \frac{(x-\xi_i)}{r} = \frac{-\Delta\Gamma_i}{2\pi}\frac{(x-\xi)}{r^2}$$

where $r_i^2 = y^2 + (x-\xi_i)^2$

Now superpose U_∞ and N vorticies to get complete velocity field

$$u = U_\infty \cos\alpha + \sum_{i=1}^{N} \frac{-\Delta\Gamma_i}{2\pi}\frac{y}{y^2+(x-\xi_i)^2} \quad \boxed{139\text{-}1}$$

$$v = U_\infty \sin\alpha + \sum_{i=1}^{N} \frac{-\Delta\Gamma_i}{2\pi}\frac{(x-\xi_i)}{y^2+(x-\xi_i)^2} \quad \boxed{139\text{-}2}$$

B.C. to satisfy no flow through plate @ $y=0$ $V=0$

$$\Rightarrow \quad U_\infty \sin\alpha = \sum_{i=1}^{N} \frac{-\Delta\Gamma_i}{2\pi} \cdot \frac{1}{x_j-\xi_i} \quad \text{at } j=1,2,\dots,N.$$

now $A_i = -\dfrac{\Delta\Gamma_i}{2\pi U_\infty \sin\alpha}$ then

$$\sum \frac{A_i}{x_j-\xi_i} = 1$$ solve for A_i $\Rightarrow$ $\Delta\Gamma_i$ is found.

and $L = \rho V_\infty \sum -\Delta\Gamma_i$

Now determine values of the $(-\Delta\Gamma_i)$'s by use of the condition: No flow through the plate at N specified points, x_j

set $y=0$, $x=x_j$ and $v=0$ in eqn. 139-2 to obtain a set of linear equations in $\Delta\Gamma_i$.

$$0 = U_\infty \sin\alpha - \frac{1}{2\pi}\sum_{i=1}^{N}\frac{(-\Delta\Gamma_i)}{(x_j-\xi_i)} \qquad \text{where } j = 1, 2 \dots N \qquad \boxed{140\text{-}1}$$

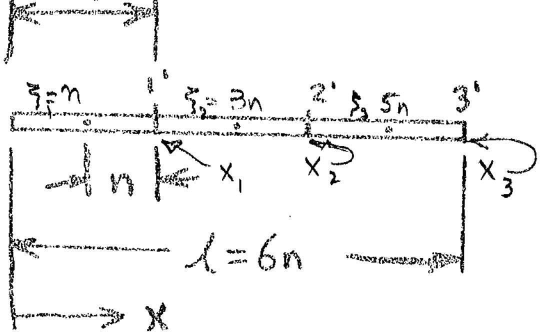
or define:

$$A_i = \frac{-\Delta\Gamma_i}{2\pi U_\infty \sin\alpha}$$

so that the above set of equations becomes a set of N linear equations in A_i

$$\sum_{i=1}^{N}\frac{A_i}{x_j-\xi_i} = 1 \qquad (j = 1, 2, \dots N) \qquad \boxed{140\text{-}2}$$

Illustrate this procedure with 3 vorticies equally spaced on a flat plate



$$\therefore \quad \frac{A_1}{n} = \frac{15}{8} = \frac{-\Delta\Gamma_1}{2\pi V_\infty \sin\alpha\, n} \qquad \therefore -\Delta\Gamma_1 = \frac{15}{4}\pi V_\infty \sin\alpha \cdot n$$

$$\frac{A_2}{n} = \frac{3}{4} = \frac{-\Delta\Gamma_2}{2\pi V_\infty \sin\alpha\, n} \qquad \therefore -\Delta\Gamma_2 = \frac{3}{2}\pi V_\infty \sin\alpha\, n$$

$$\frac{A_3}{n} = \frac{3}{8} = \frac{-\Delta\Gamma_3}{2\pi V_\infty \sin\alpha\, n} \qquad \therefore -\Delta\Gamma_3 = \frac{3}{4}\pi V_\infty \sin\alpha\, n$$

$$\therefore -\Gamma = \sum -\Delta\Gamma_i = 6\pi V_\infty \sin\alpha\, n$$

$$\therefore \quad L = -\rho V_\infty \Gamma = 6\pi\rho V_\infty^2 \sin\alpha\, n$$

now $$\frac{L}{\frac{\rho V_\infty^2}{2} \cdot 6n} = c_L = \frac{2 \cdot \cancel{6}\pi \cancel{\rho} \cancel{V_\infty^2}\, \cancel{n} \sin\alpha}{\cancel{6}\cancel{n}\, \cancel{\rho} \cancel{V_\infty^2}} = 2\pi \sin\alpha$$

Kutta Condition - u is finite at trailing edge
v is 0.

Use the condition that $V = 0$ at 1', 2' and 3'. Writing out 140-2 for these conditions

$$\left\{\begin{aligned} \frac{1}{n}A_1 - \frac{1}{n}A_2 - \frac{1}{3n}A_3 &= 1 \\ \frac{1}{3n}A_1 + \frac{1}{n}A_2 - \frac{1}{n}A_3 &= 1 \\ \frac{1}{5n}A_1 + \frac{1}{3n}A_2 + \frac{1}{n}A_3 &= 1 \end{aligned}\right\}$$

(1') $j=1$, $x_j = 2n$ | $\xi_1 = n$

(2') $j=2$, $x_j = 4n$ | $\xi_2 = 3n$

(3') $j=3$, $x_j = 6n$ | $\xi_3 = 5n$

define $B_i = \frac{A_i}{n}$ the system becomes

$$\left\{\begin{aligned} B_1 - B_2 - \tfrac{1}{3}B_3 &= 1 \\ \tfrac{1}{3}B_1 + B_2 - B_3 &= 1 \\ \tfrac{1}{5}B_1 + \tfrac{1}{3}B_2 + B_3 &= 1 \end{aligned}\right\}$$

The determinant of the coefficients is non-zero so a unique solution for the B_i's exists.

Solving the system of equations we get.

$B_1 = 15/8$, $B_2 = 3/4$, $B_3 = 3/8$

To determine the lift apply the Kutta-Joukowsky Theorem

$$L = -\rho V_\infty \Gamma$$

where Γ is the total circulation

The total circulation $\Gamma = - \sum_{i=1}^{N} [-\Delta\Gamma_i]$

$$L = -\rho U_\infty \Gamma = \rho U_\infty \sum_{i=1}^{N} [-\Delta\Gamma_i]$$

but $-\Delta\Gamma_i = 2\pi n U_\infty \sin\alpha \; B_i$

$$L = \rho U_\infty (nB_1 + nB_2 + nB_3) 2\pi U_\infty \sin\alpha \quad \boxed{142-1}$$

The lift coefficient $C_L = \dfrac{2L}{\rho U_\infty^2 l}$

$$C_L = \frac{2\rho U_\infty^2 n (B_1 + B_2 + B_3) 2\pi \sin\alpha}{\rho U_\infty^2 (6n)}$$

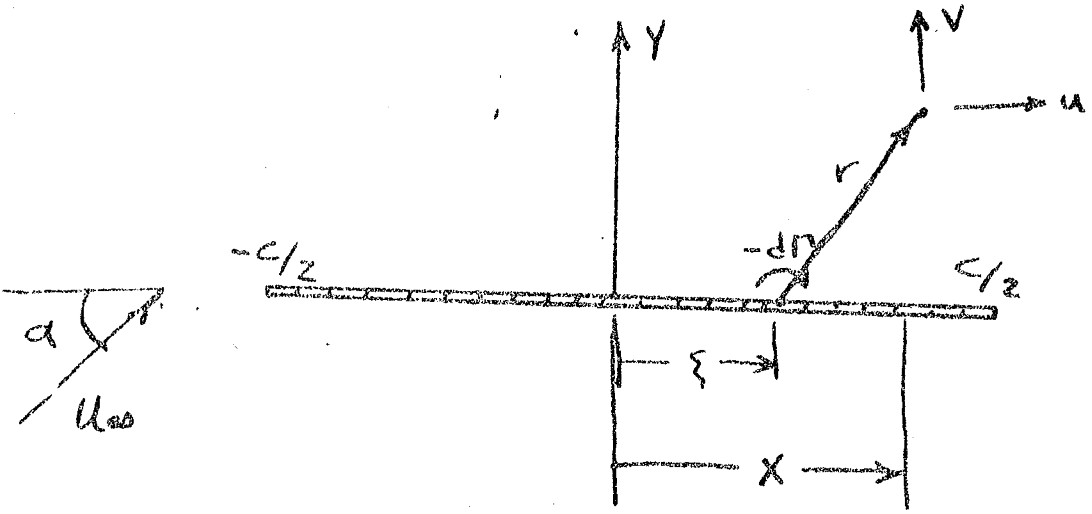
$$C_l = \frac{(B_1 + B_2 + B_3)}{3} 2\pi \sin\alpha$$

but $B_1 + B_2 + B_3 = 3$

so $C_L = 2\pi \sin\alpha$

This is the _exact_ answer for a thin plate at angle α to the flow. This is only true due to a lucky choice of x_j and ξ_i points.

Thin Plate (exact solution)



Instead of discrete vorticies at intervals along the plate we will use a continuous distribution of infinitismal vorticies. This is called a vortex sheet.

define $-\gamma(\xi) = \frac{-d\Gamma}{d\xi}$

γ is the intensity of vorticies distributed along the plate.

Using this continuous distribution of vorticies the boundary conditions ($V=0$ at $y=0$) may be satisfied for all x.

In fact we will use this boundary condition along with the Kutta condition to determine $\gamma(\xi)$.

Following equations 139-1 and 139-2

$$u = U_\infty \cos\alpha + \frac{1}{2\pi}\int_{-c/2}^{c/2} -\gamma(\xi)\,d\xi \;\frac{y}{(x-\xi)^2+y^2}$$

$$v = U_\infty \sin\alpha - \frac{1}{2\pi}\int_{-c/2}^{c/2} \frac{(x-\xi)[-\gamma(\xi)]\,d\xi}{(x-\xi)^2+y^2}$$

at the plate surface $y=0$
and $v=0$

Then

$$U_\infty \sin\alpha = -\frac{1}{2\pi}\int_{-c/2}^{c/2} \frac{-\gamma(\xi)\,d\xi}{(x-\xi)}$$

This is an integral equation for $\gamma(\xi)$ which has the solutions

$$\gamma(\xi) = -2U_\infty \sin\alpha \sqrt{\frac{c/2-\xi}{c/2+\xi}}$$

$$\gamma(\xi) = -2U_\infty \sin\alpha \sqrt{\frac{c/2+\xi}{c/2-\xi}}$$

Select the correct solution by applying the Kutta-condition. Velocity must be finite at the trailing edge ($x = c/2$)

The second solution leads to an infinite vortex intensity, since $\sqrt{c/2+\xi / c/2-\xi} \to \infty$ and thus an infinite velocity at the trailing edge.

To determine the lift use the Kutta-Joukowsky Theorem

$$L = -\rho V_\infty \Gamma_c$$

$$L = 2 \rho V_\infty^2 \sin\alpha \int_{-c/2}^{c/2} \sqrt{\frac{c/2 - \xi}{c/2 + \xi}}\, d\xi$$

$$L = \pi \rho V_\infty^2 c \sin\alpha$$

$$\boxed{C_L = 2\pi \sin\alpha}$$

DIMENSIONAL ANALYSIS AND SIMILITUDE

Motivation - can't solve complete equations
or don't know complete equations

Uses
1. Simplifying governing differential equations
2. Guidance for experimental program
3. Presentation of data
   e.g. Moody diagram for pipes
4. Model testing

Technique for obtaining dimensionless groups depends on whether or not the governing differential equation is known.

I. DIFFERENTIAL EQUATION KNOWN

technique is called normalization

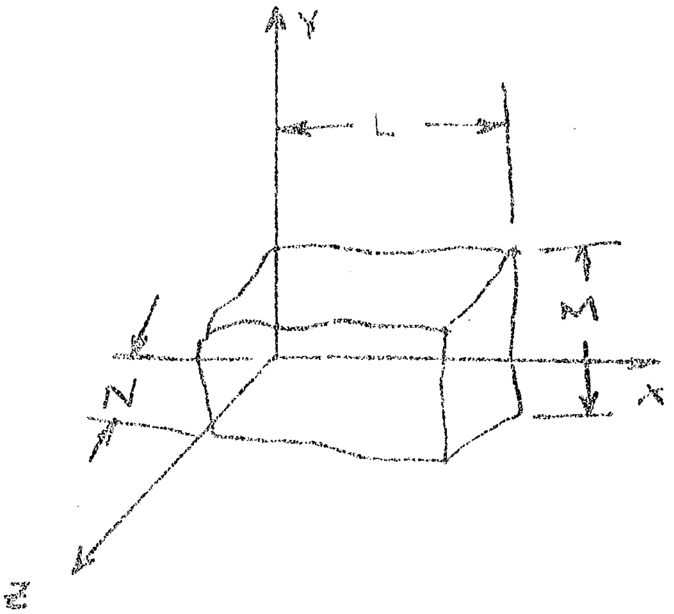
1. Make all variables non-dimensional in terms of "appropriate scales" of the problem
2. divide through by the coefficient of one term

Selection of "appropriate scales"

define non-dimensional variables such that the vary over the range 0-1 approximately.

EXAMPLE Cooking a turkey

(from Kline Similitude and Approximation Theory)



Differential Equation (heat eqn.)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

T = temp difference

t = time

α = thermal diffusivity

BOUNDARY CONDITIONS

$t = 0^-$ $T = T_i$

$t = 0^+$ $T = T_0$

[illegible]

Non-dimensionalize variables

$$T^* = \frac{T}{T_i}, \quad x^* = \frac{x}{L}$$

$$y^* = \frac{y}{M}, \quad z^* = \frac{z}{N}$$

$$t^* = \frac{t}{t_c}$$ where t_c = time constant

rewrite the differential equation with the non-dimensional variables

$$\frac{T_q}{L^2}\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{T_i}{M^2}\frac{\partial^2 T^*}{\partial y^{*2}} + \frac{T_i}{N^2}\frac{\partial^2 T^*}{\partial z^2}$$

$$= \frac{T_a}{\alpha t_c}\frac{\partial T^*}{\partial t^*}$$

now divide through the equation by $\frac{T_i}{L^2}$

$$\frac{\partial^2 T^*}{\partial x^{*2}} + \left(\frac{L}{M}\right)^2 \frac{\partial^2 T^*}{\partial y^{*2}} + \left(\frac{L}{N}\right)^2 \frac{\partial^2 T^*}{\partial z^{*2}} = \left(\frac{L^2}{t_c \alpha}\right)\frac{\partial T^*}{\partial t^*}$$

dimensionless parameters

2 geometric parameters $\frac{L}{M}, \frac{L}{N}$

third parameter = $\frac{L^2}{t_c \alpha}$

2nd example - <u>Navier-Stokes Eqns.</u>

see text p. 143

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + g$$

DEFINE $u^* = \frac{u}{u_0}$, $v^* = \frac{v}{u_0}$

$x^* = \frac{x}{\ell_0}$, $t^* = \frac{t\,u_0}{\ell_0}$

$p^* = \frac{p}{p_0}$

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}$$

$$= -\frac{p_0}{\rho u_0^2}\frac{\partial p^*}{\partial x^*} + \frac{\nu}{u_0 \ell_0}\left\{\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}}\right\}$$

$$+ \frac{g\,\ell_0}{u_0^2}$$

parameters

$$\frac{p_0}{\rho u_0^2}, \quad \frac{\nu}{u_0 \ell_0} = \frac{1}{Re}, \quad \frac{g\,\ell_0}{u_0^2} = \frac{1}{F_r^2}$$

DIMENSIONAL ANALYSIS

II. DIFFERENTIAL EQN NOT KNOWN

BUCKINGHAM Pi THEOREM

$q_1 = f(q_2, q_3 \cdots q_n)$

q_1 = dependent parameter

$q_2 \cdots q_n$ = independent parameters

The n parameters may be grouped into n-m independent dimensionless parameters. (π parameters)
where m = rank of dimensional matrix.
This is usually equal to the number independent dimensions in all of the parameters ($q_1 \cdots q_n$)

METHOD TO OBTAIN π parameters

STEP 1. List all parameters involved

STEP 2. List dimensions of all parameters in terms of primary dimensions

STEP 3. Select from the list a number of repeating parameters equal to the number of primary dimensions and including all primary dimensions.

STEP 4. Set up n-m dimensional equations combining parameters from step 3 with each of the other parameters

EXAMPLE: DRAG OF A SMOOTH SPHERE

$$F = f(\rho, V, D, \mu)$$

STEP 1.

$F \quad \rho \quad V \quad D \quad \mu \qquad n = 5$ parameters

STEP 2. Use dimensions M, L, t

F	ρ	V	D	μ
$\frac{ML}{t^2}$	$\frac{M}{L^3}$	$\frac{L}{t}$	L	$\frac{M}{Lt}$

We have $m = 3$ primary dimensions.

STEP 3. Select repeating parameters

ρ, V, D

STEP 4. We expect to obtain $n - m = 2$ dimensionless groups

Set up two dimensional equations.

$$\Pi_1 = \rho^a V^b D^c F$$

$$= \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \left(\frac{ML}{t^2}\right) = M^0 L^0 t^0$$

$M:\quad a+1=0 \Rightarrow a=-1$

$L:\quad -3a+b+c+1=0 \Rightarrow c=-2$

$t:\quad -b-2=0 \Rightarrow b=-2$

Then $\Pi_1 = \dfrac{F}{\rho V^2 D^2}$

$$\Pi_2 = \rho^d V^e D^f \mu$$

$$\left(\frac{M}{L^3}\right)^d \left(\frac{L}{t}\right)^e L^f \left(\frac{M}{Lt}\right) = M^0 L^0 t^0$$

$M:\quad d+1=0 \Rightarrow d=-1$

$L:\quad -3d+e+f-1=0 \qquad f=-1$

$t:\quad -e-1=0 \qquad e=-1$

Then $\Pi_2 = \dfrac{\mu}{\rho V D}$