Spring 1980

ME 241C
THEORY OF SHELLS

I. Differential Geometry

A. Metric (First fundamental form) — intrinsic geometry

a. Base vectors, reciprocal base vectors

b. Gradient, tensor fields

B. Normal Curvature (Second fundamental form)

a. Conditions of Gauss and Mainardi-Codazzi

b. Divergence theorem

C. Normal coordinates for E_3

II. Shell Equations

A. Stress and equilibrium

B. Displacement and compatibility

C. Static-geometric analogy

D. Strain and constitutive relations

E. Special forms for linear theory

1. Equations of Novozhilov for lines-of-curvature coordinates

2. Shell of revolution with asymmetric loads

3. Shallow-shell equations

F. Nonlinear theories — "moderately-large" rotation

III. Perturbation Methods of Solution

A. "Smooth" loads on a "smooth" shell

1. Edge-effect bending

2. Membrane stresses

3. Inextensional bending

B. "Geometric optics" solution

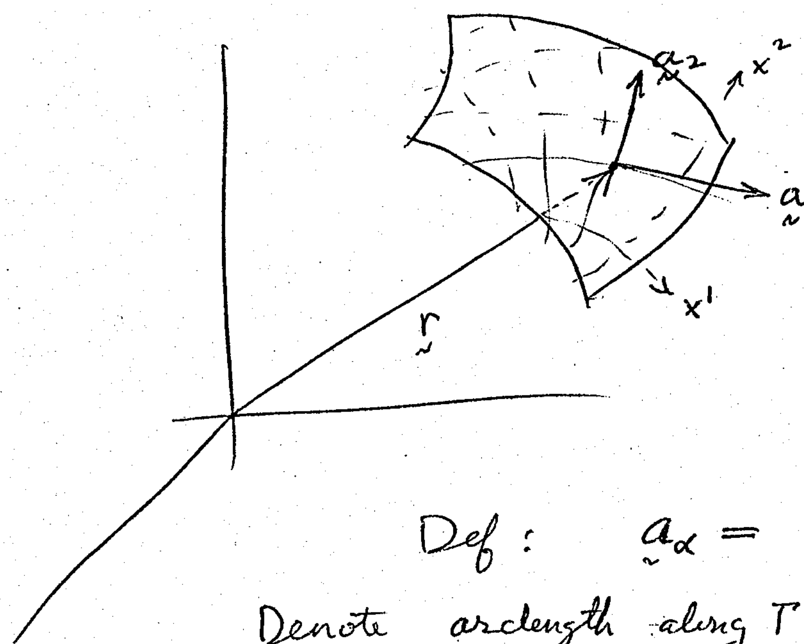
1. Variational solution of Whitham

2. Eikonal and transport equations

3. Wave propagation and stability

Geometry

We consider a surface in E_3 with curvilinear coordinates x^1, x^2 and position vector $\underset{\sim}{r} = \underset{\sim}{r}(x^1, x^2)$



For a curve Γ on the surface $x^1 = x^1(\lambda)$
$x^2 = x^2(\lambda)$

The tangent to the curve Γ is

$$\frac{d\underset{\sim}{r}}{d\lambda} = \sum_{\alpha=1,2} \frac{\partial \underset{\sim}{r}}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} = \frac{\partial \underset{\sim}{r}}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda}$$
$$= \underset{\sim}{a}_\alpha \frac{dx^\alpha}{d\lambda}$$

Def: $\underset{\sim}{a}_\alpha = \dfrac{\partial \underset{\sim}{r}}{\partial x^\alpha}$ $\alpha = 1, 2$ are <u>base vectors</u>.

Denote arclength along Γ by s. Then

$$\frac{d\underset{\sim}{r}}{d\lambda}\cdot\frac{d\underset{\sim}{r}}{d\lambda} = \left(\frac{ds}{d\lambda}\right)^2$$

That is;
$$ds^2 = (\underset{\sim}{a}_\alpha\, dx^\alpha)\cdot(\underset{\sim}{a}_\beta\, dx^\beta) = \underset{\sim}{a}_\alpha\cdot\underset{\sim}{a}_\beta\, dx^\alpha dx^\beta$$

$$\boxed{ds^2 = a_{\alpha\beta}\, dx^\alpha dx^\beta = I}$$

where I is the <u>First Fundamental Form</u> or <u>Metric</u>, and

$a_{\alpha\beta} = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta$ are the <u>Metric Coefficients</u>.

Unit normal to surface is

$$\underset{\sim}{a}_3 = \frac{\underset{\sim}{a}_1 \times \underset{\sim}{a}_2}{a}$$

where $a^{1/2} = |\underset{\sim}{a}_1 \times \underset{\sim}{a}_2| = |\underset{\sim}{a}_1|\cdot|\underset{\sim}{a}_2| \sin(\underset{\sim}{a}_1, \underset{\sim}{a}_2)$

But $a_{12} = \underset{\sim}{a}_1 \cdot \underset{\sim}{a}_2 = |\underset{\sim}{a}_1|\cdot|\underset{\sim}{a}_2| \cos(\underset{\sim}{a}_1, \underset{\sim}{a}_2) = (a_{11}a_{22})^{1/2}\cos(\underset{\sim}{a}_1, \underset{\sim}{a}_2)$

So $a^{1/2} = (a_{11}a_{22} - a_{12}^2)^{1/2}$

The differential area is

$$dA = \left| \frac{\partial \underset{\sim}{r}}{\partial x^1} \times \frac{\partial \underset{\sim}{r}}{\partial x^2} \right| dx^1 dx^2 = a^{1/2} dx^1 dx^2$$

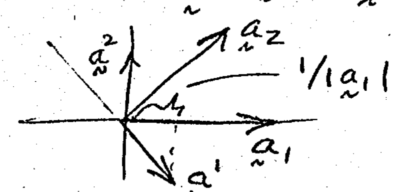
Thus the arclength of curves on the surface, the angle between curves, and the area on the surface all depend on the $a_{\alpha\beta}$. Any property of the surface dependent only on $a_{\alpha\beta}$ is an <u>intrinsic</u> property.

Any tangent vector $\underset{\sim}{V}$ may be written as a linear combination of base vectors $\underset{\sim}{V} = V^\alpha \underset{\sim}{a}_\alpha$.
To compute the coefficients V^α, take the dot product with a vector $\underset{\sim}{u}$:

$$\underset{\sim}{V} \cdot \underset{\sim}{u} = V^1 \underset{\sim}{a}_1 \cdot \underset{\sim}{u} + V^2 \underset{\sim}{a}_2 \cdot \underset{\sim}{u}$$

So choose $\underset{\sim}{u}$ such that

$$\underset{\sim}{u} \cdot \underset{\sim}{a}_1 = 1$$
$$\underset{\sim}{u} \cdot \underset{\sim}{a}_2 = 0$$



then $V^1 = \underset{\sim}{u} \cdot \underset{\sim}{V}$. This vector $\underset{\sim}{u}$ is denoted by $\underset{\sim}{a}^1$.
If we choose $\underset{\sim}{u}$ such that

$$\underset{\sim}{u} \cdot \underset{\sim}{a}_1 = 0$$
$$\underset{\sim}{u} \cdot \underset{\sim}{a}_2 = 1$$

then $V^2 = \underset{\sim}{V} \cdot \underset{\sim}{u}$. Denote this $\underset{\sim}{u}$ by $\underset{\sim}{a}^2$.
With these <u>reciprocal</u> <u>base</u> <u>vectors</u> $\underset{\sim}{a}^1$ and $\underset{\sim}{a}^2$, we have

$$\underset{\sim}{V} = V^\alpha \underset{\sim}{a}_\alpha = (\underset{\sim}{V} \cdot \underset{\sim}{a}^\alpha)\, \underset{\sim}{a}_\alpha$$

or we may expand in terms of the reciprocal base vectors

$$\underset{\sim}{V} = V_\alpha \underset{\sim}{a}^\alpha = (\underset{\sim}{V} \cdot \underset{\sim}{a}_\alpha)\, \underset{\sim}{a}^\alpha$$

The requirement for the reciprocal base vectors may be written as $\underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}_\beta = \delta^\alpha_\beta = \begin{Bmatrix} 1 & \text{when } \alpha = \beta \\ 0 & \text{when } \alpha \neq \beta \end{Bmatrix}$

Formulas of Frenet for space curve

$$\underset{\sim}{t} = \frac{d\underset{\sim}{r}}{ds}$$

$$\left\{\begin{array}{ll} \dfrac{d\underset{\sim}{t}}{ds} = \kappa\,\underset{\sim}{n} & (\underset{\sim}{n} = \text{unit normal}) \\[2ex] \dfrac{d\underset{\sim}{n}}{ds} = -\kappa\,\underset{\sim}{t} + \tau\,\underset{\sim}{b} & \underset{\sim}{b} = \text{Binormal} \\ & \quad = \underset{\sim}{t} \times \underset{\sim}{n} \\[1ex] \dfrac{d\underset{\sim}{b}}{ds} = -\tau\,\underset{\sim}{n} & \tau = \text{Torsion} \end{array}\right.$$

The curve Γ on the surface has the unit tangent vector

$$\frac{d\underset{\sim}{r}}{ds} = \frac{d\underset{\sim}{r}}{d\lambda}\frac{d\lambda}{ds}$$

and the <u>curvature</u> vector

$$\underset{\sim}{K} = \frac{d}{ds}\left(\frac{d\underset{\sim}{r}}{ds}\right)$$

which is split into the component tangent to the surface – the <u>geodesic</u> curvature – and the component normal to the surface – the <u>normal</u> curvature.

$$\underset{\sim}{K} = \underset{\sim}{K}_g + (\underset{\sim}{K}\cdot\underset{\sim}{a}_3)\,\underset{\sim}{a}_3$$

$$\underset{\sim}{K}\cdot\underset{\sim}{a}_3 = \underset{\sim}{a}_3\cdot\frac{d}{ds}\frac{d\underset{\sim}{r}}{ds} = \underset{\sim}{a}_3\cdot\frac{\partial}{\partial x^\beta}\left(\underset{\sim}{a}_\alpha\frac{dx^\alpha}{ds}\right)\frac{dx^\beta}{ds}$$

$$= \underset{\sim}{a}_3\cdot\frac{\partial\underset{\sim}{a}_\alpha}{\partial x^\beta}\,\frac{dx^\alpha}{ds}\,\frac{dx^\beta}{ds}$$

which is the <u>Second Fundamental Form</u>

$$\boxed{\mathrm{II} = \underset{\sim}{K}\cdot\underset{\sim}{a}_3\,ds^2 = b_{\alpha\beta}\,dx^\alpha dx^\beta}$$

where $b_{\alpha\beta} = \underset{\sim}{a}_3\cdot\dfrac{\partial\underset{\sim}{a}_\alpha}{\partial x^\beta} = b_{\beta\alpha}$

Consider now a function $\varphi = \varphi(x^1, x^2)$. The derivative along the curve Γ is

$$\frac{d\varphi}{d\lambda} = \frac{\partial\varphi}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} = \left(\underset{\sim}{a}_\alpha\frac{dx^\alpha}{d\lambda}\cdot\underset{\sim}{a}^\beta\right)\frac{\partial\varphi}{\partial x^\beta}$$

$$= \frac{d\underset{\sim}{r}}{d\lambda}\cdot\left(\underset{\sim}{a}^\beta\frac{\partial\varphi}{\partial x^\beta}\right) = \frac{d\underset{\sim}{r}}{d\lambda}\cdot\nabla\varphi$$

where $\nabla\varphi$ – the <u>gradient</u> of φ – is a vector field tangent to the surface, dependent only on the scalar function $\varphi(x^1, x^2)$ and the surface, and independent of the curve Γ as well as the choice for the coordinate system.

Try this for a vector field $\underset{\sim}{V}$

$$\underset{\sim}{V} = V^\alpha \underset{\sim}{a}_\alpha + V^3 \underset{\sim}{a}_3$$

The derivative along Γ is

$$\frac{d\underset{\sim}{V}}{d\lambda} = \frac{\partial x^\alpha}{\partial \lambda}\frac{\partial \underset{\sim}{V}}{\partial x^\alpha} = \left(\underset{\sim}{a}_\alpha \frac{\partial x^\alpha}{\partial \lambda} \cdot \underset{\sim}{a}^\beta\right)\frac{\partial \underset{\sim}{V}}{\partial x^\beta}$$

$$= \frac{d\underset{\sim}{r}}{d\lambda} \cdot \underset{\sim}{a}^\beta \frac{\partial \underset{\sim}{V}}{\partial x^\beta} = \frac{d\underset{\sim}{r}}{d\lambda} \cdot \nabla \underset{\sim}{V}$$

So the gradient of a vector field is, like $\nabla\varphi$, dependent only on $\underset{\sim}{V}$ and the surface, but is a <u>tensor</u> quantity. A tensor of rank 2 (a dyad) is a bilinear function of two vector spaces which, when dotted with a vector, gives a vector

$$\underset{=}{T} = \underset{\sim}{u} \otimes \underset{\sim}{v}$$

$$\underset{=}{T} \cdot \underset{\sim}{w} = \underset{\sim}{u}\,(\underset{\sim}{v} \cdot \underset{\sim}{w})$$

$$\underset{\sim}{w} \cdot \underset{=}{T} = (\underset{\sim}{w} \cdot \underset{\sim}{u})\,\underset{\sim}{v}$$

Alternate forms

$$\underset{=}{T} = u^\alpha \underset{\sim}{a}_\alpha \otimes v^\beta \underset{\sim}{a}_\beta = u^\alpha v^\beta \underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta = T^{\alpha\beta} \underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta$$

$$= u_\alpha \underset{\sim}{a}^\alpha \otimes v_\beta \underset{\sim}{a}^\beta = u_\alpha v_\beta \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta = T_{\alpha\beta} \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta$$

$$= u_\alpha \underset{\sim}{a}^\alpha \otimes v^\beta \underset{\sim}{a}_\beta = u_\alpha v^\beta \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}_\beta = T_\alpha{}^\beta \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}_\beta$$

$$= u^\alpha \underset{\sim}{a}_\alpha \otimes v_\beta \underset{\sim}{a}^\beta = u^\alpha v_\beta \underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}^\beta = T^\alpha{}_\beta \underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}^\beta$$

A tensor of rank 3 would be

$$\underset{=}{T} = \underset{\sim}{u} \otimes \underset{\sim}{v} \otimes \underset{\sim}{w}$$

ect.

Special Tensors – From $II = b_{\alpha\beta}\, dx^\alpha dx^\beta$ we may rewrite

$$II = b_{\alpha\beta}\left(\underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}_\nu\, dx^\nu\right)\left(\underset{\sim}{a}^\beta \cdot \underset{\sim}{a}_\gamma\, dx^\gamma\right)$$

$$= d\underset{\sim}{r} \cdot \left(b_{\alpha\beta}\, \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta \right) \cdot d\underset{\sim}{r}$$

So the $b_{\alpha\beta}$ are components of a tensor

$$\underset{\approx}{b} = b_{\alpha\beta}\, \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta$$

For first fundamental form

$$I = a_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$= d\underset{\sim}{r} \cdot \left(a_{\alpha\beta}\, \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta \right) \cdot d\underset{\sim}{r}$$

but $I = d\underset{\sim}{r} \cdot d\underset{\sim}{r}$, so $\underset{\approx}{\delta} = a_{\alpha\beta}\, \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta$ is the identity dyad.

$$\underset{\approx}{\delta} = a_{\alpha\beta}\, \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta = \delta_\alpha^{\ \beta}\, \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}_\beta = \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}_\alpha$$

$$= a^{\alpha\beta}\, \underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta$$

Also useful is a tensor which gives a $90°$ rotation of a tangent vector field

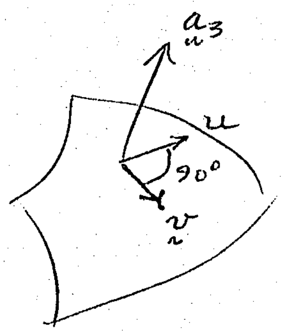
$$\underset{\approx}{\epsilon} = \epsilon_{\alpha\beta}\, \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta$$

$$\epsilon_{\alpha\beta} = \begin{cases} a^{1/2} & \text{if } \alpha = 1,\ \beta = 2 \\ -a^{1/2} & \text{if } \alpha = 2,\ \alpha = 1 \\ 0 & \text{if } \alpha = \beta \end{cases}$$

For $\underset{\sim}{v} = v^\alpha \underset{\sim}{a}_\alpha$

$$\underset{\sim}{v} \cdot \underset{\approx}{\epsilon} = \underset{\sim}{u} \qquad \underset{\sim}{u} \cdot \underset{\sim}{v} = 0 \quad |\underset{\sim}{u}| = |\underset{\sim}{v}|$$

$$\underset{\approx}{\epsilon} \cdot \underset{\sim}{v} = -\underset{\sim}{u}$$



$$\mathbf{a}_1 \times \mathbf{a}_2 = a^{1/2}\,\mathbf{a}_3 \qquad a = a_{11}a_{22} - a_{12}^2$$

$$\mathbf{a}_2 \times \mathbf{a}_1 = -a^{1/2}\,\mathbf{a}_3$$

$$\mathbf{a}_\alpha \times \mathbf{a}_\alpha = 0$$

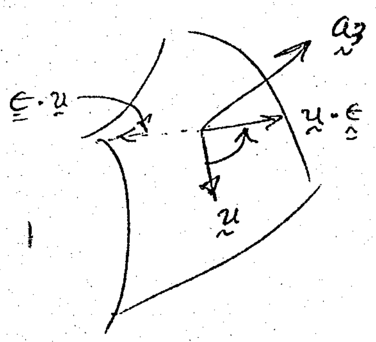
$$\mathbf{u} \times \mathbf{v} = \left(u^\alpha \mathbf{a}_\alpha \times v^\beta \mathbf{a}_\beta \right)$$

$$= \left(u^1 v^2\, \mathbf{a}_1 \times \mathbf{a}_2 - u^2 v^1\, \mathbf{a}_1 \times \mathbf{a}_2 \right)$$

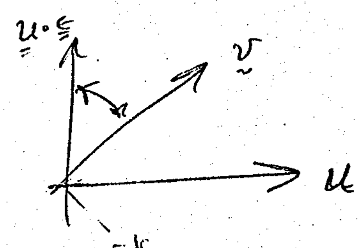
$$= \left(u^1 v^2 - u^2 v^1 \right) a^{1/2}\, \mathbf{a}_3$$

$$|\mathbf{u} \times \mathbf{v}| = \mathbf{u} \cdot \left[a^{1/2} \left(\mathbf{a}^1 \otimes \mathbf{a}^2 - \mathbf{a}^2 \otimes \mathbf{a}^1 \right) \right] \cdot \mathbf{v}$$

$$\boldsymbol{\epsilon} = a^{1/2} \left(\mathbf{a}^1 \otimes \mathbf{a}^2 - \mathbf{a}^2 \otimes \mathbf{a}^1 \right)$$



$\mathbf{u} \circ \boldsymbol{\epsilon}$



$$\mathbf{u} \times \mathbf{v} = uv \sin\theta$$

$$= uv \cos\left(\tfrac{\pi}{2} - \theta\right)$$

$$= (\mathbf{u}\cdot\boldsymbol{\epsilon}) \cdot \mathbf{v}$$

For the tensor $\nabla \mathbf{V}$ where $\mathbf{V}$ is a tangent vector field,

$$\nabla \mathbf{V} = \mathbf{a}^\alpha \otimes \frac{\partial \mathbf{V}}{\partial x^\alpha} = \mathbf{a}^\alpha \otimes \frac{\partial}{\partial x^\alpha}\left(V^\beta \mathbf{a}_\beta\right)$$

$$= \mathbf{a}^\alpha \otimes \left(\frac{\partial V^\beta}{\partial x^\alpha}\mathbf{a}_\beta + V^\beta \frac{\partial \mathbf{a}_\beta}{\partial x^\alpha} \right)$$

$$= \mathbf{a}^\alpha \otimes \left\{ \frac{\partial V^\beta}{\partial x^\alpha}\mathbf{a}_\beta + V^\nu \left(\left(\frac{\partial \mathbf{a}_\nu}{\partial x^\alpha} \cdot \mathbf{a}^\beta\right)\mathbf{a}_\beta + \left(\frac{\partial \mathbf{a}_\nu}{\partial x^\alpha} \cdot \mathbf{a}^3\right)\mathbf{a}_3 \right) \right\}$$

$$= \left(\frac{\partial V^\beta}{\partial x^\alpha} + V^\nu \frac{\partial \mathbf{a}_\nu}{\partial x^\alpha} \cdot \mathbf{a}^\beta \right) \mathbf{a}^\alpha \otimes \mathbf{a}_\beta + V^\nu b_{\nu\alpha}\, \mathbf{a}^\alpha \otimes \mathbf{a}_3$$

$$= V^\beta|_\alpha\, \mathbf{a}^\alpha \otimes \mathbf{a}_\beta + \mathbf{V} \cdot \mathbf{b} \otimes \mathbf{a}_3$$

where $V^\beta|_\alpha$ is the <u>covariant derivative</u>, and the coefficients

$$\left\{ {\beta \atop \nu\alpha} \right\} = \mathbf{a}^\beta \cdot \frac{\partial \mathbf{a}_\nu}{\partial x^\alpha}$$

are the <u>Christoffel symbols</u>. The covariant derivatives are components of a tensor which is the intrinsic part of $\nabla \mathbf{V}$

$$\nabla \mathbf{V} \cdot \boldsymbol{\delta} = V^\beta|_\alpha\, \mathbf{a}^\alpha \otimes \mathbf{a}_\beta$$

For a normal vector field

$$\mathbf{V} = V^3 \mathbf{a}_3$$

$$\nabla \mathbf{V} = \nabla V^3 \otimes \mathbf{a}_3 + V^3 \nabla \mathbf{a}_3$$

$$\nabla \mathbf{a}_3 = \mathbf{a}^\alpha \otimes \frac{\partial \mathbf{a}_3}{\partial x^\alpha} \qquad \partial a\cdot$$

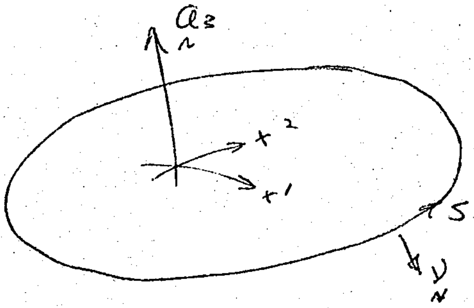
$$= \mathbf{a}^\alpha \otimes \left(\frac{\partial \mathbf{a}_3}{\partial x^\alpha} \cdot \mathbf{a}_\beta \right) \mathbf{a}^\beta = -\, \mathbf{a}^\alpha \otimes \mathbf{a}^\beta b_{\alpha\beta}$$

$$= -\,\mathbf{b}$$

So for general vector field $\mathbf{V} = V^\alpha \mathbf{a}_\alpha + V^3 \mathbf{a}_3$

$$\nabla \mathbf{V} = V^\beta|_\alpha\, \mathbf{a}^\alpha \otimes \mathbf{a}_\beta - V^3 \mathbf{b} + \left(\mathbf{V} \cdot \mathbf{b} + \nabla V^3 \right) \otimes \mathbf{a}_3$$

Divergence Theorem



For a vector u tangent to a surface Σ the divergence theorem is

$$\boxed{\iint_\Sigma \nabla\cdot u \, d\Sigma = \oint_\Gamma u\cdot\nu \, ds}$$

where ν is a unit tangent vector normal to the boundary curve Γ, which is transversed in the right-hand sense with respect to the normal $a_3 = (a_1 \times a_2)\, a^{-1/2}$

This is proven by integration by parts

$$\nabla\cdot u = a^1\cdot\frac{\partial u}{\partial x^1} + a^2\cdot\frac{\partial u}{\partial x^2} \qquad d\Sigma = a^{1/2}\,dx^1 dx^2$$

But

$$\epsilon\cdot a_2 = (\epsilon_{\mu\nu}\, a^\mu a^\nu)\cdot a_2 = \epsilon_{12}\, a^1 = a^{1/2} a^1$$

$$\epsilon\cdot a_1 = -\, a^{1/2} a^2$$

So

$$a^{1/2}\,\nabla\cdot u = \frac{\partial u}{\partial x^1}\cdot\epsilon\cdot a_2 - \frac{\partial u}{\partial x^2}\cdot\epsilon\cdot a_1$$

$$= \frac{\partial}{\partial x^1}(u\cdot\epsilon\cdot a_2) - \frac{\partial}{\partial x^2}(u\cdot\epsilon\cdot a_1)$$

$$- u\cdot\left[\frac{\partial}{\partial x^1}(\epsilon\cdot a_2) - \frac{\partial}{\partial x^2}(\epsilon\cdot a_1)\right]$$

the last term of which is zero since $\dfrac{\partial a_2}{\partial x^1} - \dfrac{\partial a_1}{\partial x^2} \equiv 0$

Thus

$$\iint \nabla\cdot u \, d\Sigma = \int_{c_1}^{c_2} u\cdot\epsilon\cdot a_2 \Big|_{f_1(x^2)}^{f_2(x^2)} dx^2 - \int_{c_3}^{c_4} u\cdot\epsilon\cdot a_1 \Big|_{f_3(x^1)}^{f_4(x^1)} dx^1$$

$$= \oint_\Gamma u\cdot\epsilon\cdot(a_2\,dx^2 + a_1\,dx^1)$$

$$= \oint_\Gamma u\cdot\epsilon\cdot dr = \oint_\Gamma u\cdot\nu\, ds$$

Furthermore, if we let $\underset{\sim}{u} = \underset{\approx}{\epsilon} \cdot \underset{\sim}{v}$, then

$$\iint_{\Sigma} \nabla \cdot (\underset{\approx}{\epsilon} \cdot \underset{\sim}{v}) \, d\Sigma = \oint \underset{\sim}{\nu} \cdot \underset{\approx}{\epsilon} \cdot \underset{\sim}{v} \, ds$$

$$= \oint d\underset{\sim}{r} \cdot \underset{\sim}{v}$$

which is Stokes theorem, since

$$\nabla \cdot (\underset{\approx}{\epsilon} \cdot \underset{\sim}{v}) = \underset{\sim}{a}^{\alpha} \cdot \frac{\partial}{\partial x^{\alpha}} (\underset{\approx}{\epsilon} \cdot \underset{\sim}{v})$$

$$\underset{\approx}{\epsilon} \cdot \underset{\sim}{v} = c$$

$$= \underset{\sim}{a}^{\alpha} \cdot \underset{\approx}{\epsilon} \cdot \frac{\partial}{\partial x^{\alpha}} (\underset{\sim}{v})$$

$$= \epsilon^{\alpha\beta} \underset{\sim}{a}_{\beta} \cdot \left[v^{\mu}|_{\alpha} \, \underset{\sim}{a}_{\mu} \right]$$

$$= \epsilon^{\alpha\beta} v_{\beta|\alpha}$$

$$= a^{1/2} \left[\frac{\partial v_2}{\partial x^1} - \frac{\partial v_1}{\partial x^2} \right]$$

$$= \underset{\sim}{a}_3 \cdot \operatorname{curl} \underset{\sim}{v}$$

For φ a scalar $\quad \oint d\varphi = 0 \quad \oint d\mathbf{r} \cdot \nabla\varphi = 0$

$\underset{\sim}{v} = \nabla\varphi$

$$\iint_{\Sigma} \nabla \cdot (\underset{=}{\epsilon} \cdot \nabla\varphi) \, d\Sigma = 0$$

Thus $\nabla \cdot (\underset{=}{\epsilon} \cdot \nabla\varphi) \equiv 0$

i.e. $\underset{\sim}{a}_3 \cdot \operatorname{curl} \nabla\varphi \equiv 0$

Now let $\underset{\sim}{v} = \underset{\approx}{T} \cdot \underset{\sim}{\lambda}$ where $\underset{\sim}{\lambda}$ is an arbitrary constant vector. Then Stokes' theorem gives

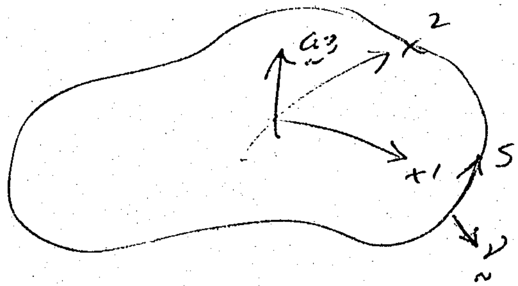
$$\iint_{\Sigma} \nabla \cdot \left(\underset{\approx}{\epsilon} \cdot \underset{\approx}{T} \cdot \underset{\sim}{\lambda} \right) d\Sigma = \oint d\underset{\sim}{r} \cdot \underset{\approx}{T} \cdot \underset{\sim}{\lambda}$$

Since $\underset{\sim}{\lambda}$ is constant and arbitrary,

$$\iint_{\Sigma} \nabla \cdot \left(\underset{\approx}{\epsilon} \cdot \underset{\approx}{T} \right) d\Sigma = \oint d\underset{\sim}{r} \cdot \underset{\approx}{T}$$

Divergence Theorem

For vector $\underset{\sim}{u}$ tangent to surface



$$\iint \nabla\cdot\underset{\sim}{u}\, d\Sigma = \oint_{\Gamma} \underset{\sim}{u}\cdot\underset{\sim}{\nu}\, ds$$

$$\underset{\sim}{\nu}\, ds = \underset{\approx}{\epsilon}\cdot d\underset{\sim}{r}$$

$$d\Sigma = a^{1/2}\, dx^1 dx^2$$

For $\underset{\sim}{u} = \underset{\approx}{\epsilon}\cdot\underset{\sim}{v}$, we obtain Stokes th.

$$\iint \nabla\cdot(\underset{\approx}{\epsilon}\cdot\underset{\sim}{v})\, d\Sigma = \oint -\underset{\sim}{v}\cdot\underset{\approx}{\epsilon}\cdot\underset{\approx}{\epsilon}\cdot d\underset{\sim}{r}$$

$$= +\oint \underset{\sim}{v}\cdot d\underset{\sim}{r}$$

$$\nabla\cdot(\underset{\approx}{\epsilon}\cdot\underset{\sim}{v}) = a^{-1/2}\left[\frac{\partial v_2}{\partial x^1} - \frac{\partial v_1}{\partial x^2}\right] = \underset{\sim}{a}_3\cdot \operatorname{curl}\underset{\sim}{v}$$

$$\oint \underset{\sim}{v}\cdot d\underset{\sim}{r} = \text{circulation}$$

For $\underset{\sim}{u} = \underset{\approx}{T}\cdot\underset{\sim}{\lambda}$ where $\underset{\sim}{\lambda}$ is a constant (3-D) vector

we obtain
$$\iint \nabla\cdot\underset{\approx}{T}\, d\Sigma = -\oint_{\Gamma} d\underset{\sim}{r}\cdot\underset{\approx}{\epsilon}\cdot\underset{\approx}{T}$$

Similarly $\underset{\sim}{v} = \underset{\approx}{T}\cdot\underset{\sim}{\lambda}$ gives

$$\iint \nabla\cdot(\underset{\approx}{\epsilon}\cdot\underset{\approx}{T})\, d\Sigma = \oint_{\Gamma} d\underset{\sim}{r}\cdot T$$

Now let $\underset{\approx}{T} = \nabla \underset{\sim}{u}$, where $\underset{\sim}{u} = u^\alpha \underset{\sim}{a}_\alpha$

$$\oint d\underset{\sim}{u} = 0 \quad \text{for any closed curve}$$

$$\oint d\underset{\sim}{r} \cdot \nabla \underset{\sim}{u} = 0$$

Therefore
$$\iint_\Sigma \nabla \cdot (\underset{=}{\epsilon} \cdot \nabla \underset{\sim}{u})\, d\Sigma = 0$$

Since Σ is arbitrary
$$\nabla \cdot (\underset{=}{\epsilon} \cdot \nabla \underset{\sim}{u}) \equiv 0$$

$$\nabla \cdot (\underset{=}{\epsilon} \cdot \nabla \underset{\sim}{u}) = \epsilon^{\alpha\beta} v_{\gamma|\beta\alpha}\, \underset{\sim}{a}^\gamma + \underset{\sim}{v} \cdot \underset{=}{b} \cdot \underset{=}{\epsilon} \cdot \underset{=}{b}$$
$$+ \epsilon^{\alpha\beta} \left(v_{\gamma|\beta} b^\gamma_\alpha - (v_\gamma b^\gamma_{\cancel{\beta}\alpha})|_\beta \right) \underset{\sim}{a}_3 = 0$$

Therefore
$$\boxed{\epsilon^{\alpha\beta} b_{\gamma\alpha|\beta} = 0} \quad \text{(Mainardi–Codazzi)}$$

For intrinsic part
$$\underset{=}{b} \cdot \underset{=}{\epsilon} \cdot \underset{=}{b} = b_\alpha^{\cancel{\alpha}\mu} \epsilon_{\mu\nu} b^{\nu\cancel{\beta}}_{\ \beta}\, \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta$$
$$= \epsilon_{\alpha\beta} K\, \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta = K \underset{\approx}{\epsilon}$$

Thus
$$-\epsilon^{\alpha\beta}\, \underset{\sim}{v}|_{\alpha\beta} + K\, \underset{\sim}{v} \cdot \underset{=}{\epsilon} = 0$$

So covariant derivatives of a vector can be interchanged only if $K \equiv 0$.

Choose now for the vector $\underset{\sim}{v}$ the base vector $\underset{\sim}{a}_\mu$ (μ fixed)

$$\underset{\sim}{v} = \underset{\sim}{a}_\mu$$

Then
$$v^\alpha|_\beta = \frac{\partial}{\partial x^\beta}(\underset{\sim}{a}_\mu \cdot \underset{\sim}{a}^\alpha) + (\underset{\sim}{a}_\mu \cdot \underset{\sim}{a}^\lambda)\left\{ {\alpha \atop \lambda\beta} \right\}$$
$$= \left\{ {\alpha \atop \mu\beta} \right\}$$

$$v^\alpha|_{\beta\gamma} = \frac{\partial}{\partial x^\gamma}\left\{ {\alpha \atop \mu\beta} \right\} + \left\{ {\lambda \atop \mu\beta} \right\}\left\{ {\alpha \atop \lambda\gamma} \right\} - \left\{ {\alpha \atop \mu\lambda} \right\}\left\{ {\lambda \atop \beta\gamma} \right\}$$

$$v_\nu \epsilon^{\nu\alpha} K = \epsilon^{\beta\gamma} v^\alpha|_{\beta\gamma} = \epsilon^{\beta\gamma}\left[\frac{\partial}{\partial x^\gamma}\left\{ {\alpha \atop \mu\beta} \right\} + \left\{ {\lambda \atop \mu\beta} \right\}\left\{ {\alpha \atop \lambda\gamma} \right\}\right]$$

Choose $\alpha = 2 \quad \mu = 1$

$$\frac{(a_{11})}{a^{1/2}} K = \epsilon^{\beta\gamma}\left[\frac{\partial}{\partial x^\gamma}\left\{ {2 \atop 1\beta} \right\} + \left\{ {\lambda \atop 1\beta} \right\}\left\{ {2 \atop \lambda\gamma} \right\}\right]$$

Thus the Gaussian curvature

$$K = b_1^1 b_2^2 - b_1^2 b_2^1$$

depends on only the metric coefficients and hence is an <u>intrinsic property</u> of the surface.

For homework, show that for orthogonal coordinates

$$ds^2 = A_1^2 (dx^1)^2 + A_2^2 (dx^2)^2$$

we have

$$K = -\frac{1}{A_1 A_2}\left[\frac{\partial}{\partial x^1}\left(\frac{\partial A_2}{A_1 \partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{\partial A_1}{A_2 \partial x^2}\right)\right]$$

Fundamental Theorem of Surface Theory —

If $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are given real functions of the real variables x^1 and x^2 (for which $a_{11}a_{22}-a_{12}^2>0$) which satisfy the condition of Gauss

$$K = b_1^1 b_2^2 - b_1^2 b_2^1 = \frac{a^{1/2}}{a_{11}} \epsilon^{\beta\gamma}\left[\frac{\partial}{\partial x^\gamma}\left\{{2 \atop 1\beta}\right\} + \left\{{\lambda \atop 1\beta}\right\}\left\{{2 \atop \lambda\gamma}\right\}\right]$$

and the conditions of Mainardi - Codazzi

$$\epsilon^{\gamma\beta} b_{\alpha\gamma|\beta} = 0$$

then there exist a real surface which has the fundamental forms

$$I = a_{\alpha\beta}\, dx^\alpha dx^\beta$$
$$II = b_{\alpha\beta}\, dx^\alpha dx^\beta$$

This surface is uniquely determined, except for an arbitrary rigid body displacement.

The curve Γ on the surface has the unit tangent vector

$$\frac{d\underset{\sim}{r}}{ds} = \frac{d\underset{\sim}{r}}{d\lambda}\frac{d\lambda}{ds}$$

and the <u>curvature</u> vector

$$\underset{\sim}{\kappa} = \frac{d}{ds}\left(\frac{d\underset{\sim}{r}}{ds}\right)$$

which is split into the component tangent to the surface - the <u>geodesic</u> curvature - and the component normal to the surface - the <u>normal</u> curvature.

$$\underset{\sim}{\kappa} = \underset{\sim}{\kappa}_g + (\underset{\sim}{\kappa}\cdot\underset{\sim}{a}_3)\,\underset{\sim}{a}_3$$

$$\underset{\sim}{\kappa}\cdot\underset{\sim}{a}_3 = \underset{\sim}{a}_3\cdot\frac{d}{ds}\frac{d\underset{\sim}{r}}{ds} = \underset{\sim}{a}_3\cdot\frac{\partial}{\partial x^\beta}\left(\underset{\sim}{a}_\alpha\frac{dx^\alpha}{ds}\right)\frac{dx^\beta}{ds}$$

$$= \underset{\sim}{a}_3\cdot\frac{\partial\underset{\sim}{a}_\alpha}{\partial x^\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}$$

which is the <u>Second Fundamental Form</u>

$$\boxed{\mathrm{II} = \underset{\sim}{\kappa}\cdot\underset{\sim}{a}_3\,ds^2 = b_{\alpha\beta}\,dx^\alpha dx^\beta}$$

where $\displaystyle b_{\alpha\beta} = \underset{\sim}{a}_3\cdot\frac{\partial\underset{\sim}{a}_\alpha}{\partial x^\beta} = b_{\beta\alpha}$

Consider now a function $\varphi = \varphi(x^1, x^2)$. The derivative along the curve Γ is

$$\frac{d\varphi}{d\lambda} = \frac{\partial\varphi}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} = \left(\underset{\sim}{a}_\alpha\frac{dx^\alpha}{d\lambda}\cdot\underset{\sim}{a}^\beta\right)\frac{\partial\varphi}{\partial x^\beta}$$

$$= \frac{d\underset{\sim}{r}}{d\lambda}\cdot\left(\underset{\sim}{a}^\beta\frac{\partial\varphi}{\partial x^\beta}\right) = \frac{d\underset{\sim}{r}}{d\lambda}\cdot\nabla\varphi$$

where $\nabla\varphi$ - the <u>gradient</u> of φ - is a vector field tangent to the surface, dependent only on the scalar function $\varphi(x^1, x^2)$ and the surface, and independent of the curve Γ as well as the choice for the coordinate system.

Spring 1980

ME 241C

THEORY OF SHELLS

Assignment 1

Due April 15, 1980

PROBLEM: A surface is defined by $z = z(x,y)$ where x,y,z are the usual cartesian coordinates in 3-D, for example the paraboloid

$$z = \tfrac{1}{2}\left(\frac{x^2}{R_1} + \frac{y^2}{R_2}\right)$$

where R_1 and R_2 are positive constants. If x and y are used for the curvilinear coordinates on the surface

$$x = x^1$$
$$y = x^2$$

then determine formulas for the base vectors $\underset{\sim}{a}_\alpha$, reciprocal base vectors $\underset{\sim}{a}^\alpha$, metric coefficients $a_{\alpha\beta}$, reciprocal coefficients $a^{\alpha\beta}$, determinant of the metric coefficients a, and unit normal to the surface $\underset{\sim}{a}_3$.

Now consider the curve on the surface defined by $x = y$, or in parametric form, $x = \lambda$, $y = \lambda$. Find the formulas for computing the arc length of this curve, the (total) curvature vector, the geodesic curvature, and normal curvature.

ME 241C
THEORY OF SHELLS
Assignment 2

Spring
1980

PROBLEM 1 - Show that for a surface with the metric

$$ds^2 = (A_1 dx^1)^2 + (A_2\, dx^2)^2$$

that the Christoffel symbols are

$$\left\{ {1 \atop 11} \right\} = \frac{\partial A_1}{A_1 \partial x^1}$$

$$\left\{ {1 \atop 12} \right\} = \frac{\partial A_1}{A_1 \partial x^2}$$

$$\left\{ {2 \atop 12} \right\} = \frac{\partial A_2}{A_2 \partial x^1}$$

$$\left\{ {2 \atop 11} \right\} = -\frac{A_1 \partial A_1}{A_2^2\, \partial x^2}$$

$$\left\{ {1 \atop 22} \right\} = -\frac{A_2 \partial A_2}{A_1^2\, \partial x^1}$$

$$\left\{ {2 \atop 22} \right\} = \frac{\partial A_2}{A_2 \partial x^2}$$

PROBLEM 2 - Show that the metric and curvature tensors for a surface are

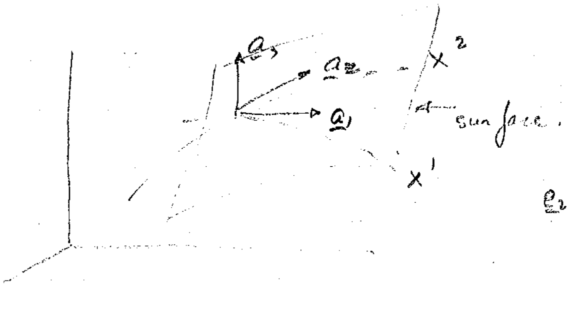
$$\underset{\approx}{\delta} = \nabla \underset{\sim}{r}$$

$$\underset{\approx}{b} = (\nabla\nabla \underset{\sim}{r}) \cdot \underset{\sim}{a}_3$$

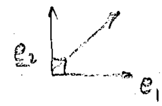
where $\underset{\sim}{r}$ is the (3-D) position vector and $\underset{\sim}{a}_3$ the unit normal to the surface.

PROBLEM 3 - How many geodesics (i.e. lines of zero geodesic curvature) connect two points on the generator of a cylinder? Sketch a few.

3 Apr 80



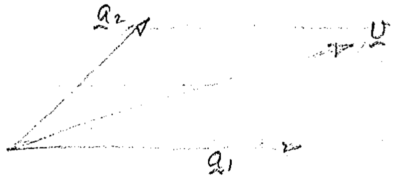
$\underset{\sim}{a}_1 = \dfrac{\partial \underset{\sim}{r}}{\partial x^1}$



$\underset{\sim}{a}_i$ are $\perp$ unit vectors $\underset{\sim}{a}_1, \underset{\sim}{a}_2$ are in surface.

$\underset{\sim}{v} = v_1 \underset{\sim}{e}_1 + v_2 \underset{\sim}{e}_2$

$\underset{\sim}{e}_1 \cdot \underset{\sim}{v} = v_1$



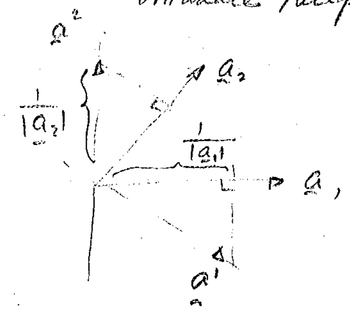
$\underset{\sim}{v} = v^1 \underset{\sim}{a}_1 + v^2 \underset{\sim}{a}_2$

$a_1 \cdot \underset{\sim}{v} = a_{11} v^1 + a_{12} v^2$

$a_2 \cdot \underset{\sim}{v} = a_{21} v^1 + a_{22} v^2$

set of algebraic eqns.

Introduce reciprocal base vector



define $\underset{\sim}{a}^1$ so that

$\underset{\sim}{a}^1 \cdot \underset{\sim}{a}_2 = 0$

$\underset{\sim}{a}^1 \cdot \underset{\sim}{a}_1 = 1$

define $\underset{\sim}{a}^2$ so that

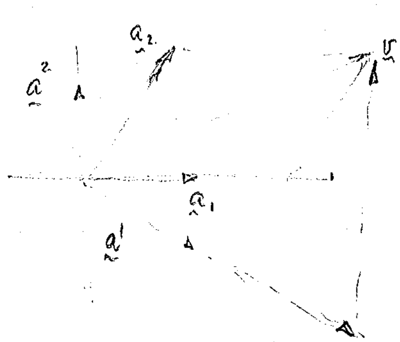
$\underset{\sim}{a}^2 \cdot \underset{\sim}{a}_1 = 0$

$\underset{\sim}{a}^2 \cdot \underset{\sim}{a}_2 = 1$

$\underset{\sim}{v} = v^1 \underset{\sim}{a}_1 + v^2 \underset{\sim}{a}_2$

$\underset{\sim}{a}^1 \cdot \underset{\sim}{v} = v^1$ $\qquad \underset{\sim}{v} = v^\alpha \underset{\sim}{a}_\alpha \qquad v^\alpha = \underset{\sim}{v} \cdot \underset{\sim}{a}^\alpha$

$\underset{\sim}{a}^2 \cdot \underset{\sim}{v} = v^2$



$\underset{\sim}{v} = c_1 \underset{\sim}{a}^1 + c_2 \underset{\sim}{a}^2$

$\underset{\sim}{v} = v_1 \underset{\sim}{a}^1 + v_2 \underset{\sim}{a}^1$

$\underset{\sim}{v} \cdot \underset{\sim}{a}_1 = v_1$ $\qquad v = v^\alpha \underset{\sim}{a}_\alpha = v_\beta \underset{\sim}{a}^\beta$

$\underset{\sim}{v} \cdot \underset{\sim}{a}_2 = v_2$

Now what happens if we change coord. system,
v^1 is highly dependent on coordinate system
but $\underset{\sim}{v}$ is <u>independent</u> of coordinate system.

The v^α are defined as the contravariant components of vector $\underset{\sim}{v}$
v_α " " " " covariant " " " $\underset{\sim}{v}$

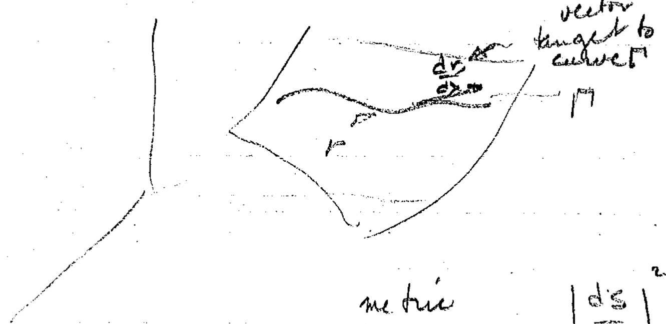
for a general vector

$$\underset{\sim}{v} = \underbrace{v^1 \underset{\sim}{a}_1 + v^2 \underset{\sim}{a}_2}_{\substack{\text{tangent to surface} \\ \text{intrinsic part of} \\ \text{vector - can be determined} \\ \text{by metric coeff.}}} + v^3 \underset{\sim}{a}_3$$

$$\underset{\sim}{a}_3 \cdot \underset{\sim}{v} = v^3 \qquad \underset{\sim}{a}_3 = \underset{\sim}{a}^3$$

since $\underset{\sim}{a}_i$ are ⊥ base vectors & are unit vectors.

Given curve Γ scribed in surface



$$\Gamma: \; x^\alpha = x^\alpha(\lambda)$$

$$\underset{\sim}{r} = \underset{\sim}{r}(x^1, x^2)$$

$$\frac{d\underset{\sim}{r}}{d\lambda} = \frac{\partial \underset{\sim}{r}}{\partial x^\alpha} \frac{dx^\alpha}{d\lambda} = \underset{\sim}{a}_\alpha \frac{dx^\alpha}{d\lambda}$$

metric $$\left|\frac{ds}{d\lambda}\right|^2 = \frac{d\underset{\sim}{r}}{d\lambda} \cdot \frac{d\underset{\sim}{r}}{d\lambda} = a_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}$$

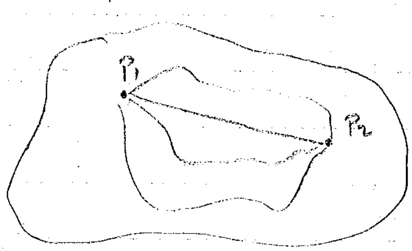
then unit tangent: $\underset{\sim}{t} = \dfrac{d\underset{\sim}{r}}{ds} = \dfrac{d\underset{\sim}{r}}{d\lambda} \dfrac{d\lambda}{ds}$

Curvature of curve Γ: $\underset{\sim}{\kappa} = \dfrac{d\underset{\sim}{t}}{ds} = \dfrac{d}{ds}\left(\dfrac{d\underset{\sim}{r}}{ds}\right)$.

$$\underset{\sim}{\kappa} = \underbrace{\kappa^\alpha \underset{\sim}{a}_\alpha}_{\substack{\text{in the surface - known} \\ \text{as intrinsic part - also} \\ \text{referred to geodesic curvature}}} + \kappa^3 \underset{\sim}{a}_3$$ ← normal curvature $\quad \kappa^3 = \underset{\sim}{\kappa} \cdot \underset{\sim}{a}^3$

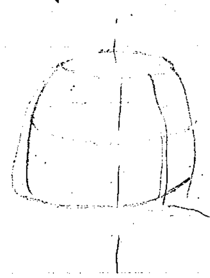
$$\kappa^\alpha \underset{\sim}{a}_\alpha = \kappa_g$$

Curves of zero geodesic curvature are min length curves that join any 2 given points (ie a straight line) Known as geodesics

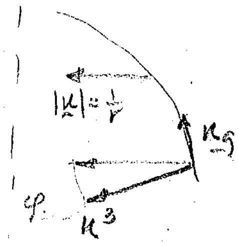


Geodesics on sphere : great circles

surface of revolution



meridian of geodesic $\kappa_g = 0$



$$|\kappa^3| = \frac{1}{r}\sin\varphi = \frac{1}{r_2}$$

$$|\kappa_g| = \frac{\cos\varphi}{r}$$

Now $\underset{\sim}{t} = \frac{\partial \underset{\sim}{r}}{\partial x^\alpha}\frac{dx^\alpha}{ds} = \underset{\sim}{a}_\alpha \frac{dx^\alpha}{ds}$

$$\underset{\sim}{\kappa} = \frac{d\underset{\sim}{t}}{ds} = \frac{\partial}{\partial x^\beta}(\underset{\sim}{t})\frac{dx^\beta}{ds} = \frac{\partial}{\partial x^\beta}\left(\underset{\sim}{a}_\alpha \frac{dx^\alpha}{ds}\right)\frac{dx^\beta}{ds}$$

$$\Rightarrow \kappa^3 = \underset{\sim}{\kappa}\cdot\underset{\sim}{a}^3 = \underset{\sim}{a}^3\cdot\frac{\partial}{\partial x^\beta}(\quad)\frac{dx^\beta}{ds}$$

$$= \underset{\sim}{a}^3\cdot\frac{\partial \underset{\sim}{a}_\alpha}{\partial x^\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} + \underbrace{\underset{\sim}{a}^3\cdot\underset{\sim}{a}_\alpha}_{=0}\left(\frac{\partial}{\partial x^\beta}\left(\frac{dx^\alpha}{ds}\right)\right)$$

$$b_{\alpha\beta} = \underset{\sim}{a}^3\cdot\frac{\partial \underset{\sim}{a}_\alpha}{\partial x^\beta} \qquad \kappa^3 ds^2 = b_{\alpha\beta}dx^\alpha dx^\beta = \mathrm{II}$$

2nd fundamental form.

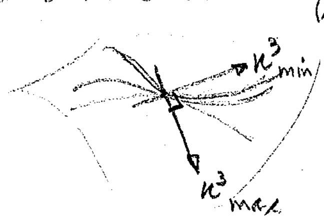
Now $b_{\alpha\beta}$ & $a_{\alpha\beta}$ can completely characterize surface if we can get some compatibility conditions

$$\underset{\sim}{a}^3\cdot\frac{\partial}{\partial x^\beta}\left(\frac{\partial \underset{\sim}{r}}{\partial x^\alpha}\right) = b_{\alpha\beta} \qquad \underset{\sim}{r} = x(x^1, x^2)e_1 + y(x^1, x^2)e_2 + \ldots$$

$$= \underset{\sim}{a}^3\cdot\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \underset{\sim}{r}}{\partial x^\beta}\right) = b_{\beta\alpha} \qquad \therefore$$

$\therefore$ $b_{\alpha\beta}$ is symmetric   depend only surface. orientation of curve is found in $dx^\alpha dx^\beta$ and κ^3 is the same for all curves through point with the same tangent

The difference will be in the geodesic curvature



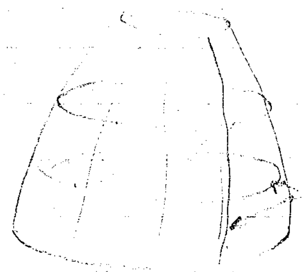
κ^3 max & min curvature direction, will be orthog. (through a given point) to each other.

Doing this for every point on surface will give rise to an orthogonal net.

lines of curvature coords.

- at every pt. are in dir of min or max K^3
- Form an orthogonal net on surface

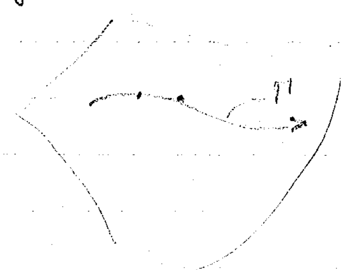
For shell of revolution



lines of curv. coords.

for sphere $\frac{1}{R}$ are min & max. K^3 & lines of curvature are any orthog set.

Given $\varphi(x^1, x^2)$ everywhere λ a parameter on Γ



$$\frac{d\varphi}{d\lambda} = \frac{\partial\varphi}{\partial x^1}\frac{dx^1}{d\lambda} + \frac{\partial\varphi}{\partial x^2}\frac{dx^2}{d\lambda}$$

$$= \underline{a}_1 \frac{dx^1}{d\lambda} \cdot \underline{a}^1 \frac{\partial\varphi}{\partial x^1} + \underline{a}_2 \frac{dx^2}{d\lambda} \cdot \underline{a}^2 \frac{\partial\varphi}{\partial x^2}$$

$$= \left(\underline{a}_1 \frac{dx^1}{d\lambda} + \underline{a}_2 \frac{dx^2}{d\lambda}\right) \cdot \left(\underline{a}^1 \frac{\partial\varphi}{\partial x^1} + \underline{a}^2 \frac{\partial\varphi}{\partial x^2}\right)$$

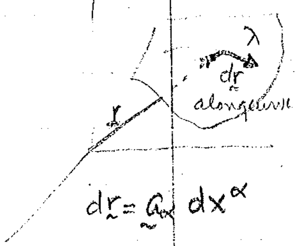
$$\frac{d\varphi}{d\lambda} = \frac{d\underline{r}}{d\lambda} \cdot \underbrace{\nabla\varphi}_{\text{grad of } \varphi} = \frac{d\underline{r}}{d\lambda} \cdot \underline{a}^\alpha \frac{\partial\varphi}{\partial x^\alpha}$$

Need Tuesday's Note

Theory of Shells. 10 April 80

$ds^2 = d\underset{\sim}{r} \cdot \underset{\sim}{\delta} \cdot d\underset{\sim}{r}$ $\quad \underset{\sim}{\delta} \cdot \underset{\sim}{v} = v^\alpha \underset{\sim}{a}_\alpha$ } intrinsic

$\underset{\sim}{v} = v^\alpha \underset{\sim}{a}_\alpha + v^3 \underset{\sim}{a}_3$ $\quad \underset{\sim}{v} \cdot \underset{\sim}{\delta} = v^\alpha \underset{\sim}{a}_\alpha$



$d\underset{\sim}{r} = \underset{\sim}{a}_\alpha dx^\alpha$

$\underset{\sim}{\delta} = a_{\alpha\beta} \, \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta$

$\mathrm{II} = (\underset{\sim}{K} \cdot \underset{\sim}{a}_3) \, ds^2 = d\underset{\sim}{r} \cdot \underset{\sim}{b} \cdot d\underset{\sim}{r}$ $\qquad \underset{\sim}{K} \cdot \underset{\sim}{a}_3$ – normal curvature

$\underset{\sim}{b} = b_{\alpha\beta} \, \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta$

dyad.

if $\underset{\sim}{u}$ is a 3-d vector & $\underset{\sim}{v}$ is also

$$\underset{\sim}{T} = (u^\alpha a_\alpha + u^3 a_3) \otimes (v^\beta a_\beta + v^3 a_3)$$
$$= u^\alpha v^\beta \underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta + u^\alpha v^3 \underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_3 + u^3 v^\beta a_3 \otimes \underset{\sim}{a}_\beta + u^3 v^3 \underset{\sim}{a}_3 \otimes \underset{\sim}{a}_3$$

Generally in 3-D use i, j, k, $\quad i = 1, 2, 3$

base vectors $\quad \underset{\sim}{r} = \underset{\sim}{r}(x^i)$ $\quad \underset{\sim}{g}_i = \dfrac{\partial \underset{\sim}{r}}{\partial x^i}$ $\qquad g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j$

coords

$$\therefore \; d\underset{\sim}{r} = \frac{\partial \underset{\sim}{r}}{\partial x^i} dx^i$$

in 3 ds $\quad \mathrm{I} = d\underset{\sim}{r} \cdot d\underset{\sim}{r} = ds^2 = g_i \, dx^i \cdot g_j \, dx^j = g_{ij} \, dx^i dx^j$

$\mathrm{II} = ?$ cannot tell – cause requires you to know info about 4th dimension

$$\underset{\sim}{u} \times \underset{\sim}{v} = u^\alpha \underset{\sim}{a}_\alpha \times v^\beta \underset{\sim}{a}_\beta = u^1 v^2 a_1 \times a_2 + u^2 v^1 a_2 \times a_1 = (u^1 v^2 - u^2 v^1) a_1 \times a_2$$

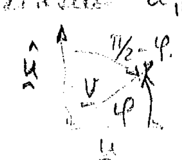
$$|\underset{\sim}{a}_1 \times \underset{\sim}{a}_2| = a^{1/2} \underset{\sim}{a}_3 \; ; \; a = a_{11} a_{22} - a_{12}^2$$

$$\underset{\sim}{u} \times \underset{\sim}{v} = (u^1 v^2 - v^2 v^1) a^{1/2} \underset{\sim}{a}_3 = \underset{\sim}{a}_3 \; \underset{\sim}{u} \cdot a^{1/2} (\underset{\sim}{a}^1 \otimes \underset{\sim}{a}^2 - \underset{\sim}{a}^2 \otimes \underset{\sim}{a}^1) \cdot \underset{\sim}{v}$$

tensor invariant

permutation tensor: $\quad \underset{\sim}{\epsilon} = a^{1/2} (\underset{\sim}{a}^1 \otimes \underset{\sim}{a}^2 - \underset{\sim}{a}^2 \otimes \underset{\sim}{a}^1)$

$\epsilon_{11} = \epsilon_{22} = 0 \quad \epsilon_{12} = -\epsilon_{21} = a^{1/2}$ $\qquad$ for cartesian $a_{11} = a_{22} = 1 \quad a_{12} = 0 \quad \therefore a = 1$



$$|\underset{\sim}{u} \times \underset{\sim}{v}| = |\underset{\sim}{u}||\underset{\sim}{v}| \sin(\overset{\varphi}{\underset{\sim}{u}, \underset{\sim}{v}})$$
$$= |\underset{\sim}{u}||\underset{\sim}{v}| \cos\left(\tfrac{\pi}{2} - (\overset{\varphi}{\underset{\sim}{u}, \underset{\sim}{v}})\right)$$

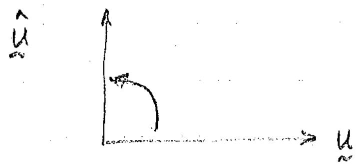
if $\underset{\sim}{u}$ is rotated through $90^\circ (= \hat{\underset{\sim}{u}}) \Rightarrow |\hat{\underset{\sim}{u}} \cdot \underset{\sim}{v}| = |\underset{\sim}{u} \times \underset{\sim}{v}|$

$|\underset{\sim}{u} \times \underset{\sim}{v}| = (\underset{\sim}{u} \cdot \underset{\sim}{\epsilon}) \cdot \underset{\sim}{v}$

$\hat{\underset{\sim}{u}} \cdot \underset{\sim}{v}$

$\underset{\sim}{u} \cdot \underset{\sim}{\epsilon}$ gives rotation in + 90° direction as defined by our r.h. rule.

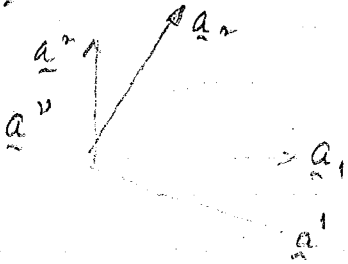
$\underset{\sim}{u} \cdot \underset{\sim}{\delta}$ takes away u_3 component.



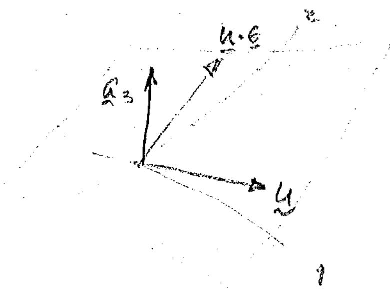
In our case $\underset{\sim}{\epsilon} = \epsilon_{ijk}\, \underset{\sim}{g}^i \otimes \underset{\sim}{g}^j \otimes \underset{\sim}{g}^k \qquad \epsilon_{123} = g^{1/2}$

$$\underset{\sim}{u} \cdot \underset{\sim}{\epsilon} = (u^\alpha \underset{\sim}{a}_\alpha)\, a^{1/2} (\underset{\sim}{a}^1 \otimes \underset{\sim}{a}^2 - \underset{\sim}{a}^2 \otimes \underset{\sim}{a}^1) = u^\alpha \epsilon_{\alpha\nu} \underset{\sim}{a}^\nu$$

$$= u^1 a^{1/2} \underset{\sim}{a}^2 - u^2 a^{1/2} \underset{\sim}{a}^1$$



$(\underset{\sim}{u} \cdot \underset{\sim}{\epsilon}) \cdot \underset{\sim}{v} = |\underset{\sim}{u}|\,|\underset{\sim}{v}| \sin\varphi$



$\underset{\sim}{\epsilon} \cdot \underset{\sim}{u} = -\underset{\sim}{u} \cdot \underset{\sim}{\epsilon}$

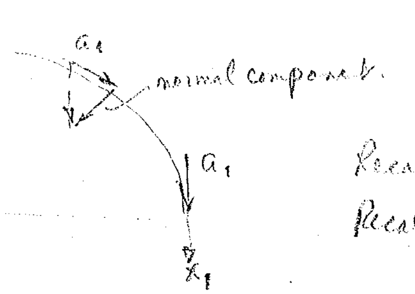
$\underset{\sim}{u} \cdot \underset{\sim}{\epsilon} \cdot \underset{\sim}{u} = 0$

$\underset{\sim}{a} \cdot \underset{\sim}{u} = |\underset{\sim}{u}|\,|\underset{\sim}{u}| \cos 90^\circ = 0$

If $\underset{\sim}{v} = v^\alpha \underset{\sim}{a}_\alpha \qquad \dfrac{d\underset{\sim}{v}}{d\lambda} = \dfrac{d\underset{\sim}{r}}{d\lambda} \cdot \nabla \underset{\sim}{v} \qquad \nabla \underset{\sim}{v} = \underset{\sim}{a}^\nu \otimes \dfrac{\partial}{\partial x^\nu}(v^\alpha \underset{\sim}{a}_\alpha)$

$$= \underset{\sim}{a}^\nu \otimes \left[\frac{\partial v^\alpha}{\partial x^\nu} \underset{\sim}{a}_\alpha + v^\alpha \frac{\partial \underset{\sim}{a}_\alpha}{\partial x^\nu}\right]$$

In any coord sys other than cartesian $\dfrac{\partial \underset{\sim}{a}_\alpha}{\partial x^\nu} \neq 0$



$$\frac{\partial \underset{\sim}{a}_\alpha}{\partial x^\nu} = \left(\frac{\partial \underset{\sim}{a}_\alpha}{\partial x^\nu} \cdot \underset{\sim}{a}^\mu\right) \underset{\sim}{a}_\mu + \left(\frac{\partial \underset{\sim}{a}_\alpha}{\partial x^\nu} \cdot \underset{\sim}{a}^3\right) \underset{\sim}{a}_3$$

Recall $\quad v^\alpha \underset{\sim}{a}_\alpha = \underset{\sim}{v} \qquad \underset{\sim}{v} \cdot \underset{\sim}{a}^\mu = v^\alpha \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}^\mu = v^\alpha \delta_{\alpha\mu} = v^\mu$

Recall $\quad \underset{\sim}{v} \cdot \underset{\sim}{a}_\beta = v^\alpha \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta = v^\alpha a_{\alpha\beta}$

thus $\quad \dfrac{\partial \underset{\sim}{a}_\alpha}{\partial x^\nu} = \left\{ {\mu \atop \alpha\nu} \right\} \underset{\sim}{a}_\mu + \left\{ {3 \atop \alpha\nu} \right\} \underset{\sim}{a}_3 = \left\{ {\mu \atop \alpha\nu} \right\} + b_{\alpha\nu} \underset{\sim}{a}_3$

this is highly dependent on coord transformation

Intrinsic part to 2-D space

from $\mathrm{II} = \underset{\sim}{\kappa} \cdot \underset{\sim}{a}_3\, ds^2 = b_{\alpha\beta}\, dx^\alpha dx^\beta$

$$b_{\alpha\beta} = \underset{\sim}{a}_3 \cdot \frac{\partial \underset{\sim}{a}_\alpha}{\partial x^\beta}$$

$\begin{Bmatrix} \mu \\ \alpha\,\nu \end{Bmatrix}$ - Christoffel symbols of 1st kind. This is not a tensor.
i.e. symbols by themselves do not form components of a tensor.

But

dummies let $\alpha \to \mu$, $\mu \to \alpha$

$$\nabla \underset{\sim}{v} = \underset{\sim}{a}^{\nu} \otimes \left[\frac{\partial v^{\alpha}}{\partial x^{\nu}} \underset{\sim}{a}_{\alpha} + v^{\alpha} \left(\begin{Bmatrix} \mu \\ \alpha\,\nu \end{Bmatrix} \underset{\sim}{a}_{\mu} + b_{\alpha\nu} \underset{\sim}{a}_{3} \right) \right]$$

$$= \underset{\sim}{a}^{\nu} \otimes \left[\left(\frac{\partial v^{\alpha}}{\partial x^{\nu}} + v^{\mu} \begin{Bmatrix} \alpha \\ \mu\,\nu \end{Bmatrix} \right) \underset{\sim}{a}_{\alpha} + v^{\alpha} b_{\alpha\nu} \underset{\sim}{a}_{3} \right] \; ; \quad \underset{\sim}{a}^{\nu} \otimes v^{\alpha} b_{\alpha\nu} \underset{\sim}{a}_{3} \Rightarrow$$

$$v^{\alpha} b_{\alpha\nu} \underset{\sim}{a}^{\nu} = \underset{\sim}{v} \cdot (b_{\alpha\nu} \underset{\sim}{a}^{\alpha} \otimes \ldots$$

$$= \underset{\sim}{v} \cdot \underset{\sim}{b}$$

define $\displaystyle v^{\alpha}\Big|_{\nu} = \frac{\partial v^{\alpha}}{\partial x^{\nu}} + v^{\mu} \begin{Bmatrix} \alpha \\ \mu\,\nu \end{Bmatrix}$

Covariant Derivative

$$\nabla \underset{\sim}{v} = v^{\alpha}\Big|_{\nu} \, \underset{\sim}{a}^{\nu} \otimes \underset{\sim}{a}_{\alpha} + \underset{\sim}{v} \cdot \underset{\sim}{b} \otimes \underset{\sim}{a}_{3}$$

Remember $d\underset{\sim}{v} = d\underset{\sim}{r} \cdot \nabla \underset{\sim}{v}$

$$d\underset{\sim}{v} \cdot \underset{\sim}{\delta} = d\underset{\sim}{r} \cdot \nabla \underset{\sim}{v} \cdot \underset{\sim}{\delta} = d\underset{\sim}{r} \cdot \left(v^{\alpha}\Big|_{\nu} \underset{\sim}{a}^{\nu} \otimes \underset{\sim}{a}_{\alpha} \right) \qquad \underset{\sim}{\delta} = a_{\alpha\beta} \underset{\sim}{a}^{\alpha} \otimes \underset{\sim}{a}^{\beta}$$

thus $\nabla \underset{\sim}{v} \cdot \underset{\sim}{\delta}$ is a tensor.

15 April 1980

Now $\underset{\sim}{u} = u^{\alpha} \underset{\sim}{a}_{\alpha}$

$$\nabla \underset{\sim}{u} = \underset{\sim}{a}^{\beta} \otimes \frac{\partial \underset{\sim}{u}}{\partial x^{\beta}}$$

$$= \underbrace{\underset{\sim}{a}^{\beta} \otimes \left(u^{\alpha}\Big|_{\beta} \underset{\sim}{a}_{\alpha} \right)}_{\text{Intrinsic part}} + \underset{\sim}{u} \cdot \underset{\sim}{b} \otimes \underset{\sim}{a}_{3}$$

$$\nabla \underset{\sim}{u} \cdot \underset{\sim}{\delta} = u^{\alpha}\Big|_{\beta} \, \underset{\sim}{a}^{\beta} \otimes \underset{\sim}{a}_{\alpha}$$

Now $\nabla \cdot \underset{\sim}{u}$: $\quad \underset{\sim}{b} \cdot \underset{\sim}{a}^{3} = 0 \qquad \nabla \cdot \underset{\sim}{u} = \underset{\sim}{a}^{\beta} \cdot \left(u^{\alpha}\Big|_{\beta} \underset{\sim}{a}_{\alpha} \right) = \delta_{\alpha\beta} \, u^{\alpha}\Big|_{\beta} = u^{\alpha}\Big|_{\alpha}$

For cartesian system. $\underset{\sim}{u} = u_x \underset{\sim}{e}_x + u_y \underset{\sim}{e}_y$

$$\nabla \underset{\sim}{u} = e_x \otimes \frac{\partial \underset{\sim}{u}}{\partial x} + e_y \otimes \frac{\partial \underset{\sim}{u}}{\partial y}$$

$$= e_x \otimes \left[\frac{\partial u_y}{\partial x} e_x + \frac{\partial u_y}{\partial x} e_y \right] + e_y \otimes \left[\frac{\partial u_x}{\partial y} e_x + \frac{\partial u_y}{\partial y} e_y \right]$$

$$= \frac{\partial u_x}{\partial x} e_x \otimes e_x + \frac{\partial u_y}{\partial x} e_x \otimes e_y + \frac{\partial u_x}{\partial y} e_y \otimes e_x + \frac{\partial u_y}{\partial y} e_y \otimes e_y$$

$$\nabla = \underset{\sim}{a}^{\alpha} \otimes \frac{\partial}{\partial x^{\alpha}}$$

$$\nabla\cdot\underline{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \mathrm{div}(\underline{u})$$

Now $u^\alpha\big|_\beta = \dfrac{\partial u^\alpha}{\partial x^\beta} + u^\nu \left\{ {\alpha \atop \beta\nu} \right\}$

$$\left\{ {\alpha \atop \beta\nu} \right\} = \frac{\partial a_\beta}{\partial x^\nu}\cdot \underset{\sim}{a}^\alpha = \left(\frac{\partial}{\partial x^\nu}\frac{\partial r}{\partial x^\beta}\right)\cdot \underset{\sim}{a}^\alpha = \left(\frac{\partial}{\partial x^\beta}\frac{\partial \underline{r}}{\partial x^\nu}\right)\cdot \underset{\sim}{a}^\alpha = \left\{ {\alpha \atop \nu\beta} \right\}.$$

$$\therefore \quad \left\{ {\alpha \atop \beta\nu} \right\} = \left\{ {\alpha \atop \nu\beta} \right\}.$$

Now $\nabla\cdot\underset{\sim}{u} = u^\alpha\big|_\alpha = \dfrac{\partial u^\alpha}{\partial x^\alpha} + u^\nu\left\{ {\alpha \atop \alpha\nu} \right\}. = \dfrac{\partial u^1}{\partial x^1} + \dfrac{\partial u^2}{\partial x^2} + u^1\left[\left\{ {1 \atop 11} \right\} + \left\{ {2 \atop 21} \right\}\right]$

$$+ u^2\left[\left\{ {1 \atop 12} \right\} + \left\{ {2 \atop 22} \right\}\right]$$

Let φ be a scalar.

$$\nabla\varphi = \underline{a}^\alpha \frac{\partial\varphi}{\partial x^\alpha} \quad ; \quad \nabla\cdot\nabla\varphi = \underline{a}^\beta\cdot\frac{\partial}{\partial x^\beta}\left(\underline{a}^\alpha \frac{\partial\varphi}{\partial x^\alpha}\right)$$

$$\underline{a}^\alpha\cdot\underline{a}_\nu = \delta^\alpha_\nu \qquad \frac{\partial}{\partial x^\mu}\left(\underline{a}^\alpha\cdot\underline{a}_\nu\right) = \frac{\partial \underline{a}^\alpha}{\partial x^\mu}\cdot\underline{a}_\nu + \underline{a}^\alpha\cdot\frac{\partial \underline{a}_\nu}{\partial x^\mu} = 0$$

$$\Rightarrow \frac{\partial \underline{a}^\alpha}{\partial x^\mu}\cdot\underline{a}_\nu = -\underline{a}^\alpha\cdot\frac{\partial \underline{a}_\nu}{\partial x^\mu} = -\left\{ {\alpha \atop \nu\mu} \right\}.$$

$$\nabla\cdot\nabla\varphi = \underline{a}^\beta\cdot\frac{\partial \underline{a}^\alpha}{\partial x^\beta}\frac{\partial\varphi}{\partial x^\alpha} + \underline{a}^\beta\cdot\underline{a}^\alpha \frac{\partial}{\partial x^\beta}\frac{\partial\varphi}{\partial x^\alpha} \quad ; \quad \frac{\partial \underline{a}^\alpha}{\partial x^\beta} = \left(\frac{\partial \underline{a}^\alpha}{\partial x^\beta}\cdot\underline{a}_\nu\right)\underline{a}^\nu + \left(\frac{\partial \underline{a}^\alpha}{\partial x^\beta}\cdot\underline{a}_3\right)$$

$$= \underline{a}^\beta\cdot\left(\frac{\partial}{\partial x^\beta}\frac{\partial\varphi}{\partial x^\alpha}\underline{a}^\alpha\right) + \frac{\partial\varphi}{\partial x^\alpha}\left(-\left\{ {\alpha \atop \beta\nu} \right\}\underline{a}^\nu + b^\alpha_\beta\,\underline{a}_3\right)$$

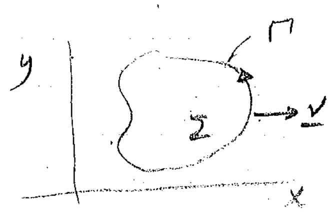
$$= \underline{a}^\beta\cdot\left[\varphi\big|_{\alpha\beta}\,\underset{\sim}{a}^\alpha + \frac{\partial\varphi}{\partial x^\alpha}\,b^\alpha_\beta\,\underset{\sim}{a}_3\right]$$

$$= a^{\alpha\beta}\varphi\big|_{\alpha\beta} \qquad \text{since } \underline{a}^\beta\cdot\underline{a}_3 = 0$$

Thus $\nabla^2\varphi = a^{\alpha\beta}\varphi\big|_{\alpha\beta}$ Laplacian.

Look at Divergence Theorem on a flat land $\underline{u} = u_x e_x + u_y e_y$

$$\iint_\Sigma \nabla\cdot\underline{u}\,dx\,dy = \iint\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)dx\,dy = \oint_C \underline{u}\cdot\underline{\nu}\,ds$$



$$\iint_\Sigma \nabla\cdot\underline{u}\, d\Sigma = \oint_\Gamma \underline{u}\cdot\underline{\nu}\, ds$$

This holds for surface.

tangent to surface

$d\underset{\sim}{r} = \underline{\nu}\cdot\underset{\sim}{\epsilon}\, ds$ $\quad \underline{\epsilon}$ is rotation tensor.

$\underline{\epsilon}\cdot d\underline{r} = \underline{\epsilon}\cdot(\underline{\nu}\cdot\underline{\epsilon})\, ds$
$= \underline{\nu}\, ds$

negative rot.

$$\oint_\Gamma \underline{u}\cdot\underline{\nu}\, ds = \oint_\Gamma \underline{u}\cdot\underline{\epsilon}\cdot d\underset{\sim}{r}$$

if $\underline{u} = \underline{\epsilon}\cdot\underline{v}$

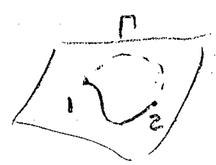
$$\iint \nabla\cdot(\underline{\epsilon}\cdot\underline{v})\, d\underline{\Sigma} = \oint_\Gamma \underbrace{(\underline{\epsilon}\cdot\underline{v})\cdot\underline{\epsilon}}_{\text{positive rot}}\cdot d\underline{r} = \oint_\Gamma \underline{v}\cdot d\underline{r}$$

negative rot

$$\iint \underline{a}_3\cdot(\nabla\times\underline{v})\, d\Sigma = \oint \underline{v}\cdot d\underline{r} \quad \text{Circulation}$$

if φ is a scalar and single valued

$$\int_{P_1}^{P_2} d\varphi = \varphi(P_2) - \varphi(P_1)$$



$$\oint_\Gamma d\varphi = \oint_\Gamma d\underset{\sim}{r}\cdot\nabla\varphi = \iint \nabla\cdot(\underline{\epsilon}\cdot\nabla\varphi)\, d\Sigma = 0$$

if $\oint_\Gamma d\varphi = 0 \Rightarrow$ for any $d\Sigma$: $\nabla\cdot(\underline{\epsilon}\cdot\nabla\varphi) = 0$
curl (grad φ)

Now let $\underset{\sim}{u} = u^\alpha \underset{\sim}{a}_\alpha$ $\quad \oint_\Gamma d\underline{u} = 0$ if $\underline{u}$ is single valued. $d\underline{u} = d\underset{\sim}{r}\cdot\nabla\underline{u}$

$\oint_\Gamma d\underline{u} = \oint d\underline{r}\cdot\nabla\underline{u}$ $\quad$ Must go back to divergence theorem

$\iint \nabla\cdot\underline{v}\, d\Sigma = \oint \underline{\nu}\cdot\underline{v}\, ds$ only if $\underline{v} = v^\alpha \underline{a}_\alpha$

To generalize if $\underline{T} = T^{\alpha\beta}\underline{a}_\alpha\otimes\underline{a}_\beta + T^{\alpha 3}\underline{a}_\alpha\otimes\underline{a}_3$

if $\underset{\sim}{v} = \underline{T}\cdot\underset{\sim}{\lambda}$ $\quad \underset{\sim}{\lambda}$ = const in 3-D space $\quad \underset{\sim}{T}\cdot\underset{\sim}{\lambda} = T^{\alpha\beta}\underset{\sim}{a}_\alpha\lambda_\beta + T^{\alpha 3}\underset{\sim}{a}_\alpha\lambda_3$

if $\underset{\sim}{\lambda}$ is const $\nabla\underset{\sim}{\lambda} = 0 \Rightarrow$

$$\iint \nabla\cdot(\underset{\sim}{T}\cdot\underset{\sim}{\lambda})\,d\Sigma = \oint \underset{\sim}{\nu}\cdot\underset{\sim}{T}\cdot\underset{\sim}{\lambda}\,dS$$

$$\left[\iint(\nabla\cdot\underset{\sim}{T})\,d\Sigma - \oint\underset{\sim}{\nu}\cdot\underset{\sim}{T}\,dS\right]\cdot\underset{\sim}{\lambda} = 0$$

since λ is const & arbitrary

$$\iint(\nabla\cdot\underset{\sim}{T})\,d\Sigma = \oint\underset{\sim}{\nu}\cdot\underset{\sim}{T}\,dS$$ only if $\underset{\sim}{T}$ has tangent vectors in 1st comp

if $\underset{\sim}{u} = u^{\alpha}\underset{\sim}{a}_{\alpha}$

$$0 = \oint_{\Gamma} d\underset{\sim}{u} = \oint_{\Gamma} d\underset{\sim}{r}\cdot\nabla\underset{\sim}{u} = \iint_{\Sigma}\nabla\cdot(\underset{\sim}{\epsilon}\cdot\nabla\underset{\sim}{u})\,d\Sigma$$

since Γ is arbitrary $\quad \nabla\cdot(\underset{\sim}{\epsilon}\cdot\underbrace{\nabla\underset{\sim}{u}}_{\text{Aha-kaser}}) = 0$ Compatibility

look at Page 12 of notes.

$$\nabla\cdot(\underset{\sim}{\epsilon}\cdot\nabla\underset{\sim}{v}) = \underbrace{\epsilon^{\alpha\beta}v_{\gamma}\big|_{\beta\alpha}\underset{\sim}{a}^{\gamma} + \underset{\sim}{v}\cdot\underset{\sim}{b}\cdot\underset{\sim}{\epsilon}\cdot\underset{\sim}{b}}_{\text{intrinsic}} + \underset{\sim}{a}_{3}\epsilon^{\alpha\beta}\left(v_{\gamma}\big|_{\beta}b^{\gamma}_{\alpha} - (v_{\gamma}b^{\gamma}_{\alpha})\right)\Big|$$

$$\nabla\underset{\sim}{v} = \underset{\sim}{a}^{\alpha}\otimes\left(v_{\gamma}\big|_{\alpha}\underset{\sim}{a}^{\gamma}\right) + \underset{\sim}{v}\cdot\underset{\sim}{b}\otimes\underset{\sim}{a}_{3}$$

$$\nabla\cdot(\underset{\sim}{\epsilon}\cdot\nabla\underset{\sim}{v}\cdot\underset{\sim}{\delta})\cdot\underset{\sim}{\delta} = \epsilon^{\alpha\beta}v_{\gamma}\big|_{\beta\alpha}\underset{\sim}{a}^{\gamma} = \epsilon^{12}\left(\underset{\sim}{v}\big|_{21} - \underset{\sim}{v}\big|_{12}\right)$$

$$\underset{\sim}{b}\cdot\underset{\sim}{\epsilon}\cdot\underset{\sim}{b} = \underset{\sim}{\epsilon}\,K \qquad K = b_1^1 b_2^2 - b_1^2 b_2^1$$ Gaussian Curvature

since $\underset{\sim}{b}, \underset{\sim}{\epsilon}, \underset{\sim}{b}$ are invariants w/coords system.

Thus $$\nabla\cdot(\underset{\sim}{\epsilon}\cdot\nabla\underset{\sim}{v}\cdot\underset{\sim}{\delta})\cdot\underset{\sim}{\delta} + K\underset{\sim}{v}\cdot\underset{\sim}{\epsilon} \equiv 0$$

$$\epsilon^{12}\left(\underset{\sim}{v}\big|_{21} - \underset{\sim}{v}\big|_{12}\right) = -K\underset{\sim}{v}\cdot\underset{\sim}{\epsilon}$$

17 April 1980

Continuation of last class

$$\underset{\sim}{v} = \underset{\sim}{v}(x^1, x^2) = v^{\alpha}\underset{\sim}{a}_{\alpha}$$

$$\oint_{\Gamma} d\underset{\sim}{v} = 0 = \oint d\underset{\sim}{r}\cdot\nabla\underset{\sim}{v} = \iint\nabla\cdot(\underset{\sim}{\epsilon}\cdot\nabla\underset{\sim}{v})\,d\Sigma$$

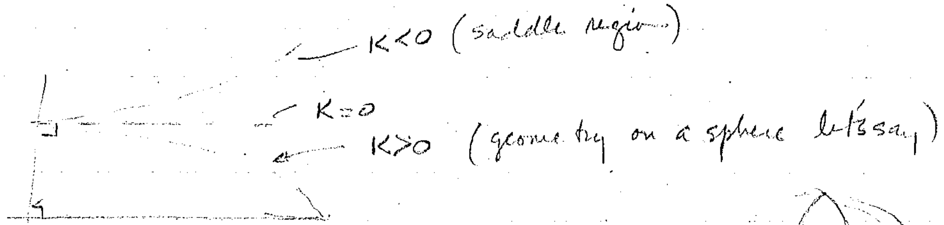
Since Γ is arbitrary $\quad \nabla\cdot(\underset{\sim}{\epsilon}\cdot\nabla\underset{\sim}{v}) = 0$

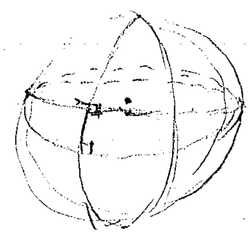
but $\nabla\cdot(\underset{\sim}{\epsilon}\cdot\nabla\underset{\sim}{v}) = \epsilon^{\alpha\beta} v^{\gamma}|_{\beta\alpha}\, \underset{\sim}{a}^{\gamma} + \underset{\sim}{v}\cdot\underset{\sim}{\epsilon}\, K + \underset{\sim}{a}_3\, \epsilon^{\alpha\beta}\left(-v^{\gamma}\, b_{\gamma\alpha}|_{\beta}\right) = 0$

$\underset{\sim}{a}_3$ component must be zero $\quad b_{\gamma 1}|_2 - b_{\gamma 2}|_1 \equiv 0 \quad \gamma = 1,2.$

These are conditions of (Mainardi Codazzi)

Thus $\epsilon^{12}\left(\underset{\sim}{v}|_{21} - \underset{\sim}{v}|_{12}\right) + \underset{\sim}{v}\cdot\underset{\sim}{\epsilon}\, K = 0 \qquad K = b_1^{\,1} b_2^{\,2} - b_1^{\,2} b_2^{\,1}$ Gaussian Curv of Surface.





These 2 lines intersect even though they are 90° to meridian

for mixed derivatives to be same $\underline{K=0}$.

example $\quad$ if $\underset{\sim}{v} = \underset{\sim}{a}_{\mu}$ (μ fixed)

$$v^{\alpha} = \underset{\sim}{v}\cdot\underset{\sim}{a}^{\alpha} = \underset{\sim}{a}^{\alpha}\cdot\underset{\sim}{a}_{\mu} = \delta^{\alpha}_{\mu}$$

$$v^{\alpha}|_{\beta} = \frac{\partial v^{\alpha}}{\partial x^{\beta}} + v^{\nu}\left\{ {\alpha \atop \nu\beta} \right\}.$$

$$v^{\alpha}|_{\beta} = v^{\nu}\left\{ {\alpha \atop \nu\beta} \right\} \quad \text{since} \quad v^{\alpha} = 1 \text{ or } 0, \ = \delta^{\alpha}_{\mu} \Rightarrow \alpha = \mu$$

$$\therefore \quad v^{\alpha}|_{\beta} = \left\{ {\alpha \atop \mu\beta} \right\}.$$

thus $v^{\alpha}|_{\beta\gamma} = \dfrac{\partial}{\partial x^{\gamma}}\left(v^{\alpha}|_{\beta}\right) + v^{\nu}|_{\beta}\left\{ {\alpha \atop \nu\gamma} \right\} + v^{\nu}|_{\gamma}\left\{ {\alpha \atop \nu\beta} \right\}.$

~~opposite~~

why? $\quad$ Now $\underset{\sim}{v} = v^{\alpha}\underset{\sim}{a}_{\alpha}$ then $\dfrac{\partial \underset{\sim}{v}}{\partial x^{\mu}} = \dfrac{\partial v^{\alpha}}{\partial x^{\mu}}\underset{\sim}{a}_{\alpha} + v^{\alpha}\dfrac{\partial \underset{\sim}{a}_{\alpha}}{\partial x^{\mu}}$

$$\text{"} \quad + v^{\alpha}\left(\left\{ {\nu \atop \mu\alpha} \right\}\underset{\sim}{a}_{\nu} + b_{\alpha\mu}\underset{\sim}{a}_3\right)$$

$$= \left(\frac{\partial v^{\alpha}}{\partial x^{\mu}} + v^{\nu}\left\{ {\alpha \atop \mu\nu} \right\}\right)\underset{\sim}{a}_{\alpha} + v^{\alpha} b_{\alpha\mu}\underset{\sim}{a}_3$$

Now $\nabla\underset{\sim}{v} = \underset{\sim}{a}^{\mu}\otimes\dfrac{\partial \underset{\sim}{v}}{\partial x^{\mu}} = v^{\alpha}|_{\mu}\,\underset{\sim}{a}^{\mu}\otimes\underset{\sim}{a}_{\alpha} + \underset{\sim}{b}\cdot\underset{\sim}{v}\otimes\underset{\sim}{a}_3$

$\nabla(\nabla\underset{\sim}{v}) = \underset{\sim}{a}^{\nu}\otimes\dfrac{\partial}{\partial x^{\nu}}(\nabla\underset{\sim}{v}) = v^{\alpha}|_{\mu\nu}\,\underset{\sim}{a}_{\alpha} + \ldots$

$$v^{\alpha}|_{\mu\nu} = \frac{\partial}{\partial x^{\nu}}\left(v^{\alpha}|_{\mu}\right) + v^{\rho}|_{\mu}\left\{ {\alpha \atop \rho\nu} \right\} - v^{\alpha}|_{\nu}\left\{ {\nu \atop \alpha\mu} \right\}.$$

$$T^{\alpha\beta}\Big|_\gamma = \frac{\partial T^{\alpha\beta}}{\partial x^\gamma} + T^{\nu\beta}\left\{ {\alpha \atop \nu\gamma} \right\} + T^{\alpha\nu}\left\{ {\beta \atop \nu\gamma} \right\}$$

$$T^{\alpha\beta\gamma}\Big|_\nu = \frac{\partial T^{\alpha\beta\gamma}}{\partial x^\nu} + \ldots$$

Returning

$$v^\alpha\Big|_{\beta\gamma} = \frac{\partial}{\partial x^\alpha}\left\{ {\alpha \atop \mu\,\beta} \right\} + \left\{ {\nu \atop \mu\beta} \right\}\left\{ {\alpha \atop \nu\gamma} \right\} - \left\{ {\alpha \atop \mu\nu} \right\}\left\{ {\nu \atop \beta\gamma} \right\}$$ involves only metric

Now since $0 = \left(\epsilon^{\alpha\beta} v^\mu\Big|_{\beta\alpha}\right) \underset{\sim}{a}_\mu + \underset{\sim}{v} \cdot \underset{\sim}{\epsilon}\, K$

Look at pg 13

Since we know $\underline{v} = \underset{\sim}{a}_\mu$ and we have $v^\mu\Big|_{\beta\alpha}$ then we can solve for K

$$K = \frac{a^{1/2}}{a_{11}} \epsilon^{\beta\gamma} \left[\frac{\partial}{\partial x^\gamma}\left\{ {2 \atop 1\,\beta} \right\} + \left\{ {\lambda \atop 1\,\beta} \right\}\left\{ {2 \atop \lambda\gamma} \right\} \right]$$

only involves metric

now $K = b^1_1 b^2_2 - b^2_1 b^1_2$ is an intrinsic property of surface

for an orthog coord system.

$$ds^2 = (A_1\, dx^1)^2 + (A_2\, dx^2)^2$$

$$a_{11} = (A_1)^2 \qquad a_{22} = (A_2)^2 \qquad a_{12} = 0$$

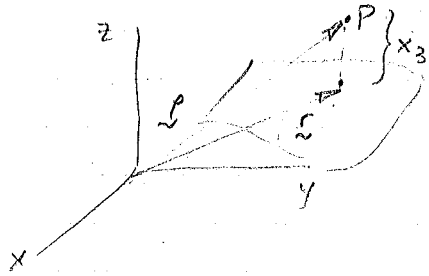
$$K = -\frac{1}{A_1 A_2}\left[\frac{\partial}{\partial x^1}\left(\frac{\partial A_2}{A_1\, \partial x^1} \right) + \frac{\partial}{\partial x^2}\left(\frac{\partial A_1}{A_2\, \partial x^2} \right) \right]$$

4/22/80

4/24/80

Next week professor Pflügge will be here to talk.

Continuation of last time 3-D Coordinates



$$\rho(x^1,x^2,x^3) = r(x^1,x^2) + x^3 a_3(x^1,x^2)$$

$$\frac{d\rho}{d\lambda} = \frac{\partial\rho}{\partial x^i}\frac{dx^i}{d\lambda} \quad (i=1,2,3) \text{ reference to 3-D space}$$

$$= \frac{\partial\rho}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} + \frac{\partial\rho}{\partial x^3}\frac{\partial x^3}{\partial\lambda}$$

$$\frac{\partial\rho}{\partial x^\alpha} = \frac{\partial r}{\partial x^\alpha} + x^3\frac{\partial a_3}{\partial x^\alpha}$$

$$= a_\alpha + x^3(-b_\alpha^\gamma\, a_\gamma)$$

$$= a_\alpha \cdot (\delta - x^3 b)$$

$\frac{\partial a_3}{\partial x^\alpha}$ lies in the surface $\frac{\partial}{\partial x^\alpha}(a_3\cdot a_3=1)$

$$\nabla a_3 = a^\alpha \otimes \frac{\partial a_3}{\partial x^\alpha} = a^\alpha \otimes(-b_{\alpha\beta}\, a^\beta) = -b$$

$$\nabla r = a^\alpha \otimes \frac{\partial r}{\partial x^\alpha} = a^\alpha \otimes a_\alpha = \delta$$

$$b_{\alpha\beta}\, a^\beta = b_\alpha^\gamma\, a_\gamma$$

$$\frac{\partial\rho}{\partial x^3} = a_3$$

Thus $$\frac{d\rho}{d\lambda} = a_\alpha\cdot(\delta - x^3 b)\frac{dx^\alpha}{d\lambda} + a_3\frac{dx^3}{d\lambda}$$

$$= \frac{dr}{d\lambda}\cdot(\delta - x^3 b) + a_3\frac{dx^3}{d\lambda}$$

symmetric

$$\frac{d\rho}{d\lambda}\cdot\frac{d\rho}{d\lambda} = \left[\frac{dr}{d\lambda}\cdot(\delta - x^3 b) + a_3\frac{dx^3}{d\lambda}\right]\cdot\left[(\delta - x^3 b)\cdot\frac{dr}{d\lambda} + a_3\frac{dx^3}{d\lambda}\right]$$

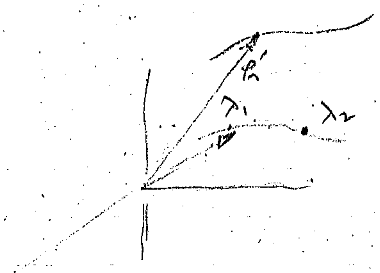
$$d\rho\cdot d\rho = \left[dr\cdot(\delta - x^3 b)\cdot(\delta - x^3 b)\cdot dr + dx^3\, dx^3\right] \quad \text{since } (\delta - x^3 b)\cdot a_3 = 0.$$

$$= dr\cdot(\delta - 2x^3 b + (x^3)^2\, b\cdot b)\cdot dr + (dx^3)^2 \quad \text{since } \delta\cdot b = b$$

$$= \underbrace{dr\cdot dr}_{\text{first fund. form of surface}} - 2x^3\underbrace{dr\cdot b\cdot dr}_{\text{2nd fund. form of surf.}} + (x^3)^2\underbrace{dr\cdot(b\cdot b)\cdot dr}_{\text{3rd fund. form for surface}} + (dx^3)^2$$

$$\underbrace{\mathbb{I}}_{\text{first fund for space}} = \mathrm{I} - 2x^3\,\mathrm{II} + (x^3)^2\,\mathrm{III} + (dx^3)^2$$

We are concerned with strains in 3-D. Hence must look a change in metric must look at changes in I, II, III. If normal is unstretched & normals unchanged then $\mathbb{I} \cong \underbrace{\mathrm{I}}_{\text{this will give membrane strains}} - 2x^3\,\underbrace{\mathrm{II}}_{\text{will give bending effects}}$



$\rho = \rho(x^i)$ undeformed surface
ρ' is deformed surface
$\rho' - \rho = u$ displ vector.

$\rho = r + a_3 x^3$ if we take $x^3 = 0$ we look only at reference surface
$\rho' = r' + x^3 A$

Generally we can't say anything about changes in a_3 unless we make some assumpt. In general A is a fn of all 3 coords.

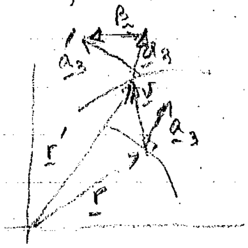
Different theories

To get a workable theory we make kinematic assumptions. If we want pts along normal to stay normal A will become fns of x^1 & x^2 only. But this also means that they will also remain unstretched. 1) To keep the stretching only then define $A = B(x^1, x^2, x^3)\, C(x^1, x^2)$. 2) For pts to remain equidistant $A = A(x^1, x^2)$.

3) If we assume normals remain normals $x^3 A = x^3 a_3'(x^1, x^2)$

With these 3 assumptions we get Love-Kirchoff assumptions

Thus $\rho' = r' + x^3 a_3'$ Thus $u = (r' - r) + x^3 (a_3' - a_3) = \rho' - \rho$
$= v + x^3 \beta$ where v & β are fns of x^1,



Now $\rho' = \rho + u$ $\quad d\rho' = d\rho + du$

$d\rho' \cdot d\rho' = d\rho \cdot d\rho + d\rho \cdot du + du \cdot d\rho + du \cdot du$
$= d\rho \cdot d\rho + 2\, du \cdot d\rho + du \cdot du$

Thus $d\rho' \cdot d\rho' - d\rho \cdot d\rho$ (gives change in metric) $= 2\, du \cdot d\rho + du \cdot du$ nonlinear
gives linear strain measures.

$du = d\rho \cdot \nabla u$ then $d\rho' \cdot d\rho' - d\rho \cdot d\rho = d\rho \cdot 2\nabla u \cdot d\rho + (d\rho \cdot \nabla u) \cdot (d\rho \cdot \nabla u)$

Now $T = T^{\alpha\beta} a_\alpha \otimes a_\beta$ $\quad T^t = T^{\alpha\beta} a_\beta \otimes a_\alpha$

Since we are dotting with same vector on either side

$$d\rho \cdot 2\nabla u \cdot d\rho = d\rho \cdot (\nabla u + \nabla u^t) \cdot d\rho$$
$$(d\rho \cdot \nabla u) \cdot (d\rho \cdot \nabla u) = (d\rho \cdot \nabla u) \cdot (\nabla u^t \cdot d\rho)$$

Thus $d\rho' \cdot d\rho' - d\rho \cdot d\rho = d\rho \cdot \left[\nabla u + \nabla u^t + \nabla u \cdot \nabla u^t\right] \cdot d\rho$
$\mathrm{I}' - \mathrm{I} \quad = 2\, d\rho \cdot e \cdot d\rho$ where $e = \frac{1}{2}\left[\nabla u + \nabla u^t + \nabla u \cdot \nabla u^t\right]$

Lagrangian

$d\rho$ is in terms of the undeformed coords. $\left(\nabla = g^i \otimes \frac{\partial}{\partial x^i}\right)$

for shell theory $\underset{\sim}{u}(x^1,x^2,x^3) = \underset{\sim}{v}(x^1,x^2) + x^3\underset{\sim}{\beta}(x^1,x^2)$

Now $d\underset{\sim}{u} = \frac{\partial \underset{\sim}{u}}{\partial x^\alpha}dx^\alpha + \frac{\partial \underset{\sim}{u}}{\partial x^3}dx^3 = \frac{\partial \underset{\sim}{v}}{\partial x^\alpha}dx^\alpha + x^3\frac{\partial \underset{\sim}{\beta}}{\partial x^\alpha}dx^\alpha + \underset{\sim}{\beta}\,dx^3$

$= d\underset{\sim}{v} + x^3 d\underset{\sim}{\beta} + \underset{\sim}{\beta}\,dx^3$

Now $d\underset{\sim}{u} = \nabla\underset{\sim}{u}\cdot d\underset{\sim}{\rho}$ and $2d\underset{\sim}{u}\cdot d\underset{\sim}{\rho} + d\underset{\sim}{u}\cdot d\underset{\sim}{u} = \mathbb{I}' - \mathbb{I}$

$\underset{\sim}{\rho} = \underset{\sim}{r} + x^3\underset{\sim}{a}_3 \quad d\underset{\sim}{\rho} = d\underset{\sim}{r} + dx^3\underset{\sim}{a}_3 + x^3 d\underset{\sim}{a}_3 \qquad d\underset{\sim}{u} = d\underset{\sim}{v} + x^3 d\underset{\sim}{\beta} + \underset{\sim}{\beta}\,dx^3$

Now $2d\underset{\sim}{u}\cdot d\underset{\sim}{\rho} + d\underset{\sim}{u}\cdot d\underset{\sim}{u} = 2(d\underset{\sim}{v} + x^3 d\underset{\sim}{\beta})\cdot(d\underset{\sim}{r} + x^3 d\underset{\sim}{a}_3) + (d\underset{\sim}{v} + x^3 d\underset{\sim}{\beta})\cdot(d\underset{\sim}{v} + x^3 d\underset{\sim}{\beta})$
$+ d\underset{\sim}{r}\cdot \underbrace{dx^3(\quad)}_{\text{gives transverse shearing strain}} + \underbrace{(dx^3)^2(\quad)}_{\text{gives stretching of normal.}}$

when we make kinematic assumption $(\underset{\sim}{\beta} = \underset{\sim}{a}_3' - \underset{\sim}{a}_3)$ then transverse & stretching terms = 0

$$\mathbb{I}' - \mathbb{I} = \underbrace{2\,d\underset{\sim}{v}\cdot d\underset{\sim}{r} + d\underset{\sim}{v}\cdot d\underset{\sim}{v}}_{\text{surface strain measure}} + x^3(2\,d\underset{\sim}{\beta}\cdot d\underset{\sim}{r} + 2d\underset{\sim}{v}\cdot d\underset{\sim}{a}_3 + 2d\underset{\sim}{v}\cdot d\underset{\sim}{\beta})$$
$$+(x^3)^2(2\,d\underset{\sim}{\beta}\cdot d\underset{\sim}{a}_3 + d\underset{\sim}{\beta}\cdot d\underset{\sim}{\beta}) = (\mathrm{I}' - \mathrm{I}) - 2x^3(\mathrm{II}' - \mathrm{II}) + (x^3)^2(\mathrm{III}' - \mathrm{III})$$