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ME 238B : Theory of Elasticity

Barnett Rm 550K

Homework – handed out on Monday, Due on Monday 1/3 grade

TA Rich King 264 Durand. TH 2-4 F 12-2

Mid-Term: Take home 1/3 grade

Final : In class $\frac{1}{3}$ grade

Rough Outline

a. 2-D Problems

Fourier Series, Integral transforms

Complex variable methods

Cracks, Inclusions

b. 3-D Problems

Green's functions methods

Boundary Integral Eqn. Method.

Well try to take a physical notion and derive p.d.e. for it.

a. $C_{ijkl} \dfrac{\partial^2 u_k}{\partial x_i \partial x_l} = 0$ (static, no. boundary forces)

or $\dfrac{\partial \sigma_{ij}}{\partial x_i} = 0 \Rightarrow \sigma_{ij} = C_{ijkl}\, e_{kl}\ ;\ e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$

uses the stress function approach; requires use of compat eqn

In 2-D we will have 2 P.D.E. and 2 unknowns → 2 B.C.

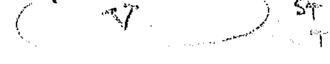
3-D " " " 3 " " 3 " → 3 B.C.

Uniqueness Thm (a review)

(1) We specify traction vector $\underset{\sim}{T}$ at all points on the boundary (T_x, T_y, T_z) This is a "dead loading" problem. $\underset{\sim}{u}$ is unique only to a rigid body motion.

(2) Specify $\underset{\sim}{u}$ everywhere on boundary – gives a unique solution

(3) Mix (1) and (2)



a. T_x, u_y, u_z

b. T_x, T_y, u_z

cannot specify T_x, u_y, u_x (cannot specify u_x & T_x)

c. for elastic foundation $T_x + k u_x = 0$

The uniqueness proof assumes that the volume is finite and "No elastic singularities".

i.e. (1) $\underline{u}$ discontinuous (shrink fit, dislocations, cracks)

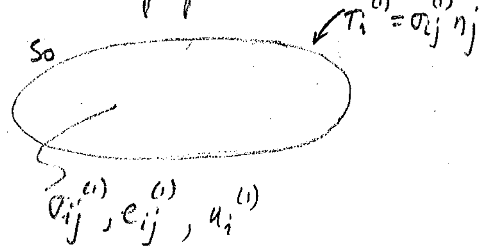
(2) $|\sigma_{ij}| \to \infty$

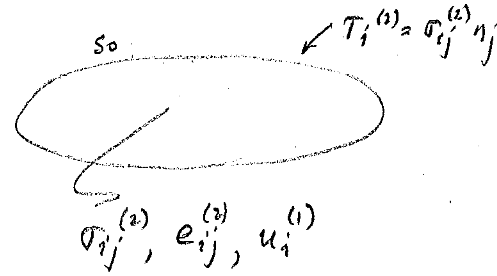
<u>Semi Inverse Method</u>

Obtain by any fashion a solution to the governing PDE (equil eqns) w/out regard for Boundary conditions. Then try to figure out what problem you have solved.

If solution is not what you need, generate another. Then linearly superpose these problems to get different solutions.

<u>Linear Superposition</u>





Equil. $\sigma_{ij,i}^{(1)} = 0$	$\sigma_{ij,i}^{(2)} = 0$
$e_{ij}^{(1)} = \frac{1}{2}(u_{i,j}^{(1)} + u_{j,i}^{(1)})$	$e_{ij}^{(2)} = \frac{1}{2}(u_{i,j}^{(2)} + u_{j,i}^{(2)})$
$\sigma_{ij}^{(1)} = C_{ijkl}\, e_{kl}^{(1)}$	$\sigma_{ij}^{(2)} = C_{ijkl}\, e_{kl}^{(2)}$
$T_i^{(1)} = \sigma_{ij}^{(1)} n_j$ on S_0	$T_i^{(2)} = \sigma_{ij}^{(2)} n_j$ on S_0

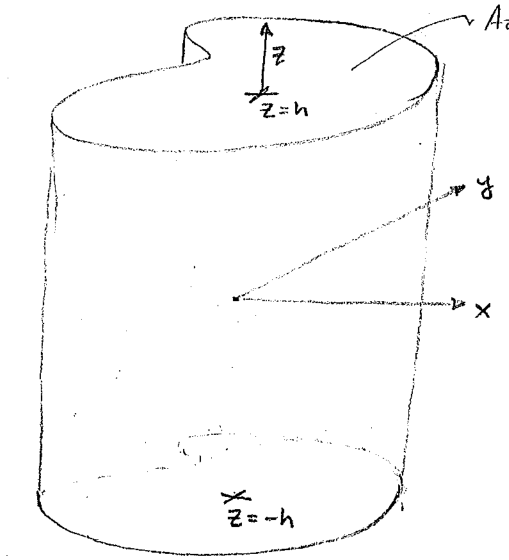
if $\Sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \Rightarrow \Sigma_{ij,i} = 0$ ← this works no matter what

if $E_{ij} = e_{ij}^{(1)} + e_{ij}^{(2)}$ and $U_i = u_i^{(1)} + u_i^{(2)} \Rightarrow E_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$
this works only for infinitesimal strains

$\Downarrow$

$\Sigma_{ij} = C_{ijkl} E_{ij}$ and $T_i = T_i^{(1)} + T_i^{(2)} = \Sigma_{ij} n_j$ on S_0

Simple extension problem



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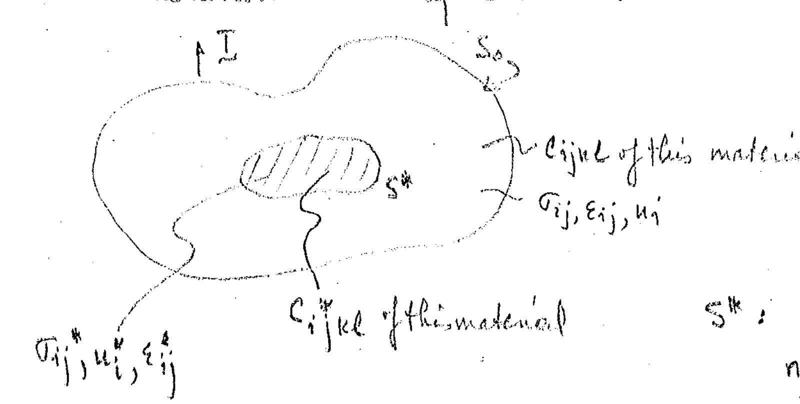
One problem not mentioned in class last time
Inclusion Boundary conditions



S_0 : $\sigma_{ij} n_j = T_i$

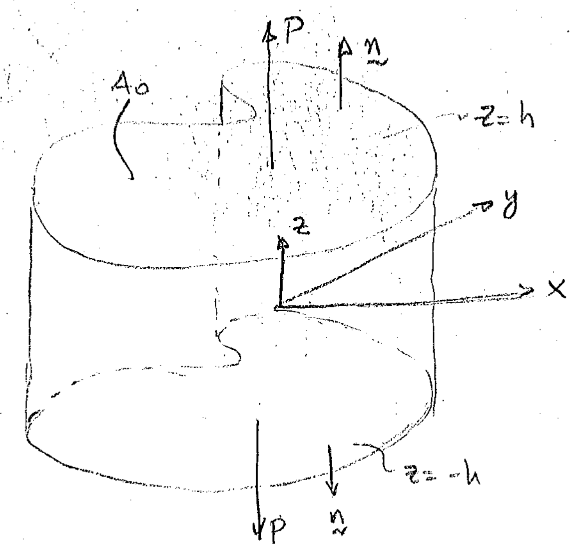
S^* : $u_i^* = u_i$ perfectly bonded
$n_j \sigma_{ij}^* = \sigma_{ij} n_j$

Back to the problem we started last time

Simple Extension

(Any constant state of stress satisfies equil eqns)

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0$$



Traction BC.

@ $z=h$ $\quad T_z = \frac{P}{A_0} = \sigma_{zj} n_j \qquad n_x = n_y = 0 \,;\; n_z = +1$

$$\boxed{T_z = \frac{P}{A_0} = \sigma_{zz} \text{ @ } z=h} \qquad \boxed{T_x = \sigma_{xj} n_j \Rightarrow \sigma_{xz} = 0 \text{ @ } z=h}$$

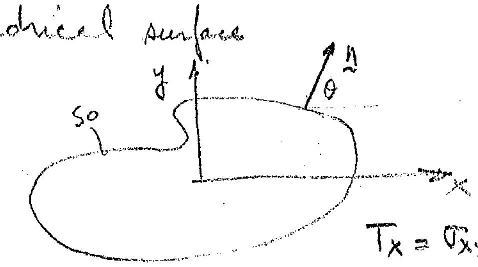
$$T_y = 0 \Rightarrow \boxed{\sigma_{yz} = 0 \text{ @ } z=h}$$

@ $z=-h$ $\qquad n_x = n_y = 0 \;;\; n_z = -1$

$$T_z = \frac{-P}{A_0} = \sigma_{zj} n_j = \sigma_{zz}(-1) \Rightarrow \boxed{\sigma_{zz} = \frac{P}{A_0} \text{ on } z=-h}$$

$$T_x, T_y = 0 \qquad \boxed{\sigma_{xz} = \sigma_{yz} = 0 \text{ on } z=-h}$$

on the cylindrical surface



$$T_x = T_y = T_z = 0$$

$$n_x = \cos\theta \quad n_y = \sin\theta \quad n_z = 0$$

$$T_x = \sigma_{xx}\cos\theta + \sigma_{xy}\sin\theta = 0$$

$$T_y = \sigma_{yx}\cos\theta + \sigma_{yy}\sin\theta = 0$$

$$T_z = \sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta = 0$$

pick $\sigma_{xx} = \sigma_{xy} = \sigma_{yy} = \sigma_{zx} = \sigma_{zy} = 0$ on S_o

Hence if we pick $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{zx} = \sigma_{zy} = 0$ everywhere and $\sigma_{zz} = \frac{P}{A}$ on $z = \pm$

Assume for this problem $\sigma_{zz} = \frac{P}{A_o}$; all others $\sigma_{ij} = 0$

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right) = \frac{-\nu}{E}\frac{P}{A_o} = \frac{\partial u_x}{\partial x}$$

$$e_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right) = \frac{-\nu P}{E A_o} = \frac{\partial u_y}{\partial y}$$

$$e_{zz} = \frac{\sigma_{zz}}{E} = \frac{P}{E A_o} = \frac{\partial u_z}{\partial z}$$

$$\sigma_{xz} = \sigma_{yz} = \sigma_{xy} = 0 \Rightarrow e_{xz} = e_{yz} = e_{xy} = 0$$

$$u_x = -\frac{\nu P}{A_o E}x + f(y,z)$$

$$u_y = -\frac{\nu P}{E A_o}y + g(x,z)$$

$$u_z = \frac{P}{E A_o}z + h(y,x)$$

$$e_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \Rightarrow \frac{\partial f}{\partial z} + \frac{\partial h}{\partial x} = 0$$

$\Rightarrow \frac{\partial f}{\partial z} = -\frac{\partial h}{\partial x} = k(y)$

$\therefore f = k_1(y)z + k_2(y)$

$h = -k_1(y)x + k_3(y)$

→ HW #1 complete and solve showing solution is/includes a rigid body rotation/translat

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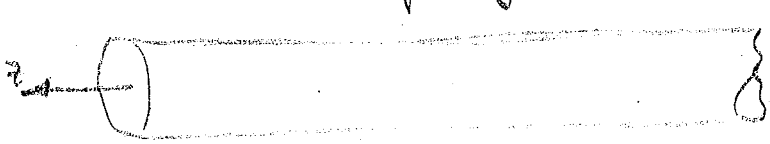
2-D elastostatic problems (isotropic materials)

Plain strain

Elastic solid very long in 1 direction



$e_{zz} = 0$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2\mu}\sigma_{yz} = C$$

$e_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{2\mu}\sigma_{xz} = 0$

$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq 0$ and also not fns of z

$\therefore$ $u_x = u_x(x,y)$; $u_y = u_y(x,y)$; $u_z = 0$
$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq 0$; $\neq$ fns of z

Since $e_{zz} = 0 = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx}+\sigma_{yy})\right) \Rightarrow \sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy}) \neq$ fn of z

each cross section has same thing happening as any other crossection. Plane strain normally simulates the effects at center of a very thick plate

The Equil Eqns reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \qquad \text{since } \sigma_{zx} = 0 \qquad (1)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \qquad \text{since } \sigma_{zy} = 0 \qquad (2)$$

since $\sigma_{zz} \neq$ fn of z $\Longrightarrow$ $\frac{\partial \sigma_{zz}}{\partial z} \equiv 0$ in third eq

<u>Solution by Airy Stress fn</u>

Define a fn ϕ $\ni$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \ ; \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \ ; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

put into equil (1) $\therefore$ $\frac{\partial^3 \phi}{\partial x \partial y^2} - \frac{\partial^3 \phi}{\partial x \partial y^2} = 0$

also same for (2) $\frac{\partial^3 \phi}{\partial x^2 \partial y} - \frac{\partial^3 \phi}{\partial x^2 \partial y} = 0$

But what does ϕ satisfy? Look at Hooke's law

$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$ assume a displ field exists $\ni$ $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

now $\sigma_{ij,i} = \lambda \delta_{ij} e_{kk,i} + 2\mu e_{ij,i} = \lambda e_{kk,j} + 2\mu e_{ij,i}$

subst. the displ gradient relationship $= \lambda (u_{k,kj}) + \mu (u_{j,ii} + u_{i,ji})$

$= (\lambda+\mu) u_{k,kj} + \mu u_{j,ii}$

now differentiate once

$$(\lambda+\mu)(u_{k,kjj}) + \mu (u_{j,iij}) = 0$$

or $(\lambda + 2\mu)(u_{k,kjj}) = 0$

or $(\lambda+2\mu) \nabla^2 e_{kk} = 0 \Rightarrow \nabla^2 \sigma_{kk} = 0 \Rightarrow \nabla^4 \phi = 0$

now $\sigma_{ii} = \sum_{i}^{3} \lambda e_{kk} + 2\mu e_{ii} \Rightarrow \sigma_{kk} = (3\lambda + 2\mu) e_{kk}$ or $\nabla^2 \sigma_{kk} = (3\lambda+2\mu)\nabla^2 e_{kk} = 0$. Next time will prove ↑
no summation

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Plane Strain — From last time

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0 \end{aligned}\right\} \quad \begin{aligned} \sigma_{xx} &= \phi_{,yy} \\ \sigma_{xy} &= -\phi_{,xy} \\ \sigma_{yy} &= \phi_{,xx} \end{aligned}$$

where ϕ is the airy stress fn

using $\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$ material relation w/

Equil $\sigma_{ij,i} = 0$ and $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$(\lambda+\mu) u_{i,ij} + \mu u_{j,ii} = 0$

now take $\frac{\partial}{\partial x_j}$ $(\lambda+\mu) u_{i,ijj} + \mu u_{j,iij} = 0$ — since dummy indices rep $i \to j$, $j \to i$ in 2nd relation but $i,jji = i,ijj$

hence $(\lambda+2\mu)(u_{i,i})_{,jj} = (\lambda+2\mu)(e_{ii})_{,jj} = 0$

$$\therefore \quad \nabla^2 e_{ii} = 0 \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i} = (\ \ )_{,ii}$$

now $\sigma_{ii} = \lambda \overset{3}{\delta_{ii}} e_{kk} + 2\mu e_{ii} = (3\lambda + 2\mu) e_{ii}$

$$\therefore \quad \nabla^2 e_{ii} \Rightarrow \nabla^2 \sigma_{ii} = 0$$

Now in plane strain $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ from $e_{zz} = 0$

$$\nabla^2 \sigma_{ii} = \nabla^2 (1+\nu)(\sigma_{xx} + \sigma_{yy}) = (1+\nu) \nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0$$

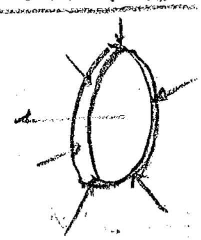
using airy stress fns

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = \nabla^2(\phi_{,yy} + \phi_{,xx}) = \nabla^2(\nabla^2 \phi) = \nabla^4 \phi = 0$$

Plane Stress

Plane $\Rightarrow \sigma_{zx} = \sigma_{zy} = 0$ Plane Stress also $\Rightarrow \sigma_{zz} = 0$

If it is small thickness & hence must vary from 0 to 0 over a small thickness: assume 0 everywhere



$$0 = \begin{pmatrix} \sigma_{zz} \\ \sigma_{zx} \\ \sigma_{zy} \end{pmatrix}$$ on flat faces

$\sigma_{zz} = 0 \Rightarrow e_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$

$\sigma_{zx} = 0 \Rightarrow e_{zx} = 0$

$\sigma_{zy} = 0 \Rightarrow e_{zy} = 0$

Tentatively Assume $\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq f(z)$ or $u_x = u_x(x,y)$, $u_y = u_y(x,y)$

Timoshenko & Goodier
Rd 274-277

HW #1b prove that this assumption is inconsistent

however we can see that u_x, u_y have z^2 component & that for $z \ll 1$ then we can assume the above w/o loss in accuracy
hence, we can define generalized displ for disc w/thickness h

$$U_x(x,y) = \frac{1}{h}\int_0^h u_x(x,y,z)\,dz$$

We will now prove that $\nabla^4\phi = 0$ is DE for plane strain & plane stress for certain conditions

Plane strain

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right) \quad \leftarrow \nu(\sigma_{xx} + \sigma_{yy})$$

$$= \frac{1}{E}\left\{\sigma_{xx}(1-\nu^2) - \sigma_{yy}\,\nu(1+\nu)\right\}$$

$$= \frac{1+\nu}{E}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$$

now $\mu = \frac{E}{2(1+\nu)}$ $\therefore e_{xx} = \frac{1}{2\mu}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$

$$e_{xy} = \frac{\sigma_{xy}}{2\mu}$$

Plane stress

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}\overset{0}{})\right) = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) = \frac{1}{2\mu}\left\{\frac{\sigma_{xx}}{1+\nu} - \frac{\nu}{1+\nu}\sigma_{yy}\right\}$$

$$e_{xy} = \frac{1}{2\mu}\sigma_{xy}$$

for these two to be same then the two conditions needed are the $\overset{\text{stress}}{\epsilon_{xx} = A\sigma_{xx} + B\sigma_{yy}} = \overset{\text{strain}}{\epsilon_{xx} = C\sigma_{xx} + D\sigma_{yy}}$

$$\frac{1}{1+\nu_\sigma} = 1-\nu_\epsilon \quad \text{and} \quad \nu_\epsilon = \frac{\nu_\sigma}{1+\nu_\sigma} \quad \Rightarrow \quad \nu_\sigma = \frac{\nu_\epsilon}{1-\nu_\epsilon}$$

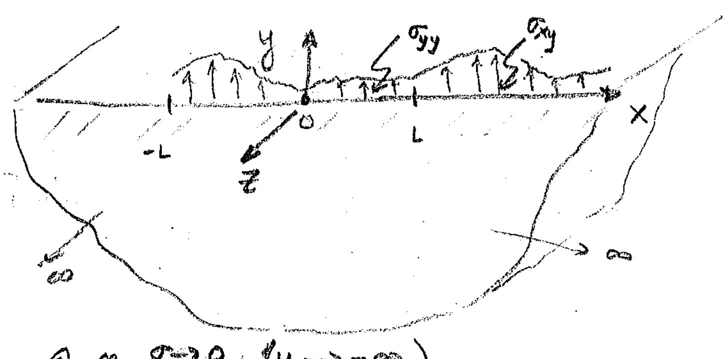
(i) Given a complete plain strain soln, get the plane stress solution by leaving μ fix and replace ν_ϵ by $\frac{\nu}{1+\nu}$

(ii) Given plane stress and want plane strain replace ν_σ by $\frac{\nu}{1-\nu}$

We now look at

2-D problems in rectangular coordinates using Fourier Series (Timo&Good pg 53 ff)
Rect, strips, half space

Problem # 1



Look at plane strain problem
Traction boundary value problem
on $y=0$ $\quad \underline{n} = e_y$

$T_x = \sigma_{xy}$
$T_y = \sigma_{yy}$
$T_z = \sigma_{zy} = 0$ since plain strain $\Leftarrow \epsilon_{yz} = 0$

@ ∞ $\sigma \to 0$ ($y \to -\infty$)

We will assume no shear loading $\sigma_{xy} = 0$. Assume $\sigma_{yy} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$

$f(x) = \sigma_{yy}(x,0) = \sum A_n \sin \frac{n\pi x}{L} \qquad A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx$

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Continuing this problem again

For $|x| < \infty$, $0 > y > -\infty$ w/ Periodic loading on $y=0 \Rightarrow \sigma_{yy}, \sigma_{xy}, \sigma_{zy}$ only loads
we look at plane strain problem i.e. $\sigma_{zy} = 0$

We now look at problem $\sigma_{yy} = \sum A_n \sin \frac{n\pi x}{L} \qquad \sigma_{xy} = 0$

Note: next problem we will look at $\sigma_{xy} = \sum B \sin \frac{n\pi x}{L} \quad \sigma_{yy} = 0$
finally look at $T_i = \eta \sigma_{yy} + \xi \sigma_{xy}$ where η, ξ are direction cosines

We also impress $\sigma_{ij} \to 0$ as $y \to -\infty$

**** <u>Problem given</u>: $\sigma_{yy}(x,0) = f(x)$ $\quad \sigma_{xy} = 0$

$$\sigma_{yy}(x, y=0) = f(x) = \sum A_n \sin \frac{n\pi x}{L} \qquad A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

PDE $\nabla^4 \phi = 0$ $\quad$ pick $\phi = g(y) \sin \frac{n\pi x}{\ell} = g(y) \sin \gamma_n x$ $\quad$ since it satisfies the form of b/c

$$\therefore \quad \nabla^4\phi = \left[(\gamma_n)^4 g - 2(\gamma_n)^2 g'' + g^{iv}\right] \sin \gamma_n x = 0 \Rightarrow \gamma_n^4 g - 2\gamma_n^2 g'' + g^{iv} = 0 \quad (1)$$

Take $g(y) = e^{sy} \Rightarrow (1) \Rightarrow (s^2 - \gamma_n^2)^2 = 0$

$\therefore \quad s = \pm\gamma_n, \pm\gamma_n$

$$\phi_n(x,y) = \sin \frac{n\pi x}{\ell}\left\{\alpha_n e^{\gamma_n y} + \beta_n e^{-\gamma_n y} + C_n y e^{\gamma_n y} + D_n y e^{-\gamma_n y}\right\}$$

$$\phi = \sum_{n=1}^{\infty} \phi_n \qquad \phi_0 = 0$$

using b.c. that $\sigma_{ij} \to 0$ as $y \to -\infty$ $\quad$ pick $\beta_n = D_n = 0$

$$\therefore \phi = \sum_{n=1}^{\infty} \phi_n = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{\ell}\left\{\alpha_n e^{\gamma_n y} + C_n y e^{\gamma_n y}\right\}$$

this satisfy compat & equil (since $\nabla^4\phi = 0$ came from these) and b.c. at $-\infty$

now look at <u>b.c. at $y=0$</u>

since $\left.\begin{matrix}\sigma_{xy} = 0 \\ @\, y=0\end{matrix}\right\} \xrightarrow{-\phi_{,xy}}$ $= -\sum_{n=1}^{\infty} \cos \frac{n\pi x}{\ell} \cdot \left(\frac{n\pi}{\ell}\right)\left\{\alpha_n \gamma_n e^{\gamma_n y} + C_n e^{\gamma_n y} + \gamma_n C_n y e^{\gamma_n y}\right\}$

$$= -\sum_{n=1}^{\infty} \cos \frac{n\pi x}{\ell}\left(\frac{n\pi}{\ell}\right)\left\{\alpha_n \gamma_n + C_n\right\} = 0$$

$$\Rightarrow \boxed{C_n = -\gamma_n \alpha_n}$$

$$\therefore \quad \phi = \sum_{n=1}^{\infty} \phi_n = \sum_{n=1}^{\infty} \alpha_n \sin \gamma_n x \,(1 - \gamma_n y)\, e^{\gamma_n y}$$

also $\sigma_{yy}(y=0) = \sum_1^{\infty} A_n \sin \frac{n\pi x}{\ell}$; $\quad \phi_{,xx} = -\sum_1^{\infty} \alpha_n \gamma_n^2 \sin \gamma_n x \,(1 - \gamma_n y)\, e^{\gamma_n y}$

at $y=0$ $\quad \phi_{,xx} = -\sum_1^{\infty} \alpha_n \gamma_n^2 \sin \gamma_n x$

$\therefore \quad A_n = -\alpha_n \gamma_n^2 \qquad$ or $\quad \boxed{\alpha_n = -\frac{A_n}{\gamma_n^2}}$

$$\therefore \phi(x,y) = -\sum_{n=1}^{\infty} \frac{A_n}{\gamma_n^2} \sin\gamma_n x \,(1-\gamma_n y)\, e^{\gamma_n y} \qquad \text{where } A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \,; \; \gamma_n = \frac{n\pi}{L}$$

to get displ $\quad \epsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{1}{E}\{\sigma_{xx} - \nu(\sigma_{yy}+\sigma_{zz})\} = \frac{-\nu}{E}\phi_{,xx} + \frac{1}{E}\phi_{,yy}$

thus $u_x = \int \frac{\partial u_x}{\partial x}\,dx + f(y,\not{z})$ now since we only want $u_x = u(x,y)$ plane strain

<u>Problem #2</u> $\quad$ on $y=0 \quad \sigma_{yy} = \frac{a_0}{2} + \sum_1^{\infty} F_n \cos\frac{n\pi x}{L}$

$\sigma_{xy} = 0$

$$\sigma_{yy}(x, y=0) = f(x) = \sum_1^{\infty} F_n \cos\gamma_n x + \frac{a_0}{2} \qquad \text{where } F_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

(2a) if $F_n = 0$ $\quad$ look at $\phi_0 = \alpha x^2 \qquad \nabla^4\phi_0 = 0$ $\quad$ and since $\sigma_{yy} = \frac{a_0}{2}$

$\sigma_{yy} = \frac{\partial^2\phi_0}{\partial x^2} = 2\alpha \Rightarrow \quad 2\alpha = \frac{a_0}{2} \qquad \alpha = \frac{a_0}{4}$

$\sigma_{xx} = 0 \quad \sigma_{xy} = 0 \quad$ since $\phi_0 \neq f(y)$

if load is periodic in direction 1 and $\nabla^4\phi = 0$ is governing pde in a half-space then $\phi \to 0$ in direction 2, as $|x_2| \to \infty$

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<u>Continuing problem #2</u>

$\sigma_{xy} = T_x = 0 \;; \quad \sigma_{yy} = T_y = \frac{a_0}{2} + \sum_{n=1}^{\infty} F_n \cos\gamma_n x \;; \quad \sigma_{yz} = T_z = 0 \quad$ on $y=0 \qquad F_n = \frac{2}{L}\int_0^L f(x)\cos\gamma_n x\,d$

if $\phi_0 = \alpha x^2$ then $\frac{\partial^2\phi_0}{\partial x^2} = 2\alpha \Rightarrow \alpha = \frac{a_0}{4}$

Try $\phi_n(x,y) = g_n(y)\cos\frac{n\pi x}{L} = g_n(y)\cos\gamma_n x$ (why? because satisfies b.c. $\frac{\partial^2\phi}{\partial x^2} \sim \cos\gamma_n x$)

$$\nabla^4\phi = \cos\gamma_n x\,(\gamma_n^4 g_n - 2\gamma_n^2 g_n'' + g^{iv}) = 0 \Rightarrow g_n(y) = \alpha_n e^{\gamma_n y} + \beta_n e^{-\gamma_n y} + E_n y e^{\gamma_n y} + H_n y e^{-\gamma_n y}$$

as $y \to -\infty \quad \sigma \to 0 \quad \Rightarrow \beta_n, H_n \to 0$

$$\phi = \sum_{n=1}^{\infty} \{\alpha_n + E_n y\} e^{\gamma_n y} \cos\gamma_n x$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y} = -\sum_1^{\infty} \{-\gamma_n \sin\gamma_n x\}\{(\alpha_n+E_n y)\gamma_n e^{\gamma_n y} + E_n e^{\gamma_n y}\}$$

$$\sigma_{xy}\Big|_{y=0} = \sum_1^{\infty} \gamma_n \sin\gamma_n x \{\alpha_n\gamma_n + E_n\} = 0 \qquad \forall x \Rightarrow \boxed{E_n = -\alpha_n\gamma_n}$$

$$\therefore \quad \phi = \sum_{n=1}^{\infty} \{1-\gamma_n y\}\alpha_n e^{\gamma_n y}\cos\gamma_n x$$

$$\sigma_{yy} = 0 = \frac{\partial^2\phi}{\partial x^2} = \sum_1^{\infty} (-\gamma_n^2)\{1-\gamma_n y\}\alpha_n e^{\gamma_n y}\cos\gamma_n x = \sum F_n \cos\gamma_n x$$

$$\sigma_{yy}\Big|_{y=0} \Rightarrow \sum_1^{\infty} (-\gamma_n^2)\alpha_n\cos\gamma_n x = \sum F_n\cos\gamma_n x$$

take $\boxed{\alpha_n = \dfrac{-F_n}{\gamma_n^2}}$

$\phi_0 = \frac{a_0 x^2}{4}$ $\qquad \phi \sim \cos\frac{n\pi x}{L}$ $\qquad \phi\sim\sin\frac{n\pi x}{L}$

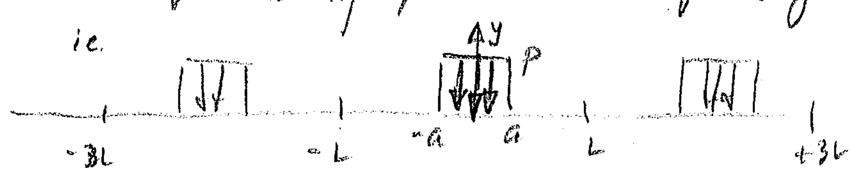
Problem 3: if $f(x) = \frac{a_0}{2} + \sum A_n\cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L} = \sigma_{yy}$
then use superposition of problem #1 & 2

if $f(x) = \frac{a_0}{2} + \sum A_n \cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L} = \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y}$

take $\phi_0 = -\frac{a_0 xy}{2}$ $\qquad \phi\sim\sin\frac{n\pi x}{L}$ $\qquad \phi\sim\cos\frac{n\pi x}{L}$ $\qquad$ since $\sigma_{xy}\sim\frac{\partial\phi}{\partial x}$ only

Consider a preliminary problem to the point fn.

i.e.



$\sigma_{yy} = -P \quad (y=0); \quad |x|<a$

$\qquad\quad 0 \qquad\qquad\quad a<|x|<L$

since this an even problem (symmetric) need only cos series

$\sigma_{yy}(y=0) = \frac{a_0}{2} + \sum_1^\infty B_n \cos\frac{n\pi x}{L} = -P$ since $T_y = \sigma_{yj} n_j = \sigma_{yy}(+1) = -P$

$\frac{a_0}{2} = \frac{1}{2L}\int_{-a}^{a}(-P)\,dx \Rightarrow \frac{2}{2L}\int_0^a -P\,dx = -\frac{Pa}{L}$

$B_n = \frac{1}{L}\int_{-a}^{a}(-P)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^a(-P)\cos\frac{n\pi x}{L}dx = -\frac{2P}{n\pi}\sin\frac{n\pi a}{L}$

$\therefore \quad \therefore \quad -P = -\left\{\frac{Pa}{L} + \sum_1^\infty \frac{2P}{n\pi}\sin\frac{n\pi a}{L}\cos\frac{n\pi x}{L}\right\}$

$\sigma_{yy}(y=0) = -\left\{\frac{2Pa}{2L} + \sum_1^\infty \frac{2Pa}{L}\frac{\sin\frac{n\pi a}{L}}{\frac{n\pi a}{L}}\cos\frac{n\pi x}{L}\right\}$

let $2Pa \to 1$ let $P \to \infty$ $a \to 0$

$\therefore \sigma_{yy} = -\left\{\frac{1}{2L} + \sum_{n=1}^\infty \frac{1}{L}\cos\frac{n\pi x}{L}\right\} = -\delta(x - 2mL)$
$m = 0, 1, 2, \ldots$

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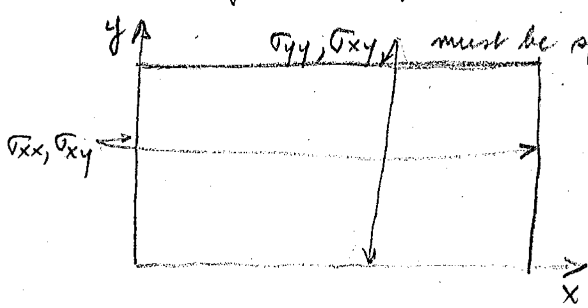
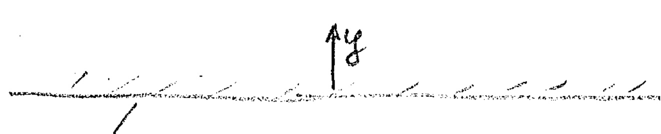
Next order of complexity in problems



P 61 & 62 in Timoshenko
Solution requires eigenfunction expansions
solutions will be of the form
$\sin\beta_n x$ where β_n are complex nos.

Aside:
why not superposition? It can be done. No reason why not.

Fourier Analysis – Fourier Integrals.
Problem



$\sigma_{xy} = g(x)$ on $y = 0$ where $g(x)$ is not periodic
$\sigma_{yy} = f(x)$ on $y = 0$

complex imbedding of cosine/sine fn. — basic solutions ∋ $|\sigma_{ij}| \to 0$ as $y \to \infty$

try $\hat{\phi}(x,y) = e^{-i\lambda x}\{A e^{-\lambda y} + B y e^{-\lambda y}\}$ w/ $\lambda > 0$ $\quad y > 0$

then

$\nabla^4 \phi = 0 \qquad \sigma_{ij}$ are bounded as $y \to +\infty$

since this is true for neg values of λ then to remove the restriction of $\lambda > 0$ take $|\lambda|$

$$\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\{A(\lambda) e^{-|\lambda| y} + B(\lambda) y e^{-|\lambda| y}\} d\lambda$$

$$\sigma_{yy}\Big|_{y=0} = \frac{\partial^2 \phi}{\partial x^2} = -\int_{-\infty}^{\infty} e^{-i\lambda x}\{A e^{-|\lambda| y} + B e^{-|\lambda| y} \cdot y\} d\lambda \Big|_{y=0} \cdot \lambda^2$$

$$= -\int_{-\infty}^{\infty} e^{-i\lambda x} \lambda^2 \cdot A(\lambda)\, d\lambda = f(x)$$

$$[\sigma_{xy}]\Big|_{y=0} = -\frac{\partial^2 \phi}{\partial x \partial y}\Big|_{y=0} = \int_{-\infty}^{\infty} -i\lambda e^{-i\lambda x}\{-A|\lambda| e^{-|\lambda| y} + B e^{-|\lambda| y} - B|\lambda| y e^{-|\lambda| y}\} d\lambda \Big|_{y=0}$$

$$= \int_{-\infty}^{\infty} \{-i\lambda\} e^{-i\lambda x}\{-|\lambda| A + B\}\Big) d\lambda = g(x)$$

Before continuing let's look at Fourier Transforms

1. Fourier Sine Transforms

consider $f(x)$ defined on $(-L, L)$ $\quad f(x) \overset{odd}{=} -f(-x)$ w/ period of $2L$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad ; \quad b_n = \frac{2}{L}\int_0^L f(s) \sin \frac{n\pi s}{L} ds$$

look at what happens as $L \to \infty$ since $\sin 0 = 0$ $\quad \sum_{n=0}^{\infty} = \sum_{n=1}^{\infty}$

$$f(x) = \sum_{n=0}^{\infty} b_n \left(\sin \frac{n\pi x}{L}\right)\Delta n \qquad (n+1) - n = 1 = \Delta n$$

Let $\xi_n = \frac{n\pi}{L}$; $\quad \Delta \xi_n = \frac{\pi \Delta n}{L}$ $\qquad$ (axis ξ with points ξ_n, ξ_{n+1}, ξ_{n+2})

then we can write that

$$f(x) = \frac{L}{\pi} \sum_{n=0}^{\infty} b_n (\sin \xi_n x)\, \Delta \xi_n \quad w/$$

$$b_n = \frac{2}{L}\int_0^L f(s) \sin \xi_n s \, ds$$

. Put this back into $f(x)$

$$f(x) = \frac{2}{L}\cdot\frac{L}{\pi}\sum_{n=0}^{\infty}\left\{\int_0^L f(s)\sin\xi_n s\,ds\right\}\sin\xi_n x\,\Delta\xi_n$$

as $L\to\infty$ $\quad\xi_n\to\xi$

$$\sum_{n=0}^{\infty}\Delta\xi_n(\quad)\longrightarrow\int_0^{\infty}(\quad)\,d\xi$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}d\xi\left(\int_0^{\infty}ds\, f(s)\sin\xi s\right)\sin\xi x$$

Let $F(\xi) = \int_0^{\infty}ds\, f(s)\sin\xi s$ then

$f(x) = \frac{2}{\pi}\int_0^{\infty}d\xi\, F(\xi)\sin\xi x$ — fourier sine transform pairs

$F(\xi)$ is the fourier sine transform of $f(x)$

2. Fourier cosine transform

Let $f(x) = f(-x)$ even fn. w/ period $2L$ on $(-L, L)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}\ ;\quad a_n = \frac{2}{L}\int_0^L f(s)\cos\frac{n\pi s}{L}\,ds$$

we now write this as a $\sum_{-\infty}^{\infty}$ since fn is even.

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty}a_n'\cos\frac{n\pi x}{L}\ ;\quad a_0' = a_0\ ;\quad a_n' = a_n\ ,\quad n\geq 1$$

with $a_{-n}' = a_n' = a_n$

let $\xi_n = \frac{n\pi}{L}$ $\quad\Delta n = 1$ $\quad\therefore$ putting a_n back into the sum

(sum $\sum_{-\infty}^{\infty}$; a_n; redef ξ_n)

$$f(x) = \frac{1}{2}\cdot\frac{2}{L}\cdot\frac{L}{\pi}\sum_{n=-\infty}^{\infty}\left\{\int_0^L f(s)\cos\xi_n s\,ds\right\}\cos\xi_n x\,\Delta\xi_n = \frac{1}{\pi}\sum_{-\infty}^{\infty}\left\{\int_0^L f(s)\cos\xi_n s\,ds\right\}\cos\xi_n x\,\Delta\xi_n$$

as $L\to\infty$ $\quad\xi_n\to\xi$ $\quad\sum_{-\infty}^{\infty}\longrightarrow\int_{-\infty}^{\infty}$

$$\therefore\ f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\cos\xi x\int_0^{\infty}ds\cos\xi s\, f(s) = \frac{2}{\pi}\int_0^{\infty}d\xi\cos\xi x\int_0^{\infty}ds\cos\xi s\, f(s)$$

since these are symmetric

$\therefore$ if $F(\xi) = \int_0^\infty ds \cos\xi s\, f(s)$

$$f(x) = \frac{2}{\pi}\int_0^\infty d\xi \cos\xi x\, F(\xi)$$ for even fn.

everything works so long as ① $\int_0^\infty |f(s)|\,ds \leq M$, ② if x_0 is a pt of discontinuity then $f(x_0) = \frac{1}{2}f(x_0^-) + \frac{1}{2}f(x_0^+)$, ③ continuous on intervals & ④ $f'(x^-), f'(x^+)$ exist

3. General Fourier Transform

Let $f(x)$ be defined on $(-\infty,\infty)$; $f(x)$ neither even nor odd - then

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = E(x) + O(x)$$

$$E(-x) = E(x) \qquad O(-x) = -O(-x)$$

Call $R(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} f(x)\,dx = \int_{-\infty}^{\infty} (\cos\xi x + i\sin\xi x)(E(x)+O(x))\,dx$

$$\int_{-\infty}^{\infty} \cos\xi x\, E(x)\,dx + i\int_{-\infty}^{\infty} \sin\xi x\, O(x)\,dx + i\int_{-\infty}^{\infty} \begin{pmatrix}(\sin\xi x)(E(x))\\(O(x))(\cos\xi x)\end{pmatrix}$$

since even product $\int_{-\infty}^{\infty} = 2\int_0^\infty$; odd · odd = even f.; odd · even = odd; = even $\Big|_{-\infty}^{\infty} = 0$

thus:

$$R(\xi) = 2\left[U(\xi) + i\,V(\xi)\right]$$

with

$$U(\xi) = \int_0^\infty \cos\xi x\, E(x)\,dx \quad \& \quad V(\xi) = \int_0^\infty \sin\xi x\, O(x)\,dx$$

we can define

$$E(x) = \frac{2}{\pi}\int_0^\infty \cos\xi x\, U(\xi)\,d\xi \quad \& \quad O(x) = \frac{2}{\pi}\int_0^\infty \sin\xi x\, V(\xi)\,d\xi$$

Next time we will look at the proof of $\int_{-\infty}^{\infty} e^{-i\xi x} R(\xi)\,d\xi = 2\pi f(x)$

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Recap:

$$R(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} f(x)\,dx \quad ; \quad f(x) = E(x) + O(x) \qquad E(x) = E(-x), \quad O(x) = -O(-x)$$

$$= 2\left\{ \int_0^{\infty} \cos\xi x\, E(x)\,dx + i\int_0 \sin\xi x\, O(x)\,dx \right\}$$

$$R(\xi) = 2\left[U(\xi) + i\,V(\xi)\right] \quad \text{with } U(\xi) = \int_0^{\infty}\cos\xi x\, E(x)\,dx, \quad V(\xi) = \int_0^{\infty} \sin\xi x\, O(x)\,dx$$

$U(\xi) = U(-\xi)$ since $\cos\xi x$ is even in ξ; $V(\xi) = -V(-\xi)$ since $\sin\xi x$ is odd in ξ

$$E(x) = \frac{2}{\pi}\int_0^{\infty} \cos\xi x\, U(\xi)\,d\xi \quad ; \quad O(x) = \frac{2}{\pi}\int_0^{\infty} \sin\xi x\, V(\xi)\,d\xi$$

Consider:

$$\int_{-\infty}^{\infty} e^{-i\xi x} R(\xi)\,d\xi = 2\int_{-\infty}^{\infty} [\cos\xi x - i\sin\xi x][U(\xi) + i\,V(\xi)]\,d\xi$$

$$= 2\int_{-\infty}^{\infty} \underbrace{\cos\xi x\, U(\xi)}_{\text{even}\times\text{even}=\text{even}}\,d\xi + 2\int_{-\infty}^{\infty} \underbrace{\sin\xi x\, V(\xi)}_{\text{odd}\times\text{odd}=\text{even}}\,d\xi + i\underbrace{\int_{-\infty}^{\infty} \left(V(\xi)\cdot\cos\xi x - \sin\xi x\, U(\xi)\right)}_{\text{odd}\times\text{even}=\text{odd}\;\to\;0}$$

$$= 4\int_0^{\infty}\left[\cos\xi x\, U(\xi)\,d\xi + \sin\xi x\, V(\xi)\,d\xi\right]$$

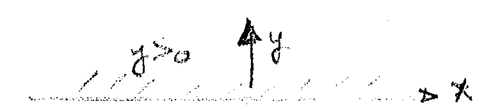
$$= 4\left\{\frac{\pi}{2}E(x) + \frac{\pi}{2}O(x)\right\} = 4\cdot\frac{\pi}{2}\left\{f(x)\right\} = 2\pi f(x)$$

$\therefore$ define $R(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} f(x)\,dx$ w/ $f(x) = E(x) + O(x)$

then $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} R(\xi)\,d\xi$$

$R(\xi)$ = Fourier Transform of $f(x)$; $f(x)$ is fourier transform of $R(\xi)$

Return to Half Space problem of 1/17/79



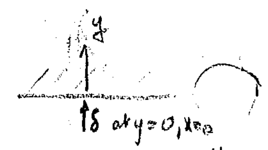
$\sigma_{yy}(y=0) = f(x)$

$\sigma_{xy}(y=0) = g(x)$

$$\phi(x,y) = \int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda x}\left[A(\lambda)\,e^{-|\lambda| y} + y\,B(\lambda)\,e^{-|\lambda| y}\right]$$

Require $\quad f(x) = -\int_{-\infty}^{\infty} \lambda^2 e^{-i\lambda x} A(\lambda)\,d\lambda \; ; \quad g(x) = -i\int_{-\infty}^{\infty} \lambda e^{-i\lambda x}\left\{-|\lambda|\,A(\lambda) + B(\lambda)\right\} d\lambda$

Look at special case where $\sigma_{xy}=0$ and σ_{yy} is the now famous Dirac Delta



take $g(x)=0$ and $\sigma_{yy}(y=0) = f(x) = -\delta(x)$ since $T_y = \sigma_{yj} n_j$ $n_y=-1$ thus

since $\delta(x) = T_y = -\sigma_{yy} \Rightarrow \sigma_{yy} = -\delta(x)$

(note to myself: that T_y the traction in + y direction is $\lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}$ hence direction of traction in this case is in same direction as ΔF_y).

Aside: Delta fns are defined by

(1) $\delta(x-x_0) = 0$ for $x \neq x_0$ $\delta(x-x_0)=\infty$ for $x=x_0$

(2) $\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = 1$

(3) $\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$

$$\therefore \int_{-\infty}^{\infty} \sigma_{yy}(y=0)\,dx = \int_{-\infty}^{\infty} -\delta(x)\,dx = -1 = -\int_{-\infty}^{\infty} T_y\,dx$$

since $f(x) = -\delta(x) = -\int_{-\infty}^{\infty} \lambda^2 e^{-i\lambda x} A\,d\lambda$ we have to represent $\delta(x) = \int_{-\infty}^{\infty} e^{-i\lambda x} R(\lambda)\,d\lambda$

thus using the fourier transforms $R(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\,\delta(x)\,dx = \frac{1}{2\pi} e^{i\lambda \cdot 0} = \frac{1}{2\pi}$

thus $R(\lambda) = \frac{1}{2\pi}$ if $f(x)=\delta(x)$ thus $\delta(x) = \int_{-\infty}^{\infty} e^{-i\lambda x} R(\lambda)\,d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,d\lambda$

and $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,d\lambda$ is the Fourier Integral Representation of the δ Function

$$-\delta(x) = f(x) = -\int_{-\infty}^{\infty} \lambda^2 e^{-i\lambda x} A(\lambda)\,d\lambda = \frac{-1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,d\lambda \Rightarrow \lambda^2 A = \frac{1}{2\pi}$$ and from $g(x)=0$

$-|\lambda| A + B = 0$ $\therefore$ $A(\lambda) = \frac{1}{2\pi\lambda^2}$ $B(\lambda) = |\lambda| A(\lambda) = \frac{1}{2\pi|\lambda|}$. Substituting into $\sigma_{yy}(x,y)$

$$\therefore \sigma_{yy}(x,y) = \frac{\partial^2\phi}{\partial x^2} = -\int_{-\infty}^{\infty} d\lambda\, \lambda^2 e^{-i\lambda x}\left[\frac{1}{2\pi\lambda^2} e^{-|\lambda| y} + y\frac{1}{2\pi|\lambda|} |\lambda| e^{-|\lambda| y}\right]$$

$$\sigma_{yy}(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda x}\, e^{-|\lambda| y}\,\{1+|\lambda| y\}$$. Since only result exists if $\int_{-\infty}^{\infty}$ even fn

($e^{-i\lambda x}$: cos + i sin; $e^{-|\lambda|y}$: even fn of λ; $\{1+|\lambda|y\}$: even as a fn of λ)

only non zero term is

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, \cos\lambda x\, e^{-|\lambda| y}\{1+|\lambda| y\} = 2\cdot\frac{-1}{2\pi}\int_{0}^{\infty} \cos\lambda x\, e^{-\lambda y}\{1+\lambda y\}$$

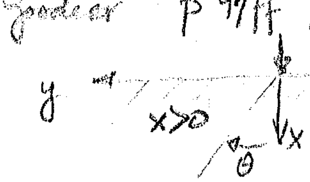
now (1): $\int_0^{\infty} d\lambda \cos\lambda x\, e^{-\lambda y} = \frac{y}{x^2+y^2}$ this is Laplace transf of $\cos\lambda x$ with $y>0$

(2): $y\int_0^\infty d\lambda \cos\lambda x\, e^{-\lambda y}\cdot\lambda$ since integration is wrt λ can take y outside integral. Notice

that $y\frac{\partial}{\partial y}(1) = -(2) = y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$ $\therefore$ $(2) = -y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$

$$\sigma_{yy} = -\frac{1}{\pi}\left\{\frac{y}{x^2+y^2} - y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\} = -\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2}$$

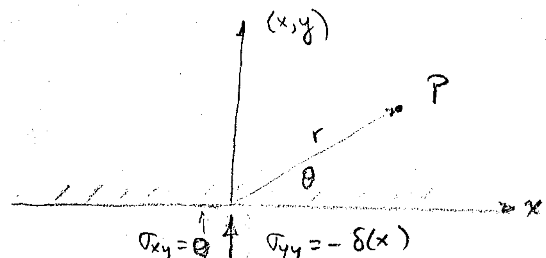
Look @ Timoshenko & Goodier p 97ff — this system looks like this



results are same as ours except w/change in x, y and mult by P
checked out it is correct.

HW find σ_{xy}, σ_{xx} and ϕ for this problem.

1/22/79



$$\sigma_{yy}(x,y) = -\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2} = -\frac{2}{\pi}\frac{\sin^3\theta}{r}$$

For $\sigma_{yy}(x,y=0) = -f(x)$ $\;= f(x)\,[-\delta(x)]$ $\;\sigma_{xy}(x,y=0) = 0$ we can get the answer based on our delta fn result. We know:

1. for a point force applied at at point $x=\xi$ on $y=0$. Then by shift of origin
$$\sigma_{yy}(x,y;\xi) = -\frac{2}{\pi}\frac{y^3}{[(x-\xi)^2+y^2]^2}$$

2. then by the principle of linear superposition with a distributed load $f(x)$
$$\sigma_{yy} = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y^3 f(\xi)\,d\xi}{[(x-\xi)^2+y^2]^2}$$

if $f(\xi) = \text{const} = p_0$ $\quad |x|\le l$

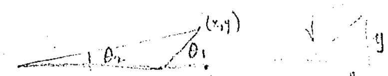
$$= -\frac{2}{\pi}p_0\int_{-l}^{l}\frac{y^3\,d\xi}{((x-\xi)^2+y^2)^2}$$

$\frac{y^3}{[(x-\xi)^2+y^2]^2}$ is the Green's fn for a half space

$$\sigma_{yy} = \frac{2}{\pi}p_0\int_{x+l}^{x-l}\frac{-y^3\,du}{(u^2+y^2)^2}$$

$$= \frac{2}{\pi}p_0 y^3\left[\frac{u}{2y^2(u^2+y^2)} + \frac{1}{2y^3}\tan^{-1}\frac{u}{y}\right]_{x+l}^{x-l}$$

$$= \frac{2}{\pi}p_0\left\{\frac{y(x-l)}{2((x-l)^2+y^2)} + \frac{1}{2}\tan^{-1}\frac{(x-l)}{y} - \frac{y(x+l)}{2((x+l)^2+y^2)} - \frac{1}{2}\tan^{-1}\frac{(x+l)}{y}\right\}$$



$\sin\theta_1\cos\theta_1$

$$\sigma_{yy} = \frac{2}{\pi}p_0\left[\left\{\frac{\sin 2\theta_1}{2} + \theta_1\right\} - \left\{\frac{\sin 2\theta_2}{2} + \theta_2\right\}\right]$$

for the delta fn. on boundary we had shown that $\delta(x-2nL)$ $\forall n$

$$-\delta(x-2nL) = \frac{1}{2L} + \frac{1}{L}\sum_{n=1}^{\infty} \cos\frac{n\pi x}{L}$$ point loads at $\pm 2nL$; $n=0,1,2,\dots$

for the case of loading $\sigma_{yy}(y=0) = -\delta(x-2nL)$ on the halfspace then for all space

$$\sigma_{yy} = -\frac{1}{L}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(1 - \frac{n\pi y}{L}\right)e^{+\frac{n\pi y}{L}}\cos\frac{n\pi x}{L}\right\} \quad y \le 0$$

for $y \geq 0$

$$\sigma_{yy} = -\frac{1}{L}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(1+\frac{n\pi y}{L}\right)e^{-\frac{n\pi y}{L}}\cos\frac{n\pi x}{L}\right\} \quad y \geq 0 \qquad (*)$$

using the green's fn approach if $f(\xi) = \sum_{n=-\infty}^{\infty}\delta(\xi - 2nL)$ we should get same result as above.

$$\sigma_{yy} = -\frac{2}{\pi}\int_{-\infty}^{\infty} y^3 \frac{\sum_{-\infty}^{\infty}\delta(\xi-2nL)}{[(x-\xi)^2+y^2]^2}d\xi = -\frac{2y^3}{\pi}\sum_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta(\xi-2nL)\,d\xi}{[(x-\xi)^2+y^2]^2}$$

$$= -\frac{2y^3}{\pi}\sum_{-\infty}^{\infty}\frac{1}{[(x-2nL)^2+y^2]^2}$$ if we can interchange the Σ & $\int$

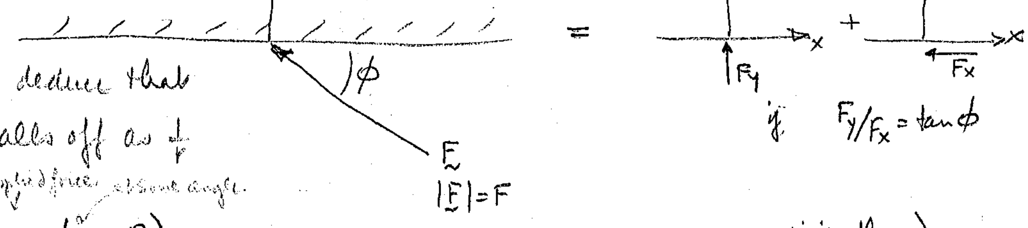
this series converges faster than (*)

$$-\frac{2}{\pi}y^3\sum_{n=-\infty}^{\infty}\frac{1}{[(x-2nL)^2+y^2]^2} = -\frac{1}{L}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(1+\frac{n\pi y}{L}\right)e^{-\frac{n\pi y}{L}}\cos\frac{n\pi x}{L}\right\}$$

This is the poisson's summation formula

Problem.

We will now deduce that the stress falls off as $\frac{1}{r}$



Normally: $\sigma_{ij} = \sigma_{ij}(F, \phi, r, \theta)$

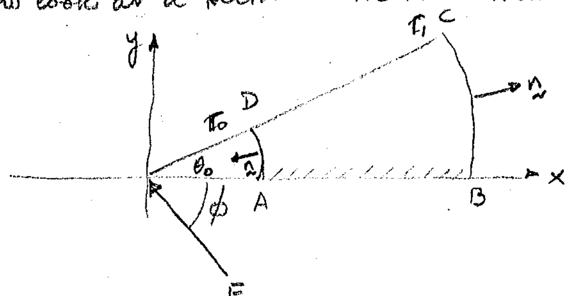
if we double the load then the stresses should double (linear elasticity theory), $\therefore \sigma_{ij} = F\sigma_{ij}(\phi, r, \theta)$

$$\sigma_{ij_{\text{units}}} = \left(\frac{\text{Force}}{\text{length}^2}\right) = \left(F\sigma_{ij}(\phi,r,\theta)\right)_{\text{units}} = \frac{\text{force}}{\text{length}}\cdot\left(\sigma_{ij}(\phi,r,\theta)\right)_{\text{units}} \quad \therefore \left(\sigma_{ij}(\phi,r,\theta)\right)_{\text{units}} = \frac{1}{\text{len}}$$

F has units of $\frac{\text{force}}{\text{length}}$

$$\Rightarrow \sigma_{ij} = \frac{F}{r} f_{ij}(\theta,\phi)$$

Now look at a sector : We know that Σ Forces $= 0$ for body to be in equilibrium



AB: $T_i = 0$ since only force is F @ origin

BC: $T_{i_{net}} = \int_{\theta=0}^{\theta_0} \sigma_{ij} n_j ds = \int_{\theta=0}^{\theta_0} \frac{F}{r_1} f_{ij}(\phi,\theta) n_j(\theta) r_1 d\theta = F \int_0^{\theta_0} f_{ij}(\phi,\theta) n_j \, d\theta$ fn only of θ

AD: $n_j = -n_{j \text{ on } bc}$ $\therefore$ $T_{i_{net AD}} = -T_{i_{net BC}}$ $\quad T_{i_{net AD}} = \int \frac{F}{r_0} f_{ij}(\phi,\theta)[-n_j(\theta)] r_0 \, d\theta$

DC: $T_{i_{net}} = \int_{r_0}^{r_1} \sigma_{ij} n_j ds = \int_{r_0}^{r_1} \frac{F}{r} f_{ij}(\phi,\theta_0) \, n_j(\theta_0) dr = F f_{ij}(\phi,\theta_0) n_j(\theta_0) \int_{r_0}^{r_1} \frac{dr}{r}$

$= F \ln(r_1/r_0) \, f_{ij}(\phi,\theta_0) \, n_j(\theta_0) \equiv 0$ since the $\Sigma F = 0$ on AB, BC, AD

$\Rightarrow f_{ij}(\theta_0,\phi) \, n_j(\theta_0) = 0 \quad \forall \, \theta_0$ (since θ_0) is arbitrary

Therefore since ds was arbitrary $\Rightarrow$ T_i <u>on radial lines</u> <u>are zero.</u>

$\sigma_{ij} n_j = 0$ $\forall$ radial planes coming out from the origin and $T_i = 0$ $\forall$ points on all radial plane.

in r, θ, z coordinates : on radial plane, $n_j(\theta) \Rightarrow n_\theta = 1, \; n_r = n_z = 0 \Rightarrow \sigma_{i\theta} = 0$ on all radial planes. $\Rightarrow \sigma_{r\theta}, \sigma_{\theta\theta}, \sigma_{z\theta} = 0$ on all radial planes

thus in r, θ, z problems only $\sigma_{rr}, \sigma_{zz} = \nu \sigma_{rr}$ (in plane strain) exist

σ_{rr} exists (in plane stress)

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380F : Prof. Barnett → Applied Math Seminar today 4:15 PM

Timoshenko 2-D problems in polar coordinates pg 132-135

① $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_{\theta\theta} + \sigma_{rr} + \sigma_{zz}$ (1st invariant) true no matter what coord.

② Plane strain solution $\quad \nabla^2(\sigma_{xx} + \sigma_{yy}) = 0 \quad$ but $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = \nu(\sigma_{rr} + \sigma_{\theta\theta})$

$\Rightarrow \nabla^2(\sigma_{rr} + \sigma_{\theta\theta}) = 0$ look at pg 65-68

Thus we can define a stress fn. $\phi(r,\theta)$ $\ni$.

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \qquad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \qquad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

thus $\nabla^2(\nabla^2\phi)=0$ where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

we will Study: $\phi(r,\theta)$ if $\nabla^2\phi=0 \Rightarrow \nabla^4\phi=0$

(1) if f is harmonic the xf, yf are biharmonic

$$\nabla^2(xf) = f(\nabla^2 x) + x(\nabla^2 f) + 2(\nabla f \cdot \nabla x)$$
$$= 0 + 0 + 2(\nabla f \cdot \underset{\sim}{i}) = 2\frac{\partial f}{\partial x}$$

$$\nabla^4(xf) = 2\nabla^2\left(\frac{\partial f}{\partial x}\right) = 2\frac{\partial}{\partial x}(\nabla^2 f) = 0$$

(2) if f is harmonic $r^2 f = (x^2+y^2)f$ is biharmonic

$$\nabla^2(x^2 f) = f\,\nabla^2(x^2) + x^2(\nabla^2 f) + 2\,\nabla f \cdot \nabla(x^2)$$
$$f\cdot 2 + 0 \quad 2(\nabla f \cdot 2x\underset{\sim}{i})$$
$$= 2f + 4x\frac{\partial f}{\partial x}$$

$$\nabla^2(\nabla^2 x^2 f) = 2\nabla^2 f + 4\nabla^2\left(x\frac{\partial f}{\partial x}\right)$$
$$= 0 \quad 4\frac{\partial f}{\partial x}(\nabla^2 x) + 4\,x\left(\nabla^2\frac{\partial f}{\partial x}\right) + 8\,\nabla\left(\frac{\partial f}{\partial x}\right)\cdot\nabla x$$
$$\qquad\qquad = 0 \qquad\qquad = 0$$

$$8\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}f\right) = 8\frac{\partial^2 f}{\partial x^2}$$

$\therefore \quad \nabla^4(x^2 f) = 8\frac{\partial^2 f}{\partial x^2}$ similarly $\nabla^4(y^2 f) = 8\frac{\partial^2 f}{\partial y^2}$

$$\nabla^4(r^2 f) = 8\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right] = \nabla^4[(x^2+y^2)f] = 8\,\nabla^2 f = 0$$

Let us look at $r^n\cos n\theta$

$$\nabla^2(r^n\cos n\theta) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right](r^n\cos n\theta)$$
$$= \left[n(n-1)r^{n-2} + n\,r^{n-2} - n^2 r^{n-2}\right]\cos n\theta = \left[n^2 - n + n - n^2\right]r^{n-2}\cos n\theta$$
$$= 0$$

thus $\nabla^2(r^n\cos n\theta)=0$ similarly $\nabla^2(r^n\sin n\theta)=0$
also $\nabla^2(r^{-n}\cos n\theta)=0$ & $\nabla^2(r^{-n}\sin n\theta)=0$ } since $\nabla^2\phi_1=0$ then $\nabla^2(\nabla^2\phi_1)=\nabla^2(0)=0$

thus if $\phi_i = r^{n+2}\cos n\theta,\ r^{n+2}\sin n\theta,\ r^{-n+2}\cos n\theta,\ r^{-n+2}\sin n\theta$
then $\nabla^4\phi=0$ assume $n\geq 2$

$$\phi^{(1)} = \sum (A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2})\cos n\theta + (E_n r^n + F_n r^{-n} + G_n r^{n+2} + H_n r^{-n+2})\sin n\theta$$

θ is harmonic $\therefore \Rightarrow x\theta, y\theta, r^2\theta$ are biharmonic

$$\phi^{(1)} = a_0'\theta + a_1' r\theta\cos\theta + a_2' r\theta\sin\theta + a_3' r^2\theta$$

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look for $\phi = \phi(r)$ only in axi-symmetric case. $\exists \psi \ni \nabla^4\psi = 0$. for use to get ϕ
look for
Harmonic fn $\nabla^2\phi = 0$; in axi sym case $\nabla^2\phi = \phi_{,rr} + \frac{1}{r}\phi_{,r} \Rightarrow (r\phi')' =$

$\therefore \quad r\phi' = k \qquad \phi = k\ln r + k_0 \qquad$ but $r^2\phi$ is biharmonic

$r^2\phi = \phi^{(2)} \qquad \phi = \phi^{(1)}$

$\therefore \quad \psi = k_3 r^2 \ln r + k_2 r^2 + k_1 \ln r + k_0 = \phi^{(3)} \qquad \nabla^4\psi = \nabla^4(r^2\phi^{(1)}) + \nabla^2(\nabla^2\phi^{(1)}) = 0$

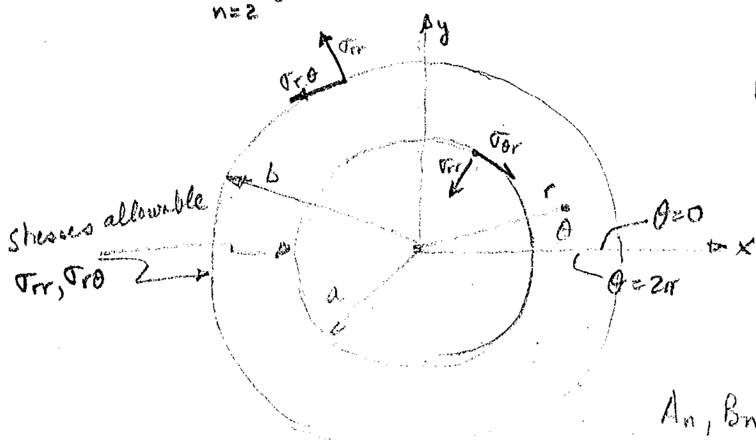
(k_0) ← we can drop this, since when we take derivs this makes no difference

also $x\phi$, $y\phi$ are biharmonic $\therefore \phi^{(4)} = \alpha_0 r\ln r\cos\theta + \beta_0 r\ln r\sin\theta + k_0 x + k y$

($k_0 x + k y$) ← we can drop this also as before

Our total stress fn is therefore: T&G p 133 (eqn 80)

$$\phi(r,\theta) = a_0\ln r + b_0 r^2 + c_0 r^2\ln r + d_0 r^2\theta + a_0'\theta + a_1 r\theta\sin\theta + (b_1 r^3 + a_1' r^{-1} + b_1' r\ln r)\cos\theta$$
$$+ c_1 r\theta\cos\theta + (d_1 r^3 + c_1' r^{-1} + d_1' \ln r)\sin\theta + \sum_{n=2}^{\infty}(a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2})\cos n\theta + \sum_{n=2}^{\infty}(c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2})\sin n\theta$$



For a complete ring θ is multivalued
$\therefore$ take $d_0 = 0$

B.C. $(\sigma_{rr})_{r=a} = A_0 + \sum_{n=1}^{\infty}(A_n\cos n\theta + B_n\sin n\theta)$

$(\sigma_{rr})_{r=b} = A_0' + \sum(A_n'\cos n\theta + B_n'\sin n\theta)$
$\forall \; 0 \le \theta \le 2\pi$

A_n, B_n, A_n', B_n' are known

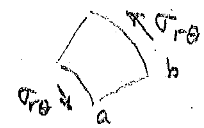
$$(\sigma_{r\theta})_{r=a} = C_0 + \sum_{n=1}^{\infty}(C_n \cos n\theta + D_n \sin n\theta)$$

C_n, D_n, C_n', D_n' are known

$$(\sigma_{r\theta})_{r=b} = C_0' + \sum_{n=1}^{\infty}(C_n' \cos n\theta + D_n' \sin n\theta)$$

To check the well posedness of the problem: now look at moment & force equilib

Moment equil about origin σ_{rr} produces no contrib



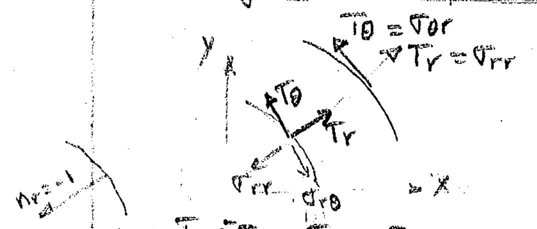
$$\therefore \sum M = 0 \text{ (about origin)} \Rightarrow \int_0^{2\pi}\{b\,\sigma_{\theta r}\}_{r=b}\cdot b\,d\theta - \int_0^{2\pi}\{a\,\sigma_{\theta r}\}_{r=a}\cdot a\,d\theta = 0 \qquad (1)$$

but $\int_0^{2\pi}\cos n\theta\,d\theta\,,\ \int_0^{2\pi}\sin n\theta\,d\theta = 0 \quad n>0$

$\therefore$ the eq (1) reduces to $\int(b^2C_0' - a^2C_0)\,d\theta = 0 \quad \therefore\ 2\pi(b^2C_0' - a^2C_0) = 0$

$\boxed{b^2C_0' \equiv a^2C_0}$ for moment equil

For a plane strain problem: tractions on the boundary



$T_r = \sigma_{rj}n_j \ \therefore\ T_r = \sigma_{rr} \quad T_\theta = -\sigma_{\theta r}$

$T_r = \sigma_{rr}n_r = -\sigma_{rr}$

$T_\theta = \sigma_{\theta j}n_j = \sigma_{\theta r}n_r = -\sigma_{\theta r}$

$$T_x = T_r\cos\theta - T_\theta\sin\theta$$

$$T_y = T_r\sin\theta + T_\theta\cos\theta$$

$$F_x^{net} = \int_0^{2\pi}\Big[\sigma_{rr}\cos\theta - \sigma_{\theta r}\sin\theta\Big]_{r=b} b\,d\theta + \int_0^{2\pi}\Big[\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\Big]_{r=a} a\,d\theta$$

($T_x|_{r=b}$, $T_x|_{r=a}$)

for equilib $F_x^{net} \equiv 0$

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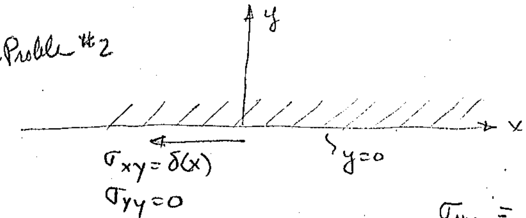
Homework Problem #2



$$\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\{A(\lambda)e^{-|\lambda|y} + B(\lambda)\,y\,e^{-|\lambda|y}\}\,d\lambda$$

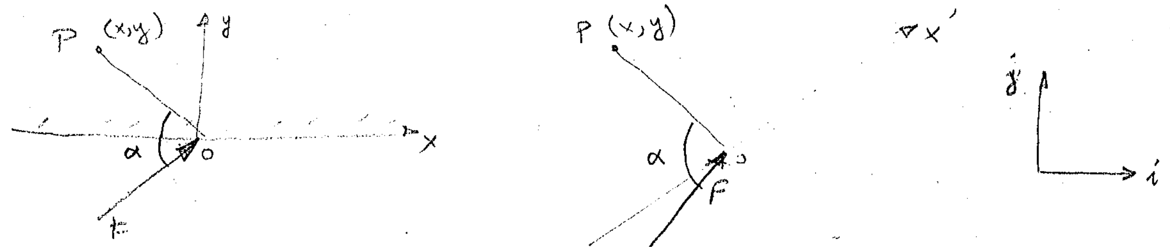
$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}\Big|_{y=0} = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x}A(\lambda)\,d\lambda = 0 \Rightarrow A(\lambda) = 0$$

$$\phi(x,y) = y\int_{-\infty}^{\infty} e^{-i\lambda x}\,B\,e^{-|\lambda|y}\,d\lambda$$

$$\sigma_{xy} = -\phi_{,xy}\Big|_{y=0} = \int_{-\infty}^{\infty} i\lambda\,e^{-i\lambda x}B(\lambda)\big[e^{-|\lambda|y} - |\lambda|y\,e^{-|\lambda|y}\big]\,d\lambda = \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,d\lambda$$

$B(\lambda) = \frac{1}{2\pi i \lambda}$ $\qquad \phi(x,y) = \frac{y}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\lambda x - |\lambda| y}}{\lambda}\, d\lambda$

$$\sigma_{xx} = \phi_{,yy} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda\, \frac{e^{-i\lambda x}}{\lambda} \frac{\partial^2}{\partial y^2}\left[y e^{-i|\lambda| y}\right].$$



$\sigma_{ij}(x,y)$ will be the same no matter what the rotation of the semi infinite medium if i,j are defined wrt a fixed set of coordinates

Now at

$$\phi(x,y) = \frac{y}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\lambda x - |\lambda| y}}{\lambda}\, d\lambda = \frac{2y}{2\pi i}\, i \int_0^{\infty} -\frac{\sin\lambda x}{\lambda} e^{-|\lambda| y} d\lambda \qquad y>0$$

Let

$$J = \int_0^{\infty} \frac{\sin\lambda x}{\lambda} e^{-\lambda y} d\lambda \qquad \frac{\partial J}{\partial x} = \int_0^{\infty} \cos\lambda x\, e^{-\lambda y} d\lambda = \frac{y}{x^2+y^2}$$

$$J = \int \frac{y\, dx}{x^2+y^2} = -\tan^{-1}\frac{y}{x} + g(y)$$

now @ $x=0$ $\quad J=0$ since $\displaystyle\int_0^{\infty} \frac{\sin\lambda x}{\lambda} e^{-\lambda y} d\lambda = x\int_0^{\infty} \left(\frac{\sin\lambda x}{\lambda x}\right) e^{-\lambda y} d\lambda \to 0$ as $x \to 0$ (with $\frac{\sin\lambda x}{\lambda x} \to 1$)

$$\therefore\ J(0) = 0 = -\tan^{-1}\frac{y}{0^+} + g(y) = -\pi/2 + g(y) \quad \forall y \quad \therefore\ g(y) = \pi/2$$

$$\therefore\ J = \pi/2 - \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{x}{y}$$

$$\therefore\ \phi = -\frac{y}{\pi}\tan^{-1}\frac{x}{y} = -\frac{y}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{y}{x}\right) = \left(-\frac{y}{2}\right) + \frac{y}{\pi}\tan^{-1}\frac{y}{x}$$

(circled term: throw out since it gives no stresses.)

Now return to problem where $\sigma_{yy}\big|_{y=0} = \delta(x)$ $\quad \sigma_{xy} = 0 \quad A = \frac{1}{2\pi\lambda^2} \quad B = \frac{1}{2\pi|\lambda|}$

and $\displaystyle \phi(x,y) = \frac{1}{\pi}\int_0^{\infty} d\lambda\, e^{-\lambda y}\left\{\frac{\cos\lambda x}{\lambda^2} + y\frac{\cos\lambda x}{\lambda}\right\}$ this fn $\to \infty$ at $\lambda = 0$

define $\phi^{(1)} = \int_0^\infty e^{-\lambda y} \frac{\cos\lambda x}{\lambda}\, d\lambda = -\int_0^\infty \left(\frac{1-\cos\lambda x}{\lambda}\right) e^{-\lambda y} + \int_0^\infty d\lambda\, e^{-\lambda y}\frac{1}{\lambda}$

let $\xi = \lambda y$

$\int_0^\infty e^{-\xi}\xi^{-1} d\xi \to \infty$

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1. Midterm on the 12/14th will be take home.

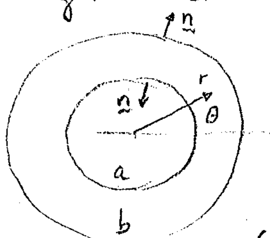
$Q = \int_0^\infty \frac{1-\cos\lambda x}{\lambda} e^{-\lambda y} d\lambda \qquad \frac{\partial Q}{\partial x} = \int_0^\infty \sin\lambda x\, e^{-\lambda y} d\lambda = \frac{x}{x^2+y^2}$

$\therefore\; Q = \frac{1}{2}\ln(x^2+y^2) + g(y)$ as $y \to \infty$ $Q \to 0$ since $e^{-\lambda y} \to 0$

$Q\big|_{y\to\infty,\ x \text{ fixed}} \to \frac{1}{2}\ln(y^2) + g(y) = 0 \quad \therefore\; g(y) = -\frac{1}{2}\ln y^2$

$\therefore\; Q = \frac{1}{2}\ln\left(\frac{x^2+y^2}{y^2}\right)$

Annular Ring - Fourier Loading



we had found the stress fn for the annular ring to be

$\Phi = a_0\ln r + b_0 r^2 + c_0 r^2\ln r + d_0 r^2\theta + a_0'\theta + a_1 r\theta\sin\theta$
$+ (b_1 r^3 + a_1' r^{-1} + b_1' r\ln r)\cos\theta + c_1 r\theta\cos\theta + (d_1 r^3 + c_1' r^{-1} +$
$d_1' r\ln r)\sin\theta + \sum_{n=2}^\infty (a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2})\cos n\theta$
$+ \sum_{n=2}^\infty (c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2})\sin n\theta$

$(\sigma_{rr})_{r=a} = A_0 + \sum_{n=1}^\infty (A_n\cos n\theta + B_n\sin n\theta) \qquad (\sigma_{rr})_{r=b}$ same with primes

$(\sigma_{r\theta})_{r=a} = C_0 + \sum_{n=1}^\infty (C_n\cos n\theta + D_n\sin n\theta) \qquad (\sigma_{r\theta})_{r=a}$ " " "

From moment equil we found $C_0 a^2 = C_0' b^2$

For force equil we found $T_x = T_r\cos\theta - T_\theta\sin\theta$

$T_y = T_r\sin\theta + T_\theta\cos\theta$

on $r=b$ $T_r = \sigma_{rr}$, $T_\theta = \sigma_{r\theta}$ $r=a$: $T_r = -\sigma_{rr}$, $T_\theta = -\sigma_{r\theta}$

$$\frac{F_x^{net}}{\text{unit length}} = \int_0^{2\pi} (\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)\Big|_{r=b} b\,d\theta - \int_0^{2\pi} (\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)\Big|_{r=a} a\,d\theta = 0$$

only contrib for σ_{rr} are $\cos\theta$ terms ; for $\sigma_{r\theta}$ are $\sin\theta$ terms

$$\Rightarrow \quad \underline{b(A_1' - D_1') = a(A_1 - D_1)}$$

for F_y^{net} a similar relation only terms involving $\sin\theta$ for σ_{rr} and $\cos\theta$ for $\sigma_{r\theta}$ give contribution $\Rightarrow$ $\underline{b(B_1' + C_1') = a(B_1 + C_1)}$

Computing σ_{rr} from ϕ ie $\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$

(for complete ring)

$$\sigma_{rr} = \frac{a_0}{r^2} + 2b_0 + c_0(1+2\ln r) + \cancel{2d_0\theta} + \frac{2a_1}{r}\cos\theta + \left(2b_1 r - \frac{2a_1'}{r^3} + \frac{b_1'}{r}\right)\cos\theta - \frac{2c_1}{r}\sin\theta$$
$$+ \left(2d_1 r - \frac{2c_1'}{r^3} + \frac{d_1'}{r}\right)\sin\theta + \sum_{n=2}^{\infty}\left[a_n n(1-n) r^{n-2} + b_n(2+n-n^2) r^n + a_n'(n^2+n) r^{-n-2}\right.$$
$$\left. + b_n'(2-n-n^2) r^{-n}\right]\cos n\theta + \sum_{n=2}^{\infty}\left[c_n \downarrow \; + d_n \downarrow \; - c_n' \downarrow \right.$$
$$\left. + d_n' \downarrow \right]\sin\theta$$

Since $\sigma_{rr}\Big|_{r=a} = A_0 + \sum (A_n \cos n\theta + B_n \sin n\theta)$ then equate term by term.

independent of θ terms

$$\therefore \; A_0 = \frac{a_0}{a^2} + 2b_0 + c_0(1+2\ln a) \;\; \& \;\; A_0' = \frac{a_0}{b^2} + 2b_0 + c_0(1+2\ln b) \qquad \text{2 eqn, 3 unkn.}$$

Look at coeff of $\cos\theta$, $\sin\theta$

$$A_1 = \frac{2a_1}{a} + 2b_1 a - \frac{2a_1'}{a^3} + \frac{b_1'}{a} \; ; \; A_1' = \frac{2a_1}{b} + 2b_1 b - \frac{2a_1'}{b^3} + \frac{b_1'}{b} \qquad \text{2 eqs, 4 unks}$$

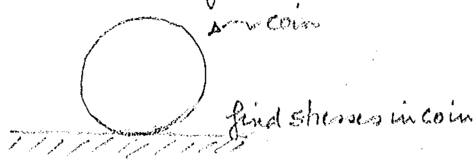
$$B_1 = -\frac{2c_1}{a} + 2d_1 a - \frac{2c_1'}{a^3} + \frac{d_1'}{a} \quad B_1' = -\frac{2c_1}{b} + 2d_1 b - \frac{2c_1'}{b^3} + \frac{d_1'}{b} \qquad \text{2 eqs, 4 unk.}$$

now doing $\sigma_{r\theta}$ we get relations for C_1, C_1' and D_1, D_1'

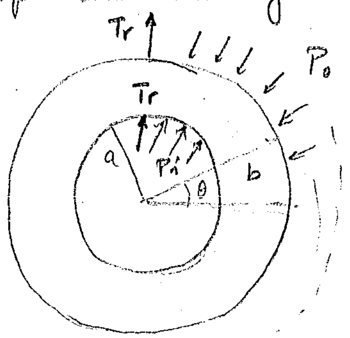
Using these we will find that there is a redundancy and $F_x^{net} = F_y^{net} = 0$ is identically satisfied; so no new info is obtained we must look at the compatibilit

2/2/79

One problem of the midterm



Example: continuation of the derivation is in the handout for rings



Axisymmetric deformation:
P_o, P_i goes all way around by symmetry no θ dependence

@ $r=b$ $\sigma_{rr} = -P_o$, $\sigma_{r\theta} = 0$ $T_r = \sigma_{rj} n_j$ since $n_j = n_r$ & $T_r = -P_o$

@ $r=a$ $\sigma_{rr} = -P_i$ $\sigma_{r\theta} = 0$ since $n_j = -n_r$ & $T_r = p_i$

Fourier loading $A_0 = -P_i$, $A_0' = -P_o$ all other fourier coeffs are $=0$

$$\frac{a_0}{a^2} + 2b_0 + c_0(1+2\ln a) = A_0$$

$$\frac{a_0}{b^2} + 2b_0 + c_0(1+2\ln b) = A_0'$$

Timo & Goodier gives that $c_0 \equiv 0$ for single valued displ (Pg 77-78)

$\phi = a_0 \ln r + b_0 r^2$ (only part of ϕ not a fn of θ) and give single valued displ & stresses

$$a_0 = \frac{(P_o - P_i)a^2b^2}{b^2-a^2}$$

$$b_0 = \frac{1}{2}\,\frac{P_i a^2 - P_o b^2}{b^2-a^2}$$

→ really only had to worry about σ_{rr} since $\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial\theta}\right)$ but $\frac{\partial}{\partial\theta} = 0$

∴ $\sigma_{r\theta} = 0$ on $r=a,b$ is identically satisfied

∴ LAMÉS Solution

$\sigma_{r\theta} = 0$; $$\sigma_{rr} = \frac{(P_o-P_i)a^2b^2}{b^2-a^2}\frac{1}{r^2} + \frac{P_i a^2 - P_o b^2}{b^2-a^2}$$

$$\sigma_{\theta\theta} = \frac{-(P_o-P_i)a^2b^2}{b^2-a^2}\frac{1}{r^2} + \frac{P_i a^2 - P_o b^2}{b^2-a^2}$$

$$\sigma_{rr} + \sigma_{\theta\theta} = 2\,\frac{P_i a^2 - P_o b^2}{b^2 - a^2} = \frac{1}{\nu}\sigma_{zz} \quad \text{for plane strain } (= \text{const})$$

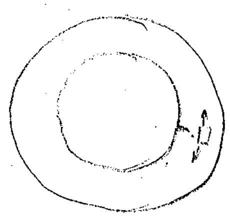
in the homework 3rd problem I ask what if I wanted plane stress soln? How could I superpose on this soln another solution ∋ $\sigma_{zz} = 0$

Problem #2

IF $P_o = 0$ then

$$\sigma_{rr} = -\frac{P_i a^2 b^2}{b^2 - a^2}\frac{1}{r^2} + \frac{P_i a^2}{b^2 - a^2} = \frac{P_i a^2}{b^2 - a^2}\left\{1 - \frac{b^2}{r^2}\right\} \le 0 \quad \text{compressi}$$

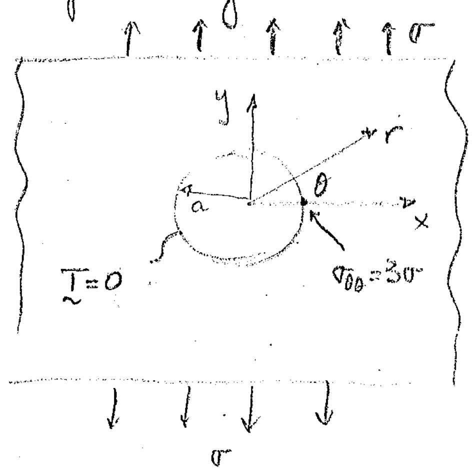
$$\sigma_{\theta\theta} = +\frac{P_i a^2 b^2}{b^2 - a^2}\frac{1}{r^2} + \frac{P_i a^2}{b^2 - a^2} = \frac{P_i a^2}{b^2 - a^2}\left\{1 + \frac{b^2}{r^2}\right\} > 0 \quad \text{tensi}$$



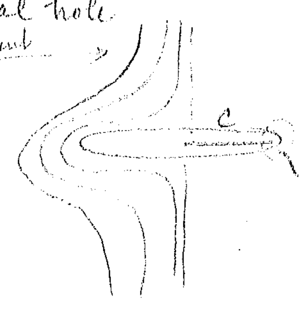
if cracks develop in radial dir } $\sigma_{\theta\theta}$ being tensile cause cracks to propagate

Problem #3

Look at an infinite body with a circular hole in a thick plate

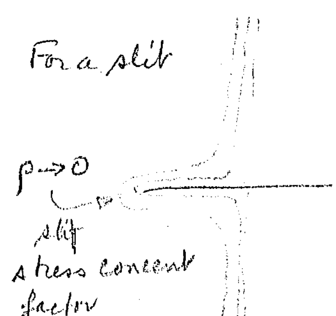


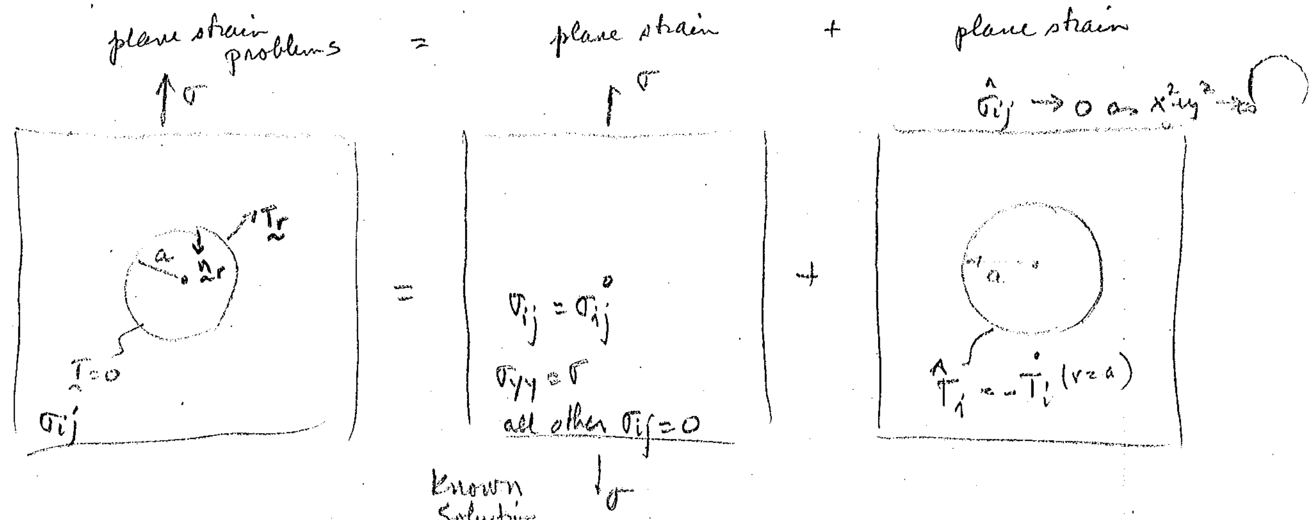
For an elliptical hole



Stress conc: $\sqrt{c/\rho}$

For a slit





Stress concentration on a Circular Hole.

$$\sigma_{ij} = \sigma^0_{ij} + \hat{\sigma}_{ij}$$

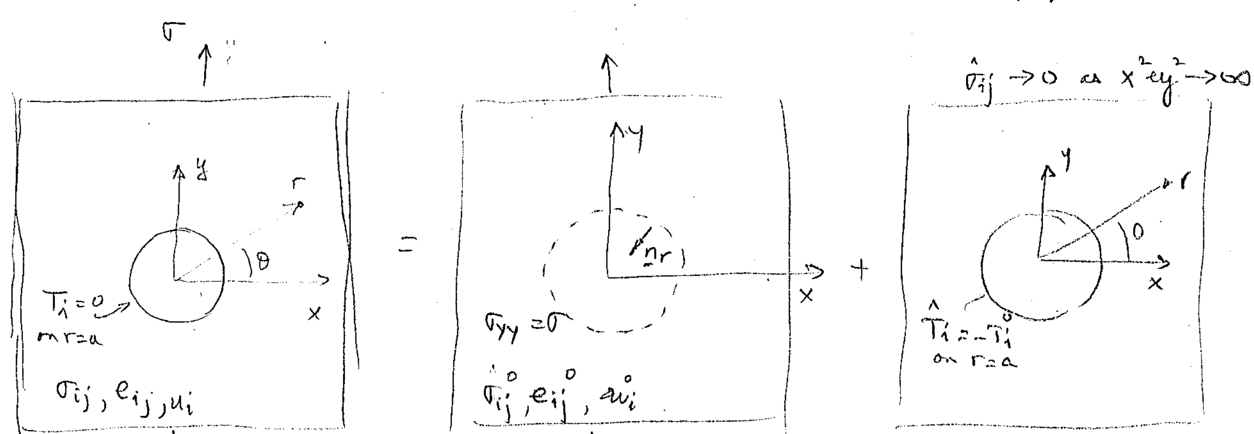
BC: $\hat{\sigma}_{ij} \to 0$ as $x^2+y^2 \to \infty$

$$T_1 = T_1^0 + \hat{T}_1 = 0 \implies \hat{T}_i = -T_i^0 \text{ on } r=a$$

$$\hat{T}_r = \hat{\sigma}_{rr} \overset{(-1)}{n_r} = -\hat{\sigma}_{rr} = -T_r^0 = -\sigma^0_{rr} n_r = \sigma^0_{rr} \qquad \therefore \quad \hat{\sigma}_{rr} = -\sigma^0_{rr} \text{ on } r=a$$

$$\hat{T}_\theta : \qquad \hat{\sigma}_{r\theta} = -\sigma^0_{r\theta} \text{ on } r=a$$

Stress concent on a Circular Hole 2/5/79



$$\sigma_{ij} = \sigma^0_{ij} + \hat{\sigma}_{ij}$$

°FIELDS $\Rightarrow$ $\sigma^0_{yy} = \sigma$ $\quad \sigma^0_{ij} = 0$ all others. from transformation

on $r=a$
$$\begin{cases} T^0_r = \sigma_{rr} n_r = -\sigma^0_{rr} = -\sigma^0_{yy} \sin^2\theta = -\frac{\sigma}{2}(1-\cos 2\theta) \\ T^0_\theta = \sigma_{r\theta} n_r = -\sigma^0_{\theta r} = -\sigma^0_{yy} \sin\theta\cos\theta = -\frac{\sigma}{2}\sin 2\theta \end{cases}$$

$$\sigma^0_{\theta\theta} = \tfrac{\sigma}{2}(1+\cos 2\theta) = \sigma/2 \cos^2\theta$$

$$\therefore \hat{T}_r = \hat{\sigma}_{rr} \overset{(-1)}{n_r} = -T^0_r = \frac{\sigma}{2}(1-\cos 2\theta) \qquad \therefore \quad \hat{\sigma}_{rr} = -\frac{\sigma}{2}(1-\cos 2\theta)$$

$$\hat{T}_\theta = \hat{\sigma}_{r\theta} n_r = -T^0_\theta = +\frac{\sigma}{2}\sin 2\theta \qquad \therefore \quad \hat{\sigma}_{r\theta} = -\sigma/2(\sin 2\theta)$$

on $r=a$

Now look at the annular ring where at $r=b$ $\sigma_{ij}(r=b)=0$ & let $b \to \infty$

Go back to handout to apply bc.

$\therefore \quad A_0 = -\frac{\sigma}{2} \qquad A_2 = \frac{\sigma}{2} \qquad D_2 = -\frac{\sigma}{2}$ all others are 0.

(from $\sigma_{rr}|_{r=a}$) (from $\sigma_{r\theta}|_{r=a}$)

for the stress fn. when $\sigma_{rr}\Big|_{r=a} = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\Big|_{r=a}$

pg 35 handout then $\frac{a_0}{a^2} + 2b_0 = A_0 = -\frac{\sigma}{2}$ ~~$+c_0(1+\ln a)$~~ must be dropped for single valued displacement.

$\frac{a_0}{b^2} + 2b_0 = -A_0' = 0$ as $b \to \infty$ $\Rightarrow b_0 = 0 \Rightarrow \boxed{a_0 = -\frac{\sigma}{2}a^2}$

(remember A_0, A_0' are from applied bc at $r=a$, $r=b \to \infty$)

now for A_2, D_2 and A_2' and $D_2' = 0$ since no stresses applied at ∞.

for $n=2$ since $\sin 2\theta$, $\cos 2\theta$ terms in A_2, D_2

from pg 36 handout

$\sigma_{rr}|_{r=a} \quad a_2 \cdot 2(1-2)a^{2-2} + b_2(2+2-4)a^2 - a_2'\,2(1+2)a^{-2-2} + b_2'(2-2-4)a^{-2} = A_2 = \frac{\sigma}{2}$

$\sigma_{rr}|_{r=b} \quad a_2 \cdot 2(1-2)b^{2-2} + b_2(2+2-4)b^2 - a_2'\,2(1+2)b^{-4} + b_2'(2-2-4)b^{-2} = A_2' = 0$

as $b \to \infty$ b_2' term $\to 0$, a_2' term $\to 0$, b_2 term $\to 0$ since $(2+2-4)=0$

$\Rightarrow \boxed{a_2 = 0} \Rightarrow$ 1st eq must reduce to $\boxed{-a_2'(6a^{-4}) + b_2'(-4)a^{-2} = \frac{\sigma}{2}}$ *

pg 37 handout

$\sigma_{r\theta}\big|_{r=a}: \quad 2\left\{a_2 \cancel{k}^{\,0} + b_2(2+1)a^2 - a_2'(2+1)a^{-4} - b_2'(2-1)a^{-2}\right\} = D_2 = -\frac{\sigma}{2}$

$\sigma_{r\theta}\big|_{r=b}: \quad 2\left\{a_2 \cancel{k}^{\,0} + b_2(3)b^2 + a_2'(2+1)b^{-4} - b_2'(2-1)b^{-2}\right\} = D_2' = 0$

$\therefore \boxed{b_2 = 0}$ (→ 0 as $b \to \infty$) (→ 0 as $b \to \infty$)

$\therefore \quad \boxed{-a_2'(3a^{-4}) - b_2'(1)a^{-2} = -\frac{\sigma}{4}}$ **

$\therefore$ * $\Rightarrow \quad \frac{3a_2'}{a^4} + 2\frac{b_2'}{a^2} = -\frac{\sigma}{4}$

** $\Rightarrow \quad \frac{3a_2'}{a^4} + \frac{b_2'}{a^2} = \frac{\sigma}{4}$

$\Rightarrow \boxed{b_2' = -\frac{\sigma}{2}a^2} \quad \boxed{a_2' = \frac{\sigma a^4}{4}}$

$$\Rightarrow \hat{\sigma}_{rr} = \frac{\sigma}{2}\left[-\frac{a^2}{r^2} + \left\{4\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right\}\cos 2\theta\right]$$

$$\sigma_{rr} = \hat{\sigma}_{rr} + \overset{\circ}{\sigma}_{rr} = \frac{\sigma}{2}(1-\cos 2\theta) + \hat{\sigma}_{rr} = \frac{\sigma}{2}\left[1-\frac{a^2}{r^2}\right] - \frac{\sigma}{2}\left[1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right]\cos 2\theta$$

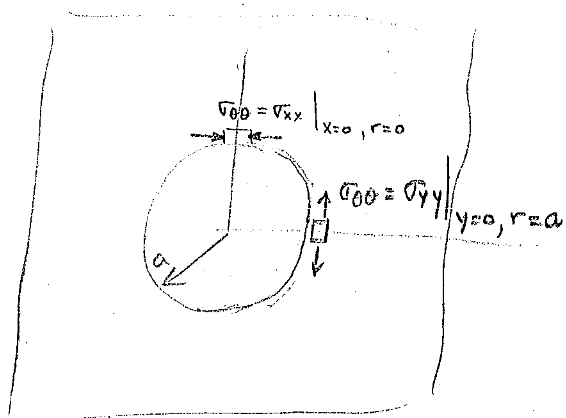
T&G pg 91 $\quad \theta_{here} = \frac{\pi}{2} - \theta_{T\&G}$

Thus $\quad \sigma_{\theta\theta} = \overset{\circ}{\sigma}_{\theta\theta} + \hat{\sigma}_{\theta\theta} = \frac{\sigma}{2}\left[1+\frac{a^2}{r^2}\right] + \frac{\sigma}{2}\left[1+\frac{3a^4}{r^4}\right]\cos 2\theta$

$$\sigma_{r\theta} = \frac{\sigma}{2}\left[1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right]\sin 2\theta$$

as $r \to \infty$
$$\left.\begin{aligned} \sigma_{rr} &\to \frac{\sigma}{2}\left[1-\cos 2\theta\right] \\ \sigma_{\theta\theta} &\to \frac{\sigma}{2}\left[1+\cos 2\theta\right] \\ \sigma_{r\theta} &\to \frac{\sigma}{2}\left[\sin 2\theta\right] \end{aligned}\right\} \overset{\circ}{\sigma}_{yy} = \sigma$$

Stress Concentration

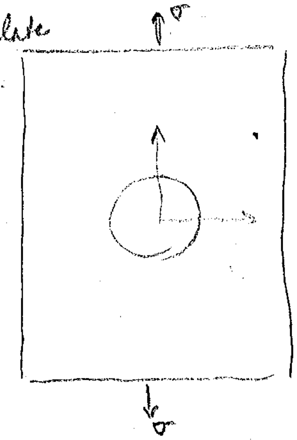


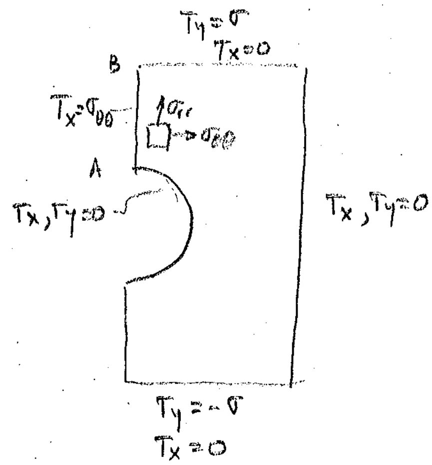
$$\sigma_{\theta\theta}\Big|_{\substack{\theta=0\\ r=a}} = \frac{\sigma}{2}(2) + \frac{\sigma}{2}(4) = 3\sigma$$

$$\sigma_{\theta\theta}\Big|_{\substack{\theta=\pi/2\\ r=a}} = \frac{\sigma}{2}(2) - \frac{\sigma}{2}(4) = -\sigma$$

2/7/79

cut a plate w/ hole



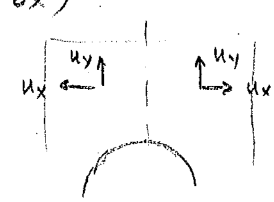


since there are no $_\wedge$ horiz stresses on bdy ⇒ no forces in x direction

$\therefore \int_A^B T_x\, ds = 0 \;\Rightarrow\; \sigma_{\theta\theta}$ is not of the same sign on $\theta = \pm \frac{\pi}{2}$

to prove $\sigma_{xy} = \mu\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial x}\right)$ is odd

$$\sigma_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$



u_y is even fn in x

u_x is odd fn in x

$\frac{\partial u_y}{\partial x}$ = is odd fn in x

$\frac{\partial u_x}{\partial y}$ is odd fn in x

$\therefore$ σ_{xy} is an odd fn in x

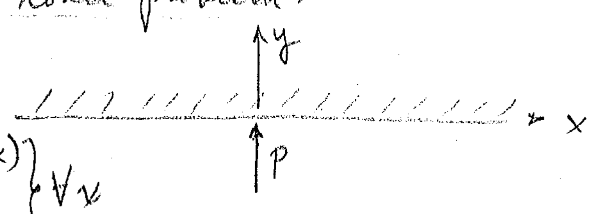
$$\text{now } \sigma_{\theta\theta}\Big|_{\theta=\pi/2} = \frac{\sigma}{2}\left(1 + \frac{a^2}{r^2} - 1 + 3\frac{a^4}{r^4}\right) = \frac{\sigma}{2}\left(\frac{a^2}{r^2}\right)\left(1 - 3\frac{a^2}{r^2}\right)$$

fn depends on sign of this term ∴ $\frac{1}{\sqrt{3}} = \frac{a}{r}$ is pt when sign changes.

pg 83-88 in Timoshenko & Goodier: Bending of Bar w/ curvature

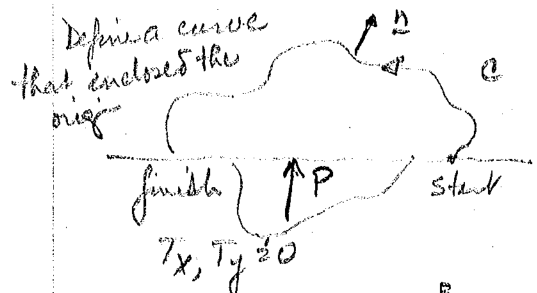
<u>Look at it!</u>

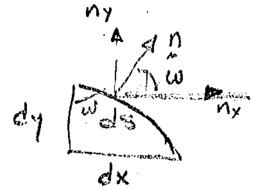
<u>Return to point load problem.</u>



y > 0 B.C.

$$\left.\begin{aligned} \sigma_{yy}\big|_{y=0} &= -P\delta(x) \\ \sigma_{xy}\big)_{y=0} &= 0 \end{aligned}\right\} \forall x$$





$$\frac{dy}{ds} = \cos\omega = n_x$$

$$\frac{dx}{ds} = -\sin\omega = -n_y$$

$$\oint_{start}^{finish} T_y\,ds = \oint_S^F (\sigma_{yx} n_x + \sigma_{yy} n_y)\,ds$$

$$= \oint_S^F \left[-\frac{\partial^2\phi}{\partial x \partial y}\frac{dy}{ds} + \frac{\partial^2\phi}{\partial x^2}\left(-\frac{dx}{ds}\right)\right] ds$$

$$= -\oint_S^F \left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\frac{dy}{dx} + \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)\frac{dy}{ds}\right] ds = -\oint_S^F \frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right) ds$$

$$= -\oint_S^F d\left(\frac{\partial\phi}{\partial x}\right) = -\left[\frac{\partial\phi}{\partial x}\Big|_F - \frac{\partial\phi}{\partial x}\Big|_S\right] = -\frac{\partial\phi}{\partial x}\Big]_S^F$$

Similarly

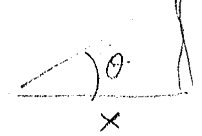
$$\oint_{start}^{fin} T_x\,ds = \frac{\partial\phi}{\partial y}\Big|_S^F = \oint_S^F (\sigma_{xx} n_x + \sigma_{xy} n_y)\,ds$$

This result is true <u>in general</u>

For our case

Look at $\phi = Ar\theta\cos\theta = Ax\theta$ for a half space this satisfies Equil and since this is not a circular region we can use it

In an infinite medium we could not use it since θ is multivalued



$\theta = \arctan\frac{y}{x}$

$\frac{d\theta}{dx} = \frac{-y}{x^2+y^2}$

$$\sigma_{\theta\theta} = \phi_{,rr} = 0$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0$$

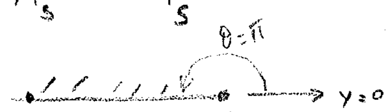
$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -\frac{2A}{r}\sin\theta$$

<u>Now</u>

$$-\frac{\partial\phi}{\partial x} = -A\left[\theta + x\frac{\partial\theta}{\partial x}\right]_S^F = -A(\pi - 0) = -A\pi \qquad \frac{\partial\theta}{\partial x}\Big|_{y=0} = 0 \quad \therefore\ x\frac{\partial\theta}{\partial x}\Big|_{y=0} = 0 \text{ at S \& F}$$

$$\frac{\partial\phi}{\partial y} = Ax\frac{\partial\theta}{\partial y} = A\frac{x^2}{x^2+y^2}\Big|_S^F = A\left(\frac{x^2}{x^2}\right)\Big|_S^F = A\Big|_S^F = 0$$

now $\int_F^S T_y\,dx = P \qquad \int_F^S T_x\,dx = 0$



or $\int_S^F T_y\,dx = -P$

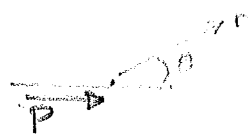
since $\int_S^F T_y\,dx = -\frac{\partial\phi}{\partial x}\Big|_S^F = -P$

want $-\frac{\partial\phi}{\partial x}\Big|_S^F = -A\pi = -P$ for equil $\quad \therefore\ A = \frac{P}{\pi}$

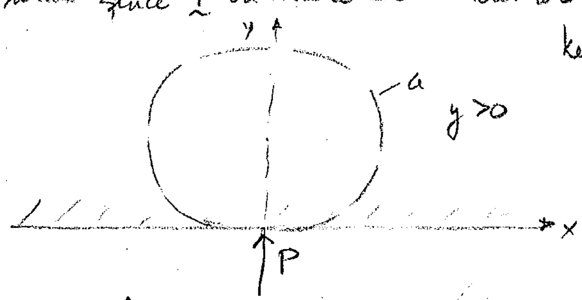
now $\oint T_y\,ds = 0 = \int_S^F T_y\,ds + \int_F^S T_y\,dx = 0$

$\therefore\ \int_S^F T_y\,ds = -\int_F^S T_y\,dx = -P = -\frac{\partial\phi}{\partial x}\Big|_S^F$ thats why

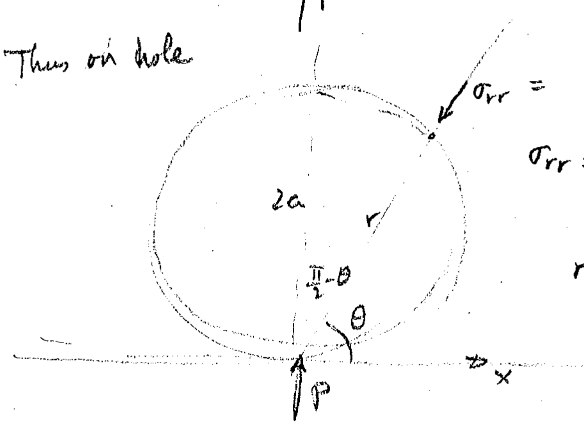
hence $\sigma_{rr} = -\frac{2A}{r}\sin\theta = -\frac{2P}{\pi r}\sin\theta \quad \& \quad \phi = \frac{P}{\pi}x\theta$

for  let $\phi = Ar\theta\sin\theta$

Look at the problem as medium. If we cut a hole from it the material would relax since T on hole = 0. But we must apply forces to bdy in order to keep body in same stress cond.



Thus on hole



$\sigma_{rr} = -\frac{2P}{\pi r}\sin\theta$

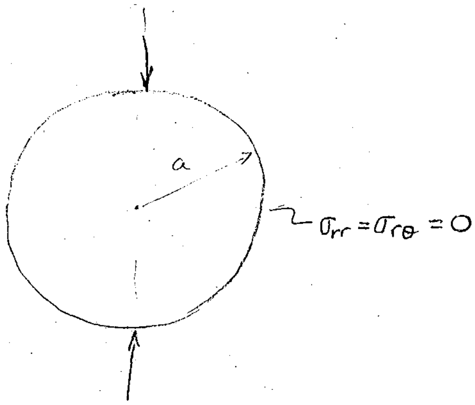
$\sigma_{rr} = -\frac{P}{\pi a}$

$r = 2a\cos\left(\frac{\pi}{2}-\theta\right) = 2a\sin\theta$

2/9/79

Exam 12-21 Feb

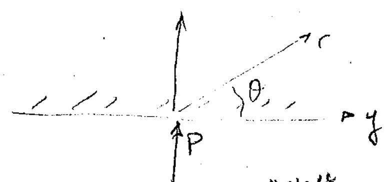
Disc loaded by equal and opposite diametrical forces



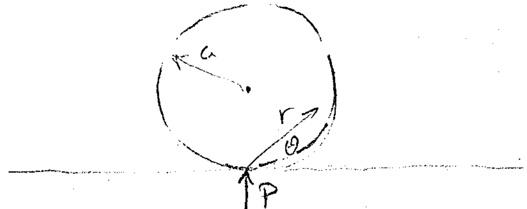
We can use half-space solution here

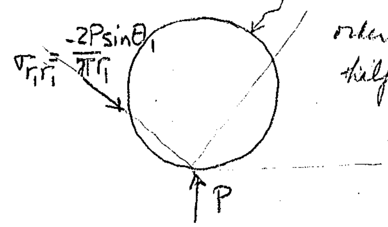
Suppose we have half space w/ point load we found

$$\sigma_{rr} = -\frac{2P}{\pi r}\sin\theta$$

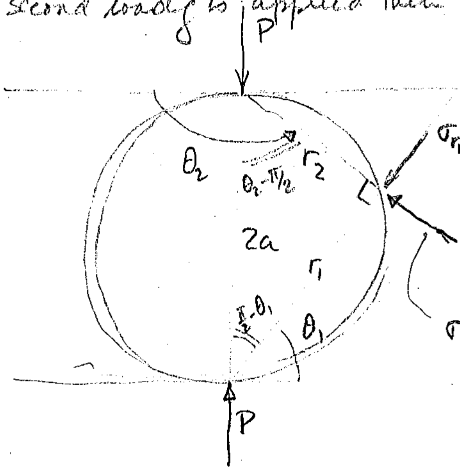


Now cut disc out of half space what BVP for disc





If a second loading is applied then



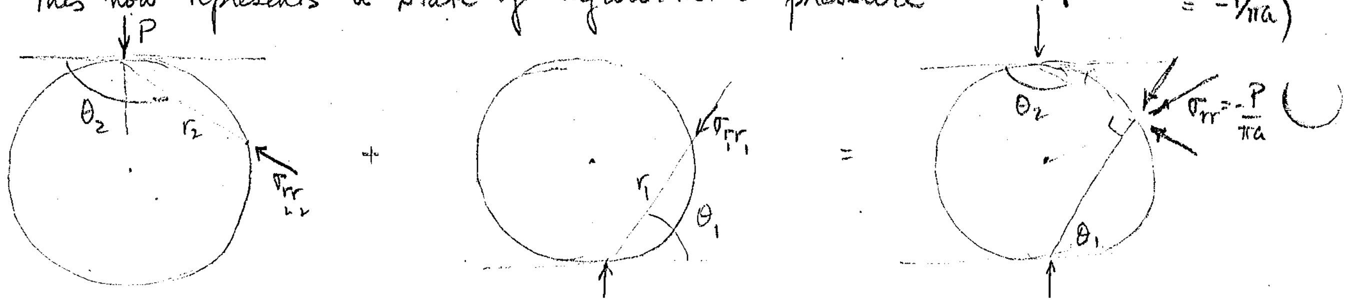
$$\sigma_{r_1r_1} = -\frac{2P}{\pi r_1}\sin\theta_1\Big|_{bdy} = -\frac{P}{\pi a}$$

since $r_1 = 2a\cos(\pi/2 - \theta_1)$ on bound $= 2a\sin\theta_1$

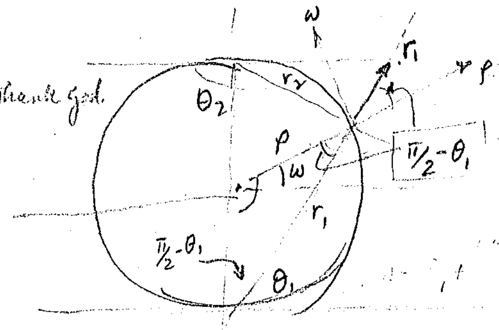
$$\sigma_{r_2r_2} = -\frac{2P}{\pi r_2}\sin\theta_2\Big|_{bdy} = -\frac{P}{\pi a}$$

but $r_2 = 2a\cos(\theta_2 - \pi/2)\Big|_{bound} = 2a\sin\theta_2$

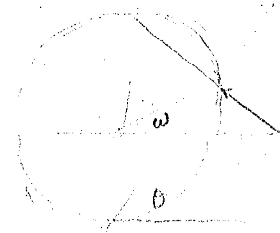
This now represents a state of hydrostatic pressure and σ_{rr} (by the 1st invar must be $= -P/\pi a$)



Doing it the long way, thank god.



In (ρ, ω) coordinate system.



$$\sigma_{\rho\rho} = \sigma_{r_1r_1} \cos^2\left(\frac{\pi}{2} - \theta_1\right) = -\frac{P}{\pi a} \sin^2\theta_1$$

$$\sigma_{\omega\omega} = \sigma_{r_1r_1} \sin^2\left(\frac{\pi}{2} - \theta_1\right) = -\frac{P}{\pi a} \cos^2\theta_1$$

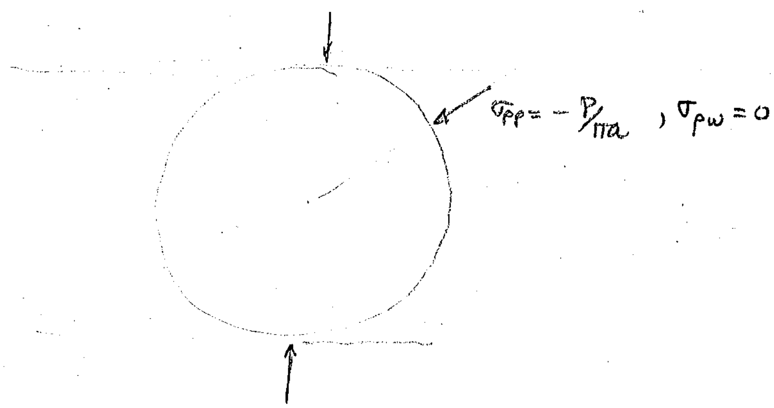
$$\sigma_{\rho\omega} = \sigma_{r_1r_1} -\sin\left(\frac{\pi}{2} - \theta_1\right)\cos\left(\frac{\pi}{2} - \theta_1\right) = -\frac{P}{\pi a} \sin\theta_1 \cos\theta_1$$

If I do this for $r_2, \theta_2 \Rightarrow \sigma_{\rho\rho} = -\frac{P}{\pi a} \cos^2\theta_1 \quad \sigma_{\omega\omega} = -\frac{P}{\pi a} \sin^2\theta_1 \quad \sigma_{\rho\omega} = \frac{P}{\pi a} \sin\theta_1 \cos\theta_1$

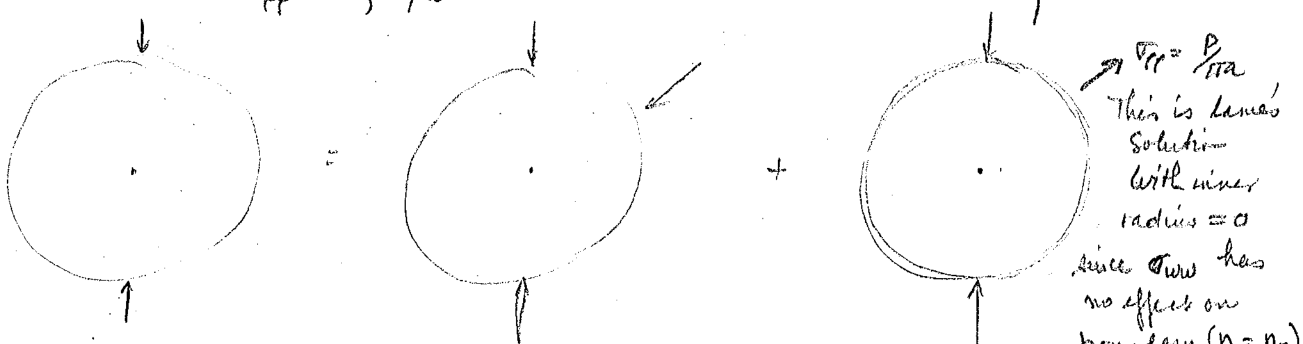
Now, we can add each of these to get

$\sigma_{\rho\rho} = -\frac{P}{\pi a}\left(\sin^2\theta_1 + \sin^2\theta_2\right)$ but $\theta_2 = \pi/2 + \theta_1 \quad \therefore \quad \sin\theta_2 = \cos\theta_1$ etc. $\cos\theta_2 = -\sin\theta_1$

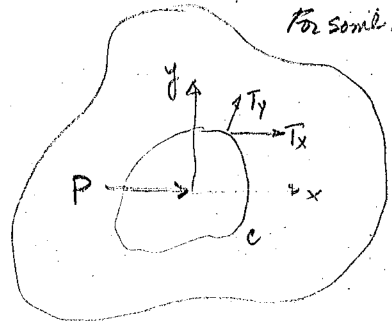
$\therefore \quad \sigma_{\rho\rho} = -\frac{P}{\pi a} \quad \sigma_{\omega\omega} = -\frac{P}{\pi a} \quad \sigma_{\rho\omega} = 0 \quad \therefore$



but we wanted $\sigma_{\rho\rho} = 0$, $\sigma_{\rho\omega} = 0 \quad \therefore$ we must add a loading $\therefore$



Green's Functions for 2-d elasticity

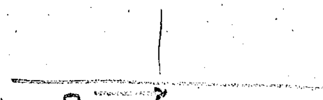


For some body: Given a load in the central portion, if we want to solve it, We might look at the half space problems.

Now from force equil

$$\oint_c T_y\,ds = 0$$

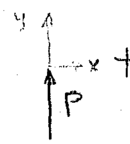
$$\oint_c T_x\,ds = -P$$

Suppose we look at half space 

This has a $\phi = a_1 r\theta\sin\theta$; this however is <u>not</u> singlevalued. To make it singlevalued we must add another fn, such as done in handout.

$\therefore$ by results of hand out $\phi = a_1 r\theta\sin\theta + b_1' r\cos\theta\ln r$

with $b_1' = -\dfrac{a_1(1-2\nu)}{2(1-\nu)}$ in plane strain

if P were  then $\phi = a_1 r\theta\cos\theta + b_1' r\sin\theta\ln r$

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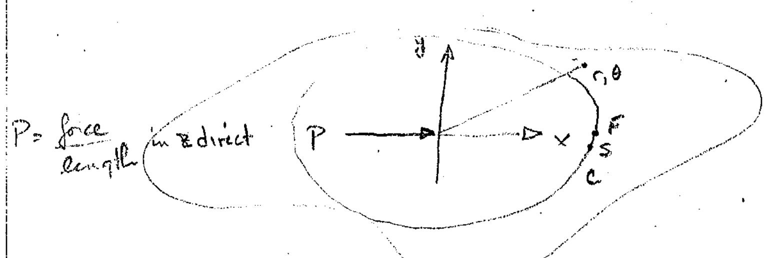
The 2-D Line force Green's fn.



$$\int_c T_y\,ds = 0$$

$$\int_c T_x\,ds = -P$$

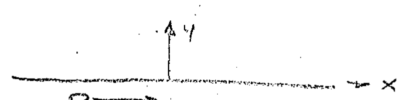
$$\Big\}\ \oint_c \underset{\sim}{T}\,ds = \underset{\sim}{F}$$

for any curve c

using $\oint_c T_y\,ds = -\left[\dfrac{\partial\phi}{\partial x}\right]_S^F = 0$

$$\oint T_x\,ds = +\left[\frac{\partial\phi}{\partial y}\right]_S^F = -P$$

$$\sigma_{ij} \sim \frac{P}{r} f_{ij}(\theta)$$

Look at $\phi = a_1 r\theta\sin\theta$ 

since we are using a full space problem ϕ is multivalued we have to add $b_1' r\cos\theta\ln r$ to get single valued displacement.

$$\phi = a_1 r\theta\sin\theta + b_1' r\cos\theta\ln r = a_1 y\theta + b_1' x\ln r$$

given force & disp mult contrib — mult disp only

For plane strain $b_1' = -a_1 \dfrac{(1-2\nu)}{2(1-\nu)}$ — thus we treat a full space as an annulus where $r_0 \to \infty$ and $r_i \to 0$.

now $\dfrac{\partial \phi}{\partial x} = a_1 y \dfrac{\partial \theta}{\partial x} + b_1' \left[\ln r + x \dfrac{\partial \ln r}{\partial x} \right]$ is cont. and single valued since $\dfrac{\partial \theta}{\partial x}$ is

$$\frac{\partial \theta}{\partial x} = \frac{\partial \tan^{-1}}{\partial x}\left(\frac{y}{x}\right) = \frac{-y/x^2}{1+(y/x)^2} = \frac{-y}{x^2+y^2} \; ; \quad \frac{\partial \theta}{\partial y} = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2}$$

$$\int T_x\, ds = \frac{\partial \phi}{\partial y} = a_1 \theta + a_1 y \frac{\partial \theta}{\partial y} + b_1' x \left[\frac{\partial \ln r}{\partial y}\right] \qquad \therefore \text{ take } \left.\frac{\partial \phi}{\partial y}\right|_S^F = \left.\frac{\partial \phi}{\partial y}\right|_{\theta=0}^{2\pi=\theta}$$

($a_1\theta$: not single valued)

$$\frac{\partial \phi}{\partial y} = 2\pi a_1 = -P \qquad \therefore a_1 = \frac{-P}{2\pi}$$

for P in the x direction $(P=1)$

$$\phi^{(x)} = -\frac{1}{2\pi}\left\{ r\theta \sin\theta - \frac{1-2\nu}{1-\nu} r\cos\theta \ln r \right\} \quad \text{plane strain}$$

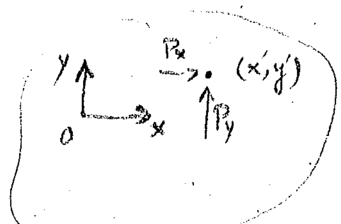
$$= -\frac{1}{2\pi}\left\{ r\theta\cos\theta - \frac{1-\nu}{2} \quad '' \quad \right\} \quad \text{plane stress}$$

for P in y direct $(P=1)$

$$\phi^{(y)} = \frac{1}{2\pi}\left\{ r\theta\cos\theta - \frac{1-2\nu}{1-\nu} r\sin\theta \ln r \right\} \quad \text{plane strain}$$

$$\frac{1}{2\pi}\left\{ r\theta\cos\theta - \frac{1-\nu}{2} r\sin\theta \ln r \right\} \quad \text{plane stress}$$

if you were interested in



thus Compute $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ in terms of x, y using $\phi^{(x)}$ & $\phi^{(y)}$
now let $x - x'$ be x & $y - y'$ be y

Can also compute $u_x^{(x)}(x, y;\, x', y')$ — The displacement in the x direction at x, y due to a unit force at x', y' applied in x direction.

(subscript of u) (superscript of u)

$u_y^{(x)}(x,y;x',y')$ the disp in y direct at (x,y) due to a unit force in x direct applied at x',y'. using $\phi^{(x)}$

we can also define $u_x^{(y)}(x;y;x',y')$ $u_y^{(y)}(x,y;x',y')$ using $\phi^{(y)}$

Define $G_{ij}(x,y;x',y')$ = displ in i^{th} direction due to a unit force applied in the j direction (x',y')

$$\begin{bmatrix} G_{11}=u_x^{(x)} & G_{12}=u_x^{(y)} \\ G_{21}=u_y^{(x)} & G_{22}=u_y^{(y)} \end{bmatrix}$$

can show that $G_{12}(x,y;x',y') = G_{21}(x',y';x,y)$
$= G_{21}(x,y;x',y')$ for an infinite medium

Basically the reciprocity theorem of Maxwell.

G_{ij} ~ the 2-D elastic Green's function.

Return to $\phi = a_1 r\theta\sin\theta + b_1' r\cos\theta\ln r$

gives net forces, mult val disp — gives no force but mult val disp.

define $\phi^{(d)} = b_1' r\cos\theta\ln r$ gives $\oint_c \hat{T}ds = 0$ but gives multivalued disp

this stress fn gives a self stress -

gives rise to Dislocation - self stress state ie mult. valued disp.

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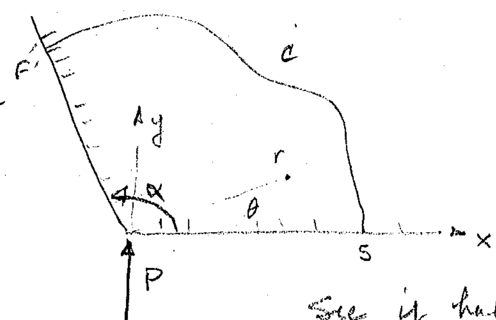
Suppose we define a wedge



$$\oint_C T_y ds = -\left[\frac{\partial\phi}{\partial x}\right]_S^F \overset{S/B}{=} -P$$

$$\oint_c T_x ds = \left[\frac{\partial\phi}{\partial y}\right]_S^F \overset{S/B}{=} 0$$

See if half space stress fn. works.

$$\therefore \quad \phi = a_1 x\theta = a_1 r\theta\cos\theta \qquad \frac{\partial\phi}{\partial x} = a_1\left[\theta + x\frac{\partial\theta}{\partial x}\right] = a_1\left[\theta - \frac{xy}{x^2+y^2}\right]$$

$$= a_1\left[\theta - \frac{\sin 2\theta}{2}\right]$$

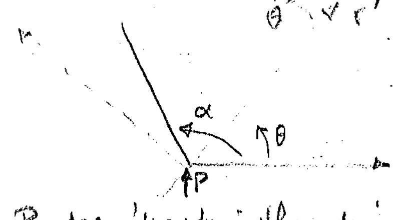
we want $P = \left[\frac{\partial\phi}{\partial x}\right]_{S=\theta=0}^{F=\theta=\alpha} = a_1\left[\alpha - \frac{1}{2}\sin 2\alpha\right]$ $\therefore$ $a_1 = \dfrac{P}{\alpha - \frac{1}{2}\sin 2\alpha}$

now from $\oint T_x \, ds = 0$

$$\left.\frac{\partial \phi}{\partial y}\right|_S^F = \left.\frac{\partial}{\partial y}\left(a_1 x \theta\right)\right|_S^F = \left. a_1 x \frac{x}{x^2+y^2}\right|_S^F = \left. a_1 \cos^2\theta \right|_S^F = a_1 \cos^2\alpha$$

In general.

This will not give zero. but if we pick coordinate system.

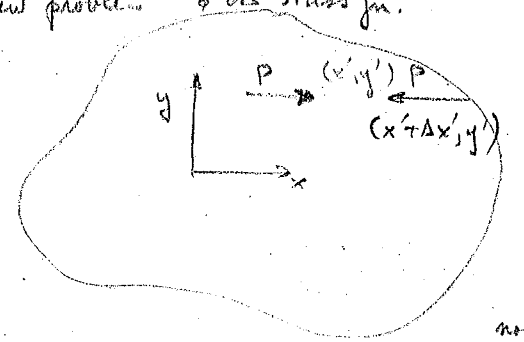


$$\left.\frac{\partial}{\partial y}(a_1 x \theta)\right|_{S=-\alpha/2}^{F=\alpha/2} = \left. a_1 \cos^2\theta\right|_S^F = 0$$

however now we have to redefine a_1 since the force P doesn't act in the direction ⊥ & ∥ to r

to get it to work out in the x, y coord system need both $\phi = a_1 r \theta \cos\theta + b_1 \theta r \sin\theta$

look at new problem & its stress fn.



$\phi^{(2)} = P f(x-x'; y-y')$ where $f = \phi^{(x)}$

if we add another force equal but opposite then

$$\phi = P\{f(x-x'; y-y') - f(x-x'-\Delta x'; y-y')\}$$
$$= P\Delta x' \left\{ \frac{f(x-x'; y-y') - f(x-x'-\Delta x'; y-y')}{\Delta x'} \right.$$

now let $\Delta x' \to 0$ $P \to \infty$ ∋. $P\Delta x' = k$.

$$\therefore \quad \phi^{(2)} = K\left[-\frac{\partial \phi^{(1)}}{\partial x'}\right] = K\left[\frac{\partial \phi^{(1)}}{\partial x}\right]$$

This is a force doublet w/out moment.

we can define $\lim_{\Delta y' \to 0}$ (P ↓, Δy', ↑ P) $K\left[\frac{\partial \phi^{(1)}}{\partial y}\right] = \phi^{(2)}$ ← doublet in y direction where $\phi^{(1)} = \phi^{(y)}$

we can also define (P ←, Δy', →) $K \frac{\partial \phi^{(1)}}{\partial y} = \tilde{\phi}^{(2)}$ ← the double force w/ moment. where $\phi^{(1)} = \phi^{(x)}$

$K = P\Delta y'$.

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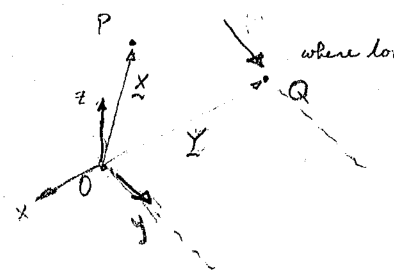
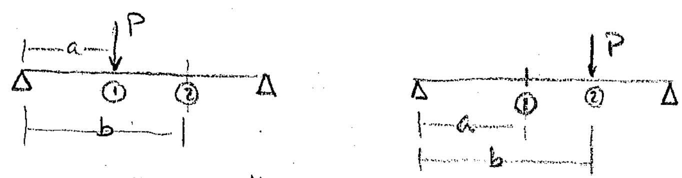


$G_{im}(\underline{x}, \underline{y})$ is defined as the displ in the i^{th} dir at $\underline{x}$ due to a load at $\underline{y}$ in the j^{th} dir

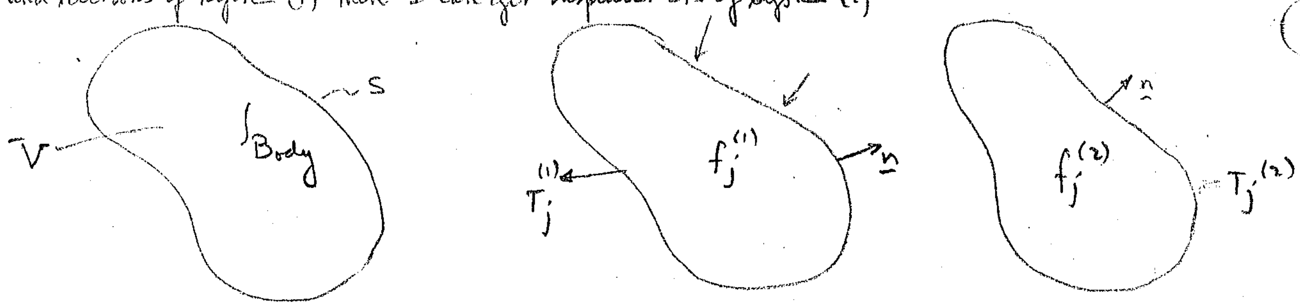
Boundary Integral Equation Method



$u_{12} = u_{21}$ Bettie's Law.

displacement at 1 due to load at 2 = displ at 2 due to load at 1

this result is due to consideration of the work of the systems. If I know system (2) and tractions of system (1) then I can get displacements of system (1)



(1) $$\int_S T_i^{(1)} u_i^{(2)}\, ds + \int_V f_i^{(1)} u_i^{(2)}\, dV = W = \int_S T_i^{(2)} u_i^{(1)}\, ds + \int_V f_i^{(2)} u_i^{(1)}\, dV$$

now $T_i = \sigma_{ij} n_j$

$$\int_S (\sigma_{ij} n_j u_i)\, dS = \int_V (\sigma_{ij} u_i)_{,j}\, dV = \int_V \sigma_{ij,j} u_i\, dV + \int_V \sigma_{ij} u_{i,j}\, dV$$

Now collect terms. & use fact that body is in equilib

$$\int_V (\sigma_{ij,j}^{(1)} + f_i^{(1)}) u_i^{(2)}\, dV + \int_V \sigma_{ij}^{(1)} u_{i,j}^{(2)}\, dV = \int (\sigma_{ij,j}^{(2)} + f_i^{(2)}) u_i^{(1)} + \int_V \sigma_{ij}^{(2)} u_{i,j}^{(1)}\, dV$$

by equil   0 by equil

$$\therefore \int_V \sigma_{ij}^{(1)} u_{i,j}^{(2)}\, dV = \int_V \sigma_{ij}^{(2)} u_{i,j}^{(1)}\, dV \qquad (2)$$

now $u_{i,j} = \varepsilon_{ij} + \omega_{ij}$ $\qquad \nabla u = \frac{1}{2}(\nabla u + u\nabla) + \frac{1}{2}(\nabla u - u\nabla)$ (3) since $\sigma_{ij}\omega_{ij} = -\sigma_{ji}\omega_{ji}$ & $\omega_{ii} = 0$

$\therefore \sigma_{ij} u_{i,j} = \sigma_{ij}\varepsilon_{ij} + \sigma_{ij}\omega_{ij}$ → 0 sym · antisym tensor = 0 put all this into (2) to get

$$\int_V \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} dV = \int_V \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)} dV \qquad (4)$$

for an anisotropic body $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$ $\therefore$

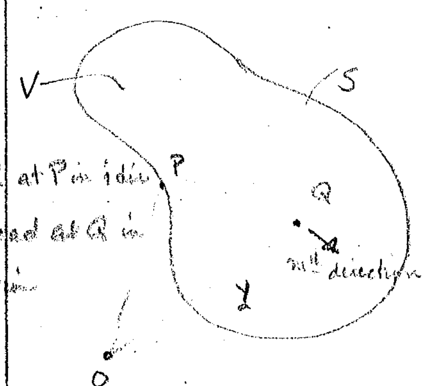
$$\int_V \left[C_{ijkl}\,\varepsilon_{kl}^{(1)}\varepsilon_{ij}^{(2)} - C_{ijkl}\,\varepsilon_{kl}^{(2)}\varepsilon_{ij}^{(1)}\right] dV = 0 \qquad \text{for the same body}$$

$$\int_V \left[C_{ijkl}\,\varepsilon_{kl}^{(1)}\varepsilon_{ij}^{(2)} - C_{klij}\,\varepsilon_{ij}^{(2)}\varepsilon_{kl}^{(1)}\right] dV = 0 \qquad \text{now } C_{ijkl} = C_{klij} \text{ from symmetry}$$

then the proof just follows

Now $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} = C_{ijkl}\,u_{k,l}$ (5)

Given: $\sigma_{ij}^{(1)}$; $\varepsilon_{ij}^{(1)}$ are wanted for some $T_i^{(1)}$, $u_i^{(1)}$ applied on boundary & no $f_i^{(1)}$



G_{im} = displ at P in i dir due to load at Q in m direction

supposedly known ‖ $\sigma_{ij}^{(2)}$; $\varepsilon_{ij}^{(2)}$ are derived from $G_{im}(\underline{x}, \underline{y})$

$u_i^{(2)} = G_{im}(\underline{x}, \underline{y})$

define $f_i^{(2)} = \delta(\underline{x} - \underline{y})\,\delta_{im}$ body force in m^{th} direction at point Q where a pt load is defined

now $\varepsilon_{ij}^{(2)} = \frac{1}{2}(G_{im,j} + G_{jm,i})$

σ_{ij} in m direction $= \sigma_{ij}^{m} = \sigma_{ij}^{(2)} = C_{ijkl}\,\varepsilon_{kl}^{(2)} = C_{ijkl}\,u_{k,l}^{(2)} = C_{ijkl}\,G_{km,l}(\underline{x}, \underline{y})$

$$T_i^{m}(\underline{x}, \underline{y}) = \sigma_{ij}^{m} n_j = C_{ijkl}\,G_{km,l}(\underline{x}, \underline{y})\,n_j = T_i^{(2)}(\underline{x}, \underline{y})$$

now $\int_S T_i^{(1)} u_i^{(2)} dS + \int_V f_i^{(1)} u_i^{(2)} dV = \int_S T_i^{(2)} u_i^{(1)} dS + \int_V f_i^{(2)} u_i^{(1)} dV$ (since $f_i^{(1)} = 0$) where $^{(1)}$ state is a fn of $\underline{x}$ only whereas $^{(2)}$ state is a fn of $(\underline{x}, \underline{y})$.

$$\int_S \left(T_i^{(1)} u_i^{(2)} - T_i^{(2)} u_i^{(1)}\right) dS - \int_V f_i^{(2)} u_i^{(1)} dV = 0$$

$f_i^{(2)}$

Now $\int \delta(\underline{x}-\underline{y})\, \delta_{im}\, u_i^{(1)}(\underline{x})\, dV = u_m^{(1)}(\underline{y})$ or $\int f_i^{(2)} u_i^{(1)} dV = u_m(\underline{y})$

∴ put in previous eq

$$\int_S \left[T_i^{(1)}(\underline{x})\, G_{im}(\underline{x},\underline{y}) - T_i^{(m)}(\underline{x},\underline{y})\, u_i^{(1)}(\underline{x}) \right] ds(\underline{x}) - u_m(\underline{y}) = 0$$

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Boundary Integral Method. - Summary

<u>Have</u> - solution to point load in infinite medium (ie Green's function)

<u>Want</u> - soln to elasticity problem in a finite body

<u>Method</u> - take advantage of Reciprocity

Reciprocal theorems (Betti's Law)

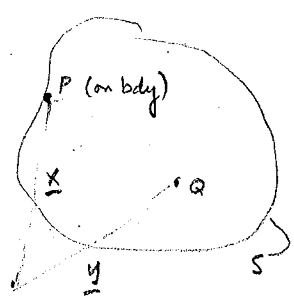
$$\int_S T_i^{(1)} u_i^{(2)} ds + \int_V f_i^{(1)} u_i^{(2)} dV = \int_S T_i^{(2)} u^{(1)} ds + \int f_i^{(2)} u_i^{(1)} dV$$

take special case where $f_i^{(1)} = 0$ $\quad f_i^{(2)} = \delta_{im}\,\delta(\underline{x}-\underline{y})$

$$T_i^{(2)} = T_i^{(m)}(\underline{x},\underline{y}) = \sigma_{ij}^{(m)} n_j = n_j C_{ijkl} G_{km,l}$$

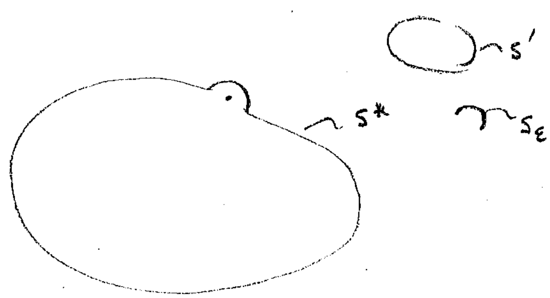
Result: $\int_S T_i^{(1)}(\underline{x})\, G_{im}(\underline{x},\underline{y}) - T_i^{(m)}(\underline{x},\underline{y})\, u_i^{(1)}(\underline{x})\, ds = u_m^{1}(\underline{y})$

Somigliana's Identity.



So given G_{im}, $T_i^{(m)}$ for infinite body and if we knew $T_i^{(1)}(x)$ and $u_i^{(1)}(\underline{x})$ on surface then we could find u_m but we cannot prescribe both $T_i^{(1)}$ & $u_i^{(1)}$ on bdy.

Lets look at what happens when $u_m(\underline{y})$ is a boundary pt. (the above Eqn is good only for interior points) we will use limiting process



$$\int_{S*} = \int_{S'} + \int_{S\varepsilon}$$

take care of this by PV theorem ($\int_{S'}$)

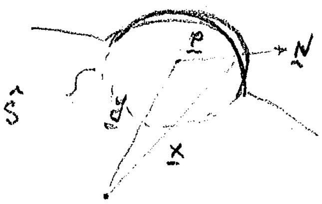
look at ($\int_{S\varepsilon}$)

$$\lim_{\epsilon \to 0} \int_{S'} f(\underline{x},\underline{y})\,ds = PV\int_S f(\underline{x},\underline{y})\,ds$$ principal value cauchy

Consider
$$\int_{S_\epsilon} T_i^{(1)}(\underline{x})\, G_{im}(\underline{x},\underline{y}) - T_i^{m}(\underline{x},\underline{y})\, u_i^{(1)}(\underline{x})\, ds$$

Tackle 2nd part first:

on S_ϵ: $u_i(\underline{x}) = u_i(\underline{y}+\underline{\rho})$ $\quad |\underline{\rho}| = \epsilon \quad \underline{\rho} = \epsilon \underline{N}$



now expand $u_i(\underline{x})$ in taylors series where $\underline{y}$ is fixed

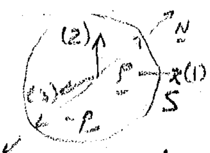
$$u_i(\underline{x}) = u_i(\underline{y}) + \left.\frac{\partial u_i(\underline{x})}{\partial x_j}\right|_{\underline{y}} \epsilon \underline{N} + \ldots$$

hence $$\int_{S_\epsilon} T_i^{(m)}(\underline{x},\underline{y})\, u_i(\underline{x})\,ds \cong u_i(\underline{y}) \int_{S_\epsilon} T_i^{(m)}(\underline{x},\underline{y})\,ds + \epsilon \underline{N}_j \frac{\partial u_i(\underline{y})}{\partial x_j} \int_{S_\epsilon} T_i^{(m)}(\underline{x},\underline{y})\,ds$$

but for a full sphere $\hat{S}$

$$\int_{\hat{S}} T_i^{(m)}(\underline{x},\underline{y})\,ds = -\delta_{im}$$

now



σ_{ij} is odd in $\underline{\rho}$, $\underline{N}$ is odd in $\underline{\rho}$ } T_i is even in $\underline{\rho}$

using even-odd argument $\quad \therefore \int_{\hat{S}} = 2\int_{S_\epsilon}$

$$\therefore \int_{S_\epsilon} T_i^{(m)}(\underline{x},\underline{y})\,ds = -\tfrac{1}{2}\delta_{im}$$ Kronecker Delta

$$\lim_{\epsilon \to 0} \epsilon N_j \frac{\partial u_i(\underline{y})}{\partial x_j} \underbrace{\int_{S_\epsilon} T_i^{(m)}(\underline{x},\underline{y})\,ds}_{-\frac{1}{2}\delta_{im}} \to 0$$ since everything is bounded

Consider (1) first integral and expand $T_i(\underline{x})$ about $T_i(\underline{y})$

$$\int_{S_\epsilon} T_i(\underline{x})\, G_{im}(\underline{x},\underline{y})\,ds$$ for 3D : $G_{im} \sim \frac{1}{\epsilon}$ $\quad ds \sim \epsilon^2$; 2D : $\sim \ln\epsilon$ $\quad ds \sim \epsilon$

$$= \int \sigma_{ij}(\underline{x})\, n_j\, G_{im}(\underline{x},\underline{y})\,ds = \int_{S_\epsilon} \left[\sigma_{ij}(\underline{y}) + \epsilon n_j \frac{\partial \sigma_{ij}}{\partial x_j}\right] n_j\, G_{im}(\underline{x},\underline{y})\,ds$$

take out of integral since $\underline{y}$ is fixed

$$= \sigma_{ij}(\underline{y}) \underbrace{\int_{S_\epsilon} n_j\, G_{im}(\underline{x},\underline{y})\,ds}_{\text{order } \epsilon \to 0} + \epsilon n_j \frac{\partial \sigma_{ij}(\underline{y})}{\partial x_j} \underbrace{\int_{S_\epsilon} n_j\, G_{im}\,ds}_{\text{order } \epsilon^2 \to 0}$$

$\therefore$ first part of integral $= 0$

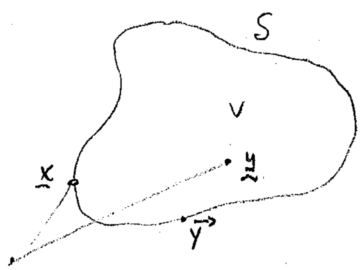
$\therefore$ Somigliana's identity leads to
$$\tfrac{1}{2} u_m(\underline{y}) + PV\int_S T_i^{(m)}(\underline{x},\underline{y})\, u_i(\underline{x})\,ds = PV\int_S T_i(\underline{x})\, G_{im}(\underline{x},\underline{y})\,ds$$

2/26/79

BIE Discretization

(1) Somigliana's Identity for $\underset{\sim}{y} \in V$ but not on ∂V

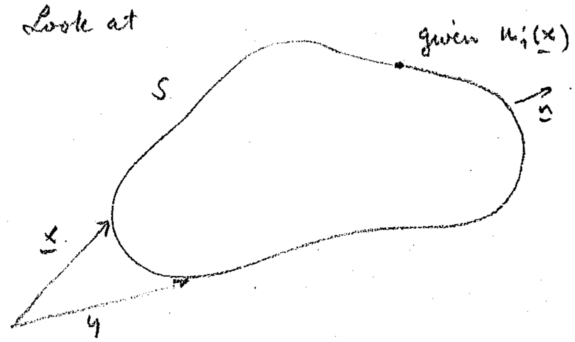
$$u_m(\underset{\sim}{y}) + \int_S u_i(\underline{x})\, \overset{\text{known}}{T_i^{(m)}}(\underline{x},\underline{y})\, dS(x) = \int_S t_i(\underline{x})\, \overset{\text{known}}{G_{im}}(\underline{x},\underline{y})\, dS(\underline{x})$$



(2) BIE Constraint ($\underset{\sim}{y} \to \underline{y}$ on S)

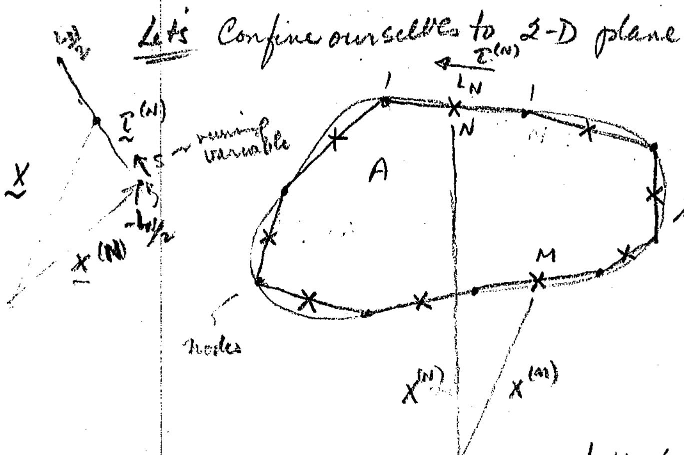
$$\frac{1}{2}u_m(\underline{y}) + PV\int_S u_i(\underline{x})\, \underset{\text{known}}{T_i^{(m)}}(\underline{x},\underline{y})\, dS(\underline{x}) = PV\int_S t_i(\underline{x})\, \overset{\text{knowing}}{G_{im}}(\underline{x},\underline{y})\, dS(\underline{x})$$

Look at



for (2) you will find that $\underline{n}$ was taken the outer normal, hence we were working interior problem.
for exterior problem $\underline{n}$ is into V'
then the $T_i^{(m)}(\underline{x},\underline{y})$ implicitly define which problem we look at (interior/exterior)

Let's Confine ourselves to 2-D plane strain



1. Let boundary be represent by polygon with N segments
2. on any segment $u_i(\underline{x})$ $t_i(\underline{x})$ are constant and given by their values at the segment mid points

Let $\underset{\sim}{y} = X^{(M)}$ $\underline{x} = X^{(N)}$ then

$$\frac{1}{2}u_m(X^{(M)}) + \sum_{seg}\int \underset{\text{Constant}}{u_i(\underline{x}^{(N)})}\, T_i^{(m)}(\underline{x},\underline{y})\, dS(\underline{x}) = \sum\int \underset{\text{const over}}{t_i(\underline{x})}\, G_{im}(\underline{x},\underline{y})\, d$$

traction in i^{th} direction at $\underset{\sim}{X}^{(N)} + \underset{\sim}{\tau}^{(N)} s$
due to unit force at $\underset{\sim}{X}^{(M)}$ in m^{th} direction

$$\frac{1}{2} u_m(\underset{\sim}{X}^{(M)}) + \sum_{N=1}^{W} u_i(\underset{\sim}{X}^{(N)}) \int_{-LN/2}^{LN/2} T_i^{(m)}\left(\underset{\sim}{X}^{(N)} + \underset{\sim}{\tau}^{(N)} s, \underset{\sim}{X}^{(M)}\right) ds$$

$$= \sum_{N=1}^{W} t_i(\underset{\sim}{X}^{(N)}) \int_{-LN/2}^{LN/2} G_{im}(\underset{\sim}{X}^{(N)} + \underset{\sim}{\tau}^{(N)} s, \underset{\sim}{X}^{(M)}) ds$$

Let $u_i(\underset{\sim}{X}^{(N)}) = u_i(N)$
$t_i(\underset{\sim}{X}^{(M)}) = t_i(M)$

$$\frac{1}{2} u_m(\underset{\sim}{X}^{(M)}) + \sum u_i(\underset{\sim}{X}^{(N)}) \, \Delta T_i^{(m)}(N,M) = \sum t_i(\underset{\sim}{X}^{(N)}) \, \Delta G_{im}(N,M)$$

or $$\frac{1}{2} u_m(M) + \sum_{N=1}^{W} u_i(N) \, \Delta T_i^{(m)}(N,M) = \sum_{N=1}^{W} t_i(N) \Delta G_{im}(N,M)$$

$M = 1, 2, 3, \ldots W \qquad i = 1, 2 \qquad m = 1, 2$

normally in one integral M is fixed and N will vary over $1, \ldots, W$
now define $\delta_{im}(N,M) = 1$ if $i = m$, $N = M$
$\qquad 0$ if otherwise

then
$$\sum_{N=1}^{W} u_i(N) \left\{ \frac{1}{2} \delta_{im}(N,M) + \Delta T_i^{(m)}(N,M) \right\} = \sum_{N=1}^{W} \left\{ t_i(N) \, \Delta G_{im}(N,M) \right\}$$

I claim we can write
$$[U]\left[\tfrac{1}{2} I + \Delta T\right] = [t][\Delta G]$$

where

$$[U] = \left[u_1(1) \;\; u_2(1) \;\vdots\; u_1(2) \;\; u_2(2) \;\vdots\; \cdots \;\vdots\; u_1(W) \;\; u_2(W) \right]$$

row matrix with 2W

$$[t] = \left[t_1(1) \;\; t_2(1) \;\vdots\; \qquad \;\vdots\; t_1(W) \;\; t_2(W) \right]$$

row matrix with 2W

$[I]$ is a $2W \times 2W$ unit matrix

over.

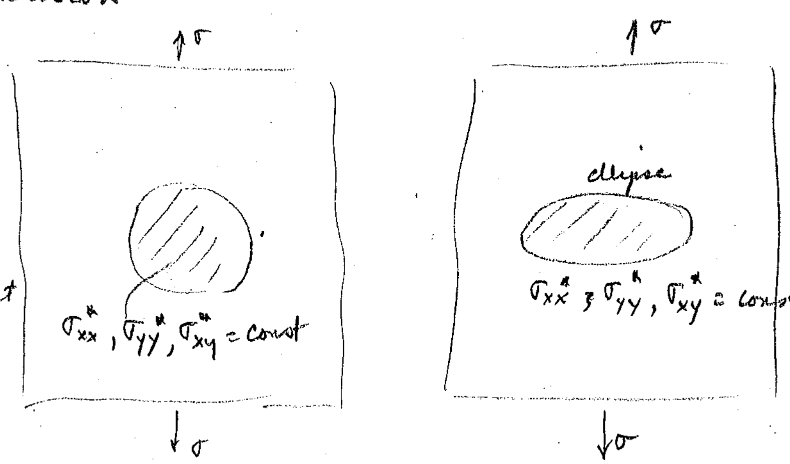
$$[\Delta T] = \left[\begin{array}{cc|cc} \Delta T_1^{(1)}(1,1) & \Delta T_1^{(2)}(1,1) & \Delta T_1^{(1)}(1,2) & \Delta T_1^{(2)}(1,2) \\ \Delta T_2^{(1)}(1,1) & \Delta T_2^{(2)}(1,1) & \Delta T_2^{(1)}(1,2) & \Delta T_2^{(2)}(1,2) \quad \cdots \\ \hline \Delta T_1^{(1)}(2,1) & \Delta T_1^{(2)}(2,1) & & \\ \Delta T_2^{(1)}(2,1) & \Delta T_2^{(2)}(2,1) & & \\ \vdots & & & \end{array}\right]$$

in traction problem: fix 2 points to get rid of rigid body rotation & displ
$[t]$ is known, ΔT, ΔG are known.

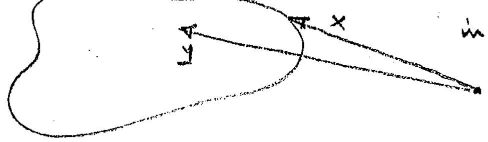
What we want to do. 2/28/79

(1) Edge dislocation Solution (Internal Stress)
(2) Solution for a line force in the interior of a half space (by method of images)
(3) Construct the Solution for a point force in 3-D solid (anisotropy)
(4) Use (3) to find line force solution in an infinite solid. $\rightarrow G_{im}(\underline{x}, \underline{y})$
(5) Inclusion

it turns out



Given



Somigliana's identity
in Interior is $u_m(\underset{\sim}{y}) + \int_S u_i(\underline{x})\, T_i^{(m)}(\underline{x}, \underline{y})\, dS(\underline{x}) = \int_S t_i(\underline{x})\, G_{im}(\underline{x}, \underline{y})\, dS(\underline{x})$

to get stresses differentiate wrt $\frac{\partial}{\partial y_p}$ because we know that $\sigma_{\alpha\beta}(\underline{y}) = C_{\alpha\beta m p}\, u_{m,p}(\underline{y})$

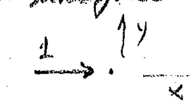
$$\frac{\partial u_m(\underline{y})}{\partial y_p} + \int_S u_i(\underline{x}) \frac{\partial}{\partial y_p}\left(T_i^{(m)}\langle \underline{x},\underline{y}\rangle\right) dS(\underline{x}) = \int_S t_i(\underline{x}) \frac{\partial G_{im}(\underline{x},\underline{y})}{\partial y_p}\, dS(\underline{x})$$

$p = 1,2,3$

and since $\underline{x}$ is fixed can move differentiation into integral

Dislocations (Edge Dislocations) state of internal stress w/o application of external stresses

For a line force in an infinite solid: $\phi^L = -\frac{1}{2\pi}\left\{ r\theta \sin\theta - \frac{1-2\nu}{2(1-\nu)} r\cos\theta \ln r \right\}$



(under $r\theta\sin\theta$:) line force at origin but multi val disp

(under the second term:) no net force but cancels out multi-val disps

If I look at just 2nd part of ϕ^L this represents the edge dislocation

$$\phi^d = b_1' \, r\cos\theta \ln r$$

in the first handout we show that ϕ^d gives

$$u_r = \frac{\cos\theta}{E}\left\{ b_1'(1-\nu-2\nu^2)\right\}\ln r + g(\theta)$$

$$u_\theta = -\int_\theta g(t)\,dt + \sin\theta\, \frac{b_1'(1-\nu-2\nu^2)}{E}(1-\ln r)$$

$$g(\theta) = \frac{(1-\nu)b_1'}{\mu}\theta \sin\theta$$

Now with $\left\{ \begin{array}{l} u_x = u_r\cos\theta - u_\theta \sin\theta \\ u_y = u_r \sin\theta + u_\theta\cos\theta \end{array}\right.$ $\Rightarrow$ to within rigid body terms

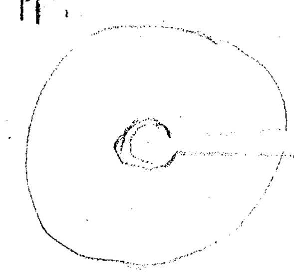
$$u_x = \frac{b_1'}{4\mu}\left\{(1-2\nu)\ln(x^2+y^2) - \frac{x^2-y^2}{x^2+y^2}\right\}$$

$$u_y = \frac{1-\nu}{\mu} b_1' \tan^{-1}\frac{y}{x} - \frac{b_1'}{2\mu}\frac{xy}{x^2+y^2}$$

(under $\tan^{-1}\frac{y}{x}$:) multi valued ness is right here.

Suppose I were to displace $u_y(\theta=2\pi) - u_y(\theta=0) = -B$ (constant, Burger's vector)

$$\frac{1-\nu}{\mu} b_1'(2\pi - 0) = -B$$



$$\boxed{b_1' = -\frac{\mu B}{2\pi(1-\nu)}}$$

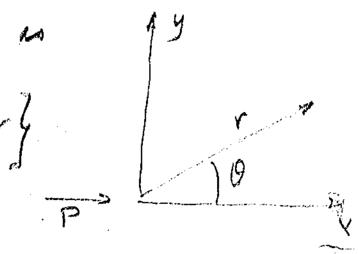
now we can get $\sigma_{ij} \sim \frac{1}{r}$ using $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$\sigma_{ij} = (\lambda + 2\mu)\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$$

Line Force in a Half-Space: Plane Strain

look at Handout for line force in Half Space

Full Infinite medium gives stress function as shown.



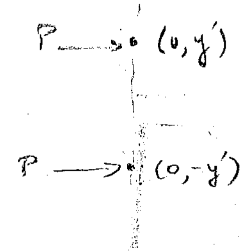
$$\phi = -\frac{P}{2\pi}\left\{(r\sin\theta)\theta - \frac{1-2\nu}{2(1-\nu)} r\cos\theta \ln r\right\}$$

This in turn gives

$$\sigma_{yy} = \frac{P}{4\pi(1-\nu)}\left[\frac{x}{(x^2+y^2)^2}\right]\left\{(1-2\nu)x^2 - (1+2\nu)y^2\right\}$$

$$\sigma_{xy} = -\frac{P}{4\pi(1-\nu)}\left[\frac{y}{(x^2+y^2)^2}\right]\left\{(3-2\nu)x^2 + (1-2\nu)y^2\right\}$$

if P is applied at $(0,y')$ let y be replaced by $y-y'$

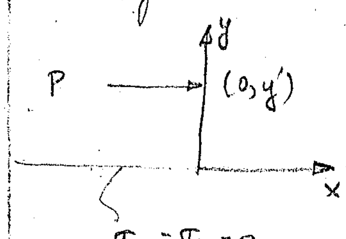


$\hat{\sigma}_{ij}$ (same as before w y replaced by $y+y'$)

Sum of 2: $\sigma_{ij}(x,y-y') + \sigma_{ij}(x,y+y')$ gives no shear stress at $\sigma_{ij}(x,0)$

3/1/79

Line forces in the interior of half space



for the full space solution $\sigma_{yy} = \dfrac{P}{4\pi(1-\nu)}\dfrac{x}{[x^2+(y-y')^2]^2}\left\{(1-2\nu)x^2 - (1+2\nu)(y-y')^2\right\}$

$$\sigma_{xy} = -\frac{P}{4\pi(1-\nu)}\frac{y-y'}{[x^2+(y-y')^2]^2}\left\{(3-2\nu)x^2 + (1-2\nu)(y-y')^2\right\}$$

this however doesn't solve problem

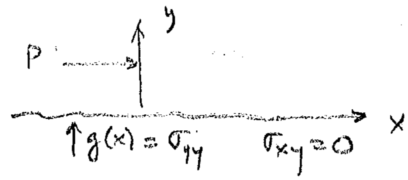
If I put an image pt at $(0,-y')$ then we note that $\sigma_{xy}=0$ on $y=0$

$\therefore\ \sigma_{xy}(x,y-y') + \sigma_{xy}(x,y+y')$ at $y=0$ then $\sigma_{xy}(x,-y') + \sigma_{xy}(x,+y') = 0$

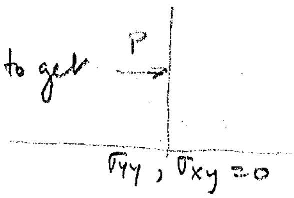
$$\sigma_{yy}(x,-y') + \sigma_{yy}(x,y') = 2\,\sigma_{yy}(x,y')$$
$$= f(x) = \sigma_{yy}(x,0)$$

Now cut the plane space in half

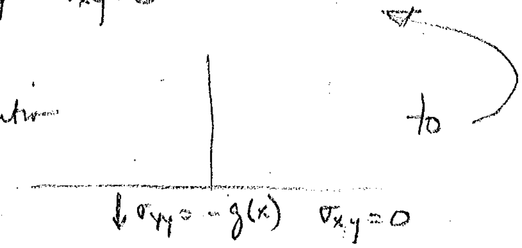
this is equivalent to



to get

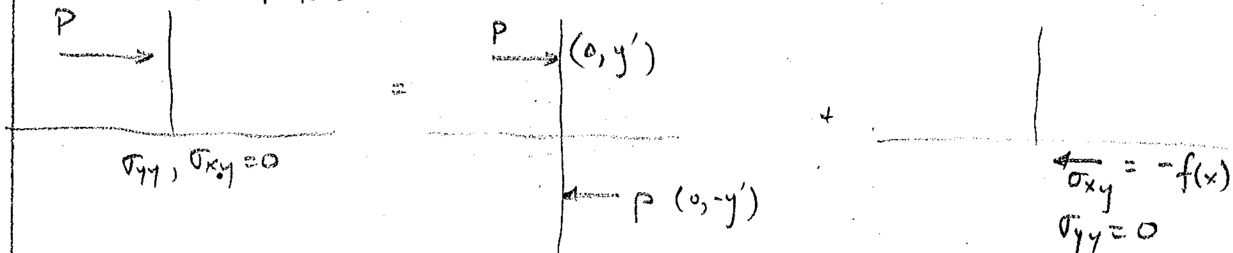


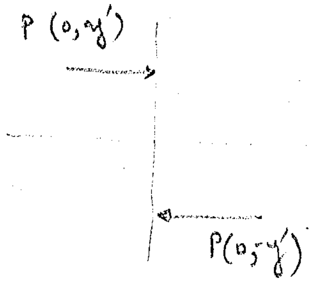
add the solution



this we've done before in fourier transform method.

another method to do this is



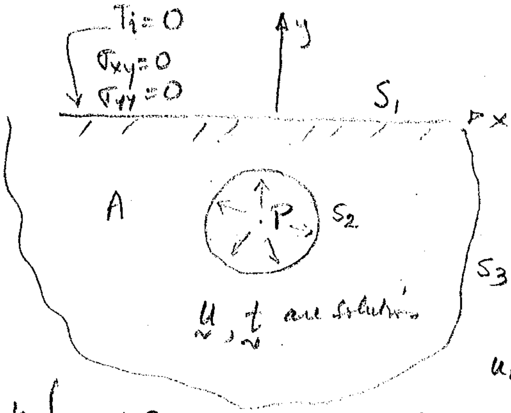


gives $\sigma_{xy}(x, -y') - \sigma_{xy}(x, y') = 2\,\sigma_{xy}(x, y') = \sigma_{xy}(x, 0) = f(x)$

$\sigma_{yy}(x, y') - \sigma_{yy}(x, -y') = 0$

and to get rid of $f(x)$ we add on a second solution ∋. $\sigma_{xy} = -f(x)$ (on $y=0$) & $\sigma_{yy} = 0$. This do by Fourier transform technique

<u>Problem # 2</u> Suppose



S_1, S_2, S_3 bound area A

define $\hat{g}_{im}(\underset{\sim}{x}, y)$ the displacement in the i^{th} dir at $\underset{\sim}{x}$ due to a load (line force mag=1) applied in the m^{th} dir at $\underset{\sim}{y}$

also $\hat{g}_{im}(\underset{\sim}{x}, \underset{\sim}{y})$ is ∋. $\hat{T}_i^{(m)}(\underset{\sim}{x}, \underset{\sim}{y}) = 0$ on S_1

$$u_m(\underset{\sim}{y}) + \int_{S_1+S_2+S_3} u_i(\underset{\sim}{x})\, \hat{T}_i^{(m)}(\underset{\sim}{x}, \underset{\sim}{y})\, dS(\underset{\sim}{x}) = \int_{S_1+S_2+S_3} t_i(\underset{\sim}{x})\, \hat{g}_{im}(\underset{\sim}{x}, \underset{\sim}{y})\, dS(\underset{\sim}{x})$$

note $\int_{S_3} \to 0$ as $r \to \infty$ $i = 1, 2$

we know this; = 0 on S_1

we know this; = 0 on S_1 since it's a bc

$\hat{g}_{im}$ is obtained by method of images + fourier transform ∋. $\hat{T}_i^{m}(x, y) = 0$ on S_1 due to $\hat{g}_{im}$

$\int$ = 0 hence all you need to do is work w/ S_2 this makes the solution easy

g_{im} is known since $\epsilon_{xx} = \frac{\partial u_y}{\partial x} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}(\sigma_{yy} + \sigma_{zz})$

gives u_x, u_y, u_z

3-D Green's function or Kelvin Solution

$G_{im}(\underset{\sim}{x}, \underset{\sim}{y})$: the displ in the i^{th} direction at $\underset{\sim}{x}$ due to a unit point force applied in the m^{th} direction at $\underset{\sim}{y}$ in an infinite solid

What are stress fields: $\sigma_{ij}^{(m)}(\underset{\sim}{x}, \underset{\sim}{y}) = C_{ijkl} \frac{\partial G_{km}}{\partial x_l}(\underset{\sim}{x}, \underset{\sim}{y})$; $\sigma_{ij} = C_{ijkl}\,\epsilon_{kl}$ $\quad \epsilon_{kl} = \frac{\partial G_{km}}{\partial x_l}$

now $\sigma_{ij,i}^{(m)} + f_j = 0$ from equil $\quad f_j$ is a unit point force

$f_j = \delta_{jm}\, \delta(\underset{\sim}{x} - \underset{\sim}{y}) = \delta_{jm}\, \delta(x_1 - y_1)\, \delta(x_2 - y_2)\, \delta(x_3 - y_3)$

delta function
$= 1 \quad \underset{\sim}{x} = \underset{\sim}{y}$
$= 0 \quad \underset{\sim}{x} - \underset{\sim}{y} \neq 0$ } defines loc of point force

put equil into stress field

$$C_{ijkl} \frac{\partial^2 G_{km}(\underset{\sim}{x}, \underset{\sim}{y})}{\partial x_i \partial x_l} + \delta_{jm}\, \delta(\underset{\sim}{x} - \underset{\sim}{y}) = 0 \qquad j = 1, 2, 3$$

27 terms $i \cdot k \cdot l = 3^3$

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we have 9 unknowns G_{km} $\quad k, m = 1, 2, 3$
9 eqns $\quad j, m = 1, 2, 3$

3rd D Fourier transform

Given $f(x_1, x_2, x_3)$ then define $\hat{F}(k_1, k_2, k_3) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3\, f(x_1, x_2, x_3)\, e^{i(k_1x_1 + k_2x_2 + k_3x_3)}$

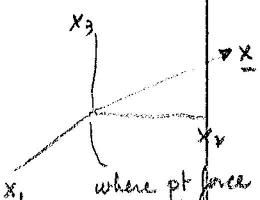
fourier transform of F: $\hat{F}(k_1, k_2, k_3) = \iiint_{-\infty}^{\infty} d^3\underset{\sim}{x}\; f(\underset{\sim}{x})\, e^{i\, \underset{\sim}{k} \cdot \underset{\sim}{x}}$

where $d^3\underset{\sim}{x} = dx_1\, dx_2\, dx_3$

$\underset{\sim}{k}$ fourier wave vector $\quad \underset{\sim}{k} = \frac{2\pi}{\lambda}$

$$f(\underline{x}) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} d^3\underline{k}\;\hat{F}(\underline{k})\,e^{-i\underline{k}\cdot\underline{x}}$$

take $c_{ijkl}\dfrac{\partial^2 G_{km}}{\partial x_i \partial x_l} + \delta_{jm}\,\delta(\underline{x}-\underline{y}) = 0$ & take 3-D Fourier transform.
Let me take $\underline{y}=0$ also



$$c_{ijkl}\iiint_{-\infty}^{\infty} d^3\underline{x}\; e^{i\underline{k}\cdot\underline{x}}\,\frac{\partial^2 G_{km}(\underline{x})}{\partial x_i \partial x_l} + \delta_{jm}\underbrace{\iiint_{-\infty}^{\infty} d^3\underline{x}\; e^{i\underline{k}\cdot\underline{x}}\,\delta(\underline{x})}_{e^{i\underline{k}\cdot 0}=1} = 0 \qquad (*)$$

Integrate part 1 by parts

Aside
$$\int_{-\infty}^{\infty} dx_1\, e^{ik_1x_1}\frac{\partial}{\partial x_1}f(x_1) = f(x_1)\,e^{ik_1x_1}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx_1\, ik_1 f(x_1)\,e^{ik_1x_1}$$

Assume $f(x_1)\to 0$ as $x_1 \to \pm\infty$ the boundary term $=0$

$$-\int_{-\infty}^{\infty} dx_1\; ik_1 f(x_1)\,e^{ik_1x_1} = -ik_1\,\hat{F}(k_1)$$

If the 2nd derivative is $\frac{\partial^2}{\partial x_1^2}f(x_1)$

$$\iiint_{-\infty}^{\infty} d^3\underline{x}\; e^{i\underline{k}\cdot\underline{x}}\,\frac{\partial^2 G_{km}(\underline{x})}{\partial x_i \partial x_l} = (-ik_i)(-ik_l)\,\hat{g}_{km}(\underline{k}) \qquad \hat{g}_{km}(\underline{k}) = \iiint_{-\infty} d^3\underline{x}\; e^{i\underline{k}\cdot\underline{x}}\,G_{km}(\underline{x})$$

Fourier transform of $G_{km}(\underline{x})$

if $G_{km}\to 0$ as $|\underline{x}|\to\infty$ & if $G'_{km}\to 0$ as $|\underline{x}|\to\infty$ the above is true.

Thus the eqn (*) becomes

$$-k_ik_l\,c_{ijkl}\,\hat{g}_{km}(\underline{k}) = -\delta_{jm} \Rightarrow k_ik_l\,c_{ijkl}\,\hat{g}_{km}(\underline{k}) = \delta_{jm}$$

now $\underline{k} = K\underline{z}$ where $K = |\underline{k}|$ $\quad |\underline{z}| = 1$ $\quad \therefore\; k_i = Kz_i \quad k_l = Kz_l$

$$\therefore\; \underbrace{z_i\,c_{ijkl}\,z_l}_{\text{matrix } M_{jk}(\underline{z})}\,\hat{g}_{km}(\underline{k}) = \frac{\delta_{jm}}{K^2}$$

depends on direction of $\underline{k}$ not on magnitude

$$M_{jk}(\underline{z})\;\hat{g}_{km}(\underline{k}) = \frac{\delta_{jm}}{K^2}$$

$M_{jk} = M_{kj}$ is a real symmetric matrix

$$M_{kj} = z_i C_{ikjl} z_l = z_l \overline{C_{lkji}} z_i \qquad \overline{} = C_{ijkl} \text{ because of sym of elastic const} \Rightarrow M_{kj} = M_{jk}$$

By use of elastic stability M_{kj} is positive definite matrix $\det M_{kj} \neq 0$
all EV's of M are > 0.

Elastic Stability $w = \frac{1}{2} C_{ijkl} e_{ij} e_{kl} > 0$

$$\therefore [M][\hat{g}] = \frac{[I]}{K^2} \qquad \therefore \quad \hat{g} = \frac{[M]^{-1}}{K^2}$$

since M doesn't depends on mag of $\underline{k}$ only direction

$$\hat{g}_{km}(\underline{k}) = \frac{M^{-1}_{km}(\underline{z})}{K^2}$$

since $\hat{g}$ depends on magn of $\underline{k}$ as $1/K^2 \Rightarrow$
G_{km} depends on $(\underset{\sim}{x})$ as $\frac{1}{\underset{\sim}{x}}$

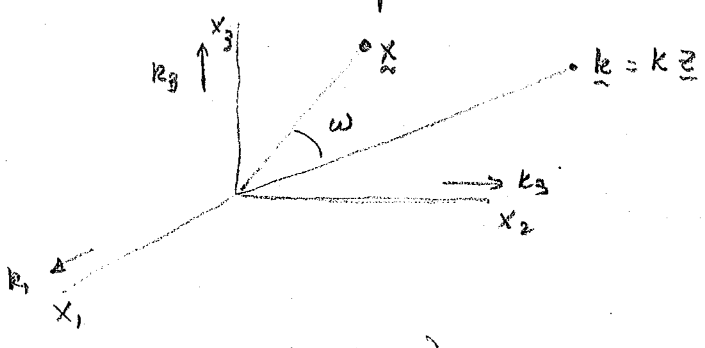
since $M_{km} = M_{mk} \Rightarrow M^{-1}_{km} = M^{-1}_{mk} \Rightarrow \hat{g}_{km} = \hat{g}_{mk} \Rightarrow G_{km}(\underset{\sim}{x}) = G_{mk}(\underset{\sim}{x})$

Give me $\underline{z}$ I will get $\hat{g}_{km}(\underline{k}) \Rightarrow G_{km}(\underline{x})$

$$G_{km}(\underline{x}) = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} d^3\underset{\sim}{k} \; \frac{M^{-1}_{km}(\underline{z})}{K^2} \; e^{-i\underline{k}\cdot\underline{x}}$$

Better to work in spherical coords

$\underline{k}\cdot\underline{x} = K|\underline{x}|\cos(\underline{k}, \underline{x})$
if we pick $\underline{x}$ as the polar axis
the $d^3\underline{k} = K^2 \sin\omega \, d\omega \, d\theta \, dK$



Questions on exam (midterm) 3/7/79

Body Force in y direction w/o regard to boundary conditions

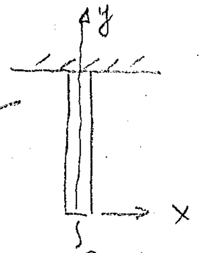
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \rho g = 0$$

Solns:
1. $\sigma_{yy} = \rho g y$, all others zero
2. $\sigma_{yy} = \rho g y + \text{const}$ all others zero
3. $\sigma_{xx} = \sigma_{yy} = \rho g y$

| w/o regard to b.c.

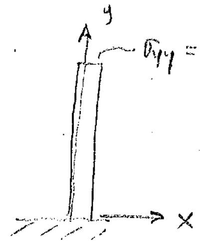
Look at the rope hanging from ceiling



$\sigma_{yy} = \rho g y$ at $y = 0$

Soln: $\sigma_{yy} = \rho g y$ all others are zero

$\sigma_{yy} > 0$



$\sigma_{yy} = p$ at $y = L$

Soln $\sigma_{yy} = \rho g (y - L)$ $y \le L$ all others $= 0$

$\sigma_{yy} < 0$

in problem #1 on exam all we wanted to do was get a stress fn that would get out the body force from the problem and not worry about b.c.

———o———o———

To return back to 3D green's fn.

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$$

we had from before

$$\hat{g}_{km}(\underline{K}) = M^{-1}_{km}(\underline{z})/K^2 \qquad \text{with} \quad G_{km}(\underline{x}) = \frac{1}{8\pi^3}\iiint_{-\infty}^{\infty} d^3\underset{\sim}{K}\, M^{-1}_{km}(\underline{z})\,\frac{1}{K^2}\,e^{-i\underline{K}\cdot\underline{x}}$$

$$M_{km} = C_{\alpha k m \beta}\, z_\alpha z_\beta$$

$$C_{\alpha k m \beta} = \lambda(\delta_{\alpha k} + \delta_{m\beta}) + \mu(\delta_{\alpha m}\delta_{k\beta} + \delta_{\alpha\beta}\delta_{km})$$

$$\begin{aligned} M_{km} &= z_\alpha z_\beta \lambda \delta_{\alpha k}\delta_{m\beta} + \mu z_\alpha z_\beta \delta_{\alpha m}\delta_{k\beta} + \mu z_\alpha z_\beta \delta_{\alpha\beta}\delta_{km} \\ &= z_k z_m \lambda + \mu z_m z_k + \mu \underbrace{z_\alpha z_\alpha}_{1} \delta_{km} \\ &= \mu\left[\delta_{km} + \frac{\lambda+\mu}{\mu} z_k z_m\right] \end{aligned}$$

1 since z_α is a unit vector

constructing inverse requires

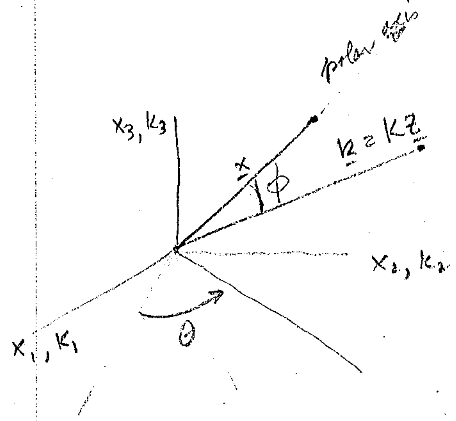
$M_{km} M^{-1}_{mn} = \delta_{mn}$. Since M_{km} is of simple form =>

Let's Guess $M^{-1}_{mn} = \frac{1}{\mu}\left\{\delta_{mn} + A z_m z_n\right\}$ where A is to be picked

$$M_{km} M^{-1}_{mn} = \left(\delta_{km} + \frac{\mu+\lambda}{\mu} z_k z_m\right)\left(\delta_{mn} + A z_m z_n\right) = \delta_{km} + A z_k z_n + \frac{\mu+\lambda}{\mu} z_k z_n + A\left(\frac{\mu+\lambda}{\mu}\right) \overset{1}{z_m z_m}\, z_k z_n$$

$$\therefore\quad A + \frac{\mu+\lambda}{\mu} + A\left(\frac{\mu+\lambda}{\mu}\right) = 0 \qquad A = \frac{-\frac{\mu+\lambda}{\mu}}{1 + \frac{\mu+\lambda}{\mu}} = -\frac{\lambda+\mu}{\lambda+2\mu}$$

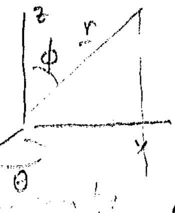
$$\therefore\quad M^{-1}_{km}(\underline{z}) = \frac{1}{\mu}\left\{\delta_{km} - \frac{\lambda+\mu}{\lambda+2\mu} z_k z_m\right\}$$



$$G_{km}(\underline{x}) = \frac{1}{8\pi^3\mu} \iiint_{-\infty}^{\infty} d^3\underset{\sim}{k} \left\{ \delta_{km} - \frac{\lambda+\mu}{\lambda+2\mu} z_k z_m \right\} \frac{e^{-i\underline{k}\cdot\underline{x}}}{k^2}$$

Better to integrate in spherical coords

Normally spherical coords are defined as follows: ⟹



If we take x in dir of z axis then

$dV = r^2 \sin\phi \, dr \, d\theta \, d\varphi$

$r: 0 \to \infty$
$\theta: 0 \to 2\pi$
$\varphi: 0 \to \pi$

$$G_{km}(\underline{x}) = \frac{1}{8\pi^3\mu} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \left\{ \delta_{km} - \frac{\lambda+\mu}{\lambda+2\mu} z_m z_k \right\} \int_0^{\infty} k^2 dk \, \frac{e^{-ik|\underline{x}|\cos\phi}}{k^2}$$

depends on ϕ, θ since z_m, z_k are only directional vectors

$\underline{k}\cdot\underline{x} = |\underline{k}||\underline{x}|\cos(\underline{k},\underline{x})$

here $dV = d^3\underset{\sim}{k} = k^2 \sin\phi \, dk \, d\theta \, d\varphi$

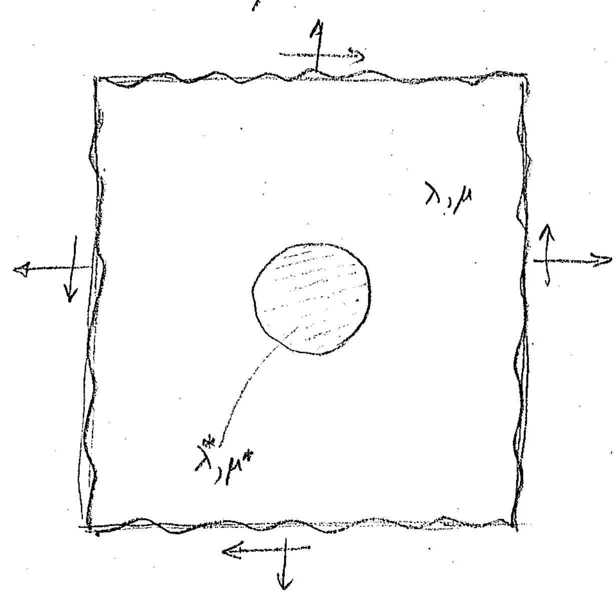
now $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} dx = \frac{1}{\pi}\int_0^{\infty} \cos\xi x \, d\xi = \delta(x) - i \, PV\left(\frac{1}{x}\right) \to 0$

now $\int_0^{\infty} dk \, e^{-i\{k|\underline{x}|\cos\phi\}} = \int_0^{\infty} \cos(k|\underline{x}|\cos\varphi) dk = \frac{1}{|\underline{x}|}\int_0^{\infty} \cos(\xi\cos\varphi) d\xi = \frac{\pi}{|\underline{x}|}\delta(\cos\varphi)$

since only even fn gives results if $k|\underline{x}| = \xi$

$$\therefore \quad G_{km}(\underline{x}) = \frac{1}{8\pi^2|\underline{x}|\mu} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi \left\{ \delta_{km} - \frac{\lambda+\mu}{\lambda+2\mu} z_m z_k \right\} \delta(\cos\varphi)$$

3/9/79

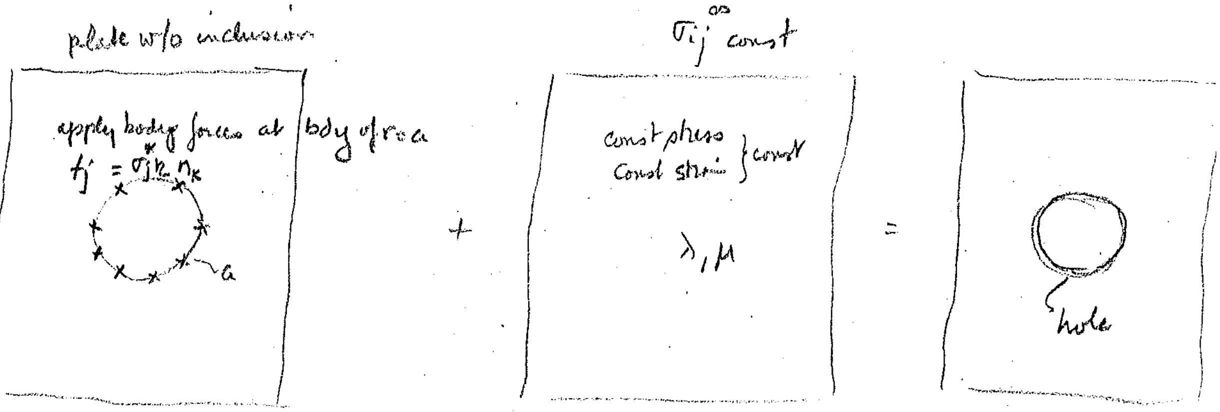


can be solved by superposition

messy

can use green's fn.

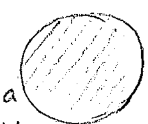
∞ plate w/ inclusion and stresses applied on bdy.



σ_{ij}, e_{ij} are const
u_i cont } on r=a
T_i discont

$\sigma_{ij}, \varepsilon_{ij}$ const
u_i contin on r=a
T_i disc on r=a

add to this an inclusion



∋. u_i = cont on r=a

we can set the problem ∋. we get the soln needed.

— o — o — o —

Back to green's fn.

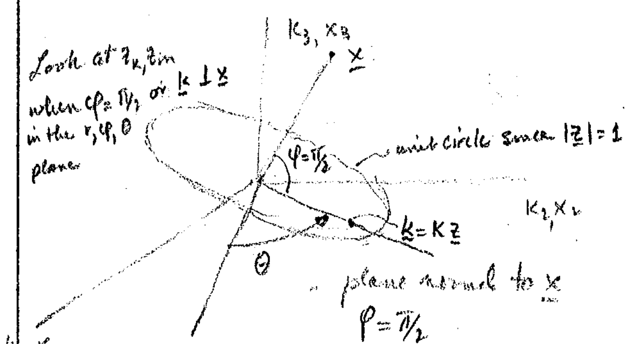
$$G_{km}(\underline{x}) = \frac{1}{8\pi^2|\underline{x}|}\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi \;\frac{1}{\mu}\left\{\delta_{km} - \frac{\lambda+\mu}{\lambda+2\mu} z_k z_m\right\}\delta(\cos\varphi)$$

value at $\varphi = \pi/2$

$$\delta(\cos\varphi) = \begin{cases} 1 & \cos\varphi = 0 \\ 0 & \cos\varphi \neq 0 \end{cases} \qquad \therefore\ \varphi = \pi/2$$

$$\mu\frac{1}{8\pi^2|\underline{x}|}\int_0^{2\pi} d\theta\left[\delta_{km} - \frac{\lambda+\mu}{\lambda+2\mu} z_k z_m\right]_{\varphi=\pi/2}$$

Look at z_k, z_m when $\varphi = \pi/2$, or $\underline{k} \perp \underline{x}$ in the r, φ, θ plane



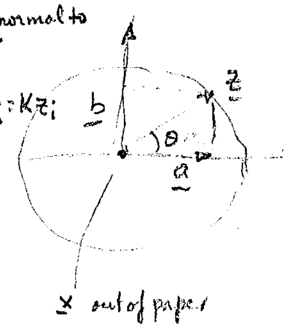
unit vectors in plane normal to $\underline{x}$

$\underline{z} = \underline{a}\cos\theta + \underline{b}\sin\theta$ but $k_i = K z_i$

$z_k = a_k\cos\theta + b_k\sin\theta$

$z_m = a_m\cos\theta + b_m\sin\theta$

where $|a_i|, |b_i| < 1$



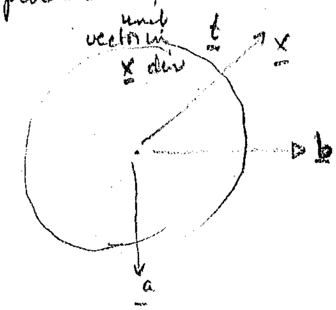
$$= \frac{1}{|\underline{x}|\,8\pi^2\mu}\left\{\delta_{km}\cdot 2\pi - \frac{\lambda+\mu}{\lambda+2\mu}\int_0^{2\pi}\left[a_k a_m\cos^2\theta + b_m b_k\sin^2\theta + (a_k b_m + b_k a_m)\sin\theta\cos\theta\right]d\theta\right\}$$

($\int = \pi$, $\int = \pi$, $\int = 0$ since odd fn over interval)

$$G_{km} = \frac{\delta_{km}}{4\pi\mu|\underline{x}|} - \frac{\lambda+\mu}{\lambda+2\mu}\,\frac{1}{8\pi\mu|\underline{x}|}\left\{a_k a_m + b_k b_m\right\}$$

tensor cannot be written as vector dot product

$$a_k a_m + b_k b_m = \delta_{km} - t_k t_m = \delta_{km} - \frac{x_k x_m}{|\underline{x}|^2}$$



now $\lambda = \dfrac{2\mu\nu}{1-2\nu}$ put into whole thing algebrate and get

$$G_{km}(\underline{x}) = \frac{1}{16\mu(1-\nu)\pi}\left\{\frac{(3-4\nu)\delta_{km}}{|\underline{x}|} + \frac{x_k x_m}{|\underline{x}|^3}\right\}$$

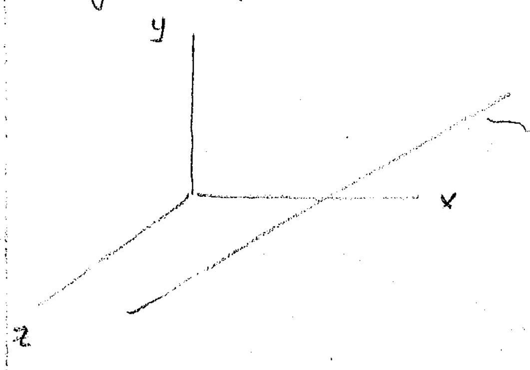
$$G_{km} = G_{mk} \; ; \; G_{km} \sim \frac{1}{|\underline{x}|}$$

for pt forces at $\underline{x}'$ then

3-D

$$G_{km}(\underline{x}-\underline{x}') = \frac{1}{16\mu(1-\nu)\pi}\left\{\frac{(3-4\nu)\delta_{km}}{|\underline{x}-\underline{x}'|} + \frac{(x_k - x_k')(x_m - x_m')}{|\underline{x}-\underline{x}'|^3}\right\}$$

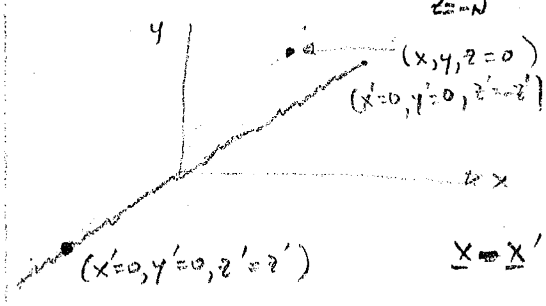
for 3-D problems use this in integral (BIE)
for 2-D problems use line force soln to be derived : use 3-D Green's fn.



// to z axis line of force can get by $\int_{-\infty}^{\infty} G_{km}\, dz$

Define 2-D Green's functions let $\underline{x}' = 0$

$$g_{km} = \lim_{N\to\infty} \frac{1}{16\pi\mu(1-\nu)} \int_{z=-N}^{z=N} \left\{\frac{3-4\nu\ \delta_{km}}{|\underline{x}|} + \frac{(x_k x_m)}{|\underline{x}|^3}\right\} dz'$$



displ at (x,y) (will be in z=0 plane due to sym due to g_{km} at $(0,0,z')$)

$$\underline{x} - \underline{x}' = x\underline{e}_x + y\underline{e}_z - z'\underline{e}_z \qquad |\underline{x}-\underline{x}'| = \sqrt{x^2+y^2+(z')^2}$$

only need $k, m = 1, 2$ since we'll get plane soln.

$$\int_{-N}^{N} \frac{dz' \,(3-4\nu)\delta_{km}}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \quad \text{is of form} \quad \int_{-N}^{N} \frac{dz'}{[x^2+y^2+(z'^2)]^{1/2}} = \ln\left\{ z' + \sqrt{x^2+y^2+z'^2} \right\}_{-N}^{N}$$

let $x^2+y^2 = \rho^2$

$$= \ln \frac{N+\sqrt{\rho^2+N^2}}{-N+\sqrt{\rho^2+N^2}} = \ln \frac{\sqrt{1+\rho^2/N^2}+1}{\sqrt{1+\rho^2/N^2}-1} = \ln \frac{2+\rho^2/N^2+2\sqrt{1+\rho^2/N^2}}{\rho^2/N^2}$$

clear denom (in comparison to)

$$\approx \ln \frac{4N^2}{\rho^2} = 2(\ln 2N - \ln \rho)$$

as $N \to \infty$

for our case all $\ln N$ gives a rigid body translation thus we can drop it (doesn't give stresses)

$\therefore$ we get $-(3-4\nu)\ln(x^2+y^2)\,\delta_{km}$ for 1st part

$$\frac{1}{16\pi\mu(1-\nu)} \int_{-N}^{N} \frac{(x_k - x_k')(x_m - x_m')\,dz'}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}$$

when $k=m=1$: x^2
$k=m=2$: y^2
$k=1, m=2$; $k=2, m=1$: xy
are not fns of z'

do integ for $x_k', x_m' = 0$

$$\frac{x_k x_m}{16\pi\mu(1-\nu)} \int_{-\infty}^{\infty} \frac{dz'}{(\rho^2+z'^2)^{3/2}} \quad \text{of form} \quad \left.\frac{1}{\rho^2}\frac{z'}{(z'^2+\rho^2)^{1/2}}\right|_{z'\to-\infty}^{z'\to\infty} = \frac{2}{\rho^2}$$

putting this into the original integral and generalize the results to a pt located at x', y'

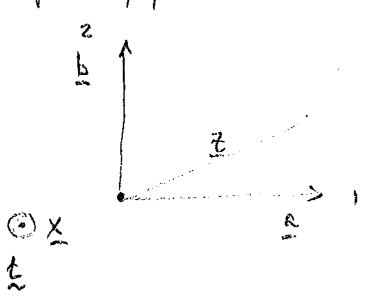
this is plane strain soln w/ $k, m = 1, 2$

$$g_{km}(\underset{\sim}{x}, \underset{\sim}{x}') = \frac{1}{8\pi\mu(1-\nu)} \left\{ -(3-4\nu)\ln|\underset{\sim}{x} - \underset{\sim}{x}'|\,\delta_{km} + \frac{(x_k - x_k')(x_m - x_m')}{|\underset{\sim}{x} - \underset{\sim}{x}'|^2} \right\}$$

where $|\underset{\sim}{x} - \underset{\sim}{x}'| = \sqrt{(x-x')^2+(y-y')^2}$

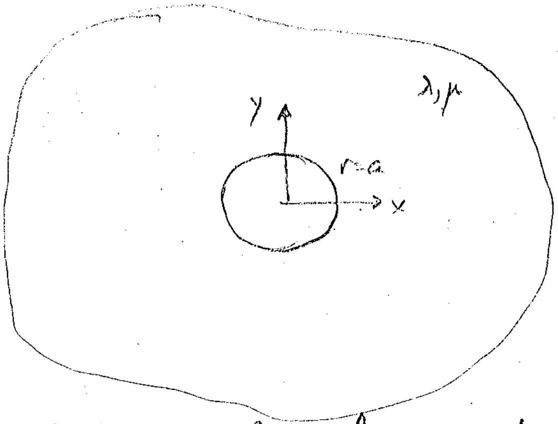
3/12/79

Up to 3/5/79 to include for couples, BIE Final.



tensor relation

$a_k a_m + b_k b_m = \delta_{km} - t_k t_m$ in x_1, x_2, x_3 space



Distribute body forces (line forces)
$f_j = \sigma^*_{jk} n_k$ on $r=a$
some ↑ constant stress fields

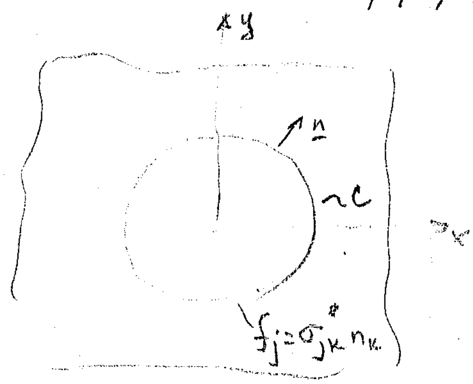
want to find what u_i's are inside & outside.

See handout. from here on (or Circular Inclusion Problems)

3/14/79

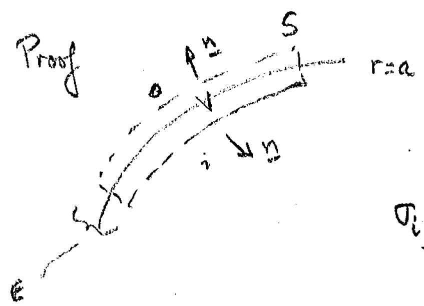
Solution to Problem #1

$u_i(\underline{x})$
$\varepsilon_{ip}(\underline{x})$ } as $r \to \infty$
$\sigma_{ij}(\underline{x})$



$\sigma_{ij}(\underline{x}) \to 0$ as $r \to \infty$
$u_i(\underline{x})$ is contin across $r=a$
$T_i(\underline{x})$ discontin across $r=a$

Proof



$$\int_V f_i dV + \int_S \sigma_{ij} n_j dS = 0 \quad \text{for equil.}$$

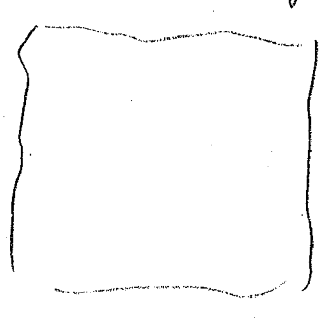
$$\sigma^o_{ij} \cdot n^o_j + \sigma^i_{ij} n^i_j + f_i = 0 \qquad n^o_j = -n^i_j$$

$$\therefore \quad (\sigma^o_{ij} - \sigma^i_{ij}) n^o_j + \sigma^*_{ij} n^o_j = 0$$

$$(T^o_i - T^{in}_i) + \sigma^*_{ij} n_j = 0$$

Go to pg 30 of handout on Circular Inclusion

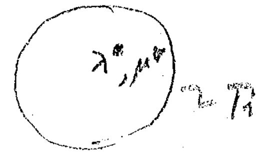
Problem #2



define displ (automatically satisfy compat)
⇒ the stresses at ∞ for original problem are satisfied

Problem #3

Take a cylinder C^* with radius $r \le a$ and elastic const λ^*, μ^*
give C^* the displ: $U_i(\underline{x})$: $r \le a$



$\mathcal{E}_{ij}(\underline{x})$ are the strains due to $U_i(\underline{x})$ = const.
but stresses are $\Sigma^*_{ij} = C^*_{ijkl}\, \mathcal{E}_{kl}$ = const

also $\Sigma^*_{ij,j} = 0$ since constant state of stress satisfies equil ; on boundary $\Sigma^*_{ij} n_j = T_i$

if we replace C by C^* we're OK as far as the displ are concerned HOWEVER
if I want tractions to be continuous across $r = a$ when C^* replaces C
or
$(\Sigma^0_{ij} - \Sigma^*_{ij})\, n_j = 0$ ∴ we must make sure that $\Sigma^{*\,in}_{ij} = \Sigma^{in}_{ij} - \sigma^*_{ij}$ (†)
const. const const

ordinarily we can satisfy u_i continuous however it is the fact that we are dealing w/constant states of stress that allows us to solve the problem.

write out
(†) $(C^*_{ijkl} - C_{ijkl})(e_{kl} + \hat{e}_{kl}) = -\sigma^*_{ij}$ $\qquad e_{ip}(r \le a) = \dfrac{\sigma^*_{jk}}{16\mu(1-\nu)}\{F_{ipjk}\}$ const.

$(\underbrace{\qquad\qquad}_{\text{known}})(F_{klmn}\sigma^*_{mn} + \underline{E_{kl}}) = -\sigma^*_{ij}$

const & known

unknown

gives 6 eqns. in 6 unknowns σ^*_{mn}

Paper on elastic inclusions & Inhomogeneities
J.D. Eshelby
Progress in Solid Mechanics, 1961

3/16/79

Final Exam in Philosophy Corner 104 & 107 Meet in 104 on Wed 8:30–11:3
Open Notes, Timo & Goodier & Table of Integrals
3 problems.

no bie on final should take $\approx 1\frac{1}{2} - 2$ hr.

series representation for $\ln\sqrt{a^2+r^2-2ar\cos(\theta-\theta')} = \frac{1}{2}\ln\left[(ae^{i\theta'}-re^{i\theta'})(ae^{-i\theta'}-re^{-i\theta})\right]$

$= \frac{1}{2}\ln\left[a^2\left(1-\frac{r}{a}e^{i(\theta-\theta')}\right)\left(1-\frac{r}{a}e^{-i(\theta-\theta')}\right)\right]$

$= \ln a + \frac{1}{2}\ln\left(1-\frac{r}{a}e^{i(\theta-\theta')}\right) + \frac{1}{2}\ln\left(1-\frac{r}{a}e^{-i(\theta-\theta')}\right)$

$\ln(1-x)$; $|x|<1$ let $z = r/a\, e^{i(\theta-\theta')}$

$|z|<1 \Rightarrow r/a < 1$

$$\ln(1-z) = -\sum_{m=1}^{\infty}\frac{z^m}{m}$$

$$S = \ln a + \frac{1}{2}\left[-\sum_{m=1}^{\infty}(r/a)\frac{1}{m}e^{i(\theta-\theta')m} - \sum(r/a)\frac{1}{m}e^{-i(\theta-\theta')m}\right]$$

$$= \ln a - \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}(r/a)^m\left\{e^{im(\theta-\theta')} + e^{-im(\theta-\theta')}\right\}$$

$2\cos(\theta-\theta')m$

ME 238 B — THEORY OF ELASTICITY

HOMEWORK # 1 — DUE MONDAY, JAN. 22, 1979

1. FOR THE PROBLEM OF SIMPLE EXTENSION CONSIDERED IN CLASS WE SHOWED

$$u_x = -\frac{\nu P}{EA_0} x + f(y,z) \; ; \quad u_y = -\frac{\nu P}{EA_0} y + g(x,z) \; ; \quad u_z = -\frac{P}{EA_0} z + h(x,y)$$

WHERE

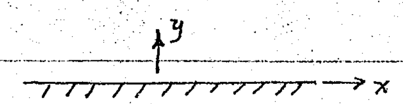
$$\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} = \frac{\partial g}{\partial z} + \frac{\partial h}{\partial y} = \frac{\partial f}{\partial z} + \frac{\partial h}{\partial x} \equiv 0.$$

FIND f, g, AND h.

2. IN TIMOSHENKO AND GOODIER, READ PGS. 235–239 AND 274–277.

FILL IN THE STEPS OF THE DISCUSSION IN PGS. 274–277 CONCERNING THE APPROXIMATE CHARACTER OF PLANE STRESS SOLUTIONS.

3. CONSIDER THE HALF-SPACE $y < 0$, $|x| < \infty$



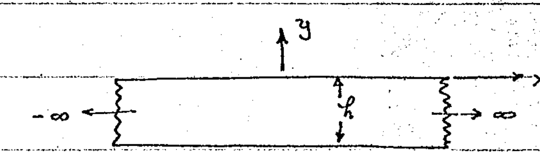
WITH THE TRACTION BOUNDARY CONDITIONS

$$\sigma_{yy} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \quad \text{ON } y = 0$$

$$\sigma_{xy} = 0 \quad \text{ON } y = 0.$$

SHOW THAT THE RECTANGLE $-L \le x \le L$, $-y_0 < y \le 0$ IS IN FORCE AND MOMENT EQUILIBRIUM AS $y_0 \to \infty$.

4. SOLVE FOR THE STRESS DISTRIBUTION IN THE SEMI-INFINITE STRIP $-\infty < x < \infty$, $0 \ge y \ge -h$, SUBJECTED TO THE FOLLOWING BOUNDARY CONDITIONS:



$$\sigma_{yy}(x, y=0) = \sigma_{yy}(x, y=-h) = 0$$

$$\sigma_{xy}(x, y=0) = 0 \; ; \quad \sigma_{xy}(x, y=-h) = 3 \sin \frac{\pi x}{L}$$

$g_{xz} + f_{yz} = 0$
$h_{yx} + g_{zx} = 0$ } $h_{yx} - f_{yz} = 0 \Rightarrow h_{yx} = f_{yz}$
$f_{yz} + h_{xy} = 0$

or h_{x}...
or g_{xz}...

$e_{xy} = 0 \Rightarrow \frac{\partial}{\partial z}\frac{\partial g(x,z)}{\partial x} + \frac{\partial}{\partial y} f(y,z) = 0$

$e_{yz} = 0 \Rightarrow \frac{\partial}{\partial x}\frac{\partial h(y,x)}{\partial y} + \frac{\partial g(x,z)}{\partial z} = 0$

$e_{zx} = 0 \Rightarrow \frac{\partial}{\partial y}\frac{\partial f(y,z)}{\partial z} + \frac{\partial h(x,y)}{\partial x} = 0$

take $\frac{\partial e_{xy}}{\partial z}$, $\frac{\partial e_{yz}}{\partial x}$, $\frac{\partial e_{zx}}{\partial y} = 0$ shuffle to finally get $\frac{\partial^2 f(y,z)}{\partial y \partial z} = 0$

$\frac{\partial f}{\partial y} = f_1'(y) \Rightarrow f = f_1(y) + f_2(z)$

this implies $f(y,z) = f_1(y) + f_2(z)$

also by similarity we can get $g(x,z) = g_1(x) + g_2(z)$ since $\frac{\partial^2 g(x,z)}{\partial x \partial z} = 0$

and $h(x,y) = h_1(x) + h_2(y)$ since $\frac{\partial^2 h(x,y)}{\partial x \partial y} = 0$

now for $e_{xy} \Rightarrow \frac{\partial u_x}{\partial y} = -\frac{\partial u_y}{\partial x} \Rightarrow \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x} = f_1'(y) = -g_1'(x) = c_1$

$\therefore f_1'(y) = c_1 \Rightarrow f_1(y) = c_1 y + c_2$

$g_1'(x) = -c_1 \Rightarrow g_1(x) = -c_1 x + c_3$

from $e_{yz} \Rightarrow \frac{\partial u_z}{\partial y} = -\frac{\partial u_y}{\partial z} \Rightarrow \frac{\partial h}{\partial y} = -\frac{\partial g}{\partial z} = h_2'(y) = -g_2'(z) = c_4$

$h_2'(y) = c_4 \Rightarrow h_2(y) = c_4 y + c_5$

$g_2'(z) = -c_4 \Rightarrow g_2(z) = -c_4 z + c_6$

from $e_{zx} \Rightarrow \frac{\partial u_z}{\partial x} = -\frac{\partial u_x}{\partial z} \Rightarrow \frac{\partial h}{\partial x} = -\frac{\partial f}{\partial z} = h_1'(x) = -f_2'(z) = c_7$

$h_1'(x) = c_7 \Rightarrow h_1(x) = c_7 x + c_8$

$f_2'(z) = -c_7 \Rightarrow f_2(z) = -c_7 z + c_9$

$\therefore f(y,z) = c_1 y - c_7 z + c_{11}$ $\quad c_{11} = c_2 + c_9$

$\therefore g(x,z) = g_1(x) + g_2(z) = -c_1 x - c_4 z + c_{10}$ $\quad c_{10} = c_3 + c_6$

$h(x,y) = h_1(x) + h_2(y) = c_7 x + c_4 y + c_{12}$ $\quad c_{12} = c_5 + c_8$

$\therefore u_x = -\frac{\nu P}{A_0 E} x + c_1 y - c_7 z + c_{11}$

$u_y = -\frac{\nu P}{A_0 E} y - c_1 x - c_4 z + c_{10}$

$u_z = \frac{P z}{A_0 E} + c_7 x + c_4 y + c_{12}$

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}(c_1 + c_1) = c_1$$

$$\omega_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) = \frac{1}{2}(-c_4 - c_4) = -c_4$$

$$\omega_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right) = \frac{1}{2}(c_7 + c_7) = c_7$$

$$\therefore \quad f(y,z) = \omega_{xy}\, y - \omega_{zx}\, z + c_{11}$$

$$g(x,z) = -\omega_{xy}\, x + \omega_{yz}\, z + c_{10}$$

$$h(x,y) = \omega_{zx}\, x - \omega_{yz}\, y + c_{12}$$

thus f, g, h are at most rotations + translation (rigid body rotations + rigid body translation)

$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq f(z)$

$\sigma_{zz} = 0 \quad \sigma_{zx} = 0 \quad \sigma_{zy} = 0$

from equil

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}^{0} = 0$$

$$\frac{\partial \sigma_{zx}}{\partial x}^{0} + \frac{\partial \sigma_{zy}}{\partial y}^{0} + \frac{\partial \sigma_{zz}}{\partial z}^{0} = 0$$

$\therefore$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \qquad \text{let } \sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

for compat if we assume $\sigma_{xx}, \sigma_{yy}, \sigma_{xy} \neq f(z)$

$$\epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] = f(x,y) = \frac{1}{E}\left[\phi_{yy} - \nu\phi_{xx}\right]$$

$$\epsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz}^{0})\right] = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] = \frac{1}{E}\left[\phi_{xx} - \nu\phi_{yy}\right]$$

$$\epsilon_{zz} = \frac{1}{E}\left[\sigma_{zz}^{0} - \nu(\sigma_{xx} + \sigma_{yy})\right] = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) = -\frac{\nu}{E}\left[\phi_{xx} + \phi_{yy}\right]$$

$$\epsilon_{xy} = \frac{\sigma_{xy}}{2G} = \frac{-1}{2G}\phi_{xy}$$

$$\epsilon_{xz} = 0$$

$$\epsilon_{yz} = 0$$

$$\psi_{xx} = -\frac{\partial^2}{\partial z^2}[\;] \qquad -\frac{\partial^2 \epsilon_{zz}}{\partial y^2} = 0 \Rightarrow +\frac{\nu}{E}\left[\frac{\partial^4 \phi}{\partial y^4} + \frac{\partial^4 \phi}{\partial x^2 \partial y^2}\right] = 0 \quad \frac{\partial^2}{\partial y^2}(\nabla^2 \phi) = 0$$

$$\psi_{yy} = +\frac{\nu}{E}\left[\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial x^4}\right] = 0 \leftarrow -\frac{\partial^2 \epsilon_{zz}}{\partial x^2} = \frac{\partial^2}{\partial x^2}(\nabla^2 \phi) = 0$$

$$\psi_{zz} = 2\left[-\frac{\partial^4 \phi}{\partial^2 x \partial y^2} \cdot \frac{1}{2G}\right] \frac{-1}{E}\left[\frac{\partial^4 \phi}{\partial y^4} - \nu\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial x^4} - \nu\frac{\partial^4 \phi}{\partial x^2 \partial y^2}\right]$$

$$-2\frac{(1+\nu)}{E} \qquad \frac{-1}{E}\left[\phi_{yyyy} - 2\nu\,\phi_{xxyy} + \phi_{xxxx}\right]$$

$$= -\frac{1}{E}\left[\nabla^4 \phi\right] = 0$$

$\frac{\partial \phi_1}{\partial x} = ay$ $\quad \phi_1 = axy + by$ $\quad \frac{\partial \phi_1}{\partial x} = ay$ $\quad \frac{\partial^2 \phi_1}{\partial x \partial y} = a \Rightarrow a = 0$

$\frac{\partial \phi_1}{\partial y} = b$ $\quad \frac{\partial^2 \phi_1}{\partial y^2} = 0$

$ax + by + c$ $\quad \phi_1 = ax + by + c$

$\frac{\partial \phi_1}{\partial x} = a$ $\quad \frac{\partial^2 \phi_1}{\partial x^2} = 0$ $\quad \frac{\partial \phi_1}{\partial y} = b$ $\quad \frac{\partial^2 \phi_1}{\partial y^2} = 0$

ϕ_1 $\quad \phi_1 = (ax + by + c)$

$\frac{\partial^2 \phi_1}{\partial x \partial y} = 0$ $\quad \frac{\partial \phi_1}{\partial x} = f'(x)$

$\phi_1 = f(x) + g(y)$

$\frac{\partial \phi_1}{\partial x} = f'$

$\frac{\partial^2 \phi_1}{\partial x^2} = f'' = 0$ $\quad f' = c_1$ $\quad f = c_1 x + c_2$

$\Rightarrow$ $\quad g = c_3 y + c_4$

$\therefore \phi_1 = c_1 x + c_3 y + c_5$

$$\psi_{yz} = \overset{\text{since not f. of } y}{0} - \frac{\partial}{\partial x}\left(\overset{0}{/} \quad \overset{0}{/} \quad \overset{0}{/} \right) = 0 \quad \text{since not f. of } z$$

$$\psi_{zx} = 0 - \frac{\partial}{\partial y}\left(0 + \overset{\text{not f. of } z}{\not{0}} \quad \not{0} \right) = 0$$

$$\psi_{xy} = -\frac{\nu}{E}\frac{\partial^2}{\partial x \partial y}(\nabla^2 \phi) - 0 \quad \text{since not f. of } z = 0$$

only if $\nabla^2\phi = 0$ and $\nabla^4\phi = 0$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{1}{E}(\phi_{yy} - \nu\phi_{xx}) \quad \therefore u_x = \int(\phi_{yy} - \nu\phi_{xx})dx + f(y,z) = (1+\nu)\int \phi_{yy}\,dx + f_1(y,z)$$

$\phi_{yy} - \nu\phi_{xx} + \nu\phi_{yy} - \nu\phi_{xx}$

$$\epsilon_{yy} = \frac{\partial u_y}{\partial y} = \frac{1}{E}[\phi_{xx} - \nu\phi_{yy}] \quad \therefore u_y = \int(\phi_{xx} - \nu\phi_{yy})dy + f(x,z) = (1+\nu)\int\phi_{xx}\,dy + f_2$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} = \frac{-\nu}{E}[\phi_{xx} + \phi_{yy}] \quad \therefore u_z = \frac{-\nu}{E}\int(\phi_{xx} + \phi_{yy})dz + f(x,y)$$

$$= \frac{-\nu}{E}(\phi_{xx} + \phi_{yy})z + f_3(x,y)$$

"0 by comp solution

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) =$$

$$\epsilon_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{2}\frac{\partial f_1}{\partial z} + \frac{1}{2}\frac{\partial f_3}{\partial x} = -\frac{1}{2G}(\phi_{xy}) \Rightarrow \frac{\partial f_1}{\partial z} = 0 \text{ or } f_1 = f(y)$$

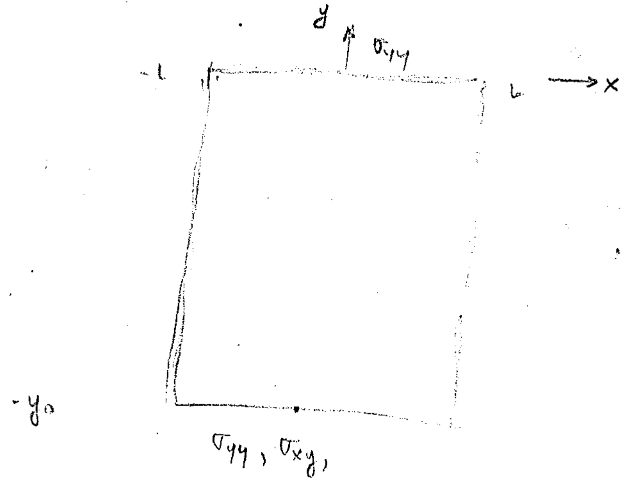
$$\text{or} \quad f_3 = -\frac{1}{G}\int \phi_{xy}\,dx + f_{3x}(y)$$

$$\epsilon_{yz} = 0 = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2}\frac{\partial f_2}{\partial z} + \frac{\partial f_3}{\partial y}$$

since $\Theta = \sigma_x + \sigma_y = \Theta_0 + kz = f(x,y) + kz \Rightarrow \Theta = (x,y,z)$

but since ~~compat~~ equil says $\frac{\partial^2\phi}{\partial x^2} = \sigma_y \quad \frac{\partial^2\phi}{\partial y^2} = \sigma_x$

$\therefore \sigma_x + \sigma_y = \nabla_1^2\phi = \Theta \leftarrow \Theta(x,y,z) \quad \therefore \phi$ must also be a fn $f(x,y,z)$



Mechanical Engineer's

- Goldsmith: Berkeley US national congress at UCLA
- Homah: SRI (Menlo Park)

This problem had the following stress fn.

$$\phi(x,y) = -\sum_{n=1}^{\infty} \frac{A_n}{\gamma_n^2} \sin \gamma_n x \, (1-\gamma_n y) e^{\gamma_n y} \quad \text{where } \gamma_n = \frac{n\pi}{L}$$

$$\sigma_{yy}(x, y=0) = \phi_{,xx}(x,0) = \sum_{n=1}^{\infty} A_n \sin \gamma_n x \, (1-\gamma_n y) e^{\gamma_n y} \quad (1-\gamma_n y)\gamma_n e^{\gamma_n y} + e^{\gamma_n y}(-\gamma_n) = -\gamma_n^2 y$$

$$\frac{\partial \phi}{\partial y} = -\sum \frac{A_n}{\gamma_n^2} \sin \gamma_n x \left[\gamma_n e^{\gamma_n y} - \gamma_n e^{\gamma_n y} - \gamma_n^2 y e^{\gamma_n y}\right] = +\sum \frac{A_n}{\gamma_n^2} \sin \gamma_n x \cdot \gamma_n^2 y e^{\gamma_n y} = \sum A_n \sin \gamma_n x \left(y e^{\gamma_n y}\right)$$

$$\frac{\partial^2 \phi}{\partial^2 y} = +\sum A_n \sin \gamma_n x \left[e^{\gamma_n y} + \gamma_n y e^{\gamma_n y}\right] \Rightarrow \sigma_{xx}\Big|_{y=0} = \sum A_n \sin \gamma_n x$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = \sum_{n=1}^{\infty} (A_n \gamma_n \cos \gamma_n x) \, y e^{\gamma_n y} \Rightarrow \sigma_{xy}\Big|_{y=0} = 0$$

Thus

$$F_{yy}\Big|_{y=0} = \int_{-L}^{L} \sigma_{yy} \, dx = \int_{-L}^{L} \sum A_n \sin \gamma_n x \, dx = \sum A_n \int_{-L}^{L} \sin \gamma_n x \, dx = \sum \frac{A_n}{\gamma_n} (-\cos \gamma_n x)\Big|_{-L}^{L}$$

$$= -\sum \frac{A_n}{\gamma_n} (\cos n\pi - \cos n\pi) = 0$$

$$F_{xy}\Big|_{y=0} = \int_{-L}^{L} \sigma_{xy} \, dx = \int_{-L}^{L} 0 \cdot dx = 0$$

on $y = -y_0$

$$\sigma_{yy} = \sum A_n \sin \gamma_n x \, (1+\gamma_n y_0) e^{-\gamma_n y_0}$$

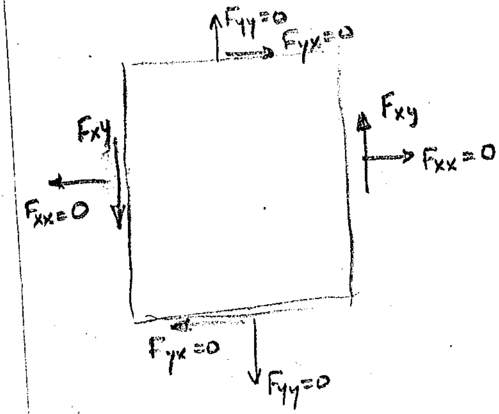
$$F_{yy}\Big|_{y=0} = \int_{-L}^{L} \sigma_{yy} \, dx = \sum A_n (1+\gamma_n y_0) e^{-\gamma_n y_0} \int_{-L}^{L} \sin \gamma_n x \, dx = 0$$

$\frac{1}{\gamma_n}\left(1 - y_0\gamma_n e^{-\gamma_n y_0} + y_0\gamma_n e^{-\gamma_n y_0} - e^{-\gamma_n y_0}\right) - \frac{y_0\gamma_n}{\gamma_n} e^{-\gamma_n y_0}$

$1 - (y_0\gamma_n + 1) e^{-\gamma_n y_0}$

$$F_{xy}\Big|_{y=-y_0} = +\sum A_n \gamma_n e^{-\gamma_n y_0} y_0 \int_{-L}^{L} \cos\gamma_n x \, dx = \sum \left(+A_n e^{-\gamma_n y_0} y_0\right) \sin\gamma_n x \Big|_{-L}^{L}$$

$$= \sum + A_n e^{-\gamma_n y_0} y_0 \left\{ \sin n\pi - \sin(-n\pi) \right\} = \sum + A_n 2 y_0 e^{-\gamma_n y_0} \sin n\pi = 0$$



$$F_{xx}\Big|_{x=L} = \int_{-y_0}^{0} \sigma_{xx} dy = \sum A_n \sin n\pi \int_{-y_0}^{0} e^{\gamma_n y}(1+\gamma_n y)\, dy = \sum A_n \sin n\pi \left\{ (1+\gamma_n y)\frac{e^{\gamma_n y}}{\gamma_n} - \frac{e^{\gamma_n y}}{\gamma_n} \right\}_{-y_0}^{0}$$

$$= \sum A_n \sin n\pi \left\{ +y_0 e^{-\gamma_n y_0} \right\} \quad \text{but } \sin n\pi = 0 \ \forall n$$

$$F_{xx}\Big|_{x=L} = 0$$

$$F_{xy}\Big|_{x=L} = \int_{-y_0}^{0} \sigma_{xy} dy = -\sum A_n \gamma_n \cos n\pi \int_{-y_0}^{0} y e^{\gamma_n y} dy = -\sum A_n \gamma_n \cos n\pi \left[y\frac{e^{\gamma_n y}}{\gamma_n} - \frac{1}{\gamma_n^2} e^{\gamma_n y} \right]_{-y_0}^{0}$$

$$-\sum A_n \cos n\pi \left[\left(-\frac{1}{\gamma_n}\right) + y_0 e^{-\gamma_n y_0} + \frac{1}{\gamma_n} e^{-\gamma_n y_0} \right] = \sum_{n=1} A_n \cos n\pi \left\{ (1 - e^{-\gamma_n y_0})\frac{1}{\gamma_n} - y_0 e^{-\gamma_n y_0} \right\}$$

$$F_{xx}\Big|_{x=-L} = \int_{-y_0}^{0} \sigma_{xx} dy = -\sum A_n \sin n\pi \left\{ y_0 e^{-\gamma_n y_0} \right\} = 0 \ (\text{since } \sin n\pi = 0) = F_{xx}\Big|_{x=L}$$

$$F_{xy}\Big|_{x=-L} = \int_{-y_0}^{0} \sigma_{xy} dy = +\sum A_n \cos(-n\pi) \left\{ \frac{1}{\gamma_n}(1 - e^{-\gamma_n y_0}) - y_0 e^{-y_0\gamma_n} \right\}$$
$$= F_{xy}\Big|_{x=L}$$

$$\frac{1}{\gamma_n}(1 - e^{-\gamma_n y_0}) = -\frac{1}{\gamma_n}\sum_{n=1} \frac{(-\gamma_n y_0)^n}{n!} =$$

$1 - 1 + \gamma_n y_0 + \frac{(\gamma_n y_0)^2}{2!} + \frac{(\gamma_n y_0)^3}{3!} \ldots = \gamma_n y_0 - \frac{(\gamma_n y_0)^2}{2!} + \frac{(\gamma_n y_0)^3}{3!}$

$$\therefore \frac{1}{\gamma}\sum \frac{(\gamma_n y_0)^j (-1)^{j+1}}{j!} = \sum_{j=1}^{\infty} \frac{\gamma_n^{j-1} y_0^j}{j!} (-1)^{j-1}$$

$$y_0 \sum_{j=0}^{\infty} \frac{(\gamma_n)^j}{j!} (y_0)^j (-1)^j$$

let $m = j+1$, $j = m-1$

$$= \sum_{j=0} \frac{(\gamma_n)^{j} y_0^{j+1}}{j!} (-1)^j$$

$$= \sum_{m=1} \frac{(\gamma_n)^{m-1} y_0^m}{(m-1)!} (-1)^{m-1}$$

$$\lim_{y_0\to\infty} F_{yy} = \lim_{y_0\to\infty} \sum_{n=1}^{\infty} -\frac{A_n}{\gamma_n}(1+\gamma_n y_0)e^{-\gamma_n y_0}\left[\underbrace{\cos n\pi - \cos(-n\pi)}_{0}\right] = \sum \lim_{y_0\to\infty} = 0$$

$$F_{xy} = \lim_{y_0\to\infty} \sum A_n e^{-\gamma_n y_0} y_0 \left\{\underbrace{\sin n\pi - \sin n\pi}_{0}\right\} = \sum \lim_{y_0\to\infty} = 0$$

$$\lim_{y_0\to\infty} F_{xx}\Big|_{x=L} = \lim_{y_0\to\infty} \sum A_n \underbrace{\sin n\pi}_{0} \left\{y_0 e^{-\gamma_n y_0}\right\} = \sum \lim_{y_0\to\infty} A_n \sin n\pi \left\{y_0 e^{-\gamma_n y_0}\right\} = 0$$

$$\lim_{y_0\to\infty} F_{xy}\Big|_{x=L} = \lim_{y_0\to\infty} \sum A_n \cos n\pi \left\{(1-e^{-\gamma_n y_0})\frac{1}{\gamma_n} - y_0 e^{-y_0\gamma_n}\right\}$$

now $\frac{1}{\gamma_n}(1-e^{-\gamma_n y_0}) - y_0 e^{-y_0\gamma_n} = \sum_{j=1}^{\infty} \gamma_n^{j-1} y_0^{j} (-1)^{j-1}\left[\frac{1}{j!} - \frac{1}{(j-1)!}\right] = \sum_{j=1}^{\infty} \frac{\gamma_n^{j-1} y_0^{j} (-1)^{j-1}}{j!}(1-j)$

this series converges since the ratio test $\Rightarrow \left|\frac{a_{j+1}}{a_j}\right| = \gamma_n y_0 (-1) \frac{(-j)}{(j+1)(1-j)} = \frac{j}{(j+1)(1-j)}\gamma_n y_0 \to 0$ as $j\to\infty$

now $\sum_{n=1}^{\infty} A_n \cos n\pi \{ \quad \}$

~~$A_n L \int \cos \gamma_n x$~~ $\quad A_n L \int \sin \quad = -A_n L \left.\frac{\cos\gamma_n x}{\gamma_n}\right|_{-L}^{L}$

$\frac{-L}{\gamma_n}$

$$\sum A_n (-1)\frac{L}{\pi} + \frac{A_2}{2}\frac{L}{\pi} + \cdots \frac{A_2}{n}\frac{L}{\pi}$$

$$\frac{L}{\pi}\sum A_n \frac{(-1)}{n}$$

$$\phi_n = \sum \sin \left(\alpha_n e^{\gamma_n y} + \beta_n e^{-\gamma_n y} + C_n y e^{\gamma_n y} + D_n y^{-\gamma_n y}\right)$$

$$y\gamma_n e^{\gamma_n y} + e^{\gamma_n y} \quad - \gamma_n y e^{-\gamma_n y} + e^{-\gamma_n y}$$

on $y = 0$ $\quad \sigma_{yy} = \sum A_n \sin \quad \sigma_{xy} = 0$

$y = -y_0$ $\quad \sigma_{yy} = 0 \quad \sigma_{xy} = 0$

$$-\frac{\partial^2 \phi}{\partial x \partial y} = -\sum \cos \gamma_n x \; \gamma_n \{\alpha_n \gamma_n + \gamma_n \beta_n + C_n + D_n\} = 0$$

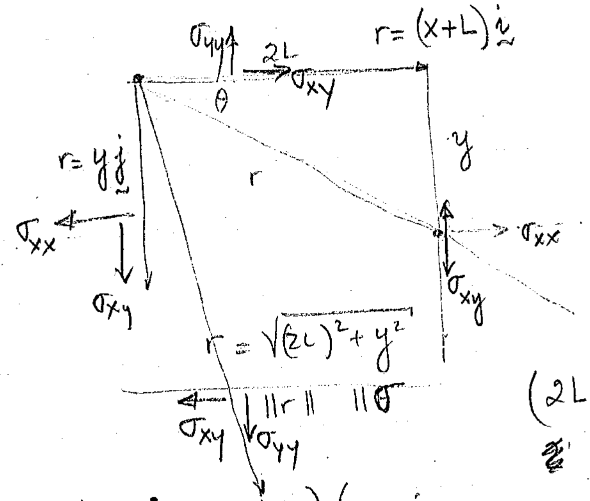
$$\Rightarrow \quad (\alpha_n - \beta_n)\gamma_n + C_n + D_n = 0$$

$1 - e^{-\gamma_n y_0} \longrightarrow 1$ as $y_0 \to \infty$ $\qquad \frac{1}{\gamma_n}$

$y_0 e^{-\gamma_n y_0} \longrightarrow 0$ as $y_0 \to \infty$

$$+\frac{y_0 e^{-\gamma_n y_0}}{\gamma_n} - \frac{1}{\gamma_n^2} e^{\gamma_n y}\Big|_{-y_0}^{0}$$

$$-\frac{1}{\gamma_n^2} + \frac{1}{\gamma_n^2} e^{-\gamma_n y_0}$$



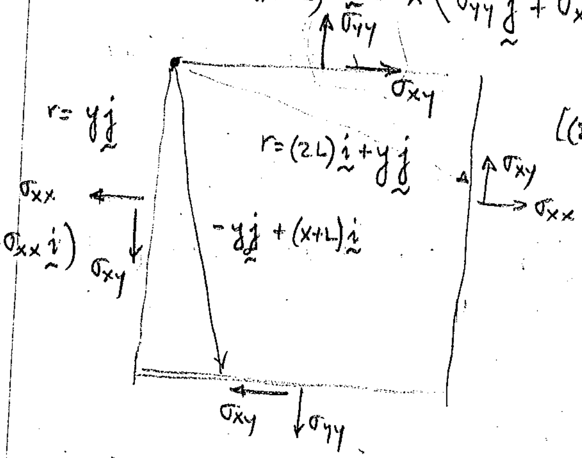
$$(2L\,\underset{\sim}{i} + y\,\underset{\sim}{j}) \times (\sigma_{xx}\,\underset{\sim}{i} + \sigma_{xy}\,\underset{\sim}{j})$$

$$2L\sigma_{xy}\,\underset{\sim}{k} - y\sigma_{xx}\,\underset{\sim}{k}$$

$$(-y_0\,\underset{\sim}{j} + (x+L)\,\underset{\sim}{i}) \times (-\sigma_{yy}\,\underset{\sim}{i} +$$

$$dM = \underset{\sim}{r} \times \underset{\sim}{\sigma}\, dA$$

$$(x+L)\underset{\sim}{i} \times (\sigma_{yy}\underset{\sim}{j} + \sigma_{xy}\underset{\sim}{i}) = (x+L)\sigma_{yy}\underset{\sim}{k}$$



$$[(2L)\underset{\sim}{i} + y\underset{\sim}{j}] \times (\sigma_{xx}\underset{\sim}{i} + \sigma_{xy}\underset{\sim}{j}) = 2L\sigma_{xy}\underset{\sim}{k} - y\sigma_{xx}\underset{\sim}{k}$$

$$-y\underset{\sim}{j} \times (\sigma_{xy}\underset{\sim}{j} + \sigma_{xx}\underset{\sim}{i})$$
$$+ y\sigma_{xx}\underset{\sim}{k}$$

$$[(x+L)\underset{\sim}{i} - y_0\underset{\sim}{j}] \times -(\sigma_{yy}\underset{\sim}{i} + \sigma_{xy}\underset{\sim}{j}) = (x+L)\sigma_{xy}\underset{\sim}{k} - y_0\sigma_{yy}\underset{\sim}{k}$$

on upper surface $\quad M_{TOP} = \int_{-L}^{L} (x+L)\sigma_{yy}\,dx = \sum A_n \int_{-L}^{L} \underset{u}{(x+L)}\, \underset{dv}{\sin\gamma_n x\, dx} = \sum \frac{2L}{\gamma_n} A_n (-1)^{n+1}$

$du = dx \qquad v = -\frac{\cos\gamma_n x}{\gamma_n}$

$$M_{TOP} = \left[-(x+L)\frac{\cos\gamma_n x}{\gamma_n}\Big|_{-L}^{L} + \frac{\sin\gamma_n x}{\gamma_n^2}\Big|_{-L}^{L} \right]$$
$$-2L\frac{(-1)^n}{\gamma_n}$$

on $x = L$ $\quad 2L\int_{-L}^{L}\sigma_{xy}\,dy = \sum_{n=1}^{\infty} 2L A_n (-1)^n \left\{ \frac{1}{\gamma_n}(1 - e^{-\gamma_n y_0}) - y_0 e^{-\gamma_n y_0} \right\}$

$$\lim_{y_0 \to \infty} = \sum_{n=1}^{\infty} \frac{2L A_n}{\gamma_n}(-1)^n = M_{RHS}$$

$$\int y\sigma_{xx}\,dy = \sum A_n \sin\gamma_n L = 0$$

on $y = -y_0$ $\quad = \int_{-L}^{L} (x+L)\sum_{n=1}^{\infty} A_n\gamma_n \cos(\gamma_n x)\, y_0 e^{-\gamma_n y_0}\,dx = \sum A_n\gamma_n y_0 e^{-\gamma_n y_0} \int_{-L}^{L} \underset{u}{(x+L)}\,\underset{dv}{\cos\gamma_n x\,dx}$

$du = dx \qquad \frac{\sin\gamma_n x}{\gamma_n} = v$

$$\therefore \int_{-L}^{L}(x+L)\sigma_{xy}\,dx = 0$$

$$\left[(x+L)\frac{\sin\gamma_n x}{\gamma_n}\Big|_{-L}^{L} + \int \frac{\cos\gamma_n x}{\gamma_n}\Big|_{-L}^{L} \right.$$

$2L$

$$y_0\int_{-L}^{L}\sigma_{yy}\,dx = 0 \qquad M_{LHS} = 0$$

$$\frac{\cos\gamma_n L}{\gamma_n} - \frac{\cos\gamma_n L}{\gamma_n}$$

on $x = -L$ $\quad \int_{-y_0}^{0} -\sum A_n \sin n\pi\, (1 + \gamma_n y) e^{\gamma_n y}\, y\,dy = 0$

$\sigma_{yy} = \sum -\gamma^2$

$\sigma_{xy} = \sum \left[\left(A_n + C_n y + \frac{D_n}{\gamma_n}\right)\cosh \gamma_n y + \gamma\left(\frac{C_n}{\gamma_n} - A_n\gamma_n^3 y\right)\sinh \gamma_n y\right]\sin \gamma x$

$\frac{1}{\sinh^2\gamma h - \gamma^2 h^2}\sum \Big[3\gamma(\gamma h\cosh\gamma h - \sinh\gamma h)\,y\cosh\gamma y + \Big[3(\gamma_1 h\cosh\gamma_1 h - \sinh\gamma_1 h) + 3h\sinh\gamma_1 h(\gamma_1^2)y\Big]\sinh$

$3\gamma^2 hy\cosh(y+h) \quad \{-3\gamma y\sinh\cosh y + 3\gamma h\cos\sin y$

$3\gamma y h[\cosh\gamma h\cos\gamma y + \ldots \sin\sin \qquad -3\gamma y\cos\ y + 3\gamma\,y\cos\sin y$

$-3\gamma y[sc + cs] + 3\gamma\sin\cos(y+h)$

$-3\gamma y[\sinh(y+h)] + 3\gamma(y+h)\cosh\gamma h\sinh\gamma y - 3\sinh\gamma_1 h\sinh\gamma_1 y$

$3\gamma_1^2 yh\cosh\gamma_1 h\cosh\gamma_1 y + 3\gamma_1(y+h)\cosh\gamma_1 h\sinh\gamma_1 y - 3\gamma_1 y\sinh\gamma_1(y+h) - 3\sinh\gamma_1 h\sinh\gamma_1 y$

$3\gamma_1^2[-h^2\cos^2\quad] + 3\sinh^2$

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$\frac{\partial\phi}{\partial y} = \sum -\gamma_n\left[(A_n + C_n y)\cosh\gamma y + \left(\frac{C_n}{\gamma_n} + D_n y\right)\sinh\gamma y\right]\cos\gamma x$

$\frac{\partial^2\phi}{\partial y^2} = \sum -\gamma_n\left[C_n\cosh + \gamma_1 C_n y\sinh\gamma y + \gamma_n\left(\frac{C_n}{\gamma_n} + D_n y\right)\cosh\gamma y + D_n\sinh\gamma y\right.$

$\sum -\gamma_n[2C_1\cosh\gamma_1 y + D_1\sinh\gamma_1 y$

$\sum -\gamma_1(2C_1 + \gamma D_1 y)\cosh\gamma_1 y + (\gamma_1 C_1 y + D_1)\sinh\gamma_1 y$

$$\phi = g(y)\cos\frac{\pi x}{L}$$

$$\nabla^4\phi = \left[\left(\frac{\pi}{L}\right)^4 g - 2\left(\frac{\pi}{L}\right)^2 g'' + g''''\right]\cos\frac{\pi x}{L} = 0 \qquad \text{let } g(y) = e^{my}$$

$$\left[\left(\frac{\pi}{L}\right)^4 - 2\left(\frac{\pi}{L}\right)^2 m^2 + m^4\right] = 0 \qquad \left\{m^2 - \left(\frac{\pi}{L}\right)^2\right\}\left\{m^2 - \left(\frac{\pi}{L}\right)^2\right\} \quad m = \pm\frac{\pi}{L}, \pm\frac{\pi}{L} = \gamma$$

$$\phi = \left[C_1 \sinh\gamma y + \cancel{C_2}\cosh\gamma y + y\,(C_3 \sinh\gamma y + \overset{-C_1\gamma}{C_4}\cosh\gamma y)\right]\cos\frac{\pi x}{L}$$

$$\frac{\partial\phi}{\partial y} = \left[C_1\gamma\cosh\gamma y + \cancel{C_2\gamma}\sinh\gamma y + (C_3\sinh\gamma y + C_4\cosh\gamma y) + y\gamma(C_3\cosh\gamma y + C_4\sinh\gamma y)\right]\cos\frac{\pi x}{L}$$

$$= \left\{[\cancel{C_1\gamma} + \cancel{C_4} + yC_3\gamma]\cosh\gamma y + [\cancel{C_2\gamma} + C_3 + C_4\gamma y]\sinh\gamma y\right\}\cos\frac{\pi x}{L}$$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = \left\{\gamma C_3\cosh\gamma y + [C_1\gamma + C_1 + y C_3\gamma]\gamma\sinh\gamma y + \gamma C_4\sinh\gamma y + [\cancel{C_2\gamma} + C_3 + C_4\gamma y]\gamma\cosh\gamma y\right\}\cos\frac{\pi x}{L}$$

$$\left\{[\cancel{C_1\gamma^2} + \cancel{C_4\gamma} + C_4\gamma + C_3\gamma^2 y]\sinh\gamma y + [\cancel{C_2\gamma^2} + C_3\gamma + \gamma C_3 + C_4\gamma^2 y]\cosh\gamma y\right\}\cos\frac{\pi x}{L}$$

($-C_1\gamma^2$; $-C_1\gamma^3 y$)

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y} = +\gamma\left\{[\cancel{C_1\gamma} + \cancel{C_4} + yC_3\gamma]\cosh\gamma y + [\cancel{C_2\gamma} + C_3 + C_4\gamma y]\sinh\gamma y\right\}\sin\frac{\pi x}{L}$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = -\gamma^2\left[(C_1 + C_3 y)\sinh\gamma y + (\cancel{C_2} + C_4 y)\cosh\gamma y\right]\cos\frac{\pi x}{L}$$

$$\sigma_{yy}(x,0) = 0 = -\gamma^2[C_2\cos\gamma x] \qquad C_2 = 0$$

$$\sigma_{xy}(x,0) = 0 \quad \gamma[C_1\gamma + C_4]\sin\frac{\pi x}{L} \Rightarrow -C_1\gamma = C_4 \qquad C_1\gamma + C_4 = 0$$

$$\therefore\ \phi = \left[C_1(\sinh\gamma y - \gamma y\cosh\gamma y) + C_3 y\sinh\gamma y\right]\cos\frac{\pi x}{L}$$

here C_1 C_3 / A_1 C_1

$$\sigma_{xx} = \left[-C_1\gamma^2(\sinh\gamma y + \gamma y\cosh\gamma y) + C_3\gamma(\gamma y\sinh\gamma y + 2\cosh\gamma y)\right]\cos\frac{\pi x}{L}$$

$$\sigma_{xy} = \gamma\left[C_3(\gamma y\cosh\gamma y + \sinh\gamma y) - C_1\gamma^2 y\sinh\gamma y\right]\sin\frac{\pi x}{L}$$

$$\sigma_{yy} = -\gamma^2\left[C_1(\sinh\gamma y - \gamma y\cosh\gamma y) + C_3 y\sinh\gamma y\right]\cos\frac{\pi x}{L}$$

$$\sigma_{yy}(x,-h) = -\gamma^2\left[C_1(-\sinh\gamma h + \gamma h\cosh\gamma h) + C_3(-h\sinh\gamma h)\right]\cos\frac{\pi x}{L} = 0$$

$$\sigma_{xy}(x,-h) = \gamma\left[C_3(-\gamma h\cosh\gamma h - \sinh\gamma h) - C_1\gamma^2 h\sinh\gamma h\right]\sin\frac{\pi x}{L} = 3\sin\frac{\pi x}{L}$$

$$\begin{bmatrix} -\sinh\gamma h + \gamma h\cosh\gamma h & h\sinh\gamma h \\ -\gamma^3 h\sinh\gamma h & \end{bmatrix}\begin{pmatrix} C_1 \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ \\ \end{pmatrix}$$

$$C_1 = \frac{\begin{pmatrix} 0 & h \sinh \gamma h \\ 3 & -\gamma^2 h \cosh \gamma h - \gamma \sinh \gamma h \end{pmatrix}}{-\gamma^3 h^2 \cosh^2 \gamma h + \gamma \sinh^2 \gamma h + \gamma^3 h^2 \sinh^2 \gamma h} = \frac{-3h \sinh \gamma h}{\gamma\left[\sinh^2 \gamma h - (\gamma h)^2\right]}$$

$$-\gamma^3 h^2 (\cosh^2 \gamma h - \sinh^2 \gamma h) + \gamma \sinh^2 \gamma h = -\gamma^3 h^2 + \gamma \sinh^2 \gamma h$$

$$C_3 = \frac{\begin{pmatrix} -\sinh \gamma h + \gamma h \cosh \gamma h & 0 \\ -\gamma^3 h \sinh \gamma h & 3 \end{pmatrix}}{\gamma(\sinh^2 \gamma h - \gamma^2 h^2)} = \frac{3(\gamma h \cosh \gamma h - \sinh \gamma h)}{\gamma(\sinh^2 \gamma h - \gamma^2 h^2)}$$

$$C_4 = -C_1 \gamma = \frac{3h \sinh \gamma h}{\sinh^2 \gamma h - (\gamma h)^2}$$

$$-3h \sinh \gamma h \left[\sinh \gamma y - \gamma y \cosh \gamma y\right] + 3(\gamma h \cosh \gamma h - \sinh \gamma h) y \sinh \gamma y$$

$$+ 3h\gamma y\left[+sc + sc\right] = 3h\gamma y\left[\sinh \gamma(h+y)\right]$$

$$- 3h \sinh \gamma h \sinh \gamma y - 3y \sinh \gamma h \sinh \gamma y$$

$$- 3(h+y) \sinh \gamma h \sinh \gamma y$$

$$-\gamma^{2} \frac{\left[3h\gamma y \sinh \gamma(h+y) - 3(h+y) \sinh \gamma h \sinh \gamma y\right]}{\cancel{\gamma} \quad \sinh^2 \gamma h - (\gamma h)^2}$$

$$\sigma_{xy} = \gamma \left\{ (3\gamma h \cosh \gamma h - 3 \sinh \gamma h)(\gamma y \cosh \gamma y + \sinh \gamma y) + 3h \sinh \gamma h \, (\gamma^2 y \sinh \gamma y) \right\}$$

$$3\gamma^2 yh \cosh\cos + 3\gamma h \cos\xi \sin\gamma y - 3\gamma y \sin\xi \cosh\gamma y - 3 \sin\sin + 3\gamma^2 hy \sin\sin$$

~~$3\gamma^2 yh\left[1 + \sinh^2\right.$~~

$$3\gamma h \sin\xi \cos y - 3\gamma h \sin\xi \cos y$$

$$\cancel{\gamma}\frac{\left\{3\gamma^2 yh \cosh \gamma(h+y) + 3\gamma h \sinh \gamma(y+h) - 3\gamma(h+y) \sinh \gamma h \cosh \gamma y - 3 \sinh \gamma y \sinh \gamma h\right\}}{\cancel{\gamma}(\sinh^2 \gamma h - (\gamma h)^2)}$$

$$3\gamma\left[\gamma yh \cosh \gamma(h+y) + h \sinh \gamma(y+h) - (h+y) \sinh \gamma h \cosh \gamma y - \frac{1}{\gamma} \sinh \gamma y \sinh \gamma h\right]$$

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Problem Set #1

1. From results in class we had found

$$e_{xy}=0 \Rightarrow \frac{\partial}{\partial x}g(x,z)+\frac{\partial}{\partial y}f(y,z)=0$$
$$e_{yz}=0 \Rightarrow \frac{\partial}{\partial y}h(x,y)+\frac{\partial}{\partial z}g(x,z)=0$$
$$e_{zx}=0 \Rightarrow \frac{\partial}{\partial z}f(y,z)+\frac{\partial}{\partial x}h(x,y)=0$$

Now take $\frac{\partial e_{xy}}{\partial z}$, $\frac{\partial e_{yz}}{\partial x}$, $\frac{\partial e_{zx}}{\partial y}$. $\frac{\partial e_{xy}}{\partial z}-\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y} \Rightarrow \frac{\partial^2}{\partial y\partial z}f(y,z)=0$ (1)

and similarly $\frac{\partial e_{xy}}{\partial z}-\frac{\partial e_{yz}}{\partial x}-\frac{\partial e_{zx}}{\partial y} \Rightarrow \frac{\partial^2}{\partial z\partial x}g(x,z)=0$ (2) and with these two conditions this implies that $\frac{\partial^2}{\partial x\partial y}h(x,y)=0$ (3)

(1) $\Rightarrow f=f_1(y)+f_2(z)$; (2) $\Rightarrow g=g_1(x)+g_2(z)$; (3) $\Rightarrow h=h_1(x)+h_2(y)$

From $e_{xy}=0 \Rightarrow \frac{\partial u_x}{\partial y}=-\frac{\partial u_y}{\partial x} \Rightarrow f_1'(y)=-g_1'(x)=c_1 \Rightarrow f_1(y)=c_1y+c_2$; $g_1(x)=-c_1x+$

From $e_{yz}=0 \Rightarrow \frac{\partial u_z}{\partial y}=-\frac{\partial u_y}{\partial z} \Rightarrow h_2'(y)=-g_2'(z)=c_4 \Rightarrow h_2(y)=c_4y+c_5$; $g_2(z)=-c_4z+c_6$

From $e_{zx}=0 \Rightarrow \frac{\partial u_z}{\partial x}=-\frac{\partial u_x}{\partial z} \Rightarrow h_1'(x)=-f_2'(z)=c_7 \Rightarrow h_1(x)=c_7x+c_8$; $f_2(z)=-c_7z+c_9$

$$\therefore f(y,z)=f_1+f_2=c_1y-c_7z+c_{11} \quad \text{where } c_{11}=c_2+c_9$$
$$g(x,z)=g_1+g_2=-c_1x-c_4z+c_{10} \quad \text{where } c_{10}=c_3+c_9$$
$$h(x,y)=h_1+h_2=c_7x+c_4y+c_{12} \quad \text{where } c_{12}=c_5+c_8$$

Now from
$$\omega_{xy}=\tfrac{1}{2}\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)=-c_1$$
$$\omega_{yz}=\tfrac{1}{2}\left(\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z}\right)=c_4$$
$$\omega_{zx}=\tfrac{1}{2}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)=-c_7$$
rotation about z,x,y axis $\Rightarrow$
$$f=-\omega_{xy}y+\omega_{zx}z+c_{11}$$
$$g=\omega_{xy}x-\omega_{yz}z+c_{10}$$
$$h=-\omega_{zx}x+\omega_{yz}y+c_{12}$$

hence f, g, h are each made up of rigid body rotation + rigid body translation.

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2. In plane stress we assume that (1) σ_z, τ_{zx}, $\tau_{zy}=0$ and furthermore that (2) remaining quantities are independent of z. We will show that solution to (1) does not imply (2) except in cases where the z dimension is very small and hence z dependence can be neglected.

We need the compatability equations $\nabla^2\sigma_{ij}+\frac{1}{1+\nu}\Theta_{,ij}=0$ where we assume 0 body forces and $\Theta=\sigma_{ii}$

Since $\sigma_{yz}, \sigma_{zx}, \sigma_{zz} = 0 \Rightarrow \Theta_{,zz} = 0, \; \Theta_{,zx} = 0, \; \Theta_{,zy} = 0$ from compatability. Thus integrating $\Theta_{,zz}$ wrt z we get $\Theta_{,z} = f(y,x)$. Using the two other compatability relations $\Rightarrow \frac{\partial f}{\partial x} = 0, \; \frac{\partial f}{\partial y} = 0 \Rightarrow \Theta_{,z} = \text{const} = k$

Thus $\quad \Theta = kz + \Theta_0(x,y) \qquad$ (a)

Now we go back to equilib $\sigma_{ij,i} = 0$; the 3rd equation is immediately satisfied since $\sigma_{zz}, \sigma_{zx}, \sigma_{zy} = 0$. The remaining two equations reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

if we define

$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \quad \sigma_{xy} = \sigma_{yx} = -\frac{\partial^2 \phi}{\partial x \partial y}$ we satisfy the above equilib equations. We now prove that ϕ must be a fn of x, y, z. Since

$\Theta = \sigma_{ii} = \sigma_{xx} + \sigma_{yy} = \nabla_1^2 \phi = kz + \Theta_0(x,y)$. The only way we can obtain a solution is if $\phi = \phi(x,y,z)$. The above equation is (e).

If we add $\nabla^2 \sigma_{xx} + \frac{1}{1+\nu}\Theta_{,xx} + \nabla^2 \sigma_{yy} + \frac{1}{1+\nu}\Theta_{,yy} + \nabla^2 \sigma_{zz} + \frac{1}{1+\nu}\Theta_{,zz} = 0$

then $\quad \nabla^2(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) + \frac{1}{1+\nu}\nabla^2\Theta$ or $\frac{2+\nu}{1+\nu}\nabla^2\Theta = 0 \qquad$ (c)

Thus $\qquad \nabla^2 \Theta = 0 \Rightarrow \nabla^2(kz + \Theta_0) = \nabla_1^2 \Theta_0 = 0 \qquad$ (d)

Now since $\sigma_{xx} = \frac{\partial^2 \phi}{\partial x^2}$ and $\Theta = kz + \Theta_0 \Rightarrow \nabla^2 \sigma_{xx} + \frac{1}{1+\nu}\Theta_{,xx} = \nabla^2 \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{1+\nu}\Theta_{0,xx}$

or $\quad (1+\nu)\nabla^2 \frac{\partial^2 \phi}{\partial y^2} + \Theta_{0,xx} = 0 \qquad$ (f)

but $\quad \nabla^2 \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2}{\partial y^2}(\nabla^2 \phi) = \frac{\partial^2}{\partial y^2}\left(\nabla_1^2 \phi + \frac{\partial^2 \phi}{\partial z^2}\right) = \frac{\partial^2}{\partial y^2}\left(kz + \Theta_0 + \frac{\partial^2 \phi}{\partial z^2}\right)$

$$= \frac{\partial^2}{\partial y^2}\left(\Theta_0 + \frac{\partial^2 \phi}{\partial z^2}\right)$$

also since $\nabla_1^2 \Theta_0 = 0 \Rightarrow \frac{\partial^2 \Theta_0}{\partial x^2} = -\frac{\partial^2 \Theta_0}{\partial y^2}$

$\therefore$ from (f) $\quad (1+\nu)\nabla^2 \frac{\partial^2 \phi}{\partial y^2} + \Theta_{0,xx} = (1+\nu)\frac{\partial^2}{\partial y^2}\left(\Theta_0 + \frac{\partial^2 \phi}{\partial z^2}\right) - \frac{\partial^2 \Theta_0}{\partial y^2} = 0$

or $\quad \nu \frac{\partial^2}{\partial y^2}(\Theta_0) + (1+\nu)\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 \phi}{\partial z^2}\right) = 0$ or

$$\frac{\partial^2}{\partial y^2}\left[\frac{\partial^2 \phi}{\partial z^2} + \frac{\nu}{1+\nu}\Theta_0\right] = 0 \qquad (g)$$

We can similarly get

$$\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2\phi}{\partial z^2}+\frac{\nu}{1+\nu}\Theta_0\right]=0 \quad \text{and} \quad \frac{\partial^2}{\partial x\partial y}\left[\frac{\partial^2\phi}{\partial z^2}+\frac{\nu}{1+\nu}\Theta_0\right]=0$$

from the compatability eqs of σ_{xx} and σ_{xy} remembering that $\frac{\partial^2}{\partial y^2}\Theta_0 = -\frac{\partial^2}{\partial x^2}\Theta_0$ with (g) and the two equations above

$$\frac{\partial}{\partial y}\left[\frac{\partial^2\phi}{\partial z^2}+\frac{\nu}{1+\nu}\Theta_0\right]= g(x,z) \quad \text{and} \quad \left[\frac{\partial^2\phi}{\partial z^2}+\frac{\nu}{1+\nu}\Theta_0\right]= y\,g(x,z)+h(x,z)$$

$$\frac{\partial}{\partial x}[\quad]= y\frac{\partial g}{\partial x}+\frac{\partial h}{\partial x} \quad \text{and} \quad \frac{\partial^2}{\partial x^2}[\quad]= y\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 h}{\partial x^2}=0$$

$$\Rightarrow \frac{\partial^2 g}{\partial x^2}=0 \qquad \frac{\partial^2 h}{\partial x^2}=0$$

also $\frac{\partial^2}{\partial y\partial x}[\quad]=\frac{\partial g}{\partial x}=0$

Using $\frac{\partial^2 g}{\partial x^2}=0 \Rightarrow g = x f_1''(z)+l_1''(z)$ but $\frac{\partial g}{\partial x}=0 \Rightarrow f_1''(z)=0$ thus

$$g(x,z)= l_1''(z)$$

Using $\frac{\partial^2 h}{\partial x^2}=0 \Rightarrow \frac{\partial h}{\partial x}=f_2''(z)$ and $h = x f_2''(z)+l_2''(z)$

$$\therefore \quad \frac{\partial^2\phi}{\partial z^2}+\frac{\nu}{1+\nu}\Theta_0 = y\,l_1''(z)+x f_2''(z)+l_2''(z) \qquad (h)$$

Now integrate ϕ twice : $\frac{\partial\phi}{\partial z} = -\frac{\nu}{1+\nu}\Theta_0 z + y\,l_1'(z)+x f_2'(z)+l_2'(z)+\phi_1(x,y)$

and $\phi = -\frac{\nu}{1+\nu}\Theta_0\frac{z^2}{2}+y\,l_1(z)+x f_2(z)+l_2(z)+\phi_1(x,y)\cdot z+\phi_0(x,y)$ (i)

we will assume that $\phi_1(x,y)$ is not linear in x, y otherwise it can be lumped into l_1, f_2, l_2

$$\sigma_{xx}=\frac{\partial^2\phi}{\partial y^2} = -\frac{\nu}{1+\nu}\frac{\partial^2\Theta_0}{\partial y^2}\frac{z^2}{2}+\frac{\partial^2\phi_1}{\partial y^2}z+\frac{\partial^2\phi_0}{\partial y^2}$$

$$\sigma_{yy}=\frac{\partial^2\phi}{\partial x^2} = -\frac{\nu}{1+\nu}\frac{\partial^2\Theta_0}{\partial x^2}\frac{z^2}{2}+\frac{\partial^2\phi_1}{\partial x^2}z+\frac{\partial^2\phi_0}{\partial x^2}$$

$$\sigma_{xy}=-\frac{\partial^2\phi}{\partial x\partial y} = \frac{\nu}{1+\nu}\frac{\partial^2\Theta_0}{\partial x\partial y}\frac{z^2}{2}-\frac{\partial^2\phi_1}{\partial x\partial y}z-\frac{\partial^2\phi_0}{\partial x\partial y}$$

hence since $\sigma_{xx}, \sigma_{yy}, \sigma_{xz}$ are not fns of l_1, f_2, l_2 we can set them to zero

If we only look at symmetric loading problems about the mid plane $z=0$

$\Rightarrow \quad \sigma_{xx}(z=+h) = \sigma_{xx}(z=-h)\;;\quad \sigma_{yy}(z=+h) = \sigma_{yy}(z=-h)\;;\quad \sigma_{xy}(z=+h) = \sigma_{xy}(z=-h)$

This can only occur iff $\dfrac{\partial^2 \phi_1}{\partial x^2} = \dfrac{\partial^2 \phi_1}{\partial y^2} = \dfrac{\partial^2 \phi_1}{\partial x \partial y} = 0$

but the solution is $\phi_1(x,y) = ax+by+c$ where a,b,c are constants. However I originally stated that ϕ_1 was not linear in x,y. Thus $\phi_1 z$ can be taken as zero.

Thus $\Theta = \nabla_1^2 \phi = \sigma_{xx}+\sigma_{yy} = -\frac{\nu}{1+\nu}\frac{z^2}{2}\nabla_1^2 \Theta_0 + \nabla_1^2 \phi_1 \cdot z + \nabla_1^2 \phi_0 = kz + \Theta_0$

but $\nabla_1^2 \Theta_0 = 0$ (d) $\Rightarrow \quad \nabla_1^2 \phi_1 = k$ but from above $\nabla_1^2 \phi_1 = 0 \;\therefore\; k=0$

and $\nabla_1^2 \phi_0 = \Theta_0$ or $\nabla_1^2(\nabla_1^2 \phi_0) = \nabla_1^2 \Theta_0 = 0 \Rightarrow \nabla_1^4 \phi_0 = 0$ (k, l)

Thus solving (l) for ϕ_0 we use this to get Θ_0 from (k). Thus

$$\phi = -\frac{\nu}{1+\nu}\Theta_0 \frac{z^2}{2} + \phi_0. \qquad (m)$$

Thus if we take the plate thin enough we can reduce the effect of the first term

$$\phi(x,y) = -\sum_{n=1}^{\infty} \frac{A_n}{\gamma_n^2} \sin\gamma_n x\,(1-\gamma_n y)e^{\gamma_n y}$$ is the solution to the half space problem as shown in class.

We will take a look at the forces on each face by integrating the stresses along to boundary.

$$\sigma_{yy} = \phi_{,xx} = \sum_{n=1}^{\infty} A_n \sin\gamma_n x\,(1-\gamma_n y)e^{\gamma_n y}$$

$$\phi_{,y} = -\sum_{n=1}^{\infty} \frac{A_n}{\gamma_n^2}\sin\gamma_n x\left[-\gamma_n e^{\gamma_n y} + (1-\gamma_n y)\gamma_n e^{\gamma_n y}\right] = \sum_{n=1}^{\infty} A_n \sin\gamma_n x\; y e^{\gamma_n y}$$

$$\sigma_{xy} = -\phi_{,xy} = -\sum_{n=1}^{\infty} A_n\gamma_n\cos(\gamma_n x)(y e^{\gamma_n y})$$

$$\sigma_{xx} = \phi_{,yy} = \sum_{n=1}^{\infty} A_n \sin\gamma_n x\left[\gamma_n y e^{\gamma_n y} + e^{\gamma_n y}\right] = \sum_{n=1}^{\infty} A_n\sin\gamma_n x\,(1+\gamma_n y)e^{\gamma_n y}$$

a. On $y=0$ the only stresses acting are σ_{yy}, σ_{xy}. If we find the forces/unit width

$$\therefore\; F_y\Big|_{y=0} = \int_{-L}^{L}\sum_{n=1}^{\infty} A_n \sin\gamma_n x\,dx = \sum_{n=1}^{\infty} A_n\int_{-L}^{L}\sin\gamma_n x\,dx = -\sum_{n=1}^{\infty}\frac{A_n}{\gamma_n}\cos\gamma_n x\Big|_{x=-L}^{x=L}$$

$$\cos\gamma_n L - \cos(-\gamma_n L) = \cos n\pi - \cos(-n\pi) = \cos n\pi - \cos n\pi = 0 \quad \therefore\; F_y\Big|_{y=0} = 0$$

$$F_x\Big|_{y=0} = \int_{-L}^{L}\sigma_{xy}\,dx = \int_{-L}^{L} 0\cdot dx = 0$$

b. On $y=-y_0$ the only stresses acting are σ_{yy}, σ_{xy}. again finding the forces/unit width

$$F_y\Big|_{y=-y_0} = \int_{-L}^{L}\sigma_{yy}\,dx = \int_{-L}^{L}\sum_{n=1}^{\infty} A_n\sin\gamma_n x\,(1+\gamma_n y_0)e^{-\gamma_n y_0}dx = \sum_{n=1}^{\infty} A_n(1+\gamma_n y_0)e^{-\gamma_n y_0}\int_{-L}^{L}\sin\gamma_n x\,dx$$

but as before $\int_{-L}^{L}\sin\gamma_n x\,dx = 0 \qquad \therefore\; F_y\Big|_{y=-y_0} = 0$

$$F_x\Big|_{y=-y_0} = \int_{-L}^{L}\sigma_{xy}\,dx = \int_{-L}^{L} +\sum_{n=1}^{\infty} A_n\gamma_n\cos\gamma_n x\, y_0 e^{-\gamma_n y_0}dx = \sum_{n=1}^{\infty} A_n y_0 e^{-\gamma_n y_0}\gamma_n\int_{-L}^{L}\cos\gamma_n x\,dx$$

$$\int_{-L}^{L}\cos\gamma_n x\,dx = \frac{\sin\gamma_n x}{\gamma_n}\Big|_{-L}^{L} = \frac{1}{\gamma_n}\left[\sin\gamma_n L - \sin\gamma_n(-L)\right] = \frac{2}{\gamma_n}\sin n\pi = 0 \quad \therefore\; F_x\Big|_{y=-y_0} = 0$$

c. On $x=L$ the only stresses acting are σ_{xx}, σ_{xy}. Finding the forces/unit width

$$F_x\Big|_{x=L} = \int_{-y_0}^{0}\sigma_{xx}\,dy = \int_{-y_0}^{0}\sum_{n=1}^{\infty} A_n\sin\gamma_n L\,(1+\gamma_n y)e^{\gamma_n y}dy \equiv 0 \text{ since } \sin\gamma_n L = \sin n\pi = 0$$

$$P_y\Big|_{x=L} = \int_{-y_0}^{0}\sigma_{xy}\,dy = \int_{-y_0}^{0} -\sum_{n=1}^{\infty} A_n\gamma_n\cos\gamma_n L\,(y e^{\gamma_n y}dy) = -\sum_{n=1}^{\infty} A_n\gamma_n\cos n\pi \int_{-y_0}^{0} y e^{\gamma_n y}dy$$

but
$$\int_{-y_0}^{0} y e^{\gamma_n y}dy = \left[y\frac{e^{\gamma_n y}}{\gamma_n}\Big|_{-y_0}^{0} - \int_{-y_0}^{0}\frac{1}{\gamma_n}e^{\gamma_n y}dy\right] = \left[y\frac{e^{\gamma_n y}}{\gamma_n} - \frac{1}{\gamma_n^2}e^{\gamma_n y}\right]_{-y_0}^{0} = -\left\{\frac{1}{\gamma_n^2}(1-e^{-\gamma_n y_0}) - y_0\frac{e^{-\gamma_n y_0}}{\gamma_n}\right\}$$

Thus
$$P_y\Big|_{x=L} = \sum_{n=1}^{\infty} A_n(-1)^n\left\{\frac{1}{\gamma_n}(1-e^{-\gamma_n y_0}) - y_0 e^{-\gamma_n y_0}\right\}$$

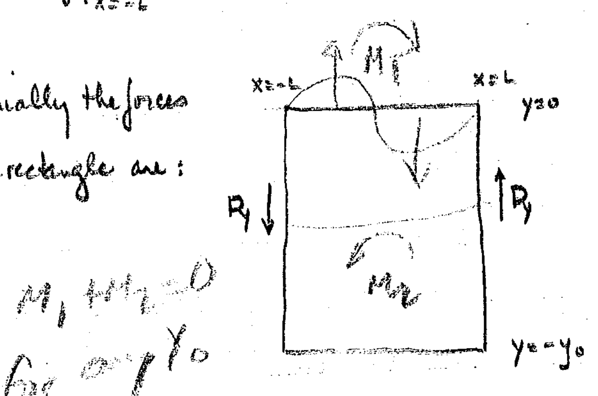
on $x=-L$ the only stresses acting are σ_{xx}, σ_{xy}. Thus

$$P_x\Big|_{x=L} = \int_{-y_0}^{0}\sigma_{xx}\,dy = \int_{-y_0}^{0}\sum_{n=1}^{\infty} A_n\sin(-\gamma_n L)\,[1+\gamma_n y]e^{\gamma_n y}dy \equiv 0 \text{ since } \sin(-\gamma_n L) = -\sin n\pi = 0$$

$$P_y\Big|_{x=-L} = \int_{-y_0}^{0}\sigma_{xy}\,dy = \int_{-y_0}^{0} -\sum A_n\gamma_n\cos(-\gamma_n L)\,(y e^{\gamma_n y}dy) = \int_{-y_0}^{0} -\sum A_n\gamma_n\cos(n\pi)\,y e^{\gamma_n y}dy \equiv P_y\Big|_{x=L}$$

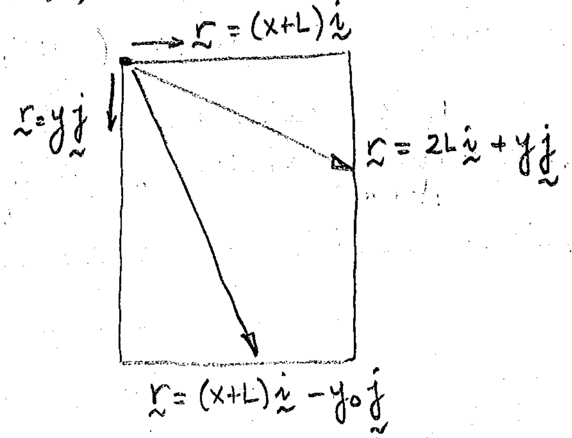
but $\sigma_{xy}\big|_{x=-L}$ is in opposite direction of $\sigma_{xy}\big|_{x=L}$ $\therefore$ $P_y\big|_{x=-L}$ is in opposite direct of $P_y\big|_{x=L}$

Pictorially the forces on the rectangle are:



Thus $\sum F_x \equiv 0$ and $\sum F_y = P_y - P_y = 0$ whether or not we take the limit as $y_0 \to \infty$. However to prove moment equilibrium we must take the limit as $y_0 \to \infty$

To prove that the moment tends to be balanced define the position vector $\underset{\sim}{r}$ from $(-L,0)$ then



$dM = \underset{\sim}{r}\times\underset{\sim}{\sigma}\,dA$ and $M = \oint\!\!\oint \underset{\sim}{r}\times\underset{\sim}{\sigma}\,dA$

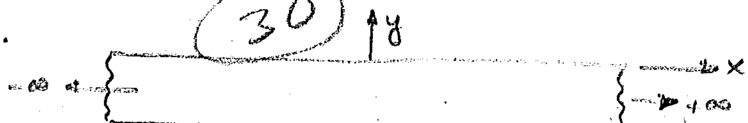
$$M = \int_{-L}^{L} [(x+L)\,\underset{\sim}{i}] \times [\sigma_{yy}\,\underset{\sim}{j} + \sigma_{xy}\,\underset{\sim}{i}]\,dx + \int_{-y_0}^{0} [2L\,\underset{\sim}{i} + y\,\underset{\sim}{j}] \times [\sigma_{xx}\,\underset{\sim}{i} + \sigma_{xy}\,\underset{\sim}{j}]\,dy$$

$$- \int_{-L}^{L} [(x+L)\,\underset{\sim}{i} - y_0\,\underset{\sim}{j}] \times [\sigma_{yy}\,\underset{\sim}{j} + \sigma_{xy}\,\underset{\sim}{i}]\,dx - \int_{-y_0}^{0} [y\,\underset{\sim}{j}] \times [\sigma_{xx}\,\underset{\sim}{i} + \sigma_{xy}\,\underset{\sim}{j}]\,dy$$

the first integral gives $\sum_{n=1}^{\infty} A_n \left(-\frac{2L}{\gamma_n}\right)(-1)^n$; the second gives $\sum_{n=1}^{\infty} A_n \left[\frac{2L(-1)^n}{\gamma_n}\right] \cdot G$

where $G = (1 - e^{-\gamma_n y_0}) - \frac{y_0}{\gamma_n} e^{-\gamma_n y_0}$. The last two integrals give zero.
Now as $y_0 \to \infty$ $G \to 1$; thus $M = \sum_{n=1}^{\infty} \frac{A_n}{\gamma_n}(-2L)(-1)^n + \sum_{n=1}^{\infty} \frac{A_n}{\gamma_n}(2L)(-1)^n = 0$

4. (30)



$\sigma_{yy}(x,0) = \sigma_{yy}(x,-h) = 0$
$\sigma_{xy}(x,0) = 0$; $\sigma_{xy}(x,-h) = 3\sin\frac{\pi x}{L}$

Governing PDE : $\nabla^4 \phi = 0$ we pick $\phi_s = g(y)\cos\frac{n\pi x}{L}$ since $\sigma_{xy} = -\frac{\partial^2\phi}{\partial x \partial y}$
and $\left.\frac{\partial^2 \phi_s}{\partial x \partial y}\right|_{y=-h} = -g'(-h)\frac{n\pi}{L}\sin\frac{n\pi x}{L}$ which is of same form as bc.

$\nabla^4 \phi_s = [g\gamma_n^4 - 2\gamma_n^2 g'' + g^{IV}]\cos\gamma_n x = 0 \Rightarrow$ for this solution to exist
$g\gamma_n^4 - 2\gamma_n^2 g'' + g^{IV} = 0$
Take $g(y) = e^{sy}$ then $[\gamma_n^4 - 2s^2\gamma_n^2 + s^4]e^{sy} = 0 \Rightarrow (s^2 - \gamma_n^2)^2 = 0$
or $s = \pm\gamma_n, \pm\gamma_n$ $\therefore$ $g_n(y) = \hat{\alpha}_n e^{\gamma_n y} + \hat{\beta}_n e^{-\gamma_n y} + \hat{c}_n y e^{\gamma_n y} + \hat{D}_n y e^{-\gamma_n y}$
Thus we can also write $g_n(y) = A_n \sinh\gamma_n y + B_n\cosh\gamma_n y + C_n y\sinh\gamma_n y + D_n y\cosh\gamma_n y$
or

$$\phi_s = \sum_{n=1}^{\infty} [A_n\sinh\gamma_n y + B_n\cosh\gamma_n y + C_n y\sinh\gamma_n y + D_n y\cosh\gamma_n y]\cos\frac{n\pi x}{L}$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = \sum -\gamma_n^2 [A_n\sinh\gamma_n y + B_n\cosh\gamma_n y + C_n y\sinh\gamma_n y + D_n y\cosh\gamma_n y]\cos\gamma_n x$$

$$\frac{\partial\phi}{\partial y} = \sum_{n=1}^{\infty} [A_n\gamma_n\cosh\gamma_n y + B_n\gamma_n\sinh\gamma_n y + C_n\sinh\gamma_n y + C_n\gamma_n y\cosh\gamma_n y + D_n\cosh\gamma_n y + D_n\gamma_n y\sinh\gamma_n y] \times \cos\gamma_n x$$

$$\frac{\partial^2\phi}{\partial x \partial y} = \sum_{n=1}^{\infty} \gamma_n \left[\gamma_n\left(A_n + C_n y + \frac{D_n}{\gamma_n}\right)\cosh\gamma_n y + \gamma_n\left(B_n + \frac{C_n}{\gamma_n} + D_n y\right)\sinh\gamma_n y\right]\sin\gamma_n x = \sigma_{xy}$$

BC

$\sigma_{yy}(x,0) = 0 = \sum -B_n \gamma_n^2 \cos\gamma_n x \quad \text{or } B_n \equiv 0$

$\sigma_{xy}(x,0) = 0 = \sum (\gamma_n A_n + D_n)\sin\gamma_n x \quad \text{or } D_n = -\gamma_n A_n$

$\sigma_{yy}(x,-h) = 0 = \sum -\gamma_n^2[-A_n \sinh\gamma_n h + C_n h \sinh\gamma_n h + \gamma_n A_n h \cosh\gamma_n h]\cos\gamma_n x$

$\sigma_{xy}(x,-h) = 3\sin\frac{\pi x}{L} = \sum \gamma_n^2[(-hC_n)\cosh\gamma_n h - (\frac{C_n}{\gamma_n} + A_n\gamma_n h)\sinh\gamma_n h]\sin\gamma_n x$

from 4th BC $\gamma_n = \frac{\pi}{L} = \gamma_1$ and $3 = -\gamma^2[hC\cosh\gamma h + (\frac{C}{\gamma} + A\gamma h)\sinh\gamma h]$

all coeffs other than A_1, C_1, D_1, γ_1 are 0.

Thus

$$\phi(x,y) = (A_1\sinh\gamma_1 y + C_1 y\sinh\gamma_1 y + D_1 y\cosh\gamma_1 y)\cos\gamma_1 x \qquad \gamma_1 = \frac{\pi}{L}$$

From the 3rd & 4th BC we get

$$A_1 = \frac{-3h\sinh\gamma_1 h}{\gamma_1(\sinh^2\gamma_1 h - \gamma_1^2 h^2)} \qquad C_1 = \frac{3(\gamma_1 h\cosh\gamma_1 h - \sinh\gamma_1 h)}{\gamma_1(\sinh^2\gamma_1 h - \gamma_1^2 h^2)}$$

$$D_1 = \frac{3h\sinh\gamma_1 h}{(\sinh^2\gamma_1 h - \gamma_1^2 h^2)}$$

$$\phi(x,y) = \frac{\cos\frac{\pi x}{L}}{\frac{\pi}{L}(\sinh^2\frac{\pi h}{L} - \frac{\pi^2 h^2}{L^2})}\left\{-3(h+y)\sinh\gamma_1 h\sinh\gamma_1 y + 3\gamma_1 hy\sinh\gamma_1(h+y)\right\}$$

$$\sigma_{yy} = \frac{-\frac{\pi}{L}\cos\frac{\pi x}{L}}{(\sinh^2\frac{\pi h}{L} - \frac{\pi^2 h^2}{L^2})}\left\{-3(h+y)\sinh\frac{\pi h}{L}\sinh\frac{\pi y}{L} + \frac{3\pi hy}{L}\sinh\frac{\pi}{L}(h+y)\right\}$$

$$\sigma_{xy} = \gamma_1[\gamma_1 C_1 y\cosh\gamma_1 y + (C_1 + A_1\gamma_1^2 y)\sinh\gamma_1 y]\sin\gamma_1 x$$

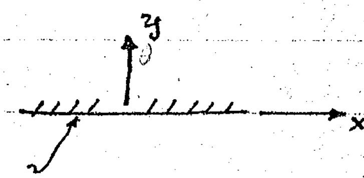
$$\sigma_{xx} = +\frac{\partial^2\phi}{\partial y^2} = \gamma_1[(2C_1 - \gamma_1^2 A_1 y)\cosh\gamma_1 y + (\gamma_1 C_1 y - \gamma_1 A_1)\sinh\gamma_1 y]\cos\gamma_1 x$$

ME 238B – HOMEWORK #2

DUE MONDAY, JANUARY 29, 1979

1. IN CLASS WE FOUND THAT

$$\sigma_{yy}(x,y) = -\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2}$$



$\sigma_{xy} = 0$, $\sigma_{yy} = -\delta(x)$ ON $y = 0$.

FOR THE CASE OF A CONCENTRATED UNIT FORCE NORMAL TO THE BOUNDARY OF THE HALF-SPACE $y \geq 0$. COMPLETE THIS SOLUTION BY FINDING

a. $\sigma_{xy}(x,y)$

b. $\sigma_{xx}(x,y)$

c. $\phi(x,y)$.

2. REDO PROBLEM 1 IF ON $y=0$, $\sigma_{yy} = 0$ AND $\sigma_{xy} = \delta(x)$.

3. FIND THE PRINCIPAL STRESSES AT A POINT (x,y) IN $y>0$ ASSOCIATED WITH PROBLEMS 1 AND 2.

#1

① $$\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x} \{ A(\lambda)\, e^{-|\lambda| y} + B y e^{-|\lambda| y} \} d\lambda$$

$$A = \frac{1}{2\pi\lambda^2} \qquad B = \frac{1}{2\pi|\lambda|}$$

$$\therefore \quad \phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x} \left\{ \frac{1}{2\pi\lambda^2} e^{-|\lambda| y} + \frac{1}{2\pi|\lambda|} y\, e^{-|\lambda| y} \right\} d\lambda$$

$$\phi_{,y} = \int_{-\infty}^{\infty} e^{-i\lambda x} \left\{ \frac{-|\lambda|}{2\pi\lambda^2} e^{-|\lambda| y} + \frac{1}{2\pi|\lambda|} \left[e^{-|\lambda| y} - |\lambda| y e^{-|\lambda| y} \right] \right\} d\lambda$$

$$\int_{-\infty}^{\infty} e^{-i\lambda x} \left\{ -\frac{1}{2\pi} y e^{-|\lambda| y} \right\} d\lambda = -\frac{1}{2\pi} y \int e^{-i\lambda x} e^{-|\lambda| y} d\lambda$$

$$\phi_{,xy} = \int_{-\infty}^{\infty} -i\lambda\, e^{-i\lambda x} \left\{ -\frac{1}{2\pi} y e^{-|\lambda| y} \right\} d\lambda$$

$$-\phi_{,xy} = \sigma_{xy} = \frac{-i}{2\pi} \int_{-\infty}^{\infty} \{ \lambda e^{-i\lambda x} y e^{-|\lambda| y} \} d\lambda = \frac{-i}{2\pi} y \int_{-\infty}^{\infty} \lambda e^{-i\lambda x} e^{-|\lambda| y} d\lambda$$

$$\int \lambda e^{-i\lambda x} e^{-|\lambda| y} d\lambda = \underset{\text{odd}}{\lambda} (\underset{\text{even}}{\cos} + i \underset{\text{odd}}{\sin\lambda})\, \underset{\text{even}}{e^{-|\lambda| y}} d\lambda$$

only non zero terms $\int \lambda \cos e = \int o = e \Big|_{-\infty}^{\infty} = 0$

$\int \lambda \sin e = \int e$

$$\therefore \quad \frac{-i y}{2\pi} \int_{-\infty}^{\infty} \{ \lambda (i \sin\lambda x) e^{-|\lambda| y} \} d\lambda = \frac{-y}{2\pi} \int_{-\infty}^{\infty} (\lambda \sin\lambda x)\, e^{-|\lambda| y} dx$$

~~note that $\frac{\partial}{\partial x} \int_{-\infty}^{\infty}$~~

$$\frac{-y}{2\pi}\, 2 \int_{0}^{\infty} \lambda \sin\lambda x\, e^{-\lambda y} d\lambda$$

note that $\frac{\partial}{\partial x} \int_0^{\infty} d\lambda \cos\lambda x\, e^{-\lambda y} = -\int_0 \lambda\, d\lambda \sin\lambda x\, e^{-\lambda y}$

$$\therefore \quad \frac{-y}{\pi} \int_0^{\infty} \lambda \sin\lambda x\, e^{-\lambda y} d\lambda = \frac{y}{\pi} \frac{\partial}{\partial x} \int_0^{\infty} d\lambda \cos\lambda x\, e^{-\lambda y}$$

$$\frac{\partial}{\partial x} \left(\frac{y}{x^2+y^2} \right)$$

$$\frac{(x^2+y^2)\cdot 0 - y \cdot 2x}{(x^2+y^2)^2} = -\frac{2yx}{(x^2+y^2)^2}$$

$$\therefore \quad \sigma_{xy} = -\frac{2}{\pi} \frac{y^2 x}{(x^2+y^2)^2}$$

$$\sigma_{xx} = \phi_{,yy} = \frac{\partial}{\partial y}\left\{ \int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{2\pi} y e^{-|\lambda| y} dx \right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\left\{ e^{-|\lambda| y} - |\lambda| y e^{-|\lambda| y}\right\} dx$$

~~$$\sigma_{xy}\Big|_{y=0} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x} dx = -\frac{1}{2\pi}\left[\frac{e^{-i\lambda x}}{-i\lambda}\right]_{-\infty}^{\infty} = +\frac{1}{2\pi i}\frac{e^{i\lambda x}}{x}\Big|_{-\infty}^{\infty}$$~~

even odd; $c - is$; even fn of λ; even fn

$$\sigma_{xx} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\{1 - |\lambda| y\} e^{-|\lambda| y} d\lambda$$ only nonzero is

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\lambda x \{1 - |\lambda| y\} e^{-|\lambda| y} d\lambda =$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} = -\frac{1}{\pi}\int_{0}^{\infty} \cos\lambda x\, e^{-\lambda y} d\lambda + \frac{1}{\pi}\int_{0}^{\infty} \lambda y \cos\lambda x\, e^{-\lambda y} d\lambda$$

$$-\frac{1}{\pi}\left[\frac{y}{x^2+y^2}\right] - \frac{1}{\pi} y \frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2}\right]$$

$$= -\frac{1}{\pi}\left\{ \frac{y}{x^2+y^2} + y\frac{(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2}\right\}$$

✓ $$\sigma_{xx} = -\frac{1}{\pi}\left\{ \frac{x^2y + y^3 + x^2y - y^3}{(x^2+y^2)^2}\right\} = \boxed{-\frac{2}{\pi}\frac{x^2 y}{(x^2+y^2)^2} = \sigma_{xx}}$$

$$\sigma_{xx} = \phi_{,yy} = -\frac{2}{\pi}\frac{x^2y}{(x^2+y^2)^2} \Rightarrow \phi_{,y} = -\frac{2}{\pi}\int\frac{x^2y\,dy}{(x^2+y^2)^2} = -\frac{1}{\pi}x^2\int\frac{du}{(x^2+u)^2} = +\frac{1}{\pi}x^2\frac{1}{(x^2+u)} + \hat{f}_1(x)$$

$$\phi_{,y} = \frac{x^2}{\pi}\frac{1}{(x^2+y^2)} + \hat{f}_1(x) \Rightarrow \phi = \frac{x^2}{\pi}\int\frac{dy}{x^2+y^2} + \hat{f}_1(x)y + \hat{f}_2(x)$$

$$= \frac{x^2}{\pi}\left[\frac{1}{x}\arctan\frac{y}{x}\right] + \hat{f}_1(x)y + \hat{f}_2(x) =$$

$$\phi_{,yx} = \frac{2x}{\pi}\frac{1}{(x^2+y^2)} - \frac{x^2}{\pi}\frac{2x}{(x^2+y^2)^2} + \hat{f}_1'(x) = \frac{2x^3+2xy^2}{\pi(x^2+y^2)^2} - \frac{2x^3}{\pi(x^2+y^2)^2} + \hat{f}_1'(x) = \frac{2xy^2}{\pi(x^2+y^2)^2} + \hat{f}_1'(x) \equiv \frac{2xy^2}{\pi(x^2+y^2)^2}$$

$$\therefore \hat{f}_1'(x) = 0 \quad \therefore \hat{f}_1(x) = c_1 \Rightarrow \phi = \frac{x^2}{\pi}\left[\frac{1}{x}\arctan\frac{y}{x}\right] + c_1 y + \hat{f}_2(x)$$

$$\phi_{,x} = \frac{1}{\pi}\arctan\frac{y}{x} - \frac{1}{\pi}\frac{yx}{(x^2+y^2)} + \hat{f}_2'$$

$$\phi_{,xx} = \frac{1}{\pi}\left(\frac{-y}{x^2+y^2}\right) - \frac{1}{\pi}\left[\frac{(x^2+y^2)y - yx(2x)}{(x^2+y^2)^2}\right] + \hat{f}_2''$$

$$= \frac{1}{\pi}\left(\frac{-x^2y - y^3 - x^2y - y^3 + 2x^2y}{(x^2+y^2)^2}\right) = \frac{-2y^3}{\pi(x^2+y^2)^2} + \hat{f}_2'' = \sigma_{yy} = -\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2} \Rightarrow \hat{f}_2'' = 0$$

or $\hat{f}_2 = c_2 x + c_3 \Rightarrow \boxed{\phi = \frac{x}{\pi}\arctan\frac{y}{x} + c_1 y + c_2 x + c_3}$

$$\phi_{,y} = \int_{-\infty}^{\infty} e^{-i\lambda x}\left\{A(-|\lambda|)e^{-|\lambda|y} + B(-|\lambda|)\,y\,e^{-|\lambda|y} + Be^{-|\lambda|y}\right\}d\lambda$$

($A(-|\lambda|)$ annotated $-B$)

$$\phi_{,yy} = \int_{-\infty}^{\infty} -e^{-i\lambda x} B|\lambda|\left\{e^{-|\lambda|y} + y(-|\lambda|)e^{-|\lambda|y}\right\}d\lambda = -\int_{-\infty}^{\infty} e^{-i\lambda x} B|\lambda| e^{-|\lambda|y}\left\{1 - |\lambda|y\right\}d\lambda$$

$$\phi_{,x} = \int_{-\infty}^{\infty} e^{-i\lambda x}\left\{Ae^{-|\lambda|y} + Bye^{-|\lambda|y}\right\}d\lambda\,(-i\lambda) \qquad A|\lambda| = B$$

$$\phi_{,xx} = \int_{-\infty}^{\infty} -\lambda^2\left\{Ae^{-|\lambda|y} + Bye^{-|\lambda|y}\right\}d\lambda\, e^{-i\lambda x}$$

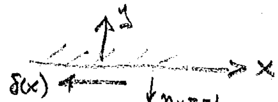
$$\int_{-\infty}^{\infty} -\lambda^2 A\, e^{-|\lambda|y}\left[1 + |\lambda|y\right]d\lambda$$

$$\phi_{,xy} = -i\int_{-\infty}^{\infty} \lambda e^{-i\lambda x}\left\{A(-|\lambda|)e^{-|\lambda|y} + Be^{-|\lambda|y} + B(-|\lambda|)ye^{-|\lambda|y}\right\}d\lambda$$

$$= -i\int_{-\infty}^{\infty} -\lambda|\lambda| e^{i\lambda x} Bye^{-|\lambda|y}\,d\lambda = -i\int_{-\infty}^{\infty} -\frac{1}{2\pi} e^{i\lambda x} y e^{-|\lambda|y}\,dy$$

#2

let $$\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\left\{Ae^{-|\lambda|y} + Bye^{-|\lambda|y}\right\}d\lambda$$



$\therefore T_y = \sigma_{yy} n_y = 0$

$-\delta = T_x = \sigma_{xy} n_y = -\sigma_{xy}$

$\therefore \sigma_{xy} = \delta$

$g(x) = \delta(x)$

if $\sigma_{xy} = \delta(x)$ $\sigma_{yy} = 0$ @ $y = 0$

$$\phi_{,x} = \int_{-\infty}^{\infty}(-i\lambda)e^{-i\lambda x}\left\{Ae^{-|\lambda|y} + Bye^{-|\lambda|y}\right\}d\lambda$$

$$\sigma_{yy} = \phi_{,xx} = \int_{-\infty}^{\infty}(-\lambda^2)e^{-i\lambda x}\left\{Ae^{-|\lambda|y} + Bye^{-|\lambda|y}\right\}d\lambda$$

if $\sigma_{yy}|_{y=0} = 0 \implies A = 0$ since $\int_{-\infty}^{\infty}\lambda^2 e^{-i\lambda x}d\lambda \neq 0$

$$\sigma_{xy} = -\phi_{,xy} = -\left[\int_{-\infty}^{\infty}(-i\lambda)e^{-i\lambda x}\left\{-A|\lambda|e^{-|\lambda|y} + Be^{-|\lambda|y} - By|\lambda|e^{-|\lambda|y}\right\}d\lambda\right]$$

$$\sigma_{xy}|_{y=0} = i\int_{-\infty}^{\infty}\lambda e^{-i\lambda x}\left\{-A|\lambda| + B\right\}d\lambda$$ but since $A = 0$ then

$$= i\int_{-\infty}^{\infty}\lambda e^{-i\lambda x}B\,d\lambda = \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}d\lambda \qquad \therefore i\lambda B = \frac{1}{2\pi}$$

$$\therefore B = \frac{1}{2\pi i\lambda}$$

hence $$\phi(x,y) = \int_{-\infty}^{\infty}e^{-i\lambda x}\left\{\frac{1}{2\pi i\lambda}\,y\,e^{-|\lambda|y}\right\}d\lambda$$

$$\therefore \quad \sigma_{yy} = \phi_{,xx} = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x} B y e^{-|\lambda| y} \, d\lambda = \frac{1}{2\pi i}\int_{-\infty}^{\infty} -\lambda e^{-i\lambda x} y e^{-|\lambda| y} d\lambda$$

$$\sigma_{xy} = -\phi_{,xy} = -\int_{-\infty}^{\infty} (-i\lambda) e^{-i\lambda x} B e^{-|\lambda| y} \{1 - |\lambda| y\} \, d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x} e^{-|\lambda| y} \{1 - |\lambda| y\} \, d\lambda$$

$$\phi(x,y) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{\lambda} \, y e^{-|\lambda| y} \, d\lambda \qquad B = \frac{1}{2\pi i \lambda}$$

$$\phi_{,y} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{\lambda} \left\{ e^{-|\lambda| y} - |\lambda| y e^{-|\lambda| y} \right\} d\lambda = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{\lambda} \{1 - |\lambda| y\} e^{-|\lambda| y} d\lambda$$

$$e^{-|\lambda| y} - |\lambda| y e^{-|\lambda| y}$$

$$\sigma_{xx} = \phi_{,yy} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{\lambda} \left\{ -|\lambda| e^{-|\lambda| y} - [1 - |\lambda| y] |\lambda| e^{-|\lambda| y} \right\} d\lambda$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{\lambda} \left\{ -|\lambda| e^{-|\lambda| y} \left[2 - |\lambda| y \right] \right\} d\lambda$$

Using the even/odd argument on λ

$$\sigma_{yy} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \underset{\text{odd}}{-\lambda} \, \underset{\text{odd}}{e^{-i\lambda x}} \, \underset{\text{even}}{y e^{-|\lambda| y}} d\lambda = \frac{1}{2\pi i}\int_{-\infty}^{\infty} -\lambda (-i \sin \lambda x) y e^{-|\lambda| y} d\lambda$$

$$= \frac{2}{2\pi}\int_0^{\infty} \lambda \sin \lambda x \, y e^{-\lambda y} d\lambda = \frac{y}{\pi}\int_0^{\infty} \lambda \sin \lambda x \, e^{-\lambda y} d\lambda$$

$$= \frac{y}{\pi} \, -\frac{\partial}{\partial x}\left(\int_0^{\infty} \cos \lambda x \, e^{-\lambda y} d\lambda \right) = -\frac{y}{\pi} \frac{\partial}{\partial x}\left(\frac{y}{x^2 + y^2} \right) = -\frac{y}{\pi}\left[\frac{-2yx}{(x^2+y^2)^2} \right]$$

$$\frac{(x^2+y^2)\cdot 0 - y 2x}{(x^2+y^2)^2}$$

$$\boxed{\sigma_{yy} = \frac{2}{\pi} \frac{y^2 x}{(x^2+y^2)^2}}$$

$$\sigma_{xy} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{\text{even}}{e^{-i\lambda x}} \, \underset{\text{even}}{e^{-|\lambda| y}} \, \underset{\text{even}}{\{1 - |\lambda| y\}} d\lambda$$

$$= \frac{1}{2\pi} \cdot 2 \int_0^{\infty} \cos \lambda x \, e^{-\lambda y} \{1 - \lambda y\} d\lambda = \frac{1}{\pi}\int_0^{\infty} e^{-\lambda y} \cos \lambda x \, d\lambda - \frac{y}{\pi}\int_0^{\infty} \lambda \cos \lambda x \, e^{-\lambda y} d\lambda$$

$$\frac{1}{\pi}\left[\frac{y}{x^2+y^2} \right] + \frac{y}{\pi} \frac{\partial}{\partial y}\left(\int_0^{\infty} \cos \lambda x \, e^{-\lambda y} d\lambda \right)$$

$$= \frac{1}{\pi}\left[\frac{y}{x^2+y^2} \right] + \frac{y}{\pi} \frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2} \right] = \frac{1}{\pi}\left\{ \frac{x^2 y + y^3}{(x^2+y^2)^2} + \frac{(x^2 y - y^3)}{(x^2+y^2)^2} \right\}$$

$$\boxed{\sigma_{xy} = \frac{2}{\pi} \frac{x^2 y}{(x^2+y^2)^2}}$$

$$\frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$\sigma_{xx} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \underbrace{\frac{e^{-i\lambda x}}{\lambda}}_{odd} \left\{ -\underbrace{|\lambda|}_{even}\, \underbrace{e^{-|\lambda| y}}_{even} \underbrace{\left[2 - |\lambda| y \right]}_{even} \right\} d\lambda = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{-i \sin\lambda x}{\lambda}\left[-|\lambda| e^{-|\lambda| y}\left[2-|\lambda| y\right]\right]$$

$$= \frac{1}{\pi}\int_0^\infty \sin\lambda x \left[e^{-\lambda y}(2-\lambda y)\right] d\lambda = \frac{2}{\pi}\int_0^\infty \sin\lambda x\, e^{-\lambda y} d\lambda - \frac{y}{\pi}\int_0^\infty \lambda \sin\lambda x\, e^{-\lambda y} d\lambda$$

$$= \frac{2}{\pi}\left[\frac{x}{x^2+y^2}\right] + \frac{y}{\pi}\frac{\partial}{\partial x}\left[\int_0^\infty \cos\lambda x\, e^{-\lambda y} d\lambda\right] = \frac{2}{\pi}\left[\frac{x}{x^2+y^2}\right] + \frac{y}{\pi}\frac{\partial}{\partial x}\left[\frac{y}{x^2+y^2}\right] = \frac{2}{\pi}\left\{\frac{2x^3+2xy^2}{} - \frac{2xy^2}{}\right.$$

$$= \frac{2}{\pi}\frac{x^3}{(x^2+y^2)^2}$$

3. $$\sigma_1, \sigma_2 = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2} \qquad \sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$$

$$= \frac{1}{2}\left\{-\frac{2}{\pi}\left[\frac{x^2 y}{(x^2+y^2)^2} + \frac{y^3}{(x^2+y^2)^2}\right]\right\} \pm \sqrt{\left\{\frac{1}{2}\left[-\frac{2}{\pi}\frac{x^2 y}{(x^2+y^2)^2} + \frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2}\right]\right\}^2 + \frac{4}{\pi^2}\frac{y^4 x^2}{(x^2+y^2)^4}}$$

$$= \frac{-y}{\pi}\frac{(x^2+y^2)}{(x^2+y^2)^2} \pm \sqrt{\frac{1}{\pi^2 (x^2+y^2)^4}\left(\{y^2 - x^2\}^2 y^2 + 4y^4x^2\right)}$$

$$y^6 - 2x^2y^4 + x^4y^2 + 4y^4x^2 = y^2(y^4 + 2x^2y^2 + x^4) = y^2(y^2+x^2)^2$$

$$\frac{-y}{\pi}\frac{1}{x^2+y^2} \pm \frac{y(x^2+y^2)}{\pi(x^2+y^2)^2}$$

$$\frac{-y \pm y}{\pi(x^2+y^2)} = 0 \text{ or } \frac{-2y}{\pi(x^2+y^2)} \qquad \sigma_1 = 0,\ \sigma_2 = \frac{-2y}{\pi(x^2+y^2)} \quad y>0 \Rightarrow \sigma_2 < 0,\ \sigma_3<0$$

problem # 1

for plane strain
$$\nu\frac{\sigma_{xx}+\sigma_{yy}}{} = \nu\frac{-2y}{\pi}\frac{1}{(x^2+y^2)} = \sigma_{zz} \qquad \sigma_3 = \frac{-2\nu y}{\pi(x^2+y^2)}$$

Check this

$$\sigma_1, \sigma_2 = \frac{1}{2}\left\{\frac{2}{\pi}\left[\frac{x^3}{(x^2+y^2)^2} + \frac{xy^2}{(x^2+y^2)^2}\right]\right\} \pm \sqrt{\left[\frac{1}{\pi}\left(\frac{x^3 - xy^2}{(x^2+y^2)^2}\right)\right]^2 + \frac{4}{\pi^2}\frac{x^4y^2}{(x^2+y^2)^4}}$$

$$= \frac{x}{\pi}\frac{1}{(x^2+y^2)} \pm \frac{x}{\pi(x^2+y^2)^2}\sqrt{(x^2-y^2)^2 + 4x^2y^2}$$

$$x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = x^4 + 2x^2y^2 + y^4 = (x^2+y^2)^2$$

$$\frac{x}{\pi}\frac{1}{(x^2+y^2)} \pm \frac{x}{\pi}\frac{1}{(x^2+y^2)} = \frac{2x}{\pi(x^2+y^2)},\ 0$$

$$\sigma_3 = \frac{2\nu x}{\pi(x^2+y^2)}$$

$$\phi_{,xx} = \frac{2}{\pi}\frac{y^2 x}{(x^2+y^2)^2} \qquad \phi_{,x} = \frac{-y^2}{\pi}\frac{1}{(x^2+y^2)} + \hat{f}_1(y) \qquad \phi = \frac{y}{\pi}\arctan\frac{x}{y} + x\hat{f}_1(y) + \hat{f}_2(y)$$

$$\frac{y}{\pi}\arctan y/x$$

$$\phi_{,xy} = \frac{-2y}{\pi}\left(\frac{1}{x^2+y^2}\right) + \frac{y^2}{\pi}\frac{2y}{(x^2+y^2)^2} + \hat{f}_1' = \frac{1}{\pi}\left[\frac{-2yx^2 - 2y^3 + 2y^3}{(x^2+y^2)^2}\right] + \hat{f}_1' = \frac{-2yx^2}{\pi(x^2+y^2)^2} + \hat{f}_1' = \frac{-2yx^2}{\pi(\ )^2}$$

$$\therefore \hat{f}_1' = 0 \qquad \hat{f}_1 = c_1$$

$$\phi_{,y} = \frac{1}{\pi}\arctan y/x + \frac{y}{\pi}\frac{x}{x^2+y^2} + \hat{f}_2' ; \quad \phi_{,yy} = \frac{2}{\pi}\frac{x}{x^2+y^2} - \frac{2y^2}{\pi}\frac{x}{(x^2+y^2)^2} + \hat{f}_2'' = \frac{2}{\pi}\left[\frac{x^3 + xy^2 - y^2x}{(x^2+y^2)^2}\right] \Rightarrow \hat{f}_2'' = 0$$

Cesar Levy
ME 238B
Prof Barnett

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Problem Set #2

1. From the lecture we had

a. $$\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\{A e^{-|\lambda| y} + By e^{-|\lambda| y}\}d\lambda \quad \text{with } A = \frac{1}{2\pi\lambda^2} \quad B = \frac{1}{2\pi|\lambda|}$$

$\therefore \phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\left\{\frac{1}{2\pi\lambda^2} e^{-|\lambda| y} + \frac{1}{2\pi|\lambda|} y e^{-|\lambda| y}\right\}d\lambda$; By differentiating we obtain

$$\frac{\partial\phi}{\partial y} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} y\, e^{-i\lambda x - |\lambda| y} d\lambda \; ; \quad \frac{\partial^2\phi}{\partial x\partial y} = \frac{i}{2\pi}\int_{-\infty}^{\infty} \lambda y\, e^{-i\lambda x - |\lambda| y} d\lambda \; ;$$

thus $\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{i}{2\pi} y \int_{-\infty}^{\infty} \lambda e^{-i\lambda x} e^{-|\lambda| y} d\lambda$. The only term that will not be zero in the integration is

$$\frac{-i}{2\pi} y \int_{-\infty}^{\infty} (-i \sin\lambda x)\lambda e^{-|\lambda| y} d\lambda = \frac{-y}{2\pi}\int_{-\infty}^{\infty} \lambda \sin\lambda x\, e^{-|\lambda| y} d\lambda = \frac{-y}{\pi}\int_{0}^{\infty} \lambda \sin\lambda x\, e^{-\lambda y} d\lambda$$

Note that $\frac{\partial}{\partial x}\int_0^{\infty} \cos\lambda x\, e^{-\lambda y} d\lambda = -\int_0^{\infty} \lambda \sin\lambda x\, e^{-\lambda y} d\lambda = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{-2yx}{(x^2+y^2)^2}$

$$\therefore \quad \frac{y}{\pi}\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{-2}{\pi}\frac{y^2 x}{(x^2+y^2)^2} = \sigma_{xy}$$

b. Since $\frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\{1 - |\lambda| y\} e^{-|\lambda| y} d\lambda$. The only terms that will not be zero in the integration is

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\lambda x\{1 - |\lambda| y\} e^{-|\lambda| y} d\lambda = -\frac{1}{\pi}\int_0^{\infty} \cos\lambda x\{1 - \lambda y\} e^{-\lambda y} d\lambda$$

$$= -\frac{1}{\pi}\left\{\frac{y}{x^2+y^2} + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\} = -\frac{2}{\pi}\frac{x^2 y}{(x^2+y^2)^2} = \sigma_{xx}$$

here we used $\int_0^{\infty} \cos\lambda x(-\lambda y e^{-\lambda y})d\lambda = y\frac{\partial}{\partial y}\int_0^{\infty} \cos\lambda x\, e^{-\lambda y} d\lambda$

c. Since

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = -\frac{2}{\pi}\frac{x^2 y}{(x^2+y^2)^2} \Rightarrow \frac{\partial\phi}{\partial y} = \frac{x^2}{\pi}\frac{1}{(x^2+y^2)} + \hat{f}_1(x) \Rightarrow \phi = \frac{x}{\pi}\arctan\frac{y}{x} + \hat{f}_1(x) y + \hat{f}_2$$

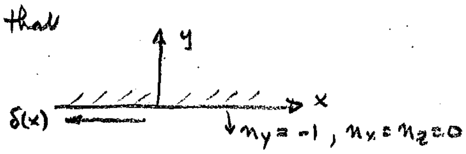
now $\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) = \frac{2xy^2}{\pi(x^2+y^2)^2} + \hat{f}_1'(x) = -\sigma_{xy} \Rightarrow \hat{f}_1'(x) = 0$ or $\hat{f}_1(x) = c_1$

now $\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{1}{\pi}\arctan y/x - \frac{1}{\pi}\frac{yx}{x^2+y^2} + \hat{f}_2'\right] = \frac{-2y^3}{\pi(x^2+y^2)^2} + \hat{f}_2'' = \sigma_{yy} \Rightarrow \hat{f}_2''(x) = 0$

or $\hat{f}_2(x) = c_2 x + c_3$ $\quad\therefore\quad \phi(x,y) = \frac{x}{\pi}\arctan\frac{y}{x} + c_1 y + c_2 x + c_3$; the last three terms don't play a role in defining the stresses $\Rightarrow$ we can take $c_1 = c_2 = c_3 = 0$ if we wish

2. again let $\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\{A+By\}e^{-|\lambda|y}\,d\lambda$

we note that



$\therefore\ T_y = \sigma_{yj}n_j = -\sigma_{yy} = 0 \qquad T_x = -\delta(x) = n_j\sigma_{xj} = -\sigma_{xy}$ thus σ_{xy} must be to the left.

Now $\dfrac{\partial\phi}{\partial x} = \int_{-\infty}^{\infty}(-i\lambda)e^{-i\lambda x}\{A+By\}e^{-|\lambda|y}d\lambda$ and $\dfrac{\partial^2\phi}{\partial x^2} = \sigma_{yy} = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x}\{A+By\}e^{-|\lambda|y}d\lambda$

since $\sigma_{yy}\big|_{y=0} = 0 \Rightarrow \quad 0 = \int_{-\infty} -\lambda^2 e^{-i\lambda x}A\,d\lambda$. It can be shown that $\int_{-\infty}^{\infty}\lambda^2 e^{-i\lambda x}d\lambda \neq 0 \ \therefore \underline{A=0}$

Now $-\dfrac{\partial}{\partial y}\left(\dfrac{\partial\phi}{\partial x}\right) = +\sigma_{xy} = i\int_{-\infty}^{\infty}\lambda e^{-i\lambda x}Be^{-|\lambda|y}\{1-|\lambda|y\}d\lambda$

But since $\sigma_{xy} = \delta(x) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}d\lambda = i\int_{-\infty}^{\infty}\lambda e^{-i\lambda x}B\,d\lambda \Rightarrow Bi\lambda = \dfrac{1}{2\pi}$ or $\underline{B = \dfrac{1}{2\pi i\lambda}}$

hence $\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\dfrac{y}{2\pi i\lambda}e^{-|\lambda|y}d\lambda$; using all this we have

$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2} = \int_{-\infty}^{\infty}\dfrac{-\lambda y}{2\pi i}e^{-i\lambda x}e^{-|\lambda|y}d\lambda$, $\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}e^{-|\lambda|y}\{1-|\lambda|y\}d\lambda$

Now since $\dfrac{\partial\phi}{\partial y} = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty}\dfrac{e^{-i\lambda x}}{\lambda}\left[1-|\lambda|y\right]e^{-|\lambda|y}d\lambda$ thus $\dfrac{\partial^2\phi}{\partial y^2} = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty}-\dfrac{|\lambda|}{\lambda}(2-|\lambda|y)e^{-i\lambda x-|\lambda|}d\lambda$

using the even/odd argument we then obtain:

a. $\sigma_{yy} = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty}-\lambda(-i\sin\lambda x)\,y\,e^{-|\lambda|y}d\lambda = \dfrac{y}{\pi}\int_0^{\infty}\lambda\sin\lambda x\,e^{-\lambda y}d\lambda = \dfrac{y}{\pi}\dfrac{-\partial}{\partial x}\left(\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}d\lambda\right)$

$= \dfrac{-y}{\pi}\dfrac{\partial}{\partial x}\left(\dfrac{y}{x^2+y^2}\right) = \dfrac{-y}{\pi}\left[\dfrac{-2yx}{(x^2+y^2)^2}\right] = \underline{\dfrac{2}{\pi}\dfrac{y^2x}{(x^2+y^2)^2}} = \sigma_{yy}$

b. $\sigma_{xy} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\cos\lambda x\,e^{-|\lambda|y}\{1-|\lambda|y\}d\lambda = \dfrac{1}{\pi}\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}[1-\lambda y]d\lambda = \dfrac{1}{\pi}\left[\dfrac{y}{x^2+y^2}\right] + \dfrac{y}{\pi}\dfrac{\partial}{\partial y}\left(\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}d\lambda\right)$

$= \dfrac{1}{\pi}\left(\dfrac{y}{x^2+y^2}\right) + \dfrac{y}{\pi}\dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2}\right) = \underline{\dfrac{2}{\pi}\dfrac{x^2y}{(x^2+y^2)^2}} = \sigma_{xy}$

c. $\sigma_{xx} = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty}\dfrac{-i\sin\lambda x}{\lambda}\left\{-|\lambda|e^{-|\lambda|y}\left[2-|\lambda|y\right]\right\}d\lambda = \dfrac{1}{\pi}\int_0^{\infty}\sin\lambda x\,e^{-\lambda y}(2-\lambda y)d\lambda$

$= \dfrac{2}{\pi}\left[\dfrac{x}{x^2+y^2}\right] + \dfrac{y}{\pi}\dfrac{\partial}{\partial x}\left[\left(\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}d\lambda\right)\right] = \dfrac{2}{\pi}\left[\dfrac{x}{x^2+y^2}\right] + \dfrac{y}{\pi}\dfrac{\partial}{\partial x}\left[\dfrac{y}{x^2+y^2}\right] = \dfrac{2}{\pi}\dfrac{x^3}{(x^2+y^2)^2}$

or $\underline{\sigma_{xx} = \dfrac{2}{\pi}\dfrac{x^3}{(x^2+y^2)^2}}$

d. To obtain ϕ :

$\dfrac{\partial^2\phi}{\partial x^2} = \sigma_{yy} = \dfrac{2}{\pi}\dfrac{y^2x}{(x^2+y^2)^2} \Rightarrow \dfrac{\partial\phi}{\partial x} = \dfrac{-y^2}{\pi}\dfrac{1}{x^2+y^2} + \hat{f}_1(y) \Rightarrow \phi = \dfrac{-y}{\pi}\arctan\dfrac{x}{y} + x\hat{f}_1(y) + \hat{f}_2(y)$

$= -\dfrac{y}{\pi}\arctan x/y + x\hat{f}_1(y) + \hat{f}_2(y)$

$$\frac{\partial^2\phi}{\partial x\partial y} = -\sigma_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = -\frac{2yx^2}{\pi(x^2+y^2)^2} + \hat{f}_1' \Rightarrow \hat{f}_1'(y)=0 \text{ or } \hat{f}_1(y) = c_1$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{\pi}\arctan y/x + \frac{y}{\pi}\frac{x}{x^2+y^2} + \hat{f}_2' \; ; \; \frac{\partial^2\phi}{\partial y^2} = \frac{2x^3}{\pi(x^2+y^2)^2} + \hat{f}_2''(y) = \sigma_{yy} \Rightarrow \hat{f}_2''(y)=0 \text{ or } \hat{f}_2 = c_2 y + c_3$$

$$\therefore \phi(x,y) = \frac{y}{\pi}\arctan y/x + c_1 x + c_2 y + c_3$$ same argument on c_1, c_2, c_3 as problem 1

3a For principal stresses in a plane strain problem $\sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$: Define λ_i to be the principal stresses $\therefore \sigma \cdot n = \lambda n$ or $\det\begin{pmatrix}\sigma_{xx}-\lambda & \sigma_{xy} & 0\\ \sigma_{xy} & \sigma_{yy}-\lambda & 0 \\ 0 & 0 & \sigma_{zz}-\lambda\end{pmatrix} = 0$ $\therefore$ $\lambda_3 = \sigma_{zz}$ and

$$\lambda_{1,2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$ after the plug in and the algebra

$$\therefore \nu(\sigma_{xx}+\sigma_{yy}) = \sigma_{zz} = \frac{-2\nu y}{(x^2+y^2)} = \lambda_3 \qquad \lambda_1 = 0 \qquad \lambda_2 = \frac{-2y}{\pi(x^2+y^2)}$$

Since $y>0$ and we assume $0<\nu<1$ the stresses are ordered

$(\lambda_1, \lambda_3, \lambda_2)$ in decreasing tension (from left to right)

3b. again we obtain for plane strain $\lambda_3 = \sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$ and

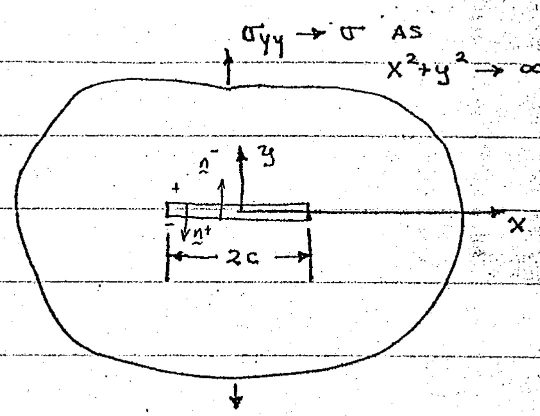
$$\lambda_{1,2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

$$\therefore \nu(\sigma_{xx}+\sigma_{yy}) = \sigma_{zz} = \lambda_3 = \frac{2\nu x}{\pi(x^2+y^2)} \qquad \lambda_1 = 0 \;, \; \lambda_2 = \frac{2x}{\pi(x^2+y^2)}$$

for $x>0$ and assuming $0<\nu<1$ the stresses are ordered $(\lambda_2, \lambda_3, \lambda_1)$ in decreasing tension (from left to right). for $x<0$ the stresses are ordered $(\lambda_1, \lambda_3, \lambda_2)$ in decreasing tension (from left to right)

ME 238B — THEORY OF ELASTICITY

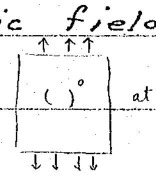
HOMEWORK #3 — DUE MONDAY, FEB. 12, 1979



1. As an example of a mixed boundary value problem in the theory of elasticity, consider an infinite isotropic linear elastic solid containing a "slit" crack of length $2c$ and loaded in remote tension at infinity as shown in the figure. The faces of the crack are traction-free. Consider plane strain deformation. Using superposition write the solution to this problem as

$$u_i = u_i^{\circ} + \hat{u}_i \; ; \quad e_{ij} = e_{ij}^{\circ} + \hat{e}_{ij} \; ; \quad \sigma_{ij} = \sigma_{ij}^{\circ} + \hat{\sigma}_{ij}$$

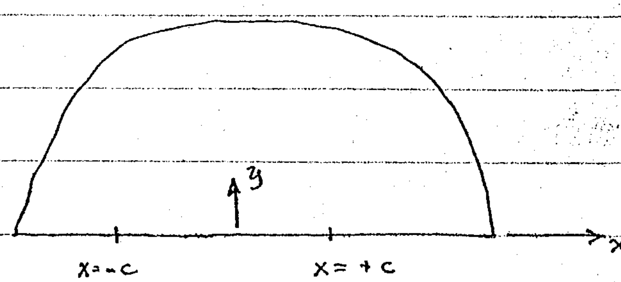
where u_i°, e_{ij}°, σ_{ij}° are the elastic fields when $c = 0$.



a. Compute u_i°, e_{ij}°, and σ_{ij}°.

b. Using symmetry arguments show that $\hat{u}_i$, $\hat{e}_{ij}$, and $\hat{\sigma}_{ij}$ can be obtained from the solution to the following <u>mixed</u> boundary value problem for the half-space $y > 0$:

$$\begin{cases} \hat{\sigma}_{xy} = 0 \; ; & -\infty < x < \infty, \; y = 0 \\ \hat{\sigma}_{yy} = -\sigma \; ; & |x| < c, \; y = 0 \\ \hat{u}_y = 0 \; ; & |x| > c, \; y = 0 \end{cases}$$



– Specify any relevant boundary conditions on $\hat{\sigma}_{ij}$ as $x^2+y^2 \to \infty$.

c. Using a stress function of the form

$$\phi(x,y) = \int_0^\infty e^{-\lambda y} \cos\lambda x \, [A(\lambda) + y\, B(\lambda)]\, d\lambda$$

show that the mixed boundary value problem in (b) may be solved provided that we can determine $A(\lambda)$ from the <u>dual integral equations</u>

$$\left.\begin{aligned} \int_0^\infty \lambda^2 A(\lambda) \cos\lambda x \, d\lambda &= \sigma \;, \quad 0 < x < c \\ \int_0^\infty \lambda\, A(\lambda) \cos\lambda x \, d\lambda &= 0 \;, \quad c < x < \infty \end{aligned}\right\}.$$

d. Given that the Bessel function of order 1 of the first kind $J_1(\xi)$ satisfies the following integral relations:

$$\int_0^\infty J_1(\alpha x) \cos\beta x \, dx = \frac{1}{\alpha} \quad , \text{if } \beta < \alpha$$

$$= -\frac{\alpha}{\sqrt{\beta^2-\alpha^2}\,\left(\beta + \sqrt{\beta^2-\alpha^2}\right)} \quad , \text{if } \beta > \alpha$$

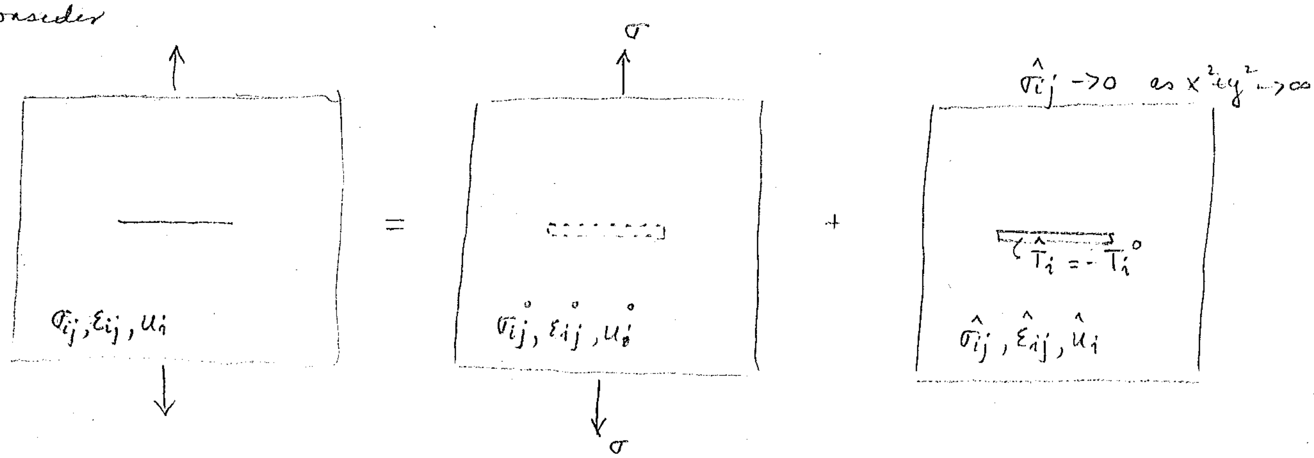
and

$$\int_0^\infty \frac{J_1(\alpha x)}{x} \cos\beta x \, dx = 0 \quad , \text{if } \beta > \alpha ,$$

deduce the function $A(\lambda)$ which solves the dual integral equations and compute $\hat{\sigma}_{yy}(x, y=0)$

Look at the problem of a body loading of σ for σ_{yy} as $x^2+y^2 \to \infty$

now consider



for ° fields $\quad \sigma^\circ_{yy} = \sigma \quad \sigma^\circ_{ij} = 0$ for all others

$\varepsilon^\circ_{ij} = \frac{1+\nu}{E}\sigma^\circ_{ij} - \frac{\nu}{E}(\sigma^\circ_{kk})\delta_{ij}$ the only non zero strains are $\varepsilon^\circ_{yy}, \varepsilon^\circ_{xx}$, since each contains σ°_{yy}

For plane strain

Now $\varepsilon^\circ_{zz} = \frac{1+\nu}{E}\sigma^\circ_{zz} - \frac{\nu}{E}(\sigma^\circ_{kk}) = 0 \Rightarrow \sigma^\circ_{zz} = \nu(\sigma^\circ_{xx}+\sigma^\circ_{yy}) = \nu\sigma \qquad \varepsilon^\circ_{xy} = 0 \Rightarrow \frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x} = 0$

$\varepsilon^\circ_{xx} = \frac{1+\nu}{E}\sigma^\circ_{xx} - \frac{\nu}{E}(\sigma^\circ_{yy}+\sigma^\circ_{xx}+\sigma^\circ_{zz}) = -\frac{\nu\sigma}{E} - \frac{\nu^2\sigma}{E} = -\frac{\nu\sigma}{E}(1+\nu) \qquad \varepsilon^\circ_{xz} = 0 \Rightarrow \frac{\partial u^\circ}{\partial z} + \frac{\partial w^\circ}{\partial x} = 0 \qquad (1)$

$\varepsilon^\circ_{yy} = \frac{1+\nu}{E}\sigma^\circ_{yy} - \frac{\nu}{E}(\sigma^\circ_{yy}+\sigma^\circ_{xx}+\sigma^\circ_{zz}) = \frac{\sigma}{E} - \frac{\nu^2\sigma}{E} = \frac{(1-\nu^2)\sigma}{E} \qquad \varepsilon^\circ_{yz} = 0 \Rightarrow \frac{\partial v^\circ}{\partial z} + \frac{\partial w^\circ}{\partial y} = 0$

Now $\frac{\partial u^\circ}{\partial x} = -\frac{\nu\sigma}{E}(1+\nu) \Rightarrow \quad u^\circ = -\frac{\nu\sigma(1+\nu)}{E}x + f(y)$

$\frac{\partial v^\circ}{\partial y} = \frac{(1-\nu^2)\sigma}{E} \qquad v^\circ = \frac{(1-\nu^2)\sigma}{E}y + g(x)$

$\frac{\partial w^\circ}{\partial z} = 0 \qquad w^\circ = h(y,x)$

$\Rightarrow f'(y) + g'(x) = 0 \quad \therefore f'(y) = -g'(x) = \text{const}$

$f(y) = cy + c_2; \; g(x) = -cx + c_1$

$\frac{\partial h}{\partial x} = 0, \; \frac{\partial h}{\partial y} = 0 \qquad h = c_4 + h_1(y) \qquad \frac{\partial h}{\partial y} = h_1'(y) = 0 \therefore h_1(y) = \text{const} \quad \therefore h = \text{const}$

Thus f, g represent rigid body rot + rigid body trans ; h rigid body translation

thus f, g, h represent rigid body rotation + rigid body translation.

Since we want no rigid body rotation or translation $f, g, h \equiv 0$

$\uparrow T_y = \sigma \qquad T_y = -\sigma$

Now $T^\circ_x = \sigma^\circ_{xj} n_j = \sigma^\circ_{xy} n_y \qquad$ now on $y = 0^+ \quad n_y = -1 \quad \therefore T^\circ_x\big|_{y^+} = -\sigma^\circ_{xy} = 0 \qquad y^+ = \lim_{y \downarrow 0^+}$

$T^\circ_y = \sigma^\circ_{yj} n_j = \sigma^\circ_{yy} n_y \qquad$ now on $y = 0^+ \quad n_y = -1 \quad -\sigma = T^\circ_y\big|_{y^+} = -\sigma^\circ_{yy} \quad \therefore \sigma^\circ_{yy} = +\sigma$

$T^\circ_x\big|_{y^-} = \sigma^\circ_{xj} n_j = \sigma^\circ_{xy} n_y = +\sigma^\circ_{xy} = 0 \qquad y^- = \lim_{y \uparrow 0^-}$

$+\sigma = T^\circ_y\big|_{y^-} = \sigma^\circ_{yj} n_y = \sigma^\circ_{yy} n_y = \sigma^\circ_{yy} \quad \therefore \sigma^\circ_{yy} = +\sigma$

now on hole $\quad \hat{T}_x = T_x - T^\circ_x = 0 - 0 = 0 = \hat{\sigma}_{xy}$

$T_y\big|_{y^+} = T^\circ_y\big|_{y^+} + \hat{T}_y\big|_{y^+} = 0 = -\sigma + \hat{T}_y\big|_{y^+} \Rightarrow \hat{T}_y\big|_{y^+} = \sigma = -\hat{\sigma}_{yy}$

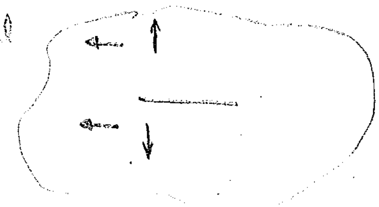
$T_y\big|_{y^-} = 0 = T^\circ_y\big|_{y^-} + \hat{T}_y\big|_{y^-} = \sigma + \hat{T}_y\big|_{y^-} \Rightarrow \hat{T}_y\big|_{y^-} = -\sigma = \hat{\sigma}_{yy}$

$\hat{T}_x\big|_{y^-} = 0$ since $T_x\big|_{y^-} = 0$ and $T^\circ_x\big|_{y^-} = 0$

also since $\overset{o}{\sigma}_{xy} = 0$ everywhere (given) then $\overset{o}{\sigma}_{xy}\Big|_{\substack{y=0\\ |x|>c}} = 0$

<u>Now</u> to prove $\hat{\sigma}_{xy} = 0 \quad |x| < \infty,\ y = 0$

In original



$$\sigma_{xy} = \frac{E\epsilon_{xy}}{(1+\nu)} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

by symm $\begin{cases} u \text{ is an even fn in } y & \therefore \dfrac{\partial u}{\partial y} \text{ must be odd in } y \\ v \text{ is an odd fn in } y & \therefore \dfrac{\partial v}{\partial x} \text{ must be an odd fn in } y \end{cases}$

$\therefore \sigma_{xy}$ must be odd in $y \quad \therefore \sigma_{xy}(x,y) = -\sigma_{xy}(x,-y)$

as $y \to 0 \implies \sigma_{xy}(x,0) = -\sigma_{xy}(x,0) \quad \forall x \quad \therefore \sigma_{xy}\Big|_{y=0} \equiv 0$

but $\sigma_{xy}\Big|_{\substack{y=0\\ |x|<\infty}} = 0 = \overset{o}{\sigma}_{xy}\Big|_{\substack{y=0\\ |x|<\infty}} + \hat{\sigma}_{xy}\Big|_{\substack{y=0\\ |x|<\infty}} \implies \hat{\sigma}_{xy}\Big|_{\substack{y=0\\ |x|<\infty}} = 0$ (with $\overset{o}{\sigma}_{xy} = 0$)

Now look at $v\Big|_{\substack{y=0\\ |x|>c}}$ since we showed that v is an odd fn of y by symm

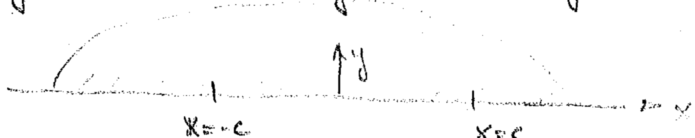
$$v(x,y) = -v(x,-y)$$

$$\text{as } y \to 0 \quad v(x,0) = -v(x,-0)$$

but since the body is continuous at $y=0$, $|x|>c \implies v(x,0) = 0 \quad |x|>c$

now $v(x,0) = 0 = \overset{o}{v}(x,0) + \hat{v}(x,0) \implies \hat{v}(x,0) = 0$, $|x|>c$ (with $\overset{o}{v} = 0$)

Thus we need only look at a body with the following ^ fields



with $\hat{\sigma}_{xy} = 0; \quad |x| < \infty,\ y = 0$

$\hat{\sigma}_{yy} = -\sigma; \quad |x| < c,\ y = 0$

$v = 0; \quad |x| > c,\ y = 0$

since $\sigma_{ij} \to \sigma$ as $x^2 + y^2 \to \infty$ and $\overset{o}{\sigma}_{ij}$: $\overset{o}{\sigma}_{yy} = \sigma$, $\overset{o}{\sigma}_{ij} = 0$ for others then $\hat{\sigma}_{ij}$: $\hat{\sigma}_{yy} \to 0$; $\hat{\sigma}_{ij} \to 0$ as $x^2 + y^2 \to \infty$

This problem has the stress fn.

$$\phi(x,y) = \int_0^\infty e^{-\lambda y}\cos\lambda x\,[A(\lambda) + y B(\lambda)]\,d\lambda$$

this satisfies plane strain conditions ie $\hat{w} = 0$, $\hat{u}, \hat{v}$ are fns of x, y only

now $\hat{\sigma}_{yy} = \dfrac{\partial^2\phi}{\partial x^2} = \displaystyle\int_0^\infty -\lambda^2 e^{-\lambda y}\cos\lambda x\,[A(\lambda) + y B(\lambda)]\,d\lambda$

and on $y = 0$, $|x| < c$: $\quad -\sigma = \displaystyle\int_0^\infty -\lambda^2\cos\lambda x\, A\, d\lambda \quad$ or $\quad \sigma = \displaystyle\int_0^\infty \lambda^2 A\cos\lambda x\, d\lambda$

now $\hat{\sigma}_{xy} = -\dfrac{\partial\phi}{\partial x\,\partial y} = -\displaystyle\int_0^\infty -\lambda\sin\lambda x\left[-\lambda e^{-\lambda y}(A + By) + e^{-\lambda y}(B)\right]d\lambda$

$= \displaystyle\int^\infty \lambda\sin\lambda x\, e^{-\lambda y}\{-A\lambda - B\lambda\ldots + B\}\,d\lambda \quad \ldots$

$$\hat{\sigma}_{xy}\Big|_{y=0} = 0 = \int_0^\infty \lambda \sin\lambda x \left[-A\lambda + B\right] d\lambda$$ since this must be true $\forall x \Rightarrow B = A\lambda$

Now $$\frac{\partial^2\phi}{\partial y^2} = \hat{\sigma}_{xx} = \frac{\partial}{\partial y}\int_0^\infty \cos\lambda x\left[-\lambda e^{-\lambda y}(A+By) + e^{-\lambda y}B\right]d\lambda = \frac{\partial}{\partial y}\int_0^\infty \cos\lambda x\, e^{-\lambda y}\left[-\lambda A + B - B\lambda y\right]d\lambda$$

$$= \frac{\partial}{\partial y}\int_0^\infty -A\lambda^2\cos\lambda x\,(e^{-\lambda y}y)\,d\lambda = \int_0^\infty -A\lambda^2\cos\lambda x\left(-\lambda e^{-\lambda y}y + e^{-\lambda y}\right)d\lambda = \int_0^\infty -A\lambda^2\cos\lambda x\, e^{-\lambda y}(1-\lambda y)\,d\lambda$$

now $$\hat{\epsilon}_{yy} = \frac{\hat{\sigma}_{yy}}{E} - \frac{\nu}{E}(\hat{\sigma}_{xx}+\hat{\sigma}_{zz}) = \frac{\hat{\sigma}_{yy}}{E} - \frac{\nu}{E}(\hat{\sigma}_{xx} + \nu\hat{\sigma}_{xx} + \nu\hat{\sigma}_{yy}) = \frac{\hat{\sigma}_{yy}(1-\nu^2)}{E} - \frac{\nu(1+\nu)}{E}\hat{\sigma}_{xx} = \frac{\partial \hat{v}}{\partial y}$$

now since $$\hat{\sigma}_{yy} = \int_0^\infty -A\lambda^2 e^{-\lambda y}\cos\lambda x\left[1+\lambda y\right]d\lambda$$

then $$\frac{\partial v}{\partial y} = \frac{1-\nu^2}{E}\int_0^\infty -A\lambda^2 e^{-\lambda y}\cos\lambda x(1+\lambda y)\,d\lambda - \left(\frac{\nu+\nu^2}{E}\right)\int_0^\infty -A\lambda^2\cos\lambda x\, e^{-\lambda y}(1-\lambda y)\,d\lambda$$

$$= \int_0^\infty -\frac{A\lambda^2}{E}e^{-\lambda y}\cos\lambda x\left\{(1+\lambda y)(1-\nu^2) - (\nu+\nu^2)(1-\lambda y)\right\}d\lambda$$

$$(1+\nu)\left[(1+\lambda y)(1-\nu) - \nu(1-\lambda y)\right]d\lambda$$

$$1+\lambda y - \nu - \nu\lambda y - \nu + \nu\lambda y$$

$$(1-2\nu+\lambda y)\,d\lambda \qquad = -\frac{1+\nu}{E}\int_0^\infty A\lambda^2 e^{-\lambda y}\cos\lambda x\,(1-2\nu+\lambda y)\,d\lambda$$

dv, $\frac{e^{-\lambda y}}{-\lambda} = v$, $-\lambda\,dy = du$

$$\therefore\ v = -\left(\frac{1+\nu}{E}\right)\int_0^\infty A\lambda^2\cos\lambda x\left[\frac{e^{-\lambda y}}{-\lambda}(1-2\nu+\lambda y) + \frac{e^{-\lambda y}}{-\lambda}\right]d\lambda + g(x)$$

$$= \left(\frac{1+\nu}{E}\right)\int_0^\infty A\lambda\cos\lambda x\, e^{-\lambda y}(2-2\nu+\lambda y)\,d\lambda + g(x)$$

$$\hat{\epsilon}_{xx} = \frac{\partial u}{\partial x} = \frac{\hat{\sigma}_{xx}}{E} - \frac{\nu}{E}(\hat{\sigma}_{xx}+\hat{\sigma}_{zz}) = \frac{\hat{\sigma}_{xx}(1-\nu^2)}{E} - \frac{\nu(1+\nu)}{E}\hat{\sigma}_{yy}$$

$$= \frac{1+\nu}{E}\int_0^\infty -A\lambda^2\cos\lambda x\, e^{-\lambda y}\left[(1-\nu)(1-\lambda y) - \nu(1+\lambda y)\right]d\lambda = -\frac{1+\nu}{E}\int_0^\infty +A\lambda^2\cos\lambda x\, e^{-\lambda y}(1-2\nu-\lambda y)\,d\lambda$$

$$1-\lambda y - \nu + \nu\lambda y - \nu - \nu\lambda y$$

$$u = -\left(\frac{1+\nu}{E}\right)\int_0^\infty A\lambda\sin\lambda x\, e^{-\lambda y}(1-2\nu-\lambda y)\,d\lambda + f(y)$$

we will prove later that $g(x)$, $f(y)$ are rigid body rot, trans

take f, g as 0.

$$\therefore\ v = \frac{1+\nu}{E}\int_0^\infty A\lambda\cos\lambda x\, e^{-\lambda y}(2-2\nu+\lambda y)\,d\lambda$$

on $y=0$ $$v = \frac{2(1-\nu)(1+\nu)}{E}\int_0^\infty A\lambda\cos\lambda x\,d\lambda$$ for $|x|\le c$ $v=0$ $\therefore \Rightarrow \int_0^\infty A\lambda\cos\lambda x\,d\lambda = 0$

In original problem: v is odd in y, even in x $\therefore \frac{\partial v}{\partial x}$ is odd in y, odd in x

u is even in y, odd in x $\therefore \frac{\partial u}{\partial y}$ is odd in y, odd in x

$\therefore\ \sigma_{xy}(x,y) = -\sigma_{xy}(-x,y)$ and $\sigma_{xy}(x,y) = -\sigma_{xy}(x,-y)$

$\Rightarrow \hat{\sigma}_{xy}(x,y) = -\hat{\sigma}_{xy}(-x,y)$ and $\hat{\sigma}_{xy}(x,y) = -\hat{\sigma}_{xy}(x,-y)$

since $\sigma^0_{xy} = 0$

$\hat{\sigma}_{xy}(x,y) \sim f'(y) + g'(x)$

$\hat{\sigma}_{xy}(x,y) = -\hat{\sigma}_{xy}(-x,y) \sim -[f'(y) + g'(-x)]$ $\forall y \Rightarrow f'(y) = 0$

$\hat{\sigma}_{xy}(x,y) = -\hat{\sigma}_{xy}(x,-y) \sim -[f'(-y) + g'(x)]$ $\forall x \Rightarrow g'(x) = 0$

or $f(y)$, $g(y)$ are const $\therefore$ they represent rigid body trans.

also since no loads are applied at ∞ $v\to 0$, $u\to 0$ as $x^2+y^2\to\infty$ $\Rightarrow f(y) = 0 = g(x)$

d. $$\int_0^\infty J_1(c\lambda)\cos(x\lambda)\,d\lambda = \frac{1}{c} \quad \text{if} \quad x<c$$

$$= -\frac{c}{\sqrt{x^2-c^2}\left(x+\sqrt{x^2-c^2}\right)} \quad \text{if} \quad x>c$$

$$\int_0^\infty \frac{J_1(c\lambda)}{\lambda}\cos(x\lambda)\,d\lambda = 0 \quad \text{if} \quad x>c$$

now if we write $\frac{1}{\sigma c}\int_0^\infty \lambda^2 A \cos\lambda x \, d\lambda = \frac{1}{c}$ for $x < c$

and if we multiply $\frac{1}{\sigma c}\int_0^\infty \lambda A \cos\lambda x \, d\lambda = 0$

then $\frac{\lambda^2 A}{\sigma c} = J_1(c\lambda) \Rightarrow \frac{\lambda A}{\sigma c} = \frac{J_1(c\lambda)}{\lambda}$

$$\therefore \quad A(\lambda) = J_1(c\lambda)\frac{\sigma c}{\lambda^2}$$

$$\hat{\sigma}_{yy} = \int_0^\infty -\lambda^2 e^{-\lambda y}\cos\lambda x \, A(\lambda)\,[1+\lambda y]\,d\lambda$$

$$= \int_0^\infty -\lambda^2 e^{-\lambda y}\cos\lambda x \; J_1(c\lambda)\frac{\sigma c}{\lambda^2}\,(1+\lambda y)\,d\lambda$$

$$= \int_0^\infty -e^{-\lambda y}\cos\lambda x \; J_1(c\lambda)\,\sigma c\,(1+\lambda y)\,d\lambda$$

$$\hat{\sigma}_{yy}\Big|_{y=0} = -\sigma c\int_0^\infty \cos\lambda x \; J_1(c\lambda)\,d\lambda$$

$$= -\sigma c\int_0^\infty \cos\lambda x \; J_1(c\lambda)\,d\lambda = c\sigma\left[\; + \frac{c}{\sqrt{x^2-c^2}\,\left(x+\sqrt{x^2-c^2}\right)}\right] \quad \text{for } x > c$$

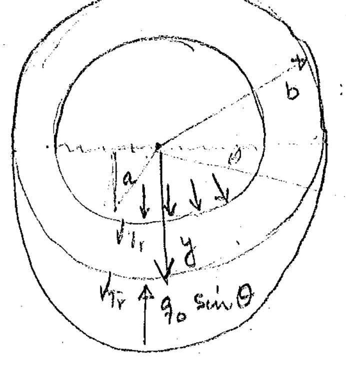
if $x = c+\varepsilon$ then

$$\sqrt{x^2-c^2} = \sqrt{x-c}\,\sqrt{x+c} \cong \sqrt{\varepsilon}\,\sqrt{2c}\,\left(c+\sqrt{2\varepsilon c}\right)$$

$$= \frac{+c^2\sigma}{c\sqrt{2\varepsilon c}+2\varepsilon c} = +\frac{c\sigma}{\sqrt{2\varepsilon c}+2\varepsilon} \approx \frac{\sigma\sqrt{c}\,\sqrt{\pi}}{\sqrt{2\varepsilon}\,\sqrt{\pi}}$$

$$= \frac{\sigma\sqrt{\pi c}}{\sqrt{2\pi\varepsilon}} = \frac{K_I}{\sqrt{2\pi\varepsilon}}$$

$\sigma_{r\theta} = 0$



$p_i = \gamma a \sin\theta \quad p_i = T_r^i = \sigma_{rj}^i n_j = \sigma_{rr}^i n_r = -\sigma_{rr}^i \quad \therefore \sigma_{rr}^i = -p_i$

$p_o = q_0 \sin\theta \quad -p_o = T_r^o = \sigma_{rj}^o n_j = \sigma_{rr}^o n_r = \sigma_{rr}^o \quad \therefore \sigma_{rr}^o = -p_o$

∴ by our results ~~$B_1 = -\gamma a \quad B_i = 0 \; i > 1 \quad A_i = 0 \; \forall i \quad B_1' = -q_0 \quad B_i' = 0 \; i>1 \quad A_i' = 0 \; \forall i \quad 0 \le \theta \le \pi$~~

~~all coeffs are 0 for $\pi \le \theta \le 2\pi$~~

since $\sigma_{r\theta}$ tractions don't exist

~~Now $C_0 a^2 = C_0' b^2 \equiv 0$ Moment Equil is satisfied~~

~~Now $b(A_1' - D_1') = a(A_1 - D_1) \equiv 0$ since $A_1, A_1', D_1', D_1 = 0$~~

~~Now~~

$$(\sigma_{rr})_{r=a} = \left.\begin{matrix} -\gamma a \sin\theta & 0 \le \theta \le \pi \\ 0 & \pi \le \theta \le 2\pi \end{matrix}\right\} = A_0 + \sum_1^\infty A_n \cos n\theta + \sum_1^\infty B_n \sin n\theta$$

$$\Rightarrow \int_0^{2\pi} \sigma_{rr} \, d\theta = \int_0^\pi -\gamma a \sin\theta \, d\theta = \int_0^{2\pi} A_0 \, d\theta + \sum_1^\infty A_n \int_0^{2\pi} \cos n\theta \, d\theta + \sum_1^\infty B_n \int_0^{2\pi} \sin n\theta \, d\theta$$

$$= -\gamma a (-\cos\theta)\Big|_0^\pi$$

$$= -\gamma a [1+1] = -2\gamma a = A_0 \cdot 2\pi + \sum A_n \left(\frac{\sin n\theta}{n}\right)\Big|_0^{2\pi} + \sum B_n \left(\frac{-\cos n\theta}{n}\right)$$

$$\Rightarrow \boxed{A_0 = -\frac{\gamma a}{\pi}}$$

∴

$$\int_0^{2\pi} \sigma_{rr} \sin n\theta \, d\theta = \int_0^\pi -\gamma a \sin\theta \sin m\theta \, d\theta = B_n \int_0^{2\pi} \sin^2 n\theta \, d\theta$$

$$= B_n \int_0^{2\pi} \frac{1 - \cos 2n\theta}{2} \, d\theta$$

$$= B_n \left[\frac{\theta}{2} - \frac{\sin 2n\theta}{4n}\right]_0^{2\pi}$$

$$-\gamma a \int_0^\pi$$

$$= \pi B_n$$

$C_m C_1 + S_m S_1 - C_m C_1 + S_m S_1$

$$\frac{1}{2}\int \left[\cos(m-1)\theta - \cos(m+1)\theta\right] d\theta$$

$$\frac{1}{2}\left[\frac{\sin(m-1)\theta}{m-1} - \frac{\sin(m+1)\theta}{m+1}\right]\Big|_0^\pi = 0 \quad m \ne 1$$

$$-\gamma a \int_0^\pi \sin^2\theta \, d\theta = -\gamma a \int_0^\pi \frac{1 - \cos 2\theta}{2} \, d\theta = -\gamma a \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^\pi$$

$$= -\frac{\pi \gamma a}{2} \quad m = 1$$

$$\therefore -\frac{\pi \gamma a}{2} = \pi B_1 \quad \therefore \boxed{B_1 = -\frac{\gamma a}{2}}$$

$$\int_0^{2\pi} \sigma_{rr} \cos n\theta \, d\theta = \int_0^\pi -\gamma a \sin\theta \cos n\theta \, d\theta = \sum A_n \int_0^{2\pi} \cos^2 n\theta \, d\theta = \sum A_n \int_0^{2\pi} \left[\frac{1 + \cos 2n\theta}{2}\right] d\theta$$

$snc + cns \quad -[snc + cns]$

$$-\gamma a \int_0^\pi \left[\sin(n+1)\theta - \sin(n-1)\theta\right] d\theta \qquad = \sum A_n \left[\frac{\theta}{2} + \frac{\sin 2n\theta}{4n}\right]_0^{2\pi}$$

$$-\frac{\gamma a}{\pi}\left[-\frac{\cos(n+1)\theta}{n+1}+\frac{\cos(n-1)\theta}{n-1}\right]_0^{\pi}$$ $\cos n\pi \cos\pi$ $(-1)\cos n\pi$ $(-1)^n$

$$-\frac{\gamma a}{2\pi}\left[-\frac{1}{n+1}(-1)^{n+1}+\frac{1}{n+1}+\frac{1}{n-1}(-1)^{n-1}-\frac{1}{n-1}\right]=A_n \quad n\geq 2$$

if n is odd $(-1)^{n+1}=1$ $(-1)^{n-1}=1$ $\therefore A_n=0$

$$-\frac{\gamma a}{2}\int_0^{\pi}\sin 2\theta\, d\theta=-\frac{\gamma a}{2\pi}\left(\frac{-\cos 2\theta}{2}\right)\Big|_0^{\pi}=-\frac{\gamma a}{2\pi}\left[\frac{-1+1}{2}\right]=0=A_1$$

if n is even $(-1)^{n+1}=-1$ $(-1)^{n-1}=-1$

$$-\frac{\gamma a}{2\pi}\left[\frac{2}{n+1}-\frac{2}{n-1}\right]=A_n \quad n \text{ is } ~~odd~~ \text{ even}$$

ie let ~~[illegible]~~ $2n-2-2n-2$

$$\frac{-4}{(n+1)(n-1)}$$

$$\sigma_{rr}\Big)_{r=a}=\sum_{\substack{n=0\\ even}}^{\infty}\frac{+2\gamma a}{(n+1)(n-1)\pi}\cos n\theta-\frac{\gamma a}{2}\sin\theta=-\gamma a\sin\theta$$

$$\sigma_{rr}\Big)_{r=b}\sum_{\substack{n=0\\ even}}^{\infty}\frac{2q_0}{(n+1)(n-1)\pi}\cos n\theta-\frac{q_0}{2}\sin\theta=-q_0\sin\theta$$

by our results $A_0=-\frac{2\gamma a}{\pi}$ $B_1=-\frac{\gamma a}{2}$ $A_0'=-\frac{q_0}{\pi}$ $B_1'=-\frac{q_0}{2}$

$A_n=\frac{2\gamma a}{\pi(n^2-1)}$ for $n=$ even $\quad A_n'=\frac{+2q_0}{\pi(n^2-1)}$ for n even

now since the tractions are radial $C_i, D_i, C_i', D_i'\equiv 0$

Equil (Moment) is identically satisfied $a^2C_0=b^2C_0'$ since $C_0, C_0'=0$

$b(A_1'-D_1')=a(A_1-D_1)\equiv 0$ since non zero A_i, A_i' are for i even.

$b(B_1'+C_1')=a(B_1+C_1)=a\left(+\frac{\gamma a}{2}\right)=b\left(+\frac{q_0}{2}\right)$ $\therefore$ $\boxed{q_0=\frac{a^2\gamma}{b}}$

we take $d_0=0$, $c_0=0$ for the ring

$\therefore \frac{a_0}{a^2}+2b_0=-\frac{\gamma a}{\pi}$ $\quad\therefore a_0+2a^2b_0=-\frac{\gamma a^3}{\pi}$

$\frac{a_0}{b^2}+2b_0=-\frac{q_0}{\pi}=-\frac{a^2\gamma}{\pi b}$ $\quad a_0+2b^2b_0=-\frac{a^2b\gamma}{\pi}$

$$(2a^2-2b^2)b_0=\frac{a^2b\gamma}{2\pi}-\frac{2\gamma a^3}{2\pi}=\frac{\gamma a^2}{2\pi}(b-a)$$

$(a^2-b^2)b_0=\frac{\gamma a^2}{2\pi}(a-b)$

$$\boxed{b_0=-\frac{\gamma a^2}{2\pi(a+b)}}$$

$$\boxed{a_0'=0}$$

$$a_0=-\frac{2a^2b\gamma}{2\pi}+\frac{2b^2\gamma a^2}{2\pi(a+b)}=\frac{a^2b\gamma}{2\pi(a+b)}\left[-2(a+b)+2b\right]=\frac{-2a^3b\gamma}{2\pi(a+b)}$$

$$a_0=-\frac{a^2b\gamma}{\pi}+\frac{2b^2\gamma a^2}{\pi(a+b)}=\frac{2\gamma a^2b}{2\pi}\left[-1+\frac{b}{a+b}\right]=-\frac{2\gamma a^3b}{2\pi(a+b)}$$

$$\boxed{a_0=-\frac{2\gamma a^3b}{2\pi(a+b)}}$$

$$\frac{2a_1}{a}=0 \Rightarrow \boxed{a_1=0\Rightarrow b_1'=0\Rightarrow b_1=a_1'=0 \quad \text{since } \frac{a}{b^3}-\frac{b}{a^3}\neq 0}$$

$$+\frac{2c_1}{a}=+\frac{\gamma a}{2} \qquad \boxed{c_1=\frac{\gamma a^2}{4}} \qquad \boxed{d_1'=-\frac{\gamma a^2}{8}\frac{(1-2\nu)}{(1-\nu)}}$$

$$2d_1a-\frac{2c_1'}{a^3}=+\frac{\gamma a}{8}\left(\frac{1-2\nu}{1-\nu}\right)$$

$$2d_1b-\frac{2c_1'}{b^3}=\frac{\gamma a^2}{8b}\left(\frac{1-2\nu}{1-\nu}\right)$$

$$\left(-\frac{2b}{a^3}+\frac{2a}{b^3}\right)c_1'=\left(\frac{\gamma ab}{8}-\frac{\gamma a^3}{8b}\right)\left(\frac{1-2\nu}{1-\nu}\right)$$

$$2\left[\frac{-b^4+a^4}{a^3b^3}\right]c_1'=\frac{\gamma a}{8b}\left[b^2-a^2\right]\left(\frac{1-2\nu}{1-\nu}\right)$$

$$2\left[\frac{(a^2-b^2)(a^2+b^2)}{a^3b^2}\right]c_1'=\frac{-\gamma a}{8}\left[a^2-b^2\right]\left(\frac{1-2\nu}{1-\nu}\right)$$

$$\boxed{c_1'=\frac{-\gamma a^4b^2}{16(a^2+b^2)}\left[\frac{1-2\nu}{1-\nu}\right]}$$

$$2d_1 a = \frac{\gamma a}{8}\left(\frac{1-2\nu}{1-\nu}\right) - \frac{2}{a^2}\left[\frac{\gamma a^4 b^2}{16(a^2+b^2)}\right]\left(\frac{1-2\nu}{1-\nu}\right) = \frac{\gamma a}{8}\left(\frac{1-2\nu}{1-\nu}\right)\left\{1 - \frac{b^2}{a^2+b^2}\right\}$$

$$\boxed{d_1 = \frac{\gamma a^2}{16(a^2+b^2)}\left(\frac{1-2\nu}{1-\nu}\right)}$$

$$\left\{a_n n(1-n)a^{n-2} + b_n \overset{(2-n)(1+n)}{(2+n-n^2)}a^n - a_n' n(1+n)a^{-n-2} + b_n' \overset{(2+n)(1-n)}{(2-n-n^2)}a^{-n}\right\} = \frac{2\gamma a}{\pi(n^2-1)}$$

$$\left\{a_n n(1-n)b^{n-2} + b_n(2+n-n^2)b^n - a_n' n(1+n)b^{-n-2} + b_n'(2-n-n^2)b^{-n}\right\} = \frac{2\gamma a^2}{b\pi(n^2-1)}$$

$$\left\{a_n(n-1)a^{n-2} + b_n(n+1)a^n - a_n'(n+1)a^{-n-2} - b_n'(n-1)a^{-n}\right\} = 0$$ for all even n

$$\left\{a_n(n-1)b^{n-2} + b_n(n+1)b^n - a_n'(n+1)b^{-n-2} - b_n'(n-1)b^{-n}\right\} = 0$$

for n odd the rhs $= 0$

det of coeffs.

$$\begin{vmatrix} n(1-n)a^{n-2} & (2-n)(1+n)a^n & -n(1+n)a^{-n-2} & (2+n)(1-n)a^{-n} \\ n(1-n)b^{n-2} & (2-n)(1+n)b^n & -n(1+n)b^{-n-2} & (2+n)(1-n)b^{-n} \\ (n-1)a^{n-2} & (n+1)a^n & -(n+1)a^{-n-2} & -(n-1)a^{-n} \\ (n-1)b^{n-2} & (n+1)b^n & -(n+1)b^{-n-2} & -(n-1)b^{-n} \end{vmatrix}$$

$$= -(1-n)^2(1+n)^2\begin{vmatrix} n a^{n-2} & (2-n)a^n & n a^{-n-2} & (2+n)a^{-n} \\ n b^{n-2} & (2-n)b^n & n b^{-n-2} & (2+n)b^{-n} \\ -a^{n-2} & a^n & a^{-n-2} & a^{-n} \\ -b^{n-2} & b^n & b^{-n-2} & b^{-n} \end{vmatrix} = -(1-n^2)^2 a^{-2n-4} b^{-2n-4}\begin{vmatrix} na^{2n} & (2-n)a^{2n+2} & n & (2+n)a^2 \\ nb^{2n} & (2-n)b^{2n+2} & n & (2+n)b^2 \\ -a^{2n} & a^{2n+2} & 1 & a^2 \\ -b^{2n} & b^{2n+2} & 1 & b^2 \end{vmatrix}$$

$$na^{2n}\begin{vmatrix} (2-n)b^{2n+2} & n & (2+n)b^2 \\ a^{2n+2} & 1 & a^2 \\ b^{2n+2} & 1 & b^2 \end{vmatrix} - nb^{2n}\begin{vmatrix} (2-n)a^{2n+2} & n & (2+n)a^2 \\ a^{2n+2} & 1 & a^2 \\ b^{2n+2} & 1 & b^2 \end{vmatrix} - a^{2n}\begin{vmatrix} (2-n)a^{2n+2} & n & (2+n)a^2 \\ (2-n)b^{2n+2} & n & (2+n)b^2 \\ b^{2n+2} & 1 & b^2 \end{vmatrix} + b^{2n}\begin{vmatrix} (2-n)a^{2n+2} & n & (2+n)a^2 \\ (2-n)b^{2n+2} & n & (2+n)b^2 \\ a^{2n+2} & 1 & a^2 \end{vmatrix}$$

$$na^{2n}\left[(2-n)b^{2n+4} + na^2b^{2n+2} + (2+n)b^2a^{2n+2} - (2+n)b^{2n+4} - na^{2n+2}b^2 - (2-n)b^{2n+2}a^2\right]$$

$$na^{2n}\left[-2nb^{2n+4} + (2n+2)a^2b^{2n+2} + 2b^2a^{2n+2}\right]$$

$$-nb^{2n}\left[2na^{2n+4} + (2-2n)b^2a^{2n+2} - 2a^2b^{2n+2}\right]$$

$$-a^{2n}\left[(2n-4)a^{2n+2}b^2 + 2n^2b^{2n+4} + (4-2n-2n^2)a^2b^{2n+2}\right]$$

$$b^{2n}\left[-2n^2a^{2n+4} + (2n^2+2n-4)b^2a^{2n+2} + (4-2n)b^{2n+2}a^2\right]$$

$$-nb^{2n}\left[(2-n)b^2a^{2n+2} + na^2b^{2n+2} + (2+n)a^{2n+4} - (2+n)a^2b^{2n+2} - na^{2n+2}b^2 - (2-n)a^{2n+4}\right]$$

$$-a^{2n}\left[n(2-n)a^{2n+2}b^2 + n(2+n)b^{2n+4} + (4-n^2)a^2b^{2n+2} - (2+n)na^2b^{2n+2} - n(2-n)b^{2n+4} - (4-n^2)a^{2n+2}b^2\right]$$

$$b^{2n}\left[(2n-n^2)a^{2n+4} + (2n+n^2)b^2a^{2n+2} + (4-n^2)b^{2n+2}a^2 - (2n+n^2)a^{2n+4} - (2n-n^2)b^{2n+2}a^2 - (4-n^2)a^{2n+2}b^2\right]$$

$$-2n^2a^{2n}b^{2n+4} + (2n^2+2n)a^{2n+2}b^{2n+2} + 2na^{4n+2}b^2$$
$$-2n^2a^{2n+4}b^{2n} + (-2n+2n^2)b^{2n+2}a^{2n+2} + 2nb^{4n+2}a^2$$
$$(4-2n)a^{4n+2}b^2 - 2n^2b^{2n+4}a^{2n} + (2n^2+2n-4)a^{2n+2}b^{2n+2} =$$
$$-2n^2b^{2n}a^{2n+4} + (2n^2+2n-4)b^{2n+2}a^{2n+2} + (4-2n)b^{4n+2}a^2$$

$$\left(-4n^2a^{2n}b^{2n+4} + (8n^2+4n-8)a^{2n+2}b^{2n+2} + 4a^{4n+2}b^2 - 4n^2a^{2n+4}b^{2n} + 4b^{4n+2}a^2\right)\cdot -(n^2-1)^2(ab)^{-2n-4}$$

note this denom $\neq 0$ $\therefore$ unique values of a_n, b_n, a_n', b_n' are $\neq 0$

Am not going to do in detail but for n odd lhs is same rhs $= 0$

since det $\neq 0$ $\Rightarrow$ $\boxed{a_n, b_n, a_n', b_n' \equiv 0 \quad \forall \text{ odd } n}$

Now for c_n, d_n, c_n', d_n' since the matrix of coeff are the same then the det of the coff are not $= 0$. However $A_n, A_n', B_n, B_n' = 0 \Rightarrow$ $\boxed{c_n, d_n, c_n', d_n' = 0}$

$\therefore$ the non zero terms are: $a_0, b_0, c_1, d_1', c_1', d_1$ $\{a_n, b_n\ a_n', b_n'$ n even$\}$

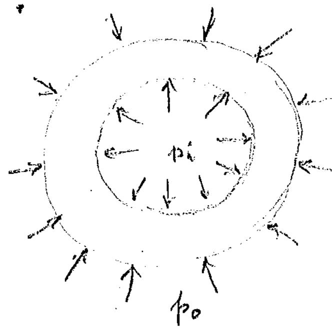
$$a_0 = \frac{-\gamma a^3 b}{\pi(a+b)} \qquad b_0 = \frac{-\gamma a^2}{2\pi(a+b)} \qquad c_1 = \frac{\gamma a^2}{4} \qquad d_1' = -\frac{\gamma a^2}{8}\left(\frac{1-2\nu}{1-\nu}\right) \qquad c_1' = \frac{-\gamma a^4 b^2}{16(a^2+b^2)}\left(\frac{1-2\nu}{1-\nu}\right)$$

$$d_1 = \frac{\gamma a^2}{16(a^2+b^2)}\left(\frac{1-2\nu}{1-\nu}\right)$$

$$\sigma_{rr} = \frac{a_0}{r^2} + 2b_0 - \frac{2c_1}{r}\sin\theta + \left(2d_1 r - \frac{2c_1'}{r^3} + \frac{d_1'}{r}\right)\sin\theta + \sum_{\substack{n=2\\ \text{even}}}^{\infty}\{a_n n(1-n)r^{n-2} + b_n(2+n-n^2)r^n - a_n' n(1+n)r^{-n-2}$$
$$+ b_n'(2-n-n^2)r^{-n}\}\cos n\theta$$

$$\sigma_{r\theta} = -\cos\theta\left\{2d_1 r - \frac{2c_1'}{r^3} + \frac{d_1'}{r}\right\} + \sum_{\substack{n=2\\ \text{even}}}^{\infty} n\sin n\theta\{a_n(n-1)r^{n-2} + b_n(n+1)r^n - a_n'(n+1)r^{-n-2} - b_n'(n-1)r^{-n}\}$$

$$\sigma_{\theta\theta} = -\frac{a_0}{r^2} + 2b_0 + \left(6d_1 r + 2c_1'/r^3 + \frac{d_1'}{r}\right)\sin\theta + \sum_{\substack{n=2\\ \text{even}}}^{\infty}\{a_n n(n-1)r^{n-2} + b_n(n+2)(n+1)r^n + n(n+1)a_n' r^{-n-2} + b_n'(n^2-3n+2)$$
$$\}\cos n\theta$$

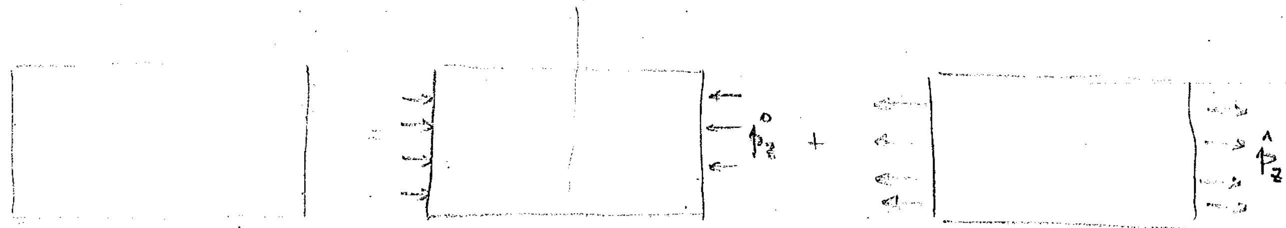


for plane strain

$$\sigma_z = \nu(\sigma_{rr} + \sigma_{\theta\theta}) = \frac{2\nu}{(b/a)^2 - 1}\left[p_i - \left(\frac{b}{a}\right)^2 p_o\right]$$

now since this is not r, θ dependent σ_z = constant

to get rid of this



$T_z = \overset{\circ}{T}_z + \hat{T}_z = 0$ $\qquad = \overset{\circ}{T}_z = \overset{\circ}{\sigma}_z$ $\qquad = \hat{T}_z = \hat{\sigma}_z$

$$\hat{T}_z = -\overset{\circ}{T}_z = -\overset{\circ}{\sigma}_z = \hat{\sigma}_z = \frac{-2\nu}{(b/a)^2 - 1}\left[p_i - \left(\frac{b}{a}\right)^2 p_o\right]$$

This means we apply to our problem a constant state of stress

ie $\hat{\sigma}_z = \dfrac{-2\nu}{(b/a)^2 - 1}\left[p_i - \left(\frac{b}{a}\right)^2 p_o\right]$ $\qquad \hat{\sigma}_n = 0 \qquad \hat{\sigma}_{\theta\theta} = 0 = \hat{\sigma}_{r\theta}$ on boundary

we had done this problem in class and this leds to
$\hat{\sigma}_{rr} = 0$ everywhere
$\hat{\sigma}_{\theta\theta} = 0$ everywhere
$\hat{\sigma}_{r\theta} = 0$ everywhere
$\hat{\sigma}_{\theta z} = 0$ "
$\hat{\sigma}_{zr} = 0$ "

if if $\sigma_{zz} \neq 0$ $\hat{\sigma}_{rr} = \hat{\sigma}_{\theta\theta} = \hat{\sigma}_{r\theta} = 0 \Rightarrow \hat{\sigma}_{xx} = \hat{\sigma}_{xy} = \hat{\sigma}_{yy} = 0$

Thus the plane stress case did not change the σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ of the original problem.

However $\hat{\sigma}_{zz} \neq 0$ leads to

$$\hat{\epsilon}_{zz} = \frac{\hat{\sigma}_{zz}}{E} = \text{constant} \quad \therefore \quad \hat{w} = \frac{\hat{\sigma}_{zz}}{E} z$$

$$\hat{\epsilon}_{xx} = -\frac{\nu}{E}\hat{\sigma}_{zz} = \text{const}$$

$$\hat{\epsilon}_{yy} = -\frac{\nu}{E}\hat{\sigma}_{zz} = \text{"}$$

thus the displacements will be modified.

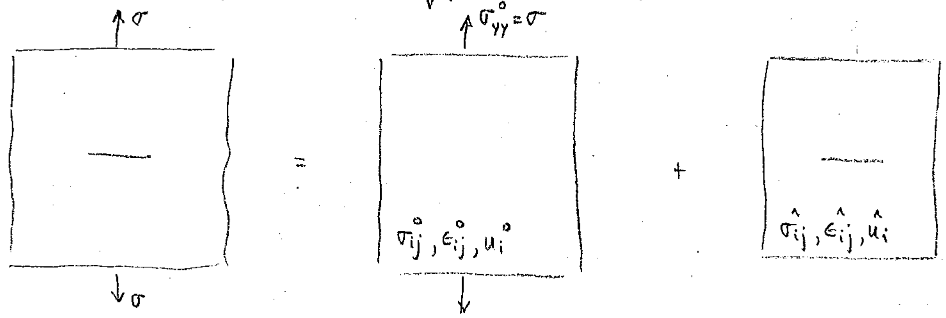
This say we can get the plane stress problem from the plane strain problem by saying that $\sigma_{r\theta}$, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ ~~also~~ is the same and setting $\sigma_{zz} = 0$.
& sol is exact.

ME238B
Prof Barnett
Cesar Levy

100

HW Set # 3

1. We solve this by superposition of plane strain problems



1a. ° fields $\sigma^\circ_{yy}=\sigma$, all $\sigma^\circ_{ij}=0$ for all others

$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma^\circ_{ij} - \frac{\nu}{E}(\sigma^\circ_{kk})\delta_{ij}$ and since we have 2 non zero strains $\epsilon^\circ_{xx}, \epsilon^\circ_{yy}$ (each contain σ°_{yy}) then

$$\epsilon^\circ_{zz} = \frac{1+\nu}{E}\sigma^\circ_{zz} + \frac{\nu}{E}(\sigma^\circ_{kk}) = 0 \Rightarrow \sigma^\circ_{zz} = \nu\sigma^\circ_{yy} = \nu\sigma \qquad \epsilon_{xy}^\circ = 0 \Rightarrow \frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x} = 0$$

$$\epsilon^\circ_{xx} = \frac{1+\nu}{E}\sigma^\circ_{xx} - \frac{\nu}{E}(\sigma^\circ_{kk}) = -\frac{\nu}{E}(\sigma^\circ_{zz}+\sigma^\circ_{yy}) = -\frac{\nu}{E}(1+\nu)\sigma \qquad \epsilon^\circ_{xz} = 0 \Rightarrow \frac{\partial w^\circ}{\partial x} = 0$$

$$\epsilon^\circ_{yy} = \frac{1+\nu}{E}\sigma^\circ_{yy} - \frac{\nu}{E}(\sigma^\circ_{kk}) = \frac{1+\nu}{E}\sigma - \frac{\nu}{E}(\sigma+\nu\sigma) = \frac{1-\nu^2}{E}\sigma \qquad \epsilon^\circ_{yz} = 0 \Rightarrow \frac{\partial w^\circ}{\partial y} = 0 \quad \Big\} \; w^\circ = \text{const}$$

Now since

$$\left.\begin{aligned}\epsilon^\circ_{xx} &= \frac{\partial u^\circ}{\partial x} = -\frac{\nu}{E}(1+\nu)\sigma \quad \text{then} \quad u^\circ = -\frac{\nu(1+\nu)}{E}\sigma x + f(y)\\ \epsilon^\circ_{yy} &= \frac{\partial v^\circ}{\partial y} = \frac{1-\nu^2}{E}\sigma \quad \text{then} \quad v^\circ = \frac{1-\nu^2}{E}\sigma y + h(x)\end{aligned}\right\} \Rightarrow \epsilon^\circ_{xy} = 0 \Rightarrow \begin{aligned} f(y) &= c_1 y + c_2 \\ g(x) &= -c_1 x + c_3\end{aligned}$$

thus f, g represent rigid body rotation and displacements and may be taken as 0.

Define $y^+ = \lim_{y \downarrow 0^+}$ $\quad y^- = \lim_{y\uparrow 0^-}$

1b. Now $T_x^\circ = \sigma^\circ_{xj}n_j$; on $y=0^+$ $n_j = n_y = -1$ $\therefore$ $T_x^\circ|_{y^+} = -\sigma^\circ_{xy}|_{y^+} = 0$

$T_y^\circ = \sigma^\circ_{yj}n_j$; on $y=0^+$ $n_j = n_y = -1$ $\therefore$ $-\sigma = T_y^\circ|_{y^+} = -\sigma^\circ_{yy}$ $\therefore$ $\sigma^\circ_{yy}|_{y^+} = \sigma$

on $y=0^-$ $n_j = n_y = 1$

$T_x^\circ = \sigma^\circ_{xj}n_j$; $\quad T_x^\circ|_{y^-} = \sigma^\circ_{xy}|_{y^-} = 0$

$T_y^\circ = \sigma^\circ_{yj}n_j$; $\quad +\sigma = T_y^\circ|_{y^-} = \sigma^\circ_{yy}$ $\quad \therefore \; \sigma^\circ_{yy}|_{y^-} = \sigma$

- Now $\hat T_x = T_x - \overset{\circ}{T}_x$ we will now prove that $\hat\sigma_{xy}(x,y) = -\hat\sigma_{xy}(x,-y)$ and hence $\hat\sigma_{xy}|_{y=0} \equiv 0$

Since v is an even fn in x, odd fn in y and u is even in y and odd in x then $\frac{\partial v}{\partial x}$ is odd in y and odd in x, while $\frac{\partial u}{\partial y}$ is odd in y, odd in x. Thus $\sigma_{xy}(x,y) \simeq \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ but this gives that $\sigma_{xy}(x,y) = -\sigma_{xy}(-x,y) = -\sigma(x,-y)$. Since $\sigma^\circ_{xy} = 0$ then $\hat\sigma_{xy}(x,y) = -\hat\sigma_{xy}(-x,y) = -\hat\sigma_{xy}(x,-y)$ QED;

- now $\overset{\circ}{T}_x \equiv 0$ on $y=0$ $\quad \therefore \quad \hat T_x = 0$ on $y=0$ or $\hat\sigma_{xy}(x,0) = 0$

on the hole $T_y = 0$ now $\hat{T}_y\big|_{y+} = T_y\big|_{y+} - \mathring{T}_y\big|_{y+} = 0 - (-\sigma) = \sigma = -\hat{\sigma}_{yy}$ since $\hat{T}_y\big|_{y+} = \hat{\sigma}_{yj} n_j$ and $n_j = n_y = -1$. Similarly on y^- $n_j = n_y = 1$ $\hat{T}_y\big|_{y-} = \hat{\sigma}_{yj} n_j = \hat{\sigma}_{yy} = T_y\big|_{y-} - \mathring{T}_y\big|_{y-} = -\sigma$

$\therefore$ $\hat{\sigma}_{yy} = -\sigma$ on both sides of the cut

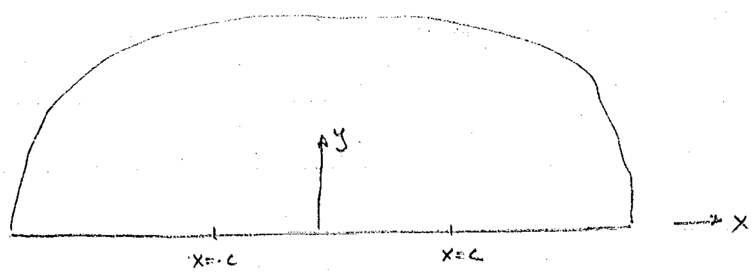
we had shown that $v(x,y) = -v(x,-y)$. Away from the cut (where v is discontinuous) the body is continuous $\therefore$ $v(x,0) = -v(x,0) = 0$ for $|x| \geq c$. Similarly $\mathring{v}(x,0) = 0 \Rightarrow |x| > c$, $\hat{v}(x,0) = 0$

- also at ∞ we know that $\hat{T}_x = T_x - \mathring{T}_x = 0$ and $\hat{T}_y = T_y - \mathring{T}_y = \sigma - \sigma = 0$

Thus at ∞ $\hat{\sigma}_{ij} \to 0$ as $x^2 + y^2 \to \infty$

- By the fact that we know what is happening for $y \geq 0$ & $|x| < \infty$ we can use these results to determine what's happening below the x axis.

We can then look at the following problem on a half space.



$$\left.\begin{array}{lll} \hat{\sigma}_{xy} = 0 & -\infty < x < \infty, & y = 0 \\ \hat{\sigma}_{yy} = -\sigma & |x| < c, & y = 0 \\ \hat{v} = 0 & |x| > c, & y = 0 \end{array}\right\} \text{B.C.}$$

and $\hat{\sigma}_{ij} \to 0$ as $x^2 + y^2 \to \infty$

c. From previous results use the stress for $\phi(x,y) = \int_0^\infty e^{-\lambda y} \cos\lambda x \left[A(\lambda) + y B(\lambda)\right] d\lambda$

$\hat{\sigma}_{yy} = \dfrac{\partial^2 \phi}{\partial x^2} = \int_0^\infty -\lambda^2 e^{-\lambda y} \cos\lambda x \left[A + yB\right] d\lambda$

on $y = 0$, $|x| < c$ then $\hat{\sigma}_{yy} = -\sigma = -\int_0^\infty \lambda^2 \cos\lambda x \, A(\lambda)\, d\lambda$ or $\sigma = \int_0^\infty \lambda^2 A \cos\lambda x \, d\lambda$ $\quad y = 0 \quad |x| < c$

$\hat{\sigma}_{xy} = -\dfrac{\partial^2 \phi}{\partial x \partial y} = \int_0^\infty \lambda \sin\lambda x \, e^{-\lambda y} \left[-A\lambda - B\lambda y + B\right] d\lambda$

on $y = 0$ $\hat{\sigma}_{xy} = 0$ $\forall x$ $\Rightarrow$ $-A\lambda + B = 0$ or $\underline{B = A\lambda}$

$\hat{\sigma}_{xx} = \dfrac{\partial^2 \phi}{\partial y^2} = -\int_0^\infty A\lambda^2 \cos\lambda x \, e^{-\lambda y} (1 - \lambda y)\, d\lambda$

and $\hat{\epsilon}_{yy} = \dfrac{\partial v}{\partial y} = \dfrac{\hat{\sigma}_{yy}}{E} - \dfrac{\nu}{E}(\hat{\sigma}_{xx} + \hat{\sigma}_{zz}) = \dfrac{\hat{\sigma}_{yy}}{E} - \dfrac{\nu}{E}\left([1+\nu]\hat{\sigma}_{xx} + \nu \hat{\sigma}_{yy}\right) = \dfrac{1-\nu^2}{E}\hat{\sigma}_{yy} - \dfrac{\nu(1+\nu)}{E}\hat{\sigma}_{xx}$

using the fact that ^ fields are to be solved for a plane strain solution

$\dfrac{\partial v}{\partial y} = \dfrac{1-\nu^2}{E} \int_0^\infty -\lambda^2 e^{-\lambda y} A \cos\lambda x (1 + \lambda y)\, d\lambda - \dfrac{\nu(1+\nu)}{E} \int_0^\infty -A\lambda^2 \cos\lambda x \, e^{-\lambda y}(1 - \lambda y)\, d\lambda$

$= -\dfrac{1+\nu}{E} \int_0^\infty A\lambda^2 e^{-\lambda y} \cos\lambda x \, (1 - 2\nu + \lambda y)\, d\lambda.$

this integrates to

$$\hat{v} = \frac{1+\nu}{E}\int_0^\infty A\lambda \cos\lambda x\, e^{-\lambda y}(2-2\nu+\lambda y)\,d\lambda + g(x)$$

on $y=0$ $\quad \hat{v} = \frac{2(1-\nu^2)}{E}\int_0^\infty A\lambda\cos\lambda x\,d\lambda + g(x)$

$$\hat{\epsilon}_{xx} = \frac{\partial \hat{u}}{\partial x} = \frac{1-\nu^2}{E}\hat{\sigma}_{xx} - \frac{\nu(1+\nu)}{E}\hat{\sigma}_{yy}$$

$$= \frac{1+\nu}{E}\int_0^\infty -A\lambda^2\cos\lambda x\, e^{-\lambda y}\left[(1-\lambda y)(1-\nu) - \nu(1+\lambda y)\right]d\lambda$$

$$= \frac{1+\nu}{E}\int_0^\infty -A\lambda^2\cos\lambda x\, e^{-\lambda y}(1-2\nu-\lambda y)\,d\lambda$$

$$\hat{u} = -\left(\frac{1+\nu}{E}\right)\int_0^\infty A\lambda\sin\lambda x\, e^{-\lambda y}(1-2\nu-\lambda y)\,d\lambda + f(y)$$

now $\hat{\epsilon}_{xy} = \frac{1}{2}\left(\frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{v}}{\partial x}\right) = -\left(\frac{1+\nu}{E}\right)\int_0^\infty A\lambda^2 \sin\lambda x\, e^{-\lambda y}\lambda y\,d\lambda + f'(y) + g'(x)$

along $y=0$ $|x|>c$ $\quad \hat{\sigma}_{xy} = 0 \Rightarrow f'(0)+g'(x) = 0 \ \forall x \Rightarrow g'(x) = -f'(0) = \text{const} = c_1$

$\therefore g(x) = c_1 x + c_2$; as $x^2+y^2 \to \infty$ $\hat{\sigma}_{xy} \to 0 \Rightarrow f'(y) + g'(x) = 0$ or $-f'(y) = g'(x) = +c = c_1$

but $\hat{v} \to 0$ as $x^2+y^2\to\infty$, $\hat{u}\to 0$ too $\therefore g(x), f(y) \to 0 \Rightarrow c_1 = 0,\ c_2 = 0 \Rightarrow f(y) = c_3$

$\Rightarrow c_3 = 0 \Rightarrow f(y), g(x) = 0$

thus $\hat{v}\Big|_{\substack{y=0\\|x|>c}} = \frac{2(1-\nu^2)}{E}\int_0^\infty A\lambda\cos\lambda x\,d\lambda = 0$ or $\int_0^\infty \lambda A\cos\lambda x\,d\lambda = 0$ $\quad y=0\ |x|>c$

d. Now let me write

$$\frac{1}{\sigma c}\int_0^\infty \lambda^2 A\cos\lambda x\,d\lambda = \frac{1}{c} \quad \text{for } x<c$$

and $$\frac{1}{\sigma c}\int_0^\infty \lambda A\cos\lambda x\,d\lambda = 0 \quad \text{for } x>0$$

thus $\frac{\lambda^2 A}{\sigma c} = J_1(c\lambda)$ or $\boxed{A = \frac{\sigma c}{\lambda^2}J_1(c\lambda)}$

thus $$\hat{\sigma}_{yy}\Big|_{y=0} = \int_0^\infty -\lambda^2 A e^{-\lambda y}\cos\lambda x(1+\lambda y)\,d\lambda\Big|_{y=0} = -c\sigma\int_0^\infty J_1(c\lambda)\cos\lambda x\,dx$$

$$= -c\sigma\frac{-c}{\sqrt{x^2-c^2}\left(x+\sqrt{x^2-c^2}\right)} \quad \text{for } x>c$$

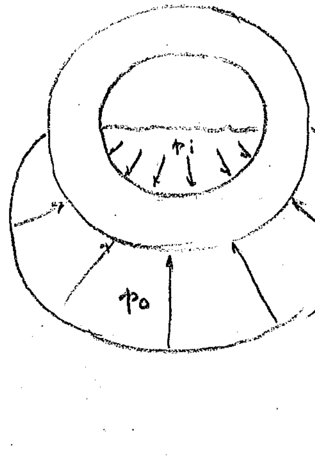
now for $x = c+\epsilon$ then $(x-c) = \epsilon$ $\quad x+c \sim 2c$

$$\hat{\sigma}_{yy} \cong \frac{c^2\sigma}{\sqrt{\epsilon}\sqrt{2c}}\left[\frac{1}{(c+\epsilon)+\sqrt{2c\epsilon}}\right] \cong \frac{c^2\sigma}{c\sqrt{2c\epsilon}+\epsilon\sqrt{2c\epsilon}+2c\epsilon} \cong \frac{c^2\sigma}{c\sqrt{2c\epsilon}} = \frac{\sigma\sqrt{c}}{\sqrt{2\epsilon}} = \frac{\sigma\sqrt{\pi c}}{\sqrt{2\pi\epsilon}}$$

$$= \frac{K_I}{\sqrt{2\pi\epsilon}}$$

PHEW!! That was long

2.



$p_i = \gamma a \sin\theta \qquad p_i = T_r^i = \sigma_{rj}^i n_j^i = -\sigma_{rr}^i \quad \therefore \sigma_{rr}^i = -p_i$

$p_0 = q_0 \sin\theta \qquad -p_0 = T_r^0 = \sigma_{rj}^0 n_j^0 = \sigma_{rr}^0 \quad \therefore \sigma_{rr}^0 = -p_0$

assume $(\sigma_{r\theta})_{r=a} = (\sigma_{r\theta})_{r=b} = 0.$

Now $(\sigma_{rr})_{r=a} = \begin{matrix} -\gamma a \sin\theta & 0 \le \theta \le \pi \\ 0 & 0 \le \theta \le 2\pi \end{matrix}$

$(\sigma_{rr})_{r=b} = \begin{matrix} -q_0 \sin\theta & 0 \le \theta \le \pi \\ 0 & 0 \le \theta \le 2\pi \end{matrix}$

let $\sigma_{rr}^i = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta$

$\sigma_{rr}^0 = A_0' + \sum_{n=1}^{\infty} A_n' \cos n\theta + \sum_{n=1}^{\infty} B_n' \sin n\theta$

using $\int_0^{2\pi} \sigma_{rr}^i \, d\theta = -2\gamma a = A_0 \cdot 2\pi \quad \therefore \quad A_0 = -\frac{\gamma a}{\pi} \qquad$ similarly $A_0' = \frac{-q_0}{\pi}$

$\int_0^{2\pi} \sigma_{rr}^i \sin n\theta \, d\theta = \begin{matrix} 0 & n > 1 \\ -\frac{\gamma a}{2}\pi & n = 1 \end{matrix} = \pi B_n \qquad B_1 = -\frac{\gamma a}{2} \qquad \text{"} \qquad B_1' = \frac{-q_0}{2}$

$\int_0^{2\pi} \sigma_{rr}^i \cos n\theta \, d\theta = \begin{matrix} 0 & n \text{ odd} \\ \frac{2\gamma a}{n^2-1} & n \text{ even} \end{matrix} = \pi A_n \qquad A_n = \frac{2\gamma a}{\pi(n^2-1)} \qquad \text{"} \qquad A_n' = \frac{2q_0}{\pi(n^2-1)}$

n even

Since radial tractions only $C_i, D_i, C_i', D_i' = 0 \quad \forall i$

Moment equilib is identically satisfied; the first force equil equation is identically satisfied

from the 2nd force equil eq $\quad b B_1' = a B_1 \quad \Rightarrow \quad \boxed{q_0 = \frac{a^2\gamma}{b}}$

we will take $\boxed{d_0 = c_0 = 0}$ for the ring

using the equations for a_0, b_0 involving A_0, A_0' gives $\boxed{b_0 = \frac{-\gamma a^4}{2\pi(a+b)} \text{ and } a_0 = \frac{-\gamma a^3 b}{\pi(a+b)}}$

Using the equation for $a_0' = a^2 c_0 \Rightarrow \boxed{a_0' = 0}$; from $\frac{2a_1}{a} = A_1 - D_1 \Rightarrow \boxed{a_1 = 0}$

since b_1' is linear in a_1 then $\boxed{b_1' = 0}$ and since $\frac{a^4 - b^4}{a^3 b^3} \neq 0$ then using $a_1, b_1' \Rightarrow \boxed{b_1 = a_1' = 0}$

Similarly from $\frac{-2c_1}{a} = B_1 + C_1$ then $\boxed{c_1 = \frac{\gamma a^2}{4}}$ and since d_1' depends on $c_1 \Rightarrow \boxed{d_1' = -\frac{\gamma a^2}{8}\left(\frac{1-2\nu}{1-\nu}\right)}$

using these into the eqns for c_1', d_1 we obtain

$\boxed{c_1' = \frac{-\gamma a^4 b^2}{16(a^2+b^2)}\left[\frac{1-2\nu}{1-\nu}\right]} \qquad \boxed{d_1 = \frac{\gamma a^2}{16(a^2+b^2)}\left[\frac{1-2\nu}{1-\nu}\right]}$

using the equations for A_n, A_n', D_n', D_n we obtain n even

$$a_n n(1-n)a^{n-2} + b_n(2-n)(1+n)a^n - a_n'(1+n)n a^{-n-2} + b_n'(2+n)(1-n)a^{-n} = \frac{2\gamma a}{\pi(n^2-1)}$$

$$a_n n(1-n)b^{n-2} + b_n(2-n)(1+n)b^n - a_n' n(1+n)b^{-n-2} + b_n'(2+n)(1-n)b^{-n} = \frac{2\gamma a^2}{\pi b(n^2-1)}$$

$$a_n(n-1)a^{n-2} + b_n(1+n)a^n - a_n'(1+n)a^{-n-2} - b_n'(n-1)a^{-n} = 0$$

$$a_n(n-1)b^{n-2} + b_n(n+1)b^n - a_n'(1+n)b^{-n-2} - b_n'(n-1)b^{-n} = 0$$

the determinant of the coeffs of a_n, b_n, a_n', b_n' is a real mess but $\neq 0$ $\therefore$ a solution exists. For odd n the rhs $= 0$ and since det $\neq 0$ $\Rightarrow$ $\boxed{a_n, b_n, a_n', b_n' = 0 \quad n \text{ odd}}$

for c_n, d_n, c_n', d_n' the rhs $= 0$ and the determinant of coeffs are identical to the one above $\Rightarrow$ $\boxed{c_n, d_n, c_n', d_n' = 0 \quad \forall n}$

Summarizing

$$a_0 = \frac{-\gamma a^3 b}{\pi(a+b)}; \quad b_0 = \frac{-\gamma a^2}{2\pi(a+b)}; \quad c_1 = \frac{\gamma a^2}{4}; \quad d_1' = -\frac{\gamma a^2}{8}\left(\frac{1-2\nu}{1-\nu}\right); \quad c_1' = \frac{-\gamma a^4 b^2}{16(a^2+b^2)}\left(\frac{1-2\nu}{1-\nu}\right);$$

$$d_1 = \frac{\gamma a^2}{16(a^2+b^2)}\left(\frac{1-2\nu}{1-\nu}\right).$$

Thus

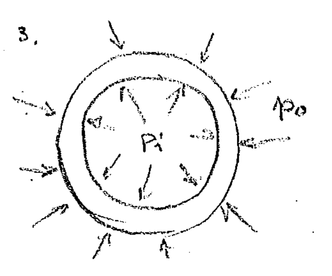
$$\sigma_{rr} = \frac{a_0}{r^2} + 2b_0 - \frac{2c_1}{r}\sin\theta + \left(2d_1 r - \frac{2c_1'}{r^3} + \frac{d_1'}{r}\right)\sin\theta + \sum_{\substack{n=2\\ \text{even}}}^{\infty}\left\{a_n n(1-n)r^{n-2} + b_n(2+n-n^2)r^n - a_n' n(1+n)\times r^{-n-2} + b_n'(2-n-n^2)r^{-n}\right\}\cos n\theta$$

$$\sigma_{r\theta} = -\cos\theta\left\{2d_1 r - \frac{2c_1'}{r^3} + \frac{d_1'}{r}\right\} + \sum_{\substack{n=2\\ \text{even}}}^{\infty} n\sin n\theta\left\{a_n(n-1)r^{n-2} + b_n(n+1)r^n - a_n'(n+1)r^{-n-2} - b_n'(n-1)r^{-n}\right\}$$

$$\sigma_{\theta\theta} = -\frac{a_0}{r^2} + 2b_0 + \left(6d_1 r + \frac{2c_1'}{r^3} + \frac{d_1'}{r}\right)\sin\theta + \sum_{\substack{n=2\\ \text{even}}}^{\infty}\left\{a_n n(n-1)r^{n-2} + b_n(n^2+3n+2)r^n + n(n+1)a_n' r^{-n-2} + b_n'(n^2-3n+2)r^{-n}\right\}\times \cos n\theta$$

$$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$$

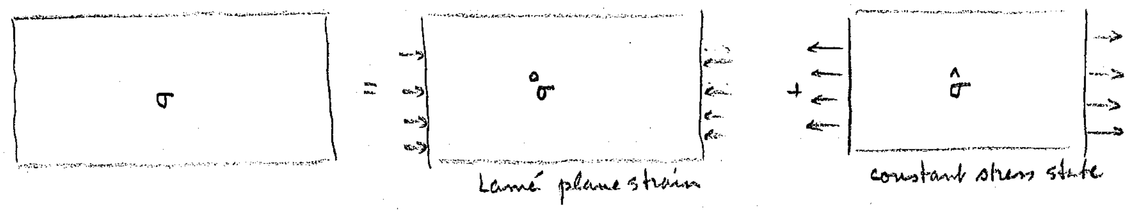
3.



for plane strain $\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) = \dfrac{2\nu}{(b/a)^2 - 1}\left[p_i - \left(\dfrac{b}{a}\right)^2 p_o\right]$

now since this is not r or θ dependent σ_{zz} = constant = p_z

to get rid of this add a solution which will produce $\sigma_{zz} = \ldots$



this is adding a constant stress state solution to the problem. We did this problem in class

pick a solution $\ni$. $\hat{\sigma}_{zz} = p_z$ and all other $\hat{\sigma}_{ij} = 0$ are zero everywhere else

$\Rightarrow \hat{\sigma}_{rr} = 0$ everywhere $\hat{\sigma}_{\theta\theta} = 0$ everywhere $\hat{\sigma}_{r\theta} = \hat{\sigma}_{\theta z} = \hat{\sigma}_{zr} = 0$ everywhere

Thus by superposing this solution $\sigma_{rr} = \mathring{\sigma}_{rr}$, $\sigma_{\theta\theta} = \mathring{\sigma}_{\theta\theta}$, $\mathring{\sigma}_{r\theta} = \sigma_{r\theta}$, $\sigma_{zz} = 0$

However $\hat{\sigma}_{zz} \neq 0 =$ constant adds a constant strain to all ϵ_{ii}

ie $\hat{\epsilon}_{zz} = \frac{\hat{\sigma}_{zz}}{E} =$ const, $\hat{\epsilon}_{xx} = -\hat{\sigma}_{zz}\frac{\nu}{E} = \hat{\epsilon}_{yy} =$ const $\therefore$ $\hat{W} =$ constant $\cdot z$

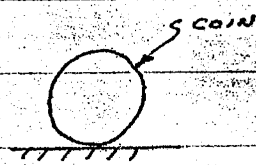
thus the displacements only have to be modified.

This says we can get the Lamé plane stress solution from the plane strain solution by saying that $\sigma_{r\theta}, \sigma_{\theta\theta}, \sigma_{rr}$ are the same but $\sigma_{zz} = 0$, and the strains are altered by a constant.

ME 238B MID-TERM EXAM

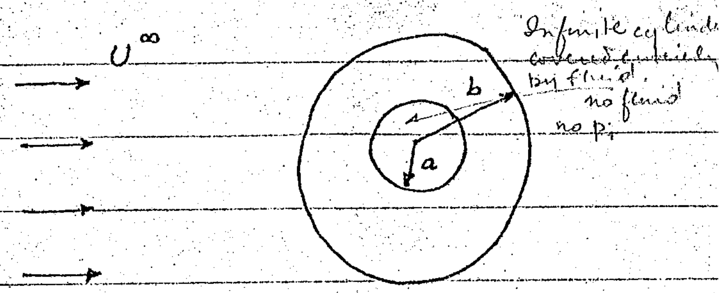
DUE FEB. 21, 1979

OPEN BOOKS; OPEN NOTES. ASSUME LINEAR ISOTROPIC ELASTICITY THROUGHOUT THE QUIZ.



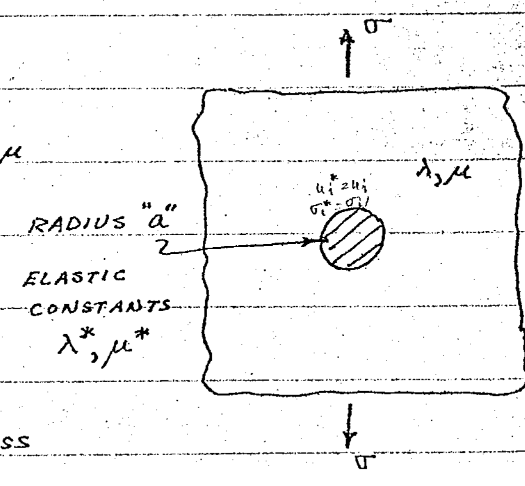
1. A uniform homogeneous coin of radius "a" stands on edge on a table. Compute the stress state in the coin. Do not go to the literature to seek a solution.

✓ 2. A hollow cylinder of inner and outer radii "a" and "b", respectively, is submerged in an infinite, incompressible, inviscid fluid flowing with a uniform velocity U^{∞} as shown. Determine the stress state in the cylinder at steady flow. Clearly state all assumptions you are making in formulating the associated boundary value problem.



use potential flow solution w/ $p_i = 0$ p_o whatever is developed

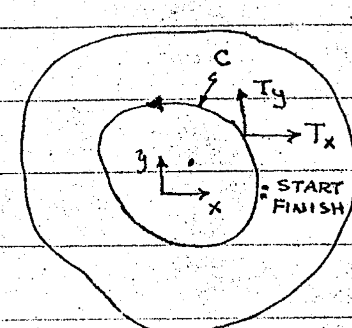
✓ 3. An infinite plate of elastic constants λ and μ contains a circular cylindrical inclusion of radius "a" and elastic constants λ^* and μ^*. The plate is subjected to a uniform tensile stress



σ at infinity. Assuming plane strain deformation, formulate a boundary value problem in elasticity to describe the deformation in this composite system, assuming that the inclusion is "perfectly welded" to the plate material. Do not attempt to solve this problem, but instead choose stress functions for the plate and for the inclusion which would, in principle, allow you to obtain a solution -- i.e., specify the stress functions but do not determine any constant coefficients which may appear therein.

Using your intuition and sound reasoning, do you expect the stress concentration for a very rigid inclusion ($\lambda^* \gg \lambda$, $\mu^* \gg \mu$) to be greater than or less than 3? Please indicate your reasoning clearly.

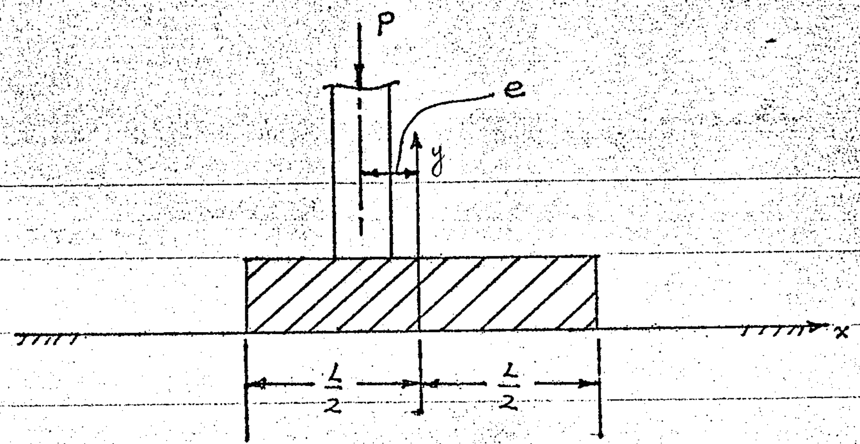
✓ 4. Consider a closed curve C within an isotropic linear elastic solid which is in a state of plane strain deformation. Derive an expression for



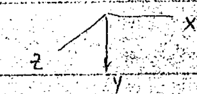
$$\oint_C M_z \, ds$$

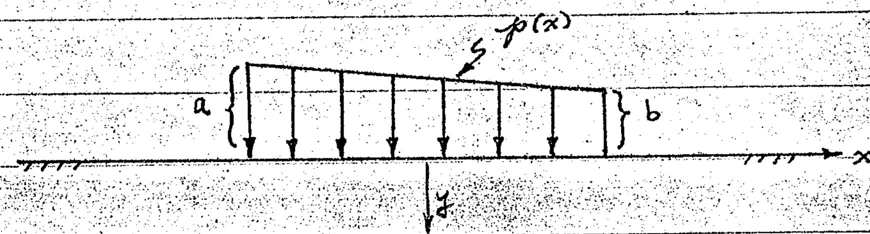
where M_z is the moment about the z-axis due to tractions T_x and T_y acting at ds on C.

5. ✓



look at Green's fn for half space w/ load





A wall, eccentrically located along a footing, carries a load P (per unit length). An assumption commonly made by foundation engineers is that the pressure under such a footing would be distributed as shown. Use this assumption, neglect the weight of the footing, and assume the structure is very long in the z-direction. (plane strain)

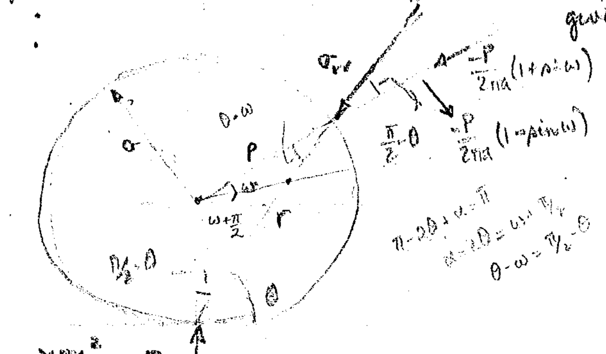
(i) Determine a and b.

(ii) Determine the stress field in the soil, assuming the soil behaves as a linear isotropic elastic continuum.

(iii) Soil borings indicate a soft clay layer is located 100 feet below the footing. It is predicted that the footing will settle an undue amount if the vertical pressure on this layer exceeds 800 psi. × 144 = psf Using the results of (ii), taking $L = 10'$, $e = 1'$, and $P = 120{,}000$ pounds per foot of wall, estimate if the settlement of the footing will be acceptable.

for the stress fn. $\phi_A = \frac{P\theta}{\pi} r\cos\theta$ we know that this is the stress fn for the half space which gives

A



$\sigma_{rr} = -\frac{2P}{\pi r}\sin\theta$, and for $r = 2a\cos(\pi/2 - \theta) = 2a\sin\theta \Rightarrow \sigma_{rr} = -P/\pi a$

$\sigma_{\theta\theta} = 0$

$\sigma_{r\theta} = 0$

$\sigma_{\rho\rho} = \sigma_{rr}\cos^2(\pi/2 - \theta) + \sigma_{\theta\theta}\sin^2(\pi/2-\theta) + 2\sigma_{r\theta}\sin(\pi/2-\theta)\cos(\pi/2-\theta)$

$\sigma_{\omega\omega} = \sigma_{rr}\sin^2(\pi/2-\theta) + \sigma_{\theta\theta}\cos^2(\pi/2-\theta) - 2\sigma_{r\theta}\sin(\pi/2-\theta)\cos(\pi/2-\theta)$

$\sigma_{\rho\omega} = (\sigma_{rr} - \sigma_{\theta\theta})\sin(\pi/2-\theta)\cos(\pi/2-\theta) + \sigma_{r\theta}\cos(\pi - 2\theta)$

$\gamma\pi a^2 = P$

$\therefore \frac{P}{2\pi a} = \frac{\gamma\pi a^2}{2\pi a} = \frac{\gamma a}{2}$

but $\theta = \frac{1}{2}(\omega + \pi/2)$

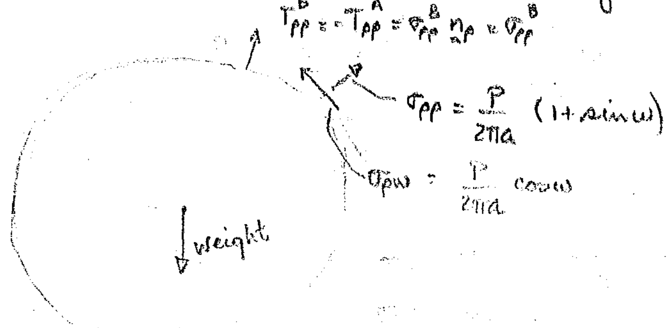
$T^A_{\rho\rho} = \sigma_{\rho\rho} n_\rho$ $\therefore$

$\sigma^A_{\rho\rho} = -\frac{P}{\pi a}\sin^2\theta = -\frac{P}{2\pi a}(1 - \cos 2\theta) = -\frac{P}{2\pi a}[1 - \cos(\omega + \pi/2)] = -\frac{P}{2\pi a}(1 + \sin\omega)$

$\sigma^A_{\omega\omega} = -\frac{P}{\pi a}\cos^2\theta = -\frac{P}{2\pi a}(1 + \cos 2\theta) = -\frac{P}{2\pi a}[1 + \cos(\omega + \pi/2)] = -\frac{P}{2\pi a}[1 - \sin\omega]$

$\sigma_{\rho\omega} = -\frac{P}{\pi a}\cos\theta\sin\theta = -\frac{P}{2\pi a}\sin 2\theta = -\frac{P}{2\pi a}[\sin(\omega + \pi/2)] = -\frac{P}{2\pi a}\cos\omega$

B Now we will look at solutions to the following



$\sigma_{\rho\rho} = -\frac{2P}{\pi r}\sin\theta$

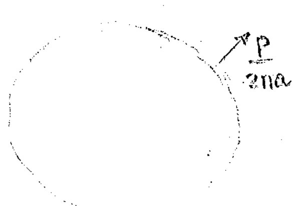
$\frac{\pi}{2} - \theta + \omega + \frac{\pi}{2} = (\pi + \omega - \theta) + \alpha = \pi$

$\therefore \alpha = \theta - \omega$

A+B give only P on the bdy so that the entire system is moment and traction self equilibrating!!

Let $B = B' + B'' + B''' + B^{iv}$

B'



let $\phi_{B'} = b_0 r^2$

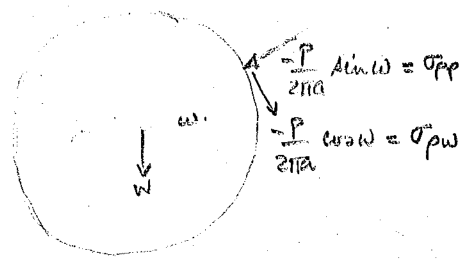
$\frac{\partial\phi'}{\partial r} = 2b_0 r$

$\frac{1}{r^2}\frac{\partial^2\phi'}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi'}{\partial r} = 2b_0 = \frac{P}{2\pi a}$ $\therefore b_0 = \frac{P}{4\pi a}$

$\phi_{B'} = \frac{Pr^2}{4\pi a}$

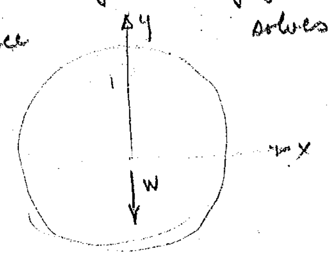
$\therefore$ A + B'

$$\phi = \phi_A + \phi_{B'}$$



Now since we must account for a body force

B'' ... the body force



solves $\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0$

$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} - \gamma = 0$

take

$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} + \gamma y \quad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} + \gamma y \quad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$

if $\phi = 0$ then $\sigma_{xx} = \gamma y \quad \sigma_{yy} = \gamma y \quad \sigma_{xy} = 0$

then $\sigma_{\rho\rho} = \gamma y \quad \sigma_{\omega\omega} = \gamma y \quad \sigma_{\rho\omega} = 0$

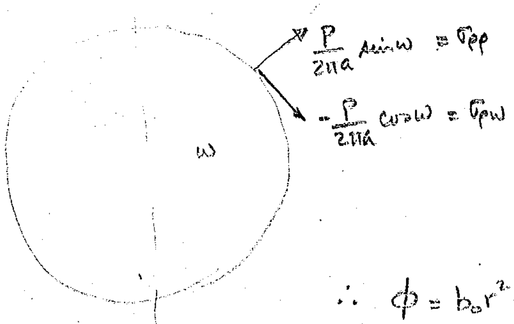
hence $\sigma_{\rho\rho} = \gamma\rho\sin\omega \quad \sigma_{\omega\omega} = \gamma\rho\sin\omega \quad \sigma_{\rho\omega} = 0$

@ $\rho = a$ then $\sigma_{\rho\rho} = \gamma a\sin\omega = \frac{P}{2\pi a}\sin\omega$

since $\gamma\pi a^2 = P$

then superpose this on A + B'

Hence A + B' + B''



now let

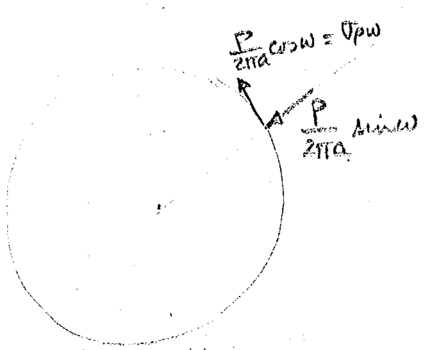
$\phi = \phi(b_0, d_0, b_1, d_1, a_n, b_n, c_n, d_n)$

$\therefore \phi = b_0 r^2 + d_0 r^2\theta + b_1 r^3\cos\theta + d_1 r^3\sin\theta +$

$\sum_{n=2}(a_n r^n + b_n r^{n+2})\cos n\theta + (c_n r^n + d_n r^{n+2})\sin n\theta$

such that

B'''



$\sigma_{rr} = 2b_0 + 2d_0\theta + 2b_1 a\cos\theta + 2d_1 a\sin\theta + \sum\{a_n(n-n^2)a^{n-2} + b_n(2+n-n^2)a^n\}\cos$

$+ \sum\{c_n(n-n^2)a^{n-2} + d_n(2+n-n^2)a^n\}\sin n\omega$

$= -\frac{P}{2\pi a}\sin\omega$

$\sigma_{r\theta} = b_1 \cdot 2a\sin\omega - d_1 2a\cos\omega + \sum n\sin n\omega\{a_n(n-1)a^{n-2} + b_n(n+1)a^n\}$

$- \sum n\cos n\omega\,[c_n(n-1)a^{n-2} + d_n(n+1)a^n]$

$= \frac{P}{2\pi a}\cos\omega$

$2d_1 a = -\frac{P}{2\pi a} \qquad d_1 = -\frac{P}{4\pi a^2}$

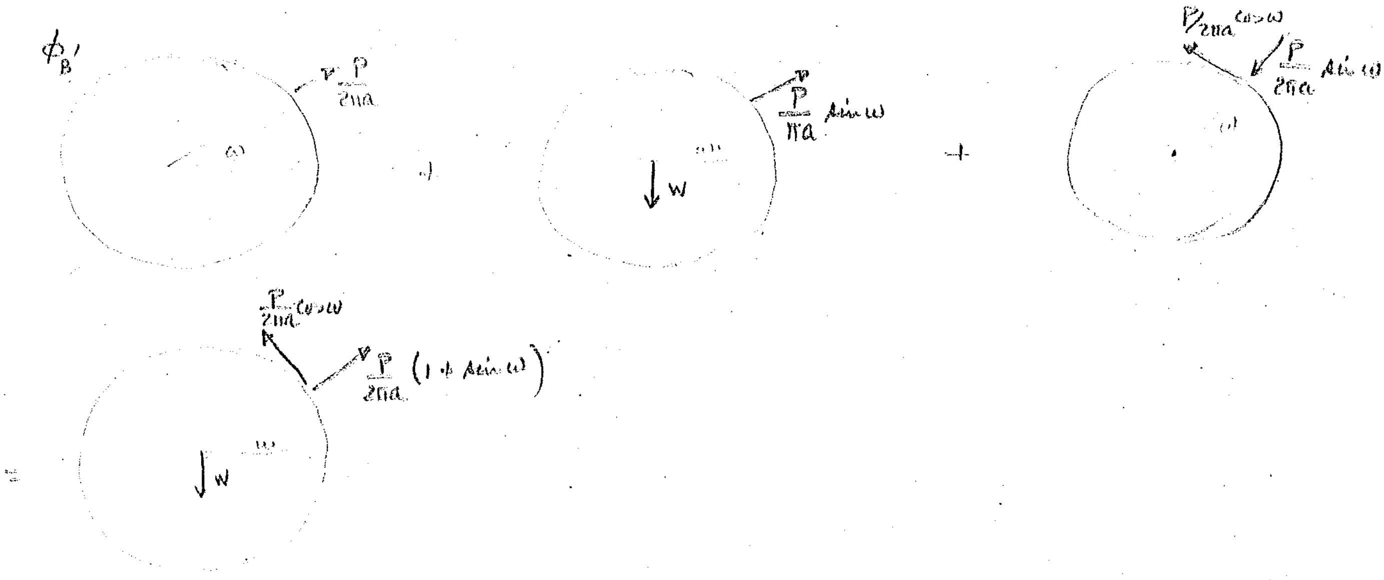
$\Rightarrow b_0 = 0, d_0 = 0 \quad b_1 = 0$

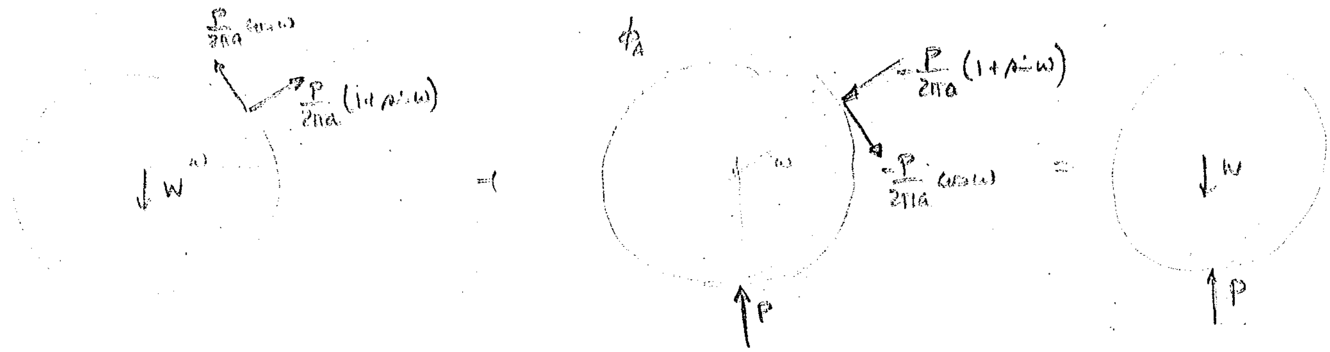
$a_n, b_n, c_n, d_n = 0$

$$\therefore \phi_{B'''} = -\frac{Pr^3}{4\pi a^2}\sin\omega$$

$-\frac{2P\rho\sin\omega}{4\pi a^2} = \sigma_{\rho\rho}$

$\frac{P}{2\pi a^2}\rho\cos\omega = \sigma_{\rho\omega}$





$\phi_A =$

$P = \gamma\pi a^2$ $\quad \phi_{B'} = \frac{P\rho^2}{4\pi a}$

$\phi_{B''} = 0$ but the body force $\Rightarrow$ $\sigma_{\rho\rho} = \gamma\rho\sin\omega$ $\quad \sigma_{\omega\omega} = \gamma\rho\sin\omega$ $\quad \sigma_{\rho\omega} = 0$

$\phi_{B'''} = -\frac{P\rho^3}{4\pi a^2}\sin\omega$

$\phi_{TOT} = \phi_A + \phi_{B'} + \phi_{B''} + \phi_{B'''}$

for $\phi_{B'}$ $\quad \sigma_{\rho\omega} = 0$ $\quad \sigma_{\rho\rho} = \frac{P}{2\pi a}$ $\quad \sigma_{\omega\omega} = \frac{2P}{4\pi a}$

$\phi_{B''}$ $\quad \sigma_{\rho\omega} = 0$ $\quad \sigma_{\rho\rho} = \frac{P}{\pi a^2}\rho\sin\omega$ $\quad \sigma_{\omega\omega} = \frac{P}{\pi a^2}\rho\sin\omega$

$\phi_{B'''}$ $\quad \sigma_{\rho\omega} = \frac{P\rho\cos\omega}{2\pi a^2}$ $\quad \sigma_{\rho\rho} = -\frac{P\rho\sin\omega}{2\pi a^2}$ $\quad \sigma_{\omega\omega} = -\frac{P}{2\pi a^2}\omega$

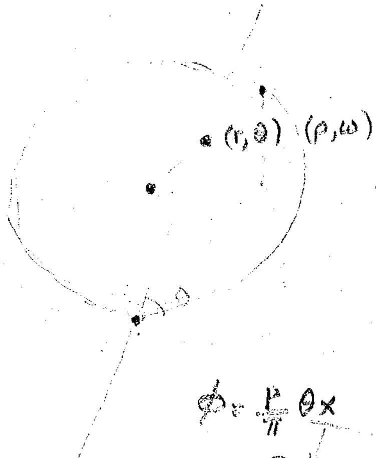
ϕ_A $\quad \sigma_{\rho\omega}$ $\quad \sigma_{\rho\rho} =$ $\quad \sigma_{\omega\omega}$

want to write
$\rho = \rho(r,\theta)$
$\omega = \omega(r,\theta)$

y' ↑ → x'

① we know
$x' = x$
$y' = y - a$

y ↑ → x



$\therefore \; y' + a = y \qquad x' = x$

$\theta = \arctan(y/x)$
$x = r\cos\theta$

$x = \rho\cos\omega$
$\omega = \arctan(y'/x')$
$\omega = \arctan\left(\dfrac{y-a}{x}\right)$
$\tan\omega = \dfrac{y-a}{x}$
$= y/x - a/x$
$= \tan\theta - a/x$

$\phi = \dfrac{P}{\pi}\theta x$
$= \dfrac{P}{\pi}\rho\cos\omega\{\omega + \arcsin(\tfrac{a}{r}\cos\omega)\}$
where $r = (\rho^2 + a^2 + 2a\rho\sin\omega)^{1/2}$

$$\frac{\sin(\omega-\theta)}{\cos\omega\cos\theta} = \tan\omega - \tan\theta = -a/x = \frac{-a}{r\cos\theta}$$

~~$\therefore \sin(\omega-\theta) = \frac{a}{\rho}\cos\omega$~~
~~$\omega - \theta = \arcsin[\frac{a}{\rho}\cos\omega]$~~
~~$\theta = \omega - \arcsin[\frac{a}{\rho}\cos\omega]$~~

$\therefore \sin(\omega-\theta) = \dfrac{-a}{r}\cos\omega$

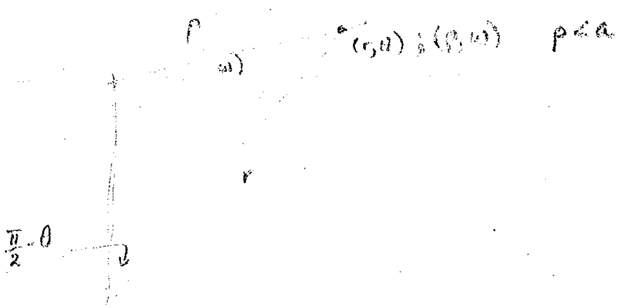
$$r = \sqrt{y^2+x^2} = \sqrt{(y'+a)^2 + x^2} = \sqrt{\rho^2 + a^2 + 2a\rho\sin\omega}$$

$$\therefore \theta - \omega = \arcsin\left[\frac{a}{r}\cos\omega\right]$$

$$\theta = \omega + \arcsin\left[\frac{a}{r}\cos\omega\right] \qquad \frac{dr}{d\omega} = \frac{1}{2r}\cdot 2a\rho\cos\omega$$

$$\frac{\partial\phi}{\partial\omega} = \frac{P}{\pi}\left\{(-\rho\sin\omega)\theta + \rho\cos\omega\left[1 + \frac{-\frac{a}{r}\sin\omega - \frac{a\cos\omega}{r^2}\frac{dr}{d\omega}}{\sqrt{1 - \frac{a^2}{r^2}\cos^2\omega}}\right]\right\}$$

$$= \frac{P}{\pi}\left\{(-\rho\sin\omega)\left(\omega + \arcsin\left[\frac{a}{r}\cos\omega\right]\right) + \rho\cos\omega\left[1 - \frac{a\sin\omega + a\cos^2\omega\cdot\frac{a\rho}{r^2}}{\sqrt{r^2 - a^2\cos^2\omega}}\right]\right.$$



for 2/ for an infinite incompressible inviscid fluid we assume steady state, 2-D irrotational flow, ρ = constant, the solid walls are streamlines, we neglect the noslip condition

ie no shear stress on cylinder and we obtain the pressure on the fluid using the Bernoulli Eqn.

the stream fn. is defined as $u = \frac{\partial \psi}{\partial y}$ $v = -\frac{\partial \psi}{\partial x}$ or $v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $v_\theta = -\frac{\partial \psi}{\partial r}$ ~~p is really a pressure difference wrt some reference pressure.~~

steady Bernoulli is $p/\rho - U + \frac{1}{2}(v_r^2 + v_\theta^2) = \text{const}$ where $\nabla U = f_b$ (body force which is conservative)

the stream fn $\psi = V_\infty r \sin\theta \left[1 - \frac{b^2}{r^2}\right]$. We note that the inner outer surfaces can be considered as streamlines and no fluid would pass through

By using the eqns for v_r, v_θ and evaluating them on the surface we obtain

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_\infty \cos\theta \left(1 - \frac{b^2}{r^2}\right)\Big|_{r=b} = 0 \qquad v_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin\theta \left(1 + \frac{b^2}{r^2}\right)\Big|_{r=b} = -2V_\infty \sin\theta$$

$\therefore$ if we neglect body forces we obtain by evaluating the bernoulli constant at ∞

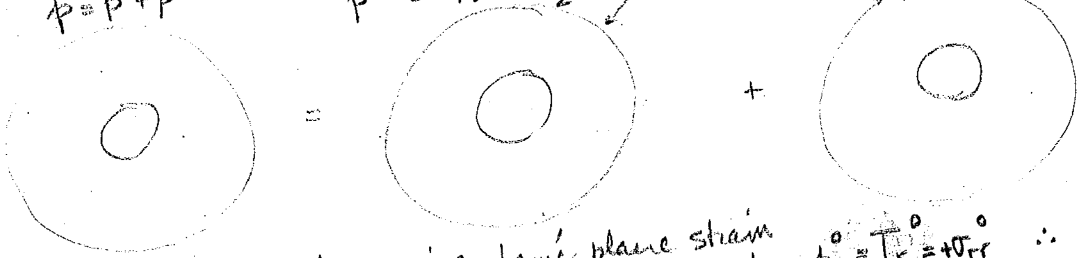
$$p = p_\infty + \rho_\infty \frac{V_\infty^2}{2} - \frac{\rho_\infty}{2} 4 V_\infty^2 \sin^2\theta$$

$$\therefore\ p = p_\infty + \frac{\rho_\infty V_\infty^2}{2}\left[1 - 4\sin^2\theta\right]; \quad -2\rho_\infty V_\infty^2 \sin^2\theta = -2\rho_\infty V_\infty^2 \frac{(1-\cos 2\theta)}{2} = -\rho_\infty V_\infty^2 + \rho_\infty V_\infty^2 \cos 2\theta$$

$$= p_\infty + \rho_\infty \frac{V_\infty^2}{2} - \rho_\infty V_\infty^2 + \rho_\infty V_\infty^2 \cos 2\theta = p_\infty - \rho_\infty \frac{V_\infty^2}{2} + \rho_\infty V_\infty^2 \cos 2\theta$$

we can solve this problem using the plane strain soln @ width of cylinder is ∞

$$p = p^0 + \hat{p} \qquad p^0 = p_\infty - \frac{\rho_\infty V_\infty^2}{2} \qquad \hat{p} = \rho_\infty V_\infty^2 \cos 2\theta$$



p^0 solution can be obtained using Lamé plane strain w/ $-p^0 = T_r^0 = +\sigma_{rr}^0$ $\therefore$ $\sigma_{rr}^0\big|_{r=b} = -p^0$, $\sigma_{r\theta}^0\big|_{r=b} = 0$, $\sigma_{rr}^0\big|_{r=a} = 0 = p_i^0$, $\sigma_{r\theta}^0\big|_{r=a} = 0$

$$\therefore\ \sigma_{rr}^0 = \frac{\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right) a^2 b^2}{b^2 - a^2}\frac{1}{r^2} - \frac{\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right) b^2}{b^2 - a^2}$$

$$\sigma_{\theta\theta}^0 = -\frac{\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right) a^2 b^2}{b^2 - a^2}\frac{1}{r^2} - \frac{\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right) b^2}{b^2 - a^2}$$

$$\sigma_{r\theta}^0 = 0$$

$$\sigma_{zz}^0 = \nu(\sigma_{rr}^0 + \sigma_{\theta\theta}^0) = \frac{-2\nu\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right) b^2}{b^2 - a^2}$$

$\hat{p}$ sol can be obtained as in class $\hat{\sigma}_{r\theta}\big|_{r=b} = \hat{\sigma}_{r\theta}\big|_{r=a} = \hat{\sigma}_{rr}\big|_{r=a} = 0$ w/ $-\hat{p} = \hat{T}_r = \hat{\sigma}_{rr}\big|_{r=b}$

$$\therefore\ \hat{\sigma}_{rr}\big|_{r=b} = -\hat{p} = -\rho_\infty V_\infty^2 \cos 2\theta$$

$\therefore$ by our results $A_0' = 0$ $A_1' = 0$, $A_2' = -\rho_\infty V_\infty^2$, $A_3' \dots A_\infty' = 0$, $B_1' = 0$

$A_0 = 0$ $A_1 = 0$, $B_1 = 0$

$C_0 = 0$, $C_1 = 0$, $D_1 = 0$, $C_0' = 0$, C_1', $D_1' = 0$

Momentum is satisfied ident, force satisfied ident.

take $d_0, c_0 = 0$, $a_0' = 0$, $a_1 = 0$, $c_1 = 0$, $b_1' = 0$, $d_1' = 0$, $b_1 = 0$, $a_1' = 0$, $c_1' = 0$, $d_1 = 0$, $a_0 = 0$, $b_0 = 0$

$a_n, b_n, a_n', b_n' = 0 \;\forall\; n = 1, 3, \ldots$

$c_n, d_n, c_n', d_n' = 0 \;\forall\; n$

only non zero terms are

$$a_2(-2) + b_2(-4)a^2 - a_2' 6 a^{-4} + b_2'(-4)a^{-2} = 0$$

$$a_2(-2) + b_2(\quad - a_2' 6 b^{-4} + b_2'(-4) b^{-2} = -\rho_\infty V_\infty^2$$

$$a_2 + b_2(3a^2) - a_2'(3)a^{-4} - b_2' a^{-2} = 0$$

$$a_2 \quad b_2(3b^2) - a_2'(3)b^{-4} - b_2' b^{-2} = 0$$

$$\begin{vmatrix} 1 & 0 & 3a^{-4} & 2a^{-2} \\ 1 & 0 & 3b^{-4} & 2b^{-2} \\ 1 & 3a^2 & -3a^{-4} & -a^{-2} \\ 1 & 3b^2 & -3b^{-4} & -b^{-2} \end{vmatrix} \begin{bmatrix} a_2 \\ b_2 \\ a_2' \\ b_2' \end{bmatrix} = \begin{pmatrix} 0 \\ +\frac{\rho_\infty V_\infty^2}{2} \\ 0 \\ 0 \end{pmatrix} =$$

since $\hat{\sigma}_{rr} = -\rho_\infty V_\infty^2 \cos 2\theta$ then $\int_0^{2\pi} \sigma_{rr}\, d\theta = \int -\rho_\infty V_\infty^2 \cos 2\theta\, d\theta = A_0' \int_0^{2\pi} d\theta + A_n' \int_0^{2\pi} \cos n\theta\, d\theta + B_n' \int_0^{2\pi} \sin n\theta\, d\theta$

$$= -\frac{\rho_\infty V_\infty^2}{2} \sin 2\theta \Big|_0^{2\pi} = 2\pi A_0' + \sum_{n=1} A_n' \frac{\sin n\theta}{n}\Big|_0^{2\pi} + \sum B_n \frac{-\cos n\theta}{n}\Big|$$

$$\frac{-\cos n 2\pi + \cos 0}{0}$$

solve for a_2, b_2, a_2', b_2'

$A_0' = 0$

$$\int_0^{2\pi} \sigma_{rr} \cos m\theta\, d\theta = \sum A_n' \int_0^{2\pi} \cos^2 n\theta\, d\theta + \sum \frac{B_n'}{2} \int_0^{2\pi} \sin 2n\theta\, d\theta$$

$$\cos^2 n\theta = \frac{1 + \cos 2n\theta}{2} \qquad \frac{-\cos 2n\theta}{4n}\Big|_0^{2\pi} = 0$$

$$-\rho_\infty V_\infty^2 \int \cos^2 2\theta\, d\theta \qquad \sum \frac{\theta}{2} + \frac{\sin 2n\theta}{4n}$$

$$\frac{\theta}{2} + \frac{\sin 4\theta}{8}\Big|_0^{2\pi} \qquad A_n' \cdot \pi$$

$$-\rho_\infty V_\infty^2 \pi = A_n' \pi \qquad A_n' = -\rho_\infty V_\infty^2$$

$$\therefore \quad \hat{\phi} = (a_2 r^2 + b_2 r^4 + a_2' r^{-2} + b_2')\cos 2\theta$$

$$\Rightarrow \quad \hat{\sigma}_{rr} = (-2a_2 - 6a_2' r^{-4} - 4b_2' r^{-2})\cos 2\theta$$

$$\hat{\sigma}_{r\theta} = 2\sin 2\theta\,\{a_2 + 3b_2 r^2 - 3a_2' r^{-4} - b_2' r^{-2})$$

$$\hat{\sigma}_{\theta\theta} = \frac{\partial^2 \hat{\phi}}{\partial r^2} = (2a_2 + 12 b_2 r^2 + 6a_2' r^{-4})\cos 2\theta$$

$$\therefore \quad \sigma_{rr} = \sigma_{rr}^0 + \hat{\sigma}_{rr}$$

$$\sigma_{r\theta} = \quad + \hat{\sigma}_{r\theta} = 2\sin 2\theta\,(a_2 + 3b_2 r^2 - 3a_2' r^{-4} - b_2' r^{-2})$$

$$\sigma_{\theta\theta} = \sigma_{\theta\theta}^0 + \hat{\sigma}_{\theta\theta}$$

$$\begin{bmatrix} -1 & 0 & 3a^{-4} & 2a^{-2} \\ 0 & 0 & 3(b^{-4}-a^{-4}) & 2(b^{-2}-a^{-2}) \\ 0 & a^2 & -2a^{-4} & -a^{-2} \\ 0 & 3b^2 & -3(a^{-4}+b^{-4}) & -(b^{-2}+2a^{-2}) \end{bmatrix} \begin{pmatrix} a_2 \\ b_2 \\ a_2' \\ b_2' \end{pmatrix} = \begin{pmatrix} 0 \\ +\frac{\rho_\infty V_\infty^2}{2} \\ 0 \\ 0 \end{pmatrix}$$

$$\text{denom.} = 3(b^{-4}-a^{-4})(-a^{-2})(3b^2) + 2(b^{-2}-a^{-2})(-3a^2)(a^{-4}+b^{-4}) + 12(b^{-2}-a^{-2})(a^{-4})b^2$$

$$+ 3(b^{-4}-a^{-4})a^2(b^{-2}+2a^{-2}) = 3(b^{-4}-a^{-4})(-3b^2a^{-2}+a^2b^{-2}-2) + (b^{-2}-a^{-2})\Big[-6a^{-2}-6a^2b^{-4}$$

$$+12a^{-8}+12a^{-4}b^{-4}\Big] = -9b^{-2}a^{-2}+3a^2b^{-6}-6b^{-4}+9b^2a^{-6}-3a^2b^{-2}+6a^{-4}-6b^{-2}a^{-2}+6a^{-4}-6a^2b^{-2}$$

$$+6b^{-4}+12b^{-2}a^{-8}-12a^{-10}+12a^{-4}b^{-6}-12a^{-6}b^{-4}$$

Need to get coeffs. to complete.

$$\begin{bmatrix} 1 & 0 & 3a^{-4} & 2a^{-2} \\ 0 & 3b^2 & -3(a^{-4}+b^{-4}) & -(b^{-2}+2a^{-2}) \\ 0 & a^2 & -2a^{-4} & -a^{-2} \\ 0 & 0 & 3(b^{-4}-a^{-4}) & 2(b^{-2}-a^{-2}) \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\rho V^2}{2} \end{pmatrix}$$

$$(b^{-2}-a^{-2})\Big[(3b^{-2}+3a^{-2})(-3b^2a^{-2}) - 6a^2(a^{-4}+b^{-4}) + 12a^{-4}b^2 + (3b^{-2}+3a^{-2})a^2(b^{-2}+2a^{-2})\Big]$$

$$\Big\{-9a^{-2}-9b^2a^{-4}-6a^{-2}-6a^2b^{-4}+12a^{-4}b^2 + 3a^2(b^{-4}+3b^{-2}a^{-2}+2a^{-4})\Big\}$$

$$(b^{-2}-a^{-2})(3a^{-4}b^2-9a^{-2}-3a^2b^{-4}+9b^{-2}) \quad \text{denom.}$$

$$\begin{vmatrix} 0 & 0 & 3a^{-4} & 2a^{-2} \\ \frac{\rho V^2}{2} & 0 & 3(b^{-4}-a^{-4}) & 2(b^{-2}-a^{-2}) \\ 0 & a^2 & -2a^{-4} & -a^{-2} \\ 0 & 3b^2 & -3(a^{-4}+b^{-4}) & -(b^{-2}+2a^{-2}) \end{vmatrix}$$

$$-\frac{\rho V^2}{2}\Big\{3a^{-4}(-a^{-2})3b^2+2a^{-2}(-3a^2)(a^{-4}+b^{-4})+12a^{-6}b^2 + 3a^{-2}(b^{-2}+2a^{-2})\Big\}$$

$$-\frac{\rho V^2}{2}\Big\{3a^{-6}b^2-6(a^{-4}+b^{-4})+3a^{-2}b^{-2}+6a^{-4}\Big\}$$

$$-\frac{\rho V^2}{2}3\left[a^{-6}b^2-2b^{-4}+a^{-2}b^{-2}\right] \quad \underline{\text{num. of } a_2}$$

$$-\rho V^2\frac{3}{2}b^{-2}\left[a^{-6}b^4-2b^{-2}+a^{-2}\right]$$

$$\begin{vmatrix} 1 & 0 & 3a^{-4} & 2a^{-2} \\ 0 & \frac{\rho V^2}{2} & 3(b^{-4}-a^{-4}) & 2(b^{-2}-a^{-2}) \\ 0 & 0 & -2a^{-4} & -a^{-2} \\ 0 & 0 & -3(a^{-4}+b^{-4}) & -(b^{-2}+2a^{-2}) \end{vmatrix}$$

$$= \frac{\rho V^2}{2}(+2a^{-4})(b^{-2}+2a^{-2}) - \frac{\rho V^2}{2}(3a^{-2})(a^{-4}+b^{-4})$$

$$= \frac{\rho V^2}{2}\left[2a^{-4}b^{-2}+4a^{-6}-3a^{-6}-3a^{-2}b^{-4}\right] = \frac{\rho V^2}{2}a^{-2}\left[2a^{-2}b^{-2}+a^{-4}-3b^{-4}\right]$$

$$a^{-6} \qquad \frac{\rho V^2}{2}a^{-2}\left[a^{-2}-b^{-2}\right]\left[a^{-2}+3b^{-2}\right]$$

$\underline{\text{Num of } b_2}$

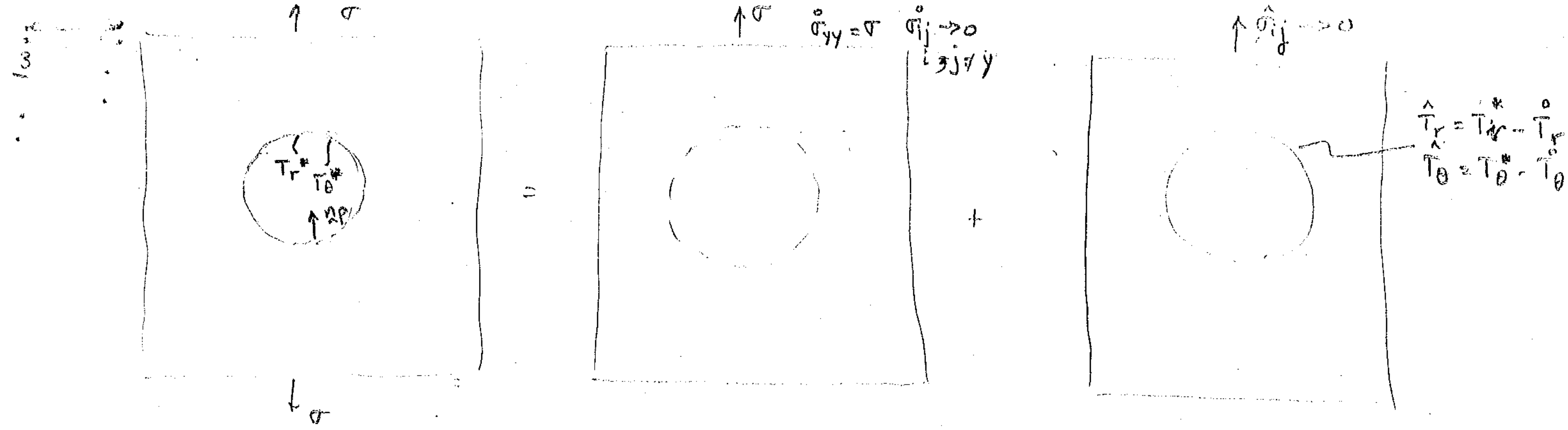
$$\begin{vmatrix} 1 & 0 & 0 & 2a^{-2} \\ 0 & 0 & \frac{\rho V^2}{2} & 2(b^{-2}-a^{-2}) \\ 0 & a^2 & 0 & -a^{-2} \\ 0 & 3b^2 & 0 & -(b^{-2}+2a^{-2}) \end{vmatrix}$$

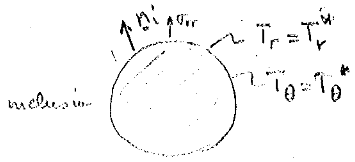
$$\frac{\rho V^2}{2}(-3a^{-2}b^2) + \frac{\rho V^2}{2}a^2(b^{-2}+2a^{-2}) = \frac{\rho V^2}{2}(-3a^{-2}b^2+a^2b^{-2}+2)$$

num of a_2'

$$\begin{pmatrix} 1 & 0 & 3a^{-4} & 0 \\ 0 & 0 & 3(b^{-4}-a^{-4}) & \rho \frac{V^2}{2} \\ 0 & a^2 & -2a^{-4} & 0 \\ 0 & 3b^2 & -3(a^{-4}+b^{-4}) & 0 \end{pmatrix} = \rho\frac{V^2}{2}\left(-3a^2(a^{-4}+b^{-4})\right) - \rho\frac{V^2}{2}(-2a^{-4})(3b^2)$$

$$= \rho\frac{V^2}{2}\left[-3a^{-2} - 3a^2b^{-4} + 6a^{-4}b^2\right] \quad \text{num of } b_2'$$





Thus we assume that due to uniform σ at ∞, there will be tractions on the boundary of inclusion/plate — we will assume we know them ⇒ let

$$T_{r_i}^* = \sigma_{rr_i}^* = A_0'^* + \sum_1^\infty A_n'^* \cos n\theta + \sum_1^\infty B_n'^* \sin n\theta$$

$$T_{\theta_i}^* = \sigma_{\theta r_i}^* = C_0'^* + \sum_1^\infty C_n'^* \cos n\theta + \sum_1^\infty D_n'^* \sin n\theta$$

since we do not expect ∞ stresses in the center of the inclusion

$$\phi_{\text{inclusion}} = b_{0_i} r^2 + b_{1_i} r^3 \cos\theta + d_{1_i} r^3 \sin\theta + \sum_{n=2}^\infty (a_{n_i} r^n + b_{n_i} r^{n+2}) \cos n\theta$$
$$+ \sum_{n=2}^\infty (c_{n_i} r^n + d_{n_i} r^{n+2}) \sin n\theta \qquad \text{where } (\;)_i = \text{inclusion}$$

Thus we can solve for $b_{0_i}, b_{1_i}, d_{1_i}, a_{n_i}, b_{n_i}, c_{n_i}, d_{n_i}$ in terms of $A_0'^*, A_n'^*, B_n'^*, C_0'^*, C_n'^*, D_n'^*$ invoking moment, force equil. We can now use the fact that $\sigma_{rr_i} = \frac{1}{r}\frac{\partial \phi_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_i}{\partial \theta^2}$, $\sigma_{r\theta_i} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi_i}{\partial \theta}\right)$, $\sigma_{\theta\theta_i} = \frac{\partial^2 \phi_i}{\partial r^2}$ to form the strains

$$\varepsilon_{\theta_i} = \frac{u_i}{r} + \frac{1}{r}\frac{\partial v_i}{\partial \theta} = \frac{1}{E^*}\left[(1-\nu^{*2})\sigma_{\theta\theta_i} - \nu^*(1+\nu^*)\sigma_{rr_i}\right] \quad (2)$$

← plug into here to get v (get indep fn $g_i(r)$ from integ.) (3)

$$\varepsilon_{r_i} = \frac{\partial u_i}{\partial r} = \frac{1}{E^*}\left[(1-\nu^{*2})\sigma_{rr_i} - \nu^*(1+\nu^*)\sigma_{\theta\theta_i}\right] \Rightarrow (1) \text{ get } u \text{ from here}$$

(get an indep fn $f_i(\theta)$ by int.)

$$\gamma_{r\theta_i} = \frac{\partial u_i}{r\,\partial\theta} + \frac{\partial v_i}{\partial r} - \frac{v_i}{r} = \frac{1}{G^*}\sigma_{r\theta_i}$$

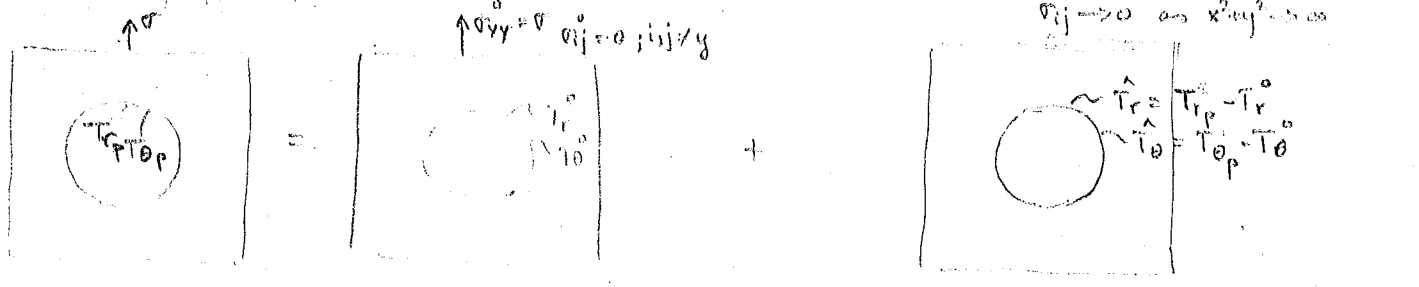
← check to see how $g_i(r)$, $f_i(\theta)$ are related here.

now put $r = a$ to define u_i^*, v_i^* at boundary. Note that u_i^*, v_i^* are fns of the unknown $A_0'^*, A_n'^*, B_n'^*, C_0'^*, C_n'^*, D_n'^*$ & E^*, ν^* where $E^* = \mu^*\left(\frac{3\lambda^* + 2\mu^*}{\lambda^* + \mu^*}\right)$ $\nu^* = \frac{\lambda^*}{2(\lambda^* + \mu^*)}$

For the plate/inclusion boundary we know that $T_{r_i} = T_{r_p} \Rightarrow \sigma_{rr_i}^* = -\sigma_{rr_p}^*$ where $(\;)_p$ = plate

$$T_{\theta_i} = T_{\theta_p} \Rightarrow \sigma_{r\theta_i}^* = -\sigma_{r\theta_p}^*$$
$$u_i^* = u_i$$
$$v_i^* = v_i^*$$

For the plate we do as we did in class except that now there will be tractions on the boundary. i.e.



in class for the constant stress at ∞ case we showed that

$$\phi^{o}_{\substack{plate\\no\,hole}} = f(b_0, a_2) = \frac{\sigma}{4}r^2 + \frac{\sigma}{4}r^2\cos 2\theta = \frac{\sigma r^2}{4}(1-\cos 2\theta)$$

in the case of the hole in the plate but $\hat{\sigma}_{ij} \to 0$ as $r \to \infty$

$$\therefore \quad \hat{\phi}_{\substack{plate\\w/hole}} = f(a_1, a_1', b_1', c_1, c_1', d_1', a_n', b_n', c_n', d_n', r, \theta)$$

$$\hat{\phi}_{\substack{plate\\w/hole}} = a_1 r\theta\sin\theta + (a_1' r^{-1} + b_1' r\ln r)\cos\theta + c_1 r\theta\cos\theta + (c_1' r^{-1} + d_1' r\ln r)\sin\theta$$
$$+ \sum_{n=2}^{\infty}\left\{ r^{-n}(a_n'\cos n\theta + c_n'\sin n\theta) + r^{-n+2}(b_n'\cos n\theta + d_n'\sin n\theta)\right\}$$

Now we can solve for the values of the coeffs in $\hat{\phi}$ since we know what the bc are at the hole. ie that

$$\hat{T}_r = T_{r_p} - T^{o}_{r_p} = -\sigma^*_{rr_p} + \sigma^{o}_{rr_p} = \sigma^*_{rr_i} + \sigma^{o}_{rr_p} = -\hat{\sigma}_{rr_p}$$
$$\hat{T}_\theta = T_{\theta_p} - T^{o}_{\theta_p} = -\sigma^*_{r\theta_p} + \sigma^{o}_{r\theta_p} = \sigma^*_{r\theta_i} + \sigma^{o}_{r\theta_p} = -\hat{\sigma}_{r\theta_p}$$

Now this defines $\hat{\sigma}_{rr_p}, \hat{\sigma}_{r\theta_p}$ in terms of $A_0'^*, A_n'^*, B_n'^*, C_0'^*, C_n'^*, D_n'^*$ of the inclusion

thus we can define $a_1, a_1', b_1', c_1, c_1', d_1', a_n', b_n', c_n', d_n'$ in terms of the above and obtain $\hat{\sigma}_{rr_p}, \hat{\sigma}_{r\theta_p}, \hat{\sigma}$ in the plate

We now use $\hat{\varepsilon}_{r_p} = \frac{\partial \hat{u}_p}{\partial r} = \frac{1}{E}\left[(1-\nu^2)\hat{\sigma}_{rr_p} - \nu(1+\nu)\hat{\sigma}_{\theta\theta_p}\right]$ to obtain $\hat{u}_p + \hat{f}_p'(\theta)$

we can also use $\hat{\varepsilon}_{\theta_p} = \frac{\hat{u}_p}{r} + \frac{1}{r}\frac{\partial \hat{v}_p}{\partial \theta} = \frac{1}{E}\left[(1-\nu^2)\hat{\sigma}_{\theta\theta_p} - \nu(1+\nu)\hat{\sigma}_{rr_p}\right]$ to obtain $\hat{v}_p + \hat{g}_p(r)$

and $\hat{\gamma}_{r\theta_p} = \frac{1}{G}\hat{\sigma}_{r\theta_p} = \frac{\partial \hat{u}_p}{r\partial\theta} + \frac{\partial \hat{v}_p}{\partial r} - \frac{\hat{v}_p}{r}$ to get the relation between $\hat{f}_p'(\theta)$ and $\hat{g}_p(r)$

We had obtained in our last HW assignment that $u^{o}_p = -\frac{\nu\sigma(1+\nu)}{E}x$

$$v^{o}_p = \frac{(1-\nu^2)\sigma}{E}y$$

thus we can now obtain $\begin{matrix} u_p = u^{o}_p + \hat{u}_p \\ v_p = v^{o}_p + \hat{v}_p \end{matrix}$ and hence $\left.\begin{matrix} u_p^* \\ v_p^* \end{matrix}\right\}$ at the boundary

but $u_i^* = u_i^*(A_0'^*, A_n'^*, B_n'^*, C_0'^*, C_n'^*, D_n'^*, \nu^*, E^*) = u_p^*(A_0'^*, A_n'^*, B_n'^*, C_0'^*, C_n'^*, D_n'^*, \nu, E)$

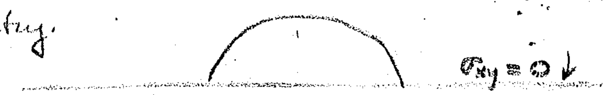
$v_i^* = v_i^*($ " $) = u_p^*($ " $)$

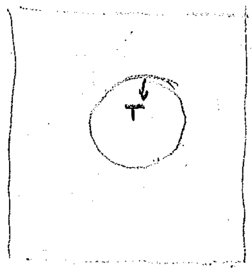
thus we can define $A_0'^*, A_n'^*, B_n'^*, C_0'^*, C_n'^*, D_n'^*$ as fns of (ν^*, E^*, ν, E, a) and hence define the stresses and displacements throughout.

$$\phi_{plate} = \phi^{o}_{\substack{plate\\w/o\,hole}} + \hat{\phi}_{\substack{plate\\w/hole}}$$

Since $\lambda^* >> \lambda$, $\mu^* >> \mu$ $\Rightarrow$ ~~$\nu^* = \frac{\lambda}{2(\lambda+\mu)}$~~ ~~$E^* = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$~~ expect stress concentration at the hole boundary to be less - Why? the material in the "hole" is relieving some of the stresses and hence decreasing stress intensity at any point on the boundary.

if we note in this problem. $\sigma_y(y=0)\ \forall x = 0$
$\sigma_{xy}(y=0)\ \forall x = 0$ } by symmetry.
$u(x,y) = u(x,-y)$
$v(x,y) = -v(x,-y)$





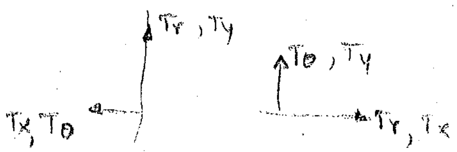
Another way to see this is suppose w/inclusion you want to create a state of 0 stress on boundary of plate/inclusion. If the only way you could load it was at ∞ then the only way 0 stress could be achieved is if there was 0 load at ∞. ⇒ no stress concentrations.

Or since $\sigma_{r\theta} = K_1\sigma$ $\sigma_{rr} = K_2\sigma$ $\sigma_{\theta\theta} = K_3\sigma$ $K_1, K_2, K_3 = 0$

whereas in the case of the plate w/hole $K_1 = K_2 = 0$ $K_3 = 3$

$$T_x^* = T_r^* \cos\theta - T_\theta^* \sin\theta$$
$$T_y^* = T_r^* \sin\theta + T_\theta^* \cos\theta$$

$$\int_0^{2\pi} (a\,\sigma_{\theta r})\, a\, d\theta = a^2 2\pi A_0'^* \quad \text{mom}$$



$$\int_0^{2\pi} T_x^* a\, d\theta = \int_0^{2\pi} = a(A_1'^* - D_1'^*)\pi = 0$$
$$\int_0^{2\pi} T_y^* a\, d\theta = \int_0^{2\pi} = a(B_1'^* + C_1'^*)\pi = 0 \quad \text{force equil}$$

$$\sigma_{rr}^* = \frac{\sigma}{2}(1-\cos 2\theta)$$
$$\sigma_{\theta r}^* = \sigma/_2 \sin 2\theta$$

$$\int \frac{\sigma}{2}\sin 2\theta \cdot a^2 d\theta = \frac{\sigma}{2}\int = 0 \quad \text{moment}$$

$$\int \frac{\sigma}{2}(1-\cos 2\theta)\, a\, d\theta$$

$$\int \frac{\sigma}{2}(1-\cos 2\theta)\cos\theta - \int \frac{\sigma}{2}\sin 2\theta \sin\theta = 0 \quad \int T_y\, ds = 0$$

$$\varepsilon_r = \frac{\partial u}{\partial r} = \frac{1}{E}\left[(1-\nu^2)\sigma_{rr} - \nu(1+\nu)\sigma_{\theta\theta}\right]$$
$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{E}\left[(1-\nu^2)\sigma_\theta - \nu(1+\nu)\sigma_r\right]$$
$$\varepsilon_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right]$$

$$\sigma_{rr} = \frac{\sigma}{2}(1-\cos 2\theta)$$
$$\sigma_{\theta\theta} = \sigma/_2(1+\cos 2\theta)$$
$$\sigma_{r\theta} = \sigma/_2 \sin 2\theta$$

$$\varepsilon_r = \frac{\partial u}{\partial r} = \frac{\sigma}{2}\frac{1+\nu}{E}\left[(1-\nu)(1-\cos 2\theta) - \nu(1+\cos 2\theta)\right]$$
$$1 - \cos 2\theta - \nu + \nu\cos 2\theta - \nu - \nu\cos 2\theta$$

$$\frac{\partial u}{\partial r} = \frac{\sigma}{2}\frac{(1+\nu)}{E}(1-2\nu-\cos 2\theta)$$
$$u = \frac{\sigma}{2}\frac{(1+\nu)}{E}(1-2\nu-\cos 2\theta)\, r + f'(\theta)$$

$$1 + \cos 2\theta - \nu - \nu\cos 2\theta - \nu + \nu\cos 2\theta$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} = \frac{\sigma}{2}\frac{(1+\nu)}{E}(1+2\nu-\cos 2\theta) + \frac{f'(\theta)}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\sigma}{2}\frac{1+\nu}{E}\left[(1-\nu)(1+\cos 2\theta) - \nu(1-\cos 2\theta)\right]$$
$$= \frac{\sigma}{2}\frac{1+\nu}{E}\left[1-2\nu+\cos 2\theta\right]$$

$$\therefore \frac{1}{r}\frac{\partial v}{\partial \theta} = -\frac{f'(\theta)}{r} + \frac{\sigma}{2}\frac{(1+\nu)}{E}\left[-1-2\nu+\cos 2\theta + 1 - 2\nu + \cos 2\theta\right] r$$
$$2(\cos 2\theta - 2\nu)$$

$$\frac{\partial v}{\partial \theta} = +\sigma\frac{(1+\nu)}{E}\left[\cos 2\theta - 2\nu\right] r - f'(\theta)$$
$$v = \frac{\sigma(1+\nu)}{E} r\left\{\frac{\sin 2\theta}{2} - 2\nu\theta\right\} - f(\theta) + g(r)$$

$$\therefore \frac{1+\nu}{2E}\sigma\sin 2\theta = \frac{1}{2}\left[2\sigma\frac{(1+\nu)}{2E}\sin 2\theta\right] + f'' + \frac{\sigma(1+\nu)}{2E}\left[-\frac{\sin 2\theta}{2} - 2\nu\theta\right] + \frac{g'(r)}{2} - \frac{\sigma(1+\nu)}{2E}\left\{\frac{\sin 2\theta}{2} - 2\nu\theta\right\} - \frac{f(\theta)}{2r} - \frac{g}{r}$$
$$= \frac{1}{2}\left\{\frac{\sigma(1+\nu)}{2E}\cdot 2\sin 2\theta + \frac{f'(\theta)}{r} + \frac{\sigma(1+\nu)}{E}\left[-\frac{\sin 2\theta}{2} - 2\nu\theta\right] + g'(r) - \frac{\sigma(1+\nu)}{E}\left\{\frac{\sin 2\theta}{2} - 2\nu\theta\right\} - \frac{f}{r}\right.$$

$$\frac{\sigma_{xx}}{E} - \frac{\nu}{E}\left(\sigma_{yy} + \nu\sigma_{xx} + \nu\sigma_{yy}\right)$$

$$-\frac{\nu}{E}\left(\sigma_{yy}\left[1+\nu\right] + \nu\sigma_{xx}\right)$$

$$\frac{1-\nu^2}{E}\sigma_{xx} - \nu\left(1+\nu\right)\sigma_{yy}$$

$$\frac{1+\nu}{E}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{1}{r}\left[\frac{\sigma}{2}\frac{(1+\nu)}{E}\,2\sin 2\theta\, r + f''\right] = \frac{\sigma(1+\nu)}{E}\sin 2\theta + \frac{f''}{r}$$

$$\frac{\partial v}{\partial r} = -\frac{\sigma(1+\nu)}{E}\left\{+\frac{\sin 2\theta}{2} - 2\nu\theta\right\} + g'(r)$$

$$-\frac{v}{r} = \frac{\sigma(1+\nu)}{E}\left\{\frac{\sin 2\theta}{2} + 2\nu\theta\right\} + \frac{f(\theta)}{r} - \frac{g(r)}{r}$$

$g' + \frac{f}{r} - \frac{g}{r}$

$$\frac{1+\nu}{2E}\sigma\sin 2\theta = \frac{\sigma(1+\nu)}{2E}(\sin 2\theta) + \frac{f''}{2r} + \frac{g'(r)}{2} + \frac{f}{2r} - \frac{g(r)}{2r}$$

$\therefore \quad f'' + g'(r)r + f - g = 0$ or

$f'' + f = +g - g'(r)r = \underline{\text{const}}$

~~$f = Ae^{\theta} + Be^{\theta} + \text{const}$~~

$e^{m\theta} = f$

$(m^2-1)e^{m\theta} = 0$

$m = \pm 1$

$f = A\cos(\theta + \varphi) + C_1$

$f = \bar{A}\cos\theta + \bar{B}\sin\theta + C_1$

for single valued ~~f~~ $A = B = 0$

$g - g' \cdot r = \text{const}$

$g - g'r = 0 \quad \therefore \frac{dg}{g} = \frac{dr}{r}$

$\ln g = \ln r$

$\underline{g = Cr}$

$g' - \frac{1}{r}g = -\frac{C_1}{r}$

$g'r - g = -C_1$

$r^2\, d\left(\frac{g}{r}\right) = -C_1$

$d(g/r) = -\frac{C_1}{r^2} \quad \therefore \quad \frac{g}{r} = \frac{C_1}{r} + C_2$

$g = C_1 + C_2 r$

$g = C_1 + C_2 r$

4.

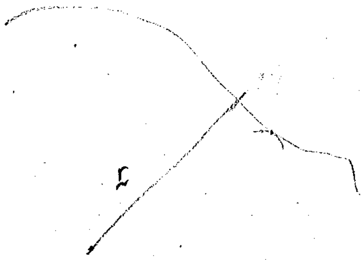
$$\int M_z\, ds = \int \underset{\sim}{r} \times \underset{\sim}{T}\, ds = \int (x\, \underset{\sim}{e}_x + y\, \underset{\sim}{e}_y) \times (T_y\, \underset{\sim}{e}_y + T_x\, \underset{\sim}{e}_x)\, ds$$

$$= \int (x T_y - y T_x)\, ds\ \underset{\sim}{e}_z$$

$$= \int \left(x \left[\sigma_{yx} n_x + \sigma_{yy} n_y \right] - y \left[\sigma_{xx} n_x + \sigma_{xy} n_y \right] \right) ds$$

$\underset{\sim}{n} = n_x \underset{\sim}{e}_x + n_y \underset{\sim}{e}_y$, $n_x^2 + n_y^2 = 1$

$$= \int x \left[-\frac{\partial^2 \phi}{\partial x \partial y} n_x + \frac{\partial^2 \phi}{\partial x^2} n_y \right] - y \left[\frac{\partial^2 \phi}{\partial y^2} n_x - \frac{\partial^2 \phi}{\partial x \partial y} n_y \right] ds$$



$$= \int x \left[-\frac{\partial^2 \phi}{\partial x \partial y} \frac{dy}{ds} + \frac{\partial^2 \phi}{\partial x^2} \left(-\frac{dx}{ds} \right) \right] - y \left[\frac{\partial^2 \phi}{\partial y^2} \left(\frac{dy}{ds} \right) - \frac{\partial^2 \phi}{\partial x \partial y} \left(-\frac{dx}{ds} \right) \right]$$

$$= \int \left[-x \frac{d}{ds} \left(\frac{\partial \phi}{\partial x} \right) \right] ds + \left[-y \frac{d}{ds} \left(\frac{\partial \phi}{\partial y} \right) \right] ds$$

$$\int \frac{d}{ds} \left[-x \frac{\partial \phi}{\partial x} \right] ds = \int -x \frac{d}{ds} \left(\frac{\partial \phi}{\partial x} \right) ds - \frac{dx}{ds} \frac{\partial \phi}{\partial x}\, ds$$

$$\int \frac{d}{ds} \left[-y \frac{\partial \phi}{\partial y} \right] ds = \int -y \frac{d}{ds} \left(\frac{\partial \phi}{\partial y} \right) - \frac{dy}{ds} \left(\frac{\partial \phi}{\partial y} \right) ds$$

$$\int \underset{\sim}{r} \times \underset{\sim}{T}\, ds = -x \frac{\partial \phi}{\partial x} - y \frac{\partial \phi}{\partial y} \Big]_S^F + \oint_S \frac{dx}{ds} \frac{\partial \phi}{\partial x}\, ds + \frac{dy}{ds} \frac{\partial \phi}{\partial y}\, ds$$

$$= -\underset{\sim}{r} \cdot \nabla \phi \Big]_S^F + \int \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \right) = \oint d\phi = \phi \Big]_S^F$$

~~$d\underset{\sim}{r} \cdot \nabla \phi \Big]_S^F$~~

$$= (d\underset{\sim}{r} - \underset{\sim}{r}) \cdot \nabla \phi \Big]_S^F$$

$$\int M_z\, ds = \left(\phi - x \frac{\partial \phi}{\partial x} - y \frac{\partial \phi}{\partial y} \right) \Big]_S^F$$

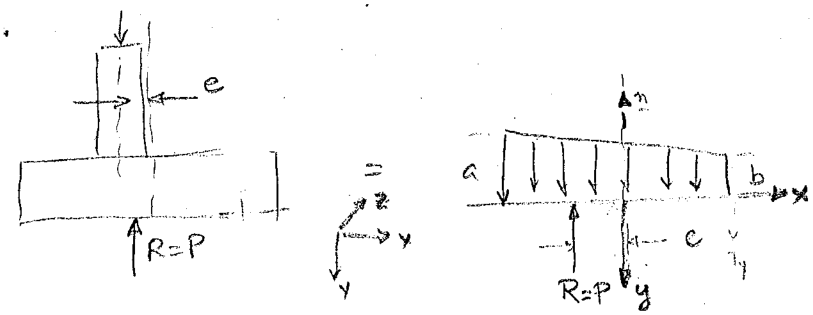
$\underset{\sim}{r} \times \underset{\sim}{t}_n$ $\quad x e_1 + y e_2 \times (n_x t_x + n_y t_y)$

$n_x [\sigma_{xx} e_x + \sigma_{xy} e_y] + n_y [\sigma_{yx} e_1 + e_y \sigma_{yy}]$

$(x e_x + y e_y) \times [\sigma_{xx} n_x + \sigma_{yx} n_y] e_x + [\sigma_{xy} n_x + \sigma_{yy} n_y] e_1$

$x(\sigma_{xy} n_x + \sigma_{yy} n_y) - y(\sigma_{xx} n_x + \sigma_{yx} n_y) =$

.5. 1



Σ Forces

$$p(x) = a + \frac{b-a}{L}\left(x+\frac{L}{2}\right)$$

$$\int_{-L/2}^{L/2} p(x)\,dx + P = 0$$

$$T_y = -\sigma_{yy} = +p(x) - P\delta(x+e)$$

$$+\int_{-L/2}^{L/2}\left\{a + \frac{b-a}{L}(x+L/2)\right\}dx + P = 0$$

$$+\left[ax + \frac{b-a}{2L}(x+L/2)^2\right]_{-L/2}^{L/2} + P = 0 \qquad +\left\{\frac{aL}{2} + \frac{b-a}{2L}\cdot L^2 + \frac{aL}{2}\right\} + P = 0$$

$$+\left\{aL + \frac{b-a}{2}L\right\} + P = +\left\{\frac{a+b}{2}L\right\} + P = 0$$

$$\text{or } \boxed{a+b = \frac{2P}{L}}$$

$$\Sigma M = \int_{-L/2}^{L/2} \underset{\sim}{r} \times \underset{\sim}{T_y}\,ds = 0 \qquad \therefore +\int_{-L/2}^{L/2}\left(x+\frac{L}{2}\right)\left\{a + \frac{b-a}{L}\left(x+\frac{L}{2}\right)\right\}dx - \int_{-L/2}^{L/2} P\left(x+\frac{L}{2}\right)\delta(x+e)\,dx = 0$$

$$+\left[\frac{a}{2}\left(x+\frac{L}{2}\right)^2 + \frac{b-a}{3L}\left(x+\frac{L}{2}\right)^3\right]_{-L/2}^{L/2} + P\left(\frac{L}{2}-e\right)$$

$$+\left[\frac{aL^2}{2} + \frac{b-a}{3L}L^3\right] + P\left(\frac{L}{2}-e\right) = 0$$

$$\frac{aL\cdot L}{2} = \frac{aL^2}{2} - P\left(\frac{L}{2}-e\right) = 0$$

$$+\left[\frac{3aL^2}{6} + \frac{2b-2a}{6}L^2\right] + P\left(\frac{L}{2}-e\right) = 0$$

$$+\frac{aL^2 + 2bL^2}{6} + P\left(\frac{L}{2}-e\right) = 0$$

$$\text{or } \boxed{(a+2b) = \frac{6P}{L^2}\left(\frac{L}{2}-e\right)}$$

$$a+b = \frac{2P}{L}$$

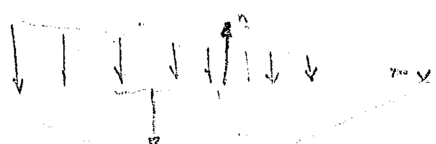
$$a+2b = \frac{6P}{L^2}\left(\frac{L}{2}-e\right)$$

$$b + \frac{2PL}{L^2} = \frac{6P}{L^2}\left(\frac{L}{2}-e\right)$$

$$b = \frac{P}{L^2}(3L - 6e - 2L) = \frac{P}{L^2}(L-6e)$$

$$a = \frac{2PL}{L^2} + \frac{P}{L^2}(6e-L) = \frac{P}{L^2}[6e - L + 2L] = \frac{P}{L^2}(6e+L)$$

Now on the surface we have



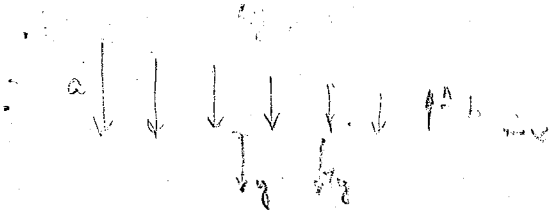
$$-p(x) = T_y = +\sigma_{yy} \qquad \therefore \quad \sigma_{yy}(x, y=0) = -p(x)$$

in class we had the formula for $\sigma_{yy}(x, y=0) = -f(x)$, $\sigma_{xy}(x, y=0) = 0$ the value of $\sigma_{yy}(x, y=0) = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y^3 f(\xi)\,d\xi}{[(x-\xi)^2+y^2]^2}$

$\therefore$ in this case $f(x) = +p(x)$ $\therefore$

$$\sigma_{yy} = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y^3 p(\xi)\,d\xi}{[(x+\xi)^2+y^2]^2} = -\frac{2}{\pi}\int_{-L/2}^{L/2}\frac{y^3\left\{a + \frac{b-a}{L}\left(\xi+\frac{L}{2}\right)\right\}d\xi}{[(x-\xi)^2+y^2]^2}$$

we also had for $\sigma_{xy} = -\frac{2}{\pi}\frac{y^2 x}{(x^2+y^2)^2}$ $\qquad \sigma_{xx} = -\frac{2}{\pi}\frac{x^2 y}{(x^2+y^2)^2}$



$p = \frac{a+b}{2} + \frac{b-a}{L}x = a + \frac{b-a}{L}(x+L/2) \quad |x|<\frac{L}{2}$

$= s + tx \qquad\qquad = 0 \quad |x|>\frac{L}{2}$

$T_y = +p(x) = -\sigma_{yy}(y=0)$

$T_x = 0$

using $\phi = \int_{-\infty}^{\infty} e^{-i\lambda x}\left[Ae^{-|\lambda|y} + By\,e^{-|\lambda|y}\right]d\lambda$

$\sigma_{yy}\Big|_{y=0} = \frac{\partial^2\phi}{\partial x^2}\Big|_{y=0} = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x} A\, d\lambda$

$\frac{\partial^2\phi}{\partial x^2} = \sigma_{yy} = -\int_{-\infty}^{\infty}\lambda^2 e^{-i\lambda x}\left[Ae^{-|\lambda|y} + Bye^{-|\lambda|y}\right]d\lambda$

$= -\int_{-\infty}^{\infty}\lambda^2 A e^{-i\lambda x}e^{-|\lambda|y}\left[1+|\lambda|y\right]d\lambda$

$\frac{\partial\phi}{\partial x} = \int_{-\infty}^{\infty} -i\lambda e^{-i\lambda x}\left[Ae^{-|\lambda|y} + Bye^{-|\lambda|y}\right]d\lambda$

$\frac{\partial\phi}{\partial x\partial y} = \int_{-\infty}^{\infty} -i\lambda e^{-i\lambda x}\left[-|\lambda|A + B + By(-|\lambda|)\right]e^{-|\lambda|y}d\lambda \qquad B(1-|\lambda|y)$

$\sigma_{xy} = -\phi_{xy} = -i\int_{-\infty}^{\infty}\lambda e^{-i\lambda x}\left[|\lambda|A + B(y|\lambda|-1)\right]e^{-|\lambda|y}d\lambda$

$\sigma_{xy}\Big|_{y=0} = 0 = -i\int_{-\infty}^{\infty}\lambda e^{-i\lambda x}\left[|\lambda|A - B\right]e^{-|\lambda|y}d\lambda$

$\therefore \boxed{A|\lambda| = B}$

when know that $\sigma_{yy} = -p(x)$

$\therefore$ if $p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}R(\lambda)\,d\lambda$

then $R(\lambda) = \int_{-\infty}^{\infty} e^{+i\lambda x}\left[s+tx\right]dx = \left\{\left[\frac{1}{+i\lambda}e^{+i\lambda x}\left[s+tx\right]\right]_{-L/2}^{L/2} - \frac{t}{+i\lambda}\int_{-L/2}^{L/2} e^{i\lambda x}dx\right\}$

$dv \quad u \qquad v = \frac{e^{+i\lambda x}}{+i\lambda} \quad du = t\,dx$

$= \left\{\left[\frac{1}{i\lambda}e^{i\lambda L/2}b - \frac{1}{i\lambda}e^{-i\lambda L/2}a\right] + \left(-\frac{t}{(i\lambda)^2}e^{i\lambda x}\Big|_{-L/2}^{L/2} = -\frac{t\,2i\sin\frac{\lambda L}{2}}{(i\lambda)^2}\right)\right\}$

$R(\lambda) = \frac{1}{i\lambda}\left\{e^{i\lambda L/2}b - e^{-i\lambda L/2}a\right\} + \frac{2it}{\lambda^2}\sin\frac{\lambda L}{2}$

$(c+is)b - (c-is)a$

$p(x) = \int_{-\infty}^{\infty} e^{-i\lambda x}\left\{\frac{-i}{2\pi\lambda}\left[(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2}\right] + \frac{2it}{2\pi\lambda^2}\sin\frac{\lambda L}{2}\right\}d\lambda \qquad t = \frac{b-a}{L}$

$\sigma_{yy} = -p(x) = +\int_{-\infty}^{\infty} e^{-i\lambda x}\left\{\frac{i}{2\pi\lambda}\left[(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2}\right] - \frac{i(b-a)}{\pi L\lambda^2}\sin\frac{\lambda L}{2}\right\}d\lambda$

$\sigma_{yy} = \int_{-\infty}^{\infty} e^{-i\lambda x}\left\{\qquad\right\}d\lambda = \int_{-\infty}^{\infty} -\lambda^2 A e^{-i\lambda x}d\lambda \rightarrow -\lambda^2 A = \left\{\frac{i}{2\pi\lambda}\left[(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2}\right] + \frac{i(a-b)}{\pi L\lambda^2}\sin\frac{\lambda L}{2}\right\}$

$\therefore\; A = \frac{-1}{\lambda^2}\left\{\frac{i}{2\pi\lambda}\left[(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2}\right] - \frac{i(b-a)}{\pi L\lambda^2}\sin\frac{\lambda L}{2}\right\}$

$B = \frac{1}{|\lambda|}\left\{\frac{i}{2\pi\lambda}\left[(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2}\right] - \frac{i(b-a)}{\pi L\lambda^2}\sin\frac{\lambda L}{2}\right\}$

$\therefore\; \frac{\partial^2\phi}{\partial x^2} = \sigma_{yy} = -\int_{-\infty}^{\infty}\lambda^2 e^{-i\lambda x} A e^{-|\lambda|y}\left\{1+|\lambda|y\right\}d\lambda$

$-\frac{\partial^2\phi}{\partial x\partial y} = \sigma_{xy} = -i\int_{-\infty}^{\infty}\lambda e^{-i\lambda x} A\,\lambda^2 e^{-|\lambda|y}\left\{y\right\}d\lambda = -i\int_{-\infty}^{\infty}\lambda^3 A e^{-i\lambda x}e^{-|\lambda|y}y\,d\lambda$

$\phi = \int_{-\infty}^{\infty} e^{-i\lambda x}\left[Ae^{-|\lambda|y} + A|\lambda|ye^{-|\lambda|y}\right]d\lambda = \int_{-\infty}^{\infty} Ae^{-i\lambda x}e^{-|\lambda|y}(1+|\lambda|y)\,d\lambda$

$\frac{\partial\phi}{\partial y} = \int_{-\infty}^{\infty} Ae^{-i\lambda x}\left[-|\lambda|e^{-|\lambda|y}(1+|\lambda|y) + e^{-|\lambda|y}|\lambda|\right]d\lambda = -\int_{-\infty}^{\infty} Ae^{-i\lambda x}e^{-|\lambda|y}|\lambda|^2 y\,d\lambda = -\int A\lambda^2 e^{-i\lambda x}e^{-|\lambda|y}y\,d\lambda$

$\frac{\partial^2\phi}{\partial y^2} = \sigma_{xx} = \int_{-\infty}^{\infty} -Ae^{-i\lambda x}\lambda^2\left[-|\lambda|e^{-|\lambda|y}y + e^{-|\lambda|y}\cdot 1\right]d\lambda = -\int Ae^{-i\lambda x}\lambda^2 e^{-|\lambda|y}\left[1-|\lambda|y\right]d\lambda$

$$\frac{+1}{\lambda^2}\left[e^{i\lambda l/2} \; e^{-i\lambda l/2}\right] = \frac{2ti}{\lambda^2}\sin\frac{\lambda l}{2}$$

$$\cos\lambda l/2 + i\sin\lambda l/2 - (\cos\lambda l/2 - i\sin\lambda l/2) = 2i\sin\lambda l/2$$

$$R(\lambda) = \frac{1}{i\lambda}\left[e^{i\lambda l/2}b - e^{-i\lambda l/2}a\right] + \frac{2ti}{\lambda^2}\sin\frac{\lambda l}{2}$$

$$R(\lambda) = \frac{-i}{\lambda}\left[(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2}\right] + \frac{2ti}{\lambda^2}\sin\frac{\lambda L}{2}$$

$$R(\lambda) = \frac{-i}{\lambda}\left[(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2}\right] - 2\left(\frac{a-b}{L}\right)\frac{i}{\lambda^2}\sin\frac{\lambda L}{2}$$

$$\therefore \; p(x) = -i\int_{-\infty}^{\infty}\frac{e^{-i\lambda x}}{2\pi\lambda}\left\{(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2} + \frac{2(a-b)}{L\lambda}\sin\frac{\lambda L}{2}\right\}d\lambda$$

$$\sigma_{yy} = -p(x) = i\int_{-\infty}^{\infty}\frac{e^{-i\lambda x}}{2\pi\lambda}\left\{(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2} + \frac{2(a-b)}{L\lambda}\sin\frac{\lambda L}{2}\right\}d\lambda = -\int_{-\infty}^{\infty}\lambda^2 e^{-i\lambda x}A(\lambda)d\lambda$$

$$\therefore \; -A\lambda^2 = \frac{i}{2\pi\lambda}\left\{(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2} + \frac{2(a-b)}{L\lambda}\sin\frac{\lambda L}{2}\right\}$$

$$A = \frac{-i}{2\pi\lambda^3}\left\{(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2} + \frac{2(a-b)}{L\lambda}\sin\frac{\lambda L}{2}\right\}; \quad B = |\lambda|A$$

$$\frac{\partial^2\phi}{\partial x^2} = \sigma_{yy} = \int_{-\infty}^{\infty}\frac{i}{2\pi}\left\{(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2} + \frac{2(a-b)}{L\lambda}\sin\frac{\lambda L}{2}\right\}\frac{e^{-i\lambda x}}{\lambda}e^{-|\lambda|y}\{1+|\lambda|y\}d\lambda$$

$$= \int_{-\infty}^{\infty}\left\langle\frac{i}{2\pi}\left\{(b-a)\cos\frac{\lambda L}{2} + \frac{2(a-b)}{L\lambda}\sin\frac{\lambda L}{2}\right\}\left(\frac{-i\sin\lambda x}{\lambda}\right) - \frac{(b+a)}{2\pi}\sin\frac{\lambda L}{2}\frac{\cos\lambda x}{\lambda}\right\rangle e^{-|\lambda|y}\{1+|\lambda|y\}d\lambda$$

$$= \int_0^{\infty}\frac{1}{\pi}\left\{\frac{(b-a)}{\lambda}\cos\frac{\lambda L}{2}\sin\lambda x + \frac{2(a-b)}{L\lambda^2}\sin\lambda x\sin\frac{\lambda L}{2} - \frac{(b+a)}{\lambda}\sin\frac{\lambda L}{2}\cos\lambda x\right\}e^{-\lambda y}\{1+\lambda y\}d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left\{\frac{b}{\lambda}\sin\lambda(x-l/2) - \frac{a}{\lambda}\sin\lambda(x+l/2) + \frac{2(a-b)}{L\lambda^2}\sin\lambda x\sin\frac{\lambda L}{2}\right\}e^{-\lambda y}\{1+\lambda y\}d\lambda$$

$$\cos\lambda(x-l/2) - \cos\lambda(x+l/2)$$

take each at a time

$$\int_0^{\infty}\frac{b}{\pi\lambda}\sin\lambda(x-l/2)e^{-\lambda y}d\lambda + \int_0^{\infty}\frac{by}{\pi}\sin\lambda(x-l/2)e^{-\lambda y}d\lambda = \frac{b}{\pi}\tan^{-1}\left(\frac{x-l/2}{y}\right) + \frac{by}{\pi}\frac{(x-l/2)}{y^2+(x-l/2)^2}$$

$y > 0$

$$\int_0^{\infty}\frac{a}{\pi\lambda}\sin\lambda(x+l/2)e^{-\lambda y}d\lambda + \frac{ay}{\pi}\int\sin\lambda(x+l/2)e^{-\lambda y}d\lambda = \frac{a}{\pi}\tan^{-1}\left(\frac{x+l/2}{y}\right) + \frac{ay}{\pi}\frac{(x+l/2)}{y^2+(x+l/2)^2}$$

$$e^{-\lambda y}(1+\lambda y)\frac{a-b}{\pi L\lambda^2}\left[2\sin\lambda x\sin\frac{\lambda L}{2}\right] = \left[\cos\lambda(x-l/2) - \cos\lambda(x+l/2)\right]\frac{(a-b)}{\pi L\lambda^2}e^{-\lambda y}(1+\lambda y)$$

$$\therefore \int_0^{\infty}\frac{a-b}{L\pi}\left[\cos\lambda(x-l/2) - \cos\lambda(x+l/2)\right]\frac{e^{-\lambda y}}{\lambda^2}d\lambda + \int_0^{\infty}\frac{a-b}{\pi L}y\left[\cos\lambda(x-l/2) - \cos\lambda(x+l/2)\right]\frac{e^{-\lambda y}}{\lambda}d\lambda$$

$$\frac{a-b}{2L\pi}y\ln\frac{y^2+(x+l/2)^2}{y^2+(x-l/2)^2}$$

Let $Q = \int_0^\infty \left[\cos\lambda(x-1/2) - \cos\lambda(x+1/2)\right] \dfrac{e^{-\lambda y}}{\lambda^2}\, d\lambda$

$$\frac{\partial Q}{\partial x} = \int_0^\infty \left[\frac{-\sin\lambda(x-1/2) + \sin\lambda(x+1/2)}{\lambda}\right] e^{-\lambda y}\, d\lambda$$

$$\frac{\partial Q}{\partial x} = -\left[\tan^{-1}\frac{x-1/2}{y} - \tan^{-1}\frac{x+1/2}{y}\right]$$

~~$\int \tan^{-1}\frac{(x-1/2)}{y}\,dx = (x-1/2)\tan^{-1}\frac{(x-1/2)}{y} - \frac{y}{2}\ln\left[y^2+(x-1/2)^2\right] + g(y)$~~

~~as $y \to \infty$ $Q \to 0$~~

~~$\therefore -\frac{y}{2}\ln y^2 + g(y) = 0 \quad \therefore g(y) = \frac{y}{2}\ln y^2$~~

~~$(x-1/2)\tan^{-1}\frac{(x-1/2)}{y} - \frac{y}{2}\ln\frac{y^2+(x-1/2)^2}{y^2}$~~ $\therefore$

$$= \left\{-(x-1/2)\tan^{-1}\left(\frac{x-1/2}{y}\right) + \frac{y}{2}\log\left[y^2+(x-1/2)^2\right] + g(y) + (x+1/2)\tan^{-1}\left(\frac{x+1/2}{y}\right) - \frac{y}{2}\ln\left[y^2+(x+1/2)^2\right]\right\}$$

since $Q \to 0$ as $y \to \infty$

$y/2 \log y^2 - y/2 \log y^2 + g(y) = 0 \Rightarrow g(y) = 0$

$$Q = -(x-1/2)\tan^{-1} + (x+1/2)\tan^{-1} + y/2 \ln\frac{y^2+(x-1/2)^2}{y^2+(x+1/2)^2}$$

$$\int_0^\infty \frac{a-b}{L\pi}\Big[\qquad \Big]\frac{e^{-\lambda y}}{\lambda^2}\,d\lambda = \frac{a-b}{L\pi}\left\{(x+1/2)\tan^{-1}\left(\frac{x+1/2}{y}\right) + \frac{y}{2}\ln\frac{y^2+(x-1/2)^2}{y^2+(x+1/2)^2} - (x-1/2)\tan^{-1}\left(\frac{x-1/2}{y}\right)\right\}$$

$$\therefore \sigma_{yy} = \frac{b}{\pi}\tan^{-1}\left(\frac{x-1/2}{y}\right) + \frac{by}{\pi}\frac{(x-1/2)}{y^2+(x-1/2)^2} - \frac{a}{\pi}\tan^{-1}\left(\frac{x+1/2}{y}\right) - \frac{ay}{\pi}\frac{(x+1/2)}{y^2+(x+1/2)^2} + \frac{a-b}{L\pi}(x+1/2)\tan^{-1}\left(\frac{x+1/2}{y}\right) - \frac{a-b}{L\pi}(x-1/2)\tan^{-1}\left(\frac{x-1/2}{y}\right)$$

$$\Longrightarrow \sigma_{yy} = \tan^{-1}\left(\frac{x-1/2}{y}\right)\left\{\frac{b}{\pi} - \frac{a-b}{L\pi}(x-1/2)\right\} + \tan^{-1}\left(\frac{x+1/2}{y}\right)\left\{\frac{a-b}{L\pi}(x+1/2) - \frac{a}{\pi}\right\} + \frac{y}{\pi}\left\{\frac{b(x-1/2)}{y^2+(x-1/2)^2} - \frac{a(x+1/2)}{y^2+(x+1/2)^2}\right\}$$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = \int_{-\infty}^{\infty} -A\lambda^2 e^{-i\lambda x} e^{-|\lambda| y}\left[1-|\lambda|y\right]d\lambda$$

$$= \int_{-\infty}^{\infty} \frac{i}{2\pi}\left\{(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2} + \frac{2(a-b)}{L\lambda}\sin\frac{\lambda L}{2}\right\}\frac{e^{-i\lambda x}}{\lambda} e^{-|\lambda| y}\left[1-|\lambda|y\right]d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \left\{\frac{b}{\lambda}\sin\lambda(x-1/2) - \frac{a}{\lambda}\sin\lambda(x+1/2) + \frac{2(a-b)}{L\lambda^2}\sin\lambda x \sin\frac{\lambda L}{2}\right\}e^{-\lambda y}\left\{1-\lambda y\right\}d\lambda$$

$$\int_0^\infty \frac{1}{\pi}\frac{b}{\lambda}\sin\lambda(x-1/2)e^{-\lambda y}d\lambda - \int_0^\infty \frac{by}{\pi}\sin\lambda(x-1/2)e^{-\lambda y}d\lambda = \frac{b}{\pi}\tan^{-1}\left(\frac{x-1/2}{y}\right) - \frac{by}{\pi}\frac{(x-1/2)}{y^2+(x-1/2)^2}$$

$$\int \frac{1}{\pi}\frac{a}{\lambda}\sin\lambda(x+1/2)e^{-\lambda y}d\lambda - \frac{ay}{\pi}\int_0^\infty \sin\lambda(x+1/2)e^{-\lambda y}d\lambda = \frac{a}{\pi}\tan^{-1}\left(\frac{x+1/2}{y}\right) - \frac{ay}{\pi}\frac{(x+1/2)}{y^2+(x+1/2)^2}$$

$$\int_0^\infty \frac{2(a-b)}{\pi L\lambda^2}\sin\lambda x\sin\frac{\lambda L}{2}e^{-\lambda y}(1-\lambda y)d\lambda = \int_0^\infty \frac{(a-b)}{\pi L\lambda^2}\left[\cos\lambda(x-1/2) - \cos\lambda(x+1/2)\right]e^{-\lambda y}(1-\lambda y)\,d\lambda$$

$$\int_0^\infty \frac{(a-b)}{\pi L\lambda^2}\left[\cos\lambda(x-1/2) - \cos\lambda(x+1/2)\right]e^{-\lambda y}d\lambda - \int_0^\infty \frac{a-b}{\pi L}y\left[\cos\lambda(x-1/2) - \cos\lambda(x+1/2)\right]\frac{e^{-\lambda y}}{\lambda}d\lambda =$$

$$-\frac{a-b}{\pi L} y \ln \frac{y^2+(x+l/2)^2}{y^2+(x-l/2)^2} + \frac{a-b}{L\pi}\left\{(x+l/2)\tan^{-1}\left(\frac{x+l/2}{y}\right) - (x-l/2)\tan^{-1}\left(\frac{x-l/2}{y}\right)\right\}$$

$$\sigma_{xx} = \frac{b}{\pi}\tan^{-1}\left(\frac{x-l/2}{y}\right) - \frac{a}{\pi}\tan^{-1}\left(\frac{x+l/2}{y}\right) - \frac{by}{\pi}\frac{(x-l/2)}{y^2+(x-l/2)^2} + \frac{ay}{\pi}\frac{x+l/2}{y^2+(x+l/2)^2}$$
$$+ \frac{a-b}{L\pi}\left\{(x+l/2)\tan^{-1}\left(\frac{x+l/2}{y}\right) - (x-l/2)\tan^{-1}\left(\frac{x-l/2}{y}\right)\right\} - \frac{a-b}{\pi L} y \ln\frac{y^2+(x+l/2)^2}{y^2+(x-l/2)^2}$$

$$\sigma_{xy} = iy\int_{-\infty}^{\infty} -A\lambda^2 \left\{\lambda e^{-i\lambda x} e^{-|\lambda| y}\, d\lambda\right\}$$

$$= iy\int_{-\infty}^{\infty} \frac{i}{2\pi\lambda}\left\{(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2} + \frac{2(a-b)}{L\lambda}\sin\frac{\lambda L}{2}\right\}\lambda e^{-i\lambda x} e^{-|\lambda| y} d\lambda$$

$$= \frac{-y}{2\pi}\int_{-\infty}^{\infty}\left\{(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2} + \frac{2(a+b)}{L\lambda}\sin\frac{\lambda L}{2}\right\} e^{-i\lambda x} e^{-|\lambda| y} d\lambda$$

$$= \frac{-y}{\pi}\int_0^{\infty}\left\{(b-a)\cos\frac{\lambda L}{2}\cos\lambda x + (i)(b+a)\sin\frac{\lambda L}{2}(-i)\sin\lambda x + \frac{2(a+b)}{L\lambda}\sin\frac{\lambda L}{2}\cos\lambda x\right\} e^{-\lambda y} d\lambda$$

$$\sigma_{xy} = \frac{-y}{\pi}\int_0^{\infty}\left\{b\cos\lambda(x-l/2) - a\cos\lambda(x+l/2) + \frac{a-b}{L\lambda}\left[\sin\lambda(x+l/2) - \sin\lambda(x-l/2)\right]\right\} e^{-\lambda y} d\lambda$$

$$S_x C_L + C_x S_L - S_x C_L + C_x S_L$$

$$= \frac{-yb}{\pi}\int_0^{\infty}\cos\lambda(x-l/2)e^{-\lambda y}d\lambda + \frac{ay}{\pi}\int_0^{\infty}\cos\lambda(x+l/2)e^{-\lambda y}d\lambda - \frac{a-b}{\pi L}y\int_0^{\infty}\left(\frac{\sin\lambda(x+l/2)-\sin\lambda(x-l/2)}{\lambda}\right)e^{-\lambda y}d\lambda$$

$$\sigma_{xy} = -\frac{yb}{\pi}\left[\frac{y}{y^2+(x-l/2)^2}\right] + \frac{ay}{\pi}\left[\frac{y}{y^2+(x+l/2)^2}\right] - \frac{(a-b)y}{\pi L}\left[\tan^{-1}\frac{(x+l/2)}{y} - \tan^{-1}\left(\frac{x-l/2}{y}\right)\right]$$

iii) $y = 100$ ft., $\sigma_{yy} \le 800$ psi, $P = 120{,}000$ #/ft, $L = 10'$, $e = 1'$

$$b = \frac{1.2\times10^5\ \#/ft}{1\times10^2\ ft^2}(4 ft) = 4.8\times10^3\ \frac{\#}{ft^2} \qquad a = \frac{1.2\times10^5\ \#}{1\times10^2\ ft^3}(16\ ft) = 1.92\times10^4\ \#/ft^2$$

$$\sigma_{yy}\Big|_{y=100} = \tan^{-1}\left(\frac{x-5}{100}\right)\left\{\frac{4.8\times10^3}{3.14159} - \frac{1.44\times10^4}{31.4159}(x-5)\right\} + \tan^{-1}\left(\frac{x+5}{100}\right)\left\{\frac{1.44\times10^4}{31.4159}(x+5) - \frac{1.92\times10^4}{3.1415}\right\}$$

$$\frac{100}{3.14159}\left\{\frac{4.8\times10^3(x-5)}{1\times10^4+(x-5)^2} - \frac{1.92\times10^4(x+5)}{1\times10^4+(x+5)^2}\right\}$$

program this to get for different values of x & get solution to part (3)

Max of 762.8 psf @ −.1 ft. ⇒ 5.3 psi we are way under limit ∴ no settling

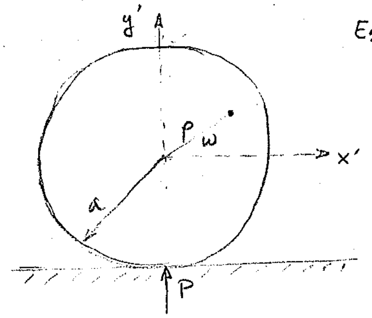
1. 20/20
2. 20/20
3. 20/20
4. 20/20
5. 20/20

100/100

Cesar Levy
ME 238B Winter Qtr 79
Prof. Barnett

Midterm Exam

I. A.



Equilib : $\gamma\pi a^2 = P$ where P = force/unit length

Since we must account for the body force the equil. equations are

$$\frac{\partial\sigma_{x'x'}}{\partial x'} + \frac{\partial\sigma_{x'y'}}{\partial y'} = 0$$
$$\frac{\partial\sigma_{x'y'}}{\partial x'} + \frac{\partial\sigma_{y'y'}}{\partial y'} - \gamma = 0$$

if we take $\sigma_{x'x'} = \frac{\partial^2\phi}{\partial y'^2} + \gamma y$ $\quad \sigma_{y'y'} = \frac{\partial^2\phi}{\partial x'^2} + \gamma y \quad \sigma_{x'y'} = -\frac{\partial^2\phi}{\partial x\partial y'}$

we satisfy equilib and from compatibility $\Delta^2\phi = 0$ where $\nabla^2 = \Delta$

we can take $\phi = 0$ now, hence all the the body force would do would give you a

$\sigma_{x'x'} = \gamma\rho\sin\omega \quad \sigma_{y'y'} = \gamma\rho\sin\omega \quad \sigma_{x'y'} = 0$ or $\sigma_{\rho\rho} = \gamma\rho\sin\omega$, $\sigma_{\omega\omega} = \gamma\rho\sin\omega$, $\sigma_{\rho\omega} = 0$

within the body.

@ $\rho = a$ then $\sigma_{\rho\rho} = \gamma a\sin\omega = \frac{P}{\pi a}\sin\omega \quad \sigma_{\omega\omega} = \frac{P}{\pi a}\sin\omega \quad \sigma_{\rho\omega} = 0$

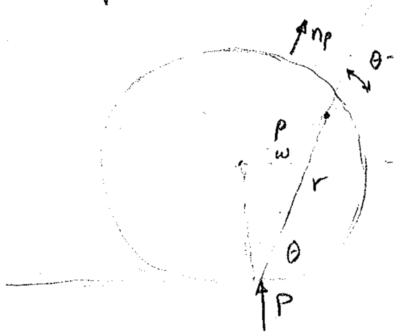
Thus if we define other stress functions, then, since we've taken care of the body force, we need only define them ∋.

$$\frac{\partial\sigma_{x'x'}}{\partial x'} + \frac{\partial\sigma_{x'y'}}{\partial y'} = 0$$
$$\frac{\partial\sigma_{x'y'}}{\partial x'} + \frac{\partial\sigma_{y'y'}}{\partial y'} = 0$$

or $\sigma_{x'x'} = \frac{\partial^2\phi}{\partial y'^2}$, $\sigma_{x'y'} = -\frac{\partial^2\phi}{\partial x'\partial y'}$, $\sigma_{y'y'} = \frac{\partial^2\phi}{\partial x'^2}$

and from compatibility $\Delta^2\phi = 0$ where $\nabla^2 = \Delta$

B. If to the above we add the solution for a half space $\phi = \frac{P}{\pi}\theta\, r\cos\theta$ and define



ρ, ω, r, θ through $r\sin\theta = \rho\sin\omega + a$; $r\cos\theta = \rho\cos\omega$

then $\sigma_{rr} = -\frac{2P}{\pi r}\sin\theta$, $\sigma_{\theta\theta} = 0$, $\sigma_{r\theta} = 0$

thus to get $\sigma_{\rho\rho} = \sigma_{rr}\cos^2(\theta-\omega)$

$\sigma_{\omega\omega} = \sigma_{rr}\sin^2(\theta-\omega)$

$\sigma_{\rho\omega} = \sigma_{rr}\cos(\theta-\omega)\sin(\theta-\omega)$ for points within the disc

on the border of the disc: $\sigma_{rr} = -\frac{P}{\pi a}$, $\theta - \omega = \pi/2 - \theta$ and hence it can be easily shown

$$\sigma_{\rho\rho} = -\frac{P}{\pi a}\cos^2(\pi/2-\theta) = -\frac{P}{2\pi a}(1+\sin\omega)$$
$$\sigma_{\omega\omega} = -\frac{P}{\pi a}\sin^2(\pi/2-\theta) = -\frac{P}{2\pi a}(1-\sin\omega)$$
$$\sigma_{\rho\omega} = -\frac{P}{2\pi a}\sin(\pi-2\theta) = -\frac{P}{2\pi a}\cos\omega$$

when we add A+B then for tractions on the boundary where $n_\rho = 1$

$\sigma_{\rho\rho} = -\frac{P}{2\pi a}(1-\sin\omega) \quad \sigma_{\rho\omega} = -\frac{P}{2\pi a}\cos\omega$

thus we must find other ϕ functions in part c $\ni$. $\sigma_{\rho\rho})_{\rho=a} = \frac{P}{2\pi a}(1-\sin\omega)$ $\sigma_{\rho\omega})_{\rho=a} = \frac{P}{2\pi a}\cos\omega$ in order for no stresses to exist on the boundary when we add A+B+C.

c. Part 1 let's look at $\phi = b_0\rho^2$ this will give a purely constant radial distribution

$$\therefore \quad \sigma_{\rho\rho} = 2b_0 \qquad \sigma_{\rho\omega} = 0 \qquad \sigma_{\omega\omega} = 2b_0 \quad ; \text{ if we set } 2b_0 = \frac{P}{2\pi a} \text{ then } b_0 = \frac{P}{4\pi a}$$

this will give the constant stress at $\rho = a$ ie $\sigma_{\rho\rho})_{\rho=a} = \frac{P}{2\pi a}$ $\sigma_{\rho\omega} = 0$ $\sigma_{\omega\omega} = \frac{P}{2\pi a}$ ✓

Part 2 we now have to find a ϕ $\ni$. $\sigma_{\rho\rho})_{\rho=a} = -\frac{P}{2\pi a}\sin\omega$ $\sigma_{\rho\omega})_{\rho=a} = \frac{P}{2\pi a}\cos\omega$

From the handout and since the disc contains the origin $\phi = \phi(b_0, d_0, b_1, d_1, a_n, b_n, c_n, d_n; \rho, \omega)$ only but evaluating the stresses $\sigma_{\rho\rho}$ and $\sigma_{\rho\omega}$ and equating the coefficients we can show $b_0 = d_0 = b_1 = a_n = b_n = c_n = d_n = 0$ and $d_1 = -\frac{P}{4\pi a^2}$

hence $\phi = -\frac{P\rho^3}{4\pi a^2}\sin\omega$ $\qquad \sigma_{\rho\rho} = -\frac{2P\rho}{4\pi a^2}\sin\omega$ $\quad \sigma_{\rho\omega} = \frac{P\rho}{2\pi a^2}\cos\omega$

$\sigma_{\rho\omega} = \frac{-3P\rho}{2\pi a^2}\sin\omega$

thus $\phi_{TOT} = \frac{P}{\pi}\theta r\cos\theta + \frac{P\rho^2}{4\pi a} - \frac{P\rho^3}{4\pi a^2}\sin\omega$ where r, θ, ρ, ω are related by $\begin{cases} r\sin\theta = \rho\sin\omega + a \\ r\cos\theta = \rho\cos\omega \end{cases}$ ✓
when $P = \gamma\pi a^2$

$\therefore$

ϕ_i	$\sigma_{\rho\rho_i}$	$\sigma_{\rho\omega_i}$	$\sigma_{\omega\omega_i}$	
0	$\frac{P}{\pi a^2}\rho\sin\omega$	0	$\frac{P}{\pi a^2}\rho\sin\omega$	
$\frac{P}{\pi}\theta r\cos\theta$	$\sigma_{rr}\cos^2(\theta-\omega)$	$\frac{\sigma_{rr}}{2}\sin(2\theta-2\omega)$	$\sigma_{rr}\sin^2(\theta-\omega)$	$\sigma_{rr} = -\frac{P}{\pi r}\sin\theta$
$\frac{P\rho^2}{4\pi a}$	$\frac{P}{2\pi a}$	0	$\frac{P}{2\pi a}$	
$-\frac{P\rho^3\sin\omega}{4\pi a^2}$	$-\frac{2P\rho}{4\pi a^2}\sin\omega$	$-\frac{3P\rho\sin\omega}{2\pi a^2}$	$\frac{P\rho}{2\pi a^2}\cos\omega$	

$$\phi_{TOT} = \sum\phi_i \qquad \sigma_{\rho\rho} = \sum\sigma_{\rho\rho_i} \qquad \sigma_{\rho\omega} = \sum\sigma_{\rho\omega_i} \qquad \sigma_{\omega\omega} = \sum\sigma_{\omega\omega_i}$$

II For an infinite incompressible inviscid flow I assumed: steady state, 2D, irrotational flow, ρ = constant solid walls are streamlines, we neglect the no slip condition (ie $\sigma_{r\theta} = 0$ at wall). We thus obtain the pressure on the cylinder by using Bernoulli's equation. We define a stream function ψ ∋, $u = \frac{\partial \psi}{\partial y}$ $v = -\frac{\partial \psi}{\partial x}$ or $v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $v_\theta = -\frac{\partial \psi}{\partial r}$

The steady state Bernoulli equation is $p/\rho - U + \frac{1}{2}(v_r^2 + v_\theta^2) = \text{const}$ where $\nabla U = f_b$ (the body force which we assume is conservative).

The stream function $\psi = V_\infty r \sin\theta \left[1 - b^2/r^2\right]$. We note that the $\left\{\begin{matrix}\text{inner}\\\text{outer}\end{matrix}\right\}$ surfaces can be considered as stream-lines and no fluid thus passes through the cylinder; thus

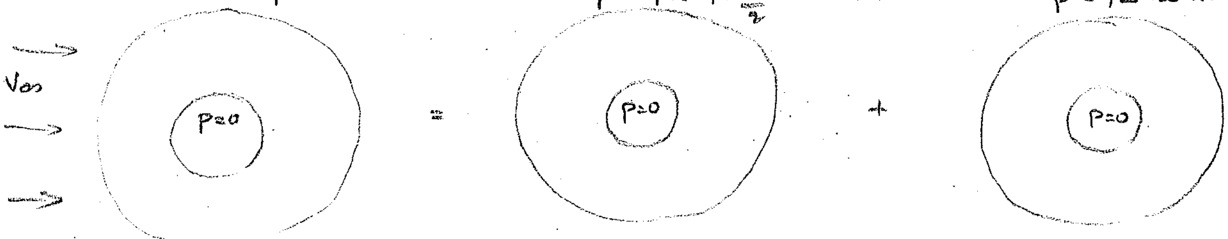
$$v_r\Big|_{r=b} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_\infty \cos\theta \left(1 - \frac{b^2}{r^2}\right)\Big|_{r=b} = 0 \qquad v_\theta\Big|_{r=b} = -\frac{\partial \psi}{\partial r} = -V_\infty \sin\theta \left(1 + \frac{b^2}{r^2}\right)\Big|_{r=b} = -2V_\infty \sin\theta$$

We will neglect the body forces and evaluate the Bernoulli constant at ∞ so that

$$p = p_\infty + \frac{\rho_\infty V_\infty^2}{2} - \frac{\rho_\infty}{2} 4 V_\infty^2 \sin^2\theta = p_\infty + \frac{\rho_\infty V_\infty^2}{2}\left[1 - 4\sin^2\theta\right] = p_\infty - \frac{\rho_\infty V_\infty^2}{2} + \rho_\infty V_\infty^2 \cos 2\theta$$

We can solve this by superposition of solutions w/ $p_{inner} = p\big|_{r=a} = 0$

$p = p^0 + \hat{p}$ $\qquad p^0 = p_\infty - \frac{\rho_\infty V_\infty^2}{2}$ $\qquad \hat{p} = \rho_\infty V_\infty^2 \cos 2\theta$



p^0 can be obtained from the plane strain solution w/ $p_{inner} = 0$

$$T_r^o = -p^o = \sigma_{rr_o} \quad \therefore \quad \sigma_{rr_o} = -p^o, \quad \sigma_{r\theta_o} = 0 \quad @ \ r = b$$
$$T_r^i = T_\theta^i = 0 \quad \therefore \quad \sigma_{rr_i} = 0 \quad \sigma_{r\theta_i} = 0 \quad @ \ r = a$$

$$\therefore \quad \sigma_{rr}^0 = \frac{\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right)a^2 b^2}{b^2 - a^2}\cdot\frac{1}{r^2} - \frac{\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right)b^2}{b^2 - a^2}$$

$$\sigma_{r\theta}^0 = 0$$

$$\sigma_{\theta\theta}^0 = -\frac{\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right)a^2 b^2}{b^2 - a^2}\cdot\frac{1}{r^2} - \frac{\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right)b^2}{b^2 - a^2}$$

$$\sigma_{zz}^0 = \nu(\sigma_{rr}^0 + \sigma_{\theta\theta}^0) = -2\nu\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right)\ b^2$$

The $\hat{p}$ solution can be obtained as in class $\hat{\sigma}_{r\theta}\big|_{r=a} = \hat{\sigma}_{rr}\big|_{r=a} = \hat{\sigma}_{r\theta}\big|_{r=b} = 0 \qquad \hat{\sigma}_{rr}\big|_{r=b} = \hat{T}_r = -\hat{p}$

by use of the stress fn and the stresses at the boundary as found in the handout

$A_0' = 0\,,\ A_1' = 0\,,\ A_2' = -\rho_\infty V_\infty^2\,,\ A_3' = A_4' = \dots = A_\infty' = 0\,,\ B_n' = 0$

$A_0 = A_n = B_n = 0\,,\ C_0 = C_n = D_n = C_0' = C_n' = D_n' = 0$

We note the due to the above coefficients the momentum equilib eqn. is satisfied identically

As in the notes for single valued displacements take $d_0 = c_0 = 0$. From the above coeffs we also get $a_0' = a_1 = c_1 = b_1' = d_1' = b_1 = a_1' = c_1' = d_1 = a_0 = b_0 = 0$

Since the only non zero term occurs for a_2, b_2, a_2', b_2' then $\forall\, n \geq 3\ \ a_n = b_n = a_n' = b_n' = 0$

Since the traction on the inner boundary $= 0 \Rightarrow c_n = d_n = c_n' = d_n' = 0 \quad \forall\, n \geq 2$.

Thus the non zero terms give rise to the following equations

$$\begin{bmatrix} -2 & 0 & -6a^{-4} & -4a^{-2} \\ -2 & 0 & -6b^{-4} & -4b^{-2} \\ 1 & 3a^2 & -3a^{-4} & -a^{-2} \\ 1 & 3b^2 & -3b^{-4} & -b^{-2} \end{bmatrix} \begin{pmatrix} a_2 \\ b_2 \\ a_2' \\ b_2' \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho_\infty V_\infty^2 \\ 0 \\ 0 \end{pmatrix}$$

hence $\quad \hat{\phi} = (a_2 r^2 + b_2 r^4 + a_2' r^{-2} + b_2')\cos 2\theta$

Now I don't know whether Prof. Barnett said not to solve this or not but here is what I found after reducing the matrix by dividing lines 1, 2 by -2.

D = denominator in the cramer rule $= (b^{-2} - a^{-2})(3a^{-4}b^2 - 9a^{-2} - 3a^2b^{-4} + 9b^{-2})$

N_1 = numerator for $a_2 = -\rho_\infty V_\infty^2 \cdot \frac{3b^{-2}}{2}(a^{-6}b^4 - 2b^{-2} + a^{-2})$

N_2 = " " $b_2 = \rho_\infty V_\infty^2 \cdot \frac{a^{-2}}{2}(a^{-2} - b^{-2})(a^{-2} + 3b^{-2})$

N_3 = " " $a_2' = \frac{\rho_\infty V_\infty^2}{2} \cdot (-3a^{-2}b^2 + a^2b^{-2} + 2)$

N_4 = " " $b_2' = \frac{\rho_\infty V_\infty^2}{2} \cdot 3(a^2b^{-4} - a^{-2} + 2a^{-4}b^2)$

$\therefore \quad a_2 = N_1/D \qquad b_2 = N_2/D \qquad a_2' = N_3/D \qquad b_2' = N_4/D$

$\therefore \quad \hat{\sigma}_{rr} = (-2a_2 - 6a_2' r^{-4} - 4b_2' r^{-2})\cos 2\theta$

$\hat{\sigma}_{r\theta} = 2\sin 2\theta\,(a_2 + 3b_2 r^2 - 3a_2' r^{-4} - b_2' r^{-2})$

$\hat{\sigma}_{\theta\theta} = (2a_2 + 12 b_2 r^2 + 6a_2' r^{-4})\cos 2\theta$

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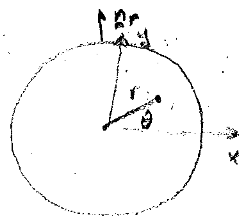
thus $\sigma_{rr} = \sigma_{rr}^0 + \hat{\sigma}_{rr} \qquad \sigma_{r\theta} = \hat{\sigma}_{r\theta} \qquad$ and $\quad \sigma_{\theta\theta} = \sigma_{\theta\theta}^0 + \hat{\sigma}_{\theta\theta}$

III For this problem we look at the inclusion plate interface using the b.c. that $T_i^* = T_p^*$ and $u_i^* = u_p^*$, $v_i^* = v_p^*$ for a perfectly welded condition at $r=a$. Note $(\ )_i$ = inclusion ; $(\ )_p$ = plate

1. We look at the inclusion first. We assume that due to uniform σ at ∞ there will be



on the boundary and lets assume we know them

$$\therefore \text{ Let } T_{r_i}^* = \sigma_{rr_i}^* = A_0'^* + \sum_1^\infty A_n'^* \cos n\theta + \sum_1^\infty B_n'^* \sin n\theta$$

$$T_{\theta_i}^* = \sigma_{r\theta_i}^* = C_0'^* + \sum_1^\infty C_n'^* \cos n\theta + \sum_1^\infty D_n'^* \sin n\theta$$

Since we do not expect ∞ stresses in the center of the inclusion and no discontinuities in stress, then

$$\phi_i = b_{0_i} r^2 + b_{1_i} r^3 \cos\theta + d_{1_i} r^3 \sin\theta + \sum_{n=2}^\infty (a_{n_i} r^n + b_{n_i} r^{n+2}) \cos n\theta + \sum_{n=2}^\infty (c_{n_i} r^n + d_{n_i} r^{n+2}) \sin n\theta$$

Thus we can solve for $b_{0_i}, b_{1_i}, d_{1_i}, a_{n_i}, b_{n_i}, c_{n_i}, d_{n_i}$ in terms of $A_0'^*, A_n'^*, B_n'^*, C_0'^*, C_n'^*, D_n'^*$.

We can now use the fact that $\sigma_{rr_i} = \frac{1}{r}\frac{\partial \phi_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_i}{\partial \theta^2}$, $\sigma_{r\theta_i} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi_i}{\partial \theta}\right)$, and $\sigma_{\theta\theta_i} = \frac{\partial^2 \phi_i}{\partial r^2}$ to form the strains

(1). $\varepsilon_{\theta_i} = \frac{u_i}{r} + \frac{1}{r}\frac{\partial v_i}{\partial \theta} = \frac{1}{E^*}\left[(1-\nu^{*2})\sigma_{\theta\theta_i} - \nu^*(1+\nu^*)\sigma_{rr_i}\right]$ where $u = u_r$, $v = u_\theta$

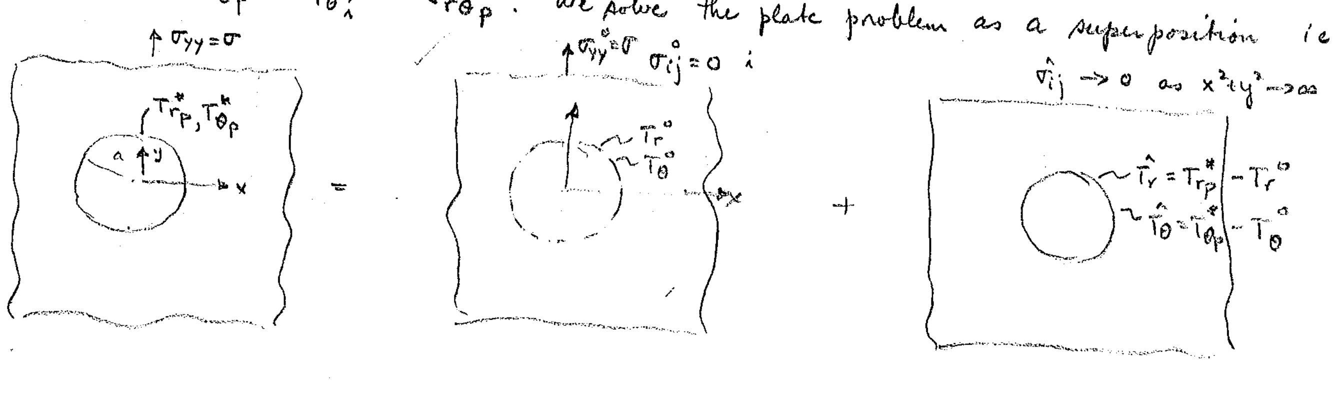
(2) $\varepsilon_{r_i} = \frac{\partial u_i}{\partial r} = \frac{1}{E^*}\left[(1-\nu^{*2})\sigma_{rr_i} - \nu^*(1+\nu^*)\sigma_{\theta\theta_i}\right]$

(3) $2\varepsilon_{r\theta_i} = \frac{\partial u_i}{r\partial \theta} + \frac{\partial v_i}{\partial r} - \frac{v_i}{r} = \frac{1}{G^*}\sigma_{r\theta_i}$ where G^* is the modulus of rigidity of the inclusion $= \frac{2(1+\nu^*)}{E^*}$

Using (2) we can get u_i to include the fn of integration $f(\theta)$. Using this in (1) we can get v_i to include the fn of integration $g(r)$. We use these in (3) to define conditions on $g(r)$ and $f(\theta)$.

We assume in doing all this plane strain deformation so that $w_i = h(r,\theta)$ only. Now setting $r=a$ we define u_i^*, v_i^* at the boundary and we note that they are still functions of the unknowns $A_0'^*, A_n'^*, B_n'^*, C_0'^*, C_n'^*, D_n'^*$.

2. We now look at the plate problem. On the plate inclusion boundary $T_{r_i}^* = T_{r_p}^* = \sigma_{rr_i}^* = -\sigma_{rr_p}^*$ and $T_{\theta_i}^* = T_{\theta_p}^* = \sigma_{r\theta_i}^* = -\sigma_{r\theta_p}^*$. We solve the plate problem as a superposition ie



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** Before I go further, I must mention that we neglect any body forces

For the ° fields as in class we pick

$$\phi_p^\circ = \frac{\sigma}{4} r^2 (1-\cos 2\theta) \qquad \sigma_{rr_p}^\circ = \frac{\sigma}{2}(1-\cos 2\theta),\ \sigma_{\theta\theta_p}^\circ = \frac{\sigma}{2}(1+\cos 2\theta),\ \sigma_{r\theta_p}^\circ = \frac{\sigma}{2}\sin 2\theta$$

For the ^ fields and with $\hat{\sigma}_{ij} \to 0$ as $r \to \infty$ we find that

$$\hat{\phi}_p = f(a_1, a_1', b_1', c_1, c_1', d_1', a_n', b_n', c_n', d_n'; r, \theta)$$ single valued displacements

Now we can solve for the values of the coeffs in $\hat{\phi}_p$ since we "know" what the bc are at the hole

ie $\hat{T}_r = T_{r_p}^* - T_{r_p}^\circ = -\sigma_{rr_p}^* + \sigma_{rr_p}^\circ = \sigma_{rr_i}^* + \sigma_{rr_p}^\circ = -\hat{\sigma}_{rr_p}$

$\hat{T}_\theta = \hat{T}_{\theta_p} - T_{\theta_p}^\circ = -\sigma_{r\theta_p}^* + \sigma_{r\theta_p}^\circ = \sigma_{rr_i}^* + \sigma_{r\theta_p}^\circ = -\hat{\sigma}_{r\theta_p}$

Now this defines $\hat{\sigma}_{rr_p}, \hat{\sigma}_{r\theta_p}$ in terms of $A_0'^*, A_n'^*, B_n'^*, C_n'^*, C_0'^*, D_n'^*$ of the inclusion.

** Again before I go further we note that the solution ϕ_p° gives no moment or net T_x, T_y hence any net moment, T_x, T_y from the inclusion portion will be canceled by that given by the solution given by $\hat{\phi}_p$ ∴ the system is self equilibrating

Thus we can define $a_1, a_1', b_1', c_1, c_1', d_1', a_n', b_n', c_n', d_n'$ in terms of the above and obtain $\hat{\sigma}_{rr_p}, \hat{\sigma}_{r\theta_p}, \hat{\sigma}_{\theta\theta_p}$ in the plate

We now use

$\hat{\varepsilon}_{r_p} = \frac{\partial \hat{u}_p}{\partial r} = \frac{1}{E}\left[(1-\nu^2)\hat{\sigma}_{rr_p} - \nu(1+\nu)\hat{\sigma}_{\theta\theta_p}\right]$ to obtain $\hat{u}_p$, to include an arbitrary fn $\hat{f}_p'(\theta)$. we can use this and $\hat{\varepsilon}_{\theta_p} = \frac{\hat{u}_p}{r} + \frac{1}{r}\frac{\partial \hat{v}_p}{\partial \theta} = \frac{1}{E}\left[(1-\nu^2)\hat{\sigma}_{\theta\theta_p} - \nu(1+\nu)\hat{\sigma}_{rr_p}\right]$ to get $\hat{v}_p$, to include an arbitrary fn $\hat{g}_p(r)$. with $\hat{u}_p, \hat{v}_p$ and $\hat{\gamma}_{r\theta_p} = \frac{1}{G}\hat{\sigma}_{r\theta_p} = \frac{\partial \hat{u}_p}{r\partial\theta} + \frac{\partial \hat{v}_p}{\partial r} - \frac{\hat{v}_p}{r}$ we can find $\hat{f}_p(\theta)$ and $\hat{g}_p(r)$.

Using this method on the ° stress fields (and barring any math errors) we get

$\varepsilon_{r_p}^\circ = \frac{\partial u_p^\circ}{\partial r} = \frac{\sigma}{2}\frac{(1+\nu)}{E}(1-2\nu-\cos 2\theta) \Rightarrow u_p^\circ = \frac{\sigma}{2}\frac{(1+\nu)}{E} r(1-2\nu-\cos 2\theta) + f'(\theta)$

$\varepsilon_{\theta_p}^\circ = \frac{\sigma}{2}\frac{(1+\nu)}{E}(1+2\nu-\cos 2\theta) \Rightarrow v_p^\circ = \frac{\sigma(1+\nu)r}{2E}\left(\frac{\sin 2\theta}{2} - 2\nu\theta\right) - f(\theta) + g(r)$

$\varepsilon_{r\theta_p}^\circ = \frac{\sigma}{2}\frac{(1+\nu)}{E}\sin 2\theta \Rightarrow f''+f = g - g'r = c_1$ const ∴ $f = c_2\cos(\theta+\varphi) + c_1 \quad g = c_1 + c_2 r$

Thus we can now obtain $u_p = u_p^\circ + \hat{u}_p$ and $v_p = v_p^\circ + \hat{v}_p$ and hence u_p^*, v_p^* on the boundary. But this must match u_i^*, v_i^* respectively with

$u_i^* = u_i^*(A_0'^*, A_n'^*, B_n'^*, C_0'^*, C_n'^*, D_n'^*; \nu^*, E^*, a) = u_p^*(A_0'^*, A_n'^*, B_n'^*, C_0'^*, C_n'^*, D_n'^*; \nu, E, a)$

and similarly $v_i^* = v_i^*($ —— " —— $) = v_p^*($ —— " —— $)$

thus we can define the coeffs in terms of the known constants (ν^*, ν, E^*, E, a) and hence define the stresses throughout as well as the displacements to within a rigid body rotation and displacement.

$$\phi_{plate} = \phi_p^{\circ} + \hat{\phi}_p$$

I expect the stress concentration factor to be less than three since, unlike the open hole, there is material there and hence it can relieve some of the stress at the boundary thereby decreasing the value of $\sigma_{\theta\theta}/\sigma$.

Another way to see this, I suppose, is to ask how can I create a zero stress on the boundary of the plate and inclusion (ie make $\sigma_{rr}, \sigma_{r\theta} = 0$ at $r=a$). The only way that would be possible is if there was 0 load at ∞, keeping the same loading method of the plate w/hole and plate w/inclusion (ie at ∞). This implies that since $\sigma_{ij} = \sigma \frac{f_{ij}(\theta)}{r}$ (using linear elasticity) then $\sigma_{\theta\theta} = 0$ or that the stress concentration factor would drop below three.

Now you have a length "a".

Please see reverse of this page for more on problem III

IV Rich, Prof. Barnett's notation is quite confusing with respect to Prof. Herrmann's. I assume he means this

$$\int_S^F \underline{M}\, ds = \int_S^F (M_x e_x + M_y e_y + M_z e_z)\, ds = \int_S^F M_z\, ds = \int_S^F \underline{r} \times \underline{T}\, ds = \int_S^F (x\,\underline{e}_x + y\,\underline{e}_y) \times (T_y\,\underline{e}_y + T_x\,\underline{e}_x)\, ds$$

$$= \int_S^F (x T_y - y T_x)\, ds \;\; \underline{e}_z$$

$$= \int_S^F \left(x\,[\sigma_{yx} n_x + \sigma_{yy} n_y] - y\,[\sigma_{xx} n_x + \sigma_{xy} n_y] \right) ds$$

$$= \int_S^F \left\{ x\left[-\frac{\partial^2\phi}{\partial x \partial y}\frac{dy}{ds} + \frac{\partial^2\phi}{\partial x^2}\left(-\frac{dx}{ds}\right)\right] - y\left[\frac{\partial^2\phi}{\partial y^2}\left(\frac{dy}{ds}\right) - \frac{\partial^2\phi}{\partial x \partial y}\left(-\frac{dx}{ds}\right)\right]\right\} ds$$

$$= \int_S^F \left\{ -x\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right) - y\frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right)\right\} ds = \left[-x\frac{\partial\phi}{\partial x} - y\frac{\partial\phi}{\partial y}\right]_S^F + \int_S^F \left(\frac{dx}{ds}\frac{\partial\phi}{\partial x}\, ds + \frac{dy}{ds}\frac{\partial\phi}{\partial y}\, ds\right)$$

$$= -\underline{r}\cdot\underline{\nabla}\phi \Big|_S^F + \int_S^F \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = -\underline{r}\cdot\underline{\nabla}\phi \Big|_S^F + \int_S^F d\phi$$

$$= -\underline{r}\cdot\underline{\nabla}\phi \Big|_S^F + \phi\Big|_S^F = \left[\phi - \underline{r}\cdot\underline{\nabla}\phi\right]_S^F$$

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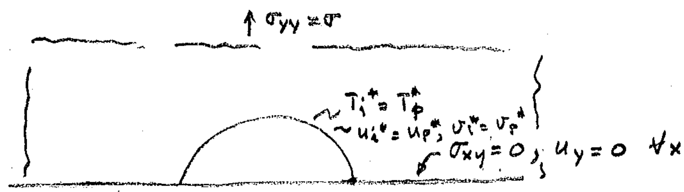
We note that 1 $u_y(y=0)\ \forall x = 0$

$$\sigma_{xy}(y=0)\ \forall x = 0$$

$$u_x(x,y) = u_x(x,-y)$$

$$u_y(x,y) = -u_y(x,-y)$$

thus we could have looked at the half plane problem



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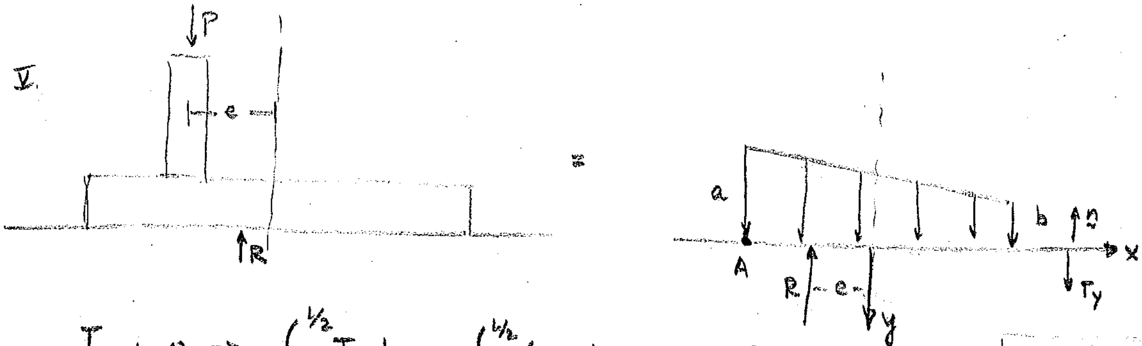


take $p(x) = a + \frac{b-a}{L}(x + L/2)$

$T_y = -\sigma_{yy} = p(x) - P\,\delta(x+e)$

$T_{y\,net} = 0 \Rightarrow \int_{-L/2}^{L/2} T_y\, ds = \int_{-L/2}^{L/2} \left\{a + \frac{b-a}{L}(x + L/2)\right\} dx - P = 0 \Rightarrow \boxed{a + b = \frac{2P}{L}}$

$T_{x\,net} \equiv 0 \quad ; \quad \Sigma M_{A})_+ = 0 \Rightarrow \int_{-L/2}^{L/2} \underline{r} \times T_y\, ds = \int_{-L/2}^{L/2} (x + L/2)\left\{a + \frac{b-a}{L}(x + L/2)\right\}dx - \int_{-L/2}^{L/2} P(x + L/2)\,\delta(x+e)\,dx = 0$

$$\Rightarrow \boxed{a + 2b = \frac{6P}{L^2}\left(\frac{L}{2} - e\right)}$$

These two equations lead to

$$\boxed{b = \frac{P}{L^2}(L - 6e) \quad , \quad a = \frac{P}{L^2}(6e + L)}$$

Now I take $y \downarrow +$ so that my integrations will be valid. Method of attack write an equation for the stress fn in a half space, take derivatives and define $\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$, $\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$, $\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$

HERE WE GO! MATHEMATICS ALL THE WAY & HOPING WE MAKE NO ERRORS!

1. Using $\phi = \int_{-\infty}^{\infty} e^{-i\lambda x}\left[A e^{-|\lambda| y} + B y\, e^{-|\lambda| y}\right] d\lambda$ with $\sigma_{yy} = -p(x)$ and $\sigma_{xy} = 0$ @ $y = 0$ we obtain that

(1) $\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -i\int_{-\infty}^{\infty} \lambda e^{i\lambda x}\left[|\lambda| A + B(y|\lambda| - 1)\right] e^{-|\lambda| y} d\lambda \Rightarrow$ @ $y = 0$ $\underline{B = A|\lambda|}$

(2) $\sigma_{yy} = +\frac{\partial^2 \phi}{\partial x^2} = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x} e^{-|\lambda| y}(A + By)\, d\lambda \Rightarrow$ @ $y = 0$ $\quad -p(x) = \int_{-\infty}^{\infty} -\lambda^2 A e^{-i\lambda x} d\lambda$

We want to write $p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x} R(\lambda)\, d\lambda \Rightarrow R(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x}(s + tx)\, dx$ where $s = \frac{a+b}{2}$, $t = \frac{b-a}{L}$

but $s + tx$ is defined $|x| \leq L/2$; we note the discont of $(s+tx)$ at $x = \pm L/2$ but no problem

$\therefore \quad R(\lambda) = \int_{-L/2}^{L/2} e^{i\lambda x}(s + tx)\, dx = \frac{1}{i\lambda}\left\{e^{i\lambda L/2} b - e^{-i\lambda L/2} a\right\} + \frac{2it}{\lambda^2}\sin\frac{\lambda L}{2}$ after consolidation of terms we get

$$\sigma_{yy}\Big|_{y=0} = -p(x) = \int_{-\infty}^{\infty} e^{-i\lambda x}\left\{\frac{L}{2\pi\lambda}\left[(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2}\right] - i\frac{(b-a)}{\pi L \lambda^2}\sin\frac{\lambda L}{2}\right\} d\lambda = \int_{-\infty}^{\infty} -\lambda^2 A e^{-i\lambda x} d\lambda$$

$$\therefore \quad A = -\frac{1}{\lambda^2}\left\{\frac{L}{2\pi\lambda}\left[(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2}\right] - i\frac{(b-a)}{\pi L \lambda^2}\sin\frac{\lambda L}{2}\right\} \quad \text{and} \quad B = A|\lambda|$$

To recap
$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = \int_{-\infty}^{\infty} -A\lambda^2 e^{-i\lambda x} e^{-|\lambda| y}\{1+|\lambda|y\}d\lambda$$
$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = \int_{-\infty}^{\infty} -A\lambda^2 e^{-i\lambda x} e^{-|\lambda| y}\{1-|\lambda|y\}d\lambda \qquad (1-3)$$
$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = \int_{-\infty}^{\infty} -A\lambda^2(+i\lambda) e^{-i\lambda x} e^{-|\lambda| y}\, y\, d\lambda$$

Taking (1)
$$\frac{\partial^2\phi}{\partial x^2} = \sigma_{yy} = \int_{-\infty}^{\infty} \frac{i}{2\pi}\left\{(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2} + \frac{2(a-b)}{L\lambda}\sin\frac{\lambda L}{2}\right\}\frac{e^{-i\lambda x}}{\lambda} e^{-|\lambda| y}\{1+|\lambda|y\}d\lambda$$
the only non zero terms after consolidation, and using the even interval results found in class, give
$$= \int_0^{\infty}\frac{1}{\pi}\left\{\frac{b}{\lambda}\sin\lambda(x-L/2) - \frac{a}{\lambda}\sin\lambda(x+L/2) + \frac{2(a-b)}{L\lambda^2}\sin\lambda x\sin\frac{\lambda L}{2}\right\}e^{-\lambda y}\{1+\lambda y\}d\lambda$$

for $y>0$
$$\frac{b}{\pi}\int_0^{\infty}\frac{\sin\lambda(x-L/2)}{\lambda}e^{-\lambda y}(1+\lambda y)\,d\lambda = \frac{b}{\pi}\tan^{-1}\left(\frac{x-L/2}{y}\right) + \frac{by}{\pi}\frac{(x-L/2)}{y^2+(x-L/2)^2} \qquad (4)$$
similarly
$$\frac{a}{\pi}\int_0^{\infty}\frac{\sin\lambda(x+L/2)}{\lambda}e^{-\lambda y}(1+\lambda y)\,d\lambda = \frac{a}{\pi}\tan^{-1}\left(\frac{x+L/2}{y}\right) + \frac{ay}{\pi}\frac{(x+L/2)}{y^2+(x+L/2)^2} \qquad (5)$$

$$\frac{a-b}{\pi L}\int_0^{\infty}\frac{2}{\lambda^2}\sin\lambda x\sin\frac{\lambda L}{2}e^{-\lambda y}(1+\lambda y)\,d\lambda = \frac{a-b}{L\pi}\left\{(x+L/2)\tan^{-1}\left(\frac{x+L/2}{y}\right) - (x-L/2)\tan^{-1}\left(\frac{x-L/2}{y}\right)\right\}$$ after simplification

the explanation of this term will be given in the evaluation of σ_{xx}

$$\therefore\ \boxed{\sigma_{yy} = \tan^{-1}\left(\frac{x-L/2}{y}\right)\left\{\frac{b}{\pi} - \frac{(a-b)}{L\pi}(x-L/2)\right\} + \tan^{-1}\left(\frac{x+L/2}{y}\right)\left\{\frac{(a-b)}{L\pi}(x+L/2) - \frac{a}{\pi}\right\} + \frac{y}{\pi}\left\{\frac{b(x-L/2)}{y^2+(x-L/2)^2} - \frac{a(x+L/2)}{y^2+(x+L/2)^2}\right\}} \qquad (A)$$

Taking (2)
$$\frac{\partial^2\phi}{\partial y^2} = \sigma_{xx} = \int_{-\infty}^{\infty}\frac{i}{2\pi}\left\{(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2} + \frac{2(a-b)}{L\lambda}\sin\frac{\lambda L}{2}\right\}\frac{e^{-i\lambda x}}{\lambda}e^{-|\lambda| y}[1-|\lambda|y]d\lambda$$
which as before gives rise to
$$= \int_0^{\infty}\frac{1}{\pi}\left\{\frac{b}{\lambda}\sin\lambda(x-L/2) - \frac{a}{\lambda}\sin\lambda(x+L/2) + \frac{2(a-b)}{L\lambda^2}\sin\lambda x\sin\frac{\lambda L}{2}\right\}e^{-\lambda y}\{1-\lambda y\}d\lambda$$
The first two integrals are the same as (4,5) except change the sign of the last term.

$$\frac{a-b}{\pi L}\int_0^{\infty}\frac{2}{\lambda^2}\sin\lambda x\sin\frac{\lambda L}{2}e^{-\lambda y}(1-\lambda y)\,d\lambda = \frac{a-b}{\pi L}\int_0^{\infty}\frac{e^{-\lambda y}}{\lambda^2}(1-\lambda y)\left[\cos\lambda(x-L/2) - \cos\lambda(x+L/2)\right]d\lambda$$
$$= \frac{a-b}{\pi L}\left\{\int_0^{\infty}\frac{e^{-\lambda y}}{\lambda^2}\left[\cos\lambda(x-L/2) - \cos\lambda(x+L/2)\right]d\lambda - y\int_0^{\infty}\frac{e^{-\lambda y}}{\lambda}\left[\cos\lambda(x-L/2) - \cos\lambda(x+L/2)\right]d\lambda\right.$$
$$= \frac{a-b}{\pi L}\cdot Q - \frac{y(a-b)}{2\pi L}\ln\frac{y^2+(x+L/2)^2}{y^2+(x-L/2)^2} \qquad (6)$$

$$Q = \int_0^\infty \left[\cos\lambda(x-1/2) - \cos\lambda(x+1/2)\right]\frac{e^{-\lambda y}}{\lambda^2}d\lambda \qquad \frac{\partial Q}{\partial x} = \int_0^\infty \left[-\sin\lambda(x-1/2) + \sin\lambda(x+1/2)\right]\frac{e^{-\lambda y}}{\lambda}d\lambda$$

$$\therefore \frac{\partial Q}{\partial x} = \left[-\tan^{-1}\left(\frac{x-1/2}{y}\right) + \tan^{-1}\left(\frac{x+1/2}{y}\right)\right] \Rightarrow Q = -(x-1/2)\tan^{-1}\left(\frac{x-1/2}{y}\right) + (x+1/2)\tan^{-1}\left(\frac{x+1/2}{y}\right)$$

$+ \frac{y}{2}\ln\left[y^2 + (x-1/2)^2\right] - \frac{y}{2}\ln\left[y^2+(x+1/2)^2\right] + g(y)$. Since $Q \to 0$ as $y \to +\infty$ then

$$Q\Big|_{y\to\infty} = \frac{y}{2}\ln y^2 - \frac{y}{2}\ln y^2 + g(y) = 0 \quad \Rightarrow g(y) = 0$$

$$\therefore \quad Q = \frac{a-b}{L\pi}\left\{(x+1/2)\tan^{-1}\left(\frac{x+1/2}{y}\right) + \frac{y}{2}\ln\frac{y^2+(x-1/2)^2}{y^2+(x+1/2)^2} - (x-1/2)\tan^{-1}\left(\frac{x-1/2}{y}\right)\right\} \qquad (7)$$

in the solution for σ_{yy} the natural log terms canceled but here they are additive, thus

$$\sigma_{xx} = \tan^{-1}\left(\frac{x-1/2}{y}\right)\left\{\frac{b}{\pi} - \frac{(a-b)}{L\pi}(x-1/2)\right\} + \tan^{-1}\left(\frac{x+1/2}{y}\right)\left\{\frac{a-b}{L\pi}(x+1/2) - \frac{a}{\pi}\right\} + \frac{y}{\pi}\left\{\frac{a(x+1/2)}{y^2+(x+1/2)^2} - \frac{b(x-1/2)}{y^2+(x-1/2)^2}\right\}$$

$$-\frac{(a-b)}{\pi L}y\ln\frac{y^2+(x+1/2)^2}{y^2+(x-1/2)^2} \qquad (B)$$

Taking (3)

$$\sigma_{xy} = iy\int_{-\infty}^{\infty}\lambda e^{-i\lambda x}e^{-|\lambda|y}\frac{i}{2\pi\lambda}\left\{(b-a)\cos\frac{\lambda L}{2} + i(b+a)\sin\frac{\lambda L}{2} + \frac{2(a-b)}{L\lambda}\sin\frac{\lambda L}{2}\right\}d\lambda$$

The only non zero terms will give rise to

$$= \frac{-y}{\pi}\int_0^\infty\left\{(b-a)\cos\frac{\lambda L}{2}\cos\lambda x + (b+a)\sin\frac{\lambda L}{2}\sin\lambda x + \frac{2(a-b)}{L\lambda}\sin\frac{\lambda L}{2}\cos\lambda x\right\}e^{-\lambda y}d\lambda$$

$$= \frac{-y}{\pi}\int_0^\infty\left\{b\cos\lambda(x-1/2) - a\cos\lambda(x+1/2) + \frac{a-b}{L\lambda}\left[\sin\lambda(x+1/2) - \sin\lambda(x-1/2)\right]\right\}e^{-\lambda y}d\lambda$$

$$\sigma_{xy} = -\frac{yb}{\pi}\left(\frac{y}{y^2+(x-1/2)^2}\right) + \frac{ay}{\pi}\left(\frac{y}{y^2+(x+1/2)^2}\right) - \frac{(a-b)}{\pi L}y\left\{\tan^{-1}\left(\frac{x+1/2}{y}\right) - \tan^{-1}\left(\frac{x-1/2}{y}\right)\right\} \qquad (C)$$

iii $y = 100$ ft., $P = 120{,}000$ #/ft $\quad L = 10'$, $e = 1'$ $\Rightarrow b = 4.8\times10^3$ #/ft^2, $a = 1.92\times10^4$ #/ft^2

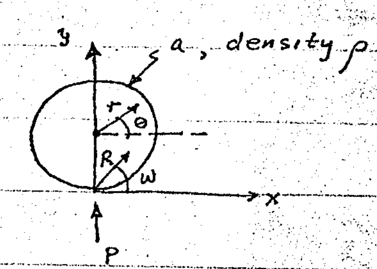
Using these into the formula for σ_{yy} (eq A) and noting that $T_y = -\sigma_{yy}$

I programmed this for values of $|x| \le 20$; the maximum attained was 762.8 psf @ $x = -.1$ ft this amounts to a $\sigma_{yy} = 5.3$ psi hence no appreciable settling. No problems!!

WHAT A LO-O-O-O-O-NG EXAM

ME 238B MID-TERM

ANSWER SHEET



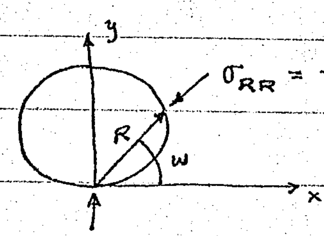
1. Body force/unit volume $= \rho g$ in negative y-direction.

Boundary conditions:

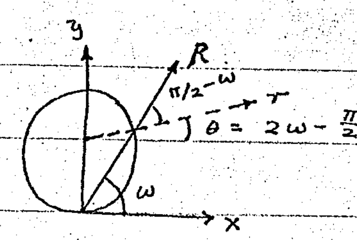
For $(x,y) \neq (0,0)$: $\sigma_{rr} = \sigma_{r\theta} = 0$ on $r = a$.

Split problem into 2 parts:

I.



$$\sigma_{RR} = -\frac{P}{\pi a} = -\rho g a$$

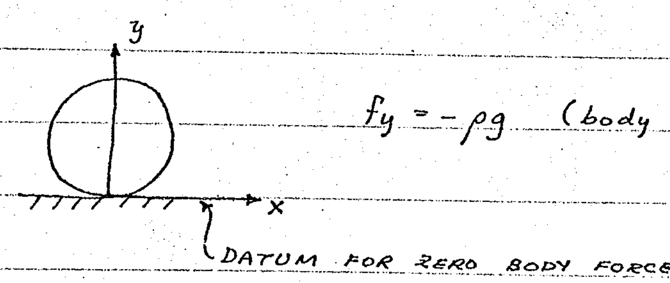


$P = \rho g \pi a^2$ (Force/unit length)

From class the solution is $\sigma_{RR} = -\frac{2P}{\pi R}\sin\omega$ everywhere. On the boundary, the only non-vanishing stress is

$$\sigma_{RR} = -\rho g a.$$

II. Pure body force problem



$f_y = -\rho g$ (body force/unit volume)

Solution is

$\sigma_{yy} = +\rho g y$ everywhere, all other in-plane stresses vanish

( ) On the boundary $y = R\sin\omega$ & $R = 2a\sin\omega \Rightarrow y = 2a\sin^2\omega$

$\therefore\ \sigma_{yy} = 2\rho g a \sin^2\omega$ on $r = a$.

Now compute the boundary tractions obtained from adding solutions (I) and (II). First find stress state on boundary in x, y, z

$$\sigma_{xx} = \sigma_{RR}\cos^2\omega = -\rho g a\cos^2\omega$$

$$\sigma_{yy} = \sigma_{RR}\sin^2\omega + 2\rho g a\sin^2\omega = -\rho g a\sin^2\omega + 2\rho g a\sin^2\omega = \rho g a\sin^2\omega$$

$$\sigma_{xy} = \sigma_{RR}\cos\omega\sin\omega = -\tfrac{1}{2}\rho g a\sin 2\omega$$

But on the boundary $r = a$, $\theta = 2\omega - \frac{\pi}{2}$

or $2\omega = \theta + \frac{\pi}{2}$

Hence on the boundary

$$\sigma_{xx} = -\rho g a\tfrac{1}{2}\{1 + \cos 2\omega\} = -\frac{\rho g a}{2}\{1 + \cos(\theta + \tfrac{\pi}{2})\} = \frac{\rho g a}{2}\{-1 + \sin\theta\}$$

( ) $$\sigma_{yy} = \rho g a\tfrac{1}{2}\{1 - \cos 2\omega\} = \frac{\rho g a}{2}\{1 + \sin\theta\}$$

$$\sigma_{xy} = -\tfrac{1}{2}\rho g a\sin(\theta + \tfrac{\pi}{2}) = -\frac{\rho g a}{2}\cos\theta$$

We must add on a stress state which annuls the tractions due to this stress state on $r = a$.

METHOD 1

Since $x = a\cos\theta$; $y = a + a\sin\theta$ on the boundary we can rewrite the stress state on the boundary due to (I) and (II) as

$$\left.\begin{aligned} \sigma_{xx} &= -\rho g a + \tfrac{1}{2}\rho g y \\ \sigma_{yy} &= \tfrac{1}{2}\rho g y \\ \sigma_{xy} &= -\tfrac{1}{2}\rho g x \end{aligned}\right\} \text{ on the boundary only}$$

( )

Consider the stress state given by

$$\left.\begin{aligned} \sigma_{xx} &= \rho g a - \tfrac{1}{2}\rho g y \ ; \quad \sigma_{xy} = \tfrac{1}{2}\rho g x \\ \sigma_{yy} &= -\tfrac{1}{2}\rho g y \end{aligned}\right\} \text{ everywhere in } r \le a$$

This stress state satisfies the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0,$$

and yields a stress state on the boundary given by

$$\left.\begin{aligned} \sigma_{xx} &= \rho g a - \tfrac{1}{2}\rho g y \\ \sigma_{yy} &= -\tfrac{1}{2}\rho g y \\ \sigma_{xy} &= \tfrac{1}{2}\rho g x \end{aligned}\right\}$$

This annuls all the boundary stresses (and hence all the boundary tractions) due to solutions (1) + (2)

The strains due to this solution are linear in x & y and thus automatically satisfy compatibility.

Hence the total solution is the sum of the following two stress states

$$\sigma_{RR} = -\frac{2\rho g \pi a^2}{\pi R} \sin\omega$$

&

$$\sigma_{xx} = \rho g (a - \tfrac{1}{2} y)$$

$$\sigma_{yy} = \rho g y - \tfrac{1}{2}\rho g y = \tfrac{1}{2}\rho g y$$

$$\sigma_{xy} = +\tfrac{1}{2}\rho g x$$

Which can be shown to match the solution on p. 219 of A.E.H. Love

METHOD 2

On the boundary of the disc the state of stress due to (I) and (II) is

$$\sigma_{xx} = \frac{\rho g a}{2}\{-1+\sin\theta\}$$

$$\sigma_{yy} = \frac{\rho g a}{2}\{1+\sin\theta\}$$

$$\sigma_{xy} = -\frac{\rho g a}{2}\cos\theta$$

The tractions on $r=a$ due to this stress state are

$$\sigma_{rr} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta$$

$$\sigma_{r\theta} = (\sigma_{yy}-\sigma_{xx})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta-\sin^2\theta)$$

or on $r=a$

$$\sigma_{rr} = \frac{\rho g a}{2}\{\langle -1+\sin\theta\rangle\cos^2\theta + \langle 1+\sin\theta\rangle\sin^2\theta - 2\cos^2\theta\sin\theta\}$$

$$\sigma_{r\theta} = \frac{\rho g a}{2}\{2\sin\theta\cos\theta - \cos\theta(\cos^2\theta-\sin^2\theta)\}$$

which is equivalent to

$$\sigma_{rr} = \frac{\rho g a}{2}\{-\cos 2\theta + \sin\theta\,(1-2\cos^2\theta)\}$$

$$= \frac{\rho g a}{2}\{-\cos 2\theta - \sin\theta\cos 2\theta\}$$

$$= -\frac{\rho g a}{2}\{\cos 2\theta + \tfrac{1}{2}[\sin(\theta+2\theta)+\sin(\theta-2\theta)]\}$$

or

$$\boxed{\sigma_{rr} = -\frac{\rho g a}{2}\{\cos 2\theta + \tfrac{1}{2}\sin 3\theta - \tfrac{1}{2}\sin\theta\}}$$

and

$$\sigma_{r\theta} = \frac{\rho g a}{2}\{\sin 2\theta - \cos\theta\cos 2\theta\}$$

$$= \frac{\rho g a}{2}\{\sin 2\theta - \tfrac{1}{2}[\cos(\theta+2\theta)+\cos(\theta-2\theta)]\}$$

or

$$\boxed{\sigma_{r\theta} = \frac{\rho g a}{2}\{\sin 2\theta - \tfrac{1}{2}\cos 3\theta - \tfrac{1}{2}\cos\theta\}}$$

Hence we must add to (I) and (II) a stress state which gives the negative of the above loading on $r=a$ with no body forces.

This is an example of Fourier loading on a disc. Since we want $\sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta}$ finite at $r = 0$ an inspection of the earlier class handout shows we want a stress function

$$\phi = d_1 r^3 \sin\theta + (a_2 r^2 + b_2 r^4)\cos 2\theta + (c_3 r^3 + d_3 r^5)\sin 3\theta .$$

We want this stress fn. to produce tractions on $r = a$ given by

$$\sigma_{rr} = \frac{\rho g a}{2}\left\{\cos 2\theta + \tfrac{1}{2}\sin 3\theta - \tfrac{1}{2}\sin\theta\right\}$$

$$\sigma_{r\theta} = -\frac{\rho g a}{2}\left\{\sin 2\theta - \tfrac{1}{2}\cos 3\theta - \tfrac{1}{2}\cos\theta\right\}$$

This requires: (from σ_{rr} B.C.)

(i) $2 d_1 a = -\frac{\rho g a}{4}$ $\Rightarrow$

(ii) $a_2 (2)(-1) + b_2 (2+2-4) a^2 = \frac{\rho g a}{2}$ $\Rightarrow$

$$\boxed{d_1 = -\frac{\rho g}{8}} \quad \boxed{a_2 = -\frac{\rho g a}{4}}$$

(iii) $c_3 (3)(-2) a + d_3 (2+3-9) a^3 = \frac{\rho g a}{4}$ $\Rightarrow$

$$\boxed{-6 c_3 a - 4 d_3 a^3 = \frac{\rho g a}{4}}$$

and (from $\sigma_{r\theta}$ B.C.)

(iv) $-2 d_1 a = \frac{\rho g a}{4}$ which is consistent

(v) $2 a_2 (1) + 2 b_2 (3) a^2 = -\frac{\rho g a}{2}$

(vi) $-3 c_3 (2) a - 3 d_3 (4) a^3 = \frac{\rho g a}{4}$

So $\boxed{d_1 = -\frac{\rho g}{8}}$, $\boxed{a_2 = -\frac{\rho g a}{4}}$

$$3 b_2 a^2 = -\frac{\rho g a}{4} - a_2 = -\frac{\rho g a}{4} + \frac{\rho g a}{4} = 0 \Rightarrow \boxed{b_2 = 0}$$

&

$$\left.\begin{aligned} -6 c_3 - 4 d_3 a^2 &= \frac{\rho g}{4} \\ -6 c_3 - 12 d_3 a^2 &= \frac{\rho g}{4} \end{aligned}\right\} \Rightarrow \boxed{d_3 = 0} \quad \boxed{c_3 = -\frac{\rho g}{24}}$$

Hence, the requisite stress function is

$$\hat{\phi} = -\frac{\rho g}{8} r^3 \sin\theta - \frac{\rho g a}{4} r^2 \cos 2\theta - \frac{\rho g}{24} r^3 \sin 3\theta$$

& the resulting stresses are

$$\hat{\sigma}_{rr} = -\frac{\rho g}{4} r \sin\theta + \frac{\rho g a}{2} \cos 2\theta + \frac{\rho g}{4} r \sin 3\theta$$

$$\hat{\sigma}_{r\theta} = \frac{\rho g}{4} r \cos\theta - \frac{\rho g a}{2} \sin 2\theta + \frac{\rho g}{4} r \cos 3\theta$$

$$\hat{\sigma}_{\theta\theta} = -\frac{3\rho g}{4} r \sin\theta - \frac{\rho g a}{2} \cos 2\theta - \frac{\rho g}{4} r \sin 3\theta$$

Express these stresses in x, y, z:

$$\hat{\sigma}_{xx} = \hat{\sigma}_{rr} \cos^2\theta + \hat{\sigma}_{\theta\theta} \sin^2\theta - 2\hat{\sigma}_{r\theta} \sin\theta\cos\theta$$

$$= \frac{\hat{\sigma}_{rr} + \hat{\sigma}_{\theta\theta}}{2} + \frac{\hat{\sigma}_{rr} - \hat{\sigma}_{\theta\theta}}{2} \cos 2\theta - \hat{\sigma}_{r\theta} \sin 2\theta$$

$$= -\frac{\rho g}{2} r \sin\theta + \cos 2\theta \left\{ \frac{\rho g a}{2} \cos 2\theta + \frac{\rho g}{4} r \sin 3\theta + \frac{\rho g}{4} r \sin\theta \right\}$$

$$- \sin 2\theta \left\{ \frac{\rho g}{4} r \cos\theta - \frac{\rho g a}{2} \sin 2\theta + \frac{\rho g r}{4} \cos 3\theta \right\}$$

$$= -\frac{\rho g}{2} r \sin\theta + \frac{\rho g a}{2} + \frac{\rho g r}{4} \underbrace{\left\{ \sin 3\theta \cos 2\theta - \cos 3\theta \sin 2\theta \right\}}_{\sin\theta}$$

$$+ \frac{\rho g r}{4} \underbrace{\left\{ \cos 2\theta \sin\theta - \sin 2\theta \cos\theta \right\}}_{-\sin\theta}$$

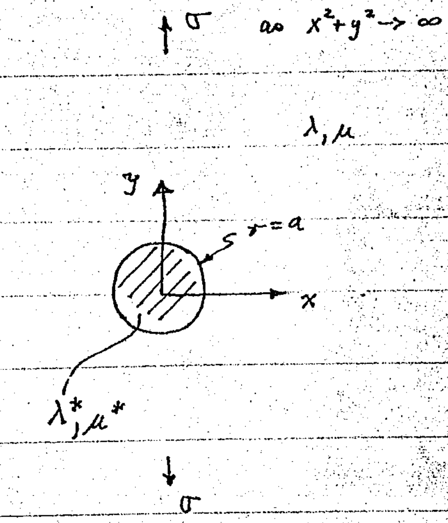
$$= \frac{\rho g a}{2} - \frac{\rho g r}{2} \sin\theta = \rho g a - \frac{\rho g}{2}(a + r\sin\theta)$$

Since $a + r\sin\theta = y$

$\hat{\sigma}_{xx} = \rho g a - \frac{1}{2}\rho g y$ which checks result obtained by method 1.

Similarly we can show $\hat{\sigma}_{yy}$, $\hat{\sigma}_{xy}$ match those obtained using method 1.

Q.E.D.



3. In plate: elastic fields are $u_i^P,\ e_{ij}^P,\ \sigma_{ij}^P$

In inclusion: " " " $u_i^I,\ e_{ij}^I,\ \sigma_{ij}^I$

Boundary conditions:

On $r=a$:

$$\sigma_{rr}^P = \sigma_{rr}^I$$

$$\sigma_{r\theta}^P = \sigma_{r\theta}^I$$

$$\left.\begin{aligned} u_r^P &= u_r^I \\ u_\theta^P &= u_\theta^I \end{aligned}\right\} \text{ or } \quad \begin{aligned} u_x^P &= u_x^I \\ u_y^P &= u_y^I \end{aligned}$$

At infinity: $\sigma_{ij}^P \to \sigma$ if $i,j=2$

$\to 0$ other

As $r \to 0$: σ_{ij}^I finite.

Want displacements single-valued and continuous everywhere

(i) Stress function for plate:

$$\phi^P = \tfrac{1}{2}\sigma x^2 + \hat{\phi}(r,\theta)$$

where (related for single-valued displacements)

$$\hat{\phi} = a_0 \ln r + a_1 r\theta \sin\theta + (a_1' r^{-1} + b_1' r \ln r)\cos\theta$$

$$+ c_1 r\theta\cos\theta + (c_1' r^{-1} + d_1' r\ln r)\sin\theta$$

$$+ \sum_{n=2}^{\infty} \left\{ a_n' r^{-n} + b_n' r^{-n+2} \right\} \cos n\theta$$

$$+ \sum_{n=2}^{\infty} \left\{ c_n' r^{-n} + d_n' r^{-n+2} \right\} \sin n\theta$$

(ii) Stress function for the inclusion

$$\phi^{I} = b_0 r^2 + b_1 r^3 \cos\theta + d_1 r^3 \sin\theta$$

$$+ \sum_{n=2}^{\infty} \left\{ (a_n r^n + b_n r^{n+2}) \cos n\theta + (c_n r^n + d_n r^{n+2}) \sin n\theta \right\}$$

By symmetry $\sigma_{rr}, \sigma_{\theta\theta}$ are even in θ and $\sigma_{r\theta}$ is odd in θ so that we expect only b_0, a_n, b_n are non-zero in ϕ^I with similar observations pertaining to ϕ^P. From the fact that $\sigma_{r\theta}$ must vanish on $\theta = \pm\frac{\pi}{2}$, the above symmetry reductions mean that $\sin\frac{n\pi}{2}$ must vanish which implies on even n remain in ϕ^I and ϕ^P.

Stress concentration effects:

For $\lambda^* = \mu^* = 0$, inclusion is a hole and S.C.F. = 3

" $\lambda^* = \lambda, \mu^* = 0$, inclusion & plate are identical & S.C.F. = 1

By continuity for $\lambda^* < \lambda, \mu^* < \mu$ expect S.C.F. < 1 and certainly less than 3. As long as $\lambda^* > 0$ and $\mu^* > 0$ the inclusion is capable of carrying some of the load and hence the stress in the plate should decrease from what it would be with $\lambda^* = \mu^* = 0$.

Problem 5 - Partial Answer Sheet

a) Force & Moment Equilibrium - everyone knew how to do that

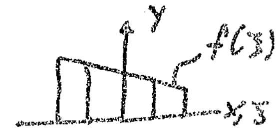
b) First determine boundary conditions:

1) $\sigma_{yx}(x,0)=0$

2) $T_y(x,0) = -p = \sigma_{yy}(x,0)\cdot 1 \rightarrow \sigma_{yy}(x,0) = -p = f(\xi)$

Determine Stresses in half-space:

3 ways to do this -



1) Easy way - we know the solution for a point load on a half space (located at 0,0):

$$\sigma_{yy} = \frac{-2}{\pi}\frac{F\,y^3}{(x^2+y^2)^2}, \quad \sigma_{xx} = \frac{-2}{\pi}\frac{F\,x^2 y}{(x^2+y^2)^2}, \quad \sigma_{xy} = \frac{-2}{\pi}\frac{F\,x\,y^2}{(x^2+y^2)^2}$$

for an arbitrary point ξ, replace x by $x-\xi$ (i.e. shift origin). The magnitude of F on an element of boundary of length $d\xi$ located at $x=\xi$ is $F = f(\xi)\,d\xi$ (1)

$$\text{where } f(\xi) = \begin{cases} \frac{b-a}{L}\xi + \frac{a+b}{2} & |\xi| \le L \\ 0 & |\xi| > L \end{cases}$$

Integrating over all possible values of ξ, and noting there will only be a contribution from $|\xi| \le L/2$, leads to

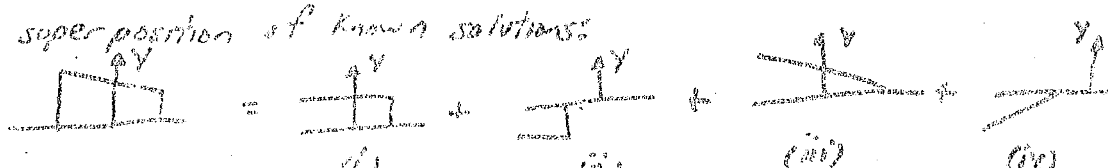
$$\sigma_{yy} = -\frac{2}{\pi}\int_{-L/2}^{L/2}\frac{f(\xi)\,y^3}{((x-\xi)^2+y^2)^2}\,d\xi; \quad \sigma_{xx} = -\frac{2}{\pi}\int_{-L/2}^{L/2}\frac{f(\xi)(x-\xi)^2 y\,d\xi}{((x-\xi)^2+y^2)^2}; \quad \sigma_{xy} = -\frac{2}{\pi}\int_{-L/2}^{L/2}\frac{f(\xi)(x-\xi)y^2\,d\xi}{((x-\xi)^2+y^2)^2}$$

The hard part is to carry out the integrations, but there are only 4 types of integrals present, all of which are tabulated in standard tables (That is, after you make the substitution $u = \xi - x$, $du = d\xi$)

other approaches

2) Directly determine solution for given loading by Fourier Transform

3) Use superposition of known solutions:

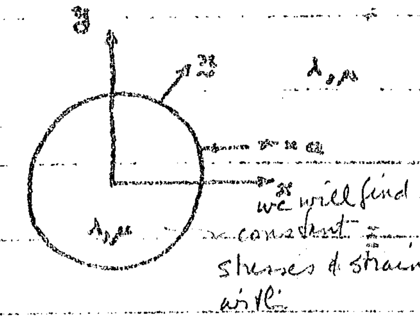


i, ii, iii, iv are tabulated in Timoshenko & Goodier

Having graded all 3 approaches, I can state unequivocally that (1) is the easiest

CIRCULAR INCLUSION PROBLEMS SOLVED

USING ELASTIC GREEN'S FUNCTIONS



Consider an infinite linear elastic solid (elastic constants λ, μ). Mark out the circle $r = a$. Apply line forces of magnitude $f_j = \sigma^*_{jk} n_k$ on $r = a$ where σ^*_{jk} is a constant state of stress. ($j,k = 1,2$). $\sigma^*_{jk} = \sigma^*_{kj}$.

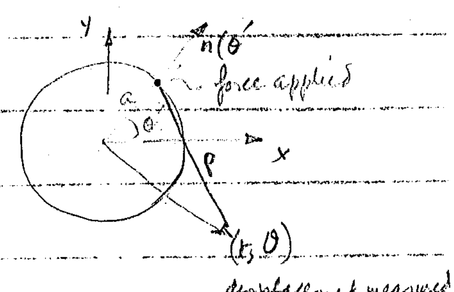
The displacement field due to this state of stress is:

displ due to one pt force = displ in the i^{th} dir at x due to line force at x' in j^{th} dir

$$u_i(\underline{x}) = \int_{r=a} ds' \, g_{ij}(\underline{x}, \underline{x}') \, f_j(\underline{x}')$$

since σ^*_{jk} is a constant matrix

$$= \sigma^*_{jk} \, a \int_0^{2\pi} d\theta' \, n_k(\theta') \, g_{ij}(r, \theta; a, \theta')$$



$n_1(\theta') = \cos\theta'$

$n_2(\theta') = \sin\theta'$

$$u_i(r, \theta) = a \sigma^*_{jk} \int_0^{2\pi} d\theta' \begin{Bmatrix} \cos\theta' \\ \sin\theta' \end{Bmatrix} \Bigg\{ -\frac{(3-4\nu)}{8\pi\mu(1-\nu)} \delta_{ij} \ln \sqrt{\underbrace{(r\cos\theta - a\cos\theta')^2}_{x - x'} + \underbrace{(r\sin\theta - a\sin\theta')^2}_{y - y'}}$$

$$+ \frac{1}{(1-\nu)} \frac{1}{8\pi\mu} \frac{(x_i - x_i')(x_j - x_j')}{r^2 + a^2 - 2ar\cos(\theta - \theta')} \Bigg\}$$

$x_1 = r\cos\theta \;;\; x_1' = a\cos\theta'$

$x_2 = r\sin\theta \;;\; x_2' = a\sin\theta'$

Now it can be shown that for $r < a$

$$\ln\sqrt{r^2+a^2-2ar\cos(\theta-\theta')} = \ln a - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r}{a}\right)^m \cos m(\theta-\theta')$$

while for $r > a$

$$= \ln r - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{a}{r}\right)^m \cos m(\theta-\theta')$$

since at $r=a$ we may get $\ln\sqrt{} \to \infty$. Compute u_i^{inside} $u_i^{outside}$ & let $r \to a$

(1) Compute one of the integrals eg $n_1(\theta') = \cos\theta'$ see if $u_i^{inside} = u_i^{outside}$. it is!!

$$I_1 = \int_0^{2\pi} d\theta' \cos\theta' \ln\sqrt{r^2+a^2-2ar\cos(\theta-\theta')} \quad ; \quad r < a$$

only contrib @ m=1   since $\int_0^{2\pi} \cos\theta' \sin m\theta' d\theta' = 0$

$$I_1 = \int_0^{2\pi} d\theta' \cos\theta' \left\{ \ln a - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{r}{a}\right)^m [\cos m\theta \cos m\theta' + \sin m\theta \sin m\theta'] \right.$$

$$= -\frac{\pi r}{a}\cos\theta = -\frac{\pi}{a}x_1 \qquad \text{for } r < a$$

for $n_k(\theta') = \sin\theta'$ $\quad I_1 = -\frac{\pi r}{a}\sin\theta = -\frac{\pi}{a}x_2$

(2) Similarly

$$I_2 = \int_0^{2\pi} d\theta' \sin\theta' \ln\sqrt{r^2+a^2-2ar\cos(\theta-\theta')}$$

$$= -\frac{\pi}{a} r\sin\theta = -\frac{\pi}{a}x_2 \qquad \text{for } r < a.$$

Thus, for $r < a$ (inside the circle)

$$u_i(r<a,\theta) = \sigma_{jk}^{\infty}\left\{ \frac{3-4\nu}{8\mu(1-\nu)}\delta_{ij}x_k + \frac{a}{8\pi\mu(1-\nu)}\int_0^{2\pi} d\theta' \left\{\begin{matrix}\cos\theta' \\ \sin\theta'\end{matrix}\right\}^{n_k(\theta')} \cdot \frac{(x_i-x_i')(x_j-x_j')}{r^2+a^2-2ar\cos(\theta-\theta')} \right\}$$

we will charc this later ∋ it solves a problem we are interested in

To evaluate the last integral look at

$$(r^2 \ln r)_{,ij} = (r^2)_{,ij} \ln r + r^2 (\ln r)_{,ij} + r^2_{,i} \ln r_{,j} + r^2_{,j} \ln r_{,i}$$

$$= 2\delta_{ij} \ln r + r^2 \left(\frac{x_i}{r^2}\right)_{,j} + 2x_i \frac{x_j}{r^2} + 2x_j \frac{x_i}{r^2}$$

$$= 2\delta_{ij} \ln r + r^2 \left\{\frac{\delta_{ij}}{r^2} - \frac{2x_i x_j}{r^4}\right\} + 4\frac{x_i x_j}{r^2}$$

$$= 2\delta_{ij} \ln r + \delta_{ij} + \frac{2x_i x_j}{r^2}$$

So with shifted origin

($r^2+a^2-2ar\cos(\theta-\theta')$; have series expansion; series expansion)

$$\frac{(x_i - x_i')(x_j - x_j')}{|\underline{x}-\underline{x}'|^2} = \frac{1}{2}\left\{\frac{\partial^2}{\partial x_i \partial x_j}\left[|\underline{x}-\underline{x}'|^2 \ln|\underline{x}-\underline{x}'|\right] - 2\delta_{ij}\ln|\underline{x}-\underline{x}'| - \delta_{ij}\right.$$

(i) $$\int_0^{2\pi} d\theta' \begin{Bmatrix}\cos\theta' \\ \sin\theta'\end{Bmatrix} \delta_{ij} = 0$$

(ii) $$-\delta_{ij}\int_0^{2\pi} \ln|\underline{x}-\underline{x}'|\, d\theta' \begin{Bmatrix}\cos\theta' \\ \sin\theta'\end{Bmatrix} = +\frac{\pi}{a}\delta_{ij} x_\alpha$$

Hence

(first log term.)

$$u_i(r<a,\theta) = \sigma^\infty_{j\alpha}\left\{\frac{3-4\nu}{8\mu(1-\nu)}\delta_{ij}x_\alpha + \frac{1}{8\mu(1-\nu)}\delta_{ij}x_\alpha\right.$$

$$+ \frac{a}{8\pi\mu(1-\nu)}\cdot\frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\int_0^{2\pi} d\theta' \begin{Bmatrix}\cos\theta' \\ \sin\theta'\end{Bmatrix} \langle\ \rangle$$

since x_i, x_j are fns of r, θ not θ'

$$\langle r^2 + a^2 - 2ar\cos(\theta-\theta')\rangle\left\langle \ln a - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r}{a}\right)^m \cos m(\theta-\theta')\right\rangle$$

Now $$\langle\ \rangle\langle\ \rangle = (r^2+a^2)\ln a - (r^2+a^2)\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r}{a}\right)^m \cos m(\theta-\theta')$$

$$-2ar\ln a\cos(\theta-\theta') + 2ar\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r}{a}\right)^m\cdot$$

$$\{\cos(m+1)(\theta-\theta') + \cos(m-1)(\theta-\theta')$$

And $\int_0^{2\pi} \begin{Bmatrix} \cos\theta' \\ \sin\theta' \end{Bmatrix} d\theta' = 0$

$$\int_0^{2\pi} d\theta' \begin{Bmatrix} \cos\theta' \\ \sin\theta' \end{Bmatrix} \cos m(\theta-\theta') = \pi \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \quad \text{if } m = 1$$

$$= 0 \quad \text{if } m \neq 1$$

$$\int_0^{2\pi} d\theta' \begin{Bmatrix} \cos\theta' \\ \sin\theta' \end{Bmatrix} \cos(\theta-\theta') = \pi \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix}$$

$$\int_0^{2\pi} d\theta' \begin{Bmatrix} \cos\theta' \\ \sin\theta' \end{Bmatrix} \cos(m+1)(\theta-\theta') = 0 \quad \text{for } m \geq 1$$

$$\int_0^{2\pi} d\theta' \begin{Bmatrix} \cos\theta' \\ \sin\theta' \end{Bmatrix} \cos(m-1)(\theta-\theta') = \pi \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \quad \text{if } m = 2$$

$$= 0 \quad \text{if } m \neq 2$$

Hence

$$u_i(r<a,\theta) = \sigma_{jk}^* \left\{ \frac{3-4\nu+1}{8\mu(1-\nu)} \delta_{ij} x_k \right.$$

ends up making no contribution

$$+ \frac{a}{16\pi\mu(1-\nu)} \cdot \pi \frac{\partial^2}{\partial x_i \partial x_j} \left[-(r^2+a^2)\frac{x_k}{a} - (2a \ln a) x_k \right.$$

$$\left. + 2a \cdot \frac{1}{4}\left(\frac{r^2}{a^2}\right) x_k \right]$$

Since $\frac{\partial^2}{\partial x_i \partial x_j} x_k = 0$

$$u_i(r<a,\theta) = \sigma_{jk}^* \left\{ \frac{1}{2\mu} \delta_{ij} x_k - \frac{1}{32\mu(1-\nu)} \frac{\partial^2}{\partial x_i \partial x_j} (r^2 x_k) \right\}$$

Now $(r^2 x_k)_{,ij} = r^2_{,ij} x_k + r^2 (x_k)_{,ij} + r^2_{,i} x_{k,j} + r^2_{,j} x_{k,i}$

$$= 2\delta_{ij} x_k + 0 + 2x_i \delta_{kj} + 2x_j \delta_{ki}$$

$$= 2\{\delta_{ij} x_k + \delta_{kj} x_i + \delta_{ki} x_j\}$$

Hence

$$u_i(r<a,\theta) = \sigma^o_{jk}\left\{ \frac{1}{2\mu}\delta_{ij}x_k - \frac{1}{16\mu(1-\nu)}\langle \delta_{ij}x_k + \delta_{kj}x_i + \delta_{ki}x_j \rangle \right\}$$

since $\sigma^o_{jk} = \sigma^o_{kj}$, these 2 terms yield same contribution.

$$= \sigma^o_{jk}\left\{ \frac{1}{2\mu}\delta_{ij}x_k - \frac{1}{16\mu(1-\nu)}\langle 2\delta_{ij}x_k + \delta_{kj}x_i \rangle \right\}$$

or

$$u_i(r<a,\theta) = \frac{\sigma^o_{jk}}{2\mu}\left\{ \delta_{ij}x_k\left(1 - \frac{1}{4(1-\nu)}\right) - \frac{1}{8(1-\nu)}\delta_{kj}x_i \right\}$$

$$= \frac{\sigma^o_{jk}}{2\mu}\left\{ \frac{3-4\nu}{4(1-\nu)}\delta_{ij}x_k - \frac{1}{8(1-\nu)}\delta_{kj}x_i \right\}$$

$$\boxed{u_i(r<a,\theta) = \frac{\sigma^o_{jk}}{8\mu(1-\nu)}\left\{ (3-4\nu)\delta_{ij}x_k - \frac{1}{2}\delta_{kj}x_i \right\}}$$

Now compute the strain for $r<a$. (Note: $x_{k,p} = \delta_{kp}$ etc.)

$$u_{i,p}(r<a,\theta) = \frac{\sigma^o_{jk}}{8\mu(1-\nu)}\left\{ (3-4\nu)\delta_{ij}\delta_{kp} - \frac{1}{2}\delta_{kj}\delta_{ip} \right\}$$

$$e_{ip} = \frac{1}{2}(u_{i,p} + u_{p,i})$$

$$\boxed{e_{ip}(r<a,\theta) = \frac{\sigma^o_{jk}}{16\mu(1-\nu)}\left\{ (3-4\nu)(\delta_{ij}\delta_{kp} + \delta_{pj}\delta_{ki}) - \delta_{kj}\delta_{ip} \right\}}$$

$\therefore$ The strain, and hence the stress, inside $r=a$ is constant !!!

Now compute the stress inside r = a:

$$\sigma_{\alpha p} = C_{\alpha p i j} e_{ij}$$

$$= \left(\lambda \delta_{\alpha p} \delta_{ij} + \mu (\delta_{\alpha i} \delta_{pj} + \delta_{\alpha j} \delta_{pi})\right) \frac{\sigma^0_{jn}}{16 \mu (1-\nu)} \left\{ (3-4\nu)(\delta_{ij} \delta_{np} + \delta_{pj} \delta_{in}) - \delta_{nj} \delta_{ip} \right\}$$

$$= \frac{\sigma^0_{jn}}{16 \mu (1-\nu)} \Big\{ (3-4\nu)(\lambda \delta_{\alpha p} \delta_{jn} + \lambda \delta_{\alpha p} \delta_{jn} + \mu \delta_{\alpha n} \delta_{jp} + \mu \delta_{\alpha j} \delta_{pn} + \mu \delta_{\alpha n} \delta_{pj} + \mu \delta_{\alpha j} \delta_{pn}) - 2\lambda \delta_{\alpha p} \delta_{jn} - \mu \delta_{\alpha p} \delta_{nj} - \mu \delta_{\alpha p} \delta_{nj} \Big\}$$

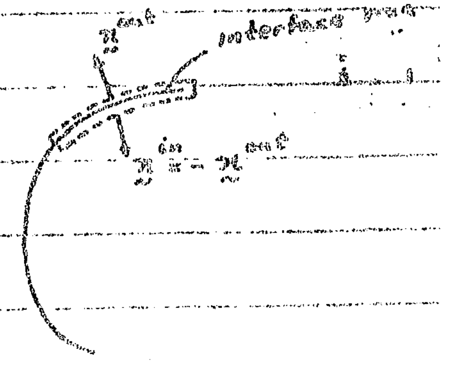
$$= \frac{\sigma^0_{jn}}{16 \mu (1-\nu)} \Big\{ \delta_{\alpha p} \delta_{jn} \left[2\lambda (3-4\nu) - (3\lambda + 2\mu) \right] + (\delta_{\alpha n} \delta_{pj} + \delta_{\alpha j} \delta_{pn}) 2\mu (3-4\nu) \Big\}$$

But

$$2\lambda(3-4\nu) - (3\lambda + 2\mu) = \lambda(6 - 8\nu - 3) - 2\mu$$
$$= \frac{2\mu\nu}{1-2\nu}(3-8\nu) - 2\mu$$
$$= 2\mu \frac{\nu(3-8\nu) - 1 + 2\nu}{1-2\nu}$$
$$= 2\mu \frac{-8\nu^2 + 5\nu - 1}{1-2\nu}$$

$$\Rightarrow \boxed{\sigma_{\alpha p} = \frac{\sigma^0_{jn}}{8(1-\nu)} \left\{ -\delta_{\alpha p} \delta_{jn} \left(\frac{1 - 5\nu + 8\nu^2}{1-2\nu} \right) + (3-4\nu)(\delta_{\alpha n} \delta_{pj} + \delta_{\alpha j} \delta_{pn}) \right\}}$$

Now the solution given by u_i, e_{ip}, and σ_{ip} above is such that u_i is continuous across $r = a$, but $\sigma_{ij} n_j$ is not.



Drawing a pillbox about an element of $r = a$ shows that at a point on the interface

or since $f_i = \sigma_{ij}^* n_j^{out}$

$$(\sigma_{ij}^{out} - \sigma_{ij}^{in}) n_j^{out} + \sigma_{ij}^* n_j^{out} = 0$$

Thus

$$(\sigma_{ij}^{out} - \sigma_{ij}^{in} + \sigma_{ij}^*) n_j^{out} = 0$$

Problem #2

Now suppose we add to the above solution the solution for the infinite homogeneous plate in a state of uniform stress, namely

$\hat{u}_i = E_{ip} x_p$; E_{ip} = constant strain matrix — since you define displ. satisfy compat.

where $E_{ip} = E_{pi}$

$\hat{e}_{ij} = E_{ip}$ = constant everywhere $\quad \hat{e}_{ij} = \frac{1}{2}\{u_{i,j} + u_{j,i}\}$

$\hat{\sigma}_{ij} = C_{ijkl} E_{kl}$ = constant everywhere $\quad = \frac{1}{2}\{E_{ip}\delta_{pj} + E_{jp}\delta_{pi}\}$

$= E_{ip}$ = const $\Rightarrow \hat{\sigma}_{ij}$ = const

The sum of the two solutions above, namely,

$$\left.\begin{aligned} U_i &= u_i + \hat{u}_i \\ \varepsilon_{ij} &= e_{ij} + E_{ij} \\ \Sigma_{ij} &= \sigma_{ij} + \hat{\sigma}_{ij} \end{aligned}\right\}$$

as $r \to \infty$ $\sigma_{ij} \to 0$ $\hat{\sigma}_{ij} \to$ const $\therefore \Sigma_{ij} \to$ const

U_i continuous on $r=a$ since both u_i, $\hat{u}_i$ are cont.

$\Sigma_{ij} n_j$ is discont. on $r=a$ since $\sigma_{ij} n_j$ is discont. $\hat{\sigma}_{ij} n_j$ is continuous

Inside $r=a$ e_{ij}, ε_{ij} are const since both components of the element are const.

is such that

U_i is continuous across $r = a$

$$(\Sigma_{ij}^{out} - \Sigma_{ij}^{in} + \sigma_{ij}^*) n_j^{out} = 0 \quad \text{on } r = a$$

As $x^2 + y^2 \to \infty$, $\Sigma_{ij} \to \hat{\sigma}_{ij}$, a remote uniform stress state.

Now suppose we take a circle of radius "a" (actually an infinitely long cylindrical inclusion) whose linear isotropic

elastic constants are λ^* and μ^*

Give this "inclusion" the displacement field

$U_i = u_i + \hat{u}_i$

so that the strains within it are

$\epsilon_{ij} = e_{ij} + \mathcal{E}_{ij}$.

and the stresses within it are

$$\tilde{\Sigma}_{ij} = C^*_{ijkl}(e_{ij} + \mathcal{E}_{ij}) = \text{constant inside } r = a \;;\; \tilde{\Sigma}_{ij,j} = 0$$

where

$$C^*_{ijkl} = \lambda^* \delta_{ij}\delta_{kl} + \mu^*(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

We can now replace the original inclusion by the * inclusion without disturbing displacement continuity across $r = a$. If we want traction continuity across $r = a$ we must be able to arrange that

$$(\Sigma^{out}_{ij} - \tilde{\Sigma}^{in}_{ij})n^{out}_j = 0 \quad \text{on} \quad r = a.$$

But we know

$$(\Sigma^{out}_{ij} - \Sigma^{in}_{ij} + \sigma^*_{ij})n^{out}_j = 0 \quad \text{on} \quad r = a.$$

Since $\tilde{\Sigma}^{in}_{ij}$, Σ^{in}_{ij}, σ^*_{ij} are constant states of stress, we must be able to arrange that

$$\boxed{\tilde{\Sigma}^{in}_{ij} = \Sigma^{in}_{ij} - \sigma^*_{ij}} \qquad 6 \text{ eqns.}$$

We can only do this because all the stresses are constant

These last 6 equations can be written as:

$$(C^*_{ijkl} - C_{ijkl})(e_{kl} + E_{kl}) + \sigma^*_{ij} = 0$$

or since

$$e_{kl} = \frac{\sigma^*_{\alpha\beta}}{16\mu(1-\nu)}\{(3-4\nu)(\delta_{k\alpha}\delta_{\beta l} + \delta_{l\alpha}\delta_{\beta k}) - \delta_{\alpha\beta}\delta_{kl}\}$$

$$= F_{kl\alpha\beta}\sigma^*_{\alpha\beta}$$

$$(C^*_{ijkl} - C_{ijkl})(F_{kl\alpha\beta}\sigma^*_{\alpha\beta}) + \sigma^*_{ij} = -(C^*_{ijkl} - C_{ijkl})E_{kl}$$

Since $\sigma^*_{ij} = \frac{1}{2}(\delta_{i\alpha}\delta_{j\beta} + \delta_{i\beta}\delta_{j\alpha})\sigma^*_{\alpha\beta}$

$$\left[\{C^*_{ijkl} - C_{ijkl}\}\{F_{kl\alpha\beta}\} + \frac{1}{2}(\delta_{i\alpha}\delta_{j\beta} + \delta_{i\beta}\delta_{j\alpha})\right]\sigma^*_{\alpha\beta} = -(C^*_{ijkl} - C_{ijkl})E_{kl}$$

This represents 6 equations in 6 unknowns, i.e., the unknown $\sigma^*_{\alpha\beta}$.

(1) Solve for $\sigma^*_{\alpha\beta}$

(2) Knowing $\sigma^*_{\alpha\beta}$, we can find all elastic fields inside and outside the inclusion

(3) Note: E_{kl} is known since the far field loading is known.

(4) For a hole, $C^*_{ijkl} = 0$ and we solve for $\sigma^*_{\alpha\beta}$ from

$$\left[C_{ijkl}F_{kl\alpha\beta} - \frac{1}{2}(\delta_{i\alpha}\delta_{j\beta} + \delta_{i\beta}\delta_{j\alpha})\right]\sigma^*_{\alpha\beta} = -C_{ijkl}E_{kl} = \sigma^{\infty}_{ij}$$

where

σ^{∞} is the remote loading at infinity

PROOF THAT THE DISPLACEMENT FIELD u_i IS CONTINUOUS ACROSS $r = a$.

Go back to

$$u_i(r,\theta) = a\,\sigma^a_{jk}\int_0^{2\pi} d\theta' \begin{Bmatrix}\cos\theta' \\ \sin\theta'\end{Bmatrix}\left\{-\frac{(3-4\nu)}{8\pi\mu(1-\nu)}\,\delta_{ij}\ln\sqrt{r^2+a^2-2ar\cos(\theta-\theta')}\right.$$

$$\left. + \frac{(x_i-x_i')(x_j-x_j')}{r^2+a^2-2ar\cos(\theta-\theta')}\right\}$$

Since $\dfrac{(x_i-x_i')(x_j-x_j')}{r^2+a^2-2ar\cos(\theta-\theta')} = N_i N_j$

where N is a unit vector along $\underline{x}-\underline{x}'$. This is a well-behaved continuous function and cannot lead to a discontinuity on $r=a$. The only possibly discontinuity could arise from the logarithmic term in the integrand.

For $r<a$ we have shown this contribution is

$$\sigma^a_{jk}\left\{\frac{3-4\nu}{8\mu(1-\nu)}\delta_{ij}x_k\right\} = \sigma^a_{jk}\frac{3-4\nu}{8\mu(1-\nu)}\delta_{ij}\,a\,n_k \quad \text{on } r=a.$$

since on $r=a$,

$$x_k = a\,n_k$$

For $r>a$ we must use the expansion

$$\ln\sqrt{r^2+a^2-2ar\cos(\theta-\theta')} = \ln r - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{a}{r}\right)^m\cos m(\theta-\theta').$$

Compute I_1 and I_2 in the manner used for $r<a$ we find the contribution from the log term is

$$-a\,\sigma^a_{jk}\frac{3-4\nu}{8\pi\mu(1-\nu)}\delta_{ij}\left(-\frac{\pi a}{r}\right)n_k$$

where $n_1 = \cos\theta$, $n_2 = \sin\theta$.

On $r = a$, this contribution is

$$\sigma_{jk}^{u} \frac{3-4\nu}{8\mu(1-\nu)} \delta_{ij}\, a\, n_k$$

so, indeed,

u_i is continuous across $r = a$.

TWO-DIMENSIONAL THEORY OF ELASTICITY FOR FINITE DEFORMATIONS

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CONTENTS

A general theory of plane stress, valid for large elastic deformations of isotropic materials, is developed using a general system of co-ordinates. No restriction is imposed upon the form of the strain-energy function in the formulation of the basic theory, which follows similar lines to the treatment by Adkins, Green & Shield (1953) of finite plane strain. The reduction of the equations to two-dimensional form subsequent to the assumption of plane stress enables the theory to be presented in complex variable notation.

A method of successive approximation is evolved, similar to that developed for problems in plane strain, which may be applied when exact solutions are not readily obtainable. The stress and displacement functions are expressed in terms of complex potential functions, and in the present paper the approximation process is terminated when the second-order terms have been obtained. The theory is formulated initially in terms of a complex co-ordinate system related to points in the deformed body, and the corresponding results for complex co-ordinates in the undeformed body are then obtained by a simple change of independent variable. Approximation methods are also applied to compressible materials in plane strain, and it is shown that the second-order terms for plane stress and plane strain can be expressed in similar forms. This leads to a general formulation of the second-order theory for two-dimensional problems, the results for plane stress or plane strain being derived by introducing the appropriate constants into the expressions thus obtained.

1. Introduction

The non-linearity of the differential equations which arise in formulating the mathematical theory of elasticity for large deformations has so far restricted the range of problems which have received satisfactory treatment to those in which marked simplifying features can be introduced. For example, in the problems of torsion, shear and flexure solved by Rivlin

(1948, 1949*a*, *b*,) for incompressible materials, and the inflation of a spherical shell examined by Green & Shield (1950), the restriction upon the form of deformation is such that, with the appropriate choice of co-ordinate system, the resulting equations can be solved quite generally for any form of strain-energy function. In other cases, as, for instance, in the analysis of the shear of a cylindrical annulus given by Rivlin (1949*b*), and in the generalizations of this problem examined by Adkins (1954), solutions have been obtained by assuming in addition a simplified form of strain-energy function, such as that postulated by Mooney (1940) for rubberlike materials.

It is evident that any general restriction upon the form of deformation is likely to produce some simplification in the form of the resulting equations. Thus if a problem can be reduced to two-dimensional form, some measure of simplicity will be achieved owing to the reduction of the number of dependent and independent variables which need to be considered, and into this category come the practically important problems of plane stress and plane strain.

The general theory of plane strain for large elastic deformations of isotropic materials has already been formulated by Adkins, Green & Shield (1953). In the present paper the corresponding theory is developed for plane stress. The undeformed body is assumed to consist of a thin plane uniform plate of isotropic elastic material which is stretched by forces in its plane so that it remains plane after deformation. When no forces act on the major surfaces of the plate, it is assumed, as in the classical theory of plane stress, that the principal stress component acting normally to the middle plane of the plate vanishes everywhere. The deformation and the stress resultants at any point are then expressed approximately as functions of position on the middle surface of the plate. Similar methods have been employed by Rivlin & Thomas (1951), and by Adkins & Rivlin (1952) in dealing with large deformations of thin sheets of incompressible materials, but in the problems there considered the resulting equations have been simplified by symmetry considerations.

In developing the general theory of plane stress, the stress resultants are expressed in terms of an Airy stress function ϕ chosen to satisfy the equations of equilibrium, and the work of §§3 and 4 is similar to that of Green & Zerna (1954) on the classical theory of thin plates. The resulting equations obtained in §§5 and 6 bear a formal resemblance to the corresponding equations for finite plane strain superposed on a uniform finite extension obtained by Adkins *et al.* (1953), but an additional unknown variable is introduced owing to the variation of thickness throughout the deformed plate.

For unsymmetrical problems, where exact solutions are not readily obtainable by orthodox methods of approach, second-order solutions, valid for a limited range of deformation, may be obtained by approximation methods. Some simple deformations of compressible materials have been investigated by such methods by Murnaghan (1937, 1951) and general formulations of the second-order theory have been given by Rivlin (1953) and by Green & Spratt (1954). Torsion problems have been similarly examined by a number of workers including Green & Shield (1951), and Green & Wilkes (1953). The method adopted in the present paper is similar to that employed by Adkins *et al.* (1953) for finite plane strain. It is assumed that the stress and displacement functions can be expressed as functions of a characteristic real parameter ϵ, the choice of which depends upon the problem under consideration. When the equations governing the deformation are expanded in terms of this parameter, the coefficients of each successive power of ϵ furnish a set of relations for the

determination of the corresponding terms in the expansions for the stress and displacement functions. In the present paper attention is limited to terms of the first and second orders.

The reduction of the theory to two-dimensional form makes possible a formulation in complex variable notation similar to that of the classical theory of elasticity (see, for example, Muskhelishvili 1953; Green & Zerna 1954). Explicit expressions for the stress and displacement functions can then be obtained in terms of complex potential functions, which are chosen to satisfy the boundary conditions in a given problem, two additional functions being introduced for each succeeding stage of the approximation process. The resulting expressions are similar in form to those derived by Adkins *et al.* in developing the theory of finite plane strain for incompressible materials, but with different values for the constant coefficients. In considering finite deformations, the complex co-ordinate system may be related either to points in the deformed body or to points in the undeformed body, the choice for any particular problem depending upon the nature of the boundary conditions. For convenience, the theory is developed in terms of complex co-ordinates in the deformed body, the corresponding formulae for co-ordinates in the undeformed body being obtained by a simple change of independent variable.

The similarity of the results for plane stress to those obtained for incompressible materials in plane strain suggests naturally the possibility of formulating in more general terms the second-order theory for two-dimensional problems. The approximate theory for compressible materials in plane strain is therefore developed in § 9; in the final section of the paper the results previously obtained are combined to yield general formulae for the determination of second-order solutions of two-dimensional problems in elasticity. These formulae express the stress and displacement functions in terms of complex potential functions, but the constant coefficients are left arbitrary. The complex potential functions may then be chosen, using these equations, to satisfy a prescribed set of boundary conditions over given contours, and the stress and displacement components evaluated in general terms. The results for plane stress or plane strain can then be obtained as special cases of the more general solution by choosing the appropriate values for the constants. Moreover, by a suitable choice of constants, the contours over which the boundary conditions are specified can form the boundaries either of the deformed body or of the undeformed body. Since the results for an incompressible material can be obtained by a limiting process from those for a compressible material, the single general solution can be made to yield, as special cases, the results for eight associated problems.

From a detailed examination of the equations for plane stress and plane strain, it is shown that the constants in the general solution may be expressed as functions of the elastic constants of the material together with two additional parameters. One of these parameters is employed to differentiate between plane stress and plane strain, while the other is chosen to distinguish between co-ordinates in the undeformed body and in the deformed body.

2. Notation and formulae

With slight modifications* we use the notation of Green & Zerna (1950) and Green & Shield (1950, 1951). The points of an unstrained and unstressed body at rest at time $t = 0$ are defined by a system of rectangular Cartesian co-ordinates x_i or by a general curvilinear

* See *Theoretical elasticity* by Green & Zerna (1954).

system of co-ordinates θ_i. The points of the deformed body may also be defined by a set of rectangular Cartesian co-ordinates y_i, and in the present paper we shall take the x_i-axes and y_i-axes to coincide. The curvilinear co-ordinates θ_i move with the body as it is deformed and form a curvilinear system in the strained body at time t. The covariant and contravariant metric tensors for the co-ordinate system θ_i in the unstrained body are denoted by g_{ij} and g^{ij} respectively, and for the co-ordinate system in the strained body, at time t, the corresponding metric tensors are G_{ij} and G^{ij} respectively. We write

$$g = | g_{ij} |, \quad G = | G_{ij} |, \tag{2·1}$$

latin indices taking the values 1, 2, 3.

For a homogeneous, isotropic, elastic material the strain-energy function W, measured per unit volume of the unstrained body, may be regarded as a function of three strain invariants I_1, I_2, I_3 given by

$$I_1 = g^{ij} G_{ij}, \quad I_2 = I_3 g_{ij} G^{ij}, \quad I_3 = G/g, \tag{2·2}$$

so that

$$W = W(I_1, I_2, I_3). \tag{2·3}$$

The contravariant stress tensor τ^{ij}, measured per unit area of the strained body, and referred to co-ordinates in the strained body may be expressed in the form

$$\tau^{ij} = g^{ij}\Phi + B^{ij}\Psi + G^{ij}p, \tag{2·4}$$

where

$$\Phi = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_1}, \quad \Psi = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_2}, \quad p = 2\sqrt{I_3}\frac{\partial W}{\partial I_3}, \tag{2·5}$$

$$B^{ij} = g^{ij} I_1 - g^{ir} g^{js} G_{rs} = \frac{1}{g} \mathrm{e}^{irm}\, \mathrm{e}^{jsn}\, g_{rs}\, G_{mn}, \tag{2·6}$$

and e^{irm} is equal to $+1$ or -1 according as i, r, m is an even or odd permutation of 1, 2, 3, and equal to 0 otherwise.

If $\mathbf{t}$ is the stress vector associated with a surface in the deformed body whose unit normal $\mathbf{u}$ is given by

$$\mathbf{u} = u_i \mathbf{G}^i, \tag{2·7}$$

then

$$\mathbf{t} = \frac{u_i \mathbf{T}_i}{\sqrt{G}} = u_i \tau^{ij} \mathbf{G}_j = \sum_i u_i \mathbf{t}_i \sqrt{G^{ii}}, \tag{2·8}$$

where

$$\mathbf{T}_i = \sqrt{(GG^{ii})}\, \mathbf{t}_i = \sqrt{(G)}\, \tau^{ij} \mathbf{G}_j. \tag{2·9}$$

$\mathbf{G}_j$, $\mathbf{G}^j$ are the covariant and contravariant base vectors in the deformed body, and $\mathbf{t}_i$ denotes the stress vector associated with the surface $\theta_i =$ constant.

With this notation the equations of equilibrium in the absence of body forces may be written in the alternative forms

$$\mathbf{T}_{i,i} = 0, \tag{2·10}$$

$$\tau^{ij} \|_i = 0, \tag{2·11}$$

where in (2·10) the comma denotes partial differentiation with respect to θ_i, and in (2·11) the double line denotes covariant differentiation with respect to the deformed body, that is, with respect to θ_i and the metric tensor components G_{ij}, G^{ij}.

PLANE STRESS

3. Stress resultants and loads

In this section the development of the theory is similar to that given by Green & Zerna (1954) for the classical theory of plates. We suppose the unstrained body to be a plate of homogeneous isotropic elastic material bounded by the plane surfaces $x_3 = \pm h_0$, where h_0 is a constant, although the results of §§ 3, 4 are also valid for an aeolotropic plate which has symmetry about the plane $x_3 = 0$. This plate undergoes a finite deformation symmetrical about the middle plane $x_3 = 0$, which thus becomes the middle plane $y_3 = 0$ in the deformed state. The major surfaces of the plate after deformation are given by $y_3 = \pm h$, where h is, in general, a function of y_1, y_2. We choose the curvilinear co-ordinate system θ_i so that

$$y_3 = \theta_3, \quad y_\alpha = y_\alpha(\theta_1, \theta_2, t), \tag{3·1}$$

greek indices taking the values 1, 2. It follows that

$$G_{ij} = \begin{pmatrix} A_{11}, & A_{12}, & 0 \\ A_{12}, & A_{22}, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \quad G^{ij} = \begin{pmatrix} A^{11}, & A^{12}, & 0 \\ A^{12}, & A^{22}, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \quad G = A, \tag{3·2}$$

with

$$A = |A_{\alpha\beta}|, \quad A^{\alpha\rho}A_{\rho\beta} = \delta^\alpha_\beta, \tag{3·3}$$

where $A_{\alpha\beta}$, $A^{\alpha\beta}$ are the covariant and contravariant metric tensors associated with co-ordinates θ_α in the middle plane $y_3 = 0$ of the deformed plate, and δ^α_β is the Kronecker delta.

The force acting on an element of area of the co-ordinate surface θ_1 = constant in the deformed body is $\mathbf{T}_1\, d\theta^2 d\theta^3$, and the length of the corresponding line element of the middle plane $y_3 = 0$ is

$$\surd(A_{22})\, d\theta^2 = \surd(AA^{11})\, d\theta^2.$$

Similar considerations apply for the other co-ordinate surface θ_2 = constant. The stress across either of the surfaces θ_α = constant may therefore be replaced by a physical stress resultant $\mathbf{n}_\alpha$, measured per unit length of the curve θ_α = constant in the plane $y_3 = 0$, where

$$\mathbf{n}_\alpha = \frac{\mathbf{N}_\alpha}{\surd(AA^{\alpha\alpha})}, \quad \mathbf{N}_\alpha = \int_{-h}^{h} \mathbf{T}_\alpha\, dy_3, \tag{3·4}$$

and we recall that h is a function of y_1, y_2 or θ_1, θ_2. Since the deformation is symmetrical about $y_3 = 0$, the corresponding stress couples are zero. From (2·9)

$$\mathbf{T}_\alpha = \surd(A)\, \tau^{\alpha j}\mathbf{G}_j,$$

so that from (3·4) we may write

$$\mathbf{n}_\alpha \surd A^{\alpha\alpha} = n^{\alpha\rho}\mathbf{G}_\rho + q^\alpha \mathbf{G}_3, \quad \mathbf{N}_\alpha = N^{\alpha\rho}\mathbf{G}_\rho + Q^\alpha\mathbf{G}_3, \tag{3·5}$$

where

$$N^{\alpha\rho} = n^{\alpha\rho}\surd A, \quad Q^\alpha = q^\alpha \surd A, \tag{3·6}$$

and

$$n^{\alpha\rho} = \int_{-h}^{h} \tau^{\alpha\rho}\, dy_3, \quad q^\alpha = \int_{-h}^{h} \tau^{\alpha 3}\, dy_3. \tag{3·7}$$

Since the deformation, and consequently the stress distribution, is symmetrical about the plane $y_3 = 0$, it follows that $q^\alpha = 0$.

The stress resultant $\mathbf{n}$ per unit length of a line drawn in the middle plane $y_3 = 0$ of the deformed plate, whose unit normal in that plane is

$$\mathbf{u} = u_\alpha \mathbf{G}^\alpha,$$

is given by

$$\mathbf{n} = \int_{-h}^{h} \mathbf{t}\, dy_3, \tag{3·8}$$

so that, from (2·8), (3·4) and (3·5),

$$\mathbf{n} = \frac{u_\alpha}{\sqrt{A}} \int_{-h}^{h} \mathbf{T}_\alpha\, dy_3 = \frac{u_\alpha \mathbf{N}_\alpha}{\sqrt{A}} = \sum_1^2 u_\alpha \mathbf{n}_\alpha \sqrt{A^{\alpha\alpha}} = u_\alpha (n^{\alpha\rho}\mathbf{G}_\rho + q^\alpha \mathbf{G}_3). \tag{3·9}$$

The functions defined in (3·7) are (plane) surface tensors. The components of the symmetrical contravariant tensor $n^{\alpha\rho}$ and the components of the contravariant tensor q^α are called stress resultants and shearing forces respectively. Mixed and covariant tensors n^α_β, $n_{\alpha\beta}$, q_α may be formed with the help of the metric tensors $A_{\alpha\beta}$, $A^{\alpha\beta}$. In order to find the physical components of $\mathbf{n}_\alpha$ we express these vectors in terms of unit base vectors along the co-ordinate curves $\theta_\alpha =$ constant. The physical stress resultants and shearing forces are denoted by $n_{(\alpha\beta)}$, $q_{(\alpha)}$ respectively, the bracket indicating that these quantities are not tensors. Thus we have

$$\mathbf{n}_\alpha = n_{(\alpha 1)} \frac{\mathbf{G}_1}{\sqrt{A_{11}}} + n_{(\alpha 2)} \frac{\mathbf{G}_2}{\sqrt{A_{22}}} + q_{(\alpha)} \mathbf{G}_3, \tag{3·10}$$

and comparison of this with (3·5) yields

$$n_{(\alpha\beta)} = n^{\alpha\beta} \sqrt{(A_{\beta\beta}/A^{\alpha\alpha})}, \quad q_{(\alpha)} = q^\alpha / \sqrt{A^{\alpha\alpha}}. \tag{3·11}$$

We now consider the external forces acting on the major surfaces of the deformed plate. The covariant components u_i of the unit normal to the surfaces $y_3 = \pm h(\theta_1, \theta_2)$ referred to the base vectors $\mathbf{G}^i$ are

$$(u_1, u_2, u_3) = k\left(-\frac{\partial y_3}{\partial \theta_1}, -\frac{\partial y_3}{\partial \theta_2}, 1\right), \tag{3·12}$$

where, remembering (3·2)

$$k = (A^{\alpha\beta} h_{,\alpha} h_{,\beta} + 1)^{-\frac{1}{2}}, \tag{3·13}$$

and at these surfaces, from (2·8),

$$\mathbf{t} = \frac{u_i \mathbf{T}_i}{\sqrt{A}} = \frac{k}{\sqrt{A}} (\mathbf{T}_3 - \mathbf{T}_\alpha y_{3,\alpha}). \tag{3·14}$$

The stress vector $\mathbf{t}$ is measured per unit area of the surfaces $y_3 = \pm h$. But

$$dS = (u_3^{-1})\, dS_3 = (\sqrt{A}/k)\, d\theta^1 d\theta^2,$$

where dS, dS_3 are corresponding elements of one of the major surfaces of the plate and of the middle plane respectively. We can thus replace $\mathbf{t}$ by $\mathbf{t}/k$ measured per unit area of the middle plane $y_3 = 0$. We now replace the surface forces by a resultant force $\mathbf{l}$ measured per unit area of this plane, where

$$\mathbf{l} = [\mathbf{t}/k]_{y_3=h} - [\mathbf{t}/k]_{y_3=-h} = [\mathbf{t}]_{-h}^{h}/k. \tag{3·15}$$

If we introduce the vector $\mathbf{L}$ where

$$\mathbf{L} = \mathbf{l}\sqrt{A}, \tag{3·16}$$

then, from (3·14),

$$\mathbf{L} = (\sqrt{A}/k)\, [\mathbf{t}]_{-h}^{h} = [\mathbf{T}_3 - \mathbf{T}_\alpha y_{3,\alpha}]_{-h}^{h}. \tag{3·17}$$

Remembering (2·9) we may now write $\mathbf{l}$, $\mathbf{L}$ in the forms

$$\mathbf{l} = l^\alpha \mathbf{G}_\alpha + l\mathbf{G}_3, \quad \mathbf{L} = L^\alpha \mathbf{G}_\alpha + L\mathbf{G}_3, \tag{3·18}$$

where

$$\left.\begin{aligned} L^\alpha = l^\alpha \surd A, \quad L = l\surd A, \\ l^\alpha = [\tau^{\alpha 3} - \tau^{\alpha\beta} y_{3,\beta}]_{-h}^{h}, \quad l = [\tau^{33} - \tau^{3\beta} y_{3,\beta}]_{-h}^{h}. \end{aligned}\right\} \tag{3·19}$$

These relations may be simplified by observing that since the deformation is symmetrical about the plane $y_3 = 0$, l and L are zero. Thus (3·18) and (3·19) yield

$$\left.\begin{aligned} \mathbf{l} = l^\alpha \mathbf{G}_\alpha, \quad \mathbf{L} = L^\alpha \mathbf{G}_\alpha, \\ [\tau^{33} - \tau^{3\beta} y_{3,\beta}]_{-h}^{h} = 0. \end{aligned}\right\} \tag{3·20}$$

4. Equations of equilibrium: Airy's stress function

If we integrate (2·10) through the thickness of the plate we obtain

$$\int_{-h}^{h} \mathbf{T}_{\alpha,\alpha} dy_3 + [\mathbf{T}_3]_{-h}^{h} = 0. \tag{4·1}$$

But, from (3·4), $$\mathbf{N}_{\alpha,\alpha} = \frac{\partial}{\partial\theta^\alpha}\int_{-h}^{h} \mathbf{T}_\alpha dy_3 = \int_{-h}^{h} \mathbf{T}_{\alpha,\alpha} dy_3 + [\mathbf{T}_\alpha y_{3,\alpha}]_{-h}^{h},$$

so that (4·1) becomes $$\mathbf{N}_{\alpha,\alpha} + [\mathbf{T}_3 - \mathbf{T}_\alpha y_{3,\alpha}]_{-h}^{h} = 0,$$

or

$$\mathbf{N}_{\alpha,\alpha} + \mathbf{L} = 0, \tag{4·2}$$

if we use (3·17). Combining this with (3·5), (3·6), (3·19) and (3·20) we have

$$n^{\alpha\beta}\,\|_\alpha + l^\beta = 0, \tag{4·3}$$

where the double line denotes covariant differentiation with respect to the plane variables θ_α in the deformed body, using Christoffel symbols formed from the metric tensors $A_{\alpha\beta}$, $A^{\alpha\beta}$. Since q^α and l are zero, the third equation of equilibrium is automatically satisfied.

We shall, from now on, assume that the major surfaces of the plate are free from applied forces so that $\mathbf{t} = 0$ when $y_3 = \pm h$. Then, from (3·16) and (3·17), we see that $\mathbf{L}$ and $\mathbf{l}$ are zero and, remembering (3·20), the equations of equilibrium (4·2), (4·3) reduce to

$$\mathbf{N}_{\alpha,\alpha} = 0 \quad \text{or} \quad n^{\alpha\beta}\,\|_\alpha = 0. \tag{4·4}$$

Equations (4·4) and (3·5), with $q^\alpha = 0$, are similar in form to the corresponding equations for $\mathbf{T}_\alpha$ and $\tau^{\alpha\beta}$ obtained by Adkins *et al.* (1953) for plane strain. The results there obtained may therefore be applied to express the stress resultants and applied forces and couples in terms of an Airy stress function ϕ. Thus

$$\left.\begin{aligned} \mathbf{N}_\alpha = \surd(A)\,\epsilon^{\gamma\alpha}\boldsymbol{\chi}_{,\gamma} = \surd(A)\,\epsilon^{\gamma\alpha}\epsilon^{\rho\beta}\phi\,\|_{\gamma\rho}\mathbf{G}_\beta, \\ n^{\alpha\beta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\rho}\phi\,\|_{\gamma\rho}, \end{aligned}\right\} \tag{4·5}$$

and

or

$$\phi\,\|_{\alpha\beta} = \epsilon_{\alpha\gamma}\epsilon_{\beta\rho}n^{\gamma\rho} = (A/a)\,({}_0\epsilon_{\alpha\gamma})\,({}_0\epsilon_{\beta\rho})\,n^{\gamma\rho}, \tag{4·6}$$

where $\boldsymbol{\chi}$ is a vector in the plane $y_3 = 0$, ϕ is a scalar invariant function of θ_1 and θ_2,

$${}_0\epsilon^{\alpha\beta}\surd a = {}_0\epsilon_{\alpha\beta}/\surd a = \epsilon^{\alpha\beta}\surd A = \epsilon_{\alpha\beta}/\surd A = \mathrm{e}_{\alpha\beta 3}, \tag{4·7}$$

and

$$\epsilon_{\alpha\rho}\epsilon^{\alpha\lambda} = \delta^\lambda_\rho.$$

The double line again indicates covariant differentiation with respect to the plane $y_3 = 0$ of the deformed body, the order of differentiation being immaterial since the Riemann-Christoffel tensor in the plane vanishes.

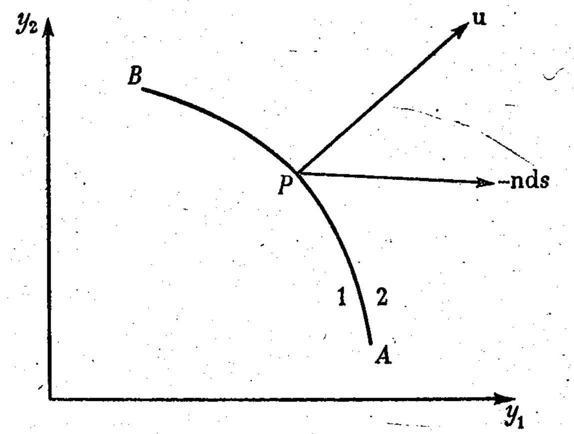

FIGURE 1

Let AP be an arc of a curve AB in the middle plane $y_3 = 0$ of the deformed body (figure 1). By an analysis similar to that used for plane strain we may obtain the resultant force across a surface in the deformed body formed by normals to $y_3 = 0$ along AB. Denoting an element of AP by $\mathrm{d}s$ and making use of (3·9) and (4·5), we obtain for the total force $\mathbf{F}$ exerted by the region 1 on the region 2, across the arc AP,

$$\mathbf{F} = -\int_A^P \mathbf{n}\,\mathrm{d}s = \boldsymbol{\chi} = \epsilon^{\rho\beta}\phi_{,\rho}\,\mathbf{G}_\beta, \tag{4·8}$$

apart from an arbitrary constant vector which may be absorbed into $\boldsymbol{\chi}$ without loss of generality. Similarly, the total moment about the y_3-axis of the forces exerted by the region 1 on the region 2 is given by

$$\mathbf{M} = \int_A^P [\mathbf{R}\wedge\boldsymbol{\chi}_{,\beta}]\frac{\mathrm{d}\theta^\beta}{\mathrm{d}s}\,\mathrm{d}s = (R^\alpha\phi_{,\alpha}-\phi)\,\mathbf{G}^3, \tag{4·9}$$

apart from an arbitrary constant vector which may again be absorbed into $\phi\mathbf{G}^3$ without affecting the stresses. In (4·9)

$$\mathbf{R} = R^\alpha\mathbf{G}_\alpha = R_\alpha\mathbf{G}^\alpha \tag{4·10}$$

is the position vector of a point on the curve AB with respect to the origin of the y_i-axes. Equation (4·9) thus represents a couple of magnitude

$$M = R^\alpha\phi_{,\alpha}-\phi \tag{4·11}$$

about the y_3-axis.

If AB is a boundary curve of the plate which is entirely free from applied forces, (4·8) and (4·11) yield the conditions

$$\boldsymbol{\chi} = 0,$$

or

$$\phi_{,1} = 0, \quad \phi_{,2} = 0, \tag{4·12}$$

at all points of AB.

5. Stress-strain relations

In the equations so far derived, no assumptions have been made regarding the thickness of the plate. We now confine our attention to plates which are thin, and write approximately

$$x_\alpha = x_\alpha(\theta_1, \theta_2), \quad x_3 = y_3/\lambda = \theta_3/\lambda, \tag{5·1}$$

where λ is a scalar invariant function of θ_1, θ_2. The metric tensors g_{ij}, g^{ij} now take the approximate forms

$$\left.\begin{aligned} g_{\alpha\beta} &= a_{\alpha\beta}, \quad g_{33} = 1/\lambda^2, \\ g^{\alpha\beta} &= a^{\alpha\beta}, \quad g^{33} = \lambda^2, \\ g &= a/\lambda^2, \quad a = |a_{\alpha\beta}|, \end{aligned}\right\} \tag{5·2}$$

where $a_{\alpha\beta}$, $a^{\alpha\beta}$ are the covariant and contravariant metric tensors associated with curvilinear co-ordinates θ_α in the plane $x_3 = 0$ of the undeformed body.

From (2·2), (3·2) and (5·2) the strain invariants are given by

$$\left.\begin{aligned} I_1 &= \lambda^2 + a^{\alpha\beta} A_{\alpha\beta}, \\ I_2 &= \lambda^2 (A/a)\, a_{\alpha\beta} A^{\alpha\beta} + A/a, \\ I_3 &= \lambda^2 A/a, \end{aligned}\right\} \tag{5·3}$$

approximately. Also, remembering (4·7), we have $a_{\alpha\beta} A^{\alpha\beta} A = a^{\alpha\beta} A_{\alpha\beta}\, a$ and hence

$$I_3 - \lambda^2 I_2 + \lambda^4 I_1 - \lambda^6 = 0. \tag{5·4}$$

These results bear a formal resemblance to the corresponding relations obtained by Adkins *et al.* (1953) for plane strain, but λ is no longer constant.

The tensor B^{ij} may be calculated from (2·6), (3·2) and (5·2), and is approximately

$$\left.\begin{aligned} B^{\alpha\beta} &= \lambda^2 a^{\alpha\beta} + A A^{\alpha\beta}/a, \\ B^{33} &= \lambda^2 (I_1 - \lambda^2). \end{aligned}\right\} \tag{5·5}$$

From (2·4), (3·2), (5·2) and (5·5) we obtain for the components of the stress tensor

$$\left.\begin{aligned} \tau^{\alpha\beta} &= (\Phi + \lambda^2 \Psi)\, a^{\alpha\beta} + (\Psi A/a + p)\, A^{\alpha\beta}, \\ \tau^{33} &= \lambda^2 \Phi + \lambda^2 (I_1 - \lambda^2)\, \Psi + p. \end{aligned}\right\} \tag{5·6}$$

Since the major surfaces of the plate are free from applied forces, from (3·15) and (3·19) we have, at $y_3 = \pm h$,

$$\tau^{\alpha 3} - \tau^{\alpha\beta} y_{3,\beta} = 0, \quad \tau^{33} - \tau^{3\alpha} y_{3,\alpha} = 0,$$

from which, by eliminating $\tau^{\alpha 3}$, we obtain

$$\tau^{33} - \tau^{\alpha\beta} y_{3,\alpha} y_{3,\beta} = 0. \tag{5·7}$$

If the thickness of the plate is sufficiently small and if $h_{3,\alpha}$ is of the same order of magnitude as h, it is evident from (5·7) that at $y_3 = \pm h$, τ^{33} is small compared with the stresses $\tau^{\alpha\beta}$. We therefore assume τ^{33} to be negligible throughout the plate, and from the last of equations (5·6) we then have approximately

$$\lambda^2 \Phi + \lambda^2 (I_1 - \lambda^2)\, \Psi + p = 0. \tag{5·8}$$

Eliminating p between (5·8) and the first equation of (5·6) we obtain

$$\tau^{\alpha\beta} = (\Phi+\lambda^2\Psi)\,a^{\alpha\beta}+\{(A/a+\lambda^4-\lambda^2 I_1)\,\Psi-\lambda^2\Phi\}\,A^{\alpha\beta}. \tag{5·9}$$

From (5·3) it is evident that the invariants, and hence also Φ and Ψ, are independent of y_3 $(=\theta_3)$. Equation (3·7) thus yields

$$n^{\alpha\beta} = 2h\tau^{\alpha\beta} = 2\lambda h_0\,\tau^{\alpha\beta}. \tag{5·10}$$

From (4·6), (5·9) and (5·10) $\qquad \phi\,\|_{\alpha\beta} = Ha_{\alpha\beta}+KA_{\alpha\beta},$ (5·11)

where, remembering (2·5) and (5·3), we have

$$\left.\begin{aligned} H &= 2h_0\lambda\frac{A}{a}\{\Phi+\lambda^2\Psi\} = 4h_0\frac{\sqrt{I_3}}{\lambda}\left\{\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right\},\\ K &= 2h_0\lambda\left\{\left(\frac{A}{a}+\lambda^4-\lambda^2 I_1\right)\Psi-\lambda^2\Phi\right\}\\ &= -4h_0\frac{\lambda}{\sqrt{I_3}}\left\{\lambda^2\frac{\partial W}{\partial I_1}+\left(\lambda^2 I_1-\lambda^4-\frac{I_3}{\lambda^2}\right)\frac{\partial W}{\partial I_2}\right\}.\end{aligned}\right\} \tag{5·12}$$

For an incompressible material $\qquad I_3 = \lambda^2 A/a = 1,$ (5·13)

and W becomes a function of I_1 and I_2 only. Equations (5·12) then reduce to

$$\left.\begin{aligned} H &= \frac{4h_0}{\lambda}\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right),\\ K &= -4h_0\lambda\left\{\lambda^2\frac{\partial W}{\partial I_1}+\left(\lambda^2 I_1-\lambda^4-\frac{1}{\lambda^2}\right)\frac{\partial W}{\partial I_2}\right\}.\end{aligned}\right\} \tag{5·14}$$

For compressible materials it is convenient to express H and K in terms of three different, mutually independent invariants J_1, J_2, J_3 defined by

$$\left.\begin{aligned} J_1 &= I_1-3,\\ J_2 &= I_2-2I_1+3,\\ J_3 &= I_3-I_2+I_1-1.\end{aligned}\right\} \tag{5·15}$$

The invariants J_1, J_2, J_3 have been employed by Rivlin (1953) and have the advantage that for small deformations they are of the first, second and third orders of smallness respectively. From (5·15) we have

$$\left.\begin{aligned} \frac{\partial W}{\partial I_1} &= \frac{\partial W}{\partial J_1}-2\frac{\partial W}{\partial J_2}+\frac{\partial W}{\partial J_3},\\ \frac{\partial W}{\partial I_2} &= \frac{\partial W}{\partial J_2}-\frac{\partial W}{\partial J_3},\\ \frac{\partial W}{\partial I_3} &= \frac{\partial W}{\partial J_3}.\end{aligned}\right\} \tag{5·16}$$

Equations (5·4) and (5·8) then yield

$$J_3+(1-\lambda^2)\,J_2+(1-\lambda^2)^2 J_1+(1-\lambda^2)^3 = 0, \tag{5·17}$$

and $$\lambda^2\frac{\partial W}{\partial J_1}+\lambda^2(J_1+1-\lambda^2)\frac{\partial W}{\partial J_2}+\{(1-\lambda^2)(J_1+1-\lambda^2)+J_2+J_3\}\frac{\partial W}{\partial J_3} = 0, \tag{5·18}$$

respectively. Expressions corresponding to (5·12) may be derived for H and K, but in place of (5·9) we now make use of the first of (5·6) from which p has not been eliminated. Thus we obtain

$$\left.\begin{aligned} H &= 4h_0\frac{\sqrt{I_3}}{\lambda}\left\{\frac{\partial W}{\partial J_1}+(\lambda^2-2)\frac{\partial W}{\partial J_2}-(\lambda^2-1)\frac{\partial W}{\partial J_3}\right\},\\ K &= 4h_0\frac{\sqrt{I_3}}{\lambda}\left\{\frac{\partial W}{\partial J_2}+(\lambda^2-1)\frac{\partial W}{\partial J_3}\right\}.\end{aligned}\right\} \tag{5·19}$$

For a compressible material these expressions are equivalent to (5·12) by virtue of (5·16), (5·17) and (5·18).

6. Formulation in terms of complex variables

With the simplifying assumptions of the previous section, and the consequent reduction of the theory to two-dimensional form, it becomes possible to employ complex variable techniques. The procedure followed in this and subsequent sections is therefore similar to that used by Adkins *et al.* (1953) in the treatment of finite plane strain. The presence of the parameter λ, however, which is now a function of the co-ordinates, renders the equations obtained more complicated in form.

For finite deformations, the complex co-ordinate reference frame may be related to points in the undeformed body or in the deformed body, and the relevant equations for either co-ordinate system may be derived by an appropriate choice for the moving system of co-ordinates θ_α in the relations of the preceding sections. Since the resulting expressions are simpler in form for complex co-ordinates in the deformed body, we shall consider this case first, and from the results thus obtained, derive the corresponding formulae for complex co-ordinates in the undeformed body by a simple change of independent variable.

We thus introduce complex co-ordinates $(\zeta, \bar{\zeta})$, $(z, \bar{z})$ in the undeformed body and in the deformed body respectively by means of the relations

$$\left.\begin{aligned} \zeta &= x_1+\mathrm{i}x_2, \quad \bar{\zeta} = x_1-\mathrm{i}x_2,\\ z &= y_1+\mathrm{i}y_2, \quad \bar{z} = y_1-\mathrm{i}y_2.\end{aligned}\right\} \tag{6·1}$$

If the components of displacement referred to the x_α-axes are (u, v), the complex displacement function D is defined by

$$D = u+\mathrm{i}v, \quad \bar{D} = u-\mathrm{i}v, \tag{6·2}$$

and since the x_α-axes and y_α-axes coincide, we have

$$z = \zeta+D, \quad \bar{z} = \bar{\zeta}+\bar{D}. \tag{6·3}$$

If we denote covariant and contravariant base vectors in the system of complex co-ordinates $(z, \bar{z})$ by $\mathbf{A}_\alpha$ and $\mathbf{A}^\alpha$ respectively, the position vector $\mathbf{R}$ of a point of the deformed body, which is given by (4·10), may be written

$$\mathbf{R} = z^\alpha\mathbf{A}_\alpha = z_\alpha\mathbf{A}^\alpha.$$

By tensor transformations

$$z^1 = \frac{\partial z}{\partial y_1}y_1+\frac{\partial z}{\partial y_2}y_2 = y_1+\mathrm{i}y_2 = z,$$

$$z^2 = \frac{\partial \bar{z}}{\partial y_1}y_1+\frac{\partial \bar{z}}{\partial y_2}y_2 = y_1-\mathrm{i}y_2 = \bar{z},$$

so that the complex co-ordinates $(z, \bar{z})$ may be denoted by z^α.

We now take the moving system of co-ordinates θ_α to coincide with the co-ordinates $(z, \bar{z})$ so that

$$\theta_1 = z, \quad \theta_2 = \bar{z}. \tag{6·4}$$

The metric tensors $A_{\alpha\beta}, A^{\alpha\beta}$ then have the values

$$\left.\begin{aligned} A_{12} = \tfrac{1}{2}, \quad A_{11} = A_{22} = 0, \quad \sqrt{A} = \tfrac{1}{2}\mathrm{i}, \\ A^{12} = 2, \quad A^{11} = A^{22} = 0. \end{aligned}\right\} \tag{6·5}$$

Remembering (5·2), the strain invariants (5·3) reduce to

$$\left.\begin{aligned} I_1 &= \lambda^2 + a^{12}, \\ I_2 &= -\lambda^2 a_{12}/a - 1/(4a), \\ I_3 &= -\lambda^2/(4a), \end{aligned}\right\} \tag{6·6}$$

where, from (5·1), (5·2), (6·1), (6·3) and (6·4), we have

$$\sqrt{a} = \frac{\partial(x_1, x_2)}{\partial(\theta_1, \theta_2)} = \frac{\mathrm{i}}{2}\left(1 - \frac{\partial D}{\partial z} - \frac{\partial \overline{D}}{\partial \bar{z}} + \frac{\partial D}{\partial z}\frac{\partial \overline{D}}{\partial \bar{z}} - \frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}\right) = \frac{\mathrm{i}\lambda}{2\sqrt{I_3}}, \tag{6·7}$$

$$\left.\begin{aligned} a_{11} = \bar{a}_{22} &= \left(\frac{\partial x_1}{\partial z}\right)^2 + \left(\frac{\partial x_2}{\partial z}\right)^2 = \frac{\partial \overline{D}}{\partial z}\left(\frac{\partial D}{\partial z} - 1\right), \\ a_{12} &= \frac{1}{2}\left\{1 - \frac{\partial D}{\partial z} - \frac{\partial \overline{D}}{\partial \bar{z}} + \frac{\partial D}{\partial z}\frac{\partial \overline{D}}{\partial \bar{z}} + \frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}\right\} \\ &= \frac{\lambda}{2\sqrt{I_3}} + \frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}, \\ a^{11} = \bar{a}^{22} &= a_{11}/a, \quad a^{12} = -a_{12}/a. \end{aligned}\right\} \tag{6·8}$$

A bar over a function indicates the complex conjugate of that function, and we have used (6·7) to simplify a_{12}. Introducing these results into (6·6) and using (5·15) we obtain

$$\left.\begin{aligned} J_1 &= \lambda^2 - 3 - \frac{a_{12}}{a} = \lambda^2 + 2\frac{\sqrt{I_3}}{\lambda} + 4\frac{I_3}{\lambda^2}\frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z} - 3, \\ J_2 &= 3 - 2\lambda^2 - (\lambda^2 - 2)\frac{a_{12}}{a} - \frac{1}{4a} \\ &= 3 - 2\lambda^2 + \frac{I_3}{\lambda^2} + 2(\lambda^2 - 2)\left\{\frac{\sqrt{I_3}}{\lambda} + 2\frac{I_3}{\lambda^2}\frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}\right\}, \\ J_3 &= (\lambda^2 - 1)\left(1 + \frac{a_{12}}{a} - \frac{1}{4a}\right) \\ &= (\lambda^2 - 1)\left\{\left(1 - \frac{\sqrt{I_3}}{\lambda}\right)^2 - 4\frac{I_3}{\lambda^2}\frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}\right\}. \end{aligned}\right\} \tag{6·9}$$

If the body is incompressible so that $I_3 = 1$, we have from (6·6), (6·7) and (6·8)

$$\sqrt{a} = \mathrm{i}\lambda/2,$$

or

$$1 - \lambda = \frac{\partial D}{\partial z} + \frac{\partial \overline{D}}{\partial \bar{z}} - \frac{\partial D}{\partial z}\frac{\partial \overline{D}}{\partial \bar{z}} + \frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}, \tag{6·10}$$

and

$$\left.\begin{aligned} a_{12} &= \tfrac{1}{2}\lambda + \frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}, \\ I_1 &= \lambda^2 + \frac{2}{\lambda} + \frac{4}{\lambda^2}\frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}, \\ I_2 &= \frac{1}{\lambda^2} + 2\lambda + 4\frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}. \end{aligned}\right\} \tag{6·11}$$

Since the components (6·5) of the metric tensor of the deformed body are constants, the corresponding Christoffel symbols are zero and therefore covariant differentiation in the deformed body reduces to partial differentiation. The equations of equilibrium (5·11) thus reduce to

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial z^2} &= Ha_{11},\\ \frac{\partial^2\phi}{\partial z\partial\bar{z}} &= Ha_{12}+\tfrac{1}{2}K,\end{aligned}\right\} \tag{6·12}$$

together with the complex conjugate of the first of these equations.

For a compressible material H and K are given by (5·19), and (6·12) then becomes

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial z^2} &= 4h_0\left(\frac{\surd I_3}{\lambda}\right)\left\{\frac{\partial W}{\partial J_1}+(\lambda^2-2)\frac{\partial W}{\partial J_2}-(\lambda^2-1)\frac{\partial W}{\partial J_3}\right\}\frac{\partial\overline{D}}{\partial z}\left(\frac{\partial D}{\partial z}-1\right),\\ \frac{\partial^2\phi}{\partial z\partial\bar{z}} &= 2h_0\left\{\frac{\partial W}{\partial J_1}+\left(\frac{\surd I_3}{\lambda}+\lambda^2-2\right)\frac{\partial W}{\partial J_2}+(\lambda^2-1)\left(\frac{\surd I_3}{\lambda}-1\right)\frac{\partial W}{\partial J_3}\right\}\\ &\quad+4h_0\left(\frac{\surd I_3}{\lambda}\right)\left\{\frac{\partial W}{\partial J_1}+(\lambda^2-2)\frac{\partial W}{\partial J_2}-(\lambda^2-1)\frac{\partial W}{\partial J_3}\right\}\frac{\partial D}{\partial\bar{z}}\frac{\partial\overline{D}}{\partial z}.\end{aligned}\right\} \tag{6·13}$$

Also, from (6·9) and (5·18), we have

$$\frac{\partial W}{\partial J_1}+2\left(\frac{\surd I_3}{\lambda}-1\right)\frac{\partial W}{\partial J_2}+\left(\frac{\surd I_3}{\lambda}-1\right)^2\frac{\partial W}{\partial J_3}+4\frac{I_3}{\lambda^2}\left(\frac{\partial W}{\partial J_2}-\frac{\partial W}{\partial J_3}\right)\frac{\partial D}{\partial\bar{z}}\frac{\partial\overline{D}}{\partial z}=0. \tag{6·14}$$

Remembering (6·7) and (6·9), the relations (6·13) and (6·14) yield four equations for the determination of ϕ, D, $\overline{D}$ and λ.

The corresponding equations for an incompressible material are obtained by introducing (5·14) and (6·11) into (6·12). Thus

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial z^2} &= \frac{4h_0}{\lambda}\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right)\frac{\partial\overline{D}}{\partial z}\left(\frac{\partial D}{\partial z}-1\right),\\ \frac{\partial^2\phi}{\partial z\partial\bar{z}} &= 2h_0\left\{(1-\lambda^3)\left(\frac{\partial W}{\partial I_1}+\frac{1}{\lambda}\frac{\partial W}{\partial I_2}\right)+2\left(\frac{1}{\lambda}\frac{\partial W}{\partial I_1}-\lambda\frac{\partial W}{\partial I_2}\right)\frac{\partial D}{\partial\bar{z}}\frac{\partial\overline{D}}{\partial z}\right\}.\end{aligned}\right\} \tag{6·15}$$

These equations, together with the incompressibility condition (6·10), are again sufficient to determine ϕ, D, $\overline{D}$ and λ.

The theory may be formulated similarly in terms of complex co-ordinates in the undeformed body by choosing the moving system of co-ordinates θ_α to coincide with the complex co-ordinate system $(\zeta,\bar{\zeta})$. Alternatively, by making use of (6·3) and (6·7) we may change the independent variables in equations (6·9) to (6·15). Thus for a compressible material we may write

$$\left.\begin{aligned}\frac{\partial}{\partial z} &= \frac{\lambda}{\surd I_3}\left\{\left(1+\frac{\partial\overline{D}}{\partial\bar{\zeta}}\right)\frac{\partial}{\partial\zeta}-\frac{\partial\overline{D}}{\partial\zeta}\frac{\partial}{\partial\bar{\zeta}}\right\},\\ \frac{\partial}{\partial\bar{z}} &= \frac{\lambda}{\surd I_3}\left\{-\frac{\partial D}{\partial\bar{\zeta}}\frac{\partial}{\partial\zeta}+\left(1+\frac{\partial D}{\partial\zeta}\right)\frac{\partial}{\partial\bar{\zeta}}\right\},\end{aligned}\right\} \tag{6·16}$$

and for an incompressible material we may put $I_3 = 1$ in these expressions. The resulting equations will not, however, be required for the approximate theory developed in §§7 to 10.

Denoting the stress resultants referred to complex co-ordinates in the deformed body by $n^{\alpha\beta}$, we have from (4·5)

$$n^{11} = \overline{n}^{22} = -4\frac{\partial^2\phi}{\partial\bar{z}^2}, \quad n^{12} = 4\frac{\partial^2\phi}{\partial z\,\partial\bar{z}}. \tag{6·17}$$

Employing (6·3) and (6·1), we may now, by simple tensor transformations, obtain expressions for the stress resultants referred to the complex co-ordinate system $(\zeta, \bar{\zeta})$, or to the real co-ordinate systems x_α, y_α, analogous to those obtained for the stress components in the corresponding theory of finite plane strain.

Also, if the resultant force $\mathbf{F}$ across any arc AP of a curve in the deformed plate has components (X, Y) along the y_1-, y_2-axes respectively, then a simple tensor transformation gives

$$\mathbf{F} = (X+\mathrm{i}Y)\,\mathbf{A}_1 + (X-\mathrm{i}Y)\,\mathbf{A}_2 = F\mathbf{A}_1 + \overline{F}\mathbf{A}_2, \tag{6·18}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$ are the covariant base vectors in the complex co-ordinate system $(z, \bar{z})$. Then, remembering (6·5), we may interpret (4·8) in complex co-ordinates to get

$$F = 2\mathrm{i}\frac{\partial\phi}{\partial\bar{z}}. \tag{6·19}$$

Similarly, for the couple M about the origin we obtain from (4·11)

$$M = z\frac{\partial\phi}{\partial z} + \bar{z}\frac{\partial\phi}{\partial\bar{z}} - \phi. \tag{6·20}$$

From (6·19), or directly from (4·12), at all points of a boundary curve which is entirely free from applied stress, we have

$$\frac{\partial\phi}{\partial z} = 0, \tag{6·21}$$

together with the complex conjugate of this equation.

By introducing (6·16) into (6·17), (6·19), (6·20) and (6·21) we may readily obtain the corresponding relations for complex co-ordinates $(\zeta, \bar{\zeta})$ in the undeformed body, but these are not required for subsequent applications.

7. Successive approximations: incompressible materials

The classical infinitesimal theory of plane stress is obtained by neglecting squares and products of the displacement D and its derivatives with respect to z, $\bar{z}$ or ζ, $\bar{\zeta}$ in the equations of the previous section. Further approximations based on the classical theory may be obtained by considering higher order terms in these relations. Taking co-ordinates $(z, \bar{z})$ in the deformed body we put

$$D = \epsilon\{{}^0D(z,\bar{z})\} + \epsilon^2\{{}^1D(z,\bar{z})\} + \ldots, \tag{7·1}$$

where ϵ is a characteristic real parameter in a given problem. Also, since λ is the ratio of the thickness of the plate after deformation to that before deformation we may write

$$\lambda = 1 + \epsilon\{{}^0\lambda(z,\bar{z})\} + \epsilon^2\{{}^1\lambda(z,\bar{z})\} + \ldots. \tag{7·2}$$

For incompressible materials we thus obtain from (6·11)

$$\left.\begin{aligned} I_1 &= 3 + \epsilon^2\left\{3({}^0\lambda)^2 + 4\frac{\partial\,{}^0D}{\partial\bar{z}}\frac{\partial\,{}^0\overline{D}}{\partial z}\right\} + \ldots, \\ I_2 &= 3 + \epsilon^2\left\{3({}^0\lambda)^2 + 4\frac{\partial\,{}^0D}{\partial\bar{z}}\frac{\partial\,{}^0\overline{D}}{\partial z}\right\} + \ldots. \end{aligned}\right\} \tag{7·3}$$

In the present paper we shall confine our attention to terms of the first and second orders in ϵ in the expansion (7·1) for D, and to this degree of approximation the form of strain-energy function suggested by Mooney (1940) is adequate. We may thus write

$$W = C_1(I_1-3)+C_2(I_2-3), \tag{7·4}$$

so that C_1, C_2 are the values of $\partial W/\partial I_1$, $\partial W/\partial I_2$ respectively at $I_1 = I_2 = 3$. Also we may put

$$\phi = {}^0H\epsilon\{{}^0\phi(z,\bar{z})+\epsilon\,{}^1\phi(z,\bar{z})+\ldots\}, \tag{7·5}$$

where ${}^0H$ is a constant, which, for convenience, we shall choose to have the value $4h_0(C_1+C_2)$ so that, from (5·14), $H = {}^0H$ when $\epsilon = 0$.

Introducing the relations (7·1) to (7·5) into (6·10) and (6·15), and equating to zero the coefficients of ϵ in the resulting equations, we obtain

$$\left.\begin{aligned} &\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}+{}^0\lambda = 0,\\ &\frac{\partial^2({}^0\phi)}{\partial z^2}+\frac{\partial\,{}^0\bar{D}}{\partial z} = 0,\\ &\frac{\partial^2({}^0\phi)}{\partial z\,\partial \bar{z}}+\frac{3\,{}^0\lambda}{2} = 0.\end{aligned}\right\} \tag{7·6}$$

Similarly the coefficients of ϵ^2 in these equations yield

$$\left.\begin{aligned} &\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\bar{D}}{\partial \bar{z}}+{}^1\lambda = \frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}-\frac{\partial\,{}^0D}{\partial \bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z},\\ &\frac{\partial^2({}^1\phi)}{\partial z^2}+\frac{\partial\,{}^1\bar{D}}{\partial z} = \frac{\partial\,{}^0\bar{D}}{\partial z}\left(\frac{\partial\,{}^0D}{\partial z}+\alpha\,{}^0\lambda\right),\\ &\frac{\partial^2({}^1\phi)}{\partial z\,\partial \bar{z}}+\frac{3\,{}^1\lambda}{2} = \alpha\frac{\partial\,{}^0D}{\partial \bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z}-\tfrac{3}{4}(1+\alpha)\,({}^0\lambda)^2,\end{aligned}\right\} \tag{7·7}$$

where $\alpha = (C_1-C_2)/(C_1+C_2)$. Similar equations may be obtained from the coefficients of higher powers of ϵ provided higher order terms than those given in (7·4) in the expansion for the strain-energy function W are taken into account. The first approximation corresponds to the classical theory, and the equations for this may be integrated in terms of complex potential functions $\Omega(z)$, $\omega(z)$. Thus, from (7·6),

$$\left.\begin{aligned} {}^0\phi(z,\bar{z}) &= \bar{z}\Omega(z)+z\bar{\Omega}(\bar{z})+\omega(z)+\bar{\omega}(\bar{z}),\\ {}^0D(z,\bar{z}) &= \tfrac{5}{3}\Omega(z)-z\bar{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z}),\\ {}^0\lambda(z,\bar{z}) &= -\tfrac{2}{3}\{\Omega'(z)+\bar{\Omega}'(\bar{z})\},\end{aligned}\right\} \tag{7·8}$$

a prime indicating the derivative of a function with respect to its argument.

Eliminating ${}^1\lambda$ between the first and third of equations (7·7) we obtain

$$\begin{aligned} \frac{2}{3}\frac{\partial^2({}^1\phi)}{\partial z\,\partial \bar{z}}-\frac{\partial\,{}^1D}{\partial z}-\frac{\partial\,{}^1\bar{D}}{\partial \bar{z}} &= \tfrac{1}{3}(2\alpha+3)\frac{\partial\,{}^0D}{\partial \bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z}-\frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}-\tfrac{1}{2}(1+\alpha)\,({}^0\lambda)^2\\ &= \tfrac{1}{3}(2\alpha+3)\,\{\bar{z}\Omega''(z)+\omega''(z)\}\{z\bar{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})\}\\ &\quad -\tfrac{1}{9}(2\alpha-13)\,\{[\Omega'(z)]^2+[\bar{\Omega}'(\bar{z})]^2\}-\tfrac{2}{9}(2\alpha+19)\,\Omega'(z)\,\bar{\Omega}'(\bar{z}), \end{aligned} \tag{7·9}$$

if we make use of (7·8). Similarly, again using (7·8), the second of equations (7·7) becomes

$$\frac{\partial^2({}^1\phi)}{\partial z^2}+\frac{\partial\,{}^1\overline{D}}{\partial z}=\tfrac{1}{3}\{\bar{z}\Omega''(z)+\omega''(z)\}\{(2\alpha-5)\,\Omega'(z)+(2\alpha+3)\,\overline{\Omega}'(\bar{z})\}. \tag{7·10}$$

This equation may be integrated to yield

$$\frac{\partial^1\phi}{\partial z}+{}^1\overline{D}=\tfrac{8}{3}\overline{\Delta}(\bar{z})+\tfrac{1}{3}(2\alpha-5)\left\{\tfrac{1}{2}\bar{z}[\Omega'(z)]^2+\int^z\Omega'(z)\,\omega''(z)\,\mathrm{d}z\right\}$$
$$-\tfrac{1}{18}(2\alpha+11)\int^{\bar{z}}\{\overline{\Omega}'(\bar{z})\}^2\,\mathrm{d}\bar{z}+\tfrac{1}{3}(2\alpha+3)\,\overline{\Omega}'(\bar{z})\,\{\bar{z}\Omega'(z)+\omega'(z)-\tfrac{5}{3}\overline{\Omega}(\bar{z})\}, \tag{7·11}$$

where $\overline{\Delta}(\bar{z})$ is an arbitrary function of $\bar{z}$ and the additional terms in $\bar{z}$ have been inserted to simplify subsequent expressions. From (7·9) and (7·11) it follows that

$$\frac{8}{3}\frac{\partial^2({}^1\phi)}{\partial z\,\partial\bar{z}}=\tfrac{8}{3}\{\Delta'(z)+\overline{\Delta}'(\bar{z})\}+\tfrac{4}{9}(2\alpha-5)\,\Omega'(z)\,\overline{\Omega}'(\bar{z})$$
$$+\tfrac{1}{3}(2\alpha+3)\,\{\Omega''(z)\,[z\overline{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})-\tfrac{5}{3}\Omega(z)]+\overline{\Omega}''(\bar{z})\,[\bar{z}\Omega'(z)+\omega'(z)-\tfrac{5}{3}\overline{\Omega}(\bar{z})]$$
$$+[\bar{z}\Omega''(z)+\omega''(z)]\,[z\overline{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})]-\tfrac{5}{3}[\Omega'(z)]^2-\tfrac{5}{3}[\overline{\Omega}'(\bar{z})]^2\},$$

and hence, by integration,

$$\frac{\partial\,{}^1\phi(z,\bar{z})}{\partial\bar{z}}=\Delta(z)+z\overline{\Delta}'(\bar{z})+\bar{\delta}'(\bar{z})+\tfrac{1}{12}(6\alpha-7)\,\Omega(z)\,\overline{\Omega}'(\bar{z})$$
$$+(2\alpha+3)\,\{\tfrac{1}{8}\Gamma_1(z,\bar{z})-\tfrac{1}{3}z[\overline{\Omega}'(\bar{z})]^2\}, \tag{7·12}$$

where

$$\Gamma_1(z,\bar{z})=\{z\overline{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})\}\{\bar{z}\Omega'(z)+\omega'(z)-\tfrac{5}{3}\overline{\Omega}(\bar{z})\}$$
$$+\{\Omega'(z)+\overline{\Omega}'(\bar{z})\}\{z\overline{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})-\tfrac{5}{3}\Omega(z)\}$$
$$=-\left\{{}^0D\frac{\partial}{\partial z}+{}^0\overline{D}\frac{\partial}{\partial\bar{z}}\right\}\frac{\partial\,{}^0\phi}{\partial\bar{z}}, \tag{7·13}$$

and $\bar{\delta}'(\bar{z})$ is a further arbitrary function of $\bar{z}$. By integration of (7·12) we may obtain an expression for ${}^1\phi$, but this is not required in applications of the theory. From (7·11) and (7·12)

$${}^1D(z,\bar{z})=\tfrac{5}{3}\Delta(z)-z\overline{\Delta}'(\bar{z})-\bar{\delta}'(\bar{z})-\tfrac{1}{12}(6\alpha-7)\,\Omega(z)\,\overline{\Omega}'(\bar{z})$$
$$-\tfrac{1}{8}(2\alpha+3)\,\Lambda_1(z,\bar{z})+\tfrac{1}{6}(6\alpha+1)\,z\{\overline{\Omega}'(\bar{z})\}^2$$
$$-\tfrac{1}{18}(2\alpha+11)\int^z[\Omega'(z)]^2\,\mathrm{d}z+\tfrac{1}{3}(2\alpha-5)\int^{\bar{z}}\overline{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,\mathrm{d}\bar{z}, \tag{7·14}$$

where

$$\Lambda_1(z,\bar{z})=\{z\overline{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})\}\{\bar{z}\Omega'(z)+\omega'(z)-\tfrac{5}{3}\overline{\Omega}(\bar{z})\}$$
$$-\{\tfrac{5}{3}\Omega'(z)-\overline{\Omega}'(\bar{z})\}\{z\overline{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})-\tfrac{5}{3}\Omega(z)\}$$
$$=\left({}^0D\frac{\partial}{\partial z}+{}^0\overline{D}\frac{\partial}{\partial\bar{z}}\right){}^0D. \tag{7·15}$$

An expression for ${}^1\lambda(z,\bar{z})$ in terms of complex potential functions may now be obtained, if required, by introducing (7·14) and (7·8) into the first of equations (7·7).

For problems which are non-dislocational in character the complex potential functions $\Omega(z)$, $\omega(z)$, $\Delta(z)$ and $\delta(z)$ must be chosen so that the stress and displacement functions are single-valued. It follows that ${}^0D$, ${}^1D$, …, ${}^0\lambda$, ${}^1\lambda$, …, and all their derivatives with respect

to z and $\bar{z}$, and similarly the second and higher order derivatives of ${}^0\phi$, ${}^1\phi$ ..., if they exist, must be single-valued at interior points of the body. Thus from (7·8)

$$[\Omega'(z)]_C = 0, \quad [\omega''(z)]_C = 0, \quad 5[\Omega(z)]_C = 3[\bar{\omega}'(\bar{z})]_C, \tag{7·16}$$

and similarly from these relations, with (7·12) and (7·14), we may infer that

$$\left.\begin{aligned} &[\Delta'(z)]_C = 0, \quad [12\delta''(z) + (6\alpha-7)\,\bar{\Omega}(\bar{z})\,\Omega''(z)]_C = 0, \\ &[5\Delta(z) - 3\bar{\delta}'(\bar{z})]_C \\ &\quad = \left[\tfrac{1}{6}(2\alpha+11)\int^z \{\Omega'(z)\}^2\,\mathrm{d}z - (2\alpha-5)\int^{\bar{z}} \bar{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,\mathrm{d}\bar{z} + \tfrac{1}{4}(6\alpha-7)\,\Omega(z)\,\bar{\Omega}'(\bar{z})\right]_C, \end{aligned}\right\} \tag{7·17}$$

where in (7·16) and (7·17), $[\,]_C$ denotes the change in value of the function inside the brackets during a complete circuit of a contour C lying entirely within the deformed body.†

For some problems it is convenient to remove the integral terms from (7·14). Replacing $\Delta(z)$ by $\Delta(z) + \frac{1}{30}(2\alpha+11)\int^z \{\Omega'(z)\}^2\,\mathrm{d}z$ and $\delta'(z)$ by $\delta'(z) + \frac{1}{3}(2\alpha-5)\int^z \Omega'(z)\,\omega''(z)\,\mathrm{d}z$ in (7·12) and (7·14) we obtain

$$\left.\begin{aligned} \frac{\partial\,{}^1\phi(z,\bar{z})}{\partial\bar{z}} &= \Delta(z) + z\bar{\Delta}'(\bar{z}) + \bar{\delta}'(\bar{z}) + \tfrac{1}{12}(6\alpha-7)\,\Omega(z)\,\bar{\Omega}'(\bar{z}) \\ &\quad + \tfrac{1}{8}(2\alpha+3)\,\Gamma_1(z,\bar{z}) - \tfrac{1}{30}(18\alpha+19)\,z\{\bar{\Omega}'(\bar{z})\}^2 \\ &\quad + \tfrac{1}{30}(2\alpha+11)\int^z \{\Omega'(z)\}^2\,\mathrm{d}z + \tfrac{1}{3}(2\alpha-5)\int^{\bar{z}} \bar{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,\mathrm{d}\bar{z}, \\ {}^1D(z,\bar{z}) &= \tfrac{5}{3}\Delta(z) - z\bar{\Delta}'(\bar{z}) - \bar{\delta}'(\bar{z}) - \tfrac{1}{12}(6\alpha-7)\,\Omega(z)\,\bar{\Omega}'(\bar{z}) \\ &\quad + \tfrac{1}{15}(14\alpha-3)\,z\{\bar{\Omega}'(\bar{z})\}^2 - \tfrac{1}{8}(2\alpha+3)\,\Lambda_1(z,\bar{z}). \end{aligned}\right\} \tag{7·18}$$

The conditions (7·17) for single-valued stress resultants and displacements now, however, reduce to

$$\left.\begin{aligned} &[\Delta'(z)]_C = 0, \quad [12\delta''(z) + (6\alpha-7)\,\bar{\Omega}(\bar{z})\,\Omega''(z)]_C = 0, \\ &[5\Delta(z) - 3\delta'(\bar{z}) - \tfrac{1}{4}(6\alpha-7)\,\Omega(z)\,\bar{\Omega}'(\bar{z})]_C = 0. \end{aligned}\right\} \tag{7·19}$$

The corresponding results for complex co-ordinates $(\zeta, \bar{\zeta})$ in the undeformed body may readily be obtained by expanding the argument z in $\phi(z,\bar{z})$, $D(z,\bar{z})$ by means of (6·3). If we express D in the form

$$D = \epsilon\{{}^0D'(\zeta,\bar{\zeta})\} + \epsilon^2\{{}^1D'(\zeta,\bar{\zeta})\} + \dots, \tag{7·20}$$

and introduce this expansion, together with (6·3), into (7·1) we obtain

$$D = \epsilon\{{}^0D(\zeta,\bar{\zeta})\} + \epsilon^2\left\{{}^1D(\zeta,\bar{\zeta}) + {}^0D'(\zeta,\bar{\zeta})\,\frac{\partial\,{}^0D(\zeta,\bar{\zeta})}{\partial\zeta} + {}^0\bar{D}'(\zeta,\bar{\zeta})\,\frac{\partial\,{}^0D(\zeta,\bar{\zeta})}{\partial\bar{\zeta}}\right\} + \dots. \tag{7·21}$$

Comparing (7·20) and (7·21) and making use of (7·15) we thus have

$$\left.\begin{aligned} {}^0D'(\zeta,\bar{\zeta}) &= {}^0D(\zeta,\bar{\zeta}), \\ {}^1D'(\zeta,\bar{\zeta}) &= {}^1D(\zeta,\bar{\zeta}) + \Lambda_1(\zeta,\bar{\zeta}), \end{aligned}\right\} \tag{7·22}$$

† The conditions (10·22) given by Adkins *et al.* (1953) for plane strain, when the resultant force on the contour is zero, are, of course, only true if the integral terms in the preceding equation (10·16) are single-valued. This is the case for the examples considered in that paper.

and similarly from (7·5) and (7·13) we may obtain

$$\left.\begin{aligned}\frac{\partial\, {}^0\phi(z,\bar{z})}{\partial \bar{z}} &= \frac{\partial\, {}^0\phi(\zeta,\bar{\zeta})}{\partial \bar{\zeta}},\\ \frac{\partial\, {}^1\phi(z,\bar{z})}{\partial \bar{z}} &= \frac{\partial\, {}^1\phi(\zeta,\bar{\zeta})}{\partial \bar{\zeta}} - \Gamma_1(\zeta,\bar{\zeta}),\end{aligned}\right\} \tag{7·23}$$

where $\Gamma_1(\zeta,\bar{\zeta})$, $\Lambda_1(\zeta,\bar{\zeta})$ are obtained by replacing $z, \bar{z}$ by $\zeta, \bar{\zeta}$ in (7·13) and (7·15) respectively. The first-order stress and displacement functions ${}^0\phi(\zeta,\bar{\zeta})$ and ${}^0D'(\zeta,\bar{\zeta})$ are thus obtained by replacing $z, \bar{z}$ by $\zeta, \bar{\zeta}$ in (7·8). Also, combining the second of equations (7·22) and (7·23) with (7·12) and (7·14) we obtain

$$\left.\begin{aligned}\frac{\partial\, {}^1\phi(z,\bar{z})}{\partial \bar{z}} &= \Delta(\zeta)+\zeta\bar{\Delta}'(\bar{\zeta})+\bar{\delta}'(\bar{\zeta})+\tfrac{1}{12}(6\alpha-7)\,\Omega(\zeta)\,\bar{\Omega}'(\bar{\zeta})\\ &\quad+\tfrac{1}{8}(2\alpha-5)\,\Gamma_1(\zeta,\bar{\zeta})-\tfrac{1}{3}(2\alpha+3)\,\zeta\{\bar{\Omega}'(\bar{\zeta})\}^2,\\ {}^1D'(\zeta,\bar{\zeta}) &= \tfrac{5}{3}\Delta(\zeta)-\zeta\bar{\Delta}'(\bar{\zeta})-\bar{\delta}'(\bar{\zeta})-\tfrac{1}{12}(6\alpha-7)\,\Omega(\zeta)\,\bar{\Omega}'(\bar{\zeta})\\ &\quad+\tfrac{1}{6}(6\alpha+1)\,\zeta\{\bar{\Omega}'(\bar{\zeta})\}^2-\tfrac{1}{18}(2\alpha+11)\int^{\zeta}\{\Omega'(\zeta)\}^2\,d\zeta\\ &\quad-(2\alpha-5)\left\{\tfrac{1}{8}\Lambda_1(\zeta,\bar{\zeta})-\tfrac{1}{3}\int^{\bar{\zeta}}\bar{\Omega}'(\bar{\zeta})\,\bar{\omega}''(\bar{\zeta})\,d\bar{\zeta}\right\},\end{aligned}\right\} \tag{7·24}$$

and similarly from (7·18) we have the alternative forms

$$\left.\begin{aligned}\frac{\partial\, {}^1\phi(z,\bar{z})}{\partial \bar{z}} &= \Delta(\zeta)+\zeta\bar{\Delta}'(\bar{\zeta})+\bar{\delta}'(\bar{\zeta})+\tfrac{1}{12}(6\alpha-7)\,\Omega(\zeta)\,\bar{\Omega}'(\bar{\zeta})\\ &\quad+\frac{1}{30}\left\{(2\alpha+11)\int^{\zeta}[\Omega'(\zeta)]^2\,d\zeta-(18\alpha+19)\,\zeta[\bar{\Omega}'(\bar{\zeta})]^2\right\}\\ &\quad+(2\alpha-5)\left\{\tfrac{1}{8}\Gamma_1(\zeta,\bar{\zeta})+\frac{1}{3}\int^{\bar{\zeta}}\bar{\Omega}'(\bar{\zeta})\,\bar{\omega}''(\bar{\zeta})\,d\bar{\zeta}\right\},\\ {}^1D'(\zeta,\bar{\zeta}) &= \tfrac{5}{3}\Delta(\zeta)-\zeta\bar{\Delta}'(\bar{\zeta})-\bar{\delta}'(\bar{\zeta})-\tfrac{1}{12}(6\alpha-7)\,\Omega(\zeta)\,\bar{\Omega}'(\bar{\zeta})\\ &\quad+\tfrac{1}{15}(14\alpha-3)\,\zeta\{\bar{\Omega}'(\bar{\zeta})\}^2-\tfrac{1}{8}(2\alpha-5)\,\Lambda_1(\zeta,\bar{\zeta}).\end{aligned}\right\} \tag{7·25}$$

The conditions for (7·24) and (7·25) to yield single-valued stress resultant and displacement components now take the forms (7·17) and (7·19) respectively with $\zeta, \bar{\zeta}$ replacing $z, \bar{z}$.

Expressions for the stress components, and for the resultant force and couple acting on a curve in the deformed body may now be obtained in terms of complex potential functions by combining the expressions obtained in the present section for ${}^0\phi$, ${}^1\phi$, ${}^0D$ and ${}^1D$ with (7·1), (7·5), (6·17), (6·19) and (6·20).

8. Successive approximations: compressible materials

Approximate solutions of equations (6·13) and (6·14) for compressible materials may be obtained without difficulty in terms of complex potential functions by the methods of the previous section. Assuming expansions of the forms (7·1) and (7·2) for D and λ, we obtain from (6·7) and (6·9)

$$\left.\begin{aligned}J_1 &= \epsilon\, {}^0J_1+\epsilon^2({}^1J_1)+\ldots,\\ J_2 &= \epsilon^2({}^1J_2)+\ldots,\\ J_3 &= O(\epsilon^3),\end{aligned}\right\} \tag{8·1}$$

where

$$\left.\begin{aligned} {}^0J_1 &= 2\left\{\frac{\partial\, {}^0D}{\partial z}+\frac{\partial\, {}^0\overline{D}}{\partial \overline{z}}+{}^0\lambda\right\},\\ {}^1J_1 &= 2\left\{\frac{\partial\, {}^1D}{\partial z}+\frac{\partial\, {}^1\overline{D}}{\partial \overline{z}}+{}^1\lambda\right\}+2\left\{\left(\frac{\partial\, {}^0D}{\partial z}\right)^2+\left(\frac{\partial\, {}^0\overline{D}}{\partial \overline{z}}\right)^2+\frac{\partial\, {}^0D}{\partial z}\frac{\partial\, {}^0\overline{D}}{\partial \overline{z}}+3\frac{\partial\, {}^0D}{\partial \overline{z}}\frac{\partial\, {}^0\overline{D}}{\partial z}\right\}+({}^0\lambda)^2,\\ {}^1J_2 &= \left(\frac{\partial\, {}^0D}{\partial z}+\frac{\partial\, {}^0\overline{D}}{\partial \overline{z}}\right)^2-4\frac{\partial\, {}^0D}{\partial \overline{z}}\frac{\partial\, {}^0\overline{D}}{\partial z}+4{}^0\lambda\left(\frac{\partial\, {}^0D}{\partial z}+\frac{\partial\, {}^0\overline{D}}{\partial \overline{z}}\right), \end{aligned}\right\} \quad (8{\cdot}2)$$

and

$$\frac{\surd I_3}{\lambda} = 1+\epsilon\left(\frac{\partial\, {}^0D}{\partial z}+\frac{\partial\, {}^0\overline{D}}{\partial \overline{z}}\right)$$
$$+\epsilon^2\left\{\frac{\partial\, {}^1D}{\partial z}+\frac{\partial\, {}^1\overline{D}}{\partial \overline{z}}+\left(\frac{\partial\, {}^0D}{\partial z}+\frac{\partial\, {}^0\overline{D}}{\partial \overline{z}}\right)^2-\frac{\partial\, {}^0D}{\partial z}\frac{\partial\, {}^0\overline{D}}{\partial \overline{z}}+\frac{\partial\, {}^0D}{\partial \overline{z}}\frac{\partial\, {}^0\overline{D}}{\partial z}\right\}+\dots \quad (8{\cdot}3)$$

Also, since $W = W(J_1, J_2, J_3)$ we may obtain from (8·1)

$$\frac{\partial W}{\partial J_r} = \left[\frac{\partial W}{\partial J_r}\right]_0+\epsilon\, {}^0J_1\left[\frac{\partial^2 W}{\partial J_1 \partial J_r}\right]_0$$
$$+\epsilon^2\left\{{}^1J_1\left[\frac{\partial^2 W}{\partial J_1\partial J_r}\right]_0+{}^1J_2\left[\frac{\partial^2 W}{\partial J_2\partial J_r}\right]_0+\tfrac{1}{2}({}^0J_1)^2\left[\frac{\partial^3 W}{\partial J_1^2\partial J_r}\right]_0\right\}+\dots \quad (r = 1, 2, 3), \quad (8{\cdot}4)$$

where the suffix 0 indicates that the quantity inside the square brackets is evaluated at $J_1 = J_2 = J_3 = 0$. It has been pointed out by Murnaghan (1937) that for a material in which the stress is zero in the undeformed state, $[\partial W/\partial J_1]_0 = 0$, a result which may readily be obtained by considering the expansion of (6·14) in powers of ϵ. Also, by considering the first approximation terms in the stress-strain relations, it has been shown by Rivlin (1953) that the Lamé constants λ and μ of the classical theory of elasticity are given by

$$\lambda = 4\left\{\left[\frac{\partial W}{\partial J_2}\right]_0+\left[\frac{\partial^2 W}{\partial J_1^2}\right]_0\right\}, \quad \mu = -2\left[\frac{\partial W}{\partial J_2}\right]_0. \quad (8{\cdot}5)$$

The Lamé constant λ in (8·5) is not used again so it need not be confused with λ used elsewhere in the pa[illegible].

By analogy with §7 we may express ϕ in the form (7·5) where ${}^0H$ now has the value $-4h_0[\partial W/\partial J_2]_0$ so that, from (5·19) we again have $H = {}^0H$ when $\epsilon = 0$. Introducing (7·1), (7·2), (7·5) and (8·1) to (8·4) into (6·13) and (6·14) and equating the coefficients of ϵ in the resulting equations to zero we obtain

$$\left.\begin{aligned} &\frac{\partial^2({}^0\phi)}{\partial z^2}+\frac{\partial\, {}^0\overline{D}}{\partial z} = 0,\\ &2\frac{\partial^2({}^0\phi)}{\partial z\partial \overline{z}}+(2c_1+1)\left(\frac{\partial\, {}^0D}{\partial z}+\frac{\partial\, {}^0\overline{D}}{\partial \overline{z}}\right)+2(c_1+1){}^0\lambda = 0,\\ &(c_1+1)\left(\frac{\partial\, {}^0D}{\partial z}+\frac{\partial\, {}^0\overline{D}}{\partial \overline{z}}\right)+c_1{}^0\lambda = 0, \end{aligned}\right\} \quad (8{\cdot}6)$$

and similarly from the coefficients of ϵ^2 we have

$$\frac{\partial^2({}^1\phi)}{\partial z^2}+\frac{\partial\, {}^1\overline{D}}{\partial z} = \frac{\partial\, {}^0\overline{D}}{\partial z}\left\{2(c_1-c_2)\frac{\partial\, {}^0D}{\partial z}+(2c_1-2c_2-1)\frac{\partial\, {}^0\overline{D}}{\partial \overline{z}}+2(c_1-c_2-c_3+1)\,{}^0\lambda\right\}, \quad (8{\cdot}7a)$$

$$2\frac{\partial^2({}^1\phi)}{\partial z\partial\bar{z}}+(2c_1+1)\left(\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\bar{D}}{\partial\bar{z}}\right)+2(c_1+1)\,{}^1\lambda$$
$$+(2c_1+3c_2+2c_4+1)\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial\bar{z}}\right)^2-(2c_1+1)\frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\bar{D}}{\partial\bar{z}}+(6c_1-4c_2-1)\frac{\partial\,{}^0\bar{D}}{\partial z}\frac{\partial\,{}^0D}{\partial\bar{z}}$$
$$+2(5c_2+c_3+2c_4)\,{}^0\lambda\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial\bar{z}}\right)+(c_1+4c_2+2c_4+1)\,({}^0\lambda)^2=0, \tag{8·7b}$$

$$2(c_1+1)\left(\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\bar{D}}{\partial\bar{z}}\right)+2c_1\,{}^1\lambda+(2c_1+5c_2+c_3+2c_4+2)\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial\bar{z}}\right)^2$$
$$-2(c_1+1)\frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\bar{D}}{\partial\bar{z}}+2(3c_1-2c_2-2c_3+3)\frac{\partial\,{}^0D}{\partial\bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z}$$
$$+4(2c_2+c_4)\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial\bar{z}}\right){}^0\lambda+(c_1+2c_4)\,({}^0\lambda)^2=0, \tag{8·7c}$$

where we have written

$$\left.\begin{aligned}\left[\frac{\partial^2W}{\partial J_1^2}\right]_0\bigg/\left[\frac{\partial W}{\partial J_2}\right]_0=c_1,\quad \left[\frac{\partial^2W}{\partial J_1\partial J_2}\right]_0\bigg/\left[\frac{\partial W}{\partial J_2}\right]_0=c_2,\\ \left[\frac{\partial W}{\partial J_3}\right]_0\bigg/\left[\frac{\partial W}{\partial J_2}\right]_0=c_3,\quad \left[\frac{\partial^3W}{\partial J_1^3}\right]_0\bigg/\left[\frac{\partial W}{\partial J_2}\right]_0=c_4.\end{aligned}\right\} \tag{8·8}$$

Equations (8·6) may be solved in terms of complex potential functions $\Omega(z)$, $\omega(z)$ to yield

$$\left.\begin{aligned}{}^0\phi(z,\bar{z})&=\bar{z}\Omega(z)+z\bar{\Omega}(\bar{z})+\omega(z)+\bar{\omega}(\bar{z}),\\ {}^0D(z,\bar{z})&=\kappa\Omega(z)-z\bar{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z}),\\ {}^0\lambda(z,\bar{z})&=\tfrac{1}{2}(\kappa-3)\{\Omega'(z)+\bar{\Omega}'(\bar{z})\},\end{aligned}\right\} \tag{8·9}$$

where
$$\kappa=\frac{5c_1+2}{3c_1+2},\quad \kappa-3=-\frac{4(c_1+1)}{3c_1+2}. \tag{8·10}$$

We may observe that the constant c_1 can be expressed in terms of Poisson's ratio η by the relation $c_1=-(1-\eta)/(1-2\eta)$ so that $\kappa=(3-\eta)/(1+\eta)$ and $\kappa-3=-4\eta/(1+\eta)$. Since, from (8·5), the modulus of rigidity μ of the material is contained in the constant ${}^0H$, equations (8·9) are equivalent to the usual formulae of the classical theory of generalized plane stress.

Introducing the expressions (8·9) into (8·7*a*), and remembering (8·10), we obtain

$$\frac{\partial^2({}^1\phi)}{\partial z^2}+\frac{\partial\,{}^1\bar{D}}{\partial z}=\{\bar{z}\Omega''(z)+\omega''(z)\}\{B_1'\Omega'(z)+B_1\bar{\Omega}'(\bar{z})\}, \tag{8·11}$$

where we have written for brevity

$$\left.\begin{aligned}B_1&=\{13c_1+6-4c_2-4(c_1+1)\,c_3\}/(3c_1+2),\\ B_1'&=\{5c_1+2-4c_2-4(c_1+1)\,c_3\}/(3c_1+2).\end{aligned}\right\} \tag{8·12}$$

Similarly, by eliminating ${}^1\lambda$ between equations (8·7*b*) and (8·7*c*) we have

$$2c_1\frac{\partial^2({}^1\phi)}{\partial z\partial\bar{z}}-(3c_1+2)\left(\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\bar{D}}{\partial\bar{z}}\right)$$
$$=\{3c_1+2+(2c_1+5)\,c_2+(c_1+1)\,c_3+2c_4\}\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial\bar{z}}\right)^2$$
$$-(3c_1+2)\frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\bar{D}}{\partial\bar{z}}+\{13c_1+6-4c_2-4(c_1+1)\,c_3\}\frac{\partial\,{}^0D}{\partial\bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z}$$
$$-2\{(c_1-4)\,c_2+c_1c_3-2c_4\}\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial\bar{z}}\right){}^0\lambda-2(2c_1c_2-c_4)\,({}^0\lambda)^2, \tag{8·13}$$

or, making use of (8·9), (8·10) and (8·12)

$$(\kappa-1)\frac{\partial^2({}^1\phi)}{\partial z\,\partial\bar{z}}-\left(\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\bar{D}}{\partial\bar{z}}\right)$$
$$=B_1\{\bar{z}\Omega''(z)+\omega''(z)\}\{z\bar{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})\}-B_2'\{[\Omega'(z)]^2+[\bar{\Omega}'(\bar{z})]^2\}-2B_2\Omega'(z)\,\bar{\Omega}'(\bar{z}), \quad (8\cdot14)$$

where

$$\left.\begin{aligned}B_2&=\{(3c_1+2)\,(13c_1^2+16c_1+4)+4[3c_1(3c_1+4)\,c_2-3c_1^2(c_1+1)\,c_3-2c_4]\}/(3c_1+2)^3,\\ B_2'&=-\{(3c_1+2)\,(19c_1^2+16c_1+4)-4[3c_1(3c_1+4)\,c_2-3c_1^2(c_1+1)\,c_3-2c_4]\}/(3c_1+2)^3.\end{aligned}\right\}\quad(8\cdot15)$$

It will be observed that (8·11) and (8·14) are similar in form to the corresponding equations (7·10) and (7·9) for an incompressible material, and an analogous procedure may therefore be adopted to obtain expressions for $\partial^1\phi/\partial\bar{z}$ and ${}^1D$ in terms of complex potential functions. Thus

$$\frac{\partial\,{}^1\phi(z,\bar{z})}{\partial\bar{z}}=\Delta(z)+z\bar{\Delta}'(\bar{z})+\bar{\delta}'(\bar{z})+B_3\Omega(z)\,\bar{\Omega}'(\bar{z})+\frac{B_1}{\kappa+1}\Gamma_2(z,\bar{z})-B_1z\{\bar{\Omega}'(\bar{z})\}^2,$$

and

$$\left.\begin{aligned}{}^1D(z,\bar{z})&=\kappa\Delta(z)-z\bar{\Delta}'(\bar{z})-\bar{\delta}'(\bar{z})-B_3\Omega(z)\,\bar{\Omega}'(\bar{z})-\frac{B_1}{\kappa+1}\Lambda_2(z,\bar{z})\\&\quad+B_1''z\{\bar{\Omega}'(\bar{z})\}^2+B_1'\int^{\bar{z}}\bar{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,\mathrm{d}\bar{z}-B_4\int^{z}\{\Omega'(z)\}^2\,\mathrm{d}z,\end{aligned}\right\}\quad(8\cdot16)$$

where

$$\left.\begin{aligned}B_1''&=B_1+\tfrac{1}{2}B_1'=\tfrac{1}{2}\{31c_1+14-12[c_2+(c_1+1)\,c_3]\}/(3c_1+2),\\ B_3&=B_1-2B_2/(\kappa+1)\\&=\{(3c_1+2)\,(39c_1^2+34c_1+8)-4c_2(21c_1^2+26c_1+4)\\&\qquad-4c_3(c_1+1)\,(9c_1^2+14c_1+4)+8c_4\}/\{2(2c_1+1)\,(3c_1+2)^2\},\\ B_4&=\tfrac{1}{2}B_1'-B_2'\\&=\{(3c_1+2)\,(53c_1^2+48c_1+12)-4c_2(27c_1^2+36c_1+4)\\&\qquad-4c_3(c_1+1)\,(3c_1^2+12c_1+4)+16c_4\}/\{2(3c_1+2)^3\},\end{aligned}\right\}\quad(8\cdot17)$$

$$\left.\begin{aligned}\Gamma_2(z,\bar{z})&=\{z\bar{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})\}\{\bar{z}\Omega'(z)+\omega'(z)-\kappa\bar{\Omega}(\bar{z})\}\\&\quad+\{\Omega'(z)+\bar{\Omega}'(\bar{z})\}\{z\bar{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})-\kappa\Omega(z)\}\\&=-\left\{{}^0D\frac{\partial}{\partial z}+{}^0\bar{D}\frac{\partial}{\partial\bar{z}}\right\}\frac{\partial\,{}^0\phi}{\partial\bar{z}},\\ \text{and}\quad\Lambda_2(z,\bar{z})&=\{z\bar{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})\}\{\bar{z}\Omega'(z)+\omega'(z)-\kappa\bar{\Omega}(\bar{z})\}\\&\quad-\{\kappa\Omega'(z)-\bar{\Omega}'(\bar{z})\}\{z\bar{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})-\kappa\Omega(z)\}\\&=\left\{{}^0D\frac{\partial}{\partial z}+{}^0\bar{D}\frac{\partial}{\partial\bar{z}}\right\}{}^0D.\end{aligned}\right\}\quad(8\cdot18)$$

An expression for ${}^1\lambda(z,\bar{z})$ in terms of complex potential functions may now be obtained from (8·7), (8·9) and (8·16). From (8·9) and (8·16) the conditions for single-valued stress resultant and displacement components become

$$[\Omega'(z)]_C=0,\quad[\omega''(z)]_C=0,\quad[\kappa\Omega(z)-\bar{\omega}'(\bar{z})]_C=0, \quad (8\cdot19)$$

$$\left.\begin{aligned}&[\Delta'(z)]_C=0,\quad[\delta''(z)+B_3\bar{\Omega}(\bar{z})\,\Omega''(z)]_C=0,\\&[\kappa\Delta(z)-\bar{\delta}'(\bar{z})]_C=\left[B_4\int^{z}\{\Omega'(z)\}^2\,\mathrm{d}z-B_1'\int^{\bar{z}}\bar{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,\mathrm{d}\bar{z}+B_3\Omega(z)\,\bar{\Omega}'(\bar{z})\right]_C.\end{aligned}\right\}\quad(8\cdot20)$$

The integral terms may, if required, be removed from the second of equations (8·16) by a process similar to that employed in §7. Thus, replacing $\Delta(z)$ by $\Delta(z)+(B_4/\kappa)\int^z\{\Omega'(z)\}^2\,dz$ and $\delta'(z)$ by $\delta'(z)+B_1'\int^z\Omega'(z)\,\omega''(z)\,dz$ we obtain

$$\left.\begin{aligned}\frac{\partial\,{}^1\phi(z,\bar z)}{\partial\bar z}&=\Delta(z)+z\bar\Delta'(\bar z)+\bar\delta'(\bar z)+B_3\Omega(z)\,\bar\Omega'(\bar z)\\&\quad+\frac{B_1}{\kappa+1}\Gamma_2(z,\bar z)-B_5 z\{\bar\Omega'(\bar z)\}^2+B_1'\int^{\bar z}\bar\Omega'(\bar z)\,\bar\omega''(\bar z)\,d\bar z+\frac{B_4}{\kappa}\int^z\{\Omega'(z)\}^2\,dz,\\{}^1D(z,\bar z)&=\kappa\Delta(z)-z\bar\Delta'(\bar z)-\bar\delta'(\bar z)-B_3\Omega(z)\,\bar\Omega'(\bar z)-\frac{B_1}{\kappa+1}\Lambda_2(z,\bar z)+B_6 z\{\bar\Omega'(\bar z)\}^2,\end{aligned}\right\}\tag{8·21}$$

where

$$\left.\begin{aligned}B_5&=B_1-B_4/\kappa\\&=\{(3c_1+2)(7c_1+2)(11c_1+6)-4c_2(3c_1^2-4c_1+4)\\&\qquad-4c_3(c_1+1)(27c_1^2+20c_1+4)-16c_4\}/\{2(5c_1+2)(3c_1+2)^2\},\\B_6&=B_1''-B_4/\kappa\\&=\{(3c_1+2)(51c_1^2+42c_1+8)-4c_2(9c_1^2+6c_1+4)\\&\qquad-4c_3(c_1+1)(21c_1^2+18c_1+4)-8c_4\}/\{(5c_1+2)(3c_1+2)^2\}.\end{aligned}\right\}\tag{8·22}$$

The conditions (8·20) for single-valued stress resultants and displacements now, however, reduce to

$$\left.\begin{aligned}&[\Delta'(z)]_C=0,\quad[\delta''(z)+B_3\bar\Omega(\bar z)\,\Omega''(z)]_C=0,\\&[\kappa\Delta(z)-\bar\delta'(\bar z)]_C=B_3[\Omega(z)\,\bar\Omega'(\bar z)]_C.\end{aligned}\right\}\tag{8·23}$$

To obtain the corresponding results for complex co-ordinates $(\zeta,\bar\zeta)$ in the undeformed body we may again assume an expansion of the form (7·20) for D, and the formulae (7·22), (7·23) may then be applied with $\Gamma_2(\zeta,\bar\zeta)$, $\Lambda_2(\zeta,\bar\zeta)$ replacing $\Gamma_1(\zeta,\bar\zeta)$, $\Lambda_1(\zeta,\bar\zeta)$ respectively. The first approximation stress and displacement functions are thus given by (8·9) with $\zeta,\bar\zeta$ replacing $z,\bar z$, and for the second-order terms we have from (8·16)

$$\left.\begin{aligned}\frac{\partial\,{}^1\phi(z,\bar z)}{\partial\bar z}&=\Delta(\zeta)+\zeta\bar\Delta'(\bar\zeta)+\bar\delta'(\bar\zeta)+B_3\Omega(\zeta)\,\bar\Omega'(\bar\zeta)+\frac{B_1'}{\kappa+1}\Gamma_2(\zeta,\bar\zeta)-B_1\zeta\{\bar\Omega'(\bar\zeta)\}^2,\\{}^1D'(\zeta,\bar\zeta)&=\kappa\Delta(\zeta)-\zeta\bar\Delta'(\bar\zeta)-\bar\delta'(\bar\zeta)-B_3\Omega(\zeta)\,\bar\Omega'(\bar\zeta)\\&\quad-\frac{B_1'}{\kappa+1}\Lambda_2(\zeta,\bar\zeta)+B_1''\zeta\{\bar\Omega'(\bar\zeta)\}^2+B_1'\int^{\bar\zeta}\bar\Omega'(\bar\zeta)\,\bar\omega''(\bar\zeta)\,d\bar\zeta-B_4\int^{\zeta}\{\Omega'(\zeta)\}^2\,d\zeta.\end{aligned}\right\}\tag{8·24}$$

Alternative expressions for $\partial^1\phi(z,\bar z)/\partial\bar z$ and ${}^1D'(\zeta,\bar\zeta)$ may be obtained from (8·21). The conditions for single-valued stress resultants and displacements are again given by (8·19), (8·20) and (8·23) with $\zeta,\bar\zeta$ replacing $z,\bar z$. The stress resultants and the resultant force and couple across a curve in the deformed body may now be obtained in terms of complex potential functions by combining the expressions obtained for ϕ and D with (6·17), (6·19) and (6·20).

By considering the uniform dilatation of a compressible material under a finite pressure, Rivlin (1953) has shown that an incompressible material may be regarded as the limiting case of a compressible material obtained by letting $[\partial^2W/\partial J_1^2]_0$ and $[\partial^2W/\partial J_1\partial J_2]_0$ tend to

infinity in such a manner that their difference remains finite. Comparison of the form of W for a compressible material with that for an incompressible material as far as the terms of the third order of smallness then yields

$$\left[\frac{\partial W}{\partial J_2}\right]_0 = -(C_1+C_2), \quad \left[\frac{\partial W}{\partial J_3}\right]_0 = -(C_1+2C_2), \tag{8·25}$$

where C_1, C_2 are the Mooney constants defined for equation (7·4). Thus, in the results of the present section the passage to the incompressible case may be achieved by inserting the conditions

$$\left.\begin{aligned} c_1\to\infty,\quad c_2\to\infty,\quad c_1/c_2\to 1,\\ c_3=\frac{C_1+2C_2}{C_1+C_2}=\tfrac{1}{2}(3-\alpha). \end{aligned}\right\} \tag{8·26}$$

For example, by introducing (8·26) into (8·16) we obtain (7·12) and (7·14), and the alternative equations for an incompressible material may be obtained from the corresponding relations for a compressible material in a similar manner.

PLANE STRAIN

9. Approximate theory for compressible materials

In the theory of finite plane strain developed by Adkins *et al.* (1953), the application of approximation methods was confined to deformations of incompressible materials. In the present section, the corresponding results will be obtained for compressible materials prior to a general formulation of the second-order theory of elasticity for two-dimensional problems.

Employing the notation of §2, we suppose the elastic body to be deformed by a uniform finite extension parallel to the x_3-axis with constant extension ratio λ_0, and that subsequently the body receives a finite plane strain parallel to the (x_1, x_2) plane. Thus if we choose the moving curvilinear co-ordinate θ_3 so that $\theta_3 = y_3$ then

$$\left.\begin{aligned} x_3 = y_3/\lambda_0 = \theta_3/\lambda_0,\\ \text{and}\quad x_\alpha = x_\alpha(\theta_1,\theta_2),\quad y_\alpha = y_\alpha(\theta_1,\theta_2,t). \end{aligned}\right\} \tag{9·1}$$

Comparing (9·1) with (3·1) and (5·1), we see that equations (3·2) and (3·3) again apply, and that the analysis given in §5 for compressible materials in plane stress may be repeated, with appropriate modifications, in the present instance. Thus, equations (5·2) to (5·6) are now satisfied exactly, provided we replace λ by λ_0 throughout. Also, from (5·6), (2·5), (5·15) and (5·16) we obtain

$$\left.\begin{aligned} \tau^{\alpha\beta} &= \frac{2}{\sqrt{I_3}}\left\{\frac{\partial W}{\partial J_1}+(\lambda_0^2-2)\frac{\partial W}{\partial J_2}-(\lambda_0^2-1)\frac{\partial W}{\partial J_3}\right\}a^{\alpha\beta}+\frac{2\sqrt{I_3}}{\lambda_0^2}\left\{\frac{\partial W}{\partial J_2}+(\lambda_0^2-1)\frac{\partial W}{\partial J_3}\right\}A^{\alpha\beta},\\ \tau^{33} &= \frac{2}{\sqrt{I_3}}\left\{\lambda_0^2\frac{\partial W}{\partial J_1}+\lambda_0^2(J_1+1-\lambda_0^2)\frac{\partial W}{\partial J_2}+[(1-\lambda_0^2)(J_1+1-\lambda_0^2)+J_2+J_3]\frac{\partial W}{\partial J_3}\right\}. \end{aligned}\right\} \tag{9·2}$$

The stress components may be expressed in terms of an Airy stress function ϕ by relations analogous to (4·5) and (4·6) so that we may write

$$\phi\,\|_{\alpha\beta} = \epsilon_{\alpha\gamma}\epsilon_{\beta\rho}\tau^{\gamma\rho} = (A/a)\,({}_0\epsilon_{\alpha\gamma})\,({}_0\epsilon_{\beta\rho})\,\tau^{\gamma\rho}, \tag{9·3}$$

and the relations (4·8) to (4·12) may also be employed if $\mathbf{F}$ and $\mathbf{M}$ now denote the force and couple respectively across the arc AP of the plane $y_3 = 0$ in the deformed body, measured per unit length of the y_3-axis.

The complex co-ordinate systems $(\zeta, \bar{\zeta})$ and $(z, \bar{z})$ may now be defined as in § 6 and it is at once evident that the relations (6·1) to (6·9) and (6·18) to (6·21) are again satisfied, with λ replaced throughout by λ_0. Moreover the equations of equilibrium may be expressed in forms analogous to (6·13). Thus we have

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial z^2} &= \frac{2\sqrt{I_3}}{\lambda_0^2}\left\{\frac{\partial W}{\partial J_1}+(\lambda_0^2-2)\frac{\partial W}{\partial J_2}-(\lambda_0^2-1)\frac{\partial W}{\partial J_3}\right\}\frac{\partial \bar{D}}{\partial z}\left(\frac{\partial D}{\partial z}-1\right),\\ \frac{\partial^2\phi}{\partial z\partial \bar{z}} &= \frac{1}{\lambda_0}\left\{\frac{\partial W}{\partial J_1}+\left(\frac{\sqrt{I_3}}{\lambda_0}+\lambda_0^2-2\right)\frac{\partial W}{\partial J_2}+(\lambda_0^2-1)\left(\frac{\sqrt{I_3}}{\lambda_0}-1\right)\frac{\partial W}{\partial J_3}\right\}\\ &\quad+\frac{2\sqrt{I_3}}{\lambda_0^2}\left\{\frac{\partial W}{\partial J_1}+(\lambda_0^2-2)\frac{\partial W}{\partial J_2}-(\lambda_0^2-1)\frac{\partial W}{\partial J_3}\right\}\frac{\partial D}{\partial \bar{z}}\frac{\partial \bar{D}}{\partial z},\end{aligned}\right\} \tag{9·4}$$

and these equations are sufficient for the determination of ϕ, D and $\bar{D}$. In applying approximation methods we shall confine our attention to plane strain for which $\lambda_0 = 1$, and equations (9·4) then reduce to

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial z^2} &= 2\sqrt{(I_3)}\left(\frac{\partial W}{\partial J_1}-\frac{\partial W}{\partial J_2}\right)\frac{\partial \bar{D}}{\partial z}\left(\frac{\partial D}{\partial z}-1\right),\\ \frac{\partial^2\phi}{\partial z\partial \bar{z}} &= \frac{\partial W}{\partial J_1}+(\sqrt{I_3}-1)\frac{\partial W}{\partial J_2}+2\sqrt{(I_3)}\left(\frac{\partial W}{\partial J_1}-\frac{\partial W}{\partial J_2}\right)\frac{\partial D}{\partial \bar{z}}\frac{\partial \bar{D}}{\partial z}.\end{aligned}\right\} \tag{9·5}$$

We now suppose the stress and displacement functions ϕ and D to be expanded in the forms (7·5) and (7·1), and we shall choose the constant ${}^0H$ to have the value $-2[\partial W/\partial J_2]_0$. From (8·25) we see that this choice is consistent with the value $2(C_1+C_2)$ employed by Adkins *et al.* in dealing with incompressible materials. From (6·9), with $\lambda = 1$, it is readily seen that $J_3 = 0$ and that the strain invariants J_1 and J_2 may be expanded in the forms (8·1), but now we have

$$\left.\begin{aligned}{}^0J_1 &= 2\left\{\frac{\partial\, {}^0D}{\partial z}+\frac{\partial\, {}^0\bar{D}}{\partial \bar{z}}\right\},\\ {}^1J_1 &= 2\left\{\frac{\partial\, {}^1D}{\partial z}+\frac{\partial\, {}^1\bar{D}}{\partial \bar{z}}+\left(\frac{\partial\, {}^0D}{\partial z}\right)^2+\left(\frac{\partial\, {}^0\bar{D}}{\partial \bar{z}}\right)^2+\frac{\partial\, {}^0D}{\partial z}\frac{\partial\, {}^0\bar{D}}{\partial \bar{z}}+3\frac{\partial\, {}^0D}{\partial \bar{z}}\frac{\partial\, {}^0\bar{D}}{\partial z}\right\},\\ {}^1J_2 &= \left(\frac{\partial\, {}^0D}{\partial z}+\frac{\partial\, {}^0\bar{D}}{\partial \bar{z}}\right)^2-4\frac{\partial\, {}^0D}{\partial \bar{z}}\frac{\partial\, {}^0\bar{D}}{\partial z}.\end{aligned}\right\} \tag{9·6}$$

Also, (8·3) and (8·4) again apply with $\lambda = 1$. The relations for the determination of ${}^0\phi$, ${}^1\phi$, ${}^0D$ and ${}^1D$ may be obtained by a procedure analogous to that employed in §§ 7 and 8. This process yields

$$\left.\begin{aligned}\frac{\partial^2({}^0\phi)}{\partial z^2}+\frac{\partial\, {}^0\bar{D}}{\partial z} &= 0,\\ 2\frac{\partial^2({}^0\phi)}{\partial z\partial \bar{z}}+(2c_1+1)\left(\frac{\partial\, {}^0D}{\partial z}+\frac{\partial\, {}^0\bar{D}}{\partial \bar{z}}\right) &= 0,\end{aligned}\right\} \tag{9·7}$$

and

$$\left.\begin{aligned}&\frac{\partial^2({}^1\phi)}{\partial z^2}+\frac{\partial\, {}^1\bar{D}}{\partial z} = \frac{\partial\, {}^0\bar{D}}{\partial z}\left\{2(c_1-c_2)\frac{\partial\, {}^0D}{\partial z}+(2c_1-2c_2-1)\frac{\partial\, {}^0\bar{D}}{\partial \bar{z}}\right\},\\ &2\frac{\partial^2({}^1\phi)}{\partial z\partial \bar{z}}+(2c_1+1)\left(\frac{\partial\, {}^1D}{\partial z}+\frac{\partial\, {}^1\bar{D}}{\partial \bar{z}}\right)+(2c_1+3c_2+2c_4+1)\left(\frac{\partial\, {}^0D}{\partial z}+\frac{\partial\, {}^0\bar{D}}{\partial \bar{z}}\right)^2\\ &\qquad-(2c_1+1)\frac{\partial\, {}^0D}{\partial z}\frac{\partial\, {}^0\bar{D}}{\partial \bar{z}}+(6c_1-4c_2-1)\frac{\partial\, {}^0\bar{D}}{\partial z}\frac{\partial\, {}^0D}{\partial \bar{z}} = 0.\end{aligned}\right\} \tag{9·8}$$

We may observe that these relations could have been obtained directly from the corresponding equations of (8·6) and (8·7) by putting $\lambda_0 = \lambda_1 = 0$. Moreover, since c_3 is absent from (9·8), we may infer from (8·25) that the stress and displacement functions for an incompressible material can only involve C_1 and C_2 in the form (C_1+C_2), a result obtained independently by Adkins *et al.*

Expressions for the stress and displacement functions in terms of the complex potential functions $\Omega(z)$, $\omega(z)$, $\Delta(z)$ and $\delta(z)$ may be obtained from (9·7) and (9·8) by a process similar to that employed for the corresponding equations in §§7 and 8. Thus from (9·7) we have

$$\left.\begin{aligned} {}^0\phi(z,\bar{z}) &= \bar{z}\Omega(z)+z\bar{\Omega}(\bar{z})+\omega(z)+\bar{\omega}(\bar{z}),\\ {}^0D(z,\bar{z}) &= \kappa\Omega(z)-z\bar{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z}), \end{aligned}\right\} \tag{9·9}$$

where

$$\kappa = \frac{2c_1-1}{2c_1+1} = 3-4\eta. \tag{9·10}$$

This definition of κ is commonly used in the classical infinitesimal theory of plane strain. By combining (9·9) with (9·8) we may express the equations for the determination of ${}^1\phi$ and ${}^1D$ in the forms (8·11) and (8·14), in which κ now has the value (9·10), and the other constants are given by

$$\left.\begin{aligned} B_1 &= (6c_1-4c_2-1)/(2c_1+1),\\ B_1' &= (2c_1-4c_2-1)/(2c_1+1),\\ B_2 &= \{(2c_1+1)\,(4c_1^2-3)-4(3c_2+2c_4)\}/(2c_1+1)^3,\\ B_2' &= -\{(2c_1+1)\,(4c_1^2+3)+4(3c_2+2c_4)\}/(2c_1+1)^3. \end{aligned}\right\} \tag{9·11}$$

The solution may then be completed as in §8 and the formulae there derived for the stress and displacement functions, from (8·16) onwards, now apply, provided we employ (9·10) and (9·11) to evaluate the remaining constants. Thus in (8·16), (8·19) to (8·21), (8·23) and (8·24) we now have

$$\left.\begin{aligned} B_1'' &= (14c_1-12c_2-3)/\{2(2c_1+1)\},\\ B_3 &= \{(2c_1+1)\,(8c_1^2-2c_1+3)-4c_2(4c_1^2+2c_1-3)+8c_4\}/\{2c_1(2c_1+1)^2\},\\ B_4 &= \{(2c_1+1)\,(12c_1^2+5)-4c_2(4c_1^2+4c_1-5)+16c_4\}/\{2(2c_1+1)^3\},\\ B_5 &= \{(2c_1+1)\,(2c_1-3)\,(6c_1+1)-4c_2(4c_1^2-4c_1+3)-16c_4\}/\{2(2c_1+1)^2\,(2c_1-1)\},\\ B_6 &= \{(2c_1+1)\,(8c_1^2-10c_1-1)-4c_2(4c_1^2-2c_1+1)-8c_4\}/\{(2c_1+1)^2\,(2c_1-1)\}, \end{aligned}\right\} \tag{9·12}$$

and in (8·18) the constant κ is given by (9·10).

The relations obtained by Adkins *et al.* (1953) for incompressible materials in plane strain may again be derived as limiting cases of these results by introduction of the conditions (8·26).

SECOND-ORDER THEORY FOR TWO-DIMENSIONAL PROBLEMS

10. General formulation

The similarity of the results obtained in §§7 to 9 suggests that the second-order theory for two-dimensional elasticity may be expressed in a more general form suitable for application to problems either in plane stress or plane strain. This has already been achieved to some extent for compressible materials in §§8 and 9. Thus the first two of equations (8·9), (8·11), (8·14), (8·16), (8·18) to (8·21), (8·23) and (8·24) apply for compressible materials in plane where we have [illegible] (10·2) and (10·3) [illegible] (13), which [illegible] and [illegible] comparing the resulting equations [illegible] expressions [illegible] B_1, B_2, [illegible] in terms of [illegible] expressions for the stress and displacement [illegible] (8·25) that

[illegible]

Remembering the definitions [illegible]

$\kappa = (3\lambda + \mu)$ [illegible]

$B_1 =$ [illegible]

$B_1' =$ [illegible]

$B_2 =$ [illegible]

$B_2' =$ [illegible]

[illegible]

Here, in (10·5) [illegible] of the deformed body. [illegible] (10·5) to (10·6) [illegible] appropriate values of μ and [illegible]

$$\left.\begin{aligned}
&\frac{\partial^2({}^1\phi)}{\partial z^2}+\frac{\partial\,{}^1\overline{D}}{\partial z}=\frac{\partial\,{}^0\overline{D}}{\partial z}\left\{2(k-c_2-1)\frac{\partial\,{}^0D}{\partial z}+(2k-2c_2-3)\frac{\partial\,{}^0\overline{D}}{\partial \bar{z}}+2(k-c_2-c_3)\,{}^0\lambda\right\},\\
&2\beta\frac{\partial^2({}^1\phi)}{\partial z\partial\bar{z}}-(2k+\beta)\left(\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\overline{D}}{\partial\bar{z}}\right)=\{2k+\beta+(2\beta+5)\,c_2+(\beta+1)\,c_3+2c_4\}\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\overline{D}}{\partial\bar{z}}\right)^2\\
&\qquad-(2k+\beta)\frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\overline{D}}{\partial\bar{z}}+\{6k+7\beta-4c_2-4(\beta+1)\,c_3\}\frac{\partial\,{}^0D}{\partial\bar{z}}\frac{\partial\,{}^0\overline{D}}{\partial z}\\
&\qquad-2\{(\beta-4)\,c_2+\beta c_3-2c_4\}\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\overline{D}}{\partial\bar{z}}\right){}^0\lambda-2(2\beta c_2-c_4)\,({}^0\lambda)^2,
\end{aligned}\right\}\qquad(10\cdot3)$$

where we have written $c_1+1=k$ to simplify the form of subsequent expressions. From (10·2) and (10·3) we may, by putting $\beta=k-1=c_1$ obtain (8·9), (8·7a) and (8·13) which are appropriate to the case of plane stress, while the value $\beta=-1$ yields the corresponding equations (9·9) and (9·8) for plane strain. By introducing (10·2) into (10·3) and comparing the resulting equations with (8·11) and (8·14) we may thus express the constants B_1, B_1', B_2, B_2' in terms of the parameter β. To evaluate the remaining constants which occur in the expressions for the stress and displacement functions we may observe from (8·17) and (8·22) that

$$\left.\begin{aligned}
&B_1''=B_1+\tfrac{1}{2}B_1',\quad B_3=B_1-2B_2/(\kappa+1),\quad B_4=\tfrac{1}{2}B_1'-B_2',\\
&B_5=B_1-B_4/\kappa,\quad B_6=B_1''-B_4/\kappa.
\end{aligned}\right\}\qquad(10\cdot4)$$

Remembering the definition of γ we thus obtain

$$\kappa=(2k+3\beta)/(2k+\beta),$$

$$\left.\begin{aligned}
&B_1=\{6k+7\beta-4c_2-4(\beta+1)\,c_3\}/(2k+\beta),\\
&B_1'=\{2k+3\beta-4c_2-4(\beta+1)\,c_3\}/(2k+\beta),\\
&B_2=\{(2k+\beta)\,[4(k+\beta)^2-3\beta^2]+12\beta(3\beta+4)\,c_2-12\beta^2(\beta+1)\,c_3-8c_4\}/(2k+\beta)^3,\\
&B_2'=-\{(2k+\beta)\,[4(k+\beta)^2+3\beta^2]-12\beta(3\beta+4)\,c_2+12\beta^2(\beta+1)\,c_3+8c_4\}/(2k+\beta)^3,
\end{aligned}\right\}\qquad(10\cdot5)$$

$$\left.\begin{aligned}
B_1''&=\{14k+17\beta-12c_2-12(\beta+1)\,c_3\}/\{2(2k+\beta)\},\\
B_3&=\{(2k+\beta)\,(8k^2+18k\beta+13\beta^2)-4[4k^2+6(k+2)\,\beta+11\beta^2]\,c_2\\
&\qquad-4(\beta+1)\,(4k^2+6k\beta-\beta^2)\,c_3+8c_4\}/\{2(k+\beta)\,(2k+\beta)^2\},\\
B_4&=\{(2k+\beta)\,(12k^2+24k\beta+17\beta^2)-4[4k^2+4(k+6)\,\beta+19\beta^2]\,c_2\\
&\qquad-4(\beta+1)\,(4k^2+4k\beta-5\beta^2)\,c_3+16c_4\}/\{2(2k+\beta)^3\},
\end{aligned}\right\}\qquad(10\cdot6)$$

$$\left.\begin{aligned}
B_5&=\{(2k+\beta)\,(2k+5\beta)\,(6k+5\beta)-4[4k^2+12(k-2)\,\beta-13\beta^2]\,c_2\\
&\qquad-4(\beta+1)\,(4k^2+12k\beta+11\beta^2)\,c_3-16c_4\}/\{2(2k+\beta)^2\,(2k+3\beta)\},\\
B_6&=\{(2k+\beta)\,(8k^2+26k\beta+17\beta^2)-4[4k^2+2(5k-6)\,\beta-5\beta^2]\,c_2\\
&\qquad-4(\beta+1)\,(4k^2+10k\beta+7\beta^2)\,c_3-8c_4\}/\{(2k+\beta)^2\,(2k+3\beta)\},
\end{aligned}\right\}\qquad(10\cdot7)$$

$$\gamma=\{2k+3\beta+4v(k+\beta)-4c_2-4(\beta+1)\,c_3\}/\{4(k+\beta)\},\qquad(10\cdot8)$$

where, in (10·8), $v=0$ for co-ordinates in the undeformed body and $v=1$ for co-ordinates in the deformed body. The constants for the incompressible case may be determined from (10·5) to (10·8) in any particular instance by proceeding to the limit, using (8·26), after the appropriate values of β and v have been inserted.

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Note added in proof (14 *October* 1954). The conditions (8·20) are simplified if, remembering (8·9), we replace $B_3\Omega(z)\overline{\Omega}'(\bar{z})$ by $B_3\overline{\Omega}'(\bar{z})\{{}^0D(z,\bar{z})+z\overline{\Omega}'(\bar{z})+\overline{\omega}'(\bar{z})\}/\kappa$ and $\bar{\delta}'(\bar{z})$ by $\bar{\delta}'(\bar{z})-(B_3/\kappa)\overline{\Omega}'(\bar{z})\overline{\omega}'(\bar{z})$. The terms $B_3\Omega(z)\overline{\Omega}'(\bar{z})-B_1z\{\overline{\Omega}'(\bar{z})\}^2$ and

$$-B_3\Omega(z)\overline{\Omega}'(\bar{z})+B_1''z\{\overline{\Omega}'(\bar{z})\}^2$$

in the first and second equations of (8·16) are then replaced by

$$(B_3/\kappa)\overline{\Omega}'(\bar{z})\,{}^0D(z,\bar{z})+B_3'z\{\overline{\Omega}'(\bar{z})\}^2$$

and
$$-(B_3/\kappa)\overline{\Omega}'(\bar{z})\,{}^0D(z,\bar{z})-B_3''z\{\overline{\Omega}'(\bar{z})\}^2$$

respectively, where $B_3'=B_3/\kappa-B_1$ and $B_3''=B_3/\kappa-B_1''$. The conditions (8·20) then reduce to

$$[\Delta'(z)]_C=0,\quad [\bar{\delta}''(\bar{z})]_C=0,$$

$$[\kappa\Delta(z)-\bar{\delta}'(\bar{z})]_C=\left[B_4\int^z\{\Omega'(z)\}^2dz-B_1'\int^{\bar{z}}\overline{\Omega}'(\bar{z})\overline{\omega}''(\bar{z})\,d\bar{z}\right]_C.$$

A similar remark applies to (7·12), (7·14), (7·18), (7·24), (7·25), (8·21), (10·1) and the associated conditions for single-valuedness.

References

Adkins, J. E. 1954 *Proc. Camb. Phil. Soc.* **50**, 334.
Adkins, J. E., Green, A. E. & Shield, R. T. 1953 *Phil. Trans.* A, **246**, 181.
Adkins, J. E. & Rivlin, R. S. 1952 *Phil. Trans.* A, **244**, 505.
Green, A. E. & Shield, R. T. 1950 *Proc. Roy. Soc.* A, **202**, 407.
Green, A. E. & Shield, R. T. 1951 *Phil. Trans.* A, **244**, 47.
Green, A. E. & Spratt, E. B. 1954 *Proc. Roy. Soc.* A, **224**, 347.
Green, A. E. & Wilkes, E. W. 1953 *Quart. Appl. Math. Mech.* **6**, 240.
Green, A. E. & Zerna, W. 1950 *Phil. Mag.* **41**, 313.
Green, A. E. & Zerna, W. 1954 *Theoretical elasticity.* Oxford University Press.
Mooney, M. 1940 *J. Appl. Phys.* **11**, 582.
Murnaghan, F. D. 1937 *Amer. J. Math.* **59**, 235.
Murnaghan, F. D. 1951 *Finite deformation of an elastic solid.* New York: John Wiley and Sons.
Muskhelishvili, N. I. 1953 *Some basic problems of the mathematical theory of elasticity* (translated from Russian by J. R. M. Radok). Groningen-Holland: P. Noordhoff Ltd.
Rivlin, R. S. 1948 *Phil. Trans.* A, **241**, 379.
Rivlin, R. S. 1949*a* *Proc. Camb. Phil. Soc.* **45**, 485.
Rivlin, R. S. 1949*b* *Phil. Trans.* A, **242**, 173.
Rivlin, R. S. 1953 *J. Rat. Mech. Anal.* **2**, 53.
Rivlin, R. S. & Thomas, A. G. 1951 *Phil. Trans.* A, **243**, 289.

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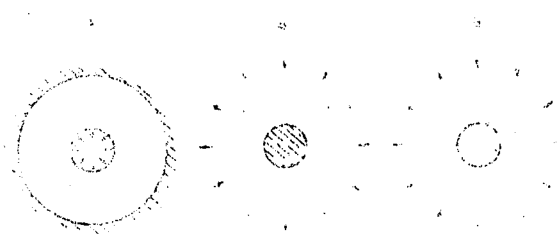

Fig. 1 [illegible] stretching of circular rubber sheet

Introduction

Among [illegible] theories of nonlinear elasticity, a well known one is [illegible] by Green and Zerna [illegible]. The framework [illegible] the Gaussian type of tensorial [illegible] relation expressed by [illegible] function which is a function of the strain invariants [illegible] for the [illegible] materials.

On the basis of [illegible] deforma[tion] in the plane of [illegible] this paper [illegible] equation governing the deformed [illegible] types of boundary [illegible]

[illegible]

... of the [illegible] variables. Since the stretch ratios are [illegible] of the differential equation is over a finite [illegible] or infinitely large sheet. In the plane of the [illegible] stretch ratios, the regions of solution for various [illegible] are clearly defined. In this plane, the direction [illegible] can be shown that the independent [illegible] or decreases. Furthermore, the [illegible] independent variable [illegible] be interchanged [illegible] foregoing observations provide the advantage of [illegible] integration from either the inner or the outer [illegible].

The [illegible] are presented for all three types of [illegible] with the strain-energy density function [illegible] of [illegible] the form

$$W = C_1(I_1 - 3) + C_2(I_2 - 3)$$

$$= C_1[(I_1 - 3) + \alpha(I_2 - 3)]$$

where [illegible] and C_1, C_2 are material constants [illegible] and $\alpha = C_2/C_1$. Various [illegible] graphically.

The [illegible] case when α is [illegible] differential equation is in a simpler [illegible] which are obtained [illegible] the nonlinear integral [illegible] Approximate Solutions.

Analysis

For a thin sheet with [illegible] must [illegible] should [illegible] of the problem the [illegible] of the hole or inclusion [illegible] are described [1].

[illegible]

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Stress Concentration in a Rubber Sheet Under Axially Symmetric Stretching

A class of axially symmetric problems, concerning a highly elastic, circular rubber sheet with (a) a centered circular hole, (b) a rigid circular inclusion under outward radial loading at outer boundary, and (c) a rigid outer boundary and a concentric hole under inward radial loading around the hole, is solved. The solution of (a) has been obtained by Rivlin and Thomas [1][1] by solving simultaneously a set of differential equations numerically. In this paper, their equations are reduced to a single second-order differential equation governing the deformation function $\rho(r)$. This is further reduced to two decoupled first-order equations after introducing the phase plane ($\lambda_1 - \lambda_2$ plane). The solutions are obtained conveniently in the phase plane by Picard's method and by straightforward numerical integration.

Introduction

Among various theories of nonlinear elasticity, a well-known one is presented comprehensively by Green and Zerna [2]. The framework of this theory is based on the Cauchy-Green type of tensorial strain measure and the constitutive relation expressed by a strain-energy density function which is a function of the strain invariants. A special case of the theory is for the incompressible materials.

On the basis of this special case, the axially symmetrical deformations in the plane of a thin rubber sheet are reformulated in this paper in terms of a second-order nonlinear differential equation governing the deformation. The solutions are obtained for three types of boundary conditions which correspond to the problems of a circular imperfection (a hole or a rigid inclusion) in the rubber sheet under outward radial stretching, and the problem of a circular sheet fixed along the outer boundary under inward radial stretching by forces applied along the boundary of a center hole, as shown in Fig. 1 (*a*, *b*, *c*). The results reveal some interesting features of stress concentration under large elastic deformations.

Rivlin and Thomas [1] have analyzed the strain distribution around a hole in a sheet under axially symmetrical deformation but have given no results on stress concentration. Their numerical solution is obtained by a forward integration method. The results in their paper have been computed for the sheet with radius three times greater than the radius of the hole in the undeformed state. In order to obtain the solution of an infinitely large sheet (or a very small hole in a large sheet), the range of integration is from the radius of the hole to a large value. Their method seems inefficient and possibly error-accumulating when a numerical integration over a large range is required.

There are definite advantages to introducing the combined single differential equation for the deformation function. By changing the dependent and the independent variables in the differential equation to a new set of variables, namely, the principal stretch ratios in the radial and circumferential directions, the equation reduces to two decoupled first-order differential equations. One of them has the two stretch ratios as the dependent and the independent variables. Since the stretch ratios are bounded, the integration of the differential equation is over a finite region even for an infinitely large sheet. In the plane of the principal stretch ratios, the regions of solution for various problems of interest are clearly defined. In this plane, the direction of integration can be chosen such that the independent variable either increases or decreases. Furthermore, the dependent and the independent variables can be interchanged arbitrarily. The foregoing observations provide the advantage of starting the integration from either the inner or the outer boundary.

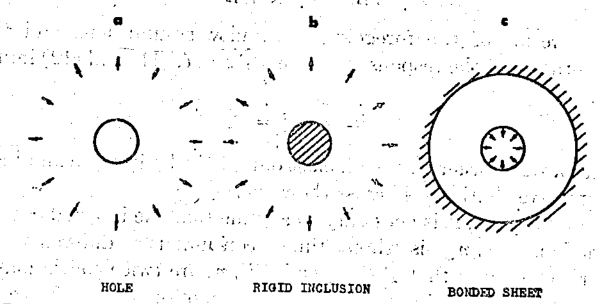

Fig. 1 Three cases of plane stretching of circular rubber sheet

The numerical solutions are presented for all three types of problems with the strain-energy density function, suggested by Mooney [3], which has the form

$$\begin{aligned} W(I_1, I_2) &= C_1(I_1 - 3) + C_2(I_2 - 3) \\ &= C_1[(I_1 - 3) + \alpha(I_2 - 3)] \end{aligned} \quad (1)$$

where I_1, I_2 are strain invariants and C_1, C_2 are material constants with dimension of force per unit area and $\alpha = C_2/C_1$. Various quantities of interest are presented graphically.

The neo-Hookean material is a special case when α is equal to zero. For this material, the differential equation is in a simpler form. Several closed-form approximate solutions are obtained for various regions in the sheet by solving the nonlinear integral equation developed in the later section, Approximate Solutions.

Analysis

For a thin sheet with dimensions of the hole or inclusion much greater than its thickness, the plane-stress assumption should lead to a good approximation. By symmetry of the problem, the cylindrical coordinate with origin at the center of the hole or inclusion is employed. The deformations under consideration are described by the mapping [1]

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[1] Numbers in brackets designate References at end of paper.

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$$\rho = \rho(r)$$
$$\Theta = \theta \qquad (2)$$
$$\eta = z\lambda_3(r)$$

where (ρ, Θ, η) and (r, θ, z) are deformed and undeformed coordinates, respectively. The principal stretch ratios in the radial and circumferential directions are, respectively,

$$\lambda_1 = \rho' = \frac{d}{dr}(\rho)$$
$$\lambda_2 = \rho/r \qquad (3)$$

and the stretch ratio in the z-direction is

$$\lambda_3 = \frac{r}{\rho\rho'} \qquad (4)$$

since, for incompressible materials, we must satisfy $\lambda_1\lambda_2\lambda_3 = 1$.

The strain invariants are

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} \qquad (5)$$
$$I_3 = 1$$

The nonzero stress components, measured per unit deformed area, are given by [1]

$$t_1 = 2(\lambda_1^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_2^2\frac{\partial W}{\partial I_2}\right)$$
$$t_2 = 2(\lambda_2^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_1^2\frac{\partial W}{\partial I_2}\right) \qquad (6)$$

in the radial and circumferential directions, respectively.

The resultants of these stresses, measured per unit length along the circumference in the respective directions, are

$$T_1 = 2h\lambda_3(\lambda_1^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_2^2\frac{\partial W}{\partial I_2}\right)$$
$$T_2 = 2h\lambda_3(\lambda_2^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_1^2\frac{\partial W}{\partial I_2}\right) \qquad (7)$$

where h is the thickness of the sheet in the undeformed state. These stress resultants must satisfy the equations of equilibrium. Two of the equations are automatically satisfied. The equation in the radial direction, without the presence of body forces, takes the form

$$\frac{d}{d\rho}(\rho T_1) = T_2 \qquad (8)$$

in the deformed coordinate. Expressing it in terms of the undeformed coordinate r, equation (8) becomes

$$\frac{d}{dr}(\rho T_1) = \rho' T_2 \qquad (9)$$

Substituting equations (7), (4), (3), and (1) into equation (9), the differential equation governing $\rho(r)$ is obtained as follows

$$\rho' - \frac{3r^2}{\rho^2\rho'^3} + \frac{3r^3}{\rho^3\rho'^2} + r\rho'' + \frac{3r^3\rho''}{\rho^2\rho'^4} - \frac{\rho}{r} + \alpha\left\{\frac{\rho^2\rho''}{r} - \frac{1}{\rho'^3} + \frac{3r\rho''}{\rho'^4} + \frac{\rho\rho'^2}{r} - \frac{\rho^2\rho'}{r^2} + \frac{r^3}{\rho^3}\right\} = 0 \qquad (10)$$

By the use of equation (3), the differential equation is reduced to one of the first order

$$\frac{d\lambda_1}{d\lambda_2} = -\frac{\lambda_1}{\lambda_2}\,\frac{3 + \lambda_1^3\lambda_2^3 + \alpha(\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 + \lambda_1^4\lambda_2^4)}{3 + \lambda_1^4\lambda_2^2 + \alpha(3\lambda_2^2 + \lambda_1^4\lambda_2^4)} \qquad (11)$$

with the companion equation

$$\frac{d\lambda_2}{\lambda_1 - \lambda_2} = \frac{dr}{r} \qquad (12)$$

The equations (11) and (12) are two decoupled first-order differential equations. After solving equation (11), for a given boundary condition, the solution of the form

$$\lambda_1 = \lambda_1(\lambda_2) \qquad (13)$$

can be used for the integration of equation (12), giving the relation of λ_2 and r. Thus the relation of ρ and r is determined, which is the solution to equation (10).

Equation (11) has no singularity. It can be integrated easily by any numerical procedure for initial value problems in ordinary differential equations. A well-known technique is the Runge-Kutta method [4]. The error of this procedure is of $0(\delta^5)$ where δ is the increment of integration.

Since λ_1 and λ_2 are nonnegative, the solution of equation (11) lies in the first quadrant of $\lambda_1 - \lambda_2$ plane. There are four special curves in this quadrant representing the solutions: (1) On the boundary of the hole; (2) at infinity; (3) on the boundary of the rigid inclusion and a fixed boundary; (4) on a circle in the rubber sheet where t_2 vanishes. These curves are shown in Fig. 2 and labeled accordingly.

The condition at the hole is $t_1 = 0$ which gives the curve (1), $\lambda_1 = \lambda_2^{-1/2}$. At infinity $\lambda_1 = \lambda_2$ which is the curve (2). For a fixed boundary, $\rho = r$ which gives the curve (3), $\lambda_2 = 1$. The function $t_2 = 0$ yields $\lambda_1 = \lambda_2^{-2}$ which is the curve (4). The solutions, in the region between curves (1) and (2), describe the

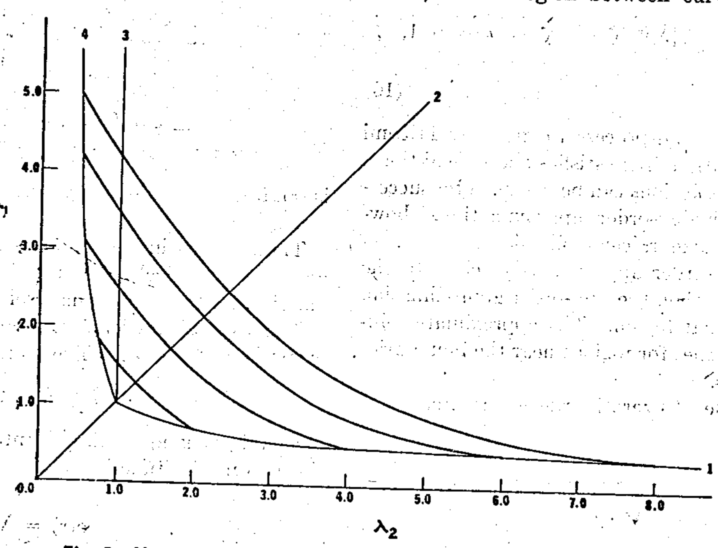

Fig. 2 Numerical solutions of the form $\lambda_1 = \lambda_1(\lambda_2)$ for $\alpha = 0.1$

$$\lambda_t^{(1)}(\lambda_\theta) = \frac{2}{3}\left(\frac{1}{\sqrt{\lambda_\theta}} - \frac{\lambda_\theta}{\sqrt{\lambda_\theta}}\right) + \frac{1}{4}(\lambda_\theta - \lambda_t) \quad \text{[illegible]}$$

is obtained from equation (16), where λ_t is a value of [illegible] [illegible] of the hole.

In the region [illegible] to the [illegible] the approximation [illegible] is chosen as

$$\lambda_t^{(0)}(\lambda_\theta) = \lambda_t$$

The first-order approximation is

$$\lambda_t^{(1)}(\lambda_\theta) = 2\lambda_\theta - \lambda_e$$

where λ_e is a value of λ_θ at $r = a$.

The curves (5) and (4) in Fig. 3 have large slopes. Th[illegible] on a fixed boundary or a summary of the [illegible]. [illegible] the [illegible] of λ_t is sensitive to the variation of λ_θ. It is desirable to [illegible] change the dependent and independent variables in equation [illegible]. This leads to the integral equation

$$\lambda_\theta(\lambda_t) = \lambda_e + \int_{\lambda_e}^{\lambda_t} F^{-1}[\xi, \lambda_\theta(\xi)]\, d\xi$$

where [illegible] is a constant value [illegible].

For the region near a fixed boundary, a zeroth-order approximation is taken as

$$\lambda_\theta^{(0)}(\lambda_t) = 1$$

which gives the first-order approximation

$$\lambda_\theta^{(1)}(\lambda_t) = 1 + \lambda_t - \lambda_f + \ln\left\{\text{[illegible]}\right\}$$

[illegible]

where λ_f is a value of λ_t on the fixed boundary [illegible]

For the region [illegible] boundary [illegible] order approximation is chosen as

$$\lambda_t^{(0)}(\lambda_\theta) = \text{[illegible]}$$

The first-order approximation

$$\lambda_t^{(1)}(\lambda_\theta) = \text{[illegible]}$$

[illegible]

is obtained [illegible] the [illegible] of [illegible] boundary [illegible] same.

These approximate solutions are plotted in Fig. [illegible] accurate numerical solutions. [illegible] that the approximate solutions are [illegible] large.

The [illegible] elasticity solution [illegible] $\lambda_\theta = 1$. The [illegible]

$$\lambda_t^{(1)} = 2\lambda_\theta - \lambda_e$$

[illegible] equation (16) [illegible] approximation [illegible]

$$a(\xi) = \text{[illegible]} \quad (5)$$

Problems in the [illegible]

[illegible] portion of a free [illegible] the sheet [illegible] [illegible]. The solution in the region [illegible] [illegible] the [illegible] wrinkled region [illegible] as shown in Fig. [illegible] [illegible] no [illegible] can be [illegible] plane stress formulation [illegible]. Physically the sheet is wrinkled in such a region. Therefore the curve [illegible] also gives the boundary [illegible] wrinkled zone.

The integration of equation [illegible] can start from any point in the first quadrant [illegible]. However, the starting point is usually on one [illegible] of the four curves where boundary conditions are given. The integration can be made in either direction (increasing or decreasing [illegible] of λ_θ). If a check in the accuracy is desired, a [illegible] integration can be made. The results shown in Fig. [illegible] obtained in this manner. [illegible] with increment [illegible]. The solutions are [illegible].

[illegible] of equation [illegible] provide the information [illegible] and strain concentration for all the problems considered. [illegible] on the [illegible], further integration of equation [illegible] required.

[illegible] for various [illegible] of interest are shown in the [illegible] Numerical [illegible].

[illegible] Solution

[illegible] (as [illegible] equation (11) [illegible]

$$\frac{d\lambda_t}{d\lambda_\theta} = -\frac{\lambda_t}{\lambda_\theta}\,\frac{3 - \lambda_t^2\lambda_\theta^2}{3 + \lambda_t^2\lambda_\theta^2} = F(\lambda_t, \lambda_\theta) \quad \text{[illegible]}$$

[illegible] equation can be changed [illegible] integral [illegible] form

$$\lambda_t(\lambda_\theta) = \lambda_t(\lambda_e) + \int_{\lambda_e}^{\lambda_\theta} F[\lambda_t(\xi), \xi]\, d\xi \quad (15)$$

[illegible] constant value of λ_θ [illegible] previous section.

[illegible] of equation (15) can be obtained by the method of [illegible] approximations,

$$\lambda_t^{(n)}(\lambda_\theta) = \lambda_t^{(n)}(\lambda_e) + \int_{\lambda_e}^{\lambda_\theta} F[\lambda_t^{(n-1)}(\xi), \xi]\, d\xi \quad \text{[illegible]}$$

(16)

[illegible] convergence conditions for this process are given by Tricomi [illegible]. The function F in equation (15) satisfies those conditions [illegible], uniformly valid solutions can be obtained by [illegible] approximations. For higher-order approximations, [illegible] integrations involve algebraic complications.

If we can choose a zeroth-order approximation close to the [illegible] solution in some region, then the first-order approximation will be accurate at least in that region. The approximate solutions are obtained in this manner for regions near the boundaries of the three types of problems.

For the region near the hole, the zeroth-order approximation is chosen as

$$\lambda_t^{(0)}(\lambda_\theta) = \frac{\text{[illegible]}}{\text{[illegible]}}$$

[illegible]

[illegible]

problem of a hole in the sheet under outward stretching. The solutions, in the region between curves (2) and (3), describe the problem of a rigid inclusion in the sheet under outward stretching. Between curves (3) and (4), there exist solutions of a circular sheet with fixed outer boundary under inward stretching forces applied around a center hole. This can be done also by considering a circular sheet with fixed outer boundary, being pulled downward at the center, through a frictionless small ring, as shown in Fig. 4. The solutions on the left side of curve (4), where $\lambda_1 < \lambda_2^{-2}$, give negative t_2. Since a thin rubber sheet has practically no resistance to in-plane compression, the plane stress formulation ceases to describe this phenomenon. Physically, the sheet is wrinkled in such a region. Therefore, the curve (4) also gives the boundaries of the wrinkled zone.

The integration of equation (11) can start from any point in the first quadrant of $\lambda_1 - \lambda_2$ plane; however, the starting point is usually on one (except $t_2 = 0$) of the four curves where boundary conditions are given. The integration can be made in either direction (increasing or decreasing direction of λ_2). If a check on the accuracy is desired, a round-trip integration can be made. The results, shown in Fig. 2, have been obtained in this manner. The error introduced after a round-trip integration with increment $\delta = 0.01$ is of the order $0(10^{-8})$. The solutions are practically exact.

The solutions of equation (11) already provide the information of stress and strain concentrations for all the problems considered. For information on the deformations, further integration of equation (12) is required.

The results for various quantities of interest are shown in the later section, Numerical Results.

Approximate Solutions

For neo-Hookean material ($\alpha = 0$), equation (11) reduces to the simpler form

$$\frac{d\lambda_1}{d\lambda_2} = -\frac{\lambda_1}{\lambda_2}\frac{3 + \lambda_1^3\lambda_2^3}{3 + \lambda_1^4\lambda_2^2} = F(\lambda_1, \lambda_2) \tag{14}$$

This differential equation can be changed to a nonlinear integral equation of the form

$$\lambda_1(\lambda_2) = \lambda_1(\lambda_0) + \int_{\lambda_0}^{\lambda_2} F[\lambda_1(\zeta), \zeta]d\zeta \tag{15}$$

where λ_0 is a constant value of λ_2 in the domain of solutions discussed in the previous section.

The solutions of equation (15) can be obtained by the method of successive approximations,

$$\lambda_1^{(n+1)}(\lambda_2) = \lambda_1^{(n)}(\lambda_0) + \int_{\lambda_0}^{\lambda_2} F[\lambda_1^{(n)}(\zeta), \zeta]d\zeta \qquad n = 0, 1, 2. \tag{16}$$

The convergence conditions for this process are given by Tricomi [5]. The function F in equation (15) satisfies those conditions. In principle, uniformly valid solutions can be obtained by successive approximations. For higher-order approximations, however, the integrations involve algebraic complications.

If we can choose a zeroth-order approximation, close to the exact solution in some region, then the first-order approximation will be accurate at least in that region. The approximate solutions are obtained in this manner for regions near the boundaries of the three types of problems.

For the region near the hole, the zeroth-order approximation is chosen as

$$\lambda_1^{(0)}(\lambda_2) = \frac{1}{\sqrt{\lambda_2}} \tag{17}$$

The first-order approximation

$$\lambda_1^{(1)}(\lambda_2) = \frac{1}{2}\left(\frac{3}{\sqrt{\lambda_2}} - \frac{1}{\sqrt{\lambda_h}}\right) + \frac{1}{4}(\lambda_h - \lambda_2) \tag{18}$$

is obtained from equation (16), where λ_h is a value of λ_2 on the boundary of the hole.

In the region, corresponding to the large value of r, a zeroth-order approximation can be chosen as

$$\lambda_1^{(0)}(\lambda_2) = \lambda_2 \tag{19}$$

The first-order approximation is

$$\lambda_1^{(1)}(\lambda_2) = 2\lambda_\infty - \lambda_2 \tag{20}$$

where λ_∞ is a value of λ_2 at $r = \infty$.

The curves (3) and (4) in Fig. 2 have large slopes. Therefore, on a fixed boundary or a boundary of the wrinkle zone, the value of λ_1 is sensitive to the variation of λ_2. It is desirable to interchange the dependent and independent variables in equation (14). This leads to the integral equation

$$\lambda_2(\lambda_1) = \lambda_2(\lambda_0) + \int_{\lambda_0}^{\lambda_1} F^{-1}[\zeta, \lambda_2(\zeta)]d\zeta \tag{21}$$

where λ_0 here is a constant value of λ_1.

For the region near a fixed boundary, a zeroth-order approximation is taken as

$$\lambda_2^{(0)}(\lambda_1) = 1 \tag{22}$$

which gives the first-order approximation

$$\lambda_2^{(1)}(\lambda_1) = 1 + \lambda_f - \lambda_1 + \ln\left\{\frac{\lambda_f}{\lambda_1}\left(\frac{3+\lambda_1^3}{3+\lambda_f^3}\right)^{1/3} \times \left(\frac{k+\lambda_1}{k+\lambda_f}\sqrt{\frac{k^2 - k\lambda_f + \lambda_f^2}{k^2 - k\lambda_1 + \lambda_1^2}}\right)^{k^{-2}}\right\} + k^{-1/2}\tan^{-1}\frac{\sqrt{3}\,k(\lambda_1 - \lambda_f)}{2k^2 - k(\lambda_1 + \lambda_f) + 2\lambda_1\lambda_f} \tag{23}$$

where λ_f is a value of λ_1 on the fixed boundary and $k = 3^{1/3}$.

For the region near the boundary of wrinkle zone, a zeroth-order approximation is chosen as

$$\lambda_2^{(0)}(\lambda_1) = \frac{1}{\sqrt{\lambda_1}} \tag{24}$$

The first-order approximation

$$\lambda_2^{(1)}(\lambda_1) = \lambda_w - \lambda_1 + \frac{2}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\lambda_w}} + \frac{8}{3k}\ln\left(\frac{k+\lambda_w}{k+\lambda_1}\sqrt{\frac{k^2 - k\lambda_1 + \lambda_1^2}{k^2 - k\lambda_w + \lambda_w^2}}\right) + 8\sqrt{3}\tan^{-1}\frac{\sqrt{3}\,k(\lambda_1 - \lambda_w)}{2k^2 - k(\lambda_1 + \lambda_w) + 2\lambda_1\lambda_w} \tag{25}$$

is obtained where λ_w is a value of λ_1 on the boundary of wrinkle zone.

These approximate solutions are plotted in Fig. 3 with the accurate numerical solutions. The results show that regions of validity, for the approximate solutions, are quite large.

The classical elasticity solutions are contained in a small region where $\lambda_1 \approx 1$, $\lambda_2 \approx 1$. The solution in the $\lambda_1 - \lambda_2$ plane is

$$\lambda_1(\lambda_2) = 2\lambda_\infty - \lambda_2, \tag{26}$$

Substituting equation (26) into equation (12) and integrating, the deformation function

$$\rho(r) = \lambda_\infty r + \frac{C}{r} \tag{27}$$

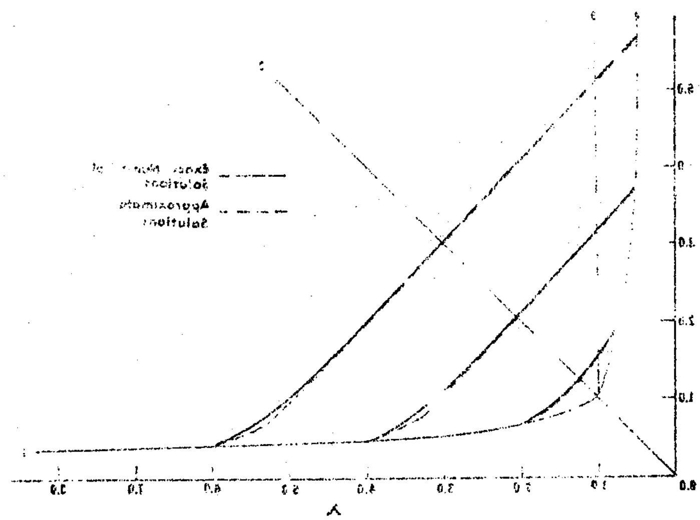

Fig. 3 Approximate and exact numerical solutions of $\lambda_1 = \lambda(\lambda_2)$ for $\alpha = 0$

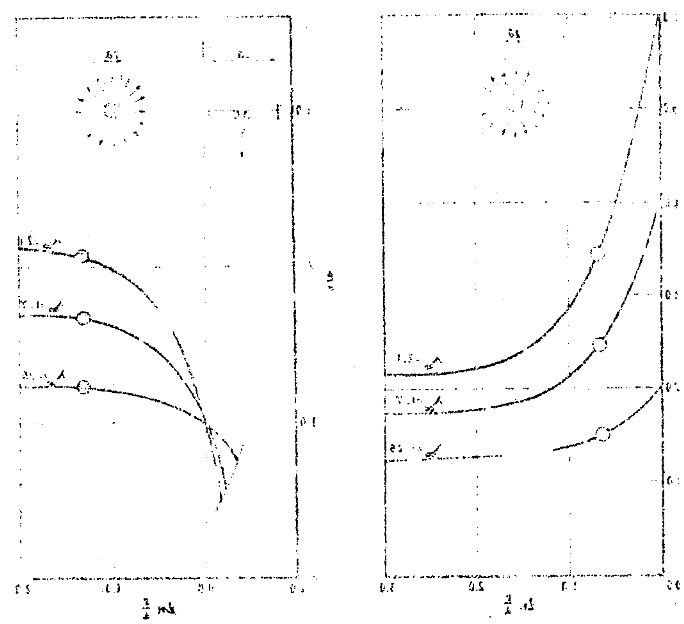

Fig. 4 Deformation $r(\rho)/\rho$ for λ = 1.25, 1.7, and 2.1 (α = 0.1)

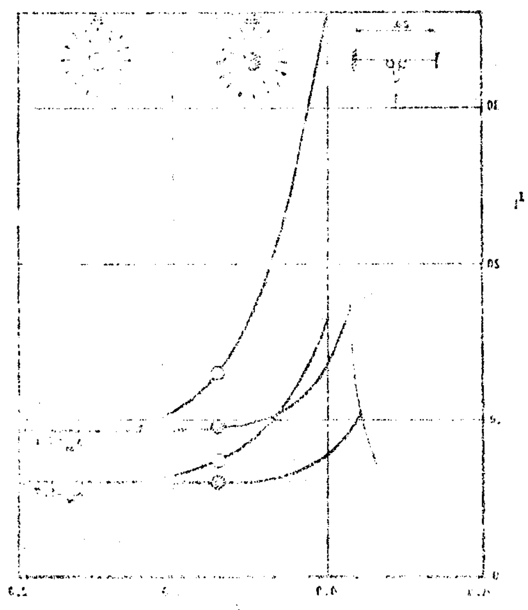

Fig. 5 Strain invariant $I_1(\rho)$ for λ = 1.25, 1.7 and 2.1 (α = 0.1)

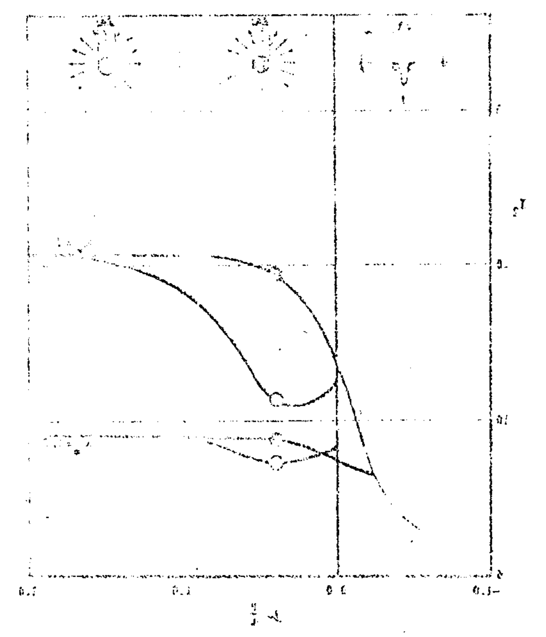

Fig. 6 Strain invariant $I_2(\rho)$ for λ = 1.25, 1.7 and 2.1 (α = 0.1)

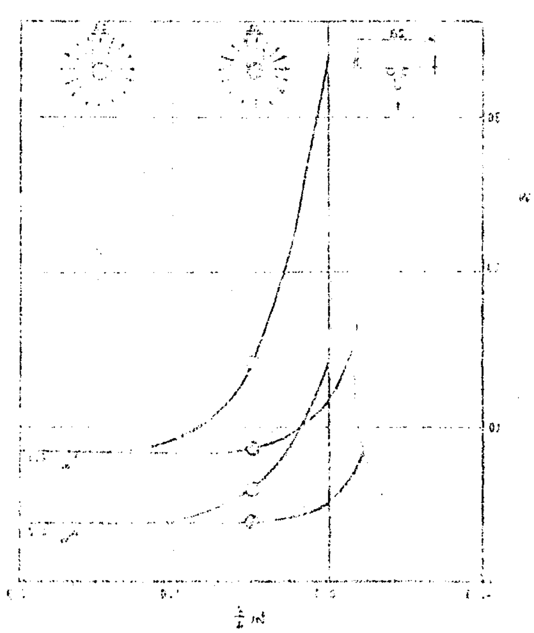

Fig. 7 Strain-energy density function $W(\rho)$ for λ = 1.25, 1.7 and 2.1 (α = 0.1)

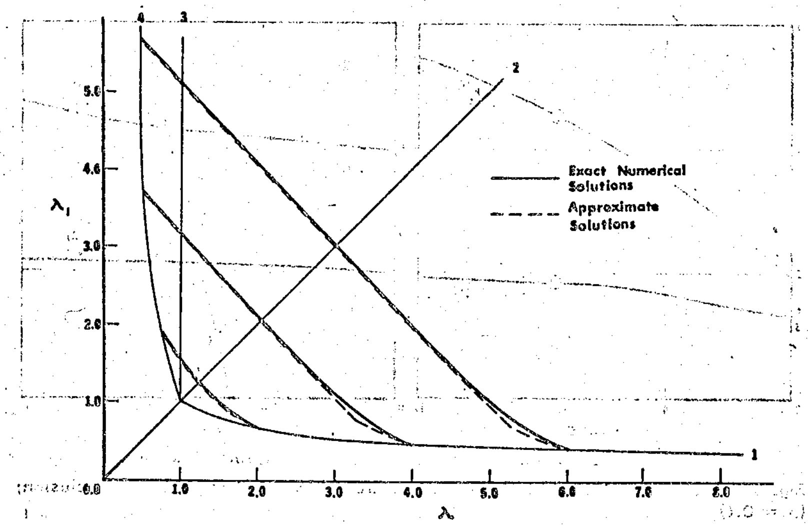

Fig. 3 Approximate and exact numerical solutions of $\lambda_1 = \lambda_1(\lambda_2)$ for $\alpha = 0$

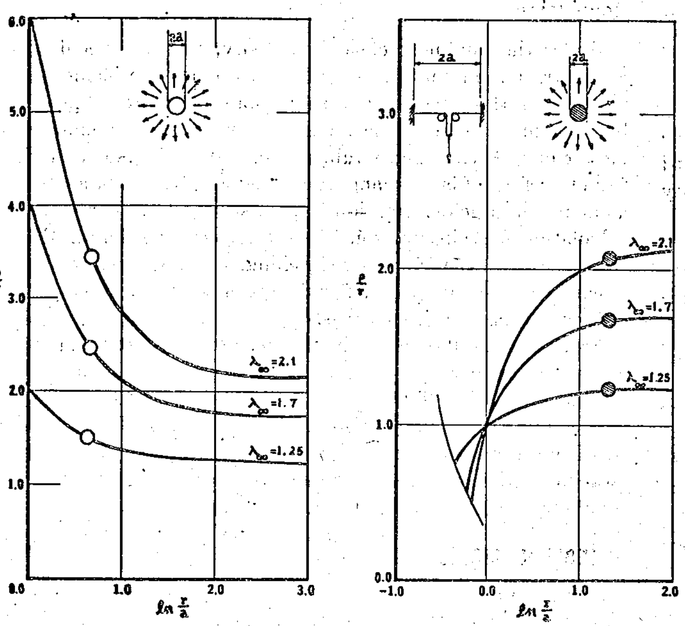

Fig. 4 Deformations $\rho(r)/r$ for $\lambda_\infty = 1.25, 1.7$, and 2.1; ($\alpha = 0.1$)

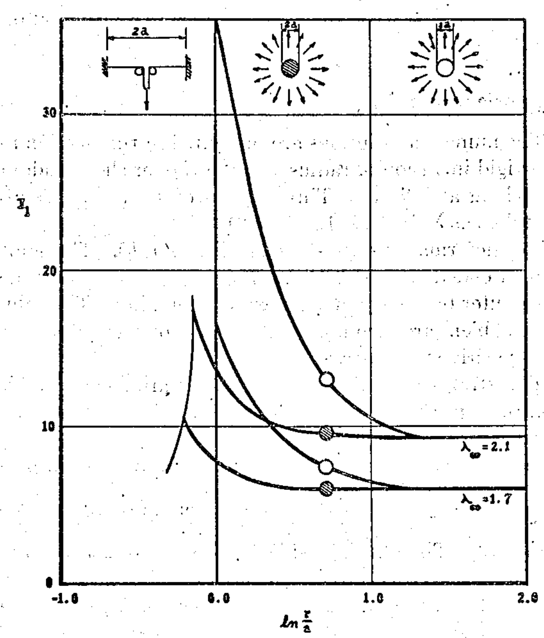

Fig. 5 Strain invariant $I_1(r)$ for $\lambda_\infty = 1.7$ and 2.1; ($\alpha = 0.1$)

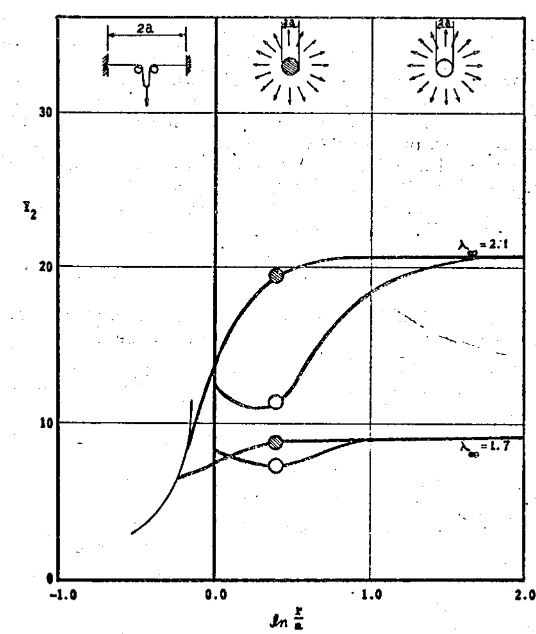

Fig. 6 Strain invariant $I_2(r)$ for $\lambda_\infty = 1.7$ and 2.1; ($\alpha = 0.1$)

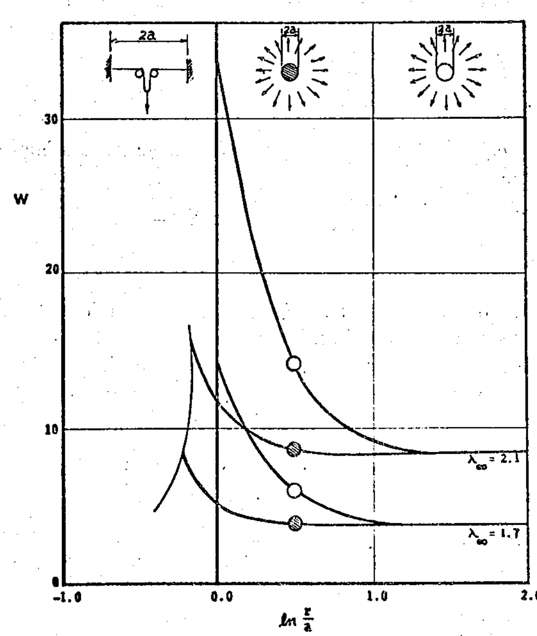

Fig. 7 Strain-energy density function $W(r)$ or $\lambda_\infty = 1.7$ and 2.1; ($\alpha = 0.1$)

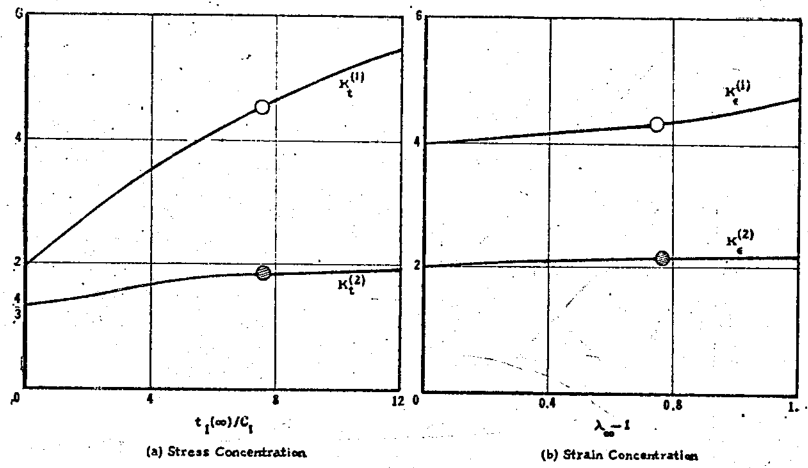

Fig. 8 Stress and strain-concentration factors around the hole and the rigid inclusion; ($\alpha = 0.1$)

for classical elasticity is recovered where C is a constant of integration.

Numerical Results

The numerical solutions are presented in this section for a hole or a rigid inclusion of radius a in the rubber sheet under outward stretching at infinity. Three values of stretch ratios at infinity are taken as $\lambda_\infty = 1.25, 1.7$, and 2.1.

The deformations are shown in Fig. 4(a, b). The solutions are also extended to the problem of a circular sheet of radius a with fixed outer boundary under inward stretching. The solutions for this problem are terminated at an envelope where t_2 changed sign from positive to negative.

The strain invariants $I_1(r)$ and $I_2(r)$ are shown in Fig. 5 and Fig. 6, respectively. The strain-energy density function $W(r)$ is shown in Fig. 7.

For the case of the hole, the results shown are in good agreement $\left(\text{for } 1 \leq \frac{r}{a} \leq 3\right)$ with the numerical solutions obtained by Rivlin and Thomas [1]. The results for larger values of r are verified by the experiments of Chu [6]. For very large value of r, the results approach the exact solution for a sheet without imperfection.

The stress-concentration factors at the boundaries of the hole and the rigid inclusion are defined as

$$K_t^{(1)} = t_2(a)/t_1(\infty)$$
$$K_t^{(2)} = t_1(a)/t_1(\infty) \tag{28}$$

respectively. The corresponding strain-concentration factors are defined as

$$K_\epsilon^{(1)} = [\lambda_2(a) - 1]/[\lambda_\infty - 1]$$
$$K_\epsilon^{(2)} = [\lambda_1(a) - 1]/[\lambda_\infty - 1] \tag{29}$$

These factors are plotted in Fig. 8(a, b) as functions of applied stress or stretch ratio at infinity.

Conclusion

Under the nonlinear elasticity theory, the stress and strain concentration factors in the plane-stretching problems depend not only on the type of the imperfections and material properties but the magnitude of the loading as well. The nonlinear behavior of the stress and strain concentration is also contributed from the effect of large change in geometry. The thickness change alone under large deformation is quite significant. It requires the calculation of redistribution of stresses over the deformed area. In the case of asymmetric stretching, other geometry changes (such as a circular hole, deforming to a noncircular shape) will further complicate the analysis.

From the results, shown in this paper, the stress and strain-concentration factors increase with increasing load for the Mooney material. The dependence of these factors on the value of α can be seen by comparison of the solutions in Fig. 2 and Fig. 3. The differences between the cases $\alpha = 0$ and $\alpha = 0.1$ are more marked when either higher-stretch ratio or load is prescribed at infinity.

Acknowledgments

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References

1 Rivlin, R. S., and Thomas, A. G., "Large Elastic Deformations of Isotropic Materials VIII. Strain Distribution Around a Hole in a Sheet," *Philosophical Transactions of the Royal Society*, London, Series A, Vol. 243, 1951, pp. 289–298.

2 Green, A. E., and Zerna, W., *Theoretical Elasticity*, The Clarendon Press, Oxford, 1954.

3 Mooney, M., *Journal of Applied Physics*, Vol. 11, 1940, p. 582.

4 Davis, H. T., *Introduction to Nonlinear Differential and Integral Equations*, Dover edition, 1962, pp. 482–486.

5 Tricomi, F. G., *Integral Equations*, Interscience, 1957, pp. 197–212.

6 Chu, B., GALCIT Report SM 65-21, Aug. 1965, Graduate Aeronautical Laboratory, California Institute of Technology.

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HONORABLE CONDUCT

in academic work is the spirit of conduct in this University.

In recognition of and in the spirit of the Honor Code, I certify that I will neither receive nor give unpermitted aid on this examination and that I will report, to the best of my ability, all Honor Code violations observed by me.

(signed) Cesar Levy
Name

SUGGESTIONS FOR CONDUCT

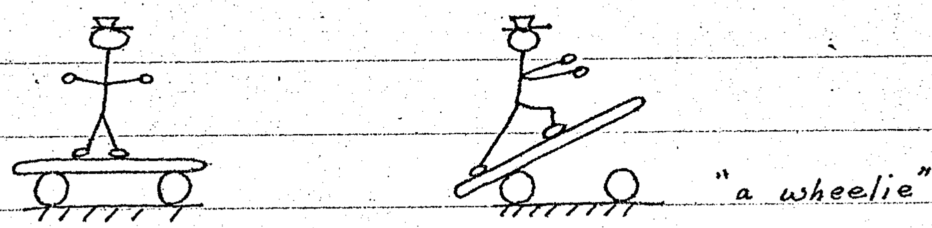
1. Occupy alternate seats where possible.
2. When in doubt as to the meaning of a question, consult the instructor, who will be found in his or her office.

M.E. 238B FINAL EXAM

MARCH 21, 1979

Open class notes and Timoshenko and Goodier's book only!

1. A playground toy consists of a slab of wood resting on two metal rollers 6" in diameter. A child weighing 100 pounds was injured when the rear roller of the toy failed while he was attempting a "wheelie."



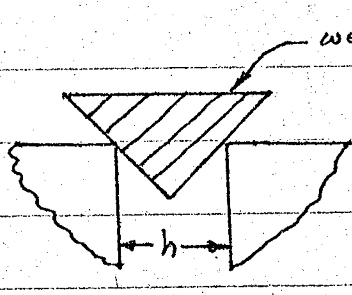
The city is being sued by the child's parents on grounds of negligence; the parents claim that the rear roller was incapable of withstanding the stresses produced in it during a "wheelie" attempt. The city hires you (for a modest sum) to perform a stress analysis of the rear roller during a "wheelie". You are told that

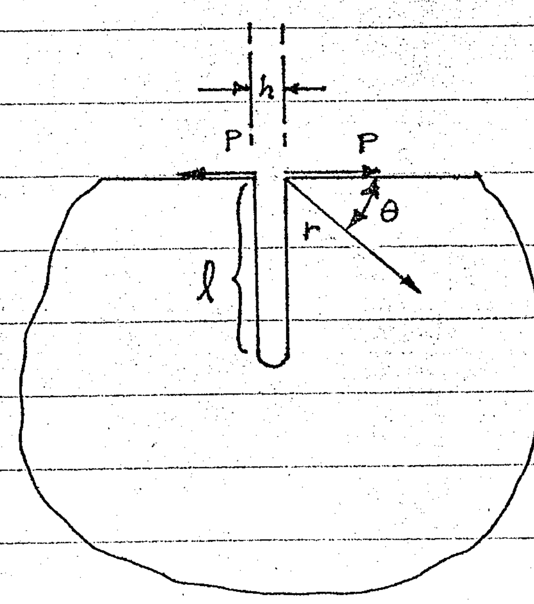
$$E_{wheel} = 29 \times 10^6 \text{ psi} \; ; \; \nu_{wheel} = 0.3 .$$

(i) If you have to appear in court tomorrow to present the results of your analysis on behalf of the city, what linear elastic problem would form the basis for your analysis. Present your model clearly and comment on how you would solve the problem

(ii) After you present the results of your analysis on the witness stand, the prosecuting attorney concedes that although your analysis may be all right for a roller made from a material whose Young's modulus is 29×10^6 psi, simple tensile tests of the actual rear roller show that its Young's modulus is only 16×10^6 psi. He claims to have available an expert witness, Professor Del Forfee, who will testify that your results are not valid for the lower modulus material. How would you defend yourself and your analysis against these charges?

2.



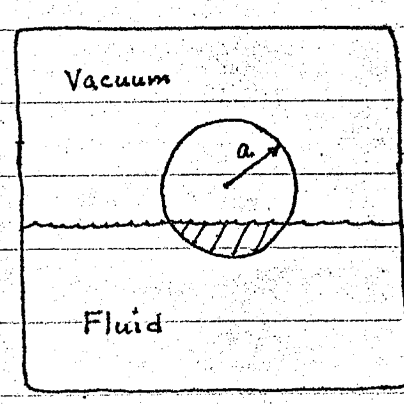


A thick, deep beam contains a slender crack of length l and width h ($h << l$). A wedge is driven into the crack to force the crack surfaces to spread apart. An approximate model of the beam might be an elastic half space containing the crack

with equal and opposite line force loads (of magnitude P per unit length normal to the paper) acting at each intersection of the crack surfaces with the half-space boundary.

(i) Compute the stress state at (r, θ) when $r \lll \ell$.

(ii) Compute the stress state at (r, θ) when $r \ggg \ell$.



3. A solid sphere of radius "a" and density ρ_o floats in an inviscid fluid of density ρ_f. Formulate a linear elastic boundary value problem from which one could, in principle, obtain the stress distribution in the sphere. Do not attempt to obtain a solution! You may consider the space above the fluid and outside the sphere to be a vacuum, i.e., neglect atmospheric pressure.

Grading:

Problem 1: 30 points

Problem 2: 40 points

Problem 3: 30 points

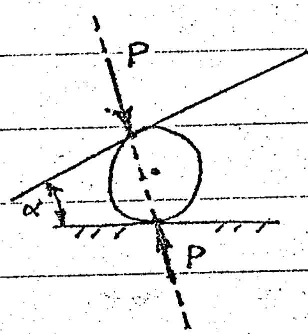
ME 238 B FINAL EXAM

ANSWER SHEET

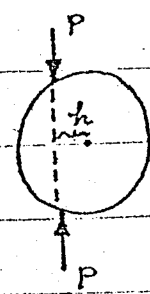
MARCH 21, 1979

1. Perform a plane strain stress analysis

(i) Draw free body diagram of rear roller during a wheelie. For simplicity neglect weight of the roller. (They aren't paying you enough to include this effect.)



For a given angle of attack "α" and weight distribution on the slab (which you would have to assume) the loadi on the roller boundary consists of two line forces which are equal and opposite to one another and which act along a line joining the point of contact of th slab and the roller with the point of contact between t roller and the ground. Hence the basic elastic probl you wish to solve is:



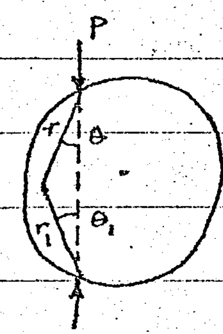
P, h, α would have to be determined by you, but their

actual values are not of importance for earning credit on this problem. For those students who did some reading of their own, this problem is treated in Timoshenko and Goodier on pages 123-126. If you did not read this section, you could still solve the problem by the same method we used in class when $h = 0$, i.e., begin by taking the solution due to a line force applied on a half-space surface,

$$\sigma_{rr} = -\frac{2P}{\pi}\frac{\cos\theta}{r}$$

and

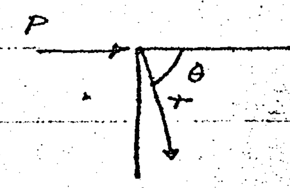
$$\sigma_{r_1 r_1} = -\frac{2P}{\pi}\frac{\cos\theta_1}{r_1}$$

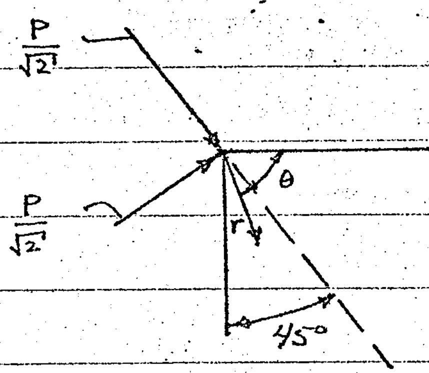


Evaluate the tractions due to the sum of these two stress states on the boundary (it will turn out to be uniform pressure on the boundary), and then remove these tractions by adding on an appropriate solution.

(ii) Since the entire problem as described in part (i) is a traction boundary value problem with self-equilibrating loading on the roller boundary, the stress solution is independent of elastic constants and your solution is good for all E, ν (The displacements in the roller will depend on E and ν.) You may wish to write the university employing Prof. Forfee urging that he be fired or retired at an early age.

2\. (i) For $r \lll l$ the field at (r, θ) is that for a wedge of apex angle $\frac{\pi}{2}$ loaded at its vertex as shown. One can write down the solution directly from the class handout or use the wedge solutions on pages 109 – 111 in Timoshenko and Goodier.





Using the T & G solutions there is only a σ_{rr} stress and

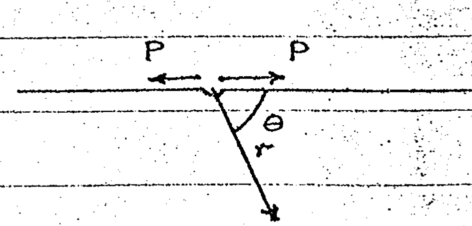
$$\sigma_{rr} = -\frac{P}{\sqrt{2}} \frac{\cos(\theta - \frac{\pi}{4})}{r\left(\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2}\right)}$$

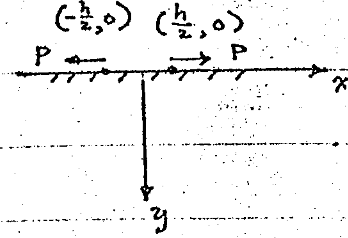
$$-\frac{P}{\sqrt{2}} \frac{\cos(\theta + \frac{\pi}{4})}{r\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right)}$$

or

$$\boxed{\sigma_{rr} = -\frac{P}{\sqrt{2}\, r}\left\{\frac{\cos(\theta - \frac{\pi}{4})}{\left(\frac{\pi}{4} + \frac{1}{2}\right)} + \frac{\cos(\theta + \frac{\pi}{4})}{\left(\frac{\pi}{4} - \frac{1}{2}\right)}\right\}}$$

(ii) For $r \ggg l$ we don't even see the crack so that the stress field at (r, θ) is the field of a double force separated by a distance $h \ll l \lll r$.





The stress function for the loading at $(\frac{h}{2},0)$ is

$$\phi^{(1)} = -\frac{P}{\pi}\, y\, \theta^{(1)} = -\frac{P}{\pi}\, y \tan^{-1}\frac{y}{x-\frac{h}{2}}$$

and for the loading at $(-\frac{h}{2},0)$ the stress function is

$$\phi^{(2)} = +\frac{P}{\pi}\, y \tan^{-1}\frac{y}{x+\frac{x}{2}}$$

The total stress fn. is (for $r \ggg \ell \gg h$)

$$\phi^{T} = \phi^{(1)} - \phi^{(2)} = -\frac{Ph}{\pi}\, y \left\{ \frac{\tan^{-1}\frac{y}{x-h/2} - \tan^{-1}\frac{y}{x+h/2}}{h} \right\}$$

$$\simeq -\frac{Ph}{\pi}\, y \left\{ \frac{\tan^{-1}\frac{y}{\xi} - \tan^{-1}\frac{y}{\xi+h}}{h} \right\} ; \quad \xi = x - \frac{h}{2}$$

$$\rightarrow +\frac{Ph}{\pi}\, y \frac{\partial}{\partial \xi}\tan^{-1}\frac{y}{\xi}$$

$$\rightarrow +\frac{Ph}{\pi}\, y \frac{(-y)}{y^2+\xi^2} \rightarrow -\frac{Ph}{\pi}\frac{y^2}{x^2+y^2} \quad \text{for } x^2+y^2 \gg h$$

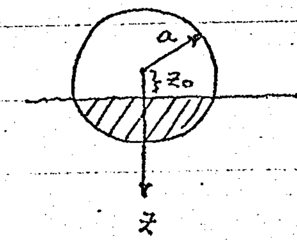
$$\text{or} \quad \phi^{T} = -\frac{Ph}{\pi}\sin^2\theta = -\frac{Ph}{2\pi}(1-\cos 2\theta)$$

$\therefore$ For $r \ggg \ell$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi^T}{\partial r^2} = 0$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi^T}{\partial \theta}\right) = \frac{Ph}{2\pi r^2}\frac{\partial}{\partial \theta}(+\cos 2\theta) = -\frac{Ph}{\pi r^2}\sin 2\theta$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi^T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = -\frac{Ph}{2\pi r^2}\, 4\cos 2\theta = -\frac{2Ph}{\pi r^2}\cos 2\theta$$



3. We need to solve the equations of equilibrium with a body force in the z-direction equal to $\rho_0 g$.

It remains only to write down the boundary conditions and determine z_0.

(i) Above the water line, i.e., $z < z_0$, the tractions vanish on $r = a$. In a spherical polar coordinate system

$$\sigma_{\rho\rho} = \sigma_{\theta\rho} = \sigma_{\varphi\rho} = 0 \quad \text{on } \rho = a \text{ for } \varphi_0 < \varphi < \pi$$

where

$$z_0 = a\cos\varphi_0 . \quad (z_0 \text{ and } \varphi_0 \text{ need to be determined})$$

(ii) Below the water line, i.e., $z > z_0$, the shear tractions on $r = a$ vanish and the normal traction is that due to the fluid pressure, i.e.,

$$\sigma_{\theta\rho} = \sigma_{\varphi\rho} = 0 \quad \text{on } \rho = a \quad \text{for } 0 < \varphi < \varphi_0$$

$$\sigma_{\rho\rho} = -P = -\rho_f g(z - z_0) \text{ on } \rho = a \quad \text{for } 0 < \varphi < \varphi_0$$

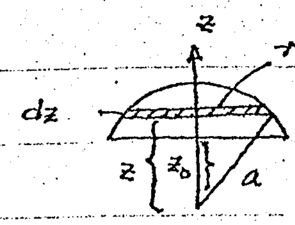
or

$$\sigma_{\rho\rho} = -\rho_f g a(\cos\varphi - \cos\varphi_0) \text{ on } \rho = a \quad \text{for } 0 < \varphi < \varphi_0 .$$

(iii) $z_0 = a\cos\varphi_0$ is determined by overall equilibrium of the sphere, i.e. the total bouyant force must equal the weight of the sphere, i.e.,

$$\frac{4}{3}\pi a^3 \rho_0 g = \rho_f g V^* ; \quad V^* = \text{volume of the shaded region in the figure}$$

$$V^* = \pi \int_{z_0}^{a} r^2 dz \quad ; \quad r^2 = a^2 - z^2$$



$$= \pi \int_{z_0}^{a} (a^2 - z^2)\, dz = \pi \left[a^2 z - \frac{z^3}{3} \right]_{z_0}^{a}$$

$$= \pi \left\{ \frac{2a^3}{3} - a^2 z_0 + \frac{z_0^3}{3} \right\}$$

or z_0 (and hence φ_0) is determined by solving the cubic equation

$$\frac{4}{3} \pi a^3 \rho_0 g = \rho_f g \pi \frac{1}{3} \left\{ z_0^3 - 3a^2 z_0 + 2a^3 \right\}$$

or

$$\left\{ z_0^3 - 3a^2 z_0 + 2a^3 \right\} = 4a^3 \frac{\rho_0}{\rho_f}$$

or

$$\boxed{\left\{ \cos^3 \varphi_0 - 3\cos\varphi_0 + 2 \right\} = 4 \frac{\rho_0}{\rho_f}}$$

One may note that the last equation may be written as

$$(1 - \cos\varphi_0)^2 (2 + \cos\varphi_0) = 4 \frac{\rho_0}{\rho_f}$$

Letting $x = 1 - \cos\varphi_0$; $0 \le x \le 2$, the eqn. becomes

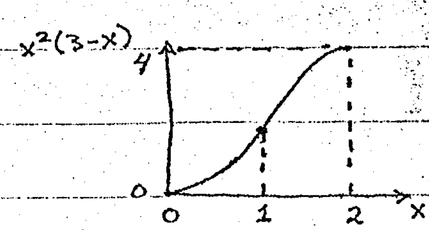
$$x^2(3 - x) = 4 \frac{\rho_0}{\rho_f}$$

Plot $x^2(3-x)$ vs. x for $0 \le x \le 2$

$f(x) = x^2(3-x)$

$f'(x) = 6x - 3x^2 = 0$ at $x = 0$; $x = 2$

$f''(x) = 6 - 6x$; inflection pt. at $x = 1$



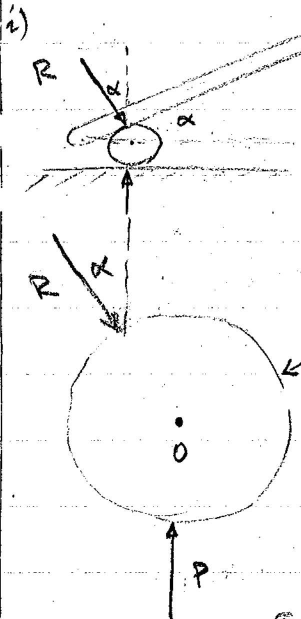
Since $4 \frac{\rho_0}{\rho_f}$ lies between 0 and 4

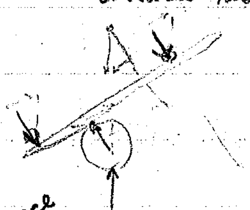
there is only <u>one</u> real solution for x

and φ_0 in the physical regime of interest.

1. The problem

i)

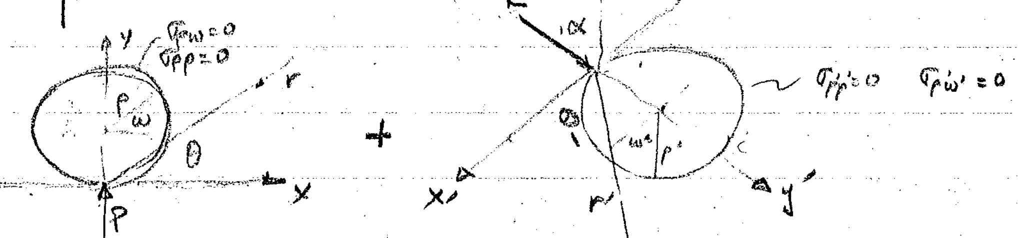




Assume the reaction R : will depend on angle of tilt of the slab to the roller and the location of the childs feet so that equilib of the bar is attained and that overall moment of system = 0

← not in force or moment equilibrium as you've shown it

can solve this problem by superposition of two problems noting that R and P act through the center of the roller



$\phi = \frac{P}{\pi}\theta r\cos\theta$ $\qquad\qquad$ $\phi = \frac{R}{\pi}\theta' r'\cos\theta'$

as we did on the mid term and in your handout of the midterm then for $\rho g \pi a^2 = P$

$$\sigma_{xx} = \rho g(a - \tfrac{1}{2}y)$$
$$\sigma_{yy} = \tfrac{1}{2}\rho g y \qquad + \quad \sigma_{rr} = \frac{-2P}{\pi r}\sin\theta$$
$$\sigma_{xy} = \tfrac{1}{2}\rho g x$$

where r = radial distance from origin

Similarly we can also get that

$$\sigma_{x'x'} = \tilde{\rho}\tilde{g}(a - \tfrac{1}{2}y') \qquad + \sigma_{r'r'} = -\frac{2R}{\pi r'}\sin\theta'$$
$$\sigma_{y'y'} = \tfrac{1}{2}\tilde{\rho}\tilde{g}y'$$
$$\sigma_{x'y'} = \tfrac{1}{2}\tilde{\rho}\tilde{g}x'$$

where now $\tilde{\rho}\tilde{g}\pi a^2 = R$
$R = R(\alpha, \text{loc of childs feet})$

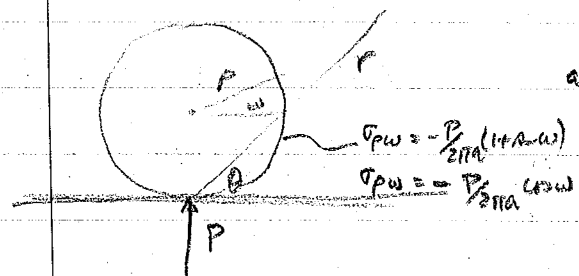
where we can now connect x' and x, y' and y by transformation of coordinates once we know the load point R on the cylinder

hence we can get that

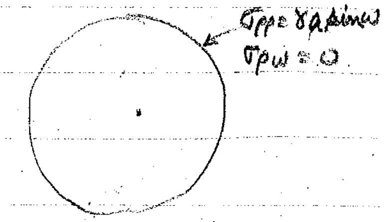
$$\sigma_{xx_{TOT}} = \Sigma\sigma_{xx}, \quad \sigma_{yy} = \Sigma\sigma_{yy} \qquad \sigma_{xy} = \Sigma\sigma_{xy}$$

where σ_{rr}, $\sigma_{r'r'}$, $\sigma_{x'x'}$, $\sigma_{y'y'}$, $\sigma_{x'y'}$ are all transformed to get the components in x y space before we take the sums

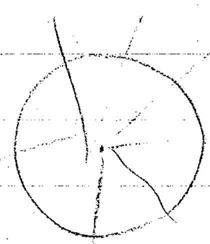
Each problem in itself was solved by taking the half space problem



adding the body force problem



adding a purely radial distribution ($\sigma_{\rho\rho} = \frac{P}{2\pi a}$ $\sigma_{\rho\omega} = 0$ $\Phi = \frac{P}{4\pi a}\rho^2$)



then adding a stress distrib function $\Phi = -\frac{P\rho^3}{4\pi a^2}\sin\omega$

that gave a $\sigma_{\rho\rho} = \frac{-2P\rho}{4\pi a^2}\sin\omega$ $\sigma_{\rho\omega} = \frac{P\rho\cos\omega}{2\pi a^2}$

to cancel the $\sigma_{\rho\rho}$ $\sigma_{\rho\omega}$ on the boundary

ii) ~~by taking a look at the maximum stress that exists on the boundary of the roller we can see that $\sigma_{\rho\rho}$ and $\sigma_{\rho\omega}$~~

~~Plane stress problem = tensile test, must convert it to plane strain by means of the fact that $E = \frac{E_0}{1-\nu^2}$~~

7/20

X

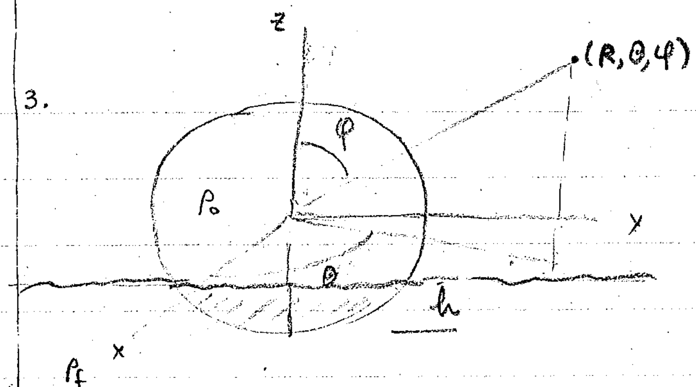
Check sample failure in a compression test

0/10

7/30

22/30

3.



6/10

1. Using archimedes principle we can do it obtain how far the sphere had sunk and hence we could find h. Since the fluid is in a vacuum we could therefore say that the pressure on any point of the immersed surface is $p_0(\theta,\varphi) = -\rho_f g(z-h)$ ✓. We also know that the outward normal is n_R ∴ $T_{iR} = 0$ above the immersed surface ∴ $T_{\theta R}, T_{\varphi R}, T_{RR} = 0$ ✓

below the immersed surface since the pressure acts ⊥ to the surface again we have that $T_{\theta R}, T_{\varphi R} = 0$ but ✓ $T_{RR} = \sigma_{RR} = -p_0(\theta,\varphi)$ ~~what is the~~

B.C.'s 9/10

$\kappa=\theta;\ \varphi=\theta$

We now ask ourselves what is the solution to the full equilib eqns:

$$\frac{\partial\sigma_{RR}}{\partial R} + \frac{1}{R\sin\varphi}\frac{\partial\sigma_{R\theta}}{\partial\theta} + \frac{1}{R}\frac{\partial\sigma_{R\varphi}}{\partial\varphi} + \frac{2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi} + \sigma_{R\varphi}\cot\varphi}{R} + F_R = 0$$

$$\frac{\partial\sigma_{R\theta}}{\partial R} + \frac{1}{R\sin\varphi}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{1}{R}\frac{\partial\sigma_{\theta\varphi}}{\partial\varphi} + \frac{3\sigma_{R\theta} + 2\sigma_{\theta\varphi}\cot\varphi}{R} + F_\theta = 0$$

$$\frac{\partial\sigma_{R\theta}}{\partial R} + \frac{1}{R\sin\varphi}\frac{\partial\sigma_{\theta\varphi}}{\partial\theta} + \frac{1}{R}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi} + \frac{3\sigma_{R\varphi} + (\sigma_{\varphi\varphi} - \sigma_{\theta\theta})\cot\varphi}{R} + F_\varphi = 0$$

what are these

7/10

with the boundary conditions $\sigma_{RR} = -p_0(\theta,\varphi)$ where θ, φ are chosen ∋ $-a \le z \le -(a-h)$.

We can apply some symmetries here since it is symmetric w r t θ ∴ all derivs w rt θ can be dropped.

THE EASIER WAY.

Another way is by using the BIE since we know what the 3-D green's functions and what the tractions are over the entire surface of the sphere.

We also know the tractions due to the Green's functions since $\sigma_{ij}^{(m)}(\underline{x},\underline{y}) = c_{ijkl}\frac{\partial G_{km}(\underline{x},\underline{y})}{\partial x_l}$. Hence we can use the descretized form of Somegliana's 2nd identity to find all the $u_m(\underline{y})$'s on the boundary. From this we can go to Somegliana's 1st identity to get the $u_m(y)$ everywhere in the sphere.

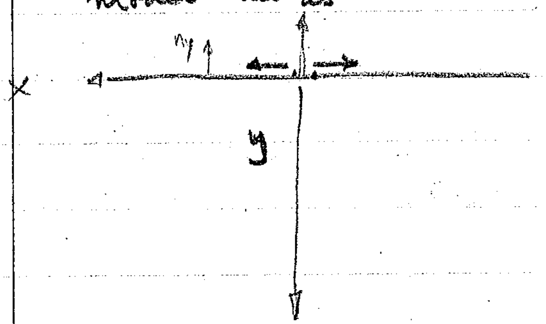
then by taking $\frac{\partial}{\partial y_p}$ of the $u_m(\underline{y})$ we can get the stresses since

$$C_{\alpha\beta mp} \frac{\partial u_m}{\partial p}(\underline{y}) = \tau_{\alpha\beta}(\underline{y})$$

20/40

2.b

for $r \gg\gg l$ the crack's influence would be almost nonexistent so we can model this as



$P\delta(x-\frac{h}{2})$ $\quad -P\delta(x+\frac{h}{2})$

at bdy
$$\begin{cases} -\sigma_{xy} = T_x = P\delta(x-\frac{h}{2}) \\ -\sigma_{xy} = -P\delta(x+\frac{h}{2}) \\ \sigma_{yy} = 0 \end{cases} \quad \therefore \quad \begin{matrix} \sigma_{xy} = -P\delta(x-h/2) = g(x) \\ \sigma_{xy} = P\delta(x+h/2) = g(x) \end{matrix} \Bigg\} \quad g(x) = P\delta(x+h/2) - P\delta(x-h/2)$$

$$\sigma_{yy}(x,y;\xi) = \frac{2P}{\pi}\int_{-\infty}^{\infty} \frac{y^2(x-\xi)\left[\delta(\xi+h/2) - \delta(\xi-h/2)\right]d\xi}{\left[(x-\xi)^2+y^2\right]^2}$$

okay

$$= \frac{2Py^2}{\pi}\left\{\frac{x+h/2}{\left[(x+h/2)^2+y^2\right]^2} - \frac{x-h/2}{\left[(x+h/2)^2+y^2\right]^2}\right\}$$

20/20

$$\cong \frac{2Py^2h}{\left[(x^2)+y^2\right]^2} = \frac{2Py^2h}{r^4} = \frac{2Ph\, r^2\sin^2\theta}{r^4}$$

$$\boxed{\sigma_{yy} = 2Ph\frac{\sin^2\theta}{r^2}}$$

where Ph = constant

✓ okay basically

$$\sigma_{xy} = \frac{2Py}{\pi}\int_{-\infty}^{\infty}\frac{(x-\xi)^2\left[\delta(\xi+h/2) - \delta(\xi-h/2)\right]d\xi}{\left[(x-\xi)^2+y^2\right]^2}$$

$$= \frac{2Py}{\pi}\frac{(x+h/2)^2-(x-h/2)^2}{\left[\quad\right]} \cong \frac{2Py}{\pi}\frac{(2x\cdot h)}{(x^2+y^2)^2}$$

$$= \frac{4Ph}{\pi}\frac{r^2\cos\theta\sin\theta}{r^4} = \boxed{\frac{2Ph}{\pi}\frac{\sin 2\theta}{r^2} = \sigma_{xy}}$$

Ph = const

$$\sigma_{xx} = \frac{2P}{\pi}\int_{-\infty}^{\infty}\frac{(x-\xi)^3\left[\delta(\xi+h/2) - \delta(\xi-h/2)\right]d\xi}{\left[(x-\xi)^2+y^2\right]^2}$$

$(a-b)^3 = (a-b)(a^2+ab+b^2)$

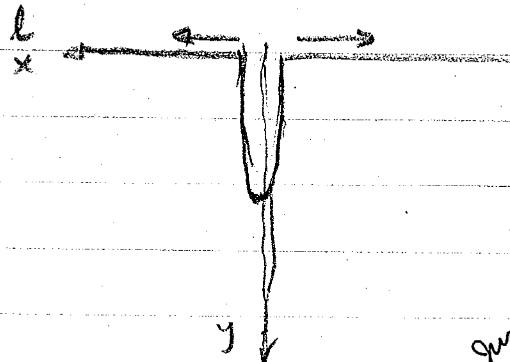
$x^2 + xh + h^2/4 + x^2 - xh + h^2/4 + x^2 - h^2/4 = 3x^2$

$$\frac{2P}{\pi}\left[\frac{(x+h/2)^3 - (x-h/2)^3}{[\quad '' \quad]^2}\right] = \frac{2P}{\pi r^4} h(3x^2)$$

$$\cong \frac{2Ph}{\pi} \cdot 3\frac{r^2\cos^2\theta}{r^4}$$

$$\boxed{\sigma_{xx} \cong \frac{6Ph}{\pi}\frac{\cos^2\theta}{r^2}}$$ $Ph = const.$

for $r \ll l$



just wedge soln.

0/20

I haven't got time to do this part

THUS WE CAN WRITE A SOLUTION TO $\nabla^4 \phi = 0$ IN POLAR COORDINATES AS: (TIMOSHENKO & GOODIER, p. 133, eq. 80)

sol for stress distr symm about origin — Vertical load on straight bdy — pure shear

$$\phi = a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0 r^2 \theta + a_0' \theta$$

radial dist along $\theta = 0, \pi$ — Portion of circular ring bent by radial force along $\theta = 0, \pi$

$$+ a_1 r \theta \sin\theta + (b_1 r^3 + a_1' r^{-1} + b_1' r \ln r)\cos\theta$$

entire line → force acting on infinite plate along $\theta = 0, \pi$; for pt force take term 1 and last term

$\theta = \pm\pi/2$

radial dist along $\theta = \pm\frac{\pi}{2}$

$$+ c_1 r \theta \cos\theta + (d_1 r^3 + c_1' r^{-1} + d_1' r \ln r)\sin\theta$$

entire line → force acting on infinite plate along $\theta = \pm\pi/2$; for pt force take term 1 and last term

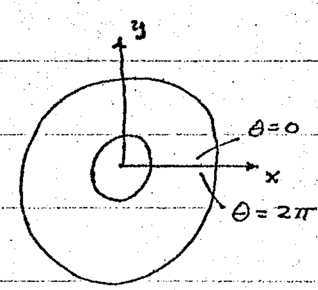
$$+ \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2}) \cos n\theta$$

shearing on ring

$$+ \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2}) \sin n\theta$$

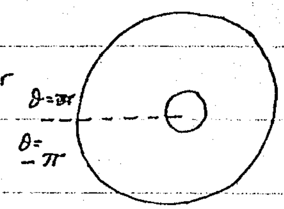
normal force on ring

LET US SEE IF WE CAN USE THIS STRESS FUNCTION (OR PIECES OF IT) TO SOLVE PROBLEMS OF LOADING ON A COMPLETE ANGULAR RING.



(1) SOMETIMES WE WILL FIND WE HAVE TO EXAMINE DISPLACEMENTS ALSO TO CHECK ON ALLOWABLE TERMS.

or



(2) SUPPOSE WE ARE INTERESTED IN SOLVING THE TRACTION BOUNDARY VALUE PROBLEM FOR A COMPLETE RING.

SUPPOSE THE BOUNDARY CONDITIONS ARE:

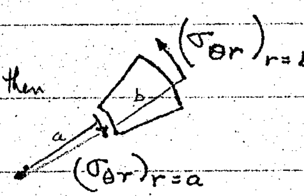
$$(\sigma_{rr})_{r=a} = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta$$

$$(\sigma_{rr})_{r=b} = A_0' + \sum_{n=1}^{\infty} A_n' \cos n\theta + \sum_{n=1}^{\infty} B_n' \sin n\theta$$

$$(\sigma_{r\theta})_{r=a} = C_0 + \sum_{n=1}^{\infty} C_n \cos n\theta + \sum_{n=1}^{\infty} D_n \sin n\theta$$

$$(\sigma_{r\theta})_{r=b} = C_0' + \sum_{n=1}^{\infty} C_n' \cos n\theta + \sum_{n=1}^{\infty} D_n' \sin n\theta$$

I. BOUNDARY LOADS MUST BE SELF-EQUILIBRATING !!



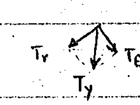
MOMENT EQUILIBRIUM since T_r gives no moments but T_θ does then

about origin
$$\int_0^{2\pi} \{ b\,\sigma_{\theta r} \}_{r=b}\, b\, d\theta - \int_0^{2\pi} \{ a\,\sigma_{\theta r} \}_{r=a}\, a\, d\theta = 0$$

NOW
$$\int_0^{2\pi} \cos n\theta\, d\theta = \int_0^{2\pi} \sin n\theta\, d\theta = 0 \quad \text{FOR } n \geq 1$$

EQUILIBRIUM OF MOMENTS REQUIRES:

$$\boxed{C_0 a^2 = C_0' b^2}$$



FORCE EQUILIBRIUM

$$T_x = T_r \cos\theta - T_\theta \sin\theta$$

$$T_y = T_r \sin\theta + T_\theta \cos\theta$$

$$T_r = \sigma_{rr}, \quad T_\theta = \sigma_{r\theta}$$

$$\frac{T_x^{NET}}{\text{unit length}} = 0 = \int_0^{2\pi} \{ \sigma_{rr} \cos\theta - \sigma_{r\theta} \sin\theta \}_{r=b} \, b \, d\theta$$

$$- \int_0^{2\pi} \{ \sigma_{rr} \cos\theta - \sigma_{r\theta} \sin\theta \}_{r=a} \, a \, d\theta$$

Only the $n=1$ terms can contribute: $\int_0^{2\pi} \cos^2\theta \, d\theta = \int_0^{2\pi} \sin^2\theta \, d\theta = \pi$

$\int_0^{2\pi} \sin\theta \cos\theta \, d\theta = 0$

$$\therefore \quad 0 = (A_1' - D_1') b - (A_1 - D_1) a$$

so

$$\boxed{b (A_1' - D_1') = a (A_1 - D_1)}$$

$T_y^{NET} = 0$ REQUIRES

$$\boxed{b (B_1' + C_1') = a (B_1 + C_1)}$$

NOW ASSUME OUR CONSTANTS ARE SUCH THAT THESE RELATIONSHIPS HOLD.

COMPUTE σ_{rr} AND $\sigma_{r\theta}$ FROM $\phi(r,\theta)$.

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

MUST CHOOSE $d_0 = 0$ FOR A COMPLETE RING

$$\sigma_{rr} = \frac{a_0}{r^2} + 2b_0 + c_0(1 + 2\ln r) + 2d_0\theta$$

$$+2\frac{a_1}{r}\cos\theta + \left(2b_1 r - \frac{2a_1'}{r^3} + \frac{b_1'}{r}\right)\cos\theta$$

$$-2\frac{c_1}{r}\sin\theta + \left(2d_1 r - \frac{2c_1'}{r^3} + \frac{d_1'}{r}\right)\sin\theta$$

$$+\sum_{n=2}^{\infty}\left\{a_n n(1-n)r^{n-2} + b_n(2+n-n^2)r^n - a_n' n(1+n)r^{-n-2} + b_n'(2-n-n^2)r^{-n}\right\}\cos n\theta$$

$$+\sum_{n=2}^{\infty}\left\{c_n n(1-n)r^{n-2} + d_n(2+n-n^2)r^n - c_n' n(1+n)r^{-n-2} + d_n'(2-n-n^2)r^{-n}\right\}\sin n\theta$$

BOUNDARY CONDITIONS ON $(\sigma_{rr})_{r=a}$ AND $(\sigma_{rr})_{r=b}$

(1) $\frac{a_0}{a^2} + 2b_0 + c_0(1 + 2\ln a) = A_0$

(2) $\frac{a_0}{b^2} + 2b_0 + c_0(1 + 2\ln b) = A_0'$

2 EQNS. — 3 UNKNOWNS

(3) $\frac{2a_1}{a} + 2b_1 a - \frac{2a_1'}{a^3} + \frac{b_1'}{a} = A_1$

(4) $+\frac{2a_1}{b} + 2b_1 b - \frac{2a_1'}{b^3} + \frac{b_1'}{b} = A_1'$

2 EQNS. — 4 UNKNOWNS

(5) $-\frac{2c_1}{a} + 2d_1 a - \frac{2c_1'}{a^3} + \frac{d_1'}{a} = B_1$

(6) $-\frac{2c_1}{b} + 2d_1 b - \frac{2c_1'}{b^3} + \frac{d_1'}{b} = B_1'$

2 EQNS — 4 UNKNOWNS

(2−n)(1+n)

$$(7)\quad a_n n(1-n)a^{n-2} + b_n(2+n-n^2)a^n - a_n' n(1+n)a^{-n-2} + b_n'(2-n-n^2)a^{-n} = A_n$$

$$(8)\quad a_n n(1-n)b^{n-2} + b_n(2+n-n^2)b^n - a_n' n(1+n)b^{-n-2} + b_n'(2-n-n^2)b^{-n} = A_n'$$

$$(9)\quad c_n n(1-n)a^{n-2} + d_n(2+n-n^2)a^n - c_n' n(1+n)a^{-n-2} + d_n'(2-n-n^2)a^{-n} = B_n$$

$$(10)\quad c_n n(1-n)b^{n-2} + d_n(2+n-n^2)b^n - c_n' n(1+n)b^{-n-2} + d_n'(2-n-n^2)b^{-n} = B_n'$$

(7) − (10): FOR EACH n, 4 EQUATIONS & 8 UNKNOWNS

NOW LOOK AT $\sigma_{r\theta}$ BOUNDARY CONDITIONS: FIRST COMPUTE $\sigma_{r\theta}$ FROM ϕ.

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right\}$$, TAKING $d_0 = 0$.

$$\sigma_{r\theta} = \frac{a_0'}{r^2}$$

$$+ \sin\theta\left\{b_1\cdot 2r - \frac{2a_1'}{r^3} + \frac{b_1'}{r}\right\}$$

$$- \cos\theta\left\{d_1\cdot 2r - \frac{2c_1'}{r^3} + \frac{d_1'}{r}\right\}$$

$$+\sum_{n=2}^{\infty} n\sin n\theta\left\{a_n(n-1)r^{n-2} + b_n(n+1)r^n - a_n'(n+1)r^{-n-2} - b_n'(n-1)r^{-n}\right\}$$

$$-\sum_{n=2}^{\infty} n\cos n\theta\left\{c_n(n-1)r^{n-2} + d_n(n+1)r^n - c_n'(n+1)r^{-n-2} - d_n'(n-1)r^{-n}\right\}$$

APPLYING BOUNDARY CONDITIONS ON $\sigma_{r\theta}$ ON $r = a, b$.

$$\frac{a_0'}{a^2} = C_0 \quad ; \quad \frac{a_0'}{b^2} = C_0' \quad \Rightarrow \quad a^2 C_0 = b^2 C_0'$$ WHICH WE FOUND BEFOR

$$\therefore \quad \boxed{a_0' = a^2 C_0 = b^2 C_0'}$$

(11) $2 b_1 a - \dfrac{2a_1'}{a^3} + \dfrac{b_1'}{a} = D_1$

(12) $2 b_1 b - \dfrac{2a_1'}{b^3} + \dfrac{b_1'}{b} = D_1'$

(3), (4), (11), (12) GIVES 4 EQNS. FOR FOUR UNKNOWNS b_1, a_1', b_1', a_1

(13) $2 d_1 a - \dfrac{2c_1'}{a^3} + \dfrac{d_1'}{a} = -C_1$

(14) $2 d_1 b - \dfrac{2c_1'}{b^3} + \dfrac{d_1'}{b} = -C_1'$

(5), (6), (13), (14) GIVES 4 EQNS. FOR FOUR UNKNOWNS d_1, c_1', d_1', c_1

(15) $n \left\{ a_n (n-1) a^{n-2} + b_n (n+1) a^n - a_n' (n+1) a^{-n-2} - b_n' (n-1) a^{-n} \right\} = D_n$

(16) $n \left\{ a_n (n-1) b^{n-2} + b_n (n+1) b^n - a_n' (n+1) b^{-n-2} - b_n' (n-1) b^{-n} \right\} = D_n'$

(7), (8), (15), (16) GIVES 4 EQNS. FOR a_n, b_n, a_n', b_n'

(17) $n \left\{ c_n (n-1) a^{n-2} + d_n (n+1) a^n - c_n' (n+1) a^{-n-2} - d_n' (n-1) a^{-n} = -C_n \right.$

(18) $n \left\{ c_n (n-1) b^{n-2} + d_n (n+1) b^n - c_n' (n+1) b^{-n-2} - d_n' (n-1) b^{-n} = -C_n' \right.$

(9), (10), (17), (18) GIVES 4 EQNS. FOR c_n, d_n, c_n', d_n'

TWO PROBLEMS: (1) & (2) STILL YIELD ONLY 2 EQNS. IN 3 UNKNOWNS

ARE WE GUARANTEED THAT:

$$b(A_1' - D_1') = a(A_1 - D_1)$$

$$b(B_1' + C_1') = a(B_1 + C_1)$$

LOOK @ SECOND QUESTION FIRST:

$$\left.\begin{array}{ll} (3)-(11) \Rightarrow & \dfrac{2a_1}{a} = A_1 - D_1 \\ (4)-(12) \Rightarrow & \dfrac{2a_1}{b} = A_1' - D_1' \end{array}\right\} \text{OKAY}$$

$$\left.\begin{array}{ll} (5)-(13) \Rightarrow & -\dfrac{2c_1}{a} = B_1 + C_1 \\ (6)-(14) \Rightarrow & -\dfrac{2c_1}{b} = B_1' + C_1' \end{array}\right\} \text{OKAY}$$

∴ WE KNOW a_1 AND c_1 FROM LOADING BOUNDARY CONDITIONS AND THIS MEANS THAT (3) & (11) REPRESENT ONLY 1 EQUATION AND (4) & (12) REPRESENT ONLY ONE EQUATION. SIMILARLY FOR (5) & (13) & (6) & (14). THUS

$$\left.\begin{array}{ll} (3)\,\&\,(11) \rightarrow & 2b_1 a - \dfrac{2a_1'}{a^3} + \dfrac{b_1'}{a} = D_1 \\ (4)\,\&\,(12) \rightarrow & 2b_1 b - \dfrac{2a_1'}{b^3} + \dfrac{b_1'}{b} = D_1' \end{array}\right\} \text{2 EQNS} \leftrightarrow \text{3 UNKNOWNS}$$

$$\left.\begin{array}{ll} (5)\,\&\,(13) \rightarrow & 2d_1 a - \dfrac{2c_1'}{a^3} + \dfrac{d_1'}{a} = -C_1 \\ (6)\,\&\,(14) \rightarrow & 2d_1 b - \dfrac{2c_1'}{b^3} + \dfrac{d_1'}{b} = -C_1' \end{array}\right\} \text{2 EQNS.} \leftrightarrow \text{3 UNKNOWN[S]}$$

$$\left.\begin{array}{ll} (1) & \dfrac{a_0}{a^2} + 2b_0 + c_0(1 + 2\ln a) = A_0 \\ (2) & \dfrac{a_0}{b^2} + 2b_0 + c_0(1 + 2\ln b) = A_0' \end{array}\right\} \text{2 EQNS.} \leftrightarrow \text{3 UNKNOW[NS]}$$

TIMOSHENKO AND GOODIER, PAGES 77-78, SHOW THAT $c_0 = 0$ FOR A COMPLETE RING OR ELSE u_θ IS MULTI-VALUED!

DERIVATION OF THE RELATION BETWEEN a_1 & b_1' AND a_1 AND d_1' BY EXAMINING THE MULTI-VALUED NATURE OF THE DISPLACEMENT FIELD (PLANE STRAI

LOOK AT THE STRESS FUNCTION

$$\phi = a_1 r\theta \sin\theta + b_1' r \ln r \cos\theta$$

THEN

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = \frac{b_1'}{r}\cos\theta$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \frac{b_1'}{r}\sin\theta$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{2a_1}{r}\cos\theta + \frac{b_1'}{r}\cos\theta = \frac{2a_1+b_1'}{r}\cos\theta$$

$$\sigma_{zz} = \nu(\sigma_{rr}+\sigma_{\theta\theta}) = \nu\,\frac{2(a_1+b_1')}{r}\cos\theta$$

NOW

$$e_{rr} = \frac{\partial u_r}{\partial r} = \frac{1}{E}\left\{\underbrace{\frac{2a_1+b_1'}{r}\cos\theta}_{\sigma_{rr}} - \nu\left[\underbrace{\frac{b_1'}{r}\cos\theta}_{\sigma_{\theta\theta}} + \nu\,\underbrace{\frac{2(a_1+b_1')}{r}\cos\theta}_{\sigma_{zz}}\right]\right\}$$

OR

$$(1)\quad \frac{\partial u_r}{\partial r} = \frac{1}{Er}\cos\theta\left\{2a_1+b_1' - \nu\left(b_1' + 2\nu(a_1+b_1')\right)\right\}$$

THUS

$$(2)\quad \frac{\partial u_r}{\partial r} = \frac{\cos\theta}{Er}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}$$

AND

$$(3)\quad u_r = \frac{\cos\theta}{E}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}\ln r + g(\theta)$$

FURTHERMORE

$$e_{rr}+e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_r}{\partial r} = \frac{1}{E}\left\{\sigma_{rr}+\sigma_{\theta\theta} - \nu\langle\sigma_{rr}+\sigma_{\theta\theta}+2\ \sigma_{zz}\rangle\right\}$$

$$= \frac{1}{E}\left\{(1-\nu)(\sigma_{rr}+\sigma_{\theta\theta}) - 2\nu^2(\sigma_{rr}+\sigma_{\theta\theta})\right\}$$

$$= \frac{1-\nu-2\nu^2}{E}(\sigma_{rr}+\sigma_{\theta\theta})$$

$$= \frac{1-\nu-2\nu^2}{E}\cdot\frac{2(a_1+b_1')}{r}\cos\theta$$

HENCE

$$\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} = \underbrace{\frac{1-\nu-2\nu^2}{E}\,\frac{2(a_1+b_1')}{r}\cos\theta}_{e_{rr}+e_{\theta\theta}} - \underbrace{\frac{\cos\theta}{Er}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}}_{\partial u_r/\partial r}$$

$$\underbrace{-\frac{\cos\theta}{Er}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}\ln r - \frac{g(\theta)}{r}}_{u_r/r}$$

OR (3) $$\frac{\partial u_\theta}{\partial \theta} = -g(\theta) + \frac{\cos\theta}{E}\left\{ a_1 \left\langle \left[-2\nu - 2\nu^2\right] - 2(1-\nu^2)\ln r \right\rangle + b_1' \left\langle 1-\nu-2\nu^2 \right\rangle \left\langle 1 - \ln r \right\rangle \right\}$$

AND

(4) $$u_\theta = -\int_\theta g(t)\,dt + f(r) + \frac{\sin\theta}{E}\left\{ a_1\left[-2\nu - 2\nu^2 - 2(1-\nu^2)\ln r\right] + b_1'(1-\nu-2\nu^2)(1-\ln r)\right\}$$

WE HAVE THE ADDITIONAL RELATION $e_{r\theta} = \frac{1}{2\mu}\sigma_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta}$

THIS REQUIRES THAT

(5) $$\frac{1+\nu}{E}\,\frac{b_1'}{r}\sin\theta = \frac{1}{2}\left\{ \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right\}$$

HENCE

$$\sin\theta\,\frac{2(1+\nu)}{E}\,\frac{b_1'}{r} = -\frac{\sin\theta}{Er}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}\ln r + \frac{1}{r}\frac{dg(\theta)}{d\theta}$$
$$+ \frac{df(r)}{dr} + \frac{\sin\theta}{E}\left\{ a_1\left(\frac{-2(1-\nu^2)}{r}\right) - \frac{b_1'}{r}(1-\nu-2\nu^2)\right\}$$
$$+ \frac{\int_\theta g(t)\,dt}{r} - \frac{f(r)}{r} - \frac{\sin\theta}{Er}\left\{ a_1\left[-2\nu - 2\nu^2 - 2(1-\nu^2)\ln\right. \right.$$
$$\left.\left. + b_1'(1-\nu-2\nu^2)(1-\ln r\right.\right.$$

AND

$$\left\{\frac{dg(\theta)}{d\theta} + \int_\theta g(t)\,dt\right\} + \left\{ r\frac{df}{dr} - f\right\}$$
$$+ \frac{\sin\theta}{E}\left\{ a_1\left[-2(1-\nu^2) + 2\nu + 2\nu^2\right] + b_1'\left[-2(1-\nu-2\nu^2) - 2(1+\nu)\right]\right\} = 0.$$

THIS IS POSSIBLE IF AND ONLY IF ??? (why not = const. *yes but only adds rigid body translat*

$$r\frac{df}{dr} - f = 0 \Rightarrow f = \alpha r$$ [THIS IS A RIGID ROTATION TERM]

AND IF

(6) $$g'(\theta) + \int_\theta g(t)\,dt + \frac{\sin\theta}{E}\left\{ a_1(-2 + 2\nu + 4\nu^2) + b_1'(-4 + 4\nu^2)\right\} = 0$$

DIFFERENTIATING (6) WITH RESPECT TO θ YIELDS

$$g''(\theta) + g(\theta) + \frac{\cos\theta}{E}\left\{ a_1\left[-2(1-2\nu)(1+\nu)\right] - 4b_1'(1-\nu^2)\right\} = 0$$

OR

$$g''(\theta) + g(\theta) = \frac{2(1+\nu)\cos\theta}{E}\left\{ a_1(1-2\nu) + 2b_1'(1-\nu)\right\} = J\cos\theta$$

WHERE

$$J = \frac{2(1+\nu)}{E}\left[a_1(1-2\nu) + 2b_1'(1-\nu)\right]$$

THE SOLUTION TO THIS DIFFERENTIAL EQUATION IS

$$g(\theta) = \alpha_0\cos\theta + \beta_0\sin\theta + \frac{J}{2}\theta\sin\theta$$

SO $g(\theta)$ AND u_r AND u_θ WILL BE MULTI-VALUED FOR A COMPLETE RING UNLESS $J \equiv 0$.

THUS

$$\boxed{b_1' = -\frac{a_1(1-2\nu)}{2(1-\nu)} \quad \text{IN PLANE STRAIN}}$$

SIMILARLY

$$\boxed{d_1' = -\frac{c_1(1-2\nu)}{2(1-\nu)} \quad \text{IN PLANE STRAIN}}$$

FOR PLANE STRESS REPLACE ν BY $\frac{\nu}{1+\nu}$ SO THAT

$$\frac{1-2\nu}{1-\nu} \rightarrow \frac{1-\frac{2\nu}{1+\nu}}{1-\frac{\nu}{1+\nu}} \rightarrow 1-\nu.$$

SINCE MY a_1 AND c_1 ARE HALF OF TIMOSHENKO AND GOODIER'S a_1 AND c_1, THE PLANE STRESS RELATIONS CHECK WITH THOSE OF T & G ON PAGE 135.

HENCE $a_1, b_1, b_1', c_1, d_1, d_1'$ CAN BE UNIQUELY DETERMINED from last 3 eq

pa 38

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = -\frac{a_0}{r^2} + 2b_0 + c_0\left[3 + 2\ln r\right] + 2d_0\theta + \left(6b_1 r + \frac{2}{r^3}a_1' + \frac{b_1'}{r}\right)\cos\theta + \left(6d_1 r + \frac{2}{r^3}c_1' + \frac{d_1'}{r}\right)\sin\theta +$$

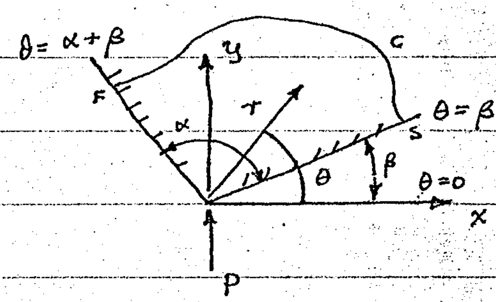
$$\sum_{n=2}^{\infty}\left[a_n(n)(n-1)r^{n-2} + b_n(n+2)(n+1)r^n + a_n'(n)(n+1)r^{-n-2} + b_n'(n-2)(n-1)r^{-n}\right]\cos n\theta$$

$$+ \sum_{n=2}^{\infty}\left[c_n(n^2-n)r^{n-2} + d_n(n^2+3n+2)r^n + c_n'(n^2+n)r^{-n-2} + d_n'(n^2-3n+2)r^{-n}\right]\sin n\theta$$

General Soln for wedge

ME 238 B HANDOUT

CONCENTRATED FORCE ACTING AT THE APEX OF A WEDGE OF ANGLE α



Pick y-direction in direction of application of the concentrated force.

Look @ the stress function

$$\phi = a_1 r\theta\cos\theta + c_1 r\theta\sin\theta$$

$$= (a_1 x + c_1 y)\theta$$

~~$\phi = a_1 r\theta\sin\theta + c_1 r\theta\cos\theta$~~

We want

$$\int_s^F T_y\, ds = -P = -\left[\frac{\partial\phi}{\partial x}\right]_s^F \qquad \Leftarrow \qquad \int_s^F T_y\, ds + P = 0$$

$$\int_s^F T_x\, ds = 0 = +\left[\frac{\partial\phi}{\partial y}\right]_s^F$$

$$(1) \qquad P = \left\{ a_1\theta + \langle a_1 x + c_1 y\rangle \frac{\partial\theta}{\partial x} \right\}_{\theta=\beta}^{\theta=\alpha+\beta}$$

$$(2) \qquad 0 = \left\{ c_1\theta + \langle a_1 x + c_1 y\rangle \frac{\partial\theta}{\partial y} \right\}_{\theta=\beta}^{\theta=\alpha+\beta}$$

Now $\dfrac{\partial\theta}{\partial x} = -\dfrac{y}{x^2+y^2} = -\dfrac{\sin\theta}{r}$; $\dfrac{\partial\theta}{\partial y} = \dfrac{x}{x^2+y^2} = \dfrac{\cos\theta}{r}$

$$x = r\cos\theta \; ; \; y = r\sin\theta \; ; \; r^2 = x^2+y^2$$

So (1) and (2) reduce to

$$(3) \qquad \left\{ a_1\theta - \langle a_1\sin\theta\cos\theta + c_1\sin^2\theta\rangle \right\}_{\theta=\beta}^{\theta=\alpha+\beta} = P \qquad 0$$

$$(4) \qquad \left\{ c_1\theta + \langle a_1\cos^2\theta + c_1\sin\theta\cos\theta\rangle \right\}_{\theta=\beta}^{\theta=\alpha+\beta} = 0 \qquad -P$$

or (5) $a_1\{\alpha+\beta-\beta\} - \frac{1}{2}\{a_1(\sin 2(\alpha+\beta) - \sin 2\beta) + c_1(1-\cos 2(\alpha+\beta)$

$-1+\cos 2\beta)\} = P$

(6) $c_1\{\alpha+\beta-\beta\} + \frac{1}{2}\{a_1[1+\cos 2(\alpha+\beta) - 1 - \cos 2\beta] + c_1[\sin 2(\alpha+\beta)$

$-\sin 2\beta] = 0$

(5) & (6) can be rewritten as

$$\left.\begin{aligned} a_1\left[\alpha - \tfrac{1}{2}K\right] + \tfrac{1}{2}c_1 J &= -P \\ \tfrac{1}{2}a_1 J + c_1\{\alpha + \tfrac{1}{2}K\} &= 0 \end{aligned}\right\} \begin{matrix} 0 \\ -P \end{matrix}$$

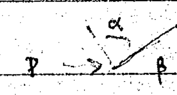
$$a_1 = \begin{vmatrix} 0 & \frac{1}{2}J \\ -P & \alpha+\frac{1}{2}K \end{vmatrix} \qquad c_1 = \begin{vmatrix} \alpha-\frac{1}{2}K & 0 \\ \frac{1}{2}J & -P \end{vmatrix}$$

where $K = \sin 2(\alpha+\beta) - \sin 2\beta$

$J = \cos 2(\alpha+\beta) - \cos 2\beta$

The solution is:

$$\boxed{\begin{aligned} a_1 &= P\,\frac{\alpha + \frac{1}{2}K}{\alpha^2 - \frac{1}{4}(K^2+J^2)} \\ c_1 &= P\,\frac{-\frac{1}{2}J}{\alpha^2 - \frac{1}{4}(K^2+J^2)} \end{aligned}}$$



$a_1 = +P\frac{1}{2}J$

$c_1 = -P(\alpha - \frac{1}{2}K)$

This can be simplified a bit since

$$\begin{aligned} K^2 + J^2 &= 2 - 2[\sin 2(\alpha+\beta)\sin 2\beta + \cos 2(\alpha+\beta)\cos 2\beta] \\ &= 2\{1 - \cos 2(\alpha+\beta-\beta)\} \\ &= 2\{1-\cos 2\alpha\} \\ &= 4\sin^2\alpha \end{aligned}$$

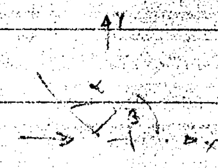
Hence $\boxed{\alpha^2 - \frac{1}{4}(K^2 + J^2) = \alpha^2 - \sin^2\alpha}$ independent of β

Furthermore

$$K = \sin 2(\alpha+\beta) - \sin 2\beta$$

Now $\sin A - \sin B = 2 \cos\frac{A+B}{2} \sin\frac{A-B}{2}$

$$\therefore \quad K = 2 \sin\frac{2(\alpha+2\beta)}{2} \cos\frac{2(\alpha+\beta-\beta)}{2} = 2 \cos(\alpha+2\beta) \sin\alpha$$



Thus $\boxed{a_1 = P\,\frac{\alpha + \cos(\alpha+2\beta)\sin\alpha}{\alpha^2 - \sin^2\alpha}}$ $\qquad a_1 = \frac{-P\sin(\alpha+2\beta)\sin\alpha}{\alpha^2 - \sin^2\alpha}$

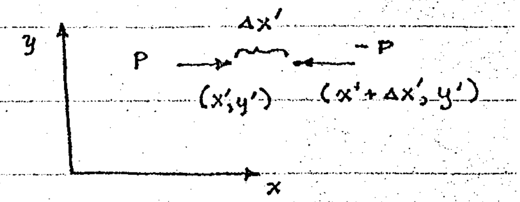
Similarly since $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$,

$$J = \cos 2(\alpha+\beta) - \cos 2\beta = -2\sin(\alpha+2\beta)\sin\alpha$$

$\boxed{c_1 = P\,\frac{\sin(\alpha+2\beta)\sin\alpha}{\alpha^2 - \sin^2\alpha}}$ $\qquad c_1 = -P\,\frac{\left[\alpha - \cos(\alpha+2\beta)\sin\alpha\right]}{\alpha^2 - \sin^2\alpha}$

ME 238B HANDOUT

LINE FORCE DIPOLES



(1) Double Force without Moment

Stress function for a line force of strength P applied at (x', y') in an infinite medium is $P\,\phi^{(x)}(x-x';\, y-y')$.

$$\phi^{(x)}_{(r,\theta)} = -\frac{1}{2\pi}\left\{ r\theta\sin\theta - \frac{1-2\nu}{1-\nu}\, r\cos\theta \ln r \right\} = -\frac{1}{2\pi}\left\{ y\tan^{-1}\frac{y}{x} - \frac{1-2\nu}{1-\nu}\,\frac{x}{2}\ln(x^2+y^2) \right.$$

Call $\hat{\phi}^{(x)}$ the solution for the Double Force configuration shown above.

Clearly

$$\phi^{(x)}(x,y;x',y') = -\frac{1}{2\pi}\left\{ (y-y')\tan^{-1}\frac{(y-y')}{(x-x')} - \frac{1-2\nu}{1-\nu}(x-x')\ln\sqrt{(x-x')^2+(y-y')^2} \right\}$$

$$\hat{\phi}^{(x)} = P\left\{ \phi^{(x)}(x-x';\, y-y') - \phi^{(x)}(x-x'-\Delta x';\, y-y') \right\}$$

$$= P\Delta x' \left\{ \phi^{(x)}(x-x';\, y-y') - \phi^{(x)}(x-x'-\Delta x';\, y-y') \right\} / \Delta x'$$

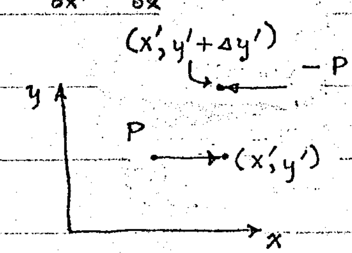
Let $P \to \infty$, $\Delta x' \to 0$ so that $P\Delta x' \to K$ a constant

$$\hat{\phi}^{(x)} = K\left\{ -\frac{\partial \phi^{(x)}}{\partial x'} \right\} = K\,\frac{\partial \phi^{(x)}}{\partial x}$$

since $\dfrac{\partial\phi}{\partial x'} = \dfrac{\partial\phi}{\partial(x-x')}\,\dfrac{\partial(x-x')}{\partial x'} = -\dfrac{\partial\phi}{\partial(x-x')}$ when $x' \to 0$ $x-x' \to$

$\therefore \dfrac{\partial\phi}{\partial x'} = -\dfrac{\partial\phi}{\partial x}$

K is the dipole strength.



(2) Double Force with Moment

Call $\tilde{\phi}^{(x)}$ the solution for this configuration.

Clearly

$$\tilde{\phi}^{(x)} = P\left\{ \phi^{(x)}(x-x';\, y-y') - \phi^{(x)}(x-x';\, y-y'-\Delta y') \right\}$$

$$= P\Delta y' \left\{ \phi^{(x)}(x-x';\, y-y') - \phi^{(x)}(x-x';\, y-y'-\Delta y') \right\} / \Delta y'$$

Let $P \to \infty$, $\Delta y' \to 0$ so that $P\Delta y' = M$

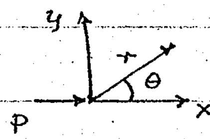
$$\tilde{\phi}^{(x)} = -M\,\frac{\partial \phi^{(x)}}{\partial y'} = M\,\frac{\partial \phi^{(x)}}{\partial y}$$

M is the dipole strength

Line Force in a Half-Space

Consider an infinite solid (plane strain deformation)

The stress function for the line of force (in z dir) shown is



$$\phi = -\frac{P}{2\pi}\left\{ r\theta\sin\theta - \frac{1-2\nu}{2(1-\nu)}\, r\cos\theta \ln r \right\}$$

as shown in annulus handout and discussion in class.

Compute the stresses:

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = +\frac{P}{2\pi}\frac{1-2\nu}{2(1-\nu)}\cos\theta\left\{ r\frac{\partial^2}{\partial r^2}\ln r + 2\frac{\partial \ln r}{\partial r}\right\}$$

$$= \frac{P}{2\pi}\frac{1-2\nu}{2(1-\nu)}\cos\theta\left\{ r\left(-\frac{1}{r^2}\right) + 2\frac{1}{r}\right\}$$

or

$$\boxed{\sigma_{\theta\theta} = \frac{P}{2\pi}\frac{1-2\nu}{2(1-\nu)}\frac{\cos\theta}{r}}$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$= -\frac{P}{2\pi}\left\{ \frac{1}{r}(\theta\sin\theta) + \frac{1}{r}(-\theta\sin\theta + 2\cos\theta) - \frac{1-2\nu}{2(1-\nu)}\cos\theta\left[\frac{1}{r}(\ln r + 1) - (r\ln r)\frac{1}{r^2}\right]\right\}$$

$$\sigma_{rr} = -\frac{P}{2\pi}\left\{ \frac{2}{r} - \frac{1-2\nu}{2(1-\nu)r}\right\}\cos\theta$$

$$\boxed{\sigma_{rr} = -\frac{P}{2\pi}\frac{1}{1-\nu}\frac{\cos\theta}{r}}$$

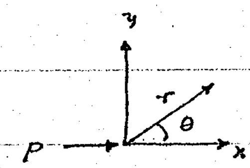
$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \frac{P}{2\pi}\frac{\partial}{\partial r}\left(-\frac{1-2\nu}{2(1-\nu)}(-\sin\theta)\right)$$

$$\boxed{\sigma_{r\theta} = \frac{P}{2\pi}\frac{1-2\nu}{2(1-\nu)}\frac{\sin\theta}{r}}$$

ME 238B HANDOUT

Line Force in a Half-Space : Plane Strain

1. Line Force at origin in an infinite medium



$$\phi = -\frac{P}{2\pi}\left\{(r\sin\theta)\theta - \frac{1-2\nu}{2(1-\nu)}\, r\cos\theta \ln r\right\}$$

$$= -\frac{P}{2\pi}\left\{y\theta - \frac{1-2\nu}{2(1-\nu)}\, x \ln r\right\}$$

$$\frac{\partial\phi}{\partial x} = -\frac{P}{2\pi}\left\{y\left(\frac{-y}{x^2+y^2}\right) - \frac{1-2\nu}{2(1-\nu)}\left[\ln r + x\cdot\frac{x}{x^2+y^2}\right]\right\}$$

$$\frac{\partial^2\phi}{\partial x^2} = -\frac{P}{2\pi}\left\{\frac{2xy^2}{(x^2+y^2)^2} - \frac{1-2\nu}{2(1-\nu)}\left[\frac{x}{x^2+y^2} + \frac{2xy^2}{(x^2+y^2)^2}\right]\right\}$$

$$= -\frac{P}{4\pi(1-\nu)}\,\frac{x}{(x^2+y^2)^2}\left[2y^2\cdot 2(1-\nu) - (1-2\nu)(x^2+y^2+2y^2)\right]$$

$$= -\frac{P}{4\pi(1-\nu)}\,\frac{x}{(x^2+y^2)^2}\left\{-(1-2\nu)x^2 + y^2\,\overset{1+2\nu}{\overbrace{\left[4(1-\nu) - 3(1-2\nu)\right]}}\right\}$$

Thus

$$\sigma_{yy} = \frac{P}{4\pi(1-\nu)}\left[\frac{x}{(x^2+y^2)^2}\right]\left\{(1-2\nu)x^2 - (1+2\nu)y^2\right\}$$

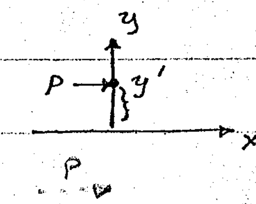
$$\frac{\partial^2\phi}{\partial x\,\partial y} = -\frac{P}{2\pi}\left\{-\frac{2x^2y}{(x^2+y^2)^2} - \frac{1-2\nu}{2(1-\nu)}\left[\frac{y}{x^2+y^2} - \frac{2x^2y}{(x^2+y^2)^2}\right]\right\}$$

$$= -\frac{P}{4\pi(1-\nu)}\,\frac{y}{(x^2+y^2)^2}\left\{-4x^2(1-\nu) - (1-2\nu)\overset{y^2-x^2}{\overbrace{(x^2+y^2-2x^2)}}\right\}$$

$$= -\frac{P}{4\pi(1-\nu)}\,\frac{y}{(x^2+y^2)^2}\left\{x^2\overset{-3+2\nu}{\overbrace{(-4+4\nu+1-2\nu)}} - (1-2\nu)y^2\right\}$$

$$\therefore\ \sigma_{xy} = -\frac{P}{4\pi(1-\nu)}\,\frac{y}{(x^2+y^2)^2}\left\{(3-2\nu)x^2 + (1-2\nu)y^2\right\}$$

Now suppose the force is applied at $(0, y')$



$$\sigma_{yy} = \frac{P}{4\pi(1-\nu)}\left[\frac{x}{\{x^2+(y-y')^2\}^2}\right]\left\{(1-2\nu)x^2 - (1+2\nu)(y-y')^2\right\}$$

$$\sigma_{xy} = -\frac{P}{4\pi(1-\nu)}\frac{y-y'}{\{x^2+(y-y')^2\}^2}\left\{(3-2\nu)x^2 + (1-2\nu)(y-y')^2\right\}$$

2. Now put a line force of strength $+P$ at the image point $(0, -y')$. Its stress field is

$$\hat{\sigma}_{yy} = \frac{P}{4\pi(1-\nu)}\frac{x}{\{x^2+(y+y')^2\}^2}\left\{(1-2\nu)x^2 - (1+2\nu)(y+y')^2\right\}$$

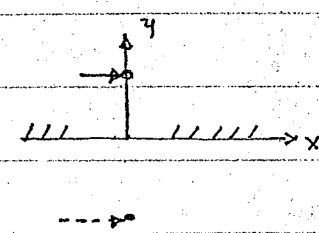
$$\hat{\sigma}_{xy} = -\frac{P}{4\pi(1-\nu)}\frac{y+y'}{\{x^2+(y+y')^2\}^2}\left\{(3-2\nu)x^2 + (1-2\nu)(y+y')^2\right\}$$

Now $\sigma^*_{xy} = \sigma_{xy} + \hat{\sigma}_{xy} = 0$ on $y = 0$

$$\sigma^*_{yy} = \sigma_{yy} + \hat{\sigma}_{yy} = \frac{P}{2\pi(1-\nu)}\frac{x}{\{x^2+y'^2\}^2}\left\{(1-2\nu)x^2 + (1+2\nu)(y')^2\right\}$$

$$= g(x) \quad \text{on} \quad y = 0$$

3. Now cut the medium along $y=0$ and apply tractions $\sigma^*_{yy} = g(x)$ on $y = 0$.



The half-space is in equilibrium.

Add to σ^*_{yy} the solution for the half-space (say $\tilde{\sigma}_{yy}$, etc.) such that

$$\tilde{\sigma}_{yy}(x, y=0) = -g(x) \quad \text{on } y = 0, \text{ and } \tilde{\sigma}_{xy}(x, y=0) = 0.$$

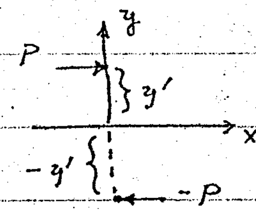
We know how to solve this problem. use Fourier transform or 2-D Green's fn

4. Another method would have been to use

$$\overset{*}{\sigma}_{yy} = \sigma_{yy} - \hat{\sigma}_{yy} \quad ; \quad \overset{*}{\sigma}_{xy} = \sigma_{xy} - \hat{\sigma}_{xy} ,$$

i.e., put a line force of strength $-P$ at the image point $(0, -y')$.

One can easily verify that



$$\overset{*}{\sigma}_{yy} = \sigma_{yy} - \hat{\sigma}_{yy} = 0 \quad \text{on} \quad y = 0$$

$$\overset{*}{\sigma}_{xy} = \sigma_{xy} - \hat{\sigma}_{xy} = \frac{2P}{4\pi(1-\nu)} \frac{y'}{(x^2+y'^2)^2} \left\{ (3-2\nu)x^2 + (1-2\nu)(y')^2 \right\}$$
$$= g(x) \quad \text{on} \quad y = 0.$$

Add to $\overset{*}{\sigma}_{xy}$, etc, the stress field $\overset{\approx}{\sigma}_{xy}$, etc., which is the solution for the half-space with

$$\overset{\approx}{\sigma}_{xy}(x, y=0) = -g(x)$$
$$\overset{\approx}{\sigma}_{yy}(x, y=0) = 0.$$

We also know how to solve this problem.

5. The net result is that we can, in principle, easily determine the stress field

$$\sigma_{ij}^{TOTAL} = \sigma_{ij} + \hat{\sigma}_{ij} + \tilde{\sigma}_{ij}$$
$$= \sigma_{ij} - \hat{\sigma}_{ij} + \overset{\approx}{\sigma}_{ij}$$

so that the half-space $y \geq 0$ contains a line of force of strength P applied at $(0, y')$, and

$$\sigma_{ij}^{TOTAL} n_j = 0 \quad \text{on} \quad y = 0$$

(i.e., $\sigma_{xy}^{TOTAL} = \sigma_{yy}^{TOTAL} = 0$ on $y = 0$)