ME 238 C

3-D Problems in Elasticity

most problems are defined in displacements.

Most 3-D problems require more mathematics and that is what we will do.

Review: Green's theorems

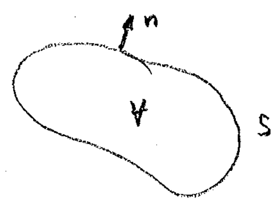
$$\int_V \nabla\cdot\underline{A}\, dV = \int_S \underline{A}\cdot\underline{n}\, ds \qquad \text{Let: } \underline{A} = g\nabla h \qquad \begin{array}{l} g = g(x,y,z) \\ h = h(x,y,z) \end{array}$$

$$\nabla\cdot(g\nabla h) = \nabla g\cdot\nabla h + g\nabla^2 h$$

$$\underline{A}\cdot\underline{n} = g\,\underline{n}\cdot\nabla h = g\frac{\partial h}{\partial n}$$

remember $\frac{\partial\varphi}{\partial l}$ (deriv of φ along any line whose unit vector is $\underline{e}$) $= \nabla\varphi\cdot\underline{e}$ where $\underline{e} = \frac{\underline{l}}{|\underline{l}|}$

$\underline{A}$ must be C_1 in V & on S.



$\therefore$ Green's first identity

$$\int_V \left(g\nabla^2 h + \nabla g\cdot\nabla h\right) dV = \int_S g\frac{\partial h}{\partial n}\, ds \qquad (1)$$

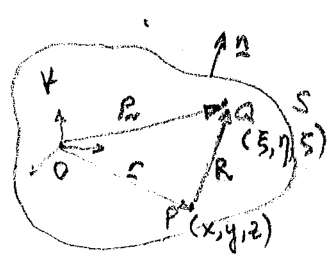
if we let $\underline{A} = h\nabla g$ then $\int_V \left(h\nabla^2 g + \nabla h\cdot\nabla g\right) dV = \int_S h\frac{\partial g}{\partial n}\, ds \qquad (2)$

Green's 2nd indentity : (1) − (2) or Green's theorem.

$$\int_V \left(g\nabla^2 h - h\nabla^2 g\right) dV = \int_S \left(g\frac{\partial h}{\partial n} - h\frac{\partial g}{\partial n}\right) ds$$

g, h must be C^2 everywhere in V singlevalued & continuous w/ their first partial deriv on boundary.

$$g(P) = \frac{1}{4\pi}\left[\int_S \left\{\frac{1}{R(P,Q)}\frac{\partial g}{\partial n} - g\frac{\partial}{\partial n}\left(\frac{1}{R(P,Q)}\right)\right\} ds - \int_V \frac{1}{R(P,Q)}\nabla^2 g\, dV\right]$$



$\rho^2 = \xi^2 + \eta^2 + \zeta^2 = \rho(\xi,\eta,\zeta)$ where $Q \in V$ or S, $P \in V$ only

$r^2 = x^2 + y^2 + z^2 = r(x,y,z)$

$R^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta^2)$ $\qquad R = R(P,Q) = R(x,y,z,\xi,\eta,\zeta)$

choose $h = \frac{1}{R}$ in green's 2^{nd} identity where $h(Q) = \frac{1}{R(P,Q)}$ where P is fixed

using green's 2^{nd} ident

$$\int_V \left(g\, \nabla^2 \frac{1}{R} - \frac{1}{R}\nabla^2 g\right) dV = \int_S \left(g\, \frac{\partial(\frac{1}{R})}{\partial n} - \frac{1}{R}\frac{\partial g}{\partial n}\right) dS$$

to find $\nabla^2(\frac{1}{R})$: look at $\nabla^2(R^m)$ and this requires $\nabla(R^m)$

$$\nabla(R^m) = \frac{d}{dR}(R^m) \cdot \nabla R = mR^{m-1}\frac{\underline{R}}{|\underline{R}|} = mR^{m-1}\frac{\underline{R}}{R} = mR^{m-2}\underline{R}$$ where $\underline{R}$ is a unitvector in $\underline{R}$ direction

$$\nabla \cdot \nabla(R^m) = \nabla \cdot (mR^{m-2}\underline{R}) = mR^{m-2}\nabla \cdot \underline{R} + m(m-2)R^{m-4}\frac{\underline{R}\cdot\underline{R}}{R^2}$$ using the above

now $\nabla \cdot \underline{R} = 3$ since $\underline{R} = (x-\xi, y-\eta, z-\zeta)$

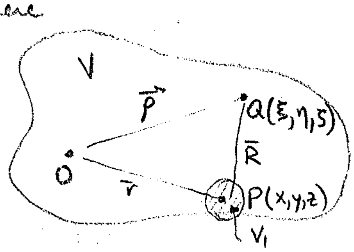
$\therefore\ \nabla^2(R^m) = R^{m-2}(m+1)m$ when $m=-1$ $\nabla^2(\frac{1}{R}) = 0$ w/ $R \neq 0$

4/6/79

Quick recap

$$g(P) = \frac{1}{4\pi}\left[\int_S \left(\frac{1}{R(P,Q)}\frac{\partial g}{\partial n} - g\frac{\partial}{\partial n}\left(\frac{1}{R(P,Q)}\right)\right)ds - \int_V \frac{1}{R(P,Q)}\nabla^2 g\, dV\right]$$

where



we also notice that $\int_V \left[g\, \nabla^2(\frac{1}{R}) - \frac{1}{R}\nabla^2 g\right] dV = \int_S \left(g\, \frac{\partial(\frac{1}{R})}{\partial n} - \frac{1}{R}\frac{\partial g}{\partial n}\right) ds$

we also showed that $\nabla^2(\frac{1}{R}) = 0$ for $R \neq 0$; ie we can draw a sphere surface S_1 volume V_1 around point P and define $V - V_1$ where $V_1 = \frac{4\pi\varepsilon^3}{3}$

then $\int_{V-V_1} g\, \nabla^2(\frac{1}{R})\, dV = 0$ now $\int_{V_1} \nabla^2(\frac{1}{R})\, dV = \int_{V_1} \nabla \cdot (\nabla \frac{1}{R})\, dV = \int_{S_1} \nabla \frac{1}{R} \cdot \underline{n}\, dS$

where $\underline{n}$ is normal to surface S_1 : but $\underline{n} \cdot \nabla = \frac{\partial}{\partial n} = \frac{\partial}{\partial r}$ on S_1

thus $\int_{S_1} \frac{\partial}{\partial n}\left(\frac{1}{R}\right) ds = \int_{S_1} \frac{\partial}{\partial r}\left(\frac{1}{R}\right) ds = -\int_{S_1} \frac{1}{R^2}\, ds = -\int_{S_1} \frac{1}{R^2} \cdot R^2 d\Omega = -4\pi$

where Ω is the solid angle $\therefore \int_{S_1} \frac{\partial}{\partial n}\left(\frac{1}{R}\right) ds = -4\pi$

$\therefore$ in general for any V $\quad \int_V \nabla^2\left(\frac{1}{R}\right) dV = \begin{cases} 0 & \{R=0\} \notin V \\ -4\pi & \{R=0\} \in V \end{cases}$

$$\therefore \quad \int_V = \int_{V-V_1} + \int_{V_1} = -4\pi$$

thus we can then say that $\nabla^2\left(\frac{1}{R}\right) = -4\pi\delta(R)$ where $\delta(R) = \begin{cases} 0 & R \neq 0 \\ 1 & R = 0 \end{cases}$

or $\nabla^2\left(\frac{1}{|\underline{x}-\underline{x}'|}\right) = -4\pi\delta(\underline{x}-\underline{x}')$ and $\int_V g(\underline{x})\,\delta(\underline{x}-\underline{a})\,dV = g(\underline{a})$

$$\therefore \int_{V_1} g\,\nabla^2\left(\tfrac{1}{R}\right) dV = -4\pi\, g(P) \int_{V_1} \delta(R)\,dV = -4\pi g(P)$$ now take $V_1 \to 0 \Rightarrow V - V_1 \to V$

another interesting property of the dirac delta fn is $\int_V g(\underline{x})\,\delta'(\underline{x}-\underline{a})\,dV = -g'(\underline{a})$

$$\therefore \int_V \left[g\,\nabla^2\left(\tfrac{1}{R}\right) - \tfrac{1}{R}\nabla^2 g\right] dV = \int_S \left(g\,\frac{\partial\left(\frac{1}{R}\right)}{\partial n} - \frac{1}{R}\frac{\partial g}{\partial n}\right) dS$$

$$-4\pi\, g(P) - \int_V \tfrac{1}{R}\nabla^2 g\, dV = \int_S (\qquad\qquad) \,dS$$

or $$g(P) = \frac{1}{4\pi}\int\left[\frac{1}{R}\frac{\partial g}{\partial n} - g\left(\frac{\partial\left(\frac{1}{R}\right)}{\partial n}\right)\right] dS - \frac{1}{4\pi}\int_V \frac{1}{R}\nabla^2 g\, dV \qquad *$$

Dirac Delta fns are generalized functions

Now if g and $\frac{\partial g}{\partial n} = 0$ on the surface the first integral $= 0$

$$\therefore\ g(P) = -\frac{1}{4\pi}\int_V \frac{1}{R(P,Q)}\nabla^2 g\, dV$$

in most cases: $g \to 0$ and $\frac{\partial g}{\partial n} \to 0$ as $R \to \infty$ and suppose $\nabla^2 g = f$ in V then

$$g(P) = -\frac{1}{4\pi}\int_V \frac{f}{R(P,Q)}\, dV$$

Are these solutions unique: look at b.c. and (*)

Boundary cond. $\quad g|_S = g_0 \quad$ Dirichlet

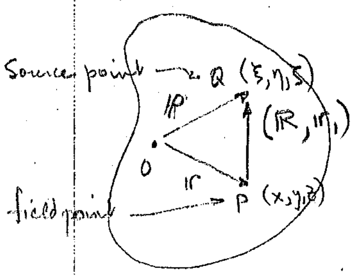
$\qquad\qquad\qquad \partial g/\partial n|_S = f_0 \quad$ Neumann

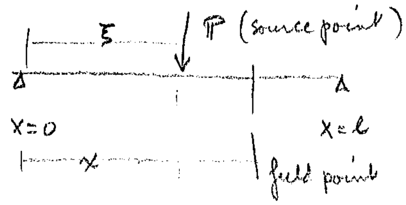
4/19/79

Review

Green's theorem: $\int_S \left(g\frac{\partial h}{\partial n} - h\frac{\partial g}{\partial n}\right)ds = \int \left(g\nabla^2 h - h\nabla^2 g\right)dV$ (*)

Green's formula $g(P) = \frac{1}{4\pi}\left[\int_S \left\{ \frac{1}{R(P,Q)}\frac{\partial g(Q)}{\partial n} - g(Q)\frac{\partial}{\partial n}\frac{1}{R(P,Q)}\right\}ds - \int_V \frac{1}{R(P,Q)}\nabla^2 g(Q)\,dV\right]$





where you measure defl due to load at ℙ

Poisson's Equation is the general problem we need to solve $\nabla^2 g = f$

BC: a.) $g = g_0$ on S (Dirichlet Condition)
b) $\frac{\partial g}{\partial n} = r_0$ on S (Neumann condition)

Prescribe one but not both then get unique soln to Poisson Eq (b) is to with arbitrary const.

Proof of Uniqueness

Assume g_1, g_2 satisfy DE & BC then $g_1 - g_2 = u$ is also a soln

$\therefore$ $\nabla^2 u = 0$ and $u\big|_S = 0$ or $\frac{\partial u}{\partial n}\big|_S = 0$

Remember $\int_V h\nabla^2 g\,dV = -\int \nabla g\cdot\nabla h\,dV + \int_S h\frac{\partial g}{\partial n}ds$ if $u = h = g$ then $\Rightarrow \int_V |\nabla u|^2 dV = 0$

or $\nabla u = 0$, u = constant but since $u\big|_S = 0 \Rightarrow u \equiv 0$ everywhere $\therefore g_1 = g_2$

better way for 1st bc: is since u must attain its max & min on bdy $\Rightarrow u \equiv 0$ everywhere $\therefore g_1 = g_2$

(*) if $g \to 0$ & $\partial g/\partial n \to 0$ as $p \to \infty$ $\Rightarrow$ $\boxed{g(P) = -\frac{1}{4\pi}\int_V \frac{\nabla^2 g}{R}dV}$

HELMHOLTZ RESOLUTION

suppose we define $\mathbb{G} = \underbrace{\nabla\phi}_{\text{lamellar}} + \underbrace{\nabla\times\mathbb{H}}_{\text{solenoidal}}$ where ϕ is scalar & $\mathbb{H}$ is vector point fn

also pick $(\nabla\cdot\mathbb{H} = 0)$ to remove the indeterminacy since ϕ, H_1, H_2, H_3 are 4 scalar fns (of $\mathbb{H}$) but G_1, G_2, G_3 are 3 fns (of $\mathbb{G}$)

to determine ϕ, $\mathbb{H}$

take $\nabla\cdot G = \nabla^2\phi + \nabla\cdot\nabla\times\mathbb{H} = \nabla^2\phi$ since we can rearrange $\therefore$ $\underbrace{\nabla\cdot\mathbb{H}\times\nabla}_{=0}$

we can thus use in $g(P) = -\frac{1}{4\pi}\int \frac{\nabla^2 g}{R}\, dV$ ∴ if $g = \phi \Rightarrow \nabla^2 g = \nabla^2\phi = \nabla\cdot G$

$$\boxed{\phi = -\frac{1}{4\pi}\int_V \frac{\nabla\cdot G}{R}\, dV \quad \text{if } G \text{ vanishes at } \infty}$$

take $\nabla\times G = \underbrace{\nabla\times\nabla\phi}_{=0} + \nabla\times\nabla\times H = \nabla\nabla\cdot H - \nabla^2 H = -\nabla^2 H$ since $\nabla\cdot H = 0$

∴ use again in the above formula to get

$$\boxed{H = \frac{1}{4\pi}\int \frac{\nabla\times G}{R}\, dV \quad \text{if } G \text{ vanishes at } \infty}$$

rewrite to get

$$G = \nabla\phi + \nabla\times H = -\frac{1}{4\pi}\left[\nabla\int_V \frac{\nabla\cdot G}{R}\, dV - \nabla\times\int_V \frac{\nabla\times G}{R}\, dV\right]$$

where $G = G(x,y,z)$
$R = R(x,y,z,\xi,\eta,\zeta)$
$\nabla = \nabla(x,y,z)$
$dV = dV(\xi,\eta,\zeta)$

Remember that the displacement equations of equilib for an isotropic, homog, linear elastic body is

$$(\lambda+\mu)\nabla\nabla\cdot u + \mu\nabla^2 u + f = 0$$

or

$$\nabla^2 u + \left(\frac{\lambda+\mu}{\mu}\right)\nabla\nabla\cdot u + f/\mu = 0$$

recall also $\nu = \frac{\lambda}{2(\lambda+\mu)}$ $\quad 1-2\nu = \mu/_{\lambda+\mu}$ where μ is the shear modulus

So $\nabla^2 u + \frac{1}{(1-2\nu)}\nabla\nabla\cdot u + f/\mu = 0$

Introduce displacements for 3-D elasticity to solve the above. This was done by Boussinesq, Papcovitch, Neuber (1930–1932)

thus

position vector of field point P

$$\boxed{u = B - \frac{\nabla}{4(1-\nu)}\left[r\cdot B + \beta\right] \quad \text{where } \nabla^2 B = -f/\mu \text{ and } \nabla^2\beta = \frac{r\cdot f}{\mu}}$$

position vector of field point where f exists

To prove their validity

Let $\mathbb{u} = \nabla\phi + \nabla\times\mathbb{H}$ w/ $\nabla\cdot\mathbb{H} = 0$

$$\therefore \nabla^2\mathbb{u} + (1-2\nu)\nabla\nabla\cdot\mathbb{u} + \frac{f}{\mu} = \nabla^2(\nabla\phi + \nabla\times\mathbb{H}) + \frac{1}{(1-2\nu)}\Big[\nabla\nabla\cdot\nabla\phi + \underbrace{\nabla\nabla\cdot\nabla\times\mathbb{H}}_{=0}\Big] + \frac{f}{\mu} = 0$$

$$\nabla^2(\nabla\phi + \nabla\times\mathbb{H}) + \frac{1}{1-2\nu}(\nabla\nabla^2\phi) + \frac{f}{\mu} = \nabla^2(\alpha\nabla\phi + \nabla\times\mathbb{H}) = -\frac{f}{\mu} = \underline{\nabla^2\mathbb{B}}$$

where $\alpha = 1 + \frac{1}{1-2\nu} = \frac{2(1-\nu)}{1-2\nu}$ and $\mathbb{B} = \alpha\nabla\phi + \nabla\times\mathbb{H}$

now define β s.t. $\phi = \frac{1}{2\alpha}(\mathbb{r}\cdot\mathbb{B} + \beta) = \frac{1}{2\alpha}(x_jB_j + \beta)$

now $2\alpha\nabla\phi = e_i(x_jB_j+\beta)_{,i} = e_i(x_{j,i}B_j + x_jB_{j,i} + \beta_{,i}) = e_i(\delta_{ji}B_j + x_jB_{j,i} + \beta_{,i})$

$$= e_i(B_i + x_jB_{j,i} + \beta_{,i})$$

$$2\alpha\nabla\cdot\nabla\phi = e_i\cdot e_i(B_i + x_jB_{j,i} + \beta_{,i})_{,i} = (B_{i,i} + B_{i,i} + x_jB_{j,ii} + \beta_{,ii})$$
$$= 2\nabla\cdot\mathbb{B} + \mathbb{r}\cdot\nabla^2\mathbb{B} + \nabla^2\beta$$

Review 4/11/79

Green's formula says - value of fn for interior point depends on ∇^2fn in the volume and fn and $\frac{\partial}{\partial n}$fn on the boundary.

we are working to show

$$\mathbb{u} = \mathbb{B} - \frac{\nabla}{4(1-\nu)}[\mathbb{r}\cdot\mathbb{B} + \beta] \quad \text{where} \quad \nabla^2\mathbb{B} = -f/\mu, \quad \nabla^2\beta = \frac{\mathbb{r}\cdot f}{\mu}$$

to prove this we use the helmholtz resolution

$\mathbb{u} = \nabla\phi + \nabla\times\mathbb{H}$ with $(\nabla\cdot\mathbb{H} = 0)$

and by manipulation $\phi = \frac{1}{2\alpha}(\mathbb{r}\cdot\mathbb{B} + \beta) = \frac{1}{2\alpha}(x_jB_j + \beta)$ $\alpha = \frac{2(1-\nu)}{1-2\nu}$

we also showed $2\alpha\nabla\phi = e_i(B_i + x_jB_{j,i} + \beta_{,i})$ (*)

and $2\alpha\nabla^2\phi = 2\nabla\cdot\mathbb{B} + \mathbb{r}\cdot\nabla^2\mathbb{B} + \nabla^2\beta$

remember $\mathbb{B} = \alpha\nabla\phi + \nabla\times\mathbb{H}$ $\therefore$ $\nabla\cdot\mathbb{B} = \alpha\nabla^2\phi + 0$

and then $2\nabla\cdot\mathbb{B} = 2\nabla\cdot\mathbb{B} + \mathbb{r}\cdot\nabla^2\mathbb{B} + \nabla^2\beta$ $\therefore$ $\underline{\nabla^2\beta = -\mathbb{r}\cdot\nabla^2\mathbb{B}}$

but $\nabla^2\beta = \frac{\mathbb{r}\cdot f}{\mu}$ since $\nabla^2\mathbb{B} = -f/\mu$

also $\mathbb{B} - \alpha\nabla\phi = \nabla\times\mathbb{H} = \mathbb{B} - \frac{1}{2}\nabla(\mathbb{r}\cdot\mathbb{B} + \beta)$ using (*)

$\therefore$ $\mathbb{u} = \nabla\times\mathbb{H} + \nabla\phi = \mathbb{B} - \alpha\nabla\phi + \nabla\phi$ using the preceding result for $\nabla\times\mathbb{H}$

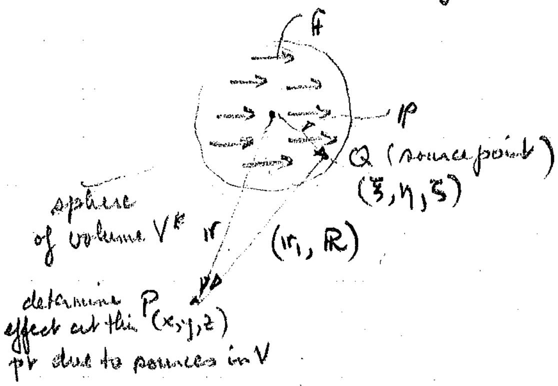
$u = \mathbb{B} + \frac{1-\alpha}{2\alpha}\nabla(\mathbf{r}\cdot\mathbb{B}+\beta)$ $\qquad \frac{1-\alpha}{2\alpha} = -\frac{1}{4(1-\nu)}$ $\qquad$ using $\nabla\phi = \frac{1}{2\alpha}\nabla(\mathbf{r}\cdot\mathbb{B}+\beta)$

$u = \mathbb{B} - \frac{1}{4(1-\nu)}\nabla(\mathbf{r}\cdot\mathbb{B}+\beta)$ $\qquad$ hence if we can find $\mathbb{B}$, β we can determine u

Let's apply this to Lord Kelvin's Problem

Kelvin's Problem – stress distribution in an ∞ elastic body subject to concentrated force.

To define concentrated force: distribute forces over a δV let $\delta V \to 0$ in such a manner that magnitude of the forces remain const.



$$\mathbb{P} = \lim_{V^*\to 0}\int_{V^*} f\,dV$$

$\nabla^2\mathbb{B} = -f/\mu$ for points inside V^*

$\nabla^2\mathbb{B} = 0$ " " outside O

since we know $\nabla^2\mathbb{B}$ we can use Green's formula
we will use simpler formula w/o surface integral then plug back into most general to see if omission is justified

Use Green's formula w/o surface integral: if $\int\frac{1}{R}\frac{\partial\mathbb{B}}{\partial n}ds \to 0$, $\mathbb{B}\to 0$ as $R\to\infty$

$ds \sim R^2$

$$\mathbb{B} = -\frac{1}{4\pi}\int\frac{\nabla^2\mathbb{B}(\xi,\eta,\zeta)}{R}\,d\xi\,d\eta\,d\zeta = -\frac{1}{4\pi}\int_{V-V^*} - \frac{1}{4\pi}\int_{V^*} = -\frac{1}{4\pi}\int_{V^*}$$ since $\nabla^2\mathbb{B}=0$ in $V-V^*$

More general form is

$$g(P) = \frac{1}{4\pi}\left\{\int_S\left[\frac{1}{R}\frac{\partial g}{\partial n} - g\frac{\partial(\frac{1}{R})}{\partial n}\right]ds - \int_V\frac{1}{R}\nabla^2 g\,dV\right\}$$

as $V^*\to 0$, $r \leftarrow R$ $\qquad$ but $\mathbb{B} = \frac{1}{4\pi}\int_{V^*}\frac{f}{\mu R}dV = \frac{\mathbb{P}}{4\pi\mu r}$

Note: as $r\to\infty$ $\mathbb{B}\to 0$ $\qquad \frac{\partial\mathbb{B}}{\partial n} = \frac{\partial\mathbb{B}}{\partial r} \lesssim \frac{1}{r^2}$ $\quad\therefore \lim_{R\to 0}\int\frac{1}{R}\frac{\partial\mathbb{B}}{\partial n}ds \le \frac{1}{R}\cdot\frac{1}{R^2}\cdot R^2 \to 0$

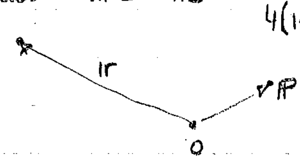
remember $\beta = -\frac{1}{4\pi}\int_{V^*}\frac{\nabla^2\beta}{R}\,d\xi\,d\eta\,d\zeta$ $\qquad$ but $\nabla^2\beta = \frac{\rho\cdot f}{\mu}$ since we are inside V^*

$$\beta = \frac{-1}{4\pi\mu}\int_{V^*}\frac{\rho\cdot f}{R}dV = -\frac{1}{4\pi\mu}\int_{V^*}\frac{\rho\cdot f}{R}dV$$ since ρ = position vector of point inside V^* sources since we are $\rho = r$

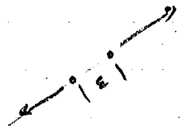
as $R\to\infty$ & $dV\to 0$ $\rho\to 0$ $\therefore \beta = -\frac{1}{4\pi\mu}\cdot\frac{0}{\infty}\mathbb{P} = 0$

Thus $u = \mathbb{B} - \frac{1}{4(1-\nu)}\nabla(\overset{\text{field point}}{\mathbf{r}}\cdot\mathbb{B}+\beta)$ $\qquad \mathbb{B} = \frac{\mathbb{P}}{4\pi\mu r}$ $\qquad \beta = 0$

$\mathbf{r} = (x,y,z)$ $\quad r = |\mathbf{r}|$



For a double force



use superposition // for each we know the answer // take their solution and take limit as $\varepsilon \to 0$

we can also get problem that produces a couple

and we can also do this in all 3 planes.

4/13/79

For 1st Homework find stresses & strains from $\mathbb{u}$ for the Kelvin Problem.

Continuing: $u_i = B_i - \frac{1}{4(1-\nu)}(x_j B_j)_{,i} = B_i - \frac{1}{4(1-\nu)}(B_j \delta_{ij} + x_j B_{j,i})$

Let $\mathbb{P} = P\mathbb{e}_3 \Rightarrow B_1 = B_2 = 0 \quad B_3 = \frac{P}{4\pi\mu r}$ since $\mathbb{B} = B_i \mathbb{e}_i = \frac{\mathbb{P}}{4\pi\mu r} = \frac{P\mathbb{e}_3}{4\pi\mu r}$

thus $u_1 = -\frac{1}{4(1-\nu)}\left[x_3 B_{3,1}\right] \qquad B_{3,1} = \frac{\partial B_3}{\partial r}\frac{\partial r}{\partial x_1} = \frac{-P}{4\pi\mu r^2}\cdot\frac{x_1}{r} = -\frac{Px_1}{4\pi\mu r^3}$

$$\boxed{u_1 = \frac{1}{4(1-\nu)}\frac{x_1 x_3 P}{4\pi\mu r^3}}$$

Similarly $\boxed{u_2 = \frac{Px_2x_3}{16\pi\mu(1-\nu)r^3}}$ since $B_1 = B_2 = 0$

$$\boxed{u_3 = \frac{P}{4\pi\mu r} - \frac{1}{4(1-\nu)}\left[\frac{P}{4\pi\mu r} - \frac{x_3^2 P}{4\pi\mu r^3}\right] \quad \text{since } B_3 \neq 0}$$

Alternate Derivation of Papcovitch-Boussinesq-Neuber Functions

Start w/ Equil in $\mathbb{u}$

$$\mu\nabla^2\mathbb{u} + \frac{\mu}{(1-2\nu)}\nabla(\nabla\cdot\mathbb{u}) + \mathbb{f} = 0 \qquad (1)$$

Now define $\mathbb{u} = \nabla\phi + \nabla\times\mathbb{H}$ w/ $(\nabla\cdot\mathbb{H} = 0)$ (2) Helmholtz Resolution

Put (2) into (1) to get $\mu\nabla^2(\alpha\nabla\phi + \nabla\times\mathbb{H}) + f = 0$ (3) where $\alpha = \frac{2(1-\nu)}{(1-2\nu)}$

define $\mathbb{B} = \alpha\nabla\phi + \nabla\times\mathbb{H}$ (4)

Put (4) into (3) to get $\boxed{\nabla^2\mathbb{B} = -\frac{\mathbb{f}}{\mu}}$ (5)

Put (4) into (2) to get $\mathbb{u} - \mathbb{B} = (1-\alpha)\nabla\phi$ (6)

take $\nabla\cdot(4)$ to get $\nabla\cdot\mathbb{B} = \alpha\nabla^2\phi$ since $\nabla\cdot\mathbb{H} = 0$ (7)

Consider next $\nabla^2(\mathbf{r}\cdot\mathbf{B}) = 2\,\nabla\cdot\mathbf{B} + \mathbf{r}\cdot\nabla^2\mathbf{B}$ (8)

$= 2\alpha\nabla^2\phi - \mathbf{r}\cdot\dfrac{\mathbf{f}}{\mu}$ (8')

define $\boxed{\nabla^2\beta = \dfrac{\mathbf{r}\cdot\mathbf{f}}{\mu}}$ so that all terms of (8') have ∇^2 (9)

Using (9) $\nabla^2(\mathbf{r}\cdot\mathbf{B}) = 2\alpha\,\nabla^2\phi - \nabla^2\beta$ integrate to get

$\nabla(\mathbf{r}\cdot\mathbf{B}+\beta) = 2\alpha\nabla\phi$ taking constant of integration to be zero (10)

Put (10) into (6) $\mathbf{u} - \mathbf{B} = \dfrac{1-\alpha}{2\alpha}\,\nabla(\mathbf{r}\cdot\mathbf{B}+\beta)$ and define $\dfrac{1-\alpha}{2\alpha} = -\dfrac{1}{4(1-\nu)}$

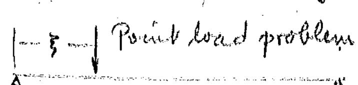
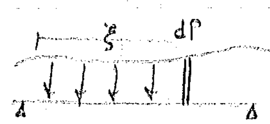
then $\boxed{\mathbf{u} = \mathbf{B} - \dfrac{1}{4(1-\nu)}\,\nabla(\mathbf{r}\cdot\mathbf{B}+\beta)}$

We will now solve Kelvin's Problem using green's functions (shades of Point forces & BIE)

if $Lu = f$ where L is a differential operator relating the displacement to the forcing function, then

$L^{-1}f = u$ where L^{-1} is the green's function

We can use green's functions only for linear problems because of superposition principle. Hence if we can get the solution to


Point load problem $= G$ then  $= G * f$ where

f is the load distribution function.

since $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$ and $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$ then

$\sigma_{ij} = C_{ijkl}\,u_{k,l}$

Now the Equil Eqns. $\sigma_{ij,j} + f_i = C_{ijkl}\,u_{k,lj} + f_i = 0$

for isotropy $C_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

Define the displacement $U_{ij}(\mathbf{r})$ – displacement $u_i(\mathbf{r})$ produced by unit force in x_j direction acting at origin for ∞ body

If you substitute into Equil Eqn.

$$C_{ijkl}\,U_{km,lj}(\mathbf{r}) + \underbrace{\delta_{im}}_{\text{Kronecker }\delta}\underbrace{\delta(\mathbf{r})}_{\text{dirac }\delta\text{ fun.}} = 0$$

where f_i = unit force $= \begin{cases}1 & i=m\\ 0 & i\neq m\end{cases}$

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Green's Tensor Function to Solve Kelvin Problem

$$c_{ijkl}\, u_{k,lj} + f_i = 0$$

for an infinite body

$$c_{ijkl}\, U_{km,lj}(\mathbf{r}) + \delta_{im}\,\delta(\mathbf{r}) = 0 \quad (\dagger)$$

δ_{im} is needed $\Rightarrow$ force acts only in one direction

B.C. $U_{ij} \to 0$ as $\mathbf{r} \to \infty$ Note: $\delta(\mathbf{r}) = \delta(x_1)\delta(x_2)\delta(x_3)$

take equil $c_{ijkl}\, u_{k,lj} = -f_i(\mathbf{r}) = -\int f_i(\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}')\,dV' = -\int \delta_{im}\,\delta(\mathbf{r}-\mathbf{r}')\, f_m(\mathbf{r}')\,dV'$

$\therefore$ substitute for $\delta_{im}\,\delta(\mathbf{r})$, $-c_{ijkl}\, U_{km,lj}(\mathbf{r})$

$$\therefore \quad c_{ijkl}\, u_{k,lj} = \int c_{ijkl}\, U_{km,lj}(\mathbf{r}-\mathbf{r}')\, f_m(\mathbf{r}')\,dV'$$

$$\therefore \quad \boxed{u_k(\mathbf{r}) = \int U_{km}(\mathbf{r}-\mathbf{r}')\, f_m(\mathbf{r}')\,dV'}$$

throw out constants (since they represent rigid body rotation & displ) from integration & divide by elasticity tensor.

This is solution to the general problem with body forces distributed in V'

Determination of Green's tensor funct by means of Fourier transform.

def: $\bar{U}_{km}(\mathbf{k}) = \int U_{km}(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, dV_{\mathbf{r}}$ $dV_{\mathbf{r}}$ is the space where $\mathbf{r}$ is measured

$U_{km}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int \bar{U}_{km}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, dV_{\mathbf{k}}$ $dV_{\mathbf{k}}$ " " " " $\mathbf{k}$ is measured

$\mathbf{r} = x_i \mathbf{e}_i$

Consider $c_{ijkl}\, k_l k_j\, \bar{U}_{km}(\mathbf{k}) = c_{ijkl} \int k_l k_j\, U_{km}(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, dV_{\mathbf{r}}$

integrate twice to get (neglecting surface terms since they go to 0 as $S \to \infty$)

$$= -c_{ijkl}\int U_{km,lj}(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, dV_{\mathbf{r}} = \int \delta_{im}\,\delta(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, dV = \delta_{im}$$

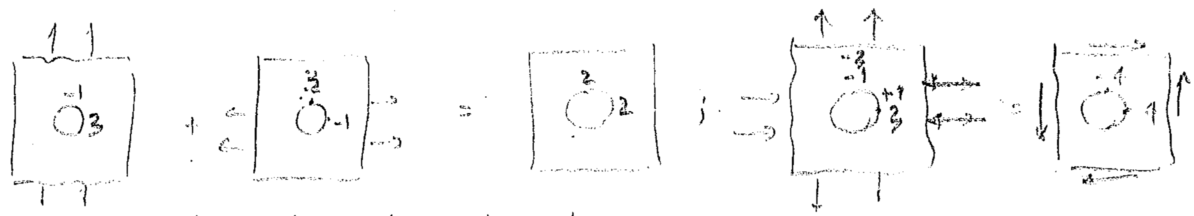
using Equil Eqns above $(\dagger)$

up to now we used anisotropic results — now specify to isotrop.

Now $c_{ijkl}\, k_l k_j\, \bar{U}_{km}(\mathbf{k}) = (\lambda+\mu)\, k_i k_k\, \bar{U}_{km} + \mu k^2\, \bar{U}_{im} = \delta_{im}$ $(**)$

Mult by k_i and sum to give

$$(\lambda+2\mu)\, k^2 k_k\, \bar{U}_{km} = k_m \quad (*)$$



Sec 13.7 | Southwell first solved the problem in 1926.

For stresses w/o cavity

$\sigma'_{zz} = T$ all others are zero

transform to R, θ, ϕ coords

$e_R \cdot e_z = \cos\phi \qquad e_\phi \cdot e_z = -\sin\phi$

thus $\sigma_{RR} = T\cos^2\phi \qquad \sigma_{R\phi} = -T\sin\phi\cos\phi \qquad \sigma_{\phi\phi} = T\sin^2\phi$

$\sigma_{R\theta} = 0$ everywhere since $e_z \cdot e_\theta = 0$

The stress system necessary is made up of

1. double force in the z direction
2. center of compression
3. Stress for $\Phi = Cz(r^2+z^2)^{-3/2}$
4. Simple Tension

This gives rise to $B_1, B_2 = 0 \qquad B_3 = A\frac{\partial}{\partial z}\left(\frac{1}{R}\right)$

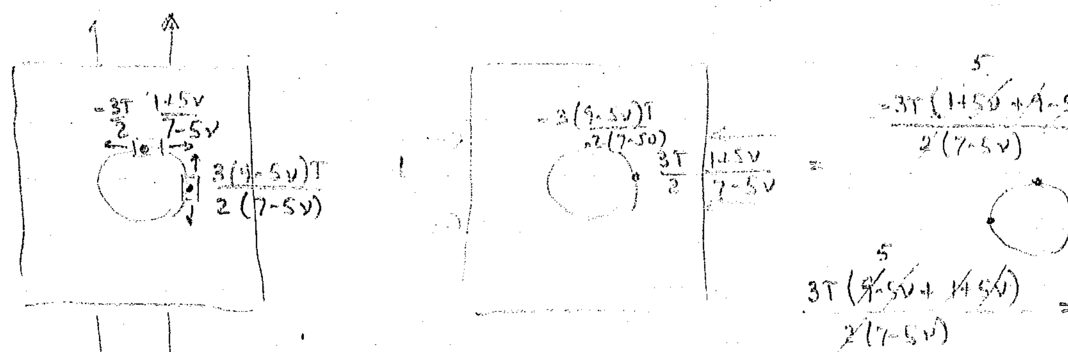
$$\beta = \frac{D}{R} + \frac{\partial^2}{\partial z^2}\left(\frac{C}{R}\right)$$

$$\sigma_{\phi\phi} = \frac{a^3 T}{2(7-5\nu)R^5}\left\{3a^2(3-7\cos^2\phi) + R^2\left[(9-5\nu)+5(1-2\nu)\cos^2\phi\right] + \frac{2(7-5\nu)}{a^3}R^5\sin^2\phi\right\}$$

when $\phi = 0 \qquad \sigma_{\phi\phi} = \frac{a^3 T}{2(7-5\nu)R^5}\left\{-12a^2 + R^2\left[9-15\nu\right]\right\}$

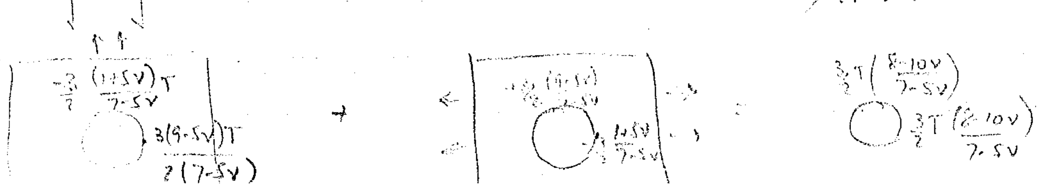
$-12a^2 + 9a^2 - 15\nu a^2 = -3a^2 - 15\nu a^2 = -3a^2(1+5\nu)$

$$\sigma_{\phi\phi}\Big|_{\phi=0,\ \phi=\pi} = \frac{-3a^5 T(1+5\nu)}{2(7-5\nu)a^5} = -\frac{3T}{2}\frac{(1+5\nu)}{7-5\nu}$$



$\frac{-3T(1+5\nu+9-5\nu)}{2(7-5\nu)} = -\frac{15T}{7-5\nu}$

$\frac{3T(9-5\nu+1+5\nu)}{2(7-5\nu)} = \frac{15T}{7-5\nu}$



Eleminate $k_m \bar{U}_{km}$ using (*) & (**) to obtain

$$\frac{\lambda+\mu}{\lambda+2\mu}\,\frac{k_i k_m}{k^2} + \mu k^2\, \bar{U}_{im} = \delta_{im}$$

Solve for $\bar{U}_{im}$ (or $\bar{U}_{km}$)

$$\bar{U}_{km} = \frac{1}{\mu}\left[\frac{\delta_{km}}{k^2} - \frac{\lambda+\mu}{\lambda+2\mu}\,\frac{k_k k_m}{k^4}\right]$$

to get $U_{km}(\mathbf{r})$ we take

$$U_{km}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int \bar{U}_{km}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, dV_k$$

the only integral we need is

$$\frac{1}{\pi^2}\int \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^4}\, dV_k = -r$$ show this to be true

Differentiate both side w/r k_m, k_m to get

$$\frac{1}{\pi^2}\int \frac{k_k k_m\, e^{i\mathbf{k}\cdot\mathbf{r}}}{k^4}\, dV_k = r_{,km}$$

if $k=m$ then $\frac{1}{\pi^2}\int \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2} = r_{,pp}$ Laplacian w/ $k_m k_m = k^2$

$$\boxed{U_{km}(\mathbf{r}) = \frac{1}{8\pi\mu}\left[\delta_{km}\, r_{,pp} - \frac{\lambda+\mu}{\lambda+2\mu}\, r_{,km}\right]}$$

This is the green funct in ∞ space

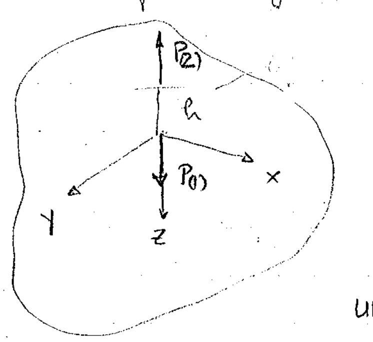
show this is the same as the result we had worked out by means of the potentials - see pg 4-back ; u_1, u_2, u_3 w/P=

4/18/79

We will now construct solutions to some problems : NUCLEI OF STRAIN — SINGULARITIES

Rich King Th 2-4 Fr 12:30-2:30. (Rm 264 Durand.)

Given an infinite body — Double Force



define the strength of singularity of $\lim_{h\to 0} Ph = A$

We want to find the stress field for a double force w/out moment.

Let $u_{(1)}$ be the soln to the problem for unit force $P_{(1)}$

$u_{(2)}$ be " " " " " " " " $P_{(2)}$

$$u_2 = -\left[u_1 + \frac{\partial u_{1(1)}}{\partial z}h + h.o.t.\right]$$ since u_2 is in opposite dir

Total solution $\frac{u}{P} = u_{(1)} + u_{(2)} = -\frac{\partial u_1}{\partial z}\, h$

$$\lim_{h\to 0} u = -A\,\frac{\partial u_{(1)}}{\partial z}$$ using the Kelvin solution if forces are tensile $A>0$, Compressive $A<0$

Now what in fact is the displ field

From Kelvin Prob

$$\mathbb{B} = \frac{\mathbb{P}}{4\pi\mu r}\ ;\ \beta = 0 \qquad B_z = \frac{P}{4\pi\mu r} \qquad B_x = B_y = \beta = 0$$

$$u_{(1)} = \left[\mathbb{B} - \frac{1}{4(1-\nu)}\nabla(\mathbf{r}\cdot\mathbb{B})\right]\frac{1}{P} \qquad \text{(we divide by } P \text{ since we want a unit force)}$$

$$= \left[\frac{P e_z}{4\pi\mu r} - \frac{1}{4(1-\nu)}\nabla\left(\frac{zP}{4\pi\mu r}\right)\right]\frac{1}{P} = \frac{1}{4\pi\mu}\left[\frac{e_z}{r} - \frac{1}{4(1-\nu)}\nabla\frac{z}{r}\right]$$

$$\frac{\partial u_{(1)}}{\partial z} = \frac{1}{4\pi\mu}\left[\frac{-e_z z}{r^2\cdot r} - \frac{1}{4(1-\nu)}\nabla\left(\frac{1}{r} - \frac{z^2}{r^3}\right)\right]$$

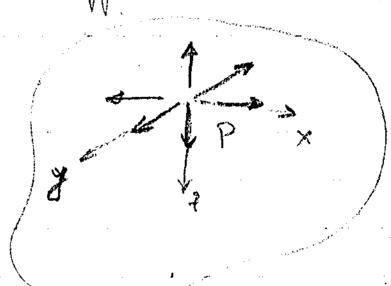
In general $\quad u = \mathbb{B} - \frac{1}{4(1-\nu)}\nabla(\mathbf{r}\cdot\mathbb{B} + \beta)$

here: $\quad \mathbb{B} = \frac{A z e_z}{4\pi\mu r^3} \qquad \beta = \frac{-A}{4\pi\mu r} \qquad \mathbf{r}\cdot\mathbb{B} = \frac{A z^2}{4\pi\mu r^3}$

Double Force Soln

$$u = \frac{A}{4\pi\mu}\left[\frac{z\,e_z}{r^3} - \frac{1}{4(1-\nu)}\nabla\left(\frac{z^2}{r^3} - \frac{1}{r}\right)\right] = -A\frac{\partial u_{(1)}}{\partial z}$$

Suppose we look at Center of Dilatation (3 limiting double forces along x, y, z axes) in an ∞ medium where each of the forces are of strength P



By superposition we obtain

$$\mathbb{B} = A\left[\frac{z\,e_z + x\,e_x + y\,e_z}{4\pi\mu r^3}\right] = \frac{A\mathbf{r}}{4\pi\mu r^3}$$

$$\beta = \frac{-3A}{4\pi\mu r}$$

thus $\quad \mathbf{r}\cdot\mathbb{B} = \frac{A r^2}{4\pi\mu r^3} = \frac{A}{4\pi\mu r}$

$$u = \frac{A\mathbf{r}}{4\pi\mu r^3} - \frac{1}{4(1-\nu)}\nabla\left[\frac{A}{4\pi\mu r} - \frac{3A}{4\pi\mu r}\right] \qquad \nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$$

$$= \frac{A\mathbf{r}}{4\pi\mu r^3} - \frac{1}{4(1-\nu)}\,\frac{-2A}{4\pi\mu}\cdot\frac{-\mathbf{r}}{r^3}$$

$$= \frac{A\mathbf{r}}{4\pi\mu r^3}\left[1 - \frac{2}{4(1-\nu)}\right] = \frac{A\mathbf{r}}{4\pi\mu r^3}\left[1 - \frac{1}{2(1-\nu)}\right] = \frac{C\mathbb{R}}{4(1-\nu)R^3}$$

Center of Dilatation

$$= u = \frac{-(1-2\nu)}{8(1-\nu)\pi\mu}\nabla\left(\frac{1}{r}\right)$$

$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x_i}\ ;\ \frac{\partial r}{\partial x_i} = \frac{1}{2}\cdot\frac{2x_i}{r} = \frac{x_i}{r}$

$\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$

if $C = \frac{(1-2\nu)A}{2\pi\mu}$ strength of center of dilation $\left| C \sim L^3 = \frac{lb \cdot L}{lb \cdot /L^2} \right|$

$u = -\frac{1}{4(1-\nu)} \nabla\left(\frac{C}{r}\right) \sim O\left(\frac{1}{L}\cdot\frac{L^3}{L}\right) = O(L)$

hence using the Papcovitch-Neuber potentials we can define $\Rightarrow \mathbb{B} = 0 \quad \beta = \frac{C}{r}$

hence we note that the same displ field is obtained from $\mathbb{B} = 0 \quad \beta = \frac{C}{r}$

and $\mathbb{B} = \frac{A r}{4\pi\mu r^3} \quad \beta = -\frac{3A}{4\pi\mu r}$

hence soln is really not unique

Note that u is not dependent on orientation of axes and only on distance from origin, and hence is isotropic

Also the field is divergence free everywhere since

$$\nabla \cdot u = \frac{-C}{4(1-\nu)} \nabla^2\left(\frac{1}{r}\right) = 0 \quad \text{because} \quad \nabla^2\left(\frac{1}{r}\right) = 0 \quad r \neq 0.$$

Since it is indep of orientation we can take the spherical coord system

$$u = \frac{C\mathbb{R}}{4(1-\nu)R^3} \quad \text{using} \quad \nabla\left(\frac{1}{R}\right) = -\frac{\mathbb{R}}{R^3} \quad \text{where } \mathbb{R} = R e_R$$

$$u_R = \frac{C}{4(1-\nu)R^2}; \quad u_\theta = 0 \quad u_\phi = 0 \qquad \text{Displ in } R, \theta, \varphi \text{ system.}$$

$$\varepsilon_{RR} = \frac{\partial u_R}{\partial R} = -\frac{C}{2(1-\nu)R^3} \qquad \varepsilon_{R\phi} = 0 \qquad \varepsilon_{R\theta} = 0 \qquad \varepsilon_{\theta\varphi} = 0$$

$$\varepsilon_{\phi\phi} = \varepsilon_{\theta\theta} = \frac{u_R}{R} = \frac{C}{4(1-\nu)R^3}$$

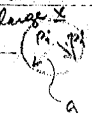
$$\sigma = \lambda \nabla\cdot u\, \mathbb{I} + \mu(\nabla u + u\nabla)$$
$$= 0 + 2\mu\,\varepsilon$$

$$\therefore \sigma = \begin{pmatrix} 2\mu\epsilon_{RR} & 0 & 0 \\ 0 & 2\mu\epsilon_{\theta\theta} & 0 \\ 0 & 0 & 2\mu\epsilon_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \sigma_{RR} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\phi\phi} \end{pmatrix}$$

$$\therefore \sigma_{RR} = \frac{-C\mu}{(1-\nu)R^3} \qquad \sigma_{\theta\theta} = \frac{C\mu}{2(1-\nu)R^3} = \sigma_{\phi\phi} \qquad \sigma_{\theta\varphi} = \sigma_{\theta R} = \sigma_{\phi R} = 0$$

Problem: given a spherical cavity under pressure p_i: assume cavity is small in comparison to body.

since p_i is same in all direct: assume at large r this looks like hydrostatic pressure



BC $\sigma_{RR} = -p_i$ at $R = a$ $\quad \sigma_{R\theta} = \sigma_{R\phi} = 0$

we can use center of dilatation soln since bc are isotropic

take $\sigma_{RR} = -\frac{C\mu}{(1-\nu)a^3} = -p_i$ $\quad$ thus $C = p_i a^3 \frac{(1-\nu)}{\mu}$

then $\sigma_{RR} = -p_i \dfrac{a^3}{R^3}$ $\quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \dfrac{p_i a^3}{2R^3}$ $\quad \sigma_{\theta\phi} = \sigma_{\theta R} = \sigma_{\phi R} = 0$

$u_R = \dfrac{p_i a^3}{4\mu R^2}$ $\quad u_\theta = u_\phi = 0$ $\quad$ w/. $u_R = \dfrac{cR\,e_R}{4(1-\nu)R^3}$ $\quad c = \dfrac{p_i (1-\nu) a^3}{\mu}$

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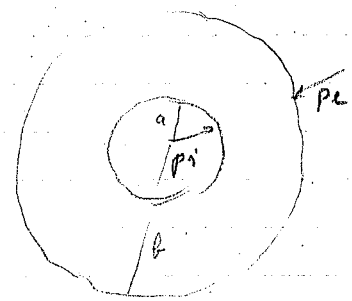
Hollowsphere solutions



$\sigma_{RR}\big|_{R=a} = -p_i$ $\qquad \sigma_{r\phi} = \sigma_{r\theta} = 0$ at $r=a, r=b$

$\sigma_{RR}\big|_{R=b} = -p_e$

For center of dilation $\quad \sigma_{RR} = \dfrac{-A\mu}{(1-\nu)R^3}$ $\qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \dfrac{A\mu}{2(1-\nu)R^3}$

$\sigma_{R\phi} = \sigma_{R\phi} = \sigma_{\theta\phi} = 0$

since this soln has only one variable A we cannot satisfy both bc. Superpose a uniform state of hydrostatic tension $\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = B$ since p_e & p_i are const.

we can also do this by putting uniform compress at ∞ & supperposing on cavity soln at $r=a$. $\sigma_{RR}\big|_{R=a} = -p_i + p_e$

BC. $\quad -\dfrac{A\mu}{(1-\nu)a^3} + B = -p_i$ $\qquad A = \dfrac{(1-\nu)a^3 b^3 (p_e - p_i)}{\mu\,(a^3 - b^3)}$

$-\dfrac{A\mu}{(1-\nu)b^3} + B = -p_e$ $\qquad B = \dfrac{b^3 p_e - a^3 p_i}{a^3 - b^3}$

$\therefore \quad \sigma_{RR} = (p_i - p_e)\dfrac{a^3 b^3}{(a^3 - b^3)R^3} + \dfrac{b^3 p_e - a^3 p_i}{a^3 - b^3}$

$\sigma_{\phi\phi} = \sigma_{\theta\theta} = \dfrac{a^3 b^3 (p_e - p_i)}{2(a^3 - b^3)R^3} + \dfrac{b^3 p_e - a^3 p_i}{a^3 - b^3}$

for a very thin shell $a \sim b$

for $p_i = 0$ then $\quad \sigma_{RR} = \dfrac{p_e b^3 (R^3 - a^3)}{(a^3 - b^3)R^3}$ $\qquad \sigma_{\phi\phi} = \sigma_{\theta\theta} = \dfrac{b^3 p_e (2R^3 + a^3)}{2(a^3 - b^3)R^3}$

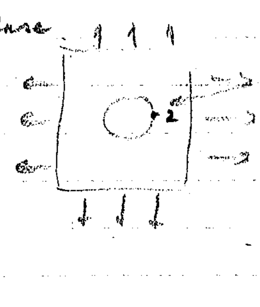
if $a \ll b$ then $\quad \sigma_{RR} \rightarrow p_e\left(-1 + (a/R)^3\right)$ $\qquad \sigma_{RR} = p_e \dfrac{(1 - (a/R)^3)}{-(1 - (a/b)^3)} \rightarrow p_e\left(-1 + (a/R)^3\right)$

$\sigma_{\theta\theta} = \sigma_{\phi\phi} = -\left[+1 + \tfrac{1}{2}(a/R)^3\right] p_e$

if we let $b \rightarrow \infty$ then we have an ∞ body w/ uniform stress w/o applied stresses on $R=a$

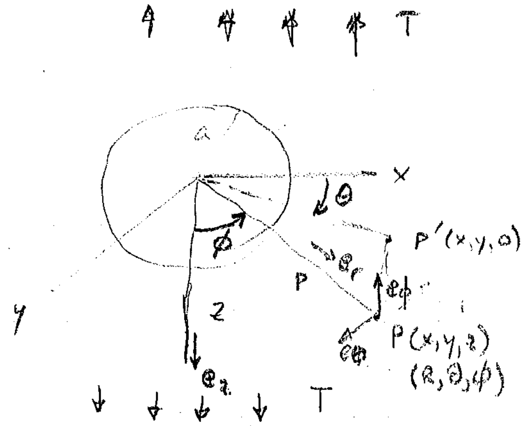
then. for $R = a$ $\quad \sigma_{RR} = 0$ $\quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -\tfrac{3}{2} p_e$

for 2d case



$\sigma_{\theta\theta} = 2 p_e$

Suppose we have an infinite body w/ Spherical Cavity
uniaxial Remote uniform tension



Use the method of superposition
1. Use uniform stress in infinite body soln.
2. Find σ's on $R = a$
3. Superpose soln. $\ni$. σ's on $R=a$ are cancelled

For body w/o cavity ('state)

$\sigma_{zz}' = T \qquad \sigma_{xx}' = \sigma_{yy}' = \sigma_{zy}' = \sigma_{yx}' = \sigma_{xz}' = 0$

transform σ's in ('state) on cavity in R, ϕ, θ coords.

given $\sigma = T\, e_z e_z$ we need σ_{RR}', $\sigma_{R\phi}'$, $\sigma_{\phi\phi}$

$e_R \cdot \sigma' \cdot e_R = \sigma_{RR}$ thus we need $e_R \cdot e_z = \cos\phi \qquad \sigma_{RR} = T\cos^2\phi$

$e_R \cdot \sigma' \cdot e_\phi = \sigma_{R\phi} \qquad e_R \cdot e_z = \cos\phi$, $e_\phi \cdot e_z = -\sin\phi \qquad \sigma_{R\phi} = -T\sin\phi\cos\phi$

$e_\phi \cdot \sigma' \cdot e_\phi = \sigma_{\phi\phi}$ thus $\sigma_{\phi\phi} = +T\sin^2\phi$

$e_R \cdot \sigma' \cdot e_\theta = \sigma_{R\theta} = 0$ throughout the body since $e_z \cdot e_\theta = 0$ & $\sigma_{\theta\theta} = 0$

$\therefore$ on $R = a$ $\sigma_{RR}', \sigma_{R\phi}'$ are $\neq 0$ $\therefore$ we must ~~suppose~~ superpose a soln. $\ni$.

$\sigma_{RR}'' = -T\cos^2\phi \qquad \sigma_{R\phi}'' = T\sin\phi\cos\phi$ on $R = a$ and $\sigma_{ij}'' \to 0$ as $R \to \infty$

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Continuing the above - if you try $B_1, B_2 = 0 \qquad B_3 = A\dfrac{\partial}{\partial z}\left(\dfrac{1}{R}\right)$

$$\beta = \frac{D}{R} + \frac{\partial^2}{\partial z^2}\left(\frac{C}{R}\right)$$

This is a solution that gives the following stresses $(B, \beta) \Rightarrow u \Rightarrow \epsilon \Rightarrow \sigma$

$$\sigma_{RR}'' = \frac{\mu\cos^2\phi}{(1-\nu)R^3}\left[(5-\nu)A - \frac{18C}{R^2}\right] + \frac{\mu}{(1-\nu)R^3}\left[-\nu A - D + \frac{6C}{R^2}\right]$$

$$\sigma_{R\phi}'' = \frac{\mu\sin\phi\cos\phi}{(1-\nu)R^3}\left[(1+\nu)A - \frac{12C}{R^2}\right]$$

$$\sigma_{R\theta}'' = 0$$

BC Equating results at r=a gives

$$\sigma''_{RR}\Big|_{R=a} = \frac{\mu\cos^2\phi}{(1-\nu)a^3}\left[(5-\nu)A - \frac{18C}{a^2}\right] + \frac{\mu}{(1-\nu)a^3}\left[-\nu A - D + \frac{6C}{a^2}\right] = -T\cos^2\phi$$

$$\Rightarrow \quad -\nu A - D + \frac{6C}{a^2} = 0 \quad \text{and} \quad \frac{\mu}{(1-\nu)a^3}\left[(5-\nu)A - \frac{18C}{a^2}\right] = -T$$

$$\sigma''_{R\phi}\Big|_{R=a} = \frac{\mu}{(1-\nu)a^3}\cos\phi\sin\phi\left[(1+\nu)A - \frac{12C}{a^2}\right] = T\sin\phi\cos\phi$$

$$\Rightarrow \quad \frac{\mu}{(1-\nu)a^3}\left[(1+\nu)A - \frac{12C}{a^2}\right] = T$$

Solving for A, C, D:

$$A = -\frac{5(1-\nu)a^3T}{\mu(7-5\nu)} \qquad D = -\frac{(1-\nu)(6-5\nu)a^3T}{\mu(7-5\nu)} \qquad C = -\frac{(1-\nu)a^5T}{\mu(7-5\nu)}$$

$\sigma_{\phi\phi}$ will create stress concentration along the rim thus

$$\sigma_{\phi\phi} = \frac{a^3T}{2(7-5\nu)R^5}\left\{3a^2(3-7\cos^2\phi) + R^2\left[(4-5\nu) + 5(1-2\nu)\cos^2\phi\right] + \frac{2(7-5\nu)}{a^3}R^5\sin^2\phi\right\}$$

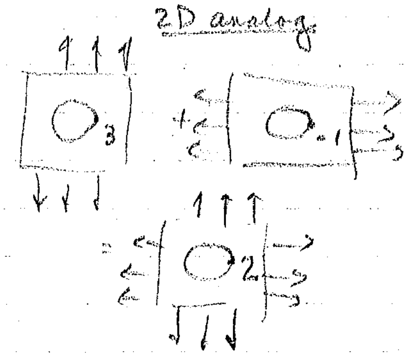
$\sigma_{\phi\phi_{max}}$ occurs @ $R=a$ and $\phi = \pi/2$

$$\sigma_{\phi\phi_{max}} = \frac{3(9-5\nu)}{2(7-5\nu)}T = KT \qquad \text{for } \nu = .3 \quad K = 2$$

In plane problems most are independent of ν
except 1. Multiply connected region where loads is not self equil on the same boundary.

2D analog

find 3D analogy for all around tension or comp. to



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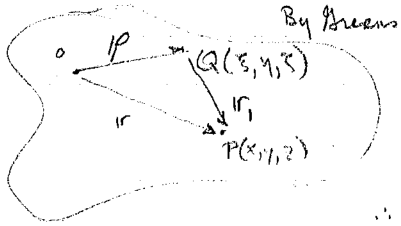
Half-Space Problems

If we want solutions in the near field of applied loads we can assume the body is semi infinite solve problem & get solution in the near field - we then care not what bc or thickness of the actual problem.

Consider



By Greens 2nd ident: $\int_S \left(g\frac{\partial h}{\partial n} - h\frac{\partial g}{\partial n}\right)ds - \int_V \left(g\nabla^2 h - h\nabla^2 g\right)dV = 0$

Pick $g \ni \nabla^2 g = 0$ in V

$g = 1/r_1$ on S.

$$\therefore \int_S \left(g\frac{\partial h}{\partial n} - h\frac{\partial g}{\partial n}\right)ds - \int_S g\nabla^2 h\, dV = 0 \qquad (A)$$

We had also shown that $4\pi h = \int_S \left[\frac{1}{r_1}\frac{\partial h}{\partial n} - h\frac{\partial(1/r_1)}{\partial n}\right]ds - \int_V \frac{1}{r_1}\nabla^2 h\, dV$ (B)

here we need h, $\frac{\partial h}{\partial n}$ on S and $\nabla^2 h$ in V

Now subtr (A) from (B)

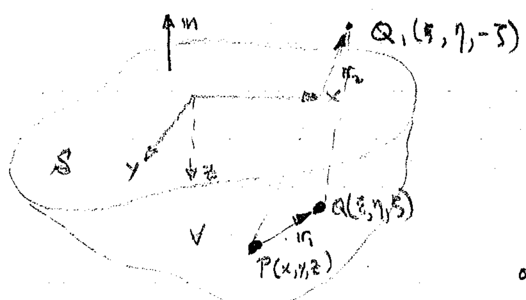
$$4\pi h = \int_S h\frac{\partial}{\partial n}\left(g - \frac{1}{r_1}\right)ds + \int_V \left(g - \frac{1}{r_1}\right)\nabla^2 h\, dV$$

Let $g - \frac{1}{r_1} \stackrel{\text{def}}{=} -G$

$$4\pi h = -\left[\int_S h\frac{\partial}{\partial n}G\, ds + \int_V G\nabla^2 h\, dV\right]$$

this requires only the value of $\nabla^2 h$ in the volume and h on the surface.

Note: $\nabla^2 G = 0$ in V and $G = 0$ on S



(mirror image of Q)

$r_1 = |\mathbf{r}_1| = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$

$r_2 = |\mathbf{r}_2| = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}$

$\nabla^2\left(\frac{1}{r_2}\right) = 0$

on S $\quad r_1 = r_2 = r_0 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}$

We now set $g = \frac{1}{r_2}$, $\quad G = \frac{1}{r_1} - \frac{1}{r_2}$ $\quad \therefore \nabla^2 G = 0$ in V and $G = 0$ on S

For the half space $\frac{\partial}{\partial n} = -\frac{\partial}{\partial \zeta}$ $\quad \therefore -\frac{\partial G}{\partial n} = +\frac{\partial}{\partial \zeta}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

$$= \frac{z-\zeta}{r_1^3} + \frac{z+\zeta}{r_2^3}$$

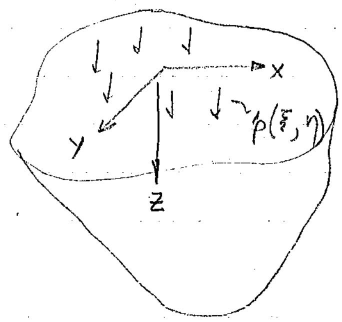
but $-\frac{\partial G}{\partial n}\bigg|_{\substack{\text{surf}\\ \zeta=0}} = \frac{z-0}{r_0^3} + \frac{z+0}{r_0^3} = \frac{2z}{r_0^3} = -2\frac{\partial}{\partial z}\left(\frac{1}{r_0}\right)$

$$\therefore \quad 4\pi h(x,y,z) = -2\frac{\partial}{\partial z}\int_S \frac{h(\xi,\eta,0)}{r_0}\,d\xi\,d\eta - \int_V \left(\frac{1}{r_1}-\frac{1}{r_2}\right)\nabla^2 h(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta \qquad (*)$$

$$\therefore \quad G = G(x,y,z,\xi,\eta,\zeta) = \text{greens fn.}$$

Example: of the results above

1. Distributed load on the half space



B.C. on z $\quad \sigma_{zx}=\sigma_{zy}=0 \qquad \sigma_{zz}=-p(\xi,\eta)$

Let $B_x = 0 = B_y \qquad B_z \neq 0 \qquad \beta \neq 0$

$$u = B_z e_z - \frac{1}{4(1-\nu)}\nabla\left[zB_z + \beta\right]$$

Since $f=0 \Rightarrow \nabla^2 B_z = 0$ & $\nabla^2\beta = 0$

$$\sigma_{zx} = \frac{\mu(1-2\nu)}{2(1-\nu)}\frac{\partial B_z}{\partial x} - \frac{\mu}{2(1-\nu)}\left(z\frac{\partial^2 B_z}{\partial x\partial z} + \frac{\partial^2\beta}{\partial x\partial z}\right)$$

$$\sigma_{zy} = \frac{\mu(1-2\nu)}{2(1-\nu)}\frac{\partial B_z}{\partial y} - \frac{\mu}{2(1-\nu)}\left(z\frac{\partial^2 B_z}{\partial y\partial z} + \frac{\partial^2\beta}{\partial y\partial z}\right)$$

$$\sigma_{zz} = \mu\frac{\partial B_z}{\partial z} - \frac{\mu}{2(1-\nu)}\left(z\frac{\partial^2 B_z}{\partial z^2} + \frac{\partial^2\beta}{\partial z^2}\right)$$

on Boundary $\quad \sigma_{zx}=0 \Rightarrow (1-2\nu)\frac{\partial B_z}{\partial x} - \left(\cancel{z\frac{\partial^2 B_z}{\partial x\partial z}} + \frac{\partial^2\beta}{\partial x\partial z}\right) = 0$

since at surface $z=0$

$$\sigma_{zy}=0 \Rightarrow (1-2\nu)\frac{\partial B_z}{\partial y} - \left(\cancel{z\frac{\partial^2 B_z}{\partial y\partial z}} + \frac{\partial^2\beta}{\partial y\partial z}\right) = 0$$

$$\sigma_{zz} = 2(1-\nu)\frac{\partial B_z}{\partial z} - \left(\underbrace{\cancel{z\frac{\partial^2 B_z}{\partial z^2}}}_{=0} + \frac{\partial^2\beta}{\partial z^2}\right) = -\frac{2(1-\nu)}{\mu}p$$

try $h = (1-2\nu)\frac{\partial B_z}{\partial x} - \frac{\partial^2\beta}{\partial x\partial z}$ since $h=0$ on Bdy & $\nabla^2 B_z, \nabla^2\beta = 0$ in $V \Rightarrow \nabla^2 h = 0$ in V

$\Rightarrow h=0$ everywhere: trivial result

if $h = (1-2\nu)\frac{\partial B_z}{\partial y} - \frac{\partial^2\beta}{\partial y\partial z}$ same trivial result

$\therefore$ take $h = 2(1-\nu)\frac{\partial B_z}{\partial z} - \frac{\partial^2\beta}{\partial z^2}$ now $\nabla^2 h = 0$ in V but $h(\xi,\eta,0)\neq 0$

$$\therefore \quad 4\pi h(x,y,z) = -2\frac{\partial}{\partial z}\int_S -\frac{2(1-\nu)}{\mu}\frac{p}{r_0}\,d\xi\,d\eta \qquad \text{using } (*) \text{ above}$$

$$\text{or} \quad h(x,y,z) = \frac{4(1-\nu)}{4\pi\mu}\frac{\partial}{\partial z}\int_S \frac{p(\xi,\eta)}{r_0}\,d\xi\,d\eta$$

$$\therefore \quad 2(1-\nu)\frac{\partial B_z}{\partial z} - \frac{\partial^2 \beta}{\partial z^2} = \frac{(1-\nu)}{\pi\mu}\frac{\partial}{\partial z}\int_S \frac{p(\xi,\eta)}{r_0}\partial\xi\, d\eta \qquad (1)$$

integrate BC. wrt x, y

$$(1-2\nu)\, B_z - \frac{\partial\beta}{\partial z} = f(y,z) = \text{rigid body displ} \quad \text{set} = 0$$

$$(2) \qquad (1-2\nu)\, B_z - \frac{\partial\beta}{\partial z} = f(x,z) = \text{'' '' ''} \quad \text{set} = 0$$

$$\therefore (3) \quad 2(1-\nu)B_z - \frac{\partial\beta}{\partial z} = \frac{1-\nu}{\pi\mu}\int_S \frac{p(\xi,\eta)}{r_0}\partial\xi\, d\eta + f(x,y) = \text{rigid body disp set} = 0$$

solving we get fr (2) & (3)

$$B_z = \frac{1-\nu}{\pi\mu}\int \frac{p(\xi,\eta)}{r_0}\, d\xi\, d\eta \qquad r_0 = \sqrt{(x-\xi)^2+(y-\eta)^2+z^2}$$

$$\frac{\partial\beta}{\partial z} = \frac{(1-2\nu)(1-\nu)}{\pi\mu}\int \frac{p(\xi,\eta)}{r_0}\, d\xi\, d\eta \Rightarrow \beta = \frac{(1-2\nu)(1-\nu)}{\pi\mu}\int\left[\int_S \frac{p(\xi,\eta)\,d\xi\,d\eta}{r_0}\right]dz$$

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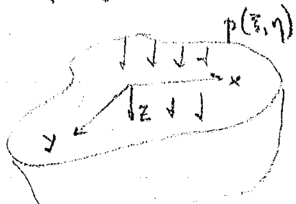
Review of half space problem:



B.C. $\sigma_{zx} = \sigma_{zy} = 0$

$\sigma_{zz} = -p$

$$4\pi\, h(x,y,z) = -2\frac{\partial}{\partial z}\int_S \frac{\partial h(\xi,\eta,0)}{r_0}d\xi\, d\eta - \int_V \left(\frac{1}{r_1}-\frac{1}{r_2}\right)\nabla^2 h\, d\xi\, d\eta\, d\zeta$$

$$r_0 = \sqrt{(x-\xi)^2+(y-\eta)^2+z^2}$$

using Papcovitch Neuber fns w/ $B_z \neq 0$ $\beta \neq 0$
we obtained

$$(1-2\nu)\frac{\partial B_z}{\partial x} - \frac{\partial^2\beta}{\partial x\partial z} = 0$$

$$(*) \quad 2(1-\nu)\frac{\partial B_z}{\partial z} - \frac{\partial^2\beta}{\partial z^2} = -2\frac{(1-\nu)}{\mu}p$$

$$(1-2\nu)\frac{\partial B_z}{\partial y} - \frac{\partial^2\beta}{\partial y\partial z} = 0$$

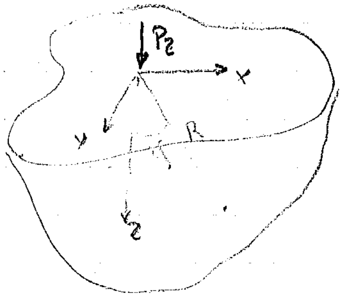
and $\nabla^2 B_z = 0$ $\nabla^2\beta = 0$
since there are no body forces

$\therefore$ take ∇^2 of $(*)$ w/ $h = 2(1-\nu)\dfrac{\partial B_z}{\partial z} - \dfrac{\partial^2\beta}{\partial z^2}$

$$B_z = \frac{1-\nu}{\pi\mu}\int \frac{p(\xi,\eta)\,d\xi\,d\eta}{r_0(x,y,z,\xi,\eta)}$$

$$\beta = \int \frac{(1-2\nu)(1-\nu)}{\pi\mu}\int \frac{p(\xi,\eta)\,d\xi\,d\eta}{r_0}\,dz$$

Boussinesq problem – Point force at origin



$$P_z = \lim_{A\to 0}\int P\, dA$$

$$r_0 = \sqrt{(x-\xi)^2+(y-\eta)^2+z^2} \to \sqrt{x^2+y^2+z^2}$$
when $\xi, \eta \to 0$

$$\therefore B_z = \frac{1-\nu}{\pi\mu}\frac{P_z}{R}$$

$$\beta = \frac{(1-2\nu)(1-\nu)P_z}{\pi\mu}\int \frac{dz}{R}$$

$$= (1-2\nu)(1-\nu)P_z \ln(R+z)$$

Consider now Distributed Normal Loads

$$u_z = B_z - \frac{1}{4(1-\nu)}\frac{\partial}{\partial z}\left(zB_z+\beta\right)$$

$$= B_z - \frac{1}{4(1-\nu)}\left[B_z + z\frac{\partial B_z}{\partial z} + \beta_{,z}\right]$$

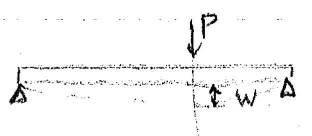
using (2,3) page 10 front side

we also showed that $\frac{\partial\beta}{\partial z} = (1-2\nu)B_z$ for $\sigma_{xz}=\sigma_{yz}=0$

$$= \frac{B_z}{2} - \frac{1}{4(1-\nu)}\left[z\frac{\partial B_z}{\partial z}\right] \quad \text{w/ } B_z = \frac{1-\nu}{\pi\mu}\int\frac{P(\xi,\eta)}{r_0}\,d\xi\,d\eta$$

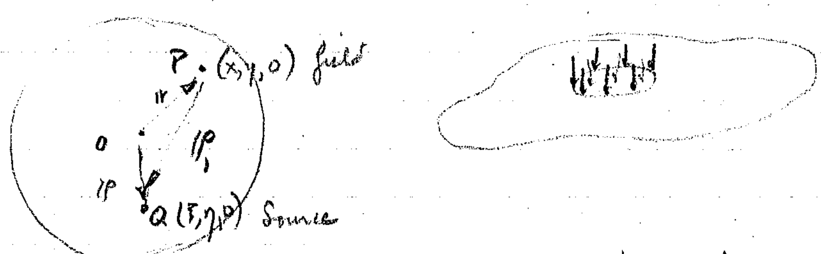
define $w = u_z(x,y,0)$
Settlement due to load.

$$\therefore\ w = \frac{B_z}{2} = \frac{1-\nu}{2\pi\mu}\int\frac{P(\xi,\eta)\,d\xi\,d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}$$

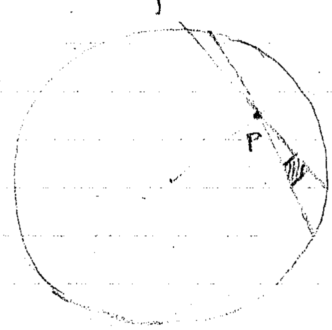


relates the stiffness $Sw = P$

Load distributed over a circle



we will keep P fixed and sweep out Q - to do this we draw a line thru P intersecting circle and then sweep line through 180° to get entire circle
like the ~~discrete~~ continuous analog of the beam



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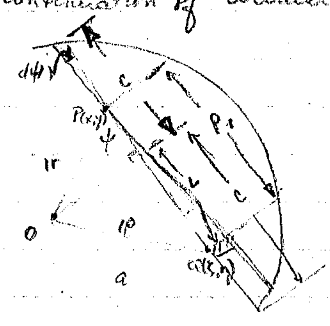
Continuation of Circular Normal load on half space



$v = \rho_1 - r\cos\psi$

$c^2 = a^2 - r^2\sin^2\psi$ (1)

$\rho^2 = v^2 + r^2\sin^2\psi$ (2)

$dA = \rho_1\,d\rho_1\,d\psi$ $\quad d\rho_1 = dv\big|_{\text{const}\,\psi}$ $\quad -c \le v \le c$, $\ -\pi/2 \le \psi \le \pi/2$

$\rho_1 = \sqrt{(x-\xi)^2+(y-\eta)^2}$

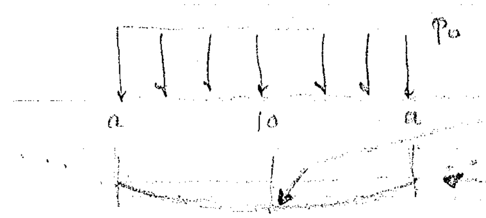
$$w(x,y) = \frac{1-\nu}{2\pi\mu}\int_{-\pi/2}^{\pi/2}\int_{-c}^{c} p(v,\psi)\,dv\,d\psi = \frac{1-\nu}{2\pi\mu}\int_{-\pi/2}^{\pi/2}\int_{-c}^{c}\frac{p(\xi,\eta)\,\rho_1\,d\rho_1\,d\psi}{\rho_1}$$

Case 1 $\quad p = \text{const} = p_0 \Rightarrow \int_{-c}^{c} dv = 2c$ valid under load only

$$w(x,y) = \frac{1-\nu}{\pi\mu}\int_{-\pi/2}^{\pi/2} p_0 c\,d\psi = \frac{2(1-\nu)p_0}{\pi\mu}\int_0^{\pi/2}\sqrt{a^2-r^2\sin^2\psi}\,d\psi$$

Now $E(r/a) = \int_0^{\pi/2}\sqrt{1-(r/a)^2\sin^2\psi}\,d\psi$ complete elliptic integral of the 2nd kind

w/ $E(0) = \pi/2 \quad E(1) = 1$



$$w(0,0,0) = \frac{(1-\nu)\,p_0 a}{\mu}$$

$$w(a,0,0) = \frac{2(1-\nu)p_0 a}{\pi\mu}$$

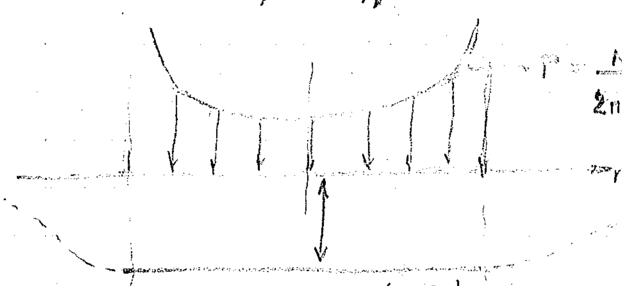
Case 2 $\quad p = \dfrac{N}{2\pi a\sqrt{a^2-\rho^2}} = \dfrac{\text{load}}{\text{unit area}}$ Like a heavy stamp over the area of a small circle

Total force $\quad \int p\,dA = \int_0^{2\pi}\int_0^{a}\frac{N\rho\,d\rho\,d\theta}{2\pi a\sqrt{a^2-\rho^2}} = \frac{N}{2\pi a}\int_0^{2\pi}-\left[\sqrt{a^2-\rho^2}\right]_0^a d\theta = N$

$$w = \frac{(1-\nu)N}{4\pi^2\mu a}\int_{-\pi/2}^{\pi/2}\int_{-c}^{c}\frac{dv\,d\psi}{\sqrt{a^2-\rho^2}} \qquad a^2-\rho^2 = c^2-v^2 \quad \text{using } (1)+(2)$$

$$w = \frac{(1-\nu)N}{4\pi^2\mu a}\int_{-\pi/2}^{\pi/2}\sin^{-1}v/c\,\Big]_{-c}^{c}d\psi = \frac{1-\nu}{4\pi^2\mu a}(\pi/2+\pi/2)\int_{-\pi/2}^{\pi/2}d\psi = \frac{(1-\nu)N}{4\mu a} = \text{constant}$$

$0 \le r \le a$



$$p = \frac{N}{2\pi a\sqrt{a^2-\rho^2}}$$

$$w = \frac{(1-\nu)N}{4\mu a}$$

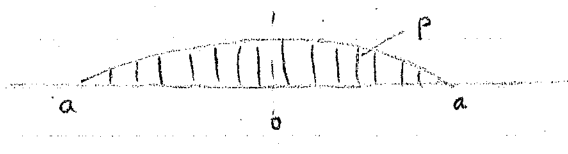
This is like a die applied to an area where the displ. is kept constant but leads to some stress distrib which is same as that shown.

might use this locally on a punch problem of a finite thickness medium if $t \gg 2a$

Case 3 $\quad p = \frac{3N}{2\pi a^2}\sqrt{a^2-\rho^2}$ load/unit area

Total force : $\int p\,dA = \int_0^{2\pi}\int_0^{a} \frac{3N}{2\pi a^3}\sqrt{a^2-\rho^2}\,\rho\,d\rho\,d\theta = N$

$$w = \frac{3(1-\nu)N}{4\pi^2\mu\, a^3}\int_{-\pi/2}^{\pi/2}\int_{-c}^{c}\sqrt{c^2-v^2}\,dv\,d\psi = \frac{3(1-\nu)N}{16\mu a^3}\left(2a^2-r^2\right)$$



center is twice rim
$w(0) = 2\times w(a)$

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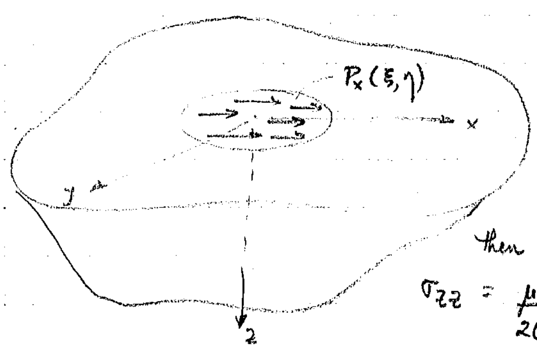
Tangential tractions over surface of Half space



BC. @ $z=0 \quad \sigma_{zz} = \sigma_{zy} = 0 \quad \sigma_{zx} = -p_x$

$T_x = p_x = \sigma_{xz} n_z = -\sigma_{xz} = -\sigma_{zx}$ hence

Let $B_y = 0$

then

$$\sigma_{zz} = \frac{\mu}{2(1-\nu)}\left\{2(1-\nu)\frac{\partial B_z}{\partial z} + 2\nu\frac{\partial B_x}{\partial x} - \left(z\frac{\partial^2 B_z}{\partial z^2} + x\frac{\partial^2 B_x}{\partial z^2} + \frac{\partial^2\beta}{\partial z^2}\right)\right\}$$

$$\sigma_{yz} = \frac{\mu}{2(1-\nu)}\left\{(1-2\nu)\frac{\partial B_z}{\partial y} - \left(z\frac{\partial^2 B_z}{\partial y\partial z} + x\frac{\partial^2 B_x}{\partial y\partial z} + \frac{\partial^2\beta}{\partial y\partial z}\right)\right\}$$

$$\sigma_{zx} = \frac{\mu}{2(1-\nu)}\left\{(1-2\nu)\left(\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right) - \left(z\frac{\partial^2 B_z}{\partial x\partial z} + x\frac{\partial^2 B_x}{\partial x\partial z} + \frac{\partial^2\beta}{\partial x\partial z}\right)\right\}$$

Using the theorem relating values on surface to those inside volume, we find we must try several functions to find something that works. (top of pg 10)

Consider $\quad H = \frac{\partial\sigma_{zx}}{\partial y} - \frac{\partial\sigma_{yz}}{\partial x} = \frac{\mu}{2(1-\nu)}\left\{(1-2\nu)\frac{\partial^2 B_x}{\partial y\partial z} + \frac{\partial^2 B_x}{\partial y\partial z}\right\} = \mu\frac{\partial^2 B_x}{\partial y\partial z}$

For $z=0 \quad \sigma_{yz}=0 \Rightarrow \frac{\partial\sigma_{zy}}{\partial x} = 0$ thus $H(x,y,0) = \frac{\partial\sigma_{zx}}{\partial y} = -\frac{\partial p_x}{\partial y}$

But $\quad H(x,y,0) = -\frac{\partial p_x}{\partial y} = \mu\left.\frac{\partial^2 B_x}{\partial y\partial z}\right|_{z=0}$

integrate wrt $y \quad \therefore \quad -p_x = \mu\left.\frac{\partial B_x}{\partial z}\right|_{z=0}$ (+ fn (which only gives rise to rigid body rot + trans let = 0)

Recall $\nabla^2 B_x = 0$ since we have no body forces $\quad \therefore \frac{\partial}{\partial z}\nabla^2 B_x = \nabla^2\frac{\partial B_x}{\partial z} = 0$
(since $\nabla^2 \mathbf{B} = -\mathbf{F}/\mu$)

Use theorem that $4\pi h(x,y,z) = -2\frac{\partial}{\partial z}\int_S \frac{h(\xi,\eta,0)}{r_0}\,d\xi d\eta - \int_V \left(\frac{1}{r_1}-\frac{1}{r_2}\right)\nabla^2 h(\xi,\eta,\zeta)\,d\xi d\eta d\zeta$ ⊛

with $h = \frac{\partial B_x}{\partial z}$. Then $\frac{\partial B_x}{\partial z}(x,y,z) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int \frac{\left.\frac{\partial B_x}{\partial z}\right|_{z=0}}{r_0}\,ds$ since $\nabla^2 \frac{\partial B_x}{\partial z} = 0$

$\therefore \quad \frac{\partial B_x}{\partial z} = -\frac{1}{2\pi\mu}\frac{\partial}{\partial z}\int \frac{-p_x(\xi,\eta)}{r_0}\,ds$

Now integrate $\quad B_x = \frac{1}{2\pi\mu}\int \frac{p_x(\xi,\eta)}{r_0}\,d\xi d\eta$

Consider $h_1' = (1-2\nu)\frac{\partial B_z}{\partial y} - \left(x\frac{\partial^2 B_x}{\partial y \partial z} + \frac{\partial^2 \beta}{\partial y \partial z}\right)$

$h_1'(x,y,0) = 0$ since this is essentially $\sigma_{yz}|_{z=0} = 0$

now $\nabla^2 h_1' = (1-2\nu)\frac{\partial}{\partial y}\underbrace{\nabla^2 B_z}_{\substack{=0\\ \text{since } f=0}} - \Big(x\frac{\partial^2}{\partial y\partial z}\underbrace{\nabla^2 B_x}_{\substack{=0\\ \text{since } f=0}} + \underbrace{\nabla^2 x}_{=0}\cdot\frac{\partial^2 B_x}{\partial y\partial z} + 1\cdot\frac{\partial^3 B_x}{\partial x\partial y\partial z} + \frac{\partial^2}{\partial y\partial z}\underbrace{\nabla^2\beta}_{=0 \text{ since } f=0}\Big)$

$= -\frac{\partial^3 B_x}{\partial x\partial y\partial z}$

Trick to make h_1' zero everywhere we will use the theorem again by adding $z\frac{\partial^2 B_x}{\partial x\partial y}$ to h_1'

$\therefore \boxed{h_1 = h_1' + z\frac{\partial^2 B_x}{\partial x\partial y}}$

Consider $h_1 = (1-2\nu)\frac{\partial B_z}{\partial y} - \left(-z\frac{\partial^2 B_x}{\partial x\partial y} + x\frac{\partial^2 B_x}{\partial y\partial z} + \frac{\partial^2\beta}{\partial y\partial z}\right)$ (1)

$h_1(x,y,0) = 0$ and $\nabla^2 h_1 = 0$ then by ⊛ $h_1(x,y,z) \equiv 0$

Consider $h_2' = 2(1-\nu)\frac{\partial B_z}{\partial z} + 2\nu\frac{\partial B_x}{\partial x} - \left(x\frac{\partial^2 B_x}{\partial z^2} + \frac{\partial^2\beta}{\partial z^2}\right)$

$h_2'(x,y,0) = 0$ since this is essentially $\sigma_{zz}|_{z=0} = 0$

now $\nabla^2 h_2' = -\frac{\partial B_x}{\partial x\partial z^2}$ as before; to make h_2' zero everywhere again by the theorem add $z\frac{\partial^2 B_x}{\partial x\partial z}$

then $\quad h_2 = h_2' + z\frac{\partial^2 B_x}{\partial x\partial z}$

$\therefore \quad h_2 = 2(1-\nu)\frac{\partial B_z}{\partial z} + 2\nu\frac{\partial B_x}{\partial x} - \left(-z\frac{\partial^2 B_x}{\partial x\partial z} + x\frac{\partial^2 B_x}{\partial z^2} + \frac{\partial^2\beta}{\partial z^2}\right)$ (2)

$\therefore \quad h_2(x,y,0) = 0 \quad$ w/ $\nabla^2 h_2 = 0 \quad \Rightarrow h_2(x,y,z) \equiv 0$

Integrate (1) wrt y

$\int h_1\,dy = (1-2\nu)B_z - \left(-z\frac{\partial B_x}{\partial x} + x\frac{\partial B_x}{\partial z} + \frac{\partial\beta}{\partial z}\right) = 0$ (3) (+ fn of (x,y) r.b.r or tion)

differentiate (3) wrt z

$\therefore (1-2\nu)\frac{\partial B_z}{\partial z} - \left(x\frac{\partial^2 B_x}{\partial z^2} + \frac{\partial^2\beta}{\partial z^2} - \frac{\partial B_x}{\partial x} - z\frac{\partial^2 B_x}{\partial x\partial z}\right) = 0$ (4)

Subtract (2) from (4) to give

$-\left(\frac{\partial B_z}{\partial z} + 2\nu\frac{\partial B_x}{\partial x} - \frac{\partial B_x}{\partial x}\right) = 0$ (5)

or $\dfrac{\partial B_z}{\partial z} = (1-2\nu)\dfrac{\partial B_x}{\partial x}$

from (3) solve for $\dfrac{\partial \beta}{\partial z}$

$$\frac{\partial \beta}{\partial z} = (1-2\nu) B_z - x\frac{\partial B_x}{\partial z} + z\frac{\partial B_x}{\partial x}$$

One Hour midterm - Friday May 11 Open notes - no books
HW : Due May 16.

Knowing B_x we can get $\dfrac{\partial B_x}{\partial x}$ $\Rightarrow$ we know $\dfrac{\partial B_z}{\partial z}$ $\Rightarrow$ integrate to get B_z

knowing B_x we can get $\dfrac{\partial B_x}{\partial x}$, $\dfrac{\partial B_x}{\partial z}$. Knowing B_z also $\Rightarrow$ we know $\dfrac{\partial \beta}{\partial z}$ $\Rightarrow$ integrate to get β.

5-4-79

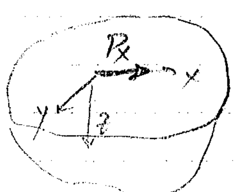
if we have tangential forces on a half space then
if $p_x(\xi,\eta)$ is given at $z=0$

$$B_x = \frac{1}{2\pi\mu}\int \frac{p_x(\xi,\eta)}{r_0}\,d\xi\,d\eta$$

$$\frac{\partial B_z}{\partial z} = (1-2\nu)\frac{\partial B_x}{\partial x}\ ; \quad B_y = 0$$

$$\frac{\partial \beta}{\partial z} = (1-2\nu) B_z - x\frac{\partial B_x}{\partial z} + z\frac{\partial B_x}{\partial x}$$

Concentrated force in tangential direction on a half space (Cerruti's Problem)



$$r_0^2 = (x-\xi)^2 + (y-\eta)^2 + z^2$$

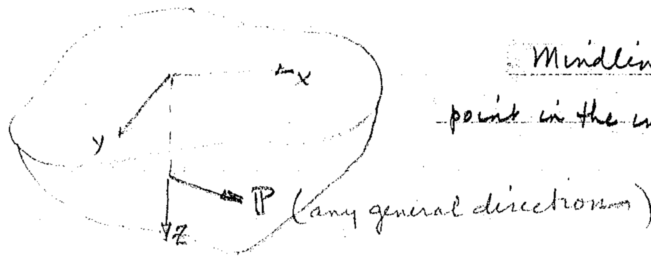
$$P_x = \lim_{A\to 0}\int_A p_x\,d\xi\,d\eta \qquad r_0^2 \to R^2 = x^2+y^2+z^2$$

$$\therefore\ B_x = \frac{P_x}{2\pi\mu R} \qquad \therefore\ \frac{\partial B_x}{\partial x} = -\frac{P_x\,x}{2\pi\mu R^3} \qquad \frac{\partial B_x}{\partial z} = -\frac{P_x\,z}{2\pi\mu R^3}$$

$$\frac{\partial B_z}{\partial z} = -\frac{(1-2\nu)P_x\,x}{2\pi\mu R^3} \Rightarrow \boxed{B_z = \frac{(1-2\nu)P_x\,x}{2\pi\mu R\,(R+z)}}$$

after subst: $\dfrac{\partial \beta}{\partial z} = \dfrac{(1-2\nu)^2 P_x\,x}{2\pi\mu R\,(R+z)} \Rightarrow \boxed{\beta = -\dfrac{(1-2\nu)^2 P_x\,x}{2\pi\mu\,(R+z)}}$

Caution: All these integrals give integrating fns which we hope involve only rigid body rotation/translation & can be dropped.

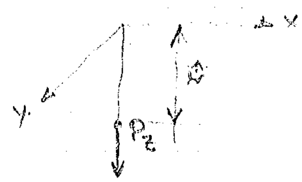


Mindlin Problem: A concentrated force at any point in the interior of a half-space

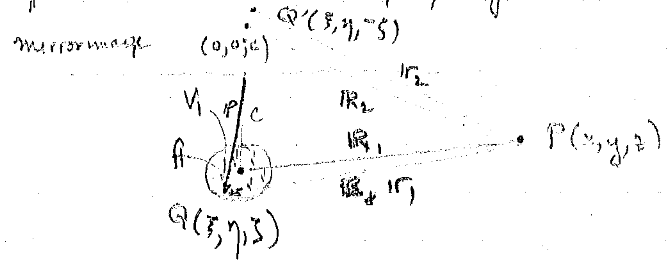
write $\mathbb{P} = P_z\, e_z + P_{x\text{-}y}\, e_{x\text{-}y}$

$e_{x\text{-}y}$ is the unit vector in the x-y plane

Look at P_z portion of prob.



Assume like a Kelvin solution with a distrib body forces $\mathbb{f}$ in a small volume V_1 w/surface area S



$$r_1^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$$
$$r_2^2 = (x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2$$
$$R_1^2 = (x^2+y^2) + (z-c)^2$$
$$R_2^2 = (x^2+y^2) + (z+c)^2$$
$$\rho^2 = \xi^2+\eta^2+\zeta^2 \quad \text{and} \quad \mathbb{f} = f_z e_z$$

as $V_1 \to 0$ $\quad r_1 \to R_1 \quad r_2 \to R_2 \quad \rho \to c$

define limit as $V_1 \to 0$ $\quad \int_{V_1} f_z\, d\xi\, d\eta\, d\zeta = P_z$

Let $B_x, B_y = 0$ and $B_z, \beta \neq 0$: But $\nabla^2 B_z = -\dfrac{\mathbb{f}}{\mu} = \begin{cases} -\dfrac{f_z}{\mu} & \text{in } V_1 \\ 0 & \notin V_1 \end{cases}$

also $\nabla^2 \beta = \begin{cases} 2f_z/\mu & \text{in } V_1 \\ 0 & \notin V_1 \end{cases}$

Calculate stresses

$$\sigma_{zz} = \frac{\mu}{2(1-\nu)}\left\{2(1-\nu)\frac{\partial B_z}{\partial z} - z\frac{\partial^2 B_z}{\partial z^2} - \frac{\partial^2 \beta}{\partial z^2}\right\}$$
$$\sigma_{zy} = \frac{\mu}{2(1-\nu)}\left\{(1-2\nu)\frac{\partial B_z}{\partial y} - z\frac{\partial^2 B_z}{\partial y\,\partial z} - \frac{\partial^2 \beta}{\partial y\,\partial z}\right\}$$
$$\sigma_{zx} = \frac{\mu}{2(1-\nu)}\left\{(1-2\nu)\frac{\partial B_z}{\partial x} - z\frac{\partial^2 B_z}{\partial x\,\partial z} - \frac{\partial^2 \beta}{\partial x\,\partial z}\right\}$$

on bdy $z=0$ $\quad \sigma_{zz}=0, \; \sigma_{zy}=0, \; \sigma_{zx}=0$

Let $h = 2(1-\nu)\frac{\partial B_z}{\partial z} - \frac{\partial^2 \beta}{\partial z^2} \quad \stackrel{ie}{\Rightarrow} \sigma_{zz} = 0$

Now $\nabla^2 h = -\frac{2(1-\nu)}{\mu}\frac{\partial f_z}{\partial z} - \frac{1}{\mu}\frac{\partial^2}{\partial z^2}(z f_z)$ in V_1

now $h = 0$ at $z = 0$ and $\nabla^2 h = 0$ outside V_1

for h inside V_1 then

$$\therefore \quad h(x,y,z) = \frac{-1}{4\pi\mu}\int_{V_1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\nabla^2 h \, dV_1 = \frac{1}{4\pi\mu}\int_{V_1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\left[\frac{+2(1-\nu)}{\mu}\frac{\partial f_z(\xi,\eta,\zeta)}{\partial \zeta} + \frac{\partial^2}{\partial \zeta^2}(\zeta f_z)\right] d\xi d\eta d\zeta$$

Now carry out the operation & let $V_1 \to 0$

$$h = \lim_{V_1 \to 0} \frac{1}{4\pi\mu}\int_{V_1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\left[2(1-\nu)\frac{\partial f_z}{\partial \zeta} + \frac{\partial^2}{\partial \zeta^2}(\zeta f_z)\right] d\xi d\eta d\zeta$$

$$= \frac{P_z}{4\pi\mu}\left[2(1-\nu)\frac{\partial}{\partial z}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + c\frac{\partial^2}{\partial z^2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right]$$

$$\Rightarrow 2(1-\nu)\frac{\partial B_z}{\partial z} - \frac{\partial^2 \beta}{\partial z^2} = \frac{P_z}{4\pi\mu}\left[2(1-\nu)\frac{\partial}{\partial z}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + c\frac{\partial^2}{\partial z^2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right]$$

$$\therefore \quad 2(1-\nu)B_z - \frac{\partial \beta}{\partial z} = \frac{P_z}{4\pi\mu}\left[2(1-\nu)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + c\frac{\partial}{\partial z}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right]$$

5/7/79

Continuation of the previous problem.

Consider $\sigma_{zx}\Big|_{z=0} = 0 = (1-2\nu)\frac{\partial B_z}{\partial x} - \frac{\partial^2 \beta}{\partial x \partial z}$

let $h = (1-2\nu)\frac{\partial B_z}{\partial x} - \frac{\partial^2 \beta}{\partial x \partial z}$ for $z=0$ $h=0$ $\qquad \nabla^2 B_z = -f_z/\mu \quad \nabla^2 \beta = \frac{z f_z}{\mu}$

$\nabla^2 h = -\frac{(1-2\nu)}{\mu}\frac{\partial f_z}{\partial x} - \frac{1}{\mu}\frac{\partial^2}{\partial x \partial z}(z f_z)$ Now using Green's theorem

thus $h(x,y,z) = \frac{+1}{4\pi\mu}\int_{V_1} -G\left[(1-2\nu)\frac{\partial f_z(\xi,\eta,\zeta)}{\partial \xi} + \frac{\partial^2(\zeta f_z)}{\partial \xi \partial \zeta}\right] d\xi d\eta d\zeta$

Now $\int_{V_1} G\frac{\partial f_z}{\partial \xi} d\xi d\eta d\zeta = \underbrace{\int_S G f_z \, d\eta d\zeta}_{=0 \text{ on surface since } f_z = 0 \text{ on } S} - \int_{V_1} f_z \frac{\partial G}{\partial \xi} d\xi d\eta d\zeta \qquad G = \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

$\frac{\partial G}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{x-\xi}{r_1^3} - \frac{x-\xi}{r_2^3}$ since $r_{1,2} = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z \mp \zeta)^2}$

$\lim_{V_1 \to 0} -\int f_z\left[\frac{x-\xi}{r_1^3} - \frac{x-\xi}{r_2^3}\right] d\xi d\eta d\zeta \quad$ (w/ $\xi \to 0$ $r_1 \to R_1$ $r_2 \to R_2$ $\int f_z \, d\xi d\eta d\zeta \to P_z$)

$\Rightarrow -P_z\left[\frac{x}{R_1^3} - \frac{x}{R_2^3}\right] = P_z\frac{\partial}{\partial x}\left[\frac{1}{R_1} - \frac{1}{R_2}\right] = \int_{V_1} G\frac{\partial f_z}{\partial \xi} d\xi d\eta d\zeta$

integration by parts in 2 dim.

we must now find $\int G \frac{\partial^2(\zeta f_z)}{\partial\xi\,\partial\zeta}\, d\xi\, d\eta\, d\zeta = \int_S G \underbrace{\frac{\partial}{\partial\zeta}(\zeta \cancel{f_z})}_{=0}\, d\eta\, d\zeta - \int_V \frac{\partial G}{\partial\xi}\frac{\partial(\zeta f_z)}{\partial\zeta}\, d\xi\, d\eta\, d\zeta$

on surface since $f_z = 0$

integrate again $= -\int_S \frac{\partial G}{\partial\xi}(\zeta \underbrace{\cancel{f_z}}_{=0})\, d\xi\, d\eta + \int_{V_1} \zeta f_z \frac{\partial^2 G}{\partial\xi\,\partial\zeta}\, d\xi\, d\eta\, d\zeta$

$f_z = 0$ on surface

$$\frac{\partial^2 G}{\partial\xi\,\partial\zeta} = \frac{\partial}{\partial\zeta}\left(\frac{\partial G}{\partial\xi}\right) = \frac{\partial}{\partial\zeta}\left[\frac{x-\xi}{r_1^3} - \frac{x-\xi}{r_2^3}\right] = 3(x-\xi)\left[\frac{z-\zeta}{r_1^5} + \frac{z+\zeta}{r_2^5}\right]$$

take limit as $V_1 \to 0$ of $\int_{V_1} \zeta f_z \frac{\partial^2 G}{\partial\xi\,\partial\zeta}\, d\zeta\, d\eta\, d\xi$ $\left(\text{w/ } \zeta \to c \quad \int f_z\, dV = P_z \right.$
$\left. \xi \to 0 \quad r_1 \to R_1 \quad r_2 \to R_2\right)$

$$\therefore \Rightarrow 3x\left[\frac{z-c}{R_1^5} + \frac{z+c}{R_2^5}\right] P_z \cdot c = -cP_z \frac{\partial}{\partial z}\left[\frac{x}{R_1^3} + \frac{x}{R_2^3}\right] = cP_z \frac{\partial^2}{\partial z\,\partial x}\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

$$\therefore h(x,y,z) = (1-2\nu)\frac{\partial B_z}{\partial x} - \frac{\partial^2 \beta}{\partial x\,\partial z} = \frac{P_z}{4\pi\mu}\left\{(1-2\nu)\frac{\partial}{\partial x}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + c\frac{\partial^2}{\partial x\,\partial z}\left[\frac{1}{R_1} + \frac{1}{R_2}\right]\right\}$$

. 3^{rd} BC gives something w/ $\frac{\partial}{\partial y}$ replacing $\frac{\partial}{\partial x}$ giving same result.

from above $\int$ w/r $x \Rightarrow (1-2\nu)B_z - \frac{\partial\beta}{\partial z} = \frac{P_z}{4\pi\mu}\left\{(1-2\nu)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + c\frac{\partial}{\partial z}\left[\frac{1}{R_1} + \frac{1}{R_2}\right]\right\}$

use this w/ $2(1-\nu)B_z - \frac{\partial\beta}{\partial z} = \frac{P_z}{4\pi\mu}\left[2(1-\nu)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + c\frac{\partial}{\partial z}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right]$ from before

to get $\boxed{B_z = \frac{P_z}{4\pi\mu}\left[\frac{1}{R_1} + \frac{3-4\nu}{R_2} - 2c\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)\right]}$

using one of the other eqns to get

$$\boxed{\frac{\partial\beta}{\partial z} = \frac{P_z}{4\pi\mu}\left[\frac{4(1-2\nu)(1-\nu)}{R_2} - c\frac{\partial}{\partial z}\left(\frac{1}{R_1}\right) - (3-4\nu)c\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)\right]}$$

integrate w/rt z to get

$$\boxed{\beta = \frac{P_z}{4\pi\mu}\left[-\frac{c}{R_1} - \frac{c(3-4\nu)}{R_2} + 4(1-2\nu)(1-\nu)\log(R_2 + z + c)\right]}$$

as $c \to \infty$ we should recover Kelvin Soln. (Bdry effects no longer important)
$c \to 0$ " " " Boussinesq Soln / or Cerruti

Mindlin solved problem in different manner – assume a force at $(0,0,c)$ Kelvin Problem + nuclei of strain in $-z$ region $\ni$ we annihilate $\sigma_{zx}, \sigma_{zy}, \sigma_{zz}$ at $z=0$

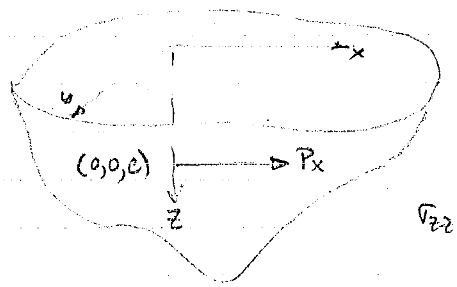
this is what Mindlin found

1. A single force at 0, 0, +c
2. A " " " 0, 0, −c
3. " double " " 0, 0, −c
4. a center of compression at (0, 0, −c)
5. A line of centers of compression extending from $z = -c$ to $z = -\infty$
6. A doublet at (0, 0, −c)

5/9/75

<u>Exam</u>: Friday – Notes & HW only allowed

Mindlin Problem – Part II – Force ∥ to X axis



No longer have axial symmetry ⇒ only $B_y = 0$
need B_x, B_z, β
we write tractions on surface

$$\sigma_{zz} = \frac{\mu}{2(1-\nu)}\left[2(1-\nu)\frac{\partial B_z}{\partial z} + 2\nu\frac{\partial B_x}{\partial x} - x\frac{\partial^2 B_x}{\partial z^2} - \frac{\partial^2\beta}{\partial z^2}\right] = 0$$

$$\sigma_{zx} = \frac{\mu}{2(1-\nu)}\left[(1-2\nu)\left(\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right) - x\frac{\partial^2 B_x}{\partial x\partial z} - \frac{\partial^2\beta}{\partial z\partial x}\right] = 0$$

(*) $$\sigma_{zy} = \frac{\mu}{2(1-\nu)}\left[(1-2\nu)\left(\frac{\partial B_z}{\partial y}\right) - x\frac{\partial^2 B_x}{\partial y\partial z} - \frac{\partial^2\beta}{\partial y\partial z}\right] = 0$$

Consider P_x is a distributed force f_x over a small volume $P_x = \lim_{V\to 0}\int f_x\,dV$

remember $\nabla^2 B_x = -f_x/\mu$ $\quad \nabla^2 B_z = 0 \quad \nabla^2\beta = -\frac{f_x x}{\mu}$

Now take $\frac{\partial\sigma_{zx}}{\partial y} - \frac{\partial\sigma_{zy}}{\partial x} = 0 \Rightarrow \frac{\partial^2 B_x}{\partial y\partial z} = 0$ on $z = 0$ $\Rightarrow \frac{\partial B_x}{\partial z} = 0$

In general $\nabla^2 B_x = -f_x/\mu \Rightarrow \nabla^2\frac{\partial B_x}{\partial z} = -\frac{1}{\mu}\frac{\partial f_x}{\partial z}$

use green's theorem on $\frac{\partial B_x}{\partial z}$; $\nabla^2\left(\frac{\partial B_x}{\partial z}\right) = -\frac{1}{\mu}\frac{\partial f_x}{\partial z}$ in V & $\frac{\partial B_x}{\partial z} = 0$ on S

$\therefore \quad \frac{\partial B_x}{\partial z}(x,y,z) = \frac{1}{4\pi\mu}\int_V G\frac{\partial f_x}{\partial\zeta}\,d\xi\,d\eta\,d\zeta \qquad G = 1/r_1 - 1/r_2$ since we integrate by parts

now take $\lim V\to 0 \Rightarrow \frac{\partial B_x}{\partial z} = \frac{1}{4\pi\mu}P_x\frac{\partial}{\partial z}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \Rightarrow \boxed{B_x = \frac{P_x}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$

look at (*) since $\frac{\partial B_x}{\partial z} = 0$ on bdy ⇒ $\sigma_{zy} = (1-2\nu)\frac{\partial B_z}{\partial y} - \frac{\partial^2\beta}{\partial y\partial z} = 0$ or

$(1-2\nu)B_z = \frac{\partial\beta}{\partial z}$ integrating w.r.t. y.

in z $\nabla^2\left[(1-2\nu)B_z - \frac{\partial\beta}{\partial z}\right] = -\frac{x}{\mu}\frac{\partial f_x}{\partial z}$ since $\nabla^2 B_z = 0$ as shown before

now use greens fn. on $(1-2\nu)B_z - \frac{\partial\beta}{\partial z} = h$ w/ $h=0$ on $z=0$ & $\nabla^2 h = -\frac{x}{\mu}\frac{\partial f_x}{\partial z}$

$\therefore (1-2\nu)B_z - \frac{\partial\beta}{\partial z} = \frac{1}{4\pi\mu}\int_V G\,\xi\frac{\partial f_x}{\partial \zeta}\,d\xi\,d\eta\,d\zeta$

in limit as $V\to 0$ $\xi\to 0$ $G\to 0$ $\int f_x\,d\xi\,d\eta\,d\zeta \to P_x$ do this by parts

(†) $\therefore (1-2\nu)B_z - \frac{\partial\beta}{\partial z} = 0$ everywhere

need another relation between B_z, β use σ_{zz}

let $\chi = 2(1-\nu)\frac{\partial B_z}{\partial z} + 2\nu\frac{\partial B_x}{\partial x} - x\frac{\partial^2 B_x}{\partial z^2} - \frac{\partial^2\beta}{\partial z^2}$ $h=0$ on $z=0$

(**) rewrite $\chi = 2(1-\nu)\frac{\partial B_z}{\partial z} - \frac{\partial^2\beta}{\partial z^2} - (1-2\nu)\frac{\partial B_x}{\partial x} - \frac{P_x c}{2\pi\mu}\frac{\partial^2}{\partial x\partial z}\left(\frac{1}{R_2}\right)$ using B_x

$\nabla^2\chi = -\frac{x}{\mu}\frac{\partial^2 f_x}{\partial z^2} + \frac{(1-2\nu)}{\mu}\frac{\partial f_x}{\partial x}$

In z using green's fn: $\chi = \frac{1}{4\pi\mu}\int_V G\,\xi\frac{\partial^2 f_x}{\partial\zeta^2}\,d\xi\,d\eta\,d\zeta - \frac{(1-2\nu)}{4\pi\mu}\int_V G\frac{\partial f_x}{\partial \xi}\,d\xi\,d\eta\,d\zeta$

$=0$ in limit after IBP $\xi\to 0$ and $G=0$ on Bdry

integrate by parts & take limit $V\to 0$

$\chi = \frac{1-2\nu}{4\pi\mu}\int_V f_x\frac{\partial G}{\partial\xi}\,d\xi\,d\eta\,d\zeta$: take limit $= -\frac{(1-2\nu)P_x}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$

put this back into (**) and integrate wrt to get

(††) $2(1-\nu)B_z - \frac{\partial\beta}{\partial z} = \frac{(1-2\nu)P_x x}{2\pi\mu R_2(R_2+z+c)} - \frac{P_x c x}{2\pi\mu R_2^3}$

using (†) and (††) we can get

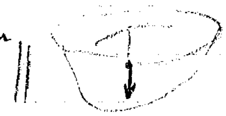
$$\boxed{B_z = \frac{P_x x}{2\pi\mu R_2}\left[\frac{1-2\nu}{R_2+z+c} - \frac{c}{R_2^2}\right]}$$

$$\boxed{\beta = \frac{P_x(1-2\nu)x}{2\pi\mu(R_2+z+c)}\left[\frac{c}{R_2} - (1-2\nu)\right]}$$

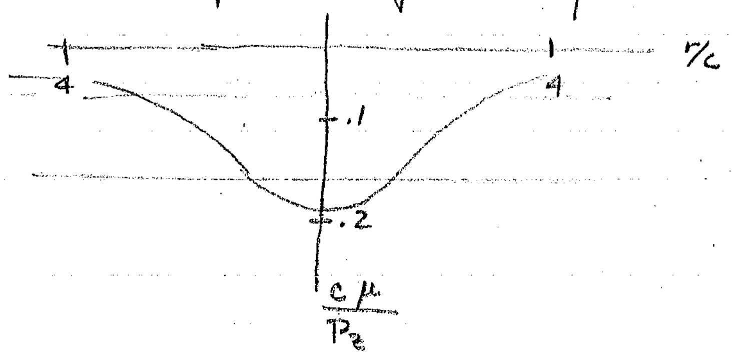
Mindlin solved this using following singularities (superposition of nuclei of strain)

1) Single force at $(0,0,c)$
2) " " at $(0,0,-c)$ same dir
3) Doublet at $(0,0,-c)$
4) Semi infinite line of doublets from $z=-c$ to $z=-\infty$ w/ strength proportional to distance from $-c$

5) Double force w/moment at (0,0,−c)
6) Semi infinite line of double forces w/moment extending from (−c, −∞)

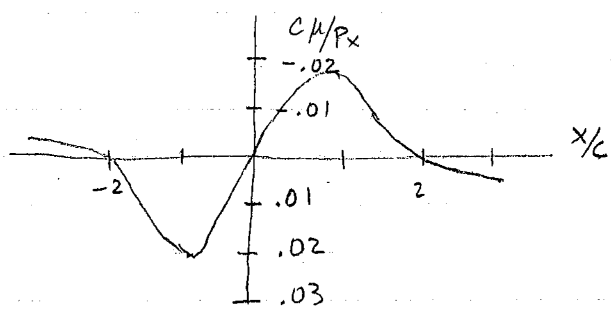
In first problem of mindlin  suppose you want settlement.

Vertical displacement of Boundary in Part I



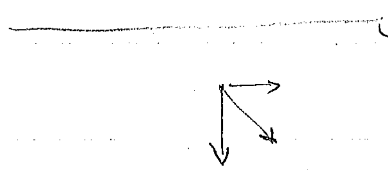
5/14/79

Part II Settlement − Vertical displ. of Boundary in Part II



Why wasn't mindlin's problem solved by the following

Kelvin Problem



calculate tractions
& apply opposite tractions
by use of ~~Kelvin~~ Boussinesq/ Cerruti Problem.

Mindlin's background was limited in math − he tried it this way first but the integrals were hard so he abandoned it.

Stresses & displ are found in his paper "Force at a Point in the Interior of a Semi-Infinite Solid" − R. D. Mindlin Physics, Vol 7, May 1936 Pg 195−202.

Mindlin problem is an intermediate case of the Kelvin (w/ BC far away) and the Boussinesq problem (w/ BC in fact plays a part).

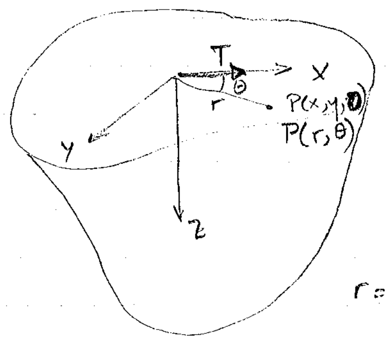
Last few lectures - possible topics

Thermo elasticity

- Dynamic next fall class
- Large Deformation
- Complex potential } together
- Fracture Mechanics

Displacements of $z=0$ plane (x-y plane) for Cerruti Problem.

Consider tangential forces distributed over a circle in the surface of half space as a prelude to contact problem. (using Cerruti Problem)



Using Cerruti Problem $u_x(x,y,0) = \dfrac{T}{2\pi\mu r} - \nu \dfrac{\partial}{\partial x}\left(\dfrac{Tx}{2\pi\mu r}\right)$

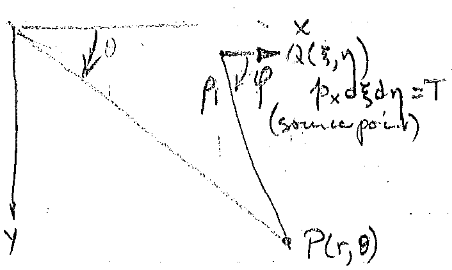
$r = \sqrt{x^2+y^2}$ $\qquad u_y(x,y,0) = -\nu \dfrac{\partial}{\partial y}\left(\dfrac{Tx}{2\pi\mu r}\right)$

thus

$$\boxed{\begin{aligned} u_x(x,y,0) &= \frac{T}{2\pi\mu r}\left[1-\nu+\underbrace{\frac{\nu x^2}{r^2}}_{\frac{\nu(1+\cos 2\theta)}{2}}\right] \\ u_y(x,y,0) &= \frac{T\nu xy}{2\pi\mu r^3}\end{aligned}} \quad \boxed{\begin{aligned} \text{in polar coords.} &= \frac{T}{4\pi\mu r}\left[2-\nu+\nu\cos 2\theta\right] \\ &= \frac{T\nu}{2\pi\mu r}\,\frac{\sin 2\theta}{2}\end{aligned}}$$

Now the above is only good when T is applied at (0,0,0)

if T is applied at (ξ,η) then



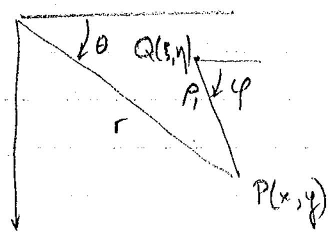
since p_x is an elementary load then we can write $du_x = \dfrac{p_x}{4\pi\mu\rho_1}\left[2-\nu+\nu\cos 2\varphi\right]d\xi d\eta$

$$du_y = \frac{p_x \nu\, d\xi d\eta}{4\pi\mu r_1}\sin 2\varphi.$$

$$\left.\begin{aligned} u_x &= \int_A \frac{p_x\left[2-\nu+\nu\cos 2\varphi\right]}{4\pi\mu\rho_1}\,d\xi d\eta \\ u_y &= \int_A \frac{p_x\,\nu\sin 2\varphi}{4\pi\mu\rho_1}\,d\xi d\eta \end{aligned}\right\} \text{ where } p_x = p_x(\xi,\eta)$$

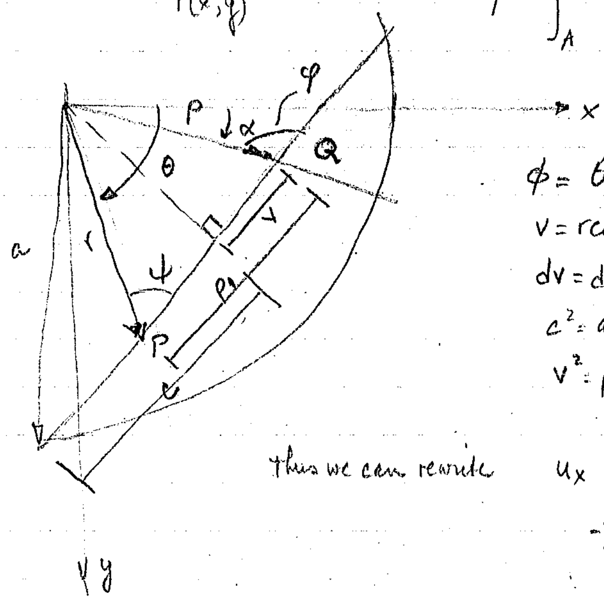
Take Home final — 5/16/79

Continuation of the last class



$$u_x = \int_A \frac{1}{4\pi\mu}\, p_x\, \frac{2-\nu+\nu\cos 2\phi}{\rho_1}\, d\xi\, d\eta$$

$$u_y = \int_A \frac{\nu}{4\pi\mu}\, p_x \frac{\sin 2\phi}{\rho_1}\, d\xi\, d\eta$$



$$\phi = \theta + \psi$$
$$v = r\cos\psi - \rho_1$$
$$dv = d\rho_1$$
$$c^2 = a^2 - r^2\sin^2\psi$$
$$v^2 = \rho^2 - r^2\sin^2\psi$$

$$d\xi\, d\eta = \rho_1\, d\rho_1\, d\psi = \rho_1\, dv\, d\psi$$

thus we can rewrite
$$u_x \int_{-\pi/2}^{\pi/2} \frac{1}{4\pi\mu} \int_{-c}^{c} p_x \left[2-\nu+\nu\cos 2\phi\right] dv\, d\psi$$

$$u_y = \frac{\nu}{4\pi\mu} \int_{-\pi/2}^{\pi/2} \int_{-c}^{c} p_x \left[\sin 2\phi\right] dv\, d\psi$$

the above is for any p_x

for $p_x = \dfrac{T}{2\pi a} \dfrac{1}{\sqrt{a^2-\rho^2}}$ w/ $\int p_x\, dA = T$

$$\therefore\ u_x = \frac{T}{8\pi^2\mu a} \int_{-\pi/2}^{\pi/2} \int_{-c}^{c} \frac{+2-\nu+\nu\cos 2(\theta+\psi)}{\sqrt{c^2-v^2}}\, d\psi\, dv$$

$$= \frac{T}{8\pi^2\mu a} \int_{-\pi/2}^{\pi/2} \left[2-\nu+\nu\cos 2(\theta+\psi)\right] \sin^{-1}\frac{v}{c} \Big|_{-c}^{c} d\psi$$

$$= \frac{T\,(2-\nu)}{8\mu a} + \frac{T\nu}{8\pi\mu a}\left[\frac{1}{2}\sin 2(\theta+\psi)\right]_{-\pi/2}^{\pi/2}$$

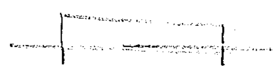
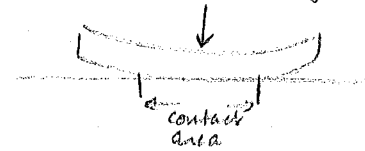
$$u_x = \frac{(2-\nu)T}{8\mu a}$$

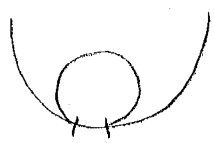
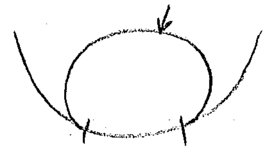
displ in direction of load is a constant ie circle moves as a rigid body in x direction

$$u_y = \frac{\nu T}{8\pi^2 \mu a}\int_{-\pi/2}^{\pi/2} \frac{\sin 2\phi}{\sqrt{c^2-v^2}}\, dv\, d\psi = 0$$

thus the circle will displace in x direction as a rigid body only. No displacement in y direction. Similar to the Case 2 of Boussinesq problem.

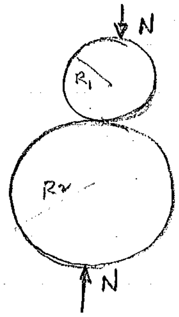
This is important when we talk about CONTACT problems.
In contact problems what we need to know is the contact surface
ie

PRE LOAD  POST LOAD 

pre load  post load 

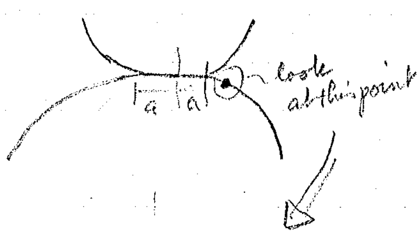
the larger to load the smaller the contact area in beam problem

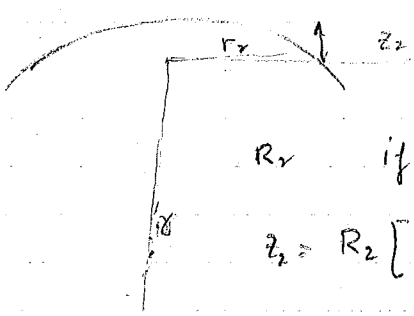
Contact Problems - Hertz about 100 years ago
Ball Bearings under applied loads



we need to know contact area as a fn of N
& tractions in contact area

We assume
1. Contact area is flat
2. " " " circular
3. " circle radius $\ll R_1, R_2$ (look at local problem)
4. N is applied remotely & how does not affect contact problem.





$$\cos\gamma = \sqrt{\cos^2\gamma} = \sqrt{1-\sin^2\gamma} = \sqrt{1-\left(\frac{r_2}{R_2}\right)^2}$$

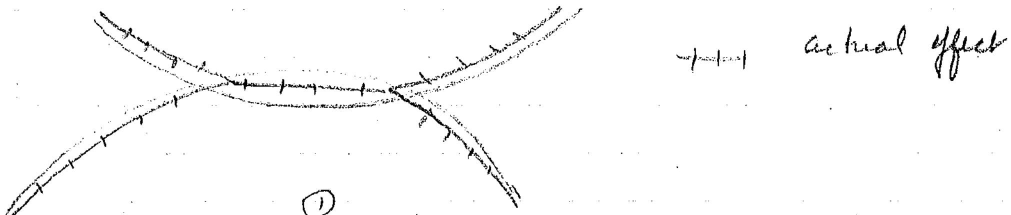
$$z_2 = R_2(1-\cos\gamma) = R_2\left(1-\sqrt{1-\left(\frac{r_2}{R_2}\right)^2}\right)$$

if we assume z_2 and r_2 are small use binomial

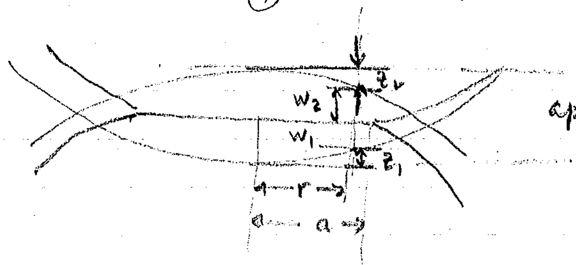
$$z_2 = R_2\left[1-\left(1-\frac{1}{2}\left(\frac{r_2}{R_2}\right)^2+\dots\right)\right] = \frac{r_2^2}{2R_2}$$

define similarly z_1 on upper sphere $z_1 = \dfrac{r_1^2}{2R_1}$

Now sketch sphere if they penetrated & did not deform



①



②

w_2, w_1 are settlements

approach $= \alpha = z_1 + w_1 + z_2 + w_2$

$w_1 + w_2 = \alpha - (z_1 + z_2)$

We assume in the local that bearing ② looks like a half space to ① and no shear on the face. We can thus assume contact area of sphere is that of problem in a half space thus we can use settlements

5/18/79

Friday - 1 June Take Home will be handed out and due the following Monday or Tuesday

$$w_1 + w_2 = \alpha - (z_1 + z_2) \qquad \text{Recall} \quad w = \frac{1-\nu}{2\pi\mu}\int_{-\pi/2}^{\pi/2}\int_{-c}^{c} p_z(v,\psi)\,dv\,d\psi$$

$$\therefore \left[\frac{1-\nu_1}{2\mu_1\pi} + \frac{1-\nu_2}{2\pi\mu_2}\right]\int_{-\pi/2}^{\pi/2}\int_{-c}^{c} p_z(v,\psi)\,dv\,d\psi = \alpha - \frac{r^2}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

look at other side for how z_1, z_2 were obtained

look at case 3 of boussinesq problem : Recall for $p_z = \dfrac{3N\sqrt{a^2 - r^2}}{2\pi a^3}$

where $\iint p_z\,dA = N$ then $w_i = \dfrac{3(1-\nu_i)N}{16a^3\mu_i}\left(2a^2 - r^2\right) = \dfrac{3k_iN\pi}{8a^3}\left(2a^2 - r^2\right)$

if we let $\dfrac{1-\nu_i}{2\pi\mu_i} = k_i$ thus $(k_1 + k_2)\left[\dfrac{3N\pi}{8a^3}\left(2a^2 - r^2\right)\right] = \alpha - \dfrac{r^2}{2}\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)$

$$(k_1 + k_2)\frac{3N\pi}{4a} = \alpha \quad \text{and} \quad (k_1 + k_2)\frac{3N\pi}{8a^3} = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{2}\left(\frac{R_1 + R_2}{R_1R_2}\right)$$

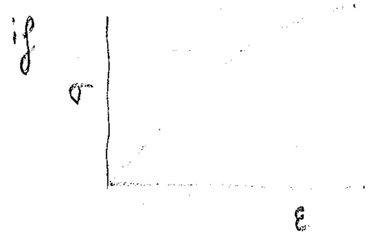
k_1, k_2 are material constants, R_1 & R_2 are physical const and N is applied load

Solve for a

$$\therefore \boxed{a = \sqrt[3]{\frac{(k_1 + k_2)\,3N\pi\,R_1R_2}{4(R_1 + R_2)}} = K_1 N^{1/3}}$$

$$\boxed{\alpha = \sqrt[3]{\frac{9N^2\pi^2(k_1 + k_2)^2(R_1 + R_2)}{16R_1R_2}} = K_2 N^{2/3}}$$

if



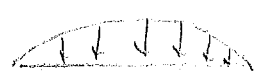
$\sigma = E\varepsilon$ E is stiffness

$\varepsilon = \frac{1}{E}\sigma$ $\frac{1}{E}$ is compliance

we can define compliance $\frac{d\alpha}{dN} = \frac{2}{3}K_2 N^{-1/3}$

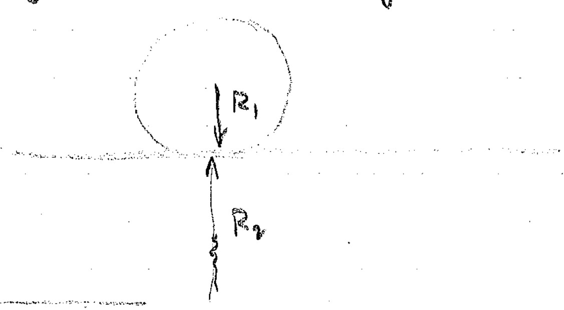
$$C = \frac{d\alpha}{dN} = \sqrt[3]{\frac{\pi^2(k_1+k_2)^2(R_1+R_2)}{6R_1R_2N}}$$

pressure distrib at the contact surface



to find stresses below surface we can use boussinesq problem superposed – & use boussinesq problem to get stresses below surface ~~but we~~ up to what depth can we use it?

Problem for a sphere on a flat surface take one of the radii to be ∞



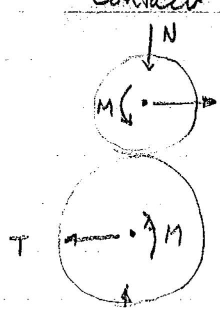
Problem: Two like spheres $k_1 = k_2 = k$, $R_1 = R_2$

$$C = \sqrt[3]{\frac{4\pi^2 k^2\cdot 2R}{6R^2N}} = \sqrt[3]{\frac{4}{3}\frac{\pi^2k^2}{RN}} = \frac{\pi k}{a}$$

$$a = \sqrt[3]{\frac{3N\pi(2K)\cdot R^2}{8R}} = \sqrt[3]{\frac{3N\pi RK}{4}} = \frac{1-\nu}{2\mu a}$$

An extension of the contact problem

Contact of 2 spheres under Normal & Tangential Loads



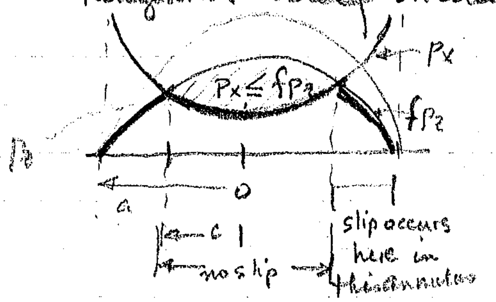
First N is applied giving rise to a contact circle. Now we apply T

Look at The T problem only

BC. ① For $r<a$ $\quad \sigma_{zz}|_{z=0} = 0$

② assume displ (produced by T) in y direction $u_y = 0$ (no warping of contact circle)
ε_{xx} must be antisymmetric due to T; in the contact circle $\varepsilon_{xx} = 0 \Rightarrow u_x = \text{const}$
all this wrt $z=0$

Recall that for $u_y = 0$ and $u_x = \text{const}$ was produced on a half plane with tangential loading on circle (after Cerruti proble) $p_x = \dfrac{T}{2\pi a}(a^2 - r^2)^{-1/2}$



is this a correct loading? No since as $p_x \uparrow$ the two bodies will slip. How do we find the areas of slip? Recall law of static friction

$$p_x \le f\, p_z$$

p_z = normal to surface of contact.

draw p_x vs. $f p_z$

where f is dependent on surface texture only

for $r < c$ $\quad \sigma_{zz} = 0, \quad u_y = 0 \quad u_x = \text{const}$
$c < r \le a$ $\quad \sigma_{zz} = 0, \quad \sigma_{zx} = f p_z, \quad \sigma_{zy} = 0$ < since slip has taken place no tractions
$r > a$ $\quad \sigma_{zz} = \sigma_{zy} = \sigma_{zx} = 0$ traction free b.c.

if $p_x = f p_z$ call this the (′ problem)

$$p_x' = f p_z = \frac{3fN}{2\pi a^3}\sqrt{a^2 - r^2} \qquad \text{let } fN = T' \qquad \text{For } \underline{c < r < a}$$

$$u_x' = \frac{3T'}{64 a^3 \mu}\left[2(2-\nu)(2a^2 - r^2) + \nu r^2 \cos 2\theta\right]$$

$$u_y' = \frac{3T'\nu r^2 \sin 2\theta}{64 a^3 \mu}$$

For $\underline{r < c}$

to correct for $r < c$ add (″ problem) such that $u_x' + u_x'' = \text{const}$
$u_y' + u_y'' = 0$

Let $p_x'' = -\dfrac{3T''}{2\pi c^3}(c^2 - r^2)$

$$u_x'' = -\frac{3T''}{64 c^3 \mu}\left[2(2-\nu)(2c^2 - r^2) + \nu r^2 \cos 2\theta\right]$$

$$u_y'' = -\frac{3T''\nu r^2 \sin 2\theta}{64 c^3 \mu} \qquad \text{take } T'' = T'\frac{c^3}{a^3} \text{ so that } u_y' + u_y'' = 0$$

$$u_x = u_x' + u_x'' = \frac{3T'}{64 a^3 \mu}\left[2(2-\nu)(2a^2 - 2c^2)\right] = \frac{3T'}{16 a \mu}(2-\nu)\left(1 - \frac{c^2}{a^2}\right) = \text{const}$$

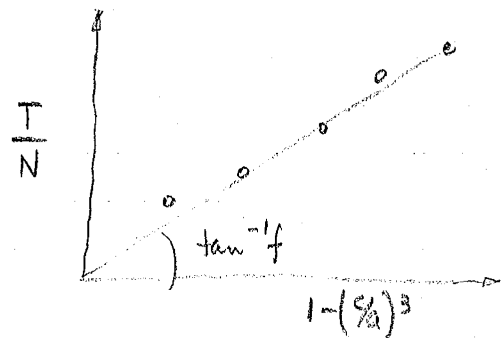
$$u_y = u_y' + u_y'' = 0$$

$$p_x = p_x' + p_x'' = \frac{3fN}{2\pi a^3}\left[(a^2 - r^2)^{1/2} - (c^2 - r^2)^{1/2}\right]$$

$$T = \int p_x dA = \frac{3fN}{2\pi a^3}\iint p_x\, r\,dr\,d\theta = \frac{3fN}{a^3}\left[\int_0^a r\sqrt{a^2-r^2}\,dr - \int_0^c r\sqrt{c^2-r^2}\,dr\right]$$

$$T = \frac{fN}{a^3}\left[a^3 - c^3\right] = fN\left(1 - \frac{c^3}{a^3}\right)$$

solve for c $\boxed{c = a\sqrt[3]{1 - \frac{T}{fN}}}$ for f $\boxed{f = \frac{T}{N\left(1-(c/a)^3\right)}}$

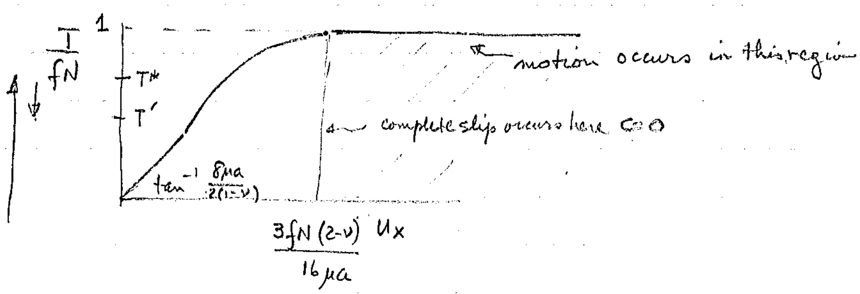


5/23/79

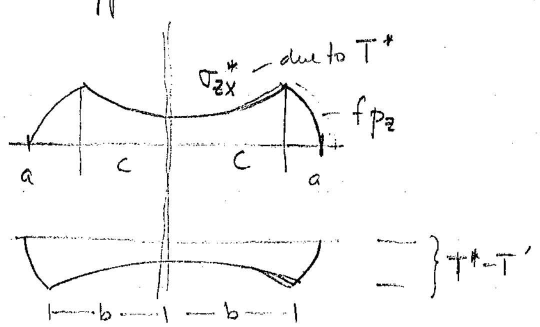
Now continuing
the horiz disp $u_x = \frac{3fN(2-\nu)}{16\mu a}\left[1 - \frac{c^2}{a^2}\right] = \frac{3fN(2-\nu)}{16\mu a}\left[1 - \left\{1 - \frac{T}{fN}\right\}^{2/3}\right]$

Calculate the compliance in shear

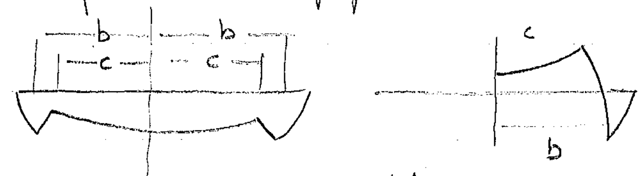
$$\frac{du_x}{dT} = \frac{2-\nu}{8\mu a}\left[1 - \frac{T}{fN}\right]^{-1/3}$$



Now suppose we increase T to T^* and then decrease from T^* to T'



We now argue that by apply T to T^* and then decreasing to T' is a superposition

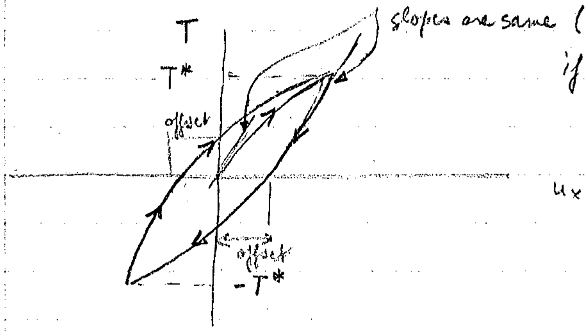


results in either one of these depending on the magnitudes of T^* and T^*-T'

by superposition we can find $b = a\left(1 - \frac{T^* - T'}{2fN}\right)^{1/3}$

$$u_x = \frac{3(2-\nu)fN}{16\mu a}\left[2\left(1 - \frac{T^*-T'}{2fN}\right)^{2/3} - \left(1 - \frac{T^*}{fN}\right)^{2/3} - 1\right]$$

displacement of adhered section of contact ~~surf~~ circle in horiz direction.



slopes are same (starting slopes & downward slope)

if we put an oscillating tangential load we form a hysteresis loop and energy loss/cycle is the area enclosed by loop

experiment energy ~ T^{*2}
theory energy ~ T^{*3}

Energy dissipated per cycle (for small T/fN)

$$E_d = \frac{(2-\nu)T^{*3}}{36\mu a fN} = \int T\,du_x$$

all these problems are dependent on order of application

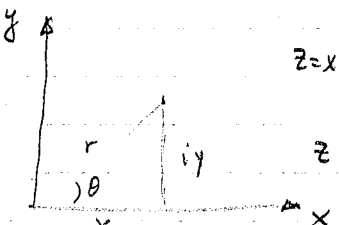
Mechanics of Contact between Deformable Bodies DePater, Kalker 1975

5/25/79

Fracture Problems

Fathers of Fracture Problems : Kolosov Muskhelishvili

Prelude : plane problems given $f(z)$



$z = x + iy$

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\frac{\partial}{\partial x}f(z) = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial z} \qquad \frac{\partial}{\partial y}f = \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = i\frac{\partial f}{\partial z} = i\frac{\partial f}{\partial x}$$

$f(z) = \alpha(x,y) + i\beta(x,y) \implies$ $\frac{\partial f}{\partial x} = \frac{\partial \alpha}{\partial x} + i\frac{\partial \beta}{\partial x}$

conjugate harmonic fn. of α $\qquad \frac{\partial f}{\partial y} = \frac{\partial \alpha}{\partial y} + i\frac{\partial \beta}{\partial y}$

$\Big\}\ i\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial x} = \frac{\partial \alpha}{\partial y} + i\frac{\partial \beta}{\partial y}$

$\implies$ Cauchy Riemann Eqns: $\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y}$; $\frac{\partial \alpha}{\partial y} = -\frac{\partial \beta}{\partial x}$ satisfaction implies f is analytic and has all derivatives

and α, β satisfy $\nabla^2\alpha = \nabla^2\beta = 0$

Consider $\psi(x,y)$ -
what is $\nabla^2(x\psi) = x\nabla^2\psi + 2\frac{\partial\psi}{\partial x}$
if $\nabla^2\psi = 0$ then $x\psi$ is biharmonic
also $y\psi$ is biharmonic
$r^2\psi = (x^2+y^2)\psi$ is biharmonic

Now we define the airy stress fn. ϕ define $P \equiv \nabla^2\phi = \sigma_{xx}+\sigma_{yy}$
$\sigma_{xx} = \phi_{,yy}$ $\sigma_{yy} = \phi_{,xx}$ $\sigma_{xy} = -\phi_{,xy}$

to satisfy compat $\nabla^4\phi = 0 \Rightarrow \nabla^2 P = 0$ P is harmonic

Let Q be the conjugate harmonic fn of P Let $F(z) = P + iQ$

consider $\frac{1}{4}\int F(z)\,dz \equiv \psi(z) = p + iq$ $\therefore$ $\frac{d\psi}{dz} = \frac{1}{4}f(z) = \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x} = \frac{P+iQ}{4}$

$\Rightarrow \frac{\partial p}{\partial x} = \frac{1}{4}P$ $\frac{\partial q}{\partial y} = \frac{Q}{4}$ from cauchy-riemann $\Rightarrow \frac{\partial p}{\partial x} = \frac{1}{4}P$ $\frac{\partial q}{\partial x} = \frac{Q}{4} = -$

$\Longrightarrow p_1 = \phi - xp - yq$ is biharmonic

$\therefore$ $\phi = p_1 + xp + yq$ is biharmonic

using both p and q is not necessary $\Rightarrow \nabla^2(\phi - 2xp) = \nabla^2\phi - 4\frac{\partial p}{\partial x} = 0$
if $\nabla^2 p = 0$

$\therefore$ $\phi - 2xp$ is biharmonic Let $p_2 = \phi - 2xp$ $\Rightarrow \phi = 2xp + p_2$ is biharm

$\phi = 2yq + p_3$ is also biharmonic

define $\chi(z) = p_1 + iq_1$ q_1 is conjug harmonic to p_1

Consider $(x-iy)(p+iq) + p_1 + iq_1$
Real part $xp + yq + p_1$; thus any stress fn ϕ can be expressed as
$\phi = \mathrm{Re}[\bar{z}\psi + \chi]$ ψ, χ are analytic fns. of a complex variable

$\bar{z}\psi + \chi = r^2\frac{\psi}{z} + \chi = r^2 p_4 + p_5$

Despl in terms of stress fn ϕ

For plane stress
$$E\frac{\partial u_x}{\partial x} = \sigma_{xx} - \nu\sigma_{yy}$$
$$E\frac{\partial u_y}{\partial y} = \sigma_{yy} - \nu\sigma_{xx} \qquad \sigma_{zz}=0$$
$$\mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) = \sigma_{xy}$$

$$E\frac{\partial u_x}{\partial x} = \phi_{,yy} - \nu\phi_{,xx} = (P-\phi_{,xx}) - \nu\phi_{,xx} = P - (1+\nu)\phi_{,xx}$$
$$E\frac{\partial u_y}{\partial y} = -(1+\nu)\phi_{,yy} + P$$

Let $P = 4\frac{\partial p}{\partial x}$ and $4\frac{\partial q}{\partial y}$

and integrate
$$2\mu\frac{\partial u_x}{\partial x} = -\phi_{,xx} + \frac{4}{1+\nu}\frac{\partial p}{\partial x}$$
$$2\mu\frac{\partial u_y}{\partial y} = -\phi_{,yy} + \frac{4}{1+\nu}\frac{\partial q}{\partial y}$$

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For any stress fn $\phi = Re[\bar z\psi(z) + \chi(z)]$
$$2\mu\frac{\partial u}{\partial x} = -\phi_{,xx} + \frac{4}{1+\nu}\frac{\partial p}{\partial x} \rightarrow 2\mu u = -\frac{\partial\phi}{\partial x} + \frac{4}{1+\nu}p + f(y)$$
$$2\mu\frac{\partial v}{\partial y} = -\phi_{,yy} + \frac{4}{1+\nu}\frac{\partial q}{\partial y} \rightarrow 2\mu v = -\frac{\partial\phi}{\partial y} + \frac{4}{1+\nu}q + f_1(x)$$

now $\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \sigma_{xy}$
$$\Rightarrow \underbrace{-\phi_{,xy}}_{\sigma_{xy}} + \frac{2}{1+\nu}\underbrace{\left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right)}_{=0 \text{ by Cauchy Reimann}} + \frac{1}{2}\frac{df}{dy} + \frac{1}{2}\frac{df_1}{dx} = \sigma_{xy}$$
$$\Rightarrow \frac{df}{dy} + \frac{df_1}{dx} = 0 \Rightarrow \frac{df}{dy} = \text{const}$$ (rigid body ~~rotation~~ displacement)

$P \stackrel{\text{def}}{=} \nabla^2\phi$ $\qquad$ Q from C.R. relations can be obtained

Now $f(z) = \alpha + i\beta$ $\quad \bar f = \alpha - i\beta$ then $f + \bar f = 2\alpha = 2 Re f$

$$2\phi = \bar z\psi(z) + \chi(z) + z\bar\psi(\bar z) + \bar\chi(\bar z)$$
$$2\frac{\partial\phi}{\partial x} = \bar z\psi'(z) + \psi(z) + \chi'(z) + z\bar\psi'(\bar z) + \bar\psi(\bar z) + \bar\chi'(\bar z) \qquad (1)$$
$$2\frac{\partial\phi}{\partial y} = i\left[\bar z\psi'(z) - \psi(z) + \chi'(z) - z\bar\psi'(\bar z) + \bar\psi(\bar z) - \bar\chi'(\bar z)\right] \qquad (2)$$
$$\Rightarrow \text{take } \frac{1}{2}\left[2\frac{\partial\phi}{\partial x} + i2\frac{\partial\phi}{\partial y}\right] = \psi(z) + z\bar\psi'(\bar z) + \bar\chi'(\bar z)$$

Now take $2\mu\,(u+iv) = -\left[\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right] + \frac{4}{1+\nu}\left[p+iq\right]$

Now $\psi(z) = p+iq \Rightarrow \boxed{2\mu(u+iv) = \frac{3-\nu}{1+\nu}\,\psi(z) - z\,\overline{\psi'}(\bar z) - \overline{\chi'}(\bar z)}$

for plane stress

for plane strain replace ν by $\frac{\nu}{1-\nu}$

This gives the displ in terms of complex fn.

We now want stresses

Now $\frac{\partial}{\partial x}\left[\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right] = \psi'(z) + z\,\overline{\psi''}(\bar z) + \overline{\psi'}(\bar z) + \overline{\chi''}(\bar z) \qquad (3)$

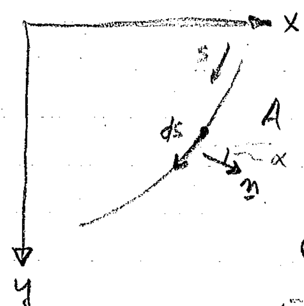
and $i\frac{\partial}{\partial y}\left[\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right] = -\psi'(z) + z\,\overline{\psi''}(\bar z) - \overline{\psi'}(\bar z) + \overline{\chi''}(\bar z) \qquad (4)$

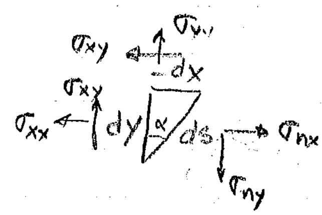
Now $\sigma_{xx}+\sigma_{yy} \quad \nabla^2\phi = (3)-(4) = 2\psi'(z) + 2\overline{\psi'}(\bar z) = 4\,\mathrm{Re}\,\psi'(z)$

Now $\sigma_{yy}-\sigma_{xx} - 2i\sigma_{xy} = (3)+(4) = 2\left[z\,\overline{\psi''}(\bar z) + \overline{\chi''}(\bar z)\right]$

this gives 3 eqns for $\sigma_{xx}+\sigma_{yy}$, $\sigma_{yy}-\sigma_{xx}$, $-2\sigma_{xy}$

B.C.





$\sigma_{nx} = \sigma_{xx}\cos\alpha + \sigma_{xy}\sin\alpha$

$\sigma_{ny} = \sigma_{yy}\sin\alpha + \sigma_{xy}\cos\alpha$

$\cos\alpha = \frac{dy}{ds} \quad \sin\alpha = -\frac{dx}{ds}$

$\sigma_{nx} = \phi_{,yy}\frac{dy}{ds} - \phi_{,xy}\left(-\frac{dx}{ds}\right) = \frac{\partial}{\partial y}(\phi_{,y})\frac{dy}{ds} + \frac{\partial}{\partial x}(\phi_{,y})\frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right)$

$\boxed{\sigma_{ny} = -\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right) \qquad \sigma_{nx} = \frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right)}$

Elliptic Coordinates

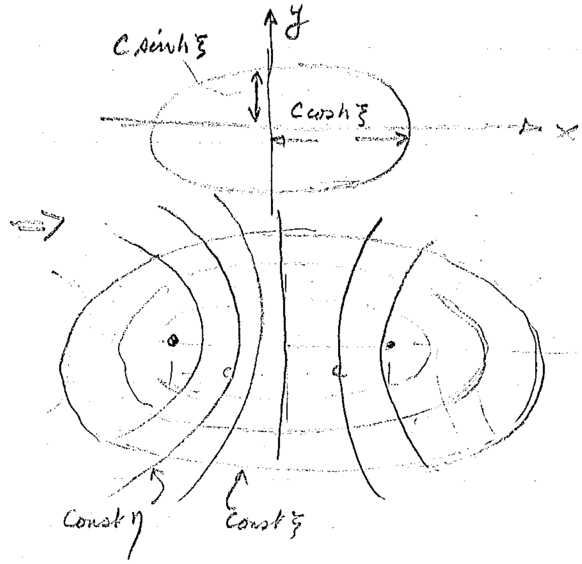
$x = c\cosh\xi\cos\eta$

$y = c\sinh\xi\sin\eta$

const ξ curves give ellipses $\frac{x^2}{c^2\cosh^2\xi} + \frac{y^2}{c^2\sinh^2\xi} = 1$

const η curves give hyperbs $\frac{x^2}{c^2\cos^2\eta} - \frac{y^2}{c^2\sin^2\eta} = 1$

$\Longrightarrow$



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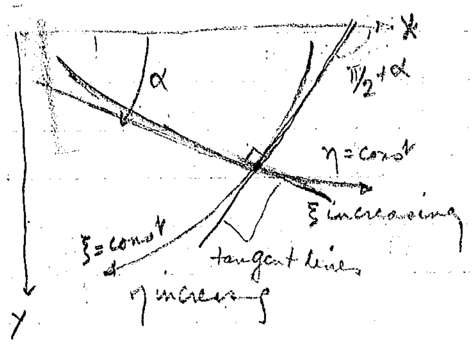
$$x = c\cosh\xi \cos\eta,\quad y = c\sinh\xi \sin\eta \quad \Big\}\ \text{elliptical coords}$$

$x+iy = c\cosh(\xi+i\eta) \Rightarrow$ since $\cosh(\alpha+i\beta) = \cosh\alpha\cos\beta + i\sinh\alpha\sin\beta$

thus $z = c\cosh\zeta$ where $\zeta = \xi + i\eta$



$$\sigma_{\xi\xi} = \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) + \tfrac{1}{2}(\sigma_{xx}-\sigma_{yy})\cos 2\alpha + \sigma_{xy}\sin 2\alpha$$

$$\sigma_{\eta\xi} = -\tfrac{1}{2}(\sigma_{xx}-\sigma_{yy})\sin 2\alpha + \sigma_{xy}\cos 2\alpha$$

$$\sigma_{\eta\eta} = \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) - \tfrac{1}{2}(\sigma_{xx}-\sigma_{yy})\cancel{\sin}\cos 2\alpha - \sigma_{xy}\sin 2\alpha$$

Whew!
ciao!

$$\sigma_{\xi\xi}+\sigma_{\eta\eta} = \sigma_{xx}+\sigma_{yy}$$

$$\begin{aligned}\sigma_{\eta\eta}-\sigma_{\xi\xi}+2i\sigma_{\xi\eta} &= -(\sigma_{xx}-\sigma_{yy})\cos 2\alpha - 2\sigma_{xy}\sin 2\alpha \\ &\quad -\tfrac{2i}{2}(\sigma_{xx}-\sigma_{yy})\sin 2\alpha + 2i\sigma_{xy}\cos 2\alpha \\ &= -(\sigma_{xx}-\sigma_{yy})e^{i2\alpha} + 2i\sigma_{xy}e^{i2\alpha} \\ &= e^{i2\alpha}[\sigma_{xx}-\sigma_{yy}+2i\sigma_{xy}]\end{aligned}$$

$\sigma_{\xi\xi}+\sigma_{\eta\eta} = 4\,\mathrm{Re}\,\Psi'(z) = 2[\Psi'(z)+\bar{\Psi}'(\bar{z})]$ as shown in last lecture

$\sigma_{\eta\eta}-\sigma_{\xi\xi}+2i\sigma_{\xi\eta} = 2e^{i2\alpha}[\bar{z}\Psi''(z)+\chi''(z)]$ using results from previous lect.

displ

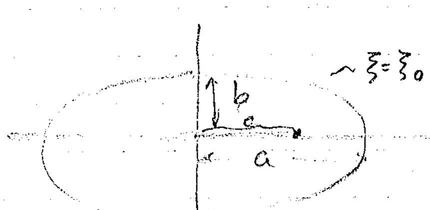
$$\left.\begin{aligned}u_\xi &= u\cos\alpha + v\sin\alpha \\ u_\eta &= v\cos\alpha - u\sin\alpha\end{aligned}\right\} \Rightarrow u_\xi + iu_\eta = e^{-i\alpha}[u+iv]$$

Note that $f'(\zeta) = J\cos\alpha + iJ\sin\alpha = Je^{i\alpha}$ $\quad J, \alpha$ real

$\bar{f}'(\bar{\zeta}) = Je^{-i\alpha}$ $\quad\therefore\quad f'(\zeta)/\bar{f}'(\bar{\zeta}) = e^{2i\alpha}$

$z = c\cosh\zeta$

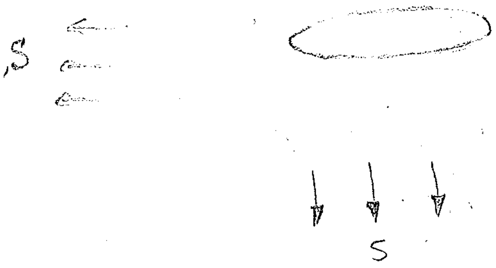
$dz/d\zeta = c\sinh\zeta$ $\qquad e^{2i\alpha} = \dfrac{\sinh\zeta}{\sinh\bar{\zeta}}$



$a = c\cosh\xi_0$

$b = c\sinh\xi_0$

look at problem #1 remote uniform tension S



BC : at ∞ $\sigma_{xx} = \sigma_{yy} = S$ as $\xi \to \infty$

$\Rightarrow 2\,\mathrm{Re}\,\psi'(z) = S$ also $\bar{z}\psi''(z) + \chi''(z)$

BC : at $\xi = \xi_0$ $\sigma_{\xi\xi}, \sigma_{\xi\eta} = 0$

must have periodicity in η

To obtain χ, ψ we must use trial & error

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we note that $\sinh n\zeta$, $\cosh n\zeta$, $Bc^2\zeta$ qualify for fns. ψ, χ

Let $\psi(z) = Ac\sinh\zeta$

$\psi'(z) = Ac\cosh\zeta \dfrac{d\zeta}{dz} = Ac\cosh\zeta \left(\dfrac{dz}{d\zeta}\right)^{-1}$; since $z = c\cosh\zeta$ $\dfrac{dz}{d\zeta} = c\sinh\zeta$

$= A\coth\zeta$ as $\zeta \to \infty$ $\coth\zeta \to 1$

$\Rightarrow$ if $2A = S$ we satisfy 1st BC. $\therefore$ $2\,\mathrm{Re}\,\psi'(z) = 2A = S$ as $\xi \to \infty$

$\psi''(z) = -\dfrac{A}{c}\dfrac{1}{\sinh^2\zeta}$ $\Rightarrow$ $\bar{z}\psi''(z) = \dfrac{-A\cosh\bar{\zeta}}{\sinh^3\zeta}$; Let $\chi(z) = Bc^2\zeta \Rightarrow \chi'(z) = \dfrac{Bc}{\sinh\zeta}$

$\Rightarrow \chi''(z) = -B\dfrac{\cosh\zeta}{\sinh^3\zeta}$ then $\bar{z}\psi''(z) + \chi''(z) = 0$ as $\xi \to \infty$

since as $\xi \to \infty$ $\bar{z}\psi'' \to 0$, $\chi'' \to 0$

thus we satisfy 2nd BC

go back to the general expressions

$$(*) \quad \sigma_{\xi\xi} - i\sigma_{\xi\eta} = \psi'(z) + \overline{\psi'(z)} - e^{2i\alpha}\left[\bar{z}\psi''(z) + \chi''(z)\right]$$

$$e^{2i\alpha} = \sinh\zeta / \sinh\bar{\zeta}$$

Now substitute the obtained expressions and algebrate in (*)

$$(**) \quad \sigma_{\xi\xi} - i\sigma_{\xi\eta} = \frac{1}{\sinh^2\zeta\,\sinh\bar{\zeta}}\left\{A\left[\sinh\zeta\,\sinh(\underbrace{\zeta+\bar{\zeta}}_{2\xi}) + \cosh\bar{\zeta}\right] + B\cosh\zeta\right\}$$

we will now get A & B by using that $\sigma_{\xi\xi} = \sigma_{\xi\eta} = 0$ on $\xi = \xi_0$ $2\xi_0 = \zeta + \bar{\zeta} \Rightarrow \bar{\zeta} = 2\xi_0 - \zeta$

plug into the (**) to get

$$\sigma_{\xi\xi} - i\sigma_{\xi\eta} = \frac{1}{\sinh^2\zeta\,\sinh\bar{\zeta}}\left(A\cosh 2\xi_0 + B\right)\cosh\zeta$$

but $A = S/2 \Rightarrow B = -\dfrac{S\cosh 2\xi_0}{2}$

$\Psi(z) = \frac{1}{2} S c \sinh \zeta$

$\chi(z) = -\frac{1}{2} S c^2 (\cosh 2\xi_0)\, \zeta$

On the rim $\sigma_{\xi\xi} = \sigma_{\xi\eta} = 0$

to find $\sigma_{\eta\eta}$ use first invariant : since $\sigma_{\xi\xi} + \sigma_{\eta\eta} = \sigma_{\eta\eta}$ on rim

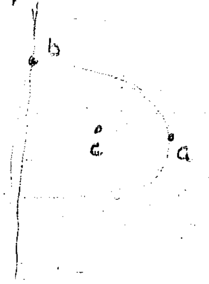
but ~~and~~ $\sigma_{\xi\xi} + \sigma_{\eta\eta} = 4 \operatorname{Re} \Psi'(z) = 2S \operatorname{Re}(\coth \zeta)$

$$\coth \zeta = \frac{\sinh 2\xi - i \sin 2\eta}{\cosh 2\xi - \cos 2\eta} \Rightarrow \operatorname{Re} \coth \zeta = \frac{\sinh 2\xi}{\cosh 2\xi - \cos 2\eta}$$

$$\Rightarrow \sigma_{\eta\eta} = \frac{2S \sinh 2\xi}{\cosh 2\xi - \cos 2\eta}$$

when $\xi = \xi_0$ then $\sigma_{\eta\eta}\Big|_{\xi=\xi_0} = \dfrac{2S \sinh 2\xi_0}{\cosh 2\xi_0 - \cos 2\eta}$

$$(*) \quad \max_{\eta = 0, \pi} \sigma_{\eta\eta} = \frac{2S \sinh 2\xi_0}{\cosh 2\xi_0 - 1}$$



since : $c^2 = a^2 - b^2$

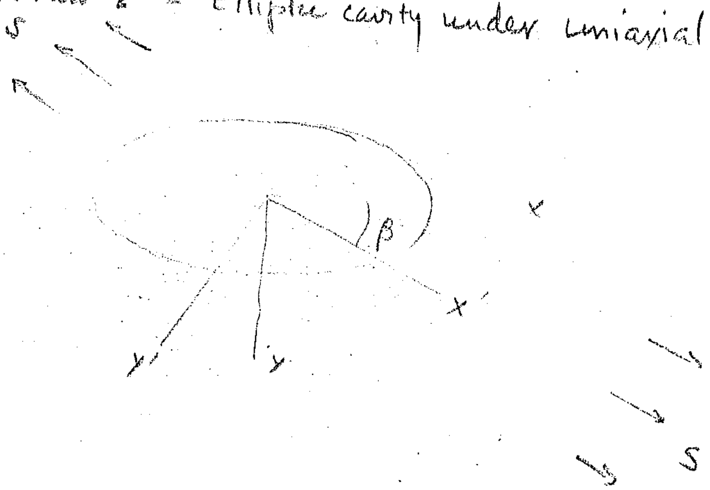
and $\sinh 2\xi_0 = 2 \sinh \xi_0 \cosh \xi_0$

$\sinh \xi_0 = b/c$ $\qquad \sinh 2\xi_0 = \dfrac{2ab}{c^2}$

$\cosh \xi_0 = a/c$ $\qquad \cosh 2\xi_0 = \dfrac{2a^2}{c^2} - 1$

plug into $(*)$ to get $\sigma_{\eta\eta}\Big|_{max} = 2S \dfrac{a}{b}$

Problem #2 - Elliptic cavity under uniaxial tension



Let $4\Psi(z) = A c \cosh \zeta + B c \sinh \zeta$

$4\chi(z) = C c^2 \zeta + D c^2 \cosh 2\zeta + E c^2 \sinh 2\zeta$

$\sigma_{x'x'} + \sigma_{y'y'} = \sigma_{xx} + \sigma_{yy}$

$\sigma_{y'y'} - \sigma_{x'x'} + 2i \sigma_{x'y'} = e^{2i\beta}(\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy})$

at ∞ $\quad \sigma_{x'x'} = S \quad \sigma_{y'y'} = \sigma_{x'y'} = 0$

$\Rightarrow 4 \operatorname{Re} \Psi'(z) = S \qquad 2\left[\bar{z} \Psi''(z) + \chi''(z)\right] = -S e^{-2i\beta}$ at ∞

at $\xi = \xi_0$ $\quad \sigma_{\xi\xi} = \sigma_{\xi\eta} = 0$

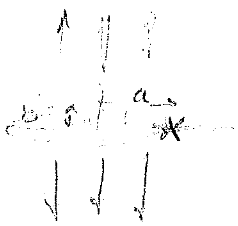
again we use periodicity to obtain $A \to 0$

also by periodicity that C & A must real after algebrating

$$A = S e^{2\xi_0} \cos 2\beta \qquad B = S'(1 - e^{2\xi_0 + 2i\beta}) \qquad C = -S(\cosh 2\xi_0 - \cos 2\beta)$$

$$D = -\tfrac{1}{2} S' e^{2\xi_0} \cosh 2(\xi_0 + i\beta) \qquad E = \tfrac{1}{2} S e^{2\xi_0} \sinh 2(\xi_0 + i\beta)$$

we now use the first stress invariant to get $\sigma_{\eta\eta}\big|_{\xi=\xi_0} = S \dfrac{\sinh 2\xi_0 + \cos 2\beta - e^{2\xi_0} \cos 2(\beta - \eta)}{\cosh 2\xi_0 - \cos 2\eta}$

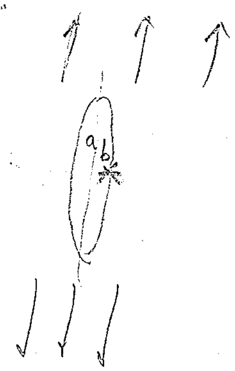


if $\beta = \pi/2$ $\quad \sigma_{\eta\eta}\big|_{\xi=\xi_0} = S e^{2\xi_0} \left[\dfrac{\sinh 2\xi_0 (1 + e^{-2\xi_0})}{\cosh 2\xi_0 - \cos 2\eta} - 1 \right]$

$$\max \sigma_{\eta\eta}\big|_{\xi=\xi_0} = S(1 + 2a/b)$$

Continuation 6/6/79

if $\beta = 0$



$$S = \left(1 + \frac{2b}{a}\right)$$

for pure shear $\quad S' @ \beta = \pi/4$

$S @ \beta = \dfrac{3\pi}{4}$ $\qquad \sigma_{\eta\eta}\big|_{\xi=\xi_0} = 2S' \dfrac{e^{2\xi_0} \sin 2\eta}{\cosh 2\xi_0 - \cos 2\eta}$

$$\max_{\tan\eta = b/a} \sigma_{\eta\eta} = \pm S \frac{(a+b)^2}{ab}$$

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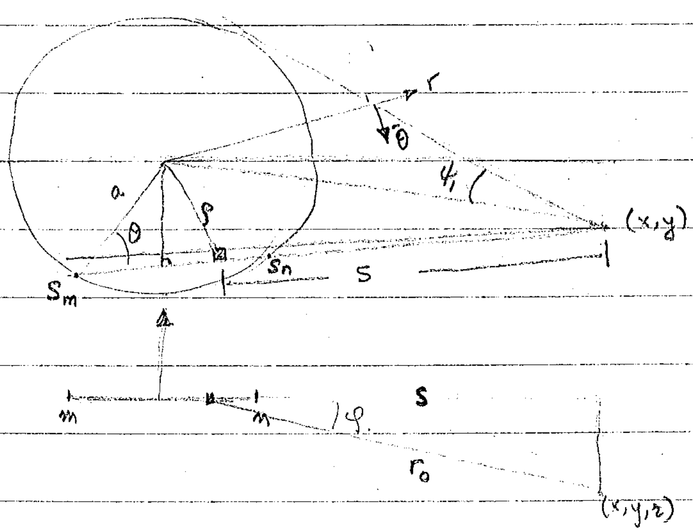
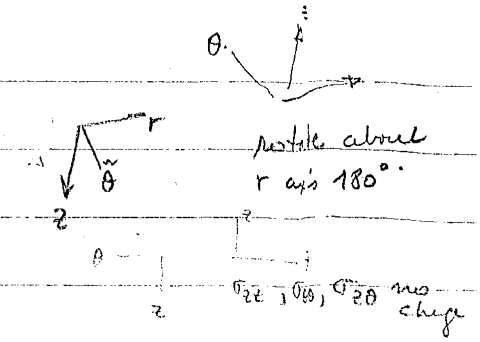




$s = r_0 \cos\varphi$.

$d\xi d\eta = (s\,d\psi)\,ds$.

$\int \frac{p\, d\xi d\eta}{r_0} = p \iint \frac{s\,ds\,d\psi}{\sqrt{s^2+z^2}}$

$r = \sqrt{x^2+y^2}$.

$$2p \int_0^{\psi_1} \sqrt{s^2+z^2}\Big|_{s=s_n}^{s=s_m} d\psi.$$

$$s_m = r\cos\psi + \sqrt{a^2 - r^2\sin^2\psi} \qquad \sqrt{s_m^2+z^2}$$

$$s_n = r\cos\psi - \sqrt{a^2 - r^2\sin^2\psi}$$

Now if we remember that $r\sin\psi = a\sin\theta$

then $d\psi = \frac{a\cos\theta\, d\theta}{r\cos\psi} = \frac{a\cos\theta\, d\theta}{r\sqrt{1-\sin^2\psi}} = \frac{a\cos\theta\, d\theta}{r\sqrt{1-(a/r)^2\sin^2\theta}}$

$$\text{or} \quad 2p \int_0^{\pi/2} \left(\sqrt{s_m^2+z^2} - \sqrt{s_n^2+z^2}\right) \cdot \frac{a\cos\theta\, d\theta}{r\sqrt{1-(a/r)^2\sin^2\theta}}$$

and $s_m = r\sqrt{1-(a/r)^2\sin^2\theta} + a\cos\theta$

$s_n = r\sqrt{1-(a/r)^2\sin^2\theta} - a\cos\theta$

by symmetry of situation look along $r = x$ then.

$$\iint P\,\delta(x-\xi)\,\delta(y-\eta)\, d\xi d\eta.$$

$$\sigma_z = -\frac{3q}{2\pi}\int \frac{z^3 \cdot s\,d\psi\,ds}{r_0^{\,5}} = -\frac{3q}{2\pi} z^3 \left[\frac{s\,ds}{\left(\sqrt{s^2+z^2}\right)^5}\, d\psi\right]$$

$$= -\frac{3q}{2\pi} z^3 \int \frac{d\psi}{(s^2+z^2)^{3/2}}\Big|_{s=s_n}^{s=s_m} = \frac{q}{2\pi} z^3 \int d\psi \left[\frac{1}{(s_m^2+z^2)^{3/2}} - \frac{1}{(s_n^2+z^2)}\right]$$

$$\tau_{rz} = -\frac{3qz^2}{2\pi}\iint \frac{s \cdot s\,d\psi\,ds}{(s^2+z^2)^{+5/2}} \qquad \text{let } s = u \quad \frac{s\,ds}{(\quad)^{5/2}} = dv$$

$du = ds \qquad -\frac{1}{3}(\quad)^{-3/2} = v$

$$A \int d\psi\left[-\frac{s}{3}(s^2+z^2)^{-3/2} + \frac{1}{3}\int \frac{ds}{(s^2+z^2)^{3/2}}\right] = A\int d\psi\left[-\frac{s}{3}(z^2+s^2)^{-3/2} + \frac{1}{3z^2}\frac{s}{(z^2+s^2)^{1/2}}\right]_{s_n}^{s_m}$$

$$\tau_{rz} = -\frac{q}{2\pi}\int d\psi\left[\frac{s^3}{(s^2+z^2)^{3/2}}\right]_{s_n}^{s_m}$$

$$\sigma_r = \frac{q}{2\pi}(1-2\nu) \int s\,d\psi\,ds \left\{ \frac{1}{s^2} - \frac{z}{s^2}\frac{1}{(s^2+z^2)^{1/2}} - \frac{3s^2 z}{(s^2+z^2)^{5/2}} \right\}$$

$$\int d\psi \left\{ \ln s + \frac{1}{2}\ln\left(\frac{z+(s^2+z^2)^{1/2}}{(s^2+z^2)^{1/2}-z}\right) - 3z\left[-\frac{1}{(s^2+z^2)^{1/2}} + \frac{z^2}{3(s^2+z^2)^{3/2}}\right]\right\}_{s_n}^{s_m}$$

$$\sigma_\theta = \frac{q}{2\pi}(1-2\nu) \int s\,d\psi\,ds \left\{ -\frac{1}{s^2} + \frac{z}{s^2}\frac{1}{(s^2+z^2)^{1/2}} + \frac{z}{(s^2+z^2)^{+3/2}} \right\}.$$

$$\int d\psi \left\{ -\ln s - \frac{1}{2}\ln\left(\frac{(s^2+z^2)^{1/2}+z}{(s^2+z^2)^{1/2}-z}\right) - \frac{z}{(s^2+z^2)^{1/2}} \right\}_{s_n}^{s_m.}$$

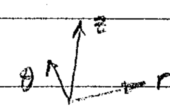
Now.

$$\sigma_r = \frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \phi_{,rr}\right) \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{1}{r}\phi_{,r}\right)$$

Now if we want

$$\sigma_z = \frac{\partial}{\partial z}\left((2-\nu)\nabla^2\phi - \phi_{,zz}\right)$$



$\Rightarrow$ $\tilde{z} = -z$, $\tilde{\theta} = -\theta$

$$\tau_{rz} = \frac{\partial}{\partial r}\left((1-\nu)\nabla^2\phi - \phi_{,zz}\right).$$

$\nabla^2(r,\theta,z) = \nabla^2(r,+\tilde{\theta},+\tilde{z})$

$\tilde{\sigma}_r = -\sigma_r$

$\tilde{\sigma}_\theta = -\sigma_\theta$

$\tilde{\sigma}_z = -\sigma_z$

$\tilde{\tau}_{rz} = \tau_{rz}$

$$\therefore \quad \tilde{\sigma}_z = +\frac{qz^3}{2\pi}\int d\psi\left[\frac{1}{(s^2+z^2)^{3/2}}\right]_{s_n}^{s_m}$$

$$\tilde{\tau}_{rz} = -\frac{q}{2\pi}\int d\psi\left[\frac{s^3}{(s^2+z^2)^{3/2}}\right]_{s_n}^{s_m}$$

$$\tilde{\sigma}_r = -\frac{q}{2\pi}(1-2\nu)\int d\psi\left\{\ln s - \frac{1}{2}\ln\left[\frac{z+(s^2+z^2)^{1/2}}{(s^2+z^2)^{1/2}-z}\right] + 3z\left[-\frac{1}{(s^2+z^2)^{1/2}} + \frac{z^2}{3(s^2+z^2)^{3/2}}\right]\right\}_{s_n}^{s_m}$$

$$\sigma_\theta = -\frac{q}{2\pi}(1-2\nu)\int d\psi\left\{-\ln s + \frac{1}{2}\ln\left[\frac{(s^2+z^2)^{1/2}+z}{(s^2+z^2)^{1/2}}\right.\right.$$

$\lambda = \frac{2\mu\nu}{(1-2\nu)}$ $\quad \sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$; $\quad \varepsilon_{ij} = \frac{(1+\nu)}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$ 1

IF g and $\frac{\partial g}{\partial n} = 0$ on S then $g(P) = -\frac{1}{4\pi}\int \frac{1}{R}\nabla^2 g \, dV$ IF $\nabla^2 g = f$ in $dV \Rightarrow g(P) = -\frac{1}{4\pi}\int \frac{f}{R} dV$

IF NOT $g(P) = \frac{1}{4\pi}\left[\int \left(\frac{1}{R}\partial g/\partial n - g\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right)ds - \int \frac{1}{R}\nabla^2 g \, dV\right]$

IF $\mathbb{G} = \nabla\phi + \nabla\times\mathbb{H}$ w/ $\nabla\cdot\mathbb{H} = 0$ then $\phi = -\frac{1}{4\pi}\int_V \frac{\nabla\cdot\mathbb{G}}{R}dV$ $\quad \mathbb{H} = -\frac{1}{4\pi}\int_V \frac{\nabla\times\mathbb{G}}{R}dV$
and $\mathbb{G}\to 0$ as $R\to\infty$

~~IF $\mathbb{u} = \nabla\phi + \nabla\times\mathbb{H}$ then $\mathbb{G} = \mathbb{u}$~~

Boussinesq - Papcovitch - Neuber fn: $\mathbb{u} = \mathbb{B} - \frac{1}{4(1-\nu)}\nabla(\mathbb{r}\cdot\mathbb{B} + \beta)$ with $\nabla^2\mathbb{B} = -\mathbb{f}/\mu$ $\quad \nabla^2\beta = \frac{\mathbb{r}\cdot\mathbb{f}}{\mu}$

A. ∞ Body Problems $\quad G_{km}(\mathbb{r}) = \frac{1}{8\pi\mu}\left[\delta_{km}r_{,pp} - \frac{\lambda+\mu}{\lambda+2\mu}r_{,km}\right]$

1. Kelvin problem - stress distrib in an ∞ elastic body subject to concentrated force

$\mathbb{B} = \frac{\mathbb{P}}{4\pi\mu r}$ $\quad \beta = 0$ $\quad$ $\mathbb{u}$ - found in HW #1 $\quad \sigma$ also

2. Double force (in tension) in z dir $\quad A = \lim_{h\to 0} Ph$

$\mathbb{B} = \frac{Az\,\mathbb{e}_z}{4\pi\mu r^3}$ $\quad \beta = -\frac{A}{4\pi\mu r}$

3. Center of Dilatation $\quad \nabla\cdot\mathbb{u} = 0$

$\mathbb{B} = \frac{A\mathbb{r}}{4\pi\mu r^3}$ $\quad \beta = \frac{-3A}{4\pi\mu r}$ $\quad$ or $\quad \mathbb{u} = -\frac{1}{4(1-\nu)}\nabla\left(\frac{c}{r}\right)$ $\quad C = \frac{(1-2\nu)A}{2\pi\mu}$

if $\mathbb{u} = \frac{C\mathbb{R}}{4(1-\nu)R^3}$ $\quad u_\theta = u_\phi = 0$

4. Spherical Cavity under pressure p_i (assume center of dilatation) $\quad C = p_i a^3(1-\nu)/\mu$

$\sigma_{RR} = -p_i(a/R)^3$; $\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{p_i}{2}(a/R)^3$; $\sigma_{\theta\phi} = \sigma_{\theta R} = \sigma_{\phi R} = 0$ $\quad u_R = \frac{p_i a}{4\mu}(a/R)^2$ $\quad u_\theta = u_\phi = 0$

5. Hollow sphere under internal & external pressure (assume center of dilatation + hydrostatic tension)

$\sigma_{RR} = (p_i - p_e)\frac{a^3b^3}{(a^3-b^3)R^3} + \frac{b^3p_e - a^3p_i}{a^3-b^3}$ $\quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{a^3b^3(p_e-p_i)}{2(a^3-b^3)R^3} + \frac{b^3p_e - a^3p_i}{a^3-b^3}$

More on pg. 7:

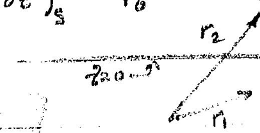
Rigid inclusion in a sphere under external pressure - see midterm

6. Spherical cavity under uniaxial tension at ∞

see pg 8 and HW #2 and 3

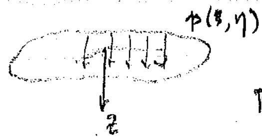
$$4\pi h(x,y,z) = -2\frac{\partial}{\partial z}\int_S \frac{h(\xi,\eta,0)}{r_0}\,d\xi\,d\eta - \int_V \left(\frac{1}{r_1}-\frac{1}{r_2}\right)\nabla^2 h(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta$$

B. Half Space Problem



$G = \frac{1}{r_1} - \frac{1}{r_2}$ Green's fn. $\nabla^2 G = 0$ in V, $G = 0$ on S

1. Distributed load in half space

w/ 

$\sigma_{zx} = \sigma_{zy} = 0 \quad \sigma_{zz} = -p(\xi,\eta) \qquad r_0 = \sqrt{(x-\xi)^2+(y-\eta)^2+z^2}$

$$\mathbb{B} = B_z \mathbf{e}_z = \frac{1-\nu}{\pi\mu}\int \frac{p(\xi,\eta)}{r_0}\,d\xi\,d\eta\,\mathbf{e}_z \qquad \beta = \int \frac{(1-2\nu)(1-\nu)}{\pi\mu}\,dz \int \frac{p(\xi,\eta)\,d\xi\,d\eta}{r_0}$$

2. Boussinesq - point force P_z at origin (in z dir)

$$B_z = \frac{1-\nu}{\pi\mu}\frac{P_z}{R} \qquad \beta = \frac{(1-2\nu)(1-\nu)}{\pi\mu}P_z \ln(R+z)$$ settlement - see midterm

3. Distributed Normal loads settlement $w = u_z(x,y,0) = \frac{B_z}{2}$ where B_z

loads over circle

a. $p(\xi,\eta) = p_0$ over a circle of rad $= a$. VALID UNDER LOAD ONLY !!!

$$w(x,y) = \frac{2(1-\nu)}{\pi\mu}p_0 a\int_0^{\pi/2}\sqrt{1-(r/a)^2\sin^2\psi}\,d\psi$$ ~ complete elliptic integral of the 2nd kind.

b. $p = \frac{N}{2\pi a\sqrt{a^2-\rho^2}}$ where $N = \int p(\xi,\eta)\,d\xi\,d\eta \qquad w(x,y) = \frac{(1-\nu)N}{4\pi\mu a} =$ const.

like die or stamp.

for u_z outside load see HW #2.

c. $p = \frac{3N}{2\pi a^2}\sqrt{a^2-\rho^2}$ where $N = \int p(\xi,\eta)\,d\xi\,d\eta \qquad w(x,y) = \frac{3(1-\nu)N}{16\mu a^3}(2a^2-r^2)$

4. tangential tractions over half space .∂. $\sigma_{zz} = \sigma_{zy} = 0 \quad \sigma_{zx} = -p_x$ or $\mathbf{e}_\varphi \cdot \mathbb{T}$

$$B_y = 0 \qquad B_x = \frac{1}{2\pi\mu}\int \frac{p_x(\xi,\eta)}{r_0}\,d\xi\,d\eta \qquad B_z = \int dz\,(1-2\nu)\frac{\partial B_x}{\partial x} \qquad r_0^2 = (x-\xi)^2+(y-\eta)^2+z^2$$

$$\beta = \int dz\left[(1-2\nu)B_z - x\frac{\partial B_x}{\partial z} + z\frac{\partial B_x}{\partial x}\right]$$

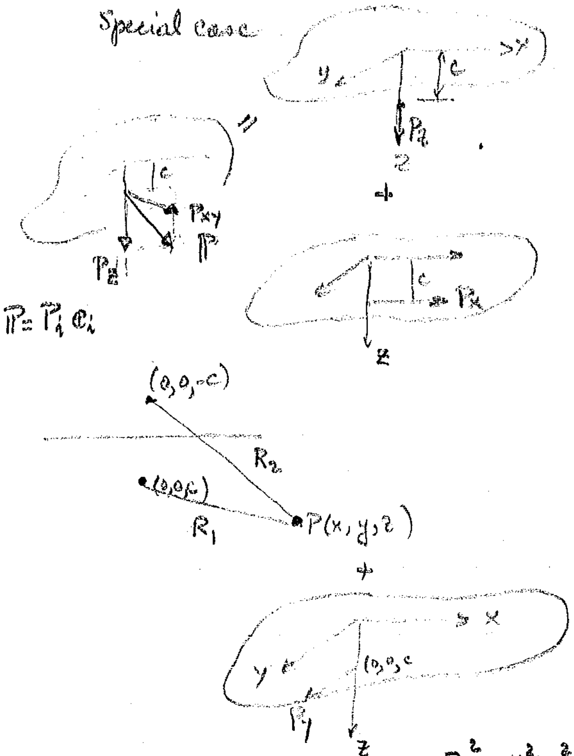
5. Cerruti Problem: concentrated force in tangential direction on half space (using 4)

$$B_x = \frac{P_x}{2\pi\mu R} \qquad B_z = \frac{(1-\nu)P_x X}{2\pi\mu R(R+z)} \qquad \beta = -\frac{(1-2\nu)^2 P_x X}{2\pi\mu(R+z)}$$

$u_x(x,y,0)$ and $u_y(x,y,0)$ are given on Pg 16

2. *Mindlin problem*: a concentrated force at any point in the interior of a half space

Special case



$$B_x, B_y = 0 \qquad B_z = \frac{P_z}{4\pi\mu}\left[\frac{1}{R_1} + \frac{3-4\nu}{R_2} - 2c\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)\right]$$

$$\beta = \frac{P_z}{4\pi\mu}\left[-\frac{c}{R_1} - \frac{c(3-4\nu)}{R_2} + 4(1-2\nu)(1-\nu)\log(R_2+z+c)\right]$$

$$B_y = 0 \qquad B_x = \frac{P_x}{4\pi\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \qquad B_z = \frac{P_x x}{2\pi\mu R_2}\left[\frac{1-2\nu}{R_2+z+c} - \frac{c}{R_2^2}\right]$$

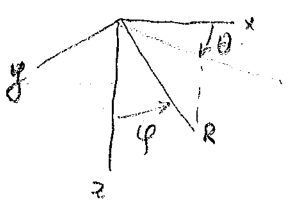
$$\beta = \frac{P_x(1-2\nu)x}{2\pi\mu(R_2+z+c)}\left[\frac{c}{R_2} - (1-2\nu)\right]$$

$$B_x = 0 \qquad B_y = \frac{P_y}{4\pi\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \qquad B_z = \frac{P_y y}{2\pi\mu R_2}\left[\frac{1-2\nu}{R_2+z+c} - \frac{c}{R_2^2}\right]$$

$$\beta = \frac{P_y\, y(1-2\nu)}{2\pi\mu(R_2+z+c)}\left[\frac{c}{R_2} - (1-2\nu)\right]$$

$$R_2^2 = x^2+y^2+(z+c)^2 \qquad R_1^2 = x^2+y^2+(z-c)^2$$

Mindlin problem: if $c \to \infty$ we approach the Kelvin problem; as $c \to 0$ we should approach Cerruti/Boussinesq



$$\frac{x}{R} = \sin\varphi\cos\theta \qquad \frac{y}{R} = \sin\varphi\sin\theta \qquad \frac{z}{R} = \cos\varphi$$

$$\begin{pmatrix} e_r \\ e_\varphi \\ e_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \\ -\sin\theta & \cos\theta & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}\ ; \quad \varepsilon_{\theta\theta} = \frac{1}{r\sin\varphi}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} + u_\varphi\frac{\cot\varphi}{r}\ ; \quad \varepsilon_{\varphi\varphi} = \frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r}\ ; \quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r\sin\varphi}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right)$$

$$\varepsilon_{\theta\varphi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\theta\cos\varphi}{r} + \frac{1}{r\cos\varphi}\frac{\partial u_\varphi}{\partial\theta}\right)\ ; \quad \varepsilon_{\varphi r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\varphi} - \frac{u_\varphi}{r} + \frac{\partial u_\varphi}{\partial r}\right)$$

$$\nabla\psi = \left[e_r\frac{\partial}{\partial r} + e_\theta\frac{1}{r\sin\varphi}\frac{\partial}{\partial\theta} + e_\varphi\frac{1}{r}\frac{\partial}{\partial\varphi}\right]\psi$$

in cartesian $\sigma_{\ell k} = \frac{\mu\nu}{(1-\nu)}B_{m,m}\delta_{\ell k} + \frac{\mu}{2(1-\nu)}\left\{(1-2\nu)\left[B_{\ell,k} + B_{k,\ell}\right] - \left[x_i B_{i,\ell k} + \beta_{,k\ell}\right]\right\}$

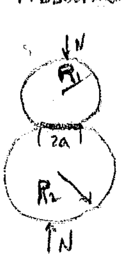
Contact Problems   See Timoshenko pg 409–421

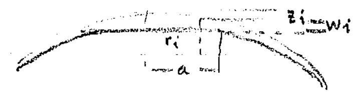
- Prelude: for $p_x = \frac{T}{2\pi a}\frac{1}{\sqrt{a^2-\rho^2}}$ on a circle (distributed tangential load on a circle on a half space)

  $u_x = \text{const} = \frac{(2-\nu)T}{8\mu a}$ rigid body disp

  $u_y = 0$

- Assume
  1. Contact area flat & circular
  2. contact area radius << radius of either bodies
  3. N applied remotely & doesn't affect problem (St Venant Principle)
  4. For local problem Rest of bodies are half spaces & no shear applied
  5. We can use settlement formula





$\alpha = (z_1+z_2) + w_1 + w_2$; if $k_i = \frac{(1-\nu_i)}{2\pi\mu_i}$ $w_i = \frac{3k_i N\pi}{8a^3}(2a^2 - r^2)$; $z_i = \frac{r_i^2}{2R_i}$

$a = \sqrt[3]{\frac{(k_1+k_2)3N\pi R_1R_2}{4(R_1+R_2)}} = K_1 N^{1/3}$; $\alpha = \sqrt[3]{\frac{9N^2\pi^2(k_1+k_2)^2(R_1+R_2)}{16R_1R_2}} = K_2 N^{2/3}$; $C_N = \frac{d\alpha}{dN} = \sqrt[3]{\frac{\pi^2(k_1+k_2)^2(R_1+R_2)}{6R_1R_2N}} = \frac{2K_2}{3N^{1/3}}$

contact radius   approach   compliance in Normal load

- Contact due to applic of tangential after normal loading using static friction law

Tangential Problem only

1. Assume for $r<a$ for tangential problem only $\sigma_{zz}\big|_{z=0} = 0$
2. $u_y = 0$ $u_x' = \text{constant} \Rightarrow p_x' = \frac{T}{2\pi a}\frac{1}{\sqrt{a^2-r^2}}$ and $p_x \le f p_z$
3. no slip radius is c ; slip occurs for $a < r < c$

$p_x = \frac{3fN}{2\pi a^3}\left[(a^2-r^2)^{1/2} - (c^2-r^2)^{1/2}\right]$ $u_y = 0$ $u_x = \frac{3fN}{16a\mu}(2-\nu)\left(1-\frac{c^2}{a^2}\right) = \frac{3fN}{16a\mu}(2-\nu)\left[1-\left(1-\frac{T}{fN}\right)^{2/3}\right]$

$\therefore\ c = a\sqrt[3]{1-\frac{T}{fN}}$ or $f = \frac{T/N}{(1-[c/a]^3)}$ see back of page 19

no slip radius

$C_s = \frac{du_x}{dT} = \frac{2-\nu}{8\mu a}\left[1-\frac{T}{fN}\right]^{-1/3}$ ← compliance in Shear

- If we increase T to T^* and then decrease to T'

then a radius $b = a\left(1 - \frac{T^*-T'}{2fN}\right)^{1/3}$ and a $u_x = \frac{3(2-\nu)fN}{16\mu a}\left[2\left(1-\frac{T^*-T}{2fN}\right)^{2/3} - \left(1-\frac{T^*}{fN}\right)^{2/3} - 1\right]$

Energy dissip $= E_d = \frac{(2-\nu)T^{*3}}{36\mu a f N}$

displacement of adhered section of contact circle in horiz direction

$u_x = 3(2-\nu)fN\left[2(b/a)^2 - (c/a)^2 - 1\right]/16\mu a$

me238c HO#2

CHAPTER III

ELASTIC INCLUSIONS AND INHOMOGENEITIES

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§ 1. Introduction

This review is concerned with the two following problems in the infinitesimal theory of elasticity, and with their inter-relation and generalization.

(i) The transformation problem.

A region (the 'inclusion') in a homogeneous elastic medium undergoes a permanent change of form which, in the absence of the constraint imposed by its surroundings (the 'matrix'), would be a prescribed uniform strain. To find the elastic field in matrix and inclusion.

(ii) The inhomogeneity problem.

A region in an otherwise homogeneous elastic medium has elastic constants differing from those of the remainder. To find how an applied stress, uniform at large distances, is disturbed by the inhomogeneity.

We shall not consider two-dimensional problems, where complex variable methods can be used, and the number of special cases which may be solved is unlimited. The three-dimensional inhomogeneity problem has been discussed extensively, particularly the special case of a cavity, i.e. an 'inhomogeneity' with vanishing elastic constants. The inclusion problem has received less attention, but is encountered in the discussion of various phenomena in solid-state physics, for example martensitic transformations and the formation of precipitates. Sternberg [1958] has given an excellent annotated bibliography of the three-dimensional inhomogeneity (and inclusion) problems which have been solved.

We shall be largely concerned with the special case where the inclusion or inhomogeneity takes the form of the general ellipsoid with three unequal axes. There are two reasons for this. First, it appears to be the most general case whose solution can be given in a manageable form. Secondly, in this particular case there is a close connexion between the transformation and inhomogeneity problems. It stems from the fact that, as we shall see, the stress is constant throughout an ellipsoidal inclusion which has undergone a uniform transformation. As an illustration of this connexion suppose that it is required to solve the inhomogeneity problem for an ellipsoidal cavity. On the stress-field due to the inclusion superimpose everywhere a uniform stress equal and opposite to the uniform stress in the inclusion. The inclusion is then free of stress and may be removed. We are left with a stress-field which becomes uniform at infinity and which gives zero traction on the surface of the ellipsoid, as required. The general problem of an ellipsoidal inhomogeneity can be handled by an extension of this argument.

Among closed surfaces the ellipsoid alone has this convenient property. It shares it with other second-degree surfaces, and the analysis of §§ 3,4 can in fact be applied with trivial modifications to hyperboloids and paraboloids. However, the properties of such infinite 'inclusions' are not of much interest, and we shall not consider them. The corresponding in homogeneity problem is essentially the problem of stress-concentration by hyperboloidal notches. The cases which are of practical use have been discussed by NEUBER [1958].

We begin (§ 2) by finding a solution to problem (i) for an inclusion of arbitrary shape. The argument used is somewhat intuitive, but it is verified that the solution which is obtained does in fact satisfy the conditions of the problem. In § 3 the special case of an ellipsoidal inclusion is worked out to a point where numerical calculation is possible. Section 4 begins with a discussion of the inhomogeneity problem for an inhomogeneous region of arbitrary shape (though not much progress can be made) and the solution for the ellipsoidal inclusion is obtained from the results of § 2 in the way already indicated. By taking advantage of the connexion between inclusions and Somigliana dislocations (§ 5) it is possible to solve the problem of an ellipsoidal inhomogeneity perturbing a non-uniform stress-field. In § 6 we present some selected physical applications of the theory.

We shall use the familiar suffix notation. Repeated suffixes are to be summed over the values 1, 2, 3 and suffixes following a comma will denote differentiation with respect to the Cartesian coordinates x_1, x_2, x_3, e.g. $u_{i,j} = \partial u_i/\partial x_j$, $\psi_{,ijk} = \partial^3\psi/\partial x_i \partial x_j \partial x_k$.

Displacement u_i and strain e_{ij} are related by

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})\,. \tag{1.1}$$

Stress p_{ij} and strain are related by

$$p_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \tag{1.2}$$

in an isotropic medium with Lamé constants λ, μ. If a second pair of quantities p'_{ij}, e'_{ij} satisfy

$$p'_{ij} = \lambda e'_{kk}\delta_{ij} + 2\mu e'_{ij}$$

then

$$p_{ij}e'_{ij} = p'_{ij}e_{ij}\,. \tag{1.3}$$

A set of quantities bearing a common affix, e.g. $\overset{\mathrm{C}}{u}_i$, $\overset{\mathrm{C}}{e}_{ij}$, $\overset{\mathrm{C}}{p}_{ij}$ will be supposed to be related by (1.1) and (1.2) unless otherwise stated. It will sometimes be convenient to use the notation

$$f = f_{kk}\,, \qquad 'f_{ij} = f_{ij} - \tfrac{1}{3}f_{kk}\delta_{ij} \tag{1.4}$$

to denote the scalar and deviatoric parts of a second-order tensor.

We shall often make use of the formula

$$\nabla^2(pq) = p\nabla^2 q + q\nabla^2 p + 2p_{,k}q_{,k} \tag{1.5}$$

for calculating the Laplacian of a product, and of Gauss's theorem in the form

$$\int_S A_{\ldots}\mathrm{d}S_k = \int_V A_{\ldots,k}\mathrm{d}v \tag{1.6}$$

where S is a closed surface enclosing the volume V. Here and elsewhere $\mathrm{d}S_k$ is an abbreviation for $n_k\mathrm{d}S$, where dS is an element of surface and n_k is its normal.

§ 2. The General Transformed Inclusion

2.1. The elastic field

In this section we give a formal solution of the following problem:

A region bounded by a closed surface S in a homogeneous isotropic elastic medium undergoes a change of form which but for the constraint imposed by the surrounding material would be an arbitrary homogeneous strain $\overset{\mathrm{T}}{e}_{ij}$. To find the resulting elastic field inside and outside S.

It will be convenient to refer to the material inside S as the 'inclusion' and to the material outside S as the 'matrix'. The strain will be called the 'transformation strain', or, following ROBINSON [1951], the 'stress-free strain'. We assume that adjacent points immediately inside and outside S suffer no relative displacement; in other words the inclusion is 'bonded' to the matrix before, during and after the transformation. The problem can also be formulated as follows. To find a state of self-stress in an infinite body, with the following property: on making a cut over a prescribed closed surface S we are left with a stress-free cavity and a stress-free solid bounded by surfaces S_1, S_2 such that S_2 is transformed into S_1 by the homogeneous strain $-\overset{\mathrm{T}}{e}_{ij}$.

TIMOSHENKO and GOODIER [1951] have given a method for finding the elastic field in a material in which each volume element alters its unconstrained shape. Our basic inclusion problem is merely the special case in which the change of shape is identical for all the volume elements inside a certain surface S and is zero for all elements outside S. The method of calculation we shall use (ESHELBY [1957]) is essentially equivalent to theirs.

It is first necessary to decide how to define the displacement. Let us fix our attention on some marked point $\boldsymbol{r}$ in the material, with coordinates $x_i(\boldsymbol{r})$ and suppose that, as we watch, the transformation takes place gradually, by some physical mechanism unspecified. Every point of the medium moves, and when the transformation is over the marked point will have different coordinates, $x_i(\boldsymbol{r}) + \overset{\mathrm{C}}{u}_i(\boldsymbol{r})$ say. We take $\overset{\mathrm{C}}{u}_i$ as our displacement function. The state of zero displacement is the state of the material before the transformation has occurred.

We may contrast this definition of the displacement with another, perhaps equally natural, one. Suppose that after the transformation has occurred we make a cut over the surface S. Points on either side of the cut will move relatively as the stresses relax. For simplicity suppose that the two faces of the cut shrink away from each other everywhere, leaving a gap. During the relaxation every point of the matrix or inclusion suffers a certain displacement, $-\overset{\mathrm{D}}{u}_i$ say. In place of $\overset{\mathrm{C}}{u}_i$ we might take $\overset{\mathrm{D}}{u}_i$ as our displacement function. In the matrix both $\overset{\mathrm{C}}{u}_i$ and $\overset{\mathrm{D}}{u}_i$ are measured from a state in which the matrix is free of stress. Consequently in the matrix $\overset{\mathrm{C}}{u}_i = \overset{\mathrm{D}}{u}_i$. In the inclusion $\overset{\mathrm{C}}{u}_i$ is measured from a state in which the inclusion is untransformed and unstressed, whereas $\overset{\mathrm{D}}{u}_i$ is measured from a state in which the inclusion has transformed but is free of stress because the constraint due to the matrix has been removed. Because of the gap which appears when matrix and inclusion are cut apart $\overset{\mathrm{D}}{u}_i$ is discontinuous across S, whereas $\overset{\mathrm{C}}{u}_i$ is continuous. It is generally more convenient to use $\overset{\mathrm{C}}{u}_i$, but we shall refer to $\overset{\mathrm{D}}{u}_i$ again in § 5.

The displacement $\overset{\mathrm{C}}{u}_i$ will be calculated with the help of a sequence of imaginary cutting, straining and welding operations. This approach is somewhat alien to the usual methods of applied mathematics; the argument can, of course, be considered to be a purely heuristic one

which points to a result (eq. (2.8)) whose validity has to be tested.

We shall suppose, to begin with, that the matrix extends to infinity. In the unstrained medium mark out the boundary S of the proposed transformed inclusion. Make a cut over S and remove the inclusion. Allow the transformation to occur. After it has suffered the uniform transformation $\overset{\mathrm{T}}{e}_{ij}$ the inclusion can no longer be fitted without strain into the cavity from which it was taken. Apply surface tractions $-\overset{\mathrm{T}}{p}_{ij}n_j$ to the surface of the inclusion, where

$$\overset{\mathrm{T}}{p}_{ij} = \lambda \overset{\mathrm{T}}{e}_{mm}\delta_{ij} + 2\mu \overset{\mathrm{T}}{e}_{ij}\,. \tag{2.1}$$

This produces a strain $-\overset{\mathrm{T}}{e}_{ij}$ in the inclusion and restores it to the form it had before transformation. Put the inclusion back in the cavity, still maintaining the surface tractions. Weld the material together across S. The surface tractions thus become an embedded layer of body force of amount

$$\mathrm{d}F_i = -\overset{\mathrm{T}}{p}_{ij}n_j\mathrm{d}S \tag{2.2}$$

on each element dS of S. The matrix is now unstressed and there is a uniform stress $-\overset{\mathrm{T}}{p}_{ij}$ in the inclusion. Further, every point in matrix or inclusion has the same position as it had before the transformation. That is, as regards displacement, the material is in the initial state from which we have agreed to measure $\overset{\mathrm{C}}{u}_i$. This state only differs from the required final state by the presence of the layer of body force (2.2). To get rid of this unwanted layer we apply an equal and opposite layer of body force

$$\mathrm{d}F_i = +\overset{\mathrm{T}}{p}_{ij}n_j\mathrm{d}S = \overset{\mathrm{T}}{p}_{ij}\mathrm{d}S_j \tag{2.3}$$

over S. The displacement induced by this operation is the displacement which we are trying to calculate.

A point force F_i at $\boldsymbol{r}'$ produces a displacement

$$u_i(\boldsymbol{r}) = U_{ij}F_j$$

at $\boldsymbol{r}$, where† (LOVE [1954])

$$U_{ij} = \frac{1}{4\pi\mu}\frac{\delta_{ij}}{|\boldsymbol{r}-\boldsymbol{r}'|} - \frac{1}{16\pi\mu(1-\sigma)}\frac{\partial^2}{\partial x_i \partial x_j}|\boldsymbol{r}-\boldsymbol{r}'|$$
$$= \frac{1}{4\pi\mu}\left(\tfrac{1}{3}\nabla^2 - \frac{1}{4(1-\sigma)}\frac{\partial^2}{\partial x_i \partial x_j}\right)|\boldsymbol{r}-\boldsymbol{r}'| \tag{2.4}$$

δ_{ij}

since

$$|\boldsymbol{r}-\boldsymbol{r}'|^{-1} = \tfrac{1}{2}\nabla^2|\boldsymbol{r}-\boldsymbol{r}'|\,. \tag{2.5}$$

† σ is Poisson's ratio.

Hence

$$u_i^C(r) = \int_S dS_k p_{jk}^T U_{ij}(|r - r'|) . \tag{2.6}$$

But by Gauss's theorem

$$\int |r - r'| \, dS_k = \int \frac{\partial}{\partial x'_k} |r - r'| \, dv' = - \frac{\partial}{\partial x_k} \int |r - r'| dv$$

λ'

and so

$$u_i^C = \frac{1}{16\pi\mu(1-\sigma)} p_{jk}^T \psi_{,ijk} - \frac{1}{4\pi\mu} p_{ik}^T \varphi_{,k} \tag{2.7}$$

or

$$u_i^C = \frac{1}{8\pi(1-\sigma)} e_{jk}^T \psi_{,ijk} - \frac{1}{2\pi} e_{ik}^T \varphi_{,k} - \frac{\sigma}{4\pi(1-\sigma)} e^T \varphi_{,i} \tag{2.8}$$

where

$$\varphi(r) = \int_V \frac{dv'}{|r - r'|} \quad \text{and} \quad \psi(r) = \int_V |r - r'| \, dv' . \tag{2.9}$$

this

φ is the ordinary (harmonic or Newtonian) potential of attracting matter of unit density filling the volume V bounded by S. ψ is the corresponding biharmonic potential. Geometrically, ψ/V is the mean distance of the point r from all the points inside S.

From (2.5)

$$\nabla^2 \psi = 2\varphi . \quad \nabla^2 r = \frac{2}{r} \tag{2.10}$$

The following results follow from the theory of attraction (Poincaré [1899], MacMillan [1958]).

reversed

$\frac{1}{|r-r'|} = \delta(r-r')$

$$\nabla^4 \psi = 2\nabla^2 \varphi = \begin{cases} -8\pi & \text{outside S} \\ 0 & \text{inside S} \end{cases} \tag{2.11}$$

$$\varphi, \ \varphi_{,i} \text{ are continuous across S} \tag{2.12}$$

$$\varphi_{,ij}(\text{out}) - \varphi_{,ij}(\text{in}) = 4\pi n_i n_j . \tag{2.13}$$

The last equation gives the difference in the second derivative at two adjacent points immediately inside and outside S at a point where the normal to S is n_i. We shall use a similar notation for other quantities which are discontinuous across S. Eq. (2.13) is a re-statement of the result that the discontinuity in attraction across a double layer is 4π times its moment. More generally, a function satisfying

$$\nabla^2 U = -4\pi\varrho \tag{2.14}$$

suffers a discontinuity in its second derivatives

$$U_{,kl}(\text{out}) - U_{,kl}(\text{in}) = -4\pi\{\varrho(\text{out}) - \varrho(\text{in})\} n_k n_l \tag{2.15}$$

on crossing a surface across which ϱ is discontinuous. But $\psi_{,ij}$ satisfies (2.14) with $\varrho = -2\varphi_{,ij}/4\pi$ and so from (2.15) we obtain the relation

$$\psi_{,ijkl}(\text{out}) - \psi_{,ijkl}(\text{in}) = 8\pi n_i n_j n_k n_l . \tag{2.16}$$

By similar arguments one finds that

$$\psi, \ \psi_{,i}, \ \psi_{,ijk} \text{ are continuous across S.} \tag{2.17}$$

The stress in the matrix is

$$p_{ij}^C = \lambda u_{m,m}^C \delta_{ij} + \mu(u_{i,j}^C + u_{j,i}^C) . \tag{2.18}$$

Since the inclusion was already subject to a uniform stress $-p_{ij}^T$ before the body force (2.3) was applied, the stress in the inclusion is

$$p_{ij}^I = p_{ij}^C - p_{ij}^T \tag{2.19}$$

with p_{ij}^C derived from u_{ij}^C as in (2.18).

We outline a method of verifying formally that the proposed solution (2.8), (2.18), (2.19) does in fact solve the inclusion problem. From (2.7) it follows that u_i^C satisfies the equilibrium equation

$$\mu\nabla^2 u_i + (\lambda + \mu) u_{m,mi} = 0$$

and from (2.12), (2.17) that it is continuous across S. If the stress is defined by (2.18) and (2.19) the relations (2.13) and (2.16) show that $p_{ij}n_j$ is continuous across S. Let an additional displacement $-u_i^C(r')$ be imposed on all points r' or the inner boundary of the matrix. By the uniqueness theorem of the theory of elasticity (Love [1954]) it will produce an additional displacement $-u_i^C(r)$ throughout the matrix, and so leave it stress-free. Likewise an additional displacement $-u_i^C(r')$ imposed on all points r' of the surface of the inclusion induces a displacement $-u_i^C(r)$ throughout its interior and so by (2.19) leaves it in a state of uniform stress $-p_{ij}^T$. At this point the inner surface of the matrix and the outer surface of the inclusion still fit perfectly, because of the continuity of u_i^C. If the tractions $-p_{ij}^T n_j$ on the inclusion are reduced to zero the inclusion suffers a uniform strain e_{ij}^T and we are left with an unstressed matrix and an unstressed inclusion between whose surfaces there is the required misfit.

Equation (2.7) can be written in the Boussinesq-Papkovich-Neuber form

$$u_i^C = B_i - \frac{1}{4\kappa(1-\sigma)} (x_m B_m + \beta)_{,i} \tag{2.20}$$

with harmonic B_i and β:

$$4\pi\mu B_i = -\overset{\mathrm{T}}{p}_{ik}\,\varphi_{,k}\,, \qquad 4\pi\mu\beta = \overset{\mathrm{T}}{p}_{jk}\,f_{jk} \tag{2.21}$$

where

$$f_{ij} = x_i\varphi_{,j} - \psi_{,ij}\,. \tag{2.22}$$

That f_{ij} (and hence β) is harmonic inside and outside the inclusion follows from (1.5) and (2.10). Further f_{ij} behaves like r^{-1} for large r, while its normal derivative

$$\frac{\partial F}{\partial x_k} n_k = \nabla F\cdot\hat{n} = \partial f_{ij}/\partial n = \varphi_{,j}n_i + x_i\varphi_{,jk}n_k - \psi_{,ijk}n_k$$

suffers a discontinuity $4\pi x_i n_j n_k n_k = 4\pi x_i n_j$ on passing through S, by (2.13) and (2.16). Hence f_{ij} is the harmonic potential of a layer of attracting matter distributed over S with surface density $x_i n_j$. In this way the biharmonic potential ψ can be replaced by the harmonic potential β of a certain surface layer.

It is interesting to see how much information can be obtained when our knowledge of φ and ψ is incomplete. We know in any case that φ and ψ behave as V/r and Vr for large r, and hence, from (2.7) that the field at large distance from the inclusion is given by

$$\overset{\mathrm{C}}{u}_i(r) = \frac{V\overset{\mathrm{T}}{e}_{jk}\,g_{ijk}}{8\pi(1-\sigma)r^2} \tag{2.23}$$

where

$$g_{ijk} = (1-2\sigma)(\delta_{ij}l_k + \delta_{ik}l_j - \delta_{jk}l_i) + 3l_il_jl_k$$

and l_i is a unit vector joining the origin to the point of observation $\boldsymbol{r}$. If only φ is known we can find the dilatation and rotation:

$$e^{\mathrm{C}} = -\frac{1-2\sigma}{8\pi\mu(1-\sigma)}\overset{\mathrm{T}}{p}_{ik}\,\varphi_{,ik} \tag{2.24}$$

$$4\pi\overset{\mathrm{C}}{\omega}_{il} = 2\pi(\overset{\mathrm{C}}{u}_{i,l} - \overset{\mathrm{C}}{u}_{l,i}) = \overset{\mathrm{T}}{e}_{ik}\,\varphi_{,kl} - \overset{\mathrm{T}}{e}_{lk}\,\varphi_{,ki}\,. \tag{2.25}$$

If $\overset{\mathrm{T}}{e}_{ij}$ happens to be a pure dilatation we can find the complete field in terms of φ:

$$\overset{\mathrm{C}}{u}_i = -\frac{1+\sigma}{12\pi(1-\sigma)}e^{\mathrm{T}}\varphi_{,i} \qquad (\overset{\mathrm{T}}{e}_{ij} = \tfrac{1}{3}e^{\mathrm{T}}\delta_{ij}) \tag{2.26}$$

(GOODIER [1937]). In this special case the dilatation has the constant value $e^{\mathrm{T}}(1+\sigma)/3(1-\sigma)$ inside the inclusion. In the matrix the dilatation is zero. Consequently the value of the bulk modulus $\varkappa^*$ of the matrix is irrelevant and (2.26) applies also to the case of an in-

clusion of elastic constants $\mu,\varkappa$ in a matrix with constants $\mu,\varkappa^*$ if we give to σ the value appropriate to the inclusion (CRUM, quoted by NABARRO [1940]; ROBINSON [1951]).

Again, it may happen that it is easier to calculate φ and ψ for points within the inclusion than for points outside it. (We shall see that this is so for an ellipsoidal inclusion.) Since the strains involve $\varphi_{,ij}$ and $\psi_{,ijkl}$ we can use (2.13) and (2.16) to find the strain $\overset{\mathrm{C}}{e}_{ij}(\text{out})$ (and hence the stress) at a point immediately outside the inclusion from the values $\overset{\mathrm{C}}{e}_{ij}(\text{in})$ at the adjacent point immediately inside the inclusion. Expressed in terms of the dilatational and deviatoric parts of $\overset{\mathrm{C}}{e}_{ij}$ the result is

$$e^{\mathrm{C}}(\text{out}) = e^{\mathrm{C}}(\text{in}) - \frac{1}{3}\frac{1+\sigma}{1-\sigma}e^{\mathrm{T}} - \frac{1-2\sigma}{1-\sigma}{}'\overset{\mathrm{T}}{e}_{ij}\,n_in_j$$

$$\begin{aligned} {}'\overset{\mathrm{C}}{e}_{il}(\text{out}) = {}&{}'\overset{\mathrm{C}}{e}_{il}(\text{in}) + \frac{1}{1-\sigma}{}'\overset{\mathrm{T}}{e}_{jk}\,n_jn_kn_in_l - {}'\overset{\mathrm{T}}{e}_{ik}\,n_kn_l - {}'\overset{\mathrm{T}}{e}_{lk}\,n_kn_i \\ &+ \frac{1-2\sigma}{3(1-\sigma)}{}'\overset{\mathrm{T}}{e}_{jk}\,n_jn_k\,\delta_{il} - \frac{1}{3}\frac{1+\sigma}{1-\sigma}e^{\mathrm{T}}(n_in_l - \tfrac{1}{3}\delta_{il})\,. \end{aligned} \tag{2.27}$$

The foregoing results all refer to a transformed inclusion in an infinite matrix. If the matrix has a finite boundary the displacements will be of the form

$$\overset{\mathrm{F}}{u}_i = \overset{\mathrm{C}}{u}_i + \overset{\mathrm{im}}{u}_i$$

and the stress will be

$$\overset{\mathrm{F}}{p}_{ij} = \overset{\mathrm{C}}{p}_{ij} + \overset{\mathrm{im}}{p}_{ij}$$

in the matrix and

$$\overset{\mathrm{F}}{p}_{ij} - \overset{\mathrm{T}}{p}_{ij}$$

in the inclusion. The 'image field' $\overset{\mathrm{im}}{u}_i$, $\overset{\mathrm{im}}{p}_{ij}$ is free of singularities in the medium and is determined by the requirement that the sum of the C-field and the image-field shall satisfy whatever boundary conditions are imposed on the outer surface of the matrix. If the outer boundary S_0 is stress-free we must have

$$\overset{\mathrm{im}}{p}_{ij}\,n_j = -\overset{\mathrm{C}}{p}_{ij}n_j \quad \text{on} \quad \mathrm{S}_0$$

that is, the image-field is the field produced by surface tractions $-\overset{\mathrm{C}}{p}_{ij}n_j$ acting on the outer surface. If the outer surface is held immovable we must have

$$\overset{\mathrm{im}}{u}_i = -\overset{\mathrm{C}}{u}_i \quad \text{on} \quad \mathrm{S}_0\,. \tag{2.28}$$

Thus when the C-quantities are known the determination of the image quantities reduces to a standard boundary-value problem.

It is sometimes convenient to have a formal expression analogous to (2.6) exhibiting u_i^{F} as the displacement induced by the layer of body force (2.3). When the boundary condition is (2.28) (rigidly held boundary) we may evidently write

$$u_i^{\mathrm{F}}(r) = \int_{\mathrm{S}} \mathrm{dS}_k\, p_{jk}^{\mathrm{T}}\, U_{ij}(\boldsymbol{r}, \boldsymbol{r}') \tag{2.29}$$

where $F_j U_{ij}(\boldsymbol{r},\boldsymbol{r}')$ is the displacement at $\boldsymbol{r}$ due to a point-force $\boldsymbol{F} = (F_1, F_2, F_3)$ acting at $\boldsymbol{r}'$ in a body at whose surface the displacement is required to be zero.

With a suitable alternative definition of $U_{ij}(\boldsymbol{r}, \boldsymbol{r}')$ (2.29) also applies to the case where the outer boundary is stress-free. However, we cannot simply say that $F_j U_{ij}$ is the displacement at $\boldsymbol{r}$ due to a point-force at $\boldsymbol{r}'$ in a body with a stress-free surface. No such solution of the elastic equations exists, since the integral of the surface traction over S_0 must be equal to $-\boldsymbol{F}$. Instead we define $U_{ij}(\boldsymbol{r}, \boldsymbol{r}')$ as the displacement in the body with stress-free surfaces due to a point-force $\boldsymbol{F}$ at $\boldsymbol{r}'$, a point-force $-\boldsymbol{F}$ at the origin and a double force (LOVE [1954]) at the origin of moment $-\boldsymbol{F} \times \boldsymbol{r}'$. Since the resultant and moment of this set of forces are both zero the condition of zero traction over S_0 can be satisfied. It can easily be shown that when the elementary forces (2.3) are summed over S the corresponding auxiliary forces and moments at the origin add up to zero. Consequently, with this definition of U_{ij} the expression (2.29) actually gives the displacement due to the layer of body force (2.3) alone.

2.2. Energy relations

The elastic energy associated with the inclusion (i.e. the energy in the inclusion plus the energy in the matrix) can be calculated very simply by following the energy changes which occur during the imaginary operations leading to (2.6). Suppose first that the matrix is infinite. When the inclusion has been welded back into the matrix but is still held in its untransformed shape by the layer of body force (2.2) the energy in the matrix is zero and the energy in the inclusion is

$$\tfrac{1}{2} \int_V p_{ij}^{\mathrm{T}} e_{ij}^{\mathrm{T}}\, \mathrm{d}v\,. \tag{2.30}$$

When the layer of body force is relaxed each element of S moves

$w = \frac{1}{2} \int_V p_{ij} (-e_{ij} + e_{ij})\, dV + \frac{1}{2} \int p_{ij} e_{ij}\, dV_{ext}$ cont. | co...

$= \frac{1}{2} \int p_{ij}^{I} e_{ij}^{T} dV + \frac{1}{2} \int_S p_{ij} u_i^{c(in)} dS_j + \frac{1}{2} \int_{S_0} p_{ij} u_i^{c(out)} (-dS_j)$

through a distance u_i^{C} as the force on it falls from $\mathrm{d}F_i$ to zero. The amount of energy extracted from the elastic solid during the relaxation is thus

$$-\tfrac{1}{2} \int u_i^{\mathrm{C}}\, \mathrm{d}F_i = \tfrac{1}{2} \int_{\mathrm{S}} p_{ij}^{\mathrm{T}} u_i^{\mathrm{C}}\, \mathrm{dS}_j = \tfrac{1}{2} \int p_{ij}^{\mathrm{T}} e_{ij}^{\mathrm{C}}\, \mathrm{d}v\,. \tag{2.31}$$

The energy remaining in the medium is found by subtracting (2.31) from (2.30):

$$E_\infty = -\tfrac{1}{2} \int_V p_{ij}^{\mathrm{T}} (e_{ij}^{\mathrm{C}} - e_{ij}^{\mathrm{T}})\, \mathrm{d}v = -\tfrac{1}{2} \int_V p_{ij}^{\mathrm{I}} e_{ij}^{\mathrm{T}}\, \mathrm{d}v\,. \tag{2.32}$$

OKAY

The suffix ∞ emphasizes that this is the energy for an inclusion in an infinite matrix. Evidently we only need to know the elastic field inside the inclusion. If the stress-free strain is a pure dilatation $e_{ij}^{\mathrm{T}} = \frac{1}{3} e^{\mathrm{T}} \delta_{ij}$ we have by (2.26)

$$E_\infty = \frac{2}{9} \mu V \frac{1+\sigma}{1-\sigma} (e^{\mathrm{T}})^2 \tag{2.33}$$

for an inclusion of any shape. According to the argument following (2.26) the expression is still correct if the bulk moduli of matrix and inclusion differ (NABARRO [1940], ROBINSON [1951]). Doesn't really m[...] sense s[...] must use e^{T*} av[...] $e^T = e^T($

Suppose next that the inclusion is situated in a finite matrix bounded by the surface S_0. No difference is made to (2.30) if we cut away the part of the unstrained matrix exterior to S_0. The energy removed by relaxing the layer of body force is given by (2.31) with u_i^{C} replaced by $u_i^{\mathrm{C}} + u_i^{\mathrm{im}}$. The elastic energy due to the stress-field of the inclusion is thus

$$E_{\mathrm{inc}} = E_\infty + E_{\mathrm{im}}$$

where the 'image term'

$$E_{\mathrm{im}} = -\tfrac{1}{2} \int_V p_{ij}^{\mathrm{T}} e_{ij}^{\mathrm{im}}\, \mathrm{d}v = -\tfrac{1}{2} \int_V p_{ij}^{\mathrm{im}} e_{ij}^{\mathrm{T}}\, \mathrm{d}v$$

varies with the position of the inclusion. As an example we consider the case of a sphere of volume $V(1+e^{\mathrm{T}})$ forced into a spherical cavity of volume V in a semi-infinite solid. In the infinite solid u_i^{C} is given by (2.26) with $\varphi = V/r$. E_∞ is given by (2.33). To find E_{im} we have to calculate e_{kk}^{im}. It is the dilatation produced by surface tractions $-p_{ij}^{\mathrm{C}} n_j$ acting on the free surface of the semi-infinite solid, and can be found by well-known methods (LOVE [1954]). As the dilatation is harmonic we need only find its value at the centre of the inclusion, since the mean value of harmonic function of a sphere is equal to its value at the centre. The result is

$$E_{\mathrm{inc}} = E_\infty \left\{1 - \frac{1}{4}(1+\sigma)\frac{a^3}{h^3}\right\} \tag{2.34}$$

I get $E_{inc} = E_\infty \left\{1 - \frac{1}{6}(1+\sigma)\frac{a^3}{h^3}\right\}$

where a is the radius of the sphere and h is the distance of its centre from the free surface. If $\sigma = \frac{1}{3}$ the elastic energy is reduced to $\frac{2}{3}E_\infty$ if the inclusion just reaches the surface ($h = a$).

It is worth recalling at this point that, in thermodynamic terms, 'elastic energy' represents internal energy under adiabatic conditions and Helmholtz free energy under isothermal conditions (SOKOLNIKOFF [1946]). A calculation such as the foregoing covers both cases; the distinction only appears when we decide to insert either the adiabatic or isothermal values of the elastic constants.

In some applications it is necessary to consider the changes of energy which occur when an inclusion is formed in a body which is already stressed by externally applied loads. For simplicity we shall suppose that the body is stressed by surface tractions which do not vary when the outer surface S_0 of the body is slightly deformed by the introduction of the inclusion; in engineering language this is the case of 'dead loading'. The stress due to the inclusion must satisfy $\overset{F}{p}_{ij} n_j = 0$ on S_0.

Let the external loads produce stress and strain $\overset{A}{p}_{ij}$, $\overset{A}{e}_{ij}$, not necessarily uniform. Before the inclusion has transformed the elastic energy in the material is

$$E_A = \tfrac{1}{2} \int \overset{A}{p}_{ij} \overset{A}{e}_{ij} \, dv \tag{2.35}$$

with the integral extending over the whole volume of the material. Suppose next that the body is subject to the combined action of the stresses due to the load and the internal stresses due to the inclusion. We might expect that the elastic energy would be the sum of (2.35), (2.34) and a cross term, representing an 'interaction energy'. But in fact the cross term is zero. To see this, suppose that the transformation occurs first, in the absence of external loads. The elastic energy is E_{Inc}. Now let the load be applied. Within the limits of the usual infinitesimal theory of elasticity the body responds to external forces just as it would if it were not self-stressed by the transformed inclusion. The work done on the body by the load is thus

$$\tfrac{1}{2} \int_{S_0} \overset{A}{p}_{ij} \overset{A}{u}_i \, dS_j = \tfrac{1}{2} \int \overset{A}{p}_{ij} \overset{A}{e}_{ij} \, dv = E_A$$

and so the total elastic energy is simply $E_{\text{Inc}} + E_A$. The same conclusion can be reached analytically. If the volume integral of the energy density is converted into a surface integral over S and the outer surface of the matrix it will be found that the cross term between the A- and F-terms vanishes because $\overset{F}{p}_{ij} n_j = 0$ on S_0.

Despite the lack of a cross term in the elastic energy it is possible to define a physically meaningful interaction energy. (For a general discussion in the context of solid state theory cf. PEACH [1951], ESHELBY [1951, 1956].) To introduce the concept of interaction energy we begin by enquiring whether in the presence of the stresses arising from the external load it is energetically possible for the inclusion to form spontaneously. At first sight it appears as if the answer is no, since the elastic energy increases by the necessarily positive quantity E_{Inc} when the inclusion is introduced. However, we have to consider not simply the elastic energy of the material, but rather the total energy, E_{tot} say, of the closed system made up of the body and the loading mechanism. When the transformation occurs in the presence of the external load the increase in the potential energy of the loading mechanism is equal to the work which the surface tractions $\overset{A}{p}_{ij} n_j$ do on the body as the surface displacements change from $\overset{A}{u}_i$ to $\overset{A}{u}_i + \overset{F}{u}_i$. Thus the increase in the energy of the whole system (inclusion plus matrix plus loading mechanism) is

$$\Delta E_{\text{tot}} = E_{\text{Inc}} + E_{\text{Int}} \tag{2.36}$$

where

$$E_{\text{Int}} = -\int_{S_0} \overset{A}{p}_{ij} \overset{F}{u}_i \, dS_j \,. \tag{2.37}$$

ah - he calls E_{ext} the interaction energy since he put loading mech. in the system

We may write

$$E_{\text{tot}} = E_0 + E_A + E_{\text{Inc}} + E_{\text{Int}} \tag{2.38}$$

where E_0 is the potential energy of the loading mechanism in the absence of the inclusion. In (2.38) the first two terms depend only on the elastic field due to the load and the third only on the field due to the inclusion. The last term, which depends on both these fields, may be appropriately called the interaction energy.

We can now answer the question posed in the preceding paragraph. If the applied stress is chosen so that $\Delta E_{\text{tot}} < 0$ (and this can always be done) it is energetically possible for the transformation to take place spontaneously. Generally ΔE_{tot}, whether it is positive or negative, may be called the energy of formation of the inclusion in the applied stress-field. This concept is familiar in thermodynamics. In fact, if the transformation leading to the formation of the inclusion occurs under adiabatic conditions ΔE_{tot} represents, in thermodynamic language, the increase in the enthalpy of the body, while if the transformation takes place under isothermal conditions ΔE_{tot} is the increase in its Gibbs free energy. This follows at once from the definition of these quantities. The increase in enthalpy associated with some change in the state of a thermally isolated system is equal to the increase in its internal energy plus the work which it does on its environment during the course of the change. For a change at constant temperature the Gibbs free energy is similarly defined, replacing 'internal energy' by 'Helmoltz free energy'. That ΔE_{tot} is the change in enthalpy or Gibbs free energy follows from the fact that, as we have seen, the 'elastic energy' has to be identified with the internal energy in adiabatic processes and with Helmholtz free energy in isothermal processes. Thus ΔE_{tot} is the enthalpy or Gibbs free energy of formation of the inclusion.

The expression (2.39) for the interaction energy can be put into a more useful form by the following artifice. We re-write (2.37) as

$(u^c + u^{im})$, $(p^c + p^{im})$

$$E_{int} = -\int_{S_0} (\overset{A}{p}_{ij}\overset{F}{u}_i - \overset{F}{p}_{ij}\overset{A}{u}_i)\,dS_j\,. \qquad (2.39)$$

Because $\overset{F}{p}_{ij}\overset{A}{u}_i = 0$ on S_0 the added term is zero. Consider the divergence of the integrand. It can be split into two terms

$$(\overset{im}{p}_{ij}\overset{A}{u}_i - \overset{A}{p}_{ij}\overset{im}{u}_i)_{,j} = \overset{im}{p}_{ij}\overset{A}{e}_{ij} - \overset{A}{p}_{ij}\overset{im}{e}_{ij} \qquad (2.40)$$

and

$$(\overset{C}{p}_{ij}\overset{A}{u}_i - \overset{A}{p}_{ij}\overset{C}{u}_i)_{,j} = \overset{C}{p}_{ij}\overset{A}{e}_{ij} - \overset{A}{p}_{ij}\overset{C}{e}_{ij}\,. \qquad (2.41)$$

By (1.3) the expression (2.40) is zero both in the matrix and the inclusion and so the image terms make no contribution. On the other hand, (2.41) vanishes in the matrix ~~but not~~ and in the inclusion, ~~since~~ but $\overset{C}{p}_{ij}$ is discontinuous across S. Hence in (2.39) we may replace $\overset{F}{p}_{ij}$, $\overset{F}{u}_i$ by $\overset{C}{p}_{ij}$, $\overset{C}{u}_i$ and carry out the integration over the boundary of the inclusion instead of over the outer surface of the body; that is

c e_{ij},

i.e. - volume integral over matrix

$$E_{int} = \int_S (\overset{C}{p}_{ij}\overset{A}{u}_i - \overset{A}{p}_{ij}\overset{C}{u}_i)\,dS_j\,. \qquad (2.42)$$

In (2.43) we may replace $\overset{C}{p}_{ij}$ by $\overset{I}{p}_{ij}$ since $\overset{C}{p}_{ij}n_j = \overset{I}{p}_{ij}n_j$ on S. The integral can then be converted to a volume integral over the inclusion:

$$E_{int} = \int_V (\overset{I}{p}_{ij}\overset{A}{e}_{ij} - \overset{A}{p}_{ij}\overset{C}{e}_{ij})\,dS_j\,.$$

dV, $\overset{c}{p}_{ij}\overset{A}{e}_{ij}$

By (1.3) the integrand is equal to $(\overset{I}{p}_{ij} - \overset{C}{p}_{ij})\overset{A}{e}_{ij}$ and so by (2.19)

$$E_{int} = -\int_V \overset{T}{p}_{ij}\overset{A}{e}_{ij}\,dv = -\int_V \overset{A}{p}_{ij}\overset{T}{e}_{ij}\,dv\,. \qquad (2.43)$$

$= -\overset{T}{e}_{ij}\int_V \overset{A}{p}_{ij}\,dV$

Evidently to find the interaction energy we need only know the stress-free strain $\overset{T}{e}_{ij}$; it is unnecessary to solve the elastic problems associated with the determination of $\overset{C}{u}_i$ or $\overset{im}{u}_i$.

§ 3. The Ellipsoidal Inclusion

3.1. The elastic field

When the inclusion is bounded by the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

the elastic field may be found explicitly. The form of the harmonic potential φ is well-known (Kellogg [1929]). There is an analogous expression for the biharmonic potential ψ (Eshelby [1959b]), but in fact all the derivatives of ψ which enter (2.7) can be found in terms of derivatives of φ. Let us compare f_{12} (eq. (2.7)) with the function $g = a^2(x_1\varphi_{,2} - x_2\varphi_{,1})/(a^2 - b^2)$ which plays a role in the hydrodynamic theory of rotating ellipsoids. Each of these functions is harmonic inside and outside the ellipsoid, is continuous across its surface and falls to zero at infinity. Across the surface there is a discontinuity $4\pi x_1 n_2$ in the normal derivative of f_{12} (eq. (2.22)). The corresponding discontinuity for g is $4\pi a^2(x_1n_2 - x_2n_1)/(a^2 - b^2)$. This is simply $4\pi x_1 n_2$ in view of the relation $b^2x_1n_2 = a^2x_2n_1$ which follows from the expressions

$$n_1 = x_1/a^2h\,,$$
$$n_2 = x_2/b^2h\,, \quad (h^2 = x^2/a^4 + y^2/b^4 + z^2/c^4)$$
$$n_3 = x_3/c^2h\,,$$

for the components of the normal to an ellipsoid. Hence f_{12} and g are identical, being harmonic potentials of the same surface distribution. We therefore have

$$f_{12} = x_1\varphi_{,2} - \psi_{,12} = \frac{a^2}{a^2 - b^2}(x_1\varphi_{,2} - x_2\varphi_{,1})$$

and similarly for the other f_{ij} with $i \neq j$. Thus

$$\psi_{,12} = \frac{a^2}{a^2 - b^2}x_2\varphi_{,1} + \frac{b^2}{b^2 - a^2}x_1\varphi_{,2}$$
$$\psi_{,23} = \frac{b^2}{b^2 - c^2}x_3\varphi_{,2} + \frac{c^2}{c^2 - b^2}x_2\varphi_{,3}$$
$$\psi_{,31} = \frac{c^2}{c^2 - a^2}x_1\varphi_{,3} + \frac{a^2}{a^2 - c^2}x_3\varphi_{,1}\,.$$

It is more difficult to derive expressions for f_{11}, f_{22}, f_{33} and hence for $\psi_{,11}$, $\psi_{,22}$, $\psi_{,33}$. However, there is no need to obtain them, since all the third derivatives $\psi_{,ijk}$ appearing in (2.8) can be made to depend on φ, $\psi_{,12}$, $\psi_{,23}$, $\psi_{,31}$. We may write, for example,

$$\psi_{,112} = (\psi_{,12})_{,1} \qquad \psi_{,111} = 2\varphi_{,1} - (\psi_{,12})_{,2} - (\psi_{,13})_{,3}\,.$$

The first relation is trivial; the second is obtained by differentiating $\nabla^2\psi = 2\varphi$ with respect to x_1.

Substitution in (2.8) gives

$$\begin{aligned} 8\pi(1-\sigma)u_1^{\mathrm{C}} = {} & \frac{e_{22}^{\mathrm{T}} - e_{11}^{\mathrm{T}}}{a^2-b^2}\frac{\partial}{\partial x_2}(a^2x_2\varphi_{,1} - b^2x_1\varphi_{,2}) + \\ & \frac{e_{33}^{\mathrm{T}} - e_{11}^{\mathrm{T}}}{c^2-a^2}\frac{\partial}{\partial x_3}(c^2x_1\varphi_{,3} - a^2x_3\varphi_{,1}) - \\ & 2\{(1-\sigma)e_{11}^{\mathrm{T}} + \sigma(e_{22}^{\mathrm{T}} + e_{33}^{\mathrm{T}})\}\varphi_{,1} - \\ & 4(1-\sigma)(e_{12}^{\mathrm{T}}\varphi_{,2} + e_{13}^{\mathrm{T}}\varphi_{,3}) + \frac{\partial}{\partial x_1}\bar{\beta} \end{aligned} \qquad (3.1)$$

where

$$\begin{aligned} \bar{\beta} = {} & \frac{2e_{12}^{\mathrm{T}}}{a^2-b^2}(a^2x_2\varphi_{,1} - b^2x_1\varphi_{,2}) + \\ & \frac{2e_{23}^{\mathrm{T}}}{b^2-c^2}(b^2x_3\varphi_{,2} - c^2x_2\varphi_{,3}) + \\ & \frac{2e_{31}^{\mathrm{T}}}{c^2-a^2}(c^2x_1\varphi_{,3} - a^2x_3\varphi_{,1})\,. \end{aligned}$$

u_2^{C} and u_3^{C} are found by cyclic permutation of (1, 2, 3), (a, b, c).

At an internal point

$$\varphi = \tfrac{1}{2}(a^2 - x^2)I_a + \tfrac{1}{2}(b^2 - y^2)I_b + \tfrac{1}{2}(c^2 - z^2)I_c \qquad (3.2)$$

where I_a, I_b, I_c are constants depending only on the axial ratios of the ellipsoid. Consequently u_i^{C} is a linear function of the x_i and the strain and stress are uniform within the inclusion, as stated in the Introduction. The constant strains e_{ij}^{C} are linear functions of the e_{ij}^{T} and we may write

$$e_{ij}^{\mathrm{C}} = S_{ijkl}\,e_{kl}^{\mathrm{T}}\,. \qquad (3.3)$$

The S_{ijkl} are symmetric in ij and in kl, but in general S_{ijkl} is different from S_{klij}. It is easy to verify that these coefficients vanish unless they are of the form S_{iiii}, S_{iijj} or S_{ijij} ($i \neq j$; no summation). That is,

unlike shears are not coupled, and shears are not coupled to extensions. From (3.1) we obtain

$$\begin{aligned} 8\pi(1-\sigma)S_{1111} &= \frac{a^2I_a - b^2I_b}{a^2-b^2} + \frac{a^2I_a - c^2I_c}{a^2-c^2} + 2(1-\sigma)I_a \\ 8\pi(1-\sigma)S_{1122} &= -b^2\frac{I_a - I_b}{a^2-b^2} - (1-2\sigma)I_a \\ 8\pi(1-\sigma)S_{1212} &= -\frac{1}{2}\frac{a^2+b^2}{a^2-b^2}(I_a - I_b) + \tfrac{1}{2}(1-2\sigma)(I_a + I_b)\,; \end{aligned} \qquad (3.4)$$

okay

the remaining coefficients may be obtained by cyclic interchange. (For an alternative way of obtaining these coefficients see ESHELBY [1957].) The rotation inside the inclusion is also constant. We have at once from (2.25) and (3.2)

$$\begin{aligned} 4\pi\omega_{12}^{\mathrm{C}} &= (I_b - I_a)\,e_{12}^{\mathrm{T}}, \\ 4\pi\omega_{23}^{\mathrm{C}} &= (I_c - I_b)\,e_{23}^{\mathrm{T}}, \\ 4\pi\omega_{31}^{\mathrm{C}} &= (I_a - I_c)\,e_{31}^{\mathrm{T}}. \end{aligned} \qquad (3.5)$$

In their role of demagnetising factors I_a, I_b, I_c have been plotted as functions of b/a and c/a by OSBORN [1945] with the notation $I_a = L$, $I_b = M$, $I_c = N$. They may also be found from tables of elliptic integrals $F(\theta,k)$, $E(\theta,k)$ using the relations

$$\begin{aligned} I_a &= \frac{4\pi abc}{(a^2-b^2)(a^2-c^2)^{\frac{1}{2}}}(F - E), \\ I_b &= 4\pi - I_a - I_c, \\ I_c &= \frac{4\pi abc}{(b^2-c^2)(a^2-c^2)^{\frac{1}{2}}}\left\{\frac{b(\cancel{b^2}\,a^2 - c^2)^{\frac{1}{2}}}{ac} - E\right\} \\ k^2 &= \frac{a^2-b^2}{a^2-c^2}, \qquad \theta = \sin^{-1}\left(1 - \frac{c^2}{a^2}\right)^{\frac{1}{2}}. \end{aligned} \qquad (3.6)$$

(Compare (3.7) below.)

For a point outside the ellipsoid the potential takes the form (KELLOGG [1929], MACMILLAN [1958])

$$\varphi = \frac{2\pi abc}{l^3}\left\{\left[l^2 - \frac{x^2}{k^2} + \frac{y^2}{k^2}\right]F(\theta,k) + \left[\frac{x^2}{k^2} - \frac{y^2}{k^2k'^2} + \frac{z^2}{k'^2}\right]E(\theta,k) + \frac{l}{k'^2}\left[\frac{C}{AB}y^2 - \frac{B}{AC}z^2\right]\right\} \qquad (3.7)$$

where

$$\left.\begin{aligned} A &= (a^2+\lambda)^{\frac{1}{2}}, \quad B = (b^2+\lambda)^{\frac{1}{2}}, \quad C = (c^2+\lambda)^{\frac{1}{2}} \\ l &= (a^2-c^2)^{\frac{1}{2}}, \quad k^2 = 1 - k'^2 = \frac{a^2-b^2}{a^2-c^2} \end{aligned}\right\}, \qquad (3.8)$$

Sphere (Isotropy):

$S_{1111} = \frac{7-5\nu}{15(1-\nu)}$; $S_{1122} = -\frac{1-5\nu}{15(1-\nu)}$; $S_{1212} = \frac{4-5\nu}{15(1-\nu)}$

$$a^2 > b^2 > c^2\,, \tag{3.9}$$

and F, E are elliptic integrals of modulus k and argument

$$\theta = \sin^{-1}(l/A)\,. \tag{3.10}$$

λ is the greatest (and in fact the only positive) root of

$$x^2/A^2 + y^2/B^2 + z^2/C^2 = 1\,. \tag{3.11}$$

Equation (3.7) also gives the potential at an internal point if we put $\lambda = 0$; this gives (3.6).

To carry out the differentiations necessary to find the displacement or stress outside the inclusion one can make repeated use of

$$\partial F/\partial\lambda = -\tfrac{1}{2}l/ABC, \qquad \partial E/\partial\lambda = -\tfrac{1}{2}lB/A^3C$$

$$\partial\lambda/\partial x = 2x/Ah, \ldots, \qquad h^2 = x^2/A^4 + y^2/B^4 + z^2/C^4\,.$$

In forming the first (but not the higher) derivatives of φ, λ may be treated as a constant. The condition (3.9) is not really necessary. It ensures that $0 < k^2 < 1$ and $0 < \theta < \frac{1}{2}\pi$. If it is violated $F(\theta,k)$, $E(\theta,k)$ can be made to depend on $F(\theta_1,k_1)$, $E(\theta_1,k_1)$ with $0 < k_1^2 < 1$, $0 < \theta_1 < \frac{1}{2}\pi$ with the help of known transformations. (See, for example, BYRD and FRIEDMAN [1954].) This is useful if it becomes convenient to ignore (3.9) at a late stage in a calculation.

The results we have obtained can only be applied to the sphere after a tedious passage to the limit. However, we may use the following expressions for the potentials of a sphere of radius a:

$$\left.\begin{aligned} \varphi &= \frac{4}{3}\pi a^2\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{a^2}\right), & r < a \\ &= \frac{4}{3}\pi a^2\frac{a}{r}, & r > a \end{aligned}\right\} \tag{3.12}$$

$$\left.\begin{aligned} \psi &= \frac{4}{3}\pi a^4\left(\frac{3}{4} + \frac{1}{2}\frac{r^2}{a^2} - \frac{1}{20}\frac{r^4}{a^4}\right), & r < a \\ &= \frac{4}{3}\pi a^4\left(\frac{1}{5}\frac{a}{r} + \frac{r}{a}\right), & r > a \end{aligned}\right\}. \tag{3.13}$$

The expression for φ is well-known. ψ may be found by integrating

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\psi}{\mathrm{d}r}\right) = 2\varphi$$

and calculating $\psi(0)$ by direct integration:

$$\psi(0) = 4\pi\int_0^a r\cdot r^2\,\mathrm{d}r = \pi a^4\,.$$

For the sphere (3.3) reduces to

$$e^{\mathrm{C}} = \alpha e^{\mathrm{T}}, \qquad e_{ij}'^{\mathrm{C}} = \beta e_{ij}'^{\mathrm{T}}$$

with

$$\alpha = \frac{1}{3}\,\frac{1+\sigma}{1-\sigma}, \qquad \beta = \frac{2}{15}\,\frac{4-5\sigma}{1-\sigma}\,. \tag{3.14}$$

KRÖNER [1958b] has given the S_{ijkl} (with the notation $\overset{-1}{w}_{ijkl}$) for prolate and oblate spheroids.

For some purposes it may be unnecessary to deal with the relatively complex field outside the inclusion. A knowledge of the e_{ij}^{T} alone is enough to give the field far from the inclusion (2.23) or the interaction energy with an applied field (2.43). When the numerical coefficients S_{ijkl} have been computed we can find the elastic field inside the inclusion, and also, with the help of (2.27), the field at points in the matrix immediately outside the inclusion.

The displacement (3.1) at points external to the ellipsoid may be regarded as the solution to one or other of the following boundary-value problems:

(i) To find the elastic field falling to zero at infinity and having the displacement

$$u_i = (e_{ij}^{\mathrm{C}} + \omega_{ij}^{\mathrm{C}})x_j \qquad (e_{ij}^{\mathrm{C}},\ \omega_{ij}^{\mathrm{C}}\ \text{constant})$$

over the surface of an ellipsoid.

(ii) To find the elastic field falling to zero at infinity and giving surface tractions

$$T_i = p_{ij}^{\mathrm{I}} n_j \qquad (p_{ij}^{\mathrm{I}}\ \text{constant})$$

on the surface of an ellipsoid.

The solution (3.1) is designed to give the constrained elastic field directly when the stress-free strain e_{ij}^{T} is known. Thus in using it to solve (i) or (ii) the first step would have to be the solution of (3.3), or (2.19) and (3.3) to find the e_{ij}^{T} appropriate to the prescribed u_i^{C} or p_{ij}^{I}.

Regarded as a solution of (i) the solution (3.1) is closely related to one already given by DANIELE [1911]. Again, if a constant stress-field $p_{ij}^{\mathrm{A}} = -p_{ij}^{\mathrm{I}}$ is superimposed on the solution of (ii) the ellipsoidal surface is free of stress and we are left with a solution representing an ellipsoidal cavity perturbing a uniform stress p_{ij}^{A}. The solution has been given by SADOWSKY and STERNBERG [1949] for the case where $p_{12}^{\mathrm{A}} = p_{23}^{\mathrm{A}} = p_{31}^{\mathrm{A}} = 0$.

Daniele determined the displacements in an infinite medium outside an ellipsoidal surface over which the displacement is required to be

$$u_i = \xi_{ij} x_j \tag{3.15}$$

with constant coefficients ξ_{ij}. He assumed that there was a solution of the form

$$\left.\begin{aligned} u_i &= \alpha_{ij}\varphi_{,j} + \alpha_{ijkl}x_j\varphi_{,kl} \\ e &= \lambda_{ij}\varphi_{,ij} \end{aligned}\right. \quad (\alpha_{ij},\ \alpha_{ijkl},\ \lambda_{ij}\ \text{constant})\,.$$

The α_{ij} and α_{ijkl} were determined by substituting in the equilibrium equations, equating the coefficients of $\varphi_{,ij}$ and $\varphi_{,ijk}$ to zero (subject to the conditions $\nabla^2\varphi_{,ij} = 0$, $\nabla^2\varphi_{,ijk} = 0$) and applying the boundary condition (3.15). Actually Daniele's solution is more general than (3.1) since we cannot prescribe $\overset{\mathrm{C}}{e}_{ij}$ and $\overset{\mathrm{C}}{\omega}_{ij}$ independently. If the $\overset{\mathrm{C}}{e}_{ij}$ are given the $\overset{\mathrm{T}}{e}_{ij}$ follow from (3.3) and the $\overset{\mathrm{C}}{\omega}_{ij}$ are then fixed by (3.5). This connection between the $\overset{\mathrm{C}}{e}_{ij}$ and $\overset{\mathrm{C}}{\omega}_{ij}$ is due to the fact that there is no external couple acting on the inclusion. Thus, in physical terms, Daniele's solution gives the field about a rigid embedded inclusion which suffers a homogeneous deformation and is then rotated from its equilibrium position by an external couple. (The case where the ellipsoid is rotated but not deformed was discussed earlier by EDWARDES [1893].) When the requirement of zero couple is imposed Daniele's solution agrees with (3.1).

Sadowsky and Sternberg presented their solution in ellipsoidal coordinates. The problem they had in mind was that of an ellipsoidal cavity perturbing a stress field which at infinity is uniform and has its principal axes parallel to those of the ellipsoid; ROBINSON [1951] and NIESEL [1953] have pointed out that it can be applied to the inclusion problem. Robinson treats the case where $\overset{\mathrm{T}}{e}_{ij}$ is a pure dilatation; Niesel considers the more general case where $\overset{\mathrm{T}}{e}_{11}$, $\overset{\mathrm{T}}{e}_{22}$, $\overset{\mathrm{T}}{e}_{33}$ are unequal and $\overset{\mathrm{T}}{e}_{12}$, $\overset{\mathrm{T}}{e}_{23}$, $\overset{\mathrm{T}}{e}_{31}$ are zero.

Sadowsky and Sternberg's solution is expressed in the form (2.20) with

$$\left.\begin{aligned} B_1 = \mathscr{C}A_1X, \quad B_2 = \mathscr{C}A_2Y, \quad B_3 = \mathscr{C}A_3Z, \\ \beta = \mathscr{C}A_4F_1 + \mathscr{C}A_5F_2 \end{aligned}\right\}. \tag{3.16}$$

(We have introduced the constant $\mathscr{C} = -2(1-\sigma)/\mu$ to make our notation agree with Sadowsky and Sternberg's.) The A_n are numerical coefficients and X, Y, Z, F_1, F_2 are harmonic functions chosen so that the boundary conditions can be satisfied. They are related to the potential φ (eq. (3.7)) as follows:

$$X = -vk\,\varphi_{,1}, \qquad Y = +vkk'^2\,\varphi_{,2}, \qquad Z = -v\frac{k'^2}{k}\varphi_{,3}$$

$$F_1 = -vk^3 f_{ii} = vk^3(2\varphi - x_i\varphi_{,i})$$

$$F_2 = \gamma_{bc}f_{11} + \gamma_{ca}f_{22} + \gamma_{ab}f_{33}$$

with

$$\gamma_{qr} = \frac{2(p-1)}{3\pi abc}(d^2-q^2)(d^2-r^2) \qquad (q,r = a,b,c)$$

$$v = \frac{l^3}{4\pi abc}, \qquad d^2 = a^2 - 2\frac{a^2-b^2}{p(1+k^2)}$$

$$p = 2 + \frac{2(k'^2+k^4)^{\frac{1}{2}}}{1+k^2}\,.$$

For completeness we have given the values of the numerical coefficients connecting φ and X, Y, Z, F_1, F_2. We note that f_{11}, f_{22}, f_{33} (eq. (2.22)) only enter the solution through the two particular linear combinations F_1 and F_2. This is a result of Sadowsky and Sternberg's requirement that F_1 and F_2 (and also X, Y, Z) shall take the form

$$f(\alpha_1)\, g(\alpha_2)\, h(\alpha_3) \tag{3.17}$$

when expressed in terms of their ellipsoidal coordinates $\alpha_1, \alpha_2, \alpha_3$. Thus the A_n cannot be found by direct comparison of (3.17) and (2.21). For example, if we choose A_4 and A_5 so that the coefficients of f_{11} and f_{22} agree with (2.7), the coefficient of f_{33} is fixed and so $\overset{\mathrm{T}}{p}_{33}$ cannot be prescribed at will. This could be remedied by adding to β a further harmonic function, $\mathscr{C}A'_5F'_2$ say, where F'_2 is derived from F_2 by changing the sign of the radical in the equation defining p. (Cf. SADOWSKY and STERNBERG [1949] eq. (41).) There is, however, no need to do this. In a representation such as (2.20) it is possible to modify B_i and β simultaneously and leave $\overset{\mathrm{C}}{u}_i$ unchanged. Sadowsky and Sternberg give a matrix relation (their eq. (50)) which enables the A_n to be determined in such a way that the resulting stress annuls the surface tractions on the ellipsoidal surface due to uniform stresses $p_{11} = \sigma_1$, $p_{22} = \sigma_2$, $p_{33} = \sigma_3$.

According to NIESEL [1953] it is convenient to include the solution F'_2 in β if we wish to pass to the limiting case of a spheroid. To extend Sadowsky and Sternberg's solution to the case where the principal axes of the stress at infinity are not parallel to the axes of the cavity (or to apply it to the inclusion problem with non-vanishing $\overset{\mathrm{T}}{e}_{12}$, $\overset{\mathrm{T}}{e}_{23}$, $\overset{\mathrm{T}}{e}_{31}$) it would be necessary to add to β terms $\mathscr{C}A_6(f_{12}+f_{21})$, $\mathscr{C}A_7(f_{23}+f_{32})$, $\mathscr{C}A_8(f_{31}+f_{13})$. These terms have the required form (3.17) when transcribed into ellipsoidal coordinates.

The use of ellipsoidal coordinates does not seem to offer much advantage. As we have seen, the problem may be set up and solved formally in Cartesian coordinates. To find the elastic field at a given point (x, y, z) outside the ellipsoid we have to solve the cubic (3.11) for λ. We appear to be spared this when ellipsoidal coordinates are used. But, in fact, we must have already prepared an ellipsoidal coordinate network in order to be able to locate points relative to the ellipsoid, and, further, a different network is required for each value of the axial ratio b/a.

Lurie [1952] has given a solution to Sadowsky and Sternberg's problem in the disconcertingly simple form

$$B_1 = (M/a^2)\varphi_{,1}, \qquad B_2 = (M/b^2 - N)\varphi_{,2}, \qquad B_3 = (M/c^2 - N)\varphi_{,3}$$
$$\beta = N(x_i\varphi_{,i} - 2\varphi) + P\varphi$$

where M, N, P are disposable constants. (Our M, N, P differ from Lurie's by a common multiplicative factor.) Unfortunately the contributions to u_i^{C} from the terms in N and P are both of the form const. $\varphi_{,i}$. (This illustrates the fact mentioned above that different choices of B_i, β can give the same elastic field.) Consequently there are really only two disposable constants, and so only two of p_{11}^{T}, p_{22}^{T}, p_{33}^{T} (or, in the cavity problem, σ_1, σ_2, σ_3) can be prescribed independently.

3.2. The inhomogeneous inclusion

So far we have been concerned with a homogeneous inclusion, that is, one which has the same elastic constants as the matrix. We consider next the case where the elastic constants of matrix and inclusion are different. As in § 2 we imagine that the inclusion undergoes a transformation specified by a uniform stress-free strain whilst constrained by the matrix, and try to calculate the resulting elastic field. This must be distinguished from the problem considered in § 4 below. There we have, in effect, a *perfectly fitting* inhomogeneous insertion cemented into a cavity in the matrix, and consequently the material is everywhere stress-free in the absence of applied forces.

We have seen that (2.26) still applies when the bulk moduli of matrix and inclusion differ. This seems to be the only simple statement one can make for an inclusion of arbitrary shape. For the ellipsoid, on the other hand, we may solve the general problem very simply by taking advantage of the fact that the stress is constant inside a homogeneous ellipsoidal inclusion (Robinson [1951], Niesel [1953], Eshelby [1957]).

For brevity let E denote the ellipsoidal inclusion we have been considering hitherto. It has the same elastic constants λ, μ as the matrix, and it has suffered a permanent change of shape characterized by the stress-free strain e_{ij}^{T} while embedded in the matrix. The strain e_{ij}^{C} relates its final form to its form before transformation. Take a second ellipsoid E^* which to begin with has the same form as E had before its transformation and which has elastic constants λ^*, μ^*. Let E^* undergo a stress-free strain $e_{ij}^{\mathrm{T}*}$. To E^* apply surface tractions chosen so as to produce a uniform elastic strain $e_{ij}^{\mathrm{C}} - e_{ij}^{\mathrm{T}*}$ in it. It then has precisely the same form as the embedded inclusion E. If this treatment should happen also to produce in E^* stresses identical with those in E we can replace E by E^* without upsetting the continuity of displacement and surface traction across the interface. The condition for this to be possible is (cf. eq. (2.19))

$$p_{ij}^{\mathrm{I}} = \lambda(e^{\mathrm{C}} - e^{\mathrm{T}})\delta_{ij} + 2\mu(e_{ij}^{\mathrm{C}} - e_{ij}^{\mathrm{T}})$$
$$= \lambda^*(e^{\mathrm{C}} - e^{\mathrm{T}*})\delta_{ij} + 2\mu^*(e_{ij}^{\mathrm{C}} - e_{ij}^{\mathrm{T}*}). \qquad (3.18)$$

When the values of λ^*, μ^* and $e_{ij}^{\mathrm{T}*}$ for the inhomogeneous inclusion are given we can solve (3.18) for the e_{ij}^{T}. The elastic field inside and outside the inhomogeneous inclusion is then identical with that of a homogeneous inclusion with the stress-free strain e_{ij}^{T}.

To find e_{ij}^{T} for the equivalent homogeneous inclusion we use (3.3) to express e_{ij}^{C} in terms of e_{ij}^{T}. For the non-diagonal components we have simply

$$e_{12}^{\mathrm{T}} = \frac{\mu^*}{2(\mu^* - \mu)S_{1212} + \mu}\, e_{12}^{\mathrm{T}*}, \qquad e_{23}^{\mathrm{T}} = \ldots .$$

To find e_{11}^{T}, e_{22}^{T}, e_{33}^{T} we have to solve the set of three simultaneous equations

$$(\lambda^* - \lambda)e^{\mathrm{C}} + \lambda e^{\mathrm{T}} + 2(\mu^* - \mu)\overbrace{S_{ijkl}e_{kl}^{\mathrm{T}}}^{=\, e_{ij}^{\mathrm{C}}} = \lambda^* e^{\mathrm{T}*} + 2\mu^* e_{ij}^{\mathrm{T}*}$$
$$+2\mu e_{ij}^{\mathrm{T}} \qquad (ij = 11, 22, 33).$$

Only e_{11}^{T}, e_{22}^{T}, e_{33}^{T} enter the kl-summation, and in the first term we have

$$e^{\mathrm{C}} = \frac{1 - 2\sigma}{4\pi(1 - \sigma)}(I_a e_{11}^{\mathrm{T}} + I_b e_{22}^{\mathrm{T}} + I_c e_{33}^{\mathrm{T}}) + \frac{\sigma}{1 - \sigma} e^{\mathrm{T}} \qquad (3.19)$$

by (2.24) and (3.2).

It is a simple matter to calculate the elastic energy E_∞^* associated with an inhomogeneous (ellipsoidal) inclusion in an infinite matrix. For the equivalent homogeneous inclusion we have from (2.32)

$$E_\infty = -\tfrac{1}{2}V p_{ij}^{\mathrm{I}} e_{ij}^{\mathrm{T}} \qquad (V = \tfrac{4}{3}\pi abc). \qquad (3.20)$$

The energy in the matrix is the same for E and E^*. The internal stress is $\overset{\mathrm{I}}{p}_{ij}$ for both, but the effective strain is $\overset{\mathrm{C}}{e}_{ij} - \overset{\mathrm{T}}{e}_{ij}$ for E and $\overset{\mathrm{C}}{e}_{ij} - \overset{\mathrm{T*}}{e}_{ij}$ for E^*. Thus we have to add $\frac{1}{2}V\overset{\mathrm{I}}{p}_{ij}(\overset{\mathrm{C}}{e}_{ij} - \overset{\mathrm{T*}}{e}_{ij}) - \frac{1}{2}V\overset{\mathrm{I}}{p}_{ij}(\overset{\mathrm{C}}{e}_{ij} - \overset{\mathrm{T}}{e}_{ij})$ to (3.20), which gives the simple result (cf. ROBINSON [1951])

$$E^*_\infty = -\tfrac{1}{2}V\overset{\mathrm{I}}{p}_{ij}\overset{\mathrm{T*}}{e}_{ij}. \tag{3.21}$$

The results of this section may be adapted to the case of an ellipsoidal cavity containing a fluid under pressure. If the pressure P of the fluid is prescribed it is only necessary to put $\overset{\mathrm{I}}{p}_{ij} = -P\delta_{ij}$ in (3.18) and solve for the $\overset{\mathrm{T}}{e}_{ij}$ or $\overset{\mathrm{T*}}{e}_{ij}$. If instead we are given the excess volume v of fluid introduced (measured at zero pressure) we have to put $e^{\mathrm{T}} = v$ and require $\overset{\mathrm{I}}{p}_{ij}$ to have the form $\frac{1}{3}p^{\mathrm{I}}\delta_{ij}$. The solution of (3.18) then gives p^{I}, $'\overset{\mathrm{T}}{e}_{ij}$ and $\overset{\mathrm{T*}}{e}_{ij}$.

§ 4. The Ellipsoidal Inhomogeneity

4.1. THE ELASTIC FIELD

In this section we shall take up the second problem mentioned in the introduction. It may be formulated as follows:

An infinite solid has elastic constants λ^*, μ^* inside a region bounded by a closed surface S (the 'inhomogeneity') and elastic constants λ, μ in the region outside S (the 'matrix'). To find the elastic field everywhere when the strain is required to reduce to the constant value $\overset{\mathrm{A}}{e}_{ij}$ far from S.

Although the problem can only be solved in detail for an ellipsoidal inhomogeneity it is convenient to start from the case where the form of the inhomogeneity is arbitrary. The problem can be reduced to the determination of the elastic field produced by a certain layer of body-force distributed over S. To see this, suppose that the strain $\overset{\mathrm{A}}{e}_{ij}$ is impressed throughout the medium. The displacement is then

$$\overset{\mathrm{A}}{u}_i = \overset{\mathrm{A}}{e}_{ij}x_j$$

plus an inessential rigid-body displacement. The equilibrium equations are satisfied inside and outside S. The traction on the inner boundary of the matrix is $(\lambda e^{\mathrm{A}}\delta_{ij} + 2\mu\overset{\mathrm{A}}{e}_{ij})n_j$, but the traction on the outer surface of the inhomogeneity is $(\lambda^* e^{\mathrm{A}}\delta_{ij} + 2\mu^*\overset{\mathrm{A}}{e}_{ij})n_j$. Consequently the required state of strain can only be maintained if there is a layer of body-force of surface density $\{(\lambda^* - \lambda)e^{\mathrm{A}}\delta_{ij} + 2(\mu^* - \mu)\overset{\mathrm{A}}{e}_{ij}\}n_j$ spread

over S. To find the actual elastic field we apply an equal and opposite layer of body force of surface density

$$T_i = \{(\lambda - \lambda^*)e^{\mathrm{A}}\delta_{ij} + 2(\mu - \mu^*)\overset{\mathrm{A}}{e}_{ij}\}n_j \tag{4.1}$$

and calculate the displacement $\overset{\mathrm{C}}{u}_i$ which it induces in the medium. The final displacement is then

$$u_i = \overset{\mathrm{A}}{e}_{ij}x_j + \overset{\mathrm{C}}{u}_i.$$

$\overset{\mathrm{C}}{u}_i$ is evidently given by the expression

$$\overset{\mathrm{C}}{u}_i(r) = \{(\lambda - \lambda^*)e^{\mathrm{A}}\delta_{kj} + 2(\mu - \mu^*)\overset{\mathrm{A}}{e}_{kj}\}\int_{\mathrm{S}} U_{ik}(r, r')\,\mathrm{dS}_j\,, \tag{4.2}$$

where $U_{ik}(r, r')$ is the i-component of the displacement at r when a unit point-force is applied at r' parallel to the x_k-axis. Because the medium is inhomogeneous U_{ik} depends on r and r' separately and not simply on $|r - r'|$ as does the corresponding quantity in (2.4). It is not possible to determine U_{ik} for an arbitrary form of S, and hence a transformation of (9.2) corresponding to the step from (2.6) to (2.7) cannot be made. However, we shall find that the formulation in terms of a layer of force is useful in deriving certain energy relations.

When the inhomogeneity has the form of an ellipsoid the solution can be found from the solution for the ellipsoidal inclusion by making use of the fact that the stress in the inclusion is uniform. For the special case of a cavity ($\lambda^* = 0$, $\mu^* = 0$) the method has already been outlined in § 1. The general ellipsoidal inhomogeneity is handled similarly. On the elastic field of a homogeneous inclusion with stress-free strain $\overset{\mathrm{T}}{e}_{ij}$ superimpose a uniform strain $\overset{\mathrm{A}}{e}_{ij}$. Let

$$\overset{\mathrm{A}}{p}_{ij} = \lambda e^{\mathrm{A}}\delta_{ij} + 2\mu\overset{\mathrm{A}}{e}_{ij}$$

be the corresponding stress. The stress in the inclusion is now

$$p^{\mathrm{inc}}_{ij} = \overset{\mathrm{I}}{p}_{ij} + \overset{\mathrm{A}}{p}_{ij} = \overset{\mathrm{C}}{p}_{ij} - \overset{\mathrm{T}}{p}_{ij} + \overset{\mathrm{A}}{p}_{ij}\,,$$

and the strain in the inclusion is

$$e^{\mathrm{inc}}_{ij} = \overset{\mathrm{C}}{e}_{ij} + \overset{\mathrm{A}}{e}_{ij}\,. \quad \text{(TOTAL STRAIN)}$$

On account of the term $-\overset{\mathrm{T}}{p}_{ij}$ in (2.19) (which appears because there is no stress associated with the stress-free transformation strain $\overset{\mathrm{T}}{e}_{ij}$) p^{inc}_{ij} and e^{inc}_{ij} are not related by Hooke's law for material with elastic constants λ, μ. They are, however, related by Hooke's law for a material with constants λ^*, μ^* provided these satisfy

$$p^{\mathrm{inc}}_{ij} = \lambda^* e^{\mathrm{inc}}\delta_{ij} + 2\mu^* e^{\mathrm{inc}}_{ij}\,,$$

that is, if

$$(\lambda-\lambda^*)(e^{\mathrm{C}}+e^{\mathrm{A}})\delta_{ij}+2(\mu-\mu^*)(e^{\mathrm{C}}_{ij}+e^{\mathrm{A}}_{ij}) = (\lambda e^{\mathrm{T}}\delta_{ij}+2\mu e^{\mathrm{T}}_{ij})\equiv p^{\mathrm{T}}_{ij}\,. \qquad (4.3)$$

An ellipsoid made of material with these constants can be used to replace the transformed inclusion with continuity of stress and displacement across the interface provided that in its unstressed state this ellipsoid coincides in shape and size with the untransformed inclusion. This replacement does not alter the stresses inside or outside the ellipsoid; they remain the same as the stresses due to a homogeneous transformed inclusion with stress-free strain e^{T}_{ij}, together with the original applied stress p^{A}_{ij}.

The argument has been presented as if e^{A}_{ij} and e^{T}_{ij} were given and λ^*, μ^* were to be found. In the actual inhomogeneity problem λ^*, μ^* and e^{A}_{ij} (or p^{A}_{ij}) are given and we have to determine e^{T}_{ij}. To do this we express p^{C}_{ij} and e^{C}_{ij} in terms of S_{ijkl} and e^{T}_{kl} and substitute in (4.3). This gives

k=1, l=2 ; k=2, l=1

$$(\lambda^*-\lambda)S_{mmkl}\,e^{\mathrm{T}}_{kl}\,\delta_{ij}+2(\mu^*-\mu)S_{ijkl}\,e^{\mathrm{T}}_{kl}+\lambda e^{\mathrm{T}}\delta_{ij}+2\mu e^{\mathrm{T}}_{ij}$$
$$= (\lambda-\lambda^*)e^{\mathrm{A}}\delta_{ij}+2(\mu-\mu^*)e^{\mathrm{A}}_{ij}\,. \qquad (4.4)$$

As in the case of (3.18) the solution for the non-diagonal e^{T}_{ij} is immediate,

$$e^{\mathrm{T}}_{12} = \frac{\mu-\mu^*}{2(\mu^*-\mu)S_{1212}+\mu}\,e^{\mathrm{A}}_{12}, \qquad e^{\mathrm{T}}_{23} = \ldots,$$

while for e^{T}_{11}, e^{T}_{22}, e^{T}_{33} we have the three simultaneous equations

$$(\lambda^*-\lambda)e^{\mathrm{C}}+2(\mu^*-\mu)S_{ijkl}\,e^{\mathrm{T}}_{kl}+\lambda e^{\mathrm{T}}+2\mu e^{\mathrm{T}}_{ij} = (\lambda-\lambda^*)e^{\mathrm{A}}+2(\mu-\mu^*)e^{\mathrm{A}}_{ij} \qquad (4.5)$$

(ij = 11, 22, 33) with the value (3.19) for e^{C}.

For a sphere (4.4) reduces to

$$e^{\mathrm{T}} = Ae^{\mathrm{A}}, \qquad {}'e^{\mathrm{T}}_{ij} = B\,{}'e^{\mathrm{A}}_{ij} \qquad (4.6)$$

where

$$A = A\{\varkappa,\varkappa^*\} = \frac{\varkappa^*-\varkappa}{(\varkappa-\varkappa^*)\alpha-\varkappa}\,,$$
$$B = B\{\mu,\mu^*\} = \frac{\mu^*-\mu}{(\mu-\mu^*)\beta-\mu} \qquad (4.7)$$

with the values (3.14) for α, β.

The e^{T}_{ij} found in this way is the stress-free strain of an 'equivalent homogeneous inclusion' from which the elastic field can be calculated. Note that e^{T}_{ij} goes to zero with e^{A}_{ij}. Consequently the solution does correspond to a perfectly fitting inhomogeneous ellipsoid in a body which is stress-free when not acted on by external forces.

The displacement inside and outside the ellipsoid is

$$u_i = u^{\mathrm{A}}_i + u^{\mathrm{C}}_i\,. \qquad (4.8)$$

The term u^{A}_i represents the unperturbed displacement; it is equal to $e^{\mathrm{A}}_{ij}x_j$ plus an arbitrary rigid-body displacement. The term u^{C}_i represents the perturbation due to the presence of the ellipsoid; it is calculated from (3.1) with the e^{T}_{ij} of the equivalent inclusion. The stress is

$$p_{ij} = p^{\mathrm{A}}_{ij} + p^{\mathrm{C}}_{ij}$$

outside the ellipsoid and

$$p_{ij} = p^{\mathrm{A}}_{ij} + p^{\mathrm{I}}_{ij}$$

inside it. ~~The form~~

$$\cancel{p_{ij} = p^{\mathrm{A}}_{ij} + \lambda e^{\mathrm{C}}\delta_{ij} + 2\mu e^{\mathrm{C}}_{ij}}\,.$$

~~is valid inside and outside the ellipsoid; in calculating the interior field we use the 'wrong' elastic constants λ, μ in place of λ^*, μ^* and this compensates for the change from p^{C}_{ij} to p^{I}_{ij}.~~

It is perhaps worth mentioning that the results of this section and of § 3.2 for the inhomogeneous inclusion can be generalized to the case where the matrix is isotropic but the interior of the ellipsoid is anisotropic and has elastic constants c^*_{ijkl}, say. For the transformation problem it is only necessary to replace (3.18) by

$$\lambda(e^{\mathrm{C}}-e^{\mathrm{T}})\delta_{ij}+2\mu(e^{\mathrm{C}}_{ij}-e^{\mathrm{T}}_{ij}) = c^*_{ijkl}\,(e^{\mathrm{C}}_{ij}-e^{\mathrm{T}*}_{ij})\,. \qquad (4.9)$$

For the inhomogeneity problem (4.4) becomes

$$\lambda(S_{mmkl}\,e^{\mathrm{T}}_{kl}-e^{\mathrm{T}}+e^{\mathrm{A}})\delta_{ij}+2\mu(S_{ijkl}\,e^{\mathrm{T}}_{kl}-e^{\mathrm{T}}_{ij}+e^{\mathrm{A}}_{ij})$$
$$= c^*_{ijpq}\,(S_{pqkl}\,e^{\mathrm{T}}_{kl}+e^{\mathrm{A}}_{pq})\,. \qquad (4.10)$$

By solving (4.9) or (4.10) the e^{T}_{ij} of the equivalent inclusion can be found.

The case of an anisotropic ellipsoid in an isotropic medium may seem rather artificial. It finds an application, however, in the theory of aggregates of anisotropic crystals (cf. § 6).

The argument leading to the expression (2.6) for the displacement u_i^C due to a transformed inclusion of any shape applies equally to an anisotropic material. To obtain an explicit expression for u_i^C analogous to (2.7) we should have to know the form of the displacement due to a point force in an anisotropic material. Unfortunately an expression explicit enough for our purpose cannot be obtained (FREDHOLM [1900], LIFSHITZ and ROSENZWEIG [1947], KRÖNER [1953b]). It is, however, possible to carry the analysis far enough to show that the stress is uniform inside a transformed anisotropic ellipsoid in an anisotropic matrix (ESHELBY [1957]).

We may give a picturesque interpretation to (4.8) by saying that the applied field 'induces' an inclusion in the inhomogeneity, with a stress-free strain e_{ij}^T proportional to the applied stress (cf. KRÖNER [1958a]).

If the inhomogeneity is in a finite body we may write the perturbing field of the inclusion in the form

$$u_i^F = u_i^C + u_i^{im}, \qquad p_{ij}^F = p_{ij}^C + p_{ij}^{im}$$

as in § 2.1, with the image terms chosen so that the boundary conditions on the outer surface S_0 of the body are satisfied. They are not, of course, quite identical with the image terms for the equivalent inclusion in a homogeneous medium.

4.2. Energy relations

It is sometimes necessary to compare the elastic behaviour of a body containing an inhomogeneity with the behaviour of a similar body which is homogeneous. In such cases it is convenient to imagine that we have a single body and that the inhomogeneity may be introduced or removed at will.

When an inhomogeneity is introduced into a body already stressed by some external mechanism there will be a change in its elastic energy. At the same time there may be a change in the potential energy of the loading mechanism. As in § 2.2 we can define a change in total energy (or in enthalpy or Gibbs free energy). We shall calculate this quantity for an ellipsoidal inhomogeneity in a body subjected to two types of loading, rigidly imposed surface displacements and constant surface tractions.

Consider first the case where a constant displacement is imposed on the outer boundary by a perfectly rigid external mechanism, producing uniform stress and strain p_{ij}^A, e_{ij}^A in the absence of the inhomogeneity.

The perturbing field when the inhomogeneity is introduced must thus satisfy the condition

$$u_i^F = u_i^C + u_i^{im} = 0 \quad \text{on} \quad S_0 . \tag{4.11}$$

The elastic energy E_A in the medium when the inclusion is absent is found by integrating the (constant) energy density over the whole volume of material. Let the inhomogeneity be introduced and, as in § 4.1, let the material be held in a state of uniform strain e_{ij}^A by a layer of body force $-T_i$ (eq. (4.9)). The elastic energy is found by replacing λ, μ by λ^*, μ^* in that part of the volume integral for E_A which refers to the interior of S. Thus at this stage the elastic energy is $E_A + W_1$ where

$$W_1 = \tfrac{1}{2}V[(\lambda^* - \lambda)e^A\delta_{ij} + 2(\mu^* - \mu)e_{ij}^A]e_{ij}^A .$$

If the layer of force $-T_i$ is relaxed to zero each element dS of S suffers a displacement u_i^F and an amount of energy

$$W_2 = \tfrac{1}{2}\int_S T_i u_i^F \mathrm{d}S$$

is withdrawn from the medium. By Gauss's theorem and (4.11) this can be put in the form

$$W_2 = \tfrac{1}{2}V[(\lambda - \lambda^*)e^A\delta_{ij} + 2(\mu - \mu^*)e_{ij}^A]e_{ij}^C - E_{im}$$

where

$$E_{im} = \tfrac{1}{2}[(\lambda - \lambda^*)e^A\delta_{ij} + 2(\mu - \mu^*)e_{ij}^A]\int_V e_{ij}^{im}\,\mathrm{d}v .$$

The increase in the elastic energy when the inhomogeneity is introduced is thus

$$\Delta E_{el} = W_1 - W_2 .$$

This is also the increase in the total energy, ΔE_{tot}, since the rigid loading mechanism does no work. With the help of (4.3) we have

$$W_1 - W_2 = -\tfrac{1}{2}Vp_{ij}^A e_{ij}^T + E_{im} .$$

We consider next the case where the body is loaded by constant surface tractions $p_{ij}^A n_j$. The perturbing field due to the introduction of the inhomogeneity must now satisfy

$$p_{ij}^F n_j = (p_{ij}^C + p_{ij}^{im})n_j = 0 \quad \text{on} \quad S_0 .$$

The elastic energy when the inhomogeneity is introduced and the layer of force $-T_i$ is present is again given by $E_A + W_1$, and W_2 still represents the energy removed on relaxing the layer of force. But in addition there will now be a movement of the outer boundary in which the surface tractions do an amount of work

$$W_3 = \int_{S_0} p_{ij}^{A} u_i^{F} \, dS_j$$

on the body. Thus

$$\Delta E_{el} = W_1 - W_2 + W_3 .$$

If we are interested in the change in total energy we do not need to know the value of W_3; evidently W_3 also represents the decrease in the energy of the loading mechanism and so

$$\Delta E_{tot} = \Delta E_{el} - W_3 = W_1 - W_2 .$$

It is in fact not difficult to establish the relation

$$W_3 = 2W_2 - W_1 .$$

We make use of the same device as in (2.39) and write

$$W_3 = \int_{S_0} (p_{ij}^{A} u_i^{F} - p_{ij}^{F} u_i^{A}) dS_j , \qquad (4.12)$$

change the surface of integration from S_0 to S and convert to a volume integral over the inhomogeneity. The relation (4.12) then follows if we note that in the inhomogeneity

$$p_{ij}^{A} = \lambda e^{A} \delta_{ij} + 2\mu e_{ij}^{A}$$

by definition, but that, contrary to the general rule laid down in § 1

$$p_{ij}^{im} = \lambda^* e^{im} \delta_{ij} + 2\mu^* e_{ij}^{im}$$

since p_{ij}^{im}, e_{ij}^{im} represent the actual image field in the inhomogeneity.

If we introduce the notation

I get a $-E_{im}$

$$E_{int} = -\tfrac{1}{2} V p_{ij}^{A} e_{ij}^{T} + E_{im} \qquad (4.13)$$

these results may be summarised as follows:

(i) for a rigidly-held boundary

$$\Delta E_{tot} = E_{int}, \qquad \Delta E_{el} = \Delta E_{tot} , \qquad (4.14)$$

(ii) for a boundary subject to constant loads

$$\Delta E_{tot} = E_{int}, \qquad \Delta E_{el} = -\Delta E_{tot} . \qquad (4.15)$$

The difference in sign between (4.14) and (4.15) may be illustrated by considering the case where the inhomogeneity takes the form of a crack, that is, a narrow region in which the elastic constants are zero. Introduction of the crack into a body strained by rigidly-imposed

surface displacements will obviously reduce the elastic energy. On the other hand the presence of the crack makes the body more 'flexible'. Consequently given applied loads will deform it more, do more work on it and so increase its elastic energy.

The first term in (4.13) is precisely half the interaction energy (2.43) for the equivalent homogeneous inclusion. The image term E_{im} will have a different value in (i) and (ii) since e_{ij}^{im} is derived from different boundary conditions. If the inhomogeneity is far from the boundaries it will be small compared with $-\frac{1}{2}V p_{ij}^{A} e_{ij}^{T}$, and we may say that provided the initial stress p_{ij}^{A} is the same in each case, the total energy changes in cases (i) and (ii) are the same, but that the changes in elastic energy are equal and opposite. It is possible to extend the analysis to the mixed case where a constant displacement is imposed on part of S_0 and a constant load on the remainder. As one might expect, ΔE_{tot} is still given by (4.14) (with the appropriate e_{ij}^{im} inserted in E_{im}), but no general statement can be made about the relation between ΔE_{el} and ΔE_{tot}. This is no drawback since it is ΔE_{tot} which is the physically important quantity.

§ 5. Relation to the Theory of Dislocations

There is a close relation between the inclusion problem and the theory of dislocations, more particularly with the general type of dislocation introduced by SOMIGLIANA [1914, 1915]). It will be convenient, however, to begin with the more familiar type of dislocation which plays a part in the physical theory of plasticity (NABARRO [1952], SEEGER [1955]). For want of a better name we may refer to these as VOLTERRA [1907] dislocations, though in fact they only correspond to the first three of his six classes. For our purposes a Volterra dislocation may be defined as a state of self-stress in which the displacement has discontinuity b_i, the Burgers vector, across a surface bounded by an open or closed curve, the dislocation line. If the dislocation line forms a closed loop and lies in a plane, the dislocation is characterised by giving the form and orientation of the loop and the value of the Burgers vector.

The displacement at large distances from a Volterra dislocation loop of area A, Burgers vector b_i and normal n_i situated at the origin is

$$u_i = \frac{A b_j n_k g_{ijk}}{8\pi(1-\sigma)r^2} \qquad (5.1)$$

with the notation of (2.23) (BURGERS [1939], NABARRO [1951]). This can be put in the alternative form

$$u_i = \frac{A b_j n_k}{8\pi(1-\sigma)} \vartheta_{ijk}\, r \tag{5.2}$$

where

$$\vartheta_{ijk} = \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} - \left\{\sigma \delta_{jk} \frac{\partial}{\partial x_i} + (1-\sigma)\delta_{ij}\frac{\partial}{\partial x_k} + (1-\sigma)\delta_{ik}\frac{\partial}{\partial x_j}\right\} \nabla^2 . \tag{5.3}$$

okay ~~looks like a → σ should be here~~

The limiting process

$$A \to 0, \quad b_j \to \infty, \quad A b_j \to s_j \tag{5.4}$$

gives an elementary infinitesimal dislocation loop of strength s_j and normal n_k. The displacement at any distance from it is given by (5.1) or (5.2) if we identify $A b_j$ with s_j.

Comparison of (5.1) with (2.23) shows that the remote field of a (finite) dislocation loop is the same as the remote field of an inclusion of arbitrary shape whose volume and stress-free strain satisfy

$$V e^{\mathrm{T}}_{ij} = \tfrac{1}{2}(b_i n_j + b_j n_i) . \tag{5.5}$$

To an elementary dislocation loop there corresponds an elementary inclusion. It is natural to imagine it as a platelet coinciding with the loop. Its thickness and stress-free strain may be so chosen that in the limit (5.5) agrees with (5.4). A finite plane dislocation loop may be formed by spreading elementary loops over a plane surface. When each loop is replaced by an equivalent elementary inclusion we obtain an inclusion in the form of a thin disc, and the elastic field which it produces is the same as that of the dislocation. It is, in fact, possible to build up directly an expression for the elastic field of a finite plane dislocation loop starting from an inclusion in the form of a disc of small (and ultimately vanishing) thickness. The quantity B_i (2.7) is then the potential of a plane lamina, and the f_{ij} (2.22) are the potentials of certain double layers. The discontinuities in potential and attraction on crossing them can be found by elementary electrostatics, and the e^{T}_{ij} may then be so adjusted that there is a discontinuity b_i in the displacement u^{C}_i between points on opposite faces of the disc. This procedure, however, does not completely determine the e^{T}_{ij}; it is necessary to impose further conditions to ensure, for example, that there is not a line of dilatation running round the rim of the disc.

We turn now to the more fruitful connection between inclusions and

the general Somigliana dislocation. A Somigliana dislocation can be constructed as follows. Make a cut over a surface S (open or closed) and give the two faces of the cut an arbitrary small relative displacement, removing material where there would be interpenetration. Fill in any gaps and weld the material together again. Let $b_i(\boldsymbol{r})$ be the relative displacement at the point $\boldsymbol{r}$ on the cut. There are several ways of calculating the resulting state of self-stress. The simplest is to regard the Somigliana dislocation as equivalent to a distribution of elementary dislocations over S, the strength of the loop associated with the element of area $\mathrm{d}S(\boldsymbol{r})$ being $b_j(\boldsymbol{r})\mathrm{d}S$. By (5.2) the displacement is

$$u^{\mathrm{D}}_i = \frac{1}{8\pi(1-\sigma)} \vartheta_{ijk} I_{jk} \tag{5.6}$$

looks like opposite of Hirth & Lothe but could be his displ. is −

where

$$I_{jk} = \int_{\mathrm{S}} b_j(\boldsymbol{r}')|\boldsymbol{r}-\boldsymbol{r}'|\,\mathrm{d}S_k .$$

This can be shown to be equivalent to the expression given by SOMIGLIANA [1915].

To see the connection with the inclusion problem suppose that a cut is made over the surface S of the Somigliana dislocation. The faces spring apart leaving a gap $b_i(\boldsymbol{r})$ (in some places the 'gap' may in fact be an interpenetration of material). As we saw in § 2.1 the inclusion behaves in just this fashion. If we make a cut over the interface a gap

$$b_j(\boldsymbol{r}) = e^{\mathrm{T}}_{jl} x_l \tag{5.7}$$

appears. (To (5.7) we might possibly add a rigid body displacement; for the moment we ignore it.) Consequently the inclusion is equivalent to a Somigliana dislocation in which S is a closed surface coinciding with the interface and having the discontinuity (5.7). We should expect the displacement calculated from (5.7) and (5.6) to be not u^{C}_i but rather the quantity which, in anticipation, we called u^{D}_i in § 2.1. That is, it should coincide with u^{C}_i outside the inclusion and differ from u^{C}_i by the displacement $e^{\mathrm{T}}_{il} x_l$ inside the inclusion. This may be verified analytically. With (5.7) we have, using Gauss's theorem,

$$I_{jk} = e^{\mathrm{T}}_{jl} \int_{\mathrm{S}} x'_l\, |\boldsymbol{r}-\boldsymbol{r}'|\,\mathrm{d}S_k = e^{\mathrm{T}}_{jl} \int_V \frac{\partial}{\partial x'_k} \{x'_l |\boldsymbol{r}-\boldsymbol{r}'|\}\,\mathrm{d}v$$

$$= e^{\mathrm{T}}_{jk}\, \psi + e^{\mathrm{T}}_{jl} X_{lk}$$

where ψ is the biharmonic potential (2.9) and

$$X_{lk} = \int_V x'_l \frac{\partial}{\partial x'_k} |\boldsymbol{r}-\boldsymbol{r}'|\,\mathrm{d}v .$$

When inserted in (5.6) the term in ψ gives u_i^{C} in the form (2.8). It can be verified that

$$\vartheta_{ijk} X_{lk} = 2(1-\sigma)\delta_{ij} \nabla^2 \int_V \frac{x_l'}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}v \,. \tag{5.8}$$

The integral is the harmonic potential of matter of density x_l filling S, and so its Laplacian is $-4\pi x_k$ inside S and zero outside. Consequently

$$\begin{aligned} \vartheta_{ijk} X_{lk} &= -8\pi(1-\sigma)\delta_{ij} x_l \quad \text{inside} \quad \mathrm{S} \\ &= 0 \quad \text{outside} \quad \mathrm{S} \end{aligned} \tag{5.9}$$

and so

$$\begin{aligned} u_i^{\mathrm{D}} &= u_i^{\mathrm{C}} - e_{il}^{\mathrm{T}} x_l \quad \text{inside} \quad \mathrm{S} \\ &= u_i^{\mathrm{C}} \quad \text{outside} \quad \mathrm{S} \end{aligned} \tag{5.10}$$

as expected. If we had included a rigid-body rotation $\omega_{jl}^{\mathrm{T}} x_l$ in (5.7) I_{jk} would have contained the additional terms $\omega_{jk}^{\mathrm{T}}\psi + \omega_{jl}^{\mathrm{T}} X_{lk}$. The first contributes nothing to u_i^{D}, since ϑ_{ijk} is symmetric in jk. According to (5.8) the second term leaves u_i^{D} unchanged outside S. Inside S it gives a rigid-body rotation which in a structureless medium produces no observable effect.

It is clear that treatment in terms of a Somigliana dislocation offers no computational advantages when applied to an inclusion which has undergone a homogeneous stress-free strain. However, we may use the formula (5.6) to extend our results to the case where the inclusion suffers any small change of shape, not necessarily a homogeneous deformation.

Suppose that the inclusion undergoes a permanent change of form which, if the matrix were absent, would not be associated with stress. To specify this 'stress-free change of form' it is only necessary to give the displacement of each point of its surface,

$$u_i^{\mathrm{T}} = u_i^{\mathrm{T}}(\boldsymbol{r}), \qquad \boldsymbol{r} \quad \text{on} \quad \mathrm{S}\,. \tag{5.11}$$

Alternatively, to maintain the analogy with § 2.1, we could introduce a variable stress-free strain $e_{ij}^{\mathrm{T}}(\boldsymbol{r})$ defined throughout the interior of the inclusion and giving a displacement agreeing with (5.11) on S. (We recall (SOKOLNIKOFF [1946]) that specification of a (compatible) strain throughout a region fixes the displacement in it to within a rigid body displacement.) Since e_{ij}^{T} would be largely arbitrary it seems better to work with u_i^{T} in the general case.

If we try to put the transformed inclusion into the matrix there will be a misfit u_i^{T} at each point of S. When this misfit is removed by suitably straining the material and welding together corresponding points on either side of S we are left with a Somigliana dislocation for which $b_i = u_i^{\mathrm{T}}$. Thus to find the elastic field when the inclusion is constrained by the matrix we have only to put $b_j(\boldsymbol{r}') = u_j^{\mathrm{T}}(\boldsymbol{r}')$ in (5.6). The resulting u_i^{D} is the displacement measured from an initial state in which the matrix is unstressed and the inclusion is transformed and stress-free. Consequently the stress is given by

$$p_{ij}^{\mathrm{D}} = \lambda u_{m,m}^{\mathrm{D}} \delta_{ij} + \mu(u_{i,j}^{\mathrm{D}} + u_{j,i}^{\mathrm{D}})$$

both inside and outside the inclusion.

This problem can also be solved by the method of § 2.1. Let

$$u_i^{\mathrm{T}} = u_i^{\mathrm{T}}(\boldsymbol{r}), \ \boldsymbol{r} \text{ inside or on S}\ ,$$

be any convenient continuous displacement which coincides with (5.11) on S, and put

$$p_{ij}^{\mathrm{T}} = \lambda u_{m,m}^{\mathrm{T}} \delta_{ij} + \mu(u_{i,j}^{\mathrm{T}} + u_{j,i}^{\mathrm{T}})\ .$$

Remove the transformed inclusion from the matrix and apply surface tractions $-p_{jk}^{\mathrm{T}} n_j$ and a distribution of body force of amount $+p_{jk,k}^{\mathrm{T}}$ per unit volume. This produces a displacement $-u_i^{\mathrm{T}}$ and so restores the inclusion to its untransformed shape. Cement it back in the matrix and relax the unwanted forces. This gives the displacement (cf. (2.6))

$$u_i^{\mathrm{C}} = \int_{\mathrm{S}} \mathrm{dS}_k\, p_{jk}^{\mathrm{T}}(\boldsymbol{r}')\, U_{ij} - \int_V \mathrm{d}v \frac{\partial p_{jk}}{\partial x_k'}(\boldsymbol{r}')\, U_{ij}\,.$$

It can be shown by the same sort of analysis as led to (5.10) that $u_i^{\mathrm{C}} = u_i^{\mathrm{D}}$ in the matrix and that $u_i^{\mathrm{C}} = u_i^{\mathrm{D}} + u_i^{\mathrm{T}}$ in the inclusion. Since u_i^{T} is by definition a stress-free displacement the stresses calculated from u_i^{C} and u_i^{D} are identical.

The solution for the elastic field due to an inclusion which has suffered a non-uniform transformation does not seem to have any obvious applications. However, for an ellipsoidal inclusion the solution has a property which enables it to be used to find how an ellipsoidal cavity or inhomogeneity perturbs a non-uniform stress-field of fairly general type. This property generalizes the result that a uniform transformation strain leads to a uniform strain in the constrained ellipsoid. It may be stated thus: if the stress-free transformation displacement is a polynomial in x_1, x_2, x_3 of degree N, then the displacement is also a polynomial of degree N inside the constrained ellipsoid. We shall only outline its derivation and application. The details of the calculation can easily be filled in.

The analysis involves a number of polynomials in x_i or x'_i with constant coefficients. We shall denote them by script capitals and indicate only the argument and degree. Thus, for example, $\mathscr{P}(M,x)$ stands for a polynomial in x_1, x_2, x_3 whose highest term is of degree M. Similarly $\mathscr{G}_{ijk}(M,x')$, $i,j = 1, 2, 3$ denotes a set of twenty-seven polynomials in x'_1, x'_2, x' all of the same degree, and so forth. We shall also use R as an abbreviation for $|r - r'|$.

Let the ellipsoidal inclusion undergo the transformation specified by

$$u_i^{\mathrm{T}}(r) = \mathscr{T}_i(N,x) \qquad (r \text{ on S}) .$$

If we put $b_i = u_i^{\mathrm{T}}$ in (5.6) and use Gauss's theorem we find that I_{jk} has the form

$$I_{jk} = \int_V \mathscr{E}_{jk}(N-1,x')R\,\mathrm{d}v + \int_V \mathscr{T}_i(N,x')\frac{\partial R}{\partial x'_k}\,\mathrm{d}v\,. \qquad (5.12)$$

In the first integrand introduce the factor

$$1 = \frac{(x_m - x'_m)(x_m - x'_m)}{|r-r'|^2}$$

and in the second write $\partial R/\partial x'_k$ as $(x'_k - x_k)/R$. After a little re-arrangement (5.12) takes the form

$$I_{jk} = x_m x_m \int_V \mathscr{F}_{jk}(N-1,x')\frac{\mathrm{d}v}{R} + x_p \int_V \mathscr{G}_{pjk}(N,x')\frac{\mathrm{d}v}{R}$$
$$+ \int_V \mathscr{H}_{jk}(N+1,x')\frac{\mathrm{d}v}{R}\,. \qquad (5.13)$$

The integrals are the harmonic potentials of solid ellipsoids whose densities are polynomial functions of the co-ordinates. FERRERS [1877] and DYSON [1891] have discussed the potentials of inhomogeneous ellipoids of this type. Their results show that when the density is of the form $\mathscr{P}(M,x)$ the potential is of the form $\mathscr{Q}(M+2,x)$ inside the ellipsoid. The coefficients of the polynomial $\mathscr{Q}$ can be calculated from the coefficients of $\mathscr{P}$ and the three quantities I_a, I_b, I_c (eq. (3.2)). Outside the ellipsoid the potential is a similar polynomial in x_1, x_2, x_3 but with coefficients which are themselves functions of position. (Compare the relation between (3.2) and (3.7).) These variable coefficients can be expressed in terms of the coefficients of $\mathscr{P}$ and the quantities

A, B, C, F, E of (3.8), (3.10). Consequently, by (5.13) and (5.6) the constrained displacement is a polynomial of the form

$$u_i^{\mathrm{D}}(r) = \mathscr{D}_i(N,x)$$

inside the ellipsoid. Thus the constrained and unconstrained displacements of the inclusion are similar polynomials with calculable relations between their coefficients.

The solution of the cavity problem follows the lines of § 4.1. Superimpose a displacement

$$u_i^{\mathrm{A}}(r) = -\mathscr{D}_i(N,x) \qquad (5.14)$$

everywhere. The total displacement $u_i^{\mathrm{A}} + u_i^{\mathrm{D}}$ is zero inside the ellipsoid. Hence the inclusion is unstressed, and it can be removed without disturbing the matrix. The coefficients in $\mathscr{T}_i$ can be chosen so as to make (5.14) any required polynomial of degree N. From these coefficients the coefficients of $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ (eq. (5.13)) can be found. The field $u_i^{\mathrm{A}} + u_i^{\mathrm{D}}$ in the matrix then follows from (5.13) and Ferrers' and Dyson's results. The extension to the ellipsoidal inhomogeneity follows as in § 4.1.

The calculations are not too unwieldy for small N. With $N = 2$ the displacement

$$u_i^{\mathrm{A}} = \alpha_{ij}^{\mathrm{A}} x_j + \beta_{ijk}^{\mathrm{A}} x_j x_k$$

covers the case of an applied stress which is a combination of simple torsion, bending and flexure. Solutions, based on other methods, have been given for a number of such problems involving spheroids and spheres (NEUBER [1958], SEN [1933], DAS [1953]).

§ 6. Applications

The application of the results reviewed here to the actual calculation of stress in and about inclusions and inhomogeneities calls for no special comment. The engineering application of the theory of cavities to the calculation of stress concentrations has been thoroughly treated by NEUBER [1958]. In this section we indicate some of the applications to physical problems.

Expressions for the elastic energy and interaction energy of an inclusion find a use in the theory of martensitic transformations. (For a general review see KAUFMAN and COHEN [1958].) Suppose that a metal can exhibit two crystal structures, γ and α, the former stable at high temperatures and the latter at low temperatures. Ideally we might expect that on cooling the whole of a single crystal of the metal would undergo a uniform stress-free strain in the sense of § 2, the e^{T}_{ij} being the deformation which carries the γ-lattice into the α-lattice. In a martensitic transformation, however, the low-temperature phase first appears in the form of inclusions of α embedded in the γ matrix. Thus, in considering the thermodynamics of the transformation, one must take into account the elastic energy of the misfitting inclusions of α and also, if there is an externally applied stress, their interaction energy with the latter.

The Gibbs free energy change associated with the formation of a martensitic inclusion may be written in the form

$$\Delta G = \Delta G_{\mathrm{chem}} + \Delta G_{\mathrm{surf}} + E_{\mathrm{inc}} + E_{\mathrm{int}}\,. \tag{6.1}$$

ΔG_{chem} is the 'chemical' contribution, the free energy change which would occur if the inclusion underwent its stress-free strain in the absence of the matrix. The theory of elastic inclusions can tell us nothing about it. ΔG_{surf} is a contribution due to the interface between matrix and inclusion. It may be estimated by using the theory of dislocations. E_{inc} is the elastic energy associated with the inclusion, and E_{int} is the interaction energy with any externally applied stress which may be present. According to the discussion in § 2.2 the terms E_{inc} and E_{int} taken together make up the elastic contribution to the change of Gibbs free energy.

It will usually be accurate enough to identify E_{int} with E_∞, the value in an infinite medium, and to ignore the difference between the elastic constants of matrix and inclusion. If the inclusion can be considered to be some form of ellipsoid E_∞ can be calculated from (3.3) and (2.32) when the transformation strain e^{T}_{ij} is known. Suppose, for example, that the inclusion takes the form of a plate in the x_1x_2-plane and that the transformation is made up of a shear parallel to this plane through an angle s, a uniform dilatation Δ and an extension ξ perpendicular to the plane of the plate. Then

$$e^{\mathrm{T}}_{13} = e^{\mathrm{T}}_{31} = \tfrac{1}{2}s, \qquad e^{\mathrm{T}}_{11} = e^{\mathrm{T}}_{22} = \tfrac{1}{3}\Delta\,, \qquad e^{\mathrm{T}}_{33} = \xi + \tfrac{1}{3}\Delta \tag{6.2}$$

and the remaining components are zero. If the plate has the form of a thin oblate spheroid with $c \ll a = b$ we have

$$\frac{1-\sigma}{\mu V}E_\infty = \frac{1}{8}\pi(2-\sigma)\frac{c}{a}s^2 + \frac{2}{9}(1+\sigma)\Delta^2 + \frac{1}{4}\pi\frac{c}{a}\xi^2 + \frac{1}{3}\pi(1+\sigma)\frac{c}{a}\xi\Delta \tag{6.3}$$

(CHRISTIAN [1958]). There is also a fairly simple expression for E_∞ when the inclusion is a flat ellipsoid with $a > b \gg c$ (ESHELBY [1957]). The expression (6.3) has been used by CHRISTIAN [1958, 1959] and KAUFMAN [1959] to discuss the nucleation of martensite.

The interaction energy may be found from (2.43). In an actual experiment the applied stress will usually be a uniaxial tension; p^{A}_{ij} then takes the form $\tau n_i n_j$, where τ is the magnitude of the tension and n_i is its direction. If the transformation strain is given by (6.2) we have

$$E_{\mathrm{int}} = -\tau C V \tag{6.4}$$

with

$$C = s\cos\beta\sin\beta + \xi\cos^2\beta + \tfrac{1}{3}\Delta$$

where β is the angle between the x_3-axis and the direction of the tension, and the latter is assumed to lie in the x_1x_3-plane. A result equivalent to (6.4) was first obtained (with $\Delta = 0$) by PATEL and COHEN [1953] (cf. also MACHLIN and WEINIG [1953]). FISHER and TURNBULL [1953] have also discussed the effect of an applied stress on the free energy change associated with martensite formation. In effect, they take the interaction energy to be the cross-term in the elastic energy between the applied field and the field due to the inclusion. As we have seen (§ 2.2), this quantity should be zero. Since they only integrate the energy density over the immediate neighbourhood of the inclusion they obtain a finite result which, however, is not quite correct numerically and which, in addition, has the wrong sign. To correct this they have to make an arbitrary reversal of sign in the relation between applied stress and applied strain.

When a crystal having the high-temperature γ-structure is cooled, ΔG decreases. According to elementary thermodynamics a martensitic inclusion of the α-phase can form when a temperature is reached at which ΔG is zero. In fact a nucleation barrier has to be overcome and the transformation will occur when ΔG reaches some finite negative value, say ΔG_{nuc}. Let T_s be the temperature at which $\Delta G = \Delta G_{\mathrm{nuc}}$ in the absence of external stress, i.e. with $E_{\mathrm{int}} = 0$ in (6.1). When there is an applied stress the term E_{int} will alter the temperature at which $\Delta G_{\mathrm{nuc}} = \Delta G$ to, say, $T_s + \Delta T_s$. If ΔG_{nuc} is independent of temperature and ΔG_{chem} is the only strongly temperature-dependent term in (6.1) we must have

$$\frac{\mathrm{d}\Delta G_{\mathrm{chem}}}{\mathrm{d}T}\Delta T_s + E_{\mathrm{int}} = 0\,,$$

and so from (6.4) we obtain the expression

$$\frac{\mathrm{d}T_s}{\mathrm{d}\tau} = C \Big/ \frac{1}{V} \frac{\mathrm{d}\Delta G_{\mathrm{chem}}}{\mathrm{d}T} \tag{6.5}$$

for the rate of change of transformation temperature with applied tensile stress. C can be found from crystallographic measurements, and the rate of change of chemical free energy with temperature may be obtained from thermodynamic data. Patel and Cohen found excellent agreement between theory and experiment for certain iron-nickel and iron-nickel-carbon alloys. They also showed how (6.5) should be modified when ΔG_{nuc} varies with temperature.

For martensitic inclusions the transformation strain is essentially a pure shear. For particles of precipitate formed by diffusion it is essentially a dilatation. For an ellipsoidal precipitate the strain energy can be found from (3.21) with $e_{ij}^{\mathrm{T}*} = \frac{1}{3}e^{\mathrm{T}}\delta_{ij}$. ROBINSON [1951] has given a detailed treatment. In these calculations it is assumed that the inclusion is coherent with the matrix. NABARRO [1940] and KRÖNER [1954] have treated the problem of finding the minimum strain energy due to an inclusion which has broken away from the matrix. The displacement need not be continuous across the interface, but the matrix and inclusion are supposed to be in contact everywhere. The volume misfit is prescribed, but the stress-free shape of the inclusion has to be determined so as to minimize the energy. According to KRÖNER [1953a, 1954] the energy is a minimum for the state in which the stress in the inclusion is purely hydrostatic. (Matrix and inclusion may be made of different anisotropic materials.)

The theory of cracks, i.e. empty cavities one of whose dimensions is evanescent, has played a part in Griffiths' treatment of rupture (see, for example, SNEDDON [1951]) and, more recently, in the theory of the brittle and ductile fracture of metals. The crack is supposed to have associated with it a surface energy γA, where A is its surface area and γ is a constant representing true surface energy or an effective surface energy associated with plastic deformation. According to the discussion in § 4.2 the total energy of the system made up of the body containing the crack and the external loading mechanism is

$$E_{\mathrm{int}} + \gamma A + \text{const.}$$

By studying the way in which E_{int} and A vary when the form of the crack is altered slightly one can find whether it is energetically favourable for the crack to spread or contract.

We may find the necessary properties of an elliptical crack from the results of §§ 4.1, 4.2 by putting $\lambda^* = \mu^* = 0$ and letting the c-axis tend to zero. The equivalent stress-free strain e_{ij}^{T} approaches infinity for a fixed applied stress p_{ij}^{A}, but the products Ve_{ij}^{T} (and hence also the potentials φ, ψ) remain finite. okay, Ve_{ij}^{T}

There exist a number of calculations for the interaction energy of a circular or two-dimensional crack (INGLIS [1913], STARR [1928], SACK [1946], SEGEDIN [1951]). They may be deduced from the following results for the elliptical crack

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1\,.$$

(i) The applied stress is a pure tension p_{33}^{A} normal to the plane of the crack (tensions p_{11}^{A}, p_{22}^{A} evidently have no effect). Then

$$E_{\mathrm{int}} = -\frac{2\pi(1-\sigma)ab^2}{3E(k)} \frac{(p_{33}^{\mathrm{A}})^2}{\mu}\,. \tag{6.6}$$

This is easily deduced from the results of GREEN and SNEDDON [1949].
(ii) The applied stress is a pure shear p_{13}^{A} in the plane of the crack. Then (ESHELBY [1957])

$$E_{\mathrm{int}} = -\frac{2\pi ab^2}{3\eta} \frac{(p_{13}^{\mathrm{A}})^2}{\mu}$$

where

$$\eta = E(k) + \frac{\sigma}{1-\sigma} \frac{K(k) - E(k)}{k^2} \left(\frac{b}{a}\right)^2 \tag{6.7}$$

In (6.6) and (6.7) E, K are complete elliptic integrals of modulus $k = (1 - b^2/a^2)^{\frac{1}{2}}$. Eq. (6.7) is valid for both $a > b$ and $a < b$. In the latter case the elliptic integrals may be reduced to real form with the help of the relation

$$K(k_1) = (1-k^2)^{\frac{1}{2}}K(k), \qquad E(k_1) = E(k)/(1-k^2)^{\frac{1}{2}},$$
$$\text{where} \quad k_1 = ik/(1-k^2)^{\frac{1}{2}}\,.$$

(Compare the remark following (3.6).) For the circular crack $a = b$, $E = \frac{1}{2}\pi$, $\eta = \pi(2-\sigma)/4(1-\sigma)$. STROH [1958] has calculated the interaction energy for two-dimensional cracks in anisotropic materials.

The concept of the interaction energy of an inhomogeneity with an applied stress simplifies the calculation of the bulk elastic constants of elastically inhomogeneous aggregates. Suppose, for example, that we wish to calculate the effective elastic constants of a material of bulk modulus $\varkappa$ and shear modulus μ containing a dispersion of spherical inhomogeneities with elastic constants $\varkappa^*$, μ^* (ESHELBY [1957]). In the absence of the inhomogeneities unit volume of the material has elastic energy

$$E_0 = \frac{1}{2}\Big(\frac{1}{9\varkappa} p^{\mathrm{A}} p^{\mathrm{A}} + \frac{1}{2\mu}\, 'p_{ij}^{\mathrm{A}}\, 'p_{ij}^{\mathrm{A}}\Big) \tag{6.8}$$

if it is subjected to the uniform stress

$$p_{ij}^{\mathrm{A}} = \tfrac{1}{3} p^{\mathrm{A}} \delta_{ij} + 'p_{ij}^{\mathrm{A}}$$

(for the notation see § 1). If the spheres are introduced the elastic energy becomes

$$E = E_1 - \Sigma E_{\mathrm{int}} \tag{6.9}$$

by (4.15), where Σ denotes summation over all the spheres. If we ignore the image term in (4.13) and also the interaction between the spheres (so that we limit ourselves to a dilute dispersion) (6.9) may be written as

$$E = \frac{1}{2}\Big\{\frac{1}{9\varkappa}(1 + Av) p^{\mathrm{A}} p^{\mathrm{A}} + \frac{1}{2\mu}(1 + Bv)\, 'p_{ij}^{\mathrm{A}}\, 'p_{ij}^{\mathrm{A}}\Big\}$$

where v is the fraction of the volume of material occupied by the inhomogeneous spheres and A, B are given by (4.7). Comparing with (6.8) we see that the effective elastic constants are

$$\varkappa_{\mathrm{eff}} = \varkappa/(1 + Av), \qquad \mu_{\mathrm{eff}} = \mu/(1 + Bv)$$

or, since the analysis is only valid for $v \ll 1$,

$$\varkappa_{\mathrm{eff}} = \varkappa(1 - Av), \qquad \mu_{\mathrm{eff}} = \mu(1 - Bv)\,. \tag{6.10}$$

These expressions have been obtained in another manner by HASHIN (REINER [1958]).

Much attention has been given to a related problem: namely, to determine the macroscopic elastic constants, $\overset{0}{c}_{ijkl}$, $\overset{0}{s}_{ijkl}$ say, of a polycrystalline aggregate whose actual constants c_{ijkl}, s_{ijkl} vary from grain to grain, not because the material is inhomogeneous, but because the crystal orientation varies. The estimates $\overset{0}{c}_{ijkl} = \bar{c}_{ijkl}$ and $\overset{0}{s}_{ijkl} = \bar{s}_{ijkl}$ are associated with the names of Voigt and Reuss. Here the bar denotes an average over all crystal orientations, weighted if necessary according to the relative frequency of each orientation. HILL [1952] has shown that

$$\bar{s}'_{ijkl} \leqslant \overset{0}{s}_{ijkl} \leqslant \bar{s}_{ijkl}$$
$$\bar{c}'_{ijkl} \leqslant \overset{0}{c}_{ijkl} \leqslant \bar{c}_{ijkl}$$

where $\bar{c}'_{ijkl}$ and $\bar{s}'_{ijkl}$ are, respectively, the tensors inverse to $\bar{s}_{ijkl}$ and $\bar{c}_{ijkl}$. For an aggregate of cubic crystals the upper and lower estimates of the bulk modulus coincide and we have exactly

$$\varkappa = \tfrac{1}{3}(c_{11} + 2c_{12})\,. \tag{6.11}$$

Hill's relation seems to mark the limit of what can be proved precisely. To go further it is necessary to make assumptions which may be physically plausible, but cannot be proved conclusively. Most of the attempts in this direction are rather outside the scope of this article. HERSHEY's [1954] and KRÖNER's [1958b] calculations, however, make use of the concept of an anisotropic inhomogeneity in an isotropic matrix. We shall not give their physical arguments, but instead present a simplified treatment which involves essentially the same mathematics.

We confine ourselves to an aggregate of cubic crystals which is macroscopically isotropic. Let the effective elastic constants be $\varkappa$, μ. Imagine that the material of each grain is replaced by isotropic material with constants $\varkappa$, μ, but that the boundaries between grains can still be distinguished. We thus obtain, trivially, an 'equivalent isotropic aggregate' with the same overall elastic constants as the polycrystalline aggregate. It seems clear that this equivalent aggregate must pass the following test: if any representative sample of its grains have their original anisotropic constants restored the bulk elastic constants are unaltered. In applying the test we may take the collection of retransformed grains to be so far apart that the effect of the perturbing field of one grain on another may be neglected. Of course the isotropic aggregate must also pass the more severe test that its bulk elastic constants are unchanged when *every* grain is restored to its original anisotropic state. However, we may hope that if the weaker test gives definite values for $\varkappa$ and μ, then these will be the same as those which would result from applying the severer test.

We shall suppose that the set of test grains have their crystal axes oriented at random and that they may be treated as spheres. This second assumption is hard to justify rigorously, but without it not much progress can be made. In the presence of a uniform applied strain e_{ij}^{A} the e_{ij}^{T} of the equivalent inclusion for one of the grains can be written in the form

$$e_{ij}^{\mathrm{T}} = D_{ijkl}\, e_{kl}^{\mathrm{A}}\,;$$

the D_{ijkl} depend on the orientation of the grain. When the anisotropic test grains are introduced into the equivalent isotropic material its elastic constants are changed by an amount proportional to the sum of their interaction energies with the applied field, by the same argument as led to (6.9). Thus the test requires that

$$\Sigma(\overset{\mathrm{A}}{p}_{ij}\overset{\mathrm{T}}{e}_{ij}) = \Sigma(\overset{\mathrm{A}}{p}_{ij}\overset{\mathrm{A}}{e}_{kl}D_{ijkl}) = 0 \tag{6.12}$$

where Σ implies summation over all the re-transformed grains. If we choose fixed coordinate axes, $\overset{\mathrm{A}}{p}_{ij}\overset{\mathrm{A}}{e}_{kl}$ is the same for each term of the summation but D_{ijkl} varies from term to term. It is therefore more convenient to choose for each term axes parallel to the crystal axes of the grain in question. Then D_{ijkl} is the same for each term, but $\overset{\mathrm{A}}{p}_{ij}\overset{\mathrm{A}}{e}_{kl}$ is not. The calculation of the sum $\Sigma\overset{\mathrm{A}}{p}_{ij}\overset{\mathrm{A}}{e}_{kl}$ could be carried out as follows. Assign fixed values to the principal strains of $\overset{\mathrm{A}}{e}_{ij}$, form $\overset{\mathrm{A}}{p}_{ij}\overset{\mathrm{A}}{e}_{kl}$, average the result over all orientations of the principal axes of the strain and multiply by the number of terms. The resulting expression will have the form of the most general isotropic tensor which has the symmetry of the suffixes in $\overset{\mathrm{A}}{p}_{ij}\overset{\mathrm{A}}{e}_{kl} = \overset{\mathrm{A}}{p}_{kl}\overset{\mathrm{A}}{e}_{ij}$. Consequently it must have the same form as the elastic constant tensor c_{ijkl} of an isotropic medium, that is

$$\Sigma\overset{\mathrm{A}}{p}_{ij}\overset{\mathrm{A}}{e}_{kl} = c_1\delta_{ij}\delta_{kl} + c_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

with arbitrary c_1, c_2. Inserting this in (6.12) we get $D_{iijj} = 0$ and $D_{ijij} = 0$, or in view of the symmetry (spherical grains, cubic crystal referred to its principal axes)

$$D_{1111} + 2D_{1122} = 0 \tag{6.13}$$

and

$$D_{1111} + 2D_{1212} = 0\,. \tag{6.14}$$

The values of the D_{ijkl} are easily calculated from (4.6) and (4.7) if we bear in mind that when extended along its axes a cubic crystal behaves like an isotropic material with $\varkappa^* = \frac{1}{3}(c_{11}+2c_{12})$, $\mu^* = \frac{1}{2}(c_{11}-c_{22})$ and that for a shear of the type e_{12} it behaves like an isotropic material with $\mu^* = c_{44}$. Eq. (6.14) gives $A\{\varkappa, \frac{1}{3}(c_{11}+2c_{12})\} = 0$ or $\varkappa = \frac{1}{3}(c_{11}+2c_{12})$ in agreement with the already known result (6.11). The expression for the shear modulus is more interesting. Eq. (6.14) gives

$$\tfrac{2}{3}B\{\mu, \tfrac{1}{2}(c_{11}-c_{12})\} - B\{\mu, c_{44}\} = 0\,.$$

On inserting the value of β (eq. (3.14)) and clearing of fractions we are left with a cubic equation:

$$16\mu^3 + 2(5c_{11} + 4c_{12})\mu^2 - 2c_{44}(7c_{11} - 4c_{12})\mu - c_{44}(c_{11} - c_{12})(c_{11} + 2c_{12}) = 0 \tag{6.15}$$

for determining μ. It has only one real positive root.

Hershey [1954] obtained a quartic equation for μ which in fact is what one obtains on multiplying (6.15) by $8\mu + 9\varkappa$. Evidently this introduces no new positive root. Kröner [1958b] obtained both the quartic and cubic and gave an argument to show that their respective positive roots are upper and lower bounds for μ, and that since these roots coincide the value so obtained is exact apart from statistical uncertainties due to the fact that the aggregate contains only a finite number of grains.

Goodier [1936] has emphasized that there is a useful analogy between problems in the slow motion of viscous liquids and elastic problems for incompressible solids. In general special care has to be taken in extrapolating elastic solutions to the incompressible case $\sigma = \frac{1}{2}$, since we have, in effect, to make the simultaneous transition $u_{k,k} \to 0$, $\lambda \to \infty$. However, in the case of the solution (2.8) to the general inclusion problem there is no trouble. With $\sigma = \frac{1}{2}$ it satisfies

$$\mu\nabla^2\boldsymbol{u}^{\mathrm{C}} = 0\,, \qquad \operatorname{div}\boldsymbol{u}^{\mathrm{C}} = 0$$

outside the inclusion. These are just the Stokes equations governing the velocity $\boldsymbol{u}^{\mathrm{C}}$ in an incompressible fluid of viscosity μ when inertial effects may be neglected and the hydrostatic pressure, p_0, say, is independent of position. Consequently (2.8) with $\sigma = \frac{1}{2}$ and $\overset{\mathrm{C}}{u}_i$ interpreted as a velocity represents a certain state of viscous flow. The associated stress is

$$p_{ij} = \mu(\overset{\mathrm{C}}{u}_{i,j} + \overset{\mathrm{C}}{u}_{j,i}) - p_0\delta_{ij}\,.$$

Eq. (2.8) describes the flow about a solid which is deforming in such a way that each point $\boldsymbol{r}$ of its surface S has a velocity $\overset{\mathrm{C}}{u}_i(\boldsymbol{r})$. Since we cannot prescribe $\overset{\mathrm{C}}{u}_i(\boldsymbol{r})$, but only the constants $\overset{\mathrm{T}}{e}_{ij}$ the analogy is not of much use in the general case. Eq. (3.1), however, (with $\sigma = \frac{1}{2}$) describes the viscous flow around an ellipsoid which is undergoing a change of shape specified by the constant rate-of-strain tensor $\overset{\mathrm{C}}{e}_{ij}$, and the $\overset{\mathrm{T}}{e}_{ij}$ appropriate to prescribed $\overset{\mathrm{C}}{e}_{ij}$ can be found be solving (3.3).

The solution to the problem of an ellipsoidal inhomogeneity perturbing a uniform strain $\overset{\mathrm{A}}{e}_{ij}$ has a simple viscous interpretation when the ellipsoid is perfectly rigid and incompressible ($\lambda^* \to \infty$, $\mu^* \to \infty$). Eq. (4.3) gives $\overset{\mathrm{C}}{e}_{ij} + \overset{\mathrm{A}}{e}_{ij} = 0$ on S or equivalently

$$\overset{\mathrm{C}}{u}_i + \overset{\mathrm{A}}{u}_i = 0 \quad \text{on} \quad \mathrm{S}\,. \tag{6.16}$$

Translated into terms of velocity this is the condition that the liquid shall adhere to the surface of the ellipsoid, and so the field $u_i = u_i^{\mathrm{C}} + u_i^{\mathrm{A}}$ gives the velocity about a solid ellipsoid immersed in the uniform flow specified by the rate-of-strain tensor e_{ij}^{A}. The elastic energy density translates into half the rate of dissipation of energy per unit volume. (In the one case we are concerned with the familiar half stress times strain, in the other with stress times rate of deformation.) Consequently $2E_{\mathrm{int}}$ (eq. (4.13)) is equal to the additional rate of dissipation of energy, $\dot{E}_{\mathrm{diss}}$ say, when a solid is introduced into a viscous liquid at whose boundary constant velocities are maintained.

If the image term may be neglected in (4.13) the interaction energy for an inhomogeneity is half the interaction energy for the equivalent transformed inclusion, and it follows that $\dot{E}_{\mathrm{diss}}$ is given by the right hand side of (2.42), or, in view of (6.16), by

$$\dot{E}_{\mathrm{diss}} = -\int_{\mathrm{S}} (p_{ij}^{\mathrm{C}} + p_{ij}^{\mathrm{A}}) u_i^{\mathrm{C}} \mathrm{d}S_j$$
$$= \int_{\mathrm{S}} \boldsymbol{p}_n \cdot \boldsymbol{u}^{\mathrm{C}} \,\mathrm{d}S$$

where $\boldsymbol{p}_n$ is the actual surface traction on the solid (with the sign convention usual in hydrodynamics) and $\boldsymbol{u}^{\mathrm{C}}$ is the perturbation of velocity due to the presence of the solid. This is identical with a result of BRENNER's [1958]. The expression (6.10) for the effective shear modulus of a dispersion of spheres gives the Einstein viscosity formula $\mu_{\mathrm{eff}} = (1 + \frac{5}{2}v)\mu$ when we put $\mu^* = \infty$, $\sigma = \frac{1}{2}$. The difference in sign between (4.14) and (4.15) has the following interpretation in terms of viscosity. The viscosity is always increased by the introduction of solid particles. Consequently a viscometer working at constant load produces a lower rate of deformation and so less energy dissipation, while a viscometer working at constant speed has to work harder to maintain the prescribed rate of strain.

The applications considered above have all been essentially macroscopic. Elastic inclusions and inhomogeneities have also been found useful as models of lattice defects in crystals. (For a general account see, for example, FRIEDEL [1956] or ESHELBY [1956].)

Suppose that one atom of a crystal lattice is replaced by a foreign atom. The foreign atom will generally have a different size from the host atoms. As a simple elastic model we may take a spherical hole in an isotropic continuum into which a misfitting elastic sphere is inserted (BILBY [1950]). Since the elastic constants of the foreign atom (in so far as one can speak of them for a single atom) will differ from

those of the host atoms the misfitting elastic sphere will in general have to be assigned elastic constants differing from those of the matrix. It is useful to consider two limiting cases. (i) A pure inclusion; the sphere and inclusion have the same elastic constants, and the misfit gives rise to a permanent elastic field. (ii) A pure inhomogeneity; there is no misfit, but the elastic constants of sphere and matrix differ. There is no permanent elastic field, but a field can be 'induced' by an applied field. The general case, intermediate between (i) and (ii), corresponds to the inhomogeneous inclusion of § 3.2.

It is not at all obvious that such a crude model will be of any use in discussing the behaviour of lattice defects. We shall discuss some of the consequences of taking the model seriously and then try to indicate the reasons for its success.

The elastic field of a pure inclusion is given by (2.26) with the value (3.12) for φ and e^{T} equal to the fractional volume misfit between hole and sphere. The stress falls off as r^{-3} and its hydrostatic component is zero. The field of a pure inhomogeneity in an applied field e_{ij}^{A} is found by calculating the equivalent e_{ij}^{T} from (4.6) and inserting them in (2.8) with the values (3.12), (3.13) for φ and ψ.

It may be shown that the expressions (2.43) and (4.13) for the interaction energies of inclusions and inhomogeneities remain valid when p_{ij}^{A}, e_{ij}^{A} refer to the field produced by some source of internal stress, another inclusion or a dislocation for example. Thus the interaction energy between two pure inclusions is proportional to $p_{ij}^{\mathrm{A}} e_{ij}^{\mathrm{T}}$, where p_{ij}^{A} is the stress produced by one inclusion and e_{ij}^{T} is the stress-free strain associated with the other. Since $e_{ij}^{\mathrm{T}} = \frac{1}{3}e^{\mathrm{T}}\delta_{ij}$ the interaction energy is proportional to p^{A}. But neither defect produces a hydrostatic pressure, and so the interaction energy is zero (BITTER [1931]). Again, let p_{ij}^{A} refer to the field of a pure inclusion and let e_{ij}^{T} be the equivalent stress-free strain which it induces in a pure inhomogeneity at a distance r from it. Since the p_{ij}^{A} are proportional to r^{-3} and the e_{ij}^{T} are proportional to the p_{ij}^{A} it follows that the interaction energy between a pure inclusion and a pure inhomogeneity is proportional to the inverse sixth power of the distance between them. The interaction energy between two defects each of which is represented by a misfitting and inhomogeneous sphere is also proportional to r^{-6}. Its numerical value may be found by a detailed calculation (ESHELBY [1958]).

The stress-field at a distance r from a dislocation line is proportional to r^{-1}. It follows by the same kind of argument as before that the dislocation has an interaction energy proportional to r^{-1} with a pure inclusion but an interaction energy proportional to r^{-2} with a pure inhomogeneity. This difference allows one to distinguish experimentally between the two types of defect. Since the interaction energy is a function of position there is an effective force

$$F = -\operatorname{grad} E_{\text{int}}(r) \tag{6.17}$$

on the defect. If the defect is capable of diffusing through the lattice, a drift velocity

$$v = DF/kT$$

is superimposed on its random motion, where D is the diffusion coefficient, k is Boltzmann's constant and T is the absolute temperature. Thus, if dislocations are introduced into a crystal containing defects, the latter will be attracted to the dislocations. The resulting depletion of defects in the bulk of the material can be detected by measuring suitable physical properties (electrical resistance, internal friction). Calculation shows that for an r^{-1} interaction the number of defects drawn into the dislocations is proportional to $t^{\frac{2}{3}}$, where t is the time since the interaction between defects and dislocations began. If, on the other hand, there is an r^{-2} interaction the expression $t^{\frac{2}{3}}$ is replaced by $t^{\frac{1}{2}}$ (FRIEDEL [1959]).

For a defect which can be represented by an inserted sphere which is both misfitting and inhomogeneous the interaction energy with a dislocation will evidently have the form $Ar^{-1} + Br^{-2}$. Even if the coefficient B is relatively large the A-term will dominate at large distances and we should expect the $t^{\frac{2}{3}}$ law to be most nearly obeyed. The $t^{\frac{2}{3}}$ law has, in fact, been verified for carbon and nitrogen diffusing in iron. We might, perhaps, regard a vacant lattice site as a pure inhomogeneity ($A = 0$). However, such calculations as have been made (e.g. TEWORDT's [1958] for copper) indicate that there is an appreciable displacement of the atoms bordering the vacant site. On the elastic model this means that there is a stress-field associated with the vacancy even in the absence of an applied field, and so $A \neq 0$.

Rather surprisingly WINTENBERGER's [1957] measurements on vacancies in aluminium follow the $t^{\frac{1}{2}}$ law, which indicates that, in aluminium at least, lattice vacancies behave as pure inhomogeneities.

We shall now try to indicate why the simple misfitting sphere model has been relatively successful. In the first place it is reasonable to suppose that sufficiently far from a lattice defect the crystal can be treated as an elastic continuum. The displacement representing the disturbance due to the defect can be expanded in ascending inverse

powers of r (the distance from the defect) each provided with a suitable angular factor. If we treat the material as isotropic the leading term of the expansion has precisely the form (2.23) with arbitrary symmetric e^{T}_{ij}. Thus, if the material is isotropic, (2.23) gives the elastic field at large distances from the most general type of point defect. In many of the common metals the elastic field of the defect will have cubic symmetry. This physical condition, combined with the artificial limitation to isotropy, requires that e^{T}_{ij} shall be of the form $\frac{1}{3}e^{\mathrm{T}}\delta_{ij}$. According to (2.26) the remote field is independent of the shape of the inclusion and we may take it to be a sphere. In this way we recover the misfitting sphere model. In some cases (in particular, interstitial carbon and nitrogen in iron) the assumption that e^{T}_{ij} is a pure dilatation is inadequate, but the displacement may still be taken to have the form (2.23) with suitably chosen values for the e^{T}_{ij}.

The interaction energy (2.42) can be re-written in the form

$$E_{\text{int}} = \int_{\mathrm{S}'} (p^{\mathrm{C}}_{ij} u^{\mathrm{A}}_i - p^{\mathrm{A}}_{ij} u^{\mathrm{C}}_i) \mathrm{d}S_j \tag{6.18}$$

where S′ is any surface drawn in the material and enclosing the inclusion. (The expressions (2.42) and (6.18) are identical because the divergence of the integrand is zero between S and S′.) The expression for the interaction energy was derived on the assumption that the infinitesimal theory could be applied everywhere. This is obviously not true near an inclusion representing an atomic defect. However, it may be shown that for (6.18) to be correct it is only necessary that the infinitesimal theory be valid in the neighbourhood of the surface S′, a much less severe requirement. The following is an outline of the argument (cf. ESHELBY [1959a]). We may form an expression for the x_l-component of the effective force (6.17) by subtracting from (6.18) the corresponding expression with p^{C}_{ij}, u^{C}_i replaced by $p^{\mathrm{C}}_{ij} + p^{\mathrm{C}}_{ij,l}\,\varepsilon$, $u^{\mathrm{C}}_i + u^{\mathrm{C}}_{i,l}\,\varepsilon$, dividing by ε and letting ε approach zero. The result is

$$F_l = \int_{\mathrm{S}'} (p^{\mathrm{C}}_{ij,l} u^{\mathrm{A}}_i - p^{\mathrm{A}}_{ij} u^{\mathrm{C}}_{i,l}) \mathrm{d}S_j\,. \tag{6.19}$$

It is possible (ESHELBY [1956]) to derive a general expression for F_l which is valid for an arbitrary non-linear stress-strain relation and for finite deformation. Like (6.19) this expression takes the form of an integral over a surface surrounding the defect on which the force is to be calculated. If it is permissible to apply the infinitesimal theory on this surface the general expression reduces to (6.19). Hence (6.19), (6.18) or the simple formula (2.43) can be used whenever it is possible to draw a surface S′ enclosing the defect and no other source of internal stress and far enough from it for the strains on S′ to be reasonably small. Applied to the interaction of two point defects, for example, this means that (2.42) can be used if the linear theory is approximately obeyed halfway between them.

The treatment of lattice defects by continuum methods sometimes gives useful results even in cases too extreme for the above considerations to apply. FRIEDEL [1954] has shown, for example, that in some cases it is possible to get a reasonable estimate for the energy of solution of atoms of a metal X in a metal Y by associating with each dissolved X-atom a strain energy (3.21), taking for the volumes of hole and inclusion the atomic volumes of X and Y and for (λ, μ), (λ^*, μ^*) the ordinary elastic constants of the metals Y and X. An amusing example is provided by JACOBS' [1954] calculation of the effect of hydrostatic pressure on the frequency of the absorption band of an F-centre in an alkali halide crystal. An F-centre is an electron trapped at a negative-ion vacancy. An empirical rule of MOLLWO's [1933] states that in passing from one alkali halide crystal with the sodium chloride structure to another the product νa^2 remains constant; ν is the frequency of the maximum of the F-absorbtion band and a is the lattice parameter. It is reasonable to suppose that the same relation will govern the behaviour of a given alkali halide when its lattice parameter is changed by compression. We should then have the relation $-(\mathrm{d}\nu/\nu)/(\mathrm{d}a/a) = 2$. In place of 2, Jacobs' experiments gave values between 4.4 and 3.4. This discrepancy can be reconciled if we admit that the characteristics of an F-centre are determined by the positions of the atoms bordering the vacancy. Then Mollwo's rule must be interpreted as $\nu R^2 = \text{const.}$, where R is, say, the distance of a neighbouring atom from the centre of the vacancy, so that

$$-\frac{\mathrm{d}\nu/\nu}{\mathrm{d}R/R} = 2\,. \tag{6.20}$$

On passing from one type of crystal to another $(\mathrm{d}R/R)/(\mathrm{d}a/a)$ is unity, since R/a depends only on the lattice geometry. However, when a given crystal is compressed, $\mathrm{d}a/a$ and $\mathrm{d}R/R$ are not identical. In fact, if we idealize the vacancy as a spherical hole in an isotropic continuum we have

$$\frac{\mathrm{d}R/R}{\mathrm{d}a/a} = \frac{e^{\mathrm{A}} + e^{\mathrm{C}}}{e^{\mathrm{A}}} \tag{6.21}$$

in the notation of (4.6), with $\varkappa^* = 0$ in A. For the alkali halides we

may put $\sigma = \frac{1}{4}$; the ratio (6.21) is then 2.25, and from (6.20) and (6.21) we have

$$-\frac{\mathrm{d}\nu/\nu}{\mathrm{d}a/a} = 4.5$$

in much better agreement with experiment. The above is admittedly rather a travesty of Jacobs' argument, but in his more rigorous calculation he also found it necessary to introduce the magnification factor (6.21).

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ME 238C — THEORY OF ELASTICITY

TENTATIVE SCHEDULE FOR THE FIRST 9 LECTURES

1. THE 3-D ELASTIC GREEN'S FUNCTION
2. ELLIPSOIDAL INHOMOGENEITIES AND INCLUSIONS

REFERENCE — J.D. ESHELBY, ELASTIC INCLUSIONS AND INHOMOGENEITIES in PROGRESS IN SOLID MECHANICS, v. II., edited by I.N. SNEDDON AND R. HILL, NORTH-HOLLAND CO., AMSTERDAM. CHAPTER 3, p. 89-140 (1961).

THE 3-D ELASTIC GREEN'S FUNCTIONS

FROM LAST QUARTER, WE KNOW THAT THE 2-D GREEN'S FUNCTION $G_{im}(x, y; x', y')$ IS DEFINED TO BE THE i^{th} COMPONENT OF THE ELASTIC DISPLACEMENT AT (x', y') IN AN INFINITE MEDIUM DUE TO A UNIT LINE FORCE APPLIED IN THE m^{th} DIRECTION AT (x', y'). IN 3-D WE DEFINE THE INFINITE MEDIUM GREEN'S FUNCTION

$$G_{im}(x, y, z\, ; x', y', z')$$

AS THE i^{th} COMPONENT OF THE ELASTIC DISPLACEMENT AT (x, y, z) DUE TO A UNIT POINT FORCE APPLIED AT (x', y', z') IN THE m^{th} (x_m) DIRECTION. $i, m = 1, 2, 3$. FOR BREVITY WE WRITE

$$G_{im}(\underset{\sim}{x}, \underset{\sim}{x}').$$

NOW THE G_{im} SATISFY THE EQUATIONS OF EQUILIBRIUM WRITTEN IN TERMS OF DISPLACEMENTS, NAMELY

$$C_{ijk\ell} \frac{\partial^2 G_{km}}{\partial x_\ell \partial x_i} + \delta_{jm}\, \delta(\underset{\sim}{x} - \underset{\sim}{x}') = 0$$

DIVERGENCE OF $\underline{\underline{\sigma}}$ POINT FORCE (BODY FORCE).

WE EXPECT THAT $G_{km}(\underset{\sim}{x}, \underset{\sim}{x}') \to 0$ AS $|\underset{\sim}{x} - \underset{\sim}{x}'| \to \infty$ AND SIMILARLY FOR ITS DERIVATIVES.

<u>SOLUTION FOR G_{km} USING FOURIER TRANSFORMS</u>

<u>GENERAL</u> <u>REMARKS</u>: THE FOURIER TRANSFORM $g(k)$ OF THE FUNCTION $G(x)$ IN ONE-DIMENSION IS DEFINED BY

$$g(k) = \int_{-\infty}^{\infty} dx\, e^{ikx}\, G(x). \qquad (1)$$

IT CAN BE SHOWN THAT THE FOLLOWING <u>INVERSION</u> RELATION IS VALID, NAMELY,

$$G(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{-ikx}\, g(k). \qquad (2)$$

THE GENERALIZATION OF EQUATIONS (1) AND (2) TO 3-D IS

$$g(k_1, k_2, k_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, dz\, e^{i(k_1 x + k_2 y + k_3 z)}\, G(x, y, z) \qquad (3)$$

$$G(x, y, z) = \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_1\, dk_2\, dk_3\, e^{-i(k_1 x + k_2 y + k_3 z)}\, g(k_1, k_2, k_3). \qquad (4)$$

CALLING $x = x_1$, $y = x_2$, $z = x_3$ AND NOTING

$$\underset{\sim}{k} = k_i \underset{\sim}{e}_i = k_1 \underset{\sim}{e}_1 + k_2 \underset{\sim}{e}_2 + k_3 \underset{\sim}{e}_3 \tag{5}$$

$$\underset{\sim}{x} = x_i \underset{\sim}{e}_i = x_1 \underset{\sim}{e}_1 + x_2 \underset{\sim}{e}_2 + x_3 \underset{\sim}{e}_3$$

SO THAT

$$\underset{\sim}{k} \cdot \underset{\sim}{x} = k_1 x_1 + k_2 x_2 + k_3 x_3 \tag{6}$$

WHERE $\underset{\sim}{e}_i$ IS A UNIT VECTOR ALONG THE x_i - AXIS, WE CAN WRITE

$$g(\underset{\sim}{k}) = \iiint_{-\infty}^{\infty} d^3\underset{\sim}{x}\; e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}}\, G(\underset{\sim}{x}) \tag{7}$$

$$G(\underset{\sim}{x}) = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} d^3\underset{\sim}{k}\; e^{-i \underset{\sim}{k} \cdot \underset{\sim}{x}}\, g(\underset{\sim}{k}) \tag{8}$$

$\underset{\sim}{k}$ IS CALLED THE WAVE VECTOR.

COROLLARY: SUPPOSE $G(x, y, z) = \delta(x)\,\delta(y)\,\delta(z) = \delta(\underset{\sim}{x})$. FROM (3)

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,dz\; e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}}\, \delta(\underset{\sim}{x}) \tag{9}$$

SO THAT THE FOURIER TRANSFORM OF A DELTA FN. IS ONE.

THEN FROM (4)

$$\delta(\underset{\sim}{x}) = \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3\underset{\sim}{k}\; e^{-i \underset{\sim}{k} \cdot \underset{\sim}{x}} \tag{10}$$

NOW LET US USE THESE RESULTS TO SOLVE FOR THE ELASTIC GREEN'S FUNCTIONS.

RETURN TO

$$C_{ijkl} \frac{\partial^2}{\partial x_i \partial x_l} G_{km}(\underset{\sim}{x}, \underset{\sim}{x}') + \delta_{jm} \delta(\underset{\sim}{x} - \underset{\sim}{x}') = 0. \qquad (11)$$

WE CAN ALWAYS SHIFT OUR ORIGIN SO THAT $\underset{\sim}{x}' = 0$ (i.e., PT. FORCE IS APPLIED AT ORIGIN). THUS, WE MUST SOLVE

$$C_{ijkl} \frac{\partial^2}{\partial x_i \partial x_l} G_{km}(\underset{\sim}{x}) + \delta_{jm} \delta(\underset{\sim}{x}) = 0. \qquad (12)$$

CALL

$g_{km}(\underset{\sim}{k})$ = FOURIER TRANSFORM OF $G_{km}(\underset{\sim}{x})$, i.e.,

$$g_{km}(\underset{\sim}{k}) = \iiint_{-\infty}^{\infty} d^3\underset{\sim}{x} \; e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}} \, G_{km}(\underset{\sim}{x}) \qquad (13)$$

SO THAT

$$G_{km}(\underset{\sim}{x}) = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} d^3\underset{\sim}{k} \; e^{-i \underset{\sim}{k} \cdot \underset{\sim}{x}} \, g_{km}(\underset{\sim}{k}). \qquad (\;)$$

USING (14) AND (10), WE CAN REWRITE (12) AS

$$C_{ijkl} \frac{\partial^2}{\partial x_i \partial x_l} \left\{ \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} d^3\underset{\sim}{k} \; e^{-i \underset{\sim}{k} \cdot \underset{\sim}{x}} \, g_{km}(\underset{\sim}{k}) \right\}$$

$$+ \delta_{jm} \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} d^3\underset{\sim}{k} \; e^{-i \underset{\sim}{k} \cdot \underset{\sim}{x}} = 0 \qquad (15)$$

SINCE $$\frac{\partial^2}{\partial x_i \partial x_l} e^{-i \underset{\sim}{k} \cdot \underset{\sim}{x}} = (-ik_i)(-ik_l) e^{-i \underset{\sim}{k} \cdot \underset{\sim}{x}}$$

$$= -k_i k_l \, e^{-i \underset{\sim}{k} \cdot \underset{\sim}{x}}, \qquad (16)$$

$$\iiint_{-\infty}^{\infty} d^3\underset{\sim}{k} \; e^{-i \underset{\sim}{k} \cdot \underset{\sim}{x}} \left\{ -C_{ijkl} k_i k_l \, g_{km}(\underset{\sim}{k}) + \delta_{jm} \right\} = 0 \qquad (17)$$

EQUATION (17) IS EASILY SATISFIED IF $g_{km}(\underset{\sim}{k})$ SATISFIES

$$\left(C_{ijkl}\, k_i k_l\right) g_{km}(\underset{\sim}{k}) = \delta_{jm}. \tag{18}$$

NOW THE UNIT VECTOR $\underset{\sim}{\xi}$ ALONG $\underset{\sim}{k}$ IS DEFINED BY

$$\underset{\sim}{\xi} = \frac{\underset{\sim}{k}}{|\underset{\sim}{k}|} \tag{19}$$

SO THAT

$$k_i = |\underset{\sim}{k}|\, \xi_i \tag{20}$$

HENCE

$$|\underset{\sim}{k}|^2 C_{ijkl}\, \xi_i \xi_l\, g_{km}(\underset{\sim}{k}) = \delta_{jm}. \tag{21}$$

DEFINE THE SYMMETRIC MATRIX $[M]$ BY

$$M_{jk} = C_{ijkl}\, \xi_i \xi_l. \tag{22}$$

THEN

$$\boxed{g_{km}(\underset{\sim}{k}) = \frac{M^{-1}_{km}(\underset{\sim}{\xi})}{|\underset{\sim}{k}|^2}} \tag{23}$$

NOTE:

M_{km} & HENCE M^{-1}_{km} (THE INVERSE OF $[M]$)

DEPEND ONLY ON $\underset{\sim}{\xi}$ (THE DIRECTION OF $\underset{\sim}{k}$) BUT NOT ON $|\underset{\sim}{k}|$!!

IT TURNS OUT THAT EQUATION (23) IS ALL WE NEED TO KNOW TO SOLVE ELLIPSOIDAL INCLUSION PROBLEMS. NEVERTHELESS WE WILL GO AHEAD AND SOLVE FOR $G_{km}(\underset{\sim}{x})$ ANYWAY.

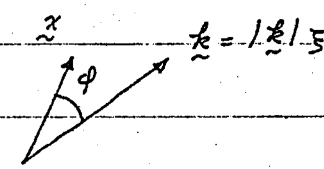
NOW, USING (14),

$$G_{km}(\underset{\sim}{x}) = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} dk_1\, dk_2\, dk_3\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}} \frac{M_{km}^{-1}(\underset{\sim}{\xi})}{|\underset{\sim}{k}|^2}. \tag{24}$$

THE INVERSION INTEGRAL IS MOST EASILY DONE IN SPHERICAL POLAR COORDINATES:

$$dk_1\, dk_2\, dk_3 = |\underset{\sim}{k}|^2 \sin\varphi\, dk\, d\vartheta\, d\varphi \tag{25}$$

WHERE φ IS THE ANGLE BETWEEN $\underset{\sim}{k}$ AND $\underset{\sim}{x}$.



i.e., $\underset{\sim}{k}\cdot\underset{\sim}{x} = |\underset{\sim}{k}||\underset{\sim}{x}|\cos\varphi$

THEN

$$G_{km}(\underset{\sim}{x}) = \frac{1}{8\pi^3} \int_0^{2\pi} d\vartheta \int_0^{\pi} \sin\varphi\, d\varphi \int_0^{\infty} d|\underset{\sim}{k}|\, M_{km}^{-1}(\underset{\sim}{\xi})\, e^{-i|\underset{\sim}{k}||\underset{\sim}{x}|\cos\varphi}$$

OR

LETTING $u = |\underset{\sim}{x}||\underset{\sim}{k}|$; $du = |\underset{\sim}{x}|\, d|\underset{\sim}{k}|$

$$G_{km}(\underset{\sim}{x}) = \frac{1}{8\pi^3|\underset{\sim}{x}|} \int_0^{2\pi} d\vartheta \int_0^{\pi} \sin\varphi\, d\varphi\, M_{km}^{-1}(\underset{\sim}{\xi}) \int_0^{\infty} du\, e^{-iu\cos\varphi}. \tag{27}$$

Now $$e^{-iu\cos\varphi} = \cos(u\cos\varphi) - i\sin(u\cos\varphi) \tag{28}$$

AND CLEARLY $G_{km}(\underset{\sim}{x})$ IS REAL, SO WE NEED ONLY WORRY ABOUT THE REAL PART OF THE INTEGRAL IN (27).

THUS $$G_{km}(\underset{\sim}{x}) = \frac{1}{8\pi^3 |\underset{\sim}{x}|} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi \, M_{km}^{-1}(\underset{\sim}{\xi}) \int_0^{\infty} du \, \cos(u\cos\varphi). \quad (29)$$

NOW,

$$\boxed{\int_0^{\infty} du \, \cos(u\cos\varphi) = \pi\,\delta(\cos\varphi)} \quad (30)$$

PROOF: USING THE 1-D FORM OF (10),

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{-ikx}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left\{ \cos kx - i \sin kx \right\}$$

BUT

$$\int_{-\infty}^{\infty} dk \, \sin kx = 0 \quad \text{SINCE } \sin kx \text{ IS ODD IN } k$$

$$\int_{-\infty}^{\infty} dk \, \cos kx = 2 \int_0^{\infty} dk \, \cos kx$$

$\therefore$

$$\delta(x) = \frac{1}{2\pi} \cdot 2 \int_0^{\infty} dk \, \cos kx$$

OR $$\int_0^{\infty} dk \, \cos kx = \pi\,\delta(x).$$

LETTING

$k \to u$, $x \to \cos\varphi$, WE GET EQUATION (30)

HENCE

$$G_{km}(\underset{\sim}{x}) = \frac{1}{8\pi^2 |\underset{\sim}{x}|} \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \, \sin\varphi \, \delta(\cos\varphi) \, M_{km}^{-1}(\underset{\sim}{\xi}) \quad (31)$$

NOW $\delta(\cos\varphi) = 0$ UNLESS $\varphi = \frac{\pi}{2}$, $0 \le \varphi \le \pi$

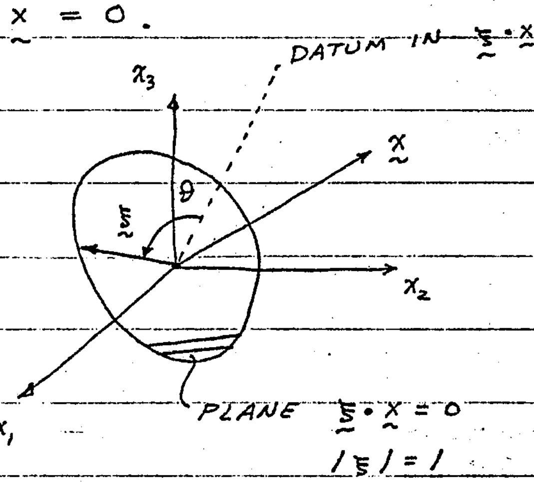
AND

$$\int_0^{\pi} d\varphi \, f(\varphi)\, \delta(\cos\varphi) = f\left(\varphi = \tfrac{\pi}{2}\right).$$

$$\therefore \quad G_{km}(\underset{\sim}{x}) = \frac{1}{8\pi^2 |\underset{\sim}{x}|} \int_0^{2\pi} d\vartheta \left[M^{-1}_{km} \right]_{\varphi = \frac{\pi}{2}}$$

$$\boxed{G_{km}(\underset{\sim}{x}) = \frac{1}{8\pi^2 |\underset{\sim}{x}|} \int_0^{2\pi} d\vartheta \left\{ M^{-1}_{km}(\underset{\sim}{\xi}) \right\}_{\substack{\text{~~ALL ξ SUCH THAT~~} \\ \underset{\sim}{\xi}\cdot\underset{\sim}{x} = 0}}} \qquad (32)$$

$^*\ \varphi = \frac{\pi}{2} \Rightarrow \underset{\sim}{k}\cdot\underset{\sim}{x} = 0$ AND $\underset{\sim}{\xi}\cdot\underset{\sim}{x} = 0$.



EQUATION (32) IS VALID FOR ARBITRARY ANISOTROPY! LET US SPECIALIZE TO ELASTIC ISOTROPY. WE CAN PERFORM THE INTEGRAL OVER ϑ EXACTLY.

<u>ISOTROPIC SOLIDS</u>

$$C_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\left(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}\right) \qquad (33)$$

$$M_{jk} = C_{ijkl}\, \xi_i\, \xi_l = \lambda\, \xi_j\, \xi_k + \mu\left(\xi_j\, \xi_k + \xi_l\, \xi_l\, \delta_{jk}\right) \qquad (34)$$

BUT

$$\xi_l \xi_l = \underset{\sim}{\xi} \cdot \underset{\sim}{\xi} = 1 \quad \text{SINCE } \underset{\sim}{\xi} \text{ IS A UNIT VECTOR.} \tag{35}$$

THUS

$$\boxed{M_{jk} = \mu \left\{ \delta_{jk} + \frac{\lambda+\mu}{\mu} \xi_j \xi_k \right\}}. \tag{36}$$

GUESS

$$M^{-1}_{km} = \frac{1}{\mu} \left\{ \delta_{km} + A \xi_k \xi_m \right\} \tag{37}$$

AND SEE IF WE CAN FIND A VALUE OF A SUCH THAT

$$M_{jk} M^{-1}_{km} = \delta_{jm}. \tag{38}$$

NOW

$$M_{jk} M^{-1}_{km} = \left\{ \delta_{jk} + \frac{\lambda+\mu}{\mu} \xi_j \xi_k \right\} \left\{ \delta_{km} + A \xi_k \xi_m \right\}$$

$$= \delta_{jk}\delta_{km} + A \xi_j \xi_m + \frac{\lambda+\mu}{\mu} \xi_j \xi_m + \frac{\lambda+\mu}{\mu} A \xi_j \xi_m (\overset{''1}{\xi_k \xi_k})$$

$$= \delta_{jm} + \xi_j \xi_m \left\{ A + \frac{\lambda+\mu}{\mu} + A \frac{\lambda+\mu}{\mu} \right\}$$

EQUATION (38) IS SATISFIED IF

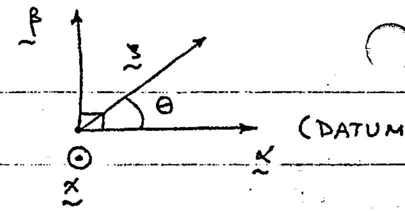
$$A \left\{ 1 + \frac{\lambda+\mu}{\mu} \right\} = - \frac{\lambda+\mu}{\mu}$$

OR

$$A = - \frac{\lambda+\mu}{\lambda+2\mu} \tag{39}$$

THUS FOR ISOTROPY

$$\boxed{M^{-1}_{km} = \frac{1}{\mu} \left\{ \delta_{km} - \frac{\lambda+\mu}{\lambda+2\mu} \xi_k \xi_m \right\}} \tag{40}$$



IN THE PLANE $\tilde{\xi} \cdot \tilde{x} = 0$ WE CAN WRITE

$$\tilde{\xi} = \tilde{\alpha} \cos\theta + \tilde{\beta} \sin \underset{\text{ANY}}{\vartheta}$$

WHERE $\tilde{\alpha}$ AND $\tilde{\beta}$ ARE TWO FIXED ORTHOGONAL UNIT VECTORS IN THE PLANE $\tilde{\xi} \cdot \tilde{x} = 0$. THUS

$$\xi_k = \alpha_k \cos\vartheta + \beta_k \sin\vartheta$$

$$(\xi_k \xi_m) = \alpha_k \alpha_m \cos^2\vartheta + \beta_k \beta_m \sin^2\vartheta + (\alpha_k \beta_m + \alpha_m \beta_k) \sin\vartheta\cos\vartheta$$

NOW

$$\int_0^{2\pi} d\vartheta = 2\pi \quad ; \quad \int_0^{2\pi} \cos^2\vartheta \, d\vartheta = \int_0^{2\pi} \sin^2\vartheta \, d\vartheta = \pi$$

$$\int_0^{2\pi} \sin\vartheta \cos\vartheta \, d\vartheta = 0$$

THUS, FORMULA (32) REDUCES TO

$$G_{km}(\tilde{x}) = \frac{1}{8\pi |\tilde{x}|} \frac{1}{\mu} \left\{ 2\,\delta_{km} - \frac{\lambda+\mu}{\lambda+2\mu} \left[\alpha_k \alpha_m + \beta_k \beta_m \right] \right\}. \quad (41)$$

$$\text{NOW} \quad \alpha_k \alpha_m + \beta_k \beta_m + \tau_k \tau_m = \delta_{km} \quad (42)$$

WHERE

$\tilde{\tau}$ IS A UNIT VECTOR ALONG $\tilde{x}$, i.e.,

$$\tau_k = \frac{x_k}{|\tilde{x}|}.$$

PROOF OF (42): PICK 1, 2, 3 AXES ALONG $\tilde{\alpha}$, $\tilde{\beta}$, AND $\tilde{\tau}$ ($\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 0$; $\beta_1 = 0$, $\beta_2 = 1$, $\beta_3 = 0$; $\tau_1 = \tau_2 = 0$ & τ_3). FORMULA WORKS AND, BEING A RELATION AMONG TENSORS, IS TRUE IN ANY COORDINATE SYSTEM.

HENCE

$$G_{km}(\underset{\sim}{x}) = \frac{1}{8\pi\mu|\underset{\sim}{x}|}\left\{\left(2 - \frac{\lambda+\mu}{\lambda+2\mu}\right)\delta_{km} + \frac{\lambda+\mu}{\lambda+2\mu}\tau_k\tau_m\right\}$$

$$= \frac{1}{8\pi\mu|\underset{\sim}{x}|}\left\{\frac{\lambda+3\mu}{\lambda+2\mu}\delta_{km} + \frac{\lambda+\mu}{\lambda+2\mu}\frac{x_k x_m}{|\underset{\sim}{x}|^2}\right\} \tag{43}$$

NOW

$$\frac{\lambda+3\mu}{\lambda+2\mu} = \frac{3-4\nu}{2(1-\nu)} \tag{44}$$

$$\frac{\lambda+\mu}{\lambda+2\mu} = \frac{1}{2(1-\nu)} \tag{45}$$

$$\therefore \quad \boxed{G_{km}(\underset{\sim}{x}) = \frac{1}{16\pi\mu(1-\nu)|\underset{\sim}{x}|}\left\{(3-4\nu)\delta_{km} + \frac{x_k x_m}{|\underset{\sim}{x}|^2}\right\}} \tag{46}$$

HENCE $G_{km}(\underset{\sim}{x}) = G_{mk}(\underset{\sim}{x})$ AND

$$G_{km}(\underset{\sim}{x}, \underset{\sim}{x}') = \frac{1}{16\pi\mu(1-\nu)|\underset{\sim}{x}-\underset{\sim}{x}'|}\left\{(3-4\nu)\delta_{km} + \frac{(x_k - x_k')(x_m - x_m')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^2}\right\}$$

$$* \quad |\underset{\sim}{x}-\underset{\sim}{x}'| = \left\{(x_1-x_1')^2 + (x_2-x_2')^2 + (x_3-x_3')^2\right\}^{1/2}$$

EXAMPLES OF THE USE OF $G_{km}(\underset{\sim}{x}, \underset{\sim}{x}')$

1. CONSTRUCTION OF THE 2-D LINE FORCE GREEN'S FUNCTIONS (WE FOUND THEM LAST QUARTER)

   CALL $g_{km}(x,y;x',y')$ THE DISPLACEMENT IN THE k^{th} DIRECTION AT (x,y) DUE TO A LINE OF FORCE OF UNIT STRENGTH / UNIT LENGTH APPLIED IN THE m^{th} DIRECTION AT (x',y').

CLEARLY

$$g_{km}(x,y;\ x',y') = \lim_{R\to\infty}\int_{-R}^{R} dz'\ G_{km}(x,y,z\ ;\ x',y',z').$$

WE NEED

$$I_1 = \lim_{R\to\infty}\int_{-R}^{R}\frac{dz'}{\{(x-x')^2+(y-y')^2+(z-z')^2\}^{1/2}}$$

AND

$$I_{km} = \lim_{R\to\infty}\int_{-R}^{R}\frac{(x_k-x_k')(x_m-x_m')\ dz'}{\{(x-x')^2+(y-y')^2+(z-z')^2\}^{3/2}}$$

NOW, CALLING $\rho^2 = (x-x')^2+(y-y')^2$

$$I_1 = \lim_{R\to\infty}\int_{-R}^{R}\frac{dz'}{\sqrt{\rho^2+(z-z')^2}}$$

OR

LETTING $z' = u + z$

$$I_1 = \lim_{R\to\infty}\int_{-R-z}^{R-z}\frac{du}{(\rho^2+u^2)^{1/2}}$$

$$= \lim_{R\to\infty}\left\{\int_0^{R-z}\frac{du}{\sqrt{\rho^2+u^2}} + \int_0^{R+z}\frac{du}{\sqrt{\rho^2+u^2}}\right\}$$

$$= \lim_{R\to\infty}\left\{\ln\frac{R-z+\sqrt{\rho^2+(R-z)^2}}{\rho} + \ln\frac{R+z+\sqrt{(R+z)^2+\rho^2}}{\rho}\right\}$$

$$= \lim_{R\to\infty}\left\{\ln\frac{2R}{\rho} + \ln\frac{2R}{\rho}\right\}$$

$$= \lim_{R\to\infty}\{2\ln 2R\} - 2\ln\rho$$

AN INFINITE CONSTANT $\Rightarrow$ RIGID TRANSLATION WHICH WE CAN DISCARD

FOR $K, m = 1$ OR 2 $(x_k - x_k')(x_m - x_m')$ IS CONSTANT AND CAN BE PULLED OUT OF THE INTEGRAL FOR I_{km}.

$[K, m = 1 \text{ OR } 2]$:

$$I_{km} = (x_k - x_k')(x_m - x_m') \lim_{R\to\infty} \int_{-R}^{R} \frac{dz'}{\{\rho^2 + (z-z')^2\}^{3/2}}$$

$$= (x_k - x_k')(x_m - x_m') \lim_{R\to\infty} \left\{ \int_0^{R-z} \frac{du}{(\rho^2+u^2)^{3/2}} + \int_0^{R+z} \frac{du}{(\rho^2+u^2)^{3/2}} \right\}$$

$$= (x_k - x_k')(x_m - x_m') \lim_{R\to\infty} \left\{ \frac{R-z}{\rho^2\sqrt{\rho^2+(R-z)^2}} + \frac{R+z}{\sqrt{\rho^2+(R+z)^2}\,\rho^2} \right\}$$

$$= \frac{2(x_k - x_k')(x_m - x_m')}{\rho^2}$$

THUS, FOR $K, m = 1$ OR 2 (DISCARDING THE INFINITE CONSTANT)

$$g_{km}(x, y\,;\,x', y') = -\frac{3-4\nu}{8\pi\mu(1-\nu)} \ln\rho + \frac{1}{8\pi\mu(1-\nu)} \frac{(x_k - x_k')(x_m - x_m')}{\rho^2}$$

THIS IS PRECISELY THE EQUATION FOR THE 2-D GREEN'S FUNCTIONS WE DERIVED LAST QUARTER.

NOW SUPPOSE $K = 1$ OR 2 AND $m = 3$ OR $K = 3$ AND $m = 1$ OR 2.

THEN g_{13}, g_{23}, g_{31}, AND g_{32} WILL BE PROPORTIONAL TO

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{(z-z')\,dz'}{\{\rho^2 + (z-z')^2\}^{3/2}} = J$$

OR

$$-J = \lim_{R\to\infty} \left\{ \int_0^{R-z} \frac{u\,du}{(\rho^2+u^2)^{3/2}} - \int_0^{R+z} \frac{u\,du}{(\rho^2+u^2)^{3/2}} \right\}$$

$$-J = \lim_{R\to\infty} \left\{ -\frac{1}{\sqrt{\rho^2+(R-z)^2}} + \frac{1}{\rho} + \frac{1}{\sqrt{\rho^2+(R+z)^2}} - \frac{1}{\rho} \right\} = 0$$

THUS $g_{13} = g_{23} = g_{31} = g_{32} = 0$

THE ONLY REMAINING TERM IS $g_{33}(x,y;x',y')$. APART FROM THE TERM PROPORTIONAL TO δ_{km} (WHICH WE HAVE ALREADY EVALUATED) THE REMAINING TERM IN g_{33} IS PROPORTIONAL TO

$$Q = \lim_{R\to\infty} \int_{-R}^{R} \frac{(z-z')^2\,dz'}{\{\rho^2+(z-z')^2\}^{3/2}} = \lim_{R\to\infty} \left\{ \int_0^{R-z} \frac{u^2\,du}{(\rho^2+u^2)^{3/2}} + \int_0^{R+z} \frac{u^2\,du}{(\rho^2+u^2)^{3/2}} \right\}$$

$$Q = \lim_{R\to\infty} \left\{ -\frac{R-z}{\sqrt{\rho^2+(R-z)^2}} + \ln\left\{R-z+\sqrt{\rho^2+(R-z)^2}\right\} - \ln\rho - \frac{R+z}{\sqrt{\rho^2+(R+z)^2}} + \ln\left\{R+z+\sqrt{\rho^2+(R+z)^2}\right\} - \ln\rho \right\}$$

OR

$$Q = \underbrace{-2}_{\text{CONSTANT– RIGID TRANSLATION}} - 2\ln\rho + \underbrace{\lim_{R\to\infty} 2\ln 2R}_{\text{INFINITE CONSTANT – RIGID TRANSLATION}}$$

HENCE, APART FROM A RIGID TRANSLATION (WHICH IS INFINITE)

$$g_{33}(x,y;x',y') = \frac{1}{16\pi\mu(1-\nu)} \left\{ (3-4\nu)(-2\ln\rho) - 2\ln\rho \right\}$$

$$= \frac{-2\left[4(1-\nu)\right]}{16\pi\mu(1-\nu)} \ln\rho$$

$$g_{33}(x,y\,;\,x',y') = -\frac{1}{2\pi\mu}\ln\rho$$ (JUST LIKE PROBLEM 2 ON FINAL EXAM)

A

Question	Score
1	20
2	20
3	20
4	
5	
6	
7	
8	
9	
10	
Total	60/60

STANFORD UNIVERSITY

OFFICIAL EXAMINATION BOOK

24 PAGE RULED

Name of Student CESAR LEVY

Date of Examination 1 JUNE 79

Subject ME 238 C ELASTICITY

HONORABLE CONDUCT

in academic work is the spirit of conduct in this University.

In recognition of and in the spirit of the Honor Code, I certify that I will neither receive nor give unpermitted aid on this examination and that I will report, to the best of my ability, all Honor Code violations observed by me.

(signed) Cesar Levy
Name

SUGGESTIONS FOR CONDUCT

1. Occupy alternate seats where possible.
2. When in doubt as to the meaning of a question, consult the instructor, who will be found in his or her office.

Due ~~Monday~~ Tuesday 12 Noon

DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING

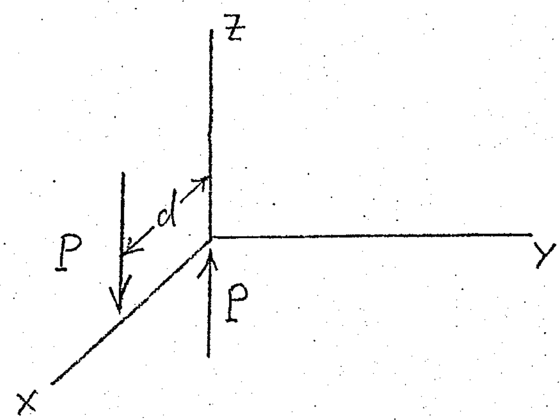
STANFORD UNIVERSITY

ME 238C Theory of Elasticity

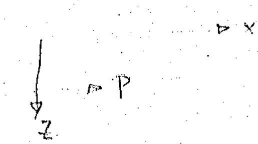
Spring 1979

Final Exam

1. Find the displacement field of an infinite body subjected to a couple of magnitude M at the origin, as shown.



2. Consider a semi-infinite elastic solid bounded by the plane $z = o$, the positive z-axis being directed into the body. A force P is applied at (0, 0, + c) and acts in the positive x-direction. Show that the displacement component u in the x-direction at the point (0, 0, 0) is given by

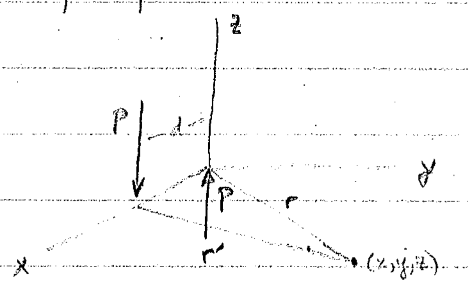


$$u_x(0,0,0) = \quad u = \frac{P(3 - 2\nu)}{8\pi\mu c}$$

where μ is the shear modulus and ν is Poisson's ratio.

3. A rigid sphere of radius R is impressed with a normal force N into an elastic half-space of shear modulus μ and Poisson's ratio ν. Determine the radius of contact, the maximum pressure at the interface and the settlement.

1. For the Kelvin Problem we had found that a force of magnitude P at the origin produced a displacement field $\mathbb{u} = \mathbb{B} - \frac{1}{4(1-\nu)}\nabla(\mathbb{r}\cdot\mathbb{B}+\beta)$ where $\mathbb{B} = \frac{\mathbb{P}}{4\pi\mu r}$ and $\beta = 0$ with: $\mathbb{P}$ is the vector describing the force and $r = \|\mathbb{r}\|$ is the distance from the origin to the field point. For a force located at the point $(d,0,0)$ we again find a $\mathbb{u}' = \mathbb{B}' - \frac{1}{4(1-\nu)}\nabla(\mathbb{r}'\cdot\mathbb{B}'+\beta')$ where $\mathbb{B}' = \frac{\mathbb{P}'}{4\pi\mu r'}$ and $\beta' = 0$ with: $\mathbb{P}'$ is the vector describing the force and $r' = \|\mathbb{r}'\|$ is the distance from the point of application of the force to the field point



$$\mathbb{u} = \frac{P e_z}{4\pi\mu r} - \frac{1}{4(1-\nu)}\nabla\left(\frac{zP}{4\pi\mu r}\right) \qquad r^2 = x^2+y^2+z^2, \quad \mathbb{r} = x e_x + y e_y + z e_z$$

$$\mathbb{u}' = \frac{-P e_z}{4\pi\mu r'} + \frac{1}{4(1-\nu)}\nabla\left(\frac{zP}{4\pi\mu r'}\right) \qquad r'^2 = (x-d)^2+y^2+z^2, \quad \mathbb{r}' = (x-d)e_x + y e_y + z e_z$$

The displacement field due to these two forces are

$$\mathbb{u} = \mathbb{u}+\mathbb{u}' = \frac{P e_z}{4\pi\mu}\left[\frac{1}{r} - \frac{1}{r'}\right] - \frac{P}{16\pi\mu(1-\nu)}\nabla\left[z\left(\frac{1}{r}-\frac{1}{r'}\right)\right]$$

if we let $d \to 0$ but keep $Pd = M$ fixed then using the taylor expansion we can find that

$$\frac{1}{r} - \frac{1}{r'} = d\frac{\partial}{\partial x}\left(\frac{1}{r'}\right)\bigg|_{d=0} - \frac{d^2}{2!}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r'}\right)\bigg|_{d=0} + \cdots$$

then the

$$\lim_{\substack{d\to 0\\ Pd = M}} \mathbb{u} = \frac{M e_z}{4\pi\mu}\left\{\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right\} - \frac{M}{16\pi\mu(1-\nu)}\nabla\left[z\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right]$$

or

$$\mathbb{u} = \frac{-M e_z}{4\pi\mu}\frac{x}{r^3} + \frac{M}{16\pi\mu(1-\nu)}\nabla\left[\frac{zx}{r^3}\right] \qquad \text{using } \frac{\partial}{\partial x}\frac{1}{r} = \frac{-x}{r^3}$$

now

$$\frac{\partial}{\partial x}\left(\frac{zx}{r^3}\right) = \frac{z}{r^3} + zx\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) = \frac{z}{r^3} - \frac{3zx^2}{r^5}$$

$$\frac{\partial}{\partial y}\left(\frac{zx}{r^3}\right) = zx\frac{\partial}{\partial y}\left(\frac{1}{r^3}\right) = \frac{-3zxy}{r^5}$$

$$\frac{\partial}{\partial z}\left(\frac{zx}{r^3}\right) = \frac{x}{r^3} + zx\frac{\partial}{\partial z}\left(\frac{1}{r^3}\right) = \frac{x}{r^3} - \frac{3z^2x}{r^5}$$

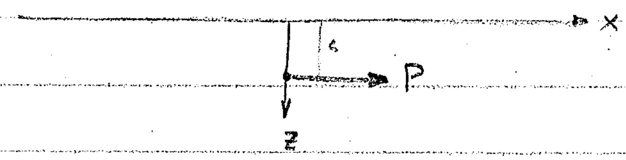
thus

$$u_x = +\frac{M}{16\pi\mu(1-\nu)}\frac{z}{r^3}\left[1 - \frac{3x^2}{r^2}\right]; \quad u_y = \frac{M}{16\pi\mu(1-\nu)}\left(\frac{-3xzy}{r^5}\right)$$

20/20

$$u_z = \frac{-Mx}{4\pi\mu r^3} + \frac{M}{16\pi\mu(1-\nu)}\left[\frac{x}{r^3} - \frac{3z^2x}{r^5}\right] = \frac{Mx}{16\pi\mu(1-\nu)r^3}\left[(4\nu-3) - \frac{3z^2}{r^2}\right]$$

2.



Show that $u_x(0,0,0) = \dfrac{P(3-2\nu)}{8\pi\mu c}$

The problem above was given as Mindlin's Problem Part II with the following

$$\mathbf{u} = \mathbf{B} - \frac{1}{4(1-\nu)}\nabla(\mathbf{r}\cdot\mathbf{B}+\beta) \quad \text{where} \quad \mathbf{B} = B_i \mathbf{e}_i \quad \text{and} \quad B_y = 0\ ;\ B_x = \frac{P}{4\pi\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

$$B_z = \frac{Px}{2\pi\mu R_2}\left[\frac{1-2\nu}{R_2+z+c} - \frac{c}{R_2^2}\right];\quad \beta = \frac{P(1-2\nu)x}{2\pi\mu(R_2+z+c)}\left[\frac{c}{R_2} - (1-2\nu)\right]$$

and $R_1^2 = x^2+y^2+(z-c)^2$ $\quad R_2^2 = x^2+y^2+(z+c)^2$ derived and defined

To get $u_x(x,y,z)$ we need $\quad u_x = B_x - \dfrac{1}{4(1-\nu)}\dfrac{\partial}{\partial x}\left[xB_x + zB_z + \beta\right]$

$$u_x = B_x - \frac{1}{4(1-\nu)}\left\{x\frac{\partial B_x}{\partial x} + B_x + z\frac{\partial B_z}{\partial x} + \frac{\partial\beta}{\partial x}\right\}$$

Now we need $\quad \dfrac{\partial B_x}{\partial x} = \dfrac{P}{4\pi\mu}\dfrac{\partial}{\partial x}\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right) = \dfrac{P}{4\pi\mu}\left[-\dfrac{x}{R_1^3} - \dfrac{x}{R_2^3}\right]$

$$\frac{\partial B_z}{\partial x} = \frac{P}{2\pi\mu R_2}\left[\frac{1-2\nu}{R_2+z+c} - \frac{c}{R_2^2}\right] - \frac{Px^2}{2\pi\mu R_2^3}\left[\frac{1-2\nu}{R_2+z+c} - \frac{c}{R_2^2}\right] + \frac{Px}{2\pi\mu R_2}\left[-\frac{1-2\nu}{(R_2+z+c)^2}\frac{x}{R_2} + 2\frac{cx}{R_2^4}\right]$$

$$\frac{\partial\beta}{\partial x} = \frac{P(1-2\nu)}{2\pi\mu(R_2+z+c)}\left[\frac{c}{R_2} - (1-2\nu)\right] - \frac{P(1-2\nu)x^2}{2\pi\mu(R_2+z+c)^2 R_2}\left[\frac{c}{R_2} - (1-2\nu)\right] + \frac{P(1-2\nu)x}{2\pi\mu(R_2+z+c)}\left[-\frac{c}{R_2^3}x\right]$$

Now at $x=0, y=0, z=0$ $\quad R_1^2 = c^2 = R_2^2 \Rightarrow$ then $\left.\dfrac{\partial B_x}{\partial x}\right|_{x=y=z=0} = 0$;

$$\left.\frac{\partial B_z}{\partial x}\right|_{x=y=0} = \frac{P}{2\pi\mu c}\left[\frac{1-2\nu}{2c} - \frac{1}{c}\right] = \frac{-P(1+2\nu)}{4\pi\mu c^2};\quad \frac{\partial\beta}{\partial x} = \frac{P(1-2\nu)}{4\pi\mu c}[2\nu]$$

$$B_x = \frac{P}{4\pi\mu}\left(\frac{2}{c}\right);\ \Rightarrow u_x = \frac{P}{2\pi\mu c} - \frac{1}{4(1-\nu)}\Big[\underbrace{0\cdot 0}_{x\frac{\partial B_x}{\partial x}} + \underbrace{\frac{P}{2\pi\mu c}}_{B_x} + \underbrace{0\cdot\cancel{\frac{-P(1+2\nu)}{4\pi\mu c^2}}}_{z\frac{\partial B_z}{\partial x}} + \underbrace{\frac{P(1-2\nu)\nu}{2\pi\mu c}}_{\partial\beta/\partial x}\Big]$$

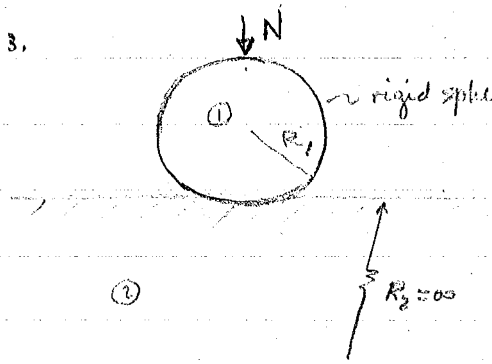
$$u_x = \frac{P}{2\pi\mu c} - \frac{1}{4(1-\nu)}\left[\frac{P}{2\pi\mu c}\left\{1 + (1-2\nu)\nu\right\}\right] = \frac{P}{2\pi\mu c}\left\{1 - \frac{1}{4(1-\nu)}\left[(1-\nu)(1+2\nu)\right]\right\}$$

$$= \frac{P}{2\pi\mu c}\left\{1 - \frac{1+2\nu}{4}\right\} = \frac{P}{2\pi\mu c}\,\frac{3-2\nu}{4} = \frac{(3-2\nu)P}{8\pi\mu c}$$

QED

20/20

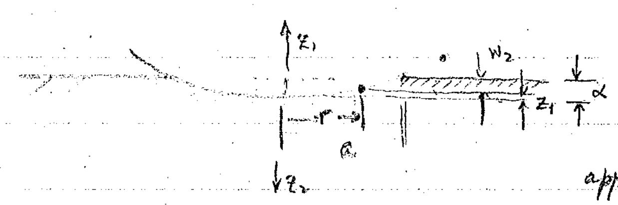
3.



~ rigid sphere implies no settlement here in the sphere ie $w_1 = 0$

$w_2 \neq 0$

μ, ν $\quad \frac{1}{R_2} = 0 \implies z_2 = 0$



If we assume that the contact area is circular with radius $a << R_1$ or R_2 and N is applied remotely and doesn't affect contact problem then

(1) $\boxed{w_2 = \alpha - z_1 = \alpha - \frac{r^2}{2R_1}}$ as shown in class. Now since w_2 is the settlement then

(2) $\boxed{w_2 = \frac{1-\nu}{2\pi\mu}\int_{-\pi/2}^{\pi/2}\int_{-a}^{c} p_z(v,\psi)\,dv\,d\psi}$. In writing this we are assuming that the contact area is flat and to a first approximation this is correct. Now since w_2 is proportional to r^2

we can use the 3rd case of normal circular loads studied in class ie

(3) $\boxed{p_z = \frac{3N\,(a^2-r^2)^{1/2}}{2\pi a^3}} \implies \boxed{w_2 = \frac{3(1-\nu)N}{16a^3\mu}(2a^2-r^2)}$ ✓ (4)

thus equating (4) and (1) we find that $\alpha = \frac{3(1-\nu)N\cdot 2a^2}{16a^3\mu} = \frac{3(1-\nu)N}{8a\mu}$

and $\frac{1}{2R_1} = \frac{3(1-\nu)N}{16a^3\mu} \implies \boxed{a = \left(\frac{3(1-\nu)NR_1}{8\mu}\right)^{1/3}}$ ✓ (5)

to find the maximum pressure we go to equation (3) and find that when

$r = 0$ $\boxed{p_z = p_{z_{max}} = \frac{3N}{2\pi a^2}}$ ✓ or if $\exists$ a $p_{z\,average} = \frac{N}{\pi a^2}$ then $p_{z_{max}} = \frac{3}{2}p_{z_{av}}$

SUMMARY

- thus knowing R_1, μ, ν, N we can find the radius of the contact circle $a = \left[\frac{3(1-\nu)NR_1}{8\mu}\right]^{1/3}$
- then knowing μ, ν, N and a we can find the settlement $w_2 = \frac{3(1-\nu)N}{16a^3\mu}(2a^2-r^2)$
- then knowing N and a we find that at $r=0$ $p_z = p_{z_{max}} = 3N/2\pi a^2$ at the interface

STANFORD UNIVERSITY

OFFICIAL EXAMINATION BOOK

24 Page Ruled

Question	Score
1	18/20
2	12/20
3	14/20
4	
5	
6	
7	
8	
Total	44/60

read the question

$\frac{44}{60} \cdot 1\frac{2}{3} = 73$

Name of student: Cesar Levy

Date of examination: 11 May 1979

Course: ME 238 C

THE STANFORD UNIVERSITY HONOR CODE

A. The Honor Code is an undertaking of the students, individually and collectively:

(1) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

(2) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

I acknowledge and accept the Honor Code.

(Signed) Cesar Levy

Interpretations and applications of the Honor Code appear on the back cover of this examination book.

$\frac{x}{r} = \sin\varphi \cos\theta$

$\longrightarrow \quad \sigma_{RR} = \frac{-3 P_x \sin\varphi \cos\theta}{4\pi r^2}$

$\sigma_{RR} = 0$ when $\varphi = 0$ or $\theta = 90^\circ$

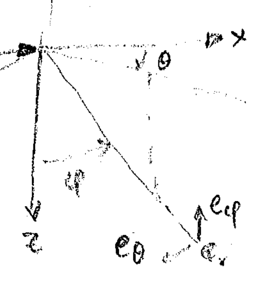
what I didn't do!!

$\therefore$ tractions on $z=0$ normal to $z=0$ is e_φ

$\Rightarrow e_\varphi \cdot \sigma = \sigma_{\varphi r} e_r + \sigma_{\varphi\theta} e_\theta + \sigma_{\varphi\varphi} e_\varphi$

but $\sigma_{\varphi r} = \sigma_{\varphi\theta} = \sigma_{\varphi\varphi} = 0$ everywhere

thus on plane of applied load traction is zero, & since load is tangential this is Cerruti problem!!



Cesar Levy

DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING
STANFORD UNIVERSITY

ME 238C Theory of Elasticity Spring 1979

Midterm Examination

1. Consider the solution of the Kelvin problem and show that if Poisson's ratio is 1/2, then this solution also provides the solution to the Boussinesq and Cerruti problems.

   (Hint: evaluate components of stress in spherical coordinates)

2. In the Boussingesq problem, calculate and sketch the settlement, as well as the displacement of points along the line of action of the force in the direction of the force.

3. An elastic sphere of radius b contains a fixed spherical inclusion of radius a and is subjected to a uniform external pressure p. Find the traction on the surface of the inclusion.

18/20

1.

From HW #1 we found $\sigma_{\varphi r} = \frac{P\sin\varphi}{8\pi r^2}\frac{(1-2\nu)}{1-\nu}$ $\quad \sigma_{\theta\varphi} = \sigma_{r\theta} = 0$

$$\sigma_{rr} = \frac{P\cos\varphi}{4\pi r^2}\left(\frac{\nu-2}{1-\nu}\right) \qquad \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \frac{P\cos\varphi}{8\pi r^2}\left(\frac{1-2\nu}{1-\nu}\right)$$

for Kelvin problem.

Now with $\nu = 1/2$ $\quad \sigma_{\varphi r} = \sigma_{\theta\varphi} = \sigma_{r\theta} = \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = 0$

$$\sigma_{rr} = \frac{P\cos\varphi}{4\pi r^2}\left[-3\right] = \frac{-3P\cos\varphi}{4\pi r^2} \qquad \sigma_{rr} = 0 \text{ when } \varphi = 90^\circ$$

~~2. Since $B_z = \frac{1-\nu}{\pi}\frac{P_z}{R}$ and $\beta = \frac{(1-2\nu)(1-\nu)}{\pi\mu}P_z \ln(R+z)$~~

~~$\sigma_{11} = e_1\cdot\sigma\cdot e_1 = \cos^2\theta\sin^2\varphi\,\sigma_{RR} = -3P$~~

$\sigma_{RR} = \frac{-3Pz}{4\pi r^3}$ note that since problem in class was taken along z direction we find $P = P_z$.

~~where P is in any direction since $\sigma_{RR} = 0$ on $z = 0$~~

~~we recover~~

~~Since for boussinesq $B_x, B_y = 0$ $\quad B_z = \frac{1-\nu}{\pi\mu}\frac{P_z}{R}$ $\quad \beta = \frac{(1-2\nu)(1-\nu)P_z}{\pi\mu}\ln(R+z)$.~~

~~for $\nu = 1/2$ $\quad B_z = \frac{P_z}{2\pi\mu R}$ $\quad \beta = 0$~~

Kelvin problem to rigid body rotation if $P_x, P_y = 0$ $\quad B_z = \frac{P_z}{4\pi\mu r}$ $\quad \beta = 0$

when $z=0$ except at $r=0$ $\sigma_{RR} = 0$ but all other stresses are zero on plane $z=0$ this is same as boussinesq problem i.e. point load in z dir on a half space

why is this equivalent to cerruti?

on what plane does it give zero tractions?

-2

if we took Kelvin problem in x direct we would have found

$\sigma_{RR} = \frac{-3Px}{4\pi r^3}$ & all other $\sigma_{ij} = 0$ where $P = P_x$

this would have been equivalent to Cerruti problem i.e. tangential load on a half space

$\frac{12}{20}$

2. a. $\cancel{B_z} =$ $w = u_z(x,y,0)$

$$\mathbf{u} = B_z \mathbf{e}_z - \frac{1}{4(1-\nu)} \nabla(\mathbf{r}\cdot\mathbf{B} + \beta)$$

$$= \frac{1-\nu}{\cancel{4}\pi\mu}\frac{P_z}{R}\mathbf{e}_z - \frac{1}{4(1-\nu)}\nabla\left(\frac{1-\nu}{\pi\mu}\frac{P_z \cdot z}{R} + \frac{(1-2\nu)(1-\nu)}{\pi\mu}P_z \ln(R+z)\right)$$

$$= \frac{1-\nu}{\pi\mu}\frac{P_z \mathbf{e}_z}{R} - \frac{P_z}{4\pi\mu}\nabla\left(\frac{z}{R} + (1-2\nu)\ln(R+z)\right)$$

$$\nabla\left(z/R + (1-2\nu)\ln(R+z)\right) = z\nabla\left(\tfrac{1}{R}\right) + \frac{\mathbf{e}_z}{R} + (1-2\nu)\nabla \ln R+z)$$
$$= -z\frac{\mathbf{R}}{R^3} + \frac{\mathbf{e}_z}{R} + (1-2\nu)\frac{\partial}{\partial x}\left[\frac{1}{R+z} \quad \text{etc} \quad\right]$$

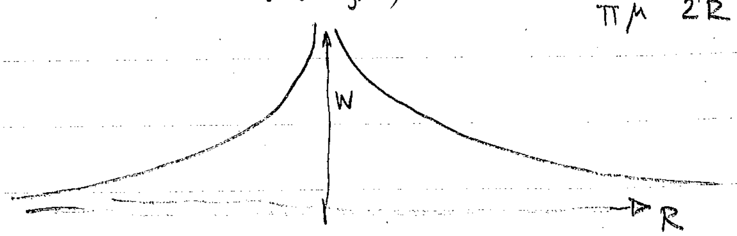
for $u_z(x,y,z) = \frac{1-\nu}{\pi\mu}\frac{P_z}{R} - \left(-\frac{z^2}{R^3} + \frac{1}{R} + (1-2\nu)\cancel{\frac{1}{R+z}}\cdot\left(\cancel{\frac{z}{R}+1}\right)\frac{1}{R}\right)\frac{P_z}{4\pi\mu}$

$$= \frac{1-\nu}{\pi\mu}\frac{P_z}{R} - \frac{P_z}{4\pi\mu}\left(\frac{(1-2\nu)}{R} + \frac{1}{R} - \frac{z^2}{R^3}\right)$$

$$\left(\frac{2(1-\nu)}{R} - z^2/R^3\right)$$

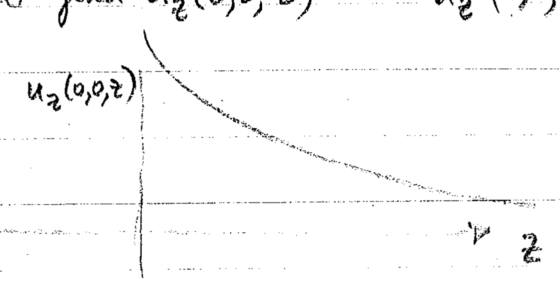
$$= \frac{(1-\nu)P_z}{\pi\mu}\left[1/2R\right] - \frac{P_z}{4\pi\mu}\left(-z^2/R^3\right)$$

@ $z=0$ $\quad u_z(x,y,0) = w = \frac{(1-\nu)}{\pi\mu}\frac{P_z}{2R}$ ✓ $\quad R = \sqrt{x^2+y^2}$



along line of action
-8

also find $u_z(0,0,z)$ $\quad u_z(0,0,z) = \frac{(1-\nu)P_z}{\pi\mu}\left[\frac{1}{2z}\right] - \frac{P_z}{4\pi\mu}\left[-\frac{1}{z}\right]$



$$= \frac{P_z}{\pi\mu}\left\{\frac{2(1-\nu)}{2\cdot 2z} + \frac{1}{4z}\right\}$$

$$\frac{P_z}{\pi\mu}\left\{\frac{3-2\nu}{4z}\right\}$$

what I didn't do !!

$$+p_i = \frac{(p_i - p)(a^3 b^3)}{(a^3 - b^3)\cancel{ab}} - \frac{b^3 p - a^3 p_i}{a^3 - b^3}$$

$$= \cancel{(p_i - p)}\,\cancel{b^3} \qquad p_i\left(\cancel{\frac{b^3 - a^3}{a^3 - b^3}}\right)^{-1} + p\left(\cancel{\frac{b^3}{a^3 - b^3}}\right)^{+1} = 0$$

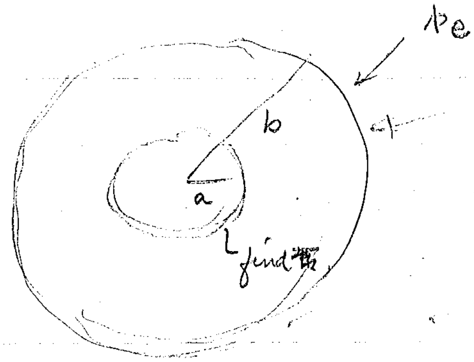
$$+p_i \cancel{= -p_i - p}$$

$$+2p_i = -p \qquad \boxed{p_i = \cancel{-p}} = 0$$

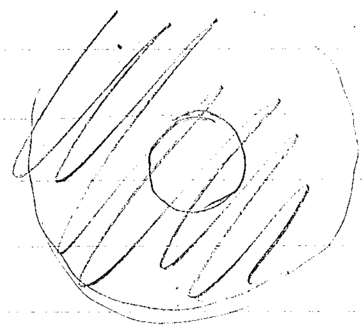
I can't find the sign somewhere

14/20

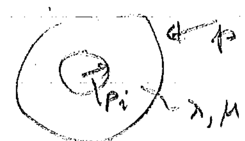
3.



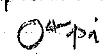
Since it's an inclusion $u|_{sphere_a} = u|_{sphere_b}$ at interface $=0$ rigid inclusion
tractions are continuous also



let use use ~~of~~ concentric sphere soln.



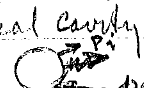
+ spherical cavity



$\lambda^*, \mu^* = \infty$

$\therefore$ u_i are continuous across bound. tractions are contin. across bdry

Spherical cavity



$u_R = \frac{p_i a^3}{4\mu R^2}$ $\quad u_\theta = u_\phi = 0$

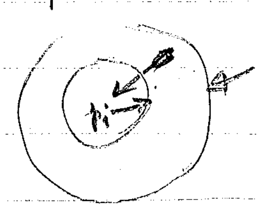
$T_r = \sigma_{rr}$ $\qquad T_r$

$T_\theta = \sigma_{\theta r}$

$T_\phi = \sigma_{\phi r}$

$\sigma_{RR} = +p_i \frac{a^3}{R^3}$ $\qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{p_i a^3}{2R^3}$ $\qquad \sigma_{\theta\phi} = \sigma_{\theta R} = \sigma_{\phi R} = 0$

Sphere



~~$\sigma_{RR}\Big|_{r=a} = (p_i - p)\frac{a^3 b^3}{(a^3 - b^3)a^3} + \frac{b^3 p - a^3 p_i}{a^3 - b^3} = -p_i$~~

~~$= (p_i$~~

$$\sigma_{RR} = \frac{(p_i - p)a^3 b^3}{(a^3 - b^3)R^3} + \frac{b^3 p - a^3 p_i}{a^3 - b^3}$$

$u_r = ?$

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} = \frac{a^3 b^3 (p - p_i)}{2(a^3 - b^3)R^3} + \frac{b^3 p - a^3 p_i}{a^3 - b^3}$$

equate $u_r|_{incl}$ to $u_r|_{sph.}$

$T_r = -\sigma_{rr}$

$T_\theta = -\sigma_{R\theta}$

$T_\phi = -\sigma_{R\phi}$

$\therefore$ I expect symmetry $\Rightarrow$

$$+p_i = \frac{(p_i - p)(a^3 b^3)}{(a^3 - b^3)R^3} + \frac{b^3 p - a^3 p_i}{a^3 - b^3}$$

$$-S(1+\nu)(ab^3) - a^4S(2-4\nu)$$

$$pab^3(1+\nu) + pab^3(2-4\nu) = 3(1-\nu)pab^3$$

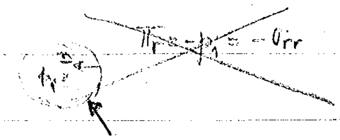
u_r of cavity$\Big|_{r=a}$ = u_r of hollow sphere$\Big|_{r=a}$

and T_r are continuous across boundary in this case $T_r = \sigma_{rr}$ only

$T_\theta = 0 \quad T_\varphi = 0$

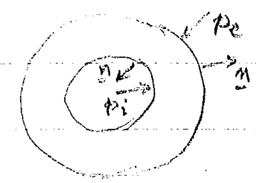
~~for a spherical cavity under p_i we showed~~

$$\cancel{u_R = -\frac{p_i a^3}{4\mu R^2} = -\frac{S a^3}{4\mu R^2}} \quad \text{~~and~~} \quad \cancel{\sigma_{rr} = +p_i \frac{a^3}{R^3} = \frac{S a^3}{R^3}}$$



for a hollow sphere with $\overset{\sigma_{rr}|_{r=a}}{p_i = -S}$ and $\overset{\sigma_{rr}|_{r=e}}{p_e = p}$

$$\sigma_{rr} = (S - p)\frac{a^3 b^3}{(a^3 - b^3)R^3} + \frac{b^3 p - a^3 S}{a^3 - b^3}$$



now σ_{rr} on $r = a$ $\quad \sigma_{rr} = (S - \cancel{p})\frac{b^3}{a^3 - b^3} + \frac{\cancel{b^3 p} - a^3 S}{a^3 - b^3} = -S$ ✓

Now we will find the displacement of $r = a$ in the hollow sphere

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{\sigma_{rr}}{E} - \frac{\nu}{E}(\sigma_{\theta\theta} + \sigma_{\varphi\varphi})$$

with $\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \dfrac{a^3 b^3 (p - S)}{2(a^3 - b^3)R^3} + \dfrac{b^3 p - a^3 S}{a^3 - b^3}$

$$\therefore \sigma_{\theta\theta} + \sigma_{\varphi\varphi} = \frac{a^3 b^3 (p - S)}{(a^3 - b^3)R^3} + 2\left(\frac{b^3 p - a^3 S}{a^3 - b^3}\right)$$

$$\therefore \varepsilon_{rr} = \frac{(S - p)}{E}\frac{a^3 b^3}{(a^3 - b^3)R^3} + \frac{b^3 p - a^3 S}{E(a^3 - b^3)} - \frac{\nu}{E}\frac{a^3 b^3 (p - S)}{(a^3 - b^3)R^3} - \frac{2\nu}{E}\left(\frac{b^3 p - a^3 S}{a^3 - b^3}\right)$$

$$\frac{\partial u_r}{\partial r} = \frac{S - p}{E}\frac{a^3 b^3}{(a^3 - b^3)R^3}\left[1 + \nu\right] + \frac{b^3 p - a^3 S}{E(a^3 - b^3)}\left[1 - 2\nu\right]$$

$$\therefore u_r = \frac{S - p}{E}\frac{a^3 b^3}{(a^3 - b^3)R^2}\left[\frac{1 + \nu}{-2}\right] + \frac{b^3 p - a^3 S}{E(a^3 - b^3)}\left[1 - 2\nu\right]R + C$$

since problem is symmetric all around C is a part const. = 0 since it is rigid body translation

$$u_r\Big|_{r=a} = \frac{S - p}{E}\left[\frac{a b^3}{a^3 - b^3}\right]\left(\frac{1 + \nu}{-2}\right) + \frac{a b^3 p - a^4 S}{E(a^3 - b^3)}(1 - 2\nu)$$

$$= -\frac{S - p}{4\mu}\left(\frac{a b^3}{a^3 - b^3}\right)\frac{(1 + \nu)}{(1 + \nu)} + \frac{a b^3 p - a^4 S}{a^3 - b^3}\,\frac{2 - 4\nu}{4\mu(1 + \nu)} = \frac{3(1 - \nu) p a b^3 - S\left[(1 + \nu) a b^3 + a^4 (2 - 4\nu)\right]}{4\mu(1 + \nu)(a^3 - b^3)}$$

for rigid inclusion $u_r\big|_{r=a} = 0$

$$u_r = \frac{3pab^3(1-\nu) - S\left[ab^3(1+\nu) + a^4(2-4\nu)\right]}{4\mu\,(1+\nu)(a^3+b^3)} = 0$$

$$\therefore \quad \frac{3pab^3(1-\nu)}{4\mu(1+\nu)(a^3-b^3)} = \frac{+\;S\left[ab^3(1+\nu) + a^4(2-4\nu)\right]}{4\pi\,(1+\nu)(a^3-b^3)}$$

$$\therefore \quad S = \frac{3pab^3(1-\nu)}{ab^3(1+\nu) + 2a^4(1-2\nu)}$$

ME 220C   Midterm   Spring 1979

1\. Boussinesq Problem: Normal Point Load on a half space

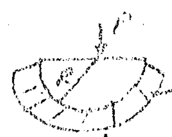
Pts Cerruti Problem: Shear point load on a half space

Show Kelvin gives solution to Boussinesq & Cerruti problems

if $\nu = .5$

5 Bous — This will be true if the Kelvin problem gives no traction

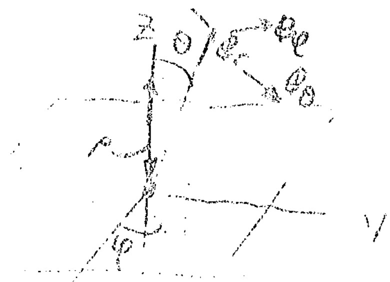
5 Cerr in a plane in which the load lies and in a plane normal to

the load. Also for Boussinesq  $\int$ traction $= P$ Cerruti $P = \int$ traction 

— In spherical coordinates, referring to prob. 6 of HW1 and setting

$\nu = \frac{1}{2}$, $\quad \sigma_{R\varphi} = \sigma_{\theta\varphi} = \sigma_{R\theta} = \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = 0$

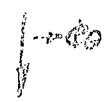
5 $\quad \sigma_{RR} = \dfrac{-3P\cos\theta}{4\pi R^2}$



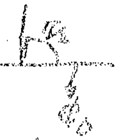
The traction on a plane whose normal

is $\vec{e}_\theta$ is $\vec{e}_\theta \cdot \sigma = \sigma_{\theta R}\vec{e}_R + \sigma_{\theta\theta}\vec{e}_\theta + \sigma_{\theta\varphi}\vec{e}_\varphi$

5

Taking $\theta = 0$,  This plane contains the load

and therefore we have proven the Cerruti problem is solved

Taking $\theta = \frac{\pi}{2}$ , This plane is normal to the load

and we've proven the Boussinesq problem is solved.

(It is easily verified that the integrated traction conditions are also satisfied in both cases)

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HS 2. Boussinesq problem. a) settlement $= u_z\big|_{z=0}$

Given in notes: $B_z = \frac{1-\nu}{\pi\mu R} P$, $B_x = B_y = 0$

$$\beta = \frac{(1-2\nu)(1-\nu)}{\pi\mu} P \ln(R+z)$$

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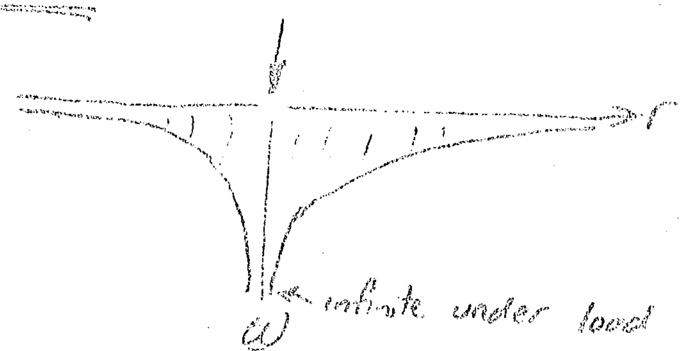
$$u = B - \frac{\nabla}{4(1-\nu)}\left[x\cdot B + \beta\right] = B_z e_z - \frac{1}{4(1-\nu)}\nabla\left(zB_z + \beta\right)$$

$$u_z = B_z - \frac{1}{4(1-\nu)}\frac{\partial}{\partial z}(zB_z+\beta) = B_z - \frac{1}{4(1-\nu)}\left(B_z + z\frac{\partial B_z}{\partial z} + \frac{\partial \beta}{\partial z}\right)$$

$$= \underline{\underline{\frac{P}{4\pi\mu}\left[\frac{3-4\nu}{R} + \frac{z^2}{R^3} - \frac{1-2\nu}{R}\right]}} = \frac{P}{4\pi\mu}\left[\frac{2(1-\nu)}{R} + \frac{z^2}{R^3}\right]$$

settlement $w = u_z\big|_{z=0}$

$$= \underline{\underline{\frac{P}{2\pi\mu}\frac{(1-\nu)}{r}}} \quad \text{where } r = \sqrt{x^2+y^2}$$



b) on line of force (z-axis) , x=y=0 , R=z

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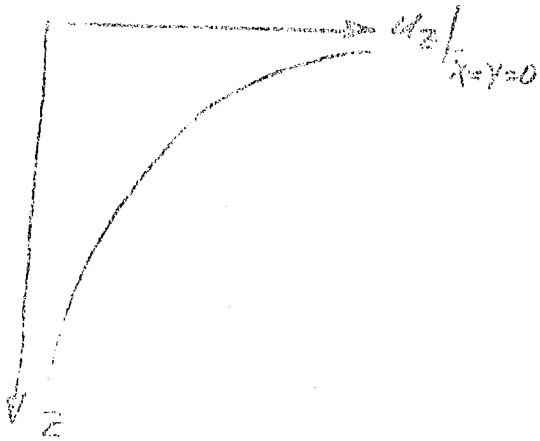
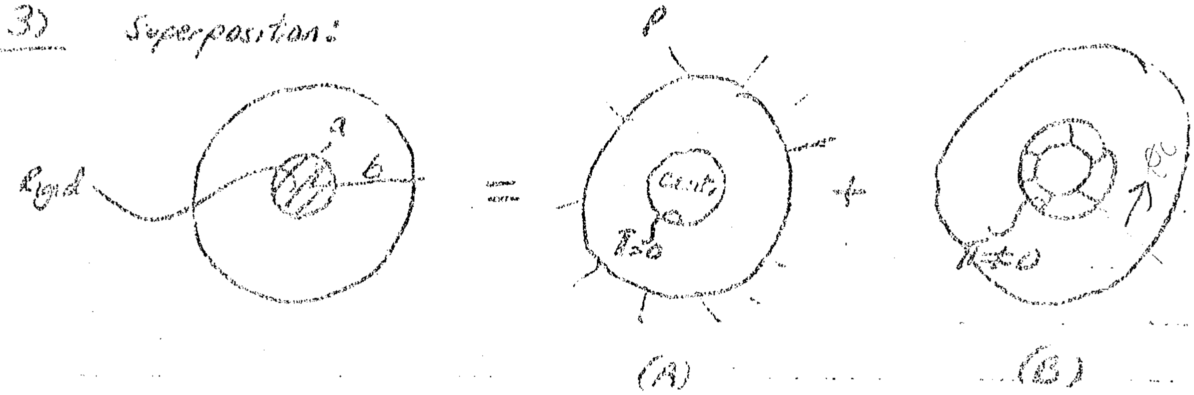
$$\Rightarrow \underline{\underline{u_z = \frac{P}{4\pi\mu}\left(\frac{3-2\nu}{z}\right)}}$$



Pts

3) Superposition:



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Interface condition: $u_A + u_B \big|_{r=a} = 0$

A.) determine displacement at $r=a$ if no inclusion were present

In notes it is shown solution to spherical cavity under internal and external pressure is given by superposition of center of dilation of strength A and hydrostatic tension B, where A, B were found to be

$$A = \frac{(1-\nu)\, a^3 b^3 (P_e - P_i)}{\mu(a^3-b^3)} \qquad B = \frac{b^3 P_e - a^3 P_i}{a^3-b^3}$$

5 displacement: Center of dilatation $u_r = \frac{A}{4(1-\nu)R^2}$ $u_\theta = u_\varphi = 0$

Hydrostatic tension $\varepsilon_{rr} = \frac{1}{E}(\sigma_r - \nu(\sigma_\theta + \sigma_\varphi)) = \frac{B}{E}(1-2\nu) = \frac{B(1-2\nu)}{2(1+\nu)\mu}$
$= \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi}$

Integrate strain-displacement relation $\Rightarrow$ $u_r = \frac{B(1-2\nu)}{2(1+\nu)\mu} R$ $u_\theta = u_\varphi = 0$

superposing, $u_r = \frac{A}{4(1-\nu)R^2} + \frac{B(1-2\nu)}{2(1+\nu)\mu} R$

Subst. for A & B their values when $P_e = P$, $P_i = 0$

$$\Rightarrow u_r = \frac{a^3 b^3 P}{4\mu(a^3-b^3)R^2} + \frac{(1-2\nu)\, b^3 P}{2\mu(1+\nu)\, a^3 - b^3} R$$

$$u_r\big|_{r=a} = \frac{a\, b^3 P}{4\mu(a^3-b^3)} + \frac{(1-2\nu)\, a b^3 P}{\ldots} = \frac{3Pab^3(1-\nu)}{\ldots(a^3-b^3)}$$

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B) determine traction at $r=a$ needed to push the interface back to where it belongs

for $P_2=0$, $P_1 = -T_r$; This is the spherical cavity in infinite space

5 $$u_r\big|_{r=a} = \frac{a b^3 T_r}{4\mu(a^3-b^3)} + \frac{T_r a^4(1-2\nu)}{(a^3-b^3)2\mu(1+\nu)}$$

$$= T_r\left[\frac{ab^3(1+\nu)+2a^4(1-2\nu)}{4\mu(1+\nu)(a^3-b^3)}\right]$$

n $\Pi_r = -T_r = -\sigma_r$

$$= \frac{3Pab^3(1-\nu)}{4\mu(1+\nu)(a^3-b^3)}$$

$$\boxed{T_r = \frac{3Pab^3(1-\nu)}{ab^3(1+\nu)+2a^4(1-2\nu)}} = \frac{3Pb^3(1-\nu)}{b^3(1+\nu)+2(1-2\nu)a^3}$$

and $T_\theta = T_\phi = 0$

$\sigma_{\theta\theta}$ $\sigma_{\phi\phi}$ σ_{rr}

$\sigma_{\theta\theta}$ $\sigma_{\phi\phi}$

e_r

Π^θ

$T_\theta = e_\theta\cdot\Pi_r = e_\theta\cdot\sigma\cdot e_r$

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ME 238C Theory of Elasticity

Problem Set No. 1

1. Evaluate the Cartesian components of stress for the Kelvin problem.

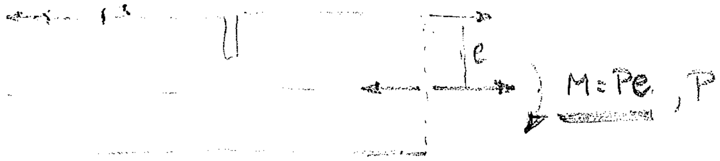
2. Evaluate the components of traction on a spherical surface of arbitrary radius with its origin at the point at which the concentrated force is acting in the Kelvin problem.

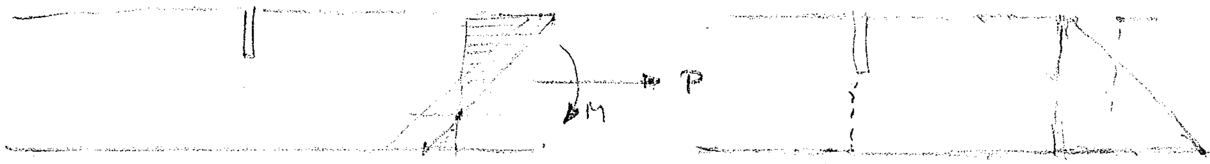
3. Show that the system of tractions evaluated in (2) above is statically equivalent to a single force.

4. Evaluate the normal strain along the line of action of the force in the Kelvin problem. What does this mean

5. Evaluate the components of displacement in spherical coordinates for the Kelvin problem.

6. Evaluate the spherical components of stress for the Kelvin problem and show in particular that the components $\sigma_{\theta\phi}$ and $\sigma_{\phi r}$ vanish.





$$\nu = \frac{\lambda}{2(\lambda+\mu)} \qquad 1-\nu = \frac{2(\lambda+\mu)-\lambda}{2(\lambda+\mu)} = \frac{\lambda+2\mu}{2(\lambda+\mu)} \qquad \frac{1}{4(1-\nu)} = \frac{\cancel{2}(\lambda+\mu)}{2\cancel{4}(\lambda+2\mu)}$$

$$1 - \frac{1}{4(1-\nu)} = \frac{2(\lambda+2\mu)-(\lambda+\mu)}{2(\lambda+2\mu)} = \frac{\lambda+3\mu}{2(\lambda+2\mu)}$$

$$\therefore \quad \frac{P_i}{8\pi\mu r}\,\frac{\lambda+3\mu}{(\lambda+2\mu)} \qquad \text{if } P_1 = 0 = P_2 \quad P_3 = P_3$$

$$\therefore \quad u_i = \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\,\frac{P_3}{r}\,\delta_{i3} + \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\,\frac{x_3 x_i}{r^3}\,P_3 \qquad \text{let } \frac{(\lambda+\mu)P}{8\pi\mu(\lambda+2\mu)}$$

$$u_i = c\left[\frac{(\lambda+3\mu)}{(\lambda+\mu)}\,\frac{\delta_{i3}}{r} + \frac{x_3 x_i}{r^3}\right] \qquad \frac{\partial}{\partial x_k}\left(\frac{1}{r}\right) = -\frac{1}{r^3}x_k \qquad \frac{\partial r}{\partial x_k} = \frac{x_k}{r} \quad \frac{2x_k}{2r}$$

$$u_{i,k} = c\left[\frac{\lambda+3\mu}{(\lambda+\mu)}\,\delta_{i3}\left(\frac{-x_k}{r^3}\right) + \frac{\delta_{3k}x_i}{r^3} + \frac{x_3\delta_{ik}}{r^3} - \frac{3}{r^5}x_3x_ix_k\right]$$

$$u_{k,i} = c\left[\frac{\lambda+3\mu}{\lambda+\mu}\,\delta_{k3}\left(\frac{-x_i}{r^3}\right) + \frac{\delta_{3i}x_k}{r^3} + \frac{x_3\delta_{ki}}{r^3} - \frac{3x_3x_ix_k}{r^5}\right]$$

$$\text{add} = c\left\{\frac{\lambda+3\mu}{\lambda+\mu}\left[-\frac{x_k\delta_{i3}}{r^3} - \frac{x_i\delta_{k3}}{r^3}\right] + \left[\frac{\delta_{3k}x_i}{r^3} + \frac{\delta_{3i}x_k}{r^3} + \frac{2x_3\delta_{ik}}{r^3} - \frac{6x_3x_ix_k}{r^5}\right]\right\}$$

$$u_{i,i} = c\left[\frac{\lambda+3\mu}{\lambda+\mu}\,\delta_{i3}\left(-\frac{x_i}{r^3}\right) + \frac{\delta_{3i}x_i}{r^3} + \frac{\overset{3}{\delta_{ii}}x_3}{r^3} - \frac{3x_3x_i\,x_i}{r^5}\right]$$

$$= c\left[\frac{\lambda+3\mu}{\lambda+\mu}\left\langle-\frac{\delta_{i3}x_i}{r^3}\right\rangle + \frac{\delta_{3i}x_i}{r^3} + \cancel{\frac{3x_3}{r^3}} - \cancel{\frac{3x_3r^2}{r^5}}\right]$$

$$= c\left[\frac{\delta_{3i}x_i}{r^3}\left\{1 - \frac{\lambda+3\mu}{\lambda+\mu}\right\}\right] = c\left[\frac{-2\mu}{\lambda+\mu}\right]\frac{\delta_{3i}x_i}{r^3}$$

$$\therefore \quad \sigma_{ij} = c\lambda\left(\frac{-2\mu}{\lambda+\mu}\right)\left[\frac{\delta_{3k}x_k}{r^3}\right]\delta_{ij} + \mu c\left\{\frac{\lambda+3\mu}{\lambda+\mu}\left\{\frac{-x_j\delta_{i3} - x_i\delta_{j3}}{r^3}\right\} + \left[\frac{\delta_{3j}x_i}{r^3} + \frac{\delta_{3i}x_j}{r^3} + \frac{2x_3\delta_{ij}}{r^3} - \frac{6x_3x_ix_j}{r^5}\right]\right.$$

$$= c\left\{\frac{x_3}{r^3}\left[\underbrace{\frac{-2\lambda\mu}{\lambda+\mu} + 2\mu}_{\frac{2\mu^2}{\lambda+\mu}}\right]\delta_{ij} - \frac{x_j\delta_{i3}}{r^3}\left[\underbrace{\frac{\mu\lambda+3\mu^2}{\lambda+\mu} - \mu}_{\frac{2\mu^2}{\lambda+\mu}}\right] - \frac{x_i\delta_{j3}}{r^3}\left[\underbrace{\frac{\mu\lambda+3\mu^2}{\lambda+\mu} - \mu}_{\frac{2\mu^2}{\lambda+\mu}}\right] - \mu\frac{6x_3x_ix_j}{r^5}\right\}$$

$$\int T_3 r^2 \sin\varphi \, d\varphi \, d\theta = -\frac{2c\mu^2}{\lambda+\mu}\int_0^{2\pi}\int_0^{\pi} \sin\varphi \, d\varphi \, d\theta - 6c\mu \int_0^{2\pi}\int_0^{\pi} \cos^2\varphi \sin\varphi \, d\varphi \, d\theta$$

$$+ \frac{4\pi c\mu^2}{\lambda+\mu} \cos\varphi \Big|_0^{\pi} + 6c\mu \cdot 2\pi \left[\frac{\cos^3\varphi}{3} \Big|_0^{\pi} \right.$$

$$-1-1 \qquad -\tfrac{2}{3}$$

$$-\frac{8\pi c\mu^2}{\lambda+\mu} \qquad -4c\mu \cdot 2\pi$$

$$-8c\mu\pi$$

$$-8\pi c\mu \left[\frac{\mu}{\lambda+\mu} + 1 \right]$$

$$-8\pi c\mu \cdot \frac{\lambda+2\mu}{\lambda+\mu} = -\frac{\cancel{8\pi\mu}\cancel{(\lambda+2\mu)}}{\cancel{\lambda+\mu}} \cdot \frac{\cancel{(\lambda+\mu)}P}{\cancel{8\pi\mu(\lambda+2\mu)}} = -P.$$

5. $\mathbb{u} = u_i e_i = u_r e_r + u_\theta e_\theta + u_\varphi e_\varphi$

$= u_x e_x + u_y e_y + u_z e_z$

$$\begin{matrix} e_r = \\ e_\varphi = \\ e_\theta = \end{matrix} \begin{pmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \\ -\sin\theta & \cos\theta & \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

then $u_r = \mathbb{u}\cdot e_r \quad u_\theta = \mathbb{u}\cdot e_\theta \quad u_\varphi = \mathbb{u}\cdot e_\varphi$

$u_r = \frac{c\cos\varphi}{r} \qquad u_\theta = 0 \qquad u_\varphi = 0$

6. Using soholnikoff pg 184 $\quad \varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \frac{u_r}{r} \qquad \varepsilon_{r\varphi} = \varepsilon_{r\theta} = \varepsilon_{\theta\varphi} = 0$

$\sigma_{rr} = \frac{\lambda - 2\mu}{r^2} c\cos\varphi \qquad \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \frac{\lambda+2\mu}{r^2} c\cos\varphi \qquad \sigma_{r\theta} = \sigma_{r\varphi} = \sigma_{\theta\varphi} = 0$

4. hermann said find $\frac{\partial u_z}{\partial z}$ $\qquad \epsilon_3 = u_{3,3} = c\left[-\frac{\lambda+3\mu}{\lambda+\mu}\frac{x_3}{r^3} + \frac{x_3}{r^3} + \frac{x_3}{r^3} - \frac{3x_3^3}{r^5} \right]$

$$\frac{2x_3}{r^3}$$

$$c\left[\frac{x_3}{r^3}\left(\frac{2\lambda+2\mu-\lambda-3\mu}{\lambda+\mu} \right) - \frac{3x_3^3}{r^5} \right]$$

$$\frac{cx_3}{r^3}\left[\frac{\lambda-\mu}{\lambda+\mu} - 3\left(\frac{x_3}{r}\right)^2 \right]$$

2

$$\therefore \ \sigma_{ij} = \frac{2C}{r^3}\mu\left\{\left(x_3\delta_{ij} - x_j\delta_{i3} - x_i\delta_{j3}\right)\frac{\mu}{\lambda+\mu} - \frac{3x_3x_ix_j}{r^2}\right\}$$

$$\sigma_{11} = \frac{2c}{r^3}\mu\left\{x_3\frac{\mu}{\lambda+\mu} - 3x_3\left(\frac{x_1}{r}\right)^2\right\}; \ \sigma_{31} = \sigma_{13} = \frac{2c}{r^3}\mu\left\{\left(-x_1\frac{\mu}{\lambda+\mu}\right) - \frac{3x_3x_1x_3}{r^2}\right\}$$

$$\sigma_{21} = \sigma_{12} = \frac{2c}{r^3}\mu\left(-\frac{3x_3x_1x_2}{r^2}\right); \quad \sigma_{22} = \frac{2c}{r^3}\mu\left\{x_3\frac{\mu}{\lambda+\mu} - 3x_3\left(\frac{x_2}{r}\right)^2\right\}; \quad \sigma_{23} = \sigma_{32} = \frac{2c}{r^3}\mu\left\{-\frac{x_2\mu}{\lambda+\mu} - \frac{3x_3^2x_2}{r^2}\right\}$$

$$\sigma_{33} = \frac{2c}{r^3}\mu\left\{\left(x_3 - 2x_3\right)\frac{\mu}{\lambda+\mu} - 3x_3\left(\frac{x_3}{r^2}\right)^2\right\} = -\frac{2c}{r^3}\mu x_3\left\{3\left(\frac{x_3}{r}\right)^2 + \frac{\mu}{\lambda+\mu}\right\}$$

2. $T_i = \sigma_{ij}n_j$ now $\underline{n} = e_r = \frac{x_j}{r}e_j = n_je_j$

$$\cancel{T_1 = \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 = -\frac{2c}{r^3}\mu x_3\left[3\left(\frac{x_1}{r}\right)^2 - \frac{\mu}{\lambda+\mu}\right]\frac{x_1}{r} - \frac{2c}{r^3}\mu\left(\frac{3x_3x_1x_2}{r^2}\right)\frac{x_2}{r} - \frac{2cx_1}{r^3}\mu\left\{3\left(\frac{x_3}{r}\right)^2 + \frac{\mu}{\lambda+\mu}\right\}\frac{x}{r}}$$

$$\cancel{= -\frac{2c}{r^4}\mu\left\{3\left(\frac{x_1}{r}\right)^2x_3x_1 - \frac{x_3x_1\mu}{\lambda+\mu} + 3x_3x_1\left(\frac{x_2}{r}\right)^2 + 3\left(\frac{x_3}{r}\right)^2x_1x_3 + \frac{\mu}{\lambda+\mu}x_1x_3\right\}}$$

$$\cancel{-\frac{2c}{r^4}\mu\left\{3x_3x_1\right\} =}$$

$$T_i = \sigma_{ij}n_j = \frac{2c}{r^4}\mu\left\{\left(x_3x_j\delta_{ij} - x_jx_j\delta_{i3} - x_ix_j\delta_{j3}\right)\frac{\mu}{\lambda+\mu} - 3x_3x_i\frac{\cancel{x_jx_j}}{\cancel{r^2}}^{1}\right\}$$

if $i,j \neq 3$ $\quad -\frac{6cx_3x_i\mu}{r^4} = T_i$

$$T_3 = \frac{2c}{r^4}\mu\left\{\left(\cancel{x_3^2} - \cancel{x_3^2}\cancel{-x_3^2}\right)\frac{\mu}{\lambda+\mu} - 3x_3^2\right\} = -\frac{2c}{r^4}\mu\left\{x_3^2\frac{\mu}{\lambda+\mu} + 3x_3^2\right\}$$

$x_3^2 - r^2 - x_3^2$

$$= -\frac{2c\mu^2}{\lambda+\mu r^2} - \frac{6c\mu x_3^2}{r^4} \qquad w/\ r=a$$

3. $$\int_S T_i\ dS = \int_0^{\pi}\int_0^{2\pi} T_i r^2\sin\varphi\ d\varphi\, d\theta = -\frac{6cx_3x_i\mu\ \cancel{a^2}\sin\varphi\, d\varphi\, d\theta}{\cancel{a^4}a^2}$$

$$= -6c\left(\frac{x_3}{a}\right)\cdot\left(\frac{x_i}{a}\right)\mu$$

$$= -6c\mu\cancel{a}\cos\varphi\cdot\cancel{a}\left\{\begin{matrix}\cos\theta\sin\varphi\\ \sin\theta\sin\varphi\end{matrix}\right\}\sin\varphi\, d\varphi\, d\theta$$

$$= -6c\mu\cos\varphi\left(\frac{1-\cos 2\varphi}{2}\right)\left\{\begin{matrix}\cos\theta\\ \sin\theta\end{matrix}\right\}d\varphi\, d\theta = 0$$

$$\int_0^{2\pi}\begin{matrix}\cos\theta\, d\theta\\ \sin\theta\, d\theta\end{matrix} = 0 \quad \text{for } i=1,2$$

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Room 102 Terman

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Problem Set #1

51/60

Kelvin Problem in cartesian

1. if $\nu = \frac{\lambda}{2(\lambda+\mu)}$ then $\frac{1}{4(1-\nu)} = \frac{\lambda+\mu}{2(\lambda+2\mu)}$ and $1 - \frac{1}{4(1-\nu)} = \frac{\lambda+3\mu}{2(\lambda+\mu)}$

thus

$$u_i = \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)} B_i - \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)} x_j B_{j,i}$$ using $B_i = \frac{P_i}{4\pi\mu r}$ then with $P_1 = P_2 = 0,\ P_3 = P$

$$u_i = \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)} \frac{P\delta_{i3}}{r} + \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)} \frac{x_3 x_i}{r^3} P = C\left[\frac{\lambda+3\mu}{\lambda+\mu}\frac{\delta_{i3}}{r} + \frac{x_3x_i}{r^3}\right]$$ where $C = \frac{(\lambda+\mu)P}{8\pi\mu(\lambda+2\mu)}$

Now $$u_{i,j} = C\left[-\frac{\lambda+3\mu}{\lambda+\mu}\frac{x_j}{r^3}\delta_{i3} + \frac{\delta_{3j}x_i}{r^3} + \frac{\delta_{ij}x_3}{r^3} - \frac{3x_3x_ix_j}{r^5}\right]$$

also

$$u_{i,i} = C\left[-\frac{\lambda+3\mu}{\lambda+\mu}\frac{x_i\delta_{i3}}{r^3} + \frac{\delta_{3i}x_i}{r^3} + \frac{\overset{=3}{\delta_{ii}}x_3}{r^3} - \frac{3x_3\overset{=r^2}{x_ix_i}}{r^5}\right] = C\left(\frac{-2\mu}{\lambda+\mu}\right)\frac{\delta_{3i}x_i}{r^3} = \frac{-2\mu C x_3}{(\lambda+\mu)r^3}$$

thus $\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j}+u_{j,i}) = \frac{2C\mu}{r^3}\left\{(x_3\delta_{ij} - x_j\delta_{i3} - x_i\delta_{j3})\frac{\mu}{\lambda+\mu} - \frac{3x_3x_ix_j}{r^2}\right\}$ after combining terms and simplifying.

Thus $\sigma_{11} = -\frac{2C\mu x_3}{r^3}\left\{3\left(\frac{x_1}{r}\right)^2 - \frac{\mu}{\lambda+\mu}\right\}$; $\sigma_{12} = -\frac{6C\mu}{r^2}\left(\frac{x_1x_2x_3}{r^3}\right)$; $\sigma_{13} = -\frac{2C\mu x_1}{r^3}\left\{3\left(\frac{x_3}{r}\right)^2 + \frac{\mu}{\lambda+\mu}\right\}$

$\sigma_{22} = -\frac{2C\mu x_3}{r^3}\left\{3\left(\frac{x_2}{r}\right)^2 - \frac{\mu}{\lambda+\mu}\right\}$; $\sigma_{23} = -\frac{2C\mu x_2}{r^3}\left\{3\left(\frac{x_3}{r}\right)^2 + \frac{\mu}{\lambda+\mu}\right\}$; $\sigma_{33} = -\frac{2C\mu x_3}{r^3}\left\{3\left(\frac{x_3}{r}\right)^2 + \frac{\mu}{\lambda+\mu}\right\}$ 10

2. Now $T_i = \sigma_{ij}n_j$ and the normal to the sphere is $\underline{r} = x_j e_j/\sqrt{(x_ix_i)}$ $\therefore n_j = x_j/|\underline{r}| = \frac{x_j}{r}$

$$\therefore T_i = \sigma_{ij}n_j = \frac{2C\mu}{r^4}\left\{(x_3x_j\delta_{ij} - \overset{=r^2}{x_jx_j}\delta_{i3} - x_ix_j\delta_{j3})\frac{\mu}{\lambda+\mu} - 3x_3x_i\overset{=1}{\frac{x_jx_j}{r^2}}\right\}$$

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thus $T_i = -\frac{6x_3x_i\mu}{r^4}$ for $i=1,2$ $\quad T_3 = -\frac{2C\mu^2}{(\lambda+\mu)r^2} - \frac{6C\mu x_3^2}{r^4}$

3. $$\int_S T_i\, dS = \int_0^{2\pi}\int_0^{\pi} T_i r^2 \sin\varphi\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi} -\frac{6Cx_3x_i}{r^4}\mu r^2 \sin\varphi\, d\varphi\, d\theta$$ for constant r and $i=1,2$; $0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi$

now $\frac{x_3}{r} = \cos\varphi$ $\quad \frac{x_i}{r} = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}\sin\varphi$ $i=1,2$; thus $\int_S T_i\, dS = \int_0^{2\pi}\int_0^{\pi} -6C\mu\cos\varphi\sin^2\varphi\, d\varphi \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} d\theta$

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but $\int_0^{2\pi}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} d\theta = 0$ $\quad\therefore \int_S T_i\, dS = 0 \quad i=1,2$

over

$$\int_S T_3 r^2 \sin\varphi\, d\varphi\, d\theta = -\frac{2C\mu^2}{\lambda+\mu}\int_0^{2\pi}\int_0^{\pi} \sin\varphi\, d\varphi\, d\theta - 6C\mu\int_0^{2\pi}\int_0^{\pi} \cos^2\varphi \sin\varphi\, d\varphi\, d\theta$$

where $\frac{x_3}{r} = \cos\varphi$; $\int_S T_3 dS = -\frac{8\pi C\mu^2}{\lambda+\mu} - 8C\mu\pi = -8\pi C\mu\left(\frac{\lambda+2\mu}{\lambda+\mu}\right) = -P$ when we substitute back for C.

Since the volume of the sphere contains the concentrated force P, for equil. to exist the surface must support the self equilibrating force $-P$.

Show tractions produce no moment -4

4. According to Prof. Herrmann we need to evaluate $\frac{\partial u_z}{\partial z} = \epsilon_z = e_z \cdot \epsilon \cdot e_z$

but we know

$$u_{i,j} = C\left[-\frac{\lambda+3\mu}{\lambda+\mu}\delta_{i3}\frac{x_j}{r^3} + \frac{\delta_{3j}x_i}{r^3} + \delta_{ij}\frac{x_3}{r^3} - \frac{3x_3x_ix_j}{r^5}\right]$$

for $i=j=3$

$$u_{3,3} = C\left[-\frac{\lambda+3\mu}{\lambda+\mu}\frac{x_3}{r^3} + \frac{x_3}{r^3} + \frac{x_3}{r^3} - \frac{3x_3^3}{r^5}\right] = \frac{Cx_3}{r^3}\left[\frac{\lambda-\mu}{\lambda+\mu} - 3\left(\frac{x_3}{r}\right)^2\right]$$

where $C = \frac{(\lambda+\mu)P}{8\pi\mu(\lambda+2\mu)}$ simplify = along the line $r = x_3$

along x_3 : $r = x_3$ $\therefore$ $u_{3,3} = \frac{C}{r^2}\left[\frac{\lambda-\mu}{\lambda+\mu} - 3\right]$

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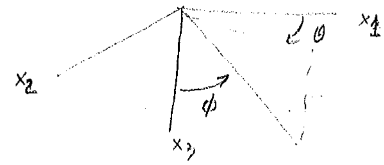
$$= \frac{C}{r^2}\left[\frac{\lambda-\mu-3\lambda-3\mu}{\lambda+\mu}\right] = -\frac{C}{r^2}\left[\frac{2(\lambda+2\mu)}{\lambda+\mu}\right]$$
$$= \frac{-P}{4\pi\mu r^2}$$

see 4th page to show tractions are 0.



5. Now $\mathbb{u} = u_1 \mathbb{e}_1 + u_2 \mathbb{e}_2 + u_3 \mathbb{e}_3 = u_r \mathbb{e}_r + u_\theta \mathbb{e}_\theta + u_\varphi \mathbb{e}_\varphi$ where

$\mathbb{e}_r = \mathbb{e}_1 \cos\theta \sin\varphi + \mathbb{e}_2 \sin\theta \sin\varphi + \mathbb{e}_3 \cos\varphi$; $\mathbb{e}_\varphi = \mathbb{e}_1 \cos\theta \cos\varphi + \mathbb{e}_2 \sin\theta \cos\varphi - \mathbb{e}_3 \sin\varphi$

$\mathbb{e}_\theta = -\mathbb{e}_1 \sin\theta + \mathbb{e}_2 \cos\theta$

Now $u_r = \mathbb{u} \cdot \mathbb{e}_r = u_1 \cos\theta \sin\varphi + u_2 \sin\theta \sin\varphi + u_3 \cos\varphi$

$u_\theta = \mathbb{u} \cdot \mathbb{e}_\theta = -u_1 \sin\theta + u_2 \cos\theta$

$u_\varphi = \mathbb{u} \cdot \mathbb{e}_\varphi = u_1 \cos\theta \cos\varphi + u_2 \sin\theta \cos\varphi - u_3 \sin\varphi$

$u_1 = \dfrac{C \cos\varphi \cos\theta \sin\varphi}{r}$ $\quad u_2 = \dfrac{C}{r} \cos\varphi \sin\theta \sin\varphi$ $\quad u_3 = \dfrac{C}{r} \cos^2\varphi$ $\quad C = \dfrac{(\lambda+\mu)P}{8\pi\mu(\lambda+2\mu)}$

$$u_r = \frac{C}{r}\left[\cos\varphi \cos^2\theta \sin^2\varphi + \cos\varphi \sin^2\theta \sin^2\varphi + \cos^3\varphi\right] = \frac{C}{r}\left[\cos\varphi\right]$$

-3

$$u_\theta = \frac{C}{r}\left[-\cos\varphi \cos\theta \sin\varphi \sin\theta + \cos\varphi \sin\theta \cos\theta \sin\varphi\right] = 0 \quad \times$$

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$$u_\varphi = \frac{C}{r}\left[\cos^2\varphi \cos^2\theta \sin\varphi + \cos^2\varphi \sin^2\theta \sin\varphi - \sin\varphi \cos^2\varphi\right] = 0$$

6. Now since we know what the components are then for the non zero displacement components

$\varepsilon_{rr} = \dfrac{\partial u_r}{\partial r} = -\dfrac{C \cos\varphi}{r^2}$ $\quad \varepsilon_{\varphi\varphi} = \dfrac{C}{r^2}[\cos\varphi] = \dfrac{u_r}{r}$ $\quad \varepsilon_{\theta\theta} = \dfrac{C}{r^2}\cos\varphi = \dfrac{u_r}{r}$

$\varepsilon_{r\theta} = 0$ $\times$ $\quad \varepsilon_{r\varphi} = 0$ $\quad \varepsilon_{\theta\varphi} = 0$

using the formulae found on pg 184 of Sokolnikov w/ θ = to my φ ; α = to my θ

Since $\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$ with $\varepsilon_{kk} = \dfrac{C}{r^2}\cos\varphi = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi}$

$= \lambda \dfrac{C}{r^2} \cos\varphi \, \delta_{ij} + 2\mu \epsilon_{ij}$

$\sigma_{rr} = \dfrac{\lambda - 2\mu}{r^2} C \cos\varphi$ $\quad \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \dfrac{\lambda + 2\mu}{r^2} C \cos\varphi$ $\quad \sigma_{r\theta} = 2\mu \epsilon_{r\theta} = 0$ $\times$ $\quad -2$

$\sigma_{r\varphi} = 2\mu \epsilon_{r\varphi} = 0$ $\quad \sigma_{\theta\varphi} = 2\mu \epsilon_{\theta\varphi} = 0.$

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see next page

Kelvin in Spherical

#5. Better $\mathbf{u} = \mathbf{B} - \frac{1}{4(1-\nu)} \nabla(\mathbf{r}\cdot\mathbf{B})$ where $\mathbf{B} = \frac{P\mathbf{e}_3}{4\pi\mu r} = \frac{P(\cos\varphi\,\mathbf{e}_r - \sin\varphi\,\mathbf{e}_\varphi)}{4\pi\mu r}$

$\mathbf{e}_3 = A\mathbf{e}_r + B\mathbf{e}_\varphi + C\mathbf{e}_\theta \quad A = \mathbf{e}_3\cdot\mathbf{e}_r \quad B = \mathbf{e}_3\cdot\mathbf{e}_\varphi \quad C = \mathbf{e}_3\cdot\mathbf{e}_\theta$

$\mathbf{r}\cdot\mathbf{B} = r\mathbf{e}_r \frac{(\cos\varphi\,\mathbf{e}_r)}{4\pi\mu r} = \frac{r\cos\varphi}{4\pi\mu r} = \frac{\cos\varphi}{4\pi\mu} \therefore \quad \mathbf{u} = \frac{P}{4\pi\mu r}\cos\varphi\,\mathbf{e}_r - \frac{P\sin\varphi}{4\pi\mu r}\mathbf{e}_\varphi - \frac{1}{4(1-\nu)}\nabla\left(\frac{\cos\varphi}{4\pi\mu}\right)$

$\nabla = \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r\sin\varphi}\frac{\partial}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r}\frac{\partial}{\partial\varphi}\mathbf{e}_\varphi \quad \therefore \quad \nabla\frac{\cos\varphi}{4\pi\mu} = \frac{-1}{4\pi\mu r}\sin\varphi\,\mathbf{e}_\varphi$

$\therefore \quad \mathbf{u} = \frac{P}{4\pi\mu r}\cos\varphi\,\mathbf{e}_r - \frac{P\sin\varphi}{4\pi\mu r}\mathbf{e}_\varphi + \frac{1}{4(1-\nu)}\frac{\sin\varphi}{4\pi\mu r}\mathbf{e}_\varphi$

$$\boxed{u_r = \frac{P\cos\varphi}{4\pi\mu r} \qquad u_\theta = 0 \qquad u_\varphi = \frac{P\sin\varphi}{4\pi\mu r}\left[-1 + \frac{1}{4(1-\nu)}\right] = \frac{4\nu-3}{4(1-\nu)}\frac{P\sin\varphi}{4\pi\mu r}}$$

Let $\frac{P}{4\pi\mu} = C$

#6 $\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = -\frac{C\cos\varphi}{r^2}$

$\varepsilon_{\theta\theta} = \frac{1}{r\sin\varphi}\cancel{\frac{\partial u_\theta}{\partial\theta}} + \frac{u_r}{r} + u_\varphi\frac{\cot\varphi}{r}$
$= \frac{P\cos\varphi}{4\pi\mu r^2} + \frac{4\nu-3}{4(1-\nu)}\frac{P\cos\varphi}{4\pi\mu r^2} = \frac{P\cos\varphi}{4\pi\mu r^2}\frac{1}{4(1-\nu)}$

$\varepsilon_{\varphi\varphi} = \frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} = \frac{P\cos\varphi}{4\pi\mu r^2}\left[1 + \frac{4\nu-3}{4(1-\nu)}\right] = \frac{P\cos\varphi}{4(1-\nu)\cdot 4\pi\mu r^2}$

$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r\sin\varphi}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) = 0$

$\varepsilon_{\theta\varphi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\theta\cos\varphi}{r} + \frac{1}{r\cos\varphi}\frac{\partial u_\varphi}{\partial\theta}\right) = 0$

$\varepsilon_{\varphi r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\varphi} - \frac{u_\varphi}{r} + \frac{\partial u_\varphi}{\partial r}\right) = \frac{1}{2}\left(-\frac{P\sin\varphi}{4\pi\mu r^2} - 2\frac{4\nu-3}{4(1-\nu)}\frac{P\sin\varphi}{4\pi\mu r^2}\right)$

$= -\frac{1}{2}\frac{P\sin\varphi}{4\pi\mu r^2}\left[\frac{4-4\nu+8\nu-6}{4(1-\nu)}\right]$

$= -\frac{1}{2}\frac{P\sin\varphi}{4\pi\mu r^2}\frac{(2\nu-1)2}{4(1-\nu)} = \frac{P\sin\varphi(1-2\nu)}{16\pi\mu r^2(1-\nu)}$

$$\boxed{\varepsilon_{rr} = -\frac{P\cos\varphi}{4\pi\mu r^2} \qquad \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \frac{P\cos\varphi}{16\pi\mu(1-\nu)r^2} \qquad \varepsilon_{r\theta} = \varepsilon_{\theta\varphi} = 0 \qquad \varepsilon_{\varphi r} = \frac{P\sin\varphi(1-2\nu)}{16\pi\mu r^2(1-\nu)}}$$

$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + \varepsilon_{ij}2\mu$

$\sigma_{\varphi r} = 2\mu\varepsilon_{\varphi r} = \frac{P\sin\varphi(1-2\nu)}{8\pi r^2(1-\nu)} \qquad \sigma_{\theta\varphi} = \sigma_{r\theta} = 0$

$\varepsilon_{kk} = \frac{P\cos\varphi}{4\pi\mu r^2}\left[-1 + \frac{2}{4(1-\nu)}\right] = \frac{P\cos\varphi}{4\pi\mu r^2}\left[\frac{-2+4\nu}{4(1-\nu)}\right] = -\frac{P\cos\varphi}{8\pi\mu r^2}\frac{(1-2\nu)}{1-\nu}$

$\sigma_{rr} = -\frac{\lambda P\cos\varphi(1-2\nu)}{8\pi\mu r^2(1-\nu)} + \frac{2\mu P\cos\varphi}{4\pi\mu r^2} = \frac{P\cos\varphi}{4\pi r^2}\left(\frac{\nu-2}{1-\nu}\right)$

$$\boxed{\sigma_{\theta\theta} = \frac{-\lambda P\cos\varphi(1-2\nu)}{8\pi\mu r^2(1-\nu)} + \frac{P\cos\varphi\ 2\mu}{16\pi\mu(1-\nu)r^2} = \frac{P\cos\varphi}{8\pi r^2}\left(\frac{1-2\nu}{1-\nu}\right) = \sigma_{\varphi\varphi}}$$

\# 3. For Torques to exist let us take moments about the origin. σ_{rr} produces no moments; $\sigma_{\theta\varphi}, \sigma_{r\theta} = 0$ "

$$\mathbb{T} = T_i e_i \quad M_i e_i = \int r \times \mathbb{T}\, dA = \int (x_1 T_2 - x_2 T_1) e_3\, dA + \int (x_2 T_3 - T_2 x_3) e_1\, dA$$

$$r = x_j e_j \quad \mathbb{T} = T_i e_i \quad r \times \mathbb{T} = x_j T_i e_{ijk} e_k \qquad + \int (x_3 T_1 - x_1 T_3) e_2\, dA$$

$$dA = r^2 \sin\varphi\, d\varphi\, d\theta \qquad x_1 = r\cos\theta\sin\varphi \quad x_2 = r\sin\theta\sin\varphi \quad x_3 = r\cos\varphi$$

$$T_1 = -\frac{6\mu}{r^2}\cos\varphi\cos\theta\sin\varphi \qquad T_2 = -\frac{6\mu}{r^2}\cos\varphi\sin\theta\sin\varphi \qquad T_3 = \frac{-2c\mu^2}{(\lambda+\mu)r^2} - \frac{6c\mu x_3^2}{r^4}$$

$$M_3 = \int (x_1 T_2 - x_2 T_1)\, dA = -6\mu \int \frac{r^2 \sin\varphi\, d\varphi\, d\theta}{r^2}\left[r\cos\theta\sin^2\varphi\cos\varphi\sin\theta - r\sin\theta\sin^2\varphi\cos\varphi\cos\theta\right] = 0$$

$$M_1 = \int r^2 \sin\varphi\, d\theta\, d\varphi \left(r\sin\theta\sin\varphi \cdot \left[\frac{-2c\mu^2}{(\lambda+\mu)r^2} - \frac{6c\mu}{r^2}\cos^2\varphi\right] + \frac{6\mu\cos\varphi}{r^2}\left[\cos\varphi\sin\theta\sin\varphi\right]\right)$$

$$\int_0^{2\pi} \sin\theta\, d\theta = 0 \quad \Rightarrow \quad M_1 = 0$$

$$M_2 = \int r^2 \sin\varphi\, d\theta\, d\varphi \left(r\cos\varphi \cdot -\frac{6\mu}{r^2}\cos\varphi\cos\theta\sin\varphi - r\cos\theta\sin\varphi \left[\quad f(\varphi)\text{ only}\right]\right)$$

$$\int_0^{2\pi} \cos\theta\, d\theta = 0 \quad \Rightarrow \quad M_2 = 0$$

DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING
STANFORD UNIVERSITY

ME 238C Theory of Elasticity | Spring 1979

Problem Set No. 2

1. Calculate the displacement associated with a doublet (double center of compression-dilatation).

2. Consider an extended elastic solid containing a spherical cavity and subjected to remote uniaxial tension. Determine the principal stresses on the surface of the cavity and show, by superposition, that if isotropic (all-around) remote tension were applied, the stress concentration factor would be 1.5.

3. Determine the settlement outside a rigid circular stamp impressed into an elastic half-space.

$$\mathbb{u} = \mathbb{B} - \frac{1}{4(1-\nu)} \nabla \left(\mathbb{r} \cdot \mathbb{B} + \beta \right)$$

$\mathbb{r} = r e_r$

$$\cancel{\frac{\partial}{\partial z} = \frac{\partial}{\partial R}\frac{\partial R}{\partial z} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z}}$$
$$\cancel{= \frac{1}{\cos\varphi}\frac{\partial}{\partial R} - \frac{1}{R\sin\varphi}\frac{\partial}{\partial \varphi}}$$

$z = R\cos\varphi$

$\cancel{1 = \frac{\partial R}{\partial z}\cos\varphi}$

$\cancel{1 = -R\sin\varphi \frac{\partial \varphi}{\partial z}}$

$\cancel{dz = -R\sin\varphi\, d\varphi + \cos\varphi\, dR}$

$\frac{\partial z}{\partial \varphi}$ $\quad$ $\frac{\partial z}{\partial r}$

$$\frac{\partial}{\partial z} = e_z \cdot \nabla$$
$$= e_r \cdot e_z \frac{\partial}{\partial r} + e_\theta \cdot e_z \frac{\partial}{r\partial \theta} \frac{1}{r\sin\varphi} + e_z \cdot e_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi}$$
$$= \cos\varphi \frac{\partial}{\partial r} + 0 \cdot + -\frac{\sin\varphi}{r}\frac{\partial}{\partial \varphi}.$$

$$e_z \cdot \nabla = e_z \cdot e_r \frac{\partial}{\partial r}$$
$$\cos\varphi \frac{\partial}{\partial r} - \frac{1}{r\sin\varphi} \cdot 0 \cdot - \frac{\sin\varphi}{r}\frac{\partial}{\partial \varphi}.$$

$$\nabla = u_r \frac{\partial}{\partial r} + \frac{u_\varphi}{r}\frac{\partial}{\partial \varphi} + \frac{u_\theta}{r\sin\varphi}\frac{\partial}{\partial \theta}$$

$$\nabla = e_x \frac{\partial}{\partial x} + e_y \frac{\partial}{\partial y} + e_z \frac{\partial}{\partial z} = e_r \frac{\partial}{\partial r} + e_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi} + e_\theta \frac{1}{r\sin\varphi}\frac{\partial}{\partial \theta}$$

$$e_z \cdot \nabla = \frac{\partial}{\partial z} = \cos\varphi \frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \cos\varphi \left[-\frac{1}{r^2}\right] = -\frac{\cos\varphi}{r^2}$$

$\mathbb{B} = B_z e_z$

$\cancel{-\frac{1}{r^2} \cdot \frac{z}{r}}$

$$\mathbb{B} = A\frac{\partial}{\partial z}\left(\frac{1}{R}\right) e_z = -A\frac{\cos\varphi}{r^2}\left[\cos\phi\, e_r - \sin\phi\, e_\phi\right]$$

$$\cancel{\mathbb{r} \cdot \mathbb{B} = A z \frac{\partial}{\partial z}\left(\frac{1}{R}\right)} \quad \mathbb{r} \cdot \mathbb{B} = r e_r \left[-A\frac{\cos\varphi}{r^2}\cos\varphi\, e_r + 0\right] = -A\frac{\cos^2\varphi}{r}$$

$$\beta = \frac{D}{R} + C\frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\left(\frac{1}{R}\right)\right] = \frac{D}{R} + C\frac{\partial}{\partial z}\left[-\frac{\cos\varphi}{r^2}\right] = \frac{D}{R} + C\left\{\frac{2\cos^2\varphi}{r^3} - \frac{\sin^2\varphi}{r^3}\right\}$$

$$\frac{\partial}{\partial z}\left(-\frac{\cos\varphi}{r^2}\right) = \cos\varphi \frac{\partial}{\partial r}\left(-\frac{\cos\varphi}{r^2}\right) - \frac{\sin\varphi}{r}\frac{\partial}{\partial \varphi}\left(-\frac{\cos\varphi}{r^2}\right)$$
$$= -\cos^2\varphi\left(-\frac{2}{r^3}\right) + \frac{\sin\varphi}{r^3}\left(-\sin\varphi\right)$$

$$\therefore \quad \mathbb{u} = +\frac{A}{r^2}\left[\frac{\sin 2\phi}{2} e_\phi - \cos^2\varphi\, e_r\right] - \frac{1}{4(1-\nu)}\nabla\left\{-\frac{A\cos^2\varphi + D}{r} + \frac{C}{r^3}\left\{\cos 2\varphi + \cos^2\varphi\right\}\right\}$$

$-2\cos\varphi\sin\varphi \quad -2\sin\varphi\cos\varphi$

$$\nabla = e_r\frac{\partial}{\partial r} + e_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi} + \cancel{e_\theta \frac{1}{r\sin\varphi}\frac{\partial}{\partial \theta}} \qquad \left[\frac{A\cos^2\varphi - D}{r^2} - \frac{3C}{r^4}\left(\cos 2\varphi + \cos^2\varphi\right)\right] e_r$$

$$+ \left[+\frac{A \cdot 2\cos\varphi\sin\varphi}{r^2} + \frac{C}{r^4}\left\{-2\sin 2\varphi - 2\cos\varphi\sin\varphi\right\}\right] e_\varphi$$

$\frac{2(1-\nu)}{2(1-\nu)}$

$$\mathbb{u} = e_\phi\left[\frac{A}{r^2}\frac{\sin 2\phi}{2} - \frac{1}{4(1-\nu)}\left\{\frac{A\sin 2\varphi}{r^2} - 3C\frac{\sin 2\varphi}{r^4}\right\}\right] + e_r\left[\cancel{-\frac{A\cos^2\varphi}{r^2}} + \frac{A\cos^2\varphi - D}{r^2} - \frac{3C}{r^4}\left(\cos 2\varphi + \cos^2\right.\right.$$

$$u_r = \cancel{-\frac{D}{r^2} - \frac{3C}{r^4}\left(\cos 2\varphi + \cos^2\varphi\right)} \qquad u_\varphi = \frac{A(1-2\nu)}{4r^2(1-\nu)}\sin 2\varphi + \frac{3C}{4(1-\nu)}\frac{\sin 2\varphi}{r^4}$$

$$u_\varphi = \frac{\sin 2\varphi}{4(1-\nu)}\left[\frac{A(1-2\nu)}{r^2} + \frac{3C}{r^4}\right] \qquad u_r = \frac{\cos^2\varphi}{4(1-\nu)}\left\{-\frac{A(5-4\nu)}{r^2} + \frac{3C}{r^4}\right\} + \frac{D}{4(1-\nu)r^2} + \frac{3C}{4(1-\nu)}\frac{\cos 2\varphi}{r^4}$$

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{\cos^2\varphi}{4(1-\nu)}\left[\frac{2A(5-4\nu)}{r^3} - \frac{24C}{r^5}\right] - \frac{2D}{4(1-\nu)r^3} \quad \frac{\cancel{+}12C}{4(1-\nu)}\,\frac{\overset{\sin^2\varphi}{\cancel{\cos 2\varphi}}}{r^5}$$

$$\varepsilon_{\theta\theta} = \frac{\cos^2\varphi}{4(1-\nu)}\left\{-\frac{A(5-4\nu)}{r^3} + \frac{3C}{r^5}\right\} + \frac{D}{4(1-\nu)r^3} + \frac{3C}{4(1-\nu)}\frac{\cos 2\varphi}{r^5} + \frac{\cos^2\varphi}{4(1-\nu)}\left[\frac{2A(1-2\nu)}{r^3} + \frac{6C}{r^5}\right] = \frac{u_r}{r} + u_\varphi\frac{\cot\varphi}{r}$$

$$= \frac{\overset{\cos^2\varphi}{\cancel{\cos 2\varphi}}}{4(1-\nu)}\left[-\frac{3A}{r^3} + \frac{9C}{r^5}\right\} + \frac{D}{4(1-\nu)r^3} + \frac{3C}{4(1-\nu)}\frac{\overset{\cos^2\varphi - \sin^2\varphi}{\cancel{\cos 2\varphi}}}{r^5} = \frac{\cos^2\varphi}{4(1-\nu)}\left[-\frac{3A}{r^3} + \frac{12C}{r^5}\right] - \frac{3C}{4(1-\nu)}\frac{\sin^2\varphi}{r^5} + \frac{D}{4(1-\nu)R^3}$$

$$\varepsilon_{\varphi\varphi} = \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} = \frac{2\cos 2\varphi}{4(1-\nu)}\left[\frac{2A(1-2\nu)}{r^3} + \frac{6C}{r^5}\right] + \frac{\cos^2\varphi}{4(1-\nu)}\left\{-\frac{A(5-4\nu)}{r^3} + \frac{3C}{r^5}\right\} + \frac{D}{4(1-\nu)r^3} + \frac{3C}{4(1-\nu)}\frac{\cos 2\varphi}{r^5}$$

$$= \frac{\cos 2\varphi}{4(1-\nu)}\left[\frac{2A(1-2\nu)}{r^3} + \frac{9C}{r^5}\right] + \frac{\cos^2\varphi}{4(1-\nu)}\left\{-\frac{A(5-4\nu)}{r^3} + \frac{3C}{r^5}\right\} + \frac{D}{4(1-\nu)r^3}$$

$$\varepsilon_{r\theta} = 0 \qquad \varepsilon_{\theta\varphi} = 0 \qquad \frac{\cos^2\varphi}{4(1-\nu)}\left[-\frac{3A}{r^3} + \frac{12C}{r^5}\right] - \frac{\sin^2\varphi}{4(1-\nu)}\left[\frac{2A(1-2\nu)}{r^3} + \frac{9C}{r^5}\right] + \frac{D}{4(1-\nu)r^3}$$

$$\varepsilon_{\varphi r} = \frac{1}{2}\left[\frac{-\sin 2\varphi}{4(1-\nu)}\left\{-\frac{A(5-4\nu)}{r^3} + \frac{3C}{r^5}\right\} - \frac{6C}{4(1-\nu)}\frac{\sin 2\varphi}{r^5} - \frac{\sin 2\varphi}{4(1-\nu)}\left\{\frac{A(1-2\nu)}{r^3} + \frac{3C}{r^5}\right\} + \frac{\sin 2\varphi}{4(1-\nu)}\left\{-\frac{2A(1-2\nu)}{r^3} - \frac{12C}{r^5}\right\}\right]$$

$$= \frac{\sin 2\varphi}{8(1-\nu)}\left\{\frac{A(5-4\nu)}{r^3} - \frac{A(1-2\nu)}{r^3} - \frac{2A(1-2\nu)}{r^3} - \frac{3C}{r^5} - \frac{6C}{r^5} - \frac{3C}{r^5} - \frac{12C}{r^5}\right\}$$

$$\frac{\sin 2\varphi}{8(1-\nu)}\left\{\frac{A(2+2\nu)}{r^3} - \frac{24C}{r^5}\right\} = \frac{\sin 2\varphi}{4(1-\nu)}\left\{\frac{A(1+\nu)}{r^3} - \frac{12C}{r^5}\right\}$$

$$\boxed{\sigma_{\varphi r} = 2\mu\varepsilon_{\varphi r} = \mu\frac{\sin\varphi\cos\varphi}{(1-\nu)}\left\{\frac{A(1+\nu)}{r^3} - \frac{12C}{r^5}\right\}} \checkmark \qquad \boxed{\sigma_{r\theta} = \sigma_{\theta\varphi} = 0 \quad \checkmark}$$

$$\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi} = \frac{\cos^2\varphi}{4(1-\nu)}\left[\frac{\overset{2A}{\cancel{10A - 8A\nu - 3A - 5A + 8A\nu}}}{r^3} + \frac{\overset{0}{\cancel{-12C + 9C + 3C}}}{r^5}\right] + \frac{D}{4(1-\nu)r^3}\left[1 - \cancel{2} + 1\right]$$

$$\frac{\cos 2\varphi}{4(1-\nu)r^3}\left[\frac{\cancel{-12C + 3C + 9C}}{r^2} + 2A(1-2\nu)\right] = \frac{\cos^2\varphi}{2(1-\nu)}\frac{A}{r^3} + \frac{\overset{\cos^2\varphi - \sin^2\varphi}{\cos 2\varphi}\,A(1-2\nu)}{2(1-\nu)r^3}$$

$$= \frac{\cos^2\varphi A}{2(1-\nu)r^3} + \frac{\cos^2\varphi A(1-2\nu)}{2(1-\nu)r^3} - \frac{\sin^2\varphi A(1-2\nu)}{2(1-\nu)r^3}$$

$$= \frac{\cos^2\varphi A}{r^3} - \frac{\sin^2\varphi A(1-2\nu)}{2(1-\nu)r^3}$$

$$\sigma_{RR} = \lambda\varepsilon_{kk} + 2\mu\varepsilon_{rr} = 2\mu\left[\frac{\nu}{(1-2\nu)}\varepsilon_{kk} + \varepsilon_{rr}\right]$$

$$2\mu\left\{\frac{\nu}{(1-2\nu)}\left\{\frac{\cos^2\varphi}{2(1-\nu)}\frac{A}{r^3} + \frac{\cos 2\varphi}{2(1-\nu)r^3}A(1-2\nu)\right\} + \frac{\cos^2\varphi}{2\cancel{4}(1-\nu)}\left[\frac{\cancel{2}A(5-4\nu)}{r^3} - \frac{\overset{6}{\cancel{12}}C}{r^5}\right] - \frac{2D}{4(1-\nu)r^3} - \frac{\overset{6}{\cancel{12}}C}{2\cancel{4}(1-\nu)}\frac{\cos 2\varphi}{r^5}\right\}$$

$$2\mu\left\{\frac{\cos^2\varphi}{2(1-\nu)r^3}\left[\frac{\nu A}{(1-2\nu)} + A(5-4\nu)\right] + \frac{\cos 2\varphi}{2(1-\nu)}\right.$$

$\cos^2\varphi - \sin^2\varphi$

$$\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\varphi\varphi} = \frac{\cos^2\varphi}{4(1-\nu)}\cdot\frac{2A}{r^3} + \frac{\cos 2\varphi\cdot 2A(1-2\nu)}{4(1-\nu)r^3} = \frac{\cos^2\varphi}{4(1-\nu)}\left[\frac{2A}{r^3}+\frac{2A(1-2\nu)}{r^3}\right] - \sin^2\varphi\cdot\frac{2A(1-2\nu)}{4(1-\nu)r^3}$$

$$= \frac{\cos^2\varphi}{r^3}A - \frac{\sin^2\varphi\cdot A(1-2\nu)}{2(1-\nu)r^3}$$

$$\sigma_{ij} = \lambda\,\varepsilon_{kk}\delta_{ij} + 2\mu\,\epsilon_{ij}$$

$$= \frac{2\mu\nu}{1-} \qquad 2\nu(\lambda+\mu)=\lambda \qquad \lambda[\cancel{1-2\nu}] = \frac{2\nu\mu}{1-2\nu}$$

$$2\mu\left[\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \epsilon_{ij}\right] \qquad \frac{\nu}{1-2\nu}\varepsilon_{rr} + \varepsilon_{rr} + \frac{\nu}{1-2\nu}(\varepsilon_{\theta\theta}+\varepsilon_{\varphi\varphi})$$

$$\frac{1-\nu}{1-2\nu}\varepsilon_{rr}$$

$$\sigma_{rr} = \frac{\cos^2\varphi}{4(1-2\nu)}\left[\frac{2A(5-4\nu)}{r^3} - \frac{12C}{r^5}\right] - \cancel{\frac{2D}{4(1-2\nu)r^3}} - \frac{12C}{4(1-2\nu)}\frac{\cos 2\varphi}{r^5} + \cancel{\frac{\nu}{1-2\nu}\left[\frac{\cos^2\varphi}{4(1-\nu)}\left\langle\frac{-A-4\nu A}{r^3}+\frac{18C}{r^5}\right\rangle\right.}$$

$$+\frac{\nu}{1-2\nu}\left\{\frac{\cos^2\varphi}{4(1-\nu)}\left[\frac{-4A(2-\nu)}{r^3} + \frac{12C}{r^5}\right] + \cancel{\frac{2D}{4(1-\nu)r^3}} + \frac{\cos 2\varphi}{4(1-\nu)}\left[\frac{12C}{r^5} + \frac{2A(1-2\nu)}{r^3}\right]\right\}$$

$\cos^2\varphi + \sin^2\varphi$

$$\frac{\cos^2\varphi}{4(1-2\nu)}\left\{\frac{2A(5-4\nu)}{r^3} - \frac{4A(2-\nu)\nu}{(1-\nu)r^3}\right\} + \cancel{2A(1-} \ \cancel{\frac{\nu}{1-2\nu}}\frac{\cos^2\varphi}{4(1-\nu)}\left[\frac{2A(1-2\nu)}{r^3}\right] \qquad 2\cos^2\varphi - 1$$

$$\frac{\cos^2\varphi\cdot 2A}{\cancel{4(1-2\nu)}r^3}\left\{\begin{matrix}(5-4\nu)(1-\nu) - 2\nu(2-\nu)\\ 5-9\nu+4\nu^2-4\nu+2\nu^2\end{matrix}\right\} = \frac{2A\cos^2\varphi}{4r^3}\,\frac{5-3\nu}{1-\nu} + \frac{\cos^2\varphi\cdot 2A}{4r^3(1-\nu)}2\nu$$

$$5-13\nu+6\nu^2$$

$$\frac{(5-3\nu)\cancel{(1-2\nu)}}{(1-\nu)} \qquad (5-\nu)A$$

$$\sigma_{\phi\phi} = 2\mu\left[\frac{1-\nu}{1-2\nu}\varepsilon_{\phi\phi} + \frac{\nu}{1-2\nu}(\varepsilon_{\theta\theta}+\varepsilon_{rr})\right]$$

$$= 2\mu\left[\frac{2\cos^2\phi-1}{4(1-2\nu)}\left\{\frac{2A(1-2\nu)}{r^3}+\frac{9C}{r^5}\right\} + \frac{\cos^2\varphi}{4(1-2\nu)}\left\{-\frac{A(5-4\nu)}{r^3}+\frac{3C}{r^5}\right\} + \frac{D}{4(1-2\nu)r^3}\right.$$

$$\left.+\frac{\nu}{1-2\nu}\left\{\frac{\cos^2\varphi}{4(1-\nu)}\left[\frac{(7-8\nu)A}{r^3} - \frac{3C}{r^5}\right] - \frac{D}{4(1-\nu)r^3} - \left(\frac{2\cos^2\varphi-1}{4(1-\nu)}\right)\cdot\frac{9C}{r^5}\right\}\right]$$

$$\frac{\cos^2\phi}{4(1-2\nu)}\left\{\frac{4A(1-2\nu)}{r^3} + \frac{(7-8\nu)\nu A}{(1-\nu)r^3} - \frac{5A+4A\nu}{r^3}\right.$$

$$-A(1+4\nu)(1-\nu) + A(7\nu-8\nu^2)$$

$$A[-1+3\nu+4\nu^2+7\nu-8\nu^2] = A[-1+4\nu-4\nu^2] = -A(1-4\nu+4\nu^2) = -A(1-2\nu)(1-2\nu)$$

$$\rightarrow 2\mu\left[\frac{\cos^2\phi}{4\cancel{(1-2\nu)}}\cdot\frac{-A(1-2\nu)^{\cancel{2}}}{(1-\nu)r^3} - \frac{2A}{4r^3} + \frac{\cos^2\phi}{4(1-2\nu)}\cdot\frac{21(1-2\nu)C}{(1-\nu)r^5} - \frac{1}{4(1-2\nu)}\cdot\frac{9C(1-2\nu)}{(1-\nu)r^5} + \frac{D}{4(1-2\nu)r^3}\frac{(1-2\nu)}{(1-\nu)}\right.$$

$$\frac{\cos^2\phi}{4(1-2\nu)}\left\{\frac{18C}{r^5}+\frac{3C}{r^5} - \frac{3C}{r^5}\frac{\nu}{(1-\nu)} - \frac{\nu}{1-\nu}\cdot\frac{18C}{r^5}\right\} \frac{-1}{4(1-2\nu)}\left[\frac{9C}{r^5} - \frac{\nu}{1-\nu}\frac{9C}{r^5}\right] + \frac{D}{4(1-2\nu)r^3}\left[-\frac{\nu}{1-\nu}+1\right]$$

$$\frac{21C}{r^5} - \frac{21\nu C}{r^5(1-\nu)} \qquad \frac{21(1-2\nu)C}{(1-\nu)r^5} \qquad = \frac{-1}{}\ \frac{9C(1-2\nu)}{}$$

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$\sigma_{\phi\phi}(\varphi = 90^\circ) + \quad = \sigma_{\phi\phi}(\varphi = 0^\circ) + \quad \sigma_{\theta\theta}(\varphi = 90^\circ)$

$$\sigma_{\phi\phi} = 2\mu\left[\frac{\cos^2\phi}{4(1-\nu)}\left(\frac{-A}{r^3}\right)(1-2\nu) + \frac{\cos^2\phi}{4(1-\nu)}\cdot\frac{21C}{r^5} - \frac{2A}{4r^3} - \frac{9C}{4(1-\nu)r^5} + \frac{D}{4(1-\nu)r^3}\right]$$

take out $4(1-\nu)r^3$

$$\sigma_{\phi\phi} = \frac{\mu}{2(1-\nu)r^3}\left\{\cos^2\phi\left[(2\nu-1)A + \frac{21C}{r^2}\right] - 2(1-\nu)A - \frac{9C}{r^2} + D\right\} + T\sin^2\phi$$

$\sigma_{\phi\phi_{TOT}} = \sigma_{\phi\phi} + T\sin^2\phi \qquad @ \quad r = a \quad \phi = 90^\circ$

$$\sigma_{\phi\phi} = \frac{\cancel{\mu}}{\cancel{2(1-\nu)a^3}}\left\{+2\cancel{(1-\nu)}\left[\frac{5(1-\nu)\cancel{a^3}T}{\cancel{\mu}(7-5\nu)}\right] + \frac{9}{\cancel{a^2}}\frac{\cancel{(1-\nu)}\,\cancel{a^5}T}{\cancel{\mu}(7-5\nu)} - \frac{\cancel{(1-\nu)}(6-5\nu)\cancel{a^3}T}{\cancel{\mu}(7-5\nu)}\right\} + T$$

$10(1-\nu) + 9 - 6 + 5\nu$

$$= \frac{T}{2(7-5\nu)}\left\{+10 - 10\nu + 9 - 6 + 5\nu \quad \cancel{7+15\nu}\right\} \qquad \frac{-7+15\nu}{2(7-5\nu)}T + \frac{2(7-5\nu)}{2(7-5\nu)}T$$

$13 - 5\nu + 14 - 10\nu$

$= 7 + 5\nu$

$$\frac{27 - 15\nu}{2(7-5\nu)} \qquad \frac{3}{2}\frac{(9-5\nu)}{7-5\nu}$$

$\varphi = 90$ $\qquad\qquad \varphi = 0$

$$\frac{3(9-5\nu)}{2(7-5\nu)} \qquad \frac{-3-15\nu}{2(7-5\nu)} \qquad -3+15\nu$$

$-5\nu A - 5D + \frac{30C}{a^2} = 0$

$$(5-\nu)A - 18\frac{C}{a^2} = -\frac{T(1-\nu)a^3}{\mu}$$

$$(1+\nu)A - 12\frac{C}{a^2} = \frac{T(1-\nu)a^3}{\mu}$$

$$6A - \frac{30C}{a^2} = 0 \qquad A = +\frac{5C}{a^2}$$

$$(5-\nu)\frac{5C}{a^2} - \frac{18C}{a^2}$$

$$-\frac{\nu 5C}{a^2} + \frac{6C}{a^2} = D \qquad \frac{7C}{a} \quad \frac{\cancel{(7-5\nu)}C}{\cancel{a^2}} = -\frac{T(1-\nu)a^5}{\mu(7-5\nu)} = C$$

$$\frac{C}{a^2}(6-5\nu)$$

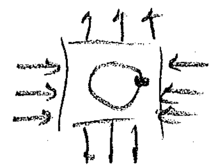
$\sigma_{\varphi\varphi}$

$$\frac{\mu\nu}{1-\nu}\, A\left(\frac{1}{R}\right)_{,zz} - \frac{\mu}{2(1-\nu)}\left[\left\{zA\left(\frac{1}{R}\right)_{,xxz} + D\left(\frac{1}{R}\right)_{,xx} + C\left(\frac{1}{R}\right)_{,xxzz}\right\}\sin^2\theta + \left\{zA\left(\frac{1}{R}\right)_{,zyy} + D\left(\frac{1}{R}\right)_{,yy} + C\left(\frac{1}{R}\right)_{,yyzz}\right\}\cos^2\theta - 2\sin\theta\cos\theta\left\{zA\left(\frac{1}{R}\right)_{,zxy} + D\left(\frac{1}{R}\right)_{,xy} + C\left(\frac{1}{R}\right)_{,xyzz}\right\}\right]$$

$$\sigma_{\theta\theta} = -\left(\frac{3-15\sigma}{14-10\sigma}\right)T = -\frac{3}{2}\left(\frac{1-5\nu}{7-5\nu}\right)T.$$

$$\sigma_{\varphi\varphi} = \frac{3(9-5\nu)}{2(7-5\nu)}T. \quad (\text{at } \varphi = 90^\circ), \quad \sigma_{\varphi\varphi}\ (\text{at } \varphi = 0) = \frac{-3-15\nu}{2(7-5\nu)}\, T = \frac{-3(1+5\nu)}{2(7-5\nu)},$$

if T is reversed $\sigma_{\varphi\varphi} = \frac{3(9-5\nu)}{2(7-5\nu)}T + \frac{3(1+5\nu)}{2(7-5\nu)}T = \frac{15T}{(7-5\nu)}$



J Goodier Pg 39 APM-55-7 Trans ASME 1933 Vol 55.

we need to add $\sigma_{\varphi\varphi_{\varphi=90^\circ}} + \sigma_{\varphi\varphi_{\varphi=0^\circ}} + \sigma_{\theta\theta} = T\frac{[27-15\nu-3-15\nu-3+15\nu]}{2(7-5\nu)}$

$$= T\left[\frac{21-15\nu}{2(7-5\nu)}\right] = \frac{3T}{2}\left(\frac{\cancel{7-5\nu}}{\cancel{7-5\nu}}\right) = \frac{3T}{2}$$ ✓✓

$$\frac{A}{a^3} = -\frac{T}{8\mu_1}\,\frac{13-10\nu}{7-5\nu}\ ; \quad \frac{B}{a^5} = \frac{T}{8\mu}\,\frac{1}{7-5\nu}\ ; \quad \frac{C}{a^3} = \frac{T}{8\mu}\,\frac{5(1-2\nu)}{7-5\nu}$$

and $\sigma_{RR} = 2\mu\left\{\frac{2A}{r^3} - \frac{2}{1-2\nu}\frac{C}{r^3} + 12\frac{B}{r^5} + \left[-\frac{2(5-\nu)}{1-2\nu}\frac{C}{r^3} + \frac{36B}{r^5}\right]\cos 2\varphi\right\}$ ~~+T sin~~ $+ T\cos^2\varphi$

$$\sigma_{\varphi\varphi} = 2\mu\left\{-\frac{A}{r^3} - \frac{2\nu}{(1-2\nu)}\frac{C}{r^3} - 3\frac{B}{r^5} + \left[\frac{C}{r^3} - 21\frac{B}{r^5}\right]\cos 2\varphi\right\} + T\sin^2\varphi$$

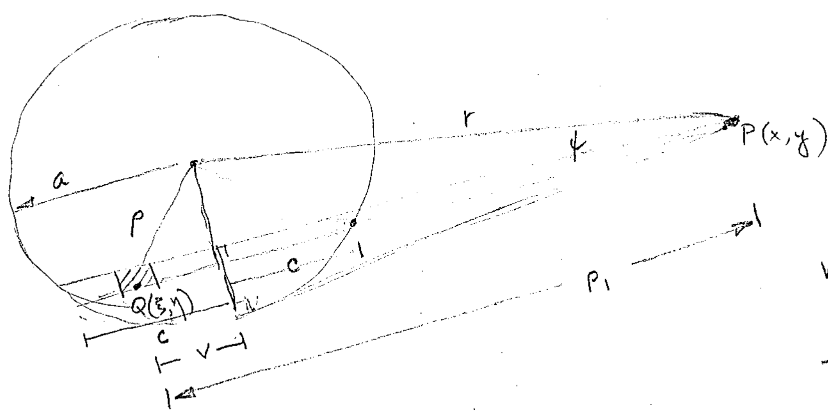
$$\sigma_{\theta\theta} = 2\mu\left\{-A/r^3 - \frac{2(1-\nu)}{1-2\nu}\frac{C}{r^3} - 9B/r^5 + \left[3C/r^3 - 15B/r^5\right]\cos 2\varphi\right\} + 0$$

$$\sigma_{r\varphi} = 2\mu\left\{-\frac{2(1+\nu)}{1-2\nu}\frac{C}{r^3} + \frac{24B}{r^5}\right\}\sin 2\varphi - \frac{T}{2}\sin 2\varphi$$

$$A^2\ = \frac{2\mu T}{8\mu(7-5\nu)}\left\{-\frac{(1+\nu)}{(1-2\nu)}\cdot 5(1-2\nu)\frac{a^3}{r^3} + 12\cdot 5(1-2\nu)\,a^5/r^5\right\}\sin 2\varphi$$

$$= \frac{5T}{2(7-5\nu)}\frac{a^3}{r^5}\left\{-(1+\nu)r^2 + 12(1-2\nu)a^2\right\}\sin 2\varphi$$

$$w = \frac{1-\nu}{2\pi\mu}\int \frac{N/2\pi a}{\sqrt{a^2-\rho^2}} \cdot \frac{d\xi\, d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}$$



$v = \rho_1 - r\cos\psi$

$a^2 - r^2\sin^2\psi = c^2$

$\rho^2 = v^2 + r^2\sin^2\psi$

$\rho_1 = \sqrt{(x-\xi)^2+(y-\eta)^2}$

$\sqrt{a^2-\rho^2} = \sqrt{c^2-v^2}$

$\cancel{\rho_1^2 = r^2\cos^2\psi + v^2}$

$$w = \frac{(1-\nu)N}{4\pi^2\mu a}\int \frac{d\xi\, d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}\,\left(\sqrt{a^2-\rho^2}\right)} = \frac{(1-\nu)N}{4\pi^2\mu a}\int \frac{\cancel{\rho_1}\, d\psi\, d\rho_1}{\cancel{\rho_1}\left(\sqrt{a^2-\rho^2}\right)}$$

$$2\,\frac{(1-\nu)N}{4\pi^2\mu a}\int_0^{\psi_1}\int_{-c}^{c} \frac{dv\, d\psi}{\sqrt{c^2-v^2}} \qquad \psi_1 = \arcsin a/r. \quad \text{ie } c=0$$

$$\cancel{4}\cdot\frac{2(1-\nu)N}{4\pi^2\mu a}\int \sin^{-1}\frac{c}{v}\Big|_{-c}^{c}\int_0^{\arcsin a/r} d\psi$$

$$\frac{(1-\nu)N}{\cancel{2}\pi\mu a}\arcsin\frac{a}{r}$$

$$\frac{2\mu}{4(1-2\nu)}\left[\frac{1}{4\cancel{(1-2\nu)}}\left\{\cos^2\varphi\left(\frac{-3A}{r^3}+\frac{12C}{r^5}\right) - 3C\frac{\sin^2\varphi}{r^5} + \frac{D}{r^3}\right\} + \frac{\nu}{\cancel{4(1-\nu)}}\left\{\frac{\cos^2\varphi}{\cancel{4}(1-\nu)}\left[\frac{-3A}{r^3}+\frac{12C}{r^5}\right] - \frac{\sin^2\varphi}{\cancel{4}(1-\nu)}\left[\frac{2A(1-2\nu)}{r^3}+\frac{9C}{r^5}\right] + \frac{D}{\cancel{4}(1-\nu)r}\right.\right.$$

$$+\frac{\cos^2\varphi}{\cancel{4}(1-\nu)}\left[\frac{2A(5-4\nu)}{r^3} - \frac{24C}{r^5}\right] + \frac{12C}{\cancel{4}(1-\nu)}\frac{\sin^2\varphi}{r^5} - \frac{2D}{\cancel{4}(1-\nu)r^3}$$

$(1-2\nu)(3-4\nu)$

$-3A - 10A\nu - 8A\nu^2 = -A(3+10\nu+8\nu^2)$

$$\frac{2\mu}{4(1-2\nu)}\left\{\frac{\cos^2\varphi}{r^3}\left[-3A - \frac{3A\nu}{1-\nu} + \frac{2A(5-4\nu)\nu}{1-\nu}\right] + \frac{\cos^2\varphi}{r^5}\left[12C + \frac{12C\nu}{1-\nu} - \frac{24C\nu}{1-\nu}\right] + \frac{\sin^2\varphi}{r^5}\left[-3C - \frac{9C\nu}{1-\nu} + \frac{12C\nu}{1-\nu}\right] - \frac{\sin^2\varphi}{r^3}\cdot\frac{2A\nu(1-2\nu)}{1-\nu}\right.$$

$-\dfrac{A(1-2\nu)(3-4\nu)}{1-\nu}$ $\quad 12C \quad -24C\nu \quad \dfrac{12C(1-2\nu)}{1-\nu} \quad 3C\nu$

$\dfrac{-3C+3C\nu+3C\nu}{1-\nu} = -3C\dfrac{(1-2\nu)}{1-\nu}$

$$+\frac{D}{r^3}\left[1 + \frac{\nu}{1-\nu} - \frac{2\nu}{1-\nu}\right] \qquad \frac{1-2\nu}{1-\nu}$$

$$\therefore \frac{\cancel{2}\mu}{2\cancel{4}\cancel{(1-2\nu)}}\left\{\frac{\cos^2\varphi}{r^3}\left[\frac{-A\cancel{(1-2\nu)}(3-4\nu)}{1-\nu}\right] + \frac{\cos^2\varphi}{r^5}\left[\frac{12C\cancel{(1-2\nu)}}{1-\nu}\right] + \frac{\sin^2\varphi}{r^5}\left[\frac{-3C\cancel{(1-2\nu)}}{1-\nu}\right] - \frac{\sin^2\varphi}{r^3}\left[\frac{2A\nu\cancel{(1-2\nu)}}{1-\nu}\right] + \frac{D}{r^3}\left(\frac{\cancel{1-2\nu}}{1-\nu}\right)\right\}$$

$$\frac{\mu}{2(1-\nu)}\left[\left\{\frac{(4\nu-3)A}{r^3} + \frac{12C}{r^5}\right\}\cos^2\varphi + \sin^2\varphi\left[\frac{-3C}{r^5} - \frac{2A\nu}{r^3}\right] + \frac{D}{r^3}\right]$$

$$\frac{\mu}{2(1-\nu)R^3}\left[-\frac{3C}{R^2} - 2A\nu + D \right]$$

$$\frac{\cancel{\mu}}{2(\cancel{1-\nu})}\left[+\frac{3(\cancel{1-\nu})T}{\cancel{\mu}(7-5\nu)} + 2\nu\cdot\frac{5(\cancel{1-\nu})T}{\cancel{\mu}(7-5\nu)} - \frac{(\cancel{1-\nu})(6-5\nu)T}{\cancel{\mu}(7-5\nu)} \right] = \frac{T}{2(7-5\nu)}\left[3+10\nu-6+5\nu \right.$$

$$-3+15\nu$$

$\cancel{3T}$

$$-3T \quad \frac{1-5\nu}{2(7-5\nu)}$$

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ME 238C
Prof Herrmann

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Problem Set #2

1. For a center of Dilatation located at (0,0,0) $u_{(1)} = -\frac{C}{4(1-\nu)} \nabla\left(\frac{1}{r}\right)$ where $r = (x_i x_i)^{1/2}$

For a center of compression located at (h,0,0) we find that for the same strength

$$u_{(2)} = -\frac{C}{4(1-\nu)} \nabla\left(\frac{-1}{r'}\right) \quad \text{where} \quad r = \left[(x_i - h\delta_{i1})(x_i - h\delta_{i1})\right]^{1/2}$$

Thus in an infinite body affected by a center of dilatation & a center of compression a distance h away gives (for a linearly elastic body where superposition holds)

$$u = u_{(1)} + u_{(2)} = -\frac{C}{4(1-\nu)} \nabla\left(\frac{1}{r} - \frac{1}{r'}\right) \;;$$

for small h we can write that $\frac{1}{r'} = \frac{1}{r} - h\frac{\partial}{\partial x}\left(\frac{1}{r'}\right)\Big|_{h=0} + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r'}\right)\Big|_{h=0} + \dots$

thus $\frac{1}{r} - \frac{1}{r'} = h\frac{\partial}{\partial x}\left(\frac{1}{r'}\right)\Big|_{h=0} - \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r'}\right)\Big|_{h=0} + \dots$

$$\therefore \; u = -\frac{C}{4(1-\nu)} \nabla\left[h\frac{\partial}{\partial x}\left(\frac{1}{r'}\right)\Big|_{h=0} - \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r'}\right)\Big|_{h=0} + \dots\right] \;; \quad \text{if: } \lim_{h\to 0} Ch \triangleq D$$

$$\text{thus} \quad u = -\frac{D}{4(1-\nu)} \nabla\left(\frac{\partial}{\partial x}\left[\frac{1}{r}\right]\right) = -\frac{D}{4(1-\nu)}\frac{\partial}{\partial x}\nabla\left(\frac{1}{r}\right) = \frac{D}{4(1-\nu)}\nabla\left(\frac{x}{r^3}\right)$$

$$\text{and} \quad u_i = \frac{D}{4(1-\nu)r^3}\left[\delta_{i1} - \frac{3x_1 x_i}{r^2}\right]$$

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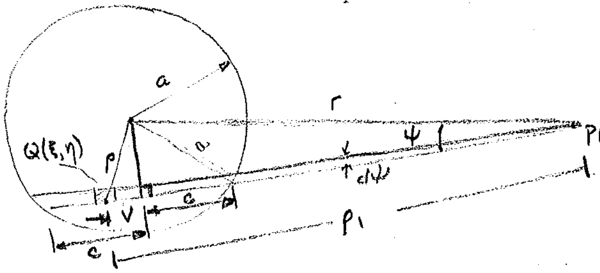
another method: if $u_2 = u_1(h,0,0)$ then $u_2 = u_1 - h\frac{\partial u_1}{\partial x}\Big|_{h=0} + \dots$ where u_1 is displ of center of dilatation. For center of compression $u_2 = -\left[u_1 - h\frac{\partial u_1}{\partial x}\Big|_{h=0} + \dots\right]$

For center of comp + center of dilatation $u_1 + u_2 = h\frac{\partial u_1}{\partial x}\Big|_{h=0} + O(h^2) = -\frac{Ch}{4(1-\nu)}\frac{\partial}{\partial x}\left(\nabla\frac{1}{r}\right) + O(h^2)$

using $\lim_{h\to 0} Ch = D$ then $u = \lim_{h\to 0}\left[u_1 + u_2\right] = -\frac{D}{4(1-\nu)}\frac{\partial}{\partial x}\left(\nabla\frac{1}{r}\right)$ as before.

3. $$w = \frac{1-\nu}{2\pi\mu}\int \frac{N/2\pi a}{\sqrt{a^2-\rho^2}} \frac{d\xi\, d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}} \qquad \text{where} \quad N = \int_{\text{circle}} p(\xi,\eta)\,d\xi\, d\eta$$

For points outside



$$v = \rho_1 - r\cos\psi$$

$$\left.\begin{array}{l} a^2 - r^2\sin^2\psi = c^2 \\ \rho^2 = v^2 + r^2\sin^2\psi \end{array}\right\} \Rightarrow \sqrt{a^2-\rho^2} = \sqrt{c^2 - v^2}$$

$$\rho_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2}$$

Now if $P(x,y)$ is a fixed point and we draw the rays as shown so that $Q(\xi,\eta)$ is any point in the circle, then the area $d\xi d\eta = \rho_1 d\psi d\rho_1$

$$\therefore \quad w = \frac{1-\nu}{4\pi^2\mu a} N \int \frac{\rho_1 d\psi d\rho_1}{\sqrt{a^2-\rho^2}\cdot\rho_1} = \frac{(1-\nu)N}{4\pi^2\mu a}\int \frac{d\psi d\rho_1}{\sqrt{a^2-\rho^2}}$$ where ψ goes from $-\psi_1$ to $+\psi_1$ and

$\psi_1 = \arcsin a/r$ along constant ψ lines. $dv = d\rho_1$ and v goes from $-c$ to $+c$. Thus

$$w = \frac{(1-\nu)N}{4\pi^2\mu a}\int_{-c}^{c}\int_{-\psi_1}^{\psi_1}\frac{dv\,d\psi}{\sqrt{c^2-v^2}} = \frac{(1-\nu)N}{4\pi^2\mu a}\int_{-\psi_1}^{\psi_1}d\psi\int_{-c}^{c}\frac{dv}{\sqrt{c^2-v^2}} = \frac{(1-\nu)N}{4\pi^2\mu a}\cdot 2\psi_1\cdot \arcsin\frac{c}{v}\Big|_{-c}^{c}$$

$$= \frac{(1-\nu)N}{4\pi^2\mu a}\cdot 2\arcsin a/r \cdot \pi = \frac{(1-\nu)N}{2\pi\mu a}\arcsin a/r$$

$$\therefore \quad w(x,y,z=0) = \frac{(1-\nu)N}{2\pi\mu a}\arcsin\frac{a}{r} \checkmark \qquad r = \sqrt{x^2+y^2}$$

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2. Starting with $\mathbf{u} = \mathbf{B} - \frac{1}{4(1-\nu)}\nabla(\mathbf{r}\cdot\mathbf{B}+\beta)$ with $\mathbf{r} = R\mathbf{e}_R$ & $\nabla = \mathbf{e}_R\frac{\partial}{\partial R} + \frac{\mathbf{e}_\varphi}{R}\frac{\partial}{\partial\varphi} + \frac{\mathbf{e}_\theta}{R\sin\varphi}\frac{\partial}{\partial\theta}$

we find that we need $\frac{\partial}{\partial z}$ in spherical coordinates

$$\nabla = \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z} \Rightarrow \mathbf{e}_z\cdot\nabla = \frac{\partial}{\partial z} = \mathbf{e}_z\cdot\mathbf{e}_R\frac{\partial}{\partial R} + \mathbf{e}_z\cdot\mathbf{e}_\varphi\frac{1}{R}\frac{\partial}{\partial\varphi} + \mathbf{e}_z\cdot\mathbf{e}_\theta\frac{1}{R\sin\varphi}\frac{\partial}{\partial\theta}$$

from a previous homework we had found $\mathbf{e}_z\cdot\mathbf{e}_R = \cos\varphi \quad \mathbf{e}_z\cdot\mathbf{e}_\varphi = -\sin\varphi \quad \mathbf{e}_z\cdot\mathbf{e}_\theta = 0$

$$\therefore \frac{\partial}{\partial z} = \cos\varphi\frac{\partial}{\partial R} - \frac{\sin\varphi}{R}\frac{\partial}{\partial\varphi} \qquad \therefore \frac{\partial}{\partial z}\left(\frac{1}{R}\right) = \cos\varphi\left(-\frac{1}{R^2}\right) = -\frac{\cos\varphi}{R^2}$$

To do this problem we need to find $\sigma_{\theta\theta}$. But we are given

$$B_3'' = A\frac{\partial}{\partial z}\left(\frac{1}{R}\right) \quad\text{and}\quad \beta'' = \frac{D}{R} + \frac{\partial^2}{\partial z^2}\left(\frac{C}{R}\right) \quad\text{where}\quad A = -\frac{5(1-\nu)a^3T}{\mu(7-5\nu)} \qquad D = -\frac{(1-\nu)(6-5\nu)a^3T}{\mu(7-5\nu)}$$

and $C = -\frac{(1-\nu)a^5T}{\mu(7-5\nu)}$; B_1 and B_2 are also given as zero

$$\mathbf{B}'' = A\frac{\partial}{\partial z}\left(\frac{1}{R}\right)\mathbf{e}_z = B_3\mathbf{e}_z = -A\frac{\cos\varphi}{R^2}\left[\cos\phi\,\mathbf{e}_R - \sin\phi\,\mathbf{e}_\phi\right]$$

$$\mathbf{r}\cdot\mathbf{B}'' = R\mathbf{e}_R\left[-\frac{A\cos\varphi}{R^2}\cos\varphi\,\mathbf{e}_R + \text{term which gives zero}\right] = -\frac{A\cos^2\varphi}{R}$$

$$\beta'' = \frac{D}{R} + C\frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\left(\frac{1}{R}\right)\right] = \frac{D}{R} + C\frac{\partial}{\partial z}\left(-\frac{\cos\varphi}{R^2}\right) = \frac{D}{R} + C\left\{\frac{2\cos^2\varphi}{R^3} - \frac{\sin^2\varphi}{R^3}\right\}$$

plugging these into $\mathbf{u}$ and using ∇ in spherical we obtain $\mathbf{u}$ and its components are

$$u_R'' = \frac{\cos^2\varphi}{4(1-\nu)}\left[\frac{-A(5-4\nu)}{R^2} + \frac{3C}{R^4}\right] + \frac{D}{4(1-\nu)R^2} + \frac{3C}{4(1-\nu)}\frac{\cos 2\varphi}{R^4} \;;\quad u_\varphi'' = \frac{\sin 2\varphi}{4(1-\nu)}\left[\frac{A(1-2\nu)}{R^2} + \frac{3C}{R^4}\right]$$

$$u_\theta'' = 0$$

Plugging into

$$\varepsilon''_{RR} = \frac{\partial u_R}{\partial R} = \frac{\cos^2\varphi}{4(1-\nu)}\left[\frac{2A(5-4\nu)}{R^3} - \frac{12C}{R^5}\right] - \frac{2D}{4(1-\nu)R^3} - \frac{12C}{4(1-\nu)}\frac{\cos 2\varphi}{R^5} \qquad \cos 2\varphi = 2\cos^2\varphi - 1$$

$$\varepsilon''_{\theta\theta} = \frac{u_R}{R} + u_\varphi \frac{\cot\varphi}{R} = \frac{\cos^2\varphi}{4(1-\nu)}\left[-\frac{3A}{R^3} + \frac{9C}{R^5}\right] + \frac{D}{4(1-\nu)R^3} + \frac{3C}{4(1-\nu)}\frac{\cos 2\varphi}{R^5}$$

$$\varepsilon''_{\varphi\varphi} = \frac{1}{R}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_R}{R} = \frac{\cos 2\varphi}{4(1-\nu)}\left[\frac{2A(1-2\nu)}{R^3} + \frac{9C}{R^5}\right] + \frac{\cos^2\varphi}{4(1-\nu)}\left[-\frac{A(5-4\nu)}{R^3} + \frac{3C}{R^5}\right] + \frac{D}{4(1-\nu)R^3}$$

$$\varepsilon''_{R\theta} = \varepsilon''_{\theta\varphi} = 0$$

$$\varepsilon''_{\varphi R} = \frac{\sin 2\varphi}{4(1-\nu)}\left[\frac{A(1+\nu)}{R^3} - \frac{12C}{R^5}\right]$$

from these we get $\sigma''_{R\theta} = 0 \quad \sigma''_{\theta\varphi} = 0 \quad \sigma''_{\varphi R} = 2\mu\,\varepsilon''_{\varphi R} = \mu\frac{\sin\varphi\cos\varphi}{(1-\nu)}\left[\frac{A(1+\nu)}{R^3} - \frac{12C}{R^5}\right]$

$$\sigma''_{RR} = \lambda\varepsilon''_{kk}\delta_{RR} + 2\mu\varepsilon''_{RR} = (\lambda+2\mu)\varepsilon''_{RR} + \lambda(\varepsilon''_{\theta\theta} + \varepsilon''_{\varphi\varphi}) = \frac{\mu\cos^2\varphi}{(1-\nu)R^3}\left[(5-\nu)A - \frac{18C}{R^2}\right] + \frac{\mu}{(1-\nu)R^3}\left[-\nu A - D + \frac{6C}{R^2}\right]$$

$$\sigma''_{\theta\theta} = 2\mu\left[\frac{1-\nu}{1-2\nu}\varepsilon_{\theta\theta} + \frac{\nu}{1-2\nu}(\varepsilon_{\phi\phi} + \varepsilon_{RR})\right] = \frac{\mu}{2(1-\nu)}\left[\left\{\frac{(4\nu-3)A}{R^3} + \frac{12C}{R^5}\right\}\cos^2\varphi + \sin^2\varphi\left\{-\frac{3C}{R^5} - \frac{2A\nu}{R^3}\right\} + \frac{D}{R^3}\right]$$

$$3\frac{(2\nu-1)A}{R^3} + \frac{15C}{R^5}\Big\}\cos^2\varphi \qquad \left\{-\frac{3C}{R^5} - \frac{2A\nu}{R^3}\right\}$$

$$\sigma''_{\varphi\varphi} = \frac{\mu}{2(1-\nu)R^3}\left[\cos^2\varphi\left[(2\nu-1)A + \frac{21C}{R^2}\right] - 2(1-\nu)A - \frac{9C}{R^2} + D\right]$$

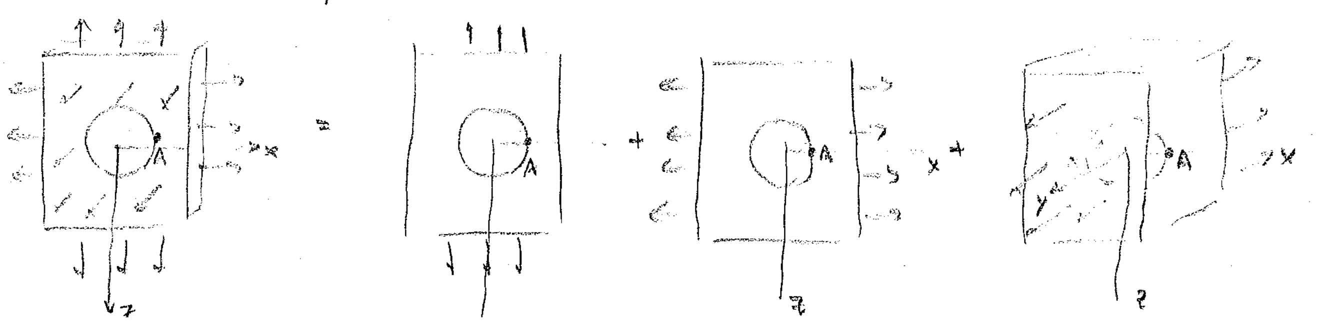
$\sigma_{RR_{TOT}} = \sigma''_{RR} + T\cos^2\varphi = 0$ on $r=a$ $\forall\varphi$ (show) $\qquad \sigma_{R\phi_{TOT}} = \sigma''_{R\phi} - T\frac{\sin 2\varphi}{2} \qquad \sigma_{\varphi\varphi_{TOT}} = \sigma''_{\varphi\varphi} + T\sin^2\varphi$ $=?$ on $r=a$ $\qquad \sigma_{\theta\theta_{TOT}} = \sigma''_{\theta\theta} = ?$ on $r=a$

$\sigma_{R\theta_{TOT}} = 0 \qquad \sigma_{\theta\varphi_{TOT}} = 0$

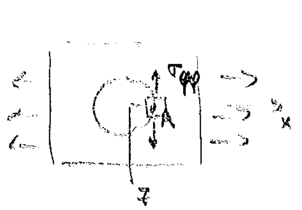
- We find that $\sigma_{\varphi\varphi_{max}}$ occurs on $\varphi = 90^\circ$ plane since both A, C are negative and $2\nu-1<0$ for $0<\nu<\frac{1}{2}$ and $\frac{\partial^2\sigma_{\varphi\varphi}}{\partial\varphi^2} = -\cos 2\varphi \cdot$ negative no. $\Rightarrow$ for max $\Rightarrow \cos 2\varphi > 0$ but $\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi} = 0 \Rightarrow \varphi = 90^\circ$ or $0^\circ \Rightarrow$ max at $\varphi = 90$
- we also find that $\sigma_{\theta\theta_{max}}$ occurs at $\varphi = 90^\circ$
- We also find that $\sigma_{R\varphi} = 0$ at $\varphi = 90^\circ$ $\quad\therefore\ \sigma_{RR}, \sigma_{\varphi\varphi}, \sigma_{\theta\theta}$ are indeed principal stresses at $\varphi = 90^\circ$

They are principal everywhere on $r=a$

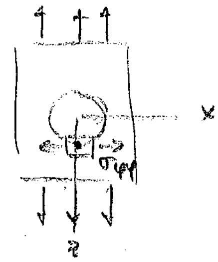
Now let us look at point A



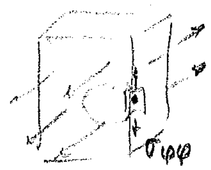
The second picture



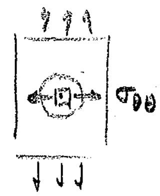
Equiv. =



The third picture



Equiv =



$$\therefore \quad \sigma_{\varphi\varphi}\Big|_{at\,A} = \sigma_{\varphi\varphi}\Big|_{\varphi=90°} + \sigma_{\varphi\varphi}\Big|_{\varphi=0°} + \sigma_{\theta\theta}\Big|_{\varphi=90°}$$

but in class we calculated $\sigma_{\varphi\varphi}\big|_{\varphi=90°} = \dfrac{3(9-5\nu)T}{2(7-5\nu)}$; also $\sigma_{\varphi\varphi}\big|_{\varphi=0°} = \dfrac{-3(1+5\nu)T}{2(7-5\nu)}$ and $\sigma_{\theta\theta}\big|_{\varphi=90°} = \dfrac{-3T}{2}\dfrac{(1-5\nu)}{7-5\nu}$

when we add these 3 together we get $3T/2$ ✓

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DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING
STANFORD UNIVERSITY

ME 238C Theory of Elasticity   Spring 1979

Problem Set No. 3

1. Show that the displacement equations of equilibrium are satisfied in terms of a vector $\mathbf{G}$ if one takes

$$\mathbf{u} = \nabla^2 \mathbf{G} - \frac{1}{2(1-\nu)} \nabla(\nabla \cdot \mathbf{G})$$

and provided $\mathbf{G}$ (Galerkin vector) is biharmonic. Express the Boussinesq-Papkovich functions in terms of the Galerkin vector.

✓ 2. Verify the expressions for the components of stress σ_{rr}, $\sigma_{r\phi}$ derived in class due to displacement potentials

done already

$$B_z = A \frac{\partial}{\partial z}\left(\frac{1}{r}\right) \quad ; \qquad \beta = \frac{D}{r} + \frac{\partial^2}{\partial z^2}\left(\frac{c}{r}\right)$$

3. Consider a half-space with a traction-free bounding plane at $z = 0$ and subjected to a concentrated force in the z direction acting at (0, 0, c). Calculate the displacement in the z direction.

k $\sigma_{zz}=0$ $\sigma_{zy}=0$ $\sigma_{zx}=0$

x

(0,0,c)   find u

z

$$\mathbb{u} = \nabla^2 \mathbb{G} - \frac{1}{2(1-\nu)} \nabla(\nabla\cdot\mathbb{G})$$

if $\mathbb{u} = \nabla\phi + \nabla\times\mathbb{H}$ then $\phi = -\frac{1}{4\pi}\int_V \frac{\nabla\cdot\mathbb{u}}{R}\,dV$ $\quad \mathbb{H} = \frac{1}{4\pi}\int_V \frac{\nabla\times\mathbb{u}}{R}\,dV$

$$\mathbb{u} = \mathbb{B} - \frac{\nabla}{4(1-\nu)}\left[\mathbb{r}\cdot\mathbb{B} + \beta\right]$$

$\Rightarrow$ $\mathbb{B} = \nabla^2\mathbb{G}$ and $\frac{\mathbb{r}\cdot\mathbb{B}+\beta}{2} = \nabla\cdot\mathbb{G}$ $\quad\therefore \frac{\mathbb{r}\cdot\ \mathbb{G}+\beta}{2} = \nabla\cdot\mathbb{G}$

$\beta = 2\nabla\cdot\mathbb{G} - \mathbb{r}\cdot\nabla^2\mathbb{G}$ $\qquad \nabla^2(\underline{a}\cdot\underline{b}) = 0$

$\nabla^2\beta = \cancel{2\nabla\cdot\mathbb{G}} - 2\nabla\cdot\nabla^2\mathbb{G} - \nabla^2(\mathbb{r}\cdot\nabla^2\mathbb{G})$

what to show that $\mathbb{u}$ satisfies $\nabla^2\mathbb{u} + \frac{1}{1-2\nu}\nabla(\nabla\cdot\mathbb{u}) = 0$

now $\nabla\cdot\mathbb{u} = \nabla\cdot(\nabla^2\mathbb{G}) - \frac{1}{2(1-\nu)}\nabla\cdot\left[\nabla(\nabla\cdot\mathbb{G})\right]$

$\frac{\partial}{\partial x_l} e_l \cdot \frac{\partial^2}{\partial x_i \partial x_i} G_j e_j \qquad \frac{\partial}{\partial x_i} e_i \cdot \frac{\partial}{\partial x_l} e_l \frac{\partial G_j}{\partial x_j}$

$= \nabla^2(\nabla\cdot\mathbb{G}) - \frac{1}{2(1-\nu)}\nabla^2(\nabla\cdot\mathbb{G}) = \nabla^2\left[\frac{1-2\nu}{2(1-\nu)}\nabla\cdot\mathbb{G}\right]$

$\frac{1}{1-2\nu}\nabla(\nabla\cdot\mathbb{u}) = \frac{\nabla^2}{2(1-\nu)}\nabla(\nabla\cdot\mathbb{G})$

$\frac{\nabla(\nabla\cdot\mathbb{u})}{1-2\nu} + \nabla^2\mathbb{u} = \nabla^4\mathbb{G} - \frac{\nabla^2}{2(1-\nu)}\nabla(\nabla\cdot\mathbb{G}) + \frac{\nabla^2}{2(1-\nu)}\nabla(\nabla\cdot\mathbb{G}) = 0$ since $\mathbb{G}$ satisfies $\nabla^4\mathbb{G} = 0$

and $f = 0$

$\nabla^2\beta = 2\nabla\cdot\nabla^2\mathbb{G} - \nabla^2(\mathbb{r}\cdot\nabla^2\mathbb{G})$

$\cancel{x_i \frac{\partial^2}{\partial x_l \partial x_l}} \cancel{x_i \frac{\partial^2 G_i}{\partial x_j \partial x_j}} = \cancel{x_i \nabla^4 G_i}$

$2\nabla\cdot\nabla^2\mathbb{G} - 2\nabla^2(\nabla\cdot\mathbb{G})$ $\quad \frac{\partial}{\partial x_k} e_k \cdot \frac{\partial}{\partial x_l} e_l \left[x_i \frac{\partial^2 G_i}{\partial x_j \partial x_j}\right]$

$+\mathbb{r}\cdot\nabla^4\mathbb{G}$

$\frac{\partial}{\partial x_k} e_k \cdot \left\{ \delta_{il} \frac{\partial^2 G_i}{\partial x_j \partial x_j} e_l + x_i \frac{\partial^3 G_i e_l}{\partial x_l \partial x_j^2}\right\}$

$\cdot\left\{ \frac{\partial^2 G_i}{\partial x_j \partial x_j} e_i + x_i \frac{\partial^3 G_i e_l}{\partial x_l \partial x_j^2}\right.$

$\frac{\partial G_i}{\partial x_k \partial x_j^2}\delta_{ik} + \delta_{ik}\frac{\partial^3 G_i}{\partial x_l \partial x_j^2}\delta_{lk}$

$+ x_i \frac{\partial^4 G_i}{\partial x_k \partial x_l \partial x_j^2}\delta_{lk}$

$\frac{\partial^3 G_k}{\partial x_k \partial x_j^2}$

$\nabla^2(\nabla\cdot\mathbb{G}) + \nabla^2(\nabla\cdot\mathbb{G}) + x_i\nabla^4 G_i$

$\mathbf{u} = \nabla^2 \mathbf{G} - \frac{1}{2(1-\nu)} \nabla (\nabla \cdot \mathbf{G})$ must satisfy $\sigma_{ij,j} = 0$ / or $\nabla^2 \mathbf{u} + \frac{1}{1-2\nu} \nabla (\nabla \cdot \mathbf{u}) = 0$

$$\nabla (\nabla \cdot \mathbf{G}) = \nabla \cdot \nabla \mathbf{G} + (\mathbf{G} \cdot \nabla)\nabla + \nabla \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \nabla)$$
$$= \nabla^2 \mathbf{G} + (\mathbf{G} \cdot \nabla)\nabla + \nabla(\nabla \cdot \mathbf{G}) - \nabla^2 \mathbf{G} + \mathbf{G} \times (\nabla \times \nabla)$$

using $\nabla \times (\nabla \times \mathbf{G}) = \nabla(\nabla \cdot \mathbf{G}) - \nabla^2 \mathbf{G}$

$\mathbf{u} =$

$$(\nabla \cdot \nabla)\mathbf{G} = \nabla(\nabla \cdot \mathbf{G}) - \nabla \times \nabla \times \mathbf{G} = \nabla(\nabla \cdot \mathbf{G}) - \left[\nabla(\nabla \cdot \mathbf{G}) - (\nabla \cdot \nabla)\mathbf{G}\right]$$

$$\mathbf{u} = \nabla(\nabla \cdot \mathbf{G}) - \nabla \times \nabla \times \mathbf{G} - \frac{1}{2(1-\nu)} \nabla(\nabla \cdot \mathbf{G})$$

$$= \frac{2(1-\nu) - 1}{2(1-\nu)} \nabla(\nabla \cdot \mathbf{G}) - \nabla \times \nabla \times \mathbf{G}$$

$$\mathbf{u} = \frac{1-2\nu}{2(1-\nu)} \nabla(\nabla \cdot \mathbf{G}) - \nabla \times \nabla \times \mathbf{G}$$

$$\nabla \cdot \mathbf{u} = \frac{1-2\nu}{2}$$

$$\nabla \cdot \mathbf{u} = \nabla^2(\nabla \cdot \mathbf{G}) - \frac{1}{2(1-\nu)} \nabla^2(\nabla \cdot \mathbf{G}) = \nabla^2(\nabla \cdot \mathbf{G}) \left[\frac{2-2\nu-1}{2(1-\nu)}\right] = \frac{1-2\nu}{2(1-\nu)} \nabla^2(\nabla \cdot \mathbf{G})$$

$$\nabla(\nabla \cdot \mathbf{u}) = \frac{1-2\nu}{2(1-\nu)} \nabla^2\left[\nabla(\nabla \cdot \mathbf{G})\right] \qquad \frac{1}{1-2\nu} \nabla(\nabla \cdot \mathbf{u}) = \frac{1}{2(1-\nu)} \nabla^2\left[\nabla(\nabla \cdot \mathbf{G})\right]$$

$\cancel{\nabla^2 \mathbf{u} = \nabla^4 \mathbf{G}}$

$$\nabla \cdot \left\{\nabla(\nabla \cdot \mathbf{G})\right\} =$$

$$\mathbf{u} = \nabla^2 \mathbf{G} - \frac{1}{2(1-\nu)} \nabla(\nabla \cdot \mathbf{G})$$

$$\nabla \cdot \mathbf{u} = \nabla^2(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\nabla^2 - \frac{1}{2(1-\nu)} \nabla^2(\nabla \cdot \mathbf{G})$$
$$= \frac{1-2\nu}{2(1-\nu)} \nabla^2(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\nabla^2$$

$$\frac{1}{1-2\nu} \nabla(\nabla \cdot \mathbf{u}) = \frac{\nabla^2 \nabla(\nabla \cdot \mathbf{G})}{2(1-\nu)} \quad \frac{\nabla^2}{1-2\nu} \nabla(\mathbf{G} \cdot \nabla)\nabla^2$$

$$\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right) = -\frac{(z+c)}{R_2^{\,3}}$$

$$\frac{\partial R_2}{\partial z} = \frac{1}{2R_2}\cdot 2(z+c) = \frac{z+c}{R_2}$$

$$\left(\frac{1}{R_2}\right) = -\frac{1}{R_2^2}\frac{\partial R_2}{\partial z}$$

$$-\frac{1}{R_2^{\,3}} - (z+c)^2\left(-\frac{3}{R_2^{\,5}}\right) = -\frac{\left[x^2+y^2+(z+c)^2\right]-3(z+c)^2}{R_2^{\,5}} = -\frac{x^2+y^2-2(z+c)^2}{R_2^{\,5}}$$

$R_2 = z+c \qquad \frac{\partial^2}{\partial z^2}\left(\frac{1}{R^2}\right) = +\frac{2R^2}{R^5} = \frac{2}{R^3}$ ✓

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{R_2}\right) = \frac{\partial}{\partial z}\left(-\frac{z+c}{R_2^{\,3}}\right) = $$

$$\frac{1}{z+c} \qquad \frac{\partial}{\partial z}\left(\frac{-1}{(z+c)^2}\right) \qquad \frac{+2}{(z+c)^3}$$

$$u_z = \frac{1}{4(1-\nu)}\frac{P_z}{4\pi\mu}\left\{\frac{9-24\nu+16\nu^2}{R_2} - \frac{4-12\nu+8\nu^2}{R_2} + \frac{2cz\,(x^2+y^2-2(z+c)^2)}{R_2^{\,5}} + \frac{(z-c)^2}{R_1^{\,3}} + \frac{3-4\nu}{R_1}\right.$$

$$\frac{5-12\nu+8\nu^2}{R_2}$$

$$+2c(3-4\nu)\frac{(z+c)}{R_2^{\,3}} + (3-4\nu)\frac{(z-c)(z+c)}{R_2^{\,3}}$$

$$\frac{(3-4\nu)(z+c)^2}{R_2^{\,3}}\ \boxed{\cancel{z+c}}$$

$$\frac{5-12\nu+8\nu^2}{z+c} + \frac{3-4\nu}{z+c} + \frac{4cz}{(z+c)^3} + \frac{4(1-\nu)}{z+c}$$

$$\frac{8(1-\nu)^2}{z+c}$$

$$\frac{8(1-\nu)^2}{z+c} + \frac{4cz}{(z+c)^3} + \frac{4(1-\nu)}{z-c}$$

$$+2c\frac{1}{(z+c)^2} \qquad -\frac{4c}{(z+c)^3}$$

$$\frac{\partial B_z}{\partial z} = \frac{P_z}{4\pi\mu}\left[-\frac{1}{(z-c)^2} - \frac{3-4\nu}{(z+c)^2} - \frac{4c}{(z+c)^3}\right]$$

$$B_z = \frac{P_z}{4\pi\mu}\left[\frac{1}{z-c} + \frac{3-4\nu}{z+c} + \frac{2c}{(z+c)^2}\right]$$

$$\frac{\partial\beta}{\partial z} = \frac{P_z}{4\pi\mu}\left[\frac{4(1-2\nu)(1-\nu)}{z+c} + \frac{c}{(z-c)^2} + \frac{(3-4\nu)c}{(z+c)^2}\right]$$

$$+\frac{z}{(z-c)^2} + \frac{(3-4\nu)z}{(z+c)^2} + \frac{4zc}{(z+c)^3} - \frac{4(1-3\nu+2\nu^2)}{z+c} - \frac{c}{(z-c)^2} - \frac{(3-4\nu)c}{(z+c)^2}$$

$$\frac{(3-4\nu)}{z-c} + \frac{(3-4\nu)^2}{z+c} + \frac{2c(3-4\nu)}{(z+c)^2}$$

$$9-24\nu+16\nu^2$$

$$\frac{5-12\nu+8\nu^2}{z+c} + \frac{\cancel{(z+c)}(3-4\nu)}{(z+c)^{\cancel{2}}} = \frac{8(1-\nu)^2}{(z+c)} + \frac{\cancel{z-c}}{(z-c)^{\cancel{2}}} + \frac{3-4\nu}{z-c} + \frac{4zc}{(z+c)^3}$$ ✓

$$\frac{8(1-\nu)^2}{z+c} + \underbrace{\frac{4(1-\nu)}{z-c}}$$

$$u_z = \frac{P_z}{16\pi\mu(1-\nu)}\left[\frac{5-12\nu+8\nu^2}{R_2} + \frac{(3-4\nu)(z+c)^2}{R_2^{\,3}} - \frac{2cz\left[R_2^{\,2}-3(z+c)^2\right]}{R_2^{\,5}} + \frac{(z-c)^2}{R_1^{\,3}} + \frac{3-4\nu}{R_1}\right]$$

Prof ...

Cesar Levy
Spring 79

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Problem Set #3

Given $\mathbb{u} = \nabla^2 \mathbb{G} - \frac{1}{2(1-\nu)} \nabla(\nabla\cdot\mathbb{G})$ and $\nabla^4 \mathbb{G} = 0$ prove that it satisfies the displacement equilibrium Eqns and express the Boussinesq papkovich functions in terms of the Galerkin vector $\mathbb{G}$.

$\nabla\cdot\mathbb{u} = \nabla^2(\nabla\cdot\mathbb{G}) - \frac{1}{2(1-\nu)}\nabla^2(\nabla\cdot\mathbb{G}) = \frac{1-2\nu}{2(1-\nu)}\nabla^2(\nabla\cdot\mathbb{G})$ since ∇^2 is a linear operator and commutes with ∇

and $\frac{1}{1-2\nu}\nabla(\nabla\cdot\mathbb{u}) = \frac{\nabla^2}{2(1-\nu)}\nabla(\nabla\cdot\mathbb{G})$ therefore

Now $\nabla^2\mathbb{u} = \nabla^4\mathbb{G} - \frac{\nabla^2}{2(1-\nu)}\nabla(\nabla\cdot\mathbb{G})$;

$\nabla^2\mathbb{u} + \frac{1}{1-2\nu}\nabla(\nabla\cdot\mathbb{u}) = \nabla^4\mathbb{G}$

but since $\mathbb{G}$ satisfies $\nabla^4\mathbb{G} = 0$ then $\nabla^2\mathbb{u} + \frac{1}{1-2\nu}\nabla(\nabla\cdot\mathbb{u}) = 0$. But this is precisely the equilibrium equations (with $\mathbb{f} = 0$) written in terms of displacements. ✓

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since $\mathbb{u} = \nabla^2\mathbb{G} - \frac{1}{2(1-\nu)}\nabla(\nabla\cdot\mathbb{G}) = \mathbb{B} - \frac{1}{4(1-\nu)}\nabla(\mathbb{r}\cdot\mathbb{B} + \beta)$

then $\boxed{\nabla^2\mathbb{G} = \mathbb{B}}$ and $\nabla\cdot\mathbb{G} = \frac{\mathbb{r}\cdot\mathbb{B}+\beta}{2}$ thus $\boxed{2\nabla\cdot\mathbb{G} - \mathbb{r}\cdot\nabla^2\mathbb{G} = \beta}$

2. Beginning with $\mathbb{u} = \mathbb{B} - \frac{1}{4(1-\nu)}\nabla(\mathbb{r}\cdot\mathbb{B} + \beta)$ and defining $\mathbb{r} = r\mathbb{e}_r$ and

$\nabla = \mathbb{e}_r\frac{\partial}{\partial r} + \frac{\mathbb{e}_\varphi}{r}\frac{\partial}{\partial\varphi} + \frac{\mathbb{e}_\theta}{r\sin\varphi}\frac{\partial}{\partial\theta}$ then $\frac{\partial}{\partial z} = \cos\varphi\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}$

$B_z'' = A\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -A\frac{\cos\varphi}{r^2}$ then $\mathbb{B} = B_z''\mathbb{e}_z = -\frac{A\cos\varphi}{r^2}\left[\cos\varphi\,\mathbb{e}_r - \sin\varphi\,\mathbb{e}_\varphi\right]$

$\beta'' = \frac{D}{r} + C\frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right] = \frac{D}{R} + C\left\{\frac{2\cos^2\varphi}{r^3} - \frac{\sin^2\varphi}{r^3}\right\}$

Plugging all these into $\mathbb{u}$ we find that $u_r'' = \frac{\cos^2\varphi}{4(1-\nu)}\left\{-\frac{A(5-4\nu)}{r^2} + \frac{6C}{r^4}\right\} + \frac{D}{4(1-\nu)r^2} + \frac{3C}{4(1-\nu)}\frac{\sin^2\varphi}{r^4}$

$u_\theta'' = 0$ $\quad u_\varphi'' = \frac{\sin 2\varphi}{4(1-\nu)}\left[\frac{A(1-2\nu)}{r^2} + \frac{3C}{r^4}\right]$

and $\varepsilon_{rr}'' = \frac{\partial u_r''}{\partial r} = \frac{\cos^2\varphi}{4(1-\nu)}\left[\frac{2A(5-4\nu)}{r^3} - \frac{24C}{r^5}\right] - \frac{2D}{4(1-\nu)r^3} + \frac{12C\sin^2\varphi}{4(1-\nu)r^5}$ $\quad$ sin $\frac{\partial u_\theta''}{\partial\theta} = 0$

$\varepsilon_{\theta\theta}'' = \frac{u_r''}{r} + u_\varphi''\frac{\cot\varphi}{r} = \frac{\cos^2\varphi}{4(1-\nu)}\left[\frac{-3A}{r^3} + \frac{12C}{r^5}\right] - \frac{3C}{4(1-\nu)}\frac{\sin^2\varphi}{r^5} + \frac{D}{4(1-\nu)r^3}$

$\varepsilon_{\varphi\varphi}'' = \frac{1}{r}\frac{\partial u_\varphi''}{\partial\varphi} + \frac{u_r''}{r} = \frac{\cos^2\varphi}{4(1-\nu)}\left[-\frac{3A}{r^3} + \frac{12C}{r^5}\right] - \frac{\sin^2\varphi}{4(1-\nu)}\left[\frac{2A(1-2\nu)}{r^3} + \frac{9C}{r^5}\right] + \frac{D}{4(1-\nu)r^3}$

$\varepsilon_{\theta\varphi}'' = \frac{1}{2}\left\{\frac{1}{r\sin\varphi}\frac{\partial u_\varphi''}{\partial\theta} + \frac{1}{r}\frac{\partial u_\theta''}{\partial\varphi} - \frac{\cot\varphi}{r}u_\theta''\right\} = 0$

$\left\{ \ldots \frac{\partial u_r}{\ldots} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right\} = 0$

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and $\varepsilon''_{\varphi r} = \dfrac{\sin\varphi\cos\varphi}{2(1-\nu)}\left[\dfrac{A(1+\nu)}{r^3} - \dfrac{12C}{r^5}\right] = \dfrac{1}{2}\left\{\dfrac{\partial u''_r}{\partial\varphi} + \dfrac{\partial u''_\varphi}{\partial r} - \dfrac{u''_\varphi}{r}\right\}$

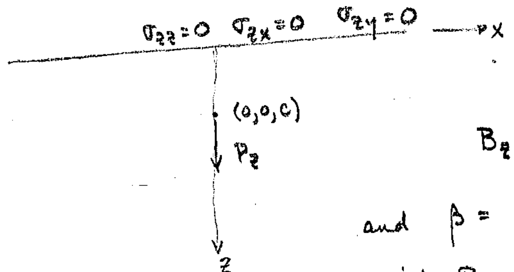
Now $\sigma''_{\varphi r} = 2\mu\,\varepsilon''_{\varphi r} = \dfrac{\mu\sin\varphi\cos\varphi}{1-\nu}\left[\dfrac{A(1+\nu)}{r^3} - \dfrac{12C}{r^5}\right]$

and $\sigma''_{rr} = (\lambda+2\mu)\,\varepsilon''_{rr} + 2\mu\left(\varepsilon''_{\theta\theta}+\varepsilon''_{\varphi\varphi}\right) = 2\mu\left[\dfrac{\nu}{1-2\nu}\,\varepsilon''_{rr} + \varepsilon''_{\theta\theta} + \varepsilon''_{\varphi\varphi}\right] = \dfrac{\mu}{(1-\nu)r^3}\left\{\cos^2\varphi\left[(5-\nu)A - \dfrac{18C}{r^2}\right] + \left[\nu A + D - \dfrac{6C}{r^2}\right]\right\}$

now at $r=a$ $\sigma''_{r\varphi} = T\sin\varphi\cos\varphi$ and $\sigma''_{rr} = -T\cos^2\varphi \Rightarrow A = \dfrac{5(1-\nu)a^3T}{\mu(7-5\nu)}$ $C = +\dfrac{a^2}{5}A$ $D = \dfrac{(6-5\nu)C}{a^2}$

Now $\sigma_{rr_{TOT}} = \sigma''_{rr} + T\cos^2\varphi$ and $\sigma_{r\varphi_{TOT}} = \sigma''_{r\varphi} - T\sin\varphi\cos\varphi$

3.



This problem is part one of Mindlin's Problem for which it was found in class

$$B_z = \frac{P_z}{4\pi\mu}\left[\frac{1}{R_1} + \frac{3-4\nu}{R_2} - 2c\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)\right]$$

and $$\beta = \frac{P_z}{4\pi\mu}\left[-\frac{c}{R_1} - \frac{c(3-4\nu)}{R_2} + 4(1-2\nu)(1-\nu)\log(R_2+z+c)\right]$$

with $B_x = B_y = 0$

$R_1^2 = x^2+y^2+(z-c)^2$

$R_2^2 = x^2+y^2+(z+c)^2$

Thus we need to find u_z or $u_z = B_z - \dfrac{1}{4(1-\nu)}\dfrac{\partial}{\partial z}\left[\mathbf{r}\cdot\mathbf{B} + \beta\right] = B_z - \dfrac{1}{4(1-\nu)}\left[B_z + z\dfrac{\partial B_z}{\partial z} + \dfrac{\partial\beta}{\partial z}\right]$

thus $u_z = \dfrac{1}{4(1-\nu)}\left[(3-4\nu)B_z - z\dfrac{\partial B_z}{\partial z} - \dfrac{\partial\beta}{\partial z}\right]$ also $\dfrac{\partial}{\partial z}\left(\dfrac{1}{R_2}\right) = -\dfrac{z+c}{R_2^3}$; $\dfrac{\partial}{\partial z}\left(\dfrac{1}{R_1}\right) = -\dfrac{z-c}{R_1^3}$; $\dfrac{\partial^2}{\partial z^2}\left(\dfrac{1}{R_2}\right) = -\left[\dfrac{x^2+y^2-2(z+c}{R_2^5}\right.$

In class we had shown $\dfrac{\partial\beta}{\partial z} = \dfrac{P_z}{4\pi\mu}\left[\dfrac{4(1-2\nu)(1-\nu)}{R_2} - c\dfrac{\partial}{\partial z}\left(\dfrac{1}{R_1}\right) - (3-4\nu)c\dfrac{\partial}{\partial z}\left(\dfrac{1}{R_2}\right)\right]$

also $z\dfrac{\partial B_z}{\partial z} = \dfrac{P_z}{4\pi\mu}\left[-2cz\dfrac{\partial^2}{\partial z^2}\left(\dfrac{1}{R_2}\right) + z\dfrac{\partial}{\partial z}\left(\dfrac{1}{R_1}\right) + (3-4\nu)z\dfrac{\partial}{\partial z}\left(\dfrac{1}{R_2}\right)\right]$

$(3-4\nu)B_z = \dfrac{P_z}{4\pi\mu}\left[\dfrac{(3-4\nu)^2}{R_2} + \dfrac{3-4\nu}{R_1} - 2c(3-4\nu)\dfrac{\partial}{\partial z}\left(\dfrac{1}{R_2}\right)\right]$

thus $u_z = \dfrac{1}{4(1-\nu)}\dfrac{P_z}{4\pi\mu}\left[\dfrac{5-12\nu+8\nu^2}{R_2} + \dfrac{(3-4\nu)(z+c)^2}{R_2^3} - \dfrac{2cz\left[R_2^2-3(z+c)^2\right]}{R_2^5} + \dfrac{(z-c)^2}{R_1^3} + \dfrac{3-4\nu}{R_1}\right]$

along the z axis $R_2 = z+c$ $R_1 = z-c$ and $u_z = \dfrac{P_z}{4\mu\pi(1-\nu)}\left[\dfrac{2(1-\nu)^2}{z+c} + \dfrac{(1-\nu)}{z-c} + \dfrac{zc}{(z+c)^3}\right]$

DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING
STANFORD UNIVERSITY

Spring 1980

ME 238C Theory of Elasticity

Midterm Examination

1. Consider a concentrated force acting normally to the surface of a half-space (Boussinesq problem). Calculate and sketch

5 for B   5 for β 5 for $u_1 = u_1(B)$
15 for final answer   5 for graph

a) the settlement of the bounding plane,

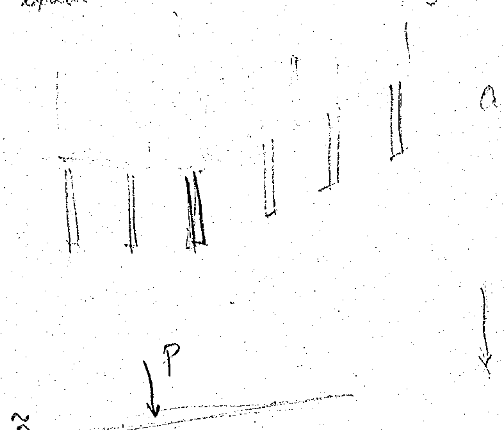
b) the displacement of points along the line of action of the face in the direction of the force.

10 for u_z 5 for graph.

2. An elastic hollow sphere of inner radius a and outer radius b is placed in a rigid spherical housing of radius b and subjected to internal pressure p_i. Determine the change in radius a and the traction exerted by the housing.

u_R & u_θ, u_φ 25 pts   15 to u_R
T_R & T_φ, T_φ 25 pts   15 to T_R

13 took exam   Akimoto. Not here



A skyscraper is to be built on legs, as shown. The legs are designed to take the loading equally. You are asked to find the settlement and the displ under each leg. How would you solve this problem and what simplifying assumptions would you make

A needle rests on a stationary record as shown and produces a settlement of $\frac{7k}{99l^3}$ ~~and~~ at a distance of $3l$ from the needle point (k being a constant having units of Force - length⁴/Fo

$w = \frac{7k}{99l^3} = \frac{(1-\nu)P}{\ \mu \cdot 3l}$ $k = \frac{99l^3}{7} \cdot \frac{1-\nu P}{6\pi\mu l}$

$\frac{3 \cdot 11 l^2 (1-\nu)P}{2 \cdot 6 \cdot 2 \cdot 11 \mu} = \frac{3l^2(1-\nu)P}{4\mu} = k$

the change in load be if the settlement

$= k/9l^3$ $P = \frac{44}{7}P$

$$\frac{\ldots}{2\pi\mu\, r} = \omega_0$$

$$\omega_0 \cdot \frac{2\pi\mu}{1-\nu} = \frac{P}{r}$$

$$\frac{12k}{108\ell^3} = \frac{k}{9\ell^3} = \frac{1-\nu}{2\pi\mu \cdot 4\ell}\hat{P} = \frac{3\ell^3(1-\nu)P}{4\mu \cdot 9\ell^3} = \frac{(1-\nu)}{12\mu\ell} = \frac{1-\nu}{8\pi\mu\ell}\hat{P}$$

$$\hat{P} = \frac{2\,8\pi\ell}{12\ell}\frac{P}{3} =$$

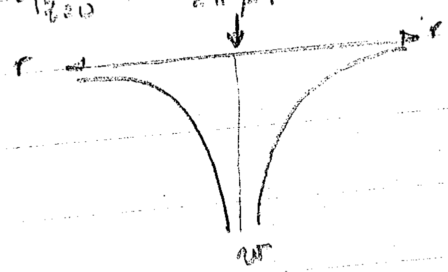
1. From Boussinesq Problem

$$B_z = \frac{1-\nu}{\pi\mu R}P \qquad B_x = B_y = 0 \qquad \beta = \frac{(1-2\nu)(1-\nu)P\ln(R+z)}{\pi\mu}$$

Using $u = \mathbb{B} - \frac{\nabla}{4(1-\nu)}\left[r\cdot\mathbb{B}+\beta\right]$ obtain

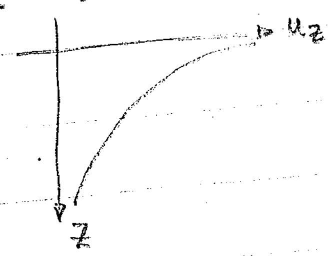
$$u_z = \frac{P}{4\pi\mu}\left[\frac{2(1-\nu)}{R} + \frac{z^2}{R^3}\right] = \frac{P}{4\pi\mu R}\left[2(1-\nu)+\left(\frac{z}{R}\right)^2\right]$$

Settlement $w = u_z\big|_{z=0} = \frac{(1-\nu)P}{2\pi\mu r}$ where $r = (x^2+y^2)^{1/2}$



along line of force $x=y=0$ $R=z$

$$\therefore \quad u_z = \frac{P}{4\pi\mu}\left[\frac{2(1-\nu)}{z} + \frac{1}{z}\right] = \frac{P(3-2\nu)}{4\pi\mu z}$$



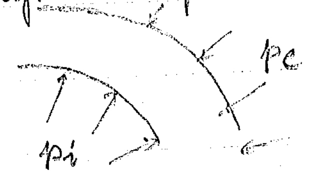
2. From symmetry of problem we expect only σ_{rr} and no $\sigma_{r\theta}$ & $\sigma_{r\varphi}$ at $r=b$. But our b.c. are

@ $r=a$ $\sigma_{rr} = -p_i$ $\sigma_{r\theta} = \sigma_{r\varphi} = 0$

@ $r=b$ $u_r = u_\theta = u_\varphi = 0$

Using the results of the hollow sphere under internal & external pressure we note that $u_\theta = u_\varphi = 0$ everywhere. Thus we satisfy the $u_\theta = u_\varphi = 0$ conditions at $r=b$.

Now all that is left is to pick $\sigma_{rr} = -p_e$ so that $u_r = 0$ at $r = b$



thus $\sigma_{RR} = \dfrac{X}{R^3} + Y$

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \frac{-X}{2R^3} + Y \quad \text{where } X = (p_i - p_e)\frac{a^3 b^3}{a^3 - b^3} \quad Y = \frac{b^3 p_e - a^3 p_i}{a^3 - b^3}$$

Now form $\epsilon_{RR} = \dfrac{\partial u_R}{\partial R} = \dfrac{1+\nu}{E}\sigma_{RR} - \dfrac{\nu}{E}(\sigma_{RR} + \sigma_{\theta\theta} + \sigma_{\varphi\varphi})$

$$= \frac{1+\nu}{E}\left(\frac{X}{R^3} + Y\right) - \frac{\nu}{E}(3Y).$$

Integrate to get $u_R = \dfrac{1-2\nu}{E} YR - \left(\dfrac{1+\nu}{E}\right)\left(\dfrac{X}{2R^2}\right) + f(\theta, \varphi)$

by symmetry $f(\theta,\varphi) \equiv 0$; thus

$$u_R = \frac{1-2\nu}{E} YR - \left(\frac{1+\nu}{2E}\right)\frac{X}{R^2}$$

since $u_R\big|_{R=b} = 0$ and $Y = Y(p_e)$, $X = X(p_e)$ solve

$$\frac{1-2\nu}{E} Yb - \left(\frac{1+\nu}{2E}\right)\frac{X}{b^2} = 0 \quad \text{for } p_e$$

thus

$$\boxed{p_e = \frac{3(1-\nu)p_i a^3}{(1+\nu)a^3 + 2(1-2\nu)b^3}}$$

Now since $\epsilon_{r\varphi}$ and $\epsilon_{r\theta}$ depend on u_θ, u_φ or $\frac{\partial}{\partial\theta}u_r$ then $\sigma_{r\theta} = \sigma_{r\varphi} = 0$ everywhere hence the traction at $r=b$ is dependent on p_e only.

b. The change in radius a just means evaluation of $u_R\big|_{R=a}$

from $u_R = \dfrac{1-2\nu}{E} YR - \left(\dfrac{1+\nu}{E}\right)\dfrac{X}{2R^2}$

let $R = a$ and use p_e in the expressions for Y and X

this leads to a value of u_R

$$\boxed{u_R\big|_{R=a} = \frac{2a(b^3 - a^3)(1-2\nu)p_i}{\mu\left[(1+\nu)a^3 + 2(1-2\nu)b^3\right]}}$$

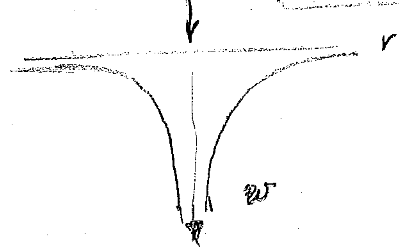
1. a   Boussinesq

$B_z = \frac{1-\nu}{\pi\mu R}P \qquad B_x = B_y = 0 \qquad \beta = \frac{(1-2\nu)(1-\nu)P\ln(R+z)}{\pi\mu}$

$$u = B - \frac{\nabla}{4(1-\nu)}\left[r\cdot B + \beta\right] = B_z e_z - \frac{1}{4(1-\nu)}\nabla\left[zB_z + \beta\right].$$

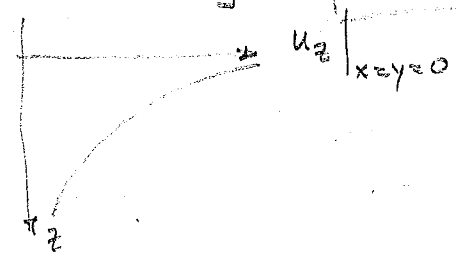
$$u_z = \frac{P}{4\pi\mu}\left[\frac{2(1-\nu)}{R} + \frac{z^2}{R^3}\right] \qquad \boxed{w = u_z\Big|_{z=0} = \frac{(1-\nu)P}{2\pi\mu r}}$$

where $r = \sqrt{x^2+y^2}$



b. along the line of force $x=y=0$ $R=z$

$$u_z = \frac{P}{4\pi\mu}\left[\frac{2(1-\nu)}{z} + \frac{1}{z}\right] = \boxed{\frac{P(3-2\nu)}{4\pi\mu z} = u_z}$$

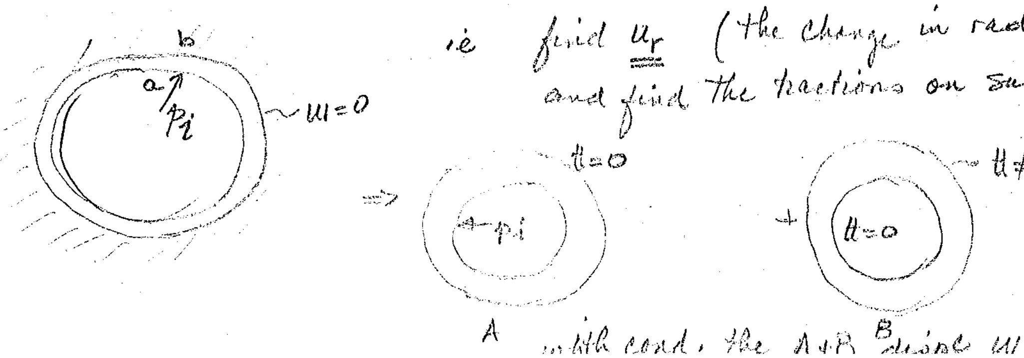


Not included c. ~~when would the displacement of b be equal to the settlement~~

$$\frac{P(3-2\nu)}{4\pi\mu z} = \frac{P(1-\nu)}{2\pi\mu r} \quad \therefore \quad z = \frac{3-2\nu}{1-2\nu}r.$$

$$\text{or} \quad r = \frac{(1-\nu)z}{3-2\nu}$$

2. ie find $\underline{\underline{u_r}}$ (the change in radius a) and find the tractions on surface



with cond. the A+B make $u = 0$

by symmetry of problem expect only a T_r and no T_θ, T_φ & $u_\theta, u_\varphi = 0$ $u_R \neq 0$.

for a sphere under internal & external pressure

$$\sigma_{RR} = (p_i - p_e)\frac{a^3b^3}{(a^3-b^3)R^3} + \frac{b^3p_e - a^3p_i}{a^3-b^3} = \frac{X}{R^3} + Y$$

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} = \frac{a^3b^3(p_e-p_i)}{2(a^3-b^3)R^3} + \frac{b^3p_e - a^3p_i}{a^3-b^3} = \frac{-X/2}{R^3} + Y$$

$$\sigma_{RR} + \sigma_{\phi\phi} + \sigma_{\theta\theta} = 3\left(\frac{b^3p_e - a^3p_i}{a^3-b^3}\right) = I_\sigma = 3Y$$

then $\varepsilon_{RR} = \frac{\partial u_R}{\partial R} = \frac{1+\nu}{E}\sigma_{rr} - \frac{\nu}{E} I_\sigma$

$$= \frac{1+\nu}{E}\left(\frac{X}{R^3} + Y\right) - \frac{\nu}{E}3Y$$

$$\therefore \quad \frac{1+\nu}{E}YR - \frac{1+\nu}{E}\frac{X/2}{R^2} - \frac{\nu}{E}3YR = u_R$$

$$\frac{1-2\nu}{E}YR - \frac{1+\nu}{E}\frac{X/2}{R^2} = u_R$$

$$\epsilon_{\theta\theta} = \epsilon_{\varphi\varphi} = \frac{u_R}{R} = \frac{1+\nu}{E}\sigma_{\theta\theta} - \frac{\nu}{E}I_\sigma = \frac{1+\nu}{E}\sigma_{\theta\theta} - \frac{\nu 3Y}{E}$$

$$= \frac{1+\nu}{E}\left(\frac{-X/2}{R^3} + Y\right) - \frac{\nu 3Y}{E}$$

$$= \frac{1-2\nu}{E}Y - \frac{1+\nu}{E}\frac{X/2}{R^3} \quad \checkmark$$

$\therefore$ $u_\theta = u_\varphi \equiv 0$ and $u_R\Big|_{R=b} = 0 \quad \therefore \quad \frac{1-2\nu}{E}\bar{Y}b - \frac{1+\nu}{E}\frac{X/2}{b^2} = 0$

$\therefore$ pick $\bar{Y}(p_e)$ and $\bar{X}(p_e)$ $\ni$ $u_R\Big|_{R=b} = 0$

$$\therefore \quad (1-2\nu)\frac{\bar{Y}b}{E} = \frac{1+\nu}{2Eb^2}X$$

$$\frac{(1-2\nu)b}{\cancel{E}}\left[\frac{b^3p_e - a^3p_i}{a^3-b^3}\right] = \frac{1+\nu}{2\cancel{E}b^{\cancel{2}}}\left[\frac{a^3\cancel{b^3}\,b\,(p_i-p_e)}{\cancel{(a^3-b^3)}}\right]$$

$$p_e\left[2(1-2\nu)b^3\right] - 2(1-2\nu)\cancel{b}a^3p_i = (1+\nu)a^3\cancel{b}p_i - (1+\nu)a^3\cancel{b}p_e$$

$$p_e\cancel{b}\left\{(2-4\nu)b^3 + (1+\nu)a^3\right\} = \left\{(1+\nu)a^3 + 2(1-2\nu)a^3\right\}p_i\cancel{b}$$

$$\boxed{p_e = T_r} = \frac{3(1-\nu)p_ia^3}{((1+\nu)a^3 + 2(1-2\nu)b^3)}$$

$$\frac{1+\nu}{\ldots}\frac{(p_i-p_e)a^3b^3}{(a^3-b^3)}$$

$$u_R\Big|_{R=a} = \frac{1-2\nu}{E}\bar{Y}a - \frac{1+\nu}{2E}\frac{\bar{X}}{a^2}$$

$$= \frac{2(1-2\nu)\bar{Y}a^3}{2Ea^2} - \frac{1+\nu}{2Ea^2}\bar{X} = \frac{1}{2Ea^2}\left\{2(1-2\nu)\frac{b^3p_e - a^3p_i}{a^3-b^3}a^3 - \frac{(1+\nu)(p_i-p_e)a^3b^3}{a^3-b^3}\right.$$

$$= \frac{1}{2Ea^2(a^3-b^3)}\left\{a^3p_e\Big[\underbrace{2(1-2\nu)[b^3]+(1+\nu)b^3}_{b^3(2-4\nu+1+\nu)=3(1-\nu)b^3}\Big] - a^3p_i\Big[2(1-2\nu)a^3+(1+\nu)b^3\Big]\right\}$$

$$\frac{1}{2E(a^3-b^3)}\left\{3(1-\nu)a\,b^3p_e - a\left[2(1-2\nu)a^3+(1+\nu)b^3\right]p_i\right\}$$

$$p_e = \frac{3(1-\nu)p_i a^3}{(1+\nu)a^3+2(1-2\nu)b^3} \Rightarrow$$

$$u_R = \frac{1}{2E(a^3-b^3)}\left\{\frac{9(1-\nu)^2a^4b^3p_i}{(1+\nu)a^3+2(1-2\nu)b^3} - a\left[2(1-2\nu)a^3+(1+\nu)b^3\right]p_i\right\}$$

$$= \frac{ap_i}{2E(a^3-b^3)}\left\{9(1-\nu)^2a^3b^3 - \Big[\underbrace{4(1-2\nu)^2a^3b^3}_{4(1-4\nu+4\nu^2)+1+2\nu+\nu^2} + 2(1-2\nu)(1+\nu)\left[b^6+a^6\right] + (1+\nu)^2a^3b^3\Big]\right.$$

$$a^3b^3(9-18\nu+9\nu^2) - (5-14\nu+17\nu^2)a^3b^3 - 2(1-2\nu)(1+\nu)(b^6+a^6)$$

$$4-4\nu-8\nu^2$$

$$\left\{a^3b^3\,4(1-2\nu)(1+\nu) - 2(1-2\nu)(1+\nu)(b^6+a^6)\right\} = 2(1-2\nu)(1+\nu)\left[b^6-2a^3b^3+a^6\right]$$

$$\frac{ap_i}{\cancel{2E(a^3-b^3)}}\left\{\frac{-\cancel{2}(1-2\nu)(1+\nu)\left[(a^3-b^3)^{\cancel{2}}\right]}{(1+\nu)a^3+2(1-2\nu)b^3}\right\} = \frac{2(b^3-a^3)ap_i(1-2\nu)}{\mu\left[(1+\nu)a^3+2(1-2\nu)b^3\right]}$$

Thus $\boxed{u_R = \dfrac{(b^3-a^3)a\cdot 2(1-2\nu)}{\mu\left[(1+\nu)a^3+2(1-2\nu)b^3\right]}p_i}$

$u_\theta = u_\varphi = 0$

$$\sigma_{\varphi\varphi} = \sigma_{\theta\theta} = \frac{a^3(p_e-p_i)}{2(a^3-b^3)} + \frac{2b^3p_e - 2a^3p_i}{2(a^3-b^3)}$$

$$= \frac{(a^3+2b^3)p_e - 3a^3p_i}{2(a^3-b^3)}$$

$$= \left[a^3+2b^3\right]\left\{\frac{3(1-\nu)p_ia^3}{(1+\nu)a^3+2(1-2\nu)b^3}\right\} - 3a^3p_i$$

$$(a^3+2b^3)3(1-\nu) - 3(1+\nu)a^3 - 6(1-2\nu)b^3$$

$$-6\nu(a^3-b^3) \qquad \frac{a^3p_i}{\ldots}$$

DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING
STANFORD UNIVERSITY

ME 238C Theory of Elasticity   Spring 1980

Problem Set No. 1

12 1. Derive the expression for the six components of stress in terms of the four Boussinesq-Papkovitch functions for an isotropic elastic solid with reference to a Cartesian system of coordinates. Use the shear modulus and Poisson's ratio as elastic constants.

12 2. Evaluate the Cartesian components of stress for the Kelvin problem.

~~12~~ 9 3. Evaluate the components of traction on a spherical surface of arbitrary radius with its origin at the point at which the concentrated force is acting in the Kelvin problem.

12 4. Show that the system of tractions evaluated in (2) above is statically equivalent to a single force.

6 ~~7~~ 9 5. Evaluate the normal strain along the line of action of the force in the Kelvin problem.

6 6. Evaluate the components of displacement in spherical coordinates for the Kelvin problem.

12 7. Evaluate the spherical components of stress for the Kelvin problem and show in particular that the components $\sigma_{\theta\phi}$ and $\sigma_{\phi r}$ vanish.

6 8. Consider an incompressible elastic medium (Poisson's ratio is 1/2) and reduce the results in (7) above to this special case.

−1 for non expansion

12 $\sigma_{ij} = 2\mu\left[\frac{\nu}{1-2\nu}\delta_{ij}\left\{\frac{1-2\nu}{2(1-\nu)}B_{k,k} - \frac{1}{4(1-\nu)}\left[x_\ell B_{\ell,kk} + \beta_{,kk}\right]\right\} + \left\{\frac{1-2\nu}{4(1-\nu)}\left[B_{i,j}+B_{j,i}\right] - \frac{1}{4(1-\nu)}\left[x_m B_{m,ij} + \beta_{,ij}\right.\right.\right.$

12 2. $\sigma_{ii} = \frac{-P}{8\pi(1-\nu)r^2}\left(\frac{x_3}{r}\right)\left[2\nu-1 + 3\left(\frac{x_i}{r}\right)^2\right]$

$-(2\nu-1)$ for σ_{33}

$\sigma_{12} = \frac{-3P}{8\pi(1-\nu)r^2}\left(\frac{x_1x_2x_3}{r^3}\right) \qquad \sigma_{i3} = \frac{-P}{8\pi(1-\nu)r^2}\left(\frac{x_i}{r}\right)\left[1-2\nu + 3\left(\frac{x_3}{r}\right)^2\right]$

9 3. $T_1 = \frac{-3P}{8\pi(1-\nu)}\frac{x_1x_3}{r^4} \qquad T_2 = \frac{-3P}{8\pi(1-\nu)}\frac{x_2x_3}{r^4} \qquad T_3 = \frac{-P}{8\pi(1-\nu)r^2}\left[(1-2\nu)+3\left(\frac{x_3}{r}\right)^2\right.$

12 4. $F_1 = F_2 = 0 \qquad F_3 = -P \qquad M_1 = M_2 = M_3 = 0$

6 5. $\epsilon_{33} = \frac{P}{4\pi\mu r^2}$ −1 for non eval. along x_3

6 6. $u_r = \frac{P}{4\pi\mu r}\cos\phi \qquad u_\theta = 0 \qquad u_\varphi = \frac{4\nu-3}{1-\nu}\frac{P\sin\phi}{16\pi\mu r}$

12 7. $\sigma_{rr} = \frac{-P\cos\phi}{\cancel{8}4\pi r^2(1-\nu)}\left(\frac{2-\nu}{\cancel{1-2\nu}}\right) \qquad \sigma_{\theta\theta} = \frac{P(1-2\nu)\cos\phi}{8\pi(1-\nu)r^2} = \sigma_{\varphi\varphi}$

$\sigma_{r\varphi} = \frac{P(1-2\nu)\sin\phi}{8\pi r^2(1-\nu)} \qquad \sigma_{r\theta} = \sigma_{\theta\varphi} = 0$

6 8. $\sigma_{rr} = \frac{-3P\cos\phi}{4\pi r^2}$ all others are zero

$$\mathbb{T} = T_i e_i = \sigma_{ij} n_j e_i = \sigma \cdot n\, e_i$$

$4(1-\nu)$ $4(1-\nu)$

$$u_{i,j} = B_{i,j} - \frac{1}{4(1-\nu)}\left(B_i + x_k B_{k,i} + \beta_{,i}\right)_{,j} = B_{i,j} - \frac{1}{4(1-\nu)}\left(B_{i,j} + B_{j,i} + x_k B_{k,ij} + \beta_{,ij}\right)$$

$$= \frac{3-4\nu}{4(1-\nu)} B_{i,j} - \frac{1}{4(1-\nu)}\left(B_{j,i} + x_k B_{k,ij} + \beta_{,ij}\right)$$

$$u_{j,i} = \frac{3-4\nu}{4(1-\nu)} B_{j,i} - \frac{1}{4(1-\nu)}\left(B_{i,j} + x_k B_{k,ji} + \beta_{,ji}\right)$$

$$u_{i,j} + u_{j,i} = \cancel{\frac{3-4\nu}{4(1-\nu)}}\left[\frac{2-4\nu}{4(1-\nu)}(B_{j,i} + B_{i,j}) - \frac{1}{4(1-\nu)}\left(2x_k B_{k,ij} + 2\beta_{,ij}\right)\right]$$

$$\varepsilon_{ij} = \frac{1-2\nu}{4(1-\nu)}\left(B_{j,i} + B_{i,j}\right) - \frac{2}{4(1-\nu)}\left(x_k B_{k,ij} + \beta_{,ij}\right)$$

$$\epsilon_{kk} = \frac{1-2\nu}{2(1-\nu)} B_{k,k} - \frac{1}{2(1-\nu)}\left(x_k B_{k,ii} + \beta_{,ii}\right)$$

$$\sigma_{ij} = \frac{2G\nu}{1-2\nu}\delta_{ij}\left[\frac{1-2\nu}{2(1-\nu)} B_{k,k} - \frac{1}{2(1-\nu)}\left(x_l B_{l,kk} + \beta_{,kk}\right)\right] + 2G\left[\frac{1-2\nu}{4(1-\nu)}\left(B_{j,i} + B_{i,j}\right) - \frac{1}{2(1-\nu)}\left(x_k B_{k,ij} + \beta_{,ij}\right)\right]$$

$$= 2G\,\delta_{ij}\left[\frac{\nu}{2(1-\nu)} B_{k,k}\right. \qquad \delta_{ki} x_l B_{l,kk} \delta_{kj} \beta_{,ij} = \beta_{,kk}\delta_{ik}\delta_{jk}$$

$$= 2G\,\mathbb{I}\left[\frac{\nu}{2(1-\nu)}\nabla\cdot\mathbb{B} - \frac{\nu}{2(1-\nu)(1-2\nu)}\left[\mathbb{r}\cdot\nabla^2\mathbb{B} + \nabla^2\beta\right]\right] + 2G\left[\frac{1-2\nu}{4(1-\nu)}\left[\nabla\mathbb{B} + \mathbb{B}\nabla\right]\right.$$

$$- \frac{1}{2(1-\nu)} \quad \cancel{(\mathbb{r}\cdot(\mathbb{B}\nabla)\nabla} + \beta$$

$$\mathbb{I}\cdot(\mathbb{r}\cdot\nabla^2\mathbb{B})\cdot\mathbb{I} + \mathbb{I}\cdot\nabla^2\beta\cdot\mathbb{I}$$

$$\frac{\nu}{1-\nu} G\, B_{k,k}\delta_{ij} + G\frac{(1-2\nu)}{2(1-\nu)}\left[B_{j,i} + B_{i,j}\right] - \frac{G\nu}{(1-2\nu)(1-\nu)}\delta_{ij}\, x_l B_{l,kk} - \frac{G}{1-\nu} x_k B_{k,ij}$$

$$- \frac{G\nu}{(1-\nu)(1-2\nu)}\delta_{ij}\,\beta_{,kk} - \frac{G}{\cancel{2}(1-\nu)}\beta_{,ij}$$

$$4-4\nu-1 = \frac{3-4\nu}{4(1-\nu)} = \frac{\lambda+3\mu}{2(\lambda+\mu)}$$

$$\cancel{\frac{3-3\nu}{4(1-\nu)}} \quad \frac{3-4\nu}{4(1-\nu)} - \nu = \frac{3\mu}{2(\lambda+\mu)}$$

$$3-4\nu-4\nu+4\nu^2$$

$$(1-2\nu)(1-\nu) \qquad = \frac{3-8\nu+4\nu^2}{\cancel{4}(1-\nu)} \quad \frac{\cancel{2}}{3}$$

$$1-3\nu+2\nu^2$$

$$1 - \frac{1}{2(1-\nu)} = \frac{\cancel{1-2\nu}}{\cancel{2}(1-\nu)} \cdot \frac{\cancel{2}\mu\nu}{\cancel{1-2\nu}}$$

$$\mu \, \frac{1-2\nu}{1-\nu} \, B_{1,1}$$

problem #2

$$w = \frac{(1-\nu)N}{2\pi\mu a}\sin^{-1}\frac{a}{r} = \frac{2(1-\nu)N}{4\pi^2\mu a}\int_0^{\psi_1}\int_{-c}^{c}\frac{dv\,d\psi}{\sqrt{c^2-v^2}} \qquad \psi_1 = \arcsin\frac{c}{r}$$

problem #1

$$\sigma_{RR} = \cancel{\frac{2A\cos^2\varphi}{4r^3}\,\frac{5-3\nu}{1-\nu} + \frac{\cos^2\varphi\, 2A}{4r^3(1-\nu)}\,2\nu}$$

$$\cancel{\sigma_{\phi\phi} = }$$

$$\cancel{\sigma_{\theta\theta}}$$

$$\sigma_{RR} = \frac{\mu\cos^2\varphi}{(1-\nu)R^3}\left[(5-\nu)A - \frac{18C}{R^2}\right] + \frac{\mu}{(1-\nu)R^3}\left[-\nu A - D + \frac{6C}{R^2}\right] + T\cos^2\varphi$$

$$\sigma_{\theta\theta} = \frac{\mu}{2(1-\nu)R^3}\left[\left\{\underset{3(2\nu-1)A}{(4\nu-3)A} + \frac{12C}{R^2}\right\}\cos^2\varphi + \overset{1-\cos^2\varphi}{\sin^2\varphi}\left\{-\frac{3C}{R^2} - 2A\nu\right\} + D\right]$$

$$\left[\left\{(6\nu-3)A + \frac{15C}{R^2}\right\}\cos^2\varphi + D - \frac{3C}{R^2} - 2A\nu\right]$$

$$\sigma_{\varphi\varphi} = \frac{\mu}{2(1-\nu)R^3}\left[\cos^2\varphi\left[(2\nu-1)A + \frac{21C}{R^2}\right] - 2(1-\nu)A - \frac{9C}{R^2} + D\right] + T\sin^2\varphi$$

$$A = -\frac{5(1-\nu)a^3T}{\mu(7-5\nu)} \qquad C = \frac{-(1-\nu)a^5T}{\mu(7-5\nu)} \qquad D = -\frac{(1-\nu)(6-5\nu)a^3}{\mu(7-5\nu)}$$

$6\nu-4+10-2\nu = 6+4\nu$

$$I_\sigma = \frac{\mu\cos^2\varphi}{(1-\nu)R^3}\left[\frac{2(5-\nu)A}{2} - \frac{36C}{2R^2} + \frac{4\nu-3}{2}A + \frac{12C}{2R^2} + \frac{(2\nu-1)A}{2} + \frac{21C}{2R^2}\right]$$

$$+ \frac{\mu}{(1-\nu)R^3}\left[\frac{-2\nu A}{2} - \frac{2D}{2} + \frac{12C}{2R^2} + \frac{D}{2} - \frac{2(1-\nu)A}{2} - \frac{9C}{2R^2} + \frac{D}{2}\right] + \frac{\mu\sin^2\varphi}{2(1-\nu)R^3}\left[-\frac{3C}{R^2} - 2\right.$$

5.17 | 7.6

$$+T$$

$$= \frac{\mu\cos^2\varphi}{(1-\nu)R^3}\left[(3+2\nu)A - \frac{3C}{R^2}\right] + \frac{\mu}{(1-\nu)R^3}\left[-A + \frac{3C}{2R^2}\right] + \frac{\mu\sin^2\varphi}{(1-\nu)R^3}\left[-\frac{3C}{2R^2} - A\right. + T$$

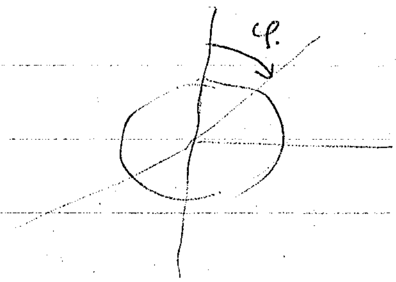
$$= \frac{\mu A}{(1-\nu)R^3}\left[(3+2\nu)\cos^2\varphi - 1 - \overset{-\nu+\nu\cos^2\varphi}{\nu\sin^2\varphi}\right] + T = 0$$

$$3(1+\nu)\cos^2\varphi - (\nu+1)$$

$$\frac{\mu A}{(1-\nu)R^3} = -\frac{5a^3T}{(7-5\nu)R^3} \qquad \frac{-TR^3(1-\nu)}{\mu A} = f(\theta)$$

$$R^3 = -f(\theta)\frac{\mu A}{T(1-\nu)}$$

$$\boxed{R = \sqrt[3]{-f(\theta)\frac{\mu A}{T(1-\nu)}}}$$



$$(3+2\nu)\left[1-\sin^2\varphi\right] - 1 - \nu\sin^2\varphi = f(\theta)$$

$$3+2\nu-1 - \sin^2\varphi\left[3+2\nu+\nu\right]$$

$$2(1+\nu) - \sin^2\varphi \; 3(1+\nu)$$

$$(1+\nu)\left[2-3\sin^2\varphi\right] = f(\theta)$$

$$+(1+\nu)\left(2-3\sin^2\varphi\right)\cdot\frac{5a^3}{7-5\nu} = R^3 \Big/ \frac{\mu A}{(1-\nu)T} = \frac{(-5a^3)}{7-5\nu}$$

$$\boxed{R = \sqrt[3]{\left(\frac{1+\nu}{7-5\nu}\right)\cdot 5a^3\left(2-3\sin^2\varphi\right)}}$$

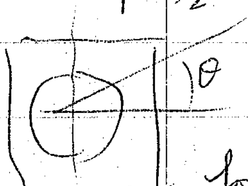
or

$$\sqrt[2]{-\frac{1}{3}\left[\frac{R^3}{5a^3}\left(\frac{7-5\nu}{1+\nu}\right)\right] + \frac{2}{3}} = \sin\varphi$$

2-D

$$\sigma_{RR} = \frac{\sigma}{2}\left(1-\frac{a^2}{r^2}\right) - \frac{\sigma}{2}\left(1 - 4\frac{a^2}{r^2} + \frac{3a^4}{r^4}\right)\cos 2\theta$$

$\varphi = \pi/2 - \theta$



$$\sigma_{\theta\theta} = \frac{\sigma}{2}\left(1+a^2/r^2\right) + \frac{\sigma}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

for plane stress $\sigma_{ZZ}=0$ if plane strain $\sigma_{ZZ} = \nu(\sigma_{RR}+\sigma_{\theta\theta})$

$\cos\varphi = \cos(\pi/2 - \theta)$
$= \sin\theta$

$$\sigma_{RR}+\sigma_{\theta\theta} = \sigma + 2\sigma\frac{a^2}{r^2}\cos 2\theta$$

$\cos\theta = \cos(\pi/2 - \varphi)$
$= \sin\varphi$

for plane stress $I_\sigma = 0 \Rightarrow 1 + \frac{2a^2}{r^2}\cos 2\theta = 0$ or

$$\frac{1+\cos 2\theta}{2} = \cos^2\theta$$

$$-\frac{r^2}{4a^2} + \frac{1}{2} = \sin^2\varphi$$

$$\frac{4a^2}{r^2}\left(1-\frac{2a^2}{r^2}\right) + \frac{4a^2}{r^2}\cos^2\theta = 0$$

$$r = a\sqrt{2-4\sin^2\varphi} \qquad \sin\varphi = \sqrt{\frac{1}{2}-\frac{r^2}{4a^2}} \qquad r^2 = 2a^2 - 4a^2\cos^2\theta$$

for plane strain $I_\sigma = \sigma_{RR} + \sigma_{\theta\theta} + \sigma_{zz} = (1+\nu)(\sigma_{RR} + \sigma_{\theta\theta}) = 0$

gives same results indep of ν

$r = a\sqrt{2 - 4\sin^2\varphi}$ $\qquad \sin\varphi = \sqrt{\frac{1}{2} - \frac{1}{4}\left(\frac{r}{a}\right)^2}$

for 3-D $\sin\varphi = \sqrt{\frac{2}{3} - \frac{1}{15}\left(\frac{7-5\nu}{1+\nu}\right)\left(\frac{R}{a}\right)^3}$ $\qquad R = a\sqrt[3]{\left(\frac{5+5\nu}{7-5\nu}\right)\left(2-3\sin^2\varphi\right)}$

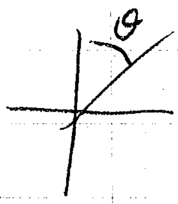
$$\left(3\cos^2\phi - 1\right) = \frac{7-5\nu}{5(1+\nu)}\left(\frac{R}{a}\right)^3$$

$2 - 3\sin^2\phi$ $\cancel{\ \ }$

$$= +\frac{2}{3} - \frac{7-5\nu}{15(1+\nu)}\left(\frac{R}{a}\right)^3$$

$$\cos\theta = \left[\frac{1}{3}\left(1 + \frac{7-5\nu}{5(1+\nu)}\left(\frac{R}{a}\right)^3\right)\right]^{1/2}$$

$$\frac{R}{a} = \left[\frac{5\left(3\cos^2\theta - 1\right)(1+\nu)}{7-5\nu}\right]^{1/3}$$



$\cos 2\theta = \cos 2(\pi/2 - \varphi) = -\cos 2\varphi$ ✓

$\cancel{2\cos 2\theta}$ $\qquad \frac{1+\cos 2\theta}{2} = \cos^2\theta$

$1 - 2\cos^2\theta = -\cos 2\theta$

$$3\left(\frac{a}{R}\right)^2 \sin^2\phi + 5\left[(2-\nu) - (\nu+4)\cos^2\phi\right] + \frac{7-5\nu}{\cancel{\ \ }}\left(\frac{R}{a}\right)^5$$

$\cancel{\ \ }$

$(\nu+4)\cos^2\phi$

$(2-\nu)$

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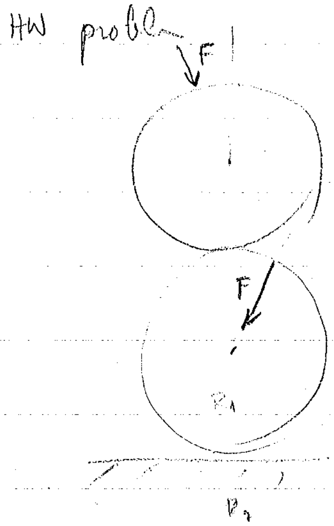
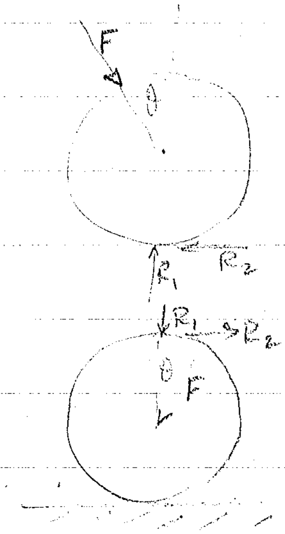
HW problem





$R_2 = F\sin\theta \cong F\theta = T$

$R_1 = F\cos\theta \cong F = N$

$N = 2F\cos\theta \cong 2F$

$T = 0$

$\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}} = \frac{R_1R_2}{R_2+R_1}$

$\frac{R_2+R_1}{R_1R_2} = \frac{1}{R_1}+\frac{1}{R_2}$

For Ball/surface $R_1 = R$ $R_2 = \infty$ $a = \sqrt[3]{\frac{(k_1+k_2)\cdot 6F\pi R}{4}} \cong \sqrt[3]{\frac{(k_1+k_2)\cdot 3F\pi R}{2}}$

$$\alpha = \sqrt[3]{\frac{\cancel{36}\,F^2\pi^2(k_1+k_2)^2}{16R}} = \sqrt[3]{\frac{9}{4}\frac{F^2\pi^2}{R}(k_1+k_2)^2} \qquad a = \sqrt[3]{\frac{(k_1+k_2)\cdot 3F\pi R}{2}}$$

for b/b contact $\frac{2}{R} = \frac{R_2+R_1}{R_1R_2}$ for a hard surface $k_2 = 0$ $\therefore$ $a = \sqrt[3]{\frac{3F\pi k_1R}{2}}$ $\alpha = \sqrt[3]{\frac{9}{4}\frac{F^2\pi^2k_1^2}{R}}$

$\sqrt[3]{\frac{3F}{4\mu}(1-\nu)R}$ $\sqrt[3]{\frac{9}{16}\frac{F^2(1-\nu)^2}{R\mu^2}}$

For ball/ball contact.

$$a = \sqrt[3]{\frac{2k_1\cdot 3F\pi R_1^2}{4\cdot 2R}} = \sqrt[3]{\frac{3k_1F\pi R}{4}} = \sqrt[3]{\frac{3(1-\nu)FR}{8\mu}}$$

$$c = a\sqrt[3]{1-\frac{T}{fN}} = a\sqrt[3]{1-\frac{\theta}{f}} \cong a\left(1-\frac{\theta}{3f}\right)$$

$$u_x = \frac{3fN}{16a\mu}(2-\nu)\left(1-\frac{c^2}{a^2}\right) \cong \frac{3fF}{16a\mu}(2-\nu)\cdot\frac{2}{3}\frac{\theta}{f} = \frac{2F\theta(2-\nu)}{16a\mu}$$

or $\frac{2F\theta(2-\nu)}{16\mu\sqrt[3]{\frac{3k_1F\pi R}{4}}} = u_x$ $\alpha =$

$$\cancel{\alpha = \sqrt[3]{\frac{9N^2\pi^2(k_1+k_2)^2R_1R_2}{16(R_1+R_2)}} = \sqrt[3]{9F^2\pi^2}}$$

$$\alpha = \sqrt[3]{\frac{9(k_1+k_2)^2N^2\pi^2(R_1+R_2)}{16R_1R_2}} = \sqrt[3]{\frac{9\cdot\cancel{4}k^2\cdot F^2\pi^2\cdot\cancel{2}R}{2\,\cancel{16}R^2}} = \sqrt[3]{\frac{9}{2}\frac{(1-\nu)^2}{4\mu^2R}F^2} = \sqrt[3]{\frac{9}{8}\frac{(1-\nu)^2F^2}{\mu^2R}}$$

1. Consider a concentrated force acting tangentially to the surface of a half space. Calculate

a. The settlement of the bounding plane, and

b. If θ is the angle measured on the surface of the half space, measured from the positive x-axis, Find those values of θ and those physically possible values of ν for which the settlement vanishes (Remember $0 \le \nu \le \frac{1}{2}$).

2. An elastic sphere of radius b contains a fixed spherical inclusion of radius a and is subjected to uniform external pressure p. Find the traction on the surface of the inclusion.

For 1

$$u = \mathbb{B} - \frac{\nabla}{4(1-\nu)}\left[\mathbf{r}\cdot\mathbb{B}+\beta\right]$$

from class $\quad \mathbb{B} = \frac{P}{2\pi\mu R}\, e_x + \frac{(1-2\nu)\,Px}{2\pi\mu R(R+z)}\, e_z$

for settlement need u_z

$$u_z = B_z - \frac{\partial}{\partial z}\frac{1}{4(1-\nu)}\left[\mathbf{r}\cdot\mathbb{B}+\beta\right]$$

$$= B_z - \frac{1}{4(1-\nu)}\left[\left(\frac{\partial}{\partial z}\mathbf{r}\right)\cdot\mathbb{B} + \mathbf{r}\cdot\frac{\partial \mathbb{B}}{\partial z} + \frac{\partial\beta}{\partial z}\right]$$

$$= \qquad \left[\underbrace{e_z\cdot\mathbb{B}}_{B_z} + x\cdot\frac{\partial B_x}{\partial z} + z\,\frac{\partial B_z}{\partial z} + \frac{\partial\beta}{\partial z}\right]$$

in class you derived $\frac{\partial B_x}{\partial z}$, $\frac{\partial B_z}{\partial z}$, $\frac{\partial \beta}{\partial z}$

plug in and let $z=0$

$$w = u_z(x,y,0) = \frac{(1-2\nu)\,Px}{2\pi\mu r(r)} - \frac{1}{4(1-\nu)}\left[\frac{(1-2\nu)Px}{2\pi\mu r^2} + x\cdot\cancelto{0}{\frac{\partial B_x}{\partial z}} + \cancelto{0}{z\frac{\partial B_z}{\partial z}} + \frac{(1-2\nu)^2 Px}{2\pi\mu r^2}\right]$$

$$= \frac{Px}{2\pi\mu r^2}\left[4(1-2\nu)(1-\nu) - (1-2\nu) - (1-2\nu)^2\right]\cdot\frac{1}{4(1-\nu)} = \frac{Px(1-2\nu)\left[2(1-\nu)\right]}{2\pi\mu r^2\, 4(1-\nu)}$$

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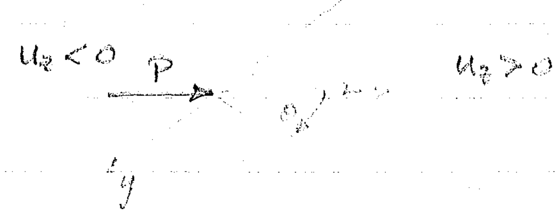
$\therefore\ w=0 \qquad \theta: \pi/2,\ 3\pi/2 \quad \nu=\frac{1}{2}$



For 2. B.C. are at $r=b$ $\quad \sigma_{RR}=-p \quad \sigma_{R\phi}=\sigma_{R\theta}=0$

at $r=a$ $\quad u_R=0,\ u_\theta=0,\ u_\varphi=0$

Use notes from hollow sphere. From symmetry this means that the only stress on inclusion walls are $\sigma_{RR}, \sigma_{\theta\theta}, \sigma_{\varphi\varphi}$; assume $\sigma_{RR}=-p_i$

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$$\sigma_{RR} = (p_i - p)\frac{a^3 b^3}{(a^3-b^3)R^3} + \frac{b^3 p - a^3 p_i}{a^3 - b^3} = \frac{X}{R^3} + Y$$

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this solution automatically satisfies $u_\theta = u_\varphi = 0$ everywhere

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$$u_R = \frac{1-2\nu}{E} Y R - \frac{1+\nu}{E}\frac{X}{2R^2} + \text{r.b. motion}$$ take as zero

$$u_R\big|_{R=a} = 0 \Rightarrow \frac{1-2\nu}{E} Y a = \frac{1+\nu}{2Ea^2} X$$

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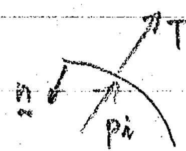

$$-\sigma_{RR} = T_r = p_i$$

For 1

$$u_1 = \mathbb{B} - \frac{\nabla}{4(1-\nu)}\left[\mathbf{r}\cdot\mathbb{B} + \beta\right]$$

from class $\quad \mathbb{B} = \dfrac{P}{2\pi\mu R}\, e_x + \dfrac{(1-2\nu)Px}{2\pi\mu R(R+z)}\, e_z$

for settlement need u_z

$$u_z = B_z - \frac{\partial}{\partial z}\frac{1}{4(1-\nu)}\left[\mathbf{r}\cdot\mathbb{B}+\beta\right]$$

$$= B_z - \frac{1}{4(1-\nu)}\left[\left(\frac{\partial}{\partial z}\mathbf{r}\right)\cdot\mathbb{B} + \mathbf{r}\cdot\frac{\partial\mathbb{B}}{\partial z} + \frac{\partial\beta}{\partial z}\right]$$

$$= \qquad \Big[\underbrace{e_z\cdot\mathbb{B}}_{B_z} + x\cdot\frac{\partial B_x}{\partial z} + z\frac{\partial B_z}{\partial z} + \frac{\partial\beta}{\partial z}\Big]$$

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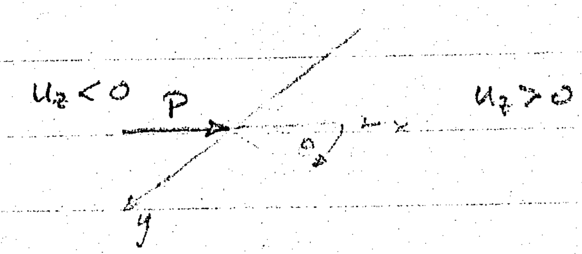

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take as zero

$$u_R\big|_{R=a} = 0 \Rightarrow \frac{1-2\nu}{E}Ya = \frac{1+\nu}{2Ea^2}X$$

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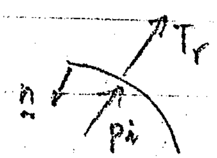

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For 1

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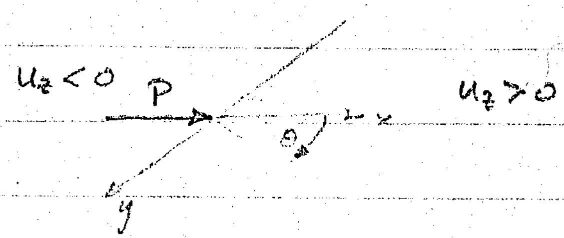

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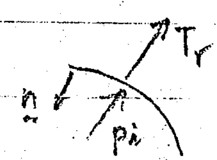
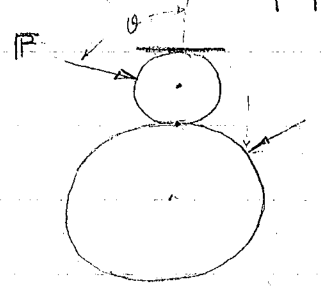
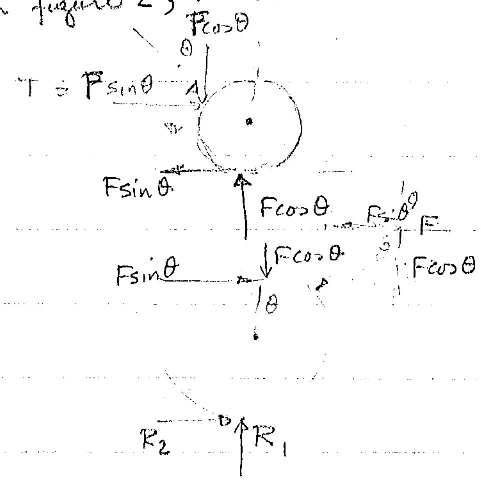


$$-\sigma_{RR} = T_r = p_i$$

now ... clamped
not ...
Free
shear + slipping

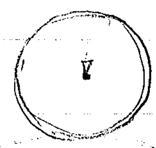
~~at some time t~~ the forces acting on the ... ...
at $t=0$
the forces on the ball bearing are moving so slowly that the problem ~~is~~ may be considered
and at time t, they appear as shown in figure 2, Find:

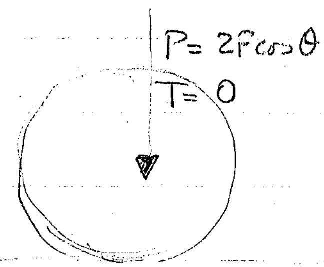




what approximations

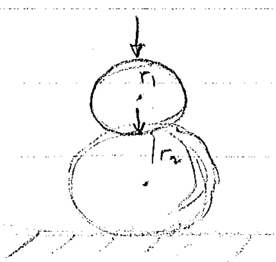
$\therefore\ R_2 = 0 \qquad R_1 = 2F\cos\theta$

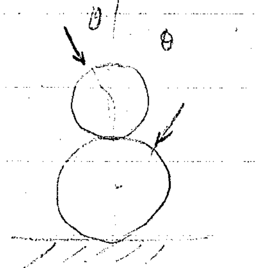




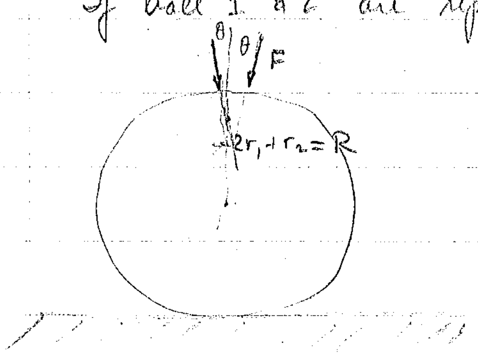
what is approach of ball 1 relative to ball 2.

what is approach of ball 2 relative to surface.
" " " " ball 1 " " surface.





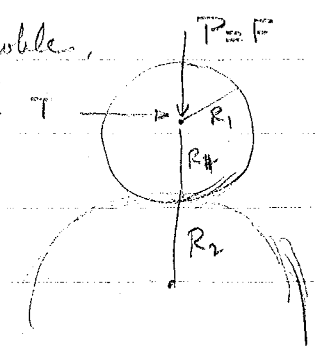
If ball 1 & 2 are replaced by 1 ball with same force system,
which system would ~~have~~ cause largest
contact surface on the half space



For small θ

for first problem,

$F\theta = T$



Use these results to find

$\alpha = (k_1 + k_2)\dfrac{3P\pi}{4a}$ $\qquad k_i = \dfrac{1-\nu_i^2}{\pi E_i}$

where

$a = \sqrt[3]{\dfrac{(k_1^b + k_2^b)\, 3P\pi R_1 R_2}{4(R_1+R_2)}}$

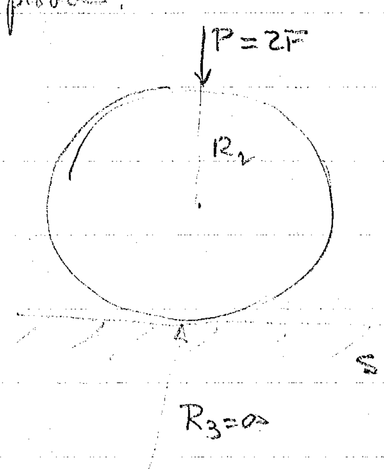
also $p_{z_{max}} = \dfrac{3P}{2\pi a^2}$

what is no slip region if coeff of friction is f_1

$c_1 = a_1 \sqrt[3]{1 - \dfrac{\theta}{f_1}}$

$u_x = \dfrac{3 f_1 F(2-\nu)}{16}\left[1 - \left\{1 - \dfrac{\theta}{f_1}\right\}^{2/3}\right]$

for second problem,



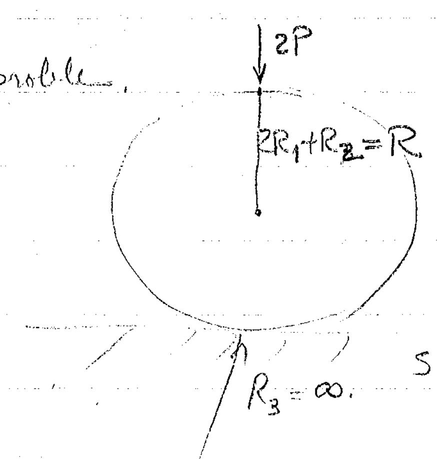
$\alpha = (k_1 + k_2)\dfrac{6P\pi}{4a}$

where

$a_2 = \sqrt[3]{\dfrac{(k_1^b + k_2^s)\, 6P\pi R_2}{4}}$

since there is no T $\quad c_2 = a_2$

for 3rd problem,



$\alpha = (k_1^b + k_2^s)\dfrac{6P}{4a}$

where $a = \sqrt[3]{\dfrac{(k_1^b + k_2^s)\, 6P\pi\, (2R_1 + R_2)}{4}}$

DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING
STANFORD UNIVERSITY

ME 238C Theory of Elasticity

Spring 1980

Final Exam

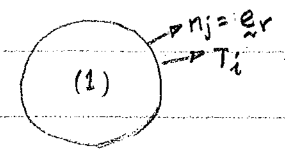
1. A sphere of radius R_1 is subjected to uniform hydrostatic pressure, such that its radius is reduced to a value R_2. Then it is placed into a hollow sphere, (which, of course, had to be cut open and then glued together again) of inner radius R_2 and outer radius R_3. The given material properties of the two spheres are different. Determine the stress distribution in the system consisting of the two spheres. Compare your results to the results of the corresponding two-dimensional problem, i.e. shrink-fitted cylinders.

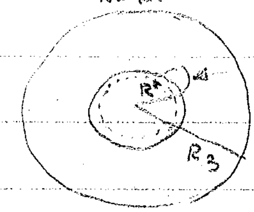
2. Determine the displacement distribution (which would lead to determination of stresses) produced by pinching the arm of a friend. Consider the arm in first approximation as a half-space.

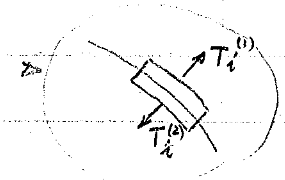
3. Determine the area of contact and the approach in the hip joint subjected to a normal load. The joint is essentially a ball (radius R_1) and socket (radius R_2) arrangement. Be explicit as to your assumptions.

1. Due to symmetry of problem the only nonzero traction will be T_R our boundary conditions will be the continuity of tractions and continuity of displacements at the interface

ie $\quad u_i^{(2)}\Big|_{R=R^*} = u_i^{(2)}\Big|_{R=R^*}$







Similarly by equilib $\quad \sigma_{ij}^{(1)} n_j = \sigma_{ij}^{(2)} n_j$; this comes from $T_i^{(1)} + T_i^{(2)} = 0 \Rightarrow \sigma_{ij}^{(1)} n_j^{(1)} = T_i^{(1)} = -T_i^{(2)}$

but $-T_i^{(2)} = -\sigma_{ij}^{(2)} n_j^{(2)}$; however $-n_j^{(2)} = n_j^{(1)} \quad \therefore \sigma_{ij}^{(1)} n_j^{(1)} = \sigma_{ij}^{(2)} n_j^{(1)}$

If we define everything from the deformed state

Assume that the traction when the syste reaches equilibrium

is p Then by results in class for sphere (2)

$$\sigma_{RR}^{(2)} = p\frac{R^{*3}R_3^3}{(R^{*3}-R_3^3)R^3} - \frac{pR^{*3}}{(R^{*3}-R_3^3)} = \frac{pR^{*3}\left[R_3^3 - R^3\right]}{(R^{*3}-R_3^3)R^3}$$

$$\sigma_{\phi\phi}^{(2)} = \sigma_{\theta\theta}^{(2)} = \frac{-pR^{*3}R_3^3}{2(R^{*3}-R_3^3)R^3} - \frac{pR^{*3}}{(R^{*3}-R_3^3)} = \frac{-pR^{*3}\left[R_3^3 + 2R^3\right]}{2R^3\left[R^{*3}-R_3^3\right]}$$

all the other stresses are zero. Now $I_\sigma^{(2)} = \sigma_{RR} + \sigma_{\theta\theta} + \sigma_{\phi\phi} = \dfrac{-3pR_2^3}{R_2^3 - R_3^3}$

For the inner sphere (1)

$$\sigma_{RR}^{(1)} = -p \qquad \sigma_{\phi\phi}^{(1)} = \sigma_{\theta\theta}^{(1)} = -p \qquad I_\sigma^{(1)} = -3p$$

$$\text{thus} \quad \varepsilon_{RR}^{(2)} = \frac{\partial u_R^{(2)}}{\partial R} = \frac{1+\nu^{(2)}}{E^{(2)}}\sigma_{RR}^{(2)} - \frac{\nu^{(2)}}{E^{(1)}} I_\sigma^{(2)} = \frac{\sigma_{RR}^{(2)}}{E^{(2)}} - \frac{\nu^{(2)}}{E^{(2)}}\left(\sigma_{\phi\phi}^{(2)} + \sigma_{\theta\theta}^{(2)}\right)$$

$$= \frac{1+\nu^{(2)}}{E^{(2)}} \frac{pR^{*3}R_3^3}{(R^{*3}-R_3^3)R^3} + \frac{2\nu^{(2)}-1}{E^{(2)}} \frac{pR^{*3}}{R^{*3}-R_3^3}$$

$$\text{thus} \quad u_R^{(2)} = -\left(\frac{1+\nu^{(2)}}{2E^{(2)}}\right)\frac{pR^{*3}R_3^3}{(R^{*3}-R_3^3)R^2} + \frac{2\nu^{(2)}-1}{E^{(2)}}\frac{pR^{*3}}{R^{*3}-R_3^3}R + \underbrace{f^{(2)}(\theta,\phi)}_{=0 \text{ by symmetry}}$$

$$\text{Similarly} \quad \varepsilon_{RR}^{(1)} = \frac{\partial u_R^{(1)}}{\partial R} = \frac{1+\nu^{(1)}}{E^{(1)}}\sigma_{RR}^{(1)} - \frac{\nu^{(1)}}{E^{(1)}} I_\sigma^{(1)} = -\frac{1+\nu^{(1)}}{E^{(1)}}p + \frac{3\nu^{(1)}}{E^{(1)}}p = \frac{2\nu^{(1)}-1}{E^{(1)}}p$$

and $u_R^{(1)} = \dfrac{2\nu^{(1)}-1}{E^{(1)}}pR + f^{(1)}(\theta,\varphi) \rightarrow 0$ by symmetry

Thus at $R = R^*$ $\quad u_R^{(1)} = u_R^{(2)} \quad \therefore \quad \dfrac{2\nu^{(1)}-1}{E^{(1)}}pR^* = -\left(\dfrac{1+\nu^{(2)}}{2E^{(2)}}\right)\dfrac{pR^*R_3^3}{(R^{*3}-R_3^3)} + \dfrac{2\nu^{(2)}-1}{E^{(2)}}\dfrac{pR^*R^{*3}}{R^{*3}-R_3^3}$

let $\frac{2\nu-1}{E} = A \qquad \frac{1+\nu}{2E} = B$

$$A^{(1)}\left[R^{*3} - R_3^3\right] + B^{(2)} \quad R_3^3 - A^{(2)} R^{*3} = 0$$

$$R^{*3}\left[A^{(1)} - A^{(2)}\right] + B^{(2)} R_3^3 - A^{(2)} R_3^3 = 0$$

$$\therefore \quad R^* = \left[\frac{A^{(2)} - B^{(2)}}{A^{(1)} - A^{(2)}}\right]^{1/3} R_3 \qquad \left(R^*/R_3\right)^3 = \frac{A^{(2)} - B^{(2)}}{A^{(1)} - A^{(2)}}$$

Now $u^{(2)} = -\left(\frac{1+\nu^{(2)}}{2E^{(2)}}\right)\frac{pR^{*3}R_3^3}{(R^{*3} - R_3^3)R^2} + \frac{2\nu^{(2)}-1}{E^{(2)}}\frac{pR^{*3}R}{(R^{*3} - R_3^3)}$

at $R = R^*$ $\quad u^{(2)} = R^* - R_2 = -\left(\frac{1+\nu^{(2)}}{2E^{(2)}}\right)\frac{pR^*R_3^3}{R^{*3} - R_3^3} + \left(\frac{2\nu^{(2)}-1}{E^{(2)}}\right)\frac{pR^{*4}}{R^{*3} - R_3^3}$

$$= \frac{-B^{(2)} pR^*}{-\left[1 - \left(\frac{A^{(2)} - B^{(2)}}{A^{(1)} - A^{(2)}}\right)\right]} + \frac{A^{(2)} \quad pR^{*4}}{-R_3^3\left[1 - \left(\frac{A^{(2)} - B^{(2)}}{A^{(1)} - A^{(2)}}\right)\right]}$$

$$= -pR^*\left\{\frac{-B^{(2)}\left(A^{(1)} - A^{(2)}\right)}{A^{(1)} - 2A^{(2)} + B^{(2)}} + \frac{A^{(2)}\left[\frac{A^{(2)} - B^{(2)}}{A^{(1)} - A^{(2)}}\right]}{A^{(1)} - 2A^{(2)} - B^{(2)}}\right\}$$

$\therefore$

$$-pR^*\frac{\left[A^{(2)}\right]^2 - B^{(2)}A^{(1)}}{A^{(1)} - 2A^{(2)} + B^{(2)}} = R^* - R_2 \qquad \therefore \quad p = -\frac{\left(1 - R_2/R^*\right)\left[A^{(1)} - 2A^{(2)} + B^{(2)}\right]}{\left[A^{(2)}\right]^2 - B^{(2)}A^{(1)}}$$

$\therefore$

$$\sigma_{RR}^{(2)} = \frac{p(R^*R_3)^3}{(R^{*3} - R_3^3)R^3} - \frac{pR^{*3}}{(R^{*3} - R_3^3)}$$

$$\sigma_{\phi\phi}^{(2)} = \sigma_{\theta\theta}^{(2)} = \frac{-p(R^*R_3)^3}{2(R^{*3} - R_3^3)R^3} - \frac{pR^{*3}}{R^{*3} - R_3^3} \qquad \sigma_{R\phi}^{(2)} = \sigma_{R\theta}^{(2)} = \sigma_{\phi\theta}^{(2)} = 0$$

and

$$\sigma_{RR}^{(1)} = \sigma_{\phi\phi}^{(1)} = \sigma_{\theta\theta}^{(1)} = -p \qquad \sigma_{R\phi}^{(1)} = \sigma_{R\theta}^{(1)} = \sigma_{\phi\theta}^{(1)} = 0$$

$u_{R_{inle}} = -\frac{p_c}{E} r (1-2\nu)(1+\nu)$

Another way of doing it is to measure everything from the undeformed state

$$\sigma_{ij} n_j = \sigma_{ij} n_j \quad \text{at } R = R_2 \quad \Rightarrow \sigma_{RR}^{(1)} = \sigma_{RR}^{(2)}$$

$$u_R^{(1)} - u_R^{(2)} = R_2 - R_1 \quad \text{at } R_2 = R$$

$$\sigma_{RR}^{(2)} = \frac{p(R_2 R_3)^3}{(R_2^3 - R_3^3) R^3} - \frac{p(R_2)^3}{(R_2^3 - R_3^3)} \qquad \sigma_{\theta\theta}^{(2)} = \sigma_{\phi\phi}^{(2)} = -\frac{p(R_2R_3)^3}{2(R_2^3 - R_3^3)R^3} - \frac{pR_2^3}{R_2^3 - R_3^3}$$

all others are zero

and $\sigma_{RR}^{(1)} = \sigma_{\theta\theta}^{(1)} = \sigma_{\phi\phi}^{(1)} = -p$ all others are zero.

Now $u_R^{(2)} = -B^{(2)} \dfrac{p(R_2R_3)^3}{(R_2^3 - R_3^3)R^2} + A^{(2)} \dfrac{pR_2^3 R}{(R_2^3 - R_3^3)}$; at $R = R_2$ $u_R^{(2)} = -p\dfrac{A^{(2)} R_2 R_3^3}{R_2^3 - R_3^3} + \dfrac{B^{(2)} p R_2^4}{R_2^3 - R_3^3}$

and $u_R^{(1)} = A^{(1)} p R$; at $R = R_2$ $A^{(1)} p R_2 = u_R^{(1)}$

$$\therefore \quad u_R^{(1)} - u_R^{(2)} = R_2 - R_1 = pR_2 \left\{ \frac{A^{(1)}(R_2^3 - R_3^3) - B^{(2)} R_2^3 + A^{(2)} R_3^3}{R_2^3 - R_3^3} \right\}$$

$$1 - \frac{R_1}{R_2} = p \left[\frac{R_2^3\{A^{(1)} - B^{(2)}\} + R_3^3\{A^{(2)} - A^{(1)}\}}{R_2^3 - R_3^3} \right]$$

$$\therefore \quad p = \left(1 - \frac{R_1}{R_2}\right) \frac{(R_2^3 - R_3^3)}{R_2^3\{A^{(1)} - B^{(2)}\} + R_3^3\{A^{(2)} - A^{(1)}\}}$$

for 2-D case $\quad \sigma_{RR} = \dfrac{R_2^2 R_3^2 (-p_i)}{r^2 \, (R_3^2 - R_2^2)} + \dfrac{p_i R_2^2}{R_3^2 - R_2^2} = -\dfrac{Ap_i}{r^2} + Bp_i$

for outer

$$\sigma_{\theta\theta} = + \frac{R_2^2 R_3^2 p_i}{r^2(R_3^2 - R_2^2)} + \frac{p_i R_2^2}{R_3^2 - R_2^2} \qquad \sigma_{zz} = \nu(\sigma_{RR} + \sigma_{\theta\theta})$$

for inner $\quad \sigma_{RR} = -p_e \quad \sigma_{\theta\theta} = -p_e \quad \sigma_{zz} = \nu(-2p_e)$

$$\varepsilon_{RR} = \frac{\sigma_{RR}}{E} - \frac{\nu}{E}((1+\nu)\sigma_{\theta\theta} + \nu\sigma_{zz}) = \frac{1-\nu^2}{E}\sigma_{RR} - \frac{\nu(1+\nu)}{E}\sigma_{\theta\theta}$$

$$= \frac{1-\nu^2}{E}\left(-\frac{Ap_i}{r^2} + Bp_i\right) - \frac{\nu(1+\nu)}{E}\left(\frac{Ap_i}{r^2} + Bp_i\right)$$

$$= \frac{Ap_i}{Er^2}\left[-(1-\nu^2) - \nu^2 - \nu\right] + \frac{Bp_i}{E}(1 - \nu^2 - \nu - \nu^2)$$

$$\frac{\partial u_R}{\partial r} = -\left(\frac{1+\nu}{E}\right)\frac{Ap_i}{r^2} + \frac{Bp_i}{E}(1-2\nu)(1+\nu)$$

$u_R = +\left(\dfrac{1+\nu}{E}\right)\dfrac{Ap_i}{r} + \dfrac{Bp_i}{E}(1-2\nu)(1+\nu)r$; for inner $\varepsilon_{RR} = -\dfrac{p_e}{E} - \dfrac{\nu}{E}(-p_e - 2\nu p_e) = -\dfrac{p_e}{E}(1 - \nu - 2\nu^2)$

$$\frac{1}{R(R+z)} + \frac{x}{(R+z)} \cdot \frac{-x}{R^3} + \frac{x}{R^2} \frac{-x}{(R+z)^2}$$

$$\frac{1}{R(R+z)} + \frac{-x^2(R+z)}{R^3(R+z)^2} - \frac{x^2}{R^2(R+z)^2}$$

$$\frac{\partial}{\partial z}\frac{1}{R^3} = \frac{-3}{R^4}\frac{z}{R} = \frac{-3z}{R^5}$$

$$\frac{\partial}{\partial z}\frac{x^2 z}{(R^2+Rz)^2} = \frac{x^2}{(R^2+Rz)^2} + \frac{-2x^2 z}{(R^2+Rz)^3} \cdot \left(2z + \frac{z^2}{R} + R\right)$$

$$= x^2(R^2 + \cancel{Rz}) - 4x^2z^2 - \frac{2x^2z^2}{R} - \cancel{2x^2zR}$$

$$x^2R^2 - x^2zR - \frac{2x^2z^2}{R} - 4x^2z^2$$

$$\frac{\partial}{\partial x}\left(\frac{x}{R(R+z)}\right) = \frac{1}{R(R+z)} + \frac{-x}{[R(R+z)]^2}\left(2x + \frac{xz}{R}\right)$$

$$\frac{-x^2(2R+z)}{R^3(R+z)^2} \qquad \frac{x^2}{R^3(R+z)} \qquad \frac{x^2}{R^2(R+z)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{R}\right) = \frac{1}{R} + \frac{-x^2}{R^3}$$

$$\frac{\partial}{\partial x}\left[\frac{xz}{R(R+z)}\right] = \frac{z}{R(R+z)} + \frac{-xz}{[R(R+z)]^2}\left(2x + \frac{zx}{R}\right)$$

$$\frac{\partial}{\partial z}\frac{x}{R+z} = \frac{1}{R+z} \qquad \frac{-x^2}{R(R+z)^2}$$

$$u_x\Big|_{z=0} = \frac{1-2\nu}{2\pi\mu}\Lambda\left[\frac{-2x^2}{R^4} + \frac{1}{R^2}\right] - \frac{\Lambda}{4(1-\nu)2\pi\mu}\left[(1-2\nu)\left[\frac{-2x^2}{R^4} + \frac{1}{R^2}\right] - (1-2\nu)^2\left(\frac{x^2}{R^4} - \frac{1}{R^2}\right)\right]$$

$$\frac{-x^2+y^2}{R^4} \qquad \frac{-x^2+y^2}{R^4} \qquad \frac{y^2 - \cancel{x^2}}{R^4}$$

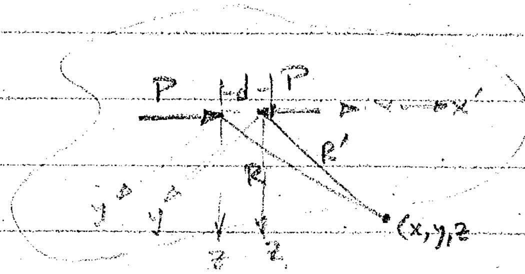
$$\frac{-x^2}{} \quad \left[\frac{}{R^2}(R+z)^{-1} + x^2(R+z)^{-3}(z/R+1)\right]$$

$$-\frac{1}{(R+z)^2}\left(\frac{z}{R}+1\right) \to -\frac{1}{R^2}$$

$$\frac{1-2\nu}{2\pi\mu}\Lambda\left[\frac{-x^2+y^2}{R^4} + \frac{1}{4(1-\nu)}\left[\frac{x^2-y^2}{R^4} - \frac{(1-2\nu)^2 y^2}{R^4}\right]\right]$$

$$-4(1-\nu) + 1 \qquad \frac{-3+4\nu}{4(1-\nu)}\frac{x^2}{R^4}$$

for concentrated forces



$$\mathbb{u}_{(1)} = \mathbb{B}_{(1)} - \frac{1}{4(1-\nu)} \nabla\left(\mathbb{r}\cdot\mathbb{B}_{(1)} + \beta_{(1)}\right)$$

$$\mathbb{B}_{(1)} = \frac{P}{2\pi\mu R} e_x + \frac{(1-2\nu)}{2\pi\mu} \frac{Px}{R(R+z)} e_z$$

$$\beta_{(1)} = -\frac{(1-2\nu)^2 P x}{2\pi\mu(R+z)}$$

$\mathbb{u}_{(2)} = \mathbb{u}_{(2)}(x', y, z; P)$ where $x' = x-d$ and $x = x'+d$

$\therefore\ \mathbb{u}^{(2)} = \mathbb{u}^{(1)} - d\frac{\partial \mathbb{u}^{(1)}}{\partial x} + \dots \quad ; \quad \mathbb{u}^{(2)}(-P) = -\mathbb{u}^{(1)} + d\frac{\partial \mathbb{u}^{(1)}}{\partial x}$

$$\mathbb{u} = \mathbb{u}_{(1)} + \mathbb{u}_{(2)} = \frac{Pe_x}{2\pi\mu}\left[\frac{1}{R} - \frac{1}{R'}\right] + \frac{1-2\nu}{2\pi\mu} P e_z \left[\frac{x}{R(R+z)} - \frac{(x-d)}{R'(R'+z)}\right] - \frac{1}{4(1-\nu)} \nabla\left\{\frac{P}{2\pi\mu}\left[\frac{x}{R} - \frac{x-d}{R'}\right]\right.$$

$$\left. + \frac{1-2\nu}{2\pi\mu} P \left[\frac{xz}{R(R+z)} - \frac{(x-d)z}{R'(R'+z)}\right] - \frac{(1-2\nu)^2 P}{2\pi\mu}\left[\frac{x}{R+z} - \frac{(x-d)}{(R'+z)}\right]\right\}$$

Now $\lim_{d\to 0} \frac{1}{R} - \frac{1}{R'} = \frac{-x}{R^3} \Rightarrow \frac{d}{RR'}\frac{R'-R}{d} = \frac{d}{RR'}\frac{\sqrt{R^2+d^2-2dx} - R}{d} = \frac{d}{RR'}\left[R\left(1+\frac{d^2-2dx}{2R^2}\right) - R\right]$

$= \frac{d}{RR'}\left[\frac{d}{2R} - \frac{x}{R}\right] \to \frac{-x}{R^3}$

$$\lim_{d\to 0} \frac{x(R'^2+R'z) - x(R^2+Rz) + d(R^2+Rz)}{RR'(R+z)(R'+z)} = \frac{xd}{R^2(R+z)^2}\left\{\left[\frac{(x-d)^2 - x^2}{d}\right] + z\frac{(R'-R)}{d}\right\} + d\frac{1}{R(R+z)}$$

$$\lim_{d\to 0}\left(\frac{x}{R(R+z)} - \frac{x-d}{R'(R'+z)}\right) = \frac{xd}{R^2(R+z)^2}\left\{-2x - \frac{zx}{R}\right\} + \frac{d}{R(R+z)}$$

$$\lim_{d\to 0}\left[\frac{x}{R+z} - \frac{(x-d)}{R'+z}\right] = \lim_{d\to 0}\frac{xR' - xR + d(R+z)}{(R'+z)^2} = \frac{dx}{(R+z)^2}\frac{(R'-R)}{d} + \frac{d}{R+z} = \frac{dx}{(R+z)^2}\left[\frac{-x}{R}\right] + \frac{d}{R+z}$$

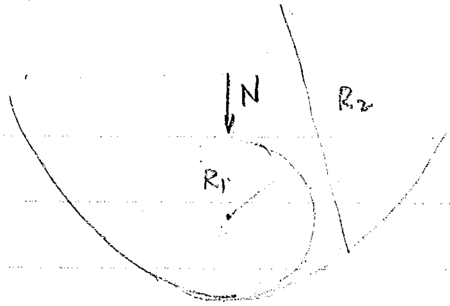
$$\therefore\ \mathbb{u} = \frac{Ae_x}{2\pi\mu}\left[\frac{-x}{R^3}\right] + \frac{1-2\nu}{2\pi\mu}Ae_z\left[\frac{-2x^2}{R^2(R+z)^2} - \frac{zx^2}{R^3(R+z)^2} + \frac{1}{R(R+z)}\right] - \frac{1}{4(1-\nu)}\nabla\left\{\frac{A}{2\pi\mu}\left(\frac{-x^2}{R^3} + \frac{1}{R}\right)\right.$$

$$\left. + \frac{1-2\nu}{2\pi\mu}A\left[\frac{-2x^2z}{R^2(R+z)^2} - \frac{z^2x^2}{R^3(R+z)^2} + \frac{z}{R(R+z)}\right] - \frac{(1-2\nu)^2}{2\pi\mu}A\left[\frac{-x^2}{R(R+z)^2} + \frac{1}{(R+z)}\right]\right\}$$

$$u_x = \frac{-A}{2\pi\mu}\frac{x}{R^3} - \frac{A}{4(1-\nu)2\pi\mu}\frac{\partial}{\partial x}\left[\left(\frac{-x^2}{R^3} + \frac{1}{R}\right) + (1-2\nu)\left[\frac{-2x^2z}{R^2(R+z)^2} - \frac{x^2z^2}{R^3(R+z)^2} + \frac{z}{R(R+z)}\right] - (1-2\nu)^2\left(\frac{-x^2}{R(R+z)^2} + \frac{1}{(R+z)}\right)\right]$$

$$u_y = \frac{-A}{4(1-\nu)2\pi\mu}\frac{\partial}{\partial y}\left[\left(\frac{-x^2}{R^3} + \frac{1}{R}\right) + (1-2\nu)\left[\frac{-2x^2z}{R^2(R+z)^2} - \frac{z^2x^2}{R^3(R+z)^2} + \frac{z}{R(R+z)}\right] - (1-2\nu)^2\left(\frac{-x^2}{R(R+z)^2} + \frac{1}{R+z}\right)\right]$$

$$u_z = \frac{1-2\nu}{2\pi\mu}A\left[\frac{-2x^2}{R^2(R+z)^2} - \frac{zx^2}{R^3(R+z)^2} + \frac{1}{R(R+z)}\right] - \frac{A}{4(1-\nu)2\pi\mu}\frac{\partial}{\partial z}\left[\left(\frac{-x^2}{R^3} + \frac{1}{R}\right) + (1-2\nu)\left[\frac{-2x^2z}{R^2(R+z)^2} - \frac{x^2z^2}{R^3(R+z)^2} + \frac{z}{R(R+z)}\right] - (1-2\nu)^2\left(\frac{-x^2}{R(R+z)^2} + \frac{1}{R+z}\right)\right]$$



$$a = \sqrt[3]{\frac{(k_1+k_2)3N\pi}{4\left[\frac{1}{R_1}-\frac{1}{R_2}\right]}} = \sqrt{\frac{(k_1+k_2)3N\pi R_1R_2}{4(R_2-R_1)}}$$

$\therefore$ area of contact $= \pi a^2 = \pi\left[\frac{(k_1+k_2)3N\pi R_2R_1}{4(R_2-R_1)}\right]^{2/3}$

R ball $= R_1$ $\quad$ R socket $= R_2$

if socket is rigid $k_2 = 0$ $\quad$ $\pi a^2 = \pi\left[\frac{1-\nu_1}{2\mu_1}\cdot\frac{3N\,R_2R_1}{4(R_2-R_1)}\right]^{2/3}$

$$= \pi\left[\frac{1-\nu_1}{\mu_1}\;\frac{3NR_2R_1}{8(R_2-R_1)}\right]^{2/3}$$

$$\alpha = \sqrt[3]{\frac{9N^2\pi^2(k_1+k_2)^2}{16\cdot\frac{1}{\frac{1}{R_1}-\frac{1}{R_2}}}} = \sqrt[3]{\frac{9N^2\pi^2(k_1+k_2)^2(R_2-R_1)}{16R_1R_2}}$$

for rigid socket $k_2 = 0$

$$\alpha = \sqrt[3]{\frac{9N^2\pi^2k_1^2(R_2-R_1)}{16R_1R_2}} = \sqrt[3]{\frac{9N^2(1-\nu_1)^2(R_2-R_1)}{64\mu_1^2R_1R_2}}$$

if same properties $\quad a = \sqrt[3]{\frac{k_1\cdot 3N\pi R_2R_1}{2(R_2-R_1)}} = \sqrt[3]{\frac{(1-\nu)}{4\mu}\;\frac{3NR_2R_1}{R_2-R_1}}$

$$\alpha = \sqrt[3]{\frac{9N^2\pi^2k_1^2(R_2-R_1)}{4R_1R_2}} = \sqrt{\frac{9N^2(1-\nu)^2(R_2-R_1)}{\mu^2\,16R_1R_2}}$$

contt area $= \pi\left[\frac{(1-\nu)^2\cdot 9N^2R_2^2R_1^2}{16\mu^2\;(R_2-R_1)^2}\right]^{1/3}$