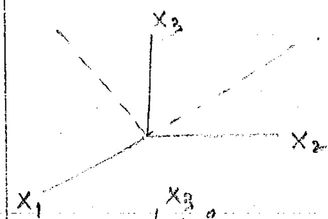
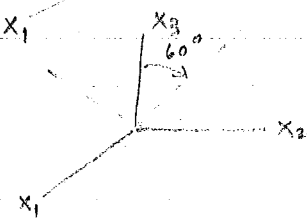


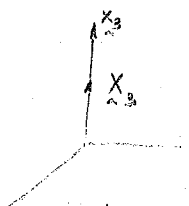
Not correct

This is axis rotation of 45°
body rotation of 45° is transp,

$$d\underset{\sim}{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} d\underset{\sim}{X} \qquad F^1 d\underset{\sim}{X}$$

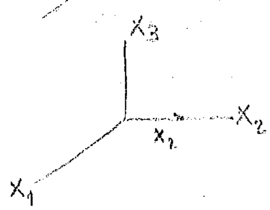


$$d\underset{\sim}{x} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} d\underset{\sim}{X} \qquad F^2 dX$$

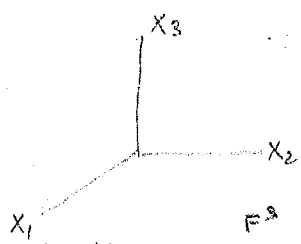


$$d\underset{\sim}{x} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} d\underset{\sim}{X} \qquad F^3$$

this is not correct
should be zero.



$$d\underset{\sim}{x} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} d\underset{\sim}{X} \qquad F^4$$



$$d\underset{\sim}{x} \quad \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} d\underset{\sim}{X} \qquad F^5$$

$$F = F^5 F^4 F^3 F^2 F^1$$

$$\begin{bmatrix} \frac{1}{2}-\sqrt{3} & \frac{-\sqrt{3}}{2\sqrt{2}} - \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{\sqrt{2}} \\ -\sqrt{3} & 0 & \frac{2}{\sqrt{2}} \\ -\sqrt{3} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \overset{F^3}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix} = \overset{F^2}{\begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}} \overset{F^1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}$$

$$\begin{bmatrix} \frac{-\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{2}} \\ \frac{3\sqrt{3}-1}{2} & \frac{\sqrt{3}+2}{2\sqrt{2}} & \frac{-(\sqrt{3}+4)}{2\sqrt{2}} \\ -\frac{3\sqrt{3}}{2} & -\frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} = \overset{F^5}{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} \frac{1-3\sqrt{3}}{2} & \frac{-(\sqrt{3}+2)}{2\sqrt{2}} & \frac{\sqrt{3}+4}{2\sqrt{2}} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{3}{2}\sqrt{3} & -\frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} = \overset{F^4}{\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}} \begin{bmatrix} \frac{1-2\sqrt{3}}{2} & \frac{-(\sqrt{3}+2)}{2\sqrt{2}} & \frac{\sqrt{3}+2}{2\sqrt{2}} \\ -\sqrt{3} & 0 & \sqrt{2} \\ -\sqrt{3} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$d\underset{\sim}{x} = F \; d\underset{\sim}{X}$$

$$\det F = \frac{dV}{dV_0} = -\frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}+2}{2\sqrt{2}}\right)\sqrt{2} + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{3\sqrt{3}-1}{2}\right) + \frac{3\sqrt{3}}{2}\left(\frac{\sqrt{3}+2}{2\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}+}{2\sqrt{}}\right.$$

$$= -\left[\frac{3+2\sqrt{3}}{4}\right] - \left[\frac{3\sqrt{3}-1}{4}\right] + \frac{9+6\sqrt{3}}{4} + \frac{3+4\sqrt{3}}{8}$$

$$= \frac{-6 - \cancel{4\sqrt{3}} - 6\sqrt{3} + 2 + 18 + 12\sqrt{3} + 3 + \cancel{4\sqrt{3}}}{8}$$

$$= \frac{12\sqrt{3}+18}{8} > 0.$$

$$+1 \cdot \frac{1}{2} \cdot 2 \cdot 1 \cdot 1$$

Division of Applied Mechanics

ME 242A — Problem Set 1 — Due Friday, Apr. 25, 1980

1.) Take fixed right-handed axes x_1 , x_2 , x_3 . Write down the deformation gradient matrix, $\dfrac{\partial x_i^2}{\partial x_j^1} = F_{ij}$, for deformation of a body from $\underset{\sim}{x}^1$ to $\underset{\sim}{x}^2$ for

a) right-handed rotation of 45° about x_1 . ie axes rotated -45° about x_1

b) left-handed rotation 60° about x_2 . " " " -60° about x_2

c) stretch by a stretch ratio 2 in the x_3 direction.

d) stretch by a stretch ratio $\frac{1}{2}$ in the x_2 direction.

e) right-handed rotation of 90° about x_3 . ie axes rotated -90° about x_3

Obtain the total deformation matrix for these motions carried out in order. Using this, check the final volume change ratio.

2.) Find B , C , U , V , R for the following mapping:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{12}{25} & \frac{18}{25} & -\frac{12}{5} \\ \frac{3}{5} & \frac{17}{5} & 0 \\ -\frac{16}{25} & -\frac{24}{25} & \frac{9}{5} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \Rightarrow \quad \underset{\sim}{x} = \underset{\sim}{F}\,\underset{\sim}{X}$$

Check that this corresponds to a permissible deformation in a continuous body.

3.) For the special case when $\underset{\sim}{F} = \underset{\sim}{I} + \underset{\sim}{\delta}$, $\underset{\sim}{\delta}$ being small so that squares and products of components can be neglected, show through use of $\underset{\sim}{F}^T\underset{\sim}{F}$ that the symmetric part of $\underset{\sim}{\delta}$ only occurs in the pure deformation part of the deformation (rotation eliminated) , and develop the usual results of infinitesimal strain theory, e.g. that for a series of deformations, strains and rotations can be added algebraically.

$F^TF = (I+\delta^T)(I+\delta) = I^TI + I^T\delta + \delta^T + \delta^T\delta = I + \delta + \delta^T$

$F^TF - I = \delta + \delta^T$

$2E$

$E = \dfrac{\delta + \delta^T}{2}$

4.) Consider the deformation:

$$x_1 = X_1$$
$$x_2 = X_2 + kX_1$$
$$x_3 = X_3$$

$$\left[\underset{\sim}{x}\right] = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\underset{\sim}{X}\right]$$

Determine $\underset{\sim}{F}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, and obtain the principal stretches. Find $\underset{\sim}{U}$, $\underset{\sim}{V}$, and $\underset{\sim}{R}$.

(1) Mc Bull. 128 4/80

ME242A <u>Intro. to Nonlinear Continuum Mechanics</u>

3 units

~~T Th 11-12:15~~ EH Lee off Durand 281

x7-

Blue Cards

Text : L.E. Malvern - Intro. to the mech. of a contin. medium

Ref : Reserve Engr. Library

Grading : 1/3 hmwk 2/3 final exam

Course Outline - Handout

<u>Introduction</u>

<u>Motivation</u>

mechanics of a continuous medium

solids
liquids
gases

stress and deformation/flow

1) disregard molecular structure of matter
ideal gas Kinetic theory OK
solids, liquids } atomic level lattice almost hopel[ess]
rules out: high altitude aerodyn[amics]
~~formation fatigue crack~~

2) piecewise continuous functions

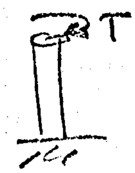
deals with materials as normally used (differing from modern physics) phenomenological theory
[illegible]

Classical ex

infinitesimal elasticity
ideal fluid th.
ideal plasticity
} very special cases

Finite elasticity
nonlinear viscosity
| very clean

inadequate for many mechanics problems

torsion paradox  does it lengthen or shorten?
linear: neither (symmetry)
nonlinear: depends on material

non linear visco (normal-stress effect) 1-D notion

elastic stability – path dependence
shock waves – wave speeds depend on stress amplitude
~~metal forming~~

needed {
polymers, plastics
soils, earth (foundations, earthquake response)
tissue (biomech)

Nonlinearities
geometrical – finite strain (can't fully understand classical theory w/o it)

material – memory, path dependence, relation between force and deformation (not strain)
{ functions $\sigma = f(\varepsilon)$
functionals $\sigma(t) = \mathcal{F}_{-\infty}^{t}[\varepsilon(t')]$

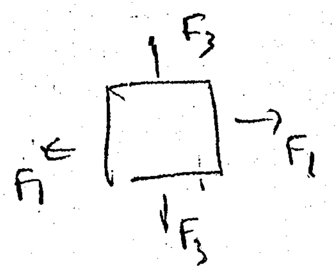
Thermodynamics

names: Cauchy, Green, Murnaghan (1937 Am Math J)
Rivlin (Rubber elasticity after WW II)
Oldroyd (nonlin. fluids)
Noll, Truesdell, Coleman (axiomatic) TW Ting

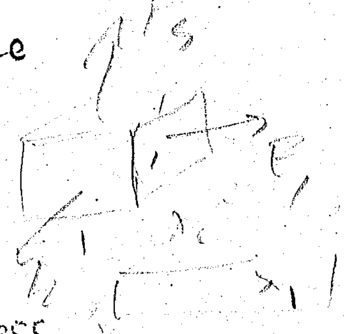
Controversy : pd following WW II

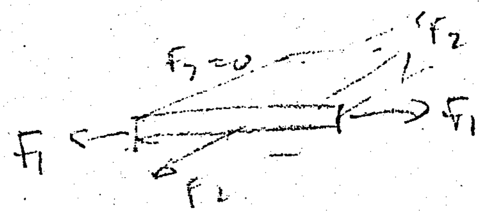
"Authorities" often wrong

1) Rivlin review
Ed Truesdell & Noll   block of isotropic hyperelastic material
in equil. - pure homogeneous deformation



equal & opposite forces uniformly distrib. over faces
claim: greater stretch in direction of greater force





sheet

now apply F_3 large area   low stress

See
Beavers of Axiomatics in
Applied Math. L C Woods
Bulletin of the Institute of Mathematics
and its Applications
Feb 1973   pp. 40 – 44

COURSE OUTLINE

ME 242A,B. E. H. Lee

Introduction to non-linear Continuum Mechanics

242A

General discussion of the need for analysis which covers finite strain, non-linear and time dependent response of materials.

The geometry of deformation. Transformation matrices, sequential deformation, polar decomposition, strain.

Constitutive relations and the constraints on them, objectivity, symmetry groups.

Applications: non-linear viscosity

rubber-elasticity

Solution to problems for simple geometrical bodies. Torsion and tension of a rod, internal pressure in a thick walled circular cylindrical tube, turning a tube inside-out (for finite deformation elasticity).

242B Thermodynamics: finite deformation thermo-elasticity, role of entropy (entropy elasticity).

Formulations of the theory using referential (Lagrange) or spacial (Euler) coordinates, various nominal stresses and their respective advantages.

Finite deformation plasticity, the theory of rate laws, formulation of boundary value problems: metal forming solutions. Finite deformation viscoelasicity.

RESERVE BOOK LIST

Please Note:

1. Only **REQUIRED READING** will be placed on reserve. This is defined as material assigned as required reading for all students and listed either in the course syllabus or reading list.
2. Optional reading will not be placed on reserve but library holdings will be checked, on request, to insure that materials are in the collection. Submit a reading list with your reserve request. Missing items will be rush ordered.
3. For prompt handling reserve book lists must be submitted at least **THREE WEEKS** before resgistration day each quarter.
4. Books are not automatically kept on reserve from one quarter to another.
5. Reserve loan periods may be **2 HOURS, 1 DAY, 2 DAYS, or 3 DAYS.**
6. Courses numbered 1–199 Meyer Library; Courses numbered 200 and above Main Library.
EXCEPTIONS: English and Humanities Courses 1–299 are at Meyer Library.

Dept. & Course No. ME 242 A

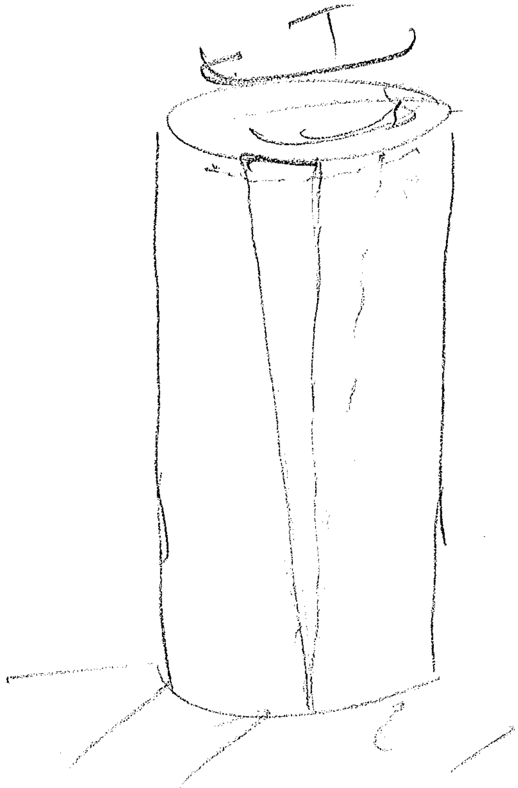
Instructor E H Lee

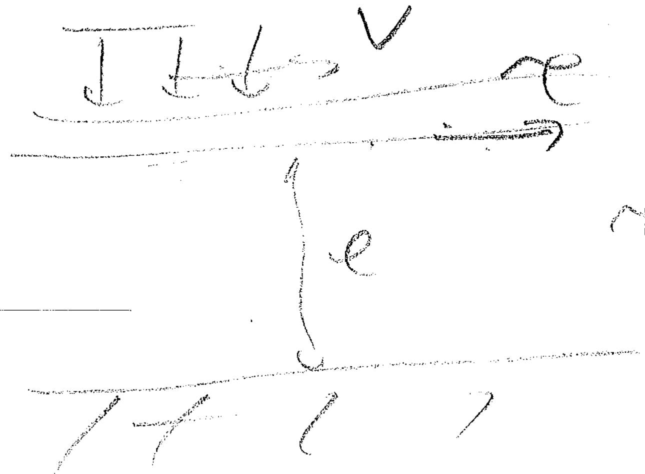
Quarter Spring 1980

Enrollment Expected 15

Call Number	Author's Full Name	Title	Suggest No. of Copies	Loan Per.	No. of Copies Placed on Reserve
		"		"	
	L. E. Malvern	Introduction to the mechanics of a cont. medium 1969	"	"	
	C. Truesdell	The Elements of Continuum Mechanics 1966	"	"	
	C Truesdell and W Noll	The Non-Linear Field Theories of Mech. Handbook der Physik III/3 1965	"	"	

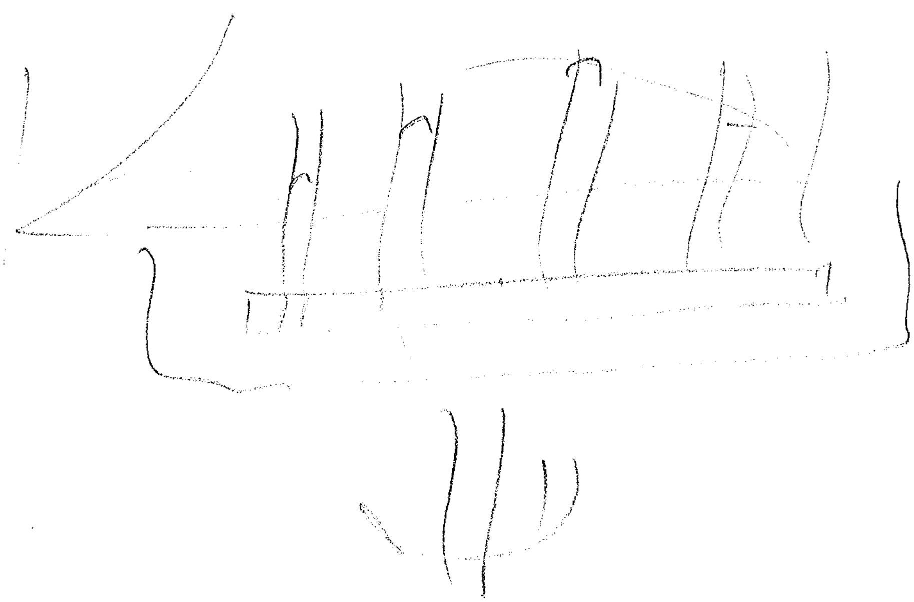
Dept. &





$\tau = \frac{V}{e}$

linea. visc



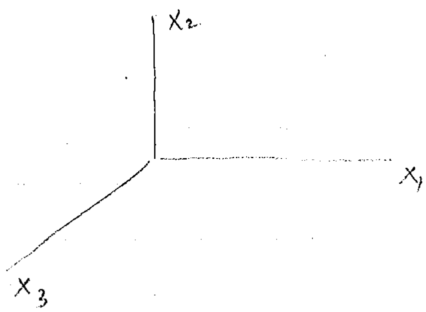
April 4, 1980

Theory of Nonlinear Contin. Mech.

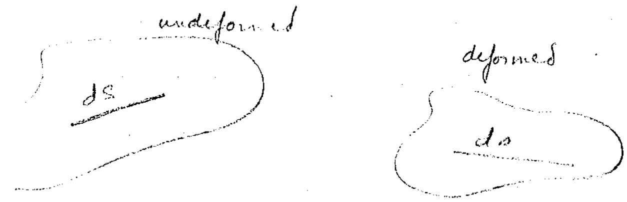
2 ways of looking at this area – < mathematical (axiomatic) not correct physi / applied

We will follow axiomatic approach but will digress in certain areas.

We will use cartesian coordinates



Constitutive Relation (Stress – deformation relation



Lagrangian:

$$ds^2 - dS^2 = 2E_{ij}\, dx_i\, dx_j$$

original coord.

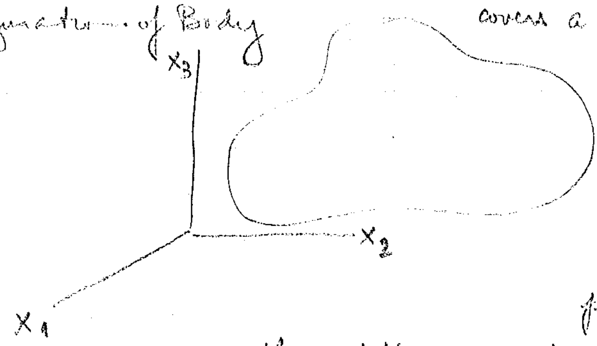
Lagrangian strain

if we rotate body only the strain ds^2 is unchanged but the stresses are different ∴ not only are stress fns of strain but also fns of rotation

<u>Principles</u> (Axioms)

Given: Material body B w/ smooth distrib of mass particles. There exists a dm measure ∋. $\int_B dm = M_B$ (conservation of mass).

Configuration of Body



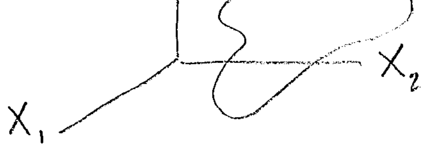
covers a region of space (Euclidean) at time t

motion is a continuous variation of configuration. Defining $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ the cartesian syst., we can use

$\underset{\sim}{x} = \underset{\sim}{\chi}(X, t)$ where X is the particle points

we can then define mass density $\int_{B_\chi} \rho_\chi\, dV = \int_B dm = M_B$ volume differential in configuration at time t

It is much easier to define a reference configuration & define change from this conf.



$\underset{\sim}{X}$ reference coord. = $\underset{\sim}{k}(X)$

This then labels each particle (mapping body points onto tuple)

$\underset{\sim}{X}$ is not necessarily achievable physically

April 9, 1980

We will deal w/ Finite Displacements.

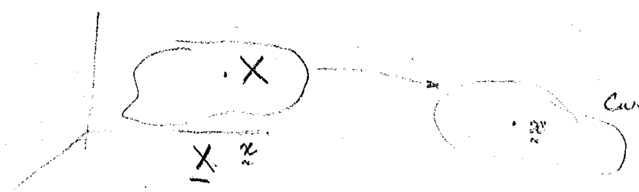
1. Must allow for deformation of body when discussing tractions applied & stress.

$\to t_i$ traction vector $\underline{\text{traction}}$ bdy conditions

T_{ij} stress tensor

2. must satisfy BC's on deformed bdy. (in classical body deformation is very small.)

Must define stresses wrt deformed state



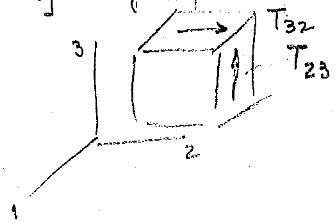
$\underset{\sim}{T} = T_{ij}$ Cauchy Stress; true stress stress wrt deformed state.

$T_{ij} = T_{ji} = \underset{\sim}{T}^T$ from conservation of angular moment. & true stress definition

We assume no moments (coupled stress in the limit).

can define stress wrt initial state. this is defined as $\underline{\text{nominal}}$ stress.

T_{ij} - force/unit area in direction j on area whose normal is i



in classical elasticity $T_i = \sigma_{ij} n_j$ not actually corrected

for us $t_j(\underset{\sim}{x}, t, \underset{\sim}{n}) = T_{ij} n_i$ Cauchy's lemma.

$$\frac{\partial T_{ij}}{\partial x_i} + \rho b_j = \rho \frac{\partial^2 x_j}{\partial t^2}\bigg|_{\underline{X}} \qquad \text{Eqns of motion}$$

body force unit mass. (in deformed state.)

$$\operatorname{div} \underset{\sim}{T} + \rho \underset{\sim}{b} = \rho \frac{\partial^2 \underset{\sim}{x}}{\partial t^2}\bigg|_{\underline{X}} = \rho \left[\frac{\partial^2 \underline{X}}{\partial t^2} + \frac{\partial \underline{X}}{\partial t} \frac{\partial}{\partial t}\left(\frac{\partial \underline{X}}{\partial \underset{\sim}{x}}\right)\right]$$

$\cong 0$ for classical theory since $\underline{X} \cong \underset{\sim}{x}$ & $\frac{\partial \underline{X}}{\partial \underset{\sim}{x}} \cong \delta$

material deriv = local + convective = substantial

These apply $\forall$ materials, Must come up w/ constitutive Eqns for each material

Constitutive Relations

$\underset{\sim}{x} = \underset{\sim}{\chi}(\underline{X}, t)$

$\underset{\sim}{T} = \underset{\sim}{T}(\underline{X}, t)$ and $\underset{\sim}{\chi}(\underline{X}, t)$

applied tractions geometry change

this defines a dynamic process

$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}$
$= \frac{\gamma+1}{2} = \frac{2.4}{2}$
$T/T_0 = \frac{5}{6} = .833$

This is possible dynamically by selecting $\underset{\sim}{b}$ (body force) $= \underset{\sim}{b}(\underset{\sim}{x}, t)$
In general theory we are interested in the totality of problems expressed by $(\underset{\sim}{\chi}, \underset{\sim}{T})$. This is possible if $\underset{\sim}{\chi}, \underset{\sim}{T}$ satisfy the constitutive relation of the material being deformed

Now what is the relation (sensible ones!) between $(\underset{\sim}{\chi}, \underset{\sim}{T})$
These are models of reality which represent ideal materials which ~~represent~~ approximates real materials. Approx. which may be good in some circumstances poor in others.

Truesdell's Principle (Working Hypothesis) of Determinism

$$\underset{\sim}{T}(X, t) = \underset{\sim}{\mathfrak{F}}\left(\underset{\sim}{\chi}(z, \tau)_{-\infty}^{t}; X, t\right) \quad \forall z \in B, \ \tau \le t \qquad B \text{ is body}$$

where stress is measured — functional (depends on history) — Volterra Notation

stress is depends on history of total deformation of body previously.

11 April 1980

ME242

Continuation: $\underset{\sim}{T}(\underset{\sim}{X},t) = \mathcal{F}(\underset{\sim}{\chi}(\underset{\sim}{z},\underset{\sim}{\tau}^{t}); X, t) \qquad \underset{\sim}{z}\in B,\ \tau\le t \qquad \underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X},t)$

$\underset{\sim}{T} = \underset{\sim}{T}^{T}$

Good Books – Pearson, Budiansky have published books in the area of NonlinCont.
Micropolar or Cosserat Materials (1903); Materials with microstructures
" " directors

There is a nonlocal effect – but Truesdell's next Principle states

Principle of Local Action

Two deformations $\underset{\sim}{\chi}$ & $\overline{\underset{\sim}{\chi}}$ differ only outside an arbitrary small neighborhood $N(\underset{\sim}{X})$ of $\underset{\sim}{X}$ for $\tau\le t$, then $\underset{\sim}{T}$ at $\underset{\sim}{X}$ is same

ie $\quad \underset{\sim}{\chi}(\underset{\sim}{z},\tau) = \overline{\underset{\sim}{\chi}}(\underset{\sim}{z},\tau) \quad$ for $\tau\le t$ for $\underset{\sim}{z}\in N(\underset{\sim}{X})$

$\mathcal{F}(\underset{\sim}{\chi},\underset{\sim}{X},t) = \mathcal{F}(\overline{\underset{\sim}{\chi}},\underset{\sim}{X},t)$ we can thus expand in a taylor series

$$\underset{\sim}{\chi}(\underset{\sim}{z},t) = \underset{\sim}{\chi}(\underset{\sim}{X}^{o},t) + \frac{\partial\underset{\sim}{\chi}}{\partial\underset{\sim}{X}}(\underset{\sim}{X}^{o},t)(\underset{\sim}{z}-\underset{\sim}{X}^{o}) \qquad |\underset{\sim}{z}-\underset{\sim}{X}^{o}| < \text{small}$$

$\Rightarrow \quad \underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{z},t) \Rightarrow$

$$x_i(\underset{\sim}{z},t) = x_i(\underset{\sim}{X}_o,t) + \frac{\partial x_i}{\partial X_j}(\underset{\sim}{X}_o,t)(z_j - X_j^{o})$$

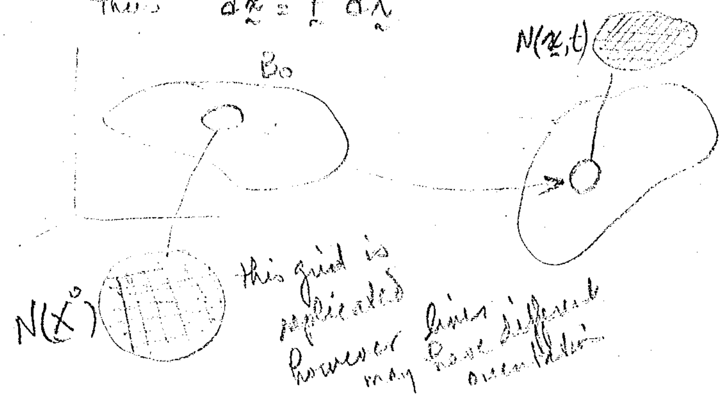
translation no effect on stress

$$\frac{\partial\underset{\sim}{\chi}}{\partial\underset{\sim}{X}} = \underset{\sim}{F} \qquad \frac{\partial x_i}{\partial X_j} = F_{ij} \quad \text{deformed grad}$$

$$\underset{\sim}{T}(\underset{\sim}{X}^{o},t) = \mathcal{G}\left[F(\underset{\sim}{X}^{o},\underset{\sim}{\tau}^{t}),t\right]$$

Covers only materials that are not cosserat materials (ie when $\underset{\sim}{T}\ne\underset{\sim}{T}^{T}$); Covers fluids & other normal materials

then $\quad d\underset{\sim}{x} = \underset{\sim}{F}\,d\underset{\sim}{X} \qquad dx_i = F_{ij}dX_j$



$$dS^2 = (dX_1, dX_2, dX_3)\begin{pmatrix} dX_1 \\ dX_2 \\ dX_3 \end{pmatrix} = d\underset{\sim}{X}^T d\underset{\sim}{X}$$

$$ds^2 = d\underset{\sim}{x}^T d\underset{\sim}{x} = d\underset{\sim}{X}^T \underset{\sim}{F}^T \underset{\sim}{F} d\underset{\sim}{X}$$ $F^T F$ is symmetric F is not sym.
6 indep. quantities

$$= d\underset{\sim}{X}^T \underset{\sim}{C}\, d\underset{\sim}{X}$$ $\underset{\sim}{C}$ Right Cauchy-Green tensor.

$$ds^2 - dS^2 = 2E_{ij}\, dX_i\, dX_j = d\underset{\sim}{X}^T (F^T F - I) d\underset{\sim}{X}$$

Lagrang. tensor $\quad \therefore \underset{\sim}{F}^T \underset{\sim}{F} = \underset{\sim}{I} + 2\underset{\sim}{E}$

$\underset{\sim}{F} = \dfrac{\partial x_i}{\partial X_j}$ $\quad \det(\underset{\sim}{F}) = J$ (Jacobian) $\Rightarrow \underbrace{dx_1 dx_2 dx_3}_{\text{deformed volume}} = J \underbrace{dX_1 dX_2 dX_3}_{\text{reference volume}} \Rightarrow J = \dfrac{dv}{dV_0}$

$\underset{\sim}{F}$ is never singular by causality (can never squeeze volume to zero).

<u>Polar Decomposition Theorem</u>: Any invertible matrix (i.e. $\underset{\sim}{F}$) can be expressed in the form $\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U} = \underset{\sim}{V}\underset{\sim}{R}$ where $\underset{\sim}{R}$ is an orthog matrix & $\underset{\sim}{U}, \underset{\sim}{V}$ are symmetric matrices.

$d\underset{\sim}{x} = U\, d\underset{\sim}{X}$ $\quad U$ has real eValues & thus stretches the reference axes

$= \lambda\, d\underset{\sim}{X}$ $\quad$ & no rotation

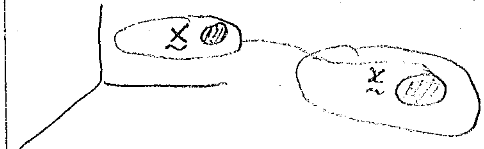
λ is the stretch ratio; U is a pure stretch

thus if $ds^2 = dS^2 \Rightarrow F^T F = I \Rightarrow F^T = F^{-1}$ F is orthog.
thus $\underset{\sim}{F}$ is a pure rotation and a stretch.

$$F^T F = U^T \underbrace{R^T R}_{I} U = U^T U = U^2 = C$$

Review $\qquad$ 14 April 1980

$$T(\underset{\sim}{x}, t) = \mathcal{G}\left[F(\underset{\sim}{X}, \underset{\sim}{\tau}), t\right] \quad \tau \le t$$



$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t)$ $\quad \underset{\sim}{F} = \dfrac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}}$ $\quad F_{ij} = \dfrac{\partial x_i}{\partial X_j}$

$d\underset{\sim}{x} = \underset{\sim}{F} d\underset{\sim}{X}$, $\quad dx_i = F_{ij}\, dX_j$

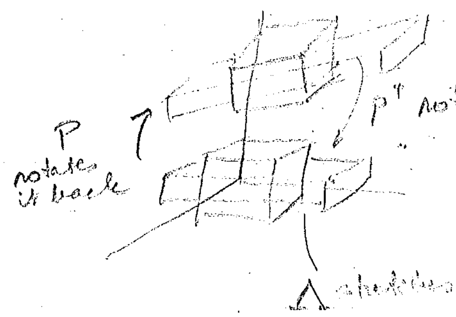
if $\underset{\sim}{F} = \underset{\sim}{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ no deformation

Next Symmetric (Real) Matrix V. $\ni$, $F = VR$ $\exists$ a full set of $\vec{EV}$ (and EV) which are orthogonal and the EV are real.

Thus $V = P\Lambda P^{-1}$ where Λ is diag & P are orthog

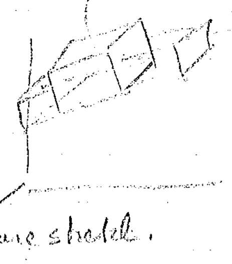
thus
$$V dX = \lambda dX$$
$$V dX^{(i)} = \lambda^{(i)} dX^{(i)} \quad \text{by placing } dX^{(i)} \text{ to form a matrix}$$
$$\underset{\text{EV matrix}}{VP} = \underset{\text{EV matrix}}{P\Lambda} \quad \text{where } \Lambda_{ij} = \delta_{ij}\lambda^{(i)} \text{ no sum}$$

$P^{-1} = P^T$ Thus we can choose direction of EV so that P is proper orthog.



$$d\underset{\sim}{x}^{(2)} = \Lambda\, d\underset{\sim}{x}^{(1)}$$

Thus in actuality all we need to do is just stretch that we can choose λ to be + and this is just pure stretch.



$$\underset{\sim}{F} = RU = VR$$

16 April 1980

<u>Polar Decomposition</u>

A real invertible matrix $\underset{\sim}{F}$ can be written as the product of a symmetric positive definite matrix + an orthog matrix
$$\hat{F} = \underset{\sim}{R}\,\underset{\sim}{U} = \underset{\sim}{V}\,\underset{\sim}{R}$$
where $F^T F = U^T R^T R U = U^2 = C$ Right Cauchy-Green Tensor
$F F^T = V R R^T V^T = V^2 = B$ Left " " "

positive definite matrix has + real evalues

$$F^T F dX = C dX = \lambda dX$$
$$dX^T (F^T F dX) = \lambda dX^T dX = \lambda dS^2 \quad \Rightarrow \lambda > 0$$
$$C = C^T$$

Thus for each EV then if $P = (\bar{EV}_1, \bar{EV}_2, \bar{EV}_3)$ then $CP = P\Lambda$ where Λ is the diagonal matrix of EValues

18 April 1980

$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} = \underset{\sim}{V}\,\underset{\sim}{R}$ where $F^T F = C$ $\quad U^2 = C$

$F F^T = B \quad V^2 = B$

$$dS^2 = dX^T dX$$

$$ds^2 = dx^T dx = dX^T F^T F\, dX$$

$$ds^2 - dS^2 = dX^T dX - dX^T F^T F dX = dX^T 2 E^{Lag} dX = dX^T (F^T F - I) dX$$

Lagrange Strain

$$dx^T dx = ds^2$$

$$dS^2 = dx^T F^{-1\,T} F^{-1} dx = dx^T (F F^T)^{-1} dx \quad \text{where } d\underset{\sim}{x} = F\, d\underset{\sim}{X}$$

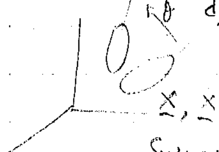
$$d\underset{\sim}{X} = F^{-1} d\underset{\sim}{x}$$

$$ds^2 - dS^2 = dx^T dx - dx^T (F F^T)^{-1} dx = dx^T \left(I - (FF^T)^{-1}\right) dx = dx^T 2 E^{Euler} dx$$

Euler Strain

Suppose we rotated body by amount θ

$R\theta \quad d\underset{\sim}{x}^{(2)} = \underset{\sim}{R}\, dx^{(1)}$

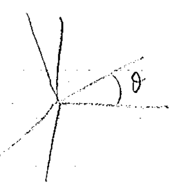


$\underset{\sim}{X}, \underset{\sim}{x}$

Suppose we rotated axes by amount θ.

$$d\underset{\sim}{x}^{(\theta)} = \underset{\sim}{R}^T d\underset{\sim}{x}$$

$$R^T = R^{-1}$$



For rotation of axes $\quad dx_i' = l_{ij}\, dx_j$

For a tensor $\quad \sigma_{ij}' = l_{im}\, l_{jn}\, \sigma_{mn}$.

$d\underset{\sim}{x} = \underset{\sim}{R}\, d\underset{\sim}{X}$ (body motion) $\quad$ for axis motion

Thus $\sigma^* = R \sigma R^T \quad\quad \sigma^* = R^T \sigma R$

Any constitutive relation $\quad \underset{\sim}{T}(X,t) = \underset{\sim}{G}\left(F(\tau_{-\infty}^{t}, \underset{\sim}{X}), t\right)$

must satisfy the following

Suppose we superpose a rotation (rigid body). ie R is a proper orthog. may fix

$$T^*(\underset{\sim}{X},t) = R(t)\, T(\underset{\sim}{X},t)\, R^T(t)$$

$$F^* = \frac{\partial \underset{\sim}{x}^*}{\partial \underset{\sim}{X}} = RF \quad\quad d\underset{\sim}{x} = \underset{\sim}{F}\, dX$$

$$dx^* = R \underset{\sim}{F}\, dX = R\, d\underset{\sim}{x}$$

Must be that $T^* = \mathcal{G}\left(F^*(\underset{\sim}{X}, \tau_{-\infty}^{t}), t\right)$

ie r.b. rotation should just rotate stress

Green & Rivlin Vol1, No1 Archives for Rational Mechanics 1958

This results has led to the Principle of Objectivity the rotation must be on the deformed frames of reference <u>not</u> on the inertial reference frame.

In general $\quad \underset{\sim}{X} \neq \underset{\sim}{\tilde{x}}$

and $T^* = \mathcal{G}(F^* \ldots)$ is true wrt this frame only.

and the fnl relationship must not depend on ~~frame~~ axes of reference.

21 April 1980

Handbuch der Physik III/1 Truesdell & Toupin Classical Field Theory

" III/3 " & Noll Nonlinear Field Theories

here we don't worry about inertia forces since balancing them w/ body forces will remove them from problem.

Returning to our result if we apply in general a proper orthog group to T $\Rightarrow$ $QTQ^T = \mathcal{G}$

$$T(\underset{\sim}{X}, t) = Q^{-1} \mathcal{G}[\underbrace{QR}_{} U, t]\, Q^{-1^T} \quad \forall Q \text{ . In partic}$$

$$\text{let } Q = R^T \quad \Rightarrow \quad R^T T R = \mathcal{G}[U(X,t)] \qquad \text{since } F = RU$$

Then

$$T(t) = \underset{\sim}{R}(t)\, \mathcal{G}[U(\tau), t]\, R^T$$

guarantees objectivity is a "restricted" form of constitutive relation w/ symmetric matrix argument only

(only 6 indep coeff of argument.)

$$T = R\, U U^{-1}\, \mathcal{G}[U(\tau), t]\, U^{-1}\, U^T R^T$$

$$= F U^{-1}\, \mathcal{G}[U(\tau), t]\, U^{-1} F^T \quad \text{where we can define}$$

$$U^{-1} \mathcal{G} U^{-1} = H \quad \text{and} \quad U^2 = C \quad \therefore \quad T = F H[C^{(\tau)}, t] F^T.$$

where T, C is at some point where we want measurement.

Thus in Elasticity $\quad T = F H[C(t)] F^T$

Elasticity $\Rightarrow$ stresses are fns of deformation

Hyperelasticity $\Rightarrow$ " " " " " & $\exists$ a strain energy fn.

we can choose $X_i^T X_j = I$ $\therefore$ $C = P\Lambda P^{-1} = P\Lambda P^T$ Set of orthonormalized EV is an orthog matrix

$$\therefore \quad C = \underbrace{P\Lambda^{1/2}\, P^T}_{U}\underbrace{P\,\Lambda^{1/2}\, P^T}_{U}$$

$$\therefore \longrightarrow \underline{U = P\Lambda^{1/2} P^T}$$

$F = RU$ $\therefore$ $R = FU^{-1}$

$R^T = U^{-1^T} F^T$ $\qquad U^T = P\Lambda^{1/2}P^T$

$R^T R = U^{-T} F^T F U^{-1} = U^{-T} U^T U U^{-1} = I$

if $FF^T = V^2$ $\qquad d\underline{x} = F d\underline{X}$ $\qquad d\underline{X} = F^{-1} d\underline{x}$

$$d\underline{X}^T d\underline{X} = d\underline{x}^T F^{-1^T} F^{-1} d\underline{x} = d\underline{x}^T (FF^T)^{-1} d\underline{x}$$

then if $AX = \lambda X$ $\quad \frac{1}{\lambda} X = A^{-1} X$

We can then say that the symmetric part corresponds to the symmetric part in infinitesimal theory & Orthog corresponds to anti symmetric part.

Now we can also find V from $F = RU = VR$ $\therefore$ $V = RUR^{-1} = RUR^T$

thus V has same EValues but different $\vec{EV}$.

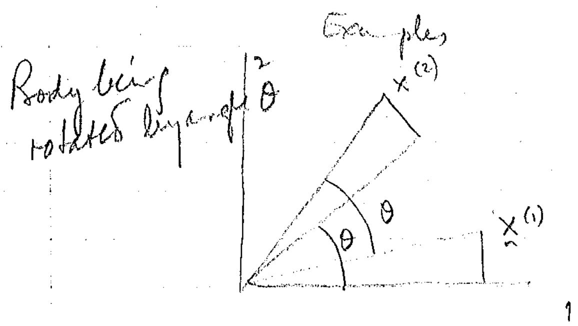
Proof

$$UX = \lambda^i X \implies UR^T RX = \lambda X$$
$$RUR^T(RX) = \lambda(RX)$$
$$V \quad Y = \lambda Y$$

$\therefore$ V has same EV but different $\vec{EV}$.

thus $\underline{RX}$ is the EV of $\underline{V}$.

Examples



$$X_1^{(2)} = X_1^{(1)}\cos\theta - X_2^{(1)}\sin\theta + 0\, X_3^{(1)}$$
$$X_2^{(2)} = X_1^{(1)}\sin\theta + X_2^{(1)}\cos\theta + 0\, X_3^{(1)}$$
$$X_3^{(2)} = 0\, X_1^{(1)} + 0\, X_2^{(1)} + X_3^{(1)}$$

$$\begin{bmatrix} \underline{x}^{(2)} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \end{bmatrix} \begin{bmatrix} \underline{x}^{(1)} \end{bmatrix}$$

$$= \uparrow F_{ij} = \frac{\partial x_i^{(2)}}{\partial x_j^{(1)}}$$

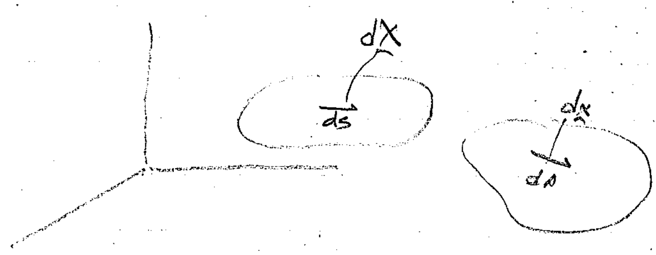
Axes rotated by angle θ (body fixed)

then $F_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ transpose of above

$\underset{\sim}{F}$ is orthog $\quad \det(F) \neq 0 \quad$ since $\det(F) = \dfrac{dV}{dV_0} = \dfrac{\rho}{\rho}$

Jacobian $\quad F$ never singular $\quad > 0$

always an invertible matrix



$F^T = F^{-1}$

$F^T F = I \quad d\underset{\sim}{x} = F\, dX$

$d\underset{\sim}{x}^T d\underset{\sim}{x} = dA^2 = dX^T \bar{F}^T F \bar{d}X$ generally

for our case $\quad = dX^T dX = dS^2$ true.

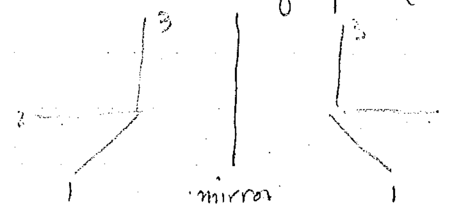
Thus Rigid body motion $F^T F = I \quad \det(F^T F) = 1 = \det(F^T)\det(F)$

$= (\det F)^2$

Rotation $\det F = 1$ (Proper Orthog Group). $\quad \Rightarrow \det F = \pm 1$

These two results are called a Total Orthog Group

$\det F = -1$ is not a group (Improper Orthog Matrices) Rigid body rotation + a mirror reflection



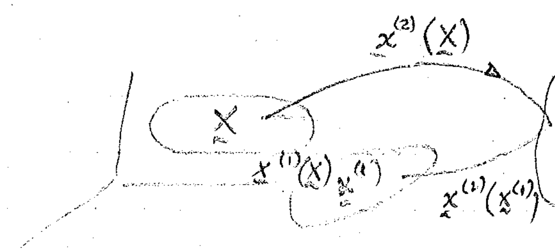
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

mirror reflection

Not possible deformation

Sequential Deformation

$\underset{\sim}{X} \to \underset{\sim}{x}^{(1)} \longrightarrow \underset{\sim}{x}^{(2)}$



$X \to x^{(2)} \quad F_{ij} = \dfrac{\partial x_i^{(2)}}{\partial X_j}$

$= \dfrac{\partial x_i^{(2)}}{\partial x_k^{(1)}} \dfrac{\partial x_k^{(1)}}{\partial X_j}$ by chain rule.

$\underset{\sim}{F} = \underset{\sim}{F}^{(2)}_{(1)} \underset{\sim}{F}^{(1)}$ thus must premultiply by each new deformati

Thus given a set of deformations $F_1, F_2, F_3, \ldots$

$\underset{\sim}{F} = \ldots\ F_3 \cdot F_2 \cdot F_1$

Principle of Objectivity – rigid body rotation should only change direction of stress not stress itself.

Since we assume single valued mapping $\underset{\sim}{k}$ then $\underset{\sim}{k}^{-1}(\underline{X}) = X$

then

$$\underset{\sim}{x} = \underset{\sim}{\chi}\left(\underset{\sim}{k}^{-1}(\underline{X}), t\right) = \underset{\sim}{\chi}_{\underset{\sim}{k}}(\underline{X}, t) \Rightarrow \underset{\sim}{x} = \underset{\sim}{x}(\underline{X}, t)$$

To discuss deformation we can do it 4 ways

1. Material Description using X, t — no reference state needed; not useful for continuum. Useful for rigid body & particle dynamics – since have finite # of degrees of freedom

2. Referential Descript (Lagrangian special case) use $\underset{\sim}{X}, t$ where $\underset{\sim}{X} = \underset{\sim}{k}(X)$

For Lagrangian variables $\underset{\sim}{X} = \underset{\sim}{x}(X, 0)$. Useful in elasticity & solid mechanics in general where stresses are important in unstressed state. Or special config is important – past history is important