ME 241 Steele, C. Theory of Plates

Prereqs: CE 114 (Strength of Materials)

Math: ODE; PDE; $\Delta\varphi = 0$ $\Delta^2\varphi = p$

Approach will be mathematical

Objectives $\rightarrow$ fundamental understanding & Computational capability

Method : solve problems weekly assignments

Grade : 50% HW 50% in-class Final (<u>not</u> open book)

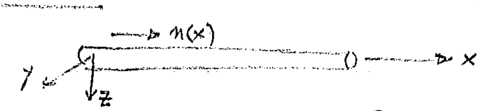
<u>What is a plate :</u>

first 3-D Structures are usually not efficient usage of structure

1-D wire / beam } one dim is >> than other 2 dimensions $EIw''' = p(x)$

2-D plate one dim is small wrt the other 2 $EI\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = p(x,y)$

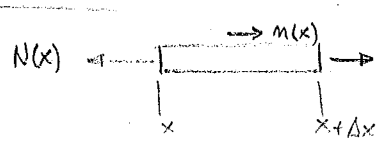
<u>Stretching of a rod</u>



$n(x)$ = distributed load / unit length

This load produces a 3-D stress. Since the length >> the radius we can define a stress resultant $N(x) = \iint \sigma_x \, dA$ to get rid of 3 dimensionality

<u>looking at equilibrium</u>



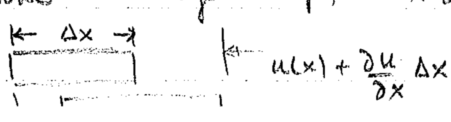
$$N(x+\Delta x) = N(x) + \left(\frac{\partial N}{\partial x}\right)_x \Delta x + O(x)$$

thus $\left(\frac{\partial N}{\partial x}\right)_x \Delta x + \dots + n(x)\Delta x = 0$; taking limit as $\Delta x \to 0$ is $\boxed{\frac{dN}{dx} = -n(x)}$

if $n(x)$ is known then $\boxed{N(x) = -\int_{x_0}^{x} n(x)\,dx + N(x_0)}$

Normally we don't know $N(x_0)$ but we do know what displacements are

thus define average disp, in x direction $u(x) = \frac{1}{A}\int_A u \, dA$

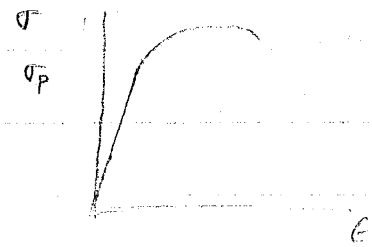


we can thus define the engineering strain $\epsilon = \dfrac{\text{final length} - \text{init length}}{\text{initial length}} = \dfrac{\frac{\partial u}{\partial x}\Delta x}{\Delta x} = \dfrac{\partial u}{\partial x}$

$$\boxed{\epsilon = \frac{\partial u}{\partial x}}$$

This is relation that defines the displacement in terms of the strain

Now we use the stress strain relation to relate the stresses to the strain



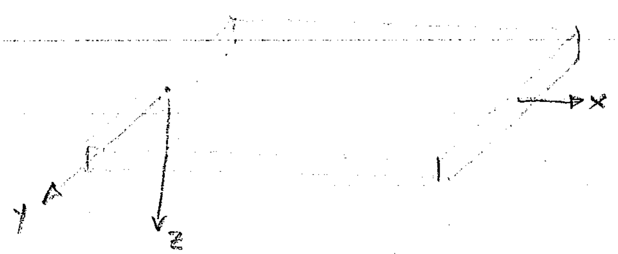
Generally $|\sigma| < |\sigma_p|$ in design – we thus use Hooke's law $\sigma = E\epsilon$ E is Young's modulus where σ is the 3-D stress

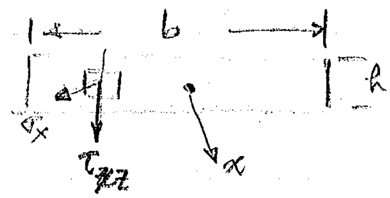
now $N(x) = \int \sigma dA = EA\epsilon$ if σ is a constant

now $\dfrac{dN}{dx} = -n(x) \Rightarrow \boxed{\dfrac{\partial}{\partial x}\left(EA\dfrac{\partial u}{\partial x}\right) = -n(x)}$

this is the relation between the distributed load and the displacement

9/28 Beams in bending



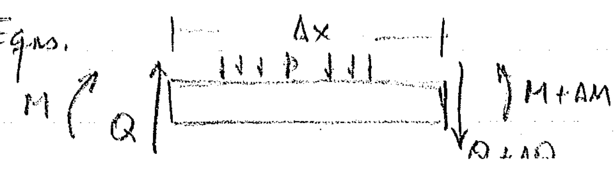


Define stress Resultants

$$Q = \int_{-h/2}^{h/2} \tau_{zx}\, dz \qquad \text{force per unit width}$$

$$M = \int_{-h/2}^{h/2} \sigma_x z\, dz \qquad \text{moment/unit width}$$

Equilib Eqns.



Force: $\Delta Q + p\,\Delta x \sim 0$ $\qquad$ p = load/unit width

$$\lim_{\Delta x \to 0} \frac{\Delta Q}{\Delta x} = -p \quad \text{or} \quad \boxed{\frac{dQ}{dx} = -p} \qquad (1)$$

Moment $p\Delta x \cdot \frac{\Delta x}{2} + (Q + \Delta Q)\Delta x - M - \Delta M + M \sim 0$ or $\Delta M \sim Q\Delta x$

$$\Rightarrow \boxed{\frac{dM}{dx} = Q} \qquad (2)$$

$$(1)\ \&\ (2) \Rightarrow \boxed{\frac{d^2M}{dx^2} = -p} \qquad (3)$$

<u>Displacement of midsurface</u>

w – in z-dir

0 in x-dir

For the displacement of a point a distance z from the midsurface we must make a kinematic hypothesis

For Beams – Bernoulli or Euler

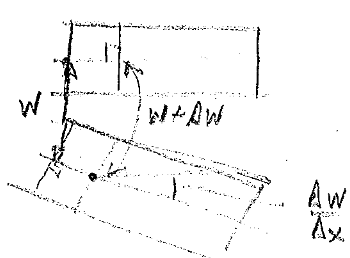
For plate – Kirchhoff

For shell – Kirchhoff, Love

<u>Assume:</u>

(1) A normal to the neutral surface remains straight & unstretched (plane surfaces remain plane)

(2) normals remain ⊥ to the neutral surface



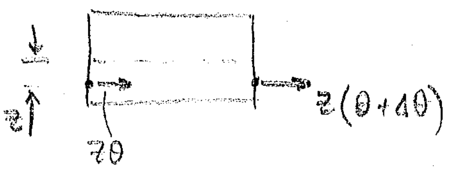
Displacement of pt a distance z from the midsurface is:

w – in z direction

zθ – in x direction

Thus rotation of the midsurface $\boxed{\theta = -\frac{dw}{dx}}$ (4)

Strains



$$\epsilon_x \cong z\,\frac{\Delta\theta}{\Delta x}$$

or $$\boxed{\epsilon_x = z\,\frac{d\theta}{dx}} \qquad (5)$$

Using the stress strain law

$$\sigma_x = E\epsilon_x$$

$$M = \int_{-h/2}^{h/2} \sigma_x z\,dz = \int E\,\frac{d\theta}{dx} z^2 dz = E\,\frac{d\theta}{dx}\,\frac{h^3}{12} \qquad \text{using (5)}$$

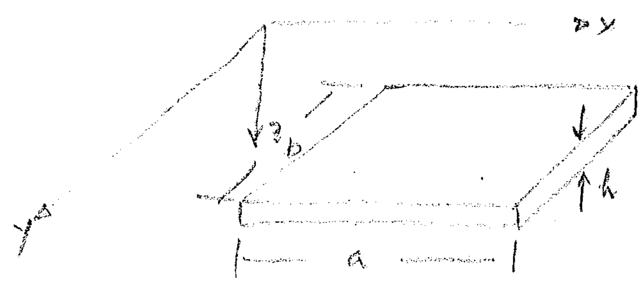
$$Mb = E\frac{d\theta}{dx}\,\frac{bh^3}{12} = EI\,\frac{d\theta}{dx}$$ for a rectangular beam.

or $$\boxed{Mb = EI\left(-\frac{d^2w}{dx^2}\right)} \quad \text{using (4)} \qquad (6)$$

but $$\frac{d^2M}{dx^2} = -p \quad (3) \qquad \therefore \quad \boxed{pb = \frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right)} \qquad (7)$$

beam bending in terms of displacement

10/1/79 Definition of a plate



Assume: $h \ll a, b$

(2) plate is elastic, homog, isotropic

(3) small deflections $w \le \frac{h}{5}$

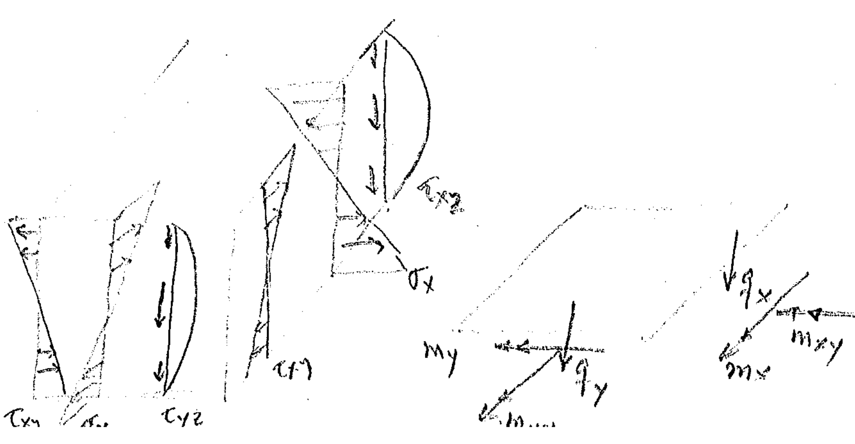
(4) $\frac{\partial w}{\partial x}$ & $\frac{\partial w}{\partial y} \ll 1$

(5) Kirchhoff Hypothesis
- normals remain straight & unstrec and normal
- $w = w(x,y)$ only

(6) Inextensional theory - no stretching

(7) no initial imperfections

(8) stresses normal to midsurface are negligible $\sigma_z \ll \sigma_x, \sigma_y$

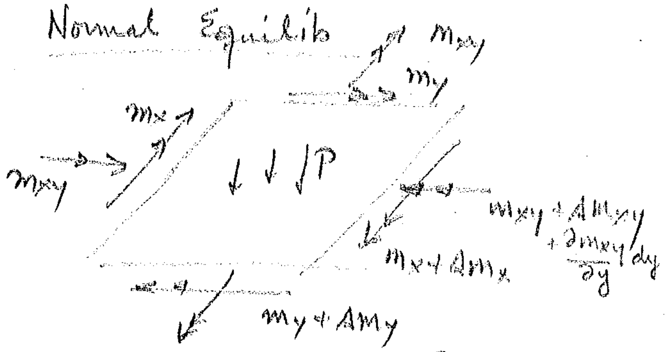


Define our stress resultants

$$q_x = \int_{-h/2}^{h/2} \tau_{xz}\,dz \qquad q_y = \int_{-h/2}^{h/2} \tau_{yz}\,dz \qquad m_x = \int_{-h/2}^{h/2} \sigma_x z\,dz \qquad m_y = \int_{-h/2}^{h/2} \sigma_y z\,dz$$

$$m_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\,dz \qquad m_{yx} = \int_{-h/2}^{h/2} \tau_{yx} z\,dz$$

Normal Equilib



$$m_{xy} + \Delta m_{xy} = m_{xy} + \frac{\partial m_{xy}}{\partial x}\,dx$$

$$\frac{\partial q_y}{\partial y} \Delta y \Delta x + \frac{\partial q_x}{\partial x} \Delta x \Delta y + p \Delta x \Delta y \cong 0$$

or $$\boxed{\frac{\partial q_y}{\partial y} + \frac{\partial q_x}{\partial x} + p = 0} \qquad (1)$$

Moment Equil in x-direction

$$\frac{\partial m_y}{\partial y} \Delta x \Delta y + \frac{\partial m_{xy}}{\partial x} \Delta x \Delta y - q_y \Delta x \Delta y \cong 0$$

or $$\boxed{\frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} - q_y = 0} \qquad (2)$$

Moment Equil in y dir:

$$\boxed{\frac{\partial m_{yx}}{\partial y} + \frac{\partial m_x}{\partial x} - q_x = 0} \qquad (3)$$

take $\frac{\partial}{\partial y}(2)$ $\frac{\partial}{\partial x}(3)$ add & substitute (1) to give

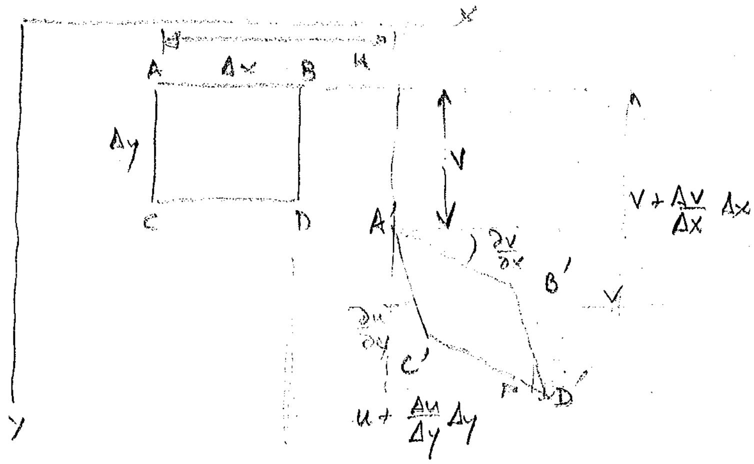
$$\boxed{\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} + P = 0} \qquad (4)$$

Displacements $\quad w = w(x, y)$

$$u = z\,\theta_x = -z\frac{\partial w}{\partial x} \qquad v = z\,\theta_y = -z\frac{\partial w}{\partial y}$$

Strains

$$\epsilon_x = z\frac{\partial \theta_x}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \qquad \epsilon_y = z\frac{\partial \theta_y}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}$$



$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

Stress - Strain relations

$$\epsilon_x = \tfrac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

neglect $\sigma_z << \sigma_x, \sigma_y$

$$\epsilon_y = \tfrac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\epsilon_x + \nu\epsilon_y\right)$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\epsilon_y + \nu\epsilon_x\right)$$

obtained by inverting ϵ_x, ϵ_y for σ_x, σ_y

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

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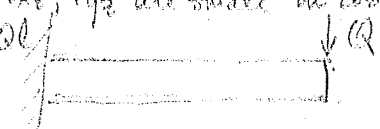
to show τ_{xz}, τ_{yz} are small in comparison to σ_x



$$\sigma_x = \frac{Mc}{I} = \frac{6M}{h^2} = \frac{6Mh}{2 \cdot h^3/12} = \frac{6Q}{h}\left(\frac{l}{h}\right)$$

$$\text{aver } \tau_{xz} = \frac{Q}{h} \Rightarrow \sigma_x >> \tau_{xz}, \tau_{yz}$$ thus neglect τ_{xz}, τ_{yz}

however we do not neglect q_x, q_y

$$m_x = \int_{-h/2}^{h/2} \sigma_x z\,dz = \frac{E}{1-\nu^2}\left(\frac{\partial\theta_x}{\partial x} + \nu\frac{\partial\theta_y}{\partial y}\right)\int_{-h/2}^{h/2} z^2 dz = \boxed{\frac{-EI}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = m_x}$$

in doing this σ_x is assumed constant across cross section $D = \frac{EI}{1-\nu^2} = \frac{Eh^3}{12(1-\nu^2)}$ Bending Rigidity

Constitutive Eqs

Thus $\boxed{m_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)}$ $\boxed{m_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]}$

$$m_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\,dz = \frac{E}{2(1+\nu)} \int_{-h/2}^{h/2} -2 \frac{\partial^2 w}{\partial x \partial y} z^2 dz = -\frac{EI}{1+\nu} \frac{\partial^2 w}{\partial x \partial y} = \boxed{-D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = m_{xy}}$$

now $q_x = m_{x,x} + m_{xy,y}$

$$= -D\left\{\frac{\partial}{\partial x}\left[w_{,xx} + \nu w_{,yy}\right] + (1-\nu)\frac{\partial}{\partial y}(w_{,xy})\right\} = -D\frac{\partial}{\partial x}(\Delta w) = \boxed{-D\frac{\partial}{\partial x}(\nabla^2 w) = q_x}$$

$$\therefore \quad \boxed{q_y = -D\frac{\partial}{\partial y}(\nabla^2 w)}$$

Normal equil eqn.

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + p = 0 \quad \Rightarrow \quad -D\left(\nabla^2\left(\nabla^2 w\right)\right) + p = 0$$

or $\boxed{\Delta^2 w = \nabla^4 w = p/D}$ beam bending eq in terms of displacement

Nonhomogeneous Biharmonic Eqn.

$\nabla^2 w = 0$ Laplace $\qquad \nabla^2 w \neq 0$ Poisson $\qquad \nabla^4 w = 0$ Biharmonic

Define Curvature strain measures

$$K_x = -\frac{\partial^2 w}{\partial x^2} \qquad K_y = -\frac{\partial^2 w}{\partial y^2} \qquad K_{xy} = -\frac{\partial^2 w}{\partial x \partial y}$$

Stress Resultants in terms of strain measures (Kinematic relations)

$$m_x = D(K_x + \nu K_y) \qquad m_y = D(K_y + \nu K_x) \qquad m_{xy} = D(1-\nu)K_{xy}$$

<u>Summary</u> of how we got governing eqns.

- Eqns of Equil.
- Strain Displ relations
- Stress - Strain Relations
  * stress resultants & strain measures
- kinematic relations

Boundary Conditions

Since 4^{th} order ODE need 4 bc. (2 on each end)

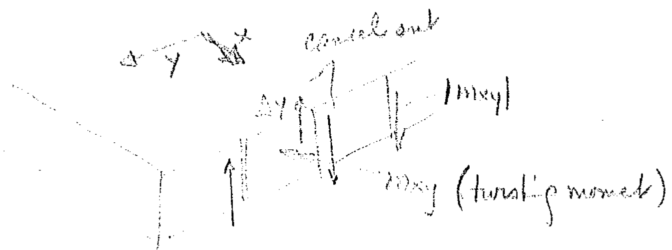
<u>Geometric BC</u>

$$w, \frac{\partial w}{\partial x} \quad \text{on } x = \text{const}$$

$$w, \frac{\partial w}{\partial y} \quad \text{on } y = \text{const}$$

– fixed edge $\quad w = 0 \quad \frac{\partial w}{\partial x} = 0 \quad$ at $x = a$

<u>Stress BC</u> $\Longrightarrow$ m_x q_x & m_{xy} on $x = \text{const.}$



can be replaced by a free couple of magnitude $|m_{xy}|$. This adds to the vertical shear

Net Shear resultant on edge is $\quad v_x = q_x + \frac{\partial m_{xy}}{\partial y}$

$$v_y = q_y + \frac{\partial m_{xy}}{\partial x}$$

Thus our BC is v_x & m_x on $x = \text{const.}$

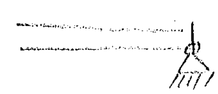
on a free edge $\quad m_x = v_x = 0$

$$v_x = f(w) = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]$$

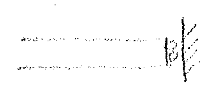
$$v_y = g(w) = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]$$

<u>Mixed boundary conditions</u>

Simply - Supported Edge $\quad w = 0, \quad m_x = 0 \quad$ at $x = \text{const.}$



$$v_x = 0 \quad \frac{\partial w}{\partial x} = 0$$



Summary

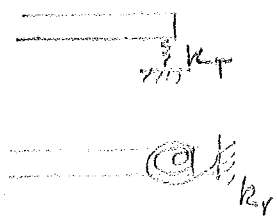
fixed $w = \frac{\partial w}{\partial x} = 0$

free $m_x = v_x = 0$

Simply Supported $w = 0$, $m_x = 0$

elastic support $m_x = 0$ $w = \frac{v_x}{k_t}$

$v_x = 0$ $\frac{\partial w}{\partial x} = \frac{m_x}{k_r}$



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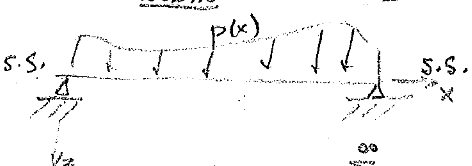
Fourier Series method of sol.

Beam $EI\, w^{IV} = p(x)$



simple support $w=0$ $M=0$ or $w=0$ & $\frac{\partial^2 w}{\partial x^2} = 0$

look at sol. $w = \sum_{n=1}^{\infty} w_n \sin \frac{n\pi x}{L}$ this satisfies B.C. : any sufficiently smooth fn. can be represented by this series

put w into ODE each term satisfies BC identically & doesn't necessarily solve de ident.

$$\therefore \quad EI \sum_{n=1}^{\infty} w_n \left(\frac{n\pi}{L}\right)^4 \sin \frac{n\pi x}{L} = p(x) \quad \Rightarrow \quad \frac{2EI}{L}\int_0^L \sum w_n \left(\frac{n\pi}{L}\right)^4 \sin\frac{n\pi x}{L} \sin \frac{k\pi x}{L}\, dx = \frac{2}{L}\int_0^L \sin\frac{k\pi x}{L}\, p(x)$$

$$\Rightarrow \quad EI\, w_k \left(\frac{k\pi}{L}\right)^4 = \frac{2}{L}\int_0^L p(x) \sin\frac{k\pi x}{L}\, dx \qquad \therefore \boxed{w_k = \frac{2}{EIL}\left(\frac{L}{k\pi}\right)^4 \int_0^L p(x) \sin \frac{k\pi}{L}}$$

$$\therefore \quad \boxed{w_k = \frac{2}{EIL}\left(\frac{L}{k\pi}\right)^4 \int_0^L p(x) \sin \frac{k\pi x}{L}\, dx}$$

alternate write $p(x) = \sum p_n \sin\frac{n\pi x}{L}$ $\quad p_n = \frac{2}{L}\int_0^L p(x) \sin\frac{n\pi x}{L}\, dx$

$$\Rightarrow \quad EI\left(\frac{n\pi}{L}\right)^4 w_n = p_n \quad \Rightarrow \quad \boxed{w_n = \frac{p_n}{EI}\left(\frac{L}{n\pi}\right)^4}$$

very good convergence in comparison to p series

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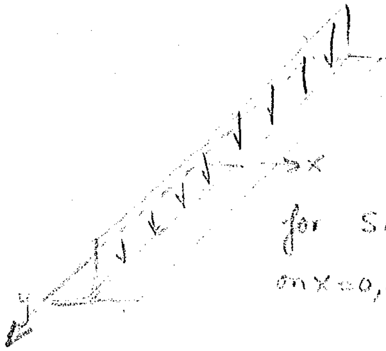
Long Plate strip



for S.S. on $x=0,a$

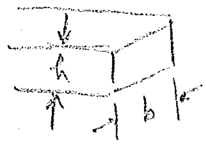
Plate Eq: $D\Delta\Delta w = p$

for plate strip $p = p(x)$

then we expect $w = w(x)$

Eqn reduces $D\dfrac{d^4w}{dx^4} = p$

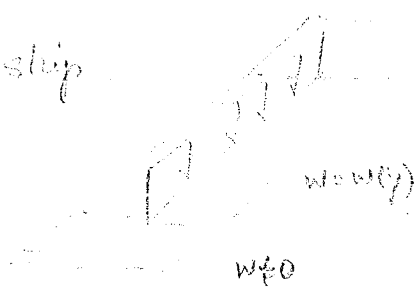
Similar to beam problem $EI\dfrac{d^4w}{dx^4} = pb$



$I = \dfrac{bh^3}{12}$ $\Rightarrow$ $\dfrac{Eh^3}{12}\dfrac{d^4w}{dx^4} = p$ but $D = \dfrac{Eh^3}{12(1-\nu^2)}$

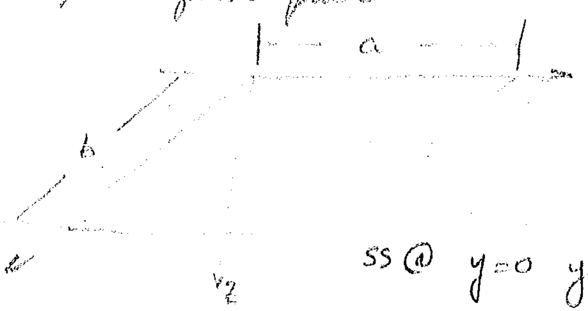
thus only need to modify beam solution by $1-\nu^2$

for a plate strip



cut plate, solve problem & then try to match displacements at end .ꞏ. we get cont. displ rotation & shears.

For a rectangular plate



SS @ $x=0$ $x=a$ $\quad w=0$, $M_x=0$

$M_x = D(K_x + \nu K_y) = -D(w_{xx} + \nu w_{yy})$

$\Rightarrow w_{xx} = w_{yy}$ (from $w=0 \Rightarrow w_y = w_{yy} = 0$) $=0$

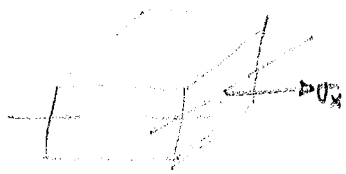
SS @ $y=0$ $y=b$ $\quad \left.\begin{matrix} w=0 \\ M_y=0 \end{matrix}\right\} \Rightarrow w_{yy}=0$

Solution for $p(x,y) = p_0 \sin\dfrac{\pi x}{a}\sin\dfrac{\pi y}{b}$

Let $w = w_0 \sin\dfrac{\pi x}{a}\sin\dfrac{\pi y}{b}$ $\Rightarrow$ $\Delta(w) = \left[-\left(\dfrac{\pi}{a}\right)^2 - \left(\dfrac{\pi}{b}\right)^2\right]\cdot w$

$$\Delta\Delta w = \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]^2 w = p/D \quad \Rightarrow \quad \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]^2 w_0 = p_0/D \quad \Rightarrow w_0 = \frac{p_0}{D\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]^2}$$

Stresses



3-D Stress Strain

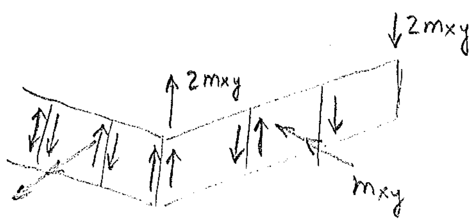
$$E\varepsilon_x = \sigma_x - \nu(\sigma_y + \sigma_z)$$

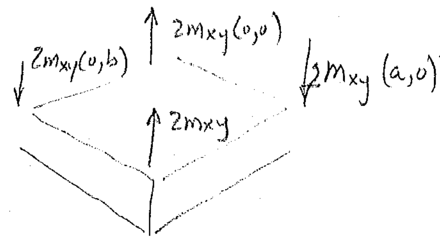
$$E\varepsilon_y = \sigma_y - \nu(\sigma_x + \sigma_z) \qquad \text{if plate is thin } \sigma_z \cong 0$$

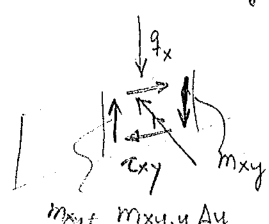
$$\Rightarrow \quad \sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) \qquad \varepsilon_x = z\frac{\partial\theta_x}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} = z\kappa_x$$

$$\sigma_x = \frac{Ez}{1-\nu^2}\left(\kappa_x + \nu\kappa_y\right) = \frac{-Ez}{1-\nu^2}\left(w_{xx} + \nu w_{yy}\right)$$

10/12/79







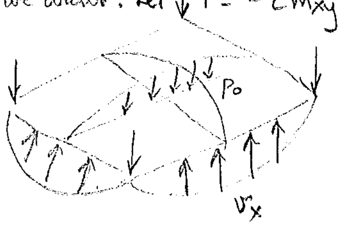
$$\Rightarrow \quad V_x = q_x + \frac{\partial m_{xy}}{\partial y} \qquad m_{xy} = \int_{-h/2}^{h/2} z\,\tau_{xy}\,dz$$

Continuation of previous result 10/8

$$m_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = -D(1-\nu)W_0\frac{\pi^2}{ab}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}$$

$$m_{xy}(0,0) = -D(1-\nu)\frac{\pi^2}{ab}W_0 \qquad |F| = |2m_{xy}|$$

if we want: Let $\downarrow F = -2m_{xy}(0,0)$



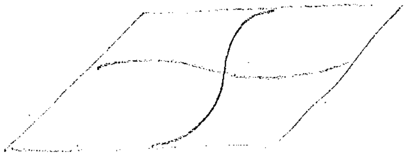
now $\dfrac{\partial m_{xy}}{\partial y} \sim +\sin\dfrac{\pi y}{b}\cos\dfrac{\pi x}{a}$

$$q_x = -D\frac{\partial}{\partial x}\Delta w \quad \sim +\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

V_x is + since $q_x + \dfrac{\partial m_{xy}}{\partial y}$ as $+\sin\dfrac{\pi y}{b}\cos\dfrac{\pi x}{a}$

Navier - Double Fourier Series soln For general $p = p(x,y)$ s.s. edges on plate $a \times b \times l$

Look for soln



$$w(x,y) = \sum_n \sum_m w_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}.$$

$$\therefore\ p(x,y) = \sum_n \sum_m p_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

$$p_{mn} = \frac{2}{a}\int_0^a \frac{2}{b}\int_0^b p(x,y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}\, dx\,dy$$

$$\therefore\ D\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^2 w_{mn} = p_{mn}$$

for $p_0 = p(x,y)$ $\qquad p_{mn} = \dfrac{16 p_0}{mn\pi^2}$ $\qquad w(a/2, b/2) = \displaystyle\sum_{n=1,3,\ldots}\sum_{m=5} \frac{16 p_0 \,(-1)^{n+m}}{\pi^2 D\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^2 mn}$

10/15/79

<u>Soln of PDE / ODE</u>

$\dfrac{d^2u}{dx^2} + ku = 0$ if $k =$ const. Let $u = e^{\lambda x}\cdot A$ $\qquad A =$ const

$\Rightarrow$ char eqn $[\lambda^2 + k = 0]\cdot Ae^{\lambda x}$ $\qquad \lambda^2 = -k$ $\qquad$ <u>if</u> $k = -p^2$ $\quad p > 0$

$\lambda^2 = p^2$ $\qquad \lambda_{1,2} = \pm p$

$\therefore\ A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x} = u$ $\qquad u = A_1 e^{px} + A_2 e^{-px}$

for a semi infinite region let $A_1 = 0$ $\qquad$ (bdd @ $+\infty$)

<u>if</u> $k = p^2$ $\qquad \lambda^2 = -p^2$ $\qquad \lambda_{1,2} = \pm ip$

→ for a poly w/real coeffs if we have complex roots they must appear in conjugate pairs.

→ if $\lambda_2 = \bar{\lambda}_1 \Rightarrow A_2 = \bar{A}_1$ for u to be real $\Rightarrow$ $u = 2\,\mathrm{Re}\,(A_1 e^{\lambda_1 x})$

$= 2[\mathrm{Re}\,A_1 \cos px - \mathrm{Im}\,A_1 \sin px]$ $\qquad$ let $2\,\mathrm{Re}\,A_1 = B_1$

$u = B_1 \cos px + B_2 \sin px$ $\qquad B_1\ \&\ B_2$ are real $\qquad -2\,\mathrm{Im}\,A_1 = B_2$

for n^{th} order system with real coeffs. if you find one complex soln $u = u_1$, $\exists$ another solution $u = u_2 = \bar{u}_1$

$\Delta u = 0 \qquad (\partial_x^2 + \partial_y^2)u = 0 \qquad$ Let $u = Ae^{\lambda x + \mu y}$

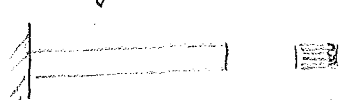
$\Rightarrow (\lambda^2 + \mu^2)u = 0 \Rightarrow \lambda^2 + \mu^2 = 0$ Let $\lambda = ip \quad p > 0$

$\mu^2 = -\lambda^2 = p^2 \quad \mu = \pm p$

$\therefore \quad u = A_1 e^{ipx + py} + A_2 e^{+ipx - py} \Rightarrow \cos px\, e^{-py}, \sin e^{py}$ are soln $\therefore$ u is bdd $(A_1 = 0)$

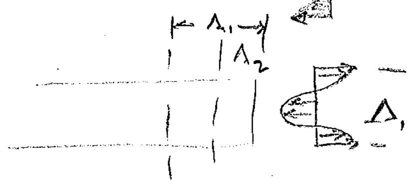
if Λ (one cycle) $= \frac{2\pi}{p} \Rightarrow$ soln decay at dist Λ: $e^{-p\Lambda} = e^{-2\pi} = .002$

Look at beam w/ following distrib

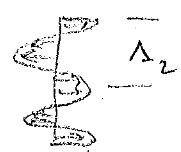


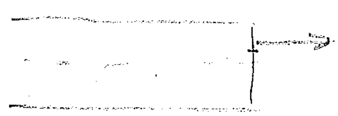
caused by extensional loading (global effect) felt at all stations

moment at end global effect felt at all stations



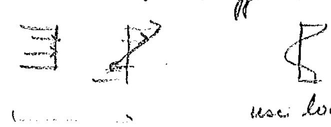
Λ_1 no load & moment (total) causes only local effect





offset will have load, moment + local effects

for large distances

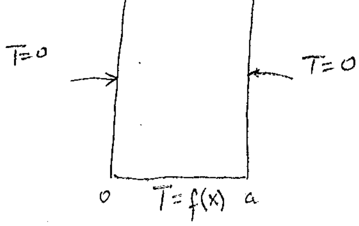


use element beam theory

use local description to define what goes on.

10/17/79

PDE : Steady State Heat Conduction $(\Delta T = 0)$ in a strip



Can have a solutions of form $T = e^{\lambda x + \mu y}$

put into DE $\quad \lambda^2 + \mu^2 = 0 \quad$ choose $\lambda = \frac{in\pi}{a} \Rightarrow \mu = \pm \frac{n\pi}{a}$

we want fn to $\to 0$ as $y \to \infty \Rightarrow \mu = -\frac{n\pi}{a}$

$T = A_n e^{\frac{in\pi x}{a}} e^{-\frac{n\pi y}{a}}$ since $T = 0$ at $x = 0$ & a take $\sin \frac{n\pi x}{a}$

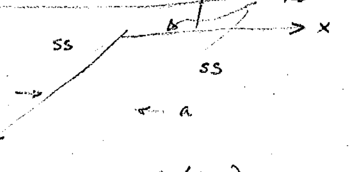
$\therefore \quad T = \sum A_n \sin \frac{n\pi x}{a} e^{-\frac{n\pi y}{a}}$

let $f(x) = \sum f_n \sin \frac{n\pi y}{a}$ $\quad \therefore \quad T(x,0) = \sum A_n \sin \frac{n\pi x}{a} = \sum f_n \sin \frac{n\pi y}{a} \Rightarrow f_n = A_n$

This PDE & all bdy conds. also.

Analogue in Plate theory $\quad$ Let $w(x,0) = 0$ & $m_y(x,0) = f(x)$ $\quad$ hinged edge

Strip



Choose a solution $w(x,y) = Y(y) \sin \lambda x$ $\quad$ since $\lambda = \frac{n\pi}{a}$

$$\Delta(\Delta w) = \Delta(-\lambda^2 Y + Y'') \sin \lambda x = 0 \quad \text{since no load/unit length}$$
$$= (\lambda^4 Y - 2\lambda^2 Y'' + Y^{IV}) \sin \lambda x = 0$$

Solving $(Y^{IV} - 2\lambda^2 Y'' + \lambda^4 Y) = 0$ $\quad Y = (e^{\lambda y}, e^{-\lambda y})$ in general try $e^{\mu y}$

$$\mu^4 - 2\lambda^2 \mu^2 + \lambda^4 \quad \text{or} \quad (\mu^2 - \lambda^2)^2 = 0$$

$$\therefore \quad Y = (A_1 + B_1 y) e^{\lambda y} + (C_1 + D_1 y) e^{-\lambda y}$$

for bdd solns at $y = +\infty$ $\quad$ choose $A_1, B_1 = 0$ $\quad \lambda = \frac{n\pi}{a}$ $\quad$ Choose $D_1 = B_n \lambda_n \quad C_1 = A_n$

$$\therefore \quad \left| \; w(x,y) = \sum_{n=1}^{\infty} (A_n + B_n \lambda_n y) e^{-\lambda_n y} \sin \lambda_n x \; \right|$$

This satisfies PDE & S.S. at $x = 0, a$.

for $w(x,0) = 0 \Rightarrow A_n \equiv 0 \;\; \forall n$.

$$w_{,y}(x,y) = \sum \lambda_n B_n (-\lambda_n y + 1) e^{-\lambda_n y} \sin \lambda_n x$$

Since $w(x,0) = 0 \Rightarrow \frac{\partial w}{\partial x} = 0 \Rightarrow \frac{\partial^2 w}{\partial x^2} = 0$. Thus $M_y(x,0) = f(x) = -D \frac{\partial^2 w}{\partial y^2}(x,0)$

$$\therefore \quad w_{,yy} = \sum \lambda_n^2 B_n \left[-1 - 1 + \lambda_n y \right] e^{-\lambda_n y} \sin \lambda_n x$$
$$= \sum \lambda_n^2 B_n \left[\lambda_n y - 2 \right] e^{-\lambda_n y} \sin \lambda_n x$$

@ $y = 0$: $-D \left.\frac{\partial^2 w}{\partial x}\right|_{x, y=0} = f(x) = +2D \sum \lambda_n^2 B_n \sin \lambda_n x$ $\quad$; if $f(x) = \sum f_n \sin \lambda_n x \Rightarrow$

$$\boxed{\frac{+f_n}{2D\lambda_n^2} = B_n}$$

10/18/79

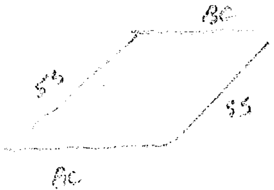
Since we have a 4^{th} order pde ; on each bdy we require 2 bc $\Rightarrow$

for the case where we have M_y, M_{xy}, q_y are prescribed we have too many bc

but M_{xy} & q_y are coupled & we can prescribe $V_y = q_y + \frac{\partial M_{xy}}{\partial x}$ & $\frac{\partial w}{\partial y}$

or w and M_y

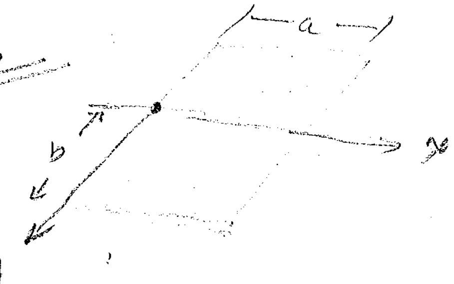
Given



$$w(x,y) = \sum \sin\frac{n\pi x}{a}\left\{\left(A_n + B_n\frac{n\pi y}{a}\right)e^{-\frac{n\pi y}{a}} + \left(C_n + D_n\frac{n\pi y}{a}\right)e^{\frac{n\pi y}{a}}\right\}$$

this is very tedious: must try to use symmetry of problem.

thus

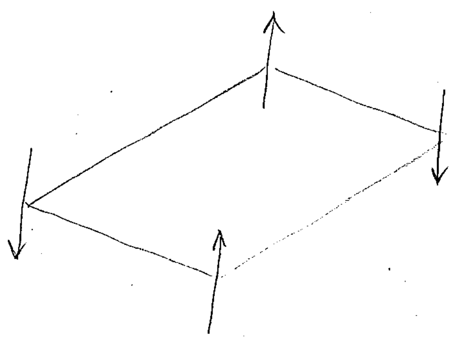


$|y| \le b \quad 0 < x \le a$

use $\cosh z$, $\sinh z$ to help in the symmetry.

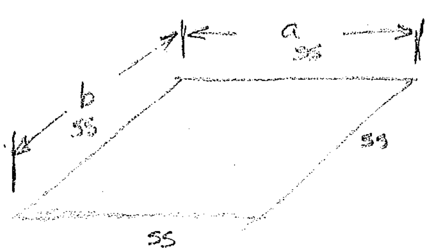
$$\therefore \underset{w \text{ even about } y=0}{w(x,y)} = \sum \sin\lambda_n x\left[A_n\cosh\lambda_n y + B_n\lambda_n y\sinh\lambda_n y\right]$$

$$\underset{w \text{ odd about } y=0}{w(x,y)} = \sum \sin\lambda_n x\left[A_n\sinh\lambda_n y + B_n\lambda_n y\cosh\lambda_n y\right]$$



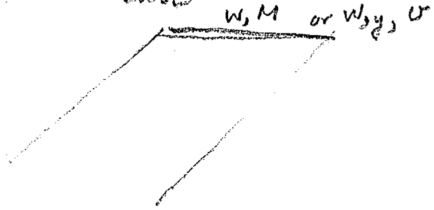
directions of arrow are + sign convention of corner forces

10/22/79



for $D\Delta^2 w = p$ we assumed w was a double fourier series and assumed p could be expanded as a double fourier series $p = \sum p_{mn}\sin(\ )\sin(\ )$

The 2nd case $\quad p = 0 \quad \Delta^2 w = 0$ with given B.C.



Non Homog diff eqn. $L[w]=0$

Homog if w_1 is a soln then $w_2 = Cw_1$ is also a soln.

$D\Delta^2 w = p$ is a non homog diff eq due to pressure load.

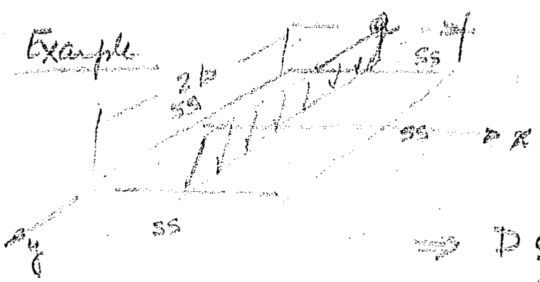
for a non hom. eq: General Procedure

$w = w_p + w_c$ where $D\Delta^2 w_p = p$ but need not satisfy b.c.

and the find a w_c $\ni$ $\Delta^2 w_c = 0$ which satisfies the bc. $w_{c_{BC}} = w_{BC} - w_{p_{BC}}$

This only works for a linear system

Example



for $P = P(x)$ only w/ BC $x = 0, a$

then let $W_p = \sum W_n \sin\left(\frac{n\pi x}{a}\right)$

$P = \sum P_n \sin\left(\frac{n\pi x}{a}\right)$

$$\Rightarrow D\frac{d^4 w_p}{dx^4} = p(x) \Rightarrow D\left(\frac{n\pi}{a}\right)^4 W_{p_n} = p_n \quad \therefore \quad W_{p_n} = \frac{p_n}{D\left(\frac{n\pi}{a}\right)^4}$$

this is a solution to cylindrical bending.

Now we look for solution $w_c = \sum w_{mn_c} \sin(\quad)\sin(\quad)$

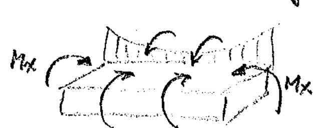
since we have SS on $y = 0, a$ $\Rightarrow$ $w = 0$ & $M_y = 0$

$$\Rightarrow w_p \neq 0 \text{ \& } M_{y_p} = -D\left(\frac{\partial^2 w_p}{\partial y^2} + \nu\frac{\partial^2 w_p}{\partial x^2}\right) \neq 0 \quad \text{on } y = 0 \text{ \& } a$$

Thus we expect w_p to contribute to the bc.

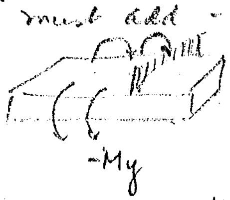
$\therefore$ $w = 0 \Rightarrow w_c = -w_p$

$M_y = 0 \Rightarrow M_{y_c} = -M_{y_p} = D\nu\frac{\partial^2 w_p}{\partial x^2}$



So bars with free lateral faces undergoes curvature change in y-direction of opposite sign to the x-dir curvature

for free lateral faces must add $-M_y$



we can now choose $w_c = \sum w_n \sin\frac{n\pi x}{a}\left[A_n \cosh\frac{n\pi}{a}y + B_n \frac{n\pi y}{a}\sinh\frac{n\pi y}{a}\right]$

we have used the symmetry of soln to choose the y dependence of w

at $y=b$ $\quad w = w_c + w_p = 0$

$M_y = M_{y_p} + M_{y_c} = 0$

$$\partial w_c/\partial y = \sum \sin \lambda_n x \,(\lambda_n)\left[A_n \sinh \lambda_n y + B_n\left(\sinh \lambda_n y + \lambda_n y \cosh \lambda_n y\right)\right]$$

$$\frac{\partial^2 w_c}{\partial y^2} = \sum \sin \lambda_n x \,(\lambda_n)^2\left[A_n \cosh \lambda_n y + B_n\left(2\cosh \lambda_n y + \lambda_n y \sinh \lambda_n y\right)\right]$$

$$(M_y)_c = -D \sum \sin \frac{n\pi x}{a} \,(\lambda_n)^2\left[(1-\nu) A_n \cosh \lambda_n y + B_n\left\{(\lambda_n y \sinh \lambda_n y)(1-\nu) + 2\cosh \lambda_n y\right\}\right]$$

$$w_c = -w_p = -\sum w_{pn} \sin \frac{n\pi}{a} x$$

$$\frac{(M_y)_c}{-D} = -\frac{(M_y)_p}{-D} = \nu \sum w_{pn} \lambda_n^2 \sin \frac{n\pi x}{a}$$

$\Rightarrow$ from $w_c\Big|_{y=b}$: $\left[A_n \cosh \frac{n\pi b}{a} + B_n \lambda_n b \sinh \lambda_n b\right] = -w_{pn}$

for $\frac{(M_y)_c}{-D}\Big|_{y=b}$: $\cancel{\lambda_n^2}\left[(1-\nu) A_n \cosh \lambda_n b + B_n\left(\lambda_n b \sinh \lambda_n b (1-\nu) + 2\cosh \lambda_n b\right)\right]$

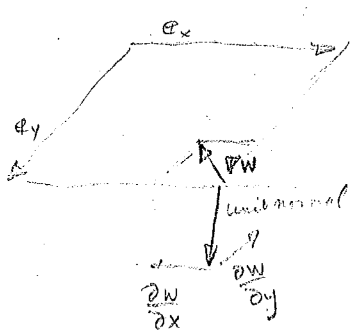
$$= \nu \cancel{\lambda_n^2} w_{pn}$$

$$-(1-\cancel{\nu}) w_{pn} + B_n \cdot 2\cosh \lambda_n b = \cancel{\nu w_{pn}} \qquad B_n = \frac{w_{pn}}{2\cosh \lambda_n b}$$

$$\Rightarrow A_n = \frac{-w_{pn}}{\cosh \lambda_n b} - B_n \lambda_n b \tanh \lambda_n b = \frac{-w_{pn}}{\cosh \lambda_n b} - \frac{w_{pn}}{2\cosh \lambda_n b} \lambda_n b \tanh \lambda_n b$$

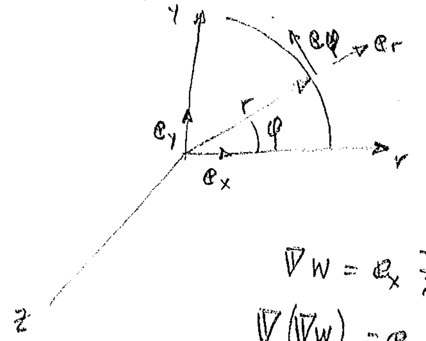
$$= \frac{-w_{pn}}{2\cosh \lambda_n b}\left[2 - \lambda_n b \tanh \lambda_n b\right]$$

10/24/79



$\nabla w = e_{x_i} \frac{\partial w}{\partial x_i}$ $\quad$ vector ∇w gives magnitude & direction of the displ of unit normal if $w = w(x,y)$

If we choose a different coordinate system ∇w will not change but its representation will no longer be the same as before.



$$\therefore \quad e_r \frac{\partial w}{\partial r} + e_\varphi \frac{\partial w}{r\partial \varphi} = \nabla w\,(r,\varphi).$$

$$\Delta w = \nabla^2 w = \nabla\cdot\nabla w = \text{Divergence of the grad of } w$$

$$\nabla w = e_x \frac{\partial w}{\partial x} + e_y \frac{\partial w}{\partial y}$$

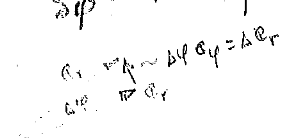
$$\nabla(\nabla w) = e_x \otimes \left[\frac{\partial \nabla w}{\partial x}\right] + e_y \otimes \left[\frac{\partial \nabla w}{\partial y}\right] = \underbrace{e_x e_x \frac{\partial^2 w}{\partial x^2}}_{-\kappa_x} + (e_x e_y + e_y e_x)\underbrace{\frac{\partial^2 w}{\partial x \partial y}}_{-\kappa_{xy}} + e_y e_x \underbrace{\frac{\partial^2 w}{\partial y^2}}_{\kappa_y}$$

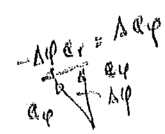
$$\therefore \quad \nabla(\nabla w) = -\left[e_x e_x \kappa_x + (e_x e_y + e_y e_x)\kappa_{xy} + e_y e_y \kappa_y\right] = -\mathbb{K} \text{ curvature change tensor in } x,y$$

Curvature Change tensor $\nabla(\nabla w) = -\kappa_r e_r e_r - (e_\varphi e_r + e_r e_\varphi)\kappa_{r\varphi} - \kappa_\varphi e_\varphi e_\varphi$

$$\therefore \quad \kappa_r = -\frac{\partial^2 w}{\partial r^2} \qquad \kappa_{r\varphi} = -\frac{\partial}{\partial r}\left(\frac{\partial w}{r\partial\varphi}\right) \qquad \kappa_\varphi = -\left[\frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right]$$

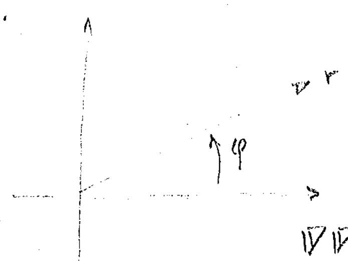
$$\frac{\partial}{\partial\varphi}(e_r) = e_\varphi \qquad \frac{\partial e_\varphi}{\partial \varphi} = -e_r$$





10/26/79

Polar Coordinate



we had shown $\nabla w = \underline{e}_x \frac{\partial w}{\partial x} + \underline{e}_y \frac{\partial w}{\partial y}$;

$$\nabla w = e_r \frac{\partial}{\partial r} w + \frac{e_\varphi}{r}\frac{\partial w}{\partial \varphi}$$

$$\nabla\nabla w = e_x e_x w_{,xx} + (e_x e_y + e_y e_x) w_{,xy} + e_y e_y w_{,yy}$$

$$= e_r e_r w_{,rr} + (e_r e_\varphi + e_\varphi e_r)\frac{\partial}{\partial r}\left(\frac{\partial w}{r\partial\varphi}\right) + e_\varphi e_\varphi \left(\frac{\partial^2 w}{r^2 \partial\varphi^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)$$

$$\nabla\cdot\nabla w = w_{,rr} + \frac{1}{r^2} w_{,\varphi\varphi} + \frac{1}{r}\frac{\partial w}{\partial r} = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial\varphi^2}\right] w$$

Axisym Problems $\frac{\partial}{\partial\varphi} = 0$ $\quad D\Delta\Delta w = p(r,\varphi) = p(r)$ only

B.C. are given on 2 concentric circles $r = a, b$

$\therefore\ w = w(r)$ only $\qquad \Delta\Delta w = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)w\right\}\right] = p/D$

Example: $p = \text{const} = p_0$ $\quad$ We need to find the particular

$$\Rightarrow \quad r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}w\right)\right) = \frac{p_0}{D}\left[\frac{r^2}{2} + c_1\right]$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}w\right)\right) = p_0/D\left[\frac{r^2}{4} + c_1 \ln r + c_2\right]$$

$$r\frac{\partial}{\partial r}w = p_0/D\left[\frac{r^4}{16} + c_1\left(\frac{r^2\ln r}{2} - \frac{r^2}{4}\right) + c_2\frac{r^2}{2} + c_3\right]$$

$$w = p_0/D\left[\frac{r^4}{64} + c_1\left(\frac{r^2\ln r}{4} - \frac{2r^2}{8}\right) + c_2\frac{r^2}{4} + c_3\ln r + c_4\right]$$

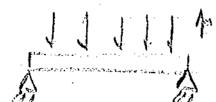
total sol. $w = w_p + w_c = p_0/D\left[\frac{r^4}{64} + \underbrace{c_1 r^2\ln r + c_3\ln r + c_4 + c_2 r^2}_{w_H} + \cancel{c_5 r}\right]$

since we want boundedness in the $w \Rightarrow c_1, c_3 = 0 \Rightarrow w = p_0/D\left[\frac{r^4}{64} + c_2 r^2 + c_4\right]$

particular is $p_0/D\,\frac{r^4}{64} = w_{partic}$

homog $w_c \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}w\right)\right] = 0$

$$w_{Hom} = c_1 r^2\ln r + c_2\ln r + c_3 r^2 + c_4$$

Problem  $0 \le r \le b$ @ $r = b$ $w(b) = 0$, $M_r(b) = 0$

if w is bdd $(c_1 = 0)$ if M_r is bdd $(c_2 = 0)$ $\Rightarrow w_{TOT} = \frac{p_0 r^4}{64D} + c_3 r^2 + c_4$

$$\Rightarrow \frac{dw}{dr} = \frac{p_0 r^3}{16D} + 2c_3 r \qquad \frac{d^2w}{dr^2} = \frac{3p_0 r^2}{16D} + 2c_3$$

Since $M_r = -D\left[w_{,rr} + \nu\left(\frac{1}{r}w_{,r} + \cancel{\frac{1}{r}w_{,\varphi r}}\right)\right] \Rightarrow M_r(b) = 0 \therefore$

$$\frac{3p_0 b^2}{16D} + 2c_3 + \frac{\nu}{b}\left[\frac{p_0 b^3}{16D} + 2c_3 b\right] = 0 \Rightarrow \frac{(3+\nu)b^2 p_0}{16D} = -2c_3(1+\nu)$$

$$c_3 = -\frac{(3+\nu)b^2 p_0}{32(1+\nu)D} \qquad \therefore \quad M_r = -D\left[\frac{3p_0 r^2}{16D} - \frac{(3+\nu)b^2 p_0}{16(1+\nu)D} + \nu\left(\frac{p_0 r^2}{16D} - \frac{(3+\nu)b^2 p_0}{16(1+\nu)D}\right)\right]$$

$$= -D\left[\frac{p_0(3+\nu)r^2}{16D} - \frac{(3+\nu)b^2 p_0}{16D}\right] = \frac{3+\nu}{16}\cdot p_0\,(b^2 - r^2)$$

$\therefore$ $M_r(0) \geq 0$ which intuitively is correct $M_r(0) = \frac{3+\nu}{16}p_0 b^2$

From elementary beam theory $\sigma = \frac{Mc}{I} = \frac{M_r h/2}{h^3/12} = \frac{6M_r}{h^2}$

From plate theory: $\sigma_r = \frac{6}{h^2} \frac{(3+\nu)b^2 p_0}{16}$

from 3-D elasticity : Timo & Goodier (p. 385)

$$\sigma_r = p_0 \cdot \frac{3(3+\nu)b^2 z}{32\,(h/2)^3} \cdot f(z) \qquad \sigma_r(z) = p_0 \cdot \frac{3}{4}\frac{(3+\nu)a^2 z}{h^3}$$

$$f(z) = 1 - \left(\frac{h}{2b}\right)^2 \left\{ \frac{4(2+\nu)}{15(3+\nu)} \left(3 - 5\left(\frac{2z}{h}\right)^2 \right) \right\}$$

first term = plate theory & rest represents distrib due to 3-D theory

note that if $z = h/2$ $\nu = .3$ $f(z) = 1 + (h/b)^2(.092)$

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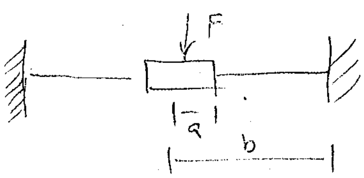
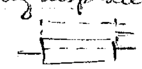


BC. no rotation where plate & plug meet

μ plug >> μ plate . plug displaces as a whole



$\frac{dw}{dr} = 0$ at $r = a$

$q_r = -\frac{F}{2\pi a}$ at $r = 0$

$$r = b \begin{cases} w = 0 \\ \frac{dw}{dr} = 0 \end{cases}$$

no surface loads - need only complimentary soln.

$$W(r) = \bar{C}_1 + \bar{C}_2 r^2 + \bar{C}_3 \ln r + \bar{C}_4 r^2 \ln r$$

define $\rho = r/b$

$$w(r) = C_1 + C_2 \rho^2 + C_3 \ln\rho + C_4 \rho^2 \ln\rho$$

$$\frac{dw}{dr} = \frac{dw}{d\rho}\frac{d\rho}{dr} = \frac{1}{b}\frac{dw}{d\rho} = \frac{1}{b}\left[2C_2\rho + C_3/\rho + C_4\left(2\rho\ln\rho + \rho\right)\right]$$

$$\frac{d^2w}{dr^2} = \frac{1}{b}\frac{d}{d\rho}\left(\frac{1}{b}\frac{dw}{d\rho}\right) = \frac{1}{b^2}\left[2C_2 - C_3/\rho^2 + C_4\left(2\ln\rho + 3\right)\right]$$

$$q_r = -D\frac{\partial}{\partial r}\left(\nabla^2 w\right) = -D\frac{d}{dr}\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = -\frac{D}{b^3}\frac{d}{d\rho}\left[4C_2 + C_4(4\ln\rho + 4\right.$$

$$= -\frac{D}{b^3}\left[\frac{4C_4}{\rho}\right] = -\frac{F}{2\pi r} \Rightarrow \boxed{C_4 = \frac{Fb^2}{8\pi D}}$$

Thus

$$w(a) = \frac{b^2 F}{8\pi D}\left[\frac{1-\rho^2}{2} + \rho^2\ln\rho - \rho^2(\ln\rho)(1+\rho) - \frac{2\rho^2\ln^2\rho}{1-\rho^2}\right]$$

$$M_r(a) = \frac{F}{2\pi}\left(\frac{\ln\rho}{1-\rho^2} + 1/2\right)$$ In the limit as $a/b \to \infty$

$$\lim_{\rho\to 0} w(\rho) = \frac{b^2 F}{16\pi D} \qquad M_r(0) \to -\infty.$$

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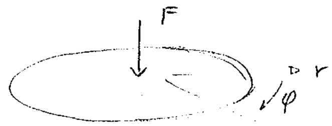


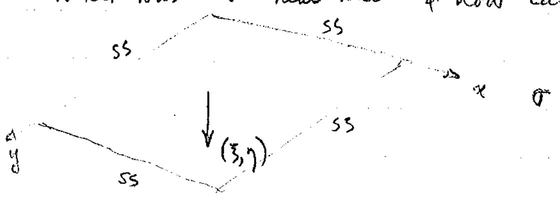
Axisym Sol for unit circle

$$w(r) = C_1 + C_2 r^2 + C_3 \log r + C_4 r^2 \log r$$

$$C_4 = \frac{F b^2}{8\pi D}$$

$$\sigma\Big|_{r=0} \sim M \sim \log r\Big|_{r=0}$$

what was σ near load & how can we use this singular term for rect. plates



Remember $p(x,y) = \sum\sum p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

$$p_{mn} = \frac{4}{ab}\iint p(x,y) \sin \sin \qquad p(x,y) = F\delta(x-\xi)\delta(y$$

$$= \frac{4}{ab} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}$$

$$\therefore \quad w(x,y) = \sum\sum w_{mn} \sin \sin \qquad w_{mn} = \frac{p_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2}$$

$$w_{mn} = \frac{4}{abD} \frac{\sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \sim m^{-4} \text{ or } n^{-4}$$

$$w_{xx} \sim -\sum\sum \left(\frac{m\pi}{a}\right)^2 . w_{mn} \sin \sin \qquad \therefore \quad w_{xx}\Big|_{\xi,\eta} \sim -\sum\sum f(m^{-2}, n^{-4}) . \sin^2 \sin^2$$

which is of much slower convergence

Methods of Sol. – very fast if fns are smooth; when not smooth (singular/near sing) either use finer mesh or many terms or ignore problem
HIGH COST or HIGH ERROR

- Fourier Series
- Finite Elements
- Finite Difference
- Energy Methods

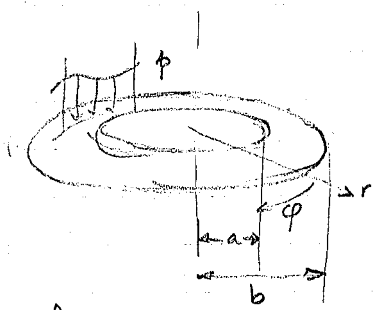
Now we can use singular sol in r ie $r^2 \log r$ get value of fn on bdy then use complementary sol. & non zero bdy

For particular solution choose $w_p(x,y) = \frac{F}{8\pi D} r^2 \log(r/a) \qquad r^2 = (x-\xi)^2 + (y-\eta)^2$
where a is one of the rectangular plate lengths
Then $w = w_c + w_p$ but B.C. for w_c are "smooth" when load not too close to bdy. The series or direct num. sol for w_c will be efficient & converge very quickly.

for a load over an area; replace F by $\iint_A p(\xi,\eta)\,d\xi\,d\eta$ thus we would choose

$$w_p(x,y) = \frac{1}{8\pi D} \iint_A p(\xi,\eta)\, r^2 \log r/a \; d\xi\, d\eta \qquad r^2 = (x-\xi)^2 + (y-\eta)^2$$

look at non axisymmetric (asymmetric) loading of circular plate
$a \le r \le b$



Solution must be periodic φ.
$w(r, \varphi + 2\pi) = w(r, \varphi)$

For this case we can use $w(r,\varphi) = \sum_{n=0,1,\dots}^{\infty} w_n(r) \cos n\varphi + \sum_{n=1,\dots}^{\infty} \bar{w}_n(r) \sin n\varphi$.

We want to look at $\Delta\Delta w_c = 0$ $\qquad \Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$

$$\Rightarrow \sum\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) w_n \cos n\varphi = \Delta w \quad \therefore \Delta\Delta w = \sum\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)\left(\qquad\right) w_n = 0$$

for the ode w/ equidimensionality $\qquad$ for constant coeff $\quad w_n \sim e^{\lambda r}$
$\qquad$ for equidi. $\quad w_n \sim r^{\lambda}$

$\frac{d}{dr} w_n = \lambda r^{\lambda - 1} \qquad \frac{d^2}{dr^2} = \lambda(\lambda - 1) r^{\lambda - 2}$

$$\Delta w = \left(\lambda(\lambda-1) + \lambda - n^2\right] r^{\lambda-2} \cos = (\lambda^2 - n^2)\, r^{\lambda-2} \begin{smallmatrix}\cos(\,)\\ \sin(\,)\end{smallmatrix}$$

$$\Delta\Delta w = \left[(\lambda-2)^2 - n^2\right]\left[\lambda^2 - n^2\right] r^{\lambda-4} \begin{smallmatrix}\cos(\,)\\ \sin(\,)\end{smallmatrix} = 0 \Rightarrow \begin{matrix}\lambda - 2 = \pm n & \lambda = \pm n + 2\\ \lambda = \pm n & \lambda = \pm n\end{matrix}$$

$$w_n(r) = r^{-n+2},\ r^{-n},\ r^{n+2},\ r^{n}$$

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$\rightarrow$ SCF = (Stress w/ difficulty) / Stress without

Continuation of results from previous class:

now $w_n(r) = c_{1n} r^n + c_{2n} r^{-n} + c_{3n} r^{n+2} + c_{4n} r^{-n+2}$

for $n=0$ (axisymm) $\qquad w_0(r) = c_{10} + c_{20} \log r + c_{30} r^2 + c_{40} r^2 \log r$

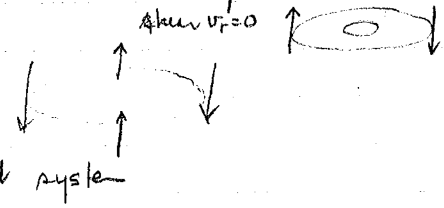
for $n=1$ $\qquad w_1(r) = c_{11} r + c_{21} r^{-1} + c_{31} r^3 + c_{41} r \log r$

for $n \ge 2$ $\qquad w_n(r) = c_{1n} r^n + c_{2n} r^{-n} + c_{3n} r^{n+2} + c_{4n} r^{-n+2}$

for $n=0$ we cannot prescribe shear at both boundaries since this might not ~~solve~~ satisfy equil

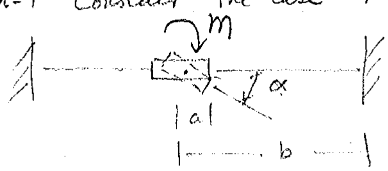
for $n=1$ shear load produces a moment hence we cannot prescribe moments at both boundaries

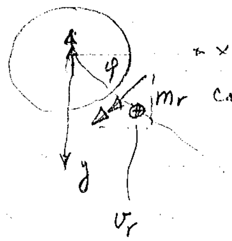
shear $v_r = 0$



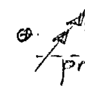
for $n \ge 2$ shear loading $\int^{2\pi} v_r \, r \, d\varphi = 0$
and resultant moment $= 0$
thus we get self equil. force & moment system

for $n=1$ Consider the case : Before we considered the case with F↓ acting on plug.





m_r caused a tension on $+x$ side



produces mom.

$m_r \, r \, d\varphi \times \cos\varphi$ — mom/unit length × length; resultant component in y dir

$$m = \int_0^{2\pi} m_r \, r \, d\varphi \cos\varphi - \int_0^{2\pi} \underbrace{v_r}_{\text{shear/unit length}} r \, d\varphi \cdot \underbrace{\cos\varphi}_{\text{proj in y dir}} \cdot \underbrace{r}_{\text{moment arm}} = \int_0^{2\pi} (m_r - v_r r) \, r \cos\varphi \, d\varphi$$

in general form only those terms which depend on $\cos\varphi$ will give any type of contribution.

$$m_r = -D\left[w_{,rr} + \nu\left(\frac{w_r}{r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2}\right)\right] \qquad \text{but } \frac{\partial^2 w}{\partial\varphi^2} = -w \quad \therefore$$

$$\Longrightarrow m_r = -D\left[w_{,rr} + \nu\left(\frac{w_r}{r} - \frac{w}{r^2}\right)\right] \qquad ; \text{ also } v_r = q_r + \frac{\partial m_{r\varphi}}{r\partial\varphi}$$

$$q_r = -D\frac{\partial}{\partial r}\Delta w$$

$$m_{r\varphi} = -D(1-\nu)\frac{\partial}{\partial r}\left(\frac{\partial w}{r\partial\varphi}\right) \qquad \frac{\partial m_{r\varphi}}{r\partial\varphi} \sim \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\varphi^2}\right) = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{w}{r}\right)$$

$$m_r - v_r r = -D\left[w_{rr} + \cancel{\nu\left(\frac{w_r}{r} - \frac{w}{r^2}\right)} - r\frac{\partial}{\partial r}(\Delta w) + (1-\cancel{\nu})\cancel{\left(\frac{w_r}{r} - \frac{w}{r^2}\right)}\right]$$

$$= -D\left[\Delta w - r\frac{\partial}{\partial r}(\Delta w)\right] = Dr^2\frac{\partial}{\partial r}\left(\frac{\Delta w}{r}\right)$$

but $w_1 = C_{11}r + C_{21}r^{-1} + C_{31}r^3 + C_{41}r\log r$ now since r & r^{-1} are soln to $\Delta w = 0$

$w_{1,rr} = 2C_{21}/r^3 + 6C_{31}r + C_{41}\left(\frac{1}{r}\right)$ then ⟹ C_{11}, C_{21} will drop out

the moment will only depend on C_{41}

$$\Delta w = rC_{31}[6 + 3 - 1] + C_{41}/r\,[1 + \log r + 1 - \log r]$$

$$= 8rC_{31} + 2C_{41}/r$$

$$m_r - v_r r = -Dr^2\left[+4\,C_{41}/r^3\right] = -\frac{4DC_{41}}{r}$$

QED.

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continuation of last problem.

$r=b \quad w=0 \quad \frac{\partial w}{\partial r}=0$

at $r=a$ we only want rigid body rotation $\therefore$ take $w = \alpha x = \alpha r\cos\varphi$

since the plug rotates & plate is clamped to rigid plug: $\frac{\partial w}{\partial x} = \alpha\cos\varphi$

$w\big|_a = \alpha a\cos\varphi$

$\frac{\partial w}{\partial r}\Big|_a = \alpha\cos\varphi.$

but we've introduced α : to get rid of it look at K_φ (curvature change)

$$K_\varphi = -\left(\frac{\partial w}{r\partial r} + \frac{\partial^2 w}{r^2\partial\varphi^2}\right) = -\left(\frac{\partial w}{r\partial r} - \frac{w}{r^2}\right) = 0 \text{ for } n=1 \text{ along bdy.}$$

this is due to the fact that plug is rigid and cannot undergo any relative displ in the φ direction

$\therefore \quad \sigma_r = \frac{E}{1-\nu^2}\left(\epsilon_r + \nu\cancel{\epsilon_\varphi}\right) \qquad \sigma_\varphi = \frac{E}{1-\nu^2}\left(\nu\epsilon_r + \cancel{\epsilon_\varphi}\right) = \nu\sigma_r$ always occurs on an attached boundary.

$\Rightarrow \quad M_\varphi = \nu M_r$ since $\int \sigma_\varphi z = \nu\int\sigma_r z$

Thus we only have 3 kinematic conditions (since K_φ connects $\frac{\partial w}{\partial r}\Big|_{r=a}$ & $w\big|_{r=a}$).

the fourth const C_4 is related to moment $\therefore$ $\boxed{C_4 = \frac{mb}{4\pi D}}$

$C_4 = \frac{F}{8\pi D}$ $\qquad 2qb = m \qquad F = 2q$

at $\rho=1$ $\quad \boxed{\begin{aligned} &C_1 + C_2 + C_3 = 0 \\ &C_1 - C_2 + C_3 + C_4 = 0\end{aligned}}$

$$\rho = \frac{a}{b} < 1 \quad \frac{1}{\rho}\left[C_1 - C_2\rho^{-2} + 3C_3\rho^{2} + C_4(1+\log\rho)\right] - \frac{C_1}{\rho} - \frac{C_2}{\rho} - C_3\rho^{2} - C_4\log\rho/\rho = 0$$

$$\boxed{-2C_2/\rho^2 + 2C_3\rho^2 + C_4 = 0}$$

Solving : this gives $\qquad$ solving for C_2, C_3 in terms of C_4

$$\begin{bmatrix} -1 & 1 \\ -\rho^{-2} & \rho^2\end{bmatrix}\begin{pmatrix} C_2 \\ C_3\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}\left(-C_4/2\right) \qquad C_2 = \frac{C_4}{2\rho^{-2}}\,\frac{(1-\rho^2)}{1-\rho^4} = \frac{C_4\rho^2}{2(1+\rho^2)}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}^{-1} = \frac{1}{\det}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11}\end{pmatrix}. \qquad C_3 = \frac{-C_4\,\cancel{\rho^{-2}}(1-\rho^2)}{2\cancel{\rho^{-2}}(1-\rho^4)} = \frac{-C_4}{2(1+\rho^2)}$$

$$C_1 = -(C_2 + C_3) = \frac{C_4}{2(1+\rho^2)}(1-\rho^2)$$

$$\alpha = \frac{w(a,\varphi)}{a\cos\varphi} = \frac{1}{a}\left[C_1\rho + C_2\rho^{-1} + C_3\rho^3 + C_4\,\rho\log\rho\right]$$

$$= \frac{C_4\rho}{2a(1+\rho^2)}\left[1-\rho^2 + 1 - \rho^2 + 2(1+\rho^2)\log\rho\right]$$

$$= \frac{m}{4\pi D}\left[\frac{1-\rho^2}{1+\rho^2} + \log\rho\right]$$

as $\rho\to 1 \quad \alpha\to 0$ ie rigid body rot (when body is clamped) $=0$

For $\dfrac{m_r(a,\varphi)}{\cos\varphi} = -\dfrac{\mathcal{M}}{2\pi b}\,\dfrac{1-\rho^2}{\rho(1+\rho^2)}$ $\qquad$ $m_r \to 0$ as $\rho \to 1$

as $\rho \to 0$ $\quad \dfrac{m_r}{\cos\varphi} \to -\dfrac{\mathcal{M}}{2\pi a}$

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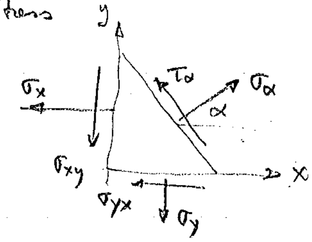
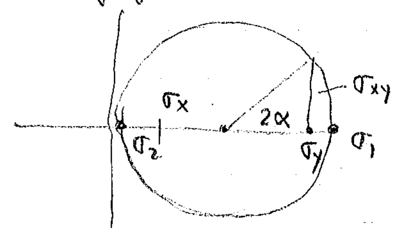
Transformation from (x,y) to (R,ϕ) coords

I. Stress



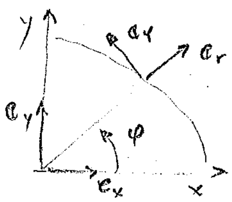
Equilib of forces leads to Mohr's Circle



$$\tan 2\alpha = \frac{\sigma_{xy}}{\frac{(\sigma_y - \sigma_x)}{2}}$$

To derive eqns in terms of moments. use moment tensor

$$\underset{\sim}{M} = e_x M_{xx} e_x + (e_x e_y + e_y e_x) M_{xy} + e_y M_y e_y$$



$e_x = e_r \cos\varphi - e_\varphi \sin\varphi$
$e_y = e_r \sin\varphi + e_\varphi \cos\varphi$ $\Big\}$ from Geometry

Subst. into mom tensor & assume $M_{xy} = 0$

$$\underset{\sim}{M} = M_x (e_r\cos\varphi - e_\varphi\sin\varphi)(e_r\cos\varphi - e_\varphi\sin\varphi) + M_y (e_r\sin\varphi + e_\varphi\cos\varphi)(e_r\sin\varphi + e_\varphi\cos\varphi)$$

thus $\underset{\sim}{M} = e_r e_r (M_x\cos^2\phi + M_y\sin^2\phi) + (e_r e_\varphi + e_\varphi e_r)(M_y - M_x)\dfrac{\sin 2\varphi}{2}$
$\qquad + e_\varphi e_\varphi (M_x \sin^2\varphi + M_y\cos^2\varphi)$

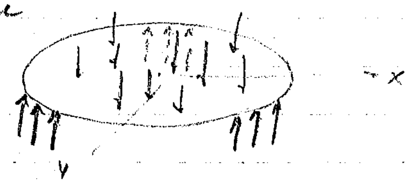
but $\underset{\sim}{M} = e_r M_{rr} e_r + (e_r e_\varphi + e_\varphi e_r) M_{r\varphi} + e_\varphi M_{\varphi\varphi} e_\varphi \Rightarrow$

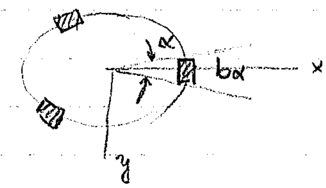
$M_r = M_x\cos^2\varphi + M_y\sin^2\varphi$
$M_\varphi = M_x\sin^2\varphi + M_y\cos^2\varphi$ $\Big\}$ note $M_x + M_y = M_r + M_\varphi$ invariant
$M_{r\varphi} = \dfrac{M_y - M_x}{2}\sin 2\varphi$ $\qquad \cos^2\varphi = \dfrac{1+\cos 2\varphi}{2}$ $\qquad \sin^2\varphi = \dfrac{1-\cos 2\varphi}{2}$

$\therefore$ $M_r = \dfrac{M_x + M_y}{2} + \dfrac{M_x - M_y}{2}\cos 2\varphi$
$M_\varphi = \dfrac{M_x + M_y}{2} - \left(\dfrac{M_x - M_y}{2}\right)\cos 2\varphi$ $\qquad$ when $M_x = M_y$: $\quad M_r = M_\varphi =$ average of sum of $M_x + M_y$ & $M_{r\varphi} = 0$
$M_{r\varphi} = -\left(\dfrac{M_x - M_y}{2}\right)\sin 2\varphi$

Example: Given a circular plate loaded with constant p. Supposed by n columns of width $b\alpha$ (equally spaced) with b = plate radius

$n=3$ case

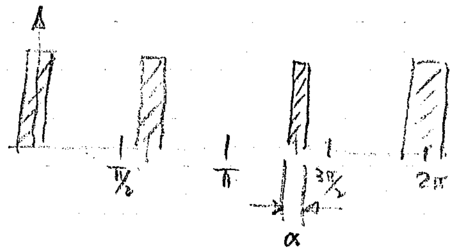




We will approx the columns by the boundary conds.

$$v_r(b,\varphi) = \text{const}=A \quad \text{at supports}$$
$$= 0 \quad \text{away from supports}$$

$$m_r(b,\varphi) = 0$$



3 supports — load/support area — load/unit area

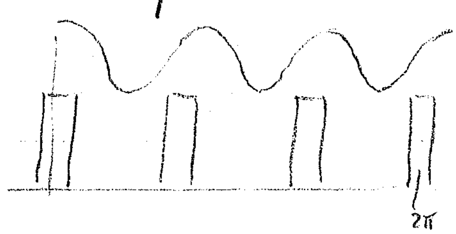
$$3b\alpha A = \pi b^2 p \qquad \text{equil in } w \text{ dir}$$

$$A = \frac{\pi b p}{3\alpha}$$

Since the distrib of columns is even fn let $v_r(b,\varphi) = \sum_{n=0}^{\infty} v_{r_n}(b)\cos n\varphi$

$$= v_{r_0} + \sum_{n=1}^{\infty} v_{r_n}(b)\cos n\varphi.$$

Need only 3 and harmonics to describe the shear.



so $$v_r(b,\varphi) = v_{r_0} + \sum_{n=3,6,9\ldots} v_{r_n}(b)\cos n\varphi$$

and $$\frac{1}{\pi}\int_0^{2\pi} \cos n\varphi \; v_r(b,\varphi) = v_{r_n}\cdot\pi$$

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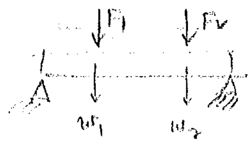
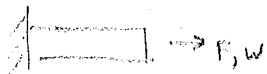
Maxwell-Betti Reciprocity Laws.

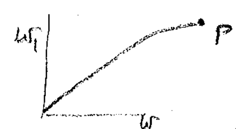


$$F_1 = F_1(w_1, w_2)$$
$$F_2 = F_2(w_1, w_2)$$



$$\delta \text{work} = F\,dw$$
$$\text{work} = \int_0^w F(w')\,dw'$$

For F_1 & F_2, work $= \int_0^P F_1\,dw_1 + F_2\,dw_2$



In a conservative system the work done is indep. of path. Thus $F_1\,dw_1 + F_2\,dw_2$ must be an exact derivative ∴

$$d\mathcal{W} = F_1\,dw_1 + F_2\,dw_2 \quad \text{where} \quad \mathcal{W} = \mathcal{W}(F_1, F_2)$$

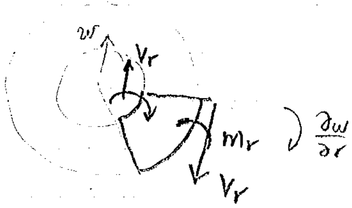
Thus $F_1 = \dfrac{\partial \mathcal{W}}{\partial w_1}$ $\quad F_2 = \dfrac{\partial \mathcal{W}}{\partial w_2}$

$$\frac{\partial F_1}{\partial w_2} = \frac{\partial F_2}{\partial w_1} = \frac{\partial^2 \mathcal{W}}{\partial w_1 \partial w_2}$$

If the forces are linear fns of displ they can be written as a matrix

$$\left(\underset{\sim}{F}\right) = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \left(\underset{\sim}{w}\right)$$

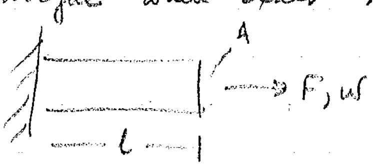
Since $\dfrac{\partial F_1}{\partial w_2} = k_{12}$ $\quad \dfrac{\partial F_2}{\partial w_1} = k_{21}$, the matrix (stiffness matrix) must be sym.



$$\delta \mathcal{W} = M_r\,\delta\left(\frac{\partial w}{\partial r}\right) + V_r\,\delta(-w)$$

<u>Strain Energy Method</u>

Useful when exact solution cannot be obtained



$$\mathcal{W} = \int_0^w F\,dw'$$

If F is linear in w then $\mathcal{W} = \frac{1}{2} F w$

For uniaxial case

$$\sigma = F/A = E\epsilon$$
$$\epsilon = w/l$$

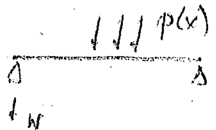
$$\therefore \quad \mathcal{W} = \tfrac{1}{2} A \sigma \epsilon l = \frac{1}{2}\underbrace{(Al)}_{\text{Volume}} E \epsilon^2 \quad \text{this is strain energy}$$

For beam $\quad \epsilon = Kz \quad \sigma = \dfrac{Mz}{I} \quad \mathcal{W} = \dfrac{1}{2}\displaystyle\int Kz\,\frac{Mz}{I}\,dA\,dx$

$\mathcal{W} = \frac{1}{2}\kappa M$ strain energy density

Thus for entire beam strain energy $= \int_0^L \frac{EI}{2}\kappa^2 dx$

The potential energy U is the strain energy in beam + potential of external loads.

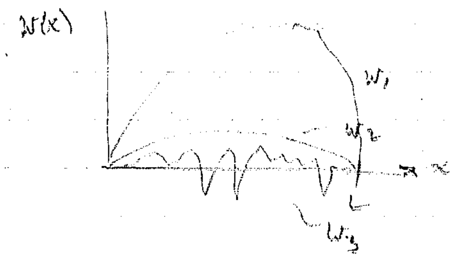


$$U = \int_0^L \frac{1}{2} EI \kappa^2 dx - \int_0^L p(x)\, w(x)\, dx$$

$$U = \int_0^L \left\{ \frac{1}{2} EI \left[w''(x) \right]^2 - pw \right\} dx$$

U is a scalar that depends on a fn $w(x)$ on the interval $0 \leq x \leq L$. Thus U is a fnl and we denote

$$U(w(x)) = \int_0^L \{ \quad \} dx$$



thus

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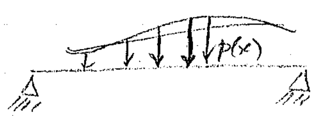
Approx. Methods

Continuation of last time



$$U[w(x)] = \int_0^L \{ \underbrace{\tfrac{1}{2} EI [w'']^2}_{\text{internal pot}} - \underbrace{pw}_{\text{external pot}} \} dx$$

out of all possible w that satisfy B.C.'s, the soln of equil eqn makes U a min.

assume that w is the fn. that minimizes U. Then $U[w(x) + \varepsilon f(x)] = g(\varepsilon)$ is a fn. of ε with w & f fixed. $\Rightarrow \left.\frac{\partial g}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$

$$U(w+\varepsilon f) = \int_0^L \left[\frac{1}{2} EI \left(w'' + \varepsilon f''\right)^2 - P(w+\varepsilon f) \right] dx$$

$$\frac{\partial U}{\partial \varepsilon} = \int_0^L \left[\frac{1}{2} EI\, 2\left(w'' + \varepsilon f''\right) f'' - Pf \right] dw$$

$$\lim_{\varepsilon \to 0} \frac{\partial U}{\partial \varepsilon} = \int_0^L \left(EI\, w'' f'' - pf \right) dx = -\int_0^L (EI\, w'')' f' dx - \int_0^L pf\, dx + EI w'' f' \Big|_0^L$$

$$= \int_0^L \left[(EI w'')'' - p \right] f\, dx + EI w'' f' \Big|_0^L - (EI w'')' f \Big|_0^L$$

f & w must satisfy BC

$$\therefore \quad \underbrace{EIw''}_{\text{mom.}}\underbrace{f'}_{\text{rot}}\Big|_0^L - \underbrace{(EIw'')'}_{\text{shear}}\underbrace{f}_{\text{displ}}\Big|_0^L$$

either mom or rot = 0 either shear or displ = 0 at Boundary.

BC. 1) either (EIw'') or $w' = 0$

2) " $(EIw'')'$ or $w = 0$ $\therefore \lim_{\epsilon \to 0} \frac{dU}{d\epsilon} = \int_0^L \{(EIw'')'' - p\} f\,dx = 0$

$\forall f$ arbitrary $\Rightarrow \boxed{(EIw'')'' - p = 0}$ $\forall x$ — Euler Beam Eqn.

we've used stress-strain rel. to obtain result.

this result is a necessary condition for minim not a sufficient condition since $\frac{dU}{d\epsilon} = 0$ can also produce a max or even a pt of inflection.

suppose $U = \int [w, w', w'', w''', w^{iv}, x]\,dx$. The necessary condition for $w(x)$ to give a stationary value of U is the Euler eqn.

$$\frac{d^4}{dx^4}\left(\frac{\partial \mathcal{L}}{\partial w^{iv}}\right) - \frac{d^3}{dx^3}\left(\frac{\partial \mathcal{L}}{\partial w'''}\right) + \frac{d^2}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial w''}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial w'}\right)$$

$$+ \frac{\partial \mathcal{L}}{\partial w} = 0$$

For example $\mathcal{L} = \mathcal{L}(w, w'', x) = \frac{1}{2} EI(w'')^2 - px$

$$\frac{\partial \mathcal{L}}{\partial w''} = EIw''$$

$$\frac{\partial \mathcal{L}}{\partial w} = -p$$

$$\therefore \quad \frac{d^2}{dx^2}(EIw'') - p = 0$$

Ex

Given SS beam with loads.

choose as 1st approx $w(x) = A \sin \frac{\pi x}{L}$

for clamped ends $w(x) = A\left(1 - \cos \frac{2\pi x}{L}\right)$ 1st app.

$w' = A \frac{2\pi}{L} \sin \frac{2\pi x}{L}$ $\quad w'' = A\left(\frac{2\pi}{L}\right)^2 \cos \frac{2\pi x}{L}$

$$\mathcal{L} = \frac{1}{2} EI(w'')^2 - p = \frac{1}{2} EI \left\{A^2\left(\frac{2\pi}{L}\right)^4 \cos^2 \frac{2\pi x}{L}\right\} - pA\left[1 - \cos \frac{2\pi x}{L}\right]$$

$$U = \frac{A^2}{2}\int_0^L EI\left(\frac{2\pi}{L}\right)^4 \cos^2 \frac{2\pi x}{L}\,dx - A\int_0^L p(x)\left(1 - \cos \frac{2\pi x}{L}\right)dx$$

but $U(A) = U$ only

$\therefore \dfrac{\partial U}{\partial A} = 0$

$$\frac{\partial U}{\partial A} = \frac{2A}{2}\int_0^L EI\left(\frac{2\pi}{L}\right)^4 \cos^2\frac{2\pi x}{L}\,dx - 1\int_0^L p\left(1-\cos\frac{2\pi x}{L}\right)dx = 0$$

$$\therefore\ A = \frac{\int_0^L p(1-\cos\ )\,dx}{\int_0^L EI\left(\frac{2\pi}{L}\right)^2\cos^2} = \frac{pL}{EI\left(\frac{2\pi}{L}\right)^4\cdot\frac{1}{2}L}$$ for const P & EI

$$\therefore\ A = \frac{pL^4}{EI\,8\pi^4}$$

what is $w(x = L/2) = \dfrac{2pL^4}{EI\,8\pi^4} = \dfrac{pL^4}{4EI\pi^4}$

exact soln:

$$w(L/2) = \frac{pL^4}{EI\cdot 384} \qquad 4\pi^4 = 389.64$$

Calculate bending stress

moment$|_{x=0}$ $\quad w'' = A\cdot\left(\dfrac{2\pi}{L}\right)^2 \Rightarrow \quad M = -EIw'' = -EI\cdot\dfrac{pL^4}{EI\,8\pi^4}\cdot\dfrac{4\pi^2}{L^2}$

$$M = -\frac{pL^2}{2\pi^2}$$

exact sol.

$$\text{Mom} = \frac{pL^2}{12} \qquad 2\pi^2 \simeq 19.7$$

to get better mom. resolution take $w = w_0 + w_1$

$$w_1 = A_1\left(1-\cos\frac{4\pi x}{L}\right)$$

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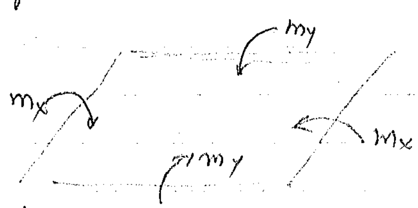
HW proble



BC $v_r = m_r = 0$

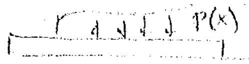
$W_{TOT} = W_I + W_{II}$

W_I: sol. w/o hole — known soln

W_{II}: sol. due to the hole

1. Localized soln
2. BC on edge of hole $(m_r, v_r)_{II} = (-m_r, -v_r)_I$

Continuation of Energy Methods



$$U[w(x)] = \int_0^L \left\{ \tfrac{1}{2} EI (w'')^2 - pw \right\} dx$$

Plate from beams: $\frac{1}{2} EI(w'')^2 = \frac{1}{2} MK = \frac{1}{2} EI K^2$

thus for plate $\frac{1}{2}(M_x K_x + 2 M_{xy} K_{xy} + M_y K_y)$

$M_x = D(K_x + \nu K_y) \qquad M_y = D(K_y + \nu K_x) \qquad M_{xy} = D(1-\nu) K_{xy}$

Invariant condition

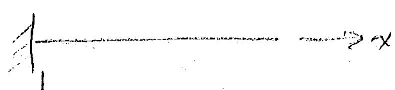
$$\therefore \quad \frac{D}{2}\left[(K_x + \nu K_y) K_x + 2(1-\nu) K_{xy}^2 + (K_y + \nu K_x) K_y \right] = \frac{D}{2}\left[(K_x + K_y)^2 + 2(1-\nu)\left\{ K_{xy}^2 - K_x K_y \right\} \right]$$

second invar.

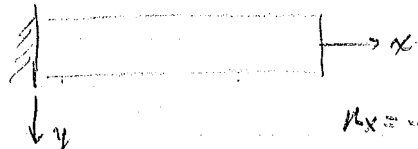
for Plate $\quad U[w(x,y)] = \iint \left(\frac{D}{2}[\quad\quad] - pw \right) dx\,dy.$

For plate

Example



BC clamped at $x=0$ free everywhere else.



$w(x,y) = c_1 x^2$ — bending approx only gives ~~conditions~~ in x direction

$+ c_2 x^2 y^2$ (gives bending approx in y direction)

$K_x = -w_{xx} = -2c_1 + 2c_2 y^2$

K_y

Since $U = fn(K_x^2, K_{xy}, K_x K_y, K_y^2) = fn(c_1^2, c_1 c_2, c_2^2, c_1, c_2)$

$$= \frac{1}{2}\left(c_1^2 k_{11} + 2 c_1 c_2 k_{12} + c_2^2 k_{22} - L_1 c_1 - L_2 c_2 \right)$$

to get min $\Rightarrow \quad \dfrac{\partial U}{\partial c_1} = 0 = c_1 k_{11} + c_2 k_{12} - L_1 = 0$

$\dfrac{\partial U}{\partial c_2} = 0 = c_1 k_{12} + c_2 k_{22} - L_2 = 0$

thus we get a linear system of eqns.

$$\therefore \quad c_1 = \frac{\begin{pmatrix} L_1 & k_{12} \\ L_2 & k_{22} \end{pmatrix}}{\begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}} \qquad c_2 = \frac{\begin{pmatrix} k_{11} & L_1 \\ k_{12} & L_2 \end{pmatrix}}{\begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}}$$

if $\underset{\sim}{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \underset{\sim}{k} = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix} \quad \underset{\sim}{L} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$

$$U = \frac{1}{2} \underset{\sim}{c}^T \underset{\sim}{k} \underset{\sim}{c} - \underset{\sim}{c}^T \underset{\sim}{L} \qquad \frac{\partial U}{\partial \underset{\sim}{c}} = \underset{\sim}{k}\underset{\sim}{c} - \underset{\sim}{L} = 0$$

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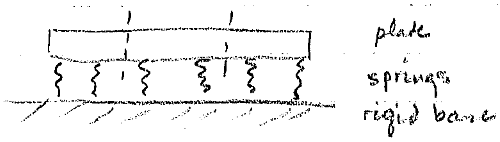
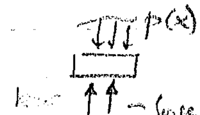
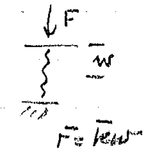
for brittle material look for max tensile stress to define SCF
for material that fails in shear look for max shearing stress to define SCF
for material that flows plastically look for value that will bring onset of plasticity

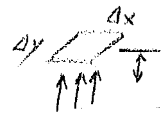
Consider Plate w/ Elastic Foundation (Winkler foundation for soils)



Take a Δx out of plate

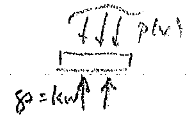






$F_{tot} = \wp\, \Delta x \Delta y$

equivalent pressure



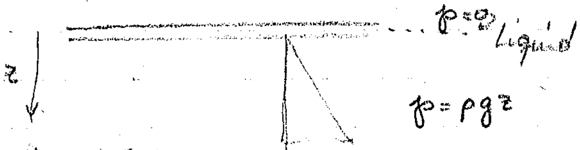
w/out found. $D\Delta\Delta w = p$

$k \; [=] \; \dfrac{\text{force}}{\text{unit area} \times (\text{unit disp.})}$

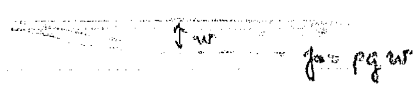
to include found: replace $p \rightarrow p - kw$ $\therefore$ $D\Delta\Delta w + kw = p$

foundation doesn't affect defn. of m_x, m_y, m_{yx} & also doesn't affect b.c.

Example: Case of an exact elastic foundation – plate floating on liquid (ice over water)



for case of deflected plate

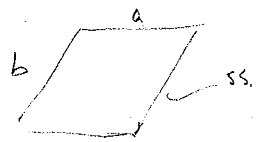


equivalent $k = \rho g$.

For soils, use elastic foundation (Winkler) for a first approx. Good only when foundation is softer than beam (plate).

Solution of $D\Delta\Delta w + kw = p$

Return to Rectangular plate with $p = p_0 \sin\frac{\pi x m}{a} \cdot \sin\frac{\pi y n}{b}$



$w = w_{mn} \sin \sin$

now put into DE: $\left[D\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}^2 + k \right] w_{mn} = p_{mn}$

- as the values $m, n \uparrow$ then $D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 >> Dk$ very. Short wavelength foundation effect non existent
- if $m, n = 0$ then $w_{mn} = \frac{p_{mn}}{DK}$ Long wavength, foundation effect very dominant
- k will not affect convergence of series. Foundation will help reduce stress in plate

for p being const, linear, quadratic, cubic : $w_{partic} = P/k$

Complementary Sol. $D\Delta\Delta w + pw = 0$
Considering $w = w(x)$ only $Dw'''' + kw = 0$ $w = e^{mx}$

$\therefore (Dm^4 + k)w = 0 \quad \therefore m = (-1)^{1/4}\left(\frac{k}{D}\right)^{1/4}$ $(-1)^{1/4} = e^{i\pi/4} \cdot e^{in\pi/2} = \frac{\sqrt{2}}{2}(1+i) \begin{Bmatrix} 1 \\ i \\ -1 \\ -i \end{Bmatrix}$

Roots are $\frac{\sqrt{2}}{2}(1+i)$; $-\frac{\sqrt{2}}{2}(1-i)$; $-\frac{\sqrt{2}}{2}(1+i)$; $+\frac{\sqrt{2}}{2}(1-i)$

Elastic foundation continued 11/19/79

for $p = \text{const}$ $w_{Partic} = P/K$
for complimentary sol. $w = e^{mx} \Rightarrow Dm^4 + k = 0$ $m = (-1)^{1/4}\left(\frac{k}{D}\right)^{1/4}$

$m = \left(\frac{K}{D}\right)^{1/4}(-1)^{1/4} = \frac{1+i}{\sqrt{2}}\left(\frac{k}{D}\right)^{1/4}$ Roots are $m, \bar{m}, -m, -\bar{m}$

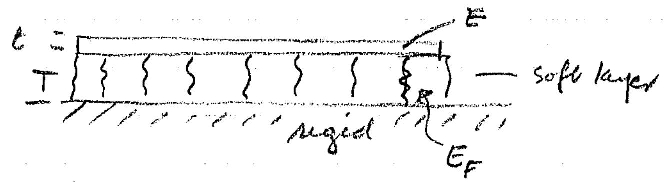
Complem Solution $w = c_1 e^{mx} + c_2 e^{\bar{m}x} + c_3 e^{-mx} + c_4 e^{-\bar{m}x}$

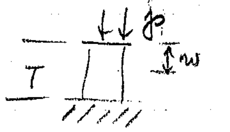
Since w is real $\Rightarrow$ if a complex fn is a sol so must its complex conj
$\Rightarrow c_2 = \bar{c}_1 \quad c_4 = \bar{c}_3 \quad \therefore \quad w = 2\,\mathrm{Re}\left[c_1 e^{mx} + c_3 e^{-mx}\right]$
$c_{1,3} = \mathrm{Re}\, c_{1,3} + i\, \mathrm{Im}\, c_{1,3}$ Thus we have 4 real consts

the $\xi = \frac{1}{\sqrt{2}}\left(\frac{K}{D}\right)^{1/4} x \quad \therefore \; e^{mx} = e^{(1+i)\xi} = e^{\xi}(\cos\xi + i\sin\xi)$

We take $e^{-\pi}$ As a decay distance $\therefore$ if $\xi = \pi$ then we find that $e^{\xi} \sim .04$
hence we find $\delta = x_{decay} = \pi\sqrt{2}\left(D/K\right)^{1/4}$ ie for $x > \delta$ soln will be less than 4 % of what it was at $x = 0$.

Suppose we have



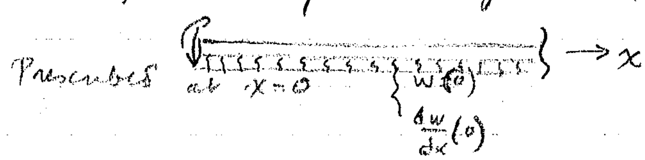


$$\epsilon = \frac{-w}{T} = \frac{\sigma}{E_f} = -p/E_f \quad \text{or} \quad p = \frac{E_f}{T} w = kw.$$

then $$\delta = \pi\sqrt{2}\left(\frac{D}{k}\right)^{1/4} = \pi\sqrt{2}\left[\frac{Et^3 T}{12(1-\nu^2)E_f}\right]^{1/4} = \frac{\pi\sqrt{2}}{[12(1-\nu^2)]^{1/4}}\left(\frac{E}{E_f}\frac{T}{t}\right)^{1/4} t$$

$$\therefore \quad \frac{\delta}{t} = \frac{\pi\sqrt{2}}{[12-(1-\nu^2)]^{1/4}}\left(\frac{E}{E_f}\frac{T}{t}\right)^{1/4}$$ for theory to be valid $\frac{\delta}{t} \gg 1$.

Next Look for a soln of Semi-Inf. Plate (beam)



for a prescribed disp & rot. $\Rightarrow$ a shear & moment exist at $x=0$. Because we want bdd soln at $x \to \infty$ $\Rightarrow C_1 = 0$ $\quad w(x) = 2\,\mathrm{Re}(C_3 e^{-mx})$

$$w(0) = 2\,\mathrm{Re}\, C_3$$

$$\frac{\partial w}{\partial x}(x) = \mathrm{Re}\left\{2C_3\,(-(1+i))\,\frac{d\xi}{dx}\cdot e^{-(1+i)\xi}\right.$$

$$\frac{\partial w}{\partial x}(0) = 2\xi'(\mathrm{Re}\,C_3 - \mathrm{Im}\,C_3) \quad \text{and} \quad 2\,\mathrm{Im}\,C_3 = \frac{w'(0)}{\xi'} + w(0)$$

$$w''(x) = \mathrm{Re}\left\{2C_3\,(1+i)^2\left(\frac{d\xi}{dx}\right)^2 e^{-(1+i)\xi} + 2C_3\left(-(1+i)\frac{d^2\xi}{dx^2}\right)^{0} e^{-(1+i)\xi}\right)$$

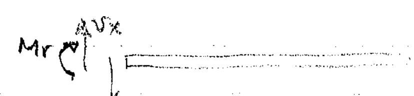
$$M_x(0) = -Dw''(0) = D(\xi')^2\, 4\,\mathrm{Im}\,C_3 = 2D(\xi')^2\left[\frac{w'(0)}{\xi'} + w(0)\right]$$

$$w'''(x) = \mathrm{Re}\; 2C_3\; 2i\,(-(1+i))\; e^{-(1+i)\xi}\,(\xi')^3$$

$$w'''(0) = 2(\xi')^3(\mathrm{Re}\,2C_3 + \mathrm{Im}\,2C_3)$$

$$V_x = -D\left(\frac{\partial}{\partial x}\Delta w + \underbrace{\frac{\partial M_{xy}}{\partial y}}_{=0}\right) = -D\frac{d^3 w}{dx^3} \quad \& \quad V_x(0) = -D\,2(\xi')^3\left(2w(0) + \frac{w'(0)}{\xi'}\right)$$

$$\begin{pmatrix} M_x \\ V_x \end{pmatrix} = 2D\begin{bmatrix} \xi' & \xi'^2 \\ -\xi'^2 & -2(\xi')^3 \end{bmatrix}\begin{pmatrix} w'(0) \\ w(0) \end{pmatrix}$$



since disp is ↓ & V_x ↑
then rewrite 2nd eq as $-V_x = + \quad +$
2 in Matrix

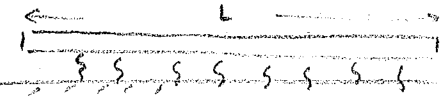
Continuation of Previous 11/21/79

$D\nabla^4 w + kw = p$ Beam of elastic foundation

For $w = w(x)$ a beam or cylindrical bending of plates we get $Dw^{iv} + kw = p$

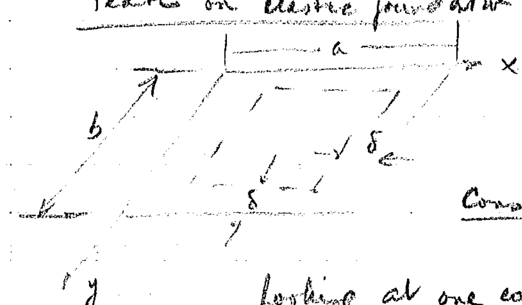
For the composite sol. we have a decay distance $\Rightarrow \delta = \pi\sqrt{2}\,(D/k)^{1/4}$

In the case of a railroad track for $L \gg \delta$



We can consider effect of conditions at the left end negligible (4%) on right end & vice versa. The BC @ each edge are "uncoupled" & may be considered separately. The foundation is then significant in the problem. When $L \ll \delta$ say $L \leq \delta/10$, the effect of found. is very much negligible & problem can be simplified. The case $\delta/10 < L < \delta$ is most difficult since both foundation & bc must be accounted for.

<u>Plate on elastic foundation</u>

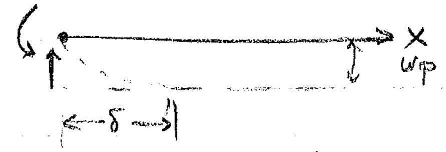


If $a, b \leq \delta/10$ the foundation K is negligible

If $a, b \geq \delta$ have boundary layers

<u>Consider</u> the case $p = \text{const}$ & $a, b \geq \delta$, clamped edges all around.

Looking at one edge we can use simple beam problem



$w(x,y) = w_p + w_c$; to satisfy bc $w_c(0,y) = -p/k$ $\quad \frac{\partial w_c}{\partial x}(0,y) = 0$

For $w_c(x,y) = \mathrm{Re}\left\{ C e^{-(1+i)\zeta} \right\}$ we found $\zeta = \frac{1}{\sqrt{2}}\left(\frac{k}{D}\right)^{1/4} x$ and on edge

$$\begin{bmatrix} M_x(0,y) \\ -V_x(0,y) \end{bmatrix} = 2D \begin{bmatrix} \zeta' & (\zeta')^2 \\ (\zeta')^2 & 2(\zeta')^3 \end{bmatrix} \begin{pmatrix} w_x(0,y) \\ w(0,y) \end{pmatrix}$$

This gives result for max. mo. as $\quad M_x(0,y) = 2D(\zeta')^2(-p/k)$

shear $\quad -V_x(0,y) = 4D(\zeta')^3(-p/k)$

A finite element sol. would require a very fine mesh spacing since the boundary layer is small & the fn. rapidly changing there.

<u>Plate on elastic foundation - hinged edge</u>

$w_p = p/k \quad w_c(0,y) = -p/k \quad w_{c,xx}(0,y) = 0$ (moment condition) start w/ soln previously obtained:

$$w_c(x,y) = Re\{ C e^{-(1+i)\xi} \}$$

$$w_{c,xx} = Re\left\{ C\left[-(1+2i-1)\xi'\right]^2 \exp\left[-(1+i)\xi\right]\right\}$$

$$= -2\xi'^2 \, Re\{ i C \exp[\;\;] \}$$

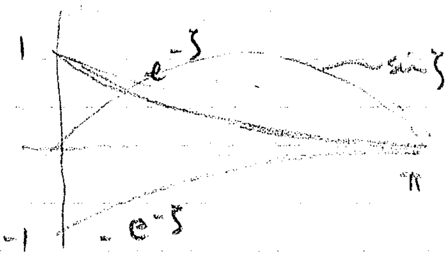
$$= 2\xi'^2 \, Im\, C e^{-(1+i)\xi}$$

so $Im\, C = 0$; $Re\, C = w_c(0,y)$ — on edge

$$= -P/k$$

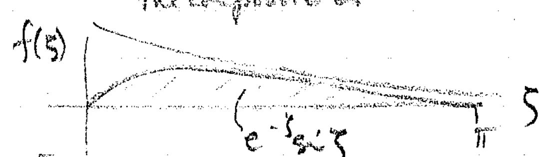
$$M_x = -Dw_{xx} = -D\,2(\xi')^2\left(-P/k\right) Im\left(e^{-(1+i)\xi}\right)$$

$$= -D\cdot 2(\xi')^2\left(-P/k\right)\underbrace{e^{-\xi}\sin\xi}_{f(\xi)}$$



$e^{-\xi}\sin\xi$ has peaks when $f'(\xi) = 0$ @ $\pi/4$

the composite is



$$f_{max} = e^{-\pi/4}\sin\pi/4 = \frac{1}{\sqrt{2}}e^{-\pi/4}$$

11/26/79

Failure Analysis Assoc – Prof. Ross

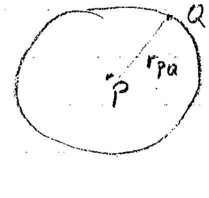
11/28/79

Exam – Open Books & Notes

Boundary Integral Method. – can handle very general problems.

Given a boundary to produce some given displ boundary conditions we can impose loads & moments on bdy [as shown in HW #6].

How do we define loads & mom. Remember a concentrated force is a fn. of $r^2(\log r/a - 1)$ since $\Delta^2 w = P\delta(\underline{x})$ has a soln $w = w(r^2\log r/a - r^2)$

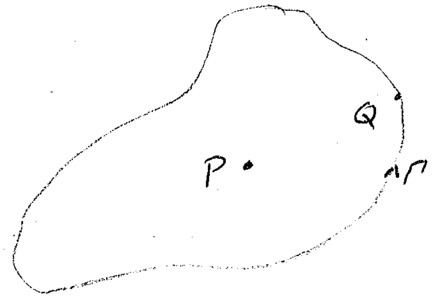


Displ at Q due to force at P

$$W(Q,P) = \frac{F}{8\pi D}\left(r^2(\log r/a - 1)\right) \quad \text{where } r = r_{PQ} = r_{QP}$$

$\therefore W(P,Q) = W(Q,P)$ thus

given a boundary curve Γ



$$w(P) = \oint \rho(Q)\, W(Q,P)\, ds_Q$$

where $\rho(Q)$ is a weighting fn. of magn of load at Q

For concentrated moment take $\frac{d}{dx} P\delta(\underline{x})$

So by use of maxwell Betty relation given 2 sol. of plate eqn $w^{(1)}, w$

$$\oint_\Gamma \left[M_n^{(1)}\left(-\frac{\partial w}{\partial n}\right) + V_n^{(1)} w + M_n \left(+\frac{\partial w}{\partial n}\right)^{(1)} - V_n w^{(1)} \right] ds_Q = 0$$

where w is what we want & $w^{(1)}$ is some fn. we assume known

boundary curve Γ w/outward normal $\underset{\sim}{n}$

if we assume $w^{(1)}$ solution is known & then let w to be the fn. we want on bdy we can prescribes w, w' or M_n & V_n

If we discretize the integral then $\oint M_n^{(1)}$ is a matrix $\left(-\frac{\partial w}{\partial n}\right)$ is a vector

$\oint V_n^{(1)}$ " " " w " " "

If we write $V_n^{(1)} = q_n^{(1)} + \frac{\partial M_{ns}^{(1)}}{\partial s}$

$$\oint V_n^{(1)} w\, ds = \oint q_n\, w\, ds + \oint \frac{\partial M_{ns}}{\partial s} w\, ds$$

but $\oint \frac{\partial M_{ns}}{\partial s} w\, ds = \oint \frac{\partial}{\partial s}(M_{ns} w) ds - \int M_{ns} \frac{\partial w}{\partial s} ds = - \int M_{ns} \frac{\partial w}{\partial s} ds$ since $M_{ns} w \Big|_{Q_1}^{Q_1} = 0$

$$\therefore \oint V_n w\, ds \equiv \int q_n w\, ds + \int M_{ns}\left(\frac{\partial w}{\partial s}\right) ds \quad \text{also.}$$

now $M_n' = D(K_n + \nu K_s) = D(K_n + K_s - (1-\nu)K_s)$

$M_{ns}' = D(1-\nu)\left(-\frac{\partial}{\partial s}\frac{\partial w}{\partial n}\right) \qquad K_s = -\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial s}\right)$

This only for n, s rectangular cartesian system but results hold for general boundary ~~bdy~~ curve.

Integral reduces

other part of integral

$$\oint D\Big[\underbrace{(\kappa_n^{(1)}+\kappa_s^{(1)})}_{-\Delta w}\Big(-\frac{\partial w}{\partial n}\Big) - \frac{\partial}{\partial n}(\Delta w^{(1)})\, w - \dots \quad = 0$$

$\therefore$ $D = \text{const}$

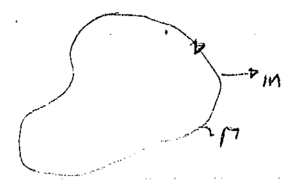
$$\oint \Big[\, \Delta w^{(1)}\Big(-\frac{\partial w}{\partial n}\Big) + \frac{\partial}{\partial n}(\Delta w^{(1)})\, w + \Delta w\Big(\frac{-\partial w^{(1)}}{\partial n}\Big) - \frac{\partial}{\partial n}(\Delta w)\, w^{(1)}\Big] ds = 0$$

hence Poisson's Ratio effect drops out.

11/30/79

Continuing our previous results.

$$\oint_\Gamma \Big[M_n^{(1)}\Big(-\frac{\partial w}{\partial n}\Big) + V_n^{(1)} w - M_n\Big(-\frac{\partial w}{\partial n}\Big)^{(1)} - V_n\, w^{(1)} \Big] ds = 0$$

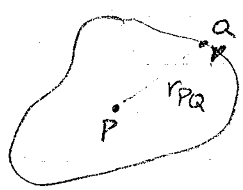


which reduces to

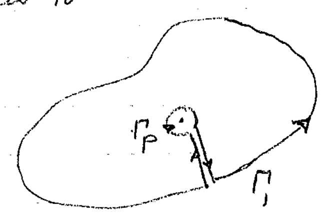
$$\oint \Big[\, \Delta w^{(1)}\Big(-\frac{\partial w}{\partial n}\Big) + \frac{\partial}{\partial n}(\Delta w^{(1)})\, w - \Delta w\Big(-\frac{\partial w}{\partial n}\Big)^{(1)} - \frac{\partial}{\partial n}(\Delta w)\, w^{(1)}\Big] ds = 0$$

Both $w, w^{(1)}$ have no singularities in domain bounded by Γ

Choose $\log r = w^{(1)}$ and consider P inside Γ & Q on the boundary



$\log r$ is singular at P & make a modification of contour to



$$\oint_\Gamma = \oint_{\Gamma_P} + \oint_{\Gamma_1} = 0 \implies \oint_{\Gamma_1} = \oint_{\Gamma_P}$$

$$w^{(1)} = \ln r_{PQ} \quad \underline{\text{but}} \quad \frac{\partial w^{(1)}}{\partial n} = \frac{\partial w^{(1)}}{\partial r} = \frac{\partial \ln r}{\partial r} = \frac{1}{r} \quad \text{on } \Gamma_P$$

now $\Delta w^{(1)} \equiv 0$ everywhere except at $r = 0$ $\implies \dfrac{\partial \Delta w^{(1)}}{\partial n} = 0$

$$\therefore \oint_{\Gamma_P} = \lim_{r\to 0}\int_0^{2\pi} \Big[\Delta w\Big(\frac{\partial w^{(1)}}{\partial r}\Big) - \frac{\partial}{\partial n}(\Delta w)\, w^{(1)}\Big] r\, d\theta$$

$$\int \Big[\Delta w \cdot \frac{1}{r} r\, d\theta - \frac{\partial}{\partial n}(\Delta w)\cdot \underbrace{r \ln r}_{\to 0}\, d\theta\Big] = \lim_{r\to 0}\int_0^{2\pi} \Delta w\, d\theta = (\Delta w)_P \cdot 2\pi$$

and thus $\int_{\Gamma'} \left[\Delta w \left(\frac{\partial w^{(1)}}{\partial n} \right) - \left(\frac{\partial}{\partial n} \Delta w \right) w^{(1)} \right] ds = 2\pi (\Delta w)_P$

where $\Delta w = \Delta w$ of point Q $\quad \frac{\partial w^{(1)}}{\partial n} = \frac{\partial w^{(1)}}{\partial n}$ of P to Q

$$\therefore \int_{\Gamma} \left[(\Delta w)_Q \left(\frac{\partial w^{(1)}}{\partial n} \right)_{PQ} - \left(\frac{\partial}{\partial n} (\Delta w)_Q \right) w^{(1)}_{PQ} \right] ds_Q = 2\pi (\Delta w)_P$$

if P is close to Q then

$$\oint_{\Gamma'} \left[(\Delta w)_Q \left(\frac{\partial w^{(1)}}{\partial n} \right)_{PQ} - \left(\frac{\partial}{\partial n} (\Delta w)_Q \right) w^{(1)}_{PQ} \right] ds_Q = \pi (\Delta w)_P$$

if we are given Δw on the bdry this eqn can give $\frac{\partial \Delta w}{\partial n}$ on bdy as an integral eqn.

Now let $w^{(2)} \to \frac{1}{4} r^2 (\log r - 1)$ $\quad r = r_{PQ}$ $\quad$ then $\Delta w^{(2)} = \ln r$

Results $\oint_{\Gamma_P} \left[\Delta w^{(2)} \left(-\frac{\partial w}{\partial n} \right) + \left(\frac{\partial}{\partial n} \Delta w^{(2)} \right) w + \Delta w \left(\frac{\partial w^{(2)}}{\partial n} \right) - \frac{\partial}{\partial n} (\Delta w) w^{(2)} \right] ds$

but $\frac{\partial w^{(2)}}{\partial n} \sim r \log r$ $\quad \therefore$ as $r \to 0$ $\quad \Delta w^{(2)} ds \to r \ln r \to 0$

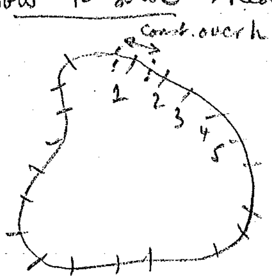
$$\frac{\partial}{\partial n} (\Delta w^{(2)}) ds \to d\theta$$

$$\therefore \oint_{\Gamma_P} = \begin{matrix} 2\pi w_P & \text{if } P \text{ not on } \Gamma \\ \pi w_P & P \text{ on } \Gamma \end{matrix} = \oint \left[\Delta w^{(2)} \left(-\frac{\partial w}{\partial n} \right) + \frac{\partial}{\partial n} (\Delta w^{(2)}) w + (\Delta w)_Q \left(\frac{\partial w^{(2)}}{\partial n} \right)_{PQ} - \left(\frac{\partial}{\partial n} (\Delta w)_Q \right) w^{(2)}_{PQ} \right] ds$$

thus these 2 integral eqns $\quad \underbrace{w, \frac{\partial w}{\partial n}}_{\text{are prescribed}} \quad \underbrace{\Delta w, \frac{\partial}{\partial n} (\Delta w)}_{\text{to determine these}}$

why? since shear & moment on Γ are fns of Δw & $\frac{\partial}{\partial n} (\Delta w)$.

How to solve: Reduce integral eqn to matrix equations.



Assume $w, \frac{\partial w}{\partial n}, \ldots$ are constant in each segment

or $w = w_j$ for $s_j - h/2 \le s \le s_j + h/2$

look at one part of the integral

$$\oint_{\Gamma} (\Delta w)_Q \left(\frac{\partial w^{(1)}}{\partial n} \right)_{PQ} ds_Q = \sum_{j=1}^{N} (\Delta w)_j \int_{s_j - h/2}^{s_j + h/2} \left(\frac{\partial w^{(1)}}{\partial n} \right)_{ij} ds$$

P at pt i
Q at pt j

$$= \sum_{j=1}^{N} a_{ij} (\Delta w)_j = \underset{-}{A} \cdot (\underset{\sim}{\Delta w})$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \ddots & \\ & & a_{nn} \end{bmatrix} \begin{bmatrix} \Delta w_1 \\ \Delta w_2 \\ \vdots \\ \Delta w_n \end{bmatrix}$$

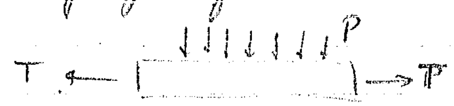
$$\therefore \oint_{\Gamma} \left[\qquad \right] = \pi (\Delta w)_p$$

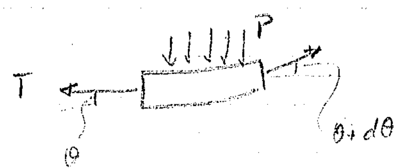
$$\Rightarrow \underset{-}{A}\, \underset{\sim}{\Delta w} - \underset{-}{B} \frac{\partial}{\partial n} (\Delta \underset{\sim}{w}) = \pi\, \underset{\sim}{I}\, \Delta \underset{-}{w} \quad \text{or} \quad (\underset{-}{A} - \pi \underset{-}{I}) \Delta \underset{-}{w} = \underset{-}{B} \left(\frac{\partial}{\partial n} \Delta \underset{-}{w} \right)$$

B^{-1} always exists

12/3/97

Effect of inplane forces on beams & plates





$$\theta = -\frac{dw}{dx}$$

$$dx P = T \left[\sin(\theta + d\theta) - \sin\theta \right]$$

for small θ $\quad P dx = T d\theta$

$$\therefore \quad p + T \frac{d^2 w}{dx^2} = 0 \qquad \text{this is string equation}$$

for original DE for beam

$$D w^{IV} = p \qquad \text{but now let } p = p_{\text{effective}} = p + T w''$$

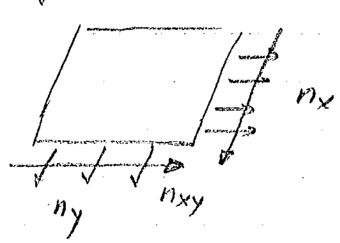
$$\therefore \quad D w^{IV} - T w'' = p$$

If no pressure load let $w = e^{\lambda x}$ $\quad (D\lambda^4 - T\lambda^2) w = 0$

or $\lambda = 0, 0, \pm\sqrt{T/D}$

$$\therefore \quad w = c_1 + c_2 x + c_3 e^{-\sqrt{T/D}\, x} + c_4 e^{\sqrt{T/D}\, x}$$

for the plate



$$P_{\text{effective}} = P + n_x w_{,xx} + n_y w_{,yy} + 2 n_{xy} w_{,xy}$$

$\therefore$ plug into plate eqn: $D \nabla^2 \nabla^2 w - n_x w_{,xx} - n_y w_{,yy} - 2 n_{xy} w_{,xy} = p$

Assume $n_{xy} = 0$

let $p = \sum_{m,n=1}^{\infty} p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$ (suppose plate is SS)

$w = \sum_{m,n} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$ put all this into DE

$$\Rightarrow \left\{ D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 + n_x\left(\frac{m\pi}{a}\right)^2 + n_y\left(\frac{n\pi}{b}\right)^2 \right\} W_{mn} = p_{mn}$$

note if $n_x, n_y > 0 \Rightarrow$ for same $p \Rightarrow$ coeff on w is larger $\Rightarrow w$ is smaller — hence tensile stress strengthens plate. If $n_x, n_y < 0$ we can made coeff of $W_{mn} = 0$ (buckling)

Consider case where $n_x = n_y = n$ $n_{xy} = 0$

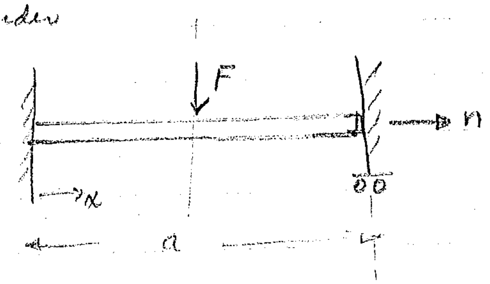
$$\Rightarrow D\Delta^2 W - n\Delta W = p \qquad (D\Delta - n)\Delta W = p$$

if $p = 0$ then a soln is $\Delta W = 0$ a second sol. is $(D\Delta - n)W = 0$

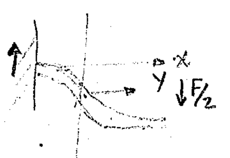
for 1-D $\frac{d^2 w}{dx^2} = 0 \Rightarrow w_1 = c_1 + c_2 x$ & $D\frac{d^2 w}{dx^2} - nw = 0 \Rightarrow w_2 = c_1 e^{-\sqrt{\frac{n}{D}}x} + c_2 e^{+\sqrt{\frac{n}{D}}x}$
$= c_3 \cosh\sqrt{\ } x + c_4 \sinh\sqrt{\ } x$

we can define a decay distance $\pi = \sqrt{\frac{n}{D}}\,\delta$ or $\delta = \sqrt{\frac{D}{n}}\,\pi$

Consider



by symmetry we will consider only half the beam



$y = x - a/4$

$w = c_1 + c_2 x + c_3 \cosh y + c_4 \sinh y$ by antisym at $y = 0$ $c_3 = 0$

BC $w\big|_{y=-a/4} = 0$ $V\big|_{y=a/4} = F/2$

$\frac{dw}{dy}\Big|_{y=a/4} = 0$ only 3 bc since we've already used symmetry.

$\therefore$ $w = c_1 + c_2[y + a/4] + c_4 \sinh y$

$w' = c_2 + c_4\sqrt{\frac{n}{D}} \cosh\sqrt{\frac{n}{D}}\, y$ $w''' = c_4 \left(\frac{n}{D}\right)^{3/2} \cosh\sqrt{\frac{n}{D}}\, y$

$w'' = c_4 \left(\sqrt{\frac{n}{D}}\right)^2 \sinh\sqrt{\frac{n}{D}}\, y$

$V\big|_{y=a/4} = -Dw''' = F/2 = -Dc_4\left(\frac{n}{D}\right)^{3/2} \cos\sqrt{\frac{n}{D}}\frac{a}{4}$ $\boxed{c_4 = \frac{-F}{2D}\left(\frac{D}{n}\right)^{3/2} \frac{1}{\cosh\sqrt{\frac{n}{D}}\frac{a}{4}}}$

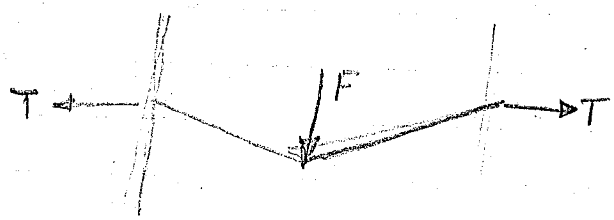
$w'\big|_{a/4} = 0 \Rightarrow \boxed{c_2 = +\frac{F}{2n}}$ $w\big|_{y=-a/4} = 0 \Rightarrow c_1 = \frac{-F}{2D}\left(\frac{D}{n}\right)^{3/2} \tanh\sqrt{\frac{n}{D}}\frac{a}{4}$
$+ Fa/8n$

$$w\big|_{y=a/4} = 2w\big|_{y=0} = 2c_1 = -\frac{F}{D}\left(\frac{D}{n}\right)^{3/2}\tanh\sqrt{\frac{n}{D}}\,\frac{a}{4} + \frac{Fa}{4n}$$

$$\Delta = \frac{Fa}{4n}\left\{1 - \frac{\tanh\sqrt{\frac{n}{D}}\,\frac{a}{4}}{\sqrt{\frac{n}{D}}\,\frac{a}{4}}\right\}$$ defl under load

$$\delta = \pi\sqrt{D/n} \Rightarrow \sqrt{\frac{n}{D}}\,\frac{a}{4} = \frac{\pi a}{4\delta} \equiv z \qquad z \gg 1 \qquad \delta \ll a$$

for large z $\quad \Delta = \frac{Fa}{4n}\left[1 - \frac{\tanh z}{z}\right] \qquad \frac{\tanh z}{z} \approx \frac{1}{z} \to 0 \quad \therefore \Delta = \frac{Fa}{4n}$



from string theory

for $z \ll 1 \qquad \delta \gg a \qquad$ since $\delta = \pi\sqrt{D/n} \Rightarrow n$ must be small

we should get same results for beam theory.

$\tanh z \sim z - z^3/3 + \ldots \qquad \frac{\tanh z}{z} \sim 1 - z^2/3 + \ldots$

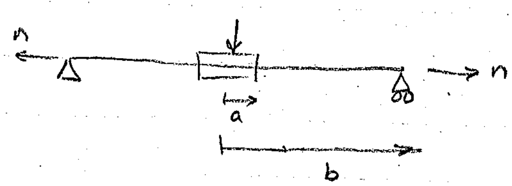
$1 - \frac{\tanh z}{z} \sim \frac{z^2}{3}$ and $\Delta = \frac{Fa}{4n}\cdot\frac{n a^2}{3 \cdot D\, 16} = \frac{Fa^3}{192 D}$ Brooks 497-218

12/5/79

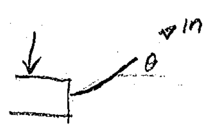
Circular Plates under uniform tension

$D\Delta^2 w - n\,\Delta w = p \qquad$ assume $p=0 \qquad \Delta w_I = 0 \qquad D\Delta^2 w_{II} - n\, w_{II} = 0$

$w_I = C_1 \ln r + C_2$



$w_I = C_1 \ln r/b$



$P = 2\pi r\theta n = 2\pi r\left(-\frac{dw}{dr}\right)n = -\frac{C_1}{r}\cdot 2\pi r n$

$P = 2\pi r\left(-\frac{C_1}{r}\right)n = -2\pi C_1 n \qquad C_1 = -\frac{P}{2\pi n} \qquad \therefore\ w = \frac{P}{2\pi n}\ln\frac{b}{r}$

for $D w_{xx} - n w = 0$ $\quad W = A e^{\sqrt{n/D}\,x} + B e^{-\sqrt{n/D}\,x}$ since we note that

On the boundary we don't feel effect of P if b is large

$\therefore$

$$w = C_1 \ln b/r + A\left(e^{\sqrt{n/D}(r-b)} - 1\right)$$ for large r note w/ r=b $\quad w = 0$ at bdy

now $\dfrac{dw}{dr} = -C_1/r + A\sqrt{\tfrac{n}{D}}\, e^{\sqrt{n/D}(r-b)}$ $\quad$ from last time $\delta = \dfrac{\pi}{\sqrt{n/D}} = \dfrac{\pi}{\lambda}$

at $r=b$ $\quad \theta = 0$ $\quad \therefore \quad 0 = -\dfrac{C_1}{b} + A\sqrt{\tfrac{n}{D}}$ $\qquad A = \dfrac{C_1}{\sqrt{n/D}\, b} = \dfrac{C_1\delta}{\pi b}$

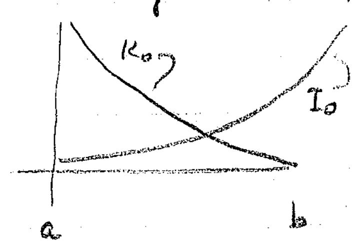
$$\left.\frac{d^2w}{dr^2}\right|_{r=b} = C_1\left[\frac{1}{r^2} + \frac{A(n/D)}{C_1} e^{\sqrt{n/D}(r-b)}\right] = \frac{C_1}{b^2}\left[1 + \frac{\pi b}{\delta}\right]$$

$\dfrac{b}{\delta} >> 1$ we have large curvature

at r=a we can do the same if $a >>> \delta$

if a is not $>> \delta$ the solns are bessel fns

if $n>0$ $\quad D\nabla^2 w_{II} - n\, w_{II} = 0$ $\qquad w_{II} = A\, I_0\left(\sqrt{\tfrac{n}{D}}\,r\right) + B\, K_0\left(\sqrt{\tfrac{n}{D}}\,r\right)$



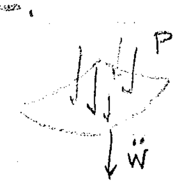
$K_0 \sim \ln r$ for $r<<1$

if $n<0$ $\quad$ sol $\quad w_{II} = A\, J_0\left(\sqrt{\tfrac{n}{D}}\,r\right) + B\, Y_0\left(\sqrt{\tfrac{n}{D}}\,r\right)$ $\qquad Y_0 \sim \ln r \quad r<<1$

$J_0 = 1 \quad r=0$

Consider Plate vibrations problem.

$D\Delta^2 w = p$



look at $D\Delta^2 w = \dfrac{\text{mass}}{\text{Area}} \cdot \text{accel.}$

inertia force $= \rho h \dfrac{\partial^2 w}{\partial t^2}$

$\therefore \quad D\Delta^2 w = p - \rho h \dfrac{\partial^2 w}{\partial t^2}$ $\quad$; consider free vibes

set $P=0$ & look at harmonics $\quad w(x,y,t) = W(x,y)\sin\omega t$

$\longrightarrow \quad D\nabla^2\nabla^2 W = +\rho h \omega^2 W$ $\quad$ like foundation w/ negative spring const

Look at SS plates $\quad$ Let $W(x,y) = A \sin\dfrac{m\pi x}{a} \sin\dfrac{n\pi y}{b}$

this satisfies bc on W & moment. put into DE

$$\therefore \quad D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 - \rho h\omega^2 = 0$$

$$\Longrightarrow \quad \omega = \sqrt{\frac{D}{\rho h}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{+1/2}$$

for a square plate $\omega_{mn} = \sqrt{\dfrac{D}{\rho h}}\left(\dfrac{\pi}{a}\right)^2 (m^2$

Forced Vibration $p(t)$

$$D\nabla^2\nabla^2 w = p - \rho h \ddot{w} \qquad \text{let } p(t,x,y) = \sum\sum p_{mn}(t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$w(x,y,t) = \sum\sum w_{mn}(t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

put into DE

look at m,n term: $D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 w_{mn}(t) + \rho h \ddot{w}_{mn}(t) = p_{mn}(t)$

let $\omega_{mn}^2 = D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 / \rho h \quad \therefore \quad \ddot{W}_{mn}(t) + \omega_{mn}^2 W_{mn} = \frac{p_{mn}(t)}{\rho h}$

natural freq of m,n mode

let $p_{mn}(t) = A_{mn} \sin \Omega t$

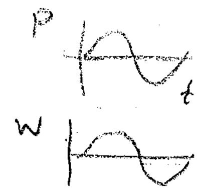
let $W_{mn}(t) = B_{mn} \sin \Omega t$

if $\Omega = \omega_{mn}$ then we have resonance.

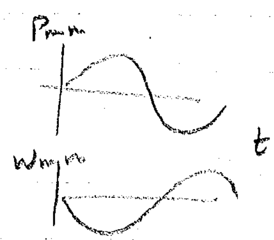
put into $\ddot{W}_{mn} + \omega_{mn}^2 W = p_{mn}/\rho h$

$$-B_{mn}\Omega^2 + \omega_{mn}^2 B_{mn} = A_{mn}/\rho h$$

$$B_{mn} = \frac{A_{mn}}{\rho h(\omega_{mn}^2 - \Omega^2)} \qquad \text{if } \omega_{mn} > \Omega$$



if $\omega_{mn} < \Omega$



when we're at resonance PDE is $\ddot{W} + \omega^2 W = f(t)$ let $f(t) = e^{i\omega t}$

$= e^{i\omega t}$

Guess $W = \hat{f}(t) e^{i\omega t}$

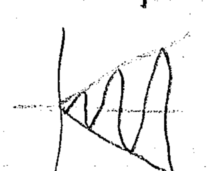
$$\dot{W} = \dot{\hat{f}} e^{i\omega t} + i\omega \hat{f} e^{i\omega t}$$

$$\ddot{W} = \ddot{\hat{f}} e^{i\omega t} + 2\dot{\hat{f}} i\omega e^{i\omega t} + (i\omega)^2 \hat{f} e^{i\omega t}$$

put into DE

$$\ddot{f} + 2i\omega\dot{f} = 1 \qquad \text{let } f = t/2i\omega \quad \dot{f} = 1/2i\omega \quad \ddot{f} = 0$$

$$W = \frac{t}{2i\omega} e^{i\omega t} + W_h$$



linear theory is good for small time.

RAYLEIGH QUOTIENT

12/7/79

Rod $\longleftarrow$ [ ] $\longrightarrow F$ $\qquad \mathcal{U} = \int_0^{\delta} F\,d\delta'$ strain energy

$$u = \int_0^{\epsilon} \sigma\, d\epsilon' = \frac{E\epsilon^2}{2} = \frac{\sigma\epsilon}{2} \quad \text{linear elasticity only.}$$

now $\quad U = \int_0^{\ell} u A\,dx$

for Bending $\;$ ([ ]) M $\qquad K = \dfrac{d\theta}{dx} \qquad u = \int_0^{K} m\,dK' = \frac{1}{2} E I K^2$ for linear elasticity

For plate theory w/o inplane forces.

$$u = \int_{\text{Init}}^{\text{final}} m_x\, dK_x + m_y\, dK_y + m_{xy}\, d\chi \qquad m_x = D(K_x + \nu K_y) \text{ etc.} \quad \chi = 2K_{xy}? \text{ or } K_{xy}$$

$$u = \frac{D}{2}\left[(K_x + K_y)^2 - 2(1-\nu)(K_x K_y - K_{xy}^2)\right] \qquad \mathcal{U} = \iint u\,dA$$

$$V = -\iint p(x,y)\, w(x,y)\, dA \qquad \therefore \quad \text{Total energy } \Pi = U + V$$

$$KE = T = \tfrac{1}{2} m v^2 \text{ for particle} \qquad T = \iint \tfrac{1}{2}\rho h\, dA\, w_{,t}^2 = \tfrac{1}{2}\rho h \iint w_{,t}^2\, dA$$

Free vibration $\quad$ Let $p = 0 \Rightarrow V = 0$

$\therefore \quad \Pi = U + T = \text{const.}$

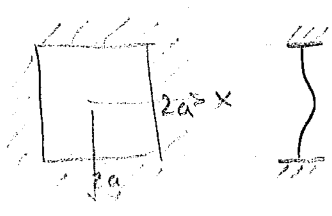
let $w(x,y,t) = W(x,y)\cos\omega t \qquad \frac{\partial w}{\partial t} = -W\omega \sin\omega t \qquad T = \omega^2 \sin^2\omega t\, F(W(x,y))$

since $\; F[W] = \iint \frac{\rho h}{2} W^2(x,y)\, dx\,dy$

$U = \cos^2\omega t\; \mathcal{U}[W(x,y)]$ where $\mathcal{U}[W] = \frac{1}{2}\iint D\left[(K_x + K_y)^2 - 2(1-\nu)(K_x K_y - K_{xy}^2)\right] dx\,dy$

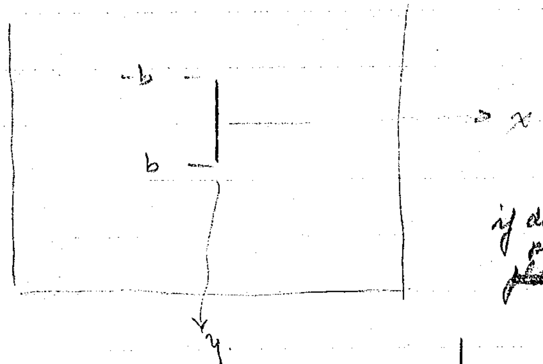
$$\therefore \quad T_{max} = \omega^2 F[W] = U_{max} = \mathcal{U}[W] \qquad \therefore \quad \omega^2 = \frac{\mathcal{U}[W]}{F[W]}$$

idea: pick w close to actual mode shapes so that we satisfy bc. This method will give $\sim \omega_{lowest}$ $\quad \omega$ from this will always be larger.

Example: look at  $\qquad w(x,y) = c\,(x^2 - a^2)^2 (y^2 - b^2)^2.$

$$w = \frac{.909}{a^2}\sqrt{\frac{D}{\rho h}}$$ good to with 4%

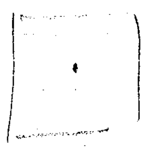
Fall 76
Problem #2



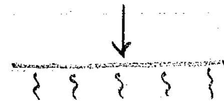
if b is very small use point load solution on plate ie decay distance large

if decay distance is small, ie b is large use ~~plate strip~~ plate with infinite line load.

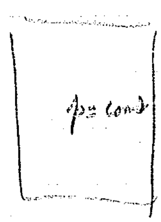
b small



b large



Fall 76
Problem #1



$$p = \sum P_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

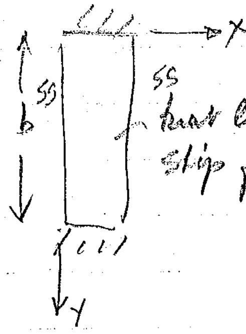
$$w = \sum W_{mn} \sin \quad \sin$$

$$W_{mn} = \frac{P_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2}$$ w/o hole.

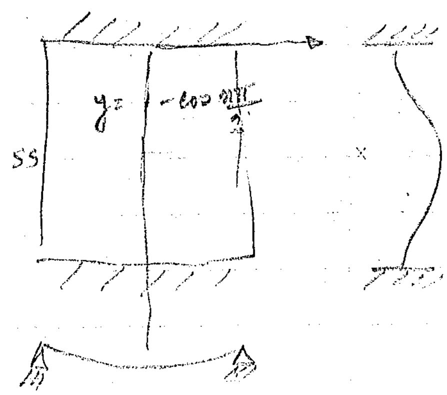
get mom & shear from this

Use HW results for circular hole in an ∞ plate w/ results from above to get ~~___~~ moment & shear to be zero on hole.

Fall 75
Problem #1



beam like plate strip for large b



Use energy method

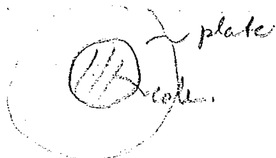
Problem #2 Axisym. loading

$$\Delta\Delta w = \left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\right] w\right) = p(r)$$

use homog. sol.

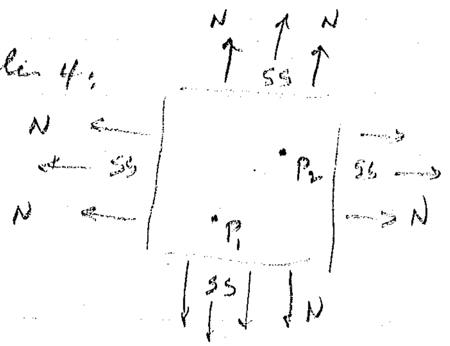
Problem 3:

make plate = decay distance



$$\delta = \sqrt{2}\ \pi \sqrt[4]{\frac{D}{k}} = d.$$

Problem 4:



superpose bending sol. near p_1, p_2 point load on the tension solution.

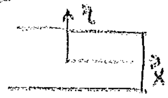
point load sol. $\to 0$ as $R \to \infty$.

Problem #5 $D\Delta\Delta w + \rho h \ddot{w} = 0$

take $w = A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \omega t$ put into DE

$$\omega = \sqrt{\frac{D}{\rho h}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

Problem #3 Fall 76

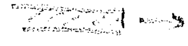


for a plate $\int_{-t/2}^{t/2} \sigma_x z\, dz = m_x$

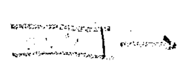
$\sigma_x = \frac{E}{1-\nu^2}\left[\epsilon_x + \nu \epsilon_y\right]$ $\epsilon_x = \kappa_x z$ $\epsilon_y = \kappa_y z$

$\therefore\ m_x = D(\kappa_x + \nu \kappa_y)$

Approx



$\sigma_{x_{top}} \, t/4 + \sigma_{x_{bot}} \, t/4 = m_x$



$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu \epsilon_y)$ $\epsilon_x = \kappa_x z$
$\epsilon_y = \kappa_y z$

$m_x = C\,(\kappa_x + \nu \kappa_y)$

or $$D' = \frac{t^3 E}{12(1-\nu^2)} - \frac{(t/2)^3 E}{12(1-\nu^2)} = \frac{7}{8} D$$

ME 241A Theory of Plates (& Shells) 4-4-79

R.L. Mallett : Durand 259 X 7-3565

Course Outline

Intro. to Plates

Small Deformation theory (Assumptions)

Methods of Soln - EXACT & APPROX.

Applic : Rectangular / Circular plates.

Vibration & Buckling of Plates.

Large Deformation of plates.

→ Intro to Shells

Membrane theory for Shells of Revolutions

No text :

Reference texts – Timoshenko & Wenofsky Krieger : Theory of Plates/Shells

Donnell : Beams, Plates, Shells. @ Reference section of Bookstore

Mansfield :

Marguerre :

McFarland :

Jaeger:

Szilard:

} only on Plates.

} on reserve in lib

Grading : Midterm 20% ; Final 40% (open notes only) ; Homework 40% due a wk from day handed out

Beam Theory see handout

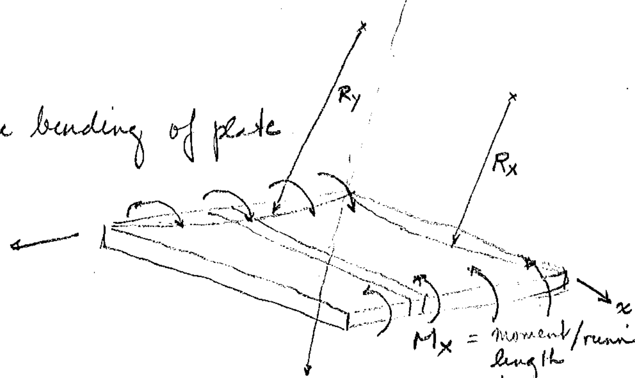
Meet Tuesday @ 12:50 - 2:05 T-Th from now on in 550D

results 4/6/79

to neglect axial load displacements of beam << depth of beam

slopes of deflection curve will also have to be very small

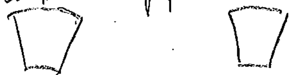
Pure bending of plate



M_x = moment/running length

where $M_x = \int_{-h/2}^{h} \sigma_x z\,dz$

each section can be considered a beam, then distortion of beams are



we must now fit them together

also $\frac{1}{R_x} = \frac{M_x}{EI}$ $\qquad \frac{1}{R_y} = -\nu\frac{M_x}{EI}$

if we had an M_y superimposed then

$$M_y = \int_{-h/2}^{h/2} \sigma_y z\,dz \qquad \text{also} \quad \frac{1}{R_y} = \frac{M_y}{EI} \qquad \frac{1}{R_x} = -\nu\frac{M_y}{EI}$$

$$\therefore \quad \frac{1}{R_{x_{TOT}}} = \frac{1}{EI}\left(M_x - \nu M_y\right) = -\frac{\partial^2 w}{\partial x^2} \; ; \quad \frac{1}{R_y} = \frac{1}{EI}\left(-\nu M_x + M_y\right) = -\frac{\partial^2 w}{\partial y^2}$$

where $I = h^3/12$

I = mom of inertia/unit width

$$M_x = -\frac{Eh^3}{12(1-\nu^2)}\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]$$

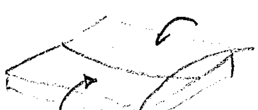
$$M_y = -\frac{Eh^3}{12(1-\nu^2)}\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]$$

define the flexural rigidity $D = \frac{Eh^3}{12(1-\nu^2)}$

For <u>Cylindrical Bending</u> must apply an M_y to get rid of curvature in y direction

$$\therefore \quad M_x = -D\frac{\partial^2 w}{\partial x^2} \qquad M_y = -D\nu\frac{\partial^2 w}{\partial x^2}$$

$\therefore$ $M_y = \nu M_x$ this is what must be applied to keep plate from stretching & curving in y direction.

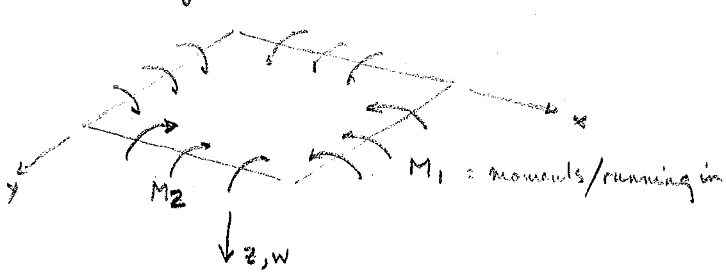


M_y applied

4/10/79

Pure Bending of Uniform Plates - no transverse shearing stresses

Rectangular



$\frac{\partial^2 w}{\partial x \partial y} = 0$ "no twist" - (change of slope at const. x, y) but see example 3 where state of deformation is twisted even though twisting curvature = 0.

$$\frac{1}{R_x} = \frac{M_1 - \nu M_2}{Eh^3/12} \qquad \frac{1}{R_y} = \frac{M_2 - \nu M_1}{Eh^3/12}$$

solve the above eqn $\frac{1}{R_x} = -W_{xx}$, $\frac{1}{R_y} = -W_{yy}$

Example 1 $\quad M_1 = M \quad M_2 = \nu M \longrightarrow \frac{1}{R_y} = 0 \Rightarrow$ Cylindrical Bending of plate ie midplane is cylindrical

$$\frac{1}{R_x} = \frac{M(1-\nu^2)}{Eh^3/12} = \frac{12(1-\nu^2)}{Eh^3} M = \frac{M}{D} \qquad D = \frac{Eh^3}{12(1-\nu^2)}$$ this is the flexural rigidity

$$\therefore \quad W_{xx} = -\frac{M}{D} \quad \therefore \quad W = -\frac{M}{D} x^2 + C_1 + C_2 x$$

Example 2. Spherical Bending $\quad M_1 = M_2 = M$

$$\frac{1}{R_x} = \frac{1}{R_y} = \frac{M(1-\nu)}{Eh^3/12} = \frac{M}{(1+\nu)D}$$ midplane is spherical - more rigid than cylindrical

$$\therefore \frac{M}{(1+\nu)D} = -\frac{\partial^2 W}{\partial x^2} \quad \text{where } W = W(x,y) \qquad \text{and} \quad \frac{M}{(1+\nu)D} = -\frac{\partial^2 W}{\partial y^2}$$

$$W = \frac{-M}{(1+\nu)D}\frac{x^2}{2} + f(y)x + g(y)$$

$$\therefore f''x + g'' = \frac{-M}{(1+\nu)D} \Rightarrow f'' = 0$$

$$f = C_2 + C_3 y \; ; \; g = \frac{-M}{(1+\nu)D}\frac{y^2}{2} + C_0 + C_1 y$$

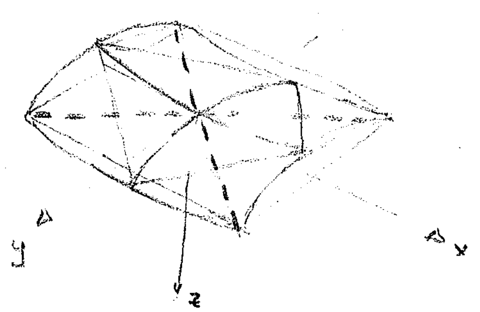
$$\therefore \quad W = \frac{-M}{(1+\nu)D}\frac{x^2+y^2}{2} + \underbrace{C_0 + C_1 y + C_2 x}_{\text{rigid body rotation}} + C_3 xy$$

also for no twisting $\frac{\partial^2 W}{\partial x \partial y} = 0 \Rightarrow C_3 = 0$ ie slopes don't change for const x or y.

Example 3. $\quad M_1 = M \quad M_2 = -M \longrightarrow \frac{1}{R_x} = -\frac{1}{R_y} = \frac{M(1+\nu)}{Eh^3/12} = \frac{M}{(1-\nu)D}$ less rigid than cylindrical

$$W = \frac{-M}{(1-\nu)D}\frac{x^2-y^2}{2} + C_0 + C_1 x + C_2 y$$ using same procedure as Ex 2.

Plot case for $C_0 = C_1 = C_2 = 0$



Saddle surface twisted deformation w/ $W_{xy} = 0$

$y = x$ straight lines

$y = x + c \qquad x^2 - y^2 = -2cx - c^2$ straight lines twisted wrt each other

Question	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

STANFORD UNIVERSITY

OFFICIAL EXAMINATION BOOK

24 Page Ruled

Name of student ______________________

Date of examination ______________________

Course ______________________

THE STANFORD UNIVERSITY HONOR CODE

A. The Honor Code is an undertaking of the students, individually and collectively:

(1) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

(2) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

I acknowledge and accept the Honor Code.

(Signed) ______________________

Interpretations and applications of the Honor Code appear on the back cover of this examination book.

HW

#1 a. Beam problem : clamped with a) distrib load
b) point force

b. Cantilever with axial & transverse forces a) $\frac{u(\ell)}{w(\ell)}$, $\frac{E\frac{\partial u}{\partial x}\big|_\ell}{\frac{-EI\,\partial^2 w/\partial x^2\big|_\ell \cdot \frac{h}{2}}{I}}$

#2 a. Fourier series for SS beam with point load.

b. Plate w/o ~~surface~~ surface loads. What edge tractions if $w = Axy$ [loads at 4 corners]

#3 a. SS plate with constant load – find anchorage force

b. SS plate with point load

#4 a. Semi infinite strip : SS on ∞ edges w/ given w & $\frac{\partial w}{\partial y}$ on one face
finding M_y & V_y

b. Infinite strip : SS on ∞ edges with q on $y=0$ find M_y & w

#5 a. Circular plate : clamped outer edge & given $p = Ar$ (axisymmetric)
find M_r, M_θ & w at $r=0$

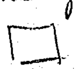
b. Circular plate : w/ M_r at outer edge (axisym. problem) & inner bdy 1. Clamped 2. Free & find (M_{max}) & SCF if inner bdy $\to 0$

#6 a. hole in an infinite plate w/ BC on hole dependent on periodic fn : find M_r, V_r
show $\begin{pmatrix} M_r \\ V_r \cdot a \end{pmatrix} = [\text{sym matrix}] \begin{pmatrix} a \cdot \frac{\partial w}{\partial r} \\ w \end{pmatrix}$

#7 a. Given M_x, M_y, M_{xy} in cartesian convert to polar

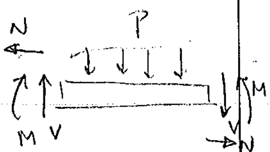
b. if a hole is put into plate w/ bending resultants of a find SCF (use superposition

#8 a. Using energy method to find max moment at some point in a clamped (3 edge) 1 free plate



#9 a. Infinite beam on elastic foundation w/ point load at center find M & w

b. Infinite plate on foundation subjected to point load – find decay distance



Beam Theory $EIw^{IV} = p(x)$ where $-EIw'' = M$ $\quad -EIw''' = V$ $\quad \theta = \frac{dw}{dx}$

Beam on Foundation $EIw^{IV} + kw = p(x)$; Beam with inplane force $EIw^{IV} - Nw'' = p$

1. Assume deflections small in comparison to length ; $\left|\frac{dw}{dx}\right| \ll 1$
2. Normals to neutral surface remain normal after deformation
3. Normals remain plane & unstretched.

→ BC either w or V and w' or M specified

defn $\quad Q = V = \int_{-h/2}^{h/2} \sigma_{zx}\,dz \qquad M = \int_{-h/2}^{h/2} \sigma_{xx}\, z\, dz \; ; \; \frac{dM}{dx} = Q \; ; \; \frac{dQ}{dx} = -p$

Plate Theory $D\Delta^2 w = p(x)$ where $-D\left[w_{,xx} + \nu w_{,yy}\right] = m_x$, $-D\left[w_{,yy} + \nu w_{,xx}\right] = m_y$

$-D(1-\nu)\, w_{,xy} = m_{xy}$

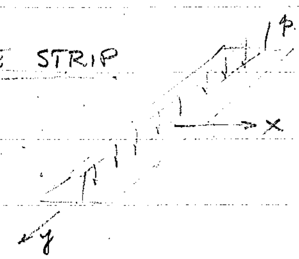
$q_x = -D\frac{\partial}{\partial x}(\Delta w) \qquad q_y = -D\frac{\partial}{\partial y}(\Delta w) \; ; \quad v_x = q_x + \frac{\partial}{\partial y} m_{xy} \; ; \; v_y = q_y + \frac{\partial}{\partial x} m_{xy}$

CAN REPLACE M_{xy} with Resultant at Edges

→ BC either w ~~...~~ or v_i and $\frac{\partial w}{\partial x_i}$ or M_{x_i}

SS $\quad w$ & M are zero $\qquad$ clamped $\; w$ & $\frac{\partial w}{\partial x_i} = 0 \qquad$ free $\; v_i$ and M are zero

See pg 5 ∞ PLATE STRIP

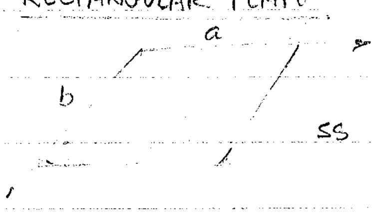


Assume no dependence on y.

$D\, w^{IV} = p(x)$

SAME SOLN AS BEAM : REPLACE EI BY D

RECTANGULAR PLATE



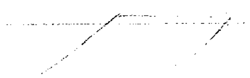
$D(\Delta^2 w) = p(x,y)$

SS soln $\quad w(x,y) = \sum A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{L}$

$p(x,y) = \sum p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

$A_{mn} = \dfrac{p_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2}$

PG 7 Homogeneous plate strip semi infinite



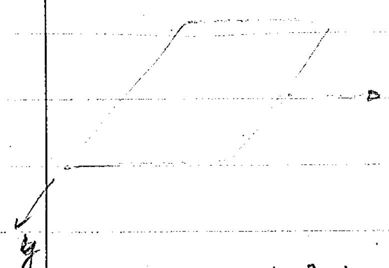
$$w(x,y) = \sum (A_n + B_n \lambda_n y) e^{-\lambda_n y} \sin \lambda_n x \qquad \text{for SS}$$

$$\lambda_n = \frac{n\pi}{a}$$

if $w(x,o) = 0$ & $m_y(x,o) = f(x) \Rightarrow A_n = 0$ and $f(x) = \sum f_n \sin \frac{n\pi x}{a}$

then $B_n = \frac{f_n}{2D\lambda_n^2}$ $\qquad f_n = \frac{2}{a}\int_0^a f(x) \sin \frac{n\pi x}{a}$

PG 8 FOR GENERAL PLATE



- use separation of variables

for SS on $x = \text{const}$ faces use $w(x,y) = \sum \left[(A_n + B_n \lambda_n y) \times e^{-\lambda_n y} + (C_n + D_n \lambda_n y) e^{\lambda_n y} \right] \sin \frac{n\pi x}{a}$

If $D(\Delta^2 w) = p$ Let $w = w_c + w_p$

Choose $w_p(x,y) \ni D\Delta^2 w_p = p$ but doesn't satisfy BC.

Choose $w_c(x,y) \ni D\Delta^2 w_c = 0$ and bc $\ni w_{c_{BC}} = w_{BC} - w_{p_{BC}}$

→ POLAR COORD $D\Delta^2 w = p(r,\theta)$

$$\Delta = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial^2 \theta} \right] \qquad \text{for axisym} \quad \Delta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$$

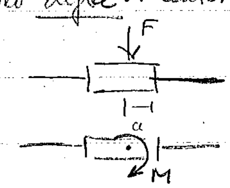
see reverse side for more

if $W_c = C_1 r^2 \ln r + C_2 \ln r + C_3 r^2 + C_4 \Rightarrow D\Delta^2 W_c = 0$ $\qquad W_c$ is homog. soln.

if $p = p_0$ partic soln is $w = \frac{p_0 r^4}{64D}$

if we want bounded deflec at center $C_2 = 0$; for bounded shear $C_1 = 0$

PAGE 10 - PLUG problem



μ plug $\gg \mu$ plate $\qquad$ plate is clamped at plug

PAGE 12 - and $v_r = \frac{-F}{2\pi a}$

HW# 6 has full soln. for non axisymm. $W_c = \sum_{n=2} \left(\frac{A_n}{r^n} + \frac{B_n}{r^{n+2}} + C_n r^n + D_n r^{-n+2} \right) \sin n\theta +$

$$\left(\frac{A_n'}{r^n} + \frac{B_n'}{r^{n+2}} + C_n' r^n + D_n' r^{-n+2} \right) \cos n\theta +$$

$$\left(c_1 r + c_2/r + c_3 r^3 + c_4 r \ln r \right) \sin \theta + \left(c_1' r + c_2'/r + c_3' r^3 + c_4' r \ln r \right) \cos \theta$$

$+ (C_1'' + C_2'' \ln r + C_3'' r^2 + C_4'' r^2 \ln r)\, \theta$

for $n = 0$ cannot prescribe shear at both bdy – must satisfy equil.

for $n = 1$ Cannot prescribe moments, since shear loads produce moments

for $n \geq 2$ self equilib.

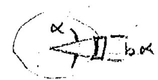
$\rightarrow \quad m_r - v_r r = -\dfrac{4 D C_4'}{r}$ where C_4' is coeff of $r \ln r \cos\theta$

General

$$m_r = -D \left[w_{,rr} + \nu \left(\frac{w_{,r}}{r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right] \qquad m_\theta = -D \left[\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial r^2} \right]$$

$$v_r = q_r + \frac{1}{r} \frac{\partial}{\partial \theta} m_{r\theta} \qquad v_\theta = q_\theta + \frac{\partial}{\partial r} m_{r\theta} \qquad m_{r\theta} = -D(1-\nu) \frac{\partial}{\partial r} \left(\frac{w_{,\theta}}{r} \right)$$

$$q_r = -D \frac{\partial}{\partial r} (\Delta w) \qquad q_\theta = -D \frac{1}{r} \frac{\partial}{\partial \theta} (\Delta w)$$

PG 13 COLUMNS supporting a plate  

N columns equally spaced ; constant load Approx $v_r(b,\theta) = A \quad |\theta| = \alpha$ $\quad N b \alpha \cdot A = p \cdot \pi b^2$

$= 0$ $\quad A = \dfrac{\pi p b}{N \alpha}$

Strain Energy due to bending $\displaystyle\int \frac{EI}{2} w_{,xx}^2 \, dx$

PG 14 – 16 For beam $\displaystyle U = \int_0^L \left\{ \tfrac{1}{2} EI \left[w'' \right]^2 - p w \right\} dx$

For plate $\displaystyle U = \int_0^b\!\!\int_0^a \left\{ \frac{D}{2} \left[(\Delta w)^2 + 2(1-\nu) \left\{ \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial y^2} \right) \right\} \right] - p w \right\} dx\, dy$

Maxwell betty relationships $\dfrac{\partial U}{\partial w} = F \qquad \dfrac{\partial \bar{U}}{\partial F} = w$ need to find U for system

can always write $\begin{pmatrix} M \\ v \end{pmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{pmatrix} w \\ w' \end{pmatrix}$ where $\dfrac{\partial M}{\partial w} = A_{11} \quad \dfrac{\partial M}{\partial w'} = \dfrac{\partial v}{\partial w} = A_{12} = A_{21} \quad \dfrac{\partial v}{\partial w'} = A_{22}$

Rayleigh Methods pick w to satisfy bc ; put into U & minimize wrt constants used in w ie pick $w(x) = \sum A_n x^n$ put in U then take $\dfrac{\partial U}{\partial A_n} = 0$ & find A^n

Pg 16 – Beam on foundation for general sol. let $w = \sum A_{mn} \sin \ \sin$ for SS plate

$p = \sum p_{mn} \sin \ \sin$

then $A_{mn} = \dfrac{p_{mn}}{D \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 + k \right]}$

if p is const, linear, square, cubic $\quad w_{part} = P/K \quad$ & $w = w_H + w_p$

w_H solves inhomog biharmonic $\quad D\Delta^2 w + kw = 0 \quad$ Plate

$$EI\frac{d^4w}{dx^4} + kw = 0 \quad \text{beam}$$

Decay distance $\delta = \pi\sqrt{2}\left(\frac{D}{K}\right)^{1/4}$ for a beam or plate w/no dependence on y.

for $x > \delta$ load will not be felt $\qquad D = \frac{Et^3}{12(1-\nu^2)}$

for solution to be valid $\delta/t >> 1 \qquad t$ = thickness of beam or plate

Pg 18 → BIE

Let $w^{(1)} = \ln r \qquad w^{(2)} = (r^2\ln r - r^2)/4$

then if we are given w and $\frac{\partial w}{\partial n}$ on the boundary then

$$\oint_{\Gamma}\left[(\Delta w)_Q\left(\frac{\partial w^{(1)}}{\partial n}\right)_{PQ} - \left(\frac{\partial}{\partial n}(\Delta w)_Q\right)w^{(1)}_{PQ}\right]ds_Q = \pi(\Delta w)_P$$

$$\oint\left[(\Delta w^{(2)})\left(-\frac{\partial w}{\partial n}\right) + \left(\frac{\partial}{\partial n}\Delta w^{(2)}\right)w + \Delta w_Q\left(\frac{\partial w}{\partial n}^{(2)}\right)_{PQ} - \frac{\partial}{\partial n}(\Delta w_Q)\, w^{(2)}_{PQ}\right]ds = \pi w_p$$

P 20 Effect of inplane forces

$$D\Delta^2 w - N_x w_{,xx} - N_y w_{,yy} - 2N_{xy} w_{,xy} = p.$$

if $p = \sum p_{mn} \sin \sin \qquad w = \sum A_{mn} \sin \sin \qquad A_{mn} = \dfrac{p_{mn}}{D\left\{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 + n_x\left(\frac{m\pi}{a}\right)^2 + n_y\left(\frac{n\pi}{b}\right.\right.}$

if $n_{xy} = 0$ also.

P22 Vibrations

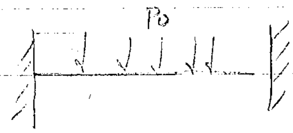
find U & $T = \iint \frac{1}{2}\rho h \dot{w}(x,y,t)^2\, dA \qquad V = -\iint p\, w\, dA$

$(PE + \text{Strain energy})_{max} = (V + U)_{max} = (KE)_{max}$ gives relation for ω^2 lowest frequency

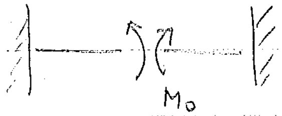
HANDOUTS

<u>Mallek</u> Portion of ME241A

1. Showing that u, v are small in comparison to w
2. Solution for an SS beam w/ $p=0$
3. What are requirements to neglect axial load.



$$w = \frac{p_0}{64D}(R^2-r^2)\left[(R^2-r^2) - \frac{4R^2}{1+\nu}\right]$$



$$w = \frac{M_0}{2(1+\nu)D}\,\frac{\left[b^2 - r^2 + 2b^2 \ln r/b\right]}{\left[1 + \frac{1-\nu}{1+\nu}\, b^2/a^2\right]}$$

ME 241A (AA246) Fall 1979

Assignment 1

Due Wednesday, October 3

PROBLEM 1: The differential equation for a beam is

$$EI \frac{d^4 w}{dx^4} = q$$

$w^{IV} = q/EI$ $w'' = q/EI\, x^2/2 + c_1 x + c_2$
$w''' = q/EI\, x + c_1$ $w' = q/EI\, x^3/6 + c_1 x^2/2 + c_2 x + c_3$
$w = q/EI\, x^4/24 + c_1 x^3/6 + c_2 x^2/2 + c_3 x + c_4$

For a beam of length L which is clamped at the ends, find the deflection at midspan due to

$w = \frac{\partial w}{\partial x} = 0$ @ $x = 0, L$

(a) uniformly distributed load q

(b) concentrated force P at midspan.

P $P/2$ $P/2$ $EI\, w''' = P/2$

PROBLEM 2: A cantilever beam of length L is loaded at the free end by an axial force N_o and a transverse force Q_o. For the case $N_o = Q_o$, find:

(a) the ratio of axial to transverse deflection

(b) the ratio of "axial" stress to "bending" stress

Problem Set #1

Problem #1a since $EIw^{IV} = q$ then

$$w''' = \frac{q}{EI}x + C_1 \qquad w' = \frac{q}{EI}\frac{x^3}{6} + C_1\frac{x^2}{2} + C_2x + C_3$$

$$w'' = \frac{q}{EI}\frac{x^2}{2} + C_1x + C_2 \qquad w = \frac{q}{EI}\frac{x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4$$

B.C. for clamped beam are $w = w' = 0$ at $x = 0, L$

$\therefore$ (1) $w(0) = 0 \Rightarrow C_4 = 0$; (2) $w(L) = 0 \Rightarrow \frac{q}{EI}\frac{L^4}{24} + C_1\frac{L^3}{6} + C_2\frac{L^2}{2} + \cancel{C_3L} = 0$

(3) $w'(0) = 0 \Rightarrow C_3 = 0$; (4) $w'(L) = 0 \Rightarrow \frac{q}{EI}\frac{L^3}{6} + C_1\frac{L^2}{2} + C_2L = 0$

multiplying (4) by $L/2$ and subtract for (2) gives $-\frac{q}{EI}\frac{L^4}{24} + C_1L^3\left(\frac{1}{6} - \frac{1}{4}\right) = 0 \Rightarrow -\frac{q}{EI}\frac{L^4}{24} - \frac{2L^3}{24}C_1 = 0$

or $\boxed{C_1 = -\frac{L}{2}\frac{q}{EI}}$ and $\boxed{C_2 = \frac{L^2}{12}\frac{q}{EI}}$

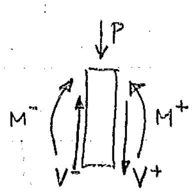
thus

$$w = \frac{q}{24EI}\left(x^4 - 2x^3L + x^2L^2\right) \qquad w(L/2) = \frac{q}{24EI}\left(\frac{L^4}{16} - \frac{2L^4}{8} + \frac{L^4}{4}\right) = \frac{qL^4}{384EI}$$

Ans: $\boxed{w(L/2) = \frac{qL^4}{384EI}}$

Problem #1b since $EIw^{IV} = q(x)$ and $q(x) = 0$ our basic equation is $EIw^{IV} = 0$

Our boundary conditions are that $w(0) = w'(0) = w(L) = w'(L) = 0$ (1-4) and that at the midspan we must have equilibrium



$V^+ + P - V^- = 0 \Rightarrow -EIw^{+'''} + P + EIw^{-'''} = 0$ @ $x = L/2$ (5)

$M^+ = M^- \Rightarrow w^{-''} = w^{+''}$ @ $x = L/2$ (6)

and $w^+ = w^-$ @ $x = L/2$ (7)

$w^{+'} = w^{-'}$ @ $x = L/2$ (8)

for $0 \le x \le L/2$ $\quad w''' = C_1 \quad w' = C_1\frac{x^2}{2} + C_2x + C_3$

$w'' = C_1x + C_2 \quad w^- = C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4$ (9-12)

for $\frac{L}{2} \le x \le L$ $\quad w''' = C_5 \quad w' = C_5\frac{x^2}{2} + C_6x + C_7$ (13-16)

$w'' = C_5x + C_6 \quad w^+ = C_5\frac{x^3}{6} + C_6\frac{x^2}{2} + C_7x + C_8$

$w(0) = w'(0) = 0 \Rightarrow$ (17,18) $C_4 = C_3 = 0$ using (1,2) in (11,12)

$w(L) = w'(L) = 0 \Rightarrow$ (19,19) $C_5\frac{L^2}{2} + C_6L + C_7 = 0$ and $C_5\frac{L^3}{6} + C_6\frac{L^2}{2} + C_7L + C_8 = 0$ using (3,4) in (15,16)

$w^+ = w^-$ @ $x = L/2 \Rightarrow$ (20) $C_1\frac{L^3}{48} + C_2\frac{L^2}{8} = C_5\frac{L^3}{48} + C_6\frac{L^2}{8} + C_7\frac{L}{2} + C_8$ using (7) in (12,16)

$w^{+'} = w^{-'}$ @ $x = L/2$ $\Rightarrow$ (21) $C_1 \frac{L^2}{8} + C_2 \frac{L}{2} = C_5 \frac{L^2}{8} + C_6 \frac{L}{2} + C_7$ using (8) in (11,15)

$W^{+''} = W^{-''}$ @ $x = L/2$ $\Rightarrow$ (22) $C_1 L/2 + C_2 = C_5 L/2 + C_6$ using (6) in (10,14)

$W^{+'''} - W^{-'''} - P/EI = 0$ @ $x = L/2$ $\Rightarrow$ (23) $C_5 - C_1 - P/EI = 0$ $\Rightarrow$ $\boxed{C_5 - C_1 = P/EI}$ using (5) in (9,13)

using (22) $C_6 - C_2 = (C_1 - C_5) L/2 = \boxed{\frac{-PL}{2EI} = C_6 - C_2}$

using (21-23) $C_7 = (C_1 - C_5)\frac{L^2}{8} + (C_2 - C_6) L/2 = \frac{-PL^2}{8EI} + \frac{PL^2}{4EI} = \boxed{+\frac{PL^2}{8EI} = C_7}$ (24)

using (22,23,24) in (20)

$$C_8 = \frac{-PL^3}{48EI} + \frac{PL^3}{16EI} - \frac{PL^3}{16EI} = \boxed{\frac{-PL^3}{48EI} = C_8} \quad (25)$$

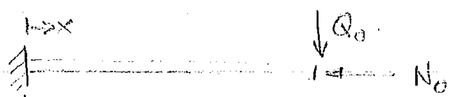
using (19,19') with (24,25) yields $\boxed{C_5 = +P/2EI}$ and $\boxed{C_6 = \frac{-3PL}{8EI}}$ (26,27)

thus $\boxed{C_1 = -P/2EI}$ and $\boxed{C_2 = +\frac{PL}{8EI}}$ (28,29)

and $W(L/2) = C_1 L^3/48 + C_2 L^2/8 = -\frac{PL^3}{96EI} + \frac{PL^3}{64EI} = \frac{PL^3}{192EI}$

Ans: $\boxed{W(L/2) = \frac{PL^3}{192EI}}$ ✓ Using symetry would save a lot of effort.

Problem 2:



under this applied loading the DE becomes

$EI W^{IV} + PW'' = 0$ where P is the applied axial load

the BC are $w(0) = w'(0) = 0$ and $w''(L) = 0$ (since no moment exists at that end)

& $W'''(L) + \frac{N_0}{EI} W'(L) = -\frac{Q_0}{EI}$ (since there is a load there) For this problem you should have decoupled the 2 loads.

The most general solution to our diff eqn is $W = Ax + B + C\sin kx + D\cos kx$ $k = \left(\frac{N_0}{EI}\right)$

since $W(0) = 0 \Rightarrow D + B = 0$ $W'(0) = 0 \Rightarrow A + kC = 0$

$W''(L) = 0 \Rightarrow (C\sin kL + D\cos kL)k^2 = 0$

and $-k^3 C\cos kL + k^3 D\sin kL + k^2\left[A + Ck\cos kL - Dk\sin kL\right] = -\frac{Q_0}{EI} \Rightarrow \boxed{A = -\frac{Q_0}{EIk^2}}$

hence $C = -A/k = \boxed{\dfrac{Q_o}{EIk^3} = C}$ but $D = -C\tan kL = \boxed{-\dfrac{Q_o}{EIk^3}\tan kL = D}$

and $B = -D = \boxed{\dfrac{Q_o}{EIk^3}\tan kL = B}$

when $Q_o = N_0 \Rightarrow \quad A = -1 \quad B = \dfrac{\tan kL}{k} \quad C = \dfrac{1}{k} \quad D = -\dfrac{\tan kL}{k}$

$$\therefore \quad w = -x + \frac{\tan kL}{k} + \frac{\sin kx}{k} - \frac{\tan kL \cos kx}{k}$$

$$w(L) = -L + \frac{\tan kL}{k} + \cancel{\frac{\sin kL}{k}} - \cancel{\frac{\sin kL}{k}} = \frac{\tan kL - kL}{k}$$

We also know that $\epsilon = \dfrac{\Delta l}{l} = \dfrac{\sigma}{E} = \dfrac{N_0}{AE}$ or $\Delta l = \dfrac{N_0 L}{AE} = \dfrac{IL}{A}\cdot\dfrac{N_0}{EI} = k^2\dfrac{IL}{A}$

if h is the depth of the beam and b is the width of the beam then

$I_z = bh^3/12 \quad A = bh \quad \therefore \quad \dfrac{IL}{A} = \dfrac{h^2 L}{12} \quad \therefore \quad \boxed{\dfrac{\Delta l}{w(L)} = \dfrac{k^3 h^2 L/12}{\tan kL - kL}}$

To find the axial stress $\sigma_a = E\epsilon = N_0/A = -k^2\dfrac{EI}{A}$

To find the bending stress $\sigma_b = \dfrac{My}{I} = -EIw''(0)\,y/I = -Eyw''(0) = -Ey\,(k\tan kL) = \mp\dfrac{Eh}{2}(k\tan kL)$

where max σ occurs when $y = \pm h/2$

$$\therefore \quad \frac{\sigma_a}{\sigma_b} = \frac{-k^2\,EI/A}{\mp\frac{Eh}{2}\,k\tan kL} = \frac{-k\,h^2/12}{\mp\frac{h}{2}\tan kL} = \boxed{\frac{\pm kh}{6\tan kL} = \frac{\sigma_a}{\sigma_b}}$$

for small $N_0, Q_0 \Rightarrow \quad \tan kL - kL \sim k^3L^3/3 \quad$ thus $\dfrac{\Delta l}{w(L)} \cong h^2/4L^2$

and $\dfrac{\sigma_a}{\sigma_b} \cong \pm\dfrac{kh}{6kL} = \pm\dfrac{h}{6L}$

$$\tan kL = \frac{\sin kL}{\cos kL} = \frac{kL - (kL)^3/3! + (kL)^5/5!}{1 - (kL)^2/2! + (kL)^4/4!}$$

$$= kL + \frac{(kL)^3}{3!} + \frac{(kL)^5}{5!} + \frac{(kL)^3}{2!}$$

$$\tan kL - kL = \frac{(kL)^3}{3}$$

$$\therefore \quad w(L) = \frac{k^2L^3}{3} \qquad \Delta l = k^2\frac{IL}{A} \quad \Rightarrow \quad \boxed{\frac{\Delta l}{w} = \frac{3I}{AL^2}}$$

ME 241A

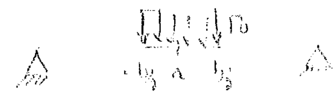
THEORY OF PLATES

Assignment 2

Due Wednesday, October 10

Problem 1: (a) Give the Fourier series solution for the deflection $W(x)$ of a simply supported beam of length L and stiffness EI with the load distribution

$$p(x) = \begin{cases} 0 & 0 \leq x < a - b/2 \\ p_0 & a-b/2 < x < a + b/2 \\ 0 & a+b/2 < x \leq L \end{cases}$$



(b) Obtain the deflection W due to a concentrated load of magnitude P at $x = a$ by taking the limit of the solution in part (a) as $b \to 0$ and $p_0 \to \infty$ such that $P = bp_0$ remains constant.

Problem 2: Consider a rectangular plate without surface loads. What are the edge tractions corresponding to the solution $W = A\,x\,y$?

(A is a constant.) Solution to homogeneous $D\Delta^2 w = 0$

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Problem Set #2

1. a. Since it is simply supported, then $w = \frac{\partial w}{\partial x} = 0$ at $x = 0, L$ $\Rightarrow$ $w(x) = \sum_{n=1}^{\infty} w_n \sin\frac{n\pi x}{L}$

where

$$w_n = \frac{2}{EIL}\left(\frac{L}{n\pi}\right)^4 \int_0^L p(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{EIL}\left(\frac{L}{n\pi}\right)^4 \int_{a-b/2}^{a+b/2} p_0 \sin\frac{n\pi x}{L}dx = \frac{2p_0L^3}{EIn^4\pi^4}\int_{a-b/2}^{a+b/2}\sin\frac{n\pi x}{L}dx$$

$$\int_{a-b/2}^{a+b/2}\sin\frac{n\pi x}{L}dx = -\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{a-b/2}^{a+b/2} \quad \therefore \quad w_n = +\frac{2p_0L^4}{EIn^5\pi^5}\left(2\sin\frac{n\pi a}{L}\sin\frac{n\pi b}{2L}\right) = +\frac{4p_0L^4}{EIn^5\pi^5}\sin\frac{n\pi a}{L}\sin\frac{n\pi b}{2L}$$

$$\therefore \boxed{w(x) = \sum_{n=1}^{\infty} +\frac{4p_0L^4}{EIn^5\pi^5}\sin\frac{n\pi a}{L}\sin\frac{n\pi b}{2L}\sin\frac{n\pi x}{L}}$$

1b. as $p_0 \to \infty$ & $b \to 0$ s.t. $P = p_0 b$ $\Rightarrow$ $\sin\frac{n\pi b}{2L} \approx \frac{n\pi b}{2L}$

$$w(x) \cong \sum_{n=1}^{\infty} +\frac{4p_0L^4}{EIn^5\pi^5}\cdot\frac{n\pi b}{2L}\sin\frac{n\pi a}{L}\sin\frac{n\pi x}{L} \cong \boxed{\sum_{n=1}^{\infty} +\frac{2PL^3}{EIn^4\pi^4}\sin\frac{n\pi a}{L}\sin\frac{n\pi x}{L} = w(x)}$$

2. Consider a rectangular plate without surface loads. What are the edge reactions corresponding to the solution $w = Axy$?

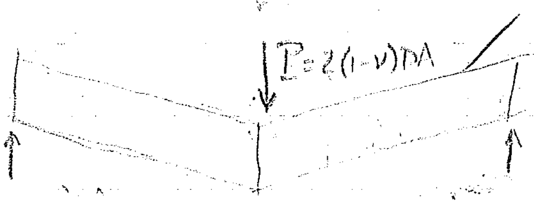
$$\epsilon_x = -z\frac{\partial^2 w}{\partial x^2} = 0 \qquad \sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) = 0$$

$$\epsilon_y = -z\frac{\partial^2 w}{\partial y^2} = 0 \qquad \sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) = 0$$

$$\gamma_{xy} = -2z\frac{\partial^2 w}{\partial x\partial y} = -2zA \Rightarrow \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = -\frac{EzA}{1+\nu}$$

$m_x = 0 = m_y$ since $\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = 0$; $m_{xy} = -\frac{EI}{1+\nu}A = \text{const.}$

$v_x = v_y = 0$ since $w_{xxx} = w_{yyy} = w_{xyy} = w_{yyx} = 0$



Fall 1979

Prof. C. Steele

ME 241A

THEORY OF PLATES

Assignment 3

Due Wednesday, October 17

Problem 1: A simply-supported, rectangular plate ($a \times b \times t$) has a uniform normal pressure p_o. Determine the series solution for the anchorage force at one corner.

Problem 2: Derive the double Fourier series solution for a simply-supported, rectangular plate ($a \times b \times t$) with a normal pressure p_o acting on a square area

$$\xi - \delta/2 < x < \xi + \delta/2$$

$$\eta - \delta/2 < y < \eta + \delta/2 .$$

Take the limit as $\delta \to o$ but with the resultant force held constant $P = \delta^2 p_o$ to obtain the solution for the concentrated load P at the point (ξ, η).

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ME 241A
AUDIT
Prof. Steele : Fall 79

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Problem Set #3

Problem 1: Simply Supported Rectangular Plate has uniform load of p_0. Determine series soln of the anchorage force at one corner

Eqn governing: $\Delta^2 w = p_0/D$ B.C. $w = \frac{\partial^2 w}{\partial y^2} = 0$ on $y = 0, b$ $w = \frac{\partial^2 w}{\partial x^2} = 0$ on $x = 0, a$

from b.c. $w = \sum_m \sum_n w_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}$

let $p(x,y) = p_0 = \sum p_{mn} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}$

$$p_{mn} = \frac{2}{a}\int_0^a \frac{2}{b}\int_0^b p_0 \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\,dx\,dy$$
$$= \frac{4p_0}{ab}\left(-\cos\frac{n\pi x}{a}\Big|_0^a\right)\left(-\cos\frac{m\pi y}{b}\Big|_0^b\right)\frac{ab}{mn\pi^2}$$
$$p_{mn} = (1-\cos n\pi)(1-\cos m\pi)\frac{4p_0}{mn\pi^2}$$

Now $\Delta^2 w = \sum_{n=1}\sum_{m=1}\left[\left(\frac{n\pi}{a}\right)^4 + 2\left(\frac{n\pi}{a}\right)^2\left(\frac{m\pi}{b}\right)^2 + \left(\frac{m\pi}{b}\right)^4\right] w_{mn} \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} = \sum\sum (1-\cos n\pi)(1-\cos m\pi)\frac{4p_0}{mn\pi^2 D}\sin\frac{m\pi y}{b}\sin\frac{n\pi x}{a}$

$$\therefore\quad w_{mn} = \frac{(1-\cos n\pi)(1-\cos m\pi)}{\pi^4\left[\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2\right]^2}\frac{4p_0}{mn\pi^2 D}$$

let $m = (2M-1)$ $n = 2N-1$ $M, N = 1, 2, \ldots$

$$\therefore\quad \boxed{w_{MN} = \frac{16p_0}{\pi^6 D(2M-1)(2N-1)}\frac{1}{\left[\left(\frac{2N-1}{a}\right)^2+\left(\frac{2M-1}{b}\right)^2\right]^2}}$$

$$w(x,y) = \sum_{M=1}\sum_{N=1} w_{MN}\sin\frac{(2N-1)\pi x}{a}\sin\frac{(2M-1)\pi y}{b}$$

Now Reaction at corners $= 2(M_{xy}) = -2(1-\nu)D\frac{\partial^2 w}{\partial x\partial y}\Big|_{(x,y)=(0,0),(a,0),(a,b),(0,b)}$; $\frac{\partial^2 w}{\partial x \partial y} = \sum\sum w_{MN}\frac{(2N-1)\pi}{a}\frac{(2M-1)\pi}{b}\cos\frac{(2N-1)\pi x}{a}\cos\frac{(2M-1)\pi y}{b}$

$$\therefore\quad R = -\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{32p_0(1-\nu)}{\pi^4 ab}\frac{1}{\left[\left(\frac{2N-1}{a}\right)^2+\left(\frac{2M-1}{b}\right)^2\right]^2}\cos\frac{(2N-1)\pi x}{a}\cos\frac{(2M-1)\pi y}{b}$$
evaluated at $(0,0)$ or $(a,0)$ or (a,b) $(0,b)$

Example: $R(0,0) = -\sum_{N=1}\sum_{M=1}\frac{32p_0(1-\nu)}{\pi^4 ab\left[\left(\frac{2N-1}{a}\right)^2+\left(\frac{2M-1}{b}\right)^2\right]^2}$; $R(a,b) = +\sum_{N=1}\sum_{M=1}\frac{32p_0(1-\nu)}{\pi^4 ab\left[\left(\frac{2N-1}{a}\right)^2+\left(\frac{2M-1}{b}\right)^2\right]^2} = -R(0,b) = -R(a,0)$

Problem 2: We use the results from above but modify $p(x,y)$ $\Rightarrow$ $p_{mn} = \frac{2}{a}\int_{\xi-\delta/2}^{\xi+\delta/2}\frac{2}{b}\int_{\eta-\delta/2}^{\eta+\delta/2} p_0 \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy$

thus $p_{mn} = \frac{4\cdot 4 p_0}{ab}\frac{a}{n\pi}\frac{b}{m\pi}\sin\frac{n\pi\xi}{a}\sin\frac{n\pi\delta}{2a}\sin\frac{m\pi\eta}{b}\sin\frac{m\pi\delta}{2b}$

or $\boxed{p_{mn} = \frac{4\cdot 4p_0}{\pi^2 nm}\sin\frac{n\pi\xi}{a}\sin\frac{m\pi\eta}{b}\sin\frac{n\pi\delta}{2a}\sin\frac{m\pi\delta}{2b}}$; use this and define $w_{mn} = \frac{p_{mn}}{\pi^4 D\left[\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)\right]}$

so that $w(x,y) = \sum_m\sum_n w_{mn}\sin\frac{n\pi x}{a}\sin\frac{n\pi y}{b}$

Now as $\delta \to 0$ & $p_0 \to \infty$ $\Rightarrow$ $P = \delta^2 p_0$; $p_{mn} \to \frac{4\cdot 4p_0}{\pi^2 nm}\cdot\frac{n\pi\delta}{2a}\cdot\frac{m\pi\delta}{2b}\sin\frac{n\pi\xi}{a}\sin\frac{n\pi\eta}{b} = \frac{4P}{ab}\sin\frac{n\pi\xi}{a}\sin\frac{m\pi\eta}{b}$

$$\boxed{p_{mn} = \frac{4P}{ab}\sin\frac{n\pi\xi}{a}\sin\frac{m\pi\eta}{b}}$$

thus $\omega_{mn} = \dfrac{4P}{ab\pi^4 D\left[\left(\frac{m}{b}\right)^2+\left(\frac{n}{a}\right)^2\right]^2} \sin\dfrac{n\pi\xi}{a} \sin\dfrac{m\pi\eta}{b}$ and $\omega(x,y) = \sum\limits_{n}\sum\limits_{m} \omega_{mn} \sin\dfrac{n\pi x}{a} \sin\dfrac{m\pi y}{b}$

ME 264A

Fall 1979

THEORY OF PLATES

Assignment 4

Due Mon. Oct. 24

PROBLEM 1: A semi-infinite plate strip $(0 \le y < \infty)$ is simply-supported on the edges $x = 0$ and $x = a$.

The edge $y = 0$ is given the displacement and rotation

$$w(x,0) = A \sin \pi x/a$$

$$w_y(x,0) = B \sin \pi x/a$$

Determine the bending moment M_y and shearing force V_y at the edge $y = 0$.

PROBLEM 2: The infinite plate strip $(-\infty < y < \infty)$ is simply supported at $x = 0, a$. A line load of constant intensity q acts on the line $y = 0$. Determine the Fourier series for the bending moment M_y and deflection w at the center $(x = 0, y = 0)$. Sum enough terms of the series for M_y and w to give three place accuracy. Compare your "exact" sum with the first term alone.

ME 241A
Prof. Steele Fall 79
Cesar Levy
AUDIT

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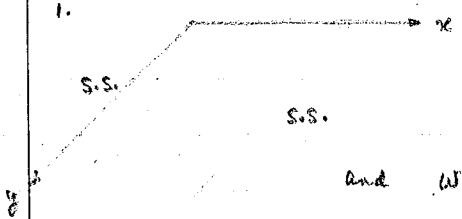
Problem Set # 4

1.



s.s. edges $\Rightarrow w(0,y) = w(a,y) = 0$

$$\frac{\partial^2 w}{\partial x^2}(0,y) = \frac{\partial^2 w}{\partial x^2}(a,y) = 0$$

and $w(x,0) = A \sin\frac{\pi x}{a}$

$w_y(x,0) = B \sin\frac{\pi x}{a}$

$\rightarrow$ find $m_y = -D(w_{,yy} + \nu w_{,xx})$ and $v_y = q_y + \frac{\partial m_{xy}}{\partial x} = -D(\Delta w)_{,y} - D(1-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}$

on $y=0$

Assume that $w(x,y) = Y(y)\sin\lambda x$ and for suitable λ it will solve the PDE & BC

put into PDE to get $(Y^{IV} - 2\lambda^2 Y'' + \lambda^4 Y)\sin\lambda x = 0$ and we find that

$Y(y) = (A_1 + B_1 y)e^{\lambda y} + (C_1 + D_1 y)e^{-\lambda y}$. But for boundedness at $\infty \Rightarrow A_1, B_1$ are zero. Thus

$Y(y) = (C_1 + D_1 y)e^{-\lambda y}$; for soln to satisfy s.s. conditions $\lambda = \lambda_n = \frac{n\pi}{a}$

Thus $$w(x,y) = \sum_{n=1}^{\infty}(C_n + D_n y)e^{-\lambda_n y}\sin\frac{n\pi x}{a}$$

Now $w(x,0) = A\sin\frac{\pi x}{a} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a} \Rightarrow C_1 = A$ and $C_i = 0 \ \forall i \geq 2$

$$w_{,y}(x,0) = \sum_{n=1}^{\infty}\left[D_n + (C_n + D_n y)(-\lambda_n)\right]e^{-\lambda_n y}\Big|_{y=0}\sin\frac{n\pi x}{a} = \sum_{n=1}^{\infty}\left[D_n - C_n\lambda_n\right]\sin\frac{n\pi x}{a}$$

$$= B\sin\frac{\pi x}{a} \Rightarrow B = D_1 - A\frac{\pi}{a} \text{ or } D_1 = B + \frac{A\pi}{a} \quad D_i = 0 \ \forall i \geq 2$$

$$\therefore \boxed{w(x,y) = \left[A + \left(B + \frac{A\pi}{a}\right)y\right]e^{-\pi y/a}\sin\frac{\pi x}{a}}$$

$m_y\big|_{y=0} = -D(w_{,yy} + \nu w_{,xx})\big|_{y=0}$; $w_{,y} = \left[\left(B + \frac{A\pi}{a}\right) - \frac{\pi}{a}\left(A + \left[B + \frac{A\pi}{a}\right]y\right)\right]e^{-\pi y/a}\sin\frac{\pi x}{a}$

$= \left[B(1 - \frac{\pi}{a}y) - A\frac{\pi^2}{a^2}y\right]e^{-\pi y/a}\sin\frac{\pi x}{a}$

$w_{,yy} = \left[-\frac{\pi B}{a} - \frac{A\pi^2}{a^2} - \frac{\pi}{a}\left\{B(1 - \frac{\pi}{a}y) - A\frac{\pi^2}{a^2}y\right\}\right]e^{-\pi y/a}\sin\frac{\pi x}{a}$ $\qquad w_{,yy}\big|_{y=0} = \left\{-\frac{2\pi B}{a} - \frac{A\pi^2}{a^2}\right\}\sin\frac{\pi x}{a}$

$w_{,xx} = -\frac{\pi^2}{a^2}\left[A + \left(B + \frac{A\pi}{a}\right)y\right]e^{-\pi y/a}\sin\frac{\pi x}{a}$ $\qquad w_{,xx}\big|_{y=0} = -\frac{\pi^2 A}{a^2}\sin\frac{\pi x}{a}$

$$m_y\big|_{y=0} = +D\left[\frac{2\pi B}{a} + (1+\nu)\frac{\pi^2 A}{a^2}\right]\sin\frac{\pi x}{a} = \boxed{\frac{\pi D}{a}\left[2B + (1+\nu)\frac{\pi A}{a}\right]\sin\frac{\pi x}{a} = m_y\big|_{y=0}}$$

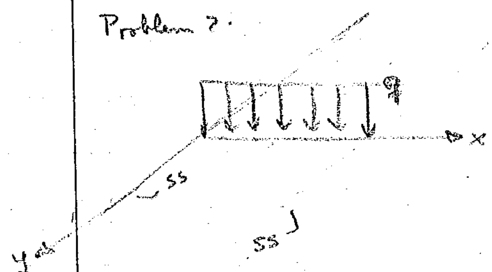
$\Delta w = \left[-\frac{2\pi B}{a} - \frac{2A\pi^2}{a^2}\right]e^{-\pi y/a}\sin\frac{\pi x}{a}$ $\qquad q_y = -D(\Delta w)_{,y} = +D\frac{\pi}{a}\left[-\frac{2\pi B}{a} - \frac{2A\pi^2}{a^2}\right]e^{-\pi y/a}\sin\frac{\pi x}{a}$

$$\frac{\partial m_{xy}}{\partial x} = +D(1-\nu)\frac{\pi^2}{a^2}\left\{\left(B+\frac{A\pi}{a}\right)-\frac{\pi}{a}\left[A+\left(B+\frac{A\pi}{a}\right)y\right]\right\}e^{-\pi y/a}\sin\frac{\pi x}{a}$$

$$q_y\Big|_{y=0} = \frac{-2\pi^2 D}{a^2}\left[B+\frac{A\pi}{a}\right]\sin\frac{\pi x}{a} \; ; \quad \frac{\partial m_{xy}}{\partial x} = D(1-\nu)\frac{\pi^2}{a^2}B\sin\frac{\pi x}{a} \qquad \therefore \; v_y = \frac{\pi^2}{a^2}D\left\{-(1+\nu)B - \frac{2A\pi}{a}\right\}\sin\frac{\pi x}{a}$$

$$\text{or} \quad \boxed{v_y = -\frac{\pi^2 D}{a^2}\left[(1+\nu)B + \frac{2A\pi}{a}\right]\sin\frac{\pi x}{a}}$$

Problem 7.



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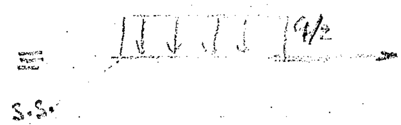


with $\frac{\partial w}{\partial y}(x,0)=0$

and $\frac{\partial^2 w}{\partial x \partial y}(x,0)=0 \Rightarrow M_{xy}=0$

$v_y = -q/2 = q_y + \partial m_{xy}/\partial x = -D(\Delta w)_{,y} = -D\frac{\partial^3 w}{\partial y^3}$

and $w(x,y)$ must be bounded at $y \to \infty$

we can now use the eqn $\Delta^2 w = 0$

if we assume $w(x,y) = Y(y)\sin\frac{n\pi x}{a} = Y(y)\sin\lambda_n x$

this will satisfy the s.s. conditions on $x=0$ & $x=a$. Now put into PDE

$$\Delta^2 w = \left[Y^{IV} - 2\lambda_n^2 Y'' + \lambda_n^4 Y\right]\sin\frac{n\pi x}{a} = 0 \qquad \forall x$$

$\Rightarrow \quad Y^{IV} - 2\lambda_n^2 Y'' + \lambda_n^4 Y = 0$ bounded solns of this give $Y_n(y) = (A_n + B_n y)e^{-\lambda_n y}$

$$\therefore \quad w(x,y) = \sum (A_n + B_n y)e^{-\lambda_n y}\sin\lambda_n x$$

$w_{,y} = \sum\left\{B_n - \lambda_n(A_n + B_n y)\right\}e^{-\lambda_n y}\sin\lambda_n x$ & $w_{,y}(x,0) = \sum\left\{B_n - \lambda_n A_n\right\}\sin\lambda_n x = 0$

or $B_n = \lambda_n A_n$

$$\therefore \quad w(x,y) = \sum A_n(1+\lambda_n y)e^{-\lambda_n y}\sin\lambda_n x$$

now $w_{,y} = \sum\left\{A_n\lambda_n - \lambda_n A_n(1+\lambda_n y)\right\}e^{-\lambda_n y}\sin\lambda_n x = \sum -\lambda_n^2 A_n y e^{-\lambda_n y}\sin\lambda_n x$

$w_{,yy} = \sum\left(-\lambda_n^2 A_n + \lambda_n^3 A_n y\right)e^{-\lambda_n y}\sin\lambda_n x$

$-D w_{,yyy}\Big|_{y=0} = -D\sum\left(2\lambda_n^3 A_n - \lambda_n^4 A_n y\right)e^{-\lambda_n y}\sin\lambda_n x\Big|_{y=0} = -2D\sum\lambda_n^3 A_n\sin\lambda_n x = -q/2$

$$\therefore \quad A_n = \frac{1}{+2D\lambda_n^3}\frac{2}{a}\int_0^a \frac{q}{2}\sin\lambda_n x\,dx = \frac{-q}{2aD\lambda_n^4}\left[\cos\lambda_n x\right]_0^a = \frac{-q}{2aD\lambda_n^4}\left\{\cos n\pi - 1\right\}$$

$$A_n = 0 \qquad \forall n \text{ even}$$

$$= \frac{+q}{aD\lambda_n^4} \qquad \forall n \text{ odd.}$$

$$\therefore \quad w(x,y) = \sum_{n=1,3,5,\ldots}\frac{q a^3}{Dn^4\pi^4}\left(1+\frac{n\pi}{a}y\right)e^{-\frac{n\pi y}{a}}\sin\frac{n\pi x}{a}$$

$$w(x,0) = \sum_{n=odd} \frac{q a^3}{D n^4 \pi^4} \sin \frac{n\pi x}{a}$$

$$M_y\Big|_{y=0} = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)\Big|_{y=0} = -D\left\{\sum -\lambda_n^2 A_n + \nu \lambda_n^2 A_n\right\} \sin\frac{n\pi x}{a} = (1+\nu)\sum \frac{q\,a}{n^2\pi^2} \sin \frac{n\pi x}{a}$$

now at $x = a/2$ $\quad w(a/2, 0) = \sum \frac{q a^3}{D n^4 \pi^4} \sin \frac{n\pi}{2} = \frac{q a^3}{D\pi^4} \cdot \left(1 - \frac{1}{3^4} + \frac{1}{5^4} - \frac{1}{7^4} + \dots\right)$

$$M_y(a/2, 0) = (1+\nu)\frac{q a}{\pi^2} \cdot \left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots\right)$$

$\frac{w(a/2,0)}{q a^3/D\pi^4} \sim .9889$ w/ 5 terms $\qquad \frac{w(a/2,0)}{q a^3/D\pi^4} = 1 \qquad$ % error = ~~11.4%~~ 1%

$\frac{M_y(a/2,0)}{(1+\nu) q a/\pi^2} \sim .916$ w/ 17 terms $\qquad \frac{M_y(a/2,0)}{(1+\nu) q a/\pi^2} = 1 \qquad$ % error 8.4%

ME 241A

Fall Quarter
1979

THEORY OF PLATES

Assignment 5

Due Wednesday, October 31

Problem 1: A circular plate is clamped at the outer edge $r = b$ and loaded with the pressure distribution $p = Ar$. $\Delta^2 w = p/D$ with only $w = w(r)$ only

Find the bending moment and deflection at the plate center $r = 0$.

$$\Delta w = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right] w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) w = p/D$$

Problem 2: A circular plate is loaded only by a moment $M_r = M$ at the outer boundary $r = b$. Find the exact solution in each of the following two cases of an inner boundary at $r = a$ which is

(a) clamped

(b) free

Find the formula for the maximum moment at the inner boundary, then take the limit for $a \ll b$ to obtain the local stress concentration factor in each case.

Cesar Levy
ME241A Fall 79
Prof. Steele
AUDIT

Problem Set #5

1. $\Delta\Delta w = p/D$ where $\Delta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$ and $p = Ar$

BC. is that at $w(b)=0$ and $\frac{\partial w}{\partial r}(b)=0$ (clamped conditions)

Thus $\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\right]\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}w\right)\right] = \frac{Ar}{D}$

$$\therefore \left(r\frac{d}{dr}\right)\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = \frac{A}{D}\left[\frac{r^3}{3}+\tilde{C}_1\right]$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = \frac{A}{D}\left[\frac{r^3}{9}+\tilde{C}_1\ln r+\tilde{C}_2\right]$$

$$r\frac{dw}{dr} = \frac{A}{D}\left[\frac{r^5}{45}+\tilde{C}_1\left(\frac{r^2\ln r}{2}-\frac{r^2}{4}\right)+\tilde{C}_2\frac{r^2}{2}+\tilde{C}_3\right]$$

$$w = \frac{A}{D}\left[\frac{r^5}{225}+\tilde{C}_1\left(\frac{1}{2}\left[\frac{1}{2}r^2\ln r-\frac{r^2}{4}\right]-\tilde{C}_1\frac{r^2}{8}+\tilde{C}_2\frac{r^2}{4}+\tilde{C}_3\ln r+\tilde{C}_4\right]$$

Now the most general form is $w(r)=\frac{A}{D}\left[\frac{r^5}{225}+C_1r^2\ln r+C_2r^2+C_3\ln r+C_4\right]$

Now for boundedness of the solution at $r=0$ $C_3=0$ and $C_1=0$

Thus $w(r)=\frac{A}{D}\left[\frac{r^5}{225}+C_2r^2+C_4\right]$

Apply BC: $w(b)=\frac{A}{D}\left[\frac{b^5}{225}+C_2b^2+C_4\right]=0$

$$\frac{\partial w}{\partial r}(b)=\frac{A}{D}\left[\frac{b^4}{45}+2C_2b\right]=0 \Rightarrow C_2=-\frac{b^3}{90} \Rightarrow C_4=\frac{b^5}{150}$$

Thus

$$w(r)=\frac{A}{D}\left[\frac{2r^5}{450}-\frac{5b^3r^2}{450}+\frac{3b^5}{450}\right]=\frac{A}{450D}\left[2r^5-5b^3r^2+3b^5\right]$$

Now $\boxed{w(0)=\frac{b^5A}{150D}}$

$m_r=-D\left[\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right]$ and $m_\varphi=-D\left[\nu\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right]$

$$\left.\frac{1}{r}\frac{dw}{dr}\right|_{r=0}=\frac{A}{450D}\left[10r^3-10b^3\right]\Big|_{r=0}=\frac{-10b^3A}{450D} \qquad \left.\frac{d^2w}{dr^2}\right|_{r=0}=\frac{A}{450D}\left[40r^3-10b^3\right]\Big|_{r=0}=\frac{-10b^3A}{450D}$$

thus $$m_r\Big|_{r=0}=-D\left[\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right]\Big|_{r=0}=-D(1+\nu)\left(\frac{-10b^3A}{450D}\right)=\boxed{\frac{(1+\nu)b^3A}{45}=m_r\Big|_{r=0}}$$

$$m_\varphi\Big|_{r=0}=-D\left[\nu\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right]\Big|_{r=0}=-D(\nu+1)\left(\frac{-10b^3A}{450D}\right)=\boxed{\frac{(1+\nu)b^3A}{45}=m_\varphi\Big|_{r=0}}$$

2. Given a circular plate is loaded only by a moment M_r at the outer boundary what is the exact solution if at $r=a$ (the inner boundary)

a. Clamped conditions exist

Since there is no distributed load and $M_r = M$, a constant, the solution will be independent of φ. Thus the general displacement solution is the homogeneous solution

$$w(r)=C_1r^2\ln r+C_2r^2+C_3\ln r+C_4$$

now $\frac{dw}{dr}=C_1(2r\ln r+r)+2C_2r+C_3/r$ and $\frac{d^2w}{dr^2}=C_1(2\ln r+3)+2C_2-C_3/r^2$

also $\frac{d^3w}{dr^3} = \frac{2C_1}{r} + \frac{2C_3}{r^3}$.

BC: @ $r=a$ $\quad w(a)=0$ $\qquad r=b \quad M_r = -D\left[\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right] = M$

$r=a \quad \frac{\partial w}{\partial r}(a)=0 \qquad r=b \quad v_r = q_r = -D\frac{d}{dr}\left[\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right] = 0$

Since $q_\varphi \equiv 0$ & $m_{\varphi r} = m_{r\varphi} = 0$.

$\therefore \quad w(a) = C_1 a^2 \ln a + C_2 a^2 + C_3 \ln a + C_4 = 0 \quad (1)$; using (3) $C_2 a^2 + C_3 \ln a + C_4 = 0$

$\left.\frac{dw}{dr}\right|_{r=a} = C_1(2a\ln a + a) + 2C_2 a + C_3/a = 0 \quad (2)$; using (3) $\quad 2C_2 a^2 + C_3 = 0 \Rightarrow C_3 = -2C_2 a^2$

$\left.v_r\right|_{r=b} = -D\frac{4C_1}{b} = 0 \Rightarrow C_1 = 0 \quad (3)$

$\left.M_r\right|_{r=b} = -D\left[2(1+\nu)C_2 - (1-\nu)C_3/b^2\right] = M \quad (4)$ using (3).

using (2) and (4)

$$C_2 = \frac{-Mb^2}{2D\left[(1+\nu)b^2 + a^2(1-\nu)\right]} \quad (5) \qquad C_3 = \frac{2Ma^2b^2}{2D\left[(1+\nu)b^2 + a^2(1-\nu)\right]} \quad (6)$$

Using (3) and (6) into (1) $\qquad C_4 = \frac{Ma^2b^2\left[1 - 2\ln a\right]}{2D\left[(1+\nu)b^2 + a^2(1-\nu)\right]} \quad (7)$

Thus $\quad w(r) = \frac{-Mb^2}{2D\left[(1+\nu)b^2 + a^2(1-\nu)\right]}\left[r^2 - 2a^2\ln r - a^2(1 - 2\ln a)\right]$

now $m_{r\varphi} = 0$ everywhere $\quad m_r = -\frac{\left[-Mb^2(1+\nu) - \frac{1}{r^2}(1-\nu)Ma^2b^2\right]}{\left[(1+\nu)b^2 + a^2(1-\nu)\right]} = \frac{Mb^2\left[(1+\nu)r^2 + (1-\nu)a^2\right]}{r^2\left[(1+\nu)b^2 + a^2(1-\nu)\right]}$

at $r=a \quad m_r = \frac{2Mb^2}{\left[(1+\nu)b^2 + a^2(1-\nu)\right]}$

$m_\varphi = \frac{Mb^2}{r^2}\frac{\left[(1+\nu)r^2 - a^2(1-\nu)\right]}{\left[(1+\nu)b^2 + a^2(1-\nu)\right]} \qquad$ at $r=a \quad m_\varphi = \frac{2\nu Mb^2}{\left[(1+\nu)b^2 + a^2(1-\nu)\right]}$

thus $m_r > m_\varphi$ at $r=a \quad \therefore \quad m_r = \frac{2M}{\left[(1+\nu) + \frac{a^2}{b^2}(1-\nu)\right]}$ and $\lim\limits_{\text{as } b\to\infty} m_r = \frac{2M}{1+\nu}$

$|\sigma| = \frac{M_r h/2}{h^3/12} = \frac{12M}{(1+\nu)h^2} \qquad$ normally $\quad |\sigma| = \frac{M h/2}{h^3/12} = \frac{6M}{h^2} \qquad \therefore$ Stress intensity factor $= \frac{2}{1+\nu}$

b. free support conditions exist

BC. @ $r=a \quad v_r = q_r = -D\frac{d}{dr}(\Delta w) = 0 \qquad r=b \quad m_r = -D\left[\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right] = M$

@ $r=a \quad m_r = -D\left[\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right] = 0 \qquad r=b \quad v_r = q_r = -D\frac{d}{dr}(\Delta w) = 0$

from $\left.v_r\right|_{r=a} = 0 \Rightarrow C_1 = 0$ which identically satisfies $\left.v_r\right|_{r=b} = 0$

now $\left.m_r\right|_{r=a} = 0 \Rightarrow 2(1+\nu)C_2 - (1-\nu)C_3/a^2 = 0$

$\left.m_r\right|_{r=b} = M \Rightarrow -D\left[2(1+\nu)C_2 - (1-\nu)C_3/b^2\right] = M$

$$\Bigg\} \quad C_3 = \frac{-M}{D(1-\nu)}\frac{a^2b^2}{(b^2-a^2)} \qquad C_2 = \frac{-Mb^2}{2D(1+\nu)(b^2-a^2)}$$

Thus $w(r) = \dfrac{-Mb^2r^2}{2D(1+\nu)(b^2-a^2)} - \dfrac{Ma^2b^2\ln r}{D(1-\nu)(b^2-a^2)} + C_4$

Thus, to an additive constant (which represents a rigid body translation) we find that

$$w(r) = \frac{-Mb^2}{D(b^2-a^2)}\left[\frac{r^2}{2(1+\nu)} + \frac{a^2\ln r}{1-\nu}\right],$$ if we neglect this rigid trans.

now $m_{r\varphi} = 0$ and $m_r = 0$ are $r = a$

$$m_\varphi\Big|_{r=a} = -\left[-\frac{2Mb^2}{b^2-a^2}\right] = 2M\left[\frac{1}{1-a^2/b^2}\right]$$ now $m_\varphi > m_r$

and $\lim_{b\to\infty} m_\varphi = 2M$; now $|\sigma| = \dfrac{m_\varphi \cdot h/2}{I} = \dfrac{2M\, h/2}{I}$ -

normally $|\sigma| = \dfrac{M\, h/2}{I}$ $\therefore$ stress intensity factor is 2

If this follows in the line of what was learned in elasticity it is expected that the stress intensity factor on the free edge be larger than that of the clamped edge. Physically the clamped edge has some of its stress "relieved" by the "clamping material".

Fall 1979

ME 241A

THEORY OF PLATES

Assignment 6

Due Wednesday, Nov. 7

PROBLEM: The edge of a hole in a large flat plate ($a \le r < \infty$) is given the displacement and rotation

$$W(a, \phi) = A_2 \cos n\phi \qquad (n = 2, 3, \ldots)$$

$$\frac{\partial W}{\partial r}(a, \phi) = A_1 \cos n\phi$$

(a) Show that the edge moment and shear are

$$v_r(a, \phi) = F_2 \cos n\phi$$

$$m_r(a, \phi) = F_1 \cos n\phi$$

where

$$\begin{array}{c} m_r \\ \text{mom. about under} \end{array} \begin{pmatrix} F_1 \\ aF_2 \end{pmatrix} = -\frac{D}{a^2} \begin{pmatrix} -2n + 1 + \nu & (1+\nu)n^2 - 2n \\ (1+\nu)n^2 - 2n & (-2n + 1 + \nu)n^2 \end{pmatrix} \begin{pmatrix} -\cancel{a}A_1 \\ A_2 \end{pmatrix} \begin{array}{c} \Delta w \\ w \end{array}$$

(b) Why should this matrix be symmetric?

(c) Determine by physical argument whether or not the sign of each term in the matrix is correct.

$$\begin{array}{c} m_r \\ v_r \end{array} \begin{pmatrix} F_1 \\ F_v \end{pmatrix} = \frac{D}{a^2} \begin{pmatrix} -a(1+\nu-2n) & (1+\nu)n^2-2n \\ -[(1+\nu)n^2-2n] & (\nu+1-2n)\frac{n^2}{a} \end{pmatrix} \begin{pmatrix} A_1 \\ A_v \end{pmatrix} \begin{array}{c} \theta \\ w \end{array}$$

Cesar Levy
ME 241A Fall '79
Prof. Steele
AUDIT

Problem Set #6

Problem given a large flat plate with a hole and the following displacement and rotation

$$w(a,\varphi) = A_2 \cos n\varphi \qquad \text{and} \qquad \frac{\partial w}{\partial r}(a,\varphi) = A_1 \cos n\varphi \qquad (n = 2, 3, \dots)$$

(a) Find $V_r(a,\varphi)$ and $m_r(a,\varphi)$

(b) explain why the matrix found in (a) is symmetric

(c) Determine by physical argument whether or not the sign of each term in the matrix is correct

(a) Starting with $\Delta^2 w = p/D$ with $p=0$ then the most general solution is

$$w = a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0 r^2 \varphi + a_0' \varphi + a_1 r \varphi \sin\varphi + (b_1 r^3 + a_1' r^{-1} + b_1' r \ln r)\cos\varphi$$
$$+ c_1 r \varphi \cos\varphi + (d_1 r^3 + c_1' r^{-1} + d_1' r \ln r)\sin\varphi + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2})\cos n\varphi$$
$$+ \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2}) \sin n\varphi$$

for finite displacement at $r=\infty$ $\quad a_0 = b_0 = c_0 = d_0 = a_1 = b_1 = b_1' = c_1 = d_1 = d_1' = a_n = b_n = c_n = d_n = 0$ for $n = 2, \dots \infty$

Thus $w = a_0'\varphi + \frac{a_1'}{r}\cos\varphi + \frac{c_1'}{r}\sin\varphi + \sum_{n=2} \left(\frac{a_n'}{r^n} + \frac{b_n'}{r^{n-2}}\right)\cos n\varphi + \sum_{n=2}^{\infty}\left(\frac{c_n'}{r^n} + \frac{d_n'}{r^{n-2}}\right)\sin n\varphi$

for periodicity $a_0' = 0$;

Now since $w(a,\varphi) = f(n\varphi)$ $n = 2,3,4,\dots \Rightarrow a_1' = c_1' = 0$; since $w(a,\varphi) = f(\cos n\varphi) \Rightarrow$ $c_n' = d_n' = 0$ $\quad n \geq 2$ $\quad \therefore \quad w(r,\varphi) = \sum_{n=2}^{\infty}\left(\frac{a_n'}{r^n} + \frac{b_n'}{r^{n-2}}\right)\cos n\varphi$

since $w(a,\varphi) = A_2 \cos n\varphi$ for $n = 2, \dots \Rightarrow$ $\boxed{w(r,\varphi) = \left(\frac{a_n' + r^2 b_n'}{r^n}\right)\cos n\varphi}$

Thus $A_2 = \dfrac{a_n' + a^2 b_n'}{a^n}$ (1) and $\dfrac{\partial w}{\partial r}(a,\varphi) = \left[\dfrac{-n a_n' - (n-2) b_n' \cdot a^2}{a^{n+1}}\right]\cos n\varphi = A_1 \cos n\varphi$

Thus $A_1 = -\left[\dfrac{n a_n' + (n-2) a^2 b_n'}{a^{n+1}}\right]$ (2). Using (1) & (2) $\boxed{b_n' = \dfrac{a^{n-2}}{2}\left[n A_2 + A_1 a\right] \qquad a_n' = -\dfrac{a^n}{2}\left[(n-2)A_2 + a A_1\right]}$

now $M_r = -D\left[w_{,rr} + \nu\left(\frac{1}{r^2} w_{,\varphi\varphi} + \frac{1}{r} w_{,r}\right)\right]$ and $V_r = -D\left[\frac{\partial}{\partial r}(\Delta w) + \frac{1-\nu}{r}\frac{\partial}{\partial\varphi}\left(\frac{1}{r} w_{,r\varphi} - \frac{1}{r^2} w_{,\varphi}\right)\right]$

$w_{,r} = \left(-n\frac{a_n'}{r^{n+1}} - \frac{(n-2)b_n'}{r^{n-1}}\right)\cos n\varphi$; $w_{,\varphi} = -n\left(\frac{a_n'}{r^n} + \frac{b_n'}{r^{n-2}}\right)\sin n\varphi$; $w_{,r\varphi} = n\left[\frac{n a_n'}{r^{n+1}} + \frac{(n-2) b_n'}{r^{n-1}}\right]\sin n\varphi$

$w_{,rr} = \left[n(n+1)\frac{a_n'}{r^{n+2}} + \frac{(n-2)(n-1)}{r^n} b_n'\right]\cos n\varphi$; $w_{,\varphi\varphi} = -n^2\left(\frac{a_n'}{r^n} + \frac{b_n'}{r^{n-2}}\right)\cos n\varphi$

Thus $\left| m_r\Big|_{r=a} = -D\left[\dfrac{(1-\nu)n(n+1)}{a^{n+2}} a_n' + \dfrac{(n-2)(n-1) - \nu(n-2)(n-1)}{a^n} b_n'\right]\cos n\varphi \right|$; applying the above for a_n', b_n' :

$$m_r\Big|_{r=a} = \frac{D}{a^2}\left\{\left[(1+\nu)n^2 - 2n\right]A_2 - aA_1\left[1+\nu-2n\right]\right\}\cos n\varphi = F_1 \cos n\varphi$$

now $\Delta w = -\dfrac{4 b_n' (n-1)}{r^n} \cos n\varphi \qquad \dfrac{\partial}{\partial r}(\Delta w) = \dfrac{4n(n-1)}{r^{n+1}} b_n' \cos n\varphi$

Thus $$\boxed{V_r \Big|_{r=a} = -D\left[\frac{n^2(n+1)(1-\nu)a_n'}{a^{n+2}} + \frac{\{4n(n-1) + n^2(n-1)(1-\nu)\} b_n'}{a^{n+1}} \right] \cos n\varphi}$$

after applying a_n' & b_n' then

$$V_r \Big|_{r=a} = \frac{D}{a^3}\left[A_2(\nu+1-2n)n^2 - aA_1\{(1+\nu)n^2 - 2n\}\right] \cos n\varphi = F_2 \cos n\varphi$$

thus we can write

$$\begin{bmatrix} F_1 \\ aF_2 \end{bmatrix} = \frac{D}{a^2}\begin{bmatrix} (1+\nu-2n) & \{(1+\nu)n^2-2n\} \\ \{(1+\nu)n^2-2n\} & (\nu+1-2n)n^2 \end{bmatrix}\begin{bmatrix} -aA_1 \\ A_2 \end{bmatrix}$$

for $0 \leq \nu \leq .5$ and $n = 2, 3, \ldots$

$$\begin{bmatrix} F_1 \\ aF_2 \end{bmatrix} = \frac{D}{a^2}\begin{bmatrix} - & + \\ + & - \end{bmatrix}\begin{bmatrix} -aA_1 \\ A_2 \end{bmatrix} = \frac{D}{a^2}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} -aA_1 \\ A_2 \end{pmatrix}$$

(b) Due to the model used (ie small deflections) and because of how the strain energy function was picked we find that the applied loads (& moments) must necessarily be linear in the displacements (& rotations). By the Maxwell-Betti Reciprocity Theory, that implies that a load group (i.e. moment & shear) at point 1 producing a displacement group (load & rotation) at point 2, has the same effect of applying the load group at point 2 and looking at the displacement groups at point 1. This necessarily forces the matrix relating the loads & displacements to be symmetric. Now look at each line separat

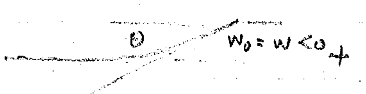
(c) A positive shear on the hole produces a negative displacement, thus $S_{22} < 0$. A positive M_r produces an edge displacement in the negative direction, but since $w \to 0$ as $r \to \infty$, that means that w must increase as r increases or the rotation $\theta = \frac{\Delta w}{\Delta r} > 0$; hence $S_{11} < 0$ since $-aA_1 < 0$. A positive shear produces a negative edge displacement as before, but also produces a positive rotation and thus a positive M_r. But if we only applied a positive shear, to get rid of this M_r we would have to induce a negative rotation hence $S_{21} > 0$ which is what we have since $-aA_1 < 0$. Similarly a positive moment produces a positive rotation but a negative displacement is also produced. If we only want a rotation & no displacem we would have to induce a positive displacement hence $S_{12} > 0$ as required.

over

$m_r = \tilde{S}_{11}\,\theta \qquad \tilde{S}_{11} > 0$

to get this

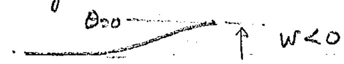
must add



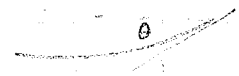
+ rigid body displ of amount $|w_0|$ in + direction

$q_r = w\,\tilde{S}_{22} \qquad \tilde{S}_{22} < 0$

to get this :



must add



+ rigid body rotation of amount $-|\theta|$

Cesar

Fall [illegible]

C.R. Steele

ME 241A

THEORY OF PLATES

Assignment 7

Due Wednesday, Nov. 14

PROBLEM 1: Show that the flat plate, bending stress resultants expressed in $x - y$ rectangular, cartesian coordinates

$$M_x = \bar{M}$$

$$M_y = -\bar{M}$$

$$M_{xy} = 0$$

where $\bar{M}$ is constant, when transformed into polar coordinates r, ϕ have the components

$$M_r = \bar{M} \cos 2\phi$$

$$M_\phi = -\bar{M} \cos 2\theta$$

$$M_{r\phi} = -\bar{M} \sin 2\phi$$

PROBLEM 2: A hole of radius $r = a$ is inserted into the plate with the bending stresses of Problem 1. If the hole radius is small in comparison with the distance from the hole to any other boundary of the plate, find the stress concentration factor, i.e. the ratio of maximum bending stress in the vicinity of the hole to that in the plate without the hole. (Also, the results from Assignment 6 can be used.)

Cesar Levy
ME 241A Fall 79
Prof. Steele

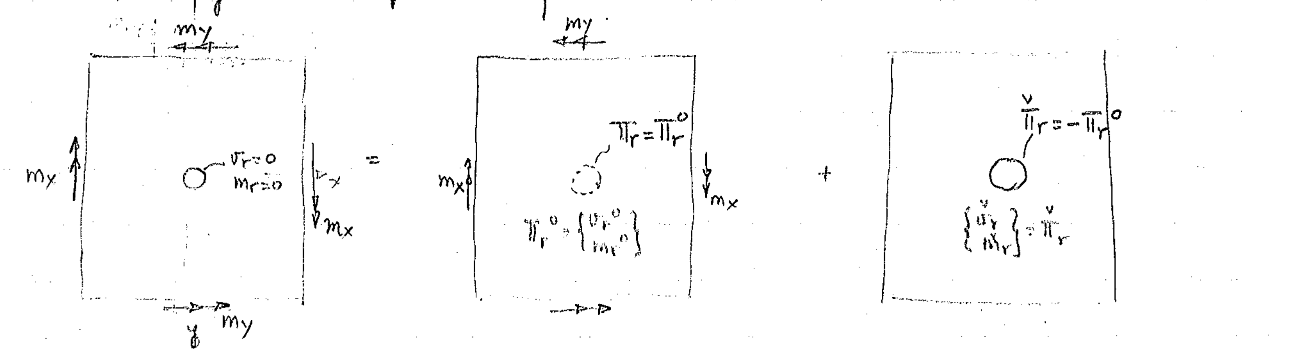
Problem Set #7

1. Given $M_x = \bar{M}$, $M_y = -\bar{M}$, $M_{xy} = 0$ find M_r, M_φ, $M_{r\varphi}$

$$M_r = M_x \cos^2\varphi + M_y \sin^2\varphi = \bar{M}(\cos^2\varphi - \sin^2\varphi) = \bar{M}\cos 2\varphi$$
$$M_\varphi = M_x \sin^2\varphi + M_y \cos^2\varphi = \bar{M}(\sin^2\varphi - \cos^2\varphi) = -M_r = -\bar{M}\cos 2\varphi$$
$$M_{r\varphi} = \frac{M_y - M_x}{2}\sin 2\varphi = -\frac{2\bar{M}}{2}\sin 2\varphi = -\bar{M}\sin 2\varphi$$

QED

2. We can solve this problem by superposition – in the following manner: We solved in problem Set 6, the problem of applied displacement and rotation to the edge of a hole in an ∞ radius plate. If we look at the solution, due to the boundedness of solution at ∞, we find that M_r, M_φ, $M_{r\varphi}$; V_r, V_φ all $\to 0$ as $r \to \infty$. But because of the given displacement and rotation, there will be moments and shears on the inner surface. From the results of this set we note that if we have applied moments which do not depend on r, then at $r=a$ the moments will be $M_r = \bar{M}\cos 2\varphi$, $M_\varphi = -M_r$ and $M_{r\varphi} = -\bar{M}\sin 2\varphi$. For the case of the hole in an ∞ plate with the applied moments $M_r = \bar{M}\cos 2\varphi$, $M_\varphi = -M_r$ and $M_{r\varphi} = -\bar{M}\sin 2\varphi$ at the edges, there are no tractions in the $\underset{\sim}{r}$ direction at the hole $\Rightarrow$ $\sigma_{r\varphi} = \sigma_{rz} = \sigma_{rr} = 0$ @ $r=a$ from 3D theory this translates to $M_{r\varphi}=0$, $q_r=0$, $M_r=0$. or that $\underline{M_r=0}$ & $\underline{V_r=0}$ at the inner boundary for the 2-D plate theory.



Hence to obtain solution use results from problem 1 this set and find tractions at $r=a$ (ie $V_r = V_r^0$, $M_r = M_r^0$) then superpose the results of problem #2 Set #6 so that $\check{V}_r = -V_r^0$ and $\check{M}_r = -M_r^0$. But $V_r^0 = q_r^0 + \frac{\partial M_{r\varphi}^0}{r\partial\varphi}$ and $M_r^0 = \bar{M}\cos 2\varphi$;
$q_r^0 = 0$ since q_x and q_y are zero and under transformation $q_r^0 = q_r^0(q_x, q_y, \varphi) = 0$
now $M_{r\varphi}$ is only a fn of φ $\therefore$ $\boxed{V_r^0\Big|_{r=a} = \frac{1}{a}\left[-2\bar{M}\cos 2\varphi\right]} = -\frac{2\bar{M}}{a}\cos 2\varphi$ and $\boxed{M_r^0\Big|_{r=a} = \bar{M}\cos}$

thus $\boxed{\check{V}_r = \frac{2\bar{M}}{a}\cos 2\varphi}$ $\boxed{\check{M}_r = -\bar{M}\cos 2\varphi}$ $\Rightarrow$ $\boxed{F_2 = \frac{2\bar{M}}{a}}$ and $\boxed{F_1 = -\bar{M}}$ using results of Set #6

using the matrix from Problem Set #6 and letting $n=2$

then
$$\begin{pmatrix} -\bar{M} \\ 2\bar{M} \end{pmatrix} = \frac{D}{a^2}\begin{pmatrix} -3+\nu & 4\nu \\ 4\nu & 4(-3+\nu) \end{pmatrix}\begin{pmatrix} -aA_1 \\ A_2 \end{pmatrix}$$

or
$$\begin{pmatrix} -aA_1 \\ A_2 \end{pmatrix} = \frac{a^2}{D}\,\frac{1}{-12(\nu+3)(\nu-1)}\begin{pmatrix} 4(-3+\nu) & -4\nu \\ -4\nu & -3+\nu \end{pmatrix}\begin{pmatrix} -\bar{M} \\ 2\bar{M} \end{pmatrix} = \frac{a^2}{D}\,\frac{1}{-12(\nu+3)(\nu-1)}\begin{pmatrix} 12(1-\nu)\bar{M} \\ -6(1-\nu)\bar{M} \end{pmatrix}$$
$$= \frac{a^2}{D}\,\frac{1}{\nu+3}\begin{pmatrix} \bar{M} \\ -\bar{M}/2 \end{pmatrix}$$

from 3D $M_{r\varphi}=0$, 2-D $M_{r\varphi}\neq 0$,

now since $v_r = 0$ on $r=a$ $\Rightarrow$ V_1 there also. So this implies max. stresses are m_φ:

$m_\varphi = m_\varphi^{0} + \check{m}_\varphi$ $\quad \boxed{m_\varphi^{0} = -\bar{M}\cos 2\varphi}$

$$\check{m}_\varphi = -D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \nu\frac{\partial^2 w}{\partial r^2}\right]$$

from my previous work with $n=2$: $\boxed{b_2' = \left[\frac{2A_2 + aA_1}{2}\right]}$ $\boxed{a_2' = -\frac{a^3}{2}A_1}$

and $w_{,r}\big|_{r=a} = -\dfrac{2a_2'}{a^3}\cos 2\varphi$ $\quad w_{,\varphi\varphi}\big|_{r=a} = -4\left(\dfrac{a_2'}{a^2} + b_2'\right)\cos 2\varphi$

$w_{,rr}\big|_{r=a} = \left[\dfrac{6a_2'}{a^4}\right]\cos 2\varphi$

$$\therefore\ \check{m}_\varphi = -D\left[-\frac{2a_2'}{a^4} - 4\left(\frac{a_2'}{a^4} + \frac{b_2'}{a^2}\right) + 6\nu\frac{a_2'}{a^4}\right]\cos 2\varphi$$
$$= -D\left[\frac{6(\nu-1)}{a^4}a_2' - \frac{4b_2'}{a^2}\right]\cos 2\varphi$$
$$= -D\left[\frac{-6(\nu-1)}{2a^2}aA_1 - \frac{2}{a^2}(2A_2 + aA_1)\right]\cos 2\varphi$$

$\therefore$ $\boxed{aA_1 = -\dfrac{a^2\bar{M}}{D(\nu+3)}}$ and $\boxed{A_2 = -\dfrac{a^2\bar{M}}{2D(\nu+3)}}$ from the matrix and $\boxed{\check{m}_\varphi = \dfrac{-(3\nu+1)}{\nu+3}\bar{M}\cos 2\varphi}$

$$\therefore\ m_\varphi = -\bar{M}\cos 2\varphi - \frac{(3\nu+1)}{\nu+3}\bar{M}\cos 2\varphi = \boxed{-\frac{4(\nu+1)}{\nu+3}\bar{M}\cos 2\varphi = m_\varphi}$$

thus $\boxed{K = \left|\dfrac{m_\varphi}{m_\varphi^{0}}\right| = \dfrac{4(\nu+1)}{\nu+3}}$

$M_\theta(r,\theta)\Big|_{max}$ is not necessarily at $r=a$.

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C. R. Steele

Assignment 8

PROBLEM: Below is a table of numerical results from Szilard (Theory and Analysis of Plates, 1974) for rectangular plates supported on three sides and loaded by constant pressure. Use the principle of minimum potential energy to obtain an approximate solution for the case of rigid clamping on three sides with a/b = 0.5 . Assume a reasonable deflection shape which satisfies kinematic w & $\frac{dw}{dx}, \frac{dw}{dy}$ boundary conditions, and see how close you come to the table value for maximum moment resultant $\alpha = -0.2004$.

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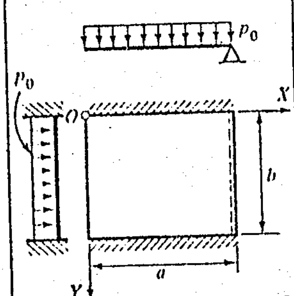
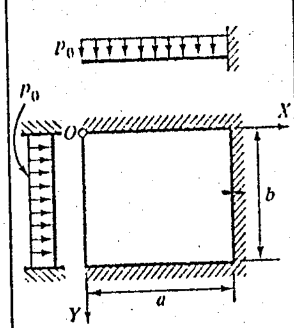
Case No.	Structural System and Static Loading	Deflection and Internal Forces	Constants	Source	Notes
84		$(m_x)_{x=a/2,y=b/2} = c_1 p_0 b^2$ $(m_y)_{x=a/2,y=b/2} = c_2 p_0 b^2$ $(m_y)_{x=0,y=b/2} = c_3 p_0 b^2$ $(m_y)_{x=a/2,y=0,b} = \beta p_0 b^2$ $(m_y)_{x=0,y=0,b} = \gamma p_0 b^2$	see table below	[A17], and [A19]	$\nu = 0.15$.
85		As case 84, and $(m_x)_{x=a,y=b/2} = \alpha p_0 a^2$	see table below	[A17], and [A19]	$\nu = 0.15$. For more extensive tables, see [A9] and [A33].

Case 84 constants:

a/b	0.30	0.40	0.50	0.60	0.70	0.80
c_1	0.0077	0.0090	0.0098	0.0104	0.0103	0.0096
c_2	0.0117	0.0171	0.0217	0.0251	0.0284	0.0315
β	−0.0377	−0.0472	−0.0557	−0.0612	−0.0662	−0.0707
c_3	0.0230	0.0307	0.0361	0.0403	0.0426	0.0438
γ	−0.0693	−0.0782	−0.0846	−0.0852	−0.0854	−0.0853

a/b	0.90	1.00	1.20	1.50	2.00
c_1	0.0086	0.0073	0.0055	0.0033	0.0011
c_2	0.0336	0.0351	0.0378	0.0403	0.0415
β	−0.0744	−0.0771	−0.0805	−0.0828	−0.0833
c_3	0.0442	0.0445	0.0447	0.0449	0.0450
γ	−0.0850	−0.0848	−0.0846	−0.0845	−0.0845

Case 85 constants:

a/b	0.30	0.40	0.50	0.60	0.70	0.80
c_1	−0.0048	−0.0014	0.0015	0.0044	0.0062	0.0076
c_2	0.0026	0.0070	0.0118	0.0170	0.0208	0.0236
α	−0.3833	−0.2783	−0.2004	−0.1476	−0.1106	−0.0865
β	−0.0131	−0.0242	−0.0335	−0.0416	−0.0493	−0.0561
c_3	0.0078	0.0173	0.0268	0.0333	0.0384	0.0413
γ	−0.0333	−0.0545	−0.0709	−0.0798	−0.0837	−0.0848

a/b	0.90	1.00	1.20	1.50	2.00
c_1	0.0087	0.0084	0.0077	0.0052	0.0025
c_2	0.0257	0.0275	0.0316	0.0362	0.0402
α	−0.0691	−0.0559	−0.0387	−0.0248	−0.0139
β	−0.0616	−0.0664	−0.0734	−0.0793	−0.0830
c_3	0.0426	0.0435	0.0443	0.0449	0.0450
γ	−0.0850	−0.0851	−0.0848	−0.0846	−0.0845

Let $w = A y^2 (y-b)^2 x^2$

$w = 0$ @ $x = 0$
$\frac{\partial w}{\partial x} = 0$ @ $x = 0$

$w = 0$ @ $y = 0,\ y = b$
$\frac{\partial w}{\partial y} = 0$ @ $y = 0$ $y = b$

$$U(w(x,y)) = \frac{D}{2}\int_0^b\int_0^a \left[(\kappa_x + \kappa_y)^2 + 2(1-\nu)\left\{ \kappa_{xy}^2 - \kappa_x\kappa_y \right\} - pw \right] dx\,dy. \qquad a/b = .5$$

$$w = A y^2 (y-b)^2 x^2 \qquad w \text{ satisfies b.c.}$$

$$\kappa_x = -w_{,xx} = 2A y^2 (y-b)^2$$

$$\kappa_y = -w_{,yy} = A x^2 \left[2(y-b)^2 + 2y^2 \right]$$

$$\kappa_{xy} = -w_{,xy} = 2xA\left[2y(y-b)^2 + 2y^2(y-b) \right]$$

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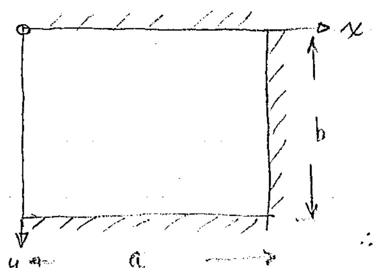
Problem Set #8

1. $$U[w(x)] = \int_0^a\!\!\int_0^b \left\{ \frac{D}{2}\left[(\kappa_x+\kappa_y)^2 + 2(1-\nu)\left\{ \kappa_{xy}^2 - \kappa_x\kappa_y \right\} \right] - pw \right\} dx\,dy$$

where $\kappa_x = -w_{,xx}$ $\quad \kappa_y = -w_{,yy}$ $\quad \kappa_x\kappa_y = -w_{,xy}$ $\quad$ and $p(x,y) = p_0$

For the case of the plate



I will take as my trial fn $\boxed{w(x,y) = A(x-a)^2 y^2 (y-b)^2}$

note that $w(a,y)=0$ and $\frac{\partial w}{\partial x}(a,y)=0$; $w(x,y=0)=0$, $w(x,y=b)$

and $\frac{\partial w}{\partial y}(y=0,x)=0$ and $\frac{\partial w}{\partial y}(x,y=b)=0$

$\therefore$ $\kappa_x = -2Ay^2(y-b)^2$ $\quad \kappa_y = -A(x-a)^2\left[2(y-b)^2 + 8y(y-b) + 2y^2\right]$

$\kappa_{xy} = -4A\left[(x-a)y(y-b)^2 + (x-a)y^2(y-b)\right]$

now $$\frac{\partial U}{\partial A} = \int_0^a\!\!\int_0^b \left\{ \frac{D}{2}\left[2(\kappa_x+\kappa_y)\left(\frac{\partial \kappa_x}{\partial A} + \frac{\partial \kappa_y}{\partial A}\right) + 2(1-\nu)\left\{ 2\kappa_{xy}\frac{\partial \kappa_{xy}}{\partial A} - \frac{\partial \kappa_x}{\partial A}\kappa_y - \frac{\partial \kappa_y}{\partial A}\kappa_x \right\} \right] - p_0\frac{\partial w}{\partial A} \right\} dx\,dy = 0$$

$$= \int_0^a\!\!\int_0^b \left\{ D\left[\kappa_x\kappa_{x,A} + \kappa_y\kappa_{y,A} + 2(1-\nu)\kappa_{xy}\kappa_{xy,A} + \nu\kappa_{x,A}\kappa_y + \nu\kappa_{y,A}\kappa_x \right] - p_0 w_{,A} \right\} dx\,dy = 0$$

now $\frac{\partial \kappa_x}{\partial A} = -2y^2(y-b)^2$ $\quad \frac{\partial \kappa_y}{\partial A} = -2(x-a)^2\left[(y-b)^2 + 4y(y-b) + y^2\right]$ $\quad \frac{\partial \kappa_{xy}}{\partial A} = -4(x-a)\left[y(y-b)^2 + y^2(y-b)\right]$

now $$\frac{\partial U}{\partial A} = 0 = \int_0^b\!\!\int_0^a \{ \cdots \}\,dx\,dy = D\left\{ 4Aa\left(\frac{b^9}{630}\right) + \frac{4a^5}{5}A\left[\frac{b^5}{5} - \frac{8b^5}{20} + \frac{18}{30}b^5 - \frac{8}{20}b^5 + \frac{b^5}{5}\right] \right.$$

$$\left. + \frac{32}{3}A(1-\nu)a^3\left[\frac{b^7}{105} - \frac{2}{140}b^7 + \frac{b^7}{105}\right] + \frac{4\nu Aa^3}{3}\left[\frac{b^7}{105} - \frac{4}{140}b^7 + \frac{b^7}{105}\right] + \frac{4\nu Aa^3}{3}\left[\frac{b^7}{105} - \frac{4}{140}b^7 + \frac{b^7}{105}\right] \right\}$$

$$- p_0\frac{a^3}{3}\left[\frac{b^5}{30}\right] = D\left\{ \frac{4Aab^9}{630} + \frac{4}{25}Aa^5b^5 + \frac{32}{3}A\frac{(1-\nu)a^3b^7}{35}\cdot\frac{1}{6} + \frac{4\nu Aa^3b^7}{3\cdot 35}\left(\frac{-2}{3}\right) + \frac{4\nu Aa^3b^7}{3}\left(\frac{-2}{105}\right) \right.$$

$$- p_0\frac{a^3b^5}{90} = 0 \qquad \text{after a lot of integrations}$$

for $a/b = .5$ or $b = 2a$

$$\therefore \quad D\left[a^{10}A\frac{2048}{63} + \frac{128}{25}Aa^{10} + \frac{32A(1-\nu)a^{10}}{3\cdot 210}(128) + \frac{4\nu(-2)}{3\cdot 105}(128)Aa^{10} + \frac{4\nu(-2)(128)}{3\cdot 105}Aa^{10}\right] - p_0\frac{32a^8}{90}$$

$$Da^{10}A\left[3.25 + 5.12 + (1-\nu)(6.50) - \nu(3.25) - \nu(3.25)\right] - p_0a^8(.36) = 0$$

$\therefore$ for $\nu = .15$ $\quad Da^{10}A[13.92] - .36p_0a^8 = 0$ $\quad$ or $\quad \boxed{A = +.028\frac{p_0}{Da^2}}$

now $m_x = D(\kappa_x + \nu\kappa_y)$; $\quad \kappa_y\Big|_{x=a,\,y=b/2} = 0$ $\qquad \kappa_x = -2A\left(\frac{b^2}{4}\right)\left(\frac{b^2}{4}\right) = -2Aa^4$

Thus $\quad m_x\Big|_{x=a,\,y=b/2} = D\kappa_x = -2\left[+.028\right]\frac{p_0}{Da^2}\cdot Da^4 = -.0558\,p_0a^2$ $\qquad$ off by a factor of ~ 3.6

$$\boxed{m_x = -.0558\,p_0a^2}$$

Fall 1979
C R Steele

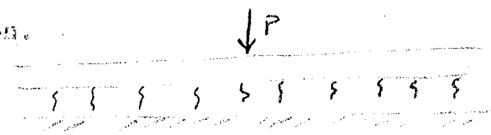
ME 241A

THEORY OF PLATES

Assignment 8

Due December 5

PROBLEM 1: An infinite beam on an elastic foundation is subjected to a point load P at $x = 0$.
Evaluate and sketch the distribution of bending moment and deflection.



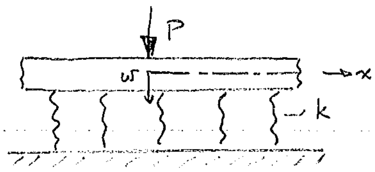
PROBLEM 2: Estimate the radius of the region of significant deflection and stress in a sheet of ice ($E = 10^9\ N/m^2$) which is 3m thick and floating in sea water due to a concentrated load (i.e. a building).

Cesar Levy
ME 241A
Theory of Plates Fall 79
Prof. Steele
AUDIT

$9\frac{1}{2}$

Problem Set # 9

1. PROBLEM 1: An infinite beem on an elastic foundation is subjected to a point load at $x=0$. Evaluate and sketch the distribution of bending moment and deflection.



The governing PDE is $EIw^{IV}+kw=0$ with $w\to 0$ as $|x|\to\infty$ and $\frac{dw}{dx}\to 0$ as $|x|\to\infty$.

We break up the problem into 2 solns $w=w_1(x,t)$? for $x>0$ and $w=w_2(x,t)$ for $x<0$ so that $w_1(x)$ satisfies $EIw_1^{IV}+kw_1=0$ and $w_1\to 0$, $\frac{dw_1}{dx}\to 0$ when $x\to\infty$ also $w=w_2(x)$ satisfies $EIw_2^{IV}+kw_2=0$ and $w_2\to 0$, $\frac{dw_2}{dx}\to 0$ when $x\to-\infty$ and the matching conditions are $w_1(0)=w_2(0)$; $w_1'(0)=w_2'(0)$; $w_1''(0)=w_1''(0)$, $w_2'''(0)-w_1'''(0)=-P/EI$

– if we define $\frac{k}{EI}=\lambda^4$ then let $w(x)=Ce^{mx}$. Putting into PDE $(m^4+\lambda^4)Ce^{mx}=0$

$\therefore\ m=\pm\lambda\sqrt{i},\ \pm\lambda i\sqrt{i}$ or $m=\pm\lambda\frac{\sqrt{2}}{2}(1+i),\ \pm\lambda\frac{\sqrt{2}}{2}(1-i)$

we can then admit, using conditions at ∞, $w_1(x)=C_1e^{-\lambda\frac{\sqrt{2}}{2}(1+i)x}+C_2e^{-\lambda\frac{\sqrt{2}}{2}(1-i)x}$

and $w_2(x)=C_3e^{\frac{\lambda\sqrt{2}}{2}(1+i)x}+C_4e^{\frac{\lambda\sqrt{2}}{2}(1-i)x}$

Thus $w_1(0)=w_2(0)\Rightarrow C_1+C_2=C_3+C_4$; $w_1'(0)=w_2'(0)\Rightarrow -C_1(1+i)-C_2(1-i)=C_3(1+i)+C_4(1-i)$

$w_1''(0)=w_2''(0)\Rightarrow C_1(1+i)^2+C_2(1-i)^2=C_3(1+i)^2+C_4(1-i)^2$; $w_2'''(0)-w_1'''(0)=-P/EI\Rightarrow$

$C_3(1+i)^3+C_4(1-i)^3+C_1(1+i)^3+C_2(1-i)^3=\frac{-P}{EI\lambda^3}\cdot 2\sqrt{2}$

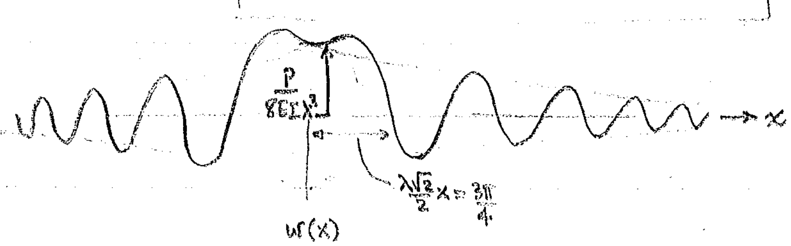
from these $C_1=C_3$ and $C_2=C_4$, $C_1=iC_2$ and $C_2=\frac{+P}{16EI\lambda^3}(1-i)\Rightarrow C_1=\frac{+P}{16EI\lambda^3}(1+i)$

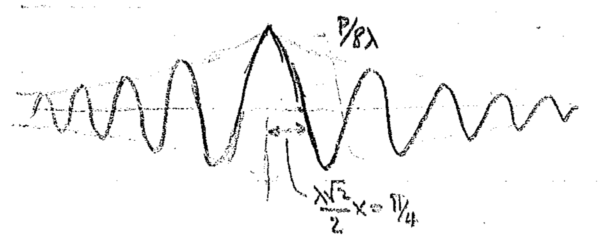
Thus $\boxed{w_1(x)=\frac{+Pe^{-\frac{\lambda\sqrt{2}}{2}x}}{8EI\lambda^3}\left\{\cos\frac{\lambda\sqrt{2}}{2}x+\sin\frac{\lambda\sqrt{2}}{2}x\right\}2\sqrt{2}}$ and $\boxed{w_2(x)=\frac{+Pe^{\frac{\lambda\sqrt{2}}{2}x}}{8EI\lambda^3}\left\{\cos\frac{\lambda\sqrt{2}}{2}x-\sin\frac{\lambda\sqrt{2}}{2}x\right\}2\sqrt{2}}$

for $x>0$ $M_1=-EIw_1''=\frac{-P}{8\lambda}e^{-\frac{\lambda\sqrt{2}}{2}x}\left[\sin\frac{\lambda\sqrt{2}}{2}x-\cos\frac{\lambda\sqrt{2}}{2}x\right]$ and for $x<0$ $M_2=-EIw_2''=\frac{+P}{8\lambda}e^{\frac{\lambda\sqrt{2}x}{2}}\left[\cos\frac{\lambda\sqrt{2}}{2}x+\sin\frac{\lambda\sqrt{2}}{2}\right]$

$\boxed{M_1=\frac{+P}{8\lambda}e^{-\frac{\lambda\sqrt{2}x}{2}}\left[\cos\frac{\lambda\sqrt{2}x}{2}-\sin\frac{\lambda\sqrt{2}x}{2}\right]2\sqrt{2}}$ $\boxed{M_2=\frac{+P}{8\lambda}e^{\frac{\lambda\sqrt{2}x}{2}}\left[\cos\frac{\lambda\sqrt{2}x}{2}+\sin\frac{\lambda\sqrt{2}x}{2}\right]2\sqrt{2}}$

$-\frac{1}{2}$





Problem 2: Estimate the radius of the region of significant deflection and stress in a sheet of ice ($E = 10^9 N/m^2$) which is 3m thick and floating in sea water due to a concentrated load.

I perceived this to be a very simple problem using

$$\delta = \pi\sqrt{2}\left(\frac{EI}{k}\right)^{1/4} = \pi\sqrt{2}\left[\frac{Et^3}{12(1-\nu^2)\rho g}\right]^{1/4}$$

here $\rho = 1000\ kg/m^3$ $g = 9.81\ m/sec^2$ $t = 3m$ and $E = 10^9 N/m^2$: ASSUME $\nu = 0$

$$\delta = \pi\sqrt{2}\left[\frac{10^9 \times 27}{12 \cdot 9810} \quad \frac{\frac{N}{m^2} \cdot m^3}{\frac{kg}{m^2 sec^2}}\right]^{1/4} = \pi\sqrt{2}\left[\frac{27 \times 10^9}{12 \cdot 9810} \quad \frac{\frac{kg\text{-}m}{sec^2} \cdot m^3}{\frac{kg}{sec^2}}\right]^{1/4}$$

$$= \pi\sqrt{2}\left(\frac{27\times10^9}{12(9810)}\right)^{1/4} m = 97.23\ m \qquad \text{now } \frac{\delta}{t} \sim 33 >> 1.$$

for $\nu = .5$ $\qquad \delta = \left[\frac{27\times10^9}{12(.75)(9810)}\right]^{1/4} m = 115.62\ m$

$\therefore$ Since I don't know ν_{ice} all I can say is that $97.23 \leq \delta \leq 115.62\ m$

<u>if</u> $\rho_{sea\ water} = 1000\ kg/m^3$.

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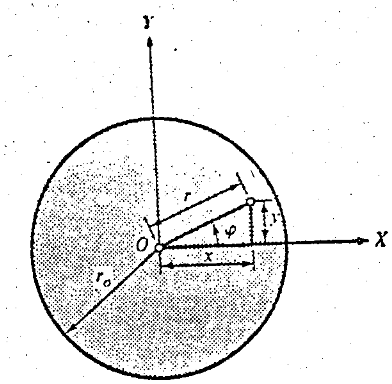
1.11 Differential Equation of Circular Plates

When circular plates are analyzed, it is convenient to express the governing differential equation (1.2.30) in a polar coordinate system, shown in Fig. 1.11.1a. This can be readily accomplished by a coordinate transformation. An alternative approach, based on the equilibrium condition of an infinitesimally small plate element (Fig. 1.11.1b), is analogous to the derivation given in Sec. 1.2 [2].

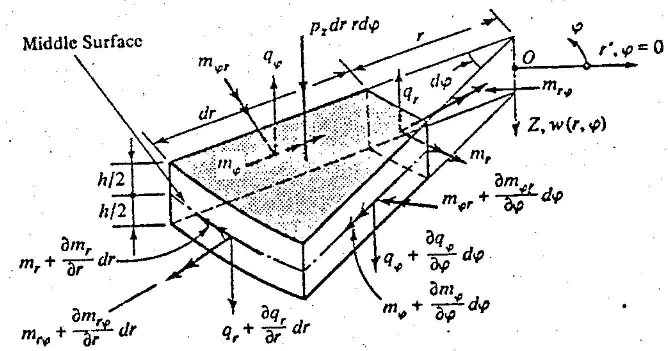
If the coordinate transformation technique is used, the following geometrical relationships between the Cartesian and polar coordinates are applicable (Fig. 1.11.1a):

$$x = r\cos\varphi, \qquad y = r\sin\varphi \tag{1.11.1a}$$

and



(a) Relationship between Cartesian and Polar Coordinates



(b) Plate Element

Fig. 1.11.1. Circular plate.

$$r = \sqrt{x^2 + y^2}, \qquad \varphi = \tan^{-1}\left(\frac{y}{x}\right). \tag{1.11.1b}$$

Since x is a function of r and φ, the derivative of $w(r, \varphi)$ with respect to x can be transformed into derivatives with respect to r and φ. Thus, we can write

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial\varphi}\frac{\partial\varphi}{\partial x}. \tag{1.11.2}$$

Using Eq. (1.11.1) we obtain

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\varphi \quad \text{and} \quad \frac{\partial \varphi}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{1}{r}\sin\varphi; \tag{1.11.3}$$

therefore

$$\frac{\partial w}{\partial x} = \cos\varphi\frac{\partial w}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial w}{\partial \varphi} = \left[\cos\varphi\frac{\partial(\ )}{\partial r} - \sin\varphi\frac{1}{r}\frac{\partial(\ )}{\partial\varphi}\right]w \tag{1.11.4}$$

from which

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \left(\cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial\varphi}\right)\cdot\left(\cos\varphi\frac{\partial w}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial w}{\partial\varphi}\right). \tag{1.11.5}$$

Equation (1.11.5) can be reduced to

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2} = \cos^2\varphi\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\sin^2\varphi\frac{\partial^2 w}{\partial\varphi^2} + \frac{1}{r}\sin^2\varphi\frac{\partial w}{\partial r}\\ -\frac{1}{r}\sin 2\varphi\frac{\partial^2 w}{\partial r\,\partial\varphi} + \frac{1}{r^2}\sin 2\varphi\frac{\partial w}{\partial\varphi}.\end{aligned} \tag{1.11.6}$$

In a similar manner we obtain

$$\frac{\partial w}{\partial y} = \sin\varphi\frac{\partial w}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial w}{\partial\varphi} \tag{1.11.7}$$

and

$$\begin{aligned}\frac{\partial^2 w}{\partial y^2} = \sin^2\varphi\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\cos^2\varphi\frac{\partial^2 w}{\partial\varphi^2} + \frac{1}{r}\cos^2\varphi\frac{\partial w}{\partial r}\\ +\frac{1}{r}\sin 2\varphi\frac{\partial^2 w}{\partial r\,\partial\varphi} - \frac{1}{r^2}\sin 2\varphi\frac{\partial w}{\partial\varphi}.\end{aligned} \tag{1.11.8}$$

Furthermore, Eqs. (1.11.4) and (1.11.7) give

$$\begin{aligned}\frac{\partial^2 w}{\partial x\,\partial y} = \frac{1}{2}\sin 2\varphi\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\cos 2\varphi\frac{\partial w}{\partial\varphi} - \frac{1}{2r^2}\sin 2\varphi\frac{\partial^2 w}{\partial\varphi^2}\\ -\frac{1}{2r}\sin 2\varphi\frac{\partial w}{\partial r} + \frac{1}{r}\cos 2\varphi\frac{\partial^2 w}{\partial r\,\partial\varphi}.\end{aligned} \tag{1.11.9}$$

Therefore the Laplace operator ∇^2 (1.1.34), in terms of polar coordinates, becomes

$$\boxed{\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r}\frac{\partial}{\partial r}.} \tag{1.11.10}$$

When the Laplacian operator ∇^2 is replaced by ∇_r^2 in Eq. (1.2.31), the plate equation in polar coordinates is obtained:

$$\nabla_r^2\nabla_r^2 w = \frac{p_z(r,\varphi)}{D}. \tag{1.11.11}$$

The expressions for internal moments and shear forces, derived in Sec. 1.2, can be also transformed into polar coordinates; thus, we can write

$$m_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\right], \tag{1.11.12}$$

$$m_\varphi = -D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2} + \nu\frac{\partial^2 w}{\partial r^2}\right), \tag{1.11.13}$$

$$m_{r\varphi} = m_{\varphi r} = -(1-\nu)D\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\varphi}\right) = -(1-\nu)D\left[\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\varphi} - \frac{1}{r^2}\frac{\partial w}{\partial\varphi}\right], \tag{1.11.14}$$

and

$$q_r = -D\frac{\partial}{\partial r}\nabla_r^2 w \quad \text{and} \quad q_\varphi = -D\frac{1}{r}\frac{\partial}{\partial\varphi}\nabla_r^2 w. \tag{1.11.15}$$

Similarly, transformation of Eq. (1.3.3) into polar coordinates gives the lateral edge forces:

$$v_r = q_r + \frac{1}{r}\frac{\partial m_{r\varphi}}{\partial\varphi} \quad \text{and} \quad v_\varphi = q_\varphi + \frac{\partial m_{\varphi r}}{\partial r}. \tag{1.11.16}$$

These expressions for an edge with outward normal in the r or φ direction become

$$\begin{aligned}v_r &= -D\left[\frac{\partial}{\partial r}\nabla_r^2 w + \frac{1-\nu}{r}\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\varphi} - \frac{1}{r^2}\frac{\partial w}{\partial\varphi}\right)\right],\\ v_\varphi &= -D\left[\frac{1}{r}\frac{\partial}{\partial\varphi}\nabla_r^2 w + (1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\varphi} - \frac{1}{r^2}\frac{\partial w}{\partial\varphi}\right)\right].\end{aligned} \tag{1.11.17}$$

Summary. The governing differential equation of circular plates, expressed in polar coordinates, was derived by transformation of the Laplacian operator ∇^2 from Cartesian to polar coordinates. In the same way, expressions for moments and shear forces could be found in terms of r and φ. Exact solutions of Eq. (1.11.11) for nonsymmetric loading and boundary conditions are extremely tedious, or, in many cases, are impossible to obtain. With symmetric boundary conditions, however, solutions for nonsymmetric loading can often be achieved by separating the loading into symmetric and antisymmetric parts, as discussed in subsequent sections.

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1.12 Rigrous Solution of Circular Plates Subjected to Rotationally Symmetric Loading

If a circular plate is under the action of lateral loads, which are radially symmetric with respect to the origin of the polar coordinate system, the deflected plate surface is also rotationally symmetric (Fig. 1.12.1), provided that

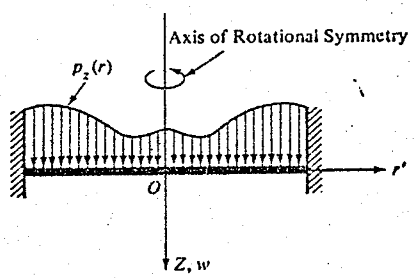

Fig. 1.12.1. Rotational symmetry.

the support has the same type of symmetry. In this case w is independent from φ: thus the Laplacian operator (1.11.10) becomes

$$\nabla_r^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}. \tag{1.12.1}$$

Consequently, the differential equation of the circular plate has the following form*:

$$\boxed{\nabla_r^2\nabla_r^2 w(r) = \frac{d^4w}{dr^4} + \frac{2}{r}\frac{d^3w}{dr^3} - \frac{1}{r^2}\frac{d^2w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr} = \frac{p_z(r)}{D},} \tag{1.12.2}$$

where D is the flexural rigidity of the plate given by Eq. (1.2.28).

Since the deflections of the circular plate under radially symmetric loading are independent of φ, the expressions of the internal forces given in Sec. 1.11 become

* The same expression can be obtained from the equilibrium of the plate element [7], also see Sec. 1.17.

$$m_r = -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right), \tag{1.12.3}$$

$$m_\varphi = -D\left(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right), \tag{1.12.4}$$

$$m_{r\varphi} = m_{\varphi r} = 0, \tag{1.12.5}$$

$$q_r = v_r = -D\left(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right) \quad \text{and} \quad q_\varphi = 0. \tag{1.12.6}$$

The rigorous solution of Eq. (1.12.2) is obtained again as the sum of the solution of the homogeneous differential equation (1.4.1) and one particular solution. Thus, we can write

$$w = w_H + w_P. \tag{1.12.7}$$

The solution of the homogeneous differential equation $\nabla_r^2\nabla_r^2 w_H = 0$ can be given by

$$w_H = C_1 + C_2r^2 + C_3\ln\frac{r}{r_0} + C_4r^2\ln\frac{r}{r_0}, \tag{1.12.8}$$

where C_1, C_2, C_3, and C_4 are constants which can be determined from the boundary conditions. If the deflections at the center of the plate are not infinitely large, C_3 and C_4 must be equal to zero, and Eq. (1.12.8) becomes

$$w_H = C_1 + C_2r^2. \tag{1.12.9}$$

The particular solution w_P is obtained by direct integration of Eq. (1.12.2). For this purpose we split the differential equation into two parts, in a way similar to that described by Eq. (1.2.34) in Sec. 1.2. Let us express first the *moment-sum* in terms of polar coordinates:

$$\mathfrak{M} = \frac{m_r + m_\varphi}{1+\nu}. \tag{1.12.10}$$

Substituting the expression of the internal moments (1.12.3) and (1.12.4) into Eq. (1.12.10), we obtain

$$\begin{aligned}\mathfrak{M} &= -\frac{D}{1+\nu}\left(\frac{d^2w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr} + \frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right)\\ &= -D\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = -D\,\nabla_r^2 w.\end{aligned} \tag{1.12.11}$$

When we apply the differential operator, ∇_r^2, to Eq. (1.12.11), then a comparison with Eq. (1.12.2) gives

$$\nabla_r^2\mathfrak{M} = -p_z(r). \tag{1.12.12}$$

Equations (1.12.11) and (1.12.12) can be given in somewhat different forms:

$$\nabla_r^2\mathfrak{M} = \frac{d^2\mathfrak{M}}{dr^2} + \frac{1}{r}\frac{d\mathfrak{M}}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathfrak{M}}{dr}\right) = -p_z(r) \tag{1.12.13}$$

thus,

$$w(r,\varphi) = w_H + w_P. \tag{1.13.4}$$

The particular solution of the nonhomogeneous differential equation (1.11.11),

$$\nabla_r^2\nabla_r^2 w = \frac{p_1}{D}\rho\cos\varphi, \tag{1.13.5}$$

may be taken in the form of

$$w_P = C_P\rho^5\cos\varphi. \tag{1.13.6}$$

Substituting Eq. (1.13.6) into Eq. (1.13.5), the unknown constant C_P can be determined:

$$C_P = \frac{p_1 r_0^4}{192D}. \tag{1.13.7}$$

The solution of the homogeneous form of the differential equation of the plate can be given as*

$$w_H = (C_1\rho + C_2\rho^3)\cos\varphi. \tag{1.13.8}$$

The general solution of the deflection of the circular plate, according to Eq. (1.13.4), is

$$w(r,\varphi) = \left(\frac{p_1 r_0^4}{192D}\rho^5 + C_1\rho + C_2\rho^3\right)\cos\varphi. \tag{1.13.9}$$

From the boundary conditions at the edge, the unknown coefficients, C_1 and C_2, can be obtained in the usual manner.

Another common type of the antisymmetric loading is a moment applied to a rigid disk located at the center of the circular plate, as shown in Fig. 1.13.4. In this case, we may state that there is no external lateral load; thus the

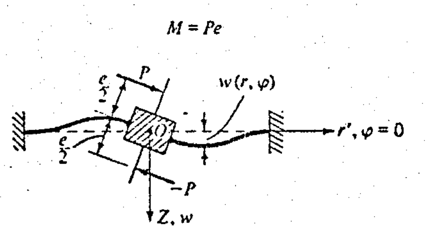

Fig. 1.13.4. Circular plate with moment acting at the center.

solution of the homogeneous form of the differential equation yields the solution of the problem. Consequently, the equation of the deflected plate surface has the form

$$w(r,\varphi) = \left(C_1\rho + C_2\frac{1}{\rho} + C_3\rho^3 + C_4\rho\ln\rho\right)\cos\varphi. \tag{1.13.10}$$

* Other types of functions can be obtained from Refs. [14] and [1.11.3]

The constants C_1, C_2, C_3, and C_4 are determined from the boundary conditions.

c. Concentrated Load Acting at the Center. A special case of symmetric bending of circular plates is when the concentrated lateral load acts at the center (Fig. 1.13.5). Because of the discontinuous character of the lateral loading,

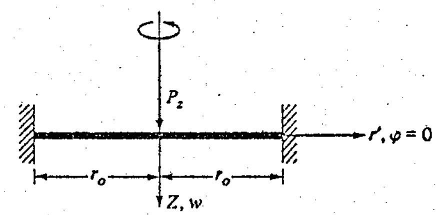

Fig. 1.13.5. Concentrated force at the center of a circular plate.

an approach other than that described in Sec. 1.12 is utilized. The solution in this case is obtained from the equilibrium of the vertical forces, which states that

$$-2\pi r q_r = P_z, \tag{1.13.11}$$

and by expressing q_r using Eq. (1.11.15); thus, we obtain

$$D\left(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right) = D\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = \frac{P_z}{2\pi r}. \tag{1.13.12}$$

Repeated integration of Eq. (1.13.12) yields

$$w(r) = \frac{P_z}{2\pi D}\left(\frac{r^2}{4}\ln r + C_1r^2 + C_2\ln r + C_3\right). \tag{1.13.13}$$

Since for $r = 0$ the deflection should remain finite, we must use $C_2 = 0$. The other two coefficients, C_1 and C_3, are to be determined from the boundary conditions of the specific problem. In the case of simply supported edges, for instance, the deflection of the plate becomes

$$w(r) = \frac{P_z r_0^2}{16\pi D}\left[\frac{3+\nu}{1+\nu}(1-\rho^2) + 2\rho^2\ln\rho\right], \tag{1.13.14}$$

while for clamped edges we obtain

$$w(r) = \frac{P_z r_0^2}{16\pi D}(1 - \rho^2 + 2\rho^2\ln\rho). \tag{1.13.15}$$

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1.14 Circular Plate Loaded by an Eccentric Concentrated Force

Since any lateral load acting on a plate can be replaced by a finite number of concentrated forces, the solution of the circular plates under an arbitrary concentrated load can be considered one of the fundamental problems in the theory of plates. Although it is assumed that the boundary condition is rotationally symmetric, the analysis of the deflections due to such nonsymmetric loads, as outlined in Sec. 1.11, is quite tedious. Certain simplifications can be achieved, however, by considering that the deflected plate surface is symmetric with respect to the line connecting the origin of the polar coordinate system ($\varphi = 0$) and the point of application, A, of the force (Fig. 1.14.1).

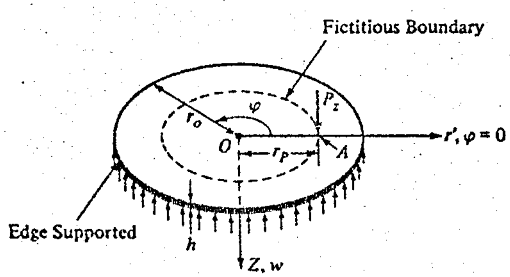

Fig. 1.14.1. Eccentric force.

The general solution of the governing differential equation of circular plates (1.11.11) was obtained by Clebsch [1.11.1] in the form

$$w(r, \varphi) = w_P + F_0(r) + \sum_{m=1}^{\infty} F_m(r) \cos m\varphi + \sum_{m=1}^{\infty} F_m'(r) \sin m\varphi. \qquad (1.14.1)$$

where w_P represents a particular solution of Eq. (1.11.11) and $F_0(r)$, $F_1(r)$, $F_2(r), \ldots, F_1'(r), F_2'(r), \ldots$ are functions of r only.

Let us divide the circular plate into two cylindrical parts by the radius $\overline{OA} = r_P$. In the case of the outer plate ring, the load P is an edge load; consequently, only the terms which represent the complementary part of the solution* must be considered. Furthermore, utilizing the above-mentioned symmetry of the deflected plate surface, Eq. (1.14.1), representing the deflection of the outer part, can be written as

$$w_{H1} = w_1(r, \varphi) = F_0(r) + \sum_{m=1}^{\infty} F_m(r) \cos m\varphi, \qquad (1.14.2)$$

where

$$\left.\begin{aligned} F_0(r) &= C_{10} + C_{20}r^2 + C_{30}\ln\frac{r}{r_0} + C_{40}r^2\ln\frac{r}{r_0} \\ F_1(r) &= C_{11}r + C_{21}r^3 + C_{31}r^{-1} + C_{41}\,r\ln\frac{r}{r_0} \\ F_2(r) &= C_{12}r^2 + C_{32}r^{-2} + C_{22}r^4 + C_{42}\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(r) &= C_{1m}r^m + C_{3m}r^{-m} + C_{2m}r^{m+2} + C_{4m}r^{-m+2}{}_{(m>1)}. \end{aligned}\right\} \qquad (1.14.3)$$

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Similarly, the deflections of the inner part of the plate can be expressed in the following form:

$$w_{H2} = w_2(r, \varphi) = f_0(r) + \sum_{m=1}^{\infty} f_m(r) \cos m\varphi, \qquad (1.14.4)$$

where the functions $f_0(r), f_1(r), f_2(r), \ldots$ are similar to those given in Eq. (1.14.3). Thus we may write

$$\left.\begin{aligned} f_0(r) &= c_{10} + c_{20}r^2 + c_{30}\ln\frac{r}{r_0} + c_{40}r^2\ln\frac{r}{r_0} \\ f_1(r) &= c_{11}r + c_{21}r^3 + c_{31}r^{-1} + c_{41}r\ln\frac{r}{r_0} \\ f_2 &\cdots = c_{12}r^2 + c_{22}r^4 + c_{3m}r^{-2} + c_{4m}\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m(r) &= c_{1m}r^m + c_{3m}r^{-m} + c_{2m}r^{m+2} + c_{4m}r^{-m+2}{}_{(m>1)}. \end{aligned}\right\} \qquad (1.14.5)$$

Since at the center of the plate the deflection, the slope, and the internal moments are not infinite, we obtain

$$\left.\begin{aligned} c_{30} &= c_{40} = 0 \\ c_{31} &= c_{41} = 0 \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ c_{3m} &= c_{4m} = 0. \end{aligned}\right\} \qquad (1.14.6)$$

The remaining unknown constants in Eqs. (1.14.3) and (1.14.5) are determined from the boundary conditions of the inner and outer ring plates.

* Solution of the homogeneous form of the differential equation.

Let us assume, for instance, that the edge of the circular plate is fixed; thus the two boundary conditions are

$$(w_1)_{r=r_0} = 0 \quad \text{and} \quad \left(\frac{\partial w_1}{\partial r}\right)_{r=r_0} = 0. \tag{1.14.7}$$

Because of the continuity of the plate at $r = r_p$, the deflections and the slopes of the inner and outer parts are the same; therefore

$$(w_1)_{r=r_p} = (w_2)_{r=r_p} \quad \text{and} \quad \left(\frac{\partial w_1}{\partial r}\right)_{r=r_p} = \left(\frac{\partial w_2}{\partial r}\right)_{r=r_p}. \tag{1.14.8}$$

Furthermore, another equation in the form of the compatibility of the internal forces can be used; thus

$$(m_{1r})_{r=r_p} = (m_{2r})_{r=r_p}, \tag{1.14.9}$$

or, in another form,

$$\left(\frac{\partial^2 w_1}{\partial r^2}\right)_{r=r_p} = \left(\frac{\partial^2 w_2}{\partial r^2}\right)_{r=r_p}. \tag{1.14.10}$$

The sixth equation for determination of the unknown constants is obtained by stating that at the point of application of the load the difference of the transverse shear forces is equal to the external load, which gives

$$(q_{1r} - q_{2r})_{r=r_p,\varphi=0} = P, \tag{1.14.11}$$

expressing the discontinuity of the internal shear forces at $r = r_p$ and $\varphi = 0$. This equation is familiar to the readers from elementary treatment of mechanics, since a similar equation is used for construction of the shear diagrams of beams.

To utilize Eq. (1.14.11) effectively, we must now express the concentrated load in a form similar to Eqs. (1.14.2) and (1.14.4), representing a cosine series expansion. In the Fourier series expression of the concentrated load P in polar coordinates, we first replace the concentrated force with a partial line load distributed over an arc of length $r_p\, 2\,\Delta\varphi$, as shown in Figure 1.14.2. The intensity of this line load between $+\Delta\varphi$ and $-\Delta\varphi$ is

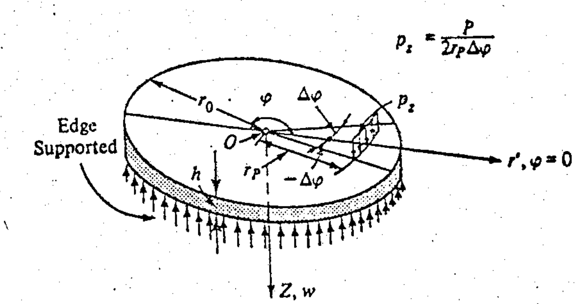
$$p_z = \frac{P}{2r_p\,\Delta\varphi} = \text{constant}. \tag{1.14.12}$$

As discussed in Sec. 1.5, a cosine series expansion of an arbitrary function can be obtained by making the function even (Fig. 1.14.2b) and using a period of $T = 2L$, which, in the case of polar coordinates, becomes $T = 2\pi$. Thus Eq. (1.5.23) can be written in the following form:

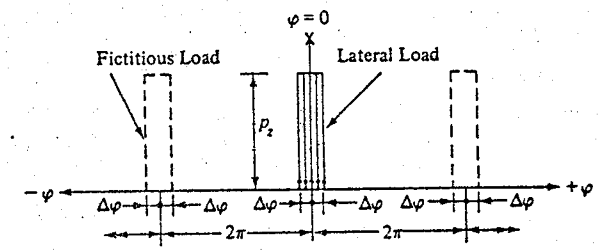
$$p_z(\varphi) = \frac{1}{2}A_0 + \sum_1^\infty A_m \cos m\varphi. \tag{1.14.13}$$

Equation (1.5.22) gives

$$A_0 = \frac{2}{\pi}\int_0^\pi p_z\,d\varphi = \frac{2}{\pi}\int_0^{+\Delta\varphi} \frac{P}{2r_p\,\Delta\varphi}\,d\varphi = \frac{P}{\pi r_p} \tag{1.14.14}$$



(a) Partial Line Load



(b) Arbitrary Continuation of the Loading Function

Fig. 1.14.2. Arbitrary continuation of the loading function.

and

$$A_m = \frac{2}{\pi}\int_0^\pi p_z \cos m\varphi\,d\varphi = \frac{2}{\pi}\int_0^{+\Delta\varphi} \frac{P}{2r_p\,\Delta\varphi}\cos m\varphi\,d\varphi = \frac{P}{\pi r_p m\,\Delta\varphi}\sin m\,\Delta\varphi. \tag{1.14.15}$$

Hence the cosine series expression of the line load in terms of polar coordinates becomes

$$\begin{aligned} p_z(\varphi) &= \frac{P}{2\pi r_p} + \sum_1^\infty \frac{P}{\pi r_p m\,\Delta\varphi}\sin m\,\Delta\varphi \cos m\varphi \\ &= \frac{P}{\pi r_p}\left(\frac{1}{2} + \sum_1^\infty \frac{\sin m\,\Delta\varphi}{m\,\Delta\varphi}\cos m\varphi\right). \end{aligned} \tag{1.14.16}$$

If $\Delta\varphi$ is approaching zero, the limit approach, described by Eq. (1.8.21), can be used.

Therefore, Eq. (1.14.16) can be written as

$$p_z(\varphi) = \frac{P}{\pi r_p}\left(\frac{1}{2} + \sum_1^\infty \cos m\varphi\right). \tag{1.14.17}$$

Substituting Eq. (1.14.17) into Eq. (1.14.11) and expressing the shear forces in

accordance with Eq. (1.11.15), we obtain

$$D\left(\frac{\partial}{\partial r}\nabla_r^2 w_1 - \frac{\partial}{\partial r}\nabla_r^2 w_2\right) = \frac{P}{\pi r_p}\left(\frac{1}{2} + \sum_1^\infty \cos m\varphi\right). \qquad (1.14.18)$$

The previously discussed boundary conditions in connection with Eq. (1.14.18) yield all the unknown coefficients in the expression of the plate deflections.

The results of these tedious computations are given in Refs. [1.1.4] and [14]. Solution for simply supported edge conditions is discussed in Refs. [2] and [1.1.4]. The maximum deflection for a clamped plate is

$$w_{\max} = w(r_p, 0) = \frac{P}{16\pi D}\frac{(r_0^2 - r_p^2)^2}{r_0^2}. \qquad (1.14.19)$$

With $r_p = 0$, Eq. (1.14.19) represents the solution for the concentrated load acting at the center.

A similar approach can be taken to include arbitrary loadings acting on circular and annular plates. This general application of Eq. (1.14.1), however, is more tedious.

Summary. It is evident that the classical solution of circular plate problems, which do not have rotational symmetry, requires extensive mathematical manipulations and can be rather time-consuming. Consequently, this approach is not recommended for everyday application by the design engineer.

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1.15 Continuous Plates

a. Introduction. Plates which are continuous over one or more intermediate supports are of considerable practical interest. As in the case of other statically indeterminate* structures, the methods used in the analysis of continuous plates fall into two distinct categories: the force and deformation methods

* Strictly speaking, continuous plates are externally (as well as internally) statically indeterminate.

[1.15.1][1.15.2]. Following either approach, the continuous plate is subdivided into individual, simple-span panels. The analysis is based on (1) equilibrium conditions of the individual panels and (2) the compatibility of displacements or force at the adjoining edges.

If the *force method* is used, we remove all redundant restraints (Fig. 1.15.1)

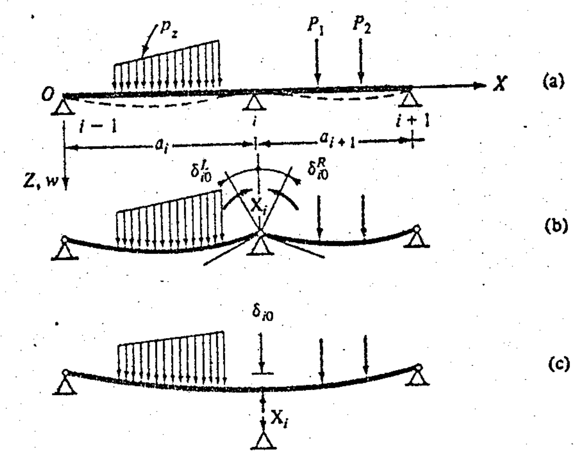


Fig. 1.15.1. Elimination of redundant forces.

and determine the deformation characteristics, δ_{i0}, of the *statically determinate* components. The unknown redundant forces, X_i, are found from the compatibility condition of displacements,

$$\delta_{i0} + \sum_k X_k \delta_{ik} = 0, \quad (i, k = 1, 2, 3, \ldots), \qquad (1.15.1)$$

where δ_{ik} represents the displacement at "i" produced by the unit redundant force ($X_k = 1$) at "k". Equation (1.15.1) states that the sum of all displacements at i is zero.

The analysis of statically indeterminate structures by the *deformation method* is somewhat different. In this case the joint displacements (usually the angles of rotation ϕ_k) are the unknowns from which the joint forces can be calculated. In applying the deformation method, the individual components are first made *kinematically determinate*. This condition is usually obtained by imposing fixed boundary conditions at the joints. Next, the fixed-end moments, M_{i0}^*, produced by the loads, are calculated for each component. The governing equation

$$M_{i0}^* + \sum_k M_{ik}^* \phi_k = 0 \quad (i, k = 1, 2, 3, \ldots) \qquad (1.15.2)$$

states that the sum of all moments at "i" is zero. In this expression, M_{ik}^* represents the fixed-end moments at "i" produced by unit rotation ($\phi_k = 1$) at "k". Equation (1.15.2) is the well-known *slope-deflection* equation, used for the analy-

2 | 85   2 | 98
43 + 49 = 92 A

Question	Score
1	33
2	33
3	18
4	1
5	
6	
7	
8	
Total	85

STANFORD UNIVERSITY

OFFICIAL EXAMINATION BOOK

24 Page Ruled

HW 98

Course A

Name of student Cesar Levy

Date of examination 12/13/79

Course ME 241 A Theory of Plates

THE STANFORD UNIVERSITY HONOR CODE

A. The Honor Code is an undertaking of the students, individually and collectively:
   (1) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;
   (2) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

I acknowledge and accept the Honor Code.

(Signed) Cesar Levy

Interpretations and applications of the Honor Code appear on the back cover of this examination book.

ME 241A | THEORY OF PLATES FINAL EXAM | C. R. Steele Fall, 1979

Problem 1: A semi-infinite plate strip ($0 \leq y < \infty$) is simply supported on the edges $x = 0$ and $x = a$. The edge $y = 0$ is given the displacement and rotation

$$w(x,0) = 0$$

$$w_y(x,0) = B \sin(\pi x/a)$$

where B is a constant. Find the <u>magnitude</u> and <u>direction</u> of the corner force at $x = y = 0$.

Problem 2: A circular plate is clamped at the outer boundary $r = b$, and at the inner boundary $r = a$ is acted upon by a uniform distribution of moment $M_r = A$ with zero transverse shearing force. Find the formula for the deflection at the inner boundary.

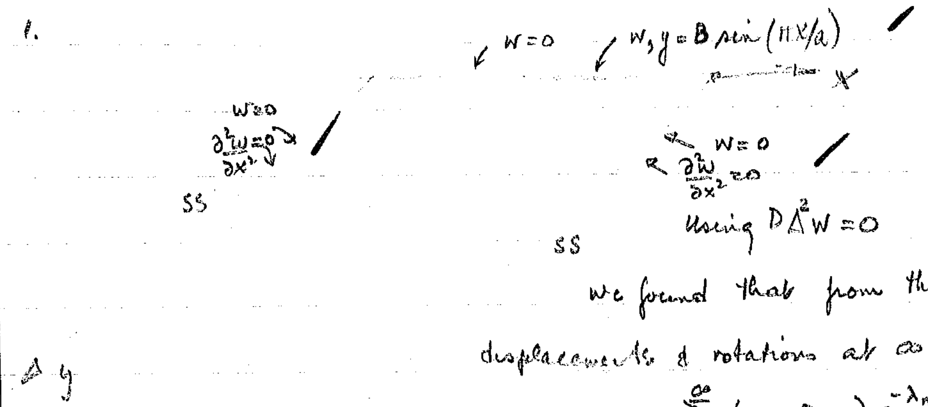
Problem 3: An infinite plate with bending stiffness D has an elastic foundation with modulus k. A line load of constant intensity q acts on an elliptical path with semi-major axis a and semi-minor axis b. The parameters are related by

$$a = 2b$$

$$k = (32)^2 D/b^4$$

Find a good approximation for the deflection at each point of the load path.

1.



Using $D\nabla^4 w = 0$

we found that from this our solns for bounded displacements & rotations at ∞ gave

$$w(x,y) = \sum_{n=1}^{\infty} (C_n + D_n y)\, e^{-\lambda_n y} \sin\frac{n\pi x}{a} \qquad \lambda_n = \frac{n\pi}{a}$$

since at $y=0$ $\quad w=0 \Rightarrow C_n = 0 \quad \forall n$

and since $\quad w_{,y}(x,0) = B\sin \pi x/a \Rightarrow \quad w_{,y}(x,y) = \sum_{n=1}^{\infty} D_n\left[e^{-\lambda_n y} - \lambda_n y e^{-\lambda_n y}\right] \sin\frac{n\pi x}{a}$

$$= \sum_{n=1}^{\infty} D_n e^{-\lambda_n y}\left[1 - \lambda_n y\right] \sin\frac{n\pi x}{a}$$

$$w_{,y}(x,0) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{a} = B \sin\frac{\pi x}{a} \Rightarrow \quad D_1 = B \quad D_n = 0 \ \forall\, n \geq 2$$

$$\therefore \quad w(x,y) = B y e^{-\frac{\pi y}{a}} \sin\frac{\pi x}{a}$$

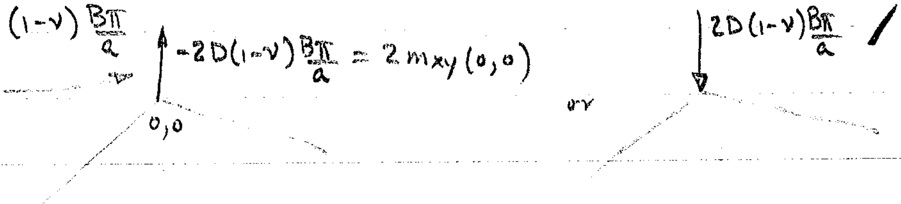
$$w_{,x} = \frac{\pi}{a} B y e^{-\frac{\pi y}{a}} \cos\frac{\pi x}{a} \qquad w_{,xy} = \frac{\pi}{a} B\left\{e^{-\frac{\pi y}{a}} - \frac{\pi}{a} e^{-\frac{\pi y}{a}} y\right\} \cos\frac{\pi x}{a}$$

$$\therefore \quad \frac{m_{xy}}{-D(1-\nu)} = \frac{B\pi}{a} e^{-\frac{\pi y}{a}} \left\{1 - \frac{\pi y}{a}\right\} \cos\frac{\pi x}{a}$$

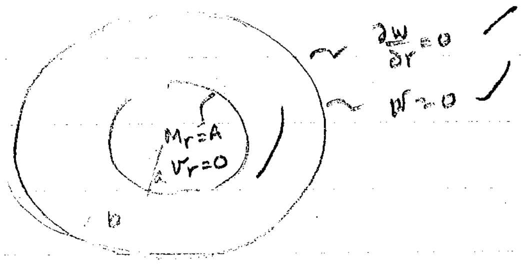
$$\therefore \quad m_{xy} = -D(1-\nu)\frac{B\pi}{a} e^{-\frac{\pi y}{a}} \left\{1 - \frac{\pi y}{a}\right\} \cos\frac{\pi x}{a}$$

$$m_{xy}(0,0) = -D(1-\nu)\frac{B\pi}{a}$$

$\therefore$ at the corner $\longrightarrow$



2.



Since $p=0$ this is also an axisymmetric problem $\therefore D\Delta^2 w=0$

$\therefore$ most general solution is $w = c_1 r^2 \ln r + c_2 r^2 + c_3 \ln r + c_4$

$w(r=b)=0$ also $V_r=0$ on inner bdy

$\frac{\partial w}{\partial r}(r=b)=0$ $M_r=A$ on inner bdy

$$V_r = q_r = -D\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}w\right)\right] \qquad m_r = -D\left[w_{,rr} + \frac{\nu w_{,r}}{r}\right]$$

$$w\Big|_{r=b} = c_1 b^2 \ln b + c_2 b^2 + c_3 \ln b + c_4 = 0$$

$$\frac{\partial w}{\partial r}\Big|_{r=b} = c_1\left[2b\ln b + b\right] + 2c_2 b + c_3/b = 0$$

$$\frac{\partial^2 w}{\partial r^2} = c_1\left[2\ln r + 3\right] + 2c_2 - c_3/r^2$$

Δw

$$\frac{dw}{dr} = c_1\left[2r\ln r + r\right] + 2c_2 r + \frac{c_3}{r}$$

$$r\frac{dw}{dr} = c_1\left[2r^2\ln r + r^2\right] + 2c_2 r^2 + c_3$$

$$\frac{d}{dr}(\ ) = c_1\left[4r\ln r + 4r\right] + 4c_2 r$$

$$\Delta w = \frac{1}{r}\frac{d}{dr}(\ ) = c_1\left[4\ln r + 4\right] + 4c_2$$

$$\Delta w = 4c_1 \ln r + 4c_1 + 4c_2$$

$$\frac{d}{dr}(\Delta w) = \frac{4c_1}{r}$$

$$\therefore V_r\Big|_{r=a} = 0 \Rightarrow -D\cdot\frac{4c_1}{a} = 0 \Rightarrow c_1 = 0$$

$$m_r\Big|_{r=a} = A = -D\left[2c_2 - \frac{c_3}{a^2} + \frac{\nu}{a}\left(2c_2 a + c_3/a\right)\right]$$

$$A = -D\left[c_2(2+2\nu) + \frac{c_3}{a^2}(-1+\nu)\right] = -D\left[2(1+\nu)c_2 + \frac{c_3}{a^2}(\nu-1)\right]$$

also $2c_2 b^2 + c_3 = 0 \Longrightarrow c_3 = -2c_2 b^2$

also $c_2 b^2 + c_3 \ln b + c_4 = 0$

$$A = -D\left[2(1+\nu) - \frac{2b^2}{a^2}(\nu-1)\right]c_2$$

$$\therefore \quad c_2 = \frac{A}{-2D\left[(1+\nu) - \frac{b^2}{a^2}(\nu-1)\right]} \qquad c_3 = \frac{b^2 A}{D\left[(1+\nu) - b^2/a^2(\nu-1)\right]}$$

$$c_4 = -c_2 b^2 - c_3 \ln b = \frac{+Ab^2}{2D\left[(\ ) - b^2/a^2(\nu-1)\right]} - \frac{2b^2 \ln b\, A}{2D\left[(\ ) - (\ )\right]}$$

$$= -c_2 b^2 - (-2c_2 b^2)\ln b$$

$$= -c_2 b^2\left[1 - 2\ln b\right]$$

$$c_4 = \frac{Ab^2}{2D\left[(\ ) - b^2/a^2(\nu-1)\right]}\left[1 - 2\ln b\right]$$

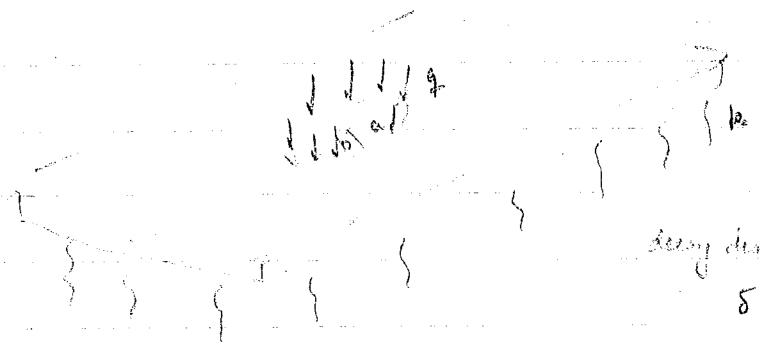
$$\therefore \quad w = \frac{A}{-2D\left[(1+\nu) - b^2/a^2(\nu-1)\right]}\left[r^2 - 2b^2\ln r - b^2(1-2\ln b)\right] = \frac{A}{2(1+\nu)D}\left[\frac{b^2 - r^2 + 2b^2\ln r/b}{1 + \frac{1-\nu}{\cdots}\, b^2/a^2}\right]$$

$$(r^2 - b^2) - 2b^2(\ln r - \ln b)$$

for inner boundary

$$\left. w \right|_{r=a} = \frac{A}{2(1+\nu)D}\left[\frac{(b^2-a^2) + 2b^2 \ln\left(a/b\right)}{1+\left(\frac{1-\nu}{1+\nu}\right)\frac{b^2}{a^2}}\right]$$

3.



$a = 2b$

$k = (32)^2 D / b^4$

$k = \dfrac{(32)^2 D}{(a/2)^4} = \dfrac{(32)^2 16 D}{a^4}$

decay distance will be on the order of

$\delta = \pi\sqrt{2}\,(D/k)^{1/4} \sim \dfrac{a\pi\sqrt{2}}{\cdots} \sim \dfrac{a\pi}{8}$

Since a will be small in comparison to the dimensions of the plate then we may assume a point load acting on plate in the overall picture (faraway)

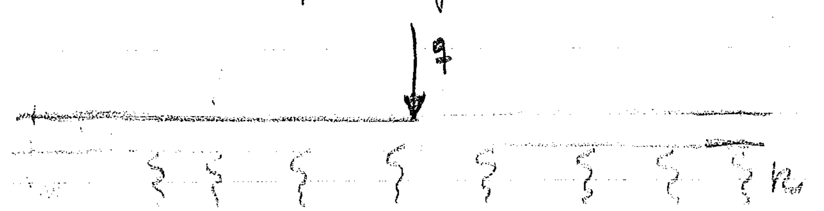
~~We can then replace this by a force $F = q\,x$ (line that it is acting on)~~

~~$F = q\pi(a+b)$~~ (~~Don't remember formula for circumference of ellipse~~)

$\delta \sim .4a \qquad \delta \sim .8b$ $\qquad a \sim 2.2\delta \qquad b \sim 1.2\delta$

When looking at each point under the load, & we want local effect, each point will look at its loading as if it is an ∞ line source. Since a & b are $> \delta$ the foundation is significant. Therefore we must solve the problem of the beam in cylindrical bending

The local curvature is the important parameter - not just $a+b$.



$w, \dfrac{dw}{dx} = 0$ as $x \to \infty$ $\qquad w, \dfrac{dw}{dx} = 0$ as $x \to \infty$

-7

~~from HW #8 we had found that $w = \frac{P\sqrt{2}}{4\lambda}$ under a point load~~

$\therefore$ from HW #8 we had found that $w = \dfrac{P\sqrt{2}}{4\lambda}$ where P = point load

point load on a beam analagous to line load on a plate. Length of ellipse is unimportant.

here let $P = q\pi(a+b)$ $\qquad \lambda = \sqrt[4]{\dfrac{K}{D}}$ $\qquad \therefore \; w = \dfrac{q\pi(a+b)\sqrt{2}}{4\,(K/D)^{1/4}} = \dfrac{q\pi(a+b)\,a\,2}{4 \cdot 16} = \dfrac{q\pi(a+b)a}{32}$

$(K/D)^{1/4} = \dfrac{(32^2 \cdot 16)^{1/4}}{a} = \dfrac{16}{a(4)^{1/4}}$

$\therefore \; w\Big|_{\text{under } q} = \dfrac{q\pi(a+b)a}{32}$ $\qquad$ The dimensions are wrong

-8

ME 241A
THEORY OF PLATES
Final Examination

Fall
1976

60% **PROBLEM 1:** A square plate whose sides have the length a is simply supported and loaded with constant pressure. At the center of the plate is a circular hole of radius b. Determine an approximation for the stress concentration due to the hole, i.e. the ratio of maximum bending stress with and without the hole, for the case of a small hole $b \ll a$.

20% **PROBLEM 2:** An infinite plate on an elastic foundation is subjected to a line load of intensity q on the segment $x = 0$, $-b < y < b$. Find the maximum deflection when the decay distance in comparison with b is (a) large and (b) small.

20% **PROBLEM 3:** What is the plate bending stiffness D for a sandwich plate of total thickness t made of a material with Young's modulus E, consisting of two homogeneous sheets of thickness $t/4$ bonded to a "honeycomb" core?

ME 241A Fall 1975

THEORY OF PLATES

Final

PROBLEM 1: A rectangular plate is simply-supported on the edges $x = 0$, a and clamped on the edges $y = 0$, b. The pressure is constant. Give an efficient formulation of the solution for the deflection in each of the following two situations:

(a) The lengths a, b are the same order of magnitude.

(b) The length b is much larger than a.

PROBLEM 2: A circular plate is clamped at the boundary $r = b$ and is loaded by the axisymmetric pressure distribution

$$p = p_o[1 - (r/b)^4] \qquad (p_o = \text{const.})$$

Find the deflection.

PROBLEM 3: The footing for a column is a circular plate with bending stiffness D, and the soil has the equivalent elastic foundation stiffness k. What plate diameter would you recommend?

PROBLEM 4: A simply supported rectangular plate has the constant membrane tension n and is loaded by a concentrated force P_1 at the point (x_1, y_1) and a force P_2 at (x_2, y_2). Give an efficient formulation for the solution when the tension is large.

PROBLEM 5: A simply-supported rectangular plate is given an impulsive loading at time $t = 0$ which causes a velocity of the plate given by

$$\frac{\partial w}{\partial t}(x, y, 0+) = A \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

while the initial displacement is zero

$$w(x, y, 0+) = 0$$

Determine the consequential response.

ME 241A (AA246)

Theory of Plates

Prof: Charles Steele

Office: 355 Durand

Hours: M W 2:15 - 4:00

Grader: Greg Brooks

Office: 390 Durand

Fall 1979

MWF 1:15

Room SK 191

I. Basic Relationships - Beam Bending

II. Rectangular Plate - Navier's Method (Double Fourier Series)

III. Rectangular Plate with various boundary conditions - Levy's Method (Single Fourier Series)

IV. Circular Plate

V. Potential Energy Methods

VI Membrane Forces - Stability

VII. Vibration

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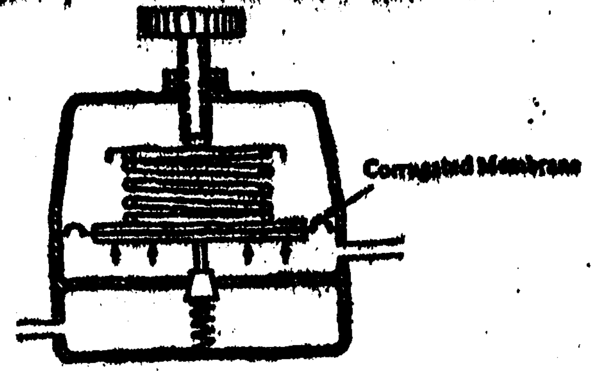
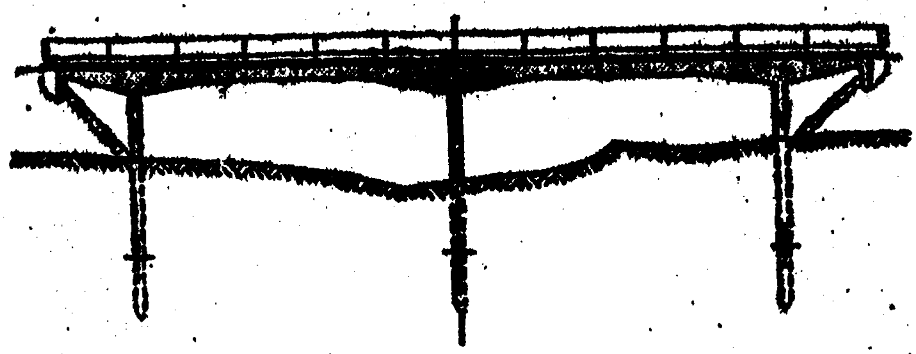
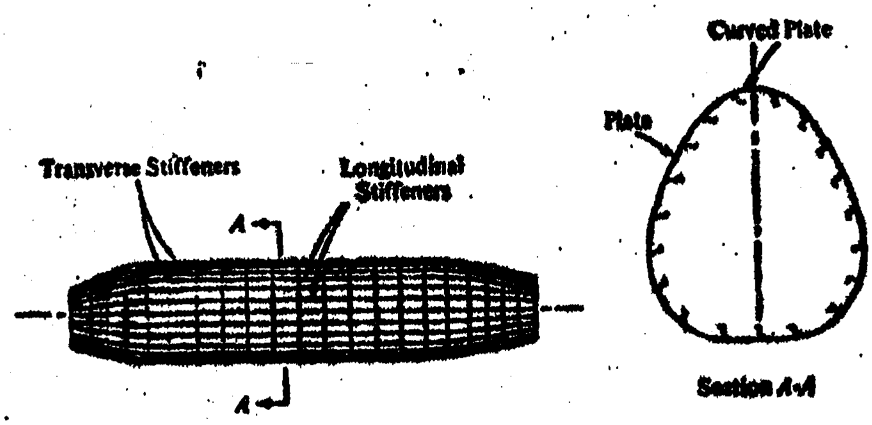


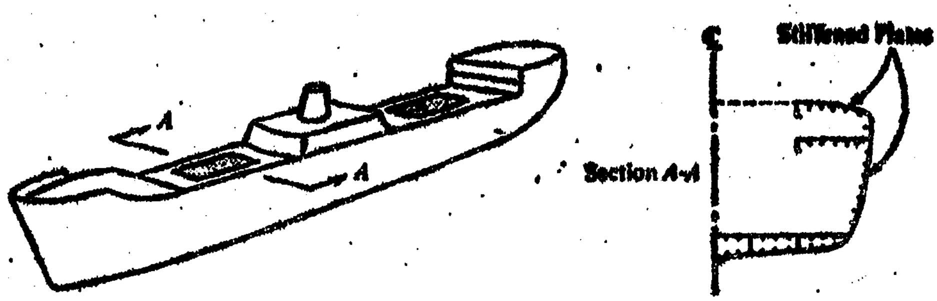
(a) Pressure Regulator



(b) Slab Bridge

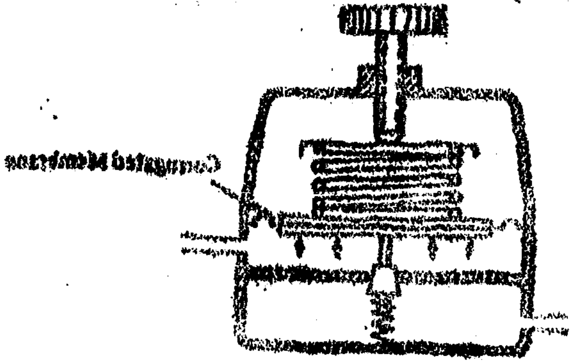


(c) Aerospace Structure

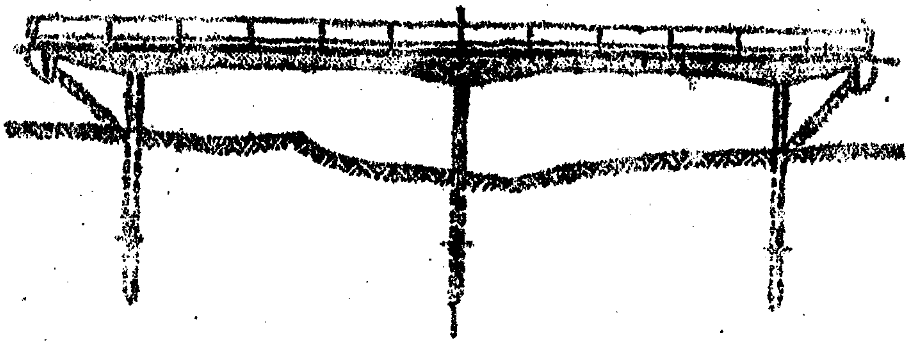


(d) Merchant Ship

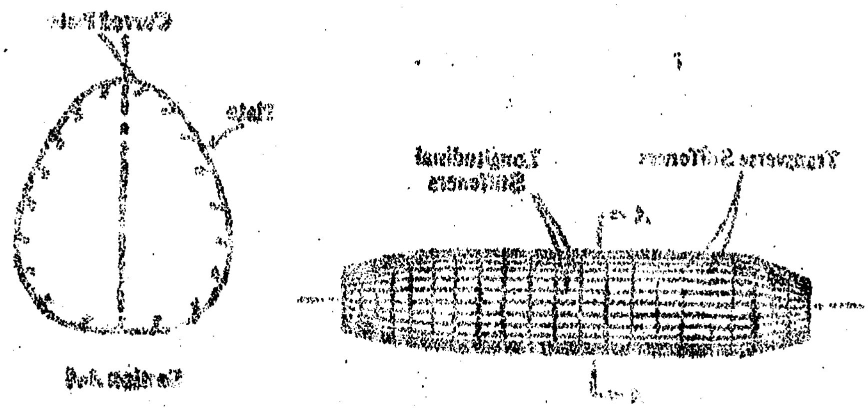
Fig. 1.2. Use of plates in various fields of engineering.



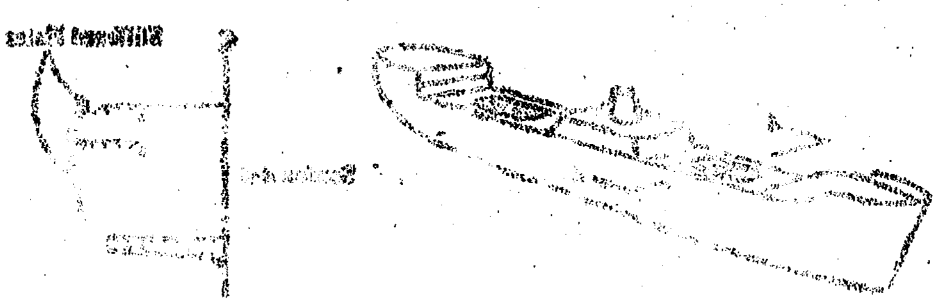
(a) Pressure Regulator



(b) Slab Bridge



(c) Aerospace Structure



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.2 *Differential Equation of Plates in Cartesian Coordinate System*

Coordinate System and Sign Convention. The shape of a plate is adequate- defined by describing the geometry of its *middle surface*, which is a surface at bisects the plate thickness h at each point (Fig. 1.2.1). The small deflec- on plate theory, generally attributed to Kirchhoff and Love [1][2],* is based n the following assumptions.

1. The material of the plate is elastic, homogeneous, and isotropic.
2. The plate is initially flat.
3. The thickness of the plate is small compared to its other dimensions. The smallest lateral dimension of the plate is at least ten times larger than its thickness.
4. The deflections are small compared to the plate thicknesses. A maximum deflection from one tenth to one fifth of the thickness is considered as the limit for small-deflection theory. This limitation can also be stated in terms of length; i.e., the maximum deflection is less than one fiftieth of the smaller span length.
5. The slopes of the deflected middle surface are small compared to unity.
6. The deformations are such that straight lines, initially normal to the middle surface, remain straight lines and normal to the middle surface (deformations due to transverse shear will be neglected).
7. The deflection of the plate is produced by displacement of points of the middle surface normal to its initial plane.
8. The stresses normal to the middle surface are of a negligible order of magnitude.

Many of these assumptions are familiar to the reader since they have their equivalent counterparts in elementary beam theory. Small- and large-scale

* Other contributors include Bernoulli, Navier, Poisson, Saint-Venant, and Lagrange.

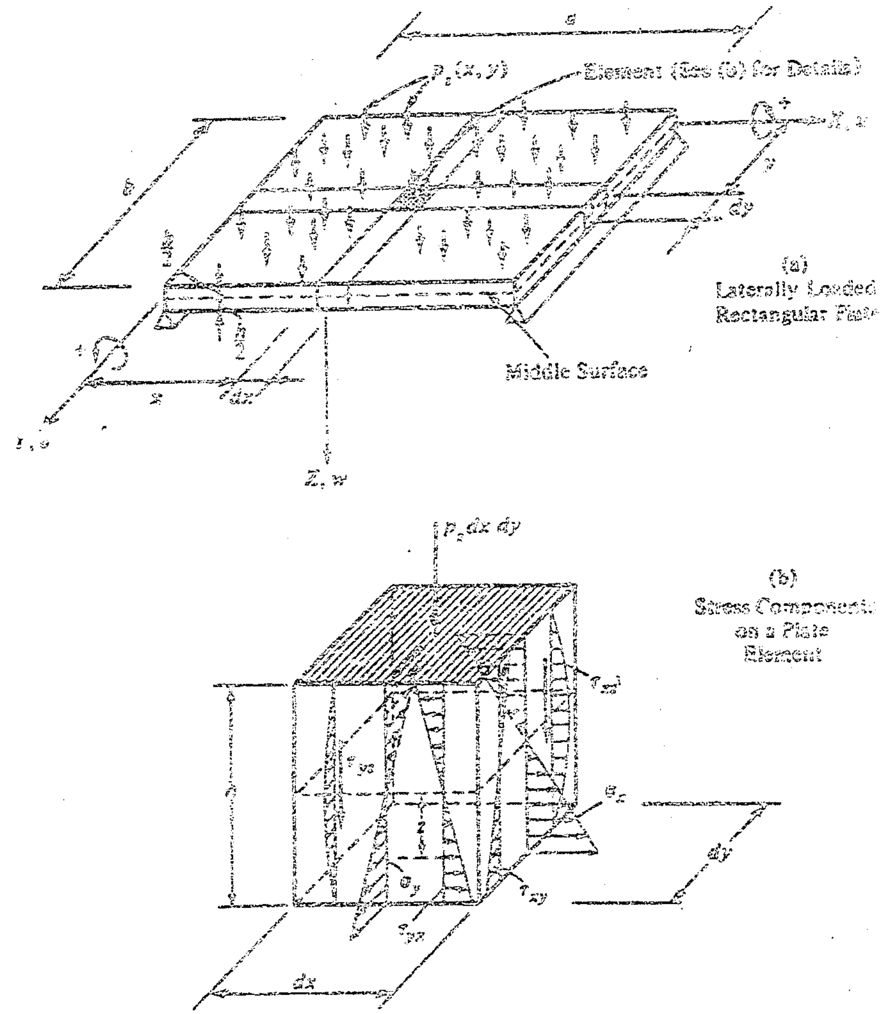


Fig. 1.2.1. Laterally loaded rectangular plate.

tests have proved the validity of these assumptions. In the case of plates with bending resistance, an additional simplifying assumption can be introduced:

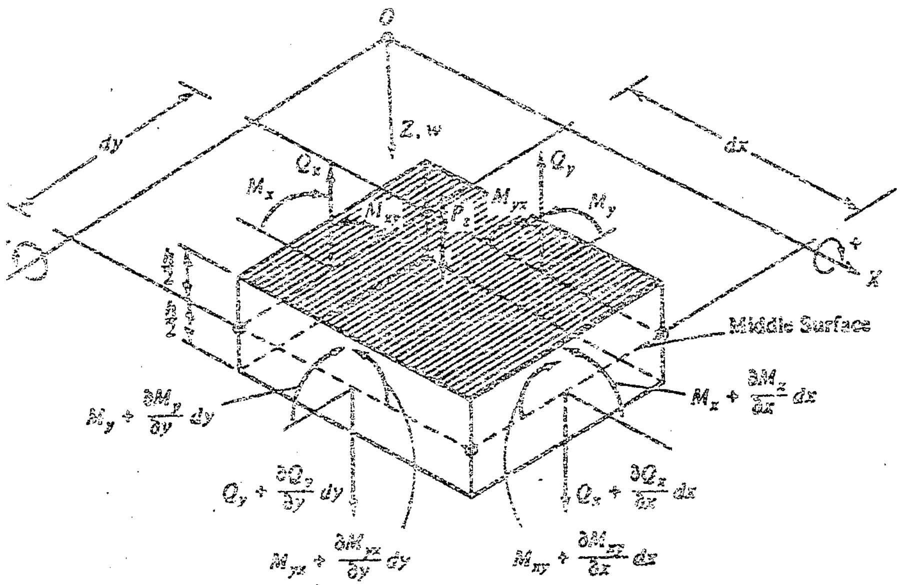
9. The strains in the middle surface produced by inplane forces can usually be neglected in comparison with strains due to bending (*inextensional plate theory*).

Unless otherwise stated, the validity of the *Kirchhoff-Love plate theory* is assumed throughout this book.

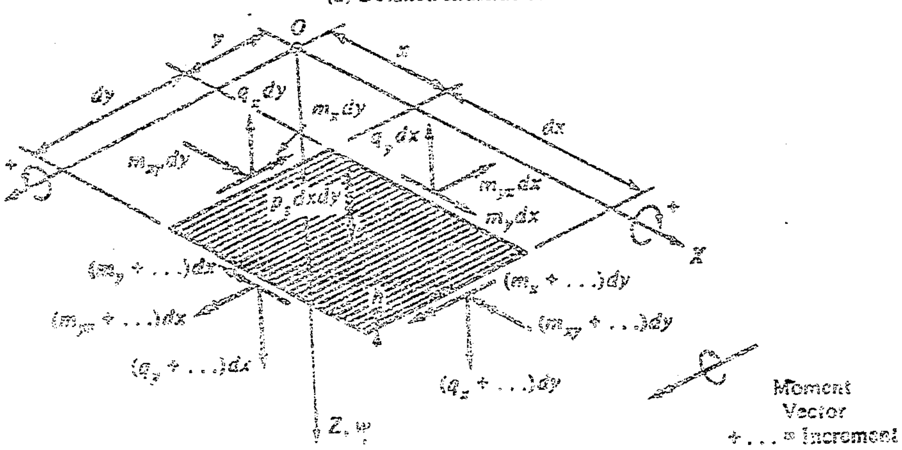
For rectangular plates, the use of Cartesian coordinate system is the most convenient (Fig. 1.2.1). The external and internal forces and the deflection components u, v, and w are considered positive when they point toward the

...sitive direction of the coordinate axes X, Y, and Z. In general engineering practice, positive moments produce tension in the fibers located at the bottom part of the structure. This sign convention is maintained for plates.

Considering an elemental parallelepiped cut out of the plate, as shown in Fig. 1.2.2, we assign positive internal forces and moments to the near faces. To satisfy the equilibrium of the element, negative internal forces and moments must act on its far sides. The first subscript of the internal forces indicates the



(a) Detailed Illustration



(b) Schematic Illustration

Fig. 1.2.2. External and internal forces on the element of the middle surface.

direction of the surface-normal pertinent to the section on which the force or the moment acts.

b. Equilibrium of the Plate Element. Assuming that the plate is subjected to lateral forces only, from the six fundamental equilibrium equations, the following three can be used:

$$\sum M_x \doteq 0, \qquad \sum M_y \doteq 0, \qquad \text{and} \qquad \sum P_z \doteq 0. \tag{1.2.1}$$

The behavior of the plate is in many respects analogous to that of a two-dimensional gridwork of beams. Thus the external load P_z is carried by Q_x and Q_y transverse shear forces and by M_x and M_y bending moments. The significant deviation from the two-dimensional gridwork action of beams is the presence of the twisting moments M_{xy} and M_{yx} (Fig. 1.2.2a). In the theory of plates it is customary to deal with internal forces and moments per unit length of the middle surface (Fig. 1.2.2b). To distinguish these internal forces from the above-mentioned resultants, the notations q_x, q_y, m_x, m_y, m_{xy}, and m_{yx} are introduced.

The procedure involved in setting up the differential equation of equilibrium is as follows:

1. Select a convenient coordinate system and draw a sketch of a plate element (Fig. 1.2.2).
2. Show all external and internal forces acting on the element.
3. Assign positive internal forces with increments ($q_x + \cdots$, $q_y + \cdots$ etc.) to the near sides.
4. Assign negative internal forces to the far sides.
5. Express the increments by a truncated Taylor's series* in the form

$$q_x + dq_x = q_x + \frac{\partial q_x}{\partial x} dx, \quad m_y + dm_y = m_y + \frac{\partial m_y}{\partial y} dy, \quad \text{etc.} \tag{1.2.2}$$

6. Express the equilibrium of the internal and external forces acting on the element.

Let us express, for instance, that the sum of the moments of all forces around the Y axis is zero (Fig. 1.2.2b). This gives

$$\left(m_x + \frac{\partial m_x}{\partial x} dx\right) dy - m_x\, dy + \left(m_{yx} + \frac{\partial m_{yx}}{\partial y} dy\right) dx - m_{yx}\, dx$$
$$- \left(q_x + \frac{\partial q_x}{\partial x} dx\right) dy \frac{dx}{2} - q_x\, dy \frac{dx}{2} = 0. \tag{1.2.3}$$

After simplification, we neglect the term containing $\frac{1}{2}(\partial q_x/\partial x)(dx)^2\, dy$, since it is a small quantity of higher order. Thus Eq. (1.2.3) becomes

$$\frac{\partial m_x}{\partial x} dx\, dy + \frac{\partial m_{yx}}{\partial y} dy\, dx - q_x\, dx\, dy = 0, \tag{1.2.4}$$

and, after division by $dx\, dy$, we obtain

* $f(a + h) = f(a) + (h/1!)f'(a) + (h^2/2!)f''(a) + \cdots + (h^n/n!)f^{(n)}(a) + \cdots$.

$$\frac{\partial m_x}{\partial x} + \frac{\partial m_{yx}}{\partial y} = q_x. \tag{1.2.5}$$

In a similar manner the sum of the moments around the X axis gives

$$\frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} = q_y. \tag{1.2.6}$$

The summation of all forces in the Z direction yields the third equilibrium equation:

$$\frac{\partial q_x}{\partial x}\,dx\,dy + \frac{\partial q_y}{\partial y}\,dx\,dy + p_z\,dx\,dy = 0, \tag{1.2.7}$$

which, after division by $dx\,dy$, becomes

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = -p_z. \tag{1.2.8}$$

Substituting Eqs. (1.2.5) and (1.2.6) into (1.2.8) and observing that $m_{xy} = m_{yx}$, we obtain

$$\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x\,\partial y} + \frac{\partial^2 m_y}{\partial y^2} = -p_z(x, y). \tag{1.2.9}$$

The bending and twisting moments in Eq. (1.2.9) depend on the strains, and the strains are functions of the displacement components (u, v, w), as discussed in Sec. 1.1. Thus, in the next steps, relations between the internal moments and displacement components are sought.

c. Relation Between Stress, Strain, and Displacements. The assumption that the material is elastic permits the use of the two-dimensional Hooke's law,[5]

$$\sigma_x = E\epsilon_x + \nu\sigma_y \tag{1.2.10a}$$

and

$$\sigma_y = E\epsilon_y + \nu\sigma_x \tag{1.2.10b}$$

which relates stress and strain in a plate element. Substituting (1.2.10b) into (1.2.10a), we obtain

$$\sigma_x = \frac{E}{1 - \nu^2}(\epsilon_x + \nu\epsilon_y). \tag{1.2.11}$$

In a similar manner

$$\sigma_y = \frac{E}{1 - \nu^2}(\epsilon_y + \nu\epsilon_x) \tag{1.2.12}$$

can be derived.

The torsional moments m_{xy} and m_{yx} produce in-plane shear stresses τ_{xy} and τ_{yx} (Fig. 1.2.3), which are again related to the shear strain γ by the pertinent Hookean relationship (1.1.21), giving

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1 + \nu)}\gamma_{xy} = \tau_{yx}. \tag{1.2.13}$$

[5] Obtained from Eq. (1.1.19) with $\sigma_z = 0$.

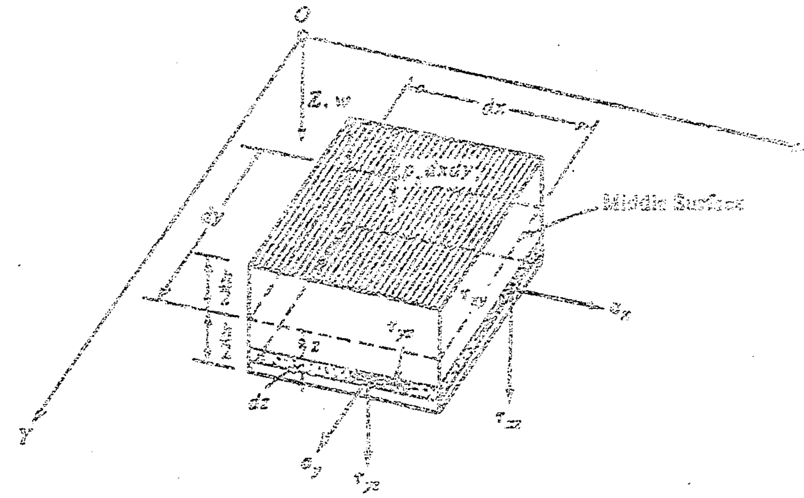


Fig. 1.2.3. Stresses on a plate element.

Next, we consider the geometry of the deflected plate to express the strain in terms of the displacement coefficients. Taking a section at a constant y, as shown in Fig. 1.2.4, we compare the section before and after deflection. Using assumptions 5 and 6, introduced earlier in this section, we express the angle of rotation of lines I-I and II-II by

$$\vartheta = -\frac{\partial w}{\partial x} \qquad \text{and} \qquad \vartheta + \cdots = \vartheta + \frac{\partial \vartheta}{\partial x}\,dx, \tag{1.2.14}$$

respectively. After the deformation the length $\overline{AB}$ of a fiber, located at z distance from the middle surface, becomes $\overline{A'B'}$ (Fig. 1.2.4). Using the definition of strain given by Eq. (1.1.13), we can write

$$\epsilon_x = \frac{\Delta dx}{dx} = \frac{\overline{A'B'} - \overline{AB}}{\overline{AB}} = \frac{[dx + z(\partial\vartheta/\partial x)\,dx] - dx}{dx} = z\frac{\partial \vartheta}{\partial x}. \tag{1.2.15}$$

Substituting into this expression the first of the equations (1.2.14), we obtain

$$\epsilon_x = -z\frac{\partial^2 w}{\partial x^2}. \tag{1.2.16}$$

A similar reasoning yields ϵ_y, the strain due to normal stresses in the Y direction; thus

$$\epsilon_y = -z\frac{\partial^2 w}{\partial y^2}. \tag{1.2.17}$$

Let us now determine the angular distortion[6] $\gamma_{xy} = \gamma' + \gamma''$ by comparing an $ABCD$ rectangular parallelogram (Fig. 1.2.5), located at a constant distance z from the middle surface, by its deformed shape $A'B'C'D'$ on the deflected

[6] See Sec. 1.1 for the definition of shear strain.

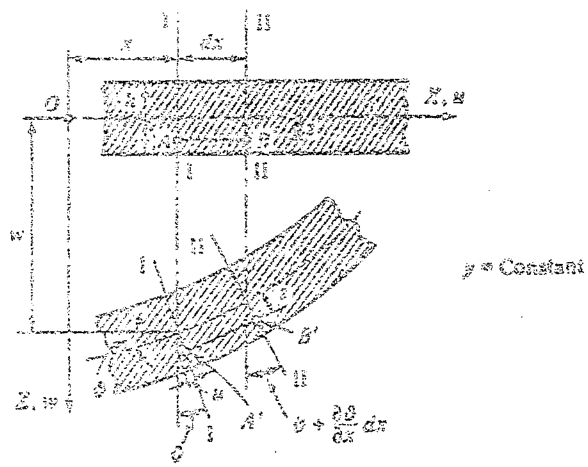


Fig. 1.2.4. Section before and after deflection.

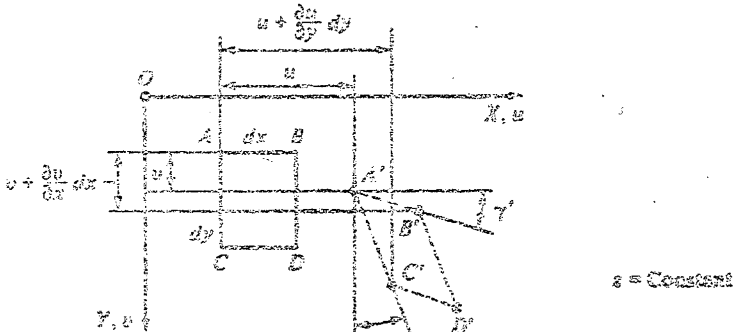


Fig. 1.2.5. Angular distortion.

plate surface. From the two small triangles in Fig. 1.2.5 and from Eq. (1.1.14) it is evident that

$$\gamma' = \frac{\partial v}{\partial x} \qquad \text{and} \qquad \gamma'' = \frac{\partial u}{\partial y}; \tag{1.2.18}$$

but from Fig. 1.2.4

$$u = z\vartheta = -z\frac{\partial w}{\partial x};$$

similarly,

$$v = -z\frac{\partial w}{\partial y}, \tag{1.2.19}$$

consequently,

$$\gamma_{xy} = \gamma' + \gamma'' = -2z\frac{\partial^2 w}{\partial x\,\partial y}. \tag{1.2.20}$$

The curvature changes of the deflected middle surface are defined by

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \qquad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \qquad \text{and} \qquad \chi = -\frac{\partial^2 w}{\partial x\,\partial y}, \tag{1.2.21}$$

where χ represents the *warping* of the plate.

d. Internal Forces Expressed in Terms of w. The stress components σ_x and σ_y (Fig. 1.2.3), produce bending moments in the plate element in a manner similar to that in elementary beam theory. Thus, by integration of the normal stress components, the bending moments, acting on the plate element, are obtained:

$$m_x = \int_{-(h/2)}^{+(h/2)} \sigma_x z\,dz \qquad \text{and} \qquad m_y = \int_{-(h/2)}^{+(h/2)} \sigma_y z\,dz. \tag{1.2.22}$$

Similarly, the twisting moments produced by the shear stresses $\tau = \tau_{xy} = \tau_{yx}$ can be calculated from

$$m_{xy} = \int_{-(h/2)}^{+(h/2)} \tau_{xy} z\,dz \qquad \text{and} \qquad m_{yx} = \int_{-(h/2)}^{+(h/2)} \tau_{yx} z\,dz; \tag{1.2.23}$$

but $\tau_{xy} = \tau_{yx} = \tau$, and therefore $m_{xy} = m_{yx}$.

If we substitute Eqs. (1.2.16) and (1.2.17) into (1.2.11) and (1.2.12), the normal stresses σ_x and σ_y are expressed in terms of the lateral deflection w. Thus, we can write

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{1.2.24}$$

and

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right). \tag{1.2.25}$$

Integration of Eqs. (1.2.22), after substitution of the above expressions for σ_x and σ_y, gives

$$\begin{aligned} m_x &= -\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \\ &= -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = D(\kappa_x + \nu\kappa_y) \end{aligned} \tag{1.2.26}$$

and

$$m_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = D(\kappa_y + \nu\kappa_x), \tag{1.2.27}$$

where

$$\boxed{D = \frac{Eh^3}{12(1-\nu^2)}} \tag{1.2.28}$$

the expression of twisting moments in terms of the angle of distortion χ becomes:

$$m_{xy} = m_{yx} = \int_{-(h/2)}^{+(h/2)} \tau z \, dz = -2G \int_{-(h/2)}^{+(h/2)} \frac{\partial^2 w}{\partial x \, \partial y} z^2 \, dz = -(1-\nu) D \frac{\partial^2 w}{\partial x \, \partial y} = D(1-\nu)\chi. \tag{1.2.29}$$

The substitution of Eqs. (1.2.26), (1.2.27), and (1.2.29) into Eq. (1.2.9) yields the *governing differential equation*[*] of the plate subjected to lateral loads:

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \, \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_z(x, y)}{D}} \tag{1.2.30}$$

or using the two-dimensional Laplacian operator [Eq. (1.1.34)], we can write:

$$D \nabla^2 \nabla^2 w = p_z. \tag{1.2.31}$$

This equation is a fourth-order, nonhomogeneous, partial differential equation of the elliptic type with constant coefficients, often called a *nonhomogeneous biharmonic equation*. Equation (1.2.30) is *linear* since the derivatives of w do not have exponents higher than one.

We would like to express also the transverse shear forces in terms of the lateral deflections. The substitution of Eqs. (1.2.26), (1.2.27), and (1.2.29) into Eqs. (1.2.5) and (1.2.6) gives

$$q_x = \frac{\partial m_x}{\partial x} + \frac{\partial m_{yx}}{\partial y} = -D \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D \frac{\partial}{\partial x} \nabla^2 w \tag{1.2.32a}$$

$$q_y = \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} = -D \frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D \frac{\partial}{\partial y} \nabla^2 w. \tag{1.2.32b}$$

The plate problem is considered solved if a suitable expression for the deflected plate surface $w(x, y)$ is found which simultaneously satisfies the differential equation of equilibrium (1.2.30) and the boundary conditions. Consequently, it can be stated that the solution of plate problems is a specific case of a boundary value problem of mathematical physics.

Finally, it should be mentioned that occasionally[†] it might be of advantage to introduce the so-called *moment-sum* in a form as follows:

$$\mathfrak{M} = \frac{1}{1+\nu}(m_x + m_y) = -D \nabla^2 w. \tag{1.2.33}$$

The right-hand side of this equation is obtained from (1.2.26) and (1.2.27). The introduction of this moment-sum permits us to split the governing fourth-order differential equation of the plate into two second-order differential equations:

$$\nabla^2 \mathfrak{M} = -p_z \tag{1.2.34a}$$

and

$$\nabla^2 w = -\frac{\mathfrak{M}}{D}. \tag{1.2.34b}$$

Provided that the boundary conditions for $\mathfrak{M}$ are known, we may solve the first differential equation. Then utilizing $\mathfrak{M}$, the lateral deflection $w(x, y)$ can be determined from Eq. (1.2.34b). It is interesting to note that Eqs. (1.2.34a) and (1.2.34b) have a form similar to that of the differential equation of the membrane (Sec. 1.19).

Summary. It has been shown that in setting up the differential equation of an elastic plate the following basic steps are required:

1. The expression of equilibrium of external and internal forces acting on a plate element.
2. Linking strains and stresses by Hooke's law.
3. The use of certain geometrical relationships, obtainable from the shape of the deflected surface, which enables us to express strains in terms of displacement components.
4. The expression of internal forces in terms of stresses and strains and finally in terms of displacement components.

Plate problems, such as the previously discussed two- and three-dimensional stress problems of elasticity, are *internally statically* indeterminate; i.e., the three equations of equilibrium [(1.2.5), (1.2.6), and (1.2.8)] contain five unknowns, i.e. three moments and two shear forces. To be able to obtain a solution, additional equations of elasticity have been utilized. In this way the governing differential equation of laterally loaded plates [Eq. (1.2.30)], which contains only one unknown (w), is derived.

1.3 Boundary Conditions of the Bending Theory

An exact solution of the governing plate equation (1.2.30) must simultaneously satisfy the differential equation and the boundary conditions of any given plate problem. Since Eq. (1.2.30) is a fourth-order differential equation, two boundary conditions, either for the displacements or for the internal forces, are required at each boundary. In the bending theory of plates, three internal force components are to be considered: bending moment, torsional moment, and transverse shear. Similarly, the displacement components to be used in formulating the boundary conditions are lateral deflections and slope. Boundary conditions of plates in bending can be generally classified as one of the following.

[*] This equation was obtained in 1811 by Lagrange; thus it is sometimes called Lagrange's equation.

[†] A simply supported boundary condition is one of these special cases.

vided by the magnitude of displacements (translation and rotation) can be used to formulate the boundary conditions in mathematical form. At fixed edges (Fig. 1.3.1a), for instance, the deflection and the slope of the deflected plate surface are zero. Thus we can write

$$(w)_x = 0, \qquad \left(\frac{\partial w}{\partial x}\right)_x = 0 \qquad (x = 0 \text{ or } x = a)$$

and $\qquad\qquad\qquad\qquad\qquad\qquad$ (1.3.1)

$$(w)_y = 0, \qquad \left(\frac{\partial w}{\partial y}\right)_y = 0 \qquad (y = 0 \text{ or } y = b).$$

Such boundary conditions are called *geometrical.*

b. Statical Boundary Conditions (Free Edges). For *statical* boundary conditions the edge forces provide the required mathematical expressions. At an

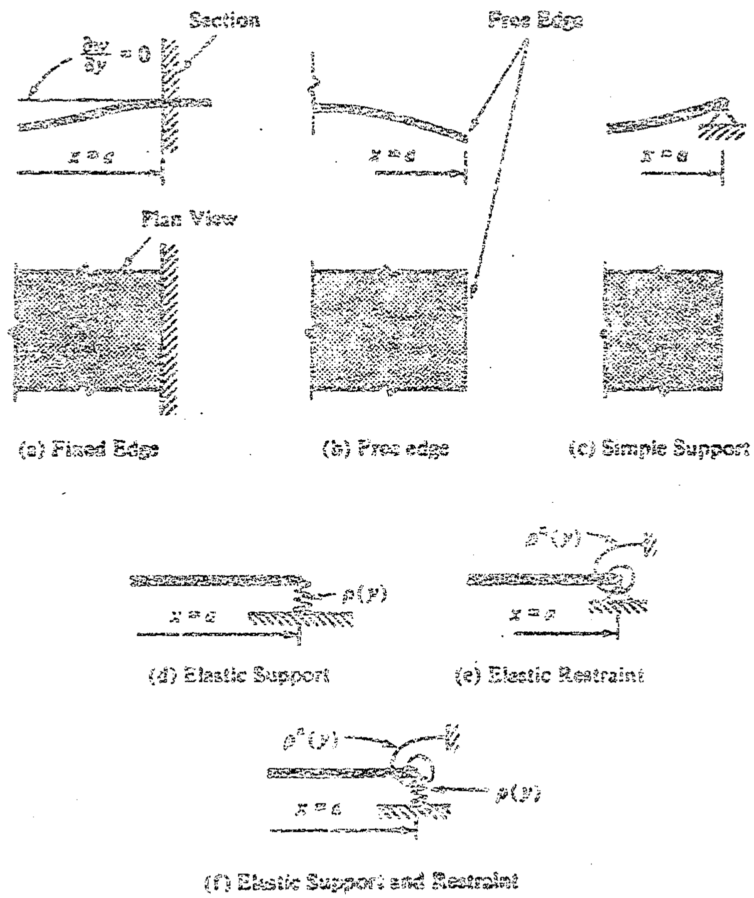


Fig. 1.3.1. Various boundary conditions.

moment and the transverse shear force (v) are zero, giving

$$(m_x)_x = (v_x)_x = 0 \qquad \text{at } x = 0, a,$$

or $\qquad\qquad\qquad\qquad\qquad\qquad$ (1.3.2)

$$(m_y)_y = (v_y)_y = 0 \qquad \text{at } y = 0, b.$$

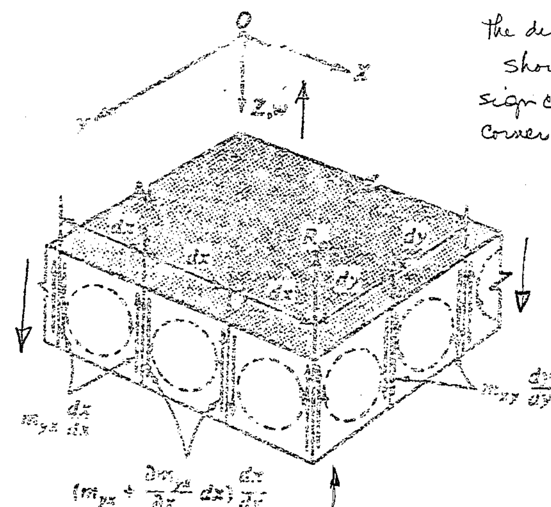
The shearing force at the edge of the plate consists of two terms, i.e., transverse shear and the effect of the torsional moment. Considering plate edges having normals in the X and Y directions, respectively, the vertical edge forces per unit length, can be written as

$$v_x = q_x + \frac{\partial m_{xy}}{\partial y} = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right]$$
$$v_y = q_y + \frac{\partial m_{yx}}{\partial x} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2\,\partial y}\right], \qquad (1.3.3)^a$$

where q_x and q_y are the lateral shear forces [Eq. (1.2.32)]. The second terms, $\partial m_{xy}/\partial y$ and $\partial m_{yx}/\partial x$, in Eq. (1.3.3) represent additional shearing forces at the edges, produced by the torsional moments $m_{xy} = m_{yx}$. Replacing the torsional moments by statically equivalent couples $m_{xy}\,dy/dy$ and $m_{yx}\,dx/dx$, respectively (Fig. 1.3.2), these forces cancel out at the adjoining elements, except for their incremental parts:

$$\frac{\partial m_{xy}}{\partial y}\,dy \qquad \text{and} \qquad \frac{\partial m_{yx}}{\partial x}\,dx.$$

Dividing these expressions by dy and dx, respectively, the additional shearing



the directions shown are positive sign convention of corner forces

Fig. 1.3.2. Edge effect of torsional moments.

[a] To obtain the reactive edge forces at $x = 0$ and $y = 0$, respectively, Eq. (1.3.3) should be used with opposite sign.

orces per unit length are obtained:

$$\bar{q}_x^* = \frac{\partial m_{xy}}{\partial y} \quad \text{and} \quad \bar{q}_y^* = \frac{\partial m_{yx}}{\partial x}. \tag{1.3.4}$$

hese forces are called *Kirchhoff's supplementary forces* (*Kirchhoffsche Ersatz-räfte*). Replacing the torsional moments by these equivalent shear forces, Kirchhoff reduced the number of internal forces to be considered from three o two. Thus from Eqs. (1.2.26), (1.2.27), (1.3.2) and (1.3.3) the boundary con-itions at the free edges are

$$\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)_x = 0, \qquad \left[\frac{\partial^3 w}{\partial x^3} + (2 - \nu)\frac{\partial^3 w}{\partial x\, \partial y^2}\right]_x = 0 \tag{1.3.5}$$

nd

$$\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)_y = 0, \qquad \left[\frac{\partial^3 w}{\partial y^3} + (2 - \nu)\frac{\partial^3 w}{\partial x^2\, \partial y}\right]_y = 0 \tag{1.3.6}$$

. *Mixed Boundary Conditions.* A simply supported edge (Fig. 1.3.1c) repre-ents *mixed* boundary conditions. Since the deflection and the bending moment long the boundary are zero, formulation of this type of boundary condition nvolves statements concerning displacements and forces. Thus,

$$(w)_x = 0, \qquad (m_x)_x = \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)_x = 0$$

nd (1.3.7)

$$(w)_y = 0, \qquad (m_y)_y = \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)_y = 0.$$

At the corners of rectangular plates the above-discussed action of torsional noments add up instead of canceling (because $m_{xy} = m_{yx}$), producing an addi-ional corner force

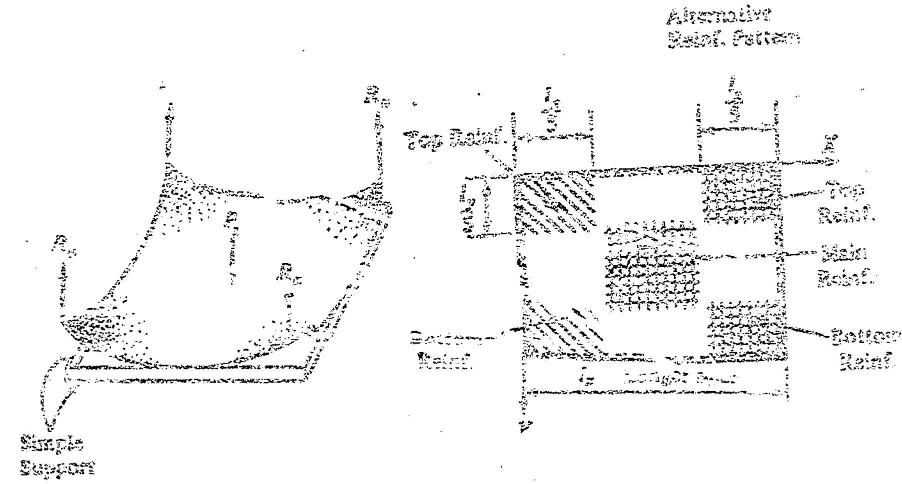
$$R_0 = 2m_{xy} = -2D(1 - \nu)\frac{\partial^2 w}{\partial x\, \partial y}. \tag{1.3.8}$$

f no anchorage is provided, these forces can lift up the corners, as illustrated Fig. 1.3.3a. Since this condition is generally undesirably, it should be avoided y holding down the edges of simply supported plates. For reinforced con-rete slabs, when lifting up the corners is not prevented, special corner rein-orcing is required to eliminate local failures (Fig. 1.3.3b).

When two adjacent edges are fixed, the additional corner force is zero $R_0 = 0$), since along these edges no torsional moment exists. The case at the ntersection of two free edges is similar.

. *Elastic Support and Restraint.* A special case of the *mixed* boundary con-ition occurs when the plate is resting on yielding support (Fig. 1.3.1d), such as rovided by an edge beam without torsional rigidity. Assuming that the sup-ort is elastic and that its translational stiffness† or spring constant is ρ, the oundary conditions, for edges having normals in the X direction, for instance,

† See Secs. 2.5 and 2.8 for the definition of stiffness coefficients.



(a)
No Edge Anchorage

(b)
Required Corner Reinforcings

Fig. 1.3.3. Up-lift of corners.

are

$$(w)_{x=0,a} = \pm\left[\frac{\bar{v}_x}{\rho(y)}\right]_{x=0,a} = \mp \frac{D}{\rho(y)}\left[\frac{\partial^3 w}{\partial x^3} + (2 - \nu)\frac{\partial^3 w}{\partial x\, \partial y^2}\right]_{x=0,a}$$

and (1.3.9)

$$(m_x)_x = \left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]_{x=0,a} = 0.$$

For continuous beam support, the first equation of (1.3.9) becomes

$$\left[EI_b \frac{\partial^4 w}{\partial y^4}\right]_{x=0,a} = \mp D\left[\frac{\partial^3 w}{\partial x^3} + (2 - \nu)\frac{\partial^3 w}{\partial x\, \partial y^2}\right]_{x=0,a}, \tag{1.3.9a}$$

where I_b represents the moment of inertia of the edge beam. The force transmit-ted from the plate to the beam is $-v_x$.

If the edge beam has torsional rigidity ρ^φ, it also provides partial restraint (fixity), as shown in Fig. 1.3.1f. Since the edge beam and the plate are built monolithically, they must have the same displacements. Consequently, the geo-metrical boundary conditions of elastic support and restraint along the edges at $x = 0$ and $x = a$ are

$$(w)_{x=0,a} = \mp \frac{D}{\rho(y)}\left[\frac{\partial^3 w}{\partial x^3} + (2 - \nu)\frac{\partial^3 w}{\partial x\, \partial y^2}\right]_{x=0,a}$$

and (1.3.10)

$$\left(\frac{\partial w}{\partial x}\right)_{x=0,a} = \mp \frac{D}{\rho^\varphi(y)}\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]_{x=0,a}.$$

Similar expressions can be written for edges at $y = 0$ and $y = b$.

The analytical formulation of the various boundary conditions, illustrated in Fig. 1.3.1, is given in Table 1.3.1.

Table [illegible] Summary of Boundary Conditions

Type of Support at $x = a$	Mathematical Expressions
Simple support (Fig. 1.3.1a)	$(w)_{x=a} = 0; \quad (m_x)_{x=a} = \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)_{x=a} = 0$
Fixed edge (Fig. 1.3.1c)	$(w)_{x=a} = 0; \quad \left(\frac{\partial w}{\partial x}\right)_{x=a} = 0$
Free edge (Fig. 1.3.1b)	$(m_x)_{x=a} = \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)_{x=a} = 0$ $(v_x)_{x=a} = \left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]_{x=a} = 0$
Partially fixed edge (Fig. 1.3.1e)	$(w)_{x=a} = 0$ $\left(\frac{\partial w}{\partial x}\right)_{x=a} = (\rho^{\phi})^{-1} D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)_{x=a}$
Elastic support (Fig. 1.3.1d)	$(m_x)_{x=a} = \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)_{x=a} = 0$ $(w)_{x=a} = \rho^{-1} D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]_{x=a}$
Elastic support and restraint (Fig. 1.3.1f)	$(w)_{x=a} = \rho^{-1} D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]_{x=a}$ $\left(\frac{\partial w}{\partial x}\right)_{x=a} = (\rho^{\phi})^{-1} D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)_{x=a}$

Note: ρ = translational stiffness of the support; ρ^{ϕ} = rotational stiffness of the support.

Summary. Boundary conditions of plates subjected to bending require the specification of *any two* of the following quantities:

$$w, \qquad \frac{\partial w}{\partial n}, \qquad m_n, \qquad v_n,$$

where n represents the direction of the normal vector of the boundary.

The analytical treatment of certain boundary conditions may create serious mathematical difficulties. In this case, numerical techniques, such as finite difference and finite element methods, should be applied.

1.4 Rigorous Solution of the Governing Differential Equation*

Mathematically, the differential equation of plates (1.2.30) is classified as a linear partial differential equation of the fourth order having constant coefficients [1.4.1][1.4.2]. Its homogeneous form,

$$\nabla^2\nabla^2 w = 0, \tag{1.4.1}$$

is called the *biharmonic equation*.

* In a Cartesian coordinate system.

In general, there are four types of mathematically rigorous solutions possible for plate problems:

1. Closed-form solution.
2. Solution of the biharmonic equation upon which a *particular solution* of the governing differential equation of plate (1.2.30) is superimposed.
3. Double trigonometric series solution.
4. Single series solution.

Since the boundaries of these categories are not rigid, some overlap is possible.

The rigorous solution of Eq. (1.2.30) must satisfy the boundary conditions characterizing each problem and the governing differential equation of the plate. Consequently, the rigorous solution of plate problems is essentially a *boundary value* problem of mathematical physics. Since the fulfillment of the boundary conditions usually presents considerable mathematical difficulties, in general, rigorous solutions of plate problems are rare. In the few cases which lend themselves to exact analysis, the linearity of Eq. (1.2.30) can be used to permit the linear combination of solutions in the form of superposition. Thus, the most general form of the rigorous solution of the governing differential equation can be written as

$$w(x, y) = w_H(x, y) + w_P(x, y), \tag{1.4.3}$$

where w_H represents the solution of the homogeneous equation (1.4.1), and w_P is a particular solution of the nonhomogeneous differential equation of the plate (1.2.30). There are a few cases, however, when the solution can be obtained directly, without employing the above-mentioned superposition principle.

Certain boundary conditions permit the use of special solutions, such as the Navier solution, described in Sec. 1.5. In the Navier solution $w_H = 0$; thus

$$w(x, y) = w_P(x, y). \tag{1.4.2}$$

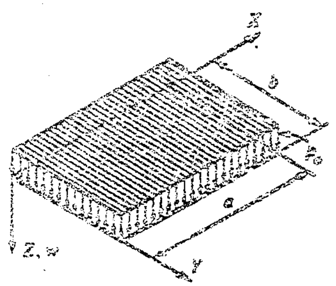
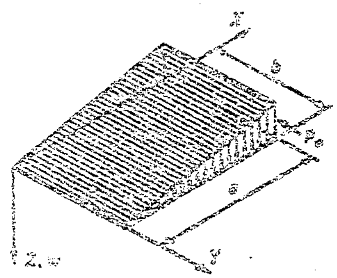
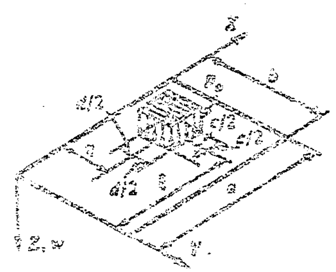
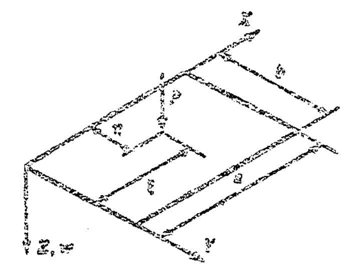
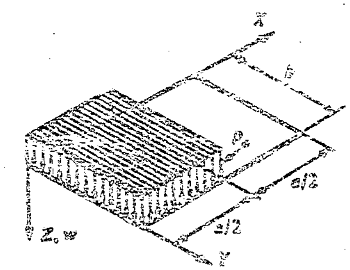
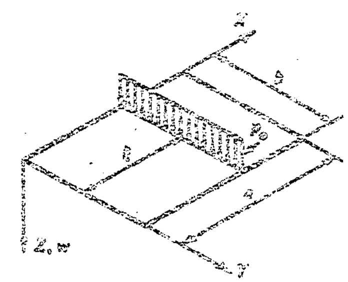
Furthermore, if the boundary conditions for the edge moments are known, we may use a two-step solution involving Eq. (1.2.34).

a. Solution of the Homogeneous Equation. The physical interpretations of the solution of the biharmonic equation ($\nabla^4 w = 0$) is to obtain the deflection of the plate $w_H(x, y)$ when only edge forces are acting. Consequently, the solution of the homogeneous equation fulfills the prescribed boundary conditions and maintains the equilibrium with the external boundary forces. The fundamental difficulty of the rigorous solution is the proper choice of functions $w_H(x, y)$ for a given problem.

The biharmonic equation (1.4.1) permits the use of the solutions of the Laplace equation $\nabla^2 w = 0$, which are

$$x, \qquad xy, \qquad \cos \alpha x, \qquad \cosh \alpha y, \qquad x^2 - y^2, \qquad \text{and} \qquad x^3 - 3xy^2, \tag{1.4.4}$$

where α represents an arbitrary constant. In Eq. (1.4.4) we may interchange x and y and replace cos by sin and cosh by sinh, respectively. If $w_1(x, y)$ and

No.	LOAD $p_z(x,y)=\sum_m \sum_n P_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$	EXPANSION COEFFICIENTS P_{mn} [Determined from (Eq. (1.5.34)]	NOTES
1		$P_{mn}=\frac{16p_0}{\pi^2 mn}$ $(m, n = 1, 3, 5, \ldots)$	Eq. (1.5.35)
2		$P_{mn}=-\frac{8p_0 \cos m\pi}{\pi^2 mn}$ $(m, n = 1, 3, 5, \ldots)$	Eq. (1.6.19)
3		$P_{mn}=\frac{16p_0}{\pi^2 mn}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b} \times \sin\frac{m\pi c}{2a}\sin\frac{n\pi d}{2b}$ $(m, n = 1, 2, 3, \ldots)$	Eq. (1.6.13)
4		$P_{mn}=\frac{4P}{ab}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$ $(m, n = 1, 2, 3, \ldots)$	Eq. (1.5.22)
5		$P_{mn}=\frac{8p_0}{\pi^2 mn}$ for $m, n = 1, 3, 5, \ldots$ $P_{mn}=\frac{16p_0}{\pi^2 mn}$ for $\begin{cases} m = 2, 6, 10, \ldots \\ n = 1, 3, 5, \ldots \end{cases}$	
6		$P_{mn}=\frac{4p_0}{bn\pi}\sin\frac{m\pi\xi}{a}$ $(m, n = 1, 2, 3, \ldots)$ $n = 1, 3, 5, \ldots$	

ILLUSTRATIVE EXAMPLE* (Double Trigonometric Series Expansion)

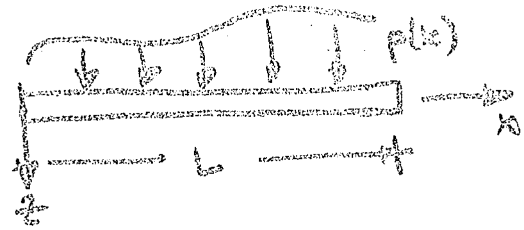
Let us expand into a function which is constant over the whole area double Fourier series, bounded by $x = 0$, $x = a$ and $y = 0$, $y = b$ in a rectangular Cartesian coordinate system.

Thus the expression of the function is $f(x, y) = f_0$; then from Eq. (1.5.34) the coefficients of expansion can be calculated:

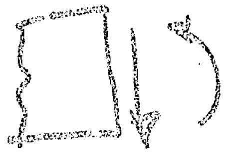
$$P_{mn}=\frac{4f_0}{ab}\int_0^a \sin\frac{m\pi x}{a}\,dx\int_0^b \sin\frac{n\pi y}{b}\,dy$$
$$=\frac{4f_0}{ab}\,\frac{a}{m\pi}\left[-\cos\frac{m\pi x}{a}\right]_0^a\cdot\frac{b}{n\pi}\left[-\cos\frac{n\pi y}{b}\right]_0^b$$
$$=\begin{cases}\frac{16f_0}{\pi^2 mn} & \text{for odd } m, n \\ 0, & \text{otherwise.}\end{cases} \qquad (1.5.37)$$

* For further examples, see Secs. 1.6 and 1.8.

Review - Engineering Beam Theory



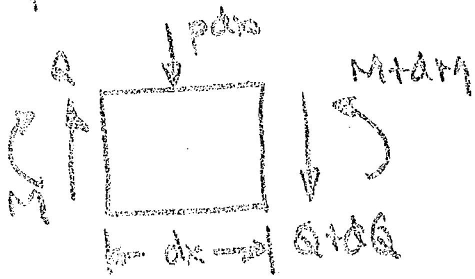
Resultant of ~~stresses~~ tractions acting on "x = constant" cross-section



M = bending ~~moment~~ couple

Q = shearing force

Equilibrium:



$\Sigma F_z = 0 \Rightarrow \frac{dQ}{dx} = -p(x)$

$\Sigma M_y = 0 \Rightarrow \frac{dM}{dx} = Q$ since these are first order only

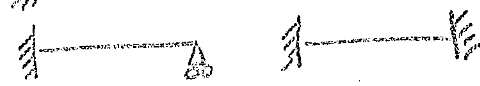
$$\therefore \boxed{\frac{d^2M}{dx^2} = -p(x)}$$

note: need 2 values of M or $Q = \frac{dM}{dx}$ to obtain M(x)

Boundary Conditions:

Statically determinate cases:

Statically indeterminate cases:



If the beam supports are such that 2 of the 4 end resultants: Q(0), M(0), Q(L), M(L) are given, then M(x) is easily determined by simple integration

$$M(x) = M(a) + Q(b)(x-a) - \int_a^x \int_b^x p(x)\,dx\,dx$$

Otherwise, the beam deformation is required.

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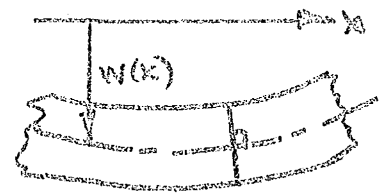
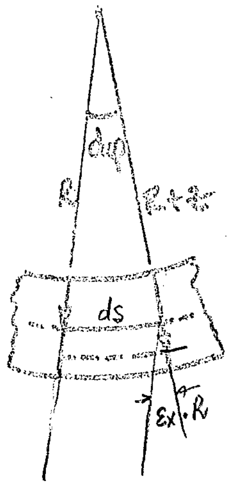
Deformation:

assume cross-sections remain plane and normal to longitudinal fibers





$$\varepsilon_x = \frac{\overset{l}{(R+z)d\varphi} - \overset{l_0}{R\,d\varphi}}{R\,d\varphi \; l_0} = \frac{z}{R}$$

$$\sigma_x = E\,\varepsilon_x = \frac{E}{R} z \qquad (z=0 \text{ is neutral axis})$$

$$\int \sigma_x \, dA = 0 \rightarrow \int z \, dA = 0 \qquad (z=0 \text{ is centroid})$$

$$M = \int \sigma_x \cdot z \, dA = \frac{E}{R}\int z^2 dA \quad \rightarrow \quad \boxed{\frac{M}{EI} = \frac{1}{R}} \quad (\text{curvature})$$

$$\sigma_x = \frac{Mz}{I}$$

now, all that remains is to determine R in terms of w(x)

if we take thin beam small slope, small deformation

$$\frac{1}{R} = \frac{d\varphi}{ds} = \frac{-\frac{d^2w}{dx^2}}{\left[1+\left(\frac{dw}{dx}\right)^2\right]^{3/2}} \approx -\frac{d^2w}{dx^2}$$

thus we obtain :

$$\boxed{\frac{M}{EI} = -\frac{d^2w}{dx^2}} \rightarrow \boxed{\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = p(x)}$$

Note The support conditions give 2 boundary conditions at each end of the beam. Thus w(x), and thus M(x), can be determined completely.

$$\frac{dM}{dx} = \frac{d}{dx}\left(-EI\frac{d^2w}{dx^2}\right) = Q$$

at each end we either have to ~~get~~ know
w or Q
or $\frac{dw}{dx}$ or M

everything up to here is exact for just pure bending
but approximate for case when we had shearing forces

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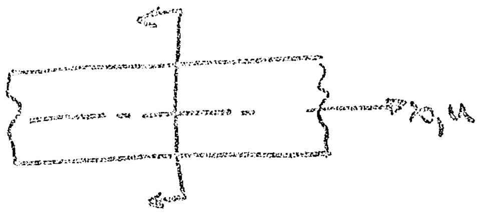
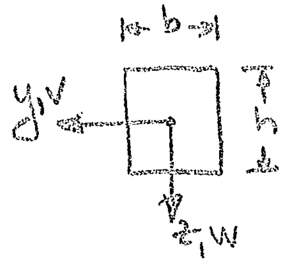
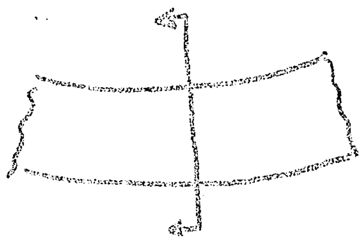
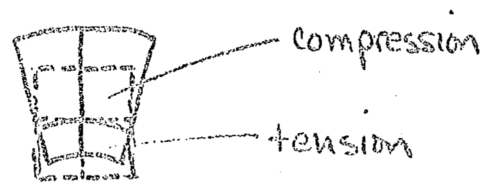
Distortion of Cross-section





$$I = \int_{-h/2}^{h/2} z^2 \underset{b\,dz}{dA} = \frac{bh^3}{12}$$





$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{z}{R} \qquad \varepsilon_y = \frac{\partial v}{\partial y} = -\nu \frac{z}{R} \qquad \varepsilon_z = \frac{\partial w}{\partial z} = -\nu \frac{z}{R}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0$$

For $R(x) =$ constant (pure bending):

$$u = \frac{M}{EI} xz + \underline{c_1 + c_5 z - c_6 y}$$

$$v = \frac{M}{EI}(-\nu y z) + \underline{c_2 + c_6 x - c_4 z}$$

$$w = \frac{M}{EI}\left[-\frac{x^2}{2} + \nu \frac{y^2 - z^2}{2}\right] + \underline{c_3 + c_4 y - c_5 x}$$

Rigid body displacements

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Proof $M(x) = \text{constant}$ (Pure Bending)

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{M}{EI} z \rightarrow u = \frac{M}{EI} xz + f(y,z)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -\nu \frac{M}{EI} z \rightarrow v = -\nu \frac{M}{EI} yz + g(x,z)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = -\nu \frac{M}{EI} z \rightarrow w = -\nu \frac{M}{EI} \frac{z^2}{2} + h(x,y)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \rightarrow \frac{\partial f(y,z)}{\partial y} + \frac{\partial g(x,z)}{\partial x} = 0$$

$$\rightarrow \begin{cases} f = y\,l(z) + p(z) \\ g = -x\,l(z) + q(z) \end{cases}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \rightarrow -\nu \frac{M}{EI} y - x\,l'(z) + q'(z) + \frac{\partial h(x,y)}{\partial y} = 0$$

$$\rightarrow \begin{cases} l'(z) = \alpha_0 \rightarrow l = \alpha_0 z + \alpha_2 \\ q'(z) = \alpha_1 \rightarrow q = \alpha_1 z + \alpha_3 \\ h = \nu \frac{M}{EI} \frac{y^2}{2} + (\alpha_0 x - \alpha_1) y + r(x) \end{cases}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \rightarrow \alpha_0 y + r'(x) + \frac{M}{EI} x + \alpha_0 y + p'(z) = 0$$

$$\rightarrow \begin{cases} \alpha_0 = 0 \\ p'(z) = \alpha_4 \rightarrow p = \alpha_4 z + \alpha_5 \\ r'(x) = -\frac{M}{EI} x - \alpha_4 \rightarrow r = -\frac{M}{EI} \frac{x^2}{2} - \alpha_4 x + \alpha_6 \end{cases}$$

rigid body displ & rotation for small displ

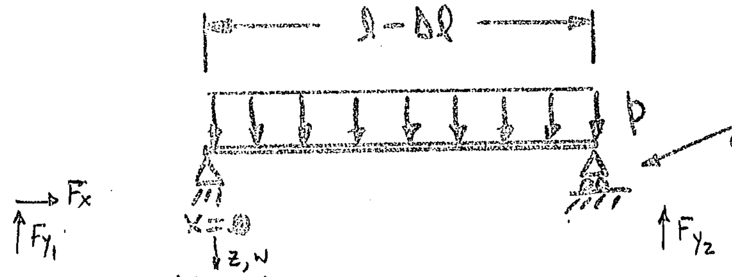
$$\therefore \quad u = \frac{M}{EI} xz + \boxed{\alpha_2 y + \alpha_4 z + \alpha_5}$$

$$v = -\nu \frac{M}{EI} yz \boxed{- \alpha_2 x + \alpha_1 z + \alpha_3}$$

$$w = \frac{M}{EI} \left[-\frac{x^2}{2} + \nu \frac{y^2 - z^2}{2} \right] \boxed{- \alpha_1 y - \alpha_4 x + \alpha_6}$$

Simply supported, uniformly loaded beam

ME241A
4-6-79 (2



from free body $F_x = 0$

Equilibrium

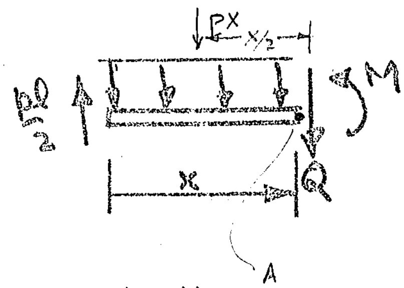
$$\frac{d^2M}{dx^2} = -p = \text{constant} \quad \Rightarrow \quad M = -\frac{px^2}{2} + c_1 x + c_2$$

B.C.

$$M(0) = 0 \Rightarrow c_2 = 0, \qquad M(l) = 0 \Rightarrow c_1 = \frac{pl}{2}$$

$$\therefore \boxed{M(x) = \frac{1}{2} p x (l-x)}$$

alternatively:



$$@A: M + px\frac{x}{2} - \frac{pl}{2}x = 0$$

$$\therefore M = \frac{1}{2} p x (l-x)$$

now $\sigma_{xx} = \frac{Mz}{I}$

Displacement

$$M = -EI\frac{d^2w}{dx^2} = \frac{1}{2}px(l-x) \Rightarrow w = -\frac{p}{2EI}\left(\frac{x^3}{6}l - \frac{x^4}{12} + c_3 x + c_4\right)$$

$$w(0) = 0 \Rightarrow c_4 = 0, \qquad w(l) = 0 \Rightarrow c_3 = -\frac{l^3}{12}$$

$$\therefore \boxed{w(x) = \frac{p}{24EI}(x^4 - 2lx^3 + l^3 x)}$$

Note: $w_{max} = w\left(\frac{l}{2}\right) = \frac{5}{16}\frac{pl^4}{24EI} = \frac{5}{16}\cdot\frac{pl^3}{24EI}\cdot l$

if w is small $\Rightarrow \frac{pl^3}{24EI}$ must be small; $\left.\frac{dw}{dx}\right|_{max} = w'(0) = \frac{pl^3}{24EI}$

dimensionless

ME241A
4-6-79 (3

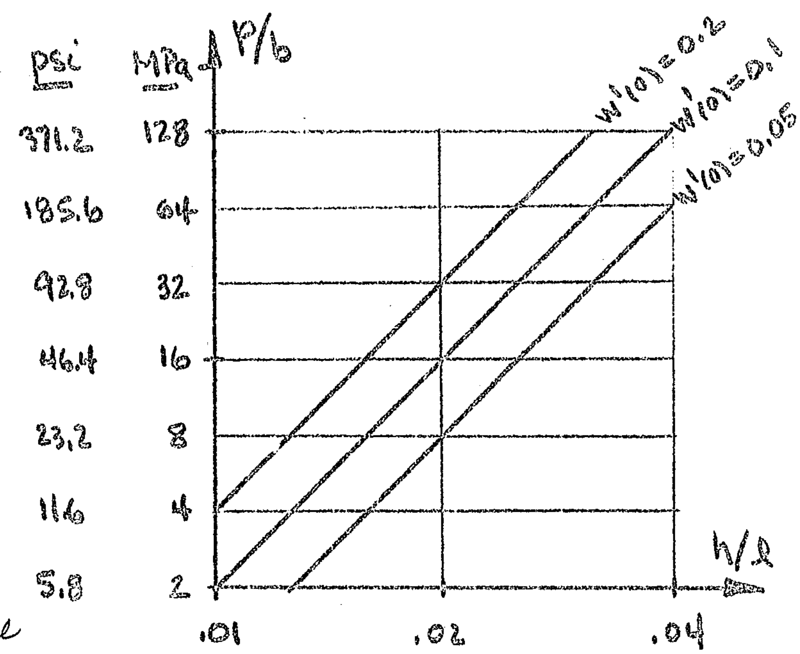
<u>Small slope assumption</u> is what allowed us to use simple for for curvature.

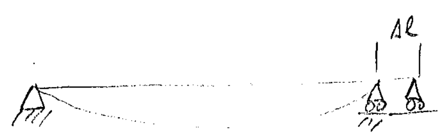
$$K = \frac{1}{R} = -\frac{w''}{[1+(w')^2]^{3/2}} \approx -w''$$

$$(w')^2 \ll 1 \rightarrow \boxed{\left(\frac{p\ell^3}{24EI}\right)^2 \ll 1} \rightarrow \left(\frac{w}{\ell}\right)^2 \ll 1$$

<u>Ex</u> $I = \frac{bh^3}{12} \rightarrow w'(0) = \frac{1}{2}\,\frac{P/b}{E}\left(\frac{\ell}{h}\right)^3$

$E = 29\times10^6$ psi
$= 2\times10^5$ MPa





We want to find out what axial load we'll need to keep to rolling end fixed

<u>Calculate Δℓ</u>

$$\Delta\ell = \int_0^\ell ds(x) - dx = \int_0^\ell \left[\sqrt{1+\left(\frac{dw}{dx}\right)^2} - 1\right]dx$$

taylors theorem $\rightarrow 1 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + \ldots$

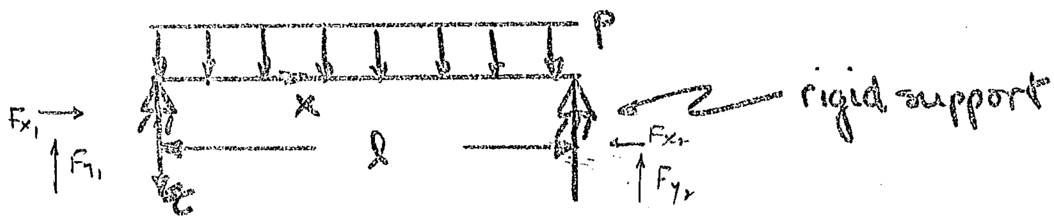
$$\approx \int_0^\ell \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx = \frac{17}{70}\left(\frac{p\ell^3}{24EI}\right)^2 \ell$$

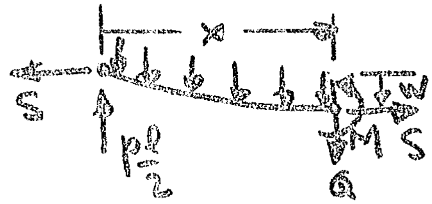
where $\frac{dw}{dx} = \frac{P}{24EI}(4x^3 - 6\ell x^2 + \ell^3)$

$$\therefore \boxed{\frac{\Delta\ell}{w_{max}} = \frac{272}{350}\left(\frac{p\ell^3}{24EI}\right)} \leftarrow \text{small}$$

Effect of axial loading

ME244A
4-6-79 (4)





$$M = \frac{pl}{2}x - px\frac{x}{2} - SW = -EI\frac{d^2w}{dx^2}$$

DE $\quad \dfrac{d^2w}{dx^2} - \dfrac{S}{EI}w = -\dfrac{p}{2EI}x(l-x)$

BC $\quad w(0) = w(l) = 0$

Solution: $\quad w = \dfrac{pl^4}{16u^4EI}\left[\dfrac{\cosh u(1-2x/l)}{\cosh u} - 1\right] + \dfrac{pl^2}{8u^2EI}x(l-x)$

where $u^2 = \dfrac{Sl^2}{4EI}$

S must be sufficient to restore support point to its original position:

$\sigma_{xx} = \dfrac{S}{bh} = E\dfrac{\Delta l}{l} = E\epsilon_{xx}$, $\quad \Delta l = \dfrac{1}{2}\displaystyle\int_0^l \left(\dfrac{dw}{dx}\right)^2 dx$ using w from

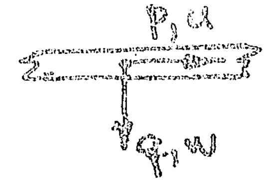
this gives a very complicated transcendental equation for S

Condition for S to be negligible

$$SW \ll \frac{1}{2}px(l-x) \rightarrow \overbrace{\underbrace{Ebh}_{E\,A}\overbrace{\left(\frac{\Delta l}{l}\right)}^{\epsilon_{xx}}\,\overbrace{\frac{5}{16}\frac{pl^3}{24EI}l}^{W_{max}\text{ from previous proble}}}^{SW} \ll \frac{pl^2}{8} \leftarrow M_{max} = M(l/2)$$

to neglect axial load
displ of beam << depth of beam $\quad \dfrac{\Delta l}{l} \sim \left(\dfrac{w}{l}\right)^2 \rightarrow \boxed{\left(\dfrac{w}{l}\right)^2 \ll \left(\dfrac{h}{l}\right)^2 \ll 1}$

Thin 1-dim. elastic continuum



longitudinal deform.

$$(AEu')' = p(x)$$
(rod)

$$T' = p(x)$$

transverse deformation

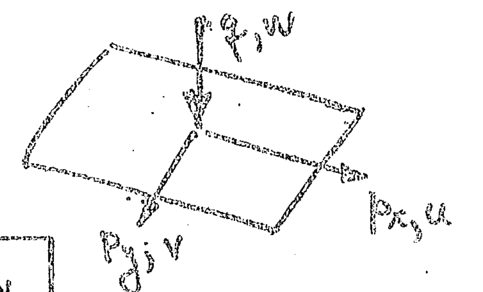
$$-(Tw')' = q(x)$$
(string)

$$(EIw'')'' = q(x)$$
(beam)

↓

$$(EIw'')'' - (Tw')' = q(x)$$
(beam with axial loads)

Thin 2-dim. elastic continuum



in-plane

a pair of 2nd order P.D.E. for u, v

or

$$\frac{1}{Eh}\nabla^4\varphi = f(x,y)$$
(plane stress)

out-of-plane

$$-N_x\frac{\partial^2 w}{\partial x^2} - 2N_{xy}\frac{\partial^2 w}{\partial x\partial y} - N_y\frac{\partial^2 w}{\partial y^2} = q(x,y)$$
(sheet)

$N_{xy}=0 \rightarrow$ membrane

$$D\nabla^4 w = q(x,y)$$
(plate)

↓

$$D\nabla^4 w - N_x\frac{\partial^2 w}{\partial x^2} - 2N_{xy}\frac{\partial^2 w}{\partial x\partial y} - N_y\frac{\partial^2 w}{\partial y^2} = q(x,y)$$
(plate with in-plane loads)

↓

$$\frac{1}{Eh}\nabla^4\varphi = \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \qquad \text{(Gaussian curvature)}$$

$$D\nabla^4 w = q(x,y) + \frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

(Von Karman Plate equations)

General Case for Rectangular Plates

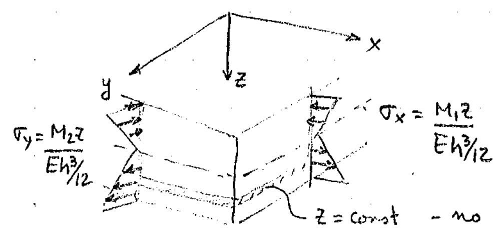
bending curvatures

$$-w_{,xx} = \frac{M_1 - \nu M_2}{D(1-\nu^2)} \qquad -w_{,yy} = \frac{M_2 - \nu M_1}{D(1-\nu^2)}$$

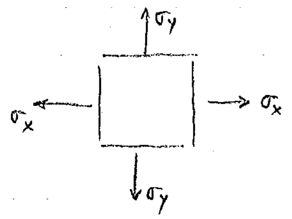
twisting curvature

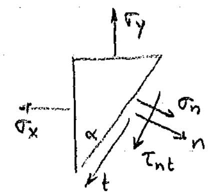
$$\frac{\partial^2 w}{\partial x \partial y} = 0$$

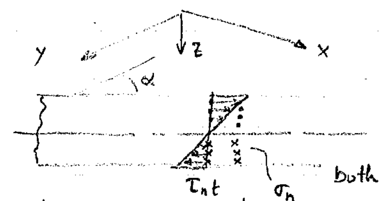
$$w(x,y) = -\frac{M_1 - \nu M_2}{D(1-\nu^2)}\frac{x^2}{2} - \frac{M_2 - \nu M_1}{D(1-\nu^2)}\frac{y^2}{2} + c_0 + c_1 x + c_2 y$$



- no transverse shear, no σ_z on faces → case of plane strain





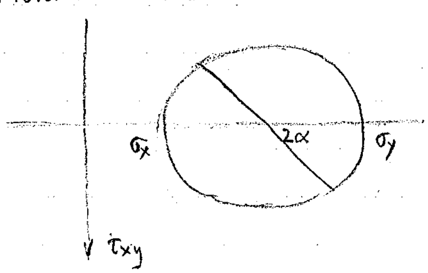


both vary linearly

$$M_n = \int_{-h/2}^{h/2} \sigma_n z \, dz$$ bending moment per unit width

$$M_{nt} = \int_{-h/2}^{h/2} \tau_{nt} z \, dz$$ twisting moment per unit width

Mohr's Circle



$$\sigma_n = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + \underbrace{\ldots}_{=0}$$

$$\tau_{nt} = \frac{(\sigma_y - \sigma_x)}{2}\cos 2\alpha + \underbrace{\ldots}_{=0}$$

$$M_n = M_1 \cos^2\alpha + M_2 \sin^2\alpha$$

$$M_{nt} = \frac{(M_2 - M_1)}{2}\sin 2\alpha$$

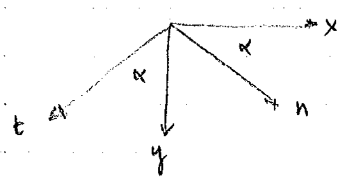
$\therefore$ since $-w_{,xx} = \dfrac{M_1 - \nu M_2}{D(1-\nu^2)}$

$-w_{,yy} = \dfrac{M_2 - \nu M_1}{D(1-\nu^2)}$

$$\boxed{\begin{aligned} M_1 &= -D(w_{,xx} + \nu w_{,yy}) \\ M_2 &= -D(w_{,yy} + \nu w_{,xx}) \end{aligned} \qquad M_{12} = -D(1-\nu)\, w_{,xy}}$$

$$M_2 - M_1 = -D(1-\nu)(w_{,yy} - w_{,xx})$$

$$M_{nt} = -D\frac{(1-\nu)}{2}(w_{,yy} - w_{,xx})\sin 2\alpha$$



$$x = n\cos\alpha - t\sin\alpha$$

$$y = n\sin\alpha + t\cos\alpha$$

$$\frac{\partial}{\partial n} = \cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y} = \underset{\sim}{n} \cdot \nabla$$

$$\frac{\partial}{\partial t} = -\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y} = \underset{\sim}{t} \cdot \nabla$$

now put W in $$\frac{\partial^2 W}{\partial n \partial t} = \frac{\sin 2\alpha}{2}\left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2}\right) + \cos^2 2\alpha \underbrace{\cancel{\frac{\partial^2 W}{\partial x \partial y}}}_{=0}$$

now to have twisting curvature

$$M_{nt} = -D(1-\nu)\frac{\partial^2 W}{\partial n \partial t}$$

likewise $$M_{xy} = -D(1-\nu)\frac{\partial^2 W}{\partial x \partial y}$$

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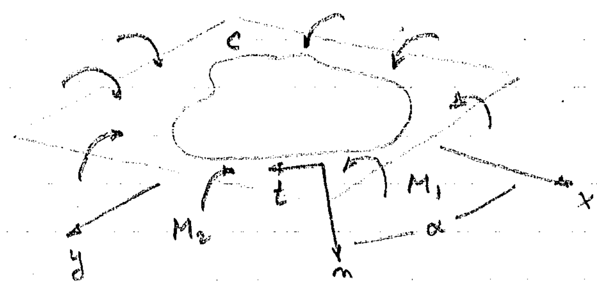


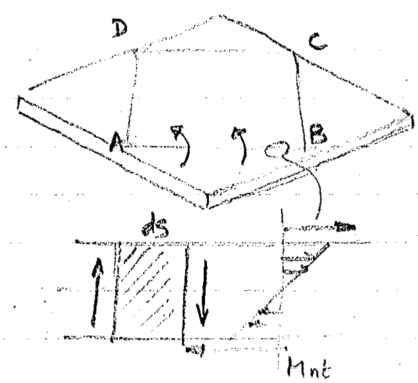
Suppose a plate is subjected to M_1 & M_2 along the y & x sides. Then if we cut out a curve C we must apply along its boundaries

$M_n = M_1\cos^2\alpha + M_2\sin^2\alpha$

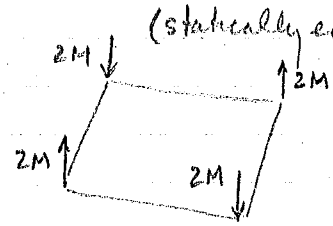
$M_{nt} = (M_2 - M_1)\frac{\sin 2\alpha}{2}$

for $M_1 = M_2 = M$ $\quad M_n \to M \quad M_{nt} = 0$: This is spherical bending for arbitrary shaped plates

for $M_1 = -M_2 = M$ $\quad M_n = 0 \quad M_{nt} = -M$



This is equivalent to the stress distribution shown. This distrib produces a force couple $= M_{nt}$ which can also be resolved into a ⊥ force couple this is true along line AB. For each adjacent piece the forces will cancel except at the ends A & B. This is equivalent to the following (statically equivalent that is)



The stresses & displ are accurate except at the corners where stress concentrations occur due to point loads but their effect is dissipated with 2 plate thicknesses

ME 241 A
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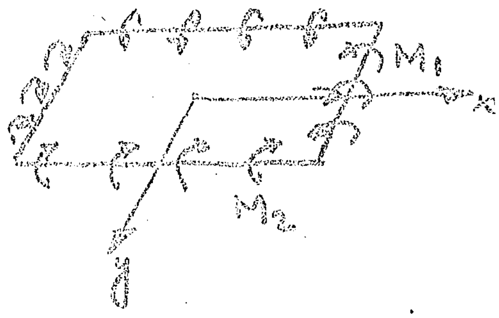
Pure Bending of Uniform Plates

Consider a rectangular plate of thickness h which is subjected to uniformly distributed bending couples along its edges



M_1, M_2 = ~~moment~~ torque per unit length

Since our solution for pure bending of a beam placed no restriction on the beam width, the deformation of the plate's midplane caused by M_1 acting alone is clearly such that:

$$\frac{M_1}{E(h^3/12)} = \frac{1}{R_x} \approx -\frac{\partial^2 w}{\partial x^2}, \quad -\nu\frac{M_1}{EI} = \frac{1}{R_y} \approx -\frac{\partial^2 w}{\partial y^2}$$

$$\text{and also} \quad \frac{1}{R_{xy}} \approx -\frac{\partial^2 w}{\partial x \partial y} = 0$$

Similar results are produced by M_2 acting alone and superposition then gives:

$$\frac{\partial^2 w}{\partial x^2} = -\frac{12}{Eh^3}(M_1 - \nu M_2)$$

$$\frac{\partial^2 w}{\partial y^2} = -\frac{12}{Eh^3}(M_2 - \nu M_1)$$

$$\frac{\partial^2 w}{\partial x \partial y} = 0$$

Note Since we are focusing attention on the plate's midplane $z=0$, $w \equiv w(x,y,0)$

An expression for w can be determined by "simple" integration or by setting $z=0$ in the results obtained earlier for pure beam bending and using superposition

$$w = \frac{M_1}{E(h^3/12)} \cdot \frac{-x^2+\nu y^2}{2} + \frac{M_2}{E(h^3/12)} \frac{-y^2+\nu x^2}{2} + \text{Rigid body terms}$$

$$= -\frac{12}{Eh^3}(M_1-\nu M_2)\frac{x^2}{2} - \frac{12}{Eh^3}(M_2-\nu M_1)\frac{y^2}{2} + c_0+c_1x+c_2y$$

Ex 1) $M_1 = M$, $M_2 = \nu M$ $\rightarrow \frac{1}{R_y} = 0$ $\therefore$ midplane deforms into a cylindrical surface

$$\frac{1}{R_x} = -\frac{\partial^2 w}{\partial x^2} = \frac{12(1-\nu^2)}{Eh^3} M = \frac{M}{D} \qquad D = \frac{Eh^3}{12(1-\nu^2)}$$

$$w = -\frac{M}{D}\frac{x^2}{2} + c_0 + c_1x + c_2y$$

Ex 2) $M_1 = M_2 = M$ $\rightarrow \frac{1}{R_x} = \frac{1}{R_y} = \frac{1}{R}$ $\therefore$ midplane becomes spherical

$$\frac{1}{R} = \frac{12(1-\nu)}{Eh^3} M = \frac{M}{(1+\nu)D}$$

using approx. expressions for curvature this becomes

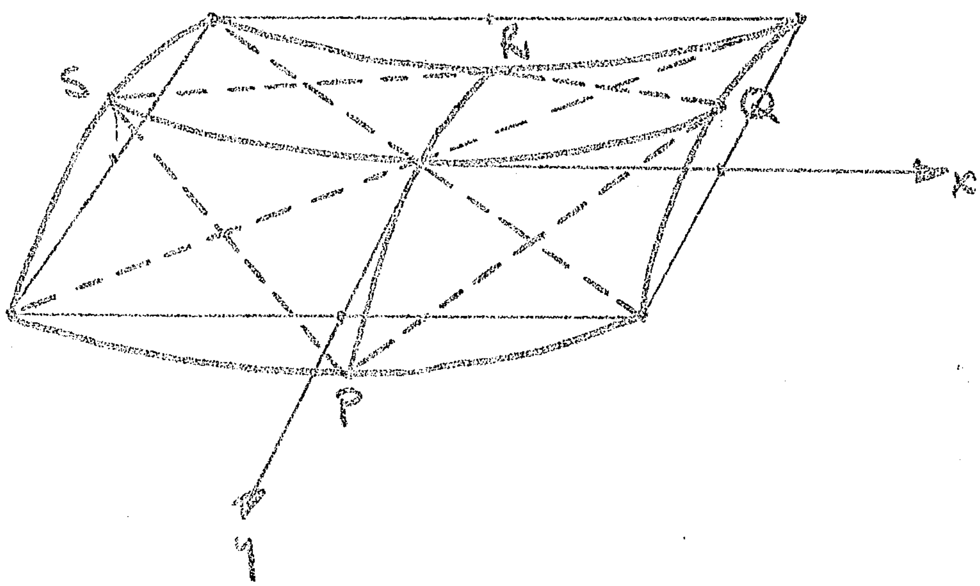
$$w = -\frac{M}{(1+\nu)D}\frac{x^2+y^2}{2} + c_0 + c_1x + c_2y \rightarrow \text{paraboloid of revolution}$$

Ex 3) $M_1 = -M_2 = M$ $\rightarrow \frac{1}{R_x} = -\frac{1}{R_y} = \frac{1}{R}$ $\therefore$ midplane becomes a hyperbolic paraboloid

$$\frac{1}{R} = \frac{12(1+\nu)}{Eh^3} M = \frac{M}{(1-\nu)D}$$

$$w = -\frac{M}{(1-\nu)D}\frac{x^2-y^2}{2} + c_0 + c_1x + c_2y$$

Further examination of this case is revealing
The deformed middle surface is illustrated below:
for a square plate where we have taken the reference plane (xy plane) tangent to the surface at the origin.

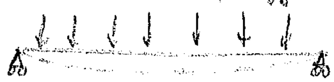


lines parallel to x-axis → parabolas concave upward
lines parallel to y-axis → parabolas concave downward
the lines $x = \pm y$ → remain undeformed
lines parallel to $x = \pm y$ → remain straight but tilt (rotate) either up or down

∴ deformed midplane → "ruled" surface

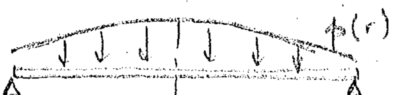
The portion PQRS of the plate would clearly be described as in a state of <u>pure twisting</u>

Circular Plate Analogy to deformation of rectangular plates.



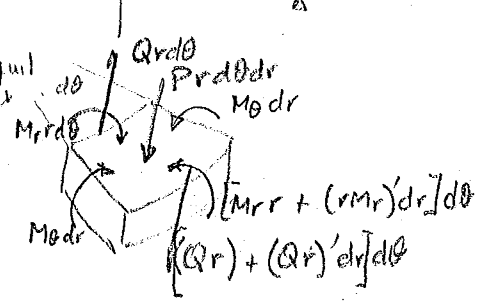
deformed state leads to compressive hoop stress (neglected in analysis so far) since either end moves in due to loading. There are in-plane stresses. We must determine when magnitude of displacement lead to significant errors in analysis ie when they are on the order of the plates thickness

Axisymmetric Loading of Circular Plates   Timoshenko Ch#3 Plates & Shells.



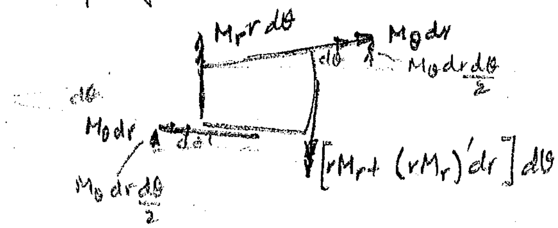
We will use same moment curvature relation

Look at equil of small element



since this is axisymmetric, no shear exists on the dr faces; since adjacent sides will have equal & opposite sign & $P = P(r)$ only we need not include them in analysis

Look at equil from above



Sum vertical forces: $(Qr)'drd\theta + Prd\theta dr = 0$

$$\therefore \boxed{\frac{1}{r}(rQ)' = -p}$$

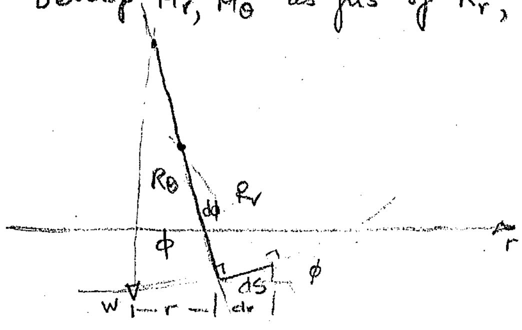
Moment Equil gives

$$(rM_r)'\cancel{drd\theta} - 2M_\theta dr\frac{d\theta}{2} - \{Qr + (Qr)'dr\}\cancel{drd\theta}$$
$$- Prd\theta dr\frac{dr}{2} = 0$$

$\lim dr \to 0$ gives $(rM_r)' - M_\theta - Qr = 0$ or

$$\boxed{rQ = (rM_r)' - M_\theta}$$

Develop M_r, M_θ as fn's of R_r, R_θ



$$\phi \approx \tan\phi = -\frac{dw}{dr}$$

$$\therefore R_r d\phi = ds = \sqrt{1 + \left(\frac{dw}{dr}\right)^2}\,dr \approx dr$$

$$\therefore \frac{1}{R_r} = \frac{d\phi}{dr} = -\frac{d^2w}{dr^2}$$

$$R_\theta \sin\phi = r \approx R_\theta \phi \qquad \therefore \qquad \frac{1}{R_\theta} = \frac{\phi}{r} = -\frac{1}{r}\frac{dw}{dr}$$

note that R_θ for const ϕ will trace out cone hence center of curvature must line on center axis of plate. R_r is the local radius of curv in a const r plate & thus will <u>not</u> normally lie on the axis of symmetry (center axis of plate)

$$\therefore \boxed{\begin{aligned} M_r &= D\left(\frac{1}{R_r} + \nu\frac{1}{R_\theta}\right) = -D\left(w'' + \nu\frac{w'}{r}\right) \\ M_\theta &= D\left(\frac{1}{R_\theta} + \nu\frac{1}{R_r}\right) = -D\left(\frac{w'}{r} + \nu w''\right) \end{aligned}}$$

Put these into eqns for rQ

$$\text{then } rQ = (rM_r)' - M_\theta = -D\left[\left(rw'' + \cancel{\nu w'}\right)' - \left(\frac{w'}{r} + \cancel{\nu w''}\right)\right] = -D\left(rw''' + w'' - \frac{w'}{r}\right)$$

$$\text{then} \qquad \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rw')\right) = -\frac{Q}{D} \qquad \text{but } rw' = r\frac{d}{dr}w$$

$$\therefore \boxed{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}w\right)\right] = -\frac{Q}{D}}$$

Put this into eqn for P

$$\text{then } \frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}w\right)\right\}\right] = \frac{p(r)}{D}$$

$$\text{or} \boxed{\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\right]^2 w = \frac{p(r)}{D}}$$

This is a non homog eq. We can get homogeneous soln if we denote $W_T = W_H + W_p$ and W_H solves

$$\Big[\overbrace{\frac{1}{r}\frac{d}{dr}}^{L_1}\Big(\overbrace{r\frac{d}{dr}}^{L_2}\Big)\Big]^2 w_H = 0 \quad \Rightarrow \quad (L_1L_2)^2 w_H = 0$$

Solution to this $\quad L_2L_1L_2 w_H = c_1$

$$L_1L_2 w_H = c_1 \ln r + c_2$$

$$L_2 w_H = c_1\left(\frac{r^2}{2}\ln r - \frac{r^2}{4}\right) + c_2\frac{r^2}{2} + c_3$$

$$w_H = \frac{c_1}{2}\left(\frac{r^2}{2}\ln r - \frac{r^2}{4}\right) + \left(\frac{c_2}{2} - \frac{c_1}{4}\right)\frac{r^2}{2} + c_3\ln r + c_4$$

$$\text{or} \qquad w_H = \tilde{c}_1 r^2 \ln r + \tilde{c}_2 r^2 + \tilde{c}_3 \ln r + \tilde{c}_4$$

for solution to exist at $r=0 \Rightarrow \tilde{c}_3 = 0$

for shear to exist at center $\Rightarrow \tilde{c}_1 = 0$

$$\therefore \quad w_H = \tilde{c}_2 r^2 + \tilde{c}_4$$

Example 1) Clamped Plate at $r=R$, uniformly loaded

$$w = \tilde{C}_2 r^2 + \tilde{C}_4 + w_p \qquad \text{where} \quad L_1 L_2 L_1 L_2 w_p = \frac{p_0}{D} = \text{constant}$$

$$\text{and } w_p = \frac{p_0}{D}\frac{r^4}{64}$$

then let

$$w = \frac{p_0}{64D}\left[r^4 + c_2 r^2 + c_4\right] \qquad w' = \frac{p_0}{64D}\left[4r^3 + 2c_2 r\right]$$

w/ $w(R)=0 \quad w'(R)=0 \quad \Rightarrow \quad R^4 + c_2 R^2 + c_4 = 0$

$\Rightarrow \quad 4R^3 + 2c_2 R = 0 \quad \Rightarrow \quad c_2 = -2R^2$

$c_4 = R^4$

$$\therefore \quad w = \frac{p_0}{64D}\left[r^4 - 2R^2r^2 + R^4\right] = \frac{p_0}{64D}\left[r^2 - R^2\right]^2$$

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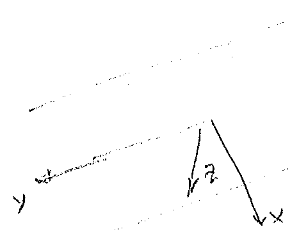
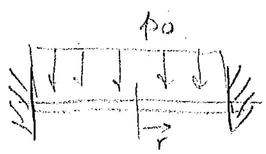
HW for long rectangular cases the effect of bc are felt w/in one or two thicknesses ∴ close to center the problem acts as if one is of cylindrical bending when the load on the plate is constant.



Circular Plates - Axisymmetric loading

$$\left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\right)^2 w = \frac{p(r)}{D}$$



regularity for a full plate requires that for $r=0$

$$w_H = C_1 + C_2 \ln r + r^2(C_3 + C_4 \ln r)$$

must reduce to $w_H = C_1 + r^2 C_3$ since $\ln r\big|_{r=0} \to -\infty$

and since $Q\big|_{r=0} \to -\infty$ unless $C_4 = 0$

for uniform pressure $p(r) = p_0 \qquad w_c = \dfrac{p_0 r^4}{64D}$

then $w = w_H + w_c = \dfrac{p_0}{64D}\left[r^4 + c_1 + c_3 r^2\right]$ w/ $w(r=R)=0$, $w'(r=R)=0 \Rightarrow w(r) = \dfrac{p_0}{64D}\left(R^2 - r^2\right)^2$

we need 4 bc's since we have four constants for full plate we have 1 bdy and hence 2 bc. The remaining 2 must come from regularity

max disp occurs at origin then $w(0) = \frac{P_0 R^4}{64D} = w_{max}$

this really isn't what we need, to get stresses $\sigma_r = \frac{M_r z}{h^3/12}$ thus we need M_r

But $M_r = -D\left(w'' + \nu \frac{w'}{r}\right)$

$w' = \frac{-P_0}{16D}(R^2 - r^2)r \qquad \frac{w'}{r} = \frac{-P_0}{16D}(R^2 - r^2)r$

$w'' = \frac{-P_0}{16D}(R^2 - 3r^2)$

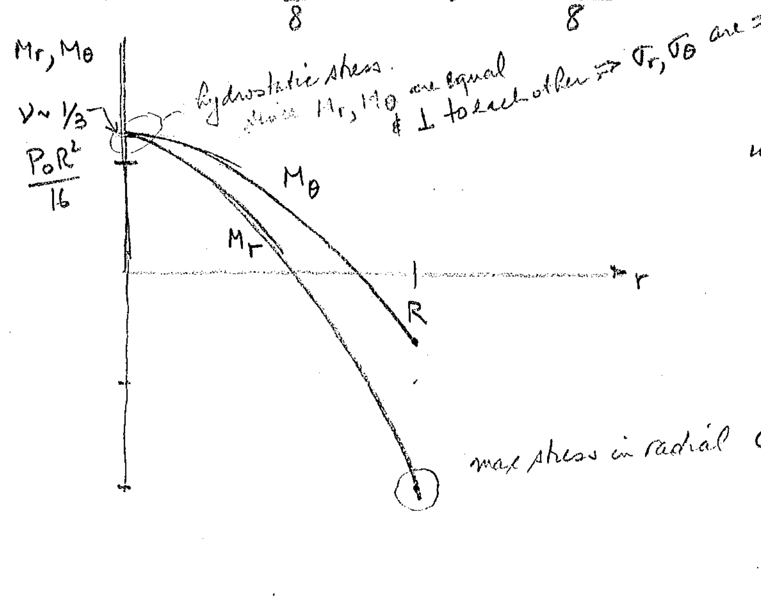
$M_r = -D\left(w'' + \nu \frac{w'}{r}\right) = \frac{P_0}{16}\left[(1+\nu)R^2 - (3+\nu)r^2\right]$

$M_\theta = \frac{P_0}{16}\left[(1+\nu)R^2 - (1+3\nu)r^2\right]$

to a proportionality constant then $M_r \sim \sigma_r \quad M_\theta \sim \sigma_\theta$

@ $r = 0 \quad M_r = M_\theta = (1+\nu)R^2 P_0/16$

$r = R \quad M_r = -\frac{P_0 R^2}{8} \qquad M_\theta = -\frac{P_0 \nu R^2}{8}$



use Mohr's circle to determine max. stresses.

max stress in radial $\sigma_{max} = \sigma_r = \frac{M_r z}{h^3/12} = \frac{\frac{-P_0 R^2}{8} \cdot \frac{-h}{2}}{h^3/12} = \frac{3P_0 R^2}{4h^3}$

To show why C_4 in homogeneous soln must be zero even though $r^2 \ln r \big|_{r=0} = 0$ for axisymmetric plate w/no hole

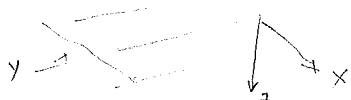
$w = C_1 + C_2 \ln r + r^2(C_3 + C_4 \ln r)$ Soln w/no surface loading only edge loading

$w(0) < \infty \Rightarrow C_2 = 0$ finite displ at origin

ME 241A   Homework Problems   Spring 1979

① A long uniform rectangular plate is clamped along its two long edges and is subjected to a uniform pressure p_0. Determine the maximum bending stress and deflection at the center of the plate.

cylindrical bending behavior
treat like beam problem
const x lines remain straight & ||.



② A uniform circular plate of radius R is simply supported along its edge and is subjected to an axisymmetric pressure loading $p(r) = p_0 \left(\frac{r}{R}\right)^n$. Determine the deflection $w(r)$ of the plate's middle surface. Use your result to evaluate the central deflection and maximum bending stress for the case $p(r) = p_0 \left(1 - \frac{r^2}{2R^2}\right)$

$w' = 2rc_3 + c_4 r + 2c_4 r \ln r$

for axisymmetric plate $w'(0) < \infty$ & $w'(0) = 0$ which is satisfied

$\left.\frac{w'}{r}\right|_{r=0} = K_\theta < \infty \Rightarrow c_4 = 0$ since curvature must be finite

$w'' = 2c_3 + c_4 + 2c_4 \ln r + 2c_4 \qquad c_4 = 0$ for $\left. w''\right|_{r=0} < \infty$

ie M_r, M_θ must be finite at center

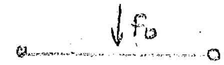
if we look at shear $Q = -D \frac{d}{dr}\left(w'' + \frac{1}{r} w'\right)$

$$= -D\frac{d}{dr}\left[2c_3 + 3c_4 + 2c_4 \ln r + 2c_3 + c_4 + 2c_4 \ln r\right]$$

$$Q(r) = -D\frac{d}{dr}\left[4c_3 + 4c_4 + 4c_4 \ln r\right] = -D\left[\frac{4c_4}{r}\right]$$

$\Rightarrow Q(r)\,r = -4Dc_4 =$ const. $\quad c_4 \neq 0$ implies a point axial load at $r=0$

if we want a point load at origin



now $F_0 = -2\pi r\, Q(r)$
$= -2\pi\left[-D \cdot 4c_4\right]$

$$\therefore \boxed{\frac{F_0}{8\pi D} = c_4}$$

$w = \underbrace{c_1}_{\text{rigid body tran}} + \underbrace{r^2}_{\text{rigid body rotation}} (c_3 + c_4 \ln r)$

spherical bending

point load.

Suppose $w(R) = 0 \Rightarrow 0 = c_1 + R^2(c_3 + c_4 \ln R)$

$\therefore w = c_3(r^2 - R^2) + c_4(r^2 \ln r - R^2 \ln R)$

Suppose $w'(R) = 0$

$w'(R) = c_3 \cdot 2R + c_4 \cdot 2R \ln R + c_4 R = 0$

$c_3 = -\frac{c_4}{2}(1 + 2\ln R)$

$$\therefore w = -\frac{c_4}{2}(1 + 2\ln R)(r^2 - R^2) + c_4(r^2 \ln r - R^2 \ln R)$$

$$= \frac{-F_0}{8\pi D}\left\{\frac{(r^2 - R^2)}{2} + \ln R\,(r^2 - \cancel{R^2}) - r^2 \ln r + \cancel{R^2 \ln R}\right\}$$

$$= \frac{+F_0}{8\pi D}\left\{\frac{R^2 - r^2}{2} + r^2 \ln\left(\frac{r}{R}\right)\right\} = R\,\frac{F_0 R}{16\pi D}\left\{1 - \left(\frac{r}{R}\right)^2 + 2\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)\right\}$$

Suppose for clamped plate w/uniform load $W_{max}\big|_{R=0} = \frac{P_0 R^4}{64D}$

for clamped plate w/point load $W_{max}\big|_{R=0} = \frac{F_0 R^2}{16\pi D}$

if $P_0 = F_0/\pi R^2$ statically equiv $W = \frac{P_0 R^4}{64D} = \frac{F_0 R^2}{64\pi D}$

∴ point load gives 4x max deflect of uniform loading.

for point load – simply supported we find for $\nu = \frac{1}{3}$ $W\big|_{max} = 5/2\; W\big|_{max}$ for distributed loading

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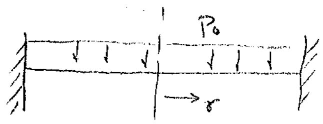
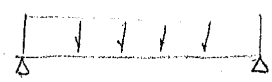


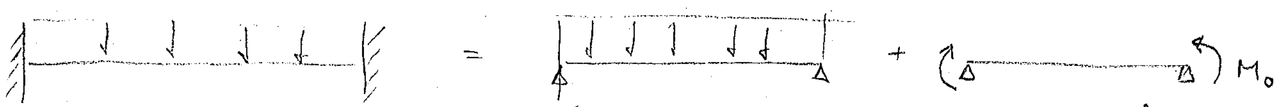
$$w = \frac{P_0(R^2-r^2)}{64D}$$

$$M_r = \frac{P_0}{16}\left[(1+\nu)R^2 - (3+\nu)r^2\right]$$

$$M_\theta =$$

We can use superposition to get





$w_{mom} = c(R^2-r^2)$ satisfies bc

$$M_r = -Dc(-2+\nu(-2)) = 2Dc(1+\nu)$$

$$\therefore c = \frac{M_0}{2(1+\nu)D}$$

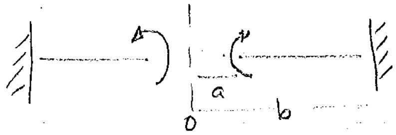
$$M_r(r=R) = -\frac{P_0 R^2}{8}$$

Now $M_r = 0 = -\frac{P_0 R^2}{8} + M_0$ $\quad M_0 = \frac{P_0 R^2}{8}$

$\therefore\ c = \frac{P_0 R^2}{16(1+\nu)D}$ $\quad\therefore\ \boxed{w_{mom} = \frac{P_0 R^2}{16(1+\nu)D}\left[R^2-r^2\right]}$

$$W_{simply\,sup} = W_{clamped} - W_{mom} = \frac{P_0(R^2-r^2)^2}{64D} - \frac{4P_0 R^2}{64(1+\nu)D}\left[R^2-r^2\right]$$

Suppose we load at the following



$$w = C_1 + C_2 \ln r + r^2 (C_3 + C_4 \ln r)$$

Since no point load at 0 and $rQ = \text{const}$ and $\Rightarrow C_4 = 0$: $Q = \text{const}$ throughout plate $\neq$ applied load $\Rightarrow Q = 0$

$$w(b) = 0 \quad = C_1 + C_2 \ln b + b^2 (C_3 + \cancel{C_4 \ln b}) \Rightarrow A(b^2 - r^2) + B \ln r/b = w(b) = 0$$

$$w'(b) = 0 \quad = C_2/r + 2r(C_3 + \cancel{C_4 \ln r}) + \cancel{\tfrac{C_4 r^2}{r}}\Big|_{r=b} = -2Ab + B/b = 0 \qquad \underline{B = 2Ab^2}$$

$$\therefore \quad w = A\left[b^2 - r^2 + 2b^2 \ln r/b\right]$$

$$w' = A\left(-2r + 2b^2/r\right)$$

$$w'' = A\left(-2 - 2b^2/r^2\right)$$

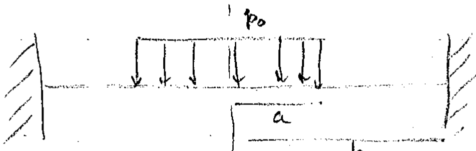
$$M_r = -DA\left[-2(1+\nu) - 2(1-\nu)\, b^2/r^2\right]$$

$$= 2(1+\nu) DA\left[1 + \frac{1-\nu}{1+\nu} \, b^2/r^2\right]$$

$$M_r\Big|_{r=a} = M_0 \quad \text{to get} \quad A = A(M_0, b, a, D, \nu) \qquad A = \frac{M_0}{2(1+\nu)D\left[1 + \frac{1-\nu}{1+\nu}\frac{b^2}{a^2}\right]}$$

$$\therefore \quad \boxed{w = \frac{M_0}{2(1+\nu)D\left[1 + \frac{1-\nu}{1+\nu} b^2/a^2\right]}\left[b^2 - r^2 + 2b^2 \ln r/b\right]}$$

Consider a circular plate loaded as shown



$w(b) = 0$
$w'(b) = 0$

Matching

$w_L(a) = w_R(a)$
$w_L'(a) = w_2'(a)$
M_R : $M_L(a) = M_R(a) \Rightarrow w_L''(a) = w_R''(a)$
$Q_L(a) = Q_R(a) \Rightarrow$ shear $\Rightarrow w_L'''(a) = w_R'''(a)$

$\frac{p_0 R^2}{64D}$ is partic soln for constant loading

Using known solns:

for $r \le a$ $\qquad w_L = \underbrace{\frac{p_0 R^2}{64D}}_{\text{particular}} + \underbrace{C_1 + C_2 r^2}_{\text{homog}}$

for $r > a$ $\qquad w_R = \underbrace{0}_{\text{particular}} + \underbrace{C_3 \ln r/b + C_4 (b^2 - r^2)}_{\text{homog}} + C_5 r^2 \ln r/b$

satisfies bc at $r = b$

must keep since origin remains in problem.

Now at $r = a$

$$w_L(a) = w_R(a)$$

$$w_L'(a) = w_R'(a)$$

$$M_L(a) = M_R(a)$$

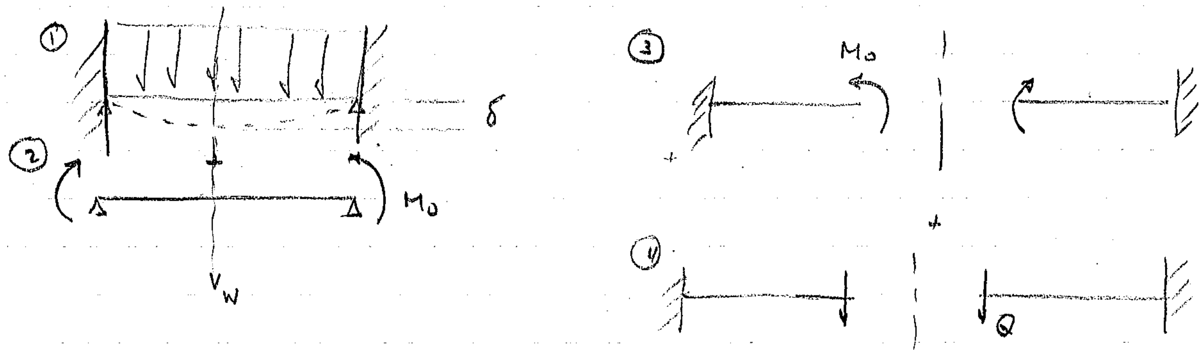
$$2\pi r Q + p_0 \pi a^2 = 0 \quad \forall\, r > a \qquad \therefore \quad rQ = -\frac{p_0 a^2}{2} = -4DC_5$$

$$\boxed{C_5 = \frac{p_0 a^2}{8D}}$$

We can then apply rest of bc to get C_1, C_2, C_3, C_4

<u>or</u>

we can use superposition of problems like ① + ② + ③ + ④



All this makes moments & shears cont.

Must now use the fact that w & w' must be equal at $r=a$ $\Rightarrow$ we can now get δ, M_0

HW#3. Solve this for w

Classical small - deflection theory

Assume Euler Bernoulli hypothesis

① normals to the midplane remain normal $\gamma_{zx} = \gamma_{zy} = 0$

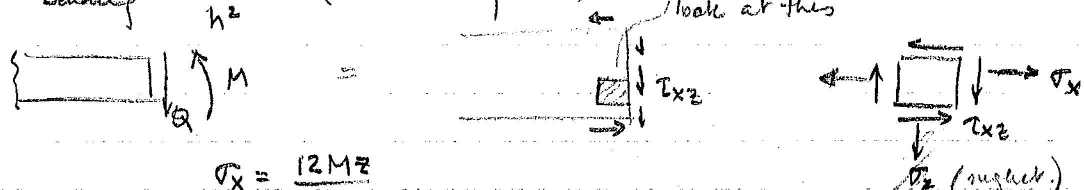
② Stresses normal to the midplane are negligible ie $\sigma_z << \sigma_x, \sigma_y$

③ Slopes are small $w_{,x}^2$ & $w_{,y}^2 << 1$

④ midplane is a neutral surface not true unless $(w/h)^2 << 1$

Stress estimates for thin plates

$$\sigma_{Bending} = \frac{6M}{h^2} \quad \text{(max bending stress)}$$



$$\sigma_x = \frac{12Mz}{h^3}$$

Using equil

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad \therefore \frac{\partial \tau_{xz}}{\partial z} = -\frac{12}{h^3} \underbrace{\frac{\partial M}{\partial x}}_{=Q} z \quad \therefore \tau_{xz} = -\frac{12}{h^3}\frac{z^2}{2}Q + c(x,y)$$

Now $\tau_{xz} = \tau_{zx}$ $= 0$ on $z = \pm h/2$

$$\tau_{xz} = -\frac{3}{2h}Q + c(x,y) = 0 \qquad c(x,y) = \frac{3Q}{2h}$$

$$\therefore \tau_{zx} = \frac{3}{2h}Q\left(1 - \left(\frac{z}{(h/2)}\right)^2\right)$$

$$\tau_{zx}\Big|_{z=0} = \frac{3Q}{2h}$$

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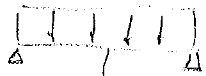
Continuation of last time

$\sigma_z \sim p \qquad \sigma_x, \sigma_y, \sigma_{xy} \approx \left(\frac{L}{h}\right)^2 p$

For  $\tau_{xz}, \tau_{yz} \approx \left(\frac{L}{h}\right) p$

$M=0 \quad M=M_{max} \quad M=0$

$$\boxed{\sigma_z \sim \frac{h}{L}(\tau_{xz}, \tau_{yz}) \sim \left(\frac{h}{L}\right)^2 (\sigma_x, \sigma_y, \sigma_{xy}) \sim p}$$

Neglect σ_z over $\sigma_x, \sigma_y, \sigma_{xy}$ since it is "2 orders of mag" smaller hence (2) assumption is satisfied.

Strains estimate $E\varepsilon_x = \sigma_x - \nu(\sigma_y + \cancel{\sigma_z})$ neglect

$G\gamma_{xz} = \tau_{xz} \implies$ since τ_{xz} is $\left(\frac{h}{L}\right)\sigma_x \Rightarrow \varepsilon_{xz} \sim \frac{h}{L}\varepsilon_x$

$\therefore \boxed{\frac{1}{2}(\gamma_{xz}, \gamma_{yz}) \sim \frac{h}{L}(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}) \sim \frac{P/E}{h/L}}$ hence (1) assumption is satisfied

When small slopes: $\left(\frac{hL^3}{24D}\right)^2 \ll 1$ if $\frac{hL^3}{24D} = .1 \Rightarrow \frac{P/E}{h/L} \approx .2\left(h/L\right)^2$

(developable surfaces – cylindrical bending only)

$$\boxed{\gamma_{xz}, \gamma_{yz} \sim (h/L)^2 \quad \text{and} \quad \varepsilon_x \varepsilon_y, \varepsilon_z, \gamma_{xy} \sim (h/L)}$$

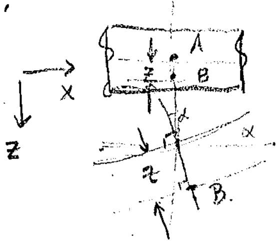
For a neutral surfaces $\left(\frac{hl^3}{24D}\right)^2 \ll \left(\frac{h}{L}\right)^2 \ll 1$ if $\frac{Pl^3}{24D} = .1\frac{h}{L}$ $\quad \frac{P/E}{h/L} \cong .2\left(h/L\right)^3$

nondevelopable surfaces.

$$\Rightarrow \boxed{\gamma_{xz}, \gamma_{yz} \sim (h/L)^3 \quad \epsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy} \sim (h/L)^2}$$ Most typical case

Consequences of Assumptions 1-4 of last lecture

deformations:



let $w_0 = w\big|_{z=0}$

$u =$ displ of B in x dir $= z \sin\alpha$ by (1)

$\sin\alpha \sim \tan\alpha \sim \alpha \quad \therefore u = z\tan\alpha$ by (3)

$$\boxed{= -z\frac{\partial w_0}{\partial x}}$$

also v displ $= \frac{\partial w_0}{\partial y}(-z) = \boxed{-z\frac{\partial w_0}{\partial y} = v}$

Another way $u\big|_z = \underbrace{\cancel{u\big|_{z=0}}}_{\substack{=0 \text{ since} \\ \text{mid is neutral surface by (4)}}} + \frac{\partial u}{\partial z}\bigg|_{z=0} z + \ldots$ if thickness is small $\frac{\partial u}{\partial z}\Big|_{z=0} z$ is large in comparison to rest of terms

Now $\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$ by ① ~~by~~ but $\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x}$ $\therefore$ $u = -\frac{\partial w}{\partial x}\Big|_{z=0} z$

also $\boxed{w = w_0}$

Strains $\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w_0}{\partial x^2}$

$\varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w_0}{\partial y^2}$

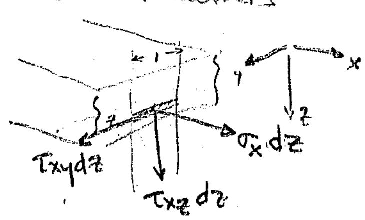
$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w_0}{\partial x \partial y}$ or $\varepsilon_{xy} = -z\frac{\partial^2 w_0}{\partial x \partial y}$

Stress: $\sigma_x = \frac{E}{(1-\nu^2)}\left[\varepsilon_x + \nu\varepsilon_y\right] = -\frac{Ez}{(1-\nu^2)}\left[w_{0,xx} + \nu w_{0,yy}\right]$

$\sigma_y = \frac{E}{(1-\nu^2)}\left[\varepsilon_y + \nu\varepsilon_x\right] = \frac{-Ez}{(1-\nu^2)}\left[w_{0,yy} + \nu w_{0,xx}\right]$

$\tau_{xy} = G\gamma_{xy} = \frac{-Ez}{(1+\nu)} w_{0,xy}$

Stress Resultants

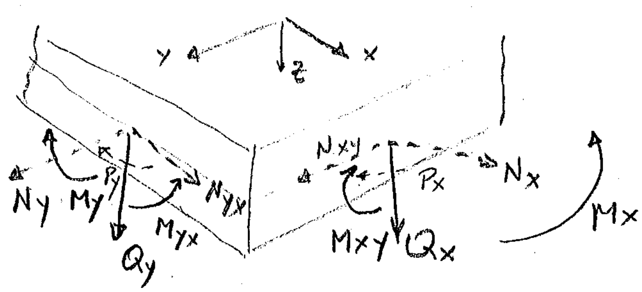


Stress Resultants $\begin{Bmatrix} N_x \\ N_{xy} \\ Q_x \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} \sigma_x \\ \tau_{xy} \\ \tau_{xz} \end{pmatrix} dz$

Stress Couple $\begin{Bmatrix} M_x \\ M_{xy} \\ P_x \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \tau_{xy} \\ 0 \end{Bmatrix} z\,dz$

in classical theory: inplane & out of plane problem decouple ie. N_x, N_{xy}, P_x can be neglected.

---- are neglected resultants



set $N_x = N_y = N_{xy} = N_{yx} = \underset{\text{By dfn}}{P_x = P_y} = 0$

$$M_x = \int \sigma_x z\,dz = -\frac{E}{(1-\nu^2)} \int z^2 dz \left(\frac{\partial^2 w_0}{\partial x^2} + \nu \frac{\partial^2 w_0}{\partial y^2}\right) = \frac{-Eh^3}{12(1-\nu^2)}\left[w_{0,xx} + \nu w_{0,yy}\right]$$

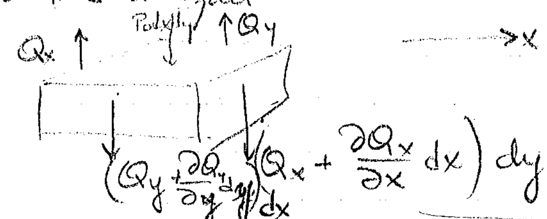
$$= -D\left(\frac{\partial^2 w_0}{\partial x^2} + \nu \frac{\partial^2 w_0}{\partial y^2}\right)$$

$$M_y = -D\left(w_{0,yy} + \nu w_{0,xx}\right) \qquad M_{xy} = M_{yx} = -(1-\nu)D\frac{\partial^2 w_0}{\partial x \partial y} \qquad D = \frac{Eh^3}{12(1-\nu^2)}$$

remember z is not in these formulae anymore, and 0 denotes midplane of plate.

How are these stress resultants in Equil

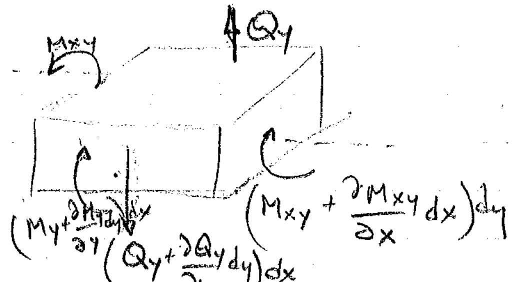
$\underline{\Sigma F_z = 0}$



$$\therefore \quad Q_{x,x}\,dx\,dy + P_0\,dx\,dy + Q_{y,y}\,dx\,dy = 0 \qquad \text{or} \quad \boxed{\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + P_0 = 0}$$

Σ mom in y = Σ mom in x = 0

Σ mom in x dir's



$$\therefore \quad \boxed{-\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0}$$

$$\boxed{-\frac{\partial M_x}{\partial x} - \frac{\partial M_{yx}}{\partial y} + Q_x = 0}$$

$\therefore$ using the results & getting rid of ~~M_{xy}~~ Q_x, Q_y we get.

$$\boxed{\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p}$$

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$$M_x = -D(w_{,xx} + \nu w_{,yy})$$
$$M_y = -D(w_{,yy} + \nu w_{,xx})$$
$$M_{xy} = -(1-\nu)D\,w_{,xy}$$
Equil $\quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0$
$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$
$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{yx}}{\partial x}$$
$$M_{yx} = M_{xy}$$

lead to $\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p$

$$Q_x = \frac{\partial}{\partial x}\left[-D(w_{,xx} + w_{,yy}) + (1-\nu)D\frac{\partial^2 w}{\partial y^2}\right] + \frac{\partial}{\partial y}\left[-D(1-\nu)w_{,xy}\right] = \frac{\partial}{\partial x}\left[-D\nabla^2 w + D^*\frac{\partial^2 w}{\partial y^2}\right] - \frac{\partial}{\partial y}D^* w_{,xy}$$

$$= -\frac{\partial}{\partial x}(D\nabla^2 w) + \frac{\partial D^*}{\partial x}\frac{\partial^2 w}{\partial y^2} - \frac{\partial D^*}{\partial y}\frac{\partial^2 w}{\partial x \partial y}$$

$$Q_y = -\frac{\partial}{\partial y}(D\nabla^2 w) + \frac{\partial D^*}{\partial y}\frac{\partial^2 w}{\partial x^2} - \frac{\partial D^*}{\partial x}\frac{\partial^2 w}{\partial x \partial y}$$

Now put into $Q_{x,x} + Q_{y,y} + p = 0$

$$-\frac{\partial^2}{\partial x^2}(D\nabla^2 w) + \frac{\partial^2 D^*}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \cancel{\frac{\partial D^*}{\partial x}\frac{\partial^3 w}{\partial x\partial y^2}} - \frac{\partial^2 D^*}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} - \cancel{\frac{\partial D^*}{\partial y}\frac{\partial^3 w}{\partial x^2\partial y}}$$

$$-\frac{\partial^2}{\partial y^2}(D\nabla^2 w) + \frac{\partial^2 D^*}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \cancel{\frac{\partial D^*}{\partial y}\frac{\partial^2 w}{\partial y\partial x^2}} - \frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} - \cancel{\frac{\partial D^*}{\partial x}\frac{\partial^3 w}{\partial y^2\partial x}} + p = 0$$

or

$$\boxed{+\nabla^2(D\nabla^2 w) - \frac{\partial^2 D^*}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 D^*}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial^2 D^*}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = p}$$

$D^* = (1-\nu)D$

varying the material properties ~~are~~ is similar to inplane forces. If E, h, ν are independent of x, y, z then

$$D\nabla^2(\nabla^2 w) = p$$

$Q_x = -D\frac{\partial}{\partial x}\nabla^2 w \qquad Q_y = -D\frac{\partial}{\partial y}\nabla^2 w$

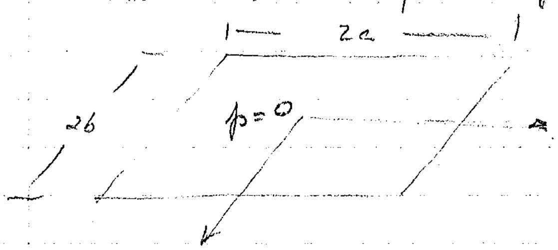
if $w = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + \dots$

	w	# of ind. poly.	M_x, M_y, M_{xy}	Q_x, Q_y	p	
a_0	const	1	0	0	0	} Rigid body
a_1, a_2	linear	2	0	0	0	
a_3, a_4, a_5	quadratic	3	const	0	0	Pure Bending/Twist
a_6, a_7, a_8, a_9	cubic	4	linear	const	0	
	quartic	5	quadratic	linear	const	

if $w = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$

now for $p=0$ $\quad D\nabla^4 w = D(4!a + 4c + 4!e) = p = 0 \quad \Rightarrow \quad c = -6(a+e)$

This then constrains quartic if $p = 0$



HW #4 find most general cubic w for the case when $p=0$ and $y = \pm b$ is a free edge (no edge loading)

sketch loading on edge $x = +a$.

leave out rigid body terms include quadratic & cubic terms.

Solutions to $\nabla^4\varphi = 0$ can be obtained from $\nabla^2\varphi = 0$

since if $\nabla^2\varphi = 0 \Rightarrow \nabla^4\varphi = 0$

$\nabla^2\varphi = 0 \Rightarrow \nabla^4(x\varphi) = \nabla^4(y\varphi) = \nabla^4(r^2\varphi) = 0.$

B.C.

e.g. on $x = $ const edge.

① Built in: clamped. $w(x=\text{const}, y) = 0 \qquad \frac{\partial w}{\partial x}(x=\text{const}, y) = 0$

or at $x = $ const $\quad w, \frac{\partial w}{\partial x} = 0$

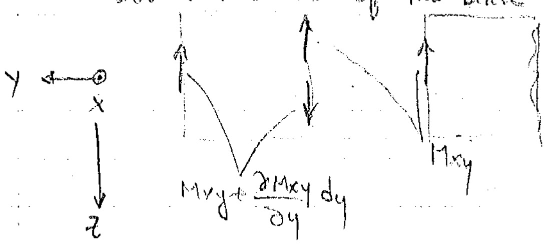
② Simply Supported edge: $w(x=\text{const}, y)=0$ $\quad M_x(x=\text{const},y) = \left.\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right|_{x=\text{con}} = 0$

$$w(c,y)=0 \Rightarrow \frac{\partial w}{\partial y}(c,y)=0 \Rightarrow \frac{\partial^2 w}{\partial y^2}(c,y)=0$$

$$\Rightarrow \boxed{w, \frac{\partial^2 w}{\partial x^2} = 0 \quad @\ x=c.}$$

③ free edge
a) "Stress free": $M_x, M_{xy}, Q_x = 0$ @ $x=c$ gives rise to too many b.c.

Kelvin & Tait in (1883) showed that M_{xy} and Q_x are related and we should see a combination of the two should give you an effective shearing stress.
Look at each cell of the beam



∴ we would get $\frac{\partial M_{xy}}{\partial y} dy \downarrow = Q_x'$

hence the effective transverse shearing is $V_x = Q_x + \frac{\partial M_{xy}}{\partial y}$

thus for free edge must give bc on M_x & V_x

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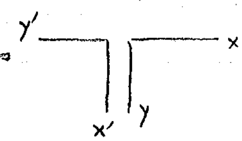
at $x=\text{const.}$ we can have the following BC.

① Clamped Edge
② Simply Supported $\quad w = M_x = 0 \Rightarrow w = w_{xx} = 0 \quad$ but $M_x = w_{,xx} + \nu w_{,yy} = 0 \Rightarrow \nabla^2 w = 0$
③ Free Edge

$$M_x = 0 \quad \text{and} \quad V_x = 0 = Q_x + \frac{\partial M_{xy}}{\partial y} = -D\frac{\partial}{\partial x}(\nabla^2 w) - (1-\nu)D\frac{\partial^3 w}{\partial x \partial y^2} = 0$$

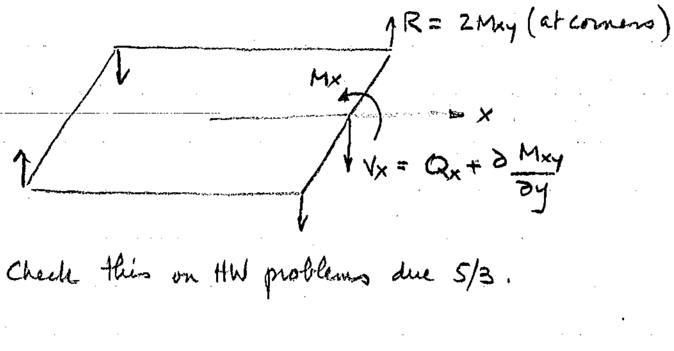
$$\therefore \frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + (2-\nu)\frac{\partial^2 w}{\partial y^2}\right] = 0$$

B.C. on $y=\text{const}$ edge rotate coordinates



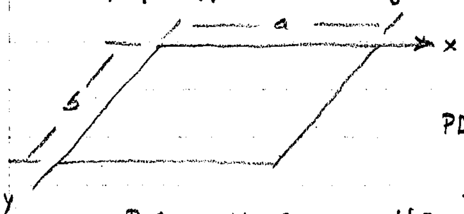
$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rightarrow -\frac{\partial}{\partial y}, \frac{\partial}{\partial x}$$

Plates with corners.



HW Check this on HW problems due 5/3.

simply supported rectangular plate



PDE $\quad \left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right)^2 w = \frac{p(x,y)}{D}$

B.C. $\quad x=0,a \qquad w = \frac{\partial^2 w}{\partial x^2} = 0$

$\qquad\quad y=0,b \qquad w = \frac{\partial^2 w}{\partial y^2} = 0$

plates flat in corners $\Rightarrow$ concentrated forces

Can't separate variables

Look at $\quad w = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad$ satisfies bc

use $\quad w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad$ satisfies all homogeneous b.c.

Substitute into PDE to get.

$$\sum_{m=1}\sum_{n=1} A_{mn}\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+\left(\frac{n\pi}{b}\right)^4\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = \frac{p(x,y)}{D}$$

$$\sum\sum DA_{mn}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = \sum\sum p_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

where $\quad p_{mn} = \frac{4}{ab}\int_0^b\int_0^a p(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$

$\therefore$

$$A_{mn} = \frac{1}{D\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}\frac{4}{ab}\int_0^a\int_0^b p(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

Uniform load $\quad p(x,y)=p_0$

$$P_{mn} = \frac{4}{ab}p_0\underbrace{\int_0^a \sin\frac{m\pi x}{a}\,dx}_{-\frac{a}{m\pi}\cos\frac{m\pi x}{a}\Big|_0^a}\int_0^b \sin\frac{n\pi y}{b}\,dy$$

$$-\frac{b}{n\pi}\cos\frac{n\pi y}{b}\Big|_0^b$$

$= 0 \quad$ if m,n even

$= \frac{4}{ab}p_0\cdot\frac{4ab}{\pi^2 mn} \qquad m,n$ odd

$$A_{mn} = \frac{\dfrac{16 p_0 ab}{ab\pi^2 mn}}{D\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2\pi^4}$$

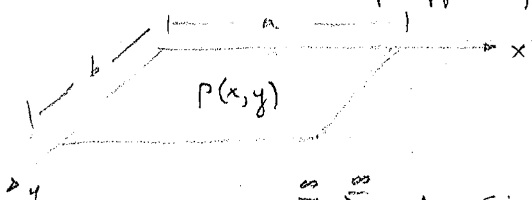
$\therefore$

$$\boxed{w = \frac{16 p_0}{D\pi^6}\sum_{m}\sum_{n\ \text{odd}}\frac{1}{mn}\frac{1}{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}$$

simple support
uniform load

5-3-75

Navier Soln for simply supported plate from last time



$$D\nabla^4 w = p(x,y)$$

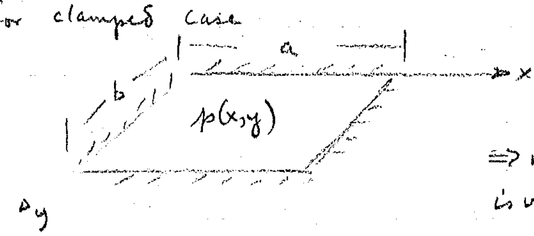
$$w = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$ satisfies bc but not DE

also write $p(x,y) = \sum\sum p_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

put into DE and equate like terms $\Rightarrow A_{mn} = \dfrac{p_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2}$

for clamped case

Aside



$$w_{mn} = \left(1-\cos\frac{2m\pi x}{a}\right)\left(1-\cos\frac{2n\pi y}{b}\right) = 4\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}$$

$\Rightarrow$ we must expand $p(x,y)$ in terms of the $\sin^2$ which is very hard to do.

These two methods are ad hoc: let's see if we can find a systematic method of finding the soln.

PDE: $D\nabla^4 w = p(x,y)$

$w = \nabla^2 w = 0$ for simply supported case: $M_x = M_y = 0 \Rightarrow w_{xx} = 0,\ w_{yy} = 0 \Rightarrow \nabla^2 w = 0$

since $\nabla^2 w$ is "floating" all over the place we can split problem, i.e.

Let $M = D\nabla^2 w = -\dfrac{M_x + M_y}{1+\nu}$

Then $\nabla^2 M = p(x,y)$ w/ $M = 0$ on bdy (since $\nabla^2 w = 0$)

$\nabla^2 w = \dfrac{M}{D}$ w/ $w = 0$ on bdy. (1)

Solving (1): find a particular soln. $w = w_p + w_H$ $\therefore$ $\nabla^2 w_H = 0$ $w_H = -w_p$ on bdy.

if $w_H \Rightarrow \nabla^2 w_H = 0$ use separation of var $\therefore$ $X(x)\,Y(y) = w_H$

$\Rightarrow X'' - \alpha^2 X = 0 \quad Y'' + \alpha^2 Y = 0$ if $\dfrac{X''}{X} = -\dfrac{Y''}{Y} = \alpha^2$

normally we won't be able to satisfy $w_H = -w_p$ on bdy totally $\therefore$ split problem

$\therefore$ solve & satisfy bc on some bdy

(1) $w_{H_1}(x,0) = 0 = w_{H_1}(x,b) = 0 \Rightarrow Y(0) = 0 = Y(b) = 0 \Rightarrow Y_{H_1}(y) = \sin\frac{n\pi y}{b}$

$X_{H_1}(x) = A_n\sinh\frac{n\pi x}{b} + B_n\cosh\frac{n\pi x}{b} \Rightarrow w_{H_1} = \sum_n X_{H_1}(x;n)\,Y_{H_1}(y;n)$

(2) $w_{H_2}(0,y) = 0 = w_{H_2}(a,y) = 0 \Rightarrow X(0) = 0 \quad X(a) = 0 \Rightarrow X_{H_2}(x) = \sin\frac{m\pi x}{a}$

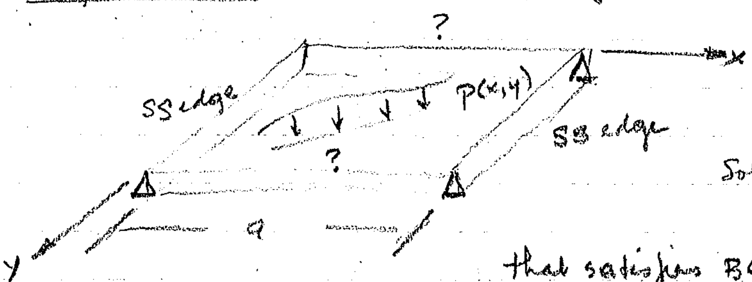
$Y_{H_2}(y) = C_m\sinh\frac{m\pi y}{a} + D_m\cosh\frac{m\pi y}{b} \Rightarrow w_{H_2} = \sum_m X_{H_2}(x;m)\,Y_{H_2}(y;m)$

now we can get A_n, B_n, C_m, D_m from fact that

$w_{H_1}(0,y) = -w_p(0,y)$ and $w_{H_1}(a,y) = -w_p(a,y)$

$w_{H_2}(x,0) = -w_p(x,0)$ and $w_{H_2}(x,b) = -w_p(x,b)$

<u>Lévy Solutions</u> to $D\nabla^4 w = p(x,y)$



? - don't care what are on these edges.

Solution $w = \sum_{n=1} \varphi_n(y) \sin\frac{n\pi x}{a}$

that satisfies BC on SS edges. Then we will let de tell me DE of φ_n

$$D\sum_{n=1}^{\infty}\left[\varphi_n^{IV}(y) - 2\left(\frac{n\pi}{a}\right)^2\varphi_n''(y) + \left(\frac{n\pi}{a}\right)^4\varphi_n(y)\right]\sin\frac{n\pi x}{a} = p(x,y) = \sum_{n=1}^{\infty} f_n(y)\sin\frac{n\pi x}{a}$$

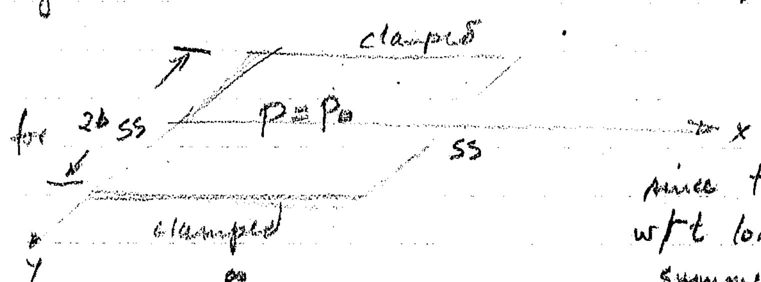
Thus φ_n satisfies the following

$$\therefore \quad \frac{f_n(y)}{D} = \varphi_n^{IV} - 2\left(\frac{n\pi}{a}\right)^2\varphi_n'' + \left(\frac{n\pi}{a}\right)^4\varphi_n = \frac{2}{aD}\int_0^a p(x,y)\sin\frac{n\pi x}{a}\,dx$$

Now $\varphi_n = \varphi_n^P + \varphi_n^H$

thus $\varphi_n^H = \left(y e^{\frac{n\pi}{a}y},\ y e^{-\frac{n\pi}{a}y},\ e^{\frac{n\pi}{a}y},\ e^{-\frac{n\pi y}{a}}\right)$ or $\left(\sinh\frac{n\pi y}{a},\ \cosh\frac{n\pi y}{a},\ y\sinh\frac{n\pi y}{a},\ y\cosh\frac{n\pi y}{a}\right)$

this gives 4 constants and we need 4 bc $\Rightarrow$ 2 bc on $y=b$, $y=0$.



since the problem is symmetric w/r/t load & bc $\Rightarrow$ we need only symmetric fns & 2 bc on $y=b$

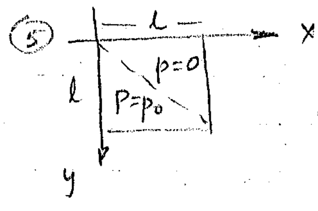
$$w_n = \sum_{n=1}^{\infty}\left(A_n\cosh\frac{n\pi y}{a} + B_n\frac{n\pi y}{a}\sinh\frac{n\pi y}{a}\right)\sin\frac{n\pi x}{a} + \sum_{n=1}^{\infty}\varphi_n^P(y)\sin\frac{n\pi x}{a}$$

w/ $w(x,b) = 0$ & $\frac{\partial w}{\partial y}(x,b) = 0$

5/8/79

<u>HW prob</u>

(5)



all edges are simply supported

find Navier Solution for $w(x,y)$

solve as a suppose of $\frac{p_0}{2}$ everywhere + $p_0/2$ antisymmetric problem.

simply supported plate

Navier Suggested $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = w(x,y)$.

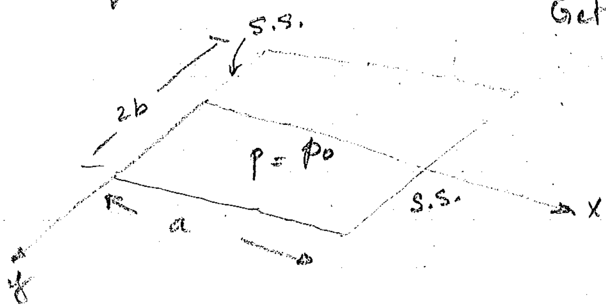
this is good when $w, \frac{\partial^2 w}{\partial x^2} = 0$ at $x=0,a$ $y=0,b$ are given; but try any other bc & problems crop up

Lévy Suggested $w = \sum_{n=1}^{\infty} Y_n(y) \sin\frac{n\pi x}{a}$ satisfy $w(0,y) = w(a,y) = 0$

$Y_n(y)$'s were found as $e^{\pm n\pi y/a}$ $y e^{\pm n\pi y/a}$

Ex

Uniformly loaded s.s. plate using Lévy Solution Let $w = w_p + w_H$



Get partic soln. so we remove load p_0 effect in w_H sol

if particular solution so at $x=0, x=a$ $w=0$ and in x direction the lines deform into cylinder bending $w_p = \frac{p_0}{24D}(x^4 - 2ax^3 + a^3 x)$

$$= \frac{p_0 a^4}{D}\frac{4}{\pi^5}\sum_{n=odd}^{\infty}\frac{1}{n^5}\sin\frac{n\pi x}{a}$$

Now $w = w_p + w_H$ put into PDE $\Rightarrow D\nabla^4 w_H = 0$

$\Rightarrow$ bc give $w_H = \frac{\partial^2 w_H}{\partial x^2} = 0$ on $x = 0, +a$ since $w = w_{,xx} = 0$ & $w_p, w_{p,xx} = 0$

cylindrical bending gives rise to moment about x axis on edges.

since $w = 0$ on $y = \pm b \Rightarrow w_H = -w_p(x, \pm b)$

since $w_{,yy} = 0$ on $y = \pm b \Rightarrow w_{H,yy} = -w_{p,yy}(x, \pm b)$

Now let w_H be a Lévy Solution $\therefore$

$w = w_p + \sum_{n=1}^{\infty} Y_n(y)\sin\frac{n\pi x}{a} \Rightarrow w, \frac{\partial^2 w_H}{\partial x^2} = 0$ on $x = 0, a$ are satisfied

$Y_n(y) = A_n \cosh\frac{n\pi y}{a} + D_n y \sinh\frac{n\pi y}{a}$ satisfy symmetry cond. & satisfy DE

ie look only for solutions satisfying bc at $y = b$ only

$w(y=b) = 0 \Rightarrow \left(A_n \cosh\frac{n\pi b}{a} + D_n b \sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a} = -\frac{p_0 a^4}{D}\cdot\frac{4}{\pi^5 n^5}\sin\frac{n\pi x}{a}$ n is odd

$\Rightarrow \boxed{A_n = D_n = 0 \text{ for } n \text{ even}}$

$w''(y=b) = 0 \Rightarrow \left(\frac{n^2\pi^2}{a^2}\right)\left[A_n\cosh\frac{n\pi b}{a} + D_n b\sinh\frac{n\pi y}{a}\right] + 2D_n\cosh\frac{n\pi b}{a}\cdot\frac{n\pi}{a} = 0$

$\Rightarrow -\frac{a 2 D_n}{n\pi}\cosh\frac{n\pi b}{a} = -\frac{p_0 a^4}{D}\cdot\frac{4}{\pi^5 n^5}$ $\therefore$ $D_n = \frac{2a^3 p_0}{D\pi^4 n^4}\frac{1}{\cosh\frac{n\pi b}{a}}$

$A_n = -\frac{p a^4}{D}\frac{2(\alpha_n \tanh\alpha_n + 2)}{\pi^5 n^5\cosh\alpha_n}$ $\alpha_n = n$

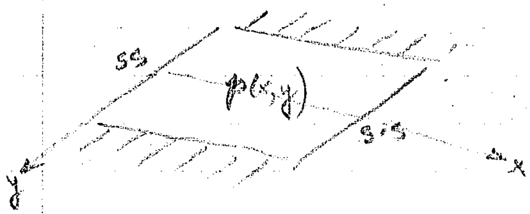
$A_n \sim -\frac{pa^4}{D}\,\frac{4b/a}{(n\pi)^4}\,e^{-n\pi b/a}$ as $n \to \infty$ $A_n \to 0$ as e^{-n}

$D_n \sim \frac{pa^4}{D}\,\frac{4}{a(n\pi)^4}\,e^{-n\pi b/a}$ as $n \to \infty$ $D_n \to 0$ as e^{-n}

$\therefore\ w_{max} = \frac{5}{384}\frac{p_0 a^4}{D} - \frac{4 p_0 a^4}{\pi^5 D}\left(.68562 - .00025 + \dots\right) \sim .00406\,\frac{pa^4}{D}$

convergence of A_n, D_n is very fast and w_{max} converges quickly
to compute stresses again converges quickly but not as quickly as deflection since $A_n, D_n \sim \frac{1}{(n\pi)^2}e^{-n\pi b/a}$

Look at S.S. – Clamped Lévy soln



w_p can again be assumed ~~as~~ cylindrical bending

$w = w_p + \sum Y_n(y)\sin\frac{n\pi x}{a}$

$Y_n(y)$ are all 4 Eigenfunctions which depend on $p(x,y)$ if $p(x,y) \neq p(x)$ only

if we find that we can't get a w_p then

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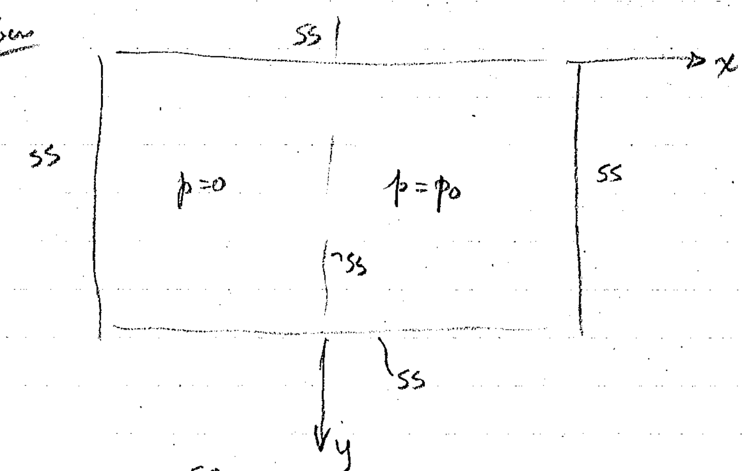
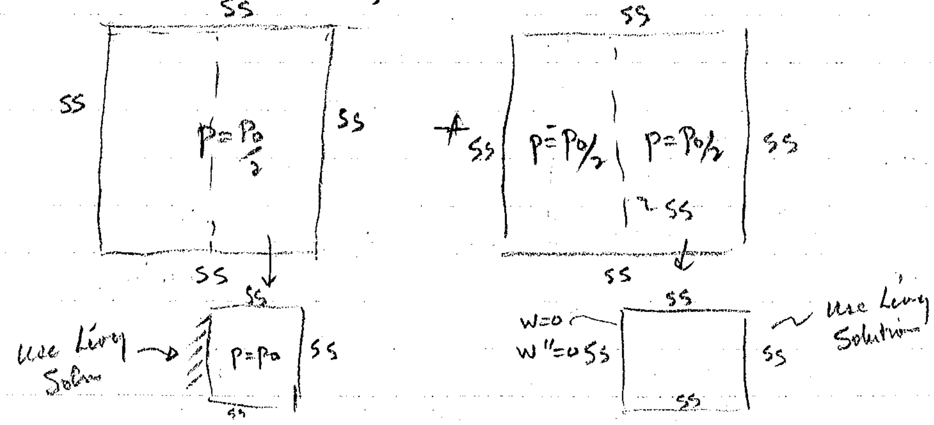
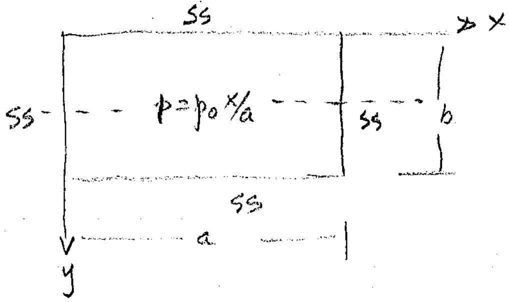
Given



How would you use Superposition



Suppose we are given   like a plate inserted in the ocean



Since it is SS we can use

1. Navier Soln $w = \sum\sum \dfrac{p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{D\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}$

2. Lévy soln.

Since it's symmetric wrt $y = b/2$ we only need n odd

$$p(x,y) = \underbrace{p_0(x/a)}_{x\text{ variation}}\underbrace{(1)}_{y\text{ variation}}$$

using the results of the fourier sine series sheet

$$p(x,y) = p_0 \cdot \frac{x}{a} \cdot 1 = p_0 \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{m\pi} \sin\frac{m\pi x}{a} \sum_{n=odd}^{\infty} \frac{4}{n\pi} \sin\frac{n\pi y}{b}$$

$$p_{mn} = \frac{2(-1)^{m+1}}{m\pi} \cdot \frac{4}{n\pi} = \frac{8p_0(-1)^{m+1}}{\pi^2 mn} \qquad n = \text{odd}.$$

Lévy: $W_p \overset{\text{pick}}{=} f(x) \Rightarrow$ cylindrical bending soln $D\nabla^4 W_p = p$

$D f'''' = p_0 x/a$

$$f = \frac{p_0}{Da}\left[\frac{x^5}{5!} + c_0 x + c_1 x + c_3 x^3\right]$$

B.C. @ $x=0 \Rightarrow c_0 = c_2 = 0 \Leftarrow \begin{cases} w=0 \\ w''=0 \end{cases}$

B.C. @ $x=a \Rightarrow \left.\begin{matrix} \frac{a^5}{5!} + c_1 a + c_3 a^3 = 0 & w=0 \\ \frac{a^3}{3!} + 6c_3 a = 0 & w''=0 \end{matrix}\right\} c_3 = -a^2/36,\ c_1 = \frac{7}{360}a^4$

$$W_p = \frac{p_0 a^4}{360D}\left[3\left(\frac{x}{a}\right)^5 - 10\left(\frac{x}{a}\right)^3 + 7\left(\frac{x}{a}\right)\right]$$

$$= \frac{p_0 a^4}{360D} \sum A_n \sin\frac{n\pi x}{a}$$

where $A_n = 3b_n^{(5)} - 10b_n^{(3)} + 7b_n^{(1)}$ from Fourier Sine handbook

$$b_n^{(5)} = b_n^{(1)} - \frac{20}{(n\pi)^2} b_n^{(3)} \qquad b_n^{(3)} = b_n^{(1)} - \frac{6}{(n\pi)^2} b_n^{(1)} \qquad b_n^{(1)} = \frac{2(-1)^{n+1}}{n\pi}$$

$$\therefore\ A_n = \frac{360}{(n\pi)^4} \frac{2(-1)^{n+1}}{n\pi}$$

Now $W = W_p + \sum_{n=1}^{\infty}\left(B_n \sinh\frac{n\pi y}{a} + C_n \frac{n\pi y}{a}\cosh\frac{n\pi y}{a}\right)\frac{p_0 a^4}{360D}\sin\frac{n\pi x}{a}$

such that bc. at $x=0,a$ are satisfied & PDE is satisfied

Not true!! at $y=0$ we satisfy bc $\Rightarrow$ only need now to satisfy bc at $y = b$

let $n\pi b/a = \alpha_n$ using $W(x,b) = 0$, $W(x,b)_{,yy} = 0$

this gives $\quad A_n + B_n \sinh \alpha_n + C_n \alpha_n \cosh \alpha_n = 0$

and $\quad 0 + \alpha_n^2 \left[B_n \sinh \alpha_n + C_n (2 \sinh \alpha_n + \alpha_n \cosh \alpha_n) \right] = 0$

subtraction gives C_n, B_n

<u>H.W. no. 6</u> fix this soln since @ $y=0$ $\quad w \neq 0$

$\Rightarrow$ we should keep all 4 y solutions sinh, y sinh, cosh, y cosh

another solution of w_p $\quad$ let $w_p = x f(y)$

$$\text{PDE} = D \nabla^4 w_p = D x f^{IV} = p_0 x/a \quad \Rightarrow \quad f = \frac{p_0}{Da}\left[\frac{y^4}{4!} + c_1 y + c_2 y^2 + c_3 y^3\right]$$

this must satisfy bc also $\quad$ at $x=0$ $\quad w_p = 0 \quad w_{p,xx} = 0$

$y = 0 \quad w_p = 0 \quad w_{p,yy} = 0 \Rightarrow c_0, c_2 = 0$

$y = b \quad w_p = 0 \quad w_{p,yy} = 0 \Rightarrow b^4/4! + c_1 b + c_3 b^3 = 0$

$b^2/2 + 6 c_3 b = 0$

or $c_3 = -b/12 \quad c_1 = \ldots$

we now need

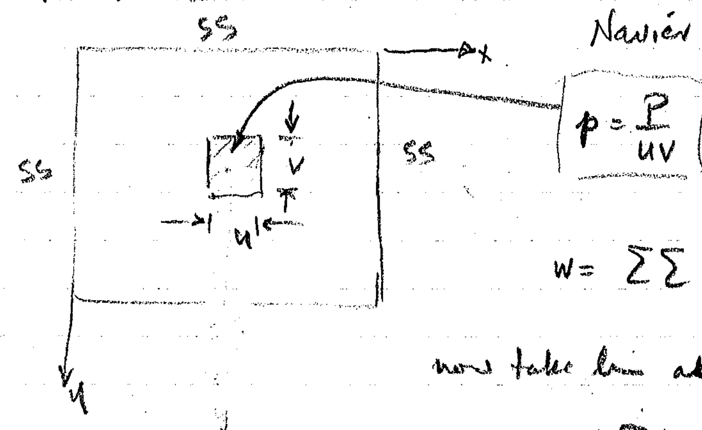
$$w = w_p + \sum_{n=1}^{\infty} \left(B_n \sinh \frac{n\pi x}{b} + C_n \frac{n\pi x}{b} \cosh \frac{n\pi x}{b}\right) \sin \frac{n\pi y}{b}$$

this satisfies at $x=0$, $y=0$, $y=b$ all bc.

@ $x=a$ we can define w_p using fourier sine sheet as fn of $(y/b)^n$ and do as we did before to solve for B_n, C_n in terms of A_n

Midterm will cover everything up to 15 May - Midterm 17 May

Point Load Soln



Navier Soln would give you for a distrib load $\to 0$

$$w = \sum\sum \frac{P_{mn}}{D[\quad]^2} \sin \sin$$

now take lim at $u, v \to 0$

$$P_{mn} = 4\,P/ab \; \frac{\sin \frac{m\pi u}{a}}{m\pi u/a} \; \frac{\sin \frac{n\pi v}{b}}{n\pi v/b}$$

$$\lim_{\substack{u \to 0 \\ v \to 0}} w \Rightarrow \lim_{\substack{u \to 0 \\ v \to 0}} P_{mn} = \frac{4P}{ab}$$

Lévy Soln (using soln for a circular plate)

$$w_p = \frac{-P}{8\pi D} r^2 \ln r = \frac{-P}{16\pi D}(x^2+y^2)\ln(x^2+y^2)$$

$$\bar{w}_p = w_p + c_1 + c_2(x^2+y^2) \qquad c_1, c_2 \rightarrow \; w\,(\text{corners})\ \& \ \text{Mom}\,(\text{corners}) = 0$$

rigid body disp

$$\bar{\bar{w}}_p = -\bar{w}_{BC} + \sum \text{levy soln} \Leftarrow BC\ @\ x = \pm a/2$$

$$w = \bar{\bar{w}}_p + \sum \text{Levy soln} \Leftarrow BC @\ y = \pm b/2$$

For HW #6 a simple change of coordinates will give the soln needed

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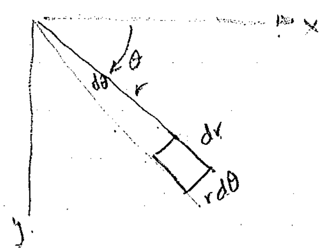
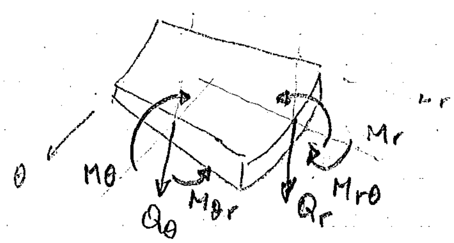
Circular Plate - To derive equilib



PDE $\quad D\nabla^2\nabla^2 w = p(r,\theta)$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{r^2\partial\theta^2}$$



$$M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \nu\left[\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right]\right)$$

$$M_\theta = -D\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \nu\frac{\partial^2 w}{\partial r^2}\right)$$

$$M_{r\theta} = M_{\theta r} = -(1-\nu)D\,\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)$$

$$Q_r = -D\frac{\partial}{\partial r}\nabla^2 w$$

$$Q_\theta = -D\frac{1}{r\partial\theta}\nabla^2 w$$

BC

r=constant edge

1) SS $\quad w(r=\text{const},\theta)=0$ and $M_r(r=\text{const},\theta)=0 \Rightarrow$ all θ deriv are zero

or $\dfrac{\partial^2 w}{\partial r^2} + \dfrac{\nu}{r}\dfrac{\partial w}{\partial r} = 0$

2) clamped $\quad w(r=\text{const},\theta)=0 \quad \dfrac{\partial w}{\partial r}(r=\text{const},\theta)=0 \Rightarrow$ all θ deriv are zero

$\Rightarrow w=0\ \&\ \nabla^2 w = 0$

3) free edge $\quad M_r = 0\ \&\ V_r = 0 = Q_r + \dfrac{1}{r}\dfrac{\partial M_{r\theta}}{\partial\theta} = 0$

Simple Soln for $p(r,\theta)=0$

Rect coord: poly soln in x and y   make subst $x=r\cos\theta$ $y=r\sin\theta$

ie quadratic $w=c(x^2+y^2)$, spherical bending case, converts to to $w=cr^2$ in polar

also $w=c(x^2-y^2) = c(r^2\cos 2\theta)$; $w=cxy = Cr^2\sin 2\theta$

if we add $c(x^2+y^2)+c(x^2-y^2) = 2cx^2 = Cr^2(1+\cos 2\theta)$ cylindrical bending

we can thus generate solutions.

if φ satisfies $\nabla^2\varphi=0 \Rightarrow \nabla^4\varphi=0$ also $x\varphi$ solves $\nabla^4\varphi=0$

$y\varphi$ " " $=0$

$r^2\varphi$ " " $=0$

BC. for complete plate @ $r=0$ @ $r=$ const and periodicity $w(r,\theta)=w(r,\theta+2\pi)$. We cannot use Navier idea here - we can use levy soln.

let $w=f(r)g(\theta)$: $g(\theta)=\cos n\theta$ or $\sin n\theta$ where n is integer to satisfy periodicity. we can write in general $w=f(r)e^{in\theta}$ $n\geq 0$

Put into de $\Rightarrow \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)^2 f=0$ Euler Eqn.

let $f(r)=r^k$ put in de $\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right) r^{k-2}\left[k(k-1)+k-n^2\right]=0$
$[k^2-n^2]$

$\therefore \left[(k-2)^2-n^2\right]\left[k^2-n^2\right]r^{k-4}=0$ $\forall r \Rightarrow$ $n=\pm(k-2), \pm k$
$k=2\pm n, \pm n$

$$W^{(H)} = \text{homog sol} = F_0(r)+\sum_{n=1}^{\infty} F_n(r)\cos n\theta + \bar{F}_n(r)\sin n\theta$$

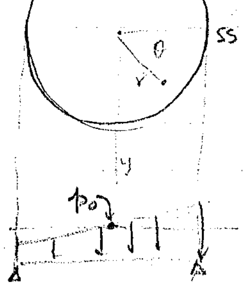
$F_0(r) = A_0+B_0\ln r + C_0r^2+D_0r^2\ln r$

$F_1(r) = A_1r+C_1r^3+D_1 r\ln r + B_1\frac{1}{r}$

$F_n(r) = A_nr^n+B_nr^{-n}+C_nr^{2+n}+D_nr^{2-n}$

$\bar{F}_n(r)$ same as these with $\bar{\ }$ over const

Example: Circular plate w/ non axisymmetric load.



$$\nabla^4 w = \frac{p(r,\theta)}{D} = p_0+p_1(x/a) = p_0+p_1\frac{r\cos\theta}{a}$$

$w(r=a,\theta)=0$ $M_r = \left[\frac{\partial^2 w}{\partial r^2}+\frac{\nu}{r}\frac{\partial w}{\partial r}\right]_{r=a}=0$

regularity at $r=0$ & periodic in θ

$W = W_p + W^{(H)}$

Let $W_p = W_p^{(1)} + W_p^{(2)}$ $\quad W_p^{(1)}$ where $p(r,\theta) = p_0$ & SS bc

$W_p^{(1)} = \frac{p_0}{64D}\left(\frac{5+\nu}{1+\nu}a^2 - r^2\right)(a^2 - r^2)$ solves DE + bc

$W_p^{(2)} = f(r)\cos\theta$ ~~where $f(r) = A_1 r + B_1 \frac{1}{r} + C_1 r^3 + D_1 r \ln r$ from before~~

~~@ $r=0 \Rightarrow B_1 = 0$ for finiteness~~

~~@ $r=0$ $\frac{\partial w}{\partial r}$ must be finite $\Rightarrow D_1 = 0$~~ } ok

~~Now $W^{(2)}(a,\theta) = 0$ and~~

Now put $W^{(2)}$ into de $\Rightarrow D\nabla^4 f(r)\cos\theta = p_1 \frac{r\cos\theta}{a}$

$$D\left[\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)^2 f\right]\cos\theta = p_1 \frac{r\cos\theta}{a}$$

Let $Cr^k = f(r)$ $\quad C\cdot D\left[(k-2)^2 - 1\right]\left[k^2 - 1\right] r^{k-4} = p_1 r/a$ $\quad \Rightarrow k = 5$

$C\cdot D\cdot 24\cdot 8 = p_1/a$ $\quad \therefore \quad f(r) = \frac{p_1}{192Da} r^5$

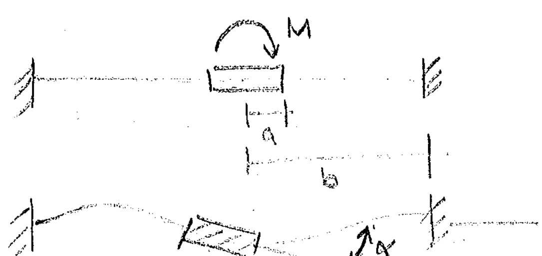
$$\therefore \quad W = W_p^{(1)} + \frac{p_1 r^5 \cos\theta}{192Da} + (A_1 r + C_1 r^3)\cos\theta$$

$W(a,\theta) = 0 \Rightarrow \frac{p_1 a^5}{192Da} + A_1 a + C_1 a^3 = 0$ since $W_p^{(1)}(r=a) = 0$

$M_r = 0 \Rightarrow \left[W_{,rr} + \frac{\nu}{r} W_{,r}\right]\Big|_{r=a} = \frac{p_1 \cdot 20a^3\cos\theta}{192Da} + 6C_1\cos\theta + \frac{\nu}{a}\left[p_1 \frac{a^4\cdot 5\cos\theta}{192Da} + (A_1 + 3C_1 a^2)\cos\theta\right]$

since $W_p^{(1)}$ satisfies $M_r = 0$: using these 2 eqns get C_1, A_1

HW #7



BC @ $r=a$; $W = \alpha x|_{r=a} = \alpha a \cos\theta$

Find $W(r)$

$k = M/\alpha$

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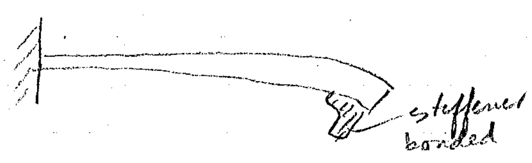
Answers to (5) HW

$$p_{mn} = \frac{4p_0}{\pi^2}\frac{2}{mn(m^2-n^2)}\begin{cases} 0 & m \text{ even} \quad n \text{ even} \\ m^2 & m \text{ even} \quad n \text{ odd} \\ n^2 & m \text{ odd} \quad n \text{ even} \\ m^2 - n^2 & m \text{ odd} \quad n \text{ odd} \end{cases}$$

Problem #6 on exam



$\Rightarrow \overbrace{\frac{\partial^2 w}{\partial x \partial y} = 0}^{\text{no change in slope in y dir}}$ & $\overbrace{\frac{\partial^2 w}{\partial y^2} = 0}^{\text{no curvature in y dir}}$

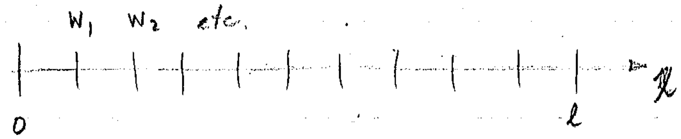
• Approximate methods.

•• Finite Differences

••• 1-D case for beam $\dfrac{d^2}{dx^2}\left(EI\,\dfrac{d^2w}{dx^2}\right) = p(x) \qquad 0 \le x \le l$

Suppose clamped at both ends $w(0) = w(l) = 0 \qquad w'(0) = w'(l) = 0$

if $EI = E(x)I(x) = S(x)$



$i\Delta x = x$

$$w(x+\Delta x) = w(x) + \Delta x\, w'(x) + \frac{\Delta x^2}{2!} w''(x) + \ldots\ldots$$
$$w(x-\Delta x) = \quad - \qquad + \qquad -$$

$$w''(x) = \frac{w(x+\Delta x) + w(x-\Delta x) - 2w(x)}{\Delta x^2} = \frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

$$\frac{d^2}{dx^2}\left(S(x)\frac{d^2w}{dx^2}\right) = \frac{S_{i+1}w''_{i+1} - 2S_i w''_i + S_{i-1}w''_{i-1}}{\Delta x^2} = p_i \qquad i = 1, \ldots, N-1$$

$$w''_{i+1} = \frac{w_{i+2} - 2w_{i+1} + w_i}{\Delta x^2} \qquad w''_{i-1} = w_{i-2} - 2w_{i-1} + w_i$$

at $i = 0, N \qquad w_1 = w_{-1} \qquad w_{N-1} = w_{N+1}$

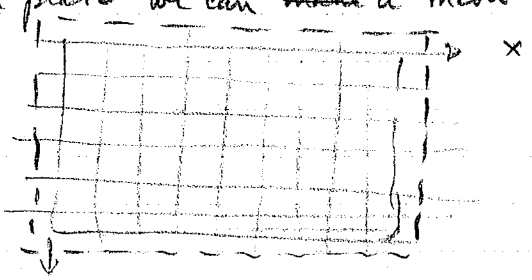
this leads to

$$\begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ \times & \times & \times & \times & \times & \\ & \ddots & & & & \ddots \end{bmatrix} \begin{pmatrix} w_{-1} \\ w_0 \\ w_1 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_1 \\ p_2 \\ \vdots \end{pmatrix}$$

quidiagonal banded

Use gaussian elimination

••• for a plate we can ~~make~~ define a mesh



Example: $D\nabla^4 w = +p$

clamped along edges

we need to introduce dashed lines to obtain

$$\nabla^2 w = \frac{(w_{i-1j} - 2w_{ij} + w_{i+1j})}{\Delta x^2} + \frac{(w_{ij-1} - 2w_{ij} + w_{ij+1})}{\Delta y^2} = \frac{1}{h^2}\left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\}$$

if $\Delta x = \Delta y = h$

$$\nabla^4 w = \frac{1}{h^4}\left\{\begin{matrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{matrix}\right\}$$

$$\Big[\ \text{quidiag}\ \Big] \begin{pmatrix} w_i \end{pmatrix} = \begin{pmatrix} p_i \Delta x^4 \end{pmatrix}$$

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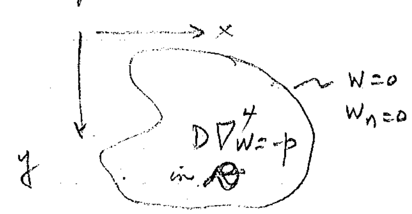
Other methods that have led to Finite Diff Eqns.

<u>Functional Approx.</u> Look at uniform plate

Represent exact soln by some fn.



Given the following problem

Let $\hat{w}$ (approx sol) $= \sum_{n=1}^{N} c_n \underbrace{\varphi_n(x,y)}_{\text{known fns satisfying bc}}$

Looks like the same thing we do in seismology

These give rise to $\sum_{n=1}^{N} \alpha_{in} c_n = \beta_i \quad i = 1, \dots, N$

(α_{in}: known; c_n: unkns; β_i: known)

collocation is one

1. Weighted Residual method.

make residual everywhere $R(x,y) = p - D\nabla^4 \hat{w}$

a. <u>Collocation</u> – pick N points in region & make $R = 0$ at these points

$\Rightarrow$ N eqns in N unknowns.

$$\underbrace{\sum c_n D\nabla^4 \varphi_n \Big|_{P_i}}_{\alpha_{in}} = \underbrace{p\Big|_{P_i}}_{\beta_i} \qquad \text{N eqns for } c_n$$

b. Galerkin method.

$$\int \varphi_i R \, d\Theta = 0 \qquad \text{N eqns } i = 1, \dots, N$$

$$\alpha_{in} = D\int_{\Theta} \varphi_i \nabla^4 \varphi_n \, d\Theta \qquad \beta_i = \int_{\Theta} \varphi_i \, p \, d\Theta$$

c. Least squares

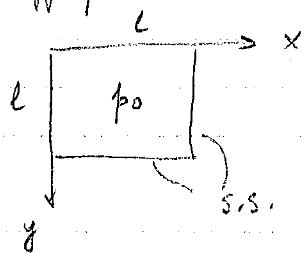
minimize R by picking C's

$$\min_{c_i} \Phi = \iint_{D} R^2\, dD \;\Rightarrow\; \frac{\partial \Phi}{\partial c_i} = 0 \;\rightarrow\; \int R \frac{\partial R}{\partial c_i}\, dD = 0$$

$$\Rightarrow \alpha_{in} = D \iint \nabla^4 \varphi_i\, \nabla^4 \varphi_n\, dx\,dy \qquad \beta_i = \iint p\, \nabla^4 \varphi_i\, dx\,dy$$

coeffs are symmetric hence we can use certain ~~for~~ matrix routines

To apply these to a simple uniformly loaded S.S. plate



pick $\hat{w} = C\left(\sin\frac{\pi x}{\ell}\right)\sin\frac{\pi y}{\ell}$ satisfies bc.

one term navier soln. $W(\text{center}) = \dfrac{p_0}{D\cdot\left(2\frac{\pi^2}{\ell^2}\right)^2} = .0026\dfrac{p_0\ell^4}{D}$

(37% error)

Weighted residual - Collocation

if we pick center, since residual increases $\Rightarrow$ larger at the edges $\therefore$ center is not good place let $R = 0$ at $\left(\frac{2\ell}{\pi^2}, \frac{\ell}{2}\right)$ then $W(\ell/2, \ell/2) = .00432\,\dfrac{p_0\ell^4}{D}$ (1.6% er)

Galerkin Method $\varphi = \sin \pi x/\ell \;\&\; \pi y/\ell$

$$\int_0^{\ell}\int_0^{\ell} \varphi R\, dx\,dy = p_0\left(\frac{2\ell}{\pi}\right)^2 - DC\left(\frac{2\pi^2}{\ell^2}\right)^2\left(\ell/2\right)^2 = 0$$

solve for C = $W(\ell/2, \ell/2) = C = \dfrac{4}{\pi^6}\dfrac{p_0\ell^4}{D} = .00416\,\dfrac{p_0\ell^4}{D}$

Galerkin didn't need us to pick a point & gives better results

for clamped cond Let $\hat{w} = C\left(1 - \cos\frac{2\pi x}{\ell}\right)\left(1 - \cos\frac{2\pi y}{\ell}\right)$

Least Squares $\rightarrow$ gives same result.

Stationary function methods min energy method is one (Ritz Method)

go from PDE ~~to a~~ functional (like Pot. energy) & minimize it

$$\Phi(w) = U(w) - \iint p(x,y)\, w\, dx\,dy$$

Let $\Phi(\hat{w})$ be the function we need to minimize $\min_{\hat{w}} = \Phi \rightarrow \dfrac{\partial \Phi}{\partial c_i} = 0$

we need to know $U(w) = \frac{1}{2}\iint dV\, \sigma_{ij}(x)\, \varepsilon_{ij}(x)$

since σ_z, γ_{xz}, γ_{yz} are negligible in plate theory

then $U(w) = \frac{1}{2}\int\left(\sigma_x \varepsilon_x + \varepsilon_y \sigma_y + \tau_{xy}\gamma_{xy}\right) dV$

$\varepsilon_x = z \cdot K_x$ etc where K_x is the curvature

$$U(w) = \frac{1}{2} D \iint (\nabla^2 w)^2 dx dy + (1-\nu) D \iint \left[(w_{,xy})^2 - (w_{,xx})(w_{,yy})\right] dx dy$$

Now for our simply supported plate uniformly load $\hat{w} = C \sin\frac{\pi x}{\ell} \sin\frac{\pi y}{\ell}$

$$\alpha_{in} = \iint D(\nabla^2 \varphi_i)\nabla^2 \varphi_n \, dx dy + (1-\nu) D \iint \ldots$$

$$\beta_i = \iint p \varphi_i \, dx dy$$

since only ∇^2 we get less differentiation. Major advantage can relax bc. (natural ones).

geometric bc w, w_n

natural bc Moment, Shear

Finite Element is a composition of Ritz method + automatic procedures to generate test fn.

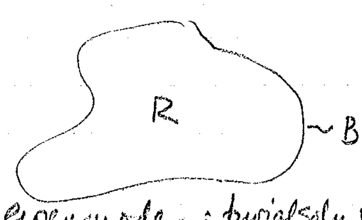
5/29/79

<u>Plate Vibrations</u> - General Problem.

inertial loading $p(\underline{r},t) = -\rho h \frac{\partial^2 w}{\partial t^2}$

PDE $\Rightarrow$

$$D\nabla^4 w + \rho h w_{,tt} = 0 \quad \text{in } R$$



w/ BC ie $w = \frac{\partial w}{\partial n} = 0$ on $\partial R = B$

Not well specified since IC not specified - look for eigenmodes; trivial soln $w=0$

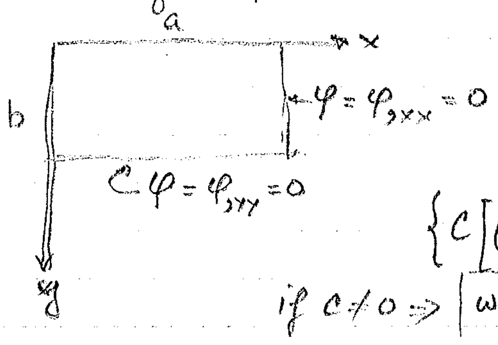
Look for soln $w = \varphi(\underline{r})e^{i\omega t}$

$$\Rightarrow D\nabla^4\varphi - \rho h \omega^2 \varphi = 0 \text{ in } R \text{ w/ } \varphi = \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial R$$

$\varphi = 0$ is trivial sol.

this is an eigenvalue problem for ω

Rectangular problem - SS BC.



$$\nabla^4 \varphi - \frac{\rho h}{D}\omega^2 \varphi = 0$$

using Navier soln method: then $\varphi = C \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

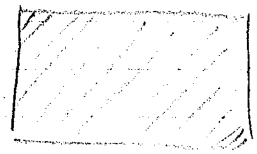
φ satisfies bc then

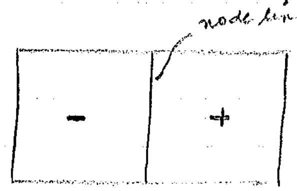
$$\left\{ C\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 - \frac{\rho h}{D}\omega^2 C \right\} \sin(\ ) \sin(\ ) = 0$$

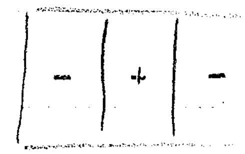
if $C \neq 0 \Rightarrow \boxed{\omega_{mn}^2 = \frac{\pi^4 D}{\rho h}\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^2}$ $\quad \varphi_{mn} = C \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

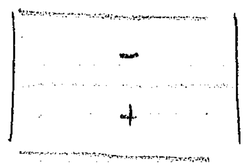
this doesn't take into account damping

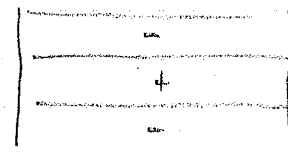
C can only be found if IC are specified ; we really don't care about it

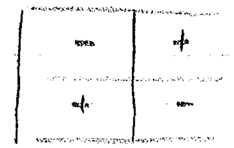
φ_{11} = 

φ_{21} = 

φ_{31} = 

φ_{12} = 

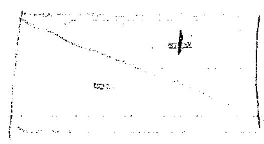
φ_{13} = 

φ_{22} = 

for a square plate $\varphi_{12} = \varphi_{21}$ $\varphi_{13} = \varphi_{31}$ this leads to degenerate modes (where 2 modes collapse into 1) $\omega_{21} = \frac{5\pi^2}{a^2}\sqrt{D/\rho h}$

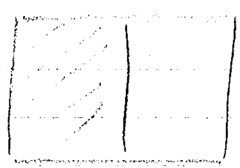
Look at

$$w = C(\varphi_{12}\cos\omega_{12}t + \varphi_{21}\cos\omega_{21}t) = C(\varphi_{12} + \varphi_{21})\cos\omega_{12}t$$ this is an eigenmode

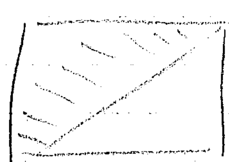


$$w = C(\varphi_{12}\cos\omega_{12}t - \varphi_{21}\cos\omega_{21}t) = C(\varphi_{12} - \varphi_{21})\cos\omega_{12}t$$ thus this is also an eigenmode

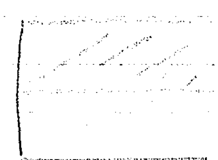
if $w = C(\varphi_{12}\cos\omega_{12}t \pm \varphi_{12}\sin\omega_{12}t)$ then



$t = 0$



$t = T/8$

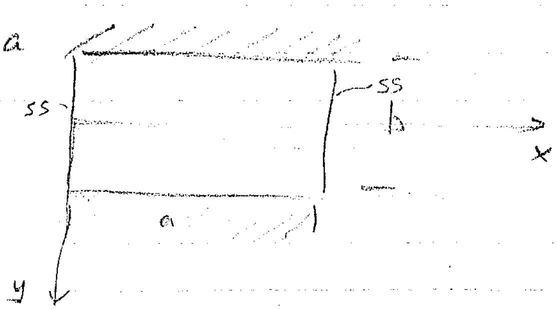


$t = T/4$

Thus the nodal line is rotating

HW #9 Look at plate with $b \times a$

find eigensoln for symmetric modes about centerline



General Case $\nabla^2(\nabla^2 w) - \frac{\rho h}{D}\omega^2 \varphi = 0$

$$\Rightarrow \left(\nabla^2 - \sqrt{\tfrac{\rho h}{D}}\,\omega\right)\left(\nabla^2 + \sqrt{\tfrac{\rho h}{D}}\,\omega\right)\varphi = 0$$ not separable in most cases

if $\left(\nabla^2 - \sqrt{\frac{\rho h}{D}}\,\omega\right)\varphi = 0$ or $\left(\nabla^2 + \sqrt{\frac{\rho h}{D}}\,\omega\right)\varphi = 0$ then original eqn is satisfied. This is the helmholtz eqn. which is separable in many case.

Circular plate $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$; with $k^2 = \sqrt{\frac{\rho h}{D}}\,\omega$

$\therefore\ (\nabla^2 \pm k^2)\varphi = 0$ using $\varphi = R(r)T(\theta) \Rightarrow \frac{r^2R'' + rR' \pm k^2r^2R}{R} = -\frac{T''}{T} = \lambda_{const}$

R = bessel fns, T are trig fn.
for a complete plate causality requires at $r=0$ R must be finite and single valuedness requires periodicity of T. $\Rightarrow \lambda$ must be integer

$\therefore\ T = \sin n\theta, \cos n\theta$ and $r^2R'' + rR' - (n^2 \mp k^2r^2)R = 0$
$\therefore\ R = A_nJ_n(kr) + B_nI_n(kr)$ I_n is modified Bessel fns.

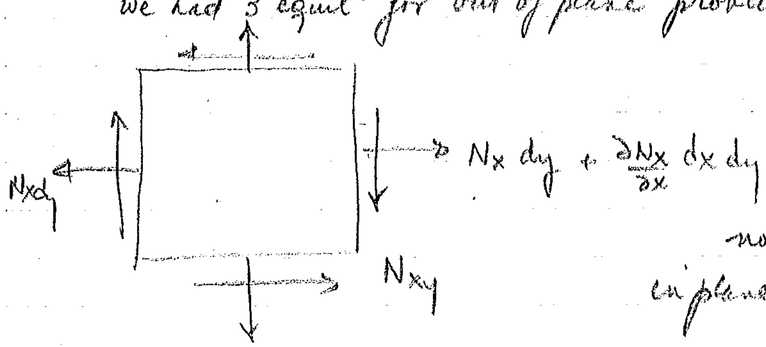
for a clamped plate $R(r=a) = 0$ $\frac{dR}{dr}(r=a) = 0$ since $w(a,\theta,t) = 0 = \left.\frac{\partial w}{\partial r}\right|_{r=a} = 0$

for nontrivial solns requires Wronskian of $\left(R, \frac{dR}{dr}\right) = 0$

Wronskian gives $\boxed{\frac{J_{n+1}(ka)}{J_n(ka)} = \frac{I_{n+1}(ka)}{I_n(ka)}}$ since R_n' can be written interms of R_{n+1}

This is our characteristic eqn that gives the value of ω through k

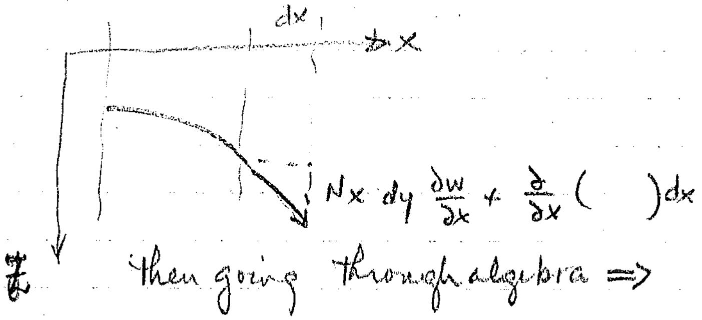
Effects of midplane forces — ie Buckling
we had 3 equil for out of plane problem $\Sigma F_x, \Sigma F_y, \Sigma M_z = 0$



ie $\Sigma F_x = 0 \Rightarrow \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$

now we want to find how these inplane affect out of plane deformation

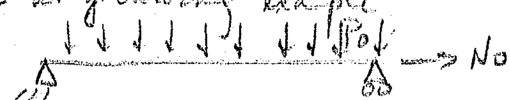
THUS:



then going through algebra $\Rightarrow$ $D\nabla^4 w = p(x,y) + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}\left(N_{xy}\frac{\partial w}{\partial y}\right)$
$+ \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y}\right)$

Normally since the inplane & out of plane problem are decoupled solve inplane problem to get N_x, N_{xy}, N_y put here then solve for w the out of plane deformation

Look at following example



SS plate $\Rightarrow$ Navier soln is OK here

PDE $\quad D\nabla^4 w - N_0 \frac{\partial^2 w}{\partial x^2} = p(x,y)$

Navier sol $\quad w = \sum\sum C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

$$p(x,y) = \frac{16 p_0}{\pi^2} \sum_{\substack{mn \\ odd}} \frac{1}{mn} \sin(\quad)\sin(\quad)$$

Put w & p into PDE to get

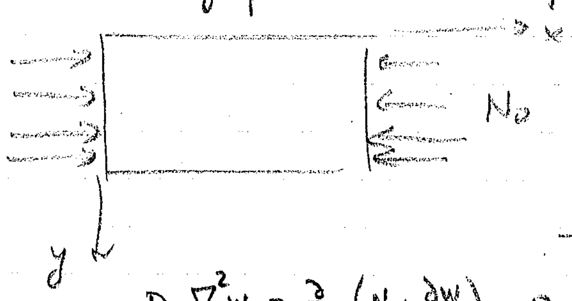
$$\sum\sum\left[C_{mn} D\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + N_0\left(\frac{m\pi}{a}\right)^2 - \frac{16 p_0}{\pi^2 mn}\right] \sin\frac{m\pi x}{a} \sin\frac{n\pi x}{b} = 0$$

$\Rightarrow C_{mn}$ can be obtained $\qquad C_{mn} =$ ______

$$\therefore \quad w = \sum_{n,m\ odd}\sum \frac{16 p_0}{\pi^6 D} \frac{\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{nm\left\{\left[\left(\frac{m^2}{a^2}\right) + \frac{n^2}{b^2}\right]^2 + \frac{N_0 m^2}{D\pi^2 a^2}\right\}}$$

as $N_0 \uparrow$ $\quad w \downarrow$ since denom $\uparrow$
for buckling for $N_0 < 0$ $\Rightarrow$ denom can be made $= 0$.

Buckling of SS plate w $p = 0$



Continuation $\qquad$ 5/31/79

$$D\nabla^2 w - \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) = 0$$

BC $\quad w = w_{xx} = 0 \quad x = 0, a \qquad w = w_{yy} = 0 \quad y = 0, b$

$N_x = - P/b$

$W = \sum\sum C_n \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$

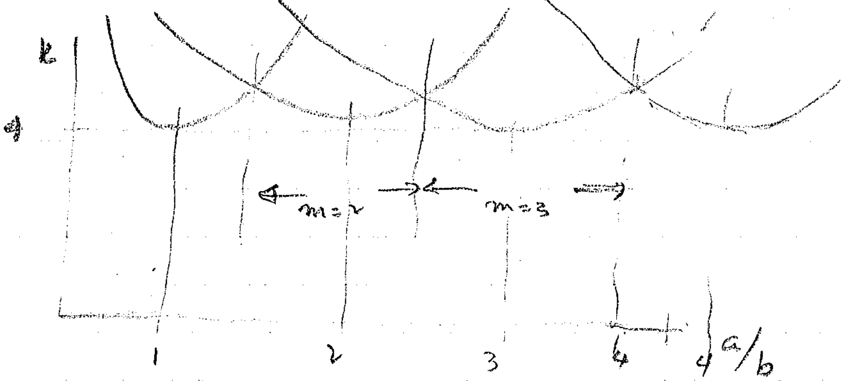
PDE $\longrightarrow \quad \sum\sum C_{mn} \left[D \left\{ \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right\}^2 - \frac{P}{b}\left(\frac{m\pi}{a}\right)^2 \right] \sin \sin = 0$

Nontrivial sol (buckling) if

$\Rightarrow P/b = \left(\frac{a}{m\pi}\right)^2 D \left[\quad \right]^2 = \frac{\pi^2 D}{b^2}\left(\frac{mb}{a} + n^2 \frac{a}{mb}\right)^2$

$P_{cr} = \frac{\pi^2 D}{b^2} \min_{m,n} \left(\frac{mb}{a} + n^2\frac{a}{mb}\right)^2 \Longrightarrow n = 1$ for min

$P_{cr} = \frac{\pi^2 D}{b^2} \min_{m} \left(\frac{mb}{a} + \frac{a}{mb}\right)^2 = \frac{\pi^2 D}{b^2} K(a/b)$



euler results $\quad \frac{4\pi^2 EI}{b^2(1-\nu^2)} = \frac{4\pi^2 D}{b} \Rightarrow \ell^2 = b^2\frac{(1-\nu^2)}{4}$

for large deflection we normally get 8^{th} order problem since two 4^{th} order problems are coupled (non linear terms)

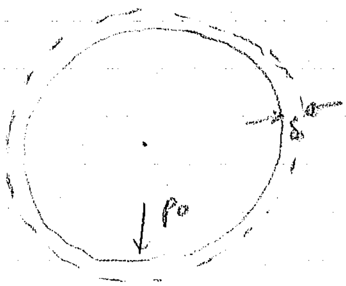
for small deflection we can uncouple the in/out of plane problems

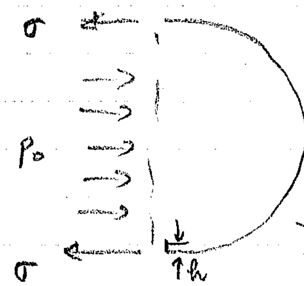
In thin shells stretching & bending are coupled : for the case where $w \ll h$ we get an eight order linear

Approximation 1. Membrane (bending is negligible) $\Rightarrow$ 4^{th} order eq
2. Inextensional buckling (stretching is negligible $\Rightarrow$ 4^{th} order eq

we will look at membranes using strength of materials approach

Look at pressurized spherical shell must give thickness, E, ν and shape





Equilib

$$\underbrace{p_0 \pi R^2}_{\text{force}} = \underbrace{\sigma \cdot 2\pi R h}_{\text{force}}$$

$$\sigma = \tfrac{1}{2} p_0 \frac{R}{h}$$

there is no shearing stress

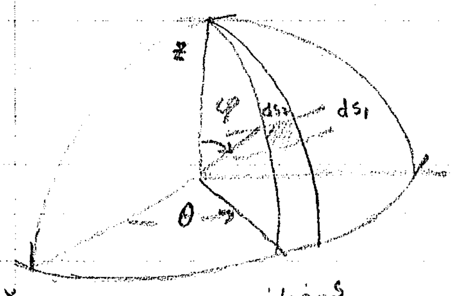
$$\left.\begin{aligned} \varepsilon_1 &= \frac{1}{E}(\sigma_1 - \nu\sigma_2) \\ \varepsilon_2 &= \frac{1}{E}(\sigma_2 - \nu\sigma_1) \end{aligned}\right\}$$

strain in any two directions
by symmetry: $\varepsilon_1 = \varepsilon_2 = \varepsilon$, $\sigma_1 = \sigma_2 = \sigma$ $\Rightarrow$ $\varepsilon = \frac{1-\nu}{E}\sigma$

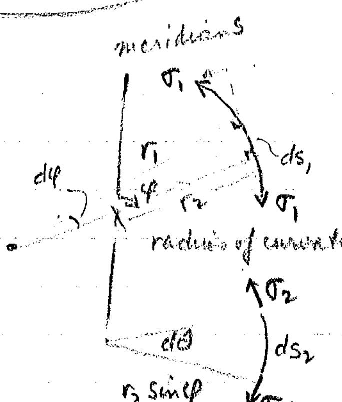
$$\therefore \quad \varepsilon = \frac{1-\nu}{2} \frac{p_0 R}{E h} \ll 1 \quad \Rightarrow \quad \frac{p_0}{E} \ll \frac{h}{R} \ll 1$$

$$\varepsilon = \frac{\text{circum after press} - \text{circum before press}}{\text{circum before press}} = \frac{2\pi(R+\delta) - 2\pi R}{2\pi R} = \frac{\delta}{R}$$

Shells of Revolution – midsurface of shell – Axisymmetric

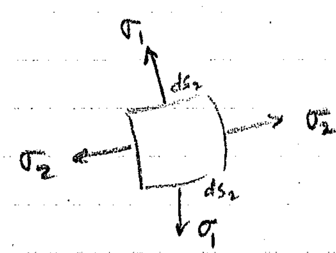


θ = longitude
φ = colatitude and for axis which is normal to shell at surface



$$r_1\, d\varphi = ds_1$$

$$r_2 \sin\varphi\, d\theta = ds_2$$



Equil for axisym pressure loading normal to shell

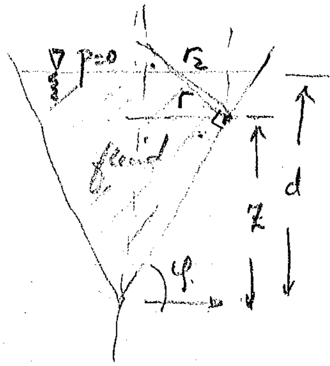
$$\begin{aligned} p_0\, ds_1\, ds_2 &= \sigma_1 h\, ds_2\, d\varphi + \sigma_2 h\, ds_1\, d\theta \sin\varphi \\ &= \sigma_1 h\, ds_2 \frac{ds_1}{r_1} + \sigma_2 h\, ds_1 \frac{ds_2}{r_2} \end{aligned}$$

$$\boxed{\frac{p_0}{h} = \frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2}}$$

p_0 can vary with φ but not θ

for spherical shell $r_1 = r_2$ $\quad \sigma_1 = \sigma_2 \quad \Rightarrow \quad \frac{p_0}{h} = \frac{2\sigma}{r}$ or $\sigma = \frac{1}{2}\frac{p_0 r}{h}$ ✓

exple: Conical Tank — one of the radii of curv = ∞



use r, z, θ to locate points on shell

$\tan\varphi = \frac{z}{r}$

$\frac{1}{r_1}$ (curv of meridian) $= 0$

$r = r_2 \sin\varphi \qquad r_2 = \frac{r}{\sin\varphi} = \frac{z}{\sin\varphi\tan\varphi}$

$p = \rho g(d - z)$ if p

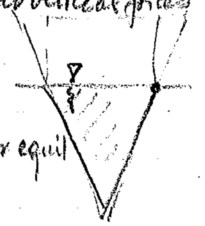
$$\frac{p}{h} = \frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} \Rightarrow \frac{\sigma_2}{r_2} = \frac{p}{h} \quad \text{or} \quad \sigma_2 = \frac{p r_2}{h} = \frac{\rho g(d-z)}{h}\frac{z}{\sin\varphi\tan\varphi}$$

hoop tension $\sigma_2 = \frac{\rho g}{h}(d-z)z\frac{\cos\varphi}{\sin^2\varphi} = \rho g d\frac{\cos\varphi}{\sin^2\varphi}\frac{d}{h}\left(1 - \frac{z}{d}\right)\frac{z}{d}$

$\sigma_2 = 0$ at top & bottom, ∴ max at mid point

meridian stress use equil analysis

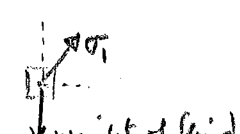
Look at vertical forces



consider equil

$$\sin\varphi\,\sigma_1 \cdot 2\pi r h = \rho g\underbrace{\left[\tfrac{1}{3}\pi r^2 z\right]}_{\text{volume of cone ①}} + \rho g\underbrace{\left[\pi r^2(d-z)\right]}_{\text{volume of cylinder}}$$ neglect weight of shell it is small

$r^2 = \frac{z^2}{\tan^2\varphi} \quad \Rightarrow \quad \boxed{\sigma_1 = \rho g d\frac{\cos\varphi}{2\sin^2\varphi}\frac{d}{h}\left(1 - \frac{2}{3}\frac{z}{d}\right)\frac{z}{d}}$



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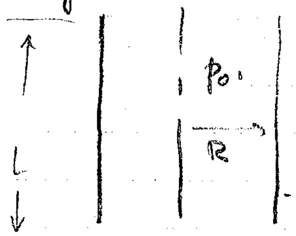
Continuation & review

Membrane soln (internal pressure)

$\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = p/h$

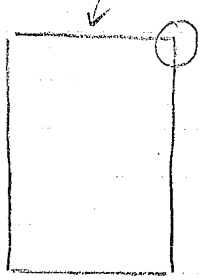
Sphere $\quad \sigma_1 = \sigma_2 = \frac{1}{2}\frac{R}{h}p \qquad \frac{\delta}{R} = \varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) = \frac{1-\nu}{2E}\frac{R}{h}p_0$

Cylinder



$r_1 = \infty$ $\quad r_2 = R$ $\qquad \sigma_2 = \frac{Rp}{h}$ $\qquad \sigma_1 = 0$

with end plates $\quad p_0 \pi R^2 = \sigma_1 \cdot 2\pi R h \qquad \sigma_1 = \frac{p_0 R}{2h} \qquad \sigma_2 = \frac{R}{h} p_0$

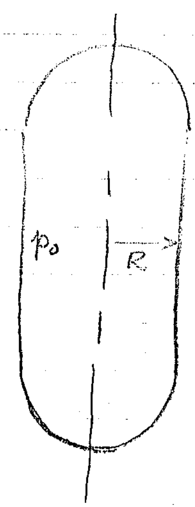


bending is important here; results not valid here

$$\frac{\delta}{R} = \varepsilon_2 = \frac{\sigma_2 - \nu\sigma_1}{E} = \frac{2-\nu}{E}\sigma_1 = \frac{2-\nu}{E}\frac{p_0 R}{2h}$$

Inextensional Bending (internal pressure)

If materials are same

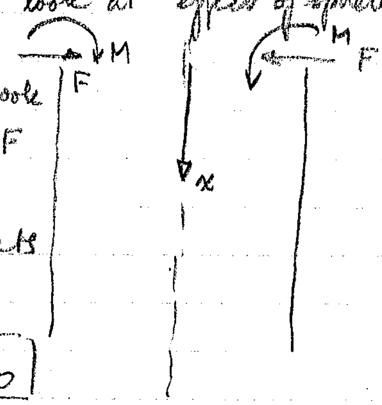


$\delta_{sphere} = \frac{1-\nu}{2E} p_0 \frac{R^2}{h}$ $\qquad$ from before

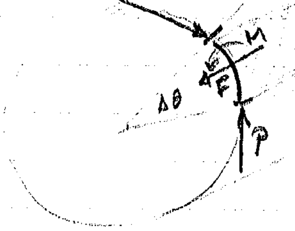
$\delta_{cyl} = \frac{2-\nu}{2E} p_0 \frac{R^2}{h}$ $\qquad$ missmatch is $\frac{1}{2}\frac{p_0}{E}\frac{R^2}{h}$ $\quad$ ie $\delta_s = \delta_c +$

Thus must look at effect of sphere on cylinder

Thus we look at & pick F and M .э. the displacements match



Look at section of cylinder $\boxed{\text{Let } M=0}$



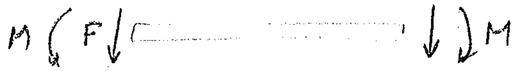
Since $\sigma_1 = 0$ then $\varepsilon_2 = \frac{\sigma_2}{E} = \frac{-u}{R}$

$\therefore \sigma_2 = \frac{-uE}{R}$ $\qquad u = u_{radial}$

$$p = |\sigma_2| h = \frac{uEh}{R}$$

$\therefore$ radial dis(?) $= p\Delta\theta = \frac{uEh}{R}\Delta\theta$

Now treat strip like beam. $\quad \frac{EI}{1-\nu^2}\frac{d^4u}{dx^4} = -\frac{Eh}{R}u\,\Delta\theta$



$$I = \frac{bh^3}{12} = \frac{R\Delta\theta\, h^3}{12} \qquad \therefore \qquad \frac{d^4u}{dx^4} + 4\alpha^4 u = 0 \qquad \alpha^4 = \frac{Eh}{4DR^2} = \frac{3(1-\nu^2)}{h^2R^2}$$

if $u = e^{sx} \Rightarrow s^4 + 4\alpha^4 = 0 \qquad s = \alpha(1 \pm i) \qquad s = \alpha(-1 \pm i)$

since we want soln to die off as we go to center of cylinder $\Rightarrow s = \alpha(-1 \pm i)$

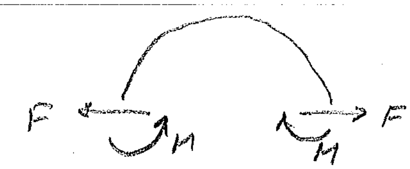
$u = e^{-\alpha x}(C_1 \cos\alpha x + C_2 \sin\alpha x)$

$\therefore$ @ $x = 0$ BC $M = 0$ $V = -F$

$$\Rightarrow u''(0) = 0 \qquad u'''(0) = F/D$$

$$C_2 = 0 \qquad C_1 = \frac{E}{2D\alpha^3}$$

we now look at spherical cap



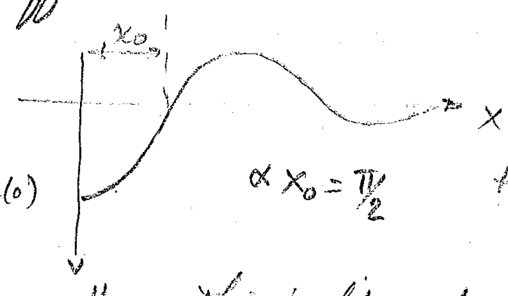
Result: F produces same u, u' but outward $\therefore M = 0$

now pick F so that $u(0)_{\text{cylinder}} = \frac{1}{2}\left(\frac{1}{2}\frac{p_0}{E}\frac{R^2}{h}\right)$

and $u(0)_{\text{sphere}} = \frac{1}{2}\left(\frac{1}{2}\frac{p_0}{E}\frac{R^2}{h}\right)$

but $u(0) = C_1 = \frac{1}{4}\frac{p_0}{E}\frac{R^2}{h}$

if 2 different materials then $\Rightarrow M \neq 0$



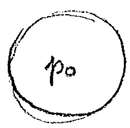
$\alpha x_0 = \pi/2$ take $\nu = .3$ $\qquad x_0 = 1.2\sqrt{Rh} \qquad \frac{x_0}{R} = 1.2\sqrt{\frac{h}{R}}$

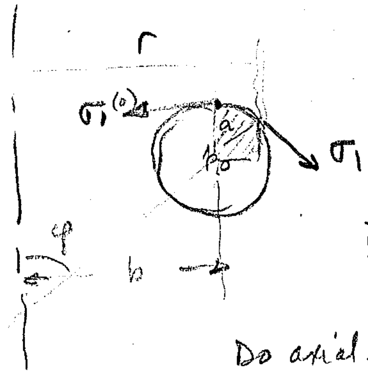
This is like a boundary layer effect.

this is true for any length of cylinder

Normally we make cylinders thicker so that shell/sphere move out/in same amount.

pressurized tire





$r_1 = a \qquad r_2 = r\sin\varphi \qquad r = b + a\sin\varphi$

$$\frac{\sigma_1}{R_1} + \frac{\sigma_2}{R_2} = \frac{p_0}{h}$$ gives φ dependence

h = thickness

Do axial equil on [▨] i.e. in z dir: $\sigma_1 \sin\varphi \cdot 2\pi r \cdot h = p_0(\pi r^2 - \pi b^2)$

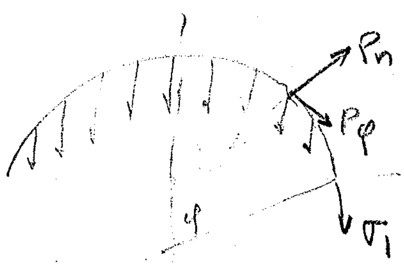
these two give

$$\sigma_1 = \frac{a}{h} p_0 \frac{1 + \frac{1}{2}\frac{a}{b}\sin\varphi}{1 + \frac{a}{b}\sin\varphi}$$

$$\sigma_2 = \tfrac{1}{2} p_0 \, a/h$$

if $a \ll b$ then torus $\Longrightarrow$ cylinder

membrane solution (axisym loading) Shells of revolution



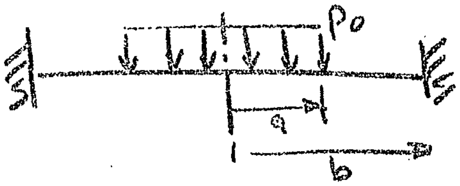
1) $$\frac{\sigma_1}{R_1} + \frac{\sigma_2}{R_2} = \frac{p_n}{h}$$

2) $$\sigma_1 \sin\varphi \, 2\pi r h = \int_0^\varphi \underbrace{(p_n \cos\varphi - p_\varphi \sin\varphi)}_{p_z} 2\pi r \cdot R_1 \, d\varphi$$

$r = R_2 \sin\varphi$

$$\sigma_1 = \frac{1}{h r_2 \sin^2\varphi} \int_0^\varphi R_1 R_2 \, p_z \sin\varphi \, d\varphi$$

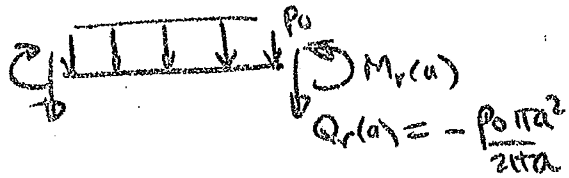
SOLUTION : PROB. 3



Solve by superimposing solutions developed in lecture

Find $w(0)$

$\underline{0 \le r \le a}$:



$$w = \delta + \frac{p_0}{64D}(a^2-r^2)^2 + \frac{M_1}{2(1+\nu)D}(a^2-r^2)$$

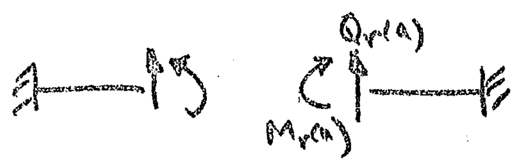
rigid body translation

clamped uniformly loaded

edge couple

$$\rightarrow \quad w(a) = \delta \qquad w'(a) = -\frac{M_1 a}{(1+\nu)D} \qquad M_r(a) = M_1 - \frac{p_0 a^2}{8}$$

$\underline{a \le r \le b}$:



$$w = \frac{p_0 a^2}{16D}(b^2 - r^2 + 2r^2 \ln r/b) + \frac{M_2}{2D}\,\frac{b^2 - r^2 + 2b^2 \ln r/b}{1+\nu+(1-\nu)b^2/a^2}$$

clamped with edge shearing force on inner edge (pt. load at center)

clamped with couple on inner edge

$$w(a) = \frac{p_0 a^2}{16D}\left(b^2 - a^2 + 2a^2 \ln \frac{a}{b}\right) + \frac{M_2}{2D}\,\frac{b^2 - a^2 + 2b^2 \ln a/b}{1+\nu+(1-\nu)b^2/a^2}$$

$$w'(a) = \frac{p_0 a^3}{4D} \ln \frac{a}{b} + \frac{M_2 a}{D}\,\frac{b^2/a^2 - 1}{1+\nu+(1-\nu)b^2/a^2}$$

$$M_r(a) = -\frac{p_0 a^2}{4}\left[1 + (1+\nu)\ln \frac{a}{b}\right] + M_2$$

Determine δ, M_1, M_2 by requiring continuity at $r=a$

M_r cont. $\rightarrow$ $\quad M_1 - \frac{p_0 a^2}{8} = M_2 - \frac{p_0 a^2}{4}\left[1+(1+\nu)\ln\frac{a}{b}\right]$

w' cont. $\rightarrow$ $\quad -M_1 = M_2 \frac{(1+\nu)(b^2/a^2-1)}{1+\nu+(1-\nu)b^2/a^2} + (1+\nu)\frac{p_0 a^2}{4}\ln\frac{a}{b}$

Add to eliminate M_1 and then solve for M_2. Substitution gives M_1

$$\underline{M_2 = \frac{p_0 a^2}{16}\frac{a^2}{b^2}\left[1+\nu+(1-\nu)b^2/a^2\right]}$$

$$\underline{M_1 = -\frac{p_0 a^2}{16}(1+\nu)\left(1-\frac{a^2}{b^2}+4\ln\frac{a}{b}\right)}$$

w cont. $\rightarrow$ $\quad \delta = \frac{p_0 a^2}{16D}\left(b^2-a^2+2a^2\ln\frac{a}{b}\right) + \underbrace{\frac{M_2}{2D}\frac{b^2-a^2+2b^2\ln a/b}{1+\nu+(1-\nu)b^2/a^2}}_{=\ \frac{p_0 a^2}{32D}\frac{a^2}{b^2}\left(b^2-a^2+2b^2\ln\frac{a}{b}\right)}$

$$\underline{\delta = \frac{p_0 a^2}{32D}\left[\left(2+\frac{a^2}{b^2}\right)(b^2-a^2)+6a^2\ln\frac{a}{b}\right]}$$

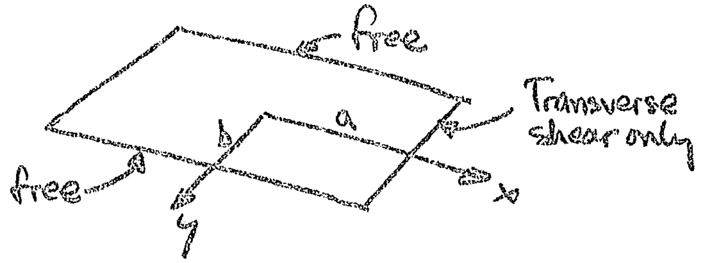
Substitution gives:

$$w = \begin{cases} \delta + \frac{p_0 a^4}{32D}\left(1-\frac{r^2}{a^2}\right)\left[\frac{1}{2}\left(1-\frac{r^2}{a^2}\right)-\left(1-\frac{a^2}{b^2}+4\ln\frac{a}{b}\right)\right] & 0\le r\le a \\ \frac{p_0 a^4}{32D}\left[\left(1+2\frac{b^2}{a^2}\right)\left(1-\frac{r^2}{b^2}\right)+\left(1+2\frac{r^2}{a^2}\right)\ln\frac{r^2}{b^2}\right] & a\le r\le b \end{cases}$$

Evaluating $w(0)$:

$$w(0) = \delta + \frac{p_0 a^4}{32D}\left(\frac{1}{2}-1+\frac{a^2}{b^2}-4\ln\frac{a}{b}\right) = \underline{\frac{p_0 a^4}{16D}\left(\frac{b^2}{a^2}-\frac{3}{4}-\ln\frac{b}{a}\right)}$$

SOLUTION : PROB (4) ME241A



w cubic

Let $w = C_1x^3 + C_2x^2y + C_3xy^2 + C_4y^3$

$$M_y = -D(w_{,yy} + \nu w_{,xx}) = -D\left[(2C_3 + 6\nu C_1)x + (6C_4 + 2\nu C_2)y\right]$$

$$M_{xy} = -(1-\nu)D\,w_{,xy} = -(1-\nu)D\,(2C_2x + 2C_3y)$$

$$Q_y = -D\frac{\partial}{\partial y}\nabla^2 w \quad = -D(6C_4 + 2C_2)$$

BC on $y = \pm b$: $M_y = 0 \rightarrow (C_3 + 3\nu C_1)x \pm (3C_4 + \nu C_2)b = 0$

$$\rightarrow C_3 = -3\nu C_1 \qquad C_4 = -\frac{\nu}{3}C_2$$

$$V_y = Q_y + M_{xy,x} = 0 \rightarrow 6C_4 + 2C_2 + (1-\nu)2C_2 = 0 \rightarrow C_4 = -\frac{2-\nu}{3}C_2$$

$\nu = 2-\nu \Rightarrow \nu = 1$ is impossible $\therefore C_2 = C_4 = 0 \rightarrow \boxed{w = C_1(x^3 - 3\nu xy^2)}$

BC on $x = a$: $M_x = -6(1-\nu^2)DC_1a \qquad V_x = -6(1-\nu)^2DC_1$

corner force $R = -2M_{xy}(a,b) = -12\nu(1-\nu)DC_1b$

To eliminate M_x on $x=a$ edge add pure bending

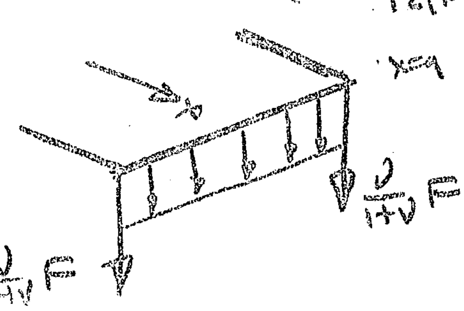
$$w = C_1(x^3 - 3\nu xy^2) + B(x^2 - \nu y^2)$$

$$M_x\big|_{x=a} = -6(1-\nu^2)DC_1a - 2(1-\nu^2)DB = 0 \rightarrow B = -3C_1a$$

Let F denote the total shearing force on edge $x=a$. Evaluating C_1

$$F = 2\left[-12\nu(1-\nu)DC_1b\right] + 2b\left[-6(1-\nu)^2DC_1\right] \rightarrow C_1 = -\frac{F}{12(1-\nu^2)Db}$$

$$\therefore \boxed{w = -\frac{F}{Eh^3b}\left[x^3 - 3\nu xy^2 - 3a(x^2 - \nu y^2)\right]}$$



A Fourier Sine Series Expansion

ME244A
5-10-79

$$\left(\frac{x}{L}\right)^N = \sum_{n=1}^{\infty} b_n^{(N)} \sin\frac{n\pi x}{L} \quad (0 < x < L)$$

$$b_n^{(N)} = \frac{2}{L}\int_0^L \left(\frac{x}{L}\right)^N \sin\frac{n\pi x}{L}\,dx = \frac{2}{(n\pi)^{N+1}}\int_0^{n\pi} \theta^N \sin\theta\,d\theta$$

integrate twice by parts

$$= \frac{2}{(n\pi)^{N+1}}\left[-\theta^N\cos\theta + N\theta^{N-1}\sin\theta - N(N-1)\int\theta^{N-2}\sin\theta\,d\theta\right]_0^{n\pi}$$

$$= -\frac{2}{n\pi}\cos n\pi - \frac{N(N-1)}{(n\pi)^2}\, b_n^{(N-2)}$$ ← recurrence formula

this recurrence formula can be used to generate the Fourier coefficients for $N \geq 2$ from those for $N=0$ and $N=1$.

$N=0$: $b_n^{(0)} = \frac{2}{n\pi}\int_0^{n\pi}\sin\theta\,d\theta = \frac{2}{n\pi}\left[-\cos\theta\right]_0^{n\pi} = \frac{2}{n\pi}(1-\cos n\pi)$

$N=1$: $b_n^{(1)} = \frac{2}{(n\pi)^2}\int_0^{n\pi}\theta\sin\theta\,d\theta = \frac{2}{(n\pi)^2}\left[-\theta\cos\theta + \sin\theta\right]_0^{n\pi}$

$$= -\frac{2}{n\pi}\cos n\pi$$

$N=2$: $b_n^{(2)} = -\frac{2}{n\pi}\cos n\pi - \frac{2}{(n\pi)^2}b_n^{(0)}$

$$= -\frac{2}{n\pi}\cos n\pi - \frac{4}{(n\pi)^3}(1-\cos n\pi)$$

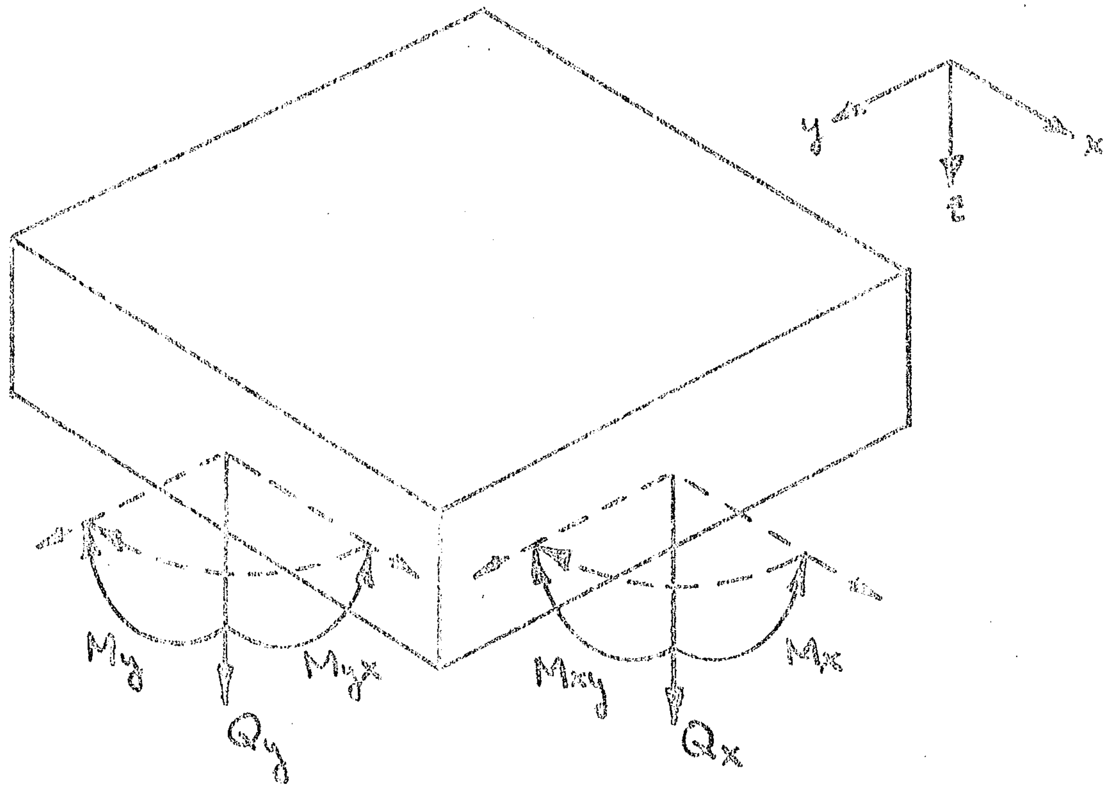
$N=3$: $b_n^{(3)} = b_n^{(1)} - \frac{6}{(n\pi)^2}b_n^{(1)}$

$$= -\frac{2}{n\pi}\left[1 - \frac{6}{(n\pi)^2}\right]\cos n\pi$$

etc

note: $\cos n\pi = (-1)^n$

Plate Stress Resultants



$$M_x \equiv \int_{-h/2}^{h/2} \sigma_x z\,dz = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y \equiv \int_{-h/2}^{h/2} \sigma_y z\,dz = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

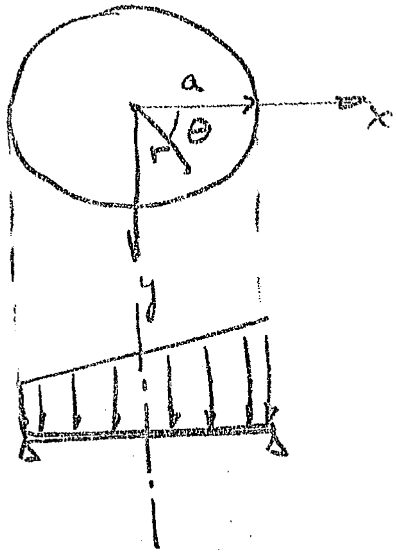
$$M_{xy} = M_{yx} \equiv \int_{-h/2}^{h/2} \tau_{xy} z\,dz = -(1-\nu) D \frac{\partial^2 w}{\partial x \partial y}$$

$$Q_x \equiv \int_{-h/2}^{h/2} \tau_{xz}\,dz = -D \frac{\partial}{\partial x}(\nabla^2 w)$$

$$Q_y \equiv \int_{-h/2}^{h/2} \tau_{yz}\,dz = -D \frac{\partial}{\partial y}(\nabla^2 w)$$

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad \nabla^2 \nabla^2 w = \frac{p}{D}$$

Circular Plate with linearly varying load



$$p(r) = p_0 + p_1 \frac{x}{a} = p_0 + p_1 \frac{r}{a}\cos\theta$$

sol'n for p_0 : $w_0 = \frac{5+\nu}{1+\nu}\frac{p_0}{64D}(a^2-r^2)^2$

particular sol'n for p_1 : $w_1 \sim r^{1+4}\cos\theta$

$$\nabla^2 r^5\cos\theta = 24 r^3\cos\theta$$

$$\nabla^4 r^5\cos\theta = 192 r\cos\theta$$

$$\therefore \quad w_1 = \frac{1}{192}\frac{p_1}{Da} r^5\cos\theta$$

$$W = W_0 + W_1 + W_H = W_0 + \frac{1}{192}\frac{p_1}{Da}\left(r^5 + A_1 r + B_1\frac{1}{r} + C r^3 + D r\ln\frac{r}{a}\right)\cdot\cos\theta$$

$w(0,\theta)$ finite $\rightarrow B_1 = 0$ $\qquad$ $Q_r(0,\theta)$ finite $\rightarrow D = 0$

$$w(a,\theta) = 0 \rightarrow a^5 + Aa + Ca^3 = 0 \rightarrow A_1 = -a^4 - Ca^2$$

$$M_r(a,\theta) = 0 \rightarrow \frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\Big|_{r=a} = 0$$

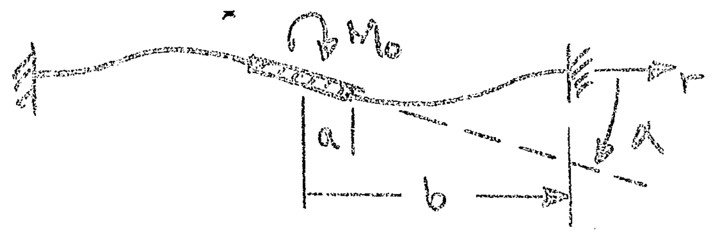
$$\rightarrow \left[20r^3 + 6Cr + \nu(-r^3 - Cr + 5r^3 + 3Cr)\right]_{r=a}\cos\theta = 0$$

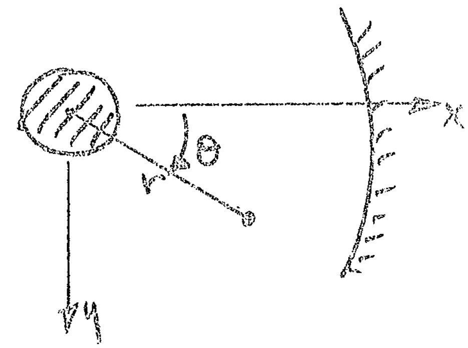
$$\rightarrow \quad C = -2\frac{5+\nu}{3+\nu}a^2$$

$$\therefore \quad w = w_0 + \frac{1}{192}\frac{p_1}{Da}\underbrace{\left(r^5 - 2\frac{5+\nu}{3+\nu}a^2 r^3 + \frac{7+\nu}{3+\nu}a^4 r\right)}_{= \; r(a^2-r^2)\left(\frac{7+\nu}{3+\nu} - r^2\right)}$$

ME241A

SOLUTION: Prob 7





BC $w(a,\theta) = \alpha x\big|_{r=a} = \alpha r\cos\theta\big|_{r=a} = \alpha a\cos\theta$ $\qquad w(b,\theta)=0$

$\frac{\partial w}{\partial r}(a,\theta) = \frac{\partial}{\partial r}(\alpha r\cos\theta)\big|_{r=a} = \alpha\cos\theta$ $\qquad \frac{\partial w}{\partial r}(b,\theta)=0$

let $w = f(r)\cos\theta$, $f(r) = Ar + Br^3 + C/r + Er\ln r$

satisfies PDE and has appropriate θ dependence for BC

BC $\rightarrow$
$$Aa + Ba^3 + C/a + Ea\ln a = \alpha a$$
$$A + 3Ba^2 - C/a^2 + E(1+\ln a) = \alpha$$
$$Ab + Bb^3 + C/b + Eb\ln b = 0$$
$$A + 3Bb^2 - C/b^2 + E(1+\ln b) = 0$$

Solving $\rightarrow$ let $\Delta = (a^2+b^2)\ln b/a - (b^2-a^2)$

$A = \left[(a^2+b^2)\ln b - \frac{1}{2}(b^2-a^2)\right]\frac{\alpha}{\Delta}$ $\qquad B = \frac{1}{2}\frac{\alpha}{\Delta}$

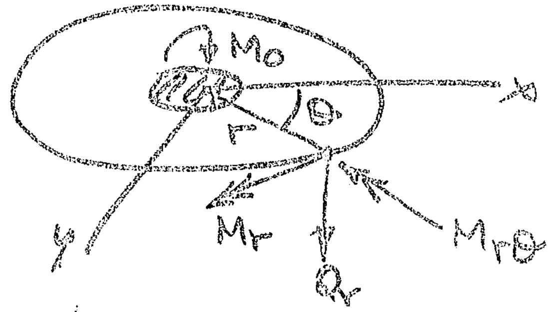
$C = -\frac{1}{2}a^2b^2\frac{\alpha}{\Delta}$ $\qquad E = -(a^2+b^2)\frac{\alpha}{\Delta}$

$$\therefore \boxed{w(r,\theta) = \frac{\alpha}{\Delta}\left[(a^2+b^2)\ln\frac{b}{r} - \frac{1}{2}\left(1+\frac{a^2}{r^2}\right)(b^2-r^2)\right]r\cos\theta \qquad a\le r\le b,\ 0\le\theta<2\pi}$$

Inspection shows that $w(a,\theta) = \alpha a\cos\theta$, $w(b,\theta)=0$

(In developing the relation between M_0 and α we will encounter another strong check on the solution)

Determine the relation between M_0 and α:



Requiring the resultant torque about the y axis to be zero gives:

$$M_0 = \int_0^{2\pi} (M_r \cos\theta - M_{r\theta} \sin\theta - Q_r\, r\cos\theta)\, r\, d\theta$$

$$M_r = -D\left[f'' + \nu\left(-\tfrac{1}{r^2}f + \tfrac{1}{r}f'\right)\right]\cos\theta$$

$$M_{r\theta} = +D(1-\nu)\left(\tfrac{1}{r}f' - \tfrac{1}{r^2}f\right)\sin\theta$$

$$Q_r = -D\left(f'' + \tfrac{1}{r}f' - \tfrac{1}{r^2}f\right)'\cos\theta$$

$$\therefore\ M_0 = -D\left[f'' + \nu\left(\tfrac{f'}{r} - \tfrac{f}{r^2}\right)\right] r\int_0^{2\pi}\cos^2\theta\, d\theta - D(1-\nu)\left(\tfrac{f'}{r} - \tfrac{f}{r^2}\right) r\int_0^{2\pi}\sin^2\theta\, d\theta$$
$$+ D\left(f''' + \tfrac{1}{r}f'' - \tfrac{2}{r^2}f' + \tfrac{2}{r^3}f\right) r^2 \int_0^{2\pi}\cos^2\theta\, d\theta$$

$$= \pi D\left(\cancel{-rf''} - f' + \tfrac{f}{r} + r^2 f''' + \cancel{rf''} - 2f' + \tfrac{2}{r}f\right)$$

$$= \pi D\left(r^2 f''' - 3f' + \tfrac{3}{r}f\right)$$ ← note: this must turn out to be independent of r

$$f' = A + 3Br^2 - C/r^2 + E(1 + \ln r) \qquad f'' = 6Br + 2C/r^3 + E/r$$
$$f''' = 6B - 6C/r^4 - E/r^2$$

$$\therefore\ M_0 = \pi D\left(\cancel{6Br^2} - \cancel{6C/r^2} - E - \cancel{3A} - \cancel{9Br^2} + \cancel{3C/r^2} - 3E(1+\ln r)\right.$$
$$\left. + \cancel{3A} + \cancel{3Br^2} + \cancel{3C/r^2} + 3E\ln r\right)$$

$$= -\pi D \cdot 4E$$ ← indep. of r (a strong solution check)

$$M_0 = k\alpha, \quad \boxed{k = D\,\frac{4\pi(a^2+b^2)}{\Delta} = D\,\frac{4\pi}{\ln\frac{b}{a} - \frac{b^2-a^2}{b^2+a^2}}}$$

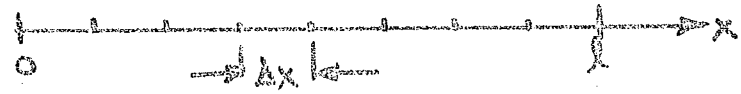
Approximate Solutions : Finite Differences

Attempt to find values of transverse displacement w only at a finite number of isolated points of the plate midplane.

Ex Beam $\quad \dfrac{d^2}{dx^2}\left[EI(x) \dfrac{d^2w}{dx^2} \right] = p(x) \quad (0 \le x \le \ell)$

$$w(0) = w(\ell) = 0, \quad w'(0) = w'(\ell) = 0$$

1) introduce a sequence of equally spaced points



want to find $w_i = w(i\Delta x) \quad i = 0, 1, 2, \cdots, N = \dfrac{\ell}{\Delta x}$

2) replace the governing ODE by an FDE by introducing finite difference approximations for derivatives – first we need an approximation for the 2nd derivative (Taylor Series)

$$w(x+\Delta x) = w(x) + \frac{dw}{dx}\Delta x + \frac{d^2w}{dx^2}\frac{(\Delta x)^2}{2} + \frac{d^3w}{dx^3}\frac{(\Delta x)^3}{6} + O(\Delta x)^4$$

$$w(x-\Delta x) = \qquad - \qquad + \qquad - \qquad +$$

$$w_{i+1} + w_{i-1} = 2w_i + \left(\frac{d^2w}{dx^2}\right)_i (\Delta x)^2 + O(\Delta x)^4$$

$$\therefore \quad \left(\frac{d^2w}{dx^2}\right)_i = \frac{w_{i-1} - 2w_i + w_{i+1}}{(\Delta x)^2} + O(\Delta x)^2$$

ODE $\Rightarrow \quad \dfrac{S_{i-1}w''_{i-1} - 2S_i w''_i + S_{i+1}w''_{i+1}}{(\Delta x)^2} \approx p_i$

$S \equiv EI(x)$

$$i = 1, 2, \cdots, N-1$$

using the formula again gives

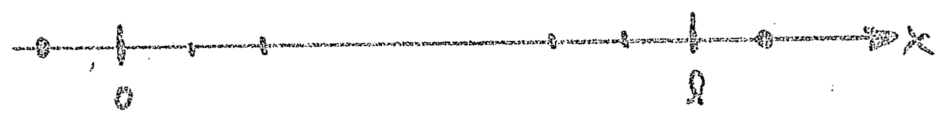
$$\boxed{\begin{aligned} &S_{i-1}w_{i-2} - 2(S_{i-1}+S_i)w_{i-1} + (S_{i-1}+4S_i+S_{i+1})w_i \\ &\quad - 2(S_i+S_{i+1})w_{i+1} + S_{i+1}w_{i+2} \approx p_i(\Delta x)^4 \qquad i = 1, 2, \cdots, N-1 \end{aligned}}$$

but for $i=1$: we would need W_{-1} ?

it looks like we should have used a different formula for w_0'' of the form $w_0'' = \alpha w_0 + \beta w_0' + \gamma w_1$

(because the BC do tell us the value of w_0')

however, a simpler (and actually more accurate) procedure is to add a fictitious point beyond each end of the beam



3) now, we have N-1 linear algebraic equations for N+3 unknowns. We clearly need 4 more equations and they are provided by the BC

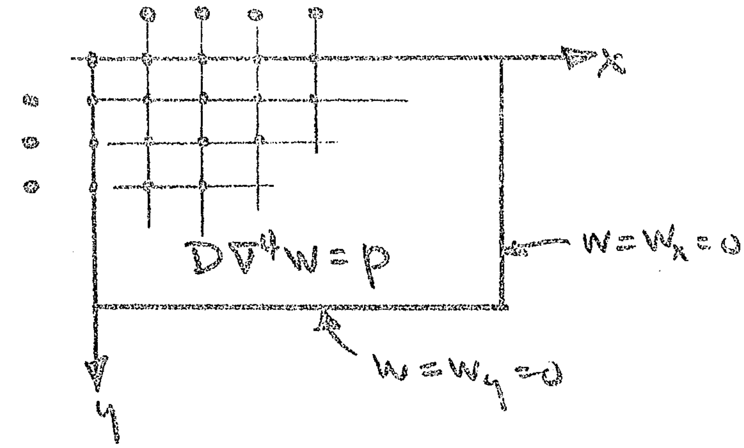
BC @ $x=0$ $\rightarrow$ $\boxed{W_0 = 0, \quad \dfrac{W_1 - W_{-1}}{2\Delta x} = 0}$

BC @ $x=\ell$ $\rightarrow$ $\boxed{W_N = 0, \quad \dfrac{W_{N+1} - W_{N-1}}{2\Delta x} = 0}$

4) finally, it remains to solve the set of equations. They can be caste in matrix form : $A\{w_i\} = \{b_i\}$ and the coefficient matrix A is banded and almost symmetric.

$$
\begin{bmatrix}
0 & 1 & & & & \\
-1 & 0 & 1 & & & \\
s_0 & -2(s_0+s_1) & s_0+4s_1+s_2 & -2(s_1+s_2) & s_2 & \\
 & s_1 & -2(s_1+s_2) & s_1+4s_2+s_3 & -2(s_2+s_3) & s_3 \\
 & & s_2 & -2(s_2+s_3) & \cdots & \\
 & & & & & \\
 & & & & &
\end{bmatrix}
\begin{Bmatrix} W_{-1} \\ W_0 \\ W_1 \\ \vdots \\ \vdots \\ W_N \\ W_{N+1} \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ p_1(\Delta x)^4 \\ p_2(\Delta x)^4 \\ \vdots \\ 0 \\ 0 \end{Bmatrix}
$$

Ex Plate



Again, introducing a uniform network of isolated points we wish to find: $w_{ij} = w(i\Delta x, j\Delta y)$

Finding a finite difference approximation for ∇^2 first

$$(\nabla^2 w)_{ij} = \left(\frac{\partial^2 w}{\partial x^2}\right)_{ij} + \left(\frac{\partial^2 w}{\partial y^2}\right)_{ij}$$

$$\approx \frac{w_{i-1,j} - 2w_{ij} + w_{i+1,j}}{(\Delta x)^2} + \frac{w_{i,j-1} - 2w_{i,j} + w_{i,j+1}}{(\Delta y)^2}$$

which simplifies if $\Delta x = \Delta y = h$ and is often written symbolically as follows:

$$\nabla^2 = \frac{1}{h^2}\left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\}$$

Superimposing 5 times to obtain ∇^4 gives

$$\nabla^4 = \frac{1}{h^4}\left\{\begin{matrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{matrix}\right\}$$

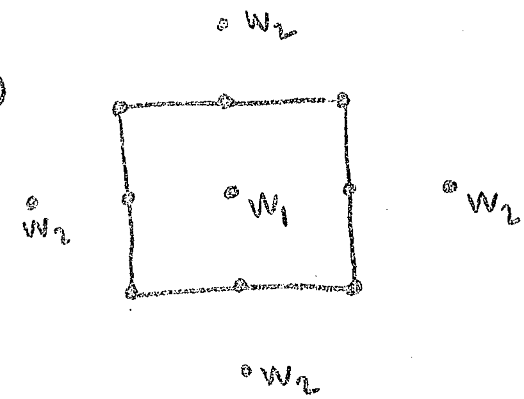
this is called the 13 point formula for ∇^4

Now the PDE can be converted into a FDE

Illustrative Example

Consider a uniformly loaded square plate of dimension $l \times l$ with simply supported edges. Lets first try a very crude (course) network of points.

1)



BC → all the boundary points have $w = 0$ and

$$\frac{w_1 - 2(0) + w_2}{h^2} = 0$$

i.e. $w_2 = -w_1$

thus we have only one unknown to determine

FDE → $$20w_1 - 8(0+0+0+0) + 2(0+0+0+0) + (-w_1 - w_1 - w_1 - w_1) = \frac{h^4}{D} p_0$$

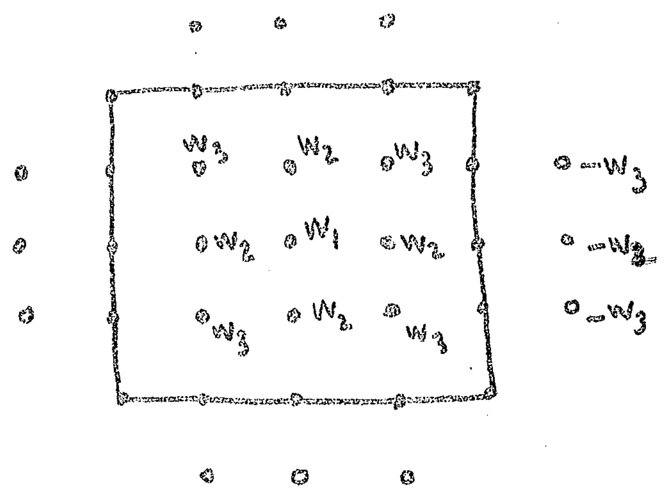
solving : $$w_1 = \frac{1}{16}\frac{p_0 h^4}{D} \underset{h = l/2}{=} \frac{1}{256}\frac{p_0 l^4}{D} \approx \underline{.00391}\frac{p_0 l^4}{D}$$

comparing with the exact solution : $w_1 = \underline{.00406}\dfrac{p_0 l^4}{D}$

% error = 3.7%

Now lets refine the mesh by cutting the mesh spacing in half

2)



$h = \frac{l}{4}$

now we have 3 unknowns

FDE $\rightarrow$ $$20w_1 - 8(4w_2) + 2(4w_3) = \frac{p_0 h^4}{D}$$

$$20w_2 - 8(w_1 + 2w_3) + 2(2w_2) + (w_2 - w_2) = \frac{p_0 h^4}{D}$$

$$20w_3 - 8(2w_2) + 2w_1 + (2w_3 - 2w_3) = \frac{p_0 h^4}{D}$$

or $$\begin{bmatrix} 20 & -32 & 8 \\ -8 & 24 & -16 \\ 2 & -16 & 20 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = \frac{p_0 l^4}{256D} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

solving gives: $w_1 = \frac{33}{32} \frac{p_0 l^4}{256D} \approx \underline{.00403} \frac{p_0 l^4}{D}$

3) Richardson Extrapolation:

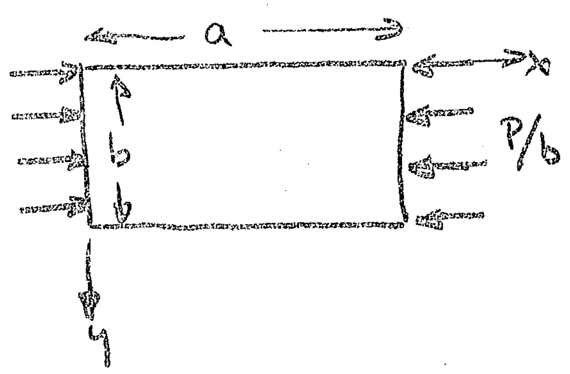
Think of w_1 as a function of h; assume $w_1(h) = w_1(0) + \frac{c}{h^2}$

$w_1(0) + \frac{4c}{l^2} = w_1(\frac{l}{2})$ | eliminate c, solve for $w(0)$

$w_1(0) + \frac{16c}{l^2} = w_1(\frac{l}{4})$ | $w_1(h=0) = \frac{4w_1(h=\frac{l}{2}) - w_1(h=\frac{l}{4})}{3}$

$$= \underline{.00407} \frac{p_0 l^4}{D}$$

Buckling of a simply supported rectangular plate, ME241A



$$D\nabla^4 w - \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) = 0$$

$$N_x = -P/b$$

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } x = 0, a$$

$$w = \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at } y = 0, b$$

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad \text{satisfies BC}$$

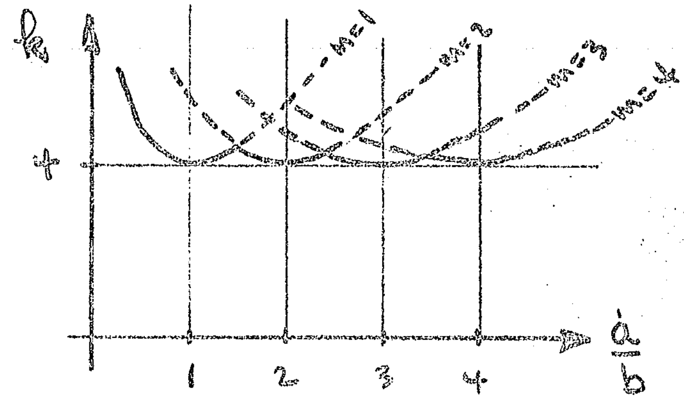
$$\text{PDE} \rightarrow \sum\sum C_{mn}\left\{ D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 - \frac{P}{b}\left(\frac{m\pi}{a}\right)^2 \right\} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = 0$$

non trivial solution (buckling) if

$$\frac{P}{b} = \left(\frac{a}{m\pi}\right)^2 D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 = \frac{\pi^2 D}{b^2}\left(\frac{mb}{a} + n^2\frac{a}{mb}\right)^2$$

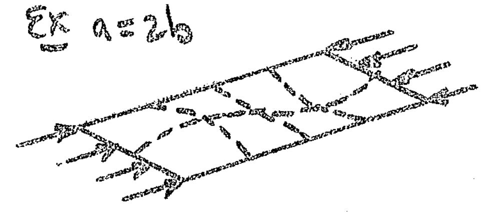
$$\therefore P_{cr} = \min_{m,n} \frac{\pi^2 D}{b}\left(\frac{mb}{a} + n^2\frac{a}{mb}\right)^2 \Rightarrow n = 1$$

$$= \frac{\pi^2 D}{b} \min_m \left(\frac{mb}{a} + \frac{a}{mb}\right)^2 = \frac{\pi^2 D}{b} k\left(\frac{a}{b}\right)$$

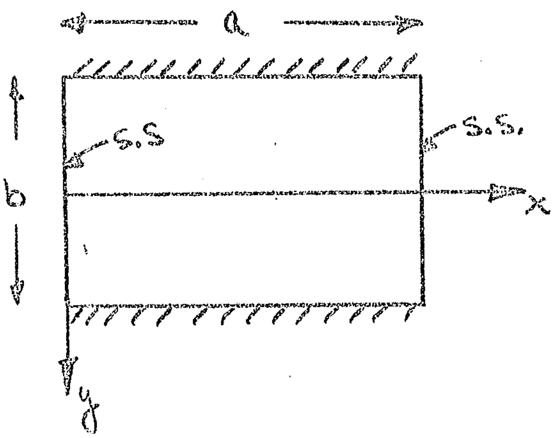


$$P_{cr} \geq \frac{4\pi^2 D}{b}$$

Ex $a = 2b$



SOLUTION : PROB.



Determine natural vibration frequencies

$$w(x,y,t) = \varphi(x,y)e^{i\omega t}$$

$$\text{PDE} \rightarrow \nabla^4\varphi = \frac{\rho h\omega^2}{D}\varphi$$

$$\text{BC} \rightarrow \begin{cases} \varphi = \varphi_{,xx} = 0 \text{ at } x = 0, a \\ \varphi = \varphi_{,y} = 0 \text{ at } y = \pm b/2 \end{cases}$$

Use Lévy idea to reduce PDE to ODE : let n = integer

$$\varphi = F(y)\sin\frac{n\pi x}{a} \rightarrow F'''' - 2\left(\frac{n\pi}{a}\right)^2 F'' + \left[\left(\frac{n\pi}{a}\right)^4 - \frac{\rho h\omega^2}{D}\right]F = 0$$

$$F = e^{\beta y} \rightarrow \beta^4 - 2\left(\frac{n\pi}{a}\right)^2\beta^2 + \left[\left(\frac{n\pi}{a}\right)^4 - \frac{\rho h\omega^2}{D}\right] = 0$$

$$\rightarrow \beta = \pm\sqrt{\sqrt{\frac{\rho h\omega^2}{D}} + \left(\frac{n\pi}{a}\right)^2}, \quad \pm i\sqrt{\sqrt{\frac{\rho h\omega^2}{D}} - \left(\frac{n\pi}{a}\right)^2}$$

<u>Note</u> it may be assumed that $\frac{\rho h\omega^2}{D} > \left(\frac{n\pi}{a}\right)^4$ because we know that the mode shapes must be oscillatory in y.

Let $\lambda^2 = \sqrt{\frac{\rho h\omega^2}{D}} + \left(\frac{n\pi}{a}\right)^2 \qquad \mu^2 = \sqrt{\frac{\rho h\omega^2}{D}} - \left(\frac{n\pi}{a}\right)^2$

<u>Modes symmetric about the x-axis</u>

$$F = A\cosh\lambda y + B\cos\mu y$$

$$\text{BC} \rightarrow F\left(\tfrac{b}{2}\right) = 0 \rightarrow A\cosh\lambda b/2 + B\cos\mu b/2 = 0$$

$$F'\left(\tfrac{b}{2}\right) = 0 \rightarrow A\lambda\sinh\lambda b/2 - B\mu\sin\mu b/2 = 0$$

nontrivial solution $\rightarrow \mu\cosh\lambda b/2\,\sin\mu b/2 + \lambda\sinh\lambda b/2\,\cos\mu b/2 =$

let $\alpha_n = \frac{n\pi b}{2a}$, $\frac{\mu b}{2} = \theta \rightarrow \frac{\lambda b}{2} = \sqrt{\theta^2 + 2\alpha_n^2}$

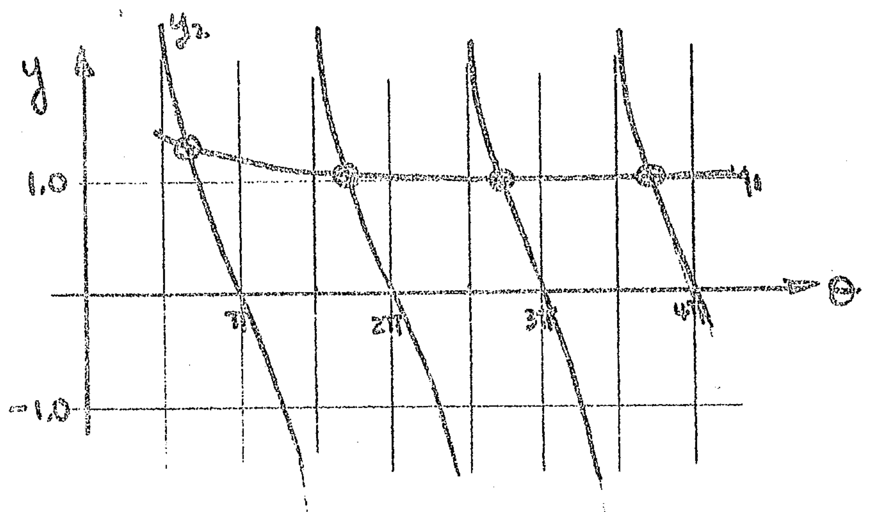
Frequency eq. $\rightarrow \frac{\sqrt{\theta^2+\alpha_n^2}}{\theta}\tanh\sqrt{\theta^2+\alpha_n^2} = -\tan\theta$

For a given value of b/a, choose n (and thus $\alpha_n = \frac{n\pi b}{2a}$)

Plot $y_1 = \frac{\sqrt{\theta^2+\alpha_n^2}}{\theta} \tanh\sqrt{\theta^2+\alpha_n^2}$ and $y_2 = -\tan\theta$

The intersections of these two curves give the natural frequencies associated with the chosen value of n : $\theta_{n1}, \theta_{n2}, \theta_{n3}, \cdots$

$$\omega_{nm} = \sqrt{\frac{D}{\rho h (b/2)^4}\left(\theta_{nm}^2+\alpha_n^2\right)} \qquad (m=1,2,3,\cdots)$$



This is the plot for $n=1$ and $b/a=1$ (a square plate). For larger values of n the y_1 curve simply becomes flatter

Note that $\theta_{nm} = (m-\frac{1}{4})\pi$ is very accurate for $m \geq 3$

A rapidly converging iteration formula is easily developed :

$$\theta_{nm}^{(k+1)} = m\pi - \tan^{-1}\left[\frac{\sqrt{\theta_{nm}^{(k)2}+\alpha_n^2}}{\theta_{nm}^{(k)}} \tanh\sqrt{\theta_{nm}^{(k)2}+\alpha_n^2}\right]$$

which should be used with the initial guess: $\theta_{nm}^{(0)} = (m-\frac{1}{4})\pi$

<u>Modes antisymmetric about the x axis</u>

$$F = A\sinh\lambda y + B\sin\mu y \rightarrow A\sinh\frac{\lambda b}{2} + B\sin\frac{\mu b}{2} = 0$$

$$A\lambda\cosh\frac{\lambda b}{2} + B\mu\cos\frac{\mu b}{2} = 0$$

freq eq : $\mu\sinh\frac{\lambda b}{2}\cos\frac{\mu b}{2} - \lambda\cosh\frac{\lambda b}{2}\sin\frac{\mu b}{2} = 0$

$$\rightarrow \frac{\theta}{\sqrt{\theta^2+\alpha_n^2}}\tanh\sqrt{\theta^2+\alpha_n^2} = \tan\theta$$