Elasticity – deals with actions of body under applied loads. We assume that when loads are removed, body returns to its original shape without hysteresis.

[Strength & materials assumption: plane sections remain plane and are $\perp$ to $\mathbb{E}$ of beam
its a 1-D analysis – we don't take into account contraction in the direction $\perp$ to the plane of motion

Assume $\lim\limits_{\delta A \to 0} \dfrac{\delta \mathbb{P}}{\delta A} = \mathbb{t}_n \qquad \lim\limits_{\delta A \to 0} \dfrac{\delta \mathbb{M}}{\delta A} = 0$ $\qquad \mathbb{M}$ is the moment vector

where $\mathbb{t}_n$ is the traction on the surface whose normal is $\mathbb{n}$.

$\mathbb{t}_j = \sigma_{ji}\mathbb{e}_i \qquad \boldsymbol{\sigma} = \mathbb{e}_j \sigma_{ji} \mathbb{e}_i \qquad \therefore \boldsymbol{\sigma} = \mathbb{e}_j \mathbb{t}_j$

State of stress at a point is said to be known if traction on every plane passed through that point is known

traction on body whose normal is $\mathbb{n}$

P.L.M. Σ forces acting on body = the time rate of change of momentum $\int_S \mathbb{t}_n \, dS + \int_V \mathbb{f} \, dV = \int_V \rho \dfrac{\partial^2 \mathbb{u}}{\partial t^2} dV$

P.A.M. Σ moments about some point = " " " " of angular momentum $\int_S \mathbb{r} \times \mathbb{t}_n \, dS + \int_V \mathbb{r} \times \mathbb{f} \, dV = \int_V \mathbb{r} \times \rho \dfrac{\partial^2 \mathbb{u}}{\partial t^2} dV$

PLM $\Rightarrow \nabla \cdot \boldsymbol{\sigma} + \mathbb{f} = \rho \dfrac{\partial^2 \mathbb{u}}{\partial t^2} \qquad$ PAM $\Rightarrow \sigma_{ij} = \sigma_{ji}$

Compatibility – depends on geometry of system; $\nabla \times \boldsymbol{\varepsilon} \times \nabla = 0$ pointwise and $\oint$ relation over multiply connected regions "holes" for compatibility

(1) If stresses are given must satisfy compat. to get unique displ. field (to w/in rigid body motion). (2) If displ. are given, compat are uniquely satisfied.

Constitutive Eqns needed: since ε_{ij} (6 unknowns) are fns of u_i (3 indep variables). Must get 3 more eqns. Use conserv of energy.

$$\underbrace{\int_S \mathbb{t}_n \cdot \mathbb{v} \, dS}_{\text{surface traction work}} + \underbrace{\int_V \mathbb{f} \cdot \mathbb{v} \, dV}_{\text{volume work}} = \underbrace{\frac{D}{Dt}\int_V \tfrac{1}{2}\rho \mathbb{v}^2 \, dV}_{KE} + \underbrace{\frac{D}{Dt}\int \rho U_0 \, dV}_{\text{internal energy/unit mass}} \qquad \mathbb{v} = \frac{\partial \mathbb{u}}{\partial t}$$

using PLM then we get that $\int_V (\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{U}) \, dV = 0$ or $\int_V \left(\sigma_{kj} \dfrac{\partial \varepsilon_{kj}}{\partial t} - \dfrac{\partial U}{\partial t}\right) dV = 0 \qquad U = \rho U_0$

postulate $U = U(\boldsymbol{\varepsilon})$ only $\quad \therefore \dfrac{\partial U}{\partial t} = \dfrac{\partial U}{\partial \boldsymbol{\varepsilon}} \dfrac{\partial \boldsymbol{\varepsilon}}{\partial t} = \dfrac{\partial U}{\partial \varepsilon_{ij}} \dfrac{\partial \varepsilon_{ij}}{\partial t} \quad \therefore \dfrac{\partial U}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{\sigma}$

also $U(\boldsymbol{\varepsilon}) = \underbrace{U(0)}_{\text{reference state take as 0}} + \underbrace{U'(0)}_{0}\boldsymbol{\varepsilon} + \dfrac{U''(0)}{2} \boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} + \text{h.o.t}$ $\qquad U(\boldsymbol{\varepsilon}) = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}$

$\Rightarrow$ residual stresses $\exists$ when in equil state unless we take as 0

h.o.t produce non linear terms

$\therefore U(\boldsymbol{\varepsilon}) = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}$ or $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$ where $\frac{1}{2}\dfrac{\partial U}{\partial \varepsilon_{ij}} = \sigma_{ij} \;\; i \neq j \qquad \dfrac{\partial U}{\partial \varepsilon_{ij}} = \sigma_{ij} \;\; i = j$

$C_{ijkl} = 81$ const since $U = C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = C_{klij}\varepsilon_{kl}\varepsilon_{ij} = C_{klij}\varepsilon_{ij}\varepsilon_{kl} \Rightarrow C_{ijkl} = C_{klij}$

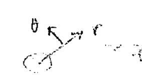
also since $\varepsilon_{ij} = \varepsilon_{ji} \Rightarrow C_{ijkl} = C_{jikl} = C_{klij} = C_{lkij} \Rightarrow$ 21 independent constants for general material

can reduce to 2 independent material constants for isotropic material

Bulk Mod $K = \frac{3\lambda + 2\mu}{3} = \frac{\sigma_{kk}}{3\varepsilon_{kk}} = \frac{\text{average pressure}}{\text{unit volume change}}$ average pressure needed to cause unit volume change.

Beltrami condition is the stress equiv of ~~biharmonic~~ compatability $\nabla^2\Theta = \nabla^2(\sigma_{kk}) = 0$ or $\nabla^4\sigma_{ij} = 0$
if no body force or accel.

Colomb Torsion assume displ field $u_r = 0$ $u_z = 0$, $u_\theta = \alpha r z$ α = twist

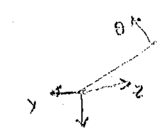

" torsion is end loaded.

St. Venant Principle – application of load produces local effect; far from pts of application two systems with equivalent static loads will have same stress field

St Venant Torsion assume end loaded stress field $\sigma_{xz}, \sigma_{yz} \neq 0$ all others $= 0$

use equil $\Rightarrow$ define ϕ $\Rightarrow$ use BC on surface to show ϕ = const on bdy / each cavity. $\Rightarrow F_x, F_y = 0$ on end,

Compatability gives $\nabla_1^2\phi = -2\mu\alpha$ α = is the twist $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$



for cavities we also have $\oint_{C_i} \frac{\partial\phi}{\partial n} ds = 2\mu\alpha A_i$ A_i = area of cavity enclosed by C
μ = rigidity

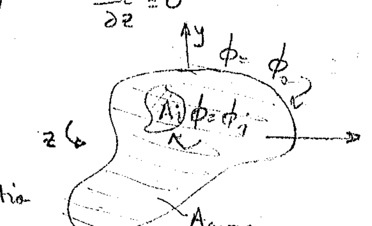
$u_x = -\alpha y z$ $u_y = \alpha x z$ $u_z = f(x,y)$

$u_r = -u_x \sin\theta + u_y \cos\theta$ $y = r\cos\theta$ $x = -r\sin\theta$ $u_x = \alpha r\cos\theta z$ $u_y = -\alpha r\sin\theta z$ $\Big\} \Rightarrow 0 = u_r$

$u_\theta = -u_x\cos\theta + u_y\sin\theta = -\alpha r\cos^2 z - \alpha r\sin^2 z = -\alpha r z$

Also $\frac{\partial u_z}{\partial x} = \frac{1}{\mu}\frac{\partial\phi}{\partial y} + \alpha y$ $\frac{\partial u_z}{\partial y} = -\left(\frac{1}{\mu}\frac{\partial\phi}{\partial x} + \alpha x\right)$ $\frac{\partial u_z}{\partial z} = 0$

Also $T = 2\left[\iint \phi\, dA - \phi_o A_{gross} + \Sigma\phi_i A_i\right]$
$= \mu\alpha J$ J = polar moment of inertia of beam about z axis



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PRINCIPAL STRESSES ARE ALWAYS OF THE FORM. $\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$

shear = 0 on planes ⊥ to principal directions

Formulas

$\mathbb{t}_i = \sigma_{ij}\,\mathbb{e}_j$ where $\sigma_{ij} = \sigma_{ji}$ $\quad \mathbb{t}_i = \mathbb{e}_i \cdot \mathbb{T}$ $\quad \mathbb{T} = \sigma_{ij}\,\mathbb{e}_i \mathbb{e}_j$ tensor

$\mathbb{t}_n = \mathbb{n}\cdot\mathbb{T} = \mathbb{T}\cdot\mathbb{n} = \mathbb{n}\cdot[\mathbb{e}_i\,\mathbb{t}_i]$ STRESS ON FACE WHOSE NORMAL IS $\mathbb{n}$

$\mathbb{T} = \mathbb{e}_i\,\mathbb{t}_j = \sigma_{ij}\,\mathbb{e}_i\mathbb{e}_j$

$\mathbb{n}\dfrac{\partial\phi}{\partial n} = \nabla\phi$ or $\dfrac{\partial\phi}{\partial n} = \mathbb{n}\cdot\nabla\phi$ RATE OF CHANGE OF ϕ WRT $\mathbb{n}$ // $\dfrac{\partial\phi}{\partial x_k} = \nabla\phi\cdot\mathbb{e}_k$

$\nabla\cdot\mathbb{T} = \nabla\cdot(\mathbb{e}_j\,\mathbb{t}_j) = \dfrac{\partial}{\partial x_j}\mathbb{t}_j = \sigma_{jk,j}\,\mathbb{e}_k$

$d\mathbb{u} = d\mathbb{r}\cdot\nabla\mathbb{u}$ and $d\phi = d\mathbb{r}\cdot\nabla\phi$

PLM $\nabla\cdot\mathbb{T} + \mathbb{f} = \rho\dfrac{\partial^2\mathbb{u}}{\partial t^2}$ or $\sigma_{jk,j} + f_k = \rho\dfrac{\partial^2 u_k}{\partial t^2}$

$\overbrace{\mathbb{e}'_m\cdot\mathbb{e}_i\sigma_{ij}\mathbb{e}_j\cdot\mathbb{e}'_n}^{\sigma}$
$\mathbb{T}' = l_{mi}\,\sigma_{ij}\,l_{nj}$

$\sigma'_{mn} = \sigma_{ij}\,l_{mi}\,l_{nj}$ where $l_{mi} = \mathbb{e}'_m\cdot\mathbb{e}_i$ STRESS TENSOR IN NEW COORD SYSTEM $l_{mi} \neq l_{im}$

$\mathbb{T}' = \mathbb{L}\,\mathbb{T}\,\mathbb{L}^T$

INVARIANTS IN TERMS OF GENERAL STRESSES <u>FROM</u> $-\lambda^3 + \mathrm{I}\lambda^2 - \mathrm{II}\lambda + \mathrm{III} = 0$ for PRINCIPALS $-\lambda^3 + \mathrm{I}'\lambda^2 - \mathrm{II}'\lambda + \mathrm{III}' = 0$

$\mathrm{I} = \sigma_{kk} = \sigma_{xx}+\sigma_{yy}+\sigma_{zz}$ $\qquad \mathrm{I}' = \mathrm{I} = \sigma_1+\sigma_2+\sigma_3$

$\mathrm{II} = \frac{1}{2}e_{mik}\,e_{njl}\,\sigma_{ij}\sigma_{kl}\,\delta_{mn} = \sigma_{xx}\sigma_{yy}+\sigma_{yy}\sigma_{zz}+\sigma_{zz}\sigma_{xx} - (\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2)$ $\qquad \mathrm{II}' = \mathrm{II} = \sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1$

$\mathrm{III} = \frac{1}{6}e_{mik}\,e_{njl}\,\sigma_{ij}\sigma_{kl}\sigma_{mn} = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - (\sigma_{xx}\sigma_{yz}^2 + \sigma_{yy}\sigma_{xz}^2 + \sigma_{zz}\sigma_{yx}^2)$ $\qquad \mathrm{III}' = \mathrm{III} = \sigma_1\sigma_2\sigma_3$

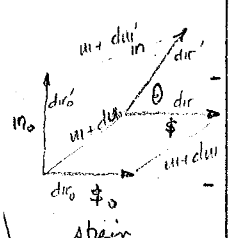
TO FIND $\lambda^{(k)}$ solve the eigenvalue equation and look at back of pg 8 notes for method to find eigenvectors (ie principal directions)

relates undeformed state to deformed state & vise versa

$(\nabla\mathbb{u})_c = \mathbb{u}\nabla = \mathbb{e}_k\mathbb{e}_j\dfrac{\partial u_k}{\partial x_j}$ $\qquad \nabla\mathbb{u} = \mathbb{e}_j\mathbb{e}_k\dfrac{\partial u_k}{\partial x_j}$ and $\hat{\mathbb{I}} - \nabla\mathbb{u} = (\delta_{ij} - \dfrac{\partial u_j}{\partial x_i})\mathbb{e}_i\mathbb{e}_j$

$d\mathbb{r}_o = d\mathbb{r}\cdot[\mathbb{I} - \nabla\mathbb{u}] = [\mathbb{I} - \mathbb{u}\nabla]\cdot d\mathbb{r}$ EULERIAN DEFORMATION TENSOR old length in terms of new.

$d\mathbb{r} = d\mathbb{r}_o\cdot[\mathbb{I} + \nabla_o\mathbb{u}] = [\mathbb{I} + \mathbb{u}\nabla_o]\cdot d\mathbb{r}_o$ LAGRANGIAN " " new length in terms of old

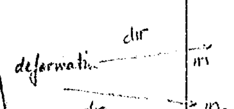


$\eta_{LL} = \dfrac{L^2 - L_o^2}{2L^2}$ EULERIAN STRETCH $\qquad \epsilon_{LL} = \dfrac{L^2 - L_o^2}{2L_o^2}$ LAGRANGIAN STRETCH $\qquad \eta_{LL} \to \epsilon_{LL} \to \epsilon = \dfrac{\Delta L}{L}$ for small strains

$= (d\mathbb{r}\cdot d\mathbb{r} - d\mathbb{r}_o\cdot d\mathbb{r}_o)/2\,d\mathbb{r}\cdot d\mathbb{r}$ $\qquad = \dfrac{d\mathbb{r}\cdot d\mathbb{r} - d\mathbb{r}_o\cdot d\mathbb{r}_o}{2\,d\mathbb{r}_o\cdot d\mathbb{r}_o}$

$\eta_{rr} = \mathbb{n}\cdot\Phi\cdot\mathbb{n}$ $\qquad \Phi = \frac{1}{2}(\nabla\mathbb{u} + \mathbb{u}\nabla - \nabla\mathbb{u}\cdot\mathbb{u}\nabla) = \frac{1}{2}\left(\dfrac{\partial u_j}{\partial x_i} + \dfrac{\partial u_i}{\partial x_j} - \dfrac{\partial u_k}{\partial x_i}\dfrac{\partial u_k}{\partial x_j}\right) = \phi_{ij}\,\mathbb{e}_i\mathbb{e}_j$

$\mathbb{n} = \dfrac{d\mathbb{r}}{|d\mathbb{r}|}$ EULERIAN STRAIN TENSOR $\phi_{ij} = \phi_{ji}$ contains rotations & extensions



$\frac{1}{2}\cos\theta = \mathbb{n}\cdot\Phi\cdot\$$ where $\mathbb{n} = \dfrac{d\mathbb{r}'}{|d\mathbb{r}'|}$ $\quad \$ = \dfrac{d\mathbb{r}}{|d\mathbb{r}|}$ $\qquad \mathbb{n}\cdot\Phi\cdot\$ = \frac{1}{2}\dfrac{d\mathbb{r}'\cdot d\mathbb{r}}{|d\mathbb{r}'||d\mathbb{r}|}$

$\epsilon_{rr} = \mathbb{n}_o\cdot\mathcal{E}\cdot\mathbb{n}_o$ $\qquad \mathcal{E} = \frac{1}{2}[\nabla_o\mathbb{u} + \mathbb{u}\nabla_o + \nabla_o\mathbb{u}\cdot\mathbb{u}\nabla_o]$ LAGRANGIAN Strain TENSOR $\qquad \mathbb{n}_o = \dfrac{d\mathbb{r}}{|d\mathbb{r}|}$

$\frac{1}{2}\cos\theta = \dfrac{\mathbb{n}_o\cdot\mathcal{E}\cdot\$_o}{\sqrt{1+2\epsilon_{s_os_o}}\sqrt{1+2\epsilon_{n_on_o}}}$ $\qquad = \frac{1}{2}\left[\dfrac{\partial u_j}{\partial x_{o_i}} + \dfrac{\partial u_i}{\partial x_{o_j}} + \dfrac{\partial u_k}{\partial x_{o_i}}\dfrac{\partial u_k}{\partial x_{o_j}}\right] = \varepsilon_{ij}$

$= \dfrac{d\mathbb{r}'\cdot d\mathbb{r}}{2\,|d\mathbb{r}'||d\mathbb{r}|}$

for small extensions / deformations then $\bar{\bar{\Phi}} \to \mathbb{\epsilon} \to \mathbb{E}$ where $\mathbb{E} + \omega = \nabla_0 u$, $\mathbb{E} - \omega = u \nabla_0$

Dilatation $\Delta = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = u_{k,k} = \epsilon_{kk} = I_\epsilon = \nabla \cdot u$

$\epsilon = \frac{1}{2}(\nabla u + u\nabla) = \epsilon_{ij} e_i e_j = \frac{1}{2}(u_{i,j} + u_{j,i}) e_i e_j$; $\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$ $\quad \omega = \omega_{ij} e_i e_j \quad \nabla u = \epsilon + \omega$

Incompatability tensor $\Psi = \nabla \times \epsilon \times \nabla$ should be $= 0$ (Eqs. Pg 14) pointwise compatibility
For multiply connected regions: $\oint_{c_i} dr \cdot (\epsilon \times \nabla) = 0$ and $\oint_{c_i} [\epsilon + (r - r_0) \times (\nabla \times \epsilon)] \cdot dr = 0$

assume $U(\epsilon) = \underbrace{U_0}_{\substack{\text{init state}\\=0}} + \underbrace{\frac{\partial U}{\partial \epsilon_L} \Delta\epsilon}_{\substack{=0 \text{ otherwise}\\ \text{imply prestrained state, if no stresses}}} + \frac{\partial^2 U}{\partial \epsilon^2} \frac{\Delta\epsilon^2}{2} + \dots$ only

obtained from conservation of energy + PLM $\Rightarrow$ $\frac{\partial U}{\partial t} = \sigma_{kj} \frac{\partial \epsilon_{kj}}{\partial t}$ where U = internal energy / volume or $\dot{U} = \sigma : \dot{\epsilon}$ $\quad \frac{\partial U}{\partial t} = \frac{\partial U}{\partial \epsilon_{ij}} \frac{\partial \epsilon_{ij}}{\partial t}$

if $U = U(\epsilon)$ only $\sigma_{ii} = \frac{\partial U}{\partial \epsilon_{ii}}$ (no summation) $\sigma_{ij} = \frac{1}{2}\frac{\partial U}{\partial \epsilon_{ij}}$ $i \neq j$

$\sigma_{ii} = \frac{\partial U}{\partial \epsilon_{ii}}$

and $U = \frac{1}{2} c_{ijkl} \epsilon_{ij} \epsilon_{kl} > 0$

Remember that $2\epsilon_{ij} = \gamma_{ij} \quad \forall i \neq j$

for isotropic material $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$ $\quad \frac{E}{2(1+\nu)} = G = \mu$ the shear modulus of elast.

$\frac{E\nu}{(1+\nu)(1-2\nu)} = \lambda$

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right]$$

See pg 18-19

- For unique soln we need that from specified init cond. that $\epsilon_0 = 0$
- $f_j^* = 0$ from body forces specified $\quad$ * is difference soln
- $\int_{S_d} t_j^* \Delta \dot{u}_j^* dS dt = 0$ compatability must be specified for discontinuities in displ field
- $\int_{S_r} t_j^* u_j^* dS dt = 0$ must specify singularities

- must specify how $\int t_j^* \dot{u}_j^* dS\, dt = 0$ ie $t_j^* = 0$ or $\dot{u}_j^* = 0$ or combination of $t_j^*, \dot{u}_j = 0$

if t_j are specified solution is specified up to rigid body effects for static problems

if t_j " " " is unique for dynamic case

now $\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i})$ $\quad \nabla \times \epsilon \times \nabla = \epsilon_{jl,sk} + \epsilon_{sk,jl} - \epsilon_{sl,jk} - \epsilon_{jk,sl} = 0$

$\nabla \times \epsilon \times \nabla \Rightarrow$ Beltrami Michell Conditions on stresses.

Given $\Theta = \sigma_{kk}$

$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu}\Theta_{,ij} = 0$ for constant body force & no accel $\quad$ Pg 20 full dynamic eqs

Beltrami $\quad$ when work w/ body force & accel we work w/ Beltrami-Michell

ME 238A

also $\nabla^2 \Theta = 0$ where Θ is harmonic

also $\nabla^4 \sigma_{ij} = 0$ equivalent to $\nabla \times \boldsymbol{\varepsilon} \times \nabla = 0$

In orthog curvilinear $ds_{\alpha_i} = h_{\alpha_i} d\alpha_i$ where $h_{\alpha_i} = \sqrt{\sum_j \left(\frac{\partial x_j}{\partial s_{\alpha_i}}\right)^2}$

$$\nabla = \sum_i \frac{1}{h_{\alpha_i}} \mathbf{e}_{\alpha_i} \frac{\partial}{\partial \alpha_i} \qquad \text{and} \quad \mathbf{u} = u_{\alpha_i} \mathbf{e}_{\alpha_i}$$

and
$$\varepsilon_{\alpha\alpha} = \mathbf{e}_\alpha \cdot \boldsymbol{\varepsilon} \cdot \mathbf{e}_\alpha = \frac{1}{h_\alpha}\left[\frac{\partial u_\alpha}{\partial \alpha} + \frac{u_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial \beta} + \frac{u_\gamma}{h_\gamma}\frac{\partial h_\alpha}{\partial \gamma}\right]$$

$$\varepsilon_{\alpha\beta} = \mathbf{e}_\alpha \cdot \boldsymbol{\varepsilon} \cdot \mathbf{e}_\beta = \frac{1}{2}\left[\frac{h_\beta}{h_\alpha}\frac{\partial}{\partial \alpha}\left(\frac{u_\beta}{h_\beta}\right) + \frac{h_\alpha}{h_\beta}\frac{\partial}{\partial \beta}\left(\frac{u_\alpha}{h_\alpha}\right)\right]$$

$$\Delta = \frac{1}{h_\alpha h_\beta h_\gamma}\left\{\frac{\partial}{\partial \alpha}(u_\alpha h_\beta h_\gamma) + \frac{\partial}{\partial \beta}(u_\beta h_\alpha h_\gamma) + \frac{\partial}{\partial \gamma}(u_\gamma h_\alpha h_\beta)\right\} = \nabla \cdot \mathbf{u}$$

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu}\Theta_{,ij} - \frac{\nu}{1+\nu}\delta_{ij}\nabla^2\Theta = -(f_{i,j} + f_{j,i}) + \rho(\cancel{\ddot{u}_{i,j}} + \cancel{\ddot{u}_{j,i}})$$

$$\cancel{-\frac{\nu}{1-\nu} f_{k,k}\delta_{ij} - (f_{i,j} + f_{j,i}) + \frac{\rho}{E}\left(\frac{\nu}{1-\nu}\ddot{\Theta}\delta_{ij} - 2(1+\nu)\ddot{\sigma}_{ij}\right)}$$

Aside

$$-\frac{\nu}{1-\nu}\delta_{ij} f_{k,k} + \rho\frac{\nu}{1-\nu}\delta_{ij}\ddot{u}_{k,k} - (\cancel{f_{i,j} + f_{j,i}}) + \rho(\ddot{u}_{i,j} + \ddot{u}_{j,i})$$

$$\cancel{-\frac{\nu}{1+\nu}} \qquad -\delta_{ij}\left[\frac{\nu}{1+\nu}\nabla^2\Theta + \frac{\nu}{1-\nu}f_{k,k} - \frac{\rho\nu}{1-\nu}\ddot{u}_{k,k}\right] = 0$$

B.C. can specify either tractions and/or displ but cannot specify T_x & u_x together { uniqueness assumes finite volume and no singularities

Since we are using linear elastic theory we can superpose solns.

** <u>Remember</u> always satisfy BC, Equil Eqns & Compatability

Equil $\sigma_{ij,i} + f_j = 0$

BC $\sigma_{ij} n_j = T_i$ and/or u_i is prescribed n_j is the normal outward

Compat: $\nabla \times \epsilon \times \nabla = 0$ for simply connected region

LINEAR ELASTICITY

$$\varepsilon_{ij} = \frac{(1+\nu)}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \qquad \varepsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j}) \qquad \omega_{ij} = \frac{1}{2}(u_{j,i} - u_{i,j})$$

<u>PLANE STRAIN</u> $\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij}$ only for isotropic ; normally $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$

$\epsilon_{zz} = 0 \quad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$

$\epsilon_{zx} = \epsilon_{zy} = 0$

$u_x(x,y) \ u_y(x,y)$ only $u_z = 0$

$\sigma_{xx}, \sigma_{xy}, \sigma_{xz} \neq f(z) \quad f_1, f_2 \neq f(z)$

$\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy} \neq f(z) \quad f_3 = 0$

$$\varepsilon_{xx} = \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu}{E}(1+\nu)\sigma_{yy}$$

$$\varepsilon_{yy} = \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right] = \frac{1-\nu^2}{E}\sigma_{yy} - \frac{\nu(1+\nu)}{E}\sigma_{xx}$$

$$\epsilon_{xy} = \frac{1}{2\mu}\sigma_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$

<u>PLANE STRESS</u>

$\sigma_{zz} = 0, \ \sigma_{xz} = 0, \ \sigma_{yz} = 0$

$f_3 = 0$ to satisfy equil

$\boxed{u_x(x,y) \ u_y(x,y) \text{ only} \quad \sigma_{xx}, \sigma_{xy}, \sigma_{yy}, f_1, f_2 \neq f(z)}$ <u>if very thin</u> <u>NOT EXACT USUALLY</u>

Define $\phi(x,y) \ni$, $\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2} \quad \sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y} \quad \sigma_{yy} = +\dfrac{\partial^2\phi}{\partial x^2}$ satisfies equil. To satisfy compatability we get $\nabla^4\phi = 0 = \nabla^2\epsilon_{kk} = \nabla^2\sigma_{kk} = 0$

IF body force is included then $\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2} + V \quad \sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2} + V$ when $f_j = -\dfrac{\partial V}{\partial x_j}$ V = potential fn

for weight force take $V = \pm\rho g y$ (sign depends on dfn of axes) and let $\phi = 0$ also $\nabla^2 V = 0$

this solution is good for either plane strain or plane stress.

Given: Plane Stress soln replace ν_σ by $\dfrac{\nu}{1-\nu}$ to get plane strain E_σ by $\dfrac{E}{1-\nu^2}$

Plane Strain " " ν_ε by $\dfrac{\nu}{1+\nu}$ to get plane stress E_ε by $E\left(\dfrac{1+2\nu}{(1+\nu)^2}\right)$

2-D semi infinite plane w/ periodic b.c. & $|\sigma_{ij}| \to 0$ as $y \to -\infty$ SEE PG 4 mynotes

y, x

a) if $\sigma_{xy} = 0 \quad \sigma_{yy} = f_1(x) \quad \sum A_n \sin\dfrac{n\pi x}{L} \quad A_n = \dfrac{2}{L}\displaystyle\int_0^L f_1(x)\sin\frac{n\pi x}{L}dx$ take $\phi_n = g(y)\sin\gamma_n x$

and let $\sum\phi_n = \phi(x,y)$ where $\gamma_n = \dfrac{n\pi}{L}$. If $e^{sy} = g(y)$ then $s = \pm\gamma_n$

Solve and get $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ now $u_i = \displaystyle\int \frac{\partial u_i}{\partial x_i}dx_i + f(x_j)$ only (ODD FUNC)

b) if $\sigma_{xy} = 0 \quad \sigma_{yy} = f_2(x) = \sum B_n\cos\dfrac{n\pi x}{L} + \dfrac{B_0}{2} \quad B_n = \dfrac{2}{L}\displaystyle\int_0^L f_2(x)\cos\frac{n\pi x}{L}dx$ take $\phi_n = g(y)\cos\gamma_n x$ (EVEN FUNC)

if $f(x) = f_1 + f_2$ then $\phi = \dfrac{B_0 x^2}{4} + \sum g_1(y)\sin\dfrac{n\pi x}{L} + \sum g_2(y)\cos\dfrac{n\pi x}{L}$ (ANY FUNC)

c) if $\sigma_{xy} = g(x) = \frac{D_0}{2} + \sum D_n \cos\frac{n\pi x}{L} + E_n \sin\frac{n\pi x}{L}$ take $\phi_n = -\frac{xy}{2}D_0 + \sum g_1(y)\sin\frac{n\pi x}{L} + g_2(y)\cos\frac{n\pi x}{L}$
$\sigma_{yy} = 0$

ie if $\sigma_{xy} = g_1(x) = \frac{D_0}{2} + \sum D_n \cos\frac{n\pi x}{L}$ take $\phi_n = -\frac{xy}{2}D_0 + \sum g_1(y)\sin\frac{n\pi x}{L}$

$\sigma_{xy} = g_2(x) = \sum E_n \sin\frac{n\pi y}{L}$ take $\phi_n = \sum g_2(y)\cos\frac{n\pi x}{L}$

Fourier Integrals : half space with $|\sigma_{ij}| \to 0$ as $y \to \infty$

$$\phi = \int_{-\infty}^{\infty} e^{-i\lambda x}\left[A(\lambda)e^{-|\lambda|y} + B(\lambda)y\,e^{-|\lambda|y}\right]d\lambda \qquad y>0$$

w/BC $\sigma_{yy}|_{y=0} = f(x) \Rightarrow f(x) = -\int_{-\infty}^{\infty}\lambda^2 e^{-i\lambda x}A\,d\lambda$;

$\sigma_{xy}|_{y=0} = g(x) \Rightarrow g(x) = \int_{-\infty}^{\infty} -i\lambda\, e^{-i\lambda x}\{-|\lambda|A + B\}d\lambda$

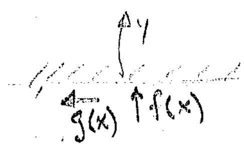
define $\bar{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\,e^{i\lambda x}dx$ and $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\lambda)\,e^{-i\lambda x}d\lambda$

we can use this iff

$\int_0^{\infty}|f|\,dx \le M$ and f is continuous on intervals $\ni$. $f'(x^-), f'(x^+)$ exist

and $f(x_0) = \frac{1}{2}[f(x_0^-) + f(x_0^+)]$ if x_0 is a pt of discontinuity

$$\boxed{\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}d\lambda}$$



if $\sigma_{xy} = 0$ and $\sigma_{yy} = -f(x)$ then
$$\sigma_{yy}(x,y;\xi) = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y^3 f(\xi)\,d\xi}{[(x-\xi)^2+y^2]^2}$$
$$\sigma_{xy}(x,y;\xi) = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y^2(x-\xi)\,f(\xi)\,d\xi}{[(x-\xi)^2+y^2]^2}$$
$$\sigma_{xx}(x,y;\xi) = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y\,(x-\xi)^2 f(\xi)\,d\xi}{[(x-\xi)^2+y^2]^2}$$

if $\sigma_{xy} = g(x)$ $\sigma_{yy} = 0$ then
$$\sigma_{yy}(x,y;\xi) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y^2(x-\xi)\,g(\xi)\,d\xi}{[(x-\xi)^2+y^2]^2}$$
$$\sigma_{xy}(x,y;\xi) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y\,(x-\xi)^2 g(\xi)\,d\xi}{[(x-\xi)^2+y^2]^2}$$
$$\sigma_{xx}(x,y;\xi) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{(x-\xi)^3 g(\xi)\,d\xi}{[(x-\xi)^2+y^2]^2}$$

if $\sigma_{xy} = g(x)$ and $\sigma_{yy} = f(x)$ take linear combination of above results

If loads is applied at origin of half space $\Rightarrow \sigma_{r\theta}, \sigma_{\theta\theta}, \sigma_{\theta z} = 0$ on all radial planes

2-D in polar coords Plane Strain soln $\nabla^2(\sigma_{kk}) = (1+\nu)\nabla^2(\sigma_{xx}+\sigma_{yy}) = (1+\nu)\nabla^2(\sigma_{rr}+\sigma_{\theta\theta}) = 0$

take $\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$; $\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}$; $\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$

where $\nabla\phi^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2}$

see ring results - handout

Lame Solution: only pressure on boundary of ring both plane strain & plane stress solutions are exact.

reaction to forces applied $= \oint T_y\, ds = -\frac{\partial\phi}{\partial x}\Big]_s^F$; reaction to forces applied $= \oint T_x\, ds = +\frac{\partial\phi}{\partial y}\Big]_s^F$ $\qquad \oint M_z\, ds = (\phi - \underline{r}\cdot\nabla\phi)\Big]_s^F$

STRESS FUNC for pt forces in ∞ plane are given on Pg 19 Prelim to G_{ij}

$\phi = b_1 r\cos\theta \ln r$ gives no traction but does give multivalued disp. Good for disloc

$\phi^{(x)}$ = stress fn for pt load in x dir at x, y (line of force in z dir.) } in infinite medium Pg 19

$\phi^{(y)}$ = " " " " " " y dir " " (line of force in z dir)

now from $\phi^{(x)}$ we can get $\sigma_{xx}^{(x)}, \sigma_{xy}^{(x)}, \sigma_{yy}^{(x)} \Rightarrow u_x^{(x)}, u_y^{(x)}$: G_{11} ; G_{21}

$\phi^{(y)}$ " " " $\sigma_{xx}^{(y)}, \sigma_{xy}^{(y)}, \sigma_{yy}^{(y)} \Rightarrow u_x^{(y)}, u_y^{(y)}$: G_{12}, G_{22}

B.I.E

$$\int_S \left[T_i^{(1)}(\underline{x})\, G_{im}(\underline{x},\underline{y}) - T_i^{(m)}(\underline{x},\underline{y})\, u_i^{(1)}(\underline{x})\right] ds = u_m(\underline{y}) \quad \text{for } \underline{y} \text{ inside body}$$

Known supposedly

$G_{im}(\underline{x}, y)$ are green's fns ; $T_i^{(m)}(\underline{x},\underline{y}) = \sigma_{ij}^{(m)} n_j$ = tractions due to the greens functions $= [\lambda\delta_{ij}\epsilon_{kk}^{(m)} + 2\mu\epsilon_{ij}^{(m)}]n_j$

and $T_i^{(1)}(\underline{x})$ & $u_i^{(1)}(\underline{x})$ on the boundary $= [\lambda\delta_{ij}G_{km,k} + \mu(G_{im,j} + G_{jm,i})]n_j$

use BIE to get $u_m(\underline{y})$ inside if you know $T_i^{(m)}, G_{im}, T_i^{(1)}(\underline{x}), u_i^{(1)}(\underline{x})$ on bdy. Normally you don't know $T_i^{(1)}(\underline{x})\ u_i^{(1)}(\underline{x})$ everywhere $\therefore$ must use to get $u_m(\underline{y})$ on bdy $\Rightarrow$

$$\tfrac{1}{2}u_m(\underline{y}) + \int_S u_i^{(1)}(\underline{x})\, T_i^{(m)}(\underline{x},\underline{y})\, dS = \int_S T_i^{(1)}(\underline{x})\, G_{im}(\underline{x},\underline{y})\, dS \quad \text{for } \underline{y} \text{ on the surface}$$

known: $T_i^{(m)}(\underline{x},\underline{y})$, $G_{im}(\underline{x},\underline{y})$ given either $T_i^{(1)}(\underline{x})$ or $u_i^{(1)}(\underline{x})$ on surface

Discretiz. see pg 22 my notes.

Dislocation see pg 24 $\quad \phi = b_1 r\cos\theta\ln r$ or $d_1 r\sin\theta\ln r$

Method of images pg 24 back etc.

Line of force solution in infinite medium

2-D $\quad G_{km}(\underset{\sim}{x}, \underset{\sim}{x}') = \frac{1}{8\pi\mu(1-\nu)} \left\{ -(3-4\nu)\ln|\underset{\sim}{x}-\underset{\sim}{x}'|\,\delta_{km} + \frac{(x_k - x_k')(x_m - x_m')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^2} \right\}$

Plane strain soln w/ k=1,2 m=1,2

where $|\underset{\sim}{x}-\underset{\sim}{x}'| = \sqrt{(x-x')^2+(y-y')^2}$ in the BIE

to attack problems use semi inverse technique: always state bc. clearly; draw free body diag

1. Pick state of stress to satisfy Equil Eq (may or may not solve BC).
2. Check to see if the solve compat.
3. Check to see what b.c. you've generated
   a. If same as bc you want then stop
   b. If not then $bc_{new} = bc_{old} - bc_{\text{stress state picked}}$

      go back to 1.

Stress fn for ~~pt~~ line of force in infinite medium pg 19

Stress fn for line of force in half space pg 25 + handout

pg 35-39
Timo

if $\phi = \frac{a_2}{2}x^2 + \frac{b_2}{2}xy + \frac{c_2}{2}y^2$: $\sigma_x = c_2$ $\sigma_y = a_2$ $\sigma_{xy} = -b_2$ state of uniform tension / compression / shear

if $\phi = \frac{a_3}{6}x^3 + \frac{b_3}{2}x^2y + \frac{c_3}{2}xy^2 + \frac{d_3}{6}y^3$: $\sigma_x = c_3x + d_3y$ $\sigma_y = b_3y + a_3x$ $\sigma_{xy} = -b_3x - c_3y$

pure bending $d_3 \neq 0$; $c_3, b_3\, a_3 = 0$

if $\phi = \frac{a_4}{12}x^4 + \frac{b_4}{6}x^3y + \frac{c_4}{2}x^2y^2 + \frac{d_4}{6}xy^3 + \frac{e_4}{12}y^4$: $\sigma_x = c_4x^2 + d_4xy - (2c_4 + a_4)y^2$

to solve compat $e_4 = -(2c_4 + a_4)$ $\sigma_y = a_4x^2 + b_4xy + c_4y^2$

$\sigma_{xy} = -\frac{b_4}{2}x^2 - 2c_4xy - \frac{d_4}{2}y^2$

Green's theorem $\oint_C P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

counterclockwise

Torsion

For arbitrary cross-section, planes do not remain plane but are warped

$$\Rightarrow \quad w = \underbrace{\alpha}_{\text{rate of twist}}\,\underbrace{\phi(x,y)}_{\text{warping function}} \qquad \text{indep. of } z \qquad (1)$$

$$\theta = \alpha z$$

and as for circular bars

$$u = -\alpha z y \qquad v = \alpha z x \qquad (2)$$

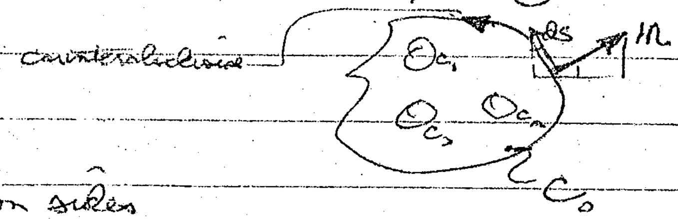
Equil. eqn. $\Rightarrow \quad \nabla^2 \phi = 0$ throughout region (3)

G.A.E

Only stress components are τ_{yz}, τ_{xz}.

B.C.

No traction on sides; and on ends of bar no resultant forces, only resultant moment



$$n_x = \frac{dx}{dn} = \frac{dy}{ds}$$

$$n_y = \frac{dy}{dn} = -\frac{dx}{ds}$$

B.C. on sides

$$\overset{n}{T_i} = \tau_{ji} n_j$$

$$T_1 = \cancel{\tau_{xx}}^{\,0} n_x + \cancel{\tau_{xy}}^{\,0} n_y + \tau_{zx} \cancel{n_z}^{\,0} = 0 = T_2$$

$$T_3 = \tau_{zx} n_x + \tau_{zy} n_y + \tau_{zz} \cancel{n_z}^{\,0} = 0$$

$$\tau_{zx} = 2\mu\left[\frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] = \mu\alpha\left(\frac{\partial \phi}{\partial x} - y\right)$$

$$\tau_{zy} = \mu\alpha\left(\frac{\partial \phi}{\partial y} + x\right)$$

$$T_3 = \mu\alpha\left[\underbrace{\left(\frac{\partial \phi}{\partial x}\frac{dx}{dn} + \frac{\partial \phi}{\partial y}\frac{dy}{dn}\right)}_{\partial\phi/\partial n} - y n_x + x n_y\right] = 0$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial n} = y n_x - x n_y = y\frac{\partial y}{\partial s} - x\left(-\frac{\partial x}{\partial s}\right)$$

$$\frac{\partial \phi}{\partial n} = y\frac{\partial y}{\partial s} + x\frac{\partial x}{\partial s} \qquad (7)$$

~~Introduce the~~ conjugate harmonic function ψ such ~~that~~

$$\Omega(z) = \phi + i\psi \qquad (8)$$

and from Cauchy-Riemann relations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad (9)$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad (10)$$

(9) (10) $\Rightarrow$ (7)

$$\frac{\partial \psi}{\partial y}\underbrace{\frac{\partial x}{\partial n}}_{\frac{\partial y}{\partial s}} - \frac{\partial \psi}{\partial x}\underbrace{\frac{\partial y}{\partial n}}_{-\frac{\partial x}{\partial s}} = y\frac{\partial y}{\partial s} + x\frac{\partial x}{\partial s} \qquad (11)$$

$$\Rightarrow \quad \frac{\partial \psi}{\partial s} = y\frac{\partial y}{\partial s} + x\frac{\partial x}{\partial s} = \frac{1}{2}\frac{\partial}{\partial s}(x^2+y^2)$$

integrating along the boundary

$$\psi = \frac{1}{2}(x^2+y^2) + k_0$$

k_0 is arbitrary, usually taken as zero

on boundary C_0 (12)

if multiply connected (12) must hold on all boundaries $C_0, C_1, C_2, \ldots C_n$

$$\underset{(12)}{\Rightarrow} \quad \psi = \frac{1}{2}(x^2+y^2) + k_i \qquad \text{on boundaries } C_i,\ i = 0,1,2,\ldots \quad (14)$$

B.C.

and since ψ is analytic

$$\nabla^2 \psi = 0 \qquad (15)$$

Can introduce the Prandtl stress function Ψ where

$$\Psi = \psi - \tfrac{1}{2}(x^2+y^2) \qquad (16)$$

$$\left.\begin{aligned} &\Rightarrow \nabla^2 \Psi = -2 \\ &\text{B.C.} \quad \Psi = k_i \quad \text{on } C_i \end{aligned}\right\} \qquad (17)$$

and satisfy (24) for compatibility in multiply

Problem was originally formulated in terms of displacement, thus for a simply connected region, compatibility is guaranteed, but for a multiply connected region in order for

$$\oint_s d\vec{u} = 0 \qquad \text{where } s \text{ is a reducible path} \qquad (18)$$

$$\Rightarrow \oint_{C_0} d\vec{u} + \oint_{C_i} d\vec{u} = 0 \qquad (19)$$

must satisfy the additional relations

$$\oint_{C_i} d\vec{u} = 0 \qquad i = 1, 2, 3, \ldots \qquad (20)$$

k_i's must be chosen so that (20) is satisfied

$$\vec{u} = u\hat{i} + v\hat{j} + w\hat{k} \qquad (21)$$

u and v as given by (6) are single valued

~~thus~~ (20) $\Rightarrow \oint_{C_i} dw = 0$

$$\Rightarrow \oint_{C_i} \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy = 0 \qquad (22)$$

$$\frac{\partial w}{\partial x} = \alpha\frac{\partial \phi}{\partial x} = \alpha\frac{\partial \psi}{\partial y} = \alpha\left(\frac{\partial \Psi}{\partial y} + y\right) \qquad (23)$$

likewise $\dfrac{\partial w}{\partial y} = \alpha\dfrac{\partial \phi}{\partial y} = -\alpha\dfrac{\partial \psi}{\partial x} = -\alpha\left(\dfrac{\partial \Psi}{\partial x} + x\right)$

(23) → (22)

$$\oint_{C_i}\left(\frac{\partial\Psi}{\partial y}\frac{dx}{ds}-\frac{\partial\Psi}{\partial x}\frac{dy}{ds}\right)ds=-\oint_{C_i} y\,dx-x\,dy$$

(by (4), but clockwise circuit) — Green's theorem

$$\oint_{C_i}\left(\frac{\partial\Psi}{\partial y}\frac{dy}{dn}-\frac{\partial\Psi}{\partial x}\left(-\frac{dx}{dn}\right)\right)ds=+\iint_{A_i}(-1-1)\,dx\,dy=-2$$

sign change because of direction of integration

$$\oint_{C_i}\frac{d\Psi}{dn}\,ds=-2A_i \qquad i=1,2,\ldots \qquad (24)$$

See also p 170 Sokolnikoff

End moment M_z may be found by integration of the stress times a suitable moment arm

$$M_z=G\alpha\left[2\iint_A\Psi\,dx\,dy-2k_0A_0+2\sum_{j=1}^{m}k_jA_j\right] \qquad (25)$$
$$=GJ\alpha$$

Membrane analogy

Membrane under uniform tension F with pressure q on surface

$$\nabla^2 z=-q/F=-2\left(\frac{q}{2F}\right) \qquad (26)$$

$$\nabla^2\left(\frac{2F}{q}z\right)=-2 \qquad (27)$$

B.C. $\quad z=c_i \quad$ on boundaries $\quad$ (28)

so $\dfrac{2F}{q}z=\Psi$

Membrane analogy is also useful as an aid in visualization of the stress distribution. Lines of $z = \text{const}$ are also lines of $\Psi = \text{const}$. The magnitude of ~~the~~ stress depends on $\frac{\partial \Psi}{\partial n}$, so max stress must occur ~~of~~ on ~~the~~ boundary. (for Ψ or ψ)

Also note solution is unaffected by cuts along a line $\Psi = \text{const}$ ($z = \text{const}$).

By this I mean if one has the ~~[illegible]~~ solution for a solid elliptical bar, ~~the~~ lines of const. Ψ are similar ellipses with ~~the~~ same center. Thus, for a hollow elliptical bar with interior boundary along a line of const Ψ, solution for Ψ is same as that for a solid elliptical bar. It is of course not as stiff (lower J) due to the decreased volume.

ω related to rigid rotation; measures avg. rotation of all line elements through a point

ϵ infinitesimal strain tensor
diagonal terms — related to change in length
off-diagonal terms — related to changes in angle

Compatibility

For single valued displacements, want du_i to be an exact differential or $\oint du_i = 0$

For a simply connected region, this is insured by

$$\nabla \times \epsilon \times \nabla = 0$$

6 equations must be satisfied at each point in the medium

$$\epsilon_{ij,kl} + \epsilon_{kl,ij} - \epsilon_{ik,jl} - \epsilon_{jl,ik} = 0$$

For multiply-connected region

$$\oint_{C_i} du_i = 0$$ where C_i are the interior contours

Stress Compatibility

Beltrami Eqn. (const. body force)

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu} \sigma_{kk,ij} = 0$$

and in this case $\nabla^2 \sigma_{kk} = 0$

Beltrami-Michell Eqn:

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu} \Theta_{,ij} = -\frac{\nu}{1-\nu} \delta_{ij} \nabla \cdot F - (F_{i,j} + F_{j,i}$$

Theory of Elasticity   Prof. Herrmann Rm 121   #521 Notes

Course supplemented by HW (3 or 4 in quarter)

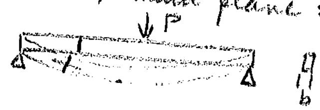
1 hr midterm + 1 final exam (Take home or in class)

Lab Assistant: Rich King   Durand 264   12-2 W; 2-4 T

Course deals with deformable bodies. Elasticity deals with what action body will take under applied loads. When loads are removed, body returns to original shape without hysteresis.

— POSSIBLE CHOICE IN MODELS —

Strength of Material assumption: plane section remain plane & are $\perp$ to ℄ of beam its a 1-D analysis



Elasticity - σ, ϵ relation exist, compatibility, equilib, boundary conditions must also be satisfied. Stress concentration

How to attack problem: What quantities do I need? Which model will I use to get this?

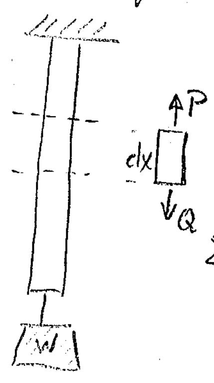
We will discuss

Stress, Strain, Rotation, Uniqueness of Solution, Problems, Torsion

__Winter quarter__ plane problems, Problems dealing with cracks.

__Spring quarter__ 3-D problems, contact stresses & elastic contact.

__Notations__ found within the literature



Look at free body diagram since body in equil means any part in equil

Use Taylors theorem to represent Q as a function of P. in 1-D analy

$$Q = P + \frac{dP}{dx}dx + \dots \quad \text{(to 1st order)}$$

$$\Sigma F_y = 0 : \text{EQUIL} \quad -P + \left[P + \frac{dP}{dx}dx\right] = 0 \quad \text{or} \quad \frac{dP}{dx} = 0 \quad \text{if no body force}$$

$$\text{if body force} \quad \frac{dP}{dx} = -F \quad \text{etc.}$$

Explicit cartesian form.

1. $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x = 0$

$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + f_y = 0$

$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0$

✓ stress equations of equil (component form)

Cartesian Tensor form (indicial notation) subscript

2. ✓ $\sigma_{jk,j} + f_k = 0$

$\frac{\partial \sigma_{jk}}{\partial x_j} + f_k = 0$

indices: if appears once free subscript
if appears twice implies summation over $j = x, y, z$
comma represents differentiations

Limitation is only to cartesian system breaks down for other curvilinear coordinates

General Tensor form

3. $T^{\beta\alpha}_{;\alpha} + f^{\beta} = 0$

$\left(T^{\alpha\beta}_{,\alpha} + \Gamma^{\beta}_{\alpha k} T^{\alpha k} + \Gamma^{k}_{\alpha k} T^{\alpha\beta} + f^{\beta} = 0 \right)$

T is stress tensor
; is differentiation wrt curvilinear coordinates
Γ Cristoffel transformations

Rm 121 from now on

Symbolic, Dyadic, Gibbs form
greek letter w/ bar will be a 2nd ran[k] tensor

✓ 4. $\nabla \cdot \bar{\sigma} + \mathbf{f} = 0$

Matrix Notation

5. $(\mathrm{div}_d \bar{T})^* + F = 0$

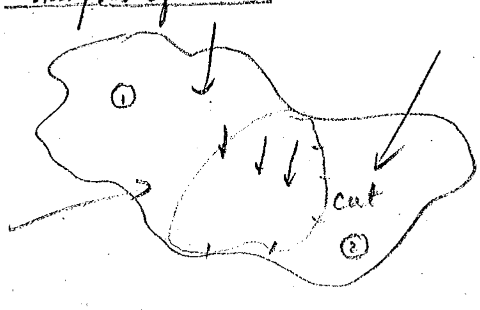
✓ notations will be used

A.E.H. Love "A Treatise on the Mathematical Theory of Elasticity" (1892) (4th Ed 1927)
(Dover Publ.)
S. Timoshenko & J.N. Goodier "Theory of Elasticity 3rd 1970 } use component form
I.S. Sokolnikoff "Math Theory of Elast" 2nd Ed 1956 uses CARTESIAN TENSOR
A.E. Green & W. ZERNA "Theoretical Elasticity" Oxford University Press 1954 uses general tensors emphasizes non-linear elast

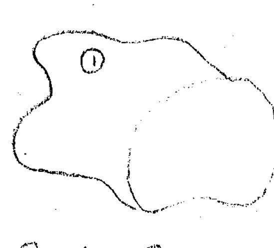
F.T. Murnaghan "Finite deformations of an elastic solids" (Wiley 1951) use matrix notation.
H.M. Westergaard Theory of Elasticity & Plasticity (1952 Harvard Univ Press) uses DYADICS

For Math Background:
DYADIC notation: C.E. Weatherburn (Open Court 1948) "Elem Vector Analy & Advanced Vector Ana"
Cartesian tensors: H Jeffreys "Cartesian Tensors" Cambridge Univ 1952
General tensors: Sokolnikoff "Tensor Analysis" Wiley 2nd 1948
Matrices: A.D. Michal "Matrix a Tensor Calculus Wiley 1947.

Analysis of Stress

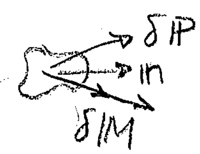


if forces acting on it.
Contact forces - 2 contiguous bodies touchin
Distance force - gravity



if body originally in equil
this part is also in equil

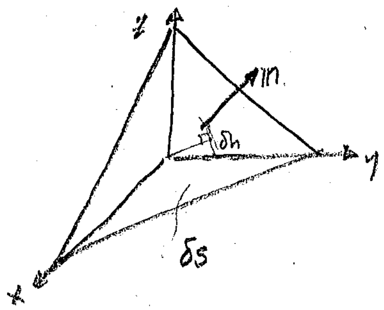
We assume that forces acting on (1) by (2) are uniform over entire surface so we only need to look at one small area & resultant force & moment acting over it. Look at $\frac{\delta \mathbb{P}}{\delta A}$, $\frac{\delta \mathbb{M}}{\delta A}$



take lim as $\delta A \to 0$ of each. $\lim_{\delta A \to 0} \frac{\delta \mathbb{P}}{\delta A} = \mathbb{t}_n$ traction (stress vector)

$\lim_{\delta A \to 0} \frac{\delta M}{\delta A} = 0$ is <u>assumed</u>

Introduce cartesian



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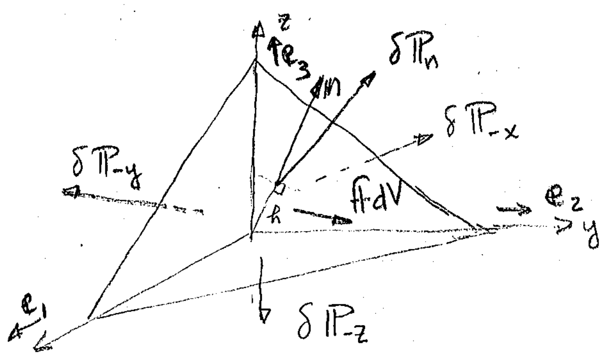


How to find the traction $t_n \left(= \lim_{\delta S \to 0} \frac{\delta P_n}{\delta S} \right)$ as a fn of the tractions in e_x, e_y, e_z

applying $\sum F = 0$

$$\delta P_{-j} = -\delta P_j \qquad (1)$$

$$\delta P_n + \delta P_{-x} + \delta P_{-y} + \delta P_{-z} + f dV = 0 \qquad (2)$$

$$\delta P_n - \delta P_x - \delta P_y - \delta P_z + f dV = 0 \qquad (1+2) \to (3)$$

$\delta S_x = \delta S \cos(n, e_x)$

$\delta S_y = \delta S \cos(n, e_y)$

$\delta S_z = \delta S \cos(n, e_z)$

projection of the plane surface onto the planes ⊥ to the x, y, z axes $\cos(n, e_{x_j}) = n \cdot e_j$

$$\delta S = \frac{\delta S_j}{n \cdot e_j} \qquad (4)$$

divide (3) by δS & sub (4) and $\delta V = \frac{\delta S}{3} \cdot \delta h$

$$\frac{\delta P_n}{\delta S} - \frac{\delta P_x}{\delta S} - \frac{\delta P_y}{\delta S} - \frac{\delta P_z}{\delta S} + \frac{f dV}{\delta S} = 0$$

$$\frac{\delta P_n}{\delta S} - n \cdot e_x \frac{\delta P_x}{\delta S_x} - n \cdot e_y \frac{\delta P_y}{\delta S_y} - n \cdot e_z \frac{\delta P_z}{\delta S_z} + f \frac{\delta S \cdot \delta h}{3 \delta S} = 0$$

now let $\delta S \to 0 \Rightarrow \delta S_x, \delta S_y, \delta S_z$ and $\delta h \to 0$ wipes out body forces.

$\therefore \; \mathbf{t}_n - \mathbf{n}\cdot\mathbf{e}_x \mathbf{t}_x - \mathbf{n}\cdot\mathbf{e}_y \mathbf{t}_y - \mathbf{n}\cdot\mathbf{e}_z \mathbf{t}_z = 0$

or $\mathbf{t}_n - \mathbf{n}\cdot\mathbf{e}_1 \mathbf{t}_1 - \mathbf{n}\cdot\mathbf{e}_2 \mathbf{t}_2 - \mathbf{n}\cdot\mathbf{e}_3 \mathbf{t}_3 = 0$

or $\mathbf{t}_n - \mathbf{n}\cdot\mathbf{e}_j \mathbf{t}_j = 0$ where j appears twice implies summation

$\mathbf{t}_n = \mathbf{n}\cdot\mathbf{e}_j \mathbf{t}_j$ Traction on a surface whose normal is $\mathbf{n}$, as a linear combination of the tractions on the cartesian surfaces

$\mathbf{t}_x$ has a general direction and can be resolved.

so
$$\mathbf{t}_x = \sigma_{xx}\mathbf{e}_x + \sigma_{xy}\mathbf{e}_y + \sigma_{xz}\mathbf{e}_z$$
$$\mathbf{t}_y = \sigma_{yx}\mathbf{e}_x + \sigma_{yy}\mathbf{e}_y + \sigma_{yz}\mathbf{e}_z$$
$$\mathbf{t}_z = \sigma_{zx}\mathbf{e}_x + \sigma_{zy}\mathbf{e}_y + \sigma_{zz}\mathbf{e}_z$$

State of stress at a point, is said to be known if traction on every plane passed through that point is known

The 9 quantities called components of stress

$$\mathbf{t}_j = \sigma_{jx}\mathbf{e}_x + \sigma_{jy}\mathbf{e}_y + \sigma_{jz}\mathbf{e}_z = \sigma_{jk}\mathbf{e}_k$$

or $\mathbf{t}_j = \sigma_{jk}\mathbf{e}_k$

$$\mathbf{t}_n = \mathbf{n}\cdot[\mathbf{e}_1\mathbf{t}_1 + \mathbf{e}_2\mathbf{t}_2 + \mathbf{e}_3\mathbf{t}_3] = \mathbf{n}\cdot\boldsymbol{\sigma}$$

this is in terms of $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ in original coordinate syst.

$\boldsymbol{\sigma}$ is called the stress dyadic and is a second rank tensor

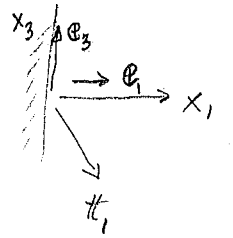
note: when vectors positioned next to each other, no operation.

10/4/78

$$\mathbb{t}_n = \mathbb{n}\cdot\mathbb{e}_1\mathbb{t}_1 + \mathbb{n}\cdot\mathbb{e}_2\mathbb{t}_2 + \mathbb{n}\cdot\mathbb{e}_3\mathbb{t}_3$$

these do not mean same

$\mathbb{t}_1$ acts across the plane whose normal is $\mathbb{e}_1$



$$\mathbb{t}_1 = \mathbb{e}_1\cdot\sigma_1 = \sigma_{xx}\mathbb{e}_x + \sigma_{xy}\mathbb{e}_y + \sigma_{xz}\mathbb{e}_z$$

$$\mathbb{t}_n = \mathbb{n}\cdot[\mathbb{e}_1\mathbb{t}_1 + \mathbb{e}_2\mathbb{t}_2 + \mathbb{e}_3\mathbb{t}_3] = \mathbb{n}\cdot\sigma$$

σ is now defined as a stress dyadic (second rank tensor)

$$\mathbb{t}_x = \sigma_{xx}\mathbb{e}_x + \sigma_{xy}\mathbb{e}_y + \sigma_{xz}\mathbb{e}_z$$

σ_{ij} — i: orientation of traction; j: direction it is acting in

$\mathbb{t}_j = \sigma_{jk}\,\mathbb{e}_k$ subscript notation

now since $\mathbb{t}_n = \mathbb{n}\cdot\sigma$ $\quad\mathbb{t}_j = \mathbb{e}_j\cdot\sigma$

really means traction in the $\mathbb{e}_j$ direction

now subst $\mathbb{t}_{x,y,z}$ for $\mathbb{t}_{1,2,3}$

$$\mathbb{t}_n = \mathbb{n}\cdot[\mathbb{e}_x(\sigma_{xx}\mathbb{e}_x + \sigma_{xy}\mathbb{e}_y + \sigma_{xz}\mathbb{e}_z) + \mathbb{e}_y(\quad) + \mathbb{e}_z(\quad + \sigma_{zz}\mathbb{e}_z)]$$

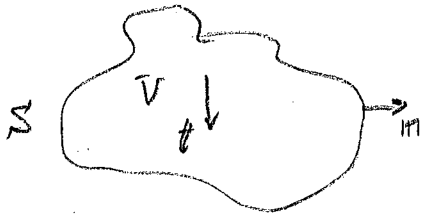
$$= \mathbb{n}\cdot\sigma \quad \text{or} \quad \boxed{\sigma = \sigma_{ij}\,\mathbb{e}_i\mathbb{e}_j}$$

we define normal stress σ_{ij} $i=j$
shear stress σ_{ij} $i \neq j$

Principle of linear momentum [must know volume V and a given S (surface) with a given set of surface tractions and the unit body force]

$$\Sigma \text{ force} = \frac{d}{dt} \text{ of momentum.}$$

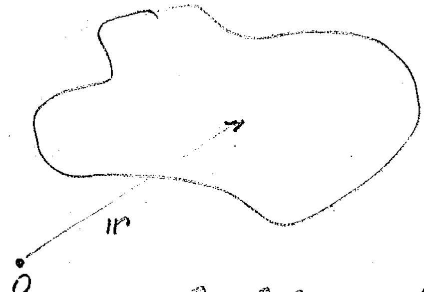
Here we assume material ρ = const & V is fixed



$$\int_S \mathbf{t}_n \, ds + \int_V \mathbf{f} \, dV = \int_V \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \, dV$$

$\mathbf{u}$ = displacement vector

Principle of Angular Momentum Σ moment about some given pt is the time rate of change of angular mom.



$$\int_S \mathbf{r} \times \mathbf{t}_n \, dS + \int \mathbf{r} \times \mathbf{f} \, dV = \int \mathbf{r} \times \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

$\mathbf{a} = a_i \mathbf{e}_i$ $\quad \mathbf{b} = b_j \mathbf{e}_j$ $\quad \mathbf{a} \cdot \mathbf{b} = a_i b_j \delta_{ij} = a_i b_i$

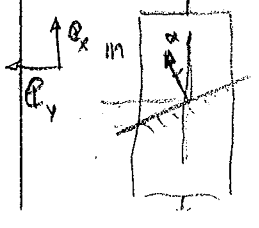
$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{e}_1 + (a_3 b_1 - a_1 b_3)\mathbf{e}_2 + (a_1 b_2 - a_2 b_1)\mathbf{e}_3$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \mathbf{a} \times \mathbf{b} = a_i b_j e_{ijk} \mathbf{e}_k$$

<u>Definition</u>

Alternating symbol $e_{ijk} = \begin{cases} +1 & i,j,k \text{ are cyclic} \\ 0 & \text{if others.} \\ -1 & i,j,k \text{ are anti cyclic} \end{cases}$

(HW due 2 wks) given in class $\qquad$ 10/6/78



$\boldsymbol{\sigma} = ?$

$\boldsymbol{\sigma} = \sigma_{xx} \mathbf{e}_x \mathbf{e}_x$

$\mathbf{n} = \mathbf{n} \cdot \mathbf{e}_x \mathbf{e}_x + \mathbf{n} \cdot \mathbf{e}_y \mathbf{e}_y$
$= \cos\alpha \, \mathbf{e}_x + \sin\alpha \, \mathbf{e}_y$

$\mathbf{n} = n_x \mathbf{e}_x + n_y \mathbf{e}_y$ in terms of the original $\mathbf{e}_x, \mathbf{e}_y$

aside $\mathbb{a} = a_x \mathbb{e}_x + a_y \mathbb{e}_y$

$\mathbb{a} \cdot \mathbb{e}_x = a_x \quad \therefore \quad \mathbb{a} = \mathbb{a} \cdot \mathbb{e}_x \mathbb{e}_x + \mathbb{a} \cdot \mathbb{e}_y \mathbb{e}_y$

$$= \mathbb{a} \cdot [\mathbb{e}_x \mathbb{e}_x + \mathbb{e}_y \mathbb{e}_y]$$

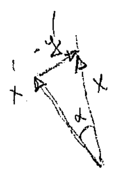
$$= \mathbb{a} \cdot \mathbb{I}$$

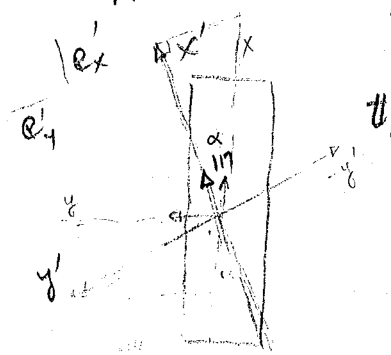
$\mathbb{I}$ identity dyadic
idemfactor

$$\mathbb{t}_n = \mathbb{n} \cdot \mathbb{T}$$

$$= T_{xx} \cos\alpha \; \mathbb{e}_x$$

$$\mathbb{t}_n = \frac{P}{A} \cos\alpha \; \mathbb{e}_x$$





Now since we want to find $\mathbb{t}_n$ in terms of $\mathbb{e}_x', \mathbb{e}_y'$ & we know that $\mathbb{t}_n = (\text{comp of } \mathbb{t}_n \text{ in } \mathbb{e}_x')\mathbb{e}_x' + (\text{comp of } \mathbb{t}_n \text{ in } \mathbb{e}_y')\mathbb{e}_y'$ then

$$\mathbb{t}_n = \mathbb{t}_n \cdot \mathbb{e}_{x'} \mathbb{e}_{x'} + \mathbb{t}_n \cdot \mathbb{e}_{y'} \mathbb{e}_{y'}$$

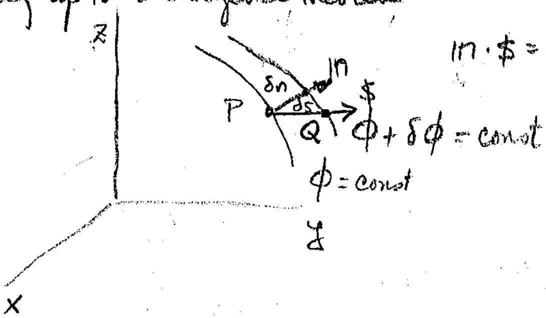
$$= \underbrace{\frac{P}{A}}_{(\mathbb{n}\cdot\mathbb{e}_x)} \cos\alpha \; \underbrace{\mathbb{e}_x \cdot \mathbb{e}_{x'}}_{\cos\alpha \; (\mathbb{e}_x \cdot \mathbb{e}_{x'})} \mathbb{e}_{x'} + \underbrace{\frac{P}{A}\cos\alpha}_{\mathbb{t}_n} \underbrace{(-\sin\alpha)}_{\mathbb{e}_x \cdot (-\mathbb{e}_{y'})} \mathbb{e}_{y'}$$

To work a problem

1. Introducing new coordinate system if convenient & work problem

Divergence Theorem $\int \underline{q} \cdot \underline{n} \, dA = \int_V \nabla \cdot \underline{q} \, dV$

working up to divergence theorem



$\mathbb{n} \cdot \$ = \cos\alpha$

0. let ϕ be a scalar fn.
1. @ P define tangent plane & the normal to that plane
2. draw $ in arbitrary direction let it intersect $\phi + \delta\phi$ @ Q
3. look at $\frac{\delta\phi}{\delta s}$

$\lim_{\delta s \to 0} \frac{\delta\phi}{\delta s} = \frac{\partial\phi}{\partial s}$ (directional derivative)

$\lim_{\delta n \to 0} \frac{\delta\phi}{\delta n} = \frac{\partial\phi}{\partial n}$ (normal derivative) note $\frac{\partial\phi}{\partial n} \geq \frac{\partial\phi}{\partial s}$ since $\partial\phi$ is the same & $\partial s > \partial n$

$$\frac{\partial \phi}{\partial s} = \lim_{\partial s \to 0} \frac{\delta \phi}{\delta n}\frac{\delta n}{\delta s} = \frac{\partial \phi}{\partial n} \cos(\mathbb{n}, \mathbb{s}) = \frac{\partial \phi}{\partial n}\left(\frac{\mathbb{n}\cdot\mathbb{s}}{|\mathbb{n}||\mathbb{s}|}\right) = \mathbb{n}\frac{\partial \phi}{\partial n}\cdot \mathbb{s} = \nabla\phi\cdot\mathbb{s}$$

$\mathbb{n}\frac{\partial \phi}{\partial n} = \nabla\phi$ this is in the directio of greatest or fastest change

$\frac{\partial \phi}{\partial s} = \nabla\phi\cdot\mathbb{s}$ the $\frac{\partial \phi}{\partial x_k} = \nabla\phi\cdot\mathbb{e}_k$

now write $\nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbb{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbb{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbb{e}_3$

define $\nabla(\ ) = \frac{\partial(\ )}{\partial x_1}\mathbb{e}_1 + \frac{\partial(\ )}{\partial x_2}\mathbb{e}_2 + \frac{\partial(\ )}{\partial x_3}\mathbb{e}_3 = \mathbb{e}_i\frac{\partial(\ )}{\partial x_i}$

– now look @ $\mathbb{F} = \mathbb{F}(x,y,z)$
and $\nabla\cdot\mathbb{F}$ (Divergence of $\mathbb{F}$)
$\nabla\times\mathbb{F}$ (cross product or curl of $\mathbb{F}$)

Let $\mathbb{F} = F_1\mathbb{e}_1 + F_2\mathbb{e}_2 + F_3\mathbb{e}_3$

$$\nabla\cdot\mathbb{F} = \mathbb{e}_j\frac{\partial}{\partial x_j}\cdot(F_i\mathbb{e}_i) = \frac{\partial F_i}{\partial x_i} = F_{i,i} \quad \delta_{ij}\frac{\partial F_i}{\partial x_j} = \frac{\partial F_i}{\partial x_i}$$

$$\nabla\times\mathbb{F} = \left(\mathbb{e}_j\frac{\partial}{\partial x_j}\right)\times F_i\mathbb{e}_i = \mathbb{e}_1\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) + \mathbb{e}_2\left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right)$$
$$+\ \mathbb{e}_3\left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) = \mathbb{e}_j\times\mathbb{e}_i\frac{\partial F_i}{\partial x_j} = e_{jil}\,\mathbb{e}_l\frac{\partial F_i}{\partial x_j}$$
$$= e_{ijl}\,\mathbb{e}_l\frac{\partial F_j}{\partial x_i}$$
$$= e_{ijk}\,\mathbb{e}_k\frac{\partial F_j}{\partial x_i}$$

$$\nabla\times\mathbb{F} = e_{ijk}F_{k,j}\mathbb{e}_i$$

$$\nabla\cdot\sigma = \nabla\cdot(\mathbb{t}_k\mathbb{e}_k) = \frac{\partial}{\partial x_1}\mathbb{t}_1 + \frac{\partial}{\partial x_2}\mathbb{t}_2 + \frac{\partial}{\partial x_3}\mathbb{t}_3 \qquad \mathbb{t}_k = \sigma_{kj}\mathbb{e}_j$$

$\nabla\cdot(\mathbb{t}_k\mathbb{e}_k) = \mathbb{e}_i\frac{\partial}{\partial x_i}\cdot(\mathbb{e}_k\sigma_{kj}\mathbb{e}_j)$ $\sigma_{kj,i}\delta_{ik}\mathbb{e}_j = \sigma_{kj,k}\mathbb{e}_j$ or since indices are dummies $\sigma_{jk,j}\mathbb{e}_k$

$$\nabla\cdot\sigma = \sigma_{jk,j}\mathbb{e}_k$$

10/9/78

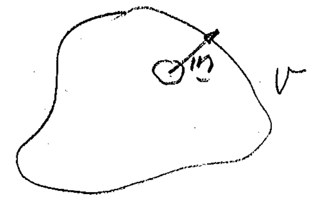
$\nabla\phi,\ \nabla\cdot\mathbb{F},\ \nabla\times\mathbb{F}$

we can extend ∇ to dyadic $\Rightarrow$ $\nabla\cdot\sigma = \sigma_{jk,j}\mathbb{e}_k$

Divergence Theorem (Gauss) ·

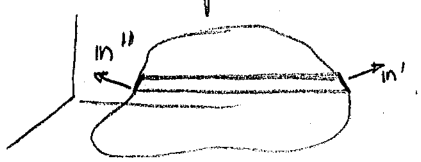
volume V enclosed by surface S Vector point function $\mathbb{G} = G(x,y,z)$

$$\int_V \nabla\cdot\mathbb{G}\, dV = \int_S \mathbb{G}\cdot\mathbb{n}\, ds$$



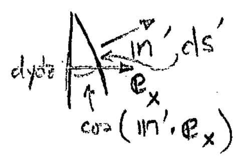
$$\int_V G_{i,i}\, dV = \int_S n_i G_i\, ds$$

look at prism $\parallel$ to x axis

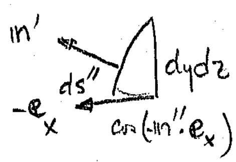


$$\iiint_V \frac{\partial G_x}{\partial x}\, dx\, dy\, dz = \iint (G_x' - G_x'')\, dy\, dz$$

$$dy\,dz = ds'\, \mathbb{n}'\cdot\mathbb{e}_x$$



$$\therefore\quad dy\,dz = ds' \cos(\mathbb{n}', \mathbb{e}_x) = ds' \frac{\mathbb{n}'\cdot\mathbb{e}_x}{\|\mathbb{n}'\|\|\mathbb{e}_x\|} = ds'\, \mathbb{n}'\cdot\mathbb{e}_x$$



$$dy\,dz = +ds'' \cos(\mathbb{n}'', -\mathbb{e}_x) = -ds'' \frac{\mathbb{n}''\cdot\mathbb{e}_x}{\|\mathbb{n}''\|\|-\mathbb{e}_x\|} = -ds''\, \mathbb{n}''\cdot\mathbb{e}_x$$

$$\therefore\quad \iint G_x'\, ds'\, \mathbb{n}'\cdot\mathbb{e}_x + G_x''\, ds''\, \mathbb{n}''\cdot\mathbb{e}_x = \int_S G_x\, \mathbb{n}\cdot\mathbb{e}_x\, ds$$

$$= \int_S n_x G_s\, ds$$

to show $\displaystyle\int_V \nabla\cdot\boldsymbol{\sigma}\, dV = \int_S \mathbb{n}\cdot\boldsymbol{\sigma}\, ds$

$$\int \nabla\cdot\boldsymbol{\sigma}\, dV = \int_V \nabla\cdot(\mathbb{t}_k \mathbb{e}_k)\, dV = \mathbb{e}_k \int_V \nabla\cdot\mathbb{t}_k\, dV$$

$$\mathbb{t}_j n_j = (\mathbb{t}_j \mathbb{e}_j)\cdot\mathbb{n}$$

$$n_j \mathbb{e}_j \cdot \sigma_{kl}\mathbb{e}_l = n_j \sigma_{kl}\delta_{jl} = n_j \sigma_{kj} \qquad \overbrace{\mathbb{e}_k \sigma_{kj} n_j}^{\sigma}$$

$$= \mathbb{e}_k \int_S \mathbb{n}\cdot\mathbb{t}_k\, ds = \mathbb{e}_k \int n_j \sigma_{kj}\, ds = \int \mathbb{e}_k n_j \sigma_{kj}\, ds$$

since $\mathbb{t}_k = \sigma_{kj}\mathbb{e}_j$ $\boxed{\nabla\cdot\mathbb{t}_k = \sigma_{kj,k}\mathbb{e}_j \text{ or } \sigma_{jk,j}\mathbb{e}_k}$ $\displaystyle\int_S \boldsymbol{\sigma}\cdot\mathbb{n}\, ds =$

or $\displaystyle\int_V \sigma_{jk,j}\, dV = \int n_j \sigma_{jk}\, ds$

to Principle of Linear Momentum (PLM) [to prove $(\sigma_{ij} = \sigma_{ji})$]

$$\int_S \mathbb{t}_n \, ds = \int_S \mathbb{n} \cdot \sigma \, ds = \int_V \nabla \cdot \sigma \, dV$$

$$\int_S \mathbb{t}_n \, ds + \int_V \mathbb{f} \, dV = +\int \rho \frac{\partial^2 \mathbb{u}}{\partial t^2} \, dV$$

or $$\int_V \left[\nabla \cdot \sigma + \mathbb{f} - \rho \frac{\partial^2 \mathbb{u}}{\partial t^2} \right] dV = 0 \Rightarrow \nabla \cdot \sigma + \mathbb{f} = \rho \frac{\partial^2 \mathbb{u}}{\partial t^2}$$

we will only look @ statics w/ $\frac{\partial^2}{\partial t^2}$ terms out

$$\nabla \cdot \sigma + \mathbb{f} = 0 \qquad \sigma_{jk,j} + f_k = 0$$ will be our basic equation

to Principle of Angular Mom

take PLM $\times$ $\mathbb{r}$ with $\mathbb{r} = x_i \mathbb{e}_i$

$$\int_S \mathbb{t}_n \times \mathbb{r} \, ds + \int_V \mathbb{f} \times \mathbb{r} \, ds = \int_V \rho \frac{\partial^2 \mathbb{u}}{\partial t^2} \times \mathbb{r} \, dV$$

recall that $\mathbb{t}_n = \sigma_{nk} \mathbb{e}_k$ and write out eqn. wrt z-direction This arises from x cross y or y cross x term

$$\int_S (\sigma_{nx} y - \sigma_{ny} x) \, ds + \int_V (f_x y - f_y x) \, dV = \int_V \rho \left(\frac{\partial^2 u_x}{\partial t^2} y - \frac{\partial^2 u_y}{\partial t^2} x \right) dV$$

$$\sigma_{nx} = \mathbb{n} \cdot \sigma \cdot \mathbb{e}_x = \mathbb{n} \cdot (\sigma_{ij} \mathbb{e}_i \mathbb{e}_j) \cdot \mathbb{e}_x = \mathbb{n} \cdot (\sigma_{ix} \mathbb{e}_i) = \sigma_{xx} \, \mathbb{n} \cdot \mathbb{e}_x + \sigma_{yx} \, \mathbb{n} \cdot \mathbb{e}_y + \sigma_{zx} \, \mathbb{n} \cdot \mathbb{e}_z$$

and $\mathbb{n} \cdot \mathbb{e}_i = n_i$ (component of $\mathbb{n}$ in $\mathbb{e}_i$ direction)

$$\int_S \left([n_x \sigma_{xx} + n_y \sigma_{yx} + n_z \sigma_{zx}] y - [n_x \sigma_{xy} + n_y \sigma_{yy} + n_z \sigma_{zy}] x \right) ds$$

Apply $\int_V \sigma_{jk,j} \, dV = \int_S n_j \sigma_{jk} \, ds$

$$\int_V \left\{ \left[\frac{\partial(\sigma_{xx} y)}{\partial x} + \frac{\partial(\sigma_{yx} y)}{\partial y} + \frac{\partial(\sigma_{zx} y)}{\partial z} \right] - \left[\frac{\partial}{\partial x}(\sigma_{xy} x) + \frac{\partial}{\partial y}(\sigma_{yy} x) + \frac{\partial}{\partial z}(\sigma_{zy} x) \right] \right.$$

or $$(\sigma_{xx,x} + \sigma_{yx,y} + \sigma_{zx,z}) y + \sigma_{yx} - \sigma_{xy} - (\sigma_{xy,x} + \sigma_{yy,y} + \sigma_{zy,z}) x + (f_x) y - \left(\rho \frac{\partial^2 u_x}{\partial t^2}\right) y$$
$$- f_y x + \frac{\partial^2 u_y}{\partial t^2} x = 0$$ but $\nabla = (f_x + \rho \frac{\partial^2 u_x}{\partial t^2}) y$ which cancels w/

look at coeff of y that is PLM in x dir $\not{c} = 0$
of x that is PLM in y dir $\not{d} = 0$

$$\Rightarrow \int_V (-\sigma_{xy} + \sigma_{yx})\, dV = 0 \quad \Rightarrow \sigma_{xy} = \sigma_{yx}$$ since any dV is to be used

$\sigma_{jk} = \sigma_{kj} \quad k \neq j$

10/10/78

repeated derivation in indicial notation

P.A.M.

$$\int_S e_{ijk}\, \sigma_{nj}\, x_k\, ds + \int_V e_{ijk} f_j x_k\, dV = \int_V \rho e_{ijk} \frac{\partial^2 u_j}{\partial t^2} x_k\, dV$$

$$\int_S e_{ijk}\, n_\ell \sigma_{\ell j}\, x_k\, ds = \int_S n_\ell [\sigma_{\ell j} x_k e_{ijk}]\, ds$$

applying divergence theorem $\int_V [\sigma_{\ell j} x_k e_{ijk}]_{,\ell}\, dV = \int_S n_\ell [\sigma_{\ell j} x_k e_{ijk}]\, ds$

or

$$\int e_{ijk} \Big[\sigma_{\ell j,\ell}\, x_k + \underbrace{\sigma_{\ell j} \overset{x_{k,\ell}}{\delta_{k\ell}}}_{\sigma_{kj}} + \sigma_{\ell j} x_k (\overset{e_{ijk,\ell}}{\text{zero}}) + f_j x_k - \rho \frac{\partial^2 u_j}{\partial t^2} x_k \Big] dV = 0$$

but $\sigma_{\ell j,\ell} + f_j - \rho \frac{\partial^2 u_j}{\partial t^2} = 0$ by principle of linear mom $\therefore$

$$\int_V e_{ijk}\, \sigma_{kj}\, dV = 0 \quad \text{and} \Rightarrow e_{ijk} \sigma_{kj} = 0 \quad \Rightarrow \sigma_{kj} = \sigma_{jk}$$

now for non orthogonal directions m, n is $\sigma_{mn} = \sigma_{nm}$?

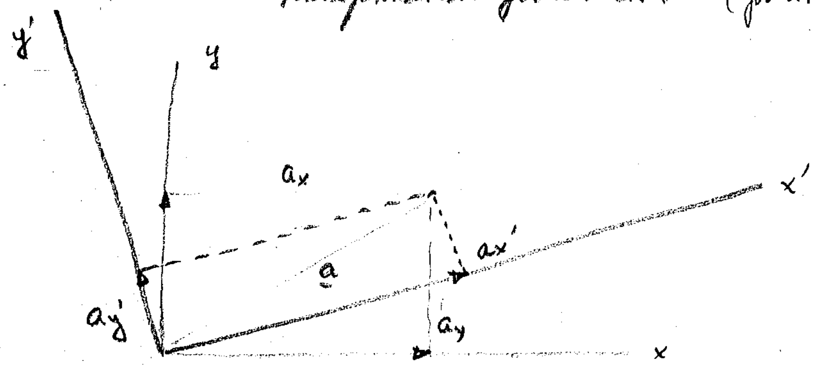
$$\sigma_{mn} = t_m \cdot n = m \cdot \sigma \cdot n = m \cdot e_k t_k \cdot n = m_k t_k \cdot n$$

this from angular momentum

or $$m_k \sigma_{kj} n_j = m_k \sigma_{jk} n_j = n_j \sigma_{jk} m_k = n \cdot t_k m_k = n \cdot t_m = n \cdot \sigma \cdot m$$

$$= \sigma_{nm} \quad \therefore \quad \sigma_{mn} = \sigma_{nm}$$

Coordinate transformation for a vector (for orthogonal systems)



Note that magnitude of $\bar{a}$ is same in all coordinate systems $\therefore a_x^2 + a_y^2 = a_x'^2 + a_y'^2$

transformation of coordinate

$$\sigma_{kl}' = \mathbf{e}_k' \cdot \boldsymbol{\sigma} \cdot \mathbf{e}_l' \qquad \text{where } \boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i\mathbf{e}_j$$

then $\sigma_{kl}' = \mathbf{e}_k' \cdot \sigma_{ij}\mathbf{e}_i\mathbf{e}_j \cdot \mathbf{e}_l'$

define for $\mathbf{e}_j' \cdot \mathbf{e}_i = l_{ji} = \cos(\mathbf{e}_j', \mathbf{e}_i)$ note that $l_{ij} \neq l_{ji}$

$$\therefore \mathbf{e}_1' = \underbrace{(\mathbf{e}_1' \cdot \mathbf{e}_1)}_{l_{11}}\mathbf{e}_1 + \underbrace{(\mathbf{e}_1' \cdot \mathbf{e}_2)}_{l_{12}}\mathbf{e}_2 + \underbrace{(\mathbf{e}_1' \cdot \mathbf{e}_3)}_{l_{13}}\mathbf{e}_3$$

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_1'$	l_{11}	l_{12}	l_{13}
$\mathbf{e}_2'$	l_{21}	l_{22}	l_{23}
$\mathbf{e}_3'$	l_{31}	l_{32}	l_{33}

$\mathbf{e}_1' \cdot \mathbf{e}_1' = l_{11}^2 + l_{12}^2 + l_{13}^2 = 1$ 3 equations of this type exist

3 orthogonal relations $\mathbf{e}_1' \cdot \mathbf{e}_2' = l_{11}l_{21} + l_{12}l_{22} + l_{13}l_{23} = 0$

$\therefore \mathbf{e}_j' \cdot \mathbf{e}_k' = \delta_{jk}$

To prove by indicial notation

$$\text{or } \mathbf{e}_j' \cdot \mathbf{e}_k' = (l_{j\alpha}\mathbf{e}_\alpha) \cdot l_{k\beta}\mathbf{e}_\beta) = l_{j\alpha}l_{k\beta}\delta_{\alpha\beta} = l_{j\alpha}l_{k\alpha} = \delta_{jk}$$

now $\mathbf{a} = a_i\mathbf{e}_i = a_i'\mathbf{e}_i'$

$$a_i' = \mathbf{a} \cdot \mathbf{e}_i' = a_k\mathbf{e}_k \cdot \mathbf{e}_i' = a_k l_{ik}$$

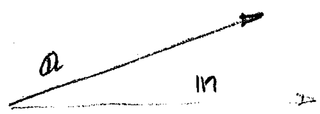
$$a_i = \mathbf{a} \cdot \mathbf{e}_i = a_k'\mathbf{e}_k' \cdot \mathbf{e}_i = a_k' l_{ki}$$

back to

$$\sigma_{mn}' = \mathbf{e}_m' \cdot \boldsymbol{\sigma} \cdot \mathbf{e}_n' = \sigma_{ij}(\mathbf{e}_m' \cdot \mathbf{e}_i)(\mathbf{e}_j \cdot \mathbf{e}_n') \quad \text{or} \quad \boldsymbol{\sigma}' = \mathbb{L}\boldsymbol{\sigma}\mathbb{L}^T$$

$$\boxed{\sigma_{mn}' = \sigma_{ij}\, l_{mi}\, l_{nj}}$$

10/13/78



if we want component of $\mathbf{a}$ along $\mathbf{n}$

take $\mathbf{a} \cdot \mathbf{n} = a_n$

$\mathbf{a} \cdot \mathbf{n}$ is max when $\mathbf{a} \parallel \mathbf{n}$

min when $\mathbf{a} \perp \mathbf{n}$

ask what about $\boldsymbol{\sigma}$

we know that $\mathbf{t}_n = \boldsymbol{\sigma} \cdot \mathbf{n}$; The stress tensor on the plane whose normal is $\mathbf{n}$

lets look $\sigma_{nn} = \mathbf{n} \cdot \mathbf{t}_n = \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}$ how does it vary (this is the component of $\mathbf{t}_n$ in $\mathbf{n}$ dir

- look at $\sigma_{nn} = \sigma_{nn}(n)$ then $d\sigma_{nn} = d\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) + \mathbf{n} \cdot (\boldsymbol{\sigma} \cdot d\mathbf{n})$
- note that since $\boldsymbol{\sigma}$ is invariant $\Rightarrow d\boldsymbol{\sigma} = 0$
- since $\boldsymbol{\sigma}$ is symmetric then $d\sigma_{nn} = 2 d\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}$

for a maximum set $d\sigma_{nn} = 2 d\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) = 0$

since $\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}_n \Rightarrow d\mathbf{n} \cdot \mathbf{t}_n = 0$ $\therefore$ $\mathbf{t}_n$ must be $\perp d\mathbf{n}$

hence $\mathbf{t}_n$ must be $\parallel \mathbf{n}$ since $\mathbf{n}$ is unit vector can only change in direction but $d\mathbf{n} \perp \mathbf{n}$

$d\mathbf{n} \perp \mathbf{n}$

now $\sigma_{nn} = \sigma_{ij} n_i n_j$ & let λ = a parameter; since $\mathbf{t}_n \parallel \mathbf{n}$ then $\mathbf{t}_n = \lambda \mathbf{n}$

$$d\sigma_{nn} = \sigma_{ij}\, dn_i\, n_j + \sigma_{ij}\, n_i\, dn_j = 2\sigma_{ij}\, dn_j\, n_i$$

since $\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}_n = \lambda \mathbf{n} \Rightarrow \sigma_{ij} n_i = \lambda n_j$

let $\lambda \mathbf{n} = \lambda \hat{\mathbb{I}} \cdot \mathbf{n}$ or the Kronecker δ

$$\therefore \quad \sigma_{ij} n_i = \lambda \delta_{ij} n_i$$

$$\therefore \quad (\sigma_{ij} - \lambda \delta_{ij}) n_i = 0 \quad \text{or} \quad (\boldsymbol{\sigma} - \lambda \hat{\mathbb{I}}) \cdot \mathbf{n} = 0$$

$$\left.\begin{aligned}
(\sigma_{xx} - \lambda) n_x + \sigma_{yx} n_y + \sigma_{zx} n_z &= 0 \\
\sigma_{xy} n_x + (\sigma_{yy} - \lambda) n_y + \sigma_{zy} n_z &= 0 \\
\sigma_{xz} n_x + \sigma_{yz} n_y + (\sigma_{zz} - \lambda) n_z &= 0
\end{aligned}\right\}$$ for a non trivial solution is that det $= 0$

$$\begin{vmatrix} \sigma_{xx} - \lambda & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} - \lambda & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \lambda \end{vmatrix} = 0$$

Next hw assign expand this and get the cubic

$$-\lambda^3 + I\lambda^2 - II\lambda + III = 0$$

$$I = \sigma_{kk} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$II = \tfrac{1}{2} e_{mik}\, e_{njl}\, \sigma_{ij}\sigma_{kl}\, \delta_{mn} = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{zx}^2$$

$$III = \tfrac{1}{6} e_{mik}\, e_{njl}\, \sigma_{ij}\sigma_{kl}\sigma_{mn} = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{xz}^2 - \sigma_{zz}\sigma_{yx}^2$$

these are stress invariants

Next time we will show that all λ are Re and $\lambda > 0$ and will be in 3 $\perp$ direction.
and that I, II, III are stress invariants.

10/16/78

6 Nov – midterm 1 hr. open notes

we were working with $\sigma \cdot n = \lambda n$

$$\sigma_{ij} n_i = \lambda n_j = \lambda \delta_{ij} n_i \quad \text{or} \quad (\sigma_{ij} - \lambda\delta_{ij}) n_i = 0$$

for non triv solution $\det|\sigma_{ij} - \lambda\delta_{ij}| = 0 \implies -\lambda^3 + I\lambda^2 - II\lambda + III = 0$

We will now prove that $\lambda^{(k)}$ are real (obvious for a symmetric dyadic).

Solutions will be $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$; for each $\lambda^{(k)}$ we can associate $n^{(k)}$

Assume $\lambda^{(k)} = \alpha^{(k)} + i\beta^{(k)}$ ie $\lambda^{(k)}$ is complex

$n^{(k)} = a^{(k)} + i\, b^{(k)}$ ie $n^{(k)}$ is complex

$$n^{(k)} \cdot \sigma = \lambda^{(k)} n^{(k)} \quad \text{or} \quad (a^{(k)} + i b^{(k)}) \cdot \sigma = (\alpha^{(k)} + i\beta^{(k)})(a^{(k)} + i\, b^{(k)})$$

$$a^{(k)} \cdot \sigma + i\, b^{(k)} \cdot \sigma = \alpha^{(k)} a^{(k)} + i\left[a^{(k)}\beta^{(k)} + b^{(k)}\alpha^{(k)}\right] - \beta^{(k)} b^{(k)}$$

1. equate real & imaginary parts
2. mult $a^{(k)} \cdot \sigma$ by $b^{(k)}$ and then $b^{(k)} \cdot \sigma$ by $a^{(k)}$ and look at real part of $n^{(k)} \cdot \sigma$

same since σ is symmetric

$$a^{(k)} \cdot \sigma \cdot b^{(k)} = \alpha^{(k)} a^{(k)} \cdot b^{(k)} - \beta^{(k)} b^{(k)} \cdot b^{(k)}$$

$$b^{(k)} \cdot \sigma \cdot a^{(k)} = a^{(k)} \beta^{(k)} \cdot a^{(k)} + b^{(k)} \alpha^{(k)} \cdot a^{(k)}$$

Now subtract 1st from 2nd $\implies 0 = \beta^{(k)}\left[a^{(k)} \cdot a^{(k)} + b^{(k)} \cdot b^{(k)}\right] = \beta^{(k)} n^{(k)} \cdot n^{(k)}$

$= \beta^{(k)} \cdot 1$ since magnitude $n = 1$

$\therefore \beta^{(k)} = 0$ & $\lambda^{(k)}$ must therefore be real.

Look at eigenvector $n^{(1)} \cdot \sigma = \lambda^{(1)} n^{(1)}$, $n^{(2)} \cdot \sigma = \lambda^{(2)} n^{(2)}$

$n^{(1)} \cdot \sigma \cdot n^{(2)} = \lambda^{(1)} n^{(1)} \cdot n^{(2)}$ also $n^{(2)} \cdot \sigma \cdot n^{(1)} = \lambda^{(2)} n^{(2)} \cdot n^{(1)}$

since σ is symmetric $n^{(1)} \cdot \sigma \cdot n^{(2)} - n^{(2)} \cdot \sigma \cdot n^{(1)} = 0$

$\therefore \quad n^{(1)} \cdot n^{(2)} \left[\lambda^{(1)} - \lambda^{(2)}\right] = 0$

if $\lambda^{(i)} \neq \lambda^{(k)} \Rightarrow \mathbf{n}^{(i)} \perp \mathbf{n}^{(k)}$

if $\lambda^{(i)} = \lambda^{(k)}$ we can no longer say that $\mathbf{n}^{(i)} \perp \mathbf{n}^{(k)}$

but in general $\lambda^{(i)} \neq \lambda^{(k)}$.

$$\begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}$$

$\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}$ are perpendicular $\therefore$ only the normal components exist and the shear components are zero on planes $\perp$ to principal directions at every point we can define the principal axes axes along $\mathbf{n}^{(1)}, \mathbf{n}^{(2)}$ and $\mathbf{n}^{(3)}$. The stress tensor along $\mathbf{n}^{(1,2,3)}$ is given by

$$\begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}$$

if all 3 λ are equal we have hydrostatic pressure state.

TO OBTAIN THE EIGENVECTORS

Look for solutions of $\mathbf{n}^{(k)}$ where ${n_x^{(k)}}^2 + {n_y^{(k)}}^2 + {n_z^{(k)}}^2 = 1$

$$(\sigma_{xx} - \lambda^{(k)}) n_x^{(k)} + \sigma_{xy} n_y^{(k)} = -\sigma_{xz} n_z^{(k)}$$

$$\sigma_{yx} n_x^{(k)} + (\sigma_{yy} - \lambda^{(k)}) n_y^{(k)} = -\sigma_{yz} n_z^{(k)}$$

$$n_x^{(k)} = \frac{n_z^{(k)} \begin{vmatrix} -\sigma_{xz} & \sigma_{xy} \\ -\sigma_{yz} & \sigma_{yy} - \lambda^{(k)} \end{vmatrix}}{\begin{vmatrix} \sigma_{xx} - \lambda^{(k)} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} - \lambda^{(k)} \end{vmatrix}} = \frac{D_x n_z^{(k)}}{D}$$

$$n_y^{(k)} = \frac{n_z^{(k)} \begin{vmatrix} \sigma_{xx} - \lambda^{(k)} & -\sigma_{xz} \\ -\sigma_{yx} & -\sigma_{yz} \end{vmatrix}}{D} = \frac{D_y n_z^{(k)}}{D}$$

put into $n_x^2 + n_y^2 + n_z^2 = 1 \Rightarrow {n_z^{(k)}}^2 \underbrace{\left[\frac{D_x^2}{D^2} + \frac{D_y^2}{D^2} + 1 \right]}_{>1} = 1$

$$\therefore \quad n_z^{(k)} = \pm \sqrt{\frac{D^2}{D_x^2 + D_y^2 + D^2}}$$

since D_x, D_y, D are real $\Rightarrow n_z^{(k)}, n_y^{(k)}, n_x^{(k)}$ are real

Geometric representation of Stress

$\mathbf{t}_n = \sigma \cdot \mathbf{n}$ this is the stress tensor on the surface whose normal is $\mathbf{n}$

$\sigma_{nn} = \mathbf{n} \cdot \mathbf{t}_n = (\mathbf{n} \cdot \sigma \cdot \mathbf{n})$ this is the normal component of $\mathbf{t}_n$

let $N \equiv \sigma_{nn} = \sigma_{ij} n_i n_j$

introduce a local coordinate system w/origin at the point P we are at and define $\mathbb{A} \parallel \mathbb{n}$ and let $X_i = A n_i$ (ie $\mathbb{A}$ starts at the pt we're looking at)

$NA^2 = \sigma_{ij} X_i X_j$ defines a quadratic surface and if we restrict $\mathbb{A}$ to have its tip lie on the surface then

$\sigma_{ij} X_i X_j = \pm k^2 \implies N = \pm \frac{k^2}{A^2}$ this is Cauchy's stress quadric

Since $A^2 > 0 \implies$ if $N > 0$ choose + sign.

the Reciprocal surface (Lame's stress director quadric)

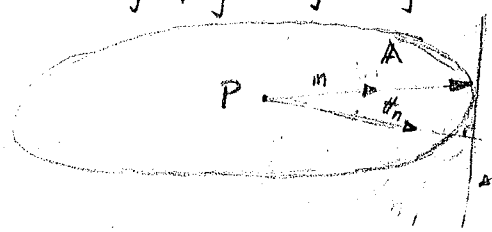
$$\sigma_{ij}^{-1} X_i X_j = \pm k^2 \quad \text{where} \quad \sigma_{ij}^{-1} \sigma_{jk} = \delta_{ik} \quad \text{or} \quad \sigma^{-1} \cdot \sigma = \hat{I}$$

now $|\mathbb{t}_n|^2 = \mathbb{t}_n \cdot \mathbb{t}_n = \mathbb{n} \cdot \sigma \cdot \sigma \cdot \mathbb{n} = (\mathbb{n} \cdot \sigma^2 \cdot \mathbb{n})$

if $\mathbb{n} \cdot \sigma^2 \cdot \mathbb{n} = k^2$ ellipsoid (Cauchy's stress ellipse)

$\mathbb{n} \cdot (\sigma^{-1})^2 \cdot \mathbb{n} = k^2$ Lames' stress ellipsoid

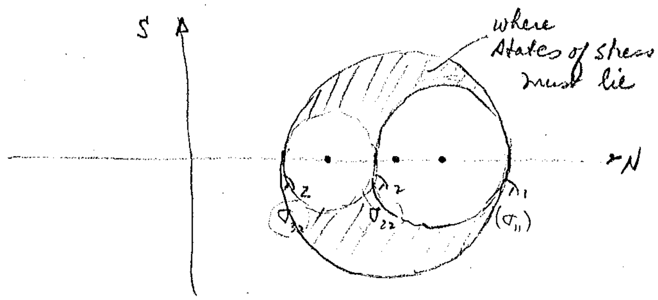
Note that $\sigma_{ij} n_i n_j = \sigma'_{ij} n'_i n'_j = \pm k^2$ (invariant)



tangent plane to stress quadric at P_0. $\mathbb{t}_n$ lies on line $\perp$ to tangent plane and P

10/18/78

TA T 3-5; W 12^{30}-2^{30} Rich King



Mohrs Circle

$S_{max} = \frac{\sigma_1 - \sigma_3}{2}$ where $N = \frac{\sigma_1 + \sigma_3}{2}$

$N_{max} = \sigma_1$, $N_{min} = \sigma_3$ $\quad S = 0$

Examples of States of Stress

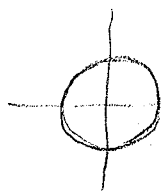
suppose $\sigma_{11} = \sigma_{22} = \sigma_{33}$ HYDROSTATIC or ISOTROPIC state (properties are independent of direction)

stress ellipsoid $\Rightarrow$ a sphere.

simple tension (or compression) $\sigma_{11} \neq 0$ $\sigma_{22} = \sigma_{33} = 0$

pure shear principle stresses $\sigma_{33} = 0$, $\sigma_{22} = -\sigma_{11}$

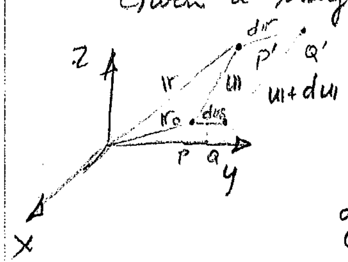
mohrs circle



Plane stress $\sigma_{33} = 0$

Analysis of Deformation

Given a body w/a pt before deformation & then after



$d\mathbf{r}_0$ is infinitesimal with respect to $\mathbf{r}_0$

$\mathbf{u}$ is displ vector of P to P'

$\mathbf{u} + d\mathbf{u}$ is change in displ of Q to Q'

given that $P(x_0, y_0, z_0)$ & $P'(x, y, z)$ then

$$\mathbf{r}_0 = X_{0i}\,\mathbf{e}_i \qquad \mathbf{r} = X_i\,\mathbf{e}_i$$

note vectorially $\mathbf{u} + d\mathbf{r} = d\mathbf{r}_0 + \mathbf{u} + d\mathbf{u}$ and that $d\mathbf{r} = d\mathbf{r}_0 + d\mathbf{u}$

remember $\nabla\phi = \mathbf{e}_j \dfrac{\partial\phi}{\partial x_j}$ and $d\mathbf{r} = dX_k\,\mathbf{e}_k$

hence $d\mathbf{r}\cdot\nabla\phi = \delta_{jk}\dfrac{\partial\phi}{\partial x_j}dX_k = \dfrac{\partial\phi}{\partial x_j}dx_j = d\phi$

thus $d\phi = d\mathbf{r}\cdot\nabla\phi$ differential of pt fn = inner product of ∇ and differential of position vector

also $\nabla\mathbf{u} = \left(\mathbf{e}_j\dfrac{\partial}{\partial x_j}\right)\left(\mathbf{e}_k u_k\right) = \mathbf{e}_j\mathbf{e}_k\dfrac{\partial u_k}{\partial x_j}$ – second rank tensor

$$d\mathbf{r}\cdot\nabla\mathbf{u} = (\mathbf{e}_j\,dx_j)\cdot\left(\mathbf{e}_k\mathbf{e}_\ell\frac{\partial u_\ell}{\partial x_k}\right) = \delta_{jk}\,\mathbf{e}_\ell\frac{\partial u_\ell}{\partial x_k}dx_j = \mathbf{e}_\ell\frac{\partial u_\ell}{\partial x_k}dx_k$$

$$= \mathbf{e}_\ell\,du_\ell = d\mathbf{u}$$

$d\mathbf{u} = (d\mathbf{r} \cdot \nabla)\mathbf{u}$ $\qquad$ $\nabla \mathbf{u}$ is the displacement gradient; second rank tensor which is not symmetric [ie $\frac{\partial u_k}{\partial x_j} e_j e_k \neq \frac{\partial u_j}{\partial x_k} e_k e_j$ in general

now substitute into $d\mathbf{r}_0 + d\mathbf{u} = d\mathbf{r}$ or $\boxed{d\mathbf{r}_0 = d\mathbf{r} - d\mathbf{u} = d\mathbf{r} \cdot [\hat{\mathbb{I}} - \nabla \mathbf{u}]}$

relates undefm. state to defm. state $\qquad$ Eulerian Deformati tensor.

$$\hat{\mathbb{I}} - \nabla \mathbf{u} = \delta_{ij} - \frac{\partial u_j}{\partial x_i}$$ since ∇ operation is first $\Rightarrow$ 1st subscript of $\frac{\partial u_j}{\partial x_i}$ must be associated with differentiation

Conjugate dyadic of $\mathbf{a}\mathbf{b}$ is $\mathbf{b}\mathbf{a}$ or $(\mathbf{a}\mathbf{b})_c = \mathbf{b}\mathbf{a}$

$(\nabla \mathbf{u})_c$ = Conjugate dyadic of $\nabla \mathbf{u} = \mathbf{u}\nabla = (e_i u_i)(e_j \frac{\partial}{\partial x_j}) = e_i e_j \frac{\partial u_i}{\partial x_j}$

one can show that $d\mathbf{u} = (\mathbf{u}\nabla) \cdot d\mathbf{r}$ $\qquad$ $d\mathbf{u}_c = d\mathbf{u} = (d\mathbf{r})_c \cdot (\nabla \mathbf{u})_c = (\nabla \mathbf{u})_c \cdot (d\mathbf{r})_c = (\mathbf{u}\nabla) \cdot (d\mathbf{r}) = \mathbf{u}\nabla \cdot d\mathbf{r}$

$\therefore \quad d\mathbf{r} = d\mathbf{r}_0 + (\mathbf{u}\nabla) \cdot d\mathbf{r}$; $\boxed{d\mathbf{r}_0 = d\mathbf{r} - (\mathbf{u}\nabla) \cdot d\mathbf{r} = [\hat{\mathbb{I}} - \mathbf{u}\nabla] \cdot d\mathbf{r}}$

If we define differentiation wrt fixed reference state then

$$\nabla_0 = e_{x_0} \frac{\partial}{\partial x_0} + e_{y_0} \frac{\partial}{\partial y_0} + e_{z_0} \frac{\partial}{\partial z_0}$$

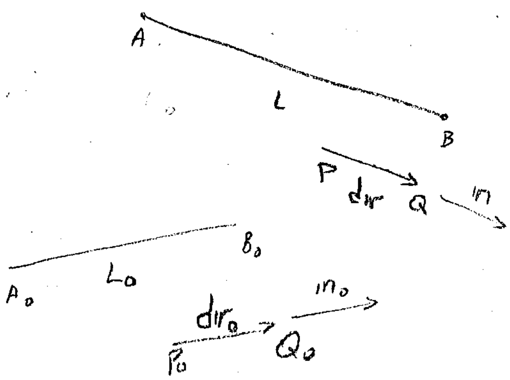
$$\therefore \quad d\mathbf{r}_0 \cdot \nabla_0 (\,) = d_0 (\,) \quad \Rightarrow d\mathbf{r}_0 \cdot \nabla_0 \phi = d_0 \phi$$

$$\therefore \quad d\mathbf{r} = d\mathbf{r}_0 + d\mathbf{u} = d\mathbf{r}_0 \cdot (\hat{\mathbb{I}} + \nabla_0 \mathbf{u}) = d\mathbf{r}_0 + (d\mathbf{r}_0 \cdot \nabla)\mathbf{u}$$

$$\boxed{d\mathbf{r} = \underbrace{(\hat{\mathbb{I}} + \mathbf{u}\nabla_0)}_{\text{Lagrangian deformation tensor}} \cdot d\mathbf{r}_0 = d\mathbf{r}_0 \cdot (\hat{\mathbb{I}} + \nabla_0 \mathbf{u})} \quad : \quad d\mathbf{r}_0 \cdot \left[\delta_{ij} + \frac{\partial u_j}{\partial x_{0i}}\right] e_i e_j$$

relates deformed state to undeformed state

10/20/78



Eulerian stretch

$$\eta_{LL} = \frac{L^2 - L_0^2}{2L^2}$$

Lagrangian stretch

$$\epsilon_{LL} = \frac{L^2 - L_0^2}{2L_0^2}$$

$$\therefore \eta_{LL} = \left(\frac{L_0}{L}\right)^2 \epsilon_{LL}$$

Strength of materials definition of elongation

$$E_{LL} = \frac{L-L_0}{L_0} = \frac{\Delta L}{L_0} \qquad \text{if } \frac{\Delta L}{L_0} \ll 1 \text{ then } \eta_{LL} = \epsilon_{LL} = E_{LL}$$

$$\eta_{LL} = \frac{(L_0+\Delta L)^2 - L_0^2}{2(L_0+\Delta L)^2} = \frac{L_0^2 + 2\Delta L\, L_0 + \Delta L^2 - L_0^2}{2(L_0+\Delta L)^2} = \frac{2\left(\frac{\Delta L}{L_0}\right) + \left(\frac{\Delta L}{L_0}\right)^2}{2\left(1 + \frac{2\Delta L}{L_0} + \left(\frac{\Delta L}{L_0}\right)^2\right)} \approx \frac{2\frac{\Delta L}{L_0}}{2} = \frac{\Delta L}{L_0}$$

if we neglect all second order terms then

$$\eta_{rr} = \frac{d\mathbf{r}\cdot d\mathbf{r} - d\mathbf{r}_0 \cdot d\mathbf{r}_0}{2\, d\mathbf{r}\cdot d\mathbf{r}} = \frac{d\mathbf{r}\cdot d\mathbf{r} - d\mathbf{r}\cdot(\hat{\mathbb{I}} - \nabla\mathbf{u})\cdot(\hat{\mathbb{I}} - \mathbf{u}\nabla)\cdot d\mathbf{r}}{2\, d\mathbf{r}\cdot d\mathbf{r}}$$

$$(\hat{\mathbb{I}} - \nabla\mathbf{u})\cdot(\hat{\mathbb{I}} - \mathbf{u}\nabla) = \hat{\mathbb{I}} - \nabla\mathbf{u} - \mathbf{u}\nabla + \nabla\mathbf{u}\cdot\mathbf{u}\nabla$$

$$d\mathbf{r}\cdot d\mathbf{r} = d\mathbf{r}\cdot\hat{\mathbb{I}}\cdot d\mathbf{r}$$

$$\therefore \quad \eta_{rr} = \frac{d\mathbf{r}\cdot\left(\nabla\mathbf{u} + \mathbf{u}\nabla - \nabla\mathbf{u}\cdot\mathbf{u}\nabla\right)\cdot d\mathbf{r}}{2|d\mathbf{r}^2|}$$

note $\frac{d\mathbf{r}}{|d\mathbf{r}|} = \mathbf{n}$ unit vector in the $d\mathbf{r}$ direction

$$\eta_{rr} = \mathbf{n}\cdot\tfrac{1}{2}\left(\nabla\mathbf{u} + \mathbf{u}\nabla - \nabla\mathbf{u}\cdot\mathbf{u}\nabla\right)\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{\Phi}\cdot\mathbf{n}$$ defines change in length

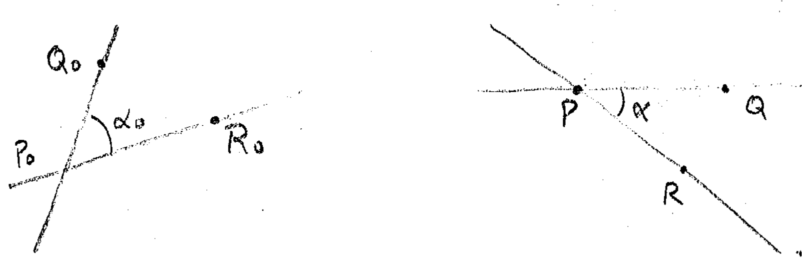
$$\tfrac{1}{2}\left(\nabla\mathbf{u} + \mathbf{u}\nabla - \nabla\mathbf{u}\cdot\mathbf{u}\nabla\right) = \mathbf{\Phi}$$

$$\nabla\mathbf{u}\cdot\mathbf{u}\nabla = \frac{\partial}{\partial x_i}u_j e_i e_j \cdot \frac{\partial u_k}{\partial x_\ell}e_k e_\ell = \frac{\partial u_j}{\partial x_i}\frac{\partial u_k}{\partial x_\ell}e_i e_\ell \delta_{jk} = \frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_\ell}e_i e_\ell$$

$$\boxed{\phi_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)}$$

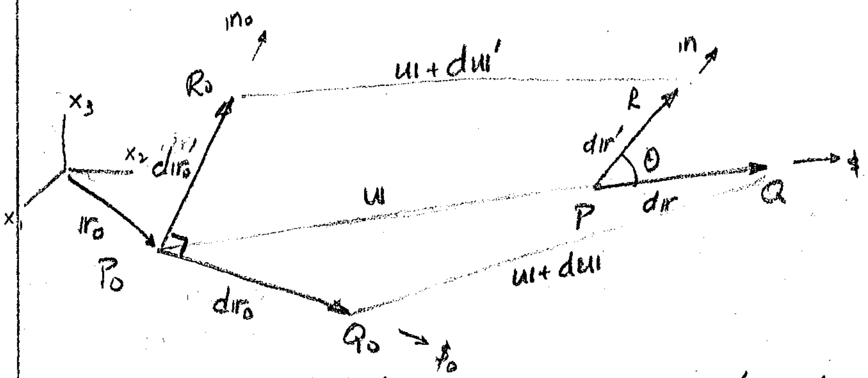
Eulerian Stretch Tensor

$$\mathbf{\Phi} = e_i \phi_{ij} e_j$$



look at angles that change w/ deformation

and let us restrict our attention to angles that were initially $\perp$



$$\frac{1}{2}\cos\theta = \frac{1}{2}\frac{d\mathbf{r}'\cdot d\mathbf{r}}{|d\mathbf{r}'||d\mathbf{r}|} \qquad d\mathbf{r}' = d\mathbf{r}_0' + d\mathbf{u}' \qquad d\mathbf{r} = d\mathbf{r}_0 + d\mathbf{u}$$

$$d\mathbf{u}' = d\mathbf{r}'\cdot(\nabla\mathbf{u}) \qquad d\mathbf{u} = (\mathbf{u}\nabla)\cdot d\mathbf{r}$$

$$d\mathbf{r}'\cdot d\mathbf{r} = \left[d\mathbf{r}_0' + d\mathbf{r}'\cdot(\nabla\mathbf{u})\right]\cdot\left[d\mathbf{r}_0 + \mathbf{u}\nabla\cdot d\mathbf{r}\right]$$

note $d\mathbf{r}_0'\cdot d\mathbf{r}_0 = 0$ since $d\mathbf{r}_0' \perp d\mathbf{r}_0$

$$= d\mathbf{r}_0'\cdot\mathbf{u}\nabla\cdot d\mathbf{r} + d\mathbf{r}'\cdot(\nabla\mathbf{u})\cdot d\mathbf{r}_0 + (d\mathbf{r}'\cdot\nabla\mathbf{u})\cdot(\mathbf{u}\nabla\cdot d\mathbf{r})$$

now substitute for $d\mathbf{r}_0'$ and $d\mathbf{r}_0$ in terms of $d\mathbf{r}'$ and $d\mathbf{r}$ i.e. $d\mathbf{r}_0' = d\mathbf{r}' - d\mathbf{u}' = d\mathbf{r}'\cdot[\mathbf{I} - \nabla\mathbf{u}]$, $d\mathbf{r}_0 = d\mathbf{r} - d\mathbf{u} = [\mathbf{I} - \mathbf{u}\nabla]\cdot d\mathbf{r}$

$$= d\mathbf{r}'\cdot\left[\nabla\mathbf{u} + \mathbf{u}\nabla - \nabla\mathbf{u}\cdot\mathbf{u}\nabla\right]\cdot d\mathbf{r}$$

and
$d\mathbf{r}'\cdot[\mathbf{u}\nabla - \nabla\mathbf{u}\cdot\mathbf{u}\nabla]\cdot d\mathbf{r}$
$+ d\mathbf{r}'\cdot[\nabla\mathbf{u} - \nabla\mathbf{u}\cdot\mathbf{u}\nabla]\cdot d\mathbf{r}$
$+ d\mathbf{r}'\cdot[\nabla\mathbf{u}\cdot\mathbf{u}\nabla]\cdot d\mathbf{r} =$

now let $\mathbf{n} = \dfrac{d\mathbf{r}'}{|d\mathbf{r}'|}$ $\quad \mathbf{s} = \dfrac{d\mathbf{r}}{|d\mathbf{r}|}$

$$\boxed{\therefore\ \tfrac{1}{2}\cos\theta = \mathbf{n}\cdot\mathbf{\Phi}\cdot\mathbf{s}}$$

defined change in angle

$\mathbf{\Phi}$ contains not only changes in length but also changes in angle

to find what has happened to two $\perp$ lines in space, find the two unit vectors along the two $\perp$ lines $\mathbf{n}_1$, $\mathbf{n}_2$ then take $\mathbf{n}_1\cdot\mathbf{\Phi}\cdot\mathbf{n}_2$

is $\mathbf{\Phi}$ symmetric? yes even if $\nabla\mathbf{u}$ is asymmetric then $\nabla\mathbf{u} + \mathbf{u}\nabla$ is symmetric and $\mathbf{u}\nabla\cdot\nabla\mathbf{u}$ is symmetric; therefore $\mathbf{\Phi}$ is symmetric

Lagrangian strain tensor $\boxed{\boldsymbol{\epsilon} = \frac{1}{2}\left[\nabla_0\mathbf{u} + \mathbf{u}\nabla_0 + \nabla_0\mathbf{u}\cdot\mathbf{u}\nabla_0\right]}$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_{0i}} + \frac{\partial u_i}{\partial x_{0j}} + \frac{\partial u_k}{\partial x_{0i}}\frac{\partial u_k}{\partial x_{0j}}\right)\mathbf{e}_{i_0}\mathbf{e}_{j_0}$$

and $\boldsymbol{\epsilon} = \mathbf{e}_{i_0}\,\epsilon_{ij}\,\mathbf{e}_{j_0}$

10/23/78

What are the corresponding Lagrangian representation

$$d\mathbf{r}' = d\mathbf{r}_0' \cdot (\hat{\mathbb{I}} + \nabla_0 \mathbf{u})$$

$$d\mathbf{r} = (\hat{\mathbb{I}} + \mathbf{u}\nabla_0) \cdot d\mathbf{r}_0$$

$$\frac{1}{2}\cos\theta = \frac{\frac{1}{2}\, d\mathbf{r}_0' \cdot (\hat{\mathbb{I}} + \nabla_0\mathbf{u}) \cdot (\hat{\mathbb{I}} + \mathbf{u}\nabla_0) \cdot d\mathbf{r}_0}{|d\mathbf{r}'||d\mathbf{r}|} = \frac{d\mathbf{r}' \cdot d\mathbf{r}}{2|d\mathbf{r}'||d\mathbf{r}|}$$

recall that $d\mathbf{r}_0' \cdot d\mathbf{r}_0 = 0$

$$\frac{1}{2}\cos\theta = \frac{|d\mathbf{r}_0'|}{|d\mathbf{r}'|} \cdot \mathbf{\epsilon} \cdot \frac{d\mathbf{r}_0}{|d\mathbf{r}|} \quad \text{where} \quad \mathbf{\epsilon} \equiv \frac{1}{2}\left[\nabla_0\mathbf{u} + \mathbf{u}\nabla_0 + \nabla_0\mathbf{u} \cdot \mathbf{u}\nabla_0\right]$$

Lagrangian deformation tensor

Recall

$$\epsilon_{rr} = \frac{d\mathbf{r} \cdot d\mathbf{r} - d\mathbf{r}_0 \cdot d\mathbf{r}_0}{2|d\mathbf{r}_0||d\mathbf{r}_0|} \quad \text{thus this will lead} \quad \epsilon_{rr} = \frac{d\mathbf{r}_0}{|d\mathbf{r}_0|} \cdot \mathbf{\epsilon} \cdot \frac{d\mathbf{r}_0}{|d\mathbf{r}_0|}$$

now if $\mathbf{n}_0 = \dfrac{d\mathbf{r}_0}{|d\mathbf{r}_0|}$ thus $\boxed{\epsilon_{rr} = \mathbf{n}_0 \cdot \mathbf{\epsilon} \cdot \mathbf{n}_0}$ this gives changes in length of line in direction of $d\mathbf{r}_0$

Common definition of elongation $E_{LL} = \sqrt{1 + 2\epsilon_{LL}} - 1 = \dfrac{L - L_0}{L_0}$ for Lagrangian

$$\frac{|d\mathbf{r}|}{|d\mathbf{r}_0|} = E_{rr} + 1 = \sqrt{1 + 2\epsilon_{rr}} \qquad \frac{|d\mathbf{r}_0|}{|d\mathbf{r}|} = \frac{1}{\sqrt{1+2\epsilon_{rr}}}$$

$$\frac{|d\mathbf{r}'|}{|d\mathbf{r}_0'|} = \sqrt{1 + 2\epsilon_{r'r'}} \qquad \frac{|d\mathbf{r}_0'|}{|d\mathbf{r}'|} = \frac{1}{\sqrt{1+2\epsilon_{r'r'}}}$$

now define $\dfrac{d\mathbf{r}_0}{|d\mathbf{r}_0|} = \mathbf{s}_0$ $\quad \dfrac{d\mathbf{r}_0'}{|d\mathbf{r}_0'|} = \mathbf{n}_0$; Since $\frac{1}{2}\cos\theta = \dfrac{d\mathbf{r}_0'}{|d\mathbf{r}'|} \cdot \mathbf{\epsilon} \cdot \dfrac{d\mathbf{r}_0}{|d\mathbf{r}|} = \underbrace{\dfrac{d\mathbf{r}_0'}{|d\mathbf{r}_0'|}}_{\mathbf{n}_0} \dfrac{|d\mathbf{r}_0'|}{|d\mathbf{r}'|} \cdot \mathbf{\epsilon} \cdot \underbrace{\dfrac{d\mathbf{r}_0}{|d\mathbf{r}_0|}}_{\mathbf{s}_0} \dfrac{|d\mathbf{r}_0|}{|d\mathbf{r}|}$

Thus
$$\boxed{\frac{1}{2}\cos\theta = \frac{\mathbf{n}_0 \cdot \mathbf{\epsilon} \cdot \mathbf{s}_0}{\sqrt{1 + 2\epsilon_{s_0s_0}}\,\sqrt{1 + 2\epsilon_{n_0n_0}}} = \frac{\epsilon_{n_0s_0}}{\sqrt{(1 + 2\epsilon_{s_0s_0})(1 + 2\epsilon_{n_0n_0})}}}$$
this gives the angular strain

$$\boxed{\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_{i0}} + \frac{\partial u_i}{\partial x_{j0}} + \frac{\partial u_k}{\partial x_{i0}}\frac{\partial u_k}{\partial x_{j0}}\right] \quad \text{Lagrangian stretch tensor}}$$

Look at Volume changes using the Lagrangian method for principal direction

$dx_i = dx_{0i}(1+2\epsilon_{ii})$ for principal direction.

$$dV = dV_0 \sqrt{(1+2\epsilon_{11})(1+2\epsilon_{22})(1+2\epsilon_{33})}$$

Define the dilatation $\Delta = \dfrac{\delta V - \delta V_0}{\delta V_0} = \sqrt{(1+2\epsilon_{11})(1+2\epsilon_{22})(1+2\epsilon_{33})} - 1$

$$\Delta = \sqrt{1 + 2I_\epsilon + 4II_\epsilon + 8III_\epsilon} - 1$$ where $I_\epsilon, II_\epsilon, III_\epsilon$ are the Lagrangian invariants

for small strains $\sqrt{1+K} = 1 + \frac{1}{2}K$; $\sim 1 + I_\epsilon = \frac{\partial u_i}{\partial x_i} + 1$

Define $\epsilon = \frac{1}{2}(\nabla_0 u + u\nabla_0)$ symmetric part $\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_{0i}} + \frac{\partial u_i}{\partial x_{0j}}\right)$ extension

$\omega = \frac{1}{2}(\nabla_0 u - u\nabla_0)$ anti symmetric part $\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_{0i}} - \frac{\partial u_i}{\partial x_{0j}}\right)$ rotation of rigid body

$\epsilon_{xx} = \dfrac{\partial u_x}{\partial x_0}$ $\quad \epsilon_{yz} = \frac{1}{2}\left(\dfrac{\partial u_z}{\partial y_0} + \dfrac{\partial u_y}{\partial z_0}\right)$

ROTATION MATRIX

$\omega_{xx} = 0$ $\quad \omega_{yz} = \frac{1}{2}\left(\dfrac{\partial u_z}{\partial y_0} - \dfrac{\partial u_y}{\partial z_0}\right)$ $\quad \omega = \begin{pmatrix} 0 & \omega_{xy} & \omega_{xz} \\ -\omega_{xy} & 0 & \omega_{yz} \\ -\omega_{xz} & \omega_{yz} & 0 \end{pmatrix}$

note $\epsilon_{ij} = \epsilon_{ji}$ $\quad \omega_{ij} = -\omega_{ji}$ and $\left.\begin{array}{l}\nabla_0 u = \epsilon + \omega \\ u\nabla_0 = \epsilon - \omega\end{array}\right\}$

$$\mathcal{E} = \frac{1}{2}\Big(\underbrace{[\epsilon+\omega]}_{\nabla_0 u} + \underbrace{[\epsilon-\omega]}_{u\nabla_0} + \underbrace{\epsilon\cdot\epsilon + \omega\cdot\epsilon - \epsilon\cdot\omega - \omega\cdot\omega}_{\nabla_0 u \cdot u\nabla_0}\Big)$$

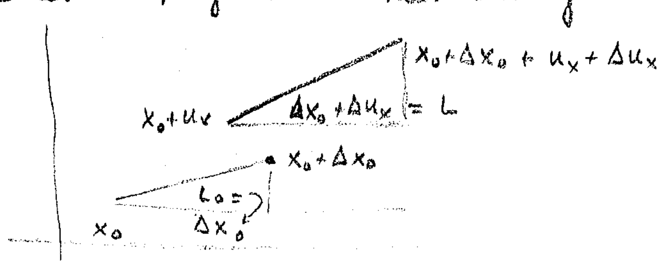
Lagrangian

$$\mathcal{E} = \epsilon + \frac{1}{2}\left(\epsilon\cdot\epsilon + \omega\cdot\epsilon - \epsilon\cdot\omega - \omega\cdot\omega\right)$$

if we are in the realm of infinitesimal elasticity $\Rightarrow (\epsilon, \omega \ll 1)$ then $\epsilon \cong \mathcal{E}$ and we won't need to make distinctions between Eulerian and Lagrangian extensional tensors

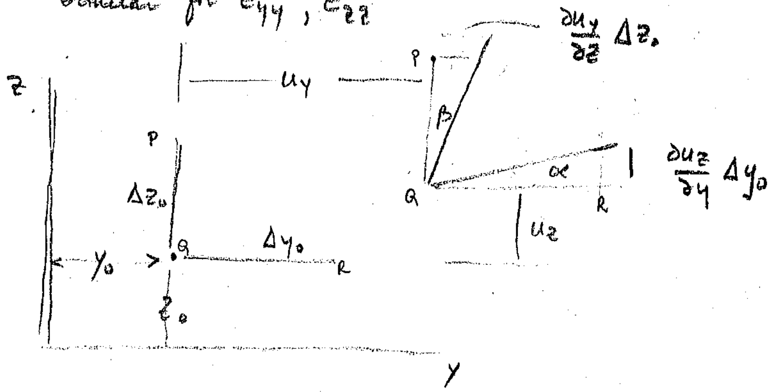
Geometric Significance of Quantities

Look at a single strand that undergoes extension & rotation



$$\epsilon_{xx} = \frac{\partial u_x}{\partial x_0} \simeq \frac{\Delta u_x}{\Delta x_0} = \frac{(x_0+\Delta x_0+\overbrace{u_x+\Delta u_x)-(x_0}^{L}+\Delta u_x)-(\Delta x_0)}{\Delta x_0} \quad \frac{\Delta x_0+\Delta u_x-\Delta x_0}{\Delta x_0} = \frac{L-L_0}{L_0}$$

Similar for ϵ_{yy}, ϵ_{zz}



If we assume small strains, rotations $\epsilon_{yy} \ll 1$, $\alpha \ll 1$, $\beta \ll 1$, $\epsilon_{zz} \ll 1$

$$\tan\alpha \approx \alpha = \frac{\partial u_z}{\partial y} \qquad \tan\beta \approx \beta = \frac{\partial u_y}{\partial z}$$

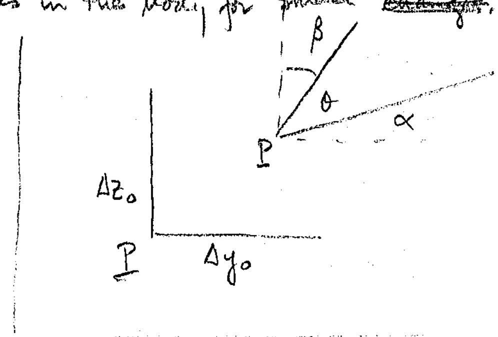
$$\cos\theta = \cos\left[\frac{\pi}{2} - (\alpha+\beta)\right] = \sin(\alpha+\beta) \approx \alpha+\beta = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}$$

$$= 2\,\epsilon_{yz} \quad \text{engineering strain } \gamma_{yz}$$

10/25/78

So far we have made no constraints on the body (ie elastic) except to say that the body must be continuous.

we have noted that $\nabla_0 u = (\epsilon + \omega)$ contained on its main diagonals the extensional changes for small ~~changes~~ deformation & on the off diag terms we have the angular changes in the body for small ~~changes~~ deformations.



$$\frac{1}{2}\cos\theta = \frac{\alpha+\beta}{2} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) = \epsilon_{yz}$$

$$\omega_{yz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) = \frac{1}{2}(\alpha-\beta)$$

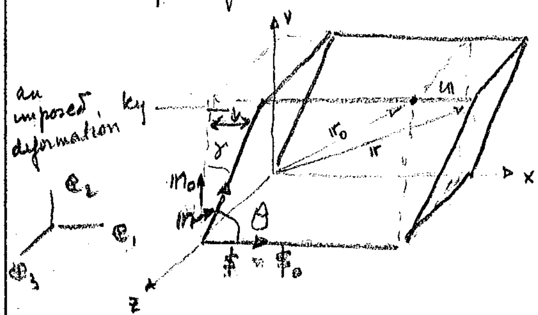
represents an average rotation of all lines passing through point P (w_{ij} represents rigid body rotation only).

we can then say that for small strains (infinitesimal deformation)

$$\mathbb{D} \Rightarrow \not{\epsilon} \Rightarrow \epsilon$$

<u>Dilatation</u> $\Delta = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = u_{k,k} = \epsilon_{kk} = I_\epsilon$

Example of Finite strains



$$\mathbb{u} = \overset{u_x}{ky\,\mathbb{e}_1} + 0 + 0$$
$$\mathbb{r} = x\mathbb{e}_1 + y\mathbb{e}_2 + z\mathbb{e}_3$$
$$\mathbb{r}_0 = x_0\mathbb{e}_1 + y_0\mathbb{e}_2 + z_0\mathbb{e}_3$$
$$\mathbb{r} = \mathbb{r}_0 + \mathbb{u}$$
$$d\mathbb{r} = d\mathbb{r}_0 + d\mathbb{u}$$
$$x = x_0 + ky \qquad y = y_0,\ z = z_0$$
$$dx = dx_0 + \left[\frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy + \frac{\partial u_x}{\partial z}dz\right] + 0 \text{ for other terms}$$
$$= dx_0 + k\,dy$$

$$\frac{\partial u_j}{\partial x_i}\mathbb{e}_i\mathbb{e}_j = \nabla\mathbb{u} = k\,\mathbb{e}_y\mathbb{e}_x : \begin{Bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$
tensor

Eulerian Deformation Tensor $\quad \hat{\mathbb{I}} - \nabla\mathbb{u} = \begin{Bmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$
tensor

$$= \mathbb{e}_x\mathbb{e}_x - k\mathbb{e}_y\mathbb{e}_x + \mathbb{e}_y\mathbb{e}_y + \mathbb{e}_z\mathbb{e}_z$$

$$\mathbb{u}\nabla = k\mathbb{e}_x\mathbb{e}_y$$

Note that $\frac{\partial}{\partial y} = \frac{\partial}{\partial y_0}$, $\frac{\partial}{\partial z} = \frac{\partial}{\partial z_0}$

Lagrangian Deformation Tensor (in this example $\nabla\mathbb{u} = \nabla_0\mathbb{u}$) since $\mathbb{e}_x = \mathbb{e}_{x_0}$, $\mathbb{e}_y = \mathbb{e}_{y_0}$

$$\hat{\mathbb{I}} + \nabla_0\mathbb{u} = \begin{Bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

Eulerian strain tensor.
$$\Phi = \frac{1}{2}\left(\nabla\mathbb{u} + \mathbb{u}\nabla - \nabla\mathbb{u}\cdot\mathbb{u}\nabla\right) = \begin{Bmatrix} 0 & \frac{k}{2} & 0 \\ k/2 & -k^2/2 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$
$$\Phi = \frac{k}{2}\mathbb{e}_y\mathbb{e}_x + \frac{k}{2}\mathbb{e}_x\mathbb{e}_y - \frac{k^2}{2}\mathbb{e}_y\mathbb{e}_y$$

Lagrangian Strain tensor $\quad \phi = \frac{1}{2}\left(\nabla_0\mathbb{u} + \mathbb{u}\nabla_0 + \nabla_0\mathbb{u}\cdot\mathbb{u}\nabla_0\right) = \begin{Bmatrix} 0 & \frac{k}{2} & 0 \\ k/2 & k^2/2 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$
$$\phi = \frac{k}{2}\mathbb{e}_{y_0}\mathbb{e}_{x_0} + \frac{k}{2}\mathbb{e}_{x_0}\mathbb{e}_{y_0} + \frac{k^2}{2}\mathbb{e}_{y_0}\mathbb{e}_{y_0}$$

now $\frac{1}{2}\cos\theta = \mathbb{n}\cdot\Phi\cdot\mathbb{s}$ where $\mathbb{n} = \sin\gamma\,\mathbb{e}_1 + \cos\gamma\,\mathbb{e}_2$
$$\mathbb{s} = \mathbb{e}_1$$

$$\frac{1}{2}\cos\theta = \overset{\mathbb{n}\cdot\Phi\cdot\mathbb{s}}{\cos\gamma\,\frac{k}{2}}\ ; \text{ but } \tan\gamma = k \text{ (from sketch) thus}$$
$$= \cos\gamma\cdot\frac{1}{2}\tan\gamma \qquad = \frac{1}{2}\sin\gamma$$

$$\frac{1}{2}\cos\theta = \frac{\mathbf{n}_0 \cdot \boldsymbol{\epsilon} \cdot \mathbf{s}_0}{\sqrt{(1+2\epsilon_{s_0 s_0})(1+2\,\epsilon_{n_0 n_0})}}$$

Infinitesimal $\boldsymbol{\epsilon} = \frac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla) = \begin{Bmatrix} 0 & \frac{k}{2} & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$ note this doesn't have any diagonal elements (unlike finite strain theory)

<u>Problem</u>: Find displacements from given strains (if you are given strain you know stresses since $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$)

Example $\epsilon_{xy} = xy$, all others $\epsilon_{ij} = 0$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = 0 \qquad \epsilon_{yy} = \frac{\partial u_y}{\partial y} = 0 \qquad \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$u_x = f(y,z) \qquad u_y = g(x,z) \qquad \epsilon_{xy} = \frac{1}{2}(g' + f') = xy$$

this is an incompatible displacement field since product $xy \neq$ sum of two functions of different variables. Thus we must find <u>compatability</u> conditions.

10/27/78

In infinitesimal strains we can forget about the differences between Lagrange/Euler. Thus we can use

$$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla) \qquad \epsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j})$$

$$\boldsymbol{\omega} = \frac{1}{2}(\nabla\mathbf{u} - \mathbf{u}\nabla) \qquad \omega_{ij} = \frac{1}{2}(u_{j,i} - u_{i,j})$$

now we are defining 6 dependent quantities (ϵ_{ij}) in terms of 3 indep quantities u_x, u_y, u_z,
$\therefore$ there must be compatability conditions by which the functions become determined uniquely.

$$\epsilon_{xx} = u_{x,x} \qquad \epsilon_{xy} = \tfrac{1}{2}(u_{x,y} + u_{y,x}) \qquad \epsilon_{xx,yy} = u_{x,xyy}$$

$$\epsilon_{yy} = u_{y,y} \qquad \epsilon_{yx} = \epsilon_{xy} \qquad \epsilon_{yy,xx} = u_{y,xxy}$$

$$\epsilon_{xy} = \frac{1}{2}(u_{x,y} + u_{y,x}) \qquad 2\epsilon_{xy,xy} = u_{x,xyy} + u_{y,xxy}$$

$$\therefore \quad \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} - 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = 0 \qquad \text{Compatibility}$$

in the same manner we get

$$\psi_{xx} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} - \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} - \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 0$$

$$\psi_{yy} = 2\frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} - \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} - \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = 0$$

$$\psi_{zz} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} - \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} - \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 0$$

and

$$\psi_{yz} = \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} - \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) = 0$$

$$\psi_{zx} = \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} - \frac{\partial}{\partial y}\left(-\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x}\right) = 0$$

$$\psi_{xy} = \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} - \frac{\partial}{\partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right) = 0$$

EQUATIONS OF COMPATIBILITY

Cesaro proved that these are the equations which are unique by saying

$\oint_c du = 0$ along any space curve. (basically this is more severe than in a complex plane).

<u>Look at handout</u> $\Psi = \nabla \times \varepsilon \times \nabla$ is the incompatibility tensor

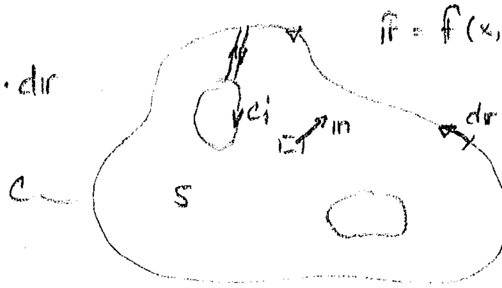
it is a symmetric tensor.

ε is symmetric and curl on both sides keeps the symmetry.

Stokes theorem

$$\int_S \mathbf{n} \cdot \nabla \times \mathbf{f} \, ds = \oint_c \mathbf{f} \cdot d\mathbf{r}$$

$\mathbf{F} = f(x,y,z)$



or

$$\int_S n_r e_{rmn} f_{n,m} \, ds = \oint_c f_k \, dx_k$$

10/31/78

Continue looking at handout

① - Compatibility $\nabla \times \epsilon \times \nabla = 0$ must be satisfied everywhere & must be satisfied pointwise.

② - For multiply connected region in addition
$$\oint_{C_i} d\mathbf{r} \cdot (\epsilon \times \nabla) = 0$$

③ - and $\oint_{C_i} \left[\epsilon + (\mathbf{r} - \mathbf{r}_0) \times (\nabla \times \epsilon) \right] \cdot d\mathbf{r} = 0$ these are integral relations that must be satisfied, not on a pointwise basis though.

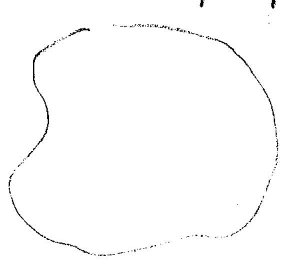
For simply connected region: must satisfy ①
For multiply connected region: must satisfy ①, ②, ③
strain field requires compatibility based only on geometric considerations <u>not</u> on what loads produced this

<u>Stress - Strain Relationships</u>

Right now we only have 3 equations w/ 6 unknowns. To get the other 3 equations must look at the material itself (ie 3 eqns of motion & 6 stresses)

Start with principles of conservation of energy

Given



the rate at which tractions do work ~~area~~ on the boundary + the rate of which body forces do work = change of internal energy + time change of energy flux across boundary or

$$\int_S \mathbf{t}_n \cdot \mathbf{v}\, ds + \int_V \mathbf{f} \cdot \mathbf{v}\, dV = \frac{D}{Dt} \int \tfrac{1}{2} \rho \mathbf{v}^2 dV + \frac{D}{Dt} \int \rho U_0\, dV$$

where U_0 is the internal energy density/unit mass and $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$

Using linear theory (Eulerian = Lagrange) we obtain

$$\int_V \rho \frac{\partial}{\partial t} \Big(\underbrace{\tfrac{1}{2} \mathbf{v}^2}_{KE} + \underbrace{U_0}_{\text{internal}} \Big) dV = \int_V \rho \left(\frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial^2 \mathbf{u}}{\partial t^2} + \frac{\partial U_0}{\partial t} \right) dV$$

Equations of Compatibility

I. Identities needed:

1) $a \times (b \times c) = a \cdot cb - a \cdot bc = a \cdot (cb - bc)$

2) $\mathbb{I} \times (b \times c) = \mathbb{I} \cdot cb - \mathbb{I} \cdot bc$
$= \mathbb{I} \cdot (cb - bc) = cb - bc$

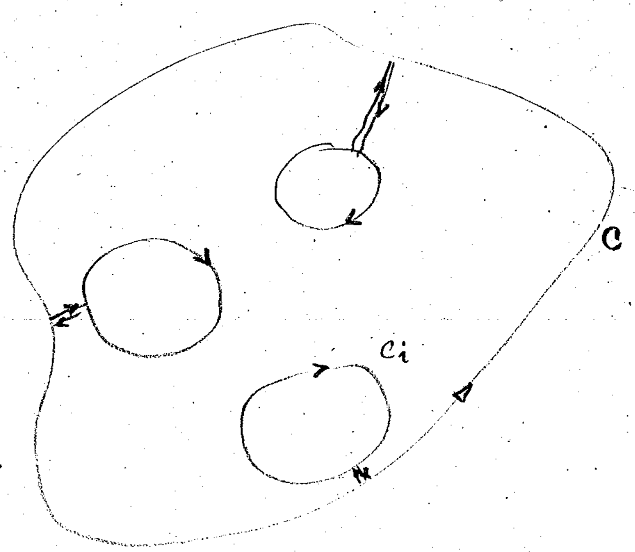
3) $\nabla \times \nabla a = 0 \quad ; \quad a\nabla \times \nabla = 0$

4) $\nabla \times (\mathbb{I} \times a) = a\nabla - \mathbb{I}\nabla \cdot a$

5) $\nabla \cdot (\nabla \times a) = 0$ — $\nabla \times a$ is a vector $\perp$ to ∇ & a $\therefore$ dotting it with any vector in the plane of ∇ & $a = 0$ since they are $\perp$

6) $a \cdot (\mathbb{I} \times b) = a \times b$ — $a \cdot (I \times b) = I \cdot (a \times b) = a \times b$

7) $da = dr \cdot \nabla a$



$\oint du = 0$

II. To show that $\underline{\nabla \times \Phi \times \nabla = 0}$

$$\text{Let} \quad \nabla \mathbf{u} = \tfrac{1}{2}(\nabla \mathbf{u} + \mathbf{u}\nabla) + \tfrac{1}{2}(\nabla \mathbf{u} - \mathbf{u}\nabla)$$

$$= \Phi - \tfrac{1}{2}\mathbf{I} \times (\nabla \times \mathbf{u}) \quad \text{relationships 2}$$

Define $\mathbf{w} = \frac{1}{2}\nabla \times \mathbf{u}$ vorticity if fluid mech. Note: $\nabla \cdot \mathbf{w} = 0$ by relation 5

Thus $\nabla \mathbf{u} = \Phi - \mathbf{I} \times \mathbf{w}$

Curl: relation 3 $\nabla \times \nabla \mathbf{u} = \nabla \times \Phi - \nabla \times (\mathbf{I} \times \mathbf{w}) = \nabla \times \Phi - \mathbf{w}\nabla + \mathbf{I}(\nabla \cdot \mathbf{w})$ relation 4

$0 = \nabla \times \Phi - \mathbf{w}\nabla + 0$ ↑ above.

Thus: $\nabla \times \Phi = \mathbf{w}\nabla$

Curl $\nabla \times \Phi \times \nabla = \mathbf{w}\nabla \times \nabla = 0$ (2nd formula relation 3), q.e.d.

III. Multiply - Connected Regions

value is assigned — using relation 7 — using $\nabla \mathbf{u} = \Phi - \mathbf{I} \times \mathbf{w}$

$$\Delta \mathbf{u} = \oint d\mathbf{u} = \oint d\mathbf{r} \cdot \nabla \mathbf{u} = \oint d\mathbf{r} \cdot \Phi - \oint d\mathbf{r} \cdot (\mathbf{I} \times \mathbf{w})$$

look at 2nd term: — choosing a ref point $\mathbf{r}_0$ — relation 6

$$\oint d\mathbf{r} \cdot (\mathbf{I} \times \mathbf{w}) = \oint d(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{I} \times \mathbf{w}) = \oint d(\mathbf{r} - \mathbf{r}_0) \times \mathbf{w}$$

integrate by parts

$$= -\oint \mathbf{w} \times d(\mathbf{r} - \mathbf{r}_0) = -\left[\mathbf{w} \times (\mathbf{r} - \mathbf{r}_0)\right]_{\mathbf{r}_0}^{\mathbf{r}_0} + \oint d\mathbf{w} \times (\mathbf{r} - \mathbf{r}_0)$$

$$= \oint \overbrace{d\mathbf{r} \cdot \nabla \mathbf{w}}^{d\mathbf{w}\ \text{relation 7}} \times (\mathbf{r} - \mathbf{r}_0)$$

$\int f\,dg = fg - \int df\, g$

$\mathbf{w}\, d(\mathbf{r} - \mathbf{r}_0) = \mathbf{w} \times (\mathbf{r} - \mathbf{r}_0)\Big|_{\mathbf{r}_0}^{\mathbf{r}_0} - \int d\mathbf{w} \times (\mathbf{r} - \mathbf{r}_0)$

Note: $\mathbf{w}\nabla = \nabla \times \Phi$ (from above); $\nabla \mathbf{w} = (\nabla \times \Phi)_{conj} = -(\Phi \times \nabla)$

now remember $\Delta \mathbf{u} = \oint d\mathbf{u} = \oint d\mathbf{r} \cdot \Phi - \oint d\mathbf{r} \cdot \nabla \mathbf{w} \times (\mathbf{r} - \mathbf{r}_0)$

$$\Delta \mathbf{u} = \oint d\mathbf{r} \cdot \Phi + \oint d\mathbf{r} \cdot (\Phi \times \nabla) \times (\mathbf{r} - \mathbf{r}_0) \qquad (\ ) \cdot d\mathbf{r} = \mathbf{n} \cdot \nabla \times (\ )$$

$$= \iint \mathbf{n} \cdot \nabla \times \left[\Phi + (\Phi \times \nabla) \times (\mathbf{r} - \mathbf{r}_0)\right] ds + \oint_{c_i} d\mathbf{u}$$ ← If we allow multiply connected region

$$= \int \mathbf{n} \cdot \left[\nabla \times \Phi \times \nabla\right] \times (\mathbf{r} - \mathbf{r}_0)\, ds + \oint_{c_i} d\mathbf{u} = 0 \text{ if } \nabla \times \Phi \times \nabla = 0 \text{ \& } \oint_{c_i} d\mathbf{u}$$

show that $\int \mathbf{n} \cdot \nabla \times \Phi\, ds = \int \nabla \cdot (\nabla \times \Phi)\, dV = 0$ since $\nabla \cdot \nabla \times \Phi = 0$

Another starting point: but $\oint_{C_i} d\mathbf{u} = \oint_{C_i} d\mathbf{r}\cdot\mathbb{S} + \oint_{C_i} d\mathbf{r}\cdot(\mathbb{S}\times\nabla)\times(\mathbf{r}-\mathbf{r}_0)$

$$\oint_{C_i} d\mathbf{u} = \oint_{C_i} d\mathbf{r}\cdot\left[\mathbb{S} + (\mathbb{S}\times\nabla)\times(\mathbf{r}-\mathbf{r}_1)\right] = 0$$

whether $\mathbf{r}_0$ or $\mathbf{r}_1$ doesn't matter

Subtract: $$\oint_{C_i}\left[d\mathbf{r}\cdot(\mathbb{S}\times\nabla)\right]\times(\mathbf{r}_0-\mathbf{r}_1) = 0$$

or $$\oint_{C_i} d\mathbf{r}\cdot(\mathbb{S}\times\nabla) = 0$$ 3 indep. eqns.

3 further indep. eqns. obtained as follows:

We had
$$\oint_{C_i}\left[\mathbb{S}\cdot d\mathbf{r} - d(\mathbf{r}-\mathbf{r}_0)\cdot(\mathbb{I}\times\mathbf{W})\right] = 0$$
relation #6

or $$\oint_{C_i}\left[\mathbb{S}\cdot d\mathbf{r} - d(\mathbf{r}-\mathbf{r}_0)\times\mathbf{W}\right] = 0$$

$\int d(\mathbf{r}-\mathbf{r}_0)\times\mathbf{W} = (\mathbf{r}-\mathbf{r}_0)\times\mathbf{W} - \int \mathbf{r}-\mathbf{r}_0\times d\mathbf{W}$

$\oint(\mathbf{r}-\mathbf{r}_0)\times\mathbf{W} = 0$

After integration by parts:
$$\oint_{C_i}\left[\mathbb{S}\cdot d\mathbf{r} + (\mathbf{r}-\mathbf{r}_0)\times d\mathbf{W}\right] = 0$$

or $$\oint_{C_i}\left[\mathbb{S} + (\mathbf{r}-\mathbf{r}_0)\times\mathbf{W}\nabla\right]\cdot d\mathbf{r} = 0$$

or $$\oint_{C_i}\left[\mathbb{S} + (\mathbf{r}-\mathbf{r}_0)\times(\nabla\times\mathbb{S})\right]\cdot d\mathbf{r} = 0$$ 3 indep. eqns.

the left hand side of the conserv eqn.

$$\int_s \mathbb{t}_n \cdot \frac{\partial \mathbb{u}}{\partial t} ds = \int_s \mathbb{n} \cdot \sigma \cdot \frac{\partial \mathbb{u}}{\partial t} ds = \int_v \nabla \cdot \left(\sigma \cdot \frac{\partial \mathbb{u}}{\partial t}\right) dv = \int_v \left(\sigma_{kj} \frac{\partial u_j}{\partial t}\right)_{,k} dv$$

$$\int_V \left(\sigma_{kj} \frac{\partial u_j}{\partial t}\right)_{,k} dv = \int_V \left(\sigma_{kj,k} \frac{\partial u_j}{\partial t} + \sigma_{kj} \frac{\partial u_{j,k}}{\partial t}\right) dv = \int_V \left[\sigma_{kj,k} \frac{\partial u_j}{\partial t} + \underbrace{\sigma_{kj} \frac{\partial}{\partial t}(\varepsilon_{kj} + \omega_{kj})}_{\sigma_{kj} \frac{\partial}{\partial t} \varepsilon_{kj}}\right.$$

since $\nabla \mathbb{u} = \varepsilon + \omega$

now $\sigma_{kj} \frac{\partial}{\partial t} \omega_{kj} = 0$ since σ is sym ω is antisym $\Rightarrow$ there will be like terms w/opposite signs causing the terms to cancel

11/1/78

Compatibility $\mathbb{H}$ is only good for small deformations <u>not</u> finite strains.

Normally we have 6 equations relating 9 unknowns $\therefore$ we must define 3 more equations from the material properties

$$\int_s \mathbb{t}_n \cdot \frac{\partial \mathbb{u}}{\partial t} ds = \int_s \mathbb{n} \cdot \left(\sigma \cdot \frac{\partial \mathbb{u}}{\partial t}\right) ds = \int_V \nabla \cdot \left(\sigma \cdot \frac{\partial \mathbb{u}}{\partial t}\right) dv = \int_V \left(\sigma_{kj} \frac{\partial u_j}{\partial t}\right)_{,k} dv$$

$$\int_V \left(\sigma_{kj} \frac{\partial u_j}{\partial t}\right)_{,k} dv = \int_V \left(\sigma_{kj,k} \frac{\partial u_j}{\partial t} + \sigma_{kj} \frac{\partial u_{j,k}}{\partial t}\right) dv = \int_V \left\{(\nabla \cdot \sigma) \cdot \frac{\partial \mathbb{u}}{\partial t} + \sigma : \frac{\partial \nabla \mathbb{u}}{\partial t}\right\}$$

but $\nabla u = \varepsilon + \omega = \underbrace{\tfrac{1}{2}(\nabla \mathbb{u} + \mathbb{u}\nabla)}_{\text{symm}} + \underbrace{\tfrac{1}{2}(\nabla \mathbb{u} - \mathbb{u}\nabla)}_{\text{anti sym.}}$

but $\sigma : \nabla \mathbb{u} = \sigma : (\varepsilon + \omega) = \sigma : \varepsilon + \sigma : \omega = \sigma : \varepsilon$ where $\sigma : \omega = 0$ since σ sy[m]
ω an[ti]

$\therefore$ we can write the conservation of energy

$$\int_V \left(\underbrace{\frac{\partial u_j}{\partial t} \sigma_{kj,k} + \frac{\partial u_j}{\partial t} f_j - \rho \frac{\partial u_j}{\partial t} \frac{\partial^2 u_j}{\partial t^2}}_{\frac{\partial u_j}{\partial t} \cdot [\text{P.L.M.} = 0]} + \sigma_{kj} \frac{\partial \varepsilon_{kj}}{\partial t} - \rho \frac{\partial U_0}{\partial t}\right) dV = 0$$

$$\therefore \int_V \left(\sigma_{kj} \frac{\partial \varepsilon_{kj}}{\partial t} - \rho \frac{\partial U_0}{\partial t}\right) dV = 0 \Rightarrow \sigma_{kj} \frac{\partial \varepsilon_{kj}}{\partial t} - \rho \frac{\partial U_0}{\partial t} = 0 \quad \text{for any } dV$$

if we set U = internal energy per volume then $\sigma_{kj} \frac{\partial \varepsilon_{kj}}{\partial t} = \frac{\partial U}{\partial t}$

or $\sigma : \dot{\varepsilon} = \dot{U}$

we now postulate (assume) that $U = U(\text{strain only}) = U(\epsilon) = U(\epsilon_{ij})$

$$\dot{U} = \frac{\partial U}{\partial \epsilon_{11}}\dot{\epsilon}_{11} + \frac{\partial U}{\partial \epsilon_{22}}\dot{\epsilon}_{22} + \frac{\partial U}{\partial \epsilon_{33}}\dot{\epsilon}_{33} + \frac{\partial U}{\partial \epsilon_{23}}\dot{\epsilon}_{23} + \frac{\partial U}{\partial \epsilon_{31}}\dot{\epsilon}_{31} + \frac{\partial U}{\partial \epsilon_{12}}\dot{\epsilon}_{12}$$

since $\dot{U} = \sigma_{kj}\dfrac{\partial \epsilon_{kj}}{\partial t} = \sigma_{11}\dot{\epsilon}_{11} + \sigma_{22}\dot{\epsilon}_{22} + \sigma_{33}\dot{\epsilon}_{33} + 2\sigma_{23}\dot{\epsilon}_{23} + 2\sigma_{13}\dot{\epsilon}_{13} + 2\sigma_{12}\dot{\epsilon}_{12}$

$$\Rightarrow \sigma_{ij} = \frac{\partial U}{\partial \epsilon_{ij}} \qquad \text{if } i=j$$

$$= \frac{1}{2}\frac{\partial U}{\partial \epsilon_{ij}} \qquad \text{if } i \neq j$$

in equil state we want no residual stress ∴ we want no strain or stress

if we expand U as a fn of ϵ then $U(\epsilon) = \cancel{U(0)} + \cancel{U'(0)}\epsilon + \frac{1}{2}U''(0)\cdot\epsilon^2 + \text{h.o.t.}$

will be dropped since its only a reference state

∴ $U = \frac{1}{2}U''(0)\cdot\epsilon^2$ in most general form $\epsilon^2 = \epsilon_{ij}\epsilon_{kl}$ then $U''(0) = C_{ijkl}$

∴ $\boxed{U = \frac{1}{2}C_{ijkl}\,\epsilon_{ij}\,\epsilon_{kl}}$ this C_{ijkl} is something we pick to represent $U''(0)$

We drop the h.o.t. because they produce non linear terms
From this we see that there should be 81 C_{ijkl}; however this can be reduced to 36 since ϵ_{ij} has only 6 independent terms (& so does ϵ_{kl} ∴ $6 \times 6 = 36$).
Of these 36, 21 will be unique (6 on main diag + 15 off diag terms).

11/3/78

Suppose we define $\sigma_p = \sigma_{ij}$ where $\sigma_1 = \sigma_{xx}, \sigma_2 = \sigma_{yy}, \sigma_3 = \sigma_{zz}, \sigma_4 = \sigma_{yz}, \sigma_5 = \sigma_{zx}, \sigma_6 = \sigma_{xy}$
and $\epsilon_p = \epsilon_{ij}$ $\quad \epsilon_1 = \epsilon_{xx}, \epsilon_2 = \epsilon_{yy}, \epsilon_3 = \epsilon_{zz}, \epsilon_4 = \gamma_{yz} = 2\epsilon_{yz}, \epsilon_5 = \gamma_{zx} = 2\epsilon_{zx}, \epsilon_6 = \gamma_{xy} = 2\epsilon_{xy}$

$$\sigma_{ij} = \begin{cases} \dfrac{\partial U}{\partial \epsilon_{ij}} & i=j \\ \dfrac{1}{2}\dfrac{\partial U}{\partial \epsilon_{ij}} & i \neq j \end{cases} \qquad \text{now} \quad \sigma_p = \frac{\partial U}{\partial \epsilon_p} \quad p = 1, \ldots, 6$$

Thus the strain energy

$$U = \frac{1}{2}C_{pq}\,\epsilon_p\,\epsilon_q$$

$C_{12} = (C_{12} + C_{21})\,\varepsilon_1\varepsilon_2$

$$U = \tfrac{1}{2}C_{11}\varepsilon_1^2 + C_{12}\varepsilon_1\varepsilon_2 + C_{13}\varepsilon_1\varepsilon_3 + \dots + C_{16}\varepsilon_1\varepsilon_6$$
$$+\tfrac{1}{2}C_{22}\varepsilon_2^2 + C_{23}\varepsilon_2\varepsilon_3 + \dots + C_{26}\varepsilon_2\varepsilon_6$$
$$\vdots$$
$$+\tfrac{1}{2}C_{66}\varepsilon_6^2$$

Now we note that

$\sigma_1 = \dfrac{\partial U}{\partial \varepsilon_1} = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + \dots C_{16}\varepsilon_6$ and this is true for any σ_r $(r = 1, \dots, 6)$

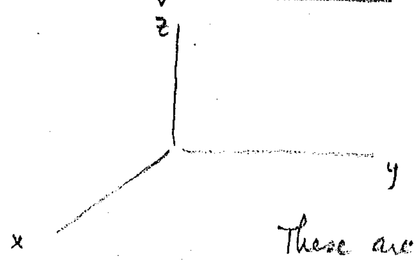
Now $\sigma_r = C_{rp}\varepsilon_p \quad (r, p = 1 \cdots 6) = \dfrac{\partial U}{\partial \varepsilon_p}$

Most general stress-strain relation of a linearly elastic solid involves 21 independent constants C_{pr}

Material scientists & physicists provided theory of crystals in order to determine C

Crystal theory	Triclinic	involves 21 constants	
	Monoclinic	13	1 symmetry
	Orthotropic/Orthorhombic	9	2 symmetries
	Tetragonal / Trigonal	7, 6	
	Hexagonal	5	
	Cubic	3	
	Isotropic	2	material properties in all directions are exactly the same.

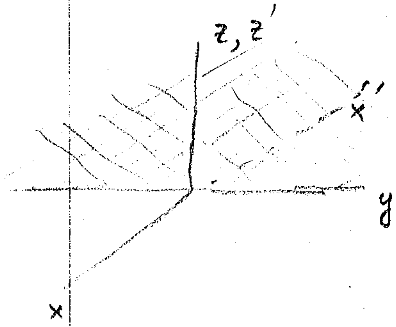
To reduce from <u>21 to 2</u> we introduce cartesian plane & use symmetries of the crystal



These are original relations involving σ_i, c_{ij} & ε_j

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 + \dots + C_{16}\varepsilon_6$$
$$\sigma_2 = C_{12}\varepsilon_1 + \dots + C_{26}\varepsilon_6$$
$$\vdots$$
$$\sigma_6 = C_{16}\varepsilon_6 + \dots + C_{66}\varepsilon_6$$

Now let the yz plane be the plane of symmetry ie (x, x')



$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & -1 & 0 & 0 \\ y' & 0 & 1 & 0 \\ z' & 0 & 0 & 1 \end{array}$$

remember $\sigma_{mn}' = \sigma_{ij}\, l_{mi}\, l_{nj}$ $\qquad \varepsilon_{mn}' = \varepsilon_{ij}\, l_{mi}\, l_{nj}$

$$\therefore \quad \begin{array}{ll|ll} \sigma_1' = \sigma_1 & \sigma_4' = \sigma_4 & \varepsilon_1' = \varepsilon_1 & \varepsilon_4' = \varepsilon_4 \\ \sigma_2' = \sigma_2 & \sigma_5' = -\sigma_5 & \varepsilon_2' = \varepsilon_2 & \varepsilon_5' = -\varepsilon_5 \\ \sigma_3' = \sigma_3 & \sigma_6' = -\sigma_6 & \varepsilon_3' = \varepsilon_3 & \varepsilon_6' = -\varepsilon_6 \end{array}$$

now $\sigma_1' = C_{11}\varepsilon_1' + \dots + C_{16}\varepsilon_{16}'$ ok: C_{ij} is invariant under transformation.

$\sigma_1 = C_{11}\varepsilon_1 + \dots - C_{15}\varepsilon_5 - C_{16}\varepsilon_6$ Substituting for $\sigma_1', \varepsilon_1', \dots, \varepsilon_6'$

but originally $\sigma_1 = C_{11}\varepsilon_1 + \dots + C_{15}\varepsilon_5 + C_{16}\varepsilon_6$ must also be true $\therefore C_{15} = C_{16} = 0$

$\sigma_2 = \sigma_2'$ from 2nd relation $\Rightarrow C_{25} = C_{26} = 0$

$\sigma_3 = \sigma_3'$ from 3rd relation $\Rightarrow C_{35} = C_{36} = 0$ $\Big\}$ reduced const C_{ij} by 8

$\sigma_4 = \sigma_4'$ from 4th relation $\Rightarrow C_{45} = C_{46} = 0$

$\sigma_5' = -\sigma_5$ no change $\Rightarrow -\sigma_5 = -C_{55}\varepsilon_5 - C_{56}\varepsilon_6$

$\sigma_6' = -\sigma_6$ no change $\quad -\sigma_6 = -C_{65}\varepsilon_5 - C_{66}\varepsilon_6$

These are the coeff ($\neq 0$) for monoclinic $C_{56}, C_{66}, C_{11}, C_{12}, C_{13}, C_{14}, C_{55}$, $C_{22}, C_{23}, C_{24}, C_{33}, C_{34}, C_{44}$

Now we look at symmetry in the (y, y')

then for orthorhombic $C_{14} = C_{24} = C_{34} = C_{56} = 0$ $\qquad$ Non zero coeff $C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}$, C_{66}, C_{55}, C_{44}

$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & 1 & 0 & 0 \\ y' & 0 & -1 & 0 \\ z' & 0 & 0 & 1 \end{array}$$

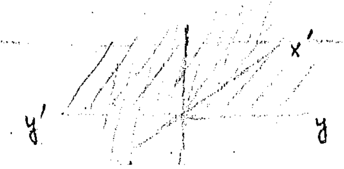
$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & & \\ C_{12} & C_{22} & C_{23} & C_{24} & & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & & \\ C_{14} & C_{24} & C_{34} & C_{44} & & \\ & & 0 & & C_{55} & C_{56} \\ & & & & C_{56} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \vdots \\ \\ \\ \\ \sigma_6 \end{pmatrix}$$

monoclinic

$$\begin{aligned} \sigma_1 &= C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 \\ \sigma_2 &= C_{12}\varepsilon_2 + C_{22}\varepsilon_2 + C_{23}\varepsilon_3 \qquad 0 \\ \sigma_3 &= C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{33}\varepsilon_3 \\ \sigma_4 &= \qquad\qquad\qquad\qquad C_{44}\varepsilon_4 \quad 0 \quad 0 \\ \sigma_5 &= \qquad 0 \qquad\qquad 0 \quad C_{55}\varepsilon_5 \quad 0 \\ \sigma_6 &= \qquad\qquad\qquad\qquad 0 \quad 0 \quad C_{66}\varepsilon_6 \end{aligned}$$

orthorhombic

$$\begin{array}{ll} \sigma_1' = \sigma_1 & \sigma_4' = -\sigma_4 \\ \sigma_2' = \sigma_2 & \sigma_5' = \sigma_5 \\ \sigma_3' = \sigma_3 & \sigma_6' = -\sigma_6 \end{array} \quad \Bigg\} \quad \begin{array}{ll} \varepsilon_1' = \varepsilon_1 & \varepsilon_4' = -\varepsilon_4' \\ \varepsilon_2' = \varepsilon_2 & \varepsilon_5' = \varepsilon_5 \\ \varepsilon_3' = \varepsilon_3 & \varepsilon_6' = -\varepsilon_6' \end{array}$$



Stress Strain Relations and Elastic Symmetry

References: Sokolnikoff, *Mathematical Theory of Elasticity* pp. 56-71
Malvern, *Introduction To The mechanics of A Deformable Medium*, pp 273-29

Triclinic Crystal (Most General Anisotropic material)

21 constants.

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}}_{C\text{ (Stiffness) Matrix}} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \rightarrow$$

Monoclinic Crystal

one plane of elastic symmetry. e.g. yz plane. Reflection of x a[xis] leaves constants unchanged

Direction Cosines of Transformation:

$$\begin{array}{c|c} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} x' \\ y' \\ z' \end{matrix} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

"C" Matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix}$$

Orthorhombic (orthotropic) Material

3 mutually orthogonal planes of symmetry. Reflection of x, y, and z leaves constants unchanged

Direction cosines (reflect y axis)

$$\begin{array}{c|c} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} x' \\ y' \\ z' \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

"C" Matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66} \end{bmatrix} \rightarrow$$

Cubic Material

Interchange of 2 axes (i.e. rotate 90° then reflect) leaves constan[ts] unchanged.

Direction cosines (interchange y & z)

$$\begin{array}{c|c} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} x' \\ y' \\ z' \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

"C" matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & C_{44} \end{bmatrix}$$

Isotropic Material:

Any coordinate transformation leaves Elastic Constants unchang[ed]

"C" matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{44}^{*} & & \\ & & & & C_{44}^{*} & \\ & & & & & C_{44}^{*} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \mu \end{bmatrix}$$

$$^{*}C_{44} = \frac{C_{11}-C_{12}}{2}$$

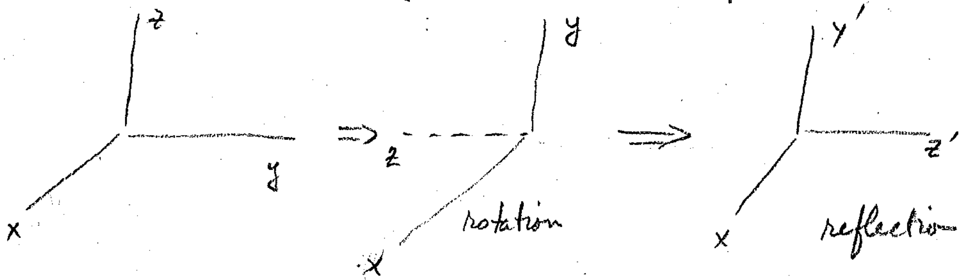
λ, μ, are the Lamé constants

$\underline{\sigma} = C\,\underline{\varepsilon}$ may be written $\quad \sigma_{ij} = \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}$

where ε_{kk} is the dilatation

11/9/78

cubic material - interchange of 2 axes rotate by 90° & reflect.



$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & 1 & 0 & 0 \\ y' & 0 & 0 & 1 \\ z' & 0 & 1 & 0 \end{array}$$

$\sigma_1' = \sigma_1$ $\quad \sigma_4' = \sigma_4$

$\sigma_2' = \sigma_3$ $\quad \sigma_5' = \sigma_6$

$\sigma_3' = \sigma_2$ $\quad \sigma_6' = \sigma_6$

$\epsilon_1' = \epsilon_1$ $\quad \epsilon_4' = \epsilon_4$

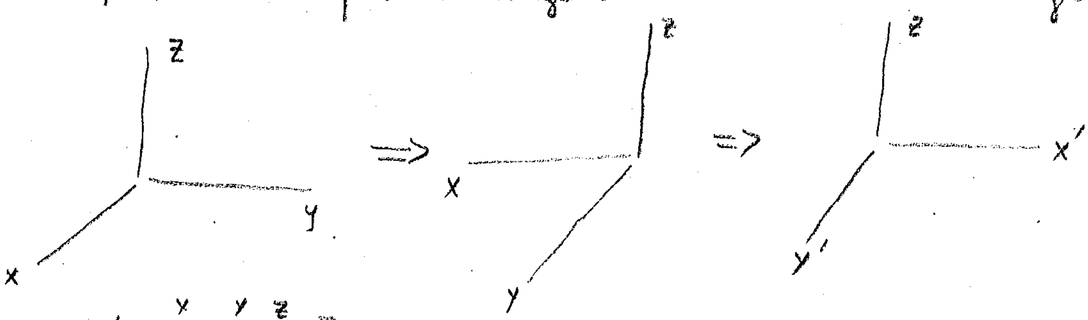
$\epsilon_2' = \epsilon_3$ $\quad \epsilon_5' = \epsilon_6$

$\epsilon_3' = \epsilon_2$ $\quad \epsilon_6' = \epsilon_6$

$$\begin{pmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_3' \\ \sigma_4' \\ \sigma_5' \\ \sigma_6' \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_3 \\ \sigma_2 \\ \sigma_4 \\ \sigma_6 \\ \sigma_5 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{13} & c_{12} & & & \\ c_{12} & c_{23} & c_{22} & & & \\ c_{13} & c_{33} & c_{23} & & & \\ & & & c_{44} & & \\ & & & & & c_{55} \\ & & & & c_{66} & \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{22} & c_{23} & & & \\ c_{13} & c_{23} & c_{33} & & & \\ & & & c_{44} & & \\ & & & & c_{55} & \\ & & & & & c_{66} \end{pmatrix}$$

$$\Rightarrow \quad c_{13} = c_{12}, \quad c_{22} = c_{33}, \quad c_{55} = c_{66}$$

now since this is only 1 interchange we can do another interchange



$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & 0 & 1 & 0 \\ y' & 1 & 0 & 0 \\ z' & 0 & 0 & 1 \end{array}$$

using this we will get $c_{11} = c_{22}$, $c_{12} = c_{23}$, $c_{44} = c_{55}$ such that

$$\begin{pmatrix} c_{11} & c_{12} & c_{12} & & & \\ c_{12} & c_{11} & c_{12} & & & \\ c_{12} & c_{12} & c_{11} & & & \\ & & & c_{44} & & \\ & & & & c_{44} & \\ & & & & & c_{44} \end{pmatrix}$$

to go to isotropic just rotate about the x axis

$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & 1 & 0 & 0 \\ y' & 0 & \cos\theta & \sin\theta \\ z' & 0 & -\sin\theta & \cos\theta \end{array}$$

$\sigma_4' = \left(\frac{\sigma_3 - \sigma_2}{2}\right) \sin 2\theta + \sigma_4 \ (\cos 2\theta)$ but $\sigma_4' = C_{44}\varepsilon_4'$

and $\varepsilon_4' = (\varepsilon_3 - \varepsilon_2) \sin 2\theta + \varepsilon_4 \cos 2\theta$

from the cubic $\sigma_3 - \sigma_2 = (C_{11} - C_{12})(\varepsilon_3 - \varepsilon_2)$ and $\sigma_4 = C_{44}\varepsilon_4$ then we obtain

$$\frac{C_{11} - C_{12}}{2} (\varepsilon_3 - \varepsilon_2) \sin 2\theta + C_{44}\varepsilon_4 \cos 2\theta = C_{44}(\varepsilon_3 - \varepsilon_2) \sin 2\theta + C_{44}\varepsilon_4 \cos 2\theta$$

since θ is arbitrary $\Rightarrow$ $C_{44} = \frac{C_{11} - C_{12}}{2}$

if we define $C_{44} = \mu$, $C_{12} = \lambda \Rightarrow C_{11} = \lambda + 2\mu$ $\quad$ λ, μ are Lamé constants.

and $\boxed{\sigma_{ij} = \lambda \varepsilon_{KK} \delta_{ij} + 2\mu \varepsilon_{ij}}$ Generalized Hooke's Law

ε_{KK} is the dilatation change in volume.

We can go directly to isotropic from the general strain energy density
U being a scalar it is invariant under any transformation for an isotropic material

Normally $U = f'(\varepsilon_{ij}) = \frac{1}{2} c'_{ijkl} \varepsilon'_{ij} \varepsilon'_{kl}$ for anisotropic
but $U = f(\varepsilon_{ij})$ for isotropic and since it is invariant under any transformation we can pick U as a fn of the strain invariants or

$U = f(I, II, III)$; since III involves cubic terms of ε_{ij} we can drop it to the first approx $\therefore$

$$U = f(I, II) \cong \bar{A} I^2 + B\, II$$

where $I = \varepsilon_{qq}$

$II = \varepsilon_{rs}\varepsilon_{rs} - \varepsilon_{qq}\varepsilon_{rr}$

$$U \cong \bar{A} \varepsilon_{qq}\varepsilon_{rr} + B\varepsilon_{rs}\varepsilon_{rs} - B\varepsilon_{qq}\varepsilon_{rr}$$
$$= A\varepsilon_{qq}\varepsilon_{rr} + B\varepsilon_{rs}\varepsilon_{rs}$$

[illegible] of Isotropic Stress-Strain Relations

Strain Energy Density ($U = f(\epsilon_{ij})$)

U is a scalar and is invariant under transformation even for an anisotropic material. However for an isotropic material it must have the same f regardless of coordinate system, thus

$$U = f(\epsilon_{ij}) = f(\epsilon'_{ij})$$

To assure this U must depend only on the invariants of strain

$$U = f(I, II, III)$$

for quadratic form

$$U = A(I)^2 + B(II)$$

$I = \epsilon_{qq}$, $I^2 = \epsilon_{pp}\epsilon_{rr}$

$II = \epsilon_{rs}\epsilon_{rs} - \epsilon_{qq}\epsilon_{rr}$

$$U = A\,\epsilon_{qq}\epsilon_{rr} + B\,\epsilon_{rs}\epsilon_{rs} - B\,\epsilon_{qq}\epsilon_{rr}$$

$$= a\,\epsilon_{qq}\epsilon_{rr} + b\,\epsilon_{rs}\epsilon_{rs} \qquad \text{where } a = A - B$$

$\sigma_{ij} = \dfrac{\partial U}{\partial \epsilon_{ij}}$; note ϵ's are all independent

e.g. $\dfrac{\partial \epsilon_{11}}{\partial \epsilon_{22}} = 0$, $\dfrac{\partial \epsilon_{12}}{\partial \epsilon_{12}} = 1 \Rightarrow \dfrac{\partial \epsilon_{pq}}{\partial \epsilon_{ij}} = \delta_{pi}\delta_{qj}$

$$\sigma_{ij} = \frac{\partial U}{\partial \epsilon_{ij}} = a(\delta_{pi}\delta_{qj}\epsilon_{rr} + \epsilon_{qq}\delta_{ri}\delta_{rj}) + b(\delta_{ri}\delta_{sj}\epsilon_{rs} + \epsilon_{rs}\delta_{ri}\delta_{sj})$$

$$= a(\delta_{ij}\epsilon_{rr} + \delta_{ij}\epsilon_{qq}) + 2b\,\epsilon_{ij}$$

$$= 2a\,\epsilon_{qq}\delta_{ij} + 2b\,\epsilon_{ij}$$

define $\lambda = 2a$, $\mu =$ [redacted] b (Lamé Constants)

$$\boxed{\sigma_{ij} = (\lambda\,\epsilon_{qq}\delta_{ij} + 2\mu\,\epsilon_{ij})}$$

Stress-Strain Relations for Isotropic Material

Indicial form $\sigma_{ij} = \lambda \varepsilon_{kk} + 2\mu\varepsilon_{ij}$ (Hooke's Law for Isotropy)

Expanded form

$$\sigma_{11} = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{11}$$

$$\sigma_{22} = \lambda(\quad " \quad) + 2\mu\varepsilon_{22} \qquad (1)$$

$$\sigma_{33} = \lambda(\quad " \quad) + 2\mu\varepsilon_{33}$$

$$\sigma_{12} = 2\mu\varepsilon_{12}\ ; \quad \sigma_{23} = 2\mu\varepsilon_{23}\ ; \quad \sigma_{13} = 2\mu\varepsilon_{13}$$

λ, μ, are the Lamé constants

$\mu = G$ = shear modulus or modulus of rigidity

Bulk modulus: sum first (3) of eqns (1) to give

$$\sigma_{kk} = 3\Theta = (3\lambda+2\mu)\Delta \qquad (\Delta = \varepsilon_{kk} = \text{dilatation})$$

$$\text{or} \quad \Theta = \left(\lambda+\frac{2\mu}{3}\right)\Delta \qquad K = \frac{\Theta}{\Delta} = \frac{\sigma_{ii}}{3\varepsilon_{ii}}$$

Bulk modulus $K = \lambda + \frac{2\mu}{3}$ (pressure required to cause unit change in volume)

Inverted σ-ε relations:

Last (3) of (1) invert directly $\Rightarrow \varepsilon_{12} = \frac{1}{2\mu}\sigma_{12}$, $\varepsilon_{13} = \frac{1}{2\mu}\sigma_{13}$, $\varepsilon_{23} = \frac{1}{2\mu}\sigma_{23}$

1st (3): Eliminate Δ using K

$$\Rightarrow \sigma_{11} = \frac{\lambda\sigma_{kk}}{3K} + 2\mu\varepsilon_{11}$$

solve for $\varepsilon_{11} = \frac{1}{2\mu}\left[\sigma_{11} - \frac{\lambda\sigma_{kk}}{3\lambda+2\mu}\right]$; $\varepsilon_{22} = \ldots$ $\varepsilon_{33} = \ldots$

Indicial inverted form: $\varepsilon_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda\sigma_{kk}}{3\lambda+2\mu}\delta_{ij}\right]$

Relation between λ, μ and E, ν:

$$\underset{\text{(Young's modulus)}}{E} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\ ; \quad \underset{\text{(Poisson's ratio)}}{\nu} = \frac{\lambda}{2(\lambda+\mu)}$$

$$\text{or} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

In terms of E, ν: $\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu(\sigma_{22}+\sigma_{33}))$ $\quad \varepsilon_{22} = \ldots$ $\quad \varepsilon_{33} = \ldots$ (cyclic)

$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}$ $\quad \varepsilon_{23} = \ldots$ $\quad \varepsilon_{13} = \ldots$ $\quad$ and $\gamma_{ij} = 2\varepsilon_{ij} = \frac{2(1+\nu)}{E}\sigma_{ij}$

TABLE 2.1

Relations between the elastic constants of an isotropic body.

Constant ↓ / in terms of → a and b	E, ν	E, μ	E, κ	ν, λ	ν, μ	ν, κ	λ, μ	λ, κ	μ, κ
$E =$	E	E	E	$\dfrac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$2\mu(1+\nu)$	$3\kappa(1-2\nu)$	$\dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\dfrac{9\kappa(\kappa-\lambda)}{3\kappa-\lambda}$	$\dfrac{9\kappa\mu}{3\kappa+\mu}$
$\nu =$	ν	$\dfrac{E}{2\mu}-1$	$\dfrac{1}{2}-\dfrac{E}{6\kappa}$	ν	ν	ν	$\dfrac{\lambda}{2(\lambda+\mu)}$	$\dfrac{\lambda}{3\kappa-\lambda}$	$\dfrac{3\kappa-2\mu}{2(3\kappa+\mu)}$
$\lambda =$	$\dfrac{E\nu}{(1+\nu)(1-2\nu)}$	$\dfrac{\mu(E-2\mu)}{3\mu-E}$	$\dfrac{3\kappa(3\kappa-E)}{9\kappa-E}$	λ	$\dfrac{2\mu\nu}{1-2\nu}$	$\dfrac{3\kappa\nu}{1+\nu}$	λ	λ	$\dfrac{3\kappa-2\mu}{3}$
$\mu =$	$\dfrac{E}{2(1+\nu)}$	μ	$\dfrac{3\kappa E}{9\kappa-E}$	$\dfrac{\lambda(1-2\nu)}{2\nu}$	μ	$\dfrac{3\kappa(1-2\nu)}{2(1+\nu)}$	μ	$\dfrac{3(\kappa-\lambda)}{2}$	μ
$\kappa =$	$\dfrac{E}{3(1-2\nu)}$	$\dfrac{\mu E}{3(3\mu-E)}$	κ	$\dfrac{\lambda(1+\nu)}{3\nu}$	$\dfrac{2\mu(1+\nu)}{3(1-2\nu)}$	κ	$\dfrac{3\lambda+2\mu}{3}$	κ	κ

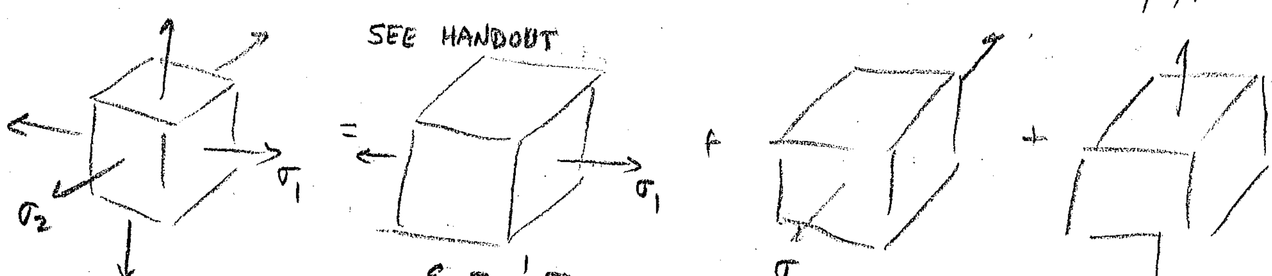
now $\sigma_{ij} = \dfrac{\partial U}{\partial \varepsilon_{ij}} = A \dfrac{\partial \varepsilon_{qq}}{\partial \varepsilon_{ij}} \varepsilon_{rr} + A\, \varepsilon_{qq} \dfrac{\partial \varepsilon_{rr}}{\partial \varepsilon_{ij}} + 2B \dfrac{\partial \varepsilon_{rs}}{\partial \varepsilon_{ij}} \varepsilon_{rs}$

$$A\left(\delta_{iq}\delta_{jq}\varepsilon_{rr} + \delta_{ir}\delta_{jr}\varepsilon_{qq}\right) + 2B\, \delta_{ir}\delta_{js}\varepsilon_{rs}$$

$$= 2A\,\varepsilon_{kk}\delta_{ij} + 2B\,\varepsilon_{ij} \qquad \text{LAMÉ} \quad \lambda = 2A \quad B = \mu$$

11/11/78

SEE HANDOUT

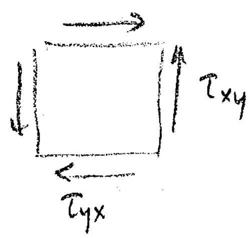


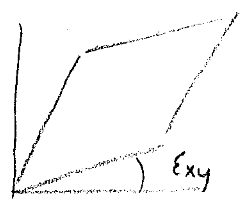
ε_{11_T} $\qquad \varepsilon_{11_a} = \frac{1}{E}\sigma_{11}$ $\qquad \varepsilon_{11_b} = -\nu\,\varepsilon_{22} = \frac{-\nu}{E}\sigma_{22}$ $\qquad \varepsilon_{11_c} = -\nu\,\varepsilon_{33} = \frac{-\nu}{E}\sigma_{33}$

by linearized theory of elast

$$\varepsilon_{11_{TOT}} = \varepsilon_{11_a} + \varepsilon_{11_b} + \varepsilon_{11_c}$$

$$\therefore \quad \varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right)$$





$$2\varepsilon_{xy} = \gamma_{xy}$$

$$\gamma_{xy} = \frac{1}{\mu}\tau_{xy} \qquad \mu = G \text{ shear modulus}$$

$$\varepsilon_{xy} = \frac{1}{2\mu}\tau_{xy}$$

$$U = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} = \tfrac{1}{2}\left(\lambda\Delta\delta_{ij} + 2\mu\,\varepsilon_{ij}\right)\varepsilon_{ij} \qquad \text{where } \Delta = \varepsilon_{kk} = \nabla\cdot u$$

$$= \tfrac{1}{2}\left(\lambda\Delta\delta_{ij}\varepsilon_{ij} + 2\mu\,\varepsilon_{ij}\varepsilon_{ij}\right) = \tfrac{1}{2}\left(\lambda\Delta^2 + 2\mu\,\varepsilon_{ij}\varepsilon_{ij}\right); \; \varepsilon_{jj} = \delta_{ij}\varepsilon_{ij}$$

$$U = \tfrac{1}{2}\lambda\Delta^2 + \mu\left[\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + 2\left(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{13}^2\right)\right]$$

- U will be shown to be positive definite ; we can rewrite U as

$$U = \tfrac{1}{2}\left\{\left(\lambda + \tfrac{2\mu}{3}\right)\Delta^2 + \mu\left[\tfrac{1}{3}\left(2\varepsilon_{22} - \varepsilon_{33} - \varepsilon_{11}\right)^2 + \left(\varepsilon_{33} - \varepsilon_{11}\right)^2 + 4\left(\varepsilon_{21}^2 + \varepsilon_{31}^2 + \varepsilon_{12}^2\right)\right.$$

note that $\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$ are independent; since $K = \lambda + \frac{2\mu}{3} \geq 0$ & $\mu \geq 0$
we still need to prove that Δ, $2\varepsilon_{22} - \varepsilon_{11} - \varepsilon_{33}$, $-\varepsilon_{11} + \varepsilon_{33}$ are indep.
– look at det of coeff; it is $= 6$ $\therefore$ they are indep. $\Rightarrow$ U is positive definite.

Now to prove uniqueness of solution

15 eqs.

$$\begin{cases} \sigma_{ij,i} + f_j = \rho \ddot{u}_j & (3) \\ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) & (6) \\ \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} & (6) \\ & \overline{(15)} \end{cases}$$

$3 u_j$, $6 \varepsilon_{ij}$, $6 \sigma_{ij}$ = 15 unknowns

Assume 2 solutions let $\sigma'_{ij}, \varepsilon'_{ij}, u'_j$ be solution #1
$\sigma''_{ij}, \varepsilon''_{ij}, u''_j$ be solution #2
$\sigma^*_{ij}, \varepsilon^*_{ij}, u^*_j$ be sol #1 – sol #2

by linear theory if sol #1, sol #2 solve eqs so do any linear comb of Sol #1 & Sol #2 $\Rightarrow$ Sol * is also a solution

11/13/78

Kirchoff: proof of uniqueness – assume 2 solns to same problem; since difference must also be a solution, one can use the energy equation and try to find conditions on the solution.

Uniqueness of solution of a typical problem

1. Assume 2 solutions $(\sigma'_{ij}, \varepsilon'_{ij}, u'_i)$ $(\sigma''_{ij}, \varepsilon''_{ij}, u''_i)$

Let $\sigma^*_{ij} = \sigma'_{ij} - \sigma''_{ij}$
$\varepsilon^*_{ij} = \varepsilon'_{ij} - \varepsilon''_{ij}$
$u^*_i = u'_i - u''_i$

} must also satisfy stress strain strain energy

The "difference" solution must satisfy (because of linearity) the stress eqs of equil $(\sigma^*_{ij,i} + f^*_j = \rho \ddot{u}^*_i)$, the strain relation $\varepsilon^*_{ij} = \frac{1}{2}(u^*_{j,i} + u^*_{i,j})$ and the stress strain $\sigma^*_{ij} = \lambda \Delta^* \delta_{ij} + 2\mu \epsilon^*_{ij}$ where $f^*_j = f'_j - f''_j$

We postulate that the total energy of the "difference" solution should vanish at all times. Let us derive the energy for the system

$$E = E(t_0) + \int_{t_0}^{t}\int_V (\dot{K} + \dot{U})\, dV\, dt$$

$$U = (\lambda + \tfrac{2}{3}\mu)\Delta^{*2} + \mu\left[\tfrac{1}{3}\left(2\epsilon_{22}^* - \epsilon_{33}^* - \epsilon_{11}^*\right)^2 + \left(\varepsilon_{33}^* - \varepsilon_{11}^*\right)^2 + 4\left(\varepsilon_{23}^{*2} + \varepsilon_{31}^{*2} + \varepsilon_{12}^{*2}\right)\right]$$

$$K = \tfrac{1}{2}\rho\, \dot{u}_K^* \dot{u}_K^* = \tfrac{1}{2}\rho\left(\dot{u}_1^{*2} + \dot{u}_2^{*2} + \dot{u}_3^{*2}\right)$$

at $t = t_0$ $\quad E = E(t_0) = 0 \quad \sigma_0^*, \varepsilon_0^*, \dot{u}_0^* = 0 \quad$ I.C. must also be specified (displacements u_i, velocities $\dot{u}_i$)

Since $K + U$ are quadratic fns, the only way that E can be zero is if each term in the quadratic is zero (since U is positive definite)

at any time: $E = \displaystyle\int_{t_0}\int_V \Big[\underbrace{\rho(\dot{u}_1^*\ddot{u}_1^* + \dot{u}_2^*\ddot{u}_2^* + \dot{u}_3^*\ddot{u}_3^*)}_{\dot{K}} + \underbrace{\sigma_{ij}^*\dot{\varepsilon}_{ij}^*}_{\dot{U}}\Big]\, dV dt = 0$

$$\sigma_{ij}^*\dot{\varepsilon}_{ij}^* = \sigma_{ij}^*\dot{u}_{j,i}^* \quad (\text{since } \dot{\varepsilon}_{ij}^* = \dot{u}_{j,i}^* - \dot{\omega}_{ij}^* \text{ and since } u_{j,i} \text{ was symmetric part of } \varepsilon_{ij})$$

$$= (\sigma_{ij}\dot{u}_j^*)_{,i} - \sigma_{ij,i}\dot{u}_j^* \quad \text{using differentiation by parts}$$

$$\int_{t_0}^{t}\int_V \left[\rho\ddot{u}_j^* - \sigma_{ij,i}^*\right]\dot{u}_j^* + \underbrace{\overbrace{(\sigma_{ij}^*\dot{u}_j^*)_{,i}}^{\nabla\cdot(\sigma\cdot u)}}_{\int n_i\sigma_{ij}^* u_j^* dS}\Big]\, dV dt = 0 \quad \text{after replacing the above}$$

$\int n_i \sigma_{ij}^* u_j^* dS$ by divergence theorem

but $\sigma_{ij,i}^* + f_j^* = \rho\ddot{u}_j^*$ and $n_i\sigma_{ij}^* = t_j^*$ and to be even more general we take into account singularities, discont. b.c.

$$E = E_0 + \int_{t_0}^{t}\Big[\int_V f_j^*\dot{u}_j^* dV + \int_S t_j^*\dot{u}_j^* ds + \underbrace{\int_{S^*}}_{\substack{\text{singularities}\\\text{in the region}}} + \underbrace{\int_{S_d} t_j^*\Delta\dot{u}_j^* ds}_{\substack{\text{discontinuities}\\\text{occurring in the displacement}\\\text{field}}}\Big]dt$$

How $E = 0 \Rightarrow$

1) $E_0 = 0$ from specified initial conditions
2) $f_i^* = 0$ from specified body forces.
3) Compatibility conditions must be specified to insure $\int_{S_d} = 0$.
4) Must specify singularities – (how they become ∞) and therefore $\int_{S^*} = 0$
5) Boundary conditions must be specified so that $\int t_j^*\dot{u}_j^* = 0$
   a. either t_j^* are all zero (ie specify traction vector at each point on the surface of body
   b. either $\dot{u}_j^*$ are all zero (ie specify displacement rate at each surface point
   c. either $t_j^* / \dot{u}_j^*$ combinations are zero ie $t_1^* = \dot{u}_2^* = \dot{u}_3^* = 0$ or $t_1^*, t_2^*, \dot{u}_3^* = 0$

therefore we have 8 different ways to insure a unique solution $(a=1, b=1, c=3,3)$

11/15/78

1) For unique soln $\varepsilon_{ij}^*, u_i^* = 0$

2) $\varepsilon_{ij}^* = u_i^* = 0$ if $E = 0$ (because $E = 0$ only if every square of the positive definite fn. E vanishes separately)

3) But E can also be written as $E = E_0 + \int_{t_0}^{t} \left[\int_V f_j^* \dot{u}_j^* dV + \int_S t_j^* \dot{u}_j^* ds \right] dt$

(4) $f_j^* = 0$, which means f_j is specified
and (for example)
$t_j^* = 0$ " " t_j " "

In statics if tt are specified uniqueness of disp are specified up to rigid body effects (disp, rotation)
In dynamics uniqueness completely specifies solution.

<u>Formulation of Typical Problems in Elast.</u>

1) Disp. (3-D problem)
2) Stress formul (2-D problem)

$$\varepsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j})$$

$$\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{j,i} + u_{i,j}) \quad (2)$$

$$\sigma_{ij,i} + f_j = \rho \ddot{u}_j \quad (3)$$

substitute σ_{ij} from (2) into (3) to get Displacement Formulation

$$\mu u_{j,ii} + (\lambda + \mu) u_{i,ij} + f_j = \rho \ddot{u}_j$$

$$\mu \nabla^2 \mathbf{u} + (\lambda+\mu) \nabla(\nabla \cdot \mathbf{u}) + \mathbf{f} = \rho \ddot{\mathbf{u}}$$

$$\mu \nabla \cdot (\nabla \mathbf{u}) + (\lambda+\mu) \nabla(\nabla \cdot \mathbf{u}) + \mathbf{f} = \rho \ddot{\mathbf{u}}$$

Div (grad u) + grad (div u)

Stress formulation (we must also satisfy $\nabla \times \varepsilon \times \nabla = 0$)

$$\varepsilon_{jl,sk} + \varepsilon_{sk,jl} - \varepsilon_{sl,jk} - \varepsilon_{jk,sl} = 0$$

this is the most convenient form to be used here. Beltrami & Michell converted them to stress forms

let us define them

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$ substitute into the above equation giving

$$\sigma_{ij,kl} + \sigma_{kl,ij} - \sigma_{ik,jl} - \sigma_{jl,ik} = \frac{\nu}{\nu+1}\left(\delta_{ij}\sigma_{\gamma\gamma,kl} + \delta_{kl}\sigma_{\alpha\alpha,ij} - \delta_{ik}\sigma_{\alpha\alpha,jl} - \delta_{jl}\sigma_{\beta\beta,ik}\right)$$

There are 6 independent equations

$k=l=1$	$k=l=2$	$k=l=3$	$k=l=1$	$k=l=2$	$k=l=3$
$i=j=2$	$i=j=3$	$i=j=1$	$i=2, j=3$	$i=3\ j=1$	$i=1, j=2$

11/17/78

We will try to express $\nabla \times \varepsilon \times \nabla$ in terms of stress.

Derivation of Beltrami - Michell Cond.

$$\sigma_{ij,kl} + \sigma_{kl,ij} - \sigma_{ik,jl} - \sigma_{jl,ik} = \frac{\nu}{\nu+1}\left(\delta_{ij}\sigma_{\alpha\alpha,kl} + \delta_{kl}\sigma_{\alpha\alpha,ij} - \delta_{ik}\sigma_{\alpha\alpha,jl} - \delta_{jl}\sigma_{\beta\beta,ik}\right) \quad (A)$$

let $k=l$ and add also let $\sigma_{kk} = \Theta$

$$\underline{\sigma_{ij,kk}} + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} = \frac{\nu}{\nu+1}\left(\delta_{ij}\sigma_{\alpha\alpha,kk} + \delta_{kk}\sigma_{\alpha\alpha,ij} - \delta_{ik}\sigma_{\alpha\alpha,jk} - \delta_{jk}\sigma_{\beta\beta,ik}\right)$$

$$\nabla^2\sigma_{ij} + \Theta_{,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} = \frac{\nu}{\nu+1}\left(\delta_{ij}\nabla^2\Theta + 3\Theta_{,ij} - \underbrace{\delta_{ik}\Theta_{,jk} - \delta_{jk}\Theta_{,ik}}_{-2\Theta_{,ij}}\right)$$

$$= \frac{\nu}{\nu+1}\left(\delta_{ij}\nabla^2\Theta + \Theta_{,ij}\right)$$

from Eqs of motion

$$\sigma_{ij,i} = \rho\ddot{u}_j - f_j$$ now take $\frac{\partial}{\partial k}$ (eq of motion)

$$\sigma_{ij,ik} = \rho\ddot{u}_{j,k} - f_{j,k}$$ put this into the compat eqn.

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\Theta_{,ij} - \frac{\nu}{1+\nu}\delta_{ij}\nabla^2\Theta = -\left(f_{i,j} + f_{j,i}\right) + \rho\left(\ddot{u}_{i,j} + \ddot{u}_{j,i}\right) \quad (B)$$

Next in (A) let $k=i$, $l=j$ and add

then $$2\sigma_{ij,ij} - \sigma_{ii,jj} - \sigma_{jj,ii} = \frac{\nu}{1+\nu}\left(2\delta_{ij}\Theta_{,ij} - \underline{\delta_{ii}\Theta_{,jj}} - \delta_{jj}\Theta_{,ii}\right)$$

[illegible]

$$\sigma_{ij,ij} - \nabla^2\Theta = \frac{\nu}{1+\nu}\left(\overbrace{\nabla^2\Theta - 3\nabla^2\Theta}^{-2\nabla^2\Theta}\right) \quad \text{or} \quad \sigma_{ij,ij} = \frac{1-\nu}{1+\nu}\nabla^2\Theta$$

if we go back to eq of motion and take $\frac{\partial}{\partial x_j}$ $\therefore$ $\sigma_{ij,ij} = \rho\ddot{u}_{j,j} - f_{j,j}$

substitute for $\sigma_{ij,ij}$ into the above $\rho\ddot{u}_{j,j} - f_{j,j} = \frac{1-\nu}{1+\nu}\nabla^2\Theta$

$$\therefore \quad \nabla^2\Theta = -\left(\frac{1+\nu}{1-\nu}\right)\left[f_{j,j} - \rho\ddot{u}_{j,j}\right] \qquad \text{(C)}$$

Substitute into (B) eqn (C)

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\Theta_{,ij} = -\frac{\nu}{1-\nu}\delta_{ij}\left[f_{k,k} - \underbrace{\rho\ddot{u}_{k,k}}_{\rho\ddot{\varepsilon}_{ii}}\right] - \left(f_{i,j} + f_{j,i}\right) + \rho\underbrace{\left(\ddot{u}_{i,j} + \ddot{u}_{j,i}\right)}_{2\ddot{\varepsilon}_{ij}}$$

Using the stress strain relation $\sigma_{ij} = \lambda\Delta\delta_{ij} + 2\mu\epsilon_{ij}$ or $\ddot{\sigma}_{ij} = \lambda\ddot{\Delta}\delta_{ij} + 2\mu\ddot{\epsilon}_{ij}$

$$\boxed{\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\Theta_{,ij} = -\frac{\nu}{1-\nu}f_{k,k}\delta_{ij} - f_{i,j} - f_{j,i} - \frac{\rho}{E}\left(\frac{\nu}{1-\nu}\ddot{\Theta}\delta_{ij} - 2(1+\nu)\ddot{\sigma}_{ij}\right)}$$

for the case of const body force & no accel. then

$$\boxed{\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\sigma_{kk,ij} = 0}$$

Beltrami Eqns.

Look at the (B) eq w/no accel or body forces.

then (B) reduces to $\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\Theta_{,ij} - \frac{\nu}{1+\nu}\delta_{ij}\nabla^2\Theta = 0$ but the first 2 terms are zero by Beltrami then $\Rightarrow$

$$\boxed{\nabla^2\Theta = 0 \quad \text{or } \sigma_{kk} = \Theta \text{ is a harmonic fn}}$$

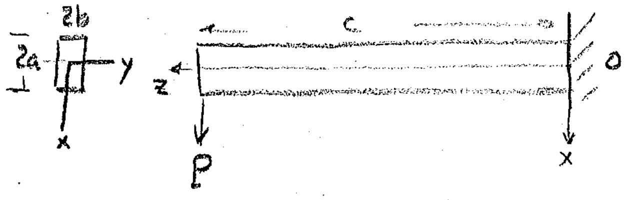
Now take ∇^2(Beltrami)

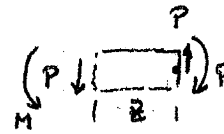
$$\nabla^2(\nabla^2\sigma_{ij}) + \frac{1}{1+\nu}\underbrace{\nabla^2\Theta_{,ij}}_{0 \text{ since } \nabla^2\Theta = 0} = 0$$

$$\therefore \boxed{\nabla^4\sigma_{ij} = 0 \quad \text{or} \quad \sigma_{ij} \text{ must be biharmonic}}$$

∴ only biharmonic fns are the only admissible stress field

Consider a cantilevered beam under an end load. Quest: Is strength of material solution also an elasticity soln.



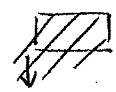


$$M + Pz = Pc$$
$$M = P(c-z)$$
$$\therefore\ -M - Pz = -Pc \qquad M = P(c-z)$$
$$\sigma_{zz} = -\frac{Mx}{I}$$

$\sigma_{xx},\ \sigma_{yy},\ \sigma_{zy},\ \sigma_{xy} = 0$

$$\sigma_{zz} = \frac{Mx}{I} = \frac{P(z-c)x}{I}$$

$$\sigma_{zx} = \frac{VQ}{Ib} = \frac{P(a^2-x^2)}{2I}$$



$$Q = \int_{-a}^{x} 2b\,x\,dx = \frac{(a^2-x^2)2b}{2}$$
$$Q = \int_{x}^{-a} b\,dA$$

Must satisfy equil
Compat
B.C.

11/21/78

CONTINUATION OF LAST LECTURE on Strength of Material Solution

Equil: (1) & (2) satisfied identically

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \qquad -\frac{Px}{I} + \frac{Px}{I} = 0 \quad ✓✓$$

check on BC

$\forall z$ for $x = \pm a,\ y = \pm b$ there must be no tractions

$$e_1 \cdot e_j \sigma_{ji} e_i = \sigma_{1i} e_i$$

For $x = \pm a$ $\quad t_n = (\sigma_{xx}, \sigma_{xy}, \sigma_{xz}) = \left.\frac{P(a^2-x^2)}{2I}\right|_{x=\pm a} = 0$ ✓ $(n = e_x)$

$y = \pm b$ $\quad t_n = (\sigma_{xy}, \sigma_{yy}, \sigma_{yz}) = 0$ ✓ $(n = e_y)$

$$e_2 \cdot e_j \sigma_{ji} e_i = \sigma_{2i} e_i$$

on the end $z = c$ $\quad t_n = (\sigma_{zx}, \sigma_{zy}, \sigma_{zz}) = \left(\frac{P(a^2-x^2)}{2I}, 0, 0\right)$ $(n = e_z)$

$$\therefore \int_{-a}^{a}\int_{-b}^{b} \frac{P(a^2-x^2)}{2I}\,dx\,dy = P \quad \therefore \text{ B.C. is satisfied}$$

$$= \frac{P\cdot 2b}{2I}\left(a^2x - \frac{x^3}{3}\right)\Big|_{-a}^{a} = \frac{Pb}{I}\left\{\frac{2a^3}{3} - \left(-a^3 + \frac{a^3}{3}\right)\right\} = \frac{4Pa^3b}{3I}$$

$$I = \frac{4a^3b}{3}$$

We do not worry about the built in end.

we will check compatibility using Beltrami Eqns.

$$\nabla^2\sigma_{xx} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} = 0$$

$$\nabla^2\sigma_{yy} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2} = 0 \qquad \text{where } \Theta = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$\nabla^2\sigma_{zz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2} = 0$$

$$\nabla^2\sigma_{yz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z} = 0$$

$$\nabla^2\sigma_{zx} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial z} = 0$$

$$\nabla^2\sigma_{xy} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial y} = 0$$

$$\Theta = \frac{P(z-c)x}{I}$$

(1–4, 6) will be zero

$(5) = \frac{-P}{I} + \frac{1}{1+\nu}\left(\frac{P}{I}\right) \neq 0 \qquad (5) = -\frac{\nu}{1+\nu}\frac{P}{I}$ thus a Poisson's effect is evident in elasticity that is not found by strength of materials

Calculating the strains

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\varepsilon_{xx} = \varepsilon_{yy} = -\frac{Px(z-c)\nu}{EI} \quad ; \quad \varepsilon_{zz} = \frac{Px(z-c)}{EI}$$

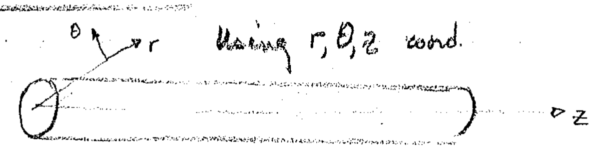
$$\varepsilon_{yz} = \varepsilon_{xy} = 0 \quad ; \quad \varepsilon_{zx} = \frac{1+\nu}{E}\frac{P(a^2-x^2)}{2I}$$

$$\frac{\partial^2\varepsilon_{yy}}{\partial z\partial x} + \frac{\partial}{\partial y}\left(\underbrace{\frac{\partial\varepsilon_{zx}}{\partial y}}_{=0} - \underbrace{\frac{\partial\varepsilon_{xy}}{\partial z}}_{=0} - \underbrace{\frac{\partial\varepsilon_{yz}}{\partial x}}_{=0}\right) = \frac{\partial^2\varepsilon_{yy}}{\partial z\partial x} = \frac{-P\nu}{EI}$$

at the top of beam fibers will stretch causing the ⊥ fibers to shorten
at "bottom" " " " contract " " " " " length
this causes transverse shears to be set up which are not taken into account by strength of materials.

We next take up torsion of beams.

Coulomb Torsion   Check this again



Coulomb assumed a displacement solution & checked the compat, equil, etc.

He assumed $u_r = 0$, $u_z = 0$, $u_\theta = \alpha r z$ where α = proportionality factors.

$\Rightarrow$ $\varepsilon_{rr} = 0$, $\varepsilon_{\theta\theta} = 0$, $\varepsilon_{zz} = 0$, $\varepsilon_{r\theta} = 0$, $\varepsilon_{zr} = 0$, $\varepsilon_{\theta z} = \frac{1}{2}\alpha r$

let's get the ~~strains~~ stresses

$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu \varepsilon_{ij} \Rightarrow \sigma_{\theta z} = \mu \alpha r$ only non vanishing shear

with this $\sigma_{\theta z}$ we satisfy equil. We also assume torsion is end loaded not surface area loaded.

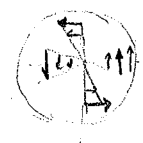
BC Cylindrical surface free of traction ie $e_r \cdot \sigma \stackrel{?}{=} 0$

but $e_r \cdot \sigma = (\sigma_{r\theta}, \sigma_{rz}, \sigma_{rr}) = 0$ thus we satisfy the BC on surface

on the ends $T = \int_0^a\int_0^{2\pi} \sigma_{\theta z} r \cdot r\,dr\,d\theta = \int_0^a \int_0^{2\pi} \mu\alpha r^3\,dr\,d\theta = \frac{\pi a^4}{2}\mu\alpha$

define polar moment of inertia $J = \iint r^2\,dA = \frac{\pi a^4}{2}$ $\therefore$ $T = J\mu\alpha$ (torque)

load on the end $e_z \cdot \sigma = (\sigma_{zr}, \sigma_{z\theta}, \sigma_{zz}) = \sigma_{\theta z}$



shear go in opposite direction opposite sides $\therefore$ they cancel out

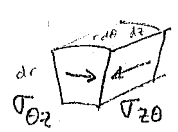
$$\omega_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial(r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] = \alpha z$$

α (is the twist) rotation / unit of length of cylinder

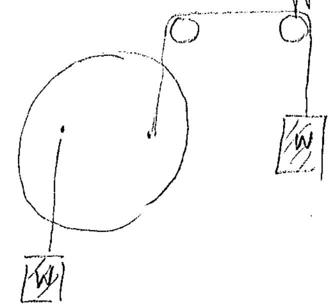
$$\alpha = \frac{\partial}{\partial z}(\omega_{r\theta})$$

11/22/78

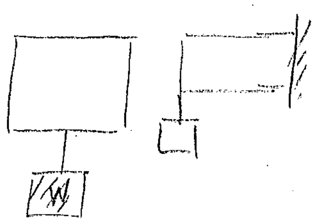
each cross section rotates as a body in Coulomb torsion



Suppose we look at different problems



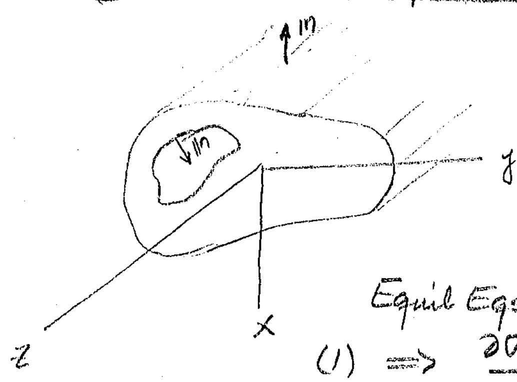
such that this system $\ni$ T. Is this system the same as the last torsion problem



If $W = P$ is this system the same as a uniformly end loaded beam such that $\int \sigma_{xz} dA = P$.

The answer is given by <u>St. Venant's principle</u>. If the two systems are statically equivalent then the stress distribution will be the same everywhere <u>except</u> near the loading point (points, faces etc) where the stress distribution produces a local effect. (This has not been proven - there are contradictive case.

<u>St. Venant Torsion problem.</u> No tractions on surface of the body except on the ends.



Let $\sigma_{zz}, \sigma_{xx}, \sigma_{yy}, \sigma_{xy} = 0$ and no body forces
Let $\sigma_{zx} \neq 0$; $\sigma_{zy} \neq 0$

Equil Eqs:

$$(1) \Rightarrow \frac{\partial \sigma_{zx}}{\partial z} = 0 \Rightarrow \sigma_{zx} = \sigma_{zx}(x,y)$$

$$(2) \Rightarrow \frac{\partial \sigma_{zy}}{\partial z} = 0 \Rightarrow \sigma_{zy} = \sigma_{zy}(x,y)$$

$$(3) \Rightarrow \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0$$

Consider $\phi = \phi(x,y)$ such that $\sigma_{xz} = \dfrac{\partial \phi}{\partial y}$ $\quad \sigma_{zy} = -\dfrac{\partial \phi}{\partial x}$

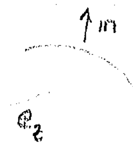
Hence we let $\phi(x,y)$ Stress function for torsion.

~~on cylindrical surface~~

B.C. for any surface $\mathbb{n}\cdot\mathbb{T}=0$

$$\mathbb{n}\cdot\left[\sigma_{zx}\,\mathbb{e}_z\mathbb{e}_x+\sigma_{zy}\,\mathbb{e}_z\mathbb{e}_y+\sigma_{zx}\,\mathbb{e}_x\mathbb{e}_z+\sigma_{zy}\,\mathbb{e}_y\mathbb{e}_z\right]=0$$

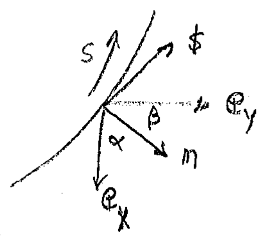
BC for cylindrical surface: for a cylindrical surface $\mathbb{n}\cdot\mathbb{e}_z=0$



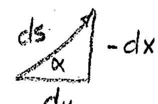
$$\therefore\ \mathbb{n}\cdot\mathbb{T}=0\ \Rightarrow\ \left(\mathbb{n}\cdot\mathbb{e}_x\,\sigma_{zx}+\mathbb{n}\cdot\mathbb{e}_y\,\sigma_{zy}\right)\mathbb{e}_z=0$$

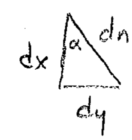
or $\quad\mathbb{n}\cdot\mathbb{e}_x\,\sigma_{zx}+\mathbb{n}\cdot\mathbb{e}_y\,\sigma_{zy}=0$

Look at the following section of the surface



$$\mathbb{n}\cdot\mathbb{e}_x=\$\cdot\mathbb{e}_y=\frac{dy}{ds}=\frac{dx}{dn}$$



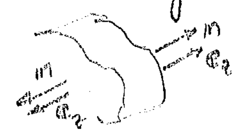


$$\mathbb{n}\cdot\mathbb{e}_y=\$\cdot(-\mathbb{e}_x)=-\frac{dx}{ds}=\frac{dy}{dn}$$

$$\therefore\quad \mathbb{n}\cdot\mathbb{e}_x\,\sigma_{xz}+\mathbb{n}\cdot\mathbb{e}_y\,\sigma_{zy}=0$$

$$\frac{dy}{ds}\cdot\frac{\partial\phi}{\partial y}+-\frac{dx}{ds}\,-\frac{\partial\phi}{\partial x}=\frac{d\phi}{ds}=0\ \Rightarrow\ \phi=\text{constant along the boundary.}$$

BC. at the end faces $\quad\mathbb{n}\cdot\mathbb{T}=\pm\,\mathbb{e}_z\cdot\mathbb{T}=\pm\left[\sigma_{zx}\mathbb{e}_x+\sigma_{zy}\mathbb{e}_y\right]$



Now on each end we want the resultant forces — we will show them to be $\equiv 0$

$$\therefore\quad\iint_A\sigma_{zx}\,dA=\iint_A\frac{\partial\phi}{\partial y}\,dA=F_x$$

$$\iint_A\sigma_{zy}\,dA=-\iint\frac{\partial\phi}{\partial x}\,dA=F_y$$

Recalling green's theorem

$$\iint_A\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=\oint_{\text{boundary}}P\,dx+Q\,dy$$

$\Rightarrow$ let $Q=0\quad -P=\phi\quad$ for F_x

$\phi=Q\quad P=0\quad$ for F_y

Evaluation of F's $\Rightarrow -\oint_{\text{boundary}} \phi\, dx = A_1 \cdot \oint dx;\ -\oint_{\text{boundary}} \phi\, dy = A_1 \cdot \oint dy$ but $\oint dx = 0$ $\oint dy = 0$ since ϕ = const on surface $\Rightarrow F_x = F_y = 0$

On the end faces there will be no resultant force

Resultant torque

$$\mathbb{T} = \iint \mathbb{r} \times (\mathbb{n} \cdot \sigma)\, dA$$

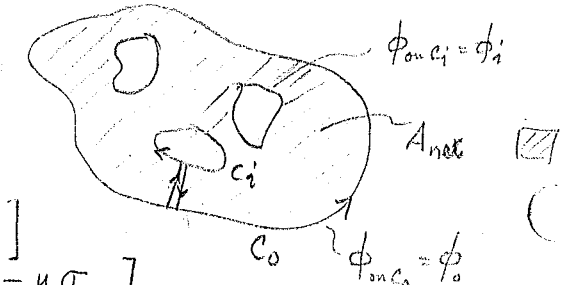
$$\mathbb{r} = x\mathbb{e}_x + y\mathbb{e}_y$$

$$\mathbb{r} \times (\mathbb{n} \cdot \sigma) = \pm \left[x\sigma_{zy} - y\sigma_{zx} \right] \mathbb{e}_z$$

11/27/78

Resultant Torque

$$\mathbb{T} = \iint_A \mathbb{r} \times (\mathbb{n} \cdot \sigma\, dA)$$



$$\mathbb{r} = x\mathbb{e}_x + y\mathbb{e}_y$$

$$\pm \mathbb{e}_z \cdot \sigma = \mathbb{n} \cdot \sigma = \pm \left[\mathbb{e}_x \sigma_{zx} + \mathbb{e}_y \sigma_{zy} \right]$$

$$\mathbb{r} \times (\mathbb{n} \cdot \sigma) = \pm \mathbb{e}_z \left[x\sigma_{zy} - y\sigma_{zx} \right]$$

$$\therefore \quad T = \int_A (x\sigma_{zy} - y\sigma_{zx})\, dA = -\iint \left(x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} \right) dx\, dy$$

we rewrite

$$T = \iint_A \left[2\phi - \frac{\partial}{\partial x}(\phi x) - \frac{\partial}{\partial y}(\phi y) \right] dx\, dy$$

now we use green's theorem $\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\, dx + Q\, dy$

$$T = \iint_{A_{net}} 2\phi\, dA - \oint_{C_0} \phi_0\, x\, dy + \oint_{C_0} \phi_0\, y\, dx - \sum_i \phi_i \oint_{C_i} x\, dy + \sum_i \phi_i \oint_{C_i} y\, dx$$

$$- \phi_0 \oint_{C_0} [x\, dy - y\, dx] \qquad - \sum \phi_i \left[\oint_{C_i} (x\, dy - y\, dx) \right]$$

and by using greens theorem in reverse — since this is a clockwise integration of Green's theorem

$$- \phi_0 \iint_{A_{gross}} 2\, dA \qquad - \sum \phi_i \iint_{A_i} -2\, dA$$

$$T = 2\left[\iint_{A_{net}} \phi\, dA - \phi_0 A_{gross} + \sum_i \phi_i A_i \right]$$ where $A_{gross} = A_{net} + \sum A_i$

Strains and Displacements – we will now get strain and displacement solutions

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = 0 \quad \text{and} \quad \varepsilon_{yz} = \frac{1+\nu}{E}\sigma_{zy}, \quad \varepsilon_{zx} = \frac{1+\nu}{E}\sigma_{zx}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad \varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0 \Rightarrow u_x = u_x(y,z)$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0 \Rightarrow u_y = u_y(x,z)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0 \Rightarrow u_z = u_z(x,y)$$

$$2\varepsilon_{xy} = (u_{x,y} + u_{y,x}) = 0 \Rightarrow \quad u_{x,y} = -f(z) \text{ only} \quad u_{y,x} = +f(z)$$

integration of $u_{x,y}$ and $u_{y,x}$ gives

$$\therefore \quad u_y(x,z) = x f(z) + g_y(z) \qquad u_x(y,z) = -y f(z) + g_x(z)$$

look at $\dfrac{\partial u_x}{\partial z} = 2\varepsilon_{zx} - \dfrac{\partial u_z}{\partial x}$ (note that $\varepsilon_{zx} = \frac{1}{2}(u_{x,z} + u_{z,x})$) Now take $\frac{\partial}{\partial z}$ of this equation

$$\frac{\partial^2 u_x}{\partial z^2} = \frac{\partial}{\partial z}\left(2\varepsilon_{zx} - \frac{\partial u_z}{\partial x}\right)$$

since $u_z = u_z(x,y)$ and $\frac{\partial}{\partial z} u_z = 0$ and since $\sigma_{zx} = \sigma_{zx}(x,y)$ only $\Rightarrow \varepsilon_{zx} = \varepsilon_{zx}(x,y)$ only and $\frac{\partial \varepsilon}{\partial z} \equiv 0$.

$$\therefore \frac{\partial^2 u_x}{\partial z^2} = 0 = -y f'' + g_x'' = 0 \quad \Rightarrow \quad f'' = 0 \text{ and } g_x'' = 0 \quad \forall y$$

hence $f(z) = az + b \quad g_x(z) = cz + d.$

We can get an analogous result for ε_{yz}: since $\varepsilon_{zy} = \varepsilon_{zy}(x,y)$ from fact that $\sigma_{zy}(x,y)$ only $\therefore \dfrac{\partial \varepsilon_{zy}}{\partial z} = 0$ and $\dfrac{\partial^2 u_z}{\partial y \partial z} = 0 \Rightarrow$

$$u_y = axz + bx + ez + f \qquad u_x = -ayz - by + cz + d.$$

B.C.

At the origin as a reference point $u_x, u_y = 0 \Rightarrow f, d = 0$ No r. b. transl.

At origin slopes wrt $z = 0 \quad u_{x,z} = u_{y,z} = 0 \Rightarrow c, e = 0$ No. r. b. rot

Specify that rotation about the z-axis will be measured from the origin $\omega_{xy}\big|_{z=0} = 0$

$$\omega_{xy}\Big|_{z=0} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \frac{1}{2}\left(az+b+az+b\right) = az+b\Big|_{z=0} = 0 \Rightarrow b=0$$

Let $a = \alpha$ (the twist) hence we go to displ. eqns and finally get.

$u_y = \alpha x z \quad u_x = -\alpha y z$

note that $\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = 2\varepsilon_{yz} = -2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial x}$

$$\frac{\partial u_z}{\partial y} = -2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial x} - \alpha x \qquad \left(\text{since } \frac{\partial u_y}{\partial z} = -\alpha x\right)$$

$$\text{also } \frac{\partial u_z}{\partial x} = 2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial y} + \alpha y$$

$$\frac{\partial u_z}{\partial z} = 0 \text{ since } u_z = u_z(x,y) \text{ only}$$

we can now get u_z by integrating

Since we have a multiply connected region we must use cesaro's theorem.

$$\therefore \oint du_i = 0 \Rightarrow \oint \cancel{\frac{\partial u_x}{\partial x}}^{\,0} dx + \frac{\partial u_x}{\partial y} dy + \frac{\partial u_x}{\partial z} dz = \oint du_x = 0$$

$$-\alpha z\, dy - \alpha y\, dz = -\alpha \oint z dx + y dz = -\alpha \cdot 0 = 0$$

$\iint (1-1)\,dx\,dz$

by Green's Theorem

hence by Cesaro's theorem the displ are single valued in x direction
we can do same in y direction

11/29/78

we have found so far $u_x = -\alpha y z$ and $u_y = \alpha x z$

$$\text{now } \frac{\partial u_z}{\partial y} = -2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial x} - \alpha x$$

$$\frac{\partial u_z}{\partial x} = 2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial y} + \alpha y$$

$$\frac{\partial u_z}{\partial z} = 0$$

For single valuedness of u_z

$$\oint_c du_z = \oint_c \left(\frac{\partial u_z}{\partial x} dx + \frac{\partial u_z}{\partial y} dy + \cancelto{0}{\frac{\partial u_z}{\partial z} dz} \right) = \oint_{C_0} du_z + \sum_i \oint_{C_i} du_z$$

$$= \oint_c \left[\left(\frac{1}{\mu} \frac{\partial \phi}{\partial y} + \alpha y \right) dx - \left(\frac{1}{\mu} \frac{\partial \phi}{\partial x} + \alpha x \right) dy \right]$$

using Stokes Thm. the line integral is

$$-\iint_{A_{gross}} \left[\left(\frac{1}{\mu} \frac{\partial^2 \phi}{\partial x^2} + \alpha \right) + \left(\frac{1}{\mu} \frac{\partial^2 \phi}{\partial y^2} + \alpha \right) \right] dA - \underbrace{\sum_{i=0} \oint_{C_i} du_z}_{\text{cavities}}$$

For compatibility we must have

$$\frac{1}{\mu} \nabla_1^2 \phi + 2\alpha = 0 \quad \text{and} \quad \oint_{C_i} du_z = 0$$

or $\boxed{\nabla_1^{\,2} \phi = -2\mu\alpha}$

now let us look at $\oint_{C_i} du_z = 0$

$$\oint_{C_i} du_z = \frac{1}{\mu} \oint \left(\underbrace{\frac{\partial \phi}{\partial y} dx}_{\frac{\partial u_z}{\partial x} dx} - \underbrace{\frac{\partial \phi}{\partial x} dy}_{\frac{\partial u_z}{\partial y} dy} \right) + \alpha \oint (y\,dx - x\,dy)$$

From our previous lectures we related n, s, e_x, e_y at the boundary so that

$$\frac{dy}{ds} = \frac{dx}{dn} \;①, \quad -\frac{dx}{ds} = \frac{dy}{dn} \;②$$ solve for dy & dx in ① & ② respectively

hence the first part of the integral

$$\oint \frac{\partial \phi}{\partial y} dx - \frac{\partial \phi}{\partial x} dy = \oint \left(-\frac{\partial \phi}{\partial y} \frac{dy}{dn} - \frac{\partial \phi}{\partial x} \frac{dx}{dn} \right) ds = \oint -\frac{\partial \phi}{\partial n} ds$$

$$\oint y\,dx - x\,dy = -\oint (y\,dx - x\,dy) = 2 \oint_{A_i} dA = 2A_i$$

$$\therefore \oint_{C_i} du_z = -\frac{1}{\mu} \oint \frac{\partial \phi}{\partial n} ds + 2\alpha A_i = 0 \qquad \therefore \boxed{\oint_{C_i} \frac{\partial \phi}{\partial n} ds = 2\alpha A_i \mu}$$ on cavity bdy

Summary of the St Venant Torsion

<u>Equilib</u> - automatically satisfied by introducing $\phi(x,y)$ $\ni$ $\sigma_{zx} = \frac{\partial\phi}{\partial y}$, $\sigma_{zy} = -\frac{\partial\phi}{\partial x}$

<u>B.C.</u> we assumed that surface tractions on the surface $= 0$

$$\phi_i = K_i \;(\text{const}) \qquad (i = 0, 1, \dots, N)$$

K_0 on outer bdy
K_1 on 1st cavity etc.

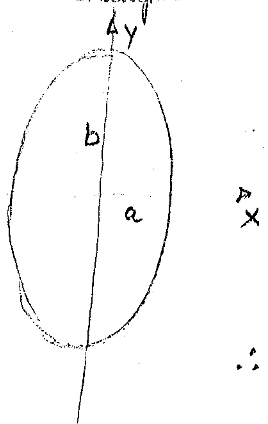
<u>Torque</u> $$T = 2\Big[\iint_{A_{net}} \phi\, dA - \underbrace{K_0 A_0}_{\text{gross area}} + \sum_{i=1}^{N} K_i A_i\Big]$$ $\ni$ $A_0 - \sum_i A_i = A_{net}$

<u>Compatibility</u> $\nabla^2\phi = -2\mu\alpha$

<u>Additional conditions for shaft w/ cavities</u> $\oint_{C_i} \frac{\partial\phi}{\partial n}\, ds = 2\mu A_i \alpha$

<u>Displacements</u> $u_x = -\alpha y z \qquad u_y = \alpha x z \qquad u_z = f(x,y)$ the warping fn.

• 1st Example Torsion of a bar of elliptical cross section w/ <u>NO</u> cavities



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{Eqn of contour w/ } b > a$$

since ϕ = const on the ellipse surface and since $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ on contour let us take $\phi = B\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

$\therefore$ Try $\phi = B\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$ const on bdy

$\therefore$ since $\nabla^2\phi = -2\mu\alpha = B\left[\frac{2}{a^2} + \frac{2}{b^2}\right]$ $\therefore$ $\boxed{B = -\frac{\mu\alpha a^2 b^2}{a^2 + b^2}}$

$$T = 2\iint \phi\, dA - K_0 A_0 = \iint \phi\, dA \quad \text{since } K_0 = 0 \text{ on bdy}$$

$$T = 2\iint \phi\, dA = -B\pi ab = \frac{\pi a^3 b^3 \mu\alpha}{a^2 + b^2} = D\alpha = \mu\alpha J$$

D ~ torsional stiffness

J = polar moment of inertia about z axis

12/1/78

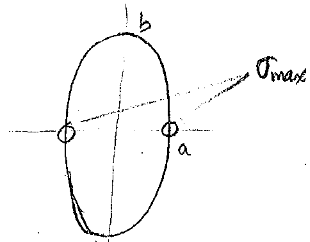
Final Exam : is due Monday the 11th of Dec   will be given out next week

$$\sigma_{zx} = \frac{\partial \phi}{\partial y} = \frac{-2Ty}{\pi a b^3} \qquad \sigma_{zy} = -\frac{\partial \phi}{\partial x} = \frac{2Tx}{\pi a^3 b}$$

$$\sigma_{res} = \left(\sigma_{zx}^2 + \sigma_{zy}^2\right)^{1/2} = \frac{2T}{\pi ab}\left(\frac{y^2}{b^4} + \frac{x^2}{a^4}\right)^{1/2} = \frac{2T}{\pi a^3 b^3}\left(b^4x^2 + a^4y^2\right)^{1/2}$$

$$\boxed{\sigma_{max}\ (@\ x = \pm a) = \frac{2T}{\pi a^2 b}}$$

∴ if yield occurs at pts on surface where it is closest to origin



Now $u_x = -\alpha yz = -\dfrac{T(a^2+b^2)}{\pi a^3 b^3 \mu} yz$

$u_y = \alpha xz = \dfrac{T(a^2+b^2)}{\pi a^3 b^3 \mu} xz$

from our summary $\dfrac{\partial u_z}{\partial x} = \dfrac{1}{\mu}\dfrac{\partial \phi}{\partial y} + \alpha y = \dfrac{1}{\mu}\left(\dfrac{-2Ty}{\pi a b^3}\right) + \left(-\dfrac{T(a^2+b^2)}{\pi a^3 b^3 \mu}\right) y$

$$\boxed{\frac{\partial u_z}{\partial x} = \frac{b^2-a^2}{b^2+a^2}\alpha y}$$

$$\frac{\partial u_z}{\partial y} = -\left(\frac{1}{\mu}\frac{\partial \phi}{\partial x} + \alpha x\right) = \frac{b^2-a^2}{b^2+a^2}\alpha x$$

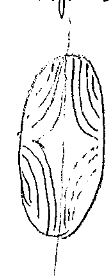
integrate   integrate

Now $u_z = \dfrac{b^2-a^2}{b^2+a^2}\alpha xy + f_1(y)$

$u_z = \dfrac{b^2-a^2}{b^2+a^2}\alpha xy + f_2(x)$

∴ $f_1(y) = f_2(x) = \text{const.}$ represents a rigid body disp in the z direct

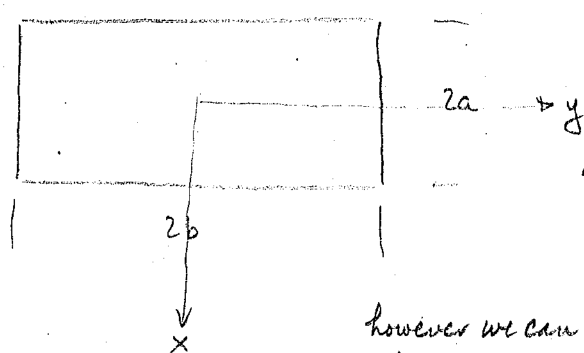
∴ $u_z = \dfrac{b^2-a^2}{b^2+a^2}\alpha xy$



warping function

$u_x = u_x(x,y)$
$u_y = u_y(x,y)$ } plane strain problems.
$u_z = 0$

Torsion of a bar of rectangular crossection



can we define a stress function ϕ .э. $\nabla^2\phi = -2\mu\alpha$?
we cannot find a single fn. of ϕ constant on body satisfying eq.

however we can construct solutions $\phi_0 = \mu\alpha(a^2 - x^2)$
.э. $\phi_0 = 0$ on $x = \pm a$
and we can construct a second ϕ_1 $\therefore$ $\nabla^2\phi = \nabla^2\phi_0 + \nabla^2\phi_1 = -2\mu\alpha$
thus $\nabla^2\phi_1 = 0$ since $\nabla^2\phi_0 = -2\mu\alpha$

The B.C. @ $x = \pm a$ $\therefore$ $\phi(\pm a, y) = \phi_1(\pm a, y) = 0$
$y = \pm b$ $\quad \phi(x, \pm b) = \mu\alpha(a^2 - x^2) + \phi_1(x, \pm b) = 0$

since we want ϕ on bdy $= 0$

if we assume $\phi_1(x,y) = F_1(x)\,F_2(y)$ then $\nabla^2\phi_1 = F_1''F_2 + F_1F_2'' = 0$

thus $\dfrac{F_1''}{F_1} = -\dfrac{F_2''}{F_2} = \text{const} = -\beta^2$

$\therefore$ $F_1'' + \beta^2 F_1 = 0 \implies F_1 = A_1\cos\beta x + B_1\sin\beta x$ $\quad \forall y$
$F_2'' - \beta^2 F_2 = 0 \implies F_2 = A_2\cosh\beta y + B_2\sinh\beta y$ $\quad \cancel{\forall x}$

Symmetry w/t x, y $\implies$ $B_1 = 0$ and $B_2 = 0$ $\quad$ $(\sin\beta(-x) = -\sin\beta x)$
$(\sinh\beta(-x) = -\sinh\beta x)$

$\therefore$ $\phi_1 = C\cos\beta x\cosh\beta y$

$$\phi_1(\pm a, y) = C\cos\beta a \cosh\beta y = 0 \qquad \therefore \quad \beta a = (2k+1)\frac{\pi}{2} \qquad (k = 0, 1, 2, \dots)$$

$$\therefore \quad \phi_1(x,y) = \sum_{k=0}^{\infty} C_k \cos\frac{(2k+1)\pi}{2a} x \cosh\frac{(2k+1)\pi}{2a} y$$

other BC. $\phi_1(x, \pm b) = -\mu\alpha(a^2 - x^2)$

$$\therefore \quad -\mu\alpha(a^2 - x^2) = \sum C_k \cos\frac{(2k+1)\pi}{2a} x \cosh\frac{(2k+1)\pi}{2a} b$$

now
$$C_k \cosh\frac{(2k+1)\pi b}{2a} = \frac{2}{a}\int_0^a -\mu\alpha(a^2 - x^2)\cos\frac{(2k+1)\pi}{2a} x \, dx$$

$$= \frac{-32\mu a^2 (-1)^k}{(2k+1)^3\pi^3}$$

since only terms which will give results in fourier series is when $m = 2k+1$

$$\therefore \quad C_k = \frac{-32\mu\alpha a^2(-1)^k}{(2k+1)^3\pi^3}$$

$$\therefore \quad \phi = \phi_0 + \phi_1 = -\mu\alpha\left[(x^2 - a^2) + \frac{32a^2}{\pi^3}\sum_{k=0}^{\infty}(-1)^k \frac{\cos\frac{(2k+1)\pi x}{2a}\cosh\frac{(2k+1)\pi y}{2a}}{(2k+1)^3\cosh\frac{(2k+1)\pi b}{2a}}\right]$$

σ_{max} occurs at $x = \pm a$, $y = 0$

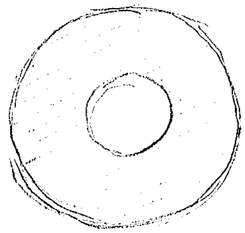
$$\sigma_{yz} = -\left.\frac{\partial\phi}{\partial y}\right|_{\substack{x=a \\ y=0}} = 2\mu\alpha a \zeta \qquad \text{where } \zeta = 1 - \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2\cosh\frac{(2k+1)\pi b}{2a}}$$

b/a	ζ
1.0	.675
1.2	.759
1.5	.848
2.0	.930
2.5	.968
3.0	.985
5.	.999
∞	1.0

for engineering purposes if $b/a > 2.5$ $\zeta \approx 1$

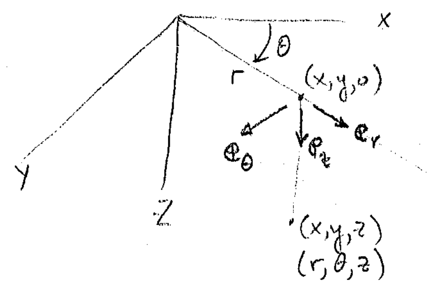
$$\therefore \quad \sigma_{yz} = 2\mu\alpha a$$

12/4/78



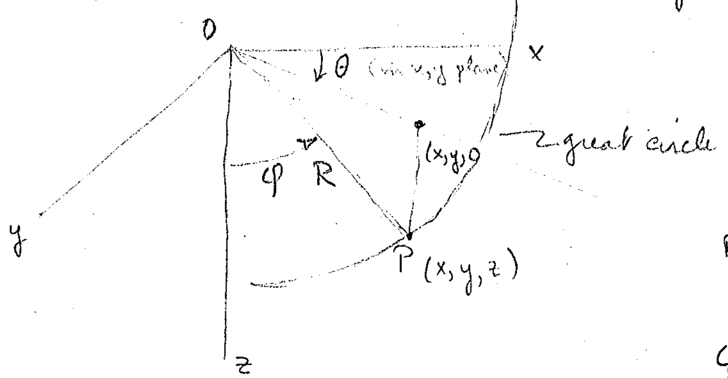
Orthogonal Curvilinear Coordinates will be discussed in order to do torsional problem of a hollow surface.

Cylindrical Coordinate systems.



	invertible
$x = r\cos\theta$	$r = \sqrt{x^2+y^2}$
$y = r\sin\theta$	$\theta = \arctan\left(y/x\right)$
$z = z$	$z = z$

Spherical Coordinate System.



$x = R\sin\varphi\cos\theta$

$y = R\sin\varphi\sin\theta$

$z = R\cos\varphi$

$R = \sqrt{x^2+y^2+z^2}$

$\theta = \tan^{-1}(y/x)$

$\varphi = \arctan\left(\dfrac{\sqrt{x^2+y^2}}{z}\right)$

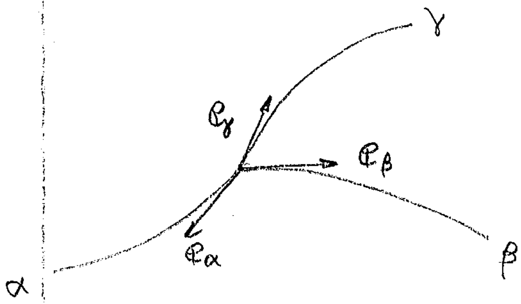
General orthogonal curvilinear coordinates (α,β,γ)

$\alpha = \alpha(x,y,z)$ Look at $\alpha =$ const this defines a surface

$\beta = \beta(x,y,z)$ Look at $\beta =$ " " " " "

$\gamma = \gamma(x,y,z)$ " " $\gamma =$ " " " " "

the intersection of these 3 surfaces defines a point p.



$$e_{\alpha_i} \cdot e_{\alpha_j} = \delta_{ij} \quad , \quad e_{\alpha_i} \times e_{\alpha_j} = e_{\alpha_k}\, e_{ijk}$$

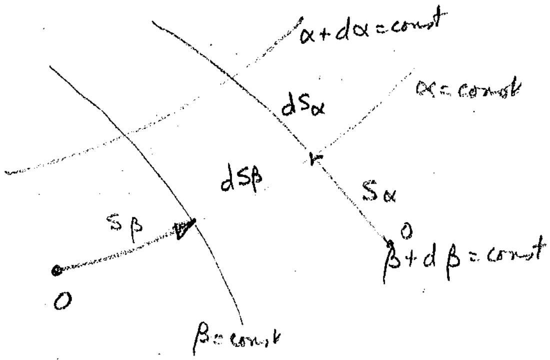
define the displacement vector $\mathbf{u} = u_\gamma \mathbf{e}_\gamma + u_\beta \mathbf{e}_\beta + u_\gamma \mathbf{e}_\gamma$

define a tensor $\boldsymbol{\sigma} = \sigma_{\alpha\alpha} \mathbf{e}_\alpha \mathbf{e}_\alpha + \sigma_{\alpha\beta} \mathbf{e}_\alpha \mathbf{e}_\beta + \dots = \sigma_{\alpha_i \alpha_j} \mathbf{e}_{\alpha_i} \mathbf{e}_{\alpha_j}$

	x	y	z
α	$\mathbf{e}_\alpha \cdot \mathbf{e}_x$	$\mathbf{e}_\alpha \cdot \mathbf{e}_y$	
β		$\mathbf{e}_\beta \cdot \mathbf{e}_y$	
γ	$\mathbf{e}_\gamma \cdot \mathbf{e}_x$		$\mathbf{e}_\gamma \cdot \mathbf{e}_z$

This transforms items in (x,y,z) plane to (α,β,γ) plane

Now look at surface $\gamma = $ const.



stretch function or stretch factor

$ds_\alpha = h_\alpha(\alpha,\beta,\gamma)\, d\alpha$ $\qquad \dfrac{d\alpha}{ds_\alpha} = \dfrac{1}{h_\alpha}$

$ds_\beta = h_\beta(\alpha,\beta,\gamma)\, d\beta$ $\qquad$ since, $\alpha, \beta,$ or γ may not be a physical length (ie θ)

$ds_\gamma = h_\gamma(\alpha,\beta,\gamma)\, d\gamma$

Looking at the position vector in the cartesian plane $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ then

$$d\mathbf{r} = dx\,\mathbf{e}_x + dy\,\mathbf{e}_y + dz\,\mathbf{e}_z = \mathbf{e}_\alpha\, ds_\alpha + \mathbf{e}_\beta\, ds_\beta + \mathbf{e}_\gamma\, ds_\gamma$$

now we note that $\dfrac{\partial \mathbf{r}}{\partial s_\alpha} = \mathbf{e}_\alpha$

We can take $$\frac{\partial \mathbf{r}}{\partial s_\alpha} = \overset{\mathbf{e}_\alpha\cdot\mathbf{e}_x}{\frac{\partial x}{\partial s_\alpha}} \mathbf{e}_x + \overset{\mathbf{e}_\alpha\cdot\mathbf{e}_y}{\frac{\partial y}{\partial s_\alpha}} \mathbf{e}_y + \overset{\mathbf{e}_z\cdot\mathbf{e}_\alpha}{\frac{\partial z}{\partial s_\alpha}} \mathbf{e}_z = \mathbf{e}_\alpha$$

now take the dot product of $$\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha = 1 = \left(\frac{\partial x}{\partial s_\alpha}\right)^2 + \left(\frac{\partial y}{\partial s_\alpha}\right)^2 + \left(\frac{\partial z}{\partial s_\alpha}\right)^2$$

$$= \left(\frac{\partial x}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial s_\alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial s_\alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial s_\alpha}\right)^2$$

$$= \frac{1}{h_\alpha^2}\left[\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2\right]$$

$$\therefore \quad h_\alpha = \sqrt{\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2}$$

$$\therefore \quad \boxed{h_{\alpha_j} = \sqrt{\frac{\partial x_i}{\partial \alpha_j}\frac{\partial x_i}{\partial \alpha_j}} \quad \text{sum over } i, \text{ no sum over } j}$$

$$h_r = \sqrt{\cos^2 + \sin^2} = 1$$

$$h_\theta = \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2} = r$$

$$h_z = 1$$

$$ds_\theta = h_\theta \, d\theta = r\,d\theta$$
$$ds_r = dr$$
$$ds_z = dz$$

for spherical $h_R = 1$, $h_\theta = R\sin\varphi$, $h_\varphi = R$ $\therefore$ $ds_R = dr$, $ds_\theta = R\sin\varphi\, d\theta$, $ds_\varphi = R\,d\varphi$

Direction cosines

$$l_{\alpha y} = e_\alpha \cdot e_y = \frac{\partial y}{\partial s_\alpha} = \frac{1}{h_\alpha} \cdot \frac{\partial y}{\partial \alpha}$$

$$\therefore \quad l_{\alpha_i x_j} = e_{\alpha_i} \cdot e_j = \frac{1}{h_{\alpha_i}} \frac{\partial x_j}{\partial \alpha_i} \qquad i \text{ not summed.}$$

$$e_\alpha = \frac{1}{h_\alpha}\frac{\partial x}{\partial \alpha} e_x + \frac{1}{h_\alpha}\frac{\partial y}{\partial \alpha} e_y + \frac{1}{h_\alpha}\frac{\partial z}{\partial \alpha} e_z$$

for cylindrical coord.

$$e_z = e_z \qquad \frac{\partial e_r}{\partial \theta} = e_\theta$$
$$e_r = \cos\theta\, e_x + \sin\theta\, e_y$$
$$e_\theta = -\sin\theta\, e_x + \cos\theta\, e_y \qquad \frac{\partial e_\theta}{\partial \theta} = -e_r$$

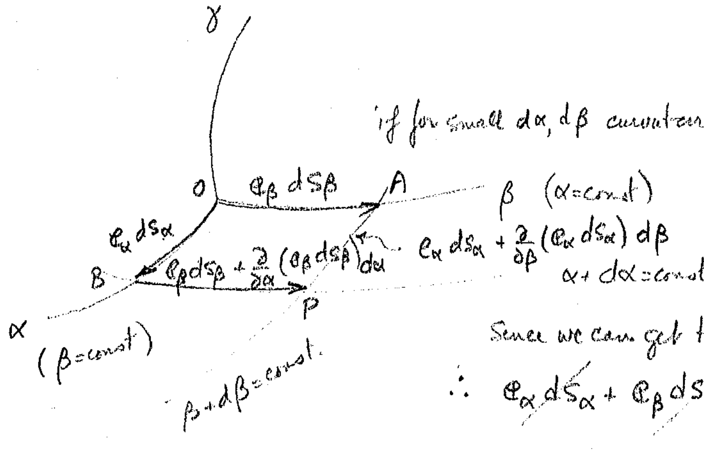
for spherical coordinates

$$e_R = \sin\varphi\cos\theta\, e_x + \sin\varphi\sin\theta\, e_y + \cos\varphi\, e_z$$
$$e_\theta = -\sin\theta\, e_x + \cos\theta\, e_y$$
$$e_\varphi = \cos\varphi\cos\theta\, e_x + \cos\varphi\sin\theta\, e_y - \sin\varphi\, e_z$$

$$\frac{\partial e_R}{\partial \theta} = \sin\varphi\, e_\theta, \quad \frac{\partial e_R}{\partial \varphi} = e_\varphi$$

12/6/78

Back to orthogonal curvilinear coord. on constant γ surfaces



if for small $d\alpha, d\beta$ curvature can be neglected

Since we can get to P from O along OA + OP or along OB + BP

$$\therefore\ \cancel{e_\alpha ds_\alpha} + e_\beta ds_\beta + \frac{\partial}{\partial\alpha}(e_\beta\, ds_\beta)d\alpha = \cancel{e_\beta ds_\beta} + e_\alpha ds_\alpha + \frac{\partial}{\partial\beta}(e_\alpha\, ds_\alpha)d\beta$$

now divide both sides by $d\alpha\, d\beta$ and remembering that $ds_\alpha = h_\alpha d\alpha$ we get

$$\frac{\partial}{\partial\alpha}(e_\beta h_\beta d\beta)d\alpha = \frac{\partial}{\partial\alpha}(e_\beta h_\beta)\, d\beta\, d\alpha = \frac{\partial}{\partial\beta}(e_\alpha\ h_\alpha d\alpha)d\beta = \frac{\partial}{\partial\beta}(e_\alpha h_\alpha)\, d\alpha\, d\beta$$

hence we get $\quad h_\alpha \frac{\partial}{\partial\beta} e_\alpha + e_\alpha \frac{\partial}{\partial\beta} h_\alpha = e_\beta \frac{\partial h_\beta}{\partial\alpha} + h_\beta \frac{\partial e_\beta}{\partial\alpha}$

now take dot product w/ e_β and remembering that $\frac{\partial e_\beta}{\partial\alpha} \perp e_\beta$ and $\underset{\text{since orthog}}{e_\alpha \cdot e_\beta = 0}$

then $\quad \frac{\partial h_\beta}{\partial\alpha} = h_\alpha\ e_\beta \cdot \frac{\partial e_\alpha}{\partial\beta} \qquad$ remembering that $\frac{ds_\alpha}{d\alpha} = h_\alpha$

rewrite as $\quad \frac{1}{h_\alpha}\frac{\partial h_\beta}{\partial\alpha} = e_\beta \cdot \frac{\partial e_\alpha}{\partial\beta}$ or $\frac{\partial h_\beta}{\partial s_\alpha} = e_\beta \cdot \frac{\partial e_\alpha}{\partial\beta}\frac{h_\beta}{h_\beta} = h_\beta\, e_\beta \cdot \frac{\partial e_\alpha}{\partial s_\beta}$ and dividing by h_β then $\frac{1}{h_\beta}\frac{\partial h_\beta}{\partial s_\alpha} = \frac{\partial(\ln h_\beta)}{\partial s_\alpha}$, then

$$e_\beta \cdot \frac{\partial e_\alpha}{\partial s_\beta} = \frac{\partial(\ln h_\beta)}{\partial s_\alpha}$$

and by cyclic permutation

$$e_\gamma \cdot \frac{\partial e_\beta}{\partial s_\gamma} = \frac{\partial(\ln h_\gamma)}{\partial s_\beta}$$

$$e_\alpha \cdot \frac{\partial e_\gamma}{\partial s_\alpha} = \frac{\partial(\ln h_\alpha)}{\partial s_\gamma}$$

also we can get that since $e_\alpha \cdot e_\beta = 0$ then $\frac{\partial}{\partial s_\alpha}$ of $e_\alpha \cdot e_\beta = 0$ or

$$e_\alpha \cdot \frac{\partial e_\beta}{\partial s_\alpha} + \frac{\partial e_\alpha}{\partial s_\alpha}\cdot e_\beta = 0$$

$$e_\beta \cdot \frac{\partial e_\alpha}{\partial s_\alpha} + \frac{\partial e_\beta}{\partial s_\alpha}\cdot e_\alpha = 0$$

$$\Bigg\}\quad e_\beta \cdot \frac{\partial e_\alpha}{\partial s_\alpha} = -e_\alpha \cdot \frac{\partial e_\beta}{\partial s_\alpha} = -\frac{\partial(\ln h_\alpha)}{\partial s_\beta}$$

since $e_\alpha \cdot e_\beta = e_\beta \cdot e_\alpha$ we can using the above & interchange α, β get.

and by cyclic permutation $\quad e_\beta \cdot \frac{\partial e_\alpha}{\partial s_\beta} = \frac{\partial(\ln h_\beta)}{\partial s_\alpha}$ also $e_\beta \cdot \frac{\partial e_\alpha}{\partial s_\gamma} = 0$

since $\perp e_\gamma$

Now since

$$\nabla = e_x \frac{\partial}{\partial x} + e_y \frac{\partial}{\partial y} + e_z \frac{\partial}{\partial z}$$

since the s_j represent length changes we can write $\nabla = e_\alpha \frac{\partial}{\partial s_\alpha} + e_\beta \frac{\partial}{\partial s_\beta} + e_\gamma \frac{\partial}{\partial s_\gamma}$ or using the stretch factors

$$\boxed{\nabla = \frac{1}{h_\alpha} e_\alpha \frac{\partial}{\partial \alpha} + e_\beta \frac{1}{h_\beta} \frac{\partial}{\partial \beta} + \frac{1}{h_\gamma} e_\gamma \frac{\partial}{\partial \gamma}}$$

to get any general strain we know that $\varepsilon_{\alpha_i \alpha_j} = e_{\alpha_i} \cdot \varepsilon \cdot e_{\alpha_j}$

we also know that $\varepsilon = \frac{1}{2}(\nabla u + u\nabla)$

$$= \frac{1}{2}\left(e_\alpha \frac{\partial u}{\partial s_\alpha} + e_\beta \frac{\partial u}{\partial s_\beta} + e_\gamma \frac{\partial u}{\partial s_\gamma} + \frac{\partial u}{\partial s_\alpha} e_\alpha + \frac{\partial u}{\partial s_\beta} e_\beta + \frac{\partial u}{\partial s_\gamma} e_\gamma\right)$$

now if we look at one particular term (viz $\varepsilon_{\alpha\alpha}$)

$\varepsilon_{\alpha\alpha} = e_\alpha \cdot \varepsilon \cdot e_\alpha$ and let $u = u_\alpha e_\alpha + u_\beta e_\beta + u_\gamma e_\gamma$

Substitution and differentiation w/some conditions on the derivatives of the unit vectors gives for example for $\frac{\partial u}{\partial s_\alpha} = \frac{\partial(u_\alpha e_\alpha + u_\beta e_\beta + u_\gamma e_\gamma)}{\partial s_\alpha}$

$$\frac{\partial u}{\partial s_\alpha} = \frac{\partial u_\alpha}{\partial s_\alpha} e_\alpha + \frac{\partial u_\beta}{\partial s_\alpha} e_\beta + \frac{\partial u_\gamma}{\partial s_\alpha} e_\gamma + u_\alpha \frac{\partial e_\alpha}{\partial s_\alpha} + u_\beta \frac{\partial e_\beta}{\partial s_\alpha} + u_\gamma \frac{\partial e_\gamma}{\partial s_\alpha}$$

hence we get $\varepsilon_{\alpha\alpha} = \frac{1}{2}\left(\frac{\partial u}{\partial s_\alpha} \cdot e_\alpha + e_\alpha \cdot \frac{\partial u}{\partial s_\alpha}\right) = \frac{1}{2}\left[2\frac{\partial u_\alpha}{\partial s_\alpha} + 2u_\beta \frac{\partial e_\beta}{\partial s_\alpha} \cdot e_\alpha + 2u_\gamma \frac{\partial e_\gamma}{\partial s_\alpha} \cdot e_\alpha\right]$

$$\varepsilon_{\alpha\alpha} = \frac{\partial u_\alpha}{\partial s_\alpha} + u_\beta \frac{\partial(\ln h_\alpha)}{\partial s_\beta} + u_\gamma \frac{\partial(\ln h_\alpha)}{\partial s_\gamma}$$

and $\boxed{\varepsilon_{\alpha\alpha} = \frac{1}{h_\alpha}\left[\frac{\partial u_\alpha}{\partial \alpha} + \frac{u_\beta}{h_\beta} \frac{\partial h_\alpha}{\partial \beta} + \frac{u_\gamma}{h_\gamma} \frac{\partial h_\alpha}{\partial \gamma}\right]}$ using $\partial s_{\alpha_i} = h_{\alpha_i} \partial \alpha_i$

and for the shear strains

$\boxed{\varepsilon_{\alpha\beta} = \frac{1}{2}\left[\frac{h_\beta}{h_\alpha} \frac{\partial}{\partial \alpha}\left(\frac{u_\beta}{h_\beta}\right) + \frac{h_\alpha}{h_\beta} \frac{\partial}{\partial \beta}\left(\frac{u_\alpha}{h_\alpha}\right)\right]}$ can cyclic permute

obtained from $2\varepsilon_{\alpha\beta} = e_\alpha \cdot \varepsilon \cdot e_\beta = \frac{\partial u}{\partial s_\alpha} \cdot e_\beta + e_\alpha \cdot \frac{\partial u}{\partial s_\beta} = \frac{\partial u_\beta}{\partial s_\alpha} - \frac{\partial(\ln h_\beta)}{\partial s_\alpha} + \frac{\partial u_\alpha}{\partial s_\beta} - \frac{\partial(\ln h_\alpha)}{\partial s_\beta}$

Now we define the dilatation $\Delta = \nabla \cdot u = \varepsilon_{\alpha\alpha} + \varepsilon_{\beta\beta} + \varepsilon_{\gamma\gamma}$ using cyclic permutation to get $\varepsilon_{\beta\beta}$, $\varepsilon_{\gamma\gamma}$

$$\therefore \Delta = \frac{1}{h_\alpha}\left[\frac{\partial u_\alpha}{\partial \alpha} + \frac{u_\beta}{h_\beta} \frac{\partial h_\alpha}{\partial \beta} + \frac{u_\gamma}{h_\gamma} \frac{\partial h_\alpha}{\partial \gamma}\right] + \frac{1}{h_\beta}\left[\frac{\partial u_\beta}{\partial \beta} + \frac{u_\gamma}{h_\gamma} \frac{\partial h_\beta}{\partial \gamma} + \frac{u_\alpha}{h_\alpha} \frac{\partial h_\beta}{\partial \alpha}\right]$$
$$+ \frac{1}{h_\gamma}\left[\frac{\partial u_\gamma}{\partial \gamma} + \frac{u_\alpha}{h_\alpha} \frac{\partial h_\gamma}{\partial \alpha} + \frac{u_\beta}{h_\beta} \frac{\partial h_\gamma}{\partial \beta}\right] = \frac{1}{h_\alpha h_\beta h_\gamma}\left[\frac{\partial}{\partial \alpha}[u_\alpha h_\beta h_\gamma] + \frac{\partial}{\partial \beta}[u_\beta h_\alpha h_\gamma] + \frac{\partial}{\partial \gamma}[u_\gamma h_\alpha h_\beta\right.$$

We summarize relations (34), (37) and a third relation which is proved in the homework. The following relations are also valid for a cyclic permutation of the subscripts α, β, γ. We have

$$e_\beta \cdot \frac{\partial e_\alpha}{\partial s_\alpha} = -\frac{\partial (\ln h_\alpha)}{\partial s_\beta} \quad (38)$$

$$e_\beta \cdot \frac{\partial e_\alpha}{\partial s_\beta} = \frac{\partial (\ln h_\beta)}{\partial s_\alpha} \quad (39)$$

$$e_\beta \cdot \frac{\partial e_\alpha}{\partial s_\gamma} = 0 \quad (40)$$

In Cartesian coordinates we define the operator

$$\nabla = e_x \frac{\partial}{\partial x} + e_y \frac{\partial}{\partial y} + e_z \frac{\partial}{\partial z} \quad (41)$$

We may show (see homework) that the operator transformed to the orthogonal curvilinear coordinate system becomes

$$\nabla = e_\alpha \frac{\partial}{\partial s_\alpha} + e_\beta \frac{\partial}{\partial s_\beta} + e_\gamma \frac{\partial}{\partial s_\gamma} \quad (42)$$

We recall the definition of the small strain tensor

$$\epsilon = \frac{1}{2}\left(\nabla U + U \nabla\right) \quad (43)$$

and any component of strain in orthogonal curvilinear coordinates is

$$\epsilon_{\alpha_i \alpha_j} = e_{\alpha_i} \cdot \epsilon \cdot e_{\alpha_j} \quad (44)$$

From the form (42) for ~~a vector~~ the operator and from definition (43) we write the small strain tensor in orthogonal curvilinear coordinates as

$$\epsilon = \frac{1}{2}\left[e_\alpha \frac{\partial U}{\partial s_\alpha} + e_\beta \frac{\partial U}{\partial s_\beta} + e_\gamma \frac{\partial U}{\partial s_\gamma} + \frac{\partial U}{\partial s_\alpha} e_\alpha + \frac{\partial U}{\partial s_\beta} e_\beta + \frac{\partial U}{\partial s_\gamma} e_\gamma \right. \quad (45)$$

Note that each term in (45) is a product of two vectors (a dyad) and hence the order must be preserved. We now write one component of the ϵ tensor, $\epsilon_{\alpha\alpha}$, thus

$\epsilon_{\alpha\alpha} = \mathbf{e}_\alpha \cdot \epsilon \cdot \mathbf{e}_\alpha$ with ϵ as given by (45). Note also

$$\mathbf{U} = U_\alpha \mathbf{e}_\alpha + U_\beta \mathbf{e}_\beta + U_\gamma \mathbf{e}_\gamma$$

and

$$\begin{aligned}\epsilon_{\alpha\alpha} = \frac{1}{2}\Bigg\{\Bigg[\frac{\partial \mathbf{U}}{\partial S_\alpha} &+ \left(\frac{\partial U_\alpha}{\partial S_\alpha} + U_\beta\, \mathbf{e}_\alpha \cdot \frac{\partial \mathbf{e}_\beta}{\partial S_\alpha} + U_\gamma\, \mathbf{e}_\alpha \cdot \frac{\partial \mathbf{e}_\gamma}{\partial S_\alpha}\right)\mathbf{e}_\alpha \\ &+ \left(\frac{\partial U_\alpha}{\partial S_\beta} + U_\beta\, \mathbf{e}_\alpha \cdot \frac{\partial \mathbf{e}_\beta}{\partial S_\beta} + U_\gamma\, \mathbf{e}_\alpha \cdot \frac{\partial \mathbf{e}_\gamma}{\partial S_\beta}\right)\mathbf{e}_\beta \\ &+ \left(\frac{\partial U_\alpha}{\partial S_\gamma} + U_\beta\, \mathbf{e}_\alpha \cdot \frac{\partial \mathbf{e}_\beta}{\partial S_\gamma} + U_\gamma\, \mathbf{e}_\alpha \cdot \frac{\partial \mathbf{e}_\gamma}{\partial S_\gamma}\right)\mathbf{e}_\gamma\Bigg] \cdot \mathbf{e}_\alpha\Bigg\}\end{aligned} \tag{46}$$

Note that $$\frac{\partial \mathbf{U}}{\partial S_\alpha} = \frac{\partial U_\alpha}{\partial S_\alpha}\mathbf{e}_\alpha + \frac{\partial U_\beta}{\partial S_\alpha}\mathbf{e}_\beta + \frac{\partial U_\gamma}{\partial S_\alpha}\mathbf{e}_\gamma + U_\beta \frac{\partial \mathbf{e}_\beta}{\partial S_\alpha} + U_\gamma \frac{\partial \mathbf{e}_\gamma}{\partial S_\alpha} + U_\alpha \frac{\partial \mathbf{e}_\alpha}{\partial S_\alpha} \tag{47}$$

so that (46) becomes

$$\epsilon_{\alpha\alpha} = \frac{1}{2}\left[\frac{\partial U_\alpha}{\partial S_\alpha} + U_\beta\, \mathbf{e}_\alpha \cdot \frac{\partial \mathbf{e}_\beta}{\partial S_\alpha} + U_\gamma\, \mathbf{e}_\alpha \cdot \frac{\partial \mathbf{e}_\gamma}{\partial S_\gamma} + \frac{\partial U_\alpha}{\partial S_\alpha} + U_\beta\, \mathbf{e}_\alpha \cdot \frac{\partial \mathbf{e}_\beta}{\partial S_\alpha} + U_\gamma\, \mathbf{e}_\alpha \cdot \frac{\partial \mathbf{e}_\gamma}{\partial S_\alpha}\right]$$

We make use of (39) and its permutations to write

$$\epsilon_{\alpha\alpha} = \frac{\partial U_\alpha}{\partial S_\alpha} + U_\beta \frac{\partial (\ln h_\alpha)}{\partial S_\beta} + U_\gamma \frac{\partial (\ln h_\alpha)}{\partial S_\gamma} \tag{48}$$

Finally we write the components of normal strain as

$$\epsilon_{\alpha\alpha} = \frac{1}{h_\alpha}\left[\frac{\partial U_\alpha}{\partial \alpha} + \frac{U_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial \beta} + \frac{U_\gamma}{h_\gamma}\frac{\partial h_\alpha}{\partial \gamma}\right] \tag{49}$$

$$\epsilon_{\beta\beta} = \frac{1}{h_\beta}\left[\frac{\partial U_\beta}{\partial \beta} + \frac{U_\gamma}{h_\gamma}\frac{\partial h_\beta}{\partial \gamma} + \frac{U_\alpha}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha}\right] \tag{50}$$

$$\epsilon_{\gamma\gamma} = \frac{1}{h_\gamma}\left[\frac{\partial U_\gamma}{\partial \gamma} + \frac{U_\alpha}{h_\alpha}\frac{\partial h_\gamma}{\partial \alpha} + \frac{U_\beta}{h_\beta}\frac{\partial h_\gamma}{\partial \beta}\right] \tag{51}$$

Also, we have $$\epsilon_{\alpha\beta} = \frac{1}{2}\left[\frac{h_\beta}{h_\alpha}\frac{\partial}{\partial \alpha}\left(\frac{U_\beta}{h_\beta}\right) + \frac{h_\alpha}{h_\beta}\frac{\partial}{\partial \beta}\left(\frac{U_\alpha}{h_\alpha}\right)\right] \tag{52}$$

We write the dilatation using the components of normal strain as

$$\Delta = \nabla \cdot U = U_{k,k} = \epsilon_{\alpha\alpha} + \epsilon_{\beta\beta} + \epsilon_{\gamma\gamma}$$

or

$$\Delta = \frac{1}{h_\alpha}\frac{\partial U_\alpha}{\partial \alpha} + \frac{U_\alpha}{h_\alpha h_\beta}\frac{\partial h_\beta}{\partial \alpha} + \frac{U_\alpha}{h_\alpha h_\gamma}\frac{\partial h_\gamma}{\partial \alpha}$$
$$+ \frac{1}{h_\beta}\frac{\partial U_\beta}{\partial \beta} + \frac{U_\beta}{h_\beta h_\alpha}\frac{\partial h_\alpha}{\partial \beta} + \frac{U_\beta}{h_\beta h_\gamma}\frac{\partial h_\gamma}{\partial \beta}$$
$$+ \frac{1}{h_\gamma}\frac{\partial U_\gamma}{\partial \gamma} + \frac{U_\gamma}{h_\gamma h_\alpha}\frac{\partial h_\alpha}{\partial \gamma} + \frac{U_\gamma}{h_\gamma h_\beta}\frac{\partial h_\beta}{\partial \gamma} \quad \ldots \quad (1)$$

If we multiply the top and bottom of each term above by the proper combination of h's such that the denominator becomes $h_\alpha h_\beta h_\gamma$, we get,

$$\Delta = \frac{1}{h_\alpha h_\beta h_\gamma}\left[\frac{\partial(h_\beta h_\gamma U_\alpha)}{\partial \alpha} + \frac{\partial(h_\gamma h_\alpha U_\beta)}{\partial \beta} + \frac{\partial(h_\alpha h_\beta U_\gamma)}{\partial \gamma}\right] \quad \ldots \quad (2)$$

We have written the gradient in curvilinear coordinates as

$$\nabla = \frac{1}{h_\alpha}\frac{\partial}{\partial \alpha} e_\alpha + \frac{1}{h_\beta}\frac{\partial}{\partial \beta} e_\beta + \frac{1}{h_\gamma}\frac{\partial}{\partial \gamma} e_\gamma \quad (3)$$

and

$$\nabla^2 \phi = \nabla \cdot \nabla \phi \quad \text{by definition.}$$

Using the above definition and relation (3) we may write

$$\nabla^2 \phi = \frac{1}{h_\alpha h_\beta h_\gamma}\left[\frac{\partial}{\partial \alpha}\left(\frac{h_\beta h_\gamma}{h_\alpha}\frac{\partial \phi}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{h_\gamma h_\alpha}{h_\beta}\frac{\partial \phi}{\partial \beta}\right) + \frac{\partial}{\partial \gamma}\left(\frac{h_\alpha h_\beta}{h_\gamma}\frac{\partial \phi}{\partial \gamma}\right)\right] \quad \ldots \quad (4)$$

We write the stress equations of equilibrium as

$$\nabla \cdot \underset{\sim}{\tau} + f = \rho \ddot{U} \quad \ldots \quad (5)$$

We now write expressions for the divergence of a vector and a dyadic in orthogonal curvilinear coordinates as

$$\nabla \cdot U = \nabla \cdot (U_\alpha e_\alpha + U_\beta e_\beta + U_\gamma e_\gamma) \quad (6)$$

and

$$\nabla \cdot \tau = \nabla \cdot \Big[\sigma_{\alpha\alpha}\, e_\alpha e_\alpha + \sigma_{\alpha\beta}\, e_\alpha e_\beta + \sigma_{\alpha\gamma}\, e_\alpha e_\gamma + \sigma_{\beta\alpha}\, e_\beta e_\alpha + \sigma_{\beta\beta}\, e_\beta e_\beta + \sigma_{\beta\gamma}\, e_\beta e_\gamma + \sigma_{\gamma\alpha}\, e_\gamma e_\alpha + \sigma_{\gamma\beta}\, e_\gamma e_\beta + \sigma_{\gamma\gamma}\, e_\gamma e_\gamma \Big] \quad (7)$$

$$= \nabla \cdot \Big[e_\alpha(\sigma_{\alpha\alpha}\, e_\alpha + \sigma_{\alpha\beta}\, e_\beta + \sigma_{\alpha\gamma}\, e_\gamma) + e_\beta(\sigma_{\beta\alpha}\, e_\alpha + \sigma_{\beta\beta}\, e_\beta + \sigma_{\beta\gamma}\, e_\gamma) + e_\gamma(\sigma_{\gamma\alpha}\, e_\alpha + \sigma_{\gamma\beta}\, e_\beta + \sigma_{\gamma\gamma}\, e_\gamma) \Big] \quad (8)$$

We use the definition of ∇ as given in (3) and apply $\nabla \cdot$ to the expression (8) in [ ] to give us

$$\nabla \cdot \tau = \frac{1}{h_\alpha h_\beta h_\gamma} \Big[\frac{\partial}{\partial \alpha}\{h_\beta h_\gamma(\sigma_{\alpha\alpha}\, e_\alpha + \sigma_{\alpha\beta}\, e_\beta + \sigma_{\alpha\gamma}\, e_\gamma)\} + \frac{\partial}{\partial \beta}\{h_\gamma h_\alpha(\sigma_{\beta\alpha}\, e_\alpha + \sigma_{\beta\beta}\, e_\beta + \sigma_{\beta\gamma}\, e_\gamma)\} + \frac{\partial}{\partial \gamma}\{h_\alpha h_\beta(\sigma_{\gamma\alpha}\, e_\alpha + \sigma_{\gamma\beta}\, e_\beta + \sigma_{\gamma\gamma}\, e_\gamma)\} \Big] \quad (9)$$

We expand (9) and write it as a sum of three vectors in the α, β, and γ directions. Several terms are broken into components. For components in the α direction of two typical terms we use the relations

$$e_\alpha \cdot \frac{\partial e_\beta}{\partial \alpha} = h_\alpha\, e_\alpha \cdot \frac{\partial e_\beta}{\partial s_\alpha} = h_\alpha \left[\frac{\partial (\ln h_\alpha)}{\partial s_\beta} \right] = \frac{1}{h_\beta} \frac{\partial h_\alpha}{\partial \beta} \quad (10)$$

and

$$e_\alpha \cdot \frac{\partial e_\beta}{\partial \beta} = -\frac{h_\beta}{h_\alpha} \frac{\partial (\ln h_\beta)}{\partial \alpha} = -\frac{1}{h_\alpha} \frac{\partial h_\beta}{\partial \alpha} \quad (11)$$

The expression (9) reduces to

$$\nabla \cdot \tau = \frac{1}{h_\alpha h_\beta h_\gamma} \Big\{ e_\alpha \Big[\frac{\partial (h_\beta h_\gamma \sigma_{\alpha\alpha})}{\partial \alpha} + \frac{\partial}{\partial \beta}(h_\gamma h_\alpha \sigma_{\beta\alpha}) + \frac{\partial}{\partial \gamma}(h_\alpha h_\beta \sigma_{\gamma\alpha}) + h_\gamma \sigma_{\alpha\beta} \frac{\partial h_\alpha}{\partial \beta} + h_\beta \sigma_{\alpha\gamma} \frac{\partial h_\alpha}{\partial \gamma} - h_\gamma \sigma_{\beta\beta} \frac{\partial h_\beta}{\partial \alpha} - h_\beta \sigma_{\gamma\gamma} \frac{\partial h_\gamma}{\partial \alpha} \Big]$$
$$+ e_\beta \Big[\frac{\partial (h_\gamma h_\alpha \sigma_{\beta\beta})}{\partial \beta} + \frac{\partial}{\partial \gamma}(h_\alpha h_\beta \sigma_{\gamma\beta}) + \frac{\partial}{\partial \alpha}(h_\beta h_\gamma \sigma_{\alpha\beta}) + h_\alpha \sigma_{\beta\gamma} \frac{\partial h_\beta}{\partial \gamma} + h_\gamma \sigma_{\beta\alpha} \frac{\partial h_\beta}{\partial \alpha} - h_\alpha \sigma_{\gamma\gamma} \frac{\partial h_\gamma}{\partial \beta} - h_\gamma \sigma_{\alpha\alpha} \frac{\partial h_\alpha}{\partial \beta} \Big]$$
$$+ e_\gamma \Big[\frac{\partial (h_\alpha h_\beta \sigma_{\gamma\gamma})}{\partial \gamma} + \frac{\partial}{\partial \alpha}(h_\beta h_\gamma \sigma_{\alpha\gamma}) + \frac{\partial}{\partial \beta}(h_\gamma h_\alpha \sigma_{\beta\gamma}) + h_\beta \sigma_{\gamma\alpha} \frac{\partial h_\gamma}{\partial \alpha} + h_\alpha \sigma_{\gamma\beta} \frac{\partial h_\gamma}{\partial \beta} - h_\beta \sigma_{\alpha\alpha} \frac{\partial h_\alpha}{\partial \gamma} - h_\alpha \sigma_{\beta\beta} \frac{\partial h_\beta}{\partial \gamma} \Big] \Big\} \quad (1$$

$$\nabla \cdot \sigma = \nabla \cdot \left[\sigma_{\alpha\alpha} \mathbf{e}_\alpha \mathbf{e}_\alpha + \sigma_{\alpha\beta} \mathbf{e}_\alpha \mathbf{e}_\beta + \ldots\ldots \right]$$

since order α, β matters we cannot say that $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$

We will get notes on this part later.

In cylindrical coords.

$h_r = 1, \quad h_\theta = r, \quad h_z = 1$

Grad $\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}$; if $\mathbb{H} = H_r \mathbf{e}_r + H_\theta \mathbf{e}_\theta + H_z \mathbf{e}_z$

$$\text{Div}(\mathbb{H}) = \nabla \cdot \mathbb{H} = \frac{1}{r}\frac{\partial (r H_r)}{\partial r} + \frac{1}{r}\frac{\partial H_\theta}{\partial \theta} + \frac{\partial H_z}{\partial z}$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$

<u>Equil</u>

$$\nabla \cdot \sigma + \mathbb{f} = \rho \frac{\partial^2 \mathbb{u}}{\partial t^2}$$ This is independent of coordinate system

in curvilinear coord.

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = \rho \frac{\partial^2 u_r}{\partial t^2}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2}{r}\sigma_{r\theta} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + f_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = \rho \frac{\partial^2 u_z}{\partial t^2}$$

12/8/78

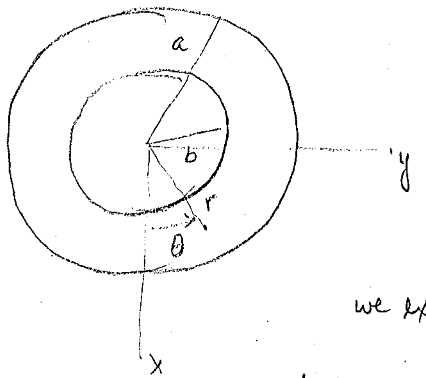
Cylindrical Coordinates Continued

Components of strain

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) \qquad \varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right)$$

Torsion of a Hollow circular cylinder



Compatibility demands that ϕ satisfies

$$\nabla_1^2 \phi = -2\mu\alpha$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -2\mu\alpha$$

we expect θ symmetry $\therefore$ $\nabla_1^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -2\mu\alpha$

$$\therefore \quad \frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -2\mu\alpha r \quad \text{or} \quad r\frac{d\phi}{dr} = -\mu\alpha r^2 + C_1$$

since $\phi \neq \phi(\theta, r)$ but only $\phi(r)$

now $$\boxed{\phi = -\mu\alpha\frac{r^2}{2} + C_1 \ln r + C_2}$$

for a multiply connected region

$$\oint_{C_i} \frac{d\phi}{dn}\,ds = 2\mu\alpha A_i$$

$$\left.\frac{d\phi}{dn}\right|_{C_i} = -\frac{d\phi}{dr} \quad \text{and} \quad ds = -b\,d\theta$$

$$\oint_0^{-2\pi} -\left[-\mu\alpha r + \frac{C_1}{r}\right]_{r=b}(-b\,d\theta) = 2\mu\pi\alpha b^2 - 2\pi C_1 = 2\mu\alpha\overset{2\mu\alpha A_i}{\underline{\pi b^2}} \Rightarrow C_1 = 0$$

now $$\phi = -\frac{1}{2}\mu\alpha r^2 + C_2$$

on $r = a$ $\quad K_0 = -\frac{1}{2}\mu\alpha a^2 + C_2$ & $r = b$ $\quad K_1 = -\frac{1}{2}\mu\alpha b^2 + C_2$

now $K_0 - K_1 = -\frac{1}{2}\mu\alpha(a^2 - b^2)$ or $K_0 = -\frac{1}{2}\mu\alpha(a^2 - b^2) + K_1$

hence $$\boxed{C_2 = K_0 + \frac{1}{2}\mu\alpha a^2 = K_1 + \frac{1}{2}\mu\alpha b^2}$$

hence $$\boxed{\phi = \frac{1}{2}\mu\alpha(a^2 - r^2) + K_0 \quad \text{or} \quad = \frac{1}{2}\mu\alpha(b^2 - r^2) + K_1}$$

Now looking at the stresses to get the Torque.

$$\sigma_{zr} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \rightarrow \sigma_{zr} = 0$$

$$\sigma_{z\theta} = -\frac{\partial\phi}{\partial r} = \mu\alpha r$$

$$T = 2\left[\iint_{net} \phi\, dA - k_0 A_0 + k_1 A_1\right]$$

$$\frac{T}{2} = \int_0^{2\pi}\int_b^a \left[\frac{1}{2}\mu\alpha(a^2-r^2) + k_0\right] r\,dr\,d\theta - k_0\pi a^2 + k_1\pi b^2$$

$$\rightarrow k_0(\pi a^2 - \pi b^2) + \quad = (k_1-k_0)\pi b^2 = \frac{\pi b^2}{2}\mu\alpha(a^2-b^2)$$

$$\boxed{T = \frac{\pi}{2}\mu\alpha(a^4-b^4)}$$

Alternately $T = \iint \sigma_{z\theta}\, r\, dA = \int_0^{2\pi}\int_a^b \mu\alpha r^2 (r\,dr\,d\theta) = \mu\alpha \cdot 2\pi \int_a^b r^3 dr =$

$T = \mu\alpha J$

J = polar moment of inertia

$$\boxed{\sigma_{max} = \mu\alpha a}$$

Now looking at displacements we obtain

$$\left.\begin{aligned} u_x &= -\alpha y z \\ u_y &= \alpha x z \end{aligned}\right\} \rightarrow \begin{aligned} u_r &= 0 \\ u_\theta &= \alpha r z \end{aligned}$$

without derivation we have

$$\frac{\partial u_z}{\partial s_\alpha} = \frac{1}{\mu}\frac{\partial\phi}{\partial s_\beta} + \frac{\alpha}{2}\frac{\partial r^2}{\partial s_\beta}$$

$$\frac{\partial u_z}{\partial s_\beta} = \frac{-1}{\mu}\frac{\partial\phi}{\partial s_\alpha} - \frac{\alpha}{2}\frac{\partial r^2}{\partial s_\alpha}$$

let $s_\alpha = r$ let $s_\theta = r\theta$

$$\therefore \frac{\partial u_z}{\partial r} = \frac{1}{\mu}\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \frac{\alpha}{2}\frac{1}{r}\frac{\partial r^2}{\partial\theta} = 0 \quad \text{because } \phi \neq \phi(\theta) \text{ and } r \neq r(\theta)$$

$$\therefore u_z = u_z(\theta)$$

$$\frac{\partial u_z}{r\,\partial\theta} = \frac{-1}{\mu}\frac{\partial\phi}{\partial r} - \frac{\alpha}{2}\frac{\partial r^2}{\partial r} = \alpha r - \alpha r = 0$$

$\therefore u_z \neq u_z(\theta)$ hence u_z = const we can take $u_z = 0$ since its only a rigid body displacement

B

Question	Score
1	20/20
2	17/20
3	20
4	20
5	16
6	10
7	
8	
9	
10	
Total	103/120 = 86%

STANFORD UNIVERSITY

OFFICIAL EXAMINATION BOOK

24 PAGE RULED

Name of Student CÉSAR LEVY

Date of Examination 11 Dec 78

Subject ME238A

HONORABLE CONDUCT
in academic work is the spirit of conduct in this University.

In recognition of and in the spirit of the Honor Code, I certify that I will neither receive nor give unpermitted aid on this examination and that I will report, to the best of my ability, all Honor Code violations observed by me.

(signed) Cesar Levy
Name

SUGGESTIONS FOR CONDUCT

1. Occupy alternate seats where possible.
2. When in doubt as to the meaning of a question, consult the instructor, who will be found in his or her office.

Turn in at Prof. Herrmann's office

B

[illegible]

[illegible]

[illegible]

DIVISION OF APPLIED MECHANICS

DEPARTMENT OF MECHANICAL ENGINEERING

ME 238A Theory of Elasticity   Autumn 1978

Final Examination

1. At a point the components of stress are:

OK $$\sigma_{zx} = \sigma_{yz} = 0 \; ; \; \sigma_{xx} = 2\text{Pa} \; ; \; \sigma_{yy} = 4\text{Pa} \; ; \; \sigma_{zz} = -2\text{Pa} \; ; \; \sigma_{xy} = 2\sqrt{2}\text{Pa}$$

   a) Determine the principal components of stress.

   b) Determine the unit vector in the direction of the largest principal components of stress.

2. Consider the following stress field in a simply-connected medium with no body forces present:

$$\sigma_{xx} = a(x^2 - y^2) \; ; \; \sigma_{yy} = bx^2 + cy^2 \; ; \; \sigma_{zz} = d(x^2 + y^2) \; ;$$

$$\sigma_{xy} = exy \; ; \; \sigma_{xz} = \sigma_{yz} = 0$$ use Equil + Beltrami Michell

Determine conditions upon the constants a, b, c, d, and e so that this stress field represents an acceptable elasticity solution.

3. The following state of strain exists in a simply-connected body:

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = T(x,y,z) \quad ; \quad \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0$$ use compat

Show that if this strain field is to correspond to a single-valued displacement field, $T(x,y,z)$ is at most a linear function of x,y and z.

4. Consider a bar with constant cross-section A, length L, mass density ρ, Young's modulus E, and Poisson's ratio ν. It is hung in the vertical position and the upper end is appropriately fixed, while the lower end is free. The bar is subjected to a weight W hung a distance a from the upper end, as well as gravity. Neglect local effects near the point where the weight W is attached. Solve this elasticity problem as follows:

   a) Assume a state of stress.

   b) Verify the satisfaction of equilibrium equations.

   c) Verify the satisfaction of the boundary conditions.

   d) Verify the satisfaction of the compatibility equations.

   e) Calculate the components of strain.

   f) Find the components of displacement by integrating the components of strain.

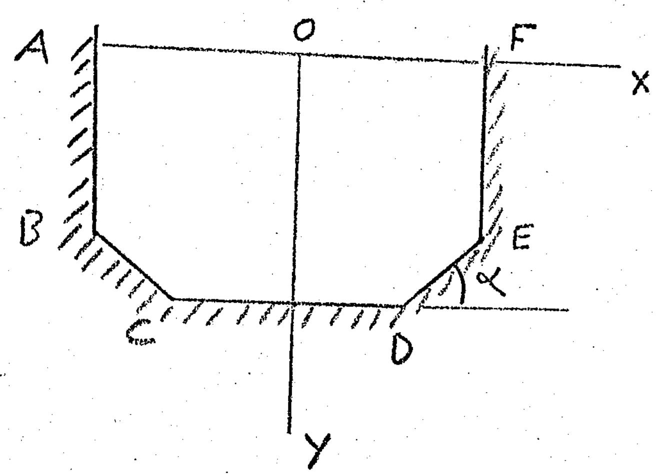
5. An elastic rod (Young's modulus E, Poisson's ratio ν) of length L and constant cross-sectional area A is placed in a tightly fitting, absolutely rigid tube, and it is compressed by an evenly distributed force F on the ends.

a) Determine the shortening ΔL of the rod.

b) Assuming $0 \leq \nu \leq \frac{1}{2}$, what is the range of ΔL ?

c) Determine the lengthening of the rod if the force F is tensile.

6. An elastic body is placed in a rigid container with lubricated walls A, B, C, D, E, F and is subjected to some body forces (not specified in detail). Write down the boundary conditions along the portion DE in terms of components of displacement referred to the coordinate system Oxy shown. Consider quantities in the xy-plane only.



$7/20$

1. Given

$$\begin{pmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \sigma$$

a. Find the principal stresses

To find principal stresses form $\sigma - \lambda \mathbb{I}$ and set $\det(\sigma - \lambda \mathbb{I}) = 0$. This gives λ_i's the EIGENVALUES.

Thus

$$\det \begin{pmatrix} 2-\lambda & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 4-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{pmatrix} = -(2-\lambda)(4-\lambda)(2+\lambda) + 8(2+\lambda) = 0$$

$$= (2+\lambda)\left[-(8-6\lambda+\lambda^2)+8\right] = (2+\lambda)\left[\lambda(6-\lambda)\right] = 0$$

hence $\underline{\lambda_1 = 6 \text{ Pa}}$ $\underline{\lambda_2 = 0 \text{ Pa}}$ $\underline{\lambda_3 = -2 \text{ Pa}}$ ✓

Hence the principal stress tensor is

$$\sigma_p = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

b. Determine the unit vector in the direction of the largest principal stress

To do this let $\lambda = 6$ and substitute into $(\sigma - \lambda_1 \mathbb{I})\,\underline{v}^{(1)} = 0$ and $n^{(1)} = \dfrac{\underline{v}^{(1)}}{\|\underline{v}^{(1)}\|}$

Therefore

$$\begin{pmatrix} -4 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & -2 & 0 \\ 0 & 0 & -8 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \underline{0}$$

This can be reduced to

$$\begin{pmatrix} \sqrt{2} & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \underline{0} \quad \text{by linear algebra}$$

This shows that $V_z \equiv 0$ and $\sqrt{2}\, V_x = V_y$ hence we assume V_x to be the variable hence let it $= 1$ $\therefore$ $V_y = \sqrt{2}$

thus

$$\underset{\sim}{V}^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 0 \end{pmatrix} \quad \text{now} \quad \|\underset{\sim}{V}^{(1)}\| = \sqrt{1 + (\sqrt{2})^2} = \sqrt{3}$$

let's take $\|\underset{\sim}{V}\| = \sqrt{3}$ hence

$$\mathbb{n}^{(1)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \sqrt{2} \\ 0 \end{pmatrix}$$

17/20

2. Given:

$$\sigma = \begin{pmatrix} a(x^2-y^2) & exy & 0 \\ exy & bx^2+cy^2 & 0 \\ 0 & 0 & d(x^2+y^2) \end{pmatrix}$$ in a simply connected region with no body forces

find conditions on a, b, c, d and e so that this stress field is an acceptable solution.

Use the stress compatability equations of Beltrami (simply connected, no body forces and I will assume no accelerations)

$$\nabla_1^2 \sigma_{ij} + \frac{1}{1+\nu}\sigma_{kk,ij} = 0$$

$\sigma_{kk} = (a+b+d)x^2 + (c+d-a)y^2$ hence

$$\nabla_1^2\sigma_{xx} + \frac{1}{1+\nu}\sigma_{kk,xx} = 2a-2a+\frac{1}{1+\nu}2(a+b+d) = 0 \Rightarrow a+b+d=0 \quad (1)$$

$$\nabla_1^2\sigma_{xy} + \frac{1}{1+\nu}\sigma_{kk,xy} = 0 + \frac{1}{1+\nu}\cdot 0 \equiv 0 \quad (2)$$

$$\nabla_1^2\sigma_{xz} + \frac{1}{1+\nu}\sigma_{kk,xz} = 0 + \frac{1}{1+\nu}\cdot 0 \equiv 0 \quad (3)$$

$$\nabla_1^2\sigma_{yy} + \frac{1}{1+\nu}\sigma_{kk,yy} = 2b+2c+\frac{1}{1+\nu}2(c+d-a) = 0 \quad (4)$$

$$\nabla_1^2\sigma_{yz} + \frac{1}{1+\nu}\sigma_{kk,yz} = 0 + \frac{1}{1+\nu}\cdot 0 \equiv 0 \quad (5)$$

$$\nabla_1^2\sigma_{zz} + \frac{1}{1+\nu}\sigma_{kk,zz} = 2d+2d+\frac{1}{1+\nu}\cdot 0 = 4d = 0 \quad (6)$$

in doing the above we assume $\frac{1}{1+\nu} \neq 0$

from (6) $\underline{d=0}$ and from (1) $\underline{a=-b}$. from (4) $(2b+2c)\left(1+\frac{1}{1+\nu}\right)=0$;

hence $b=-c$ or $\underline{c=a}$

not arbitrary see

$$\boxed{\therefore\ a, e \text{ are arbitrary} \quad c=a=-b \quad \text{and} \quad d=0}$$

hence 1 solution is to take $a=1$ $b=-1$ $c=1$ $d=0$ $e=0$

you need to check equilibrium eqs. −3

$\sigma_{xx,x} + \sigma_{yx,y} + \sigma_{zx,z} = 2ax + ex + 0 = 0$ $\quad\therefore\ 2a+e=0 \quad \forall x,y,z$

$\sigma_{xy,x} + \sigma_{yy,y} + \sigma_{zy,z} = ey + 2cy = 0$ $\quad\therefore\ e+2c=0 \quad \forall y,x,z$

$\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} = 0+0+0=0$ $\quad\therefore\ e=-2a$ and $e=-2c$

20

3. Given

$$\boldsymbol{\varepsilon} = \begin{pmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{pmatrix}$$

where $T = T(x,y,z)$. If this is to correspond to a single valued displacement field, T is a linear function of x, y, z in a simply connected body

Proof

$\boldsymbol{\varepsilon}$ must satisfy the compatability condition $\nabla \times \boldsymbol{\varepsilon} \times \nabla = 0$

hence

$$\psi_{xx} = 2\varepsilon_{yz,yz} - \varepsilon_{yy,zz} - \varepsilon_{zz,yy} = 0 - T_{,yy} - T_{,zz} = 0 \quad \text{or} \quad \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad (1)$$

$$\psi_{yy} = 2\varepsilon_{xz,xz} - \varepsilon_{xx,zz} - \varepsilon_{zz,xx} = 0 - T_{,zz} - T_{,xx} = 0 \quad \text{or} \quad \frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial x^2} = 0 \qquad (2)$$

$$\psi_{zz} = 2\varepsilon_{xy,xy} - \varepsilon_{xx,yy} - \varepsilon_{yy,xx} = 0 - T_{,yy} - T_{,xx} = 0 \quad \text{or} \quad \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} = 0 \qquad (3)$$

$$\psi_{yz} = \varepsilon_{xx,yz} - \frac{\partial}{\partial x}\left(-\varepsilon_{yz,x} + \varepsilon_{zx,y} + \varepsilon_{xy,z}\right) = \varepsilon_{xx,yz} = T_{,yz} \quad \text{or} \quad \frac{\partial^2 T}{\partial y \partial z} = 0 \qquad (4)$$

$$\psi_{zx} = \varepsilon_{yy,zx} - \frac{\partial}{\partial y}\left(-\varepsilon_{zx,y} + \varepsilon_{xy,z} + \varepsilon_{yz,x}\right) = \varepsilon_{yy,zx} = T_{,zx} \quad \text{or} \quad \frac{\partial^2 T}{\partial z \partial x} = 0 \qquad (5)$$

$$\psi_{xy} = \varepsilon_{zz,xy} - \frac{\partial}{\partial z}\left(-\varepsilon_{xy,z} + \varepsilon_{yz,x} + \varepsilon_{zx,y}\right) = \varepsilon_{zz,xy} = T_{,xy} \quad \text{or} \quad \frac{\partial^2 T}{\partial x \partial y} = 0 \qquad (6)$$

From (1) we get that $\frac{\partial^2 T}{\partial y^2} = -\frac{\partial^2 T}{\partial z^2}$ and from (2) That $-\frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial x^2} \Rightarrow \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial x^2}$

combining the above result and (3) gives that

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial y^2} = 0 \text{ and } \Rightarrow \frac{\partial^2 T}{\partial z^2} = 0 \qquad (7a,b,c)$$

$$\text{From } \frac{\partial^2 T}{\partial x^2} = 0 \Rightarrow \frac{\partial T}{\partial x} = f(y,z) \text{ and } T = x f(y,z) + g(y,z) \qquad (8)$$

Now put (8) into $\frac{\partial^2 T}{\partial y^2} = 0$ gives that $x \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 g}{\partial y^2} = 0$. Since this must be

... $\partial^2 f$ ... $= 0$ and $\partial^2 g = 0$

$\therefore$ from $\frac{\partial^2 f(y,z)}{\partial y^2} = 0$ we get $f(y,z) = y\,h(z) + l(z)$ (9)

from $\frac{\partial^2 g(y,z)}{\partial y^2} = 0$ we get $g(y,z) = y\,p(z) + q(z)$ (10)

Now using $\frac{\partial^2 T}{\partial z^2} = 0$ gives that $xy\,h''(z) + x\,l''(z) + y\,p''(z) + q''(z) = 0$. Since this must be true $\forall x$ and $y \Rightarrow h''(z) = 0$, $l''(z) = 0$, $p''(z) = 0$, $q''(z) = 0$

this leads to

$h(z) = a_1 z + b_1$; $l(z) = a_2 z + b_2$; $p(z) = a_3 z + b_3$; $q(z) = a_4 z + b_4$ (11a-d)

Summary : so far

$T(x,y,z) = xy\,h(z) + x\,l(z) + y\,p(z) + q(z)$ (12)

Now applying (4) to (12) gives that $\frac{\partial^2 T}{\partial y \partial z} = x a_1 + a_3 = 0$. Since this must be true $\forall x \Rightarrow a_1 = 0,\ a_3 = 0$ (13a, b)

Now applying (5) to (12) incorporating (13a, b) gives that

$$\frac{\partial^2 T}{\partial z \partial x} = a_2 = 0 \qquad (14)$$

Now applying (6) to (12) incorporating (13a, b) and (14) gives that

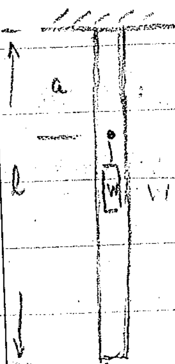
$$\frac{\partial^2 T}{\partial x \partial y} = b_1 = 0 \qquad (15)$$

hence we obtain the final result that

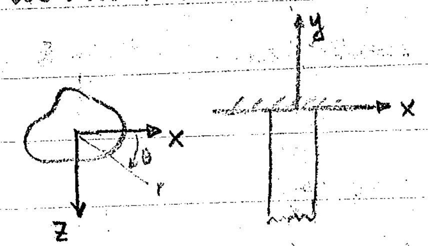
$T(x,y,z) = b_2 x + b_3 y + a_4 z + b_4$ which is a linear fn of x, y, z

QED!

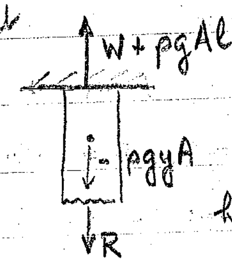
4. Consider a bar with cross section A, length L, mass density ρ, Young's Modulus E, and Poisson's ratio ν.



We will take the coordinate axis and we look at two cases, stresses above the weight and below the weight



From the free body diagram we get that the reaction at the built in end is $W + \rho g A l$. Hence for the section above the weight

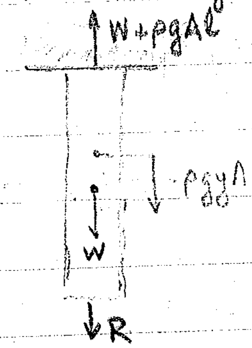


(Note: $f_y = -\rho g$, $f_x = 0$, $f_z = 0$)

hence $R = W + \rho g A l + \rho g y A = W + \rho g A(l+y)$

and $\sigma_{yy} = \frac{W}{A} + \rho g(l+y)$ ✓ $-a \le y \le 0$

also the section below the weight



Note: $f_y = -\rho g$, $f_x = 0$, $f_z = 0$

hence $R = W + \rho g A l - \cancel{W} + \rho g y A = \rho g A(l+y)$

and $\sigma_{yy} = \rho g(l+y)$ ✓ $-l \le y \le -a$

a) In defining the deformation caused by this

$$\sigma_{yy} = \frac{W}{A} + \rho g(l+y) \qquad -a \le y \le 0$$

$$\sigma_{yy} = \rho g(l+y) \qquad -l \le y < -a$$

we are assuming that each cross section is in uniform tension due to reaction force R and this leads to

$$\sigma_{zz} = \sigma_{xx} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \quad ✓ \ \forall x,y,z \text{ in the body}$$

Thus we notice that $\sigma_{yy\,max}$ is at $y = 0$ and $\sigma_{yy\,min}$ is at $y = -l$.

b) To see if it satisfies the equilib equations

$\sigma_{xx,x} + \sigma_{yx,y} + \sigma_{zx,z} + f_x \equiv 0$ identically

$\sigma_{xy,x} + \sigma_{yy,y} + \sigma_{zy,z} + f_y = \rho g - \rho g = 0$ since $\sigma_{yy,y} = \rho g$ for both sections

the weight produces a constant stress which causes a stress jump to occur at $y = -a$.

$\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} + f_z \equiv 0$

Hence we satisfy equilib

why do you complicate?

c) To satisfy the boundary conditions

- $\mathbf{n} = \cos\theta\, \mathbf{e}_x + \sin\theta\, \mathbf{e}_z$ on the surface. Hence $\mathbf{n}\cdot\boldsymbol{\sigma} = \cos\theta(\sigma_{xx}\mathbf{e}_x + \sigma_{xy}\mathbf{e}_y + \sigma_{xz}\mathbf{e}_z) + \sin\theta(\sigma_{zx}\mathbf{e}_x + \sigma_{zy}\mathbf{e}_y + \sigma_{zz}\mathbf{e}_z) = 0$ since $\sigma_{zx}, \sigma_{xy}, \sigma_{xx}, \sigma_{zy}, \sigma_{zz} = 0$. Thus we satisfy the boundary condition of no tractions on the surface.
- $\mathbf{n} = -\mathbf{e}_y$ on the bottom. Hence $\mathbf{n}\cdot\boldsymbol{\sigma} = -\sigma_{yx}\mathbf{e}_x - \sigma_{yy}\mathbf{e}_y - \sigma_{yz}\mathbf{e}_z = 0$. Thus we satisfy the no traction b.c. on the end. Thus we satisfy the bc

d) To satisfy compatability the only equation that will be affected from the Beltrami - Michell equations is

$$\nabla_1^2 \sigma_{yy} + \frac{1}{1+\nu}\Theta_{,yy} = 0$$

; The others are satisfied identically. We note that since σ_{yy} is linear in y the $\nabla^2(\text{linear fn}) = 0$ and $\Theta_{,yy} = 0$; hence we satisfy this equation also. We note that since $f_x = 0$, $f_y = -\rho g = \text{const}$ and $f_z = 0$, the differentials of the body forces that would appear on the right side of all the Beltrami - Michell equations give 0 right hand sides. Hence we satisfy the compatibility equations

e) Now

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \implies \varepsilon_{xx} = -\frac{\nu}{E}\sigma_{yy} = \varepsilon_{zz} \text{ and } \varepsilon_{yy} = \sigma_{yy}/E \text{ only}$$

thus

for $-a \le y \le 0$ $\quad \varepsilon_{xx} = \varepsilon_{zz} = -\frac{\nu}{E}\left[\frac{W}{A} + \rho g(l+y)\right] \qquad \varepsilon_{yy} = \frac{W}{AE} + \frac{\rho g}{E}(l+y)$

for $-l \le y < -a$ $\quad \varepsilon_{xx} = \varepsilon_{zz} = -\frac{\nu}{E}\rho g(l+y)$ and $\varepsilon_{yy} = \frac{\rho g}{E}(l+y)$

f) Find the components of displacement

for $-a \le y \le 0$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -\frac{\nu}{E}\left[\frac{W}{A} + \rho g(l+y)\right] \Rightarrow u_x = -\frac{\nu x}{E}\left[\frac{W}{A} + \rho g(l+y)\right] + \tilde{f}_1(y,z)$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = \frac{1}{E}\left[\frac{W}{A} + \rho g(l+y)\right] \Rightarrow u_y = \frac{y}{E}\left[\frac{W}{A} + \rho g(l+\frac{y}{2})\right] + \tilde{f}_2(x,z)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = -\frac{\nu}{E}\left[\frac{W}{A} + \rho g(l+y)\right] \Rightarrow u_z = -\frac{\nu z}{E}\left[\frac{W}{A} + \rho g(l+y)\right] + \tilde{f}_3(x,y)$$

We must use the fact that $\varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz} = 0$ to get $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$

for $-l \le y \le -a$

$$\varepsilon_{xx} = -\frac{\nu}{E}\rho g(l+y) = \frac{\partial u_x}{\partial x} \Rightarrow u_x = -\frac{\nu x}{E}\rho g(l+y) + \tilde{f}_4(y,z)$$

$$\varepsilon_{yy} = \frac{\rho g}{E}(l+y) = \frac{\partial u_y}{\partial y} \Rightarrow u_y = \frac{\rho g}{E}(ly + y^2/2) + \tilde{f}_5(x,z)$$

$$\varepsilon_{zz} = -\frac{\nu}{E}\rho g(l+y) = \frac{\partial u_z}{\partial z} \Rightarrow u_z = -\frac{\nu z}{E}\rho g(l+y) + \tilde{f}_6(x,y)$$

we must use that fact that $\varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz} = 0$ to get f_4, f_5, f_6 <u>AND</u> the fact that the displacements are continuous at the pt $y=-a$ $\forall x,z$ to relate $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ to $\tilde{f}_4, \tilde{f}_5, \tilde{f}_6$.

with conditions @ $(0,0,0)$ that $u_x, u_y, u_z = 0$ and no rotation about the y axis we get that

$$\tilde{f}_1(y,z) = \tilde{f}_2(x,z) = \tilde{f}_3(x,y) \equiv 0$$

hence for $-a \le y \le 0$

$$u_x = -\frac{\nu}{E}\left[\frac{W}{A} + \rho g(l+y)\right]x \checkmark$$

$$u_y = \rho g\frac{Wy}{EA} + \frac{\rho g}{E}\left[(ly + y^2/2) + \frac{\nu}{2}(x^2+z^2)\right]$$

$$u_z = -\frac{\nu}{E}\left[\frac{W}{A} + \rho g(l+y)\right]z \checkmark$$

before the application of the above boundary conditions

$$\tilde{f}_1(y,z) = c_{13}z + c_{14} - yc_9$$

$$\tilde{f}_2(x,z) = \frac{\nu\rho g}{2E}(x^2+z^2) + c_9x + c_{11}z + c_{16}$$

$$\tilde{f}_3(x,y) = -c_{13}x + c_{15} - yc_{11}$$

for $-l \leq y \leq -a$

$$\tilde{f}_4(y,z) = c_5 z + c_6 - c_1 y$$
$$\tilde{f}_5(x,z) = \frac{\nu \rho g}{2E}(x^2+z^2) + c_1 x + c_3 z + c_8$$
$$\tilde{f}_6(x,y) = -c_5 x + c_7 - y c_3$$

if we assume continuous displacements at $y = -a$ then

$$-\frac{\nu x}{E}\frac{W}{A} = c_5 z + c_6 + c_1 a$$
$$-\frac{aW}{EA} = \frac{\nu \rho g}{2E}(x^2+z^2) + c_1 x + c_3 z + c_8$$
$$-\frac{\nu z}{E}\frac{W}{A} = -c_5 x + c_7 + a c_3$$

if $\underline{x=0, z=0}$ $\quad -c_1 a = c_6 \quad$ or $\quad c_1 = -\frac{c_6}{a}$

$$c_8 = -\frac{aW}{EA}$$
$$c_7 = -a c_3 \quad \text{or} \quad c_3 = -\frac{c_7}{a}$$

hence
$$\tilde{f}_4(y,z) = c_5 z + \frac{c_6}{a}(a+y)$$
$$\tilde{f}_5(x,z) = \frac{\nu \rho g}{2E}(x^2+z^2) - \frac{1}{a}(c_6 x + c_7 z) - \frac{aW}{EA}$$

$$\tilde{f}_6(x,y) = -c_5 x + \frac{c_7}{a}(a+y)$$

thus we satisfy the displacement field at $(0,-a,0)$. If we want to define the displacement throughout at $(x,-a,0)$ we have to define a relationship between x and W. If not, we get a discontinuity of displacement at $y=-a$.

enclosed in the sheets are the scratch papers that give the step by step evaluation of $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4, \tilde{f}_5, \tilde{f}_6$

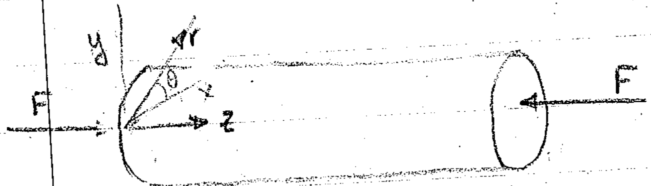
16

5. An elastic rod of length L and cross-section A is placed in a tight fitting, absolutely rigid tube, and it is compressed by an evenly distributed load F on the ends

a. Determine the shortening ΔL of the rod

b. Assuming $0 \le \nu \le \frac{1}{2}$, what is the range of ΔL?

c. Determine the lengthening of the rod if the force F is tensile.



The boundary conditions on the rod are that u_r at $r = r_0 = 0$

The problem is symmetric with respect to θ $\therefore$ all derivatives (written in polar coordinates) with respect to $\theta = 0$.

We will use superposition

1. We will assume problem #1 has only σ_{zz} acting on it and check the values of $\varepsilon_{rr}, \varepsilon_{\theta\theta}$. We will then look at problem #2 whose loads are such to cause the cancellation of strains caused by problem #1

Problem #1 $\therefore$ $\varepsilon_{zz} = \frac{\sigma_{zz}}{E}$ $\Rightarrow$ $\varepsilon_{rr} = -\frac{\nu\sigma_{zz}}{E}$ $\quad \varepsilon_{\theta\theta} = -\frac{\nu\sigma_{zz}}{E}$ $\quad \sigma_{r\theta} = \sigma_{rz} = \sigma_{\theta z} = 0$, $\sigma_{\theta\theta}, \sigma_{rr} = 0$

now in problem 2 we define $\sigma_{rr}, \sigma_{\theta\theta}$ $\ni$.

$$\left.\begin{aligned} \varepsilon_{rr} &= \frac{\nu\sigma_{zz}}{E} = \frac{\sigma_{rr}}{E} - \frac{\nu\sigma_{\theta\theta}}{E} \\ \varepsilon_{\theta\theta} &= \frac{\nu\sigma_{zz}}{E} = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu\sigma_{rr}}{E} \end{aligned}\right\} \quad \begin{aligned} \sigma_{rr} &= \sigma_{zz}\frac{(1+\nu)\nu}{1-\nu^2} \\ \sigma_{\theta\theta} &= \sigma_{zz}\frac{(1+\nu)\nu}{1-\nu^2} \end{aligned}$$

hence $\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{rr} + \sigma_{\theta\theta}) = \frac{(1-2\nu)(1+\nu)}{1-\nu}\frac{\sigma_{zz}}{E}$

but $\varepsilon_{zz} = \frac{\Delta L}{L}$ $\quad \sigma_{zz} = -\frac{F}{A}$ $\quad \therefore$

$$\Delta L = -\frac{FL}{AE}\frac{(1-2\nu)(1+\nu)}{1-\nu}$$

hence for $\nu = 0$ $\quad \Delta L = -\frac{FL}{AE}$ $\quad$ and $\nu = +\frac{1}{2}$ $\quad \Delta L = 0$ $\quad \therefore -\frac{FL}{AE} \le \Delta L \le 0$

This solution allows for no strains in the r and θ directions. Note that this solution for $\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}$ satisfies compatability & boundary conditions and equil

in the part c

if F is tensile then $\Delta L = \frac{FL}{AE}\frac{(1-2\nu)(1+\nu)}{1-\nu}$ since the development was for any general σ_{zz} regardless of sign × For tensile F

$\varepsilon_{rr}, \varepsilon_{\theta\theta} \neq 0$

If the body were allowed to have σ_θ stresses then $\varepsilon_{zz} = \frac{(1-\nu^2)\sigma_{zz}}{E}$, $\varepsilon_{rr} = 0$ and $\varepsilon_{\theta\theta} = -\frac{\nu}{E}(\sigma_{zz} + \nu\sigma_{zz}) = -\nu\frac{(1+\nu)}{E}\sigma_{zz}$ and for $\sigma_{zz} = -F/A$

and $\Delta L = \varepsilon_{zz} \cdot L = -(1-\nu^2)\frac{FL}{AE}$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} \qquad \varepsilon_{rr} = -\frac{\nu\sigma_{zz}}{E} \quad \text{and} \quad \varepsilon_{\theta\theta} = -\frac{\nu\sigma_{zz}}{E}$$

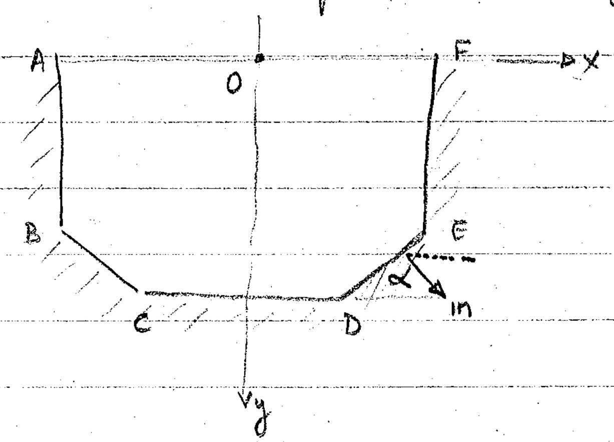
$$\therefore \quad \frac{\Delta L}{L} = \frac{F}{AE} \quad \text{or} \quad \Delta L = \frac{FL}{AE}$$

$$\cancel{\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{rr} + \sigma_{\theta\theta})}$$

$$\cancel{\frac{\Delta l}{l} = \frac{\sigma_{zz}}{E}(1-\nu^2) = \frac{F}{AE}(1-\nu^2)}$$

$$\cancel{\therefore \quad \Delta l = \frac{Fl}{AE}(1-\nu^2)}$$

6. An elastic body is placed in a rigid container with lubricated walls and is subjected to some body forces. Write down the bdry conditions along DE in terms of the components of displacements



This will be considered a plane strain problem since each cross section will be same as others $\therefore$ $\varepsilon_{zz}=0 \Rightarrow \sigma_{zz}=\nu(\sigma_{xx}+\sigma_{yy})$

$\therefore$ u_z = function of (x,y) only

B.C. displacement in the m direction must be zero

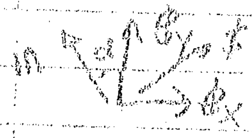
ie $\mathbf{u}\cdot\mathbf{m}=0$ or $u_x \sin\alpha + u_y\cos\alpha = 0$ $\forall x,y,z$ (except possibly the end points)

other B.C. $\sigma_{ns}=0$

I hope this is what you mean. [illegible] expressible in terms [illegible] of displacement [illegible]

(6)

6) a) on OE, rigid boundary $\Rightarrow u \cdot n = 0$



$n = -\sin\alpha\, e_x + \cos\alpha\, e_y$

$u = u_x e_x + u_y e_y \quad u \cdot n = \boxed{-\sin\alpha\, u_x + \cos\alpha\, u_y = 0 \text{ on OE}}$

b) on OE no shear component of traction (since walls are lubed)

$n \cdot \sigma \cdot s = n \cdot \sigma \cdot s = 0 \Rightarrow \sigma_{ns} = \sigma'_{xy}$ (letting $s' = e'_x$ & $n = e'_y$)

$$\begin{bmatrix} e_x' \\ e_y' \\ e_z' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \qquad \sigma'_{xy} = 0 = \mu \varepsilon'_{xy} \Rightarrow \varepsilon'_{xy} = 0$$

$$\varepsilon'_{xy} = a_{xp} a_{yq} \varepsilon_{pq} = -\sin\alpha\cos\alpha\, \varepsilon_{xx} + \cos^2\alpha\, \varepsilon_{xy}$$
$$- \sin^2\alpha\, \varepsilon_{yx} - \sin\alpha\cos\alpha\, \varepsilon_{yy} = 0$$

$$-\sin\alpha\cos\alpha \left(\frac{\partial u_x}{\partial x}\right) + \frac{1}{2}(\cos^2\alpha - \sin^2\alpha)\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - \sin\alpha\cos\alpha \left(\frac{\partial u_y}{\partial y}\right) = 0$$

mult through by 2 $\Rightarrow \boxed{\cos 2\alpha \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - \sin 2\alpha \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0}$

$n \cdot \sigma \cdot s = 0 \qquad n = \cancel{\sin 2\alpha\, j + \cos\alpha} \quad \cos\alpha\, \underline{j} + \sin\alpha\, \underline{i}$

$s = \cos\alpha\, \underline{i} + \sin\alpha\, \underline{j}$

$$L = \begin{bmatrix} \cos & \sin & 0 \\ -\sin & \cos & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \qquad L^T = \begin{bmatrix} \cos & -\sin & 0 \\ \sin & \cos & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} c + \sigma_{xy} s & -\sigma_{xx} s + \sigma_{xy} c & \cancel{\sigma_{xz}} \\ \sigma_{yx} c + \sigma_{yy} s & -\sigma_{yx} s + \sigma_{yy} c & \cancel{\sigma_{yz}} \\ \cancel{\sigma_{zx} c + \sigma_{zy} s} & \cancel{-\sigma_{zx} s + \sigma_{zy} c} & \sigma_{zz} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} c^2 + \sigma_{xy} cs + \sigma_{yx} cs + \sigma_{yy} s^2 & -\sigma_{xx} sc + \sigma_{xy} c^2 - \sigma_{yx} s^2 + \sigma_{yy} cs & 0 \\ -\sigma_{xx} sc - \sigma_{xy} s^2 + \sigma_{yx} c^2 + \sigma_{yy} sc & \sigma_{xx} s^2 - \sigma_{xy} sc - \sigma_{yx} cs + \sigma_{yy} c^2 & \\ \cancel{\sigma_{zx} c + \sigma_{zy} s} & \cancel{-\sigma_{zx} s + \sigma_{zy} c} & \sigma_{zz} \end{bmatrix}$$

$$\sigma_{xx}' = \sigma_{xx} c^2 + \sigma_{yy} s^2 + 2\sigma_{xy} cs$$

$$\sigma_{xy}' = (\sigma_{yy} - \sigma_{xx}) sc + \sigma_{xy} (c^2 - s^2)$$

$$\sigma_{yy}' = \sigma_{xx} s^2 + \sigma_{yy} c^2 - 2\sigma_{xy} sc$$

Question	Score
1	
2	
3	
4	
5	
6	
7	
8	
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10	
Total	83/100

STANFORD UNIVERSITY

OFFICIAL EXAMINATION BOOK

24 PAGE RULED

Name of Student Cesar Levy

Date of Examination 6 Nov 78

Subject Elasticity ME 238A

HONORABLE CONDUCT
in academic work is the spirit of conduct in this University.

In recognition of and in the spirit of the Honor Code, I certify that I will neither receive nor give unpermitted aid on this examination and that I will report, to the best of my ability, all Honor Code violations observed by me.

(signed) Cesar Levy
Name

SUGGESTIONS FOR CONDUCT

1. Occupy alternate seats where possible.
2. When in doubt as to the meaning of a question, consult the instructor, who will be found in his or her office.

DIVISION OF APPLIED MECHANICS

DEPARTMENT OF MECHANICAL ENGINEERING

238A Theory of Elasticity — Autumn 1978

Midterm Exam

1. Consider a state of stress in a body for which $\sigma_{xx} = 3A$; $\sigma_{zz} = -A$; $\sigma_{yy} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$ and where $A > 0$. Determine

(15)

[15] a) The maximum normal stress 3A

(15)

[15] b) The shear component of traction in the x-z plane on principal planes ⊥ to princip dir $\tau = 0$

[20] c) The traction across a plane equally inclined to the positive x- and z-axes and containing the y-axis.

2. Consider the displacement field

(24)

$$u_x = By \; ; \; u_y = u_z = 0$$

[25] Determine the relative elongation (stretch) at 45° to both the x- and y-axes.

3. Consider a state of strain in a simply-connected body given by

$$\varepsilon_{xx} = k_1(x^2 - y^2)$$

$$\varepsilon_{yy} = k_2 xy$$

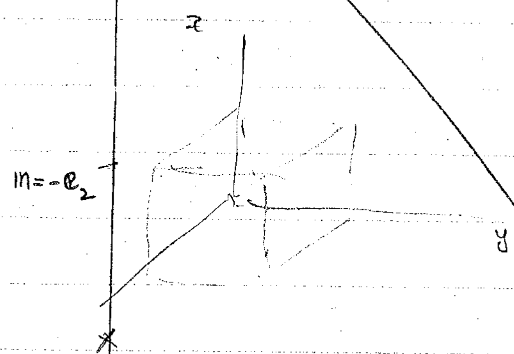
$$\varepsilon_{xy} = k_3 xy$$

All other $\varepsilon_{ij} = 0$

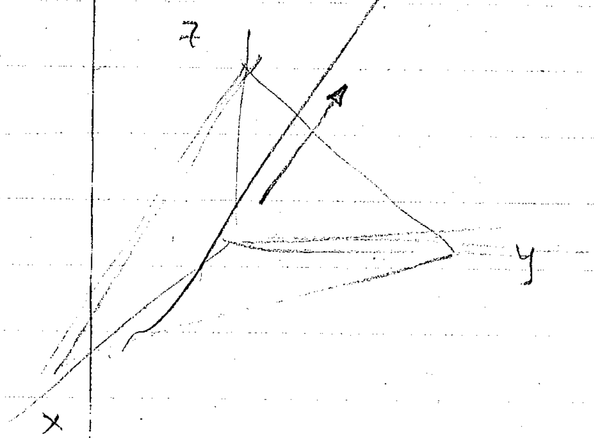
(24)

[25] For this strain field to be associated with a single-valued displacement field, what relation(s) must be satisfied among the coefficients k_1, k_2, k_3 ?

$$\sigma = \begin{pmatrix} 3A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -A \end{pmatrix}$$



the shear comp on x,z plane $\quad n \cdot \sigma = -t_2 = -\left(\sigma_{21} e_1 + \sigma_{22} e_2 + \sigma_{23} e_3\right)$



24

3. To solve use compat

$\Psi_{xx} = 2\cdot 0 - 0 - 0 = 0$ ✓

$\Psi_{yy} = 2\cdot 0 - 0 - 0 = 0$ ✓

$\Psi_{zz} = 2\cdot K_3 + 2K_1 - 0 = 0$ only if $K_1 = -K_3$ ✓

$\Psi_{yz} = 0 + 0 - 0 - 0 = 0$ ✓

$\Psi_{zx} = 0 + 0 - 0 - 0 = 0$

$\Psi_{xy} = 0 - 0 + 0 \quad 0 = 0$

no restriction of K_2 but $K_1 = K_3$

2. $$\Phi = \phi_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \qquad \frac{\partial u_1}{\partial x_2} \quad \frac{\partial u_k}{\partial x_1}$$

24

and $\mathbb{n}\cdot\Phi\cdot\mathbb{n}$ $\qquad \mathbb{n} = \frac{1}{\sqrt{2}}(\mathbb{e}_1 + \mathbb{e}_2)$

$$\phi_{11} = \frac{1}{2}\left(0 + 0 - [\,0\,]\right) = 0$$
$$\phi_{12} = \frac{1}{2}\left(B + 0 - 0\cdot\right) = \frac{1}{2}B$$
$$\phi_{13} = \frac{1}{2}\left(0 + 0 - B\cdot 0\right) = 0$$
$$\phi_{21} = \frac{1}{2}\left(B - 0\right) = \frac{1}{2}B$$
$$\phi_{22} = \frac{1}{2}\left(0 + 0 - B^2\right) = -B^2/2$$
$$\phi_{23} = \frac{1}{2}\left(0 + 0 + 0\right) = 0$$

$$\begin{pmatrix} 0 & \frac{1}{2}B & 0 \\ \frac{1}{2}B & -\frac{B^2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

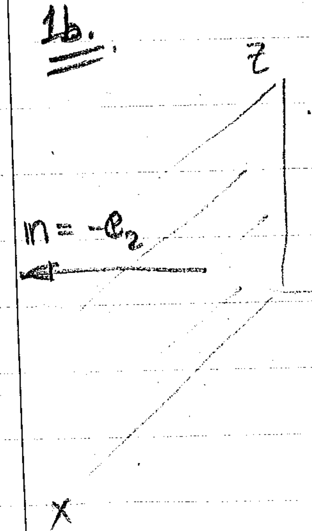
$$\phi_{ij} = \frac{1}{2}B\mathbb{e}_y\mathbb{e}_x + \frac{1}{2}B\mathbb{e}_x\mathbb{e}_y - \frac{B^2}{2}\mathbb{e}_y\mathbb{e}_y$$

$$\mathbb{n}\cdot\Phi\cdot\mathbb{n} = \frac{1}{\sqrt{2}}(\mathbb{e}_1 + \mathbb{e}_2)\cdot\left[\frac{1}{2}B\mathbb{e}_1\mathbb{e}_2 + \frac{1}{2}B\mathbb{e}_2\mathbb{e}_1 - \frac{B^2}{2}\mathbb{e}_2\mathbb{e}_2\right]\cdot\frac{1}{\sqrt{2}}(\mathbb{e}_1 + \mathbb{e}_2) =$$

$$= \frac{1}{\cancel{8}\,4}\left[B\mathbb{e}_2 + B\mathbb{e}_1 - B^2\mathbb{e}_2\right]\cdot(\mathbb{e}_1 + \mathbb{e}_2)$$

$$\mathbb{n}\cdot\Phi\cdot\mathbb{n} = \frac{1}{\cancel{8}\,4}\left[B + B - B^2\right] = \frac{B}{\cancel{8}\,4}\left[2 - B\right]$$

1b.



15

$$\therefore \quad t_n = n \cdot \sigma = -t_2 = -\left[\overset{0}{\cancel{\sigma_{21}}} e_1 + \overset{0}{\cancel{\sigma_{22}}} e_2 + \overset{0}{\cancel{\sigma_{23}}} e_3\right]$$

$\therefore \quad t_n = 0 \qquad \therefore$ <u>shear traction = 0</u>

since $\quad n \cdot t_n$ = normal component of traction $= 0$

if $t_n = 0 = a\,n + b\,\pi \;\Rightarrow\; a = b = 0$

1a

$n_x^{(1)} = n_z^{(1)} \begin{vmatrix} 0 & 0 \\ 0 & -\lambda \end{vmatrix} = 0$

$n_y^{(2)} = n_z \begin{vmatrix} 3A-\lambda^{(1)} & 0 \\ 0 & 0 \end{vmatrix} = 0$

$n_z^{(3)} = n_z$

Use invariants

$I = \sigma_{11} + \sigma_{33} = \sigma_1 + \sigma_2 + \sigma_3 = 2A$

15 $II = \sigma_{11}\sigma_{33} = -3A^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$

$III = 0 \Rightarrow \sigma_1\sigma_2\sigma_3 = 0 \Rightarrow$ one of the $\sigma_3 = 0$, assume $\sigma_3 = 0$

$\therefore \sigma_1 + \sigma_2 = 2A \qquad \sigma_1 + \sigma_2 = \sigma_1 - \frac{3A^2}{\sigma_1} = 2A$

$\sigma_1\sigma_2 = -3A^2$

$\therefore \sigma_1^2 - 3A^2 - 2A\sigma_1 = 0$

or $\frac{2A \pm \sqrt{4A^2 - 4(-3A^2)}}{2} = \sigma_{1,2}$

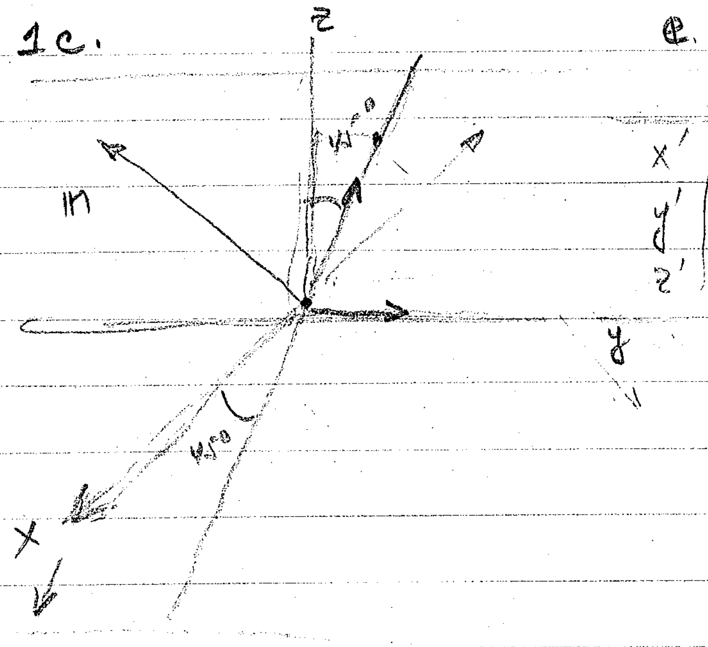
$\sigma_{1,2} = \frac{2A \pm \sqrt{16A^2}}{2} = \frac{2A \pm 4A}{2} = \frac{6A}{2}$

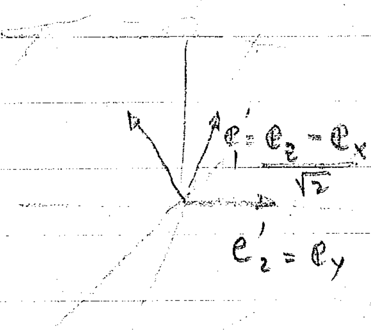
$3A$ & $-A$ $\qquad \therefore$ max stress $= \underline{3A}$ ✓

min stress $= -A$

5

1c.





$$\mathbf{n} \cdot e_1' \times e_2' = \begin{vmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ a & b & c \end{vmatrix} = 0$$

for non coplanarity

$$c\left(-\frac{1}{\sqrt{2}}\right) - \frac{a}{\sqrt{2}} = 0$$

$\therefore$

$$c = -a \quad \therefore$$

$$\mathbf{n} = \frac{1}{\sqrt{2}}\left(e_1 + e_2\right) \quad ✓$$

$$\therefore \mathbf{n} \cdot \mathbf{t}_n = \frac{1}{\sqrt{2}} t_1 + \frac{1}{\sqrt{2}} t_2 = \frac{1}{\sqrt{2}}\left(e_1 + e_2\right) \left[e_i\, t_i\right]$$

$$= \frac{1}{\sqrt{2}}\left(3A\, e_1\right) + \frac{1}{\sqrt{2}}\, 0 = \frac{3A}{\sqrt{2}} e_1$$

$$-\frac{c}{\sqrt{2}} - \frac{a}{\sqrt{2}} = 0 \implies c = -a$$

$$\therefore \; m = a\left[\frac{1}{\sqrt{2}}, \; \frac{-1}{\sqrt{2}}\right]$$

$$\underbrace{m \cdot e_1 \times e_2}_{m} = 1 \quad \therefore$$

$$\begin{vmatrix} -1/\sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ a & b & c \end{vmatrix} = c\left(\frac{-1}{\sqrt{2}}\right) - \frac{a}{\sqrt{2}} = 1$$

$$\therefore \; (-c-a) = \sqrt{2}$$

$$\therefore \; (c+a) = -\sqrt{2}$$

also $c^2 + a^2 = 1 \quad \therefore \quad c^2 + a^2 + 2ca = 2$

$$2ca = 1$$

— $m \cdot e_1' = 0 \qquad a\left(-\frac{1}{\sqrt{2}}\right) + c\left(\frac{1}{\sqrt{2}}\right) = 0 \qquad a = -c$

— $m \cdot e_2' = 0 \qquad b = 0$

— $m \cdot m = 1 \qquad a^2 + c^2 = 1 \qquad \therefore \quad 2c^2 = 1 \quad c = \pm\frac{1}{\sqrt{2}}$

$$a = \mp\frac{1}{\sqrt{2}}$$

$$\therefore \; \boxed{m = \mp\frac{1}{\sqrt{2}} \; \pm\frac{1}{\sqrt{2}}}$$

DIVISION OF APPLIED MECHANICS

DEPARTMENT OF MECHANICAL ENGINEERING

238A Theory of Elasticity | Autumn 1978

Problem Set No. 1

DEFINITIONS

1. Kronecker Delta: $$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

2. Idemfactor (Identity Dyadic): $\mathbb{I} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3 = \mathbf{e}_i\mathbf{e}_i$

   where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are mutually orthogonal unit vectors.

3. Direct Product of Two Dyadics: If $\Phi = \mathbf{a}\mathbf{b}$ and $\Psi = \mathbf{c}\mathbf{d}$, then $\Phi \cdot \Psi = (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}$

4. Double Dot Product of Two Dyadics: If $\Phi = \mathbf{a}\mathbf{b}$ and $\Psi = \mathbf{c}\mathbf{d}$,

   then $\Phi : \Psi = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})$

VERIFY THESE EXPRESSIONS FOR YOURSELF (Don't hand in)

1. $\nabla\cdot(\nabla\phi) = \nabla^2\phi = \phi_{,ii}$ (ϕ a scalar)
2. $\nabla\times(\nabla\phi) = 0$
3. $V_i\,\delta_{ij} = V_j$
4. $\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$ where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are mutually orthogonal unit vectors.
5. $\mathbb{I} = \delta_{ij}\,\mathbf{e}_i\mathbf{e}_j$
6. $\mathbf{a}\times\mathbf{b} = a_i b_j\, e_{ijk}\,\mathbf{e}_k$ $\quad \mathbf{a} = a_i\mathbf{e}_i$, $\mathbf{b} = b_j\mathbf{e}_j$ $\quad a_i b_j(\mathbf{e}_i\times\mathbf{e}_j) = e_{ijk}\,\mathbf{e}_k\, a_i b_j$

PROBLEMS

1. Let $\mathbf{f} = f_i\,\mathbf{e}_i$

   a. Show that $\mathbf{f}\cdot\mathbb{I} = \mathbf{f}$ b. Show that $\mathbb{I}\cdot\mathbf{f} = \mathbf{f}$

2. Let $\Psi = \Psi_{ij}\,\mathbf{e}_i\mathbf{e}_j$

   a. Show that $\Psi\cdot\mathbb{I} = \Psi$ b. Show that $\mathbb{I}\cdot\Psi = \Psi$

3. Show that $e_{ijk}\,a_i a_j = 0$

4. Show that $e_{ijk}\,e_{kji} = -6$

5. Let $\Psi = \Psi_{ij}\,\mathbf{e}_i\mathbf{e}_j$

   a. Show that $\nabla\times\Psi = \Psi_{\ell j,k}\,e_{k\ell i}\,\mathbf{e}_i\mathbf{e}_j$ b. Show that $\Psi\times\nabla = \Psi_{i\ell,k}\,e_{\ell kj}\,\mathbf{e}_i\mathbf{e}_j$? ?

6. Let $\Phi = \phi_{ij}\,\mathbf{e}_i\mathbf{e}_j$ and $\Psi = \Psi_{k\ell}\,\mathbf{e}_k\mathbf{e}_\ell$

   a. Show that $\Phi\cdot\Psi = \phi_{\cdot\cdot}\,\Psi_{\cdot\cdot}\,\mathbf{e}\,\mathbf{e}$ b. Show that $\Phi : \Psi = \phi\;\;\Psi$

1. $\mathbb{f} = f_i \mathbb{e}_i$

$\therefore \quad \mathbb{f}\cdot\mathbb{e}_1 = \mathbb{e}_1\cdot\mathbb{f} = \mathbb{e}_1\cdot(f_i\mathbb{e}_i) = f_i(\mathbb{e}_1\cdot\mathbb{e}_i) = (f_i\mathbb{e}_i)\cdot\mathbb{e}_1 = f_1$

a. $\therefore \quad \mathbb{f} = f_i\mathbb{e}_i = f_1\mathbb{e}_1 + f_2\mathbb{e}_2 + f_3\cdot\mathbb{e}_3 = (\mathbb{f}\cdot\mathbb{e}_1)\mathbb{e}_1 + (\mathbb{f}\cdot\mathbb{e}_2)\mathbb{e}_2 + (\mathbb{f}\cdot\mathbb{e}_3)\mathbb{e}_3$

$= \mathbb{f}\cdot[\mathbb{e}_1\mathbb{e}_1 + \mathbb{e}_2\mathbb{e}_2 + \mathbb{e}_3\mathbb{e}_3] = \mathbb{f}\cdot\mathbb{I}$

or

b. $\mathbb{e}_i f_i = \mathbb{f} = \mathbb{e}_1[\mathbb{e}_1\cdot\mathbb{f}_1] + \mathbb{e}_2[\mathbb{e}_2\cdot\mathbb{f}] + \mathbb{e}_3[\mathbb{e}_3\cdot\mathbb{f}] = [\mathbb{e}_1\mathbb{e}_1 + \mathbb{e}_2\mathbb{e}_2 + \mathbb{e}_3\mathbb{e}_3]\cdot\mathbb{f} = \mathbb{I}\cdot\mathbb{f}$

2.a. $\Psi = \psi_{ij}\mathbb{e}_i\mathbb{e}_j$

$\Psi\cdot\mathbb{e}_1 = \psi_{ij}\mathbb{e}_i\mathbb{e}_j\cdot\mathbb{e}_1 = \psi_{i1}\mathbb{e}_i$

$\Psi = \psi_{i1}\mathbb{e}_i\mathbb{e}_1 + \psi_{i2}\mathbb{e}_i\mathbb{e}_2 + \psi_{i3}\mathbb{e}_i\mathbb{e}_3$

$= [\Psi\cdot\mathbb{e}_1]\mathbb{e}_1 + [\Psi\cdot\mathbb{e}_2]\mathbb{e}_2 + [\Psi\cdot\mathbb{e}_3]\mathbb{e}_3$

$= \Psi\cdot[\mathbb{e}_1\mathbb{e}_1 + \mathbb{e}_2\mathbb{e}_2 + \mathbb{e}_3\mathbb{e}_3] = \Psi\cdot\mathbb{I}$

b. $\Psi = \mathbb{e}_i\psi_{ij}\mathbb{e}_j$

$\mathbb{e}_1\cdot\Psi = \mathbb{e}_1\cdot\mathbb{e}_i\psi_{ij}\mathbb{e}_j = \psi_{1j}\mathbb{e}_j$

$\Psi = \mathbb{e}_1\psi_{1j}\mathbb{e}_j + \mathbb{e}_2\psi_{2j}\mathbb{e}_j + \mathbb{e}_3\psi_{3j}\mathbb{e}_j$

$\Psi = \mathbb{e}_1[\mathbb{e}_1\cdot\Psi] + \mathbb{e}_2\cdot[\mathbb{e}_2\cdot\Psi] + \mathbb{e}_3[\mathbb{e}_3\cdot\Psi]$

$= [\mathbb{e}_1\mathbb{e}_1 + \mathbb{e}_2\mathbb{e}_2 + \mathbb{e}_3\mathbb{e}_3]\cdot\Psi = \mathbb{I}\cdot\Psi$

3. $e_{ijk}\, a_i a_j = 0$ Show this is true

note that $e_{ijk} = \begin{cases} +1 & ijk \text{ cyclic} \\ 0 & \text{others} \\ -1 & ijk \text{ anticyclic} \end{cases}$

the only terms that will contribute are the cyclic and anti cyclic terms

ie (1 2 3, 2 3 1, 3 1 2) cyclic and (3 2 1, 1 3 2, 2 1 3) anticyclic

note that $e_{123}a_1a_2 + e_{213}a_2a_1 = 0$ since $e_{123} = -e_{213}$

$e_{231}a_2a_3 + e_{321}a_3a_2 = 0$ " $e_{231} = -e_{321}$

thus $e_{ijk}\, a_i a_j = 0$

4. $e_{ijk}\, e_{kji} = -6$ show this

note that $e_{kji} = -e_{ijk}$ since its indices are anticyclic wrt e_{ijk}.
thus only the indicial terms of e_{ijk} equal to 123, 312, 231 and 321, 132, 213 give any contribution and this turns out to be true of e_{kji} also. Hence $e_{ijk}\, e_{kji}$

$$= e_{ijk}(e_{ijk}) = -(1\cdot1 + 1\cdot1 + 1\cdot1) - [(-1\cdot-1) + (-1\cdot-1) + (-1\cdot-1)] = -6$$

5. $\nabla \times \psi$ for $\psi = \psi_{ij}\,\mathbf{e}_i\mathbf{e}_j$

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3}$$

$$\therefore\ \nabla \times \psi = \left(\mathbf{e}_k \frac{\partial}{\partial x_k}\right) \times \psi_{ij}\,\mathbf{e}_i\mathbf{e}_j = \left[\mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3}\right] \times \left(\psi_{11}\mathbf{e}_1\mathbf{e}_1 + \psi_{12}\mathbf{e}_1\mathbf{e}_2 + \psi_{13}\mathbf{e}_1\mathbf{e}_3\right.$$

$$\left. + \psi_{21}\mathbf{e}_2\mathbf{e}_1 + \psi_{22}\mathbf{e}_2\mathbf{e}_2 + \psi_{23}\mathbf{e}_2\mathbf{e}_3 + \psi_{31}\mathbf{e}_3\mathbf{e}_1 + \psi_{32}\mathbf{e}_3\mathbf{e}_2 + \psi_{33}\mathbf{e}_3\mathbf{e}_3\right)$$

$$\psi_{21,1}\mathbf{e}_3\mathbf{e}_1 + \psi_{22,1}\mathbf{e}_3\mathbf{e}_2 + \psi_{23,1}\mathbf{e}_3\mathbf{e}_3 - \psi_{31,1}\mathbf{e}_2\mathbf{e}_1 - \psi_{32,1}\mathbf{e}_2\mathbf{e}_2 - \psi_{33,1}\mathbf{e}_2\mathbf{e}_3$$
$$+ \psi_{31,2}\mathbf{e}_1\mathbf{e}_1 + \psi_{32,2}\mathbf{e}_1\mathbf{e}_2 + \psi_{33,2}\mathbf{e}_1\mathbf{e}_3 - \psi_{11,2}\mathbf{e}_3\mathbf{e}_1 - \psi_{12,2}\mathbf{e}_3\mathbf{e}_2 - \psi_{13,2}\mathbf{e}_3\mathbf{e}_3$$
$$+ \psi_{11,3}\mathbf{e}_2\mathbf{e}_1 + \psi_{12,3}\mathbf{e}_2\mathbf{e}_2 + \psi_{13,3}\mathbf{e}_2\mathbf{e}_3 - \psi_{21,3}\mathbf{e}_1\mathbf{e}_1 + \psi_{22,3}\mathbf{e}_1\mathbf{e}_2 - \psi_{23,3}\mathbf{e}_1\mathbf{e}_3$$

note any typical term ie $-\psi_{11,2}\,\mathbf{e}_3\mathbf{e}_1 = \psi_{11,2}(\mathbf{e}_2 \times \mathbf{e}_1)\mathbf{e}_1 = \psi_{11,2}(e_{213}\mathbf{e}_3)\mathbf{e}_1$

or in general $\mathbf{e}_k \frac{\partial}{\partial x_k} \times \psi_{ij}\,\mathbf{e}_i\mathbf{e}_j = \frac{\partial \psi_{ij}}{\partial x_k}\,\underbrace{(\mathbf{e}_k \times \mathbf{e}_i)}_{e_{kil}\mathbf{e}_l}\,\mathbf{e}_j$

since indices are dummy then $\psi_{ij,k}\, e_{kil}\, \mathbf{e}_l\mathbf{e}_j$ can be written as $\psi_{lj,k}\, e_{kli}\, \mathbf{e}_i\mathbf{e}_j$
when $i \to l$ & $l \to i$

$$\psi \times \nabla = \psi_{ij}\,\mathbf{e}_i\mathbf{e}_j \times \mathbf{e}_k\frac{\partial}{\partial x_k} = \mathbf{e}_i(\mathbf{e}_j \times \mathbf{e}_k)\frac{\partial \psi_{ij}}{\partial x_k} = \mathbf{e}_i(e_{jkl}\,\mathbf{e}_l)\frac{\partial \psi_{ij}}{\partial x_k}$$

since indices are dummy then $\psi_{ij,k}\, e_{jkl}\, \mathbf{e}_i\mathbf{e}_l$ can be written as $\psi_{il,k}\, e_{lkj}\, \mathbf{e}_i\mathbf{e}_j$

and since $e_i e_j = e_j e_i$ we get final result $\psi_{il,k}\, e_{lkj}\, e_i e_j$

6. $\Phi = \phi_{ij}\, e_i e_j$ and $\Psi = \psi_{kl}\, e_k e_l$

by definition $\Phi \cdot \Psi = \phi_{ij}\, e_i e_j \cdot \psi_{kl}\, e_k e_l$ or $e_i(\phi_{ij}\psi_{kl}\, e_l)\, e_j \cdot e_k$

$e_j \cdot e_k = 1$ when $j = k = m$ $\quad e_j \cdot e_k = 0 \quad j \neq k \quad \therefore$

$\Phi \cdot \Psi = \phi_{im}\psi_{ml}\, e_i e_l$ ∴ since indices are dummy replace $l \longmapsto m$

$\therefore \quad \phi_{im}\psi_{mj}\, e_i e_j$

$\Phi = \phi_{ij}\, e_i e_j$ and $\Psi = \psi_{kl}\, e_k e_l$

$\Phi : \Psi = e_i \phi_{ij}\, e_j \cdot e_k \psi_{kl}\, e_l = \phi_{ij}\psi_{kl}(e_i \cdot e_k)(e_j \cdot e_l)$

$e_j \cdot e_k = 1$ when $j = k = m$ $\quad$ & $= 0$ when $j \neq k$

and $e_j \cdot e_l = 1$ when $j = l = n$ or 0 when $j \neq l$

$\therefore \quad \Phi : \Psi = \phi_{nm}\psi_{mn}$

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ME 238A
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100/100

Problem set # 1.

1. Let $\mathbf{f} = f_i \mathbf{e}_i$

   a. Show that $\mathbf{f} \cdot \mathbb{I} = \mathbf{f}$

10 Since $f_i = \mathbf{f} \cdot \mathbf{e}_i$ then $\mathbf{f} = (\mathbf{f} \cdot \mathbf{e}_i)\mathbf{e}_i = \mathbf{f} \cdot \mathbf{e}_i \mathbf{e}_i = \mathbf{f} \cdot \delta_{ij} \mathbf{e}_i \mathbf{e}_j = \mathbf{f} \cdot \mathbb{I}$ ✓

10 b. Show that $\mathbb{I} \cdot \mathbf{f} = \mathbf{f}$

Since $f_i = \mathbf{f} \cdot \mathbf{e}_i = \mathbf{e}_i \cdot \mathbf{f}$ and $\mathbf{f} = f_i \mathbf{e}_i = \mathbf{e}_i f_i$ then $\mathbf{f} = \mathbf{e}_i (\mathbf{e}_i \cdot \mathbf{f}) = \mathbf{e}_i \mathbf{e}_i \cdot \mathbf{f} = \mathbb{I} \cdot \mathbf{f}$ ✓

2. Let $\Psi = \psi_{ij} \mathbf{e}_i \mathbf{e}_j$

10 a. Show that $\Psi \cdot \mathbb{I} = \Psi$

note that $\Psi \cdot \mathbf{e}_1 = \psi_{i1} \mathbf{e}_i$ thus $\Psi = [\Psi \cdot \mathbf{e}_1]\mathbf{e}_1 + [\Psi \cdot \mathbf{e}_2]\mathbf{e}_2 + [\Psi \cdot \mathbf{e}_3]\mathbf{e}_3$

and $\Psi = \Psi \cdot [\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3] = \Psi \cdot \mathbb{I}$ ✓

10 b. Show that $\Psi = \mathbb{I} \cdot \Psi$ ✓

note that since $\Psi = \psi_{ij} \mathbf{e}_i \mathbf{e}_j$ then $\Psi = \mathbf{e}_i \psi_{ij} \mathbf{e}_j$ and $\mathbf{e}_1 \cdot \Psi = \psi_{1j} \mathbf{e}_j$

thus $\Psi = \mathbf{e}_1[\mathbf{e}_1 \cdot \Psi] + \mathbf{e}_2[\mathbf{e}_2 \cdot \Psi] + \mathbf{e}_3[\mathbf{e}_3 \cdot \Psi] = [\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3] \cdot \Psi = \mathbb{I} \cdot \Psi$

3. Show that $e_{ijk}\, a_i\, a_j = 0$

10 since $e_{ijk} = \begin{cases} +1 & \text{for } i,j,k \text{ cyclic} \\ 0 & \text{for all others} \\ -1 & \text{for } i,j,k \text{ anticyclic} \end{cases}$

The only terms to contribute are (123, 231, 312) for the cyclic terms and (321, 132, 213) for the anticyclic terms. Thus look at all $a_i a_j$ and $a_j a_i$ terms (eg $e_{123}\, a_1 a_2$ and $e_{213}\, a_2 a_1$ for $k=3$). We note that they cancel, since the factor e for one term will be negative of the other term and when summed give zero. Thus for each cyclic terms there is an anticyclic term. Hence $e_{ijk}\, a_i a_j \equiv 0$.

note: there are 3 scalar eq'ns for $k = 1,2,3$

4. Show that $e_{ijk}\, e_{kji} = -6$

10 note that $e_{kji} = -e_{ijk}$ thus $e_{ijk}\, e_{kji} = -e_{ijk}\, e_{ijk}$. Since only 6 terms contribute [viz (123, 231, 312) and (321, 132, 213)], we can see that $-(e_{ijk})^2$ will give -1 for each of these terms. Since there are 6, $e_{ijk}\, e_{kji} = -6$. All other terms

give zero since either one of the terms e_{ijk} or e_{kji} will be zero. Thus $e_{ijk}e_{kji} = -6$.

5. Let $\Psi = \psi_{ij}\, e_i e_j$

a. Show that $\nabla \times \Psi = \psi_{lj,k}\, e_{kli}\, e_i e_j$

10 $\nabla = e_k \dfrac{\partial}{\partial x_k}$ thus $\nabla \times \Psi = e_k \dfrac{\partial}{\partial x_k} \times \psi_{ij}\, e_i e_j = \psi_{ij,k}\,(e_k \times e_i) e_j$

but $e_k \times e_i = e_{kil}\, e_l$ $\therefore$ $\nabla \times \Psi = \psi_{ij,k}\, e_{kil}\, e_l e_j$

since indices are dummies then if I interchange all i's with all l's I get

$\nabla \times \Psi = \psi_{lj,k}\, e_{kli}\, e_i e_j$

b. Show that $\Psi \times \nabla = \psi_{il,k}\, e_{lki}\, e_i e_j$

10 $\nabla = e_k \dfrac{\partial}{\partial x_k}$ thus $\Psi \times \nabla = \psi_{ij}\, e_i (e_j \times e_k) \dfrac{\partial}{\partial x_k} = e_i\,(e_j \times e_k)\, \psi_{ij,k}$

but $e_j \times e_k = e_{jkl}\, e_l$ $\therefore$ $\Psi \times \nabla = e_i e_l\, e_{jkl}\, \psi_{ij,k}$

since indices are dummies then if I interchange all j's with all l's I get

$\Psi \times \nabla = e_i e_j\, e_{lkj}\, \psi_{il,k}$

6. Let $\Phi = \phi_{ij}\, e_i e_j$ and $\Psi = \psi_{kl}\, e_k e_l$

10 a. Show that $\Phi \cdot \Psi = \phi_{im} \psi_{mj}\, e_i e_j$

$\Phi \cdot \Psi = \phi_{ij}\, e_i e_j \cdot \psi_{kl}\, e_k e_l = \phi_{ij} \psi_{kl}\, e_i e_l\, (\delta_{jk}) = \phi_{ij} \psi_{jl}\, e_i e_l$

since j and l are dummy variables replace all j's by m's and then all l's by j's;

hence $\phi_{im} \psi_{mj}\, e_i e_j = \Phi \cdot \Psi$

10

b. Show that $\Phi : \Psi = \phi_{mn} \psi_{mn}$

$\Phi : \Psi = (\phi_{ij}\, e_i e_j) : (\psi_{kl}\, e_k e_l) = \phi_{ij} \psi_{kl}\, (e_i \cdot e_k)(e_j \cdot e_l)$

$= \phi_{ij} \psi_{kl}\, \delta_{ik} \delta_{jl} = \phi_{ij} \psi_{ij}$

since j and i are dummy variables, replace all i's by m's and all j's by n's for result

$\therefore \Phi : \Psi = \phi_{mn} \psi_{mn}$

DIVISION OF APPLIED MECHANICS

DEPARTMENT OF MECHANICAL ENGINEERING

238A Theory of Elasticity — Autumn 1978

Problem Set No. 2 — Due 1 Nov. 78

1. The components of stress at a point in a body referred to a rectangular Cartesian system of coordinates are given by

$$\sigma_{xx} = 5 \times 10^6 \text{ Pa} \qquad \sigma_{xy} = 5 \times 10^6 \text{ Pa}$$
$$\sigma_{yy} = 0 \qquad \sigma_{yz} = -7.5 \times 10^6 \text{ Pa}$$
$$\sigma_{zz} = -3 \times 10^6 \text{ Pa} \qquad \sigma_{zx} = 8 \times 10^6 \text{ Pa}$$

Show that the magnitude of traction acting across a plane [whose normal has direction cosines $1/2$, $1/2$, $1/\sqrt{2}$ with respect to the x, y, z axes, respectively] is 11.2×10^6 Pa and that the normal and tangential components of traction across the same plane have the magnitudes of 2.65×10^6 Pa and 10.9×10^6 Pa, respectively.

2. At a point of a body located on the boundary whose direction cosines are ℓ, m and n, with respect to a rectangular Cartesian coordinate system x, y, z, the surface traction is known to have a magnitude a and be parallel to x. Furthermore, it is known that $\sigma_{xy} = \sigma_{xz} = \sigma_{zz} = 0$.

Show that the remaining components of stress at that point are $\sigma_{zy} = 0$, $\sigma_{yy} = 0$, $\sigma_{xx} = a/\ell$.

3. Verify the expressions for the three invariants of stress in terms of components of stress using indicial tensor notation.

4. Show that if the traction at a point across any three planes [whose normals are not coplanar] is in each case perpendicular to the plane, then the stress is hydrostatic (i.e., isotropic). Write the equation of motion for such a state of stress in indicial notation.

✓ 5. In several applications of analysis of stress at a point, it is shown that a special role is played by traction acting across planes which have like orientation with respect to all three principal directions. There are 8 such planes and they form an octahedron. The normal and shear components of traction across these planes are referred to as the octahedral normal stress σ_{oct} and octahedral shear stress τ_{oct}. Show that

need to write these 2 up

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$= \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$= \frac{1}{3}\sqrt{(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\sigma_{xy}{}^2 + \sigma_{yz}{}^2 + \sigma_{zx}{}^2)}$$

where σ_1, σ_2, σ_3 are the principal components of stress.

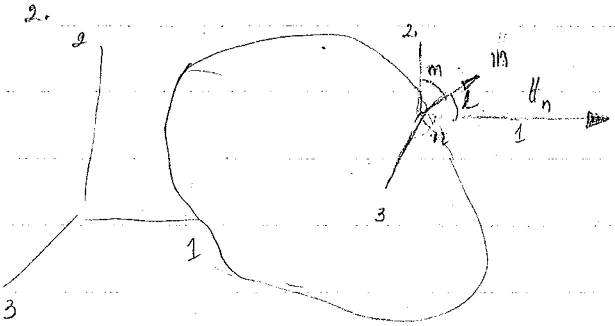
✓ 6. It is sometimes useful to decompose the stress tensor into its isotropic (or spherical) and its deviatoric (or distortional) components. The isotropic component is defined as having a matrix whose 3 terms on the main diagonal are all equal and whose terms off the diagonal all vanish. Show that the invariants of the stress deviator I_o, II_o, III_o are expressed in terms of the invariants I, II, III of the complete stress tensor as

$$I_o = 0$$

$$II_o = II - \frac{1}{3} I^2$$

$$III_o = III - \frac{1}{3} I \left(II - \frac{2}{9} I^2\right)$$

2.



$$t_n = n \cdot [t_j e_j]$$

where $t_1 = \sigma_{xx} e_x + \cancel{\sigma_{xy} e_y} + \cancel{\sigma_{xz} e_z} = \sigma_{xx} e_x$

$t_2 = \cancel{\sigma_{yx} e_x} + \sigma_{yy} e_y + \sigma_{yz} e_z = \sigma_{yy} e_y + \sigma_{yz} e_z$

$t_3 = \cancel{\sigma_{zx} e_x} + \sigma_{zy} e_y + \cancel{\sigma_{zz} e_z} = \sigma_{zy} e_y$

$\therefore\ t_n = n \cdot \sigma = l t_1 + m t_2 + n t_3$

$= l \sigma_{xx} e_x + m(\sigma_{yy} e_y + e_z \sigma_{yz}) + n \sigma_{zy} e_y$

now since $t_n \parallel e_x \implies t_n = \alpha e_x + \beta e_y + \gamma e_z$ where $\beta = \gamma = 0$

$\implies (m\sigma_{yy} + n\sigma_{zy}) = 0$ and $\sigma_{yz} = 0$ but $\sigma_{yz} = \sigma_{yz}$

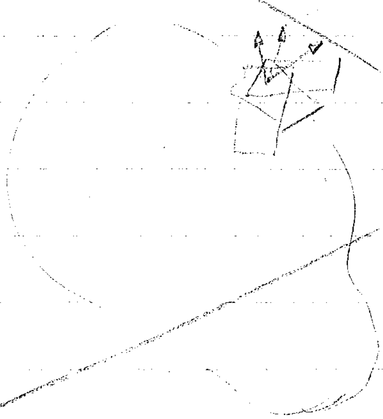
$\implies \sigma_{yy} = 0$

$\therefore\ \|t_n\|^2 = a^2 = \alpha^2 + \beta^2 + \gamma^2 = l^2 \sigma_{xx}^2$ or $\boxed{\sigma_{xx} = \frac{a}{l}}$

~~in order to find σ_{yy} let me take $t_n \cdot e_y = [l\sigma_{xx} e_x + m(\sigma_{yy} e_y) + m$~~

4

$t_{n_i} = n_i \cdot [t_j e_j] =$ $\quad \ni.\ n_i \cdot n_j \neq 0$



$$[t_1 l_1 + t_2 m_1 + t_3 n_1] = \{\alpha n_1\}$$
$$[t_1 l_2 + t_2 m_2 + t_3 n_2] = \{\beta n_2\}$$
$$[t_1 l_3 + t_2 m_3 + t_3 n_3] = \{\gamma n_3\}$$

$$\begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = ($$

3

$\sigma_{11}\sigma_{22}\sigma_{33} - \lambda(\sigma_{11} + \sigma_{33})\sigma_{22} + \sigma_{22}\lambda^2 - \sigma_{11}\sigma_{33}\lambda + \lambda^2(\sigma_{11} + \sigma_{33}) - \lambda^3$

$\sigma_{11}\sigma_{33} - \lambda(\sigma_{11} + \sigma_{33}) + \lambda^2$

$$\begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = (\sigma_{11} - \lambda)(\sigma_{33} - \lambda)(\sigma_{22} - \lambda) + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{32}\sigma_{21} - \sigma_{13}(\sigma_{22} - \lambda)(\sigma_{31}) - \sigma_{12}\sigma_{21}(\sigma_{33} - \lambda)$$

$$-(\sigma_{11} - \lambda)\sigma_{32}\sigma_{23} = 0$$

$-\lambda^3 + \lambda^2(\sigma_{11} + \sigma_{22} + \sigma_{33}) - \lambda(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33}) + [\sigma_{11}\sigma_{22}\sigma_{33} + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{32}\sigma_{21} - \sigma_{13}\sigma_{22}\sigma_{31} - \sigma_{12}\sigma_{21}\sigma_{33}$

$- \sigma_{11}\sigma_{32}\sigma_{23}] + \lambda[\sigma_{13}\sigma_{31} + \sigma_{12}\sigma_{21} + \sigma_{32}\sigma_{23}] = 0$ or $-\lambda^3 + I\lambda^2 - II\lambda + III = 0$

or $I = (\sigma_{11} + \sigma_{22} + \sigma_{33}) = \sigma_{kk}$

$II = [-\sigma_{13}\sigma_{31} - \sigma_{12}\sigma_{21} - \sigma_{32}\sigma_{23} + \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33}]$

$III = [\sigma_{11}\sigma_{22}\sigma_{33} + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{32}\sigma_{21} - \sigma_{13}\sigma_{22}\sigma_{31} - \sigma_{12}\sigma_{21}\sigma_{33} - \sigma_{11}\sigma_{32}\sigma_{23}]$

go to 2

now $II = \frac{1}{2} e_{mik} e_{njl} \sigma_{ij} \sigma_{kl} \delta_{mn}$

$e_{123}, e_{231}, e_{312}$ are +

$e_{321}, e_{213}, e_{132}$ are −

δ_{11}

$$II = \frac{1}{2} e_{mik} e_{mjl} \sigma_{ij} \sigma_{kl}$$

$$\frac{1}{2}\Big[e_{123}\big(e_{123}\sigma_{22}\sigma_{33} + e_{231}\sigma_{23}\sigma_{31} + e_{312}\sigma_{21}\sigma_{32} + e_{321}\sigma_{22}\sigma_{31} + e_{213}\sigma_{21}\sigma_{33} + e_{132}\sigma_{23}\sigma_{32}\big) + e_{231}\big(e_{123}\sigma_{32}\sigma_{13} + e_{231}\sigma_{33}\sigma_{11} + e_{312}\sigma_{31}\sigma_{12} + e_{321}\sigma_{32}\sigma_{11} + e_{213}\sigma_{31}\sigma_{13} + e_{132}\sigma_{33}\sigma_{12}\big)$$

$$\frac{1}{2}\Big[\big(\sigma_{22}\sigma_{33} + \sigma_{23}\sigma_{31} + \sigma_{21}\sigma_{32} - \sigma_{22}\sigma_{31} - \sigma_{21}\sigma_{33} - \sigma_{23}\sigma_{32}\big) + \big(\sigma_{32}\sigma_{13} + \sigma_{33}\sigma_{11} + \sigma_{31}\sigma_{12} - \sigma_{32}\sigma_{11} - \sigma_{31}\sigma_{13} - \sigma_{33}\sigma_{12}\big)$$
$$+ \big(\sigma_{12}\sigma_{23} + \sigma_{13}\sigma_{21} + \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21} - \sigma_{11}\sigma_{23} - \sigma_{13}\sigma_{22}\big) - \big(\sigma_{32}\sigma_{13} + \sigma_{23}\sigma_{11} + \sigma_{21}\sigma_{12} - \sigma_{22}\sigma_{11} - \sigma_{21}\sigma_{13} - \sigma_{23}\sigma_{12}\big)$$
$$- \big(\sigma_{12}\sigma_{33} + \sigma_{13}\sigma_{31} + \sigma_{11}\sigma_{32} - \sigma_{12}\sigma_{31} - \sigma_{11}\sigma_{33} - \sigma_{13}\sigma_{32}\big) - \big(\sigma_{32}\sigma_{23} + \sigma_{33}\sigma_{21} + \sigma_{31}\sigma_{22} - \sigma_{32}\sigma_{21} - \sigma_{31}\sigma_{23} - \sigma_{33}\sigma_{22}$$

$$\frac{1}{2}\Big[2\sigma_{22}\sigma_{33} + 2\sigma_{33}\sigma_{11} + 2\sigma_{11}\sigma_{22} - 2\sigma_{13}^2 - 2\sigma_{12}^2 - 2\sigma_{32}^2\Big] = II$$

now $\frac{1}{6} e_{mik} e_{njl} \sigma_{ij} \sigma_{kl} \sigma_{mn} = \frac{1}{6}\Big[\big(\sigma_{11}\sigma_{22}\sigma_{33} + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{21}\sigma_{32} - \sigma_{13}\sigma_{22}\sigma_{31} - \sigma_{12}\sigma_{21}\sigma_{33} - \sigma_{11}\sigma_{23}\sigma_{32}\big)$

$$+ \big(\sigma_{21}\sigma_{32}\sigma_{13} + \sigma_{22}\sigma_{33}\sigma_{11} + \sigma_{23}\sigma_{31}\sigma_{12} - \sigma_{23}\sigma_{32}\sigma_{11} - \sigma_{22}\sigma_{31}\sigma_{13} - \sigma_{21}\sigma_{33}\sigma_{12}\big) + \big(\sigma_{31}\sigma_{12}\sigma_{23} + \sigma_{32}\sigma_{13}\sigma_{21} + \sigma_{33}\sigma_{11}\sigma_{22}$$
$$- \sigma_{33}\sigma_{12}\sigma_{21} - \sigma_{32}\sigma_{11}\sigma_{23} - \sigma_{31}\sigma_{13}\sigma_{22}\big) - \big(\sigma_{31}\sigma_{22}\sigma_{13} + \sigma_{32}\sigma_{23}\sigma_{11} + \sigma_{33}\sigma_{21}\sigma_{12} - \sigma_{33}\sigma_{22}\sigma_{11} - \sigma_{32}\sigma_{21}\sigma_{13} - \sigma_{31}\sigma_{23}\sigma_{12}\big)$$
$$- \big(\sigma_{21}\sigma_{12}\sigma_{33} + \sigma_{22}\sigma_{13}\sigma_{31} + \sigma_{23}\sigma_{11}\sigma_{32} - \sigma_{23}\sigma_{12}\sigma_{31} - \sigma_{22}\sigma_{11}\sigma_{33} - \sigma_{21}\sigma_{13}\sigma_{32}\big) - \big(\sigma_{11}\sigma_{32}\sigma_{23} + \sigma_{12}\sigma_{33}\sigma_{21} + \sigma_{13}\sigma_{33}\sigma_{22}$$
$$- \sigma_{13}\sigma_{32}\sigma_{21} - \sigma_{12}\sigma_{31}\sigma_{23} - \sigma_{11}\sigma_{33}\sigma_{22}\big)\Big]$$

$$\frac{1}{6}\Big[6\sigma_{11}\sigma_{33}\sigma_{22} + 6\sigma_{12}\sigma_{23}\sigma_{31} + 6\sigma_{13}\sigma_{32}\sigma_{21} - 6\sigma_{13}\sigma_{22}\sigma_{31} - 6\sigma_{12}\sigma_{21}\sigma_{33} - 6\sigma_{11}\sigma_{23}\sigma_{32}\Big] = III$$

problem #1 on other side

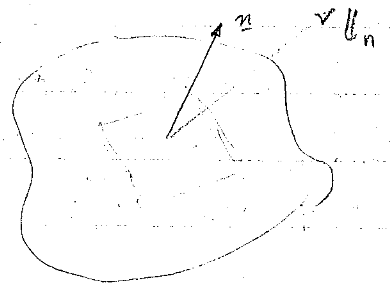
[redacted]

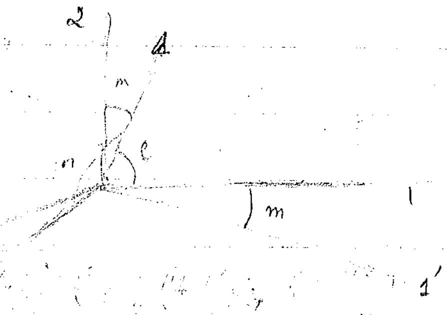
$n =$

1. $t_x = (5\times10^6)\,e_x + (5\times10^6)\,e_y + (8\times10^6)\,e_z$

$t_y = (5\times10^6)\,e_x + (0)\,e_y + (-7.5\times10^6)\,e_z$

$t_z = (8\times10^6)\,e_x + (-7.5\times10^6)\,e_y + (-3\times10^6)\,e_z$





$$t_n = \frac{1}{2}t_1 + \frac{1}{2}t_2 + \frac{1}{\sqrt{2}}t_3 = n\cdot\sigma$$

$$\frac{t_n}{10^6} = \left[2.5 + 2.5 + 4\sqrt{2}\right]e_1 + \left[2.5 + \left(\frac{-7.5}{\sqrt{2}}\right)\right]e_2 + \left[4 + \left(\frac{-7.5}{2}\right) + \left(\frac{-3}{\sqrt{2}}\right)\right]e_3$$

$$= \underset{10.7\times10^6}{(10.657\times10^6)}\,e_1 + \underset{-2.8\times10^6}{\left[-2.803\times10^6\right]}e_2 + \underset{-1.9\times10^6}{\left[-1.871\times10^6\right]}e_3$$

$\|t_n\| = 11.177\times10^6\ pa$ or $11.2\times10^6\ pA$ 11.2

now we know that

$$t_n = (t_n\cdot e_x')\,e_x' + (t_n\cdot e_y')\,e_y' + (t_n\cdot e_z')\,e_z'$$

$=$

$n\cdot t_n =$ now $t_n = p\,n + (q)\,\pi$ $\qquad n\cdot n = \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right) = 1$

$\underline{n\cdot t_n} = p + q\,(n\cdot\pi) = p$

$|t_n| = (p^2 + q^2)^{1/2}$

$$n\cdot t_n = \left\{\frac{1}{2}\overset{10.7}{(10.657)} + \frac{1}{2}\overset{-2.8}{(-2.803)} + \frac{\sqrt{2}}{2}\overset{-1.9}{(-1.871)}\right\}\times10^6$$

$= 2.605\times10^6$ or 2.61

$\therefore\ (|t_n|^2 - p^2)^{1/2} = q = 10.869\times10^6\ Pa$

since t_n is the plane is $\perp$ to the plane $\Rightarrow$ t_n is proportional to n the normal to the plane $\therefore$

4. $t_{n_1} = \alpha n_1 \Rightarrow \sigma \cdot n_1 - \alpha n_1 = (\sigma - I\alpha) \cdot n_1 = 0$

$t_{n_2} = \beta n_2 \Rightarrow \sigma \cdot n_2 - \beta n_2 = (\sigma - I\beta) \cdot n_2 = 0$

$t_{n_3} = \gamma n_3 \Rightarrow \sigma \cdot n_3 - \gamma n_3 = (\sigma - I\gamma) \cdot n_3 = 0$

use $t_{n_i} = n_i \cdot \sigma = \alpha_j \delta_{ji} n$ and take

$n_k \cdot n_\ell \times n_i \sigma = \alpha_j \delta_{ji} (n_k \cdot n_\ell \times n$

now take approp $n_k \cdot n_\ell \times n_i \cdot \sigma - n_\ell \cdot n_k \times n_i \sigma = (\alpha_j - \alpha_i) n_k \cdot n_\ell \times n_i =$

$\therefore$ ~~$n_2 \cdot (\sigma - I\alpha) \cdot n_1 = n_1 \cdot (\sigma - I\alpha) \cdot n_2 = 0$~~ this gives the component of t_{n_1} in n_2

~~$n_1 \cdot (\sigma - I\beta) \cdot n_2 = n_1 \cdot (\sigma - I\beta) \cdot n_2 = 0$~~ this gives the component of t_{n_2} in n_1

since $n_k \cdot n_\ell \times n_i = n_\ell \cdot n_k \times n_i \Rightarrow$ for $n_k \cdot n_\ell \times n_i \neq 0$ (non coplanarity condition) $\alpha_j = \alpha_i$

~~$\therefore$ subtract them and remembering that σ + I are symmetric~~

~~$n_1 \cdot [(\sigma - I\alpha) - (\sigma - I\beta)] \cdot n_2 = 0$~~

~~$n_1 \cdot [I(\beta - \alpha)] \cdot n_2 = (\beta - \alpha) n_1 \cdot I \cdot n_2 = (\beta - \alpha) n_1 \cdot n_2 = 0$~~

~~since n_1 is not coplanar with n_2 $\Rightarrow$ $\beta - \alpha = 0$ or $\beta = \alpha$~~

we can then do this with the others ~~n_1 + n_3~~ to prove $\alpha = \gamma$ $\therefore$ $\alpha = \beta = \gamma$

Thus we have shown that for that for planes having n_1, n_2, n_3 as normals and that they are noncoplanar $\sigma = I\alpha$. But we didn't choose the planes in any special manner $\therefore$ the results can be generalized to any 3 planes whose normals are not coplanar. Thus $\sigma = I\alpha$ or

$$\sigma = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

this is equivalent to hydrostatic stress

from the principle of linear momentum and

$$\int_s t_n \, ds = \int_s n \cdot \sigma \, ds = \int_V \nabla \cdot \sigma \, dV = \alpha \int_V \nabla \cdot n \, dV$$

$$\therefore \int_s t_n \, ds + \int_V f \, dV = \int \rho \frac{\partial^2 u}{\partial t^2} dV \Rightarrow \int_V \left[\alpha \nabla \cdot n + f - \rho \frac{\partial^2 u}{\partial t^2}\right] dV = 0$$

$\therefore$ since $\nabla \cdot n = 0$ $\Rightarrow$ for any volume $f = \rho \frac{\partial^2 u}{\partial t^2}$ or $f_i = \rho \frac{\partial^2 u_i}{\partial t^2}$

$a n_1 + b n_2 + c n_3 = 0$

$a + bq + cr = 0$

$qa + b + cs = 0$

$ar + bs + c = 0$

$\begin{pmatrix} 1 & q & r \\ q & 1 & s \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad$ $\det A = (1 + 2qsr - r^2 - q^2 - s^2)$ $\quad$ $\begin{pmatrix} 1 & q & r \\ 0 & 1-q^2 & s \cdot qr \end{pmatrix}$

$\sigma \cdot n_1 \times n_2 = \alpha (n_1 \times n_2)$

$n_3 \cdot (\sigma \cdot n_1 \times n_2) =$

$t_{n_1} = \sigma \cdot n_1 = \alpha n_1$

$t_{n_1} \times n_2 = \sigma \cdot n_1 \times n_2 = \alpha\, n_1 \times n_2$

$n_3 \cdot t_{n_1} \times n_2 = (n_3 \cdot \sigma) \cdot n_1 \times n_2 = \alpha (n_3 \cdot n_1 \times n_2)$

$n_1 \cdot t_{n_2} \times n_3 = n_1 \cdot \sigma \cdot n_2 \times n_3 = \beta (n_1 \cdot n_2 \times n_3) = \beta (n_3 \cdot n_1 \times n_2)$

for $n_1 \cdot n_2 \times n_3 = - n_1 \cdot n_3 \times n_2 = n_3 \cdot n_1 \times n_2$

$\therefore \quad n_3 \cdot t_{n_1} \times n_2 - n_1 \cdot t_{n_2} \times n_3 = (\alpha - \beta)(n$

$n_3 \cdot t_{n_1} \times n_2 - n_1 \cdot t_{n_2} \times n_3 = (\alpha - \beta)(n_3 \cdot n_1 \times n_2)$

$n_1 \cdot n_2 \qquad \sigma \cdot ([n_3 \cdot n_1 \times n_2] - \sigma \cdot ($

$\sigma \cdot [n_3 \cdot n_1 \times n_2 - n_1 \cdot n_2 \times n_3]$

$0 = (\alpha - \beta)\, n_3 \cdot n_1 \times n_2$

but for non coplanarity $n_3 \cdot n_1 \times n_2 \neq 0 \Rightarrow \alpha - \beta =$

if coplanar $\underset{\cos a_1}{n_1 \cdot n_2}, \underset{\cos a_2}{n_2 \cdot n_3}, \Rightarrow \underset{\cos a_3}{n_1 \cdot n_2}$

$\therefore a_1 + a_2 = a_3$

$t_{n_1} = \sigma \cdot n_1 = \alpha n_1$

$t_{n_1} \cdot n_2 = \sigma \cdot n_1 \cdot n_2 = \alpha\, n_1 \cdot n_2$

$\sigma \cdot [n_1 \cdot n_2 \times n_3] = \alpha (n_1 \cdot n_2 \times n_3)$

$\sigma \cdot [n_2 \cdot n_1 \times n_3] = \beta (n_2 \cdot n_1 \times n_3)$

$\cos(a_1 + \cos a_2) = \cos a_3$

$\cos a_1 \cos a_2 - \sin a_1 \sin a_2 = \cos a_3 =$

$(n_1 \cdot n_2)(n_2 \cdot n_3) - (n_1 \times n_2) \cdot (n_2 \times n_3)$

$\phi = n_1 \cdot n_2 \quad \psi = n_2 \cdot n_3$

$\phi : \psi -$

$a_1\sigma_{xx} + a_2\sigma_{xy} + a_3\sigma_{xz} = \alpha a_1$

$a_1 t_1 + a_2 t_2 + a_3 t_3 = \alpha a_1 e_1 + \alpha a_2$

$t, e \quad a_1 e_1 + a_2 e_2 + a_3 e_3$

$\sigma \cdot n_1 = \alpha n_1$

$\sigma \cdot n_1 \cdot n_2 =$

$\Rightarrow \quad (\quad)(\quad) =$

$1 = ap + bq$

$q = ar + b$

$p = a + br$

$1 = a^2 + abr + abr + b^2$

$1 = a^2 + 2abr + b^2$

$\beta = \alpha$ or $n_1 \cdot n_2 = 0$ — if $n_1 \cdot n_2 = 0 \Rightarrow n_1 \perp n_2$

$\gamma = \beta$ or $n_3 \cdot n_2 = 0$ — $n_2 \perp n_3$

$\alpha = \gamma$ or $n_1 \cdot n_3 = 0$ — $n_1 \perp n_3$

$n_1 = a n_2 + b n_3$ coplanar

$\therefore n_1 \cdot n_1 = 1 = a (n_2 \cdot n_1) + b (n_1 \cdot n_3)$

$n_2 \cdot n_1 \quad 1 = a p + b q$

$p = a + b\, n_2 \cdot n_3 = a + br$

$q = ar + b$

the

if coplanar $(n_1 \cdot n_2)$

Assume $n_1 =$ must be a linear combin of the other 2

$\therefore (\alpha - \beta)\, n_1 \cdot n_2 = 0$

$\{a + b\, n_2 \cdot n_3\}$

$\therefore n_1 = m\, n_2 + p\, n_3$

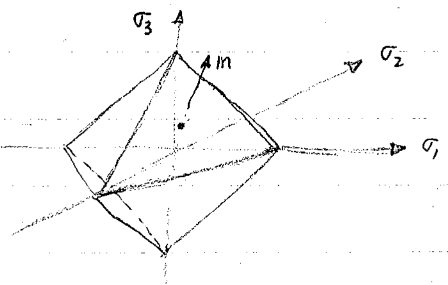
$(\alpha + \beta) a$

$\therefore n_1 \cdot n_2 = m + p\, n_3 \cdot n_2$ but $n_i \cdot n_j = \delta_{ij} \Rightarrow m, p = 0 \Rightarrow n$

$\Rightarrow n_1 = a n_2 + b n_3 \Rightarrow n_1 \cdot n_2 = a + b\, n_2 \cdot n_3)$ or $n_2 \cdot n_3 = -\frac{a}{b}$

$\Rightarrow 0 = a(\alpha - \beta) + b(\alpha - \beta)\, n_2\, n_3$

Let us look at



For an octahedral plane the normal $n = a_1 e_1 + a_2 e_2 + a_3 e_3$ where e_i are the unit vectors in the principal directions so that $|n|^2 = a_1^2 + a_2^2 + a_3^2$ and $a_i = n \cdot e_i$; but all directional cosines are the same in all directions $\therefore$ $a_i^2 = \frac{1}{3}$ $i = 1,2,3$

or $n = \frac{1}{\sqrt{3}}(e_1 + e_2 + e_3)$

now we know that $\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$ $\therefore$ $t_n = \sigma \cdot n = \frac{1}{\sqrt{3}}(\sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3)$

now since $t_n = a\, n + b\, \pi$ then $n \cdot t_n = a = \frac{1}{\sqrt{3}}(1,1,1) \cdot \frac{1}{\sqrt{3}}(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$
$= \sigma_{oct}\, n + \tau_{oct}\, \pi$

and since $|t_n|^2 = a^2 + b^2$ $\therefore$ $b = \sqrt{|t_n|^2 - a^2}$

$|t_n|^2 = \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$ $\therefore$ $b = \sqrt{\frac{3}{9}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2}$ after expansion

$\tau_{oct} = b = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$

since from the problem #3 we will assume that we can the cubic equation in terms of the principle stresses $\sigma_1, \sigma_2, \sigma_3$ the $(\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma - \sigma_3) = 0$ or $\sigma^3 - (\sigma_1 + \sigma_2 + \sigma_3)\sigma^2 + (\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)\sigma - (\sigma_1\sigma_2\sigma_3) = 0$ or $I_1 = \sigma_1 + \sigma_2 + \sigma_3$, $II_2 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3$, $III_3 = \sigma_1\sigma_2\sigma_3$

since $-\lambda^3 + I\lambda^2 - II\lambda + III = -\sigma^3 + I_1\sigma^2 - II_2\sigma + III_3$ $\Rightarrow$ $I = I_1$, $II = II_2$, $III = III_3$

or from above

$a = \frac{1}{3} I_1 = \frac{1}{3} I = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \sigma_{oct}$

now $9\tau_{oct}^2 = 3(\cancel{\sigma_1^2} + \cancel{\sigma_2^2} + \cancel{\sigma_3^2}) - \left[(\sigma_1 + \sigma_2)^2 + 2\sigma_3(\sigma_1 + \sigma_2) + \sigma_3^2 = \cancel{\sigma_1^2} + \cancel{\sigma_2^2} + 2\sigma_1\sigma_2 + 2\sigma_3\sigma_1 + 2\sigma_3\sigma_2 + \cancel{\sigma_3^2}\right]$

$= -2(\sigma_1\sigma_2 + \sigma_3\sigma_1 + \sigma_3\sigma_2) + 2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$

$(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2) + (\sigma_1^2 + \sigma_3^2 - 2\sigma_3\sigma_1) + (\sigma_2^2 + \sigma_3^2 - 2\sigma_3\sigma_2)$

$9\tau_{oct}^2 = 2I_1^2 - 6I_2 = 2(\sigma_1 + \sigma_2 + \sigma_3)^2 - 6(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)$

$= 2\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_3\sigma_1 + 2\sigma_3\sigma_2\right] - 6(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)$

$= (\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2) \cdots$ etc.

$\therefore$ $9\tau_{oct}^2 = 2I_1^2 - 6I_2 = 2I^2 - 6II = 2(\sigma_{11} + \sigma_{22} + \sigma_{33})^2 - 6(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{13}^2 - \sigma_{12}^2 - \sigma_{32}^2)$

$= 2\left[\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{11}\sigma_{22} + 2\sigma_{11}\sigma_{33} + 2\sigma_{22}\sigma_{33}\right] - 6\sigma$

or $9\tau_{oct}^2 = 2(\sigma_{11}^2+\sigma_{22}^2+\sigma_{33}^2) - 2(\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{11}\sigma_{33}) + 6(\sigma_{13}^2+\sigma_{12}^2+\sigma_{32}^2)$

$$= (\sigma_{11}^2+\sigma_{22}^2-2\sigma_{11}\sigma_{22}) + (\sigma_{11}^2+\sigma_{33}^2-2\sigma_{11}\sigma_{33}) + (\sigma_{22}^2+\sigma_{33}^2-2\sigma_{11}\sigma_{33}) + 6(\quad)$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_{11}-\sigma_{22})^2+(\sigma_{11}-\sigma_{33})^2+(\sigma_{22}-\sigma_{33})^2 + 6(\sigma_{13}^2+\sigma_{12}^2+\sigma_{32}^2)} \quad \text{qed.}$$

6. if we define $\tau_{ij} = \sigma_0\delta_{ij} + \tau_{ij}'$ where $\sigma_0 = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3) = \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) = \frac{1}{3}I_1$

then $\tau_{ij}' = \tau_{ij} - \sigma_0\delta_{ij}$ To determine the principal stress deviations $\Rightarrow \tau_{ij}' n_i = \lambda\delta_{ij}n_j$

Find $\tau_{ij}' - \lambda\delta_{ij} = \tau_{ij} - \sigma'\delta_{ij}$ where $\sigma' = \sigma_0 + \lambda$

$$\therefore \tau_{ij}' = \begin{pmatrix} \sigma_{11}'-\lambda' & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}'-\lambda' & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}'-\lambda' \end{pmatrix} = (\sigma_{11}'-\lambda')(\sigma_{22}'-\lambda')(\sigma_{33}'-\lambda') + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{32}\sigma_{21}$$
$$- \sigma_{13}^2(\sigma_{22}'-\lambda') - \sigma_{12}^2(\sigma_{33}'-\lambda') - \sigma_{23}^2(\sigma_{11}-\lambda')$$

$$-\lambda^3 + \lambda^2(\sigma_{11}'+\sigma_{22}'+\sigma_{33}')^{\to 0} + [\sigma_{11}'\sigma_{22}'\sigma_{33}' + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{32}\sigma_{21} - \sigma_{22}'\sigma_{13}^2 - \sigma_{12}^2\sigma_{33}'$$
$$- \sigma_{23}^2\sigma_{11}'] + \lambda[-\sigma_{13}\sigma_{31} - \sigma_{12}\sigma_{21} - \sigma_{32}\sigma_{23} + \sigma_{11}'\sigma_{22}' + \sigma_{22}'\sigma_{33}' + \sigma_{33}'\sigma_{11}'] = 0$$
$$-\lambda[II - 2\sigma_0 I + 3\sigma_0^2]$$

where
$$\sigma_{11}' = \sigma_{11} - \sigma_0$$
$$\sigma_{22}' = \sigma_{22} - \sigma_0$$
$$\sigma_{33}' = \sigma_{33} - \sigma_0$$
$$\sigma_{KK}' = \sigma_{KK} - 3\sigma_0 = \sigma_{KK} - \sigma_{KK} = 0$$

$$I_0 = 0 \qquad II_0 = II - 3\sigma_0^2$$
$$= II - 3\left(\frac{1}{9}I^2\right) = II - \frac{1}{3}I^2$$

$$\sigma_{11}'\sigma_{22}' = \sigma_{11}\sigma_{22} - \sigma_0(\sigma_{11}+\sigma_{22}) + \sigma_0^2$$
$$\sigma_{22}'\sigma_{33}' = \sigma_{22}\sigma_{33} - \sigma_0(\sigma_{22}+\sigma_{33}) + \sigma_0^2$$
$$\sigma_{33}'\sigma_{11}' = \sigma_{33}\sigma_{11} - \sigma_0(\sigma_{33}+\sigma_{11}) + \sigma_0^2$$
$$\sigma_{KK}'\sigma_{jj}' = \sigma_{KK}\sigma_{jj} - 2\sigma_0(\sigma_{11}+\sigma_{22}+\sigma_{33}) + 3\sigma_0^2$$
$$-6\sigma_0^2 + 3\sigma_0^2 = -3\sigma_0^2$$

$$\therefore -\lambda^3 + I_0^{\to 0}\lambda^2 - II_0\lambda + III_0 = 0$$

now $\sigma_{11}'\sigma_{22}'\sigma_{33}' = [\sigma_{11}\sigma_{22} - \sigma_0(\sigma_{11}+\sigma_{22}) + \sigma_0^2][\sigma_{33}-\sigma_0] = \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_0(\sigma_{11}+\sigma_{22})\sigma_{33} + \sigma_0^2\sigma_{33} - \sigma_0\sigma_{11}\sigma_{22} + \sigma_0^2(\sigma_{11}+\sigma_{22}) - \sigma_0^3$

$$= \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_0(\sigma_{11}\sigma_{33} + \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33}) + \sigma_0^2(\sigma_{11}+\sigma_{22}+\sigma_{33}) - \sigma_0^3$$
$$+ 3\sigma_0^3 - \sigma_0^3$$

then $III_0 = \{\sigma_{11}\sigma_{22}\sigma_{33} + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{32}\sigma_{21} - \sigma_{22}\sigma_{13}^2 - \sigma_{12}^2\sigma_{33} - \sigma_{23}^2\sigma_{11}\}$

$$- \sigma_0(\sigma_{11}\sigma_{33} + \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33}) + \sigma_0(\sigma_{13}^2 + \sigma_{12}^2 + \sigma_{23}^2) + 2\sigma_0^3$$

$$III - \sigma_0[II] + 2\sigma_0^3 = III - \frac{1}{3}I\cdot II + \frac{2}{27}I^3 = III - \frac{1}{3}I\left(II - \frac{2}{9}I^2\right)$$

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120/100

Problem Set #2.

1. Given σ at a point for a cartesian system and the normal n at the given point. Find the magnitude of the traction acting at the point and the magnitude of the normal and tangential components on the plane whose normal is n.

20

For

$$\sigma = \begin{pmatrix} 5 & 5 & 8 \\ 5 & 0 & -7.5 \\ 8 & -7.5 & -3 \end{pmatrix} \times 10^6$$

then we can write $t_n = n \cdot \sigma$ where t_n is the traction on the plane whose normal is n. $\sigma = e_i t_i$ & $t_i = \sigma_{ij} e_j$

$$\therefore \quad t_n = \tfrac{1}{2} t_1 + \tfrac{1}{2} t_2 + \tfrac{1}{\sqrt{2}} t_3 \quad \text{where } n = \tfrac{1}{2}[e_1 + e_2 + \sqrt{2}\, e_3]$$

or $t_n = (10.657 \times 10^6) e_1 + (-2.803 \times 10^6) e_2 + (-1.871 \times 10^6) e_3 = \alpha_i e_i$

Thus $|t_n| = (\alpha_i \cdot \alpha_i)^{1/2} = [(10.657)^2 + (-2.803)^2 + (-1.871)^2]^{1/2} \times 10^6$

$= 11.177 \times 10^6$ Pa $\approx 11.2 \times 10^6$ Pa.

Since $t_n = (n \cdot t_n) n + (\pi \cdot t_n) \pi$ where π is the unit vector tangent to the plane now if we take $n \cdot t_n$ = magnitude of t_n in the normal direction. To find $\pi \cdot t_n$ = mag. of t_n in the tangential direction use the fact that $|t_n|^2 = (n \cdot t_n)^2 + (\pi \cdot t_n)^2$ and

$$(\pi \cdot t_n) = \{|t_n|^2 - (n \cdot t_n)^2\}^{1/2}$$

$$\therefore \quad n \cdot t_n = \tfrac{1}{2}(10.657 \times 10^6) + \tfrac{1}{2}(-2.803 \times 10^6) + \tfrac{1}{\sqrt{2}}(-1.871 \times 10^6)$$

$= 2.605 \times 10^6$ Pa $\approx 2.61 \times 10^6$ Pa

$$\therefore \quad \pi \cdot t_n = \{(11.2 \times 10^6)^2 - (2.61 \times 10^6)^2\}^{1/2} = 10.892 \times 10^6 \text{ Pa} \approx 10.9 \times 10^6 \text{ Pa}$$

2.

Given $\begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & \sigma_{yz} \\ 0 & \sigma_{zy} & 0 \end{pmatrix}$ in the cartesian coordinate system for a given point on the boundary whose direction cosines are l, m, n. The magnitude of the surface traction is $\underline{\underline{a}}$ and is parallel to x. Find σ_{zy}, σ_{yy}, σ_{xx}

We note that at the surface the normal $\mathbb{n}\left(= l\,\mathbb{e}_x + m\,\mathbb{e}_y + n\,\mathbb{e}_z\right)$ can be dotted unto σ to get the traction at the point in the $\mathbb{n}$ dir

$$\therefore \quad t_n = \mathbb{n}\cdot\sigma = l\,\mathbb{t}_1 + m\,\mathbb{t}_2 + n\,\mathbb{t}_3 \qquad \text{where } \sigma = \mathbb{t}_i\,\mathbb{e}_i$$

$$= l\sigma_{xx}\,\mathbb{e}_x + m\left(\sigma_{yy}\,\mathbb{e}_y + \sigma_{yz}\,\mathbb{e}_z\right) + n\,\sigma_{zy}\,\mathbb{e}_y$$

now since $\mathbb{t}_n \parallel \mathbb{e}_x \Rightarrow \quad \mathbb{t}_n = \alpha\,\mathbb{e}_x + \beta\,\mathbb{e}_y + \gamma\,\mathbb{e}_z$ where $\beta = \gamma = 0$

but $\beta = m\sigma_{yy} + n\sigma_{zy} = 0$ and $m\sigma_{yz} = \gamma = 0$ or $\sigma_{yz} = 0$ if $m \neq 0$ (which will be true in general unless the normal through the boundary is $\perp$ to the y axis). But $\sigma_{yz} = \sigma_{zy}$ and $\Rightarrow \sigma_{yy} = 0$.

We therefore have $\boxed{\sigma_{yy} = \sigma_{yz} = \sigma_{zy} = 0}$. We are also told that $|\mathbb{t}_n| = a$. But

$$|\mathbb{t}_n| = a = l\,\sigma_{xx} \quad \Rightarrow \quad \boxed{\sigma_{xx} = a/l}$$

3. Verify I, II, III in terms of stress using indicial notation.

We are going to use the case where $\sum_{i=1}^{3}$ (ie the three coordenate axes)

From $\sigma\cdot\mathbb{n} - \lambda\,\mathbb{I}\cdot\mathbb{n} = 0$ we obtained the following determinant

$$\begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda \end{vmatrix} = 0$$

; When expanding we obtain: $(\sigma_{11}-\lambda)(\sigma_{33}-\lambda)(\sigma_{22}-\lambda) + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{32}\sigma_{21} - \sigma_{13}\sigma_{31}(\sigma_{22}-\lambda) - \sigma_{12}\sigma_{21}(\sigma_{33}-\lambda) - \sigma_{32}\sigma_{23}(\sigma_{11}-\lambda) = 0$

when expanded and rearranged we obtain $-\lambda^3 + I\lambda^2 - II\lambda + III = 0$, where

$I = (\sigma_{11} + \sigma_{22} + \sigma_{33})$ or $\boxed{I = \sigma_{kk}}$

$II = -\sigma_{13}\sigma_{31} - \sigma_{12}\sigma_{21} - \sigma_{32}\sigma_{23} + \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33}$

$III = \sigma_{11}\sigma_{22}\sigma_{33} + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{32}\sigma_{21} - \sigma_{13}\sigma_{22}\sigma_{31} - \sigma_{12}\sigma_{21}\sigma_{33} - \sigma_{11}\sigma_{32}\sigma_{23}$

now $II = \frac{1}{2}\,e_{mik}\,e_{njl}\,\sigma_{ij}\,\sigma_{kl}\,\delta_{mn}$ and we will prove this by expanding it

$\mathrm{II} \overset{?}{=} \frac{1}{2} e_{mik} e_{njl} \sigma_{ij} \sigma_{kl} \delta_{mn} = \frac{1}{2} e_{mik} e_{mjl} \sigma_{ij} \sigma_{kl}$ since $\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$

now note that e_{mik}, e_{mjl} only give terms when mik or mjl are $(123, 231, 312, 321, 213, 132)$

Thus the only terms that come into being are:

$$\frac{1}{2}\left[e_{123} e_{123} \sigma_{22}\sigma_{33} + e_{123} e_{132} \sigma_{23}\sigma_{32} + e_{231} e_{231} \sigma_{33}\sigma_{11} + e_{231} e_{213} \sigma_{31}\sigma_{13} + e_{312} e_{312} \sigma_{11}\sigma_{22} + e_{312} e_{321} \sigma_{12}\sigma_{21} \right.$$
$$\left. + e_{321} e_{312} \sigma_{21}\sigma_{12} + e_{321} e_{321} \sigma_{22}\sigma_{11} + e_{213} e_{231} \sigma_{13}\sigma_{31} + e_{213} e_{213} \sigma_{11}\sigma_{33} + e_{132} e_{123} \sigma_{32}\sigma_{23} + e_{132} e_{132} \sigma_{33}\sigma_{22} \right]$$

Knowing that $e_{123,231,312} = 1$ and that $e_{321,213,132} = -1$ we obtain the result that

$$\frac{1}{2} e_{mik} e_{njl} \sigma_{kl} \sigma_{ij} \delta_{mn} = \frac{1}{2}\left[2\sigma_{22}\sigma_{33} + 2\sigma_{33}\sigma_{11} + 2\sigma_{11}\sigma_{22} - 2\sigma_{13}\sigma_{31} - 2\sigma_{12}\sigma_{21} - 2\sigma_{32}\sigma_{23} \right] \equiv \mathrm{II}$$

$$\therefore \quad \boxed{\mathrm{II} = \frac{1}{2} e_{mik} e_{njl} \sigma_{kl} \sigma_{ij} \delta_{mn}}$$

Now $\mathrm{III} = \frac{1}{6} e_{mik} e_{njl} \sigma_{ij} \sigma_{kl} \sigma_{mn}$ and we will prove this by expansion also

again we note that e_{mik}, e_{njl} only give nonzero terms when mik, njl are $(123, 231, 312, 321, 213, 132)$

Thus we obtain for $\frac{1}{6} e_{mik} e_{njl} \sigma_{ij} \sigma_{kl} \sigma_{mn}$:

$$\frac{1}{6}\left[\left(\sigma_{11}\sigma_{22}\sigma_{33} + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{21}\sigma_{32} - \sigma_{13}\sigma_{22}\sigma_{31} - \sigma_{12}\sigma_{21}\sigma_{33} - \sigma_{11}\sigma_{23}\sigma_{32} \right) + \left(\sigma_{21}\sigma_{32}\sigma_{13} + \sigma_{22}\sigma_{33}\sigma_{11} + \sigma_{23}\sigma_{31}\sigma_{12} \right. \right.$$
$$\left. - \sigma_{23}\sigma_{32}\sigma_{11} - \sigma_{22}\sigma_{31}\sigma_{13} - \sigma_{21}\sigma_{33}\sigma_{12} \right) + \left(\sigma_{31}\sigma_{12}\sigma_{23} + \sigma_{32}\sigma_{13}\sigma_{21} + \sigma_{33}\sigma_{11}\sigma_{22} - \sigma_{33}\sigma_{12}\sigma_{21} - \sigma_{32}\sigma_{11}\sigma_{23} - \sigma_{31}\sigma_{13}\sigma_{22} \right)$$
$$- \left(\sigma_{31}\sigma_{22}\sigma_{13} + \sigma_{32}\sigma_{23}\sigma_{11} + \sigma_{33}\sigma_{21}\sigma_{12} - \sigma_{33}\sigma_{22}\sigma_{11} - \sigma_{32}\sigma_{21}\sigma_{13} - \sigma_{31}\sigma_{23}\sigma_{12} \right) - \left(\sigma_{21}\sigma_{12}\sigma_{33} + \sigma_{22}\sigma_{13}\sigma_{31} + \sigma_{23}\sigma_{11}\sigma_{32} \right.$$
$$\left. \left. - \sigma_{23}\sigma_{12}\sigma_{31} - \sigma_{22}\sigma_{11}\sigma_{33} - \sigma_{21}\sigma_{13}\sigma_{32} \right) - \left(\sigma_{11}\sigma_{32}\sigma_{23} + \sigma_{12}\sigma_{33}\sigma_{21} + \sigma_{13}\sigma_{31}\sigma_{22} - \sigma_{13}\sigma_{32}\sigma_{21} - \sigma_{12}\sigma_{31}\sigma_{23} - \sigma_{11}\sigma_{33}\sigma_{22} \right) \right]$$

$$\therefore \quad \frac{1}{6} e_{mik} e_{njl} \sigma_{ij} \sigma_{kl} \sigma_{mn} = \frac{1}{6}\left[6\sigma_{11}\sigma_{22}\sigma_{33} + 6\sigma_{12}\sigma_{23}\sigma_{31} + 6\sigma_{13}\sigma_{32}\sigma_{21} - 6\sigma_{13}\sigma_{22}\sigma_{31} - 6\sigma_{12}\sigma_{21}\sigma_{33} - 6\sigma_{11}\sigma_{23}\sigma_{32} \right] \equiv \mathrm{III}$$

$$\therefore \quad \boxed{\mathrm{III} = \frac{1}{6} e_{mik} e_{njl} \sigma_{kl} \sigma_{ij} \sigma_{mn}}$$

+20 ex credit

4. Given that the tractions on 3 separate planes are perpendicular to the plane and that the normals of these 3 planes are not coplanar.

Since $\mathbf{t}_n$ is given $\perp$ to its plane, then $\mathbf{t}_n$ is proportional to $\mathbf{n}$, the normal to the plane. Therefore if $\mathbf{t}_{n_i} = \mathbf{n}_i \cdot \boldsymbol{\sigma} = \alpha_j \mathbf{n}_i \delta_{ij}$ $\quad i = 1,2,3$ then

$$\mathbf{n}_2 \times \mathbf{t}_{n_1} = \mathbf{n}_2 \times \mathbf{n}_1 \cdot \boldsymbol{\sigma} = \alpha_1 \mathbf{n}_2 \times \mathbf{n}_1 \quad \underline{\text{and}} \quad \mathbf{n}_3 \cdot \mathbf{n}_2 \times \mathbf{t}_{n_1} = \mathbf{n}_3 \cdot \mathbf{n}_2 \times \mathbf{n}_1 \cdot \boldsymbol{\sigma} = \alpha_1 \mathbf{n}_3 \cdot \mathbf{n}_2 \times \mathbf{n}_1$$

but we can write

$$\mathbf{n}_3 \times \mathbf{t}_{n_2} = \mathbf{n}_3 \times \mathbf{n}_2 \cdot \boldsymbol{\sigma} = \alpha_2 \mathbf{n}_3 \times \mathbf{n}_2 \quad \underline{\text{and}} \quad \mathbf{n}_1 \cdot \mathbf{n}_3 \times \mathbf{t}_{n_2} = \mathbf{n}_1 \cdot \mathbf{n}_3 \times \mathbf{n}_2 \cdot \boldsymbol{\sigma} = \alpha_2 \mathbf{n}_1 \cdot \mathbf{n}_3 \times \mathbf{n}_2$$

$\underline{\text{but}}$ $\quad \mathbf{n}_3 \cdot \mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{n}_1 \cdot \mathbf{n}_3 \times \mathbf{n}_2$

$$\therefore \quad \mathbf{n}_3 \cdot \mathbf{n}_2 \times \mathbf{t}_{n_1} - \mathbf{n}_1 \cdot \mathbf{n}_3 \times \mathbf{t}_{n_2} = (\mathbf{n}_3 \cdot \mathbf{n}_2 \times \mathbf{n}_1 - \mathbf{n}_1 \cdot \mathbf{n}_3 \times \mathbf{n}_2) \cdot \boldsymbol{\sigma} = (\alpha_1 - \alpha_2)\, \mathbf{n}_3 \cdot \mathbf{n}_1 \times \mathbf{n}_2 = 0 \cdot \boldsymbol{\sigma} = 0$$

however since $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are not coplanar $\mathbf{n}_3 \cdot \mathbf{n}_1 \times \mathbf{n}_2 \neq 0^*$ $\Rightarrow$ $\alpha_1 = \alpha_2$

We can also show in a similar manner that $\alpha_2 = \alpha_3$ and $\alpha_1 = \alpha_3$ or $\alpha_1 = \alpha_2 = \alpha_3 = -p$

Since we did not choose the planes in any particular manner, the results shown are true for any 3 planes whose normals are not coplanar. Thus $\boldsymbol{\sigma} = -p\mathbf{I}$. This result is exactly that of hydrostatics. Now from the principle of linear momentum

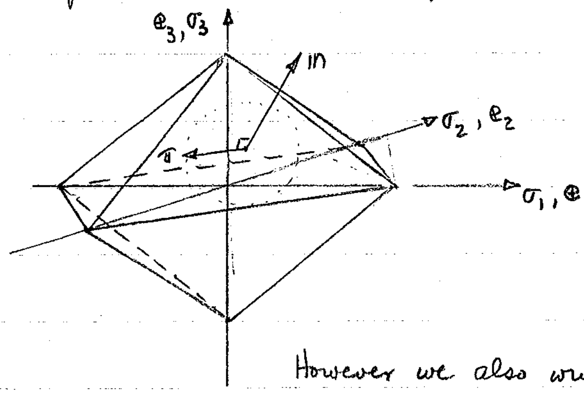
* to show this we note that $\mathbf{n}_1 \times \mathbf{n}_2 = \underset{\sim}{\xi}$ a vector $\perp$ to both $\mathbf{n}_1$ & $\mathbf{n}_2$. Thus if $\mathbf{n}_3$ were coplanar then $\underset{\sim}{\xi}$ would also be $\perp$ to $\mathbf{n}_3$. Hence $\mathbf{n}_3 \cdot \mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{n}_3 \cdot \underset{\sim}{\xi} = |\mathbf{n}_3||\underset{\sim}{\xi}| \cos(\mathbf{n}_3, \underset{\sim}{\xi})$ where $\cos(\mathbf{n}_3, \underset{\sim}{\xi}) = \cos\frac{\pi}{2} = 0$; thus $\mathbf{n}_3 \cdot \mathbf{n}_1 \times \mathbf{n}_2 = 0$ for coplanarity.

Using the principle of linear momentum that the rate of change of momentum in a given control volume = sum of all the forces acting on the volume. Since the rate of change of momentum is $\frac{\partial}{\partial t}\int_V \rho u_i \, dV$, and the forces acting on the volume in general are pressure forces and body forces, then we can write, noting that ρ for materials we talk about is constant and for fixed volume

$$\int_V \rho \ddot{u}_i \, dV = \int_V -\frac{\partial p}{\partial x_i} dV + \int f_i \, dV \quad \text{or} \quad \int_V \left[\rho \ddot{u}_i + \frac{\partial p}{\partial x_i} - f_i\right] dV = 0$$

or $\rho \ddot{u}_i = -\frac{\partial p}{\partial x_i} + f_i$ for arbitrary dV.

5. Find the normal and shear stress on an octahedron. (A general outline of the results is given in <u>Foundations of Solid Mechanics</u> by Fung)



For an octahedral, the normal has equal directional cosines with the principal directions and thus $\mathbb{n} = \frac{1}{\sqrt{3}}(\mathbb{e}_1 + \mathbb{e}_2 + \mathbb{e}_3)$

Since we know that for this case

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \quad \text{then} \quad \mathbb{t}_n = \mathbb{n}\cdot\boldsymbol{\sigma} = \frac{1}{\sqrt{3}}(\sigma_1 \mathbb{e}_1 + \sigma_2 \mathbb{e}_2 + \sigma_3 \mathbb{e}_3$$

which gives the stress tensor on the face of the octahedral

However we also write $\mathbb{t}_n = a\,\mathbb{n} + b\,\pi$ where $a = \sigma_{oct}$ and $b = \tau_{oct}$

Thus $a = \mathbb{n}\cdot\mathbb{t}_n = \frac{1}{\sqrt{3}}(\mathbb{e}_1 + \mathbb{e}_2 + \mathbb{e}_3)\cdot\frac{1}{\sqrt{3}}(\sigma_1 \mathbb{e}_1 + \sigma_2 \mathbb{e}_2 + \sigma_3 \mathbb{e}_3) = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$

$$\therefore \quad \boxed{\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)}$$

we pick π to be a unit vector $\perp$ to $\mathbb{n}$. Now as in problem 1 we can find b from

$$b = \sqrt{|\mathbb{t}_n|^2 - a^2} \quad \text{where} \quad |\mathbb{t}_n|^2 = \mathbb{t}_n\cdot\mathbb{t}_n = \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

Thus $b = \sqrt{\frac{1}{3}(\sigma_1^2+\sigma_2^2+\sigma_3^2) - \frac{1}{9}(\sigma_1+\sigma_2+\sigma_3)^2}$. After expansion and rearrangement of terms

$$\boxed{\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}}$$

To obtain the other expressions we go back to problem #3 and realize that we can write the equation $-\lambda^3 + I\lambda^2 - II\lambda + III = 0$ as $-(\lambda-\sigma_1)(\lambda-\sigma_2)(\lambda-\sigma_3) = 0$

By expansion of this equation we get that $-\lambda^3 + I_1\lambda^2 - II_2\lambda + III_3 = 0$ where

$I_1 = \sigma_1 + \sigma_2 + \sigma_3$; $II_2 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3$; $III_3 = \sigma_1\sigma_2\sigma_3$. Now since $f(\lambda, I, II, III)$ equals $g(\lambda, I_1, II_2, III_3)$ then $I = I_1$, $II = II_2$, $III = III_3$

$$\therefore \quad \boxed{\text{since } \sigma_{oct} = \frac{1}{3}I_1 = \frac{1}{3}I = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})}$$

Since $\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}$ then we will show that $9\tau_{oct}^2 = 2I_1^2 - 6II_2$

but $9\tau_{oct}^2 = 2\sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 - 2(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3) = 2[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)] - 6(\sigma_1\sigma_3 + \sigma_1\sigma$ $+ \sigma_2\sigma_3) = 2I_1^2 - 6II_2$. Since $I_1 = I$ and $II_2 = II$ then $\boxed{9\tau_{oct}^2 = 2I^2 - 6II = 2I_1^2 - 6II_2}$

or $9\tau_{oct}^2 = 2(\sigma_{11} + \sigma_{22} + \sigma_{33})^2 - 6(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2)$. When we expand this, cancel and collect terms we get $9\tau_{oct}^2 = (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2 + 6(\sigma_{13}^2 + \sigma_{12}^2 + \sigma_{3}$

20

solving for τ_{oct} we get

$$\boxed{\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{11}-\sigma_{33})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2)}}$$

6. Find the invariants of the stress deviator (again found in Fung)

If we decompose $\boldsymbol{\sigma} = \sigma_0 \hat{\mathbb{I}} + \mathbb{T}$ where $\sigma_0 = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3) = \frac{1}{3}\mathrm{I}$, then $\mathbb{T} = \boldsymbol{\sigma} - \sigma_0 \hat{\mathbb{I}}$ where $\mathbb{T}$ is the stress deviator and $\hat{\mathbb{I}}$ is the idemfactor. In indicial notation $\tau_{ij} = \sigma_{ij} - \sigma_0 \delta_{ij}$

Again we do what we did to find the principal stress – we find the principal deviations. We will note that the principal stress deviation occurs when parallel to the vector normal to the plane where the stress deviator acts. Thus $\mathbb{T}\cdot \mathbf{n} = \lambda \mathbf{n}$ or $\tau_{ij} n_i = \lambda n_j = \lambda \delta_{ij} n_i$

Hence $(\tau_{ij} - \lambda \delta_{ij}) n_i = 0$ or $(\mathbb{T} - \lambda \hat{\mathbb{I}})\cdot \mathbf{n} = 0$

For a solution to exist $\det(\mathbb{T} - \lambda \hat{\mathbb{I}}) = 0$ which will in a cubic equation in λ of the form $-\lambda^3 + \mathrm{I}_0 \lambda^2 - \mathrm{II}_0 \lambda + \mathrm{III}_0 = 0$ where, if we define $\sigma'_{ii} = \sigma_{ii} - \sigma_0$ (ie $\tau_{ii} = \sigma'_{ii} - \lambda$),

$$\mathrm{I}_0 = (\sigma'_{11} + \sigma'_{22} + \sigma'_{33}) = \sigma_{11}+\sigma_{22}+\sigma_{33} - 3\sigma_0 = \mathrm{I} - 3\left(\tfrac{1}{3}\mathrm{I}\right) = 0 \qquad \therefore \boxed{\mathrm{I}_0 = 0};$$

$$\mathrm{II}_0 = \sigma'_{11}\sigma'_{22} + \sigma'_{22}\sigma'_{33} + \sigma'_{33}\sigma'_{11} - (\sigma_{13}^2 + \sigma_{12}^2 + \sigma_{23}^2) = \left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - [\sigma_{13}^2+\sigma_{12}^2+\sigma_{23}^2]\right) + 3\sigma_0^2$$
$$- 2\sigma_0(\sigma_{11}+\sigma_{22}+\sigma_{33})$$ when we expand. But the first bracket is II and $\sigma_{11}+\sigma_{22}+\sigma_{33} = 3\sigma_0$

Thus $\mathrm{II}_0 = \mathrm{II} + 3\sigma_0^2 - 2\sigma_0(3\sigma_0) = \mathrm{II} - 3\sigma_0^2 = \mathrm{II} - 3\left(\frac{1}{9}\mathrm{I}^2\right) = \mathrm{II} - \frac{1}{3}\mathrm{I}^2$

$$\therefore \boxed{\mathrm{II}_0 = \mathrm{II} - \frac{1}{3}\mathrm{I}^2};$$

$$\mathrm{III}_0 = \sigma'_{11}\sigma'_{22}\sigma'_{33} + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{32}\sigma_{21} - \sigma'_{22}\sigma_{13}^2 - \sigma_{12}^2\sigma'_{33} - \sigma_{23}^2\sigma'_{11} = [\sigma_{11}\sigma_{22}\sigma_{33}] - \sigma_0(\sigma_{11}\sigma_{33} + \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33})$$
$$+ \sigma_0^2(\sigma_{11}+\sigma_{22}+\sigma_{33}) - \sigma_0^3 + \left[\sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{32}\sigma_{21} - (\sigma_{22}\sigma_{13}^2 + \sigma_{12}^2\sigma_{33} + \sigma_{23}^2\sigma_{11})\right] + \sigma_0^2(\sigma_{13}^2+\sigma_{12}^2+\sigma_{23}^2)$$
$$= \mathrm{III} - \sigma_0\mathrm{II} + \sigma_0^2(3\sigma_0) - \sigma_0^3 = \mathrm{III} - \sigma_0\mathrm{II} + 2\sigma_0^3 = \mathrm{III} - \sigma_0(\mathrm{II} - 2\sigma_0^2) = \mathrm{III} - \frac{1}{3}\mathrm{I}\left(\mathrm{II} - \frac{2}{9}\mathrm{I}^2\right)$$

$$\therefore \boxed{\mathrm{III}_0 = \mathrm{III} - \frac{1}{3}\mathrm{I}\left(\mathrm{II} - \frac{2}{9}\mathrm{I}^2\right)}$$

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1. Components of σ with respect to Cartesian coordinates: $\sigma = 10^6 Pa \begin{bmatrix} 5 & 5 & 8 \\ 5 & 0 & -7.5 \\ 8 & -7.5 & -3 \end{bmatrix}$

also given $n = \frac{1}{2} e_x + \frac{1}{2} e_y + \frac{\sqrt{2}}{2} e_z$

where n is the unit normal to the plane under study

a) $t^n = n \cdot \sigma$

<u>In Dyadic Form</u> $t^n = \left(\frac{1}{2} e_x + \frac{1}{2} e_y + \frac{\sqrt{2}}{2} e_z\right) \cdot \overset{10^6 Pa}{\wedge}\left(5 e_x e_x + 5 e_x e_y + 8 e_x e_z + 5 e_y e_x - 7.5 e_y e_z + 8 e_z e_x - 7.5 e_z e_y - 3 e_z e_z\right)$

$= (10.66\, e_x - 2.8\, e_y - 1.87\, e_z) 10^6 Pa$

<u>In indicial form</u>

$t_i^n = \sigma_{ji} n_j \qquad t_1^n = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$

$= 10^6 Pa \left(\frac{5}{2} + \frac{5}{2} + \frac{8}{\sqrt{2}}\right) = 10.66 \times 10^6 Pa$

$t_2^n = \sigma_{j2} n_j = 10^6 Pa \left(\frac{5}{2} + 0 - \frac{7.5}{\sqrt{2}}\right) = -2.8 \times 10^6 Pa$

$t_3^n = \sigma_{j3} n_j = 10^6 Pa \left(\frac{8}{2} - \frac{7.5}{2} - \frac{3}{\sqrt{2}}\right) = -1.87 \times 10^6$

<u>In Matrix Form</u>

$t^n = n \cdot \sigma = [n]^T [\sigma] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 5 & 5 & 8 \\ 5 & 0 & -7.5 \\ 8 & -7.5 & -3 \end{bmatrix} 10^6 Pa$

$= \begin{bmatrix} 10.66 \\ -2.8 \\ -1.87 \end{bmatrix} \times 10^6 Pa$

$|t^n| = \sqrt{t^n \cdot t^n} = \left(\sqrt{10.66^2 + 2.8^2 + 1.87^2}\right) \times 10^6 Pa = \underline{\underline{11.2 \times 10^6 Pa}}$

b) $t_n^n \equiv \sigma_{nn} = t^n \cdot n$

<u>in Dyadic Form</u> $t^n \cdot n = (t_1^n e_1 + t_2^n e_2 + t_3^n e_3) \cdot \left(\frac{1}{2} e_1 + \frac{1}{2} e_2 + \frac{1}{\sqrt{2}}\right.$

$= 10^6 Pa \left(5.33 - 1.4 - \frac{1.87}{\sqrt{2}}\right) = 2.61 \times 10^6 Pa$

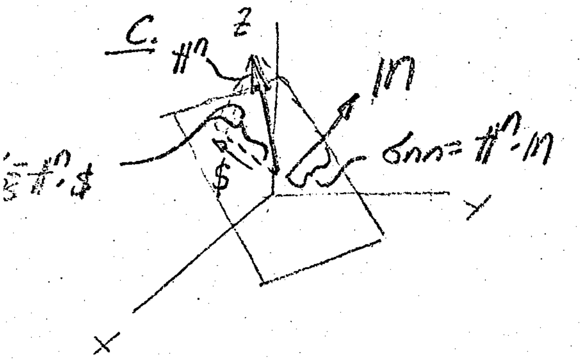
<u>In Indicial Form</u>

$t^n \cdot n = t_i^n n_i = t_1^n \underbrace{n_1}_{1/2} + t_2^n \underbrace{n_2}_{1/2} + t_3^n \underbrace{n_3}_{\sqrt{2}/2} = 2.61 \times 10^6 Pa$

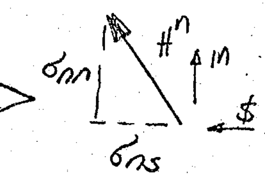
in matrix form $t^n \cdot n = [t^n]^T [n]$ (2

$$= 10^6 Pa \begin{bmatrix} 10.66 & -2.8 & -1.87 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ \sqrt{2}/2 \end{bmatrix} = 2.61 \times 10^6 Pa$$

C.



$\Rightarrow$



$$t^{n\,2} = \sigma_{ns}^2 + \sigma_{nn}^2 \qquad \sigma_{ns} = \sqrt{(t^n)^2 - \sigma_{nn}^2} = \underline{\underline{10.9 \times 10^6 Pa}}$$

If direction were also desired, note vector sum

$$\sigma_{nn}\, n + \sigma_{ns}\, s = t^n \quad \text{or} \quad (t^n \cdot n)\, n + (t^n \cdot s)\, s = t^n$$

$$\cancel{10^6 Pa}\,(2.61)\left(\tfrac{1}{2} e_1 + \tfrac{1}{2} e_2 + \tfrac{\sqrt{2}}{2} e_3\right) + 10.9 \times \cancel{10^6} Pa\,(s_1 e_1 + s_2 e_2 + s_3 e_3) = \cancel{10^6 Pa}\,(10.66 e_1 - 2.8 e_2 - 1.87 e_3)$$

$$s_1 = \frac{1}{10.9}\left(10.66 - \frac{2.61}{2}\right) = +.858$$

$$s_2 = \frac{1}{10.9}\left(-2.8 - \frac{2.61}{2}\right) = -.376$$

$$s_3 = \frac{1}{10.9}\left(-1.87 - \frac{2.61}{\sqrt{2}}\right) = -.341$$

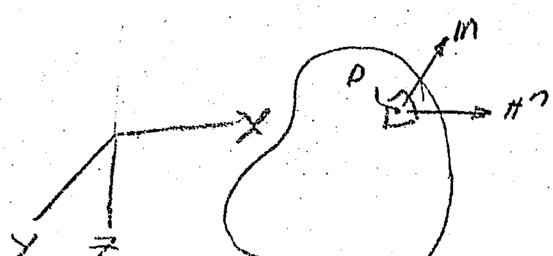
$$s = +.858\, e_1 - .376\, e_2 - .341\, e_3$$

2.

Given $n = l e_x + m e_y + n e_z$, $t^n = a e_x$

and $\Phi = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & \sigma_{zy} \\ 0 & \sigma_{zy} & 0 \end{bmatrix}$

assume l, m, n do not $= 0$



Dyadic Form: $t^n = n \cdot \Phi = (l e_x + m e_y + n e_z) \cdot (\sigma_{xx} e_x e_x + \sigma_{yy} e_y e_y + \sigma_{zy} e_y e_z + \sigma_{zy} e_z e_y)$

$$= l \sigma_{xx} e_x + (m \sigma_{yy} + n \sigma_{yz}) e_z + m \sigma_{yz} e_y = a e_x$$

$\therefore\ l \sigma_{xx} = a$, $\sigma_{xx} = a/l$; $m \sigma_{yz} = 0 \Rightarrow \sigma_{yz} = 0$

$m \sigma_{yy} + n \sigma_{yz} = 0 \Rightarrow \sigma_{yy} = 0$

Indicial Form: $t_i^n = \sigma_{ji} n_j$ (noting $x = x_1$, $y = x_2$, $z = x_3$)

$t_1^n = \sigma_{j1} n_j = l \sigma_{11} = a$ $\quad t_3^n = \sigma_{j3} n_j = m \sigma_{23} = 0$ $\quad t_2^n = \sigma_{j2} n_j = m \sigma_{22} + n \sigma_{23}$

$\Rightarrow \sigma = a/l$

(3

in matrix form, $\mathbb{n} = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

$\mathbb{t}^n = [a \; 0 \; 0]$ and $\quad \sigma = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & \sigma_{yz} \\ 0 & \sigma_{yz} & 0 \end{bmatrix}$

$$\mathbb{t}^n = \mathbb{n} \cdot \sigma = \mathbb{n}^T \sigma = [l \; m \; n] \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & \sigma_{yz} \\ 0 & \sigma_{yz} & 0 \end{bmatrix} = \begin{bmatrix} l\sigma_{xx} & m\sigma_{yy} + n\sigma_{yz} & m\sigma_{yz} \end{bmatrix}$$

$$= [a \quad 0 \quad 0]$$

$$\Rightarrow \sigma_{xx} = \frac{a}{l} \quad \sigma_{yy} = \sigma_{yz} = 0$$

3.)

$$\det \begin{vmatrix} \sigma_{xx} - \lambda & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \lambda & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \lambda \end{vmatrix} = 0$$

$$= (\sigma_{xx} - \lambda)\left[(\sigma_{yy} - \lambda)(\sigma_{zz} - \lambda) - \sigma_{zy}\sigma_{yz}\right] + \sigma_{xz}\left[\sigma_{yx}\sigma_{zy} - \sigma_{zx}(\sigma_{yy} - \lambda)\right]$$
$$- \sigma_{xy}\left[\sigma_{yx}(\sigma_{zz} - \lambda) - \sigma_{zx}\sigma_{yz}\right]$$

multiplying out, $\lambda^3 + \lambda^2 \overbrace{(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})}^{I} + \lambda \overbrace{(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{zx}^2)}^{II}$

$$+ \overbrace{(\sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2)}^{III} = 0$$

4) Given: a) for arbitrary unit vectors $\mathbb{a}, \mathbb{b}, \mathbb{c}$, normal to planes A, B, C, respectively,

or $\quad \mathbb{a} \cdot \sigma = \sigma_a \mathbb{a}; \quad \mathbb{b} \cdot \sigma = \sigma_b \mathbb{b}; \quad \mathbb{c} \cdot \sigma = \sigma_c \mathbb{c}$

(4)

∴ . . or in indicial form

$$\mathbf{a}\cdot\boldsymbol{\phi} = a_k \mathbf{e}_k \cdot \sigma_{ij}\mathbf{e}_i \mathbf{e}_j = a_k \sigma_{ij} \delta_{ki} \mathbf{e}_j$$
$$= a_k \sigma_{kj} \mathbf{e}_j = \sigma_1 a_j \mathbf{e}_j$$

$$\Rightarrow a_k \sigma_{kj} = \sigma_1 a_j \Rightarrow (\sigma_{kj} - \delta_{kj}\sigma_1) a_k = 0 \quad (1)$$

similarly, $b_k \sigma_{kj} = \sigma_2 b_j \Rightarrow (\sigma_{kj} - \delta_{kj}\sigma_2) b_k = 0 \quad (2)$

$c_k \sigma_{kj} = \sigma_3 c_j \Rightarrow (\sigma_{kj} - \delta_{kj}\sigma_3) c_k = 0 \quad (3)$

∴ multiply (1) · b_j, (2) · a_j, and subtract

$$\Rightarrow (\sigma_{kj} - \delta_{kj}\sigma_1) a_k b_j - (\sigma_{kj} - \delta_{kj}\sigma_2) b_k a_j = 0$$

$$\Rightarrow (\sigma_2 - \sigma_1)\delta_{jk} a_k b_j = 0 \quad (\text{using } \sigma_{jk} = \sigma_{kj})$$

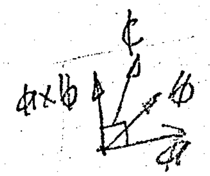
$$\Rightarrow (\sigma_2 - \sigma_1) a_j b_j = 0$$

$\mathbf{a}, \mathbf{b}$ are arbitrary ∴ $\mathbf{a}\cdot\mathbf{b} = a_j b_j$ is not necessarily zero

∴ $\sigma_2 = \sigma_1$

similarly $\sigma_3 = \sigma_2 \Rightarrow \sigma_1 = \sigma_2 = \sigma_3 = \sigma \quad (4)$

(b) $\mathbf{c}$ is not in plane formed by $\mathbf{a}$ & $\mathbf{b}$



$$\therefore (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \neq 0$$

$$\Rightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0 \quad (5)$$

subst from (4) back into (1), (2), (3)

$$\Rightarrow (\sigma_{kj} - \delta_{kj}\sigma) a_k = 0 \quad (6)$$
$$(\sigma_{kj} - \delta_{kj}\sigma) b_k = 0 \quad (7)$$
$$(\sigma_{kj} - \delta_{kj}\sigma) c_k = 0 \quad (8)$$

for $j=1$, (6); (7); (8) become

$$(\sigma_{11}-\sigma)a_1 + \sigma_{12}a_2 + \sigma_{13}a_3 = 0$$

$$(\sigma_{11}-\sigma)b_1 + \sigma_{12}b_2 + \sigma_{13}b_3 = 0$$

$$(\sigma_{11}-\sigma)c_1 + \sigma_{12}c_2 + \sigma_{13}c_3 = 0$$

or
$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} \sigma_{11}-\sigma \\ \sigma_{12} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (7)$$

3 homogeneous eqns. Nontrivial solution only if

$$\det\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$ but this is not true by (5)

∴ only solution to (7) is the "trivial" solution

$$\sigma_{11}-\sigma=0 \Rightarrow \sigma_{11}=\sigma \quad \sigma_{12}=0 \quad \sigma_{13}=0$$

similarly for $j=2$ & $j=3$ $\Rightarrow \sigma_{22}=\sigma_{33}=\sigma$, $\sigma_{23}=0$

<u>Equation of motion</u>

In general $\sigma_{ij,i} + \rho f_j = \rho \ddot{u}_j$ (8)

here $\sigma_{ij} = \sigma\delta_{ij}$

into (8) $\Rightarrow (\sigma\delta_{ij})_{,i} + \rho f_j = \rho\ddot{u}_j$

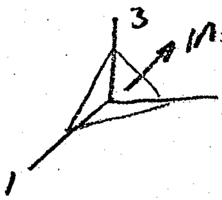
$$\sigma_{,i}\delta_{ij} + \sigma\cancel{\delta_{ij,i}}^{0} + \rho f_j = \rho\ddot{u}_j$$

$$\boxed{\sigma_{,j} + \rho f_j = \rho\ddot{u}_j}$$ which are the Euler Equations

5. The equation for a unit vector with like orientation with respect to the three principal directions is

$$\mathbf{n} = \pm\frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$$

Choosing the plus sign places this vector in the first "octant"



$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$

$$t^n = \mathbf{n}\cdot\boldsymbol{\sigma}$$

$$= \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)\cdot(\sigma_1\mathbf{e}_1\mathbf{e}_1 + \sigma_2\mathbf{e}_2\mathbf{e}_2 + \sigma_3\mathbf{e}_3\mathbf{e}_3)$$

$$= \frac{1}{\sqrt{3}}(\sigma_1\mathbf{e}_1 + \sigma_2\mathbf{e}_2 + \sigma_3\mathbf{e}_3)$$

$$\sigma_{oct} = \sigma_{nn} = \mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n} = \frac{1}{\sqrt{3}}(\sigma_1\mathbf{e}_1 + \sigma_2\mathbf{e}_2 + \sigma_3\mathbf{e}_3)\cdot\frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$$

$$= \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \qquad (a)$$

noting that $I = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{kk}$ is invariant under rotation of coordinates,

(a) is equivalent to $\sigma_{oct} = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$

$$\tau_{oct}^2 = \sigma_{ns}^2 = t^{n2} - (\sigma_{oct})^2$$

$$= \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2$$

$$= \frac{1}{9}(3\sigma_1^2 + 3\sigma_2^2 + 3\sigma_3^2 - \sigma_1^2 - \sigma_2^2 - \sigma_3^2 - 2\sigma_1\sigma_2 - 2\sigma_2\sigma_3 - 2\sigma_1\sigma_3$$

$$= \frac{1}{9}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right]$$

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right]^{1/2};$$

to express in terms of x, y, z coord

examine invariants: characteristic equation

$\sigma^3 - I\sigma^2 + I$

finding roots of this is equivalent to factoring it:

$$(\sigma-\sigma_1)(\sigma-\sigma_2)(\sigma-\sigma_3)=0 \qquad (2)$$

expanding, $\sigma^3-\sigma^2(\sigma_1+\sigma_2+\sigma_3)+\sigma(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)-\sigma_1\sigma_2\sigma_3=0$ (3)

comparing (3) & (1),

$$I_1=\sigma_1+\sigma_2+\sigma_3, \quad I_2=\sigma_2\sigma_1+\sigma_2\sigma_3+\sigma_1\sigma_3 \quad I_3=\sigma_1\sigma_2\sigma_3 \qquad (4)$$

using (4), it can be verified that

$$(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_2)^2=2I^2-6\,II$$

now substituting

$$I=\sigma_x+\sigma_y+\sigma_z \quad \& \quad II=\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_x\sigma_z-\sigma_{xy}^2-\sigma_{yz}^2-\sigma_{xz}^2$$

into $\tau_{oct}=\sqrt{2I^2-6II}$

yields $\tau_{oct}=\sqrt{(\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2+6(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{xz}^2)}$

6) by definition, $\phi_{spherical}=\begin{bmatrix}\sigma&&\\&\sigma&\\&&\sigma\end{bmatrix}=\sigma\delta_{ij}e_ie_j$

where $\sigma=\dfrac{\sigma_{kk}}{3}$

$$[\sigma]=[\sigma_{Dev}]+[\sigma_{sper}]=\begin{bmatrix}\sigma_{11}-\sigma & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma\end{bmatrix}+\begin{bmatrix}\sigma&&\\&\sigma&\\&&\sigma\end{bmatrix}$$

or $\sigma_{ij}^{Dev}=\sigma_{ij}-\dfrac{\sigma_{kk}}{3}\delta_{ij}$

invariants of σ_{ij}^{Dev} are determined by solving eigenvalue problem

$$\det\begin{vmatrix}\sigma_{11}^{Dev}-\sigma-\lambda & \sigma_{12}^{Dev} & \sigma_{13}^{Dev}\\ \sigma_{21}^{Dev} & \sigma_{22}^{Dev}-\sigma-\lambda & \sigma_{23}^{Dev}\\ \sigma^{Dev} & \sigma^{Dev} & \sigma^{Dev}\end{vmatrix}=0$$

expanding this leads to a cubic in λ:

$$\lambda^3 - I_0 \lambda^2 + II_0 \lambda - III_0 = 0$$

where $I_0 = 0$

$$II_0 = II - \frac{1}{3} I^2$$

$$III_0 = III - \frac{1}{3} I \left(II - \frac{2}{9} I^2 \right)$$

DEPARTMENT OF APPLIED MECHANICS
STANFORD UNIVERSITY

238A Theory of Elasticity — Autumn 1978

27 Nov 78

Problem Set No. 3

1. Consider the displacement vector

$$
\begin{aligned}
U &= (dx + 6y + 5z)\,\phi_x \\
&\quad + (ax + ey + 3z)\,\phi_y \\
&\quad + (bx + cy + fz)\,\phi_z
\end{aligned}
$$

and determine the constants a, b, c, d, e and f such that U represents a rigid body rotation only.

2. Assume that the components of strain ε_{ij} are expressed in terms of a single function $\phi(x,y)$ as follows:

$$
\varepsilon_{xx} = \frac{\partial^2\phi}{\partial y^2} \qquad \varepsilon_{zz} = 0
$$

$$
\varepsilon_{yy} = \frac{\partial^2\phi}{\partial x^2} \qquad \varepsilon_{yz} = 0
$$

$$
\varepsilon_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \qquad \varepsilon_{zx} = 0
$$

Can the function ϕ be considered arbitrary or does it have to satisfy some conditions to insure single-valuedness of displacement?

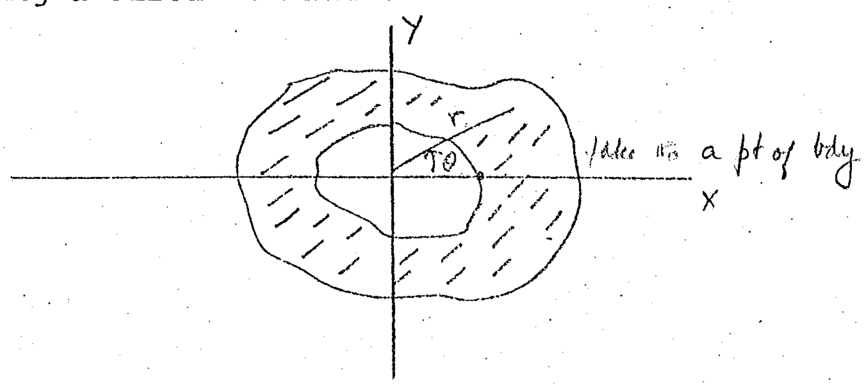
3. Consider the doubly connected plane region and assume the components of strain to be

$$
\varepsilon_{yz} = \frac{kx}{x^2+y^2} = \frac{k\cos\phi}{r}
$$

$$
\varepsilon_{zx} = \frac{-ky}{x^2+y^2} = -\frac{k\sin\phi}{r}
$$

All other $\varepsilon_{ij} = 0$

Determine the conditions for single-valued displacements and determine k for single-valued displacements, for the special case of the inner boundary being a circle of radius a.



1. $\mathbb{U} = \underbrace{(dx+6y+5z)}_{u_x} \mathbf{e}_x + \underbrace{(ax+ey+3z)}_{u_y}\mathbf{e}_y + \underbrace{(bx+cy+fz)}_{u_z}\mathbf{e}_z$

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ $\quad$ $\varepsilon_{xx} = \frac{1}{2}(u_{x,x} + u_{x,x})$ $\quad$ $\omega_{ij} = \frac{1}{2}(u_{j,i} - u_{i,j})$

$$\varepsilon_{ij} = \begin{pmatrix} d & \frac{6+a}{2} & \frac{5+b}{2} \\ \frac{a+6}{2} & e & \frac{3+c}{2} \\ \frac{b+5}{2} & \frac{c+3}{2} & f \end{pmatrix} \qquad \omega_{ij} = \begin{pmatrix} 0 & \frac{a-6}{2} & \frac{b-5}{2} \\ \frac{6-a}{2} & 0 & \frac{c-3}{2} \\ \frac{5-b}{2} & \frac{3-c}{2} & 0 \end{pmatrix}$$

for rigid body rotation only the extensional strains $\varepsilon_{ij} = 0$

$\varepsilon_{ij} = 0 \implies d = e = f = 0$, $a = -6$, $b = -5$, $c = -3$

$$\therefore \begin{pmatrix} 0 & -6 & -5 \\ 6 & 0 & -3 \\ 5 & 3 & 0 \end{pmatrix} = \omega_{ij}$$

2. Must satisfy compatibility conditions namely that $\nabla \times \phi \times \nabla = 0$

$$\Psi_{xx} = 2\cdot 0 - \phi_{xxzz} - 0 \equiv 0 \quad \text{since} \quad \phi_{xxzz} = \phi_{zzxx} = 0$$

$$\Psi_{yy} = 2\cdot 0 - \phi_{yyzz} - 0 \equiv 0 \quad ''$$

$$\Psi_{zz} = 2(-\phi_{xxyy}) - \phi_{yyyy} - \phi_{xxxx} = \Delta^2\phi = 0 \quad \text{for compat.}$$

$$\Psi_{yz} = \phi_{yyyz} - \frac{\partial}{\partial x}(0 + 0 - \phi_{xyz}) \equiv 0 \quad \text{since} \quad \phi_z = 0$$

$$\Psi_{xz} = \phi_{xxxz} - \frac{\partial}{\partial y}(0 - \phi_{xyz} + 0) = 0 \quad \text{since} \quad \phi_z = 0$$

$$\Psi_{xy} = 0 - \frac{\partial}{\partial z}(\phi_{xyz} + 0 + 0) = 0 \quad \text{since} \quad \phi_z = 0$$

3. for single valuedness of displacements in a simply connected domain $\Delta^2\phi = \nabla^4\phi \equiv 0$

$$\phi \times \nabla = \varepsilon_{il,k}\, e_{lkj}\, e_i e_j \qquad \text{only for } e_{123}\ \overset{+1}{e_{231}}\ e_{312} \text{ and } e_{321}\ \overset{-1}{e_{213}}\ e_{132}$$

$$= \varepsilon_{i1,2}\, e_i e_3 + \varepsilon_{i2,3}\, e_i e_1 + \varepsilon_{i3,1}\, e_i e_2 - \underline{\varepsilon_{i3,2}\, e_i e_1} - \varepsilon_{i2,1}\, e_i e_3 - \varepsilon_{i1,3}\, e_i e_2$$

only ε_{23}, ε_{31} are not zero

$$\Psi_{xx} = 0 \quad \text{since} \quad \varepsilon_{yz} = f(x,y) \text{ only} \quad \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} = 0$$

$$\Psi_{yy} = 0 \quad \text{since} \quad \varepsilon_{xz} = g(x,y) \text{ only} \quad \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} = 0$$

$$\Psi_{zz} \equiv 0$$

$$\Psi_{yz} = \frac{\partial^2}{\partial x^2}\varepsilon_{yz} - \frac{\partial^2 \varepsilon_{zx}}{\partial x \partial y} = 0$$

$$\Psi_{zx} = \frac{\partial^2}{\partial y^2}\varepsilon_{zx} - \frac{\partial^2 \varepsilon_{yz}}{\partial x \partial y} = 0$$

$$\Psi_{xy} = 0 \quad \text{same as (1) (2)}$$

$$\varepsilon_{yz,x} = \frac{(x^2+y^2)k - kx\,2x}{(x^2+y^2)^2} = \frac{k(y^2-x^2)}{(x^2+y^2)^2} = \frac{k\left[r^2\sin^2\phi - r^2\cos^2\phi\right]}{r^4} = \frac{k}{r^2}(1 - 2\cos^2\phi)$$

$$\varepsilon_{yz,xx} = \frac{(x^2+y^2)^2(-2kx) - 2k(y^2-x^2)(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} = (x^2+y^2)(-2kx)\left\{\overbrace{(x^2+y^2) + 2(y^2-x^2)}^{3y^2-x^2}\right\}$$

$$= \frac{(3y^2-x^2)(-2kx)}{(x^2+y^2)^3}$$

$$\varepsilon_{zx,y} = \frac{(x^2+y^2)(-k) + 2ky^2}{(x^2+y^2)^2} = \frac{k(y^2-x^2)}{(x^2+y^2)^2} = -\frac{k\cos 2\phi}{r^2}$$

$$\varepsilon_{zx,x} = \frac{(x^2+y^2)\cdot 0 + ky(2x)}{(x^2+y^2)^2} = \frac{2kyx}{(x^2+y^2)^2} =$$

$$\varepsilon_{zx,xy} = \frac{(x^2+y^2)^2\, 2kx - 2kyx\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4} = \frac{(x^2+y^2)\cdot 2kx\left[\overbrace{(x^2+y^2) - 4y^2}^{x^2-3y^2}\right]}{(x^2+y^2)^{4\,3}}$$

$$\therefore\ \varepsilon_{yz,xx} - \varepsilon_{zx,xy} = \frac{(3y^2-x^2)(-2kx)}{(x^2+y^2)^3} - \frac{(-2kx)(-x^2+3y^2)}{(x^2+y^2)^3} = 0$$

since $-\varepsilon_{zx}$ can be obtained by replacing the x by y in ε_{yz} then $-\varepsilon_{zx,yy}$ can be obtained by replacing the x by y in $\varepsilon_{yz,xx}$.
similarly $-\varepsilon_{yz}$ can be obtained by replacing the y by x in $+\varepsilon_{zx}$ then $-\varepsilon_{yz,xy}$ can be obtained by replacing the y by x in $+\varepsilon_{zx,xy}$.

Hence $\quad \varepsilon_{yz,xy} - (-\varepsilon_{zx,yy}) = \varepsilon_{zx,yy} - \varepsilon_{yz,xx} = 0$

$$d\mathbf{r}\cdot\boldsymbol{\varepsilon}\times\nabla = (dx_j e_j)\cdot\left[(\varepsilon_{i1,2}\,e_ie_3 - \varepsilon_{i2,1}\,e_ie_3) + (\varepsilon_{i2,3}\,e_ie_1 - \varepsilon_{i3,2}\,e_ie_1) + (\varepsilon_{i3,1}\,e_ie_2 - \varepsilon_{i1,3}\,e_ie_2)\right]$$
$$= dx_j\cdot(\varepsilon_{i1,2}-\varepsilon_{i2,1})\,\delta_{ij}\,e_3 + dx_j\cdot(\varepsilon_{i2,3}-\varepsilon_{i3,2})\,\delta_{ij}\,e_1 + dx_j\cdot(\varepsilon_{i3,1}-\varepsilon_{i1,3})\,\delta_{ij}\,e_2$$

$\Rightarrow$

① $\oint dx_i\cdot(\varepsilon_{i1,2}-\varepsilon_{i2,1}) = 0 \Rightarrow \int dx_1(\cancel{\varepsilon_{11,2}-\varepsilon_{12,1}}) + dx_2(\cancel{\varepsilon_{21,2}-\varepsilon_{22,1}}) + dx_3\,\overbrace{(\varepsilon_{31,2}-\varepsilon_{32,1})}^{\text{ok}} = 0$

② $\oint dx_i\cdot(\varepsilon_{i2,3}-\varepsilon_{i3,2}) = 0 \Rightarrow \int dx_1(\overbrace{\cancel{\varepsilon_{12,3}}}^{0} - \overbrace{\varepsilon_{13,2}}^{\text{ok}}) + dx_2(\overbrace{\cancel{\varepsilon_{22,3}}}^{0} - \overbrace{\varepsilon_{23,2}}^{\text{ok}}) + dx_3(\overbrace{\varepsilon_{32,3}}^{\text{ok}} - \overbrace{\cancel{\varepsilon_{33,2}}}^{0}) = 0$

③ $\oint dx_i\cdot(\varepsilon_{i3,1}-\varepsilon_{i1,3}) = 0 \Rightarrow \int dx_1(\overbrace{\varepsilon_{13,1}}^{\text{ok}} - \cancel{\varepsilon_{11,3}}) + dx_2(\overbrace{\varepsilon_{23,1}}^{\text{ok}} - \overbrace{\cancel{\varepsilon_{21,3}}}^{0}) + dx_3(\cancel{\varepsilon_{33,1}} - \overbrace{\varepsilon_{31,3}}^{\text{ok}}) = 0$

Now if we are integrating in the plane $dx_3 = 0 \Rightarrow$ ① is satisfied

now $dx_1 = -r\sin\phi\, d\phi \qquad dx_2 = r\cos\phi\, d\phi$

② $\Rightarrow \int -\varepsilon_{13,2}\,dx_1 - \varepsilon_{23,2}\,dx_2 = 0 = \oint -\varepsilon_{13,2}(-\sin\phi\cdot r\,d\phi) - \overbrace{\varepsilon_{23,2}(r\cos\phi\, d\phi)}^{0} = 0$

③ $\Rightarrow \int \varepsilon_{13,1}\,dx_1 + \varepsilon_{23,1}\,dx_2 = 0 = \oint \overbrace{+\varepsilon_{13,1}(-\sin\phi\, r\,d\phi)}^{=0} + \varepsilon_{23,1}(r\cos\phi\, d\phi) = 0$

③ $\int \varepsilon_{23,1}(r\cos\phi\, d\phi) = \int \frac{-k}{r^2}(\cos 2\phi)(r\cos\phi\, d\phi) = \frac{k}{a}\oint \cos 2\phi\,\cos\phi\, d\phi = 0$

② $\int \varepsilon_{13,2}(r\sin\phi\, d\phi) = -\frac{k}{a}\oint_0^{2\pi} \frac{\cos 2\phi}{r^2}\sin\phi\, d\phi = 0$

$$\oint_{C_i'} \left[\varepsilon + (\mathbb{r} - \mathbb{r}_0) \times (\nabla \times \varepsilon) \right] \cdot d\mathbb{r} \Big] = 0 \qquad \text{let } \mathbb{r}_0 = 0$$

$$p = \nabla \times \varepsilon = \varepsilon_{\ell j, k}\, e_{k\ell i}\, e_i\, e_j \qquad \varepsilon_{23} = \varepsilon_{yz} = \frac{kx}{x^2+y^2} = \frac{k\cos\phi}{r} \qquad \varepsilon_{ij} = 0 \text{ all others}$$

$$\mathbb{r} \times p = r_m p_i\, e_{min}\, e_n \qquad \varepsilon_{31} = \varepsilon_{zx} = \frac{-ky}{x^2+y^2} = \frac{-k\sin\phi}{r}$$

$e_{k\ell i} \neq 0$ for $k\ell i = 123, 231, 312$ & $321, 213, 132$

$$\therefore\ \varepsilon_{2j,1}\, e_3 e_j + \varepsilon_{3j,2}\, e_1 e_j + \varepsilon_{1j,3}\, e_2 e_j - \varepsilon_{2j,3}\, e_1 e_j - \varepsilon_{1j,2}\, e_3 e_j - \varepsilon_{3j,1}\, e_2 e_j$$ this yields 18 terms

if $j = 1, 2, 3$

$$\varepsilon_{31,2}\, e_1 e_1 - \varepsilon_{31,1}\, e_2 e_1 + \varepsilon_{32,2}\, e_1 e_2 - \varepsilon_{32,1}\, e_2 e_2 + \varepsilon_{32,1}\, e_3 e_3 + \cancel{\varepsilon_{31,3}\, e_2 e_3} - \cancel{\varepsilon_{32,3}\, e_1 e_3} - \varepsilon_{31,2}\, e_3 e_3$$

since $\frac{\partial}{\partial z} = 0$ this reduces to 6

$$\mathbb{r} \times \nabla \times \varepsilon = x_i e_i \times (\nabla \times \varepsilon) = -x_2 \varepsilon_{31,2}\, e_3 e_1 - x_1 \varepsilon_{31,1}\, e_3 e_1 - x_2 \varepsilon_{32,2}\, e_3 e_2 - x_1 \varepsilon_{32,1}\, e_3 e_2 - x_1 \varepsilon_{32,1}\, e_2 e_3$$

$$\varepsilon_{23} e_2 e_3 + \varepsilon_{31} e_3 e_1 + \varepsilon_{23} e_3 e_2 + \varepsilon_{31} e_1 e_3 \quad + x_2 \varepsilon_{32,1}\, e_1 e_3 + x_1 \varepsilon_{31,2}\, e_2 e_3 - x_2 \varepsilon_{31,2}\, e_1 e_3$$ since $x_3 = 0$

$$\oint_{C_i} (\varepsilon + \mathbb{r} \times \nabla \times \varepsilon)\, (dx_j e_j) =$$

$$\oint \Big[(\varepsilon_{31} - x_2 \varepsilon_{31,2} - x_1 \varepsilon_{31,1})\, e_3 e_1 + (\varepsilon_{32} - x_2 \varepsilon_{32,2} - x_1 \varepsilon_{32,1})\, e_3 e_2 + (\varepsilon_{32} + x_1 \varepsilon_{32,1} + x_1 \varepsilon_{31,2})\, e_2 e_3$$

$$+ (\varepsilon_{31} + x_2 \varepsilon_{32,1} - x_2 \varepsilon_{31,2})\, e_1 e_3 \Big] \cdot dx_i e_i = 0 \quad \text{since}$$

or $\qquad -r\sin\phi\, d\phi \qquad r\cos\phi\, d\phi$

① e_3 $\quad \oint (\varepsilon_{31} - x_2 \varepsilon_{31,2} - x_1 \varepsilon_{31,1})\, dx_1 + (\varepsilon_{32} - x_2 \varepsilon_{32,2} - x_1 \varepsilon_{32,1})\, dx_2 = 0$

② e_2 $\quad \oint (\varepsilon_{32} - x_1 \varepsilon_{32,1} + x_1 \varepsilon_{31,2})\, dx_3 = 0$ identically

③ e_1 $\quad \oint (\varepsilon_{31} + x_2 \varepsilon_{32,1} - x_2 \varepsilon_{31,2})\, dx_3 = 0$ identically

$$\oint \varepsilon_{31} dx_1 + \oint \varepsilon_{32} dx_2 - \Big[\oint \overset{r\sin\phi}{x_2} \varepsilon_{31,2}\, dx_1 + \overset{r\cos\phi}{x_1} \varepsilon_{32,1}\, dx_2 + \oint \overset{u}{x_1} \overset{dv}{\varepsilon_{31,1} dx_1} + \oint x_2 \varepsilon_{32,2}\, dx_2 \Big]$$

$$x_1 \varepsilon_{31} \Big|_0^{2\pi} - \oint \varepsilon_{31} dx_1 + x_2 \varepsilon_{32} - \oint \varepsilon_{32} dx_2$$

$$= 2\oint \varepsilon_{31} dx_1 + 2\oint \varepsilon_{32} dx_2 - (x_1 \varepsilon_{31} + x_2 \varepsilon_{32}) \Big|_0^{2\pi} - \Big[\oint \Big[(r\sin\phi)(\tfrac{-k}{r^2}\cos 2\phi)(-r\sin\phi\, d\phi) + (r\cos\phi)(\tfrac{-k}{r^2}\cos 2\phi)(r\cos\phi\, d\phi)$$

$$\varepsilon_{31,2} = \varepsilon_{zx,y} = -\frac{k}{r^2}\cos 2\phi \qquad \Big[-k \oint \cos^2 2\phi\, d\phi \Big] = \cancel{\frac{-k}{2}\sin 2\phi \Big|_0^{2\pi}} = 0 \quad k\pi$$

$$\varepsilon_{23,1} = \varepsilon_{yz,x} = \frac{-k}{r^2}\cos 2\phi$$

$$= 2\oint \varepsilon_{31} \cdot -r\sin\phi\, d\phi + 2\oint \varepsilon_{32} \cdot r\cos\phi\, d\phi - \left(-\frac{k}{2}\sin 2\phi + \frac{k}{2}\sin 2\phi \right)$$

$$= +2k \oint \sin^2\phi\, d\phi + 2k \oint \cos^2\phi\, d\phi = 2k(2\pi) = 4\pi k \quad \Rightarrow k \equiv 0.$$

$$\int x_1 d\,\varepsilon_{31} = \int x_1 \varepsilon_{31,1}\, dx_1 + \int x_1 \varepsilon_{31,2}\, dx_2$$

$$x_1\varepsilon_{31} - \int \varepsilon_{31}\, dx_1$$

then $$-\int x_1\, \varepsilon_{31,1}\, dx_1 = \int x_1\, \varepsilon_{31,2}\, dx_2 - \underbrace{\int x_1\, d\varepsilon_{31}}$$

$$-\int x_1 \varepsilon_{31,1}\, dx_1 = \int x_1\, \varepsilon_{31,2} dx_2 - x_1 \varepsilon_{31} + \int \varepsilon_{31}\, dx_1$$

$$-\int x_2\, \varepsilon_{32,2} dx_2 = \int x_2\, \varepsilon_{32,1} dx_1 - x_2\, \varepsilon_{32} + \int \varepsilon_{32}\, dx_2$$

$$\int\left(2\varepsilon_{31}\, dx_1 - x_2\, \varepsilon_{31,2}\, dx_1\right) + \int \left(x_1\, \varepsilon_{31,2}\, dx_2\right) - x_1\varepsilon_{31}\Big|_0^{2\pi} + \int\left(2\varepsilon_{32}\, dx_2 - x_1\, \varepsilon_{32,1}\, dx_2\right) - x_2\varepsilon_{32}\Big|_0^{2\pi}$$

$$+ \int x_2\, \varepsilon_{32,1}\, dx_1$$

$$\int\left(2\varepsilon_{31}\, dx_1 + 2\,\varepsilon_{32} dx_2\right) - \int\left(x_2\, \varepsilon_{31,2}\, dx_1 + x_1\, \varepsilon_{32,1} dx_2\right) - \left(x_1\varepsilon_{31} + x_2 \varepsilon_{32}\right)\Big|_0^{2\pi} \to 0$$

$$(r\cos\phi)\left(-\frac{k}{r^2}\cos 2\phi\right)\left(dr\sin\phi + r\cos\phi\, d\phi\right) + (r\sin\phi)\left(-\frac{k}{r^2}\cos 2\phi\right)\left(dr\cos\phi - r\sin\phi\, d\phi\right)$$

$$\int\left(x_1\, \varepsilon_{31,2} dx_2 + x_2\, \varepsilon_{32,1}\, dx_1\right)$$

$$\int -\frac{k}{r}\cos 2\phi\, \sin 2\phi\, dr + \int -k\cos 2\phi\, \cos^2\phi\, d\phi + k\cos 2\phi\, \sin^2\phi\, d\phi$$

$$-k\int_0^{2\pi} \cos 2\phi\, \cos 2\phi\, d\phi = -k\pi$$

$5\pi k - \pi k = 4\pi k.$

$$\oint - \varepsilon_{13,2}\, dx_1 - \varepsilon_{23,2}\, dx_2.$$

$$d\varepsilon_{23} = \varepsilon_{23,1}\, dx_1 + \varepsilon_{23,2}\, dx_2$$

$$-\int d\varepsilon_{23} + \int \varepsilon_{23,1} dx_1 = -\int \varepsilon_{23,2} dx_2$$

$$\int - \varepsilon_{13,2} dx_1 + \varepsilon_{23,1}\, dx_1 = \varepsilon_{23}\Big|_0$$

$$\int\left(-\varepsilon_{31,2} + \varepsilon_{32,1}\right) dx_1 = \varepsilon_{23}\Big|_0^{2\pi} = 0$$

$$0.$$

$$\oint \varepsilon_{13,1} dx_1 = \oint d\varepsilon_{13} - \oint \varepsilon_{13,2} dx_2$$

$$d\varepsilon_{13} = \varepsilon_{13,1} dx_1 + \varepsilon_{13,2} dx_2$$

$$d\varepsilon_{13} - \varepsilon_{13,2} dx_2$$

$$x_2^B \left(-\varepsilon_{31,2} + \varepsilon_{32,1}\right) dx_1 + x_1 \left(-\varepsilon_{32,1} + \varepsilon_{31,2}\right) dx_2$$

Prob. Set. No. 3 Solution 11-14-78

1. $u_x = dx + 6y + 5z \qquad u_y = ax + ey + 3z \qquad u_z = bx + cy + fz$

For Rigid body ~~displacement~~ rotation only, all strain components $= 0$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0 \Rightarrow d = 0$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0 \Rightarrow e = 0$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0 \Rightarrow f = 0$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}(6+a) = 0 \Rightarrow a = -6$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{2}(5+b) = 0 \Rightarrow b = -5$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2}(3+c) = 0 \Rightarrow c = -3$$

$$\underline{u = (6y+5z)\,e_x + (-6x+3z)\,e_y + (-5x-3y)\,e_z}$$

There are no constant terms in this displacement so it represents rotation only, no translation

components of rotation:

(Rotation about X) $\quad \omega_{yz} = \left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) = 6$

(Rotation about Y) $\quad \omega_{zx} = \left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right) = -10$

(Rotation about Z) $\quad \omega_{xy} = \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) = 12$

Motion is sum of 3 simple rotations about X_1, X_2, X_3

2. Assume Simply Connected region. Check compatibility

5 out of 6 equations trivially satisfied

$$6^{th}: \; \psi_{zz} = 0 = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} - \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} - \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}$$

$$\varepsilon_{xx} = \frac{\partial^2 \phi}{\partial y^2} \qquad \frac{\partial \varepsilon_{xx}}{\partial y} = \frac{\partial^3 \phi}{\partial y^3} \qquad \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^4 \phi}{\partial y^4}$$

$$\varepsilon_{yy} = \partial^2 \phi \qquad \partial \varepsilon_{yy} \quad \partial^3 \phi \qquad \partial^2 \varepsilon_{yy} = \partial^4 \phi$$

Prob. Set No. 3 Solution

$$\varepsilon_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \qquad \frac{\partial \varepsilon_{xy}}{\partial x} = -\frac{\partial^3 \phi}{\partial x^2 \partial y} \qquad \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = -\frac{\partial^4 \phi}{\partial x^2 \partial y^2}$$

$$0 = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \qquad (1)$$

using the 2 dimensional Laplacian operator

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,)$$

(1) can be written $\nabla^4 \phi = \nabla^2 \nabla^2 \phi = 0$ which is the biharmonic equation: $\nabla^2 \nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} =$

3. (a) $$\Psi_{yz} = \frac{\partial^2 \cancel{\varepsilon_{xx}}^{\,0}}{\partial y \partial z} - \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \cancel{\frac{\partial \varepsilon_{xy}}{\partial z}}^{\,0}\right) = 0$$

$$\frac{\partial \varepsilon_{yz}}{\partial x} = \frac{K}{x^2+y^2} - \frac{2Kx^2}{(x^2+y^2)^2}$$

$$\frac{\partial \varepsilon_{zx}}{\partial y} = \frac{-K}{x^2+y^2} + \frac{2Ky^2}{(x^2+y^2)^2}$$

$$-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} = \frac{2Kx^2}{(x^2+y^2)^2} - \frac{K}{x^2+y^2} + \frac{2Ky^2}{(x^2+y^2)^2} - \frac{K}{x^2+y^2}$$

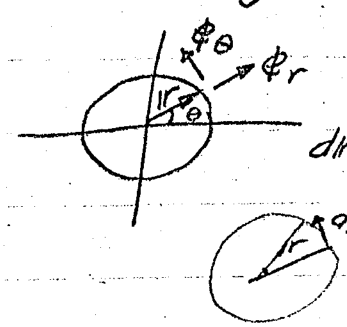
$$= \frac{2K}{x^2+y^2} - \frac{2K}{x^2+y^2} = 0 \quad \therefore \text{ no information about } K$$

similar $\Psi_{zx} = 0$ is identically satisfied with no restriction on K

$\Psi_{xx} = 0$, etc. are trivially satisfied

Check compatibility for multiply-connected region

b) $$\oint_{C_i} d\mathbf{r} \cdot (\boldsymbol{\varepsilon} \times \nabla) = 0$$



$\mathbf{r} = a\mathbf{e}_r$

$d\mathbf{r} = a\,d\theta\,\mathbf{e}_\theta$

here C_i is a circle of radius a:

$$\varepsilon_{zy} = \frac{K\cos\theta}{r} = \frac{Kx}{x^2+y^2}$$

$$\varepsilon_{zx} = \frac{-K\sin\theta}{r} = \frac{-Ky}{x^2+y^2}$$

$$\mathbf{\varepsilon}\times\nabla = (\varepsilon_{zy}\mathbf{e}_z\mathbf{e}_y + \varepsilon_{yz}\mathbf{e}_y\mathbf{e}_z + \varepsilon_{zx}\mathbf{e}_z\mathbf{e}_x + \varepsilon_{xz}\mathbf{e}_x\mathbf{e}_z)\times\left(\mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}\right)$$

$$= \mathbf{e}_x(\cancel{\varepsilon_{zy,z}}^{\,0}\mathbf{e}_z - \varepsilon_{yz,y}\mathbf{e}_y - \varepsilon_{xz,y}\mathbf{e}_x)$$
$$+ \mathbf{e}_y(-\cancel{\varepsilon_{zx,z}}^{\,0}\mathbf{e}_z + \varepsilon_{yz,x}\mathbf{e}_y + \varepsilon_{xz,x}\mathbf{e}_x)$$
$$+ \mathbf{e}_z(\varepsilon_{zx,y}\mathbf{e}_z - \varepsilon_{zy,x}\mathbf{e}_z)$$

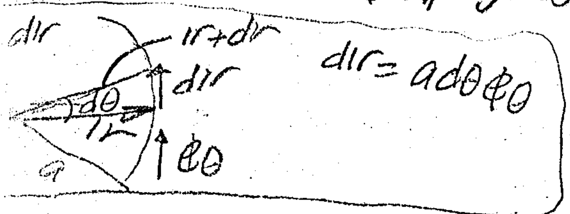
$$\varepsilon_{yz,y} = \frac{-2kxy}{(x^2+y^2)^2} = \frac{-2k\sin\theta\cos\theta}{r^2} \qquad \varepsilon_{xz,y} = \frac{-k}{x^2+y^2} + \frac{2ky^2}{(x^2+y^2)^2} = \frac{-k}{r^2} + \frac{2k\sin^2\theta}{r^2}$$

$$\varepsilon_{yz,x} = \frac{k}{x^2+y^2} - \frac{2kx^2}{(x^2+y^2)^2} = \frac{-k}{r^2} + \frac{2k\cos^2\theta}{r^2} \qquad \varepsilon_{xz,x} = \frac{2kxy}{(x^2+y^2)^2} = \frac{2k\sin\theta\cos\theta}{r^2}$$

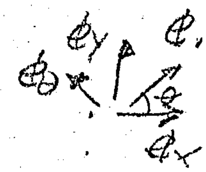
subst. into $\mathbf{\varepsilon}\times\nabla$ gives

$$\underline{\underline{\mathbf{\varepsilon}\times\nabla}} = \mathbf{e}_x\left(\frac{2k\sin\theta\cos\theta}{r^2}\mathbf{e}_y + \left(\frac{k}{r^2} - \frac{2k\sin^2\theta}{r^2}\right)\mathbf{e}_x\right)$$
$$+ \mathbf{e}_y\left(\left(\frac{k}{r^2} - \frac{2k\cos^2\theta}{r^2}\right)\mathbf{e}_y + \frac{2k\sin\theta\cos\theta}{r^2}\mathbf{e}_x\right)$$

($\mathbf{e}_z$ term cancels)



$d\mathbf{r} = a\,d\theta\,\mathbf{e}_\theta$



$d\mathbf{r}\cdot\mathbf{e}_z = 0 \quad \mathbf{e}_x\cdot d\mathbf{r} = -a\sin\theta\,d\theta \quad \mathbf{e}_y\cdot d\mathbf{r} = a\cos\theta\,d\theta$

$$d\mathbf{r}\cdot(\mathbf{\varepsilon}\times\nabla) = \left[\mathbf{e}_x\left(\frac{k\sin\theta}{r^2}\right) + \mathbf{e}_y\left(-\frac{k\cos\theta}{r^2}\right)\right]a\,d\theta$$

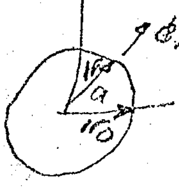
note: $\mathbf{r}_0$ is the position vector of the point at which you start & finish the line integral $\oint_{C_i}$.. for circle centered at origin, the magnitude of $\mathbf{r}_0$ must be $\underline{\underline{a}}$

on C_i; $r = a$ $\;\therefore\; \int_0^{2\pi}\left(\mathbf{e}_x\frac{k\sin\theta}{a} - \mathbf{e}_y\frac{k\cos\theta}{a}\right)d\theta = 0$

$$= -\frac{k\cos\theta}{a}\Big|_0^{2\pi}\mathbf{e}_x - \frac{k\sin\theta}{a}\Big|_0^{2\pi}\mathbf{e}_y \equiv 0 \quad \underline{\text{with no restriction on } k}$$

c) $\oint_{C_i}[\mathbf{\varepsilon} + (\mathbf{r}-\mathbf{r}_0)\times\nabla\times\mathbf{\varepsilon}]\cdot d\mathbf{r} = 0$

$$\mathbf{\varepsilon}\cdot d\mathbf{r} = (\varepsilon_{yz}\mathbf{e}_y\mathbf{e}_z + \varepsilon_{zy}\mathbf{e}_z\mathbf{e}_y + \varepsilon_{zx}\mathbf{e}_z\mathbf{e}_x + \varepsilon_{xz}\mathbf{e}_x\mathbf{e}_z)\cdot a\,d\theta(-\mathbf{e}_x\sin\theta + \mathbf{e}_y\cos\theta)$$



$$= \mathbf{e}_z\frac{k\,a\,d\theta}{r} = \mathbf{e}_z k\,d\theta \quad \text{on } C_i \text{ (where } r = a)$$

pick $\mathbf{r}_0 = a\mathbf{e}_x$; $\mathbf{r} = a\mathbf{e}_r = a\cos\theta\,\mathbf{e}_x + a\sin\theta\,\mathbf{e}_y$

$\mathbf{r}-\mathbf{r}_0 = a(\cos\theta-1)\mathbf{e}_x + a\sin\theta\,\mathbf{e}_y$

$$\nabla\times\mathbf{\varepsilon} = -\mathbf{\varepsilon}^T\times\nabla = -\mathbf{\varepsilon}\times\nabla \underset{\text{(see above)}}{=} -\begin{bmatrix} \frac{k}{r^2} - \frac{2k\sin^2\theta}{r^2} & \frac{2k\sin\theta\cos\theta}{r^2} & 0 \\ \frac{2k\sin\theta\cos\theta}{r^2} & \frac{k}{r^2} - \frac{2k\cos^2\theta}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(\mathbf{r}-\mathbf{r}_0)\times(\nabla\times\mathbf{\varepsilon}) = \mathbf{e}_z\left[\mathbf{e}_x\left(a(\cos\theta-1)\left(\frac{-2k\sin\theta\cos\theta}{r^2}\right) + a\sin\theta\left(\frac{k}{r^2} - \frac{2k\sin^2\theta}{r^2}\right)\right)\right.$$
$$\left. + \mathbf{e}_y\left(a(\cos\theta-1)\left(\frac{-k}{r^2} + \frac{2k\cos^2\theta}{r^2}\right) + a\sin\theta\left(\frac{2k\sin\theta\cos\theta}{r^2}\right)\right)\right]$$

$$(\mathbf{r}-\mathbf{r}_0)\times(\nabla\times\mathbf{\varepsilon})\cdot d\mathbf{r} = \frac{\mathbf{e}_z k\,d\theta}{a}\left[+2a(\cos\theta-1)\sin^2\theta\cos\theta - a\sin^2\theta(1-2\sin^2\theta) + a(\cos\theta-1)(\cos\theta-2\cos^3\theta) + 2a\sin^2\theta\cos^2\theta\right]$$

$$= \mathbf{e}_z k\,d\theta\,[1-\cos\theta]$$

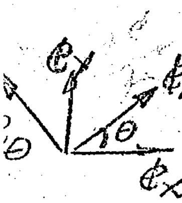
$$\oint_{C_1} \boldsymbol{\varepsilon}\cdot d\mathbf{r} = \oint_0^{2\pi} \mathbf{e}_z K\, d\theta = 2\pi K \mathbf{e}_z$$

$$\therefore \oint_{C_1} [\boldsymbol{\varepsilon}+(\mathbf{r}-\mathbf{r}_0)\times(\nabla\times\boldsymbol{\varepsilon})]\cdot d\mathbf{r} = \mathbf{e}_z(2\pi K + 2\pi K) = \underline{\underline{4\pi K}} \Rightarrow \underline{K=0}$$

$\therefore$ The given strain field will not be compatible except for the trivial case $K=0$ (i.e. no strain)

<u>Alternate approach</u> : Use polar coordinates

transform strain to polar : direction cosines –



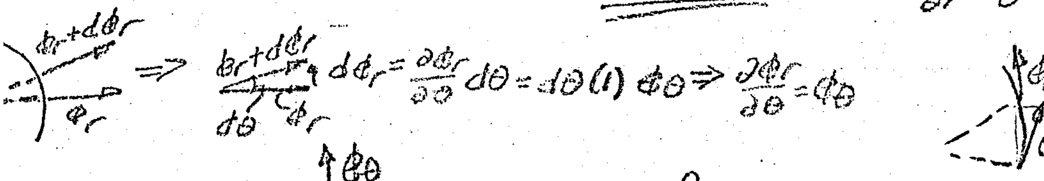
$$\begin{bmatrix}\mathbf{e}_r\\ \mathbf{e}_\theta\\ \mathbf{e}_z\end{bmatrix} = \underbrace{\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}}_{L}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\\ \mathbf{e}_z\end{bmatrix} \qquad [\boldsymbol{\varepsilon}_{polar}] = [\boldsymbol{\varepsilon}'] = [L][\boldsymbol{\varepsilon}][L^T]$$

$$= \begin{bmatrix} 0 & 0 & (\cos\theta)\varepsilon_{xz}+(\sin\theta)\varepsilon_{yz}\\ 0 & 0 & (-\sin\theta)\varepsilon_{xz}+\cos\theta\,\varepsilon_{yz}\\ (\cos\theta)\varepsilon_{xz}+(\sin\theta)\varepsilon_{yz} & (-\sin\theta)\varepsilon_{xz}+(\cos\theta)\varepsilon_{yz} & 0\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&K/r\\0&K/r&0\end{bmatrix}$$

Polar $\nabla = \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)$

$$\nabla\times\boldsymbol{\varepsilon} = \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\times\left(\frac{K}{r}\mathbf{e}_\theta\mathbf{e}_z + \frac{K}{r}\mathbf{e}_z\mathbf{e}_\theta\right)$$

note all unit vectors do not vary with r or z thus $\frac{\partial}{\partial r}$ & $\frac{\partial}{\partial z}$ of unit vectors $=0$ also $\frac{\partial \mathbf{e}_z}{\partial\theta}=0$. However $\underline{\frac{\partial}{\partial\theta}\mathbf{e}_r = \mathbf{e}_\theta}$ and $\frac{\partial}{\partial\theta}\mathbf{e}_\theta = -\mathbf{e}_r$:



$\Rightarrow$ $d\mathbf{e}_r = \frac{\partial\mathbf{e}_r}{\partial\theta}d\theta = d\theta(1)\,\mathbf{e}_\theta \Rightarrow \frac{\partial\mathbf{e}_r}{\partial\theta} = \mathbf{e}_\theta$

$d\mathbf{e}_\theta = \frac{\partial\mathbf{e}_\theta}{\partial\theta}d\theta = d\theta(1)$ $\therefore \frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r$

$$\nabla\times\boldsymbol{\varepsilon} = \mathbf{e}_r\Big[\underbrace{\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{K}{r}\mathbf{e}_\theta\right)}\ -\ \overset{0}{\cancel{\frac{\partial}{\partial z}\left(\frac{K}{r}\mathbf{e}_z\right)}}\Big] + \mathbf{e}_\theta\Big[-\frac{\partial}{\partial r}\left(\frac{K}{r}\mathbf{e}_\theta\right)\Big] + \mathbf{e}_z\Big[\frac{\partial}{\partial r}\left(\frac{K}{r}\mathbf{e}_z\right)\Big]$$

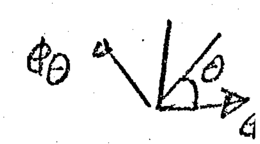
$$= -\frac{K}{r^2}\mathbf{e}_r\mathbf{e}_r + \frac{K}{r^2}\mathbf{e}_\theta\mathbf{e}_\theta - \frac{K}{r^2}\mathbf{e}_z\mathbf{e}_z$$

$\mathbf{r} = a\mathbf{e}_r$; pick $\mathbf{r}_0 = a\mathbf{e}_x$ $\qquad d\mathbf{r} = a\,d\theta\,\mathbf{e}_\theta$

$\mathbf{r}\times(\nabla\times\boldsymbol{\varepsilon}) = -\frac{Ka}{r^2}(-\mathbf{e}_z\mathbf{e}_\theta - \mathbf{e}_\theta\mathbf{e}_z)$ $\quad \mathbf{r}\times(\nabla\times\boldsymbol{\varepsilon})\cdot d\mathbf{r} = \frac{Ka^2}{r^2}d\theta\,\mathbf{e}_z$

$-\mathbf{r}_0\times(\nabla\times\boldsymbol{\varepsilon}) = \frac{Ka}{r^2}[(\mathbf{e}_x\times\mathbf{e}_r)\mathbf{e}_r - (\mathbf{e}_x\times\mathbf{e}_\theta)\mathbf{e}_\theta + (\mathbf{e}_x\times\mathbf{e}_z)\mathbf{e}_z]$

$-\mathbf{r}_0\times(\nabla\times\boldsymbol{\varepsilon})\cdot d\mathbf{r} = -\frac{Ka^2}{r^2}d\theta(\mathbf{e}_x\times\mathbf{e}_\theta) = -\frac{Ka^2}{r^2}d\theta\sin(\theta+\frac{\pi}{2})\mathbf{e}_z$



$$= -\frac{Ka^2\cos\theta}{r^2}\mathbf{e}_z$$

$\boldsymbol{\varepsilon}\cdot d\mathbf{r} = \frac{Ka}{r}\mathbf{e}_z$

$$\oint_{C_1}[\boldsymbol{\varepsilon}+(\mathbf{r}-\mathbf{r}_0)\times(\nabla\times\boldsymbol{\varepsilon})]\cdot d\mathbf{r} = \mathbf{e}_z\int_0^{2\pi}\left(\frac{Ka}{r}+\frac{Ka^2}{r^2}-\frac{Ka^2}{r^2}\cos\theta\right)d\theta \ ; \text{ on } C_1\ r=a$$

$$\Rightarrow \mathbf{e}_z\int_0^{2\pi}(K+K-K\cos\theta)\,d\theta = K\mathbf{e}_z\int_0^{2\pi}(2-\cos\theta)\,d\theta$$

$$= 4\pi K = 0$$

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ME 238A
Prof. Herrmann

100

96

Problem Set # 3

1. Consider $\mathbf{u} = (dx + 6y + 5z)\mathbf{e}_x + (ax + ey + 3z)\mathbf{e}_y + (bx + cy + fz)\mathbf{e}_z = u_i \mathbf{e}_i$

Find a, b, c, d, e, f for $\mathbf{u}$ to represent rigid body rotation

For small displacements then $\boldsymbol{\varepsilon} + \boldsymbol{\omega} = \nabla\mathbf{u}$ or $\boldsymbol{\omega} = \frac{1}{2}(\nabla\mathbf{u} - \mathbf{u}\nabla)$, $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla)$

and $\varepsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j})$, $\omega_{ij} = \frac{1}{2}(u_{j,i} - u_{i,j})$. Thus

$$\varepsilon_{ij} = \begin{pmatrix} d & \frac{6+a}{2} & \frac{5+b}{2} \\ \frac{a+6}{2} & e & \frac{3+c}{2} \\ \frac{b+5}{2} & \frac{c+3}{2} & f \end{pmatrix} \qquad \omega_{ij} = \begin{pmatrix} 0 & \frac{a-6}{2} & \frac{b-5}{2} \\ \frac{6-a}{2} & 0 & \frac{c-3}{2} \\ \frac{5-b}{2} & \frac{3-c}{2} & 0 \end{pmatrix}$$

For rigid body rotation $\varepsilon_{ij} = 0 \ \forall i,j$ hence $d,e,f = 0 \quad a = -6,\ b = -5,\ c = -3$

hence

$$\omega_{ij} = \begin{pmatrix} 0 & -6 & -5 \\ 6 & 0 & -3 \\ 5 & 3 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{u} = (6y + 5z)\mathbf{e}_x + (-6x + 3z)\mathbf{e}_y + (-5x - 3y)\mathbf{e}_z$$

2. given that $\boldsymbol{\varepsilon}$ is given by $\begin{pmatrix} \phi_{,yy} & -\phi_{,xy} & 0 \\ -\phi_{,xy} & \phi_{,xx} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ can ϕ be considered arbitrary or does it have to satisfy some conditions in order to insure single valuedness of displacements

- Assumption: we will be working in a simply connected region

To insure single valued displacements ϕ must check satisfaction of the compatibility equations $\nabla \times \boldsymbol{\varepsilon} \times \nabla = 0$

$\therefore$ Since $\phi = \phi(x,y)$ only, it will satisfy $\Psi_{xx}, \Psi_{yy}, \Psi_{yz}, \Psi_{zx}, \Psi_{xy} = 0$ identically because they will involve taking derivatives of ϕ with respect to z.

The only non vanishing element of $\nabla \times \boldsymbol{\varepsilon} \times \nabla = \boldsymbol{\Psi}$ is

$$\Psi_{zz} = 2(-\phi_{,xxyy}) - \phi_{,yyyy} - \phi_{,xxxx} = -\Delta^2\phi = \nabla^4\phi \quad \left(\text{note } \Delta\phi = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

For single valued displacements ϕ must satisfy $\Delta^2\phi = \nabla^4\phi = 0$ pointwise.

- We note that this will be the only condition of ϕ in a simply connected region

For multiply connected regions we must also check $\oint_{C_i} d\mathbf{r} \cdot \boldsymbol{\varepsilon} \times \nabla = 0$ and $\oint_{C_i} [\boldsymbol{\varepsilon} + (\mathbf{r} - \mathbf{r}_0) \times (\nabla \times \boldsymbol{\varepsilon})] \cdot d\mathbf{r} = 0$

Since no other information about the region is given we will not check these integrals.

3. Given a doubly connected region with ϵ given by $\begin{pmatrix} 0 & 0 & A \\ 0 & 0 & B \\ A & B & 0 \end{pmatrix}$ where $A = \frac{kx}{x^2+y^2} = \frac{k\cos\phi}{r}$ and $B = \frac{-ky}{x^2+y^2} = -\frac{k\sin\phi}{r}$. Determine the conditions for single-valued displacements and determine the value of K for the special case of the inner boundary being a circle of radius a.

Solution by CESARO'S theorem we must check $\nabla \times \epsilon \times \nabla = \Psi = 0$ pointwise and $\oint_{C_i} d\mathbf{r} \cdot [\epsilon \times \nabla] = 0$,

$\oint_{C_i} [\epsilon + (\mathbf{r} - \mathbf{r}_0) \times (\nabla \times \epsilon)] \cdot d\mathbf{r} = 0$ over any curve enclosing the inclusions in the region

A $\underline{\nabla \times \epsilon \times \nabla = \Psi}$: We note that $\Psi_{xx} = \Psi_{yy} = \Psi_{zz} = \Psi_{xy} = 0$ identically, since both ε_{yz}, ε_{zx} are fns of x, y only and these values of Ψ_{ij} require taking derivatives with respect to z. But

$$\Psi_{yz} = \frac{\partial^2}{\partial x^2} \varepsilon_{yz} - \frac{\partial^2}{\partial x \partial y} \varepsilon_{zx} \quad \text{and} \quad \Psi_{zx} = \frac{\partial^2}{\partial y^2} \varepsilon_{zx} - \frac{\partial^2}{\partial x \partial y} \varepsilon_{yz}$$

Now $\frac{\partial \varepsilon_{yz}}{\partial x} = \frac{k(y^2-x^2)}{(x^2+y^2)^2} = \frac{-k}{r^2}\cos 2\phi$; $\frac{\partial^2 \varepsilon_{yz}}{\partial x^2} = \frac{(3y^2-x^2)(-2kx)}{(x^2+y^2)^3}$; $\frac{\partial \varepsilon_{zx}}{\partial x} = \frac{2kyx}{(x^2+y^2)^2}$;

$$\frac{\partial^2 \varepsilon_{zx}}{\partial x \partial y} = \frac{2kx(x^2-3y^2)}{(x^2+y^2)^3}$$

Thus we can see that $\Psi_{yz} \equiv 0$.

We also note that $-\varepsilon_{zx}$ can be obtained by replacing the $\binom{x}{y}$ by $\binom{y}{x}$ in ε_{yz}. Hence $-\varepsilon_{zx,yy}$ can be obtained by replacing the $\binom{y}{x}$ by $\binom{x}{y}$ in $\varepsilon_{yz,xx}$ (This is OK to do since differentiation is with respect to the other variable). Similarly $-\varepsilon_{yz}$ can be obtained by replacing the $\binom{x}{y}$ by $\binom{y}{x}$ in ε_{zx} then $-\varepsilon_{yz,xy}$ can be obtained by replacing the $\binom{x}{y}$ by $\binom{y}{x}$ in $\varepsilon_{zx,xy}$.

$$\text{Hence } \Psi_{zx} = \frac{\partial^2}{\partial y^2}\varepsilon_{zx} - \frac{\partial^2 \varepsilon_{yz}}{\partial x \partial y} = \text{Replaced}(-\varepsilon_{yz,xx} + \varepsilon_{zx,xy}) = \text{Replaced}(-\Psi_{yz}) = 0$$

Thus $\underline{\epsilon \text{ satisfies } \nabla \times \epsilon \times \nabla \equiv 0}$

B. $\epsilon \times \nabla = \varepsilon_{il,k}\, e_{lkj}\, \mathbf{e}_i \mathbf{e}_j$ and $d\mathbf{r} \cdot \epsilon \times \nabla = dx_m \mathbf{e}_m \cdot \varepsilon_{il,k}\, e_{lkj}\, \mathbf{e}_i \mathbf{e}_j = dx_i\, \varepsilon_{il,k}\, e_{lkj}\, \mathbf{e}_i \mathbf{e}_j$

this reduces to the following: $\oint d\mathbf{r} \cdot \epsilon \times \nabla = 0 =$

$$\oint_{C_i} dx_i (\varepsilon_{i1,2} - \varepsilon_{i2,1})\mathbf{e}_3 + \oint_c dx_i (\varepsilon_{i2,3} - \varepsilon_{i3,2})\mathbf{e}_1 + \oint_c dx_i (\varepsilon_{i3,1} - \varepsilon_{i1,3})\mathbf{e}_2 = 0$$

or we get 3 independent equations

1. $\oint_c (\varepsilon_{11,2} - \varepsilon_{12,1})\, dx_1 + \oint_c dx_2 \,(\varepsilon_{21,2} - \varepsilon_{22,1}) + \oint_c (\varepsilon_{31,2} - \varepsilon_{32,1})\, dx_3 \equiv 0$ since $dx_3 = 0$ and all ε_{ij} in the other two integrals $\equiv 0$

2. $\oint_c (\varepsilon_{12,3} - \varepsilon_{13,2})\, dx_1 + \oint_c (\varepsilon_{22,3} - \varepsilon_{23,2})\, dx_2 + \oint_c (\varepsilon_{32,3} - \varepsilon_{33,2})\, dx_3 = \oint_c - \varepsilon_{13,2}\, dx_1 - \varepsilon_{23,2}\, dx_2$ since $dx_3 = 0$ and all other $\varepsilon_{ij} = 0$

3. $\oint_c (\varepsilon_{13,1} - \varepsilon_{11,3})\, dx_1 + \oint_c (\varepsilon_{23,1} - \varepsilon_{21,3})\, dx_2 + \oint_c (\varepsilon_{33,1} - \varepsilon_{31,3})\, dx_3 = \oint_c \varepsilon_{13,1}\, dx_1 + \varepsilon_{23,1}\, dx_2$ since $dx_3 = 0$ and all other $\varepsilon_{ij} = 0$

For single valuedness $\oint_c - \varepsilon_{13,2}\, dx_1 - \varepsilon_{23,2}\, dx_2 = \oint_c \varepsilon_{13,1}\, dx_1 - \varepsilon_{23,1}\, dx_2 = 0$

$\times$ -4

c. $\oint_c [\boldsymbol{\epsilon} + (\mathbf{r} - \mathbf{r}_0) \times (\nabla \times \boldsymbol{\epsilon})] \cdot d\mathbf{r} = 0$: First we will take $\mathbf{r}_0$ to be the origin and remember that

$\nabla \times \boldsymbol{\epsilon} = \varepsilon_{\ell j,k}\, e_{k\ell i}\, \mathbf{e}_i \mathbf{e}_j$ hence $\mathbf{r} \times (\nabla \times \boldsymbol{\epsilon}) = (x_m \mathbf{e}_m) \times (\varepsilon_{\ell j,k}\, e_{k\ell i}\, \mathbf{e}_i \mathbf{e}_j) = x_m \varepsilon_{\ell j,k}\, e_{k\ell i}\, e_{min}\, \mathbf{e}_n \mathbf{e}_j$

hence we get for $[(\mathbf{r} - \mathbf{r}_0) \times \nabla \times \boldsymbol{\epsilon}] \cdot d\mathbf{r} = x_m \varepsilon_{\ell j,k}\, e_{k\ell i}\, e_{min}\, \mathbf{e}_n\, dx_j$. Putting it together the integral reduces to

$$\oint [\boldsymbol{\epsilon} \times (\mathbf{r} - \mathbf{r}_0) \times (\nabla \times \boldsymbol{\epsilon})] \cdot d\mathbf{r} = \oint_c \Big[\{ (\varepsilon_{31} - x_2 \varepsilon_{31,2} - x_1 \varepsilon_{31,1})\, dx_1 + (\varepsilon_{32} - x_2 \varepsilon_{32,2} - x_1 \varepsilon_{32,1})\, dx_2 \} \mathbf{e}_3 + \{ (\varepsilon_{32} - x_1 \varepsilon_{32,1} + x_1 \varepsilon_{31,2})\, dx_3 \} \mathbf{e}_2 + \{ (\varepsilon_{31} + x_2 \varepsilon_{32,1} - x_2 \varepsilon_{31,2})\, dx_3 \} \mathbf{e}_1 \Big] = 0$$

or we get 3 independent equations

1. $\oint_c \left[(\varepsilon_{31} - x_2 \varepsilon_{31,2} - x_1 \varepsilon_{31,1})\, dx_1 + (\varepsilon_{32} - x_2 \varepsilon_{32,2} - x_1 \varepsilon_{32,1})\, dx_2 \right] = 0$

2. and 3. $\oint_c (\varepsilon_{32} - x_1 \varepsilon_{32,1} + x_1 \varepsilon_{31,2})\, dx_3 = \oint (\varepsilon_{31} + x_2 \varepsilon_{32,1} - x_2 \varepsilon_{31,2})\, dx_3 \equiv 0$ since $dx_3 = 0$

now we can rewrite equation 1 as

$$\oint (x_1 \varepsilon_{31,2}\, dx_2 + x_2 \varepsilon_{32,1}\, dx_1) + 2 \oint_c (\varepsilon_{31}\, dx_1 + \varepsilon_{32}\, dx_2) - (x_1 \varepsilon_{31} + x_2 \varepsilon_{32}) \Big|_0^{2\pi} - \oint (x_2 \varepsilon_{31,2}\, dx_1 + x_1 \varepsilon_{32,1}\, dx_2) = 0$$

and since $x_1(2\pi) = x_1(0)$ & $x_2(2\pi) = x_2(0)$ for single valuedness, we obtain

$$\oint_c \left[(x_1 \varepsilon_{31,2}\, dx_2 + x_2 \varepsilon_{32,1}\, dx_1) + 2 (\varepsilon_{31}\, dx_1 + \varepsilon_{32}\, dx_2) - (x_2 \varepsilon_{31,2}\, dx_1 + x_1 \varepsilon_{32,1}\, dx_2) \right] = 0$$

- SUMMARY we will use $x_1 = r\cos\phi$, $x_2 = r\sin\phi$ with $r = r(\phi)$ in general so that $r(0) = r(2\pi)$

$\nabla \times \boldsymbol{\epsilon} \times \nabla \equiv 0$

From B. $\oint_c -\varepsilon_{13,2}dx_1 - \varepsilon_{23,2}dx_2 = \oint_c (-\varepsilon_{31,2} + \varepsilon_{23,1})dx_1 \equiv \oint_c d\varepsilon_{23} \equiv 0$ (I)

$$\oint_c \varepsilon_{13,1}dx_1 + \varepsilon_{23,1}dx_2 = \oint_c (-\varepsilon_{13,2} + \varepsilon_{23,1})dx_2 \equiv -\oint_c d\varepsilon_{13} \equiv 0 \quad (\mathrm{II})$$

where for single valued displacements $\oint d\varepsilon_{23} \equiv 0$, $\oint d\varepsilon_{13} \equiv 0$. Note that the integrands are 0, so that it doesn't matter what the shape of the hole is (except that the 2nd derivs of ε_{ij} be continuous)

From C.

$$\oint_c \left[2(\varepsilon_{31}dx_1 + \varepsilon_{32}dx_2) - (x_2\varepsilon_{31,2}dx_1 + x_1\varepsilon_{32,1}dx_2) + (x_1\varepsilon_{31,2}dx_2 + x_2\varepsilon_{32,1}dx_1)\right] = \int_0^{2\pi} 2K\,d\phi = 0 \quad (\mathrm{III})$$

or in general $4K\pi = 0$ note that this integral is <u>not</u> dependent on r and would be equal to $4K\pi$ no matter what shape the closed curve is (Curve has to be such that 2nd derivs of ε_{ij} are cont.)

D for the specific case $r = a$

We note that I, II tell you nothing about K. III tells you that $K \equiv 0$. ✓
It is very interesting though that the results (both in general and specific terms) do not depend on the radius and hence the shape of the hole is of no consequence.

DIVISION OF APPLIED MECHANICS

DEPARTMENT OF MECHANICAL ENGINEERING

ME 238A Theory of Elasticity   Autumn 1978

Problem Set No. 4

1. Determine the transformation relations of components of strain if the yz-plane is rotated about the x-axis through an angle θ .

2. Consider a cubic crystal with three elastic constants c_{11} , c_{12} and c_{44} . Determine the relation between these constants if the stress-strain relations are to remain unchanged for a rotation of the xy-plane about the z-axis through an angle ψ .

3. a) Express Lamé's constant λ in terms of the shear modulus and the bulk modulus.

   b) Express Poisson's ratio in terms of Young's modulus and Lamé's constant μ .

3. Write $\lambda = \lambda(G, K)$ where $e = \dfrac{I\sigma}{(3\lambda + 2\mu)}$ $= -3p$ where $e = \varepsilon_x + \varepsilon_y + \varepsilon_z$

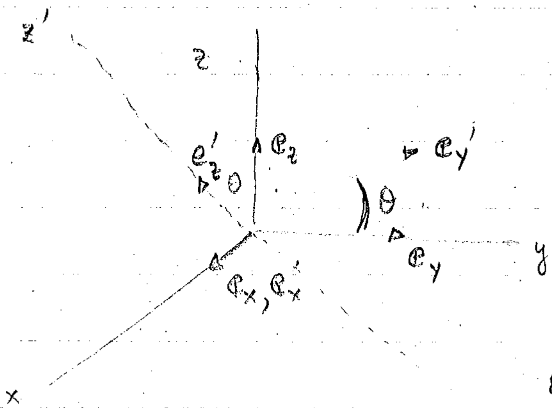
$$K = \frac{3\lambda + 2\mu}{3}$$

$$G = \frac{E}{2(1+\nu)} = \mu$$

$$3K = 3\lambda + 2G \Rightarrow \boxed{\frac{3K - 2G}{3} = \lambda}$$

$\nu = \nu(E, \mu)$ since $G = \mu = \dfrac{E}{2(1+\nu)}$ $\therefore$ $1 + \nu = \dfrac{E}{2\mu}$ or $\boxed{\nu = \dfrac{E}{2\mu} - 1 = \dfrac{E - 2\mu}{2\mu}}$

1.



find ε_{ij}' $e_m' \cdot e_i = l_{mi}$

	e_x	e_y	e_z
e_1'	1	0	0
e_2'	0	$\cos\theta$	$\sin\theta$
e_3'	0	$-\sin\theta$	$\cos\theta$

$l_{21} = l_{12} = l_{13} = l_{31} = 0$

$$\varepsilon_{mn}' = \varepsilon_{ij}\, l_{mi}\, l_{nj}$$

✓ $\varepsilon_{11}' = \varepsilon_{11}$

✓ $\gamma_{12}' = \gamma_{12}\cos\theta + \gamma_{13}\sin\theta$

✓ $\gamma_{13}' = -\gamma_{12}\sin\theta + \gamma_{13}\cos\theta$

✓ $\gamma_{21}' = \gamma_{21}\cos\theta + \gamma_{31}\sin\theta$

✓ ε_{22}' $\varepsilon_{22}\cos^2\theta + \gamma_{23}\dfrac{\sin 2\theta}{2} + \varepsilon_{33}\sin^2\theta$

✓ $\gamma_{23}' = -\varepsilon_{22}\sin 2\theta + \gamma_{23}(\cos^2\theta - \sin^2\theta) + \varepsilon_{33}\sin 2\theta$

✓ $\gamma_{31}' = -\gamma_{21}\sin\theta + \gamma_{31}\cos\theta$

✓ $\gamma_{32}' = -\varepsilon_{22}\sin 2\theta + \gamma_{23}(\cos^2\theta - \sin^2\theta) + \varepsilon_{33}\sin 2\theta$

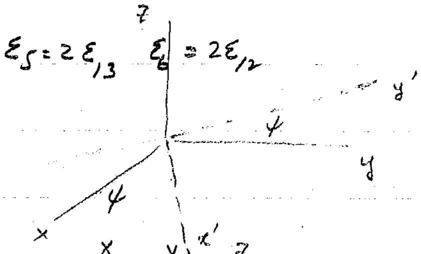
✓ $\varepsilon_{33}' = \varepsilon_{22}\sin^2\theta - \gamma_{23}\dfrac{\sin 2\theta}{2} + \varepsilon_{33}\cos^2\theta$

$x \to z$
$z \to y$
$y \to x$

$\therefore$ $\varepsilon_{11}' = \varepsilon_{11}$, $\varepsilon_{22}' = \varepsilon_{22}\cos^2\theta + \varepsilon_{33}\sin^2\theta + \gamma_{23}\dfrac{\sin 2\theta}{2}$, $\varepsilon_{33}' = \varepsilon_{22}\sin^2\theta - \gamma_{23}\dfrac{\sin 2\theta}{2} + \varepsilon_{33}\cos^2\theta$

$\gamma_{12}' = \gamma_{21}' = \gamma_{12}\cos\theta + \gamma_{13}\sin\theta$, $\gamma_{13}' = \gamma_{31}' = -\gamma_{12}\sin\theta + \gamma_{13}\cos\theta$, $\gamma_{23}' = \gamma_{32}' = \sin 2\theta(\varepsilon_{33} - \varepsilon_{22}) + \gamma_{23}\cos 2\theta$

where $\varepsilon_1 = \varepsilon_{11}$ $\varepsilon_2 = \varepsilon_{22}$ $\varepsilon_3 = \varepsilon_{33}$ $\varepsilon_4 = 2\varepsilon_{23}$ $\varepsilon_5 = 2\varepsilon_{13}$ $\varepsilon_6 = 2\varepsilon_{12}$



2. a cubic crystal

$$\begin{bmatrix} c_{11} & c_{12} & c_{12} & & & \\ c_{12} & c_{11} & c_{12} & & & \\ c_{12} & c_{12} & c_{11} & & & \\ & & & c_{44} & & \\ & & & & c_{44} & \\ & & & & & c_{44} \end{bmatrix}$$

we want

$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & \cos\psi & \sin\psi & 0 \\ y' & -\sin\psi & \cos\psi & 0 \\ z' & 0 & 0 & 1 \end{array}$$

σ_{xx} $\sigma_1' = \sigma_1 \cos^2\psi + \sigma_6 \sin 2\psi + \sigma_2 \sin^2\psi$ $\qquad$ $\varepsilon_1' = \varepsilon_1 \cos^2\psi + \varepsilon_6 \frac{\sin 2\psi}{2} + \varepsilon_2 \sin^2\psi$

σ_{yy} $\sigma_2' = \sigma_1 \sin^2\psi - \sigma_6 \sin 2\psi + \sigma_2 \cos^2\psi$ $\qquad$ $\varepsilon_2' = \varepsilon_1 \sin^2\psi - \varepsilon_6 \frac{\sin 2\psi}{2} + \varepsilon_2 \cos^2\psi$

σ_{zz} $\sigma_3' = \sigma_3$ $\qquad$ $\varepsilon_3' = \varepsilon_3$

σ_{yz} $\sigma_4' = -\sigma_5 \sin\psi + \sigma_4 \cos\psi$ $\qquad$ $\varepsilon_4' = -\varepsilon_5 \sin\psi + \varepsilon_4 \cos\psi$

σ_{zx} $\sigma_5' = \sigma_5 \cos\psi + \sigma_4 \sin\psi$ $\qquad$ $\varepsilon_5' = \varepsilon_5 \cos\psi + \varepsilon_4 \sin\psi$

σ_{xy} $\sigma_6' = -\frac{\sigma_1}{2} \sin 2\psi + \sigma_6 \cos 2\psi + \frac{\sigma_2}{2} \sin 2\psi$ $\qquad$ $\varepsilon_6' = -\varepsilon_1 \sin 2\psi + \varepsilon_6 \cos 2\psi + \varepsilon_2 \sin 2\psi$

$= (\sigma_2 - \sigma_1)\, cs + \sigma_6 (c 2\psi)$ $\qquad$ $(\varepsilon_2 - \varepsilon_1) \sin 2\psi + \varepsilon_6 \cos 2\psi$

$\sigma_1' = c_{11}\varepsilon_1' + c_{12}\varepsilon_2' + c_{12}\varepsilon_3'$ $\qquad$ & $\sigma_1 = c_{11}\varepsilon_1 + c_{12}\varepsilon_2 + c_{12}\varepsilon_3$

$$\sigma_1 \cos^2\psi + \sigma_6 \sin 2\psi + \sigma_2 \sin^2\psi = c_{11}\varepsilon_1 \cos^2\psi + c_{11}\varepsilon_6 \frac{\sin 2\psi}{2} + c_{11}\varepsilon_2 \sin^2\psi + c_{12}\varepsilon_1 \sin^2\psi - c_{12}\varepsilon_6 \frac{\sin 2\psi}{2} + c_{12}\varepsilon_2 \cos^2\psi + c_{12}\varepsilon_3$$

$\sigma_2' = c_{12}\varepsilon_1' + c_{11}\varepsilon_2' + c_{12}\varepsilon_3'$ $\qquad$ & $\sigma_2 = c_{12}\varepsilon_1 + c_{11}\varepsilon_2 + c_{12}\varepsilon_3$

$$\sigma_1 \sin^2\psi - \sigma_6 \sin 2\psi + \sigma_2 \cos^2\psi = c_{12}\left(\varepsilon_1 \cos^2\psi + \varepsilon_6 \frac{\sin 2\psi}{2} + \varepsilon_2 \sin^2\psi\right) + c_{11}\left(\varepsilon_1 \sin^2\psi - \varepsilon_6 \frac{\sin 2\psi}{2} + \varepsilon_2 \cos^2\psi\right) + c_{12}\varepsilon_3$$

$\sigma_3' = c_{12}\varepsilon_1' + c_{12}\varepsilon_2' + c_{11}\varepsilon_3'$ $\qquad$ $\sigma_3 = c_{12}\varepsilon_1 + c_{12}\varepsilon_2 + c_{11}\varepsilon_3$

$$\sigma_3 = c_{12}\left(\varepsilon_1 \cos^2\psi + \varepsilon_6 \frac{\sin 2\psi}{2} + \varepsilon_2 \sin^2\psi\right) + c_{12}\left(\varepsilon_1 \sin^2\psi - \varepsilon_6 \frac{\sin 2\psi}{2} + \varepsilon_2 \cos^2\psi\right) + c_{11}\varepsilon_3$$

$\sigma_4' = c_{44}\varepsilon_4'$ $\qquad$ $\sigma_4 = c_{44}\varepsilon_4$

$-\sigma_5 \sin\psi + \sigma_4 \cos\psi = c_{44}(-\varepsilon_5 \sin\psi + \varepsilon_4 \cos\psi)$ $\Rightarrow$ $\cancel{-\sigma_5 \sin\psi = c_{44}(-\varepsilon_5 \sin\psi)}$

$\sigma_5' = c_{44}\varepsilon_5'$ $\qquad$ $\sigma_5 = c_{44}\varepsilon_5$

$\sigma_5 \cos\psi + \sigma_4 \sin\psi = c_{44}(\varepsilon_5 \cos\psi + \varepsilon_4 \sin\psi)$ $\Rightarrow$ $\cancel{\sigma_4 \sin\psi = c_{44}\varepsilon_4 \sin\psi}$

$$\sigma_6' = C_{44}\,\varepsilon_6' \qquad \sigma_6 = C_{44}\,\varepsilon_6$$

$$-\frac{\sigma_1}{2}\sin 2\psi + \sigma_6\cos 2\psi + \frac{\sigma_2}{2}\sin 2\psi = C_{44}\left(-\varepsilon_1\sin 2\psi + \varepsilon_6\cos 2\psi + \varepsilon_2\sin 2\psi\right)$$

$$\sigma_1\cos^2\psi = C_{11}\varepsilon_1\cos^2\psi + C_{12}\varepsilon_2\cos^2\psi + C_{12}\varepsilon_3\cos^2\psi$$

$$+\sigma_6\sin 2\psi = C_{44}\varepsilon_6\sin 2\psi$$

$$+\sigma_2\sin^2\psi = C_{12}\varepsilon_1\sin^2\psi + C_{11}\varepsilon_2\sin^2\psi + C_{12}\varepsilon_3\sin^2\psi$$

$$= C_{11}\varepsilon_1\cos^2\psi + C_{12}\varepsilon_2\cos^2\psi + C_{12}\varepsilon_3\cos^2\psi + C_{44}\varepsilon_6\sin 2\psi + C_{12}\varepsilon_1\sin^2\psi + C_{11}\varepsilon_2\sin^2\psi \quad + C_{12}\varepsilon_3 \sin^2$$

$$= C_{11}\varepsilon_1\cos^2\psi + C_{12}\varepsilon_2\cos^2\psi + C_{12}\varepsilon_3 + \frac{C_{11}}{2}\varepsilon_6\sin 2\psi + C_{12}\varepsilon_1\sin^2\psi + C_{11}\varepsilon_2\sin^2\psi$$

$$- \frac{C_{12}}{2}\varepsilon_6\sin 2\psi$$

$$\Rightarrow \boxed{\cancel{C_{12}(\cos^2\psi - 1)\varepsilon_3} + \left(C_{44} - \frac{C_{11}+C_{12}}{2}\right)\varepsilon_6\sin 2\psi + \cancel{C_{12}\varepsilon_1\sin^2\psi} + \cancel{C_{11}\varepsilon_2\sin^2\psi} = 0}$$

$$C_{44} - \frac{C_{11}+C_{12}}{2} = 0$$

$C_{44} = C_{11}\ C_{12}$

$$\sigma_3\sin^2\psi = C_{12}\varepsilon_1\sin^2\psi + C_{12}\varepsilon_2\sin^2\psi + C_{11}\varepsilon_3\sin^2\psi$$

$$-\sigma_6\sin 2\psi = -C_{44}\varepsilon_6\sin 2\psi$$

$$+\sigma_2\cos^2\psi = C_{12}\varepsilon_1\cos^2\psi + C_{11}\varepsilon_2\cos^2\psi + C_{12}\varepsilon_3\cos^2\psi$$

$$= C_{12}\varepsilon_1\sin^2\psi + C_{12}\varepsilon_2\sin^2\psi + C_{11}\varepsilon_3\sin^2\psi - C_{44}\varepsilon_6\sin 2\psi + C_{12}\varepsilon_1\cos^2\psi + C_{11}\cos^2\psi\,\varepsilon_2 + C_{12}\varepsilon_3\cos^2\psi$$

$$= C_{12}\varepsilon_1\cos^2\psi + C_{12}\varepsilon_2\sin^2\psi + \cancel{C_{11}\varepsilon_3\sin^2\psi} + C_{12}\varepsilon_6\sin 2\psi \quad + C_{11}\varepsilon_2\cos^2\psi + C_{12}\varepsilon_3$$

$$- C_{11}\varepsilon_6\sin 2\psi$$

$$\Longrightarrow \boxed{-C_{12}\varepsilon_1\cos 2\psi + \varepsilon_6(-C_{44} - C_{12} + C_{11})\sin 2\psi + \varepsilon_1(C_{12}\cos^2\psi) + C_{12}(\cos^2\psi - 1)\varepsilon_3 = 0}$$

$$\sigma_3 = C_{12}\varepsilon_1 + C_{12}\varepsilon_2 + C_{11}\varepsilon_3$$

$$\sigma_3 = C_{12}\varepsilon_1\cos^2\psi + C_{12}\varepsilon_2\sin^2\psi + C_{11}\varepsilon_3 + C_{12}\varepsilon_6\sin 2\psi$$

$$C_{12}\varepsilon_1\sin^2\psi + C_{12}\varepsilon_2\cos^2\psi + \quad - C_{12}\varepsilon_6\sin 2\psi$$

$$\boxed{C_{12}(1-\cos^2\psi)\varepsilon_1 + C_{12}(-\sin^2\psi)\varepsilon_3 = 0}$$

$0 = 0$ nothing from here ~~C_{12} 2 or 2 ψ/0~~

$$-\sigma_5\sin\psi + \sigma_4\cos\psi = -\varepsilon_5 C_{44}\sin\psi + \varepsilon_4 C_{44}\cos\psi$$

$$= \varepsilon_5 C_{44}\sin\psi + \varepsilon_4 C_{44}\cos\psi$$

$$0 = 0 \quad \checkmark$$

nothing here

$$\sigma_5 \cos\psi + \sigma_4 \sin\psi = \varepsilon_5 C_{44}\cos\psi + \varepsilon_4 C_{44}\sin\psi$$
$$C_{44}\varepsilon_5\cos\psi + \varepsilon_4 C_{44}\sin\psi$$
$$0 = 0$$ ✓ nothing here.

$$-\frac{\sigma_1}{2}\sin 2\psi = -\frac{1}{2}\sin 2\psi\,(C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{12}\varepsilon_3)$$
$$+\sigma_6\cos 2\psi = C_{44}\varepsilon_6\cos 2\psi$$
$$+\frac{\sigma_2}{2}\sin 2\psi = \frac{1}{2}\sin 2\psi\,(C_{12}\varepsilon_1 + C_{11}\varepsilon_2 + \varepsilon_3 C_{12})$$
$$= -C_{11}\varepsilon_1\left(+\tfrac{1}{2}\sin 2\psi\right) - C_{12}\varepsilon_2\left(+\tfrac{1}{2}\sin 2\psi\right) + \cancel{C_{12}\varepsilon_3\left(-\tfrac{1}{2}\sin 2\psi\right)} + \cancel{C_{44}\varepsilon_6\cos 2\psi} + \tfrac{1}{2}\sin 2\psi\, C_{12}\varepsilon_1 + \tfrac{1}{2}\sin 2\psi\, C_{11}\varepsilon_2 + \tfrac{1}{2}\sin 2\psi\, C_{12}\varepsilon_3$$
$$-C_{44}\varepsilon_1(\sin 2\psi) + C_{44}\varepsilon_2(\sin 2\psi) \qquad \cancel{+C_{44}\varepsilon_6\cos 2\psi}$$

$$\boxed{\varepsilon_1(-C_{11}+C_{44}+C_{12})\frac{\sin 2\psi}{2} + \varepsilon_2(-C_{12}-C_{11}-C_{44})\frac{\sin 2\psi}{2} - \sin 2\psi\, C_{12}\varepsilon_3 = 0}$$

$$\left(-\frac{C_{11}}{2}+\frac{C_{12}}{2}+C_{44}\right)\varepsilon_1 \sin 2\psi + \left(\frac{C_{11}}{2}-\frac{C_{12}}{2}-C_{44}\right)\varepsilon_2 \sin 2\psi = 0 \qquad \left(-\frac{C_{11}+C_{12}}{2}+C_{44}\right)(\varepsilon_1-\varepsilon_2)\sin 2\psi = 0$$
$$\frac{C_{11}-C_{12}}{2} = C_{44}$$

$$\begin{bmatrix} C_{12}\sin^2\psi & C_{11}\sin^2\psi & C_{12}(-\sin^2\psi) & 0 & 0 & (C_{44}-C_{11}+C_{12})\sin 2\psi \\ C_{12}\sin^2\psi & 0 & C_{12}(-\sin^2\psi) & 0 & 0 & -(C_{44}-C_{11}+C_{12})\sin 2\psi \\ C_{12}\sin^2\psi & 0 & C_{12}(-\sin^2\psi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ (-C_{11}+C_{44}+C_{12})\frac{\sin 2\psi}{2} & (C_{12}+C_{11}+C_{44})\frac{\sin 2\psi}{2} - C_{12}\sin 2\psi & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \\ \varepsilon_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\sigma_1\sin^2\psi = C_{11}\varepsilon_1\sin^2\psi + C_{12}\varepsilon_2\sin^2\psi + C_{12}\varepsilon_3\sin^2\psi$$
$$-\sigma_6\sin 2\psi = -C_{44}\varepsilon_6\sin 2\psi$$
$$+\sigma_2\cos^2\psi = C_{12}\varepsilon_1\cos^2\psi + C_{11}\varepsilon_2\cos^2\psi + C_{12}\varepsilon_3\cos^2\psi$$
$$= C_{11}\varepsilon_1\sin^2\psi + C_{12}\varepsilon_2\sin^2\psi + C_{12}\varepsilon_3\sin^2\psi - C_{44}\varepsilon_6\sin 2\psi + C_{12}\varepsilon_1\cos^2\psi + C_{11}\varepsilon_2\cos^2\psi + C_{12}\varepsilon_2\cos^2\psi$$
$$C_{11}\varepsilon_1\sin^2\psi + C_{12}\varepsilon_2\sin^2\psi + C_{12}\varepsilon_3 \qquad + \frac{C_{12}}{2}\varepsilon_6\sin 2\psi + C_{12}\varepsilon_1\cos^2\psi + C_{11}\varepsilon_2\cos^2\psi - \frac{C_{11}}{2}\varepsilon_6\sin 2\psi$$

$$\left(-C_{44}-\frac{C_{12}}{2}+\frac{C_{11}}{2}\right)\varepsilon_6\sin 2\psi = 0 \qquad \text{or} \qquad \boxed{C_{44} - \frac{C_{11}+C_{12}}{2} = 0}$$

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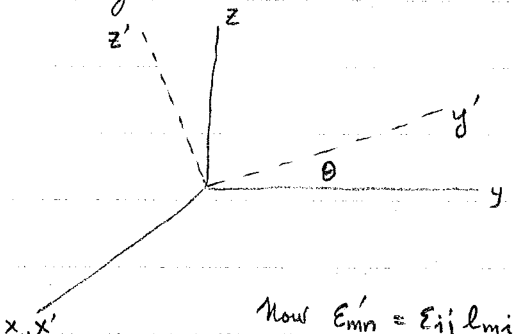
100

Problem #4

1\. Determine the strains $\varepsilon_{ij}' = \varepsilon_{ij}'(\varepsilon_{ij})$ if the yz plane is rotated about the x axis through an angle θ.



	e_x	e_y	e_z
e_x'	1	0	0
e_y'	0	$\cos\theta$	$\sin\theta$
e_z'	0	$-\sin\theta$	$\cos\theta$

and define $e_m' \cdot e_i = l_{mi}$

Now $\varepsilon_{mn}' = \varepsilon_{ij} l_{mi} l_{nj}$. By applying this using the above transformation matrix we obtain

For 2D case drop $\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{xz}$

$$\varepsilon_{xx}' = \varepsilon_{xx}$$

$$\varepsilon_{yy}' = \varepsilon_{yy}\cos^2\theta + \varepsilon_{zz}\sin^2\theta + \varepsilon_{yz}\sin 2\theta$$

$$\varepsilon_{zz}' = \varepsilon_{yy}\sin^2\theta - \varepsilon_{yz}\sin 2\theta + \varepsilon_{zz}\cos^2\theta$$

$$\varepsilon_{xy}' = \varepsilon_{yx}' = \varepsilon_{xy}\cos\theta + \varepsilon_{xz}\sin\theta$$

$$\varepsilon_{xz}' = \varepsilon_{zx}' = -\varepsilon_{xy}\sin\theta + \varepsilon_{xz}\cos\theta$$

$$\varepsilon_{yz}' = \varepsilon_{zy}' = \frac{\sin 2\theta}{2}\left(\varepsilon_{zz} - \varepsilon_{yy}\right) + \varepsilon_{yz}\cos 2\theta$$

2\. Consider a cubic crystal whose stiffness matrix looks as follows:

$$\Delta = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix}$$

Find the relation between c_{11}, c_{12}, c_{44} if after a rotation of the xy plane about the z axis through an angle ψ leaves the stress-strain relations unchanged.

A. First the rotation matrix looks as follows:

	e_1	e_2	e_3
e_1'	$\cos\theta$	$\sin\theta$	0
e_2'	$-\sin\theta$	$\cos\theta$	0
e_3'	0	0	1

we now renumber the stresses and strains so that

$\sigma_1 = \sigma_{11}$, $\sigma_2 = \sigma_{22}$, $\sigma_3 = \sigma_{33}$, $\sigma_4 = \sigma_{23}$, $\sigma_5 = \sigma_{13}$, $\sigma_6 = \sigma_{12}$

$\varepsilon_1 = \varepsilon_{11}$, $\varepsilon_2 = \varepsilon_{22}$, $\varepsilon_3 = \varepsilon_{33}$, $\varepsilon_4 = 2\varepsilon_{23}$, $\varepsilon_5 = 2\varepsilon_{13}$, $\varepsilon_6 = 2\varepsilon_{12}$

B. To find the relations between σ_i' and σ_i, ε_i' and ε_i we note we could use the equations found in problem 1 if in the double subscripted variables $x \to 3$, $y \to 1$, $z \to 2$.

therefore if we now place the result into the single subscripted variables

$$\begin{pmatrix}\sigma_1'\\ \varepsilon_1'\end{pmatrix} = \begin{pmatrix}\sigma_1\\ \varepsilon_1\end{pmatrix}\cos^2\psi + \begin{pmatrix}\sigma_6\\ \frac{\varepsilon_6}{2}\end{pmatrix}\sin 2\psi + \begin{pmatrix}\sigma_2\\ \varepsilon_2\end{pmatrix}\sin^2\psi \quad ; \quad \begin{pmatrix}\sigma_2'\\ \varepsilon_2'\end{pmatrix} = \begin{pmatrix}\sigma_1\\ \varepsilon_1\end{pmatrix}\sin^2\psi - \begin{pmatrix}\sigma_6\\ \frac{\varepsilon_6}{2}\end{pmatrix}\sin 2\psi + \begin{pmatrix}\sigma_2\\ \varepsilon_2\end{pmatrix}\cos^2\psi$$

$$\begin{pmatrix}\sigma_3'\\ \varepsilon_3'\end{pmatrix} = \begin{pmatrix}\sigma_3\\ \varepsilon_3\end{pmatrix} \; ; \; \begin{pmatrix}\sigma_4'\\ \varepsilon_4'\end{pmatrix} = -\begin{pmatrix}\sigma_5\\ \varepsilon_5\end{pmatrix}\sin\psi + \begin{pmatrix}\sigma_4\\ \varepsilon_4\end{pmatrix}\cos\psi \; ; \; \begin{pmatrix}\sigma_5'\\ \varepsilon_5'\end{pmatrix} = \begin{pmatrix}\sigma_5\\ \varepsilon_5\end{pmatrix}\cos\psi + \begin{pmatrix}\sigma_4\\ \varepsilon_4\end{pmatrix}\sin\psi$$

$$\begin{pmatrix}\sigma_6'\\ \varepsilon_6'\end{pmatrix} = -\begin{pmatrix}\sigma_1\\ \varepsilon_1\end{pmatrix}\sin 2\psi + \begin{pmatrix}\sigma_6\\ \varepsilon_6\end{pmatrix}\cos 2\psi + \begin{pmatrix}\sigma_2\\ \varepsilon_2\end{pmatrix}\sin 2\psi$$

c. Now we know that $\underset{\sim}{\sigma}' = \Delta\,\underset{\sim}{\varepsilon}'$ and $\underset{\sim}{\sigma} = \Delta\,\underset{\sim}{\varepsilon}$ for invariance under this transformation.

By manipulation of $\sigma_i' = \delta_{ij}\,\varepsilon_j'$ and $\sigma_i = \delta_{ij}\,\varepsilon_j$ we can get the following for the 6 equations

$$\left(C_{44} - \frac{C_{11}+C_{12}}{2}\right)\varepsilon_6 \sin 2\psi = 0$$

$$\left(-C_{44} - \frac{C_{12}+C_{11}}{2}\right)\varepsilon_6 \sin 2\psi = 0$$

$$0 = 0$$

$$0 = 0$$

$$0 = 0$$

$$\left(\frac{C_{11}-C_{12}}{2} - C_{44}\right)(\varepsilon_2 - \varepsilon_1)\sin 2\psi = 0$$

From all these the basic equation that pops out if $\psi \neq 0$, $\varepsilon_6 \neq 0$, $\varepsilon_1 \neq \varepsilon_2$ is

$$\boxed{\tfrac{1}{2}(C_{11} - C_{12}) = C_{44}}$$

3. Express Lamé's constant λ as $\lambda = \lambda(G, K)$ where G is the shear modulus and K is the bulk modulus.

Express Poisson's ratio ν as $\nu = \nu(E, \mu)$ where μ is the 2nd Lamé constant

a. We know that $\sigma_{ij} = \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}$ now $\sigma_{ii} = 3\lambda\,\varepsilon_{kk} + 2\mu\,\varepsilon_{kk} = (3\lambda + 2\mu)\,\varepsilon_{kk}$

if $\sigma_{ii} = -3p$ (hydrostatic pressure) then $\varepsilon_{kk} = \dfrac{-p}{K} = \dfrac{-p}{(3\lambda+2\mu)/3}$

We also know that $\mu = G$ hence $K = (3\lambda + 2G)/3$ or $\boxed{\lambda = \dfrac{3K - 2G}{3}}$

b. We know that $\varepsilon_{ij} = \dfrac{1+\nu}{E}\sigma_{ij} - \dfrac{\nu}{E} I_\sigma\,\delta_{ij}$ where $I_\sigma = \sigma_{11} + \sigma_{22} + \sigma_{33}$

when $i \neq j$ $\quad \varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij}$ $\quad$ or $\quad 2\epsilon_{ij} = \gamma_{ij} = \frac{2(1+\nu)}{E}\sigma_{ij}$ $\quad$ and $\quad \frac{2(1+\nu)}{E} = \frac{1}{G} = \frac{1}{\mu}$

hence $\quad \frac{E}{2\mu} - 1 = \nu \quad$ or $\quad \boxed{\nu = \frac{E-2\mu}{2\mu}}$