DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING
STANFORD UNIVERSITY

ME 236 WAVES AND VIBRATIONS — Autumn 1979

SUGGESTED REFERENCE MATERIAL

1. J. D. Achenbach - *Wave Propagation in Elastic Solids* (North Holland, 1973)

2. B. A. Auld - *Acoustic Fields and Waves in Solids*, Volumes I and II (Wiley-Interscience, New York, 1973)

3. L. Brillouin - *Wave Propagation in Periodic Structures* (Dover, 1953)

4. L. Brillouin - *Wave Propagation and Group Velocity* (Academic Press, 1960)

(3–4) wrk in crystal lattice.

5. W. M. Ewing, W. S. Jardetsky and F. Press - *Elastic Waves in Layered Media* (McGraw-Hill, 1957) Geophysics

6. Y. C. Fung - *Foundations of Solid Mechanics* (Prentice Hall, 1965) general mechanics + deformable bodies

#7. H. Kolsky - *Stress Waves in Solids* (Oxford, 1953) Oldest book

#8. A. H. E. Love - *Mathematical Theory of Elasticity* (Dover, 1944) one of topics in book

9. P. M. Morse - *Vibration and Sound* (McGraw-Hill, 1948) written from acoustician point of view primarily in air

10. Y.-H. Pao and C.-C. Mow - *Diffraction of Elastic Waves and Dynamic Stress Concentrations* (Crane, Russak, 1973) Special Applic.

11. Lord Rayleigh - *Theory of Sound*, Volumes I and II (Dover, 1945)(First Edition, 1877) Discrete Systems discussed.

12. J. S. Rinehart - *Stress Transients in Solids* (Hyperdynamics, Sante Fe, (197?) Very large stresses in solids

13. L. A. Viktorov - *Rayleigh and Lamb Waves* (Plenum Press, New York, 1967) Translated - EE approach.
Surface waves / waves in thin layer.

14. R. J. Wasley - *Stress Wave Propagation in Solids* (M. Dekker, New York, 1973) Same class as 1.

15. G. B. Whitham - *Linear and Nonlinear Waves* (Wiley-Interscience, New York, 1974) Applied Mathematics point of view applied to fluid problem primarily

16. K. F. Graff - *Wave Motion in Elastic Solids* (Ohio State University Press, Columbus, 1975) almost same as one.

17. J. Miklowitz - *The Theory of Elastic Waves and Wave~~grids~~guides*, (North Holland, 1978)

18. *Modern Problems in Elastic Wave Propagation* - edited by J. Miklowitz and J.D. Achenbach (Wiley-Interscience, 1978) Proceedings of a conference

I 1. if $\mathbf{u} = A e^{\phi(\mathbf{x},t)}$ then $\nabla_x \phi = \mathbf{k}$ where $\nabla_x = \mathbf{e}_{x_i}\frac{\partial}{\partial x_i}$

2. $\frac{\partial \phi}{\partial t} = -\omega$ and $\nabla_k \, \omega = \mathbf{c}_g$ (group veloc.) $\nabla_k = \mathbf{e}_{k_i}\frac{\partial}{\partial k_i}$ $\Big\}$ $\phi = \mathbf{k}\cdot\mathbf{x} - \omega t$

3. $\frac{\omega}{\|\mathbf{k}\|}\, \mathbf{n}_k = \mathbf{c}_p$ (phase velocity) $\mathbf{n}_k = \frac{\mathbf{k}}{\|\mathbf{k}\|}$

4. ENERGY propagated with $\mathbf{c}_g$; dispersion relation relates ω & $\mathbf{k}$; DISTURBANCE propagates with $\mathbf{c}_p$

II One dimensional motion $c^2 = E/\rho$ $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$ $u = f(x-ct) + g(x+ct)$ $x \pm ct$ are characteristics

along this line $E u_x = \sigma$ $\therefore$ $E\frac{\partial^3 u}{\partial x^3} - \frac{1}{c^2} E \frac{\partial^3 u}{\partial t^2 \partial x} = 0 \Rightarrow \frac{\partial^2 \sigma}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \sigma}{\partial t^2} = 0$ $\sigma = -\rho c \dot{u}$

DISPERSION occurs due to inhomogeneity, boundaries, inelasticity, nonlinearities.

III Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{kc}$ where ω is circular freq ; k is wave number ; c is 1-D phase veloc.

wavelength $\lambda = \frac{2\pi}{k}$

IV Eqns of motion w/o body force $\sigma_{ij,j} = \rho \ddot{u}_i$ $\nabla\cdot\boldsymbol{\sigma} = \rho\ddot{\mathbf{u}}$

Strain - Displ $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla)$

Strain - Stress Law $\sigma_{ij} = \lambda \delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$ $\boldsymbol{\sigma} = \lambda \mathbf{I}\Delta + 2\mu\boldsymbol{\varepsilon}$

Eqns of Motion - displ formulation $\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} = \rho\ddot{u}_i$ $\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) = \rho\ddot{\mathbf{u}}$

Wave Eqn on dilatation $\nabla\cdot\mathbf{u} = \Delta$ $(\lambda+2\mu)\nabla^2(\Delta) = \rho\frac{\partial^2}{\partial t^2}(\Delta)$ no shear

rotation $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ $\mu\nabla^2\boldsymbol{\omega} = \rho\frac{\partial^2}{\partial t^2}\boldsymbol{\omega}$ no dilatation

→ $\mathbf{u} = \nabla\phi + \nabla\times\mathbf{H}$ with $\nabla\cdot\mathbf{H} = 0$ (Helmholtz Decomposition)

$\therefore$ $(\lambda+2\mu)\nabla^2\phi = \rho\ddot{\phi}$ & $\mu\nabla^2\mathbf{H} = \rho\ddot{\mathbf{H}}$

thus

$$\mathbf{W} = -\frac{1}{4\pi}\iint \frac{\mathbf{u}(\boldsymbol{\xi})}{|\mathbf{x}-\boldsymbol{\xi}|}\, dV_\xi \qquad \phi = \nabla\cdot\mathbf{W} \qquad \mathbf{H} = -\nabla\times\mathbf{W} \qquad |\mathbf{x}-\boldsymbol{\xi}| = [(x_i-\xi_i)(x_i-\xi_i)]^{1/2}$$

$$dV_\xi = d\xi_1 d\xi_2 d\xi_3$$

V PLANE waves in ∞ medium

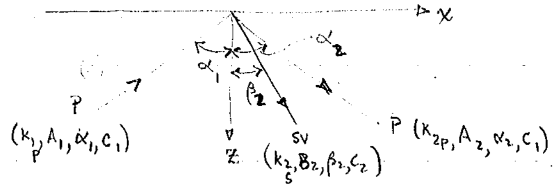
$\mathbf{u} = A\mathbf{d}\, e^{ik(\mathbf{w}\cdot\mathbf{r} - ct)}$ if $\mathbf{d} \parallel \mathbf{w}$ then $c_1 = \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}$ P waves

$\mathbf{d} \perp \mathbf{w}$ then $c_2 = (\mu/\rho)^{1/2}$ SH, SV "

if wave amplitude is not constant then it is an INHOMOGENEOUS wave ie the vector $\mathbf{w}$ is complex

$\mathbf{w}$ - is a unit vector in direction of propagation $\mathbf{d}$ is a unit vector in the direction of motion

VI Reflections in an elastic half space at traction free surfaces



$k_1 = k_{2p}$ $\frac{k_{2s}}{k_{2p}} = \frac{c_1}{c_2} = \frac{\sin\alpha_2}{\sin\beta_2}$ and $k_{2p}\sin\alpha_2 = k$

$\alpha_1 = \alpha_2$,

$k = k_1 \sin\alpha_1$ is the wave number of wave propag. along $z = 0$

$c = \omega/k = c_1/\sin\alpha_1$ phase velocity at surface

Love waves are dispersive since the eqn $f(\omega, k) = 0 \Rightarrow$

$$\tan\left\{\left[\left(\frac{c}{c_2^B}\right)^2 - 1\right]^{1/2} kH\right\} - \frac{\mu}{\mu^B}\,\frac{\left[1 - c^2/c_2^2\right]^{1/2}}{\left[1 - c^2/c_{2}^{B\,2}\right]^{1/2}} = 0$$

X. Waves in an elastic layer in plane strain take $\phi = f e^{ik(x-ct)}$ $H_3 = h e^{ik(x-ct)}$

<u>See pg 20-21</u> u_x (u_y) is sym. in x_2 if it contains cos y (sin y) Extens.

u_x (u_y) is antisym in x_2 " " " sin y (cos y) Flexural

these give rise to Rayleigh-Lamb Freq Eqns.

See pg 22 for table on extensional / Flexural motions

ME 236 – Waves & Vibs Prof. Herrmann

Propagation of disturbance created by some phenomena.

We may look at what happens when entire structure is impressed with a dynamic load or locally how the stress dyadic changes with the propagation of the waves.

Objective: to familiarize oneself where wave prog can occur not so much on how to solve the problems.

Wave propagation in continuous systems

let $u(\mathbf{x}, t)$ be the dependent variables. $\mathbf{x}$ = independent variables (space)

Let $L\left[u(\mathbf{x},t)\right] = 0$ where L is the linear operator characterizing the system.

The solution will be of the form $u = \mathbb{A}\, e^{\phi(\mathbf{x},t)}$ where $\mathbb{A}$ is the amplitude and ϕ being the phase.

since $\phi = \phi(\mathbf{x}, t)$ we can define $\nabla_{\mathbf{x}} \phi = \mathbb{k}$ = wave number

also $\frac{\partial \phi}{\partial t} = -\omega$ the frequency

Thus $L[u] = 0 \iff f(\omega, \mathbb{k}) = 0$ this is the dispersion relation which plays a central relation in wave propagation.

We can now form $\nabla_{\mathbb{K}}\, \omega = \mathbb{C}_g$ which is the group velocity. It is of importance since energy of system will be propagated with the group velocity

– we can define $\frac{\omega}{\|\mathbb{K}\|} \mathbb{n}_k = \mathbb{C}_p$ which is the phase velocity $\mathbb{n}_k = \frac{\mathbb{K}}{\|\mathbb{K}\|}$

$$\frac{\omega\, \mathbb{n}_K}{\|\mathbb{K}\|} = \frac{\omega\, \mathbb{K}}{K^2} = \mathbb{C}_p$$

Example: one dim problem – if: $u(x,t) = U \cos\phi(x,t)$

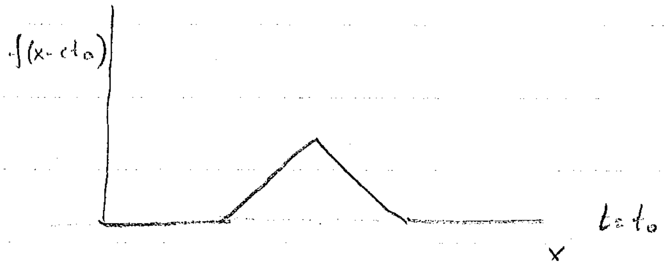
$$k = \frac{\partial \phi}{\partial x} \qquad \omega = -\frac{\partial \phi}{\partial t} \qquad f(\omega, k) = 0 \Rightarrow \underset{\text{sometimes}}{\omega = g(k)}$$

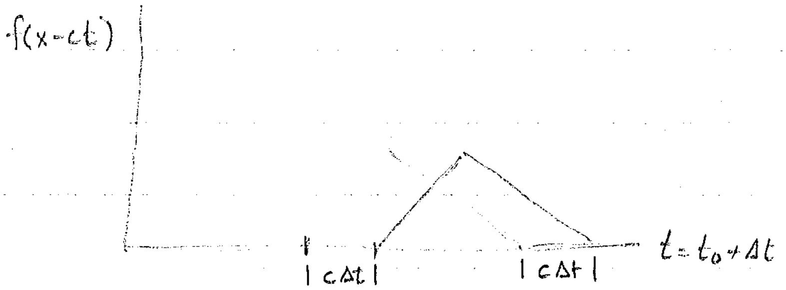
$$\omega = g(k) \Rightarrow -\frac{\partial \phi}{\partial t} = g\left(\frac{\partial \phi}{\partial x}\right)$$

$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial t} dt$: if $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial t}$ are const. then $\phi = \frac{\partial \phi}{\partial x} x + \frac{\partial \phi}{\partial t}$

then $\boxed{\phi = kx - \omega t}$ or $\frac{\phi}{k} = \tilde{\phi} = x - \frac{\omega}{k} t = x - c_p t$ c is phase velocity

if we have $f(x - ct)$ which is simple wave fn. Now @ $t = t_0$

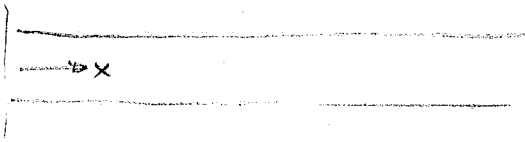


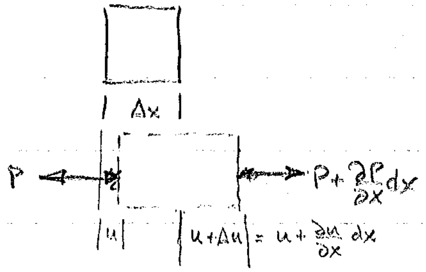


we can use the results at $t=t_0$ ⇒ wave moves to the right with velocity c

Notation: phase velocity - no subscript ; group velocity - $(\ )_g$. c dictates the speed at which disturbance propagates.

Example : Uniform bar ; crossection remain plane & moves in x direction only
one dimensional motion





Eqn of mot: $\sum F = ma$

$$-\frac{\partial P}{\partial x}dx = \rho A \frac{\partial^2 u}{\partial t^2}dx \text{ ; but } P=\sigma A = E\varepsilon A = E\frac{\partial u}{\partial x}A$$

then $$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2} \quad\text{or}\quad \frac{\partial^2\sigma}{\partial x^2} = \rho/E\,\frac{\partial^2\sigma}{\partial t^2} \quad \text{define } E/\rho = c^2 \quad\text{or}\quad \boxed{c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0}$$

(take $\frac{\partial}{\partial x}$ of $Eu_{xx}=\rho u_{tt}$ and use $Eu_x=\sigma$ to get)

1-D wave eqn.

let $v_1 = x-ct \qquad v_2 = x+ct$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v_1} + \frac{\partial u}{\partial v_2} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial v_1^2} + 2\frac{\partial^2 u}{\partial v_1 \partial v_2} + \frac{\partial^2 u}{\partial v_2^2}$$

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial v_1} + c\frac{\partial u}{\partial v_2} \qquad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial v_1^2} - 2c^2\frac{\partial^2 u}{\partial v_1\partial v_2} + c^2\frac{\partial^2 u}{\partial v_2^2}$$

$$c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 4c^2\frac{\partial^2 u}{\partial v_1 \partial v_2} = 0 \quad\text{or}\quad \boxed{\frac{\partial^2 u}{\partial v_1\partial v_2} = 0}$$

$$\frac{\partial}{\partial v_1}\left(\frac{\partial u}{\partial v_2}\right) = 0 \qquad \frac{\partial u}{\partial v_2} = \psi(v_2) \qquad u = \underbrace{\int \psi(v_2)\,dv_2}_{g(v_2)} + f(v_1)$$

$$= f(v_1) + g(v_2)$$

then $$\boxed{u = \underbrace{f(x-ct)}_{\text{wave to right}} + \underbrace{g(x+ct)}_{\text{wave to left}}}$$

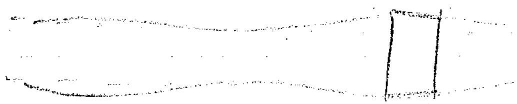
10/2/79

Now assume $f(x-ct) = A\cos k(x-ct) = A\cos(kx-\omega t)$
or in more compact form $f(x-ct) = Ae^{i(kx-\omega t)}$

Put into wave eqn then $\Rightarrow c^2 = E/\rho$: here c is constant & not a fn of k as is the case in some systems. When c is a fn of k this is dispersion

Dispersion might be due to nonlinear effects, inhomogeneity or boundaries or inelasticity

When deriving the wave eqn, we took an elementary theory - we expect that due to boundaries $c = c(k)$ but we don't get it.



bar is sheared which we don't take into account

Particle velocity

$$u(x,t) = Ae^{ik(x-ct)} \qquad \frac{\partial u}{\partial x} = Aik\, e^{ik(x-ct)}$$

$$\sigma = E\frac{\partial u}{\partial x} = EAik\, e^{ik(x-ct)}$$

$$\frac{\partial u}{\partial t} = \dot{u} = -Aikc\, e^{ik(x-ct)} \qquad \Rightarrow \qquad -\frac{E}{c}\frac{\partial u}{\partial t} = \sigma = -\frac{E}{\rho}\frac{\rho}{c}\dot{u} = -\dot{u}\rho$$

$\boxed{\sigma = -\dot{u}\rho c}$ for a harmonic fn.

Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{kc}$

Wavelength $\lambda = \frac{2\pi}{k}$. wave number represents the no. of wavelengths in period of 2π

Starting with an extended solid w/isotropic, homogeneous, elastic properties

Recall that the stress eqns of motion w/o body force

$$\sigma_{ij,j} = \rho\ddot{u}_i \qquad \nabla\cdot\boldsymbol{\sigma} = \rho\ddot{\mathbf{u}}$$

Strain - Disp relations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad \boldsymbol{\varepsilon} = \tfrac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla)$$

Hooke's law

$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$ $\qquad$ $\sigma = \lambda \Delta \mathbb{I} + 2\mu \varepsilon$

Take Hookes law & put in them the strain-displ relations. Take results & put into eqns of motion. Thus:

$\mu u_{i,jj} + (\lambda+\mu) u_{j,ji} = \rho \ddot{u}_i$ $\qquad$ $\boxed{\mu \nabla^2 \mathbf{u} + (\mu+\lambda) \nabla (\nabla \cdot \mathbf{u}) = \rho \ddot{\mathbf{u}}}\ (*)$

Reduction to wave eqns.

(1) Take $\nabla \cdot$ of the $(*)$ eqn.

$$\mu \nabla^2 (\nabla \cdot \mathbf{u}) + (\mu+\lambda) \nabla^2 (\nabla \cdot \mathbf{u}) = \rho \frac{\partial^2}{\partial t^2} (\nabla \cdot \mathbf{u})$$

thus $\nabla^2 \left[\mu \nabla \cdot \mathbf{u} + (\lambda+\mu) \nabla \cdot \mathbf{u} \right] = \rho \frac{\partial^2}{\partial t^2} (\nabla \cdot \mathbf{u})$ $\qquad$ $\Delta = \nabla \cdot \mathbf{u}$

or $\boxed{\left[(\lambda+2\mu) \nabla^2 (\Delta) \right] = \rho \frac{\partial^2}{\partial t^2} \Delta}$ 3-D wave eqn on the dilatation

→ does not involve shear

(2) Take $\nabla \times$ the $(*)$ eqn.

$$\mu \nabla^2 (\nabla \times \mathbf{u}) + (\mu+\lambda) \nabla \times \nabla \overset{\to 0}{(\nabla \cdot \mathbf{u})} = \rho \frac{\partial^2}{\partial t} (\nabla \times \mathbf{u})$$

This is 3-D wave eqn on the rotation ω : $\boxed{\mu \nabla^2 \omega = \rho \frac{\partial^2}{\partial t} \omega}$

dilatation is uncoupled from rotation → does not involve volume change.

Reduction by Helmholtz resolution w/ Gage condition

$\mathbf{u} = \nabla \phi + \nabla \times \mathbf{H}$ also take $\nabla \cdot \mathbf{H} = 0$ to take indeterminacy on

ϕ, $\mathbf{H}$ are called potentials of $\mathbf{u}$

$\nabla \phi$ is the lamellar part of $\mathbf{u}$

$\nabla \times \mathbf{H}$ is the solenoidal part of $\mathbf{u}$

put $\mathbf{u}$ into $(*)$ ⇒

$$\rho \nabla \ddot{\phi} + \rho \nabla \times \ddot{\mathbf{H}} = \mu \nabla^2 (\nabla \phi) + (\lambda+\mu) \nabla^2 (\nabla \phi) + \mu \nabla^2 (\nabla \times \mathbf{H}) + (\lambda+\mu) \nabla \overset{= 0 \text{ by Gage}}{(\nabla \cdot \nabla \times \mathbf{H})}$$

$$\therefore \quad \nabla \left[(\lambda+2\mu) \nabla^2 \phi - \rho \ddot{\phi} \right] + \nabla \times \left[\mu \nabla^2 \mathbf{H} - \rho \ddot{\mathbf{H}} \right] = 0$$

$$\Rightarrow (\lambda+2\mu) \nabla^2 \phi = \rho \ddot{\phi} \quad \& \quad \mu \nabla^2 \mathbf{H} = \rho \ddot{\mathbf{H}}$$

Given $\mathbf{u}$ how does one find ϕ, $\mathbf{H}$

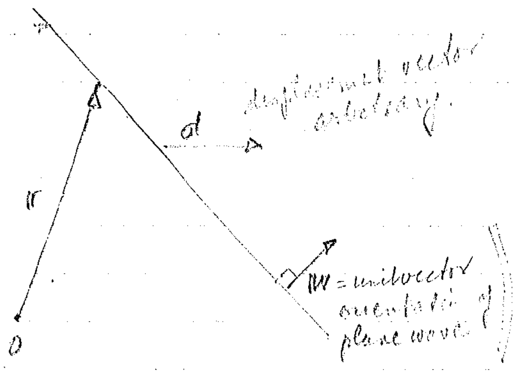
define

$$\mathbf{w}(\mathbf{x}) = -\frac{1}{4\pi}\iint \frac{\mathbf{u}(\boldsymbol{\xi})}{|\mathbf{x}-\boldsymbol{\xi}|}\,dV_\xi \qquad |\mathbf{x}-\boldsymbol{\xi}| = \left[(x_i-\xi_i)(x_i-\xi_i)\right]^{\frac{1}{2}}$$

then $\phi = \nabla\cdot\mathbf{W}$ $\mathbf{H} = -\nabla\times\mathbf{W}$

<u>Plane wave solutions that are harmonic in an ∞ body.</u>

Given: plane wave



$$\mathbf{u} = A\,\mathbf{d}\,e^{ik(\mathbf{w}\cdot\mathbf{r}-ct)}$$

$$u_x = A d_x\, e^{ik(w_x x + w_y y + w_z z - ct)}$$

$$\frac{\partial u_x}{\partial x} = ikw_x A d_x\, e^{ik(\qquad)}$$

$$\nabla\cdot\mathbf{u} = ikA(\mathbf{w}\cdot\mathbf{d})\,e^{ik(\mathbf{w}\cdot\mathbf{r}-ct)} = \frac{\partial u_i}{\partial x_i}$$

$$\nabla\times\mathbf{u} = ikA\,(\mathbf{w}\times\mathbf{d})\,e^{ik(\mathbf{w}\cdot\mathbf{r}-ct)}$$

$$\overset{ik\mathbf{w}\cdot ik\mathbf{w}}{(\nabla\cdot\nabla)\mathbf{u}} = \nabla^2\mathbf{u} = -k^2 A\,\mathbf{d}\,e^{ik(\mathbf{w}\cdot\mathbf{r}-ct)} \qquad \ddot{\mathbf{u}} = -k^2c^2 A\,\mathbf{d}\,e^{ik(\mathbf{w}\cdot\mathbf{r}-ct)}$$

$$\nabla(\nabla\cdot\mathbf{u}) = -k^2A(\mathbf{w}\cdot\mathbf{d})\,e^{ik(\mathbf{w}\cdot\mathbf{r}-ct)}\,\mathbf{w} = \nabla\left[ikA(\mathbf{w}\cdot\mathbf{d})e^{ik(\mathbf{w}\cdot\mathbf{r}-ct)}\right] = ik\mathbf{w}[\;]$$

$$\nabla\mathbf{u} = ik\mathbf{w}\,[\mathbf{u}]$$

put into law of motion ⇒ vector eqn. is defined

<u>ie</u>

$$\left.\begin{array}{l}\mu\,\mathbf{d} + (\lambda+\mu)(\mathbf{w}\cdot\mathbf{d})\mathbf{w} = \rho c^2\mathbf{d} \\ \text{or} \\ (\mu-\rho c^2)\,\mathbf{d} + (\lambda+\mu)(\mathbf{w}\cdot\mathbf{d})\mathbf{w} = 0\end{array}\right\}$$

imply that 1) either $\mathbf{w}$ is $\parallel \mathbf{d}$
2) or $\mathbf{w}\cdot\mathbf{d}$ is 0 $\mathbf{w}\perp\mathbf{d}$ & $\mu = \rho c^2$

<u>Look at these 2 cases</u>

10/4/79

Case 1) $\mathbf{d}\parallel\mathbf{W}$

$\mu - \rho c^2 + (\lambda+\mu) = 0$ since $\mathbf{W}\parallel\mathbf{d} \Rightarrow \mathbf{W}\cdot\mathbf{d} = \mathbf{d}\cdot\mathbf{d} = 1$

$\therefore c_1 = \left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}} \Rightarrow$ <u>$\nabla\times\mathbf{u} = 0$</u> irrotational wave

2) $\mu = \rho c^2$ or $c_2 = \left(\mu/\rho\right)^{\frac{1}{2}} \Rightarrow \nabla\cdot\mathbf{u} = 0$ Equivoluminal wave

suppose we want to find $\mathbf{t}_w$: $\mathbf{T}\cdot\mathbf{W} = \mathbf{W}\cdot(\lambda\,\nabla\cdot\mathbf{u}\,\mathbb{I} + 2\mu\,\mathbf{\varepsilon})$

$$= \lambda\,(\nabla\cdot\mathbf{u})\mathbf{w} + \mu\,\mathbf{w}\cdot(\nabla\mathbf{u} + \mathbf{u}\nabla))$$

$$\mathbf{W}\cdot\nabla\mathbf{u} = ikA\,\mathbf{d}\,e^{ik(\mathbf{w}\cdot\mathbf{r}-ct)} = \mathbf{w}\cdot ik\mathbf{w}\,[\mathbf{u}] = ik\,\mathbf{u}$$

$$\mathbf{W}\cdot\mathbf{u}\nabla = ikA(\mathbf{w}\cdot\mathbf{d})\,\mathbf{w}\,e^{ik(\mathbf{w}\cdot\mathbf{r}-ct)} = \mathbf{w}\cdot\nabla^c\mathbf{u}^c = \mathbf{w}\cdot\mathbf{u}\,[ik\mathbf{w}] = ikA(\mathbf{w}\cdot\mathbf{d})\mathbf{w}e^{ik(\;)}$$

Thus

$$\mathbf{w}\cdot\mathbf{T} = ikA\,e^{ik(\mathbf{w}\cdot\mathbf{r}-ct)}\left[\lambda(\mathbf{W}\cdot\mathbf{d})\mathbf{w} + \mu\,\mathbf{d} + \mu\,(\mathbf{w}\cdot\mathbf{d})\mathbf{w}\right]$$

for $\mathbb{w} = A d e^{ik(\mathbb{w}\cdot\mathbb{r} - ct)}$ $t_w = w \cdot \sigma = ikAe^{ik(w\cdot r - ct)}\left[\mu d + (\lambda + \mu)(\mathbb{w}\cdot d)w\right]$

1) when $\mathbb{w} \parallel d$ $t_w = (\lambda + 2\mu) ikAd e^{ik(\mathbb{w}\cdot r - ct)}$ is normal to plane of wave; since there would ≈ pressure stress.

2) $\mathbb{w} \perp d$ $t_w = \mu ikA e^{ik(w\cdot r - ct)} d$ is in the plane of wave; [illegible] ≈ shear stress.

→ 1) is known as pressure wave ("Primary", "Push") P wave

→ 2) is known as shear wave ("Shear", "Shake wave", "Secondary") S wave

$c_1^2/c_2^2 = \frac{\lambda + 2\mu}{\mu} = \frac{2(1-\nu)}{1-2\nu}$ for $\nu = 0$ $c_1^2/c_2^2 = 2$; $c_1^2/c_2^2 = 3.5$ $\nu = .3$

for $\nu = .5$ $c_1^2/c_2^2 = \infty$

if we extend results ⇒ $\mathbb{w} = \mathbb{w}' + i\mathbb{w}''$ where $\mathbb{w}$ is complex

$\mathbb{u} = Ade^{-k(\mathbb{w}''\cdot r)} e^{ik(\mathbb{w}'\cdot r - ct)}$

Amplitude decays (in space) if $\mathbb{w}''\cdot r > 0$ | this is a propagating wave

for inhomog. wave the amplitude is not constant ie $\mathbb{w}$ is complex.

Consider $\mathbb{w}'\cdot\mathbb{w}' - \mathbb{w}''\cdot\mathbb{w}'' = 1$ and $\mathbb{w}'\cdot\mathbb{w}'' = 0$ ⇒ $\mathbb{w}\cdot\mathbb{w} = 1$

The 2 planes are ⊥
1. Const amplitude ⇒ $\mathbb{w}''\cdot r = $ Constant c_1
2. for fixed time of locus of constant phase $\mathbb{w}'\cdot r = c_2$ (const)

planes of constant phase are ⊥ to planes of constant amplitude
amplitude is constant in direction of prop but varies exponentially in ⊥ direction

Look at 2-D Wave Eqn: $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2}$

Case 1. const amplitude

if we let $\phi = $ const $e^{i(\mathbb{k}\cdot\mathbb{x} - \omega t)}$ then for $k\mathbb{w} = k_x e_x + k_y e_y = \mathbb{k}$

Let us find sol.: $k_x^2 + k_y^2 = \frac{\omega^2}{c_1^2}$
[illegible] DE [illegible]

circles

by

const circles

Thus const amplitude waves are circular emanating from center ie $\mathbb{w}$ is real

Case 2 $\mathbb{k} = k_x e_x$ only
$\mathbb{w}$ = complex

Another example $\phi = \Phi(y)\, e^{i(k_x x - \omega t)}$ put to WE $\Delta \phi - \frac{\ddot{\Phi}}{c^2} = 0$

$$\left[-k_x^2 \Phi + \Phi'' + \frac{\omega^2}{c_1^2} \Phi \right] e^{i(k_x x - \omega t)} = 0$$

$$\Rightarrow \quad \underbrace{\left(\frac{\omega^2}{c_1^2} - k_x^2 \right)}_{\alpha^2} \Phi + \Phi'' = 0$$ ODE for $\times$ in y direction

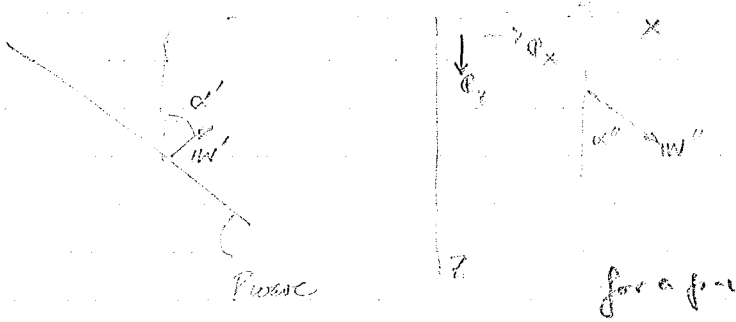
if $\alpha^2 > 0$ soln : trig func

if $\alpha^2 = 0$ soln : linear $\Phi = A + By$ $\phi = (A + By)\, e^{i(k_x x - \omega t)}$

if $\alpha^2 < 0$ soln : exponential f. total soln is not dependent on Bdry conditions

Elastic Half Space - Plane Waves reflecting off of a ~~free surface~~ $\equiv$ free surface

1) Reflection



for a p-wave $\mathbb{W} \parallel d$

thus $w' = d' = -e_z \cos\alpha' + e_x \sin\alpha'$
$\mathbb{u}' = A' d' e^{ik'(w' \cdot x - c_1 t)}$

$\mathbb{w} \cdot d$

now in general $\sigma_{ij} = \left[\lambda (w_k \cdot e_k)\, \delta_{ij} + \mu (w_j d_i + w_i d_j) \right] \times$
$ik A e^{ik(w_k x_k - ct)}$

for our case

$$\sigma_{ij} = \left[\lambda \delta_{ij} + 2\mu (w_i' w_j') \right] ik' A' e^{ik'(w_k' x_k - c_1 t)}$$

$$\sigma_{zz} = \left[\lambda + 2\mu \cos^2\alpha' \right] ik' A' e^{ik'(x \sin\alpha' - z \cos\alpha' - c_1 t)}$$

$$\sigma_{zx} = -\mu \sin 2\alpha'\, ik' A' e^{ik'(x \sin\alpha' - z \cos\alpha' - c_1 t)}$$

$\sigma_{zy} = 0$ since plane problem

on bdy $z = 0$ $\Rightarrow \sigma_{zx}, \sigma_{zz} \neq 0$ we must have reflected waves at surface to annihilate surface traction

Assume reflected P-wave.

$$u'' = A'' d'' e^{ik''(w''\cdot r - c_1 t)} \qquad w'' = d'' = e_z \cos\alpha'' + e_x \sin\alpha''$$

$$\left.\begin{aligned}\sigma_{zz}'' &= (\lambda + 2\mu\cos^2\alpha'')\, ik'' A'' e^{ik''(x\sin\alpha'' - c_1 t)} \\ \sigma_{zx}'' &= \mu \sin 2\alpha''\, ik'' A'' e^{ik''(x\sin\alpha'' - c_1 t)}\end{aligned}\right\} \text{on surface}$$

$\sigma_{zy}'' = 0$

Continuation of the above

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$A''+A'=0 \Rightarrow A''-A'=-2A'\neq 0$

the only way tractions can be annihilated $\Rightarrow$ if ① $\sigma_{zz}=0 \Rightarrow \sigma_{zx}\neq 0$

or if ② $\sigma_{zx}=0 \Rightarrow \sigma_{zz}\neq 0$

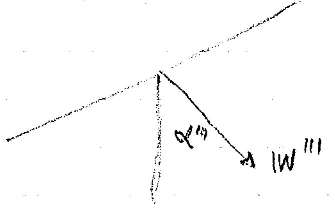
$A'-A''=0 \Rightarrow A'+A''=2A''\neq 0$

thus reflected wave cannot be <u>only</u> a P wave but must also consist of a reflected S wave.

$\therefore$ for a shear wave

2 possible waves SH is out of plane

SV in same plane as 1



$$u''' = A''' d''' e^{ik'''(w'''\cdot r - c_2 t)}$$

$$w''' = e_z \cos\alpha''' + e_x \sin\alpha'''$$
$$d''' = -e_x \cos\alpha''' + e_z \sin\alpha'''$$

<u>at $z=0$</u>

$$\sigma_{zz}''' = \mu \sin 2\alpha'''\, ik''' A''' e^{ik'''(x\sin\alpha''' - c_2 t)}$$
$$\sigma_{zx}''' = -\mu \cos 2\alpha'''\, ik''' A''' e^{ik'''(x\sin\alpha''' - c_2 t)}$$
$$\sigma_{zy}''' = 0$$

Since it must be true that $\forall x,t$ $\sigma_{zz} = \sigma_{zx} = 0 \Rightarrow k'(x\sin\alpha' - c_1 t) = k''(x\sin\alpha'' - c_1$

$= k'''(x\sin\alpha''' - c_2 t)$

$k' = k''$ & $\dfrac{k'''}{k'} = \dfrac{c_1}{c_2}$ $\qquad \alpha' = \alpha''$ $\qquad \dfrac{\sin\alpha'''}{\sin\alpha'} = \dfrac{k'}{k'''} = \dfrac{c_2}{c_1}$

$$\left.\begin{aligned}\text{Let } \alpha' &= \alpha'' = \alpha \\ \alpha''' &= \beta\end{aligned}\right\} \quad \text{since } c_1 > c_2 \qquad \alpha > \beta$$

$$\therefore \quad \sigma_{zz} = i e^{ik(x\sin\alpha - c_1 t)}\left[(\lambda + 2\mu\cos^2\alpha)(kA' + kA'') + \mu k \frac{c_1}{c_2}\sin 2\beta A'''\right] = 0$$

$$\sigma_{zx} = i e^{ik(x\sin\alpha - c_1 t)}\left[-\mu\sin 2\alpha\, kA' + \mu\sin 2\alpha\, kA'' - \mu k \frac{c_1}{c_2}\cos 2\beta A'''\right] = 0$$

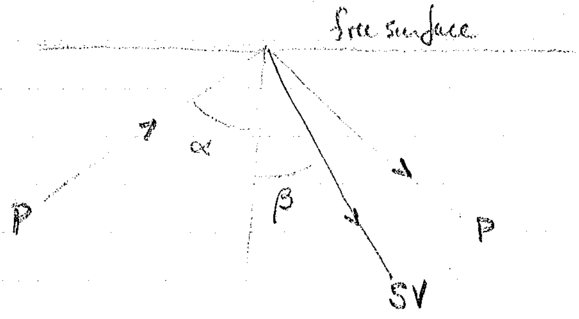
normally A', c_1, α are given $\Rightarrow$ we must write A'' & A''' in terms of A' & the compat. conditions given β in terms of α, c_1, c_2

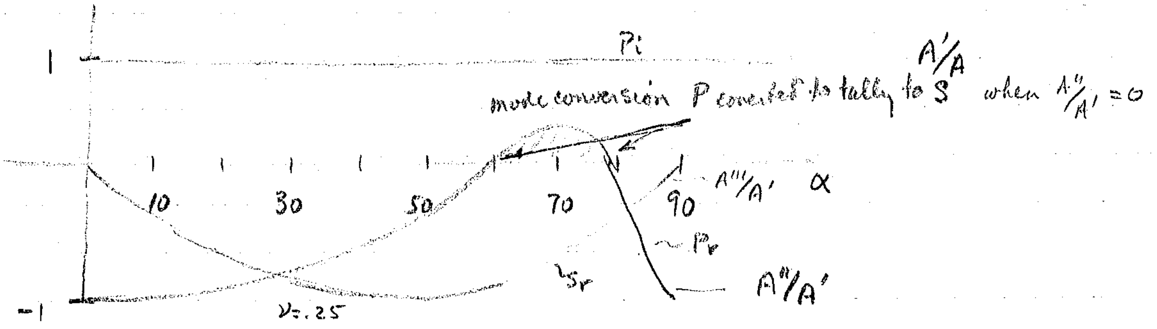
$$\therefore \quad \frac{\lambda + 2\mu\cos^2\alpha}{\rho} = c_1^2 - \frac{2\mu}{\rho}\sin^2\alpha = c_1^2 - 2c_2^2\sin^2\alpha = c_1^2 - 2c_1^2\sin^2\beta = c_1^2\cos 2\beta$$

$$\therefore \quad \frac{A''}{A'} = \frac{c_2^2\sin 2\alpha\sin 2\beta - c_1^2\cos 2\beta}{c_2^2\sin 2\alpha\sin 2\beta + c_1^2\cos 2\beta} \qquad \frac{A'''}{A'} = -\frac{2c_1c_2\sin 2\alpha\cos 2\beta}{c_2^2\sin 2\alpha\sin 2\beta + c_1^2\cos 2\beta}$$

$$\boxed{\left(\frac{A''}{A'}\right)^2 + \frac{\sin 2\beta}{\sin 2\alpha}\left(\frac{A'''}{A'}\right)^2 = 1}$$

Eqn of an ellipse





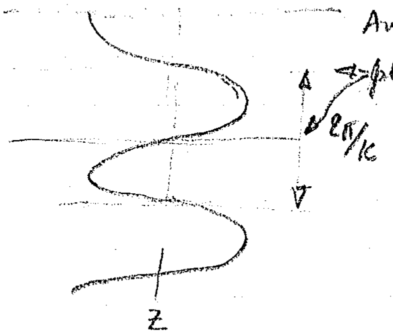
for $\alpha = 0$ normal incidence Special Case

$$\mathbf{w}' = -\mathbf{e}_z \quad \mathbf{w}'' = \mathbf{e}_z \quad \mathbf{w}''' = 0 \qquad \mathbf{d}'' = -\mathbf{d}' \quad \Rightarrow A''\mathbf{d}'' = (-A')(-\mathbf{d}') = A'\mathbf{d}'$$

$$\mathbf{u}' = \mathbf{d}'A'e^{ik(-z-c_1t)} \qquad \mathbf{u}'' = \mathbf{d}'A'e^{ik(z-c_1t)}$$

$$\mathbf{u} = \mathbf{u}' + \mathbf{u}'' = \mathbf{d}'A'\left[e^{ik(-z-c_1t)} + e^{ik(z-c_1t)}\right] = 2\mathbf{d}'A'e^{-ikc_1t}\cos kz$$

$\Rightarrow$ standing waves : vibrations motion not fn of x



$\lambda = \frac{2\pi}{k} \qquad kc_1 = \omega$

high freq $\Rightarrow$ small λ

$\alpha = 90^\circ$ (grazing incidence) is more difficult; we will defer until after full & systematic development is made.

Reflection of waves using potentials

Recall $u = \nabla\phi + \nabla\times H$ w/ $\nabla\cdot H = 0$ & the from single Eqn of motion

$$\nabla^2\phi = \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} \qquad \nabla^2 H = \frac{1}{c_2^2}\frac{\partial^2 H}{\partial t^2}$$

$$\nabla\cdot H = 0 \therefore \left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right) = 0$$

$$\nabla\times H = \left(\frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y}, \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}, \frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right)$$

Separation for 1) Plane Strain $u_x(x,y,t)$ only & $u_y(x,y,t)$; $u_z = 0$

2) Anti Plane Strain $u_x = u_y = 0$ $u_z(x,y,t)$

Plane Strain Problem $u_x = \frac{\partial\phi}{\partial x} + \frac{\partial H_z}{\partial y}$ $\qquad u_y = \frac{\partial\phi}{\partial y} - \frac{\partial H_z}{\partial x}$ $\qquad$ let $H = (0, 0, H_z)$

Let us concentrate on traction free surface $\overline{\quad\quad}\!\!\downarrow y$ $\Rightarrow \sigma_{yy}, \sigma_{yx} \neq 0$ $\sigma_{yz} = 0$

Put into strain-disp & Hooke's law $\Rightarrow$

$$\sigma_{yy} = (\lambda + 2\mu)\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) - 2\mu\frac{\partial u_x}{\partial x} = (\lambda+2\mu)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) - 2\mu\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2 H_z}{\partial x\,\partial y}\right)$$

$$\sigma_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \mu\left(2\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2 H_z}{\partial y^2} - \frac{\partial^2 H_z}{\partial x^2}\right)$$

Consider solns of waves traveling // to bounding plane: ie $\phi = f(y)e^{i(k_\phi x - \omega_\phi t)}$

and $H_z = h_z(y)e^{i(k_H x - \omega_H t)}$.

Substitute into Eqns for wave eq $\Rightarrow \frac{d^2 f}{dy^2} + \alpha^2 f = 0$ w/ $\alpha^2 = \frac{\omega_\phi^2}{c_1^2} - k_\phi^2 = k_\phi^2\left(\frac{c^2}{c_1^2} - 1\right)$

also $\frac{d^2 h_z}{dy^2} + \beta^2 h_z = 0$ w/ $\beta^2 = \frac{\omega_H^2}{c_2^2} - K_H^2 = K_H^2\left(\frac{c^2}{c_2^2} - 1\right)$

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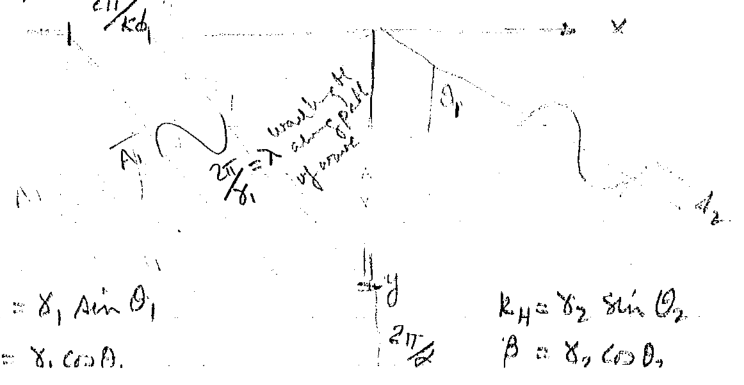
Continuation of the problem involved

Thus $\phi = A_1 e^{i(k_\phi x - \alpha y - \omega_\phi t)} + A_2 e^{i(k_\phi x + \alpha y - \omega_\phi t)}$ ~ downgoing wave

since $f'' + \alpha^2 f = 0$ has soln $f = A_1 e^{i\alpha y} + A_2 e^{-i\alpha y}$

Similarly $H_z = B_1 e^{i(k_H x - \beta y - \omega_H t)} + B_2 e^{i(k_H x + \beta y - \omega_H t)}$



apparent
γ_i = wave number = $\frac{\omega}{c_i}$
along the path of the wave
$k_\phi = \frac{\omega}{c}\phi$

$\alpha^2 = \gamma_1^2 - k_\phi^2$

$k_\phi = \gamma_1 \sin\theta_1$

$\alpha = \gamma_1 \cos\theta_1$

$k_H = \gamma_2 \sin\theta_2$

$\beta = \gamma_2 \cos\theta_2$

$$\phi = A_1 \exp\{i\gamma_1(\sin\theta_1 x - \cos\theta_1 y - c_\phi t)\} + A_2 \exp\{i\gamma_1(\sin\theta_1 x + \cos\theta_1 y - c_\phi t)\}$$

$$H_z = B_1 \exp\{i\gamma_2(\sin\theta_2 x - \cos\theta_2 y - c_2 t)\} + B_2 \exp\{i\gamma_2(\sin\theta_2 x + \cos\theta_2 y - c_2 t)\}$$

BC at $y=0$: $\sigma_{yy} = \sigma_{xy} = 0$ Let $\delta^2 = c_1^2/c_2^2$

$$\sigma_{yy} \equiv 0 = \gamma_1^2(2\sin^2\theta_1 - \delta^2)(A_1 + A_2)\exp\{i\gamma_1(\sin\theta_1 x - c_1 t)\} - \gamma_2^2 \sin\overset{2\theta_2}{\boxed{^2\theta_2}}(B_1 - B_2)\exp\{i\gamma_2(\sin\theta_2 x -$$

since σ_{yy} couples ϕ & H_z ⇒ exp must be same for any x, t ⇒ $\left[\begin{array}{l}\gamma_1 \sin\theta_1 = \gamma_2 \sin\theta_2 \\ \gamma_1 c_1 = \gamma_2 c_2\end{array}\right.$

$$\therefore \quad c_1/c_2 = \frac{\sin\theta_1}{\sin\theta_2} = \frac{\gamma_2}{\gamma_1} \Rightarrow K_H = K_\phi \quad \& \quad \omega_\phi = \omega_H$$

thus $\gamma_1^2(2\sin^2\theta_1 - \delta^2)(A_1 + A_2) - \gamma_2^2 \sin^2 2\theta_2 (B_1 - B_2) = 0$

$$\sigma_{xy} \equiv 0 = \gamma_1^2 \sin 2\theta_1 (A_1 - A_2) - \gamma_2^2 \cos 2\theta_2 (B_1 + B_2) = 0$$

for an impinging P wave only: $A_1 \rightarrow A_2, B_2$: B_1 (impinging SV) $= 0$
SV wave only: $B_1 \rightarrow A_2, B_2$: A_1 (" P) $= 0$

<u>Grazing Incidence of P waves</u>

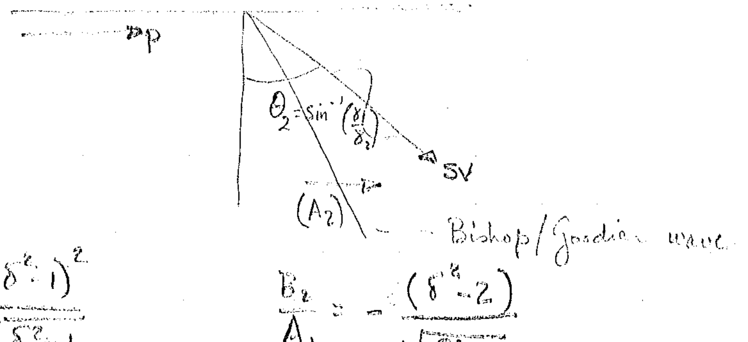
Let $\theta_1 \rightarrow 90°$ ⇒ $\cos\theta_1 \Rightarrow 0$ $\sin\theta_1 \Rightarrow 1$ $\alpha \Rightarrow 0$

$\cancel{\phi}$ $f'' + \alpha^2 f = 0 \Rightarrow f'' = 0$ $f = Ay + B$

inhom plane wave

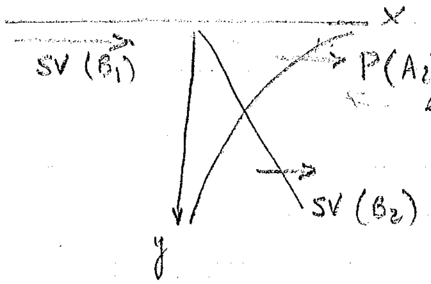
$\phi = (A_1 + A_2 y)e^{i(k_\phi x - \omega t)}$
$H_z = B_2 e^{i(k_\phi x + \beta y - \omega t)}$

put these into BC $\sigma_{yy} = \sigma_{xy} = 0$ $\frac{\gamma_1}{\gamma_2} = \sin\theta_2 = \frac{1}{\delta} = \frac{c_2}{c_1}$



$$A_2/A_1 = \frac{i(\delta^2 - 1)^2}{4\sqrt{\delta^2 - 1}} \qquad \frac{B_2}{A_1} = -\frac{(\delta^2 - 2)}{2\sqrt{\delta^2 - 1}}$$

2 P waves are out of phase by 90°

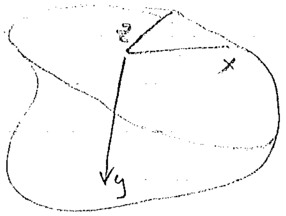


exponentially decaying in y dir (evanescent wave)

inhomogeneous wave

<u>We want to find :</u>

<u>Surface waves: ie</u>(Rayleigh waves) stay within some distance close to surface.



Want to find waves moving in x direct. ⇒ inhomog waves and want them to decay in y direct.

<u>Look for solns</u>

ie
$$u_x = A\, e^{-by}\, e^{ik(x-ct)}$$
$$u_y = B\, e^{-by}\, e^{ik(x-ct)}$$
$$u_z = 0$$

$\text{Im}\, b = 0 \quad \text{Re}\, b > 0$

can use potentials (perhaps next time)

<u>Subst. into eqns of motion.</u> ⇒ 2 hom. eqns on A & B. For non trivial sol. determinant must = 0. ⇒ $0=[c_1^2 b^2 - (c_1^2 - c^2) K^2][c_2^2 b^2 - (c_2^2 - c^2) K^2$

this eqn relates b, k & c. Quadratic in b

$$b_1 = k\left(1 - c^2/c_1^2\right)^{1/2}, \quad b_2 = k\left(1 - c^2/c_2^2\right)^{1/2}$$

for $b > 0 \Rightarrow \qquad c < c_2 < c_1$

Using this we can calculate $\left(B/A\right)_{b=b_1} = -b_1/ik \qquad \left(B/A\right)_{b=b_2} = \frac{ik}{b_2}$

$$\therefore \quad u_x = [A_1 e^{-b_1 y} + A_2 e^{-b_2 y}]\, e^{ik(x-ct)}$$
$$u_y = [-\frac{b_1}{ik} A_1 e^{-b_1 y} + \frac{ik}{b_2} A_2 e^{-b_2 y}]\, e^{ik(x-ct)}$$

They then satisfy Eqns of motion & compatability ...

<u>Now apply BC</u> to find A_1, A_2 & k

$\sigma_{yy} = 0 = \sigma_{xy} = 0$ for $y = 0$

$$\sigma_{yy} = 0 \Rightarrow 2b_1 A_1 + (2 - c^2/c_2^2) k^2 A_2/b_2 = 0$$
$$\sigma_{xy} = 0 \Rightarrow (2 - c^2/c_2^2) A_1 + 2b_2 \frac{A_2}{b_2} = 0$$

Thus det = 0 ⇒ Rayleigh Eqn.

$$\boxed{\left(2-\frac{c^2}{c_2{}^2}\right)^2-4\left(1-\frac{c^2}{c_1{}^2}\right)^{1/2}\left(1-\frac{c^2}{c_2{}^2}\right)^{1/2}=0}$$ k drops out

or if rationalized

$$\frac{c^2}{c_2{}^2}\left[\frac{c^6}{c_2{}^6}-8\frac{c^4}{c_2{}^4}+8\left(3-2\frac{c_2{}^2}{c_1{}^2}\right)\frac{c^2}{c_2{}^2}-16\left(1-\frac{c_2^2}{c_1{}^2}\right)^2\right]=0$$

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Energy Propagation

Power in a wave $p = -t_i\dot{u}_i = -\mathbf{t}\cdot\dot{\mathbf{u}}$

Consider a longit wave propag in x direct

$$u_x = A\, d_x\, e^{ik(x-c_Lt)}$$

$$\sigma_{xx} = i(\lambda+2\mu)\,Ak\,e^{ik(x-c_Lt)}$$

$$P_{longit.} = -\sigma_{xx}\dot{u}_x$$

$$u_x = A\cos k(x-c_Lt) \qquad c_L = \frac{\omega}{k}$$

$$\dot{u}_x = Akc_L\sin k(x-c_Lt)$$

$$\sigma_{xx} = -(\lambda+2\mu)\,Ak\sin k(x-c_Lt)$$

$$P_{Long} = (\lambda+2\mu)\,c_L\,k^2A^2\sin^2[k(x-c_Lt)] = (\lambda+2\mu)\frac{\omega^2}{c_L}A^2\sin^2(\omega t-kx)$$

this still looks like a wave & a better indication is the time average of P_L over one period

$$\bar{P}_{aver} = \frac{1}{T}\int_\tau^{\tau+T}P_{long}\,dt = (\lambda+2\mu)\frac{\omega^2}{c_L}A^2\underbrace{\frac{1}{T}\int_\tau^{\tau+T}\sin^2(kx-\omega t)\,dt}_{1/2}$$

$$\bar{P}_{long} = \frac{\lambda+2\mu}{2}\,\frac{\omega^2}{c_L}A^2$$

we can define velocity at which energy is propagated at is $c_e = \bar{P}_L/\bar{W}$

now. $\bar{W}$ = time average of the total energy density $= \bar{K}+\bar{U}$ kinetic + potential energy.

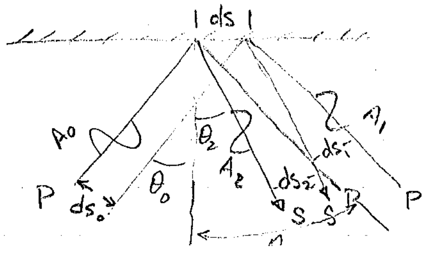
$$\bar{K} = \frac{1}{2}\frac{1}{T}\int_\tau^{\tau+T}\rho\,\dot{u}_x^2\,dt = \frac{1}{4}\rho A^2\omega^2$$

$$\bar{U} = \frac{1}{2}\frac{1}{T}\int_\tau^{\tau+T}\sigma_{xx}\varepsilon_{xx}\,dt = \frac{1}{2}\frac{1}{T}\int_\tau^{\tau+T}(\lambda+2\mu)\left(\frac{\partial u_x}{\partial x}\right)^2dt = \frac{1}{4}(\lambda+2\mu)A^2k^2$$

since $c_L = \frac{\omega}{k} = \sqrt{\frac{\lambda+2\mu}{\rho}} \Rightarrow \bar{K}=\bar{U} \quad \therefore \quad \bar{W} = 2\bar{K} = 2\bar{U} = \frac{1}{2}(\lambda+2\mu)A^2k^2$

$$\therefore c_e = \frac{\lambda+2\mu}{2}\frac{\omega^2}{c_L}A^2 \Big/ \frac{\lambda+2\mu}{2}A^2k^2 = \frac{\omega^2}{k^2c_L} = \frac{c_L{}^2}{c_L} = c_L = \frac{\bar{P}_L}{\bar{W}}$$

Consider



want to find how the energy breaks up (divides between P, S) we had shown previously $\bar{P}_L = \frac{1}{2}(\lambda+2\mu)\frac{\omega^2}{c_L}A^2$

we can show: $\bar{P}_{T_{ang}} = \frac{1}{2}\mu\frac{\omega^2}{c_T^{\,2}}A^2$

thus using conserv of energy energy in = energy out.

$$\frac{1}{2}(\lambda+2\mu)\frac{\omega^2}{c_L}A_0^{\,2}ds_0 = \frac{1}{2}(\lambda+2\mu)\frac{\omega^2}{c_L}A_1^{\,2}ds_1 + \frac{1}{2}\mu\frac{\omega^2}{c_T}A_2^{\,2}ds_2 \quad (*)$$

we use geometrics & get $ds_0 = ds_1 = ds\cos\theta_0$

$$ds_2 = ds\cos\theta_2$$

thus putting back into (*)

$$\left(\frac{A_1}{A_0}\right)^2 + \left(\frac{A_2}{A_0}\right)^2\frac{c_T}{c_L}\frac{\cos\theta_2}{\cos\theta_0} = 1$$

we also know from before that $\sin\theta_2 = K^{-1}\sin\theta_0$ where $K \equiv \left[\frac{2(1-\nu)}{1-2\nu}\right]^{1/2}$

then $\left(\frac{A_1}{A_0}\right)^2 + \left(\frac{A_2}{A_0}\right)^2\frac{1}{K\cos\theta_0}\left(1-\frac{\sin^2\theta_0}{K^2}\right)^{1/2} = 1$

should be same as boxed eqn (~~o~~page 9)

Distinction between Group & phase velocity

Given $u(x,t) = A\cos\underbrace{k(x-\overset{\text{phase vel}}{c}t)}_{\text{phase}} = A\cos(k_1x-\omega_1t)$

now add another wave with different k_2 & ω_2. $k_2 = k_1+\varepsilon$ $\omega_2 = \omega_1+\delta$

$u_1+u_2 = \phi = A\cos(k_1x-\omega_1t) + A\cos(k_2x-\omega_2t)$

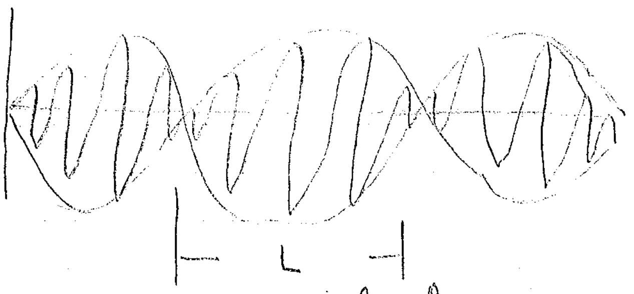
we have the trig relation $\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

thus $\phi = 2A\cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right)$

$= \phi_0\cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)$ w/ $\phi_0 = 2A\cos\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right)$

like the previous wave but different amplitude

$\frac{k_1-k_2}{2} = \pi\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right) = \pi\frac{(\lambda_2-\lambda_1)}{\lambda_1\lambda_2} = \frac{\pi}{\lambda_2\left(\frac{\lambda_1}{\lambda_2-\lambda_1}\right)} = \frac{2\pi}{L}$ $L \gg \lambda_1, \lambda_2$



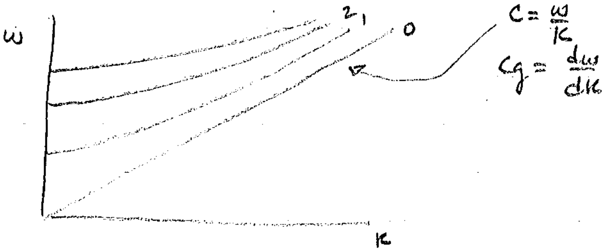
group wave length

$c_g = \frac{\omega_1-\omega_2}{k_1-k_2} = \frac{\Delta\omega}{\Delta k} \Rightarrow \frac{d\omega}{dk}$

$c = \frac{\omega_1+\omega_2}{k_1+k_2}$ phase veloc.

$c_g = \frac{d(ck)}{dk} = c + k\frac{dc}{dk}$ thus if $\frac{dc}{dk} \neq 0$ $c_g \neq c$

if we look at wave prop in a clamped plate SH



$c = \frac{\omega}{k}$

$c_g = \frac{d\omega}{dk}$

if $c > c_g$ normal dispersion
$c < c_g$ anomalous dispers

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Returning to rayleigh waves
First we postulated the displ (on pg 12) put into Eqns of Motion & applied bc to get

$$2b_1A_1 + \left(2 - \frac{c^2}{C_2^2}\right)k^2 A_2/b_2 = 0$$

$$\left(2 - \frac{c^2}{C_2^2}\right)A_1 + 2b_2A_2/b_2 = 0$$

} if A_1 & A_2 are not zero the determenat of coeff $\equiv 0$

this then gives $\left(2 - \frac{c^2}{C_2^2}\right)^2 - 4\left(1 - \frac{c^2}{C_1^2}\right)^{1/2}\left(1 - \frac{c^2}{C_2^2}\right)^{1/2} = 0$

k drops out since b_1, b_2 are fns of k $\quad b_i = k/\left(1 - c^2/c_i^2\right)^{1/2}$

When we rationalized

$$\frac{c^2}{c_2^2} f(c/c_2) = \frac{c^2}{C_2^2}\left[\frac{c^6}{c_2^6} - 8\frac{c^4}{c_2^4} + 8\left(3 - 2\frac{c_2^2}{c_1^2}\right)\frac{c^2}{c_2^2} - 16\left(1 - \frac{c_2^2}{c_1^2}\right)^2\right] = 0$$

3 roots (only one is physical) since this is a cubic in $\left(c^2/c_2^2\right)$

if $c = c_2$ then $f(c/c_2) = 1$

let $c = \varepsilon c_2$ then $f(c/c_2) = f(\varepsilon) \sim -16 + 16\, c_2^2/c_1^2 = -16\left(1 - c_2^2/c_1^2\right) < 0$

$\therefore$ a root must exist between c_2 & εc_2 since $\left(c_2/c_1\right) < 1$

Let $\nu = \frac{1}{4}$ $\lambda = \mu \Rightarrow c_1 = c_2\sqrt{3}$ $\qquad f(c/c_2) = \left[\frac{c^6}{c_2^6} - 8\frac{c^4}{c_2^4} + \frac{56}{3}\frac{c^2}{c_2^2} - \frac{32}{3}\right] = 0$

$c^2/c_2^2 = 4$; $2 \pm 2/\sqrt{3}$ $\qquad$ only soln is $c^2/c_2^2 = 2 - \frac{2}{\sqrt{3}}$

if $c > c_2$ we don't have surface waves. If there's more than one ~~us~~ roots in the interval $\frac{d^2 f}{d\left(c^2/c_2^2\right)^2} = 0$ somewhere in the interval : $\left(\frac{c^2}{c_2^2}\right) = 8/3$. there

but this is not in interval $\therefore$ only one root.

if $\nu = 1/4$ $\quad c_R/c_2 = .9194$ $\qquad u_x = A'\left(e^{-.8475ky} - .5773e^{-.3933ky}\right)\cos k(x - c_Rt)$

$u_y = A'\left(-.8475\, e^{-.8475ky} + 1.4679e^{-.3933ky}\right)\sin k(x-$

A' is some arbitrary constant.

ν	C_R/C_2
0	.862
1/4	.919
1/3	.932
1/2	.955

pronouncement of rayleigh velocity as a fn of ν

In 1908 seismic data on surface waves showed $u_y \sim 0$ & $u_x \neq f(x)$ but $u_x = f(z)$. 1911 Love showed existance of other waves – Love waves (To be discussed later)

Rayleigh surface waves using potentials

$$u = \nabla\phi + \nabla \times H \qquad \nabla \cdot H = 0$$

let $\phi = \phi(x,y,t)$ $\quad H_3 = H_3(x,y,t)$

$\phi = f(y)\, e^{ik(x-ct)}$ $\quad H_3 = h(y)\, e^{ik(x-ct)}$

Put in Eqns of Motion to get : (wave Eqn)

$f'' + k^2\left(c^2/c_1^2 - 1\right) f = 0 \quad \Rightarrow \quad \alpha = k^2\left(c^2/c_1^2 - 1\right)$ $\quad$ let $\alpha = i\alpha_1$

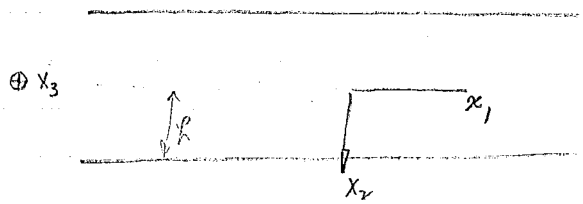
$h'' + k^2\left(c^2/c_2^2 - 1\right) h = 0 \quad \Rightarrow \quad \beta = k^2\left(c^2/c_2^2 - 1\right)$ $\quad$ let $\beta = i\beta_1$

$\Rightarrow \quad f = Ae^{-\alpha_1 y} \qquad h = Be^{-\beta_1 y}$ $\quad$ for decaying waves

this then defines u_1, u_2. Apply bc $\overset{ie}{\Rightarrow}$ $\sigma_{22}, \sigma_{12} = 0$ (as fns of $(u_{i,j})$)

$$\Rightarrow \left.\begin{array}{l} -2ik\alpha_1 A + (k^2+\beta_1^2)B = 0 \\ (k^2+\beta_1^2)A + 2ik\beta_1 B = 0 \end{array}\right\} \Rightarrow (k^2+\beta_1^2)^2 - 4k^2\alpha_1\beta_1 = 0 \quad \text{for } A \neq B \neq 0$$

Let's look at an elastic layer (& leave the half space problem.) Assume homog isotropic



Our eqns will uncouple if we assume plane motion or

$$\left.\begin{array}{l} u_1 = u_1(x_1,x_2,t) \\ u_2 = u_2(x_1,x_2,t) \\ u_3 = 0 \end{array}\right\} \left\{\begin{array}{l} u_1(x_1,x_2,t) \\ 0 \\ u_3(x_1,x_2,t) \end{array}\right)$$

OR anti plane strain

$u_1 = 0 \qquad u_2 = 0 \qquad u_3 = u_3(x_1,x_2,t)$

put into Eqn of motion $\Rightarrow$ $\quad \Delta u_3 = \frac{1}{c_2^2} u_{3,tt}$

Assume $\sigma_{22} = \sigma_{12} = 0$ at $\pm h$ (are identically satisfied) since $u_{1,2}; u_{2,1}; u_{2,2}; u_{1,1} = 0$

$\sigma_{32} = \mu \frac{\partial u_3}{\partial x_2} = 0$ at $\pm h$

Assume a sol of form $\quad u_3 = f(x_2)\, e^{ik(x_1-ct)}$

put into DE : $\quad -k^2 f + f'' = -k^2c^2 f/c_2^2$

if we let $\quad q^2 = k^2\left(c^2/c_2^2 - 1\right) = \omega^2/c_2^2 - k^2$

$f(x_2) = B_1 \sin(q x_2) + B_2 \cos(q x_2)$

from BC : $B_1 \cos qh \pm B_2 \sin qh = 0$ $\left.\frac{\partial u_3}{\partial x_2}\right|_{x_2 = \pm h} = 0$

for satisfaction (determinant $= \sin 2qh \neq 0$) $\Rightarrow B_1 = 0 = B_2$ trivial case,
So we must take 1 $B_1 = 0$ & $\sin qh = 0$ or
2 $\cos qh = 0$ & $B_2 = 0$

Case 1: $B_1 = 0$; $\sin qh = 0$ $\Rightarrow$ displacement u_3 must be symmetric wrt $x_2 = 0$
Case 2 $B_2 = 0$ $\cos qh = 0$ $\Rightarrow$ displacement u_3 must be antisym wrt $x_2 = 0$ plan

$\Rightarrow$ for each problem $qh = \frac{n\pi}{2}$ n = even gives symmetric case $n = 0, 2, 4, .$
n = odd gives antisym case $n = 1, 3, 5.$

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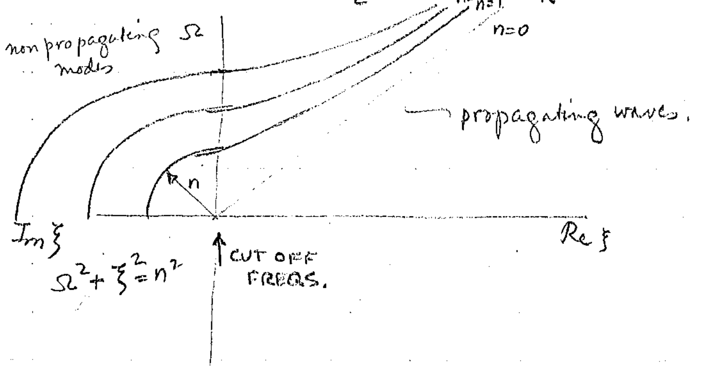
define $\Omega = \frac{2h\omega}{\pi C_2}$ $\xi = \frac{2kh}{\pi}$ $\Rightarrow \Omega^2 = n^2 + \xi^2$

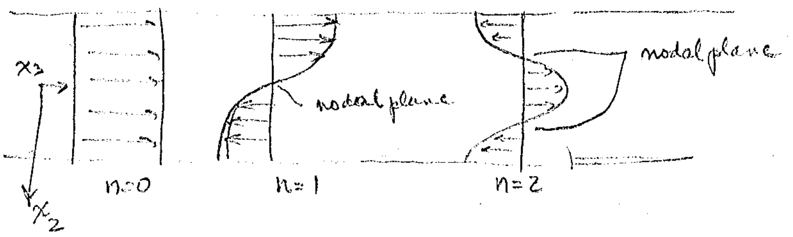


$n = 0, 2, 4, \ldots \Rightarrow$ symmetric modes
$n = 1, 3, 5, \ldots \Rightarrow$ antisym modes

if k were imaginary the $ik < 0$
and we would have non propagating wave

only for $n = 0$ $\frac{\Omega}{\xi} = \frac{2h\omega/\pi C_2}{\frac{2khC_2}{\pi C_2}} = 1 = \frac{\omega}{kC_2} \Rightarrow$ there is no dispersion
$\omega = kC$ $\therefore \frac{\omega}{kC_2} = 1 \Rightarrow \underline{\underline{C = C_2}}$

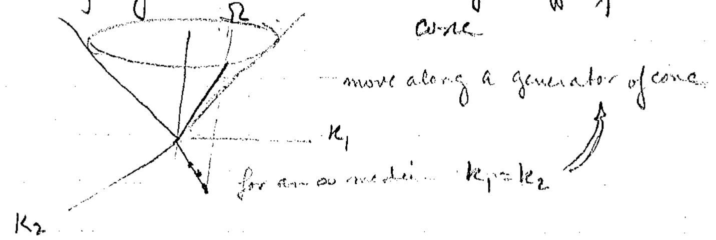
for any other $n \neq 0 \Rightarrow$ $c \neq$ const and we have dispersion



dispersiveness is due to the boundaries & reflections from bdy

Consider a wave propagating in an infinite medium $u_3 = u_3(x_1, x_2, t) = Ae^{i(\underline{k} \cdot \underline{x} - \omega t}$
then $k_1^2 + k_2^2 = \frac{\omega^2}{c^2} = \Omega^2$ putting into $\Delta u_3 = \frac{u_{3,tt}}{c^2}$

this is an eqn of a cone and soln. of diff eqn must have k_1, k_2, ω on the cone



$$\text{if} \quad \mathbb{C}_p = \frac{\omega}{|\mathbb{K}|}\mathbb{n}_k \; ; \quad \mathbb{n}_k = \frac{\mathbb{K}}{|\mathbb{K}|} \quad \Rightarrow \quad \mathbb{C}_p = \frac{\omega \mathbb{K}}{k^2}$$

$$\text{if} \quad \mathbb{K} = k_1 \mathbb{e}_1 + k_2 \mathbb{e}_2 \quad \Rightarrow \quad \mathbb{C}_p = \frac{\omega}{k^2}\left[k_1\mathbb{e}_1 + k_2 \mathbb{e}_2\right] \qquad \underline{k^2 = k_1^2 + k_2^2}$$

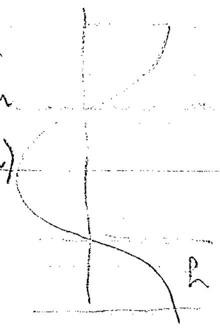
$$\text{if } k_2 = 0 \quad \Rightarrow \quad k^2 = (k_1^2 + k_2^2) = k_1^2 \;\;\&\;\; \mathbb{K} = k_1 \mathbb{e}_1 \qquad \therefore \; \mathbb{C}_p = \frac{\omega}{k_1^2} k_1 \mathbb{e}_1 = \frac{\omega \mathbb{e}_1}{k}$$

for a wave guide must have soln which has $\frac{\partial u_3}{\partial x_2} = 0$ at bdy as restricting condition

suppose $n = 1 \Rightarrow \lambda = 4h$

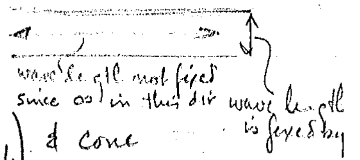
$\Rightarrow k = \frac{2\pi}{\lambda} = \frac{\pi}{2h}$

(for $n=1$ $qh = \pi/2$ thus we only get $1/4$ wave in $h \Rightarrow$ 1 full wave in $4h$)

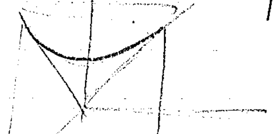


for a waveguide : $k_2 = \frac{\pi}{2h} \Rightarrow$ fixed $\Rightarrow$ k_1 can be anything

Our soln is restricted by the boundaries i.e. $\rightarrow$

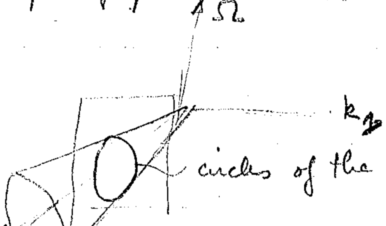


thus we have a soln of Ω^2 = intersect of plane $(k_2 = \pi/2h)$ & cone $\Rightarrow$ hyperbola and these are hyperbolas for Re ξ part of spectrum.

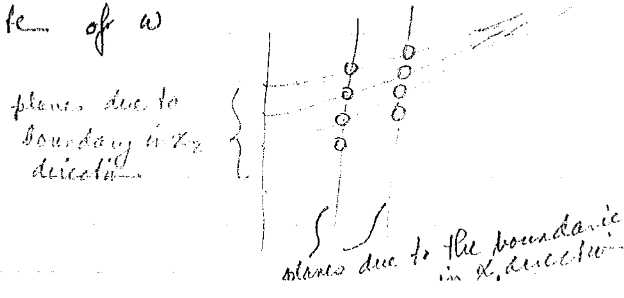


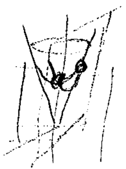
when $\xi^2 < 0$ then $\Omega^2 + \xi^2 = n^2$

again if for a waveguide let $k_2 = \frac{\pi}{2h}$ & k_1 = anything then plane intersecting cone $\Rightarrow$ circles and are the circles at the Im ξ portion of spectrum



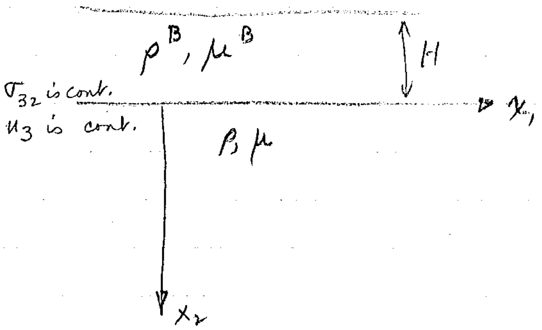
If we add boundaries in the x_1 direct $\Rightarrow$ we must do same as before i.e restrict k_1 $\Rightarrow$ discrete system of ω





• solutions are pts that are on intersection of k_1 = const, k_2 = const planes

Love waves. - a elastic layer above an ∞ elastic medium look at horiz polarized (antiplane) shear waves



In Half Space $u_3 = A e^{-bx_2} e^{ik(x_1 - ct)}$

put into DE $\Delta u_3 - \frac{1}{c_2^2} u_{3,tt} = 0$

$$\left[b^2 - k^2 + \frac{k^2 c^2}{c_2^2} \right] u_3 = 0$$

$$\Rightarrow \quad b = k\left(1 - c^2/c_2^2\right)^{1/2}$$

In the upper layer $u_3^B = \left[B_1 \sin q_B x_2 + B_2 \cos q_B x_2\right] e^{ik(x_1 - ct)}$

put into DE $\quad q_B = k\left[\frac{c^2}{c_2^{B2}} - 1\right]^{1/2}$

bc. traction free at upper surface

bonded condition at interface $\quad x_2 = -H \quad \sigma_{32}$ must be 0 $\quad \frac{\partial u_3}{\partial x_2} = 0$

boundedness at $x_2 \to \infty$

continuity of traction & disp at $x_2 = 0$

$\sigma_{32} = 0$ at $x_2 = -H \quad \Rightarrow \quad B_1 \cos q_B H + B_2 \sin q_B H = 0$

traction = continuous at $x_2 = 0$ $\qquad$ disp = cont. at $x_2 = 0$

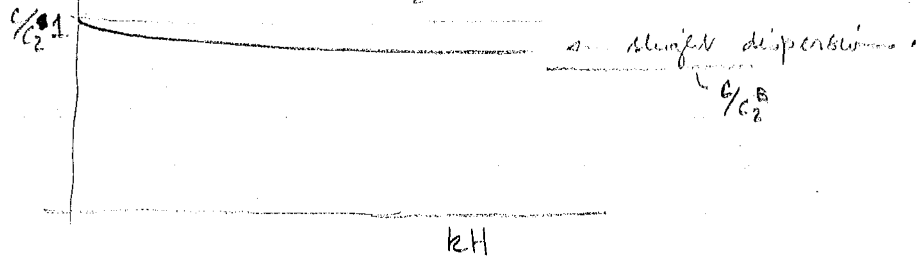
$\mu^B q_B B_1 = -\mu b A = \mu \frac{\partial u_3}{\partial x_2}\Big|_{x_2 = 0^+} = \mu \frac{\partial u_3}{\partial x_2}\Big|_{x_2 = 0^-} \qquad B_2 = A = u_3\Big|_{x_2 = 0^+} = u_3\Big|_{x_2 = 0^-}$

for non triv soln we have a transcendental Eq. in B_1 & A

$$\therefore \begin{vmatrix} \cos q_B H & \sin q_B H \\ \mu^B q_B & \mu b \end{vmatrix} = 0 \Rightarrow 0 = \tan\left\{\left[\left(\frac{c}{c_2^B}\right)^2 - 1\right]^{1/2} kH\right\} - \frac{\mu\left[1 - \left(\frac{c}{c_2}\right)^2\right]}{\mu_B\left[\left(\frac{c}{c_2^B}\right)^2 - 1\right]^{1/2}}$$

Real roots exists in interval $c_2^B < c \leq c_2$ $\qquad$ no real roots exist if $c_2 < c_2^B$

$$\frac{c_2^B}{c}\frac{c}{c_2} = \frac{c_2^B}{c_1} < \frac{c}{c_2} < 1 \qquad \text{also} \qquad 1 < \frac{c}{c_2^B} < \frac{c_2}{c_B^B}$$



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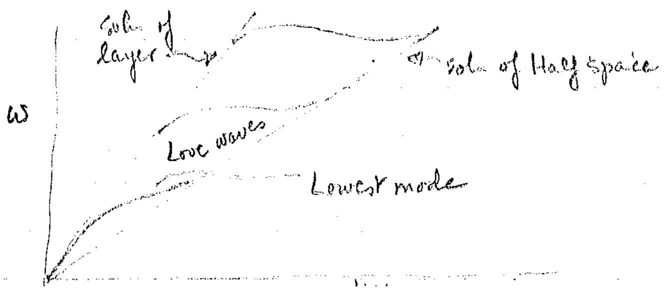
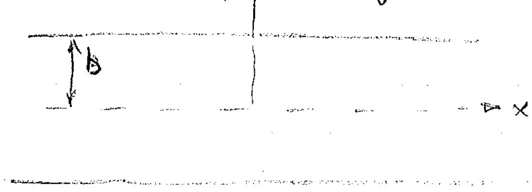


Elastic Layer, (Plane Strain)   Rayleigh-Lamb Problem

Given a plate of thickness 2b with the following Plane Strain Assumptions and we want waves for which σ_{xy} , $\sigma_{yy} = 0$ on $\pm b$



$u_x = u_x(x,y,t)$
$u_y = u_y(x,y,t)$
$u_z = 0$

$$\mathbf{u} = \nabla\phi + \nabla \times \mathbf{H} \qquad \nabla\cdot\mathbf{H} = 0$$

Let : $\phi = \phi(x,y,t)$ $H_3 = H_3(x,y,t)$ $H_1, H_2 = 0$

Since we don't know dependence in the y direction let

$$\phi = f(y) \sin \xi x \, e^{i\omega t}$$
$$H_3 = h(y) \cos \xi x \, e^{i\omega t}$$

pick this since we expect fase shift

$\xi = \frac{\omega}{c}$

we note that $\Delta\phi = \frac{1}{c_1^2}\phi_{,tt}$ and $\Delta H_3 = \frac{1}{c_2^2} H_{3,tt}$ ξ is the wave no.

are the eqns of motion. Put ϕ, H_3 into them to get

$$f'' + \alpha^2 f = 0 \quad w/ \ \alpha^2 = \xi^2\left(\frac{c^2}{c_1^2} - 1\right) \qquad \beta^2 = \xi^2\left(\frac{c^2}{c_2^2} - 1\right)$$
$$h'' + \beta^2 h = 0$$

$$\Rightarrow \quad f = A \sin \alpha y + B \cos \alpha y$$
$$h = C \sin \beta y + D \cos \beta y$$

Displacements: $u_x = \frac{\partial\phi}{\partial x} + \frac{\partial H_3}{\partial y} = (\xi f + h')\cos \xi x \, e^{i\omega t}$ sym w/rt y if f, h' are fn of evn y

$u_y = \frac{\partial\phi}{\partial y} - \frac{\partial H_3}{\partial x} = (f' + \xi h) \sin \xi x \, e^{i\omega t}$ sym w/rt y if f' & h are fn of evn y

now using stress strain relation

$$\sigma = \lambda \nabla\cdot\mathbf{u}\, \mathbb{I} + 2\mu \varepsilon \qquad \sigma_{ij} = \lambda (u_{\ell,\ell})\delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

putting these into stress-displ relations and we want σ_{xy} & $\sigma_{yy} = 0$ at $y = \pm b$. In general

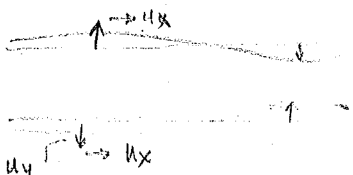
$$\sigma_{xx} = -\mu\left[(\beta^2 + \xi^2 - 2\alpha^2) f + 2\xi h'\right] \sin \xi x \, e^{i\omega t}$$
$$\sigma_{yy} = -\mu\left[(\beta^2 - \xi^2) f - 2\xi h'\right] \sin \xi x \, e^{i\omega t}$$
$$\sigma_{zz} = -\lambda(\alpha^2 + \xi^2) f \sin \xi x \, e^{i\omega t}$$
$$\sigma_{xy} = \mu\left[2\xi f' + (\xi^2 - \beta^2) h\right] \cos \xi x \, e^{i\omega t}$$
$$\sigma_{zy} = \sigma_{zx} = 0 \quad \text{due to plane strain condition}$$

We look at 2 conditions where u_y <
- symmetric about $y = 0$
- antisymmetric about $y = 0$

Let $f = B\cos\alpha y$ $h = C\sin\beta y$ $\Rightarrow u_x = (\xi B\cos\alpha y + \beta C\cos\beta y)\cos\xi x\, e^{i\omega t}$

$u_y = (-\alpha B\sin\alpha y + \xi C\sin\beta y)\sin\xi x\, e^{i\omega t}$

wrt y u_x is ~~antisymmetric~~ sym. u_y is symmetric $\Rightarrow$ we have "extensional" motion



look at B.C. $\sigma_{yy} = \sigma_{xy} = 0$ for $y = \pm b$

$$(\beta^2-\xi^2)B\cos\alpha b - 2\xi\beta C\cos\beta b = 0$$

since $$-2\xi\alpha B\sin\alpha b + (\xi^2-\beta^2)C\sin\beta b = 0$$ solve for B & C

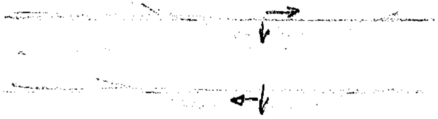
for $B, C \neq 0 \Rightarrow$ determ. $= 0 \Rightarrow$ $\boxed{\dfrac{\tan\beta b}{\tan\alpha b} = -\dfrac{4\xi^2\alpha\beta}{(\beta-\xi)^2}}$ dispersion eqn.

with this we can solve for $B/C = \dfrac{2\xi\beta\cos\beta b}{(\beta^2-\xi^2)\cos\alpha b}$

Let $f = A\sin\alpha y$ $h = D\cos\beta y$ $\Rightarrow u_x = (\xi A\sin\alpha y - \beta D\sin\beta y)\cos\xi x\, e^{i\omega t}$

$u_y = (\alpha A\cos\alpha y + \xi D\cos\beta y)$

wrty u_x = sym. u_y is antisym.



we have "flexural" motions

now we apply B.C. $\sigma_{yy} = \sigma_{xy} = 0$ for $y = 0$

$$(\beta^2-\xi^2)A\sin\alpha b + 2\xi\beta D\sin\beta b = 0$$

$$2\xi\alpha A\cos\alpha b + (\xi^2-\beta^2)D\cos\xi b = 0$$ solve for A, D

$$\boxed{\frac{\tan\beta b}{\tan\alpha b} = -\frac{(\beta^2-\xi^2)^2}{4\xi^2\alpha\beta}}$$ dispersion relation

$$\frac{A}{D} = \frac{2\xi\beta\sin\beta b}{(\xi^2-\beta^2)\sin\alpha b}$$

if we admit $c_2 < c_2 < c$ then then $\alpha = i\alpha_1$, $\beta = i\beta_1$

$$\frac{\tanh \beta_1 b}{\tan \alpha_1 b} = \frac{4\xi^2\alpha_1\beta_1}{(\beta_1^2+\xi^2)^2}$$ for extensional motion

$$\alpha_1 = \xi\sqrt{1-\frac{c^2}{c_1^2}} \qquad \beta_1 = \xi\sqrt{1-\frac{c^2}{c_2^2}}$$

if $\xi \uparrow \infty \Rightarrow$ LHS of $\rightarrow 1 \Rightarrow$ wave length $\rightarrow 0$

$$\therefore (\beta_1^2+\xi^2)^2 - 4\xi^2\alpha_1\beta_1 = 0$$ this is the rayleigh eqn.

Reason why we get this eqn. is that wave of small wavelength looks at bdy's as if they are very far away & hence must be governed in limit by the Rayleigh problem.

let $\Omega = \frac{\omega}{\omega_s}$; $\omega_s = \frac{\pi}{2b}\sqrt{\frac{\mu}{\rho}} = \frac{\pi c_2}{2b}$ non dim freq

$Z = \frac{2b}{\pi}\xi$; $= \frac{4b}{L}$ ~ wave length   non dim wave no. ; $\frac{2\pi}{\xi} = L$

ξ ~ wave number.

$$\frac{c_1^2}{c_2^2} = \frac{2(1-\nu)}{1-2\nu} = \delta^2$$

$$\frac{2\alpha b}{\pi} = \sqrt{\frac{\Omega^2}{\delta^2} - Z^2} \qquad \frac{2\beta b}{\pi} = \sqrt{\Omega^2 - Z^2}$$

Extensional Motions (Symmetric)

$f \sim B\cos\alpha y \qquad h \sim C\sin\beta y$

$$u_x \sim (B\xi\cos\alpha y + C\beta\cos\beta y)$$

$$u_y \sim (-B\alpha\sin\alpha y + C\xi\sin\beta y)$$

$$\frac{\tan\beta b}{\tan\alpha b} + \left[\frac{4\xi^2\alpha\beta}{(\xi^2-\beta^2)^2}\right] = 0$$

Case 1 $Z = 0$ ($\xi = 0$ $L = \infty$)

$$\alpha b = \frac{\pi\Omega}{2\delta}$$

$\tan\beta b / \tan\alpha b = 0$

$\beta b = \frac{n\pi}{2}$ (even) $\Rightarrow \Omega = n$

$\alpha b = \frac{m\pi}{2}$ (odd) $\Rightarrow \Omega = m\delta$

Flexural (Antisymmetric)

$f \sim A\sin\alpha y \qquad h \sim D\cos\beta y$

$$u_x \sim (A\xi\sin\alpha y - D\beta\sin\beta y)$$

$$u_y \sim (A\alpha\cos\alpha y + D\xi\cos\beta y)$$

$$\frac{\tan\beta b}{\tan\alpha b} + \left[\frac{4\xi^2\alpha\beta}{(\xi^2-\beta^2)^2}\right]^{-1} = 0$$

$Z = 0$ ($\xi = 0$, $L = \infty$)

$$\beta b = \frac{\pi\Omega}{2}$$

$\tan\beta b / \tan\alpha b = \infty$

$\beta b = \frac{n\pi}{2}$ odd $\Rightarrow \Omega = n$

$\alpha b = \frac{m\pi}{2}$ even $\Rightarrow \Omega = m\delta$

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Continuing

Assumed a layer under plane strain and the following for the potentials

Assoc w/extension $\Phi = f(y) \sin \xi x \, e^{i\omega t}$

Assoc. w/Shear $H_3 = h(y) \cos \xi x \, e^{i\omega t}$

to satisfy the eqns of motion

$$f(y) = \overbrace{A \sin \alpha y + B \cos \alpha y}^{\text{Dilatational}} \quad w/ \; \alpha^2 = \xi^2\left(\frac{c^2}{c_1^2} - 1\right)$$

satisfies $\nabla^2(\Delta) = \frac{1}{c_1^2}\frac{\partial^2 \Delta}{\partial t^2}$

$$h(y) = \underbrace{C \sin \beta y + D \cos \beta y}_{\text{equivolum.}} \quad w/ \; \beta^2 = \xi^2\left(\frac{c^2}{c_2^2} - 1\right)$$

satisfy $\Delta(\omega) = \frac{1}{c_2^2}\frac{\partial^2 \omega}{\partial t^2}$

we find then that

Stress-Displ Eqns.

$$u_x = \frac{\partial \phi}{\partial x} + \frac{\partial H_3}{\partial y} = (\xi f + h') \cos \xi x \, e^{i\omega t}$$

$$u_y = \frac{\partial \phi}{\partial y} - \frac{\partial H_3}{\partial x} = (f' + \xi h) \sin \xi x \, e^{i\omega t}$$

we want extensional motion (symmetric) ie $B, C \neq 0$ $A \,\&\, D = 0$

" " flexural motion (Antisymmetric) if $A, D \neq 0$ $B, C = 0$

BC: traction free planes BC do not couple f&g. but give the relation between ξ & ω.

$$\Longrightarrow \quad \frac{\tan \beta b}{\tan \alpha b} + \left[\frac{4\xi^2 \alpha \beta}{(\xi^2 - \beta^2)^2}\right]^{\pm 1} = 0 \qquad (*) \qquad \begin{array}{l} +1 \; : \text{extensional motion} \\ -1 \; : \text{flexural motion} \end{array}$$

now $\omega = \xi c$ thus $(*)$ is an implicit fn of ξ & ω.

To get insight into what goes on we set up table (on pg 22 these notes)

since we have shown that $\frac{\tan \beta b}{\tan \alpha b}$, which governs both motions, can result in solutions which give rise to soln to extens, indep of soln to ~~equiv~~ flexural, we can have solns that are independent of the other 3.

Thus we can have extens dilat, extens equiv, flex. dilat & flex. equiv we can look at mixed by which doesn't couple equivol & dilatational modes. We can thus look at the 4 cases stated.

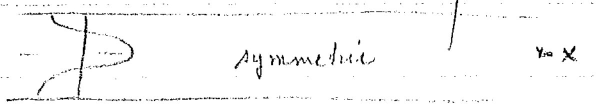
Equivoluminal waves

for infinitely long wavelength motion indep. of x_1

$\Omega = n$

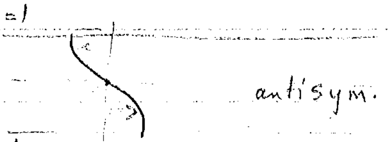
Ext. (Sym) $B, C \neq 0$	Flex (Antisy. $A, D \neq 0$
$B = 0$	$A = 0$
$u_x \sim C\beta \cos \beta y$; $u_y = 0$	$u_x \sim -D\beta \sin \beta y$; $u_y = 0$
$u_x = C\beta \cos \frac{n\pi y}{2b}$ n even $n=2$	$u_x = \cdot D\beta \sin \frac{n\pi y}{2b}$ n is odd.

for $n=2$



thickness shear $n=2$

for $n=1$



thickness shear mode
(lowest mode).

Dilational Waves

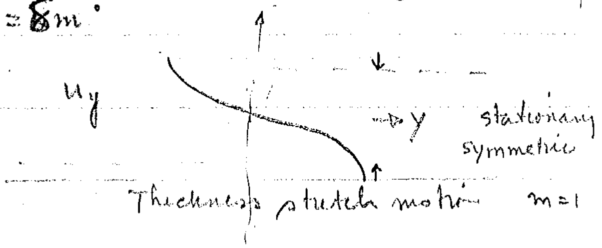
for ∞ long wavelength motion independent of x_1

Extensional
$C, B \neq 0$ (sym.)
$C=0$

$u_x = 0$; $u_y = -B\alpha \sin \alpha y$

$u_y = -B\alpha \sin \frac{m\pi y}{2b}$ m odd

$\Omega = 8m$

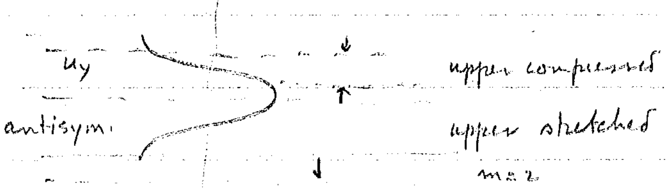


Thickness stretch motion $m=1$

Flex.
$A, D \neq 0$ (antisym.)
$D=0$

$u_x = 0$; $u_y = A\alpha\cos\alpha y$

$= A\alpha\cos m\frac{\pi y}{2b}$ (m even



Thickness stretch mode

11/1/79

Midterm: Next Thursday

Suppose $\xi = \beta \Rightarrow c/c_2 = \sqrt{2}$ This is a special case : $\beta = \xi\left(c^2/c_2^2 - 1\right)^{1/2}$

$\Rightarrow \alpha = \xi\left[\left(\frac{c}{c_1}\right)^2 - 1\right]^{1/2}$

Remember $Z = \sqrt{\Omega^2 - z^2}$

$$\alpha b = \frac{\pi}{2}\sqrt{\frac{2z^2}{\delta^2} - z^2} = \frac{\pi z}{2}\sqrt{\frac{2-\delta^2}{\delta^2}} = \frac{\pi}{2}\Omega\sqrt{\frac{2-\delta^2}{2\delta^2}} \qquad \beta b = \frac{\pi}{2} z = \frac{\pi}{2}\frac{\Omega}{\sqrt{2}}$$

Extensional $(B, C \neq 0)$

Freq eqn $\frac{\tan\beta b}{\tan\alpha b} = \infty$ see general eq (pg 22)

to satisfy the above let

$\beta b = \frac{r\pi}{2}$ r odd

or

$\alpha b = \frac{s\pi}{2}$ s even

$\Rightarrow \Omega = r\sqrt{2}$, $Z = r$ r odd

Flexural $(A, D \neq 0)$

$\frac{\tan\beta b}{\tan\alpha b} = 0$

to satisfy the above let

$\beta b = \frac{r\pi}{2}$ r even

$\alpha b = \frac{s\pi}{2}$ s odd

$\Omega = r\sqrt{2}$ $Z = r$ r even

$(C, D \neq 0)$ Equivolum. waves (no volume changes)

LAMÉ modes

Equiv Ext $B = 0$

$u_x \sim C\beta\cos\beta y\cos\beta x$

$u_y \sim C\beta\sin\beta y\sin\beta x$

substituting β for arguments

$u_x \sim C\beta\cos\frac{\pi y}{2b}\cos\frac{\pi x}{2b}$

$u_y \sim C\beta\sin\frac{\pi y}{2b}\sin\frac{\pi x}{2b}$

wavelength in x, y direct are same

Equiv Flex $A = 0$

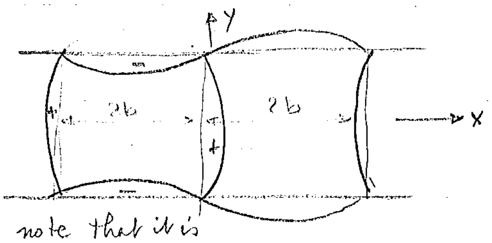
$u_x \sim -D\beta\sin\beta y\cos\beta x$

$u_y \sim D\beta\cos\beta y\sin\beta x$

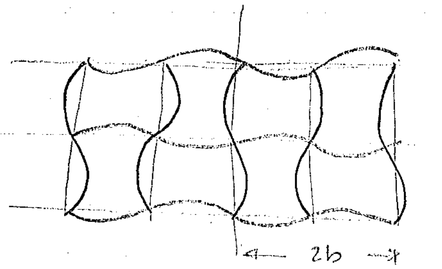
$u_x \sim -D\beta\sin\frac{\pi y}{b}\cos\frac{\pi x}{b}$

$u_y \sim D\beta\cos\frac{\pi y}{b}\sin\frac{\pi x}{b}$

How will this look



note that it is sym wrt middle plane

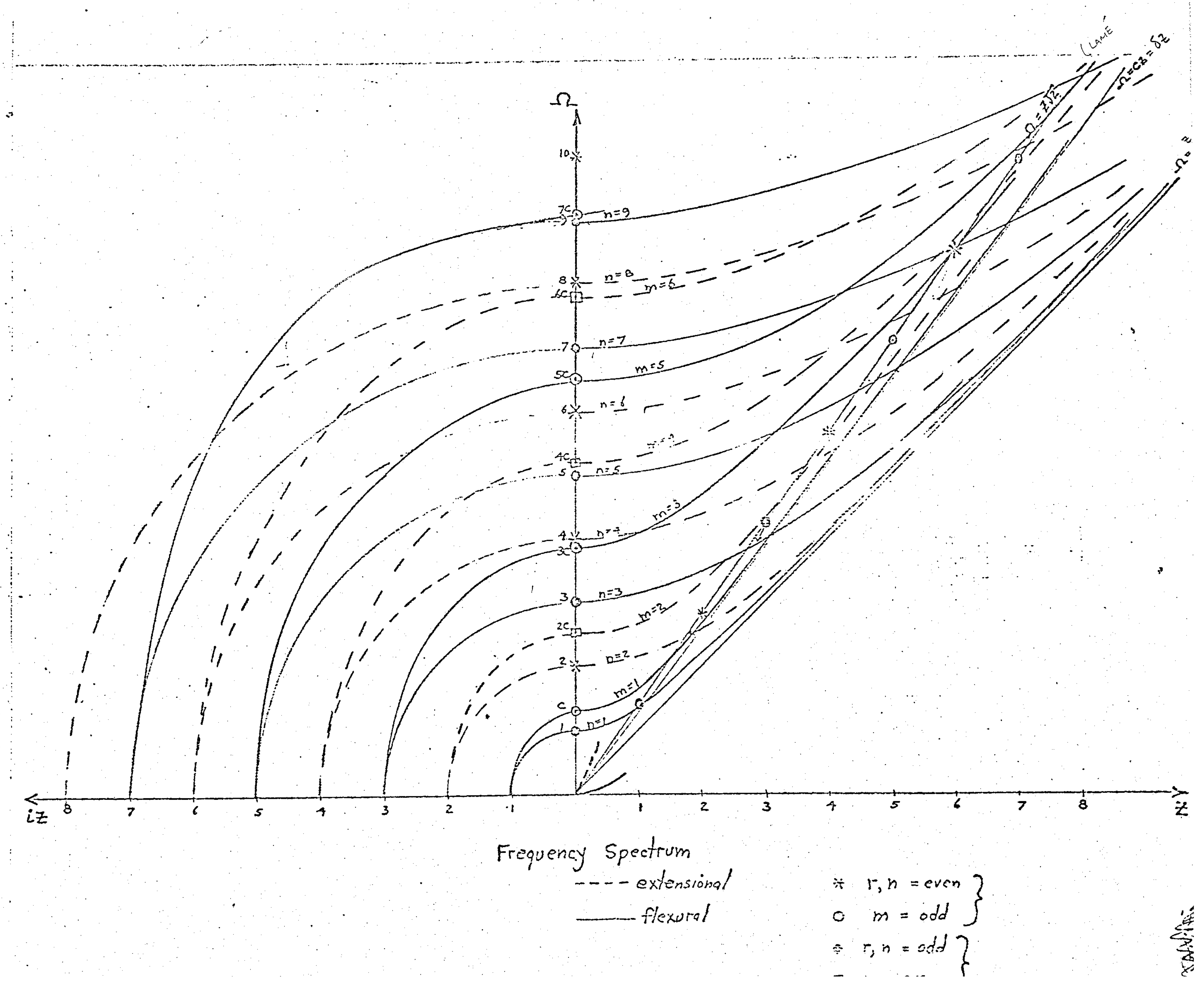


antisym. wrt middle plane

Remember $\alpha^2 = \xi^2\left(\frac{c^2}{c_1^2} - 1\right)$ $\beta^2 = \xi^2\left(\frac{c^2}{c_2^2} - 1\right)$

with the freq eq : $\frac{\tan\beta b}{\tan\alpha b} + \left[\frac{4\xi^2\alpha\beta}{(\xi^2 - \beta^2)^2}\right]^{\pm 1} = 0$ $\omega = \xi c$

this gives 3 eqns in 3 unkns α, ξ, β



Frequency Spectrum

where now $$\Omega = 2b\omega/\pi c_2, \qquad \bar{\xi} = 2b\xi/\pi. \tag{8.1.47}$$

In a similar fashion, for the SV waves we have from (8.1.37) and (8.1.41)

$$\beta b = n\pi/2 \qquad (n = 1, 2, 3 \ldots), \tag{8.1.48}$$

where n even and odd governs, respectively, the symmetric and antisymmetric SV waves. Using the second of (8.1.23), this may be put in the form

$$\Omega^2 = (n^2+\bar{\xi}^2) \qquad (n = 1, 2, 3, \ldots). \tag{8.1.49}$$

In order to plot the spectrum, Poisson's ratio must be specified, since it enters into (8.1.46) through k^2.

The curves for the real branches of the spectrum are seen to be hyperbolas. The cutoff frequencies, given by $\bar{\xi} \to 0$, are $\Omega = km, n$ for the P and SV waves. For $\Omega < km$, the wavenumbers become imaginary. Replacing $\bar{\xi}$ by $i\bar{\xi}$ in (8.1.46), it is seen that the P wave branches of the spectrum are ellipses. For $\Omega < n$, the wavenumbers of the respective SV waves become imaginary and, it is seen from (8.1.49), in the form of circles. The resulting spectrum, plotted for $\nu = 0{\cdot}31$, so that $k^2 = 1{\cdot}91$, is shown in Fig. 8.5.

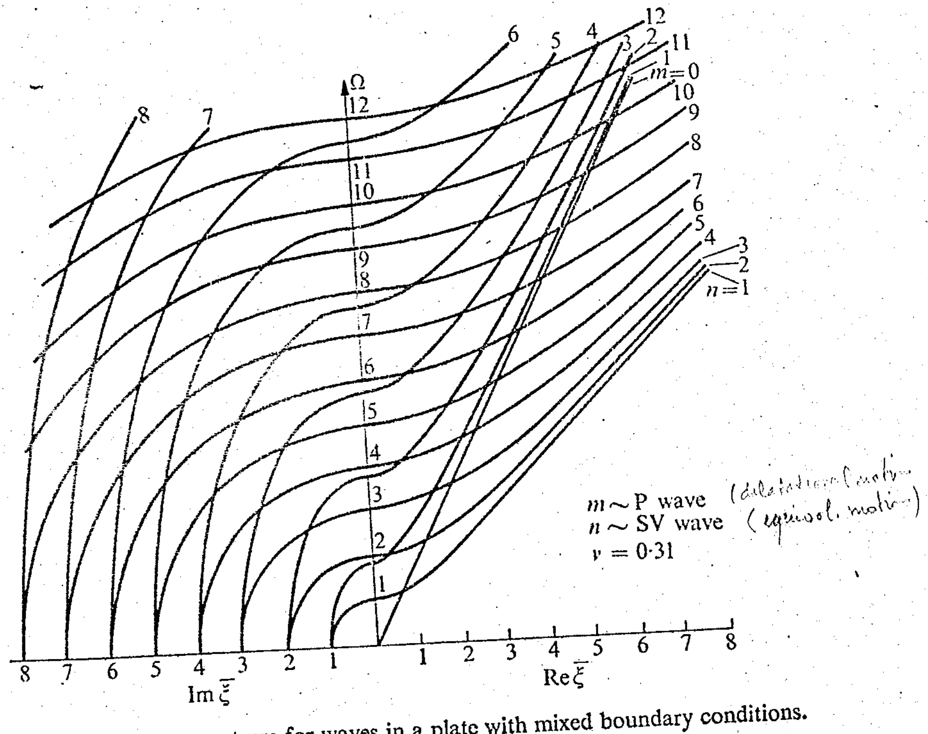


FIG. 8.5. Frequency spectrum for waves in a plate with mixed boundary conditions.

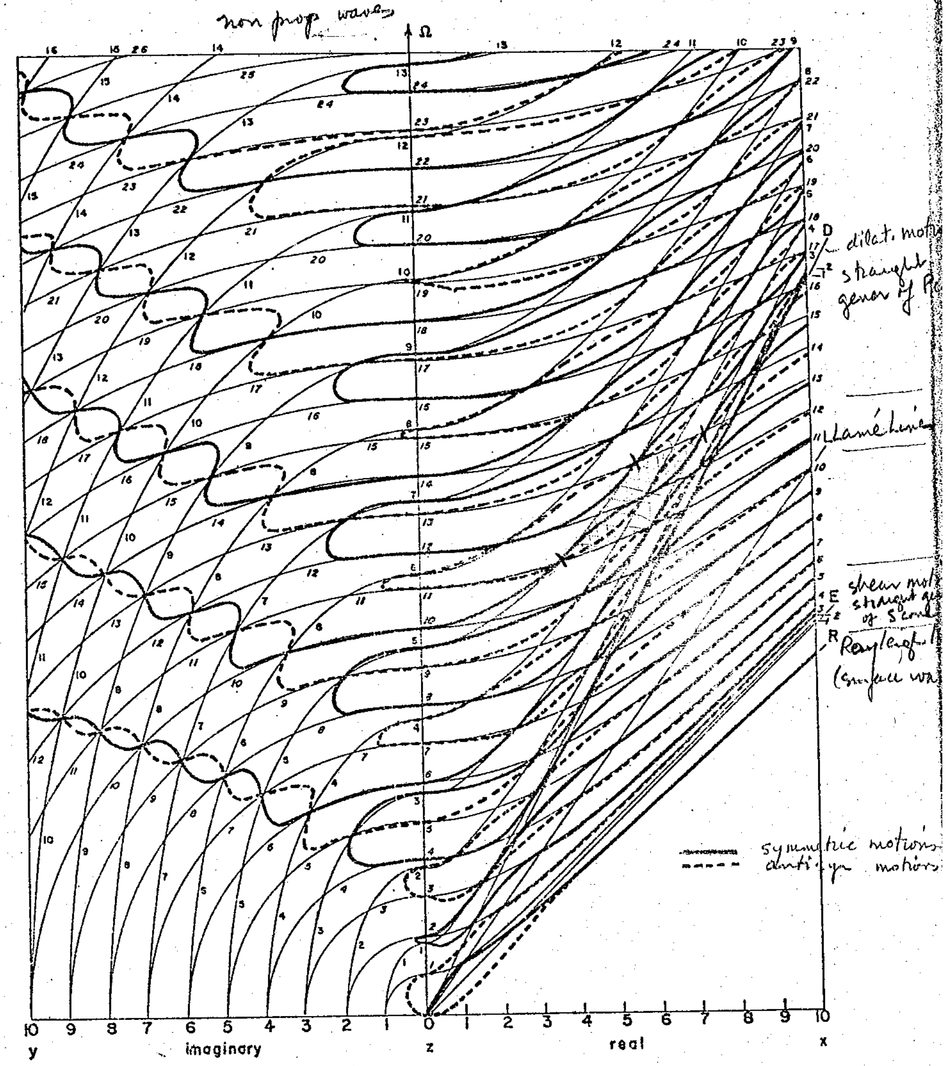


Fig. 19. Frequency spectrum of an infinite plate for real and imaginary wave-numbers ($\nu=0.31$).

quencies which fall on the curves $m=$constant and $n=$constant, in Figs. 18 and 19, only at the intersections of curves m odd with curves n odd. If $e>0$, $\sin \alpha b=0$ and $\sin \beta b=0$ are not solutions of (85) since both of (85) cannot then be satisfied by a single ratio A/D. Hence, the branches of (85) do not pass through the intersections of curves m even, n even as long as $e>0$.

The behavior of a branch of (85), as e passes from infinity (mixed boundary conditions) to zero (traction-free boundaries) is illustrated in Fig. 20 for a portion of the spectrum. In this portion, the curves $m=5$, $n=13$ correspond to mixed boundary conditions ($e=\infty$). The two dashed curves marked $0<e<\infty$ are for successive values of e, the lower curve corresponding to the

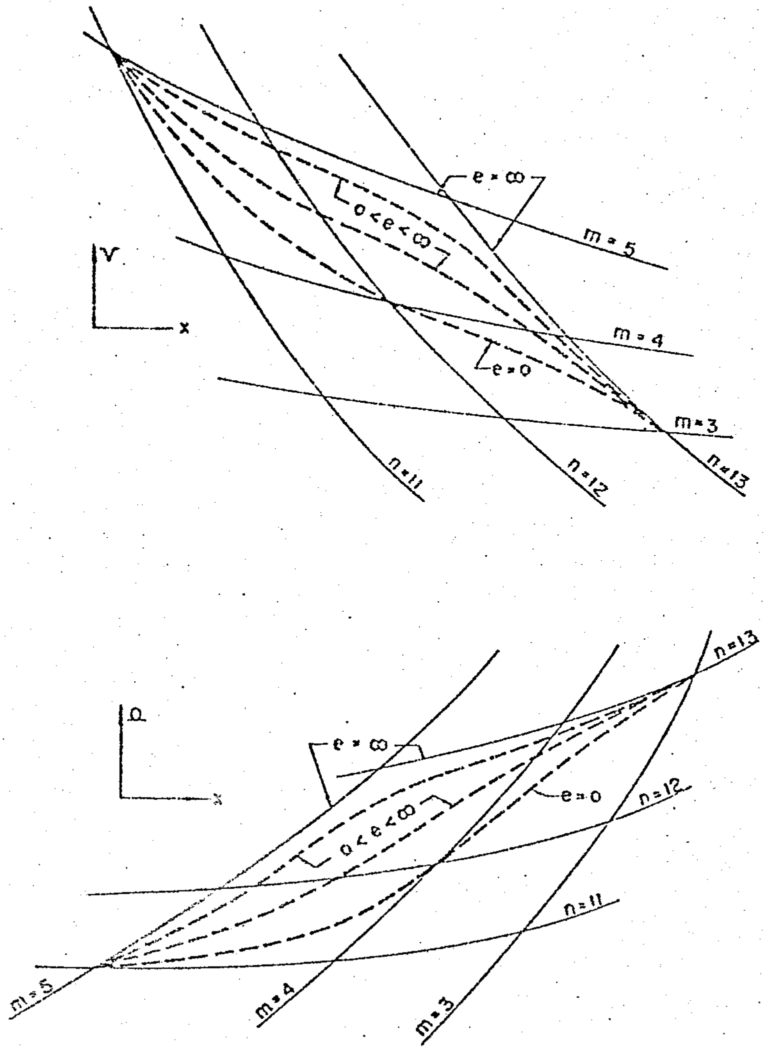
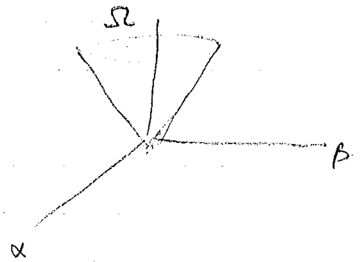


Fig. 20. Effect on the phase-velocity and frequency spectra of the antisymmetric modes of an infinite plate of the development of coupling between dilatational and equivoluminal overtones, as a result of relaxation of boundary constraint.

suppose we want to plot Ω vs. α, ξ for dilatational mode



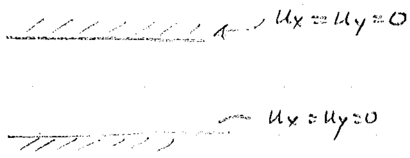
because of Bdy we only have selected values
thus $\alpha = \alpha(\xi)$ is a right circular cone.
$\alpha^2 = \xi^2\left(\frac{c^2}{c_1^2} - 1\right)$ for const α
we get sol of hyperbolas
now $\beta^2 = \xi^2\left(c^2/c_2^2 - 1\right)$ is a cone also but
because $c_1, c_2 \neq$ then it must be an
elliptic cone.

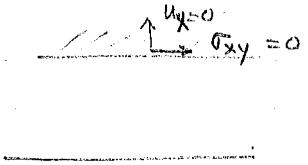
The intersection of these cones will be our solution

in the antiplane strain picking β in a good method satisfies BC. & we only have to satisfy the α eqn [ie motion of hyperbola (α = const) on a right circular cone]

Mindlin: "Waves & Vibrations in Isotropic Elastic Plates". Proc. First Symposium on Naval Struct. Mech.; Pergamon Press, 1960. Pg 199-232.

He considered





lubricated surface

if we write $e u_x = \sigma_{xy}$ & let $0 < e < \infty$ $m = 5,\ n = 11$ & $m = 3$ $n = 13$
satisfy conditions

<u>Approximate theories</u> based on freq spectrum we can restrict our freq so that we can keep only lowest modes of either sym or antisym modes.

Let $\Omega = \sum a_i \xi^i$ $a_0 = 0$. If we only keep 2 terms than
for the ~~extensional~~ case
flexural $\omega = \xi^2 b \sqrt{\frac{E}{3\rho(1-\nu^2)}}$ $\Omega = \pi z^2 \sqrt{\frac{1}{3}\left(1 - \frac{1}{\delta^2}\right)}$

for the ~~flexural~~ case
extensional $\omega = \xi \sqrt{\frac{E}{\rho(1-\nu^2)}}$

and $\Omega = 2z \sqrt{1 - 1/\delta^2}$

Using method of sep. of variables

Let $u(x,t) = X(x)\,T(t) \implies c^2 X''T = X\ddot{T}$

$$c^2 \frac{X''}{X} = \frac{\ddot{T}}{T} = \text{const} = -\omega^2$$

ω is the circular freq.

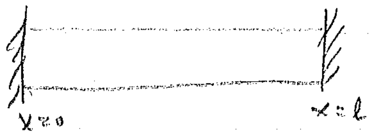
$$\implies X'' + \frac{\omega^2}{c^2} X = 0 \qquad \ddot{T} + \omega^2 T = 0$$

$$X(x) = D \sin \frac{\omega x}{c} + C \cos \frac{\omega x}{c}$$
$$T(t) = A \cos \omega t + B \sin \omega t.$$

sin, cos are principal modes of for linear system are orthog to each other

Specify BC.

of bar



fixed ends

$u(0,t) = u(l,t) = 0$

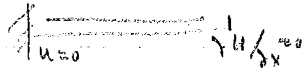
$\implies C=0 \quad \frac{\omega L}{c} = n\pi \quad : \quad \omega_n = \frac{n\pi c}{L}$

A, B will be termined as a fn of init conditions

$$\therefore u(x,t) = \sum X_n(x) T_n(t)$$

Free ends $\quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \quad D=0 \quad ; \quad \omega_n = \frac{n\pi c}{l}$

Fixed - Free



$$C=0 \quad \omega_n = \frac{n\pi c}{2l} \qquad n=1,3,5,7\ldots$$

Initial conditions $\quad u(x,0) = u_0(x)$

$\frac{\partial u}{\partial t}(x,0) = \bar{u}_0(x)$

$T(0) = \sum A_n X_n = u_0(x)$

$\dot{T}(0) = \sum + \omega_n B_n X_n = \bar{u}_0(x)$

mult both sides by $X_r(x)$ & integrate $\int_0^l$

$$= \int_0^l \sum = \sum A_n \int_0^l X_n X_r \, dx$$

0 if $r \neq n$

$$\boxed{\text{also } B_r = \frac{\int_0^l \bar{u}_0(x) X_r(x) dx}{\omega_r \int_0^l X(r^2) dx}}$$

$$\implies \boxed{A_r = \frac{\int_0^l u_0(x) X_r(x) dx}{\int_0^l X_r(x) dx}}$$

only good iff $\int_0^l X_r X_n \, dx = \begin{matrix} 0 & r \neq n \\ \neq 0 & r = n \end{matrix}$

now $\int_0^l X_r^2 dx = ?$ use D.E.

now: $\frac{-\omega_n^2}{c^2} X_n = X_n'' \Rightarrow -\frac{\omega_n^2}{c^2}\int_0^l X_n X_r dx = \int_0 X_n'' X_r dx$

$$= X_r X_n' \Big|_0^l - \int_0^l X_r' X_n' dx$$

$$= X_r X_n' \Big|_0^l - X_r' X_n \Big|_0^l + \int_0^l X_r'' X_n$$

but $X_r'' = -\frac{\omega_r^2}{c^2} X_r$ $\qquad -\frac{\omega_r^2}{c^2}\int_0^l X_r X_n$

$\therefore$

$$\cancel{\frac{+\omega_r^2 - \omega_n^2}{c^2}} \int_0^l X_n X_r dx = \left\{ X_r X_n' \Big|_0^l - X_r' X_n \Big|_0^l \right\} \frac{c^2}{\omega_r^2 - \omega_n^2}$$

free end $X_n' = X_r' = 0$
fixed end $X_r = X_n = 0$ $\Big\}$ on bdy. $\Rightarrow \int_0^l X_n X_r dx = 0$ if $\omega_r \neq \omega_n$, $r \neq n$

Forced motions

We can only allow ~~only~~ shear forces acting in the direction of centroidal axis.. define F = resultant shear/unit length.

$P \leftarrow \boxed{Fdx} \rightarrow P + \frac{\partial P}{\partial v} dx \qquad \left(\frac{\partial P}{\partial x} + F\right)\cancel{dx} = \rho A \frac{\partial^2 u}{\partial t^2}\cancel{dx}$

$$\therefore \quad E\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} = -\frac{F}{A}$$

or $$c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = -\frac{F}{\rho A}$$

11/8/79

Forced motions (continued)

$$c^2 u'' - \ddot{u} = -\frac{P(x,t)}{\rho A} \qquad \text{How to solve}$$

Assume we solve the free vib problem: Now let the solns be $X_n(x)\, q_n(t) = W_n(x,t)$

& $u(x,t) = \sum X_n \cancel{q_n} T_n(t)$ & $\sum W_n(x,t) = -P/\rho A$

put into DE to get $c^2 \sum X_n'' T_n - \sum X_n T_n'' = -\sum q_n(t) X_n(x)$

or $c^2 X_n'' T_n = (T_n'' - q_n) X_n$ $\qquad q_n = \dfrac{\int X_n P/\rho A\, dx}{\int X_n^2 dx}$

$$\therefore \frac{c^2 X_n''}{X_n} = \frac{T_n'' - q_n}{T_n} = -\omega_n^2$$

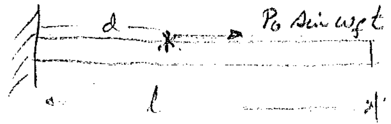
$$T_n'' + \omega_n^2 T_n = -q_n(t)$$

Solve homogeneous : $T_{n_h}(t) = A_n \cos \omega_n t + B_n \sin \omega_n t$

Particular : $T_{n_p}(t) = -\frac{1}{\omega_n} \int_0^t q_n(\tau) \sin\underset{\omega_n}{\wedge}(t-\tau)\, d\tau$

$\therefore T_n = T_{np} + T_{nh}$

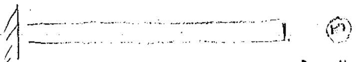
Example: Given a built-in bar free at one end with a periodic forcing function



$$q_n(t) = \frac{2P_0}{\rho A L} \sin \omega_f t \sin \frac{n\pi d}{2l} \quad n = \text{odd}$$

$$u(x,t) = \sum_{n=\text{odd}}^{\infty} \left\{ A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{2P_0 \sin \frac{n\pi d}{2l}}{\rho A L} \left[\frac{-\frac{\omega_f}{\omega_n} \sin \omega_n t - \sin \omega_f t}{\omega_n^2 - \omega_f^2} \right] \sin \frac{n\pi x}{l} \right\}$$

Let's Consider time-dependent bdry conditions



$DE = c^2 u'' - \ddot{u}$

x=0 x=l

BC' $D_l\, u(l,t) = f_2(t)$
$D_0\, u(0,t) = f_1(t)$

$D_{0,l} \equiv 1$ if displ is prescribed
$\frac{\partial}{\partial x}$ if forces pres

Let $u(x,t) = \bar{u}(x,t) + f_1(t) g_1(x) + f_2(t) g_2(x)$

apply DE $\Rightarrow c^2 \bar{u}'' - \ddot{\bar{u}} = -c^2 f_1 g_1'' - c^2 f_2 g_2'' + \ddot{f}_1 g_1 + \ddot{f}_2 g_2$

BC $\Rightarrow D_0[\bar{u}(0,t)\} = f_1 - f_1 D_0 g_1(0) - f_2 D_0 g_2(0) = f_1\{1 - D_0 g_1(0\} - f_2 D_0 g_2(0)$

$D_l[\bar{u}(l,t)] = \{f_2 - f_1 D_l g_1(l) - f_2 D_l g_2(l)\} = f_2\{1 - D_l g_2(l)] - f_1 D_l g_1(l)$

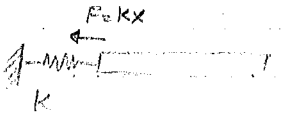
how can we cause BC to be zero and get rid of difficulty due to nonhom. BC?
since f_1 & f_2 are independent. $\Rightarrow$ each must be independent.

Then $f_1(1 - D_0 g_1(0)) = 0 \Rightarrow D_0 g_1(0) = 1$
$f_2 D_0 g_2(0) = 0$
$f_2(1 - D_l g_2(l)) = 0$
$f_1 D_l g_1(l) = 0 \Rightarrow D_l g_1(l) = 0$

Choose $\therefore g_1(x) \stackrel{\vee}{=} a_1 + b_1 x + c_1 x^2 +$

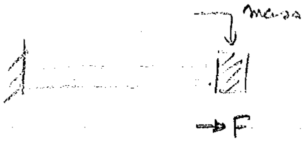
Generally choose any fn ~~to~~ satisfy BC similarly for g_2 ; put into DE . This will give a nonhomo DE on $\bar{u}$ w/ homog. boundary conditions . Now we can use our previous treatise

Example :



if k is soft assume bar is rigid & we have one-degree of freedom system.

Suppose :



the force $F = M \cdot \ddot{u}(l)$ will be taken into BC : but $\int X_n X_r \, dx \neq 0$

Bar Eqn of motion: $P' = \rho A \ddot{u}$ $\quad 0 \leq x < l$ $\quad P = AEu'$

Mass Eq of motion $-P = M\ddot{u}$ $\quad x = l$ $\quad P_n = AEX_n'$

$$P_n' = -\rho A \omega_n^2 X_n \qquad 0 < x < l \qquad (*)$$

$$-P_n = -M\omega_n^2 X_n \qquad x = l \qquad (**)$$

now take $\displaystyle\int_0^l P_n' X_r \, dx = -\rho A \omega_n^2 \int_0^l X_n X_r \, dx \qquad 0 < x < l \qquad$ from $(*)$

$$\hookrightarrow P_n X_r \Big|_0^l - \int_0^l P_n X_r' \, dx = -\rho A \omega_n^2 \int_0^l X_n X_r \, dx \qquad (\dagger)$$

$$\text{now } P_n X_r \Big|^l = +M\omega_n^2 X_n X_r \Big|^l \qquad \text{from } (**) \qquad (\dagger\dagger)$$

also $\displaystyle -\rho \omega_r^2 A \int_0^l X_n X_r \, dx = \int_0^l P_r' X_n \, dx = P_r X_n \Big]_0^l - \int_0^l P_r X_n' \, dx \quad (\dagger *)$

also $\displaystyle - M\omega_r^2 X_n X_r \Big|^l = - P_r X_n \Big|^l \qquad (*\dagger)$

taking $(\dagger)$ and $(\dagger *)$ and $(\dagger\dagger)$ & $(*\dagger)$

$$\rho A(\omega_n^2 - \omega_r^2) \int_0^l X_n X_r \, dx + M(\omega_n^2 - \omega_r^2) X_n X_r \Big|^l = - P_n X_r \Big|_0^l + \overset{①}{\int_0^l P_n X_r' \, dx}$$

$$+ P_r X_n \Big|_0^l - \overset{②}{\int_0^l P_r X_n' \, dx}$$

but $P_r = AEX_r'$ $\quad P_n = AEX_n'$

$$+ P_n X_r \Big|^l - P_r X_n \Big|^l$$

$\Rightarrow$ ① cancels ②

but rest of RHS is

$$- P_n X_r \Big|_0^l + P_r X_n \Big|_0^l + P_n X_r \Big|^l - P_r X_n \Big|^l = \overset{EAX_r'}{P_r} X_n \Big|_0 - \overset{EAX_n'}{P_n} X_r \Big|_0$$

at built-in end $\Rightarrow$ $X_r = 0$ $\quad X_n = 0$

at free end $\quad X_n' = 0 \quad X_r' = 0$

Thus for $\omega_r \neq \omega_n$ $\Rightarrow$

$$\rho A \int_0^l X_n X_r \, dx + M X_r X_n \Big|^l = 0$$

for homog. solution $T_n(t) = A_n \cos\omega_n t + B_n \sin\omega_n t$

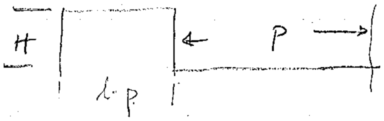
$\Rightarrow$ we can get the A_n & B_n using the relation just found if $u(x,0) = u_0(x)$

then $$A_n = \frac{\int_0^l X_n u_0 \, dx + Rl X_n(l) u_0(l)}{\int_0^l X_n^2 dx + Rl X_n^2(l)} \qquad Rl = M$$

Look at homework (Problem Set #2) Ex #1

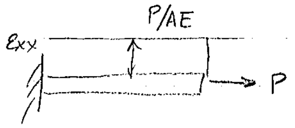
$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{P}{AE}\frac{4}{\pi}\sum_{n=1,3,\ldots}\frac{(-1)^{\frac{n-1}{2}}}{n}\cos\frac{n\pi x}{2l}\cos\frac{n\pi ct}{2l}$$

look at heaviside step fn. $\therefore$ $H(x) - H(x-p)$

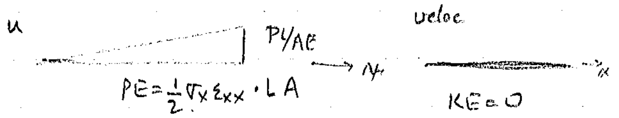


$$H = \frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n}\cos\frac{n\pi p}{2l}\cos\frac{n\pi x}{2l}$$

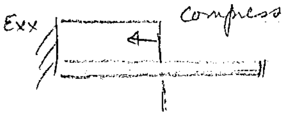
$\therefore$



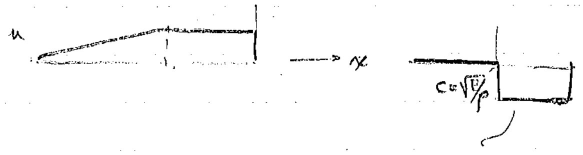
@ $t = 0^-$



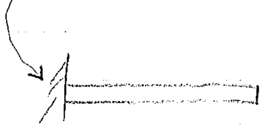
@ $t = t_0$



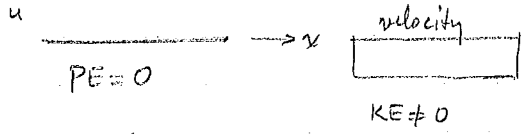
compression wave @ speed E/ρ travels in direction

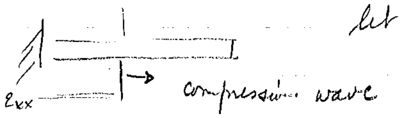


@ $t = l/c$ $\cos\frac{n\pi ct}{2l} = 0$

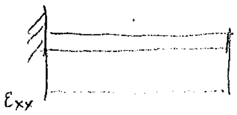


beam is undisturbed



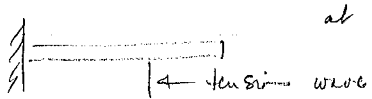


let $t = t_0 + l/c$

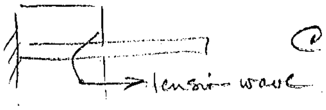


@ $t = 2l/c$ $\cos\frac{n\pi ct}{2l} = -1$

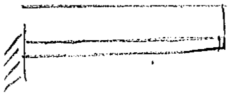
wave has reached end of bar, bar in uniform compression



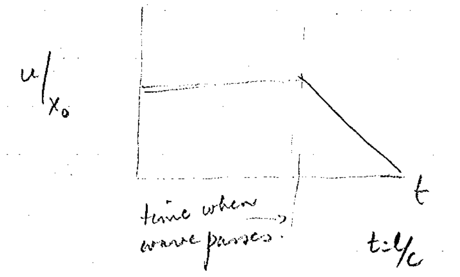
at $t = t_0\, 2l/c$



@ $t = 3l/c + t_0$



@ $t = 4l/c$ bar is in uniform tension again



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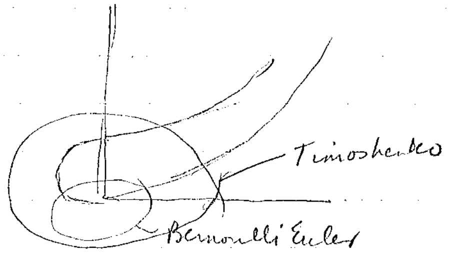
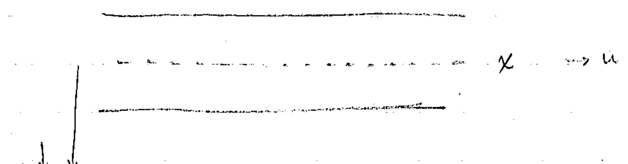
Different theories



Will use Hamilton Principle to obtain Timo Beam theory then by reducing it we get bernoulli-euler



either beam (symmetric about x-z planes so no torsion) of plate in plane strain

Introduce displ $u_i = u_i(x,y,z,t)$

Let $u_{1,2,3} = u, v, w$. Replace the exact u_i by approx $\bar{u}_i$ by restriction of possible motions

Let $\bar{u} = -z\psi(x,t)$ only

$\bar{v} = 0$

$\bar{w} = w(x,t)$

To derive the eqns of motion using Hamilton Principle. Look at Lagrangian between t_1 & t_2. The actual motion is such that $\delta\int_{t_1}^{t_2} L\,dt = 0$

$L = KE - PE = T - U$

$\delta L = \delta T - \delta U$

$\delta U = \delta V - \delta W$

elastic strain energy

work done by applied external forces.

$\delta V = \sigma_{ij}\delta\varepsilon_{ij}$

$W = \int(T_i u_i ds + B_i u_i dV)$

$\delta W = T_i \delta u_i dA + B_i \delta u_i dV$

Force/unit Area

body force/vol.

If we use the unconstrained u_i then we would get from Hamilton's Principle the eqns of motion.

$$\bar{\varepsilon}_{xx} = -z\frac{\partial \psi}{\partial x} \qquad \bar{\varepsilon}_{yy} = 0 \qquad \bar{\varepsilon}_{zz} = 0$$

$$\bar{\varepsilon}_{yz} = 0 \qquad \bar{\varepsilon}_{zx} = \frac{1}{2}(w' - \psi) \qquad \bar{\varepsilon}_{xy} = 0$$

per unit volume / unit time

$$\delta \bar{V}^* = -\bar{\sigma}_{xx}\, z\, \delta\psi' + \bar{\sigma}_{zx} \cdot \frac{1}{2}(\delta w' - \delta\psi) \cdot 2$$ since $\sigma_{zx}\varepsilon_{zx} + \sigma_{xz}\varepsilon_{xz} = 2\sigma_{xz}\varepsilon_{xz}$

$\delta V = \int \delta V^* \, dx\,dy\,dz$ dependence on z & y is explicit for $\delta \bar{V}^*$

$$\therefore \quad \delta\bar{V} = \int_0^L \Big[\underbrace{-\iint_{Area} (\bar{\sigma}_{xx} z \, dy\,dz}_{moment} \Big] \delta\psi' dx + \int_0^L \Big[\underbrace{\iint \bar{\sigma}_{zx}\, dy\,dz}_{Q - \text{total shear force.}} \Big] (\delta w' - \delta\psi)\, dx$$

$$\therefore \quad \delta\bar{V} = \int_0^L \Big[-M \delta\psi' dx + Q(\delta w' - \delta\psi) \Big] dx$$

integrate by parts $\displaystyle \int_0^L [M'\delta\psi - Q'\delta w - Q\delta\psi]\, dx - M\,\delta\psi\Big|_0^L + Q\,\delta w\Big|_0^L$

we have assumed : $\delta(\psi') = (\delta\psi)'$ variation & differentiation are interchang

$$\bar{T}^* = \frac{\rho}{2} \dot{u}_i^{\,2} \qquad \delta\bar{T}^* = \rho\, \dot{u}_i \delta\dot{u}_i \qquad \delta\bar{T}^* = \rho\left[z^2 \dot{\psi}\,\delta\dot{\psi} + \dot{w}\,\delta\dot{w}\right]$$

$\therefore \delta\bar{T} = \int_{vol} \delta\bar{T}^* \, dx\,dy\,dz$ again z & y dependence is explicit

Let $\displaystyle\int_{area} z^2 dy\,dz = I \qquad \int_{area} dy\,dz = A$

$$\delta\bar{T} = \int_0^l \left[\rho I \dot{\psi}\,\delta\dot{\psi} + A \dot{w}\,\delta\dot{w}\right] dx$$

$$\int_{t_1}^{t_2} \delta\bar{T}\, dt = \int_0^l \rho \int_{t_1}^{t_2} (I \dot{\psi}\,\delta\dot{\psi} + A\dot{w}\,\delta\dot{w})\, dx\, dt$$ use integration by part.

$$\int_{t_1}^{t_2} \delta\bar{T}\, dt = \int_0^l \rho \int_{t_1}^{t_2} (-I\ddot{\psi}\,\delta\psi - A\ddot{w}\,\delta w)\, dx\,dt + \Big[\int_0^l \rho(I\dot{\psi}\,\delta\psi + A\dot{w}\,\delta w)\, dx\Big]_{t_1}^{t_2}$$

by Ham. Principle the bdy term is zero since we allow no variation of the variables ψ, but if we keep them then we deduce to permissible BC.

to get external work done by forces (external traction + body forces)

in general $$\delta\bar{W} = \int_S \underset{\text{stress tensor} \perp \text{to surface.}}{t_{ni}}\, \delta u_i\, ds + \int_{vol} \underset{\text{body forces}}{b_i}\, \delta u_i\, dV$$

$$\delta\bar{W} = \int_A \underset{\text{end faces}}{(-F_x z\,\delta\psi + F_z\,\delta w)}\, dy\,dz\Big|_0^l + \int_0^l \Big[\oint_c \underset{\text{cylindrical surface}}{(-F_x z\,\delta\psi + F_z\,\delta w)}\, dy\,dz\Big] dx$$

$b_i = X_i$ where $X_1 = X$, $X_2 = Y$, $X_3 = Z$ body forces.

$$+ \underbrace{\int_0^{\ell} \iint_{Area} \left(- X z \, \delta\psi + Z \, \delta w \right) dx\, dy\, dz}_{}$$

$$\int_0^{\ell} \left(- \bar{X} \delta\psi + \bar{Z} \, \delta w \right) dx \qquad \bar{X} = \int X z \, dy\, dz \quad \bar{Z} = \int Z \, dy \, dz$$

but: $\oint_c f_x z \, dy\, dz = M_s$ bending moment due to applied axial shear tractions

and: $\oint_c f_z dy dz = q$ is the net transverse load on cylindrical surface $\quad \int_A F_x z\, dz\, dy = M$

$$\int_A F_z \, dz\, dy \underset{\ell}{=} Q$$

$$\therefore \; \delta \bar{W} = \Big[- \underset{tract.}{M} \delta\psi + \underset{tract.}{Q} \delta w \Big]_0^{\ell} + \int_0^{\ell} \left(- m_s \delta\psi + q \, \delta w \right) dx + \int_0^{\ell} \left(- \bar{X} \delta\psi + \bar{Z} \, \delta w \right) dx.$$

applied mom + shear due to shear on ends.

Put all these into hamilton's principle

$$0 = \int_0^{\ell} \int_{t_1}^{t_2} \left(- \rho I \ddot{\psi} \delta\psi - \rho A \ddot{w} \delta w - M' \delta\psi + Q' \delta w + Q \, \delta\psi - m_s \delta\psi + q \, \delta w - \bar{X} \delta\psi + \bar{Z} \delta w \right) dx\, dt$$

+ boundary terms. Since variations $\delta\psi$, δw are anything we want & independent of each other $\Rightarrow$ their coeffs must vanish.

$$\left.\begin{array}{ll} - M' + Q - m_s - \bar{X} = \rho I \ddot{\psi} & \delta\psi \\ Q' + q + \bar{Z} = \rho A \ddot{w} & \delta w \end{array}\right\} \text{ w/ bc \& IC}$$

normally M, Q, w, ψ are unknown thus we must find material eqns to define other 2 eqns.

$\longrightarrow \quad \sigma_{ij} = \lambda \varepsilon_{ll} \delta_{ij} + 2\mu \varepsilon_{ij}$ what must they become using approx. theory keeping in mind that result must be physically sensible

$$\sigma_{xx} = (\lambda + 2\mu)\, \varepsilon_{xx} + \lambda (\varepsilon_{yy} + \varepsilon_{zz})$$

$$\varepsilon_{yy} + \varepsilon_{zz} = \frac{\sigma_{yy} + \sigma_{zz}}{2\lambda + \mu} - \frac{\lambda}{\lambda + \mu} \varepsilon_{xx}$$

$$\Rightarrow \sigma_{xx} = \left| \frac{(\lambda + 2\mu)(\lambda + \mu) - \lambda^2}{\lambda + \mu} \right| \varepsilon_{xx} + \frac{\lambda}{2(\lambda + \mu)} \cdot (\sigma_{yy} + \sigma_{zz})$$

$3\lambda\mu + 2\mu^2$

$\mu(3\lambda + 2\mu) = E$

$\lambda + \mu$

E

we now ask which do we satisfy i.e. $\sigma_{yy} + \sigma_{zz} = 0$ or $\varepsilon_{yy} + \varepsilon_{zz} = 0$ when bar bends we expect free expansion due to poisson ratio $\Rightarrow \varepsilon_{yy} + \varepsilon_{zz} \neq 0$ $\Rightarrow \sigma_{yy} + \sigma_{zz} = 0$ but M is a fn of σ_{xx}

$\therefore$ we can use a weaker condition that $\int (\sigma_{yy} + \sigma_{zz})\, dy\, dz = 0$

$$E \cdot (-z\psi') \, z \, dy\, dz$$

$$\therefore \; M = \int_{Area} - E z^2 \psi' \, dy\, dz + \frac{\lambda}{2(\lambda + \mu)} \int_{Area} (\sigma_{yy} + \sigma_{zz}) \, dy\, dz$$

Thus we violate compatibility in order to get a better shear-strain relation.

thus $\boxed{M = -EI\psi'}$ now $\int \bar{\sigma}_{zx}\, dy\,dz = Q$

$= \iint \sigma_{xx} z\, dy\, dz = +E\iint \varepsilon_{xx} z\, dy\, dz$

$= E\iint -z^2 \psi' dy\,dz = -EI\psi'$ ✓

$\bar{\varepsilon}_{zx} = \frac{1}{2}(w' - \psi) \Rightarrow \bar{\sigma}_{zx} = 2\mu\bar{\varepsilon}_{zx}$
$= \mu(w' - \psi)$

$\int \bar{\sigma}_{zx}\, dy\, dz = Q = AG(w' - \psi)$ $\underline{\underline{G = \mu}}$ since w, ψ are not fn. of y, z

BC $-M\delta\psi\Big|_0^L + Q\delta w\Big|_0^L$ from 3-d elasticity we got $t_i\,\delta u_i$ on S.

must specify M or ψ and Q or w.

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Timoshenko Beam - revisited

$-M' + Q + m_S - \bar{X} = \rho I\ddot{\psi}$ angular momentum.

$Q' + q + \bar{Z} = \rho A\ddot{w}$

$M = -EI\psi'$

$Q = AG(w' - \psi)$

To obtain Bernoulli - Euler Eqn.

let rotatory inertia $\ddot{\psi}$ / m_S, $\bar{X}$, $\bar{Z}$ all be ignored

from 1: $-M' + Q = 0 \Rightarrow -M'' + Q' = 0$ (3)

from 2: $Q' + q = M'' + q = \rho A\ddot{w} \Rightarrow \rho A\ddot{w} - M'' - q = 0$

$\Rightarrow w' = \psi$ hence there is no shear strain but $\not\Rightarrow Q = 0$. Hence where do we get Q from (3) since $Q = M'$. (Note $Q = AG(w' - \psi)$ is inoperative Q is inoperative $\Rightarrow$ Q is shifted from stress-strain law in Timo Beam theory to a statement of equilibrium). Since $Q \neq 0 \Rightarrow G = \infty$; we assume beam is infinitely rigid.

$\therefore M = -EI\psi' = -EIw''$ $Q = M' = -EI\psi'''$

& $-M'' + \rho A\ddot{w} = q \Rightarrow EIw^{IV} + \rho A\ddot{w} = q$

let $q = 0$ then for free vibration $\frac{EI}{\rho A} w^{IV} + \ddot{w} = 0$

Note ["BC & IC remain same remember we had to specify $M \overset{or}{\psi}$ ~~or~~ $\overset{and}{}$ $Q \overset{or}{w}$

$\Rightarrow Mw'$ & Qw $\Rightarrow$ EIw'' or w' <u>and</u> EIw''' or w must be specified

Let $\frac{EI}{\rho A} = a^2$ $\therefore a^2 w^{IV} + \ddot{w} = 0$

if $w(x,t) = XT \Rightarrow a^2 X^{IV}T + XT'' = 0 \Rightarrow \frac{a^2 X^{IV}}{X} = -\frac{\ddot{T}}{T} = \omega^2$

thus $T = A\cos\omega t + B\sin\omega t$

if $\omega^2 = \frac{m^4}{\ell^4} a^2 \Rightarrow \mathrm{X}^{iv} - \frac{m^4}{\ell^4}\mathrm{X} = 0$ where ℓ = length of beam.

Let $e^{\lambda x} = \mathrm{X} \Rightarrow \lambda^4 - \frac{m^4}{\ell^4} = 0 \therefore \left(\lambda^2 - \frac{m^2}{\ell^2}\right)\left(\lambda^2 + \frac{m^2}{\ell^2}\right)$

$$\Rightarrow X = C\cos\frac{mx}{\ell} + D\sin\frac{mx}{\ell} + E\cosh\frac{mx}{\ell} + F\sinh\frac{mx}{\ell}. \quad (*)$$

Consider SS beam: $w=0$ & $w''=0$ @ $x=0,\ell$ $\Rightarrow$ @ $x=0$ $C+E=0$ ($w(0)=0$), @ $x=0$ $C+E=0$ ($w''(0)=0$) $\Rightarrow C=E=0$, det $\neq 0$, $\frac{m}{\ell} \neq 0$

@ $x=\ell$ $D\sin m + F\sinh m = 0$

$\frac{m^2}{\ell^2}\left[-D\sin m + F\sinh m\right] = 0$

det $= 2\sin m \sinh m = 0$ for non trivial sol. $\Rightarrow m_n = n\pi$

$\omega_n^2 = \frac{n^4\pi^4 a^2}{\ell^4}$

$$\boxed{\omega_n = \frac{n^2\pi^2}{\ell^2}\sqrt{\frac{EI}{\rho A}}} \qquad \boxed{\mathrm{X}_n = \sin\frac{n\pi x}{\ell}}$$

Consider Cantelever beam: @ $x=0$ $w=0$ ∴ $w'=0$ no displ or rotation

$x=\ell$ $w''=0$ $w'''=0$ no mom/no shear

using (*)

$w(0)=0 \Rightarrow C_n + E_n = 0$

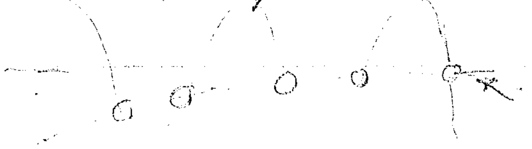
$w'(0)=0 \Rightarrow D_n + F_n = 0$

$w''(\ell)=0 \Rightarrow$ using $C_n = -E_n$ $D_n = -F_n$

$-C_n(\cos m_n + \cosh m_n) - D_n(\sin m_n + \sinh m_n) = 0$

$C_n(\sin m_n - \sinh m_n) - D_n(\cos m_n + \cosh m_n) = 0$

for nontriv $\cos m_n \cosh m_n = -1$ transcendental eq =



for large n intersection is ~ $m_n = (n - \frac{1}{2})\pi$

$m_1 = 1.875$

$m_2 = 4.694$

$m_3 = 7.855$

$m_4 = 10.996 \cong \frac{7}{2}\pi$ to get mode shapes put in m_n into eqns involving C_n & D_n to get ratio of C_n, D_n

to get A_n & B_n for $T = A_n\cos\omega_n t + B_n\sin\omega_n t$

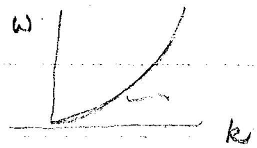
I.C. @ $t=0$ $w(x,0) = w_0 = \sum A_n X_n$ using orthog $\int_0^\ell X_n X_r\,dx = \begin{cases} 0 & n \neq r \end{cases}$

$\dot{w}(x,0) = \dot{w}_0$

$$\therefore \quad A_n = \frac{\int_0^l w_0 X_n \, dx}{\int_0^l X_n^2 \, dx} \qquad B_n = \frac{\int_0^l \dot{w}_0 X_n \, dx}{\int_0^l X_n^2 \, dx}$$

for infinite beam. $a^2 w^{IV} + \ddot{w} = 0$ let $w(x,t) = W e^{ikx} e^{i\omega t}$

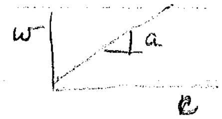
$$\Rightarrow \underset{\text{char eqn.}}{a^2 k^4 - \omega^2 = 0} \qquad \omega = \pm a k^2$$



compare to lower branch of Rayleigh Lamb prob.

to show this is dispersive

$$c = \frac{\omega}{k} = \frac{a k^2}{k} = ak$$



for the finite beam we get non dispersive sol. propagates waves w/same velocity

" " infinite " " " dispersive sol. as $k \to 0$ $\lambda \to \infty \Rightarrow c \to 0$

infinite beam contains results of ss beam but not cantilever

For Forced motions

$$a^2 w^{IV} + \ddot{w} = \frac{q}{A}(x,t)$$

1. Solve Homog. first in terms of X_n
2. Solve RHS in terms of X_n, ie expand i.e. $\frac{q}{A} = \sum X_n Q_n(t)$

$$Q_n(t) = \frac{\int_0^l \frac{q(x,t)}{\rho A} X_n(x) \, dx}{\int_0^l X_n^2 \, dx}$$

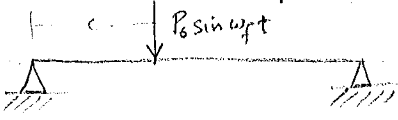
now the time dep part becomes $T'' + \omega^2 T = Q_n(t)$

Solve the homog to get $T_h = A_n \cos \omega_n t + B_n \sin \omega_n t$

$$T_p = \frac{1}{\omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) \, d\tau$$ to include init conditions

11/29/79

Forced Vibration - Example.



Assume steady state ie initial conditions play no part

$$q(x,t) = \sum Q_n(t) X_n(x) \qquad Q_n = \frac{\int_0^l X_n q \, dx}{\int_0^l X_n^2 \, dx}$$

$$X_n = \sin \frac{n\pi x}{l} \qquad \int_0^l X_n^2 \, dx = l/2 \qquad q(x,t) = P_0 \sin \omega_f t \, \delta(x-c)$$

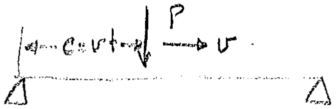
$$\therefore \quad Q_n(t) = \frac{2}{l} P_0 \sin \omega_f t \sin \frac{n\pi c}{l}$$

$$\text{Now } T_n(t) = \frac{1}{\omega_n} \int_0^{\tau} Q_n(\tau) \sin \omega_n (t-\tau) \, d\tau.$$

$$\therefore \quad w(x,t) = \frac{2 P_0 L^3}{\pi^4 EI} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi c}{l} \sin \frac{n\pi x}{l} \left(\sin \omega_f t - \frac{\omega_n}{\omega_f} \sin \omega_n t \right)}{n^4 \left(1 - \omega_f^2/\omega_n^2\right)}$$

if $\omega_f/\omega_n = 1 \Rightarrow$ no longer steady state but must give initial conditions

Suppose the load is not located at c $\forall t$ but moves w/ velocity v



replaces $P_0 \sin \omega_f t$ by P

$$P\delta(x-ct) = q(x,t) = \sum Q_n(t) X_n(x)$$

$$Q_n(t) = \frac{\int_0^l P\delta(x-ct)\, X_n(x)\, dx}{\int_0^l X_n^2\, dx} = P \cdot \frac{2}{l} \sin \frac{n\pi v t}{l}$$

$$\Rightarrow \quad w(x,t) = \frac{2Pl^3}{\pi^4 EI} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{l} \left(\sin \frac{n\pi v t}{l} - \frac{vl}{n\pi a} \sin \frac{n^2 \pi^2 a t}{l^2} \right)}{n^4 \left[1 - \left(\frac{vl}{n\pi a} \right)^2 \right]} \qquad a = \frac{EI}{\rho a}$$

To find vibrations after P moves past RH support, use the u # $u_{,t}$ that exist @ the time P moves past RH support - use them as initial conditions for free vib problem

To find max. w_1 when $v \ll 1 \Rightarrow w_{max} = w_{max\ stat}$ when $v \gg 1 \Rightarrow w_{max} = 0$ (no response time)

Contrib of first mode only. $n=1$ $\omega_1 = \pi^2 a/l^2$ if $v \gg$

$$w_1 = \frac{2P_0 l^3}{\pi^4 EI} \frac{\sin \frac{\pi x}{l} \left(\sin \frac{\pi v t}{l} - \frac{vl}{\pi a} \sin \frac{\pi^2 a t}{l^2} \right)}{1 - \left(\frac{vl}{\pi a} \right)^2}$$

$$\text{Amplitude factor } A = \frac{w_1 \text{ max dyn}}{w_1 \text{ max static}} = \frac{96}{\pi^4} \left[\frac{\sin \frac{\pi v t}{l} - \frac{\pi v}{\omega_1 l} \sin \omega_1 t}{\left[1 - \left(\frac{\pi v}{\omega_1 l} \right)^2 \right]} \right]$$

let $\beta = \frac{\omega_1 l}{v}$ (velocity) $\quad u = \frac{vt}{l}$ (position of force) $\Rightarrow A = \frac{96}{\pi^4} \dfrac{\sin \pi u - \pi/\beta \sin \beta u}{1 - (\pi/\beta)^2}$

$\frac{dA}{dk} = 0$ (where is loc of load when A is max) $= \frac{96}{\pi^4} \frac{\pi \cos \pi k - \pi \cos \beta k}{1 - (\pi/\beta)^2} = 0$

$\Rightarrow \cos \pi k = \cos \beta k \qquad \pi k = 2n\pi \pm \beta k$

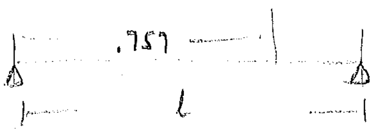
$$k = \frac{2n\pi}{\pi \mp \beta}$$

$n=0 \quad A = 48/\pi^3$

$n=1 \quad A = A(\beta) \iff k = \frac{2\pi}{\pi+\beta}$; max for $\beta = 1.64\pi \quad A = 1.743$

Time of transit is .82 times the fundamental period of beam

$k = .757$



$\Rightarrow$ dangerous velocity $= \frac{l}{t} = \frac{l}{.82 \text{ times the fund period of beam}}$,
& it occurs .757 units along beam.

Forces acting on mech. system that are not fixed will ~~[illegible]~~ perform work $= \int \mathbb{F} \cdot \mathbb{v} \, dt$
where $\mathbb{v}$ is velocity of particle of point under which the load is at, at time t

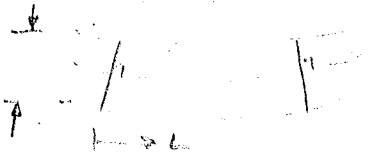
$$W = P \int_0^{l/v} \left(\frac{\partial w}{\partial t} \right)_{x=vt} dt$$

12/4/79

Take homework exam: Given Thursday 12/6 return by monday 12/10

Assumptions for elem beam theory

1. Plane sections remain plane & ⊥ to central axis deformed & undeformed.



If $\lambda \gg h$ assumptions are pretty good
if $\lambda \ll h$ " are not clean.

2. Only account for transverse inertia. If h << must include rotary inertia

for β (wave number) $\rightarrow 0 \quad \lambda \rightarrow \infty$

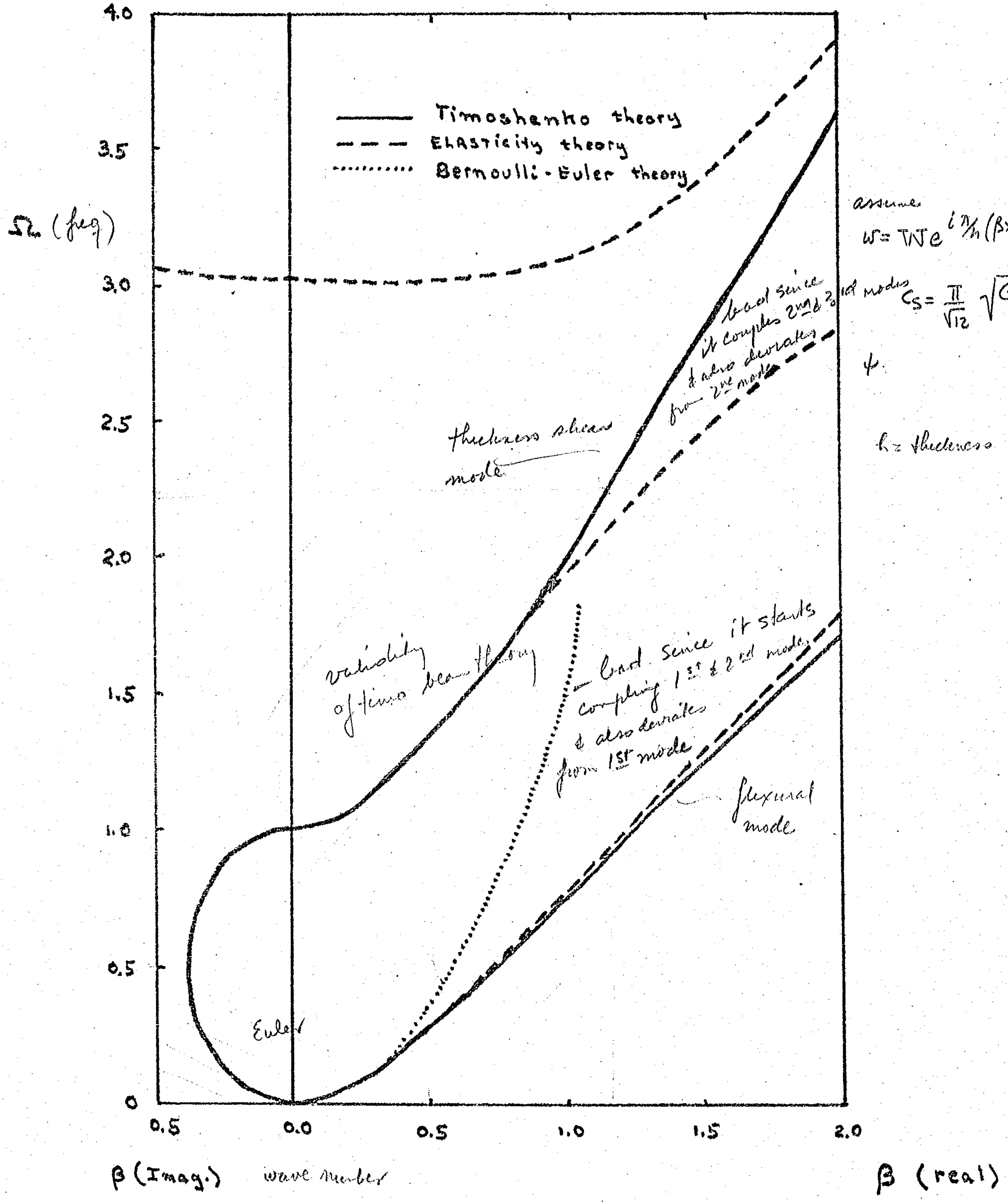
$(w'' - \psi') \frac{G}{\rho} = \ddot{w} \quad \Rightarrow w = 0$

Solns are independent of X

$\psi''/\rho + (w' - \psi) G/\rho r^2 = \ddot{\psi}$

beam doesn't deflect in w direction

$\Rightarrow \psi$ are sines & cosines Solns are harmonic fns. $\omega = \frac{1}{r}\sqrt{G/\rho}$



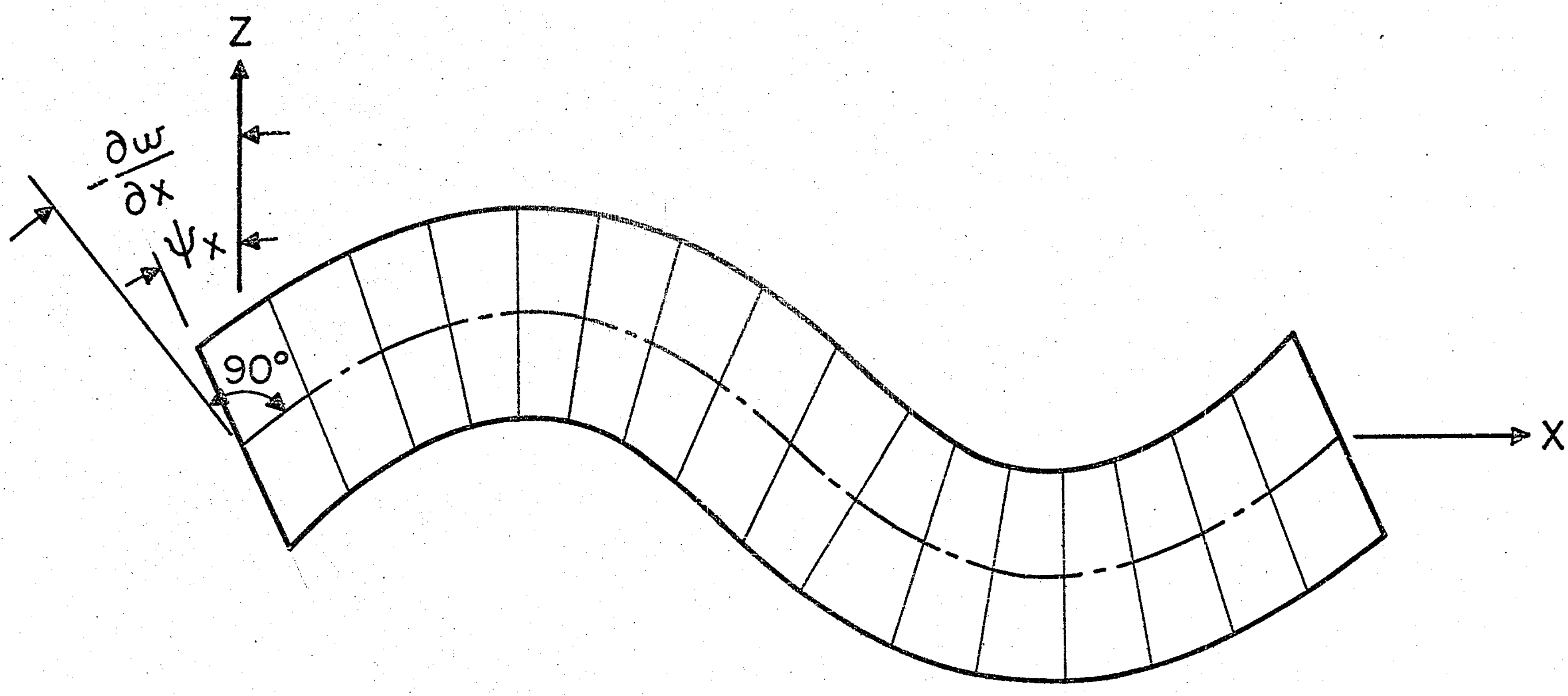
assume

$w = W e^{i \pi/h (\beta x - c_s \Omega t)}$

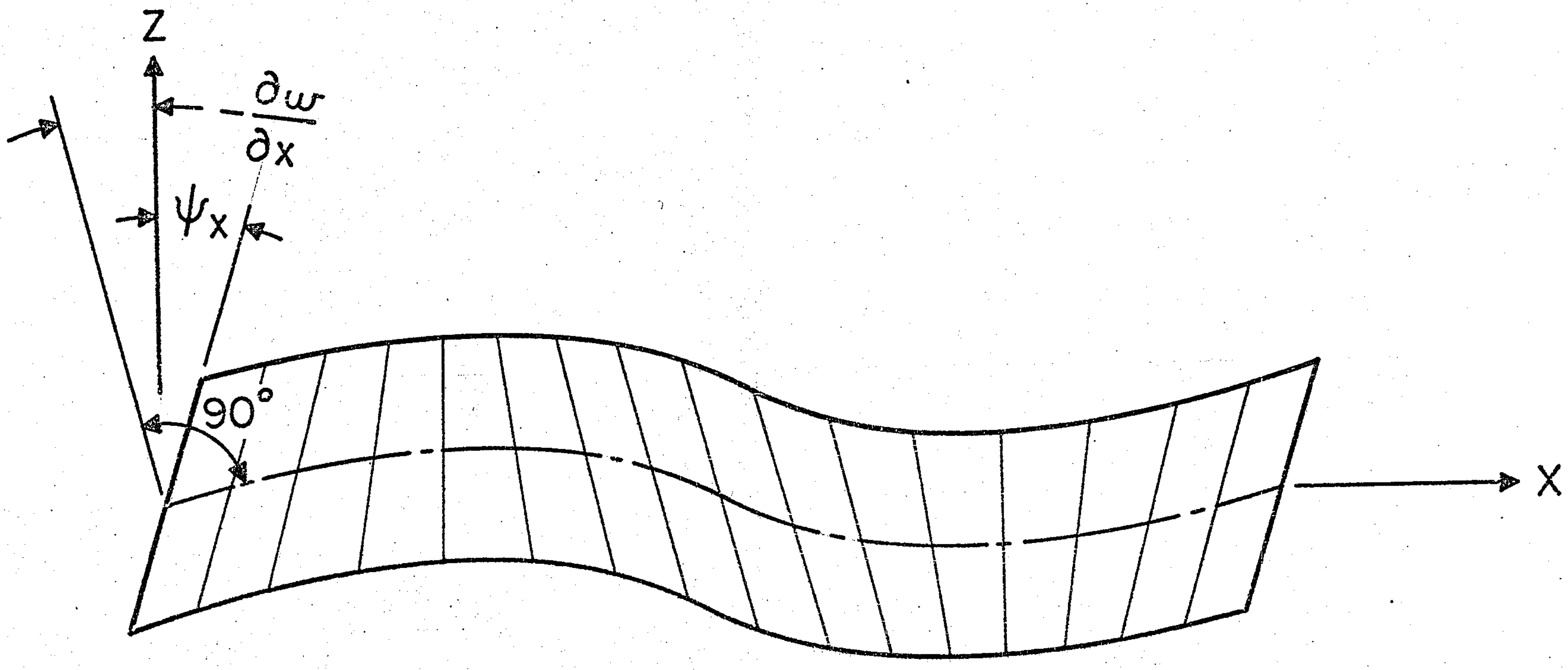
$c_s = \frac{\pi}{\sqrt{12}} \sqrt{G/\rho}$

ψ.

h = thickness



(A) FLEXURAL MODE primarily deflection

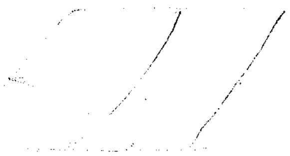


(E) THICKNESS - SHEAR MODE

prevailing shear

$$\therefore \quad \omega^2 = (j^2 + k^2)c^2 = \left(\frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{a^2}\right)c^2 \qquad \omega_{mn} = \pi c\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

$$\therefore \quad w = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_n Y_m T_{mn}$$



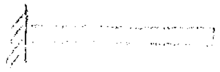
$m=1, n=1$

for given ω we can have 2 mode shapes: $\omega = \frac{\pi c}{a}(n^2+m^2)^{1/2}$ for a square

$$\omega_{9,2} = \frac{\pi c}{a}\sqrt{85} \qquad \omega_{6,7} = \frac{\pi c}{a}\sqrt{85}$$

12/6/79

Rayleigh's Method



for cantilever beam $KE = \frac{A\rho}{2}\int_0^l \dot{w}^2\,dx \qquad PE = \frac{EI}{2}\int_0^l w''^2\,dx$

based on element theory

assume a w to satisfy bc

let $w(x,t) = \bar{W}\,x^2 \sin\omega t$

$$\dot{w} = \bar{W}\omega\,x^2\cos\omega t \qquad (\dot{w})^2 = \bar{W}^2\omega^2 x^4\cos^2\omega t$$

$$KE = \frac{A\rho\bar{W}^2\omega^2}{2}\int_0^l x^4\,dx\,\cos^2\omega t = A\rho\bar{W}^2\omega^2\cos^2\omega t\,\frac{L^5}{10}$$

$$w' = 2x\bar{W}\sin\omega t \quad w'' = 2\bar{W}\sin\omega t; \quad PE = \frac{EI}{2}\cdot\int_0^l dx\,4\bar{W}^2\sin^2\omega t = 2EI\sin^2\omega t\,\bar{W}^2\,l$$

$T_{max} = KE = V_{max} = PE \qquad \therefore \qquad \omega^2 = \frac{20EI}{l^4\rho A} \qquad \omega \sim \frac{4.47}{l^2}\sqrt{\frac{EI}{\rho A}}$

take when $\sin\omega t = 1 = \cos\omega t$: $\frac{A\rho l^5}{10}\omega^2\cancel{W^2} = 2EI\,l\,\cancel{W^2}$

$$\omega^2 = \frac{20EI}{A\rho}\frac{1}{l^4}$$

$\omega_{exact}\cdot\frac{3.5}{l^2}\sqrt{}$

if $w = \bar{W}\left(\frac{3x^2}{2l^2} - \frac{x^3}{2l^3}\right)\sin\omega t \qquad \omega \sim \frac{3.55}{l^2}\sqrt{\frac{EI}{\rho A}}$

Stress Concentrations – Dynamic (Diffraction)

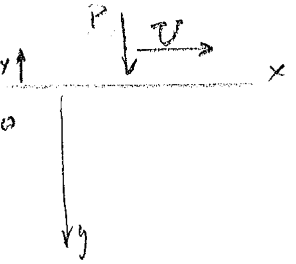
Uniform static field has a stress concentration

Stress (dynamic) at cavity will cause a stress concentration

Get Timo / Young / Weaver Vibs Prob in Eng.

$T\cdot n = \sigma_{yy}, \sigma_{xy}, \sigma_{xz}$

for plane strain $\sigma_{xz} = 0$



BC $\sigma_{xy} = 0$ $\sigma_{yy} = -P\delta(x-Ut)$

define a moving coord syst. ∋. it is always under load. $x' = x - Ut$

$y' = y$

$t' = t$

Write de in terms of pot.

$$\phi : \frac{\partial^2\phi}{\partial x'^2} + \frac{\partial^2\phi}{\partial y'^2} = \frac{U^2}{c_1^2}\frac{\partial^2\phi}{\partial t'^2}$$

only H_3 pot :

$$\frac{\partial^2 H_3}{\partial x'^2} + \frac{\partial^2 H_3}{\partial y'^2} = \frac{U^2}{c_2^2}\frac{\partial^2 H_3}{\partial t'^2}$$

Let $M_1 = \frac{U}{c_1}$ $M_2 = \frac{U}{c_2}$ let $\beta_1 = \sqrt{1-M_1^2}$ if $M_1 < 1$

$\beta_2 = \sqrt{1-M_2^2}$ $M_2 < 1$

if $M_1, M_2 > 1 \Rightarrow \bar{\beta}_1 = \sqrt{M_1^2-1}$ $\bar{\beta}_2 = \sqrt{M_2^2-1}$

now $\sigma_{xx}/\mu = (M_2^2 - 2M_1^2 - 2)\,\phi_{,x'x'} + 2H_{3,x'y'}$

$\sigma_{yy}/\mu \quad (M_2^2-2)\,\phi_{,x'x'} - 2H_{3,x'y'}$

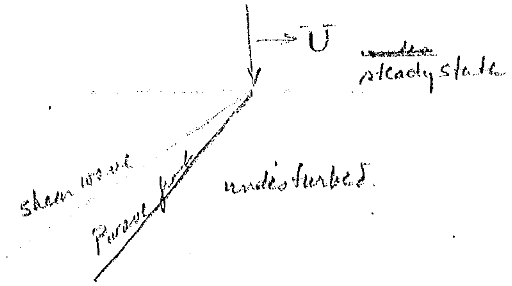
$\sigma_{xy}/\mu = 2\phi_{,x'y'} + (M_2^2-2)H_{3,x'x'}$

apply BC σ_{yy} $(M_2^2-2)\frac{\partial^2\phi}{\partial x'^2} - 2\frac{\partial^2 H_3}{\partial x'\partial y'} = -P/\mu\ \delta(x')$ can integrate wrt x'

σ_{xy} $2\phi_{,x'y'} + (M_2^2-2)H_{3,x'x'} = 0$

for supersonic case $\phi = f(x' + \bar{\beta}_1 y')$

$H_3 = g(x' + \bar{\beta}_2 y')$.



Example #2

Suppose $\sigma_{yy}\big|_{y=0} = P e^{i(\omega t + kx)}$ Can be used later for $f(x)$ written as a fn of fourier integrals

Write DE for ϕ, H_3 w/ BC $\sigma_{xy} = 0$ $\sigma_{yy} = P e^{i(\omega t + kx)}$

let $\phi = A e^{-\alpha_1 y} e^{i(\omega t + kx)}$
$H_3 = B e^{-\beta_1 y} e^{i(\omega t + kx)}$ $\Longrightarrow$ from bc $-2Ai\alpha_1 k + (\beta_1^2 + k^2) B = 0$
$A\left[\lambda(-k^2) + (\lambda + 2\mu)\alpha_1^2\right] + B\, 2\mu i k \beta_1$ P

w/ $\alpha_1^2 = k^2 - \frac{\omega^2}{c_1^2}$ $\beta_1^2 = k^2 - \frac{\omega^2}{c_2^2}$ solve for A, B toget

$$A = \frac{2k^2 - \omega^2/c_2^2}{R(k)} \frac{P}{\mu} \qquad B = \frac{2ik\beta_1}{R(k)} \frac{P}{\mu}$$

$R(k) = (k^2 + \beta_1^2)^2 - 4k^2 \alpha_1 \beta_1$ Rayleigh fn.

if $R(k) = 0$ we have rayleigh surface waves or resonance; ∴ problem blows up & nolonger steady state — must reformulate

If $2k^2 = \frac{\omega^2}{c_2^2}$ $A = 0 \Rightarrow (\phi = 0)$ stress will depend only on μ.

DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING
STANFORD UNIVERSITY

ME 236A Waves and Vibrations Autumn 1979

Problem Set No. 1

1. Study the reflection of SV-waves from a traction-free plane of an elastic half-space.

2. Sketch the instantaneous particle velocity and the particle path of Rayleigh surface waves on the surface.

3. For Rayleigh waves, sketch the dependence of amplitudes of displacements on depth. Normalize these amplitudes with respect to the normal displacement amplitude on the surface. Normalize depth coordinate with respect to wave length. ie draw $\bar{u}_x = f(\bar{y})$ where $\bar{u}_x = \frac{u_x(x,y)}{u_x(x,0)}$ $\bar{y} = y/\lambda$

4. For Rayleigh waves, sketch the dependence of the three non-vanishing stress amplitudes on depth. Normalize these amplitudes with respect to the non-vanishing normal stress component on the surface. Normalize depth as in the previous problem.

if $\sigma_{yy} = A(y)\, e^{ik(x-\omega t)}$
$\sigma_{xy} = B(y)\, e^{ik(x-\omega t)}$
$\sigma_{yz} = C(y)\, e^{ik(x-ct)}$

draw $\frac{A(y)}{C(0)}$, normalize $\bar{y} = y/\lambda$

$\frac{B(y)}{C(0)}$

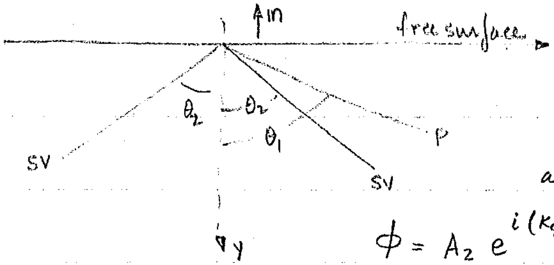
$\frac{C(y)}{C(0)}$

[redacted]

Cesar Levy
ME236A
Prof. Herrmann
Fall '79

Problem Set #1

1. Study the reflection of S-V waves from a traction-free plane of an elastic half-space



Since an S-V can be reflected off the free boundary as a P and an S-V wave take

$$\phi = A_2 e^{i(k_\phi x + \alpha y - \omega_\phi t)} \quad ; \quad H_z = B_1 e^{i(k_H x - \beta y - \omega_H t)} + B_2 e^{i(k_H x + \beta y - \omega_H t)} = H_{z_1} + H_{z_2}$$

where we can define : $k_\phi = \gamma_1 \sin\theta_1$; $k_H = \gamma_2 \sin\theta_2$; $\alpha = \gamma_1 \cos\theta_1 = k_\phi\left(\frac{c^2}{c_1^2}-1\right)^{1/2}$; $\beta = \gamma_2\cos\theta_2 = k_H\left(c^2/c_2^2-1\right)^{1/2}$

Now $\mathbf{n} = -\mathbf{e}_y$ $\therefore$ $\mathbf{t}_n = \mathbf{T}\cdot\mathbf{n} \equiv 0 = \mathbf{e}_x \sigma_{xy} + \mathbf{e}_y \sigma_{yy} + \mathbf{e}_z \sigma_{zy} \Rightarrow \sigma_{xy} = \sigma_{yy} = \sigma_{zy} = 0$ at the bdy

$$u_x = \frac{\partial\phi}{\partial x} + \frac{\partial H_z}{\partial y} \; ; \; \varepsilon_{xx} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial H_z}{\partial x\partial y} \; ; \; u_y = \frac{\partial\phi}{\partial y} - \frac{\partial H_z}{\partial x} \; ; \; \varepsilon_{yy} = \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2 H_z}{\partial x\partial y} \; ; \; u_z = 0 \Rightarrow \frac{\partial u_z}{\partial x} = \frac{\partial u_z}{\partial y} = \frac{\partial u_z}{\partial z} = 0$$

$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right] = \frac{\partial^2\phi}{\partial x\partial y} + \frac{1}{2}\left(\frac{\partial^2 H_z}{\partial y^2} - \frac{\partial^2 H_z}{\partial x^2}\right) ; \; \varepsilon_{yz} = \frac{1}{2}\left[\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right] = 0 \Rightarrow \sigma_{yz} \equiv 0 ;$$

$$\sigma_{xy} = 2\mu\,\varepsilon_{xy} = \mu\left[2\frac{\partial^2\phi}{\partial x\partial y} + \left(\frac{\partial^2 H_z}{\partial y^2} - \frac{\partial^2 H_z}{\partial x^2}\right)\right] \; ; \; \sigma_{yy} = \lambda\left[\varepsilon_{xx} + \varepsilon_{yy}\right] + 2\mu\,\varepsilon_{yy} = (\lambda+2\mu)\nabla^2\phi - 2\mu\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial H_z}{\partial x\partial y}\right)$$

at the bdy: $\sigma_{yz} \equiv 0$

$$\left\{\begin{array}{l} \frac{\partial\phi}{\partial x} = ik_\phi\,\phi \, , \; \frac{\partial^2\phi}{\partial x^2} = -k_\phi^2\,\phi \, , \; \frac{\partial^2\phi}{\partial x\partial y} = -k_\phi\alpha\,\phi \, , \; \frac{\partial\phi}{\partial y} = i\alpha\,\phi \, , \; \frac{\partial^2\phi}{\partial y^2} = -\alpha^2\phi \\ \frac{\partial H_z}{\partial x} = ik_H H_z \, , \; \frac{\partial H_z}{\partial x^2} = -k_H^2 H_z \, , \; \frac{\partial H_z}{\partial x\partial y} = k_H\beta(H_{z_1} - H_{z_2}) \, , \; \frac{\partial^2 H_z}{\partial y^2} = -\beta^2 H_z \end{array}\right.$$

and

$$\sigma_{xy}\Big|_{y=0} = \mu\left[-2k_\phi\alpha\,\phi - \beta^2 H_z + k_H^2 H_z\right] = 0 \Rightarrow \mu\left[-2k_\phi\alpha A_2 - \left(\frac{c^2}{c_2^2}-2\right)(B_1+B_2)\right]e^{i(k_H x - \omega_H t)} = 0$$

where we assume that $k_H = k_\phi$ and $\omega_H = \omega_\phi$

or $\sigma_{xy}\Big|_{y=0} = 0 \Rightarrow -2r_\alpha A_2 - (r_\beta^2-1)(B_1+B_2) = 0$ where $\alpha = k_\phi r_\alpha$ $\beta = k_H r_\beta$

or $-2r_\alpha A_2 + (1-r_\beta^2)(B_1+B_2) = 0$

$$\sigma_{yy}\Big|_{y=0} = \left[(\lambda+2\mu)(-k_\phi^2)(r_\alpha^2+1)A_2 - 2\mu\left(-k_\phi^2 A_2 + k_H^2 r_\beta(B_1-B_2)\right)\right]e^{i(k_H x - \omega_H t)} = 0$$

or $\sigma_{yy}\Big|_{y=0} = (\lambda+2\mu)(r_\alpha^2+1)A_2 - 2\mu\left(A_2 - r_\beta(B_1-B_2)\right) = 0$

or $\left[\lambda(r_\alpha^2+1) + 2\mu r_\alpha^2\right]A_2 + 2\mu r_\beta(B_1-B_2) = 0$

then $$\boxed{A_2/B_1 = \frac{4\mu r_\beta(1-r_\beta^2)}{4\mu r_\alpha r_\beta - \left[(\lambda+2\mu)(r_\alpha^2+1) - 2\mu\right](1-r_\beta^2)}}$$ and $$\boxed{\frac{B_2}{B_1} = \frac{4\mu r_\alpha r_\beta + \left[(\lambda+2\mu)(r_\alpha^2+1) - 2\mu\right](1-r_\beta^2)}{4\mu r_\alpha r_\beta - \left[(\lambda+2\mu)(r_\alpha^2+1) - 2\mu\right](1-r_\beta^2)}}$$

when $\theta_2 = \theta_{2c}$ (critical angle) $\theta_1 = 90°$ $\Rightarrow r_\alpha = 0$ $\frac{A_2}{B_1} = -\frac{4\mu}{\lambda}r_\beta$ $\frac{B_2}{B_1} = -1$

for $\theta_2 > \theta_{2c}$ we must look for solutions of P waves which decay with y $\therefore$ let $i\alpha < 0$

if $r_\alpha^* = (1 - c^2/c_1^2)^{1/2}$ then $r_\alpha = \pm i\, r_\alpha^*$ and $i\alpha = ik_\phi \cdot i r_\alpha^* < 0$

Thus $$B_2/B_1 = \frac{4\mu i r_\alpha^* r_\beta + [(\lambda+2\mu)(1-r_\alpha^{*2})+2\mu](1-r_\beta^2)}{4\mu i r_\alpha^* r_\beta - [(\lambda+2\mu)(1-r_\alpha^{*2})+2\mu](1-r_\beta^2)} = \frac{\zeta e^{i\varphi}}{-\zeta e^{-i\varphi}} = -e^{2i\varphi}$$

where $\zeta = \left\{(4\mu r_\alpha^* r_\beta)^2 + (1-r_\beta^2)^2\left[(\lambda+2\mu)(1-r_\alpha^2)+2\mu\right]^2\right\}^{1/2}$ and $\varphi = \tan^{-1}\dfrac{4\mu r_\alpha^* r_\beta}{[(\lambda+2\mu)(1-r_\alpha^{*2})+2\mu](1-}$

thus there is a phase shift in the amplitude of the reflected S-V wave

and $$A_2/B_1 = \frac{4\mu r_\beta(1-r_\beta^2)}{4\mu i r_\alpha^* r_\beta - [(\lambda+2\mu)(1-r_\alpha^{*2})-2\mu](1-r_\beta^2)}$$

These results can be converted to those found in Achenbach if we note that $r_\beta = \frac{\gamma_2 \cos\theta_2}{k_H} = \cot\theta_2$ and similarly $r_\alpha = \cot\theta_1$. Hence $r_\alpha^2+1 = 1/\sin^2\theta_1 = (c/c_1)^2$ similarly $1-r_\beta^2 = -\cos 2\theta_2/\sin^2\theta_2$ and as previously shown in class $\sin\theta_2 = \kappa\sin\theta_1$ where $\kappa = c_1/c_2$, If so then

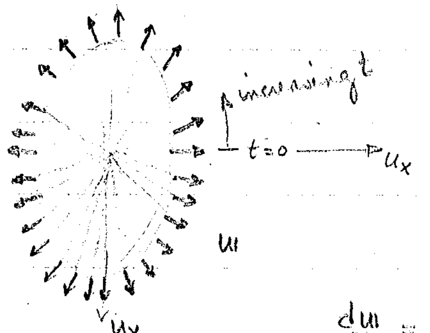
$$A_2/B_1 = -\frac{\kappa\sin 4\theta_1}{\sin 2\theta_1 \sin 2\theta_2 + \kappa^2\cos^2 2\theta_1} \quad \& \quad B_2/B_1 = \frac{\sin 2\theta_1 \sin 2\theta_2 - \kappa^2\cos^2 2\theta_1}{\sin 2\theta \sin 2\theta_2 + \kappa^2\cos^2 2\theta_1} \quad \text{for } \cot\theta_1 \text{ real}$$

**see last page for discussion.

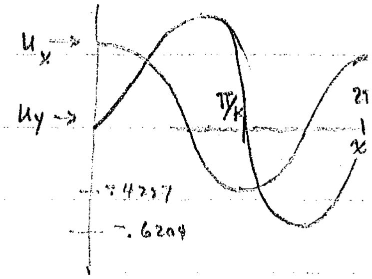
2. for Rayleigh waves at the surface with $\nu = 1/4$ $c_R/c_2 = .9194$ we find that

$$u_x = A'(.4227)\cos k(x-c_R t) \quad \text{and} \quad u_y = A'(.6204)\sin k(x-c_R t)$$

$\therefore \left(\frac{u_x}{A'.4227}\right)^2 + \left(\frac{u_y}{A'.6204}\right)^2 = 1 \Rightarrow$ a particle at the surface travels an elliptical path thus $\mathbf{u} = u_x \mathbf{e}_x + u_y \mathbf{e}_y$. Note that each of the displacement components is sinusoidal. We will look at $x=0$ (for simplicity) as $t>0$

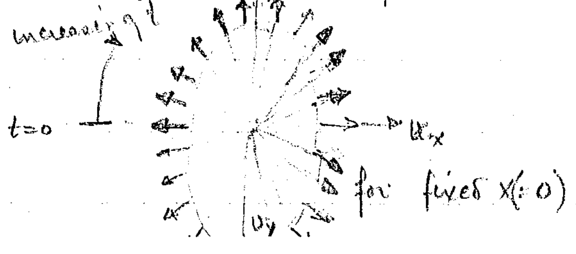


for fixed t and increasing x

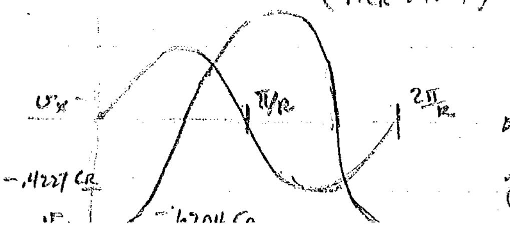


$$\frac{d\mathbf{u}}{dt} = +\underbrace{A'c_R(.4227)\sin k(x-c_R t)}_{V_x}\mathbf{e}_x \underbrace{- c_R A'(.6204)\cos k(x-c_R t)}_{V_y}\mathbf{e}_y$$

again the velocity vector traverses an elliptical path $\left(\frac{V_x}{A'c_R.4227}\right)^2 + \left(\frac{V_y}{A'.6204c_R}\right)^2 = 1$



for fixed x(=0)



x

for fixed t (assume=0

*** see last page for composite

3. We note that as the depth increases we must investigate (for $\nu = \frac{1}{4}$)

$$u_x = A'(e^{-.8475ky} - .5773e^{-.3933ky})\cos k(x - c_R t)$$

$$u_y = A'(-.8475e^{-.8475ky} + 1.4679e^{-.3933ky})\sin k(x - c_R t)$$

Normalizing with the surface amplitudes $u_x/u_y^0 = [e^{-.8475ky} - .5773e^{-.3933ky}]/.6204$

$u_y/u_y^0 = [-.8475e^{-.8475ky} + 1.4679e^{-.3933ky}]/.6204$; Now $\lambda k = 2\pi$ & $ky = k\lambda \cdot y/\lambda = 2\pi\bar{y}$

$\Rightarrow$ $u_x/u_y^\circ = [e^{-5.325\bar{y}} - .5773e^{-2.4712\bar{y}}]/.6204$ and $u_y/u_y^\circ = [-.8475e^{-5.325\bar{y}} + 1.4679e^{-2.4712\bar{y}}]/.6204$

Plotted on a separate sheet are the amplitudes of displacement as a function of normalized depth. We note that as $\bar{y} > .192$ then the direction of motion of the ellipse reverses. These plots are for $\nu = 1/4$. If we want other plots for different values of ν, we need to return to the general problem and, for a given ν, 1) obtain c_1 & c_2 ; 2) Put into eqn for $(c/c_1, c/c_2)$ to find solutions; then 3) recompute A_1 and B_1 to get u_x and u_y. 4) Normalize results and then plot.

4. For $\nu = 1/4$ $\quad u_x = A'(e^{-.8475ky} - .5773e^{-.3933ky})\cos k(x - c_R t)$ $\quad c_R = .9194c_2$

$$u_y = A'(-.8475e^{-.8475ky} + 1.46979e^{-.3933ky})\sin k(x - c_R t)$$

Now $\sigma_{yy} = (\lambda + 2\mu)\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) - 2\mu\frac{\partial u_x}{\partial x} = \left[3\rho c_2^2\{-.2817e^{-5.325\bar{y}} - .0008e^{-2.4712\bar{y}}\} + 2\rho c_2^2\{e^{-5.325\bar{y}} - .5773e^{-.24712\bar{y}}\}A'k \cdot \sin k(x-\right.$

$\sigma_{yx} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = c_2^2\rho A'k[-1.695e^{-5.325\bar{y}} + 1.695e^{-2.4712\bar{y}}]\cos k(x - c_R t)$

$\sigma_{xx} = (\lambda + 2\mu)\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) - 2\mu\frac{\partial u_y}{\partial y} = [3\rho c_2^2\{-.2817e^{-5.325\bar{y}} - .0008e^{-2.4712\bar{y}}\} - 2\rho c_2^2\{.7183e^{-5.325\bar{y}} - .5781e^{-2.4712\bar{y}}\} \cdot A'k\sin k$

@ the boundary $|\sigma_{yx}| = 0$ $\quad |\sigma_{yy}| = 0$ $\quad |\sigma_{xx}| = [3(-.2825) - 2\{.1402\}]A'\rho c_2^2 k = -1.1279A'\rho c_2^2 k$

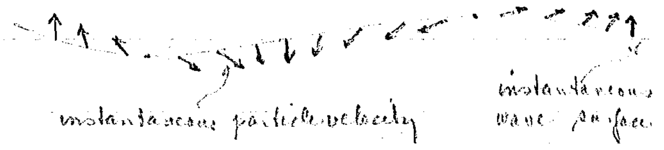
Thus $\dfrac{\sigma_{yy}}{\sigma_{xx}|_{y=0}} = 1.157(e^{-5.325\bar{y}} - e^{-2.4712\bar{y}})/-1.1279$; $\dfrac{\sigma_{yx}}{\sigma_{xx}|_{y=0}} = \dfrac{1.695}{1.1279}(e^{-5.325\bar{y}} - e^{-2.4712\bar{y}})$

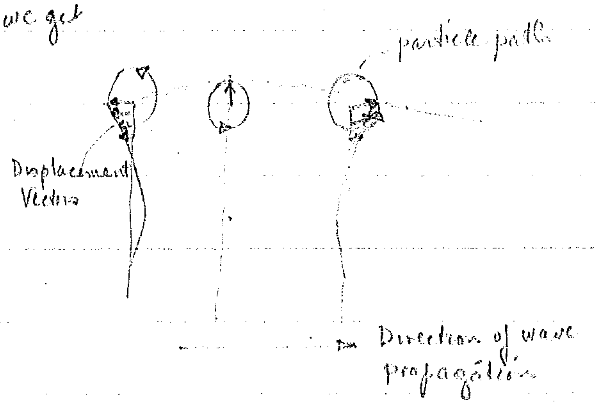
$$\frac{\sigma_{xx}}{\sigma_{xx}|_{y=0}} = (-2.2817e^{-5.325\bar{y}} + 1.1538e^{-2.4712\bar{y}})/-1.1279$$

The graphs are plotted on a separate sheet

Discussion of Problem #1 : We note that from $A_2/B_1 = 0 \Rightarrow \theta_1 = \pi/4, \pi/2$, and 0; ie the Pwave vanishes. Thus for these values SV waves are reflected as SV waves. If $B_2/B_1 = 0$, meaning that if the numerator is 0, then SV waves are totally reflected as Pwaves. For vanishing Pwaves $\theta_1 = 0$ means a vertically incident SV wave, $\theta_1 = \frac{\pi}{2}$ means a horizontally grazing incident SV wave. We must restrict the admissible values because we must consider $\sin\theta_2 = K \sin\theta_1$ and since $K > 1$ then $\exists!\ \theta_{1_c}\ \ni.\ K \sin\theta_{1_c} = 1$ ie $\theta_2 = \pi/2$. Thus only those values of $\theta_1 < \theta_{1_c}$ will be admissible values $\ni.$ incident SV waves are reflected totally as SV waves. Similarly with the reflection of SV waves as Pwaves, those values of θ_1 that cause the numerator of $B_2/B_1 = 0$ must be less than θ_{1_c}. From the figure 5.7 we note that for $\nu < .26$ there is a possibility of SV conversion to P waves but above $\nu = .26$ no such conversion is possible.

Discussion of Problem #2 : Combining both pictures we get

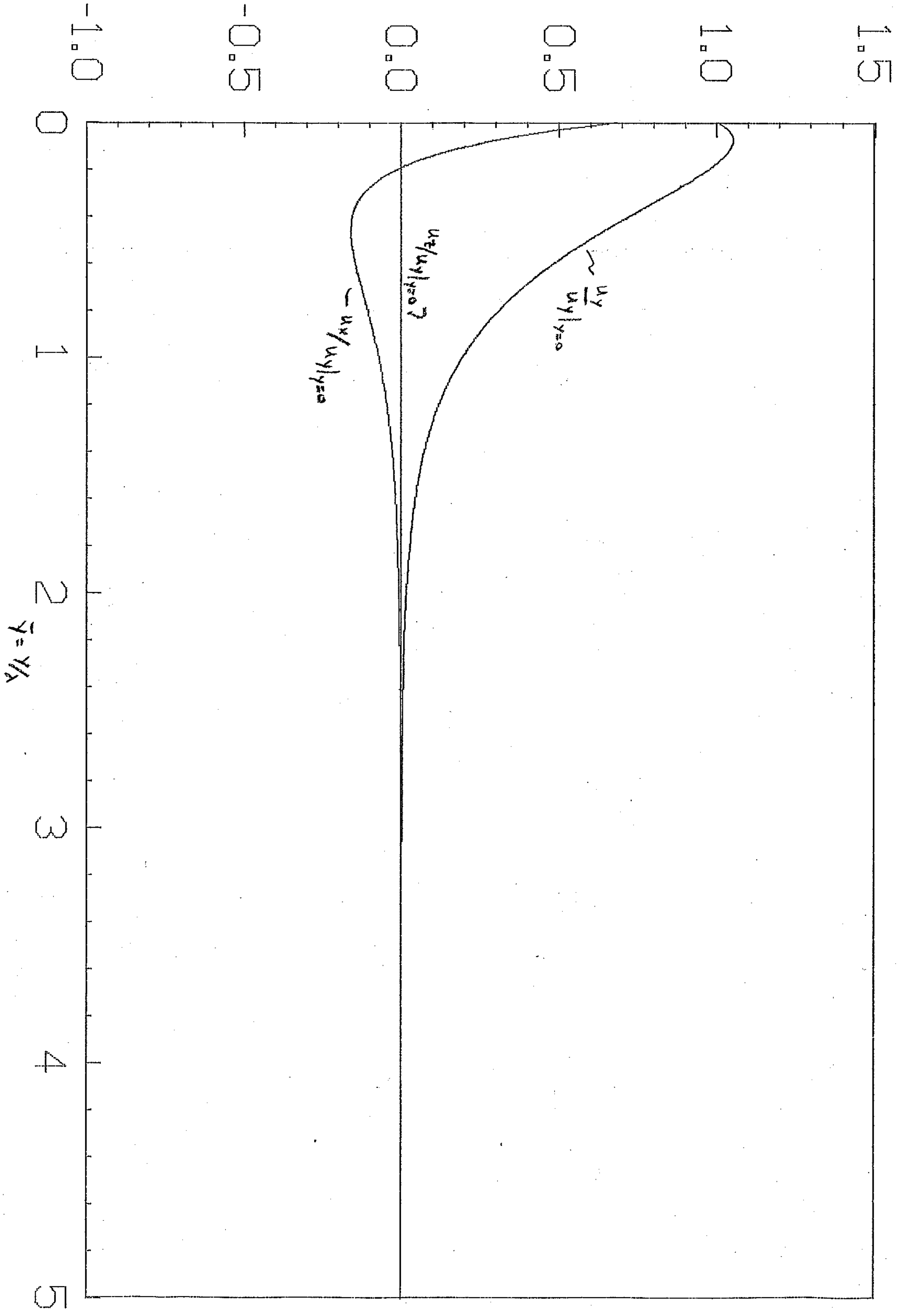




Y.C. Fung : Foundations of Solid Mech.

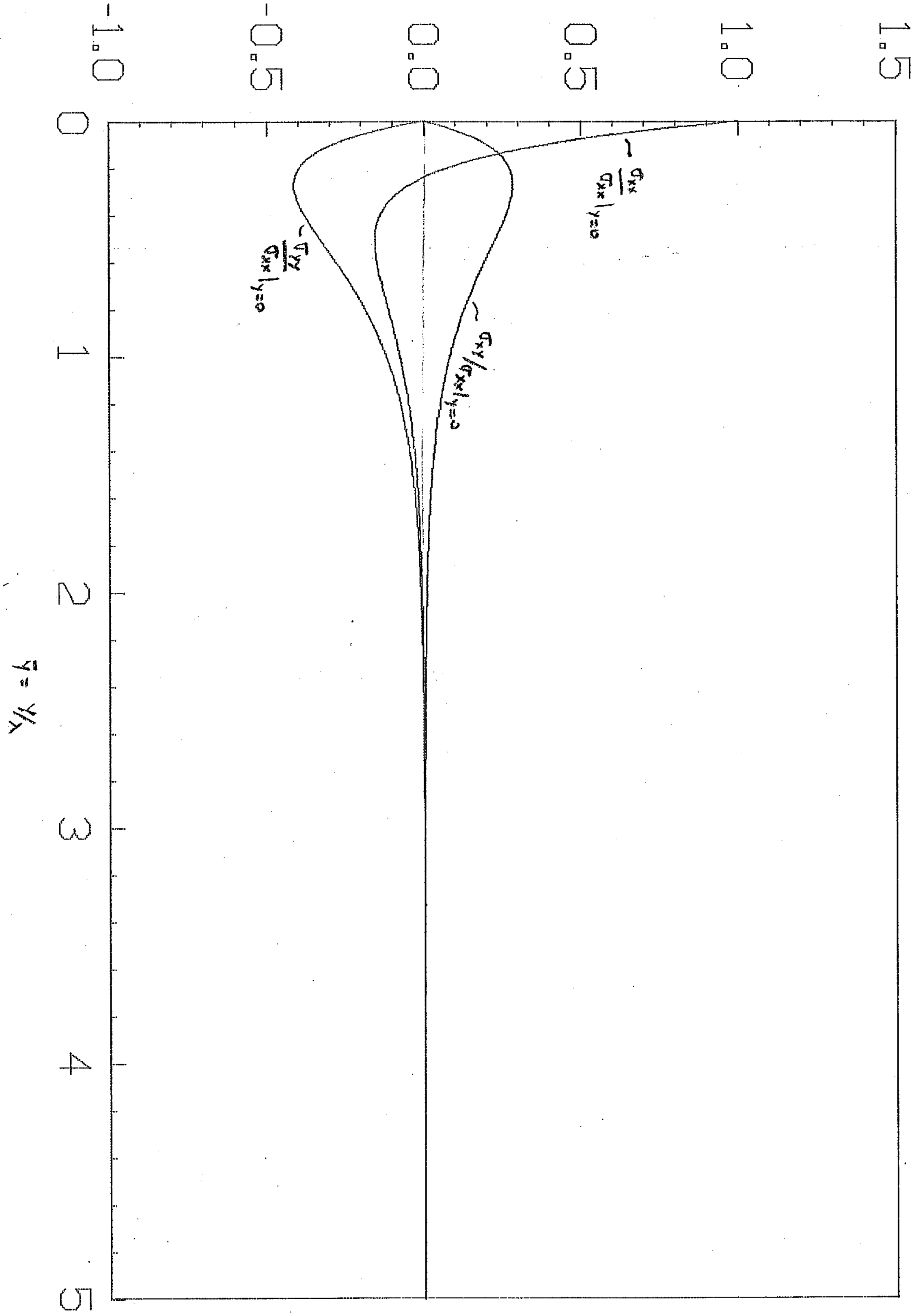
Fig 7.9:1 Pg. 181

amplitude ratio = $\dfrac{\text{amplitude of displ}}{\text{amplitude of } u_y|_{y=0}}$ = .6204



Problem #3 ($\nu = 1/4$)
Normalized amplitudes vs. normalized depth
cross over point $\bar{y} \approx .192$

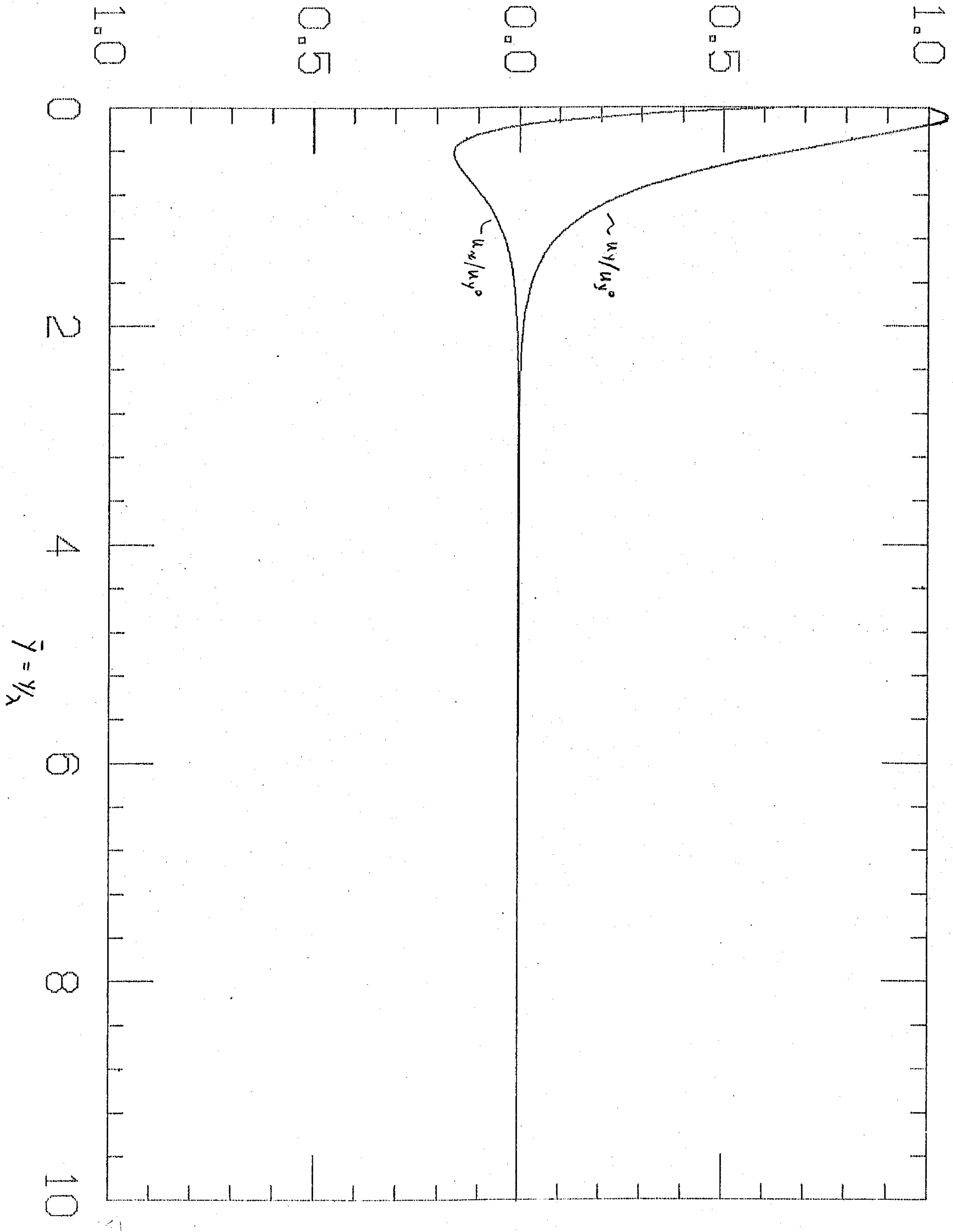
$$\text{amplitude ratio} = \frac{\text{ampl of stress}}{\text{ampl. of } \sigma_{11}|_{y=0}} = -1.1279$$



Problem #4 $(\nu = 1/4)$

Normalized stresses vs. normalized depth

C. Levy

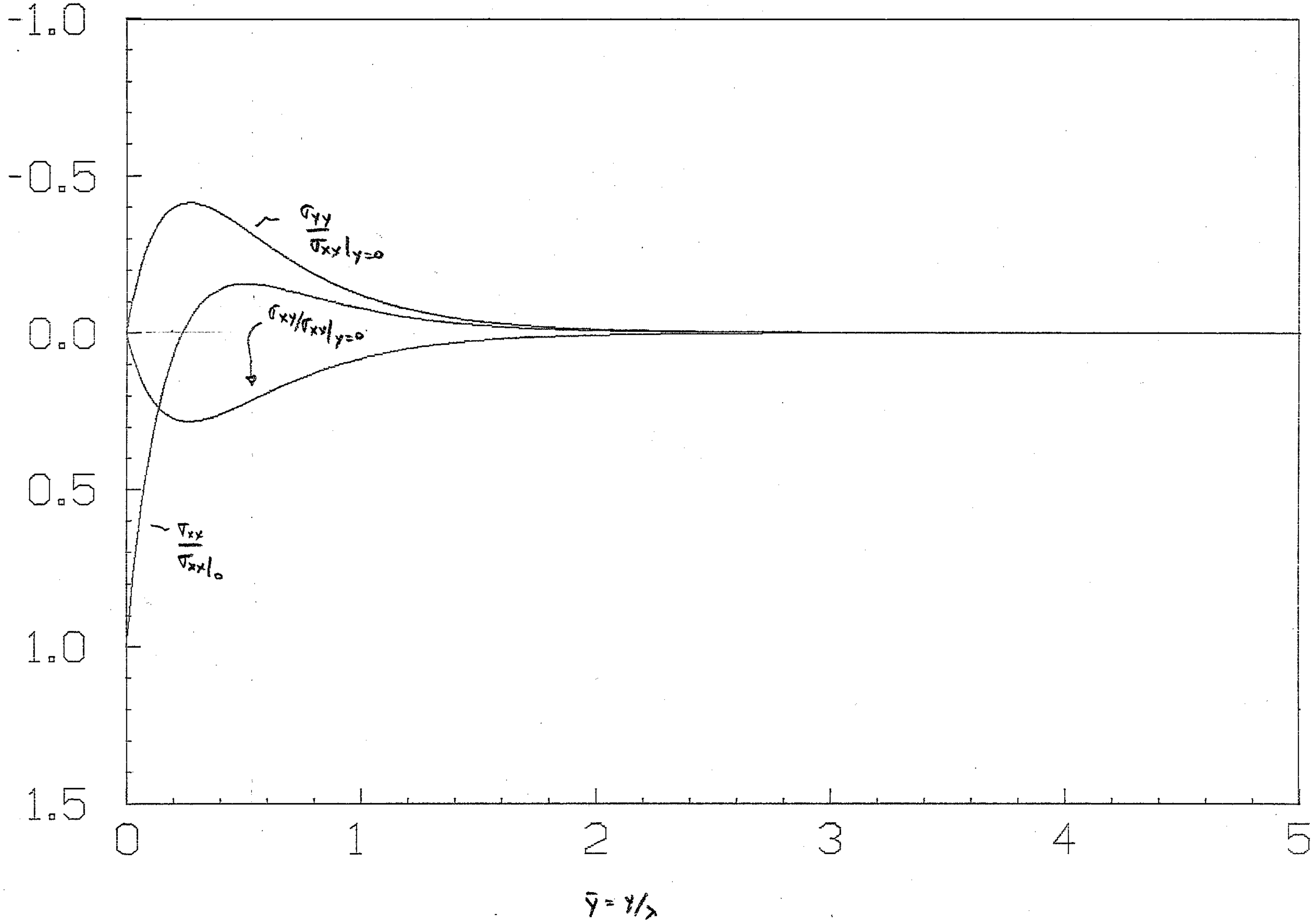


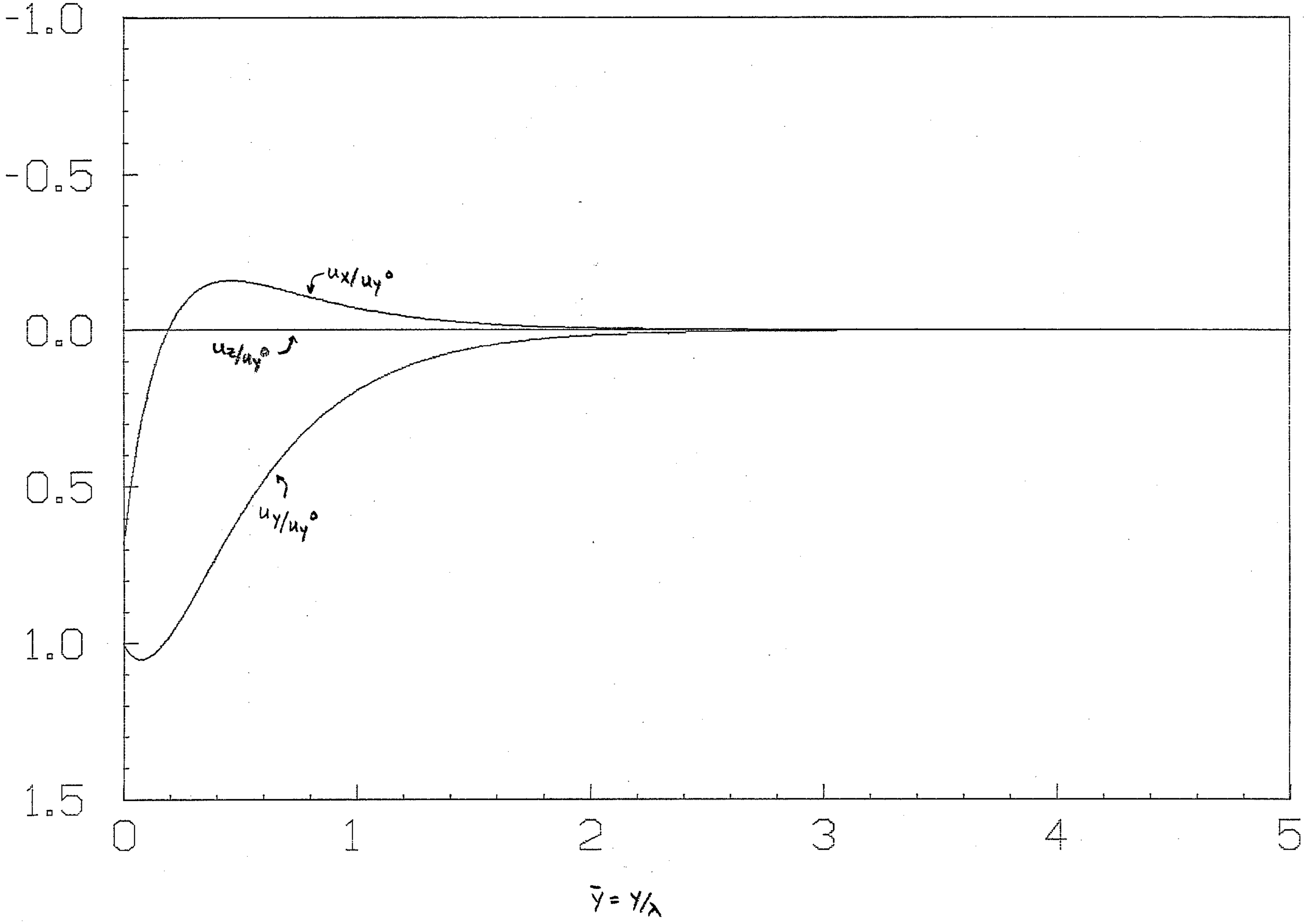
Problem #3 ($\nu = 1/4$)
Normalized amplitude versus normalized depth
cross over point is at $\bar{y} = .192$

#3

```
COLLECT
1.  //SETCL JOB
2.  /*JOBPARM  DEST=SELF
3.  //ᵇEXECᵇFORTCG
4.  //FORT.SYSINᵇDDᵇ*
      DIMENSION  S11(100), S12(100), S13(100), Y(100)
      DO10I = 1,100
       YB = .1*(I-1)
       Y(I)= YB
       EX1 = EXP(-5.325*YB)
       EX2 = EXP(-2.4712*YB)
       S22(I) = 1.157*(EX1 -EX2)/(-1.1279)              (1* ex1 -.5773* ex2)/.6204
        S12(I) = 1.695*(EX1- EX2)/(1.1279)              (-.8475* ex1 + 1.4679 *ex2)/.6204
        S11 (I)  =(-2.2817*EX1 + 1.1538*EX2)/(-1.1279).              0.
   10   CONTINUE
        DO 20 I= 1,101
         WRITE (3,25)  S11(I),Y(I)
    35   FORMAT (2(2X,F20.10))
    20   CONTINUE
         WRITE (3,26)
    26   FORMAT (_'bbbbbb JOIN')
         DO 40 I=1,101
          WRITE (3,25)  S22(I),Y(I)
    40    CONTINUE
          WRITE(3,26)
          DO 60 I=1,101
           WRITE (3,25)  S12(I), Y(I)
      60   CONTINUE
           WRITE (3,26)
           STOP
           END
// GO.FT03F001_DD_DSN=WYL.NJ.MEN.PLOTDAT, DISP=(NEW,KEEP),
// ___ VOL=SER=PUB004,UNIT=DISK.,DCB=(LRECL=80,RECFM=FB,BLKSIZE=6160),
// ____SPACE=(TRK,(5,5),RLSE)
  SAVE  PROG  ON PUB004
   RUN  HOLD

Collect
1. SET SIZE  10.5  BY  7.88
2. SET WINDOW   Y  2 TO 7     X  2 TO  9
3.  SET  AXES; SET  LIMITS   Y   1.5   TO -1.0     X  0  TO  10
4    SET  TICKS  SIZE  .03 RIGHT OFF  TOP OFF;  PLOT AXES
  SAVE   PLOTC  ON PUB004
     COPY  FROM  PLOTDAT ON PUB004  TO 5   BY  1
    EXEC  FROM  #TOPDRAW  PUB  ON  CAT.
```


-1.0
-0.5
0.0
0.5
1.0
1.5
0
1
2
3
4
5
$\bar{y} = y/\lambda$
$\frac{\sigma_{yy}}{\sigma_{xx}|_{y=0}}$
$\sigma_{xy}/\sigma_{xx}|_{y=0}$
$\frac{\sigma_{xx}}{\sigma_{xx}|_0}$


-1.0
-0.5
0.0
0.5
1.0
1.5
0
1
2
3
4
5
$\bar{y} = y/\lambda$
u_x/u_y^o
u_z/u_y^o
u_y/u_y^o

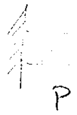
DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING
STANFORD UNIVERSITY

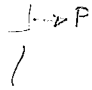
ME 236A Waves and Vibrations Autumn 1979

Problem Set No. 2 *Thursday 29th*

1. A prismatic bar is fixed at one end and subjected at the other end to an axial force P . The force is suddenly removed at time $t = 0$. Show that the ensuing displacement can be put into the form

$$u(x,t) = \frac{8PL}{AE\pi^2} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin\frac{n\pi x}{2L} \cos\frac{n\pi ct}{2L} \quad ; \quad c^2 = \frac{E}{\rho}$$

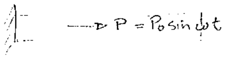




$\mathcal{E}_{xx} = \frac{P}{AE}$

2. A bar is traveling to the left with constant velocity v . At time $t = 0$ its left end strikes a rigid stop situated at $x = 0$. Under the assumption that the left end remains fixed for $t > 0$ and the right end is free, determine the resulting motion and the maximum strain.

3. Consider a prismatic bar fixed at one end and free at the other. It is subjected to a periodic axial force $P_o \sin \omega_f t$ at a distance d from the fixed end. Determine the displacement field in the bar.



$UT' : \dot{C}\lambda \cos\lambda(0) + \tilde{D}\lambda ~~A~~ \lambda b = \tilde{C}\lambda\tilde{U}$

$T(0) \quad \tilde{D}\,U(x) = \frac{Px}{AE}$

$\frac{\partial u}{\partial t}(x,0) = +\lambda\tilde{C}\,U(x) = 0 \Rightarrow \tilde{C} =$

$\frac{P}{A} = \sigma \qquad \frac{P}{AE} = \frac{\Delta u}{\Delta x} \qquad \frac{Px}{AE} = u$

$\tilde{D} \qquad U(x) \qquad \left(1 - \cos\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$

$u_{xx} = -\left(\frac{n\pi}{2L}\right)^2 \qquad u_{xx} - c^2 u_{tt} = 0. \qquad \int x \sin$

$u(0,t) = 0$

$\frac{\partial u}{\partial t}(x,0) = 0.$

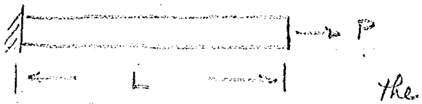
$u_{tt} = -\left(\frac{n\pi c}{2L}\right)^2 \qquad \frac{\partial u}{\partial x}(\qquad \cos\frac{n\pi x}{2L}$

$28/30$

ME 236 A Fall 77
Prof. Herrmann
Cesar Levy

Problem Set #2

1. A prismatic bar is fixed at one end and subjected to and axial force P for $t<0$. If at time $t=0$ the force is removed find $u(x,t)$



The governing PDE is $u_{xx} - \frac{1}{c^2} u_{,tt} = 0 \quad c^2 = E/\rho$

the I.C. are that $u(x,0) = \frac{Px}{AE}$ and $\frac{\partial u}{\partial t}(x,0) = 0$

The B.C. are that $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$, which correspond to fixed end & no impressed stress

Let $u(x,t) = U(x)T(t)$ then by placing into PDE, we can get the form.

$$c^2 \frac{U''}{U} = \frac{T''}{T} = -\lambda^2 \quad \text{where } \lambda^2 \text{ is is a constant} \neq 0$$

then $U'' + (\frac{\lambda}{c})^2 U = 0$ thus $U(x) = A \sin \frac{\lambda}{c} x + B \cos \frac{\lambda x}{c}$

also $T'' + \lambda^2 T = 0$ thus $T(t) = \tilde{C} \sin \lambda t + \tilde{D} \cos \lambda t$

Thus the I.C become $U(x)T(0) = U(x)\tilde{D} = \frac{Px}{AE}$ and $U(x)\lambda\tilde{C} = 0$; if $\lambda \neq 0 \Rightarrow \tilde{C} = 0$

The BC become $U(0)T(t) = 0 = BT(t) = 0 \Rightarrow B = 0 \quad \forall t$, and $A\frac{\lambda}{c} \cos\frac{\lambda L}{c} T(t) = 0 \quad \forall t$

thus if $A, \lambda \neq 0 \quad \frac{\lambda L}{c} = \frac{n\pi}{2}$ for $n = 1,3,5\ldots$ or $\lambda_n = \frac{n\pi c}{2L}$

Hence $u(x,t) = \sum_{n=1,3,5,\ldots} u(x,t) = \sum_{n=1,3,5,\ldots} A_n \sin \frac{n\pi x}{2L} \cos \frac{n\pi c t}{2L} \qquad u_n(x,t) = U_n(x) T_n(t)$

now $u(x,0) = \frac{Px}{AE} = \sum_{n=1,3,5,\ldots} A_n \sin \frac{n\pi x}{2L}$ then $A_n = \frac{2}{L}\int_0^L \frac{Px}{AE} \sin \frac{n\pi x}{2L}\,dx \qquad n=1,3,5,\ldots$

$$A_n = \frac{2P}{LAE}\int_0^L x \sin \frac{n\pi x}{2L}\,dx \;; \quad \int_0^L x \sin\frac{n\pi x}{2L}\,dx = \underbrace{x \cos\frac{n\pi x}{2L} \cdot \left(-\frac{2L}{n\pi}\right)\Big|_0^L}_{\to 0} + \frac{2L}{n\pi}\int_0^L \cos\frac{n\pi x}{2L}\,dx$$

$$= \frac{4L^2}{n^2\pi^2} \sin\frac{n\pi x}{2L}\Big|_0^L = \frac{4L^2}{n^2\pi^2} \sin\frac{n\pi}{2} = \frac{4L^2}{n^2\pi^2}(-1$$

Thus $A_n = \frac{2P}{LAE} \cdot \frac{4L^2}{n^2\pi^2}(-1)^{\frac{n-1}{2}} = \frac{8PL}{AE\pi^2} \cdot \frac{(-1)^{\frac{n-1}{2}}}{n^2}$

and hence $u(x,t) = \frac{8PL}{AE\pi^2} \sum_{n=1,3,5,\ldots} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin\frac{n\pi x}{2L} \cos\frac{n\pi c t}{2L}$ as required ✓

Note: If one investigates $\lambda^2 = 0$: $T_0(t) = \bar{A}t + \bar{B}$ and $U_0(x) = \bar{C}x + \bar{D}$. Applying BC $\Rightarrow \bar{C}, \bar{D} = 0$ & $U_0(x) = 0 \Rightarrow u_0(x,t)$

Problem 2: A bar traveling to the left with constant velocity V. At $t=0$ its left end strikes a rigid stop at $x=0$ and remains fixed at all time $t>0$ and its right end is vibrates freely. Determine $u(x,t)$ and $\left|\frac{\partial u}{\partial x}\right|_{max}$

the governing PDE is $u_{xx} - \frac{1}{c^2} u_{tt} = 0$

with IC $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = -V$

and BC $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$

Using separation of variables $u(x,t) = U(x)T(t)$ then

$$c^2 \frac{U''}{U} = \frac{T''}{T} = -\lambda^2$$

and for $\lambda \neq 0$

$$U(x) = A \cos \frac{\lambda}{c} x + B \sin \frac{\lambda}{c} x$$

$$T(t) = C \cos \lambda t + D \sin \lambda t$$

From BC $u(0,t) = 0 \Rightarrow A = 0$; $\frac{\partial u}{\partial x}(L,t) = 0 \Rightarrow \frac{\lambda L}{c} = \frac{n\pi}{2}$ n is odd or $\lambda_n = \frac{n\pi c}{2L}$ $\forall t$

From IC $u(x,0) = 0 \Rightarrow C = 0$; $\frac{\partial u}{\partial t}(x,0) = \sum_{n=1,3,5,\ldots} U_n(x) D_n \lambda_n = -V$

Thus $u(x,t) = \sum_{n=1,3,5,\ldots} \tilde{B}_n \sin \frac{n\pi x}{2L} \sin \frac{n\pi c t}{2L}$ when $\tilde{B}_n = B_n D_n$

$\therefore$ $\frac{\partial u}{\partial t}(x,0) = -V = \sum_{n=1,3,5,\ldots} \tilde{B}_n \lambda_n \sin \frac{n\pi x}{2L}$

Thus $\tilde{B}_n = \frac{-2V}{L\lambda_n} \int_0^L \sin \frac{n\pi x}{2L} \, dx = +\frac{2V}{L\lambda_n} \cdot \frac{2L}{n\pi} \cos \frac{n\pi x}{2L} \Big|_0^L = \frac{+4V}{\lambda_n n\pi}(-1) = \frac{-4V \cdot 2L}{n^2\pi^2 c} = \frac{-8VL}{n^2\pi^2 c}$

for $n = 1,3,5,\ldots$

$$\therefore u(x,t) = \frac{-8VL}{\pi^2 c} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \sin \frac{n\pi x}{2L} \sin \frac{n\pi c t}{2L}$$ ✓

$\frac{\partial u}{\partial x}(x,t) = -\frac{4V}{\pi c} \sum_{n=1,3,5,\ldots} \frac{1}{n} \cos \frac{n\pi x}{2L} \sin \frac{n\pi c t}{2L}$ and $\frac{\partial^2 u}{\partial x^2}(x,t) = +\frac{2V}{Lc} \sum_{n=1,3,5,\ldots} \sin \frac{n\pi x}{2L} \sin \frac{n\pi c t}{2L}$

$\frac{\partial^2 u}{\partial x \partial t}(x,t) = -\frac{2V}{L} \sum_{n=1,3,5,\ldots} \cos \frac{n\pi x}{2L} \cos \frac{n\pi c t}{2L}$

for a max or min $\frac{\partial \varepsilon_x}{\partial x} = \frac{\partial^2 u}{\partial x^2} = 0$ and $\frac{\partial \varepsilon_x}{\partial t} = \frac{\partial^2 u}{\partial x \partial t} = 0$ at the same point in (x,t) space.

$\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{n\pi x}{2L} = m\pi, 0 \Rightarrow x = 2L$ (not physical) or $\boxed{x = 0}$ for any n ∀t, and $\sin \frac{n\pi c t}{2L} = 0$

$\forall x$; the only values of t that satisfy this equality for any n $t = \frac{2L}{c} k$ $k = 0,1,2,3,\ldots$

From $\frac{\partial^2 u}{\partial x \partial t}(x,t) = 0 \Rightarrow \frac{n\pi x}{2L} = \frac{m\pi}{2}$ (m odd) For any n the only value of x is $x = L$ $\forall t$; also $\cos \frac{n\pi c t}{2L} = 0$

$\forall x$; the only values of t that satisfy this are $t = \frac{L}{c}$ and odd mult. of L/c

Since there are no matching points in the interior of the bar, then the two end points must be examined for absolute max.

At $x = 0$ $\varepsilon_x(0,t) = -\frac{4V}{\pi c} \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin \frac{n\pi c t}{2L}$ min occurs at $t = \frac{L}{c}, \frac{5L}{c}, \frac{9L}{c}, \ldots (4k+1)\frac{L}{c}$ $k = 0,1,\ldots$

At $x = L$ $\varepsilon_x(L,t) = 0$ $\forall t$ max occurs at $t = \frac{3L}{c}, \frac{7L}{c}, \frac{11L}{c}, \ldots (4k+3)\frac{L}{c}$ $k = 0,1,\ldots$

$\because \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin \frac{n\pi}{2} = \frac{\pi}{4}$ for min $\sum_{n=1,3,5,\ldots} \frac{1}{n} \sin \frac{3n\pi}{2} = -\pi/4$ for max

thus for max $\varepsilon_x(0, (4k+1)\frac{L}{c}) = \frac{V}{c}$ for min $\varepsilon_x(0, (4k+3)L/c) = -\frac{V}{c}$ ✓

3. Given a prismatic bar fixed at one end and free at the other end. It is subjected to a

periodic axial force $P_0 \sin \omega_f t$ at a distance d from the fixed end. Find $u(x,t)$

The governing PDE is $u_{xx} - \frac{1}{c^2} u_{tt} = 0$

and define $u = u(x,t)$

The BC are that $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(l,t) = 0$

The IC are that $u(x,0) = 0$ $\frac{\partial u}{\partial t}(x,0) = 0$ if not steady state

We attack the problem in the following manner: Look at this as being a sum of 2 problems if problem is not one of steady state

Problem #1: $u = \hat{u}(x,t)$ with BC $\hat{u}(0,t) = 0$ $\frac{\partial \hat{u}}{\partial x}(l,t) = 0$; IC unsatisfied & $P = P_0 \sin \omega_f t$ at $x=$

Proble #2: $u = \check{u}(x,t)$ with BC $\check{u}(0,t) = 0$ $\frac{\partial \check{u}}{\partial x}(l,t) = 0$; IC = - IC of proble #1 & no forcing f

Thus we solve Proble #1 with unrestricted IC. We then get the IC, whatever they are. We then solve Proble #2 with imposed IC = - IC of those found in proble 1.

$\therefore$ Soln = $\sum$ Problem 1 + Problem 2 and we also satisfy $u(0,t) = \frac{\partial u}{\partial x}(l,t) = 0$

$\frac{\partial u}{\partial t}(x,0) = u(x,0) = 0$ and $P = P_0 \sin \omega_f t$ @ $x = d$.

Problem #1:

define $\hat{u} = u_1(x,t)$ $0 \le x \le d$

$\hat{u} = u_2(x,t)$ $d \le x \le L$

w/BC $u_1(0,t) = 0$ and $\frac{\partial u_2}{\partial x}(l,t) = 0$ and open initial conditions

the matching conditions are that $u_1(d,t) = u_2(d,t)$ and $EA\left[\frac{\partial u_2}{\partial x}(d,t) - \frac{\partial u_1}{\partial x}(d,t)\right] = -P_0 \sin \omega_f$

Since the jump conditions are a fn. of $\sin \omega_f t$ then choose

$u_1(x,t) = X_1(x) \sin \omega_f t$ and $u_2(x,t) = X_2(x) \sin \omega_f t$

put into DE $\Rightarrow$ $X_1'' + \frac{\omega_f^2}{c^2} X_1 = 0$ for $0 \le x \le d$

$X_2'' + \frac{\omega_f^2}{c^2} X_2 = 0$ for $d \le x \le L$

they must satisfy $X_1(0) = 0$ $X_2'(l) = 0$ and $X_1(d) = X_2(d)$, $X_2'(d) - X_1'(d) = -\frac{P_0}{EA}$

Thus $X_1(x) = B \sin \frac{\omega_f}{c} x$ $X_2(x) = D \cos \frac{\omega_f}{c}(L - x)$

From the matching conditions $B \sin \frac{\omega_f}{c} d - D \cos \frac{\omega_f}{c}(l - d) = 0$

and $\frac{\omega_f}{c}\left[D \sin \frac{\omega_f}{c}(l - d) - B \cos \frac{\omega_f}{c} d\right] = -\frac{P_0}{EA}$

Thus $B = \frac{P_0}{EA} \frac{c}{\omega_f} \frac{\cos \frac{\omega_f}{c}(l-d)}{\cos \frac{\omega_f}{c} l}$ $D = \frac{P_0}{EA} \frac{c}{\omega_f} \frac{\sin \frac{\omega_f}{c} d}{\cos \frac{\omega_f}{c} l}$

Thus

$$\hat{u}(x,t) = \begin{cases} \dfrac{P_0}{EA}\dfrac{c}{\omega_f}\,\dfrac{\cos\frac{\omega_f}{c}(L-d)\,\sin\frac{\omega_f}{c}x}{\cos\frac{\omega_f}{c}L}\,\sin\omega_f t & 0\le x\le d \\ \dfrac{P_0}{EA}\dfrac{c}{\omega_f}\,\dfrac{\sin\frac{\omega_f}{c}d\,\cos\frac{\omega_f}{c}(L-x)}{\cos\frac{\omega_f}{c}L}\,\sin\omega_f t & d\le x\le L \end{cases}$$

Thus from this we have the following IC

$$\hat{u}(x,0) = 0 \qquad 0\le x\le L$$

$$\frac{\partial\hat{u}}{\partial t}(x,0) = \begin{cases} \dfrac{P_0 c}{EA}\,\dfrac{\cos\frac{\omega_f}{c}(L-d)\,\sin\frac{\omega_f}{c}x}{\cos\frac{\omega_f}{c}L} & 0\le x\le d \\ \dfrac{P_0}{EA}c\,\dfrac{\sin\frac{\omega_f d}{c}\,\cos\frac{\omega_f(L-x)}{c}}{\cos\frac{\omega_f}{c}L} & d\le x\le L \end{cases}$$

Let $\dfrac{P_0 c}{EA\cos\frac{\omega_f}{c}L} = \Gamma$

for prob. #2: This part is <u>not</u> necessary if the problem is one of steady state

define $\check{u}(x,t)$

W/BC $\check{u}(0,t)=0 \quad \dfrac{\partial\check{u}}{\partial x}(l,t)=0$

and IC $\check{u}(x,0) = -\hat{u}(x,0)=0$ and $\dfrac{\partial\check{u}}{\partial t}(x,0) = -\dfrac{\partial\hat{u}}{\partial t}(x,0)$

Let $\check{u} = X(x)\,T(t)$; put into PDE to get

$$X''T - \frac{1}{c^2}XT''=0 \quad \text{or} \quad \frac{X''}{X} = \frac{T''}{c^2T} = -\lambda^2$$

$\therefore \; X''+\lambda^2X=0 \quad \& \quad T''+\lambda^2c^2T=0$

$X = E\sin\lambda x + F\cos\lambda x$ and $T = G\sin\lambda ct + H\cos\lambda ct$

applying BC $\Rightarrow X(0)=0$ and $X'(l)=0 \Rightarrow F=0$ and $\cos\lambda l=0 \;\therefore\; \lambda l=\frac{n\pi}{2}$ n odd

$\therefore\; \lambda_n = \dfrac{n\pi}{2L}$ and $X_n = E_n\sin\dfrac{n\pi x}{2L}$ n odd

and also IC $\Rightarrow T(0)=0 \Rightarrow H=0 \;\forall x \quad T'(0)=\lambda cG$; Let $E_nG=U_n$

$$\therefore\; \check{u}(x,t) = \sum_{n=1,3,5,\dots} U_n\sin\frac{n\pi x}{2L}\sin\frac{n\pi ct}{2L}$$

$$\text{and}\; \frac{\partial\check{u}}{\partial t}(x,0) = \frac{\pi c}{2L}\sum_{n=1,3,5,\dots} nU_n\sin\frac{n\pi x}{2L} = \begin{cases} -\Gamma\cos\frac{\omega_f}{c}(L-d)\sin\frac{\omega_f}{c}x & 0\le x\le d \\ -\Gamma\sin\frac{\omega_f d}{c}\cos\frac{\omega_f}{c}(L-x) & d\le x\le L \end{cases}$$

$$\therefore\; \frac{\pi c}{2L}\int_0^L \sum nU_n\sin\frac{n\pi x}{2L}\sin\frac{m\pi x}{2L}\,dx = -\Gamma\left\{\int_0^d \cos\frac{\omega_f}{c}(L-d)\sin\frac{\omega_f x}{c}\sin\frac{m\pi x}{2L}\,dx + \int_d^L \sin\frac{\omega_f d}{c}\cos\frac{\omega_f}{c}(L-x)\sin\frac{m\pi x}{2L}\,dx\right\}$$

and for $n=m$ the $U_m = \dfrac{4\Gamma}{\pi c}\left\{\int_0^d \qquad dx + \int_d^L \qquad dx\right.$

m is odd here & we assume $\frac{m\pi}{2L} \ne \frac{\omega_f}{c}$. If it is we have resonance. -2

for $n\ne m$ we have an identity and its mathematically horrible to show so I won't

DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING
STANFORD UNIVERSITY

ME 236A Waves and Vibrations — Autumn 1979

Problem Set No. 3

1. Consider a uniform cantilever beam of length ℓ. Sketch the first three modes of free vibration with reasonable accuracy.

2. One end of a uniform beam is clamped and the other is subjected to a periodic displacement of the form $W = W_o \sin \omega t$. Determine the steady-state motion.

3. Calculate the phase velocities of the two free harmonic ~~e~~waves in an infinite Timoshenko beam. Construct the plot of phase velocity versus wave number and discuss the limiting values for very large and very small wave numbers. Also construct the plot of frequencies versus wave number.

4. A beam of length ℓ is resting on an elastic foundation (modulus n). Determine the lowest mode of free vibration if both ends are free.

$\frac{40}{40}$

Cesar Levy
ME 236A
Waves & Vibs. Fall 79

Problem Set #3

1. Consider a uniform cantilever beam of length l. Sketch the first 3 modes of free vibrations

PDE governing is $EIW^{IV} + \rho A \ddot{W} = 0$ w/ BC $W(0) = W'(0) = 0$ and $EIW'''(L) = 0$ & $EIW''(L)$:

thus Let $W(x,t) = X(x)T(t)$ $\Rightarrow$ $EIX^{IV}T + \rho A X\ddot{T} = 0$ $\Rightarrow$ $\frac{X^{IV}}{X} = -\frac{\ddot{T}}{T}\frac{\rho A}{EI} = +\lambda^4$

$X^{IV} - \lambda^4 X = 0$ choose $X(x) = Ce^{mx}$ $\Rightarrow$ $X = C_1\sinh\lambda x + C_2\cosh\lambda x + C_3\sin\lambda x + C_4\cos\lambda x$

$\ddot{T} + \lambda^4\frac{EI}{\rho A}T = 0$ $\Rightarrow$ $T(t) = A\cos\kappa\lambda^2 t + B\cos\kappa\lambda^2 t$ $\kappa^2 = \frac{EI}{\rho A} = c^2r^2$ where $c^2 = E/\rho$, r = rad. of g.

applying BC $\Rightarrow$ $C_2 + C_4 = 0$ $C_1 + C_3 = 0$ or $C_2 = -C_4$; $C_1 = -C_3$

and $C_1\sinh\lambda L + C_2\cosh\lambda L - C_3\sin\lambda L - C_4\cos\lambda L = 0$ if $\lambda \neq 0$

$C_1\cosh\lambda L + C_2\sinh\lambda L - C_3\cos\lambda L + C_4\sin\lambda L = 0$ if $\lambda \neq 0$

$$\therefore \begin{bmatrix}(\sinh\lambda L + \sin\lambda L) & (\cosh\lambda L + \cos\lambda L)\\ (\cosh\lambda L + \cos\lambda L) & (\sinh\lambda L + \sin\lambda L)\end{bmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix} = 0$$

$\Rightarrow$ det $\equiv 0$ for nontrivial Solns. $\Rightarrow$ $\sinh^2\lambda L + \sin^2\lambda L - \cosh^2\lambda L - 2\cosh\lambda L\cos\lambda L - \cos^2\lambda L = 0$

since $\cosh^2\lambda L - \sinh^2\lambda L = 1$ and $\cos^2\lambda L + \sin^2\lambda L = 1$ $\Rightarrow$

$$-1 - 1 - 2\cosh\lambda L\cos\lambda L = 0 \Rightarrow \cos\lambda L\cosh\lambda L = -1$$

or $\lambda_1 L = 1.875$ $\lambda_2 L = 4.694$ $\lambda_3 L = 7.855$ from class notes.

now $-\frac{[\sinh(1.875) + \sin(1.875)]}{[\cosh(1.875) + \cos(1.875)]}C_1 = C_2$ & let $C_1 = 1$ $C_2 = -1.362$
$C_3 = -1$ $C_4 = 1.362$

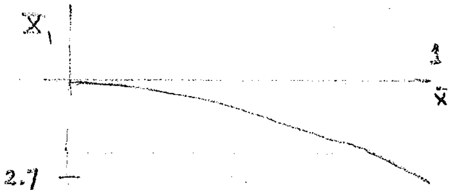
now $\boxed{X_1 = (\sinh\lambda_1 x - \sin\lambda_1 x) + 1.362(\cos\lambda_1 x - \cosh\lambda_1 x)}$ $\lambda_1 x = 1.875\bar{x}$ $\bar{x} = x/L$

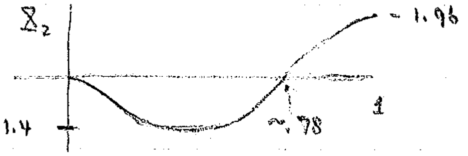
also $-\left[\frac{\sinh(4.694) + \sin(4.694)}{\cosh(4.694) + \cos(4.694)}\right]C_1 = C_2$ & let $C_1 = 1$ $C_2 = -.982$
$C_3 = -1$ $C_4 = +.982$

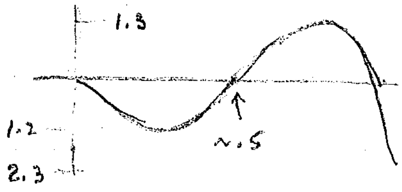
$\therefore$ $\boxed{X_2 = (\sinh\lambda_2 x - \sin\lambda_2 x) + .982(\cos\lambda_2 x - \cosh\lambda_2 x)}$ $\lambda_2 x = 4.694\bar{x}$

finally $-\left[\frac{\sinh(7.855) + \sin(7.855)}{\cosh(7.855) + \cos(7.855)}\right]C_1 = C_2$ & let $C_1 = 1$ $C_2 = -1.001$
$C_3 = -1$ $C_4 = +1.001$ $\lambda_3 x = 7.855\bar{x}$

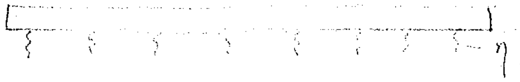
$\boxed{X_3 = (\sinh\lambda_3 x - \sin\lambda_3 x) + 1.001(\cos\lambda_3 x - \cosh\lambda_3 x)}$ $\sim -e^{-\lambda x} + \cos\lambda x - \sin\lambda x$ ✓







See reverse for numbers

4. 

The beam of length L rests on an elastic foundation of modulus η. Find the lowest modes of free vibration if the ends are free. The governing PDE is $EIw^{IV} + \eta w + \rho A\ddot{w} = 0$

The BC are that $w''(0) = w''(L) = w'''(0) = w'''(L)$

Let $w = XT \Rightarrow$ $EIX^{IV}T + \eta XT + \rho AX\ddot{T} = 0 \Rightarrow EI\frac{X^{IV}}{X} + \eta = -\rho A\frac{\ddot{T}}{T} \Rightarrow \frac{X^{IV}}{X} + \frac{\eta}{EI} = -\frac{\rho A}{EI}\frac{\ddot{T}}{T} = \lambda^4$

thus $\ddot{T} + \lambda^4\kappa^2T = 0$ $\quad\kappa^2 = c^2r^2$ where r = radius of gyration $\therefore$ $T = A\cos\kappa\lambda^2t + B\sin\kappa\lambda^2t$

also $X^{IV} - (\lambda^4 - \eta/_{EI})X = 0$ $\qquad$ Let $\lambda^4 - \eta/_{EI} = \mu^4$

then $X = C_1\sinh\mu x + C_2\cosh\mu x + C_3\sin\mu x + C_4\cos\mu x$

applying bc
$$\left.\begin{array}{ll} w''(0) = 0 = C_2 - C_4 \Rightarrow & C_2 = C_4 \\ w'''(0) = 0 = C_1 - C_3 \Rightarrow & C_1 = C_3 \end{array}\right\} X = C_1(\sinh\mu x + \sin\mu x) + C_2(\cosh\mu x + \cos\mu x)$$

$$w''(L) = C_1(\sinh\mu L - \sin\mu L) + C_2(\cosh\mu L - \cos\mu L) = 0$$

$$w'''(L) = C_1(\cosh\mu L - \cos\mu L) + C_2(\sinh\mu L + \sin\mu L) = 0$$

$\Rightarrow$ det $= 0$ or $\{\sinh^2\mu L + \sinh\mu L\sin\mu L - \sinh\mu L\sin\mu L - \sin\mu L\sin\mu L - \cosh^2\mu L + 2\cos\mu L\cosh\mu L - \cos^2\mu L\} = 0$

or $\{-2 + 2\cos\mu L\cosh\mu L\} = 0$ $\quad$ or $\cos\mu L\cosh\mu L = 1$ ✓ $\quad$ take $\mu = 0 \Rightarrow \lambda = \sqrt[4]{\eta/_{EI}}$

but does this mean we have a soln to bc & de : $\mu = 0 \Rightarrow X_0^{IV} = 0$ or $X_0 = ax^3 + bx^2 + cx + d$

$X_0' = 3ax^2 + 2bx + c$; $X_0'' = 6ax + 2b \Rightarrow w''(0) = 0 \Rightarrow b = 0$ and $w''(L) = 0 \Rightarrow a = 0$

$X_0''' = 6a = 0 \;\forall x$ $\quad\therefore$ we <u>do</u> satisfy bc and $X_0 = \underline{cx + d}$ thus $\ddot{T}_0 + \eta/_{\rho A}T_0 = 0$

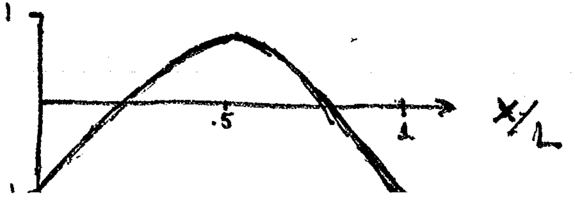
and $T_0 = A_1\cos\sqrt{\eta/_{\rho A}}\,t + B_1\sin\sqrt{\eta/_{\rho A}}\,t$

$$\therefore \quad \left| w_1(x,t) = c_1x\cos\sqrt{\eta/_{\rho A}}\,t + c_2x\sin\sqrt{\eta/_{\rho A}}\,t + d_1\cos\sqrt{\eta/_{\rho A}}\,t + d_2\sin\sqrt{\eta/_{\rho A}}\,t \right|$$

X_0 represents a rigid body translation and rotation since $\left.\frac{\partial w}{\partial x}\right| = c \quad \forall x$

$\mu = 0$ IS NOT A REALLY INTERESTING CASE

$\mu L = 4.73$ IS THE NEXT ONE. THE MODE SHAPE FOR THIS ONE LOOKS LIKE:



3. Calculate the phase velocities of the two free harmonic waves in an infinite Timoshenko Beam, Construct the plot of the phase velocity versus wave number and discuss the limiting values for a) very large and b) very small wave numbers. Also construct the plot of frequencies versus wave number

The governing PDE for the Timoshenko Beam Theory are

$$EI\psi'' + AG(w' - \psi) = \rho I \ddot{\psi} \qquad (1)$$

$$AG(w' - \psi)' = \rho A \ddot{w} \qquad (2)$$

by joining the two one can find the full pde on ψ only, namely

$$\ddot{\ddot{\psi}} - \left(\frac{E}{G}+1\right)\frac{G}{\rho}\ddot{\psi}'' + \frac{G}{\rho}\frac{E}{\rho}\psi^{IV} + \frac{AG}{\rho I}\ddot{\psi} = 0 \qquad (3)$$

Now let $\psi = \Psi e^{ikx} e^{i\omega t}$ and plug into (3) to obtain

$$\omega^4 - (E/G+1)\,G/\rho\; k^2\omega^2 + \frac{G}{\rho}\frac{E}{\rho}k^4 - \frac{AG}{I\rho}\omega^2 = 0 \qquad (4) \qquad \text{or}$$

$$c_p{}^4 - (E/G+1)\,G/\rho\; c_p{}^2 + \frac{G}{\rho}E/\rho - \frac{AG}{I\rho}\frac{c_p^2}{k^2} = 0 \qquad (5)$$

Solving (5) for c_p gives

$$c_p = \left\{\frac{1}{2}(E/G+1)G/\rho + \frac{1}{2}\frac{AG}{\rho I k^2} \pm \frac{1}{2}\left[\left\{(E/G+1)G/\rho + \frac{AG}{\rho I k^2}\right\}^2 - 4\frac{GE}{\rho^2}\right]^{1/2}\right\}^{1/2} \qquad (6)$$

or

$$\omega = \left\{\frac{1}{2}(E/G+1)G/\rho\, k^2 + \frac{1}{2}\frac{AG}{\rho I} \pm \frac{1}{2}\left[\left\{(E/G+1)G/\rho\, k^2 + \frac{AG}{\rho I}\right\}^2 - 4\frac{GE}{\rho^2}k^4\right]^{1/2}\right\}^{1/2} \qquad (7)$$

Let $k \to \infty$ for (6). Then

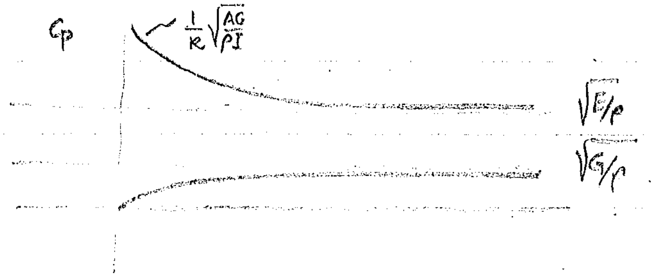
$$c_p \sim \left\{\frac{1}{2}(E/G+1)G/\rho \pm \frac{1}{2}(E/G+1)G/\rho\right\}^{1/2} \quad \text{or} \quad c_{p_1} = (E/\rho)^{1/2} \quad \text{or} \quad c_{p_2} = (G/\rho)^{1/2}$$

Let $k \to 0$ for (6) Then

$$c_p \sim \left\{\frac{1}{2}\frac{AG}{\rho I k^2} + \frac{1}{2}(E/G+1)G/\rho \pm \frac{1}{2}\frac{AG}{\rho I k^2}\left[1 + (E/G+1)\frac{I}{A}k^2 + O(k^4)\right]\right\}^{1/2}$$

$$c_{p_1} = \frac{1}{k}\sqrt{\frac{AG}{\rho I}} \quad \text{and} \quad c_{p_2} = 0$$

Discussion: For very large $k \Longleftrightarrow$ very small λ, the boundaries appear very far away. Thus to a wave the beam appears to be an infinite body. The infinite body, as shown in class, has $C_{p_1} = \sqrt{E/\rho}$ and $C_{p_2} = \sqrt{G/\rho}$ as our results indicate. For very small $k \Longleftrightarrow$ very large λ, the wave travels to the boundary in an infinitesimal time; hence the beam must have an ∞ velocity ie it is independent of x, which explains why $c_{p_1} \to \infty$. Since w, ψ are independent of x, then ψ will have a harmonic solution of $\cos\sqrt{\frac{AG}{\rho I}}\,t$ & $\sin\sqrt{\frac{AG}{\rho I}}\,t$ with $\frac{\omega}{k} = \frac{1}{k}\sqrt{\frac{AG}{\rho I}} = c_{p_1}$. But $\ddot{w} = 0 \Rightarrow$ w is a constant hence we only have rigid body motion in the transverse direction $\Rightarrow c_{p_2} = 0$ as shown. Thus



for the frequency let $k \to \infty$ for (7). Then

$$\omega \approx \left\{ \tfrac{1}{2}\left(E/G+1\right)G/\rho\, k^2 \pm \tfrac{1}{2}\left(E/G-1\right)G/\rho\, k^2 \right\} \quad \therefore \quad \omega_1 \sim \sqrt{E/\rho}\,k \qquad \omega_2 \sim \sqrt{G/\rho}\,k$$

for the frequency let $k \to 0$ for (7). Then

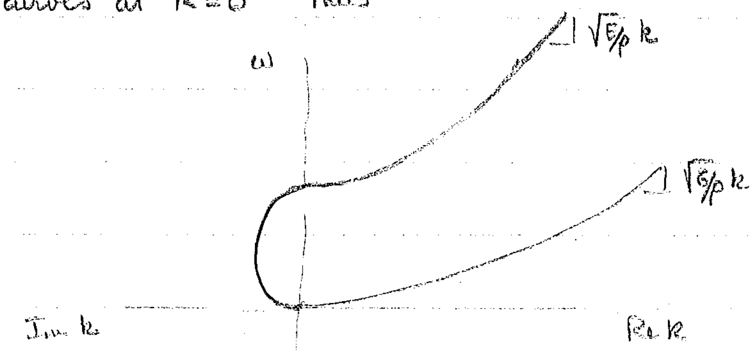
$$\omega \sim \left\{ \frac{1}{2}\frac{AG}{\rho I} \pm \frac{1}{2}\frac{AG}{\rho I} + O(k^2) \right\}^{1/2} \quad \therefore \quad \omega_1 \sim \sqrt{\frac{AG}{\rho I}} \qquad \omega_2 \sim 0$$

for k imaginary & let $x = \omega^2$ $y = k^2$

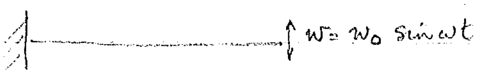
$$x^2 + (E/G+1)\,G/\rho\, xy + \frac{G}{\rho}\frac{E}{\rho}y^2 - \frac{AG}{I\rho}x = 0$$

. This of form $Ax^2 + Bxy + Cy^2 + Ex + Dy + F =$

This is an equation of an ellipse since $B^2 - 4AC > 0$. The ellipse intersects the two curves at $k = 0$ Thus



2. If one end of beam is subjected to $W = W_0 \sin\omega t$ and the other is clamped, determine the steady state motion



the governing PDE is $EI\,w^{IV} + \rho A \ddot{w} = 0$ ✓

thus will BC that $w(0) = 0$ and $w'(0) = 0$ $w(l) = w_0 \sin\omega t \;\forall t$ and $EI\,w''(l) = 0$

then let $X(x)\,T(t) = w(x,t)$; put into PDE and rearrange.

$$\text{then} \quad \frac{X^{IV}}{X} = -\frac{\ddot{T}}{T}\frac{\rho A}{EI} = \lambda^4 \quad \text{and} \implies T = A\cos K\lambda^2 t + B\sin K\lambda^2 t \quad K^2 = c^2 r^2$$

- is rad. of gyr. ; because of the boundary condition on $w(l,t) \implies \omega = K\lambda^2$ and $A = 0$

$$\therefore \quad X = C_1 \sinh\lambda x + C_2 \cosh\lambda x + C_3 \sin\lambda x + C_4 \cos\lambda x$$

applying first 2 bc $\implies C_4 = -C_2$ and $C_3 = -C_1$

Now $w_0 = C_1(\sinh\lambda L - \sin\lambda L) + C_2(\cosh\lambda L - \cos\lambda L)$ from 3rd bc

$0 = C_1(\sinh\lambda L + \sin\lambda L) + C_2(\cosh\lambda L + \cos\lambda L)$ from 4th bc

$$\text{then} \quad C_1 = \frac{\begin{bmatrix} w_0 & \cosh\lambda L - \cos\lambda L \\ 0 & \cosh\lambda L + \cos\lambda L \end{bmatrix}}{2[\sinh\lambda L\cos\lambda L - \sin\lambda L\cosh\lambda L]} = \frac{w_0[\cosh\lambda L + \cos\lambda L]}{2[\sinh\lambda L\cos\lambda L - \sin\lambda L\cosh\lambda L]}$$

$$C_2 = \frac{\begin{bmatrix} \sinh\lambda L - \sin\lambda L & w_0 \\ \sinh\lambda L + \sin\lambda L & 0 \end{bmatrix}}{2[\sinh\lambda L\cos\lambda L - \sin\lambda L\cosh\lambda L]} = \frac{-w_0(\sinh\lambda L + \sin\lambda L)}{2[\sinh\lambda L\cos\lambda L - \sin\lambda L\cosh\lambda L]}$$

$$\therefore \quad X(x)T(t) = w(x,t) = \left[C_1(\sinh\lambda x - \sin\lambda x) + C_2(\cosh\lambda x - \cos\lambda x)\right]\sin K\lambda^2 t \quad \text{where} \quad \lambda = \sqrt{\frac{\omega}{K}}$$

$$w(x,t) = \frac{w_0\left\{[\cosh\lambda L + \cos\lambda L][\sinh\lambda x - \sin\lambda x] - [\sinh\lambda L + \sin\lambda L][\cosh\lambda x - \cos\lambda x]\right\}}{2(\sinh\lambda L\cos\lambda L - \sin\lambda L\cosh\lambda L)}\sin\omega t$$

To prove this method of attack is OK look at following:

Suppose we assume $W(x,t) = \bar{w}(x,t) + f(t)\,g(x)$ $\quad f(t) = W_0 \sin\omega t$

we then must solve $EI\,w^{IV} + \rho A\ddot{w} = EI\,\bar{w}^{IV} + \rho A\ddot{\bar{w}} + EI\,f(t)\,g^{IV} + \rho A\ddot{f}g = 0$

with $w(0,t) = \bar{w}(0,t) + f(t)\,g(0) = 0 \implies g(0) = 0 \quad \bar{w}(0,t) = 0$

$w'(0,t) = \bar{w}'(0,t) + f(t)\,g'(0) = 0 \implies g'(0) = 0 \quad \bar{w}'(0,t) = 0$

$w(l,t) = \bar{w}(l,t) + f(t)\,g(l) = f(t) \implies g(l) = 1 \quad \bar{w}(l,t) = 0$

$w''(l,t) = \bar{w}''(l,t) + f(t)\,g''(l) = 0 \implies g''(l) = 0 \quad \bar{w}''(l,t) = 0$

with the given bc on $g(x)$ let us pick $g(x) = Ax^3 + Bx^2 + Cx + D$

using bc we find that $C = D = 0$ $A = -1/2\ell^3$ $B = 3/2\ell^2$ $\therefore g(x) = -\frac{1}{2}(x/\ell)^3 + \frac{3}{2}(x/\ell)$

$\therefore EI\,\bar{w}^{IV} + \rho A\ddot{\bar{w}} = -\rho A\,\ddot{f}(t)\,g(x)$ since $g(x)$ is a 3rd degree eqn only.

$$= \rho A\omega^2 f(t)\,g(x)$$

The solution to $\bar{w}$ $\ni$. $\bar{w} = \bar{w}^h + \bar{w}^p$ with $EI\,\bar{w}^{h\,IV} + \rho A\ddot{\bar{w}}^h = 0$ & $\bar{w}^h(0,t) = \bar{w}'^h(0,t)$ $= \bar{w}^h(\ell,t) = \bar{w}^{h''}(\ell,t) = 0$ gives the characteristic equation:

$$\boxed{\sinh\lambda\ell\,\cos\lambda\ell - \sin\lambda\ell\,\cosh\lambda\ell = 0}^{**}$$

λ being the separation of variable constant related to the frequencies since $\omega_n = \lambda_n^2\sqrt{\frac{EI}{\rho A}}$.

$$X_n \text{ (the eigenfns)} = \frac{(\cosh\lambda x - \cos\lambda x)(\sinh\lambda\ell - \sin\lambda\ell) - (\cosh\lambda\ell - \cos\lambda\ell)(\sinh\lambda x - \sin\lambda x)}{(\sinh\lambda\ell - \sin\lambda\ell)}$$

if we continue the procedure as outlined then $\rho A\omega^2 f(t)\,g(x) = \sum_{n=1}^{\infty} X_n(x)\,q_n(t)$

where $q_n(t) = \dfrac{\int_0^\ell \rho A\omega^2 f(t)\,g(x)\,dx}{\int_0^\ell X_n^2(x)\,dx} = A_n(\rho;A;\ell;\omega)\,f(t)$ where A_n is a const.

Then $EI\,\bar{w}^{IV} + \rho A\ddot{\bar{w}} = \sum_{n=1}^{\infty} A_n(\rho;A;\ell;\omega)\,f(t) = f(t)\sum_{n=1}^{\infty} A_n(\rho;A;\ell;\omega)$

and since $f(t)$ is periodic, we could have taken $\bar{w} = F(x)\,f(t)$

since the DE would have reduced to $EI\,F^{IV} - \omega^2\rho A F = \rho A\omega^2 g(x)$

with bc. $F(0) = F'(0) = F''(\ell) = F(\ell) = 0$.

All this means is that $w(x,t) = \bar{w}(x,t) + f(t)\,g(x) = [F(x) + g(x)]\,f(t)$ hence we could have started by assuming a form $w(x,t) = G(x)\sin\omega t$ without loss in generality. Physically the vibrational frequency of the beam at steady state when all transients are gone must be ω.

If you will note the frequency equation ** and my solution on first page this problem, you will note that the denominator are the same. Hence if ω is one of the natural frequencies, $w(x,t) \to \infty$ as expected. The reason why this works in this case is that since we are dealing in steady state solutions, the initial conditions' effect has vanished and only the bc's provide the dominant effect to the vibration of the beam.

Looks ok.

Question	Score
1	7
2	10
3	
4	
5	
6	
7	
8	
Total	17/20

STANFORD UNIVERSITY

OFFICIAL EXAMINATION BOOK

24 Page Ruled

Name of student Cesar Levy

Date of examination 13 Nov 79

Course ME236A

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(Signed) Cesar Levy

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DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING
STANFORD UNIVERSITY

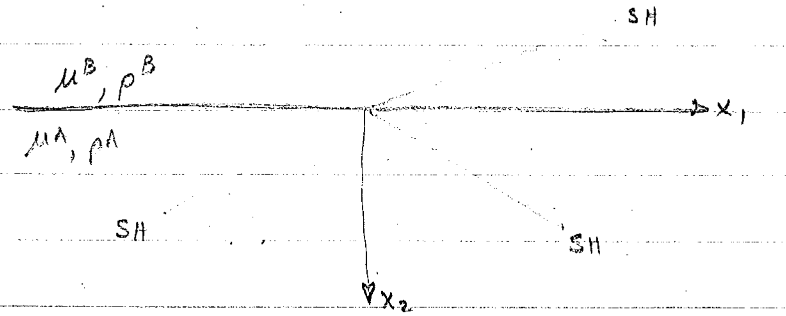
ME 236A Waves and Vibrations Autumn 1979

Midterm Examination

1. Consider 2 elastic half-spaces (with properties μ^A, ρ^A and μ^B, ρ^B, respectively) bonded at the plane interface. Consider an incident SH-wave and determine the amplitude ratios of the transmitted and reflected waves with respect to the incident wave. Also, determine the angle of incidence at which there will be no reflected, but only a transmitted wave.

see my notes on Seismology!

2. Consider a very long, elastic cylinder of rectangular cross-section (edge length a and b). It is welded to a fixed outer housing along all four faces. Determine the frequencies of free axial vibration by considering the displacement in the axial direction to be independent of the axial coordinate, while the displacement components normal to the axis vanish (antiplane strain).



for SH wave $u_3 = u_3(x_1, x_2; t)$

DE goveri'g is $\Delta u_3 = \frac{1}{c_2^2} u_{3,tt}$ take $u_3 = f(x_2) e^{ik(x_1 - ct)}$

then $\Delta u_3 - \frac{1}{c_{2_{A,B}}^2} u_{3,tt} = (f'' - \cancel{c}k^2 f) + \frac{\omega^2}{c_2^2} f = 0$ or let $\beta_{A,B}^2 = \frac{\omega^2}{c^2_{2\,A,B}} - k^2$

$\beta^2 = k^2\left(\left(\frac{c}{c_2^A}\right)_B^2 - 1\right)$

thus $f'' + \beta_{A,B}^2 f = 0$ or $f = A \sin \beta_{A,B} x_2 + B \cos \beta x_2 = f(x_2) = A e^{i\beta_{A,B} x_2}$

thus SH $u_3 = A_1 e^{i(kx_1 - \omega t + \cancel{c}\beta_A x_2)}$ for upgoing

$\checkmark$ $u_3 = A_2 e^{i(kx_1 - \omega t - \beta_A x_2)}$ for downgo'g wave

$u_3 = A_3 e^{i(kx_1 - \omega t + \beta_B x_2)}$ for transmitted wave.

at boundary $\mu_A \frac{\partial u_3}{\partial x_2}\Big|_{x_2 = 0^+} = \mu_B \frac{\partial u_3}{\partial x_2}\Big|_{x_2 = 0^-} = \sigma_{3,2}$ must be continuous

and $\mu_B \frac{\partial u_3}{\partial x_1}\Big|_{x_2 = 0^-} = \mu_A \frac{\partial u_3}{\partial x_1}\Big|_{x_2 = 0^+} = \sigma_{31}$ must be cont. Why?

continuity of displacement?

$\therefore$ $i\beta_A A_1 \mu_A - i\beta_A A_2 \mu_A = i\beta_B A_3 \mu_B$ $\beta_A = k^2\left[\left(\frac{c}{c_2^A}\right)^2 - 1\right]$

$\mu_A A_1 + \mu_A A_2 = \mu_B A_3$ $\beta_B = k^2\left[\left(\frac{c}{c_2^B}\right)^2 - 1\right]$

$\Rightarrow$

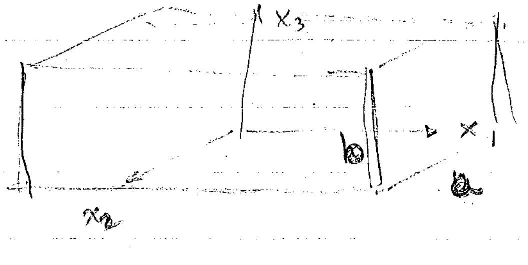
$$\begin{bmatrix} \beta_A \mu_A \\ \mu_A \end{bmatrix} = \begin{bmatrix} \beta_A \mu_A & \beta_B \mu_B \\ -\mu_A & \mu_B \end{bmatrix} \begin{bmatrix} A_2/A_1 \\ A_3/A_1 \end{bmatrix}$$

$$A_2/A_1 = \frac{\beta_A \mu_A \mu_B - \mu_A \mu_B \beta_B}{\beta_A \mu_A \mu_B + \beta_B \mu_A \mu_B} = \frac{\beta_A - \beta_B}{\beta_A + \beta_B}$$

no reflect $\Rightarrow \beta_A = \beta_B$

$$A_3/A_1 = \frac{\beta_A \mu_A^2 + \beta_B \mu_A^2}{(\beta_A + \beta_B)\mu_A \mu_B} = \frac{2\beta_A \mu_A}{\mu_B(\beta_A + \beta_B)}$$

2.



u_1 displacements in x_1 direction $f(x_2, x_3)$

$u_2 = 0$, $u_3 = 0$

antiplane strain

$$\Delta u_1 - \frac{1}{c_2^2} u_{1,tt} = 0 \qquad \text{let} \quad u_1 = f(x_2)\, g(x_3)\, e^{i\omega t}$$

Put into de.

$$f'' g + g'' f + \frac{\omega^2}{c_2^2} f g = 0$$

$$\therefore \quad \frac{f''}{f} + \frac{\omega^2}{c_2^2} = -\frac{g''}{g} = +\lambda^2 \qquad \therefore \quad g'' + \lambda^2 g = 0 \qquad \lambda \neq 0$$

$$\text{and} \quad f'' + \left(\frac{\omega^2}{c_2^2} \mp \lambda^2\right) f = 0$$

$$\therefore \quad g = \tilde{A}_1 e^{i\lambda x_3} + \tilde{A}_2 e^{-i\lambda x_3} = A \cos \lambda x_3 + B \sin \lambda x_3$$

$$g\Big|_{x_3 = \pm b/2} = 0 \qquad \text{welded condition}$$

$$\therefore \quad A \cos \frac{\lambda b}{2} + B \sin \frac{\lambda b}{2} = 0$$

$$A \cos \frac{\lambda b}{2} - B \sin \frac{b\lambda}{2} = 0$$

$$\Rightarrow 2 \cos \frac{\lambda b}{2} \sin \frac{\lambda b}{2} = 0 \qquad \text{for } A, B \neq 0$$

$$\therefore \quad \sin \lambda b = 0 \qquad \lambda b = n\pi \qquad \therefore \boxed{\lambda_n = \frac{n\pi}{b}}$$

$$f = C \sin \beta x_2 + D \cos \beta x_2 \qquad \beta^2 = \frac{\omega^2}{c_2^2} - \lambda^2$$

$$f\Big|_{x_2 = \pm a/2} = 0 \qquad \text{welded condition} \Rightarrow \qquad \beta_m = \frac{m\pi}{a}$$

$$\text{but} \quad \beta_m^2 = \frac{m^2}{a^2}\pi^2 = \frac{\omega_{mn}^2}{c_2^2} - \lambda_n^2 = \frac{\omega_{mn}^2}{c_2^2} - \frac{n^2\pi^2}{b^2}$$

$$\therefore \quad \omega_{mn}^2 = \pi^2 c_2^2 \left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right] \qquad m, n \geq 1$$

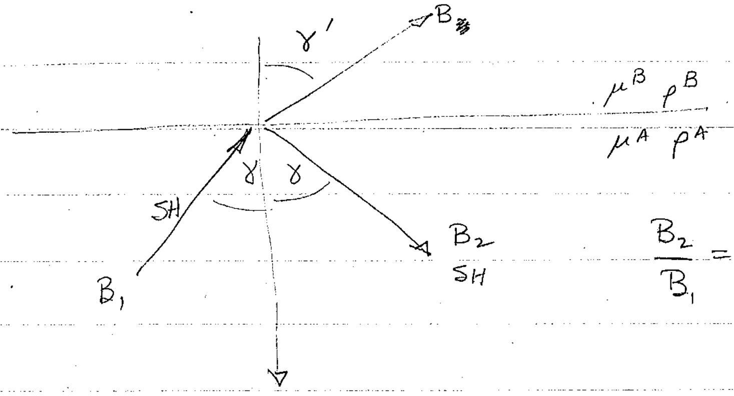
$$\text{for } \lambda = 0 \Rightarrow g'' = 0 \;\&\; \cancel{f'' = \frac{\omega^2}{c_2^2}} \qquad f = b_1 \sin \frac{\omega}{c_2} x_2 + b_2 \cos \frac{\omega}{c_2}$$

$$\hookrightarrow g = a_1 x_3 + a_2 \qquad \cancel{f = -\frac{\omega^2}{c_2^2}\frac{x^2}{2} + b_1 x + b_2}$$

$$g\Big|_{x_3 = \pm b/2} = 0 \Rightarrow \begin{matrix} a_1 b/2 + a_2 = 0 \\ -a_1 b/2 + a_2 = 0 \end{matrix} \qquad \det\begin{bmatrix} b/2 & 1 \\ -b/2 & 1 \end{bmatrix} \neq 0 \Rightarrow a_1, a_2 \equiv 0$$

$$\cancel{f\Big|_{x_2 = \pm a/2} \Rightarrow \begin{matrix} b_1 a/2 + b_2 = \frac{\omega^2}{c_2^2}\frac{a^2}{8} \\ -b_1 a/2 + b_1 = \frac{\omega^2}{c_2^2}\frac{a^2}{8} \end{matrix}} \qquad f\Big|_{x_2 = \pm a/2} \Rightarrow \begin{matrix} 0 = b_1 \sin \frac{\omega}{c_2} \frac{a}{2} + b_2 \cos \frac{\omega a}{2c_2} \\ 0 = -b_1 \sin \frac{\omega}{c_2} \frac{a}{2} + b_2 \cos \frac{\omega a}{2c_2} \end{matrix}$$

Solution



$$\frac{B_2}{B_1} = \frac{\mu_A \cos\gamma - \mu_B \left(\frac{C_S^A}{C_S^B}\right)\cos\gamma'}{\mu_A \cos\gamma + \mu_B \left(\frac{C_S^A}{C_S^B}\right)\cos\gamma'}$$

$$\frac{B'}{B_1} = \frac{2\rho_A \mu_B \cos\gamma}{\rho^B\left[\mu_A \cos\gamma + \mu_B \left(\frac{C_S^A}{C_S^B}\right)\cos\gamma'\right]}$$

for $B_2 = 0 \Rightarrow \cos\gamma = \frac{\mu_B}{\mu_A}\left(\frac{C_S^A}{C_S^B}\right)\cos\gamma'$

$$\omega^2 = C_2^2 \pi^2 \left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right]$$

Final A

Question	Score
1	10/10
2	9/10
3	10/10
4	10/10
5	10/10
6	
7	
8	
Total	49/50

STANFORD UNIVERSITY

OFFICIAL EXAMINATION BOOK

24 Page Ruled

Name of student Cesar Levy

Date of examination Dec 10, 1979

Course ME 236A Waves and Vibrations

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Prof. Herrmann: The HW assignment is included inside

12 PM Moday 10 Dec

DIVISION OF APPLIED MECHANICS
DEPARTMENT OF MECHANICAL ENGINEERING
STANFORD UNIVERSITY

ME 236A Waves and Vibrations Autumn 1979

Final Examination

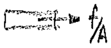
1. For the Rayleigh-Lamb problem, assume displacement potentials of the form

$$\phi = (A_1 e^{iax_2} + A_2 e^{-iax_2}) e^{i(kx_1 - \omega t)}$$

$$\psi = (A_3 e^{ibx_2} + A_4 e^{-ibx_2}) e^{i(kx_1 - \omega t)}$$

where $a = (\omega^2/c_\ell^2 - k^2)^{\frac{1}{2}}$, $b = (\omega^2/c_t^2 - k^2)^{\frac{1}{2}}$, and find the dispersion equation in terms of the roots of a 4 x 4 determinant. B~~x~~y means of simple row and column operations on the determinant, show that it may be factored into the product of two 2 x 2 determinants. Evaluate the two determinants, and show that they correspond to the symmetric and antisymmetric cases discussed in class.

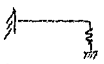
✓ 2. An elastic rod of elastic modulus E , mass density ρ and cross-sectional area A is embedded in another elastic medium, whose effect on the rod can be approximated by postulating that the axial force (per unit length) exerted by the medium on the rod is proportional to the axial displacement (constant η). Derive the equation of motion for the rod as an extension of the elementary theory and calculate (and sketch) the frequency as a function of the wave number.



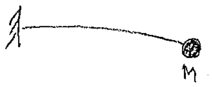
$f/A = u\eta$

✓ 3. Consider a very long beam which is resting on an elastic foundation and use elementary theory (stiffness EI , cross-sectional area A , mass density ρ , foundation modulus k). Calculate, sketch and discuss the dependence of phase velocity, group velocity and frequency on wave number for free harmonic waves.

4. Determine the frequency equation of free flexural motions of a uniform elastic beam of length ℓ , whose one end is clamped and the other supported by a spring of stiffness k . Consider limiting cases of k = 0 and k = ∞ and also the relationships for high frequencies. ? Call & ask about this: Sunday



5. A cantilever beam of length ℓ and stiffness EI carries a concentrated mass M at the free end. Choose a reasonable deflection curve and use Rayleigh's method to find an approximate value of the lowest natural frequency of bending vibration.



where $A_1 + A_2 = B_1$

$(A_1 - A_2)i = B_2$

$(A_3 + A_4) = B_3$

$(A_3 - A_4)i = B_4$

$f(x_2) = B_1 \cos ax_2 + B_2 \sin ax_2$

$g(x_2) = B_3 \cos bx_2 + B_4 \sin bx_2$

$$\phi_1 = f(x_2)e^{i(kx_1 - \omega t)} \qquad \psi = g(x_2)e^{i(kx_1 - \omega t)}$$

$$u_1 = \frac{\partial\phi}{\partial x_1} + \frac{\partial\psi}{\partial x_2} = (ikf + g')e^{i(\quad)}$$

$$u_2 = \frac{\partial\phi}{\partial x_2} - \frac{\partial\psi}{\partial x_1} = (f' - ikg)e^{i(\quad)}$$

$$u_{1,1} = (-k^2 f + ikg')e^{(\quad)} \qquad u_{1,2} = (ikf' + g'')e^{(\quad)} = (ikf' - b^2 g)e^{(\quad)}$$

$$u_{2,1} = (ikf' + k^2 g)e^{(\quad)} \qquad u_{2,2} = (f'' - ikg')e^{(\quad)} = (-a^2 f - ikg')e^{(\quad)}$$

since $\sigma_{22} = \sigma_{21} = 0$ on bdy

$$\sigma_{22} = (\lambda + 2\mu)u_{2,2} + \lambda u_{1,1} = \left[(\lambda + 2\mu)(-a^2 f - ikg') + \lambda(-k^2 f + ikg')\right] e^{(\quad)}$$

$$\sigma_{21} = \mu(u_{2,1} + u_{1,2}) = \boxed{\mu\left[2ikf' + (k^2 - b^2)g\right] e^{(\quad)} = \sigma_{21}}$$

$$\sigma_{22} = \left[f(-a^2\lambda - 2\mu a^2 - k^2\lambda) + ikg'(\lambda - \lambda - 2\mu)\right] e^{(\quad)}$$

$$\left[f\{-\lambda(a^2 + k^2) - 2\mu a^2\} - ikg'(2\mu)\right] e^{(\quad)}$$

but $\frac{\lambda}{\mu} = \frac{\lambda/\rho}{\mu/\rho} = \frac{c_1^2 - 2c_2^2}{c_2^2} = \frac{c_1^2}{c_2^2} - 2$; $\frac{c^2}{c_2^2} = b^2 + k^2$ $\quad \frac{c^2}{c_1^2} = a^2 + k^2$

$$\therefore \frac{c_1^2}{c_2^2} = \left(\frac{c}{c_2}\right)^2 \Big/ \left(c/c_1\right)^2 = \frac{b^2 + k^2}{a^2 + k^2} \quad \text{and} \quad \frac{c_1^2}{c_2^2} - 2 = \frac{b^2 + k^2 - 2a^2 - 2k^2}{a^2 + k^2}$$

$$\frac{\lambda}{\mu} = \frac{b^2 - k^2 - 2a^2}{a^2 + k^2}$$

$$\therefore -\lambda(a^2 + k^2) - 2\mu a^2 = -\frac{\lambda}{\mu}(a^2 + k^2)\mu - 2\mu a^2$$

$$= (-b^2 + k^2 + 2a^2)\mu - 2\mu a^2 = -(b^2 - k^2)\mu$$

$$\therefore \sigma_{22} = \left[-(b^2 - k^2)\mu f - ikg'(2\mu)\right] e^{(\quad)} = -\mu\left[(b^2 - k^2)f + 2ikg'\right] e^{(}$$

$$\boxed{\sigma_{22} = -\mu\left[(b^2 - k^2)f + 2ikg'\right] e^{(\quad)}}$$

Now since $\sigma_{21}(@\ x_2 = h) = 0$

$$f' = a\left[-B_1 \sin ax_2 + B_2 \cos ax_2\right] \qquad g'(x_2) = b\left[-B_3 \sin bx_2 + B_4 \cos bx_2\right]$$

since we have σ_{21} + $\sigma_{22} = 0$ on bdy we can drop $+\mu e^{(\quad)}$ from both eq.

@ $x_2 = h$ $\quad \therefore \sigma_{21} = \left[2ika(-B_1 \sin ah + B_2 \cos ah) + (k^2 - b^2)(B_3 \cos bh + B_4 \sin bh)\right] = 0$

$x_2 = -h$ $\quad = \left[2ika(B_1 \sin ah + B_2 \cos ah) + (k^2 - b^2)(B_3 \cos bh - B_4 \sin bh)\right] = 0$

$x_2 = h$ $\quad \sigma_{22} = \left[(b^2 - k^2)(B_1 \cos ah + B_2 \sin ah) + 2ikb(-B_3 \sin bh + B_4 \cos bh)\right] = 0$

$x_2 = -h$ $\quad = \left[(b^2 - k^2)(B_1 \cos ah - B_2 \sin ah) + 2ikb(B_3 \sin bh + B_4 \cos bh)\right] = 0$

Thus
$$\begin{bmatrix} -2ika\sin ah & 2ika\cos ah & (k^2-b^2)\cos bh & (k^2-b^2)\sin bh \\ 2ika\sin ah & 2ika\cos ah & (k^2-b^2)\cos bh & -(k^2-b^2)\sin bh \\ (b^2-k^2)\cos ah & (b^2-k^2)\sin ah & -2ikb\sin bh & 2ikb\cos bh \\ (b^2-k^2)\cos ah & -(b^2-k^2)\sin ah & 2ikb\sin bh & 2ikb\cos bh \end{bmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \underline{0}$$

since b, a are fns of ω & k this is a determinant that implicitly determines ω as a fn of k.

4
$$\begin{bmatrix} -2ika\sin ah & 2ika\cos ah & (k^2-b^2)\cos bh & (k^2-b^2)\sin bh \\ 0 & 2ika\cos ah & (k^2-b^2)\cos bh & 0 \\ (b^2-k^2)\cos ah & (b^2-k^2)\sin ah & -2ikb\sin bh & 2ikb\cos bh \\ 1(b^2-k^2)\cos ah & 0 & 0 & 2ikb\cos bh \end{bmatrix} \underset{\sim}{B} = \underline{0}$$

4
$$\begin{bmatrix} -2ika\sin ah & 0 & 0 & (k^2-b^2)\sin bh \\ 0 & 2ika\cos ah & (k^2-b^2)\cos bh & 0 \\ 0 & (b^2-k^2)\sin ah & -2ikb\sin bh & 0 \\ (b^2-k^2)\cos ah & 0 & 0 & 2ikb\cos bh \end{bmatrix} \underset{\sim}{B} = \underline{0}$$

↑ mult ① by $\dfrac{(b^2-k^2)}{2ika}\,\dfrac{1}{\tan ah}$ & add to ④ mult ② by $\dfrac{-(b^2-k^2)\sin ah}{2ika\cos ah}$ & add to ③

$$\left[\begin{array}{cc|cc} -2ika\sin ah & 0 & 0 & (k^2-b^2)\sin bh \\ 0 & 2ika\cos ah & 0 & 0 \\ 0 & 0 & (k^2-b^2)\cos bh & 0 \\ 0 & 0 & -2ikb\sin bh + Y & 2ikb\cos bh + X \end{array}\right]$$

$$Y = \frac{(k^2-b^2)^2\cos bh\sin ah}{2ika\cos ah} \qquad X = \frac{-(k^2-b^2)^2}{2ika\tan ah}\sin bh$$

$$\begin{matrix} 0 & \times & 0 \\ 0 & 0 & \times \\ 0 & 0 & \times \end{matrix}$$

This det in form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det A\cdot\det C.$

$$\det A = 4k^2a^2\sin ah\cos ah$$

$$\det K = (k^2-b^2)\cos bh\left\{(2ikb\cos bh) + \frac{i(k^2-b^2)^2}{2ka}\frac{\sin bh}{\tan ah}\right\}$$

$$\text{thus } \det C = \frac{i(k^2-b^2)\cos^2 bh}{2ka(k^2-b^2)^2}\left\{4k^2ab\tan ah + (k^2-b^2)^2\tan bh\right\} \quad 0$$

$$\therefore\ 4k^2ab\tan ah = -(k^2-b^2)^2\tan bh$$

This ~~are~~ is the dispersion relation for the symmetric modes.

$$\text{or } \left(\frac{\tan ah}{\tan bh} = -\frac{(k^2-b^2)^2}{4k^2ab} \quad \text{or} \quad \frac{\tan bh}{\tan ah} = -\frac{4k^2ab}{(k^2-b^2)^2}\right.$$

or

$$4k^2a^2\sin ah\cos ah\cdot i(k^2-b^2$$

$A_1\cos + A_3 i\sin + A_2\cos + iA_4\sin$

$(A_1+A_2)\cos + (A_3+A_4)i\sin$

since $A_1+A_2=B_1$; $(A_1-A_2)i=B_2$; $A_3+A_4=B_3$; $(A_3-A_4)i=B_4$

then $[\text{matrix}](\underset{\sim}{B}) = \left(\begin{bmatrix} a_{11} & a_{11} & a_{13} & a_{13} \\ a_{21} & a_{21} & a_{23} & | \\ a_{41} & a_{41} & a_{43} & a_{43} \end{bmatrix} + i\begin{bmatrix} a_{12} & -a_{12} & a_{14} & -a_{14} \\ a_{22} & | & | & | \\ a_{42} & -a_{42} & a_{44} & -a_{44} \end{bmatrix}\right)(\underset{\sim}{A}) = \begin{bmatrix} ia_{12} & a_{11} & ia_{14} & a_{13} \\ | & | & | & | \\ ia_{42} & a_{41} & ia_{44} & a_{43} \end{bmatrix}(\underset{\sim}{A})$

~~to get 2nd solution & eliminate upper left & lower right~~

$$\begin{bmatrix} -2ika\sin ah & 2ika\cos ah & (k^2-b^2)\cos bh & (k^2-b^2)\sin bh \\ 2ika\sin ah & 2ika\cos ah & (k^2-b^2)\cos bh & -(k^2-b^2)\sin bh \\ (b^2-k^2)\cos ah & (b^2-k^2)\sin ah & -2ikb\sin bh & 2ikb\cos bh \\ (b^2-k^2)\cos ah & -(b^2-k^2)\sin ah & 2ikb\sin bh & 2ikb\cos bh \end{bmatrix}$$

thus $\begin{bmatrix} a_{12}i & a_{11} & a_{14}i & a_{13} \\ | & | & | & | \\ | & | & | & | \\ i & | & i & | \end{bmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$

$$\begin{bmatrix} 2ika\cos ah & -2ika\sin ah & (k^2-b^2)\sin bh & (k^2-b^2)\cos bh \\ 2ika\cos ah & 2ika\sin ah & -(k^2-b^2)\sin bh & (k^2-b^2)\cos bh \\ (b^2-k^2)\sin ah & (b^2-k^2)\cos ah & 2ikb\cos bh & -2ikb\sin bh \\ -(b^2-k^2)\sin ah & (b^2-k^2)\cos ah & 2ikb\cos bh & 2ikb\sin bh \end{bmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \underset{\sim}{0}$$

$$-4\begin{bmatrix} 2ika\cos ah & -2ika\sin ah & (k^2-b^2)\sin bh & (k^2-b^2)\cos bh \\ 2\cancel{X}ika\cos ah & 0 & 0 & \cancel{X}(k^2-b^2)\cos bh \\ (b^2-k^2)\sin ah & (b^2-k^2)\cos ah & 2ikb\cos bh & -2ikb\sin bh \\ 0 & 2(b^2-k^2)\cos ah & 2ikb\cos bh & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2ika\sin ah & (k^2-b^2)\sin bh & 0 \\ 2ika\cos ah & 0 & 0 & (k^2-b^2)\cos bh \\ (b^2-k^2)\sin ah & 0 & 0 & -2ikb\sin bh \\ 0 & (b^2-k^2)\cos ah & 2ikb\cos bh & 0 \end{bmatrix}$$

$-\dfrac{(b^2-k^2)\sin ah}{2ika\cos ah}$

$$\begin{bmatrix} 0 & -2ika\sin ah & (k^2-b^2)\sin bh & 0 \\ 2ika\cos ah & 0 & 0 & (k^2-b^2)\cos bh \\ 0 & 0 & 0 & -2ikb\sin bh + \dfrac{(k^2-b^2)^2\cos bh\sin ah}{2ika\cos ah} \\ 0 & (b^2-k^2)\cos ah & 2ikb\cos bh & 0 \end{bmatrix}$$

$-1\cdot-1$

$\dfrac{(b^2-k^2)\cos ah}{2ika\sin ah}$

$$\left[\begin{array}{cc|cc} 2ika\cos ah & 0 & 0 & (k^2-b^2)\cos bh \\ 0 & -2ika\sin ah & (k^2-b^2)\sin bh & 0 \\ \hline 0 & 0 & 0 & -2ik\sin bh + [\quad] \\ 0 & 0 & 2ikb\cos bh - \dfrac{(k^2b^2)\sin bh\cos ah}{2ika\sin ah} & 0 \end{array}\right]$$

$$-1\left[\begin{array}{cc|cc} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ \hline & 0 & x & 0 \end{array}\right]$$

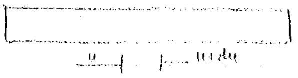
$$\therefore \quad \det A = 4k^2a^2 \cos ah \sin ah$$

$$\det B = \left[2ikb\cos bh + i\frac{(k^2-b^2)^2 \sin bh \cos ah}{2ka\sin ah} \right]\left[-2ikb\sin bh + i\frac{(k^2-b^2)^2 \cos bh \sin ah}{2ka\cos ah} \right]$$

$$\frac{\cos bh}{2ka}\left[4k^2ba + (k^2-b^2)^2 \frac{\tan bh}{\tan ah} \right]\left[\frac{\sin bh}{2ka} \right]\left[4k^2ab + (k^2-b^2)^2 \frac{\tan ah}{\tan bh} \right] = 0$$

$$\Rightarrow \frac{\tan bh}{\tan ah} = -\frac{4k^2ba}{(k^2-b^2)^2} \qquad \Rightarrow \frac{\tan bh}{\tan ah} = -\frac{(k^2-b^2)^2}{4k^2ab}$$

2.



$P \leftarrow \boxed{\ } \rightarrow f(x,t)dxA \rightarrow P+dP$ $\qquad A\sigma = E\varepsilon = AEu''dx$

$dP + f(x,t)d\cancel{x} = m\ddot{u}$ $\qquad F \propto \eta u$

$(AEu')'dx$

$\frac{\partial P}{\partial x}dx + f(x,t)dx = \rho A dx\,\ddot{u}$

$$f(x,t) = \eta u$$

$\therefore \quad E u_{,xx} - \rho u_{,tt} + \eta/_A u = 0$

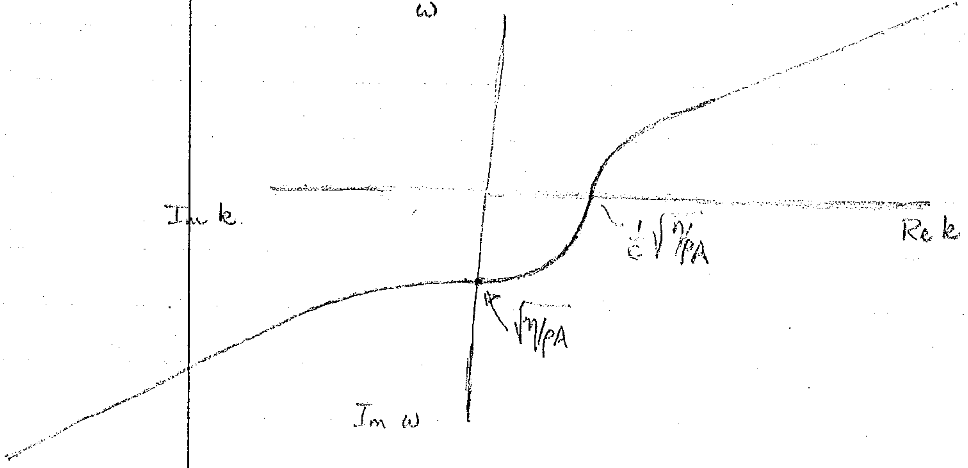
$\therefore \quad E/_\rho u_{,xx} - u_{,tt} + \eta/_{\rho A} u = 0$

if $W = e^{ikx}e^{i\omega t}$ $\qquad -E/_\rho k^2 + \omega^2 + \eta/_{\rho A} = 0$

$\therefore \quad \omega^2 = c^2k^2 - \eta/_{\rho A}$

$\omega = \sqrt{c^2k^2 - \eta/_{\rho A}}$ $\quad$ w/ $\quad k = \frac{1}{c}\sqrt{\eta/_{\rho A}}$ $\quad \omega = 0$ $\quad$ no longer harmonic

$\omega \sim ck$ $\quad$ for large k $\qquad \omega^2 \sim c^2k^2$

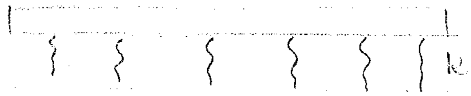


if k is 0 . $\quad c^2k^2$

$c^2k^2 - \eta/_{\rho A}$

$\omega = i\sqrt{c^2k^2 + \eta/_{\rho A}}$

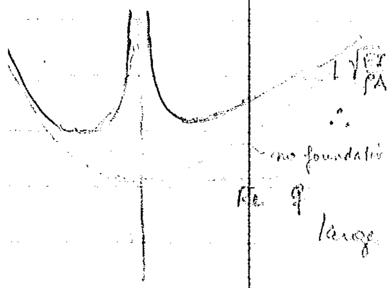
3.



$$EIw^{IV} + kw + \rho A\ddot{w} = 0$$ is DE

now if $w = e^{iqx}e^{i\omega t}$ $\quad EIq^4 + k - \rho A\omega^2 = 0$

$\therefore \quad EIq^2 + \frac{k}{q^2} - \rho A c^2 = 0$ where $\omega = \frac{c}{q}$



$\therefore \quad c = \sqrt{\frac{EI}{\rho A}q^2 + \frac{k}{\rho A q^2}}$

for very large q ⟺ small λ $\quad c \sim q\sqrt{\frac{EI}{\rho A}}$

for small q ⟺ large λ $\quad c \sim \frac{1}{q}\sqrt{\frac{k}{\rho A}}$

large λ ⟺ no x dependence $\Rightarrow$ $kw + \rho A\ddot{w} = 0$ or $\ddot{w} + \frac{k}{\rho A}w = 0$

$W = f(\cos\omega t, \sin\omega t)$

$\omega = \sqrt{\frac{k}{\rho A}}$ & $c = \frac{\omega}{q}$

$= \frac{1}{q}\sqrt{\frac{k}{\rho A}}$

$\frac{\cancel{2}EI}{\cancel{\rho A}}q - \frac{\cancel{2}k}{\rho A q^3} = 0$

$\therefore q^4 = \frac{k}{EI}$

$q = \sqrt{\frac{k}{EI}}$

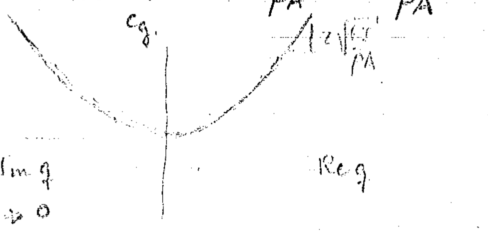
$q^2 = \frac{k}{EI}$

$\frac{EI}{\rho A}\cdot\frac{k}{\cancel{EI}} + \frac{\cancel{k}}{\rho A}\cdot\frac{EI}{\cancel{k}}$

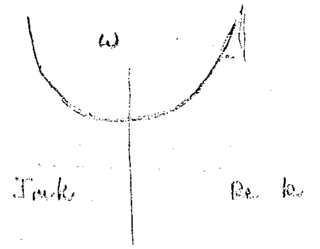
$\frac{k}{\rho A} + \frac{EI}{\rho A}$

now $\quad \omega^2 = \frac{EI}{\rho A}q^4 + \frac{k}{\rho A}$ $\qquad 2\omega\frac{d\omega}{dq} = \frac{4EI}{\rho A}q^3$ $\qquad \therefore \frac{d\omega}{dq} = \frac{\frac{2EI}{\rho A}q^3}{\sqrt{\frac{EI}{\rho A}q^4 + \frac{k}{\rho A}}}$

for large q $\quad \frac{d\omega}{dq} = c_g \approx \frac{\frac{2EI}{\rho A}q^3}{q^2\sqrt{\frac{EI}{\rho A}}} = 2\sqrt{\frac{EI}{\rho A}}\,q$



for $q \to 0$ $\quad c_g \sim \frac{\frac{2EI}{\rho A}q^3}{\sqrt{k/\rho A}} = \frac{2EI}{\sqrt{\rho A k}}q^3 \to 0$



$\omega = \sqrt{\frac{EI}{\rho A}q^4 + \frac{k}{\rho A}}$ $\qquad$ for large q $\quad \omega = q^2\sqrt{\frac{EI}{\rho A}}$

for small q $\quad \omega = \sqrt{k/\rho A}$

c vs k

for first case when we have large q the wavelength is so small the body appears infinite and the effect of what goes on the boundary. hence the velocity should be that of a beam freely oscillating, which it is. For small q, large λ the body acts as if it shows no dependence on x a just rigid body motion resisted by a spring. Hence if no x dependence, $k w + \rho A\ddot{w} = 0$ or w = fn of $\sin\omega t, \cos\omega t$ $\omega = \sqrt{k/\rho A}$ & $c = \frac{\omega}{q} = \frac{1}{q}\sqrt{k/\rho A}$
As in the freely vibrating case the fact that for large q, c depends on q is incorrect

cg vs k

for 2nd case we note that ~~the light~~ cg is the velocity with which energy ~~with~~ of cycle will be propagated hence for long wavelengths the velocity that carries their energy is very low and for the short wavelengths the velocity is a lot higher and hence large if not on elastic foundation

$\left(\frac{2h}{\pi C_2}\right)^2\omega^2-\left(\frac{2h}{\pi}\right)^2 k^2=n^2$

where ~~$\omega^2=$~~ $a=\frac{\pi c_2 n}{2h}$

$b=\pi/_2\frac{n}{h}$

ω vs. q.

This implies that if we suddenly apply a concentrated load, that it will be felt everywhere immediately since the concentrated load contains harmonic components with all orders of wavelength & for small λ → ∞ wave speed. This is physically unreasonable.

amounts of energy are carried at the higher wavenumbers (lower frequencies) Also from the results of our frequency-wave number plots we found that $\frac{d\omega}{dk}\rightarrow$ constant for large q which contradicts the results shown here for large q. If we look at the cone for the wave guide problem we obtained hyperbolas for ω versus q now $\frac{d\omega}{dq}=c_g$ are basically the slope of the asymptotes for large q and hence are virtually constant. But for ~~small~~ small q: the hyperbolas are of the form $\omega=\sqrt{\left(\frac{a}{b}q\right)^2+a^2}$ & for small q

$$\frac{d\omega}{dq}=\frac{\frac{2a^2}{b^2}q}{\sqrt{\left(\frac{a}{b}q\right)^2+a^2}}\sim \text{const}\cdot q \qquad \text{hence } c_g\rightarrow 0 \text{ as } q\rightarrow 0$$

$$2\,a/_{b^2}\,q=2\frac{\frac{\pi c_2}{2h}n}{\frac{\pi^2 n^2}{4h^2}}q=2\frac{c_2\cdot 2h}{\pi n}q$$

for large q $\omega\sim q^2$; hence we can look at an infinite body since λ is small clearly this is like having no foundation but this also leads to an incorrect result since for large q $\omega\sim q$ from the solution of the cone. For the small values of q the beam acts like a rigid body at the end of a spring & hence we find that ω = constant

1. First I will write $\Phi(x_1, x_2, t) = f(x_2)\, e^{i(kx_1 - \omega t)}$ and $\Psi(x_1, x_2, t) = g(x_2)\, e^{i(kx_1 - \omega t)}$

with $f(x_2) = B_1 \cos a x_2 + B_2 \sin a x_2$ and $g(x_2) = B_3 \cos a x_2 + B_4 \sin a x_2$

and finally $(A_1 + A_2) = B_1$, $(A_1 - A_2) i = B_2$, $(A_3 + A_4) = B_3$, $(A_3 - A_4) i = B_4$

Thus $\underline{u_1 = \phi_{,x_1} + \psi_{,x_2} = (ikf + g')\, e^{i(kx_1 - \omega t)}}$ and $\underline{u_2 = \phi_{,x_2} - \psi_{,x_1} = (f' - ikg) e^{i(kx_1 - \omega t)}}$

Placing this into $\sigma_{ij} = \lambda(u_{k,k})\delta_{ij} + \mu(u_{i,j} + u_{j,i})$, and noting that all we want are the σ_{12} and σ_{22} components for use as boundary conditions at $x_2 = \pm h$, we obtain the following:

$$\sigma_{22} = (\lambda + 2\mu) u_{2,2} + \lambda u_{1,1} = \left[(\lambda + 2\mu)(-a^2 f - ikg') + \lambda(-k^2 f + ikg') \right] e^{i(kx_1 - \omega t)}$$

$$\sigma_{21} = \mu(u_{2,1} + u_{1,2}) = \boxed{\mu\left[2ikf' + (k^2 - b^2) g \right] e^{i(kx_1 - \omega t)} = \sigma_{21}}$$

we've used the fact that $f'' + a^2 f = 0$; $g'' + b^2 g = 0$

we also use the fact that $-[\lambda(a^2 + k^2) + 2\mu a^2] = -(b^2 - k^2)\mu$ to rewrite σ_{22} as

$$\boxed{\sigma_{22} = -\mu\left[(b^2 - k^2) f + 2ikg' \right] e^{i(kx_1 - \omega t)}}$$

applying the b.c. at $x_2 = \pm h$ we obtain

$$\begin{bmatrix} -2ika \sin ah & 2ika \cos ah & (k^2 - b^2) \cos bh & (k^2 - b^2) \sin bh \\ 2ika \sin ah & 2ika \cos ah & (k^2 - b^2) \cos bh & -(k^2 - b^2) \sin bh \\ (b^2 - k^2) \cos ah & (b^2 - k^2) \sin ah & -2ikb \sin bh & 2ikb \cos bh \\ (b^2 - k^2) \cos ah & -(b^2 - k^2) \sin ah & 2ikb \sin bh & 2ikb \cos bh \end{bmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \underset{\sim}{0}$$

we now write this as $A\underset{\sim}{b} = 0$, where A is the matrix, and resubstitute the $b_i = B_i$ in terms of the A_i's, where $a_i = A_i$

Thus $[A]\underset{\sim}{b} = \left([\underset{\sim}{C}] + i[\underset{\sim}{D}] \right) \underset{\sim}{a} = 0$ $\quad \underset{\sim}{C} = \begin{bmatrix} a_{11} & a_{11} & a_{13} & a_{13} \\ \vdots & \vdots & \vdots & \vdots \\ a_{41} & a_{41} & a_{43} & a_{43} \end{bmatrix}$ $\quad \underset{\sim}{D} = \begin{bmatrix} a_{12} & -a_{12} & a_{14} & -a_{14} \\ \vdots & \vdots & \vdots & \vdots \\ a_{42} & -a_{42} & a_{44} & -a_{44} \end{bmatrix}$

by some matrix reduction techniques we can get this to

$$(\underset{\sim}{C} + i\underset{\sim}{D})\underset{\sim}{a} = \begin{bmatrix} a_{12} i & a_{11} & a_{14} i & a_{13} \\ \vdots & \vdots & \vdots & \vdots \\ a_{42} i & a_{41} & a_{44} i & a_{43} \end{bmatrix} \underset{\sim}{a} = E\underset{\sim}{a} = 0$$

Thus all we need to do is solve $\det E = 0$ for nontrivial solutions.

Note: If one solves the above determinant for $A\underset{\sim}{b} = 0$ & row reduce it to the form $\det\begin{bmatrix} M_1 & | & M_2 \\ M_3 & | & M_4 \end{bmatrix} = \det\begin{bmatrix} M_1' & | & 0 \\ 0 & | & M_4' \end{bmatrix} = 0$ we will get the symmetric mode. If we now reduce it to $\det\begin{bmatrix} 0 & | & M_2' \\ M_3' & | & 0 \end{bmatrix} = 0$ we will get the antisymmetric mode.

Thus

$$\begin{bmatrix} -2ka\cos ah & -2ika\sin ah & i(k^2-b^2)\sinh bh & (k^2-b^2)\cosh bh \\ -2ka\cos ah & 2ika\sin ah & -i(k^2-b^2)\sinh bh & (k^2-b^2)\cosh bh \\ i(b^2-k^2)\sin ah & (b^2-k^2)\cos ah & -2kb\cosh bh & -2ikb\sinh bh \\ -i(b^2-k^2)\sin ah & (b^2-k^2)\cos ah & -2kb\cosh bh & 2ikb\sinh bh \end{bmatrix} \underset{\sim}{A} = 0$$

for non trivial solns the determinant = 0 This matrix, since a & b are implicit fns of k and ω, is an implicit relation between k and ω, hence $\frac{d\omega}{dk}$ and k. Thus this is a dispersion relation.

by row reducing this matrix we can get the determinant in the final form.

$$\det \left[\begin{array}{cc|cc} 2ika\cos ah & 0 & 0 & 0 \\ 0 & -2ika\sin ah & 0 & 0 \\ \hline 0 & 0 & Y & 0 \\ 0 & 0 & 0 & X \end{array}\right] = \det \left|\begin{array}{c|c} S & 0 \\ \hline 0 & T \end{array}\right| = \det S \cdot \det T$$

where $Y = 2ikb\cosh bh - (k^2-b^2)^2 \frac{\sinh bh\cos ah}{2ika\sin ah}$ $\qquad X = -2ikb\sinh bh + (k^2-b^2)^2 \frac{\cosh bh\sin ah}{2ika\cos ah}$

or $\left[\frac{4k^2ba}{(k^2-b^2)^2} + \frac{\tanh bh}{\tan ah}\right]\left(\frac{i\cosh bh\,(k^2-b^2)^2}{2ka}\right) = Y$; $\left[\frac{\tanh bh}{\tan ah} + \frac{(k^2-b^2)^2}{4k^2ba}\right]\left(\frac{2kb\cosh bh\tan ah}{i}\right) = X$

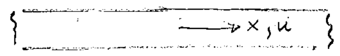
Thus the determinant can be written $\det S \cdot \det T = 0$

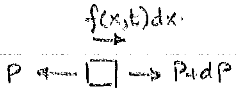
Thus $4k^2a^2\cos ah\sin ah\, XY = 0$ either $k=0$ or $ah = n\pi$ or $k = \pm b$ or $bh = \frac{n\pi}{2}$ (n odd) or $ah = m\pi$ (m even) or $Y = 0$ or $X = 0$; X and $Y = 0$ turn out to be the antisymmetric and symmetric modes respectively.

Thus $\frac{\tanh bh}{\tan ah} = -\frac{4k^2ba}{(k^2-b^2)^2}$ for symmetric modes ✓

$\frac{\tanh bh}{\tan ah} = -\frac{(k^2-b^2)^2}{4k^2ba}$ for antisymmetric modes. ✓

2.





and $f(x,t) = \overset{\swarrow \text{sign}}{\eta} u(x,t)$

thus $dP + f(x,t)dx = \rho A dx \ddot{u}$

$dP = \frac{\partial P}{\partial x} dx = (AEu')' dx$; we note that $f(x,t)$ is a force/length.

$$\therefore \quad \boxed{(AEu')' + f(x,t) - \rho A \ddot{u} = 0}$$

if A, E are not functions of x we can rewrite this as $\boxed{c^2 u'' \overset{-}{+} \frac{\eta}{\rho A} u - \ddot{u} = 0}$ $c^2 = E/\rho$

if $We^{ikx} e^{i\omega t} = u(x,t)$ then $-k^2 c^2 u'' + \frac{\eta}{\rho A} u + \omega^2 u = 0$

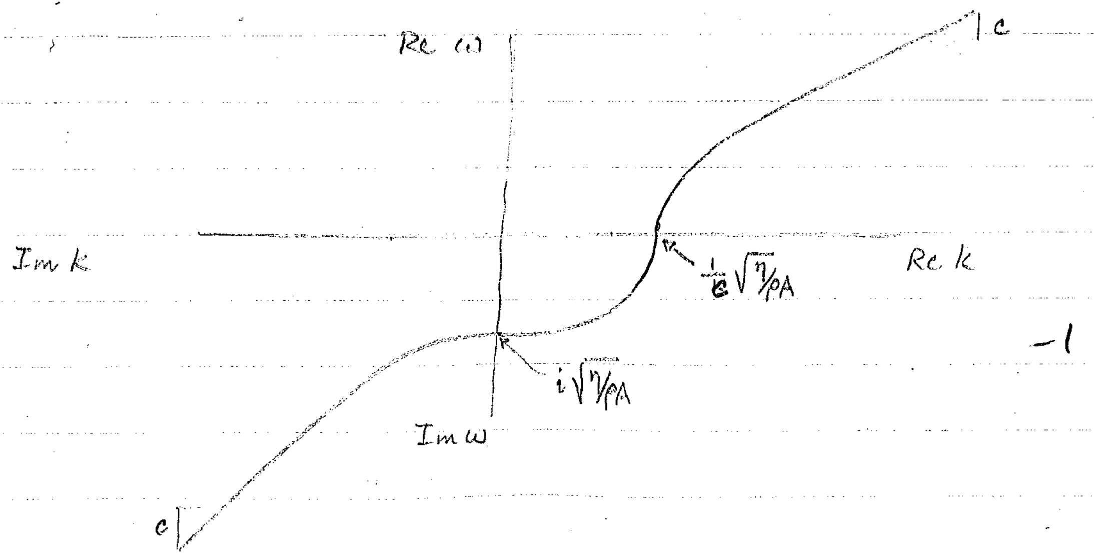
and our frequency eqn is

$$\omega^2 = k^2 c^2 \overset{+}{-} \eta/\rho A \quad \text{thus} \quad \omega = k\sqrt{c^2 - \eta/\rho A k^2}$$

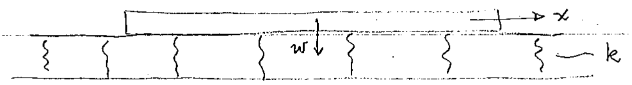
$\omega = 0 \Rightarrow k = \frac{1}{c}\sqrt{\eta/\rho A}$; as $k \to \infty$ $\omega \sim ck$; as $k \to 0$ $\omega \to i\sqrt{\eta/\rho A}$

if k is imaginary $\omega = ik\sqrt{c^2 + \eta/\rho A k^2}$ or $\omega = i\sqrt{c^2 k^2 + \eta/\rho A}$

$\omega = 0 \Rightarrow c = \frac{i}{k}\sqrt{\eta/\rho A}$; as $k \to \infty$ $\omega \sim ikc$; as $k \to 0$ $\omega \to i\sqrt{\eta/\rho A}$



3.



The DE is $EIw^{IV} + kw + \rho A\ddot{w} = 0$; assume a wave $w(x,t) = W e^{iqx} e^{i\omega t}$

$$\therefore \quad (EI q^4 + k - \rho A \omega^2)\, w = 0 \quad ; \quad \text{let } \frac{\omega}{q} = c \text{ then}$$

$$c^2 = \frac{EI}{\rho A} q^2 + \frac{k}{\rho A}\frac{1}{q^2} \quad \text{or} \quad c = \sqrt{\frac{EI}{\rho A} q^2 + \frac{k}{\rho A}\frac{1}{q^2}}$$

also $\omega = \left(\frac{EI}{\rho A} q^4 + \frac{k}{\rho A}\right)^{1/2}$

→ for very large q (small λ) $c \sim q\sqrt{\frac{EI}{\rho A}}$; for small q (large λ) $c \sim \frac{1}{q}\sqrt{\frac{k}{\rho A}}$

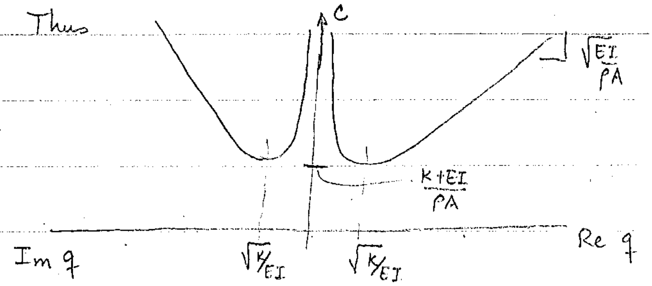
if we have imaginary q we get the same frequency equation and the same phase velocity equation and the same limiting values.

→ for very large q (small λ) $\omega \sim q^2\sqrt{\frac{EI}{\rho A}}$; for small q (large λ) $\omega \sim \sqrt{\frac{k}{\rho A}}$

To get the group velocity $c_g = \frac{d\omega}{dk} = \frac{4\frac{EI}{\rho A} q^3}{2\left(\frac{EI}{\rho A} q^4 + k/\rho A\right)^{1/2}} = \frac{2\frac{EI}{\rho A} q^3}{\left(\frac{EI}{\rho A} q^4 + k/\rho A\right)^{1/2}}$

→ for very large q (small λ) $c_g \sim 2\sqrt{\frac{EI}{\rho A}}\, q$; for small q (large λ) $c_g \sim \frac{2EI}{\sqrt{\rho A k}} q^3$

Thus



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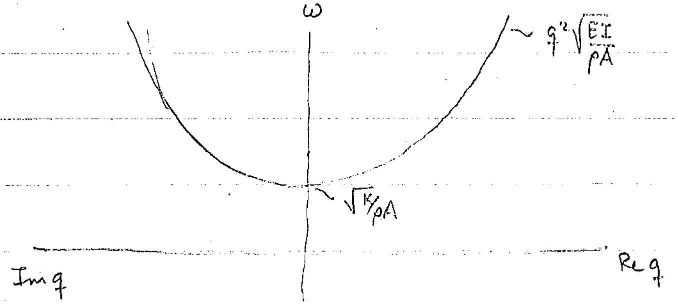

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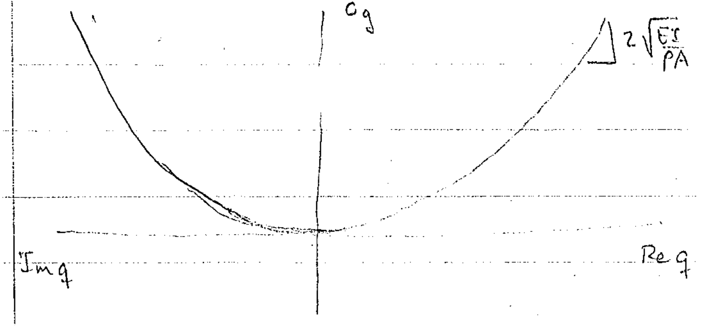


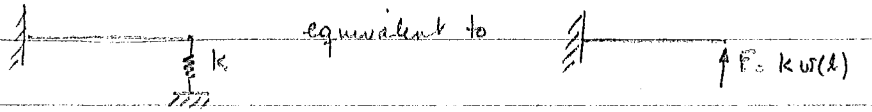
Discussion

c vs. q : When we have large q, the wavelength is so small the body appears to be infinite and the effect of the boundaries aren't felt; hence the velocity should be that of the infinite body ie $\sqrt{E/\rho} = c_1$ and $\sqrt{G/\rho} = c_2$. We note that this is not the results shown; hence the theory is invalid for large q. For small q, large λ, the body acts as if it is independent of x or just a rigid body oscillating at the end of a spring. Thus $w(x,t) = \bar{w}(t)$ which is a fn of $\cos\omega t$ & $\sin\omega t$ where $\omega = \sqrt{K/\rho A}$ and $\frac{\omega}{q} = c = \frac{1}{q}\sqrt{K/\rho A}$ and $c_g = \frac{d\omega}{dq} = 0$ Our theory is good here since we find that $c \sim \frac{1}{q}\sqrt{K/\rho A}$ as $q \to 0$.

c_g vs. q : we note that c_g is the velocity will which the energy of the system will be propagated; hence for long wavelengths the velocity that carries the energy is very low and for low wavelengths the velocity that carries the energy is very high; hence large amounts of energy are carried at the high wavenumbers (lower frequencies). This implies that if we suddenly apply a concentrated load, that it will be felt everywhere immediately, since the concentrated load has a fourier expansion containing harmonic components with all orders of wavelength. Thus for small λ (large q) we will have ∞ group speed; This is physically unreasonable. Also from the results of our frequency-wavenumber plots for the ∞ body (approximating our case when q is large) we found that $\frac{d\omega}{dq}$ tends to a constant for large q, which contradicts the results shown for large q. This is so because if we look at the cone for the waveguide problem, we obtained hyperbolas for ω versus q. Now $\frac{d\omega}{dq} = c_g$ are basically the slope of the asymptotes for large q and hence are virtually constant.

ω vs. q : for large q we found that $\omega \sim q^2$. This is an incorrect result since for large q (small λ) the body appears to be ∞. From the ω vs. q plot of the cone $\omega \sim q$ [Remember that $\Omega^2 = n^2 + \xi^2 \Rightarrow \omega = \frac{\pi c}{2h}\sqrt{\left(\frac{2h}{\pi}\right)^2 q^2 + n^2} \sim cq$]. For small values of q, the beam acts like a rigid body at the end of a spring & hence we find that $\omega = $ constant

4.



thus our DE is $EIw^{IV} + \rho A \ddot{w} = 0$ w/ BC $w(0) = w'(0) = w''(l) = 0$ and

$$-EIw''' = V = -F = -kw(l)$$

let $w(x,t) = \bar{w}(x)e^{i\omega t}$ then $\frac{EI}{\rho A}\bar{w}^{IV} - \omega^2\bar{w} = 0$

with BC / $\bar{w}(0) = \bar{w}'(0) = 0$ $\bar{w}''(l) = 0$ and $EI\bar{w}'''(l) = k\bar{w}(l)$

Let $\omega^2 \frac{\rho A}{EI} = \lambda^4$ then

$$\bar{w}(x) = C_1 \sin\lambda x + C_2\cos\lambda x + C_3\sinh\lambda x + C_4\cosh\lambda x$$

from $\bar{w}(0) = 0 \Rightarrow C_2 + C_4 = 0$

$\bar{w}'(0) = 0 \Rightarrow C_1 + C_3 = 0$ $\Big\}$ $\bar{w}(x) = C_1(\sin\lambda x - \sinh\lambda x) + C_2(\cos\lambda x - \cosh\lambda x)$

Applying the last two BC we get

$$\begin{bmatrix} \sin\lambda l + \sinh\lambda l & \cosh\lambda l + \cos\lambda l \\ -\lambda^3(\cos\lambda l + \cosh\lambda l) - \frac{k}{EI}(\sin\lambda l - \sinh\lambda l) & \lambda^3(\sin\lambda l - \sinh\lambda l) - \frac{k}{EI}(\cos\lambda l - \cosh\lambda l) \end{bmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

∴ det matrix = 0 ⟹

$$\boxed{\lambda^3(1 + \cosh\lambda l\cos\lambda l) + \frac{k}{EI}(\cosh\lambda l\sin\lambda l - \sinh\lambda l\cos\lambda l) = 0}$$ ✓

since λ is a fn of ω. this is the frequency equation

if $k \to 0$ then for nontrivial solutions $\cosh\lambda l\cos\lambda l = -1$. ✓ This is the free vibrations equation for a cantilever beam ✓

if $k \to \infty$ then for nontrivial solutions $\cosh\lambda l\sin\lambda l - \sinh\lambda l\cos\lambda l = 0$. This is the free vibrations equation for a simply supported - clamped beam.

for high frequencies, hence large λ, $\cosh\lambda l \sim \frac{e^{\lambda l}}{2}$ & $\sinh\lambda l \sim \frac{e^{\lambda l}}{2}$ then

our frequency equation $\sim \left[\lambda^3\cos\lambda l + \frac{k}{EI}(\sin\lambda l - \cos\lambda l)\right]\frac{e^{\lambda l}}{2} = 0$

or $\left(\lambda^3 - \frac{k}{EI}\right)\cos\lambda l + \frac{k}{EI}\sin\lambda l = 0 \Rightarrow \tan\lambda l = 1 - \frac{\lambda^3 EI}{k}$

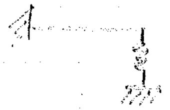
as $k \to \infty$ $\tan\lambda l \to 1$ ∴ $\lambda l \sim \frac{4n+1}{4}\pi$ n very large

or since $\lambda = \left(\frac{\omega^2}{c^2 r^2}\right)^{1/4}$ where r is the radius of gyration $\omega \sim \sqrt{\frac{EI}{\rho A}}\left(\frac{4n+1}{4}\frac{\pi}{l}\right)^2$

as $k \to 0$ $\tan\lambda l \to -\infty$ ∴ $\lambda l \sim -\frac{2n+1}{2}\pi$ n very large thus

$$\omega \sim \sqrt{\frac{EI}{\rho A}}\left(\frac{2n+1}{2}\frac{\pi}{l}\right)^2$$

4.



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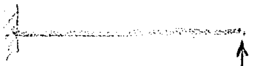


$w(0)=0$ $\qquad$ $kw(l)=V$ $\qquad$ $\therefore\ -EI\,w'''(l) = V = -k\,w(l)$

$w'(0)=0$ $\qquad$ $w''(l)=0$

$$\therefore\quad EI\,w^{IV} + \rho A\,\ddot{w} = 0 \qquad \text{or} \qquad \frac{EI}{\rho A}w^{IV} + \ddot{w} = 0$$

w/BC $\quad w(l)=0 \qquad EI\,w'''(l) = +k\,w(l)$

$\qquad w'(l)=0 \qquad w''(l)=0$

let $w(x,t) = \bar{w}(x)e^{i\omega t}$ then $EI\,\bar{w}^{IV} - \omega^2\bar{w} = 0$

$\therefore\quad \bar{w}(0)=0 \qquad EI\,\bar{w}'''(l) = \overset{\rho A}{k}\bar{w}(l)$

$\qquad \bar{w}'(0)=0 \qquad \bar{w}''(l)=0 \qquad$ let $\omega^2\frac{\rho A}{EI} = \lambda^4$

$$\therefore\quad \bar{w} = c_1\sin\lambda x + c_2\cos\lambda x + c_3\sinh\lambda x + c_4\cosh\lambda x$$

$$\left.\begin{array}{l}\bar{w}(0)=0 \Rightarrow c_2+c_4=0\\ \bar{w}'(0)=0 \Rightarrow c_1+c_3=0\end{array}\right\}\ \bar{w} = c_1(\sin\lambda x - \sinh\lambda x) + c_2(\cos\lambda x - \cosh\lambda x)$$

$$\bar{w}' = \lambda\left[c_1(\cos\lambda x - \cosh\lambda x) + c_2(-\sin\lambda x - \sinh\lambda x)\right]$$

$$\bar{w}'' = \lambda^2\left[c_1(-\sin\lambda x - \sinh\lambda x) + c_2(-\cos\lambda x - \cosh\lambda x)\right]$$

$$\bar{w}''' = \lambda^3\left[c_1(-\cos\lambda x - \cosh\lambda x) + c_2(\sin\lambda x - \sinh\lambda x)\right]$$

$\therefore$ from 3rd BC $\quad \lambda^3\left[-c_1(\cos\lambda l + \cosh\lambda l) + c_2(\sin\lambda l - \sinh\lambda l)\right] = \frac{k}{EI}\left[c_1(\sin\lambda l - \sinh\lambda l) + c_2(\cos\lambda l - \cosh\lambda l)\right]$

$c_1\left[-\lambda_3(\cos\lambda l + \cosh\lambda l) - \frac{k}{EI}(\sin\lambda l - \sinh\lambda l)\right] + c_2\left[\lambda^3(\sin\lambda l - \sinh\lambda l) - \frac{k}{EI}(\cos\lambda l - \cosh\lambda l)\right]$

$$\bar{w}''(l) = 0 = +c_1\left[\sin\lambda l + \sinh\lambda l\right] + c_2(\cosh\lambda l + \cos\lambda l)$$

$\therefore$

$$\begin{bmatrix}\sin\lambda l + \sinh\lambda l & \cosh\lambda l + \cos\lambda l\\ & \\ -\lambda^3(\cos\lambda l + \cosh\lambda l) - \frac{k}{EI}(\sin\lambda l - \sinh\lambda l) & \end{bmatrix}\begin{bmatrix}c_1\\ \\ c_2\end{bmatrix} = \underset{\sim}{0}$$

$$\downarrow$$
$$\lambda^3(\sin\lambda l - \sinh\lambda l) - \frac{k}{EI}(\cos\lambda l - \cosh\lambda l)$$

$$\lambda^3\left[(\sin\lambda l - \sinh\lambda l)(\sin\lambda l + \sinh\lambda l)\right] - \frac{k}{EI}(\cos\lambda l - \cosh\lambda l)(\sin\lambda l + \sinh\lambda l) + \lambda^3(\cos\lambda l + \cosh\lambda l)^2$$
$$+\frac{k}{EI}(\sin\lambda l - \sinh\lambda l)(\cosh\lambda l + \cos\lambda l) = 0$$

for non-trivial soln.

$$\lambda^3\left[\sin^2\lambda l - \sinh^2\lambda l\right] + \lambda^3\left[\cosh\lambda l + \cos\lambda l\right]^2 - \frac{k}{EI}\left[\cancel{cs} + cs_h - c_hs - \cancel{c_hs_h} - sc_h - \cancel{sc} + s_hc_h + s_hc\right]$$

$$\lambda^3\left[\sin^2\lambda l - \sinh^2\lambda l + \cosh^2\lambda l + 2\cosh\lambda l\cos\lambda l + \cos^2\lambda l\right] - \frac{k}{EI}\left[2cs_h - 2sc_h\right] = 0$$

$$\lambda^3(2 + 2\cosh\lambda l\cos\lambda l) + \frac{2k}{EI}\left[\cosh\lambda l\sin\lambda l - \sinh\lambda l\cos\lambda l\right] = 0$$

if $k \to 0$ then $\quad \cosh\lambda l\cos\lambda l = -1$ which is the free vib eq for a cantilever beam,

if $k \to \infty$ $\quad \cosh\lambda l\sin\lambda l - \sinh\lambda l\cos\lambda l = 0$ which is the free vib prob of a

Suppose we write $\frac{\omega^2}{c^2 r^2} = \lambda^4$ where r is the radius of gyration $\lambda = \left(\frac{\omega^2}{c^2 r^2}\right)^{1/4}$

since $\lambda \sim \omega^{1/2}$ for high frequencies $\cosh\lambda l \sim \frac{e^{\lambda l}}{2}$ $\sinh\lambda l \sim \frac{e^{\lambda l}}{2}$

$$\therefore \lambda^3\left(2 + \frac{2e^{\lambda l}}{2}\cos\lambda l\right) + \frac{2k}{EI}\frac{e^{\lambda l}}{2}\left[\sin\lambda l - \cos\lambda l\right] = 0$$

~~if~~ or $\sim \cos\lambda l\,(\lambda^3 - k/EI) + k/EI \sin\lambda l = 0$ or $\frac{-\lambda^3 + k/EI}{k/EI} = \tan\lambda l = 1 - \frac{\lambda^3 EI}{k}$

if $k \to 0$ $2\lambda^3 + e^{\lambda l}\cos\lambda l = 0$ or $\lambda l \sim \frac{n\pi}{2}$ n odd $\therefore \lambda^4 \sim \left(\frac{n\pi}{2l}\right)^4 = \frac{\omega^2}{c^2\rho^2}$

$\therefore \frac{EI}{\rho A}\left(\frac{n\pi}{2l}\right)^4 = \omega^2$ or $\omega \sim \sqrt{\frac{EI}{\rho A}}\left(\frac{n^2\pi^2}{4l^2}\right)$ high frequencies as $k \to 0$

for $k \to 0$ we have from free vib $\cosh\lambda l\cos\lambda l = -1$ ~~the following~~ ~~$2\frac{e^{\lambda l}}{2}\left[e^{i\lambda l} + e^{-i\lambda l}\right] = -$~~

or $\cos\lambda l = \frac{-1}{\cosh\lambda l} \to 0$ $\lambda l = \frac{n\pi}{2}$ n odd as shown above.

for $k \to \infty$ & large ω we have $\frac{e^{\lambda l}}{2}\left[\sin\lambda l - \cos\lambda l\right] = 0$ $\therefore \sin\lambda l = \cos\lambda l$

$\lambda l \sim \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots$ $\lambda l = \left(\frac{4n+1}{4}\right)\pi$ $\therefore \lambda \sim \frac{4n+1}{4l}\pi$ and $\lambda^4 \sim \left(\frac{4n+1}{4l}\pi\right)^4$

$\therefore \omega^2 \sim \frac{EI}{\rho A}\left(\frac{4n+1}{4}\frac{\pi}{l}\right)^4$ $\therefore \omega \sim \sqrt{\frac{EI}{\rho A}}\left(\frac{4n+1}{4}\pi/l\right)^2$

$\sim \frac{e^{\lambda l}}{2}\left[\lambda^3\cos\lambda l + \frac{k}{EI}\sin\lambda l - \frac{k}{EI}\cos\lambda l\right] = 0$

$-\cos\lambda l\left(\frac{k}{EI} - \lambda^3\right) + \frac{k}{EI}\sin\lambda l = 0 \Rightarrow$ $\tan\lambda l = 1 - \frac{\lambda^3 EI}{k}$

for large λ ~~multiples~~ and $k \to \infty$ $\tan\lambda l \to 1$

or λl must be ~~multiples~~ $\sim \frac{4n+1}{4}\pi$

for large λ & as $k \to 0$ $\tan\lambda l \to -\infty$ $\therefore \lambda l \sim -\frac{n\pi}{2}$ n odd

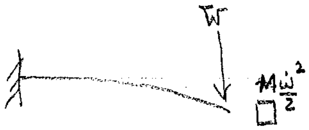
$\lambda^2 = \omega\Big/\sqrt{\frac{EI}{\rho A}} = \left(\frac{4n+1}{4}\frac{\pi}{l}\right)^2$

$\omega = \sqrt{\frac{EI}{\rho A}}\left(\frac{4n+1}{4}\frac{\pi}{l}\right)^2$

$\lambda^2 = \frac{n^2\pi^2}{2l^2} = \omega\Big/\sqrt{\frac{EI}{\rho A}}$

$\omega = \sqrt{\frac{EI}{\rho A}}\left(\frac{n\pi}{2l}\right)^2$ n odd

#.5



Equivalent problem

if we solve static deflection problem for force $\bar{W}$ on end w/ $EI\bar{w}^{IV}(x)=0$ and $\bar{w}(0)=\bar{w}'(0)=\bar{w}''(\ell)=0$ and $-EI\bar{w}'''(\ell)=\bar{W}$ we find that

$$\bar{w}(x) = \frac{\bar{W}\ell^3}{6EI}\left[3\left(\frac{x}{\ell}\right)^2-\left(\frac{x}{\ell}\right)^3\right]$$

$\therefore$ let $w(x,t) = \bar{w}(x)\sin\omega t$ then

$$KE = \int_0^\ell \frac{1}{2}\frac{m}{\ell}\dot{w}(x,t)^2dx + \frac{1}{2}M\dot{w}^2(\ell,t) \qquad PE = \int_0^\ell \frac{1}{2}EI\,w''(x,t)^2dx = \frac{1}{2}k_e\,w(\ell,t)^2$$

$m=\rho A\ell \qquad M=\bar{W}/g$

where k_e is an equivalent spring constant. since $\bar{w}(\ell)=\frac{\bar{W}\ell^3}{3EI}$ we can then say that $\bar{W}=k_e\bar{w}(\ell)$ $\therefore$ $k_e=\frac{3EI}{\ell^3}$

$$\therefore\ KE = \frac{1}{2}\frac{m}{\ell}\omega^2\cos^2\omega t\int_0^\ell\bar{w}(x)^2dx + \frac{1}{2}M\omega^2\cos^2\omega t\,\bar{w}(\ell)^2$$

$$= \frac{\omega^2\cos^2\omega t}{2}\left[\frac{m}{\ell}\int_0^\ell\bar{w}(x)^2dx + M\bar{w}(\ell)^2\right]$$

$$\int_0^\ell\bar{w}(x)^2dx = \ell\int_0^1\bar{w}(\bar{x})^2d\bar{x} \quad \text{where } \bar{x}=x/\ell \quad \therefore \int_0^\ell\bar{w}(x)^2dx = \ell\frac{\bar{W}^2\ell^6}{36EI^2}\left\{\frac{9}{5}\bar{x}^5-\bar{x}^6+\frac{\bar{x}^7}{7}\right\}_0^1$$

$$= \ell\frac{\bar{W}^2\ell^6}{36EI^2}\cdot\frac{33}{35}$$

or in terms of the $\bar{w}(\ell)$ $\quad\int_0^\ell\bar{w}^2(x)dx = \frac{33}{140}\ell\,\bar{w}(\ell)^2$

$$\therefore\ KE = \frac{\omega^2\cos^2\omega t}{2}\left[\frac{m\ell}{\ell}\cdot\frac{33}{140}\bar{w}(\ell)^2 + M\bar{w}(\ell)^2\right] = \frac{\omega^2\cos^2\omega t}{2}\left[M+\frac{33}{140}m\right]\bar{w}(\ell)^2$$

$$PE = \frac{1}{2}k_e\,w(\ell,t)^2 = \frac{1}{2}\left(\frac{3EI}{\ell^3}\right)\bar{w}(\ell)^2\sin^2\omega t$$

since $KE_{max}=PE_{max}$ $\Rightarrow$ $\frac{\omega^2}{2}\left[M+\frac{33}{140}m\right]\bar{w}(\ell)^2 = \frac{1}{2}\left(\frac{3EI}{\ell^3}\right)\bar{w}(\ell)^2$

thus $\quad\omega^2 = \frac{3EI}{\left(M+\frac{33}{140}m\right)\ell^3}\quad$ where $m=\rho A\ell$

when $M\to 0 \Rightarrow W=0$ hence $\omega^2=\frac{420}{33}\frac{EI}{m\ell^3}$ which is the result for the cantilever beam