ME 239A - Computational Plasticity (Prof Hughes 7-2040) 1/4/83

239A - Traditional Plasticity

239B - Computational side

TA: needed (maybe me?).

Conduct of Course: in class exams (probable).

1 hr midterm
3 hr final

HW's =

Prerequisite: CE114 or equiv
- must have some structural analysis background
- elasticity
- FE background for -B course, but should be familiar w/ it.

No required texts - references that can be used

- HILL, "PLASTICITY," OXFORD UNIV PRESS 1950.
- PRAGER & HODGE, "THEORY OF PERFECTLY PLASTIC SOLIDS," DOVER.
- KACHANOV, "FOUNDATIONS OF THE THEORY OF PLASTICITY," NORTH-HOLLAND, 1971.

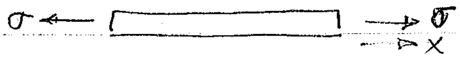
in numerical analysis

- ZIENKIEWICZ, "FEM", McGRAW-HILL, 1977. This is 3rd ed.
- OWEN & HINTON, "FE IN PLASTICITY," PINERIDGE, 1980.
- ME 235 A, B, C.

Class Begins:

Plasticity <=> Elasto-plasticity

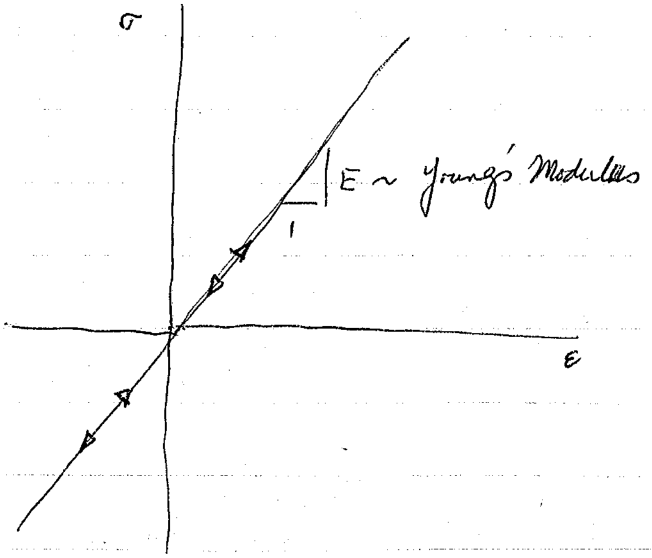
Motivation:



tensile member
force/unit area

strain due to σ $\varepsilon = du/dx$ where u is displ of some ref. point

Linear Elastic Material Response

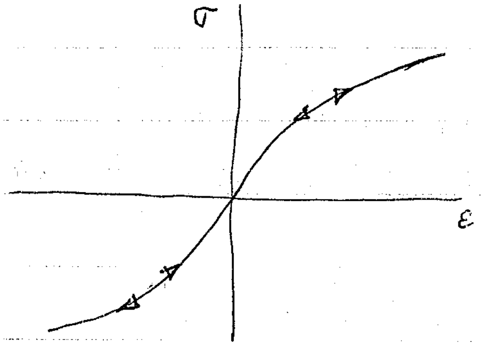


Small deformation

$\sigma = E\varepsilon$ Constitutive Eqn.

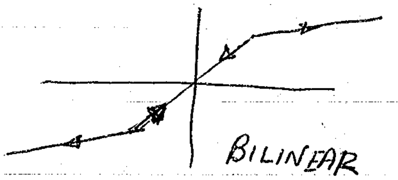
1 to 1 relationship $\sigma : \varepsilon$

Non Linear Elastic Material Response



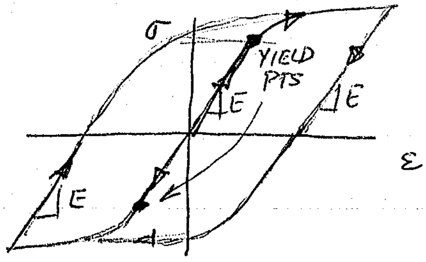
$\sigma = \sigma(\varepsilon)$ Algebraic fn. may be 1 to 1

1. Remarks: in elastic response no energy is dissipate in a loading/unloading cycle



2. Body returns to its original shape when unloaded.

Elasto-Plastic Response



yield pt - where we first go nonlin

hysteresis loop 1. energy is dissip is a loadg/unloading cycle.

2. Once you exceed yield pt, response is very differente from elastic material.

3. Cannot write $\sigma = \sigma(\varepsilon)$; ie for fixed ϵ, ∞-many possible values of σ. path is important.
4. permanent deformations can occur
5. elastic response is a fundamental part of elasto-plastic response.

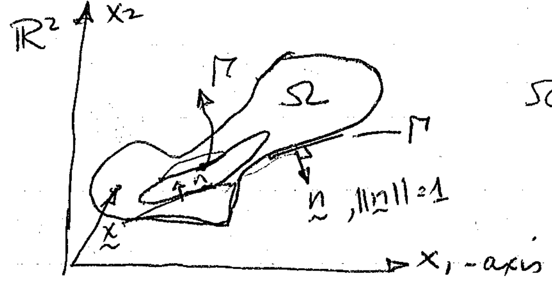
Review of classical, small deformation, primarily isotropic elasticity of eqns and BVP's.

We will use cartesian tensor notation

Elasticity References: Timoshenko - Goodier and Sokolnikoff

Preliminaries

n_{sd} = no. of space dimensions (2 or 3) $\quad \Omega \in \mathbb{R}^{n_{sd}}$



Ω is a body in $\mathbb{R}^2$ with piecewise smooth bdry Γ

Let $\bar{\Omega} = \Omega \cup \Gamma$

Let $\underset{\sim}{x}$ be any pt.

Let $\underset{\sim}{n}$ be a unit outward normal to Γ

Alternative Form of $\underset{\sim}{x}$ & $\underset{\sim}{n}$

$n_{sd} = 2$: $\underset{\sim}{x} = x_i \underset{\sim}{e}_i$ where $\underset{\sim}{e}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

assume $\phi = \Gamma_g \cap \Gamma_h$

Divergence Theorem:

$f: \bar{\Omega} \rightarrow \mathbb{R}$ contin-differentiable, then

$$\int_{\Omega} f_{,i} \, d\Omega = \int_{\Gamma} f n_i \, d\Gamma \qquad f_{,i} = \partial f / \partial x_i$$

integration by parts (IBP)

given f,g : $\bar{\Omega} \Rightarrow \mathbb{R}$ & contin differentiable

then $\displaystyle\int_\Omega f_{,i}\, g \, d\Omega = -\int_\Omega f\, g_{,i} \, d\Omega + \int_\Gamma f g\, n_i \, d\Gamma$

since $\displaystyle\int (fg)_{,i} \, d\Omega = \int_\Omega f_{,i}\, g \, d\Omega + \int_\Omega f\, g_{,i} \, d\Omega$

$\parallel$

$\displaystyle\int_\Gamma f g\, n_i \, d\Gamma$ by Div Theorem & rest follows.

1/6/83

TA - Office Hours

Tuesday 4-5PM } Rm: Durand ~~157~~ 260

Wednesday 10 AM

Arthur Muller } 7-2189

Louis Edozien

Dfns: let $1 \le i, j, k, l \le n_{sd}$

Def: σ_{ij} = Cartesian Components CAUCHY, or "TRUE", STRESS   force/unit area

it is symmetric $\sigma_{12} = \sigma_{21}$

u_i = displacement vector

f_i = prescribed body forces V.

Force / unit volume

ϵ_{ij} = (infinitesimal) strain tensor strain - displ eqs. $\varepsilon_{ij} = u_{(i,j)} = (u_{i,j} + u_{j,i})/2$ $\quad u_{i,j} = \partial u_i/\partial x_j$

$\varepsilon_{ij} = \varepsilon_{ji}$

* Constitutive eqn for linear elasticity (classical infinitesimal) - Generalized Hooke's Law.

$$\sigma_{ij} = \underbrace{C_{ijkl}}_{\text{given}} \varepsilon_{kl} \triangleq \sum_{k=1}^{n_{sd}} \sum_{l=1}^{n_{sd}} c_{ijkl}\, \varepsilon_{kl}$$

given - elastic moduli (may be fns of x)

IF C_{ijkl} are constant over entire body, body is called homogeneous

$\boxed{\text{We will work w/ Homogeneous material}}$

We assume C_{ijkl} satisfy $C_{ijkl} = C_{klij}$ Major Symmetry

$C_{ijkl} = C_{jikl} = C_{ijlk}$ Minor ''

For Uniqueness we assume that $C_{ijkl}(\underset{\sim}{x})\, \psi_{ij} \psi_{kl} \geq 0$ Positive Def

and also $C_{ijkl}(\underset{\sim}{x})\, \psi_{ij} \psi_{kl} = 0 \Rightarrow \psi_{ij} = 0$ ∀ symmetric $\underset{\sim}{\psi}$ $\forall \underset{\sim}{x} \in \Omega$

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FORMAL STATEMENT (STRONG FORM OF THE BVP of LINEAR ELASTOSTATICS

Given: $f_i : \Omega \to \mathbb{R}$ & $g_i : \Gamma_g \to \mathbb{R}$ & $h_i : \Gamma_h \to \mathbb{R}$

body force. prescribed displ. prescribed traction

Find: $u_i : \overline{\Omega} \to \mathbb{R}$ ∋

$\sigma_{ij,j} + f_i = 0$ on Ω EQILIB

$u_i = g_i$ on Γ_g

$\sigma_{ij} n_j = h_i$ on Γ_h } BC. this is a mixed value form

where $\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl} = C_{ijkl}(u_{k,l})$

Let us discuss principle of ~~[illegible]~~ Virtual work (o ~~[illegible]~~ weak fo · of the BVP)

Let $\mathcal{S}_i$ = set of trial solns (FTN's which satisfy displ BC ie $u_i = g_i$ on Γ_g)

$\mathcal{V}_i$ = set of weighting fns (ftn's satisfy homog BC's $w_i = 0$ on Γ_g)

(Also called the virtual displ")

Given : f_i, g_i, h_i (as previous); ~~[illegible]~~

Find : $u_i \in \mathcal{S}_i \ni. \ \forall w_i \in \mathcal{V}$

$$\underbrace{\int_\Omega w_{(i,j)} \sigma_{ij} \, d\Omega}_{\text{internal force}} = \underbrace{\int_\Omega w_i f_i \, d\Omega \ \cancel{+ \int_{\Gamma_h} w_i f_i \, d\Omega} + \int_{\Gamma_h} w_i h_i \, d\Gamma}_{\text{external force}}$$

Thm: Sol of Strong & weak fm ~~are~~ is the same.

$\ast \ \underset{\text{symmetric part}}{w_{(i,j)} \sigma_{ij} = w_{i,j} \sigma_{ij}}$ obtain by IBP

arguments of calculus of variation are used to obtain this.

Back to the constitutive eqn.

The body is called <u>isotropic</u> at $\underset{\sim}{x}$ if

$$c_{ijkl}(\underset{\sim}{x}) = \mu(\underset{\sim}{x}) \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \lambda(\underset{\sim}{x}) \, \delta_{ij}\delta_{kl}$$

$$\delta_{ij} = \begin{matrix} 1 & i = j \\ 0 & i \neq j \end{matrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

μ & λ are Lamé parameters and will be assumed to be constants

$\mu = G$ = shear modulus = $E/2(1+\nu)$

$\lambda = \nu E / [(1+\nu)(1-2\nu)]$

} E – Young's modulus
ν – Poisson's ratio

Exercises:

1. Start w Generalized Hooke's law, def. of isotropy, use symmetry of σ_{ij}, ε_{ij}, C_{ijkl} to derive matrix forms as follows:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \begin{matrix} \text{DIRECT} \\ \text{STRESS} \\ \\ \text{SHEAR} \\ \text{STRESS} \end{matrix} = \left[\begin{array}{ccc|ccc} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda+2\mu & & & \\ \hline & & & \mu & 0 & 0 \\ & 0 & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{array}\right] \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{32} \end{pmatrix} \begin{matrix} \text{DIRECT} \\ \text{STRAINS} \\ \\ \text{SHEAR} \\ \text{STRAINS} \end{matrix}$$

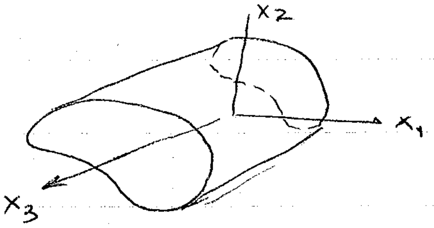
define $\gamma_{ij} = 2\varepsilon_{ij} \quad i \neq j$

engineering strains.

2. In plane strain typical slice of a long body ie let $\varepsilon_{33} = 0$ $(u_3 = 0)$ & let $u_1 = u_1(x_1, x_2)$, $u_2 = u_2(x_1, x_2)$

$$\Rightarrow \underbrace{\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} \equiv 0}$$

$$\Rightarrow \sigma_{23} = \sigma_{13} = 0$$

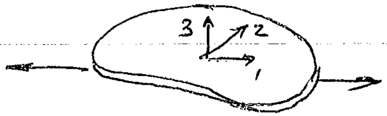


Thus.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}$$

$\sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22})$ All follow from result in 3-D.

3. In plane stress: Thin stressed plate ie loaded in its own plane



Assume $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$

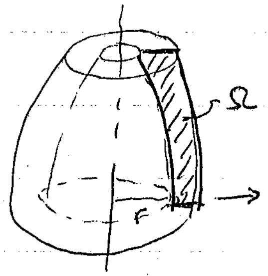
Show that $\sigma_{i3} = 0$ & the first exercise give

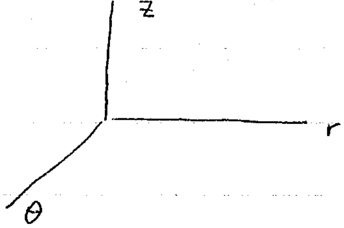
$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \bar{\lambda}+2\mu & \bar{\lambda} & 0 \\ \bar{\lambda} & \bar{\lambda}+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}$$

where $\bar{\lambda} = 2\lambda\mu/(\lambda+2\mu)$

and $\varepsilon_{33} = -\lambda/_{\lambda+2\mu}\,(\varepsilon_{11} + \varepsilon_{22})$

4. Axisymmetric case: Introduce cylindrical coordinates





$x_1 = r$ radial
$x_2 = z$ axial
$x_3 = \Theta$ circumferential

DISP's $u_1, u_2, u_3 \rightarrow u_r, u_z, u_\Theta$

Assumptions: all fn's are independent of Θ.

$u_r = u_r(r,z) \quad u_z = u_z(r,z)$

additional assumption "torsion free" $u_\theta = 0$

Consequences: $\varepsilon_{r\theta} = \varepsilon_{z\theta} = 0 \qquad \varepsilon_{\theta\theta} = u_r/r \qquad \varepsilon_{rr} = u_{r,r}$

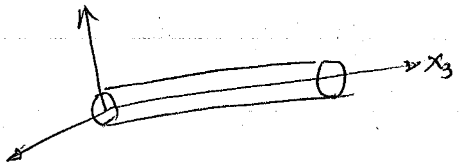
Show from exercise 1: $\varepsilon_{zz} = u_{z,z} \qquad \varepsilon_{rz} = (u_{r,z} + u_{z,r})/2$

$d\Omega = 2\pi r\,dr\,dz$

$$\begin{pmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{pmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \begin{pmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\theta\theta} \\ 2\varepsilon_{rz} \end{pmatrix}$$

Beam / Frame Case

5. Assume $\sigma_{11} = \sigma_{22} = \sigma_{12} = 0$ (1-D



$\Rightarrow$ starting w/ exercise 1

$\sigma_{33} = E\,\varepsilon_{33}$

$\sigma_{13} = 2\mu\,\varepsilon_{13} = G\,\gamma_{13}$

$\sigma_{23} = 2\mu\,\varepsilon_{23} = G\,\gamma_{23}$

$\varepsilon_{11} = \varepsilon_{22} = -\nu\,\varepsilon_{33}, \quad \varepsilon_{12} = 0.$

with approp. kinematic assumptions, the above relationships can be used to generate simple theories governing beam / frame response including: axial, bending, transverse shear effects torsional shear effects.

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Equil Eqns:

3-D $\sigma_{ij,j} + f_i = 0$

$\sigma_{1,1} + \sigma_{12,2} + \sigma_{13,3} + f_1 = 0$ etc.

2-D : plane stress / strain $\sigma_{13}, \sigma_{23} = 0$

$$\boxed{\begin{aligned} &\sigma_{11,1} + \sigma_{12,2} + f_1 = 0 \\ &\sigma_{21,1} + \sigma_{22,2} + f_2 = 0 \end{aligned}}$$

p stress: $\sigma_{33} = 0 \Rightarrow f_3 = 0$ constraint

p strain: since $\sigma_{33} = \sigma_{33}(x_1, x_2)$ only $\Rightarrow \sigma_{33,3} = 0 \Rightarrow f_3 = 0$

3-D Cylindrical

$\sigma_{rr,r} + \sigma_{rz,z} + \frac{1}{r}\sigma_{r\theta,\theta} + f_r + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$ Radial

$\sigma_{zr,r} + \sigma_{zz,z} + \frac{1}{r}\sigma_{z\theta,\theta} + \frac{1}{r}\sigma_{rz} + f_z = 0$ Axial

$\sigma_{\theta r,r} + \sigma_{\theta z,z} + \frac{1}{r}\sigma_{\theta\theta,\theta} + \frac{2}{r}\sigma_{r\theta} + f_\theta = 0$ Circum.

Axisym & torsion free: all $r\theta$, $z\theta$ terms $\Rightarrow 0$

$$\boxed{\begin{aligned} &\sigma_{rr,r} + \sigma_{rz,z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = 0 \\ &\sigma_{zr,r} + \sigma_{zz,z} + \frac{1}{r}\sigma_{rz} + f_z = 0 \end{aligned}}$$

since $\sigma_{\theta\theta} = \sigma_{\theta\theta}(r, z) \Rightarrow \sigma_{\theta\theta,\theta} = 0 \Rightarrow f_\theta = 0$

Beam / Frame Case : Start w/ Cartesian $\sigma_{ij,j} + f_i = 0$

σ_{11}, σ_{22} and $\sigma_{12} = 0$

$$\sigma_{13,3} + f_1 = 0$$ ① transverse equil eqn ⇒ torsional equil

$$\sigma_{23,3} + f_2 = 0$$

$$\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} + f_3 = 0$$ ② axial equil ⇒ Moment Equil

Recall BVP of elastostatics

Given : f_i, g_i, h_i (fns of $\underset{\sim}{x}$) over their respective domains

Want : $u_i(\underset{\sim}{x})$ ∋ equil is satisfied and so are the BC's

$$(\sigma_{ij,j} + f_i)(\underset{\sim}{x}) = 0 \qquad \underset{\sim}{x} \in \Omega$$

$$u_i(\underset{\sim}{x}) = g_i(\underset{\sim}{x}) \qquad \underset{\sim}{x} \in \Gamma_g$$

$$(\sigma_{ij} n_j)(\underset{\sim}{x}) = h_i(\underset{\sim}{x}) \qquad \underset{\sim}{x} \in \Gamma_h$$

Constit eqn $\sigma_{ij}(\underset{\sim}{x}) = c_{ijkl}(\underset{\sim}{x}) \cdot \varepsilon_{kl}(\underset{\sim}{x})$

For linear elastic tensile specimen process is independent of time but when it becomes t-dependent this makes the process elastodynamic

Must introduce inertial forces and IC's

Inertia forces : introduce VIA D'Alembert's Principle

⇒ replace f_i by $f_i - \rho \ddot{u}_i$ $\quad \dot{u}_i = \partial u_i / \partial t$. $\quad \ddot{u}_i = \ddot{u}_i(x, t)$

but $\rho = \rho(\underset{\sim}{x})$ and is assumed given

For the elastodynamic prob (Initial BVP)

Given : $f_i : \Omega \times]0,T[\rightarrow \mathbb{R}$ $\quad$ i.e. $f_i(\underset{\sim}{x}, t)$ $\qquad]0,T[\equiv (0,T)$

$$\text{BC} \begin{cases} g_i : \Gamma_g \times]0,T[\rightarrow \mathbb{R} \\ h_i : \Gamma_h \times]0,T[\rightarrow \mathbb{R} \end{cases}$$

IC's $\{$ $u_{0i} : \Omega \rightarrow \mathbb{R}$ $\quad$ i.e. $u_{0i} = u_{0i}(\underset{\sim}{x})$ only

IC's $\Big] v_{0i} : \Omega \to \mathbb{R}$ $\qquad v_{0i} = v_{0i}(\underset{\sim}{x})$

and $(\rho, C_{ijkl}$ ftns of $\underset{\sim}{x}$ only$)$

Find: $u_i : \overline{\Omega} \in [0,T] \to \mathbb{R}$

Thus $\sigma_{ij,j} + f_i = \rho \ddot{u}_i$ $\qquad$ eqns of motion $\forall\, \underset{\sim}{x} \in \Omega$ $\quad t \in (0,T)$

$$\underline{BC} \begin{cases} u_i = g_i & \underset{\sim}{x} \in \Gamma_g \quad \& \\ \sigma_{ij} n_j = h_i & \underset{\sim}{x} \in \Gamma_h \quad \& \end{cases} \qquad t \in (0,T)$$

w/ IC $\quad u_i(\underset{\sim}{x},0) = u_{0i}(\underset{\sim}{x})$ $\qquad \underset{\sim}{x} \in \Omega$

$\qquad \dot{u}_i(\underset{\sim}{x},0) = \dot{u}_{0i}(\underset{\sim}{x})$

Now $\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$ the constitutive Law.

Weak form is easy to get.

Consider the case where $\rho\ddot{u}_i$ is small, but we keep the t dependence. (we'd have to ignore the IC's). This will be a quasistatic process.

In plasticity the constit. eqns are changed. ~~eg~~ $\sigma_{ij} \Rightarrow \sigma_{ij}(\dot{\varepsilon}_{kl}, \varepsilon_{kl})$.

We will deal w/ simple idealizations of elasto-plastic behavior

Must answer what is a ~~theory~~ constitutive Theory?

View it ~~as~~ an input-output device

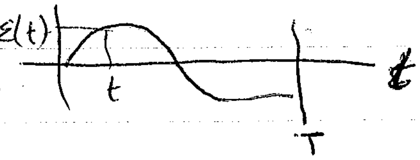
$$\xrightarrow{\varepsilon} \boxed{} \xrightarrow{\sigma}$$ roughly speak

More precisely a relation giving the stress history as a fn of strain history.

1-D Case $\quad$ What is strain history (at a fixed pt $\underset{\sim}{x}$) is the graph of ~~$\varepsilon = \varepsilon(\underset{\sim}{x})$~~

$\varepsilon = \varepsilon(x, \cdot) : [0,T] \Rightarrow \mathbb{R}$

$h_\varepsilon = \{(t, \varepsilon(t)) \mid 0 \le t \le T\}$



stress history

likewise $h_\sigma = \{(t,\sigma(t)) \mid 0 \le t \le T\}$ at the same $\underset{\sim}{x}$

Thus

$$h_{\underset{\sim}{\varepsilon}} \longrightarrow \boxed{\text{Constit Theory}} \longrightarrow h_{\underset{\sim}{\sigma}}$$

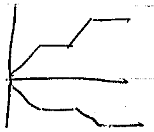
+ perhaps some ICs on $\underset{\sim}{\sigma}(\underset{\sim}{x},0)$

Ex: Simple & concrete 1-D idealization

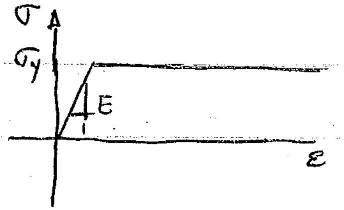
ie elastic/perfectly plastic behavior

Physical ideals incorporated in the model –

assume ϵ is monotone (either $\dot\varepsilon \geq 0$ or $\dot\varepsilon \leq 0$)

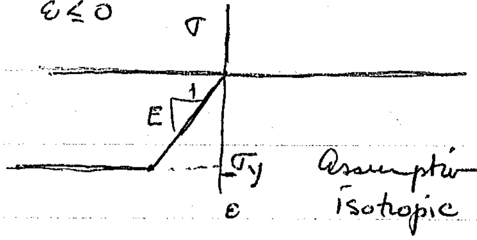


for $\dot\varepsilon \geq 0$



σ_y = yield pt

for $\dot\varepsilon \leq 0$



Assumption

Isotopic assumption in plastic ie $\sigma_{y\,\text{compress}} = -\sigma_{y\,\text{tensile}}$

for $\dot\varepsilon \geq 0$ Rate eqn form $\dot\sigma = E\dot\varepsilon$ if $\sigma < \sigma_y$

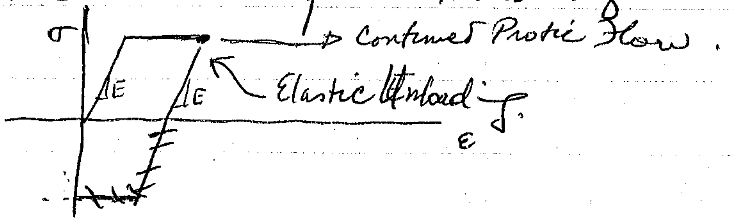
$\dot\sigma = 0$ if $\sigma = \sigma_y$

for $\dot\varepsilon \leq 0$ " $\dot\sigma = E\dot\varepsilon$ if $-\sigma \leq -\sigma_y$

$\dot\sigma = 0$ if $-\sigma = -\sigma_y$

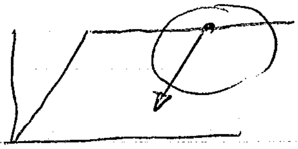
Now to explore!

non monotone histories if $\dot\varepsilon$ reverses sign

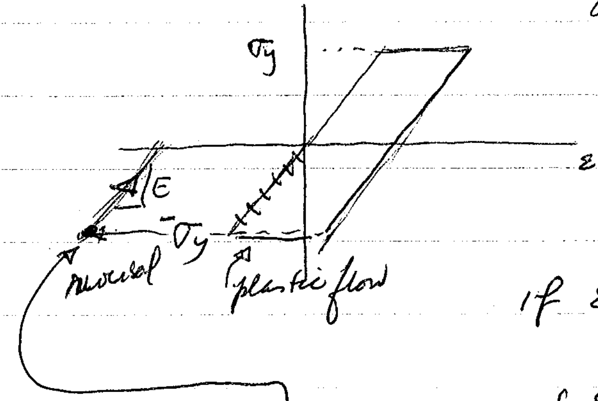


Algorithmically

let $\dot{\sigma}^{tr} = E\dot{\varepsilon}$ trial rate of stress ; $E>0$ is assumed $\operatorname{sgn} \dot{\sigma}^{tr} = \operatorname{sgn} \dot{E}$



Now on $\sigma = \sigma_y$ $\dot{\sigma} = \begin{cases} \dot{\sigma}^{tr} & \text{if } \dot{\sigma}^{tr} < 0 \\ 0 & \text{if } \dot{\sigma}^{tr} \geq 0 \end{cases}$ ie $\dot{\varepsilon} < 0$



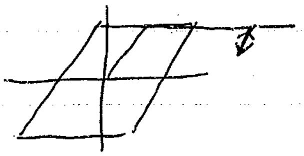
allowable σ's are $\ni$. $\sigma \in [-\sigma_y, \sigma_y]$

if $\dot{\varepsilon} > 0$, then elastic loading

if $\sigma = -\sigma_y$, then $\begin{cases} \dot{\sigma} = \dot{\sigma}^{tr} & \text{if } \dot{\sigma}^{tr} > 0 \quad \text{ie } \dot{\varepsilon} > 0 \\ 0 & \text{if } \dot{\sigma}^{tr} \leq 0 \end{cases}$

- - - - - - - - - -

Another case Whenever $|\sigma| < \sigma_y$, $\dot{\sigma} = \dot{\sigma}^{tr} = E\dot{\varepsilon}$ (elastic behavior)



Summary

$$\dot{\sigma}^{tr} = E\dot{\varepsilon}$$

$$\dot{\sigma} = \begin{cases} \dot{\sigma}^{tr} \text{ if } \boxed{\begin{cases} |\sigma| < \sigma_y \\ \underline{\text{OR}} \\ |\sigma| = \sigma_y \text{ and } \operatorname{sgn} \dot{\sigma}^{tr} = -\operatorname{sgn} \sigma \end{cases}} & \text{elastic process (E)} \\ 0 \ \boxed{\text{if } \begin{cases} |\sigma| = \sigma_y \ \& \ \operatorname{sgn} \dot{\sigma}^{tr} = \operatorname{sgn} \sigma \\ \text{or} \\ |\sigma| = \sigma_y \ \& \ \operatorname{sgn} \dot{\sigma}^{tr} = 0 \end{cases}} & \text{plastic process (P)} \end{cases}$$

thus

$$\dot{\sigma} = \begin{cases} \dot{\sigma}^{tr} & (E) \\ 0 & (P) \end{cases}$$ is our constitutive model.

$$h_\varepsilon + \sigma(0) \longrightarrow \boxed{} \longrightarrow h_\sigma$$

1/13/83

Exercise

Consider : $\varepsilon(t) = \varepsilon_0 + \varepsilon_1 t + \varepsilon_2 \sin t \qquad 0 \leq t \leq T$

Assume : $E = 1$, $\sigma_y = 1$

Sketch $\sigma(t)$

(i) $T = \pi$, $\sigma(0) = 0$, $\varepsilon_0 = \varepsilon_1 = 0$, $\varepsilon_2 = 1/2$

(ii) " " " $\varepsilon_2 = 2$

(iii) $T = 2\pi$ same as in (i) $\Big\}$ $n\pi$?

(iv) " " " (ii) $\Big\}$ $n > 2$

(v) $\sigma(0) = 1/2$, $\varepsilon_0 = 1$, redo cases (i)–(iv)

(vi) $T = 5\pi$, $\sigma(0) = 0$, $\varepsilon_0 = 0$, $\varepsilon_1 = 1$, $\varepsilon_2 = 2$

(vii) = same but let $\varepsilon_1 = -1$, $\varepsilon_2 = -2$

incremental stress-pt algorithm ($\underset{\sim}{x}$ being fixed)

let n = time step no.

Δt = " "

$\varepsilon(x, t_n) = \varepsilon_n \qquad t_n = n\Delta t$

Given : $\underbrace{\varepsilon_n, \ \Delta\varepsilon_n = (\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n)}_{\text{numerical strain history}}$, also ~~$\Delta t$~~ but don't need it, σ_n

Find σ_{n+1} :

idea : $\displaystyle\int_{t_n}^{t_{n+1}} (\dot{\sigma}^{tr} = E\dot{\varepsilon})\,\Delta t$ with σ_n being initial value

1. $\sigma_{n+1}^{tr} - \sigma_n = E\Delta\varepsilon_n$

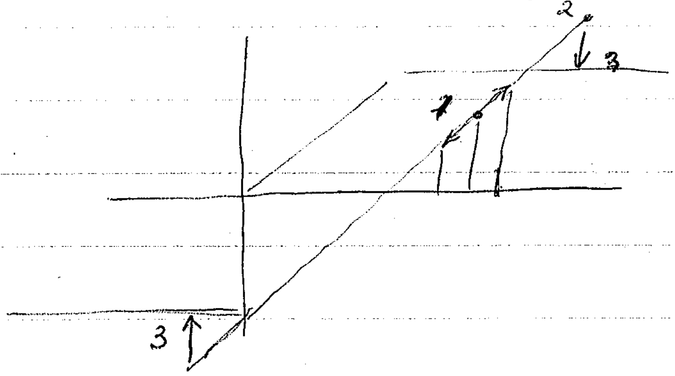
or $\sigma_{n+1}^{tr} = E\Delta\varepsilon_n + \sigma_n$

note Δt doesn't appear here.

2. if $|\sigma_{n+1}^{tr}| \le \sigma_y$, let $\sigma_{n+1} = \sigma_{n+1}^{tr}$

EXIT TO next set.

3. if not let $\sigma_{n+1} = \frac{\sigma_{n+1}^{tr}}{|\sigma_{n+1}{}^{tr}|}\sigma_y = (\underbrace{\text{sgn } \sigma_{n+1}^{tr}}_{\pm 1})\sigma_y$; exist



Exercise Continue:

Program the algo rithm to solve the above problems w $\Delta t = \pi/10, \pi/20, \pi/40$.

Remark: A theory of this kind is called "rate-independent" if for an arbitrary strain history, ε, and an arbitrary constant, α, :

if $\sigma(t)$ corresponds to $\varepsilon(t)$ & $\sigma(\alpha t)$ corresponds to $\varepsilon(\alpha t)$.

[unlike viscous phenomena ($\sigma = \mu\dot{\varepsilon}$)]. Rate-independent theories are called also inviscid. (⇒ algorithm independent of Δt.

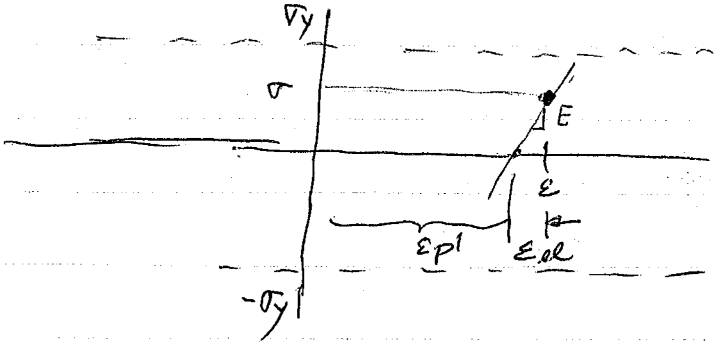
Concepts of elastic & plastic strains:

Rewrite $\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^{pl}) = E\dot{\varepsilon}^{elastic}$

$$\dot{\varepsilon}^{pl} = \begin{cases} 0 & (E) \\ \dot{\varepsilon} & (P) \end{cases}$$

We assume additive decomposition of strain rate into elastic and plastic parts.

physical interpretation



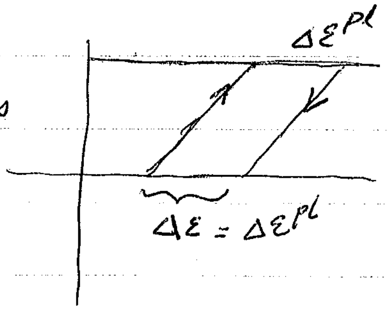
Remove σ via an elastic process

ε_{pl} is the permanent strain

ε_{el} " " recovered "

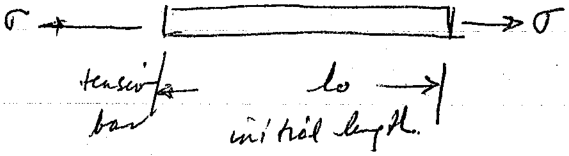
1. note that in an elastic process $\dot{\varepsilon}_{pl}=0$ since ε^{pl} = constant $\Rightarrow \dot{\varepsilon}^{pl}=0$.

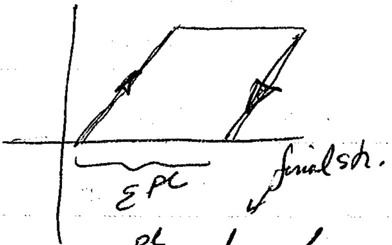
2. in a plastic process



$\Rightarrow \Delta\varepsilon = \Delta\varepsilon^{pl} \Rightarrow$
$\dot{\varepsilon} = \dot{\varepsilon}^{pl}$.

3. plastic strain is measurable.





$$\varepsilon^{pl} = \frac{l_f - l_0}{l_0}$$

Multidimensional version of elast/perf. pl).

Assume that $$\sigma_{ij} = \underbrace{C_{ijkl}\left(\dot{\varepsilon}_{kl}\right.}_{\dot{\sigma}_{ij}^{\,tr}} \underbrace{\left.- \dot{\varepsilon}^{pl}_{kl}\right)}_{\dot{\varepsilon}^{elastic}_{kl}}$$

$$\dot{\varepsilon}^{pl}_{kl} = \begin{cases} 0 & \text{in an elastic process } (E) \\ ? & \text{" " " " } (P) \end{cases}$$

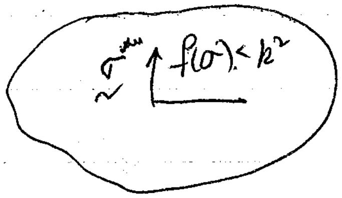
① what is an elastic process in multi dyn.

② what is a plastic process in mult. dy.

③ ______ what is dfn

1. in 1-D $\left.\begin{array}{l}|\sigma| < \sigma_y \\ |\sigma| = \sigma_y\end{array}\right)$ differentiated between

yield pt become yield surface in 3D.



Assume ~~yield~~ yield surface analytically is given by $f(\sigma)$

$\cancel{\Phi} = f(\tilde\sigma) = k^2$

$\tilde\sigma$'s will be considered ~~plastic~~ elastic for ~~in~~ inside. on the y.s. can lead to either elastic or plas. ~~pl~~ plastic processes.

1 D

$|\sigma| < \sigma_y$

$=$

$> \sigma_y$

3 D

$f(\tilde\sigma) < k^2$

$=$

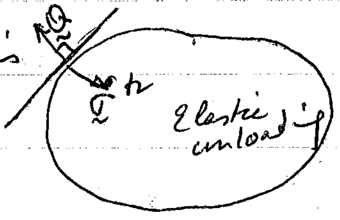
$>$ impossible

① elastic process (E)

1D: $|\sigma| < \sigma_y$ $\longleftrightarrow$ 3D: $f(\sigma) < k^2$

Elastic Unload or $|\sigma| = \sigma_y$ & sign $\dot\sigma$ = − sign σ $\longleftrightarrow$ $\boxed{\dot{\tilde\sigma} \cdot \tilde Q = \dot\sigma_{ij} Q_{ij} < 0 \text{ and } f(\tilde\sigma) = 0}$

to get to this interior of yield surface.



$\tilde Q$ is unit normal to yieldsurface.

define $\tilde Q = \dfrac{\nabla f}{\|\nabla f\|}$ $= \left(\dfrac{\partial f}{\partial\sigma_{ij}} \dfrac{\partial f}{\partial(\sigma_{ij})}\right)^{1/2}$ Euclidean length.

Thus $\nabla f = \dfrac{\partial f}{\partial\tilde\sigma} = \left[\dfrac{\partial f}{\partial\sigma_{ij}}\right]$

To get ② **Plastic Process**

1D

$|\sigma| = \sigma_y$

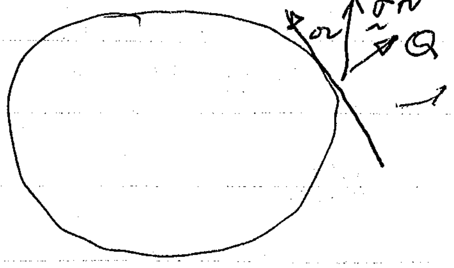
and $\operatorname{sgn} \dot{\sigma}^{tr} = \operatorname{sgn} \sigma$

$\sigma = 0$

Counterpart

3D

$f(\underset{\sim}{\sigma}) = k^2$



$\dot{\underset{\sim}{\sigma}}^{tr} : \underset{\sim}{Q} \geq 0$

before we do ③., consider a particularly simple specific yield surface.

Von Mises yield fcn: useful for (metals, undrained soils).

$f(\underset{\sim}{\sigma}) = \frac{1}{2} |\underset{\sim}{\sigma}'|^2$ $\underset{\sim}{\sigma}'$ is the deviatoric stress.

$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}' + \underset{\sim}{\sigma}''$ $\underset{\sim}{\sigma}'' \equiv \frac{1}{3} (\sigma_{ii}) \underset{\sim}{I}$

$= (\frac{1}{3} \operatorname{tr} \underset{\sim}{\sigma}) \underset{\sim}{I}$ => $\operatorname{tr} \underset{\sim}{\sigma} = \sigma_{kk}$ $\underset{\sim}{I} = \delta_{ij}$

$\underset{\sim}{\sigma}' = \underset{\sim}{\sigma} - \underset{\sim}{\sigma}''$ $\sigma_{ij}' = \sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij}$

$\frac{1}{3} \operatorname{tr} \underset{\sim}{\sigma}$ = "mean stress" p = pressure = $-\frac{1}{3} \operatorname{tr} \underset{\sim}{\sigma}$

$\underset{\sim}{\sigma}'$ is like a generalized shear type system of stresses.

In terms of (any theory von mises) for any $f(\underset{\sim}{\sigma}) = g(\underset{\sim}{\sigma}')$ the yield surface (~~&~~ plastic flow) doesn't depend on pressure).

Bridgeman's experiments on metals.

Geometry of Mises yield Surface.

Preliminaries: consider any symmetric 2nd rank tensor

we'll work w/ $\underset{\sim}{\sigma}$ (could employ $\underset{\sim}{\varepsilon}$, $\underset{\sim}{\sigma}'$ etc).

Recall eigenproblem $\det(\underset{\sim}{\sigma} - \sigma \underset{\sim}{I}) = 0$ E-values

$$\Rightarrow \sigma^3 - J_1\sigma^2 - J_2\sigma - J_3 = 0$$

were J_i's $(J_i = J_i(\underset{\sim}{\sigma}))$ are principal invariants (to a sign)

$J_1 = \mathrm{tr}\,\underset{\sim}{\sigma}$ $\quad -J_2$ = sum of principal minors of $\underset{\sim}{\sigma}$

$$= \det\begin{bmatrix}\sigma_{22} & \sigma_{23}\\ \sigma_{23} & \sigma_{33}\end{bmatrix} + \det\begin{bmatrix}\sigma_{11} & \sigma_{13}\\ \sigma_{13} & \sigma_{33}\end{bmatrix} + \det\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}$$

$J_3 = \det \underset{\sim}{\sigma}$

Jan 18, 1983

TA office hrs

Tuesday 4-5PM Wed 10-11 AM Rm: Durand 264 497-1011

9 Dim sp. $\underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon} = \sigma_{ij}\varepsilon_{ij} = \sigma_{11}\varepsilon_{11} + 2\varepsilon_{12}\sigma_{12} + 2\sigma_{13}\varepsilon_{13} + \sigma_{22}\varepsilon_{22} + 2\sigma_{23}\varepsilon_{23} + \varepsilon_{33}\sigma_{33}$

if $\underset{\sim}{\sigma}$ & $\underset{\sim}{\varepsilon}$ are symmetric

invariance: what does it mean? Consider a rotation of coords about $\underset{\sim}{x}$

$\Rightarrow$ $\bar{\underset{\sim}{x}} = \underset{\sim}{l}\,\underset{\sim}{x}$ where $\underset{\sim}{l}^{-1} = \underset{\sim}{l}^{T}$ (orthogonal).

$\underset{\sim}{l}^T\underset{\sim}{l} = \underset{\sim}{l}\,\underset{\sim}{l}^T = \underset{\sim}{l}\,\underset{\sim}{l}^{-1} = I$

Under such a transformation, any 2nd rank tensor, for example $\underset{\sim}{\sigma}$, transforms as $\underset{\sim}{\bar{\sigma}} = \underset{\sim}{l}\,\underset{\sim}{\sigma}\,\underset{\sim}{l}^T$ $\quad \bar{\sigma}_{ij} = l_{ik}\,\sigma_{km}\,l^T_{mj}$

$l_{ik}\,\sigma_{km}\,l_{jm}$

$g(\underset{\sim}{\sigma})$, a scalar value, is invariant if $g(\underset{\sim}{\bar{\sigma}}) = g(\underset{\sim}{\sigma})$.

→ exercise: verify J_i's are invariant

use $-J_2 = \frac{1}{2}\, e_{ijk}\, e_{ilm}\, \sigma_{jl}\, \sigma_{km}$

$J_3 = \frac{1}{3!}\, e_{ijk}\, e_{lmn}\, \sigma_{il}\, \sigma_{jm}\, \sigma_{kn}$

$$\begin{tikzcd} i \arrow[r] & j \arrow[ld] \\ k \arrow[u, bend left] \arrow[ur, phantom, "+"] & \end{tikzcd}$$

$e_{ijk} = +1$ if ijk is an even permutation of 123

alternating tensor. -1 if ijk " " odd " of 123

0 if repeated index.

working in eigencoordinates in which $\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_1 & & \underset{\sim}{0} \\ & \sigma_2 & \\ \underset{\sim}{0} & & \sigma_3 \end{bmatrix}$

$J_1(\underset{\sim}{\sigma}) = \sum_{i=1}^{3} \sigma_i \quad ; \quad -J_2 = \sigma_2\sigma_3 + \sigma_1\sigma_3 + \sigma_1\sigma_2 \; ; \quad J_3 = \sigma_1\sigma_2\sigma_3$

$J_i' = J_i'(\underset{\sim}{\sigma}) = J_i(\underset{\sim}{\sigma}')$

$J_1' = tr\,\underset{\sim}{\sigma}' = tr\,(\underset{\sim}{\sigma} - \frac{1}{3}\,tr\,\underset{\sim}{\sigma}\,I) = tr\,\sigma - \frac{1}{3}\,tr\,\sigma\,\underbrace{tr\,I}_{3} = tr\,\underline{\sigma} - 3\cdot\frac{1}{3}\,tr\,\underline{\sigma} = 0$

$J_2' = \sum$ principal minors

$J_3' = \det \underset{\sim}{\sigma}'$

// take $\underset{\sim}{\sigma}$ in eigencoords

$\underset{\sim}{\sigma}' = \underset{\sim}{\sigma} - \frac{1}{3}\,tr\,\underset{\sim}{\sigma}\,\underset{\sim}{I} = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} - \frac{\Sigma\sigma_i}{3}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

$\sigma_1' = (2\sigma_1 - (\sigma_2 + \sigma_3))/3$

$\sigma_2' = (2\sigma_2 - (\sigma_1 + \sigma_3))/3$

$\vdots$

$0 = J_1' = \sigma_1' + \sigma_2' + \sigma_3'$

$0 = (J_1')^2 = \sigma_1'^2 + \sigma_2'^2 + \sigma_3'^2 + 2\underbrace{(\sigma_2'\sigma_3' + \sigma_3'\sigma_1' + \sigma_1'\sigma_2')}_{-J_2'}$

$$J_2' = \tfrac{1}{2}\left[(\sigma_1')^2 + (\sigma_2')^2 + (\sigma_3')^2\right]$$

Mises yield surface : $k^2 = f(\underline{\sigma}) = \tfrac{1}{2}\,|\underline{\sigma}'|^2 = \tfrac{1}{2}\,\sigma_{ij}'\sigma_{ij}'$ this is an invariant – Euclidean length

$\therefore$ can write $\tfrac{1}{2}\,|\underset{\sim}{\sigma}'|^2 = \tfrac{1}{2}\,(\sigma_i'\sigma_i') = J_2'$

Proof of invariance $\bar{\sigma}_{ij}'\bar{\sigma}_{ij}' = l_{ik}\,\boxed{l_{jl}}\,\sigma_{kl}'\;\; l_{im}\,\boxed{l_{jn}}\,\sigma_{mn}'$

δ_{km} δ_{ln}

$$\delta_{km}\sigma_{kl}' \quad \delta_{ln}\sigma_{mn}' = \sigma_{ml}'\sigma_{ml}' = \sigma_{ij}'\sigma_{ij}'$$

QED

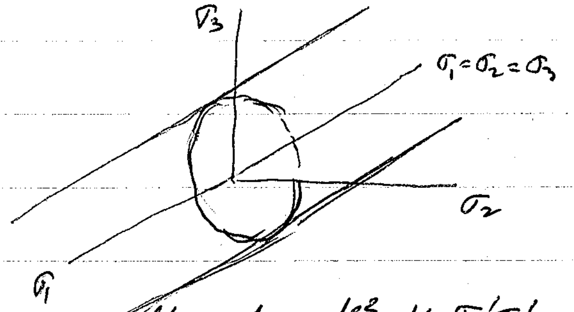
exercise: Shown $J_3' = \det \sigma_1'\sigma_2'\sigma_3' = \tfrac{1}{3}\left((\sigma_1')^3 + (\sigma_2')^3 + (\sigma_3')^3\right)$

look at Mises in 3-space $(\sigma_1, \sigma_2, \sigma_3)$:

Observe : if $(\sigma_1, \sigma_2, \sigma_3)$ is on y.s. then so is $(\sigma_1+\alpha, \sigma_2+\alpha, \sigma_3+\alpha)$

note $\sigma_i' = (\sigma_i' + \alpha)$ mean stress doesn't do anything to the deviatoric stress.

$\Rightarrow$ yield surface is a cylinder (in general sense) w/ generating axis $= \sigma_1 = \sigma_2 = \sigma_3$.



("press axis")

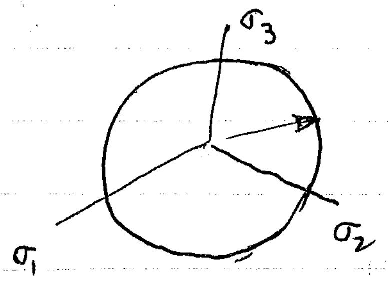
π plane : plane $\perp$ to pressure axis & through origin

Remember $k^2 = \tfrac{1}{2}\,\sigma_i'\sigma_i'$ in the π plane, $\sigma_1 + \sigma_2 + \sigma_3 = 0$

$\therefore$ the yield surface is the surface which is the intersection of the spherical surface $\tfrac{1}{2}\,\sigma_i\sigma_i = k^2$ w $\sigma_1 + \sigma_2 + \sigma_2 = 0$

ie a circle in the π-plane the radius $= \sqrt{2}k$

- Now let's view along the pressure axis



high pressure variation means you are running along the pressure generator & don't reach yield surface just like Mises criteria says — mean pressure doesn't do anything to cause yielding

Exercise: show that $f(\sigma) = k^2$ can be written as:

$$(\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + (\sigma_{11} - \sigma_{22})^2 + 6(\sigma_{13}^2 + \sigma_{23}^2 + \sigma_{12}^2) = 6k^2$$

or $$(\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_1 - \sigma_2)^2 = 6k^2$$

2. [Plane Strain or axisym cases] show if $\sigma_{23} = \sigma_{13} = 0$, then $1 \Longrightarrow$

$$k^2 = (\sigma_{12}')^2 - \sigma_{11}'\sigma_{22}' - \sigma_{22}'\sigma_{33}' - \sigma_{33}'\sigma_{11}'$$

Previously our switches for definition of $\dot{\sigma} = E\dot{\varepsilon}$ required the unit normal to define where we were on yield surface.

Now define $\underset{\sim}{Q}$ outward unit normal, $\underset{\sim}{Q} = \frac{\partial f}{\partial \underset{\sim}{\sigma}} / \left|\partial f / \partial \underset{\sim}{\sigma}\right|$

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial}{\partial \sigma_{ij}}\left(\tfrac{1}{2}\sigma_{kl}'\sigma_{kl}'\right) = \sigma_{kl}' \frac{\partial \sigma_{kl}'}{\partial \sigma_{ij}};$$ to get this remember $\sigma_{kl}' = \sigma_{kl} - \frac{\sigma_{mm}}{3}\delta_{kl}$.

$$\text{thus } \frac{\partial \sigma_{kl}'}{\partial \sigma_{ij}} = \delta_{ik}\delta_{jl} - \frac{\delta_{im}\delta_{jm}\delta_{kl}}{3} = \boxed{\delta_{ik}\delta_{jl} - \frac{\delta_{ij}}{3}\delta_{kl} = \frac{\partial \sigma_{kl}'}{\partial \sigma_{ij}}}$$

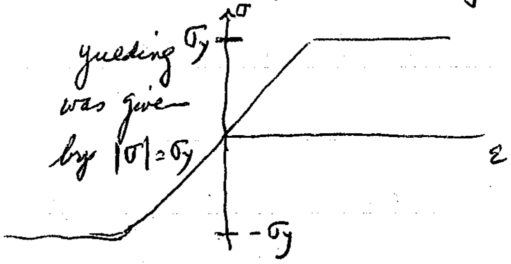
$$\frac{\partial f}{\partial \sigma_{ij}} = \sigma'_{kl}\left(\delta_{ik}\delta_{jl} - \tfrac{1}{3}\delta_{ij}\delta_{kl}\right)$$
$$= \sigma'_{ij} - \tfrac{1}{3}\delta_{ij}\sigma'_{kk}$$ but $\sigma'_{kk} = 0$ since deviatoric stress
$$= \sigma'_{ij}$$

$\therefore$ ~~on yield surface~~ : $\frac{\partial f}{\partial \underset{\sim}{\sigma}} = \underset{\sim}{\sigma}'$ $\quad\therefore\quad$ $\underset{\sim}{Q} = \dfrac{\frac{\partial f}{\partial \underset{\sim}{\sigma}}}{\left|\frac{\partial f}{\partial \underset{\sim}{\sigma}}\right|} = \underset{\sim}{\sigma}'/|\underset{\sim}{\sigma}'|$

on yield surface ~~that if~~ $f(\underset{\sim}{\sigma}) = \tfrac{1}{2}|\underset{\sim}{\sigma}'|^2 = k^2 \Rightarrow |\underset{\sim}{\sigma}'| = \sqrt{2}k = R$

thus $\boxed{\underset{\sim}{Q} = \underset{\sim}{\sigma}'/R}$ for yield surface

Ex: simple tension: $\sigma_{33} = \sigma \qquad \sigma_{ij} = 0 \quad i \neq 3, j \neq 3.$



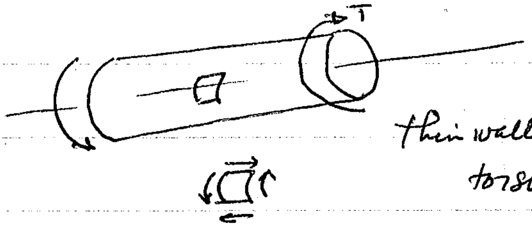
$(\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + \cdots = 6k^2$

we had $2\sigma_{33}^2 = 2\sigma^2 = 6k^2$ or $\underline{\sigma^2 = 3k^2}$ or $\sigma = \sqrt{3}k$

back in ex1 ~~from $\varepsilon(t) = \varepsilon_0 + \varepsilon_1(t) + \varepsilon_2 \sin t$~~

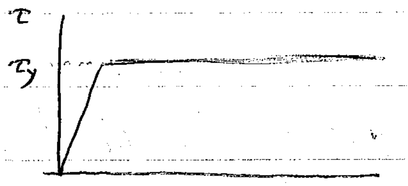
on y.s. $2|\sigma|^2 = 6k^2$ or $|\sigma| = \sqrt{3}k$ or $\underline{\sigma_y/\sqrt{3} = k}$

in Pure shear Assume $\sigma_{23} = \tau \neq 0$ & all σ_{ij} $i \neq 2, j \neq 3)$ are zero.



thin walled tube in pure torsion

if plot of τ vs γ

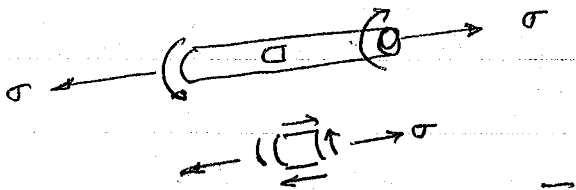


$|\tau| = \tau_y$ yield value

now exercise 1 gives $6\tau^2 = 6k^2 \quad \tau = k$

$\Rightarrow |\tau| = k \quad \Rightarrow$ but $|\tau| = \tau_y \quad \therefore \tau_y = k$

thus $\underset{\text{tension}}{\sigma_y/\sqrt{3} = k}$ & $\underset{\text{torsion}}{\tau_y = k} \Rightarrow \underline{\sigma_y/\sqrt{3} = \tau_y}$ since only one value of k exists combined tension / torsion



$\sigma_{33} = \sigma$ all others 0

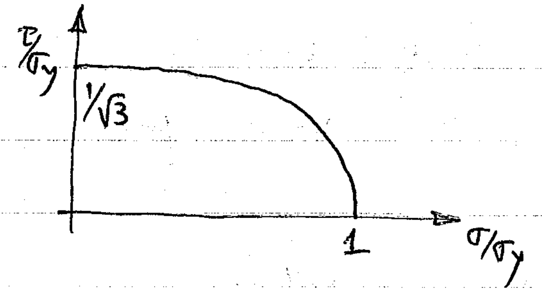
$\sigma_{23} = \tau$

$$\Rightarrow \quad 2\sigma^2 + 6\tau^2 = 6k^2$$

$$\sigma^2 + 3\tau^2 = 3k^2 = \sigma_y^2$$ but remember $3k^2 = \sigma_y^2$

$$\therefore \quad (\sigma/\sigma_y)^2 + 3\left(\frac{\tau}{\sigma_y}\right)^2 = \underline{1}$$

thus



See handout #2

exercise: consider ~~by~~ biaxial stress where the principal stresses.

$\sigma_1 \neq 0, \ \sigma_2 \neq 0, \ \sigma_3 = 0$

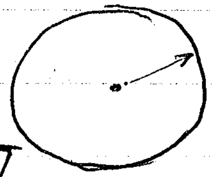
sketch the mises yield surface in $\frac{\sigma_1}{\sigma_y}$, σ_2/σ_y space.

1/20/83

Lots account - go to Ceras & open account

Distortional energy / Mises yield surface.

Assume elasticity
and write elastic strain ~~[illegible]~~
energy density for classical L.E. case : $\frac{1}{2}\sigma_{ij}\varepsilon_{ij}$



Now recall: $\sigma_{ij} = \sigma_{ij}' + \sigma_{ij}''$ where $\sigma_{ij}'' = +\frac{tr\,\underset{\sim}{\sigma}}{3}\delta_{ij}$

$\varepsilon_{ij} = \varepsilon_{ij}' + \varepsilon_{ij}''$

Handout #2
Notes
ME 239A

The yield criterion of von Mises has been shown to be in excellent agreement with experiment for many ductile metals, for example copper, nickel, aluminium, iron, cold-worked mild steel, medium carbon and alloy steels. The influence of the intermediate principal stress on yielding, and the corresponding failure of Tresca's criterion, was first clearly shown in the work of Lode[†] (1925), who stressed tubes of iron, copper,

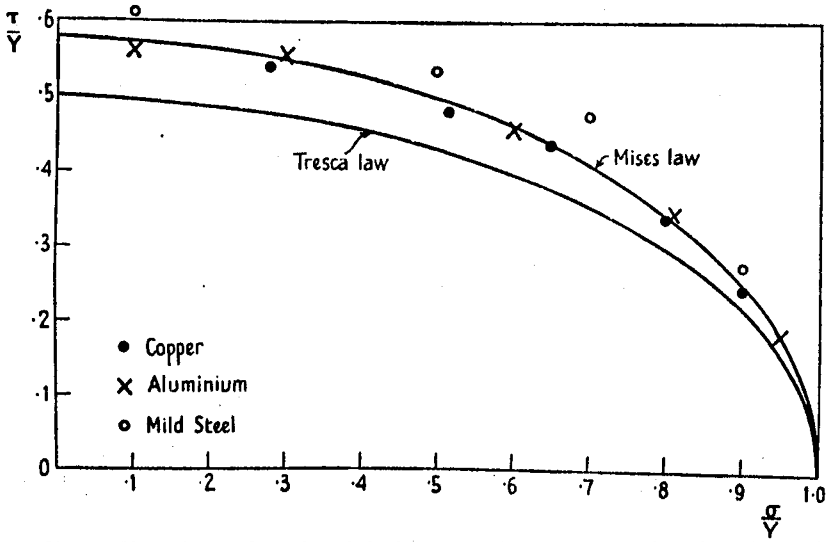


Fig. 4. Experimental results of Taylor and Quinney from combined torsion and tension tests, each metal being work-hardened to the same state for all tests. The Mises law is $\sigma^2+3\tau^2 = Y^2$, while the Tresca law is $\sigma^2+4\tau^2 = Y^2$, where σ = tensile stress, τ = shear stress, Y = tensile yield stress.

and nickel, under combined tension and internal pressure. The substantial accuracy of the Mises law was afterwards demonstrated by the work of Taylor and Quinney[‡] (1931), Lessells and MacGregor[§] (1940), and Davis[‖] (1945). As an example, the results of Taylor and Quinney are given in Fig. 4. Better agreement could occasionally be obtained by adding a small correction term in J_3', but in view of other differences between the ideal plastic body and a real metal, it would hardly be worthwhile in practical applications.

For the upper yield-point of annealed mild steel Tresca's law appears

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[†] W. Lode, *Zeits. ang. Math. Mech.* **5** (1925), 142; *Zeits. Phys.* **36** (1926), 913; *Forschungsarbeiten des Vereines deutscher Ingenieure*, **303** (1927).

[‡] G. I. Taylor and H. Quinney, *Phil. Trans. Roy. Soc.* A, **230** (1931), 323.

[§] J. M. Lessells and C. W. MacGregor, *Journ. Franklin Inst.* **230** (1940), 163.

[‖] E. A. Davis, *Trans. Am. Soc. Mech. Eng.* **62** (1940), 577; **65** (1943), A-187. See also Miller and Edwards, *Journ. Am. Petr. Inst.* (1939), 483; Marin and Stanley, *Journ. Am. Weld. Soc., Weld. Res. Suppl.* **19** (1940), 748.

to fit the data better th
by the sensitivity of the
for example, eccentricit
stress concentration in t
cated caution in acceptir
contained in the work
Putnam[‖] (1919), and R
experiments with anne
criterion of yielding app
men. It has been sugges
that the yield stress in
distribution is not unifo
pressure). This may be t
cal reasons for supposin
material has to be prod
Several attempts[‖‖] ha
polycrystal from the obs
crystal. No really satis
constraints between the

3. Strain-hardening

(i) *Dependence of the*
for a given state of the n
the whole of the previo
annealing. It will be
oriented, so that isotrop
cold work is usually negl

[†] J. L. M. Morrison, *Proc.*
144, 33.
[‡] J. J. Guest, *Phil. Mag.* **50**
Proc. Inst. Auto. Eng. **35** (194
[§] W. A. Scoble, *Phil. Mag.*
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also W. Mason, *Proc. Inst. M*
Dept. Bull. **85** (1916), 84.
[‡‡] G. Cook, *Engineering*, **13**
Proc. Roy. Soc. A, **137** (1932)
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[§§] G. Cook and A. Robertso
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[‖‖] G. Sachs, *Zeits. Ver. deut.*
Phys. Soc. **49** (1937), 134; U. I
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$$\tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} = \tfrac{1}{2}\left(\sigma_{ij}'\varepsilon_{ij}' + \underbrace{\sigma_{ij}'\varepsilon_{ij}''}_{①} + \underbrace{\sigma_{ij}''\varepsilon_{ij}'}_{②} + \underbrace{\varepsilon_{ij}''\sigma_{ij}''}_{③}\right)$$

① : $\sigma_{ij}'\left(\frac{\mathrm{tr}\,\underset{\sim}{\varepsilon}}{3}\delta_{ij}\right) = \frac{\mathrm{tr}\,\underset{\sim}{\varepsilon}}{3}\ \cancel{\sigma_{ii}'} \rightarrow 0$

② : $\sigma_{ij}''\varepsilon_{ij}' = \left(\frac{\mathrm{tr}\,\underset{\sim}{\sigma}}{3}\delta_{ij}\right)\left(\varepsilon_{ij}'\right) = \frac{\mathrm{tr}\,\underset{\sim}{\sigma}}{3}\ \cancel{\varepsilon_{ii}'} \rightarrow 0$ since ( )′( )″ are orthog.

③ : $\tfrac{1}{3}\sigma_{ij}''\varepsilon_{ij}'' = \frac{\mathrm{tr}\,\underset{\sim}{\sigma}}{3}\delta_{ij}\ \frac{1}{3}\mathrm{tr}\,\underset{\sim}{\varepsilon}\,\delta_{ij} = 3\cdot\frac{\mathrm{tr}\,\underset{\sim}{\sigma}}{3}\cdot\frac{1}{3}\mathrm{tr}\,\underset{\sim}{\varepsilon} = \frac{\mathrm{tr}\,\underset{\sim}{\sigma}\cdot\mathrm{tr}\,\underset{\sim}{\varepsilon}}{3}$

$\therefore\ \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} = \tfrac{1}{2}\sigma_{ij}'\varepsilon_{ij}' + \tfrac{1}{2}\left(\tfrac{1}{3}\mathrm{tr}\,\underset{\sim}{\sigma}\right)\mathrm{tr}\,\underset{\sim}{\varepsilon}$

↑ distortional energy = U_{dis} — A result of volume changes } akin to dilatational strain energy

want to relate Mises criterion to the above

remember $k^2 = \tfrac{1}{2}\sigma_{ij}'\sigma_{ij}'$

$= \tfrac{1}{2}\left(2\mu\,\varepsilon_{ij}'\sigma_{ij}'\right) = 2\mu\,U_{dis}$

$U_{dis} = k^2/2\mu$ for yielding

Recall $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} = \left[\mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \lambda\delta_{ij}\delta_{kl}\right]\varepsilon_{kl}$

$= \mu\,\varepsilon_{ij} + \mu\,\varepsilon_{ji} + \lambda\,\delta_{ij}\,\varepsilon_{kk} = \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}$

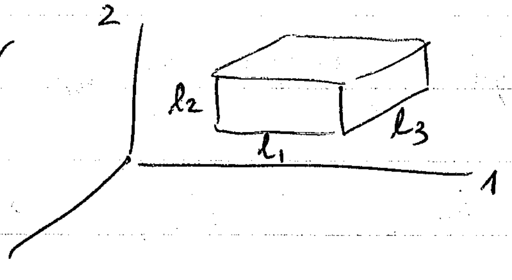
now $\frac{1}{3}\mathrm{tr}\,\underset{\sim}{\sigma} = \boxed{\lambda\,\frac{\varepsilon_{kk}\overset{=3}{\cancel{\delta_{ii}}} + 2\mu\,\varepsilon_{kk}}{3}} = \underbrace{\left(\lambda + 2/3\,\mu\right)}_{\text{Bulk Modulus.}}\varepsilon_{kk} \rightarrow \sigma_{ij}'' = \tfrac{1}{3}\mathrm{tr}\,\underset{\sim}{\sigma}\,\delta_{ij}$

$\sigma_{ij}' = 0 + 2\mu\,\varepsilon_{ij}' = \sigma_{ij} - \sigma_{ij}''$

Remember last time we had defined (E) (F) page 8B now must define $\dot{\varepsilon}_{kl}^{\,plastic}$

Observation: ~~plasting~~ plastic straining in metals (or undrained soils) <u>preserves</u> <u>volume</u> <u>locally</u>. What do we mean by the underlined?

Look at a parallelopiped



initial vol = $V = l_1 l_2 l_3$

deformed vol = $V+\Delta V = (1+\varepsilon_{11})l_1(1+\varepsilon_{22})l_2(1+\varepsilon_{33})l_3$

$$= \cancel{V}+\Delta V = \underbrace{(\varepsilon_{11}+\varepsilon_{33}+\varepsilon_{22})}_{tr\,\underset{\sim}{\varepsilon}}V + \cancel{V} + h.o.t.$$

$$\Delta V/V = tr\,\underset{\sim}{\varepsilon} + h.o.t.$$ since we assume small strains h.o.t. go to zero.

Thus if plastic strains ~~preserve~~ proportional to volume $\Rightarrow$ plastic strains are deviatoric strains

Recall $\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}' + \frac{tr\,\underset{\sim}{\varepsilon}}{3} I$ but $tr\,\underset{\sim}{\varepsilon} = 0 \Rightarrow \underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}'$

thus $\dot{\varepsilon}^{pl}_{ij}$ = deviatoric $\Rightarrow$ $\dot{\varepsilon}^{pl}_{ij} = 0$

also normality (dependent on stability): we will assume $\dot{\underset{\sim}{\varepsilon}}^{pl} = \lambda \underset{\sim}{Q}$ outward normal

factor of proportionality $\lambda(\underset{\sim}{x}, t)$.

This is the associative flow rule (like heat eqn?)

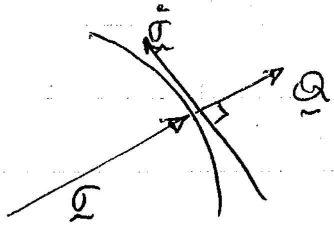
non-associative is $\dot{\underset{\sim}{\varepsilon}}^{pl} \not\propto \underset{\sim}{Q}$

Mises: $Q = \underset{\sim}{\sigma}'/R$
↑deviatoric

thus $\dot{\varepsilon} = \dot{\varepsilon}'$ which is deviatoric

∴ we are consistent

Now pick λ to satisfy consistency. During pl. flow, the stress remains on the yield surface.



$$\therefore \quad 0 = \underset{\sim}{Q} \cdot \overset{\circ}{\underset{\sim}{\sigma}} = Q_{ij} \overset{\circ}{\sigma}_{ij}$$

Now $\overset{\circ}{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot (\overset{\circ}{\underset{\sim}{\varepsilon}} - \overset{\circ}{\underset{\sim}{\varepsilon}}{}^{pl})$

$= \overset{\circ}{\underset{\sim}{\sigma}}{}^{tr} - \underset{\sim}{C}\, \overset{\circ}{\underset{\sim}{\varepsilon}}{}^{pl}$ we know direction of $\dot{\underset{\sim}{\varepsilon}}{}^{pl}$ but not app

$= \dot{\underset{\sim}{\sigma}}{}^{tr} - \lambda \underset{\sim}{C} \cdot \underset{\sim}{Q}$

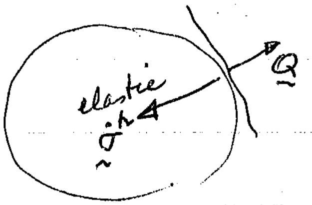
Invoking consist $\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}} = 0 = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}{}^{tr} - \lambda\, \underset{\sim}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim}{Q}$

now $\boxed{\lambda = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}{}^{tr} / \underset{\sim}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim}{Q}}$ note that $\underset{\sim}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim}{Q} > 0$

$\underset{\sim}{Q} \neq \underset{\sim}{0}$

$\therefore \quad \operatorname{sgn} \lambda = \operatorname{sgn} \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}{}^{tr}$

Recall elastic unloading $\Rightarrow \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}{}^{tr} < 0 \Rightarrow$ sign $\lambda < 0$ is elastic process



$\therefore$ $\lambda < 0 \Rightarrow$ this doesn't correspond to (P)

Now if $\lambda = 0 \Leftrightarrow \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}{}^{tr} = 0 \Rightarrow \dot{\underset{\sim}{\varepsilon}}{}^{pl} = \underset{\sim}{0}$ so we can consider it elastic

$\therefore$ for $\lambda > 0 \Rightarrow \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}{}^{tr} > 0 \Rightarrow$ (P)

thus $\lambda \leq 0$ (E) don't include this term in constit eqn since only elastic results

$\lambda > 0$ (P)

Now $\lambda = ?$, recall $\underset{\sim}{Q} = \underset{\sim}{\sigma}'/R$ thus $\boxed{\dfrac{\lambda}{R} = \dfrac{\underset{\sim}{\sigma}' \cdot \dot{\underset{\sim}{\sigma}}{}^{tr}}{\underset{\sim}{\sigma}' \cdot \underset{\sim}{C} \cdot \underset{\sim}{\sigma}'}}$

Let $\hat{\lambda}$ = lamé parameters $\underset{\sim}{C} \cdot \underset{\sim}{\sigma}' = C_{ijkl}(\sigma'_{kl}) = \hat{\lambda}\, \overset{0}{\cancel{tr\, \underset{\sim}{\sigma}'}}\, \delta_{ij} + 2\mu\, \sigma'_{ij}$

thus $\underset{\sim}{\sigma}' \cdot \underset{\sim}{C} \cdot \underset{\sim}{\sigma}' = 2\mu\, \sigma'_{ij} \sigma'_{ij} = 4\mu k^2$ on yield surface $= 2\mu R^2$

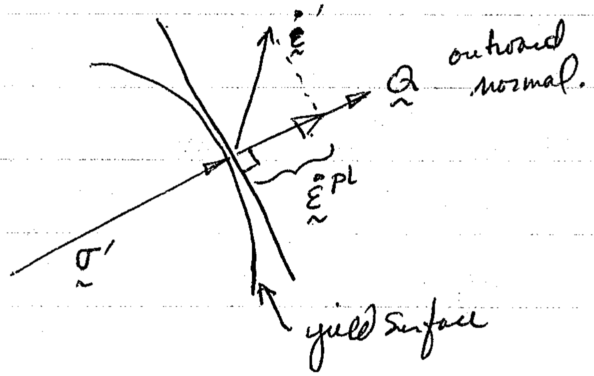
thus $\sigma' \cdot \dot{\sigma}^{tr} = \sigma_{ij}' \left(\hat{\lambda} \dot{\varepsilon}_{kk} \delta_{ij} + 2\mu \dot{\varepsilon}_{ij} \right) = \hat{\lambda} \dot{\varepsilon}_{kk} \cancelto{0}{tr\, \sigma'} + 2\mu \sigma_{ij}' \dot{\varepsilon}_{ij}$

$= 2\mu\, \sigma_{ij}' \dot{\varepsilon}_{ij}'$

since $a' \cdot b = a' \cdot b'$ ~~since~~ $\Rightarrow$ $\cancel{(a + tr\, a}$ $a'\left(b' + \frac{tr\, b}{3} I\right) = a' \cdot b' + \frac{tr\, b}{3} \cancelto{0}{tr\, a'} = a' \cdot b'$

Now $\frac{\lambda}{R} = \frac{2\mu \sigma' \cdot \dot{\varepsilon}'}{2\mu R^2} \Rightarrow \frac{\sigma' \cdot \dot{\varepsilon}'}{R} = \lambda$

thus $\dot{\varepsilon}^{pl} = \lambda Q = \frac{\sigma' \cdot \dot{\varepsilon}'}{R} \frac{\sigma'}{R} = (Q \cdot \dot{\varepsilon}') * Q$ what does this mean physically



———— × ————

Summary of elastic-perfectly plastic constit eqn w/ mises yieldsurface and associate flow rule.

$\dot{\sigma} = C(\dot{\varepsilon} - \dot{\varepsilon}^{pl}) = \dot{\sigma}^{tr} - C \dot{\varepsilon}^{pl}$

Givens for run

$\sigma, \dot{\sigma}^{tr}$

Find: $\dot{\sigma}$

$$\dot{\varepsilon}^{pl} = \begin{cases} 0 & (E) \\ \lambda Q & (P) \end{cases}$$

$\lambda Q = (Q \cdot \dot{\varepsilon}) Q = (\sigma' \cdot \dot{\varepsilon}) \sigma' / R^2$

and $\lambda = Q \cdot \dot{\varepsilon} = \left(\frac{Q \cdot \dot{\sigma}^{tr}}{Q \cdot C \cdot Q} \right)$

$$(E) = \begin{cases} f(\underset{\sim}{\sigma}) = k^2 \ \ \& \ \ \lambda \leq 0 \\ f(\underset{\sim}{\sigma}) < k^2 \end{cases}$$

$$(P) \ = \ f(\underset{\sim}{\sigma}) = k^2 \text{ and } \lambda > 0.$$

Radial Return Algorith

LLL : Hemp, dyna

SANDIA : Hondo

Convergent, resonably accurate.

Remember

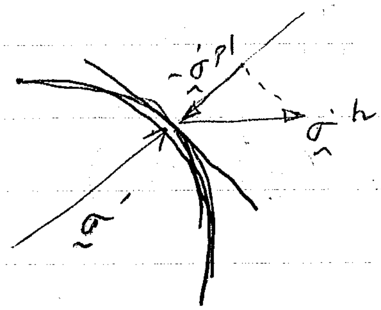
$$(E) \quad \int_{t_n}^{t_{n+1}} \left(\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C}\, \dot{\underset{\sim}{\varepsilon}} \right)$$

$$\underset{\sim}{\sigma}_{n+1}^{tr} = \underset{\sim}{\sigma}_n + \underset{\sim}{C} \cdot \Delta \underset{\sim}{\varepsilon}_n$$

(E) if $\underset{\sim}{\sigma}_{n+1}^{tr}$ is inside yield surface $\Rightarrow \sigma_{n+1} = \sigma_{n+1}^{tr}$

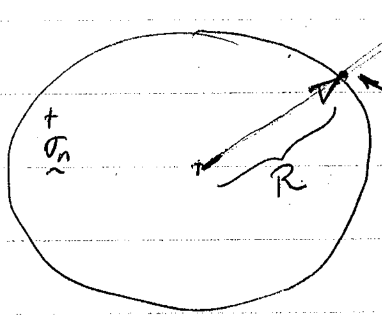
for (P) look at what theory says in the small & get its equiv in the large.

Motivation $\quad \dot{\underset{\sim}{\sigma}} = \dot{\underset{\sim}{\sigma}}^{tr} - \underbrace{\underset{\sim}{C} \cdot \dot{\underset{\sim}{\varepsilon}}^{pl}}$



"$\dot{\underset{\sim}{\sigma}}^{pl}$"-like quantity:
its trying to return
you to the yield surface.

Suppose



$+\underset{\sim}{\sigma}_{n+1}^{tr\prime}$ (this is where we were sent from $\underset{\sim}{\sigma}_n$)

let this pt $\underset{\sim}{\sigma}_{n+1}'$ be where we should be

thus $\quad \underset{\sim}{\sigma}_{n+1}' = R \ \underset{\sim}{\sigma}_{n+1}^{tr\prime} / |\underset{\sim}{\sigma}_{n+1}^{tr\prime}|$

$$f(\underset{\sim}{\sigma}_{n+1}) = \tfrac{1}{2} \, |\underset{\sim}{\sigma}_{n+1}'|^2 = \tfrac{1}{2} \underset{\sim}{\sigma}_{n+1}' \cdot \underset{\sim}{\sigma}_{n+1}'$$

$$= \frac{1}{2} R^2 \, \underset{\sim}{\sigma}_{n+1}^{tr\prime} \cdot \sigma_{n+1}^{tr\prime} / |\sigma_{n+1}^{tr\prime}|^2 = \tfrac{1}{2} \cancel{k} \cdot 2k^2 = k$$

$\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}'_{n+1} + \underset{\sim}{\sigma}^{tr\,''}_{n+1}$ ← mean trial stress

Algorith 1. $\underset{\sim}{\sigma}^{tr}_{n+1} = \underset{\sim}{\sigma}_n + \underset{\sim}{C} \cdot \Delta \underset{\sim}{\varepsilon}_n$

if $f(\underset{\sim}{\sigma}^{tr}_{n+1}) \leq k^2$, $\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1}$

if $> k^2$, $\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}'_{n+1} + \underset{\sim}{\sigma}^{tr\,''}_{n+1}$

$R\, \underset{\sim}{\sigma}^{tr\,'}_{n+1} / |\underset{\sim}{\sigma}^{tr\,'}_{n+1}|$

1/25/83

- Radial Return Algo rewritten for efficient coding

1. Calculate $\underset{\sim}{\sigma}^{tr}_{n+1} = \underset{\sim}{\sigma}_n + \underset{\sim}{C} \cdot \Delta \underset{\sim}{\varepsilon}_n$ calculate R^2 & store somewhere
2. " $\underset{\sim}{\sigma}^{tr\,'}_{n+1}$; $\underset{\sim}{\sigma}^{tr\,''}_{n+1}$
3. $\alpha = \underset{\sim}{\sigma}^{tr\,'}_{n+1} \cdot \underset{\sim}{\sigma}^{tr\,'}_{n+1}$ to determine if you are inside or outside surface
4. IF $\alpha \leq R^2$, then $\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1}$ (E)

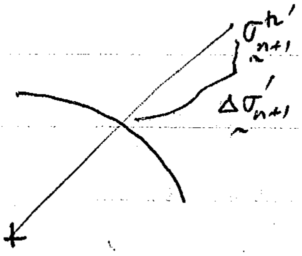
otherwise, $\underset{\sim}{\sigma}_{n+1} = \left(\frac{R}{\sqrt{\alpha}}\right) \underset{\sim}{\sigma}^{tr\,'}_{n+1} + \underset{\sim}{\sigma}^{tr\,''}_{n+1}$ (P)

Plastic strain $\Delta \underset{\sim}{\varepsilon}^{pl} = \frac{1}{2\mu} \Delta \underset{\sim}{\sigma}'_{n+1} = \frac{1}{2\mu} (\underset{\sim}{\sigma}^{tr}_{n+1} - \underset{\sim}{\sigma}_{n+1})$

ASIDE

define $\delta \underset{\sim}{\varepsilon} = \frac{1}{2\mu} \Delta \underset{\sim}{\sigma}'_{n+1}$; ↑ the above is a discrete version of the analytic formulate

$\dot{\underset{\sim}{\varepsilon}}^{pl} = (\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\varepsilon}}) \underset{\sim}{Q}$ $\dot{\underset{\sim}{\varepsilon}}$ or $\dot{\underset{\sim}{\varepsilon}}'$



now $\delta \underset{\sim}{\varepsilon} = (\delta \underset{\sim}{\varepsilon} \cdot \underset{\sim}{Q}) \underset{\sim}{Q}$

Example: "Mock" Fortan Prog

For: Plane Strain Axisym Case

Given EPSI = $\begin{Bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{33} \\ \Delta\gamma_{12} \end{Bmatrix}$ Find $\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \end{Bmatrix}$ = SIG(I)

```
C
C...  CALCULATE  TRIAL  STRESS
C
          DO   20  I = 1,4
          DO   10  J = 1,4
   10     SIG(I) = SIG(I) + C(I,J)*EPSI(J)
   20     CONTINUE
```

recode taking adv of the fact that

$$\underset{\sim}{C} = \begin{bmatrix} c_1 & C_2 & C_2 & \cdot \\ C_2 & c_1 & C_2 & \cdot \\ C_2 & C_2 & c_1 & \cdot \\ \cdot & \underline{Sym} & \cdot & C_3 \end{bmatrix}$$

$C_1 = 2\mu + \lambda$

$C_2 = \lambda$

$C_3 = \mu$

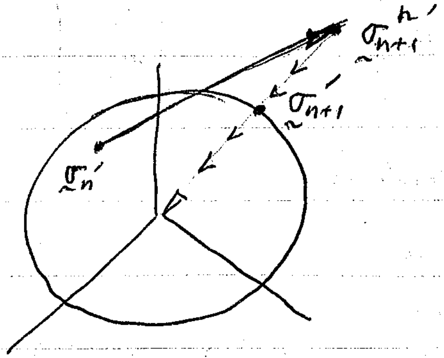
```
C
C....   CALC   STRESS  DEV
C
        SIGM= (SIG(1) + SIG(2) + SIG(3))/3.
        DO  30  I=1,3
   30   DEV(I) = SIG(I) - SIGM
        DEV (4) = SIG(4)
C
C...     CALC   ALPHA
C
        ALPHA =  DEV (1)**2  + DEV(2)**2  + DEV(3)**2 + 2* DEV(4)**2
C
C....    UPDATE  STRESS  (store R & RR=R^2 as consts)
C
        IF (ALPHA . LE. RR)   RETURN     (Elastic)
        ALPHA =  R /SQRT(ALPHA)
        DO  40   I=1,3
   40   SIG(I) = ALPHA*DEV(I) + SIGM
        SIG(4) =  ALPHA*DEV(4)
        RETURN
        END
```

exercise: generalize code to 3-D. ie ADD $\Delta\gamma_{23}$ $\Delta\gamma_{31}$

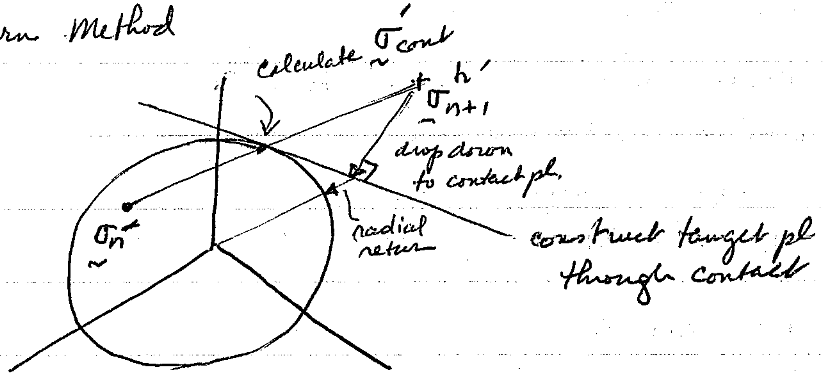
Let's compare algo's that are used in practice.

Graphical Comparison of Algos' in deviatric π- plane (elastic part is exact for all viable algorithms).

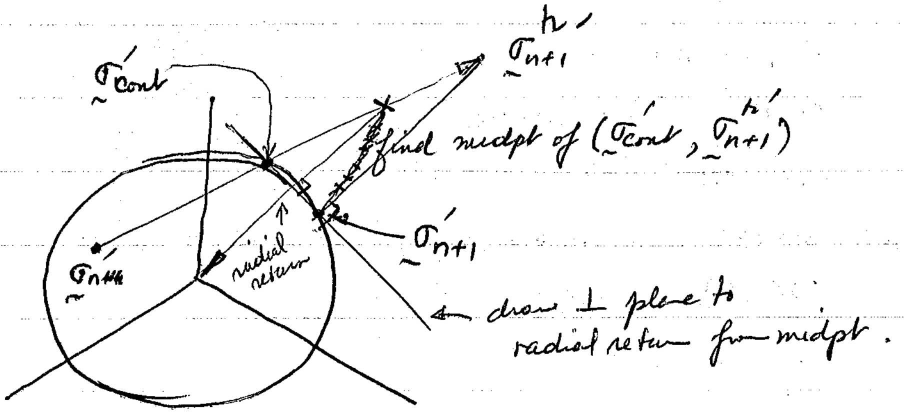
<u>Radial Return</u>



Tangent radial Return Method



<u>Secant Method</u>



How do we evaluate these algo's : tangent & secant are more complicated

<u>See Handout #1</u> : errors can be represented in terms of a single parameter in the π- plane.

no worse than 45°

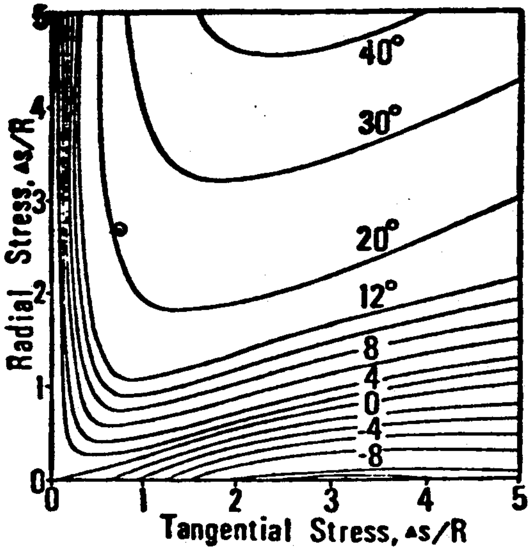


Fig. 5 Lines of constant angular error θ for the tangent-radial return method

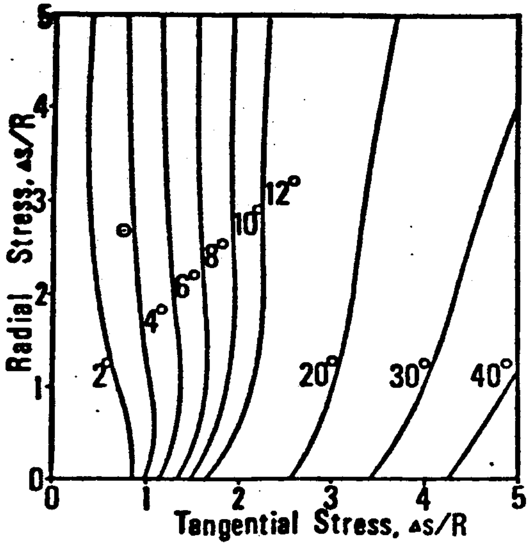


Fig. 6 Lines of constant angular error θ for the secant method

no worse than 45°

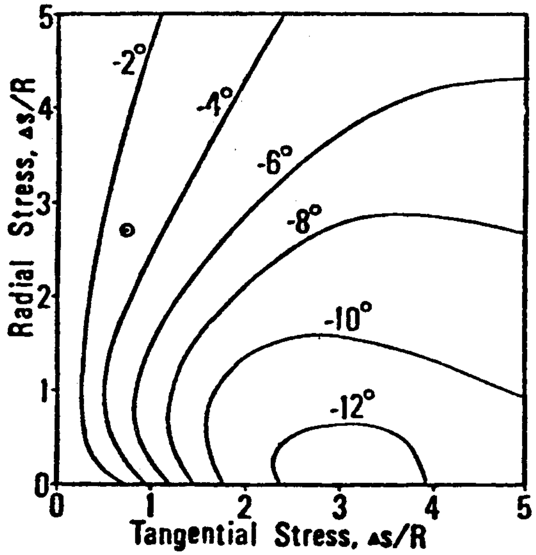


Fig. 7 Lines of constant angular error θ for the radial return method

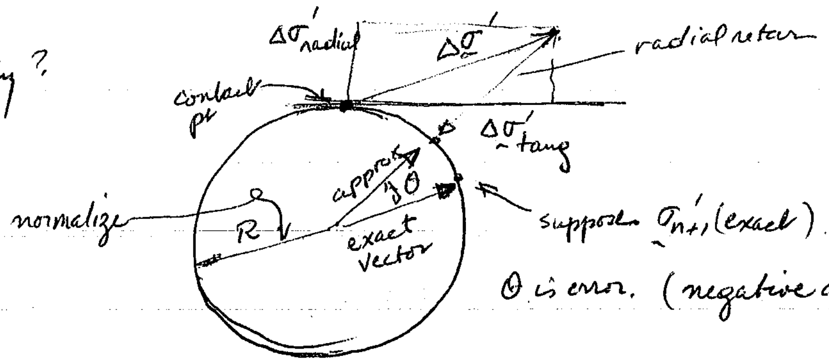
no error worse than 12.7°

1. $\underset{\sim}{S} = \underset{\sim}{\sigma}'$ our notation

2. Fig. 4 θ is neg. as shown

FROM: R.D. KRIEG & D.B. KRIEG, "ACCURACIES OF NUMERICAL SOLUTION METHODS FOR THE ELASTIC / PERFECTLY - PLASTIC MODEL," J. PRESSURE VESSEL TECHNOLOGY, 510 - 515, NOV. 1977

why ?



θ is error. (negative as shown

Remark : 1. all methods are exact for radial loading

2. Radial return seems to be the best.

<u>Exercise</u> Program the radial return algo & solve the following problems.

Take $\sigma_y = \sqrt{3}\,k = 1$; $E = 1$; $\nu = 0$; $\mu = E/2(1+\nu) = 1/2$

with nonzero strains defined by

$\varepsilon_0 + \varepsilon_1 t + \varepsilon_2 \sin t$ [Recall 1-D problems]

1. Const Vol triaxial test. $\Rightarrow \varepsilon_{33} \neq$ $\varepsilon_{11} = \varepsilon_{22} = -\frac{1}{2}\varepsilon_{33}$
rest $\equiv 0$.

Plot $(\sigma_{33} - \sigma_{11})$ vs. ε_{33} ie $(\sigma_{33} - \sigma_{22})$ vs ε_{33} is same plot.

2. Simple Shear $\gamma_{23} \neq 0 = \varepsilon_0 + \varepsilon_1 t + \varepsilon_2 \sin t$ all others $= 0$.

plot $\sqrt{3}\,\sigma_{23}$ vs $\gamma_{23} = 2\varepsilon_{23}$

Repeat for the 1-D case.

<u>Plane Stress case</u> : how to change "radial return"

Use plane strain / axisym & calculate $EPS(3) = \frac{-\lambda}{\lambda + 2\mu}\left[EPS1(1) + EPS1(2)\right]$

$$\begin{Bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\gamma_{12} \end{Bmatrix} \text{ must be expanded since } \Delta\varepsilon_{33} \neq 0 \Rightarrow \begin{Bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{33} \\ \Delta\gamma_{12} \end{Bmatrix}$$

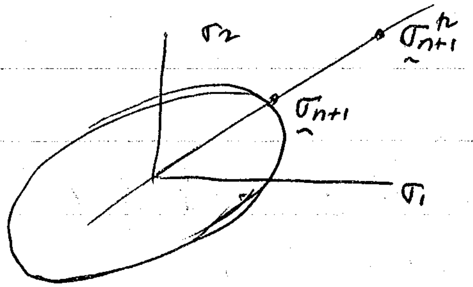
$$\underset{\sim}{\sigma}_{n+1}^{tr} = \underset{\sim}{\sigma}_n + \underset{\sim}{C} \cdot \Delta \underset{\sim}{\varepsilon}_n$$

Put $\sigma_{33} = 0$

$\Rightarrow \sigma_{33}^{tr} = 0$

1. Calculate $\underset{\sim}{\sigma}_{n+1}^{tr}$ from above (eps1(3) give from above)
2. Calc deviatoric stress.
3. IF $f(\underset{\sim}{\sigma}_{n+1}^{tr}) \le k^2$, $\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}_{n+1}^{tr}$, return

otherwise → radial return in the $\sigma_{33}=0$ subspace



formula:

$$\hat{\sigma}_{n+1} = R\, \underset{\sim}{\sigma}_{n+1}^{tr\,\prime} / |\underset{\sim}{\sigma}_{n+1}^{tr\,\prime}|, \quad \text{return.}$$

Check if $\underset{\sim}{\sigma}_{n+1}$ is on yield surface.

$$\underset{\sim}{\sigma}_{n+1}' \cdot \underset{\sim}{\sigma}_{n+1}' \overset{?}{=} R^2$$

$$\rightarrow R^2 / |\underset{\sim}{\sigma}_{n+1}^{tr\,\prime}|^2 \left(\underset{\sim}{\sigma}_{n+1}^{tr\,\prime} \cdot \underset{\sim}{\sigma}_{n+1}^{tr\,\prime} \right) = R^2 \quad \text{yes}$$

Elastic Plastic Moduli

idea here is the usefulness in calculating tangent statements.

Constit Eq is $\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} (\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}_{pl})$; want to write in the for

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot \dot{\underset{\sim}{\varepsilon}}$$

C here is $\underset{\sim}{C}^{el-pl}$

Recall $\dot{\underset{\sim}{\varepsilon}}^{pl} = (\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\varepsilon}}) \underset{\sim}{Q}$

are somewhat related

$$\dot{\sigma}_{ij} = C_{ijkl} \left(\dot{\varepsilon}_{kl} - (Q_{mn} \varepsilon_{mn}) Q_{kl} \right)$$

$$= C_{ijkl} \dot{\varepsilon}_{kl} - C_{ijkl} Q_{mn} \dot{\varepsilon}_{ml} Q_{kl}$$

$$= (C_{ijkl} - \underbrace{C_{ijkm}}_{\text{elastic plastic } c_{ijkl}} Q_{mn} Q_{kl}) \dot{\varepsilon}_{kl}$$

Tresca yield surface (1864)   (also known as "Guest Cond.")

1. Useful in solving some simple BVP by hand (piecewise flat)

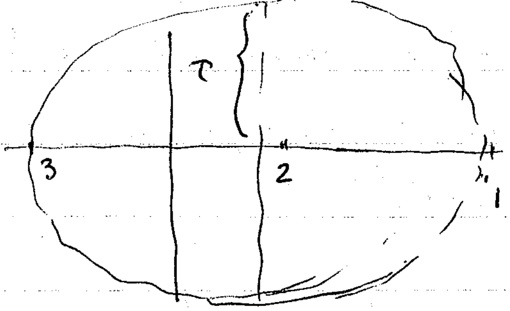
2. Some ambiguity exists due to "corners" or "vertices" of the surface.

3. It doesn't seem to fit experimental data as well as Mises: (1913) (Taylor-Quinney exper) (see handout #2).

4. Inconvenient numerically.

Tresca: plastic yielding occurs when the max. shear stress reaches a critical value, say k_T.   (from now on define k from before $= k_m =$ mises k).

Assume principal stress are ∋. $\sigma_1 \geq \sigma_2 \geq \sigma_3$



$\tau = \frac{1}{2}(\sigma_1 - \sigma_3)$

" max shear

we can also write it as $\max\limits_{i,j} \frac{1}{2}|\sigma_i - \sigma_j| = k_T$ yield condition of Tresca.

We must ask: is k_T an invariant quantity? is is a deviatoric quantity?

To answer look at $\begin{bmatrix} \sigma_1-\sigma_2 & & 0 \\ & \sigma_2-\sigma_3 & \\ 0 & & \sigma_1-\sigma_3 \end{bmatrix}$ now k_T = absolute value of max. EV of this matrix. EV of matrix are invariant

is it deviatoric $tr[\ \ ] = 0 \Rightarrow$ Deviatoric

this matrix = eigenvalues of σ = inv
– reordered EV of σ = inv   ∴ result is invariant.

$C_{ijkm} Q_{mn}$ then this equals $\lambda \overset{\text{deviatoric}}{Q_{mm}} \delta_{ij} + 2\mu Q_{ij}$

$$\Rightarrow \dot{\sigma} = \left(C_{ijkl} - 2\mu Q_{ij} Q_{kl}\right) \dot{\varepsilon}_{kl}$$

$$\underbrace{\left(\underset{\sim}{C} - 2\mu \underset{\sim}{Q} \otimes \underset{\sim}{Q}\right)}_{\underset{\sim}{C}^{el\,pl}} \dot{\underset{\sim}{\varepsilon}}$$

but Mises says $\underset{\sim}{Q} = \underset{\sim}{\sigma}'/R \Rightarrow \underset{\underset{\sim}{C}^{el\text{-}pl}}{\left(\underset{\sim}{C} - \frac{2\mu}{R^2} \underset{\sim}{\sigma}' \otimes \underset{\sim}{\sigma}'\right)}$ if we are plastic

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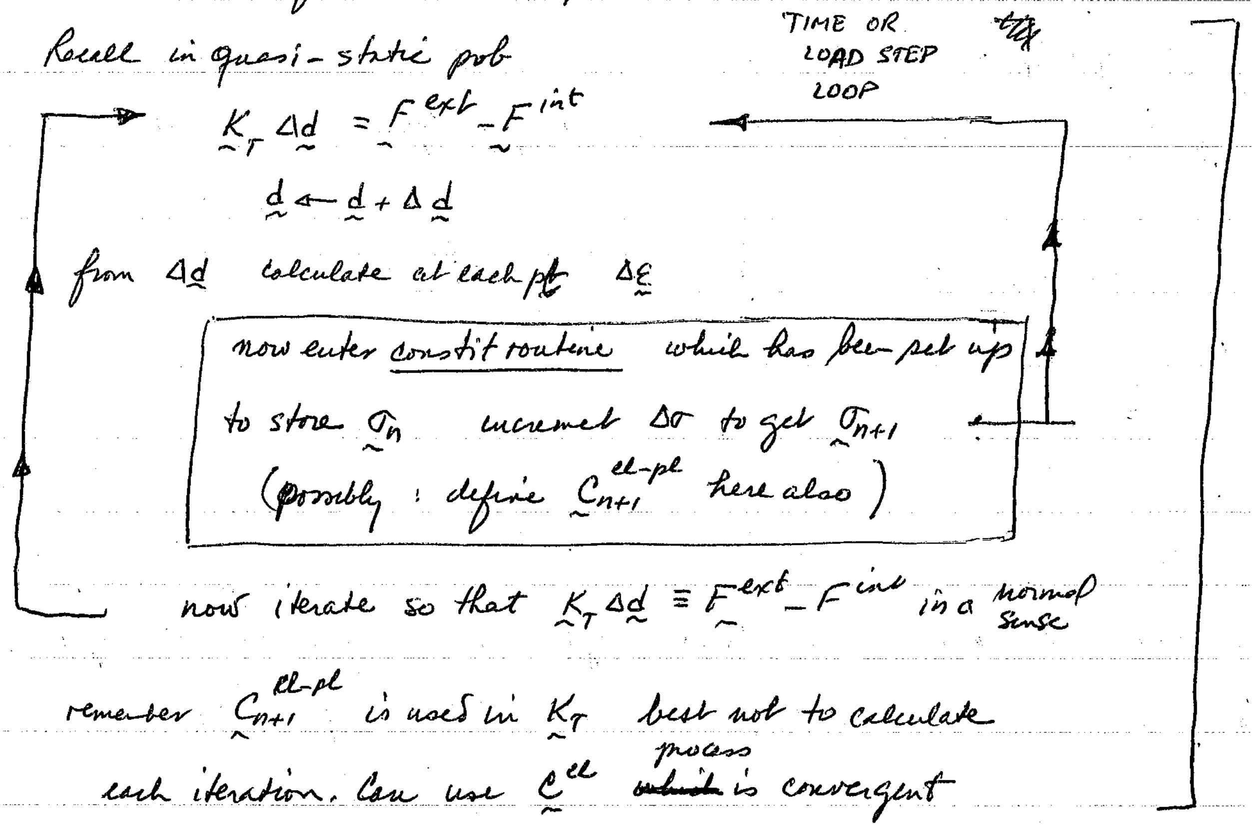
Where does all this fit into FE analysis



Recall in quasi-static prob

$$\underset{\sim}{K}_T \Delta \underset{\sim}{d} = \underset{\sim}{F}^{ext} - \underset{\sim}{F}^{int}$$

$$\underset{\sim}{d} \leftarrow \underset{\sim}{d} + \Delta \underset{\sim}{d}$$

from $\Delta \underset{\sim}{d}$ calculate at each pt $\Delta \underset{\sim}{\varepsilon}$

now enter constit routine which has been set up to store $\underset{\sim}{\sigma}_n$ increment $\Delta\sigma$ to get $\underset{\sim}{\sigma}_{n+1}$ (possibly: define $\underset{\sim}{C}_{n+1}^{el\text{-}pl}$ here also)

now iterate so that $\underset{\sim}{K}_T \Delta \underset{\sim}{d} \equiv \underset{\sim}{F}^{ext} - F^{int}$ in a normal sense

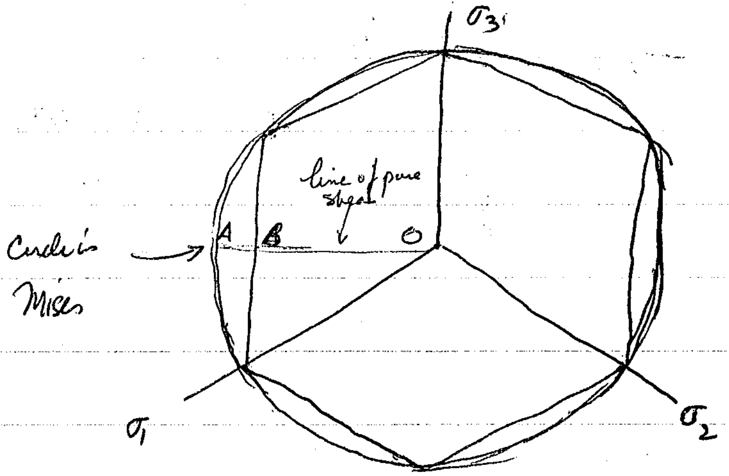
remember $\underset{\sim}{C}_{n+1}^{el\text{-}pl}$ is used in $\underset{\sim}{K}_T$ best not to calculate each iteration. Can use $\underset{\sim}{C}^{el}$ ~~which~~ process is convergent

Cautions: 1. incompressible kinematics must be well represented by means of mean pressure elements (see Ch IV of FE notes).

2. Must use sophisticated enough global iterative strategy (ie for elastic unloading use stiffer tangent modulus (ie C^{el}) in order to converge). Rember $K^{el\text{-}pl}$ is softer than $\tilde{K}^{el}$ hence its use in elastic region leads to divergence.

Deviatoric ⇒ Tresca Y.S. is a "cyl." w/ pressure axis as a generator.

look at Π-plane in the deviatoric subspace

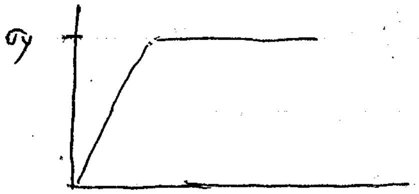


Tresca Hexagon looking along the pressure axis.

ex: if $\sigma_1 = \sigma_2 = 0$, $\sigma_3 \neq 0$

$k_T = \frac{1}{2}|\sigma_3|$.

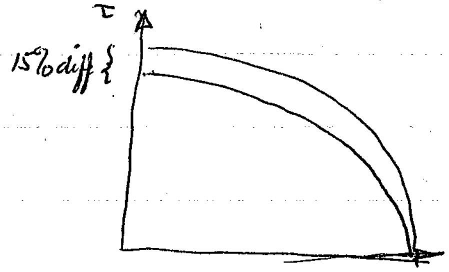
simple tension exper: $\sigma_3 = \sigma \neq 0$ rest $= 0$



$\sigma_y = \sigma \Rightarrow k_T = \frac{1}{2}|\sigma_y|$

(from Von Mises remember $k_m = \sigma_Y/\sqrt{3}$)

Thus $k_T = \frac{\sqrt{3}}{2} k_m = .866\, k_m$ Max values ⇒ $\hat{OA} = \frac{2}{\sqrt{3}}\hat{OB} = 1.155\,\hat{OB}$

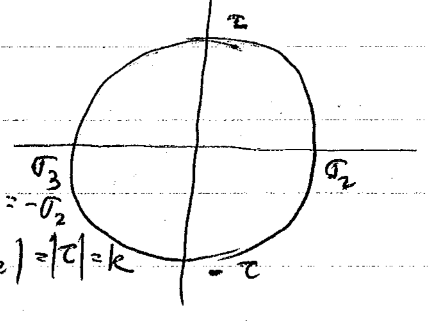


Remember Taylor Quinney data

Recall pure shear $\sigma_{23} = \tau \neq 0$ rest $= 0$ (recall $|\tau| = k_m$ from Mises)

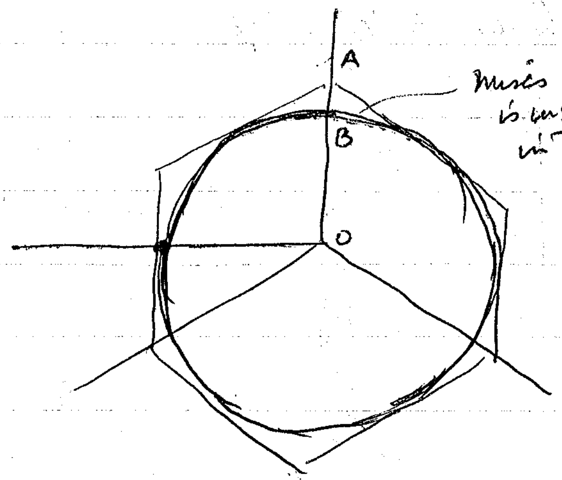
for Tresca.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \tau & 0 \end{bmatrix}$$



now $k_T = \frac{1}{2}|\sigma_2 - \sigma_3| = \frac{1}{2}|2\sigma_2| = |\sigma_2| = |\tau| = k$

identification of k_T, k_m on pure exper'' $\Rightarrow$ $k_T = k_m$



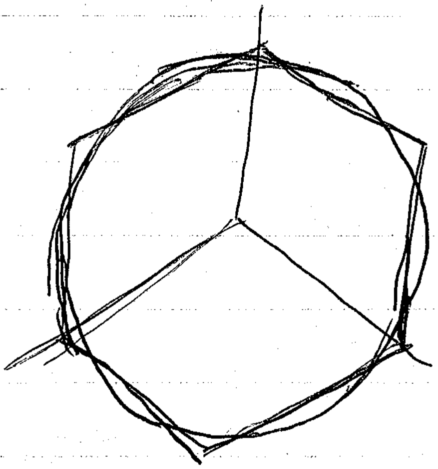
Mises is inscribed in Tresca.

now $\widehat{OA} = \frac{2}{\sqrt{3}} \cdot \widehat{OB}$
$= 1.155\ \widehat{OB}$

thus summarizing tensile : $k_T = .866 k_m$
shear : $k_T = 1.\ k_m$

some books have a "best fit" hexagon which is about 8% off. $k_T = \frac{.866+1}{2} k_m$

best fit



Hw

<u>exercise 1</u> Assume $\left(k_T = \frac{\sqrt{3}}{2} k_m\right)$

Consider combined tension-torsion

$[\sigma_{33} = \sigma \neq 0,\ \sigma_{23} = \tau \neq 0,\ \text{rest} = 0]$

recall $\sigma^2 + 3\tau^2 = \sigma_y^2$: Mises

derive for Tresca

$\sigma^2 + 4\tau^2 = \sigma_y^2$

} Taylo
Quinny
Picture

Tresca yield condition in this case is a ellipse (use Mohr's circle)

in a previous exercise: we plotted Mises in σ_1/σ_y, σ_2/σ_y space
for biaxial stress. $[\sigma_1 \neq 0, \sigma_2 \neq 0, \sigma_3 \equiv 0]$

$$\begin{bmatrix} x & x & 0 \\ x & x & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Derive & superimpose the tresca yield surface.

Calculate major & minor axis & intercepts.

3. Determine the plastic strain rate ratios $\dot{\varepsilon}_1^{pl} ; \dot{\varepsilon}_2^{pl} : \dot{\varepsilon}_3^{pl}$ for mises & following stress states :

a uniaxial tension $\sigma_{33} = \sigma_y$; rest = 0

b biaxial state : $\sigma_{11} = -\sigma_y/\sqrt{3}$, $\sigma_{22} = \sigma_y/\sqrt{3}$

Check that you are on yield surface.

c. pure shear : $\sigma_{23} = \sigma_y/\sqrt{3}$, rest = 0.

d. biaxial tension : $\sigma_1 = \sigma_{22} = \sigma_y$; rest = 0

e. Combine tension/torsion : $\sigma_{33} = \sigma_y/2$, $\sigma_{23} = \sigma_y/2$; rest = 0

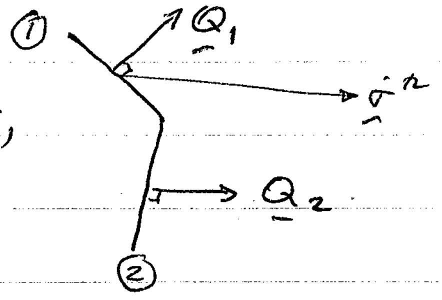
Now let's put theory together for El-Perf Pl Constitutive Theory w/ Tresca yield Surface. $\dot{\underline{\sigma}} = \underset{\sim}{C}(\dot{\underline{\varepsilon}} - \dot{\underline{\varepsilon}}^{pl})$

Remember we had to define (E), (P) $\dot{\underset{\sim}{\varepsilon}}^{pl}$: idea about the same as Mises

but there are complications at vertices. Use normality conditions as before off the vertices. Normality cannot be defined at vertices.

define $g(\underset{\sim}{\sigma}) = \frac{1}{2} \max_{ij} |\sigma_i - \sigma_j|$

(E) $g(\underset{\sim}{\sigma}) < k_T$



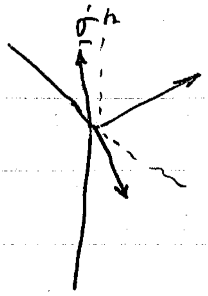
1. if we are on surface & not at vertex $g(\underset{\sim}{\sigma}) = k_T$, we want to see if $\dot{\underset{\sim}{\sigma}}^{tr} : \underset{\sim}{Q} < 0 \Rightarrow$ we are inside

2. if we are on surface & at a vertex then we want $\dot{\underset{\sim}{\sigma}}^{tr} \cdot \underset{\sim}{Q}_{(i)} < 0$ for both $i = 1, 2$.

(P) $g(\underset{\sim}{\sigma}) = k_T$ & pointing out (we're on the surface)

(1) not a vertex $\dot{\underset{\sim}{\sigma}}^{tr} : \underset{\sim}{Q} \geq 0$

(2) at a vertex : either $\dot{\sigma}^{tr} : Q \geq 0$ or $\dot{\sigma}^{tr} : Q \geq 0$



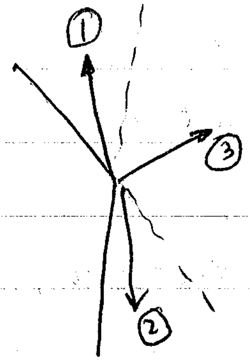
what is $\dot{\underset{\sim}{\varepsilon}}^{pl}$

$\dot{\underset{\sim}{\varepsilon}}^{pl} = 0$ (E)

if plastic :

— Suppose not a vertex

$\dot{\underset{\sim}{\varepsilon}}^{pl} = \lambda \underset{\sim}{Q}$



— pt is a vertex :

IF $\dot{\underset{\sim}{\sigma}}^{tr} \cdot \underset{\sim}{Q}_1 \geq 0$ but $\dot{\underset{\sim}{\sigma}}^{tr} \cdot \underset{\sim}{Q}_2 < 0$ ① $\dot{\underset{\sim}{\varepsilon}}^{pl} = \lambda \underset{\sim}{Q}_1$

IF $\dot{\underset{\sim}{\sigma}}^{tr} \cdot \underset{\sim}{Q}_2 \leq 0$ " " > 0 ② $\dot{\underset{\sim}{\varepsilon}}^{pl} = \lambda \underset{\sim}{Q}_2$

IF $\dot{\underset{\sim}{\sigma}}^{tr} \cdot Q_{(i)} \geq 0$ for $i = 1,2$ ③ $\dot{\underset{\sim}{\varepsilon}}^{pl} = \lambda_i \underset{\sim}{Q}_i$

this is the Prager-Koiter Condition

how to get λ : if one λ Consistency $\Rightarrow$ $\underset{\sim}{Q}_i \cdot \dot{\underset{\sim}{\sigma}} = \underset{\sim}{Q}_i (\dot{\sigma}^{tr} - \lambda \underset{\sim}{C} \cdot Q_i)$

& solve for λ as if for Mises

ie $\lambda = \underset{\sim}{Q}_i \cdot \dot{\underset{\sim}{\sigma}}^{tr} / \underset{\sim}{Q}_i \cdot \underset{\sim}{C} \cdot \underset{\sim}{Q}_i$

: if 2 lambdas Consistency must still hold w respect to both branches

$\dot{\underset{\sim}{\sigma}} = \dot{\underset{\sim}{\sigma}}^{tr} - \lambda_1 \underset{\sim}{C} \cdot \underset{\sim}{Q}_1 - \lambda_2 \underset{\sim}{C} \cdot Q_2$

now dot w/ $\underset{\sim}{Q}_i$ $i = 1,2$ & set $= 0$ to give λ_1, λ_2

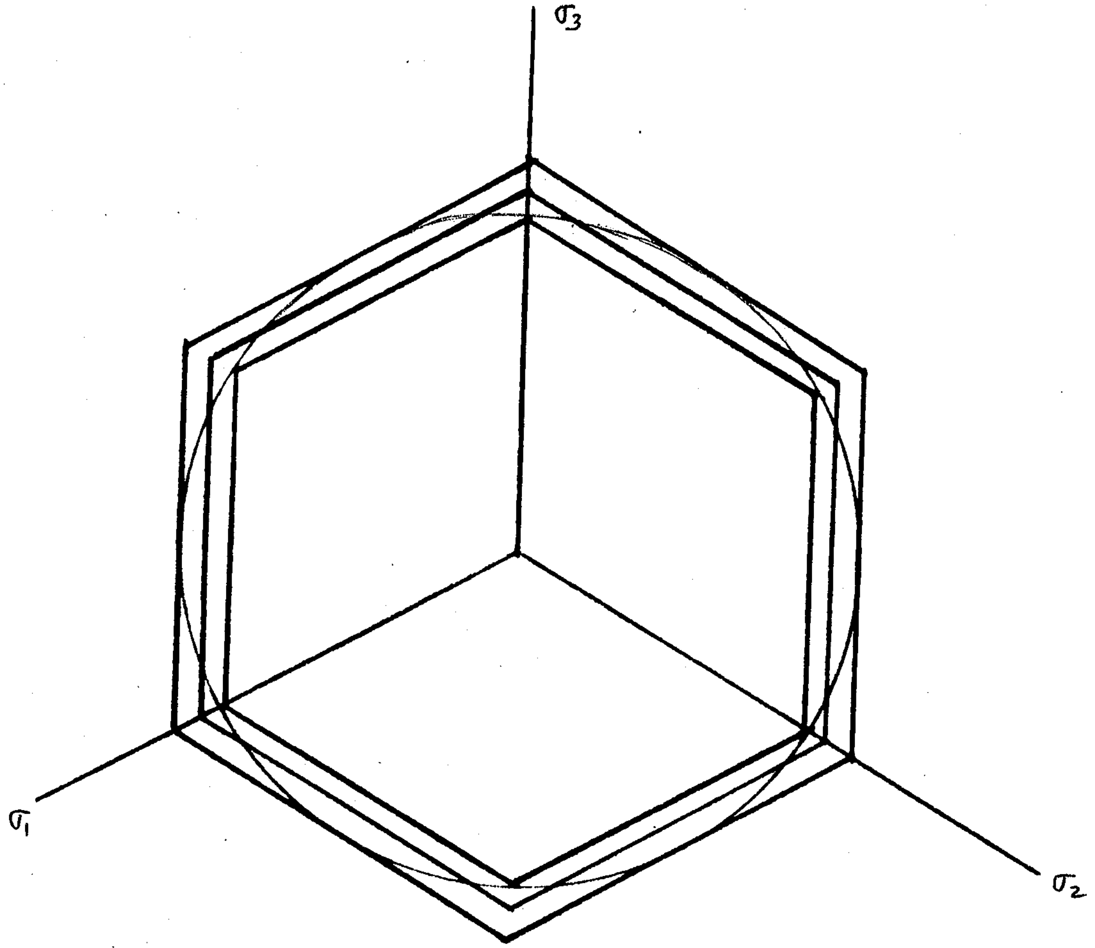
now $\Rightarrow$ $\alpha_{ij} \lambda_j = \underset{\sim}{Q}_i \cdot \dot{\underset{\sim}{\sigma}}^{tr}$ $i = 1,2$

where $\alpha_{ij} = \underset{\sim}{Q}_i \cdot \underset{\sim}{C} \cdot \underset{\sim}{Q}_j$ solve for λ_1, λ_2

<u>exercise</u> : Argue, Prove, Accept or Devise :

that $\underset{\sim}{\varepsilon}^{pl} // \underset{\sim}{\sigma}^{tr}$ for the last case.

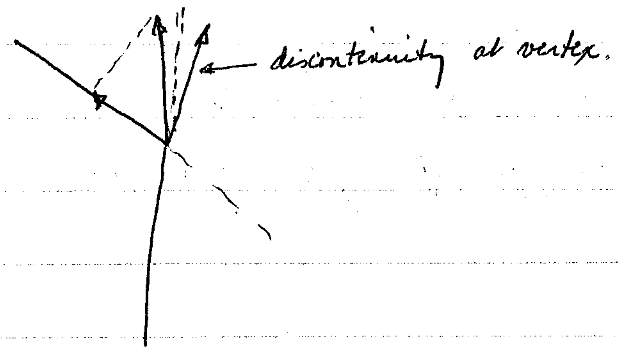
Handout #3
Notes
ME 239A



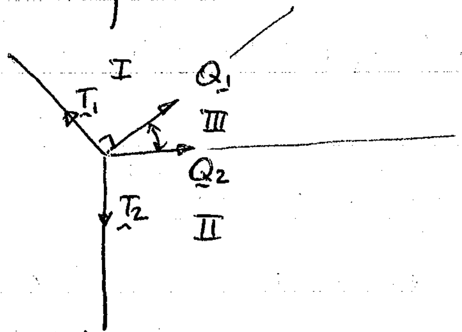
2/1/83

Omit case (iv) in the radial return algo. calculators.

Correction wrt Tresca in plastic loading at a vertex.



To remove discontinuity when we cross line ...... define new regions



Test as follows:

(E) same as before

(P) away from a vertex: same

(P) at a vertex (replace before by the following)

if in zone I, load zone I

if in " II " " II

if in " III stay at vertex

Zone

$\therefore \quad \dot{\varepsilon}^{pl} = \lambda \underset{\sim}{Q}_1 \qquad \dot{\underset{\sim}{\sigma}}^{tr} \cdot \underset{\sim}{T}_1 \geq 0$ and $\dot{\underset{\sim}{\sigma}}^{tr} \cdot \underset{\sim}{T}_2 < 0$ I

" $= \lambda Q_2 \qquad$ " $\underset{\sim}{T}_2 \geq 0$ and $\dot{\underset{\sim}{\sigma}}^{tr} \cdot \underset{\sim}{T}_1 < 0$ II

$= \lambda_1 \underset{\sim}{Q}_1 + \lambda_2 Q_2$ if both $\dot{\underset{\sim}{\sigma}}^{tr} \cdot \underset{\sim}{T}_1$ & $\dot{\underset{\sim}{\sigma}}^{tr} \cdot T_2 < 0$ III

Voluntary exercise: define @ vertex $\underset{\sim}{Q}_1, \underset{\sim}{Q}_2, \underset{\sim}{T}_1, \underset{\sim}{T}_2$

Continuing w/ Tresca and revisit "invariance & the deviatoric".

Recall yield surface $\frac{1}{2}\max_{i,j} |\sigma_i - \sigma_j| = k$,

whereas $\sigma_i' - \sigma_j' \Rightarrow$ DEV

Alternate ways of writing yield condition

Square both sides ① $\max_{i,j} (\sigma_i' - \sigma_j')^2 = 4k_T^2 \Longleftrightarrow$

② say one or 2 of the following ~~are~~ satisfied

$(\sigma_2' - \sigma_3')^2 = 4k_T^2$

$(\ 3 \quad 1\ )^2 = \text{''}$

$(\ 1 \quad 2\ )^2 \quad \text{''}$

if one or more are satisfied $\Longrightarrow 0 = [(\sigma_2' - \sigma_3')^2 - 4k_T^2][(\sigma_3' - \sigma_1')^2 - 4k_T^2][(\sigma_1' - \sigma_2')^2 - 4k_T^2]$

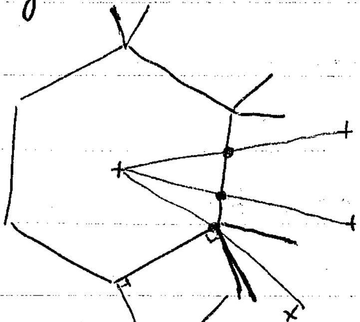
③ exercise show that means $\Longleftrightarrow 4(J_2')^3 - 27(J_3')^2 - 36k_T^2 J_2'^2 + 96k_T^4 J_2' - 64k_T^6 = 0$.

[Hint: use identities $J_2' = \frac{1}{2}\sum(\sigma_i')^2$
$J_3' = \frac{1}{3}\sum(\sigma_i')^3$]

Algo for tresca (using radial return? $\Rightarrow$ bad idea

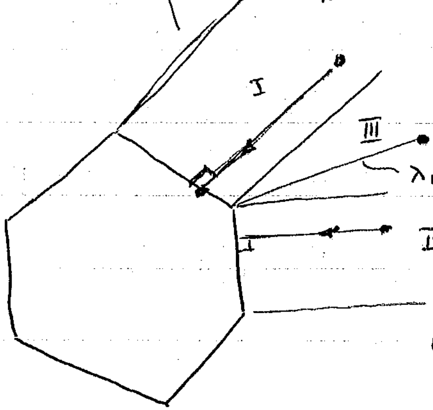
↓

exceptional to arrive at a vertex. But the vertices will be attractive



)

Better algorithm



Strategy in case $\underset{\sim n+1}{\sigma}^{tr'}$ is in I→III

$\lambda_1 Q_1 + \lambda_2 Q_2$ in I: project normally

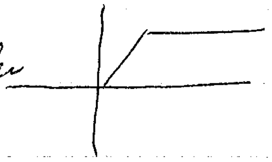
in II: " "

in III project along $\lambda_1 \underset{\sim}{Q}_1 + \lambda_2 \underset{\sim}{Q}_2$

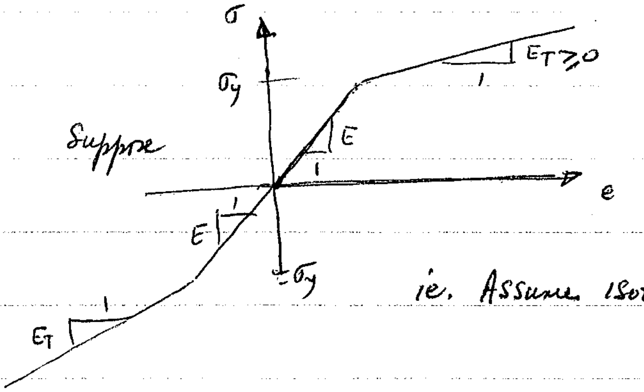
important thing here is that response is continuous across zones boundaries.

but note that the model is complicated to implement,

Now we move to hardening: so far

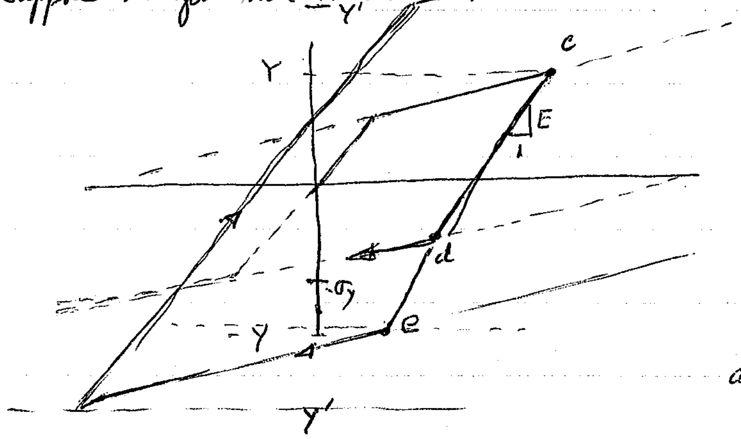


Suppose



ie. Assume isotropy

Suppose we get non-monotonic $\dot{\varepsilon}$



@ c : $\hat{\varepsilon} < 0$

when do we reload

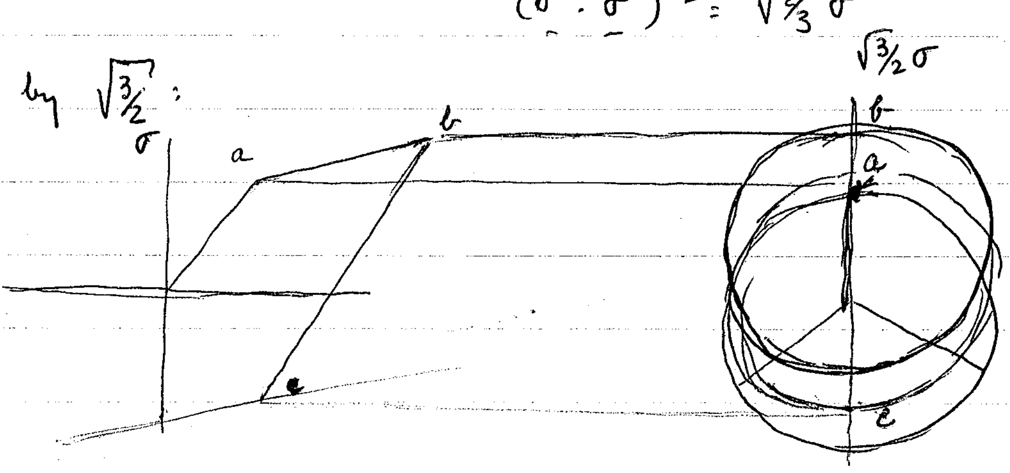
d : possibility reload at d
known @ KINEMATIC HARDENING
e : "isotropic hardening"
anywhere in between is another poss.
is a combination of kine + isotr

Multi-D –

Define uniaxial tension $\sigma_{33}\overset{\sigma}{/\!/} \neq 0$ in dev plane $\begin{bmatrix} \frac{2}{3}\sigma & & \\ & -\frac{\sigma}{3} & \\ & & -\frac{\sigma}{3} \end{bmatrix} = \sigma'$

$$(\underline{\sigma}' \cdot \underline{\sigma}')^{1/2} = \sqrt{2/3}\,\sigma$$

Scale by $\sqrt{3/2}$:



This is kinematic hardening

Size & Shape of YS is fixed
position in deviat space is not

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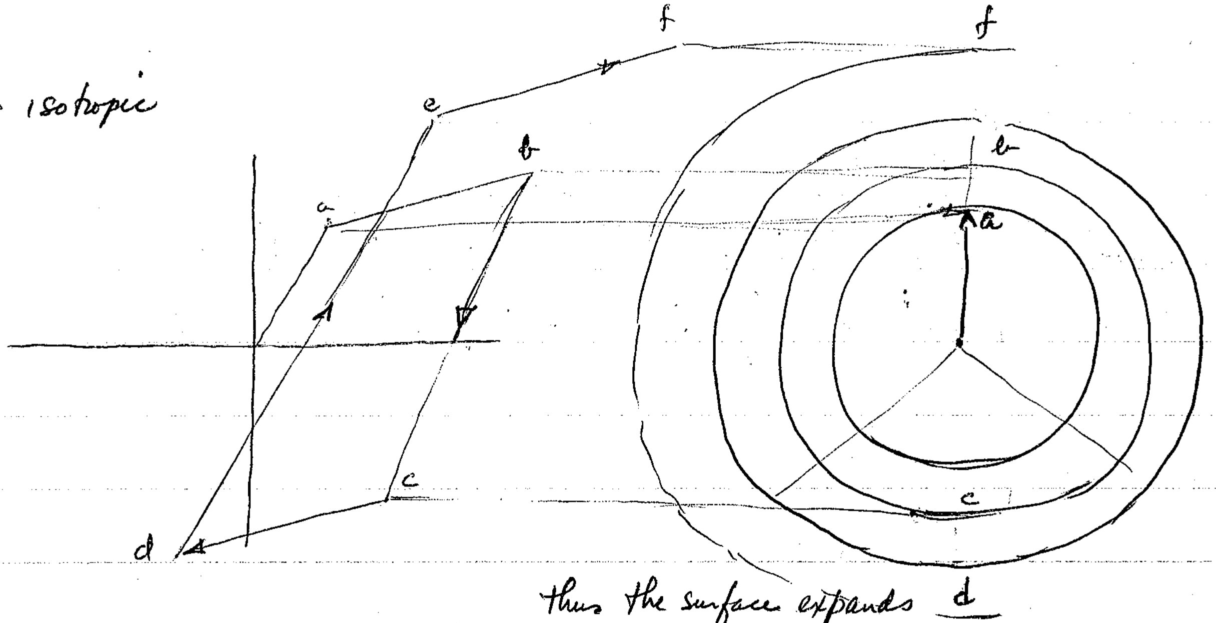
for isotropic



thus the surface expands
to accomodate ~~movement of stress~~ plastic loading
but sits there during elastic process
<u>Can never</u> return to original surface. Not good
physical model.

① for isotropic : k characterizes yield surface. here k is no longer const.

② kinematic : k is constant but center is not zero in deviatoric plane.
thus "center": $\underset{\sim}{\alpha} = [\alpha_{ij}'] \neq \underset{\sim}{0}$. $= \underset{\sim}{\alpha}'$ ie deviatoric itself

center is also known as the "back stress" :

③ in the combined theory $k \neq$ const $\underset{\sim}{\alpha} \neq 0$.

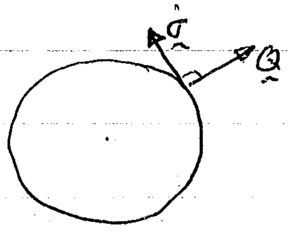
Now we will set up a const't theory isotropic hardening (mises yield surface, assoz. flow rule)

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot (\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^{pl})$$

$$\dot{\varepsilon}^{pl} = \begin{cases} 0 & (E) \\ \lambda \underset{\sim}{Q} & (P) \end{cases}$$ λ is to be determined

definition of (E) & (P) same as in elastic-perfectly plastic case

when we write $f(\underset{\sim}{\sigma}) = k^2 \neq$ const. Things that determine λ is consistency
need consistency for this new law : To do this
Recall : Consist for el-perf. pl $\dot{\underset{\sim}{\sigma}} \cdot \underset{\sim}{Q} = 0$



This is a consequence of staying on yield surface.

alternatively we can say for consistency

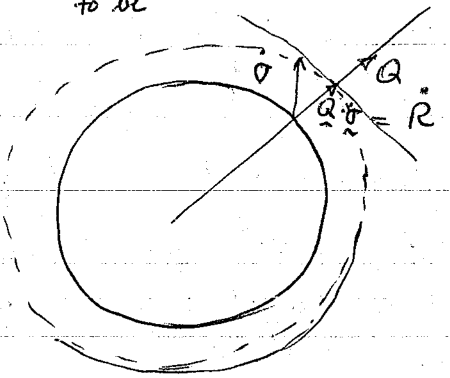
$\underset{\sim}{\sigma}$ satisfies $(f(\underset{\sim}{\sigma})-k^2)=0$ as a fn of time

time differentiate thus $\dot{f}(\underset{\sim}{\sigma}) = 2k\dot{k}$; for elastic perf plastic $\underline{\dot{k}=0}$.

but for elastic-perf pl: $\dot{f} = \frac{\partial f}{\partial \underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\sigma}} = \underset{\sim}{Q}\left|\frac{\partial f}{\partial \sigma}\right| \cdot \dot{\underset{\sim}{\sigma}} = 0 \Rightarrow \underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}} = 0$ QED

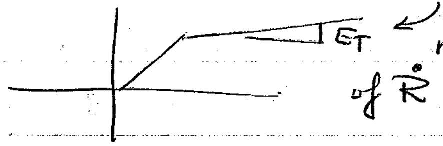
for isotropic: consistency $\Rightarrow \underbrace{\dot{f}(\underset{\sim}{\sigma})}_{\underbrace{\left|\frac{\partial f}{\partial \underset{\sim}{\sigma}}\right|}_{\text{derived this to be } \sigma' = R = \sqrt{2}k} Q\cdot\dot{\underset{\sim}{\sigma}}} = 2k\dot{k} \neq 0$

thus $\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}} = \sqrt{2}\,\dot{k} = \dot{R}$



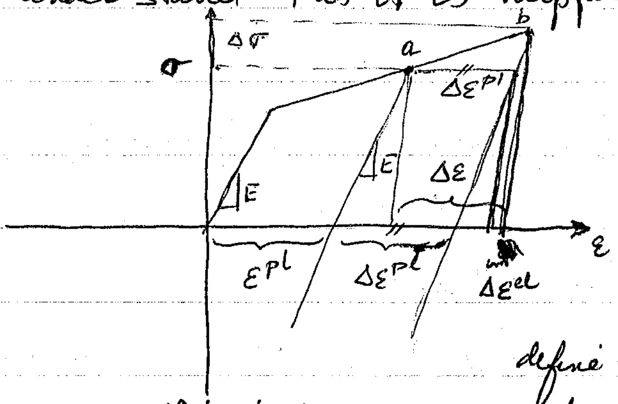
growth rate of circle

now what is $\dot{R}$? 1D picture



restricts form of $\dot{R}$

To understand this it is helpful to define the "plastic modulus" first

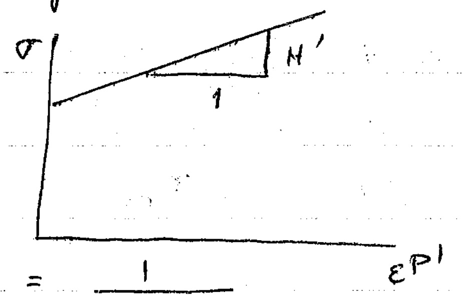


hardening $\Rightarrow$ during plastic loading $\dot{\varepsilon}^{pl} \neq \dot{\varepsilon}$

now how does σ change w ε^{pl}

define plastic modulus $= H' = \frac{d\sigma}{d\varepsilon^{pl}} = \frac{\Delta\sigma}{\Delta\varepsilon^{pl}}$

this is slope of σ vs. ε^{pl} curve



$H' > E_T$

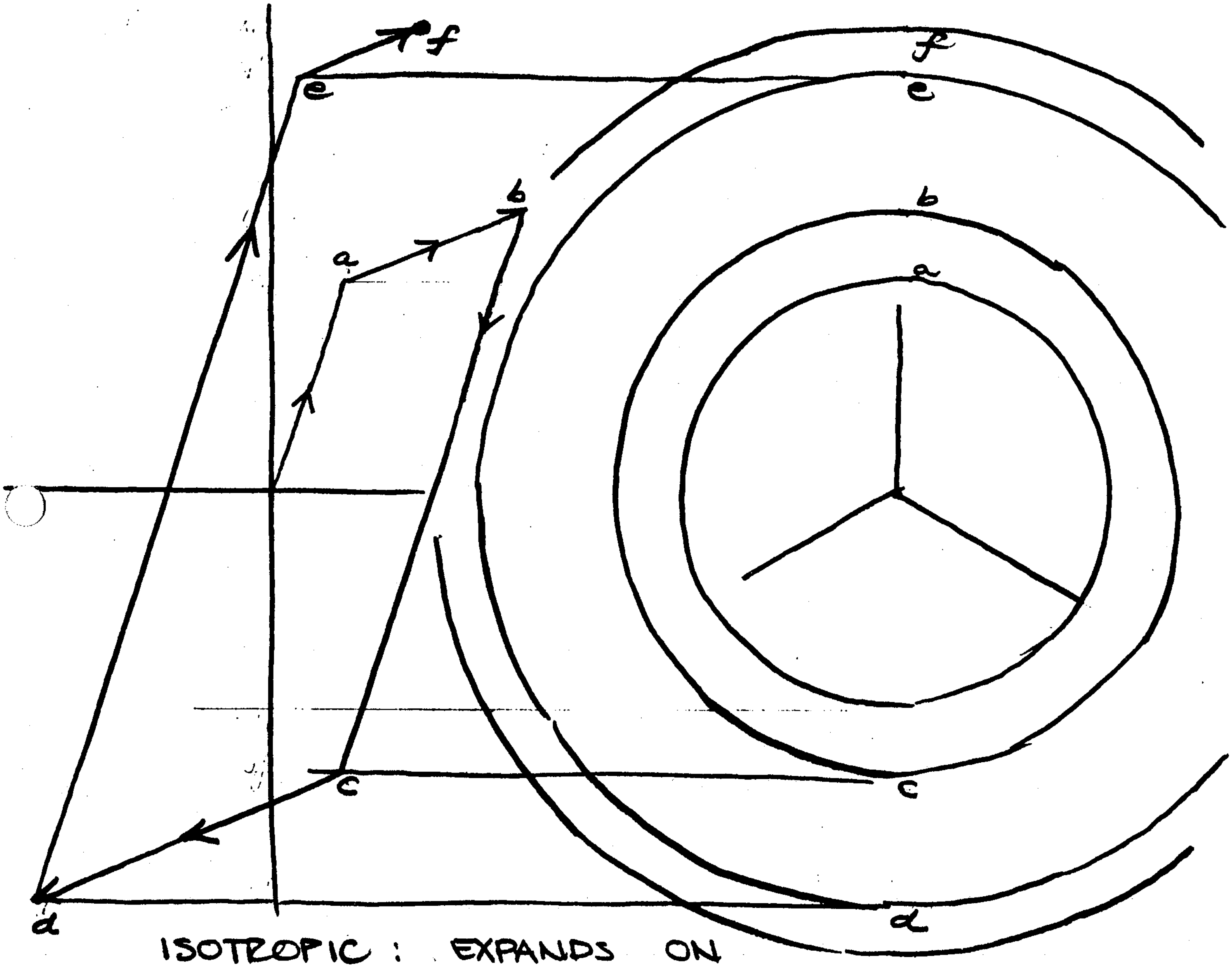
or

$$\frac{d\sigma}{d\varepsilon^{pl}} = \frac{\Delta\sigma}{\Delta\varepsilon - \Delta\varepsilon^{el}} = \frac{1}{\frac{1}{\Delta\sigma/\Delta\varepsilon} - \frac{1}{\Delta\sigma/\Delta\varepsilon^{el}}}$$

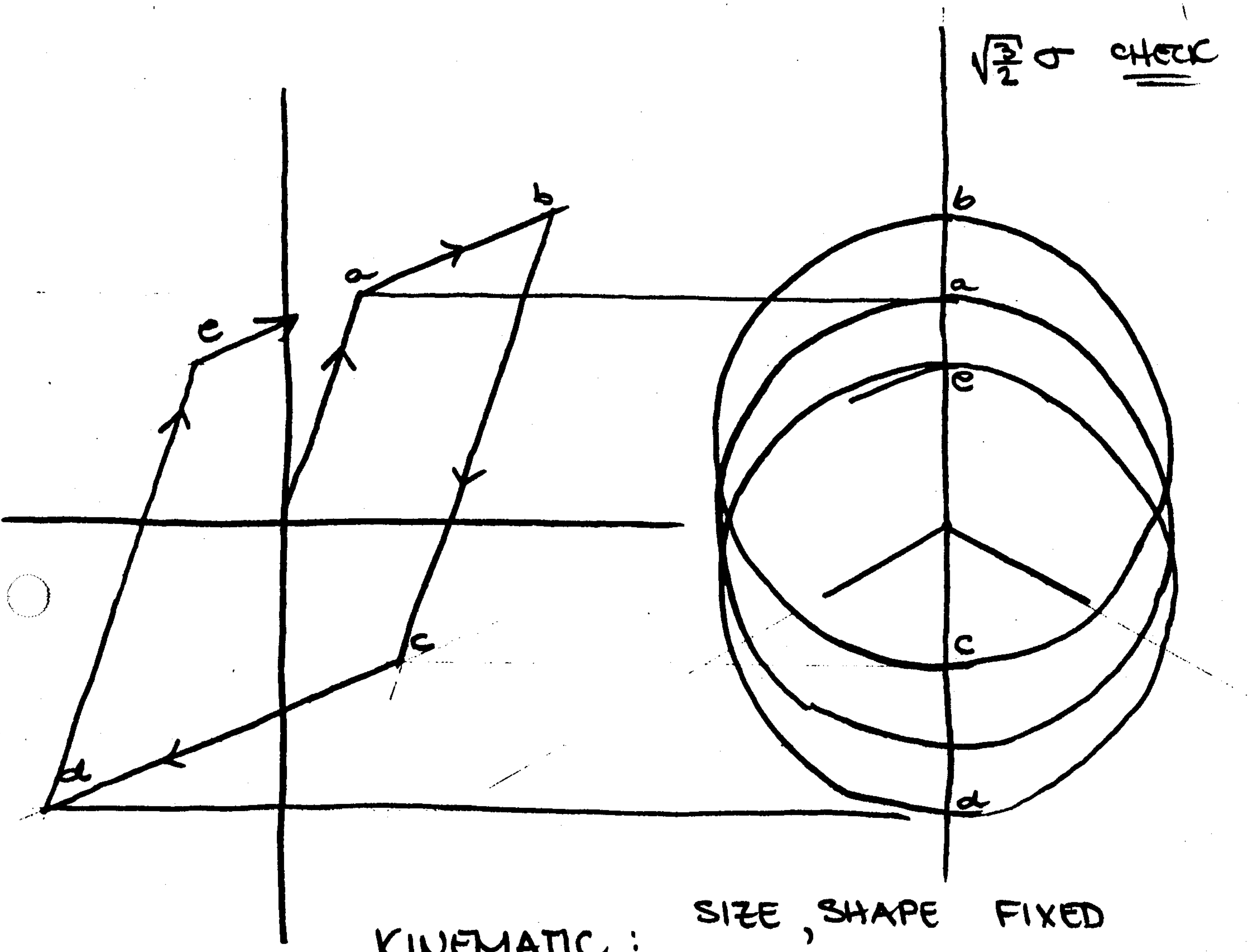
but $\Delta\sigma/\Delta\varepsilon$ on plastic pl $= E_T$

$\Delta\sigma/\Delta\varepsilon^{el}$ " $= E$

Handout #4
ME 239A
Notes



ISOTROPIC : EXPANDS ON PL. LOADING.



KINEMATIC : SIZE , SHAPE FIXED
LOCATION IN DEV. PL. CHANGES.

$$H' = \frac{d\sigma}{d\varepsilon^{pl}} = \frac{1}{1/E - 1/E_T} = \frac{E_T}{(1 - \underbrace{E_T/E}_{<1})} > E_T$$

2/3/83

What we needed was a constit eqn for $\overset{\circ}{R}$ which was consistent w/ 1-D picture

We want to generalize to 3-D and show that 1D uniaxial case is a partic. instance of the 3D case

$$(U)\begin{cases} \text{In uniaxial tension} & \sigma_{33} = \sigma \neq 0 \text{ , rest} = 0 \\ \text{"} \quad \text{"} & \varepsilon_{33}^{pl} = \varepsilon^{pl} \neq 0 \end{cases}$$

since $\underset{\sim}{\varepsilon}^{pl}$ is deviatoric $\Rightarrow$ other ε_{ij}^{pl} must not be zero

for uniax tension we will assume lateral contractions in 1 & 2 directions are equal & shear strains are all zero

ie $\varepsilon_{11}^{pl} = \varepsilon_{22}^{pl} = -\frac{1}{2}\varepsilon_{33}^{pl}$ in order to maintain the deviatoric requir-

we want to define effective stress $(\bar{\sigma}) \Rightarrow \bar{\sigma} = \sqrt{3}\,(J_2')^{1/2}$ invariant since J_2' is invar

$$(U) \Rightarrow \bar{\sigma} = \sqrt{3}\left(\tfrac{1}{2}\sigma_{ij}'\sigma_{ij}'\right)^{1/2} = \sqrt{3/2}\left\{\left[\tfrac{2}{3}\sigma\right]^2 + 2\left[-\tfrac{1}{3}\sigma\right]^2\right)^{1/2} = \sigma$$

Effective plastic strain $(\bar{\varepsilon}^{pl}) \Rightarrow \bar{\varepsilon}^{pl} = \sqrt{2/3}\left(\underset{\sim}{\varepsilon}^{pl}\cdot\underset{\sim}{\varepsilon}^{pl}\right)^{1/2} = \sqrt{2/3}\left(\varepsilon_{ij}^{pl}\cdot\varepsilon_{ij}^{pl}\right)^{1/2}$

$$(U) \Rightarrow \bar{\varepsilon}^{pl} = \sqrt{2/3}\left(1 + 2\left(-\tfrac{1}{2}\right)^2\right)^{1/2}\varepsilon^{pl} = \varepsilon^{pl}$$

Suppose we do an exper & measure $\bar{\sigma}$ $\bar{\varepsilon}^{pl}$ then H' will be the slope of the plastic loading curve. Thus H' will be ~~the~~ invariant $= d\bar{\sigma}/d\bar{\varepsilon}^{pl}$

We want to generalize def of $\bar{\varepsilon}^{pl}$ for any complicated process

Thus define

$$\dot{\bar{\varepsilon}}^{pl} = \sqrt{2/3}\,\left(\underset{\sim}{\dot{\varepsilon}}^{pl}\cdot\underset{\sim}{\dot{\varepsilon}}^{pl}\right)^{1/2} \text{ then } \quad \bar{\varepsilon}^{pl}\Big|_t = \int_0^t \dot{\bar{\varepsilon}}^{pl}\,dt.$$

Return to isotropic hardening & ask what is $\dot{R}$ as a fn of $\bar{\varepsilon}^{pl}, \bar{\sigma}$

Remember $H' \Longrightarrow \left(\underset{\sim}{Q}\cdot\underset{\sim}{\dot{\sigma}} = \underset{\sim}{\dot{\varepsilon}}\right)$

Antis Sym

since $\sigma'\cdot\sigma''_{mea} = 0$

$$\underset{\sim}{Q}\cdot\underset{\sim}{\dot{\sigma}} = \frac{\sigma'}{R}\cdot\underset{\sim}{\dot{\sigma}} \equiv \frac{\underset{\sim}{\sigma}'}{R}\cdot\underset{\sim}{\dot{\sigma}}' = \frac{1}{R}\Bigg[\underbrace{\frac{1}{2}\frac{d}{dt}(\sigma'\cdot\sigma')}_{\frac{d}{dt}\left(J_2' \;=\; \bar{\sigma}^2/3\right)}\Bigg]$$

thus also $\sqrt{2J_2'} = R$ by Mises, hence $R = \sqrt{2\bar{\sigma}^2/3}$

$$\frac{1}{R} \qquad \frac{d}{dt}(J_2') \quad = \sqrt{2/3}\,\bar{\sigma} = \text{[illegible]}\, R$$

Thus $\underset{\sim}{Q}\cdot\underset{\sim}{\dot{\sigma}} = \sqrt{3/2}\,\frac{1}{\bar{\sigma}} \cdot \frac{2}{3}\bar{\sigma}\dot{\bar{\sigma}} = \sqrt{2/3}\,\dot{\bar{\sigma}}$

$$\mathring{R} = \sqrt{2/3}\,\dot{\bar{\sigma}} = \sqrt{2/3}\,H'\dot{\bar{\varepsilon}}^{pl}$$

Try consistency again to determine λ: $\quad \underset{\wedge}{\mathring{R}} = \underset{\wedge}{Q}\cdot\underset{\wedge}{\dot{\sigma}}$

$$\sqrt{2/3}\,H'\dot{\bar{\varepsilon}}^{pl} = \underset{-}{Q}\cdot\underset{-}{\dot{\sigma}}^{tr} - \underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\boxed{\underset{\sim}{\dot{\varepsilon}}^{pl}} \text{ — } \underset{-}{\lambda}\underset{-}{Q}$$

$$= \quad '' \quad - \lambda\underbrace{\underset{-}{Q}\cdot\underset{-}{C}\cdot\underset{-}{Q}}_{2\mu}$$

remember $\underset{\sim}{C}\cdot\underset{\sim}{Q} = \hat{\lambda}\,\cancel{\mathrm{tr}\,\underset{\wedge}{Q}}\,\underset{-}{I} + 2\mu\underset{\sim}{Q}$

$= 2\mu\underset{\sim}{Q}$ unit normal

$\underset{-}{Q}\cdot\underset{-}{C}\cdot\underset{-}{Q} = 2\mu\underbrace{\underset{\sim}{Q}\cdot\underset{\sim}{Q}}_{1}$

now $\sqrt{2/3}\left(\underset{\underset{\lambda\hat{Q}}{\downarrow}}{\dot{\varepsilon}^{pl}}\cdot\underset{\sim}{\dot{\varepsilon}}^{pl}\right)^{1/2}$ for $\underset{-}{\dot{\bar{\varepsilon}}}^{pl}$

$$\dot{\bar{\varepsilon}}^{pl} = \sqrt{2/3}\,\lambda = \dot{\bar{\varepsilon}}^{pl}$$

$$\therefore \left(\sqrt{2/3}\right)^2 H'\lambda = \underset{-}{Q}\cdot\underset{-}{\dot{\sigma}}^{tr} - \lambda\cdot 2\mu \quad \text{\& solve for } \lambda$$

$$\boxed{\frac{\underset{\sim}{Q}\cdot\underset{\sim}{\dot{\sigma}}^{tr}}{\frac{2}{3}H' + 2\mu} = \lambda}$$

recall also

now $$\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}}^{tr} = \underset{\sim}{Q}\cdot(\underset{\sim}{C}\cdot\dot{\underset{\sim}{\varepsilon}})$$

$$\underbrace{\underset{\sim}{Q}\cdot\ (\hat{\lambda}\ tr\ \dot{\underset{\sim}{\varepsilon}}\ I}_{=0} + 2\mu\dot{\underset{\sim}{\varepsilon}})$$

$$2\mu\ \underset{\sim}{\hat{Q}}\cdot\dot{\underset{\sim}{\varepsilon}}$$

or $$\lambda = \frac{2\mu}{2\mu+\frac{2}{3}H'}\ \underset{\sim}{Q}\cdot\dot{\underset{\sim}{\varepsilon}}$$ since $\frac{2\mu}{2\mu+\frac{2}{3}H'} < 1$. (elastic pl case)

∴ in hardening $\dot{\underset{\sim}{\varepsilon}}^{pl}$ is not as big as before since ∃ a $\Delta\varepsilon^{el}$ in the unload'g cycle

Summary of isotropic hardening theory: (Mises, with assoc. flow rule).

Const't Eqn: $$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C}(\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^{pl}) = \dot{\underset{\sim}{\sigma}}^{tr} - \underset{\sim}{C}\cdot\dot{\underset{\sim}{\varepsilon}}^{pl}$$

also $$\dot{R} = \sqrt{\tfrac{2}{3}}\,H'\,\dot{\bar{\varepsilon}}^{pl} = \sqrt{2}\,\dot{k}$$

$$\dot{\underset{\sim}{\varepsilon}}^{pl} = \begin{cases} 0 & (E) \\ \lambda\underset{\sim}{Q} & (P)\end{cases}$$

Now we ask what ~~is~~ happening on surface (yield surface)   (E) & (P) are exactly same as for elastic-perf plastic case

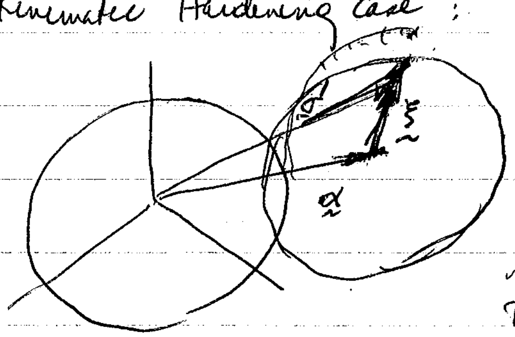
λ = formulae above.

What material prop. do we need: E, H', σ_y

Let's look at Kinematic Hardening case: $\dot{k} = 0$ $\dot{\underset{\sim}{\alpha}} \neq \underset{\sim}{0}$

yield condition when surface moves

$\underset{\sim}{\alpha}$ = back stress.



Same size & shape

$\xi = \underset{\sim}{\sigma}' - \underset{\sim}{\alpha}$

what we want is $f(\underset{\sim}{\xi}) = \frac{1}{2}\underset{\sim}{\xi}'\cdot\underset{\sim}{\xi}' = k^2$

"translated" Mises yield surface.

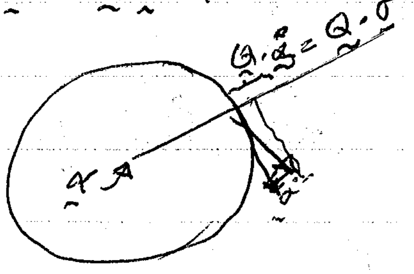
Consistency $f(\underline{\xi}) = k^2$ (as a fn of time is unchanged)

$$\dot f = 2k\cancel{\dot k} = 0 \quad = \quad \frac{\partial f}{\partial \underset{\sim}{\xi}} \cdot \dot{\underset{\sim}{\xi}}$$

$$\underset{\sim}{Q} \left|\frac{\partial f}{\partial \underset{\sim}{\xi}}\right| \qquad \Rightarrow \quad \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\xi}} = 0$$

$$R\underset{\sim}{Q} \qquad \underset{\sim}{Q} \cdot (\dot{\underset{\sim}{\sigma}}' - \dot{\underset{\sim}{\alpha}}) = 0$$

$\therefore \; \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}' = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\alpha}}$.

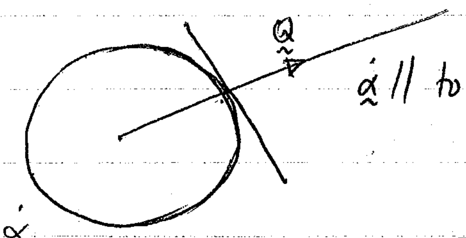


Now we ask what does $\dot\sigma = H'\dot{\bar\varepsilon}^{p'}$ do to $\dot{\underset{\sim}{\alpha}}$?

Now recall $\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}} = \overset{\text{same as}}{\text{fr isotropic}}$ $\sqrt{2/3}\,\dot{\bar\sigma} = \sqrt{2/3}\, H'\dot{\bar\varepsilon}^{p'} = 2/3\, H'\lambda \Rightarrow \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\alpha}} = 2/3\, H'\lambda$

it is a scaler condition on $\dot{\underset{\sim}{\alpha}}$. Now we must get an assumption on direction of $\dot{\underset{\sim}{\alpha}}$

(Prager Kinematic Hardening) $\Rightarrow$ $\dot{\underset{\sim}{\alpha}} = a\,\dot{\underset{\sim}{\varepsilon}}^{p'} = a\lambda\underset{\sim}{Q}$ ie $\dot{\underset{\sim}{\alpha}}$ is radial



$\dot{\underset{\sim}{\alpha}} \parallel$ to $\underset{\sim}{Q}$

a is a material paramete

& a is defined by 1-D exper.

$$\therefore \quad \underset{\sim}{Q} \cdot \overbrace{a\lambda\underset{\sim}{Q}}^{\dot{\alpha}} = 2/3\, H'\lambda \quad \Rightarrow \quad \boxed{a = 2/3\, H'}$$

from consistency we determine λ.

From: $\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\alpha}} = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}$; using $\dot{\underset{\sim}{\alpha}} = \overbrace{2/3\, H'}^{a}\lambda Q$ & $\hat{\underset{\sim}{\sigma}} = \hat{\underset{\sim}{\sigma}}^{tr} - \overbrace{\underset{\sim}{C} \cdot \hat{\underset{\sim}{\varepsilon}}}^{\underset{\sim}{C}\cdot\underset{\sim}{\varepsilon}}{}^{p'}$

$$2/3\, H'\lambda = \underset{\sim}{Q} \cdot \hat{\underset{\sim}{\sigma}} = \underset{\sim}{Q} \cdot \hat{\underset{\sim}{\sigma}}^{tr} - \lambda\underbrace{\underset{\sim}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim}{Q}}_{2\mu}$$

λQ

or $\boxed{\lambda = (\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}^{tr}) / (2/3\, H' + 2\mu) \text{ as in isotropic hardening!}}$

Will comback to other kinematic theories for multiple ~~yield~~ dimensional yielding

Summary of kinematic hardening:

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C}\,(\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^{pl}) = \dot{\underline{\sigma}}^{tr} - \underline{C}\cdot\dot{\underset{\sim}{\varepsilon}}^{pl}$$

$$\dot{\underset{\sim}{\alpha}} = \tfrac{2}{3}\,H'\,\dot{\underset{\sim}{\varepsilon}}^{pl}$$

$$\dot{\underset{\sim}{\varepsilon}}^{pl} = \begin{cases} 0 & (E) \\ \lambda \underset{\sim}{Q} & (P) \end{cases}$$ exactly same as before

$$\lambda = (\underline{Q}\cdot\dot{\underline{\sigma}}^{tr}) / (\tfrac{2}{3}H' + 2\mu)$$

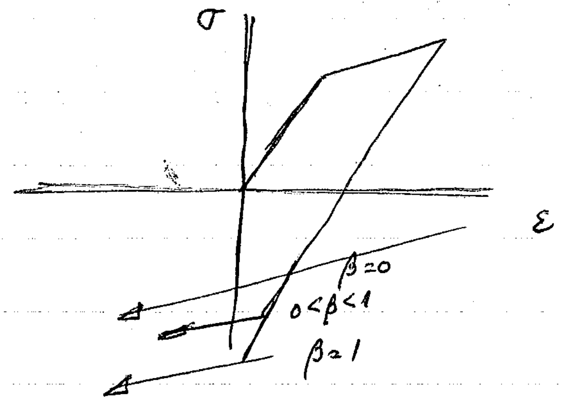
Now we combine theories (linearly combine).

introduce β = scalar, const, material param

$$\dot{k} = \beta \frac{1}{\sqrt{3}}\,H'\,\dot{\bar{\varepsilon}}^{pl} \qquad \beta \in [0,1]$$

$$\dot{\underset{\sim}{\alpha}} = \tfrac{2}{3}\,H'\,\dot{\underset{\sim}{\varepsilon}}^{pl}\,(1-\beta)$$

if $\beta = 0$ kine hardening
$\quad = 1$ iso "



Exercise: Using consistency $\underset{\sim}{Q}\cdot\overset{\circ}{\underset{\sim}{\xi}} = \dot{\underset{\sim}{R}} \neq 0$ show λ = exactly as before ie β disappear

also show that $\beta/_{1-\beta} = \dot{R}/_{\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\alpha}}}$ this will justify the above figure.

Summary of Combined theories

$$\dot{\underset{\sim}{\sigma}} = \underbrace{\underset{\sim}{C}\cdot(\dot{\underset{\sim}{\varepsilon}}}_{\dot{\underset{\sim}{\sigma}}^{tr}} - \dot{\underset{\sim}{\varepsilon}}^{pl}) \qquad \dot{k} = \beta\frac{1}{\sqrt{3}}\,H'\,\dot{\bar{\varepsilon}}^{pl}$$

$$\dot{\underset{\sim}{\alpha}} = (1-\beta)\,\tfrac{2}{3}\,H'\,\dot{\underset{\sim}{\varepsilon}}^{pl}$$

$$\underset{\sim}{Q} = \frac{\partial f}{\partial \underset{\sim}{\xi}} = \xi/R \qquad \dot{\underset{\sim}{\varepsilon}}^{pl} = \begin{cases} 0 & (E) \\ \lambda\underset{\sim}{Q} & (P) \end{cases}$$ repeat those from elastic perf plastic

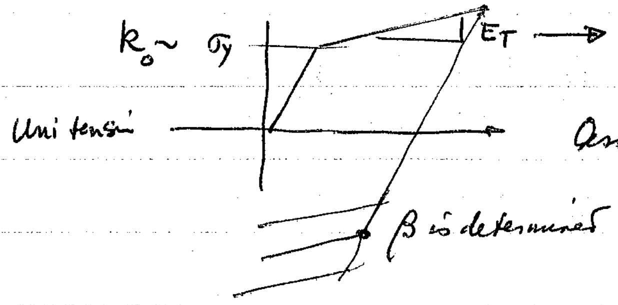
$$\lambda = \frac{\underset{\sim}{Q} \cdot \underset{\sim}{\dot{\sigma}}^{tr}}{\frac{2}{3} H' + 2\mu} = \frac{2\mu}{2\mu + \frac{2}{3} H'} \underset{\sim}{Q} \cdot \underset{\sim}{\dot{\varepsilon}}$$

Note:

$$(E) \begin{cases} f(\xi) < k^2 \text{ or} \\ f(\xi) = k^2 \text{ \& } \underset{\sim}{Q} \cdot \underset{\sim}{\dot{\sigma}}^{tr} \leq 0 \end{cases}$$

$$(P) \begin{cases} f(\underset{\sim}{\xi}) = k^2 \text{ \&} \\ \underset{\wedge}{Q} \cdot \underset{\sim}{\dot{\sigma}}^{tr} > 0 \end{cases} \qquad \underset{-}{\xi'} = \underset{\sim}{\sigma'} - \underset{\sim}{\alpha}$$

Need to measure E, ν β // we will define $\underset{\sim}{C}^{el-pl}$ next time.



Assume $\underset{\sim}{\alpha_0} = \underset{\sim}{\sigma_0} = 0$.

$H' = 0 \Longrightarrow$ elastic perf plastic case

$1 + H'/3\mu \longrightarrow \phi$

2/8 Definition of El-pl moduli

$$\dot{\underset{\sim}{\sigma}} = \overset{el\text{-}pl}{\underset{\sim}{C}} \; \dot{\underset{\sim}{\epsilon}} \qquad \dot{\underset{\sim}{\epsilon}}^{pl} = \lambda \underset{\sim}{Q} = \frac{(\underset{\sim}{Q} : \dot{\underset{\sim}{\epsilon}})}{(1+\frac{H'}{3\mu})} \underset{\sim}{Q}$$

from $\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \dot{\underset{\sim}{\epsilon}}^{elasti} = \underset{\sim}{C}(\dot{\underset{\sim}{\epsilon}} - \dot{\underset{\sim}{\epsilon}}^{pl})$

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C}\dot{\underset{\sim}{\epsilon}} - \underset{\sim}{C}\dot{\underset{\sim}{\epsilon}}^{pl}$$

$$\dot{\sigma}_{ij} = C_{ijkl}\,\dot{\epsilon}_{kl} - \left(\frac{C_{ij\cancel{kl}}^{\;\;mn}}{(1+\frac{H'}{3\mu})} Q_{\cancel{kl}\,mn}\right) \underset{kl}{Q_{mn}}\;\underset{kl}{\dot{\epsilon}_{mn}} \qquad \Rightarrow 2\mu Q_{ij}$$

do a dummy index interchange

$$= \left(C_{ijkl} - \frac{2\mu}{(1+\frac{H'}{3\mu})} Q_{ij} Q_{kl}\right)\dot{\epsilon}_{kl}$$

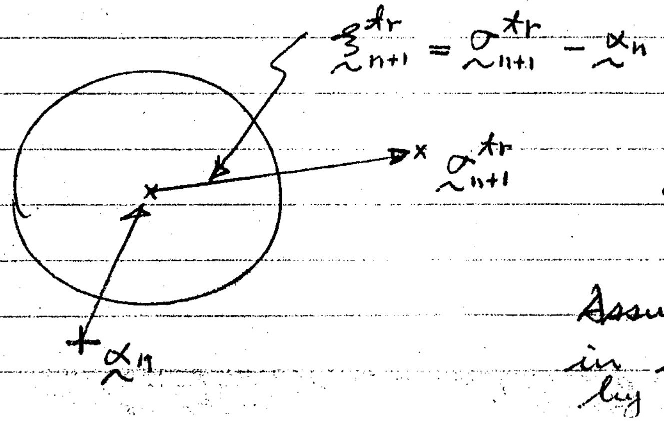
$$\dot{\underset{\sim}{\sigma}} = \underbrace{\left(\underset{\sim}{C} - \frac{2\mu}{(1+\frac{H'}{3\mu})} \underset{\sim}{Q} \otimes \underset{\sim}{Q}\right)}_{\underset{\sim}{C}^{el\text{-}pl}} \cdot \dot{\underset{\sim}{\epsilon}}$$

Algo. Idea calc. $\underset{\sim}{\sigma}^{tr}_{n+1}$, $f(\underset{\sim}{\xi}^{tr}_{n+1}) \le k_n^2$.

$$\underset{\sim}{\xi}^{tr}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - \underset{\sim}{\alpha}_n$$

yes → elastic, done

no → plastic loading



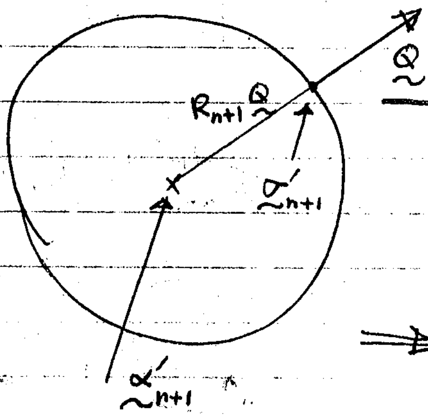
allow for expansion, movement of Y.S.

~~Assume~~ "Kinematics" occurs in the direction defined by $\underset{\sim}{\xi}^{tr}_{n+1}$ ("radial direction")

Discretized versions of all rate eqs. :

$\underset{\sim}{\dot\sigma} = \underset{\sim}{C} \cdot (\underset{\sim}{\dot\epsilon} - \lambda \underset{\sim}{Q})$	$\underset{\sim}{\sigma}_{n+1} - \underset{\sim}{\sigma}_n = \underset{\sim}{C} \cdot \Delta \underset{\sim}{\epsilon}_n - \underset{\sim}{C} \underbrace{(\Delta t \lambda)}_{\tilde\lambda} \underset{\sim}{Q}$ $\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - \tilde\lambda\, 2\mu\, \underset{\sim}{Q}$
$\dot R = \sqrt{2}\,\dot k = \beta \frac{\sqrt{2}}{\sqrt{3}} H' \underset{\Downarrow\ \sqrt{\frac{2}{3}}\lambda}{\dot\epsilon^{pl}}$ $= \beta \frac{2}{3} H' \lambda$	$R_{n+1} = R_n + \frac{2}{3}\beta H' \underbrace{(\Delta t \lambda)}_{\tilde\lambda}$
$\underset{\sim}{\dot\alpha} = (1-\beta)\frac{2}{3} H' \underset{\Downarrow\ \lambda \underset{\sim}{Q}}{\underset{\sim}{\dot\epsilon}^{pl}}$	$\underset{\sim}{\alpha}_{n+1} = \underset{\sim}{\alpha}_n + (1-\beta)\frac{2}{3} H' \tilde\lambda \underset{\sim}{Q}$

Invoke consistency "in-the-large" : S.T.



$\underset{\sim}{Q}$ has been selected to be ∥ to $\underset{\sim}{\xi}^{tr}_{n+1}$

Consistency condit

$$\Longrightarrow \boxed{\underset{\sim}{\alpha}'_{n+1} + R_{n+1} \underset{\sim}{Q} = \underset{\sim}{\sigma}'_{n+1}}$$

$$\overbrace{\underset{\sim}{\alpha}'_n + (1-\beta)\frac{2}{3}H'\cancel{\tilde{\lambda}\underset{\sim}{Q}}}^{\alpha'_{n+1}} + (\overbrace{R_n + \frac{2}{3}\cancel{\beta H'\tilde{\lambda}}}^{R_{n+1}})\underset{\sim}{Q}$$

$$= \overbrace{\underset{\sim}{\sigma}^{tr'}_{n+1} - 2\mu\tilde{\lambda}\underset{\sim}{Q}}^{\sigma'_{n+1}}$$

$$\left[(1-\cancel{\beta})\frac{2}{3}H' + \frac{2}{3}\cancel{\beta}H' + 2\mu\right]\tilde{\lambda}\underset{\sim}{Q} = \overbrace{\underset{\sim}{\sigma}^{tr'}_{n+1} - \underset{\sim}{\alpha}'_n}^{\underset{\sim}{\xi}^{tr'}_{n+1}} - R_n\underset{\sim}{Q}$$

$$\underset{\sim}{Q}\cdot\left[\underbrace{\left(2\mu + \frac{2}{3}H'\right)\tilde{\lambda}\underset{\sim}{Q}} = \underset{\sim}{\xi}^{tr'}_{n+1} - R_n\underset{\sim}{Q}\right]$$

now take $\underset{\sim}{Q}\cdot$ this ↑

$$\Downarrow \; \cancel{= \underset{\sim}{Q}\cdot\underset{\sim}{\xi}^{tr}_{n+1}}$$

$\underset{\sim}{Q}\cdot\underset{\sim}{Q} = 1$

$$\left(2\mu + \frac{2}{3}H'\right)\tilde{\lambda} = \underbrace{\underset{\sim}{Q}\cdot\underset{\sim}{\xi}^{tr'}_{n+1}} - R_n$$

$$\frac{\underset{\sim}{\xi}^{tr'}_{n+1}}{\left|\underset{\sim}{\xi}^{tr'}_{n+1}\right|} \qquad \frac{|\cdot|^{x}}{\cancel{|\cdot|}}$$

since $\underset{\sim}{Q}$ is // $\underset{\sim}{\xi}^{tr'}_{n+1}$ then $\underset{\sim}{Q}\cdot\underset{\sim}{\xi}^{tr'}_{n+1}$ is a number $= \left|\underset{\sim}{\xi}^{tr'}_{n+1}\right|$ since $\underset{\sim}{\xi}^{tr'}_{n+1} = |\,|\,\underset{\sim}{Q}$

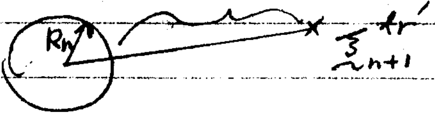
$$2\mu\left(1 + \frac{H'}{3\mu}\right)\tilde{\lambda} = \left|\underset{\sim}{\xi}^{tr'}_{n+1}\right| - R_n$$

Solving for $\tilde{\lambda} = \Delta t\,\lambda$

$$\tilde{\lambda} = \frac{1}{2\mu}\,\frac{1}{\left(1 + \frac{H'}{3\mu}\right)}\,\underbrace{\left(\left|\underset{\sim}{\xi}^{tr'}_{n+1}\right| - R_n\right)}$$

$\Downarrow$

the numerical distance from the surface



To Update information

$R_n \leftarrow R_{n+1}$

$\underset{\sim}{\alpha}_n \leftarrow \underset{\sim}{\alpha}_{n+1}$

$\underset{\sim}{\sigma}_n \leftarrow \underset{\sim}{\sigma}_{n+1}$

$$R_{n+1} = R_n + \frac{2}{3}\beta H' \tilde{\lambda}$$

$$\underset{\sim}{\alpha}_{n+1} = \underset{\sim}{\alpha}_n + (1-\beta)\frac{2}{3} H' \tilde{\lambda} \underset{\sim}{Q}$$

$$\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - 2\mu \tilde{\lambda} \underset{\sim}{Q}$$

$$\underset{\sim}{\dot{\epsilon}}^{pl} = \lambda \underset{\sim}{Q}$$

$$\Delta \underset{\sim}{\epsilon}^{pl} = \tilde{\lambda} \underset{\sim}{Q}$$

$$\dot{\bar{\epsilon}}^{pl} = \sqrt{\frac{2}{3}}\lambda$$

$$\Delta \bar{\epsilon}^{pl} = \sqrt{\frac{2}{3}}\tilde{\lambda}$$

How to implement

Step in Algo.

1. Calc. $\underset{\sim}{\sigma}^{tr}_{n+1} = \underset{\sim}{\sigma}_n + \underset{\sim}{C} : \Delta \underset{\sim}{\epsilon}_n$

2. " $\underset{\sim}{\xi}^{tr}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - \underset{\sim}{\alpha}_n$

3. " $\underset{\sim}{\xi}^{tr'}_{n+1}$

4. if $f(\underset{\sim}{\xi}^{tr}_{n+1}) \leq k_n^2$, elastic, $\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1}$, RETUR

   $\left(\Leftrightarrow \left| \underset{\sim}{\xi}^{tr'}_{n+1} \right|^2 \leq R_n^2 \right)$

   else: plasticity

5. Calc. $\underset{\sim}{Q} = \dfrac{\underset{\sim}{\xi}^{tr'}_{n+1}}{\left| \underset{\sim}{\xi}^{tr'}_{n+1} \right|}$

6. Calc. $$\tilde{\lambda} = \frac{1}{2\mu\left(1+\frac{H'}{3\mu}\right)}\left(\left|\underset{\sim}{\xi}^{tr'}_{n+1}\right| - R_n\right)$$

7. Update:

$R_{n+1} = \cdots$

$\underset{\sim}{\alpha}_{n+1} = \cdots$

$\underset{\sim}{\sigma}_{n+1} = \cdots$

$\bar{\epsilon}^{pl}_{n+1} = \bar{\epsilon}^{pl}_{n} + \cdots$

x

FORTRAN : **FOR 2-D case**

$\underset{\sim}{\sigma}$	SIGMA	arrays	$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\end{pmatrix}$
$\underset{\sim}{C}$	C		
$\Delta\underset{\sim}{\epsilon}$	EPSI		
$\underset{\sim}{\xi}^{tr}, \underset{\sim}{\xi}^{tr'}, \underset{\sim}{Q}$	XI		
$\underset{\sim}{\alpha}$	ALPHA		
R	R		
$\lvert\underset{\sim}{\xi}'_{n+1}\rvert$	X		
$\tilde{\lambda}$	A		
$\bar{\xi}^{pl}$	EBAR		
$\left[2\mu\left(1+\frac{H'}{3\mu}\right)\right]$	= C1		
$\frac{2}{3}\beta H'$	= C2		
$\frac{2}{3}(1-\beta)H'$	= C3		

$2\mu \quad = C4$

$\sqrt{\frac{2}{3}} \quad = C5$

×

RAD. RET. PROG. (Axisymm. - pl. strain)

```
§ ... CALC. SIGMA TRIAL
    DO 20 I=1, 4
    DO 10 J=1, 4
```

plane stress : EPSI(3) $= \frac{\lambda}{\lambda+2\mu}$ (EPSI(1) + EPSI(2))

$\sigma_{n+1} = \sigma_n + \underline{C}\Delta\underline{\varepsilon}$

```
10    SIGMA (I) = SIGMA (I) + C(I,J) * EPSI (J)
20    CONTINUE
§ ... CALC. XI TRIAL
    DO 30 J=1, 4
```

$\underline{\xi}^{tr}$

```
30    XI(J) = SIGMA (J) - ALPHA (J)
§ ... CALC. DEV. XI
    XIMEAN = (XI(1) + XI(2) + XI(3))/3.0 ,
   DO 40 J=1, 3
40    XI(J) = XI(J) - XIMEAN
§ ... CHECK IF ELASTIC
    X = XI(1)**2 + XI(2)**2 + XI(3)**3
        +2.0 * XI(4)**2
```

$length^2$

```
    IF (X, LE. R*R) RETURN
```

```
§---- PLASTIC   PHASE

§---- CALC.
      X = SQRT(X)
   DO 50  J=1, 4
50    XI(J) = XI(J)/X  ← normalizes ξ, ie Q compounds

§---- CALC.  LAMBDA  TILDE   ("A")
      A = C1 * (X-R)
```

$2\mu(1+H'/3\mu)[|\underset{\sim}{\xi}^n|-R]$

```
§---- UPDATE
      R = R + C2 * A
```

$\frac{2}{3}\beta H' \cdot \tilde{\lambda}$

```
~~DO~~ T1 = C3 * A
      T2 = C4 * A
```

~~T1 = C3 * A~~

If plane stress:

```
X = XIMEAN / X
DO 55 XI(J) = XI(J) + X
      J=1,4
```

```
~~DO~~ DO 60  J=1, 4
```

$\frac{2}{3}H'(1-\beta)\cdot\tilde{\lambda}$ $\quad \underset{\sim}{\xi}/|\underset{\sim}{\xi}|$

```
      ALPHA(J) = ALPHA(J) + T1 * XI(J)
```

$2\mu\tilde{\lambda}$ $\quad \underset{\sim}{\xi}/|\underset{\sim}{\xi}|$

```
60    SIGMA(J) = SIGMA(J) + T2 * XI(J)
      EBAR = EBAR + C5 * A
```

$\sqrt{2/3}\,\tilde{\lambda}$

```
      RETURN
      END
```

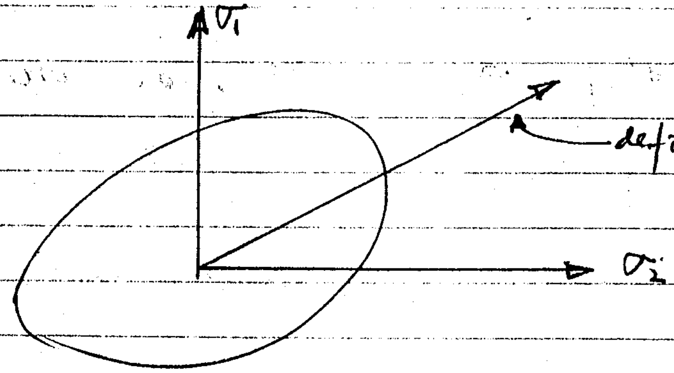
X

Exercise: 1. Modify coding for 3D

2. RERUN the 1D prob's with new algo. (omit case (iv))

$\beta = 0, \quad \beta = \frac{1}{2}, \quad \beta = 1$

$E_T = 0.1E$ Get plotted results.



defines direction in plane stress.

Multi - Y.S. theories:

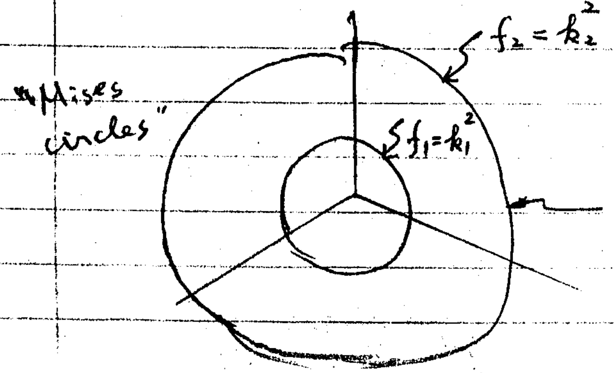
Advantage: better representation of the cyclic phenomena;
certain physical phenomena. (e.g. ~~soils~~)

Disadvantage: increased data to be stored
& processed.

MROZ, IWAN, (& PRÉVOST: soils)
(DUWEZ - BESSELING "Sublayer concept"
analogous to multi - Y.S.)

Begin with elementary 2-surface theory.
— Kinematic hardening to be emphasized

(Drop isotropic hardening for now)



π - plane

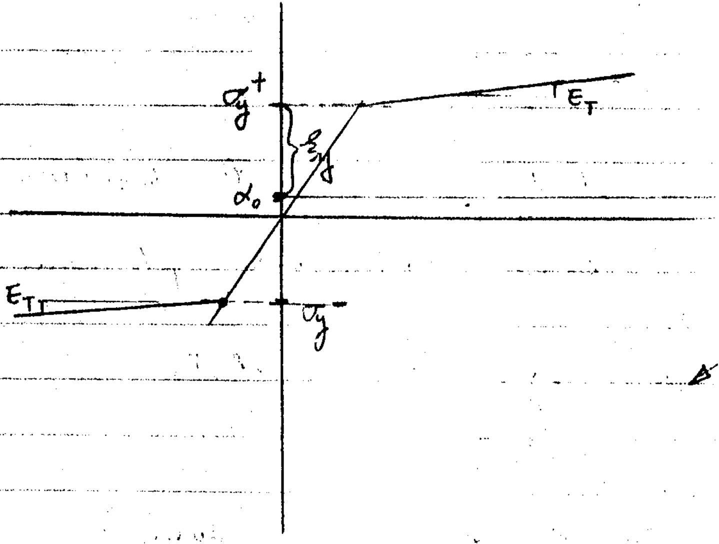
outer surface is sometimes called the "Bounding surface."

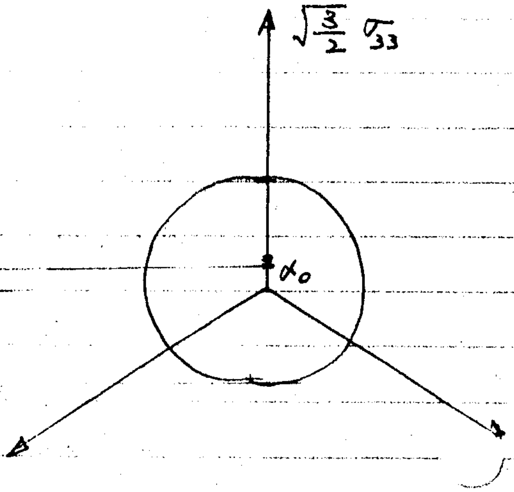
$\underset{\sim}{\sigma}$ must stay ~~a~~ inside, or on, all the Y.S.'s.

assign: H_1', H_2'

X

2/10 Interlude:





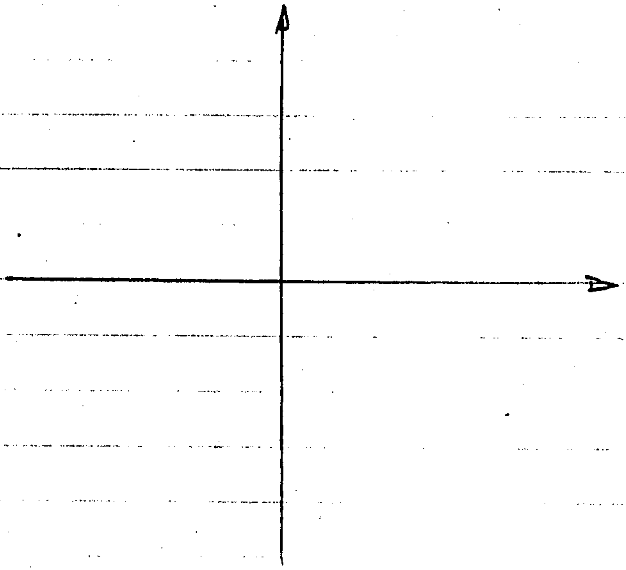
$$\alpha_0 = \frac{1}{2}\left(\sigma_y^- + \sigma_y^+\right)$$

$$\underset{\sim}{\alpha}_0 = \text{deviatoric part of } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-\alpha_0}{3} & 0 & 0 \\ 0 & \frac{-\alpha_0}{3} & 0 \\ 0 & 0 & \frac{2\alpha_0}{3} \end{bmatrix}$$

$$|\underset{\sim}{\alpha}_0| = \left(\left(\frac{1}{9} + \frac{1}{9} + \frac{4}{9}\right)\alpha_0^2\right)^{1/2}$$

$$= \sqrt{\frac{2}{3}}\,\alpha_0$$



×

$\xi_y = \sigma_y^+ - \alpha_0$

Exercise: 1. Show kinematic $z = Y - 2\xi_y$

2. " Iso $z = 2\alpha_0 - Y$

3. If $z \in [2\alpha_0 - Y, Y - 2\xi_y]$,

then:

$$z = Y - 2\xi_y + 2\beta(\alpha_0 - Y + \xi_y)$$

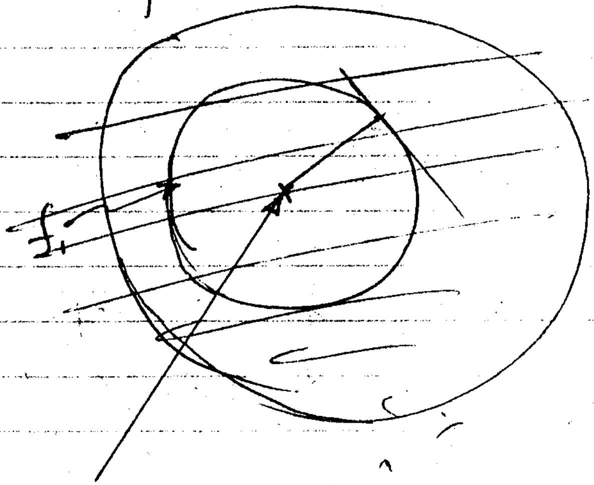
$$\Longrightarrow \beta = \frac{(z - Y + 2\xi_y)}{2(\alpha_0 - Y + \xi_y)}$$

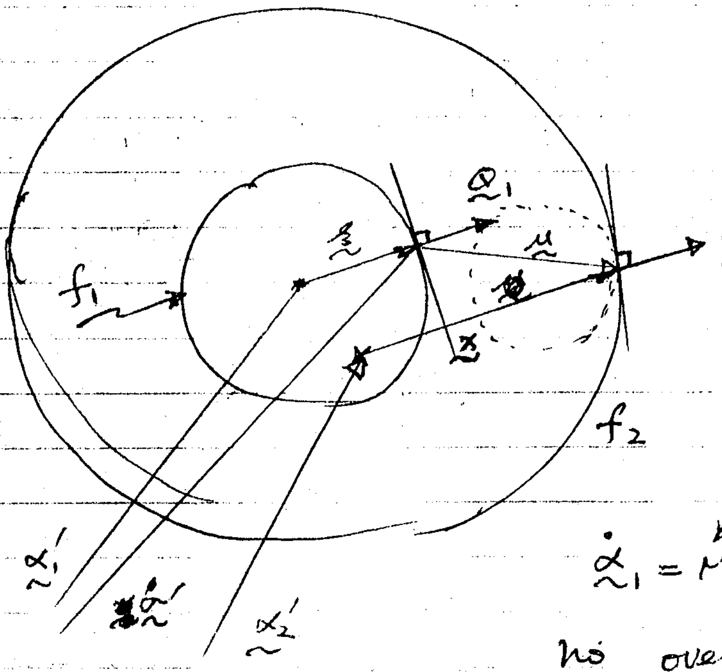
×

Return to Multi-Y.S. theories:

Kinematic ideas — theory of inner surface:

inside inner surface – elastic, when stress point impinges upon inner surface it moves towards outer surface in a special way. (Assume outer surface stationary until inner surface contacts it).





Mroz hard.

$\dot{\alpha}_1 = M\mu$ insures surface meet in osculatory fashion

no overlaps of surfaces.

$$\underset{\sim}{\mu} = \underset{\sim}{\alpha}_2' + \underset{\sim}{\chi} - \underset{\sim}{\sigma}'$$

note $\underset{\sim}{\chi} \parallel \underset{\sim}{\xi}'$

$$|\underset{\sim}{\chi}| = R_2 \; ; \quad |\underset{\sim}{\xi}| = R_1$$

$$\underset{\sim}{\chi} = \frac{R_2}{R_1}\underset{\sim}{\xi}'$$

$$\boxed{\underset{\sim}{\mu} = \underset{\sim}{\alpha}_2' - \frac{R_2}{R_1}\underset{\sim}{\xi} - \underset{\sim}{\sigma}'}$$

Determination of M:

inner Y. S

Consistency: use $\dot{\bar{\sigma}} = H_1' \, \dot{\bar{\epsilon}}^{pl}$

$$0 = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\xi}} \quad \left(\text{differentiate } f_1(\xi) = k_1^2\right)$$

$$= \underbrace{\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}} - \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\alpha}}_1$$

$\longrightarrow$ recall $\sqrt{\frac{2}{3}}\,\dot{\bar{\sigma}} = \sqrt{\frac{2}{3}}\,H_1' \, \dot{\bar{\epsilon}}^{pl}$

$$\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\alpha}}_1 = M \frac{\underset{\sim}{\xi}'}{R_1} \cdot \underset{\sim}{\mu} \quad /\!\!/$$

solve:
$$M = \frac{R_1\sqrt{\frac{2}{3}}\, H_1' \, \dot{\bar{\epsilon}}^{pl}}{(\underset{\sim}{\xi}' \cdot \underset{\sim}{\mu})_{\neq 0}}$$

~~$\dot{\underset{\sim}{Q}} = \underset{\sim}{C}(\dot{\underset{\sim}{\epsilon}} - \lambda \underset{\sim}{Q})$~~

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C}(\dot{\underset{\sim}{\epsilon}} - \lambda_1 \underset{\sim}{Q})$$ consistency:

$$\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}} = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\alpha}}_1$$

function λ_1

$$\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}^{tr} - \lambda_1 \underbrace{\underset{\sim}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim}{Q}}_{2\mu} = \sqrt{\tfrac{2}{3}}\, H_1' \, \dot{\bar{\epsilon}}^{pl}$$ as usual

$$\lambda_1 = \text{same} = \frac{1}{\left(1 + \frac{H_1'}{3\mu}\right)} \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\epsilon}}$$

Recapitulate : Theory of inner surface.

$$\dot{\underset{\sim}{\sigma}} = \underbrace{\underset{\sim}{C} \cdot (\dot{\underset{\sim}{\epsilon}}}_{\dot{\underset{\sim}{\sigma}}^{tr}} - \dot{\underset{\sim}{\epsilon}}^{pl})$$

$$\dot{k}_1 = 0$$

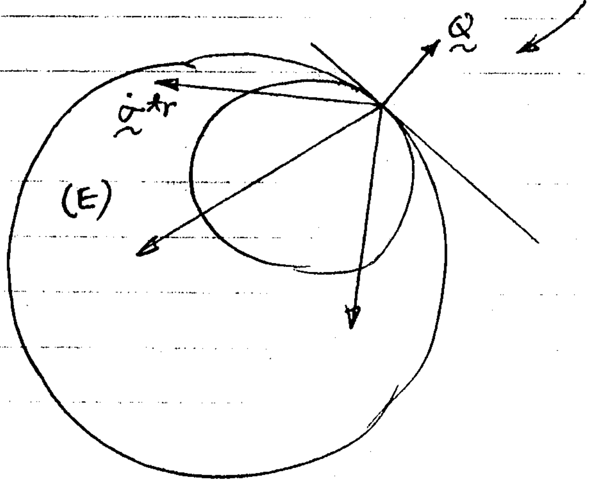
$$\dot{\underset{\sim}{\alpha}}_1 = M \underset{\sim}{\mu}$$ (def's above)

$$\dot{\underset{\sim}{\epsilon}}^{pl} = \begin{cases} 0 & (E) \\ \underset{\text{as above}}{\underset{\uparrow}{\lambda_1}} \underset{\sim}{Q} & (P) \end{cases}$$ ← as usual in terms of $\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}^{tr}$ & $f_1(\underset{\sim}{\xi}) = k_1^2$

El-pl. moduli same w. H_1'

X

Theory of outer surface:



suppose we have contact

(E)
(p) } same M

If (E) surface don't move

If (P) surface move together, in this case surface 2 is said to be "activated".

theory: $\underset{\sim}{\sigma} = \underset{\sim}{C} \cdot (\underset{\sim}{\dot{\epsilon}} - \underset{\sim}{\dot{\epsilon}}^{pl})$

$$\dot{k}_1 = 0$$

$$\cancel{\underset{\sim}{\dot{\alpha}}_2 = H_2 \underset{\sim}{Q}}$$

prager kinematic hardening

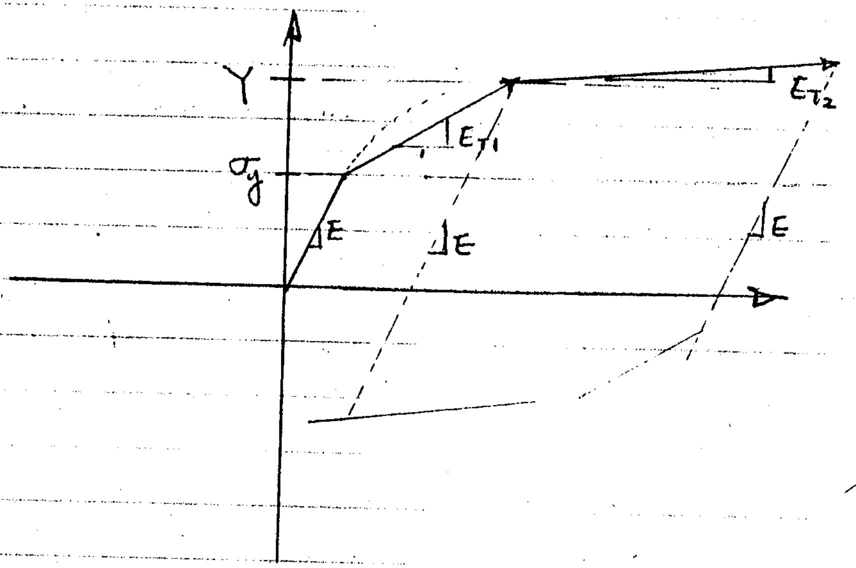
$$\underset{\sim}{\dot{\alpha}}_1 = \underset{\sim}{\dot{\alpha}}_2 = \text{Consis.} \Rightarrow \cancel{H_2} \cancel{H_2 \underset{\sim}{Q}} = \frac{2}{3} H_2' \underset{\sim}{\dot{\epsilon}}^{pl}$$

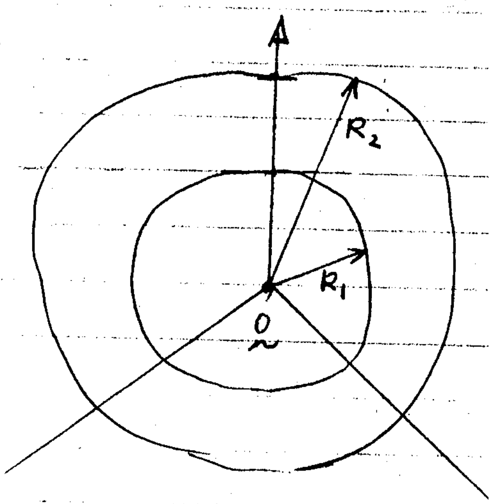
(as before)

$$\underset{\sim}{\dot{\epsilon}}^{pl} = \begin{cases} 0 & (E) \\ \lambda_2 \underset{\sim}{Q} & (P) \end{cases}$$

$$\lambda_2 = \frac{1}{\left(1 + \frac{H_2'}{3\mu}\right)} \underset{\sim}{Q} \cdot \underset{\sim}{\dot{\epsilon}}$$

X





for typical metals,
assume "symm."

$\Longrightarrow$ $\underset{\sim}{\alpha}_i'$, s $\equiv \underset{\sim}{0}$ initially

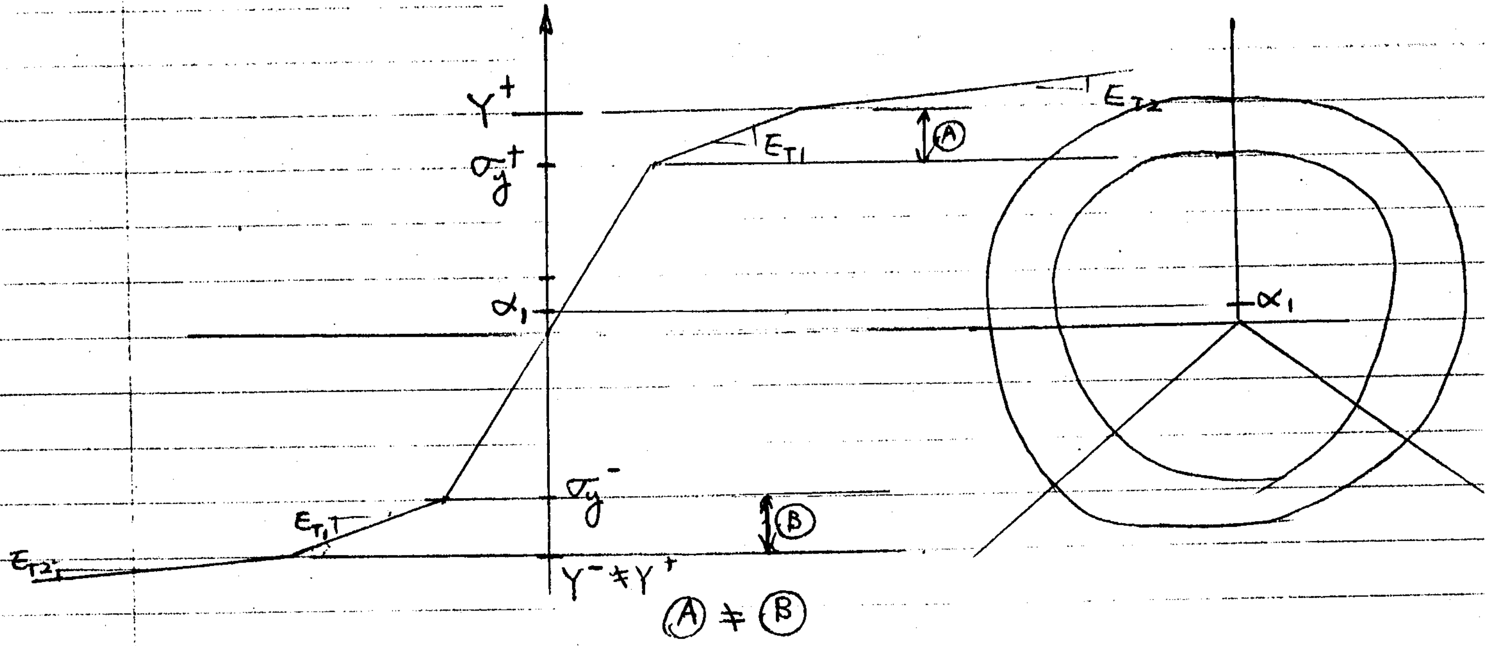
$$k_1 = \frac{\sigma_y}{\sqrt{3}}$$

$$k_2 = \frac{Y}{\sqrt{3}}$$

measure E_{T1}, E_{T2} $\Longrightarrow$ H_1', H_2'

$$H_i' = \frac{E_{Ti}}{\left(1 - \frac{E_{Ti}}{E}\right)} \qquad i = 1, 2$$

———————— × ————————



$$\alpha_1 = \frac{1}{2}(\sigma_y^+ + \sigma_y^-)$$

$$\alpha_2 = \frac{1}{2}(Y^+ + Y^-)$$

$$\underset{\sim}{\alpha}_i = \begin{bmatrix} \frac{-\alpha_i}{3} & & \\ & \frac{-\alpha_i}{3} & \\ & & \frac{2\alpha_i}{3} \end{bmatrix} \quad \text{etc.}$$

×

Masing phenomenon

· experimentally observed for metals in unloadi that the elastic range is double as are inelastic ranges; has been used as a working hypothesis.

(▸Consequence of multi-Y.S. kinematic plastic

#5

Keynote address, reprinted from the Proceedings of the Third Engineering Mechanics Division Specialty Conference ASCE / Austin, Texas / September 17-19, 1979.

MACROSCOPIC AND MICROSCOPIC CYCLIC METAL PLASTICITY

by

Egor P. Popov[1], F. ASCE, and Miguel Ortiz[2]

ABSTRACT

Classical plasticity theories for ideal plastic material, as well as for those exhibiting isotropic and kinematic hardening properties, are reviewed first. These are shown to be inadequate for describing cyclic metal plasticity, and the more accurate newer multi- and two-surface theories are considered next. The reasons for the material macroscopic behavior modeled by these theories is clarified by an appeal to the dislocations theory. A remarkable agreement between one of the phenomenological theories of macroscopic plasticity theory based on internal variables with that deduced from the dislocation theory is demonstrated for uniaxial cyclic loading. The paper is limited to the consideration of rate-independent, isothermal problem.

1. INTRODUCTION

In recent years the computational capability generally available for the plastic analysis of structures experienced dramatic improvements. Most of these advances have been made with the aid of computers. The development of reliable constitutive relations lagged behind. This is particularly true as it applies to the difficult problem of cyclic plasticity, a subject of great importance in many diverse engineering applications. These range in the need for accurate predictions from the inelastic cyclic behavior of aircraft and nuclear reactor components to joinery in buildings subjected to seismic forces. As a result, numerous papers and reviews advancing and evaluating the current status of plasticity have appeared in the technical literature [1,2,3,4]. An NSF sponsored workshop was held in 1975 to bring together the latest thinking on the subject [5]. This paper, without an attempt on completeness, pursues a more limited objective. Only those theories which in the opinion of the authors are most widely used in cyclic metal plasticity are reviewed. These phenomenological formulations are based on the macroscopic behavior of materials. The reasons for the observed phenomena are elaborated upon by examining the cyclic material behavior from the point of view of the dislocations theory. A remarkable agreement in the macroscopic material behavior predicted by one of the phenomenological plasticity theories based on internal variables with that derivable from the dislocations behavior is shown for a uniaxial case. Possible extensions to multiaxial stress states are implied. The paper is limited to the isothermal, rate-independent plasticity.

It would appear that good formulations for predicting cyclic plastic response of materials is likely to remain highly specialized. However, in the words of Otto Mohr, written in 1901, "if one should succeed in finding a few rules under which many experiences can be subordinated, no laws of nature would have been derived, but some means found for judging the probability of new results of experience" [6]. Some such rules are emerging in cyclic metal plasticity.

2. MACROSCOPIC PLASTICITY THEORIES

Total strain (deformation) theories are unsuitable for describing cyclic plasticity. One must base the required formulations on incremental (flow) theories. For constructing such theories along the classical lines the following basic requirements must be met.

1. During a loading-unloading processes a yield-loading surface* exists in the stress space. The stress space within a yield-loading surface is purely elastic.

2. For strain-hardening materials, hardening rules defining the change in size and position of the loading surfaces in the course of plastic deformation must be specified.

3. The strain rate can be decomposed into elastic and plastic components, i.e. $\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p$. Here dots over the quantities indicate differentiation with respect to time, which is used as a parameter.

4. The plastic strain rate vector at a load point is directed along an outward normal to a yield-loading surface, i.e., the strain rate vector obeys the normality rule. Strain rate vectors not conforming to this rule, but suitable for some soils and work-softening materials, are not considered in this paper.

5. The magnitude of the plastic strain rate vector is assumed to be derivable from a potential function which is defined by the yield-loading surface.

The discussion that follows is principally confined to the first two items of the above. A reader is referred to available texts for treatment of the other items [2,8,9,10].

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[1] Prof. of Civ. Engrg., Univ. of California, Berkeley, CA

[2] Research Asst., Univ. of California, Berkeley, CA

*For simplicity, no distinction between yield and loading surfaces in the sense of Eisenberg and Phillips [7] is made.

2.1 Ideal Elasto-Plastic Material If the experimental evidence for a material clearly exhibits a pronounced yield as shown in Fig. 1(a), such a material can be treated as ideally elasto-plastic. For a multiaxial state of stress either the Tresca or the von Mises yield surfaces can be used to define the onset of yield, Fig. 1(b). According to Hencky [6] the von Mises yield criterion implies that yielding begins whenever the combination of stresses is such that the strain energy of distortion is equal to the corresponding energy at yield for uniaxial stress. This condition can be expressed mathematically as

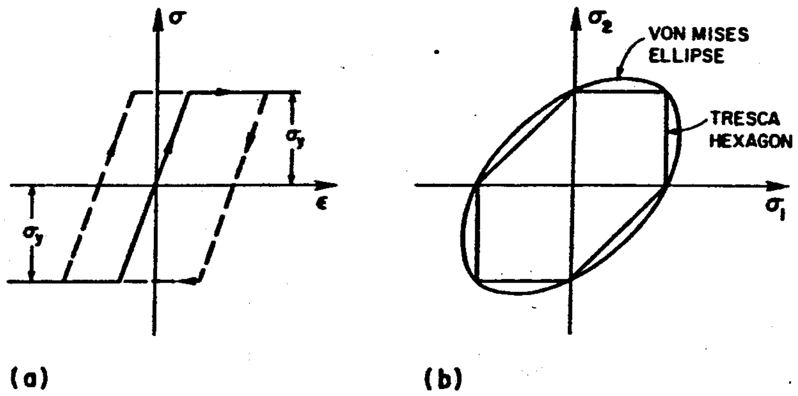


Fig. 1 Ideal Elasto-Plastic Material Behavior

$$f(\sigma_{ij}) = \frac{1}{2} s_{ij} s_{ij} = k^2 \tag{1}$$

where the function of stresses σ_{ij} is taken to be the second invariant of deviatoric stresses s_{ij}, and the constant k is the yield stress in pure shear.

Alternatively, the Tresca yield condition asserts that inelastic action begins whenever for a multiaxial stress the maximum shearing stress on any critical plane reaches a value equal to the maximum shearing stress for uniaxial stress. The relation giving this condition reads

$$f_1 = \left[(\sigma_1 - \sigma_2)^2 - 4k^2\right]\left[(\sigma_2 - \sigma_3)^2 - 4k^2\right]\left[(\sigma_3 - \sigma_1)^2 - 4k^2\right] = 0 \tag{2}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses, and k is the same constant as that in Eq. 1.

As can be seen from Fig. 1(b) the difference between the two criteria is not large. The experimental evidence is not decisive, although the von Mises criterion appears to be the more accurate one of the two. No ambiguities arise in applying these criteria for cyclic loadings.

However, since the behavior of very few materials can be idealized in this manner, other plasticity theories were introduced.

2.2 Isotropically Hardening Material Some materials, after the initial yield, strain-harden as shown in Fig. 2(a). Some of the earliest attempts to introduce this effect analytically were due to Hill [8] and Hodge [11]. According to this concept, on a complete load reversal the same stress level along an elastic path can be reached, thereby expanding the initial elastic range. In the stress space, on piercing the initial yield surface, progressively larger, geometrically similar, loading surfaces are developed, Fig. 2(b). Within any new loading surface the material is purely elastic. For example, after reaching point C on the loading surface, the unloading path CD is elastic. In this formulation the scalar constant k introduced earlier in Eqs. 1 and 2 is made to increase monotonically with increasing load as determined by an experiment. This theory contradicts the Bauschinger effect, i.e., the reduction in yield strength on complete load reversal. For this reason it is not suitable for cyclic plasticity with load reversals.

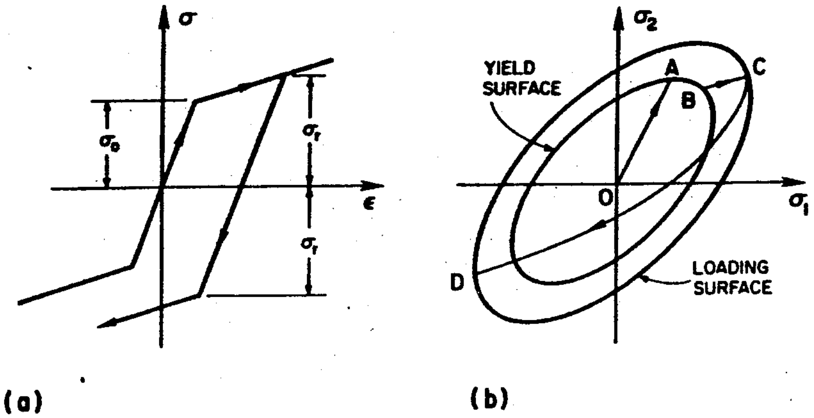
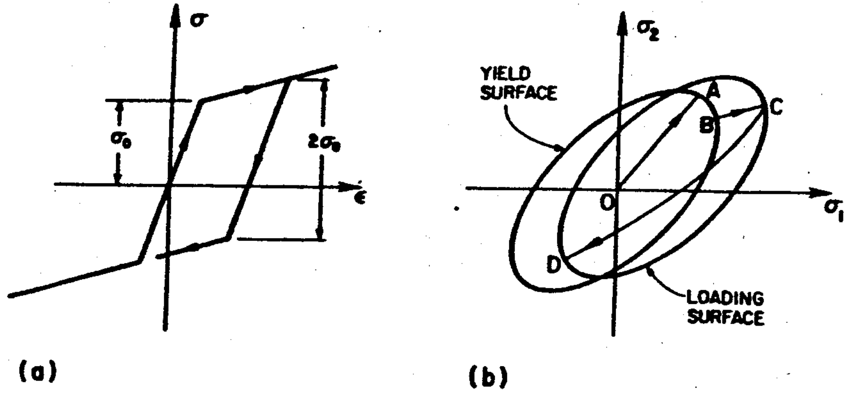


Fig. 2 Isotropically Hardening Material Behavior

2.3 Kinematically Hardening Material A much more satisfactory assumption for treating load reversals is illustrated in Fig. 3. This approach, initiated by Ishlinskii [12] and Prager [13], is referred to as kinematic hardening. Note that in this idealization the elastic region remains the same regardless of the previous strain history experienced by a material. Either the von Mises or the Tresca yield condition can be used for defining the yield surface. In a form of an equation of the von Mises type this criterion can be expressed as

$$f\left(\sigma_{ij} - \alpha_{ij}\right) = k^2 \tag{3}$$

This relation differs from Eq. 1 by the presence of α_{ij}, which defines



(a) (b)

Fig. 3 Kinematically Hardening Material Behavior

the translation of the initial yield surface. This quantity can be found by applying Prager's kinematic rule

$$\dot{\alpha}_{ij} = c\,\dot{\epsilon}^{P}_{ij} \tag{4}$$

where c is a material constant. The translation rate $\dot{\alpha}_{ij}$ of the loading surface is directed along or normal to this surface at a load point.

To achieve consistency in certain subspaces, Ziegler [14] modified Eq. 4 to read

$$\dot{\alpha}_{ij} = \dot{\mu}\left(\sigma_{ij} - \alpha_{ij}\right) \tag{5}$$

which means that the direction of motion of the center of a loading surface is along the radius vector that joins its instantaneous center α_{ij} with the load point σ_{ij}. Here $\dot{\mu} > 0$. The meaning of Eqs. 4 and 5 is illustrated in Fig. 4.

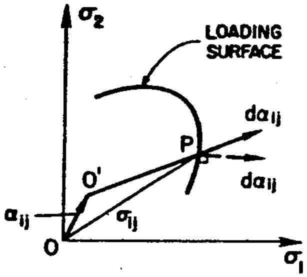


Fig. 4 Direction of Translation $d\alpha_{ij}$ of the Loading Surface: Prager's Rule Shown Dashed, Ziegler's Rule Shown Solid (After [1])

The kinematically hardening model for a material is the most commonly used theory for cyclic plasticity. For some applications it is combined with the isotropic model as illustrated with the Tresca yield condition in Fig. 5.

2.4 Refinements of Loading Surface Shapes

The material models discussed above may not be sufficiently accurate for some applications in cyclic plasticity. Whereas the isotropic model can define a gradual change from the elastic into the strain-hardening range, the generally more useful kinematically hardening model cannot. As illustrated in Fig. 6, for a kinematic hardening behavior represented by a bilinear relationship, a sharp corner develops at a point such as point A. The behavior of a real material may be more accurately defined by a curved line.

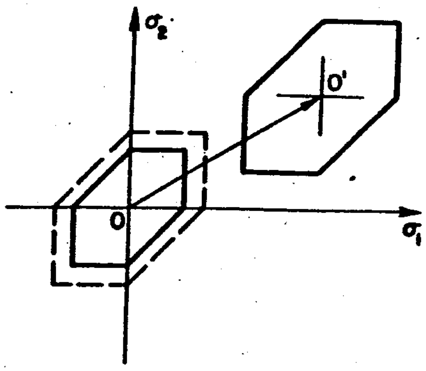


Fig. 5 Simultaneous Isotropic and Kinematic Hardening

The discrepancy on a load reversal between the classical plasticity theories and experiments is particularly pronounced. For an isotropically hardening material, the elastic range expands corresponding to a line such as BE in Fig. 6. Alternatively, according to the kinematic theory, this range remains constant as represented by line BC in Fig. 6. Neither one of these approximations corresponds to the real behavior of materials as first observed by Bauschinger in 1886. To properly account for the Bauschinger effect, the elastic range must extend say over the line BG of Fig. 6. The rounding of the response curve may also be necessary in some applications.

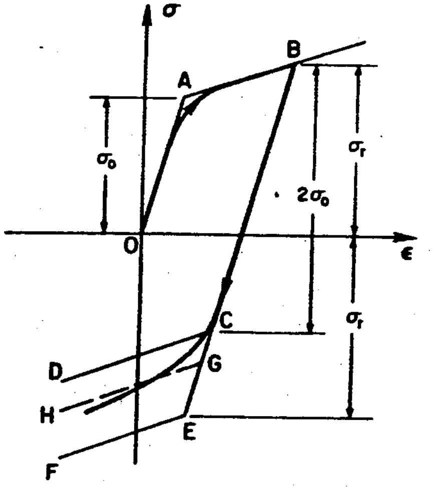


Fig. 6 Idealizations of Material Behavior on Load Reversal

A number of other discrepancies between theory and experiment are commonly neglected. Among these one could note that the stiffness of a material during a load reversal sometimes is found to be smaller than that occurring during the initial loading. Likewise, a sharp definition of a point such as B for repeated loadings and unloadings actually does not exist. Fortunately, a number of these considerations are not particularly important. It must be emphasized further, that the materials treated in this manner are considered to be rate-independent.

Some early attempts to account for the Bauschinger effect have been made by assuming piecewise linear modifications of the yield surface of the type shown in Fig. 7(a) [15]. Other modifications of the yield surfaces to account for the same effect were justified based on the concept of

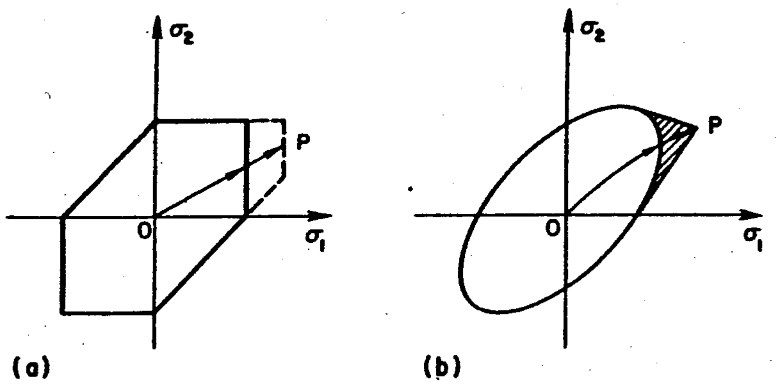


Fig. 7 Modifications of Yield Surfaces for Including Bauschinger Effect

slip along a critical plane giving rise to the development of corners on the yield surfaces [16]. No recent research activity in this direction can be noted at this time.

2.5 Mechanical Sublayer Models A development of a successful model for representing cyclic material behavior may be traced to the work of Duwez [17] with further elaborations by Besseling [18]. The basic concept for this model can be visualized by examining an arrangement with three parallel springs shown in Fig. 8(a). Two of the springs are attached to weights which slide on reaching their respective critical forces which correspond to ideal plastic behavior. On applying a monotonically increasing force the force-deformation path is as shown by the broken line OABC in Fig. 8(b). By reversing the applied force the system response is given by a series of straight lines CD, DE and EF. Note that during a load reversal the elastic range is doubled, as are the corresponding inelastic ranges. This correlates well with the experimental observations of Masing for cyclic loadings [19], and has often been applied in analysis as a working hypothesis. This kind of an approach became known in the literature as a sublayer or a subvolume model.

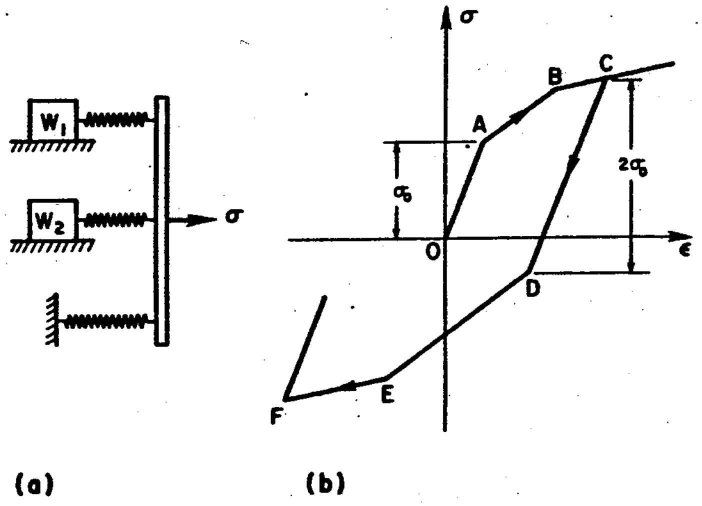


Fig. 8 Piece-wise Linear Material Idealization-Parallel or Sublayer Model

The above model has been generalized for multiaxial states of stress by Mróz [20], and, in a somewhat analogous manner by Iwan [21]. In the Mróz generalization discussed here, a field of workhardening moduli is introduced, with a constant workhardening modulus assigned to each sublayer. For simplicity, as an illustration, consider the series of concentric circles shown in Fig. 9(a). In actual applications usually a series of concentric von Mises ellipses would be used at the initiation of a loading process. For each one of these circles a constant workhardening modulus is assigned as f_0, f_1, f_2 etc. A load point within the first circle at all times remains elastic. However, on reaching the boundary of the first circle, for further loading the modulus becomes f_0. This magnitude of the modulus remains constant as the load point together with the first circle translates toward the second circle. On reaching the second circle, the coupled first and second circles continue their motion toward the third circle as shown in Fig. 9(b). During this stage of the loading process the modulus associated with the second circle is applicable.

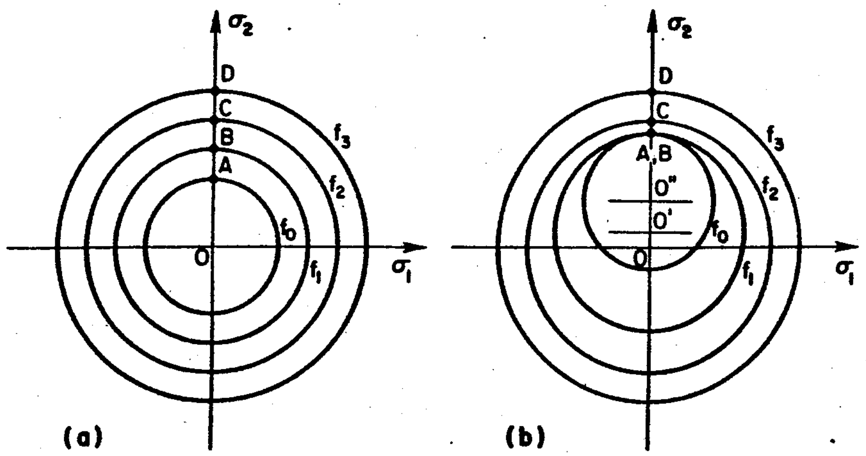


Fig. 9 Schematic Illustration of Sublayer Model

In the above discussion it was tacitly assumed that the load point moved

monotonically upward along the vertical axis. In general this is not the case, and further, since the lines corresponding to constant work-hardening moduli are not circles, an additional rule for translation of these loading surfaces must be introduced. Mróz achieves this by requiring that a load point on a given loading surface translates toward a point on the next loading surface such that the direction of the normals to two loading surfaces at the two related points is the same. This rule appears to give reasonable results when compared with the very limited available experimental data.

The Mróz model has been programmed for a computer, and a few comparative studies of its accuracy and efficiency are available [4,5].

2.6 Cyclic Plasticity Based on Internal Variables The objective of continuum mechanics is to describe the mechanical state at a material point of a body. Ordinarily this is achieved by using the observable or external variables such as deformation, temperature and stress. In plasticity, in addition to these, it is convenient to introduce also the hidden or internal variables such as inelastic strain. A phenomenological model using the internal variables formalism has been recently developed for rate-independent cyclic plasticity [22]. The rudiments of this model are described below.

Uniaxial case From a study of uniaxial stress-strain curves shown schematically in Fig. 10 three distinct regions may be observed.

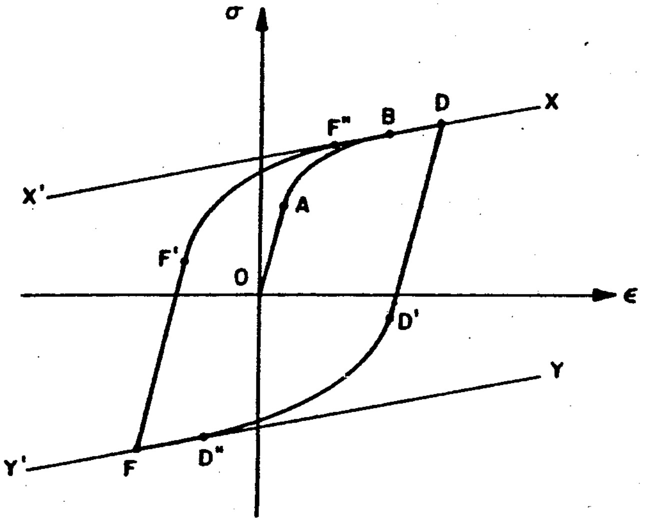


Fig. 10 Schematic Illustration of the Line Bounds in σ-ε Space [24]

Starting for example from point F, the first part FF' represents the elastic behavior. The second part F'F" represents the strongly

nonlinear plastic behavior. Finally, the curve F"X representing plastic behavior attains an essentially constant plastic modulus. Since the elastic strain can be readily determined from Hooke's law, attention need be confined only to the study of stress-plastic strain curves such as shown in Fig. 11. Note the clearly evident bounds in both diagrams.

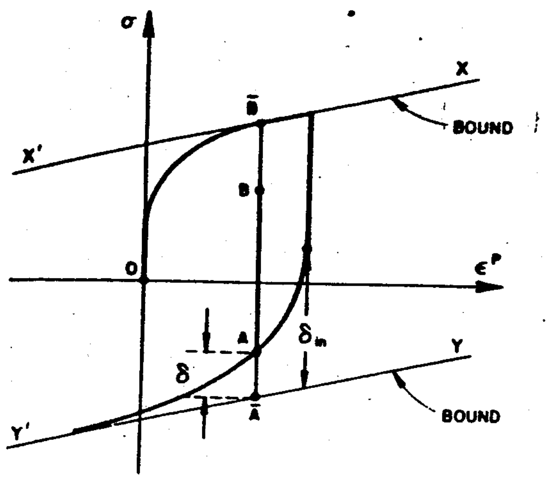


Fig. 11 Schematic Illustration of δ and δ_{in} [22,23]

A study of numerous experimental curves led to the conclusion that the material behavior can be described by considering the plastic modus E^P to be a function of the distance δ of the stress state A from the corresponding point $\bar{A}$ on the bound [23]. However, this parameter alone is not sufficient, and for each initiation of the yielding process a parameter δ_{in} must be introduced. This quantity is an indirect measure of the amount of loading which occurred during a previous cycle. A large δ_{in} implies an extensive plastic loading before a stress reversal. The δ's have the dimensions of stress.

If a loading process is reversed at some point, such as A in Fig. 11, first an elastic response is observed. This corresponds to the regions AB shown in Figs. 11 and 12. On further loading, the shape of the curve BB' depends on δ_{in} associated with this path and an instantaneous δ. This behavior of a material exhibited in the stress-strain space is not suitable for further generalization. Therefore one must re-cast the available information into the stress space which for a uniaxial case is the vertical σ-axis. This is obtained by projecting the appropriate points onto the vertical axis as shown in Fig. 12. From this diagram it can be seen that an elastic region a'b' is contained within a region $\bar{a}'\bar{b}'$ defined by the bounding lines. Both regions remain constant throughout any loading process. Since at any load point such as B' its δ as well as δ_{in} are known, sufficient information is available for determining E^P, which is the quantity sought. The rules for a coupled translation of the elastic and bounding regions have been established [22,23,24]. The choice of straight bounds and the use of δ and δ_{in} for defining the shape of the plastic curves are a matter of convenience, and do not place a restriction on the method described.

Using the above approach, the proposed form [23,24] for the plastic modulus E^P is

$$E^P = E^P_o + h\left(\frac{\delta}{\delta_{in} - \delta}\right) \qquad (6)$$

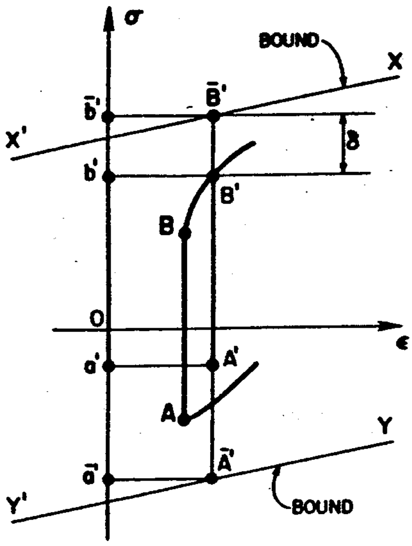


Fig. 12 Projection of σ-ε^P Space into the Stress Space for Uniaxial Loading.

where at $\delta = 0$, $E^P = E^P_0$, and h is a shape parameter determined from experimental data. An application of this method for representing random cyclic loading behavior is shown in Fig. 13. As can be seen, the agreement between theory and experiment is excellent.

Multiaxial case In a multidimensional stress space the end points such as a',b' shown in Fig. 12 lie on a yield surface, and the end points $\bar{a}'$,$\bar{b}'$ can be thought to lie on another surface enclosing the yield surface. The latter surface is called the bounding surface first introduced by Dafalias and Popov [24]. Independently, this concept was also adopted by Krieg [25] for reasons of numerical advantages in applications. The former approach is discussed here.

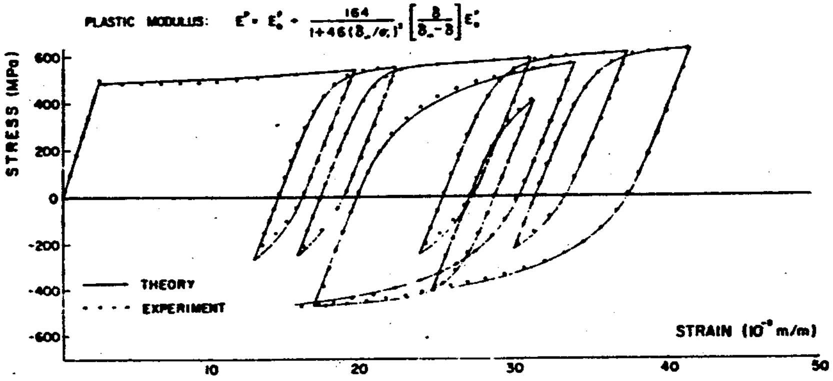


Fig. 13 Random Cyclic Loading on Grade 60 Steel Specimen [22,23]

A schematic illustration of yield and bounding surfaces is given in Fig. 14. Here the yield surface can be defined as

$$f\left(\sigma_{ij} - \alpha_{ij},\ q_n\right) = 0 \tag{7}$$

where kinematic hardening is explicitly introduced by α_{ij}; the other plastic internal variables such as plastic strain are denoted by q_n. The corresponding equation for the bounding surface is

$$F\left(\sigma_{ij} - \beta_{ij},\ q_n\right) = 0 \tag{8}$$

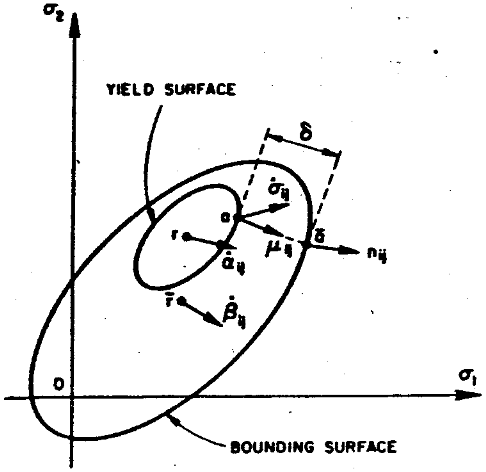


Fig. 14 Schematic Representation of the Yield and Bounding Surface and Illustration of Their Motions [22,23]

where β_{ij} is another plastic internal variable giving the coordinates of the center for the bounding surface. During a course of plastic deformation the two surfaces translate simultaneously in the stress space, and in general, may also deform. Throughout the course of plastic deformation the distance δ between two points such as a and $\bar{a}$ in Fig. 14 continuously changes. Following Mróz, the load point a on the yield surface is related to the point $\bar{a}$ on the bounding surface by having the same direction of the normals to these surfaces at these points. The distance δ is given by an Euclidean metric. From the information on δ_{in} at the initiation of a plastic loading process and an instantaneous δ, a generalized plastic modulus can be found in a manner analogous to that used in the uniaxial case. This general procedure can be readily specialized into the purely kinematic and/or isotropic hardening rules [22].

An alternative numerical implementation At a numerical implementation level the model described above may lead to inaccurate results in some cases. For example, consider the uniaxial cyclic loading pattern illustrated in Fig. 15 [26]. Here, since some load reversals take place before any plastic flow occurs in the opposite sense, the updating of the key parameter δ_{in} cannot be done correctly. This can be remedied by devising additional rules for updating δ_{in}, or, alternatively, by introducing auxiliary surfaces between the yield and the bond surfaces. The latter approach was used by Petersson and Popov [26,27] with good results. Further, in their work a procedure was devised for handling the behavior at a yield plateau, which characteristically occurs in mild steel. Some features of their approach are discussed below. It must be emphasized, however, that these refinements introduce no basic changes of the earlier model.

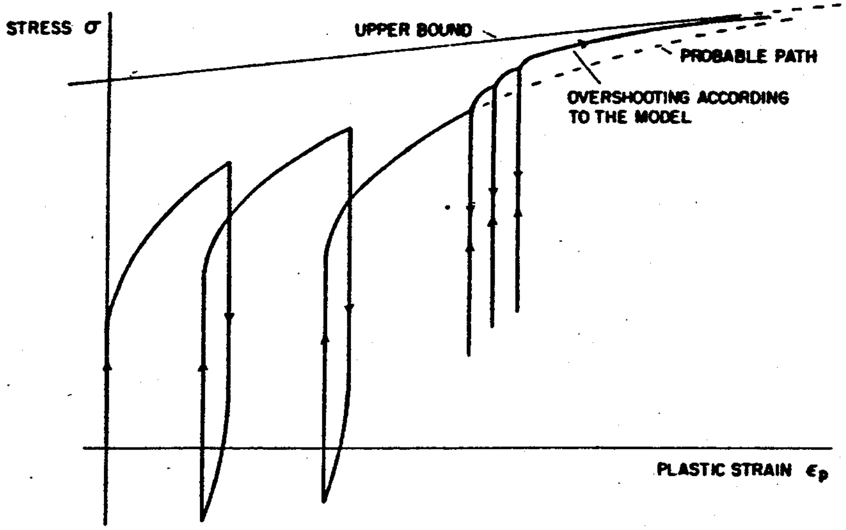
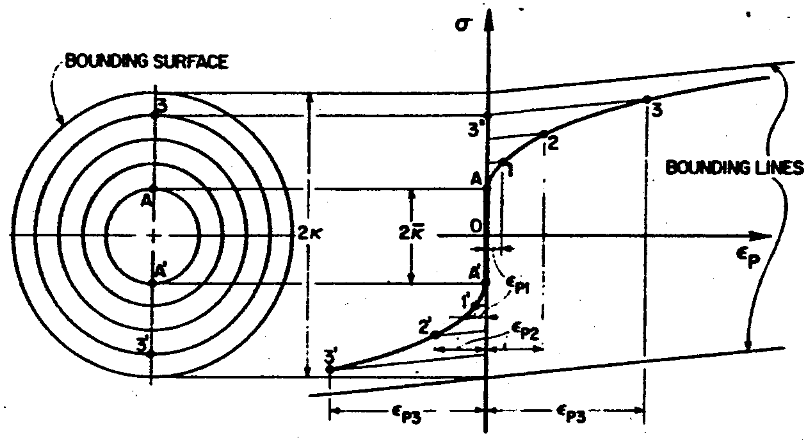


Fig. 15 Deficiency of Two-Parameter Model [26]

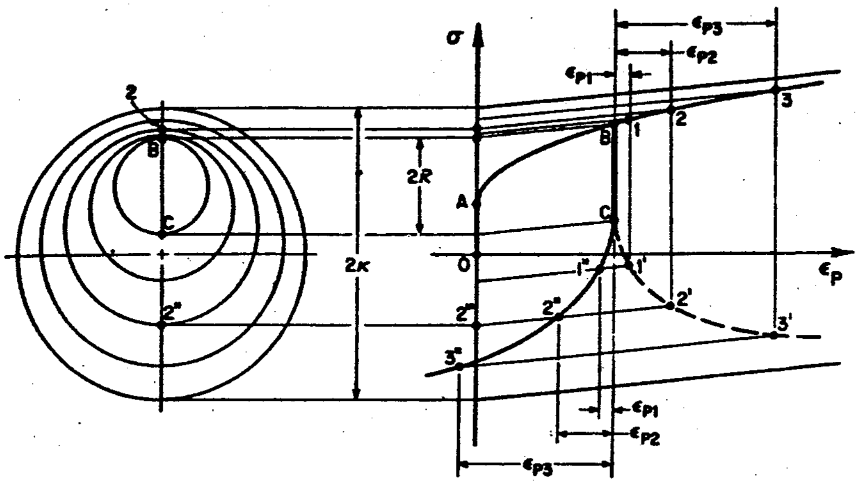
By using auxiliary surfaces between the yield and the bounding surfaces the initial stress-strain diagram can be accurately defined, Fig. 16.



(a) STRESS SPACE (b) STRESS-STRAIN SPACE

STRESS BETWEEN A AND A'

Fig. 16 Representation of Constitutive Relations: Monotonic Case [27]



(a) STRESS SPACE (b) STRESS-STRAIN SPACE

STRESS BETWEEN B AND C

Fig. 17 Representation of Constitutive Relations: First Load Reversal [27]

The intermediate surfaces are used for purposes of interpolation, and in principle are not related at all to those introduced in the Mróz model. During a load reversal the extended preceding stress-strain path, such as the path AB123 in Fig. 17(b), with the aid of the interpolating surfaces determines the forward path C1'2'3'. This procedure is analogous to the use of endochronic time. However, it is customary to reverse the strains to generate a new stress-strain path such as C1"2"3" shown in Fig. 17(b). The corresponding translation of the initial yield surface and the interpolating surfaces is shown in Fig. 17(a). At the next load reversal the curve C1"2"3" plays the same role as did the curve AB123 in the previous case.

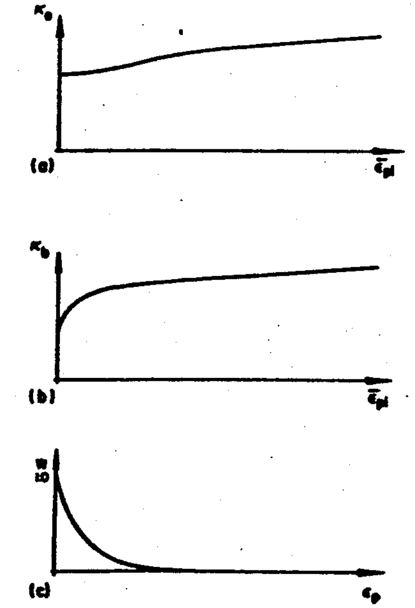


Fig. 18 Functions for Defining Load Surfaces [27]

The special problem of a yield plateau occurring in steel can be accounted for through the use of a weighting function. In this approach two stress-strain relations are used: the one defines a monotonic loading such as shown in Fig. 18(a); the other, a typical cyclic response, Fig. 18(b). By favoring initially the monotonic behavior by assigning to it a large weighting function, Fig. 18(c), which decreases with the accumulated plastic strain, a proper blend between the two functions can be achieved [26,27].

3. MICROSCOPIC APPROACH TO PLASTICITY

The macroscopic plastic behavior of metals is the result of the overall effect of structural changes that take place at a microscopic level during a loading process. Under appropriate conditions a metal may find it energetically advantageous to adopt new microstructural arrangements. Such microstructural re-arrangements occur at what became known as lattice defects. Among these, the line dislocations, as well as interstitial atoms and vacancies, play a particularly important role. The basic mechanism of plastic deformation is dependent on the motion of dislocations due to slip, climb, twinning, etc., and is considered in the sequel. The presentation is developed in such a manner as to relate the microscopic to macroscopic behavior. In the present discussion only the uniaxial state of stress is considered in detail.

3.1 Motion of Dislocations and Macroscopic Behavior Single Crystals

It is well established that dislocations in a single crystal move along well defined crystallograhic planes [28]. A set of all dislocation segments in a given glide plane with parallel Burgers vectors is referred to as a glide system. Such a glide system r is defined by a pair of orthogonal vectors $\underline{m}^{(r)}$ and $\underline{n}^{(r)}$, which represent, respectively, the direction of the Burgers vector and the normal to the glide plane. By extending a summation over all active glide systems, the infinitesimal plastic strain rate $\dot{\epsilon}^p_{ij}$ of a crystal can be related to the dislocation motion as [29]

$$\dot{\epsilon}^p_{ij} = \sum_r \dot{\gamma}^{(r)} \mu^{(r)}_{ij} \tag{9}$$

where $\mu_i^{(r)} = \left(m_i^{(r)} n_j^{(r)} + m_j^{(r)} n_i\right)/2$ is the Schmidt orientation tensor which completely describes the geometric properties of an r-th dislocation system. The quantity $\dot{\gamma}^{(r)}$ is the rate of plastic shear strain due to the motion of the dislocations in the glide system r. It is given by a relation [28]

$$\dot{\gamma}^{(r)} = b\rho_M^{(r)} v^{(r)} \tag{10}$$

where b is the magnitude of a Burgers vector, $\rho_M^{(r)}$ is the length of moving dislocations per unit volume, and $v^{(r)}$ is their average velocity.

By considering the plastic power $\dot{W}^p = \bar{\sigma}_{ij}\dot{\epsilon}^p_{ij}$, one can conclude that the conjugate quantity to $\gamma^{(r)}$ is a resolved shear stress $\bar{\tau}^{(r)}$ acting along a glide plane r in the direction of the Burgers vector. The resolved shear stress which drives the dislocations can then be expressed as

$$\bar{\tau}^{(r)} = \bar{\sigma}_{ij} \mu^{(r)}_{ij} \tag{11}$$

where $\bar{\sigma}_{ij}$ are components of a total stress tensor $\bar{\underset{\sim}{\sigma}}$ [30]. This total stress tensor consists of $\underset{\sim}{\sigma}$ due to the externally applied forces, and $\underset{\sim}{\sigma}^i$ caused by the internal stresses resulting from the local fields of dislocations. Hence

$$\bar{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma} + \underset{\sim}{\sigma}^i \tag{12}$$

Wilson and Konnan [31,32] have conclusively shown that a metal after experiencing plastic deformations contains large internal stresses. At the microscopic level, by their very nature the internal stresses are very irregular and develop sharp peaks. For the purposes at hand, however, interest lies in a smeared out average of these stresses.

A simple expression for the averaged $\underset{\sim}{\sigma}^i$ can be obtained by following Kroupa [33]. Based on Eshelby's solution for a stress field for an ellipsoidal inclusion in an elastic matrix, he worked out an explicit expression for the stress at a point created by a distribution of dislocations. The solution consists of two parts. One part of this solution shows the effect of dislocations in the neighborhood of the point under consideration. The other weaker effect due to the external dislocations is given by complicated volume integrals. The part of the solution due to the dislocations in the neighborhood of a point is particularly simple reading

$$\sigma_{ij} = - \frac{7(7-5\nu)}{15(1-\nu)} G\, \epsilon^p_{ij} \tag{13}$$

where ν is the Poisson ratio, and G is the shear modulus.

Recognizing that the external dislocations soften the material matrix in the Eshelby's solution, by analogy to Eq. 13 one can approximate the expression for the internal stress rate tensor as

$$\dot{\underset{\sim}{\sigma}}^i = - E^i \dot{\underset{\sim}{\epsilon}}^p \tag{14}$$

where E^i is an experimentally determined plastic modulus which indirectly accounts for the external dislocations. A similar approach recently has been suggested by Zaoui [34]. Equation 14 can be interpreted as providing a rule for the shift of the center of a yield surface, and is consistent with kinematic hardening.

Polycrystals Since materials are generally composed of a large number of randomly oriented grains, instead of a small number of well defined glide systems $\underset{\sim}{\mu}^{(r)}$ which occur in a single crystal, any number of possible orientations $\underset{\sim}{\mu}$ can give rise to plastic deformatin. In this case $\underset{\sim}{\mu}$ is no longer a discrete variable, but a continuous one. For this reason the treatment of this problem must be drastically simplified. A very satisfactory approach for resolving it has been made by Kelly [35]. In his approach one determines the glide system $\underset{\sim}{\mu}$ which makes the greatest contribution to plastic strain by maximizing a relation for stress $\sigma = \sigma_{k\ell}\mu_{k\ell}$ subject to the constraints $\mu_{kk} = 0$, and $\mu_{k\ell}\mu_{k\ell} = 1/2$. This leads to the result that

Fig. 19 Glide Planes in a Tension Specimen

$$\mu_{ij} = \frac{1}{2}\frac{\bar{s}_{ij}}{\sqrt{\bar{J}_2}} \qquad (15)$$

where the deviatoric stress $\bar{s}_{ij} = \bar{\sigma}_{ij} - \bar{\sigma}_{kk}\delta_{ij}/3$, and $\bar{J}_2 = \bar{s}_{ij}\bar{s}_{ij}/2$. As before, the barred stresses refer to the total stress (see Eq. 12). Whence on using Eq. 9, one has

$$\dot{\epsilon}^p_{ij} = \dot{\gamma}\,\frac{1}{2}\frac{\bar{s}_{ij}}{\sqrt{\bar{J}_2}} \qquad (16)$$

where now $\dot{\gamma}$ is the rate of shear deformation in the "principal" glide system. This equation completely coincides with the well-known Prandt-Reuss flow rule.

For the uniaxial case the above approximation indicates a deformation pattern of the type shown in Fig. 19. It also follows that here $\bar{\tau}$ and $\dot{\gamma}$ are related to the uniaxial stress $\bar{\sigma}_1$ and the uniaxial plastic strain rate $\dot{\epsilon}^p_1$, respectively, by $\bar{\tau} = \bar{\sigma}_1/\sqrt{3}$ and $\dot{\gamma} = \sqrt{3}\,\dot{\epsilon}^p_1$. An explicit expression for $\dot{\gamma}$ is established in the next section.

3.2 Motion of Dislocations in a Glide System In the preceding section, kinematic simplifications have been introduced that reduce the problem of plastic deformation to the description of the response of a single glide system at a time. As mentioned earlier, the motion of dislocations in a glide system is driven by the corresponding resolved shear stress, and is hindered by a number of different types of obstacles. Among these are: 1) Viscous forces, which become significant at very large dislocation speeds, 2) Extended obstacles, like the long-range stress field with which dislocations in multipoles interact, and which are meant to be accounted for by means of the quantity σ^i, and 3) Non-extended obstacles, with a very short range of interaction, effective only over a few atomic distances. To this group belong impurity atoms, precipitates, jogs in the glide dislocations, but mostly forest dislocations [29].

Non-extended obstacles have often been idealized as distributions of "pinning points" on a glide plane. Dislocation segments extending from one pinning point to another bow out under stress until they reach an unstable configuration and become detached and are free to proceed further, Fig. 20. This occurs at a total shear stress [36,37]

$$\bar{\tau} = \alpha_1 \frac{Gb}{\ell} \qquad (17)$$

where ℓ is the distance between the pinning points, and α_1 is a constant ranging from 0 to 1 depending on the break away angle ϕ.

Statistical approach Experimental observations [38,39] show that point-obstacles are quite randomly distributed in the glide planes.

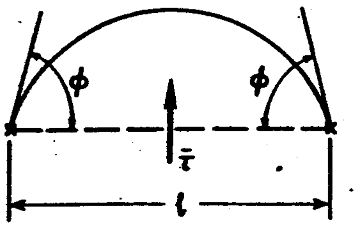


Fig. 20 Dislocation Segment Bowing-out Under Stress

Based on statistical considerations Kocks [27] derived the following expression for frequency distribution of inter-obstacle spacings ℓ:

$$f(\ell) = \frac{\ell}{\ell_m^2}\exp\left(-\frac{\ell^2}{\ell_m^2}\right) \qquad (18)$$

where ℓ_m is the average inter-obstacle spacing. This relation is in excellent agreement with the experimental measurements.

Using Eq. 17, it is convenient to change the variable ℓ to $\bar{\tau}$ giving

$$\bar{f}(\bar{\tau}) = \frac{\tau_m^2}{\bar{\tau}^3}\exp\left(-\frac{\tau_m^2}{\bar{\tau}^2}\right) \qquad (19)$$

where $\tau_m = \alpha_1 Gb/\ell_m$ is the mean flow stress. The differential $\bar{f}(\bar{\tau})\,d\bar{\tau}$ expresses the fraction of segments that become released from their pinning points when the resolved shear stress is increased from $\bar{\tau}$ to $\bar{\tau}+d\bar{\tau}$. This can also be expressed as $d\bar{P}(\bar{\tau})$, where $\bar{P}(\bar{\tau})$ is the probability distribution function associated with $\bar{f}(\bar{\tau})$. Using the Kocks model,

$$\bar{P}(\bar{\tau}) = \int_0^{\bar{\tau}} \bar{f}(s)ds = \exp\left(-\frac{\tau_m^2}{\bar{\tau}^2}\right) \qquad (20)$$

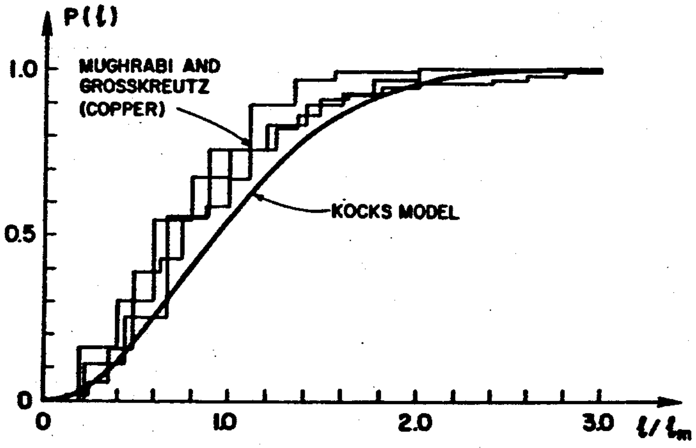


Fig. 21 Inter-obstacle Spacing Distribution Function (Based on data given in ref. [38])

A similar function $P(\ell)$, which can be found in an analogous manner by using Eq. 18, is compared with some experimental results in Fig. 21. The agreement between the theoretical result and experiments is seen to be very satisfactory. A similar curve using $\bar{\tau}$ as the independent variable is shown in Fig. 22.

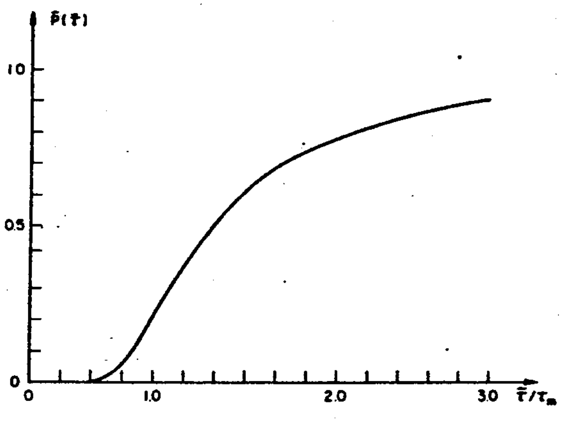


Fig. 22 Inter-obstacle Spacing Distribution Function in Terms of Required Stress to Overcome Obstacles

With the aid of the function shown in Fig. 22, a differential increment in the dislocation length density ρ_M set in motion upon an increase of the stress $\bar{\tau}$ to $\bar{\tau} + d\bar{\tau}$ can be found. Thus, if ρ is the total dislocation length density in a glide system, and β is the mobile fraction, i.e., the ratio of the mobile dislocation density to the total density, one can write

$$d\rho_M = \beta\rho \, d\bar{P}(\bar{\tau}) = \beta\rho \frac{\partial \bar{P}}{\partial \bar{\tau}} d\bar{\tau} \tag{21}$$

To calculate the increment in shear strain that these released dislocation segments produce one has to know how far they will go before being arrested. For a given stress $\bar{\tau}$ such a distance $L(\bar{\tau})$ on a statistical basis was found by Kocks [27] to be

$$L(\bar{\tau}) = \frac{\ell_m}{1 - \bar{P}(\bar{\tau})/\bar{P}(\tau_m)} \tag{22}$$

This equation shows that at $\bar{\tau} = 0$, the flight distance available for the released segments is ℓ_m; for $\bar{\tau} \sim \tau_m$, all of the glide plane becomes available and the flight distance tends to become infinite. In reality, the latter condition can never develop due to strain-hardening. A plot of the Kocks model is shown in Fig. 23.

Making use of Eqs. 10, 21 (in rate form), and 22, one finally obtains an expression for the rate of shear deformation

$$\dot{\gamma} = b\dot{\rho}_M L = b\beta\rho\ell_m \frac{\partial\bar{P}(\bar{\tau})/\partial\bar{\tau}}{1 - \bar{P}(\bar{\sigma})/\bar{P}(\sigma_m)} \dot{\bar{\tau}} \tag{23}$$

Calling for simplicity

$$H^P(\bar{\tau};\tau_m,\beta,\rho) = \frac{1 - \bar{P}(\bar{\tau})/\bar{P}(\tau_m)}{b\beta\rho\ell_m \, \partial\bar{P}(\bar{\tau})/\partial\bar{\tau}} \tag{24}$$

one can write Eq. 23 as

$$\dot{\gamma} = \dot{\bar{\tau}}/H^P(\bar{\tau};\tau_m,\beta,\rho) \tag{25}$$

Recalling that for the uniaxial case $\bar{\tau} = \bar{\sigma}_1/\sqrt{3}$ and $\gamma = \sqrt{3}\,\dot{\epsilon}_1^P$, from Eq. 25 one has

$$\dot{\epsilon}_1^P = \frac{\dot{\gamma}}{\sqrt{3}} = \frac{\dot{\bar{\tau}}}{\sqrt{3}\,H^P} = \frac{\dot{\bar{\sigma}}_1}{3H^P} = \frac{\dot{\bar{\sigma}}_1}{E^P} \tag{26}$$

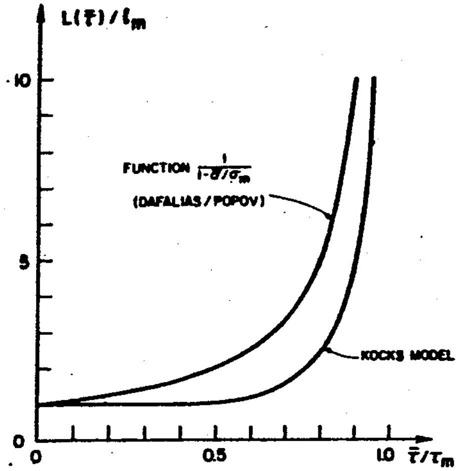


Fig. 23 Available Flight Distances at Different Stress

where E^P is the plastic tangent modulus. From this relation it can be concluded that for a uniaxial case H^P differs from the plastic tangent modulus E^P merely by a constant factor of 3.

Simplified approach At this point it is advantageous to introduce some approximations. Thus, instead of using Eq. 21, let

$$d\rho_M = \beta\rho \frac{\bar{\tau}}{\tau_m^2} d\bar{\tau} \tag{27}$$

which means that the fraction of the dislocation density which is released due to a stress increment $d\bar{\tau}$ is directly proportional to the total stress. The nature of this approximation may be seen in Fig. 24 where it is compared with the Kocks statistical model. Further, take a simple plausible expression for the flight distance $L(\bar{\tau})$ by replacing Eq. 22 with

$$L(\bar{\tau}) = \frac{\ell_m}{1 - \bar{\tau}/\tau_m} \tag{28}$$

A comparison of this function with the Kocks model is shown in Fig. 23.

On substituting the approximate Eqs. 27 and 28 into Eq. 23, one finds

$$\dot{\gamma} = \frac{b\beta\rho\ell_m}{\sigma_m} \frac{\bar{\tau}}{\tau_m - \bar{\tau}} \dot{\bar{\tau}} \tag{29}$$

which by virtue of Eqs. 25 and 26, and the relation between the uniaxial stress σ_1 and $\bar{\tau}$, yields

$$E^P = \frac{3\tau_m}{b\beta\rho\ell_m}\,\frac{\sigma_m - \bar{\sigma}_1}{\bar{\sigma}_1} = h\,\frac{\sigma_m - \bar{\sigma}}{\bar{\sigma}} \tag{30}$$

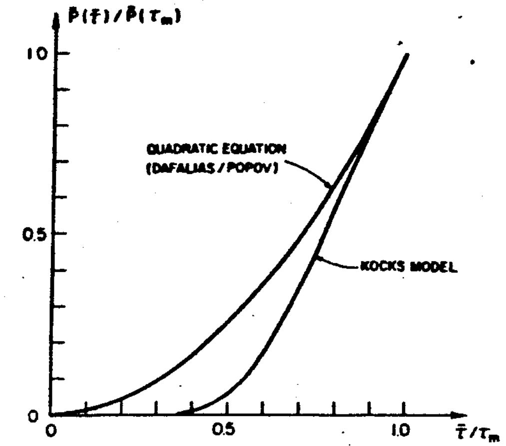


Fig. 24 Comparison of a Quadratic Function with Statistical Probability Function for Inter-Obstacle Spacing

where $\sigma_m = \sqrt{3}\,\tau_m$, and h is the shape parameter defined earlier in connection with Eq. 6.

In terms of the notation introduced in Fig. 11 (also see Fig. 23), $\delta = \sigma_m - \bar{\sigma}$, and $\delta_{in} = \sigma_m$. Further, since $\bar{\sigma}$ and σ_m are measured from a line having a slope E^I, which is equal to E^P_0, Eq. 30 completely coincides with the phenomenological model for cyclic plasticity given by Eq. 6. Note, however, that the shape parameter h now is expressed in terms of the internal variables σ_m, ℓ_m, β, and ρ. Therefore, the evolution of the plastic deformation process accounting for hardening and softening can be formulated from kinetic equations. Some such equations are given in the next section.

At this point it is significant to remark that although the difference in the form of Eqs. 6 and 30 may appear to be trivial, because of this difference there are important consequences. As pointed out in connection with Fig. 15, for load reversals taking place before any noticeable plastic flow takes place in the opposite sense, the use of Eq. 6 tends to overshoot the probable stress path. In contrast, the response predicted for similar situations by Eq. 30 tends to undershoot such a path. The resulting sawtooth pattern of the stress-strain diagram appears to be in good agreement with many experimental results. Further, no auxiliary surfaces, such as shown in Figs. 16 and 17, are needed for the condition described if Eq. 30 is employed. Therefore, the use of Eq. 30 is recommended.

3.3 Kinetic Equations for Internal Variables

The behavior of a glide system exemplified by Eq. 25 is described in terms of stress and a number of internal variables, namely, the dislocation length density ρ, the fraction of mobile dislocations β, the average shear flow stress τ_m, and the resolved shear stress due to internal stresses τ^I. The evolution of these variables defines the constitutive equations.

The simplest theory of work-hardening follows from the concepts of a mean free path of dislocations, introduced by Nabarro [40], which is usually taken as $1/\sqrt{\rho}$, and the principle of similitude [41]. With these assumptions one finds that τ_m hardens at a linear rate.

$$\dot{\tau}_m = H^P_0\,|\dot{\gamma}| \tag{31}$$

In the stress-strain space this equation traces out straight lines. Moreover, Eq. 23, or more simply Eq. 29, show that the stress-strain curve tends asymptotically toward these straight lines. Thus, the linear hardening theory predicts straight bounds. Note that in a plot such as Fig. 25 the total stress must be measured from a straight line OA, rather than from the strain axis, to account for the development of the internal stresses. Further, if one sets $H^I = aH^P_0$, for $a = 0$, the bounds represent isotropic hardening, for $a = 1$, pure kinematic hardening, and for $0 < a < 1$, mixed hardening.

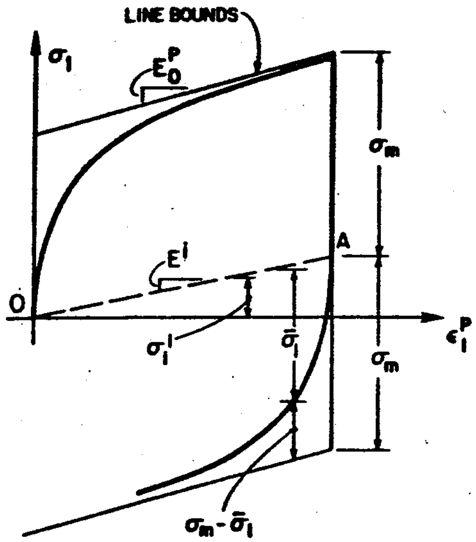


Fig. 25 Schematic Illustration of a Linear Hardening Theory

The above approach, however, is not entirely satisfactory since it cannot account for the dislocation saturation stress and the associated phenomena. On the basis of experimental and theoretical work by Li [42], and Johnston and Gilman [43], Sackett, Kelly, and Gillis [44] proposed suitable rate equations for the dislocation length density and the mobile fraction which can be used for a more refined treatment of the problem. The two needed equations are

$$\dot{\rho} = \frac{\lambda}{b}\left(1 - \frac{\rho}{\rho_s}\right)\beta|\dot{\gamma}| \tag{32}$$

$$\dot{\beta} = \frac{\kappa}{b}\left(\beta_s\,\frac{\rho_s}{\rho} - \beta\right)\beta|\dot{\gamma}| \tag{33}$$

where ρ_s is a dislocation saturation density at which the rate of dislocation annihilation becomes equal to the rate of their generation and the dislocation density remains constant. Similarly, β_s is a saturation mobile fraction. The coefficients λ and κ are material constants. A schematic illustration of the evolution of ρ and β as a function of $|\gamma|$ is shown in Fig. 26.

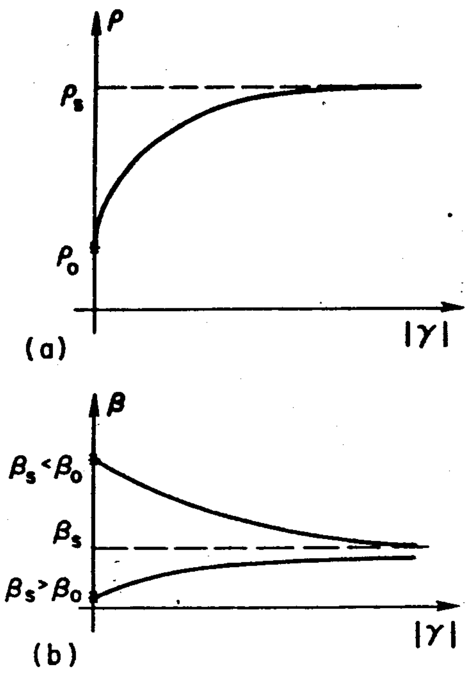


Fig. 26 Evolution of ρ and β with Increasing Dislocation Density

Assuming that the non-extended obstacles are caused primarily by the dislocations themselves, $\ell_m = 1/\sqrt{\rho}$. Whence, on recalling from Eq. 19 the definition of $\tau_m \equiv \alpha_1 Gb/\ell_m$, with the aid of Eq. 32, a rate equation for the average flow stress σ_m reduces to

$$\dot{\tau}_m = \frac{1}{2}\frac{\lambda}{b}\left(\frac{1}{\rho}-\frac{1}{\rho_s}\right)3\tau_m|\dot{\gamma}|$$
$$= H_o^P\left(\tau_m,\beta,\rho\right)|\dot{\gamma}| \qquad (34)$$

According to this equation the bounds or lines traced by σ_m ($= \sqrt{3}\ \tau_m$) are now not straight, but have a progressively decreasing slope, which is in conformity with the experimental observations. In the limit the bounds become horizontal. This condition observed in cyclic plasticity [38] is reached at a saturation stress $\underline{\sigma_S} = \sqrt{3}\ \tau_S$, where $\tau_S = \alpha_1 Gb\sqrt{\rho_S}$. When this level of saturation is reached, the hysteretic curves basically retrace themselves. This type of behavior is illustrated in Fig. 27.

A rate equation for the shape parameter h follows from Eqs. 30 and 33 to read

$$\dot{h} = \frac{\kappa}{b}\left(\beta - \frac{\rho_s}{\rho}\ \beta_s\right)\ h\ |\dot{\gamma}| \qquad (35)$$

This equation shows that if $(\beta/\beta_s) < (\rho_s/\rho)$ the stress-strain curve progressively flattens, and vice versa.

It should be noted that in the above development the yield plateau type effect was not considered.

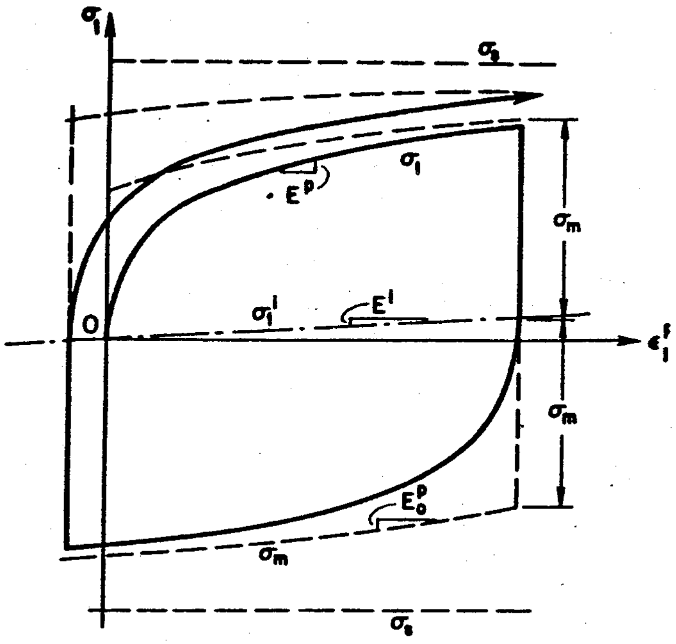


Fig. 27 Illustration of Curved Bounds and Saturation Effect in the Course of Plastic Deformation

4. CONCLUDING REMARKS

Rate-independent classical metal plasticity theories were reviewed first. For cyclic loadings the kinematically hardening theory appears to be reasonably satisfactory when accurate predictions of the material behavior are not required. The ideal elastic-plastic behavior is a special case of this type of theory, and may be useful in some applications. For a more refined analysis of cyclic material behavior, two newer theories were outlined. Because of the conceptual and analytical simplicity, the formulation given for a two-surface theory by Dafalias and Popov shows considerable promise. Although this theory initially was developed primarily on a phenomenological basis, it can be reasonably well justified, at least for the uniaxial case, by the dislocation theory. This gives further credence to the proposed approach, which, in addition to the customary use of a yield surface, brings in the concept of a bounding surface. It is believed that a satisfactory correlation between the phenomenological and the microscopic approaches has been demonstrated for the case of uniaxial cyclic loading. Additional effort should be now directed at a more rational analytic formulation of the yield plateau effect, and a more thorough treatment of the multiaxial case. It is hoped to clarify these questions in papers to follow. Extensions of the microscopic approach discussed in this paper may be logically extended into the realm of visco-plasticity.

5. ACKNOWLEDGEMENTS

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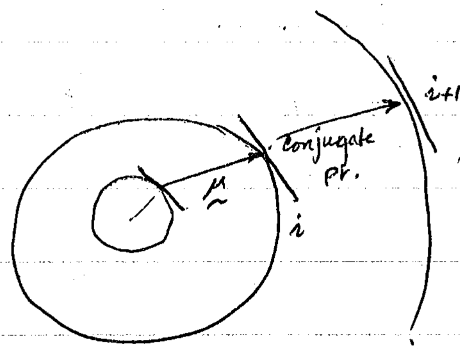
2/15/83

Multiple yield surface

2 Surface Theory		m-Surface Theory
{Interior of inside surface was elastic}	→	{inside of surface 1 was elastic}
{plastic Mroz hardening}	→	{All interior surfaces Mroz harden}
{Outside classical kinematic hardening}	→	{Surface m; same}



Moral: μ when loading surface $i<m$, points to the "conjugate pt." on surface i

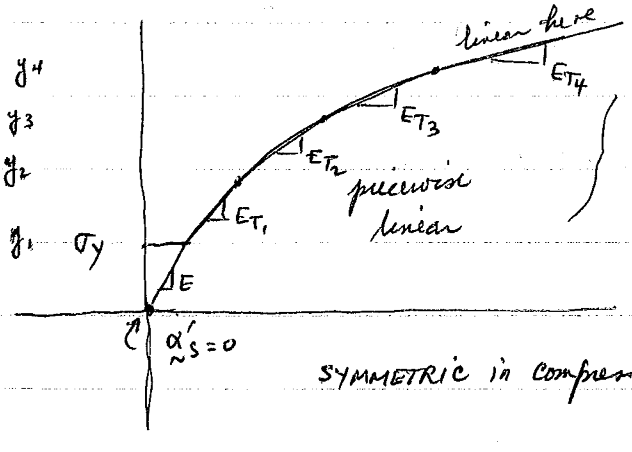
as pt moves to y.s. it moves instantaneously to next surface, dragging previous surface along.

DATA needed E, ν k_i $i = 1, 2, \dots, m$ } extremely easy to obtain

H_i' "

$\underset{\sim}{\alpha}_i$ (backstresses) "
centers of surface.

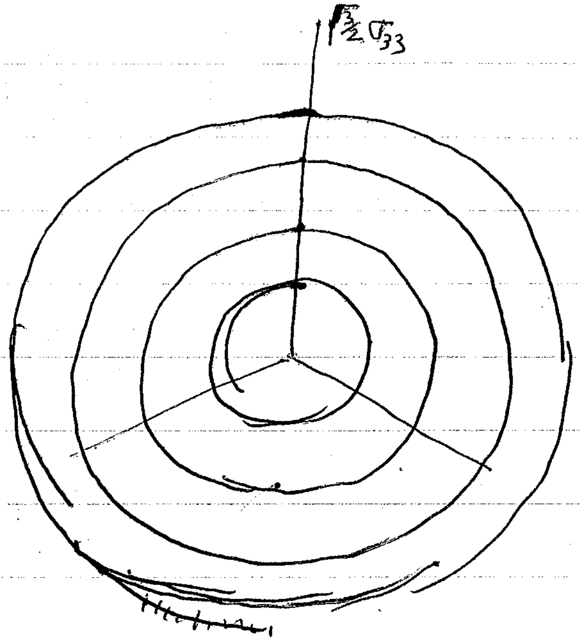
identification
in the context of metals in which $\underset{\sim}{\alpha}_i = 0$ (isotropy in initial state)



} 4 yield surfaces

now go to scaled version of Π plane

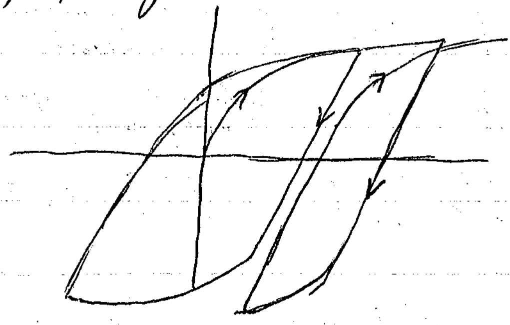
SYMMETRIC in compress or tension



now calculate $k_i = Y_i/\sqrt{3}$

and $H_i' = E_{T_i} / (1 - E_{T_i}/E)$

You'll get following type of hysteresis loop.



Masing phenom is present elastic regio(~~doubles in size~~) in plastic regime during unloading doubles in size.

<u>Summarize:</u> Multi-Yield Surface Kinetic hardening theory.

Constit eq => $\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot (\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^{pl})$

$$\dot{\underset{\sim}{\varepsilon}}^{pl} = \begin{cases} 0 & (E) \\ \lambda_i \underset{\sim}{Q}_i & (P) \end{cases}$$

these def's are same in terms of $f_i(\xi_i) = k_i$ & $\underline{Q}_i = \hat{\underline{\sigma}}^{tr}$

i is the active surface.

$i-1, \ldots\ldots, 1$ are passive in loading but not in unloading

$$\lambda_i = \frac{1}{1 + \frac{H_i'}{3\mu}} \underset{\sim}{Q}_i \cdot \dot{\underset{\sim}{\varepsilon}}$$ (normal component of strain rate.

$$\dot{\underset{\sim}{\alpha}}_i = M \underset{\sim}{\mu} \quad \text{(Prager Hardening Rule)}$$

$$\underset{\sim}{\mu} = \underset{\sim}{\alpha}'_{i+1} + \frac{R_{i+1}}{R_i} \underset{\sim}{\xi}' - \underset{\sim}{\sigma}'$$

$\nwarrow \; \underset{\sim}{\sigma} - \underset{\sim}{\alpha}_i$

$$M = R_i \sqrt{\tfrac{2}{3}} \, H_i' \, \dot{\varepsilon}^{pl} / (\underset{\sim}{\xi}' \cdot \underset{\sim}{\mu})$$

$$\dot{\underset{\sim}{\alpha}}_j = \dot{\underset{\sim}{\alpha}}_i \quad (1 \le j \le i) \quad \text{any } i < m$$

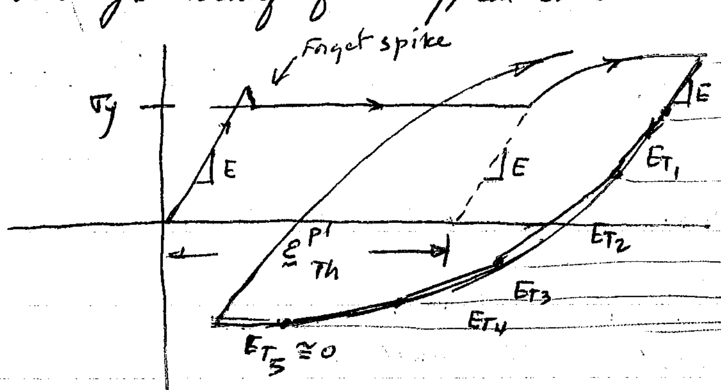
$\Bigg\}$ $i < m$

for $i = m$

$$\dot{\underset{\sim}{\alpha}}_i = \tfrac{2}{3} H_i' \, \dot{\underset{\sim}{\varepsilon}}^{pl} \quad \& \quad \dot{\alpha}_j = \dot{\alpha}_i$$

$$\underset{\sim}{C}_i^{el-pl} = \underset{\sim}{C} - \frac{2\mu}{(1 + H_i'/3\mu)} \underset{\sim}{Q}_i \otimes \underset{\sim}{Q}_i \quad \text{for surface } i$$

Consider a Mult-ys identif for a typical steel



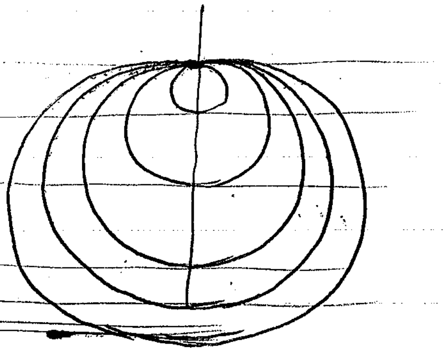


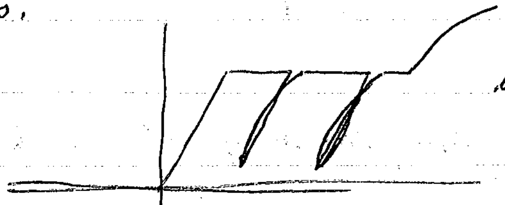
figure ident on unloading side

up to $\bar{\varepsilon}^{pl}$ threshhold set $H_i' = 0$ where surface i is ∋. σ_y defines its diameter initially; set H_j', $1 \le j \le i$, $= \infty$ (ie. elastic).

when $\bar{\varepsilon}^{pl} \geq \bar{\varepsilon}^{pl}_{\text{threshhold}}$ set all H_j''s $j = 1, \dots, m$ to the values determined fr the unloading branch as above. (see fig 2-13 in handout).

A refinement to the above ident.

It is observed that before $\bar{\varepsilon}^{pl}_{th}$ is reached unloading produces small hysteresis loops.



loops tend to thickness as $\bar{\varepsilon}^{pl}$ is accumulated.

to model this we want to make $\underset{(=\infty \text{ orig})}{H_j'}$ $j < i$ "soften" gradually as $\tilde{\bar{\varepsilon}}^{pl}$ is accumulated.

Simple Approx $H_j' = a_j \, e^{b_j / \bar{\varepsilon}^{pl}}$ $\bar{\varepsilon}^{pl} < \bar{\varepsilon}^{pl}_{Th}$ a_j, b_j are > 0

$\bar{\varepsilon}^{pl} \Rightarrow$ $H_j' =$ "∞" $j < i$

select a_j, b_j's to be ∋ the slopes of 2 consecutive hysteresis loops are matched.

Recall 2-surface theory (<u>not</u> $m=2$ in the multi-y.s. theory) have been developed to reduce data of general multi-surface theory.

idea here is to "interpolate" the plastic moduli

Krieg, Dafalias, Mroz - Zienkiewicz.

$\nwarrow$ in handout, but has problems.

Algorithmic Ingredients: Kinematic M-YS theory

- need incremental version of rate eqn.

$$\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - 2\mu\,\tilde{\lambda}\,\underset{\sim}{Q}$$

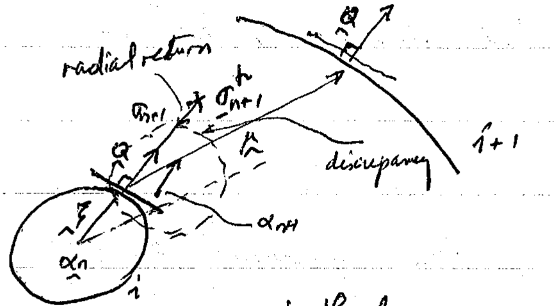
$$\underset{\sim}{\alpha}_{n+1} = \underset{\sim}{\alpha}_n + \left[R_i \frac{2}{3} H_i' \,\hat{\lambda} \Big/ \left(\underset{\sim}{\xi}' \cdot \underset{\sim}{\mu} \right) \right] \underset{\sim}{\mu}$$

surface i values

i - active surface, $< m$.

$$(i = m) \quad \underset{\sim}{\alpha}_n + \tfrac{2}{3} H' \hat{\lambda}\, \underset{\sim}{Q}, \quad \& \quad R = \text{constant}.$$

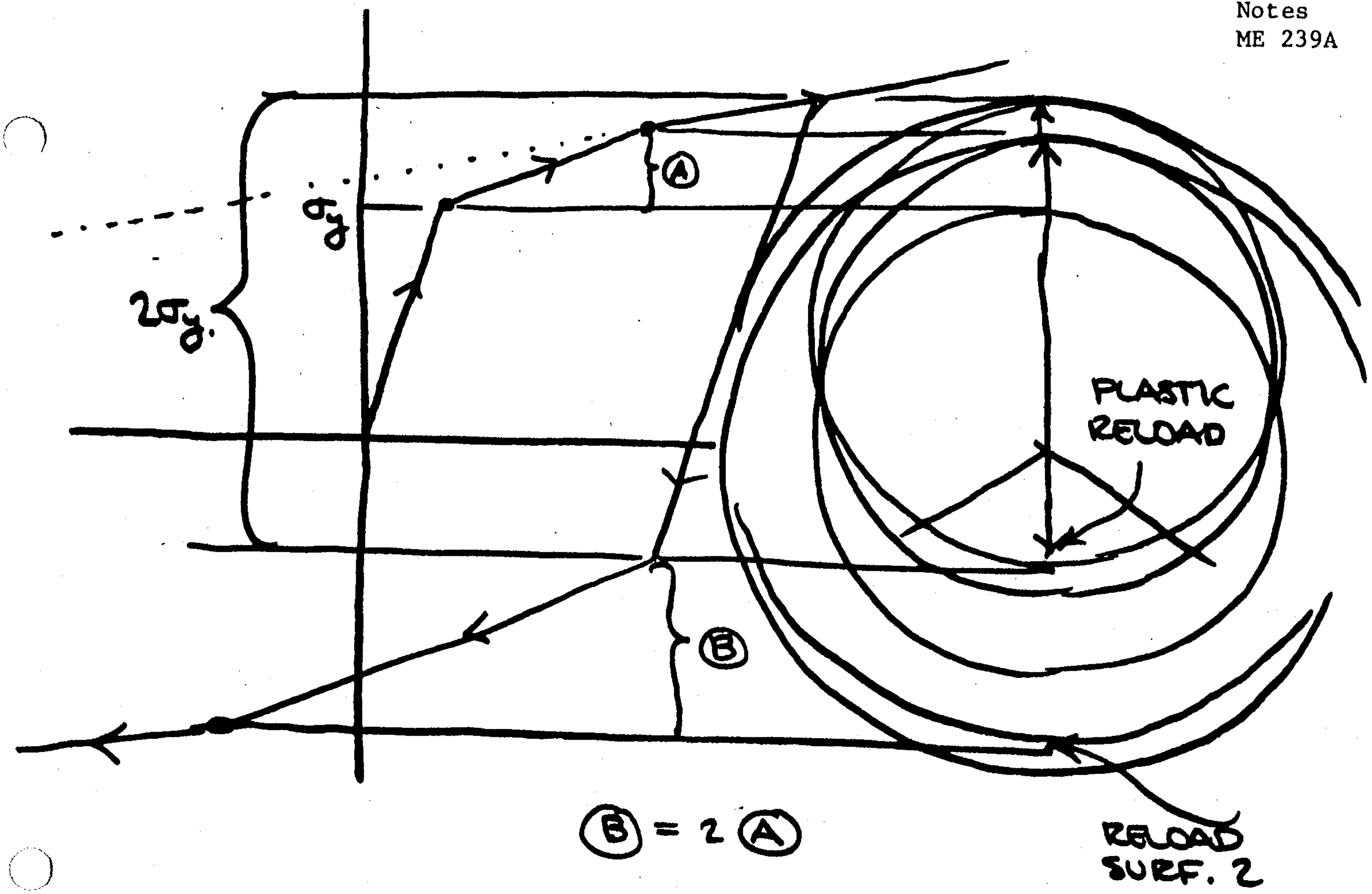
- Determine $\hat{\lambda}$ via an incrcm version of consistency:



Apply consistency wrt normal direction as indicated: "$\hat{\underset{\sim}{Q}}$"

in the large if we use this $\nearrow$ $\left(\underset{\sim}{\alpha}_{n+1} + R\,\underset{\sim}{Q}'' = \underset{\sim}{\sigma}_{n+1}\right)$ this don't work.

so fix Q & dot the eqn w/ $\underset{\sim}{Q}$ $\Rightarrow$ $\hat{\lambda}$ ≈ same as before.



Ⓑ = 2 Ⓐ

From last time we had 17 Feb 83

$$\underset{\sim}{Q} \cdot (\underset{\sim}{\alpha}_{n+1} + R\underset{\sim}{Q} = \underset{\sim}{\sigma}_{n+1})$$

For the outside surface we would get

$$\tilde{\lambda} = \frac{1}{2\mu\,(1 + H'/3\mu)} \left(|\underset{\sim}{\xi}^{tr}_{n+1}| - R \right)$$ where H' is for the outside surface
R " " " " "

for interior surface i we get:

$$\underset{\sim}{Q} \cdot \left(\underset{\sim}{\alpha}_{n+1} + R\underset{\sim}{Q} = \underset{\sim}{\sigma}_{n+1} \right)$$

$$\left(\underset{\sim}{\alpha}_n + R\,\tfrac{2}{3}\,\frac{H'\,\tilde{\lambda}}{(\underset{\sim}{\xi}' \cdot \underset{\sim}{\mu})}\,\underset{\sim}{\mu} + R\underset{\sim}{Q}^{\cancel{tr}} = \underset{\sim}{\sigma}^{tr}_{n+1} - 2\mu\tilde{\lambda}\underset{\sim}{Q} \right)$$

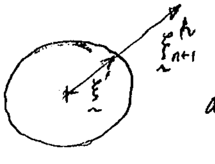
by rearranging

$$\left(R\,\tfrac{2}{3}\,H'\frac{\underset{\sim}{Q} \cdot \underset{\sim}{\mu}}{\underset{\sim}{\xi}' \cdot \underset{\sim}{\mu}} + 2\mu \right)\tilde{\lambda} = \underset{\sim}{Q} \cdot \left(\underset{\sim}{\xi}^{tr}_{n+1} - R\underset{\sim}{Q} \right)$$

$\boxed{\text{NOTE: HERE}}$ H' & R are for surface i

w/ $\underset{\sim}{Q} = \underset{\sim}{\xi}'/R \implies$

$$2\mu\left(1 + \frac{H'}{3\mu}\right)\tilde{\lambda} = \underset{\sim}{Q} \cdot \underset{\sim}{\xi}^{tr}_{n+1} - R \qquad \text{since } |\underset{\sim}{Q}| = 1$$



also $\underset{\sim}{Q} = \underset{\sim}{\xi}^{tr\,\prime}_{n+1} / |\underset{\sim}{\xi}^{tr}_{n+1}|$ $\quad\therefore\quad \underset{\sim}{Q} \cdot \underset{\sim}{\xi}^{tr}_{n+1} = |\underset{\sim}{\xi}^{tr\,\prime}_{n+1}|$

∴ we get same result as before

Obvious complications exist

ie 1. $\underset{\sim}{\sigma}_{n+1}$ is not on the YS unless $\underset{\sim}{\mu} \parallel \underset{\sim}{Q}$.

thus must increment strain (ie devise a sub incremental stategy – this is necessary)

2. After loading $\underset{\sim}{\sigma}_{n+1}$ might be outside surface $i+1$, ∴ need stategy to prevent this. ~~ie must re-initializing stress~~

<u>Remark</u>: robust routine seems to require many "BELLS & WHISTLES", ie requires many cutoffs & numerical tests to keep surfaces from overlapping etc.

Applications of Multi-yield surfaces to undrained clays. Assume plasticity is deviatoric; reason for this is assume fluid is carrying the pressure (or mean stress) elastically ie bulk modulus of fluid $B_f \gg 2\mu$ of soil. ie the material is ≅ incompressible.

Assume following symmetries on stresses $\sigma_{33} = \sigma_{11}$, $\sigma_{31} = \sigma_{32} = 0$

& also for centers of yield surfaces $\alpha_{33} = \alpha_{11}$, $\alpha_{31} = \alpha_{32} = 0$

<u>N.B.</u> $\underset{\sim}{\xi} = \underset{\sim}{\sigma} - \underset{\sim}{\alpha}$ [This will enable treatment of triaxial & simple shear tests].

ie Y.S. expression for these conditions is

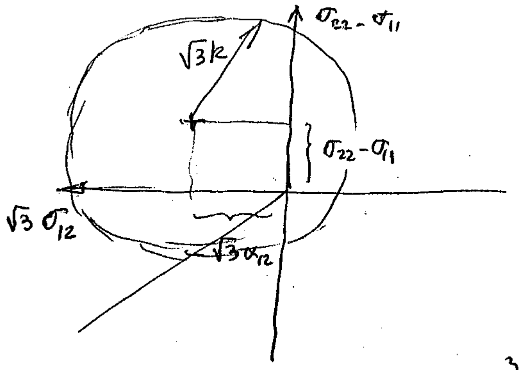
$$f(\underset{\sim}{\xi}) = k^2 \quad \text{ie} \quad (\xi_{22}-\xi_{33})^2 + (\xi_{33}-\xi_{11})^2 + (\xi_{11}-\xi_{22})^2 + 6(\xi_{23}^2 + \xi_{31}^2 + \xi_{12}^2) = 6k^2$$

since α's can move. $\Rightarrow \xi_{33} = \xi_{11}$, $\xi_{31} = \xi_{32} = 0$ ∴

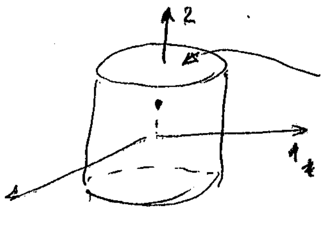
$$f(\underset{\sim}{\xi}) = k^2 \Rightarrow \cancel{2}(\xi_{22}-\xi_{11})^2 + \overset{3}{\cancel{6}}\,\xi_{12}^2 = \overset{3}{\cancel{6}}\,k^2 \qquad ⊛$$

To study these tests work in space suited for the remaining components

∴ Consider space w/ $\sigma_{22}-\sigma_{11}$, $\sqrt{3}\,\sigma_{12}$ as axes ⇒ ⊛ is a circle w/ radius $\sqrt{3}k$ & center at $(\alpha_{22}-\alpha_{11})$ & $\sqrt{3}\,\alpha_{12}$.



Consider the triaxial compression/ext test w/ 2 axis which will be strained



strain applied in 2-dir

we will assume, for this test,

1. $\sigma_{12} = \alpha_{12} = 0$
2. assume $\dot{\varepsilon}_{kk} \cong 0$ (incompress)

ie $\dot{\varepsilon}_{11} = \dot{\varepsilon}_{33} \cong -\tfrac{1}{2}\dot{\varepsilon}_{22}$

thus the vertical applied strain rate is $\dot{\varepsilon}_{22}$

Want to calculate an expression for $(\dot{\sigma}_{22}-\sigma_{11})$ as a fn of $\dot{\varepsilon}_{22}$

must in same manner as in 1-D case.

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} & \cdot & \cdot \\ \cdot & \sigma_{22} & \cdot \\ \cdot & \cdot & \sigma_{33}=\sigma_{11} \end{bmatrix}$$; want to get deviatoric $$\underset{\sim}{\sigma}' = \begin{bmatrix} \sigma_{11}-\frac{1}{3}(\sigma_{22}+2\sigma_{11}) & \cdot & \cdot \\ \cdot & \sigma_{22} = -\frac{1}{3}(\sigma_{22}+2\sigma_{11}) & \cdot \\ \cdot & \cdot & \sigma_{11}-\frac{1}{3}(\sigma_{22}+2\sigma_{11}) \end{bmatrix} \textcircled{\dagger}$$

thus $$\underset{\sim}{\sigma}' = \begin{bmatrix} \frac{1}{3}(\sigma_{11}-\sigma_{22}) & \cdot & \cdot \\ \cdot & \frac{2}{3}(\sigma_{22}-\sigma_{11}) & \cdot \\ \cdot & \cdot & \frac{1}{3}(\sigma_{11}-\sigma_{22}) \end{bmatrix}$$

for $\underset{\wedge}{\xi}$ do same $$\underset{\sim}{\xi} \begin{bmatrix} \xi_{11} & & \\ & \xi_{22} & \\ & & \xi_{33} \end{bmatrix} \Rightarrow \underset{\sim}{\xi}' = \begin{bmatrix} \frac{1}{3}(\xi_{11}-\xi_{22}) & \cdot & \cdot \\ \cdot & \frac{2}{3}(\xi_{22}-\xi_{11}) & \cdot \\ \cdot & \cdot & \frac{1}{3}(\xi_{11}-\xi_{22}) \end{bmatrix} \circledast\circledast$$

constitutive eqn becomes

$$\dot{\underset{\sim}{\sigma}} = \left(\underset{\sim}{C} - \frac{2\mu}{(1+H'/3\mu)} \underset{\wedge}{Q} \otimes \underset{\sim}{Q} \right) \cdot \dot{\underset{\sim}{\varepsilon}} \qquad \text{where } \underset{\wedge}{Q} = \underset{\sim}{\xi}'/R$$

taking deviatoric part

this is already deviatoric

$$\dot{\underset{\sim}{\sigma}}' = (\underset{\sim}{C} \cdot \dot{\underset{\sim}{\varepsilon}})' - \frac{2\mu}{1+H'/3\mu} (\underset{\wedge}{Q} \cdot \dot{\underset{\sim}{\varepsilon}}) \underset{\sim}{Q} \qquad \underset{\wedge}{Q} = \underset{\wedge}{\xi}'/R$$

gives $+2\mu\underset{\wedge}{\varepsilon}$

$(\lambda+2\mu)\,\mathrm{tr}\,\dot{\underset{\wedge}{\varepsilon}}$

by incomp 0

$$\text{now} \quad \underset{\sim}{\xi}' \cdot \dot{\underset{\sim}{\varepsilon}} = \xi'_{11}\dot{\varepsilon}_{11} + \xi'_{22}\dot{\varepsilon}_{22} + \xi'_{11}\dot{\varepsilon}_{11} = 2\xi'_{11}\cdot\dot{\varepsilon}_{11} + \xi'_{22}\dot{\varepsilon}_{22}$$

by incomp

$$= 2\xi'_{11}\cdot -\tfrac{1}{2}\dot{\varepsilon}_{22} + \xi'_{22}\dot{\varepsilon}_{22}$$

$$= \dot{\varepsilon}_{22}(\xi'_{22}-\xi'_{11})$$

$$\text{using } \dot{\underset{\sim}{\sigma}}' = 2\mu\,\dot{\underset{\sim}{\varepsilon}}' - \frac{2\mu}{\left(1+\frac{H'}{3\mu}\right)} \frac{1}{R^2} (\xi'_{22}-\xi'_{11})\,\dot{\varepsilon}_{22}\,\underset{\sim}{\xi}'$$

$$\text{taking the 22 component: } \dot{\sigma}'_{22} = 2\mu\,\dot{\varepsilon}'_{22} - \frac{2\mu}{\left(1+\frac{H'}{3\mu}\right)} \frac{1}{R^2} (\xi'_{22}-\xi'_{11})\,\dot{\varepsilon}_{22}\,\xi'_{22}$$

$$\text{now} \quad \xi'_{22} = \tfrac{2}{3}(\xi_{22}-\xi_{11}) \text{ from } \circledast\circledast = \tfrac{2}{3}(\xi'_{22}-\xi'_{11})$$

Now since the strains are incompressible $\dot{\varepsilon}'_{22} = \dot{\varepsilon}_{22}$

$$\therefore \quad \dot{\sigma}'_{22} = \tfrac{2}{3}(\dot{\sigma}_{22}-\dot{\sigma}_{11}) \text{ from } \textcircled{\dagger} = 2\mu\,\dot{\varepsilon}_{22} - \frac{2\mu}{(1+H'/3\mu)} \frac{1}{R^2}(\xi_{22}-\xi_{11})\,\dot{\varepsilon}_{22}\cdot\tfrac{2}{3}(\xi_{22}-\xi_{11})$$

Now for the yield condition $\circledast$ $\quad (\xi_{22}-\xi_{11})^2 = 3k^2 = \tfrac{3}{2}R^2$

$$\tfrac{2}{3}(\dot{\sigma}_{22}-\dot{\sigma}_{11}) = \dot{\sigma}'_{22} = 2\mu\,\dot{\varepsilon}_{22} - \frac{2\mu}{1+H'/2\mu}\,\dot{\varepsilon}_{22} = 2\mu\,\dot{\varepsilon}_{22}\left[\frac{H'/3\mu}{1+H'/3\mu}\right]$$

thus $(\dot\sigma_{22} - \dot\sigma_{11}) = \left(\frac{H'}{1 + H'/3\mu}\right) \dot\varepsilon_{22} = \hat{E}_T \dot\varepsilon_{22}$

this is a tangent mod of the $\dot\sigma_{22} - \dot\sigma_{11}$ vs $\dot\varepsilon_{22}$ curve.

Thus by measuring this value of $\hat{E}_T$ we can get H' ie $H' = \hat{E}_T / (1 - \hat{E}_T/3\mu)$

Now let's consider a shear test (idealized)

Assume: $\dot\varepsilon_{11} = \dot\varepsilon_{22} = \dot\varepsilon_{33} = 0$; $\dot\varepsilon_{31} = \dot\varepsilon_{32} = 0$

$\dot\varepsilon_{21} \neq 0$ input

from our constit eqn $\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot \dot{\underset{\sim}{\varepsilon}} - \frac{2\mu}{(1 + H'/3\mu)} \frac{1}{R^2} (\underset{\sim}{\xi}' \cdot \dot{\underset{\sim}{\varepsilon}}) \underset{\sim}{\xi}'$

for nonzero component look at the $\overset{\circ}{\sigma}_{12}$ which is automatically deviatoric.

$$\dot\sigma_{12} = 2\mu \dot\varepsilon_{12} - \frac{2\mu}{(\quad)} \frac{1}{R^2} \underbrace{\xi_{12} (\dot\varepsilon_{12} \cdot 2\xi_{12})}_{2\xi_{12}^2 \dot\varepsilon_{12}}$$

Going to ⊛ that gives that $\xi_{12}^2 = k^2 = R^2/2$ initially ; put into the above

$$\therefore \quad 2\mu \dot\varepsilon_{12} \left[1 - \frac{1}{(1 + H'/3\mu)}\right] = \dot\sigma_{12}$$

$$\dot\sigma_{12} = \frac{H'/3}{1 + H'/3\mu} \overset{\circ}{\gamma}_{12} \qquad \gamma_{12} = 2\dot\varepsilon_{12}$$

initially at least. This will produce a 2nd order error if we assume $\xi_{22} - \xi_{11} = 0$ ∀ time.

Ref: Int. Journal of Numerical & Analytical Methods in Geomechanics V. 1, 195-216 (1977) J. H. Prévost.

Example: Drammen Clay 14 surface model.

22 Feb 83

initially $3\xi_{12}^2 = 3k^2$ and afterwards. $(\xi_{22} - \xi_{11})^2$ is not zero

∴ we must use $(\xi_{22} - \xi_{11})^2 + 3\xi_{12}^2 = 3k^2$ & solve for ξ_{12}^2

now substitute for ξ_{12}^2 into

$$\overset{\circ}{\sigma}_{12} = 2\mu \left[1 - \frac{1}{(1 + H'/3\mu)} \cdot \frac{2\xi_{12}^2}{R^2}\right] \overset{\circ}{\varepsilon}_{12}$$

must therefore obtain data for $(\xi_{22} - \xi_{11})$ quantity.

1)

Handout #7
ME 239A
Notes

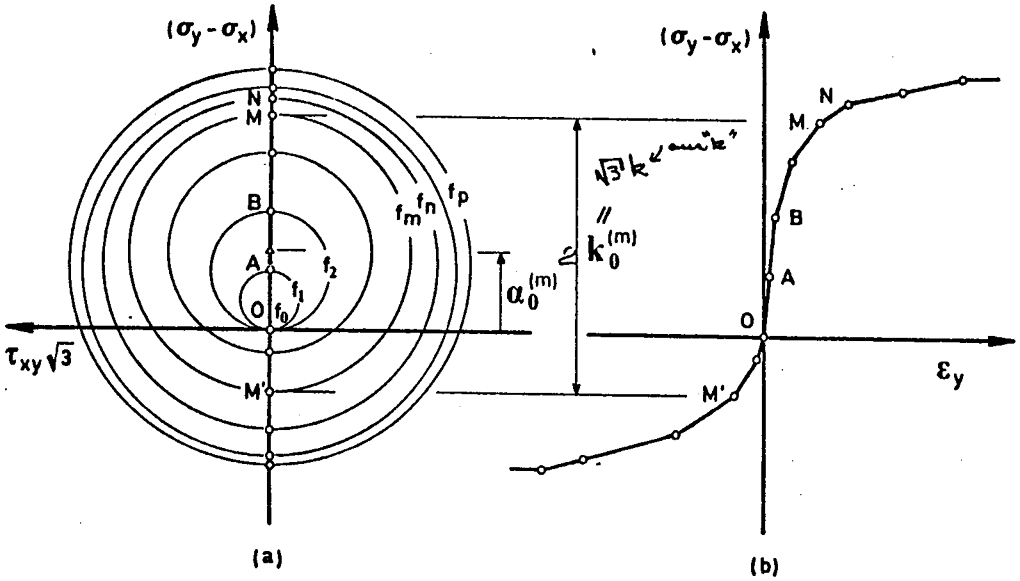


Figure 2. Monotonic triaxial compression and extension tests (a) representation in II-plane (b) stress–strain curves

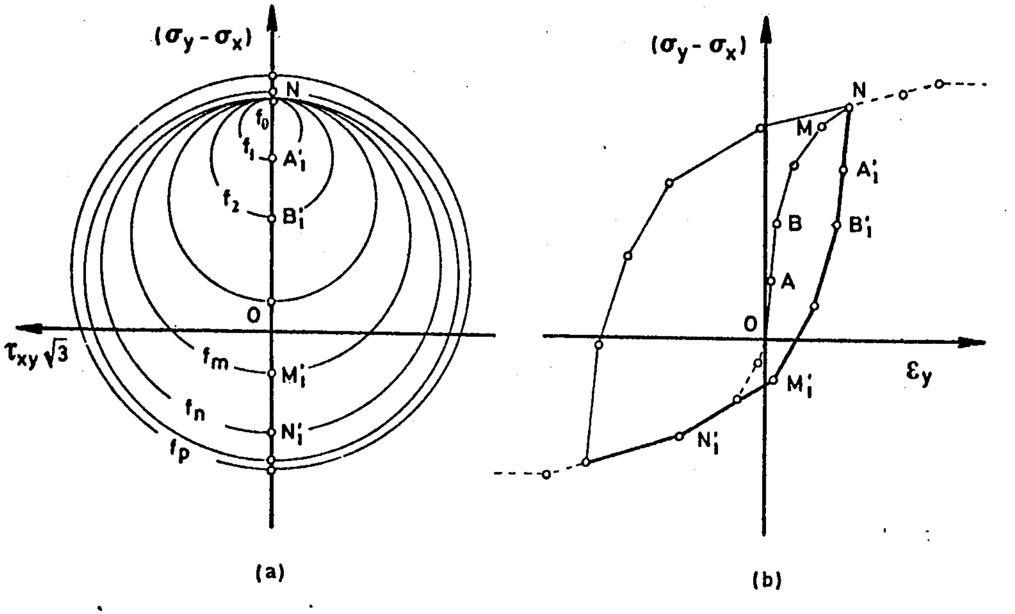


Figure 3. Cyclic triaxial test (a) field of yield surfaces upon reaching point N in compression (b) loading-reverse loading stress–strain curves

2)

$k^{(m)} = k_0^{(m)}$, is shown in Figure 3b. Variations in the $k^{(m)}(\lambda)$ functions usually start to occur upon the first loading reversal, and they are simply determined by comparing the stress differences corresponding to the segments NA'_1 and OA, $A'_1B'_1$ and AB, etc.... (Figures 2 and 3) since when $k^{(m)} = k_0^{(m)}$, $NA'_1 = 2k_0^{(1)}, \ldots, NM'_1 = 2k_0^{(m)}$, etc. Experimental deviations from these equalities are attributed to variations in $k^{(m)}(\lambda)$. For instance, under cyclic loading conditions, the stress–strain curves consist of consecutive hysteresis loops. The values of $k^{(m)}$ can be found for each branch of the loops, and the functions $k^{(m)}(\lambda)$ can be determined. Once the yield surfaces reach their ultimate limiting sizes, their associated shear moduli H_m start to vary, and these changes $H_m(\lambda)$ are determined by comparing the shape of the consecutive loops.

Interpretation of the simple shear soil tests

In the case of simple shear strain loading ($d\varepsilon_x = d\varepsilon_y = d\varepsilon_z = 0$), the stress point moves in stress space in such a way that the resulting strain increment vector does not have a normal strain component. For a soil specimen initially subjected to equal horizontal normal stresses, it is apparent from equations (A1)–(A3) that the stress path occurs in the II-plane (i.e., $\sigma_x = \sigma_z$ at all times). When the magnitude of the plastic strain increment vector is large in comparison to the magnitude of the elastic strain increment vector (i.e., $1/H'_m$ is large in comparison to $1/2G$), the stress path simply goes through the apexes of the subsequent yield surfaces, as shown in Figure 4a. The stress point and the currently attached yield surfaces then move along together in the

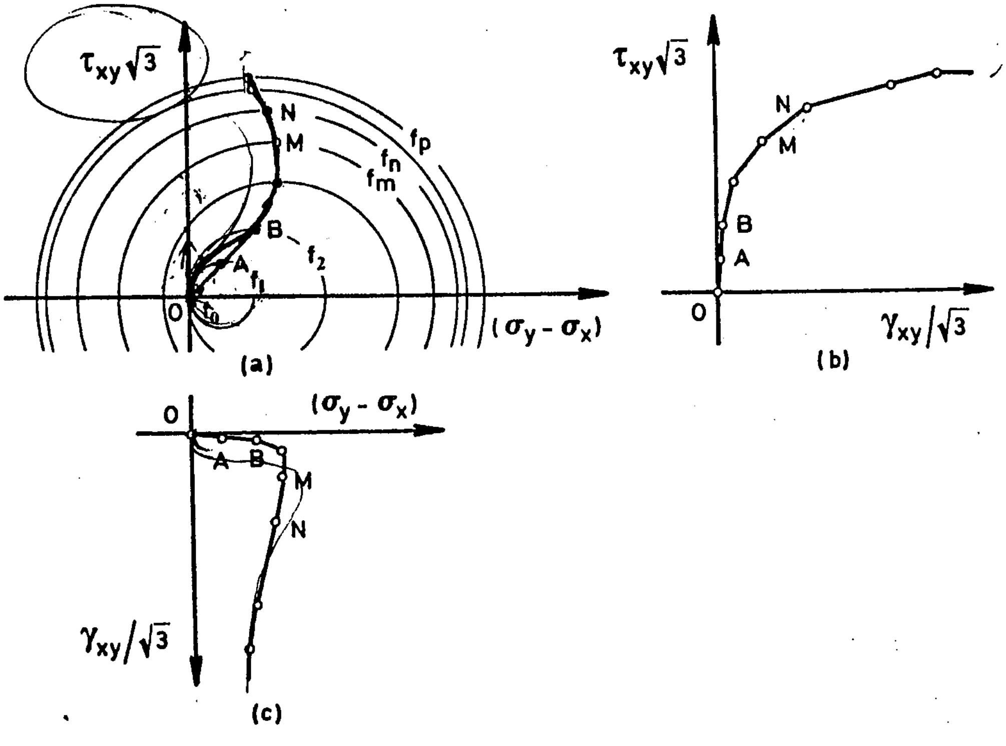


Figure 4. Monotonic simple shear test (a) representation in II-plane (b) shear stress *versus* shear strain (c) normal stress difference *versus* shear strain

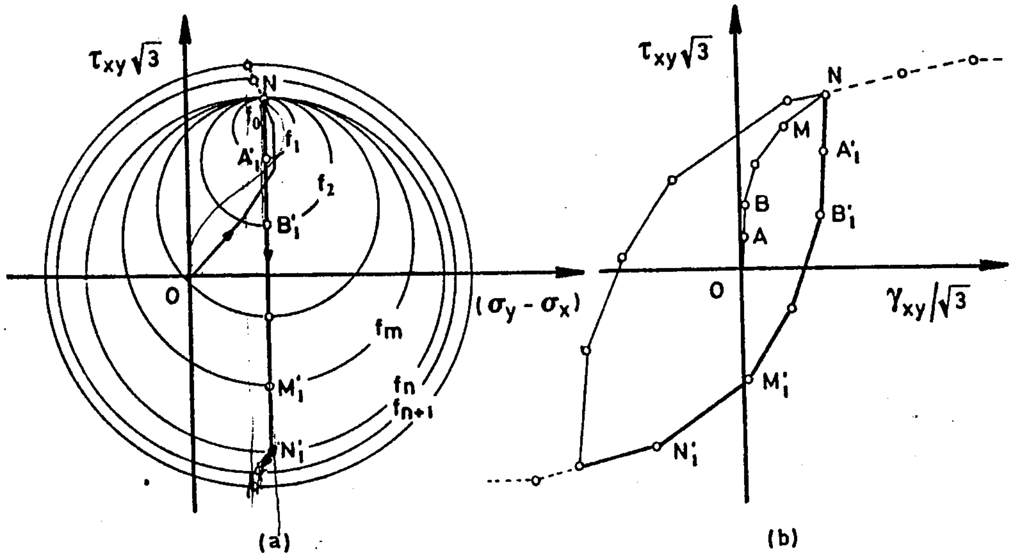
3)

same direction, and when the stress point reaches the yield surface f_m, $(\sigma_y-\sigma_x)=\alpha_0^{(m)}$ and $\tau_{xy}\sqrt{3}=k_0^{(m)}$. The stress–strain relation (equation (A1)) then simplifies to:

$$d\gamma_{xy}=\left(\frac{1}{G}+\frac{2}{H'_m}\right)d\tau_{xy}=\frac{2}{H_m}d\tau_{xy} \tag{7}$$

It is therefore apparent that by combining the experimental stress–strain curves (i.e., shear stress *versus* shear strain) obtained in a monotonic simple shear test, the initial positions, sizes and associated shear moduli of the yield surfaces may be simply determined. This is illustrated schematically in Figure 4. Note that during a simple shear test on a soil sample, the shear stress τ_{xy} acting on the top and bottom faces of the sample is not uniform and $\tau_{xy}=1{\cdot}11\tau_h$, where τ_h denotes the average horizontal shear stress measured experimentally.[8]

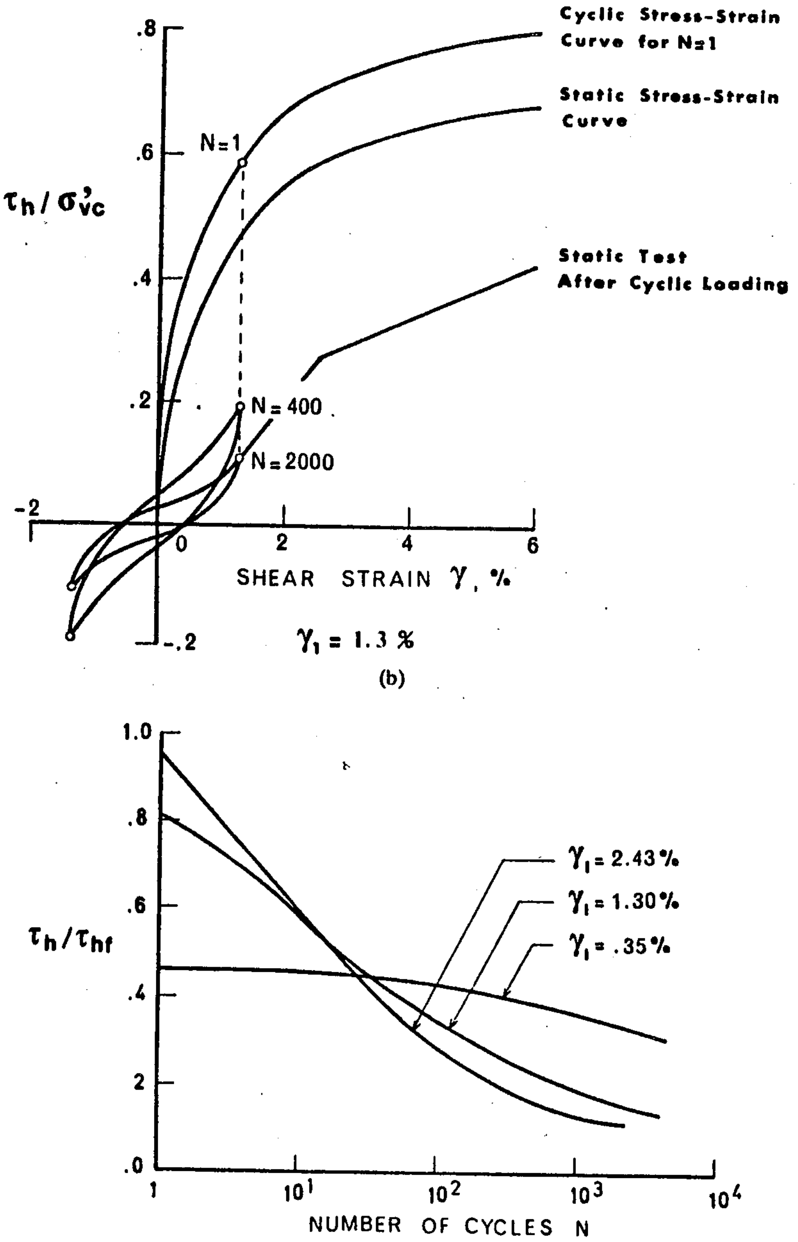
Figure 5a presents the situation upon reaching the yield surface f_n. Upon loading reversal, the stress point leaves the yield surface f_n and travels *vertically* downwards in the II-plane, pushing back the yield surfaces until it reaches the yield surface f_n once more (Figure 5a). Thereafter, the stress path bends over, and its direction is governed by the current position of f_{n+1}. If the surfaces are translated without changing in size ($k^{(m)}=k_0^{(m)}$), the model predicts that the reverse loading curve (Figure 4b) is then uniquely defined by the primary loading curve *OABMN* since during loading reversal, the stress difference corresponding to any shear modulus H_m equals twice the difference observed during primary loading, i.e. the material then exhibits a Masing[9] type behaviour. Experimental deviations from these equalities are therefore due to variations in $k^{(m)}$ or H_m, and the functions $k^{(m)}(\lambda)$ and $H_m(\lambda)$ are determined by using cyclic simple shear experimental test results, as explained previously for the triaxial tests.



(a) (b)

Figure 5. Cyclic simple shear test (a) field of yield surfaces upon reaching point *N* (b) loading-reverse loading stress–strain curves

4)



(b)

(c)

Figure 6. *(cont'd)*

measured in the slow monotonic test. It is of importance to note that the shear stress which is necessary to produce a specified strain amplitude during the first quarter cycle $N = 1$ of the cyclic tests is larger than the one observed in the slow monotonic test. The stress–strain curve constructed from the cyclic test results at $N = 1$ is shown by the upper curve in Figure 6b.

5)

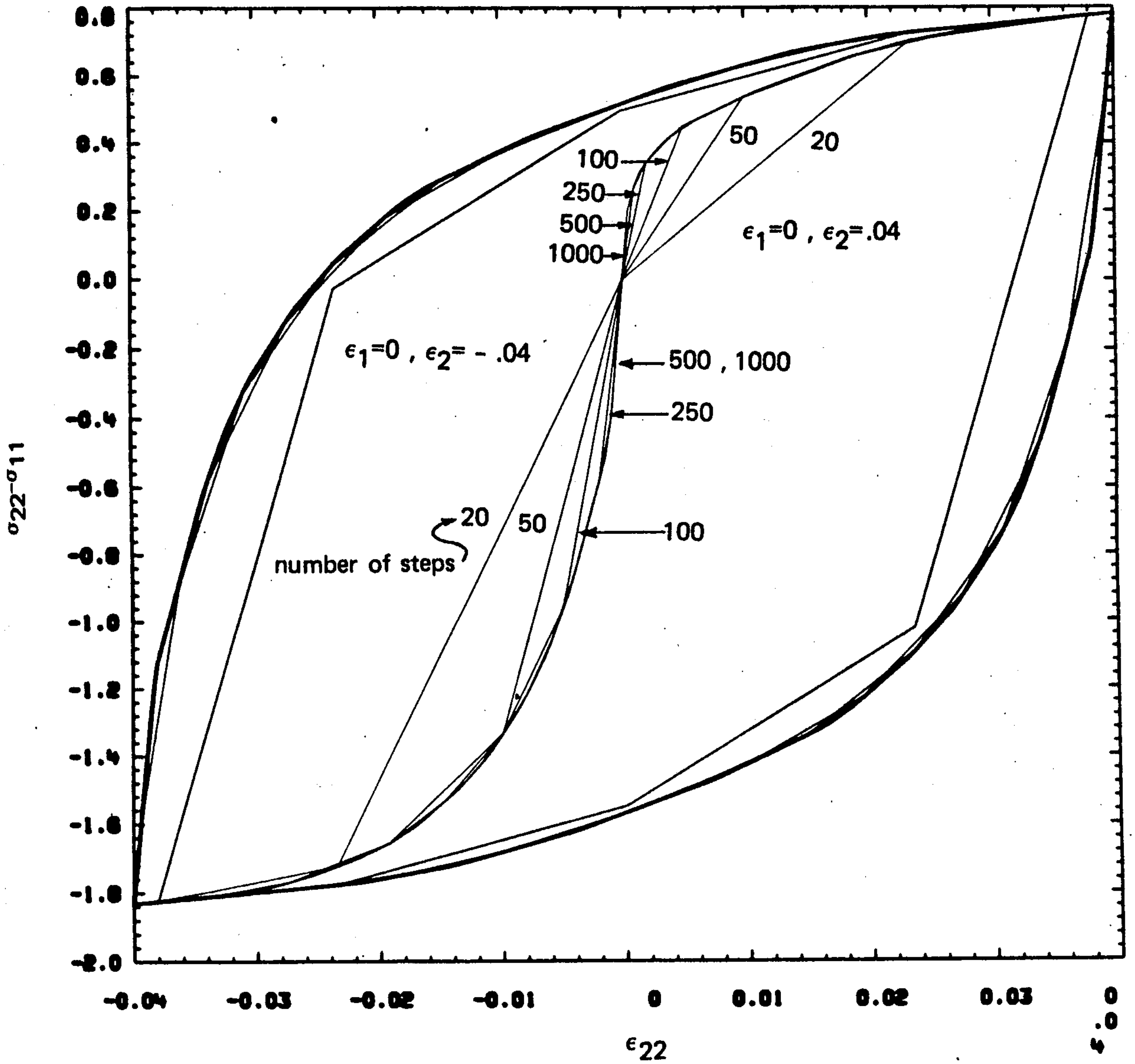


FIG. 9. Drammen clay, triaxial, 2 cycles.

(6)

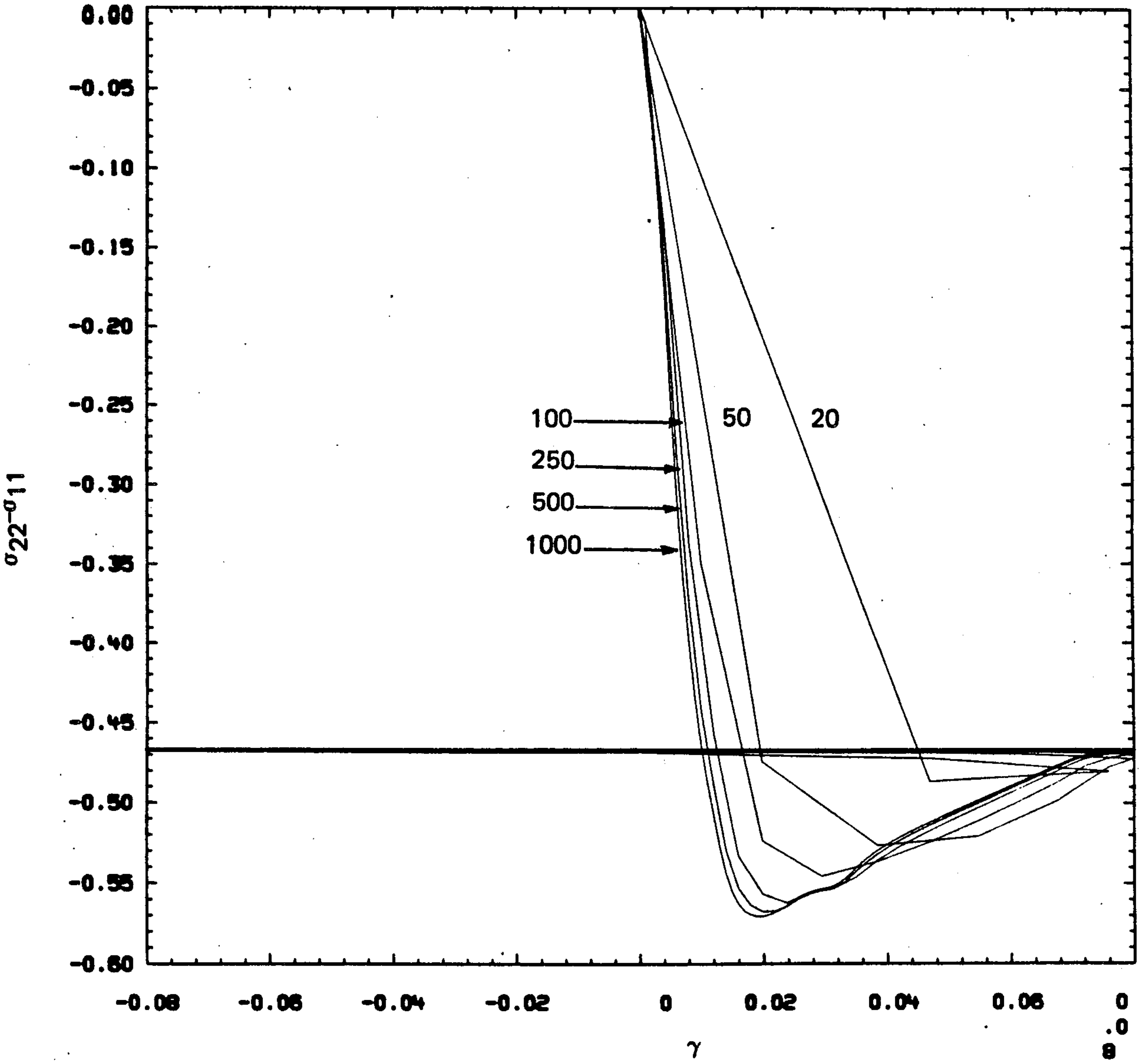


FIG. 10. Drammen clay, simple shear, 2 cycles, $\epsilon_1 = 0$, $\epsilon_2 = .08$.

η)

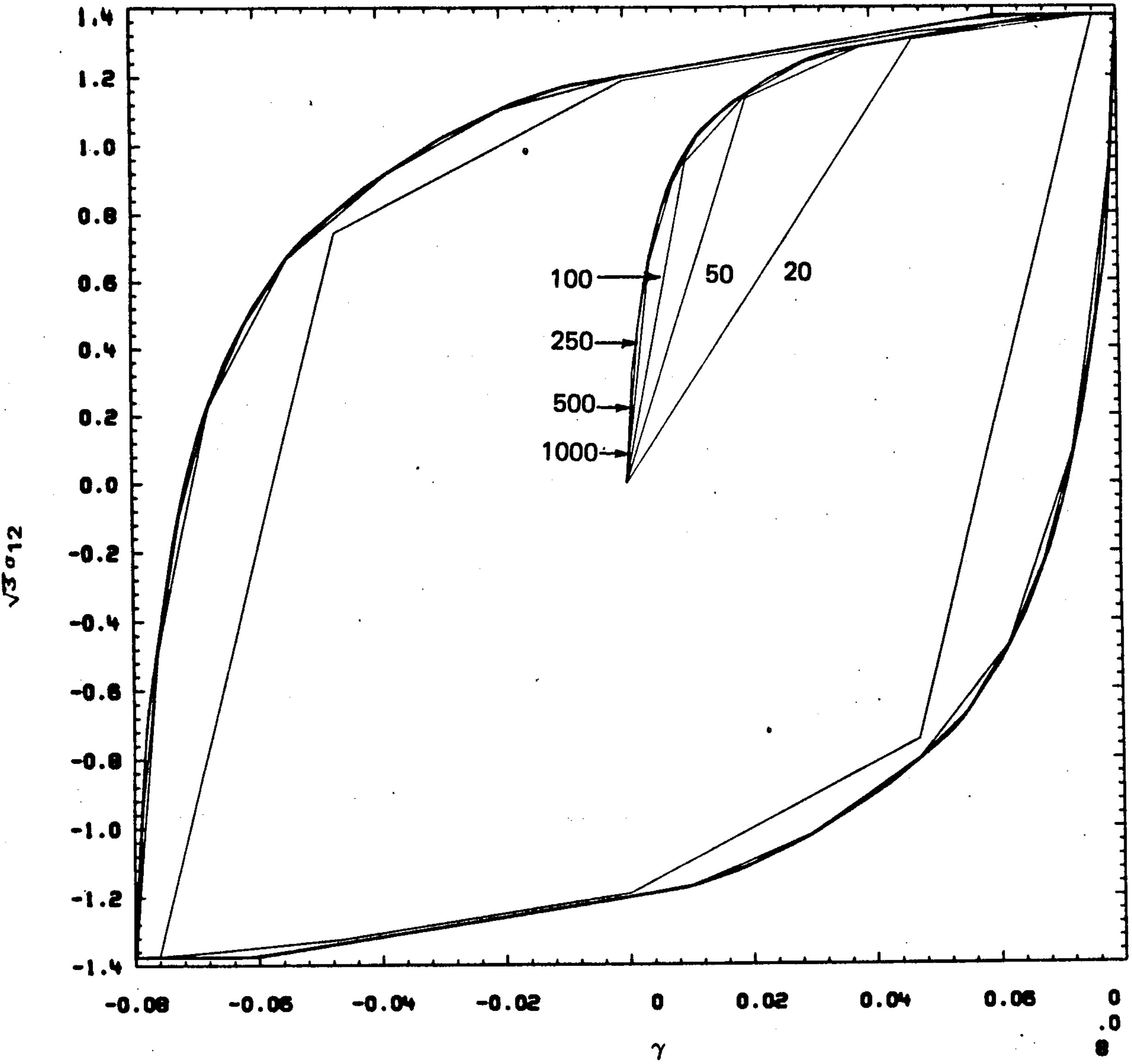


FIG. 11. Drammen clay, simple shear, 2 cycles, $\epsilon_1 = 0$, $\epsilon_2 = .08$.

8)

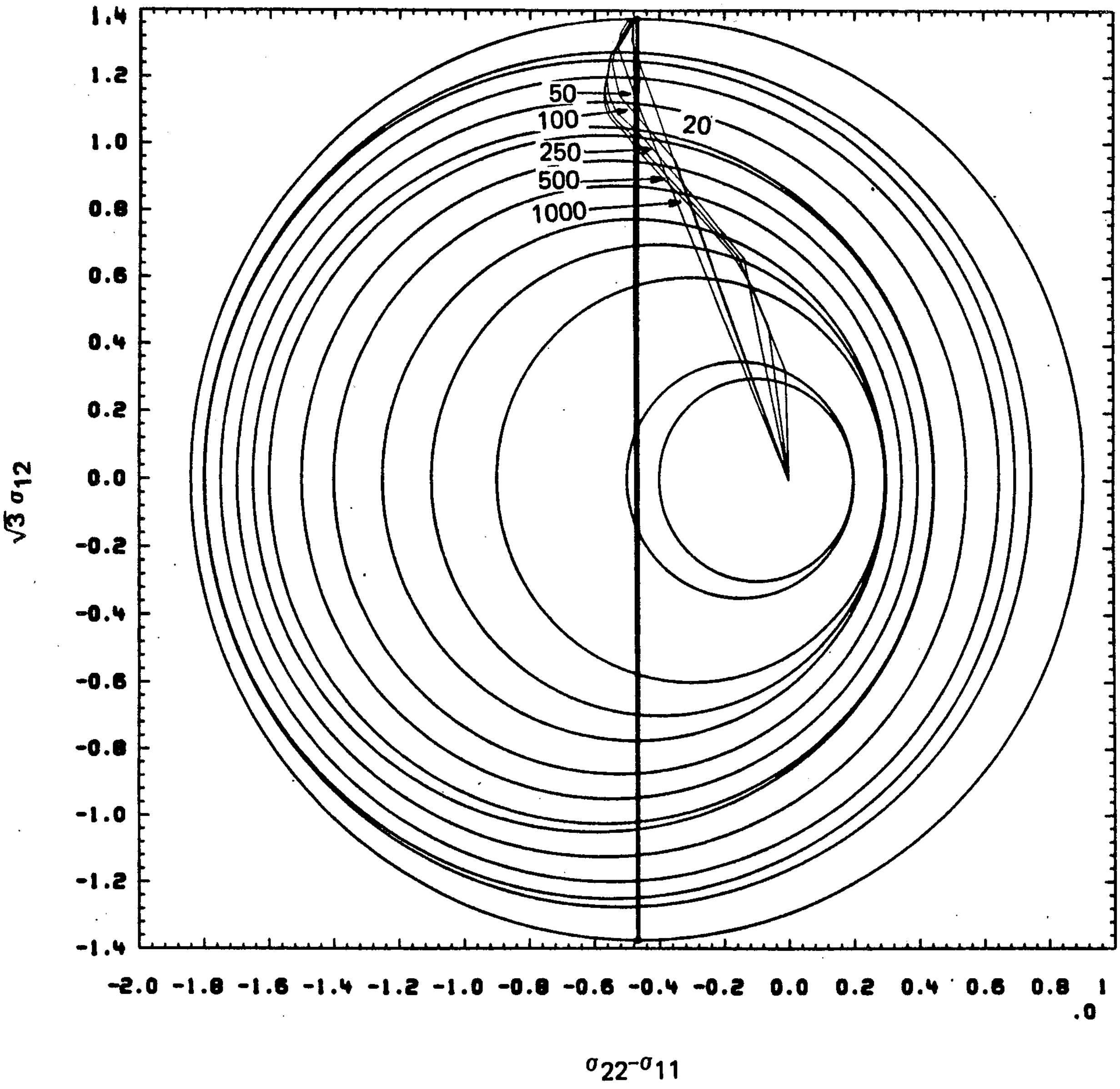


FIG. 12. Drammen clay, simple shear, 2 cycles, $\epsilon_1 = 0$, $\epsilon_2 = .08$.

9)

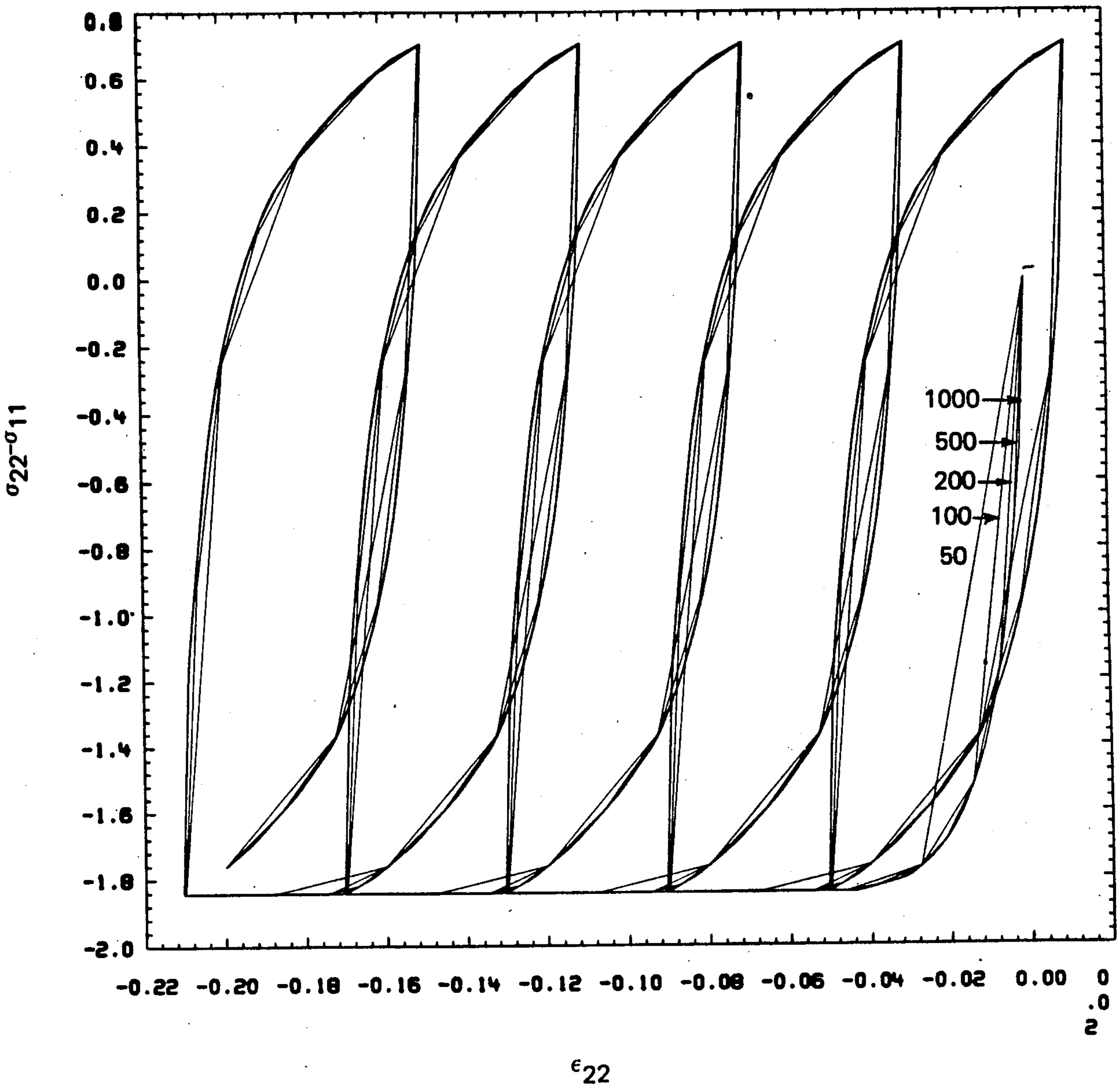


FIG. 13. Drammen clay, triaxial, 5 cycles, $\epsilon_1 = -.2$, $\epsilon_2 = -.04$

10)

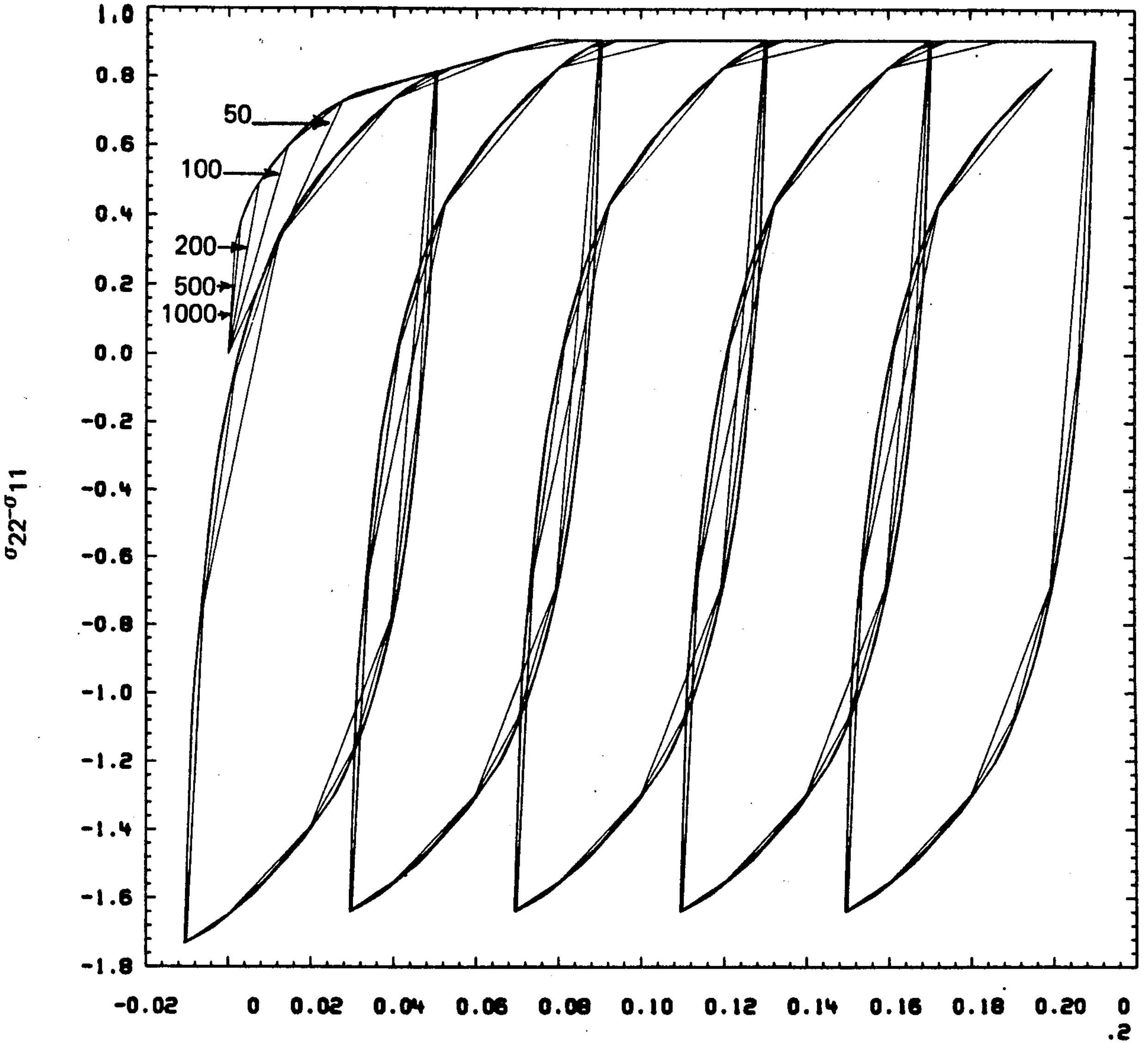


FIG. 14. Drammen clay, triaxial, 5 cycles, $\epsilon_1 = .2$, $\epsilon_2 = .04$

11)

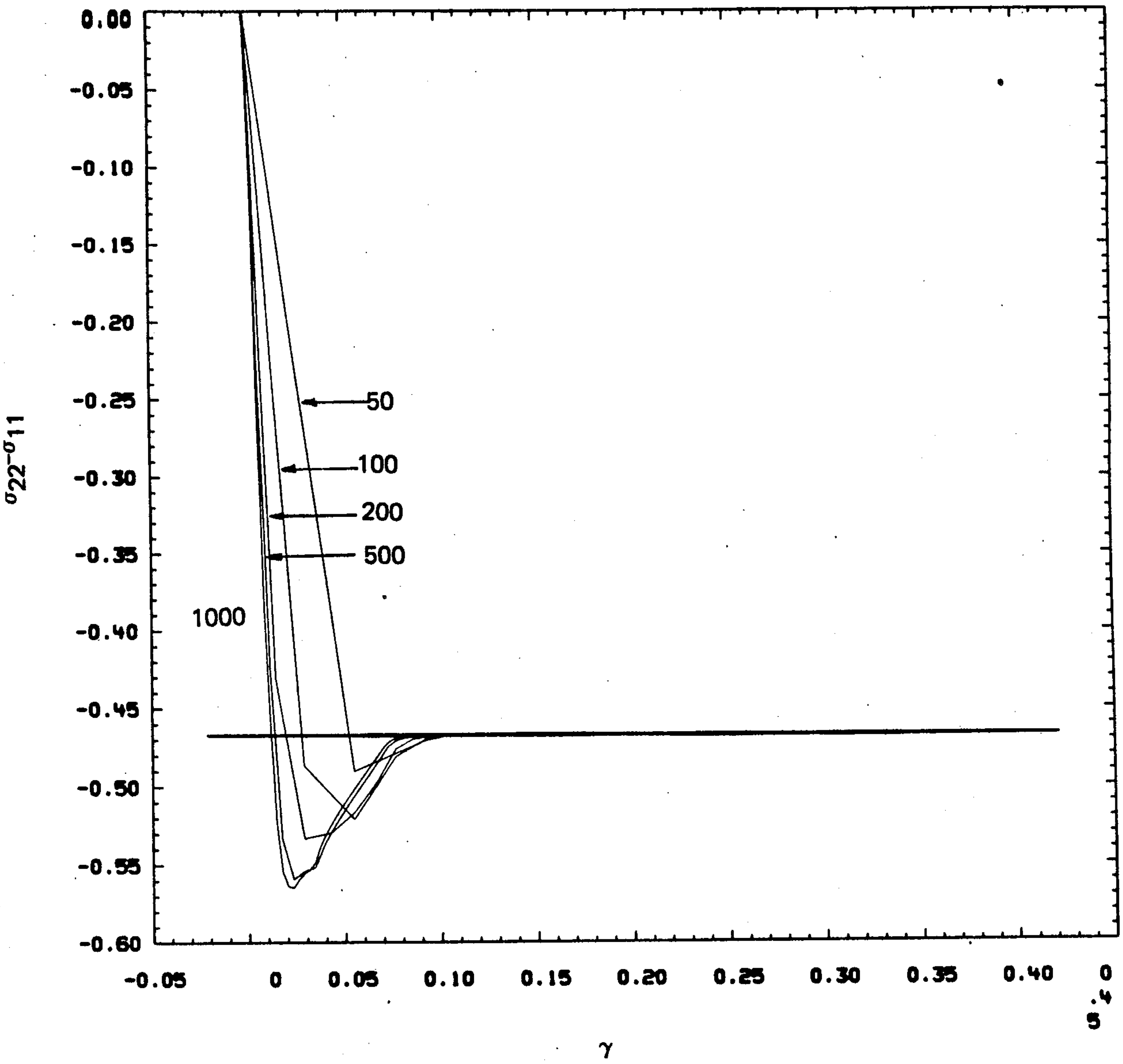


FIG. 15. Drammen clay, simple shear, 5 cycles, $\epsilon_1 = .4$, $\epsilon_2 = .08$.

(2)

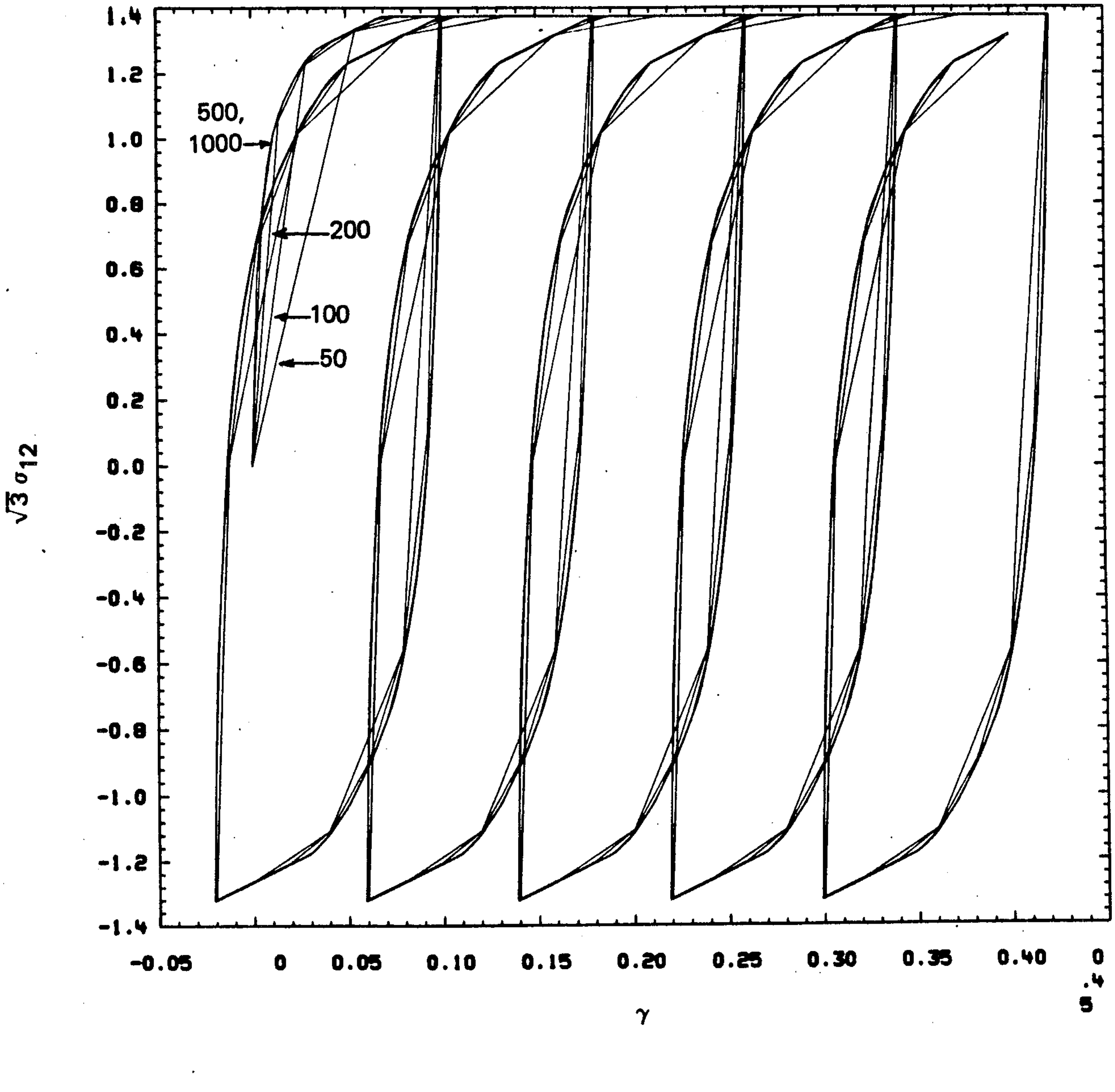


FIG. 16. Drammen clay, simple shear, 5 cycles, ϵ_1 = .4 , ϵ_2 = .08.

13)

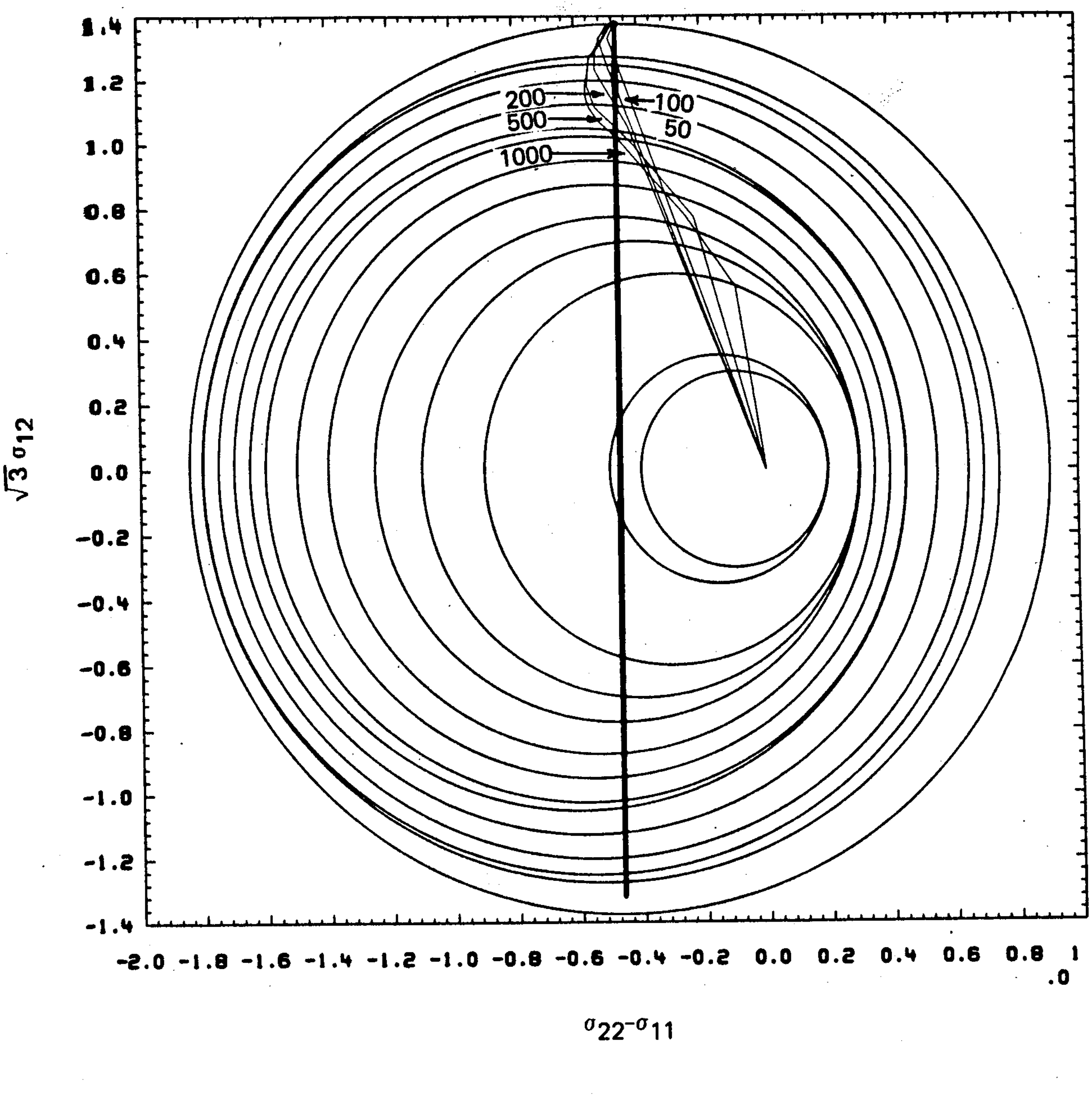
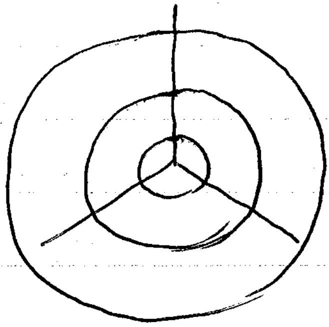


FIG. 17. Drammen clay, simple shear, 5 cycles, $\epsilon_1 = .4$, $\epsilon_2 = .08$.

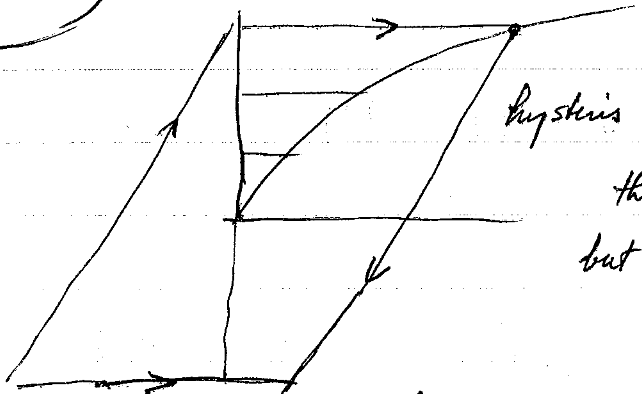
Multi - yield Surface w/ isotopic hardening.

Some work in paper give out last time.

1. Consider first the problem of non kinematic hardening



Consider one of the circles expanding to the next circle ~~—~~ activating the next etc



DIRT code.

hysteris loop after unloading

this is OK for monotonic hardening but inappropriate for cyclic phenomic

2. we can introduce multi ys concept by using the "β-type" idea by combining isotropic & kinematic system. [Could let all surface isotopically expand by the same amount], but in proportion to their current radii.

[ ] is implemented in Livermore code.

New topic - pressure dependent plasticity geologic as well as metals ie (soils)

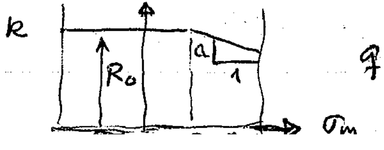
Pressure dependency of y.s. in metals : "pressure-correction model".

- Sandia Lab experiments w/ Aluminum materials

We begin w/ Mises yield surface want to account for contraction of yield surface for large tensile 3-D stresses.

For Elastic - perfectly plastic case :

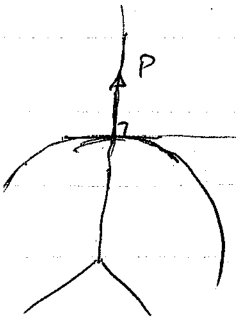
$J_2' = k^2$: we assume $k = k^0 + a\,(\sigma_m^0 - \sigma_m)$ where $a = \begin{cases} 0 & \sigma_m < \sigma_m^0 \\ \neq 0,\ \text{const} > 0 & \sigma_m \geq \sigma_m^0 \end{cases}$

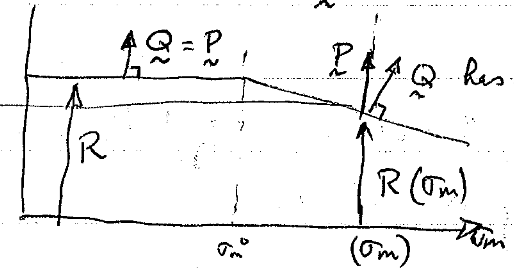


$q = |\underset{\sim}{\sigma}'|$

we assume that plastic flow is still devi & normal i deviatoric plane

$$\dot{\underset{\sim}{\varepsilon}}^{P\prime} = \begin{cases} 0 & (E) \\ \lambda \underset{\sim}{P} & (P) \end{cases}$$ must check $\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}}^{tr}$





$\underset{\sim}{Q}$ is the normal to y.s.

$\underset{\sim}{Q}$ has comp i pressure direction

• ∴ $\underset{\sim}{P} \doteq \underset{\sim}{P}'$

define $\hat{P}$ as the unit vector i deviatoric plane.

$\underset{\sim}{P}$ is normal is π plane
but not normal to y. surface.
Constit eq will have same form $\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C}\left(\dot{\hat{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^{P\prime}\right)$. Calculating an expression for $\hat{P}$ & $\hat{Q}$:

$$\underset{\sim}{P} = \frac{\partial f/\partial \underset{\sim}{\sigma}'}{|\partial f/\partial \sigma|} = \frac{\sigma'}{R} \quad \text{when } \underset{\sim}{P} = \underset{\sim}{Q}$$

$\underset{\sim}{P}$ when $\underset{\sim}{P} \neq \underset{\sim}{Q}$ $\qquad \underset{\sim}{P} = \underset{\sim}{\sigma}'/R(\sigma_m)$ $\qquad$ where $R(\sigma_m) = \sqrt{2}\,k(\sigma_m) = \sqrt{2}\left(k^0 + a(\sigma_m^0 - \sigma_m)\right)$.

must get k^0, σ_m^0 and a from expeiments.

Now $\hat{Q}$ has compo.el i pressure axis

before $f(\underset{\sim}{\sigma}') = k^2$ must now define $F(\underset{\sim}{\sigma}) = f(\underset{\sim}{\sigma}) - k^2 = 0$ since $k = k(\sigma)$

$$\underset{\sim}{Q} \equiv \frac{\partial F/\partial \sigma}{|\partial F/\partial \sigma|} \Rightarrow \frac{\partial F}{\partial \underset{\sim}{\sigma}} = \frac{\partial f}{\partial \underset{\sim}{\sigma}} - 2k\frac{\partial k}{\partial \underset{\sim}{\sigma}}$$ where $\sigma_m = \frac{1}{3}\sigma_{kk}$

$$= \underset{\sim}{\sigma}' - 2k\frac{\partial k}{\partial \sigma_m}\cdot\frac{\partial \sigma_m}{\partial \underset{\sim}{\sigma}}$$

$$= \underset{\sim}{\sigma}' - 2k(-a)\cdot\frac{1}{3}\frac{\partial \sigma_{kk}}{\partial \sigma}$$

$$-\overset{>0}{\overline{2k(-a)}}\cdot\frac{1}{3}\;\underset{\delta_{ij}}{\underline{\delta_{ki}\delta_{kj}}}$$

$$= \underset{\sim}{\sigma}' + \frac{2}{3}ka\,\underset{\sim}{I}$$ a vector pointing i the direction of the hydrostatic axis

now $$\left|\frac{\partial F}{\partial \sigma}\right|^2 = \left(\underset{\sim}{\sigma}' + \frac{2}{3}ka\,\underset{\sim}{I}\right)^2$$

$$\left|\frac{\partial \tilde{F}}{\partial \underset{\sim}{\sigma}}\right| = \left|\underset{\sim}{\sigma}'^2 + \overset{0}{\cancel{\underset{\text{devi}}{\underset{\sim}{\sigma}'}:\underset{\text{nondiv}}{\underset{\sim}{I}}}} + \frac{4}{9}k^2a^2\cdot\underset{I\cdot I}{3}\right|$$

$$\therefore \quad \left|\frac{\partial F}{\partial \underset{\sim}{\sigma}}\right| = \sqrt{|\underset{\sim}{\sigma}'|^2 + \tfrac{4}{3}k^2a^2} \qquad \therefore \quad \underset{\sim}{Q} = \frac{\underset{\sim}{\sigma}' + \tfrac{2}{3}ka\underset{\sim}{I}}{\sqrt{|\underset{\sim}{\sigma}'|^2 + \tfrac{4}{3}k^2a^2}}$$

— to get consistency

Remeber $f(\underset{\sim}{\sigma}) = k^2$

remeber $\dot{\underset{\sim}{\varepsilon}}^{pl} = \Lambda \underset{\sim}{P}$

we will change notation since volumetic effects will introduce lamé constants λ, μ.

$$\text{now } \dot{f} = 2k\dot{k} \implies \dot{f} = \underbrace{\frac{\partial f}{\partial \underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\sigma}}}$$

$$= \sqrt{2}k\sqrt{2}\dot{k}$$

$$= \underbrace{R\,\dot{R}} \qquad \underbrace{|\partial f/\partial \underset{\sim}{\sigma}|}_{R}\underset{\sim}{P}\cdot\dot{\underset{\sim}{\sigma}} = R\cdot\dot{R} \implies \boxed{\underset{\sim}{P}\cdot\dot{\underset{\sim}{\sigma}} = \dot{\underset{\sim}{R}}}$$

$$\text{now } \underset{\sim}{P}\cdot\dot{\underset{\sim}{\sigma}} = \underset{\sim}{P}\cdot\Big(\overset{C\dot{\underset{\sim}{\varepsilon}}}{\dot{\underset{\sim}{\sigma}}^{tr}} - \Lambda\overset{C\dot{\varepsilon}^{pl}}{\underset{\sim}{C}\underset{\sim}{P}}\Big) = \sqrt{2}\left(-a\dot{\sigma}_m\right) = \dot{\underset{\sim}{R}}$$

$$= \underset{\sim}{P}\cdot\dot{\underset{\sim}{\sigma}}^{tr} - \Lambda\underbrace{\underset{\sim}{P}\underset{\sim}{C}\underset{\sim}{P}}_{2\mu} = -\sqrt{2}a\dot{\sigma}_m$$

unit & deviatoric vector

$\lambda\,tr\,\dot{\underset{\sim}{\varepsilon}} + 2\mu\dot{\underset{\sim}{\varepsilon}}$ but $\underset{\sim}{P}\cdot tr\,\dot{\underset{\sim}{\varepsilon}} = 0$ (dev, nondev)

$$\therefore \implies \quad \Lambda = \tfrac{1}{2\mu}\Big[\underset{\text{deviatoric}}{\underset{\sim}{P}}\,\underset{2\mu\,\underset{\sim}{P}\cdot\dot{\underset{\sim}{\varepsilon}}}{\dot{\underset{\sim}{\sigma}}^{tr}} + \sqrt{2}a\dot{\sigma}_m\Big]$$

remeber Constit.

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C}\dot{\underset{\sim}{\varepsilon}} - \Lambda\overbrace{\underset{\sim}{C}\cdot\underset{\text{deviator.}}{\underset{\sim}{P}}}^{2\mu\underset{\sim}{P}} \qquad \therefore \quad \dot{\sigma}_m = \left(\underset{\sim}{C}\cdot\dot{\underset{\sim}{\varepsilon}}\right)_m \text{ since mean part of } \underset{\sim}{C}\cdot\underset{\sim}{P} = 0$$

$$\dot{\sigma}_m = \left[\lambda\,tr\,\dot{\underset{\sim}{\varepsilon}}\,\underset{\sim}{I} + 2\mu\dot{\underset{\sim}{\varepsilon}}\right]_m = \tfrac{1}{3}\left(\lambda\,tr\,\dot{\underset{\sim}{\varepsilon}}\,3 + 2\mu\,tr\,\dot{\underset{\sim}{\varepsilon}}\right)$$

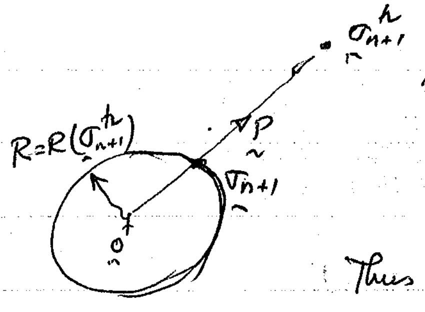
$$\dot{\sigma}_m = \underbrace{\left(\lambda + \tfrac{2}{3}\mu\right)}_{\text{Bulk modulus}}tr\,\dot{\underset{\sim}{\varepsilon}} = B\,tr\,\dot{\underset{\sim}{\varepsilon}}$$

$$\therefore \quad \boxed{\Lambda = \cancel{\tfrac{1}{2\mu}}\left[\cancel{2\mu}\,\underset{\sim}{P}\cdot\dot{\underset{\sim}{\varepsilon}} + \frac{\sqrt{2}a}{2\mu}B\,tr\,\dot{\underset{\sim}{\varepsilon}}\right]}$$

We can use everything as before but w/new Λ

Numerically: calculate $\underset{\sim}{\sigma}^{tr}_{n+1} = \underset{\sim}{\sigma}_n + \underset{\sim}{C}\cdot\Delta\underset{\sim}{\varepsilon}_n$

what does this mean

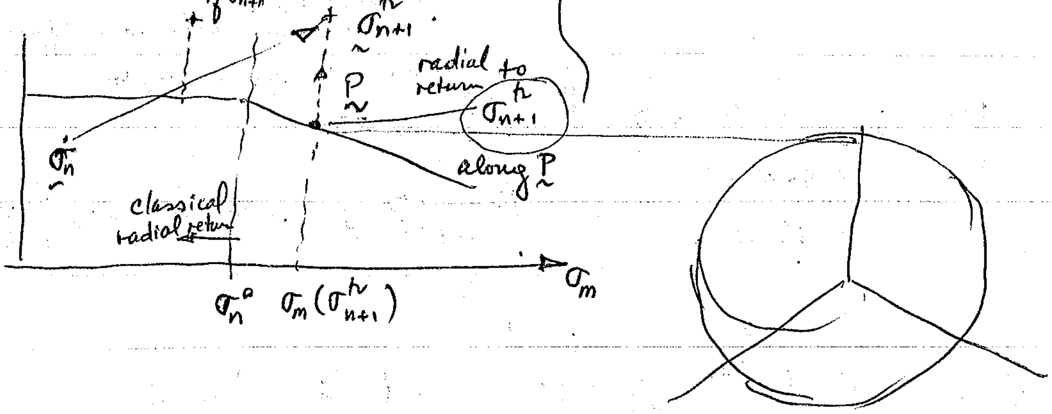


// calculate R & radial return along $\underline{P}$

$$\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - \tilde{\Delta}\, 2\mu\, \underset{\sim}{P}$$

$$\underset{\sim}{\varepsilon}^{pl}_{n+1} = \underset{\sim}{\varepsilon}^{pl}_{n} + \tilde{\underline{\Delta}}\, \underset{\sim}{P}$$

Thus



thus $\underset{\sim}{\sigma}'_{n+1} = R(\sigma^{tr}_{n+1})\, \underset{\sim}{P} \leftarrow \dfrac{\underset{\sim}{\sigma}^{tr\,\prime}_{n+1}}{|\underset{\sim}{\sigma}^{tr}_{n+1}|}$

finally $\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}'_{n+1} + \underset{\sim}{\sigma}''_{n+1}$

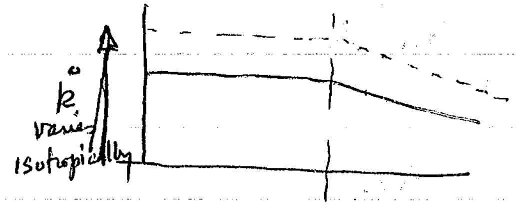
$\underset{\sim}{\sigma}''_{n+1} = 1/3(tr\, \sigma^{tr}_{n+1})\, \underset{\sim}{I}$ given the usual tsts for elasticity/Plastic

For an associative flow rule $\dot{\underset{\sim}{\varepsilon}}^{pl} = \Lambda \underset{\sim}{Q}^{\leftarrow\text{ normal}} \neq \Delta \underset{\sim}{P}$ thus this flow rule is <u>non associative</u>. Thus the elastic-plastic moduli are nonsymmetric

Now let's generalize pressure correction model to combined isotropic/kinematic hardening.
Essentially all formulas hold as previous but w/ minor modifications.

Same Constit. $\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C}\cdot(\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^{pl})$ before k^o_0, σ^o_m & a were fixed.

we will now allow k^o_0 expand ie $k = \overset{o}{k} + \underbrace{a}_{\text{constant, nonvarying}}(\sigma^o_m - \sigma_m)$



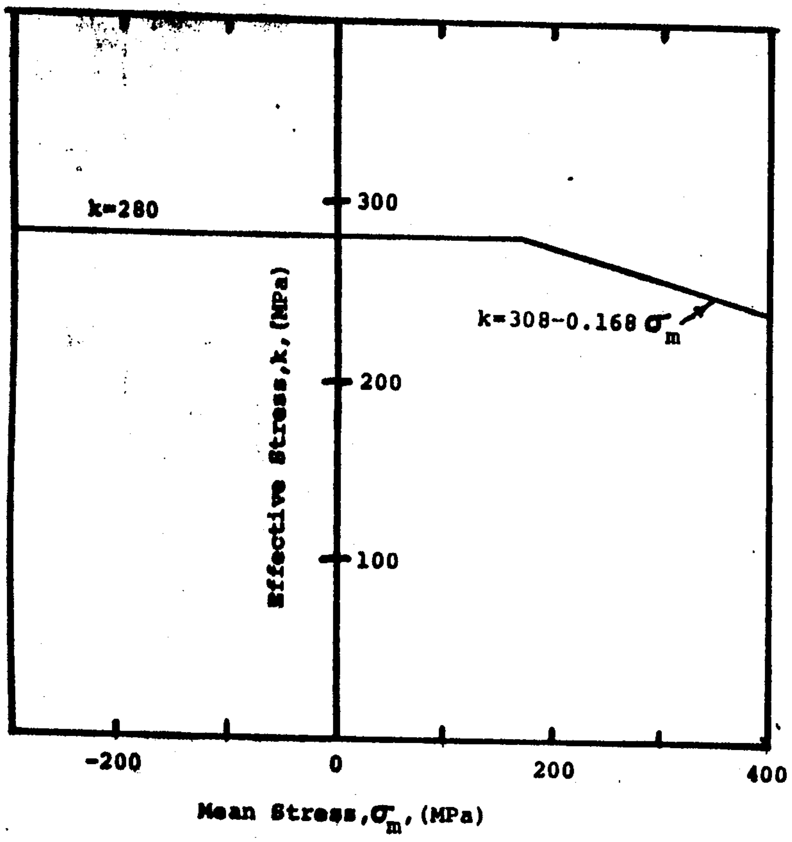


Figure 2. Bilinear representation of the pressure dependent yield of 7075-T6 aluminum.

Handout #8
ME 239A
Notes

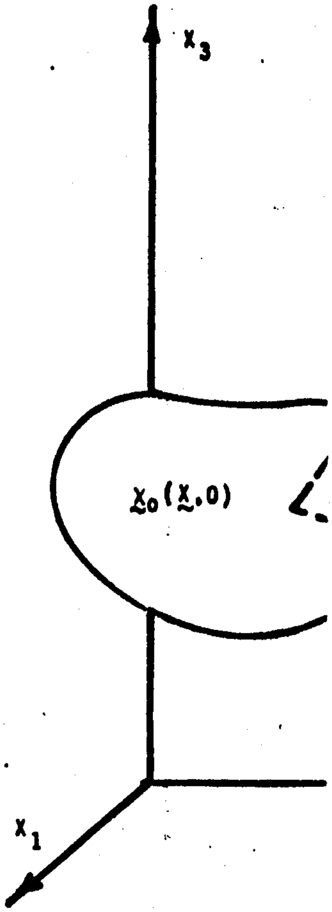


Figure 3. A body
mathema

principal axes of stress. The value of τ_{oct} is related to J_2' by

$$\tau_{oct} = \sqrt{(2J_2'/3)}. \tag{7.13}$$

Thus yielding can be interpreted to begin when τ_{oct} reaches a critical value. Hencky (1924) pointed out that the Von Mises law implies that yielding begins when the (recoverable) elastic energy of distortion reaches a critical value.

Fig. 7.1 shows the geometrical interpretation of the Von Mises yield surface to be a circular cylinder whose projection onto the π plane is a circle of radius $\sqrt{(2)}k$ as shown in Fig. 7.2(a). The two dimensional plot of the Von Mises yield surface is the ellipse shown in Fig. 7.2(b). A physical meaning of the constant k can be obtained by considering the yielding of materials under simple stress states. The case of pure shear ($\sigma_1 = -\sigma_2$, $\sigma_3 = 0$) requires on use of (7.9) and (7.10) that k must equal the yield shear stress. Alternatively the case of uniaxial tension ($\sigma_2 = \sigma_3 = 0$) requires that $\sqrt{(3)}k$ is the uniaxial yield stress.

The Tresca yield locus is a hexagon with distances of $\sqrt{(2/3)}\,Y$ from origin to apex on the π plane whereas the Von Mises yield surface is a circle of radius $\sqrt{(2)}k$. By suitably choosing the constant Y, the criteria can be made to agree with each other, and with experiment, for a single state of stress. This may be selected arbitrarily; it is conventional to make the circle pass through the apices of the hexagon by taking the constant $Y = \sqrt{(3)}k$, the yield stress in simple tension. The criteria then differ most for a state of pure shear, where the Von Mises criterion gives a yield stress $2/\sqrt{(3)}$ ($\approx 1 \cdot 15$) times that given by the Tresca criterion. For most metals Von Mises' law fits the experimental data more closely than Tresca's, but it frequently happens that the Tresca criterion is simpler to use in theoretical applications.

The Mohr–Coulomb yield criterion

This is a generalisation of the Coulomb (1773) friction failure law defined by

$$\tau = c - \sigma_n \tan\phi, \tag{7.14}$$

where τ is the magnitude of the shearing stress, σ_n is the normal stress (tensile stress is positive), c is the cohesion and ϕ the angle of internal friction. Graphically (7.14) represents a straight line tangent to the largest principal stress circle as shown in Fig. 7.3 and was first demonstrated by Mohr (1882). From Fig. 7.3, and for $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ (7.14) can be rewritten as

$$-\tfrac{1}{2}(\sigma_1-\sigma_3)\cos\phi = c-\left(\frac{\sigma_1+\sigma_3}{2}-\frac{(\sigma_1-\sigma_3)}{2}\sin\phi\right)\tan\phi, \tag{7.15}$$

or rearranging

$$(\sigma_1-\sigma_3) = 2c\cos\phi-(\sigma_1+\sigma_3)\sin\phi. \tag{7.16}$$

3)

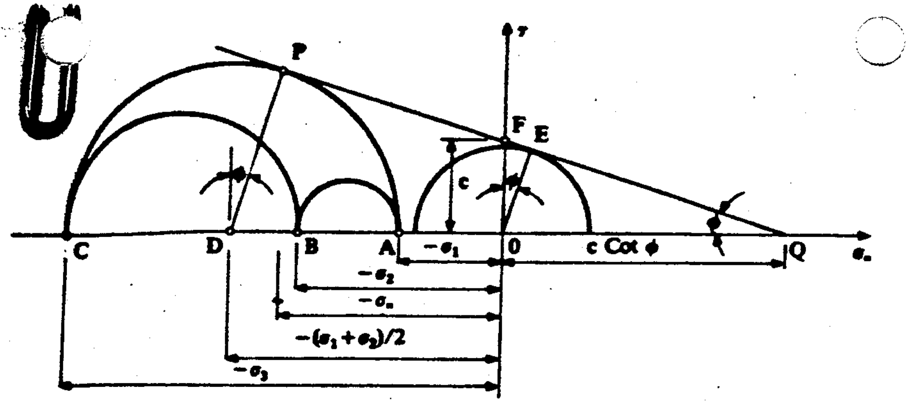


Fig. 7.3 Mohr circle representation of the Mohr–Coulomb yield criterion.

Again, as for the Tresca criterion, the complete yield surface is obtained by considering all other stress combinations which can cause yielding (e.g. $\sigma_3 \geqslant \sigma_1 \geqslant \sigma_2$). In principal stress space this gives a conical yield surface whose normal section at any point is an irregular hexagon as shown in Fig. 7.4. The conical, rather than cylindrical, nature of the yield surface is a consequence of the fact that a hydrostatic stress does influence yielding which is evident from the last term in (7.14). When $\sigma_1 = \sigma_2 = \sigma_3$ we have from (7.16) that the mean hydrostatic stress, $\sigma_m = c \cot\phi$ and therefore the apex of the hexagonal pyramid, 0, in Fig. 7.4, lies along the space diagonal at the point $\sigma_1 = \sigma_2 = \sigma_3 = c \cot\phi$. This criterion is applicable to concrete, rock and soil problems.

The Drucker–Prager yield criterion

An approximation to the Mohr–Coulomb law was presented by Drucker and Prager (1952) as a modification of the Von Mises yield criterion. The influence of a hydrostatic stress component on yielding was introduced by inclusion of an additional term in the Von Mises expression to give

$$\alpha J_1 + (J_2')^{\frac{1}{2}} = k'. \qquad (7.17)$$

This yield surface has the form of a circular cone. In order to make the Drucker–Prager circle coincide with the outer apices of the Mohr–Coulomb hexagon at any section, it can be shown that

$$\alpha = \frac{2\sin\phi}{\sqrt{(3)}(3-\sin\phi)}, \quad k' = \frac{6c\cos\phi}{\sqrt{(3)}(3-\sin\phi)}. \qquad (7.18)$$

Coincidence with the inner apices of the Mohr–Coulomb hexagon is provided by

$$\alpha = \frac{2\sin\phi}{\sqrt{(3)}(3+\sin\phi)}, \quad k' = \frac{6c\cos\phi}{\sqrt{(3)}(3+\sin\phi)}. \qquad (7.19)$$

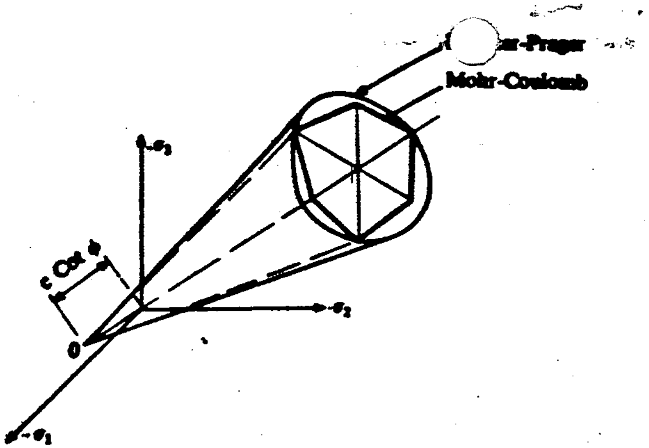


Fig. 7.4 (a) Geometrical representation of the Mohr–Coulomb and Druck Prager yield surfaces in principal stress space.

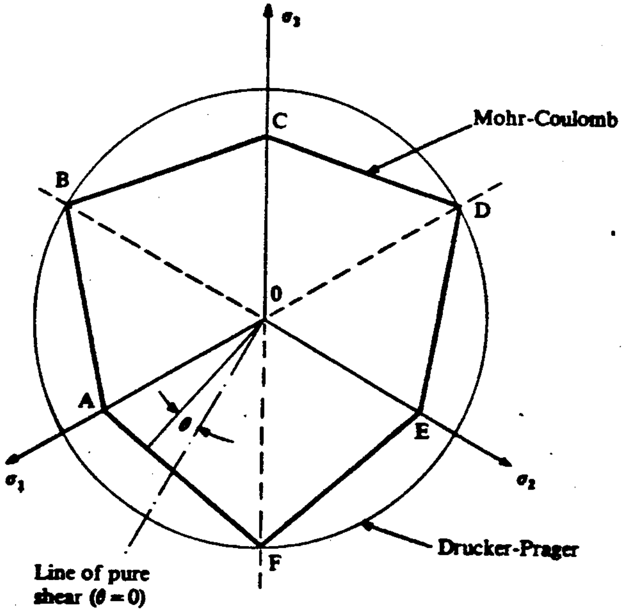


Fig. 7.4 (b) Two-dimensional, π plane, representation of the Mohr–Coulomb . Drucker–Prager yield criteria.

However, the approximation given by either the inner or outer cone to true failure surface can be poor for certain stress combinations.[5]

now $\overset{\circ\circ}{k} = \beta \frac{1}{\sqrt{3}} H' \dot{\bar{\varepsilon}}^{pl}$ same as before but rate eqn for $\overset{\circ}{k}$

$$\overset{\circ}{\underset{\sim}{\alpha}} = (1-\beta)\,{}^{2}\!/_{3}\, H' \dot{\underset{\sim}{\varepsilon}}^{pl}$$

$$\dot{\underset{\sim}{\varepsilon}}^{pl} = \left\{ \begin{array}{ll} 0 & (E) \\ \Lambda \underset{\sim}{P} & (P) \end{array} \right)$$ will be in terms of $\left[f(\underset{\underset{\underset{\sim}{\sigma}-\underset{\sim}{\alpha}}{\parallel}}{\underset{\sim}{\xi}}) = k^2 ;\ \underbrace{\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}}^{h}}_{\text{same as before}} \right]$

next consistency: remember $f(\underset{\sim}{\xi}) = k^2$

$$(\partial f/\partial \xi)\cdot \overset{\circ}{\underset{\sim}{\xi}} = 2k\overset{\circ}{k} = \underbrace{2k}\left(\overset{\circ\circ}{k} - a\,\overset{\circ}{\sigma}_m\right) \leftarrow \frac{d}{dt}\left[k = k_0 + a(\sigma_m^0 - \sigma_m)\right]$$

$$\underbrace{\left|\partial f/\partial \underset{\sim}{\xi}\right|}_{\text{as previous } \frac{|\underset{\sim}{\xi}'|}{\cancel{R(\sigma_m)}}} \underset{\sim}{P}\cdot \overset{\circ}{\underset{\sim}{\xi}} = \cancel{R}\sqrt{2}\left(\beta\frac{1}{\sqrt{3}} H' \underset{\underset{\sqrt{2/3}\,\Lambda}{\downarrow}}{\dot{\bar{\varepsilon}}^{pl}} \cancel{)} - a\dot{\sigma}_m\right)$$

$$\underset{\sim}{P}\cdot \underset{\underset{(\dot{\underset{\sim}{\sigma}}-\dot{\underset{\sim}{\alpha}})}{\sim}}{\dot{\xi}} = {}^{2}\!/_{3}\,\beta H'\Lambda - \underbrace{a\sqrt{2}\dot{\sigma}_m}$$

$$\underset{\sim}{P}\cdot\left(\dot{\underset{\sim}{\sigma}}^{h} - \Lambda \underset{\sim}{C}\underset{\sim}{P} - (1-\beta)\,{}^{2}\!/_{3}\, H' \underset{\underset{\Lambda \underset{\sim}{P}}{\swarrow}}{\dot{\underset{\sim}{\varepsilon}}^{pl}}\right) = \downarrow$$

$$\underset{\sim}{P}\cdot\dot{\underset{\sim}{\sigma}}^{h} - \Lambda 2\mu - (1-\cancel{\beta})\,{}^{2}\!/_{3}\, H'\Lambda = \cancel{{}^{2}\!/_{3}\,\beta H'\Lambda} - a\sqrt{2}\,\dot{\sigma}_m$$

$$\underset{\sim}{P}\cdot\dot{\underset{\sim}{\sigma}}^{h} + a\sqrt{2}\,\dot{\sigma}_m = \Lambda\left(2\mu + {}^{2}\!/_{3}\, H'\right)$$

$$2\mu \underset{\sim}{P}\cdot\dot{\underset{\sim}{\varepsilon}} + a\sqrt{2} B h \dot{\underset{\sim}{\varepsilon}} = \Lambda 2\mu\left(1 + \frac{H'}{3\mu}\right)$$

$$\Lambda = \frac{\underset{\sim}{P}\cdot\dot{\underset{\sim}{\varepsilon}} + {}^{a\sqrt{2}}\!/_{2\mu} B h \dot{\underset{\sim}{\varepsilon}}}{1 + H'/3\mu}$$

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Pressure-Correction Model:

$$\Lambda = \frac{1}{(1+H'/3\mu)}\left(\underset{\sim}{P}\cdot\dot{\underset{\sim}{\varepsilon}} + \frac{aB}{\sqrt{2}\mu} h \dot{\underset{\sim}{\varepsilon}}\right)$$

remember that $\underset{\sim}{C}^{el-pl}\dot{\underset{\sim}{\varepsilon}} = \underset{\sim}{C}\dot{\underset{\sim}{\varepsilon}} - \Lambda \overset{\underset{\sim}{C}\,\dot{\underset{\sim}{\varepsilon}}^{pl}}{\underset{\sim}{C}\underset{\sim}{P}}$ deviatoric

$$C_{ijkl} P_{kl} = 2\mu P_{ij}$$

thus

$$= C_{ijkl}\ \dot{\varepsilon}_{kl}$$

$$- \frac{1}{(1+H'/3\mu)}\, 2\mu P_{ij} \left[P_{kl}\ \dot{\varepsilon}_{kl} + \frac{aB}{\sqrt{2}\mu} \underbrace{\overset{tr\,\dot{\underset{\sim}{\varepsilon}}}{\varepsilon_{kk}}}_{\delta_{kl}\,\dot{\varepsilon}_{kl}} \right]$$

$$= \left(C_{ijkl} - \frac{1}{(\ )} \cdot 2\mu \left(P_{ij} P_{kl} + \frac{aB}{\sqrt{2\mu}} P_{ij}\delta_{kl} \right) \right) \dot{\varepsilon}_{kl}$$

$$\Rightarrow \underset{\sim}{C}^{el\,pl}\, \underset{\sim}{\dot{\varepsilon}} = \left[\underset{\sim}{C} - \frac{2\mu}{(\quad)} \left[\underset{\sim}{P} \otimes \underset{\sim}{P} + \frac{aB}{\sqrt{2}\mu} \underbrace{\underset{\sim}{P} \otimes \underset{\sim}{I}}_{\text{not sym}} \right] \right] \underset{\sim}{\dot{\varepsilon}}$$

remember a plastic flow rule $\underset{\sim}{\dot{\varepsilon}}^{pl} = \Lambda \hat{P} \neq \Lambda \hat{Q} \leftarrow$ normal is called ~~an~~ non assoc flow rule

$\Longrightarrow$ $\underset{\sim}{C}^{elpl}$ non symmetric

<u>ALGORITHM</u>: for this Discrete eqn.

$$\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - 2\mu \tilde{\Lambda} \underset{\sim}{P} \qquad \text{where } \tilde{\Lambda} = \Lambda \Delta t$$

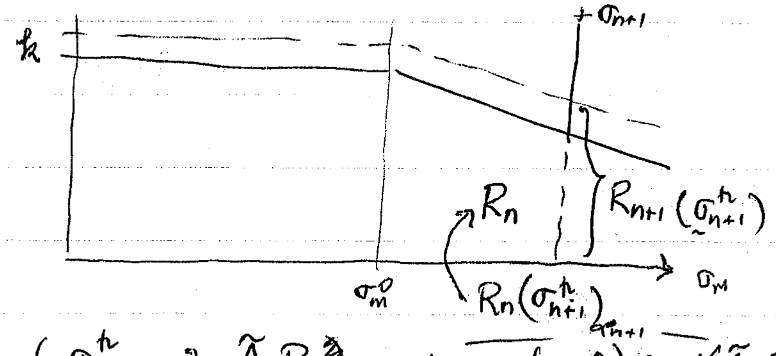
isotropic hard'g exists ~~of~~ due to k^0 : $\quad k^0_{n+1} = k^0_n + \beta \frac{1}{\sqrt{2}} H' \sqrt{2/3}\, \tilde{\Lambda}$

kinematic component : $\quad \underset{\sim}{\alpha}_{n+1} = \underset{\sim}{\alpha}_n + (1-\beta)\, 2/3\, H' \tilde{\Lambda} \hat{P}$

evolution of plastic strain tensor $\quad \underset{\sim}{\varepsilon}^{pl}_{n+1} = \underset{\sim}{\varepsilon}^{pl}_n + \tilde{\Lambda} \hat{P}$ $\qquad \hat{P}$ is assumed constant in terms of direction

Λ consistency in the large : we want $\underset{\sim}{P} \cdot \left[\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\alpha}_{n+1} + R_{n+1}(\underset{\sim}{\sigma}^{tr}_{n+1}) \underset{\sim}{P} \right]$

(rad of surf is dependent on mean stress) $\qquad$ and $\underset{\sim}{P} = \underset{\sim}{\xi}^{tr'}_{n+1} / | \underset{\sim}{\xi}^{tr'}_{n+1} |$ when $\underset{\sim}{\xi}^{tr'}_{n+1} = \underset{\sim}{\sigma}^{tr'}_{n+1} - \underset{\sim}{\alpha}_n$



remember $R = \sqrt{2}\, k$

$$\underset{\sim}{P} \cdot \left(\underset{\sim}{\sigma}^{tr}_{n+1} - 2\mu \tilde{\Lambda} \underset{\sim}{P} = \overbrace{\underset{\sim}{\alpha}_n + (1-\beta)\, 2/3\, H' \tilde{\Lambda} \underset{\sim}{P}}^{\alpha_{n+1}} + \overbrace{\sqrt{2}\, k_{n+1} (\underset{\sim}{\sigma}^{tr}_{n+1}) \underset{\sim}{P}}^{R} \right)$$

$$k_{n+1} \leftarrow k^0_{n+1} + a \left(\sigma^0_m - \sigma_m(\underset{\sim}{\sigma}^{tr}_{n+1}) \right)$$

thus letting $k^{o}_{n+1} = k^{o}_{n} + \beta \frac{1}{\sqrt{2}} H' \sqrt{2/3}\, \tilde{\Lambda}$. and solving for

$$\underset{\sim}{P}\cdot\Big(\underbrace{\underset{\sim}{\sigma}^{tr}_{n+1} - \underset{\sim}{\alpha}_{n}}_{\underset{\sim}{\xi}^{tr}_{n+1}} - \overbrace{\sqrt{2}\underset{\sim}{P}\big(k^{o}_{n} + a\,(\sigma^{o}_{m} - \sigma_{m}\,(\underline{\underset{\sim}{\sigma}^{tr}_{n+1}}))\big)}^{R_n \underset{\sim}{P}}\Big) = \underset{\sim}{P}\cdot\Big(2\mu\underset{\sim}{P} + \tfrac{2}{3}H'\underset{\sim}{P} \qquad\Big)\tilde{\Lambda}$$

$$= \left(2\mu + \tfrac{2}{3}H'\right)\tilde{\Lambda} \quad \text{since } \underset{\sim}{P}\cdot\underset{\sim}{P} = 0$$

now $\underset{\sim}{P}\cdot\big(\underset{\sim}{\xi}^{tr}_{n+1} - R_n \underset{\sim}{P}\big)\cdot =$

$$\underset{\sim}{P}\cdot\underset{\sim}{\xi}^{tr}_{n+1} - R_n =$$

and $\big|\underset{\sim}{\xi}^{tr'}_{n+1}\big| - R_n = 2\mu\left(1 + \frac{1}{3\mu}H'\right)\Lambda$ where $\underset{\sim}{\xi}^{tr}_{n+1}$ is $/\!/\ \underset{\sim}{P}$

and

$$\tilde{\lambda} = \frac{1}{2\mu\left(1 + H'/3\mu\right)}\left(\big|\underset{\sim}{\xi}^{tr'}_{n+1}\big| - R_n\right)$$

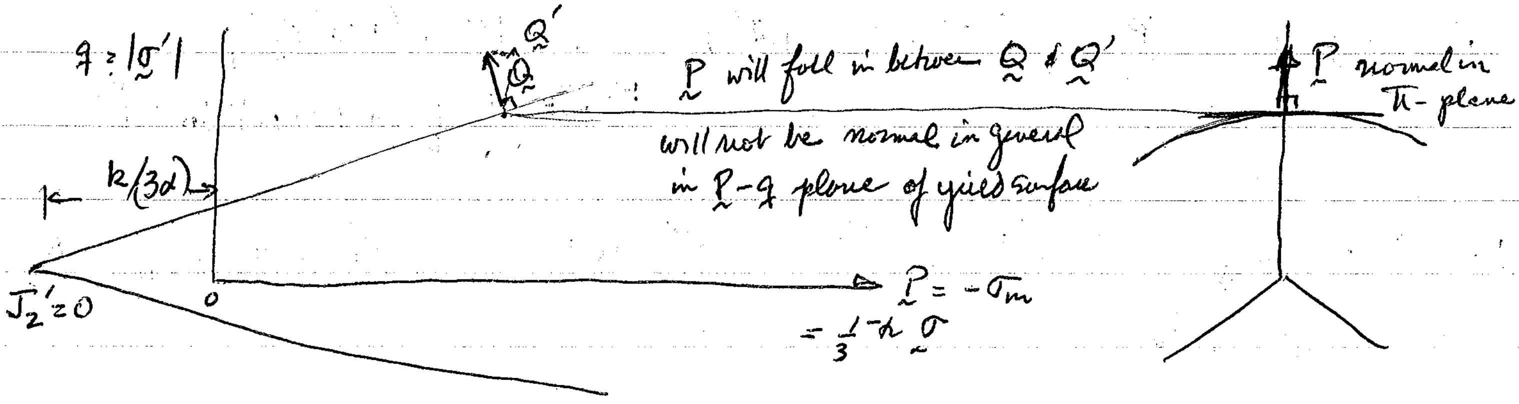
<u>Exercise</u>: Using this account for the pressure-correction effect in the radial return algo. What happens if $\sigma_m >> \sigma^{o}_{m}$ ie if the taper goes to a point. We will address that later

—— × —— × —— × ——

We consider in the next model where volumetric plastic strains are present ie $\dot{\varepsilon}_{kk} \neq 0$ Occur in geological materials.

Drucker-Prager model extends results of Mohr Coulomb. Good within experimental error. We will look at Drucker Prager elastic-perfect plastic cases (see last handout)

we assume $\dot{\underset{\sim}{\varepsilon}}^{P'} = \Lambda \underset{\sim}{P}$ $\begin{cases} \underset{\sim}{P} \neq \underset{\sim}{Q} & \text{non-assoc} \\ \underset{\sim}{P} \neq \underset{\sim}{P}' & \text{non deviatoric} \end{cases}$



we will assume $J_2'(\underset{\sim}{\sigma})^{1/2} = k - \alpha J_1(\underset{\sim}{\sigma})$ (*) ie surface is contracting linearly w mean stress $J_1 = tr\,\underset{\sim}{\sigma}$

material parameter $\overset{\shortparallel}{tr\,\underset{\sim}{\sigma}}$

$$0 = k - \alpha \underbrace{tr\,\underset{\sim}{\sigma}}_{3\sigma_m} \quad \therefore \quad \sigma_m = k/(3\alpha)$$

$$p = -k/3\alpha$$

we assume now that $\underset{\sim}{P} \parallel \underset{\sim}{Q}' + (1+A)\underset{\sim}{Q}''$ & $|\underset{\sim}{P}| = 1$; if $A = 0$ $\underset{\sim}{P} = \underset{\sim}{Q}$

↖ assoc parameter $A = -1$ deviatoric plasticity

$-1 < A < 0$ between $\underset{\sim}{Q}, \underset{\sim}{Q}'$

what is $\underset{\sim}{Q}$: for drucker-prager. Now from (*) $J_2' = [k - \alpha J_1(\underset{\sim}{\sigma})]^2 = \tilde{k}^2$

now define $F(\underset{\sim}{\sigma}) = f(\underset{\sim}{\sigma}) - \tilde{k}^2$

& use $\underset{\sim}{Q} = \partial F/\partial\underset{\sim}{\sigma} \,/\, |\partial F/\partial\underset{\sim}{\sigma}|$ $\Rightarrow$ $\dfrac{\partial f}{\partial \sigma} - 2\tilde{k}\dfrac{\partial \tilde{k}}{\partial \underset{\sim}{\sigma}}$

$$\uparrow\underset{\sim}{\sigma}' - 2\tilde{k}\left\{-\alpha \frac{\partial J}{\partial \underset{\sim}{\sigma}}\right\} \qquad \frac{\partial \sigma_{kk}}{\partial \sigma_{ij}} = \delta_{ik}\delta_{jk} = \delta_{ij}$$

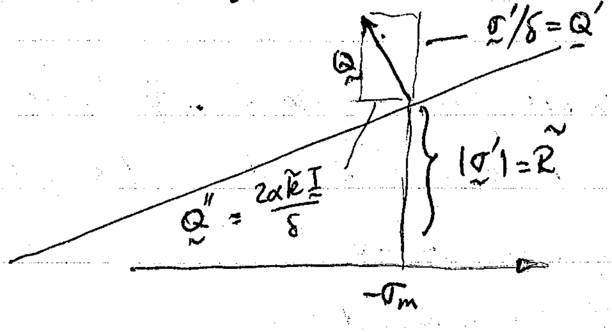
$$(-\alpha \underset{\sim}{I})$$

$$\underset{\text{devi}}{\underset{\sim}{\sigma}'} + \underbrace{2\tilde{k}\alpha\underset{\sim}{I}}_{\text{dilat.}}$$

To get $\hat{\underset{\sim}{Q}}$:

$$\underset{\sim}{Q} = \frac{1}{\delta}(\underset{\sim}{\sigma}' + 2\alpha\tilde{k}\underset{\sim}{I}) \text{ where } \delta = |(\quad)^2| = (|\underset{\sim}{\sigma}'|^2 + 4\alpha^2\tilde{k}^2 3)^{1/2}$$

$$= (|\underset{\sim}{\sigma}'|^2 + 6\alpha^2\tilde{R}^2)^{1/2}; \ \tilde{R} = \sqrt{2}\tilde{k}$$



but $J_2' = \tilde{k}^2 \Rightarrow |\underset{\sim}{\sigma}'|^2 = 2\tilde{k}^2 = R^2$

$$\therefore \Rightarrow \delta = R(1 + 6\alpha^2)^{1/2}$$

now to get $\underset{\sim}{P}$ $\underset{\sim}{P} = \frac{1}{\gamma}(\underset{\sim}{Q}' + (1+A)\underset{\sim}{Q}'')$ $\gamma = |\underset{\sim}{Q}' + (1+A)\underset{\sim}{Q}''|$

verify $= \dfrac{\tilde{R}}{\delta}\left[1 + (1+A)^2 6\alpha^2\right]^{1/2}$

From the consist eqn $\hat{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma}^{tr} - \underline{C}\,\underline{\varepsilon}^{pl}$ $\underset{\sim}{\varepsilon}^{pl} = \Lambda \underset{\sim}{P}$ and Λ is obtained from the consistency law

Consistency

now $f(\underset{\sim}{\sigma}) = \tilde{k}^2(\underset{\sim}{\sigma})$ as before $\dot{f} = 2\tilde{k}\dot{\tilde{k}} \Rightarrow \frac{\partial f}{\partial \underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma}' \cdot (\dot{\underset{\sim}{\sigma}}^{h} - \Lambda \underset{\sim}{C}\underset{\sim}{P})$

$$= 2\tilde{k}(-\alpha \dot{J}_1)$$

$$= 2\tilde{k}(-\alpha\, \mathrm{tr}\, \dot{\underset{\sim}{\sigma}}) =$$

$\therefore$

$$= -2\alpha\tilde{k}\left(\mathrm{tr}\, \dot{\underset{\sim}{\sigma}}^{h} - \mathrm{tr}(\underset{\sim}{C}\cdot\underset{\sim}{P})\right)$$

if we take all Λ-terms to one side $\Rightarrow$

$$\underset{\sim}{\sigma}' \cdot \dot{\underset{\sim}{\sigma}}^{h} + 2\alpha\tilde{k}\, \underset{\overset{\|}{\underset{\sim}{I}\cdot\dot{\underset{\sim}{\sigma}}^{h}}}{\mathrm{tr}\,\dot{\underset{\sim}{\sigma}}^{h}} = \Lambda\left(\underset{\sim}{\sigma}' \cdot \underset{\sim}{C}\cdot\underset{\sim}{P} + 2\alpha\tilde{k}\, \mathrm{tr}\,(\underset{\sim}{C}\cdot\underset{\sim}{P})\right)$$

$$\underbrace{(\underset{\sim}{\sigma}' + 2\alpha\tilde{k}\underset{\sim}{I})}_{\delta \underset{\sim}{Q}} \cdot \dot{\underset{\sim}{\sigma}}^{h} = \Lambda\left(\underset{\sim}{\sigma}' \cdot \underset{\sim}{C}\cdot\underset{\sim}{P} + 2\alpha\tilde{k}\, \underset{\sim}{I}\cdot\underset{\sim}{C}\cdot\underset{\sim}{P}\right) = \Lambda\underbrace{(\underset{\sim}{\sigma}' + 2\alpha\tilde{k}\underset{\sim}{I})}_{\delta \underset{\sim}{Q}} \cdot (\underset{\sim}{C}\cdot\underset{\sim}{P})$$

$$\therefore \boxed{\Lambda = \underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}}^{h} / \underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\underset{\sim}{P}}$$

now $\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}}^{h} = \underset{\sim}{Q}\cdot(\underset{\sim}{C}\dot{\underset{\sim}{\varepsilon}}) = \underset{\sim}{Q}\cdot(\lambda\, \mathrm{tr}\,\dot{\underset{\sim}{\varepsilon}}\, \underset{\sim}{I} + 2\mu\dot{\underset{\sim}{\varepsilon}})$

$$\dot{\underset{\sim}{\varepsilon}}' + \frac{\mathrm{tr}\,\dot{\underset{\sim}{\varepsilon}}}{3}\underset{\sim}{I} = \dot{\underset{\sim}{\varepsilon}}$$

$$= \underset{\sim}{Q}\cdot\left((\lambda + \tfrac{2}{3}\mu)\,\mathrm{tr}\,\dot{\underset{\sim}{\varepsilon}}\,\underset{\sim}{I} + 2\mu\dot{\underset{\sim}{\varepsilon}}'\right) = \underset{\sim}{Q}\cdot(B\,\mathrm{tr}\,\dot{\underset{\sim}{\varepsilon}}\,\underset{\sim}{I} + 2\mu\dot{\underset{\sim}{\varepsilon}}')$$

but $\underset{\sim}{Q} = \underset{\sim}{Q}' + \tfrac{1}{3}\mathrm{tr}\,\underset{\sim}{Q}\,\underset{\sim}{I} \Rightarrow\ ?$

$$\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}}^{h} = 2\mu\,\dot{\underset{\sim}{\varepsilon}}'\cdot\underset{\sim}{Q}' + \frac{B}{3}\mathrm{tr}\,\dot{\underset{\sim}{\varepsilon}}\,\mathrm{tr}\,\underset{\sim}{Q}$$

$\therefore\ \underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\underset{\sim}{P} = B\,\mathrm{tr}\,\underset{\sim}{Q}\,\mathrm{tr}\,\underset{\sim}{P} + 2\mu\,\underset{\sim}{Q}'\cdot\underset{\sim}{P}'$ verify

$\therefore$

$$\underset{\sim}{C}^{el\,pl}\dot{\underset{\sim}{\varepsilon}} = \underset{\sim}{C}\dot{\underset{\sim}{\varepsilon}} - \Lambda\underset{\sim}{C}\underset{\sim}{P}$$

$$\Lambda = \frac{\underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\dot{\underset{\sim}{\varepsilon}}}{\underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\underset{\sim}{P}} \quad \text{or} \quad C_{ijkl}\varepsilon_{kl} - \frac{C_{ijmn}P_{mn}Q_{pq}C_{pqkl}\dot{\varepsilon}_{kl}}{\underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\underset{\sim}{P}}$$

$$\underset{\sim}{C}^{el\,pl} = \frac{\left(\underset{\sim}{C} - (\underset{\sim}{C}\cdot\underset{\sim}{P})\otimes(\underset{\sim}{Q}\cdot\underset{\sim}{C})\right)}{\underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\underset{\sim}{P}}$$

Caution: this is for perfect plasticity

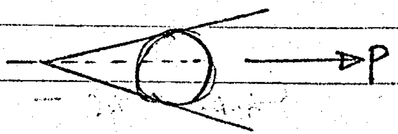
Exercise: Assume $P = P'$, $Q = Q'$ & reduce to the deviatoric case.

3/1 Last time

* Finished " pressure - correction model " for metals

* Theories accounting for volumetric plastic strains (e.g. soils)

Y.S. { Drucker - prager



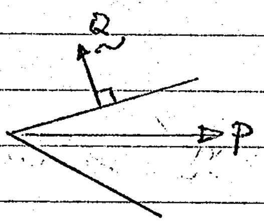
general case of Mises.

Y.S. { mohr - coulomb

← hexagon

$$\dot{\underset{\sim}{\epsilon}}^{pl} = \Lambda \underset{\sim}{P} \quad \begin{cases} \underset{\sim}{P} \neq \underset{\sim}{Q} & \text{non-assoc.} \\ \underset{\sim}{P} \neq \underset{\sim}{P}' & \text{non-dev.} \end{cases}$$

D.P.



constant

$$\underset{\sim}{Q} : \underbrace{f(\sigma)^{1/2} = \tilde{k} = k + 3\alpha p}$$

$$|\underset{\sim}{P}| = 1 \; ; \quad \underset{\sim}{P} \; // \; \underset{\sim}{Q}' + \underset{\text{constant}}{(1+A)}\underset{\sim}{Q}''$$

perf. plasticity :

$$\Lambda = \frac{\overset{\dot{\underset{\sim}{\sigma}}^{tr}}{\underset{\sim}{Q} \cdot \underset{\sim}{C} \cdot \dot{\underset{\sim}{\epsilon}}}}{\underset{\sim}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim}{P}} \; ;$$

$$\underset{\sim}{C}^{el-pl} = \left(\underset{\sim}{C} - \frac{(\underset{\sim}{C} \cdot \underset{\sim}{P}) \otimes (\underset{\sim}{Q} \cdot \underset{\sim}{C})}{\underset{\sim}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim}{P}} \right)$$

non symm. non-associative

Need to determine . k, α, A

k, α : measure Y. stress at 2 different pressure

say P_1, P_2 : solve for k, α :

$$\text{measure} \begin{cases} \tilde{k}(P_1) = k + 3\alpha P_1 \\ \tilde{k}(P_2) = k + 3\alpha P_2 \end{cases}$$

Exer. Devise experiment to det. A.

Summary : ~~Drager~~ Drucker - Prager Y. S. / perfect plasticity.

non - association flow rule

$$\dot{\underset{\sim}{\sigma}} = \underbrace{\underset{\sim}{C} \cdot (\dot{\underset{\sim}{\epsilon}}}_{\dot{\underset{\sim}{\sigma}}^{tr}} - \dot{\underset{\sim}{\epsilon}}^{pl})$$

$$\dot{\underset{\sim}{\epsilon}}^{pl} = \begin{cases} \underset{\sim}{0} \quad (E) \\ \Lambda \underset{\sim}{P} \quad (P) \end{cases}$$

as usual in terms of $\underline{\underline{\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}^{tr}}}$

~~Ap~~ Apex. ???

← produce answer in the algo.

Algo. : (Ideas)

calc. $\underset{\sim}{\sigma}^{tr}_{n+1} = \underset{\sim}{\sigma}_n + \underset{\sim}{C}\, \Delta \underset{\sim}{\epsilon}_n$

Y.S. inside cone ? $\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1}$, return

outside " ?

$$\longrightarrow \underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - \tilde{\Lambda}\, \underbrace{\underset{\sim}{C}\, \underset{\sim}{P}}$$

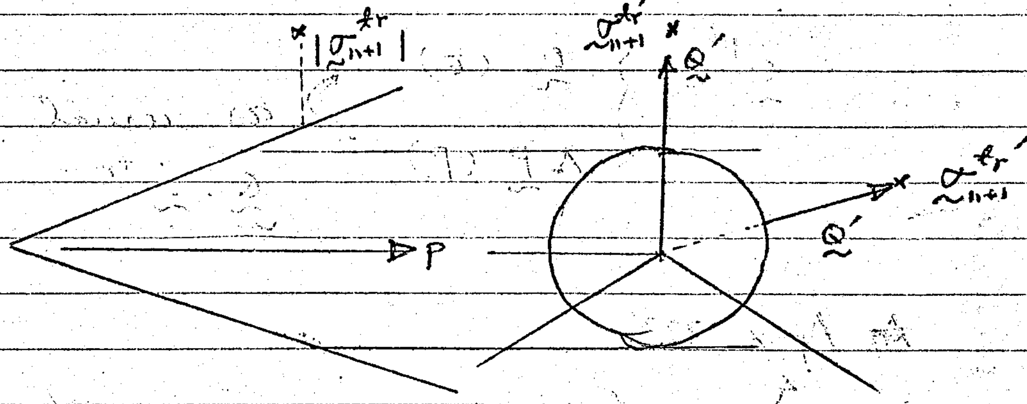
→ return to Y.S. along direction defined by $\underset{\sim}{C}\, \underset{\sim}{P}$

$$\left(\text{recall} : \underset{\sim}{Q} = \frac{1}{\delta}\left(\underset{\sim}{\sigma}' + 2\alpha \tilde{k}\, \underset{\sim}{I} \right) \right)$$

unit

$$\delta = \cancel{k}\, \tilde{k}\, (1 + 6\alpha^2)^{1/2}$$

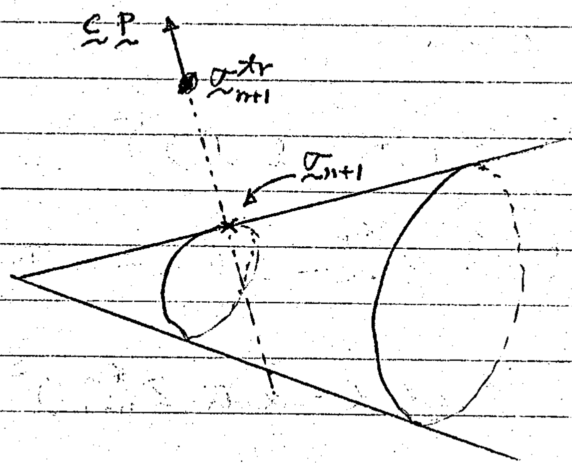
→ algo. require $\underset{\sim}{Q}'$ to point in direction $\underset{\sim}{\sigma}^{tr'}_{n+1}$



$$\longrightarrow \underset{\sim}{Q} = \frac{\underset{\sim}{\sigma}^{tr'}_{n+1} + 2\alpha \tilde{k}(\underset{\sim}{\sigma}^{tr}_{n+1})\, \underset{\sim}{I}}{\left(|\underset{\sim}{\sigma}^{tr'}_{n+1}|^2 + 4\alpha^2 \tilde{k}(\underset{\sim}{\sigma}^{tr}_{n+1})^2\, 3 \right)^{1/2}}$$

$$\underset{\sim}{P} = \underset{\sim}{Q}' + (1+A)\,\underset{\sim}{Q}''$$

$\underset{\sim}{C}\,\underset{\sim}{P}$:

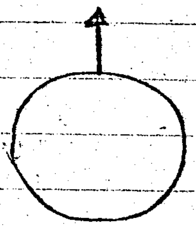


$$(\underset{\sim}{C}\,\underset{\sim}{P})' = (B \,\text{tr}\,\underset{\sim}{P}\,\underset{\sim}{I} + 2\mu\,\underset{\sim}{P})'$$

$$= 2\mu\,\underset{\sim}{P}'$$

$$\Rightarrow \boxed{\underset{\sim}{Q}' \parallel (\underset{\sim}{C}\,\underset{\sim}{P})'}$$

$$\Rightarrow (\underset{\sim}{C}\cdot\underset{\sim}{P})' \parallel \underset{\sim}{\sigma}^{tr\,\prime}_{n+1}$$



Analysis : Pick $\tilde{\lambda}$ (consistency) such that

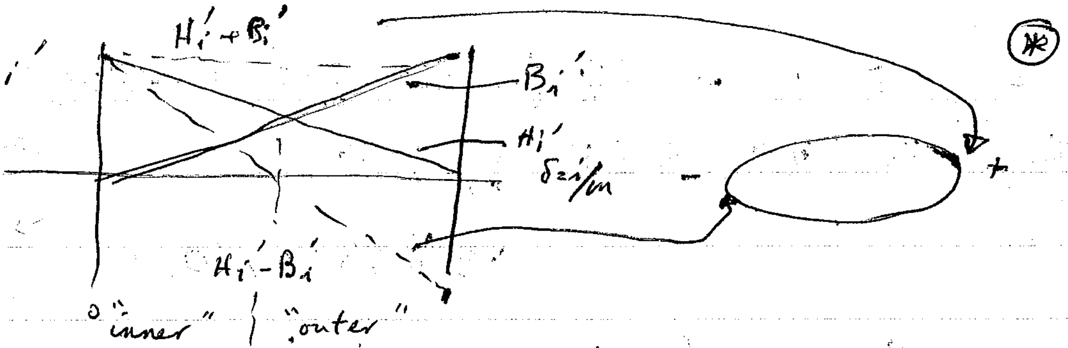
$(\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - \tilde{\lambda}\,\underset{\sim}{C}\,\underset{\sim}{P})$ is the closest point on Y. S.

intersection of straight line param. by λ and Y. F.

$$\underset{\sim\sim\sim}{f(\underset{\sim}{\sigma}_{n+1}) = \tilde{k}^2_{n+1} = \tilde{k}(\underset{\sim}{\sigma}_{n+1})^2}$$

assume $H_1' \cong H_i' =$ constant. Interpolate betwe. $i=1$, $i=m$

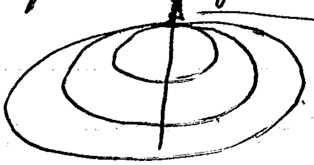
example assume H_i'



roughly speaking, inner ~~surface~~ surfaces hardening

outer " hardening & soften.

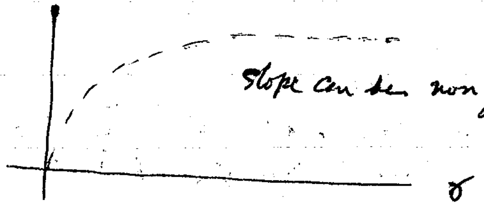
What if we load so that surfaces engage like this



$tr\ \underset{\sim}{Q} = 0$

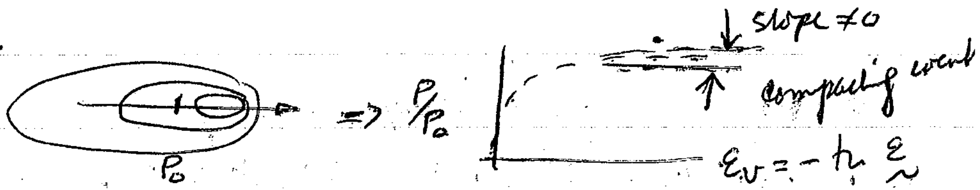
$\therefore H_i' = H_i q$

H_i' ~~degr~~ decreases w/ i



in sketch ⊛ $B_m' < H'$ $\therefore$ $H_i' + B_i'$ also decreases w/ i

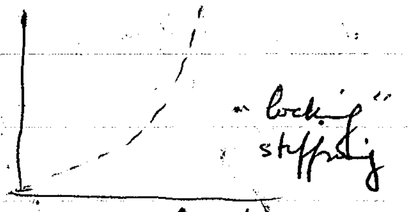
If a process exists $\ni$



but note $H_m' + B_m' \neq 0$

$\therefore$ a slope will exist

suppose we want to replace ↗ with a stiffness curve



we want moduli of surfaces to ↑ w/ i want $H_i' + B_i'$ to increase w/ i

for the outer surface, $i = m$, one of the bounding surface models considered previously.

From Consistency; $\dot{F} = 0 \Rightarrow l\hat{Q} \cdot \dot{\xi} = 0 \Rightarrow \hat{Q} \cdot \dot{\xi} = 0 \Rightarrow \hat{Q} \cdot (\dot{\sigma} - \dot{\alpha}_i) = 0$

thus $\tilde{Q}_i \cdot (\dot{\tilde{\sigma}}^h - \Lambda_i \underset{\sim}{C} \underset{\sim}{P}_i - M \hat{\mu}) = 0$ ⊛
for the ith surface

$\sqrt{2/3}$ ~~H_i~~ $\sqrt{\frac{2}{3}} \Lambda_i / (Q_i \cdot \hat{\mu})$

thus $\Lambda_i \left[\tilde{Q}_i \cdot C \cdot \tilde{P}_i + \frac{2}{3} \mathcal{H}_i' \right] = \tilde{Q} \cdot \dot{\tilde{\sigma}}^h$

$\dot{k}_j = \dot{k}_i$

For isotropic hardening inclusion replace M by $(1-\beta)M$ and let $\dot{k}_i = \beta \frac{l_i}{R_i} \frac{H_i'}{\sqrt{3}} \dot{\tilde{\varepsilon}}^{p'}$

for one surface theory $X_i = l_i/R_i$ $i=1$

Exercise: show this.

Consistency - reconsider $l_i \tilde{Q}_i \cdot \dot{\tilde{\xi}}_i = 2k_i \dot{k}_i$

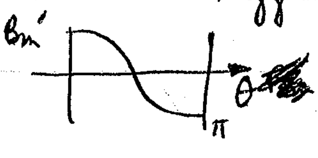
$$\hat{Q}_i \cdot \dot{\tilde{\xi}}_i = \frac{2k_i \dot{k}_i}{l_i}$$

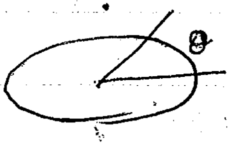
$$\Lambda_i \left(\tilde{Q}_i \cdot \underset{\sim}{C} \cdot \tilde{P}_i + (1-\beta)\tfrac{2}{3} H_i' + \overset{\sqrt{2}}{\cancel{\frac{2k_i}{l_i}}} \beta \cancel{\frac{l_i}{R_i}} \frac{H_i'}{\sqrt{3}} \sqrt{\tfrac{2}{3}} \right) = \tilde{Q}_i \cdot \dot{\tilde{\sigma}}^h)$$ from ⊛

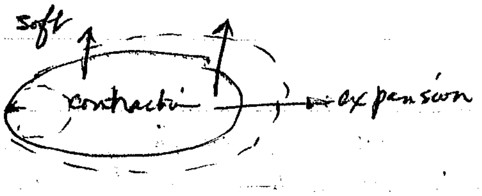
$$\Lambda_i \left(Q_i \cdot C \cdot P_i + \tfrac{2}{3} H_i' \cancel{- \tfrac{2}{3}\beta H_i'} + \cancel{\tfrac{2}{3}\beta H_i'} \right) = \tilde{Q}_i \cdot \dot{\sigma}^h$$ this is same as before

Thus this is also for the isotropic hardening.
We now define the "field of hardening/softening moduli" which will allow H_i' to vary w/ Pts. on surface, also vary with i.
say $\mathcal{H}_i' = H_i' - \frac{h \tilde{Q}_i}{3} \tilde{\beta}_i$ we used this on the bounding surface. In particular we look at case $H_m' \equiv 0 \Rightarrow \mathcal{H}_m' = \frac{-h \tilde{Q}_m}{\sqrt{3}} B_m'$

remember we suggest this so that

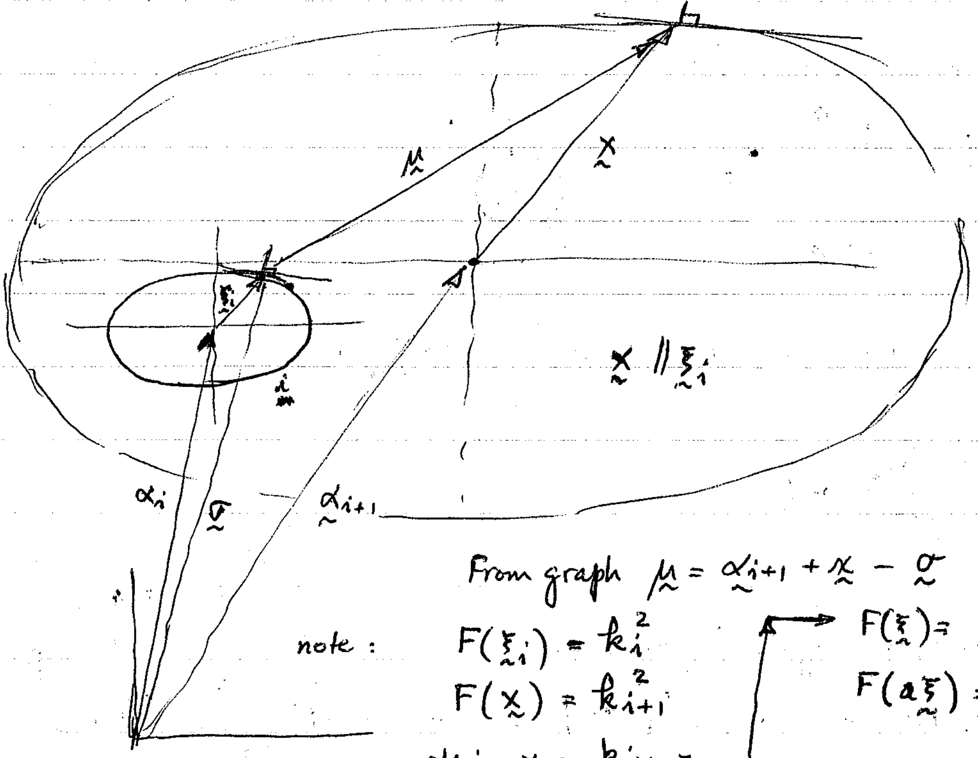






the hardening laws are defined by $\dot{\underset{\sim}{\alpha}}_i = M\underset{\sim}{\hat{\mu}}$. For deviator case $\underset{\sim}{\mu} = \underset{\sim}{\alpha}_{i+1} + \frac{\overset{\sqrt{2}\,k_{i+1}}{R_{i+1}}}{R_i}\underset{\sigma-\alpha_i}{\underset{\sim}{\xi}_i} - \underset{\sim}{\sigma}$

we claim that this also holds in this case. we will show this



$c \equiv$ axis ratio

From graph $\underset{\sim}{\mu} = \underset{\sim}{\alpha}_{i+1} + \underset{\sim}{x} - \underset{\sim}{\sigma}$

note: $F(\underset{\sim}{\xi}_i) = k_i^2$

$F(\underset{\sim}{x}) = k_{i+1}^2$

claim $\underset{\sim}{x} = \frac{k_{i+1}}{k_i}\underset{\sim}{\xi}$ ⟶ $F(\underset{\sim}{\xi}) = \frac{c^2}{3}\xi_m^2 + \frac{1}{2}|\underset{\sim}{\xi}'|^2$

$F(a\underset{\sim}{\xi}) = a^2 F(\underset{\sim}{\xi})$

now [redacted] $F(\underset{\sim}{x}) = \frac{k_{i+1}^2}{k_i^2} F(\underset{\sim}{\xi}_i) = \frac{k_{i+1}^2}{k_i^2}$, $k_i^2 \overset{?}{=} k_{i+1}^2$ ✓

$\therefore\ \underset{\sim}{\mu} = \underset{\sim}{\alpha}_{i+1} + \frac{k_{i+1}}{k_i}\underset{\sim}{\xi}_i - \underset{\sim}{\sigma} = \alpha_{i+1} + \frac{\cancel{\sqrt{2}}R_{i+1}}{\cancel{\sqrt{2}}R_i}\underset{\sim}{\xi}_i - \underset{\sim}{\sigma}$. ✓ QED

We want to modify $M = R_i\sqrt{2/3}\ \mathcal{H}_i'\ \dot{\bar{e}}^{pl}/(\underset{\sim}{\xi}_i \cdot \underset{\sim}{\hat{\mu}})$ (for deviatoric case)

in dev case $\underset{\sim}{\xi}_i = R_i \underset{\sim}{Q}_i$ but not true in this case. (since $\hat{\xi} \parallel \underset{\sim}{\hat{Q}}_i$ in dev case)

$\mapsto \frac{R_i}{\underset{\sim}{\xi}_i \cdot \underset{\sim}{\hat{\mu}}} = \frac{R_i}{R_i}\frac{1}{(\underset{\sim}{Q} \cdot \underset{\sim}{\hat{\mu}})}$

what is M in general $M = \sqrt{2/3}\ H_i'\ \dot{\bar{e}}^{pl}/(\underset{\sim}{\hat{Q}}_i \cdot \underset{\sim}{\hat{\mu}})$

ignore for the moment isotropic hardening also $\dot{\alpha}_j = \dot{\alpha}_i \quad (1 \le j \le i)$

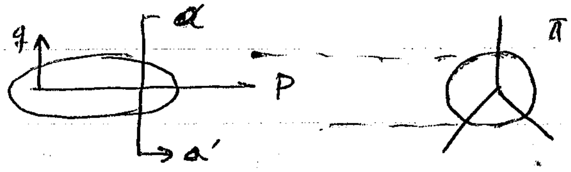
F1

10 March 83

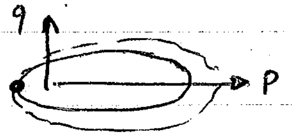
Professor A. Miller 7-3732 Computational work in Inelasticity 1/3 - 1/2 time RA.

Last time ellipsoidal yield surface

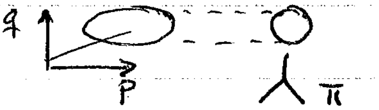
Y.S.



1. Special combined iso-kinematic hard'g. pt is fixed:

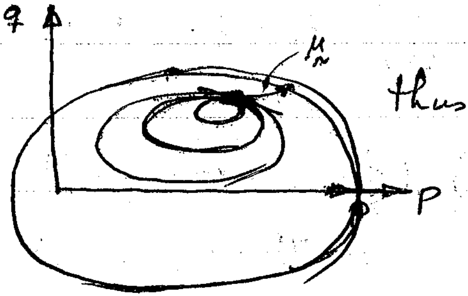


2. Combined theory analogous to deviatoric combined theory



1 & 2 can be used as "bounding surfaces" in multi y.s. theories

Now lookat multi Y.S. ellipsoidal theory:
we insist that all ellipses are "similar" (axis ratio is same).



thus the inner surfaces nest according to the Mroz hardening law : we generalize to full stress space

Development of theory of inner surfaces & perhaps a special theory for outer surface

Constit eq is same $\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot (\dot{\underset{\sim}{\epsilon}} - \dot{\underset{\sim}{\epsilon}}^{Pl})$

$$\dot{\underset{\sim}{\epsilon}}^{Pl} = \left\{ \begin{array}{ll} 0 & (E) \\ \Lambda_i P_i & (P) \end{array} \right\} \rightarrow \text{same with } F(\xi) = k_i^2 \text{ and } Q_i \cdot \dot{\underset{\sim}{\sigma}}^{tr}.$$

i: index of active yield surface.

diagram, assuming that yielding occurs. In the sketch, identify the significance of B, the elastic bulk modulus, and $\mathscr{H}'$, the plastic modulus (assumed a positive constant).

(g) (20 points) Assume that $\mathscr{H}'$ is the following function of effective plastic strain:

$$\mathscr{H}' = \begin{cases} H' , & \bar{\varepsilon}^{pl} \leq \bar{\varepsilon}^{pl}_{th} \\ H' \, \bar{\varepsilon}^{pl}_{th} / \bar{\varepsilon}^{pl} , & \bar{\varepsilon}^{pl} > \bar{\varepsilon}^{pl}_{th} \end{cases}$$

where H' and $\bar{\varepsilon}^{pl}_{th}$ are (positive) material <u>constants</u>. Generalize the sketch of part (f) to this case.

(h) (25 points) Assume an elastic-perfectly plastic solid. Delineate the steps of a "radial-return" algorithm in which the normal vector, $\underset{\sim}{Q}$, from the yield surface pointing to the trial stress, and the quantity $\ell = |\partial F/\partial \underset{\sim}{\sigma}|$ are assumed known. <u>Provide formulae for</u> $\tilde{\lambda}$, $\underset{\sim}{\sigma}_{n+1}$ and $\underset{\sim}{\varepsilon}^{pl}_{n+1}$.

(i) (25 points) Generalize the algorithm of part (h) to account for isotropic hardening as described in part (e). Assume $\mathscr{H}'$ is constant. In addition to the quantities required by part (h), provide a formula for k_{n+1}.

(ii) the π-plane; and

(iii) in σ_m , $|\underset{\sim}{\sigma}'|$ - space. Interpret the limiting cases $c \to 0$ and $c \to \infty$.

(b) (20 points) Calculate the unit outward normal vector, $\underset{\sim}{Q}$, to each portion of the yield surface. Show that $\underset{\sim}{Q}$ is continuous as a function of σ_m at $\sigma_m = \alpha_m$. Describe conditions under which $\underset{\sim}{Q} = \underset{\sim}{Q}'$ and $\underset{\sim}{Q} = \underset{\sim}{Q}''$.

?

(c) (20 points) Assume the usual constitutive equation, namely,

$$\dot{\underset{\sim}{\sigma}} = \underbrace{\underset{\sim}{C} \cdot (\dot{\underset{\sim}{\varepsilon}}}_{\dot{\underset{\sim}{\sigma}}tr} - \dot{\underset{\sim}{\varepsilon}}^{pl}) \quad ,$$

and assume we are dealing with an elastic-perfectly plastic solid. Precisely characterize an elastic process "(E)", and a plastic process "(P)". Consider the cases $\sigma_m \leq \alpha_m$ and $\alpha_m \leq \sigma_m \leq \alpha_m + \sqrt{3}\, k/c$ individually.

(d) (20 points) Again assume an elastic-perfectly plastic solid. If the flow rule is assumed to take the form,

$$\dot{\underset{\sim}{\varepsilon}}^{pl} = \begin{cases} 0 & (E) \\ \Lambda \underset{\sim}{Q} & (P) \end{cases}$$

determine Λ by the "consistency condition."

(e) (20 points) Consider isotropic hardening in which it is assumed that

$$\dot{k} = \frac{1}{\sqrt{3}} \mathscr{H}' \frac{\ell}{R} \dot{\bar{\varepsilon}}^{pl}$$

where $\dot{\bar{\varepsilon}}^{pl} = \sqrt{\frac{2}{3}} \; |\dot{\underset{\sim}{\varepsilon}}^{pl}|$, $\ell = |\partial F/\partial \underset{\sim}{\sigma}|$ and $R = \sqrt{2}\, k$. Generalize the definition of Λ to account for this effect.

(f) (20 points) Assume the initial states of stress and strain are zero. Consider a process in which $\dot{\underset{\sim}{\varepsilon}}' \equiv \underset{\sim}{0}$ and $\mathrm{tr}\, \dot{\underset{\sim}{\varepsilon}} > 0$. Determine an expression for $\dot{\sigma}_m$ in terms of $\mathrm{tr}\, \dot{\underset{\sim}{\varepsilon}}$. Sketch the σ_m vs. $\mathrm{tr}\, \underset{\sim}{\varepsilon}$

(iii) (20 points) Combine the results of parts (i) and (ii) to obtain an expression for c in terms of σ_y and τ_y. Prager found that a value of $c = .73$ fit some experimental observations well. Determine an explicit expression for the ratio τ_y/σ_y based upon this value of c. (Don't worry about carrying out the arithmetic!)

(iv) (20 points) Derive an expression for the unit outward normal vector, $\underset{\sim}{Q}$, to the Prager yield surface. Hint: The following result is useful:

$$m_{ij} = \partial J_3'/\partial\sigma_{ij} = \sigma'_{ik}\,\sigma'_{kj} - \frac{2}{3}\;J_2'\;\delta_{ij}$$

Determine expressions for $\underset{\sim}{Q}'$ and $\underset{\sim}{Q}''$.

(v) (20 points) Consider combined tension-torsion (i.e., $\sigma_{33} = \sigma$, $\sigma_{12} = \sigma_{21} = \tau$, rest = 0). Derive an expression for g where

$$g(\sigma\;,\;\tau\;,\;c) = k^2$$

(After the exam is over, computer plot this expression with $c = .73$ and compare with the analogous Mises and Tresca formulae, and the Taylor-Quinney data.)

3. A yield condition has been proposed by Berg which attempts to account for the failure of a material at sufficiently high levels of triaxial tension. The yield surface is made up of two distinct portions. For mean stress levels below a critical value, α_m, the material yields according to the Mises cylinder of radius $R = \sqrt{2}\,k$. The yield surface in the tensile range is terminated by an ellipsoidal cap whose extremity is defined by $\alpha_m + \sqrt{3}\,k/c$, where c is the axis ratio. The three constants α_m, k and c are determined experimentally. The two distinct portions of the yield surface can be expressed as

$$F_1(\underset{\sim}{\sigma}) = J_2'(\underset{\sim}{\sigma}) = k^2 \quad , \quad \sigma_m \leq \alpha_m \qquad \text{(Mises cylinder)}$$

$$F_2(\underset{\sim}{\sigma}) = \frac{c^2}{3}(\sigma_m - \alpha_m)^2 + J_2'(\underset{\sim}{\sigma}) = k^2 \quad , \quad \alpha_m \leq \sigma_m \leq \alpha_m + \sqrt{3}\,k/c$$

(ellipsoidal cap)

(a) (15 points) Sketch the yield surface in:

(i) principal stress space;

Handout #9

ME 239A Theoretical and Computational Plasticity
Winter Quarter 1983

Stanford University
Instructor: T.J.R. Hughes

Final Exam

Time = 3 hours

Total points = 300

Instructions: Open notes, homeworks and books are allowed. All results from the class notes may be used so please be brief.

1. (15 points) Let $\underset{\sim}{C}$ represent the matrix of isotropic elastic moduli. Obtain explicit expressions in terms of B, the elastic bulk modulus, and μ, the elastic shear modulus, for the following quantities:

   (i) $\underset{\sim}{Q}' \cdot \underset{\sim}{C} \cdot \underset{\sim}{Q}'$

   (ii) $\underset{\sim}{Q}'' \cdot \underset{\sim}{C} \cdot \underset{\sim}{Q}''$

   (iii) $\underset{\sim}{Q}' \cdot \underset{\sim}{C} \cdot \underset{\sim}{Q}''$.

2. Prager has suggested the following modification of the Mises yield surface:

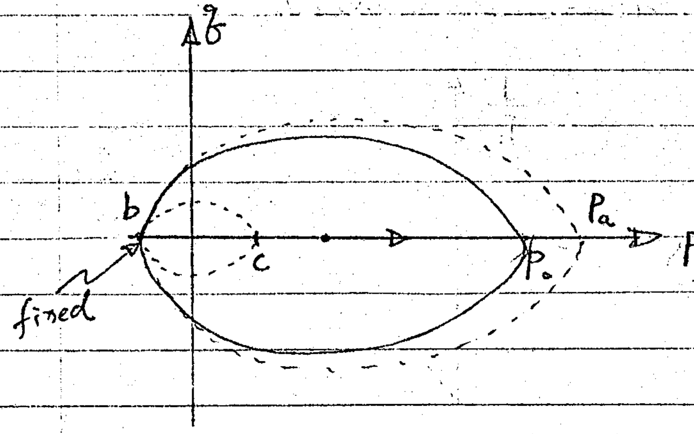
$$F(\underset{\sim}{\sigma}) = J_2' - c(J_3'/J_2')^2 = k^2$$

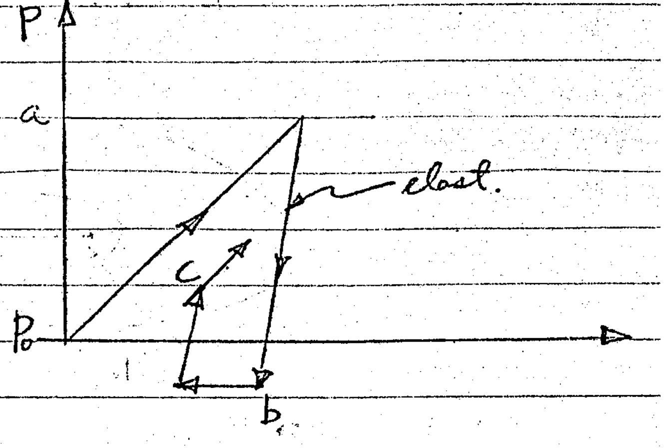
where c is a material constant. Clearly, if $c = 0$ we reduce to the Mises yield surface.

   (i) (20 points) Assume conditions of uniaxial tension (i.e., $\sigma_{33} = \sigma$, rest = 0). Determine an expression for k in terms of c and the yield stress in tension, σ_y .

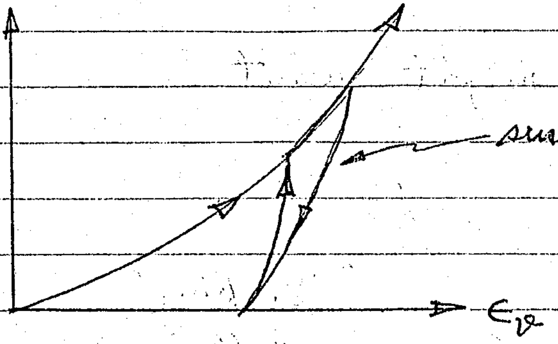
   (ii) (20 points) Assume conditions of simple shear (i.e., $\sigma_{12} = \sigma_{21} = \tau$, rest = 0). Determine an expression for k in terms of c and the yield stress in shear, τ_y .

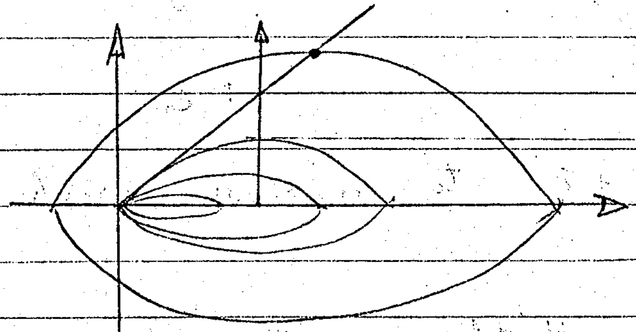
Cyclic consolidation : ($H'=0$, 1 surf.)

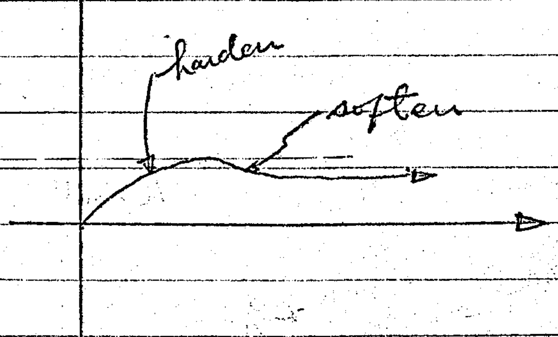


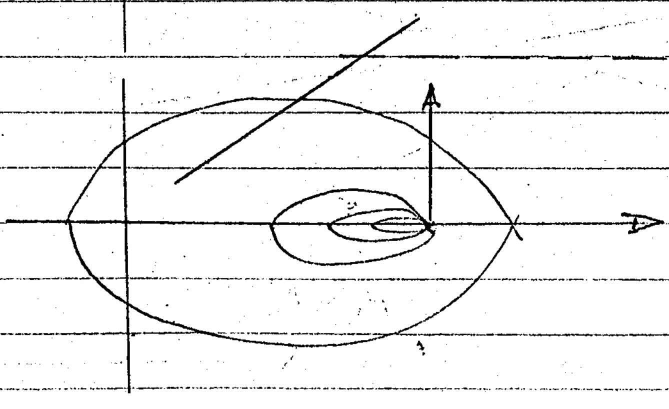


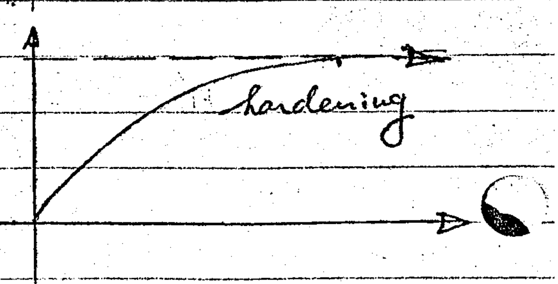
Multi-Y.S :







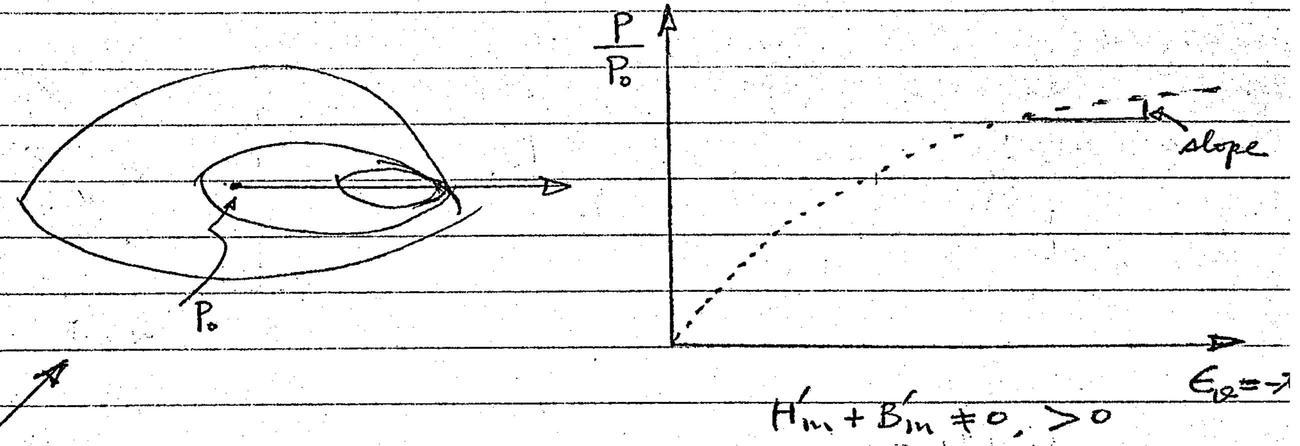




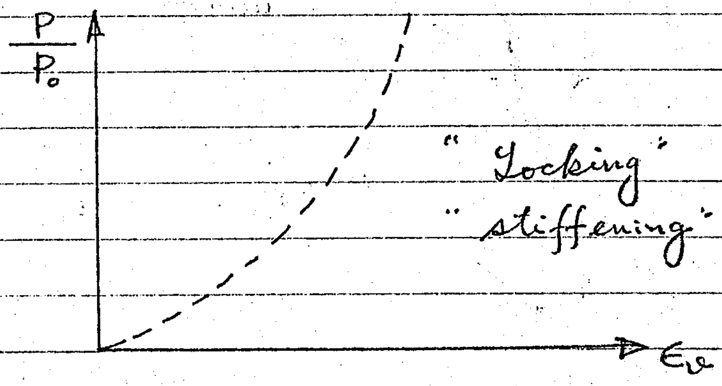
all the possibility in this theory

in sketch $B'_m < H'_1$

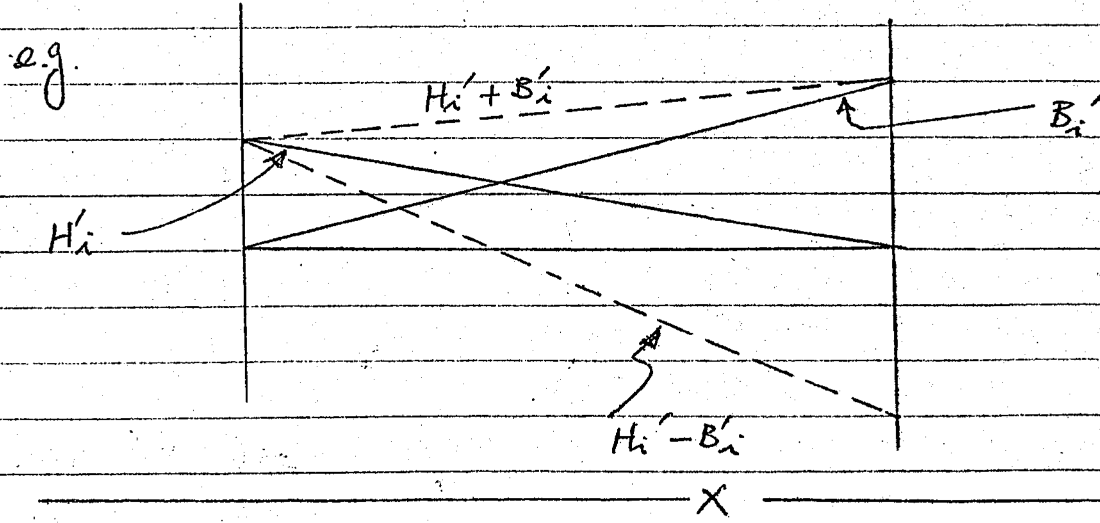
$H'_i + B'_i$ also decreases with i



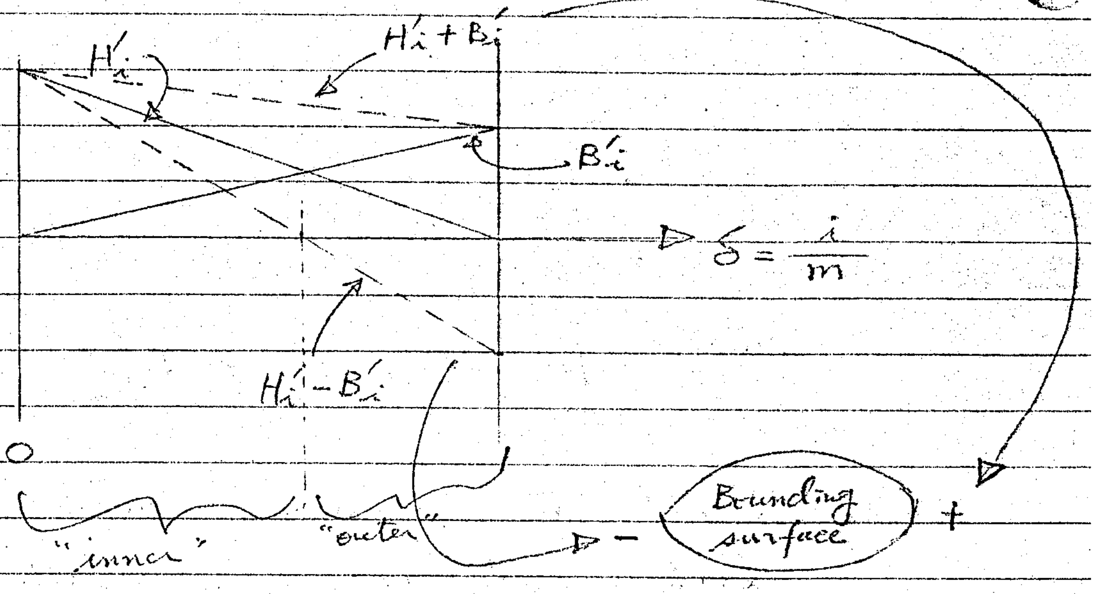
However, we might want



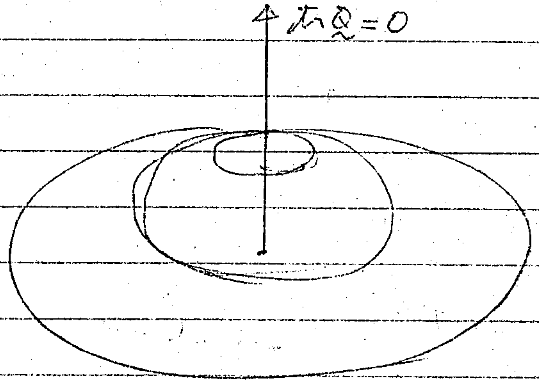
Want $H'_i + B'_i$ to increase with i



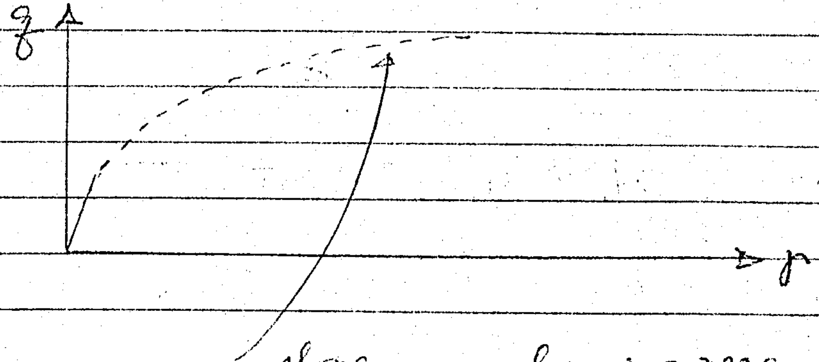
e.g.



Roughly speaking, inner surfaces harden
outer surfaces harden & softe



$H_i' = H_i'$ $\quad$ H_i' decreases with i



slope can be nonzero if $H_m' \neq 0$

Definition of "field of hard/soft moduli):

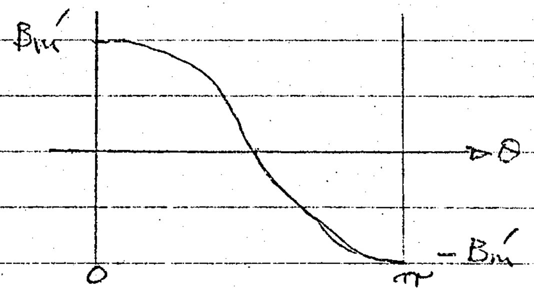
→ allow $\mathcal{H}'_i$ to vary with points on each surface

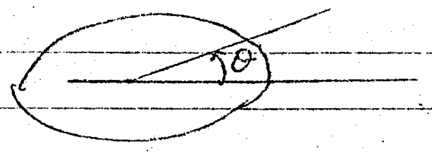
also allows vary with i

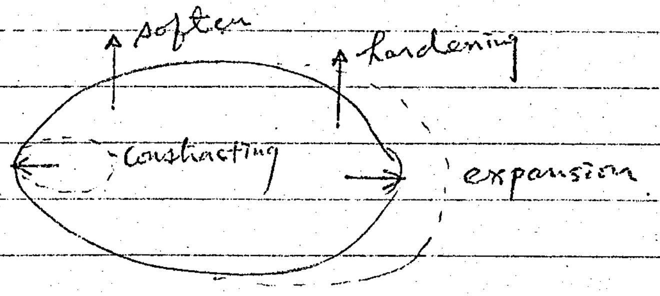
say $\mathcal{H}'_i = H'_i - \frac{\text{tr}\, Q_i}{3} B'_i$

Recall on bounding surface. In particular: looked at case in which $H'_m \equiv 0$:

$\mathcal{H}'_m = \frac{-\text{tr}\, Q_m}{\sqrt{3}} B'_m$







assume $\mathcal{H}'_i \cong H'_i \cong$ const.

intep between $i=1$, $i=m$

e.g.

Consistency: $\dot{F} = 0$

$$l \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\xi}} = 0$$

$$\underset{\sim i}{Q} \cdot (\dot{\underset{\sim}{\alpha}} - \dot{\underset{\sim i}{\alpha}}) = 0$$

$$\underset{\sim i}{Q} \cdot (\dot{\underset{\sim}{\sigma}}^{tr} - \Delta_i \underset{\sim}{C} \underset{\sim i}{P} - M \underset{\sim}{\mu}) = 0$$

$$\sqrt{\tfrac{2}{3}} \mathcal{H}_i' \sqrt{\tfrac{2}{3}} \Delta_i \Big/ (\underset{\sim i}{Q} \cdot \underset{\sim}{\mu})$$

$$\longrightarrow \boxed{\Delta_i \left(\underset{\sim i}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim i}{P} + \frac{2}{3} \mathcal{H}_i' \right) = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}^{tr}}$$

—————————— × ——————————

Generalize to account for isotopic

$$\begin{cases} M \longleftarrow (1-\beta) M \\ \dot{k}_i = \beta \left(\frac{l_i}{R_i} \right) \frac{\mathcal{H}_i'}{\sqrt{3}} \dot{\bar{\epsilon}}^{pl} \end{cases}$$

"γ"

e.g. $\dot{k}_j = \dot{k}_i$ $\quad j = 1, 2, \dots, m$

→ Check: get same expression for Δ_i

Consistency – reconsider

$$l_i \underset{\sim i}{Q} \cdot \dot{\underset{\sim i}{\xi}} = 2 k_i \dot{k}_i$$

$$\Delta_i \left(\underset{\sim i}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim i}{P} + (1-\beta) \frac{2}{3} \mathcal{H}_i' + \underbrace{\frac{2 k_i}{l_i} \frac{\beta l_i}{R_i} \frac{\mathcal{H}_i'}{\sqrt{3}} \sqrt{\frac{2}{3}}}_{\frac{2}{3} \beta \mathcal{H}_i'} \right)$$

$$= \underset{\sim i}{Q} \cdot \dot{\underset{\sim}{\sigma}}^{tr}$$

$$\therefore \Delta_i (Q_i \cdot C \cdot P_i + \tfrac{2}{3} \mathcal{H}_i') = Q \cdot \dot{\underset{\sim}{\sigma}}^{tr}$$

Note $F(\underset{\sim}{\xi}_i) = k_i^2$ $\xrightarrow{\text{P.F}}$ $F(\underset{\sim}{\xi}) = \frac{c^2}{3}\xi_m^2 + \frac{1}{2}|\underset{\sim}{\xi}'|^2$

$F(\underset{\sim}{x}) = k_{i+1}^2$ $\qquad F(a\underset{\sim}{\xi}) = a^2 F(\underset{\sim}{\xi})$

Claim: $\boxed{\underset{\sim}{x} = \frac{R_{i+1}}{R_i}\underset{\sim}{\xi}_i}$

$$k_{i+1}^2 = F(\underset{\sim}{x}) = \left(\frac{k_{i+1}}{k_i}\right)^2 F(\underset{\sim}{\xi}_i) = k_{i+1}^2$$

$\Longrightarrow$ $\therefore$ true

$$\Longrightarrow \boxed{\underset{\sim}{\mu} = \underset{\sim}{\alpha}_{i+1} + \overset{\Rightarrow \sqrt{2}R_{i+1}}{\frac{k_{i+1}}{k_i}}\underset{\sim}{\xi}_i - \underset{\sim}{\sigma}}$$

Deviatoric case

$$\left\{ M = R_i\sqrt{\tfrac{2}{3}}\,H_i'\,\dot{\epsilon}^{pl} \Big/ (\underset{\sim}{\xi}_i \cdot \underset{\sim}{\mu}) \right]$$

in deviatoric case $\underset{\sim}{\xi}_i = R_i\underset{\sim}{Q}_i \Rightarrow \frac{R_i}{(\underset{\sim}{\xi}_i\cdot\underset{\sim}{\mu})} = \frac{\cancel{R_i}}{\cancel{R_i}}\frac{1}{(\underset{\sim}{Q}_i\cdot\underset{\sim}{\mu})}$

$$M \underset{\substack{\uparrow \\ \text{in general}}}{=} \sqrt{\tfrac{2}{3}}\,H_i'\,\dot{\epsilon}^{pl} \Big/ (\underset{\sim}{Q}_i\cdot\underset{\sim}{\mu})$$

(ignore any isotropic hard.)

Theory of inner surfaces

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot \left(\dot{\underset{\sim}{\epsilon}} - \dot{\underset{\sim}{\epsilon}}^{pl} \right)$$

$$\dot{\underset{\sim}{\epsilon}}^{pl} = \left\{ \begin{matrix} 0 & (E) \\ \Delta_i \underset{\sim}{P}_i & (P) \end{matrix} \right\}$$ same with $\boxed{F(\underset{\sim}{\xi}) = k_i^2}$

and $\underset{\sim}{Q}_i \cdot \dot{\underset{\sim}{\sigma}}^{tr}$

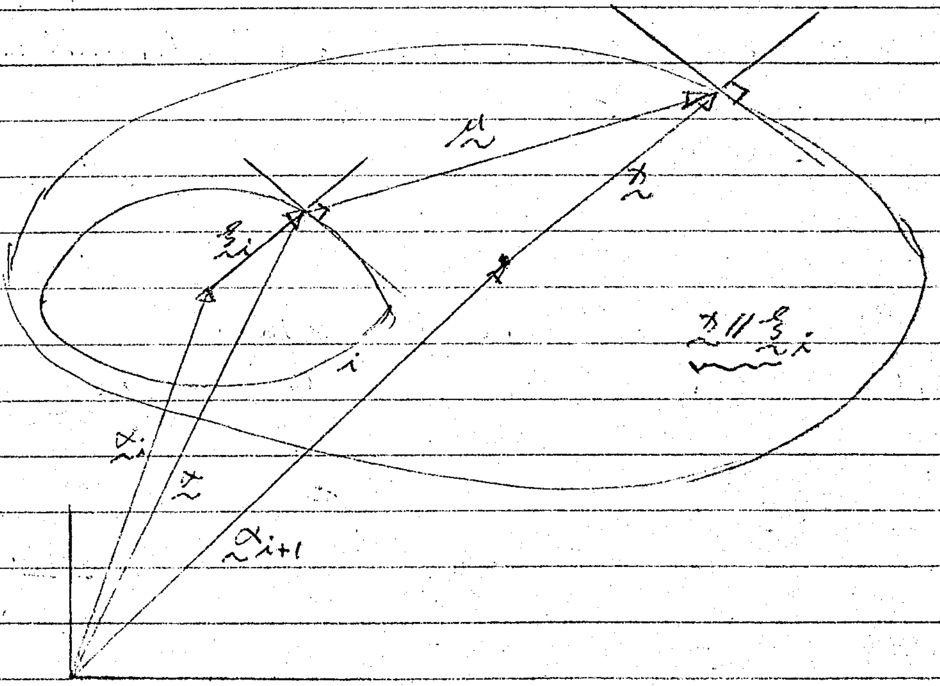
$$\rightarrow \boxed{\dot{\underset{\sim}{\alpha}}_i = M \underset{\sim}{\mu}}$$

deviatoric case $\underset{\sim}{\mu} = \underset{\sim}{\alpha}_{i+1} + \frac{\overset{\nearrow \sqrt{2} k_{i+1}}{R_{i+1}}}{R_i} \underset{\sim}{\xi}_i - \underset{\sim}{\sigma}$

$$\Downarrow$$

$$\underset{\sim}{\sigma} - \underset{\sim}{\alpha}_i$$

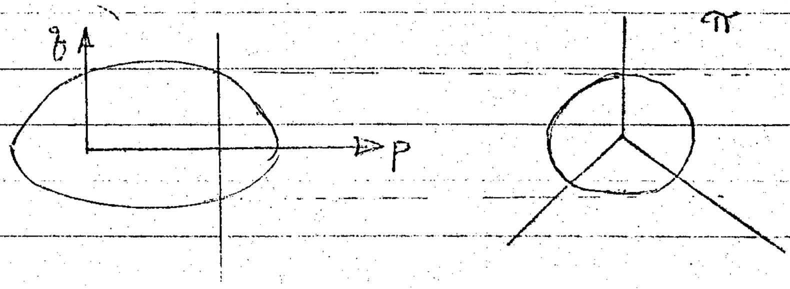
claim: also holds in present case



$$\underset{\sim}{\mu} = \underset{\sim}{\alpha}_{i+1} + \underset{\sim}{x} - \underset{\sim}{\sigma}$$

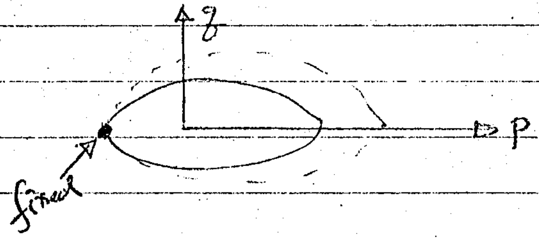
3/10 Last time

Y S.

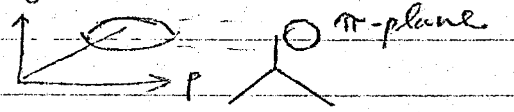


1. Special combined iso-kin hard

⇒ point is fixed



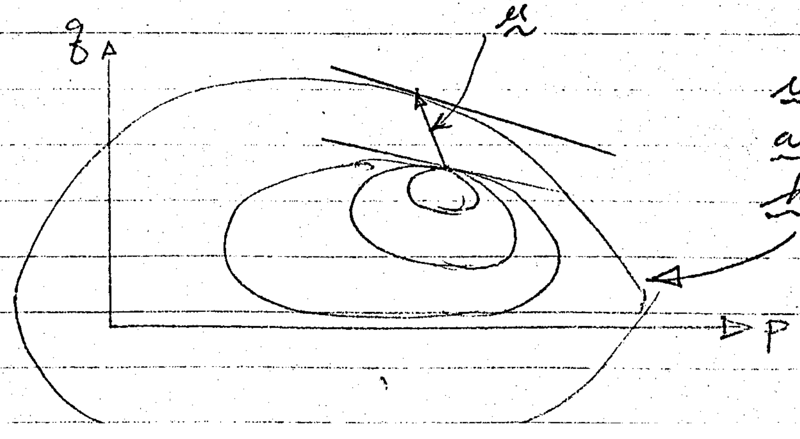
2. Combined theory ~~at~~ analogous to deviatoric combined theory



1 & 2 can be used as "bounding surfaces" in multi-Y.S. theory.

→ Multi Y.S. ellipsoidal theory

all ellipses are "similar" (axis ratio is same)



inner surfaces nest according to MROZ hardening law

Consistency: $\boxed{\dot{F}(\underset{\sim}{\xi}) = 2k\dot{k}}$

$$\underbrace{\frac{\partial F}{\partial \underset{\sim}{\xi}}} \cdot \dot{\underset{\sim}{\xi}}$$

$$l\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\xi}} = 2k\dot{k}$$

$$\Downarrow$$

$$\underset{\sim}{Q} \cdot (\dot{\underset{\sim}{\sigma}} - \dot{\underset{\sim}{\alpha}})$$

$$\underset{\sim}{Q} \cdot \left(\dot{\underset{\sim}{\sigma}}^{tr} - \Lambda \underset{\sim}{C}\underset{\sim}{P} - y(1-\beta)\frac{2}{3}H'\Lambda \underset{\sim}{P}\right) = \frac{2k}{l}\frac{l}{R}\frac{\beta}{\sqrt{3}}\sqrt{\frac{2}{3}}\Lambda$$

$$\Lambda\left(\underset{\sim}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim}{P} + y(1-\beta)\frac{2}{3}H'\underset{\sim}{Q} \cdot \underset{\sim}{P} + \frac{2}{3}\beta H'\right) = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}^{tr}$$

$$\boxed{y = \frac{1}{\underset{\sim}{Q} \cdot \underset{\sim}{P}}}$$

$$\Lambda\left(\underset{\sim}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim}{P} + \frac{2}{3}H'\right) = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}^{tr}$$

Same as iso. hardening

$$\underset{\sim}{C}^{el-pl} = \underset{\sim}{C} - \frac{(\underset{\sim}{C} \cdot \underset{\sim}{P}) \otimes (\underset{\sim}{Q} \cdot \underset{\sim}{C})}{d}$$

$$d \leftarrow \underset{\sim}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim}{P} + \frac{2}{3}H'$$

×

$$\Lambda = \frac{\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}^{tr}}{(\underset{\sim}{Q} \cdot \underset{\sim}{C} \cdot \underset{\sim}{P} + \frac{2}{3}H')}$$

Y.S $F(\underset{\sim}{\xi}) = k^2$

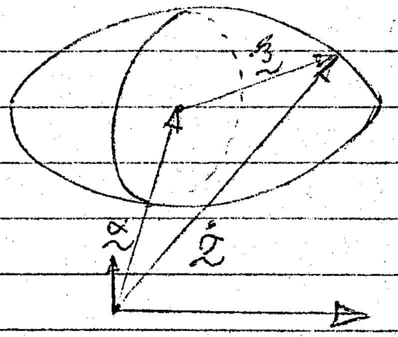
$$\uparrow \quad \frac{c^2}{3}\,\phi\,\xi_m^2 + \frac{1}{2}|\underset{\sim}{\xi}'|^2 = k^2$$

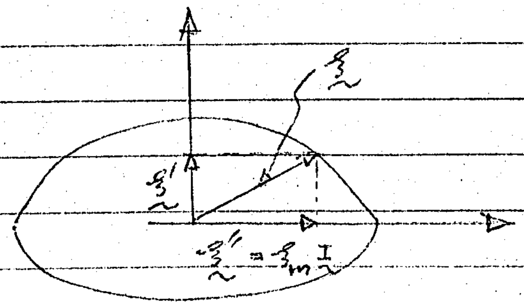
$$\underset{\sim}{\dot\sigma} = \underbrace{\underset{\sim}{C}\cdot(\underset{\sim}{\dot\epsilon}}_{\underset{\sim}{\dot\sigma}^{tr}} - \underset{\sim}{\dot\epsilon}^{pl})$$

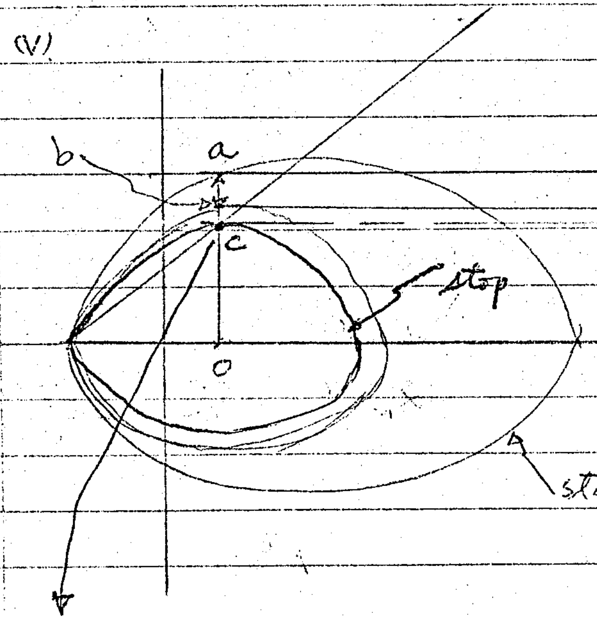
$$\longrightarrow \left\{ \begin{array}{l} \mathring{k} = \beta \frac{1}{\sqrt{3}} \underset{\underset{x}{\shortparallel}}{\left(\frac{l}{R}\right)} \mathcal{H}' \dot{\bar\epsilon}^{pl} \\ \\ \underset{\sim}{\dot\alpha} = \underset{\uparrow\,\text{float}}{y}\,(1-\beta)\frac{2}{3}\mathcal{H}' \underset{\sim}{\dot\epsilon}^{pl} \end{array} \right.$$

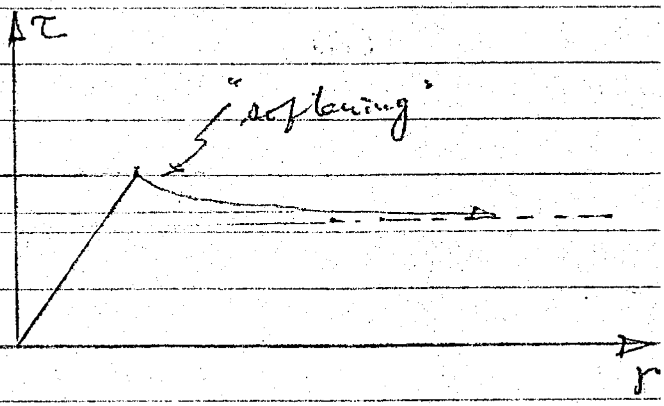
$$\underset{\sim}{\dot\epsilon}^{pl} = \left\{ \begin{array}{ll} 0 & (E) \\ \Lambda \underset{\sim}{P} & (p) \end{array} \right\} \text{ as usual in terms of } F = k^2 \;\&\; \underset{\sim}{Q}\cdot\underset{\sim}{\dot\sigma}^{tr}$$

$$\underset{\sim}{P} = \underset{\sim}{P}(\underset{\sim}{Q}) \qquad\qquad \underset{\sim}{Q} = \underset{\sim}{Q}(\underset{\sim}{\xi})$$









$$\dot{k} = \frac{x}{\sqrt{3}} \mathcal{H}' \dot{\bar{\epsilon}}^{pl}$$

$H_c' \equiv 0$

$\dot{k} = \dot{\alpha}_m = 0$
stops

$\mathcal{H}_a' < 0 \; ; \; \dot{k}_a < 0$

$\dot{\alpha}_m \big|_a > 0 \quad (\text{"}\leftarrow\text{"})$

$\mathcal{H}_b' > \mathcal{H}_a'$

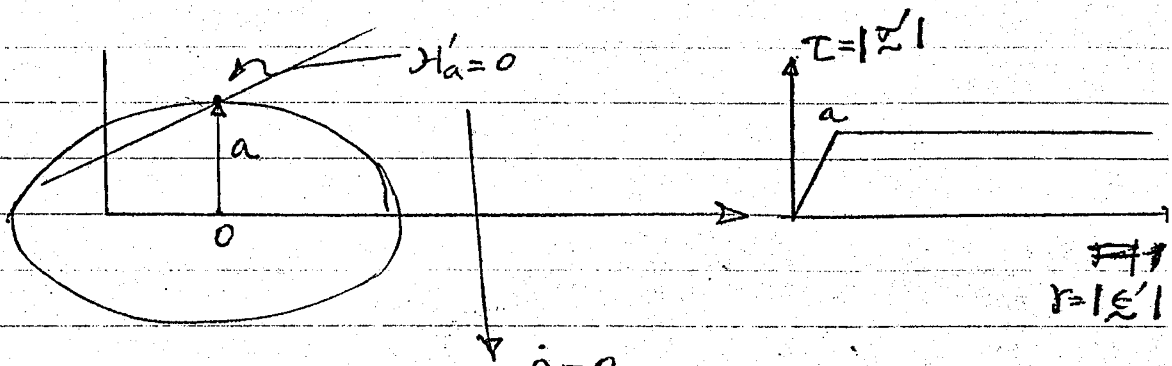
——————— × ———————

Special combined model above has been proposed as a "<u>Bounding surf.</u>" for multi-Y.S. models

——————— × ———————

"General" theory of ellipsoidal Y.S. accounting for combined iso-kin hardening analogous to the deviatoric version

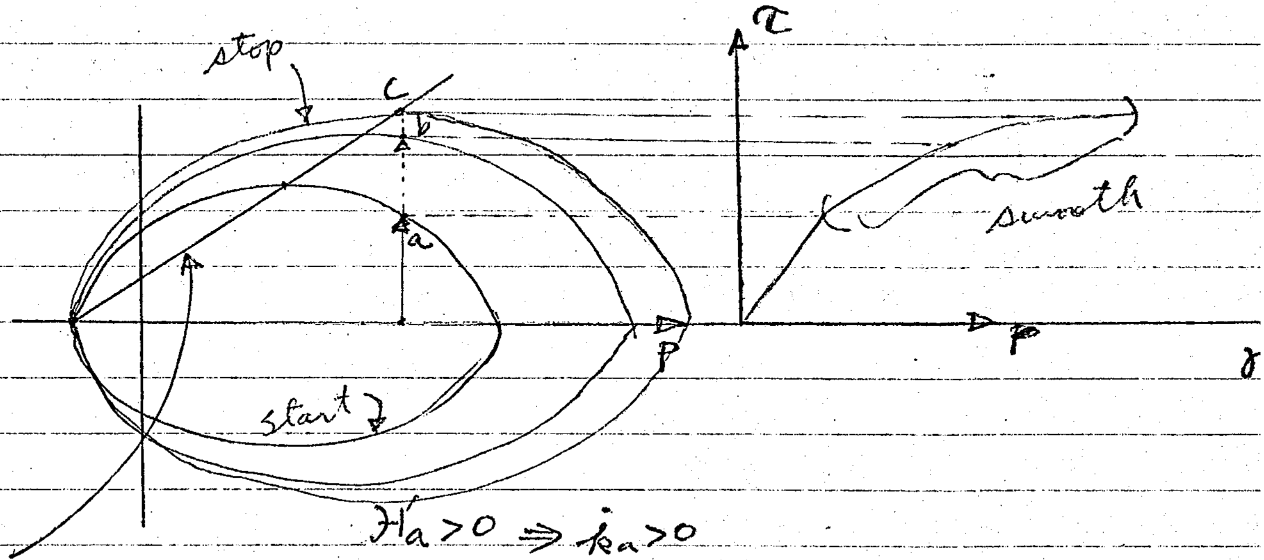
(iii)



$\dot{k} = 0$

$\dot{\alpha}_m = 0$ $\qquad \dot{\alpha}_m = -\frac{\sqrt{3}}{c}\dot{k}$

(iv)



$\mathcal{H}'_a > 0 \Rightarrow \dot{k}_a > 0$

$\dot{\alpha}_m|_a < 0$

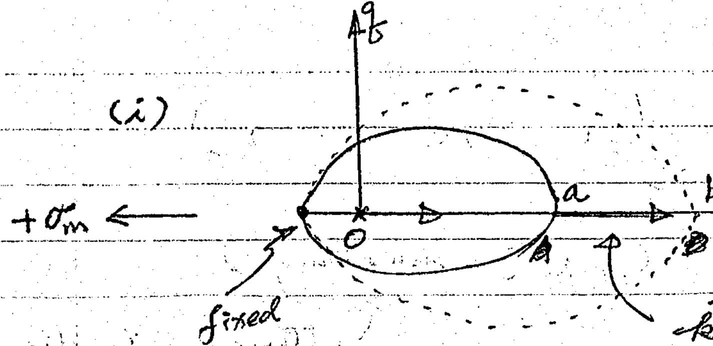
$\mathcal{H}' \equiv 0$
along crit.
state line

$0 = \mathcal{H}'_c < \mathcal{H}'_b < \mathcal{H}'_a$

$x = \frac{\ell}{k}$

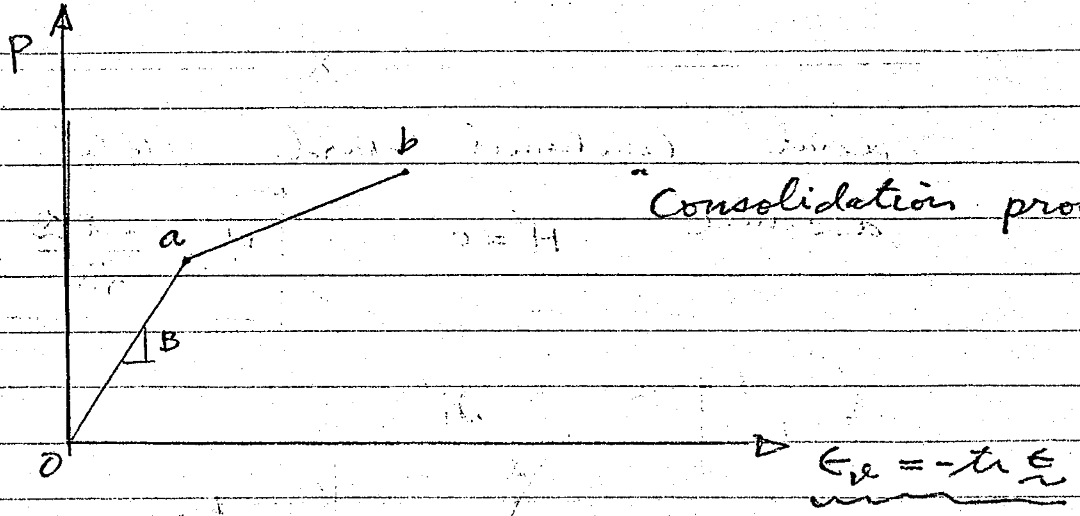
$$\dot{k} = \frac{x}{\sqrt{3}} \mathcal{H}' \dot{\epsilon}^{pl}$$

(i)



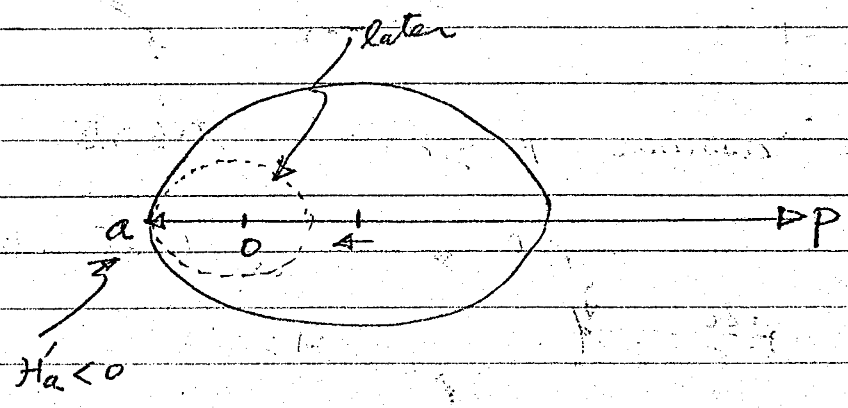
$\dot{k} > 0 \implies \mathcal{H}' > 0$

$\dot{\alpha}_m < 0 \implies$ center moves to right



"Consolidation process"

(ii)



(a fixed) $\dot{k} < 0$

$\dot{\alpha}_m > 0$ center moves to left

Continue until Y.S. shrinks to a point.

→ "The ~~po~~ party is over"
theory self-destructs.

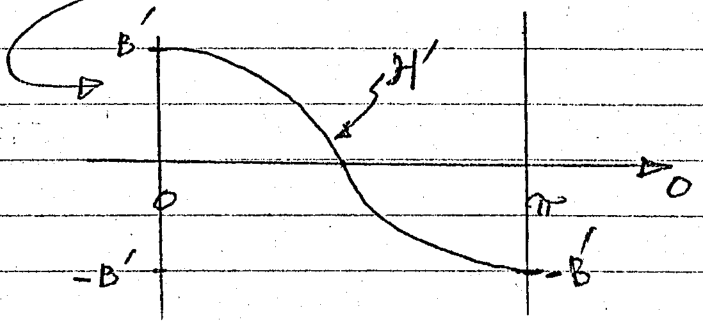
2. $\underset{\sim}{c}^{el-pl} = \left(\underset{\sim}{c} - \frac{1}{d} (\quad) \otimes (\quad) \right)$

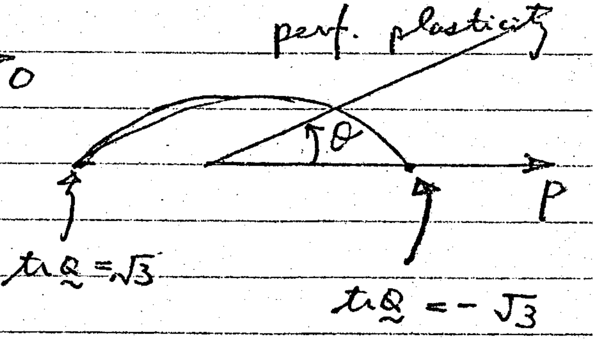
$(\underset{\sim}{Q} \cdot \underset{\sim}{c} \cdot \underset{\sim}{P} + \text{something})$

attempt to fix at $\frac{2}{3} H'$

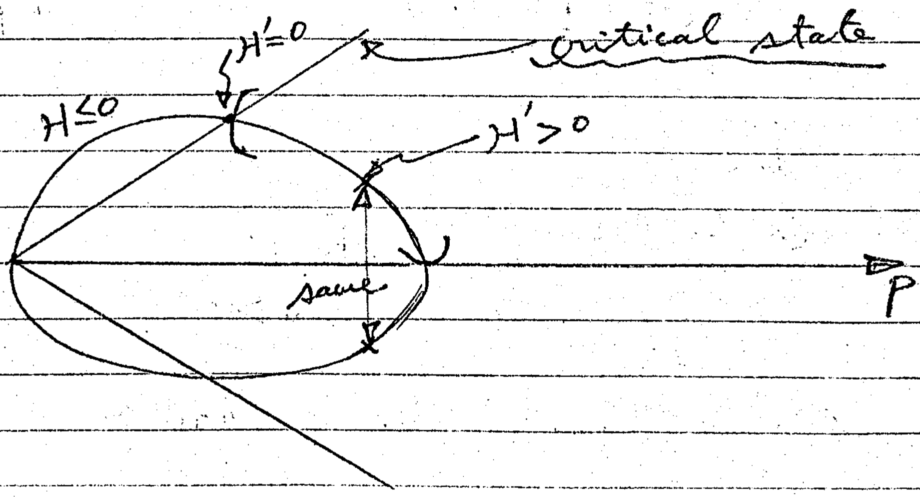
"Special" Combined hard. Model.

assume $H' = 0$ $\therefore$ $\boxed{H' = \frac{-\mathrm{tr}\underset{\sim}{Q}}{\sqrt{3}} B'}$ const.

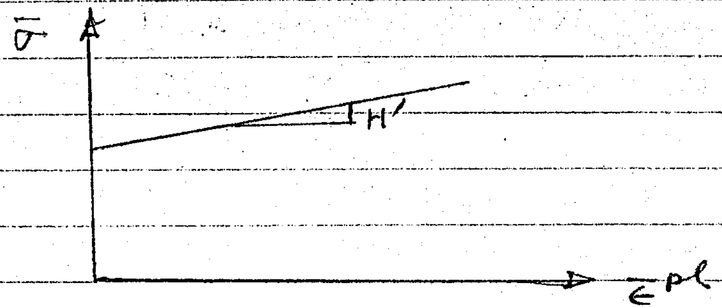




assume $B' > 0$



$$\underset{\sim}{Q} \parallel \underset{\sim}{\sigma}', \quad |\underset{\sim}{Q}| = 1 \qquad \therefore \quad \underset{\sim}{Q} = \frac{\underset{\sim}{\sigma}'}{|\underset{\sim}{\sigma}'|}$$



X

In a non-radial process:

$$\underbrace{\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}}_{} = \frac{\underset{\sim}{\xi}'}{R} \cdot \dot{\underset{\sim}{\sigma}} = \frac{(\underset{\sim}{\sigma}' - \underset{\sim}{\alpha}')}{R} \cdot \dot{\underset{\sim}{\sigma}}$$

consis. $\nearrow$

$$= \frac{1}{R}\left(\frac{d}{dt}\frac{1}{2}|\underset{\sim}{\sigma}'|^2 - \underset{\sim}{\alpha}' \cdot \dot{\underset{\sim}{\sigma}}\right)$$

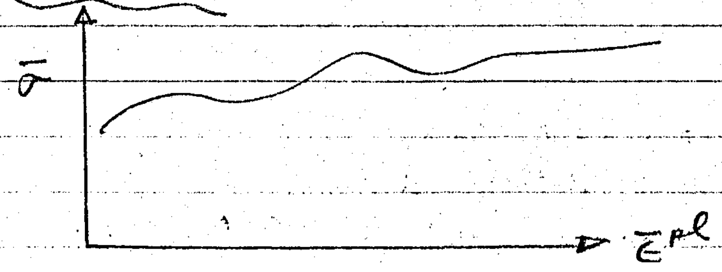
$= \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\alpha}}$

$= \sqrt{\frac{2}{3}}\, H' \dot{\bar{\epsilon}}^{pl}$

$$= \frac{1}{R}\frac{2}{3}\bar{\sigma}\dot{\bar{\sigma}} - \frac{1}{R}\underset{\sim}{\alpha}' \cdot \dot{\underset{\sim}{\sigma}}$$

$$\dot{\bar{\sigma}} = \frac{3}{2}\frac{R}{\underset{\uparrow \text{varying}}{\bar{\sigma}}}\left(\sqrt{\frac{2}{3}}\, H' \dot{\bar{\epsilon}}^{pl} + \overbrace{\frac{1}{R}\underset{\sim}{\alpha}' \cdot \dot{\underset{\sim}{\sigma}}}^{\text{extra}}\right)$$

A simple relation between $\dot{\bar{\sigma}}$ and $\dot{\bar{\epsilon}}^{pl}$ does not exist.



$$\tilde{R} = R\left(1 - \frac{l}{R}\frac{\sqrt{3}}{\sqrt{2}c}\, tr\,\underset{\sim}{Q}\right)$$

$$\cancel{\mathcal{H}'\tilde{R} = \tilde{\mathcal{H}}'}$$

$$\tilde{\mathcal{H}}' = \mathcal{H}'\left(1 - \frac{l}{R}\frac{\sqrt{3}}{\sqrt{2}c}\, tr\,\underset{\sim}{Q}\right)$$

proceed as in iso. w. $R \leftarrow \tilde{R}$

$$\longrightarrow \boxed{\Lambda = \frac{\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}}^{tr}}{\underbrace{\left(\underset{\sim}{Q}\cdot\underset{\sim}{c}\cdot\underset{\sim}{P} + \frac{2}{3}\frac{\cancel{R}}{\cancel{R}}\tilde{\mathcal{H}}'\right)}_{d}}}$$

Combined iso and special kin.

3/8 Revisits:

1. Kinematic hardening
Deviatoric etc "way back when"
Stated that we wanted

$$\dot{\bar{\sigma}} = H'\,\dot{\bar{\epsilon}}^{pl} \quad (\text{for iso. O.K.})$$

consistency

$$\overset{\Rightarrow \dot{R}}{\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}}} = \frac{\underset{\sim}{\sigma}'}{|\underset{\sim}{\sigma}'|}\cdot\dot{\underset{\sim}{\sigma}}' = \dots = \sqrt{\frac{2}{3}}\,\dot{\bar{\sigma}}$$

?

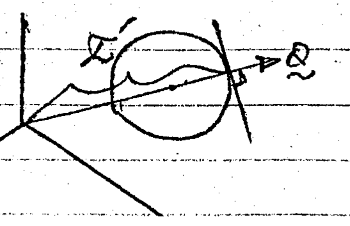
We ~~need~~ used in deriving rel's for kinematic hardening

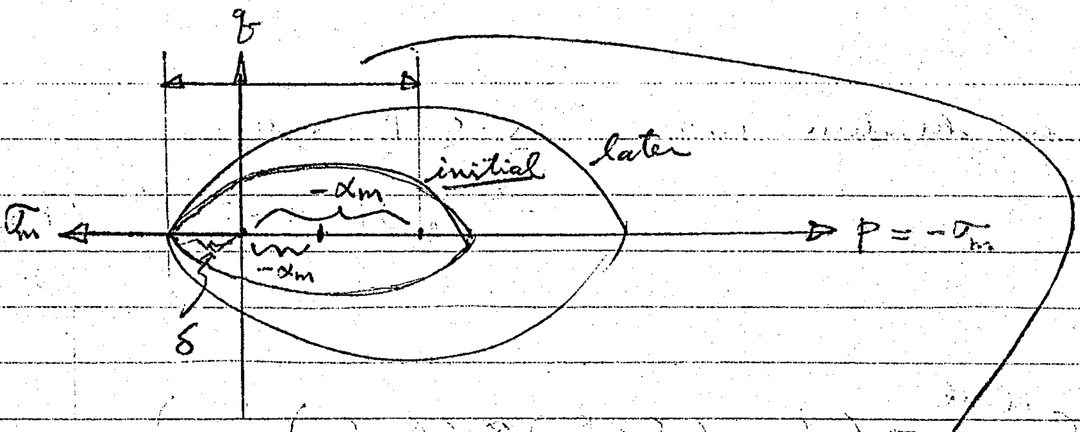
however, $\underset{\sim}{Q} = \dfrac{\underset{\sim}{\xi}'}{|\underset{\sim}{\xi}'|}$

When does $\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}} = \sqrt{\frac{2}{3}}\,\dot{\bar{\sigma}}$ for kin.?

Ans: when $\underbrace{\underset{\sim}{Q} \,//\, \underset{\sim}{\sigma}'}$

→ a process for which this condition always holds is called a radial process





$$(\delta - \alpha_m) = \frac{\sqrt{3}\,k}{c}$$

const.

$$\frac{d}{dt}: \quad -\dot{\alpha}_m = \frac{\sqrt{3}\,\dot{k}}{c}$$

$$\boxed{\dot{\alpha}_m = -\frac{\sqrt{3}}{c}\,\dot{k}}$$

$$\dot{k} > 0\,, \quad \dot{\alpha}_m < 0$$
$$< 0\,, \quad > 0$$

$$\underset{\sim}{\alpha} = \underset{\sim}{\alpha}' + \underset{\sim}{\alpha}''$$

$$\Downarrow \alpha_m \underset{\sim}{I}$$

$$? \quad \boxed{\dot{\underset{\sim}{\alpha}}' = \underset{\sim}{0}}$$

Consistency: Λ?

$$l\,\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\xi}} = 2k\dot{k}$$

$$\Downarrow$$

$$\dot{\underset{\sim}{\alpha}} - \dot{\underset{\sim}{\alpha}}$$

$$\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\alpha}} = \underset{\sim}{Q} \cdot (\dot{\alpha}_m \underset{\sim}{I})$$

$$\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\alpha}} = \frac{2k\dot{k}}{l} + \overbrace{\mathrm{tr}\,\underset{\sim}{Q}\,\dot{\alpha}_m} = \frac{R\dot{R}}{l} + \mathrm{tr}\,\underset{\sim}{Q}\,\dot{\alpha}_m = \tilde{H}'$$

$$= \frac{R}{l}\dot{R} + \frac{-\sqrt{3}}{c}\,\overset{\cancel{k}}{\dot{k}}\,\mathrm{tr}\,\underset{\sim}{Q} = \frac{R}{l}\overbrace{\left(1 - \frac{l}{R}\frac{\sqrt{3}}{\sqrt{2}c}\,\mathrm{tr}\,\underset{\sim}{Q}\right)}^{H'(x(\dots))}\dot{R}$$

$$\underset{\sim}{C}^{el-pl}\dot{\underset{\sim}{\epsilon}} = \underset{\sim}{c}(\dot{\underset{\sim}{\epsilon}} - \dot{\underset{\sim}{\epsilon}}^{pl})$$
$$= \underset{\sim}{C}\dot{\underset{\sim}{\epsilon}} - \underset{\sim}{C}\Lambda\underset{\sim}{P}$$
$$= C_{ijkl}\dot{\epsilon}_{kl} - \frac{Q_{ab}C_{abcd}\dot{\epsilon}_{cd}}{d} C_{ijkl}P_{kl}$$
$$= C_{ijkl}\dot{\epsilon}_{kl} - \frac{Q_{ab}C_{abkl}\dot{\epsilon}_{kl}}{d} C_{ijcd}P_{cd}$$

$$\Rightarrow \underset{\sim}{C}^{el-pl} = \underset{\sim}{C} - \frac{1}{d}(\underset{\sim}{C}\cdot\underset{\sim}{P})\otimes(\underset{\sim}{Q}\cdot$$

(d in deviatoric case : $\underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\underset{\sim}{P} = 2\mu$

$$\frac{2}{3}\frac{R}{\ell}\mathcal{H}' = \frac{2}{3}H' \quad (R=\ell))$$

$\mathcal{H}' = H$

Verify! $\underset{\sim}{C}^{el-pl} = (\underset{\sim}{C} - (\underset{\sim}{C}\cdot\underset{\sim}{P})\otimes(\underset{\sim}{Q}\cdot\underset{\sim}{C})\frac{1}{d})$

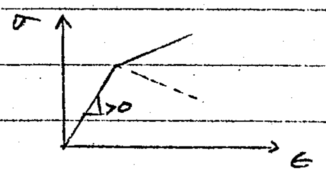
not symm. → non associativ

We allow $\mathcal{H}' < 0$

but to avoid pathologies, assume :

$$\left|\frac{2}{3}\frac{R}{\ell}\mathcal{H}'\right| < \left|\underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\underset{\sim}{P}\right|$$

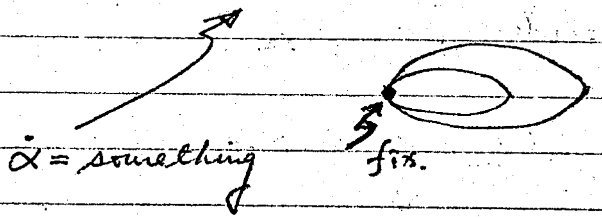
>0



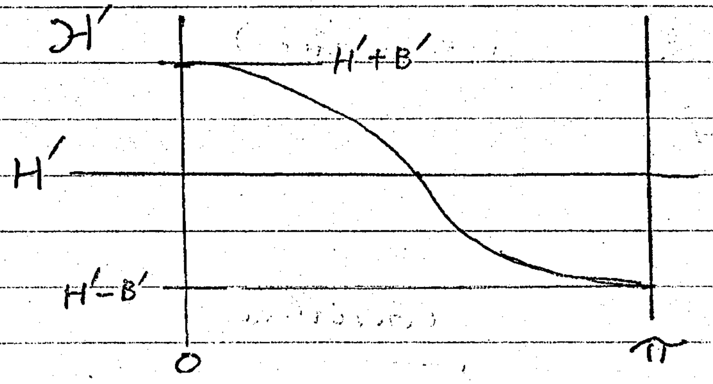
restricts P to lie near Q.

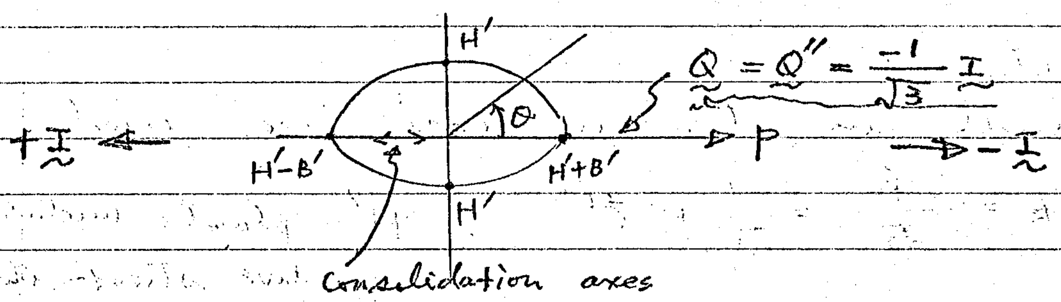
Special theory:

Combined iso + ~~Spec~~ Special kinematic :



$\text{tr}\,\underset{\sim}{Q} \sim \cos\theta$





$$\longrightarrow \mathcal{H}' = H' - \frac{\text{tr}\,\underset{\sim}{Q}}{\sqrt{3}} B'$$

assume const (may depend on $\underset{\sim}{\epsilon}^{pl}$???)

$$l\,\underset{\sim}{Q}\cdot\dot{\underset{\sim}{v}} = 2k\,\dot{k}$$

$$\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\alpha}}^{tr} - \Lambda\,\underbrace{\underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\underset{\sim}{P}}_{\text{" effective elastic modulus "}} = \frac{1}{l}R\dot{R} = \frac{1}{l}k\overset{h}{\sqrt{\tfrac{2}{3}}\mathcal{H}'}\overbrace{\sqrt{\tfrac{2}{3}}\Lambda}^{\dot{\epsilon}}$$

$$= \frac{2}{3}\frac{R}{l}\mathcal{H}'\Lambda\, \underset{\downarrow \frac{l}{R}}{h}$$

$$\longrightarrow \boxed{\Lambda = \frac{\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\epsilon}}^{tr}}{}} \Leftarrow \underset{\sim}{C}\cdot\dot{\underset{\sim}{\epsilon}}$$

Generalize theory to iso. hardening

$F(\underset{\sim}{\xi}) = k^2$ (k varies)

$\dot{F} = 2k\dot{k}$

$$\underbrace{\frac{\partial F}{\partial \underset{\sim}{\xi}}}_{l\underset{\sim}{Q}} \cdot \overset{\dot{\underset{\sim}{\xi}}}{\Downarrow}\ \dot{\underset{\sim}{\sigma}}$$

Consistency :

$$\boxed{l\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}} = 2k\dot{k}}$$

Aside:

recall in deviatoric case we used

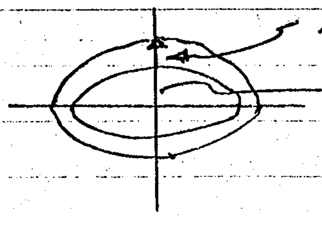
$\dot{k} = \frac{\dot{k}}{\sqrt{2}} = \frac{1}{\sqrt{3}} H' \dot{\bar{\epsilon}}^{pl}$; H' plastic modulus of effective stress-strain curve

$\dot{\bar{\epsilon}}^{pl} = \sqrt{\frac{2}{3}} \left(\dot{\underset{\sim}{\epsilon}}^{pl} \cdot \dot{\underset{\sim}{\epsilon}}^{pl} \right)^{1/2}$ (not deviatoric now!)

↑ deviatoric before

Our new pl. mod $= \underset{\sim}{\mathcal{H}'}$

~ when strains are deviatoric,



$\gamma = \frac{l}{R}$

→ def. $\boxed{\dot{k} = \frac{\gamma}{\sqrt{3}} \mathcal{H}' \dot{\bar{\epsilon}}^{pl}}$

→ e.g. $= H' - \frac{\mathrm{tr}\,\underset{\sim}{Q}}{\sqrt{3}} B'$

$$\left|\frac{\partial F}{\partial \underset{\sim}{\sigma}}\right| \boxed{\underset{\sim}{Q}\cdot\dot{\underset{\sim}{\sigma}} = 0}$$

$$\underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\dot{\underset{\sim}{\epsilon}} = \underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\underset{\sim}{P}\,\Lambda$$

$$\longrightarrow \boxed{\Lambda = \frac{\underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\dot{\underset{\sim}{\epsilon}}}{\underset{\sim}{Q}\cdot\underset{\sim}{C}\cdot\underset{\sim}{P}}}$$

×

Calculate $\underset{\sim}{Q}$: ~~(perfect~~

$$\frac{\partial F}{\partial \underset{\sim}{\sigma}} = \frac{\partial f}{\partial \underset{\sim}{\sigma}} + \frac{2c^2}{3}\xi_m\frac{\partial \sigma_m}{\partial \underset{\sim}{\sigma}}$$

$$= \underset{\sim}{\sigma}' + \frac{2c^2}{3}\xi_m\frac{1}{3}\underset{\sim}{I}$$

$$\left(\sigma_m = \frac{1}{3}\operatorname{tr}\underset{\sim}{\sigma} = \frac{1}{3}\sigma_{kk}\right)$$

$$\underset{\sim}{Q} = \frac{\frac{\partial F}{\partial \underset{\sim}{\sigma}}}{l}$$

$$l = \left|\frac{\partial F}{\partial \underset{\sim}{\sigma}}\right| = \left(|\underset{\sim}{\sigma}'|^2 + \left(\frac{2c^2}{9}\xi_m\right)^2\cdot 3\right)^{1/2}$$

Kinematic hardening

Kin. $\longrightarrow$ $F = F(\underset{\sim}{\xi}) = \frac{c^2\xi_m^2}{3} + f(\underset{\sim}{\xi})$

$$F(\underset{\sim}{\xi}) = f(\underset{\sim}{\xi}) + \frac{c^2}{3}(\sigma_m - \alpha_m)$$

$$\frac{\partial F}{\partial \underset{\sim}{\xi}} = \frac{\partial f}{\partial \underset{\sim}{\xi}} + \frac{2c^2}{3}\xi_m\cdot\frac{1}{3}\underset{\sim}{I}$$

$$= \underset{\sim}{\xi}' + \frac{2}{9}c^2\xi_m\underset{\sim}{I}$$

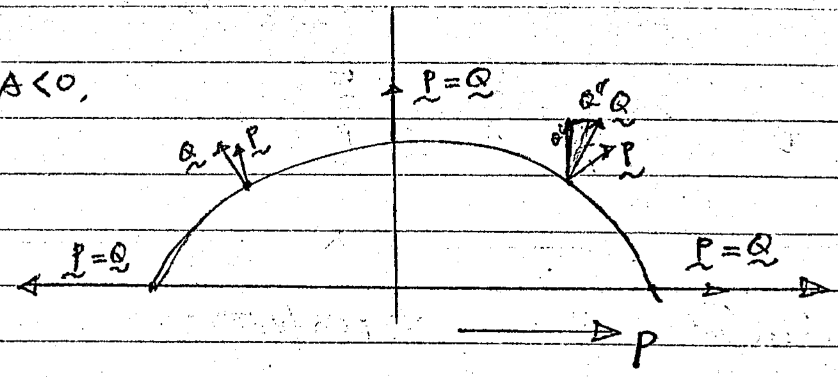
$$l = \left(|\underset{\sim}{\xi}'|^2 + \left(\frac{2c^2}{9}\xi_m\right)^2\cdot 3\right)^{1/2}$$

×

Non-associate flow rule $\underset{\sim}{P} \neq \underset{\sim}{Q}$ in general

(e.g. $\underset{\sim}{P} \parallel \underset{\sim}{Q}' + (1+A)\underset{\sim}{Q}''$)

$A < 0$,



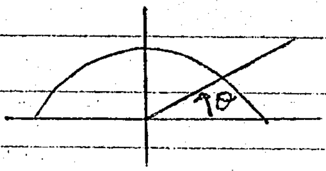
×

Plastic modulus varies on Y.S.

vary with θ

$\sim \cos\theta + \text{const.}$

more or less



×

Theory: Perfect-plasticity first

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C} \cdot (\dot{\underset{\sim}{\epsilon}} - \dot{\underset{\sim}{\epsilon}}^{pl})$$

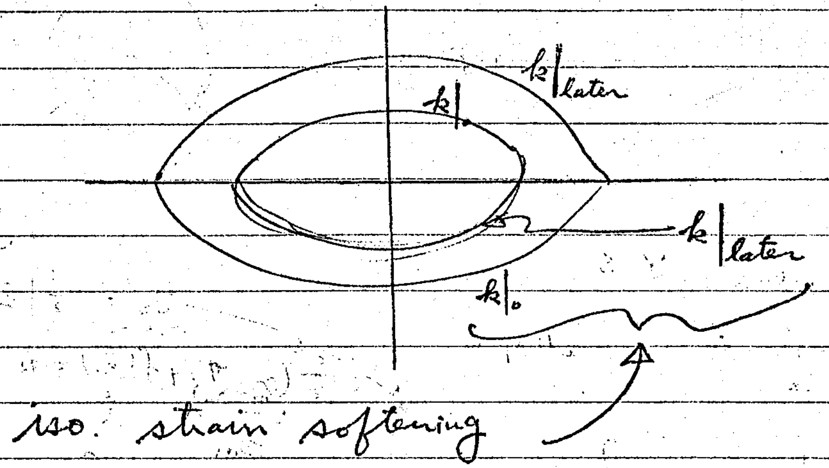
$$\dot{\underset{\sim}{\epsilon}}^{pl} = \begin{cases} 0 & (E) \\ \dot{\Lambda}\underset{\sim}{P} & (p) \end{cases}$$ as usual in terms of F, $\underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}}^{tr}$

Consistency: ~~$\dot{F}=0$~~ $\dot{F}(\underset{\sim}{\sigma}, \alpha_m) = 0$

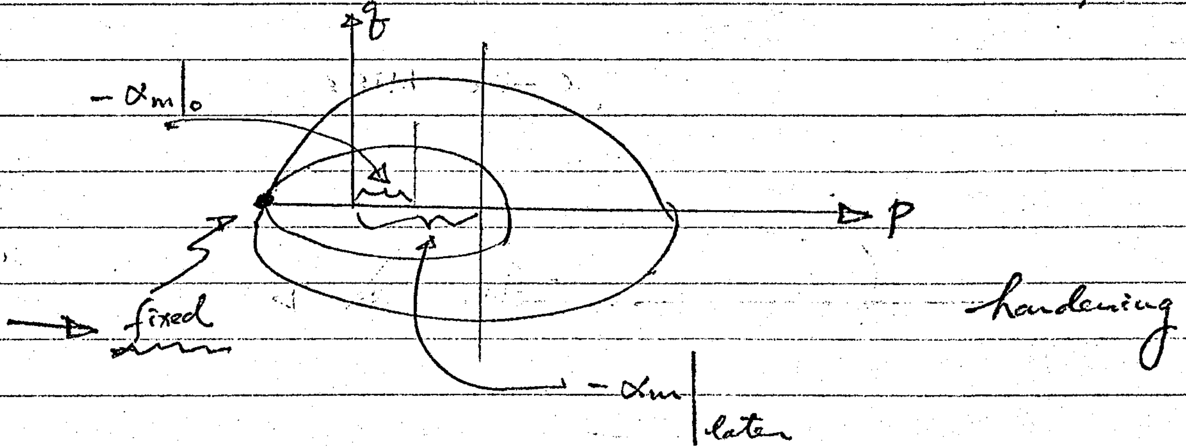
const.

$$\frac{\partial F}{\partial \underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\sigma}} = 0$$

Iso. ~~kin~~ hardening:



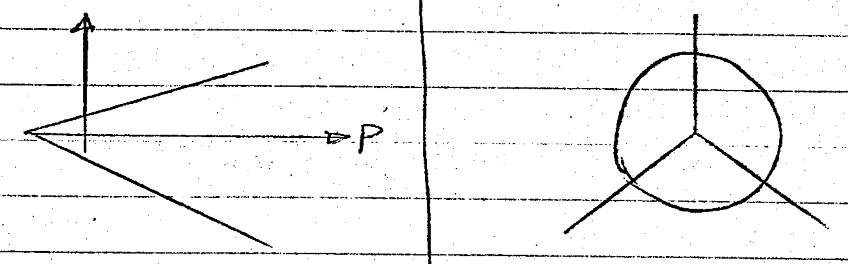
Special combination of isotropic kin hardening/(softeni



Softening : 0 ⟷ later

$$\frac{c^2}{3}(\sigma_m - \alpha_m)^2 + \underbrace{f(\tilde{\sigma})}_{J_2' = \frac{1}{2}|\tilde{\sigma}'|^2} = k^2$$

3/3 Drucker - Prager Y. S.

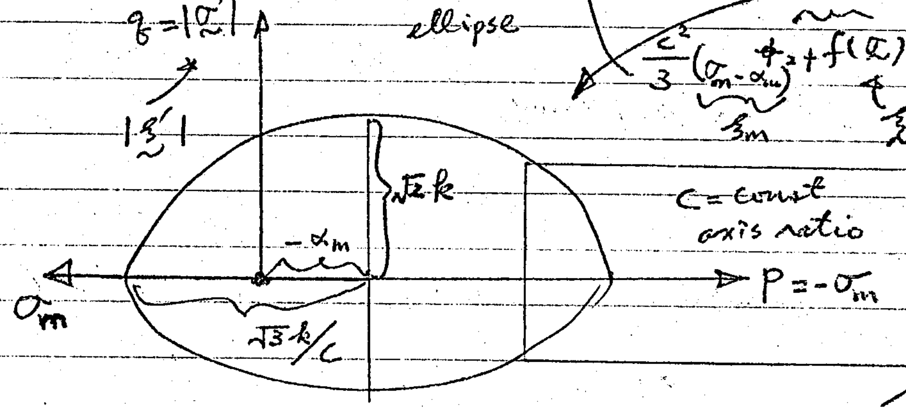


Ellipsoidal Y. S.

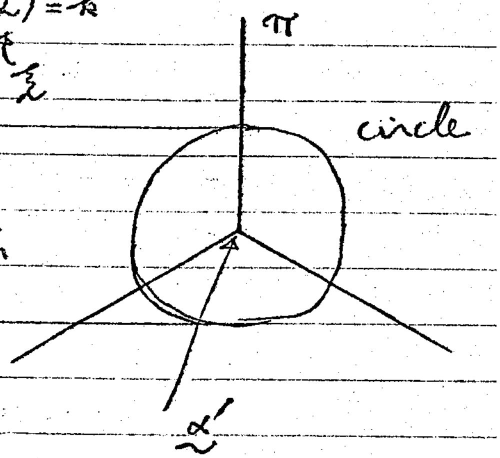
$$\overset{q}{\underbrace{J_2'}} = \frac{1}{2}|\tilde{\sigma}'|^2$$

$$\frac{c^2}{3}\underbrace{(\sigma_m - \alpha_m)}_{\xi_m}{}^2 + f(\tilde{\sigma}) = k^2$$

$\tilde{\xi}$

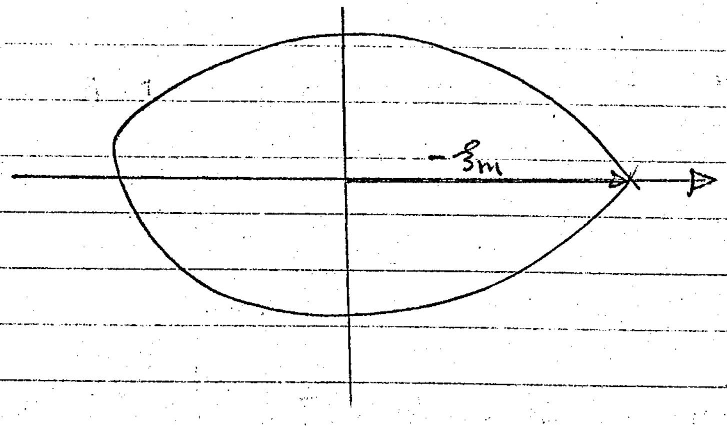


c = const
axis ratio



if ellipse is moving in $\tilde{\sigma}$-space.

c → 0 Mises

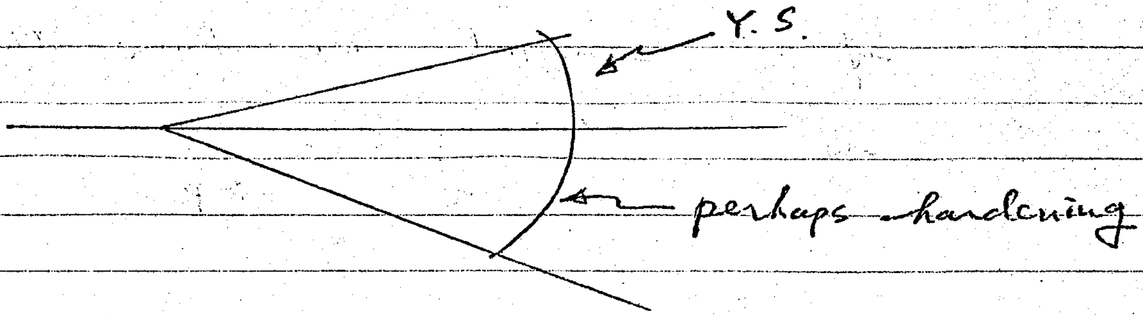


$$\boxed{F(\tilde{\xi}) = \frac{c^2}{3}(\sigma_m - \alpha_m)^2 + f(\tilde{\sigma})}$$

we can allow isotropic hardening, ~~kine~~ kin., combination.

k changes. c = const

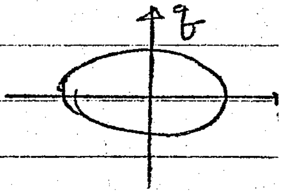
Cap Model   D.P. + "Cap"



soil-structure blast phenomena
earthquake weidlinger assoc.

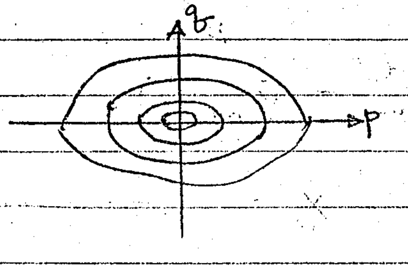
Other important surface is an ellipse
p-q space.

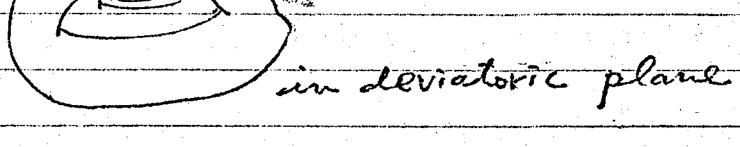
* Critical state theory — cam-clay



→ * D.P. ∩ ellipse = cap

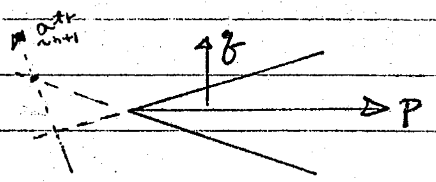
* Muliple Y.S. ellipses





i.e. $$\boxed{P_{n+1} = -\frac{1}{3}\, tr\, \underset{\sim}{\sigma}_{n+1} \geq \frac{-k}{3\alpha}}$$

if not satisfied, we know the picture is



→ In this case define $\underset{\sim}{\sigma}_{n+1} = \frac{k}{3\alpha}\underset{\sim}{I}$ Apex.

$\underset{\sim}{I}$ ← a unit vector in positive σ_m direc

$$\Delta \underset{\sim}{\epsilon}^{pl} = \underset{\uparrow\,?}{\tilde{\Lambda}}\, \underset{\sim}{P}$$

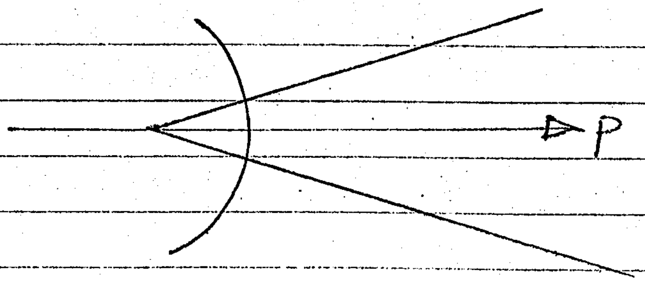
$\underset{\sim}{P}$ ← but $\underset{\sim}{P}$ in this case must be defined differently

$$\underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - \tilde{\Lambda}\, \cancel{P}\, \underset{\sim}{C}\cdot\underset{\sim}{P}$$

$$\left(\frac{k}{3\alpha}\right)\underset{\sim}{I}$$

$$\therefore\ \tilde{\Lambda}\, \underset{\sim}{P} = \underset{\sim}{C}^{-1}\cdot\left(\underset{\sim}{\sigma}^{tr}_{n+1} - \underset{\sim}{\sigma}_{n+1}\right)$$

×



What really happens when ~~p~~ P ≫ "P_{yield}" ?

plasticity happens!

$$\left\{\begin{array}{l} \underset{\sim}{\sigma}_{n+1} = \underset{\sim}{\sigma}^{tr}_{n+1} - \tilde{\Lambda}\, \underset{\sim}{C}\cdot\underset{\sim}{P} \\ \underset{\sim}{\sigma}'_{n+1} = \underset{\sim}{\sigma}^{tr\,\prime}_{n+1} - \tilde{\Lambda}(\underset{\sim}{C}\cdot\underset{\sim}{P})' \end{array}\right.$$

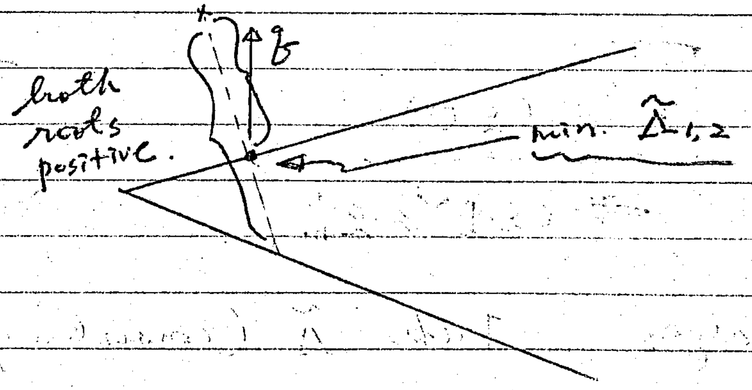
$$\rightarrow \left|\underset{\sim}{\sigma}'_{n+1}\right|^2 = 2\tilde{k}^2_{n+1} = 2\left(k - \alpha\, \mathrm{tr}\, \underset{\sim}{\sigma}_{n+1}\right)^2$$

$\longrightarrow$ Quadratic eq. for $\tilde{\Lambda}$:

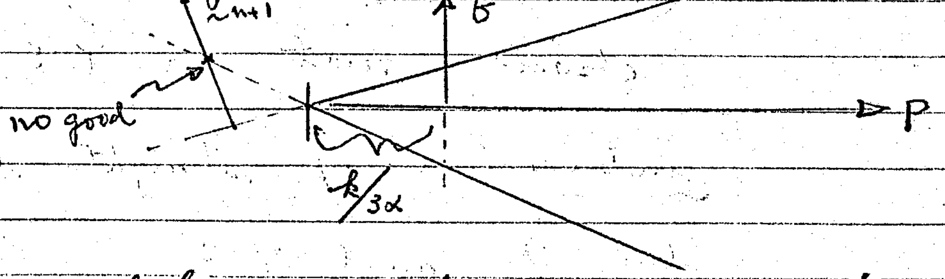
$$a\tilde{\Lambda}^2 + b\tilde{\Lambda} + c = 0$$

$$\left\{\begin{array}{l} a = \left|(\underset{\sim}{C}\cdot\underset{\sim}{P})'\right|^2 - 2\left(\mathrm{tr}\,(\underset{\sim}{C}\cdot\underset{\sim}{P})\right)^2\alpha^2 \\ b = -2\left[(\underset{\sim}{C}\cdot\underset{\sim}{P})'\cdot\underset{\sim}{\sigma}^{tr\,\prime}_{n+1} - 2\left(k - \alpha\,\mathrm{tr}\,\underset{\sim}{\sigma}^{tr}_{n+1}\right)(\mathrm{tr}\,\underset{\sim}{C}\underset{\sim}{P})\alpha\right] \\ c = \left|\underset{\sim}{\sigma}^{tr\,\prime}_{n+1}\right|^2 - 2\left(k - \alpha\,\mathrm{tr}\,\underset{\sim}{\sigma}^{tr}_{n+1}\right)^2 \end{array}\right.$$

$$\tilde{\Lambda}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



suppose :



To preclude : check mean component of $\underset{\sim}{\sigma}_{n+1}$ is to the right of $k/3$