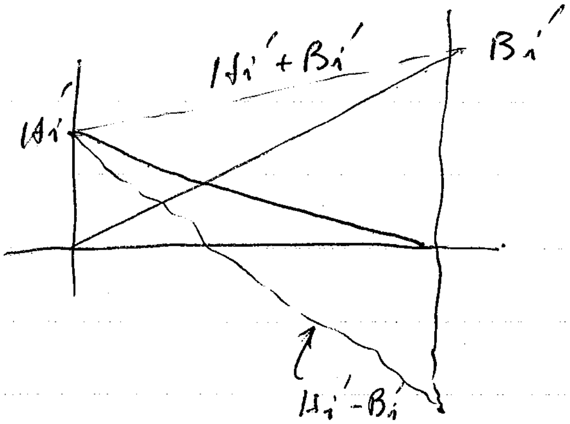

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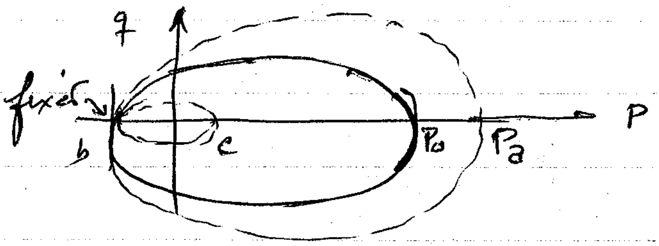

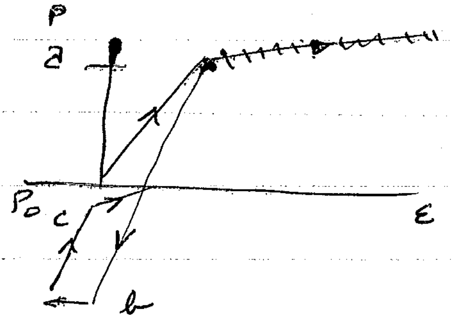

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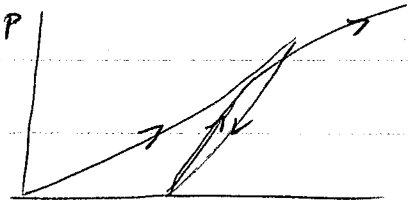

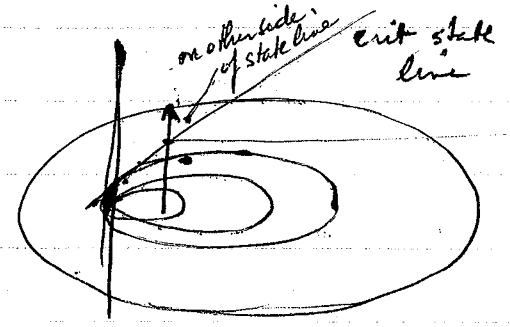

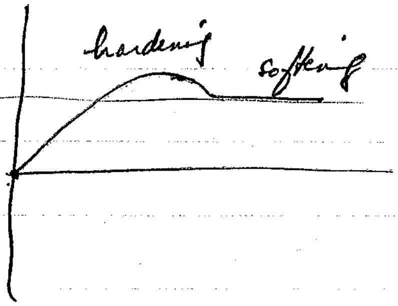

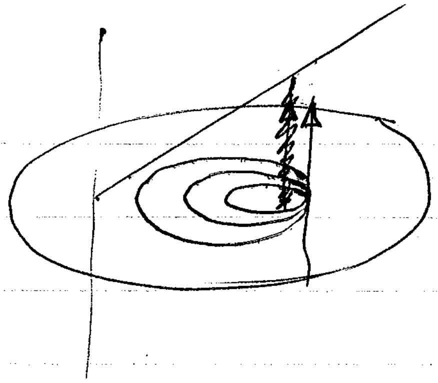

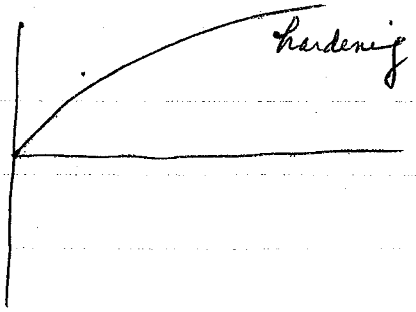

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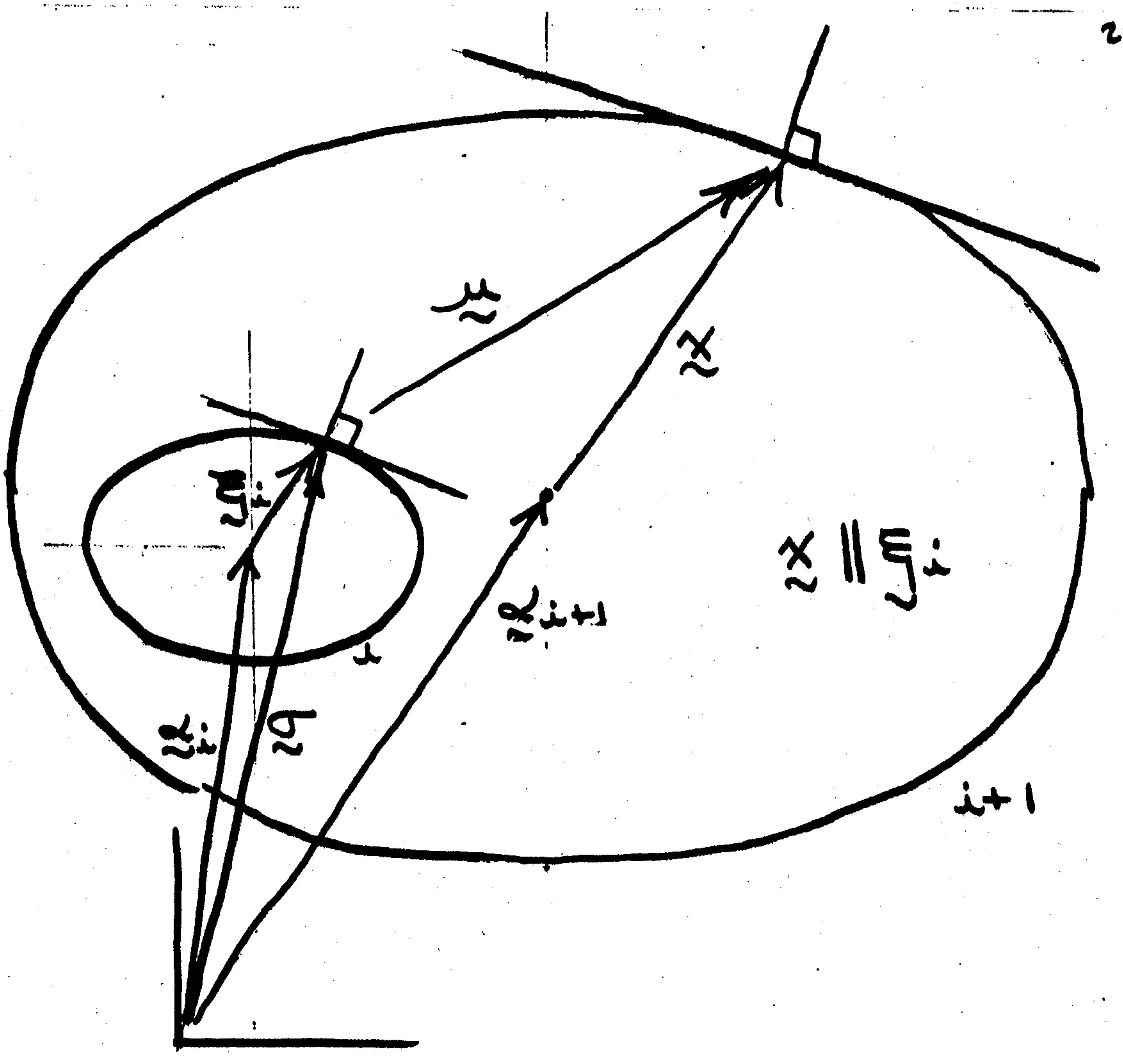

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4

PLANE STRAIN: PROBLEMS WITH AXIAL SYMMETRY

15. General relations

In this chapter we shall discuss the stresses and strains in a thick-walled circular tube which is subjected to internal pressure, the ends of the tube being restrained from motion in the axial direction. The interior radius of the tube will be denoted by a and the exterior radius by b. The internal pressure p will be regarded as a given function of time: $p = p(t)$. To describe the stresses and strains which this pressure produces in the tube, we shall use a system of cylindrical coordinates r, θ, z, the z axis of which coincides with the axis of the tube.

In the problem which is to be discussed here, the axial strain ϵ_z vanishes throughout the tube and at all times; moreover, all stresses and strains are independent of θ and z. The principal directions of stress and strain at a generic point of the tube are radial, circumferential, and axial, and the only equation of equilibrium which remains to be satisfied is

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{15.1}$$

(see, for instance, Chap. 1, Ref. 1, p. 55). Note that, though it is not subject to any equilibrium condition, $\sigma_z = \sigma_z(r, t)$ need not vanish.

If the radial displacement component is denoted by $u = u(r, t)$, we have

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r} \tag{15.2}$$

(see, for instance, Chap. 1, Ref. 1, pp. 62, 63). Since $\epsilon_z = 0$, the mean normal strain (Eq. 2.3) is given by

$$e = \frac{1}{3}(\epsilon_r + \epsilon_\theta + \epsilon_z) = \frac{1}{3}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right). \tag{15.3}$$

We shall find it convenient to use the mean normal strain as one of the dependent strain variables, and to introduce another strain variable ϕ defined by

$$\phi = \frac{1}{3}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right). \tag{15.4}$$

Elimination of u between 15.3 and 15.4 shows that e and ϕ must satisfy the equation of compatibility

$$\frac{\partial e}{\partial r} - \frac{\partial \phi}{\partial r} = 2\frac{\phi}{r}. \tag{15.5}$$

In terms of e and ϕ, the radial displacement and the principal components of the strain tensor and the strain deviation are

$$u = \tfrac{3}{2}(e - \phi)r, \tag{15.6}$$

$$\epsilon_r = \tfrac{3}{2}(e + \phi), \quad \epsilon_\theta = \tfrac{3}{2}(e - \phi), \quad \epsilon_z = 0, \tag{15.7}$$

$$e_r = \tfrac{1}{2}(e + 3\phi), \quad e_\theta = \tfrac{1}{2}(e - 3\phi), \quad e_z = -e. \tag{15.8}$$

In the elastic range, the mean normal stress is given by the last Eq. 3.2. Once an element of the tube begins to deform plastically, this equation must be replaced by the differential equation 5.11 which has to be integrated under an initial condition expressing the relation between s and e at the instant when the element enters the plastic range. However, this initial relation between s and e is precisely the last Eq. 3.2. Consequently, the relation

$$s = \tfrac{1}{3}(\sigma_r + \sigma_\theta + \sigma_z) = 3Ke \tag{15.9}$$

is valid in the plastic as well as the elastic range.

In view of 15.9, the principal stresses may be written as

$$\sigma_r = s_r + 3Ke, \quad \sigma_\theta = s_\theta + 3Ke, \quad \sigma_z = s_z + 3Ke, \tag{15.10}$$

where s_r, s_θ, s_z are the principal components of the stress deviation. Finally, the Mises yield condition 4.5 reduces to

$$J_2 = s_r^2 + s_r s_\theta + s_\theta^2 = k^2. \tag{15.11}$$

Our problem now is to determine the four quantities e, ϕ, s_r, s_θ. Once these are known, the complete solution is obtained directly from Eqs. 15.6, 15.7, 15.8, and 15.10. Four equations are necessary to determine these four unknowns. The compatibility equation 15.5 and the equilibrium equation 15.1 are valid throughout the tube. In terms of the present variables, the latter takes the form

$$\frac{\partial s_r}{\partial r} + 3K\frac{\partial e}{\partial r} = \frac{s_\theta - s_r}{r}. \tag{15.12}$$

The remaining equations are furnished by Hooke's law in the elastic domain, while the stress-strain law and the yield condition each furnish an additional relation in the plastic domain.

The radial stress σ_r must have the value $-p$ at the interior surface and vanish at the exterior surface. Thus, we have the boundary conditions

$$\begin{aligned} s_r + 3Ke &= -p \quad &&\text{for } r = a, \\ s_r + 3Ke &= 0 &&\text{for } r = b. \end{aligned} \tag{15.13}$$

In addition, all four variables must be continuous across the elastic-plastic boundary.

In the elastic range, Hooke's law furnishes the relations

$$s_r = G(e + 3\phi), \quad s_\theta = G(e - 3\phi). \tag{15.14}$$

Substituting these into 15.12, we obtain

$$\frac{G + 3K}{3G}\frac{\partial e}{\partial r} + \frac{\partial \phi}{\partial r} = -2\frac{\phi}{r}. \tag{15.15}$$

Equations 15.5 and 15.15 are readily solved to yield

$$e = B, \quad \phi = \frac{C}{r^2}, \tag{15.16}$$

where B and C are independent of the coordinates. Substituting 15.16 and the first Eq. 15.14 into the boundary conditions 15.13 and solving for B and C, we find

$$B = \frac{p'}{G + 3K}, \quad C = -\frac{p'b^2}{3G}, \tag{15.17}$$

where

$$p' = p\frac{a^2}{b^2 - a^2}. \tag{15.18}$$

With these values of B and C, we finally obtain the fully elastic solution of our problem:[1]

---

[1] This elastic solution may be obtained more directly from the equation of equilibrium and Hooke's law (see, for instance, Chap. 1, Ref. 1, Sec. 22). However, the present derivation lends itself more readily to the discussion of the elastic-plastic problem.

$$u = \frac{3}{2}p'\left(\frac{r}{G+3K} + \frac{b^2}{3Gr}\right),$$

$$s_r = p'\left(\frac{G}{G+3K} - \frac{b^2}{r^2}\right), \quad s_\theta = p'\left(\frac{G}{G+3K} + \frac{b^2}{r^2}\right), \quad s_z = -\frac{2p'G}{G+3K}, \tag{15.19}$$

$$\sigma_r = p'\left(1 - \frac{b^2}{r^2}\right), \quad \sigma_\theta = p'\left(1 + \frac{b^2}{r^2}\right), \quad \sigma_z = p'\frac{3K-2G}{3K+G}.$$

To find the range of validity of these formulas, we must determine for which pressure the left-hand side of 15.11 as evaluated from 15.19 will first reach the value k^2. Now, with 15.19 the left-hand side of 15.11 is found to be

$$J_2 = p'^2\left[3\left(\frac{G}{G+3K}\right)^2 + \frac{b^4}{r^4}\right]. \tag{15.20}$$

This is a maximum for $r = a$. Thus, yielding will begin at the inner surface when the pressure has reached the value

$$p^* = k\frac{b^2 - a^2}{a^2}\left[\frac{b^4}{a^4} + 3\left(\frac{G}{G+3K}\right)^2\right]^{-\frac{1}{2}}. \tag{15.21}$$

For a pressure somewhat greater than p^*, part of the tube will become plastic. On account of the axial symmetry, the elastic-plastic boundary must be a cylinder $r = \rho$. The quantities e and ϕ in the elastic region ($\rho \leq r \leq b$) are still given by 15.16, but the boundary conditions are now

$$\begin{aligned} s_r^2 + s_r s_\theta + s_\theta^2 &= k^2 \qquad \text{for } r = \rho, \\ s_r + 3Ke &= 0 \qquad \text{for } r = b. \end{aligned} \tag{15.22}$$

When these boundary conditions are used to determine B and C in 15.16, it is found that the solution in the elastic region of an elastic-plastic tube may still be written in the form 15.19, provided that p' is replaced throughout by

$$p'' = k\left[3\left(\frac{G}{G+3K}\right)^2 + \frac{b^4}{\rho^4}\right]^{-\frac{1}{2}}. \tag{15.23}$$

In the plastic domain the yield condition must be satisfied. This condition may be written as

$$s_\theta = \frac{-s_r \pm \sqrt{4k^2 - 3s_r^2}}{2}. \tag{15.24}$$

The proper sign of the radical is determined by observing that, in the

elastic domain,

$$2s_\theta + s_r = p''\left(\frac{3G}{G+3K} + \frac{b^2}{r^2}\right) > 0. \tag{15.25}$$

By continuity, it follows that the upper sign should be used in 15.24.

The final equation in the plastic domain is provided by the stress-strain relation. In Sec. 5, the "dot" in the stress-strain law was interpreted as indicating differentiation with respect to time. Since the stress-strain relations of Prandtl-Reuss are homogeneous in the rates of stress and strain, however, we are at liberty to replace time by any other parameter which increases monotonically with time and is, therefore, characteristic of the state of deformation. For the investigation of the elastic-plastic behavior of a thick-walled tube under monotonically increasing internal pressure, it will prove convenient to interpret the dot as indicating differentiation with respect to ρ, the radius of the elastic-plastic boundary. The stress-strain law of Prandtl-Reuss (Eq. 5.5) then gives the following relations:

$$2G\frac{\partial e_r}{\partial \rho} = \frac{\partial s_r}{\partial \rho} + \lambda s_r, \quad 2G\frac{\partial e_\theta}{\partial \rho} = \frac{\partial s_\theta}{\partial \rho} + \lambda s_\theta. \tag{15.26}$$

Eliminating λ from 15.26, we obtain

$$2G\left(\frac{\partial e_r}{\partial \rho}s_\theta - \frac{\partial e_\theta}{\partial \rho}s_r\right) = \frac{\partial s_r}{\partial \rho}s_\theta - \frac{\partial s_\theta}{\partial \rho}s_r, \tag{15.27}$$

or, in terms of the variables e, ϕ, s_r, s_θ,

$$G\left[(s_\theta - s_r)\frac{\partial e}{\partial \rho} + 3(s_r + s_\theta)\frac{\partial \phi}{\partial \rho}\right] = s_\theta\frac{\partial s_r}{\partial \rho} - s_r\frac{\partial s_\theta}{\partial \rho}. \tag{15.28}$$

Equations 15.5, 15.12, 15.24, and 15.28 define the variation of e, ϕ, s_r, s_θ in the plastic region. We may use 15.24 to eliminate s_θ from 15.12 and 15.28. We then are left with the following basic equations for e, ϕ, and s_r:

$$\begin{aligned} &\frac{\partial e}{\partial r} - \frac{\partial \phi}{\partial r} = 2\frac{\phi}{r}, \\ &\frac{\partial s_r}{\partial r} + 3K\frac{\partial e}{\partial r} = \frac{-3s_r + \sqrt{4k^2 - 3s_r^2}}{2r}, \\ &\frac{\partial s_r}{\partial \rho} - \frac{G}{4k^2}\left(4k^2 - 3s_r^2 - 3s_r\sqrt{4k^2 - 3s_r^2}\right)\frac{\partial e}{\partial \rho} \\ &\qquad - \frac{3G}{4k^2}\left(4k^2 - 3s_r^2 + s_r\sqrt{4k^2 - 3s_r^2}\right)\frac{\partial \phi}{\partial \rho} = 0. \end{aligned} \tag{15.29}$$

These equations must be solved for e, ϕ, and s_r in the triangular domain, $a \leq \rho \leq b$, $a \leq r \leq \rho$. The values of all quantities on the line $r = \rho$ are known from the elastic solution. Although a solution in closed form is impossible, Eqs. 15.29 may be replaced by the corresponding difference equations and solved numerically. The resulting stress distributions for various positions of the elastic-plastic boundary are shown in Figs. 24, 25, and 26 for the particular case $b = 2a$. In Fig. 27, the internal and external radial displacements, together with the pressure, are plotted as functions of the radius ρ of the elastic-plastic boundary.

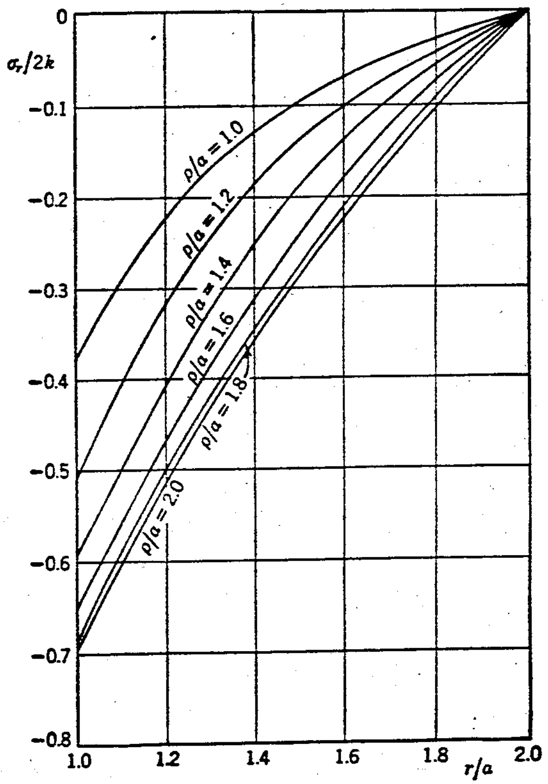

FIG. 24. Distributions of radial stress.

16. Incompressible material

The numerical integration of Eqs. 15.29 is cumbersome. It therefore is worth while to look for simplifications in the mathematical treatment of our problem. One possibility is to assume the material to be incompressible in both the elastic and the plastic ranges. According to 15.3 this assumption is expressed by the differential equation

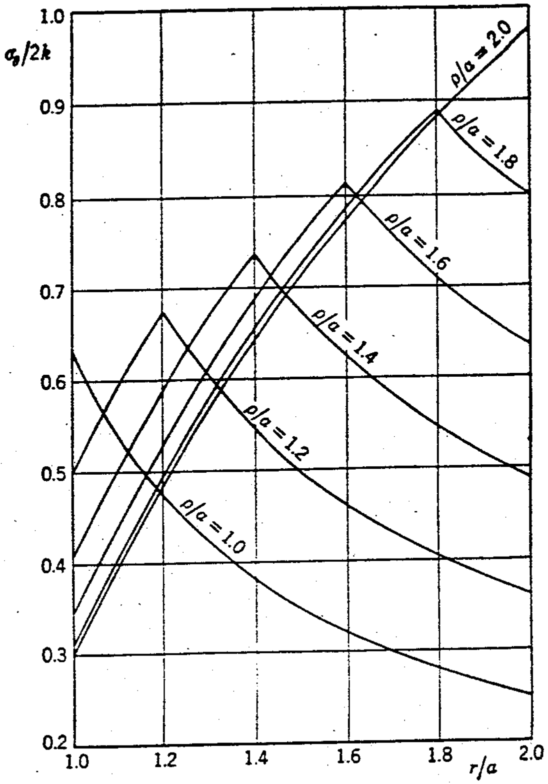

FIG. 25. Distributions of circumferential stress.

$$\frac{\partial u}{\partial r} + \frac{u}{r} = 0. \tag{16.1}$$

Integration of 16.1 yields

$$u(r, \rho) = \frac{D(\rho)}{r}, \tag{16.2}$$

independently of the stress distribution and in both the elastic and the plastic regions. In 16.2, ρ again denotes the radius of the elastic-plastic boundary and is used to characterize a state of contained plastic deformation.

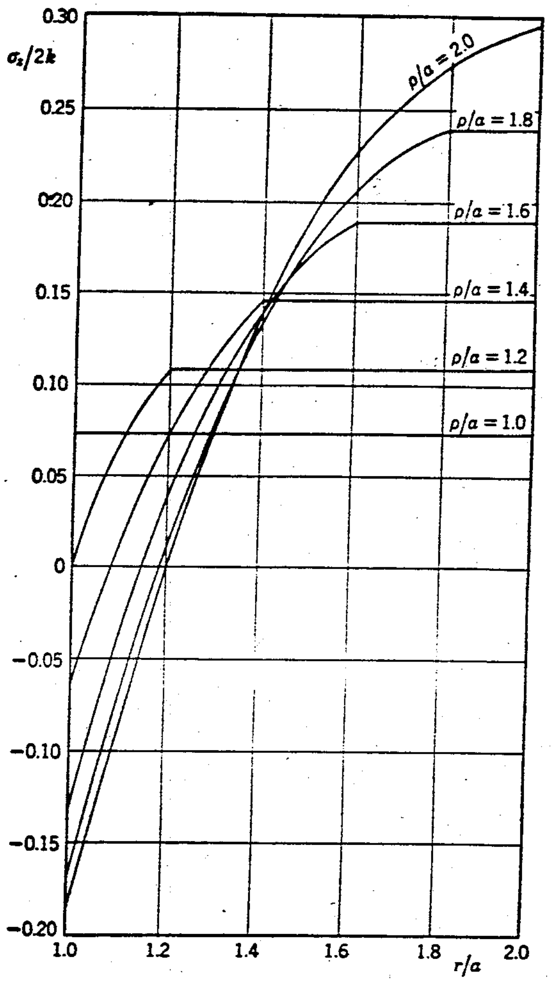

FIG. 26. Distributions of axial stress.

The principal components of the strain tensor are now found to be

$$\epsilon_r = -\frac{D}{r^2}, \quad \epsilon_\theta = \frac{D}{r^2}, \quad \epsilon_z = 0. \tag{16.3}$$

The first two equations are obtained by substituting 16.2 into 15.2. Since the material is incompressible, the strain deviation is identical with the strain tensor. According to Hooke's law (Eq. 3.2), the principal components of the stress deviation in the elastic region are therefore given by

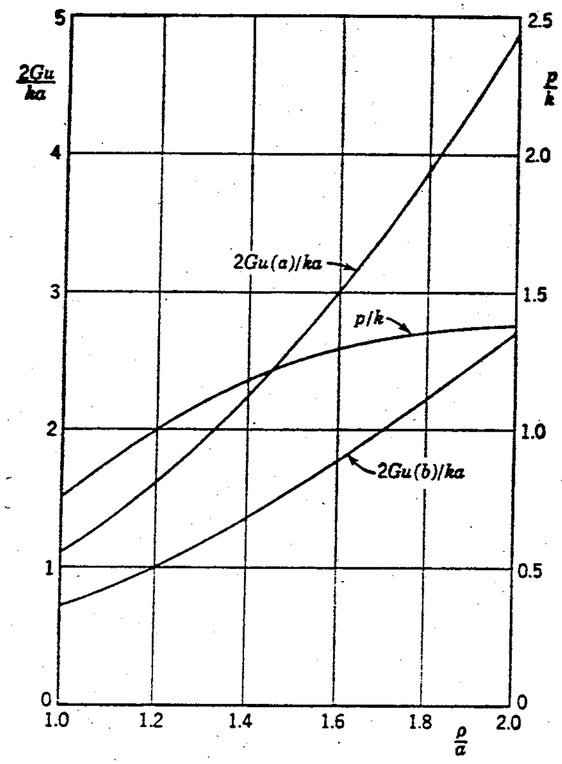

FIG. 27. Radial displacements $u(a)$, $u(b)$ and pressure p versus radius ρ of elastic-plastic boundary.

$$s_r = -2G\frac{D}{r^2}, \quad s_\theta = 2G\frac{D}{r^2}, \quad s_z = 0. \tag{16.4}$$

Denoting the mean normal stress by s, we then have

$$\sigma_r = s - 2G\frac{D}{r^2}, \quad \sigma_\theta = s + 2G\frac{D}{r^2}, \quad \sigma_z = s \tag{16.5}$$

in the elastic region. Note that, for an incompressible elastic material, s can not be found from Hooke's law because $K \rightarrow \infty$ when $e \rightarrow 0$. The equation for s is here obtained from the condition that the stresses 16.5 must satisfy the equation of equilibrium 15.1. Since D does not depend on r, this reduces to

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$$\begin{tikzcd}
 & T_X\mathcal{B} \arrow[dr, "R"] & \\
T_X\mathcal{B} \arrow[ur, "U"] \arrow[rr, "F"] \arrow[dr, "R"'] & & T_x\mathcal{S} \\
 & T_x\mathcal{S} \arrow[ur, "V"'] &
\end{tikzcd}$$

Notice that U and V operate within each fixed tangent space; that is, $U(X)$: $T_X\mathcal{B} \longrightarrow T_X\mathcal{B}$ and $V(x)$: $T_x\mathcal{S} \longrightarrow T_x\mathcal{S}$. On the other hand, R maps $T_X\mathcal{B}$ to $T_x\mathcal{S}$; that is, it shifts the base point as well as rotating.

3.13 Algorithm for Computing the Polar Decomposition Let X, x be fixed, and let F be given. To compute R and U, let $C = F^TF$, and let $\psi^1, \dots, \psi^n$ be orthonormal eigenvectors of C with eigenvalues $\lambda_1, \dots, \lambda_n$. Let

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \text{and} \quad \Psi = (\psi^1, \dots, \psi^n)$$

so that $\Lambda = \Psi^TC\Psi$. Let $U = \Psi\Lambda^{1/2}\Psi^T$, where

$$\Lambda^{1/2} = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} \qquad (\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n} \text{ are the principal stretches}),$$

and set $R = FU^{-1}$. Use a similar procedure for the left decomposition or let $V = RUR^T$. An explicit formula for U, R in the two-dimensional case is worked out in 3.15.

Observe that $b = V^2 = (RUR^T)(RUR^T) = RU^2R^T = RCR^T$. Thus the Finger deformation tensor b and the deformation tensor C are conjugate under the rotation matrix.

3.14 An Example of the Polar Decomposition[13] Let $\mathcal{B}$ be the unit circular cylinder contained in $\mathbb{R}^3$ and let $\{z^i\}$ and $\{Z^I\}$ denote coincident Cartesian coordinate systems for $\mathbb{R}^3$. Then $\mathcal{B}$ can be written as

$$\mathcal{B} = \{X \mid [Z^1(X)]^2 + [Z^2(X)]^2 \leq 1\}.$$

Consider the configuration $\phi: \mathcal{B} \longrightarrow \mathbb{R}^3$ defined explicitly by

$$z^1(X) = \sqrt{3}\, Z^1(X) + Z^2(X), \qquad z^2(X) = 2Z^2(X), \quad \text{and} \quad z^3(X) = Z^3(X).$$

---

[13] The data for this example come from Jaunzemis [1967], pp. 78–79.

This configuration may be described as biaxial stretching the in z^1, z^2-plane. The boundary of $\mathcal{B}$, given by $\partial\mathcal{B} = \{X \mid [Z^1(X)]^2 + [Z^2(X)]^2 = 1\}$, is deformed under ϕ into an ellipse: $\partial\phi(\mathcal{B}) = \{x \mid [z^1(x)]^2 + z^1(x)z^2(x) + [z^2(x)]^2 = 3\}$. In the coordinate system $\{\bar{z}^i\}$ defined by $\bar{z}^1 = \sqrt{2}\,(z^1 + z^2)$, $\bar{z}^2 = \sqrt{2}\,(z^2 - z^1)$, and $\bar{z}^3 = z^3$, the boundary of $\phi(\mathcal{B})$ can be represented by the equation $[\bar{z}^1(x)]^2/6 + [\bar{z}^2(x)]^2/2 = 1$. Thus the coordinate axes $\bar{z}^1$ and $\bar{z}^2$, which are rotated 45° counterclockwise with respect to z^1 and z^2, coincide with the major and minor axes of the ellipse.

The matrices pertient to the polar decomposition are listed below.

$$F = \begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ \sqrt{3} & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U = \frac{1}{2\sqrt{2}} \begin{bmatrix} 3 + \sqrt{3} & 3 - \sqrt{3} & 0 \\ 3 - \sqrt{3} & 1 + 3\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix}$$

$$U^{-1} = \frac{1}{4\sqrt{6}} \begin{bmatrix} 1 + 3\sqrt{3} & \sqrt{3} - 3 & 0 \\ \sqrt{3} - 3 & 3 + \sqrt{3} & 0 \\ 0 & 0 & 4\sqrt{6} \end{bmatrix}$$

$$R = \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3} + 1 & \sqrt{3} - 1 & 0 \\ 1 - \sqrt{3} & \sqrt{3} + 1 & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix}$$

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{3} + 1 & \sqrt{3} - 1 & 0 \\ \sqrt{3} - 1 & \sqrt{3} + 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

The physical interpretation of these results follows (see Fig. 1.3.1).

Right decomposition U maps the unit disk in the Z^1, Z^2-plane into an ellipse with major and minor axes rotated 60° counterclockwise with respect to the Z^1

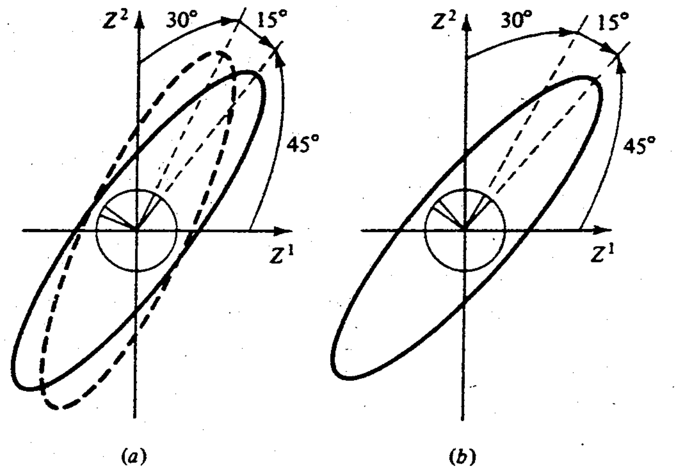

Figure 1.3.1 (a) Right decomposition of F: pure deformation (dashed curve) followed by a clockwise rotation through 15°. (b) Left decomposition of F: clockwise rotation of the circle through 15°, followed by the pure deformation V.

and Z^2 axes, respectively. R then rigidly rotates the elliptic cylinder 15° clockwise about the Z^3-axis into its final position.

Left decomposition R maps the unit disk rigidly into itself. Then V maps the unit disk into an ellipse with major and minor axes coinciding with the $\bar{z}^1$ and $\bar{z}^2$ axes, respectively.

Note that U leaves three orthogonal directions unrotated (the directions are defined by its eigenvectors). Similarly, V leaves three coordinate directions unrotated (these are given by its eigenvectors, which coincide with the axes of the coordinate system $\{\bar{z}^i\}$). The principal stretches—that is, 1, $\sqrt{6}$ and $\sqrt{2}$—which are the eigenvalues of U (or V) determine the "size" of the deformation. For example, the major axis of the deformed ellipse is $\sqrt{6}$ and minor axis is $\sqrt{2}$.

Problem 3.2 Find the polar decompositions of

$$F = \begin{bmatrix} 1 & \kappa \\ 0 & 1 \end{bmatrix}.$$

Employ the substitution $\kappa = 2 \tan \alpha$ and check that

$$U = \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & (1 + \sin^2 \alpha)/\cos \alpha \end{bmatrix}.$$

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On the Analysis of Rotation and Stress Rate in Deforming Bodies

By

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With 1 Figure

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Summary — Zusammenfassung

On the Analysis of Rotation and Stress Rate in Deforming Bodies. When a solid ement experience large deformations, the components of stress will, in general, vary as a sult of material rotation. These changes occur even in the absence of additional strain, and ed to be accounted for in formulating constitutive laws that involve the rate of change of ress. In this paper the correction terms are extended to the case when material axes become rongly skewed. An expression for the rate of material rotation as an explicit function of rticity, rate of deformation and stretch is derived. It is then shown that the rate of change stress depends on the rate of material rotation. As an example, expressions for material tation and stress are derived for a hypoelastic material undergoing uniform, rectilinear, ear. The shear stress is compared with a solution that neglects skewing of the axes, and it is und that, for the example, skewing may be neglected for strains less than 0.4. Finally, the e of these relations in numerical calculations involving finite deformation is discussed.

Zur Untersuchung der Rotations- und Spannungsgeschwindigkeit in sich defor-ierenden Körpern. Wenn ein Festkorper große Verformungen erfährt, werden sich die pannungskomponenten im allgemeinen als Folge der Materialrotation ändern. Diese Ver-nderungen treten sogar in Abwesenheit zusätzlicher Verzerrung auf, und müssen bei der ormulierung der Zustandsgleichungen, welche die [illegible] der Spannungsänderung berück-ichtigen, in Betracht gezogen werden. In dieser Arbeit werden die Korrekturglieder erweitert zu dem Fall, wenn die Materialachsen stark schräg zueinander werden. Abgeleitet wird ein Ausdruck für die Werkstoffrotationsgeschwindigkeit als eine explizite Funktion der Wirbel-, Verformungs- und Ausdehnungsgeschwindigkeit. Es wird dann gezeigt, daß die Änderung der Spannung von der Materialrotationsgeschwindigkeit abhängig ist. Als Beispiel werden Ausdrücke für Werkstoffrotation und Spannung für einen hypo-elastischen Werkstoff, der gleichförmigen, geradlinigen Schub aufweist, hergeleitet. Die Schubspannung wird mit einer Lösung verglichen, die die Schräge der Achsen vernachlässigt, und es ergibt sich, daß, für dieses Beispiel, die Schräge für Verzerrungen weniger als 0,4 zu vernachlässigen ist. Abschließend wird die Anwendung dieser Beziehungen in numerischen Rechnungen für endliche Deformationen erörtert.

I. Introduction

Constitutive laws for complex solids, such as those that exhibit elastic-plastic or viscoelastic behavior, generally involve relations between stress rate, strain rate, stress and strain. Zaremba [1], [2], in connection with a generalized viscosity theory of the Maxwell type, was apparently the first to note that in computing the rate of change of stress it is necessary to account for the effect of material

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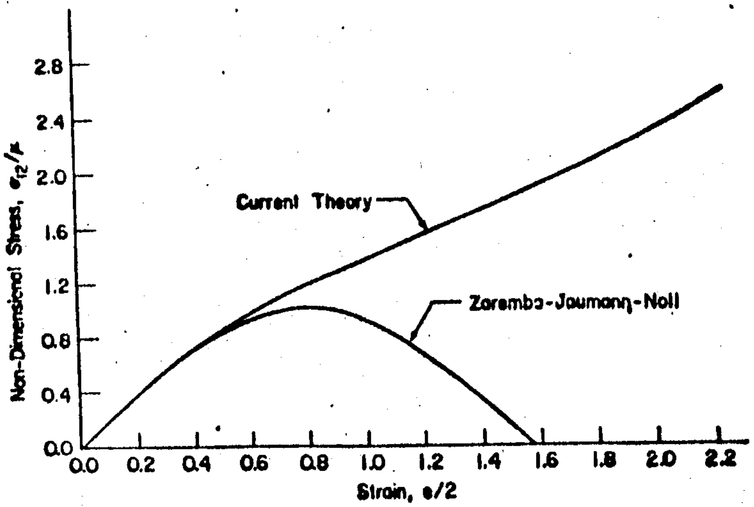

Fig. 1. A comparison between the shear stress on a hypo-elastic material in simple, rectilinear shear using the current theory and the Zaremba-Jaumann-Noll stress rate

VI. Comparison With Zaremba-Jaumann-Noll

The ZJN stress rate is often used [10], [17], [18], [19] to calculate the stresses in a material that is deforming with non-zero vorticity. This stress rate applies when the current time is equal to the reference time, so that $V = R = I$. Then

$$\check{\sigma} = \dot{\sigma} - W\sigma + \sigma W \tag{6.1}$$

in view of (3.7) and (4.3). Since ϵ is not time dependent, in view of (5.3) the simultaneous Eqs. (5.20) become

$$\begin{aligned} &\dot{\sigma}_{11} - a\sigma_{21} = 0 \\ &\dot{\sigma}_{12} - \frac{1}{2} a(\sigma_{22} - \sigma_{11}) = \mu a \\ &\dot{\sigma}_{22} + a\sigma_{21} = 0 \end{aligned} \tag{6.2}$$

where

$$a = \dot{\epsilon} = \frac{\partial v}{\partial \xi_2} \tag{6.3}$$

is a constant. It is straightforward to show that if the stresses initially vanish, then

$$\begin{aligned} &\sigma_{12} = \mu \sin at \\ &\sigma_{11} = \mu(1 - \cos at) \\ &\sigma_{22} = -\mu(1 - \cos at). \end{aligned} \tag{6.4}$$

It is clear from physical considerations that the stresses cannot actually be periodic. One conclusion of this study is that the ZJN stress rate is very accurate

Handout #5
ME 239B
Notes

FINITE ROTATION EFFECTS IN NUMERICAL INTEGRATION OF RATE CONSTITUTIVE EQUATIONS ARISING IN LARGE-DEFORMATION ANALYSIS

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SUMMARY

An improved algorithm is presented for integrating rate constitutive equations in large-deformation analysis. The algorithm is shown to be 'objective' with respect to large rotation increments.

INTRODUCTION

Rate constitutive equations are used in many theories of engineering interest. As examples we may mention plasticity, viscoelasticity and viscoplasticity. In large-deformation analysis, equations of this type must be written in terms of 'objective rates' to maintain correct rotational transformation properties. Standard time-discretization procedures when applied to rate constitutive equations typically only achieve objectivity in the limit of vanishingly small time steps—see e.g. Reference 5. This may lead to excessive error accumulation in practice. Hallquist[1] has shown, in the context of two-dimensional analysis, how to maintain objectivity for large time steps. The procedure employed, however, suffers from several drawbacks. First, deformation measures need be calculated with respect to two different configurations during each calculation. Second, trigonometric functions need be used to calculate terms in the stress-rotation formulae. Third, the generalization to three dimensions is not apparent. Despite the limitations of the procedure, significant improvements in accuracy have resulted through its use.

The purpose of the present paper is to present an algorithm for integrating rate constitutive equations which is objective for large rotation increments and does not suffer from the shortcomings cited above.

In the next section we introduce a class of rate constitutive equations of inviscid-type which is sufficiently general for the developments herein. Then we introduce the algorithm and finally we derive its main properties.

A CLASS OF RATE CONSTITUTIVE EQUATIONS

We consider rate constitutive equations of the following form:

$$\dot{\sigma}_{ij} = \bar{c}_{ijkl} v_{(k,l)} + s_{ijkl} v_{[k,l]} \tag{1}$$

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in which

$$v_{(k,l)}=(v_{k,l}+v_{l,k})/2 \tag{2}$$

$$v_{[k,l]}=(v_{k,l}-v_{l,k})/2 \tag{3}$$

$$s_{ijkl}=(\sigma_{il}\delta_{jk}+\sigma_{jl}\delta_{ik}-\sigma_{ik}\delta_{jl}-\sigma_{jk}\delta_{il})/2 \tag{4}$$

where σ_{ij} is the Cauchy stress tensor; v_i is the velocity vector; δ_{ij} is the Kronecker delta; a comma is used to denote partial differentiation (e.g. $v_{k,l}=\partial v_k/\partial y_l$, where y_l denotes the spatial Cartesian co-ordinates); a superposed dot denotes the material time derivative, in which material particles are held fixed; and the summation convention is assumed to be in effect for repeated indices.

The first term on the right-hand side of (1) represents the material response due to deformation, whereas the second term accounts for rotational effects. The tensor $\bar{c}_{ijkl}$ is a material response tensor which typically depends upon the stresses, deformation gradient and material parameters. The tensor s_{ijkl} is uniquely specified by 'objectivity' which requires that the stress-rate (i.e. $\dot{\sigma}_{ij}$) transform properly under time-dependent rigid rotations. Rate equations of the form (1) are frequently used in large-deformation finite element and finite difference computer programs. A more detailed discussion of equations of this type may be found in Reference 3.

Remark

In the present work we assume that a Lagrangian kinematical description is adopted. For the majority of nonlinear solids problems, a description of this kind is appropriate. In a numerical formulation, the stresses will be calculated at discrete points which correspond to a set of material particles. Consequently, a time-discretization of the rate constitutive equation is most conveniently facilitated if the equation is written in terms of the material time derivative, as is (1).

NUMERICAL ALGORITHM

To simplify the subsequent writing, we shall adopt bold-faced notation to represent vectors and matrices. In a typical time step, the configuration of the body at step $n+1$ may be written as a function of the configuration at step n and the step length Δt, viz.

$$\mathbf{y}^{n+1}=\mathbf{y}^{n+1}(\mathbf{y}^n,\Delta t) \tag{5}$$

The displacement increment over the step is

$$\boldsymbol{\delta}=\mathbf{y}^{n+1}-\mathbf{y}^n \tag{6}$$

Consider the following one-parameter family of configurations

$$\mathbf{y}^{n+\alpha}=(1-\alpha)\mathbf{y}^n+\alpha\mathbf{y}^{n+1} \tag{7}$$

Let $\mathbf{G}$ denote the gradient of $\boldsymbol{\delta}$ with respect to $\mathbf{y}^{n+\alpha}$. In component form

$$G_{ij}=\partial\delta_i/\partial y_j^{n+\alpha} \tag{8}$$

Strain and rotation increments may be defined in terms of $\mathbf{G}$ as follows:

$$\boldsymbol{\gamma}=(\mathbf{G}+\mathbf{G}^{\mathrm{T}})/2 \tag{9}$$

$$\boldsymbol{\omega}=(\mathbf{G}-\mathbf{G}^{\mathrm{T}})/2 \tag{10}$$

in which a superscript T denotes transpose. γ and ω represent discrete approximations to the time integrals over the step of the symmetric and skew-symmetric parts of the velocity gradients, respectively.

Consider the following algorithm for integrating the constitutive equation:

$$\boldsymbol{\sigma}^{n+1} = \bar{\boldsymbol{\sigma}}^{n+1} + \Delta\boldsymbol{\sigma} \tag{11}$$

$$\bar{\boldsymbol{\sigma}}^{n+1} = \mathbf{Q}\boldsymbol{\sigma}^{n}\mathbf{Q}^{\mathrm{T}} \tag{12}$$

$$\mathbf{Q} = (\mathbf{I} - \alpha\boldsymbol{\omega})^{-1}(\mathbf{I} + (1-\alpha)\boldsymbol{\omega}) = \mathbf{I} + (\mathbf{I} - \alpha\boldsymbol{\omega})^{-1}\boldsymbol{\omega} \tag{13}$$ †

where $\mathbf{I}$ denotes the identity matrix, and $\Delta\boldsymbol{\sigma}$ is the material response part of the stress increment. In component form

$$\Delta\sigma_{ij} = \bar{c}_{ijkl}\gamma_{kl} \tag{14}$$

The expression for $\mathbf{Q}$ is obtained by applying the generalized midpoint rule (see e.g. Reference 2, p. 2) to the generating equation $\mathrm{d}\mathbf{Q}/\mathrm{d}t = \boldsymbol{\omega}\mathbf{Q}$.

PROPERTIES OF THE ALGORITHM

Let $\mathscr{R}$ be a set of proper-orthogonal matrices. An algorithm, such as (11)–(14), will be called *incrementally objective with respect to* $\mathscr{R}$ if for all $\mathbf{R} \in \mathscr{R}$

$$\mathbf{y}^{n+1} = \mathbf{R}\mathbf{y}^{n} \tag{15}$$

implies

$$\boldsymbol{\sigma}^{n+1} = \mathbf{R}\boldsymbol{\sigma}^{n}\mathbf{R}^{\mathrm{T}} \tag{16}$$

Theorem. Assume that (a) $\alpha = \frac{1}{2}$; and (b) $\mathscr{R}$ is the set of all proper-orthogonal matrices, $\mathbf{R}$, such that $\mathbf{R} + \mathbf{I}$ is nonsingular. Then: (i) $\mathbf{Q}$ is orthogonal; and (ii) the algorithm defined by (11)–(14) is incrementally objective with respect to $\mathscr{R}$.

In proving the theorem, we shall make use of the following preliminary result.

Lemma. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{A} + \mathbf{B}$ be square, nonsingular matrices. Then

$$(\mathbf{A}^{-1} + \mathbf{B}^{-1})^{-1} = \mathbf{A}(\mathbf{A} + \mathbf{B})^{-1}\mathbf{B} = \mathbf{B}(\mathbf{A} + \mathbf{B})^{-1}\mathbf{A} \tag{17}$$

Equation (17) is a standard exercise in linear algebra (see e.g. Reference 6, p. 18).

Proof of Theorem

Part (i)

In this case (13) may be written

$$\mathbf{Q} = (\mathbf{I} - \tfrac{1}{2}\boldsymbol{\omega})^{-1}(\mathbf{I} + \tfrac{1}{2}\boldsymbol{\omega}) \tag{18}$$

Application of (17) to (18) enables us to write

$$\mathbf{Q} = (\mathbf{I} + \tfrac{1}{2}\boldsymbol{\omega})(\mathbf{I} - \tfrac{1}{2}\boldsymbol{\omega})^{-1} \tag{19}$$

Transposition of (19) results in

$$\mathbf{Q}^{\mathrm{T}} = (\mathbf{I} + \tfrac{1}{2}\boldsymbol{\omega})^{-1}(\mathbf{I} - \tfrac{1}{2}\boldsymbol{\omega}) \tag{20}$$

Comparison of (18) and (20) establishes the orthogonality of $\mathbf{Q}$.

---

† Observe that $(\mathbf{I} - \alpha\boldsymbol{\omega})$ is *always* nonsingular. This may be seen as follows: $\mathbf{x}^{\mathrm{T}}(\mathbf{I} - \alpha\boldsymbol{\omega})\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{x}$ by virtue of $\boldsymbol{\omega}$ being skew-symmetric. Thus $(\mathbf{I} - \alpha\boldsymbol{\omega})$ is positive-definite and, consequently, nonsingular.

Part (ii)

It suffices to show that $\boldsymbol{\gamma}=\mathbf{0}$ and $\mathbf{Q}=\mathbf{R}$ follows from (15) to establish the incremental objectivity of (11)–(14).

Let us first show that $\boldsymbol{\gamma}=\mathbf{0}$:

It follows from (6), (7) and (15) that

$$\boldsymbol{\delta}=2(\mathbf{R}-\mathbf{I})(\mathbf{R}+\mathbf{I})^{-1}\mathbf{y}^{n+1/2} \tag{21}$$

and therefore

$$\mathbf{G}=2(\mathbf{R}-\mathbf{I})(\mathbf{R}+\mathbf{I})^{-1} \tag{22}$$

By (9) and (17)

$$\begin{aligned}\boldsymbol{\gamma}&=(\mathbf{R}-\mathbf{I})(\mathbf{R}+\mathbf{I})^{-1}+(\mathbf{R}+\mathbf{I})^{-\mathrm{T}}(\mathbf{R}-\mathbf{I})^{\mathrm{T}}\\&=\mathbf{R}(\mathbf{R}+\mathbf{I})^{-1}+(\mathbf{R}+\mathbf{I})^{-\mathrm{T}}\mathbf{R}-(\mathbf{R}+\mathbf{I})^{-\mathrm{T}}-(\mathbf{R}+\mathbf{I})^{-1}\\&=\mathbf{R}(\mathbf{R}+\mathbf{I})^{-1}+(\mathbf{R}+\mathbf{I})^{-\mathrm{T}}\mathbf{R}-\mathbf{R}(\mathbf{R}+\mathbf{I})^{-1}-(\mathbf{R}+\mathbf{I})^{-\mathrm{T}}\mathbf{R}\\&=\mathbf{0}\end{aligned} \tag{23}$$

Now let us show that $\mathbf{Q}=\mathbf{R}$:

It follows from (22) and (23) that

$$\boldsymbol{\omega}=2(\mathbf{R}-\mathbf{I})(\mathbf{R}+\mathbf{I})^{-1} \tag{24}$$

Multiplying through by $(\mathbf{R}+\mathbf{I})$ yields

$$\boldsymbol{\omega}(\mathbf{R}+\mathbf{I})=2(\mathbf{R}-\mathbf{I}) \tag{25}$$

whereupon solving for $\mathbf{R}$ results in

$$\mathbf{R}=(\mathbf{I}-\tfrac{1}{2}\boldsymbol{\omega})^{-1}(\mathbf{I}+\tfrac{1}{2}\boldsymbol{\omega}) \tag{26}$$

By comparison of (18) with (26) we see that $\mathbf{Q}=\mathbf{R}$ which completes the proof of the theorem. ■

Remarks

1. It is thus seen from the theorem that the good behaviour of the algorithm is contingent upon the selection $\alpha=\frac{1}{2}$.

2. The only limitation on $\mathbf{R}$ emanates from hypothesis (b). Requiring $(\mathbf{R}+\mathbf{I})$ to be nonsingular is equivalent to insisting that -1 not be an eigenvalue of $\mathbf{R}$. To appreciate the meaning of this condition, consider the two-dimensional case in which $\mathbf{R}$ is a 2×2 matrix. If -1 is one eigenvalue of $\mathbf{R}$, then the other one must also be -1, by proper-orthogonality. Thus $\mathbf{R}$ represents a rotation of 180 degrees.

In the three-dimensional case, a proper-orthogonal matrix always possesses one eigenvalue equal to $+1$. Thus if it also possesses an eigenvalue equal to -1, the third eigenvalue must also equal -1 by proper-orthogonality. Consequently, $\mathbf{R}$ represents a rotation of 180 degrees in the plane perpendicular to the eigenvector associated with the $+1$ eigenvalue. The situation is thus reducible to the two-dimensional case.

The pathology which occurs under these circumstances is illustrated in Figure 1. The 'average configuration' collapses to a point. As a result the algorithm is limited to rotation increments of less than 180 degrees per step. This is far beyond the range of realistic values and therefore does not restrict the practical application of the scheme.

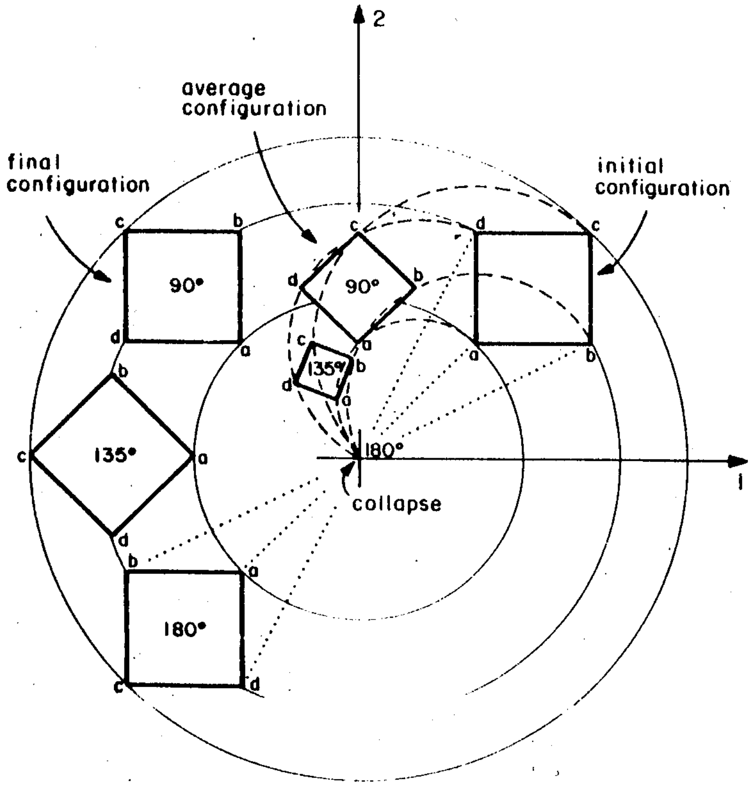

Figure 1. Behaviour of the average configuration as a function of rotation

3. Sub-incremental algorithms are often advocated for use in integrating rate constitutive equations (see e.g. Reference 7). To generalize the preceding algorithm to a sub-incremental one, we need only repeat the calculations involved in (11)–(14) within each sub-increment.

4. The polar decomposition theorem (see e.g. Reference 4) might also be used as a basis for constructing the rotational transformation. This would have the advantage of being exact for arbitrary deformation gradients. The disadvantage is the increased number of operations involved in this procedure.

CONCLUSIONS

A simple algorithm has been presented for the integration of rate-type constitutive equations. The algorithm has been shown to be 'objective' for large rotation increments.

ACKNOWLEDGEMENTS

We wish to express our gratitude to: G. L. Goudreau and J. O. Hallquist for drawing our attention to the problem treated herein; and the Electric Power Research Institute and the National Science Foundation for research support.

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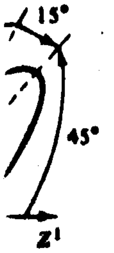
Notes
ME 239B
Handout #6



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3.15 Proposition *In the two-dimensional case, let*

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I_C = trace C, *and* $\mathrm{II}_C = \det C$.

Then $$U = \frac{1}{\sqrt{(\mathrm{I}_C + 2\sqrt{\mathrm{II}_C})}}(C - \sqrt{\mathrm{II}_C}I) \quad \textit{and} \quad R = FU^{-1}.$$

Proof By the Cayley–Hamilton theorem from linear algebra,

$$U^2 - \mathrm{I}_U U + \mathrm{II}_U I = 0.$$

But $C = U^2$, and $\det C = (\det U)^2$, so $C - \mathrm{I}_U U + \sqrt{\mathrm{II}_C} I = 0$. Taking the trace gives $\mathrm{I}_C - (\mathrm{I}_U)^2 + 2\sqrt{\mathrm{II}_C} = 0$, so $\mathrm{I}_U = \sqrt{\mathrm{I}_C + 2\sqrt{\mathrm{II}_C}}$. Solving $C - \mathrm{I}_U U + \sqrt{\mathrm{II}_C} I = 0$ for U and substituting for I_U gives the result. ∎

The reader might wish to re-work Problem 3.2 using 3.15.

> ***Problem 3.3*** The Cayley–Hamilton theorem for 3 × 3 matrices U states that
>
> $$-U^3 + \mathrm{I}_U U^2 - \mathrm{II}_U U + \mathrm{III}_U I = 0$$
>
> where $\mathrm{I}_U = \operatorname{tr} U$, $\mathrm{II}_U = \det U(\operatorname{tr} U^{-1})$, and $\mathrm{III}_U = \det U$
>
> are the principal invariants of U. Use this to work out an explicit formula for U in terms of C and its principal invariants.

Next we study how the deformation tensor and the stretch tensors measure changes in lengths and angles. Recall that if $\sigma: [a, b] \longrightarrow \mathcal{B}$ is a C^1 (or piecewise C^1) curve, its *length* is given by

$$l(\sigma) = \int_a^b \|\sigma'(\lambda)\|\, d\lambda.$$

3.16 Proposition *Let σ be a C^1 curve in $\mathcal{B}$ and let ϕ be a C^1 configuration of $\mathcal{B}$ in $\mathcal{S}$. Let $\bar{\sigma} = \phi \circ \sigma$ be the image of σ under ϕ. Then the length of $\bar{\sigma}$ depends only on σ and on the stretch tensor U.*

Proof From the chain rule $\bar{\sigma}'(\lambda) = T\phi(\sigma(\lambda))\sigma'(\lambda) = F_{\sigma(\lambda)}\sigma'(\lambda)$. Hence

$$\begin{aligned} \|\bar{\sigma}'(\lambda)\| &= \langle F_{\sigma(\lambda)}\sigma'(\lambda), F_{\sigma(\lambda)}\sigma'(\lambda)\rangle^{1/2} \\ &= \langle \sigma'(\lambda), F^T_{\sigma(\lambda)}F_{\sigma(\lambda)}\sigma'(\lambda)\rangle^{1/2} \\ &= \langle \sigma'(\lambda), C_{\sigma(\lambda)}\sigma'(\lambda)\rangle^{1/2} \\ &= \langle \sigma'(\lambda), U^2_{\sigma(\lambda)}\sigma'(\lambda)\rangle^{1/2} = \|U_{\sigma(\lambda)}\sigma'(\lambda)\|. \end{aligned}$$

Thus $$l(\bar{\sigma}) = \int_a^b \|U_{\sigma(\lambda)}\sigma'(\lambda)\|\, d\lambda. \quad ∎$$

We call $\bar{\sigma}$ the *deformation* of σ under ϕ (see Figure 1.3.2).

The 3D problem is :

$$\boxed{- \underset{\sim}{U}^3 + \mathrm{I}_u \underset{\sim}{U}^2 - \mathrm{II}_u \underset{\sim}{U} + \mathrm{III}_u \underset{\sim}{I} = 0} \qquad \text{(C-H)}$$

Using definitions :

def(*1*) : $\underset{\sim}{C} = \underset{\sim}{U}^2$

def(*2*) : $\mathrm{I}_u = \text{trace } \underset{\sim}{U}$

def(*3*) : $\mathrm{II}_u = \det \underset{\sim}{U} \text{ Trace } (\underset{\sim}{U}^{-1})$

def(*4*) : $\mathrm{III}_u = \det \underset{\sim}{U}$

Using the same approach as in 2D, we start from Cayley-Hamilton theorem and def(*1*) :

C-H becomes :

$$- \underset{\sim}{U}\,\underset{\sim}{C} + \mathrm{I}_u \underset{\sim}{C} - \mathrm{II}_u \underset{\sim}{U} + \mathrm{III}_u \underset{\sim}{I} = 0$$

we factorize $\underset{\sim}{U}$:

$$\underset{\sim}{U} (- \underset{\sim}{C} - \mathrm{II}_u \underset{\sim}{I}) + \mathrm{I}_u \underset{\sim}{C} + \mathrm{III}_u \underset{\sim}{I} = 0$$

we assume $(- \underset{\sim}{C} - \mathrm{II}_u \underset{\sim}{I})$ to be non-singular, so $\underset{\sim}{U}$ can be expressed as :

$$\boxed{\underset{\sim}{U} = (\mathrm{I}_u \underset{\sim}{C} + \mathrm{III}_u \underset{\sim}{I})\,(\underset{\sim}{C} + \mathrm{II}_u \underset{\sim}{I})^{-1}}$$

We now have to express the $\underset{\sim}{U}$ invariants in term of $\underset{\sim}{C}$ invariants.

.../..

- The first relation is again obvious from def(*1*) and def(*4*) :

$$\text{def}(1) + \text{def}(4) \implies \det \underset{\sim}{C} = (\det \underset{\sim}{U})^2$$

$$\text{or} \quad \boxed{III_u = \sqrt{III_c}}$$

- The second one is obtained using another definition of the second invariant (that can be obtain using again C-H theorem and using the trace) $II_u = \frac{1}{2}((\text{trace } \underset{\sim}{U})^2 - \text{trace } \underset{\sim}{U}^2)$

$$\text{so} \quad : \quad \boxed{II_u = \frac{1}{2}\,(I_u^2 - I_c)}$$

- The last relation needed is obtained using C-H theorem one more time : and multiplying by U^{-2} in order to make II_c apparent :

$$-\underset{\sim}{U} + I_u \underset{\sim}{I} - II_u \underset{\sim}{U}^{-1} + III_u \underset{\sim}{U}^{-2} = 0$$

Taking the trace :

$$-\text{trace } \underset{\sim}{U} + I_u \text{ trace } \underset{\sim}{I} - II_u \text{ trace } \underset{\sim}{U}^{-1} + III_u \text{ trace } \underset{\sim}{U}^{-2} = 0$$

using : trace $\underset{\sim}{U} = I_u$ [def(*2*)]

trace $\underset{\sim}{I} = 3$

trace $\underset{\sim}{U}^{-1} = \dfrac{II_u}{III_u}$ [def(*3*) + def(*4*)]

trace $\underset{\sim}{U}^{-2} = \dfrac{II_c}{III_c}$ [def(*3*) + def(*4*) on the $\underset{\sim}{C}$ matrix]

.../.

So we obtain :

$$- I_u + 3\, I_u - \frac{II_u^2}{III_u} + III_u \frac{II_c}{III_c} = 0$$

therefore the final relation is :

$$\boxed{2\, I_u\, III_u - II_u^2 + II_c = 0}$$

- We did not succeed in obtaining from these last two relations explicit formulas for U invariant in term of C invariant. The only solution we obtained for your problem was :

$$\boxed{\begin{array}{l} \underset{\sim}{U} = (I_u\, \underset{\sim}{C} + III_u\, \underset{\sim}{I})\, (\underset{\sim}{C} + II_u\, \underset{\sim}{I})^{-1} \\[2em] \text{with :} \left\{ \begin{array}{l} III_u = \sqrt{III_c} \\[1.5em] II_u = \frac{1}{2}\,(I_u^2 - I_c) \\[1.5em] 2\, I_u\, III_u - II_u^2 + II_c = 0 \end{array} \right. \end{array}}$$

but this is not an explicit formula for U in terms of C and its principal invariants, and therefore not an answer for your problem. Do you thing there is truly one solution ?.

Yes! Find it and win the Hughes Research Prize !!!

Plasticity of Metals at Finite Strain, Proceedings of a Research Workshop,
Stanford University, June 1981

Handout #8
Notes
ME 239 B

UNIFIED CONSTITUTIVE EQUATIONS FOR TIME-DEPENDENT PLASTICITY

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I. INTRODUCTION - WHY ARE UNIFIED CONSTITUTIVE EQUATIONS NECESSARY?

In current engineering practice, non-elastic strain is generally modelled as the sum of a "time-independent plastic" strain ε_p and a "time dependent creep" strain $\dot{\varepsilon}_c$, thus in one-dimensional form the total strain ε_T (excluding thermal expansion) is given in terms of the equation:

$$\varepsilon_T = \varepsilon/E + \varepsilon_p + \varepsilon_c(t) \quad . \tag{1}$$

While this approach can be rationalized on historical grounds and perhaps on computational convenience, it has serious inadequacies in some situations of current interest. In particular, situations involving mixtures of "plasticity" and "creep" or "plasticity" and other time-dependent phenomena such as recovery appear to cause difficulties.

As one such case in point, Figure 1 (taken from work at Battelle Columbus Laboratories [1]) shows the cyclic stress response of type 304 stainless steel at 593°C, for a strain range of 0.6%. Material which previously saw creep strains in the range of 0.10% to 0.24% (solid curves) has a higher initial stress amplitude than material which did not (broken curve). This effect would not be predicted by eq. (1), because the cyclic stress amplitude will be governed by the ε_p term, which is generally unaffected by the value of the creep strains associated with the ε_c term.

One solution to this inadequacy (currently being utilized in high-temperature design) is to establish an ad-hoc interaction between ε_c and ε_p. For example, the "Interim Guidelines for Detailed Inelastic Analysis of High-Temperature Reactor System Components," [2] are based on the "classical" approach of eq. (1). Time-independent plastic straining (a change in ε_p) occurs when the yield condition is satisfied, specifically:

$$\tfrac{1}{2}(\sigma'_{ij} - \alpha_{ij})(\sigma'_{ij} - \alpha_{ij}) = \kappa \qquad (2)$$

where σ'_{ij} is the deviatoric stress tensor

α_{ij} is a back stress tensor representing kinematic hardening

κ is a strength constant.

Because of the effect of prior creep on the subsequent cyclic flow stress, κ is <u>not</u> taken to be a constant. Rather, κ changes according to the level of creep strain; initially $\kappa = \kappa_o$ where κ_o corresponds to the strength of the material in its annealed state, but when an effective creep strain of 0.2% has been reached κ changes to κ_1, where κ_1 corresponds to the 10th cycle strength properties observed in cyclic straining tests.

This approach, while successful in reproducing specific behavior shown in Figure 1, is generally regarded as being ad-hoc in nature and not satisfactory as a general model for non-elastic deformation. (For example, while it treats the effect of prior creep on subsequent plastic strains, it does not treat the effect of prior cyclic hardening on subsequent creep response.) Therefore the eventual preferred methodology for these and other similar situations is seen by a number of investigators as being "unified" constitutive equations, in which ε_p and ε_c are both included within a <u>single</u> variable, hereinafter referred to as non-elastic strain and denoted as ε. Thus the "unified" constitutive equations take the form:

$$\epsilon_T = \sigma/E + \epsilon \ . \quad (3)$$

A second case in point indicating the desirability of using the unified approach is shown in Figure 2 (taken from work at Oak Ridge National Laboratory [3]). A ferritic steel (2¼Cr-1Mo) was given a tensile test (at 538°C) interrupted by hold periods of constant total strain. As would be expected, strain hardening occurs as the material is strained, and stress relaxation occurs during the hold periods. In addition to this, the noteworthy behavior is that upon each reloading, the material's flow stress is significantly lower than would be expected based on ordinary strain hardening concepts. The reason [3] is that recovery (loss of work hardening) has occurred during the hold periods. This time-dependent recovery would not ordinarily be predicted by the "classical" approach (eq. 1) but _is_ predicted by the unified approach, as will be shown below.

Additional considerations based on the controlling physical mechanisms tend to reach the same conclusion, namely that unified constitutive equations are more appropriate than the classical approach because a common set of physical processes (dislocation glide, climb, multiplication, pileups, etc.) underlies non-elastic deformation in both the "plasticity" and "creep" regimes. (This clearly excludes the contributions of purely diffusional creep as well as grain-boundary sliding.)

II. SOME UNIFIED CONSTITUTIVE EQUATIONS

In response to the above and other considerations, a number of investigators [4-13] have developed unified constitutive equations. Two features are common to all of these approaches: First, as discussed above, "creep" and "plasticity" are both included within a single variable (ε). And second, internal variables (also referred to as "state" or "structure" variables) are used within the equations

to predict transient deformation and the effects of previous history. (It is well-established that the ordinary variables of non-elastic strain ε and time t are <u>not</u> satisfactory as state variables for general-purpose constitutive equations.) Because they are history-dependent, these internal variables must be specified in terms of evolutionary equations which give the rates at which they change. There may be one or more internal variables, but assuming for the moment that they can collectively be denoted as "X", then the form of the unified constitutive equations becomes:

$$\dot{\varepsilon} = f(\sigma, T, X) \tag{3a}$$

$$\dot{X} = g(\dot{\varepsilon}, T, X) \tag{3b}$$

Figure 3 illustrates schematically how such equations can predict both "plasticity" (e.g. situations of approximately constant $\dot{\varepsilon}$) and creep (e.g. constant σ). In both cases the structure variable X changes during transient deformation but reaches a constant value (under constant boundary conditions) which causes steady-state flow to occur. Thus under constant $\dot{\varepsilon}$, as X increases the flow stress increases (strain hardening) while under constant σ, as X increases the non-elastic strain rate decreases (primary creep). It should be noted that if the initial value of X is sufficiently high (due, for example, to prior deformation under a different set of boundary conditions) then <u>decreases</u> in X are possible, which would constitute dynamic softening. Whether or not this actually occurs depends on the equations.

A major question with respect to unified constitutive equations is the proper number of internal variables X. There is probably no "correct" answer to this question; a true description of the material's "state" would require, for example, a complete representation of the configuration of all of the dislocations in the specimen, yet computational considerations limit X to comprising at most a few

specific variables. In striking a balance between these conflicting demands, a number of investigators have arrived at the conclusion that two structure variables are appropriate: one to represent isotropic hardening (such as that due to an overall increase in the dislocation density) and one to represent kinematic, or directional hardening (such as that due to dislocation pileups or other polarized features of the dislocation substructure).

Thus, for example, in the equations proposed by Krieg, Swearengen, and Rohde [7] (Figure 4) the two structure variables are α (a back stress, representing kinematic hardening) and σ_D, a "drag" stress, representing isotropic hardening. Figure 4 also shows the manner in which these variables are utilized within the equations. (This format is found in a number of approaches and appears to represent a point of consensus.) The back stress α is subtracted from the applied stress σ; in multiaxial forms a back stress tensor α_{ij} is subtracted from the applied stress tensor σ_{ij}, representing the translation of surfaces of constant effective non-elastic strain rate $\dot{\bar{\varepsilon}}$ in stress space [14]. The drag stress σ_D is divided into the magnitude of $\sigma-\alpha$; this is an appropriate means of representing isotropic hardening because increases in σ_D should cause a reduction in the magnitude of $\dot{\varepsilon}$, regardless of the signs of σ and $\dot{\varepsilon}$.

Figure 4 also illustrates the typical form for the differential questions governing α and σ_D, namely the work hardening/recovery form. In the $\dot{\sigma}_D$ equation the magnitude of $\dot{\varepsilon}$ (irrespective of its sign) is the driving force for increases in σ_D, while σ_D itself is the driving force for decreases in σ_D due to recovery; the rate of recovery depends on temperature T. In the $\dot{\alpha}$ equation the sign of $\dot{\varepsilon}$ affects the sign of $\dot{\alpha}$ as is expected for a directional hardening process while again α itself provides the driving force for thermally-activated recovery of α towards $\alpha = 0$.

The "MATMOD" constitutive equations (which are discussed further below) and others follow a similar approach for representing isotropic hardening and kinematic hardening. Krieg [15] has reviewed the extent to which a number of approaches fit the format illustrated in Fig. 4.

The equations developed by Hart, Li and colleagues [4] are illustrated in Figure 5. A directional hardening term σ_a is subtracted from the applied stress σ in determining the overall non-elastic strain rate $\dot{\varepsilon}$. A second variable σ^* (the "hardness") appears within the recovery term of the $\dot{\sigma}_a$ equation. Although in the sense of Fig. 4 σ^* is not an isotropic hardening variable, because of the algebraic details of the equations σ^* has an effect similar to isotropic hardening (for example, the flow stress during load relaxation simulations increases with increases in σ^*). The form of the recovery term in the $\dot{\sigma}_a$ equation of Figure 5 is based on observations from load relaxation experiments.

A fourth set of unified constitutive equations, developed by Robinson [6], is shown in Fig. 6. Here, isotropic hardening is deemed to be negligible for the materials and regimes of behavior considered, so that a single state variable α is utilized. The equations are perhaps unique among unified approaches in that they contain certain "if-tests" shown in Fig. 6.

III. THE "MATMOD" CONSTITUTIVE EQUATIONS

The remainder of this paper will focus on one particular approach within the unified constitutive equations category, namely the "MATMOD" (MATerials MODel) equations [5, 16-19]. While sharing the kinematic hardening-isotropic hardening framework (shown in Figure 4) with other approaches the MATMOD approach aims at a somewhat broader scope of phenomena. (For this reason it may be less accurate in specific predictions than approaches which cover

a more restricted range of phenomena.) A feature which is unique to the MATMOD equations is their explicit representation of solute strengthening effects, allowing useful predictions of flow stress plateaus, effects of dynamic strain aging on strain-rate sensitivity, inverse temperature sensitivity of the flow stress, and other related phenomena.

Figure 7 shows the central equation governing $\dot{\varepsilon}$, along with a pictorial representation of some of the major phenomena associated with the various variables and terms. Directional hardening is represented by a back stress term (R) which is similar to α in Fig. 4. Isotropic hardening is represented by a variable called F_{def} ("friction stress due to deformation") similar to σ_D in Figure 4. Solute strengthening is represented by two terms: $F_{sol,1}$, which is most important in behavior at small strains (actually small F_{def}), and $F_{sol,2}$, which is most important in behavior at large strains. Temperature enters the equations through a temperature-dependent factor Θ', similar to exp($-Q/kT$) except that Q varies with temperature.

R and F_{def}, being history-dependent, are governed by differential equations. $F_{sol,1}$ and $F_{sol,2}$ are not history dependent but do depend on temperature and $\dot{\varepsilon}$ because solute drag depends on the relative velocities of solute diffusion and dislocation motion. The full equations are given in Fig. 8.

The internal workings of these equations have been described elsewhere [5, 16-19]. Therefore the remainder of this paper will simply illustrate the kinds of predictions which can be made with this model.

A flow stress plateau can be predicted because of the solute strengthening terms (Fig. 9) [16]. Also, the model can predict a regime of inverse temperature sensitivity of the flow stress as

illustrated for 316 stainless steel in Fig. 10 [16].

A related effect is the prediction of the strain-rate sensitivity (m) as a function of temperature and strain rate. Fig. 11 [18] illustrates that a local minimum in m vs. T can be predicted. For this case (Zircaloy) and others, m can be near zero in specific regimes, so that the model's predictions appear similar to "rate-independent plasticity" even though the model is built around an equation for $\dot{\varepsilon}$. Similarly, the equations can predict the stress relaxation response (Fig. 12) [18] showing a minimum in the slope of the log σ vs. log $\dot{\varepsilon}$ curve at intermediate temperatures. This produces a behavior very similar to the data (on Zircaloy) in which the curves are concave downwards at high temperatures and concave upwards at some lower temperatures.

One of the useful capabilities of the MATMOD equations (along with some other unified approaches) is prediction of the response to complex loading histories [17,19]. This capability is associated with the differential equations governing R and F_{def}; the current values of R and F_{def} reflect the past history but the rates at which they change reflect the current conditions. With this approach, the model can successfully predict, for example, the response of 2¼Cr-1Mo steel to constant strain rate loading interupted by periods of stress relaxation (Fig. 13), for which the data was slower in Fig. 2. In the simulation, the material weakens during the stress relaxation periods (as observed in the data) because of recovery of R.

A second example is a stress-change creep test. Figure 14 [19] shows data and reasonably accurate predictions for the response of pure aluminum to a step stress increase.

It is worth noting that in the MATMOD equations (as in other approaches) the structure variables R and F_{def} govern deformation response both for "plasticity"-like situations and "creep"-like situations. Hence these structure variables provide a natural means for predicting interactions between "plasticity" and "creep". Figure 15 demonstrates this for pure aluminum which was cold worked and then creep tested. In the predictions, as in the data, the cold work substantially affects the primary creep response but does not affect the subsequent steady-state creep rate.

IV. CONCLUDING REMARKS

The above discussion has illustrated the usefulness of unified constitutive equations for predicting non-elastic deformation response in a variety of situations. As a category, the unified approach is probably most useful for predictions involving complex histories, behavior over wide ranges in conditions, or situations where data are relatively sparse. For problems involving only simple loading histories, over specific ranges of conditions, and where ample data exist, the classical methods are probably adequate for design analysis.

One of the current problems in utilizing the unified constitutive equations in structural or metal forming analyses is their numerical characteristics. In finite-element programs built to accept both "time-independent plastic" (ε_p) and "time-dependent creep" (ε_c) terms, as eq. (1), the predictions of the unified equations fit most logically into the ε_c algorithms. And yet when this is done, it is sometimes found that exceedingly small time steps must be taken to assure stability [20], which makes the analysis costly in computer time. Research has been done on the development of appropriate numerical strategies [15,21,22] but a successful general methodology

has not yet emerged. Work on this subject is continuing, for example [23].

ACKNOWLEDGEMENT

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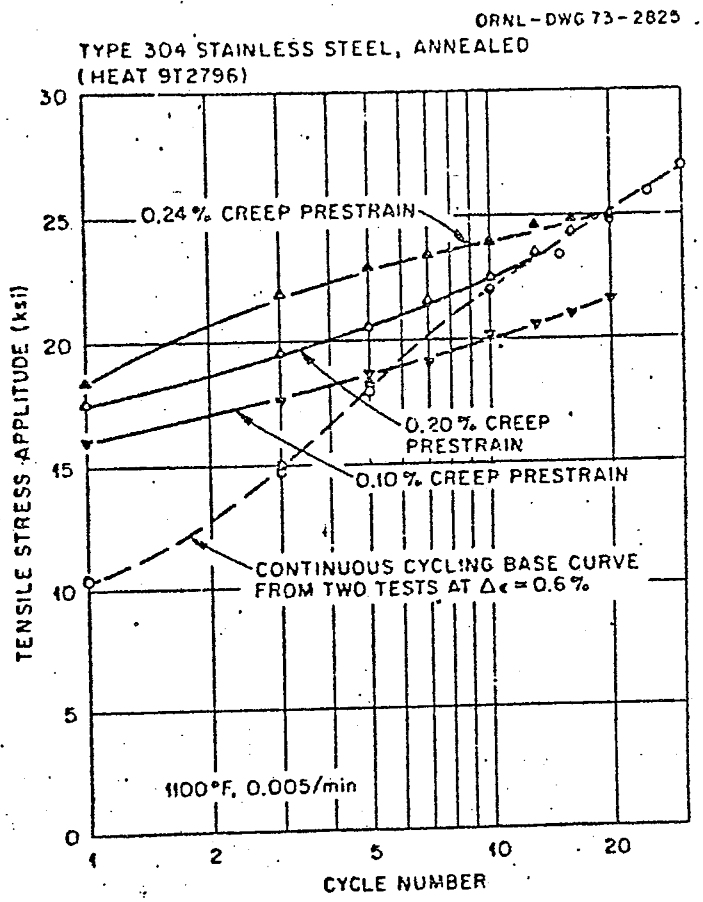


Fig. 1. Effect of prior creep on subsequent cyclic elastic-plastic behavior of annealed type 304 stainess steel at 1100°F (Battelle-Columbus Laboratories).

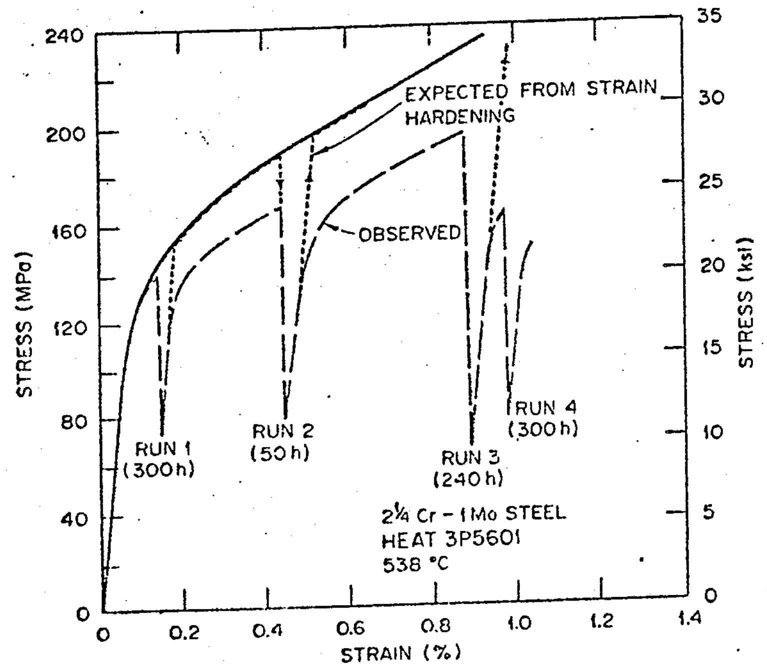


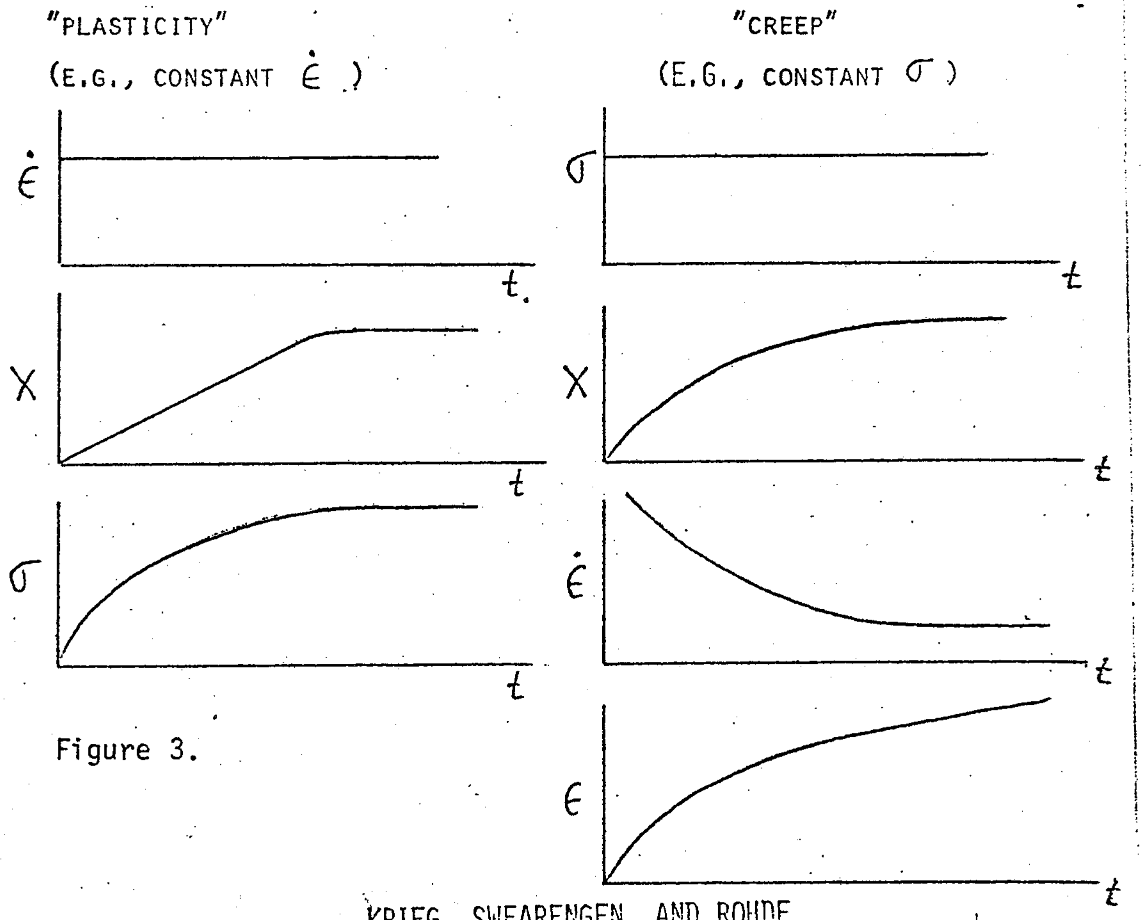
Fig. 2. Stress-Strain-Time response for Multiple-Loading Relaxation Test.

THE UNIFIED APPROACH TO PREDICTING NON ELASTIC STRAIN, ϵ

$$\left(\epsilon_{TOT} = \frac{\sigma}{E} + \epsilon \right)$$

$$\dot{\epsilon} = f(\sigma, T, X)$$

$$\dot{X} = g(\dot{\epsilon}, T, X)$$



"PLASTICITY"
(E.G., CONSTANT $\dot{\epsilon}$)

"CREEP"
(E.G., CONSTANT σ)

Figure 3.

KRIEG, SWEARENGEN, AND ROHDE CONSTITUTIVE EQUATIONS

(1 - DIMENSIONAL SIMPLIFICATION)

$$\dot{\epsilon} = \dot{\epsilon}_0(T) \left\{ \frac{|\sigma - \alpha|}{\sigma_D} \right\}^{m(T)}$$

$$\dot{\alpha} = A_\alpha \dot{\epsilon} - r_\alpha(\alpha, T)$$

$$\dot{\sigma}_D = A|\dot{\epsilon}| - r(\sigma_D, T)$$

Figure 4

HART-LI CONSTITUTIVE EQUATIONS

$$\dot{\epsilon} = A(T)(\sigma - \sigma_a)^M$$

$$\dot{\sigma}_a = \mathcal{M}\dot{\epsilon} - \frac{\mathcal{M}(\sigma^*/G)^m f\,\mathrm{Exp}(-Q/RT)}{\ln(\sigma^*/\sigma_a)^{1/\lambda}}$$

$$\dot{\sigma}^* = \begin{pmatrix}\text{WORK}\\ \text{HARDENING}\\ \text{TERM}\end{pmatrix} - \begin{pmatrix}\text{THERMALLY-ACTIVATED}\\ \text{RECOVERY TERM}\end{pmatrix}$$

Figure 5.

ROBINSON CONSTITUTIVE EQUATIONS

$$\dot{\epsilon} = A(\sigma - \alpha)|\sigma - \alpha|^{n-1} \qquad \text{IF} \quad \sigma(\sigma - \alpha) > 0$$

$$\dot{\alpha} = C\dot{\epsilon}/|\alpha|^{\beta} - D\alpha|\alpha|^{m-\beta-1}$$

$$\text{IF} \quad \alpha > \alpha_0 \quad \text{AND} \quad \sigma\alpha \geq 0$$

Figure 6.

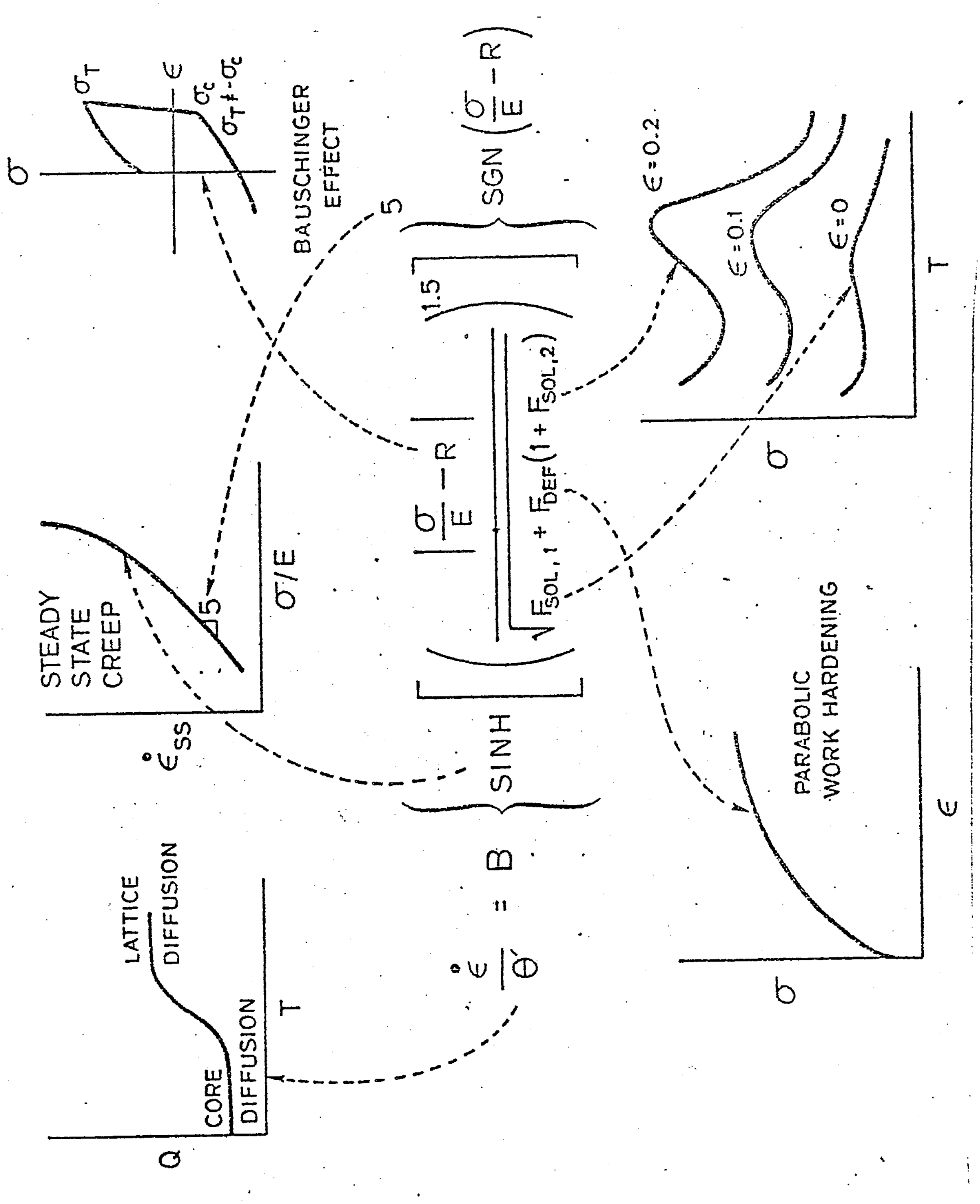


Figure 7. Basic phenomenological observations incorporated into the MATMOD strain rate equation.

MATMOD CONSTITUTIVE EQUATIONS

I. General Form:

A. Thermal strain rate:

$$\dot{\varepsilon}_{th} = f(T)\, f[\, \sigma/E - R,\ F_{def},\ F_{sol,1},\ F_{sol,2}\ ]$$

B. History-dependent state variables:

1. R ("Rest stress" or back stress):
   Directional (kinematic) hardening associated with pileups or dislocation bowing
2. F_{def} ("Friction stress due to deformation"):
   Isotropic hardening associated with subgrains, forest dislocations, irradiation

$$\dot{R} = \frac{dR}{dt} = \text{work-hardening term} - \text{thermal recovery term}$$

$$\dot{F}_{def} = \frac{dF_{def}}{dt} = \text{work-hardening term} + \text{flux-hardening term} - \text{thermal recovery term}$$

C. Temperature and strain-rate dependent solute strengthening variables:

F_{sol} ("Friction stress due to solutes"):
Isotropic strengthening associated with solute atmospheres, dynamic strain aging, etc.

$$F_{sol} = f(T, \dot{\varepsilon})$$

1. $F_{sol,1}$: independent of strain hardening (substitutional solutes?)
2. $F_{sol,2}$: synergistic with strain hardening (interstitial solutes?)

II. Specific Equations (1-dimensional form):

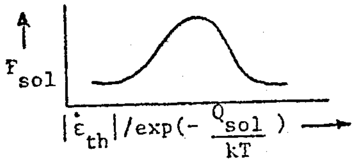
$$\dot{\varepsilon}_{th} = B\theta' \left\{ \sinh \left[\left(\frac{|\sigma/E - R|}{\sqrt{F_{sol,1} + F_{def}(1+F_{sol,2})}} \right)^{1.5} \right] \right\}^{n} \operatorname{sgn}(\sigma/E - R) \qquad (1)$$

$$\dot{R} = H_1 \dot{\varepsilon}_{th} - H_1 B\theta' [\sinh(A_1 |R|)]^{n} \operatorname{sgn}(R) \qquad (2)$$

$$\dot{F}_{def} = H_2 [C_2 + |R| - (A_2/A_1) F_{def}^{1.5}] |\dot{\varepsilon}_{th}| + H_3 \phi - H_2 C_2 B\theta' [\sinh(A_2 F_{def}^{1.5})]^{n} \qquad (3)$$

$$F_{sol,1} = f_1(|\dot{\varepsilon}_{th}|, T) \qquad (4)$$

$$F_{sol,2} = f_2(|\dot{\varepsilon}_{th}|, T) \qquad (5)$$



$$\dot{\varepsilon}_{irr} = B_2 \exp(-\frac{Q_{irr}}{kT}) (|\sigma|/E)^{n_2} (\phi)^{p} \operatorname{sgn}(\sigma) \qquad (6)$$

$$\dot{\varepsilon} = \dot{\varepsilon}_{th} + \dot{\varepsilon}_{irr} \qquad (7)$$

θ' is similar to $\exp(-\frac{Q}{kT})$

Figure 8.

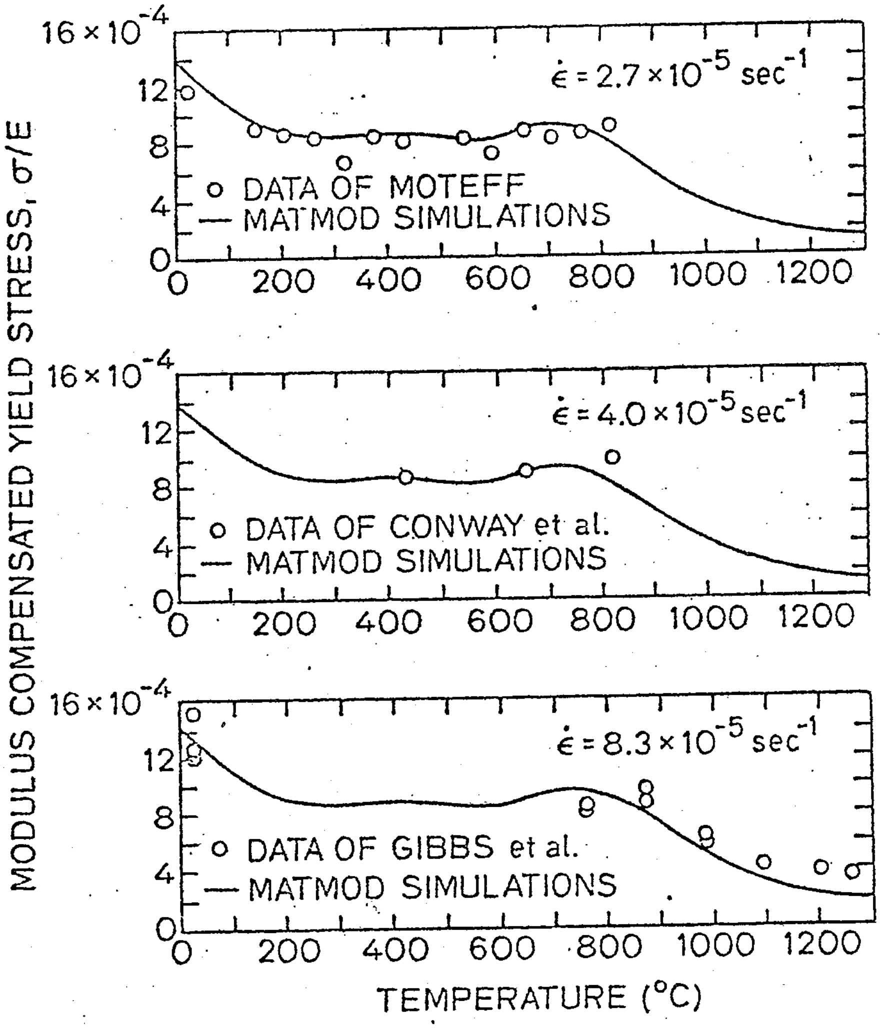


Figure 9. Yield stress data of solution annealed type 316 stainless steel as function of temperature and strain rate.

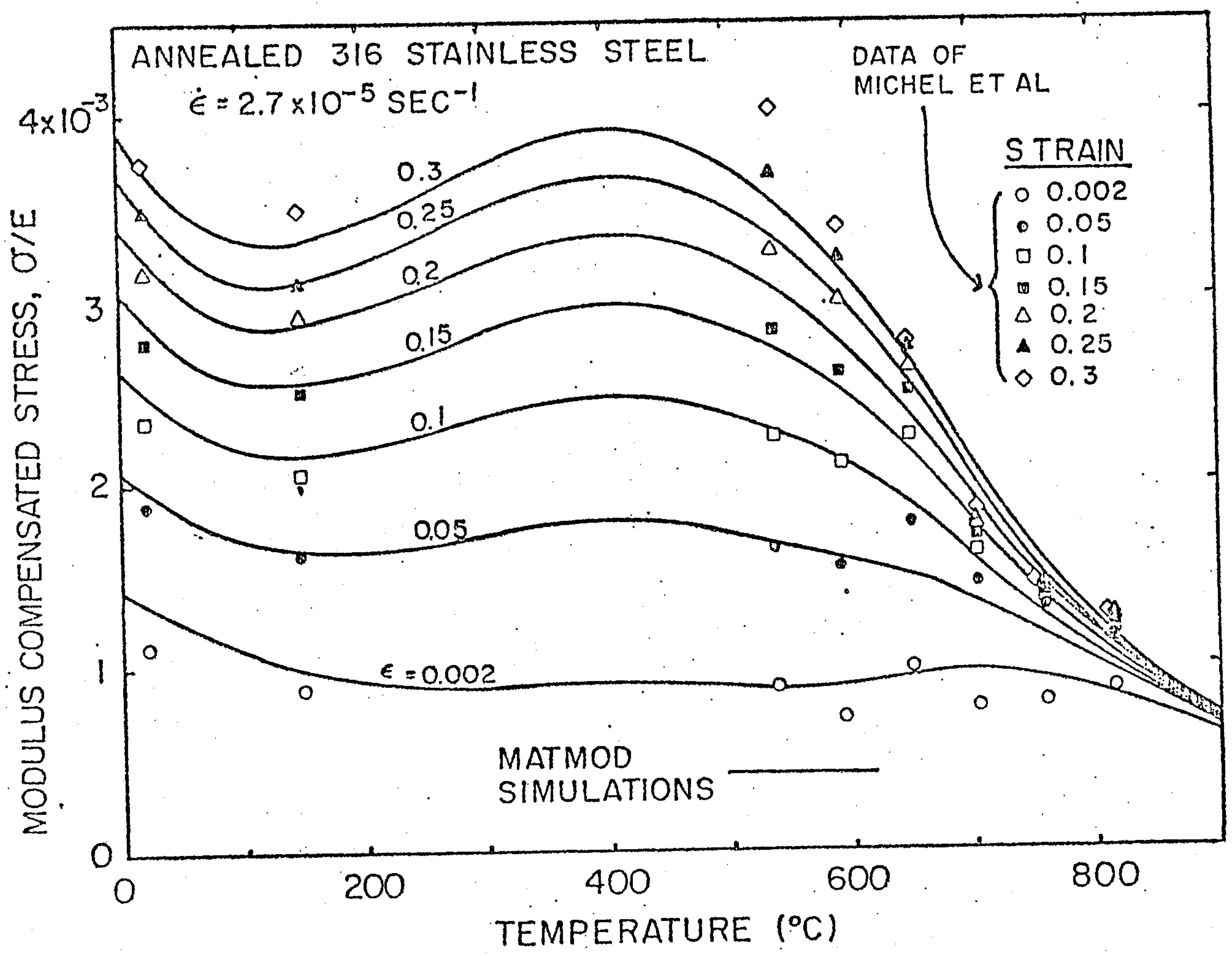


Figure 10 Monotonic stress-strain data of solution annealed type 316 stainless steel as a function of temperature at a strain rate of 2.7×10^{-5} sec^{-1}. The lines shown are the model's fit to the data. The data are from Michel, et al.

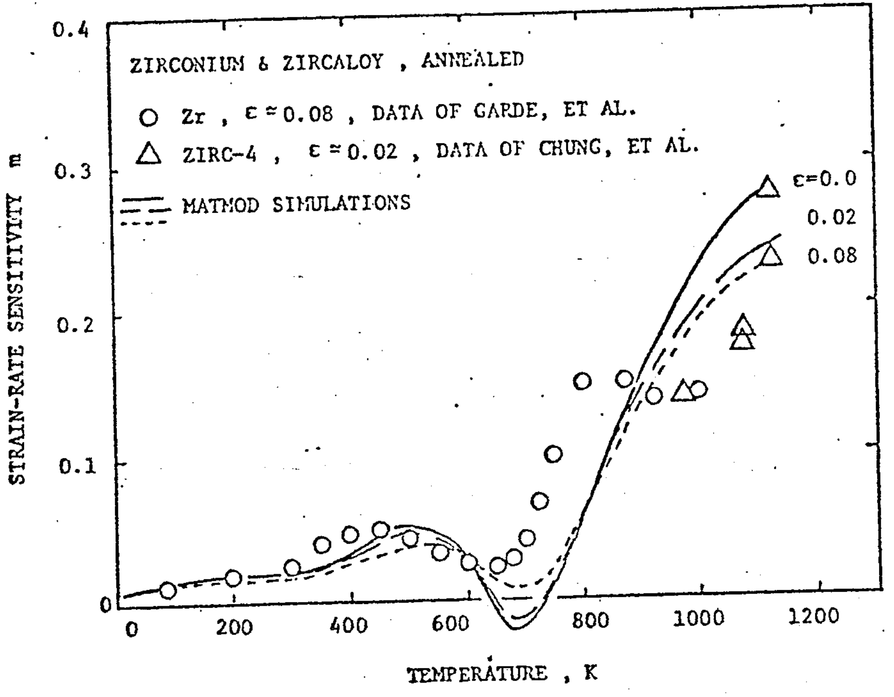


Figure 11. Strain-rate sensitivity as a function of temperature at various strain levels. The simulations utilize the final material constants and are independent of the data.

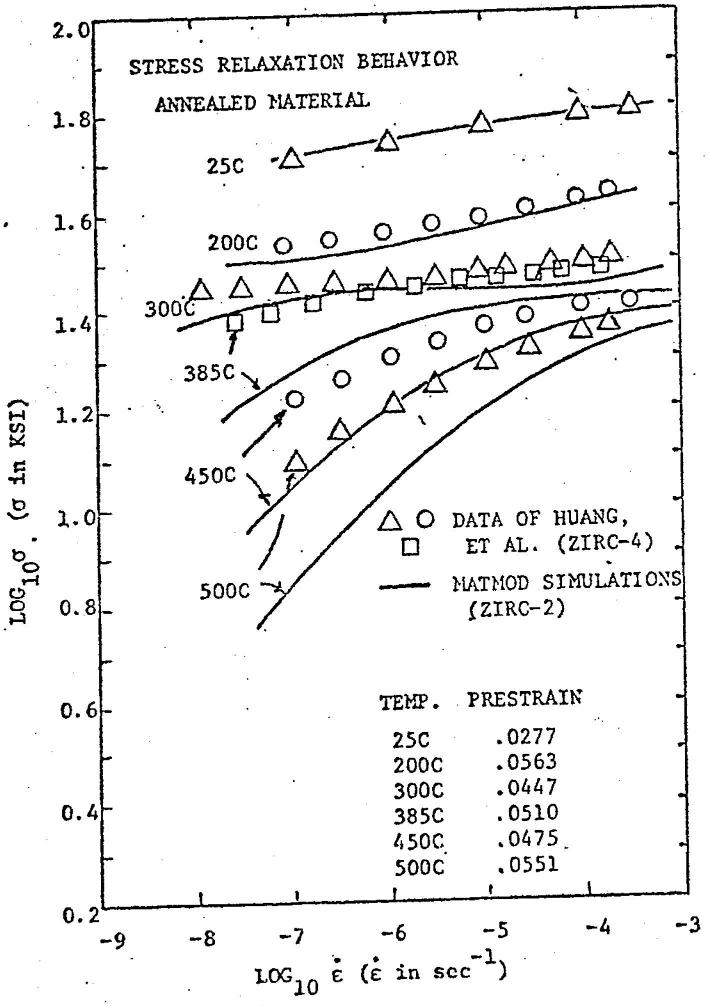


Figure 12. Stress relaxation behavior of annealed Zircaloy. The simulations (final material constants) are totally independent of the data.

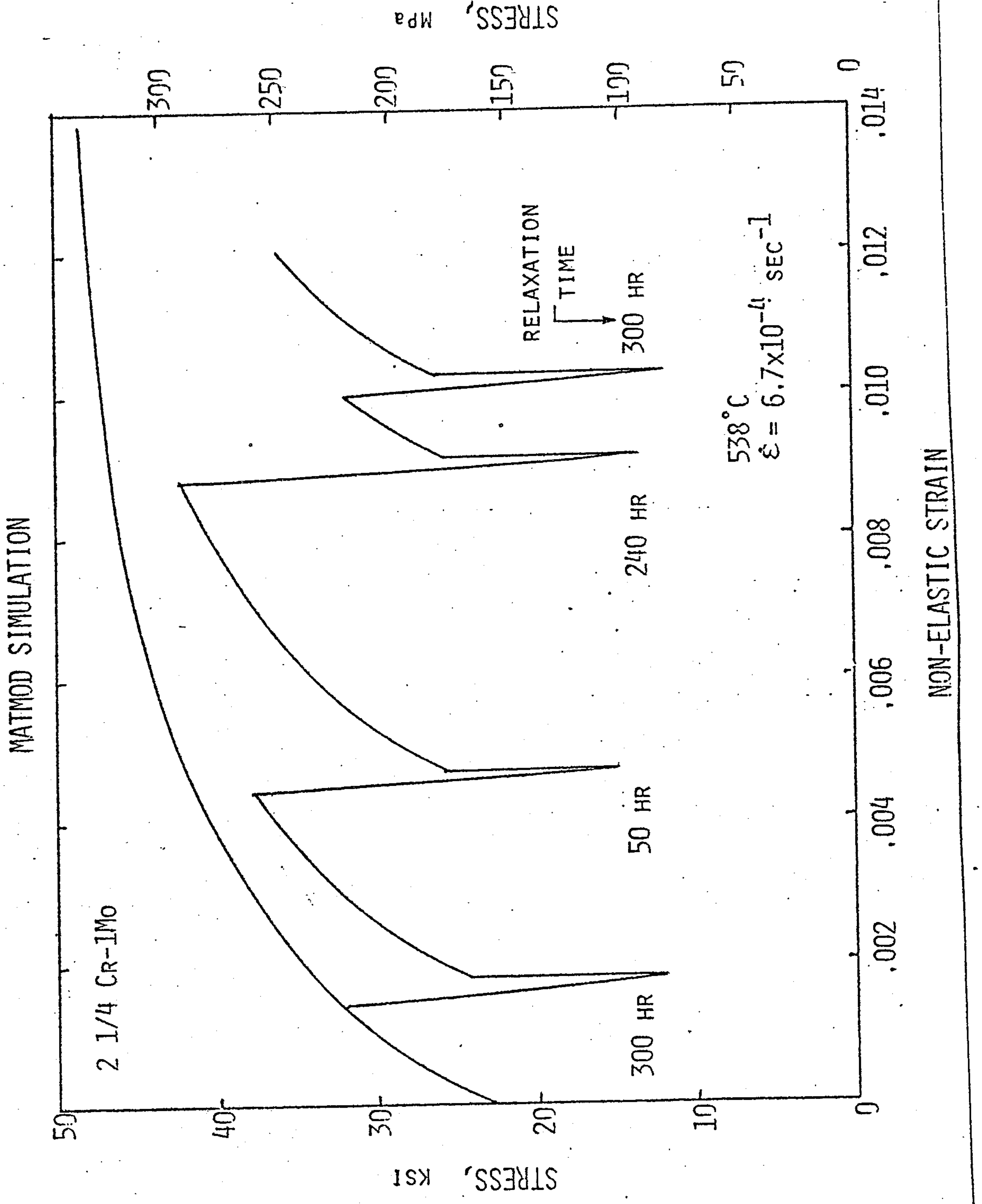


Figure 13. MATMOD simulation of the same conditions used to generate the data of Figure 2.

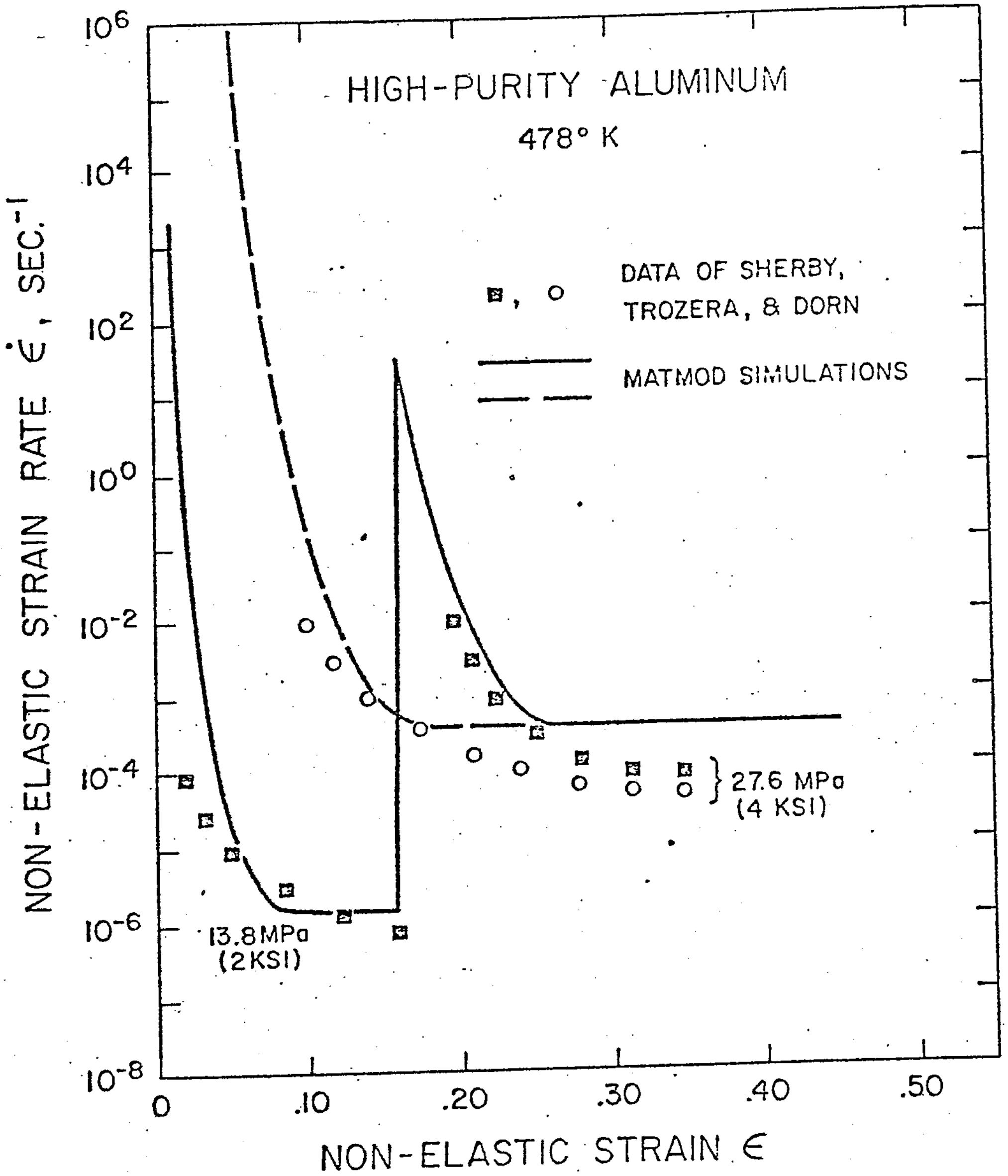
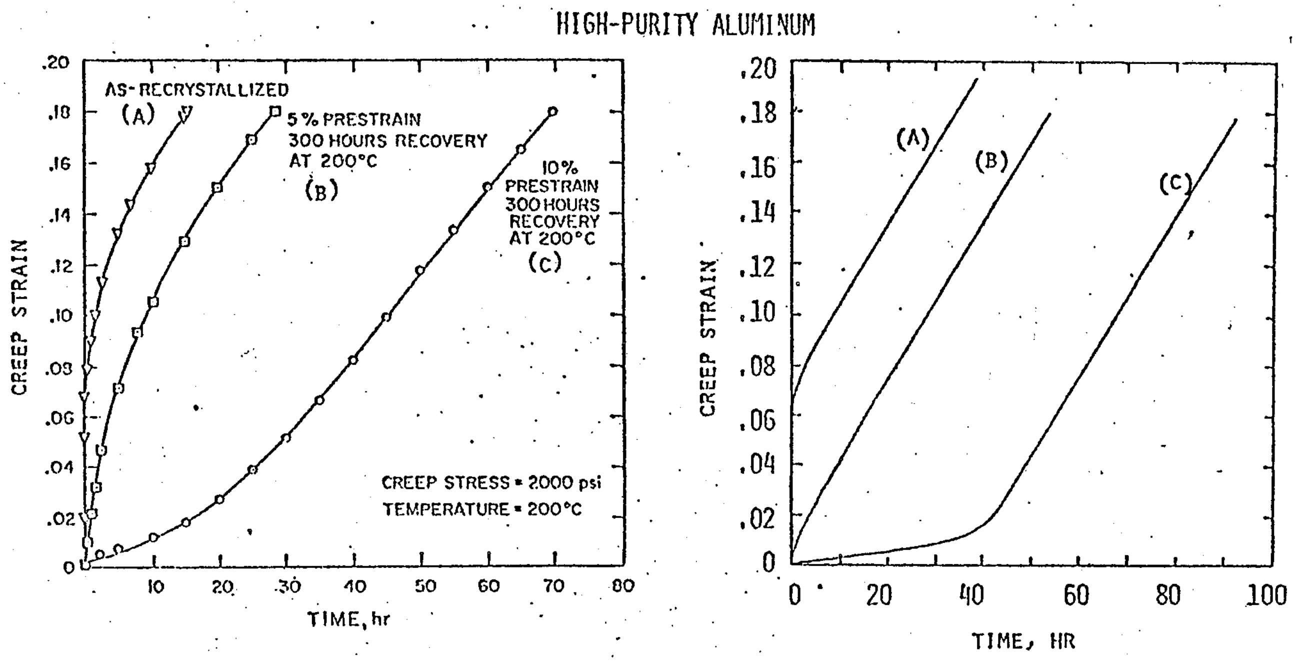


Figure 14. Predictions and data for a creep test with a sudden increase in applied stress. The behavior of the strain rate at 16% strain reflects the differences in the stress history up to that point.



(a) Data of Chen, Young, and Lytton [11]

(b) MATMOD simulations; curves A,B, and C are for the same histories and creep conditions as their experimental counterparts in Figure 8A.

Figure 15. Experimental data and independent simulations of the effect of prior room temperature straining on the subsequent creep response. Prestraining reduces the amount of primary creep; the work hardening is not removed by static annealing at 200°C, but dynamic recovery occurs upon subsequent creep testing at 200°C.

3/29 ME 239B Theoretical & Computational plasticity

→ Finite deformation pl. — a lot of research is going on in this field

Viscoplasticity.
Reinforced concrete

X

TOM HUGHES, Office 281 DURAND, x 72040
(OPEN DOOR)

T.A. LOUIS EDOZIEN

X

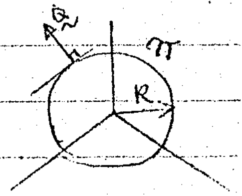
1 HR. MIDTERM maybe
3 HR. FINAL

X

Viscoplasticity:
Simple theory involving a Mises Y.S., Deviatoric ~~plastic~~ viscoplastic, ~~ab~~ analog of perfect plasticity. [Stress pt. can get outside the Y.S.]

→ Recall perfect plasticity w. Mises Y.S., deviatoric etc

$$\dot{\underset{\sim}{\sigma}} = \underbrace{\underset{\sim}{c} \cdot (\dot{\underset{\sim}{\epsilon}} - \dot{\underset{\sim}{\epsilon}}^{pl})}_{\dot{\underset{\sim}{\sigma}}^{tr}} \quad ; \quad f(\underset{\sim}{\sigma}) = J_2' = \frac{1}{2} |\underset{\sim}{\sigma}'|^2 = k^2$$



$R = \sqrt{2}\, k.$

$$\dot{\underset{\sim}{\epsilon}}^{pl} = \begin{cases} 0 & (E) \\ \Lambda \underset{\sim}{Q} & (P) \end{cases}$$

$$\underset{\sim}{Q} = \frac{\frac{\partial f}{\partial \underset{\sim}{\sigma}}}{\left| \frac{\partial f}{\partial \underset{\sim}{\sigma}} \right|} = \frac{\underset{\sim}{\sigma}'}{R}$$

$$(E) : \quad f(\underset{\sim}{\sigma}) < k^2 \quad \text{or} \quad f(\underset{\sim}{\sigma}) = k^2 \;\&\; \dot{\underset{\sim}{\sigma}}^{tr} \cdot \underset{\sim}{Q} \leq 0$$

$$(P) : \quad f(\underset{\sim}{\sigma}) = k^2 \;\&\; \dot{\underset{\sim}{\sigma}}^{tr} \cdot \underset{\sim}{Q} > 0$$

Consistency : $\quad 0 = \dot{f} = \frac{\partial f}{\partial \underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\sigma}} = R \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}$

$$0 = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}} = \underset{\sim}{Q} \cdot \underbrace{\underset{\sim}{C} \cdot (\dot{\underset{\sim}{\epsilon}}}_{\dot{\underset{\sim}{\sigma}}^{tr}} - \Lambda \underset{\sim}{Q})$$

$$\Lambda = \frac{\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}^{tr}}{\underset{\sim}{Q} \underbrace{\cdot \underset{\sim}{C} \cdot}_{2\mu} \underset{\sim}{Q}} = \frac{2\mu \, \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\epsilon}}}{2\mu} = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\epsilon}}$$

X

Viscopl. :

Viscopl. strain rate

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot (\dot{\underset{\sim}{\epsilon}} - \dot{\underset{\sim}{\epsilon}}^{vp})$$

$$\boxed{\dot{\underset{\sim}{\epsilon}}^{vp} = \underset{\sim}{\beta}(\underset{\sim}{\sigma})}$$ a given function of $\underset{\sim}{\sigma}$

additional constitutive hypothesis.

Assume : $\boxed{\underset{\sim}{\beta}(\underset{\sim}{\sigma}) = g \underbrace{\langle \phi \rangle}_{} \underset{\sim}{\sigma}'}$ (deviatoric)

dimensions = $\frac{1}{T}$ ← time units

$\langle \phi \rangle$ → dimensionless

$\underline{g > 0 \text{ , constant}}$, $g \frac{F}{L^2} = \frac{1}{T}$

dimensions $g = \frac{L^2}{F\,T}$

$$\longrightarrow \langle \phi \rangle = \begin{cases} 0 & \phi \leq 0 \\ \phi & \phi > 0 \end{cases}$$

MACAULEY BRACKET

$$\boxed{\phi = (\text{sgn } F)\,|F|^{m}}$$ → nondimensional

$$F = \frac{f(\underset{\sim}{\sigma})}{k^2 - 1}$$ ← Mises Y.F. → nondimensional

$$\begin{cases} \text{inside Y. S.} & F < 0 \\ \text{on} \quad " & F = 0 \\ \text{outside} \quad " & F > 0 \end{cases}$$

Consider : 1. $\underset{\sim}{\sigma}$ is s.t. $f(\underset{\sim}{\sigma}) \leq k^2$,

$F \leq 0$, $\phi \leq 0$, $\langle \phi \rangle = 0$, $\underset{\sim}{\beta} = \underset{\sim}{0}$

& $\underbrace{\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot \dot{\underset{\sim}{\epsilon}}}_{\text{instantaneous elastic}}$

2. $\underset{\sim}{\sigma}$ is S.T. $f(\underset{\sim}{\sigma}) > k^2$,

$F > 0$, $\Phi > 0$, $\langle \Phi \rangle = \phi$, $\underset{\sim}{p} = g \phi \underset{\sim}{\sigma}'$

& $\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot (\dot{\underset{\sim}{\epsilon}} - g \phi \underset{\sim}{\sigma}')$

↑ a measure of distance between $\underset{\sim}{\sigma}$ & Y.S.

Remark: Present theory is not homogeneous in the rates ; ∴ not ~~inviscid~~ inviscid

X

Dilatational part of ~~consist~~ constitutive eq. is still elastic:

$$\dot{\underset{\sim}{\sigma}}' + \dot{\underset{\sim}{\sigma}}'' = \underset{\sim}{C} \cdot (\dot{\underset{\sim}{\epsilon}} - g \phi \underset{\sim}{\sigma}')$$

$$= \underbrace{B \, tr \dot{\underset{\sim}{\epsilon}} \, \underset{\sim}{I}}_{\text{dilatational}} + \underbrace{2\mu (\dot{\underset{\sim}{\epsilon}}' - g \phi \underset{\sim}{\sigma}')}_{\text{deviatoric}}$$

$\dot{\underset{\sim}{\sigma}}'' = B \, tr \dot{\underset{\sim}{\epsilon}} \, \underset{\sim}{I}$ elastic

$\dot{\underset{\sim}{\sigma}}' = 2\mu (\dot{\underset{\sim}{\epsilon}}' - g \phi \underset{\sim}{\sigma}')$ viscopl.

↑ nonlinear of $\underset{\sim}{\sigma}'$

Examples: Behavior outside Y.S, $\dot{\underset{\sim}{\epsilon}}$ will be given

$\underset{\sim}{\sigma}\Big|_{t_i}$ = initial time

possibly

assume: pure shear; $\dot{\underset{\sim}{\epsilon}}'$: $\dot{\gamma} = 2\dot{\epsilon}_{12} \neq 0$

rest $\dot{\underset{\sim}{\epsilon}}' = \underset{\sim}{0}$

$\underset{\sim}{\sigma}'$ at t_i is s.t. $\tau = \sigma_{12} \neq 0$,

s.t. $f(\underset{\sim}{\sigma}) > k^2$, rest $\underset{\sim}{\sigma}' = \underset{\sim}{0}$ at t_i

$$\Longrightarrow \dot{\tau} = \dot{\sigma}_{12} = 2\mu\left(\frac{\dot{\gamma}}{2} - g\underset{\uparrow \phi(\tau)}{\phi}\tau\right)$$

$$\dot{\tau} + \left[2\mu g\phi(\tau)\right]\tau = \mu\dot{\gamma}$$

Solve O.D.E.

$\dot{\gamma}$ given, → can solve for $\tau(t)$ provide $\tau(t_i)$ given

assume $\tau(t_i) > 0$, $\dot{\gamma} = 0$

$$\dot{\tau} = -\underbrace{\left[\qquad\right]}_{>0}\tau < 0,$$

$\Longrightarrow$ τ decreases, $\phi \rightarrow 0$

at $\underbrace{\phi = 0}_{\tau \text{ is on Y.S}}$, $\dot{\tau} = 0$

stress shrinkage to Y.S ~~at~~ and then sit on Y

stop!!

so in this case τ returns to Y.S and stops

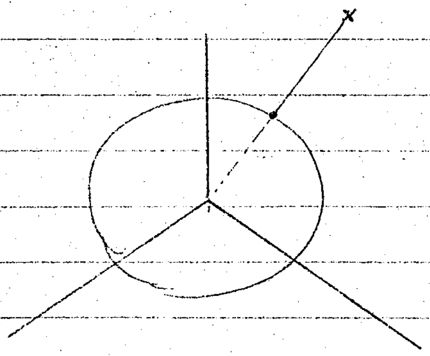
———————— X ————————

Again assume as above except $\tau < 0$

$$\dot{\tau} = -\underbrace{[\qquad]}_{>0}\underbrace{\tau}_{<0} > 0$$

τ increase (i.e. becomes less negative)

$\phi \longrightarrow 0, \quad \tau \longrightarrow$ Y.S.

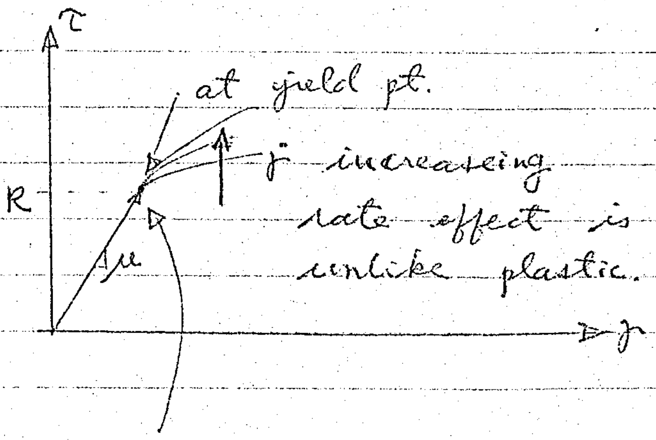


relax to the Y.S. and sit on there

———————— X ————————

Automatic mechanism in theory to return to Y.S.

τ v.s. γ, $\dot{\gamma} > 0$



rate effect is obvious unlike plastic.

$$\dot{\tau} + \underbrace{2\mu g\,\phi(\tau)\tau}_{\equiv 0 \text{ at Y.P.}} = \mu\dot{\gamma} > 0$$

at ~~Y.P.~~ yield point $\phi \equiv 0$; so at Y. P.

~~$\dot{\tau} = \mu \dot{\gamma}$~~ $\dot{\tau} = \mu \dot{\gamma}$

——————— X ———————

Stress - point algo. (Given strain history) :

A scheme : Backward difference

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \left(\dot{\underset{\sim}{\epsilon}} - g \langle \phi(\underset{\sim}{\sigma}) \rangle \underset{\sim}{\sigma}' \right)$$

evaluate at t_{n+1} : Backward diff. on $(\dot{\ })$

$$\dot{\underset{\sim}{\sigma}}_{n+1} = \underset{\sim}{C} \cdot \left(\dot{\underset{\sim}{\epsilon}}_{n+1} - g \langle \phi(\underset{\sim}{\sigma}_{n+1}) \rangle \underset{\sim}{\sigma}'_{n+1} \right)$$

$$\underset{\sim}{\sigma}_{n+1} - \underset{\sim}{\sigma}_n = \underset{\sim}{C} \cdot \left(\underbrace{\underset{\sim}{\epsilon}_{n+1} - \underset{\sim}{\epsilon}_n}_{\Delta \underset{\sim}{\epsilon}_n \text{ given}} - \Delta t \, g \underbrace{\langle \phi(\underset{\sim}{\sigma}_{n+1}) \rangle} \underset{\sim}{\sigma}'_{n+1} \right)$$

$$\underset{\sim}{\sigma}_{n+1} = \underset{\underset{\underset{\sim}{\sigma}_n + \underset{\sim}{C} \cdot \Delta \underset{\sim}{\epsilon}_n}{\Downarrow}}{\underset{\sim}{\sigma}^{tr}_{n+1}} - \underbrace{2\mu \Delta t \, g \langle \phi(\underset{\sim}{\sigma}_{n+1}) \rangle \underset{\sim}{\sigma}'_{n+1}}_{\text{maybe}}$$

Calculate $f(\underset{\sim}{\sigma}^{tr}_{n+1}) \leq k^2$, yes: $\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1}$

done!

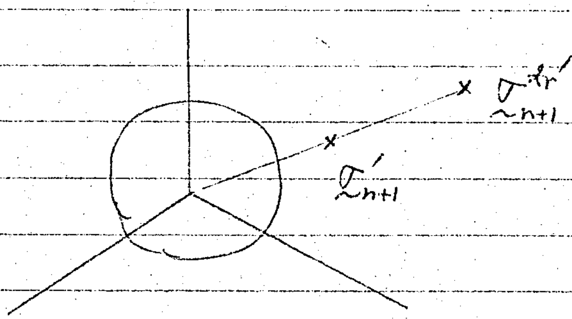
no: $\underset{\sim}{\sigma}''_{n+1} = \underset{\sim}{\sigma}^{tr''}_{n+1}$

$$\underset{\sim}{\sigma}'_{n+1} = \underset{\sim}{\sigma}^{tr'}_{n+1} - \underbrace{2\mu \Delta t \, g \langle \phi(\underset{\sim}{\sigma}_{n+1}) \rangle}_{a} \underset{\sim}{\sigma}'_{n+1}$$

determine

$$(1+a) \underset{\sim}{\sigma}'_{n+1} = \underset{\sim}{\sigma}^{tr'}_{n+1}$$

are on same line



say $\underset{\sim}{\sigma}'_{n+1} = c\,\underset{\sim}{\sigma}^{tr'}_{n+1}$

$(1+a)\,c = 1$ ◁ determine c s.t.

$$\underbrace{(1+2\mu\Delta t\, g\langle\phi(\overset{\uparrow}{c}\,\underset{\sim}{\sigma}^{tr'}_{n+1})\rangle)\overset{\uparrow}{c}}_{?} = 1$$

↑ known

$$\left(1+2\mu\Delta t\, g\left|\overset{\uparrow}{c^2} f(\underset{\sim}{\sigma}^{tr}_{n+1})/k^2 - 1\right|^m\right)\overset{\uparrow}{c} = 1$$

need an algo. to get c :

fix - pt. : $i=0,\ c_0 = 1$, ~~def~~ def: c_{i+1} by

$(1+\cdots\cdots|c_i^2\cdots\cdots|^m)\,c_{i+1} = 1$ ← check for the convergence of c_i

$i \leftarrow i+1$

$$\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}''_{n+1} + c\,\underset{\sim}{\sigma}^{tr'}_{n+1}$$

3/31 T.J.R.H. & R.L. TAYLOR " Algo's viscoplastic F.E. Analysi

Computer & Structures V. 8, 169 – 173 (1978)

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$$\dot{\underset{\sim}{\sigma}} = \underbrace{\underset{\sim}{C}^{el-pl}} \; \dot{\underset{\sim}{\epsilon}}$$

$$\Delta \underset{\sim}{\sigma} = \uparrow \; \Delta \underset{\sim}{\epsilon}$$

material tangent array → to call tang.

Viscoplastic Analog of $\underset{\sim}{C}^{el-pl}$:

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot (\dot{\underset{\sim}{\epsilon}} - \beta(\underset{\sim}{\sigma}))$$

algo. : B.W.D. – Diff's.

iteration m

$$\underbrace{\overset{\uparrow}{\underset{\sim}{\sigma}^{(i)}_{n+1}} - \underset{\sim}{\sigma}_n = \underset{\sim}{C} \cdot (\underbrace{\underset{\sim}{\epsilon}^{(i)}_{n+1} - \underset{\sim}{\epsilon}_n}_{\Delta \underset{\sim}{\epsilon}_n}) - \Delta t \, \underset{\sim}{C} \cdot \underbrace{\underset{\sim}{\beta} (\underset{\sim}{\sigma}^{(i)}_{n+1})}_{\uparrow}}_{\uparrow \text{ got}}$$

$$\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{(i)}_{n+1} + \delta \underset{\sim}{\sigma}$$

increment ?

$$\underset{\sim}{\epsilon}_{n+1} = \underset{\sim}{\epsilon}^{(i)}_{n+1} + \delta \underset{\sim}{\epsilon}$$

linear relationship $\delta \underset{\sim}{\sigma} \sim \delta \underset{\sim}{\epsilon}$?

→ we want:

$$\underset{\sim}{\sigma}_{n+1} - \underset{\sim}{\sigma}_n = \underset{\sim}{C} \cdot (\underset{\sim}{\epsilon}_{n+1} - \underset{\sim}{\epsilon}_n) - \Delta t \, \underset{\sim}{C} \cdot \underset{\sim}{\beta} (\underset{\sim}{\sigma}_{n+1})$$

$$\underset{\sim}{\sigma}^{(i)}_{n+1} + \delta \underset{\sim}{\sigma} - \underset{\sim}{\sigma}_n = \underset{\sim}{C} \cdot (\underset{\sim}{\epsilon}^{(i)}_{n+1} + \delta \underset{\sim}{\epsilon} - \underset{\sim}{\epsilon}_n) - \Delta t \, \underset{\sim}{C} \cdot \underbrace{\underset{\sim}{\beta} (\underset{\sim}{\sigma}^{(i)}_{n+1} + \delta}_{\Downarrow}$$

$$\simeq \underset{\sim}{\beta} (\underset{\sim}{\sigma}^{(i)}_{n+1})$$

$$\approx \beta(\underset{\sim}{\sigma}^{(i)}_{n+1}) + \frac{\partial \beta}{\partial \underset{\sim}{\sigma}}(\underset{\sim}{\sigma}^{(i)}_{n+1}) \cdot \delta \underset{\sim}{\sigma} + \underbrace{O(\| \delta \underset{\sim}{\sigma} \|^2)}_{\text{drop H.O.T.'s}}$$

$$\delta \underset{\sim}{\sigma} + \Delta t\, \underset{\sim}{C} \cdot \frac{\partial \beta^{(i)}_{n+1}}{\partial \underset{\sim}{\sigma}} \cdot \delta \underset{\sim}{\sigma} + \underbrace{\underset{\sim}{\sigma}^{(i)}_{n+1} - \underset{\sim}{\sigma}_n}$$

$$= \underset{\sim}{C} \cdot \delta \underset{\sim}{\epsilon} + \underbrace{\left\{ \underset{\sim}{C} \cdot (\underset{\sim}{\epsilon}^{(i)}_{n+1} - \underset{\sim}{\epsilon}_n) \underset{-}{\cancel{+}} \Delta t\, \underset{\sim}{C} \cdot \underset{\sim}{\beta}^{(i)}_{n+1} \right\}}$$

$$\delta \sigma_{ij} + \Delta t\, C_{ijkl} \frac{\partial \beta_{kl}(\underset{\sim}{\sigma}^{(i)}_{n+1})}{\partial \sigma_{pq}} \delta \sigma_{pq} = C_{ijkl}\, \delta \epsilon_{kl}$$

$$\underbrace{\left(\delta_{ip} \delta_{jq} + \Delta t\, C_{ijkl} \frac{\partial \beta_{kl}(\underset{\sim}{\sigma}^{(i)}_{n+1})}{\partial \sigma_{pq}} \right)}_{\overset{\text{def}}{=}\; b_{ipjq}(\underset{\sim}{\sigma}^{(i)}_{n+1})} \delta \sigma_{pq} = C_{ijkl}\, \delta \epsilon_{kl}$$

$$\underset{\sim}{b} \cdot \delta \underset{\sim}{\sigma} = \underset{\sim}{C} \cdot \delta \underset{\sim}{\epsilon}$$

$$\delta \underset{\sim}{\sigma} = \underbrace{\underset{\sim}{b}^{-1} \cdot \underset{\sim}{C}} \cdot \delta \underset{\sim}{\epsilon}$$

Consistent w.r.t. Taylor's formula, also w.r.t. algo

$$\boxed{\delta \underset{\sim}{\sigma} = \underbrace{\underset{\sim}{C}^{el\text{-}vp}(\underset{\sim}{\sigma}^{(i)}_{n+1})}_{\underset{\sim}{K}_T \big|^{(i)}_{n+1}} \cdot \delta \underset{\sim}{\epsilon}}$$

——— X ———

Example from paper cited above :

$E = 30 \times 10^{6}$ psi

$\nu = 0.3$

$\sigma_y = 30 \times 10^{3}$ psi

$m = 1$

$g = 10^{-8}\ {}^{1}/_{sec.}$

X

H.W. Study thick-walled cylindrical sol
pp. 95–103 from Prager–Hodge handout #2

X

Large-deformation Plasticity :

some of the difficulties : associated with large rotations $\longrightarrow$ shoot down constit. theory we've dealt with so far.

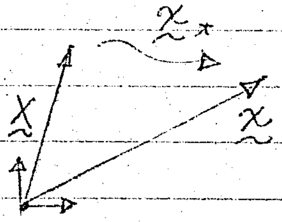
$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot \left(\dot{\underset{\sim}{\epsilon}} - \dot{\underset{\sim}{\epsilon}}^{pl} \right) = \underset{\uparrow}{\underset{\sim}{C}^{el\text{-}pl}} \; \dot{\underset{\sim}{\epsilon}}$$

even if $\equiv \underset{\sim}{C}$, as soon as rotation are "large". $\dot{\underset{\sim}{\sigma}}$, $\dot{\underset{\sim}{\epsilon}}$ are not approp.

$$(\dot{\ }) = \left.\frac{\partial}{\partial t}\right|_{X} \text{ particle fixed}$$

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t)$$

↑ particle



$$\ni \; \underset{\sim}{x}_*(\underset{\sim}{X}) = \underset{\sim}{x}(\underset{\sim}{X}, t)$$

Rigid rotation : $\underset{\sim}{x} = \underset{\sim}{R}\underset{\sim}{X}$ $\qquad \underset{\sim}{R} = \underset{\sim}{R}(t)$

$\underset{\sim}{R}$ orthogonal (proper)

$\Longrightarrow$ det +1

$$\underset{\sim}{R}^T\underset{\sim}{R} = \underset{\sim}{R}\underset{\sim}{R}^T = \text{Identity}$$

$$\underset{\sim}{R}^{-1} = \underset{\sim}{R}^T$$

should create no strain

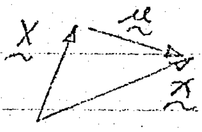
claim

$\Longrightarrow \dot{\underset{\sim}{\epsilon}}$ makes no sense $\neq 0$

$\underset{\sim}{\epsilon}$ " " " $\neq 0$

$$\underset{\sim}{u} = \underset{\sim}{x} - \underset{\sim}{X}$$

$$= (\underset{\sim}{R} - \underset{\sim}{I})\underset{\sim}{X}$$



$$\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{X}} = (\underset{\sim}{R} - \underset{\sim}{I}) \; ; \qquad \underset{\sim}{\epsilon} = \frac{1}{2}\left(\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{X}} + \left(\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{X}}\right)^T\right)$$

$$= \frac{1}{2}\left(\underset{\sim}{R} - \underset{\sim}{I} + \underset{\sim}{R}^T - \underset{\sim}{I}\right)$$

$$= \frac{\underset{\sim}{R} + \underset{\sim}{R}^T}{2} - \underset{\sim}{I}$$

2D:

$$\underset{\sim}{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\frac{\underset{\sim}{R} + \underset{\sim}{R}^{T}}{2} = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix}$$

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \dots$$

$$\underset{\sim}{\epsilon} = \begin{bmatrix} \cos\theta - 1 & 0 \\ 0 & \cos\theta - 1 \end{bmatrix} \neq 0$$

$$= - \begin{bmatrix} \frac{\theta^2}{2} & 0 \\ 0 & \frac{\theta^2}{2} \end{bmatrix} + h.o.t's \quad = O(\theta^2)$$

nonzero $\underset{\sim}{\epsilon}$ for rigid rotations

X $\dot{\underset{\sim}{\epsilon}}$ as a viable measures

——————————— X ———————————

= R.H.S. made sense

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C}^{el\text{-}pl} \; \underset{\sim}{d}$$

suppose

rate of deformation that makes sense

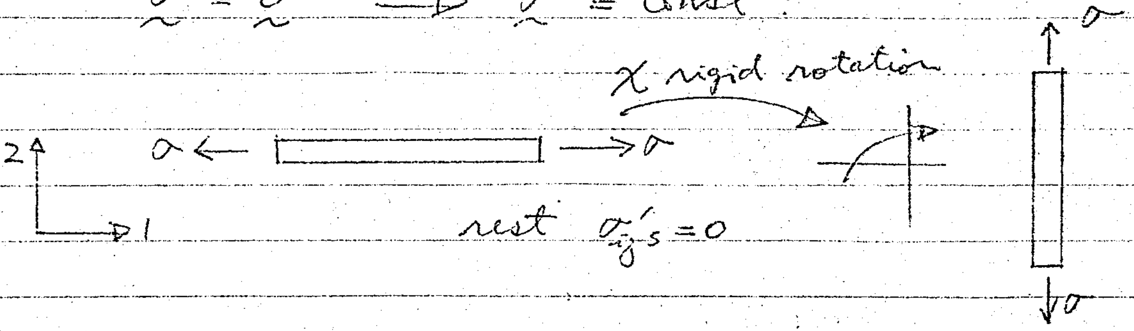
$\dot{\underset{\sim}{\sigma}}$ does not make sense

$\underset{\sim}{\sigma}$ = "True stress"

= make sense

Assume rigid rotation $\longrightarrow \underset{\sim}{d} \equiv 0$;

so $\dot{\underset{\sim}{\sigma}} = \underset{\sim}{0} \Longrightarrow \underset{\sim}{\sigma} = \text{const}$?



$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}\Bigg|_{\text{before}} \neq \underset{\sim}{\sigma} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix}\Bigg|_{\text{after}}$$

∴ $\underset{\sim}{\sigma}$ changes in a rigid rotation

×

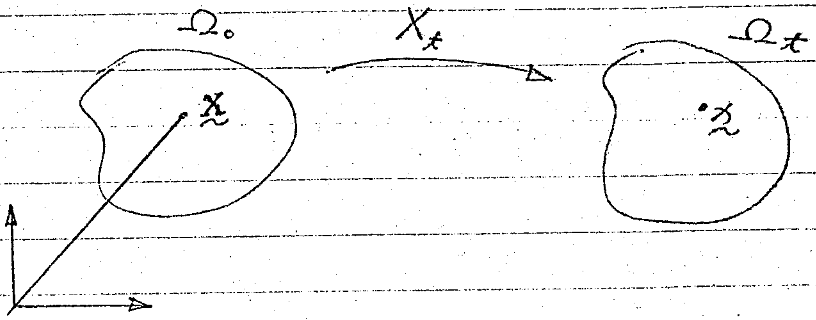
$$\underset{\sim}{\sigma}^{*} = \underset{\sim}{C}^{el\text{-}pl}\ \underset{\sim}{d}$$

"rate" ∃ transforms properly in rigid rotation

$$\underset{\sim}{\sigma}^{*} = \underset{\sim}{C}(\underset{\sim}{\sigma})\ \underset{\sim}{d} \quad \text{"Hypoelastic"}$$

×

4/5 Motion of a body:



$$\underset{\sim}{x} = \underset{\sim}{\chi}_t(\underset{\sim}{X}) = \underline{\underset{\sim}{\chi}(\underset{\sim}{X}, t)}$$

$$\underset{\sim}{X} = \underset{\sim}{\chi}(\underset{\sim}{X}, 0) = \underset{\substack{\Downarrow \\ \text{Id}}}{\underset{\sim}{\chi}_0}(\underset{\sim}{X})$$

Velocity:

$$\underset{\sim}{V} \cancel{\underset{\sim}{x}} = \frac{\partial}{\partial t} \underset{\sim}{\chi} \Big|_{\underset{\uparrow \text{ \underline{material}}}{\underset{\sim}{X}}} = \dot{\underset{\sim}{\chi}} \longleftarrow \underline{\text{material time derivative}}$$

Acceleration (likewise):

$$\underset{\sim}{A} = \dot{\underset{\sim}{V}} = \ddot{\underset{\sim}{\chi}}$$

$$\underset{\sim}{V} \cancel{\underset{\sim}{x}} = \underset{\underset{\sim}{V}}{\cancel{\underset{\sim}{x}}}(\underset{\sim}{X}, t) \quad , \quad \underset{\sim}{A} = \underset{\sim}{A}(\underset{\sim}{X}, t)$$

Want to change variables

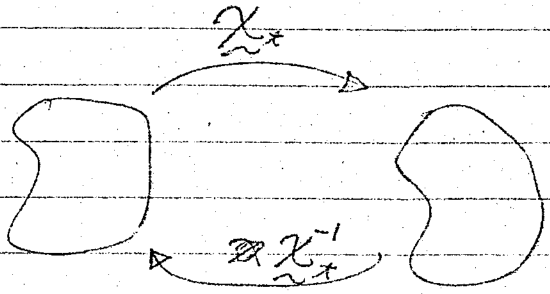
$$\underset{\sim}{X} = \underset{\sim}{\chi}_t^{-1}(\underset{\sim}{x}) \qquad \underset{\sim}{\chi}_t \text{ is assumed invertible smooth. etc}$$

$$\boxed{\underset{\sim}{v}(\underset{\uparrow}{\underset{\sim}{x}}, t) = \underset{\sim}{V}(\underset{\uparrow}{\underset{\sim}{\chi}_t^{-1}(\underset{\sim}{x})}, \underset{\uparrow}{t})}$$

↔ "spatial"   ~~sp~~ "material" ↔ "Lagrangian"

$$\dot{\underset{\sim}{v}} = \frac{\partial}{\partial t}\bigg|_{\underset{\sim}{X}} \underset{\sim}{v} = \underset{\sim}{a}$$

$$\underset{\sim}{a} = \dot{\underset{\sim}{v}} \underset{\substack{\uparrow \\ \text{sloppy}}}{=} \ddot{\underset{\sim}{X}}$$



Velocity Gradients: (Cartesian!)

$$\boxed{\underset{\sim}{l} = \underset{\sim}{\nabla}_{\underset{\sim}{x}} \underset{\sim}{v}}$$

$\underset{\sim}{x} = \{x_i\}$ i, j, k, l

$\underset{\sim}{X} = \{X_A\}$ A, B, C, D

$$\underbrace{l_{ij} = v_{i,j} = \frac{\partial v_i}{\partial x_j}}$$

$v_i = V_i$

$\underset{\sim}{l} =$ symm $\underset{\sim}{l}$ + skew symm $\underset{\sim}{l}$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\frac{\underset{\sim}{l} + \underset{\sim}{l}^T}{2} \qquad\qquad \frac{\underset{\sim}{l} - \underset{\sim}{l}^T}{2} \qquad\qquad \left\{ \begin{array}{l} \underset{\sim}{d} = \underset{\sim}{d}^T \\ \underset{\sim}{\omega} = -\underset{\sim}{\omega}^T \end{array} \right.$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\underset{\sim}{d} \qquad\qquad\qquad \underset{\sim}{\omega}$$

rate-of-deformation tensor   Vorticity, "spin"

Deformation gradient

$$\underset{\sim}{F} = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}}$$

$$F_{iA} = \frac{\partial x_i}{\partial X_A} = \chi_{i,A} \overset{\text{sloppy}}{=} x_{i,A}$$

regular χ^{-1} exists, one-to-one orientation preserving

$\det \underset{\sim}{F} > 0$, $\underset{\sim}{F}^{-1}$ exists

X

→ Polar decomposition

$$\underset{\sim}{F} \underset{\text{left decomposition}}{=} \underset{\sim}{R}\,\underset{\sim}{U}$$

$$\begin{cases} \underset{\sim}{U} = \text{symm } \overset{\text{positive definite}}{\text{pos def}} \\ \underset{\sim}{R} = \text{proper orthog.} \end{cases}$$

$$\begin{cases} \underset{\sim}{U} \text{ measure of "deformation"} \\ \underset{\sim}{R} \text{ measure of rotation of material} \end{cases}$$

$$\det \underset{\sim}{R} = \underset{\text{proper}}{+1} \qquad \underset{\sim}{R}^T \underset{\sim}{R} \overset{\text{sloppy}}{=} \underset{\sim}{R}\,\underset{\sim}{R}^T = \text{id.}$$

$$R_{iA} R_{jA} = \delta_{ij}$$

$$R_{iA} R_{iB} = \delta_{AB}$$

$$F_{iA} = \underbrace{R_{iB} U_{BA}}_{\text{sum}}$$

$$\left\{\begin{array}{l}\underset{\sim}{R}\ \text{called rotation}\\ \underset{\sim}{U}\ \text{called stretching (left)}\end{array}\right.$$

×

Right decomp.

$$\underset{\sim}{F} = \underset{\substack{\uparrow\\ \text{symm. pos. def.}}}{\underset{\sim}{V}}\,\underset{\sim}{R}$$

$$F_{iA} = V_{ij}\,R_{jA}$$

×

Establish $\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U}$

consider $\boxed{\underset{\sim}{C} = \underset{\sim}{F}^{T}\underset{\sim}{F}}$

→ claim $\underset{\sim}{C}$ is symm., positive definite

$$\underset{\sim}{C}^{T} = (\underset{\sim}{F}^{T}\underset{\sim}{F})^{T} = \underset{\sim}{F}^{T}\underset{\sim}{F} = \underset{\sim}{C}$$

→ (pos def vector $\underset{\sim}{z}$ $\quad \underset{\sim}{z}^{T}\underset{\sim}{C}\,\underset{\sim}{z} > 0 \quad \forall\, \underset{\sim}{z} \neq \underset{\sim}{0}$)

$$\underset{\sim}{z}^{T}\underset{\sim}{C}\,\underset{\sim}{z} = \underset{\sim}{z}^{T}\underset{\sim}{F}^{T}\underset{\sim}{F}\,\underset{\sim}{z} = (\underset{\sim}{F}\,\underset{\sim}{z})^{T}\underbrace{(\underset{\sim}{F}\,\underset{\sim}{z})}_{\underset{\sim}{Z}}$$

$$\underset{\sim}{0} \neq \underset{\sim}{Z} = \underset{\sim}{F}\,\underset{\sim}{z}$$

$\underset{\sim}{F}^{-1}$ exist $\qquad \underset{\sim}{Z}^{T}\underset{\sim}{Z} = \underset{\sim}{Z}\cdot\underset{\sim}{Z} \neq 0,\ > 0$

$\boxed{\underset{\sim}{U}^2 = \underset{\sim}{C}}$; $\underset{\sim}{U} \overset{def.}{=} \underset{\sim}{C}^{1/2}$

X

$\underset{\sim}{C}$ symm, pos.-def. ∴ It has real positive eigenvalues, orthonormal eigenvectors.

$$\begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} = \underset{\sim}{\Lambda}^2 = \underset{\sim}{\Psi}^T \underset{\sim}{C} \underset{\sim}{\Psi}$$

λ^2 eigenvalues of $\underset{\sim}{C}$

$$\boxed{\underset{\sim}{C} = \underset{\sim}{\Psi} \underset{\sim}{\Lambda}^2 \underset{\sim}{\Psi}^T}$$

def. $\underset{\sim}{U} = \underset{\sim}{\Psi} \underset{\sim}{\Lambda} \underset{\sim}{\Psi}^T$

$\underset{\sim}{U}^2 = \underset{\sim}{C}$? : $\underset{\sim}{U}^2 = \underset{\sim}{\Psi} \underset{\sim}{\Lambda} \underbrace{\underset{\sim}{\Psi}^T \underset{\sim}{\Psi}}_{Id} \underset{\sim}{\Lambda} \underset{\sim}{\Psi}^T$

$= \cancel{\Psi}\, \underset{\sim}{\Psi} \underset{\sim}{\Lambda}^2 \underset{\sim}{\Psi}^T = \underset{\sim}{C}$

is $\underset{\sim}{U}$ symm? yes $\left(\underset{\sim}{\Psi} \underset{\sim}{\Lambda} \underset{\sim}{\Psi}^T\right)^T = \underset{\sim}{\Psi} \underset{\sim}{\Lambda} \underset{\sim}{\Psi}^T = \underset{\sim}{U}$

positive def. λ_A's > 0

verify

X

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U}$$

$$\underset{\sim}{F}\,\underset{\sim}{U}^{-1} = \underset{\sim}{R} \longleftarrow \text{defines.}$$

$$\underset{\sim}{R}^{T}\underset{\sim}{R} = (\underset{\sim}{F}\,\underset{\sim}{U}^{-1})^{T}\,\underset{\sim}{F}\,\underset{\sim}{U}^{-1}$$

$$= \underset{\sim}{U}^{-T}\,\underbrace{\underset{\sim}{F}^{T}\,\underset{\sim}{F}}\,\underset{\sim}{U}^{-1} = \underset{\sim}{U}^{-T}\,\underset{\sim}{C}\,\underset{\sim}{U}^{-1} = \underset{\sim}{U}^{-1}\,\underset{\sim}{U}^{2}\,\underset{\sim}{U}^{-1}$$

$$= \underbrace{\underset{\sim}{U}^{-1}\,\underset{\sim}{U}}_{id}\;\underbrace{\underset{\sim}{U}\,\underset{\sim}{U}^{-1}}_{Id} = \text{Identity}$$

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Relationships between $\underset{\sim}{F}, \underset{\sim}{R}, \underset{\sim}{U}, \underset{\sim}{V},$

$\underset{\sim}{l}, \underset{\sim}{d}, \underset{\sim}{w} \longleftarrow$

$$\dot{\underset{\sim}{F}} = \dot{\underset{\sim}{R}}\,\underset{\sim}{U} + \underset{\sim}{R}\,\dot{\underset{\sim}{U}}$$

$$= \dot{\underset{\sim}{V}}\,\underset{\sim}{R} + \underset{\sim}{V}\,\dot{\underset{\sim}{R}}$$

$$l_{ij} = \frac{\partial v_i}{\partial x_j}$$

$$= \frac{\partial v_i}{\partial X_A}\,\frac{\partial X_A}{\partial x_j} \Rightarrow \chi^{-1}_{,A}$$

Chain rule

$$F_{iA} = \frac{\partial x_i}{\partial X_A}$$

$$(\underset{\sim}{F})^{-1}_{Ai} \quad \frac{\partial X_A}{\partial x_i}$$

$$\underset{\sim}{l} = \nabla_{\underset{\sim}{X}}\,\underset{\sim}{v}\;\underset{\sim}{F}^{-1}$$

$$\frac{\partial v_i}{\partial X_A} = \underset{\text{commute}}{\underset{\curvearrowright}{\frac{\partial}{\partial X_A}\,\frac{\partial}{\partial t}}}\Bigg|_{\underset{\sim}{X}} x_i(\underset{\sim}{X}, t)$$

$$= \frac{\partial}{\partial t}\,\underbrace{\frac{\partial}{\partial X_A} x_i}_{F_{iA}} \quad = \left[\dot{F}_{iA}\right]$$

$$\longrightarrow \quad \boxed{\underset{\sim}{l} = \dot{\underset{\sim}{F}}\,\underset{\sim}{F}^{-1}} \qquad\qquad \underset{\sim}{R}^{-1} = \underset{\sim}{R}^{T}$$

$$\underset{\text{symm.}}{\underset{\uparrow}{\underset{\sim}{d}}} + \underset{\substack{\text{skew}\\ \text{spin tensor}}}{\underset{\uparrow}{\underset{\sim}{w}}} = \underset{\sim}{l} = \dot{\underset{\sim}{F}}\,\underset{\sim}{F}^{-1} = (\dot{\underset{\sim}{R}}\,\underset{\sim}{U} + \underset{\sim}{R}\,\dot{\underset{\sim}{U}})(\underbrace{\underset{\sim}{R}\,\underset{\sim}{U}}_{\underset{\sim}{U}^{-1}\underset{\sim}{R}^{T}})^{-1}$$

$$= \underbrace{\dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T}}_{\text{skew}} + \underbrace{\underset{\sim}{R}\,\dot{\underset{\sim}{U}}\,\underset{\sim}{U}^{-1}\underset{\sim}{R}^{T}}_{\substack{\text{not symm.}\\ \text{in general.}}}$$

material
angular velocity

$\dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T}$ skew ?yes $\qquad -(\dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T})^{T} \overset{?}{=} \dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T}$

$$\left(\dot{\overline{\underset{\sim}{R}\,\underset{\sim}{R}^{T}}} = \underset{\sim}{I}\right)$$

$$\dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T} + \underset{\sim}{R}\,\dot{\underset{\sim}{R}}^{T} = \underset{\sim}{0}$$

$$\dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T} = -\underset{\sim}{R}\,\dot{\underset{\sim}{R}}^{T} = -(\dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T})^{T} \quad \therefore \text{ skew.}$$

$$\underset{\sim}{\mathit{w}} = \dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T} = \text{"material rate of rotation"}$$

not $\underset{\sim}{\omega}$ ← in general

$$\underset{\sim}{d} + \underset{\sim}{\omega} = \underset{\sim}{l} = \underset{\sim}{\mathit{w}} + \underset{\sim}{\delta} \Longrightarrow \underset{\sim}{R}\,\underbrace{\dot{\underset{\sim}{U}}\,\underset{\sim}{U}^{-1}}_{\substack{\uparrow \\ \text{not symm.}}}\,\underset{\sim}{R}^{T}$$

$$\underset{\sim}{\delta} = \text{symm. } \underset{\sim}{\delta} + \text{skew } \underset{\sim}{\delta}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\frac{\underset{\sim}{R}\left(\dot{\underset{\sim}{U}}\,\underset{\sim}{U}^{-1} + \underset{\sim}{U}^{-1}\dot{\underset{\sim}{U}}\right)\underset{\sim}{R}^{T}}{2} \qquad \frac{\underset{\sim}{R}\left(\dot{\underset{\sim}{U}}\,\underset{\sim}{U}^{-1} - \underset{\sim}{U}^{-1}\dot{\underset{\sim}{U}}\right)\underset{\sim}{R}^{T}}{2}$$

$$\underset{\sim}{U}^{-1}\dot{\underset{\sim}{U}} \rightarrow (\dot{\underset{\sim}{U}}\,\underset{\sim}{U}^{-1})^{T}$$

physical interpretation:

$\underset{\sim}{\omega}$ = measures angular velocity of principal axes of $\underset{\sim}{d}$

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Uniaxial stretch: $\underset{\sim}{\chi}$

$$x_1 = \lambda(t) X_1$$
$$x_2 = X_2$$
$$x_3 = X_3$$

$$x_i = \chi_i(\underset{\sim}{X}, t)$$

$$\underset{\sim}{F} = \begin{bmatrix} \lambda(t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left(= \left[\frac{\partial x_i}{\partial X_A}\right] \right)$$

$$\begin{matrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & \cdots & \\ x_{3,1} & \cdots & \end{matrix}$$

$$\underset{\sim}{U}^2 = \underset{\sim}{F}^T \underset{\sim}{F} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{U} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underset{\sim}{F} \implies \underset{\sim}{R} = \underset{\sim}{I} \quad \underset{\sim\sim\sim\sim}{\text{not rotate}}$$

$$\underset{\sim}{l} = \dot{\underset{\sim}{F}} \underset{\sim}{F}^{-1} = \begin{bmatrix} \dot{\lambda} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\dot{\lambda}}{\lambda} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\implies \underset{\sim}{d} = \underset{\sim}{l} \quad ; \quad \underset{\sim}{\omega} = \underset{\sim}{0}$$

↑ often introduced as a measure of "rate-of-strain"

$$\dot{\underset{\sim}{\epsilon}} \overset{\text{def}}{=} \underset{\sim}{d}$$

$$\underset{\sim}{\epsilon}\Big|_{t=0} = \underset{\sim}{0} \qquad \underset{\sim}{\epsilon}(t) = \int_0^t \underset{\sim}{d}(\tau)\, d\tau \qquad \underset{\sim\sim\sim\sim}{\underset{\sim}{X} \text{ fixed}}$$

$$\underset{\sim}{\epsilon}(t) = \int_0^t \begin{bmatrix} \frac{\dot{\lambda}}{\lambda} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} d\tau = \ln \frac{\lambda(t)}{\lambda(0)}$$

" Logarithmic strain measure

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4/7

$\underset{\sim}{d}$

→ necessary & sufficient condition for the motion to be rigid $\underset{\sim}{d} = \underset{\sim}{0}$

$\dot{\underset{\sim}{C}} = 2\underset{\sim}{F}^T \underset{\sim}{d}\, \underset{\sim}{F}$ $\underset{\sim}{F}^{-1}$ exists $\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F}$

$\dot{\underset{\sim}{C}} = 0 \Longleftrightarrow \underset{\sim}{d} = \underset{\sim}{0}$

one can show $\underset{\sim}{C}$ = const when the motion is rigi

$\dot{\underset{\sim}{C}} = \dot{\underset{\sim}{F}}^T \underset{\sim}{F} + \underset{\sim}{F}^T \dot{\underset{\sim}{F}}$ $\underset{\sim}{l} = \dot{\underset{\sim}{F}}\, \underset{\sim}{F}^{-1}$

$= (\underset{\sim}{l}\, \underset{\sim}{F}^{\#})^T \underset{\sim}{F} + \underset{\sim}{F}^T \underset{\sim}{l}\, \underset{\sim}{F}$

$= \underset{\sim}{F}^T \underbrace{(\underset{\sim}{l}^T + \underset{\sim}{l})}_{2\underset{\sim}{d}} \underset{\sim}{F}$

×

2. Assume a rotation

$x_1 = \cos\alpha\, X_1 + \sin\alpha\, X_2$

$x_2 = -\sin\alpha\, X_1 + \cos\alpha\, X_2$

$x_3 = X_3$

$$\underset{\sim}{F} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C} = \underset{\sim}{U}^2 = \underset{\sim}{F}^T \underset{\sim}{F} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{U} = \underset{\sim}{I} \implies \underset{\sim}{R} = \underset{\sim}{F}$$

$$\underset{\sim}{l} = \underbrace{\dot{\underset{\sim}{F}}\, \underset{\sim}{F}^{-1}}$$

$$\Downarrow$$

$$\dot{\underset{\sim}{R}}\, \underset{\sim}{R}^T = \underset{\sim}{w} = \text{skew-symm.} \longrightarrow \underset{\sim}{d} = \underset{\sim}{0} \quad \therefore \underline{\underset{\sim}{w} = \underset{\sim}{\omega}}$$

$$\rhd \begin{bmatrix} -\sin\alpha\,\dot{\alpha} & \cos\alpha\,\dot{\alpha} & 0 \\ -\cos\alpha\,\dot{\alpha} & -\sin\alpha\,\dot{\alpha} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dot{\alpha} & 0 \\ -\dot{\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longleftarrow \underline{\text{Angular velocity matrix}}$$

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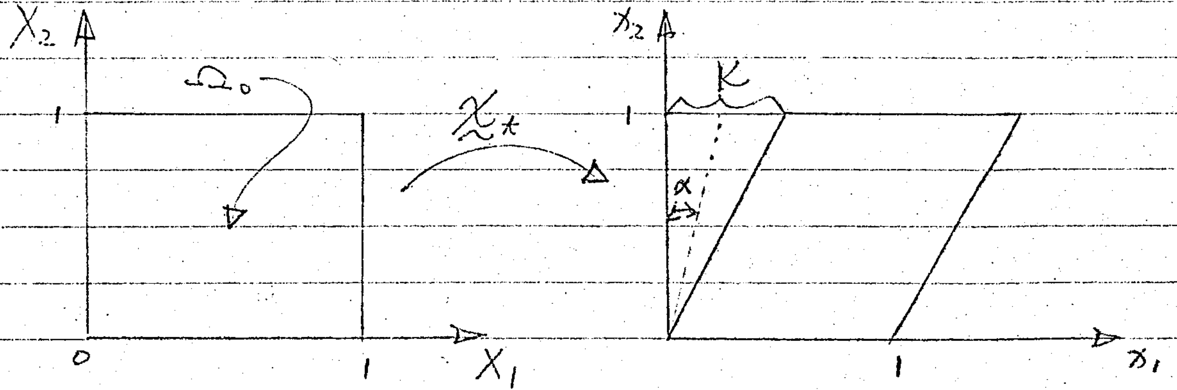
3. Shear

$x_1 = X_1 + K(t) X_2$ $\qquad K > 0$

$x_2 = X_2$ $\qquad K = c\,t$

$\qquad\qquad\qquad \uparrow$ const.

$x_3 = X_3$



def. $\alpha \longrightarrow 2\tan\alpha = K$

$0 \leq \alpha \leq \frac{\pi}{2}$; as $t \rightarrow \infty$, $\alpha \rightarrow \frac{\pi}{2}$

Calculate: $\underset{\sim}{F} = \begin{bmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\underset{\sim}{U}^2 = \underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} = \begin{bmatrix} 1 & 0 & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \overset{2\tan\alpha}{K} & 0 \\ K & 1+K^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} \qquad \underset{\sim}{U} \underset{\text{by polar decomposition}}{=} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ \sin\alpha & \frac{1+\sin^2\alpha}{\cos\alpha} & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[t\rightarrow\infty]{} \begin{bmatrix} 0 & 1 & 0 \\ 1 & \infty & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R} = \underset{\sim}{F}\,\underset{\sim}{U}^{-1} = \begin{bmatrix} \cancel{1} & \cancel{2\tan\alpha} & \cancel{0} \\ \cancel{2\tan\alpha} & \cancel{1+4\tan^2\alpha} & \cancel{0} \\ \cancel{0} & 0 & \cancel{1} \end{bmatrix}$$

$$\underset{\sim}{U}^{-1} = \begin{bmatrix} \frac{1+\sin^2\alpha}{\cos\alpha} & -\sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{11} = \left\{ \frac{1+\sin^2\alpha}{\cos\alpha} - \frac{2\sin^2\alpha}{\cos\alpha} \right] = \cos\alpha$$

$$R_{12} = -\sin\alpha + 2\sin\alpha = \sin\alpha$$

$$R_{13} = 0$$

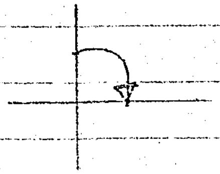
$$R_{21} = -\sin\alpha$$

$$R_{22} = \cos\alpha \qquad R_{23} = 0$$

$$R_{31} = R_{32} = 0 \quad , \quad R_{33} = 1$$

$$\underset{\sim}{R} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha \to \frac{\pi}{2} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



rotation through 90°

$$\underset{\sim}{\omega} = \underset{\sim}{\dot{R}}\,\underset{\sim}{R}^T = \begin{bmatrix} -s\dot{\alpha} & c\dot{\alpha} & 0 \\ -c\dot{\alpha} & -s\dot{\alpha} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dot{\alpha} & 0 \\ -\dot{\alpha} & 0 & 0 \end{bmatrix} \qquad \alpha \to \frac{\pi}{2} \ , \ \dot{\alpha} \to 0$$

$$\underset{\sim}{\ell} = \dot{\underset{\sim}{F}}\,\underset{\sim}{F}^{-1} = \begin{bmatrix} 0 & \dot{K} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \dot{K} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{d} = \frac{1}{2} \begin{bmatrix} 0 & \cancel{2}\dot{K} & 0 \\ \cancel{2}\dot{K} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underset{\sim}{\omega} = \frac{1}{2} \begin{bmatrix} 0 & \dot{K} & 0 \\ -\dot{K} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad = \frac{1}{2} \begin{bmatrix} 0 & c & 0 \\ -c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

const. angular velocity

Define a rotation $\underset{\sim}{P}$ by

$$\dot{\underset{\sim}{P}} = \underset{\sim}{\omega}\,\underset{\sim}{P} \longleftarrow \text{O.D.E.'s}$$

$$\underset{\sim}{\omega} = \dot{\underset{\sim}{P}}\,\underset{\sim}{P}^T$$

$$\boxed{\underset{\sim}{P}^{-1} = \underset{\sim}{P}^T}$$

Keeps on rotating ←

i.e. eigenvectors of $\underset{\sim}{d}$

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→ Objectivity: (Material Frame Indifference)
we'll assume constitutive eq's of the form

$$\underset{\sim}{\sigma}^* = \underset{\sim}{c}\,\underset{\sim}{d}$$

rate-of-deformation

"Transform property" moduli

$$\sigma^*_{ij} = c_{ijkl}\, d_{kl}$$

<u>Transformation properties</u>:
consider a superposed rigid-body motion
(recall $\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X}, t)$)

$$\boxed{\bar{\underset{\sim}{x}} = \underset{\sim}{c}(t) + \underbrace{\underset{\sim}{Q}(t)\,\underset{\sim}{x}}_{\text{rotation}}} \quad : \underset{\sim}{Q} \text{ proper orthogonal}$$

translation

(this ~~is~~ can be equivalently thought of as a time-dependent change of coordinate from $\underset{\sim}{x}$ to $\bar{\underset{\sim}{x}}$)

→ spacial coordinate! not material coordinate!

→ Definition of objectivity

$$\underset{\sim}{A} = \left[A_{\underbrace{ij,\dots k}_{\substack{\text{spatial}\\ \text{indices}}}\ \underbrace{AB,\dots C}_{\substack{\text{material}\\ \text{indices}}}} \right]$$

→ $\underset{\sim}{A}$ is said to be objective if under the coordinate change $\underset{\sim}{x} \longrightarrow \underset{\sim}{\bar{x}}$

$$\bar{A}_{ij\cdots kAB\cdots C} = Q_{il}\, Q_{jm} \cdots Q_{kn}\, A_{lm\cdots nAB\cdots C}$$

$\underset{\sim}{Q}$ is orthogonal

A is a tensor under this class of coordinate change

×

Examples 1. Def gradient

$$\underset{\sim}{F} = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}}$$

$$\underset{\sim}{\bar{F}} = \frac{\partial \underset{\sim}{\bar{x}}}{\partial \underset{\sim}{X}} = \frac{\partial \underset{\sim}{\bar{x}}}{\partial \underset{\sim}{x}} \underbrace{\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}}}_{\underset{\sim}{F}}$$

$$\underset{\sim}{\bar{x}} = \underset{\sim}{c} + \underset{\sim}{Q}\,\underset{\sim}{x} \qquad \frac{\partial \underset{\sim}{\bar{x}}}{\partial \underset{\sim}{x}} = \underset{\sim}{Q}$$

$$\underset{\sim}{\bar{F}} = \underset{\sim}{Q}\,\underset{\sim}{F}$$

$$\bar{F}_{iA} = Q_{ij} F_{jA}$$ ∴ $\underset{\sim}{F}$ is objective

×

2. $\underset{\sim}{v} = \dot{\underset{\sim}{x}}$ velocity

$$\bar{\underset{\sim}{v}} = \dot{\bar{\underset{\sim}{x}}} = \dot{\overline{\underset{\sim}{c}(t) + \underset{\sim}{Q}(t)\underset{\sim}{x}}} = \underbrace{\underset{\sim}{c}'(t) + \underset{\sim}{Q}'(t)\underset{\sim}{x}} + \underset{\sim}{Q}\dot{\underset{\sim}{x}}$$

$$\cancel{\bar{\underset{\sim}{v}}} \cancel{=} \qquad \bar{\underset{\sim}{v}} = \qquad\qquad + \underset{\sim}{Q}\underset{\sim}{v}$$

$$\bar{v}_i = \underbrace{c_i' + Q_{ij}' x_j}_{\neq 0 \text{ general}} + \boxed{Q_{ij} v_j}$$

$\xrightarrow{}$ extra terms $\Longrightarrow$ $\underset{\sim}{v}$ is not objective

X

3

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} \qquad\qquad C_{AB} = F_{iA} F_{iB}$$

$\bar{\underset{\sim}{C}} \overset{?}{=} \underset{\sim}{C} \Rightarrow$ objective $\qquad (\bar{C}_{AB} = C_{AB})$

$\rightarrow$ calculate

$$\bar{\underset{\sim}{C}} = \bar{\underset{\sim}{F}}^T \bar{\underset{\sim}{F}} = (\underset{\sim}{Q}\underset{\sim}{F})^T(\underset{\sim}{Q}\underset{\sim}{F})$$

$$= \underset{\sim}{F}^T \underbrace{\underset{\sim}{Q}^T \underset{\sim}{Q}}_{\text{Id.}} \underset{\sim}{F}$$

$$= \underset{\sim}{F}^T \underset{\sim}{F} = \underset{\sim}{C}$$

$\therefore \underset{\sim}{C}$ is objective

X

4/12 4 Rotation Matrix

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U}$$

$$\underset{\sim}{\bar{F}} = \underset{\sim}{\bar{R}}\,\underset{\sim}{\bar{U}}$$

$$\underset{\sim}{\bar{R}} = \underset{\sim}{\bar{F}}\,\underset{\sim}{\bar{U}}^{-1} = \underset{\sim}{Q}\,\underset{\sim}{F}\,\underset{\sim}{\bar{U}}^{-1} \blacktriangleleft$$

$$= \underset{\sim}{Q}\,\underset{\sim}{R}\,\underset{\sim}{U}\,\underset{\sim}{U}^{-1}$$

$$\cancel{= \underset{\sim}{Q}\,\underset{\sim}{R}}$$

Last time:

$$\bar{C}_{AB} = C_{AB}$$

$$C = U^2$$

$$\bar{U}_{AB} = U_{AB}$$

$$\underset{\sim}{\bar{R}} = \underset{\sim}{Q}\,\underset{\sim}{R}$$

$$\underset{\uparrow}{\bar{R}_{iA}} = Q_{i\underset{\uparrow}{j}}\, R_{\underset{\uparrow}{j}A} \Longrightarrow \underset{\sim}{R} \text{ objective}$$

$\times$

5.
$$\underset{\sim}{l} = \underset{\sim}{\dot{F}}\,\underset{\sim}{F}^{-1}$$

$$\underset{\sim}{\bar{l}} = \underset{\sim}{\dot{\bar{F}}}\,\underset{\sim}{\bar{F}}^{-1} \qquad \underset{\sim}{\bar{F}} = \underset{\sim}{Q}\,\underset{\sim}{F}$$

$$= (\underset{\sim}{\dot{Q}}\,\underset{\sim}{F} + \underset{\sim}{Q}\,\underset{\sim}{\dot{F}})\,\underset{\sim}{F}^{-1}\,\underset{\sim}{Q}^{T}$$

$$= \underset{\sim}{\dot{Q}}\,\underset{\sim}{Q}^{T} + \underset{\sim}{Q}\,\underbrace{\underset{\sim}{\dot{F}}\,\underset{\sim}{F}^{-1}}\,\underset{\sim}{Q}^{T}$$

$$\underset{\sim}{\bar{l}} = \underset{\text{skew}}{\underset{\uparrow}{\underset{\sim}{\dot{Q}}\,\underset{\sim}{Q}^{T}}} + \underset{\sim}{Q}\,\underset{\sim}{l}\,\underset{\sim}{Q}^{T}$$

$$\frac{d}{dt}\left[(\underset{\sim}{Q}\,\underset{\sim}{Q}^{T}) = Id\right]$$

$$\underset{\sim}{\dot{Q}}\,\underset{\sim}{Q}^{T} + (\underset{\sim}{Q}\,\underset{\sim}{\dot{Q}}^{T}) = 0$$

$$\dot{\underset{\sim}{Q}}\,\underset{\sim}{Q}^T + (\dot{\underset{\sim}{Q}}\,\underset{\sim}{Q}^T)^T = 0$$

$$\Longrightarrow \underbrace{\dot{\underset{\sim}{Q}}\,\underset{\sim}{Q}^T = \text{skew}} = -(\dot{\underset{\sim}{Q}}\,\underset{\sim}{Q}^T)^T$$

def: $\underset{\sim}{\beta} = \dot{\underset{\sim}{Q}}\,\underset{\sim}{Q}^T$ — angular velocity of $\bar{\underset{\sim}{x}}$ coordinates w.r.t. $\underset{\sim}{x}$ coordinates

$$\bar{l}_{ij} = \beta_{ij} + Q_{ik}\, l_{km}\, Q^T_{mj}$$

$$\bar{l}_{ij} = \underset{\overset{\Uparrow}{\neq 0}}{\beta_{ij}} + Q_{im} Q_{jn} l_{mn}$$

so $\underset{\sim}{l}$ not objective

———————— X ————————

6. $\underset{\sim}{d}$ = symmetric part of $\underset{\sim}{l}$

$$\bar{\underset{\sim}{d}} = \text{symm } \bar{\underset{\sim}{l}} = (\bar{\underset{\sim}{l}}^T + \underset{\uparrow}{\bar{\underset{\sim}{l}}})/2$$

$$(\underset{\sim}{\beta} + \underset{\sim}{Q}\,\underset{\sim}{l}\,\underset{\sim}{Q}^T)$$

$$= \Big(\underset{\text{skew}}{(\underset{\sim}{\beta}^T} + \underset{\sim}{Q}\,\underset{\sim}{l}^T\underset{\sim}{Q}^T) + (\underset{\sim}{\beta} + \underset{\sim}{Q}\,\underset{\sim}{l}\,\underset{\sim}{Q}^T)\Big)/2$$

$$= \underset{\sim}{Q}\,\underbrace{\left(\frac{\underset{\sim}{l} + \underset{\sim}{l}^T}{2}\right)}_{\underset{\sim}{d}}\,\underset{\sim}{Q}^T$$

$$= \underset{\sim}{Q}\,\underset{\sim}{d}\,\underset{\sim}{Q}^T$$

∴ $\underset{\sim}{d}$ is objective

7. $\underset{\sim}{w}$ $\qquad (\underset{\sim}{l} = \underset{\sim}{d} + \underset{\sim}{w})$

$$\bar{\underset{\sim}{w}} = \text{skew}\ \bar{\underset{\sim}{l}}$$

$$= \cancel{\beta}\ \frac{\underset{\sim}{\beta} - \underset{\sim}{\beta}^T}{2} + (\underset{\sim}{Q}\underset{\sim}{l}\underset{\sim}{Q}^T - \underset{\sim}{Q}\underset{\sim}{l}\underset{\sim}{Q}^T)$$

$$= \frac{\underset{\sim}{\beta} + \underset{\sim}{\beta}}{2} + \underset{\sim}{Q}\left(\frac{\underset{\sim}{l} - \underset{\sim}{l}^T}{2}\right)\underset{\sim}{Q}^T$$

$$\bar{\underset{\sim}{w}} = \underset{\substack{\uparrow \\ \neq 0}}{\underset{\sim}{\beta}} + \underset{\sim}{Q}\underset{\sim}{w}\underset{\sim}{Q}^T$$

$\therefore$ $\underset{\sim}{w}$ is not objective

X

8. $\dot{\underset{\sim}{C}}$

$$\bar{\underset{\sim}{C}} = \underset{\sim}{C}$$

$$\dot{\bar{\underset{\sim}{C}}} = \dot{\underset{\sim}{C}} \qquad \therefore \text{ Objective}$$

$\underset{\sim}{C} = \underset{\sim}{U}^2$

$$\bar{U}_{AB} = U_{AB}$$

$$\therefore \dot{\bar{U}}_{AB} = \dot{U}_{AB} \longrightarrow \underset{\sim}{U} \text{ objective}$$

X

9. $\underset{\sim}{V}$ $\quad V_{ij}$ $\qquad \underset{\sim}{F} = \underset{\sim}{V}\,\underset{\sim}{R}$

$$F_{iA} = V_{ij} R_{jA}$$

$$\bar{\underset{\sim}{F}} = \bar{\underset{\sim}{V}}\,\bar{\underset{\sim}{R}} \qquad \bar{\underset{\sim}{F}} = \underset{\sim}{Q}\,\underset{\sim}{F}$$

$$\bar{\underset{\sim}{V}} = \bar{\underset{\sim}{F}}\,\bar{\underset{\sim}{R}}^T$$

$$= \underset{\sim}{Q}\,\underset{\sim}{F}\,(\underset{\sim}{Q}\,\underset{\sim}{R})^T = \underset{\sim}{Q}\,\underset{\sim}{V}\,\underbrace{\underset{\sim}{R}\,\underset{\sim}{R}^T}_{Id}\,\underset{\sim}{Q}^T$$

$$= \underset{\sim}{Q}\,\underset{\sim}{V}\,\underset{\sim}{Q}^T$$

$\therefore$ $\underset{\sim}{V}$ objective

X

10. $\underset{\sim}{w} = \dot{\underset{\sim}{R}}\,\underset{\sim}{R}^T$

$$\bar{\underset{\sim}{w}} = \dot{\bar{\underset{\sim}{R}}}\,\bar{\underset{\sim}{R}}^T = \dot{\overline{\underset{\sim}{Q}\,\underset{\sim}{R}}}\,(\underset{\sim}{Q}\,\underset{\sim}{R})^T$$

$$= (\dot{\underset{\sim}{Q}}\,\underset{\sim}{R} + \underset{\sim}{Q}\,\dot{\underset{\sim}{R}})\,\underset{\sim}{R}^T\underset{\sim}{Q}^T$$

$$= \dot{\underset{\sim}{Q}}\,\underset{\sim}{Q}^T + \underset{\sim}{Q}\,\underbrace{\dot{\underset{\sim}{R}}\,\underset{\sim}{R}^T}\,\underset{\sim}{Q}^T$$

$$\bar{\underset{\sim}{w}} = \underset{\sim}{\beta} + \underset{\sim}{Q}\,\underset{\sim}{w}\,\underset{\sim}{Q}^T$$

$\underset{\sim}{w}$ not objective

X

11. $\underset{\sim}{\omega} - \underset{\sim}{w}$

$$\bar{\underset{\sim}{\omega}} - \bar{\underset{\sim}{w}} = \underset{\sim}{\beta} + \underset{\sim}{Q}\,\underset{\sim}{\omega}\,\underset{\sim}{Q}^T - (\underset{\sim}{\beta} + \underset{\sim}{Q}\,\underset{\sim}{w}\,\underset{\sim}{Q}^T)$$

$$= \underset{\sim}{Q}\,(\underset{\sim}{\omega} - \underset{\sim}{w})\,\underset{\sim}{Q}^T$$

$\therefore$ $(\underset{\sim}{\omega} - \underset{\sim}{w})$ is objective

X

12. $\underset{\sim}{l} = \underset{\sim}{d} + \underset{\sim}{\omega} = \underset{\sim}{w} + \underset{\sim}{\delta}$

nonsymm. not obj. — symm. objective — skew not obje. — skew not obj. — unsymm. ?

$$\bar{\underset{\sim}{\delta}} = \bar{\underset{\sim}{d}} + (\bar{\underset{\sim}{\omega}} - \bar{\underset{\sim}{w}})$$

$$= \underset{\sim}{Q}\,\underbrace{(\underset{\sim}{d} + \underset{\sim}{\omega} - \underset{\sim}{w})}\,\underset{\sim}{Q}^T$$

$$= \cancel{Q\delta}\ \underset{\sim}{Q}\,\underset{\sim}{\delta}\,\underset{\sim}{Q}^T$$

$\therefore\ \underset{\sim}{\delta}$ objective

×

Recall: $\underset{\sim}{\tau}^* = \underset{\sim}{c}\cdot\underset{\sim}{d}$

a. Assume $\underset{\sim}{c}$ is objective

i.e. $\bar{c}_{ijkl} = Q_{ia}Q_{jb}Q_{kc}Q_{ld}c_{abcd}$

(+ Fact $\underset{\sim}{d}$ is objective

$\Longrightarrow \underset{\sim}{\tau}^*$ objective)

Check: $\bar{\underset{\sim}{\tau}}^* \overset{df}{=} \bar{\underset{\sim}{c}}\cdot\bar{\underset{\sim}{d}}$

δ_{mc}

$$\bar{c}_{ijkl}\bar{d}_{kl} = Q_{ia}Q_{jb}Q_{kc}Q_{ld}c_{abcd}Q_{km}Q_{ln}d_{mn}$$

δ_{nd}

$$= Q_{ia} Q_{jb} \left(\underbrace{C_{abmn}\, d_{mn}}_{\sigma^*_{ab}} \right)$$

$$\underset{\sim}{\bar{\sigma}}^{*} = \underset{\sim}{Q}\, \underset{\sim}{\sigma}^{*}\, \underset{\sim}{Q}^{T}$$

$\longrightarrow$ $\therefore$ $\underset{\sim}{\bar{\sigma}}^{*}$ is objective

X

$\underset{\sim}{\dot{\sigma}}$ "didn't transform right"

Assume: $\underset{\sim}{\bar{\sigma}} = \underset{\sim}{Q}\, \underset{\sim}{\sigma}\, \underset{\sim}{Q}^{T}$ ($\underset{\sim}{\sigma}$ is objective)

$$\underset{\sim}{\dot{\bar{\sigma}}} = \underset{\sim}{\dot{Q}}\, \underset{\sim}{\sigma}\, \underset{\sim}{Q}^{T} + \underset{\sim}{Q}\, \underset{\sim}{\dot{\sigma}}\, \underset{\sim}{Q}^{T} + \underset{\sim}{Q}\, \underset{\sim}{\sigma}\, \underset{\sim}{\dot{Q}}^{T}$$

$$= \underset{\sim}{\dot{Q}}\, \underbrace{\underset{\sim}{Q}^{T} \underset{\sim}{Q}}_{Id}\, \underset{\sim}{\sigma}\, \underset{\sim}{Q}^{T} + \underset{\sim}{Q}\, \underset{\sim}{\dot{\sigma}}\, \underset{\sim}{Q}^{T} + \underset{\sim}{Q}\, \underset{\sim}{\sigma} \underbrace{(\underset{\sim}{Q}^{T} \underset{\sim}{Q})}_{Id} \underset{\sim}{\dot{Q}}^{T}$$

$\underset{\sim}{\beta} = \underset{\sim}{\dot{Q}}\, \underset{\sim}{Q}^{T}$

$\hookrightarrow$ skew

$$= \underset{\sim}{\beta}\, \underset{\sim}{\bar{\sigma}} + \underset{\sim}{Q}\, \underset{\sim}{\dot{\sigma}}\, \underset{\sim}{Q}^{T} + \underset{\sim}{\bar{\sigma}}\, \underset{\sim}{\beta}^{T}$$

$$\underset{\sim}{\beta}^{T} \Downarrow -\underset{\sim}{\beta}$$

$$\underset{\sim}{\dot{\bar{\sigma}}} = \underbrace{\underset{\sim}{\beta}\, \underset{\sim}{\bar{\sigma}} - \underset{\sim}{\bar{\sigma}}\, \underset{\sim}{\beta}}_{\neq 0} + \underset{\sim}{Q}\, \underset{\sim}{\dot{\sigma}}\, \underset{\sim}{Q}^{T}$$

$\longrightarrow$ $\therefore$ $\underset{\sim}{\dot{\sigma}}$ is not objective

X

→ ∴ $\underset{\sim}{\tau}^{*}$ (Jaumann) is objective.

→ $$\boxed{\dot{\underset{\sim}{\tau}} + \underset{\sim}{\tau}\,\underset{\sim}{\omega} - \underset{\sim}{\omega}\,\underset{\sim}{\tau} = \underset{\sim}{c} \cdot \underset{\sim}{d}}$$

~~Co~~ Constitutive eq. in terms of Jaumann rate

Alternative form for

(much neater, looks ~~like~~ just like infinitesimal theory, lends itself to accurate numerical algorithm's)

spin angular velocity → $$\boxed{\underset{\sim}{\omega} = \dot{\underset{\sim}{P}}\,\underset{\sim}{P}^{T}}$$ ↑ given

$\underset{\sim}{P}$ orthogonal

infinitesimal generator

$$\left\{\begin{array}{l} \dot{\underset{\sim}{P}} = \underset{\sim}{\omega}\,\underset{\sim}{P} \\[2ex] \underset{\sim}{P}\Big|_{t=0} = Id. \end{array}\right.$$

system of ODE's

Defines $\underset{\sim}{P}$ as a function of t

Consider $$\boxed{\underset{\sim}{\tau}_{P} \overset{def}{=} \underset{\sim}{P}^{T}\,\underset{\sim}{\tau}\,\underset{\sim}{P}}$$

(i.e. Cauchy stress "rotated" by $\underset{\sim}{P}^{T}$)

$$\utilde{\sigma}_P \overset{\text{def.}}{=\!=} \utilde{P}^T \utilde{\sigma} \utilde{P}$$

Claim $\boxed{\dot{\utilde{\sigma}}_P = \utilde{P}^T \utilde{\sigma}^{*(\text{Jaumann})} \utilde{P}}$

$$\dot{\utilde{\sigma}}_P = \underbrace{\dot{\utilde{P}}^T \utilde{\sigma} \utilde{P}} + \utilde{P}^T \dot{\utilde{\sigma}} \utilde{P} + \utilde{P}^T \utilde{\sigma} \dot{\utilde{P}}$$

$$= \utilde{P}^T \utilde{P} \downarrow \quad + \qquad\qquad + \utilde{P}^T \utilde{\sigma} \dot{\utilde{P}} \utilde{P}^T \utilde{P}$$

$$= \utilde{P}^T \underbrace{\underbrace{\underbrace{\underbrace{(\utilde{P} \dot{\utilde{P}}^T)}_{(\dot{\utilde{P}} \utilde{P}^T)^T}}_{\utilde{\omega}^T}}_{-\utilde{\omega}}} \utilde{\sigma} \utilde{P} + \utilde{P}^T \dot{\utilde{\sigma}} \utilde{P} + \utilde{P}^T \utilde{\sigma} (\underbrace{\dot{\utilde{P}} \utilde{P}^T}_{\utilde{\omega}}) \utilde{P}$$

$$= \utilde{P}^T (\underbrace{\dot{\utilde{\sigma}} + \utilde{\sigma} \utilde{\omega} - \utilde{\omega} \utilde{\sigma}}) \utilde{P}$$

$$\dot{\utilde{\sigma}}_P = \utilde{P}^T \qquad \utilde{\sigma}^{*(\text{Jaumann})} \utilde{P}$$

handout

4/14

$$\left\{\begin{array}{l} \underset{\sim}{\omega} = \dot{\underset{\sim}{P}}\,\underset{\sim}{P}^{T} \Leftrightarrow \dot{\underset{\sim}{P}} = \underset{\sim}{\omega}\,\underset{\sim}{P} \\[2ex] \underset{\sim}{\sigma}_{\underset{\sim}{P}} = \underset{\sim}{P}^{T}\,\underset{\sim}{\sigma}\,\underset{\sim}{P} \\[2ex] \dot{\underset{\sim}{\sigma}}_{P} = \underset{\sim}{P}^{T}\,\underset{\sim}{\sigma}^{*(\text{Jaumann})}\,\underset{\sim}{P} \end{array}\right.$$

———— × ————

B. 1. ~~[illegible]~~ $\underset{\sim}{\omega} = \dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T}$

2. def. $\underset{\sim}{\sigma}_{\underset{\sim}{R}} = \underset{\sim}{R}^{T}\,\underset{\sim}{\sigma}\,\underset{\sim}{R}$ (rotate by $\underset{\sim}{R}$)

3 $\underset{\sim}{\sigma}^{*(\text{Rotated})} \overset{\text{def.}}{=} \dot{\underset{\sim}{\sigma}} - \underset{\sim}{\omega}\,\underset{\sim}{\sigma} + \underset{\sim}{\sigma}\,\underset{\sim}{\omega}$

4 then: $\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}} = \underset{\sim}{R}^{T}\,\underset{\sim}{\sigma}^{*(\text{rotated})}\,\underset{\sim}{R}$

proof: $$\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}} = \underset{\substack{\uparrow \\ \underset{\sim}{R}^{T}\underset{\sim}{R}}}{\dot{\underset{\sim}{R}}^{T}}\,\underset{\sim}{\sigma}\,\underset{\sim}{R} + \underset{\sim}{R}^{T}\,\dot{\underset{\sim}{\sigma}}\,\underset{\sim}{R} + \underset{\sim}{R}^{T}\,\underset{\sim}{\sigma}\,\underset{\substack{\uparrow \\ \underset{\sim}{R}^{T}\underset{\sim}{R}}}{\dot{\underset{\sim}{R}}}$$

$$\underset{\sim}{R}\,\dot{\underset{\sim}{R}}^{T} = (\dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T})^{T} = \underset{\sim}{\omega}^{T} = -\underset{\sim}{\omega}$$

$$= \underset{\sim}{R}^{T}\underbrace{(\dot{\underset{\sim}{\sigma}} - \underset{\sim}{\omega}\,\underset{\sim}{\sigma} + \underset{\sim}{\sigma}\,\underset{\sim}{\omega})}_{\underset{\sim}{\sigma}^{*(\text{rotated})}}\underset{\sim}{R}$$

$$\underbrace{\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}} = \underset{\sim}{R}^{T}\,\underset{\sim}{\sigma}^{*(\text{rotated})}\,\underset{\sim}{R}}_{\uparrow\ ?\ \text{objective}}$$

———— × ————

Rotated stress rate is objective

proof: $\bar{\underset{\sim}{\sigma}}^{*\,(\text{rotated})} = \dot{\bar{\underset{\sim}{\sigma}}} - \bar{\underset{\sim}{\omega}}\,\bar{\underset{\sim}{\sigma}} + \bar{\underset{\sim}{\sigma}}\,\bar{\underset{\sim}{\omega}}$

$$\overset{?}{=} \underset{\sim}{Q}\,\underset{\sim}{\sigma}^{*\,(\text{rotated})}\,\underset{\sim}{Q}^{T}$$

$$\dot{\bar{\underset{\sim}{\sigma}}} = \frac{d}{dt}\left(\underset{\sim}{Q}\,\underset{\sim}{\sigma}\,\underset{\sim}{Q}^{T}\right) = \underset{\substack{\Downarrow \\ \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^{T}}}{\underset{\sim}{\beta}}\,\bar{\underset{\sim}{\sigma}} + \bar{\underset{\sim}{\sigma}}\,\underset{\substack{\Downarrow \\ -\underset{\sim}{\beta}}}{\underset{\sim}{\beta}^{T}} + \underset{\sim}{Q}\,\dot{\underset{\sim}{\sigma}}\,\underset{\sim}{Q}^{T} \;\Big|\; -\underbrace{\bar{\underset{\sim}{\omega}}\,\bar{\underset{\sim}{\sigma}}} + \underbrace{\bar{\underset{\sim}{\sigma}}\,\bar{\underset{\sim}{\omega}}}$$

recall

$$-\bar{\underset{\sim}{\omega}}\,\bar{\underset{\sim}{\sigma}} = -\left(\underset{\sim}{\beta} + \underset{\sim}{Q}\,\underset{\sim}{\omega}\,\underset{\sim}{Q}^{T}\right)\bar{\underset{\sim}{\sigma}}$$

$$+\,\bar{\underset{\sim}{\sigma}}\,\bar{\underset{\sim}{\omega}} = \bar{\underset{\sim}{\sigma}}\left(\underset{\sim}{\beta} + \underset{\sim}{Q}\,\underset{\sim}{\omega}\,\underset{\sim}{Q}^{T}\right)$$

$$\bar{\underset{\sim}{\sigma}}^{*\,(\text{rotated})} = \underset{\sim}{Q}\,\underbrace{\left(\dot{\underset{\sim}{\sigma}} - \underset{\sim}{\omega}\,\underset{\sim}{\sigma} + \underset{\sim}{\sigma}\,\underset{\sim}{\omega}\right)}_{\underset{\sim}{\sigma}^{*\,(\text{rotated})}}\,\underset{\sim}{Q}^{T}$$

$\Longrightarrow \therefore\ \underset{\sim}{\sigma}^{*\,(\text{rotated})}$ is objective

×

Consider Development of simple el-pl theories
(assume perf. pl. / Mises Y.S.)

1. Jaumann rate
2. "Rotated" rate

→ ① Jaumann:

iso. el. moduli

Assume: $\underset{\sim}{\sigma}^{*(\text{Jaumann})} = \underset{\sim}{C}\cdot(\underset{\sim}{d} - \underset{\sim}{d}^{pl})$

↑ plastic rate of deformation

$\underset{\sim}{\omega}$ generates $\underset{\sim}{P}$

→ Last lecture $= \underset{\sim}{P}\,\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{P}}\,\underset{\sim}{P}^T = \underset{\sim}{C}\,(\underset{\sim}{d} - \underset{\sim}{d}^{pl})$

$$\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{P}} = \underset{\sim}{P}^T\,\underset{\sim}{C}\cdot(\underset{\sim}{d} - \underset{\sim}{d}^{pl})\,\underset{\sim}{P}$$

def: $\left\{\begin{array}{l} \underset{\sim}{d}_{\underset{\sim}{P}} = \underset{\sim}{P}^T\,\underset{\sim}{d}\,\underset{\sim}{P} \\ \underset{\sim}{d}^{pl}_{\underset{\sim}{P}} = \underset{\sim}{P}^T\,\underset{\sim}{d}^{pl}\,\underset{\sim}{P} \end{array}\right\}$

$$\rightarrow \dot{\underset{\sim}{\sigma}}_{\underset{\sim}{P}} = \underset{\sim}{P}^T\left(\underset{\sim}{C}\cdot\underset{\sim}{P}\,(\underset{\sim}{d}_{\underset{\sim}{P}} - \underset{\sim}{d}^{pl}_{\underset{\sim}{P}})\,\underset{\sim}{P}^T\right)\underset{\sim}{P}$$

$$(\dot{\sigma}_{\underset{\sim}{P}})_{ij} = \underbrace{P_{ia}P_{jb}P_{kc}P_{ld}C_{abcd}}_{\text{rotated (by } \underset{\sim}{P}\text{) elastic moduli}}\left((d_{\underset{\sim}{P}})_{kl} - (d^{pl}_{\underset{\sim}{P}})_{kl}\right)$$

$\underbrace{(\text{If iso.}, = C_{ijkl})}_{\text{def. of isotropy}}$

$$\rightarrow \boxed{\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{P}} = \underset{\sim}{C}\cdot(\underset{\sim}{d}_{\underset{\sim}{P}} - \underset{\sim}{d}^{pl}_{\underset{\sim}{P}})}$$ in the rotated system

def. $\dot{\underset{\sim}{\epsilon}}_{\underset{\sim}{P}} = \underset{\sim}{d}_{\underset{\sim}{P}}$ $\quad \dot{\underset{\sim}{\epsilon}}^{pl}_{\underset{\sim}{P}} = \underset{\sim}{d}^{pl}_{\underset{\sim}{P}}$ $\quad \underset{\sim}{\epsilon}_P\Big|_{t=0} = \underset{\sim}{0}$ etc

$$\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{p}} = \underset{\sim}{C} \cdot \left(\dot{\underset{\sim}{\epsilon}}_{\underset{\sim}{p}} - \dot{\underset{\sim}{\epsilon}}^{pl}_{\underset{\sim}{p}} \right)$$

+ Mises $\quad f(\underset{\sim}{\sigma}) = k^2 \underset{\uparrow \text{perf.}}{=} \text{const.}$

$\longrightarrow$ recall f invariant $\Longrightarrow f(\underset{\sim}{\sigma}) = f(\underset{\sim}{\sigma}_p)$

unit normal:

$$\frac{\partial f}{\partial \underset{\sim}{\sigma}_{\underset{\sim}{p}}} = \underset{\sim}{\sigma}'_{\underset{\sim}{p}}$$

unit normal $\boxed{\underset{\sim}{Q}_{\underset{\sim}{p}} = \frac{\underset{\sim}{\sigma}'_{\underset{\sim}{p}}}{R}}$ $\longleftarrow$ radial $= \sqrt{2}\,k$

(don't confuse with rotation in objective transform)

plastic flow:
$$\dot{\underset{\sim}{\epsilon}}^{pl}_{\underset{\sim}{p}} = \begin{cases} 0 & (E) \\ \Lambda \underset{\sim}{Q}_{\underset{\sim}{p}} & (P) \end{cases}$$

$\longleftarrow$ same in terms of $\dot{\underset{\sim}{\sigma}}^{tr}_{\underset{\sim}{p}}$, $\underset{\sim}{Q}_{\underset{\sim}{p}}$ etc

Consistency: $\quad \dot{f}(\underset{\sim}{\sigma}_{\underset{\sim}{p}}) = 0$

$$0 = \underset{\sim}{Q}_{\underset{\sim}{p}} \cdot \dot{\underset{\sim}{\sigma}}_{\underset{\sim}{p}} = \underset{\sim}{Q}_{\underset{\sim}{p}} \cdot \left(\underset{\sim}{C} \cdot \left(\underset{\sim}{d} - \underset{\sim}{d}^{pl} \right) \right)$$

$$\Downarrow$$

$$\Lambda \underset{\sim}{Q}_{\underset{\sim}{p}}$$

$$\Lambda = \frac{Q_p \cdot \overbrace{C \cdot d_p}^{\dot{\sigma}_p^{tr}}}{Q_p \cdot C \cdot Q_p} = \frac{2\mu\,(Q_p \cdot d_p)}{2\mu}$$

$$\longrightarrow \boxed{\underbrace{\Lambda = Q \cdot d = Q_p \cdot d_p}_{\text{invariant}}}$$

$$\left(\begin{array}{l} \bar{A} \cdot \bar{B} \overset{?}{=} A \cdot B \\ \bar{A}_{ij} = P_{ia} A_{ab} P_{jb} \quad \text{etc.} \end{array} \right)$$

→ Moral: Jaumann rate form looks exactly like small-deformation form of 239A in P coordinate system (Generated by ω)

×

Is Jaumann form any good? Not really

×

Similar theory in terms of R

→ ② Rotated stress rate theory

Assume $\sigma^{*\,(\text{rotated})} = C \cdot (d - d^{pl})$

Assume. $\underset{\sim}{\sigma}^{*(\text{rotated})} = \underset{\sim}{C} \cdot (\underset{\sim}{d} - \underset{\sim}{d}^{pl})$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\underset{\sim}{R}\, \dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}}\, \underset{\sim}{R}^T =$$

$$\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}} = \underset{\sim}{R}^T \left(\underset{\sim}{C} \cdot (\underset{\sim}{d} - \underset{\sim}{d}^{pl}) \right) \underset{\sim}{R}$$

def. $\underset{\sim}{d}_{\underset{\sim}{R}} = \underset{\sim}{R}^T \underset{\sim}{d}\, \underset{\sim}{R}$

$\underset{\sim}{d}_{\underset{\sim}{R}}^{pl} = \underset{\sim}{R}^T \underset{\sim}{d}^{pl} \underset{\sim}{R}$

$$\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}} = \underset{\sim}{R}^T \left(\underset{\sim}{C} \cdot \left(\underset{\sim}{R} \left(\underset{\sim}{d}_{\underset{\sim}{R}} - \underset{\sim}{d}_{\underset{\sim}{R}}^{pl} \right) \underset{\sim}{R}^T \right) \right) \underset{\sim}{R}$$

$$\longrightarrow (\dot{\sigma}_{\underset{\sim}{R}})_{ij} = \underbrace{R_{ia} R_{jb} R_{kc} R_{ld} C_{abcd}}_{\text{el moduli rotated by } \underset{\sim}{R}} \left((d_{\underset{\sim}{R}})_{kl} - (d_{\underset{\sim}{R}}^{pl})_{kl} \right)$$

(if * iso $= C_{ijkl}$)

But, if anisotropic $= \underset{\sim}{C}_{\underset{\sim}{R}}$,

(Shear strain not so large, then O.K for anisotropic elastic moduli)

Remark: P-form make no sense for anisotropic elastic moduli

$$\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}} = \underset{\sim}{C}_{\underset{\sim}{R}} \cdot \left(\underset{\sim}{d}_{\underset{\sim}{R}} - \underset{\sim}{d}^{pl}_{\underset{\sim}{R}}\right)$$

$$\text{def.}\ \left\{ \begin{array}{l} \dot{\underset{\sim}{\epsilon}}_{\underset{\sim}{R}} = \underset{\sim}{d}_{\underset{\sim}{R}} \\ \dot{\underset{\sim}{\epsilon}}^{pl}_{\underset{\sim}{R}} = \underset{\sim}{d}^{pl}_{\underset{\sim}{R}} \end{array} \right\} + \ 0 \ \text{I.C.} \quad \text{etc}$$

Mises Y.S. $\quad f(\underset{\sim}{\sigma}) = f(\underset{\sim}{\sigma}_{\underset{\sim}{R}}) = k^2$

Consistency (Details – exercises)

$$\underset{\sim}{d}^{pl}_{\underset{\sim}{R}} = \left\{ \begin{array}{ll} 0 & (E) \\ \Lambda \underset{\sim}{Q}_{\underset{\sim}{R}} & (P) \end{array} \right.$$

same idea in terms of $\underset{\sim}{R}$-quanti

$$\Lambda = \frac{\underset{\sim}{Q}_{\underset{\sim}{R}} \cdot \dot{\underset{\sim}{\sigma}}^{tr}_{\underset{\sim}{R}}}{\underset{\sim}{Q}_{\underset{\sim}{R}} \cdot \underset{\sim}{C}_{\underset{\sim}{R}} \cdot \underset{\sim}{Q}_{\underset{\sim}{R}}} \underset{\uparrow}{=} \underset{\sim}{Q}_{\underset{\sim}{R}} \cdot \underset{\sim}{d}_{\underset{\sim}{R}} = \underset{\sim}{Q} \cdot \underset{\sim}{d}$$

if $\underset{\sim}{C}_{\underset{\sim}{R}} = \underset{\sim}{C}$ (iso.)

X

Remark if $\underset{\sim}{d} = \underset{\sim}{0}$ $\left(\Longrightarrow \underset{\sim}{d}_{\underset{\sim}{R}} = \underset{\sim}{d}_{\underset{\sim}{P}} = \underset{\sim}{0}\right)$

(constitutive eq.) $\Longrightarrow$ $\underbrace{\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}}} = \dot{\underset{\sim}{\sigma}}_{\underset{\sim}{P}} = \underset{\sim}{0}$

Stress is constant in the rotating system ($\underset{\sim}{R}$ sys. or $\underset{\sim}{P}$ sys.)

In particular, invariants of $\underset{\sim}{\tau}$ are constant

($\underset{\sim}{\tau}$ is changing)

(Prager proposed this as an hypothesis for plasticity theories)

X

What about iso/kin hardening?

combined

iso:

$$\dot{k} = \beta \frac{1}{\sqrt{3}} H' \bar{d}^{pl} \qquad \left(\bar{d}^{pl} = \sqrt{\tfrac{2}{3}} \, (\underset{\sim}{d}^{pl} \cdot \underset{\sim}{d}^{pl})^{1/2} \right)$$

$\dot{\bar{\epsilon}}^{pl} = \bar{d}^{pl}$ integrate

invariant

$\underset{\sim}{R}$ $\underset{\sim}{R}$

$\underset{\sim}{P}$ $\underset{\sim}{P}$

$$\overset{\circ}{\underset{\sim}{\alpha}}_{\underset{\sim}{R}} = (1-\beta)\frac{2}{3} H' \underset{\sim}{d}^{pl}_{\underset{\sim}{R}}$$

(or $\underset{\sim}{P}$) (or $\underset{\sim}{P}$)

$\dot{\epsilon}^{pl} = d^{pl}$

$\underset{\sim}{R} \longleftrightarrow \underset{\sim}{R}$

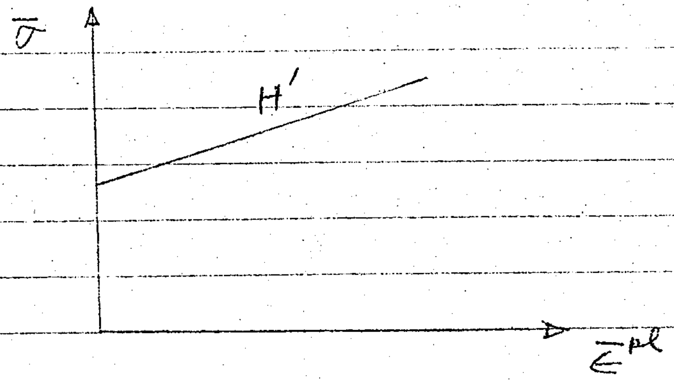
or $\underset{\sim}{P} \longleftrightarrow \underset{\sim}{P}$

Consistency (exercise):

$$\Lambda = \frac{\cancel{K}\; \underset{\sim}{Q}_{\underset{\sim}{R}} \cdot \overset{\circ}{\underset{\sim}{\tau}}_{\underset{\sim}{R}}}{\left(\underset{\sim}{Q}_{\underset{\sim}{R}} \cdot \underset{\sim}{C}_{\underset{\sim}{R}} \cdot \underset{\sim}{Q}_{\underset{\sim}{R}} + \frac{2}{3} H' \right)}$$

or $\underset{\sim}{P}$'s or nothing (∵ invariant)

Iso.



$(\ln \lambda)$
small elastic

→ Radial process also true for kinematic.

X

Guest Speaker:

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$$\dot{\underset{\sim}{\tau}}^{J} = \underset{\sim}{L}^{el-pl}\,\underset{\sim}{D}$$ Jaumann constitutive eq.

$$\dot{\underset{\sim}{\tau}}^{Jaumann} = \dot{\underset{\sim}{\tau}} - \underset{\sim}{\omega}\cdot\underset{\sim}{\tau} - \underset{\sim}{\tau}\cdot\underset{\sim}{\omega}^{T}$$ Jaumann rate

spin tensor

$$\underset{\sim}{L}^{T} = \underset{\sim}{\nabla}\,\underset{\sim}{v}$$

$$\underset{\sim}{D} = \frac{1}{2}\left(\underset{\sim}{L} + \underset{\sim}{L}^{T}\right), \qquad \underset{\sim}{\omega} = \frac{1}{2}\left(\underset{\sim}{L} - \underset{\sim}{L}^{T}\right)$$

$$\underset{\sim}{L} = \underset{\sim}{D} + \underset{\sim}{\omega}$$

$$\underset{\sim}{\Omega} = \underset{\sim}{\omega} + \underset{\sim}{f}(\underset{\sim}{D}; s)$$

anti-symmetric

material parameter (could be tensor, vector ...

$$\underset{\sim}{f}(\underset{\sim}{0}; \underset{\sim}{s}) = \underset{\sim}{0}$$

$$\underset{\sim}{f}(\lambda\underset{\sim}{D}; s) = \lambda\,\underset{\sim}{f}(\underset{\sim}{D}; s)$$

$$\dot{\underset{\sim}{\tau}}^{*} = \dot{\underset{\sim}{\tau}} - \underset{\sim}{\Omega}\cdot\underset{\sim}{\tau} - \underset{\sim}{\tau}\cdot\underset{\sim}{\Omega}^{T}$$

$$\underset{\sim}{\Omega} = \underset{\sim}{\omega} + g(I^{\alpha})\left(\underset{\sim}{D}\cdot\underset{\sim}{\alpha} - \underset{\sim}{\alpha}\cdot\underset{\sim}{D}\right)$$

$\underset{\sim}{\alpha}$: symmetric tensor describing state of materi

I^{α}: invariant of $\underset{\sim}{\alpha}$

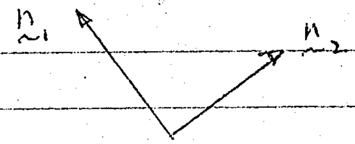
g is of order $\underset{\sim}{\alpha}^{-1}$

$$g(I^{2\underset{\sim}{\alpha}}) = \frac{1}{2} g(I^{\underset{\sim}{\alpha}})$$

$$\underset{\sim}{\Omega} = \underset{\sim}{L} + g(I^{\underset{\sim}{\alpha}})(\underset{\sim}{D}\cdot\underset{\sim}{\alpha} - \underset{\sim}{\alpha}\cdot\underset{\sim}{D}) - \underset{\sim}{D}$$

line segment with direction $\underset{\sim}{n}_1$

rotation in plane of $\underset{\sim}{n}_1$ $\underset{\sim}{n}_2$, $\underset{\sim}{n}_1\cdot\underset{\sim}{n}_2 = 0$



$$\phi = \underset{\sim}{n}_2\cdot\underset{\sim}{L}\cdot\underset{\sim}{n}_1$$

$$\Omega_{21} = \phi + g(I^{\alpha})(D_{2k}\alpha_{k1} - \alpha_{2k}D_{k1}) - D_{21}$$

* * *

Rigid plastic material behavior:

isotropic hardening rigid-plastic

yield criterion

1. $\frac{3}{2}\underset{\sim}{\tau}'\cdot\underset{\sim}{\tau}' = \sigma_y^2$ $\qquad \underset{\sim}{\sigma}' = \underset{\sim}{\sigma} - \frac{1}{3}\underset{\sim}{I}\,\underset{\sim}{I}\cdot\underset{\sim}{\sigma}$

2. Flow rule

flow $\quad \underset{\sim}{D} = \frac{\lambda\underset{\sim}{\sigma}'}{\sigma_y}$

3. Hardening rule $\quad \overset{\circ}{\sigma}_y = h\bar{D} = h\sqrt{\frac{2}{3}\underset{\sim}{D}\cdot\underset{\sim}{D}}$

$$\Longrightarrow \underset{\sim}{D} = \frac{9}{4h}\,\frac{\underset{\sim}{\sigma}'\cdot\underset{\sim}{\sigma}'}{\sigma_y^2}\cdot\dot{\underset{\sim}{\sigma}}$$

$$\underset{\sim}{D} = \frac{9}{4h}\,\frac{\underset{\sim}{\sigma}'\cdot\underset{\sim}{\sigma}'}{\sigma_y^2}\cdot\dot{\underset{\sim}{\sigma}}$$

$$\dot{\underset{\sim}{\sigma}}^* = \dot{\underset{\sim}{\sigma}} + \alpha\left(\underset{\sim}{D}\cdot\underset{\sim}{\sigma} + \underset{\sim}{\sigma}\cdot\underset{\sim}{D}\right) + \beta\left(\underset{\sim}{\Omega}\cdot\underset{\sim}{\sigma} + \underset{\sim}{\sigma}\cdot\underset{\sim}{\Omega}^T\right)$$

$$\underset{\sim}{D} = \frac{9}{4h}\,\frac{\underset{\sim}{\sigma}'}{\sigma_y^2}\left[\dot{\underset{\sim}{\sigma}}\;\dot{\underset{\sim}{\sigma}}^* - 2\alpha\,\underset{\sim}{D}:(\underset{\sim}{\sigma}\cdot\underset{\sim}{\sigma}') - 2\beta\,\underset{\sim}{\Omega}:(\underset{\sim}{\sigma}\cdot\underset{\sim}{\sigma}')\right]$$

Rewrite:

$$\underset{\sim}{D} = \frac{9}{4h}\,\frac{\underset{\sim}{\sigma}'\cdot\underset{\sim}{\sigma}'}{\sigma_y^2}:\dot{\underset{\sim}{\sigma}}^{**}$$

$$\dot{\underset{\sim}{\sigma}}^{**} = \dot{\underset{\sim}{\sigma}} + \beta\left(\underset{\sim}{\Omega}\cdot\underset{\sim}{\sigma} + \underset{\sim}{\sigma}\cdot\underset{\sim}{\Omega}^*\right)$$

$$\underset{\sim}{D}:\underset{\sim}{D} = \lambda^2\,\frac{\underset{\sim}{\sigma}':\underset{\sim}{\sigma}'}{\sigma_y^2} = \lambda^2\,\frac{2}{3}$$

$$\lambda = \frac{3}{2}\sqrt{\frac{2}{3}\underset{\sim}{D}:\underset{\sim}{D}} = \frac{3}{2}\bar{D}$$

$$\underset{\sim}{D} = \frac{\lambda\,\underset{\sim}{\sigma}'}{\sigma_y} = \frac{3}{2}\bar{D}\,\frac{\underset{\sim}{\sigma}'}{\sigma_y}$$

Differentiate yield condition

$$3\underset{\sim}{\sigma}':\dot{\underset{\sim}{\sigma}}' = 2\sigma_y\dot{\sigma}_y$$

$$\dot{\sigma}_y = \frac{3\underset{\sim}{\sigma}':\dot{\underset{\sim}{\sigma}}'}{2\sigma_y}$$

$$h\bar{D} = \frac{3\underset{\sim}{\sigma}':\dot{\underset{\sim}{\sigma}}}{2\sigma_y};$$

$$\longrightarrow \quad \underset{\sim}{D} = \frac{9}{4h}\,\frac{\underset{\sim}{\sigma}'\cdot\underset{\sim}{\sigma}'}{\sigma_y^2}:\dot{\underset{\sim}{\tau}}$$

$$\underset{\sim}{D} = \underset{\sim}{D}^e + \underset{\sim}{D}^p$$

$$\underset{\sim}{D}^p = \frac{9}{4h} \frac{\underset{\sim}{\sigma}' \cdot \underset{\sim}{\sigma}'}{\sigma_y^2} : \dot{\underset{\sim}{\sigma}}^{**}$$

$$\underset{\sim}{D}^e = \frac{1}{2\mu} \left[\dot{\underset{\sim}{\sigma}}^{*} - \frac{\lambda}{3\lambda+2\mu} \underset{\sim}{I}\, \underset{\sim}{I} : \dot{\underset{\sim}{\sigma}}^{*} \right]$$

Strain energy $$W^e = \frac{1}{4\mu} \left[\underset{\sim}{\sigma} : \underset{\sim}{\sigma} - \frac{\lambda}{3\lambda+2\mu} (\underset{\sim}{I} : \underset{\sim}{\sigma})^2 \right]$$

$$\dot{W}^e = \frac{1}{2\mu} \left[\underset{\sim}{\sigma} : \dot{\underset{\sim}{\sigma}} - \frac{\lambda}{3\lambda+2\mu} (\underset{\sim}{I} : \underset{\sim}{\sigma})(\underset{\sim}{I} : \dot{\underset{\sim}{\sigma}}) \right]$$

Objectivity requires $\beta = -1$

$$\dot{W}^e = \frac{1}{2\mu} \left[\underset{\sim}{\sigma} : \dot{\underset{\sim}{\sigma}}^{**} - \frac{\lambda}{3\lambda+2\mu} (\underset{\sim}{\sigma} : \underset{\sim}{I})(\underset{\sim}{I} : \dot{\underset{\sim}{\sigma}}^{**}) \right]$$

with condition $\beta = -1$

1. $\dot{\underset{\sim}{\sigma}}^{**} = \dot{\underset{\sim}{\sigma}} - \underset{\sim}{\Omega} \cdot \underset{\sim}{\sigma} - \underset{\sim}{\sigma} \cdot \underset{\sim}{\Omega}^T$

$$\dot{W}^e = \underset{\sim}{\sigma} : \underset{\sim}{D}^e$$

2. $\underset{\sim}{D}^e = \frac{1}{2\mu} \left[\dot{\underset{\sim}{\sigma}}^{**} - \frac{\lambda}{3\lambda+2\mu} \underset{\sim}{I} (\underset{\sim}{I} : \dot{\underset{\sim}{\sigma}}^{**}) \right]$

3 $\underset{\sim}{D}^p = \frac{9}{4h} \frac{\underset{\sim}{\sigma}' \cdot \underset{\sim}{\sigma}'}{\sigma_y^2} : \dot{\underset{\sim}{\sigma}}^{**}$

4 $\underset{\sim}{D} = \underset{\sim}{D}^e + \underset{\sim}{D}^p$

$$\dot{\underset{\sim}{\sigma}}^{**} = \lambda\, \underset{\sim}{I}\,\underset{\sim}{I} : \underset{\sim}{D} + 2\mu \left[\underset{\sim}{D} - \frac{1}{1+\frac{h}{3\mu}} \frac{3\underset{\sim}{\sigma}' \cdot \underset{\sim}{\sigma}'}{2\sigma_y^2} : \underset{\sim}{D} \right]$$

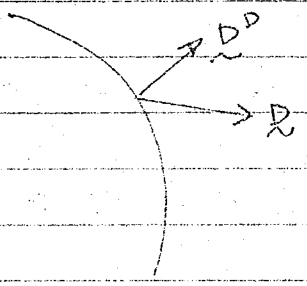
$$\dot{\underset{\sim}{\sigma}}^{**} = \dot{\underset{\sim}{\sigma}} - \underset{\sim}{\Omega} \cdot \underset{\sim}{\sigma} - \underset{\sim}{\sigma} \cdot \underset{\sim}{\Omega}^{T}$$

$$\underset{\approx}{\Omega} = \underset{\approx}{\omega} + g(I^{\sigma}) \; (\underset{\sim}{D} \cdot \underset{\sim}{\sigma} - \underset{\sim}{\sigma} \cdot \underset{\sim}{D})$$

$$\dot{\underset{\sim}{\sigma}}^{**} = \dot{\underset{\sim}{\sigma}}^{J} - g(I^{\sigma}) \left(\underset{\sim}{\sigma} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{D} + \underset{\sim}{D} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{\sigma} - 2\, \underset{\sim}{\sigma} \cdot \underset{\sim}{D} \cdot \underset{\sim}{\sigma} \right)$$

"proportional loading"

$$\underset{\sim}{D} = \alpha \underset{\sim}{I} + \beta \underset{\sim}{\sigma} \longrightarrow \dot{\underset{\sim}{\sigma}}^{**} = \dot{\underset{\sim}{\sigma}}^{J}$$



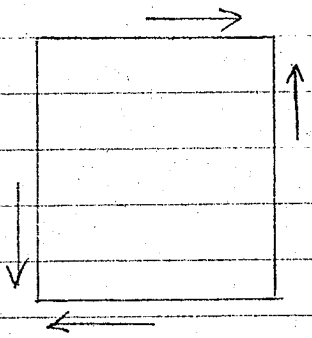
$$\dot{\underset{\sim}{\sigma}}^{J} = \lambda\, \underset{\sim}{I}\,\underset{\sim}{I} : \underset{\sim}{D} + 2\mu \left[\underset{\sim}{D} - \frac{1}{1+\frac{h}{3\mu}} \frac{2\underset{\sim}{\sigma}' \cdot \underset{\sim}{\sigma}'}{2\sigma_y^2} : \underset{\sim}{D} \right]$$

$$+ g(I^{\sigma}) \left(\underset{\sim}{\sigma} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{D} + \underset{\sim}{D} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{\sigma} - 2\, \underset{\sim}{\sigma} \cdot \underset{\sim}{D} \cdot \underset{\sim}{\sigma} \right)$$

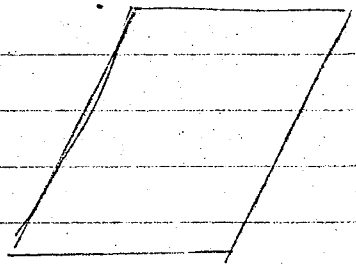
error is in order of :

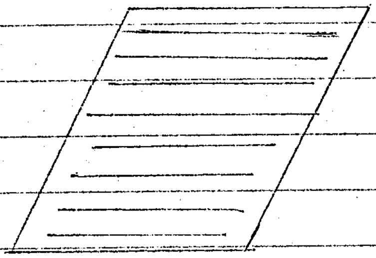
$$O\left(g(I^{\sigma})\, \underset{\sim}{\sigma} : \underset{\sim}{\sigma} / \mu \right) = O\left(\frac{\sigma_y}{\mu} \right) : O(\epsilon^{e})$$

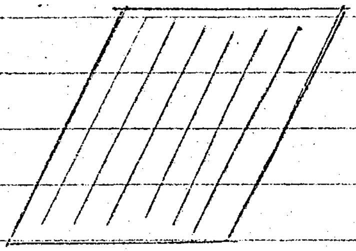
(moduli error)



infinitesimal







layer rotate

Guest Speaker.

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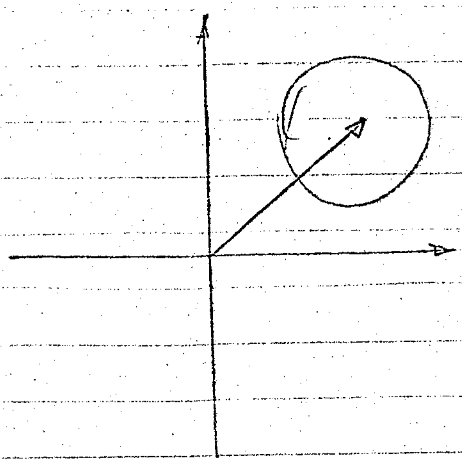

$\dot{\underset{\sim}{\alpha}} = c\,\underset{\sim}{D}^{p}$ evolution law.

$\underset{\sim}{d}^{o} = c\,\underset{\sim}{D}^{p}$

$\overset{o}{\underset{\sim}{\alpha}}_{ij} = \dot{\alpha}_{ij} - (w_{ik} + \overset{n}{\cancel{k}} D_{ik})\alpha_{kj} - \alpha_{ik}\,(w_{ik} + \overset{n}{\cancel{k}} D_{jk}$

$\underset{\sim}{L} = \underset{\sim}{D} + \underset{\sim}{W}$

$n = 1$ Truesdell

$n = 0$ Jaumann

$n = -1$ ~~Cat~~ Carter - Rivlin

Kinematic Hardening:

$3(\underset{\sim}{\sigma}' - \underset{\sim}{\alpha}) : (\underset{\sim}{\sigma}' - \underset{\sim}{\alpha}) = 2\sigma_0^2$

$\underset{\sim}{\alpha}$ = deviatoric tensor

$\dot{\underset{\sim}{\alpha}}$ = deviatoric tensor

$n = 0 \qquad \underset{\sim}{\alpha}^{T} = c\,\underset{\sim}{D}^{p}$

Guest Speaker

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NIKE 2D code

4/28 $\quad \underset{\sim}{\sigma}^{*} = \underset{\sim}{C}\,\underset{\sim}{d}$

$\uparrow$ usual linear elasticity, isotropic

(DIENES paper)

———————— X ————————

$\left\{ \begin{array}{lll} \text{Jaumann} & \underset{\sim}{\omega} & \underset{\sim}{P} \\ \text{Materially Rotated (Green - Naghdi)} & \underset{\sim}{w} & \underset{\sim}{R} \end{array} \right.$

$\longrightarrow \dot{\underset{\sim}{\Omega}} = -\dot{\underset{\sim}{\Omega}}^{T}$ (General spin)

induced rotation denoted $\underset{\sim}{R}$ (proper orthogonal)

$\boxed{\dot{\underset{\sim}{R}} = \dot{\underset{\sim}{\Omega}}\,\underset{\sim}{R}}$; $\underline{\underset{\sim}{R}(t=0) = I_d}$

$\dot{\underset{\sim}{\Omega}}$	$\underset{\sim}{R}$
$\underset{\sim}{\omega}$	$\underset{\sim}{P}$
$\underset{\sim}{w}$	$\underset{\sim}{R}$
"$\underset{\sim}{\Omega}$"	"$\underset{\sim}{R}$"

Nagtagaal

$\underset{\sim}{\sigma}^{*(\underset{\sim}{\Omega})} = \dot{\underset{\sim}{\sigma}} + \underset{\sim}{\sigma}\,\underset{\sim}{\Omega} - \underset{\sim}{\Omega}\,\underset{\sim}{\sigma}$ (assumed objective)

let $\underset{\sim}{\sigma}_{\underset{\sim}{R}} = \underset{\sim}{R}^{T}\,\underset{\sim}{\sigma}\,\underset{\sim}{R}$

$$\Longrightarrow \boxed{(\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}}) = \underset{\sim}{R}^T \underset{\sim}{\sigma}^{*(\underset{\sim}{\Omega})} \underset{\sim}{R}}$$

(everything is the same as before)

$\Longrightarrow$ Class of finite-deformation plasticity theories:

$$\underset{\sim}{\sigma}^{*(\underset{\sim}{\Omega})} = \underset{\sim}{C} \cdot (\underset{\sim}{d} - \underset{\sim}{d}^{pl})$$

$$\Downarrow$$

$$\underset{\sim}{R}(\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}})\underset{\sim}{R}^T = \underset{\sim}{C} \cdot (\underset{\sim}{d} - \underset{\sim}{d}^{pl})$$

$$\Longrightarrow \dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}} = \overbrace{\underset{\sim}{C}_{\underset{\sim}{R}} \cdot (\underset{\sim}{d}_{\underset{\sim}{R}}} - \underset{\sim}{d}_{\underset{\sim}{R}}^{pl}) \qquad \to \dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}}^{tr}$$

def. before

$$\dot{\underset{\sim}{\epsilon}}_{\underset{\sim}{R}} = \underset{\sim}{d}_{\underset{\sim}{R}} \qquad \text{etc.}$$

Mises : $f(\underset{\sim}{\sigma}_{\underset{\sim}{R}}) = k^2$

$\hookrightarrow f(\underset{\sim}{\sigma})$

Flow rule : $\underset{\sim}{d}_{\underset{\sim}{R}}^{pl} = \begin{cases} 0 & (E) \\ \Lambda \underset{\sim}{Q}_{\underset{\sim}{R}} & (P) \end{cases}$ as usual in terms of . . .

$\to$ work quantities with $\underset{\sim}{R}$ subscript

( )

Hardening: Iso. / kin. combine

$$\dot{k} = \beta \frac{1}{\sqrt{3}} H' \bar{d}^{pl}$$

$$\dot{\bar{\epsilon}}^{pl} = \bar{d}^{pl} \text{ (integrate)}$$

$$\Downarrow$$

$$\sqrt{\frac{2}{3}} \left(\underset{\sim}{d}^{pl}_{\underset{\sim}{R}} \cdot \underset{\sim}{d}^{pl}_{\underset{\sim}{R}} \right)^{1/2}$$

or drop

(∵ rotation invariant)

$$\dot{\underset{\sim}{\alpha}}_{\underset{\sim}{R}} = (1-\beta) \frac{2}{3} H' \underset{\sim}{d}^{pl}_{\underset{\sim}{R}}$$

(could write out as: $\underset{\sim}{\alpha}^{*(\underset{\sim}{\Omega})} = (1-\beta) \frac{2}{3} H' \underset{\sim}{d}^{pl}$ (

$$\Downarrow$$

$$\dot{\underset{\sim}{\alpha}} + \underset{\sim}{\alpha} \underset{\sim}{\Omega} - \underset{\sim}{\Omega} \underset{\sim}{\alpha}$$

)

Consistency condition:

$$\Lambda = \underset{\sim}{Q} \cdot \underset{\sim}{\sigma}^{tr}_{\underset{\sim}{R}} \Big/ \left(\underset{\sim}{Q}_{\underset{\sim}{R}} \cdot \underset{\sim}{C}_{\underset{\sim}{R}} \cdot \underset{\sim}{Q}_{\underset{\sim}{R}} + \frac{2}{3} H' \right)$$

$$= \underset{\sim}{Q}_{\underset{\sim}{R}} \cdot \underset{\sim}{d}_{\underset{\sim}{R}} \Big/ \left(1 + \frac{H'}{3\mu} \right)$$

isotropy

of $\underset{\sim}{C}$

×

Numerical Algo's for preceding theory:

Prelim. Ideas.:

①. Must accurately calculate strain ~~rate~~ increments & rotation measures that arise.

②. We'll use radial-return in $\underset{\sim}{R}$ - space, unaltere

③. Recall our theory is objective. We'll insist

algo. is incrementally objective.

① Assume we work over time interval $\Delta t = t_{n+1} - t$

calculate $\Delta \underset{\sim}{\epsilon}$ ($\Delta \underset{\sim}{\Omega}$ if we need it!)

Calculate $\underbrace{\underset{\sim}{R}_n, \ \underset{\sim}{R}_{n+1}}$, $\underbrace{\underset{\sim}{R}_{n+1/2}}$

e.g.: $(\underset{\sim}{\sigma}_{\underset{\sim}{R}})_n = \underset{\sim}{R}_n^T \underset{\sim}{\sigma}_n \underset{\sim}{R}_n$

(To set up calculations

for Radial-Return algo

Assume we have them!

Use them in Rad.-Return →

(ME 239 A)

Assume: $\underset{\sim}{C}$ is isotropic.

All quantities are in $\underset{\sim}{R}$ - space.

(But drop R - subscripts)

$\equiv (\underset{\sim}{\sigma}_{\underset{\sim}{R}})_n$

1. Calc. $\underset{\sim}{\sigma}^{tr}_{n+1} = \underset{\sim}{\sigma}_n + \underset{\sim}{C} \cdot \Delta \underset{\sim}{\epsilon}$

(≠ Isotropic) This is rotated into $\underset{\sim}{R}_{n+1/2}$ - space

2. Calc. $\underset{\sim}{\xi}^{tr}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - \underset{\sim}{\alpha}_n$

3. Calc. $\underset{\sim}{\xi}^{tr'}_{n+1}$

4. If $f(\underset{\sim}{\xi}^{tr}_{n+1}) \leq k_n^2$, Elastic ;

$\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1}$ return,

$(\Longleftrightarrow |\underset{\sim}{\xi}^{tr'}_{n+1}| \leq R_n^2)$

$(\underset{\sim}{\sigma}_{\underset{\sim}{R}})_{n+1}$

Cauchy $\Rightarrow \underset{\sim}{\sigma}_{n+1} = \underset{\sim}{R}_{n+1} (\underset{\sim}{\sigma}_{\underset{\sim}{R}})_{n+1} \underset{\sim}{R}^T_{n+1}$

Else ; plasticity, continue

5. Calc. $\underset{\sim}{Q} = \dfrac{\underset{\sim}{\xi}^{tr'}_{n+1}}{|\underset{\sim}{\xi}^{tr'}_{n+1}|}$

normal

6. $\tilde{\Lambda} = \dfrac{1}{2\mu(1 + \frac{H'}{3\mu})} (|\underset{\sim}{\xi}^{tr'}_{n+1}| - R_n)$

7. Updates : $R_{n+1} = R_n + \frac{2}{3}\beta H' \tilde{\Lambda}$

$$\begin{cases} \underset{\sim}{\alpha}_{n+1} = \underset{\sim}{\alpha}_n + (1-\beta)\frac{2}{3} H' \tilde{\Lambda} \underset{\sim}{Q} \\ \underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - 2\mu \tilde{\Lambda} \underset{\sim}{Q} \end{cases}$$

$$\bar{\epsilon}^{pl}_{n+1} = \bar{\epsilon}^{pl}_n + \sqrt{\tfrac{2}{3}}\,\tilde{\Lambda}$$

$$\longrightarrow \text{Reminder} = \begin{cases} (\underset{\sim}{\alpha}_{\underset{\sim}{R}})_{n+1} \\ (\underset{\sim}{\sigma}_{\underset{\sim}{R}})_{n+1} \end{cases}$$

② ② Algo's to kinematic quantities.

A. Ones for which $\underset{\sim}{R}$ will calculate from ~~a given~~ $\underset{\sim}{\Omega}$ calculated previously.
(e.g. Jaumann, Nagtegaal)

B. $\underset{\sim}{R}$ is calculated first,
(don't have to calculate $\underset{\sim}{\Omega}$)
(e.g. when $\underset{\sim}{R} = \underset{\sim}{R}$)

→ Consider separately.

A → Contributors :

WILKINS
HALLQUIST
GOUDREW
KEY
KRIEG

HUGHES.
NAGTEGAAL
⋮

Consider the motion in a neighborhood of the particle in question to be written as:

$$\underset{\sim}{x}_n = \underset{\sim}{\chi}(\underset{\sim}{X}, t_n) = \underset{\sim}{\chi}_n(\underset{\sim}{X})$$

$$\underset{\sim}{x}_{n+1} = \underset{\sim}{\chi}(\underset{\sim}{X}, t_{n+1}) = \underset{\sim}{\chi}_{n+1}(\underset{\sim}{X})$$

$$\underset{\sim}{X} = \underset{\sim}{\chi}_n^{-1}(\underset{\sim}{x}_n)$$

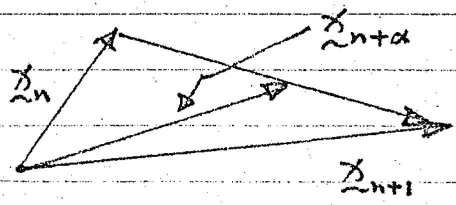
invert

$$\longrightarrow \underset{\sim}{x}_{n+1} = \underset{\sim}{\chi}_{n+1}\left(\underset{\sim}{\chi}_n^{-1}(\underset{\sim}{x}_n)\right) = \underset{\sim}{x}_{n+1}(\underset{\sim}{x}_n)$$

def. $\underset{\sim}{\delta} = \underset{\sim}{x}_{n+1} - \underset{\sim}{x}_n =$ "displacement increment"

def. $\underset{\sim}{x}_{n+\alpha} = (1-\alpha)\,\underset{\sim}{x}_n + \alpha\,\underset{\sim}{x}_{n+1}$

$\alpha \in [0,1]$ → 1 parameter (α) family of configurations.



Disp. - Inc. Gradient Tensor (w.r.t. $\underset{\sim}{x}_{n+\alpha}$) G:

$$G_{ij} = \frac{\partial \delta_i}{\partial x_j^{n+\alpha}}$$

$$\text{def.}\quad \begin{cases} \Delta\underset{\sim}{\epsilon} = (\underset{\sim}{G} + \underset{\sim}{G}^T)/2 \\ \text{"}\quad \cancel{\Delta\underset{\sim}{\Omega}}\ \underset{\sim}{\psi} = (\underset{\sim}{G} - \underset{\sim}{G}^T)/2 \end{cases}$$

(It is <u>crucial</u> for accuracy to use $\alpha = \frac{1}{2}$!)

——————— X ———————

5/3 $\quad \dot{\underset{\sim}{\epsilon}} = \underset{\sim}{d}\ ;\ \dot{\underset{\sim}{\psi}} = \underset{\sim}{w}$

$$\text{Def.}\quad \begin{cases} \Delta\underset{\sim}{\epsilon} = (\underset{\sim}{G} + \underset{\sim}{G}^T)/2 \\ \underset{\sim}{\psi} = (\underset{\sim}{G} - \underset{\sim}{G}^T)/2 \end{cases}$$

Continue with Kinematics:

(A.) $\quad \underset{\sim}{\Omega} \rightarrow \underset{\sim}{R} \quad \left(\underset{\sim}{\Omega} \neq \underset{\sim}{w} \neq \underset{\sim}{\Omega}(\underset{\sim}{w}) \right)$

objective: To calculate R's:

$$\underset{\sim}{R}_n,\ \underset{\sim}{R}_{n+\frac{1}{2}},\ \underset{\sim}{R}_{n+1}$$

assume given $\qquad$?

$\underset{\sim}{R}_{n+\frac{1}{2}}$ is needed to give radial return an accurate $\underbrace{\Delta\underset{\sim}{\epsilon}_{\underset{\sim}{R}}}$

"above" α - configuratio

$\Downarrow \frac{1}{2}$

$$(\Delta\underset{\sim}{\epsilon}_{\underset{\sim}{R}})_{n+\frac{1}{2}} = \underset{\sim}{R}^T_{n+\frac{1}{2}}\ \Delta\underset{\sim}{\epsilon}\ \underset{\sim}{R}_{n+\frac{1}{2}}$$

First calculate $\underset{\sim}{R}_{n+1}$:

recall $\dot{\underset{\sim}{R}} = \dot{\underset{\sim}{\Omega}}\,\underset{\sim}{R}$

Time discretize (via generalized midpoint rule)

$$\frac{\underset{\sim}{R}_{n+1} - \underset{\sim}{R}_n}{\cancel{\Delta t^{\nabla}}} = \underbrace{\frac{\Delta \underset{\sim}{\Omega}}{\cancel{\Delta t^{\nabla}}}\Bigg|_{n+\alpha}}_{\text{def. } \Delta\underset{\sim}{\Omega}} \left[(1-\alpha)\,\underset{\sim}{R}_n + \alpha\,\underset{\sim}{R}_{n+1}\right]$$

$(\alpha = \frac{1}{2})$

$$(\underset{\sim}{I} - \alpha\Delta\underset{\sim}{\Omega})\,\underset{\sim}{R}_{n+1} = \underbrace{(\underset{\sim}{I} + (1-\alpha)\Delta\underset{\sim}{\Omega})}_{((\underset{\sim}{I} - \alpha\Delta\underset{\sim}{\Omega}) + \Delta\underset{\sim}{\Omega})}\,\underset{\sim}{R}_n$$

$$\boxed{\underset{\sim}{R}_{n+1} = \left(\underset{\sim}{I} + (\underset{\sim}{I} - \alpha\Delta\underset{\sim}{\Omega})^{-1}\Delta\underset{\sim}{\Omega}\right)\underset{\sim}{R}_n}$$

$\Delta\underset{\sim}{\Omega} = ?$: example:

1. $\Delta\underset{\sim}{\psi}$ (Jaumann) $\dot{\underset{\sim}{\Omega}} = \dot{\underset{\sim}{\psi}} = \underset{\sim}{\omega}$

2. Nagtegaal: $\dot{\underset{\sim}{\Omega}} = \underset{\sim}{\omega} + s\,(\underset{\sim}{d}\,\underset{\sim}{\alpha} - \underset{\sim}{\alpha}\,\underset{\sim}{d})$

scalar "back stres

or $\underset{\sim}{\bar{\sigma}}$

$$\Delta\underset{\sim}{\Omega} = \Delta\underset{\sim}{\psi} + s\Big|_n \left(\Delta\underset{\sim}{\epsilon}\,\underset{\sim}{\alpha}\Big|_n - \underset{\sim}{\alpha}\Big|_n \Delta\underset{\sim}{\epsilon}\right.$$

or $\underset{\sim}{\sigma}_n$

———— X ————

How good is $\underset{\sim}{R}_{n+1}$?

Facts: (Hughes - Winget)

assume:

1. Motion of neighborhood of point we are dealing is rigid.

$$\underset{\sim}{x}_{n+1} = \underbrace{\underset{\sim}{c}}_{\substack{\text{Const. veloc} \\ \text{(translation)}}} + \underbrace{\underset{\sim}{Q}}_{\substack{\text{proper orthogonal} \\ \text{matrix (rotation)}}} \underset{\sim}{x}_n$$

2. $\alpha = \frac{1}{2}$

3. $\underset{\sim}{Q} + \underset{\sim}{I}$ is nonsingular.

then:

1. $\Delta \underset{\sim}{\epsilon} = \underset{\sim}{0}$ (no strain increment for a given rigid motion)

2. $\underline{\underset{\sim}{R}_{n+1} \underset{\sim}{R}_n^T} = \underset{\sim}{Q}$ (rotation over the ste is exactly calulated

3. Algo. is incrementally objective.

i.e. $(\underset{\sim}{\sigma}_{\underset{\sim}{R}})_{n+1} = (\underset{\sim}{\sigma}_{\underset{\sim}{R}})_n$

$$\Longleftrightarrow \underset{\sim}{R}_{n+1}^T \underset{\sim}{\sigma}_{n+1} \underset{\sim}{R}_{n+1} = \underset{\sim}{R}_n^T \underset{\sim}{\sigma}_n \underset{\sim}{R}_n$$

$$\Longleftrightarrow \underset{\sim}{\sigma}_{n+1} = \underbrace{(\underset{\sim}{R}_{n+1} \underset{\sim}{R}_n)}^T \underset{\sim}{\sigma}_n (\underset{\sim}{R}_{n+1} \underset{\sim}{R}_n)^T$$

$$\Longleftrightarrow \underset{\sim}{\sigma}_{n+1} = \underset{\sim}{Q} \underset{\sim}{\sigma}_n \underset{\sim}{Q}^T$$

proof: (H-W. paper)

———————— X ————————

$\underset{\sim}{R}_{n+\frac{1}{2}}$? (various possibilities)

work in 2D:

define: $\underset{\sim}{q} = \underset{\sim}{R}_{n+1} \underset{\sim}{R}_n^T \left(= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \right)$

"angle of rotation for the step"

define: $\underset{\sim}{P}$ s.t. it represents "half" the rotation of $\underset{\sim}{q}$

i.e. $\underset{\sim}{R}_{n+\frac{1}{2}} = \underset{\sim}{P}\, \underset{\sim}{R}_n$

$$\underset{\sim}{R}_{n+1} = \underset{\sim}{P}\, \underset{\sim}{R}_{n+\frac{1}{2}} = \underset{\sim}{P}^2\, \underset{\sim}{R}_n$$

~~half~~ have

$$\underset{\sim}{R}_{n+1} = \underset{\sim}{q}\, \underset{\sim}{R}_n$$

so $\underset{\sim}{P}^2 = \underset{\sim}{q} \implies \underset{\sim}{P}$ is "square root" of $\underset{\sim}{q}$

solve for $\underset{\sim}{P} = \underset{\sim}{P}(\underset{\sim}{q})$

How: Cayley - Hamilton theorem:

$$\underset{\sim}{P}^2 - \underset{\uparrow}{I_{\underset{\sim}{P}}}\, \underset{\sim}{P} + \underset{\uparrow}{II_{\underset{\sim}{P}}}\, \underset{\sim}{I} = \underset{\sim}{0} \qquad (*)$$

$$\left(\underset{\sim}{P},\ \underset{\sim}{Q} \text{ are proper-orthogonal} \Rightarrow \det \underset{\sim}{P} = \det \underset{\sim}{Q} = +1 \right)$$

↖ not use initially

$$\underset{\sim}{Q} = \underset{\sim}{P}^2$$

$$\det \underset{\sim}{Q} = (\det \underset{\sim}{P})^2 \Rightarrow \det \underset{\sim}{P} = \sqrt{\det \underset{\sim}{Q}}$$

↑ as long as we know it is positive

sub. into (*)

$$\underset{\sim}{Q} - (\operatorname{tr} \underset{\sim}{P})\, \underset{\sim}{P} + \sqrt{\det \underset{\sim}{Q}}\ \underset{\sim}{I} = \underset{\sim}{0} \qquad (**)$$

[1 ,]

tr (of eq.)

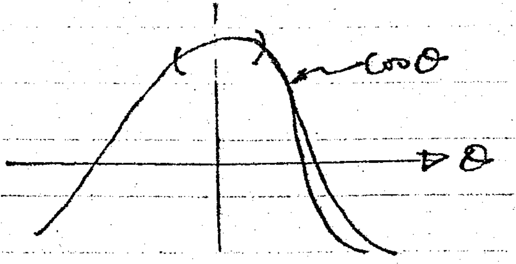
$$\operatorname{tr} \underset{\sim}{Q} - (\operatorname{tr} \underset{\sim}{P})^2 + 2\sqrt{\det \underset{\sim}{Q}} = 0$$

$$\operatorname{tr} \underset{\sim}{P} = \boxed{+} \left(\operatorname{tr} \underset{\sim}{Q} + 2\sqrt{\det \underset{\sim}{Q}} \right)^{1/2}$$

↑ sign?

recall θ is rotation in one step ∴ "small"

$$\underbrace{\operatorname{tr} \underset{\sim}{Q}}_{\approx 2} = 2 \underbrace{\cos\theta}_{\approx 1}$$



$$\underset{\sim}{P} = \begin{bmatrix} \cos\theta/2 & .. \\ .. & .. \end{bmatrix}$$

$$\operatorname{tr} P = 2\cos\theta/2 \approx 2 > 0$$

solve (**) for $\underset{\sim}{P}$:

$$\Longrightarrow \boxed{\underset{\sim}{P} = \frac{1}{\sqrt{\operatorname{tr}\underset{\sim}{q} + 2\sqrt{\det \underset{\sim}{q}}}}\left(\underset{\sim}{q} + \sqrt{\det \underset{\sim}{q}}\ \underset{\sim}{I}\right)}$$

↑ hold for prop. orthogonal (P. positive def. too)

→ Simplifies $\det \underset{\sim}{q} = +1$

$$\underset{\sim}{P} = \frac{1}{\sqrt{2 + \operatorname{tr}\underset{\sim}{q}}}\left(\underset{\sim}{q} + \underset{\sim}{I}\right)$$

then $\underset{\sim}{R}_{n+\frac{1}{2}} = \underset{\sim}{P}\,\underset{\sim}{R}_n$

→ Verify 1. $\underset{\sim}{P}^2 = \underset{\sim}{q}$ directly

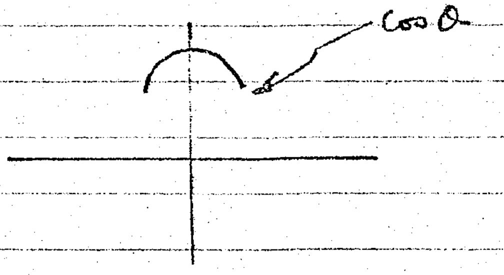
2. $\underset{\sim}{P}\,\underset{\sim}{P}^T = \underset{\sim}{I}$ 〃

——————— × ———————

Alternative approach:

$$\underset{\sim}{q} = \begin{bmatrix} \cos\theta & \sin\theta \\ \underset{\sin\theta}{-\cancel{\sin\theta}} & \cos\theta \end{bmatrix}$$

↗ have



use ½-angle form. from trig.

$$(1 \approx)\ \cos\frac{\theta}{2} = \oplus\sqrt{\frac{1+\cos\theta}{2}}$$

$$\frac{\theta}{2} = \text{arc } \cos^{-1}\left(\sqrt{\frac{1+\cos\theta}{2}}\right)$$

$$\underset{\sim}{P} = \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

X

"B." $\underset{\sim}{R} = \underset{\sim}{R}$ (of polar decomposition)

$$\dot{\underset{\sim}{R}} = \underset{\uparrow\text{ derived quantity}}{\underset{\sim}{w}}\,\underset{\sim}{R}$$

I. Calculate $\Delta\underset{\sim}{\epsilon}$ in the same way as before in "A."

II. $\left.\begin{matrix}\underset{\sim}{R}_n \\ \underset{\sim}{R}_{n+1}\end{matrix}\right\}$ via polar decomposition

$$\Longrightarrow \underset{\sim}{R}_{n+\frac{1}{2}} \text{ as in "A."}$$

Algo: 1.

$$\underset{\sim}{F}_n = \frac{\partial \underset{\sim}{x}_n}{\partial \underset{\sim}{X}} = \underset{\sim}{R}_n\,\underset{\sim}{U}_n$$

same at n+1

(drop n's)

$$\underset{\sim}{F} = \underset{\uparrow\text{ want}}{\overset{?}{\underset{\sim}{R}}}\,\underset{\sim}{U}$$

$$\underset{\sim}{U}^2 = \underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F}$$

symm positive definit

$$\underset{\sim}{U} = \underset{\sim}{U}(\underset{\sim}{C}) = \underset{\sim}{C}^{1/2}$$

Via cay-Ham. as before:

$$\underset{\sim}{U} = \frac{1}{\sqrt{\operatorname{tr}\underset{\sim}{C} + 2\sqrt{\det \underset{\sim}{C}}}}\left(\underset{\sim}{C} + \sqrt{\det \underset{\sim}{C}}\;\underset{\sim}{I}\right)$$

explicit

$$\boxed{\underset{\sim}{R} = \underset{\sim}{F}\,\underset{\sim}{U}^{-1}}$$ <u>explicitly</u>

$$\underset{\sim}{\mathcal{R}} = \underset{\sim}{R}$$

5/5 $\underset{\sim}{\mathcal{R}}_{n+1} = \underset{\sim}{R}_{n+1}$ from $\underset{\sim}{x}_{n+1}$

$\underset{\sim}{R}_{n+1/2}$: various possibilities

① Calculate $\underset{\sim}{q}\,, \Longrightarrow \underset{\sim}{P}$

$$\underset{\sim}{\mathcal{R}}_{n+\frac{1}{2}} = \underset{\sim}{P}\,\underset{\sim}{\mathcal{R}}_{n}$$

② Trig. Approach

→ ③ $\underset{\sim}{x}_{n+\alpha} = (1-\alpha)\,\underset{\sim}{x}_{n} + \alpha\,\underset{\sim}{x}_{n+1}$

$\alpha = \frac{1}{2}$

$$\frac{\partial \underset{\sim}{x}_{n+1/2}}{\partial \underset{\sim}{X}} \equiv \underset{\sim}{F}_{n+\frac{1}{2}} = \underset{\sim}{R}_{n+\frac{1}{2}}\,\underset{\sim}{U}_{n+\frac{1}{2}}$$

↑exact, explicit

$$\boxed{\underset{\sim}{\mathcal{R}}_{n+\frac{1}{2}} = \underset{\sim}{R}_{n+\frac{1}{2}} = \underset{\sim}{F}_{n+\frac{1}{2}}\,\underset{\sim}{U}^{-1}_{n+\frac{1}{2}}}$$ (Nike 2D)

×

Remark: Cay-Ham. , can be done in 3D also ; (i) $\underset{\sim}{U}^{2} = \underset{\sim}{C}$

(ii) $\underset{\sim}{P}^{2} = \underset{\sim}{q}$

×

Compare : $\Delta \underset{\sim}{\epsilon}$ $(\alpha = \frac{1}{2})$ with $\int_{t_n}^{t_{n+1}} \underset{\sim}{d}(\underset{\sim}{X}, t)\, dt$

$$\left[\begin{array}{c} "\ \dot{\sigma} = c\,\dot{\epsilon}\ " \\ "\ \underbrace{\Delta\sigma} = c\ \underbrace{\Delta\epsilon}\ " \end{array}\right]$$

start from

1. 1D uniaxial ext.

$$x = x(X, t) = \underbrace{\lambda(t)}_{\text{stretch}} X$$

$$x_0 = X \Longleftrightarrow \lambda(0) = \lambda_0 = 1$$

$$F_{,X} = \lambda$$

$$x_n = x(X, t_n)$$

$$x_{n+1} = x(X, t_{n+1})$$

$\underset{\sim}{d}$: $\quad d = v_{,x} = l$

Chain rule

exact : $$\int_{t_n}^{t_{n+1}} v_{,x}\, dt \underset{\cdot}{=} \int_{t_n}^{t_{n+1}} v_{,X}\, \underbrace{(X, x)}_{(x_{,X})^{-1}}\, dt = \int_{t_n}^{t_{n+1}} \dot{F}\, F^{-1}\, dt$$

aside : $v_{,X}$

$$v = \dot{x} = \left.\frac{\partial}{\partial t}\right|_X x(X, t)$$

$$v_{,X} = \dot{x}_{,X} \underset{\cdot}{=} (\overline{x_{,X}})^{\cdot} = \dot{F}$$ / $(\dot{\ })$ and $_{,X}$ commu

$$\int_{t_n}^{t_{n+1}} \dot{F} F^{-1} \, dt = \ln F \Big|_{t_n}^{t_{n+1}} = \ln \frac{F_{n+1}}{F_n}$$

$$= \ln \underbrace{\frac{\lambda_{n+1}}{\lambda_n}}_{1+\gamma}$$

definition of γ $\quad \gamma = \frac{\lambda_{n+1}}{\lambda_n} - 1$

exactly: $\boxed{\Delta \epsilon = \ln(1+\gamma)}$

X

Approximate: (algo.)

$$x_{n+\alpha} = \alpha x_{n+1} + (1-\alpha) x_n$$

$$= (\alpha \lambda_{n+1} + (1-\alpha) \lambda_n) X$$

$$\delta = x_{n+1} - x_n = (\lambda_{n+1} - \lambda_n) X$$

$$G = \frac{\partial \delta}{\partial x_{n+\alpha}} = \frac{\partial \delta}{\partial X} \underbrace{\frac{\partial X}{\partial x_{n+\alpha}}}$$

$$\downarrow$$

$$= \quad \text{''} \quad \left(\frac{\partial x_{n+\alpha}}{\partial X} \right)^{-1}$$

$$= (\lambda_{n+1} - \lambda_n) \left(\alpha \lambda_{n+1} + (1-\alpha) \lambda_n \right)^{-1}$$

$$= \frac{\left(\frac{\lambda_{n+1}}{\lambda_n} - 1 \right)}{\left(\alpha \frac{\lambda_{n+1}}{} + (1-\alpha) \right)}$$

$$= \frac{\gamma}{(\alpha(1+\gamma)+1-\alpha)}$$

$$\Delta\epsilon = G = \frac{\gamma}{1+\alpha\gamma}$$

G: symmetric part

$\Delta\epsilon$: algo. (approx.)

"small" $\ll 1$

exact : $\Delta\epsilon^{ex} = \ln(1+\gamma)$

$$= \gamma - \frac{\gamma^2}{2} + \frac{\gamma^3}{3} - \cdots\cdots$$

algo. : $\Delta\epsilon^{al.} = \frac{\gamma}{1+\alpha\gamma} = \gamma\left(1-\alpha\gamma+(\alpha\gamma)^2-\cdots\right)$

$$= \gamma - \alpha\gamma^2 + \alpha\gamma^3 - \cdots\cdots$$

leading term are the same

α could be anything (convergent)

(Remark: 1. Agree to $O(\gamma)$ $\forall\alpha$

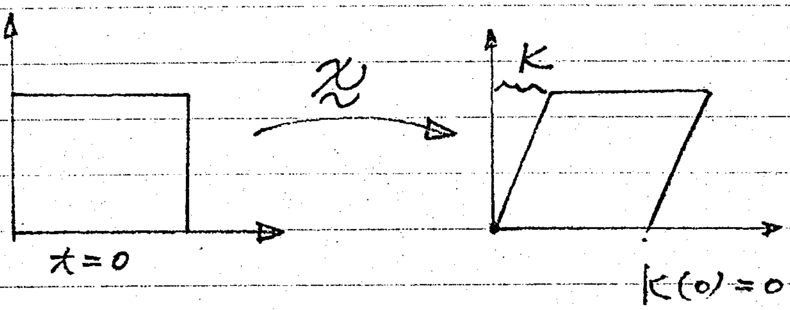
2. $\alpha = \frac{1}{2} \Longleftrightarrow O(\gamma^2)$

2nd-order accuracy)

2. Simple shear (2D)

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t) = \underset{\sim}{F}(t)\, \underset{\sim}{X}$$

$$\underset{\sim}{F}(t) \Downarrow \begin{bmatrix} 1 & K(t) \\ 0 & 1 \end{bmatrix}$$



$$\int_{t_n}^{t_{n+1}} \overset{\underset{\sim}{v}_{,\underset{\sim}{x}}}{\underset{\sim}{\ell}}\, dt$$

$$\underset{\sim}{v}_{,\underset{\sim}{x}} = \underset{\sim}{v}_{,\underset{\sim}{X}} \frac{\partial \underset{\sim}{X}}{\partial \underset{\sim}{x}} = \dot{\underset{\sim}{x}}_{,\underset{\sim}{X}} \left(\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} \right)^{-1} = \dot{\underset{\sim}{F}}\, \underset{\sim}{F}^{-1}$$

$$\int_{t_n}^{t_{n+1}} \dot{\underset{\sim}{F}}\, \underset{\sim}{F}^{-1}\, dt$$

$$= \int_{t_n}^{t_{n+1}} \underbrace{\begin{bmatrix} 0 & \dot{K} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -K \\ 0 & 1 \end{bmatrix}}_{\begin{bmatrix} 0 & \dot{K} \\ 0 & 0 \end{bmatrix}} dt$$

$$\begin{bmatrix} 0 & \dot{K} \\ 0 & 0 \end{bmatrix} \Longleftrightarrow \left(\dot{\underset{\sim}{F}}\, \underset{\sim}{F}^{-1} = \dot{\underset{\sim}{F}} \right)$$

↖ not in general

$$\text{Exact} = \begin{bmatrix} 0 & K_{n+1} - K_n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Delta K \\ 0 & 0 \end{bmatrix}$$

Algo. : $\underset{\sim}{G} \approx \int_{t_n}^{t_{n+1}} \underset{\sim}{v}_{,\underset{\sim}{x}}\, dt$

$$\underset{\sim}{G} = \frac{\partial \underset{\sim}{\delta}}{\partial \underset{\sim}{x}_{n+\alpha}} \underset{\underset{\text{chain rule}}{\uparrow}}{=} \frac{\partial \underset{\sim}{\delta}}{\partial \underset{\sim}{X}} \left(\frac{\partial \underset{\sim}{x}_{n+\alpha}}{\partial \underset{\sim}{X}} \right)^{-1}$$

$$\underset{\sim}{x}_{n+1} = \begin{bmatrix} 1 & K_{n+1} \\ 0 & 1 \end{bmatrix} \underset{\sim}{X}$$

$$\underset{\sim}{x}_{n} = \begin{bmatrix} 1 & K_{n} \\ 0 & 1 \end{bmatrix} \underset{\sim}{X}$$

$$\underset{\sim}{\delta} = \begin{bmatrix} 0 & \Delta K \\ 0 & 0 \end{bmatrix} \underset{\sim}{X}$$

$$\underset{\sim}{x}_{n+\alpha} = \left((1-\alpha) \begin{bmatrix} 1 & K_{n} \\ 0 & 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 & K_{n+1} \\ 0 & 1 \end{bmatrix} \right) \underset{\sim}{X}$$

$$= \begin{bmatrix} 1 & (1-\alpha) K_n + \alpha K_{n+1} \\ 0 & 1 \end{bmatrix} \underset{\sim}{X}$$

$$\longrightarrow \underset{\sim}{G} = \begin{bmatrix} 0 & \Delta K \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & (1-\alpha) K_n + \alpha K_{n+1} \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & \Delta K \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -(1-\alpha) K_n + \alpha K_{n+1} \\ 0 & 1 \end{bmatrix}$$

$$\text{algo.} \longrightarrow \underset{\sim}{G} = \begin{bmatrix} 0 & \Delta K \\ 0 & 0 \end{bmatrix} \quad \text{VOILA!} \quad \equiv \text{exact} \int^{t_{n+1}} \underset{\sim}{v}_{,\underset{\sim}{x}}\, dt$$

Remarks: 1. exact for all α

$\Longrightarrow$ Conclusion $\alpha = \frac{1}{2}$!

X

PROB. (Case 3) From ME 235C FINAL:

let $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t) = \underset{\sim}{F}(t)\,\underset{\sim}{X}$ (2 D)

def. $\underset{\sim}{F}(t) = \begin{bmatrix} \lambda(t) & K(t) \\ 0 & 1 \end{bmatrix}$

superposes stretching and shear.

Obtain expressions for:

$$\int_{t_n}^{t_{n+1}} \underset{\sim}{l}\, dt \qquad (\text{"exact"})$$

$$\underset{\sim}{l} \Downarrow \underset{\sim}{v}_{,\underset{\sim}{x}}$$

and $\underset{\sim}{G} = \dfrac{\partial \underset{\sim}{\delta}}{\partial \underset{\sim}{x}_{n+\alpha}}$ ("Approx.")

let $\dfrac{\lambda_{n+1}}{\lambda_n} = 1 + c_1 \Delta t$; $\Delta K = c_2 \Delta t$

constants

show that "Approx." $\approx$ "Exact"

to $O(\Delta t^2)$ when $\alpha = \frac{1}{2}$.

(Hint. observe that

$$\int_{t_n}^{t_{n+1}} K \frac{\dot{\lambda}}{\lambda}\, dt = \left(\frac{K_{n+1} + K_n}{2}\right) \int^{t_{n+1}} \frac{\dot{\lambda}}{\lambda}\, dt + O(\Delta t^2)$$

PROJECT ⟺ PROB. (Simple shear)

1. How to Calculate kinematical quantities for simple shear.

$$\Delta\underset{\sim}{\epsilon} = (\underset{\sim}{G}+\underset{\sim}{G}^{T})/2 \quad , \quad \Delta\underset{\sim}{\psi} = (\underset{\sim}{G}-\underset{\sim}{G}^{T})/2$$

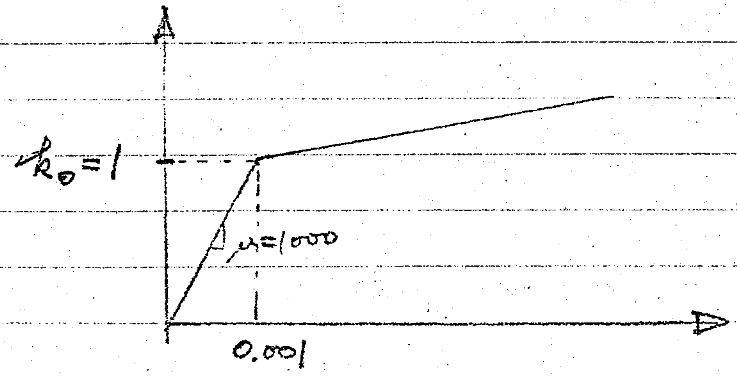
$$\uparrow \ \frac{\partial \underset{\sim}{\delta}}{\partial \underset{\sim}{x}_{n+\alpha}} \ , \quad \underset{\sim}{G} = \begin{bmatrix} 0 & \Delta K \\ 0 & 0 \end{bmatrix}$$

$$\Delta\underset{\sim}{\epsilon} \rightarrow \frac{1}{2}\begin{bmatrix} 0 & \Delta K \\ \Delta K & 0 \end{bmatrix} \qquad \Delta\underset{\sim}{\psi} \rightarrow \frac{1}{2}\begin{bmatrix} 0 & \Delta K \\ -\Delta K & 0 \end{bmatrix}$$

$$0 \le K \le 10$$

$\rightarrow$ Try $\Delta K = 1, \underline{0.1}, 0.01$

↑ accurate enough



$\lambda = 0$ $\quad$ $(\underline{\nu = 0}, \ E = 2\mu = 2000)$

↑ Lame parameter

↑ no delatation in shear

5/10 Case # 1 : Jaumann Rate

$$\underset{\sim}{R} \overset{df.}{=} \underset{\sim}{P} \qquad \dot{\underset{\sim}{\Omega}} \overset{df.}{=} \underset{\sim}{\omega} = \dot{\underset{\sim}{\psi}}$$

general rotation

say $\underset{\sim}{R}_0 = \underset{\sim}{I}$

$$\underset{\sim}{R}_{n+1} = \left(\underset{\sim}{I} + \left(\underset{\sim}{I} - \frac{1}{2} \Delta \underset{\sim}{\psi} \right)^{-1} \Delta \underset{\sim}{\psi} \right) \underset{\sim}{R}_n$$

$$\Delta \underset{\sim}{\psi} \leftarrow \frac{1}{2} \begin{bmatrix} 0 & \Delta K \\ -\Delta K & 0 \end{bmatrix}$$

$$\underset{\sim}{R}_{n+\frac{1}{2}} = \underset{\sim}{P} \, \underset{\sim}{R}_n$$

$$\underset{\sim}{q} = \underset{\sim}{R}_{n+1} \underset{\sim}{R}_n^T$$

$$\underset{\sim}{P} = \frac{1}{\sqrt{2 + tr\, \underset{\sim}{q}}} \left(\underset{\sim}{q} + \underset{\sim}{I} \right)$$

pass $(\Delta \underset{\sim}{\epsilon}_{\underset{\sim}{R}})_{n+\frac{1}{2}}$ to radial return

$$= \underset{\sim}{R}^T_{n+\frac{1}{2}} \Delta \underset{\sim}{\epsilon} \, \underset{\sim}{R}_{n+\frac{1}{2}}$$

$$\Downarrow$$

$$\frac{1}{2} \begin{bmatrix} 0 & \Delta K \\ \Delta K & 0 \end{bmatrix}$$

$$\xrightarrow{} \text{2x2 matrix} \begin{bmatrix} \cdots_{11} , & \cdots_{12} \\ \cdots_{21} , & \cdots_{22} \end{bmatrix} \rightarrow$$

EPSI (1) = $\cdots_{11}$

EPSI (2) = $\cdots_{22}$

EPSI (3) = 0

EPSI (4) = $\cdots_{12}$

Radial return calculations :

$$(\underset{\sim}{\sigma}_{\underset{\sim}{R}})_{n+1} \quad ; \quad (\underset{\sim}{\alpha}_{\underset{\sim}{R}})_{n+1}$$

SAVE

$$\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{R}_{n+1} (\underset{\sim}{\sigma}_{\underset{\sim}{R}})_{n+1} \underset{\sim}{R}^{T}_{n+1}$$

$$\underset{\sim}{\alpha}_{n+1} = \;'' \; (\underset{\sim}{\alpha}_{\underset{\sim}{R}})_{n+1} \; ''$$

Global (i.e. spatial) coordinates

Strictly for output : Don't save !!

$$\underset{\sim}{\sigma}_{n+1} = \begin{bmatrix} SIG(1) & SIG(4) \\ SIG(4) & SIG(2) \end{bmatrix}$$

$$\underset{\sim}{\alpha}_{n+1} = \begin{bmatrix} ALPHA(1) & (4) \\ (4) & (2) \end{bmatrix}$$

output (Computer plot)

$$\frac{\sigma_{11}}{\mu}, \ \frac{\sigma_{22}}{\mu}, \ \frac{\sigma_{12}}{\mu} \quad \text{v.s. } K$$

$$\frac{\alpha_{11}}{\mu}, \ \frac{\alpha_{22}}{\mu}, \ \frac{\alpha_{12}}{\mu} \quad \text{v.s. } K.$$

only $\neq 0$ in kinematic hardening

Case #2: Same except use a rate proposed by Nagtegaal's lecture.

$\Delta \underset{\sim}{\epsilon}$ same

$$\Delta \underset{\sim}{\Omega} = \Delta \underset{\sim}{\psi} + \begin{cases} \underset{\sim}{0} \text{ ; if tresca intensity} = 0 \\ \dfrac{1}{\text{TRESCA INTENSITY}} \left(\Delta \underset{\sim}{\epsilon}\, \underset{\sim}{\alpha}_n - \underset{\sim}{\alpha}_n \Delta \underset{\sim}{\epsilon} \right) \text{ ; if TRESCA IN.} \neq 0 \end{cases}$$

in global

tresca intensity

$= \alpha_{max} - \alpha_{min}$

Max. eigenvalue of $\underset{\sim}{\alpha}_n$

$$\underset{\sim}{\alpha} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ one eigenvalue = 0

Eigenvalues : $(\alpha_1, \alpha_2, 0)$

$$\underset{\sim}{R}_{n+1} = \left(\underset{\sim}{I} + \left(\underset{\sim}{I} - \tfrac{1}{2} \Delta \underset{\sim}{\Omega} \right)^{-1} \Delta \underset{\sim}{\Omega} \right) \underset{\sim}{R}_n$$

rest are the same as #1

Case 2 = Case 1 for isotropic hardening

Case # 3 : Materially Rotated rate $\underset{\sim}{R} = \underset{\sim}{R}$

(no $\Delta \underset{\sim}{\Omega}$ required)

$\Delta \underset{\sim}{\epsilon}$ same

Calculate $\underset{\sim}{F}_n = \begin{bmatrix} 1 & x_n \\ 0 & 1 \end{bmatrix}$

$$\underset{\sim}{C}_n = \underset{\sim}{F}_n^T \underset{\sim}{F}_n = \begin{bmatrix} \; \end{bmatrix}$$

$$\underset{\sim}{U}_n = \frac{1}{\sqrt{tr\, \underset{\sim}{C}_n + 2\sqrt{\det \underset{\sim}{C}_n}}} \left(\underset{\sim}{C}_n + \sqrt{\det \underset{\sim}{C}_n}\, \underset{\sim}{I}_n \right)$$

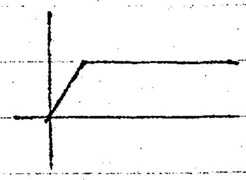
$$\underset{\sim}{R}_n = \underset{\sim}{F}_n \underset{\sim}{U}_n^{-1}$$

$$\left. \begin{matrix} \underset{\sim}{R}_{n+1} \\ \underset{\sim}{R}_{n+\frac{1}{2}} \end{matrix} \right\} \text{SAME idea}$$

$$\left\{ \begin{matrix} \underset{\sim}{F}_{n+1} = \begin{bmatrix} 1 & x_{n+1} \\ 0 & 1 \end{bmatrix} \\ \underset{\sim}{F}_{n+\frac{1}{2}} = \begin{bmatrix} 1 & x_{n+\frac{1}{2}} \\ 0 & 1 \end{bmatrix} \end{matrix} \right.$$

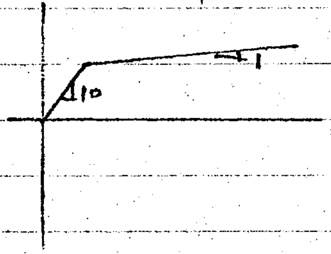
3 Cases $\begin{cases} \text{BJ.} \longrightarrow \text{Ⓐ} \\ \text{N.} \longrightarrow H' = \frac{1}{9},\ \beta = 0,\ \text{KIN.} \\ \text{R.} \longrightarrow \text{Ⓐ} \end{cases}$

Ⓐ = $\begin{cases} H' = 0 & \text{perfect plasticity} \\ H' = \frac{1}{9} \end{cases}$



3 cases.

$\frac{E_T}{E} = \frac{1}{10}$



$\begin{cases} \beta = 1 & \text{ISO.} \\ \beta = 0 & \text{KIN.} \end{cases}$

→ Total 7 cases ($\Delta X = 1,\ 0.1,\ 0.01$)

———— X ————

$\Delta \underset{\sim}{\epsilon}$ ← provide insight into the nature of this quantity

$$\begin{cases} \underset{\sim}{x}_{n+1} = \underset{\sim}{x}_{n+1}(\underset{\sim}{x}_{n+\alpha}) \\ \underset{\sim}{x}_{n} = \underset{\sim}{x}_{n}(\underset{\sim}{x}_{n+\alpha}) \end{cases}$$

$$\underset{\sim}{x}_{n+\alpha} = (1-\alpha)\underset{\sim}{x}_n + \alpha\, \underset{\sim}{x}_{n+1}$$

$$d\underset{\sim}{x}_{n+1} = \frac{\partial \underset{\sim}{x}_{n+1}}{\partial \underset{\sim}{x}_{n+\alpha}}\, d\underset{\sim}{x}_{n+\alpha}$$

$$d\underset{\sim}{x}_{n} = \frac{\partial \underset{\sim}{x}_{n}}{\partial \underset{\sim}{x}_{n+\alpha}}\, d\underset{\sim}{x}_{n+\alpha}$$

[e.g. $dx_n^i = \frac{\partial x_n^i}{\;}\, dx_{n+\alpha}^j$ (j summed)]

Change in length of a "differential" line element

$$\underset{\underset{d\underset{\sim}{x}_{n+1}\cdot d\underset{\sim}{x}_{n+1}}{\uparrow}}{d\underset{\sim}{x}_{n+1}^{2}} - \underset{\underset{\text{etc.}}{\uparrow}}{d\underset{\sim}{x}^{2}} = \left(\frac{\partial x^{k}_{n+1}}{\partial x^{i}_{n+\alpha}} \frac{\partial x^{k}_{n+1}}{\partial x^{j}_{n+\alpha}} - \frac{\partial x^{k}_{n}}{\partial x^{i}_{n+\alpha}} \frac{\partial x^{k}_{n}}{\partial x^{j}_{n+\alpha}} \right)$$

$$\cdot\, dx^{i}_{n+\alpha}\, dx^{j}_{n+\alpha}$$

sum on i, j, k

$$= \underbrace{(\;.\;.\;.\;.\;)_{ij}}_{\text{def.}}\, dx^{i}_{n+\alpha}\, dx^{j}_{n+\alpha}$$

$$= 2\left(\Delta \underset{\sim}{\epsilon}_{n+\alpha}\right)_{ij}\, dx^{i}_{n+\alpha}\, dx^{j}_{n+\alpha}$$

$$\underset{\sim}{x}_{n+\alpha} = (1-\alpha)\underset{\sim}{x}_{n} + \alpha\, \underset{\sim}{x}_{n+1}$$

$$\underset{\sim}{\delta} = \underset{\sim}{x}_{n+1} - \underset{\sim}{x}_{n}$$

$$\underset{\sim}{x}_{n+\alpha} = (1-\alpha)\underset{\sim}{x}_{n} + \alpha\left(\underset{\sim}{x}_{n} + \underset{\sim}{\delta}\right) = \underset{\sim}{x}_{n} + \alpha\, \underset{\sim}{\delta}$$

$$\underset{\sim}{x}_{n+\alpha} = (1-\alpha)\left(\underset{\sim}{x}_{n+1} - \underset{\sim}{\delta}\right) + \alpha\, \underset{\sim}{x}_{n+1}$$

$$= \underset{\sim}{x}_{n+1} - (1-\alpha)\,\underset{\sim}{\delta}$$

$$\begin{cases} \underset{\sim}{x}_{n} = \underset{\sim}{x}_{n+\alpha} - \alpha\, \underset{\sim}{\delta} \\ \underset{\sim}{x}_{n+1} = \underset{\sim}{x}_{n+\alpha} + (1-\alpha)\, \underset{\sim}{\delta} \end{cases}$$

$$\frac{\partial \underset{\sim}{x}_{n+1}}{\partial \underset{\sim}{x}_{n+\alpha}} = \underset{\sim}{I} + (1-\alpha)\frac{\partial \underset{\sim}{\delta}}{\partial \underset{\sim}{x}_{n+\alpha}} = \underset{\sim}{I} + (1-\alpha)\,\underset{\sim}{G}_{n+\alpha}$$

$$\underset{\sim}{G}_{n+\alpha} \equiv \underset{\sim}{G}$$

$$\frac{\partial \underset{\sim}{x}_{n}}{\partial \underset{\sim}{x}_{n+\alpha}} = \underset{\sim}{I} - \alpha\,\underset{\sim}{G}$$

$$2(\Delta \underset{\sim}{\epsilon}_{n+\alpha})_{ij} = \left[\underset{\sim}{I} + (1-\alpha)\underset{\sim}{G}\right]^T\left[\underset{\sim}{I} + (1-\alpha)\underset{\sim}{G}\right] - \left[\underset{\sim}{I} - \alpha\underset{\sim}{G}\right]^T\left[\underset{\sim}{I} - \alpha\underset{\sim}{G}\right]$$

$$2(\Delta \underset{\sim}{\epsilon}_{n+\alpha})_{ij} = \left(\delta_{ki} + (1-\alpha)G_{ki}\right)\left(\delta_{kj} + (1-\alpha)G_{kj}\right) - \left(\delta_{ki} - \alpha G_{ki}\right)\left(\delta_{kj} - \alpha G_{kj}\right)$$

$$= \delta_{ij} + (1-\alpha)(G_{ij} + G_{ji}) + (1-\alpha)^2 G_{ki}G_{kj} - \delta_{ij} + \alpha\,(G_{ij} + G_{ji}) - \alpha^2 G_{ki}G_{kj}$$

$(1-\alpha)^2 = 1 - 2\alpha + \alpha^2$

$$= G_{ij} + G_{ji} + (1-2\alpha)\,G_{ki}G_{kj}$$

$$\boxed{2\Delta \underset{\sim}{\epsilon}_{n+\alpha} = \underset{\sim}{G}_{n+\alpha} + \underset{\sim}{G}^T_{n+\alpha} + (1-2\alpha)\,\underset{\sim}{G}^T_{n+\alpha}\underset{\sim}{G}_{n+\alpha}}$$

α	$2\Delta \underset{\sim}{E}_{n+\alpha}$	Remarks
0	$\underset{\sim}{G}_n + \underset{\sim}{G}_n^T + \underset{\sim}{G}_n^T \underset{\sim}{G}_n$	incremental Lagrangian $\Delta \underset{\sim}{E}$
1	$\underset{\sim}{G}_{n+1} + \underset{\sim}{G}_{n+1}^T \cancel{+} - \underset{\sim}{G}_{n+1}^T \underset{\sim}{G}_{n+1}$	incremental Eulerian $\Delta \underset{\sim}{e}$
$\frac{1}{2}$	$\underset{\sim}{G}_{n+\frac{1}{2}} + G_{n+\frac{1}{2}}^T$	Midpoint

Guest Speaker : Alan Miller

5/12 1. Why are unified constitutive equations necessary?

2. Brief survey of a few approaches

3. The kinds of predictions which can be made with one approach ("MATMOD")

4. Numerical integration problems and a [illegible] new approach ("NONSS") to solving them.

Topics:

Unified Constitutive equations for time-dependent plasticity

A. K. MILLER

5/17 definition:

$$\underbrace{d\underset{\sim}{x}^2_{n+1} - dx^2_n} = \cancel{2\Delta\underset{\sim}{\epsilon}_{n+\alpha}}\ 2\,d\underset{\sim}{x}_{n+\alpha} \cdot \Delta\underset{\sim}{\epsilon}_{n+\alpha} \cdot d\underset{\sim}{x}_{n+\alpha}$$

$$\forall\alpha$$

$$= 2(\Delta\underset{\sim}{\epsilon}_{n+\alpha})_{ij}\, dx^i_{n+\alpha}\, dx^j_{n+\alpha}$$

$$= 2(\Delta\underset{\sim}{\epsilon}_{n+\alpha})_{ij}\, \underbrace{\frac{\partial x^i_{n+\alpha}}{\partial x^k_{n+\beta}}\, dx^k_{n+\beta}}\ \underbrace{\frac{\partial x^j_{n+\alpha}}{\partial x^l_{n+\beta}}\, dx^l_{n+\beta}} \qquad \forall\beta$$

$$\beta=\tfrac{1}{2};\quad = 2(\Delta\underset{\sim}{\epsilon}_{n+\alpha})_{ij}\, \frac{\partial x^i_{n+\alpha}}{\partial x^k_{n+\frac{1}{2}}}\, \frac{\partial x^j_{n+\alpha}}{\partial x^l_{n+\frac{1}{2}}}\, \underbrace{dx^k_{n+\frac{1}{2}}\, dx^l_{n+\frac{1}{2}}}$$

$$\alpha=0;\quad = 2(\Delta\underset{\sim}{\epsilon}_n)_{ij}\, \frac{\partial x^i_n}{\partial x^k_{n+\frac{1}{2}}}\, \frac{\partial x^j_n}{\partial x^l_{n+\frac{1}{2}}}\, dx^k_{n+\frac{1}{2}}\, dx^l_{n+\frac{1}{2}}$$

$$\alpha=\tfrac{1}{2};\quad = 2(\Delta\underset{\sim}{\epsilon}_{n+\frac{1}{2}})_{ij}\, \delta^i_k\, \delta^j_l\, dx^k_{n+\frac{1}{2}}\, dx^l_{n+\frac{1}{2}}$$

$$\alpha=1;\quad = 2(\Delta\underset{\sim}{\epsilon}_{n+1})_{ij}\, \frac{\partial x^i_{n+1}}{\partial x^k_{n+\frac{1}{2}}}\, \frac{\partial x^j_{n+1}}{\partial x^l_{n+\frac{1}{2}}}\, dx^k_{n+\frac{1}{2}}\, dx^l_{n+\frac{1}{2}}$$

$$(\Delta\underset{\sim}{\epsilon}_{n+\frac{1}{2}})_{kl} = (\Delta\underset{\sim}{\epsilon}_n)_{ij}\, \boxed{\frac{\partial x^i_n}{\partial x^k_{n+\frac{1}{2}}}}\, \frac{\partial x^j_n}{\partial x^l_{n+\frac{1}{2}}}$$

$$g^i_k$$

$$= (\Delta\underset{\sim}{\epsilon}_{n+1})_{ij}\, \boxed{\frac{\partial x^i_{n+1}}{\partial x^k_{n+\frac{1}{2}}}}\, \frac{\partial x^j_{n+1}}{\partial x^l_{n+\frac{1}{2}}}$$

$$f^i$$

$$\begin{array}{ccccc} \Delta\underset{\sim}{\epsilon}_{n+\frac{1}{2}} & = & \underset{\sim}{g}^{T}\,\Delta\underset{\sim}{\epsilon}_{n}\,\underset{\sim}{g} & = & \underset{\sim}{f}^{T}\,\Delta\underset{\sim}{\epsilon}_{n+1}\,\underset{\sim}{f} \\ \uparrow & & \uparrow & & \uparrow \\ \Delta\underset{\sim}{\epsilon} & & \Delta\underset{\sim}{E} & & \Delta\underset{\sim}{e} \end{array}$$

$\Delta\underset{\sim}{\epsilon} \Leftrightarrow \Delta\underset{\sim}{E}$ if $\underset{\sim}{g}$ is invertible

$\Delta\underset{\sim}{\epsilon} \Leftrightarrow \Delta\underset{\sim}{e}$ if $\underset{\sim}{f}$ " "

X

$$\Delta\underset{\sim}{\epsilon} = \int_0^t \underset{\sim}{d}\, dt + O(\Delta t^3)$$

second-order accurate

$\uparrow$

This is why we used it

X

Classical set-up:

$\underset{\sim}{x}_n \leftarrow \underset{\sim}{X}$ initial coordinates

$\underset{\sim}{x}_{n+1} \leftarrow \underset{\sim}{x}$ current "

$\underset{\sim}{x}_{n+\frac{1}{2}} \leftarrow \underset{\sim}{y}$ midpoint "

$\underset{\sim}{\delta} \leftarrow \underset{\sim}{u}$ Total displ.

$$2\underset{\sim}{\epsilon} = \text{midpoint} = \underbrace{\left(\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{y}}\right) + \left(\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{y}}\right)^T}$$

$$2\underset{\sim}{E} = \text{Lagr.} = \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{X}} + \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{X}}^{T} + \left(\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{X}}\right)^{T}\left(\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{X}}\right)$$

$$2\underset{\sim}{e} = \text{Eul.} = \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}} + \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}}^{T} - \left(\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}}\right)^{T}\left(\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}}\right)$$

$$\underset{\sim}{g} \longleftarrow \frac{\partial \underset{\sim}{X}}{\partial \underset{\sim}{y}}$$

$$\underset{\sim}{f} \longleftarrow \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}}$$

$$\underset{\sim}{\epsilon} = \underset{\sim}{g}^{T}\underset{\sim}{E}\,\underset{\sim}{g} = \underset{\sim}{f}^{T}\underset{\sim}{e}\,\underset{\sim}{f}$$

Different rates used in formulating ~~large~~ large-deformation el-pl. constitutive eq.

recall:

$$\boxed{\underset{\sim}{\sigma}^{*} = \underset{\sim}{C}^{el\text{-}pl}\,\underset{\sim}{d}}$$

some objective rate

usual

Try different objective rates; see what happen

$$\underset{\sim}{\sigma}^{*} = \underset{\sim}{C}^{elastic}\,\underset{\sim}{d}$$

classical coefficients

Hypoelastic, Grade zero

problem is infinite possibilities

$$\frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{x}}$$

$$\underset{\sim}{\sigma}^{*} \overset{①}{=} \dot{\underset{\sim}{\sigma}} - \underset{\sim}{\sigma}\,\underset{\sim}{l}^{T} - \underset{\sim}{l}\,\underset{\sim}{\sigma} \qquad (\text{OLDROYD})$$

$$\overset{②}{=} \dot{\underset{\sim}{\sigma}} - \underset{\sim}{\sigma}\,\underset{\sim}{l}^{T} + \underset{\sim}{l}^{T}\underset{\sim}{\sigma} \qquad ?$$

$$\overset{③}{=} \dot{\underset{\sim}{\sigma}} + \underset{\sim}{\sigma}\,\underset{\sim}{l} - \underset{\sim}{l}\,\underset{\sim}{\sigma} \qquad ?$$

$$\overset{④}{=} ① + \underbrace{(\operatorname{div} \underset{\sim}{v})}_{\operatorname{tr} \underset{\sim}{l}}\,\underset{\sim}{\sigma} \qquad (\text{TRUESDELL})$$

$\underset{\sim}{l} = \underset{\sim}{d} + \underset{\sim}{w}$

↑ get rid of $\underset{\sim}{d}$ in rigid body motion

$$\overset{⑤}{=} \dot{\underset{\sim}{\sigma}} + \underset{\sim}{\sigma}\,\underset{\sim}{l} + \underset{\sim}{l}^{T}\underset{\sim}{\sigma} \qquad (\text{COTTER-RIVLIN})$$

×

Exercises : 1. Verify each is objective

2. $$\frac{① + ⑤}{2} = \frac{② + ③}{2} = \text{Jaumann}$$

3. $$\underbrace{\dot{\underset{\sim}{\sigma}} + \underset{\sim}{\sigma}\,\underset{\sim}{w} - \underset{\sim}{w}\,\underset{\sim}{\sigma}}_{\text{Jaumann}} + \underset{\uparrow\ \text{scalar}}{s}\,\underbrace{(\underset{\sim}{\sigma}\,\underset{\sim}{d} - \underset{\sim}{d}\,\underset{\sim}{\sigma})}_{\downarrow}$$

could repla

$\underset{\sim}{\sigma}$ by $\underset{\sim}{\alpha}$

verify objective

$$\underset{\sim}{\omega} \leftarrow \underset{\sim}{w} \Rightarrow \dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T}$$

$$\underset{\text{or}}{\leftarrow} (1-\gamma)\,\underset{\sim}{\omega} + \gamma\,\underset{\sim}{w}$$

×

MORAL: Objective is not enough. Must consider $\underset{\sim}{d} \neq \underset{\sim}{\Omega}$ to define the constitutive eq.

Think outloud: → Scheme for devel. reasonable constitutive eqs.

Recall structure of theories so far:

1. Elastic constitutive law ←
2. Yield Function
3. Definition of elastic process } ⇒ ① + ②
   " " plastic "

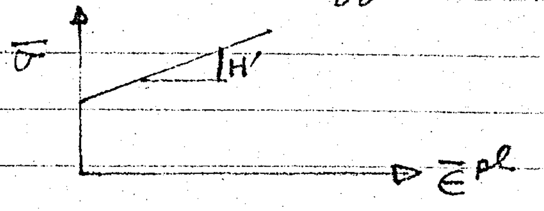
$$\underbrace{\dot{\sigma} = \dot{\sigma}(\sigma, \dot{\epsilon})}$$

4. In a plastic process, the elastic constitutive law is modified

$$\dot{\underset{\sim}{\sigma}} = \dot{\underset{\sim}{\sigma}}^{tr} + \text{something} \quad \text{assumed} \sim \dot{\underset{\sim}{\epsilon}}^{pl}$$

5. "Hardening" { 1. Perfect plasticity.
   2. Iso.
   3. Kin.

→ Effective stress - effective strain



6. $\dot{\underset{\sim}{\epsilon}}^{pl} = \Lambda \underset{\sim}{Q}$ flow rule

5/24 Guest Speaker:

5/26 Develope large-deformation plasticity theories:

desirable features:

1. Math. coherent theory

$$(\text{strain history}) \longrightarrow (\text{stress history})$$

$\Longrightarrow$ (E), (P) def's.

2. Insure objectivity
3. The elastic constitutive eq. that we start with should emanate from the theory of <u>hyperelasticity</u> ←

4. The theory should be able to reproduce basic physical response.

——————— X ———————

~~Hp~~ Hyperelasticity: (sketch)

Claim hyperelas. can be written in rate form as follows:

$$\underset{\sim}{\sigma}^{*(\text{Truesdell})} = \underset{\sim}{c} \cdot \underset{\sim}{d} \qquad (1)$$

$$\underset{\sim}{c} = \underset{\sim}{c}(\underset{\sim}{E})$$

$\underset{\sim}{c}$ ~~desif~~ derives from an assumed strain energy density function.

$\underset{\sim}{C}$ has usual symmetry:

$$\left.\begin{array}{l} C_{ijkl} = C_{klij} \\ C_{\underbrace{ij}\underbrace{kl}} = C_{\underbrace{ji}\underbrace{kl}} = C_{ijlk} \end{array}\right\} \text{for hyperelastic}$$

Exercises: show that

1. $\dot{\underset{\sim}{\sigma}} - \hat{\underset{\sim}{C}} \cdot \underset{\sim}{d} - \underset{\sim}{S} \cdot \underset{\sim}{w} = \underset{\sim}{\sigma}^{*\,(\text{Truesdell})}$

where $\hat{\underset{\sim}{C}} = \hat{\underset{\sim}{C}}(\underset{\sim}{\sigma})$, $\underset{\sim}{S} = \underset{\sim}{S}(\underset{\sim}{\sigma})$

$$\hat{C}_{ijkl} = -\sigma_{ij}\delta_{kl} + (\sigma_{il}\delta_{jk} + \sigma_{jl}\delta_{ik} + \sigma_{ik}\delta_{jl} + \sigma_{jk}\delta_{il})/$$

$$S_{ijkl} = (\sigma_{il}\delta_{jk} + \sigma_{jl}\delta_{ik} - \sigma_{ik}\delta_{jl} - \sigma_{jk}\delta_{il})/2$$

2. $\underset{\sim}{S} \cdot \underset{\sim}{w} = -\underset{\sim}{\sigma}\,\underset{\sim}{w} + \underset{\sim}{w}\,\underset{\sim}{\sigma}$

———————————— X ————————————

Exercises 7 (1) imply:

$$\Longrightarrow \dot{\underset{\sim}{\sigma}} = \bar{\underset{\sim}{C}} \cdot \underset{\sim}{d} + \underset{\sim}{S} \cdot \underset{\sim}{w}$$

(hint: 1. $\cdots = \underset{\sim}{C} \cdot \underset{\sim}{d}$ $\hookrightarrow \hat{\underset{\sim}{C}}, \underset{\sim}{S}$)

$$\bar{\underset{\sim}{C}} = \underbrace{\underset{\sim}{C}} + \underbrace{\hat{\underset{\sim}{C}}}$$

$$\hat{\underset{\sim}{C}} = \hat{\underset{\sim}{C}}(\underset{\sim}{\sigma})$$

$$\longrightarrow \underset{\sim}{C} = \underset{\sim}{C}(\underset{\sim}{F}) \hookleftarrow + \text{material constant.}$$

Exercise 2 imply:

$$\Rightarrow \underbrace{\dot{\underset{\sim}{\sigma}} - \underset{\sim}{s}\cdot\underset{\sim}{w}}_{\text{Jaumann}} = \underset{\sim}{\bar{c}} \cdot \underset{\sim}{d}$$

X

Exercise: Consider the Kirchhoff stress

$$\underset{\sim}{\tau} = \underset{\substack{\Downarrow \\ \cancel{dJ}\,\det \underset{\sim}{F}}}{J}\,\underset{\sim}{\sigma}$$

clearly: $\dot{\underset{\sim}{\tau}} = \dot{J}\,\underset{\sim}{\sigma} + J\,\dot{\underset{\sim}{\sigma}}$

→ show: $\underset{\sim}{\tau}^{*(\text{Jaumann})} = J\left(\underset{\sim}{\sigma}^{*(\text{Jaum.})} + \frac{\dot{J}}{J}\underset{\sim}{\sigma}\right)$

$$\frac{\dot{J}}{J} = \text{div}\,\underset{\sim}{v} = \text{tr}\,\underset{\sim}{d} = \text{tr}\,\underset{\sim}{D}$$

X

Hyperelasticity (summary):

$\underset{\sim}{S}$ = second Piola-Kirchhoff stress

$$\underset{\sim}{\sigma} = J^{-1}\underset{\sim}{F}\,\underset{\sim}{S}\,\underset{\sim}{F}^{T} \quad ; \quad \underbrace{\underset{\sim}{S}}_{\to\ \text{symm.}} = J\,\underset{\sim}{F}^{-1}\underset{\sim}{\sigma}\,\underset{\sim}{F}^{-T}$$

$$\to \sigma_{ij} = J^{-1} F_{iA}\underbrace{S_{AB}}_{\to\ \text{"material" - coordinate indice}} F_{jB}$$

Hyperelas. → there exist strain energy function $\Phi(\underset{\sim}{E})$

$\uparrow$

lagrangian strain

$= \frac{1}{2}(\underbrace{\underset{\sim}{F}^T \underset{\sim}{F}}_{\underset{\sim}{C}} - \underset{\sim}{I})$

$$E_{AB} = \frac{1}{2}(F_{iA} F_{iB} - \delta_{AB})$$

$$S_{AB} = \hat{S}_{AB}(\underset{\sim}{E}) = \frac{\partial \Phi}{\partial E_{AB}}$$

$$\sigma_{ij} = \underset{\uparrow}{J^{-1}} \underbrace{F_{iA}} S_{AB}(\underset{\sim}{E}) \underbrace{F_{jB}} = \hat{\sigma}_{ij}(\underset{\sim}{E})$$

$$\left\{ \begin{array}{l} J = \det \underset{\sim}{F} \\ \det(\underbrace{\underset{\sim}{F}^T \underset{\sim}{F}}_{\underset{\sim}{C}}) = \det \underset{\sim}{F}^T \cdot \det \underset{\sim}{F} = (\det \underset{\sim}{F})^2 = J^2 \\ J = (\det \underset{\sim}{C})^{1/2} \xleftarrow{\quad} \underset{\sim}{C} = 2\underset{\sim}{E} + \underset{\sim}{I} \end{array} \right.$$

$$\underset{\sim}{\sigma} = J^{-1} \underset{\sim}{F} \underset{\sim}{S} \underset{\sim}{F}^T \qquad \text{rate eq. ?}$$

take material time derivative

$\underset{\sim}{\sigma}^{*}$ (Truesdell

$$\dot{\underset{\sim}{\sigma}} = \dot{\overline{(J^{-1})}} \underset{\sim}{F} \underset{\sim}{S} \underset{\sim}{F}^T + J^{-1}(\dot{\underset{\sim}{F}} \underset{\sim}{S} \underset{\sim}{F}^T + \underbrace{\underset{\sim}{F} \dot{\underset{\sim}{S}} \underset{\sim}{F}^T}_{\downarrow\, \underset{\sim}{\bar{c}} \cdot \underset{\sim}{d}} + \underset{\sim}{F} \underset{\sim}{S} \dot{\underset{\sim}{F}}^T)$$

→ leads to (1)

sketch some steps (~~ex~~ exercise: fill in details in indices)

$$\dot{\underset{\sim}{F}} = \frac{\partial \dot{\underset{\sim}{x}}}{\partial \underset{\sim}{X}} = \frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{X}} = \frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{x}} \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} = \underset{\sim}{l}\, \underset{\sim}{F}$$

$$\overline{(J^{-1})}^{\,\cdot} = \frac{-1}{J^2} \dot{J} :$$

$$J = \det \underset{\sim}{F} = f(\underset{\sim}{F})$$

$$\dot{J} = \frac{\partial f}{\partial F_{iA}} \dot{F}_{iA} = (\text{Cofactor } F_{iA})\, \dot{F}_{iA}$$

$$= (J \underset{\sim}{F}^{-T})_{iA} \dot{F}_{iA}$$

$$= J (\underset{\sim}{F}^{-1})_{Ai} \dot{F}_{iA}$$

$$= J \operatorname{tr}(\dot{\underset{\sim}{F}} \underset{\sim}{F}^{-1}) = J \operatorname{tr}(\underset{\sim}{l})$$

$$= J \operatorname{div} \underset{\sim}{v}$$

——————— × ———————

$$\dot{\underset{\sim}{S}} = \overline{\left(\frac{\partial \Phi}{\partial \underset{\sim}{E}}\right)}^{\,\cdot} = \frac{\partial^2 \Phi}{\partial \underset{\sim}{E}\, \partial \underset{\sim}{E}} \cdot \dot{\underset{\sim}{E}}$$

$$\dot{S}_{AB} = \underbrace{\left(\frac{\partial^2 \Phi}{\partial E_{AB}\, \partial E_{CD}}\right)}_{C_{\underbrace{AB}\underbrace{CD}}(\underset{\sim}{E})} \dot{E}_{CD}$$

major symm. & minor symm.

$$\dot{\underset{\sim}{E}} = \frac{1}{2}(\dot{\underset{\sim}{F}}^T \underset{\sim}{F} + \underset{\sim}{F}^T \dot{\underset{\sim}{F}})$$

5/31 Rotational Formula : (Interlude)

$$\underset{\sim}{R}_{n+1} = \underbrace{\left(\underset{\sim}{I} + \overbrace{(\underset{\sim}{I} - \underbrace{\alpha \Delta \underset{\sim}{\Omega}}_{\longrightarrow \underset{\sim}{\theta}})}^{\underset{\sim}{r}}{}^{-1} \Delta \underset{\sim}{\Omega}\right)}_{\underset{\sim}{g}} \underset{\sim}{R}_n$$

simplify via ~~C.H. theorem~~ Cayley-Hamilton theory (2-D firs

$$\underset{\sim}{r}^2 - (\mathrm{tr}\,\underset{\sim}{r})\underset{\sim}{r} + (\det \underset{\sim}{r})\underset{\sim}{I} = \underset{\sim}{0}$$

$$\underset{\sim}{r} - (\mathrm{tr}\,\underset{\sim}{r})\underset{\sim}{I} + (\det \underset{\sim}{r})\underset{\sim}{r}^{-1} = \underset{\sim}{0}$$

$$\underset{\sim}{r}^{-1} = \frac{1}{\det \underset{\sim}{r}}\left((\mathrm{tr}\,\underset{\sim}{r})\underset{\sim}{I} - \underset{\sim}{r}\right)$$

$$\underset{\sim}{r} = \underset{\sim}{I} - \underset{\sim}{\theta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\mathrm{tr}\,\underset{\sim}{r} = 2$$

$$\det \underset{\sim}{r} = 1 + \theta^2$$

$$\underset{\sim}{r}^{-1} = \frac{1}{1+\theta^2}\left(2\underset{\sim}{I} - \underset{\underset{\underset{\sim}{I}-\underset{\sim}{\theta}}{\Downarrow}}{\underset{\sim}{r}}\right)$$

$$= \frac{1}{1+\theta^2}\left(\underset{\sim}{I} + \underset{\sim}{\theta}\right)$$

$$\boxed{\underset{\sim}{g} = \underset{\sim}{I} + \frac{1}{\alpha(1+\theta^2)} \underbrace{(\underset{\sim}{I} + \underset{\sim}{\theta})\underset{\sim}{\theta}}_{\theta + \theta^2}}$$

$$\underset{\sim}{\theta}^2 = \theta^2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \theta^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\theta^2 \underset{\sim}{I}$$

$$\underset{2-D}{\equiv} \qquad \underset{\sim}{g} = \underset{\sim}{I} + \frac{1}{\alpha(1+\theta^2)} \left(\underset{\sim}{\theta} - \theta^2 \underset{\sim}{I} \right)$$

3-D case: (Cayley-Hamilton)

$$\underset{\sim}{Y}^3 - (\mathrm{tr}\,\underset{\sim}{Y})\,\underset{\sim}{Y}^2 + \underset{\Downarrow \atop \text{sum of principal minors.}}{(II_{\underset{\sim}{Y}})}\,\underset{\sim}{Y} - (\det \underset{\sim}{Y})\,\underset{\sim}{I} = \underset{\sim}{0}$$

$$\underset{\sim}{Y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ & 0 & \theta_1 \\ \text{skew} & & 0 \end{bmatrix} \qquad \theta_1 = \alpha \Delta \Omega_1$$

... etc.

$$\mathrm{tr}\,\underset{\sim}{Y} = 3$$

$$\det \underset{\sim}{Y} = \begin{bmatrix} 1 & -\theta_3 & +\theta_2 \\ +\theta_3 & 1 & -\theta_1 \\ -\theta_2 & \theta_1 & 1 \end{bmatrix}$$

$$= 1 - \theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_3 + \underbrace{(\theta_2^2 + \theta_1^2 + \theta_3^2)}_{\text{define} = \theta^2}$$

$$= 1 + \theta^2$$

$$\mathrm{II}_{\underset{\sim}{r}} = \begin{bmatrix} 1 & -\theta_3 & \theta_2 \\ \theta_3 & 1 & -\theta_1 \\ -\theta_2 & \theta_1 & 1 \end{bmatrix} = 1+\theta_1^2+1+\theta_2^2+1+\theta_3^2$$

$$= 3+\theta^2$$

$$\underset{\sim}{r}^{-1} = \frac{1}{\det \underset{\sim}{r}} \left(\underset{\sim}{r}^2 - (\mathrm{tr}\, \underset{\sim}{r})\, \underset{\sim}{r} + \mathrm{II}_{\underset{\sim}{r}}\, \underset{\sim}{I} \right)$$

$$= \frac{1}{1+\theta^2} \left(\underset{\sim}{r}^2 - 3\underset{\substack{\Downarrow \\ \underset{\sim}{I} - \underset{\sim}{\Theta}}}{\underset{\sim}{r}} + 3\underset{\sim}{I} + \theta^2 \underset{\sim}{I} \right)$$

$$= \frac{1}{1+\theta^2} \left(\underset{\sim}{I} - 2\underset{\sim}{\Theta} + \underset{\sim}{\Theta}^2 \cancel{- 3\underset{\sim}{I}} + 3\underset{\sim}{\Theta} \cancel{+ 3\underset{\sim}{I}} + \theta^2 \underset{\sim}{I} \right)$$

$$= \frac{1}{(1+\theta^2)} \left(\underset{\sim}{I} + \underset{\sim}{\Theta} + \underbrace{\left(\underset{\sim}{\Theta}^2 + \theta^2 \underset{\sim}{I} \right)}_{\neq 0 \quad (\text{verify!})} \right)$$

However $\underset{\sim}{\Theta}^3 + \theta^2 \underset{\sim}{\Theta} = 0$ (verifi

$\underset{\sim}{\Theta}$ skew in 3-D

$$\underset{\sim}{Q} = \underset{\sim}{I} + \frac{1}{\alpha(1+\theta^2)} \left[(\underset{\sim}{I} + \underset{\sim}{\Theta})\underset{\sim}{\Theta} + \underbrace{\left(\underset{\sim}{\Theta}^3 + \theta^2 \underset{\sim}{\Theta} \right)}_{=0} \right]$$

3-D: $$\boxed{\underset{\sim}{Q} = \underset{\sim}{I} + \frac{1}{\alpha(1+\theta^2)} \left(\underset{\sim}{\Theta} + \underset{\sim}{\Theta}^2 \right)}$$

continue

$$\dot{\underset{\sim}{\sigma}} = \overset{\cdot}{\overline{J^{-1}}}\,\underset{\sim}{F}\,\underset{\sim}{S}\,\underset{\sim}{F}^{T} + J^{-1}\left(\dot{\underset{\sim}{F}}\,\underset{\sim}{S}\,\underset{\sim}{F}^{T} + \underset{\sim}{F}\,\dot{\underset{\sim}{S}}\,\underset{\sim}{F}^{T} + \underset{\sim}{F}\,\underset{\sim}{S}\,\dot{\underset{\sim}{F}}^{T}\right)$$

$$J^{-1}\,\mathrm{div}\,\underset{\sim}{v} \qquad \underset{\sim}{l}\,\underset{\sim}{F} \qquad \therefore\ \underset{\sim}{F}^{T}\underset{\sim}{l}^{T}$$

$$\dot{\underset{\sim}{S}} = \underset{\sim}{C}\cdot\dot{\underset{\sim}{E}}$$

$$\dot{\underset{\sim}{E}} = \frac{1}{2}\left(\underbrace{\dot{\underset{\sim}{F}}^{T}}_{\Downarrow\ F^{T}\underset{\sim}{l}^{T}}\underset{\sim}{F} + \underset{\sim}{F}^{T}\underbrace{\dot{\underset{\sim}{F}}}_{\Downarrow\ \underset{\sim}{l}\,\underset{\sim}{F}}\right)$$

$$= \frac{1}{2}F^{T}\underbrace{(\underset{\sim}{l}+\underset{\sim}{l}^{T})}_{\underset{\sim}{d}}\underset{\sim}{F}$$

$$\boxed{\dot{\underset{\sim}{E}} = \underset{\sim}{F}^{T}\underset{\sim}{d}\,\underset{\sim}{F}}$$

$$\dot{\underset{\sim}{\sigma}} = (-\mathrm{div}\,\underset{\sim}{v})\,\underset{\sim}{\sigma} + \underset{\sim}{l}\,\underset{\sim}{\sigma} + \underset{\sim}{\sigma}\,\underset{\sim}{l}^{T} + \underbrace{\underset{\sim}{F}\left(\underset{\sim}{C}\cdot(\underset{\sim}{F}^{T}\underset{\sim}{d}\,\underset{\sim}{F})\right)\underset{\sim}{F}^{T}J}$$

$$\underbrace{\dot{\underset{\sim}{\sigma}} - \underset{\sim}{\sigma}\,\underset{\sim}{l}^{T} - \underset{\sim}{l}\,\underset{\sim}{\sigma} + (\mathrm{div}\,\underset{\sim}{v})\,\underset{\sim}{\sigma}}_{\underset{\sim}{\sigma}^{*}\text{(truesdell)}} = \underset{\sim}{c}\cdot\underset{\sim}{d}$$

$$\Downarrow$$

$$\underset{\sim}{c}(\underset{\sim}{F})$$

$$= J^{-1}F_{iA}\left(C_{ABCD}\,F_{kC}\,d_{kl}\,F_{lD}\right)F_{jB}$$

$$= \underbrace{\left(J^{-1}\left[F_{iA}\,F_{jB}\,F_{kC}\,F_{lD}\,C_{ABCD}\right]\right)}_{c_{ijkl}(\underset{\sim}{F})}d_{kl}$$

$$\underset{\sim}{\chi}_{*}(\underset{\sim}{C})$$

push forward of $\underset{\sim}{C}$

$$\boxed{\underset{\sim}{\sigma}^{*(\text{Trues.})} = \underset{\sim}{C}(\underset{\sim}{F}) \cdot \underset{\sim}{d}}$$

$$\downarrow$$

$$= \dot{\underset{\sim}{\sigma}} - \hat{\underset{\sim}{C}}(\underset{\sim}{\sigma}) \cdot \underset{\sim}{d} - \underset{\sim}{S}(\underset{\sim}{\sigma}) \cdot \underset{\sim}{\omega}$$

$$\boxed{\dot{\underset{\sim}{\sigma}} = \bar{\underset{\sim}{C}}(\underset{\sim}{\sigma}, \underset{\sim}{F}) \cdot \underset{\sim}{d} + \underset{\sim}{S}(\underset{\sim}{\sigma}) \cdot \underset{\sim}{\omega}}$$

$$\bar{\underset{\sim}{C}} = \underset{\sim}{C}(\underset{\sim}{F}) + \hat{\underset{\sim}{C}}(\underset{\sim}{\sigma})$$

×

Exercise: Show

$$\underset{\sim}{\tau}^{*(\text{oldroyd})} = J\left(\underset{\sim}{\sigma}^{*(\text{oldroyd})} + (\text{div.}\,\underset{\sim}{v})\underset{\sim}{\sigma}\right)$$

$$= J\left(\underset{\sim}{\sigma}^{*(\text{trues.})}\right)$$

$$= J(\underset{\sim}{C} \cdot \underset{\sim}{d}) = \underset{\sim}{\chi}_{*}(\underset{\sim}{C}) \cdot \underset{\sim}{d}$$

↑ Kirchhoff

J's vanish

×

$$\dot{\underset{\sim}{\sigma}} = \bar{\underset{\sim}{C}} \cdot \underset{\sim}{d} + \underbrace{\underset{\sim}{S} \cdot \underset{\sim}{\omega}}$$

$$\underset{\sim}{R} = \underset{\sim}{P} \qquad \dot{\underset{\sim}{P}} = \underset{\sim}{\omega}\,\underset{\sim}{P}$$

$$\dot{\underset{\sim}{\Omega}} = \underset{\sim}{\omega}$$

$$\boxed{\dot{\underset{\sim}{\sigma}}_{\underset{\sim}{R}} = \bar{\underset{\sim}{C}}_{\underset{\sim}{R}} \cdot \underset{\sim}{d}_{\underset{\sim}{R}}} \longleftarrow \Rightarrow \bar{\Phi}$$

$$(\bar{\underset{\sim}{C}}_R)_{ijkl} = R_{ia} R_{jb} R_{kc} R_{ld} \bar{C}_{abcd}$$

×

Exercise: $\dot{\underset{\sim}{P}} = \underset{\sim}{\omega}\,\underset{\sim}{P}$

assume ~~&~~ $\underset{\sim}{P}\Big|_{t=0}$ = eigenvectors of $\underset{\sim}{d}\Big|_{t=0}$

$\underset{\sim}{P}$ = eigenvector's of $\underset{\sim}{d}$

$\underset{\sim}{d}_{\underset{\sim}{P}}$ = diagonal

<u>show that for simple shear 2D</u>

——————— × ———————

$$\overset{\nabla}{\underset{\sim}{\sigma}}_{\underset{\sim}{R}} = \overline{\underset{\sim}{C}}_{\underset{\sim}{R}} \cdot \underset{\sim}{d}_{\underset{\sim}{R}}$$

Make usual assumptions to generate typical plasticity theory with Mises, Kin. & Iso. combined hardening, etc.

Attributes:
1. Well-posed
2. Objective.
3. It's hyperelastic when elastic
4. Build in $\bar{\sigma}$ vs. $\bar{\epsilon}$ behavior

Negative:
1. $\overline{\underset{\sim}{C}}_{\underset{\sim}{R}}$ is a mess.

$$\text{e.g. } \overline{\underset{\sim}{C}}_{\underset{\sim}{R}} \cdot \underbrace{(\underset{\sim}{d}_{\underset{\sim}{R}} - \underset{\sim}{d}^{pl}_{\underset{\sim}{R}})}$$

perhaps better to assume $\underset{\sim}{d}^{el}_{\underset{\sim}{R}}$ objected to by Lee.

$$\overline{\underset{\sim}{C}}_{\underset{\sim}{R}} \cdot (\underset{\sim}{d}_{\underset{\sim}{R}} - \hat{\underset{\sim}{d}}_{\underset{\sim}{R}})$$

so $\underset{\sim}{\bar{C}}_{\underset{\sim}{R}} \cdot \underset{\sim}{\tilde{d}}_{\underset{\sim}{R}} = 2\mu\, \underset{\sim}{d}^{pl}_{\underset{\sim}{R}}$

← ⊥ to Y.S.

→ implementation via radial-return would be possible

$$\underset{\sim}{d} \neq \underset{\sim}{d}^{el} + \underset{\sim}{d}^{pl}$$
$$= \underset{\sim}{d}^{el} + \underset{\sim}{\tilde{d}}$$

X

2. effect of $\underset{\sim}{\hat{C}}$ on $\underset{\sim}{\bar{C}}$

3. $\underset{\sim}{\hat{C}}$ = classical iso.

is that possible? Ans. yes.

$$C_{ijkl} \overset{?}{=} \underbrace{\overset{①}{\lambda \delta_{ij} \delta_{kl}} + \overset{②}{\mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})}}$$

$$= \frac{1}{J} F_{iA} F_{jB} F_{kC} F_{lD} \underset{\substack{\uparrow \\ \text{solve for}}}{C_{ABCD}}$$

$$C_{ABCD} = J F^{-1}_{Ai} F^{-1}_{Bj} F^{-1}_{Ck} F^{-1}_{Dl} C_{ijkl}$$

→ must be a function of $\underset{\sim}{E}$

① $\lambda \delta_{ij} \delta_{kl} J F^{-1}_{Ai} F^{-1}_{Bj} F^{-1}_{Ck} F^{-1}_{Dl}$

$$F^{-1}_{Ai} F^{-1}_{Bi} F^{-1}_{Ck} F^{-1}_{Dk}$$

$$= F^{-1} \quad F^{-T}$$

$$= (\underset{\sim}{F}^{-1}\,\underset{\sim}{F}^{-T})_{AB} \;\text{-----}$$

$$= (\underbrace{\underset{\sim}{F}^{T}\,\underset{\sim}{F}^{-1}}_{})^{-1}_{AB} \;\text{--------}\quad \underset{\sim}{C}_{CD} \text{ likewise}$$

$$\underset{\sim}{C} = 2\underset{\sim}{E} + \underset{\sim}{I}$$

② likewise

$$\text{---------------}\times\text{---------------}$$

$$\underset{\uparrow}{\hat{\underset{\sim}{C}}} + \underset{\sim}{C}^{\text{usual}}$$

$$\underset{O(10^4)}{\underset{\wr}{\|\hat{\underset{\sim}{C}}\|}} << \underset{O(10^7)}{\underset{\wr}{\|\underset{\sim}{C}\|}} \quad \text{Typically}$$

$$\text{---------------}\times\text{---------------}$$

Consider other rates for kinematic hardening.

$$\dot{\underset{\sim}{\Omega}} = \underset{\sim}{\omega} + \underset{\sim}{s}(\underset{\sim}{\sigma})(\underset{\sim}{\sigma}\,\overset{\text{NAGT.}}{\underset{\sim}{d}} - \underset{\sim}{d}\,\underset{\sim}{\sigma})$$

$$+ \underset{\sim}{S}(\underset{\sim}{\alpha})(\underset{\sim}{\alpha}\,\underset{\sim}{d} - \underset{\sim}{d}\,\underset{\sim}{\alpha}) \longleftarrow$$

$$= \underset{\sim}{\omega} + \underbrace{\underset{\sim}{\zeta}(\underset{\sim}{\alpha})\cdot\underset{\sim}{d}}_{\zeta_{ijkl}\,d_{kl}}$$

$$\dot{\underset{\sim}{\sigma}} = \bar{\underset{\sim}{C}}\cdot\underset{\sim}{d} + \underset{\sim}{S}\cdot\underset{\sim}{\omega}$$

$$\text{"} \quad + \underset{\sim}{S}\cdot(\underset{\sim}{\omega} + \underset{\sim}{\zeta}\cdot\underset{\sim}{d}) - \underset{\sim}{S}\cdot\underset{\sim}{\zeta}\cdot\underset{\sim}{d}$$

$$\dot{\underset{\sim}{\sigma}} = \tilde{\underset{\sim}{C}}\cdot\underset{\sim}{d} + \underset{\sim}{S}\cdot\dot{\underset{\sim}{\Omega}}$$

$$\hookrightarrow = \left[(\bar{\underset{\sim}{C}} - \underset{\uparrow}{\underset{\sim}{S}}\cdot\underset{\nwarrow}{\underset{\sim}{\zeta}})\cdot\underset{\sim}{d}\right]$$

$$\dot{\underset{\sim}{R}} = \dot{\underset{\sim}{\Omega}}\, \underset{\sim}{R} \longrightarrow$$

$$\dot{\underset{\sim}{\tau}}_{\underset{\sim}{R}} = \hat{\underset{\sim}{C}}_{\underset{\sim}{R}} \cdot \underset{\sim}{d}_{\underset{\sim}{R}}$$ same

→ DO plast. :

$$\dot{k} = \dots$$

$$\dot{\underset{\sim}{\alpha}}_{\underset{\sim}{R}} = \dots\, \underset{\sim}{d}^{pl}_{\underset{\sim}{R}}$$

$$\underset{\sim}{d}^{pl}_{\underset{\sim}{R}} = \Lambda\, \underset{\sim}{Q}_{\underset{\sim}{R}}$$ normal

$$\dot{\underset{\sim}{\alpha}} + \underset{\sim}{\alpha}\,\dot{\underset{\sim}{\Omega}} - \dot{\underset{\sim}{\Omega}}\,\underset{\sim}{\alpha} = \dots\, \underset{\sim}{d}^{pl}$$

↑ rate ~~of~~ defined by NAGT.

What is about materially rotated rate:

$$\dot{\underset{\sim}{\Omega}} = \dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T} = \underset{\sim}{\varpi} = \underset{\Uparrow\atop \underset{\sim}{d}+\underset{\sim}{\omega}}{\underset{\sim}{l}} - \underset{\sim}{\delta} \quad \longleftarrow \text{symm. } \underset{\sim}{\delta} = \underset{\sim}{d}$$

$\underset{\sim}{\varpi}$ linear in $\underset{\sim}{d}$?

$$\underset{\sim}{\varpi} = \underset{\sim}{\omega} - \text{skew}\, \underset{\sim}{\delta}$$

$$\underset{\sim}{\delta} = \underset{\sim}{l} - \cancel{d}\, \underset{\sim}{\varpi}$$

$$= \dot{\underset{\sim}{F}}\,\underset{\sim}{F}^{-1} - \dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T}$$

$$= (\dot{\overline{\underset{\sim}{R}\underset{\sim}{U}}})\,\underset{\sim}{U}^{-1}\underset{\sim}{R}^{T} - \dot{\underset{\sim}{R}}\,\underset{\sim}{R}^{T}$$

$$= \dot{R}\,U\,U^{-1}R^{T} + R\,\dot{U}\,U^{-1}R^{T} - \dot{R}\,R^{T}$$

skew $\underset{\sim}{\delta} = \underset{\sim}{R} \left(\frac{\dot{\underset{\sim}{U}} \underset{\sim}{U}^{-1} - \underset{\sim}{U}^{-1} \dot{\underset{\sim}{U}}}{2} \right) \underset{\sim}{R}^{T}$

$$= \underset{\sim}{R} \; \underset{\sim}{y} \cdot \underset{\sim}{d} \; \underset{\sim}{R}^{T}$$

$$\Downarrow$$

$\underset{\sim}{y} (\underset{\sim}{U})$ NAGT.

$$= \underset{\sim}{\tilde{y}} \cdot \underset{\sim}{d}$$

$\underset{\sim}{w} = \underset{\sim}{\overset{3}{y}} (\underset{\sim}{R}, \underset{\sim}{U}) \cdot \underset{\sim}{d} \Longrightarrow$ Framework as above

$$\mathcal{E}_n = \mathcal{E}_0 + \mathcal{E}_1 n\Delta t + \mathcal{E}_2 \sin n\Delta t$$

$$\mathcal{E}_{n+1} = \mathcal{E}_0 + \mathcal{E}_1 (n+1)\Delta t + \mathcal{E}_2 \sin(n+1)\Delta t$$

$$\Delta\mathcal{E}_n = \mathcal{E}_1 \Delta t + \mathcal{E}_2 \left[\sin(n+1)\Delta t - \sin n\Delta t\right]$$

$$\sigma(0) = 0 \qquad \sigma_{n+1}(t) = E\,\Delta\mathcal{E}_n + \sigma_n$$

```
                DIMEN S(~~100~~ 300), ~~T(1001)~~, E(~~1001~~ 300)
10        ENTER TF, E0, E1, E2, DT, EY, SY  ——  Print all these first
            TF=PI*TF, DT=PI*DT
          N= TF/DT+1  ~  S(1)=0, T(1)=0
          DO 100 I= 2, N
            E(I)= E0+E1*(I-1)DT + E2*SIN((I-1)*DT)        $\mathcal{E}_n$      $n=i-1$
            E(I+1)=E0+E1*(I)DT + E2*SIN(I*DT)             $\mathcal{E}_{n+1}$  $n+1=i$
            DELE = E(I+1)-E(I)
            ST= S(I-1) + EY*DELE

            ~~GO TO (~~
            IF ( ABS(ST) - SY)    20, 20, 30
20          S(I) = ST
            ~~T(I) = (I-1)DT~~
            GO TO ~~100 90~~ 90
30          S(I)= SY*ST/(ABS(ST))
            GO TO ~~100 90~~ 90


90          T(I) =(I-1)*DT
100         CONTINUE
  ~~100       CONTINUE~~

            PRINT 3 COLUMNS of S(I) , I

            Let   ND = N/3
                  ND1 = ND+1
                  ND2 = 2*ND + 1
```