ACCT NO P2.K18 EKN (secretary name)

3

ME 235 C NONLIN. ANAL. (MORE EXPOSITORY THAN DETAILED, STATE-OF-THE ART MANY HANDY OUTS - PAPERS

- FINISH LIN. DYN. ANAL. { SURVEY OF INTEGRATORS IN STR. DYNAMICS; EIGENVALUES & MASS MATRICES; MATCHED METHODS }

  HW

- FORMULATION OF NONLINEAR PROBLEMS { STATIC; DYNAMIC }
  - NONLIN. HEAT.
  - " ELAST.

- SOLUTION METHODS FOR NONLIN. ALGEBRAIC SYSTEMS,
  - RESIDUAL METHODS
  - FIX-POINT
  - NEWTON-RAPHSON (TANGENT)
  - QUASI-NEWTON (SECANT)

- ~~FORMULATION OF~~

- RESULTS ON NONLIN. TRAN. ANAL. FROM NOTES (NONLIN. SYMM. SYS.)

- FLUID MECHANICS : - ADVECTION - DIFFUSION EQ$.
  - INC. NAVIER - STOKES EQ.
  - GALERKIN & UPWIND PHILOSOPHIES COMPARED.
  - RESULTS FROM NOTES ON NONSYMM. OP'S.
  - EULER - LAGRANGE MESHES.

- OTHER TOPICS IN SOLID MECHANICS ; PLASTICITY, VISCOPLASTICITY ; FINITE DEFORMATIONS (LAGRANGIAN DESCRIPTION) & SOLID & SHELLS.
- NEW ALGORITHMS
- OTHER TOPICS

T. J. R. Hughes 283 Durand x7-2040
office hrs. by appt.

Midterm In Class 1 hr. 25%
Final perhaps take-home 3hr. 75%
Required homework also handed out.

T.A.: Isaac Levit 459A Durand x7-2189 office hrs MWF 2-3

Course Text: "Analysis of Transient Algorithms with Particular Reference to Stability Behavior" to be handed out later
60 pgs covered in 235B, 90 pgs in 235C.

<u>ME235 C will cover:</u>

Continued survey of Integrators in Dynamic problems
Eigenvalue Packages, Mass Matrix development

Non-Linear Heat, Elasticity (Weak Forms, Equivalence)

Solution Techniques for non-linear algebraic systems
- residual methods
- fixed point methods
- Newton-Raphson (tangent) methods
- Quasi-Newton (secant) methods

Comparison of Symmetric and non-Symmetric problems

Fluid Mechanics Problems - Techniques, Stability, Meshes.

Finite Deformations, shells.

<u>ME235 A+B have covered:</u> (and if you believe this, I have a bridge for you.)

Fundamental Concepts in 1-D (LM, Beam Bending)
Formulating 2-D and 3-D (LaPlace eqn., ID, IEN, LM, Elasticity)
Isoparametric elements, element arrays, coding, quadrature rules
Mixed methods, Selective reduced integration, Incompatible Modes,
Penalty Methods, constraint counting, Irons patch test.

ME235C
9-30-81 #2

Solution Techniques + Features in 235 A, B

Linear 1st Order - semi-discrete Heat Equations
Trapezoidal Method
Convergence
Reduction to SDOF model problem
Stability - Consistence, - Convergence

Symmetric 2nd Order - semi-discrete Equations of Motion
Newmark Family
Analysis - SDOF - Matrix norms, Spectral Stability
Oscillatory Response, Viscous Damping,
Numerical Dissipation, Dispersion
Time Step estimates for conditionally stable algorithms

Linear Multi-Step (LMS) methods -
Reduction of Heat + Dynamics into 1st Order systems
Stability concepts

Solution Techniques + Features in 235C

Stiffly-stable Methods (Heat Conduction)
Park's method
Routh-Hurwitz Criterion

LMS on 2nd Order Equations
Houbolt method
Collocation schemes (Unconditionally Stable, Implicit)
α - Methods "

Stability via Energy Methods
Newmark, Predictor-Corrector, Implicit-Explicit Algorithms

Convergence

Non-linear - Elasticity Model Problem
linearized Energy analysis for stability
Energy-conserving Algorithm formulation
Heat Model Problem

Non-Symmetric Operators
von Neumann Method (finite difference based)
stability concerns

10/2/81

Review of ME 235 B for benefit of students on network

Office HRS 2:00 - 3:00 PM MWF Itzhak Levit Durand 459 7-2189.

Starting with $\underset{\sim}{M}\dot{\underset{\sim}{d}} + \underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F}$ $\underset{\sim}{d}(0) = \underset{\sim}{D}_0$ $\underset{\sim}{M} > 0$ $\underset{\sim}{K} \geq 0$

we use modal decomposition (source term) to get a time discrete form dependent on a parameter α

This parameter determines the characteristics of the trapezoidal rule.

We then did the same thing for $\underset{\sim}{M}\ddot{\underset{\sim}{d}} + \underset{\sim}{C}\dot{\underset{\sim}{d}} + \underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F}$ $\dot{\underset{\sim}{d}}(0) = \dot{\underset{\sim}{D}}_0$ $\underset{\sim}{d}(0) = \underset{\sim}{D}_0$

- this will lead by modal decomp (if $\underset{\sim}{C} = a\underset{\sim}{M} + b\underset{\sim}{K}$) to the time discrete modes
- on parameters γ & β where some value of γ, β correspond to the trapez. rule

ME 235C
10-5-81 #2

$\Omega_{crit} < (\gamma/2 - \beta)^{-1/2}$ ← expression with $\xi = 0$

Oscillatory Response Conditions – Spectral Stability + Complex Conjugate λ's

$\beta = 1$ $\gamma = 3/2$ backward difference $\rho(A^{\infty}) \to 0$ static solution.

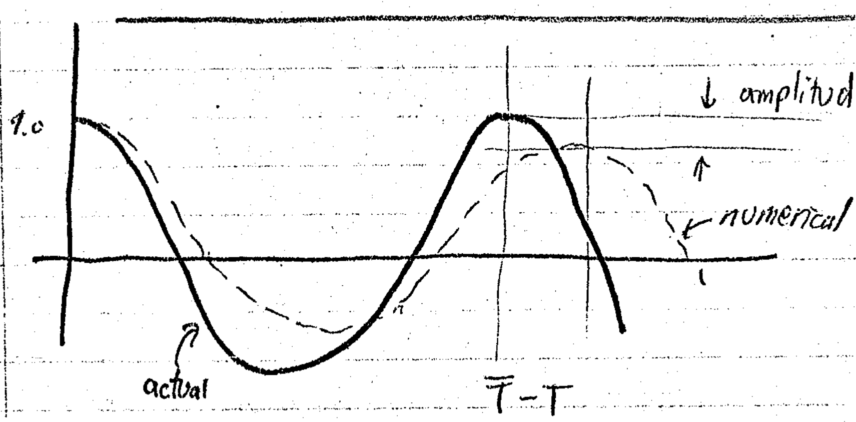
Measures of Dispersion, Dissipation

Time Step Estimates for popular elements.

LMS on 1st Order Systems – Mathematical Underpinnings

k-step methods.

10-7-81 #1

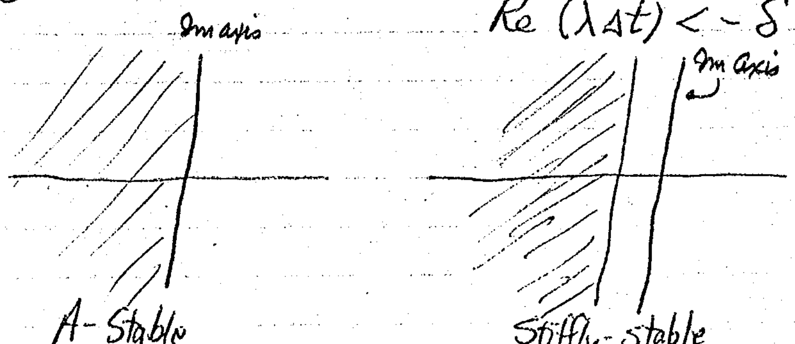


$2\pi\bar{\xi} = \bar{\delta}$

DAHLQUIST THEOREM Review

1) No A-Stable, implicit LMS
2) No 3rd Order A-stable LMS
3) Trapezoidal rule has smallest algorithmic error constant
   2-nd order, A-Stable, LMS.

Stiffly Stable Methods: Absolute Stability for $\lambda\Delta t$ s.t. $Re(\lambda\Delta t) < -\delta$



GEAR - developed K-step, K^{th} Order accurate LMS methods which are stiffly stable.

NOT APPLICABLE TO STRUCTURES PROBLEMS

very accurate for heat problems ($\lambda\Delta t$ pure real)

10-7-82 #2

Jensen - Application to Structures, possible.

PARK's METHOD linear combination of Gear's 2 + 3 step algorithms.
A-Stable (3 step)

Stiff A-Stability - A-Stable and $\frac{z_{n+1}}{z_n} \to 0$ as $Re(\lambda \Delta t) \to -\infty$

$\rho(\underset{\sim}{A}) \to 0$ as $\frac{\Delta t}{T} \to \infty$

annihilate high modes

examples: GEAR 1^{st} + 2^{nd}
PARK's

Routh-Hurwitz Criterion - useful for rank 3 or more stability problems.

LMS of 2^{nd} Order Type

$$\ddot{\mathbb{y}} = \mathbb{f}(\mathbb{y}, \dot{\mathbb{y}}, t) = \mathbb{G}_0 \mathbb{y} + \mathbb{G}_1 \dot{\mathbb{y}} + \mathbb{H}(t)$$

in dynamics: $\mathbb{y} = \mathbb{d}$, $\mathbb{G}_0 = -\mathbb{M}^{-1}\mathbb{K}$ $\mathbb{G}_1 = -\mathbb{M}^{-1}\mathbb{C}$ $\mathbb{H}(t) = \mathbb{M}^{-1}\mathbb{F}$

Geradin type (displacement difference methods)

$$\sum_{i=0}^{k} \{ \alpha_i \mathbb{y}_{n+1-i} + \Delta t \beta_i \mathbb{G}_1 \mathbb{y}_{n+1-i} + \Delta t^2 \gamma_i [\mathbb{G}_0 \mathbb{y}_{n+1-i} + H(t_{n+1-i})] \} = \underset{\sim}{0}$$

explicit for $\beta_0, \gamma_0 = 0$, backward diff $\beta_i = \gamma_i = 0$ for $i \geq 1$.
<u>all</u> common structural dynamics methods subsumed under this category.
examples - Newmark's in Zienkiewicz's book.
Parks no longer 3-step (>3)

Stability - reduction via modal analysis

$$\sum_{i=0}^{k} (\alpha_i + 2\xi\omega\Delta t \beta_i + (\omega\Delta t)^2 \gamma_i) \zeta^{n+1-i} = 0$$

pick $y_n = \zeta^n y_0$

$|\zeta| \leq 1$, multiple eigenvalues $|\zeta| < 1$

Krieg proved analog of Part 1 of Dahlquist's Theorem.
other 2 parts have been commonly assumed

Principal Roots vs. Spurious Roots
2 for 2^{nd} order equations — $n-2$ for 2^{nd} order equations
$\to 1$ as $\Omega = \omega\Delta t \to \infty$ — $\to 0$ as $\Omega \to \infty$
ACCURACY DETERMINERS

10-7-82#3 HOUBOLT'S METHOD p. 72 LMS 2nd Order type
Backward Difference Method, 2nd-Order accurate.
if $C=0$

affects low modes
high damping & period errors
Strong Numerical Damping. Stiffly A-Stable. **Historical importance only.**

COLLOCATION SCHEMES - p.73 3-step method 2nd Order type
damps low modes
Amplification Matrix Rank = 3.
$\theta = 1 \rightarrow$ Newmark ; $\beta = 1/6$, $\gamma = 1/2$ $\rightarrow$ generalized linear acceleration
Wilson θ method.

require $\gamma = 1/2$ for 2nd Order Accuracy.
if $\beta = \frac{1}{12} - \frac{\theta(\theta-1)}{2}$ 3rd Order Accurate, conditional stability.

if $\gamma = 1/2$ $\theta \geq 1$ $\frac{\theta}{2(\theta+1)} \geq \beta \geq \frac{2\theta^2-1}{4(2\theta^3-1)}$ for unconditionally stable algorithms.

10-9-81 #1.
Wilson method very sensitive to choice of θ
discuss Fig 13., Fig. 14
Know modes which you want to clobber, pick Δt appropriately.
Elongates period, note Houbolt method loses.

α-METHOD Amplification Matrix Rank 3 - 3-step LMS, 2nd Order type 2nd Order Acc.
A-Stable
Newmark Formulas $\begin{cases} d_{n+1} = \cdots & (\beta) \\ v_{n+1} = \cdots & (\gamma) \end{cases}$

$$\mathbb{M}\, a_{n+1} + (1+\alpha)\, \mathbb{C}\, v_{n+1} - \alpha\, \mathbb{C}\, v_n + (1+\alpha)\, \mathbb{K}\, d_{n+1} - \alpha\, \mathbb{K}\, d_n = (1+\alpha)\, \mathbb{F}_{n+1} - \alpha\, \mathbb{F}_n$$

"Internal Forces"

with $\alpha = 0$ have Newmark

with $\alpha \in [-1/3, 0]$, $\gamma = \frac{(1-2\alpha)}{2}$, $\beta = \frac{(1-\alpha)^2}{4}$ have canonical α-methods.

$\alpha = 0$ Trapezoidal
as α decreased to $-1/3$, numerical dissipation increased.

discussion of figures 16, 17, picking up damping at high modes only

Fig 20:
$\frac{\bar{T}-T}{T}$

10-9-81 #2

COMPARISON OF METHODS

ρ Fig 18 — damping ratio Fig 19

disastrous — Houbolt — heavy damping

Reasonable { Park — good in low modes (period) error max. high frequency dissipation — real star

Optimal Collocation

α — slightly more efficient than opt. coll. ⟶

max. acc. — Newmark (Trapezoidal) — conservative

OVERSHOOT - (not covered in notes)

$$\rho(\underset{\sim}{A}) \leq 1 \implies \| \underset{\sim}{A}^n \| \leq \text{constant}$$

But $\|\underset{\sim}{A}\| \gg 1$. can cause overshoot

H3 Analyzed this, and Collocation methods go screwy.

can mess up non-linear problems.

SPECIAL STARTING PROCEDURES

Given $\underset{\sim}{d}_o, v_o$

Houbolt, Park — needs 3 steps previous to start, somewhat messy.

also storage requirements troublesome.

Newmark's and Trapezoidal most used.

Non-linear codes coming out now have α-method.

10/9/81

Last time

Stiffly Stable Methods

Park's A-Stable method   linear combin of Gear's 2 step + 3 step

LMS methods for 2nd Order Systems.

Houbolt's method

Collocation Methods (Wilson θ scheme)

See figure 12-13-14 for discussion of Houbolt vs. Collocation

α-method   spin off from Newmark $\alpha \in [-1/3, 0]$ 2nd order accurate ↙ Parameters $\gamma = (1-2\alpha)/2$ $\beta = \frac{(1-\alpha)^2}{4}$

i.e. uses Newmark formulas $\left\{ \begin{array}{l} d_{n+1} = \ldots \beta \\ v_{n+1} = \ldots \gamma \end{array} \right.$ [involves Rank 3; 3 step LMS method of 2nd order type unconditionally stable]

+ the dynamic eqn. $\underset{\sim}{M}\,\underset{\sim}{a}_{n+1} + \left[(1+\alpha)\,\underset{\sim}{C}\,\underset{\sim}{v}_{n+1} - \alpha\,\underset{\sim}{C}\,\underset{\sim}{v}_n + (1+\alpha)\,\underset{\sim}{K}\,\underset{\sim}{d}_{n+1} - \alpha\,K\,\underset{\sim}{d}_n\right]$

$$= (1+\alpha)\underset{\sim}{F}_{n+1} - \alpha\,\underset{\sim}{F}_n$$

use these to help in dissipation : if α=0 then we get Newmark.

if α=0 trapezoidal rule   if α↓ dissipation↑

Comparison of Methods

period error   Houbolt - highly damps low modes] require special starting procedures, normally trapezoidal

period error   Park - kills high modes

Optimal Collocation

α - method

Newmark

Overshoot :   Reference H3

Stability is defined by $\rho(\hat{A}) \leq 1 \Rightarrow \|\hat{A}^n\| \leq$ constant.

But $\|\hat{A}\| >> 1$ (ie Wilson's method) causes "overshoot"

E.G. Collocation Methods $\theta >> 1$ these cause

Pathological overshoot occurs from these dissipation problems

10/12/81

10/12/81 Implicit Explicit procedures.

Newmark (implicit - anyone ie begin w/ any implicit form ie newmark)

$$M a_{n+1} + C v_{n+1} + K d_{n+1} = F_{n+1} \quad (1)$$

$$d_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2}(1-2\beta) a_n + \beta \Delta t\, a_{n+1} \quad (2)$$

$$v_{n+1} = v_n + \Delta t (1-\gamma) a_n + \Delta t\, \gamma\, a_{n+1} \quad (3)$$

now for an explicit form which is natural to this implicit form.

now let $\tilde{d}_{n+1}$ be everything known in d_{n+1} } predictors.
$\tilde{v}_{n+1}$ " " " " v_{n+1}

$$\tilde{d}_{n+1} = d_n + \Delta t v_n + \frac{\Delta t^2}{2}(1-2\beta) a_n$$

$$\tilde{v}_{n+1} = v_n + \Delta t (1-\gamma) a_n$$

we may replace the top equation (1) for v_{n+1} & d_{n+1} by the predictors then get a_{n+1} put into (2) to get d_{n+1} and v_{n+1} (in 3).

now in the element method split the operators as follows.

let $K = K^I + K^E$ I - implicit E - explicit
$C = C^I + C^E$
$M = M^I + M^E$
$F = F^I + F^E$

put these into (1)

$$M a_{n+1} + K^I d_{n+1} + \underbrace{K^E \tilde{d}_{n+1}}_{\text{move to RHS}} + C^I v_{n+1} + \underbrace{C^E \tilde{v}_{n+1}}_{\text{move to RHS}} = F_{n+1} \quad (4)$$

where $\tilde{v}_{n+1}, \tilde{d}_{n+1}$ are from (2), (3)

Stability - Energy method : Reference R1

let $[x_n] = x_{n+1} - x_n$
$\langle x_n \rangle = (x_{n+1} + x_n)/2$

we want to look at $\|x_n\| \le$ const for $n = 0, 1, 2, \ldots$ a bounded sequence

want to determine whether d_{n+1}, v_{n+1} are bounded

We use the following (Prove it! though) from the newmark method

$$a_{n+1}^T A\, a_{n+1} + v_{n+1}^T K\, v_{n+1} = a_n^T A\, a_n + v_n^T K\, v_n - (2\gamma - 1)[a_n]^T B\, [a_n] - 2\Delta t \langle a_n \rangle^T C \langle a_n \rangle$$

$A = B + \Delta t (\gamma - 1/2) C$ sym

$B = M + \Delta t (\gamma - 1/2) C + \Delta t^2 (\beta - \gamma/2) K$ sym.
> 0

thus note that if $C > 0$ then $2\Delta t \langle a_n \rangle^T C \langle a_n \rangle < 0$

& if $\gamma > 1/2$ $B > 0$ then $(2\gamma - 1)[a_n]^T B [a_n] < 0$

an the system will be stable

~~Thus~~ Thus for $\gamma \geq 1/2$ and $\underset{\sim}{B} > 0$ then $(\underset{\sim}{A} > 0)$ $\underset{\sim}{a}_{n+1}^T \underset{\sim}{A}\, \underset{\sim}{a}_{n+1} \leq \underset{\sim}{a}_n^T \underset{\sim}{A}\, \underset{\sim}{a}_n$

Thus $\begin{pmatrix} v_{n+1} \\ a_{n+1} \end{pmatrix}$ cannot amplify and we have stability, $\Rightarrow d_{n+1}$ is also bounded.

is the stability sharp? yes they give exactly newmark conditions.

for the predictor-corrector methods we use same things except the matrices $\underset{\sim}{A}$ & $\underset{\sim}{B}$ are replaced by $\hat{\underset{\sim}{A}}$ and $\hat{\underset{\sim}{B}}$

$$\bar{\underset{\sim}{B}} = \underset{\sim}{B} - \Delta t\,\gamma\,\underset{\sim}{C} - \beta\Delta t^2 \underset{\sim}{K}$$
$$\bar{\underset{\sim}{A}} = \bar{\underset{\sim}{B}} + \Delta t(\gamma - 1/2)\underset{\sim}{C}$$

The theorem then implies that $\gamma \geq 1/2$ and $\bar{B} > 0$ for stability
see figure 22

when $\xi = 0$ $\Omega_{crit} = (2/\gamma)^{1/2}$ best case is when $\gamma = 1/2$ $\Omega_{crit} = 2$ same as central difference method.

note here that as $\gamma \uparrow$ $\Omega_{crit} \downarrow \Rightarrow (\Delta t \downarrow)$ $\therefore$ max is at $\gamma = 1/2$

now accuracy of method $\underset{\sim}{C} = \underset{\sim}{0}$ $\gamma = 1/2 \iff 2^{nd}$ order accuracy
$\underset{\sim}{C} \neq 0$ 1^{st} order accuracy. } Predict Correct.

if $\underset{\sim}{K} = \underset{\sim}{0}$ $\bar{\underset{\sim}{B}} > 0 \Rightarrow \lambda\Delta t < 2$ where λ is Evalue of $\underset{\sim}{M}^{-1}\underset{\sim}{C}$ same as forward difference algorithm

For implicit-explicit (stability conditions)

$\hat{\underset{\sim}{A}}$ is replaced by $\underset{\sim}{A}^I + \bar{\underset{\sim}{A}}^E$
$\hat{\underset{\sim}{B}}$ " " $\underset{\sim}{B}^I + \bar{\underset{\sim}{B}}^E$ } eqns (25)-(29) in notes

translates to $\gamma \geq 1/2$ $\underset{\sim}{B}^I + \bar{\underset{\sim}{B}}^E > 0$ for stability.

ie suppose $\gamma = 1/2$ $\beta = 1/4$ trapezoidal rule will give unconditional stability wrt I group
this will give stability conditions for E group ie the predictor corrector cond. condition $\Omega_{crit} = (2/\gamma) = 2$

this we can see that the stability analysis uncouples the E & I parts.

Returning to the correctors $\underset{\sim}{d}_{n+1} = \tilde{\underset{\sim}{d}}_{n+1} + \beta\Delta t^2 \underset{\sim}{a}_{n+1}$
$\underset{\sim}{v}_{n+1} = \tilde{\underset{\sim}{v}}_{n+1} + \gamma\Delta t\, \underset{\sim}{a}_{n+1}$

we can use $(\tilde{\ })$ to eliminate explicit terms from eqn of motion for eqn (4)

ie $\hat{\underset{\sim}{M}}\underset{\sim}{a}_{n+1} + \underset{\sim}{C}\underset{\sim}{v}_{n+1} + \underset{\sim}{K}\underset{\sim}{d}_{n+1} = \hat{\underset{\sim}{F}}_{n+1}$

$\hat{M} = M - \Delta t\gamma C^E - \beta\Delta t^2 K^E$

this leads to

AUX Prob: $\hat{\underset{\sim}{M}}\ddot{\underset{\sim}{D}} + \underset{\sim}{C}\dot{\underset{\sim}{D}} + \underset{\sim}{K}\underset{\sim}{D} = \underset{\sim}{F}$

ie newmark problem on $\underset{\sim}{D}$

does $\underset{\sim}{D} \to \underset{\sim}{d}$ see fig 23 ~~define~~

define $\underset{\sim}{e} = \underset{\sim}{D} - \underset{\sim}{d}$ & form $\underset{\sim}{M}\ddot{\underset{\sim}{e}} + \underset{\sim}{C}\dot{\underset{\sim}{e}} + \underset{\sim}{K}\underset{\sim}{e} = \underset{\sim}{\Delta}\ddot{\underset{\sim}{D}}$

$\underset{\sim}{\Delta} = \gamma\Delta t\, \underset{\sim}{C}^E + \beta\Delta t^2 \underset{\sim}{K}^E$

if $\underset{\sim}{C}^E \neq 0$ $\underset{\sim}{e} = \underset{\sim}{D}(\Delta t)$

$\underset{\sim}{C}^E = 0$ $\underset{\sim}{e} = \underset{\sim}{D}(\Delta t^2)$

Predictor - multi corrector

$$\hat{\underset{\sim}{M}}(\ddot{e}+\ddot{d}) + \underset{\sim}{C}(\dot{e}+\dot{d}) + \underset{\sim}{K}(e+d) = F$$

$$\hat{\underset{\sim}{M}}\ddot{e} + \underset{\sim}{C}\dot{e} + \underset{\sim}{K}e + \underset{\sim}{M}\ddot{d} - \gamma\Delta t\,\underset{\sim}{C}^E\ddot{d} - \beta\Delta t^2 \underset{\sim}{K}^E\ddot{d} + \underset{\sim}{C}\dot{d} + \underset{\sim}{K}d = F$$

(crossed out: $\underset{\sim}{M}\ddot{d} = F - \underset{\sim}{C}\dot{d} - \underset{\sim}{K}d$; underbraced terms $= \hat{\underset{\sim}{M}}\ddot{d}$)

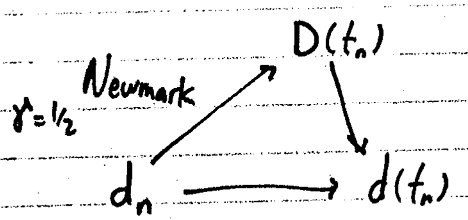
$$\therefore\ \hat{\underset{\sim}{M}}\ddot{\underset{\sim}{e}} - \gamma\Delta t\,\underset{\sim}{C}^E\ddot{d} - \beta\Delta t\,\underset{\sim}{K}^E\ddot{d}$$

$$\underset{\sim}{M}\ddot{\underset{\sim}{e}} - (\gamma\Delta t\,\underset{\sim}{C}^E + \beta\Delta t\,\underset{\sim}{K}^E)(\ddot{e}+\ddot{d}) + \underset{\sim}{C}\dot{e} + \underset{\sim}{K}e = 0 \Rightarrow \underset{\sim}{M}\ddot{\underset{\sim}{e}} + \underset{\sim}{C}\dot{\underset{\sim}{e}} + \underset{\sim}{K}\underset{\sim}{e} = \underset{\sim}{\Delta}\ddot{D}$$ ✓

$(\ddot{e}+\ddot{d}) \to \ddot{D}$

10-12-81
#3

Auxillary Problem $\hat{\underset{\sim}{M}}\,\ddot{\underset{\sim}{D}} + \mathbb{C}\,\dot{\underset{\sim}{D}} + \mathbb{K}\,\underset{\sim}{D} = \mathbb{F}$



$e = \underset{\sim}{D} - d$

$\mathbb{M}\,\ddot{e} + \mathbb{C}\,\dot{e} + \mathbb{K}\,e = \underset{\sim}{\Lambda}\,\ddot{\underset{\sim}{D}}$

$\underset{\sim}{\Lambda} = \gamma \Delta t\, \mathbb{C}^{E} + \beta \Delta t^{2}\, \mathbb{K}^{E}$

$e \sim O(\Delta t)$ if $\mathbb{C}^{E} \neq 0$

$\sim O(\Delta t^{2})$ if $\mathbb{C}^{E} = 0$

10-14-81
#1

Review + ~~Recap~~ – Operator + Mesh Partitions

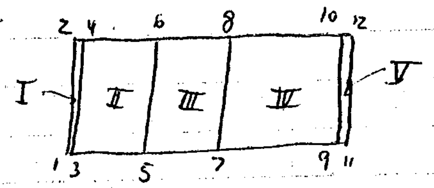
1. Implicit Method (Newmark)
2. Explicit Method (PC version of Newmark)
3. Implicit-Explicit (Predictor Corrector)

<u>Stability</u> via Energy Methods

<u>Convergence</u> via Auxiliary Problem Method.

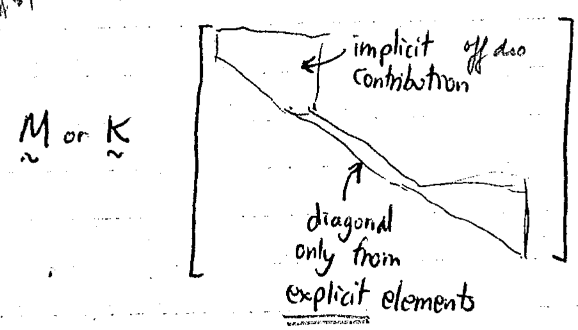
Viewgraphs on these Methods

Example of 5 element bea[m], 2 DOF/node, 12 nodes leads to 24x24 system



24x24 system.

I, V ~~explicit~~ implicit, II, III, IV explicit.



use column, skyline, profile type solvers.

band solvers nearly useless, out of date.

$$M a_{n+1}^{i+1} + C^I v_{n+1}^{(i+1)} + C^E v_{n+1}^{i} + K^I d_{n+1}^{(i+1)} + K^E v_{n+1}^{i} = F_{n+1} \qquad \tilde{v}_{n+1} = v_{n+1}^{(1)} \qquad d_{n+1} = d_{n+1}^{(1)}$$

$$C^I \tilde{v}_{n+1} + C^I \Delta t \gamma a_{n+1}^{i+1} + C^E v_{n+1}^{i} + K^I \tilde{d}_{n+1} + K^I \Delta t^2 \beta a_{n+1}^{i+1} + K^E d_{n+1}^{i} = F_{n+1}$$

$$(M + C^I \gamma \Delta t + K^I \Delta t^2 \beta)\, a_{n+1}^{(i+1)} + K^I d_{n+1}^{(i)} + K^E d_{n+1}^{(i)} = F_{n+1} - C v_{n+1}^{(i)}$$

$$\cancel{K d_{n+1}^{(i)} \quad K^E d_{n+1}^{(i)} \quad K v_{n+1}^{(i)}}$$

$$M^* \Delta a + (M + C^I \gamma \Delta t + K^I \Delta t \beta) a^i$$

$$v^{i+1} = v^i + \Delta t \gamma \Delta a \qquad d^{i+1} = d^i + \Delta t \beta^* \Delta a$$

$$(M + \Delta t \gamma + \beta^* \Delta t^2)\Delta a = F - M a^i - C v^i - K d^i$$

$$\dot{d} = \frac{d - d^0}{\Delta t}$$

$$(M + \Delta t K)\Delta d = \Delta t F - \Delta t M \dot{d} - \Delta t K d$$

$$(M + \Delta t K)\Delta d = \Delta t F - M(d - d^0) - \Delta t K d$$

$$K \Delta d = F - K d$$

$$K d^{i+1} = F$$

10-14-81
#2

PREDICTOR - MULTI-CORRECTOR METHODS

iterate within each step.

$$\mathbb{M}\, a_{n+1}^{(i+1)} + \mathbb{C}^{I}\, v_{n+1}^{(i+1)} + \mathbb{C}^{E}\, v_{n+1}^{(i)} + \mathbb{K}^{I} d_{n+1}^{(i+1)} + \mathbb{K}^{E} d_{n+1}^{(i)} = \mathbb{F}_{n+1} \quad \} A$$

((i) ← iterative counter)

$\mathbb{K}^{I} d_{n+1}^{(i+1)}$: implicit at later iteration; $\mathbb{K}^{E} d_{n+1}^{(i)}$: explicit from previous iteration

FLOW CHART:

↓

$i = 0$

$$d_{n+1}^{(i)} = \tilde{d}_{n+1} \; ; \quad v_{n+1}^{(i)} = \tilde{v}_{n+1} \; ; \quad a_{n+1}^{(i)} = \underset{\sim}{0} \qquad \leftarrow \text{Predictors}$$

$i = i+1$ next iteration →

$$\Delta \mathbb{F}^{(i)} = \mathbb{F}_{n+1} - \mathbb{M}\, a_{n+1}^{(i)} - \mathbb{C}\, v_{n+1}^{(i)} - \mathbb{K}\, d_{n+1}^{(i)} \qquad \leftarrow \text{"Residual" force}$$

$$\mathbb{M}^{*} \Delta a = \Delta \mathbb{F}^{(i)} \; ; \quad \mathbb{M}^{*} = \mathbb{M} + \gamma \Delta t\, \mathbb{C}^{I} + \beta \Delta t^{2}\, \mathbb{K}^{I} \leftarrow \text{Explicit}$$

Corrector Phase → (Newmark)

$$a_{n+1}^{(i+1)} = a_{n+1}^{(i)} + \Delta a \; ; \quad v_{n+1}^{(i+1)} = \tilde{v}_{n+1} + \Delta t\, \gamma\, a_{n+1}^{(i+1)} \; ; \quad d_{n+1}^{(i+1)} = \tilde{d}_{n+1} + \Delta t^{2} \beta\, a_{n+1}^{(i+1)}$$

↓

$n = n+1$! next time step loop

recall
$$\tilde{d}_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2}(1 - 2\beta)\, a_n$$

$$\tilde{v}_{n+1} = v_n + \Delta t\,(1-\gamma)\, a_n$$

(explicit Newmark terms)

VERIFY as an EXERCISE

DERIVATION start with A

examine corrector phase at (i) and $(i+1)$

then show $\mathbb{M}^{*}$ is correct term to get method.

look at equation 66 to show $\Delta v = v_{n+1}^{(i+1)} - v_{n+1}^{(i)} = \Delta a_{n+1}(\gamma \Delta t)$

67 to show $\Delta d = d_{n+1}^{(i+1)} - d_{n+1}^{(i)} = \beta \Delta t^{2}\, \Delta a$

1-Pass gives Implicit-Explicit method
2-Pass with Predictor Corrector gives more accuracy (regain 2^{nd} order accuracy)

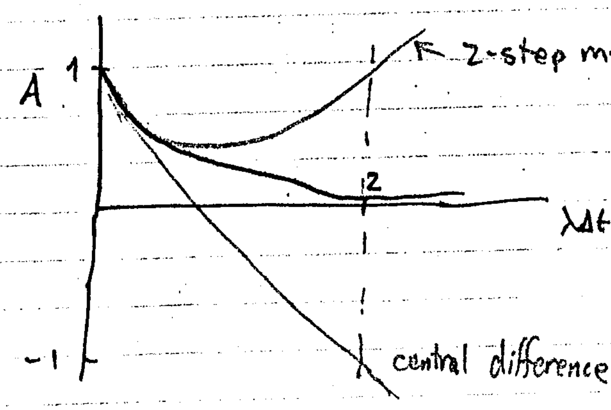
example: $\mathbb{M}\, a_{n+1}^{(i+1)} + \mathbb{C}\, v_{n+1}^{(i)} = \underset{\sim}{0}$ 1-pass

$$v_{n+1} = (I - \Delta t\, \mathbb{M}^{-1}\mathbb{C} + \gamma(\Delta t\, \mathbb{M}^{-1}\mathbb{C})^2)\, v_n$$

for $\gamma = 1/2 \rightarrow$ 2^{nd} Order Accurate

$v(t_n) = v_n + \tau$ $\tau = O(\Delta t^2)$ (use equation twice)
VERIFY as an EXERCISE

$\Omega = \lambda \Delta t < \Omega_{crit} = \frac{1}{\gamma} = 2$ same as central difference, but have second order accurate.



stability of Implicit-Explicit Partitioned operators in multi-~~step~~ corrector methods not yet attempted, neither energy nor modal decomposition works.

SUB-SPACE ITERATION TECHNIQUES NEXT TIME

Oct. 16

A review of the eigenvalue problem: if $M\ddot{d} + Kd = 0$ and $d = c_1 e^{i\omega t}\psi$ w/ $\lambda = \omega^2$ (not in the notes)

$(\underset{\sim}{K} - \lambda \underset{\sim}{M})\underset{\sim}{d} = 0$

$\underset{\sim}{K}$: symmetric, pos. semi-def
λ: eigenvalue
$\underset{\sim}{M}$: sym, pos. def.

order the eigenvalues: $0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_{neq}$
eigenvectors: $\underset{\sim}{d}_{(1)}, \dots, \underset{\sim}{d}_{(neq)}$

orthogonality property:

$$\underset{\sim}{d}^T_{(l)} \underset{\sim}{M} \underset{\sim}{d}_{(m)} = \delta_{lm}$$

$$d^T_{(l)} \underset{\sim}{K} \underset{\sim}{d}_{(m)} = \lambda_l \delta_{lm} \quad \text{(no sum on } l\text{)}$$

We are going to look at a reduced problem:

$$(\underset{\sim}{A} - \bar{\lambda}\underset{\sim}{B})\underset{\sim}{u} = \underset{\sim}{0}$$

$\underset{\sim}{A}$ & $\underset{\sim}{B}$ are usually full; A pos semi-def, sym
B sym pos. def.

There are computer algorithms today to handle the problem in this form (eg. see Bathe-Wilson) but we'll look at some other ways:

1. Static condensation

$$\underset{\sim}{M} = \begin{bmatrix} \underset{\sim}{M}_{11} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} \qquad \underset{\sim}{M}_{11} \text{ sym pos def}$$

$$\underset{\sim}{K} = \begin{bmatrix} \underset{\sim}{K}_{11} & \underset{\sim}{K}_{12} \\ \underset{\sim}{K}_{21} & \underset{\sim}{K}_{22} \end{bmatrix} \qquad \underset{\sim}{K}_{21} = \underset{\sim}{K}_{12}^T$$

$$\underset{\sim}{d} = \begin{Bmatrix} \underset{\sim}{d}_1 \\ \underset{\sim}{d}_2 \end{Bmatrix}$$

Get two eqns.:

$$\underset{\sim}{K}_{11}\underset{\sim}{d}_1 + \underset{\sim}{K}_{12}\underset{\sim}{d}_2 - \lambda \underset{\sim}{M}_{11}\underset{\sim}{d}_1 = \underset{\sim}{0}$$

$$\underset{\sim}{K}_{21}\underset{\sim}{d}_1 + \underset{\sim}{K}_{22}\underset{\sim}{d}_2 = \underset{\sim}{0} \Rightarrow \underset{\sim}{d}_2 = -\underset{\sim}{K}_{22}^{-1}\underset{\sim}{K}_{21}\underset{\sim}{d}_1$$

(never take an inverse in prac…

$$\left[\left(\underbrace{K_{11} - K_{12} K_{22}^{-1} K_{21}}\right) - \lambda M_{11}\right] d_1 = 0$$

K^*, the statically condensed stiffness matrix

Factor:

$K_{22} = U^T D U$ $\qquad$ U : upper triangular

$\qquad\qquad\qquad\qquad$ D : diagonal

$= (U^T D^{1/2})(D^{1/2} U)$ $\qquad$ $(D = (D^{1/2})^2)$

$= \bar{U}^T \bar{U}$ $\qquad$ $\boxed{\bar{U} Z = K_{21}}$

$$\boxed{K_{12} K_{22}^{-1} K_{21} = Z^T Z}$$ $\qquad$ $Z^T \bar{U}^T \cdot \bar{U}^{-T} \bar{U}^{-1} \cdot \bar{U} Z = Z^T Z$

2. Reduction via "Discrete Ritz Approach"

$\underbrace{P}_{n_{eq} \times n_{lp} \, (\leq n_{eq})}$ "static load patterns" (linearly indep.)

(n_{lp} ← load patterns)

$K R = P$ $\qquad$ ie assume $\quad P_i = \left(k_{ii}/m_{ii} \right)$

(R ← discrete Ritz vectors)

$d \approx \underbrace{R d^*}_{n_{eq} \times n_{lp}}$ ← eigenvectors of the reduced system

How do we pick the P's? The rule of thumb is to actuate the large masses and flexible portions of the structure

$(K - \lambda M) d = 0$ $\qquad$ $d \cong R d^*$

pre-mult. by R^T:

$$(\underbrace{R^T K R}_{K^*} - \lambda \underbrace{R^T M R}_{M^*}) d^* = 0$$

K^*: $n_{lp} \times n_{lp}$

$R = \underbrace{[\;\;]}_{n_{lp}} \Big\} n_{eq}$

2 Remark: It can be shown that static condensation is a special case of the discrete Ritz app. with:

$$\underset{\sim}{R} = \begin{bmatrix} \underset{\sim}{I} \\ \cdots\cdots \\ -\underset{\sim}{K}_{22}^{-1}\underset{\sim}{K}_{21} \end{bmatrix}$$

3. <u>Irons-Guyan Reduction</u>

$$\underset{\sim}{K} = \begin{bmatrix} \underset{\sim}{K}_{11} & \underset{\sim}{K}_{12} \\ \underset{\sim}{K}_{21} & \underset{\sim}{K}_{22} \end{bmatrix} \qquad \underset{\sim}{M} = \begin{bmatrix} \underset{\sim}{M}_{11} & \underset{\sim}{M}_{12} \\ \underset{\sim}{M}_{21} & \underset{\sim}{M}_{22} \end{bmatrix}$$

The idea is to retain the $\underset{\sim}{d}_1$ d.o.f.'s and eliminate the $\underset{\sim}{d}_2$ d.o.f.'s ($\underset{\sim}{d}_1$ "master", $\underset{\sim}{d}_2$ "slave")

We ignore the mass terms completely in the second eqn:

$$\underset{\sim}{K}_{21}\underset{\sim}{d}_1 + \underset{\sim}{K}_{12}\underset{\sim}{d}_2 = 0$$

define $\underset{\sim}{R} = \begin{bmatrix} \underset{\sim}{I} \\ \cdots\cdots \\ -\underset{\sim}{K}_{22}^{-1}\underset{\sim}{K}_{21} \end{bmatrix}$

$$\underset{\sim}{K}^* = \underset{\sim}{R}^T\underset{\sim}{K}\underset{\sim}{R} \;;\; \underset{\sim}{M}^* = \underset{\sim}{R}^T\underset{\sim}{M}\underset{\sim}{R} \;;\; (\underset{\sim}{K}^* - \lambda^*\underset{\sim}{M}^*)\underset{\sim}{d}^* = 0$$

$\underset{\sim}{K}^* = \underset{\sim}{K}_{11} - \underset{\sim}{K}_{12}\underset{\sim}{K}_{22}^{-1}\underset{\sim}{K}_{21}$ statically-condensed $\underset{\sim}{K}$

$\underset{\sim}{M}^* = \underset{\sim}{M}_{11} - \underset{\sim}{M}_{12}\underset{\sim}{K}_{22}^{-1}\underset{\sim}{K}_{21} - \underset{\sim}{K}_{12}\underset{\sim}{K}_{22}^{-1}(\underset{\sim}{M}_{21} - \underset{\sim}{M}_{22}\underset{\sim}{K}_{22}^{-1}\underset{\sim}{K}_{21})$

How does one work $\bar{c}$ this technique?

Say you want n e-values to be accurate.

Pick n_{ep} to be $(1 \text{ to } 5) \times n$

Decide which dof's to retain - check M_{ii}/K_{ii} and take the dof corresponding to the maximum ratios

4. <u>Subspace</u> <u>Iteration</u> <u>Techique</u>

The idea is to obtain a reduced problem via discrete Ritz but iterate to obtain the lowest eigenvalues/vectors of the original system (will converge to <u>exact</u> values)

Rough flowchart:

1. Pick $\underset{\sim}{P}$, solve for $\underset{\sim}{R}$
2. Define the reduced problem, solve for λ^*'s and $\underline{d}^*$'s
3. Calculate the "improved" load patterns, $\underline{P} \leftarrow \underline{M}\,\underline{R}\,\underline{d}^*$ { inertial load patterns

Further details:

I. Initialization

1. Form $\underset{\sim}{K}, \underset{\sim}{M}$
2. Factor $\underset{\sim}{K} = \underset{\sim}{U}^T \underset{\sim}{D}\, \underset{\sim}{U}$
3. Specify $\underset{\sim}{P}$:

   a) Pick $n_{lp} = \min(2n+1, n+9)$
      (n e-values req'd to be accurate)

   b) Set $\underset{\sim}{P}$; scan M_{ii}/K_{ii} - set 1 in dof corr. to maximum M_{ii}/K_{ii} in first vector, all others zero; 1 in 2nd M_{ii}/K_{ii} value + zero others in second vector, etc. up to last - make last vector random nos. in unit intervals

dof which max's M_{ii}/K_{ii} →

$$\begin{bmatrix} 0 & 0 & & \\ \vdots & 1 & & \\ 0 & 0 & & \\ 1 & \vdots & \cdots\cdots & \\ 0 & \vdots & & \\ \vdots & \vdots & & \\ 0 & \vdots & & \end{bmatrix}$$

(process due Bathe)

II. Iteration

1. Solve for $\underset{\sim}{R}$: $\underset{\sim}{K}\underset{\sim}{R} = \underset{\sim}{P}$; $\underline{U}^T \underline{D}\,\underline{U}\,\underline{R} = \underline{P}$
2. Compute reduced problem; solve it define $\underset{\sim}{d} = \underset{\sim}{R}\underset{\sim}{d}^*$

Convergence Check ; put off til next time
3 Calculate improved $\Phi \leftarrow M\,d$
4. Return to 1

10/19/81

what is the convergence Check
Choices : (1) Compare λ_I w/ λ_{I-1} where I is iteration no.
accuracy $\lambda_I - \lambda_{I-1}$ to 4 or 5 digits
(2) Compute $\| d_I - d_{I-1} \| \le \epsilon \| d_I \|$ where $\| \ \|$ is Euclidean
ϵ is preassigned error tol.

$\| x \| = (x \cdot x)^{1/2}$

5. Final Check is the sturm sequence check. (SSC)
Assume you want n accurate EV & $\vec{EV}$ ($n < n_{ep}$)
Have you missed any? To check
1. Take λ_n
2. Form $(K - (1+\delta)\lambda_n M)$ pick δ (= .0001 say) s.t. $(1+\delta)\lambda_n < \lambda_{n+1}$

3. Factorize it into $U^T D U$ use routines in LEARN prog.
look at D & check signs of elements of D ;
There should be n negative values by sylvester's law of inertia
If there are more than n you've missed 1 or more EV.

Other application's of sturm sequence check
- If you're curious about the frequency of structure.
Then SSC - determines the no. of ev (if any) in an interval $(\lambda_{Low}, \lambda_{HI})$

Factorize $K - \lambda_{LO} M$; look at pivots D_L $\lambda_L < \lambda_n$
$K - \lambda_{HI} M$; " " " D_{HI}

no. of negs from 1st case is L_{neg}, no. of negs in 2nd case is H_{neg}
No of frequencies = $H_{neg} - L_{neg}$.

——— x x ———

Inverse iteration technique
useful to extract one eigenvalue & corresponding vector.
E.G. stability needs lowest mode.
Thus we want subspace iteration with $n_{ep} = 1$
Converges to lowest eigenvalue.

——— x x ———

Standard EV problem is discussed.

$(A^* - \lambda I) v = 0$ $\quad (A = A^T, A \geq 0)$ — suppose your machine has routines to do this

Generalized form is $(A - \lambda B)u = 0$, $A \geq 0$, $B > 0$ — but you have this

then 1) Factorize $B = C^T C$; use Crout routine (in LEARN) gives $B = U^T D U$, then that $U^T D^{1/2} (D^{1/2} U)$

define $C = D^{1/2} U$

2. define the standard problem

$$(C^{-T} A C^{-1} - \lambda I) v = 0 \quad \text{where } v = C u$$

where λ is the same EV as original problem.

Suppose you've got a routine that doesn't handle zero EV.

1. (Estimate the smallest EIGENVALUE of $(K - \lambda M) d = 0$) say it's $\lambda_1 \leq 0$.

2. pick a shift $\alpha > 0$ s.t. $-\alpha < \lambda_1 \leq 0$

add & subtract $\pm \alpha M d$ ie $\left[\underbrace{(K + \alpha M)}_{K^*} - \underbrace{(\lambda + \alpha)}_{\lambda^*} M \right] \underbrace{d}_{\text{same EV}} = 0$

thus we have made all the EV $\lambda^* > 0$ & K^* is ≥ 0

Forced Vibration Problem

$$M\ddot{d} + C\dot{d} + K d = F$$

assume $F = e^{i\omega t} \bar{F}$ - const. vector, $d = e^{i\omega t} \bar{d}$

then $[-M\omega^2 + i\omega C + K]\bar{d} = \bar{F}$

Either define Crout's scheme as complex or else let $\bar{d} = d_1 + i d_2$, $\bar{F} = F_1 + i F_2$

$$(-M\omega^2 + K) d_1 - \omega C d_2 = F_1$$
$$(-M\omega^2 + K) d_2 + \omega C d_1 = F_2$$

then
$$\begin{bmatrix} -\underline{M}\omega^2+\underline{K} & -\omega\underline{C} \\ \omega\underline{C} & -\underline{M}\omega^2+\underline{K} \end{bmatrix} \begin{bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}.$$

Modal Analysis (To solve a transient problem)

Steps

1. Solve the Eigenproblem
   $\{\lambda_l, \underline{d}_{(l)}\}$ $1 \le l \le n_{modes} \le n_{eq}$.

how big is n_{modes}?

a) λ_l, $\underline{d}_{(l)}$ should be good approx. of the exact λ_l, $d_{(l)}$ $1 \le l \le n_m$

b) Want the spatial variation of both $\underline{d}$ & $\underline{F}$ should be well represented.

2) Reduce to SDOF. Problems.

ie Heat Eq. $\dot{d}_{(l)} + \lambda_l d_{(l)} = F_{(l)}$
$d_{(l)}(0) = D_0(l)$

ie Eqns of motion $\ddot{d}_{(l)} + 2\xi_l \omega_l \dot{d}_{(l)} + \omega_l^2 d_{(l)} = F_{(l)}$
$d_{(l)}(0) = D_0(l)$
$\dot{d}_{(l)}(0) = V_0(l)$

now solve each modal eq

solve in time accurately via any appropriate transient algorithm

3. Reconstruct soln
$$\underline{d}(t) \cong \sum_{l=1}^{n_{modes}} d_{(l)}(t)\, \underline{\psi}_{(l)} \quad \text{— EV}$$

what are ADV / DISAD vs. direct time step schemes. See next time.

10/21/81

I. Advantages of methods

A. Direct Methods (Step-by-step)
   1. Easily coded
   2. more efficient for short time calculations
   3. Generalize easily to nonlinear problems

B. Modal Analysis
   1. If many analyses of the same structure are to be performed it can save time since initial EV calc. are absorbed

2. Good for long time calc.
3. Good if only a small no. of modes are expected to particip.

Prof Hughes would certainly use direct methods unless he had $\vec{EV}$ + EV for the problem already. It is tempting nowadays to use direct methods even on non-linear analysis because they are easily coded.

II A simple problem
Wave Eqn $u_{tt} = c^2 u_{xx}$ I dim

Mass matrices: $\underset{\sim}{K}^e = \frac{c^2}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

2nd order — Consistent $\underset{\sim}{M}^e = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (upper bound in frequencies)

Lumped $\underset{\sim}{M}^e = \frac{h}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Cannot generalize result, but lumped matrices give lower frequencies

concentrate mass at translational degrees of freedom ; put zeroes at rotational dofs.

High order mass : 4th order $\frac{h}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$

$\eta = \frac{n}{n_{el}}$ ← mode no

$\frac{\omega^h}{\omega} = 1 + O(\eta^2)$ (ω^h ← finite elem, ω ← exact)

- higher freq modes are not accurate representation of the behavior of the system — that is why we want to damp them out

- Consistent mass matrix will always give upper bounds (maximize over larger class of fns).

III Matched Methods — Transient algor. + mass matrix

Consider Newmark method with $\xi = 0$, $\gamma = \frac{1}{2}$ ($\bar{\xi} = 0$) $\beta = ?$

A. Trapezoidal $\beta = \frac{1}{4}$ periods increased (freq reduced)
Central difference $\beta = 0$ " reduced (" increased)

effect in mass: Consistent mass: periods are ~~increased~~ reduced (freq. reduced)
Lumped mass: " " ~~decreased~~ increased (" increased)

So, don't use trapez w/ lumped or central w/ consistent

matched: trap + consistent ; central + lumped — used to stabilize effect on period

Cheaper to calculate a lumped mass matrix; also necessary for explicit calculations.

Next: look at errors in EV when the Galerkin recipe is allowed to λ_ℓ = exact

$$\lambda_\ell \le \lambda_\ell^{h\ (\text{f. element})} \le \lambda_h + Ch^{2(k+1-m)}$$

m = order of deriv used to derive stiffness matrix
k = order of complete poly.

m = 1 elasticity, heat
m = 2 Euler Bernoulli beam.

$\| \text{error in } \vec{EV} \|_m \le Ch^{k+1-m}$ shown in 235B

Mass Lumping via Numerical Integ. Suppose we want to use nodes for integration points, Nodal quadrature leads to diag mass. See below.

Approximation of integral
$$\int_{\square} g(\xi)\, d\square \cong \sum_{a=1}^{n_{en}} g(\xi_a) W_a$$

Consistent Mass: $m_{pq}^e = \int_{\Omega^e} \delta_{ij}\, \rho N_a N_b \, d\Omega$ evaluated at bar points (elemental node pt); δ_{ij}: coord direc

$p = n_{ed}(a-1) + i$
$q = n_{ed}(b-1) + j$
(n_{ed} = # element dof per node)

$$= \delta_{ij} \int_{\square} \rho N_a N_b\, j\, d\square$$

for nodal quadrature let ξ_c be node $\Longrightarrow$

$$\delta_{ij} \sum_{c=1}^{n_{en}} \left[\rho(\xi_c) N_a(\xi_c) N_b(\xi_c)\, j(\xi_c)\right] W_c$$

$$= \begin{cases} \delta_{ij}\, \rho(\xi_a)\, j(\xi_a)\, W_a & a = b \\ 0 & a \ne b \end{cases}$$

since $N_a(\xi_c) = \delta_{ac}$, $N_b(\xi_c) = \delta_{bc}$

10/23/81

For learn in active file
USE WYL.PZ.K91.IZL#LEARN ON PUBØ13

Last time
- Modal analysis
- EV errors
- Matched methods
Methods of Calc Diag Mass Matrices

HW # 1 : Due Nov. 6
Midterm in Class Nov. 13

Quadrature techniques & Theory } in book by Strang & Fix An Analysis of the FEM '73

1-D Quadrature need integration rule of accuracy 2(k-m) ? exactly integrate monomials of order 2(k-m)
where k - order of poly
m - order of derivatives leading to stiffness

This will ensure full rate of convergence.

examples: rod: $m^e_{ab} = \int_{\Omega} N_A N_B \, dx$ $K^e_{ab} = \int w_{,x} u_{,x} \, dx$ ∴ order m = 1.

1. Consider 2-node linear rod o——o ξ.
sampling of nodes only is the trapezoidal rule $\tilde{\xi}_1 = -1$ $\tilde{\xi}_2 = 1$ $W_1 = W_2 = 1$
trapezoidal rule is quadratic converg i.e. integrates 1, ξ accurately

$m^e_{ab} = \int_{-1}^{+1} N_A(\xi) N_B(\xi) \frac{h}{2} d\xi$ $\frac{h}{2} = j$ Using trap rule (i.e. trap. rule poly in linear k=1) $\tilde{m}^e = \frac{h}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2(k-m) = 0 for linear rod need integrator that accurately integrates constants. ∴ Trapez. Rule does this

2. for 3-node quad (k is 2 Lagrange poly). o——o——o (-1, 0, 1)

for rod k = 2 m = 1 ∴ 2(k-m) = 2 must find an integrator to accurately integrate through quadratic 1, ξ, ξ^2

Simpson's rule accurately integrates through cubics (1, ξ, ξ^2, ξ^3) has quartic converg
$\tilde{\xi} = (-1, 0, 1)$ $W = (1/3, 4/3, 1/3)$

$\tilde{m}^e = \frac{h}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ this guarantees full rate of convergence.

3. $-1 \quad -1/3 \quad +1/3 \quad -1$ — if we pick for cubic approx

$2(k-m) = 2(3-1) = 4$ must integrate $(1, \xi, \xi^2, \xi^3, \xi^4)$

Now what do we do

$g(\xi)$	$\int_{-1}^{+1} g(\xi)\,d\xi$	$\sum_{a=1}^{4} g(\xi_a)\,W_a$
1	2	① $W_1 + W_2 + W_3 + W_4$
ξ	0	② $-W_1 - \frac{1}{3}W_2 + \frac{1}{3}W_3 + W_4$
ξ^2 ✓	$2/3$	③ $W_1 + \frac{1}{9}W_2 + \frac{1}{9}W_3 + W_4$
ξ^3	0	④ $-W_1 - \frac{1}{27}W_2 + \frac{1}{27}W_3 + W_4$
ξ^4	$2/5$	$W_1 + \frac{1}{81}W_2 + \frac{1}{81}W_3 + W_4$

4th eqn can't be satisfied

unique soln for ①–④: $W_1 = 1/4$, $W_4 = 1/4$, $W_2 = W_3 = 3/4$

Try again by letting middle nodes float. this adds extra unknown: $-1 \quad -c \quad c \quad 1$

by symmetry weights are symmetric

$\xi_a = -1, 1, c, -c$ here

Rule $g(\xi)$: $W_1[g(-1) + g(+1)] + W_2[g(-c) + g(c)] = \int g(\xi)\,d\xi = \sum g(\xi_a) W_a$

$2 = \int_{-1}^{1} g(\xi)\,d\xi = 1 \cdot \xi \Big|_{-1}^{1}$ 1: $2 = 2(W_1 + W_2)$ $W_2 = -W_1 + 1$ $c = \sqrt{5}/5$

ξ ⟶ $\int g(\xi)\,d\xi = 0$ $W_1 = 1/6$ $W_2 = 5/6$

$2/3 = \int_{-1}^{1} \xi^2\,d\xi = \frac{\xi^3}{3}\Big|_{-1}^{1}$ ξ^2: $2/3 = 2(W_1 + c^2 W_2)$ → $W_1(1-c^2) + c^2 = 1/3$

ξ^3 ⟶ 0

$g(\xi) = \xi^4$: $2/5 = 2(W_1 + c^4 W_2)$

Thus convergence is to 6th order
why? because $\int g(\xi)\,d\xi = 0$ for $g(\xi) = \xi^5$
∴ next non zero term is when $\xi^6 = g(\xi)$

$$\therefore \quad \underset{\sim}{m}^e = \frac{h}{12}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(-c, c, -1, +1)$

These points are defined as Lobatto points: include end points but interior points are located to maximize accuracy & are symmetric

Cubic Lagrange

For classical cubic beam. $\int w_{,xx} u_{,xx} \, dx$ $m=2$.

Full rate of convergence was attained with the classical lumped matrix

ie $M^e = h/2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ translational mass DOF only

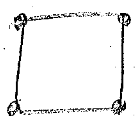
$2(k - m) = 2 \Rightarrow (1, \xi, \xi^2)$ need an integrator to accurately integrate to ξ^2 ⇒ use simpson

($k = 3$, $m = 2$)

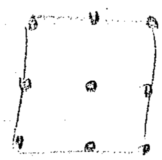
but M^e is produced by trapezoidal rule ie $(1, \xi)$.

this shows the theory provides sufficient but not necessary conditions for full rate of convergence.

Consider now 2-D 3-D cases using product rule.

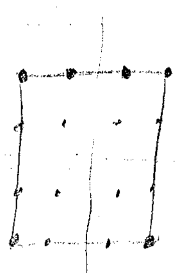
1.  product trapez rule

a	ξ_a	η_a	W_a
1	-1	-1	1
2	1	-1	1
3	1	1	1
4	-1	1	1

2.  product simpson

$W_a = \begin{cases} 1/9 & \text{for corners} \\ 4/9 & \text{for mid side} \\ 16/9 & \text{for center} \end{cases}$

this also goes for 27 node brick

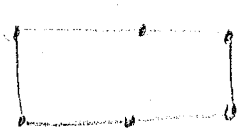
3.  Lobatto points

$c = \frac{1}{\sqrt{5}}$

$W_a =$ 1/36 corners.

5/36 mid side

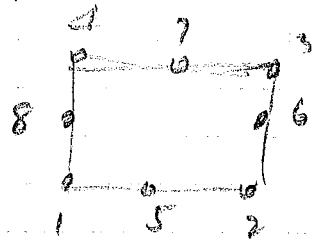
25/36 interior

4.  ↑ trapezoid (linear) × simpson (quadratic) ⟶

$W_a = 1/3$ corners

4/3 for sides

what happens when we don't have product. i.e. 8 node serendipity



Thy: must integrate a cubic poly in ξ & η.

$g(\xi)$	$\int g \, d\xi \, d\eta$	$= W_1 \left(\sum_1^4 g_a\right) + W_2 \sum_5^8 g_a$
1	4	$4(W_1 + W_2)$
ξ	0	0
η	0	0
$\xi\eta$	0	0
ξ^2	$4/3$	$4W_1 + 2W_2$
η^2	$4/3$	$4W_1 + 2W_2$ by sym.
ξ^3	0	0
$\xi^2\eta$	0	0
$\eta^2\xi$	0	0
η^3	0	0

W_1 = corner weight
W_2 = midside "

$W_1 = -1/3 \quad W_2 = 4/3$

this leads to negative masses. This leads to problems.

Not good for transient analysis

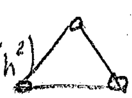
__note:__ serendipity misses no combinations

10/26/81

Lumped mass using nodal quadrature rules.

Fried & Malkus use ~~[illegible]~~ triangles for full rate of conv.

Linear triangles $2(k-m) = 2(1-1) = 0$
guarantees convergence of $\lambda^h - \lambda = O(h^2)$
$\lambda^h - \lambda = ch^{2(k+1-m)} = O(h^2)$



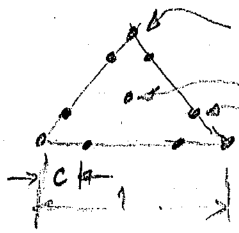
$W = 1/3$ location is nodes

2. Quadratic Triangles
$2(k-m) = 2(2-1) = 2$
$\lambda^h - \lambda = O(h^4)$
this leads to zero masses at vertices

$W = 1/3$ at mid points
$W = 0$ at nodes. (bad!)

3. cubic triangles
$2(k-m) = 2(3-1) = 4$

$c = \sqrt{3}(\sqrt{3}-1)/6$



$W = -1/60$ on nodes (bad!)
$W = 9/20$ for center
$W = 1/10$ all side nodes

$\lambda^h - \lambda = O(h^6)$

Another shortcoming of nodal quadrature for axisymm case $m^e_{pq} = 2\pi\delta_{ij} \int_{\Omega^e} \rho N_a N_b \, r \, dr \, dz$
r factor makes problems → at $r=0$, nodal quad ⟹ $m^e_{pq} = 0$

Other schemes proposed to calc. Lumped Mass Matrix (Ad Hoc in most cases).
Not nodal

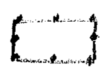
1. "Row Sum" — sum row of consistent mass matrix & lump all on diag element.

$$m_{pq}^{e} = \begin{cases} \delta_{ij} \int \rho N_a \, d\Omega ; & a = b \\ 0 & a \neq b \end{cases}$$

$$\sum_{b=1}^{n_{en}} \left(\int_{\Omega^e} \rho N_a N_b \, d\Omega \right) = \int_{\Omega^e} \rho N_a \underbrace{\sum N_b}_{=1} \, d\Omega = \int_{\Omega^e} \rho N_a \, d\Omega$$ must be integrated accurately.

Adv: this drops all zero masses in axisymm disappear.

Disadv: 8 node element in 2-D still gives negative masses at vertices



2. Scheme to remove negative elements. HINTON & ZIENKEWIC.
"Special Lumping"

$$m_{pq}^{e} = \begin{cases} \alpha \delta_{ij} \int_{\Omega^e} \rho N_a^2 \, d\Omega & a = b \\ 0, & a \neq b \end{cases}$$ diag of lumped mass

$$\alpha = \int_{\Omega^e} \rho \, d\Omega \Big/ \left(\sum_{a=1}^{n_{en}} \int_{\Omega^e} \rho N_a^2 \, d\Omega \right)$$

total mass / sum of diag elements of consistent mass.

1. All masses are positive
2. Seems to work quite well (even for 8 node element).

For simple elements in a parallelogramic geometry but non axisymmetric all approaches lead to same end.

Non Linear FE Analysis

Recall in the linear case we defined in static problem the strong form and the weak form. We approximated the weak form by FE

the finite space solution led to a matrix problem
ie

$$(S) \Leftrightarrow (W) \approx (G) \Leftrightarrow (M) \quad : \quad \underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F}$$

($\underset{\sim}{K}\underset{\sim}{d}$ = $\underset{\sim}{F}$ internal; $\underset{\sim}{F}$ = F external.)

in dynamics $(S) \Leftrightarrow (W) \approx (G) \Leftrightarrow$ system of ODE $\approx$ TIME step algorithm : $\underset{\sim}{M}^* \Delta \underset{\sim}{a} = \underset{\sim}{R}$
(G in space)

These two led to eqn systems that could be solved Algebraically

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$

For the non linear statics again we take

$$(S) \Leftrightarrow (W) \cong (G) \Leftrightarrow \text{non linear problem} : \underset{\sim}{F}^{\text{internal}} = \underset{\sim}{F}^{\text{external}}$$

$\underset{\sim}{F}$ internal is not a constant matrix times $\underset{\sim}{d}$ anymore.

ie

for elastostatics : $\underset{\sim}{F}^{\text{internal}} = \underset{\sim}{N}(\underset{\sim}{d})$ ← algebraic operator

for plasticity we can no longer do even the above.

Dynamics

For nonlinear dynamics

$(S) \Leftrightarrow (W) \approx (G) \Leftrightarrow$ non linear evolution eqns (not necessarily ODE) : $\underset{\sim}{M}\underset{\sim}{a} + \underset{\sim}{F}^{int} = \underset{\sim}{F}^{ext}$ for example in non lin structures.
(G in space)

now must introduce a time algorithm

in elastodynamics the results can be written as → $\underset{\sim}{N}^*(\underset{\sim}{a}) = \underset{\sim}{F}^*$ ← known
($\underset{\sim}{N}^*$: non linear, algebraic fn.)

$\underset{\sim}{a}$ here is acceleration

NON LINEAR STATICS

before we attack the weak forms we should look at incremental load method

$$\therefore \quad \underset{\sim}{0} = \underset{\sim}{F}^{ext} - \underset{\sim}{F}^{int} \quad ; \quad \underset{\sim}{F}^{ext} = \underset{\sim}{F}^{ext}(t)$$

t is time, or a loading parameter

$$\Longrightarrow \underset{\sim}{d} = \underset{\sim}{d}(t)$$

How do we do this (ie get $\underset{\sim}{d} = \underset{\sim}{d}(t)$). Several methods exists

1. <u>Use the incremental load method</u> $\qquad \underset{\sim}{F}_n^{int} = \underset{\sim}{N}(\underset{\sim}{d}_n)$

$$\underset{\sim}{K} \; \Delta \underset{\sim}{d}_n = \underset{\sim}{R}_{n+1} = \underset{\sim}{F}_{n+1}^{ext} - \underset{\sim}{F}_n^{int} \qquad \underset{\sim}{K} \text{ is to be defined}$$

$$\therefore \quad \underset{\sim}{d}_{n+1} = \underset{\sim}{d}_n + \Delta \underset{\sim}{d}_n$$

$\underset{\sim}{K}$ must be an approximation to the Tangent stiffness

$F_{n+1}^{ext} - F_{dn}^{int} = N(d_{n+1}) - N(d_n) = N(d_n) + \frac{\partial N}{\partial d}\Big|_{d_n}(d_n)\Delta d - N(d_n) = \frac{\partial N}{\partial d}(d_n)\,\Delta d = R_{n+1}$

$\underset{\sim}{K} = \frac{\partial N}{\partial \underset{\sim}{d}}(d_n)$

This is simplest, reasonable algo – but $\underset{\sim}{F}^{int}$ is nonlinear & must check to see if $\underset{\sim}{Q} = \underset{\sim}{F}_{n+1}^{ext} - \underset{\sim}{F}_{n+1}^{int}$

where $\underset{\sim}{d}_{n+1}$ is used in $\underset{\sim}{F}_{n+1}^{int}$

10/28/81

What is $\underset{\sim}{K}$ for nonlinear elast $\underset{\sim}{F}^{int} = \underset{\sim}{N}(\underset{\sim}{d})$

consistent tangent $\quad \underset{\sim}{K} = \partial \underset{\sim}{N} / \partial \underset{\sim}{d}$

$$K_{PQ} = \partial N_P / \partial d_Q$$

in non linear elast $\quad N_P = \partial U / \partial d_P \quad$ where U is energy $\quad \therefore K_{PQ} = \dfrac{\partial^2 U}{\partial d_Q \partial d_P}$

and it is symmetric. Problem is formulation of $\underset{\sim}{K}(\underset{\sim}{d})$ – since it depends on $\underset{\sim}{d}$

For nonlinear heat conduction takes $\quad \underset{\sim}{F}^{int} = \hat{\underset{\sim}{K}}(\underset{\sim}{d}, t) \cdot \underset{\sim}{d}$

typically $\hat{\underset{\sim}{K}} = \hat{\underset{\sim}{K}}^T$; what do we use for $\underset{\sim}{K}$?

choices:
(1) $\underset{\sim}{K} = \hat{\underset{\sim}{K}}$ – pick $\hat{K}$ based on known info, symmetric
(2) $\underset{\sim}{K} = \dfrac{\partial \underset{\sim}{F}^{int}}{\partial \underset{\sim}{d}}$ consistent tangent

if (2) $\Rightarrow K_{PQ} \underset{(1)}{=} \dfrac{\partial F_P^{int}}{\partial d_Q}$ where $F_P^{int} \underset{(2)}{=} \hat{K}_{PR} \, d_R \quad \therefore \quad K_{PQ} \underset{(1,2)}{=} \hat{K}_{PR}\,\delta_{QR} + (\partial \hat{K}_{PR} / \partial d_Q) d_R$

$$= \hat{K}_{PQ} + \frac{\partial \hat{K}_{PR}}{\partial d_Q} d_R$$

(↑ dependent on $\underset{\sim}{d}$)

now $\dfrac{\partial \hat{K}_{PR}}{\partial d_Q}$ is not symmetric (if R is fixed).

thus K_{PQ} is best representation of tangent but nonsymmetry is very costly on storage & factoring.

Now in most cases $0 \neq \underset{\sim}{F}_{n+1}^{ext} - \underset{\sim}{F}_{n+1}^{int}$ thus must iterate within step

use Newton Raphson Method

introduce an iteration counter $i = 0$

let $\underset{\sim}{d}^{(i)} = \underset{\sim}{d}_n$

(*) thus $\underset{\wedge}{K} \, \Delta \underset{\wedge}{d}^{(i)} = \underset{\wedge}{R}^{(i)} = \underset{\wedge}{F}^{ext}_{n+1} - \underset{\wedge}{F}^{int}(\underset{\sim}{d}^{(i)})$

$\underset{\sim}{d}^{(i+1)} \Leftarrow \underset{\sim}{d}^{(i)} + \Delta \underset{\sim}{d}^{(i)}$

now form $\underset{\sim}{R}^{(i+1)} = \underset{\sim}{F}^{ext}_{n+1} - \underset{\sim}{F}^{int}(\underset{\sim}{d}^{(i+1)})$

must check for convergence ie $\| \underset{\sim}{R}^{(i+1)} \| < \varepsilon_1$, $\| \Delta \underset{\sim}{d}^{(i)} \| < \varepsilon_2$ } preassigned

if they are satisfied then let $\underset{\wedge}{d}_{n+1} = \underset{\sim}{d}^{(i+1)}$

if not let $i \Leftarrow i+1$ & go to (*)

for this method (strictly speaking): $\underset{\wedge}{K} \equiv \underset{\wedge}{K}(\underset{\wedge}{d}^{(i)})$ of Newton Raph.

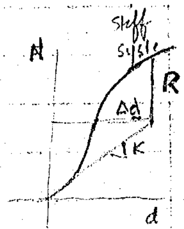
If we don't reformulate (we call it modified Newton Raph.) $\underset{\wedge}{K}$ all the time but infrequently may not converge.

10-28-81
\# 2

~~$dl^{(i+1)} = dl^{(i)} + \Delta dl^{(i)}$~~

~~$\| \mathbb{R}^{(i+1)} \| < \epsilon_1, \quad \| \Delta dl^{(i)} \| < \epsilon_2$~~

~~if satisfied, $dl_{n+1} = dl^{(i+1)}$, exit~~
~~if not, $i = i+1$, if $i > N_{max}$, exit~~
~~if $i \le N_{max}$, recalculate $\mathbb{K}$, go to (A) →~~



Modified N.R. don't re-calculate $\mathbb{K}$ at every iteration, just every few, every step, or every few steps.

→ can lead to big problems in stiffening systems.

Fix: this problem by Line Search. Equillibrium in the direction of the increment.

$$dl^{(i+1)} = dl^{(i)} + s^{(i)} \Delta dl^{(i)}$$

↖ search parameter.

have space of $\Delta dl^{(i)}$ ⊥ to space of residual $\mathbb{R}^{(i)}$

want.

$$0 = G(s^{(i)}) = \Delta dl^{(i)} \cdot (\mathbb{F}^{EXT} - \mathbb{F}^{INT}(dl^{(i)} + s^{(i)} \Delta dl^{(i)}))$$

~~We also assume~~ fixed point iteration

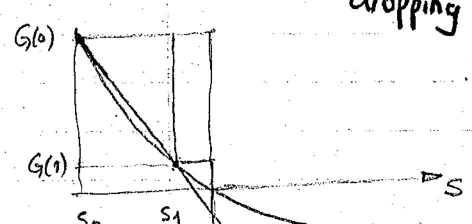
$$\tfrac{1}{2} |G(0)| \geqslant |G(s \neq 0)| \qquad G(0) = \Delta dl^{(i)} \cdot (\mathbb{F}^{EXT} - \mathbb{F}^{INT}(dl^{(i)}))$$

proposed by Strang and Mathies.

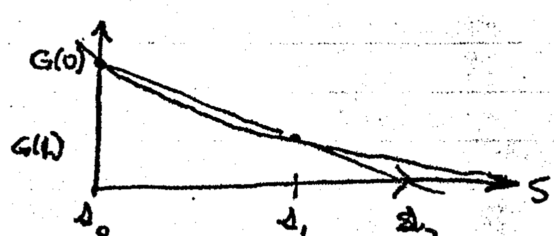
dropping i subscript for now. find $s_k^{(i)}$ iteratively

$$s_0 = 0 \Rightarrow G(0) = \Delta dl^{(i)} \cdot (\mathbb{F}^{EXT} - \mathbb{F}^{INT}(dl^{(i)}))$$

$$s_1 = 1 \Rightarrow G(1) = \Delta dl^{(i)} \cdot (\mathbb{F}^{EXT} - \mathbb{F}^{INT}(dl^{(i)} + \Delta dl^{(i)}))$$



$$s_2 = 1 - \frac{G(1)}{[G(1) - G(0)]} \quad ✓$$



$$\frac{G(0) - G(1)}{s_0 - s_1} = \frac{G(0) - 0}{s_0 - s_2}$$

$$\therefore s_0 - s_2 = \frac{G_0 (s_0 - s_1)}{G_0 - G_1}$$

$$s_2 = \cancel{s_0} - \frac{G_0 (s_0 - s_1)}{G_0 - G_1}$$

$$s_2 = \frac{G_0 s_1 - \cancel{G_1 s_0}}{G_0 - G_1} = \frac{\cancel{G_1 s_0} - G_0 \cdot s_1}{G_1 - G_0}$$

$s_2 \cdot s_1$

calculate $G(s_2)$

$$G(s_j) = \Delta dl \cdot (\mathbb{F}^{EXT} - \mathbb{F}^{INT}(dl + s_j \Delta dl))$$

$$s_{j+1} = \frac{(s_j - s_{j-1}) G(s_j)}{[G(s_{j-1}) - G(s_j)]}$$

Hallquist If $s_j > 1$, then take $s_j = 1$, quit.

$$\frac{s_2 - s_1}{G_1} = \frac{s_1 - s_0}{G_1 - G_0} \Rightarrow$$

$s_2 = G_1 (s_1 - s_0) + s_1 = 1 - G_1$

10-28-81
#3

QUASI-NEWTON UPDATES

BROYDEN update method.

define $\mathbb{R} = \mathbb{R}^{(i)}$, $d = d^{(i)}$

$\overline{\mathbb{R}} = \mathbb{R}^{(i+1)}$, $\bar{d} = d^{(i+1)}$

$\Delta\mathbb{R} = \overline{\mathbb{R}} - \mathbb{R}$ $\Delta d = \bar{d} - d$ where s – is a line search result $\Delta\Delta d = d^{i+1} - d^{i}$

want to solve $\overline{\mathbb{K}}\ \Delta d^{(i+1)} = \overline{\mathbb{R}}$ with $\overline{\mathbb{K}}$ updated version of $\mathbb{K}$ from last iteration.

formally: $\overline{\mathbb{K}}^{-1} = (\mathbb{I} + c\, s\Delta d^T)\,\mathbb{K}^{-1}$

$$c = \frac{(1+s)\Delta d - \mathbb{K}^{-1}\overline{\mathbb{R}}}{s\Delta d \cdot (\mathbb{K}^{-1}\overline{\mathbb{R}} - \Delta d)}$$

$\Delta\bar{d}$

computationally:

0. let $\mathbb{K} = \mathbb{K}$ of $d^{(i)}$ & solve $\mathbb{K}\,\Delta d = \mathbb{R} = F^{ext} - F^{int}(d)$; then let $\bar{d} = d + \Delta d$; do line search; find $\overline{\mathbb{R}} = F^{ext} - F^{int}(\bar{d})$

1. Solve $\mathbb{K}\,\Delta\bar{d} = \overline{\mathbb{R}} = F^{ext} - F^{int}(d^{i} + s\Delta d^{i})$ for $\Delta\bar{d}$

2. calc. $$c = \frac{(1+s)\Delta d - \Delta\bar{d}}{s\Delta d \cdot (\Delta\bar{d} - \Delta d)}$$ using $\Delta\bar{d}$ from (1) & $\Delta d, s$ from (0)

if $\Delta\bar{d} = 0 \Rightarrow c = \frac{1+s}{-s}\frac{\Delta d}{\Delta d^2}$

3. from $\Delta d^{(i+1)} = \Delta\bar{d} + (s\Delta d \cdot \Delta\bar{d})\,c$

$= \overline{\mathbb{K}}^{-1}\overline{\mathbb{R}} = (\mathbb{I} + c\,s\Delta d^T)\,\mathbb{K}^{-1}\overline{\mathbb{R}}$

$= (\mathbb{I} + c\,s\Delta d^T)\,\Delta\bar{d}$

$= \Delta\bar{d} + c\,(s\Delta d \cdot \Delta\bar{d})$

$x = \bar{d} + \Delta d^{(i+1)}$; find $\mathbb{R}(x) = F^{ext} - F^{int}(\bar{d} + s\,\Delta d^{(i+1)})$

do search w/ $\mathbb{R}$, $\mathbb{R}(x)$ and

10/30/81

Last time

Methods for solving Nonlinear Problems.

a. Incremental load
b. Newton – Raphson type
c. Line search
d. Quasi – newton updates
e. broyden update

6. QUASI-NEWTON UPDATES

Line searches are helpful in preventing catastrophic divergence, but to achieve fast convergence in an acceptable number of iterations, some form of update of the stiffness matrix need be performed. Since total reformulation/factorization is such an expensive proposition, other avenues have been pursued. The techniques currently gaining favor are the so-called quasi-Newton updates in which modifications are effected reflecting the changes of stiffness from iteration to iteration and step to step. However, the modifications do not involve alterations of the factorized array. There is a vast literature on these techniques [e.g., 2-4, 7, 10], although their introduction to finite element equation solving was made only quite recently [1, 5, 6, 8, 9].

Broyden Update

To illustrate the computational structure of a quasi-Newton update, in the context of finite element equation solving, we will consider the so-called inverse Broyden update. At first, to simplify matters, we will take a single iteration update. Let

$$\mathbf{R} = \mathbf{R}^{(i)} \quad , \quad \bar{\mathbf{R}} = \mathbf{R}^{(i+1)} \quad , \quad \Delta\mathbf{R} = \bar{\mathbf{R}} - \mathbf{R} \tag{19}$$

$$\mathbf{d} = \mathbf{d}^{(i)} \quad , \quad \bar{\mathbf{d}} = \mathbf{d}^{(i+1)} \quad , \quad \Delta\mathbf{d} = \bar{\mathbf{d}} - \mathbf{d} \tag{20}$$

We are interested in computing $\Delta\mathbf{d}^{(i+1)}$ from the equation

$$\bar{\mathbf{K}} \, \Delta\mathbf{d}^{(i+1)} = \bar{\mathbf{R}} \tag{21}$$

where $\bar{\mathbf{K}}$ is to represent an update of the stiffness $\mathbf{K}$ used during the previous iteration. We assume the triangular factors of $\mathbf{K}$ are known. *Formally*, we may write Broyden's update as follows:

$$\bar{\mathbf{K}}^{-1} = (\mathbf{I} + \mathbf{c} \, \Delta\mathbf{d}^{T}) \, \mathbf{K}^{-1} \tag{22}$$

where

$$\varepsilon = \frac{(1-\Delta_i)\Delta d - K^{-1}\bar{R}}{\Delta d \cdot (K^{-1}\bar{R} - \Delta d)} \tag{23}$$

To obtain $\Delta d^{(i+1)}$, however, we execute the following steps:

1. Solve $K\Delta\bar{d} = \bar{R}$ for $\Delta\bar{d}$
2. $\varepsilon = \left((1-\Delta_i)\Delta d - \Delta\bar{d}\right) / \left(\Delta d \cdot (\Delta\bar{d} - \Delta d)\right)$
3. $\Delta d^{(i+1)} = \Delta\bar{d} + (\Delta d \cdot \Delta\bar{d})\,\varepsilon$

Note that the effect of the changed inverse is entirely accounted for in Steps 2 and 3, which involve simple vector calculations.

After each iteration an update is performed, which amounts to pre-multiplying by another factor of the type indicated in Equation 22. If the number of updates accumulated at the present stage is N, then the generalization of Equation 22 is

$$\bar{K}^{-1} = (I + \varepsilon_N \Delta d_N^T)(I + \varepsilon_{N-1}\Delta d_{N-1}^T)\ldots(I + \varepsilon_1 \Delta d_1^T)\,K^{-1} \tag{24}$$

The steps in computing $\Delta d^{(i+1)}$ are:

1. Solve $K\Delta\bar{d}_o = \bar{R}$ for $\Delta\bar{d}_o$
2. Evaluate $\Delta\bar{d}_k = \Delta\bar{d}_{k-1} + (\Delta d_k \cdot \Delta\bar{d}_{k-1})\,\varepsilon_k$ where $k = 1,2,\ldots,N-1$
3. $\varepsilon_N = \left((1-\Delta_N)\Delta d_N - \Delta\bar{d}_{N-1}\right) / \left(\Delta d_N \cdot (\Delta\bar{d}_{N-1} - \Delta d_N)\right)$
4. $\Delta d^{(i+1)} = \Delta\bar{d}_{N-1} + (\Delta d_N \cdot \Delta\bar{d}_{N-1})\,\varepsilon_N$

At this point a line search may be performed to determine $s^{(i+1)}$.

Remarks

a. In large-scale calculations, the pairs of vectors $\{c_k, \Delta d_k\}$ are generally stored on secondary storage devices and retrieved as necessary.

b. In practice, a fixed number of updates is decided upon, and the earliest are discarded as new ones are created.

c. It may be worthwhile to periodically reform the stiffness and begin the update accumulation process anew.

d. Note that the Broyden update results in $\bar{K}^{-1}$ being *nonsymmetric* even when K^{-1} is symmetric. G.L. Goudreau has observed that this is of no computational consequence since $\bar{K}^{-1}$ is never formed.

e. The Broyden update is felt to be the most effective procedure for general systems. It is uniquely specified by the following two conditions:

$$-\bar{K}\Delta d = \Delta R \quad (25)$$

and

$$\bar{K}z = Kz, \quad \text{for all } z \text{ such that } z \cdot \Delta d = 0. \quad (26)$$

Equation 25 is called the *quasi-Newton equation* and amounts to insisting that $\bar{K}$ satisfy a secant relationship in the direction Δd. The second condition emanates from Broyden's argument that there is really no point in updating $\bar{K}$ in directions other than Δd.

Other requirements can be imposed to create other updates. The quasi-Newton equation appears common to all, however. The references [4,10] present interesting surveys of work to date.

Exercise: Show that (22) and (23) satisfy (25) and (

BFGS Update

One of the most successful quasi-Newton updates is the so-called BFGS (Broyden-Fletcher-Goldfarb-Shanno) update, which is designed to be symmetric and positive-definite whenever the initial stiffness, $\underset{\sim}{K}$, is symmetric and positive-definite. The BFGS update can be written in the following factored form:

$$\bar{\underset{\sim}{K}}^{-1} = (\underset{\sim}{I} + \underset{\sim}{v}\,\underset{\sim}{w}^T)\,\underset{\sim}{K}^{-1}\,(\underset{\sim}{I} + \underset{\sim}{w}\,\underset{\sim}{v}^T) \tag{27}$$

where

$$\underset{\sim}{v} = -(\Delta\underset{\sim}{d} \cdot \Delta\underset{\sim}{R})^{-1}\,\Delta\underset{\sim}{d} \tag{28}$$

$$\underset{\sim}{w} = \Delta\underset{\sim}{R} + \{ -s\Delta\underset{\sim}{d} \cdot \Delta\underset{\sim}{R}\ (\Delta\underset{\sim}{d} \cdot \underset{\sim}{R})^{-1} \}^{1/2}\,\underset{\sim}{R} \tag{29}$$

If a line search is also employed, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ can be written as

$$\underset{\sim}{v} = -\Big(G(s) - G(0)\Big)^{-1}\,\Delta\underset{\sim}{d} \tag{30}$$

$$\underset{\sim}{w} = \Delta\underset{\sim}{R} + \Big[s\Big(1 - G(s)/G(0)\Big)\Big]^{1/2}\,\underset{\sim}{R} \tag{31}$$

In Equations 30 and 31, s is the "accepted value" of the search parameter.

The solution of Equation 21 is obtained as follows:

1. $\bar{\underset{\sim}{R}}_o = \bar{\underset{\sim}{R}} + (\underset{\sim}{v} \cdot \bar{\underset{\sim}{R}})\,\underset{\sim}{w}$
2. Solve $\underset{\sim}{K}\,\Delta\bar{\underset{\sim}{d}} = \bar{\underset{\sim}{R}}_o$ for $\Delta\bar{\underset{\sim}{d}}$
3. $\Delta\underset{\sim}{d}^{(i+1)} = \Delta\bar{\underset{\sim}{d}} + (\underset{\sim}{w} \cdot \Delta\bar{\underset{\sim}{d}})\,\underset{\sim}{v}$

Exercises: 1. Show that (27) satisfies the QN equati[on]
2. Obtain (30) and (31) from (28) and (29), resp-ively.

As may be seen again, no alterations of the factors of $\underset{\sim}{K}$ are necessary. When more vectors have accumulated, the situation is not unlike that for the Broyden update. The *formal* expression for $\underset{\sim}{\bar{K}}^{-1}$ in this case is given by

$$\underset{\sim}{\bar{K}}^{-1} = \left(\underset{\sim}{I} + \underset{\sim}{v}_N \underset{\sim}{w}_N^T\right)\left(\underset{\sim}{I} + \underset{\sim}{v}_{N-1}\underset{\sim}{w}_{N-1}^T\right)\cdots\left(\underset{\sim}{I} + \underset{\sim}{v}_1 \underset{\sim}{w}_1^T\right)\underset{\sim}{K}^{-1}\left(\underset{\sim}{I} + \underset{\sim}{w}_1 \underset{\sim}{v}_1^T\right)\left(\underset{\sim}{I} + \underset{\sim}{w}_2 \underset{\sim}{v}_2^T\right)$$

$$\cdots\left(\underset{\sim}{I} + \underset{\sim}{w}_N \underset{\sim}{v}_N^T\right) \tag{32}$$

The steps in the solution of Equation 21 are

1. Evaluate $\underset{\sim}{\bar{R}}_{k-1} = \underset{\sim}{\bar{R}}_k + (\underset{\sim}{v}_k \cdot \underset{\sim}{\bar{R}}_k)\,\underset{\sim}{w}_k$ for $k = N, N-1, \ldots, 1$
   where $\underset{\sim}{\bar{R}}_N = \underset{\sim}{\bar{R}}$
2. Solve $\underset{\sim}{K}\,\Delta\underset{\sim}{\bar{d}}_o = \underset{\sim}{\bar{R}}_o$ for $\Delta\underset{\sim}{\bar{d}}_o$
3. Evaluate $\Delta\underset{\sim}{\bar{d}}_k = \Delta\underset{\sim}{\bar{d}}_{k-1} + (\underset{\sim}{w}_k \cdot \Delta\underset{\sim}{\bar{d}}_{k-1})\,\underset{\sim}{v}_k$ for $k = 1, 2, \ldots, N$
4. $\Delta\underset{\sim}{d}^{(i+1)} = \Delta\underset{\sim}{\bar{d}}_N$

At this point a new pair of vectors, $\underset{\sim}{v}$ and $\underset{\sim}{w}$, is calculated and saved (see Eqs. 28 and 29). If a line search is performed, Equations 30 and 31 are used in place of Equations 28 and 29, respectively.

Whatever the update formula being used, when $\Delta\underset{\sim}{d}^{(i+1)}$ is calculated, it is subjected to convergence tests as described under the Newton-Raphson method.

Implicit-Explicit Mesh Partitions

In the preceding description of quasi-Newton updates, it has been assumed that the stiffness matrix, $\underset{\sim}{K}$, is used as the starting matrix. The formation and factorization of $\underset{\sim}{K}$ are often very expensive propositions. Since the quasi-Newton updates tend to build up the inverse of the stiffness, it is natural to consider other starting matrices that may be formed and factorized more economically.

let $\mathbb{K} = K(\underset{\sim}{d}^0)$

solve $\mathbb{K}\Delta d^{i+1} = \mathbb{R}^i = F^{ext} - F^{int}(d^i)$ let $d^{i+1} = d^i + \Delta d^{i+1}$

define $\mathbb{R}^{i+1} = \bar{\mathbb{R}} = F^{ext} - F^{int}(d^{i+1}) = \mathbb{K}\,\bar{\Delta d}_0$ for $\bar{\Delta d}_0$

For last time we have $\bar{\underset{\sim}{K}}^{-1} = (\underset{\sim}{I} + \underset{\sim}{c}s\Delta\underset{\sim}{d}^T)\cdot\underset{\sim}{K}^{-1}$ lets do this for n steps.

i.e. suppose we have $\bar{\underset{\sim}{K}}^{-1} = (\underset{\sim}{I} + \underset{\sim}{c}_N s_N \Delta\underset{\sim}{d}_N^T)(\underset{\sim}{I} + \underset{\sim}{c}_{N-1}s_{N-1}\Delta\underset{\sim}{d}_{N-1}^T)\cdots(\underset{\sim}{I} + \underset{\sim}{c}_1 s_1\Delta\underset{\sim}{d}_1^T)\underset{\sim}{K}^{-1}$ and $\Delta\underset{\sim}{d}^{(i+1)}$

how do we get to last step

Now solve 1. $\underset{\sim}{K}\,\Delta\bar{\underset{\sim}{d}}_0 = \bar{\underset{\sim}{R}}$ for $\Delta\bar{\underset{\sim}{d}}_0$ for each step. Use same factors for $\underset{\sim}{K}$.

skip for 1st time

2. evaluate $\Delta\bar{\underset{\sim}{d}}_k = \Delta\bar{\underset{\sim}{d}}_{k-1} + (\overset{s_k}{\Delta\underset{\sim}{d}_k}\cdot\Delta\bar{\underset{\sim}{d}}_{k-1})\underset{\sim}{c}_k$ for $k = 1, 2, \ldots, n-1$

now to evaluate $(\underset{\sim}{I} + \underset{\sim}{c}_N\Delta\underset{\sim}{d}_n^T)$ factor

From before $(1+s_N)$

3. now $\underset{\sim}{c}_N = (\Delta\underset{\sim}{d}_N - \Delta\bar{\underset{\sim}{d}}_{n-1}) / (s_N\Delta\underset{\sim}{d}_N\cdot(\Delta\bar{\underset{\sim}{d}}_{n-1} - \Delta\underset{\sim}{d}_N))$ as before

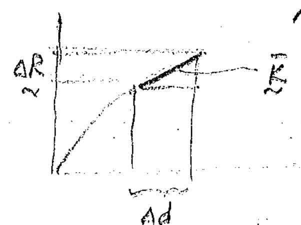
4. with $\Delta\underset{\sim}{d}^{(i+1)} = \Delta\bar{\underset{\sim}{d}}_{N-1} + (\overset{s_N}{\Delta\underset{\sim}{d}_N}\cdot\Delta\bar{\underset{\sim}{d}}_{N-1})\underset{\sim}{c}_N$

now $d^{i+1} = d^i + s^{i+1}\Delta\underset{\sim}{d}^{(i+1)}$ using line search first

Remarks

1. $\{\underset{\sim}{c}_k, s_k, \Delta\underset{\sim}{d}_k\}$ are stored on disk if line search
2. fix the strategy on the no. of updates. Discard oldest updates & redefine indices (ie replace by newest updates). Could lead to problems
3. periodically need to reform $\underset{\sim}{K}$ & start again
4. w/ regard to Broyden update. Even if K is symmetric $\bar{\underset{\sim}{K}}^{-1}$ is non symmetric. Most popular & effective procedure
5. Broyden is defined by (a) & (b)

(a) $\bar{\underset{\sim}{K}}\Delta\underset{\sim}{d} = \Delta\underset{\sim}{R}$ This is the quasi newton eqn ie $\bar{\underset{\sim}{K}}$ satisfy a secant relationships



All quasi newton relationships satisfy this

(b) $\bar{\underset{\sim}{K}}\underset{\sim}{z} = \underset{\sim}{K}\underset{\sim}{z}$ $\forall\underset{\sim}{z}$ ∋ $\underset{\sim}{z}\cdot\Delta\underset{\sim}{d} = \underset{\sim}{0}$ ie $\underset{\sim}{z}$ is ⊥ to $\Delta\underset{\sim}{d}$

this is only to Broyden

Good review of Quasi Newton update: Dennis & Moré, SIAM Review Vol 19 No. 1, Jan 77.

BFGS updates (Broyden - Fletcher - Goldfarb - Shanno) Defines $\bar{\underset{\sim}{K}}$ w/ following property: IF $\underset{\sim}{K}$ is symmetric, $\underset{\sim}{K} > 0$, then so is $\bar{\underset{\sim}{K}}$.

$$\bar{K}^{-1} = (I + v w^T) K^{-1} (I + w v^T)$$

$$\begin{cases} v = -(\Delta d \cdot \Delta R)^{-1} \Delta d & \Delta d \text{ from previous iter.} \\ w = \Delta R - \{s(\Delta d \cdot \Delta R) \quad (\Delta d \cdot R)^{-1}\}^{1/2} R \end{cases}$$

Simpler formulas exist when use this w/ line search.

$$\begin{cases} v = -(G(s) - G(0))^{-1} \Delta d \\ w = \Delta R - \left[-s\left(1 - G(s)/G(0)\right)\right]^{1/2} R \end{cases}$$

$\bar{R} = R(d^i + s\Delta d) \qquad R(0) = R$

To obtain $\Delta d^{(i+1)}$

1. Going from right to left: $\bar{R}_0 = \bar{R} + (v \cdot \bar{R}) w$
2. Solve $K \Delta \bar{d} = \bar{R}_0$ for $\Delta \bar{d}$
3. $\Delta d^{i+1} = \Delta \bar{d} + (w \cdot \Delta \bar{d}) v$

An example for n updates. $\bar{K}^{-1}$ looks like

$$\bar{K}^{-1} = (I + v_N w_N^T)(I + v_{N-1} w_{N-1}^T) \dots (I + v_1 w_1^T) K^{-1} (I + w_1 v_1^T) \dots (I + w_N v_N^T)$$

to get Δd^{i+1} for n updates

1. define $\bar{R}_{k-1} = \bar{R}_k + (v_k \cdot \bar{R}_k) w_k \quad k = N, N-1, \dots, 1$ with $\bar{R}_N = \bar{R}$
2. Now solve $K \Delta \bar{d}_0 = \bar{R}_0$ for $\Delta \bar{d}_0$ for each steps using same factors for K
3. $\Delta \bar{d}_k = \Delta \bar{d}_{k-1} + (w_k \cdot \Delta \bar{d}_{k-1}) v_k$ for $k = 1, \dots, n$
4. define $\Delta d^{(i+1)} = \Delta \bar{d}_N$

Remark on BFGS: take BFGS with $K = I$ & $N = 1$. This reduces to the conjugate gradient technique.

Behavior of the schemes –

Soft stiffness ask for trouble. Stiff stiffness matrix ⇒ slow convergence

to show this: soft / stiff tangent behavior:

look at a one DOF linear problem $kd = F$ $k > 0$

using Newton Raphson type algo

$$\tilde{k}\,\Delta d^{(i)} = F - kd^{(i)}$$

approx tangent Let $\tilde{k} = \alpha k$ if pick $0 < \alpha < 1$ soft tangent

$\alpha > 1$ stiff tangent.

Expand eqn. $\alpha k \left(d^{(i+1)} - d^{(i)} \right) = F - kd^{(i)}$

subtract $-[\alpha k (\underbrace{d - d}_{0}) = \underbrace{F - kd}_{0}]$ & define $e^{(i)} = d^{(i)} - d$

$$\alpha k \left(e^{(i+1)} - e^{(i)} \right) = -k\, e^{(i)}$$

$$\alpha k e^{(i+1)} = k(\alpha - 1)\, e^{(i)} \quad \therefore \quad e^{(i+1)} = \frac{\alpha - 1}{\alpha} e^{(i)} = A e^{(i)}$$

Now $e^{(i+1)} = \prod_{k=0}^{i+1} A\, e^{(0)} = A^{(i+1)} e^{(0)}$.

if $A = 0$ $\alpha = 1$; if $|A| < 1$ $-1 < A < 1 \Rightarrow -\alpha < \underbrace{\alpha - 1 < \alpha}_{\text{satisfied}}$

$\Rightarrow \alpha > 1/2$

$\therefore$ $|A| < 1$ converge if $\alpha > 1/2$

$|A| > 1$ diverge when $\alpha < 1/2$ ⇐ very soft stiffness!

if $|A| \approx 1$ (but less than (< 1)) $\Rightarrow \alpha >> 1$ ie too too stiff. converge slowly

$|A| \approx 0$ converge rapidly $\alpha \approx 1$.

11/2/81

Now we look at Convergence Rate of Newton Raphson

Last time we showed error satisfied $e^{(i+1)} = A e^{(i)}$

Now for non lin problems we will want to put error into same form ie

for N.R. $|e^{(i+1)}| = c\, |e^{(i)}|^2$ ← quadratic convergence

for Newton-Raphson we will show the above

To show this assume SDOF case: $N(d) = F$

Also assume

1. $N(d)$ is smooth
2. $N'(d) \neq 0 \quad \forall d$.
3. $\left| \frac{1}{N'(x)} \cdot N''(y) \right| \leq const = 2c$
4. $|c e^{(0)}| < 1$ a sufficiently close condition

Recall for Newton-Raphson (use of consistent tangent) N.R.

$$N'(d^{(i)}) \Delta d = F - N(d^{(i)})$$

we now show that N.R. converges ($e^{(i)} \to 0$, as $i \to \infty$)

Using Taylor's formula w/ remainder

$$e^{(i)} = d^{(i)} - d$$

$$F = N(d) = N(d_i) + N'(d_i) \cdot (-e^{(i)}) + \frac{N''(\bar{d})}{2} \cdot e^{(i)2} \quad d^i \leq \bar{d} \leq d \quad (*)$$

$$e^{(i+1)} = d^{(i+1)} - d \qquad \text{now } d^{(i+1)} = d^{(i)} + \Delta d = d^{(i)} + K^{-1}[F^{ext} - N(d^{(i)})]$$

$$= d^{(i)} + N'(d^{(i)})^{-1}(F - N(d^{(i)})) - d$$

N.R. approx

see pg. 19 back w/ $K = \partial N / \partial d (d^{(i)})$

$$= e^{(i)} + N'(d^{(i)})^{-1}(F - N(d^{(i)}))$$

now using the result from Taylor formula (*)

$$= \tfrac{1}{2} N'(d^{(i)})^{-1} N''(\bar{d}) e^{(i)2}$$

now use hypothesis 3.

$$|e^{(i+1)}| \leq \frac{2c}{2} |e^{(i)}|^2 = c\, e^{(i)2}$$

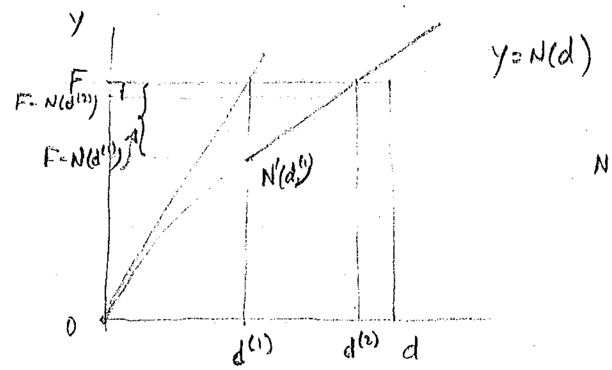
Thus using the trace back to $e^{(0)}$ & (4.) $\to$ N.R. converges.

<u>Number example</u>, let $e^{(0)} = .9$ $\quad c = 1$

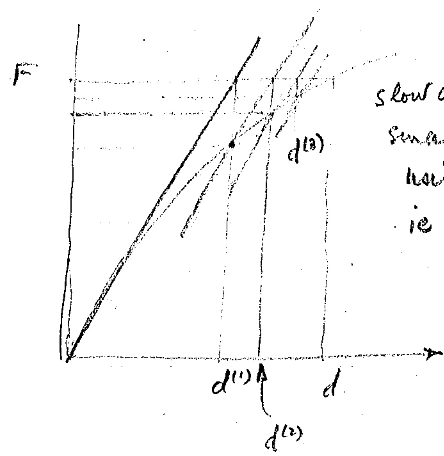
i	$\lvert e^{(i)} \rvert$ is $\leq$ than
1	.81
2	.66
3	.44
4	.19
5	.036
6	.0013
7	.000016

signif digits double each iter when you are close enough [rows 5–7]

this should be the effect of N.R.



N.R. how we get convergence



slow convergence since we're not using system info ie $K(d^{(1)}) \neq N'(d^{(1)})$

modified N.R. let $K(d) = N'(0)$

— We begin Development of Nonlinear Problems. We begin by looking at nonlinear heat as an analogous develop w/235 A linear heat conduction.

Nonlinear heat conduction

(S)

$$q_{i,i} = f \text{ in } \Omega$$

$$\text{w/ } u = g \text{ on } \Gamma_g \qquad \text{where } \overline{\Gamma_g + \Gamma_h} = \partial\Omega$$

$$-q_i n_i = h \text{ on } \Gamma_h$$

and the constit. eqn.

$$q_i = q_i(u; u_{,i})$$

$$= -\kappa_{ij}(u)\, u_{,j}$$

only nonlinearity allowed in present problem.

f, g, h are given, find u

Sidelight we can make κ_{ij} dependent on x but it won't change anything

(W) Find $u \in \mathcal{S}$ ($u = g$ on Γ_g) $\ni$ $\forall w \in \mathcal{V}$ ($w = 0$ on Γ_g)

so that $a(w, u; u) = (w, f) + (w, h)_\Gamma$

where $a(w, u; (u)) = \int_\Omega w_{,i}\, \kappa_{ij}(u)\, u_{,j}\, d\Omega$ w/ $\kappa_{ij} = \kappa_{ji}$

$(w,f) = \int_\Omega wf\,d\Omega \qquad (w,h)_\Gamma = \int_{\Gamma_h} wh\,d\Gamma$

(G) Find $u^h \in \mathcal{S}^h$, $\ni$ $\forall w^h \in \mathcal{V}^h$

$$a(w^h, u^h; u^h) = (w^h, f) + (w^h, h)_\Gamma$$

expand in terms of shape fns & nodal values. This leads to

(M) $\quad \underset{\sim}{N}(\underset{\sim}{d}) = \underset{\sim}{F} \qquad \underset{\sim}{N}(\underset{\sim}{d}) = \overset{n_{el}}{\underset{e=1}{\mathsf{A}}} (\underset{\sim}{n}^e(\underset{\sim}{d}^e))$ — temp vector in local

$\underset{\sim}{F} = \underset{\sim}{F}_{nodal} + \overset{n_{el}}{\underset{e=1}{\mathsf{A}}} (\underset{\sim}{f}^e)$ — A is always same: must define $\underset{\sim}{f}^e$, $\underset{\sim}{n}^e(\underset{\sim}{d}^e)$

define $\underset{\sim}{n}^e = \{n_a^e\}$; $\underset{\sim}{f}^e = \{f_a^e\}$

a: local node nos.

$n_a^e(\underset{\sim}{d}^e) = -\int_{\Omega^e} \underset{\sim}{B}_a^T \underset{\sim}{q}\, d\Omega$ — $\underset{\sim}{B}_a$: gradient of shape fns.

$\underset{\sim}{q} = \underset{\sim}{D}(\underset{\sim}{u}^h(\underset{\sim}{x})) \sum_{c=1}^{n_{en}} \underset{\sim}{B}_c \underset{\sim}{d}_c^e$ — $\underset{\sim}{D}$: conductivity

$f_a^e = \int_{\Omega^e} N_a f\,d\Omega + \int_{\Gamma_h^e} N_a h\,d\Gamma$ — we will not account for g boundary conditions here.

$\underset{\sim}{u}^h = \sum_{b=1}^{n_{en}} N_b d_b^e$ — Reason: first $\Delta \underset{\sim}{d}$ at boundaries is $\underset{\sim}{0}$; second: it is embedded in $\underset{\sim}{N}(\underset{\sim}{d})$

define the consistent tangent array,

$D\underset{\sim}{N}(\underset{\sim}{d}) = \overset{n_{el}}{\underset{e=1}{\mathsf{A}}} (D\underset{\sim}{n}^e(\underset{\sim}{d}^e)) \qquad D\underset{\sim}{n}^e = [\partial n_a^e / \partial d_b^e]$ — Jacobian

$N'(\underset{\sim}{d})$ ''

drop e superscripts for conven.

$$\frac{\partial n_a}{\partial d_b} = \frac{\partial}{\partial d_b} \int_{\Omega^e} \underset{\sim}{B}_a^T \underset{\sim}{D}\left(\sum N_c d_c\right)\left(\sum B_d d_d\right) d\Omega$$

$$= \int_{\Omega^e} \underset{\sim}{B}_a^T \frac{\partial \underset{\sim}{D}}{\partial u}\left(\frac{\partial u}{\partial d_b}\right) \sum B_d d_d\, d\Omega + \int_{\Omega^e} B_a^T \underset{\sim}{D}(\ )\sum B_d \delta_{bd}\, d\Omega$$

($\frac{\partial u}{\partial d_b} = \sum N_c \delta_{cb} = N_b$; $\sum B_d \delta_{bd} = B_b$)

not symmetric in a, b. — this is symmetric (and is like the linear case) in a, b

11/5/81

Continuation of last time

Remember

$$\underset{\text{local node}}{d_a^e} = \begin{cases} d_P^{(i)} & LM(a,e) \neq 0 = P \\ g_A & LM(a,e) = 0 \quad A = IEN(a,e) \end{cases}$$

prescribed bdy cond.

$$D\underset{\sim}{N}(\underset{\sim}{d}^{(i)})\ \Delta \underset{\sim}{d} = \underset{\sim}{F} - N(\underset{\sim}{d}^{(i)})$$

1-D analog of non lin heat cond.

Given the following strong form.

(S) $\quad q_{,x} = f \qquad \Omega = (0,1) \qquad\qquad \hat{n}(0) \quad h \quad g$

$u(1) = g$

$\underbrace{(-1)}_{\underset{\sim}{n}(0)} \cdot (-q(0)) = h$

w/ $\quad q(u, u_{,x}; x) = -\kappa(u; x)\, u_{,x}$

(W) $\quad \eta(w,u) = (w,f) + (w,h)_{\Gamma} \qquad\qquad \eta(w,u) = a(w,u;u)$

$$\eta(w,u) \triangleq \int_{\Omega}^{1} w_{,x}\, \kappa(u;x)\, u_{,x}\, dx$$

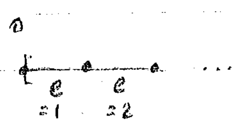
$$(w,f) = \int_0^1 wf\, dx \qquad (w,h)_{\Gamma} = w(0)\, h$$

(G) is easily obtained w/ superscripts h.

(M) is defined as follows

$$n_a^e(\underset{\sim}{d}^e) \triangleq -\int_{\Omega^e} N_{a,x}\, q\, dx \qquad \text{equiv to } \overset{\Delta}{B}{}_a^e(\xi) \text{ is not fn. of } \underset{\sim}{d}^e$$

$$f_a^e = \int_{\Omega^e} N_a\, f\, dx + \begin{cases} h\,\delta_{a1} & e = 1 \\ 0 & e \neq 1 \end{cases}$$



$$\partial n_a^e / \partial d_b^e = \int_{\Omega^e} N_{a,x} \frac{\partial \kappa}{\partial u}(u^h_{;x}) \underbrace{\sum_{c=1}^{n_{en}} N_{c,x}\, d_c}_{u_{,x}}\, N_b\, dx$$

now $\dfrac{\partial u}{\partial d_b} = \sum N_a \delta_{ab} = N_b$

now $\dfrac{\partial u_{,x}}{\partial d_b} = \dfrac{\partial \sum N_{d,x}\, d_d}{\partial d_b} = \sum N_{d,x}\, \delta_{db} = N_{b,x}$

$$+ \int_{\Omega^e} N_{a,x}\, \kappa(u^h)\, N_{b,x}\, dx$$

$$N_{a,\xi}\, \xi_{,x}\, \kappa(u^h)\, N_{b,\xi}\, \cancel{\xi_{,x}}\, \cancel{x_{,\xi}}\, d\xi$$

These are general eqns.

For special case of 2-node linear elements (1-pt Gauss quad i.e. evaluated at center, w/ W = 2) in this case $d\xi = \frac{2}{h} dx$ = d(area) of integrale

$$j = h/2 \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \cdot \frac{2}{h}$$

$$\underset{\sim}{f}^e = h^e \cdot \frac{f_1^e + f_2^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \text{we assumed } f = \sum_{a=1}^{2} N_a f_a^e \qquad \int N_a f_e \, d\Omega = N_a(0) f_e(0) \cdot 2 \cdot \frac{h}{2} = \left(\tfrac{1}{1}\right) f_e(0) h^e$$

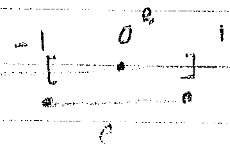
$$\underset{\sim}{n}^e(\underset{\sim}{d}^e) = -q\left(u^h(0^e),\ u^h_{,x}(0^e),\ x(0^e)\right) \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$u^h(0)^e = \sum N_d(0) d_d^e = \tfrac{1}{2}(d_1^e + d_2^e)$$

$$u^h(0^e) = \frac{d_1^e + d_2^e}{2}, \qquad u^h_{,x}(0^e) = (d_2^e - d_1^e)/h^e, \qquad x(0^e) = \frac{x_1^e + x_2^e}{2}$$

$$\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \leftarrow N_{a,\xi}\, \xi_{,x} \cdot x_{,\xi} \cdot wt \qquad u^h_{,x} = u^h_{,\xi} \cdot \frac{2}{h} = \left(\tfrac{1}{2} d_2 - \tfrac{1}{2} d_1\right) \cdot \frac{2}{h}$$

$$D\underset{\sim}{n}^e(\underset{\sim}{d}^e) = \frac{\partial \kappa}{\partial u}\left(u^h(0^e), x(0^e)\right) \cdot \frac{d_2^e - d_1^e}{h} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} + \frac{\kappa\left(u^h(0^e), x(0^e)\right)}{h^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



$$\frac{\partial u}{\partial x}$$

For Nonlinear Elastostatic Assumptions

1. Small strains
2. No initial stresses
3. Nonlinearities enter through material behavior only.

(S) $\sigma_{ij,j} + f_i = 0$ in Ω

$u_i = g_i$ on Γ_{g}

$\sigma_{ij} n_j = h_i$ on Γ_h

The constit eqn, becomes,

$$\sigma_{ij} = \partial \Phi(\underset{\sim}{\varepsilon}) / \partial \varepsilon_{ij} \qquad \varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$$

Φ is assumed given

$$c_{ijkl}(\underset{\sim}{\varepsilon}) = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial^2 \Phi}{\partial \varepsilon_{ij}\, \partial \varepsilon_{kl}}$$

this shows major symmetry $c_{ijkl} = c_{klij}$

this major symmetry $\Rightarrow$ symmetric tangent.

Problem is find the displ field

for 1 pt gauss $W = 2$

$$\int B_a^T \tilde{q}\, d\Omega = B_a^T(0)\, q(0)\, \frac{h}{2} \cdot 2 \quad = -\int N_{a,x}\, \kappa(u;x)\, u_{,x}\, d\Omega$$

$$B_a^T(0) = \frac{2}{h}\begin{Bmatrix} -\tfrac{1}{2} \\ \tfrac{1}{2} \end{Bmatrix} \qquad \begin{matrix} a=1 \\ a=2 \end{matrix} \qquad \tilde{q}(0) = -\kappa(u(0))\, \tilde{u}_{,x}(0) = -\kappa\left(\frac{d_1+d_2}{2}\right)\frac{2}{h}\left(\frac{d_2-d_1}{2}\right)$$

$B_a(x) = B_a(\xi)\, \xi_{,x}$ $\qquad \xi_{,x}\, u_{,\xi}$

$$n_a^e = -\frac{1}{2}\cdot\frac{h}{2}\cdot 2 \cdot \frac{2}{h}\, q\left(\frac{d_1+d_2}{2}; \frac{x_1+x_2}{2}\right)\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \qquad u_{,x} = \frac{d_2^e - d_1^e}{h}$$

$$\text{now} \quad \frac{\partial n_a}{\partial d_b} = -\int_{\Omega^e} N_{a,x}\, \frac{\partial \kappa}{\partial u}(u^h;x)\, u_{,x} \cdot N_b\, dx + \int_{\Omega^e} N_{a,x}\, \kappa(u^h)\, N_{b,x}\, dx$$

$$\int N_{a,\xi}\, \xi_{,x}\, \kappa(u(0))\, N_{b,\xi}\, \xi_{,x} \cdot x_{,\xi}\, d\xi$$

$$= \int_{\Omega^e} N_{a,\xi}\, \xi_{,x}\, \frac{\partial \kappa}{\partial u}(\ )\, u_{,\xi}\, \xi_{,x}\, N_b\, x_{,\xi}\, d\xi \qquad \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\cdot\frac{2}{h}\cdot\kappa \cdot \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 2 \ \text{(weight)}$$

$$= \left[\frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\cdot\frac{2}{h}\,\frac{\partial \kappa}{\partial u}(u(0);x)\, \frac{d_2-d_1}{2}\, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] 2$$

$$\frac{d_2-d_1}{2h}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\cdot\frac{\partial \kappa}{\partial u}\left(\frac{d_1+d_2}{2}; x_0\right) + \frac{2}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{2}{h^e}\,\kappa\left(\frac{d_1+d_2}{2}, x\right)$$

(weight; from gradient)

for trapez. weight j

$$n_a^e = \int_{\Omega^e} B_a^T \, \underset{\sim}{q} \, d\Omega = \left\{ [B_a^T \underset{\sim}{q}](-1) + [B_a^T \underset{\sim}{q}](1) \right\} 1 \cdot \frac{h}{2}$$

$$\underset{\sim}{q} = -\underset{\sim}{\kappa}(\underset{\sim}{u}) \, \underset{\sim}{u}_{,x}$$

where $\underset{\sim}{u} = \sum N_a d_a$ $\quad x = \sum N_a x_a \quad x_{,\xi} = \sum N_{a,\xi} x_a = \frac{h}{2}$

$$u_{,x} = u_{,\xi} \, \xi_{,x} = \sum N_{a,\xi} \, d_a \cdot \xi_{,x}$$

$$u_{,x} = \frac{2}{h} \sum N_{a,\xi} d_a = \frac{2}{h} \left(\frac{d_2 - d_1}{2} \right)$$

$$\underset{\sim}{q} = -\underset{\sim}{\kappa}\left(\sum N_a d_a\right) u_{,x} = -\kappa\left(\sum N_a d_a\right) \frac{d_2 - d_1}{h}$$

$N_1 = \frac{(1-\xi)}{2} \quad N_{1,\xi} = -\frac{1}{2}$

$N_2 = \frac{(1+\xi)}{2} \quad N_{2,\xi} = \frac{1}{2}$

$$\underset{\sim}{q}(-1) = -\kappa(d_1) \frac{d_2 - d_1}{h} \qquad \underset{\sim}{q}(+1) = -\kappa(d_2) \frac{d_2 - d_1}{h}$$

$$B_a(-1) = -\frac{1}{2} \qquad B_a(+1) = \frac{1}{2}$$

$$= \left\{ -\kappa(d_1) \frac{d_2 - d_1}{h} \left\{ -\frac{1}{2} \right\} + -\kappa(d_2) \frac{d_2 - d_1}{h} \left\{ \frac{1}{2} \right\} \right\} \cdot 2 \cdot \frac{h}{2}$$

Need to define

(W) Find $\underset{\sim}{u} \in \mathcal{S}$ (ie $u_i = g_i$ on Γ_g) $\ni$ $\forall w \in \mathcal{V}$ (ie $w_i = 0$ on Γ_g)

$\eta(w; u) = (w, f) + (w, h)_{\Gamma_h}$ $\qquad$ $\eta(w; u) = \int_\Omega w_{(i,j)}\, \sigma_{ij}(\varepsilon)\, d\Omega$

$w_{(i,j)} = \frac{1}{2}(w_{i,j} + w_{j,i})$ symmetric $\qquad$ $(\underset{\sim}{w}, \underset{\sim}{f}) = \int w_i f_i \, d\Omega$

$(\underset{\sim}{w}, \underset{\sim}{h})_\Gamma = \int_{\Gamma_h} w_i h_i \, d\Gamma$

(G) $\Rightarrow$ "h" sup everywhere

(M) $\underset{\sim}{N}(\underset{\sim}{d}) = \underset{\sim}{F}$ $\qquad$ $\underset{\sim}{N}(\underset{\sim}{d}) = \underset{e=1}{\overset{nel}{A}} \left(\underset{\sim}{n}^e(\underset{\sim}{d}^e)\right)$

internal force vector; external force

$$\underset{\sim}{F} = \underset{\sim}{F}_{nodal} + \underset{e=1}{\overset{nel}{A}} (\underset{\sim}{f}^e)$$

$\underset{\sim}{n}^e(d^e) = \{ n_p^e(\underset{\sim}{d}^e) \}$ $\qquad 1 \le p \le n_{ee}$

(p – local eqn. no.; n_{ee} – element eqn.)

$\underset{\sim}{f}^e = \{ f_p^e \}$

$$\underset{\sim}{n}_p^e(\underset{\sim}{d}^e) = \underset{\sim}{e}_i^T \int_{\Omega^e} B_a^T \underset{\sim}{\sigma}\, d\Omega \qquad p = ned(a-1) + i$$

internal force at p; unit basis vector; strain-disp vector; stress vector

in 2D for Plane Strain

$$\underset{\sim}{e}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad \underset{\sim}{e}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \qquad \underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \qquad \underset{\sim}{B}_a = \begin{bmatrix} N_{a,x} & 0 \\ 0 & N_{a,y} \\ N_{a,y} & N_{a,x} \end{bmatrix}$$

now $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}(\underset{\sim}{\varepsilon})$ $\qquad$ $\underset{\sim}{\varepsilon} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix}$

and $f_p^e = \int_{\Omega^e} N_a f_i \, d\Omega + \ldots$

Continuation of Elasticity 11/7/81

Define d^e

$\tilde{d}^e :\quad \tilde{d}^e_p = \begin{cases} d_P \\ g_{Ai} \end{cases}$ elem eqn no.

global eqn $P = LM(i, a, e) \neq 0$ (direct no, node no, elem.)

$0 = LM(\quad)$ look up global node no. A

$A = IEN(a, e)$

$p = n_{ed}(a-1)+i$ heat eqn $i=1$

$D\tilde{N}(\tilde{d}) = \mathop{A}_{e=1}^{n_{el}} (D\tilde{n}^e(\tilde{d}^e))$ drop super e.

tangent modulus array

$D\tilde{N} = [\partial n_p / \partial d_q]$ p, q are local eqn nos. for $q = n_{ed}(b-1)+j$

$\partial n_p/\partial d_q = \partial n_{ia}/\partial d_{jb} = \frac{\partial}{\partial d_{jb}} \tilde{e}_i^T \int_{\Omega^e} \tilde{B}_a^T \tilde{\sigma}(\tilde{\varepsilon})\, d\Omega$ — fixed; in infinitesimal elast this only changes

$\tilde{\varepsilon} = \sum_{c=1}^{n_{en}} \tilde{B}_c \tilde{d}_c$ where d_c in (2-D case) is $\begin{Bmatrix} d_{1c} \\ d_{2c} \end{Bmatrix}$

$\therefore \frac{\partial}{\partial d_{jb}} \tilde{e}_i^T \int_\Omega \tilde{B}_a^T \tilde{\sigma}(\tilde{\varepsilon})\, d\Omega = \tilde{e}_i^T \int_\Omega \tilde{B}_a^T \frac{\partial}{\partial d_{jb}} \sigma\, d\Omega = \tilde{e}_i^T \int \tilde{B}_a^T \frac{\partial \sigma}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial d_{jb}}\, d\Omega$

$\frac{\partial \varepsilon}{\partial d_{jb}}$: $\sum \tilde{B}_c \frac{\partial}{\partial d_{jb}} \tilde{d}_c = \sum \tilde{B}_c \delta_{cb} e_j$

$= \tilde{e}_i^T \int_\Omega \tilde{B}_a^T \frac{\partial \tilde{\sigma}}{\partial \tilde{\varepsilon}} \tilde{B}_b \tilde{e}_j\, d\Omega$

$\frac{\partial \tilde{\sigma}}{\partial \tilde{\varepsilon}}$: $\tilde{D}(\tilde{\varepsilon}) \rightarrow$ matrix of components of C_{ijkl} (I = ij, J = kl) see 235A notes
$= D_{IJ}$

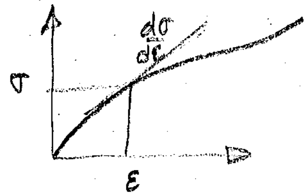
$\partial n_p/\partial d_q = \tilde{e}_i^T \left\{ \int_\Omega B_a^T \tilde{D} \tilde{B}_b\, d\Omega \right\} \tilde{e}_j$ if C_{ijkl} has major symmetry $\Rightarrow$ D must be symmetric $\therefore D_{IJ} = D_{JI}$

$\{\,\}$: $D\hat{n}^e_{ab}$ — nodal block of tangent stiffness

$= \frac{ia}{p} \quad \frac{bj}{q}$ if we were to switch indices & use major symmetry of D then $\Rightarrow \partial n_p/\partial d_q$ = symmetric we now look at 1-D version of preceding theory. (eg simple rod, string)

(S) $\sigma_{,x} + f = 0$ ⇐ DE on $\Omega = (0,1)$
$u(1) = g$
$-\sigma(0) = h$ where $\underset{\sim}{\sigma} \cdot n = -1 \cdot \underset{\sim}{\sigma}$

The constitutive eqn. $\sigma = \sigma(\varepsilon)$ the tangent array $C = C(\varepsilon) = d\sigma/d\varepsilon$



for a rod $E = E(\varepsilon)$ young's modulus.
$\varepsilon = \partial u/\partial x$

(S)⇒(W) $\eta(w,u) = (w,f) + (w,h)_\Gamma$
$= \int_0^1 w_{,x}\, \sigma(u_{,x})\, dx$

$(w,f) = \int_0^1 wf\,dx$; $(w,h)_\Gamma = h\,w(0)$

must find $u \in \delta$ ($u = g$ on $x=1$) w/ $w \in \mathcal{V}$ ($w(1)=0$)

(G) ⇒ h superscripts (W)

(M) $\underset{\sim}{n}^e(\underset{\sim}{d}^e) = \{ n_a^e(\underset{\sim}{d}^e) \}$; $\underset{\sim}{f}^e = [f_a^e]$
1 degree of freedom

e_1 = unit vector 1x, B

$n_a^e(\underset{\sim}{d}^e) = \int_{\Omega^e} N_{a,x}\, \sigma(\underset{\sim}{\varepsilon})\, dx = \int_{\Omega^e} N_{a,\xi}\, \xi_{,x}\, \sigma(\underset{\sim}{\varepsilon})\, x_{,\xi}\, d\xi \to \frac{1}{2}\binom{-1}{1} \sigma(\varepsilon(0)) \cdot 2$ ← quad

now f_a^e = is same as 1d heat (see pg 24 front) ie $= \int_0^1 N_a f\,dx + \begin{cases} N_1'(0)h \\ 0 \text{ finally} \end{cases}$

$D\underset{\sim}{n}^e(\underset{\sim}{d}^e) = [\partial n_a/\partial d_b] = \int_{\Omega^e} N_{a,x} \frac{\partial \sigma}{\partial \varepsilon}(\varepsilon) \frac{\partial \varepsilon}{\partial d_b}\, dx$ this is symmetric

$\left[\text{if } \varepsilon = u^h_{,x} = \sum N_{c,x} d_c^e \Rightarrow \frac{\partial \varepsilon}{\partial d_b} = \sum N_{c,x} \delta_{cb} = N_{b,x} \right]$

$\varepsilon(0) = \frac{d_2 - d_1}{2} \cdot \frac{2}{h}$

$N_{a,\xi}\, \xi_{,x} \frac{\partial \sigma}{\partial \varepsilon}(\varepsilon) \cdot N_{b,\xi}\, \xi_{,x}\, x_{,\xi}\, d\xi$
$\frac{1}{2}\binom{-1}{1} \cdot \frac{2}{h}\, C(\varepsilon_0) \cdot \frac{1}{2}\binom{-1}{1} \cdot 2$

in elasticity we always use consistent tangent array. In other ~~arrays~~ areas we will ~~do~~ not use consistent since might be computationally difficult.

~~Fo~~ 1-D arrays of 2-node linear element, assume 1 pt gauss quad

$\underset{\sim}{f}^e$ = looks at heat cond, pg 24 back $W = 2$ jacob $= h^e/2$

$$n_a^e = 2\frac{h^e}{2} N_{a,x}(0^e)\,\sigma(\varepsilon(0^e)) = (-1)^a \sigma(\varepsilon(0^e))$$

(quadrature rule; $\frac{h^e}{2}$ = jacobi; $N_{a,x} = (-1)^a/h^e$; $\varepsilon(0^e) = \frac{d_2^e - d_1^e}{h^e}$) see page 24 back

$$\text{Now } \partial n_a^e/\partial d_b^e = 2\frac{h^e}{2} \frac{(-1)^a}{h^e}\, C(\varepsilon(0^e))\, \frac{(-1)^b}{h^e}$$

(quad rule; B_a; $B_b = N_{b,x}$)

this affects material tangent $\underset{\sim}{D}n(\underset{\sim}{d})$

$$= \frac{1}{h^e} C\left(\frac{d_2^e - d_1^e}{h^e}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \checkmark\checkmark$$

Now consider coding aspects of nonlinear : all examples will be 2-D

<u>we will need routines</u> to obtain $\underset{\sim}{d}^e$

obtain $\underset{\sim}{\varepsilon}(\tilde{\xi}_l)$ at some quadrature point $\tilde{\xi}_l$

also define a constitutive routine to obtain

$\underset{\sim}{\sigma}(\underset{\sim}{\varepsilon}(\tilde{\xi}_l))$

also need to define $\underset{\sim}{D}(\underset{\sim}{\varepsilon}(\tilde{\xi}_l))$

if they depend explicitly on $\underset{\sim}{x}$ we need to define $\underset{\sim}{x} = \underset{\sim}{x}(\tilde{\xi}_l)$

14/9/81

Want to consider coding aspects of tangent stiffness on element level (Dn^e as well as $(\underset{\sim}{f}^e - \underset{\sim}{n}^e)$. There are differences wrt linear problem. Need to find $\underset{\sim}{d}^e$ first

-①→ 1. Loop on elements $1 \le e \le n_{el}$ (no. of elements in element group)
[Clear arrays] and get current nodal disp. vector $\underset{\sim}{d}^e$, $\underset{\sim}{x}^e$

-②→ 2. Loop on integration points $1 \le l \le n_{int}$
Call shape fn routine → gives

3. Form $\underset{\sim}{\varepsilon}$ (at integ point), $\underset{\sim}{\sigma}(\underset{\sim}{\varepsilon})$ (at integ pt), $\underset{\sim}{D}(\underset{\sim}{\varepsilon})$ at integ point } at $\tilde{\xi}_l$

also calculate $\underset{\sim}{f}$ at int pt. $= \int_{\Omega^e} N_a f \, d\Omega + \int_{\Gamma^e} N_a h \, d\Gamma$

we will redefine $\underset{\sim}{\sigma} \leftarrow j_l W_l \underset{\sim}{\sigma}$ $\quad \underset{\sim}{D} \leftarrow j_l W_l \underset{\sim}{D}$

-①→ 4. Loop on nodes $1 \le b \le n_{en}$
form $\underset{\sim}{D}\underset{\sim}{B}_b$

for 2D case $\underset{\sim}{D}\underset{\sim}{B}_b$ is 3×2 = $\begin{bmatrix} DB_{11} & DB_{12} \\ DB_{21} & DB_{22} \\ DB_{31} & DB_{32} \end{bmatrix}$

11/11/81

Vainbergs thm if $Dn(d) = (Dn(d))^T \Rightarrow \exists$ a potential $\ni$. $Dn(\underset{\sim}{d}) = \partial\Phi/\partial\underset{\sim}{d}$

→ 11/13 Exam — obj. nonlin & formulating such.
- transient analy (1) problem
- EV & $\overline{EV}$

Return to problem at hand! σ - cauchy stress tensor
σ^* - Objective rate

Objective Rates

$$\sigma_{ij}^* = \dot{\sigma}_{ij} - A_{ijkl}\, v_{[k,l]} - \overset{*}{C}_{ijkl}\, v_{(k,l)} = C_{ijkl}\, v_{(k,l)}$$ (from back of pg27)

rotation measure — deformation measure.

$v_{[k,l]}$ = antisymmetric part = $(v_{k,l} - v_{l,k})/2$

Same for all terms: $A_{ijkl}(\underset{\sim}{\sigma})$ only $= (\sigma_{il}\delta_{jk} + \sigma_{jl}\delta_{ik} - \sigma_{ik}\delta_{jl} - \sigma_{jk}\delta_{il})/2.$

$A_{ijkl} = A_{jikl}$ but $A_{ijkl} = -A_{ijlk}$
$\overset{*}{C}_{ijkl} = \overset{*}{C}_{jikl}$ $\overset{*}{C}_{ijkl} = \overset{*}{C}_{ijlk}$

— Normally $\dot{\sigma}_{ij}$ does not transform correctly under rotation ie $(\sigma_{ij}Q_{ik}Q_{jl})^{\cdot} \neq \dot{\sigma}_{ij}Q_{ik}Q_{jl}$ where Q_{ik}, Q_{jl} are rotation matrices (time dependent) & we picks up extra terms.
To make $\dot{\sigma}_{ij}$ transform right we must remove the extra terms; we add to $\dot{\sigma}_{ij}$, $-A_{ijkl}\,v_{[k,l]}$. hence under transformation C_{ijkl} & $v_{(k,l)}$ transform in standard way & $\Rightarrow$ so must σ^*_{ij}. However $\dot{\sigma}_{ij}$ does not. We however get a non symmetric tangent if we stop there $\therefore$ add this $\overset{*}{C}_{ijkl}$ terms. why?

Reason. now what is $\overset{*}{C}_{ijkl}$? Thats the rub! Choices are:

1. let $\overset{*}{C}_{ijkl} = 0 \Rightarrow \sigma^*_{ij} = \overset{\nabla}{\sigma}_{ij}$ (JAUMANN RATE of σ_{ij}) $= \dot{\sigma}_{ij} - A_{ijkl}v_{[k,l]}$
Using this leads to ~~thermodynamics~~ ~~leads to~~ objection. bc this leads to non symmetric tangent $\Rightarrow \not\exists\ \Phi$ (→←) in elasticity

2. $\tau_{ij} = J\sigma_{ij}$ (J = det $\underset{\sim}{F}$)
Kirchoff stress

if we define $\overset{*}{C}_{ijkl} = \overset{\vee}{C}_{ijkl} \triangleq -\sigma_{ij}\delta_{kl} \Rightarrow \sigma^*_{ij} = J^{-1}\cdot$ (Jaumann rate of Kirc[hoff] stress)
$= J^{-1}\overset{\nabla}{\tau}_{ij}$

$\overset{\nabla}{\tau}_{ij} = \dot{\tau}_{ij} - A_{ijkl}(\underset{\sim}{\tau})\, v_{[k,l]}$ Used by Lee & Hughes

This will lead to a symmetric tangent.

3. $\overset{*}{C}_{ijkl} = \hat{C}_{ijkl} = -\sigma_{ij}\delta_{kl} + (\sigma_{il}\delta_{jk} + \sigma_{jl}\delta_{ik} + \sigma_{ik}\delta_{jl} + \sigma_{jk}\delta_{il})/2$

to knock off non symmetricness of tangent.

This leads to Truesdell Rate.

From this nonlin elasticity naturally unfolds

Thus solving for $\overset{\circ}{\sigma}_{ij} = \overline{C}_{ijkl}(\sigma, F)\, v_{(k,l)} + A_{ijkl}\, v_{[k,l]}$ (†) (deformation, rotation)

$\overline{C}_{ijkl}(\sigma, F) + \overset{*}{C}_{ijkl}(\sigma)$

We will assume $\overset{\circ}{\sigma}_{ij}$ is given

in non lin elast. if $\sigma = \sigma(F)$ $\overset{\circ}{\sigma}$ will lead to (†) where $C^* = \hat{C}$

Nonlinear elasticity

We define $\sigma = J^{-1} P F^T$ $P = F S$ $S = S^T$

first Piola-Kirchoff stress tensor; S — 2nd Piola Kirchoff stress tensor

This $\Rightarrow \sigma_{ij} = J^{-1} P_{iA} F_{Aj}$ $P_{iA} = F_{iB} S_{BA}$ capital subscripts represents differentiation in current deformed state

Now $S_{AB} = S_{AB}(E)$ is assumed where $2E_{AB} = F_{iA}F_{iB} - \delta_{AB} = (\delta_{iA} + u_{i,A})(\delta_{iB} + u_{i,B}) - \delta_{AB}$

$= \delta_{iA}u_{i,B} + \delta_{iB}u_{i,A} + u_{i,A}u_{i,B}$

$= u_{A,B} + u_{B,A} + u_{i,A}u_{i,B}$

Lagrangian strain tensor $S_{AB} = \partial \Phi(E)/\partial E_{AB}$

$\dfrac{\partial S_{AB}}{\partial E_{CD}} = C_{ABCD}$ possesses all symmetries

$C_{ijkl}(F) = J^{-1} F_{iA} F_{jB} F_{kC} F_{lD} C_{ABCD}(E(F))$.

$\overset{*}{C}_{ijkl} = \hat{C}_{ijkl}$

11/16/81

From last time

$\overset{\circ}{\sigma}_{ij} = \overline{C}_{ijkl}(\sigma, F)\, v_{(k,l)} + A_{ijkl}\, v_{[k,l]}$ (†) $\overline{C}_{ijkl} = C_{ijkl} + \overset{*}{C}_{ijkl}$

for non linear elasticity $\sigma = \sigma(E)$

Consider another model – typifies what goes on in engineering practice today.
Example. Perfect plasticity w/ small deformations (Classical)

$c_{ijkl}(\underset{\sim}{\sigma}) = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - c\sigma'_{ij}\sigma'_{kl}$; λ, μ Lamé parameters
normally for elastic material

where σ'_{ij} (deviatoric Cauchy tensor) $= \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$ (pressure)
c is a loading/unloading param $= \begin{cases} 0 & \text{elastic loading, unloading} \\ 2\mu/(\sigma'_{ij}\sigma'_{ij}) & \text{plastic loading.} \end{cases}$

thus "$\dot{\sigma}_{ij}$" = c_{ijkl} "$\dot{\varepsilon}_{kl}$" Classical, small deformation plasticity (1)

For large deformation replace (1) by (†) (back of pg 28).
we need $c_{ijkl}(\underset{\sim}{\sigma}, \underset{\sim}{F})$ above w/ no dependence on $\underset{\sim}{F}$
we need also $\overset{*}{c}_{ijkl}$ pick examples 1–3 from last lecture
$= 0 \Rightarrow$ Jaumann
$= \tilde{c}_{ijkl} \Rightarrow$ Jaumann rate of Kirchoff
$= \hat{c}_{ijkl} \Rightarrow$ Truesdell.

For remaining work – we will select Truesdell $\hat{c}_{ijkl}$ without loss in generality.
for any thing else modify c_{ijkl} remember in (†) $\bar{c} = c + \hat{c}$

Thus if we want a rate other than truesdell then replace
$$c_{ijkl} \leftarrow c_{ijkl} - \hat{c}_{ijkl} + c^*_{ijkl} \Rightarrow \bar{c} = c + c^*$$
removes truesdell, adds whatever.

Go on to linearized variational eqns to generate algorithmic matrix eqn.
consistent tangent
We want $D\underset{\sim}{N}(\underset{\sim}{d})\Delta\underset{\sim}{d} = -\underset{\sim}{N}(\underset{\sim}{d})$: get the from weak form, Weak form gives RHS
We will want incremental version of (†). This will represent time discretization

get rid of Δt since (†) is homog. in Δt
$$\left\{ \frac{\Delta\sigma_{ij}}{\Delta t} = \bar{c}_{ijkl}\frac{\Delta u_{(k,l)}}{\Delta t} + A_{ijkl}\frac{\Delta u_{[k,l]}}{\Delta t} \right\} \cdot \Delta t$$

Now for the weak form of variational eqn. over the current conf Ω, Γ_h

$$\int_\Omega w_{(i,j)}\sigma_{ij}\, d\Omega = \underbrace{\int_\Omega w_{i,j}\sigma_{ij}\, d\Omega}_{F^{int}} = \underbrace{\int_\Omega w_i f_i\, d\Omega + \int_{\Gamma_{\underset{\sim}{h}}} w_i h_i\, d\Gamma}_{F^{ext}}$$

where $\sigma_{ij} = \sigma_{ij}(\underset{\sim}{F})$ in elasticity

we want to construct a DN directly on the $\int_\Omega$; now Ω, σ, w are all fns of ϕ
use chain rule to change variables to fix changes in body.

Now $\int_\Omega w_{i,j}\,\sigma_{ij}\,d\Omega = \int_{\Omega_0} w_{i,j}\,\sigma_{ij}\,J\,d\Omega_0$

(Ω_0: initial config; J: $\det \tilde{F}$)

now use chain rule to go from spatial to material coord.

$$= \int_{\Omega_0} w_{i,A} \frac{\partial X_A}{\partial x_j} \sigma_{ij}\, J\, d\Omega_0$$

recall $F_{jA} = \partial x_j / \partial X_A$ thus $(\tilde{F})^{-1}_{Aj} = \partial X_A / \partial x_j$ recall $F_{iA} = \delta_{iA} + u_{i,A}$ hence $\tilde{F} = \tilde{F}(\tilde{u})$

$f(\tilde{\sigma}, \tilde{u}) = \int_{\Omega_0} w_{i,A} (\tilde{F}^{-1})_{Aj}\, \sigma_{ij}\, J\, d\Omega_0$ we now have the fn wrt $\tilde{\sigma}, \tilde{u}$

for $f(\underline{\sigma} + \varepsilon \underline{\Delta\sigma}, \underline{u} + \varepsilon \underline{\Delta u})$ replace $\sigma \leftarrow \sigma + \varepsilon\Delta\sigma$, $u \leftarrow u + \varepsilon \Delta u$ to find variation of f wrt σ, u

now form $\frac{d}{d\varepsilon} f(\;\; , \;\;)\Big|_{\varepsilon=0}$ this will be directional deriv in direction $(\underline{\Delta\sigma}, \underline{\Delta u})$ and will give us what we want

thus remembering that $\int_\Omega d\Omega = \int_{\Omega_0} J\, d\Omega_0$
we get the linearized weak form:

(from σ; from $\frac{d}{d\varepsilon} J$; does not provide symmetry of ij from $\frac{d}{d\varepsilon} F^{-1}$)

$$\int_\Omega w_{i,j} \left(\Delta\sigma_{ij} + \sigma_{ij}\Delta u_{k,k} - \Delta u_{j,k}\sigma_{ki} \right) d\Omega = \int_\Omega w_i f_i\, d\Omega + \int_{\Gamma_h} w_i h_i\, d\Gamma - \int_\Omega w_{(i,j)}\sigma_{ij}\, d\Omega$$

(we have constit eqn for this [$\Delta\sigma_{ij}$]; F^{ext}; F^{int})

<u>Remark</u>: RHS is exactly same as small deformation case with Ω_φ = updated geom.
now put $\Delta\sigma$ we derived before in the above.

Nov 18, 1981

Continued from last time.
Put in constit eqn.

$$\int_\Omega w_{i,j}\, d_{ijkl}\, \Delta u_{k,l}\, d\Omega = \text{external forces} - \text{internal force}$$ (current geometry)

$d_{ijkl} = c_{ijkl} + \sigma_{jl}\delta_{ik}$ (NO BAR) due to Truesdell Rate eqn. $c^* = \hat{c}$

$d_{ijkl} = d_{klij}$ major symmetry is enforced; no minor symmetry.

tangent stiffness = symmetric matrix. Large deformation involves rotation effect which is manifested by loss in minor symmetries. Thus $\sigma_{jl}\delta_{ik}$ is defined as initial stress stiffness.

to go from large deformation to small deformation

(1) set $d_{ijkl} = c_{ijkl} + 0$ in tangent array

(2) set $s_{ijkl} = \hat{c}_{ijkl} = 0$ in constit eqn

(3) don't update geometry.

now what about integrating constit. eqn.?

First write matrix counterpart.

$$\Delta\underset{\sim}{\sigma} = \underset{\sim}{\bar{C}}\,\Delta\underset{\sim}{\gamma} + \underset{\sim}{S}\,\Delta\underset{\sim}{\theta} \qquad \text{incremental constit eqn.}$$

with indexing as discussed in 1st quarter

$\Delta\underset{\sim}{\sigma} = \{\Delta\sigma_I\} = \Delta\sigma_{ij}$

I/J	i/k	j/l
1	1	1
2	1	2
3	2	2
4	3	3
5	2	3
6	3	1

$\underset{\sim}{\bar{C}} = \underset{\sim}{C} + \underset{\sim}{\hat{C}}$ ← from handout #5 pg 2.

$$\underset{\sim}{C} = [C_{IJ}] = c_{ijkl}$$

$\Delta\underset{\sim}{\gamma}$ is a strain increment $\quad \Delta\underset{\sim}{\theta}$ is a rotation increment.

$$\underset{6\times1}{\Delta\underset{\sim}{\gamma}} = \begin{Bmatrix} \Delta u^h_{1,1} \\ \Delta u^h_{1,2} + \Delta u^h_{2,1} \\ \Delta u^h_{2,2} \\ \Delta u^h_{3,3} \\ \Delta u^h_{2,3} + \Delta u^h_{3,2} \\ \Delta u^h_{3,1} + \Delta u^h_{1,3} \end{Bmatrix} ; \quad \underset{3\times1}{\Delta\underset{\sim}{\theta}} = \begin{Bmatrix} \Delta u^h_{1,2} - \Delta u^h_{2,1} \\ \Delta u^h_{2,3} - \Delta u^h_{3,2} \\ \Delta u^h_{3,1} - \Delta u^h_{1,3} \end{Bmatrix}$$

as in 1st quarter we define the following

$$= \sum_{a=1}^{n_{en}} \underset{\sim}{B}^{\gamma}_a \,\Delta\underset{\sim}{d}^e_a \qquad\qquad = \sum_{a=1}^{n_{en}} B^{\theta}_a \,\Delta\underset{\sim}{d}^e_a$$

$$\begin{Bmatrix} \Delta d^e_{a1} \\ \Delta d^e_{a2} \\ \Delta d^e_{a3} \end{Bmatrix} \text{ direction}$$

$$\underset{\sim}{B}_a = \underbrace{\begin{bmatrix} \underset{\sim}{B}_a^{\gamma} \\ \underset{\sim}{B}_a^{\theta} \end{bmatrix}}_{9\times 3} \begin{matrix} 6\times 3 \\ 3\times 3 \end{matrix}$$

↓ throw out for small deform. theory.

Now to global eqn. $\underbrace{\underset{\sim}{DN}(\hat{d})}_{\underset{\sim}{K}} \underbrace{\underset{\sim}{\Delta d}}_{\Delta \underset{\sim}{d}} = \underset{\sim}{F} - \underset{\sim}{N}(\hat{d})$

$$\underset{\sim}{K} = \overset{n_{el}}{\underset{e=1}{\mathbf{A}}} (\underset{\sim}{k}^e) \quad : \quad \underset{3\times 3}{\underset{\sim}{k}^e} = [\underset{\sim}{k}^e_{ab}] = \left[\int_{\Omega^e} \underset{3\times 9}{\underset{\sim}{B}_a^T} \underset{9\times 9}{\underset{\sim}{D}} \underset{9\times 3}{\underset{\sim}{B}_b} \, d\Omega \right]$$

$$\text{what is } \underset{9\times 9}{\underset{\sim}{D}} = \begin{bmatrix} \underset{6\times 6}{\underset{\sim}{c}} & \underset{6\times 3}{\underset{\sim}{0}} \\ \underset{3\times 6}{\underset{\sim}{0}} & \underset{3\times 3}{\underset{\sim}{0}} \end{bmatrix} + \underset{9\times 9}{\underset{\sim}{T}} \quad \leftarrow \text{from tangent.}$$

for $\underset{\sim}{F} - \underset{\sim}{N}(\underset{\sim}{d})$ let $\underset{\sim}{f}_a^e - \underset{\sim}{n}_a^e(\underset{\sim}{d}^e) = \int_{\Omega^e} N_a \underset{\sim}{f} \, d\Omega + \int_{\Gamma_h^e} N_a \underset{\sim}{h} \, d\Gamma - \int_{\Omega^e} (\underset{\sim}{B}_a^{\gamma})^T \underset{\sim}{\sigma} \, d\Omega$

$$\text{what is } \underset{6\times 3}{\underset{\sim}{B}_a^{\gamma}} = \begin{bmatrix} B_1 & 0 & 0 \\ B_2 & B_1 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \\ 0 & B_3 & B_2 \\ B_3 & 0 & B_1 \end{bmatrix}$$

where $B_i = N_{a,i}$ as in 1st quarter.

for nearly incompressible regime this gives bad result with $\hat{B}_a^{\gamma}$. How do we modify $\hat{B}_a^{\gamma}$? (remember we have coupling between $\Delta\gamma$, $\Delta\theta$).

first define deviatoric component: $u'_{(i,j)} = u_{(i,j)} \boxed{- \frac{1}{3} u_{k,k} \delta_{ij}}$ } dilatation part

Generality of selective integration to this case. Let γ superscript be dropped. must account properly for dilatation; B_a^{γ} does not normally

$$\therefore \quad B_a^{DIL} = \frac{1}{3} \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 0 & 0 \\ B_1 & B_1 & B_3 \\ B_1 & B_2 & B_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

this leads to too much constraint $\therefore$ must use same type of selective integration $\Rightarrow \overline{\underset{\sim}{B}}_a^{DIL}$ as we did in 2nd quarter let $\underset{\sim}{B}_a^{DEV} = \underset{\sim}{B}_a - \underset{\sim}{B}_a^{DIL}$ this is OK

$\underset{\sim}{\bar{B}}_a (Good) = \underset{\sim}{B}_a^{DEV.} + \underset{\sim}{\bar{B}}_a^{DIL}$ ← to be defined

Nov 19, 1981

Defining $\underset{\sim}{\bar{B}}_a^{DIL}$ same structure as $\underset{\sim}{B}^{dil}$ where

$$\underset{\sim}{\bar{B}}_a^{DIL} = \frac{1}{3} \begin{bmatrix} \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \\ 0 & 0 & 0 \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

what are the $\bar{B}_i$'s? General theory is in Intern. Jnl of Numerical Mech Eng. Vol 15 pg. 1413-1418, 1980 by TJRH.

Example suppose 4-node quad in 2-D (8 node brick 2D) normal isoparametric element no good for nearly incompress case.
Then by either reduced integration or selective integ we can get good results. How?

1. Selective integ method take $\bar{B}_i(\underset{\sim}{\xi}) = B_i(\underset{\wedge}{0})$ wrt isoparametric coord.

2. for simple element selective integration is special case of mean-value dilatation case

Preferrable in axisymmetric: define $\bar{B}_i(\underset{\sim}{\xi}) = \int_{\Omega^e} B_i \, d\Omega \Big/ \int_{\Omega^e} d\Omega$

same in 2-D as example 1. both 1 & 2 has proper constraint counting (Remember from ME235B).

For 2-D Theory Torsionless axisymmetry, using cylindrical coords.

1. $x_1 = (r)$ radial coord 2. $x_2 = (z)$ axial coord 3. $x_3 = \theta$ circumf

Fully nonlinear finite deform.

Assume 1: All quantities are indep of x_3 (ie θ).

2: $u_3 \equiv 0$ (TORSIONLESS condition).

$\sigma_{13} = \sigma_{23} = 0$ (out of plain shear nonexistent).

Using this results leads to column two of table in handout #5 pg 1.

Consider plane strain case. $\{x_1, x_2, x_3\} = \{x, y, z\}$

Assume:

1. All quantities are not fns of x_3
2. $u_3 = 0$
3. $\sigma_{13} = \sigma_{23} = 0$.

For Plane Stress case $\{x_1, x_2, x_3\}$ cartesian system.

"Thin plate" x_3 — dir $\perp$ to surface

1. All quantities indep of x_3
2. $u_3 = 0$ (really at $z=0$).
3. $\sigma_{13} = \sigma_{23} = \sigma_{23} = 0$

what is $\underset{\sim}{C}$

$$\underset{\sim}{C} = \left[\begin{array}{c|c} \overset{3\times3}{\underset{\sim}{C}_{\sigma\sigma}} & \overset{3\times3}{\underset{\sim}{C}_{\sigma 0}} \\ \hline \underset{\sim}{C}_{0\sigma} & \underset{\sim}{C}_{00} \end{array}\right]$$

First 3 correspond to $\sigma_{11}, \sigma_{12}, \sigma_{22}$

2nd 3 $\sigma_{33}\ \sigma_{23}\ \sigma_{13}$ zeros

do the same partitioning for $\bar{\underset{\sim}{C}}$, $\hat{\underset{\sim}{C}}$

ie

$\underset{\sim}{C}_{\sigma\sigma}$ = fn of $(\sigma_{11}, \sigma_{12}, \sigma_{22})$ only. $\quad \underset{\sim}{C}_{\sigma 0}$ = fn of all 6 $\quad \underset{\sim}{C}_{00}$ = fn $(\sigma_{33}, \sigma_{23}, \sigma_{13})$

$$\Delta\underset{\sim}{\gamma} = \left\{\begin{array}{c} \Delta\underset{\sim}{\gamma}_\sigma \\ \cdots \\ \Delta\underset{\sim}{\gamma}_0 \end{array}\right\}$$

$\Delta\underset{\sim}{\gamma}_0 = -\underset{\sim}{p}\,\Delta\underset{\sim}{\gamma}_\sigma$ use static reduction to get this; p is defined on page #1 of handout

$\underset{\sim}{p} = \bar{\underset{\sim}{C}}_{00}^{-1}\,\bar{\underset{\sim}{C}}_{0\sigma}$

10/23/81

Remaining Lectures

This week: Constit algorithm

Begin Nonlinear Dynamics (symmetric systems).

Next week: Continue nonlin dynamics

Read pg. 95-98, 102-112 in dynamics notes.

Last week of Qtr: Fluids Dynamics (Expository.)

Nonsym Operators (Rd pp 113-120 of notes)

Large Handout

constitutive equation	tangent stiffness matrix
$\underbrace{\Delta\sigma}_{k\times 1} = \underbrace{\bar{C}}_{k\times k}\,\underbrace{\Delta\gamma}_{k\times 1} + \underbrace{S}_{k\times \ell}\,\underbrace{\Delta\theta}_{\ell\times 1}$	$k^e_{ab} = \int_{\Omega^e} B_a^T D B_b \, d\Omega^e$, $B_a = \begin{bmatrix} \underbrace{B_a^\gamma}_{k\times n_{dof}} \\ \hdashline \underbrace{B_a^\theta}_{\ell\times n_{dof}} \end{bmatrix}$, $D = \begin{bmatrix} \underbrace{C}_{k\times k} & \underbrace{0}_{k\times \ell} \\ \underbrace{0}_{\ell\times k} & \underbrace{0}_{\ell\times \ell} \end{bmatrix} + T$ k×ndof ℓ×ndof

	three dimensions	torsionless axisymm.	plane strain	plane stress
n_{dof}, k, ℓ	3 , 6 , 3	2 , 4 , 1	2 , 4 , 1	2 , 3 , 1
B_a^γ # stress comp # rotation comp	$\begin{bmatrix} B_1 & \cdot & \cdot \\ B_2 & B_1 & \cdot \\ \cdot & B_2 & \cdot \\ \hdashline \cdot & \cdot & B_3 \\ \cdot & B_3 & B_2 \\ B_3 & \cdot & B_1 \end{bmatrix}$ $B_i = \partial N_a / \partial x_i$	$\begin{bmatrix} B_1 & \cdot \\ B_2 & B_1 \\ \cdot & B_2 \\ \hdashline B_0 & \cdot \end{bmatrix}$ $B_0 = N_a / x_1$	same as axisymm. except $B_0 = 0$	$\begin{bmatrix} B_1 & \cdot \\ B_2 & B_1 \\ \cdot & B_2 \end{bmatrix}$
$\bar{B}_a^\gamma$	$\begin{bmatrix} B_5 & B_6 & B_8 \\ B_2 & B_1 & \cdot \\ B_4 & B_7 & B_8 \\ \hdashline B_4 & B_6 & B_9 \\ \cdot & B_3 & B_2 \\ B_3 & \cdot & B_1 \end{bmatrix}$ $B_4 = (\bar{B}_1 - B_1)/3$, $B_5 = B_1 + B_4$, $B_6 = (\bar{B}_2 - B_2)/3$, $B_7 = B_2 + B_6$, $B_8 = (\bar{B}_3 - B_3)/3$, $B_9 = B_3 + B_8$	$\begin{bmatrix} B_{12} & B_6 \\ B_2 & B_1 \\ B_{10} & B_7 \\ \hdashline B_{11} & B_6 \end{bmatrix}$ $B_{10} = B_4 + (\bar{B}_0 - B_0)/3$, $B_{11} = B_0 + B_{10}$, $B_{12} = B_1 + B_{10}$	same as axisymm. except $\bar{B}_0 = B_0 = 0$	same as above
B_a^θ	$\begin{bmatrix} B_2 & -B_1 & \cdot \\ \cdot & B_3 & -B_2 \\ -B_3 & \cdot & B_1 \end{bmatrix}$	$\begin{bmatrix} B_2 & -B_1 \end{bmatrix}$	same as axisymm.	same as axisymm.
$\bar{C}$	see (2.28)-(2.32)	first four rows and columns of $\bar{C}$	same as axisymm.	$\bar{C}_{\sigma\sigma} - \bar{C}_{\sigma 0}\, p$ where $p = \bar{C}_{00}^{-1} \bar{C}_{0\sigma}$
S Reindexing leads to this	$\begin{bmatrix} \sigma_2 & \cdot & -\sigma_6 \\ \frac{\sigma_3-\sigma_1}{2} & \frac{\sigma_6}{2} & \frac{-\sigma_5}{2} \\ -\sigma_2 & \sigma_5 & \cdot \\ \hdashline \cdot & -\sigma_5 & \sigma_6 \\ \frac{-\sigma_6}{2} & \frac{\sigma_4-\sigma_3}{2} & \frac{\sigma_2}{2} \\ \frac{\sigma_5}{2} & \frac{-\sigma_2}{2} & \frac{\sigma_1-\sigma_4}{2} \end{bmatrix}$	$\begin{Bmatrix} \sigma_2 \\ \frac{\sigma_3-\sigma_1}{2} \\ -\sigma_2 \\ \hdashline \cdot \end{Bmatrix}$	same as axisymm.	$\begin{Bmatrix} \sigma_2 \\ \frac{\sigma_3-\sigma_1}{2} \\ -\sigma_2 \end{Bmatrix}$
C	see (2.29) - (2.31)	first four rows and columns of C	same as axisymm.	$C_{\sigma\sigma} - (C_{\sigma 0} + \frac{1}{2} p^T \hat{C}_{00})\, p$
T	$\begin{bmatrix} \sigma_1 & \frac{\sigma_2}{2} & \cdot & \cdot & \cdot & \frac{\sigma_6}{2} & \frac{\sigma_2}{2} & \cdot & \frac{-\sigma_6}{2} \\ & \frac{\sigma_1+\sigma_3}{4} & \frac{\sigma_2}{2} & \cdot & \frac{\sigma_6}{4} & \frac{\sigma_5}{4} & \frac{\sigma_3-\sigma_1}{4} & \frac{\sigma_6}{4} & \frac{-\sigma_5}{4} \\ & & \sigma_3 & \cdot & \frac{\sigma_5}{2} & \cdot & \frac{-\sigma_2}{2} & \frac{\sigma_5}{2} & \cdot \\ & & & \sigma_4 & \frac{\sigma_5}{2} & \frac{\sigma_6}{2} & \cdot & \frac{-\sigma_2}{2} & \frac{\sigma_6}{2} \\ & & & & \frac{\sigma_3+\sigma_4}{4} & \frac{\sigma_2}{4} & \frac{-\sigma_6}{4} & \frac{\sigma_4-\sigma_3}{4} & \frac{\sigma_2}{4} \\ & & & & & \frac{\sigma_4+\sigma_1}{4} & \frac{\sigma_5}{4} & \frac{-\sigma_2}{4} & \frac{\sigma_1-\sigma_4}{4} \\ & \text{symm.} & & & & & \frac{\sigma_1+\sigma_3}{4} & \frac{-\sigma_6}{4} & \frac{-\sigma_5}{4} \\ & & & & & & & \frac{\sigma_3+\sigma_4}{4} & \frac{-\sigma_2}{4} \\ & & & & & & & & \frac{\sigma_4+\sigma_1}{4} \end{bmatrix}$	$\begin{bmatrix} \sigma_1 & \frac{\sigma_2}{2} & \cdot & \cdot & \frac{\sigma_2}{2} \\ & \frac{\sigma_1+\sigma_3}{4} & \frac{\sigma_2}{2} & \cdot & \frac{\sigma_3-\sigma_1}{4} \\ & & \sigma_3 & \cdot & \frac{-\sigma_2}{2} \\ \text{symm.} & & & \sigma_4 & \cdot \\ & & & & \frac{\sigma_1+\sigma_3}{4} \end{bmatrix}$	same as axisymm.	$\begin{bmatrix} \sigma_1 & \frac{\sigma_2}{2} & \cdot & \frac{\sigma_2}{2} \\ & \frac{\sigma_1+\sigma_3}{4} & \frac{\sigma_2}{2} & \frac{\sigma_3-\sigma_1}{4} \\ & & \sigma_3 & \frac{-\sigma_2}{2} \\ \text{symm.} & & & \frac{\sigma_1+\sigma_3}{4} \end{bmatrix}$

$\bar{C} = C + \hat{C}$ $C = [C\sigma I]$ $\hat{C}$ on page 2

Figure 1

'e

$$\hat{c}_{ijkl} = -\sigma_{ij}\delta_{kl} + (\sigma_{il}\delta_{jk} + \sigma_{jl}\delta_{ik} + \sigma_{ik}\delta_{jl} + \sigma_{jk}\delta_{il})/2. \tag{4.8}$$

matrix form of the $\hat{c}_{ijkl}$'s is given by

$$\hat{C} = \left[\begin{array}{ccc|ccc} \sigma_1 & \sigma_2 & -\sigma_1 & -\sigma_1 & 0 & \sigma_6 \\ 0 & \dfrac{\sigma_1+\sigma_3}{2} & 0 & -\sigma_2 & \dfrac{\sigma_6}{2} & \dfrac{\sigma_5}{2} \\ -\sigma_3 & \sigma_2 & \sigma_3 & -\sigma_3 & \sigma_5 & 0 \\ \hline -\sigma_4 & 0 & -\sigma_4 & \sigma_4 & \sigma_5 & \sigma_6 \\ -\sigma_5 & \dfrac{\sigma_6}{2} & 0 & 0 & \dfrac{\sigma_3+\sigma_4}{2} & \dfrac{\sigma_2}{2} \\ 0 & \dfrac{\sigma_5}{2} & -\sigma_6 & 0 & \dfrac{\sigma_2}{2} & \dfrac{\sigma_4+\sigma_1}{2} \end{array}\right]. \tag{4.9}$$

additional details of the derivation, see [22].

.1. Zero normal-stress projection

For application to shell analysis, the $\boldsymbol{D}$ matrix needs to be modified to account for the zero rmal-stress condition. To this end, $\sigma_4 = \sigma_{33}$ is set to zero in $\boldsymbol{T}$ and a projection operator is istructed, as suggested by (3.18), to remove the row and column of $\boldsymbol{D}$ corresponding to γ_{33},

$$\tilde{\boldsymbol{D}} = \boldsymbol{P}^{\mathrm{t}}\boldsymbol{D}\boldsymbol{P}, \tag{4.10}$$

iere

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 \\ \boldsymbol{0}_{35} & \boldsymbol{I}_3 \end{bmatrix}, \tag{4.11}$$

id

$$\boldsymbol{p}_1 = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline p_1 & p_2 & p_3 & p_5 & p_6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad \boldsymbol{p}_2 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & p_7 & p_8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]. \tag{4.12}$$

I_n is the $n \times n$ identity matrix, and

$$p_I = -\bar{C}_{4I}/\bar{C}_{44}, \quad I \le 6, \quad p_7 = \sigma_5/\bar{C}_{44}, \quad p_8 = -\sigma_6/\bar{C}_{44}. \tag{4.13}$$

now $\dot\sigma = C_{ijkl}\, \dot\epsilon_{(k,l)} = C_{ijkl}\, v_{(k,l)}$ $v_{(k,l)} = (v_{k,l} + v_{l,k})/2$ where $v_{k,l} = \frac{\partial v_k}{\partial x_l}$ ∴

in 1-D $\dot\sigma = C\, v_{1,1}$ $v_{,x} = v_{,X}\, X_{,x}$ $v_{,X} = \dot x_{,X}$

a) $\Delta\gamma$ (exact = true strain) $= \int_{t_n}^{t_{n+1}} v_{,x}\, dt = \int \dot x_{,X} (X_{,x}) dt = \int \dot F F^{-1} dt = \ln F = \ln(\lambda_{n+1}/\lambda_n)$

$\int dF/F = \ln(1+\epsilon)$

$\ln(1+\epsilon) \approx \ln 1 + \epsilon - \frac{\epsilon^2}{2!}$

$\approx \epsilon - \epsilon^2/2 + \ldots$

b) α - config $\Delta\gamma$

$x_{n+\alpha} = \alpha x_{n+1} + (1-\alpha) x_n = (\alpha \lambda_{n+1} + (1-\alpha)\lambda_n) X$

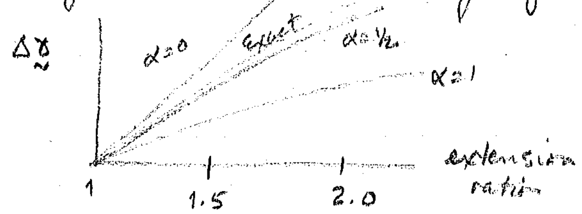
$\Delta u = x_{n+1} - x_n = (\lambda_{n+1} - \lambda_n) X$

$\Delta\gamma = \frac{\partial \Delta u}{\partial x_{n+\alpha}} = \frac{\partial \Delta u}{\partial X} \frac{\partial X}{\partial x_{n+\alpha}} = \frac{(\lambda_{n+1} - \lambda_n)}{\alpha \lambda_{n+1} + (1-\alpha)\lambda_n} = \frac{\epsilon}{1+\alpha\epsilon}$

ϵ here is $\frac{\lambda_{n+1} - \lambda_n}{\lambda_n}$ ($\frac{x_{n+1} - x_n}{x_n} = \frac{\lambda_{n+1}X - \lambda_n X}{\lambda_n X}$ final - init Δ / init)

c) Compare $\frac{\epsilon}{1+\alpha\epsilon} = \epsilon(1 - \alpha\epsilon + \alpha^2\epsilon^2 \ldots) = \epsilon - \alpha\epsilon^2 + \ldots$ with $\ln(1+\epsilon) = \epsilon - \frac{1}{2}\epsilon^2 \ldots$

So we get 2nd order accuracy if $\alpha = \frac{1}{2}$



11/25/81

HW due Monday of last wk of quarter : Dec 7,

Example 2 Simple shear

use $x = x(X, t) = F(t) X$ where $F(t) = \begin{bmatrix} 1 & A(t) \\ 0 & 1 \end{bmatrix}$ shearing parameter

assume $A(0) = 0$ ∋. $x(0) = X$ $F^{-1} = \begin{bmatrix} 1 & -A(t) \\ 0 & 1 \end{bmatrix}$

we want to find $v_{i,j} = \frac{\partial v_i}{\partial x_j} = \dot x_{,x} = \dot x_{,X}\, X_{,x} = \dot F F^{-1}$

$= \frac{\partial \dot x_i}{\partial x_j}$

Exact - true strain

now $\Delta\gamma = \int_{t_n}^{t_{n+1}} \dot F F^{-1} dt = \int \begin{bmatrix} 0 & \dot A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -A \\ 0 & 1 \end{bmatrix} dt = \int \begin{bmatrix} 0 & \dot A \\ 0 & 0 \end{bmatrix} dt = \begin{bmatrix} 0 & A_{n+1} - A_n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Delta s \\ 0 & 0 \end{bmatrix}$

α - configuration

$x_{n+\alpha} = \alpha x_{n+1} + (1-\alpha) x_n$

$= [\alpha F_{n+1} + (1-\alpha) F_n] X$

$\left(\begin{bmatrix} \alpha & \alpha A_{n+1} \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} 1-\alpha & (1-\alpha) A_n \\ 0 & 1-\alpha \end{bmatrix} \right)^{-1}$

$\begin{bmatrix} 1 & \alpha A_{n+1} + (1-\alpha) A_n \\ 0 & 1 \end{bmatrix}^{-1}$

$\Delta u = x_{n+1} - x_n = (F_{n+1} - F_n) X$

now $\frac{\partial \Delta u}{\partial x_{n+\alpha}} = \frac{\partial \Delta u}{\partial X} \left(\frac{\partial X}{\partial x_{n+\alpha}} \right) = (F_{n+1} - F_n) \cdot \frac{1}{\alpha F_{n+1} + (1-\alpha) F_n}$

Constit Algorithm

Problem is: given the deformation, find the stress increment

$$\underset{\sim}{X}^{n+1} = \underset{\sim}{X}^{n+1}(\underset{\sim}{X}^n, \Delta t) \qquad \text{motion references back to previous config}$$

$$\underset{\sim}{X}^{n+1} - \underset{\sim}{X}^n = \Delta u$$

Define a one param family of intermediate config. Want to show how good it is

$$\underset{\sim}{X}^{n+\alpha} = (1-\alpha)\underset{\sim}{X}^n + \alpha \underset{\sim}{X}^{n+1}$$

$$G_{ij} = \frac{\partial(\Delta u_i)}{\partial x_j^{n+\alpha}} \qquad \text{differentiate wrt intermediate config; this is the strain tensor}$$

$$\underset{\sim}{G} = [G_{ij}]$$

$$\Delta \underset{\sim}{\gamma} \equiv \tfrac{1}{2}(\underset{\sim}{G} + \underset{\sim}{G}^T) \qquad \text{strain increment matrix}$$

$$\Delta \underset{\sim}{\theta} = \tfrac{1}{2}(\underset{\sim}{G} - \underset{\sim}{G}^T) \qquad \text{rotation increment "}$$

we will show $\alpha = 1/2$ is essential for good results

Constit algorithm

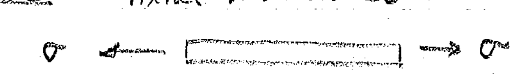
1. $\underset{\sim}{\sigma}^{n+1} = \underset{\sim}{\tilde\sigma}^{n+1} + \Delta\sigma$
2. $\underset{\sim}{\tilde\sigma}^{n+1} = \underset{\sim}{Q}\,\underset{\sim}{\sigma}^n \underset{\sim}{Q}^T$ $\qquad \underset{\sim}{Q} \stackrel{\Delta}{\equiv} \underset{\sim}{I} + (\underset{\sim}{I} - \alpha\Delta\underset{\sim}{\theta})^{-1}\Delta\underset{\sim}{\theta}$ obtained by working backwards to get correct rotation properties
3. $\Delta\underset{\sim}{\sigma}$ = material response
4. $\Delta\sigma_{ij} = C_{ijkl}\,\Delta\gamma_{kl}$

Properties of Algorithm

Let $\underset{\sim}{R}$ be a rotation matrix.

An algorithm is incrementally objective if when $\underset{\sim}{X}^{n+1} = \underset{\sim}{R}\underset{\sim}{X}^n$ (rigid rotation)
then $\underset{\sim}{\sigma}^{n+1} = \underset{\sim}{R}\,\underset{\sim}{\sigma}^n \underset{\sim}{R}^T$ (rigid rotation of body produces proper rotation of stresses)

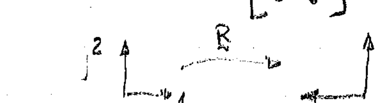
Example: Axial load on bar



$$\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix} = \begin{bmatrix}\sigma & 0\\ 0 & 0\end{bmatrix} \quad n \text{ state} \qquad \xrightarrow[\text{under rotation}]{\underset{\sim}{R}}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix}0 & 0\\ 0 & \sigma\end{bmatrix}$$

$$\underset{\sim}{R} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$



$$\begin{pmatrix}Y\\ -X\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}$$

Thm: If $\alpha = 1/2$ (and $\underset{\sim}{R} + \underset{\sim}{I}$ is nonsingular) then $\underset{\sim}{Q}$ = orthonormal matrix = $\underset{\sim}{R}$
$\Delta\underset{\sim}{\gamma} = \underset{\sim}{0}$ and the algorithm is incrementally objective

Compare $\Delta\gamma$ (for $\alpha = 1/2$) with the exact soln.

1. 1-D Uniaxial extension — current disp: $x = x(\underset{\sim}{X}, t) = \lambda(t)\,\underset{\sim}{X}$ (initial coord) — take $\lambda_0 = 1$, $x_0 = \underset{\sim}{X}$ — see pg. 27 (back) for defn of F

Remember $F = x_{,X} = \lambda$ — deformation gradient now $F_n = \begin{bmatrix} 1 & \Delta n \\ 0 & 1 \end{bmatrix}$ $F_{n+1} - F_n = \begin{bmatrix} 0 & \Delta s \\ 0 & 0 \end{bmatrix}$

$\therefore \quad \frac{\partial \Delta u}{\partial x_{n+\alpha}} = \begin{bmatrix} 0 & \Delta s \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 1 & \alpha \Delta_{n+1} + (1-\alpha)\Delta_n \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -(\alpha s_{n+1} \cdots) \\ 0 & 1 \end{bmatrix} \right\}$ using $FF^{-1} = I$

$= \begin{bmatrix} 0 & \Delta s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\alpha \cdots \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \Delta s \\ 0 & 0 \end{bmatrix}$ exact

Now we look at elastodynamics

Elastodynamics

material deriv $\partial/\partial t |_X$

$\sigma_{ij,j} + f_i = \rho \ddot{u}_i \quad$ in $\Omega \times (0,T)$

(current config

BC/ with $u_i = g_i \quad \Gamma_g \times (0,T)$

$\sigma_{ij} n_i = h_i \quad \Gamma_h \times (0,T)$

IC/

$u_i(x,0) = u_{0i}(x) \quad x \in \Omega_0$

$\dot{u}_i(x,0) = \dot{u}_{0i}(x) \quad x \in \Omega_0$ } given $\Omega_0 = \Omega$ at $t=0$

now $\sigma_{ij} = \sigma_{ij}(F)$

This leads to the variational $(w, \rho\ddot{u}) + \eta(w,u) = (w,f) + (w,h)_\Gamma$

$\int_\Omega w \cdot (\rho \ddot{u}) d\Omega$ current config $\quad + \quad \hookrightarrow F^{int} = F^{ext} = N(d)$ for elastostatics

$M\ddot{d} + F^{int} = F^{ext}$

$N(d)$

ρ = current configurational density

$M = \underset{e=1}{\overset{nel}{A}} (m^e) \qquad m^e = [m^e_{pq}] \qquad m^e_{pq} = \delta_{ij} \int_{\Omega^e} \rho N_a N_b \, d\Omega$. rewrite in initial configuration

$\longrightarrow$ then $m^e_{pq} = \delta_{ij} \int_{\Omega^e_0} \rho N_a N_b J \, d\Omega_0$ w/ $J = \det F$

but $\rho J = \rho_0 = \rho(t=0)$

$\therefore \quad m^e_{pq} = $ constant $\left(\begin{array}{l} p = ned(a-1)+i \\ q = ned(b-1)+j \end{array} \text{ remember} \right) \Longleftarrow m^e_{pq} = \delta_{ij} \int_{\Omega^e_0} \rho_0 N_a N_b \, d\Omega_0$

To define an algorithm (see pg 93 in notes)

suppose Newmark of the above generalized to viscous material

$M a_{n+1} + N(d_{n+1}, v_{n+1}) = F^{ext}_{n+1}$

$M a_{n+1} + N(d_n, v_n) + \frac{\partial N}{\partial d} \Delta d + \frac{\partial N}{\partial v} \Delta v$

$$d_n + \gamma \Delta t v_n + \frac{\Delta t^2}{2}((1-2\beta)a_n + 2\beta a_{n+1}) = \tilde{d}_{n+1} + \Delta t^2 \beta a_{n+1} \qquad v_{n+1} = v_n + \Delta t(1-\gamma)a_n + \ldots$$

also $d_{n+1} = \ldots$; $v_{n+1} = \ldots$ from newmark. $\qquad = \tilde{v}_{n+1} + \gamma \Delta t\, a_{n+1}$

Let us define $K_T = \left.\frac{\partial N}{\partial d}\right|_{v=\text{fixed}}$ $\qquad C_T = \left.\frac{\partial N}{\partial v}\right|_{d=\text{fixed}}$ both fns of d, v

write everything in iterative form.

$$a_{n+1}^{(i+1)} = a_{n+1}^{(i)} + \Delta a \qquad v_{n+1}^{(i+1)} \equiv \tilde{v}_{n+1} + \Delta t \gamma a_{n+1}^{(i+1)} = v_{n+1}^{(i)} + \Delta v \qquad v_{n+1}^{(i)} = \tilde{v}_{n+1} + \gamma \Delta t a_{n+1}^{(i)}$$

$$d_{n+1}^{(i+1)} \equiv \tilde{d}_{n+1} + \Delta t^2 \beta a_{n+1}^{(i+1)} = d_{n+1}^{(i)} + \Delta d \; ; \; d_{n+1}^{(i)} = \tilde{d}_{n+1} + \Delta t^2 \beta a_{n+1}^{(i)} \qquad \Delta v = \Delta t \gamma \Delta a \quad \Delta d = \Delta t^2 \beta \Delta a$$

thus if we define $f(\Delta a) = F_{n+1}^{ext} - M a_{n+1}^{(i+1)} - N(d_{n+1}^{(i+1)}, v_{n+1}^{(i+1)}) \overset{?}{=} 0$

Now set $f(0) + \frac{\partial f}{\partial \Delta a}(0)\, \Delta a = 0$ this will be linearized ie $M^* \Delta a = R$ this is dynamic counterpart of static eqn.

first two terms of taylor expansion about 0. What is M^* ? we will see from next time

11/30/81

This then leads to

$$\left\{ M + \gamma \Delta t\, C_T(d_{n+1}^{(i)}, v_{n+1}^{(i)}) + \beta \Delta t^2 K_T(d_{n+1}^{(i)}, v_{n+1}^{(i)}) \right\} \Delta a = R^{(i)}$$

where $R^{(i)} = F^{ext} - M a_{n+1}^{(i)} - N(d_{n+1}^{(i)}, v_{n+1}^{(i)})$ see, eqns 63, 64, 69 dynamics notes

<u>Algorithm</u>

Set $i=0$ (iter counter)

Predictors - eqns 60-62. $\quad v_{n+1}^{(i+1)} = \tilde{v}_{n+1}$; $d_{n+1}^{(i+1)} = \tilde{d}_{n+1}$, $a_{n+1}^{(i+1)} = 0$.

→ $R^{(i)}$ is calculated as above

$M^* \Delta a = R^i$

Correctors - eqns (65-67) $\quad a_{n+1}^{(i+1)} = a_{n+1}^{(i)} + \Delta a$, $v_{n+1}^{(i+1)} = v_{n+1}^{(i)} + \Delta v$, $d_{n+1}^{(i+1)} = d_{n+1}^{(i)} + \Delta d$

$i = i+1$

<u>For implicit - explicit algorithm.</u>

define new $f(\Delta a) = F_{n+1}^{ext} - M a_{n+1}^{i+1} - N^I(d_{n+1}^{(i+1)}, v_{n+1}^{(i+1)}) - N^E(d_{n+1}^{(i)}, v_{n+1}^{(i)})$

lag behind on some, ahead on others.

and find $0 = f(0) + \frac{\partial f}{\partial \Delta a}(0)\, \Delta a$

Linear operator which includes no contrib from explicit groups. Depends only on I

$$N = N^I + N^E$$

$$\Rightarrow M^* = M + \gamma \Delta t\, C_T^I + \beta \Delta t^2 K_T^I \qquad C_T^I = \frac{\partial N^I}{\partial v} \qquad K_T^I = \frac{\partial N^I}{\partial d}$$

only changes

To include "a class of viscoid rate eqns" just view K_T as defined in static nonlin case.

Notions of stability - Linearized Stability   Pp. 96 –
View the algo as fns of $(\underbrace{\tilde{d}_n, \tilde{v}_n, \tilde{a}_n, \tilde{d}_{n+1}, \tilde{v}_{n+1}, \tilde{a}_{n+1}}_{\tilde{x}})$.

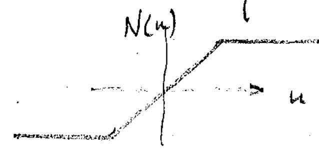
Now differentiate wrt $\tilde{x}$ all of the algorithmic eqns.
ie

$$\frac{\partial f}{\partial \tilde{x}} \cdot \delta \tilde{x} = 0$$ "This is the consistent associated linear system"

this is pg 96 eqn 9.

Pg 98. For explicit elements - linearized time step restrictions seem to be sufficient.

Example: $\ddot{u} + N(u) = 0 \quad u(0) = 0 \quad \dot{u}(0) = 25 \qquad N(u) = \begin{cases} 100u & |u| \le 2 \\ 200 \operatorname{sgn}(u) & |u| > 2 \end{cases}$



Most problems are in intermediate time step range.

Algorithms which satisfy stronger notions of stability.
Energy conserving algorithm.
1. $\tilde{N} = \tilde{N}(\tilde{d})$
2. $\tilde{N} = \frac{\partial u}{\partial \tilde{d}} = u(\tilde{d})]$
3. $\tilde{K}_T = \frac{\partial N}{\partial \tilde{d}} =$ Symm

$$\tilde{M}\ddot{\tilde{d}} + \tilde{N}(\tilde{d}) = \tilde{F}$$

from the above Energy Identity can be obtained

$$E[\tilde{d}(t), \dot{\tilde{d}}(t)] = E[\tilde{D}_0, \tilde{V}_0] + \int_0^t \dot{\tilde{d}}(\tau)^T \tilde{F}(\tau) d\tau$$

if $\tilde{F} = \tilde{0} \Rightarrow E = \text{constant} = \dot{d}^T \tilde{M} \dot{\tilde{d}}/2 + u(\tilde{d})$

a decent value of $\tilde{u} \Rightarrow E = \text{const} \Rightarrow \tilde{d}, \dot{\tilde{d}}$ are bounded.

We will start with a candidate algorithm ie Trapezoidal   37-39 in notes

$$\tilde{M}\tilde{a}_{n+1} + \tilde{N}(\tilde{d}_{n+1}) = \tilde{F}_{n+1}$$

$$\tilde{d}_{n+1} = \tilde{d}_n + \Delta t/2 (\tilde{V}_{n+1} + \tilde{V}_n)$$

$$\tilde{V}_{n+1} = \tilde{V}_n + \frac{\Delta t}{2}(\tilde{a}_{n+1} + \tilde{a}_n)$$

12/2/81

Course next quarter Tu-Th 1:15 - 2:30 PM Skilling 191
Obtain from 37-39 39→37 & 38→39

From last time $\mathcal{F}(\underset{\sim}{d}_{n+1}) = \frac{2}{\Delta t^2} \underset{\sim}{d}_{n+1}^T \underset{\sim}{M} \underset{\sim}{d}_{n+1} + U(\underset{\sim}{d}_{n+1})$

$$- \underset{\sim}{d}_{n+1}^T \left\{ \underset{\sim}{F}_{n+1} + M\left[\underset{\sim}{a}_n + \frac{4}{\Delta t^2}(\underset{\sim}{d}_n + \Delta t \underset{\sim}{v}_n)\right]\right\}$$

$\mathcal{F}'(\underset{\sim}{d}_{n+1})\delta \underset{\sim}{d}_{n+1} = \underset{\sim}{0} \Rightarrow \delta \mathcal{F} = 0$ This gives weird behavior

Thus modify algorithm

$$E_{n+1} = E_n + \tfrac{1}{2}(\underset{\sim}{d}_{n+1} - \underset{\sim}{d}_n)^T(\underset{\sim}{F}_{n+1} + \underset{\sim}{F}_n)$$

$E(d_n, v_n)$ using trapez rule on forcing term only

ie $\int dE = \int \dot{d} F^T dt \cong \frac{F^{n+1} + F_n}{2} \int \dot{d}\, dt = \frac{F^{n+1} + F^n}{2}(d_{n+1} - d_n)$

if $\underset{\sim}{F} = 0 \Rightarrow$ conservation of total energy

we want $\underset{\sim}{d}_{n+1}$ to do this we define $\mathcal{G}(\underset{\sim}{d}_{n+1})$ {in notes pg 105 eqn 46}
and we want it $= 0$

Now define augmented equns of motion $\mathcal{F}(\underset{\sim}{d}_{n+1}) + \lambda \mathcal{G}(\underset{\sim}{d}_{n+1})$ λ = lagrange mult

using variational method ~~want~~ gives.

$$\frac{\partial \mathcal{F}}{\partial d_{n+1}}(\underset{\sim}{d}_{n+1}) \cdot \delta \underset{\sim}{d}_{n+1} + \lambda \frac{\partial \mathcal{G}}{\partial \underset{\sim}{d}_{n+1}}(\underset{\sim}{d}_{n+1}) \cdot \delta \underset{\sim}{d}_{n+1} + \delta\lambda\, \mathcal{G} = 0$$

must hold $\forall\ \lambda\ \&\ \underset{\sim}{d}_{n+1} \Rightarrow$ (1) $\mathcal{G}(\underset{\sim}{d}_{n+1}) = 0$ energy ident

(2) $\mathcal{F}'(\underset{\sim}{d}_{n+1}) + \lambda\, \mathcal{G}'(\underset{\sim}{d}_{n+1}) = 0$ mod. equns of motion

unknowns: $\underset{\sim}{d}_{n+1}$, λ. This result helps remove pathological problem of trapezoidal method.

To solve these equns (ie eqn 50 pg 105) must be modified see

Hughes & Brooks' paper.

Look at Transient nonlin heat conduction

$$\kappa_{ij} = \kappa_{ij}(u, \underset{\sim}{x}, t)$$
$$c = c(u, \underset{\sim}{x}, t) \quad \text{specific heat}$$

also $q_i = -\kappa_{ij} u_{,j}$

also $q_{i,i} + f = \rho c u_{,t}$ nonlinear heat eqn.
(ρ = const)

$+ BC + IC\ (u(\underset{\sim}{x}, 0) = u_0(\underset{\sim}{x}))$.

(W) form is $(\rho c \dot{u}, w) + \eta(w, u) = (w, f) + (w, h)_\Gamma$
$a(w, u; u)$
arg of u_{ij}'s

(G) replace S by S^h; V by V^h and ^{h} superscrip on everything.

(M) $\underset{\sim}{M}(\underset{\sim}{d}, t)\dot{\underset{\sim}{d}} + \underset{\sim}{K}(\underset{\sim}{d}, t)\underset{\sim}{d} = \underset{\sim}{F}$

$$m^e_{ab} = \int_{\Omega^e} \rho\, c(u^h, \underset{\sim}{x}, t)\, N_a N_b \, d\Omega = m^e_{ba}$$
$u^h = \sum_{c=1}^{n_{el}} N_c d_c$

$$k^e_{ab} = k^e_{ba}$$

Introduce two families of algo that collapse to trapez.
General trap. - satisfy eqn at end pt.

$$\underset{\sim}{M}_{n+1} \underset{\sim}{v}_{n+1} + \underset{\sim}{K}_{n+1} \underset{\sim}{d}_{n+1} = \underset{\sim}{F}_{n+1} \qquad \underset{\sim}{M}_{n+1} = \underset{\sim}{M}(d_{n+1}, t_{n+1})$$
$$\underset{\sim}{K}_{n+1} = \underset{\sim}{K}(\qquad)$$

and
$$\left.\begin{aligned} \underset{\sim}{d}_{n+1} &= \underset{\sim}{d}_n + \Delta t\, \underset{\sim}{v}_{n+\alpha} \\ \underset{\sim}{v}_{n+\alpha} &= \underset{\sim}{v}_n(1-\alpha) + \alpha\, \underset{\sim}{v}_{n+1} \end{aligned}\right\} (*) \qquad \underset{\sim}{F}_{n+1} = \underset{\sim}{F}(t_{n+1})$$

2nd case is the generalized midpoint family satisfy eqn at some pt in interval

$$\underset{\sim}{M}_{n+\alpha} \underset{\sim}{v}_{n+\alpha} + \underset{\sim}{K}_{n+\alpha} \underset{\sim}{d}_{n+\alpha} = \underset{\sim}{F}_{n+\alpha} \quad \text{and} (*)$$

$$\underset{\sim}{M}_{n+\alpha} = \underset{\sim}{M}(d_{n+\alpha}, t_{n+\alpha})$$
$$\underset{\sim}{K}_{n+\alpha} = \underset{\sim}{K}(d_{n+\alpha}, t_{n+\alpha}) \qquad \underset{\sim}{F}_{n+\alpha} = \underset{\sim}{F}(t_{n+\alpha})$$
$$t_{n+\alpha} = t_n + \alpha \Delta t$$
$$\underset{\sim}{d}_{n+\alpha} = \underset{\sim}{d}_n(1-\alpha) + \underset{\sim}{d}_{n+1}\alpha$$

if we ~~will be concerned~~ were w/ stability set $F=0$ Both forms would collapse to the same thing for linear problems. But for nonlin this is not true.

Gourlay helped with the stability analysis
He said for "stability"
Consider SDOF model eqn: $\dot{d} + \lambda(d, t)\, d = 0$ (Scalar model of heat problem...)

Using trapezoidal ($\alpha = \frac{1}{2}$) $d_{n+1} = A d_n$
$$A = \frac{1 - \Delta t(1-\alpha)\lambda_n}{1 + \Delta t\, \alpha (\lambda_{n+1})} \qquad \lambda_n = \lambda(d_n, t_n)$$

suppose "stability" is defined $|A| \le 1$ => for $A = \frac{1 - \Delta t/2\, \lambda_n}{1 + \Delta t/2\, \lambda_{n+1}}$ => $\lambda_n > \lambda_{n+1}$ get Δt_{condi}
$\lambda_n \le \lambda_{n+1}$ OK "unconditionally stable"

ie $\Delta t \le \frac{4}{(\lambda_n - \lambda_{n+1})}$ ← this really doesn't mean much in non lin case

Using the same method
midpoint rule leads to $A = \frac{1 - \Delta t(1-\alpha)\lambda_{n+\alpha}}{1 + \Delta t\,\alpha\,\lambda_{n+\alpha}}$ exactly the same & leads to the same result as linear case

ie $|A| \le 1$ $\alpha \ge 1/2$ as in linear case. Problem w/ trapezoidal is that modal decomp <u>doesn't</u> work for it but <u>does</u> for midpoint rule.

12/4/81

<u>Exam</u>: Fri 18 December 12:15 – 3:15 Rm ERL 320

last time 1. Finish Energy Conservation Algorithm
2. Nonlin heat Cond. (transient case)
A. General trap rule
B. General midpoint rule
C. Ad Hoc Stability Analyses
via $\dot{d} + \lambda(d,t)\,d = 0$ too strong a condition for stability
=> $d_{n+1} = A\, d_n$. w/ $|A| < 1$
Trapez ($\alpha = \frac{1}{2}$) $\Delta t \le 4/(\lambda_n - \lambda_{n+1})$. stability weakened in Nlin case
mid ($\alpha \ge 1/2$) uncond "stab."
$\alpha < 1/2$ $\lambda_{n+\alpha}\,\Delta t \le 2/(1-2\alpha)$

see notes pg 109 eq (64)

For use of Global energy criterion see pg 111 eq (73)

for model problem trapezoidal rule $M = 1$ $K = \lambda$
$d_n^2 \le d_0^2 \left(1 + \lambda_0 \Delta t + \frac{\Delta t^2}{4} \lambda_0^2\right)$ this holds $\forall \Delta t$
This is different than oscillator problem. Eqn aids the algorithm unlike oscillator which destabilizes the delicate equil balance. (Ht. conduction)
Now Fluids problems / non symmetric problems pg 115 – on
leap frog on pg 113 is skipped

Rd pg. 118–120

Read pg 118–120

12/7/81

1. Advection – Diffusion eqn.] look at paper given out
2. Incompressible Navier Stokes Eqn. Brooks/Hughes.

Background Remarks

▷ Galerkin FE Method (Central Difference FD method).
Tend to be formally accurate. But they tend to produce oscillations ("wiggles").

Galerkin leads to
"Best approx." for symmetric problems which is lost for nonsymmetric problems.

To make up for this loss we use an upwind differences "robust" (ie stable, wiggle-free), but are often over diffused".

Diffusion shows up when
- a. source terms are present f – in our formulation
- b. time dependent case
- c. multidimensional cases ("crosswind diffusion")

1-D Model problem steady advection diffusion

$$\underset{\text{advection}}{u\,\phi_{,x}} = \underset{\text{diff}}{k\,\phi_{,xx}}$$

u = flow velocity & given. >0
ϕ – unknown
k – diffusivity >0

Note Convection term is $\frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x}$

want to solve this on $x \in (0, L)$ + B.C.'s

define Pe = peclet no. = uL/k

Consider classical difference technique
Central diff = simplest galerkin FE method (Piecewise LINear FE).

Let
X_A = node A

$$\underset{A-1}{\bullet} \overset{h}{\text{——}} \underset{A}{\bullet} \overset{h}{\text{——}} \underset{A+1}{\bullet}$$

then $$\left.\frac{\partial}{\partial x}\right|_A \cong \frac{(\ )_{A+1} - (\ )_{A-1}}{2h} \quad (1) \qquad \left.\frac{\partial^2}{\partial x^2}\right|_A = \frac{(\ )_{A+1} - 2(\ )_A + (\ )_{A-1}}{h^2} \quad (2)$$

PDE leads to
$u \cdot (1) = k\,(2)$ second order accurate

Upwind differences leads to the following

$\frac{\partial^2}{\partial x^2}\Big|_A$ = same as (2) $\qquad \frac{\partial}{\partial x}\Big|_A = \frac{(\ )_A - (\ )_{A-1}}{h}$ first order accurate

backward difference wrt node A.

$\longrightarrow u$

A-1 A A+1

this stabilizes result.

We must characterize result w/ element Peclet no. = $\alpha = \frac{uL}{2k}$ see paper fig 2.1/2.2

the actual results are somewhere between two figures. Must define Artificial diffusion. Interpretation of upwind difference

define CDiff in terms of upwind diff

$$\frac{u}{h}(\phi_A - \phi_{A-1}) = \frac{u}{2h}(\phi_{A+1} - \phi_{A-1}) - \frac{uh}{2}\left(\frac{\phi_{A+1} - 2\phi_A + \phi_{A-1}}{h^2}\right)$$

1st central diff term is

2nd ~ to 2nd central diff term.

artificial diffusivity $\frac{uh}{2} \gg k$ (in advection dominated case)

we added big diffusion but w/rt first central difference

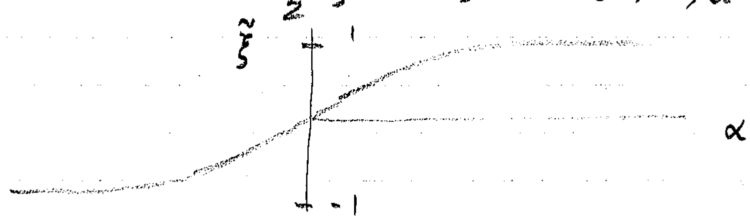
Suppose we define an "optimal upwind method"

let $u\phi_{,x}(x_A) = \frac{u}{2h}(\phi_{A+1} - \phi_{A-1}) - \tilde{k}(\phi_{A+1} - 2\phi_A + \phi_{A-1})$

let $\tilde{k} \in (0, \frac{uh}{2})$ to get optimality. How

1. Solve difference eqns exactly
2. Match nodal values with exact soln. } These lead to condition

on $\tilde{k} = \frac{uh}{2}\tilde{\xi} \qquad \tilde{\xi} = \coth(\alpha) - 1/\alpha$



Note that as $\alpha \to \infty$, $\tilde{k} \to \frac{uh}{2}$ classical upwind

$\alpha \to 0$, $\tilde{k} \to 0$ classical CD

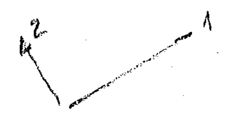
This "approx sol" is nodally exact for all α.

This is OK for model problem. But for other problems this is incorrect way to look at the numerical problems. How do we take care of this

<u>Streamline Upwind</u>

define $\tilde{k}_{ij} = \tilde{k}\,\hat{u}_i \hat{u}_j$ artificial diffusivity tensor.

$$\tilde{\underset{\sim}{k}} = \tilde{k}\begin{bmatrix} \hat{u}_1^2 & \hat{u}_1\hat{u}_2 \\ \hat{u}_2\hat{u}_1 & \hat{u}_2^2 \end{bmatrix} \quad \text{in 2D} \qquad \hat{u}_i = \text{unit velo. vector} = u_i/\|\underset{\sim}{u}\|$$



streamline

$\underset{\sim}{u}$ is tangent to streamline

$$\tilde{\underset{\sim}{k}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{along streamline}$$

acts diffusively along streamline, but not ⊥ to it.
This we add to Galerkin or central difference method. (This is like stepping off along a characteristic ?)

12/9/81

<u>Last time</u>

<u>Fluids</u>: Central Diff (Galerkin FE)
Upwind " Counteract difficulties of C.D.

Central: "wiggles" anti diffusive
Upwind: too diffusive 2D, 3D has crosswind pathology
"Optimal Upwind" 1-D, Goodelng "Streamline Upwind" (streamline)

look at problems in upwind method on
pg 40/41 Brooks paper
Move to pg 20 etc. for
~~Suppose we~~ a look at Streamline Upwind / Petrov-Galerkin.
Unsteady advection - diff eqn.

$$\phi_{,t} + \sigma_{i,i} = f \quad \text{on } \Omega$$

(f: source term prescribed)

define total flux $\sigma_i = \sigma_i^a + \sigma_i^d$ a - adv, d - diff

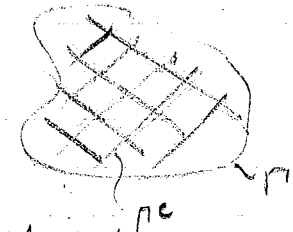
$\sigma_i^a = u_i \phi$

$\sigma_i^d = -\kappa_{ij}\phi_{,j}$

Initial - bdry val prob: BC $\phi = g$ on Γ_g & $-\sigma_n^d = h$ on Γ_h (normal component) — best way to state this bc w/ ic $\phi(x,0) = \phi_0(\underset{\sim}{x})$

<u>weighted residual formulation</u> Petro-Galerkin

Let $\bar{\Omega} = \bigcup_{e=1}^{nel} \bar{\Omega}^e \qquad \Gamma^e = \partial\Omega^e \qquad \Gamma_{int} = \bigcup_{e=1} \Gamma^e - \Gamma$



<u>Remember for Galerkin</u>, $\mathcal{S}$ trial soln space
$\mathcal{V}$ wt fn. space

with

$\mathcal{S} = \mathcal{V} \oplus$ "g" (bdry)

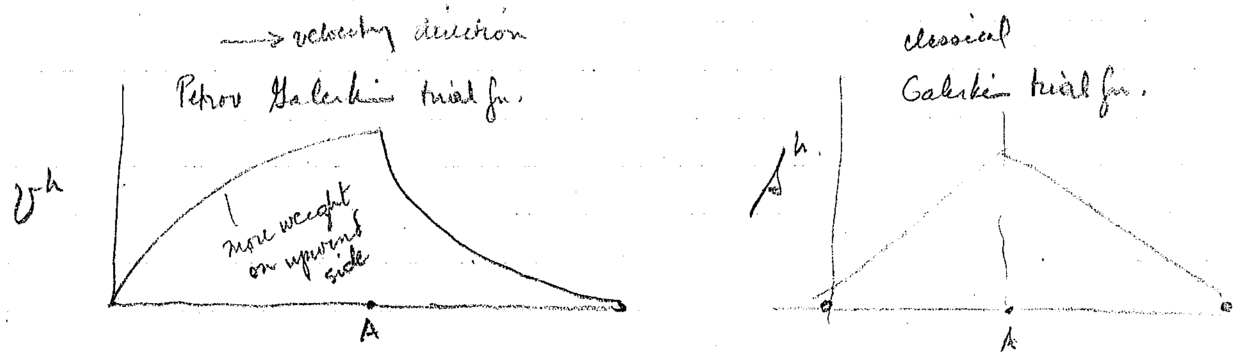
we approximate $\mathcal{V}^h \approx \mathcal{V}$

wtng fn & trial soln fns are same due to symmetric operators of heat cond & elasticity

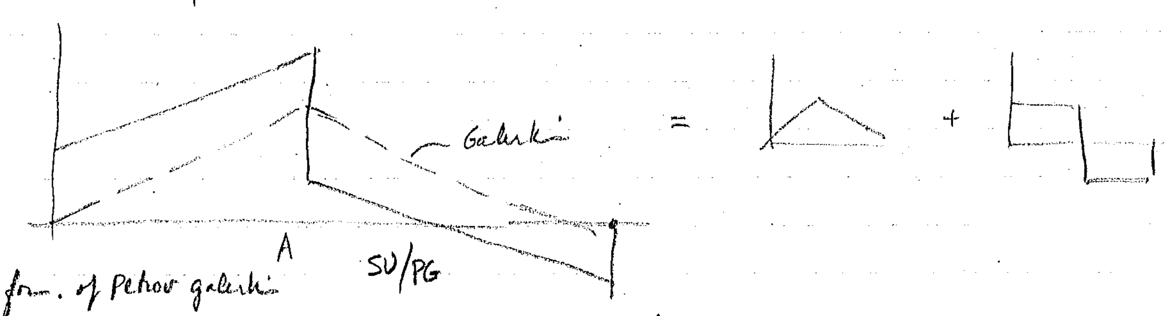
<u>Petrov-Galerkin</u>

define $\mathcal{S}^h = \mathcal{V}_1^h \oplus \ldots$ ie weighting fns & trial soln are approx
and also $\mathcal{V} \cong \mathcal{V}_2^h \qquad \mathcal{V}_1^h \neq \mathcal{V}_2^h$ by different classes of fn.

This method arises due to nonsymm operators which appear in Flow problems



approx this by



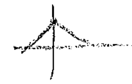
weak form. of Petrov galerkin

$$\int_\Omega w \cdot (\dot{\phi} + \sigma_{i,i}^a) d\Omega - \int_\Omega w_{,i} \sigma_i^d d\Omega = \int_\Omega w f d\Omega + \int_{\Gamma_h} w h d\Gamma$$

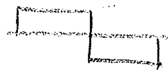
$$- \sum_e \int_{\Omega^e} P(\dot{\phi} + \sigma_{i,i} - f) d\Omega$$

interior of element e

where $\sum \Omega^e = \Omega \; ; \; \sum_e \Gamma^e = \Gamma^{int} + \Gamma_h$

let $\phi \cong$ piecewise cont FE Fns. 

w = " " " "

P = 

integrate by parts

$$0 = \sum_e \int_{\Omega^e} \tilde{w}\,(\dot{\phi} + \sigma_{i,i} - f)\,d\Omega - \int_{\Gamma_h} w(\sigma_n^d + h)\,d\Gamma - \int_{\Gamma_{int}} w\,[\sigma_n^d]\,d\Gamma$$

($\tilde{w}$ = $w + P$; the first integral is the A-D eq; the Γ_h term is the diff flux BC; the Γ_{int} term is the diff flux contin. cond.; $[\sigma_n^d]$ = jump across bdry $= (\sigma_n^d)^+ - (\sigma_n^d)^-$)

bdry

P only affects the A-D eqn on element interior Ω^e

suppose we set $P = \tilde{k}\,\hat{u}_i\,w_{,i} / \|\underset{\sim}{u}\|$ leads streamline upwind artificial diff procedure in steady case, $f = 0$.

Results: look at pg. 44 PG/SU does very well wrt classical upwind
SU has heavy damping & phase shift

Look at <u>Time dependent Case</u>
<u>Semi discrete eq.</u>

$$\underset{\sim}{M}\dot{\underset{\sim}{\phi}} + \underset{\sim}{C}\underset{\sim}{\phi} = \underset{\sim}{F} \quad \text{(prescribed data)}$$

$$\text{w/ } \underset{\sim}{\phi}(0) = \underset{\sim}{\phi}_0$$

"Mass"

since trial fns have P within $\Rightarrow M$ is nonsymm & C is also nonsymm. (Adv & Diff is in C.)

if $\tilde{k} = \frac{uh}{2}\xi$ for steady this is a boundary-layer optimiz

for dynamic case this may also work, but after some analysis $\tilde{k} = \frac{uh}{\sqrt{15}}\xi$ is better (Raymond-Garder formula) produces 4th order accuracy in phase shift

optimiz. in 1-D

Results: see pg 45 result Classical diffuses results / Petrov Galerkin - preserves

<u>Streamline Upwind Petrov Galerkin for Adv-Diff Eqn summary.</u>

no spurious - wall. (galerkin / CD pathology)
- crosswind diffusion
- source diffusion } upwind pathology.
- transient diffusion }

NS eqns Pg 70 next
how do we handle continuity eqn. see pg 71-75

12/18/81

Final Exam ERL 320 Dec 18 12:15 PM - 3:15 PM

Pg 75 NS eqns · due to convective terms, we need SU/PG routines
76 Continuity is discussed
77-on how we define the new weak form :

Can argue that for simple elements $p_i\, t_{ij,j}$ can be neglected analogous to situation for AD eqn.

Semi discrete form. pg 78
Algorithm pg 83

Final -

$\phi_{,t} + \sigma_{i,i} = f \qquad \sigma_i = \sigma_i^a + \sigma_i^d \qquad \sigma_i^a = u_i \varphi \qquad \sigma_i^d = -k_{ij}\varphi_{,j}$

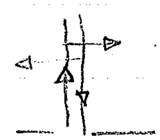
w/ $\phi = g$ on Γ_g and $-\sigma_n^d = h$ on Γ_h

pick $\phi \in \mathcal{S}$ and $w \in \mathcal{V}$ pick $\phi =$

~~$\int_\Omega w(\dot\phi + \sigma_{i,i}^a + \sigma_{i,i}^d - f)\,d\Omega = \int_\Omega w(\dot\phi + \sigma_{i,i}^a)\,d\Omega + \int$~~

$$\int \tilde w\,(\dot\phi + \sigma_{i,i} - f)\,d\Omega = 0 \implies \int_\Omega (\dot\phi + \sigma_{i,i} - f)(w + p)\,d\Omega = 0$$

$$\int_\Omega w(\dot\phi + \sigma_{i,i} - f)\,d\Omega + \int_\Omega p\,(\dot\phi + \sigma_{i,i} - f)\,d\Omega = 0$$



$$\int_\Omega w(\dot\phi + \sigma_{i,i}^a - f)\,d\Omega + \int_\Omega w\sigma_{i,i}^d\,d\Omega + \sum_e \int_{\Omega^e} p(\qquad)\,d\Omega = 0$$

$$\int_\Omega (w\sigma_i^d)_{,i}\,d\Omega - \int_\Omega w_{,i}\,\sigma_i^d\,d\Omega$$

$\downarrow$

$$\int_\Gamma (w\,\sigma_i^d\, n_i)\,d\Gamma = \int_{\Gamma_h} w h\,d\Gamma$$

$$\int_\Omega w(\dot\phi + \sigma_{i,i}^a)\,d\Omega = \int_\Omega w f\,d\Omega + \int_{\Gamma_h} w h\,d\Gamma - \int_\Omega w_{,i}\,K_{ij}\,\varphi_{,j}\,d\Omega - \sum_e \int_{\Omega^e} p(\qquad)\,d\Omega$$

$$\int_\Omega w(\dot\phi + \sigma_{i,i}^a)\,d\Omega - \int_\Omega w_{,i}\,\sigma_i^d\,d\Omega = \int_\Omega w f\,d\Omega + \int_{\Gamma_h} w h\,d\Gamma - \int_\Omega p\,(\qquad)$$

but $\Omega = \sum_e \Omega^e \quad \therefore \sum \int_{\Omega^e} = \int_\Omega$

$$\int_\Omega w(\dot\phi + \sigma_{i,i}^a)\,d\Omega - \int_\Omega w_{,i}\,\sigma_i^d\,d\Omega + \sum \int_{\Omega^e} p(\dot\phi + \sigma_{i,i} - f)\,d\Omega = \int_\Omega w f\,d\Omega + \int_{\Gamma_h} w h\,d\Gamma$$

first $\sum_e \int_{\Omega^e} = \int_\Omega$

$$-\sum_e \left[\int_{\Omega^e} \{(w\,\sigma_i^d)_{,i} - w\,\sigma_{i,i}^d\}\,d\Omega\right]$$

$$\sum_e \int_{\Omega^e} \tilde w\,(\dot\phi + \sigma_{i,i} - f)\,d\Omega - \sum \int_{\Omega^e} (w\,\sigma_i^d)_{,i}\,d\Omega - \int_{\Gamma_h} w h\,d\Gamma = 0$$

$$\sum_e \int_{\Gamma_e} (w\,\sigma_n^d)\,d\Gamma = \int_{\Gamma_{int}} + \int_{\Gamma_h}$$

over.

$$\therefore \sum \int_{\Omega^e} \tilde{\omega} (\dot{\phi} + \sigma_{i,i} - f) \, d\Omega - \int_{\Gamma_h} \omega (\sigma_n^d + h) \, d\Gamma - \int_{\Gamma_{int}} \omega [\sigma_n^d] \, d\Gamma = 0$$

where $\sum_e \int_{\Omega^e} \omega (\dot{\phi} + \sigma_{i,i} - f) \, d\Omega = \int_{\Omega} \omega (\dot{\phi} + \sigma_{i,i} - f) \, d\Omega$.

$\Rightarrow$ (1) $\sigma_n^d + h = 0$ $\quad$ $[\sigma_n^d] = 0$ across Γ_{int}

ME 235C
Handout #3
Homework

ME 235C
Nonlinear Finite Element Methods
Fall Quarter 1981

Stanford University
Instructor: T. Hughes

HOMEWORK #1

Consider the following system

$$\underset{\sim}{M}\ddot{\underset{\sim}{d}} + \underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{0}$$

$$\underset{\sim}{d} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} \qquad \underset{\sim}{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \underset{\sim}{K} = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Assume $k_1 = 1$ $k_2 = 10^4$, $m_1 = 1$ and $m_2 = 1$

(a) Determine the natural frequencies ω_1 and ω_2 of this system.

(b) Accuracy requirements dictate that, per cycle, no more than 2% amplitude decay and 5% relative period error be permitted in the fundamental mode.

Based upon stability and accuracy requirements, determine the maximum allowable time step for the following algorithms:

1. Central Difference Method ($\beta = 0$, $\gamma = 1/2$)
2. Trapezoidal Rule ($\beta = 1/4$, $\gamma = 1/2$)
3. Damped Newmark Method ($\beta = .3025$, $\gamma = .6$)
4. α-method ($\alpha = -.3$)
5. Wilson θ method ($\theta = 1.4$)
6. Houbolt method
7. Park method

(c) Consider the initial-value problem for the system above with initial data given by

$$\underset{\sim}{d}_o = \begin{Bmatrix} 1 \\ 1.1 \end{Bmatrix}$$

$$\underset{\sim}{v}_o = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Write a computer program to solve this problem employing methods 1-7 above. Run the program at $\Delta t = T_1/20$ over a time interval of $[0, 5T_1]$. For trapezoidal rule report averaged and unaveraged displacements. For the Houbolt and Park methods, use the trapezoidal rule to establish starting values. Obtain time-history plots for the displacements in each case. To facilitate data reduction use some form of computer plotting. Assuming the high-frequency response to be spurious (as is often the case in finite element systems) comment on the relative effectiveness of the algorithms.

<u>Note</u>: If you do not have access to plotting routines, the following routines can be used to produce printer plots:

```
      SUBROUTINE PLOT (X,Y,N)
      DIMENSION X(1),Y(1),LINE(101)
      DATA IBLANK,IDOT,MINUS,IZERO,IPLUS,LINE/1H ,1H*,1H-,1H0,1H+,101*1H
     1/
      JSCALE(Z) = INT(50.*(Z+1.)+.5)+1
      CALL SCALE (Y,N)
      DO 10 I=1,N
      LINE(1)=MINUS
      LINE(51)= IZERO
      LINE(101)=IPLUS
      J=JSCALE(Y(I))
      LINE(J)=IDOT
      PRINT 2000, X(I),(LINE(K),K=1,101)
   10 LINE(J)=IBLANK
      RETURN
 2000 FORMAT (1X,1PE12.5,6X,101A1)
      END
      SUBROUTINE SCALE (A,N)
      DIMENSION A(1)
      AMAX=ABS(A(1))
      DO 10 I=2,N
      TEMP=ABS(A(I))
      IF (TEMP.GT.AMAX)AMAX=TEMP
   10 CONTINUE
      IF (AMAX.EQ.0.) AMAX=1.
      DO 20 I=1,N
   20 A(I) = A(I)/AMAX
      PRINT 2000, AMAX
      RETURN
 2000 FORMAT (1H1,*SCALE FACTOR FOR THIS PLOT IS*,1PE12.5)
      END
```

$\omega_1 \le \omega_2 \le \ldots$

we have $\underset{\sim}{M}\ddot{\underset{\sim}{d}} + \underset{\sim}{K}\underset{\sim}{d} = 0$ or $[\underset{\sim}{K} - \lambda \underset{\sim}{M}]\underset{\sim}{\psi} = 0$

or $\det[\underset{\sim}{K} - \lambda\underset{\sim}{M}] = 0$ or $\det\begin{bmatrix} k_1+k_2-\lambda m_1 & -k_2 \\ -k_2 & k_2 - \lambda m_2 \end{bmatrix} = 0$

$\therefore \quad (10001-\lambda)(10000-\lambda) - 10^8 = 0 \quad \therefore$

$10^8 + 10^4 - 2\cdot10^4\lambda - \lambda + \lambda^2 - 10^8 = 0 \quad \therefore \quad \lambda^2 - \lambda(2\cdot10^4+1) + 10^4 = 0$

$\lambda = \dfrac{(2\cdot10^4+1) \pm \sqrt{(2\cdot10^4+1)^2 - 4\cdot10^4}}{2}$

$= \dfrac{(2\cdot10^4+1) \pm \sqrt{4\cdot10^8+1}}{2}$ $\qquad \lambda_2 \cong 2\times10^4 \qquad \omega_2 = \lambda_2^{1/2} = 10^2\cdot\sqrt{2} = 141.4$

$\omega_2 T_2 = 2\pi \qquad 8.89 = T_1$ use everywhere except for central use T_1 there. $\qquad \lambda_1 \cong .05 \qquad \omega_1 = \lambda_1^{1/2} = .7071$

a.) since $\beta = 0$ $\gamma = 1/2$ central difference it is conditionally stable and explicit since $\underset{\sim}{M}$ is diag

$\Omega_{crit} = 2$ where $\Omega_{crit} = \omega\Delta t$. Now highest ω defines Δt

$\therefore \quad \dfrac{2}{\omega_2} = \Delta t = \sqrt{2}\times10^{-2} = .01414$ pick $\boxed{\Delta t = .014}$

$\Delta t/T = .1$ for $\dfrac{\bar{T}-T}{T}$ Fig 21

b. trapezoidal rule $\beta = 1/4$ $\gamma = 1/2$ is unconditionally stable; can pick any Δt by ampl. by $\bar{T}-T/T$ pick $\boxed{.13 = \Delta t/T}$ unc T

$2\pi\bar{\xi} = .02 \qquad \bar{\xi} = .003$

c. Damped newmark ($\beta = .3025$ $\gamma = .6$) $\xi = 0$ from Figure 17 $\boxed{\Delta t/T = .01}$ uncond.

uncondit stable since $2\beta \geq \gamma \geq 1/2$ all

d. α-method ($\alpha = -.3$) using $\Delta t/T = .085$ ampl. decay fig 19 and $\dfrac{\bar{T}-T}{T}$ gives $\dfrac{\Delta t}{T} = .1$ fig 20 $\boxed{.1 = \dfrac{\Delta t}{T}}$

$\gamma = (1+.6)/2 = .8$

$\beta = (1.3)^2/4 = .42 \quad \therefore \quad 2\beta \geq \gamma \geq 1/2$ uncond stable

e. Wilson's θ method ($\theta = 1.4$) $\Delta t/T = 0.06$ by amp decay $\boxed{\Delta t/T = .09 \quad \dfrac{\bar{T}-T}{T}}$

uncond stable

f. Houbolt Method amplit $\Delta t/T = .03$ period $\boxed{\Delta t/T = .05}$

uncond. stable

g. Parks Method amp $\Delta t/T = .09$ period $\boxed{\Delta t/T = .085}$

A-stable backward diff & uncond. stable

$$(c_1 + i\bar{c}_1)\underset{\sim}{\psi}_1 + (c_2 + i\bar{c}_2)\underset{\sim}{\psi}_2 = \underset{\sim}{d}_0 + i\,\underset{\sim}{0}$$

$$\omega_1(-\bar{c}_1 + i c_1)\underset{\sim}{\psi}_1 + \omega_2(-\bar{c}_2 + i c_2)\underset{\sim}{\psi}_2 = \underset{\sim}{v}_0 + i\,\underset{\sim}{0}$$

$$c_1\underset{\sim}{\psi}_1 + c_2\underset{\sim}{\psi}_2 = \underset{\sim}{d}_0$$

$$\bar{c}_1\underset{\sim}{\psi}_1 + \bar{c}_2\underset{\sim}{\psi}_2 = 0 \quad \checkmark$$

$$-(\bar{c}_1\omega_1\underset{\sim}{\psi}_1 + \omega_2\bar{c}_2\underset{\sim}{\psi}_2) = \underset{\sim}{v}_0$$

$$(\omega_1 c_1\underset{\sim}{\psi}_1 + \omega_2 c_2\underset{\sim}{\psi}_2) = \underset{\sim}{0}$$

$$\begin{bmatrix} \omega_1\psi_1 & 0 & \omega_2\psi_2 & 0 \\ 0 & \omega_1\psi_1 & 0 & \omega_2\psi_2 \\ \psi_1 & 0 & \psi_2 & 0 \\ 0 & \psi_1 & 0 & \psi_2 \end{bmatrix} \begin{pmatrix} c_1 \\ \bar{c}_1 \\ c_2 \\ \bar{c}_2 \end{pmatrix} = \begin{pmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \\ \underset{\sim}{d}_1 \\ \underset{\sim}{0} \end{pmatrix}$$

$$\begin{bmatrix} \omega_1\psi_1 & 0 & \omega_2\psi_2 & 0 \\ 0 & \omega_1\psi_1 & 0 & \omega_2\psi_2 \\ 0 & 0 & (\omega_2-\omega_1)\psi_2 & 0 \\ 0 & \psi_1 & 0 & \psi_2 \end{bmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \\ -\omega_1\underset{\sim}{d}_1 \\ \underset{\sim}{0} \end{pmatrix}$$

$$\begin{bmatrix} \omega_1\psi_1 & 0 & \omega_2\psi_2 & 0 \\ 0 & \omega_1\psi_1 & 0 & \omega_2\psi_2 \\ 0 & 0 & (\omega_2-\omega_1)\psi_2 & 0 \\ 0 & 0 & 0 & (\omega_2-\omega_1)\psi_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\omega_1 d_1 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} \omega_1\psi_1 & 0 & 0 & 0 \\ 0 & \omega_1\psi_1 & 0 & 0 \\ 0 & 0 & (\omega_2-\omega_1)\psi_2 & 0 \\ 0 & 0 & 0 & (\omega_2-\omega_1)\psi_2 \end{bmatrix} \begin{pmatrix} c_1 \\ \bar{c}_1 \\ c_2 \\ \bar{c}_2 \end{pmatrix} = \begin{pmatrix} \frac{\omega_2\omega_1 d}{\omega_2-\omega_1} \\ 0 \\ -\omega_1 d_1 \\ 0 \end{pmatrix}$$

$$\bar{c}_1 = 0 \qquad \bar{c}_2 = 0$$

$$c_1\omega_1\underset{\sim}{\psi}_1 = \frac{\omega_2}{\omega_2-\omega_1}\,d_0$$

$$c_2\underset{\sim}{\psi}_2 = \frac{-\omega_1}{\omega_2-\omega_1}\,d$$

Parkes Method

$N_1 = (\underset{\sim}{M} + \Delta t^2/4\, \underset{\sim}{K}) \quad \underset{\sim}{S} = a_1 \quad l_1 = -1 \quad \underset{\sim}{b}_1 = \underset{\sim}{d}_0$

$N_2 = (\underset{\sim}{M} + \Delta t^2/4\, \underset{\sim}{K}) \quad \underset{\sim}{S} = a_2 \quad l_1 = -1 \quad \underset{\sim}{b}_1 = d_1 \quad l_2 = -\Delta t \quad \underset{\sim}{b}_2 = \underset{\sim}{v}_1 \quad l_3 = -\Delta t^2/4 \quad \underset{\sim}{b}_3 = \underset{\sim}{a}_1$

$N_3 = (\underset{\sim}{M} + (.6\Delta t)^2 \underset{\sim}{K}) \quad S = \dot{\underset{\sim}{d}}_3$

$$2\underset{\sim}{M}\, \underset{\sim}{a}_{n+1} + 2K\, \underset{\sim}{d}_{n+1} = 0$$

$$2\, \underset{\sim}{d}_{n+1} = \Delta t^2 \underset{\sim}{a}_{n+1} + 5\underset{\sim}{d}_n - 4 d_{n-1} + d_{n-2}$$

$$(2M + K\Delta t^2)\, \underset{\sim}{a}_{n+1} = -\underset{\sim}{K}\, \underset{\sim}{f}_n \qquad \underset{\sim}{f}_n = 5\underset{\sim}{d}_n - \underset{\sim}{4}\, d_{n+1} + \underset{\sim}{d}_{n-2}$$

$$\begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

det = W(1,1) W(2,2) - W(1,2) * W(2,1)

TOP1 = R(1) * W(2,2) - R(2) * W(1,2)

TOP2 = W(1,1) * R(2) - R(1) * W(2,1)

R(1) = TOP1/DET

R(2) = TOP2/DET

$$\{\underset{\sim}{M} + \underset{\sim}{K}(\theta\Delta t)^2\beta\}\, \underset{\sim}{a}_{n+\theta} = -K\, f_{n;\theta} \qquad \text{where } f_{n;\theta} = \underset{\sim}{d}_n + \theta\Delta t\, \underset{\sim}{v}_n + \left(\frac{\theta\Delta t}{2}\right)^2 (1-2\beta)\, \underset{\sim}{a}_n$$

for $[\underset{\sim}{a}_{n+\theta} - (1-\theta)\, \underset{\sim}{a}_n]/\theta = \underset{\sim}{a}_{n+1}$

$$v_{n+1} = v_n + \frac{\Delta t}{2}\left((1-\gamma)a_n + \gamma\, a_{n+1}\right)$$

$$d_{n+1} = \underset{\sim}{f}_n + (\theta\Delta t)^2 \beta\, \underset{\sim}{a}_{n+1}$$

$\underset{\sim}{M}\, a_\theta + \underset{\sim}{K}\, d_\theta$

$$-y_{n+1} + .6\Delta t\, \underbrace{G\, y_{n+1}}_{\dot{y}_{n+1}} + 1.5\, y_n - .6\, y_{n-1} + .1\, y_{n-2} = 0$$

$$-\begin{pmatrix} d \\ \dot{d} \end{pmatrix}_{n+1} + .6\Delta t \begin{pmatrix} \dot{d} \\ -M^{-1} K d \end{pmatrix}_{n+1}$$

$$-d_{n+1} + .6\Delta t\, \dot{d}_{n+1} + 1.5\, d_n - .6\, d_{n-1} + .1\, d_{n-2} = 0$$

$$-\dot{d}_{n+1} - .6\Delta t\, M^{-1} K\, d_{n+1} + 1.5\, \dot{d}_n - .6\, \dot{d}_{n-1} + .1\, \dot{d}_{n-2} = 0$$

$$-\left(M \dot{d}_{n+1} + .6\Delta t\, K\, d_{n+1}\right) + M\left(1.5\, \dot{d}_n - .6\, \dot{d}_{n-1} + .1\, \dot{d}_{n-2}\right) = 0$$

$$\therefore \quad M \dot{d}_{n+1} + .6\Delta t\, K\, d_{n+1} = M\left(1.5\, \dot{d}_n - .6\, \dot{d}_{n-1} + .1\, \dot{d}_{n-2}\right)$$

$$d_{n+1} = .6\Delta t\, \dot{d}_{n+1} + 1.5\, d_n - .6\, d_{n-1} + .1\, d_{n-2}$$

$$M\, \dot{d}_{n+1} + .6\Delta t\, K\left[.6\Delta t\, \dot{d}_{n+1} + 1.5\, d_n - .6\, d_{n-1} + .1\, d_{n-2}\right] = M\left(1.5\, \dot{d}_n - .6\, \dot{d}_{n-1} + .1\, \dot{d}_{n-2}\right)$$

$$\left[M + (.6\Delta t)^2 K\right] \dot{d}_{n+1} + .6\Delta t\, K\left(1.5\, d_n - .6\, d_{n-1} + .1\, d_{n-2}\right) = M(\qquad)$$

$$'' \qquad = 1.5\left(M \dot{d}_n - .6\Delta t\, K\, d_n\right) - .6\left(M \dot{d}_{n-1} - .6\Delta t\, K\, d_{n-1}\right)$$

$$+ .1\left(M \dot{d}_{n-2} - .6\Delta t\, d_{n-2}\right)$$

$$= R = 1.5\left(A_n\right) - .6\left(A_{n-1}\right) + .1\left(A_{n-2}\right)$$

solve for $\dot{d}_{n+1}$

$S_1 = 1 \quad G = .5 \quad B = 0 \qquad S_2 = (.6\Delta t)^2$

```
      DIMENSION  T(101), D(101,2)
      DATA D(1,1), D(1,2) /1., 1.1/, W1, W2, DT/.7071, 141.4    , .4443/ .01414
      S2 = W2-W1
      S1 = W2/S2
      S2 = W1/S2
      T(1)=0.
      DO 100 I=1,101
      DO 10 K=1,2
10    D(I+1,K) = D(1,K)*(S1*COS(W1*T(I+1)) - S2*COS(W2*T(I+1)))
100   WRITE (6,200) T(I), D(I,1), D(I,2)
200   FORMAT (3(2X,E14.7))
      WRITE (6,200) T(101), D(101,1), D(101,2)
```

C. LEVY
ME 235 C

HW # 1

(A-)

1. Given $\underset{\sim}{M}\ddot{\underset{\sim}{d}} + \underset{\sim}{K}\underset{\sim}{d} = 0$

This can be rewritten as $[\underset{\sim}{K} - \lambda \underset{\sim}{M}]\underset{\sim}{\psi} = 0$ where λ is the EV of the system and $\underset{\sim}{\psi}$ is the $\overline{EV}$ of the system. To find $\lambda, \underset{\sim}{\psi}$ we need to solve $\det(\underset{\sim}{K} - \lambda \underset{\sim}{M}) = 0$ with $\underset{\sim}{K} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$ $k_1 = 1$ $k_2 = 10^4$ and $\underset{\sim}{M} = \underset{\sim}{I}$ we find that the characteristic eqn is $\lambda^2 - \lambda(2\cdot 10^4 + 1) + 10^4 = 0$

$\therefore \lambda_1 \cong .49999 \quad \lambda_2 \cong 20001$ and $\omega = \sqrt{\lambda}$ $\therefore$ $\omega_1 = .7070997$ $\omega_2 = 141.42489$ ✓

Solving for $\underset{\sim}{\psi}_1$

$$(\underset{\sim}{K} - \lambda \underset{\sim}{M})\underset{\sim}{\psi}_1 = \begin{bmatrix} 10000.5 & -10000 \\ -10000 & 9999.5 \end{bmatrix} \underset{\sim}{\psi}_1 = 0 \Rightarrow \begin{bmatrix} 1 & -.99995 \\ 1 & -.99995 \end{bmatrix} \underset{\sim}{\psi}_1 = 0 \quad \underset{\sim}{\psi}_1^T = [.99995, 1]$$ ✓

Solving for $\underset{\sim}{\psi}_2$

$$(\underset{\sim}{K} - \lambda \underset{\sim}{M})\underset{\sim}{\psi}_2 = \begin{bmatrix} -9999.5 & -10000 \\ -10000 & -10000.5 \end{bmatrix} \underset{\sim}{\psi}_2 = \begin{bmatrix} 1 & 1.00005 \\ 1 & 1.00005 \end{bmatrix} \underset{\sim}{\psi}_2 = 0 \quad \underset{\sim}{\psi}_2^T = [1.00005, -1]$$ ✓

thus if $\underset{\sim}{d} = c\underset{\sim}{\psi} e^{i\omega t}$ $\quad \ddot{\underset{\sim}{d}} = -\lambda \underset{\sim}{d}$ $\quad c$ is $C + i\bar{C}$ (complex) $\therefore \underset{\sim}{d} = \sum_{j=1}^{2} c_j \underset{\sim}{\psi}_j e^{i\omega_j t}$ ✓

at $t=0$ ① $\underset{\sim}{d}_0 = c_1\underset{\sim}{\psi}_1 + c_2\underset{\sim}{\psi}_2$ and ② $\underset{\sim}{v}_0 = 0 = i\bar{c}_1\underset{\sim}{\psi}_1\omega_1 + i\bar{c}_2\underset{\sim}{\psi}_2\omega_2$

from this $\bar{C}_{1,2} = 0$ $\quad \underset{\sim}{d} = \frac{\underset{\sim}{d}_0}{\omega_2 - \omega_1}\left[\omega_2 e^{i\omega_1 t} - \omega_1 e^{i\omega_2 t}\right]$ ✓

① $\Rightarrow \begin{cases} \underset{\sim}{d}_0 = C_1\underset{\sim}{\psi}_1 + C_2\underset{\sim}{\psi}_2 \\ \underset{\sim}{0} = i\bar{C}_1\underset{\sim}{\psi}_1 + i\bar{C}_2\underset{\sim}{\psi}_2 \end{cases}$

② $\Rightarrow \begin{cases} i\omega_1 c_1 \underset{\sim}{\psi}_1 + i\omega_2 c_2 \underset{\sim}{\psi}_2 = \underset{\sim}{0} \\ -\omega_1 \bar{c}_1 \underset{\sim}{\psi}_1 - \omega_2 \bar{c}_2 \underset{\sim}{\psi}_2 = \underset{\sim}{0} \end{cases}$

thus we have $c_1, c_2, \underset{\sim}{\psi}_1, \underset{\sim}{\psi}_2$; to find T_1 and T_2 $\quad T_1 = 2\pi/\omega_1 = 8.885855$ ✓

$T_2 = 2\pi/\omega_2 = .0444277$ ✓

hence we have the exact solution for $\underset{\sim}{d}$.

For our case $\Delta t = T_1/20 = .4442928$ thus we need 101 points

(b) Based on stability and accuracy requirements (i.e. per cycle, no more than 2% amplitude decay and 5% relative period of the fundamental mode) find Δt for the following

1. Central Difference Method ($\beta=0$, $\gamma=\frac{1}{2}$). This is conditionally stable and explicit since $\underset{\sim}{M}$ is diagonal $\therefore$ by conditional stability $\Omega_{crit} = \omega\Delta t = 2$ where highest ω defines Δt_s $\therefore$ $2/\omega_2 = \Delta t_s = .014148$. ✓

   There is no amplitude restriction $\therefore$ Δt_a is anything. But there is a period restriction from Fig 21 $(\bar{T}-T)/T = .05 \Rightarrow \frac{\Delta t}{T}p = .131$ thus $\Delta t_p = .131T_1 = 1.164$ for fundamental.

   $\therefore$ $\Delta t = \min(\Delta t_s, \Delta t_a, \Delta t_p) = .0141$ ✓

2. Trapezoidal Rule ($\beta=\frac{1}{4}$, $\gamma=\frac{1}{2}$). This is an unconditionally stable scheme $\therefore$ Δt_s is anything. From figure 19 we can pick any Δt_a for amplitude decay of 2%. But from figure 20, for $(\bar{T}-T)/T$ of .05 $\Delta t_p/T = .127$ thus $\Delta t_p = .127T_1 = 1.129$

   $\therefore$ $\Delta t = \min(\Delta t_s, \Delta t_a, \Delta t_p) = 1.129$ ✓

3. Damped Newmark ($\beta = .3025$, $\gamma = .6$). Since $2\beta > \gamma > \frac{1}{2}$ this is an unconditionally stable scheme $\therefore$ Δt_s is anything ✓. Now $\bar{\delta} = .02$ and $\bar{\xi} = \bar{\delta}/2\pi = .00318$

   From figure 17 $\Delta t_a/T = .0077$ $\therefore$ $\Delta t_a = .0077 \times T_1 = .06835$.

   $\therefore$ take $\Delta t_{min} = .06835$ ✓

4. α-method ($\alpha = -.3$). For stability $\gamma = (1-2\alpha)/2 = .8$ $\beta = (1-\alpha)^2/4 = .42$ ✓

   since $2\beta > \gamma > \frac{1}{2}$ the method is unconditionally stable $\therefore$ Δt_s is anything ✓

   From Fig 19 for $\bar{\xi} = .00318$ $\Delta t_a/T = .09696$ $\therefore$ $\Delta t_a = \frac{\Delta t_a}{T} \cdot T_1 = .888$ ✓

   From Fig 20 for $(\bar{T}-T)/T = .05$ $\Delta t_p/T = .1$ $\therefore$ $\Delta t_p = .1 \times 8.8858 = .889$ ✓

   $\therefore$ $\Delta t = \min(\Delta t_s, \Delta t_p, \Delta t_a) = .888$ ✓

5. Wilson's θ method ($\theta = 1.4$). This is an unconditionally stable scheme with Δt_s being anything. From Fig 14 $\frac{\Delta t_a}{T} = .05757$ for $\bar{\xi} = .00318$ $\therefore$ $\Delta t_a = .5116$ ✓

From Fig 15 $\frac{\Delta t_p}{T} = .08965$ for $(\bar{T}-T)/_T = .05$ $\Delta t_p = .7957$ ✓

∴ $\Delta t = \min(\Delta t_a, \Delta t_s, \Delta t_p) = \underline{.5116}$ ✓

6. Houbolt's method. Its unconditional stability shows that Δt_s is anything

From figures 14 $\frac{\Delta t_a}{T} = .0303$ ∴ $\Delta t_a = .2693$ ✓ From figures 15 $\frac{\Delta t_p}{T} = .0567$

$\Delta t_p = .504$ ∴ $\Delta t = \min(\Delta t_a, \Delta t_s, \Delta t_p) = \underline{.2693}$ ✓

7. Park's method. It is unconditionally stable ∴ Δt_s is anything

From Fig 19 Δt_a is same as α method $= .888$. From Fig 20 $\frac{\Delta t_p}{T} = .08657$ ✓

∴ $\Delta t_p = .7692$ ∴ $\Delta t = \min(\Delta t_a, \Delta t_s, \Delta t_p) = \underline{.7692}$. ✓

Central Differences $\underset{\sim}{M}\,\underset{\sim}{a}_{n+1} + \underset{\sim}{K}\,\underset{\sim}{d}_{n+1} = 0$ w/ $\underset{\sim}{d}_{n+1} = \underset{\sim}{f}_n$ and $\underset{\sim}{v}_{n+1} = \underset{\sim}{v}_n + \frac{\Delta t}{2}\{\underset{\sim}{a}_n + \underset{\sim}{a}_{n+1}\}$

∴ and $\underset{\sim}{f}_n = \underset{\sim}{d}_n + \Delta t\,\underset{\sim}{v}_n + \frac{\Delta t^2}{2}\underset{\sim}{a}_n$ ✓ thus solve $\underset{\sim}{M}\,\underset{\sim}{a}_{n+1} = -\underset{\sim}{K}\,\underset{\sim}{d}_{n+1}$ for $\underset{\sim}{a}_{n+1}$ put into $\underset{\sim}{v}_{n+1}$

advance $\underset{\sim}{a}_{n+1} \to \underset{\sim}{a}_n$ $\underset{\sim}{v}_{n+1} \to \underset{\sim}{v}_n$ and $\underset{\sim}{d}_{n+1} (= \underset{\sim}{f}_n) \to \underset{\sim}{d}_n$

here $\underset{\sim}{a}_{n+1} = -\underset{\sim}{K}\,\underset{\sim}{d}_{n+1}$ ✓

for $n=0$ $\underset{\sim}{d}_1 = \underset{\sim}{d}_0 = \underset{\sim}{f}_0$.

Trapezoidal rule

$$\underset{\sim}{M}\,\underset{\sim}{a}_{n+1} + \underset{\sim}{K}\,\underset{\sim}{d}_{n+1} = 0 \quad ✓$$

$$\underset{\sim}{d}_{n+1} = \underset{\sim}{d}_n + \Delta t\,\underset{\sim}{v}_n + \frac{\Delta t^2}{4}(\underset{\sim}{a}_n + \underset{\sim}{a}_{n+1}) \quad ✓$$

$$\underset{\sim}{v}_{n+1} = \underset{\sim}{v}_n + \frac{\Delta t}{2}(\underset{\sim}{a}_{n+1} + \underset{\sim}{a}_n) \quad ✓$$

for $n=0$ $\underset{\sim}{M}\,\underset{\sim}{a}_1 + \underset{\sim}{K}\,\underset{\sim}{d}_1 = 0$ $\underset{\sim}{d}_1 = \underset{\sim}{d}_0 + \Delta t^2/4\,\underset{\sim}{a}_1$ $\underset{\sim}{v}_1 = \frac{\Delta t}{2}\underset{\sim}{a}_1$ where $\underset{\sim}{a}_0$ is assumed $\underset{\sim}{0}$

so solve for $\underset{\sim}{d}_1$ from $\left[\underset{\sim}{M} + \Delta t^2/4\,\underset{\sim}{K}\right]\underset{\sim}{d}_1 = \underset{\sim}{M}\,\underset{\sim}{d}_0$; then $\underset{\sim}{a}_1$ from $(\underset{\sim}{d}_1 - \underset{\sim}{d}_0)/(\Delta t^2/4) = \underset{\sim}{a}_1$

and $\underset{\sim}{v}_1 = \frac{\Delta t}{2}\underset{\sim}{a}_1$

for $n = 1, \ldots$ Solve for $\underset{\sim}{d}_{n+1}$ from $\left[\underset{\sim}{M} + \Delta t^2/4\,\underset{\sim}{K}\right]\underset{\sim}{d}_{n+1} = \underset{\sim}{M}\,\underset{\sim}{f}_n$ $\quad f_n = \underset{\sim}{d}_n + \Delta t\,\underset{\sim}{v}_n + \frac{\Delta t^2}{4}\underset{\sim}{a}_n$

then put this into $\underset{\sim}{a}_{n+1} = \frac{4}{\Delta t^2}\left[\underset{\sim}{d}_{n+1} - \underset{\sim}{f}_n\right]$ and $\underset{\sim}{v}_{n+1} = \underset{\sim}{v}_n + \frac{\Delta t}{2}(\underset{\sim}{a}_{n+1} + \underset{\sim}{a}_n)$

update $\left.\begin{array}{l}\underset{\sim}{a}_{n+1} \to \underset{\sim}{a}_n \\ \underset{\sim}{v}_{n+1} \to \underset{\sim}{v}_n \\ \underset{\sim}{d}_{n+1} \to \underset{\sim}{d}_n\end{array}\right\} \longrightarrow$ calculate f_n

Damped Newmark

$$\underset{\sim}{M}\underset{\sim}{a}_{n+1} + \underset{\sim}{K}\,\underset{\sim}{d}_{n+1} = 0$$

$$\underset{\sim}{d}_{n+1} = \underset{\sim}{f}_n + \Delta t^2 \beta \underset{\sim}{a}_{n+1} \qquad \underset{\sim}{f}_n = \underset{\sim}{d}_n + \Delta t\, \underset{\sim}{v}_n + (1-2\beta)\frac{\Delta t^2}{2}\underset{\sim}{a}_n$$

$$\underset{\sim}{v}_{n+1} = \underset{\sim}{v}_n + \Delta t/2\,(\underset{\sim}{a}_{n+1} + \underset{\sim}{a}_n)$$

thus $(M + \beta\Delta t^2 K)\,\underset{\sim}{d}_{n+1} = \underset{\sim}{M}\underset{\sim}{f}_n$ is solved for $\underset{\sim}{d}_{n+1}$ and $\underset{\sim}{a}_{n+1} = (\underset{\sim}{d}_{n+1} - \underset{\sim}{f}_n)/\beta\Delta t^2$; put into v_{n+1} ← solve

update $\left.\begin{matrix} a_{n+1} \\ v_{n+1} \\ d_{n+1}\end{matrix}\right\} \to \left.\begin{matrix} a_n \\ v_n \\ d_n\end{matrix}\right\} \to$ calculate $\underset{\sim}{f}_n$

when $n=0$ $\underset{\sim}{f}_0 = \underset{\sim}{d}_0$

α method

$$\underset{\sim}{M}\,a_{n+1} + (1+\alpha)\underset{\sim}{K}\,\underset{\sim}{d}_{n+1} - \alpha\underset{\sim}{K}\,\underset{\sim}{d}_n = 0$$

$$\underset{\sim}{d}_{n+1} = \underset{\sim}{d}_n + \Delta t\,\underset{\sim}{v}_n + \frac{\Delta t^2}{2}\left\{(1-2\beta)\,\underset{\sim}{a}_n + 2\beta\underset{\sim}{a}_{n+1}\right\}$$

$$\underset{\sim}{v}_{n+1} = \underset{\sim}{v}_n + \Delta t\left\{(1-\gamma)\,\underset{\sim}{a}_n + \gamma\,\underset{\sim}{a}_{n+1}\right\} \qquad \gamma = (1-2\alpha)/2 \qquad \beta = (1-\alpha)^2/2$$

for $n=0$ $\quad \underset{\sim}{M}\,(d_1 - d_0) + \beta\Delta t^2\left[(1+\alpha)\underset{\sim}{K}\,d_1 - \alpha\underset{\sim}{K}\,\underset{\sim}{d}_0\right] = 0$ or

$$\left[\underset{\sim}{M} + \beta\Delta t^2(1+\alpha)\underset{\sim}{K}\right]\underset{\sim}{d}_1 = \left[\underset{\sim}{M} + \beta\Delta t^2\alpha\underset{\sim}{K}\right]\underset{\sim}{d}_0$$

and $(\underset{\sim}{d}_1 - \underset{\sim}{d}_0)/\beta\Delta t^2 = \underset{\sim}{a}_1$ and $\underset{\sim}{v}_1 = \Delta t\,\gamma\,\underset{\sim}{a}_1$

for $n = 1, 2, \ldots$ let $f_n = d_n + \Delta t\,\underset{\sim}{v}_n + \Delta t^2/2\,(1-2\beta)\,\underset{\sim}{a}_n$

solve $\left[\underset{\sim}{M} + \beta\Delta t^2(1+\alpha)\underset{\sim}{K}\right]\underset{\sim}{d}_{n+1} = \alpha\beta\Delta t^2\underset{\sim}{K}\,\underset{\sim}{d}_n + \underset{\sim}{M}\,f_n$ for $\underset{\sim}{d}_{n+1}$

then $\underset{\sim}{a}_{n+1} = \frac{1}{2\beta}\left[\frac{\underset{\sim}{d}_{n+1} - \underset{\sim}{d}_n - \Delta t\,\underset{\sim}{v}_n}{\Delta t^2/2} - (1-2\beta)\,\underset{\sim}{a}_n\right]$

then

$\underset{\sim}{v}_{n+1} = \underset{\sim}{v}_n + \Delta t\left[(1-\gamma)\,\underset{\sim}{a}_n + \gamma\,\underset{\sim}{a}_{n+1}\right]$ let $\left.\begin{matrix} v_{n+1} \to v_n \\ a_{n+1} \to a_n \\ d_{n+1} \to d_n\end{matrix}\right\}$ calculate f_n

θ method

$$\underset{\sim}{M}\,\underset{\sim}{a}_{n+\theta} + \underset{\sim}{K}\,\underset{\sim}{d}_{n+\theta} = 0$$

$$\underset{\sim}{a}_{n+\theta} = (1-\theta)\,\underset{\sim}{a}_n + \theta\,\underset{\sim}{a}_{n+1}$$

$$\underset{\sim}{d}_{n+\theta} = \underset{\sim}{d}_n + \theta\Delta t\,\underset{\sim}{v}_n + \frac{(\theta\Delta t)^2}{2}\left\{(1-2\beta)\,\underset{\sim}{a}_n + 2\beta\,\underset{\sim}{a}_{n+\theta}\right\} \quad \text{w/ } d_{n+1} = d_{n+\theta}\ (\theta=1)$$

$$\underset{\sim}{v}_{n+\theta} = \underset{\sim}{v}_n + \theta\frac{\Delta t}{2}\left\{\underset{\sim}{a}_n + \underset{\sim}{a}_{n+\theta}\right\} \quad \text{w/ } v_{n+1} = v_{n+\theta}\ (\theta=1)$$

for $n=0$ $\quad \underset{\sim}{M}\,\underset{\sim}{a}_\theta + \underset{\sim}{K}\,\underset{\sim}{d}_\theta = 0 \qquad a_\theta = \theta a_1 \qquad \underset{\sim}{d}_\theta = \underset{\sim}{d}_0 + \frac{(\theta\Delta t)^2}{2}\cdot 2\beta\,\underset{\sim}{a}_\theta \qquad V_\theta = \theta\Delta t\cdot\gamma\,\underset{\sim}{a}_\theta$

thus solve $\left[\underset{\sim}{M} + \underset{\sim}{K}\,\beta\theta^2\Delta t^2\right]\underset{\sim}{d}_\theta = \underset{\sim}{M}\,\underset{\sim}{d}_0$ for $\underset{\sim}{d}_\theta$ and $\frac{d_\theta - d_0}{\beta\theta^2\Delta t^2} = \underset{\sim}{a}_\theta$; $a_\theta/\theta = \underset{\sim}{a}_1$; $V_1 = \frac{\Delta t}{2}\underset{\sim}{a}_1$,

$\beta = .1683$ for min stability; wanted us to take $\beta = 1/6$

for $n=1,2,\ldots$ solve $[M + K\,\Delta t^2\theta^2]\, d_{n+\theta} = M f_{n;\theta}$ for $d_{n+\theta}$ ← return

where $f_n = d_n + \theta\Delta t V_n + \frac{(\theta\Delta t)^2}{2}(1-2\beta)a_n$;

then solve for $a_{n+\theta} = \dfrac{d_{n+\theta} - f_n}{\beta(\Delta t\theta)^2}$; then $a_{n+1} = (a_{n+\theta} - (1-\theta)a_n)/\theta$; $V_{n+1} = V_{n+\theta}|_{\theta=1}$

and $d_{n+1} = d_n + \Delta t V_n + \frac{\Delta t^2}{2}\{(1-2\beta)a_n + 2\beta a_{n+1}\}$

now let $V_{n+1} \to V_n$, $a_{n+1} \to a_n$, $d_{n+1} \to d_n$ } calculate $f_{n;\theta}$

Houbolt's Method:

$$M a_{n+1} + K d_{n+1} = 0$$

$$a_{n+1} = (2d_{n+1} - 5d_n + 4d_{n-1} - d_{n-2})/\Delta t^2$$

$$V_{n+1} = (11 d_{n+1} - 18 d_n + 9 d_{n-1} - 2d_{n-2})/(6\Delta t)$$

for $n=0$ $\quad M a_1 + K d_1 = 0 \implies$ solve $[M + \frac{\Delta t^2}{4}K]d_1 = M d_0$ for d_1 ; $a_1 = (d_1 - d_0)/(\Delta t^2/4)$; $V_1 = \frac{\Delta t}{2}a_1$

for $n=1$ $\quad M a_2 + K d_2 = 0 \implies [M + \frac{\Delta t^2}{4}K]d_2 = M f_1$ for d_2 ; $a_2 = \dfrac{d_2 - f_1}{\Delta t^2/4}$; $V_2 = V_1 + \frac{\Delta t}{2}(a_2 + a_1)$

where $f_1 = d_1 + \Delta t V_1 + \frac{\Delta t^2}{4}a_1$

for $n \geq 2$ Solve $(2M + K\Delta t^2)\, d_{n+1} = M f$ where $f = 5d_n - 4d_{n-1} + d_{n-2}$ for d_{n+1} ←

then let $d_{n-1} \to d_{n-2}$ $\quad d_n \to d_{n-1}$ $\quad d_{n+1} \to d_n$ and calculate f

Park's Method here $y^T = (d, V)$ $\quad G = \begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix}$ and $\dot{y} = G y$ if P is the EV

matrix then $z = P^{-1}y \implies \dot{z} = \Lambda z$ where $P^{-1}GP = \Lambda$.

thus $-z_3 \oplus \Delta t(.6)\Lambda z_3 + 1.5 z_2 - .6 z_1 + .1 z_0 = 0$ &

$-y_3 \oplus \Delta t(.6) G y_3 + 1.5 y_2 - .6 y_1 + .1 y_0 = 0$

$-\begin{pmatrix} d \\ \dot{d} \end{pmatrix}_3 \oplus .6\Delta t \begin{pmatrix} \dot{d} \\ -M^{-1}Kd \end{pmatrix}_3 + 1.5\begin{pmatrix} d \\ \dot{d} \end{pmatrix}_2 - .6\begin{pmatrix} d \\ \dot{d} \end{pmatrix}_1 + .1\begin{pmatrix} d \\ \dot{d} \end{pmatrix}_0 = 0$

thus

$$d_3 = -.6\Delta t\, \dot{d}_3 + 1.5 d_2 - .6 d_1 + .1 d_0$$

$$M\dot{d}_3 \oplus .6\Delta t K d_3 = M(1.5\dot{d}_2 - .6\dot{d}_1 + .1\dot{d}_0)$$

thus $[M + (.6\Delta t)^2 K]\dot{d}_3 = M(1.5\dot{d}_2 - .6\dot{d}_1 + .1\dot{d}_0) \ominus .6\Delta t K[1.5 d_2 - .6 d_1 + .1 d_0]$ (corrected to +)

K.C. Park update formula for $\dot{y} = f(y, t)$ is

$10 y_{n+1} = 15 y_n - 6 y_{n-1} + y_{n-2}$ $\ominus$ $6\Delta t\, \dot{y}_{n+1}$

or $\left[\underset{\sim}{M} + (.6\Delta t)^2 \underset{\sim}{K}\right] \dot{\underset{\sim}{d}}_3 = 1.5\left[\underset{\sim}{M}\dot{\underset{\sim}{d}}_2 - .6\Delta t \underset{\sim}{K}\underset{\sim}{d}_2\right] - .6\left[\underset{\sim}{M}\dot{\underset{\sim}{d}}_1 - .6\Delta t \underset{\sim}{K}\underset{\sim}{d}_1\right] + .1\left[\underset{\sim}{M}\dot{\underset{\sim}{d}}_0 - .6\Delta t \underset{\sim}{K}\underset{\sim}{d}_0\right]$

w/ $\underset{\sim}{f}_0 = \underset{\sim}{d}_0$ use $\left(\underset{\sim}{M} + \frac{\Delta t^2}{4}\underset{\sim}{K}\right)\underset{\sim}{a}_1 = -\underset{\sim}{K}\underset{\sim}{f}_0$ to find $\underset{\sim}{a}_1$ $\quad \underset{\sim}{d}_1 = \underset{\sim}{f}_0 + \frac{\Delta t^2}{4}\underset{\sim}{a}_1 \quad \underset{\sim}{v}_1 = \underset{\sim}{v}_0 + \frac{\Delta t}{2}(\underset{\sim}{a}_1 + \underset{\sim}{a}_0)$

and $\underset{\sim}{f}_1 = \underset{\sim}{d}_1 + \Delta t \underset{\sim}{v}_1 + \frac{\Delta t^2}{4}\underset{\sim}{a}_1$: $\left(\underset{\sim}{M} + \frac{\Delta t^2}{4}\underset{\sim}{K}\right)\underset{\sim}{a}_2 = -\underset{\sim}{K}\underset{\sim}{f}_1$ to find $\underset{\sim}{a}_2$ $\quad \underset{\sim}{d}_2 = \underset{\sim}{f}_1 + \frac{\Delta t^2}{4}\underset{\sim}{a}_2 \quad \underset{\sim}{v}_2 = \underset{\sim}{v}_1 + \frac{\Delta t}{2}(\underset{\sim}{a}_2 + \underset{\sim}{a}_1)$

then use the above to calculate $\dot{\underset{\sim}{d}}_3$; then $\underset{\sim}{d}_3 = .6\Delta t \dot{\underset{\sim}{d}}_3 + 1.5\underset{\sim}{d}_2 - .6\underset{\sim}{d}_1 + .1\underset{\sim}{d}_0$

Next let $\underset{\sim}{y}_1 \rightarrow \underset{\sim}{y}_0 \quad \underset{\sim}{y}_2 \rightarrow \underset{\sim}{y}_1 \quad \underset{\sim}{y}_3 \rightarrow \underset{\sim}{y}_2$

Based on our Δt's and the Δt we must run the program at good results should be found for the following methods

Trapezoidal $\quad \Delta t_{req'd} = 1.129 > \Delta t_{run} = .4442928$

α - method $\quad \Delta t_{req'd} = .888$

Wilson $\quad \Delta t_{req'd} = .5116$

Parks $\quad \Delta t_{req'd} = .7692$

Bad results should be obtained for

Central $\quad \Delta t_{req'd} = .0141 < \Delta t_{run} = .4442928$

Damped Newmark $\quad \Delta t_{req'd} = .06835$

Houbolt $\quad \Delta t_{req'd} = .2693$

Central Differences $\Delta t = .4443$ - Figure 1 shows that the system is very unstable especially since $\Delta t > \Delta t_{stability}$

$\Delta t = .0141$ - Fig 2 & 3 shows that the system is stable but shows a beat phenomena Try $\Delta t = .01$

Trapezoidal Rule $\Delta t = .4443$ Fig 4 & Fig 5 (Average): Fig 4 shows that the period increases w/ time for d_1. Fig 5 shows that the averaging makes the solution much better for short time and the amplitude in both figures again increase.

A.D = 0 for trap. Fig 6 & 7: Shows the period from figure 6 increases w/ time and the amplitude decrease. We also note that Fig 7 shows better agreement for larger time than the d_1 displacement.

Damped Newmark $\Delta t = .4443$ Fig 8 & 9 : Fig 8 shows increase in period with time and similarly amplitude. Whereas Fig 9 shows decrease in amplitude with time. Thus higher frequency mode is damped slowly

α - method $\Delta t = .4443$ Fig 10 & 11 : Figure 10 & 11 show period increase with time. Amplitude decreases with time - higher mode greater decrease this is not the best one

θ - method $\Delta t = .4443$ Fig 12 & 13 : Same effect as α - method, but worse

Houbolt method $\Delta t = .4443$ Fig 14 & 15 : Same effect as α - method, so far the worse case showing great numerical damping in all modes

Central difference

$\beta = 0 \quad \gamma = 1/2$

$f_n = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2} a_n$

$M a_{n+1} = -K f_n$ solve for a_{n+1}

$v_{n+1} = v_n + \frac{\Delta t}{2}(a_{n+1} + a_n)$ $\quad d_{n+1} = f_n + \beta \Delta t^2 a_{n+1}$

$(1-\gamma) a_n + \gamma a_{n+1}$

for first step : $f_0 = d_0 \quad v_0 = 0 \quad a_0 = 0$

Trapezoidal rule

$\beta = 1/4 \quad \gamma = 1/2$

$(M + \frac{\Delta t^2}{4} K) a_{n+1} = -K f_n$ ✓ $\quad f_n = d_n + \Delta t\, v_n + \frac{\Delta t^2}{4} a_n$

solve for a_{n+1} then $d_{n+1} = f_n + \beta \Delta t^2 a_{n+1}$ ✓ $\quad$ report $\frac{d_{n+1} + d_n}{2}$ ✓

$v_{n+1} = v_n + \frac{\Delta t}{2}(a_{n+1} + a_n)$ ✓

for first step $\quad f_0 = d_0 \quad v_0 = 0 \quad a_0 = 0$

Damped newmark

$\beta = .3025 \quad \gamma = 1/2$

$(M + \Delta t^2 \beta K) a_{n+1} = -K f_n$ ✓ $\quad f_n = d_n + \Delta t\, v_n + (1-2\beta)\frac{\Delta t^2}{2} a_n$ ✓

solve for a_{n+1} then $d_{n+1} = f_n + \Delta t^2 \beta a_{n+1}$ ✓

$v_{n+1} = v_n + \frac{\Delta t}{2}(a_{n+1} + a_n)$ ✓

for first step : $f_0 = d_0 \quad v_0 = 0 \quad a_0 = 0$ ✓

α - method

$\gamma = (1-2\alpha)/2 \quad \beta = (1-\alpha)^2/2$

$(M + (1+\alpha)\Delta t^2 \beta K) a_{n+1} = -(1+\alpha) K f_n + \alpha K d_n$ where $f_n = d_n + \Delta t v_n + (1-2\beta)\frac{\Delta t}{2} a_n$

$= -K d_n - (1+\alpha) K [\Delta t\, v_n + (1-2\beta)\frac{\Delta t^2}{2} a_n]$

$d_{n+1} = f_n + \beta \Delta t^2 a_{n+1}$ $\quad v_{n+1} = v_n + \Delta t \{(1-\gamma) a_n + \gamma a_{n+1}\}$

for first step : $f_0 = d_0 \quad v_0 = 0 \quad a_0 = 0$

Wilson θ - method

$\theta = 1.4$

$[M + \theta^2 \Delta t^2 \beta K] a_{n+\theta} = -K f_{n;\theta}$ ~~$- M(1-\theta) a_n$~~ $\quad f_{n,\theta} = d_n + \theta \Delta t v_n + \frac{(\theta \Delta t)^2}{2}(1-2\beta) a_n$

Solve for a_{n+1} now $d_{n+1} = f_n \big|_{\theta=1} + \beta \Delta t^2 a_{n+1}$; $v_{n+1} = v_n + \frac{\Delta t}{2}\{a_n + a_{n+1}\}$

for first step : $f_0 = d_0 \quad v_0 = 0 \quad a_0 = 0$ $\quad a_{n+1} = [a_{n+\theta} - (1-\theta) a_n]/\theta$

Houbolt - method

$(2M + K \Delta t^2)\, a_{n+1} = -K f_n$ $\quad f_n = 5 d_n - 4 d_{n-1} + d_{n-2}$ for a_{n+1}

for first step : $(M + \Delta t^2/4\, K) a_1 = -K f_0$; $a_0 = v_0 = 0$; $d_1 = d_0 + \Delta t^2/4\, a_1$ $\quad v_1 = \frac{\Delta t}{2} a_1$ $\quad f_1 = d_1 + \Delta t v_1 + \frac{\Delta t^2}{4} a_1$

for 2nd step $(M + \Delta t^2/4\, K) a_2 = -K f_1$ $\quad d_2 = f_1 + \frac{\Delta t^2}{4} a_2$

$a_{n+1} = (2 d_{n+1} - f_n)/\Delta t^2$

Parks method.

? define

set $\underset{\sim}{f_0} = \underset{\sim}{d_0}$ → Use $(\underset{\sim}{M} + \frac{\Delta t^2}{4}\underset{\sim}{K})\underset{\sim}{a_1} = -\underset{\sim}{K}\underset{\sim}{f_0}$ to get $a_1 \rightarrow \underset{\sim}{d_1} = \underset{\sim}{f_0} + \frac{\Delta t^2}{4} a_1$ $\quad \dot{\underset{\sim}{d}}_1 = \dot{\underset{\sim}{d}}_0 + \frac{\Delta t}{2}(a_1 + a_0)$

$\underset{\sim}{f_1} = \underset{\sim}{d_1} + \Delta t \underset{\sim}{v_1} + \frac{\Delta t^2}{4}\underset{\sim}{a_1}$ Use $(\underset{\sim}{M} + \frac{\Delta t^2}{4}\underset{\sim}{K})\underset{\sim}{a_2} = -\underset{\sim}{K}\underset{\sim}{f_1}$ to get $a_2 \rightarrow \underset{\sim}{d_2} = \underset{\sim}{f_1} + \frac{\Delta t^2}{4} a_2$ $\quad \dot{\underset{\sim}{d}}_2 = \dot{\underset{\sim}{d}}_1 + \frac{\Delta t}{2}(a_2 + a_1)$

Next Calculate $\underset{\sim}{M}\dot{\underset{\sim}{d}}_i - .6\Delta t \underset{\sim}{K}\underset{\sim}{d}_i = \underset{\sim}{A}_i \quad i = 0,1,2$ and let $\underset{\sim}{R}_3 = 1.5\underset{\sim}{A}_2 - .6\underset{\sim}{A}_1 + .1\underset{\sim}{A}_0$

Solve $[\underset{\sim}{M} + (.6\Delta t)^2 \underset{\sim}{K}]\dot{\underset{\sim}{d}}_3 = \underset{\sim}{R}_3$ for $\dot{\underset{\sim}{d}}_3$

Solve $\underset{\sim}{d}_3 = .6\Delta t\, \dot{\underset{\sim}{d}}_3 + 1.5\underset{\sim}{d}_2 - .6\underset{\sim}{d}_1 + .1\underset{\sim}{d}_0$ ← this should do it!

Now $\begin{pmatrix} d_1 \\ \dot{d}_1 \end{pmatrix} \rightarrow \begin{pmatrix} d_0 \\ \dot{d}_0 \end{pmatrix}$ $\quad \begin{pmatrix} d_2 \\ \dot{d}_2 \end{pmatrix} \rightarrow \begin{pmatrix} d_1 \\ \dot{d}_1 \end{pmatrix}$ $\quad \begin{pmatrix} d_3 \\ \dot{d}_3 \end{pmatrix} \rightarrow \begin{pmatrix} d_2 \\ \dot{d}_2 \end{pmatrix}$

$\underset{\sim}{A}_1 \rightarrow \underset{\sim}{A}_0 \quad \underset{\sim}{A}_2 \rightarrow \underset{\sim}{A}_1 \quad \underset{\sim}{A}_2 = \underset{\sim}{M}\dot{\underset{\sim}{d}}_3 - .6\Delta t\, \underset{\sim}{K}\underset{\sim}{d}_3$

$$\underset{\sim}{N}\,\underset{\sim}{s} = l_1 \underset{\sim}{K}\underset{\sim}{b}_1 + l_2 \underset{\sim}{K}\underset{\sim}{b}_2 + l_3 \underset{\sim}{K}\underset{\sim}{b}_3 + l_4 \underset{\sim}{M}\underset{\sim}{b}_4 + l_5 \underset{\sim}{M}\underset{\sim}{b}_5 + l_6 \underset{\sim}{M}\underset{\sim}{b}_6$$

Central diff $\quad \underset{\sim}{N} = \underset{\sim}{M} \quad \underset{\sim}{s} = \underset{\sim}{a}_{n+1} \quad l_1 = -1 \quad \underset{\sim}{b}_1 = \underset{\sim}{d}_n \quad l_2 = -\Delta t \quad \underset{\sim}{b}_2 = \underset{\sim}{v}_n \quad l_3 = -\frac{\Delta t^2}{4} \quad \underset{\sim}{b}_3 = \underset{\sim}{a}_n$

Trapez $\quad \underset{\sim}{N} = (\underset{\sim}{M} + \frac{\Delta t^2}{4}\underset{\sim}{K}) \quad \underset{\sim}{s} = \underset{\sim}{a}_{n+1} \quad l_1 = -1 \quad \underset{\sim}{b}_1 = \underset{\sim}{d}_n \quad l_2 = -\Delta t \quad \underset{\sim}{b}_2 = \underset{\sim}{v}_n \quad l_3 = -\frac{\Delta t^2}{4} \quad \underset{\sim}{b}_3 = \underset{\sim}{a}_n$

Damp $\quad \underset{\sim}{N} = \underset{\sim}{M} + \Delta t^2 \beta \underset{\sim}{K} \quad \underset{\sim}{s} = \underset{\sim}{a}_{n+1} \quad l_1 = -1 \quad \underset{\sim}{b}_1 = \underset{\sim}{d}_n \quad l_2 = -\Delta t \quad \underset{\sim}{b}_2 = \underset{\sim}{v}_n \quad l_3 = -(1-2\beta)\frac{\Delta t^2}{2} \quad \underset{\sim}{b}_3 = \underset{\sim}{a}_n$

α - Meth $\quad \underset{\sim}{N} = \underset{\sim}{M} + (1+\alpha)\Delta t^2 \beta \underset{\sim}{K} \quad s_n = a_{n+1} \quad l_1 = -1 \quad \underset{\sim}{b}_1 = \underset{\sim}{d}_n \quad l_2 = -(1+\alpha)\Delta t \quad \underset{\sim}{b}_2 = \underset{\sim}{v}_n \quad l_3 = -(1+\alpha)(\frac{1-2\beta}{2})\Delta t^2 \quad \underset{\sim}{b}_3 = \underset{\sim}{a}_n$

$(1-.3)(1.69)/2$

$.7(1.69)/2$

Wilson $\quad \underset{\sim}{N} = \theta\underset{\sim}{M} + (\theta\Delta t)^2 \beta \underset{\sim}{K} \quad \underset{\sim}{s} = \underset{\sim}{a}_{n+1} \quad l_1 = -1 \quad \underset{\sim}{b}_1 = \underset{\sim}{d}_n \quad l_2 = -\theta\Delta t \quad \underset{\sim}{b}_2 = \underset{\sim}{v}_n \quad l_3 = -\frac{(\theta\Delta t)^2}{2}(1-2\beta) \quad \underset{\sim}{b}_3 = \underset{\sim}{a}_n$

$l_4 = -(1-\theta) \quad \underset{\sim}{b}_4 = \underset{\sim}{a}_n$

Houbolt

1st step $\quad \underset{\sim}{N}_1 = (\underset{\sim}{M} + \frac{\Delta t^2}{4}\underset{\sim}{K}) \quad \underset{\sim}{s} = \underset{\sim}{a}_1 \quad l_1 = -1 \quad \underset{\sim}{b}_1 = \underset{\sim}{d}_0$

2nd $\quad \underset{\sim}{N}_2 = (\underset{\sim}{M} + \frac{\Delta t^2}{4}\underset{\sim}{K}) \quad \underset{\sim}{s} = \underset{\sim}{a}_2 \quad l_1 = -1 \quad \underset{\sim}{b}_1 = \underset{\sim}{d}_1 \quad l_2 = -\Delta t \quad \underset{\sim}{b}_2 = \underset{\sim}{v}_1 \quad l_3 = -\frac{\Delta t^2}{4} \quad \underset{\sim}{b}_3 = \underset{\sim}{a}_1$

all others $\quad \underset{\sim}{N} = (2\underset{\sim}{M} + \underset{\sim}{K}\Delta t^2) \quad \underset{\sim}{s} = \underset{\sim}{a}_{n+1}$

$l_4 = 5 \quad b_4 = \underset{\sim}{d}_n \quad l_5 = +4 \quad b_5 = d_{n-1} \quad l_6 = 1 \quad \underset{\sim}{b}_6 = d_{n-2}$

Parks method - could not get it to work.

Best case of all are trapezoidal of all for the lowest mode as well as the higher modes.

1. For the future homeworks use solid lines and dots ('DØTS*') instead of point symbols ☺

2. You didn't say anything about no. of operations, storage requirements, starting method etc..
   (You have to weight your results with all the ~~diff.~~ above factors in order to know how to pick an integrator

D1 DISPLACEMENT

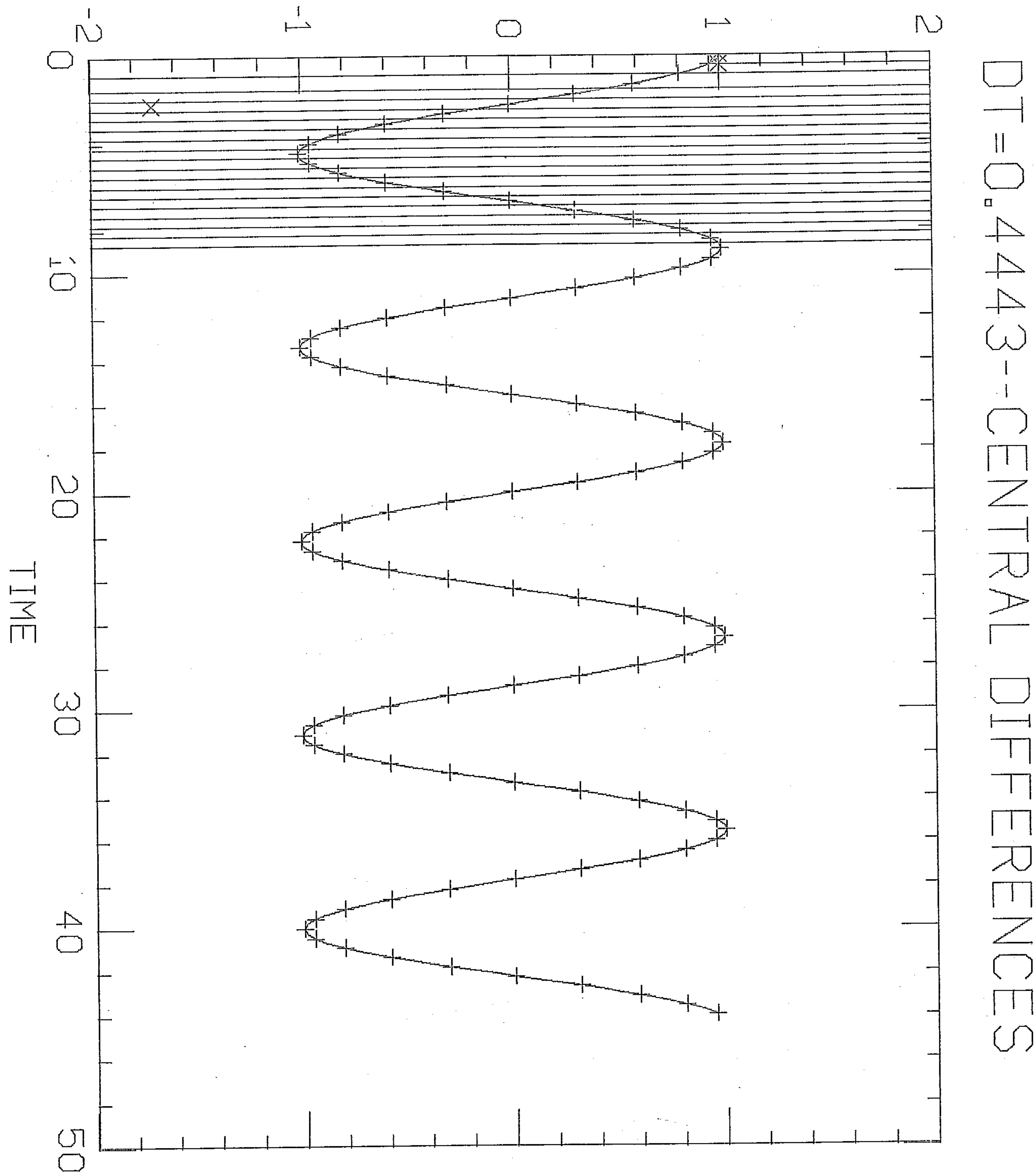


Fig. 1

D2 DISPLACEMENT

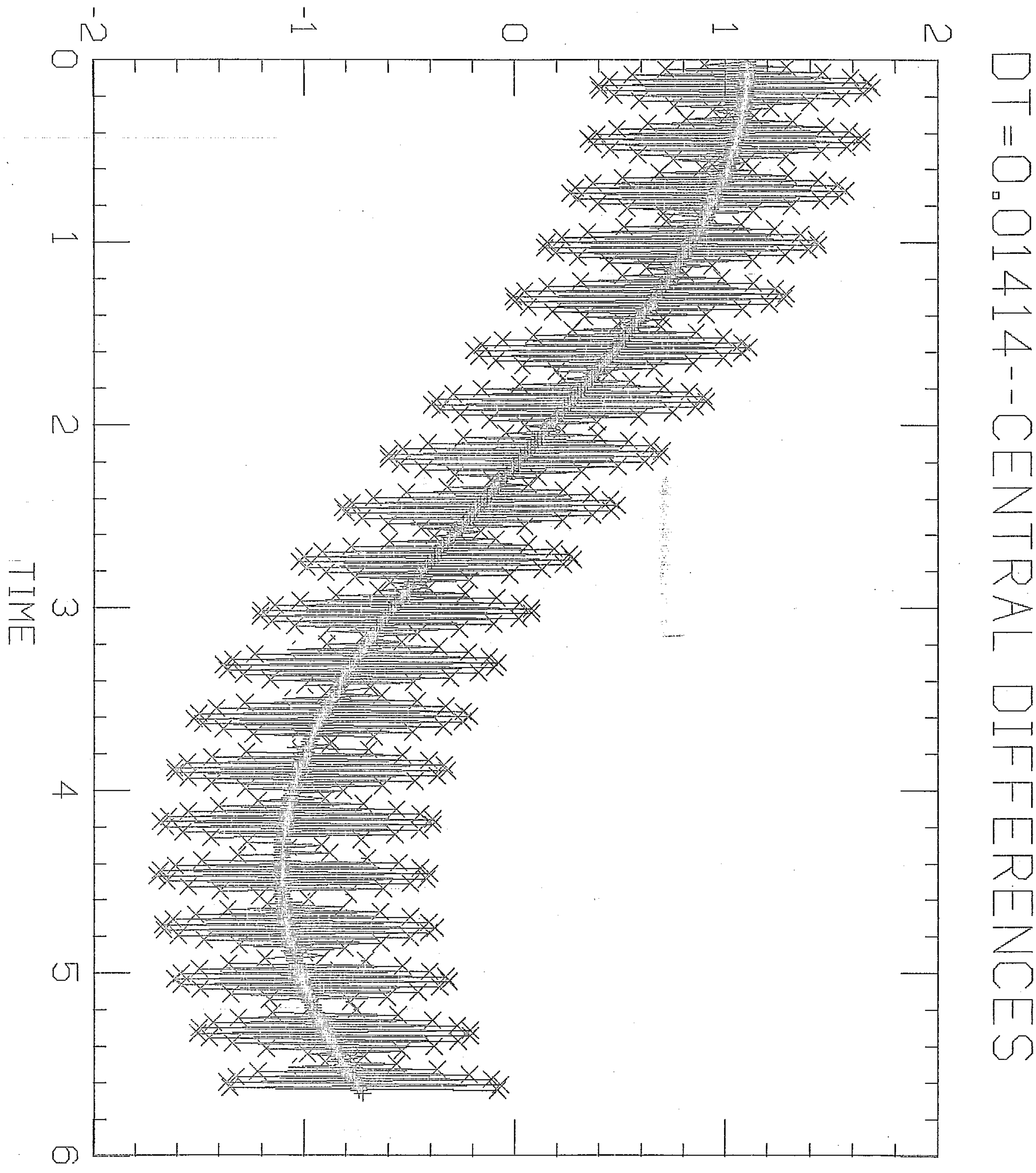


Fig 2

D1 DISPLACEMENT

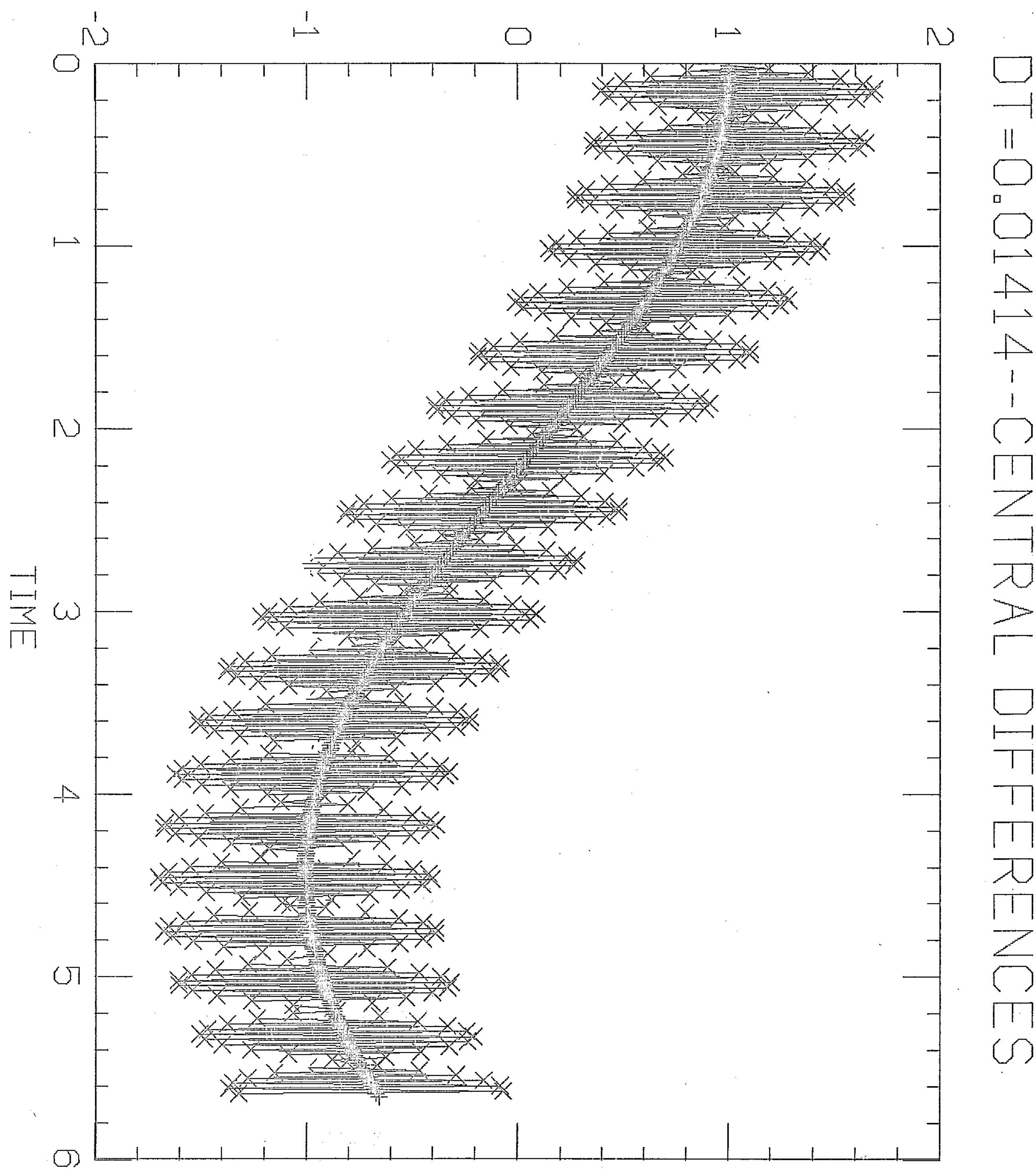


Fig 3

D1 DISPLACEMENT

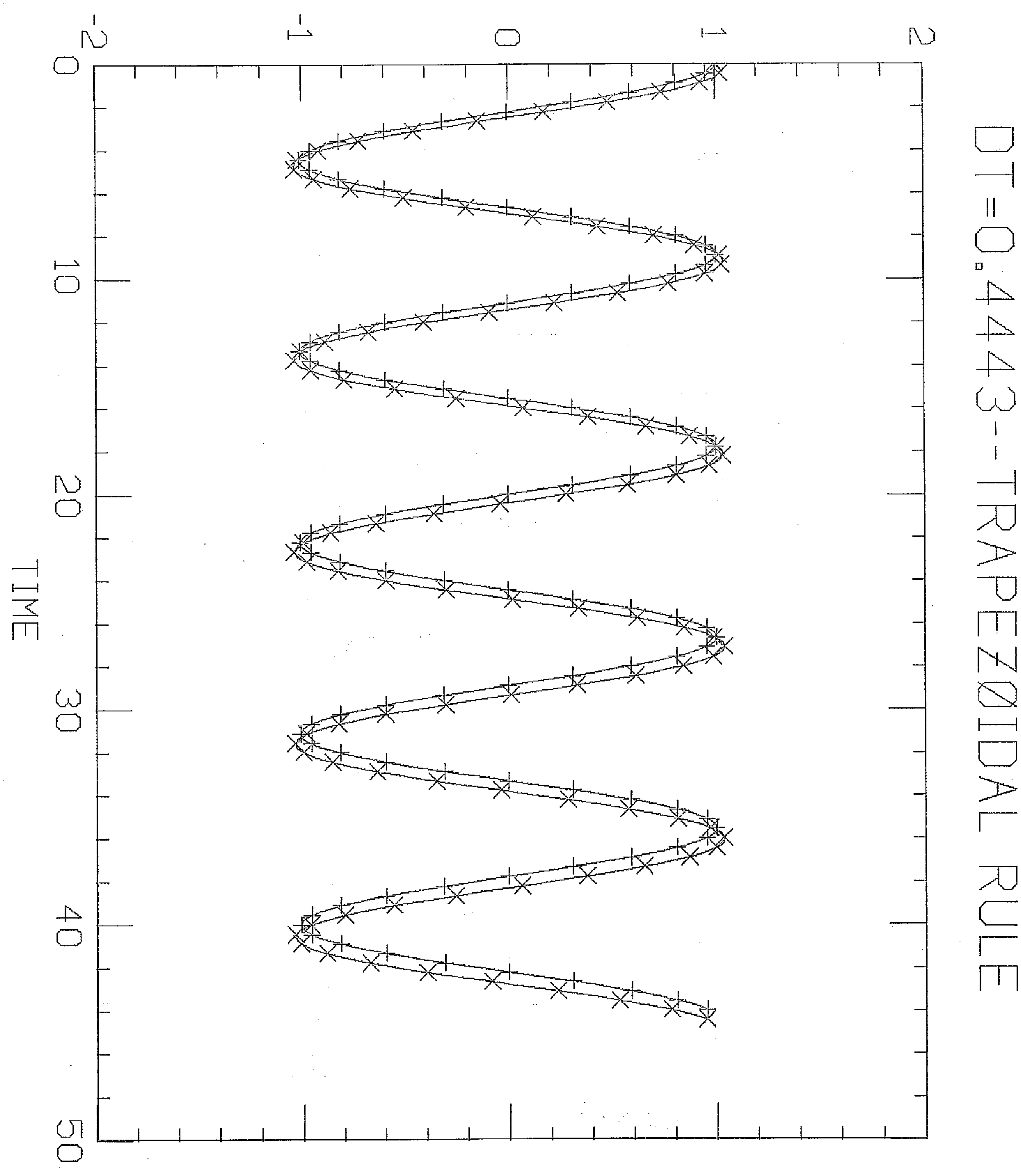


Fig 4

D1 DISPLACEMENT

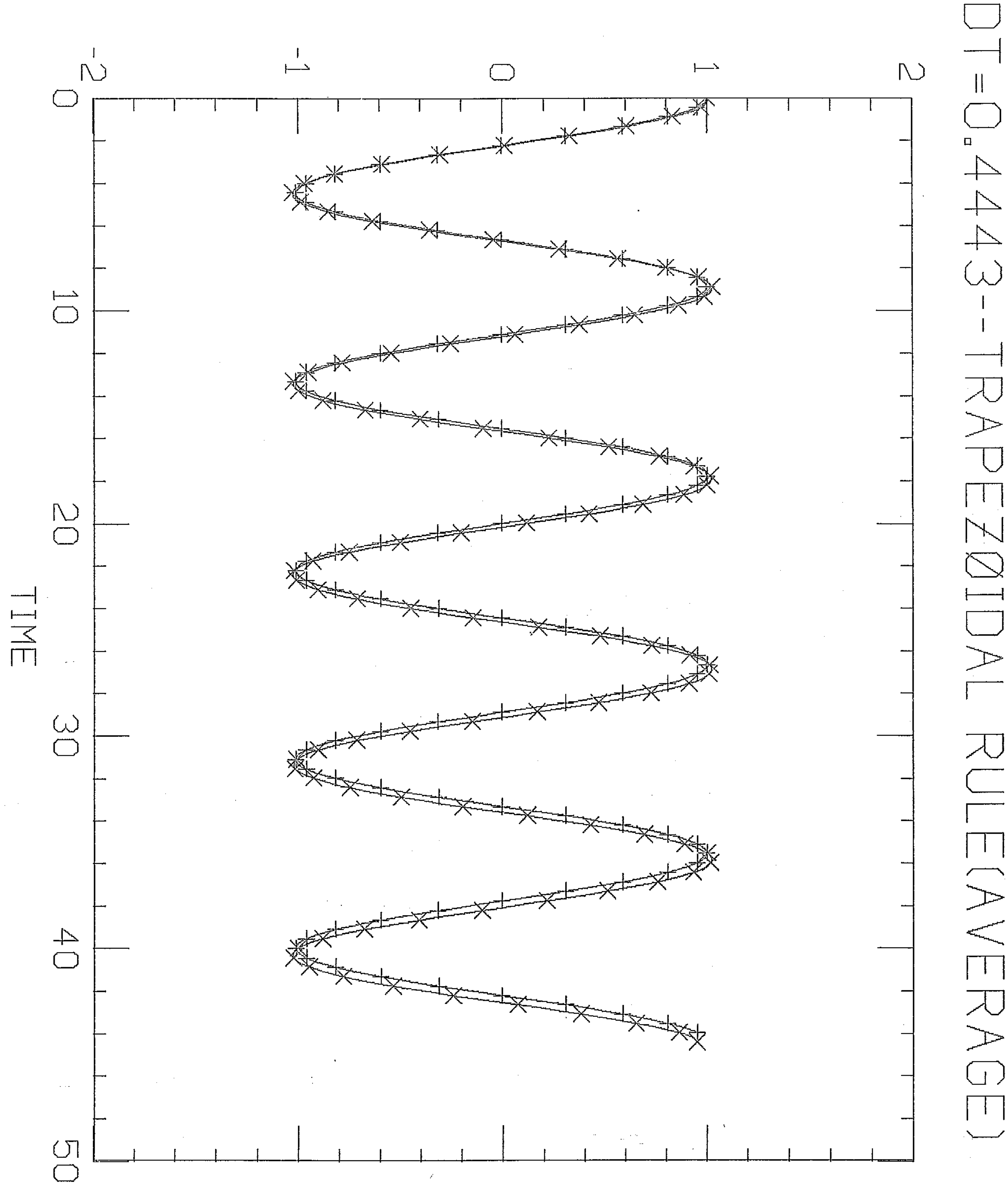


Fig 5

D2 DISPLACEMENT

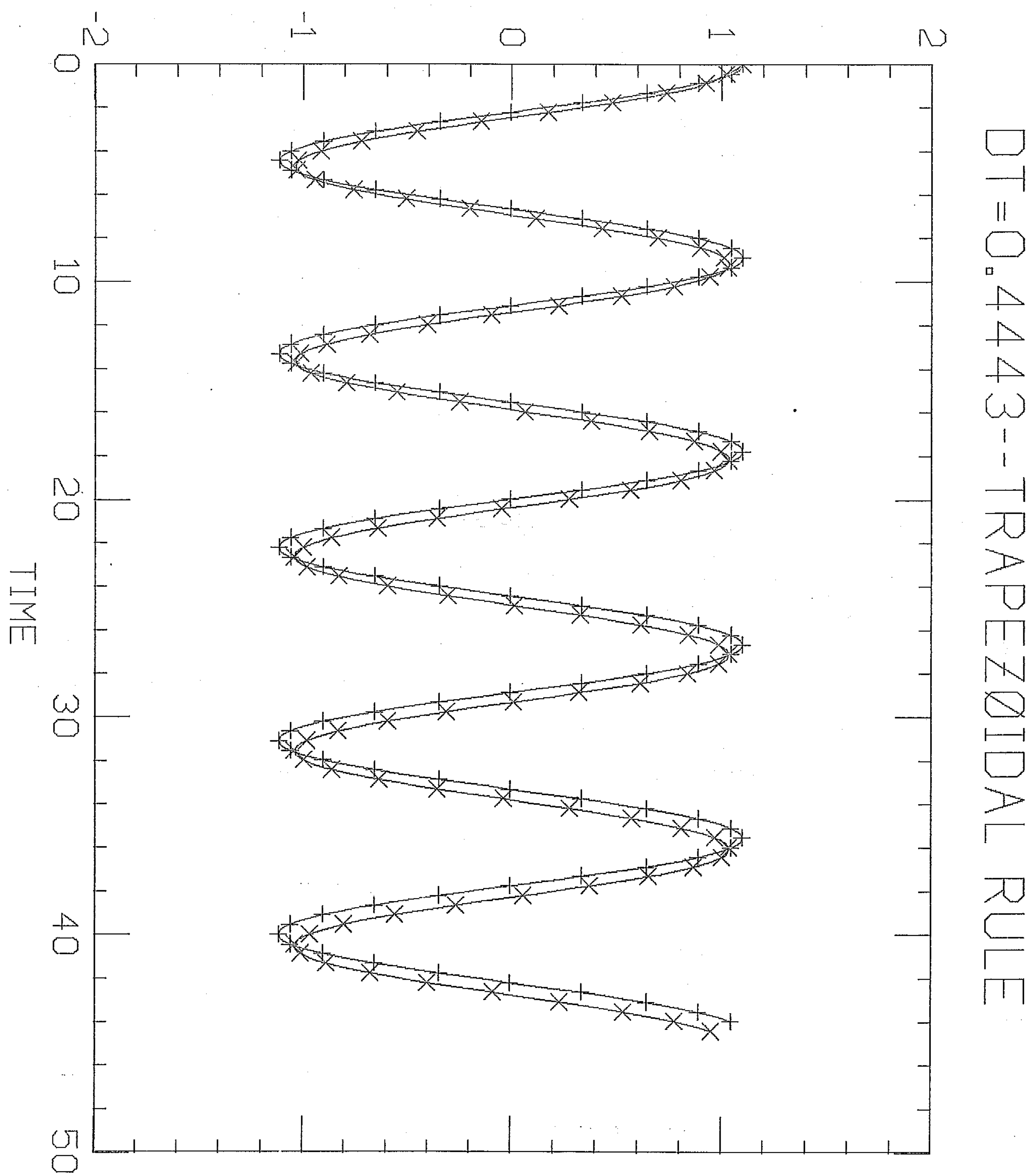


Fig 6

D2 DISPLACEMENT

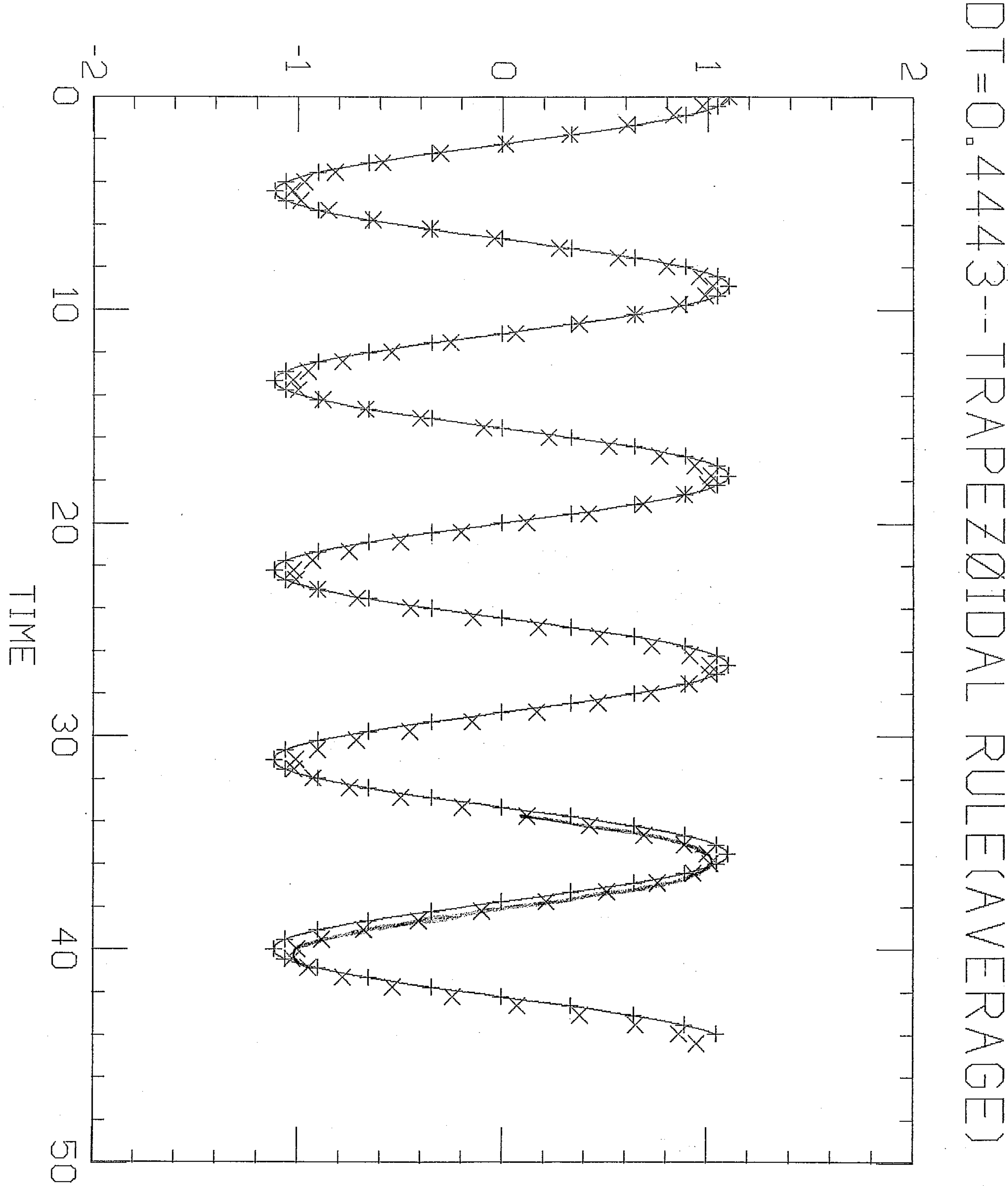


Fig 7

D1 DISPLACEMENT

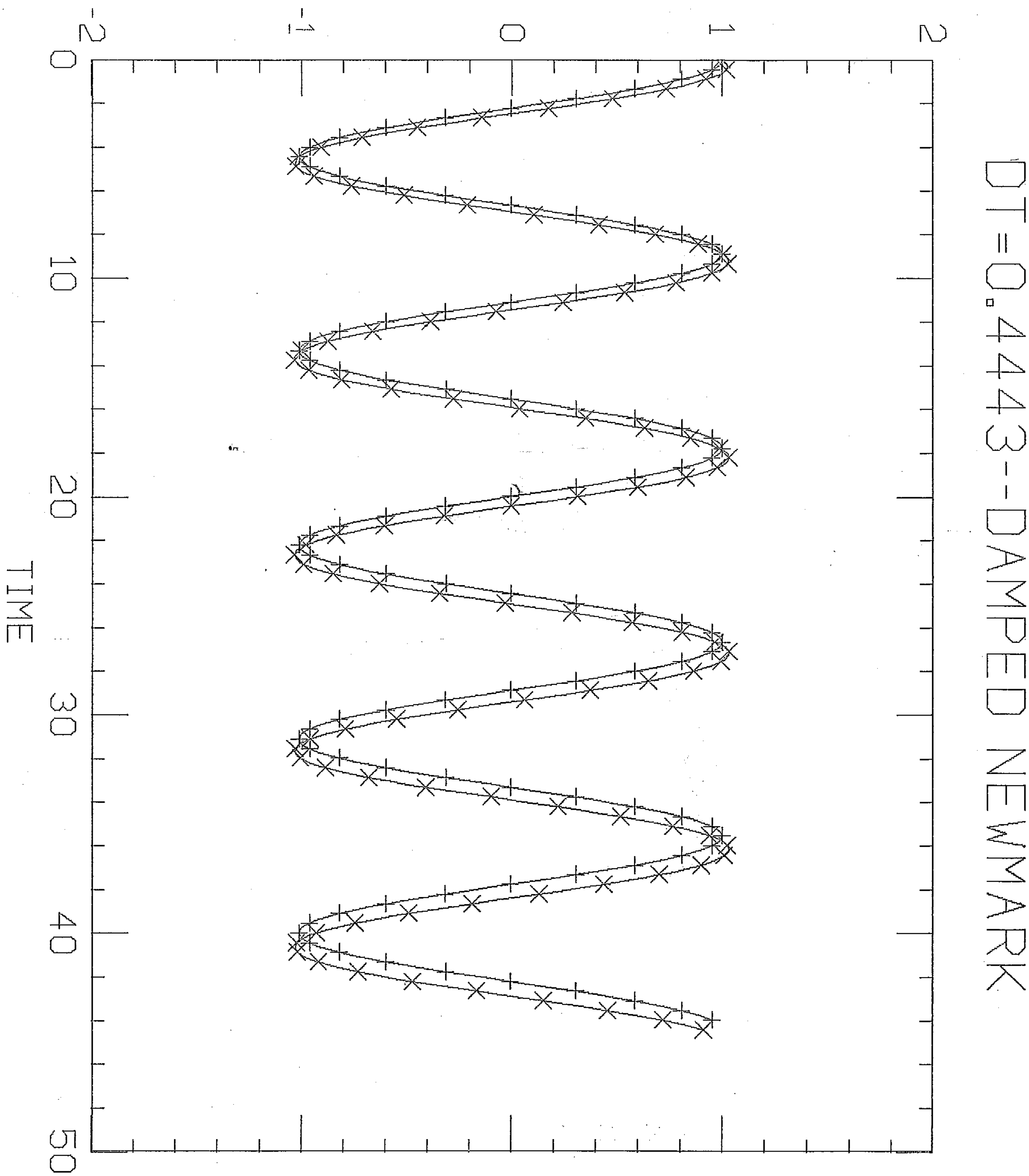


Fig 8

D2 DISPLACEMENT

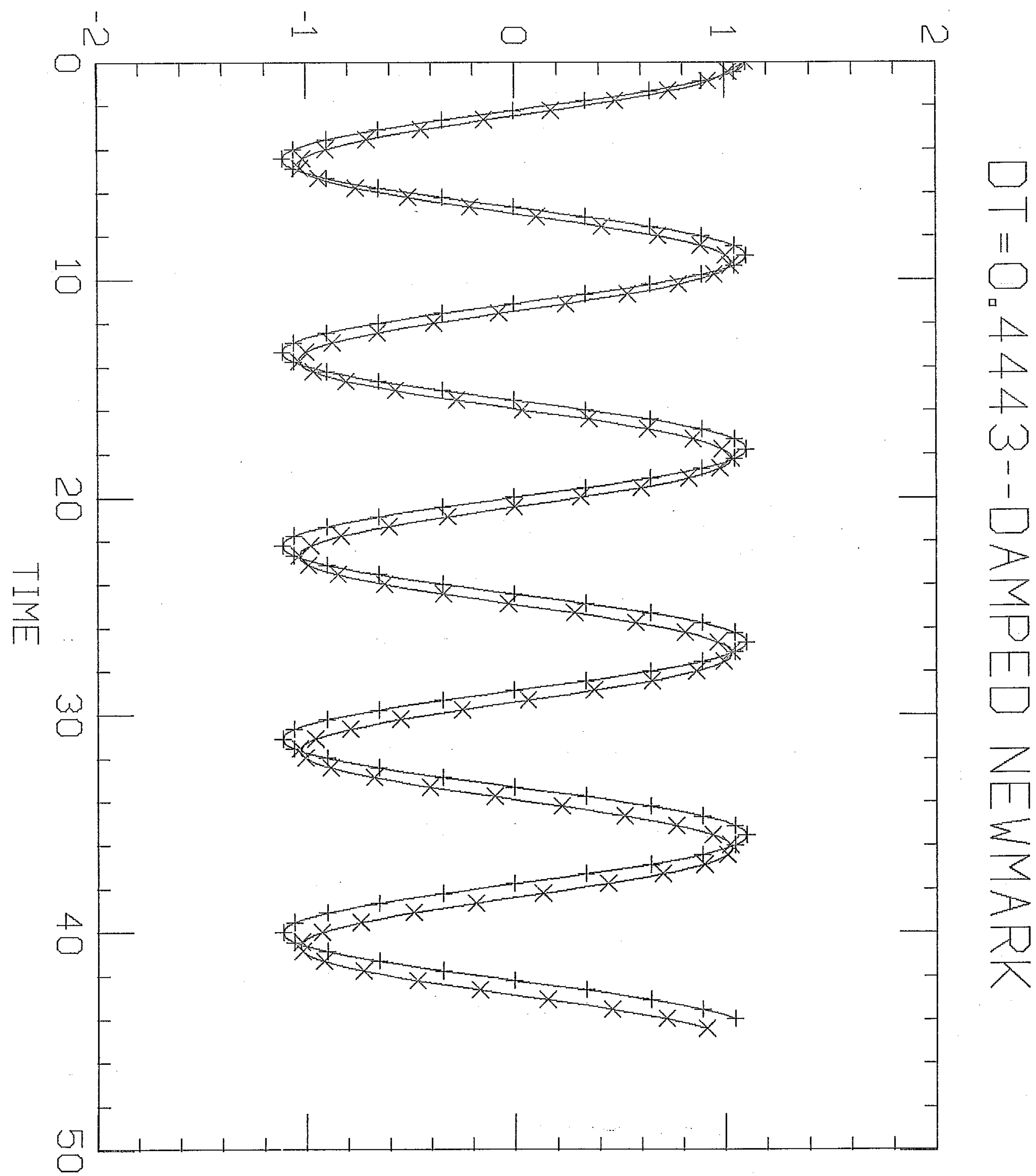


Fig 9

D1 DISPLACEMENT

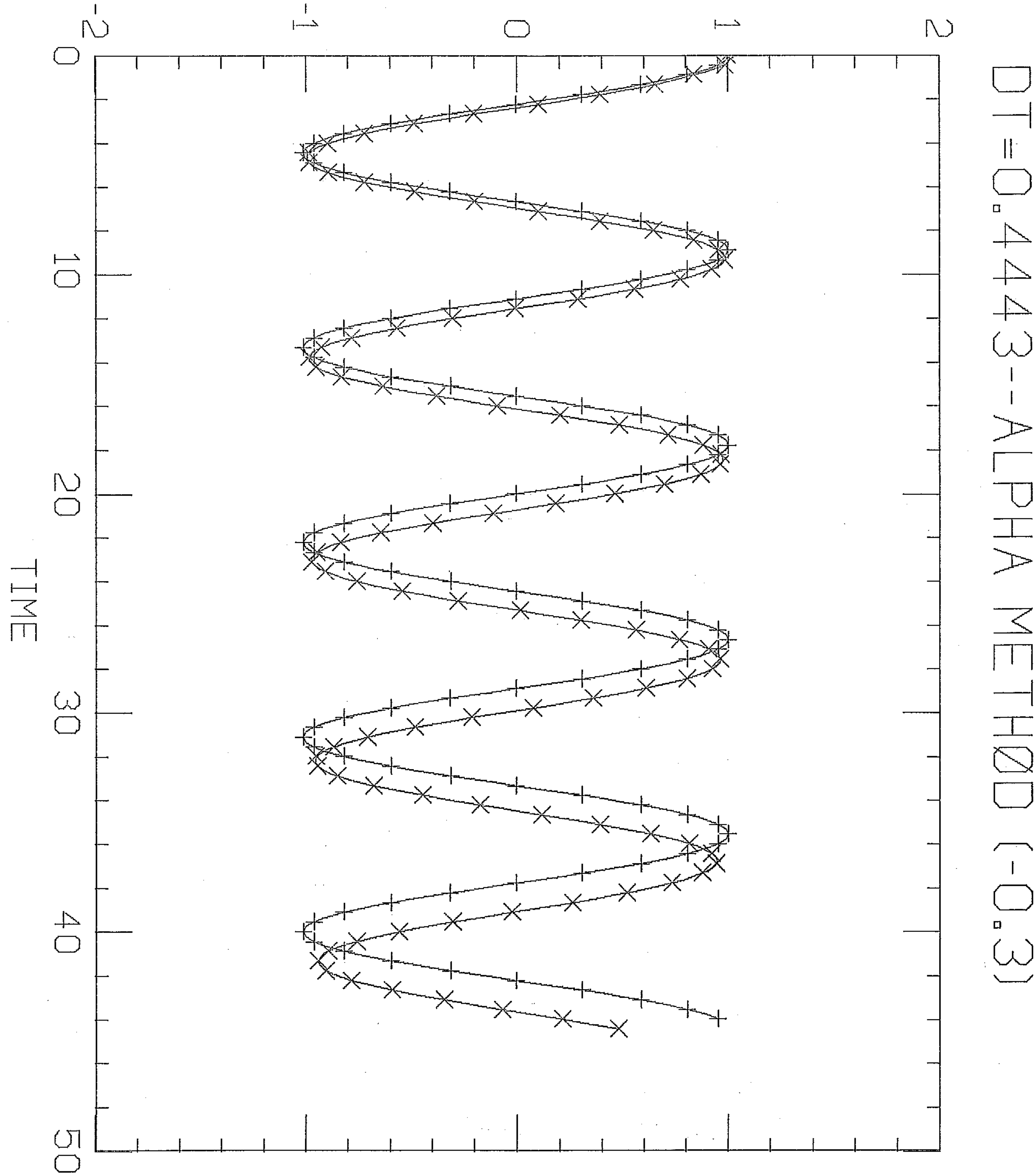


Fig 10

D2 DISPLACEMENT

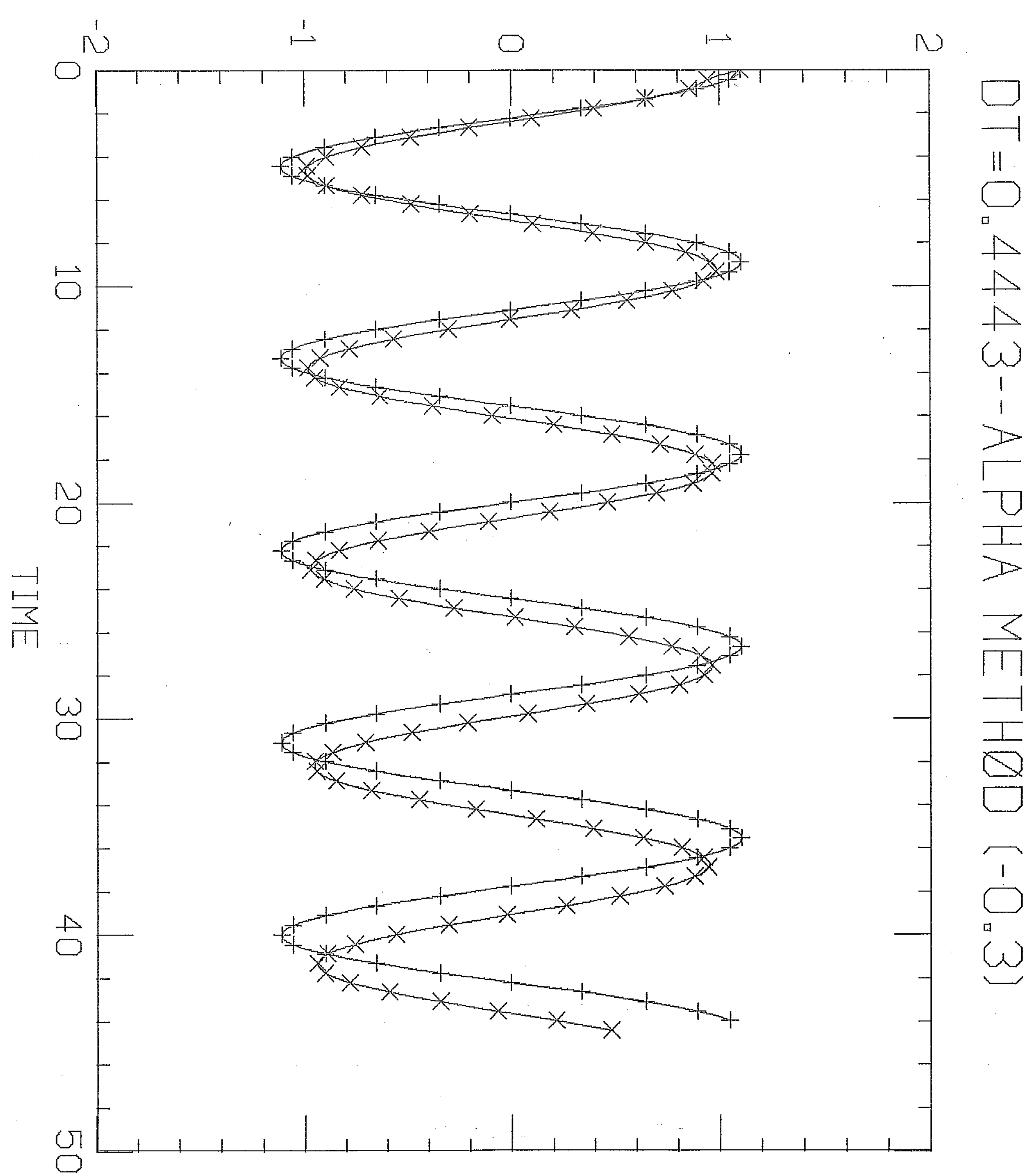


Fig. 11

DT=0.4443--WILSØN THETA METHØD (1.4)

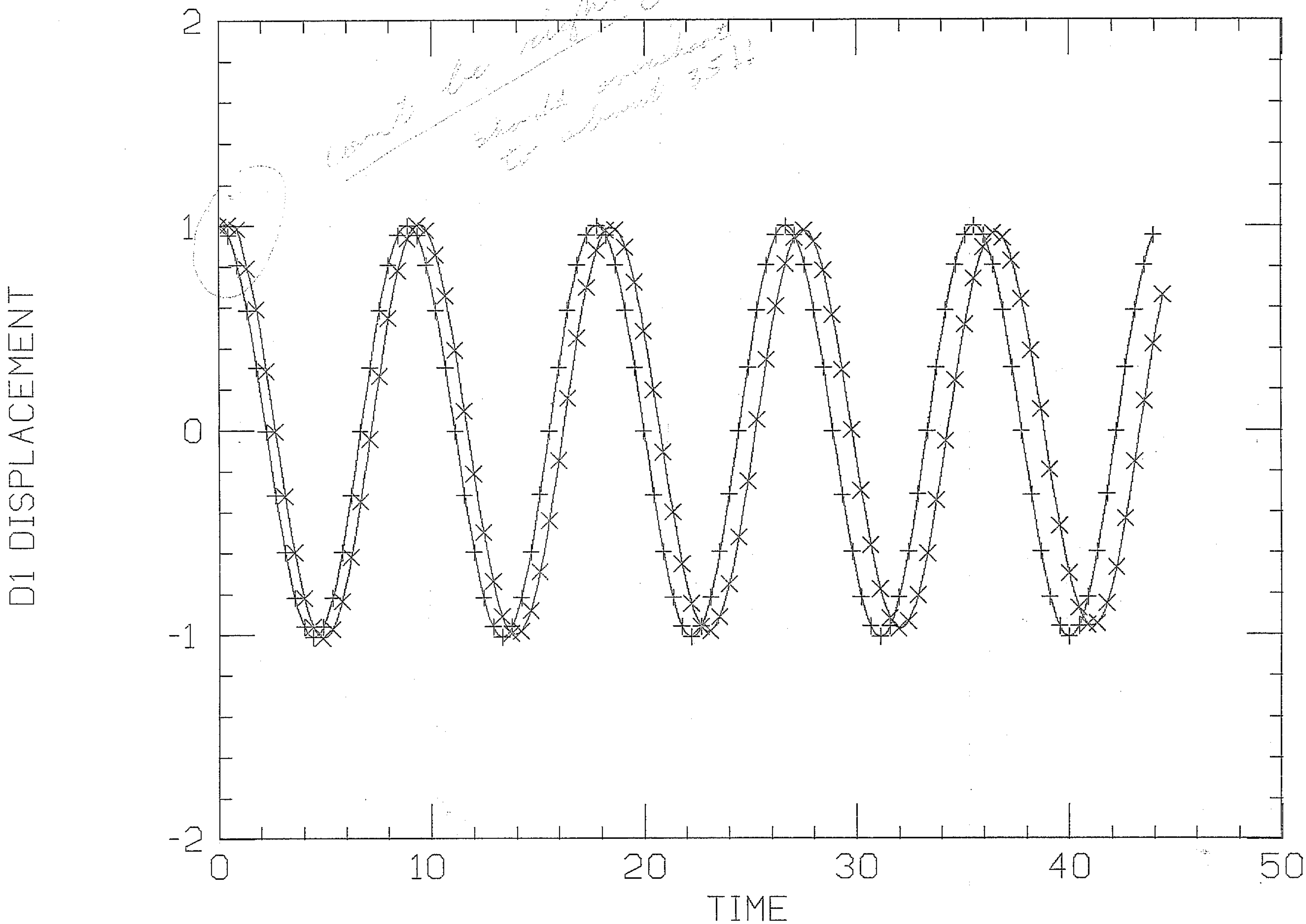


Fig 12

DT=0.4443--WILSØN THETA METHØD (1.4)

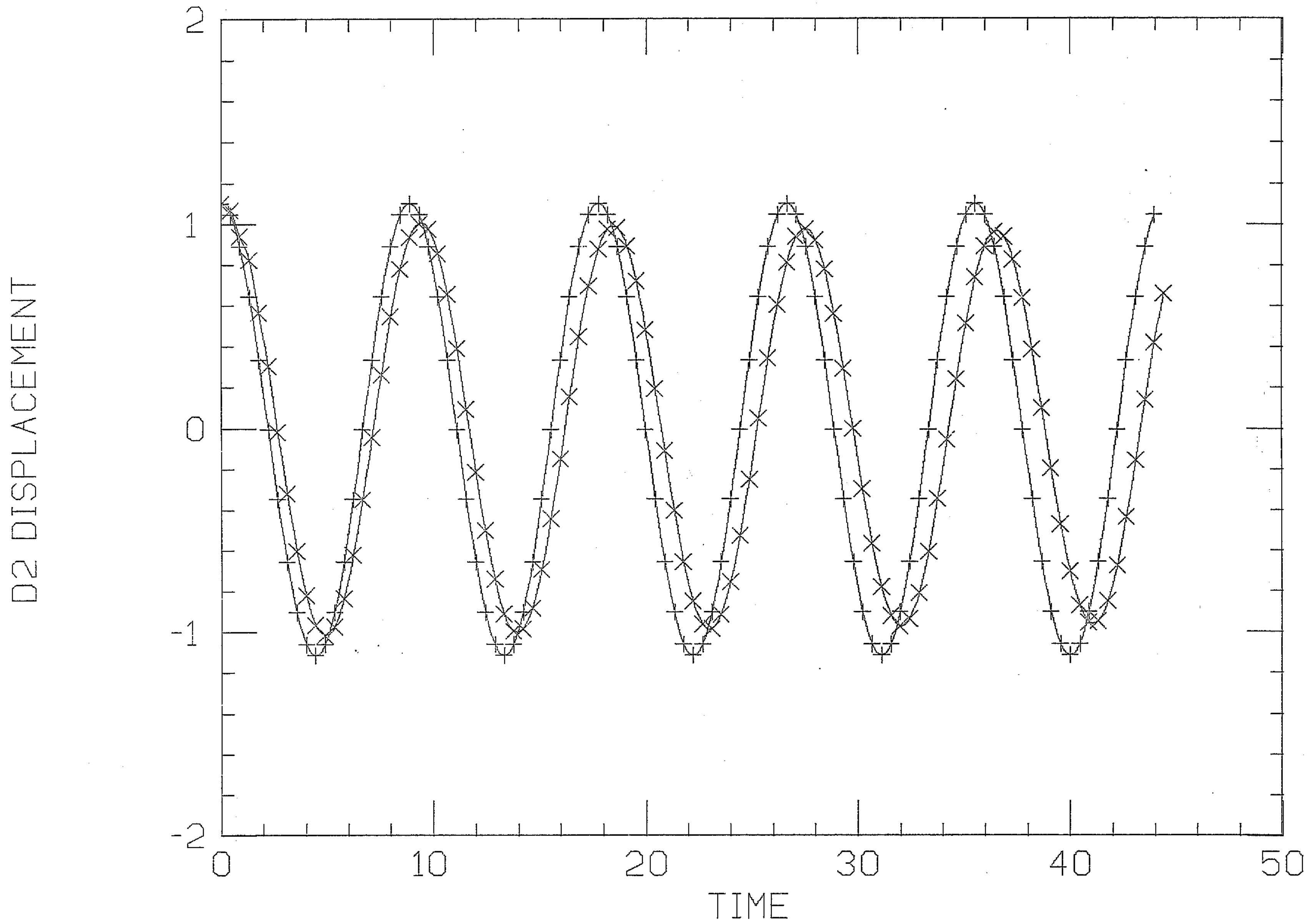


Fig 13

D1 DISPLACEMENT

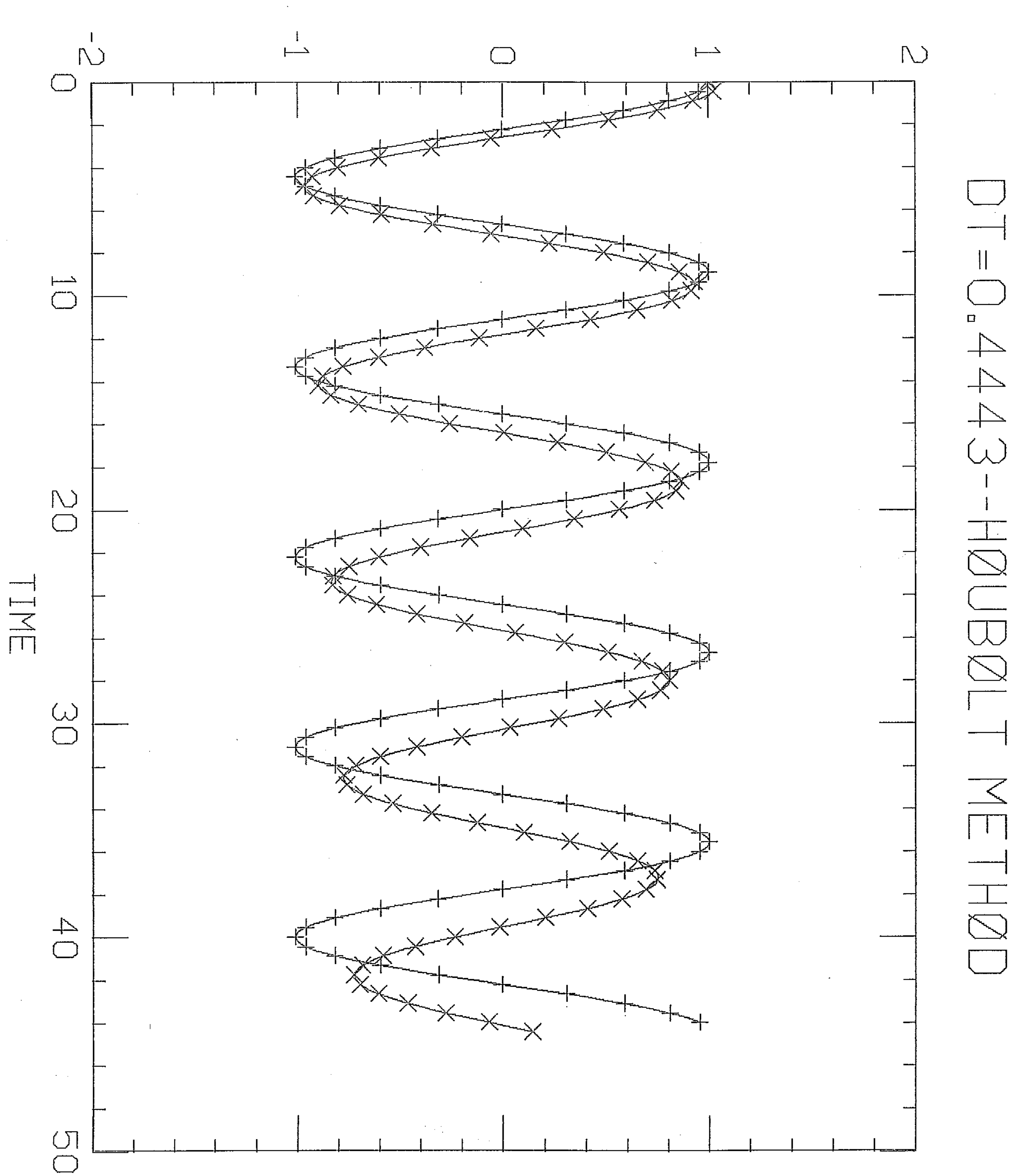


Fig 14

D2 DISPLACEMENT

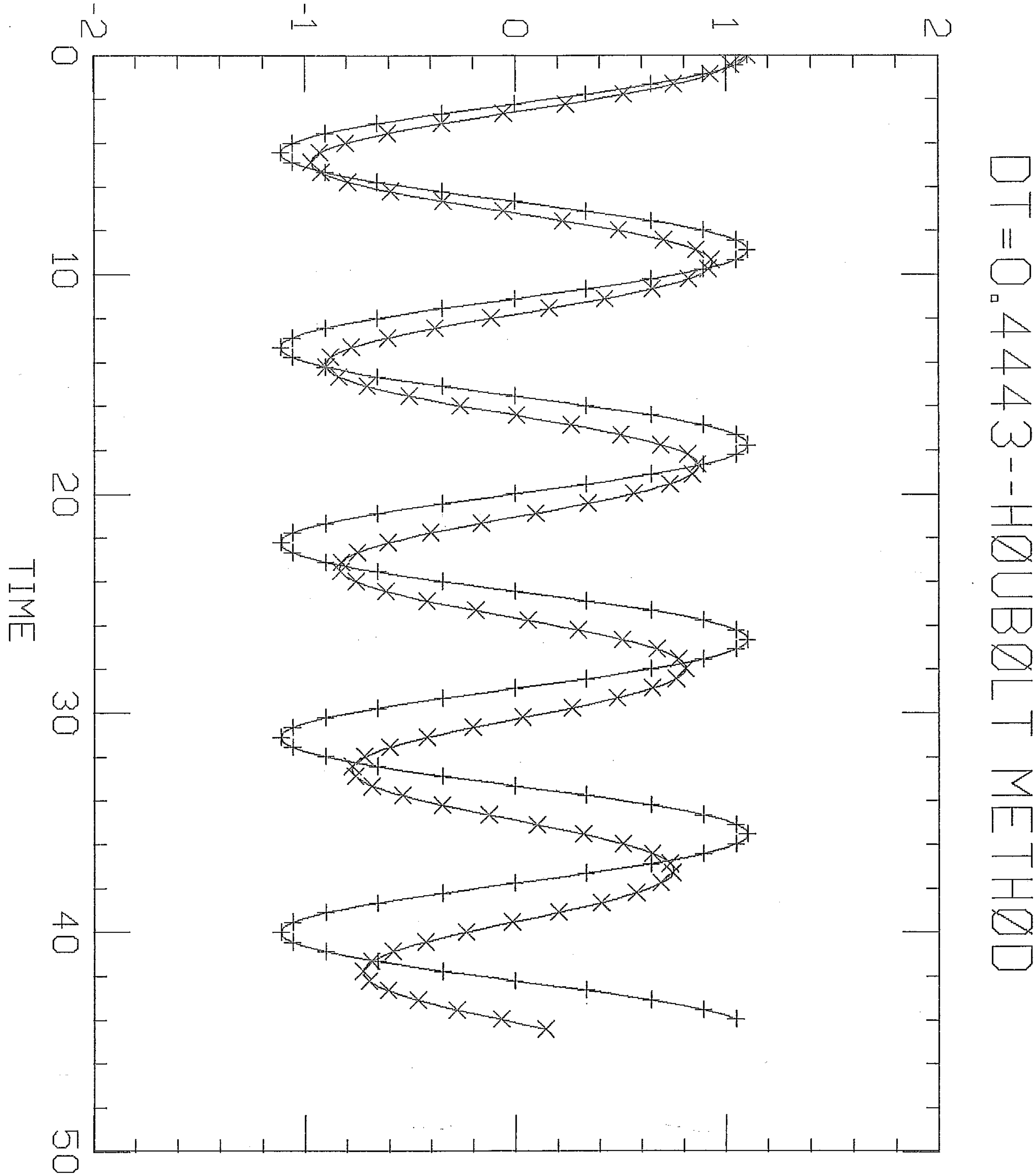


Fig 15

C. LEVY
ME235 C
Fall 1981

(A)

HW #2

Given: $\underset{\sim}{N}(\underset{\sim}{d}) = \underset{\sim}{F}$ for two cases of $\underset{\sim}{N}(\underset{\sim}{d})$ use the following to solve the problem:

(1) Newton Raphson (N.R.)

(2) Modified N.R. (mod NR)

(3) Incremental NR (Inc NR)

(4) Modified Incremental NR (mod inc NR)

(5) Mod NR with line search (mod NR LS)

(6) mod inc NR with line search (mod inc NR LS)

At enclosure #5 is the program to solve the problems given. The results obtained are for single precision arithmetic. The program was later modified to double precision arithmetic. Also checked was

(1b) NR w/ LS

(3b) inc NR w/LS

At enclosures #1-4 are the error charts

The following table is given

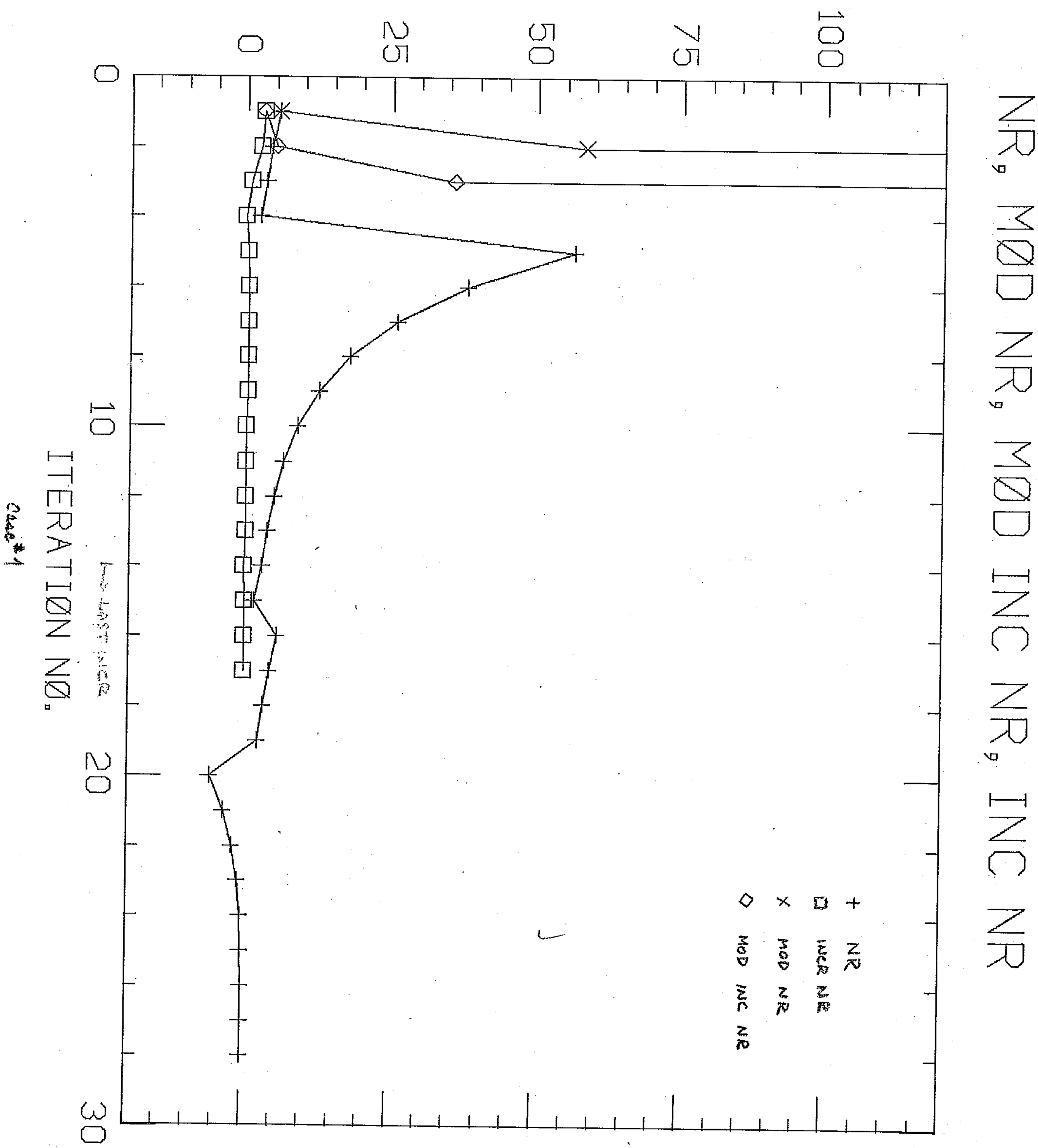
METHOD \ # iterations		Case 1	Case 2	Case 3
(1)	NR	28 ✓	7 ✓	5
(1b)	NR/LS	8 ✓	2 ✓	5
(2)	mod NR	4 (blew up) ✓	4 (blew up) ✓	30
(5)	mod NR/LS	3 ✓	2 ✓	30
(3)	inc NR	17 (4) ✓	14 (4) ✓	12
(3b)	inc NR/LS	6 (2) ✓	6 (2) ✓	12
(4)	mod inc NR	5 (blew up) ✓	7 (blew up) ✓	31
(6)	mod inc NR/LS	6 (2) ✓	6 (2) ✓	31

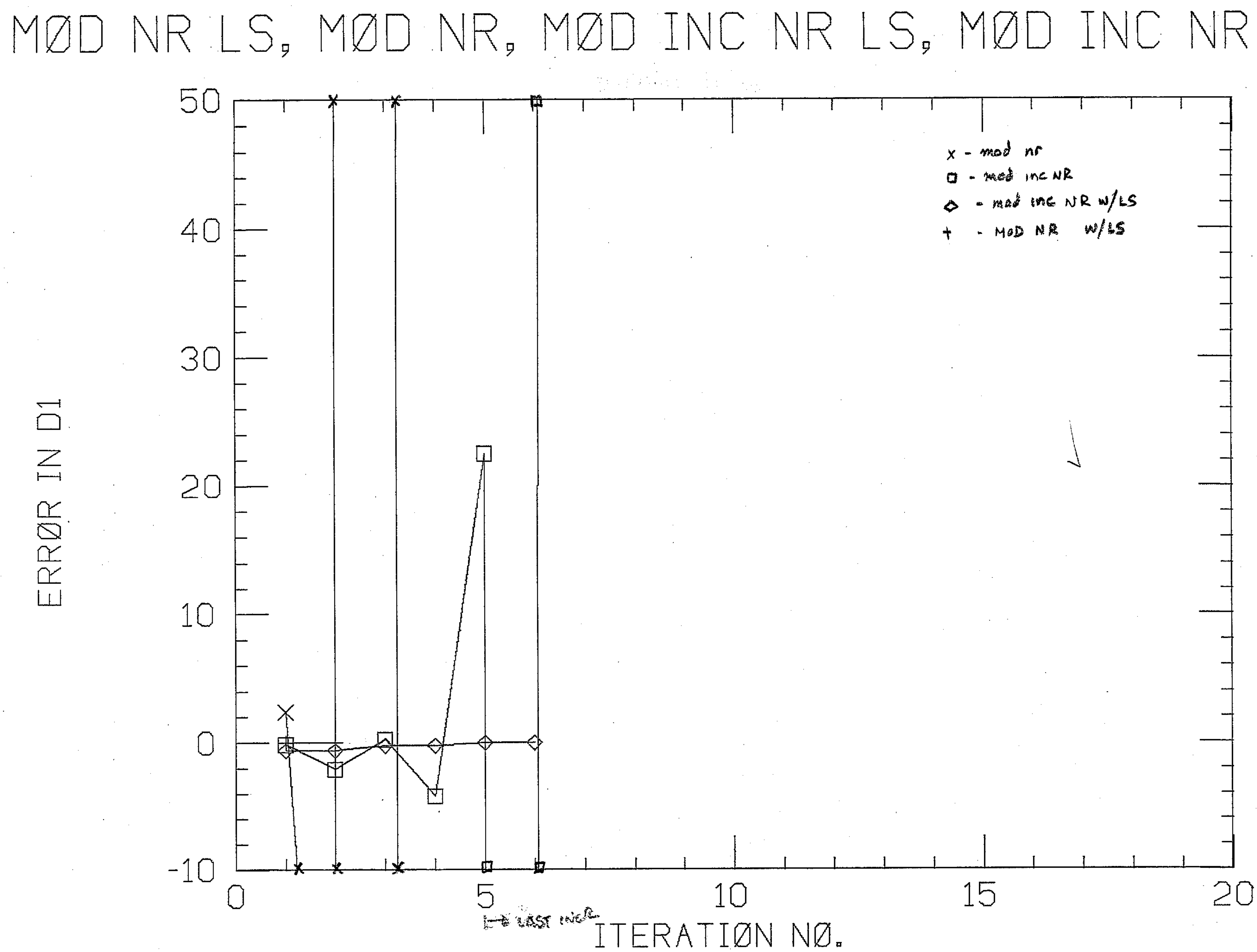
We note that within each group LS and incremental helped for case 1 & 2. However you could not combine the two to get optimal results. The no. in parenthesis represent the number of iterations for the last increment.

The third case was with $N^{(1)}(x) = N^{(2)}(x) = 40 * \tanh(x)$ to check for results of a soft stiffness. Note that if $\underset{\sim}{F}^{ext}$ is close to the upper limit of $\underset{\sim}{N}(\underset{\sim}{d})$ the number of iterations increase dramatically for anything but NR and LS doesn't help. If anything incremental does help.

Solutions	Case #1	-1.76699 & -3.533979	✓
	Case #2	1.340610 & 2.681220	✓

ERRØR IN D1


NR, MØD NR, MØD INC NR, INC NR
100
75
50
25
0
0
10
20
30
ITERATIØN NØ.
+ NR
□ INCR NR
× MOD NR
◇ MOD INC NR
Case #1


MØD NR LS, MØD NR, MØD INC NR LS, MØD INC NR
ERRØR IN D1
ITERATIØN NØ.
50
40
30
20
10
0
-10
0
5
10
15
20
x - mod nr
□ - mod inc NR
◇ - mod inc NR w/LS
+ - MOD NR w/LS
LAST INCR

$N(d) = \underset{\sim}{F}$ $\quad F^{ext}_{n+1} = (0, 37.5)^T$

$\underset{\sim}{F}^{int}(\underset{\sim}{d}^{(0)}) = \underset{\sim}{N}(\underset{\sim}{d}^{(0)})$

$\underset{\sim}{d}^{(0)} = (0,0)^T$ $\quad \underset{\sim}{N}(\underset{\sim}{d}^{(i)}) = \begin{Bmatrix} 1 & -1 \\ 0 & 1 \end{Bmatrix} \cdot \left[N^{(1)}\left(\frac{d_1^{(i)}}{h}\right), N^{(2)}\left(\frac{d_2^{(i)} - d_1^{(i)}}{h}\right) \right]^T$

where

Case 1 $\quad N^1(x) = 10(x - x^3) \quad N^2(x) = 10(x - x^3)$

let $\quad N(d_{n+1}) = F_{n+1} \quad N(d_n) = F_n$

$N(d_{n+1}) - N(d_n) = F_{n+1} - F_n$

$N(d_n) + \left.\frac{\partial N(d_n)}{\partial d}\right|_{d_n} (d_{n+1}) - (d_n)$ $\quad$ thus $\frac{\partial N(d)}{\partial d} = K.$

$$\frac{\partial \underset{\sim}{N}}{\partial d_1} = \frac{1}{h}\begin{Bmatrix} 1 & -1 \\ 0 & 1 \end{Bmatrix} \left[\frac{\partial N^{(1)}}{\partial x}\left(\frac{d_1^{(i)}}{h}\right), -\frac{\partial N^{(2)}}{\partial x}\left(\frac{d_2 - d_1}{h}\right) \right]^T$$

$$\frac{\partial \underset{\sim}{N}}{\partial d_2} = \frac{1}{h}\begin{Bmatrix} 1 & -1 \\ 0 & 1 \end{Bmatrix} \left[0, \frac{\partial N^{(2)}}{\partial x}\left(\frac{d_2 - d_1}{h}\right) \right]^T$$

Now let $\frac{\partial N^{(1)}}{\partial x} = 10(1 - 3x^2)$

$$K = \partial \underset{\sim}{N} / \partial \underset{\sim}{d} = \frac{1}{h}\begin{Bmatrix} 1 & -1 \\ 0 & 1 \end{Bmatrix} \begin{Bmatrix} \frac{\partial N^{(1)}}{\partial x}(d_1/h) & 0 \\ -\frac{\partial N^{(2)}}{\partial x}\left(\frac{d_2 - d_1}{h}\right) & \frac{\partial N^{(2)}}{\partial x}\left(\frac{d_2 - d_1}{h}\right) \end{Bmatrix}$$

$\frac{\partial N}{\partial x}$

thus $\quad \partial N_1 / \partial d_1 = K_{11} \quad \partial N_1 / \partial d_2 = K_{12}$

$\partial N_2(d) / \partial d_1 = K_{21} \quad \partial N_2 / \partial d_2 = K_{22}$

set AK(1,1) = AK(2,2) = 1 , AK(2,1) = 0 ; AK(1,2) = -1

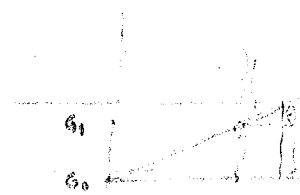
```
      SUBROUTINE FORM (W, D, I)
      DIMENSION W(1), D(2),
      COMMON H,LS,IN

      X1 = D(1)/H
      X2 = (D(2) - D(1))/H
      GO TO (10,10,20,20), I
C.... FORM N(D)
   10 W(1) = EN(1,X1) - EN(2,X2)
      W(2) = EN(2,X2)
      RETURN
C.... FORM DN(D)
   20 W(4) = (EN(4,X2)/H
      W(2) = -W(4)
      W(3) = -W(4)
      W(1) = EN(3,X1)/H + W(4)
      RETURN
      END
```

```
      FUNCTION EN (I,X)
      GO TO (1,1,2,2), I
    1 N = 10.*X*(1.-X*X)
      RETURN
    2 N = 10.*(1-3.*X*X)
      RETURN
      END
```

Case 2
N = 10*X*(1+X*X)
N = 10*(1+3*X*X

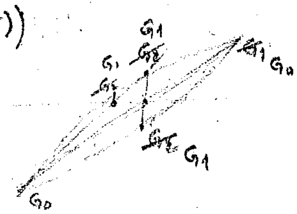
```
      SUBROUTINE TRANS (W,W1)
      DIMENSION W(1),W1(1)
      DO 10 I=1,4
   10 W(I) = W1(I)
      RETURN
      END
```

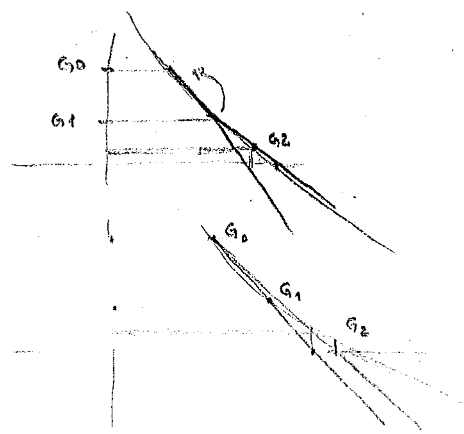
```
                    CALL LNSRCH ( D, DI, DD, FXT, FNT )
C ....  FNT ENTERS WITH DI      SUBROUTINE HLQUST (D,DI,DD,FXT,FNT)
        DIMENSION X(2)  K=0    COMMON H,LS,IN
        DO 10 J=1,2  G0=0.  IFL=0
  10    G0 = DD(J) * (NC* FXT(J) - FNT(J)) +G0.
C ....  FNT UPDATED WITH D = DI + DD
        CALL FORM (FNT, D, 1)
C ....   IF (LS.EQ.0) RETURN       IF LINE SEARCH NOT IN EFFECT RETURN
         DO 20 J=1,2    G1=0.
                  G1+
  20     G1 = DD(J) * (NC* FXT(J) - FNT(J))
C ...   check on requirement for line search
                                     C... IF (ABS(G0).LT. 2.* ABS(G1)) GO TO 100
                                            HAVE WE OVER ITERATED
         S = 1 - G0/(G1-G0)            25   K=K+1
C ...   check value of S                    IF (K.GT.50) GO TO 100
         IF (S.GT.1) GO TO 100
C ...   find new D, FNT(D) AS FN OF S
         DO 30 J=1,2
  30     X(J) = DI(J) + S * DD(J)
         CALL FORMS (FNT, X, 1).
         G2=0.
         DO 40 J=1,2
  40     G2= G2 + DD(J) * (NC* FXT(J) - FNT(J))
C ....  find new G  IF (ABS(G2).LE 1.0E-4) GO TO 60
                    IF (G2* G1) 50, 60, 70
C ....  a zero exists in interval.
  50     G0=G1
         G1=G2    IFL=1
         GO TO 25
C ....  a zero is found, define D, DD
  60     DO 65 J=1,2
         DD(J) = S* DD(J)
  65     D(J) = X(J)
         RETURN
C ....  IFL=1 IMPLIES CROSS OVER
  70     IF (IFL.EQ.0) G0=G1
         G1=G2
         GO TO 25
C ...   set S=1 & give up.
 100     CALL FORM (FNT, D, 1)
         RETURN
         END
```



$$\frac{G_1 - G_0}{0 - G_1} = \frac{1-0}{S-1}$$

$$S = 1 - G_1/(G_1 - G_0)$$





-2G(S) ... G(S) ≥ 2G(S)

```
Dimension  W(2,2), FXT(2), FNT(2), DI(2), DD(2), D(2), E(2)
Data  DI(1),DI(2),FXT(1),FXT(2),H /0.,0.,0.,12.5,1./
Common H,LS,IN
C...  set up initial displ        method type
C...  MNR=1 Newton Raphson, MNR=100 - mod newton raph, MNR=3 Mod NR w/update, IN=1 INCR LOAD, IN=3 FULL LOAD
      MNR=1 ; IN=3 ; LS=0
      DO 10 J=1,2
   10 D(J) = DI(J)
C...  set up internal force, rhs
      CALL FORM (FNT,DI,1)
      DO 20 J=1,2
   20 R(J) = FXT(J) - FNT(J)
C...  form tangent stiffness & get determinant
      CALL FORM (W,DI,3)
      DET = W(1,1)*W(2,2) - W(1,2)*W(2,1)
C...  Start iteration
      DO 100 I = 1,101   <------  K = MOD(I+1,MNR).
C...  solve W*DD = R
      CALL SOLVE (W,DD,R,DET)
C...  update displ, errors
      DO 40 J=1,2
   40 D(J) = DI(J) + DD(J)    d^{i+1}    DI = d^i    DD = Δd^i
C.... call Dalquist line search & update Finternal within Dalquist; find D, DD, FNT
      CALL LNSRCH (D,DI,DD,FXT,FNT)

      DO 50 J=1,2
      E(J) = ABS (DD(J))/ABS(D(J))
      R(J) = IN*FXT(J) - FNT(J)
   50 DI(J) = D(J)
      WRITE (6,55) I,E(1),E(2),D(1),D(2)
   55 FORMAT (2X,I5,4(2X,E14.7))
C...  Check convergence ---- DO 60 J=1,2
      IF (E(J) .GE. 1.E-4) GO TO 70
      IF (R(J) .GE. 1.E-4) GO TO 70
   60 continue
C...  soln. converged
C...  not converged, check if mod NR.
   70 IF (K .NE. 0) GO TO 100
C...  CALL FORM (W,DI,3)
      DET = W(1,1)*W(2,2) - W(1,2)*W(2,1)
  100 continue
      write (6,110)
  110 format (2X, 'iteration stop')
      stop
```

let MNR=1   INCR=3 ← Full load NR (1)
    1   1,2,3 ← inc load NR (3)
MNR=100   INCR=3 ← Full load MNR (2)
MNR=100   1,2,3 ← incl load MNR (4)

```
   60 continue
C...  SOLN CONV ; INCREMENT LOAD IF NECC
      IN = IN+1
      IF (IN .GT. 3) STOP  --- reach only for increm load
C...  REDEF RHS
      DO 65 J=1,2
C...  COMPUTE R(J) = IN*FXT(J) - FNT(J)
      REQ'D
   70 continue
      IF (K.NE.0) GO TO 100
      CALL FORMS (W,DI,3)
      DET = ...
  100
```

let $w = \sum_{a=1}^{2} N_a(\xi) c_a$ $\qquad d\xi = \xi_{,x} dx = \frac{2}{h} dx$

$$\int w_{,x} \, \kappa(u) \, u_{,x} \, d\Omega = \sum c_a \int N_{a,\xi}(\xi) \, \xi_{,x} \, \kappa\left(\sum_{a=1}^{2} N_a d_a\right) u_{,\xi} \, d\xi$$

$$= \int N_{a,\xi}(\xi) \frac{2}{h} \kappa(u) \, u_{,\xi} \, d\xi$$

$u = \sum N_a(\xi) d_a, \quad u_{,\xi} = \sum N_{a,\xi} d_a = \frac{d_2 - d_1}{2}$

wt, $u(0)$

$$= 2 \, N^e_{a,\xi}(0) \, \frac{2}{h} \, \kappa\left(\frac{d^e_2 + d^e_1}{2}\right) \frac{d^e_2 - d^e_1}{2}$$

$$= 2 \cdot \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \frac{2}{h} \cdot \frac{d^e_2 - d^e_1}{2} \, \kappa\left(\frac{d^e_2 + d^e_1}{2}\right)$$

$$= \frac{d^e_2 - d^e_1}{h} \, \kappa\left(\frac{d^e_2 + d^e_1}{2}\right) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{at } e=1 \quad d_1 = 0$$

$$= \frac{d^1_2}{h} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \kappa\left(\frac{d^1_2}{2}\right) + \frac{d^2_2 - d^2_1}{h} \kappa\left(\frac{d^2_2 + d^2_1}{2}\right) \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

let $d^1_2 = D_1 \quad d^2_1 = D_1 \quad d^2_2 = D_2$

$$N(\underset{\sim}{D}) = \frac{1}{h}\left[D_1 \, \kappa\left(\frac{D_1}{2}\right) + (D_2 - D_1) \cdot \kappa\left(\frac{D_2 + D_1}{2}\right) \right] \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \frac{D_1}{h} \left\{ \kappa\left(\frac{D_1}{2}\right) - \kappa\left(\frac{D_2 + D_1}{2}\right) \right\}$$

built in end $u_{,x} = 0 \qquad u = 0 \qquad u_{,x} = \frac{d_2 - d_1}{h} = 0$

$e=1$ for $a=1$ $\quad -\frac{D_1}{h} \kappa\left(\frac{D_1}{2}\right)$ $\qquad$ for $a=2$ $\quad \frac{D_1}{h} \kappa\left(\frac{D_1}{2}\right)$

$e=2$ $\qquad$ for $a=1$ $\quad -\frac{D_2 - D_1}{h} \kappa\left(\frac{D_1 + D_2}{2}\right)$ $\qquad$ for $a=2$ $\quad \frac{D_2 - D_1}{h} \kappa\left(\frac{D_1 + D_2}{2}\right)$

$$-\int w_{,x} \, q(u_{,x}) dx = -\int N_{a,x} \, q(u_{,x}) dx = -2 \cdot N_{a,\xi} \, \xi_{,x} \, q(u_{,\xi} \, \xi_{,x}) \, x_{,\xi} \, d\xi$$

$$= -2 \cdot \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} q\left(\frac{d_2 - d_1}{h}\right)$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} q\left(\frac{d^e_2 - d^e_1}{h}\right)$$

$d^1_2 = D_1 \quad d^1_1 = 0$

$d^2_1 = D_1 \quad d^2_2 = D_2$

$$= q\left(\frac{d_2}{h}\right) - q\left(\frac{d^e_2 - d^e_1}{h}\right)$$

$$= q\left(\frac{D_1 - 0}{h}\right)$$

$$-q^2\left(\frac{D_1}{h}\right) + q^1\left(\frac{D_2 - D_1}{h}\right)$$

$$-q^2\left(\frac{D_2 - D_1}{h}\right)$$

ME 235C
Handout #4
Homework

Nonlinear Finite Element Methods
Fall Quarter 1981

Stanford University
Instructor: T.J.R. Hughes

Homework Assignment No. 2

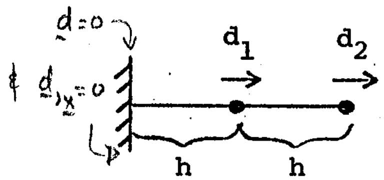
(due 3 weeks after handout)

$\begin{pmatrix}-1\\1\end{pmatrix} g(u_{,x})^{e=2} + \begin{pmatrix}-1\\1\end{pmatrix} g(u_{,x})^{e=1}$

Consider the following 2 dof system:

let $N(d) \Rightarrow \int_\Gamma w_{,x}\, g(u_{,x})\,dx$

$u = \sum N_a d_a$, $u_{,x} = \sum N_{a,\xi} d_a \xi_{,x}$



$$\underset{\sim}{N}(\underset{\sim}{d}) = \underset{\sim}{F}$$

$\int N_{a,x}\, g(u_{,x})\,dx$

$$\underset{\sim}{N}(\underset{\sim}{d}) = \begin{Bmatrix} N_1(d_1,d_2) \\ N_2(d_1,d_2) \end{Bmatrix} = \begin{Bmatrix} -N^{(2)}\left(\frac{d_2-d_1}{h}\right) + N^{(1)}\left(\frac{d_1}{h}\right) \\ N^{(2)}\left(\frac{d_2-d_1}{h}\right) \end{Bmatrix}$$

$$\underset{\sim}{F} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Assume: $h = 1$, $F_1 = 0$, $F_2 = 37.5$

case 1. $N^{(1)}(x) = N^{(2)}(x) = 10(x - x^3)$ $x(1-x^2)$ $x=0$ $x=\pm 1$

case 2. $N^{(1)}(x) = N^{(2)}(x) = 10(x + x^3)$ $x=0$

Program and attempt to solve the problem for cases 1 and 2 using each of the following algorithms.   Soln is Case #1: −1.76699, −3.533979   Case 2: 1.340610, 2.681220

(1) Newton-Raphson ✓   28 iter conv | 7 iter conv.

(2) Modified Newton-Raphson ✓   4 iter & blow up | 4 blow up

(3) Incremental Newton-Raphson (use 3 equal increments) ✓ 17 iter conv | 14 conv.

(4) Modified/Incremental Newton-Raphson (use 3 equal increments and update $\underset{\sim}{DN}$ only at the beginning of each new increment) ✓ 5 blow up | 7 blow up.

(5) Case (2) with line searches ✓   3 iter conv | 2 iter conv.

(3b) Incr NR w/ LS   6 iter conv / 6 iter conv

(1b) NR w/LS   8 iter conv. / 1 iter conv.

md m nrto (6) Case (4) with line searches ✓ 6 iters & converged. / 6 iter

~~(7) Case (2) with Broyden updates (max = 3) and line searches~~

(8) Case (2) with BFGS updates (max = 3) and line searches

SKIP

(9) Case (4) with Broyden updates (max = 3) and line searches

~~(10) Case (4) with BFGS updates (max = 3) and line searches~~

Assume in each case $\underset{\sim}{d}^{(0)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ and permit no more than 100 iterations. Terminate computations when

$$\left(\left| d_\alpha^{(i+1)} - d_\alpha^{(i)} \right| / \left| d_\alpha^{(i+1)} \right| \right) < .0001$$

Δd_α

and

$$\left| F_\alpha - N_\alpha(\underset{\sim}{d}) \right| < .0001 \quad , \qquad \alpha = 1,2 \ .$$

Report the solutions and the total number of iterations required for each algorithm. Plot the convergence of the error $e_\alpha^{(i)} = d_\alpha^{(i)} - d_\alpha$, $\alpha = 1,2$. (In the case of the incremental schemes, plot only the iterates of the last load increment.) Comment on the asymptotic rate of convergence.

ME 235C
Handout #2
Notes

FIGURES FOR

"ANALYSIS OF TRANSIENT ALGORITHMS WITH PARTICULAR REFERENCE TO STABILITY BEHAVIOR"

by

Thomas J. R. Hughes*
Division of Applied Mechanics
Durand Building
Stanford University
Stanford, California 94305

To be published as a chapter in:
Computational Methods in Transient Analysis,
edited by T. Belytschko and T. J. R. Hughes,
North-Holland Publishing Company, Amsterdam.

Babylon Preprint Series No. 4
May 1981

*Associate Professor of Mechanical Engineering

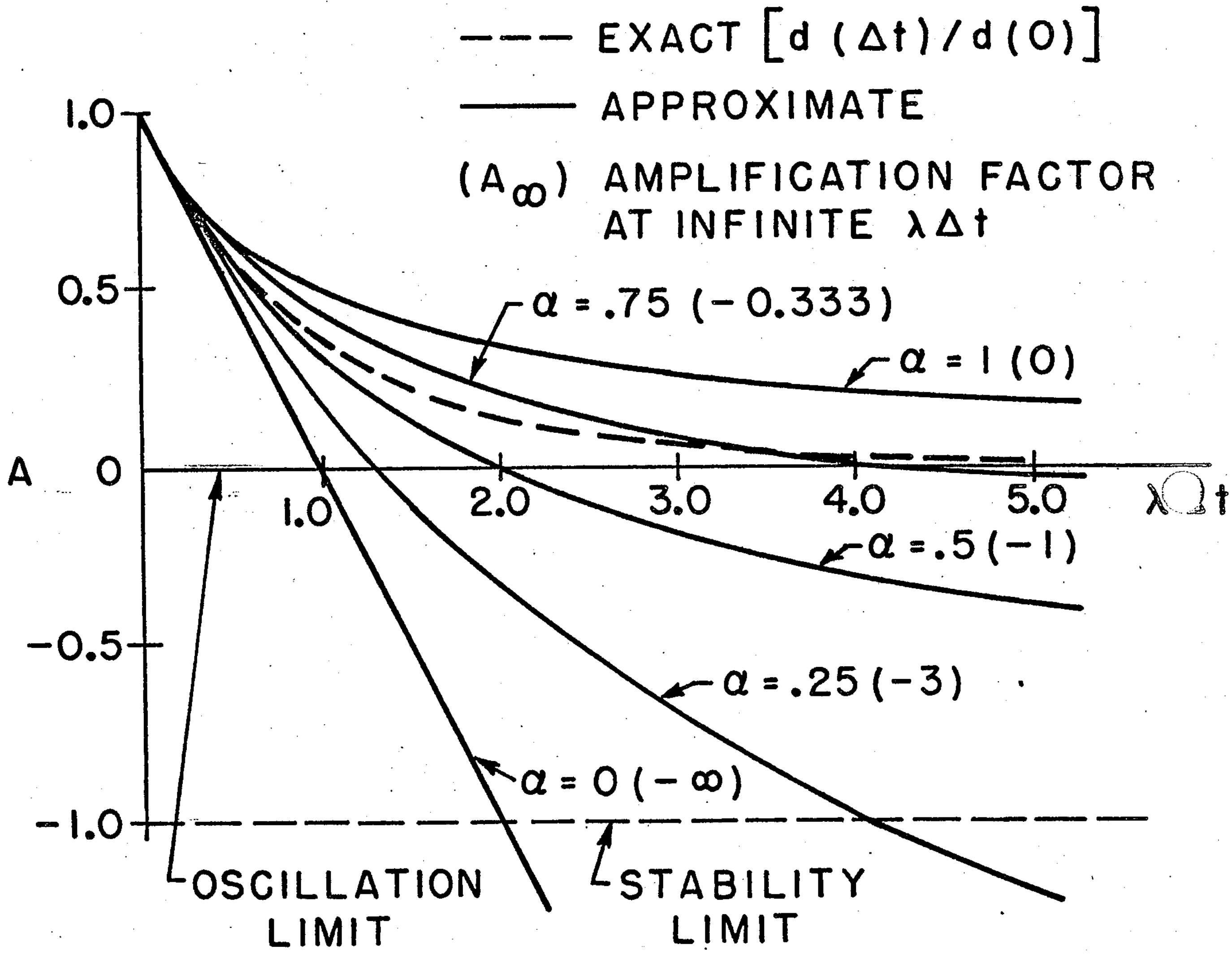


Figure 1. Amplification factor for typical one-step methods

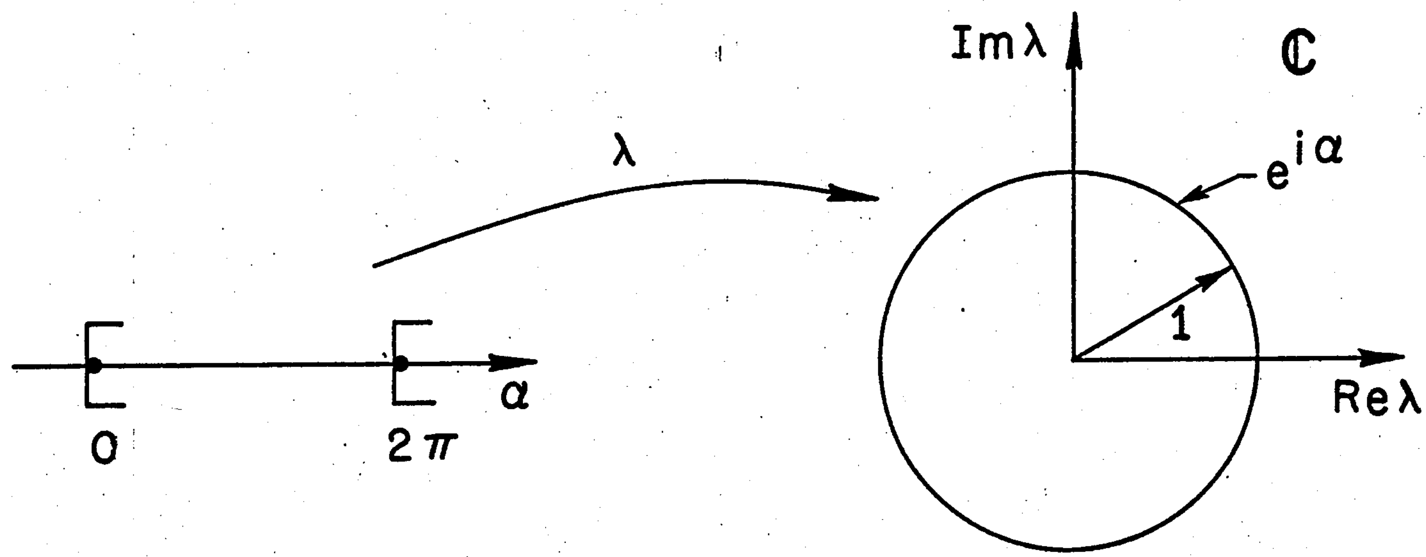


Figure 2. Roots of unit modulus

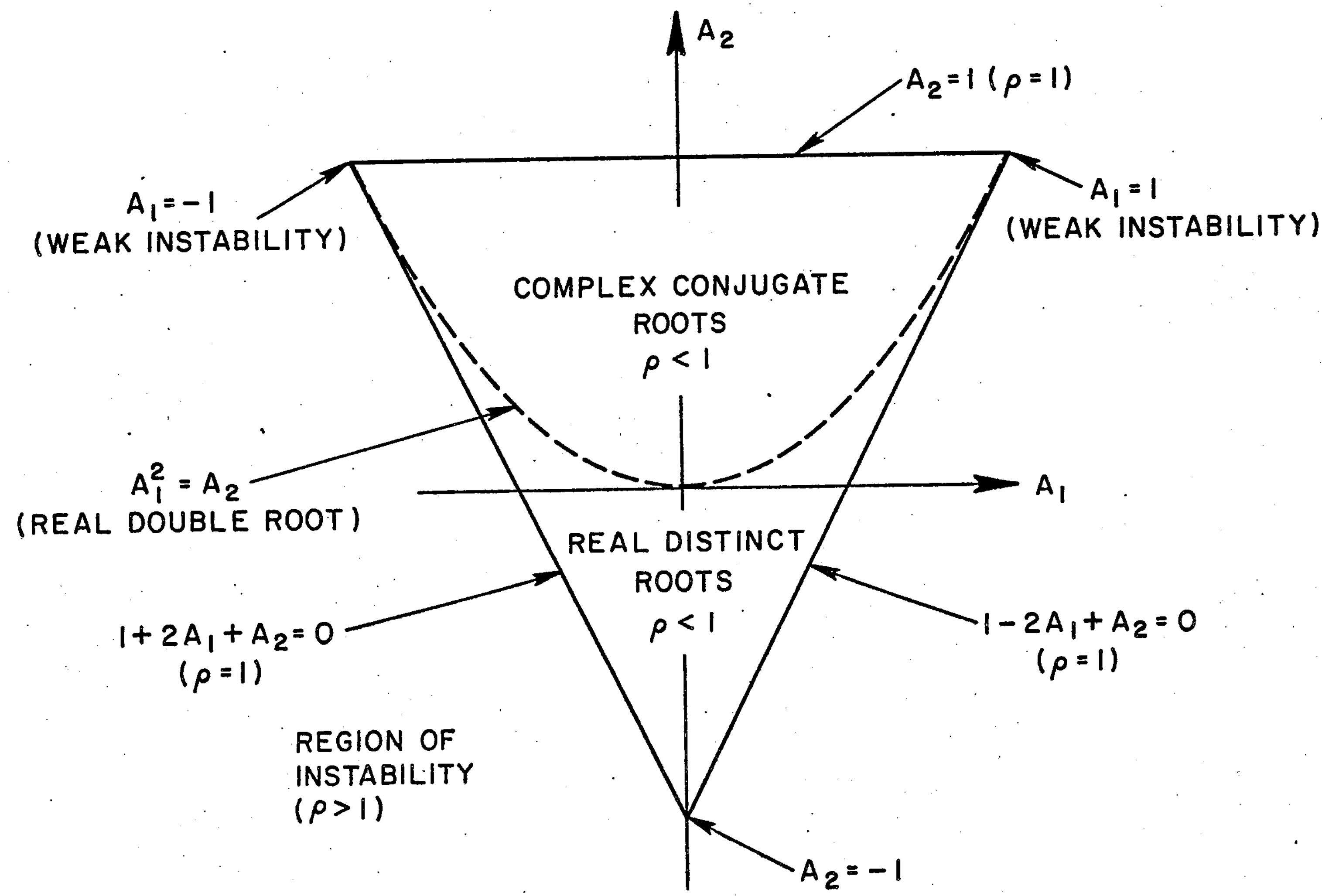


Figure 3. Region of stability in the A_1, A_2 - plane [H2].

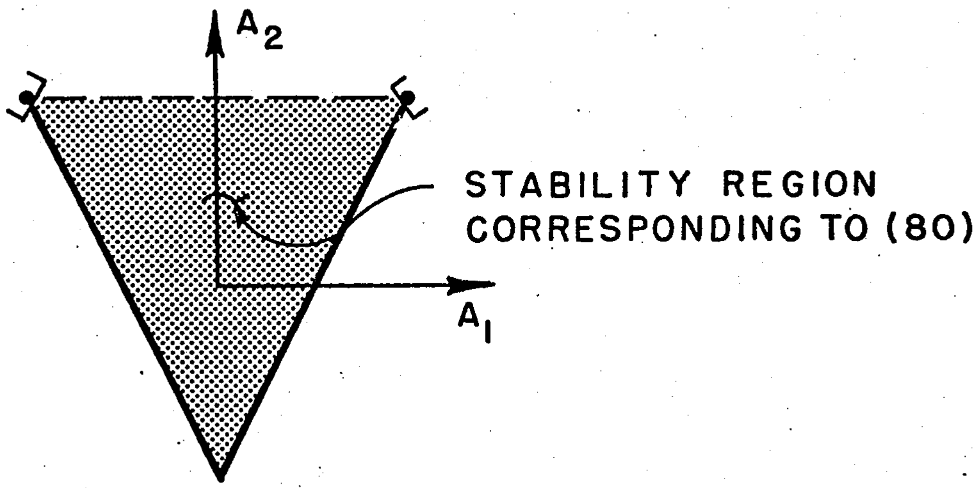


Figure 4

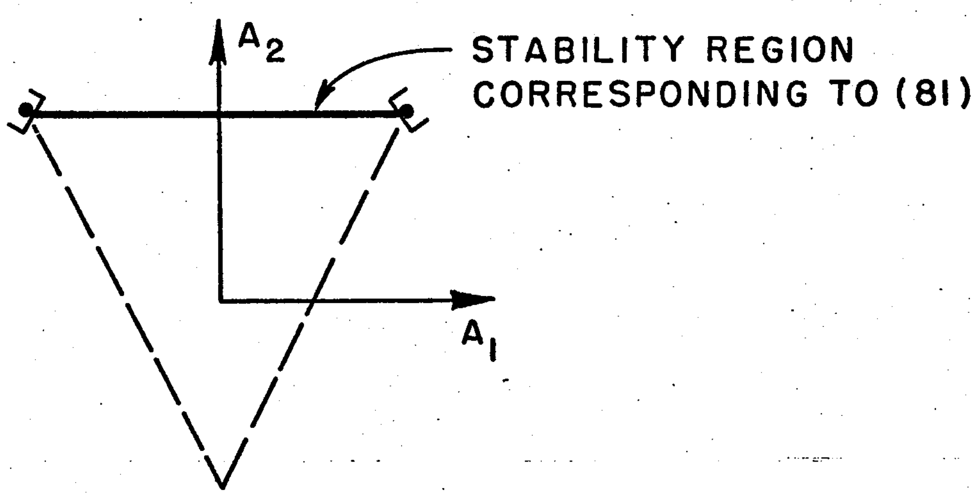


Figure 5

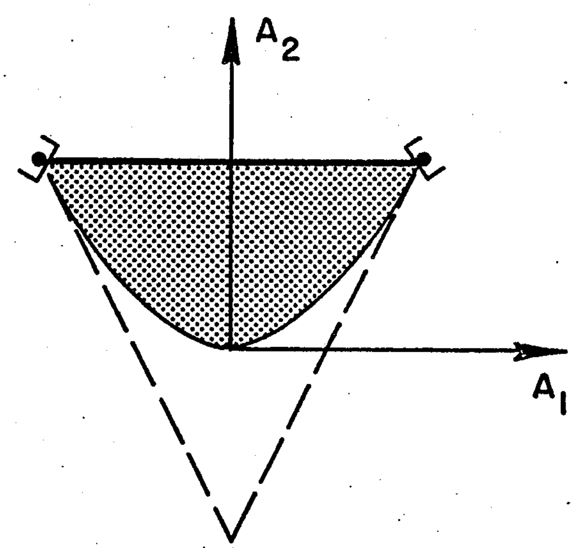


Figure 6. Stability region for which roots of the amplification matrix are complex conjugate

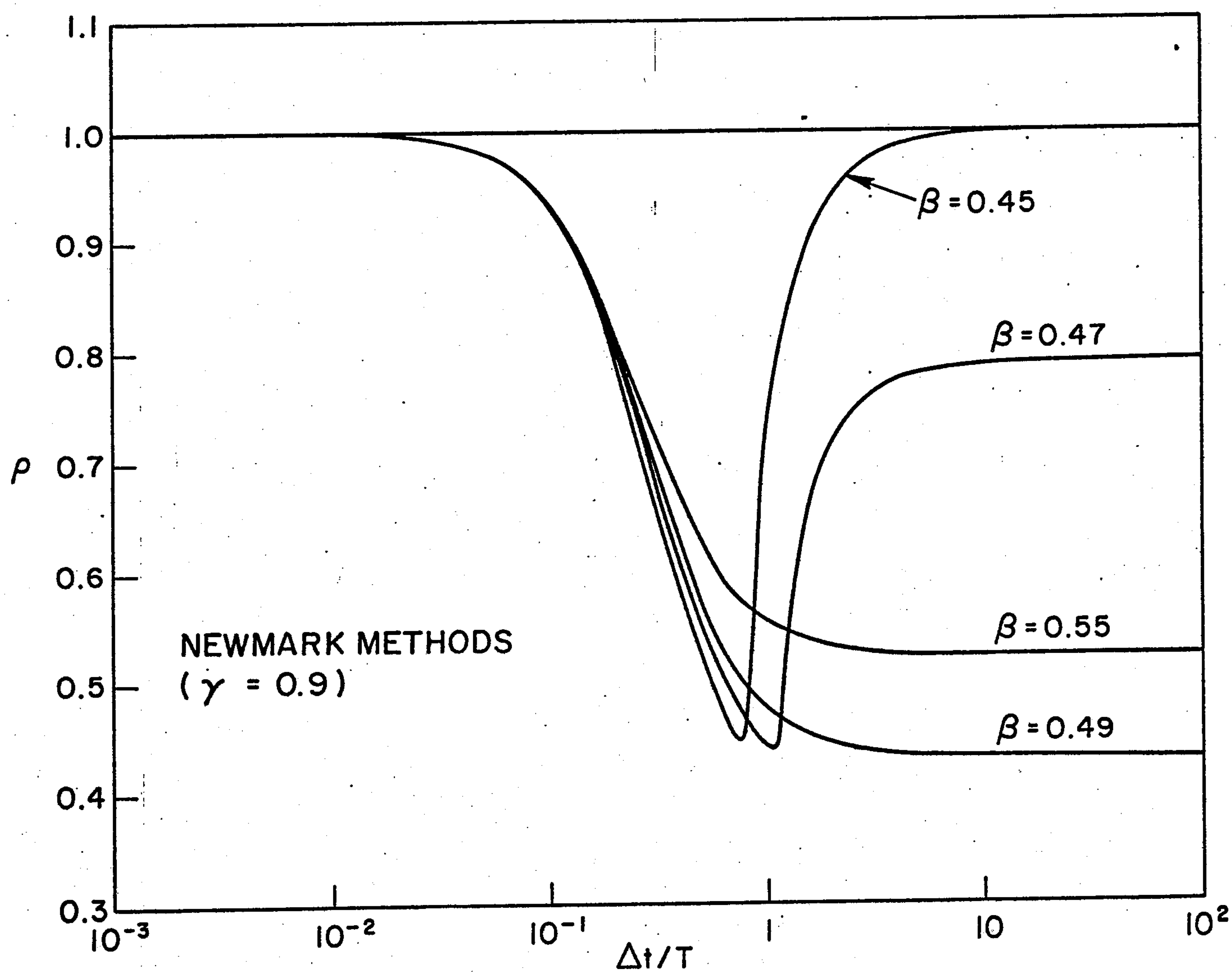


Figure 7. Spectral radii for Newmark methods for varying β [H2].

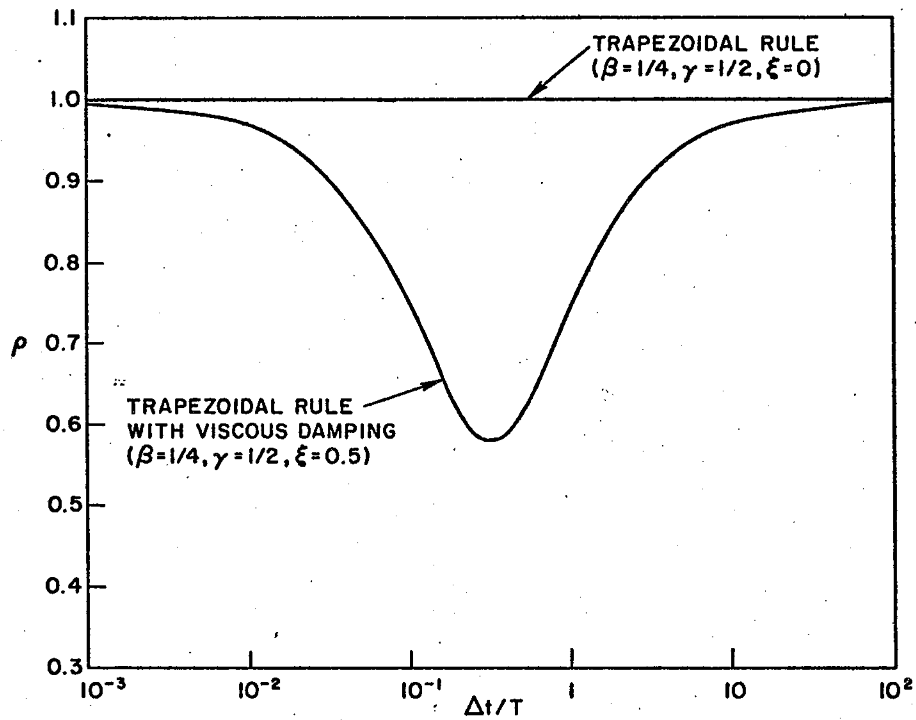


Figure 8. Trapezoidal rule with and without viscous damping [H2].

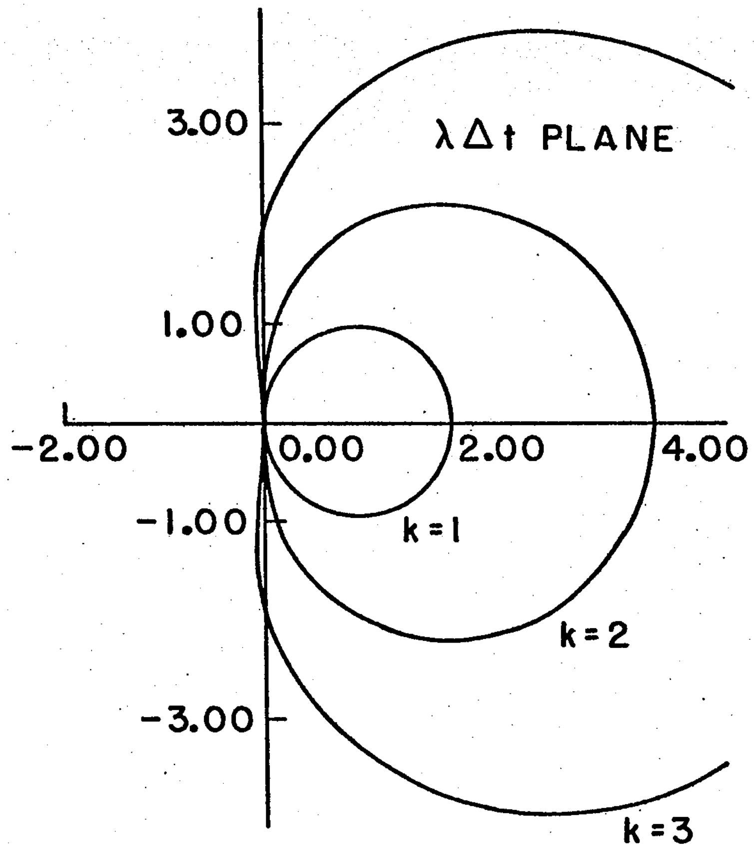


Figure 9. Regions of absolute stability for stiffly stable methods of orders one through three. Methods are stable outside of closed contours [G1].

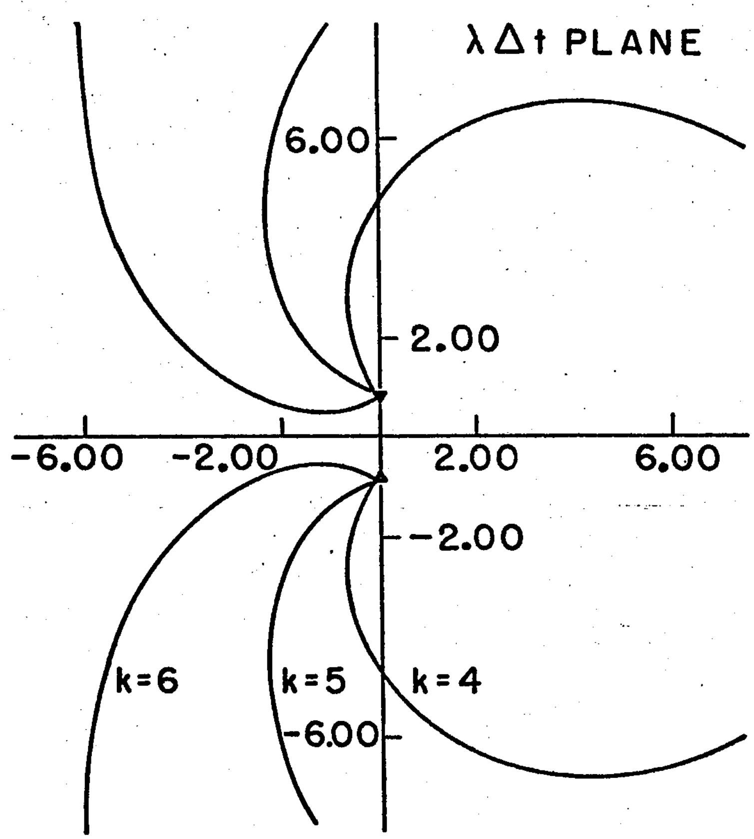


Figure 10. Regions of absolute stability for stiffly stable methods of orders four through six [G1].

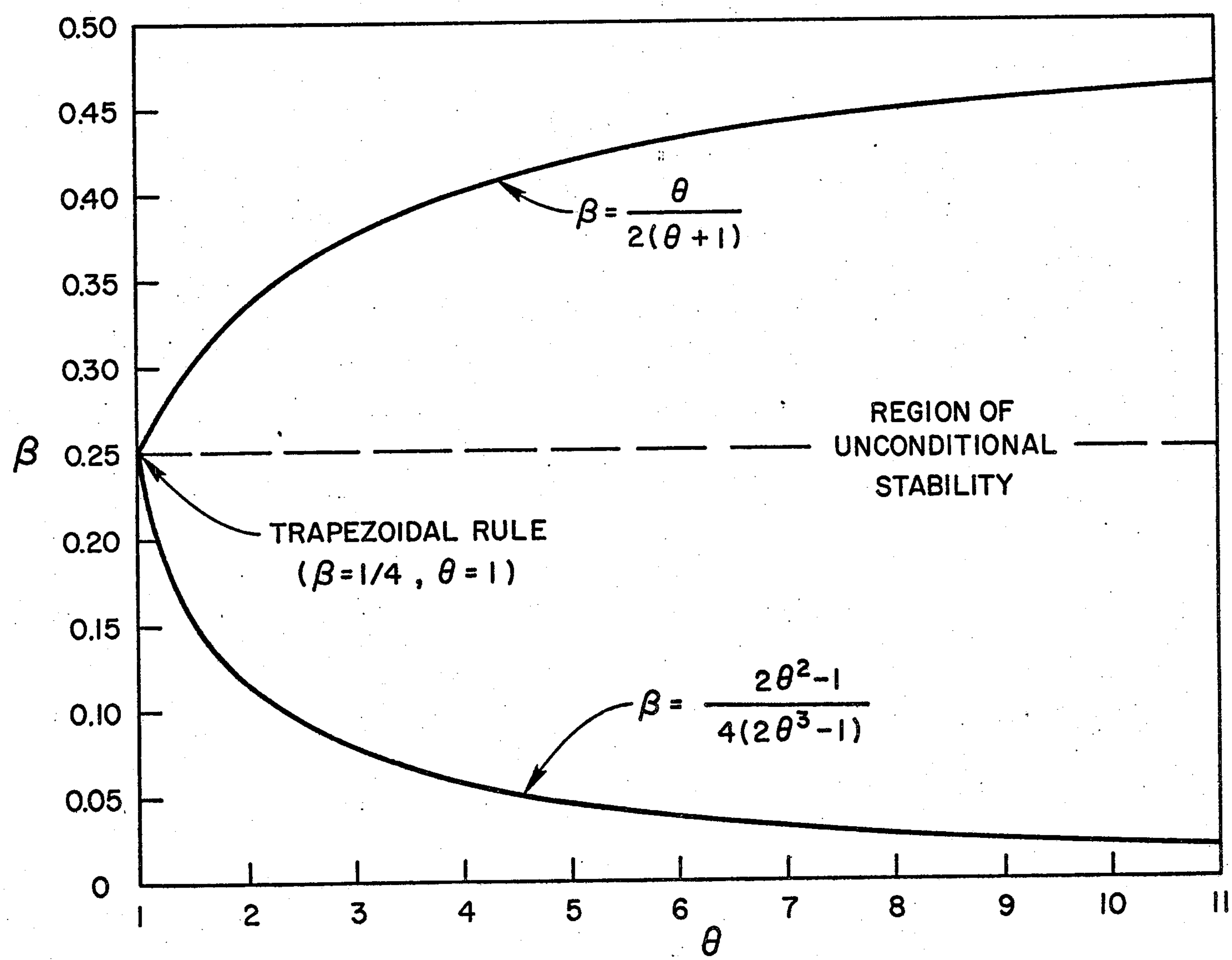


Figure 11. Region of unconditional stability for collocation schemes [H3].

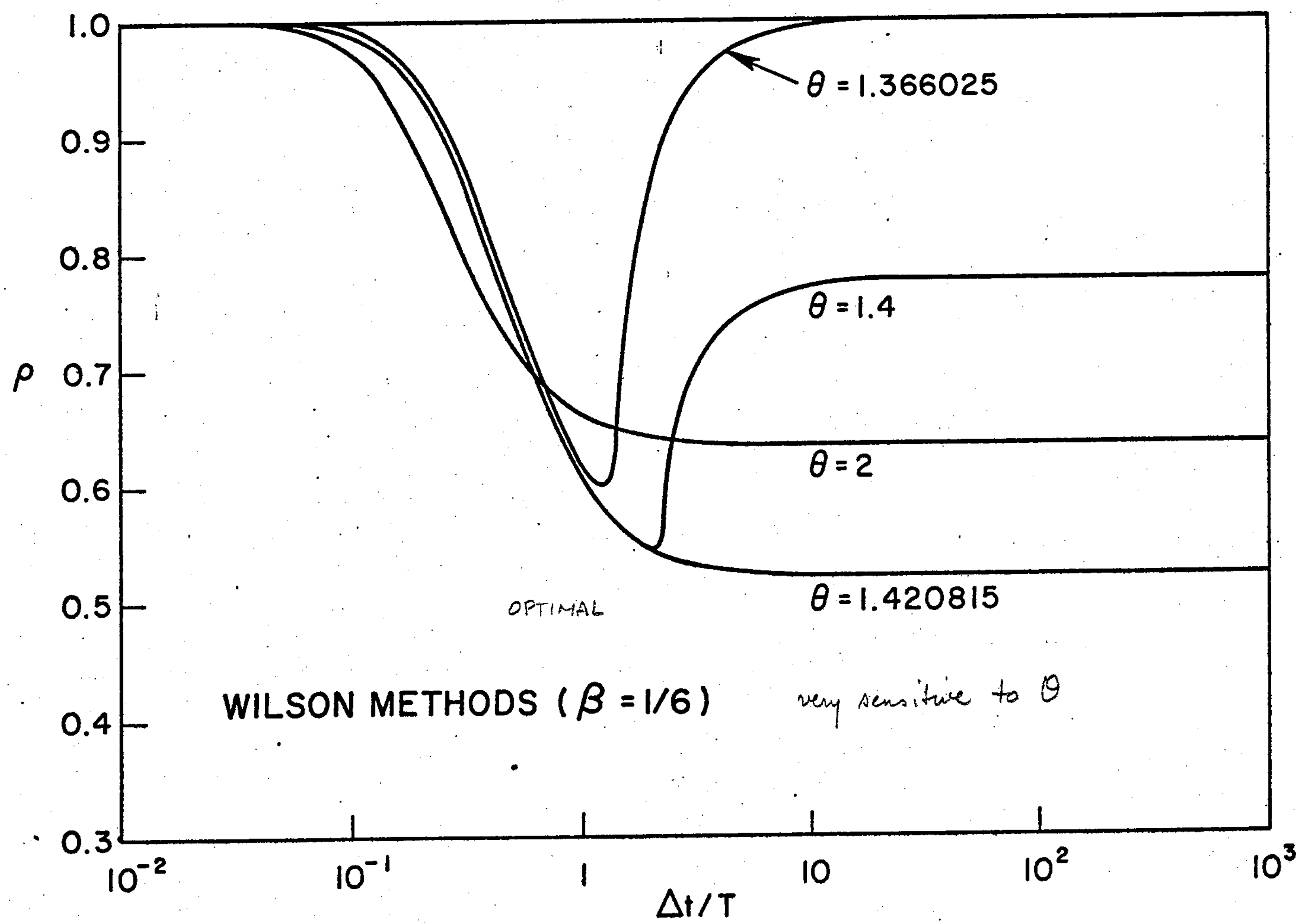


Figure 12. Spectral radii for Wilson θ methods [H3].

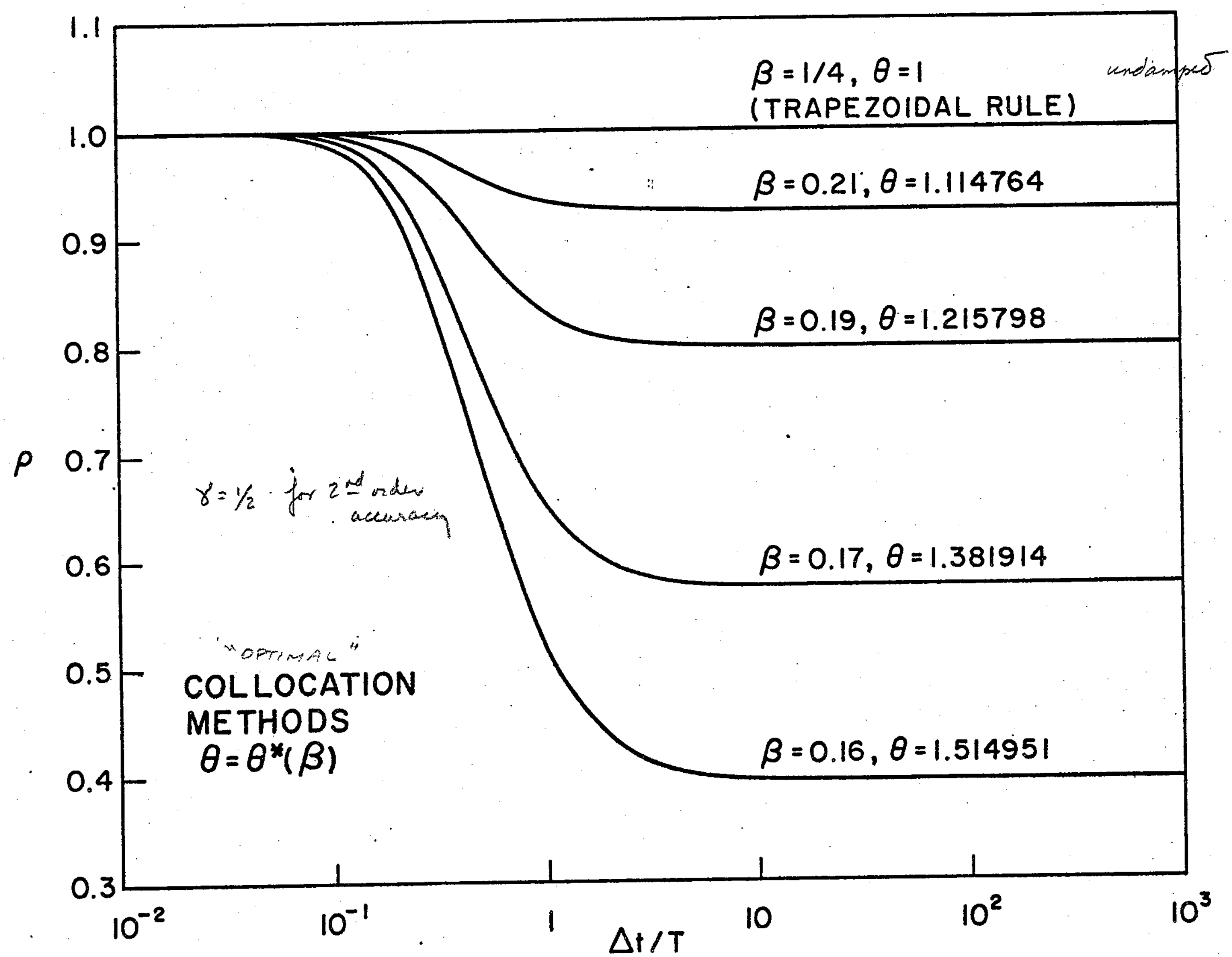


Figure 13. Spectral radii for optimal collocation schemes [H3].

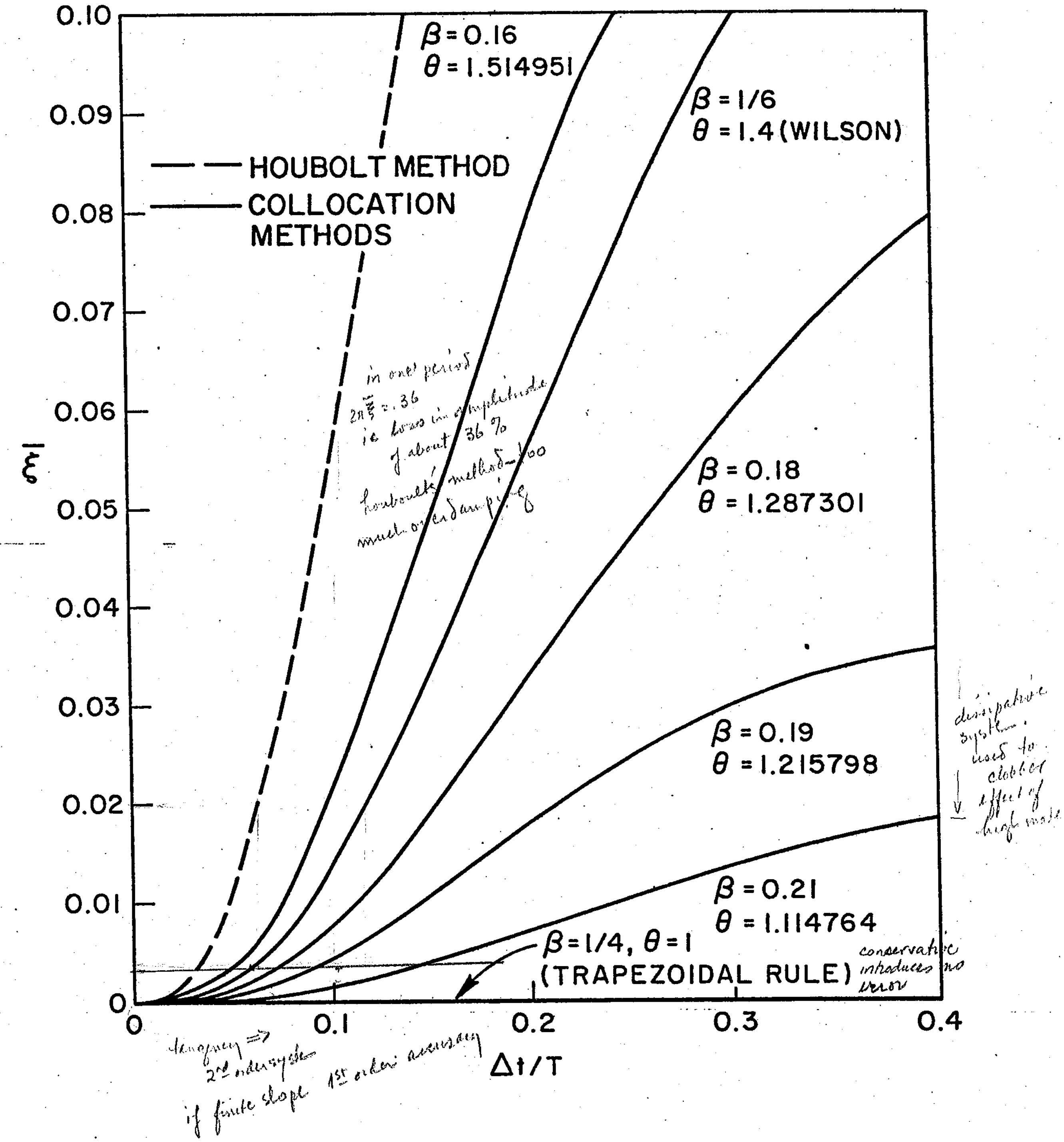


Figure 14. Algorithmic damping ratios for collocation schemes and Houbolt method [H3].

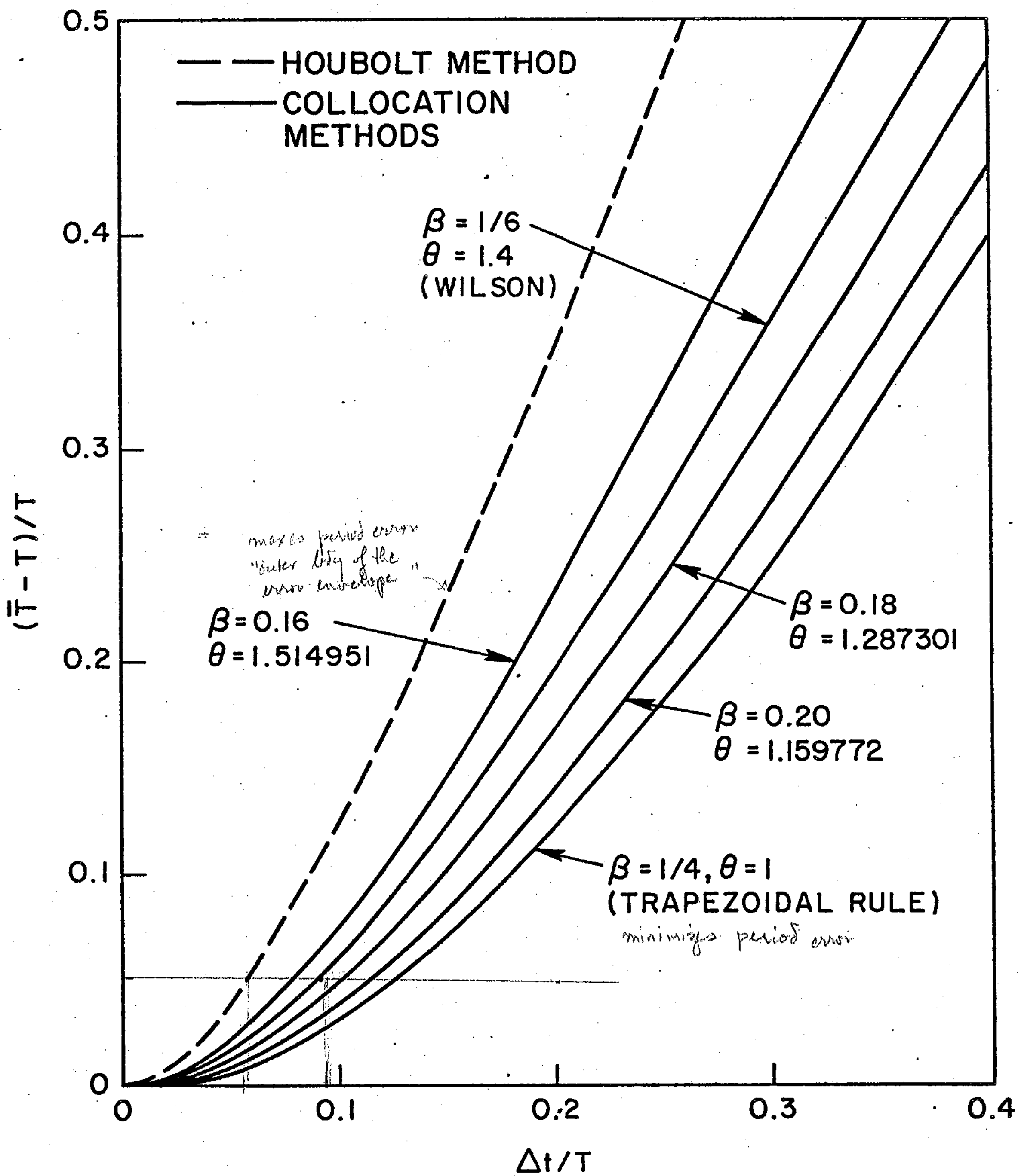


Figure 15. Relative period errors for collocation schemes and Houbolt method [H3].

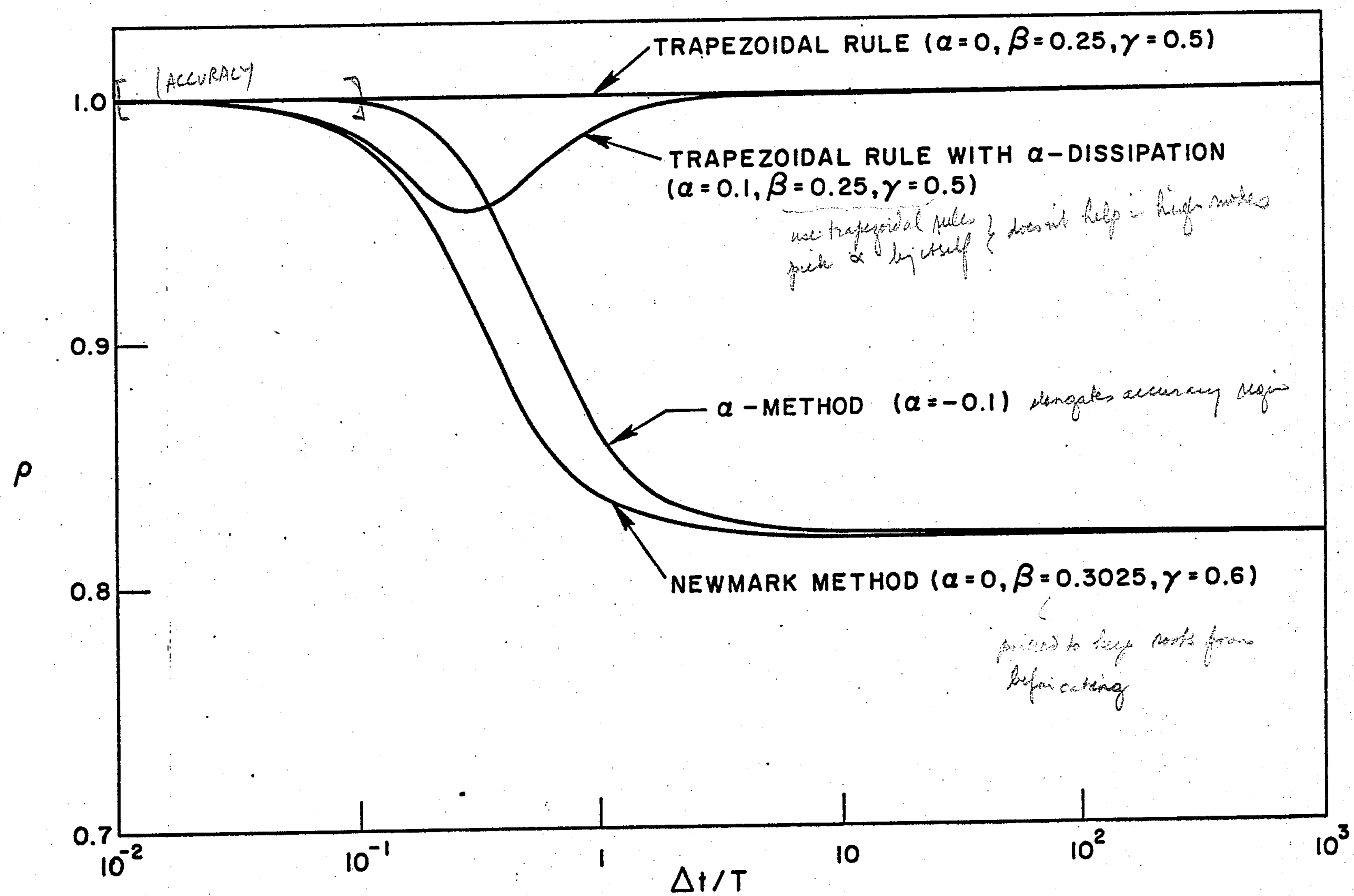


Figure 16. Spectral radii versus $\Delta t/T$ for α method and Newmark schemes [H4].

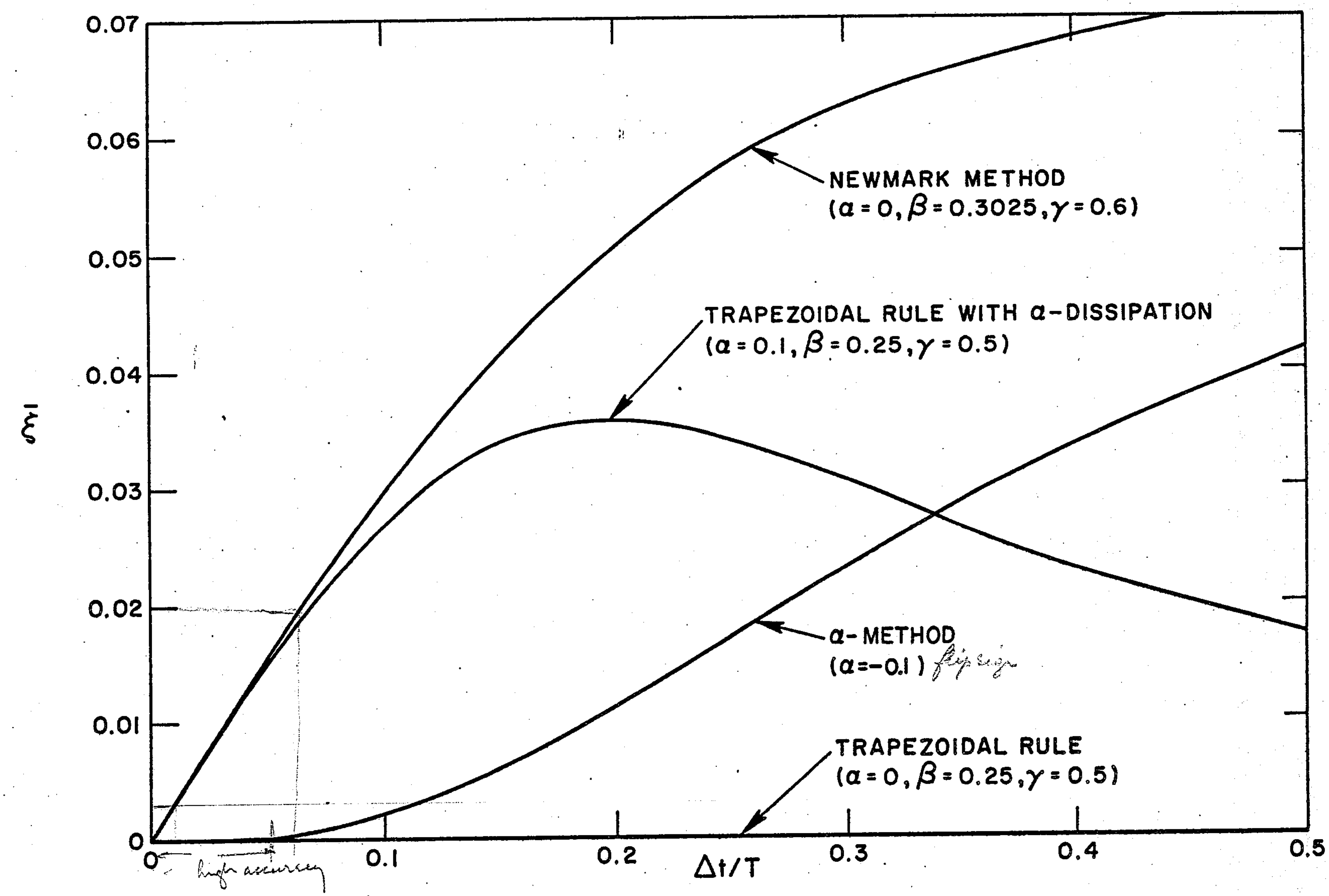


Figure 17. Damping ratios versus $\Delta t/T$ for α method and Newmark schemes [H4].

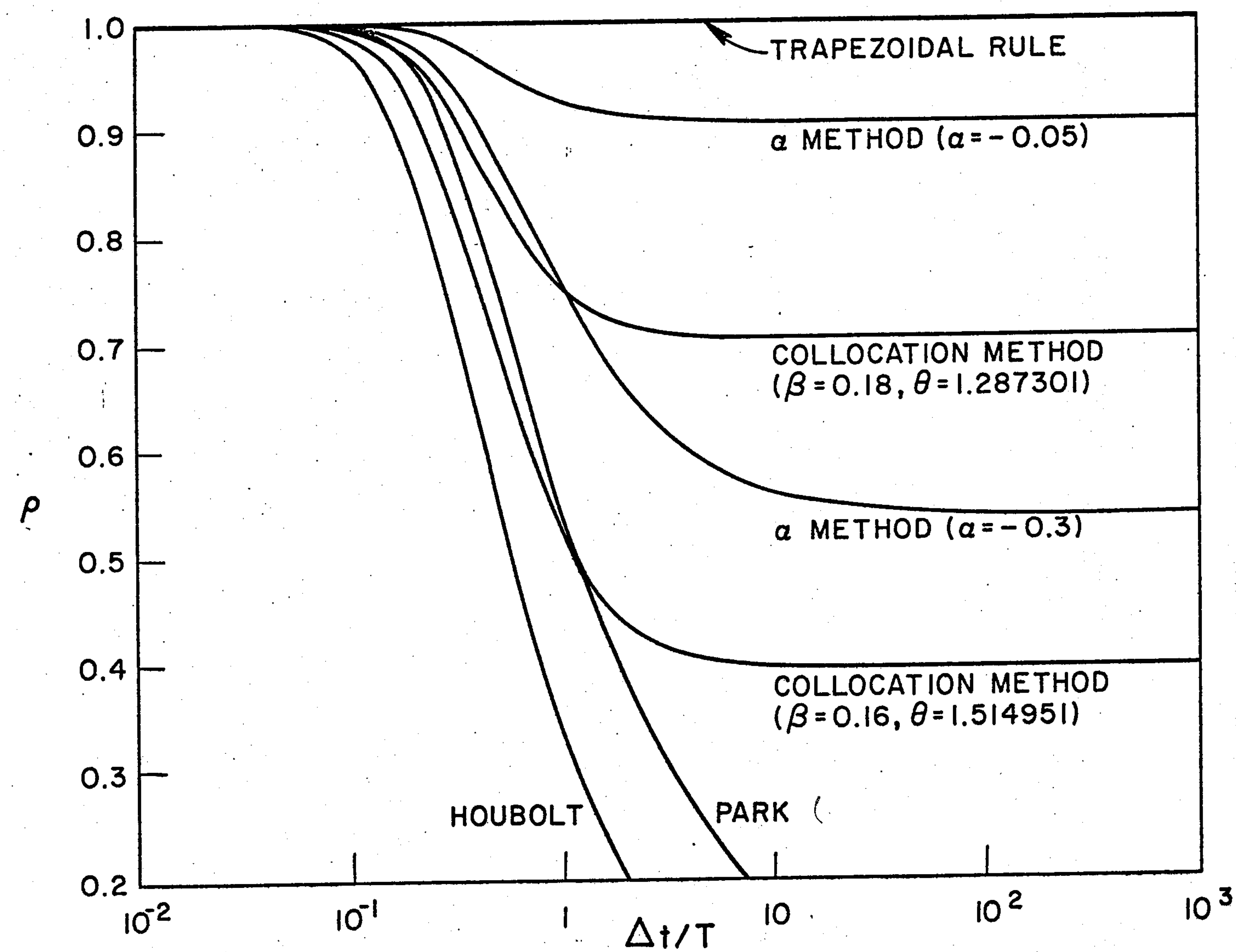


Figure 18. Spectral radii for α methods, optimal collocation schemes and Houbolt and Park methods [H3].

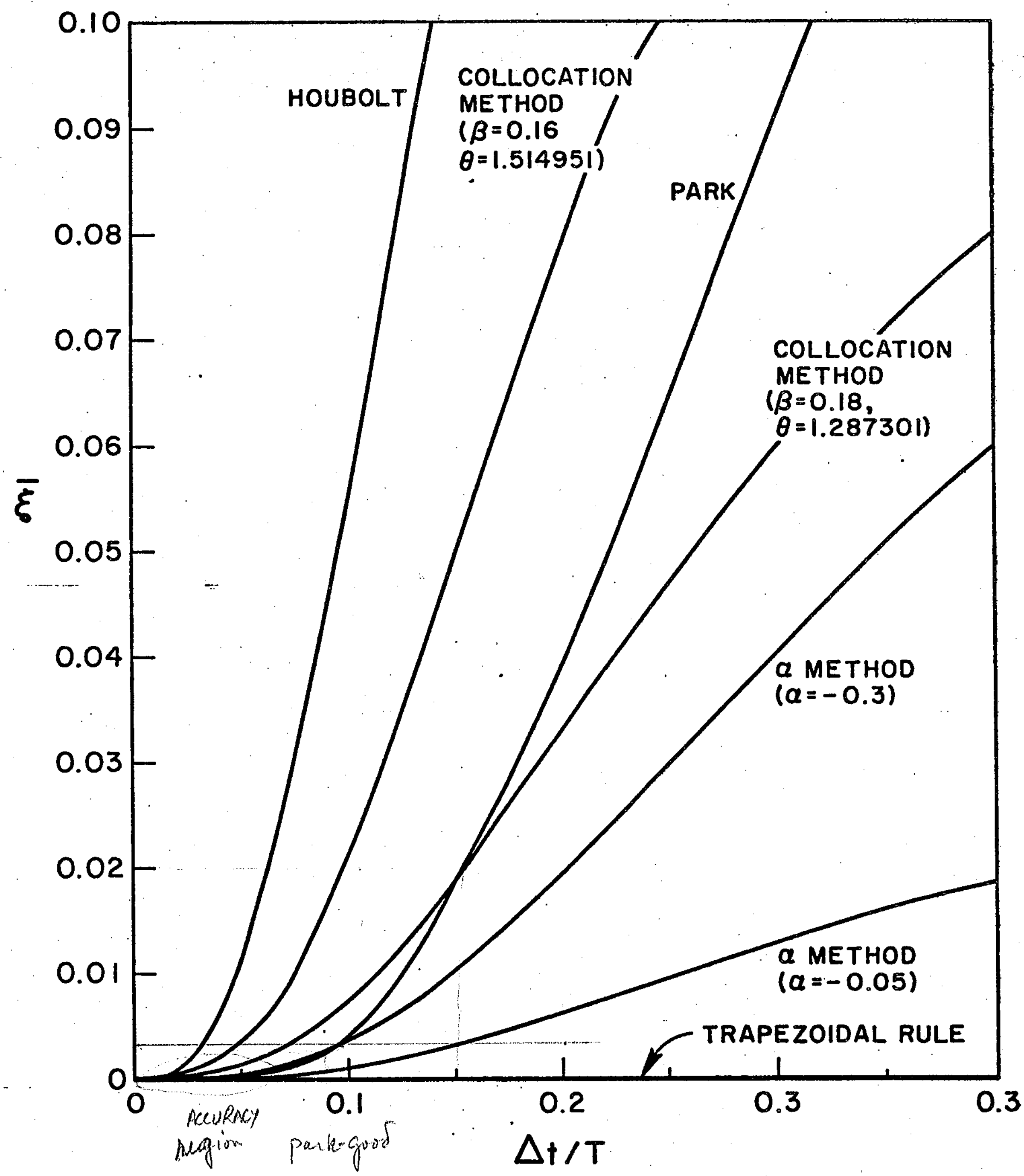


Figure 19. Algorithmic damping ratios for α-methods, optimal collocation schemes, and Houbolt and Park methods [H3].

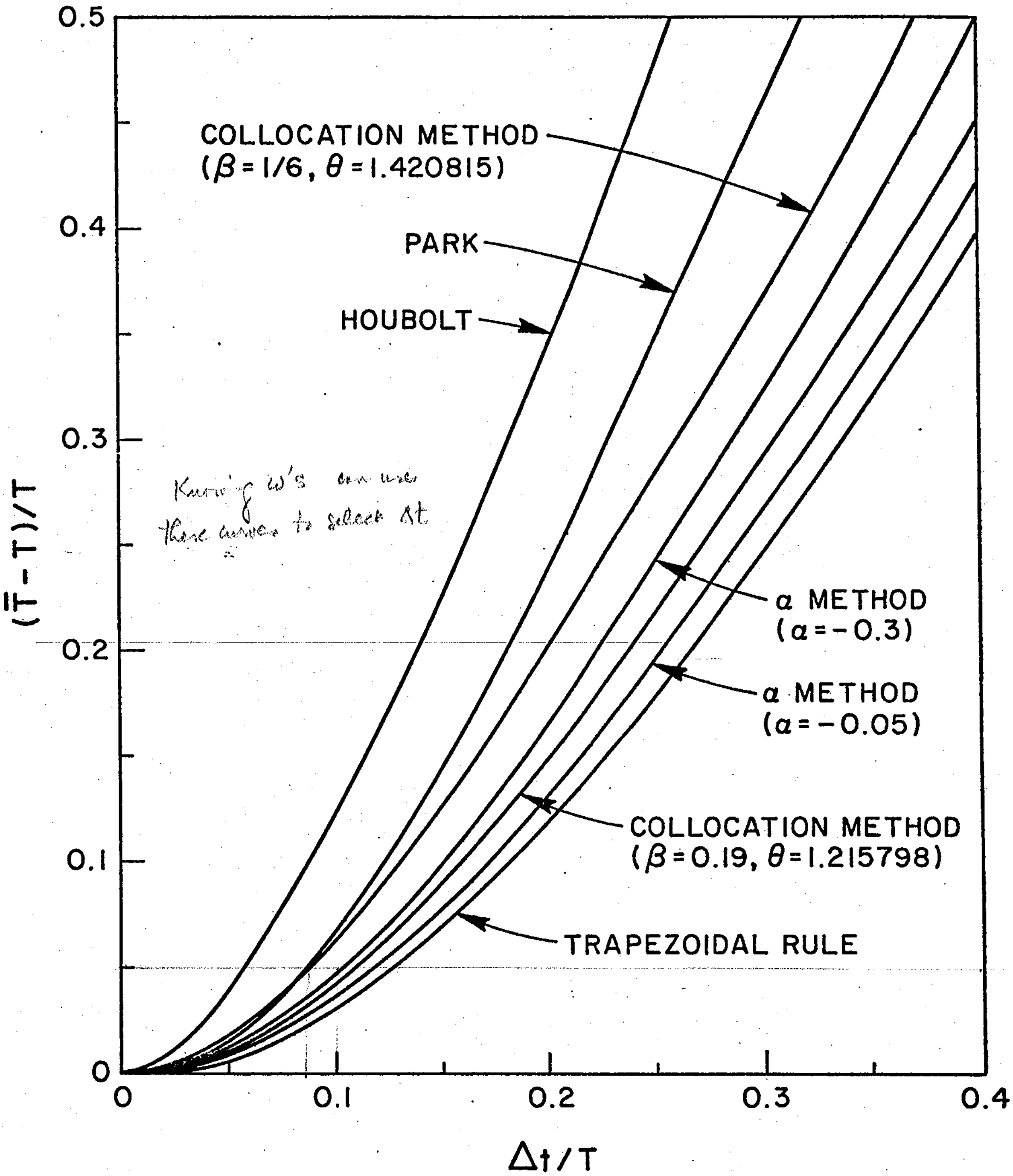


Figure 20. Relative period errors for α-methods, optimal collocation schemes, and Houbolt and Park methods [H3].

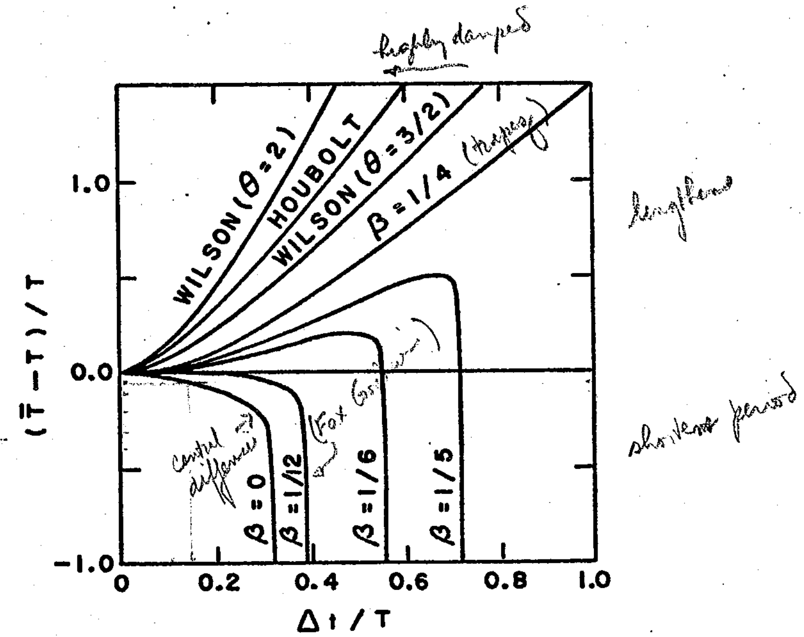


Figure 21. Period errors for undamped Newmark methods ($\gamma = \frac{1}{2}$) compared with Wilson and Houbolt schemes [G3].

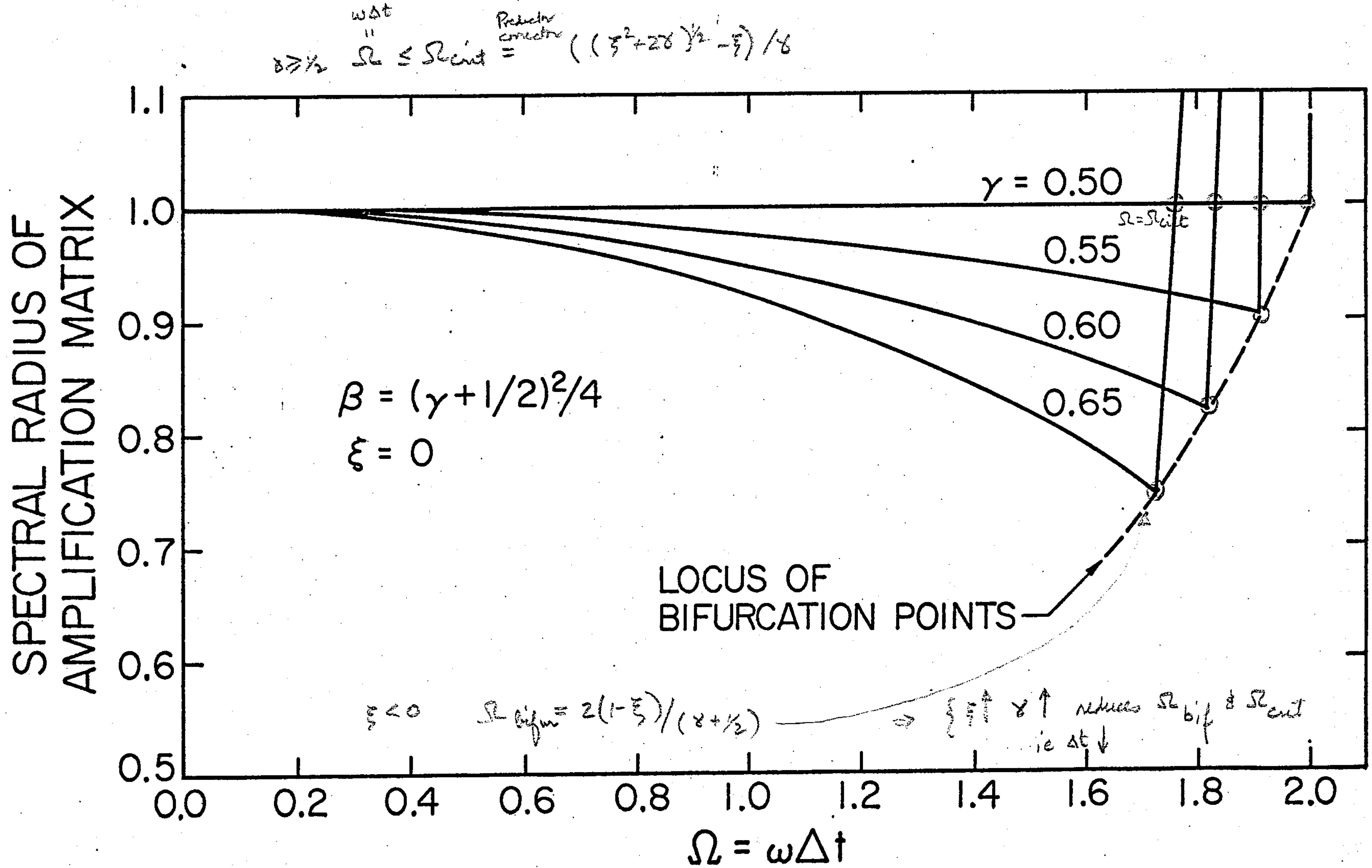


Figure 22. Spectral radius of amplification matrix for predictor-corrector algorithms [H17].

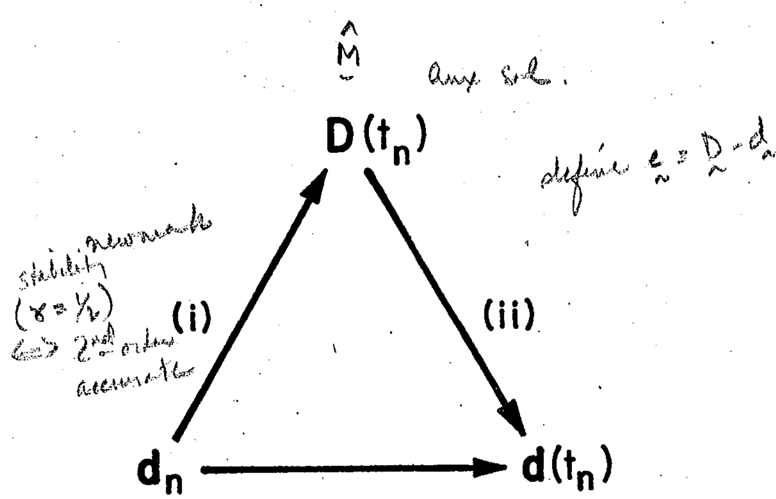


Figure 23. Schematic of convergence proof [H23].

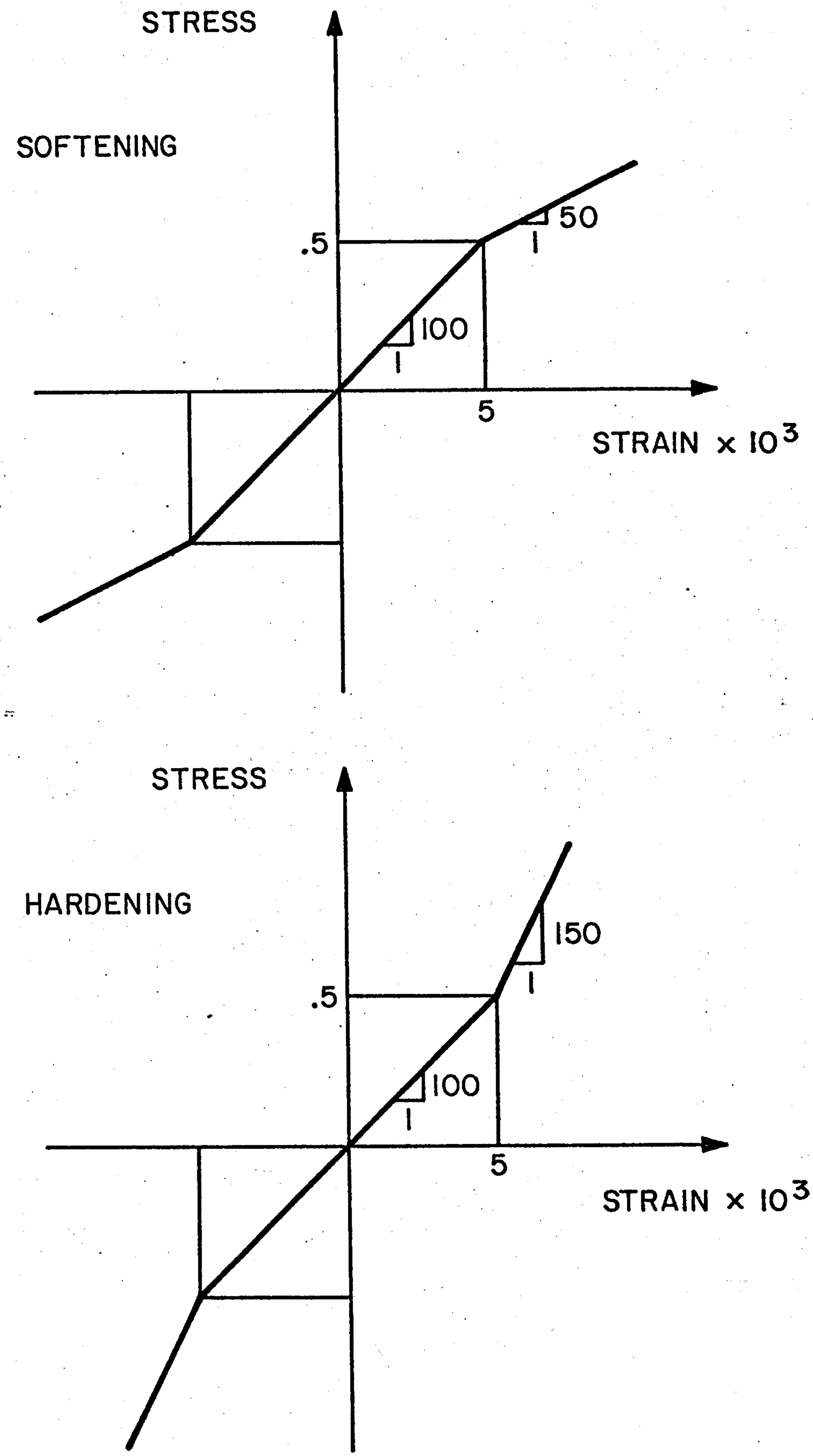


Figure 24. Bilinear material laws used in the numerical calculations [H15].

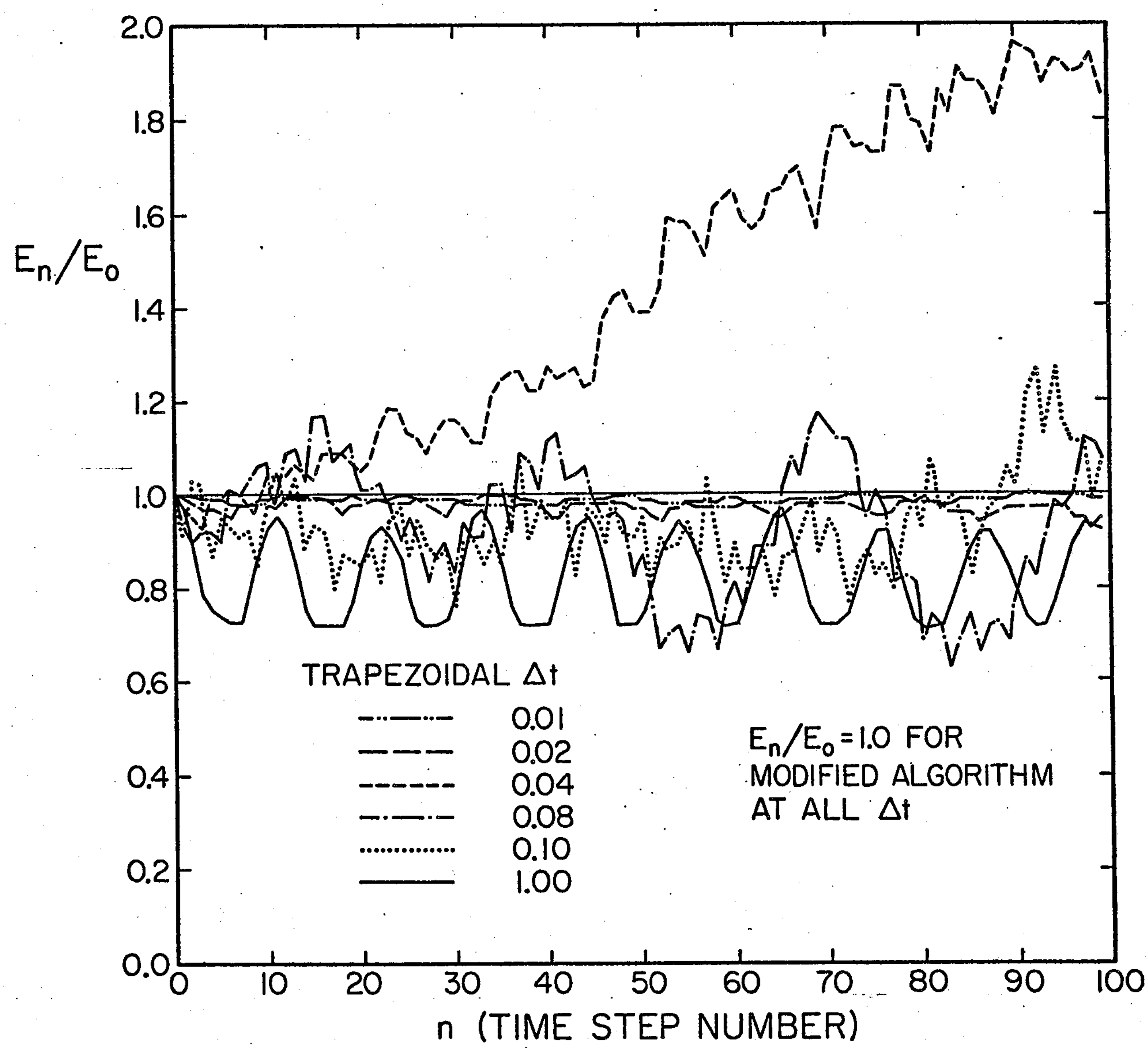


Figure 25. Energy growth and decay for softening material law [H15].

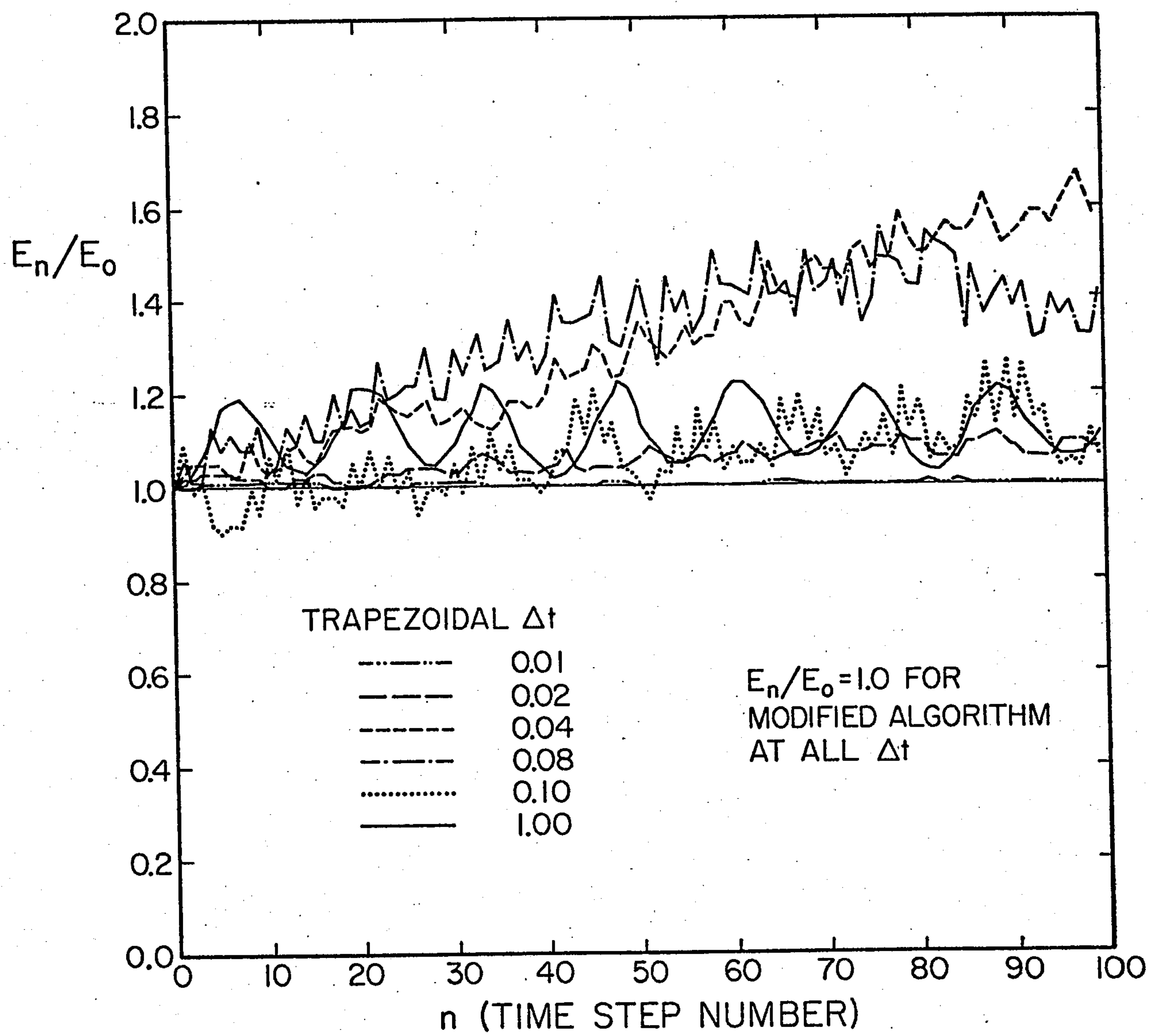


Figure 26. Energy growth and decay for hardening material law [H15].

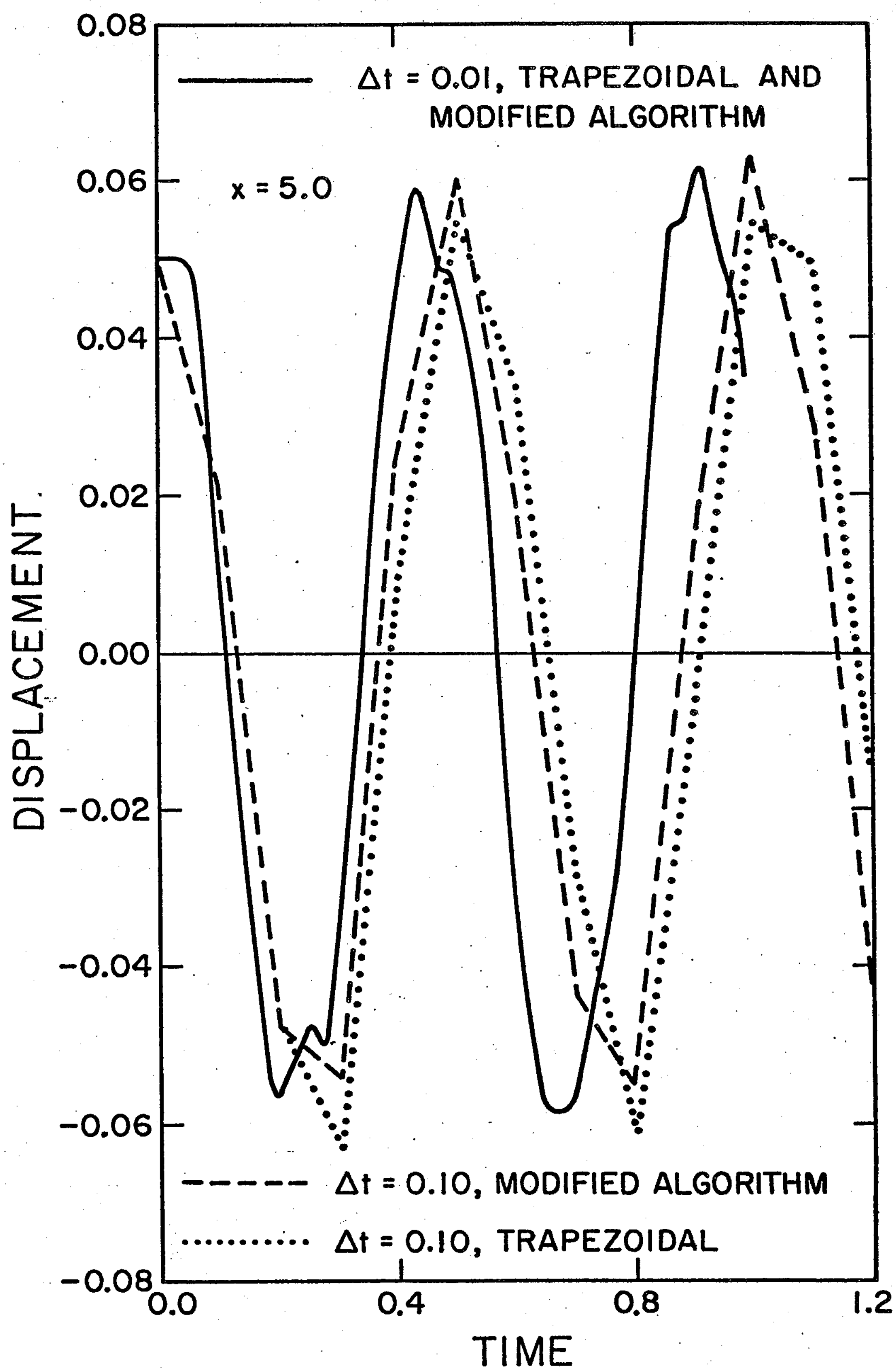


Figure 27. Displacement response for softening material law [H15].

me235

#4

COMPUTER METHODS IN APPLIED MECHANICS AND ENGINEERING 00 (1982) 000–000
NORTH-HOLLAND PUBLISHING COMPANY

AN ELEMENT-BY-ELEMENT SOLUTION ALGORITHM FOR PROBLEMS OF STRUCTURAL AND SOLID MECHANICS

Thomas J.R. HUGHES, Itzhak LEVIT and James WINGET
Division of Applied Mechanics, Durand Building, Stanford University, Stanford, CA 94305, U.S.A.

Received ~~?????~~ August 1, 1982
~~Revised ??????~~

It is proposed to solve large-scale finite-element equation systems arising in structural and solid mechanics by way of an element-by-element approximate factorization technique which obviates the need for a global coefficient matrix. The procedure has considerable operation count and I/O advantages over direct elimination schemes and it is easily implemented. Numerical results demonstrate the effectiveness of the method and suggest its potential for the analysis of large-scale systems.

1. Introduction

In principle, the majority of linear and nonlinear structural and solid mechanics problems faced by engineers can be solved by existing finite element methodology. However, if the finite element equation system is very large, the storage requirements and computational cost involved may preclude actual solution. In this paper we present a new approach to finite element equation solving which, for large systems, enjoys significant advantages over standard direct elimination techniques and, in particular, requires no storage of a global array. In its most primitive form the equation system is solved by way of an approximate factorization technique in which element arrays are individually factorized. Solution proceeds on an element-by-element basis.

Approximate factorization schemes have proven to be very effective in finite differences where multi-dimensional operators are approximately factorized into one-dimensional operators [3, 22, 25]. Techniques of this kind have been used to calculate three-dimensional, turbulent, unsteady flows in which 128^3 mesh points were employed [23]. Unfortunately, these procedures are inherently restricted to topologically regular meshes. The method proposed herein imposes no geometrical or topological restriction, thus the full versatility of finite element modelling is retained. In addition, the splitting into element factors is very easy to implement within existing finite element code architectures.

The first example of an element-by-element scheme of this kind was presented in [13]. There the main thrust was the development of an implicit, unconditionally stable, second-order, time-accurate procedure for transient heat conduction. In the present work the emphasis is on using the element-by-element concept as the essential ingredient in solving linear algebraic equations. Element factors are computationally convenient to employ and appear to retain the accuracy characteristics of the global coefficient matrix [13].

IMPORTANT
1. Please correct the proofs carefully: the responsibility for correcting errors rests with the author.
2. Restrict corrections to instances in which the proof is at variance with the manuscript.
3. Recheck all reference data.

Various hardware and software trends suggest the potential of the approach proposed herein. For example, todays supercomputers, such as the STARs and CRAYs, have significant CPU-speed advantages for vectorizable operations over previous generations of scientific computers and also possess considerable increases in available high-speed core. All CPU-intensive aspects of finite element calculations are amenable to vectorization (i.e., equation solving, element formation, constitutive function evaluation) [6] thus exploiting this feature. However, I/O (i.e., disk file read/write) costs have not dropped significantly and it is I/O which is still the major cost constituent of large-scale, finite-element problem solving. This is due to the enormous data base requirements of finite element equation systems which still often exceed core limitations, thus requiring non-sequential matrix assembly algorithms and blocked out-of-core equation-solving techniques which make heavy I/O demands. It is often not realized by the practitioner experienced only with small systems how quickly matrix storage increases for large systems, especially three-dimensional ones. For example, the symmetric, banded, stiffness matrix of a three-dimensional finite element model of a cube with only 20 nodes along each edge and 3 degrees-of-freedom per node (~24 thousand equations) requires about 29 *million* words of storage, far exceeding the core limitations of any machine in existence. The finite difference calculation described earlier involving 128^3 mesh points is clearly outside the realm of consideration of matrix-based techniques. In this case, a non-symmetric, banded matrix with 4 degrees-of-freedom per node (velocities and pressure) would require about 11 *trillion* words of storage!

Until fairly recently, a major impediment to large-scale three-dimensional finite-element analysis of complex structures was the massive, and often error prone, data preparation process required to define the model. Advances in the development of three-dimensional interactive color graphics, finite element pre- and post-processors, and the direct interfacing of sophisticated CAD systems with finite element programs augers a future in which even the most intricate engineering design may be expediently and reliably processed. Thus the only remaining major impediment is the deficiency of direct equation solving techniques.

It is also believed that techniques of the kind proposed herein will open the way to solving much larger systems on smaller machines, such as the ubiquitous VAXs and PRIMEs, in which physical speed/storage limits play the predominant role in restricting problem size. The ability to solve larger systems directly translates into the ability of obtaining more precise information about engineering designs within budgetary limitations. All areas of engineering analysis, research and design should thereby profit from the methods proposed herein.

A description of the remainder of the paper follows. In Section 2 we describe the class of semi-discrete problems considered. In Section 3 a time-discretization/iterative algorithm is presented. The procedures are applicable to linear and nonlinear, static and dynamic problems of structural and solid mechanics in which the tangent operator is symmetric. The linear algebraic equations system which needs to be solved during each step/iteration is the focal point of subsequent developments. In Section 4 a parabolic equation is introduced whose asymptotic solution is the same as the solution of the linear algebraic equation system. The solution of the parabolic equation is obtained by way of a standard implicit (pseudo) time-discretization scheme in Section 5. The coefficient matrix of the implicit scheme is factorized approximately into element arrays in Section 6. It is shown how to improve the approximate factors by way of the line search and BFGS update techniques. Numerical results which indicate the good behavior of the methodology are presented in Section 7 and conclusions are drawn in Section 8.

2. Semi-discrete equations of nonlinear mechanics

Consider the following semi-discrete system

$$\boldsymbol{M}\boldsymbol{a} = \boldsymbol{F} \tag{2.1}$$

where $\boldsymbol{M}$, $\boldsymbol{a}$ and $\boldsymbol{F}$ represent the (generalized) mass matrix, acceleration vector and force vector, respectively. Equation (2.1) may be thought of as arising from a finite element discretization of a solid, fluid, structure or combined system. In general, $\boldsymbol{M}$, $\boldsymbol{a}$ and $\boldsymbol{F}$ each depend on time (t). Explicit characterization of $\boldsymbol{M}$, $\boldsymbol{a}$ and $\boldsymbol{F}$ may be given for particular systems under consideration.

In the present work we are particularly interested in nonlinear structural and solid mechanics applications in which the Lagrangian kinematical description is adopted. In this case the important kinematical quantities are $\boldsymbol{d}$, the material-particle displacement from a reference configuration; $\boldsymbol{v} = \dot{\boldsymbol{d}}$, the particle velocity; and $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{d}}$, the particle acceleration. Dots indicate the Lagragian time-derivative in which the material particle is held fixed. The forces are assumed to take the form

$$\boldsymbol{F} = \boldsymbol{F}^{\text{ext}} - \boldsymbol{N} \tag{2.2}$$

where $\boldsymbol{F}^{\text{ext}}$ is the vector of given external forces and $\boldsymbol{N}$ denotes the vector of internal forces, which may depend upon $\boldsymbol{d}$, $\dot{\boldsymbol{d}}$ and *their histories*. To make the dependence precise, one need introduce equations which define the constitutive (i.e., stress-deformation) behavior of the materials in question. These equations vary widely in type and complexity. For example, they may be algebraic equations, differential equations or integro-differential equations. In addition, inequality constraints may be present, such as in plasticity theory.

3. Time discretization

To solve the semi-discrete problem, a time-discretization algorithm needs to be introduced. For this purpose we shall employ the Newmark family of methods [20]. Generalization to other time integrators, such as the Hilber–Hughes–Taylor algorithm [8, 11, 12] which possesses improved properties, may be easily facilitated without essential alteration to the following formulation. [8-12]

The Newmark 'predictors' are given by

$$\tilde{\boldsymbol{d}}_{n+1} = \boldsymbol{d}_n + \Delta t\, \boldsymbol{v}_n + \tfrac{1}{2}\Delta t^2 (1 - 2\beta)\boldsymbol{a}_n\,, \tag{3.1}$$

$$\tilde{\boldsymbol{v}}_{n+1} = \boldsymbol{v}_n + \Delta t (1 - \gamma)\boldsymbol{a}_n \tag{3.2}$$

where subscripts refer to the step number; Δt is the time step; $\boldsymbol{d}_n$, $\boldsymbol{v}_n$ and $\boldsymbol{a}_n$ are the approximations to $\boldsymbol{d}(t_n)$, $\dot{\boldsymbol{d}}(t_n)$ and $\ddot{\boldsymbol{d}}(t_n)$, respectively; and β and γ are parameters which govern the accuracy and stability of the method [7, 9, 18].

Calculations commence with the given initial data (i.e., $\boldsymbol{d}_0$ and $\boldsymbol{v}_0$) and $\boldsymbol{a}_0$ which may be calculated from

$$Ma_0 = F_0^{\text{ext}} - N_0 . \tag{3.3}$$

If M is diagonal, as is common in structural dynamics, the solution of (3.3) is rendered trivial. Otherwise, a factorization, forward reduction and back substitution are necessary to obtain a_0.

In the sequel we are only interested in members of the Newmark family for which $\beta > 0$.

In each time step a nonlinear algebraic problem arises which may be solved by Newton–Raphson and quasi-Newton-type iterative procedures. There are several ways of going about this. Perhaps the most generally useful implementation for structural dynamics is to form an 'effective static problem' in terms of the unknown d_{n+1}, which is in turn linearized. The calculations may then be performed as follows:

displacement formulation

$$i = 0 \qquad (i \text{ is the iteration counter}) \tag{3.4}$$

$$d_{n+1}^{(i)} = \tilde{d}_{n+1} \tag{3.5}$$

$$v_{n+1}^{(i)} = \tilde{v}_{n+1} \qquad \text{(predictor phase)} \tag{3.6}$$

$$a_{n+1}^{(i)} = 0 \tag{3.7}$$

$$R = F_{n+1}^{\text{ext}} - N_{n+1}^{(i)} - M_{n+1}^{(i)} a_{n+1}^{(i)} \qquad \text{(residual, or out-of-balance, force)} \tag{3.8}$$

$$K^* = \frac{1}{\Delta t^2 \beta} M_{n+1}^{(i)} + \frac{\gamma}{\Delta t \beta} C_{n+1}^{(i)} + K_{n+1}^{(i)} \qquad \text{(effective stiffness)} \tag{3.9}$$

$$\boxed{K^* \Delta d = R} \tag{3.10}$$

$$d_{n+1}^{(i+1)} = d_{n+1}^{(i)} + \Delta d \tag{3.11}$$

$$a_{n+1}^{(i+1)} = (d_{n+1}^{(i+1)} - \tilde{d}_{n+1})/(\Delta t^2 \beta) \qquad \text{corrector phase)}$$

$$v_{n+1}^{(i+1)} = \tilde{v}_{n+1} + \Delta t \gamma a_{n+1}^{(i+1)} \tag{3.13}$$

If additional iterations are to be performed, i is replaced by $i + 1$, and calculations resume with (3.8). Either a fixed number of iterations may be performed, or iterating may be terminated when Δd and/or R satisfy preassigned convergence conditions. When the iterative phase is completed, the solution at step $n + 1$ is defined by the last iterates (viz. $d_{n+1} = d_{n+1}^{(i+1)}$; $v_{n+1} = v_{n+1}^{(i+1)}$; and $a_{n+1} = a_{n+1}^{(i+1)}$). At this point, n is replaced by $n + 1$, and calculations for the next time step may begin.

In practice, d_n, v_n and a_n are generally saved during the iterative phase, along with $d_{n+1}^{(i+1)}$; but $v_{n+1}^{(i+1)}$ and $a_{n+1}^{(i+1)}$ may be computed as needed, on the element level.

The matrices C and K are the tangent damping and tangent stiffness matrices, respectively. These are linearized operators associated with N. For example, if N is an algebraic function of d and $\dot{d}$, then

$$K = \partial N / \partial d \tag{3.14}$$

and

$$C = \partial N / \partial \dot{d} . \tag{3.15}$$

We shall assume that M, K and C are symmetric; M and K are positive-definite; and C is positive semi-definite.

So-called implicit–explicit mesh partitions [1, 2, 14–17] may be encompassed by the above formulation simply by excluding explicit element/node contributions from the definitions of $\boldsymbol{C}$ and $\boldsymbol{K}$. A totally explicit formulation is attained by ignoring $\boldsymbol{C}$ and $\boldsymbol{K}$. In these cases it is necessary to employ a diagonal mass matrix in explicit regions to attain full computational efficiency.

It may be observed that the preceding algorithm may be [illegible] linear dynamics and statics.

Nonlinear statics. In this case ignore $\boldsymbol{M}$ and $\boldsymbol{C}$ and set $\boldsymbol{v}$ and $\boldsymbol{a}$ to zero throughout.

Linear dynamics. In this case $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are constant and

$$\boldsymbol{N} = \boldsymbol{C}\boldsymbol{v} + \boldsymbol{K}\boldsymbol{d}. \tag{3.16}$$

Linear statics. In this case ignore $\boldsymbol{M}$ and $\boldsymbol{C}$, set $\boldsymbol{v}$ and $\boldsymbol{a}$ to zero throughout, $\boldsymbol{K}$ is constant and

$$\boldsymbol{N} = \boldsymbol{K}\boldsymbol{d}. \tag{3.17}$$

In implicit time integration and is static cases computational effort associated with the formulation of $\boldsymbol{K}^*$ and solution of (3.10) can be formidable. In large-scale systems the major storage demands are due to $\boldsymbol{K}^*$. Typically, core storage of even the largest computers is exceeded, requiring heavy use of disk files. Only small portions of $\boldsymbol{K}^*$ may fit in core at one time necessitating non-sequential-access techniques for assembly and blocked out-of-core equation-solving procedures. In these circumstances I/O time tends to dominate CPU time. In the sequel we describe a procedure to solve (3.10) which does not require formation of $\boldsymbol{K}^*$. To simplify the subsequent writing we shall adopt the following notations in place of (3.10):

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \tag{3.18}$$

Thus during each step, at each iteration, we wish to solve (3.18) in which the coefficient matrix $\boldsymbol{A}$ is symmetric and positive-definite. Furthermore, $\boldsymbol{A}$ is assembled from element arrays, that is

$$\boldsymbol{A} = \sum_{e=1}^{n_{el}} \boldsymbol{A}^e \tag{3.19}$$

where each $\boldsymbol{A}^e$ is symmetric and positive semi-definite, and n_{el} is the number of elements.

4. Parabolic regularization

The first step in developing the element-by-element solution procedure is to replace the discrete elliptic problem, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, by an associated parabolic problem. To this end consider

$$\boldsymbol{W}\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\tau} + \boldsymbol{A}\boldsymbol{y} = H(\tau)\boldsymbol{b}, \tag{4.1}$$

$$\boldsymbol{y}(0) = \boldsymbol{0} \tag{4.2}$$

where W is a positive-definite diagonal matrix; $H(\tau)$ is a Heaviside function (i.e., $H(\tau) = 1$ for $\tau > 0$ and $H(\tau) = 0$ otherwise); and τ is a non-dimensional 'pseudo time'. The asymptotic solution of (4.1) coincides with the solution of the original problem, that is,

$$x = \lim_{\tau\to\infty} y(\tau)\,. \tag{4.3}$$

Several possibilities exist for the selection of W. A simple, but promising, choice is to define W to be the diagonal of A:

$$W = A_{\text{diag}}\,. \tag{4.4}$$

5. Discrete algorithm for the parabolic problem

The second step in the development of the element-by-element procedure is to introduce a (pseudo-) time discretization scheme. For this purpose we employ the classical generalized trapezoidal algorithm:

$$(W + \alpha\Delta\tau A)y_{m+1} = (W - (1-\alpha)\Delta\tau A)y_m + \Delta\tau b\,, \tag{5.1}$$

$$y_0 = 0\,. \tag{5.2}$$

Because we are interested in the asymptotic solution of (4.1), it behooves us to select $\alpha = 1$ (i.e., the backward-difference method) which maximizes algorithmic dissipation [12]. In this case (5.1) becomes

$$(W + \Delta\tau A)y_{m+1} = Wy_m + \Delta\tau b\,. \tag{5.3}$$

It is well known that (5.3) is unconditionally stable and, for large $\Delta\tau$, the solution of (5.3) converges to the asymptotic solution very quickly.

For future reference it is convenient to rewrite (5.3) as

$$y_{m+1} - y_m = \Delta\tau W^{-1/2} V W^{-1/2} r_m \tag{5.4}$$

where

$$r_m = b - Ay_m \quad \text{(residual)} \tag{5.5}$$

and

$$V = (I + \Delta\tau W^{-1/2} A W^{-1/2})^{-1} \tag{5.6}$$

in which I is the identity matrix.

6. Element-by-element algorithms for the parabolic problem

The idea is to approximate V by a product. For example, let

$$V \approx V_1 = \prod_{e=1}^{n_{el}} V_1^e = V_1^1 V_1^2 \cdots V_1^{n_{el}} \tag{6.1}$$

where

$$V_1^e = V^e(\Delta\tau) = (I + \Delta\tau W^{-1/2} A^e W^{-1/2})^{-1} \tag{6.2}$$

With V_1 in place of V in (5.4), calculations may be performed one element at a time thus obviating the need to form, store and operate upon the global array V.

It may be shown that

$$V_1 = V + O(\Delta\tau^2) \tag{6.3}$$

That is, V_1 is a first-order approximation of V. However, accuracy is not the main issue here as we are only interested in computing the asymptotic solution of (4.1). For this purpose it is important to take as large a $\Delta\tau$ as possible. In this case the approximation of V by V_1 begins to deteriorate. A somewhat better conditioned approximation may be attained by a two-pass formulation in which

$$V \approx V_2 = \Big(\prod_{e=1}^{n_{el}} V_2^e\Big)\Big(\prod_{e=n_{el}}^{1} V_2^e\Big) \tag{6.4}$$

where

$$V_2^e = V^e(\tfrac{1}{2}\Delta\tau)\,. \tag{6.5}$$

It may be shown that

$$V_2 = V + O(\Delta\tau^3) \tag{6.6}$$

and, in addition, V_2 is symmetric and positive-definite. Thus it represents a qualitatively more faithful approximation to V than does V_1.

REMARKS. (1) The ideas presented in this section emanate from [13]. However, there are subtle, but important, differences between the algorithms presented in [13] and those described herein. In [13], unconditionally stable, second-order time-accurate procedures are described for both linear and nonlinear problems. Due to the residual framework employed herein, (5.4), which differs from the formulation of [13], unconditional stability cannot be guaranteed without further embellishment. This is dealt with shortly.

(2) The notion of a finite element here may be generalized to include a subassembly of elements (i.e., subdomain model or substructure). In certain applications it may be advantageous to allow limited assembly.

(3) The potential operation count advantages for large systems may be seen by considering $N \times N$ meshes of four-node quadrilaterals and $N \times N \times N$ meshes of eight-node bricks. Cost of the element-by-element procedures is proportional to the product of the number of elements (i.e., N^2 and N^3, resp.) times number of iterations required (n_{its}). The order of the operation counts is presented in Table 1. The results do not account for the I/O penalty of direct equation solving techniques.

(4) If fast core is available, the factorized element arrays should be saved for subsequent iterations. Whether one should use disk file to store factorized element arrays, or should reform/factorize as needed, depends upon the tradeoff between I/O and CPU costs.

Table 1
Order of number of operations for element-by-element and direct equation solving procedures

	2D	3D
Direct factorization	$O(N^4)$	$O(N^7)$
Direct backsubstitution	$O(N^3)$	$O(N^5)$
Element-by-element	$O(n_{its}N^2)$	$O(n_{its}N^3)$

(5) In many iterative solution algorithms which do not employ global matrices, such as the conjugate gradient method, information propagates only to neighbouring elements during an iteration. For example, in a one-dimensional mesh of n_{el} elements subjected to a source at one end, it takes n_{el} iterations for the source to influence the opposite end of the mesh. Such an algorithm is thus seen to possess 'explicit' character. On the other hand, the two-pass element-by-element algorithm has 'implicit' character in that a source anywhere influence every element in the mesh in one iteration.

6.1. Line search

To insure the stability of the approximation scheme, we insist that at each step the new solution represents an 'improved' approximation to the asymptotic solution. To this end we use the element-by-element algorithm to define a 'search direction' Δy. The updating of the solution is then done in a manner which minimizes the total potential energy in the direction of Δy.

Let

$$\Delta y = \Delta \tau W^{-1/2} V_j W^{-1/2} r_m \tag{6.7}$$

where $j = 1$ or 2. Then the $(m+1)$st step solution is defined by

$$y_{m+1} = y_m + s\Delta y \tag{6.8}$$

where

$$s = \Delta y^t r_m / \Delta y^t A \Delta y \quad \text{(search parameter)}. \tag{6.9}$$

Clearly, the value of s defined by (6.9) minimizes the potential:

$$P(s) = (y_m + s\Delta y)^t (b - \tfrac{1}{2}A(y_m + s\Delta y)). \tag{6.10}$$

By substituting (6.7) into (6.9), it can be seen that if V_j is positive-definite (which is guaranteed for $j = 2$), then $r_m \neq 0$ implies $s \neq 0$, and thus the solution necessarily improves in every iteration.

6.2. BFGS update

The search direction can be further improved by use of the BFGS update. This technique is a quasi-Newton (i.e., secant) procedure which preserves the symmetry and positive-definite-

CMA 902-9.

Table 2
Flowchart of the element-by-element algorithm with line search and BFGS update

1. Initialization
   $m = 0, \quad y_0 = 0$
   $f_k = g_k = 0$ (loop: $k = 1, 2, \ldots, n_{BFGS}$)
   $\Delta y = \Delta_T W^{-1/2} V_j W^{-1/2} b$

2. $s = \Delta y^t r_m / \Delta y^t A \Delta y$
   $y_{m+1} = y_m + s\Delta y$

3. Convergence check
   $\|r_{m+1}\| < \varepsilon$?
   Yes: return, no: continue

4. Relabel BFGS vectors
   $f_{k-1} = f_k, \; g_{k-1} = g_k$ (loop: $k = 2, 3, \ldots, n_{BFGS}$)

5. $f_{n_{BFGS}} = (\Delta y^t r_m)^{-1} \Delta y$
   $g_{n_{BFGS}} = r_{m+1} - (1 - s^{1/2}) r_m$

6. $z = r_{m+1}$
   $z \leftarrow z + (f_k^t z) g_k$ (loop: $k = n_{BFGS}, n_{BFGS} - 1, \ldots, 1$)
   $z \leftarrow \Delta_T W^{-1/2} V_j W^{-1/2} z$
   $z \leftarrow z + (g_k^t z) f_k$ (loop: $k = 1, 2, \ldots, n_{BFGS}$)
   $\Delta y = z$

7. $m \leftarrow m + 1$, go to step 2

ness of the approximating matrix [4, 5, 19]. The implementational aspects are summarized in Table 2. The number of updates is fixed at n_{BFGS}. The BFGS vectors (i.e., f_k's and g_k's) are stored in a revolving data pool in which the oldest pair of vectors (i.e., f_1, g_1) is discarded as the latest pair is added. In practice this is simply achieved by redefining pointers.

7. Numerical results

The computed results were obtained on a VAX computer using single precision (32 bits per floating point word). Throughout, (4.4) was in force, and $\Delta_T = 1$. The following methods were compared (each is obtained by specialization of the flow chart in Table 2):

(i) *Jacobi.* The classical Jacobi iteration is obtained by setting $V_j \equiv I$, $s \equiv 1$, and ignoring the BFGS vectors (i.e., setting them to zero).

(ii) *Jacobi + line search.* This is the same as case (i) except the line search is in effect (step 2 of the flowchart).

(iii) *Jacobi + line search + BFGS.* In this case, the only simplification in Table 2 is that $V_j = I$.

(iv) *Element-by-element.* We employ $V_j = V_2$ (see (6.4)), but set $s \equiv 1$ and ignore the BFGS vectors.

(v) *Element-by-element + line search.* This is the same as case (iv) except the line search is in effect.

(vi) *Element-by-element + line search + BFGS.* This case follows the flowchart verbatim with $V_j = V_2$.

No limit was set on the number of BFGS vectors (i.e., '$n_{BFGS} = \infty$', cf. Table 2).

period

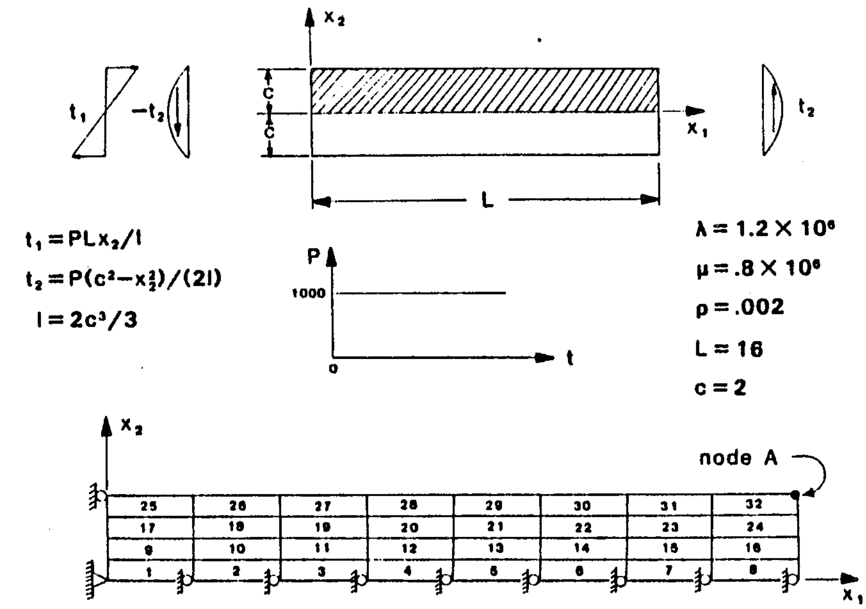


Fig. 1. Problem definition and finite element mesh.

The configuration analyzed is shown in Fig. 1. It represents one-half of a plane strain beam modelled with 32 bilinear quadrilateral elements. A lumped mass matrix was employed. The loading and boundary conditions are set in accord with an exact, static linear elasticity solution (see [24, p. 35–39]). However, here the problem is forced dynamically. The beam is assumed initially at rest and all loads are applied instantaneously at $t = 0+$. In formulating the problem, the Newmark algorithm is employed with $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$ (see Section 3). With these parameters, unconditional stability is attained and no algorithmic damping is introduced [7, 9, 12].

The numerical solution is dominated by response in the fundamental mode. This is illustrated in Fig. 2. At a time step of $\Delta t = 2.5 \times 10^{-4}$, an essentially exact solution is obtained. At a larger step of $\Delta t = 2.5 \times 10^{-3}$, a very crude approximation of the response is obtained. It is interesting to relate the sizes of these steps to the critical time step for explicit integration, $\Delta t_{cr} = h_{min}/C_D = h_{min}/\sqrt{(\lambda + 2\mu)/\rho} = 1.336 \times 10^{-4}$, and the approximate period of the fundamental node, $T_1 \approx 0.0122$ (see Table 3). As may be seen, both time steps are far outside the range of explicit integration. The larger time step resolves the fundamental mode with only 5 steps, and thus is larger than the maximum feasible for this problem.

In comparing the results of the various methods it is important to keep in mind that *all methods give identical solution.*[1] Consequently, the primary basis of comparison is the number of iterations needed to attain the solution. It was found that the number of iterations per time step did not vary significantly from one time step to another for a given method and specific step size. ~~However, it was observed that convergence was generally achieved more rapidly in the neighborhood of peaks and valleys of the fundamental mode than at zero crossings.~~

[1] The convergence criterion, ε in step 3 of the flowchart, was taken to be $0.01\,\|b\|$.

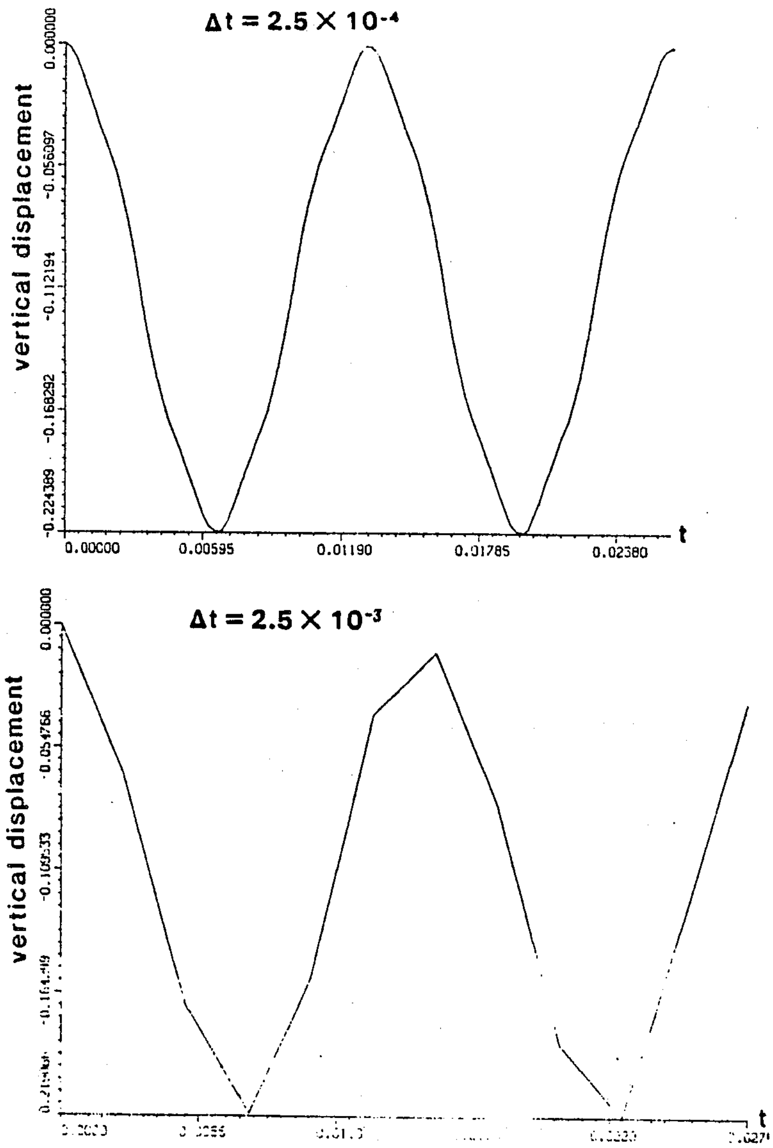


Fig. 2. Vertical displacement of [illegible]e A.

[illegible]

Com[illegible]arison of time steps used in [illegible]tions with characteristic [illegible]me scales

[illegible]	2.5×10^{-4}	2.5×10^{-3}
[illegible]	18.71	187.1
[illegible]	48.9	4.89

Table 4
Number of iterations required for convergence for the problem illustrated in Fig. 1; LS = line search; EBE = element-by-element

Method \ Δt	2.5×10^{-4} ($=18.71\Delta t_{cr}$)	2.5×10^{-3} ($=187.1\Delta t_{cr}$)
(i) Jacobi	99	∞[a]
(ii) Jacobi + LS	38	75
(iii) Jacobi + LS + BFGS	15	21
(iv) EBE	14	16
(v) EBE + LS	9	6
(vi) EBE + LS + BFGS	5	4

[a] No convergence attained after 150 iterations.

Results for the first time step, ~~effectively a zero crossing~~, are presented in Table 4. The following observations may be made. In general the element-by-element results are superior to Jacobi. Use of line search and BFGS updates accelerate convergence. The best results are attained by the element-by-element procedure with line search and BFGS updates. We feel that these results are fairly impressive.

It is somewhat surprising that methods (v) and (vi) converge faster at the larger time step than at the smaller. At this point we have no explanation for this phenomenon.

Some other numerical experiments that we have run indicated that fixing the root, which creates a singularity, slows convergence, but use of other updating procedures represented an improvement over BFGS. This will be reported upon in future work.

8. Conclusions

In this paper an element-by-element solution procedure has been proposed for finite element equation systems which arise in the linear and nonlinear, static and dynamic analysis of solids and structures. The procedure does not entail a global matrix and thus offers significant operation count and I/O advantages when compared with direct elimination schemes. The methodology may be simply incorporated in many existing finite element software systems. Numerical results have demonstrated the effectiveness of the procedure on a test problem and suggest its considerable potential for large-scale finite element equation solving.

9. Epilog

We believe it is important for the reader to realize that the element-by-element procedure proposed herein is essentially an iterative *linear-equation solving technique* and *not* a *time discretization algorithm*, such as the original element-by-element method proposed in [13]. The method of [13] achieves unconditional stability and second-order time accuracy, however, spatial truncation errors are sometimes unacceptably large at time steps at which globally implicit methods perform reasonably. Generalizations of [13] by us to structural dynamical systems (unpublished) never achieved what we felt to be a satisfactory combination of stability and accuracy properties. Based on this background, Ortiz et al. [21] proposed an element-by-element time discretization algorithm for dynamics which achieved unconditional stability. The approach is novel in that velocities and stresses are considered primary unknowns. However, the problem of unacceptably large spatial truncation errors remains.

The element-by-element algorithm proposed herein circumvents questions of accuracy and stability entirely. These properties are inherited without alteration from the chosen globally implicit method. The only question here is the *speed* at which the element-by-element algorithm is able to obtain the globally implicit solution. Based upon the results presented and work in progress we are very encouraged by the performance of our method so far and its potential for further improvement.

References

[1] T. Belytschko and R. Mullen, Mesh partitions of explicit–implicit time integration, in: K.J. Bathe et al., eds., Formulations and Computational Algorithms in Finite Element Analysis (M.I.T. Press, Cambridge, MA 1977).

[2] T. Belytschko and R. Mullen, Stability of explicit–implicit mesh partitions in time integration, Internat. J. Numer. Meths. Engrg. 12(10) (1979) 1575–1586.

[3] J. Douglas, On the numerical integration of $u_{xx} + u_{yy} = u_t$ by implicit methods, J. Soc. Indust. Appl. Math. 3 (1955) 42–65.

[4] J.E. Dennis and J.J. Moré, Quasi-Newton methods: motivation and theory, SIAM Rev. 19(1) (1977) 46–89.

[5] M. Geradin and M.A. Hogge, Quasi-Newton iteration in nonlinear structural dynamics, Paper No. M7/1, Transactions of the 5th Structural Mechanics in Reactor Technology Conference, Berlin (West), Germany (August 1979).

[6] G.L. Goudreau and J.O. Hallquist, Personal communication (1981).

[7] G.L. Goudreau and R.L. Taylor, Evaluation of numerical integration methods in elastodynamics, Comput. Meths. Appl. Mech. Engrg. 2 (1972) 69–97.

[8] H.D. Hibbitt and B.I. Karlsson, Analysis of pipe whip, ASME Paper No. 79-PVP-122, presented at the Pressure Vessels and Piping Converence, San Francisco, California, June 25–29 (1979).

[9] H.M. Hilber, Analysis and design of numerical integration methods in structural dynamics, Report No. EERC76-29, Earthquake Engineering Research Center, University of California, Berkeley, CA (1976).

[10] H.M. Hilber and T.J.R. Hughes, Collocation, dissipation and 'overshoot' for time integration schemes in structural dynamics, Earthquake Engineering and Structural Dynamic, 6 (1978) 99–118.

[11] H.M. Hilber, T.J.R. Hughes and R.L. Taylor, Improved numerical dissipation for time integration algorithms in structural dynamics, Earthquake Engrg. Structural Dynamics 5 (1977) 283–292.

[12] T.J.R. Hughes, Analysis of transient algorithms with particular reference to stability behavior, in: T. Belytschko and T.J.R. Hughes, eds., Computational Methods in Transient Analysis (North-Holland, Amsterdam, to appear).

[13] T.J.R. Hughes, I. Levit and J. Winget, Implicit, unconditionally stable, element-by-element algorithms for heat conduction analysis, J. Engrg. Mech. Div. ASCE (to appear).

[14] T.J.R. Hughes and W.K. Liu, Implicit–explicit finite elements in transient analysis: stability theory, J. Appl. Mech. 45 (1978) 371–374.

[15] T.J.R. Hughes and W.K. Liu, Implicit–explicit finite elements in transient analysis: implementation and numerical examples, J. Appl. Mech. 45 375–378.

[16] T.J.R. Hughes, K.S. Pister and R.L. Taylor, Implicit–explicit finite elements in nonlinear transient analysis, Comput. Meths. Appl. Mech. Engrg. 17/18 (1979) 159–182.

[17] T.J.R. Hughes and R.A. Stephenson, Convergence of implicit–explicit algorithms in nonlinear transient analysis, Internat. J. Engrg. Sci. 19(2) (1981) 295–302.

[18] R.D. Krieg and S.W. Key, Transient shell response by numerical time integration, Internat. J. Numer. Meths. Engrg. 7 (1973) 273–286.

[19] H. Matthies and G. Strang, The solution of nonlinear finite element equations, Internat. J. Numer. Meths. Engrg. 14(11) (1979) 1613–1626.

[20] N.M. Newmark, A method of computation for structural dynamics, J. Engrg. Mech. Div. ASCE (1959) 67–94.

[21] M. Ortiz, P.M. Pinsky and R.L. Taylor, Unconditionally stable element-by-element algorithms for dynamic problems, Comput. Meths. Appl. Mech. Engrg. (to appear).

[22] D.W. Peaceman and H.H. Rachford, The numerical solution of parabolic and elliptic differential equations, J. Soc. Indust. Appl. Math. 3 (1955) 28–41.

[23] R.S. Rogallo, Numerical experiments in homogeneous turbulence, NASA TN 81315 (September 1981).

[24] S. Timoshenko and J.N. Goodier, Theory of Elasticity (McGraw-Hill, New York, 2nd ed., 1951).

[25] R.F. Warming and R.M. Beam, On the construction and application of implicit factored schemes for conservation laws, SIAM-AMS Proc. 11 (1978) 85–129.

#7

me235B

A COURSE IN THE FINITE ELEMENT METHOD

VOL. I LINEAR FINITE ELEMENT ANALYSIS

Chapter 4

Mixed and Penalty Methods, Reduced and Selective Integration, and Sundry Variational Crimes

Thomas J.R. Hughes*
Division of Applied Mechanics
Stanford University
Stanford, CA 94305

*Associate Professor of Mechanical Engineering

§2. Incompressible Elasticity and Stokes Flow

Many problems of physical importance involve motions which essentially preserve volumes locally. That is, after deformation each small portion of the medium has the same volume as before deformation. Media that behave in this fashion are termed incompressible. Rubber is often modelled as an incompressible elastic material and many fluid flows are assumed incompressible.

In isotropic linear elasticity the condition of incompressibility may be expressed in terms of Poisson's ratio ν . As ν approaches $\frac{1}{2}$ resistance to volume change is greatly increased whereas resistance to shearing remains constant. This may be seen by calculating the ratio of bulk modulus, B , to shearing modulus, μ (Soholnikoff [1], p. 71):

$$\frac{B}{\mu} = \frac{2(1+\nu)}{3(1-2\nu)} \tag{1}$$

Clearly, as $\nu \to \frac{1}{2}$ the ratio approaches infinity. The value $\nu = \frac{1}{2}$ thus represents incompressibility.

This limit creates problems in the equations of (compressible) elasticity. Recall that the constitutive equation in the isotropic case may be written as (see Chapter 2, §6, eqs. 1, 2 and 31)

$$\sigma_{ij} = \lambda\, u_{k,k}\, \delta_{ij} + 2\mu\, u_{(i,j)} \tag{2}$$

where

$$\lambda = 2\nu\mu/(1-2\nu) \tag{3}$$

Thus the Lamé parameter, λ , also becomes unbounded in the incompressible limit and an alternative formulation of the theory is therefore necessary.

In this formulation the constitutive equation is written as

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu\, u_{(i,j)} \tag{4}$$

where $p = p(x)$ is the <u>hydrostatic pressure</u>. The pressure must be determined as part of the solution to the boundary-value problem and thus represents an additional unknown. The additional equation which needs to be introduced is the <u>kinematic condition of incompressibility</u>, namely

$$\operatorname{div} \underset{\sim}{u} = u_{i,i} = 0 \tag{5}$$

The boundary-value problem may then be stated as follows:

<u>incompressible isotropic elasticity</u>

Given ℓ_i, g_i and h_i (as in Chapter 2, §6), find the displacement, u_i, and pressure, p, such that

$$\left.\begin{aligned} \sigma_{ij,j} + \ell_i &= 0 \quad (6) \\ u_{i,i} &= 0 \quad (7) \end{aligned}\right\} \text{in } \Omega$$

$$u_i = g_i \text{ on } \Gamma_{g_i} \tag{8}$$

$$\sigma_{ij} n_j = h_i \text{ on } \Gamma_{h_i} \tag{9}$$

where σ_{ij} is given by (4).

<u>Remark</u>

In the case of the displacement boundary-value problem, in which $\Gamma = \Gamma_{g_i}$ and $\Gamma_{h_i} = \emptyset$, a consistency condition on g_i follows from incompressibility:

$$\begin{aligned} 0 &= \int_{\Omega} u_{i,i}\, d\Omega && \text{(by eq. 7)} \\ &= \int_{\Gamma} u_i n_i\, d\Gamma && \text{(divergence theorem, eq. 11, §1, Chapter 2)} \\ &= \int_{\Gamma} g_i n_i\, d\Gamma && \text{(by eq. 8)} \end{aligned} \tag{10}$$

The given g_i must satisfy (10) or else no solution to the boundary-value problem can exist. In the displacement boundary-value problem the pressure is determinable only up to an arbitrary constant.

Stokes flow

The equations of Stokes flow are identical to the equations of isotropic incompressible elasticity. Only the physical interpretation of the variables is different. In Stokes flow $\underset{\sim}{u}$ is the velocity of the fluid and μ is the dynamic viscosity. Stokes flow governs highly viscous phenomena, often referred to as "creeping flow".

Exercise Use (3), above, and (35) of §6, Chapter 2 to show that the coefficients of the isotropic plane stress constitutive equation remain bounded as $\nu \to \frac{1}{2}$. Thus a special formulation is not required for the incompressible plane stress case.

§2.1 Prelude to mixed and penalty methods

The subject of finite element approximations to incompressible elasticity problems is replete with so-called "mixed" and "penalty" formulations. In order to introduce the essential aspects of these methods in a context which is as simple as possible we shall consider the example of appending a constraint to a linear algebraic system. Suppose we wish to solve our standard matrix problem

$$\underset{\sim}{K}\,\underset{\sim}{d} = \underset{\sim}{F} \,, \tag{11}$$

where as usual $\underset{\sim}{K}$ is symmetric and positive-definite, subject to a constraint on one of the degrees-of-freedom, namely

$$d_Q = \mathcal{g} \tag{12}$$

where the subscript Q represents the equation number in the global ordering and $\mathcal{g}$ is a given constant. Physically, we can think of (12) as perhaps a modification to an original design. For example, the position of a boundary node may be altered to stiffen a structure for some purpose. The modified problem consisting of (11) and (12) may be formulated as a so-called constrained variational problem. The essential character of mixed methods is exhibited in this framework. To develop this idea, it is helpful to think of $\underset{\sim}{K}\,\underset{\sim}{d} = \underset{\sim}{F}$ as arising from the minimization of a function:

$$\mathcal{F}(\underset{\sim}{d}) = \underset{\sim}{d}^T\underset{\sim}{K}\,\underset{\sim}{d}/2 - \underset{\sim}{d}^T\underset{\sim}{F} \tag{13}$$

which is called the total potential energy function. The vector which minimizes $\mathcal{F}$ satisfies (11). This can be seen as follows:

Let ε be a real parameter. Form the one-parameter family of displacement vectors

$$\underset{\sim}{d} + \varepsilon \underset{\sim}{c} \tag{14}$$

where $\underset{\sim}{c}$ is arbitrary. $\mathscr{F}$ is minimized by $\underset{\sim}{d}$ if

$$0 = \left(\frac{d}{d\varepsilon} \mathscr{F}(\underset{\sim}{d} + \varepsilon \underset{\sim}{c})\right)_{\varepsilon=0} \tag{15}$$

for all vectors $\underset{\sim}{c}$. This calculation is carried out as follows:

$$\left(\frac{d}{d\varepsilon} \mathscr{F}(\underset{\sim}{d} + \varepsilon \underset{\sim}{c})\right)_{\varepsilon=0} = \left(\frac{d}{d\varepsilon}\left((\underset{\sim}{d} + \varepsilon \underset{\sim}{c})^T \underset{\sim}{K}(\underset{\sim}{d} + \varepsilon \underset{\sim}{c})/2 - (\underset{\sim}{d} + \varepsilon \underset{\sim}{c})^T \underset{\sim}{F}\right)\right)_{\varepsilon=0}$$

$$= \underset{\sim}{c}^T \underset{\sim}{K}\, \underset{\sim}{d}/2 + \underset{\sim}{d}^T \underset{\sim}{K}\, \underset{\sim}{c}/2 - \underset{\sim}{c}^T \underset{\sim}{F}$$

$$= \underset{\sim}{c}^T(\underset{\sim}{K}\, \underset{\sim}{d} - \underset{\sim}{F}) \qquad \text{(symm. of } \underset{\sim}{K}) \tag{16}$$

Thus we see that (16) must be zero for arbitrary $\underset{\sim}{c}$ which implies $\underset{\sim}{K}\, \underset{\sim}{d} = \underset{\sim}{F}$. Thus we have equivalent alternative characterizations of $\underset{\sim}{d}$: It is the solution of problem (11) and also the minimizer of function (13). The fact that $\underset{\sim}{d}$ minimizes $\mathscr{F}$ (i.e., solves a "variational problem") allows us to introduce standard calculus of variations methodology in order to formulate the modified problem consisting of (11) and (12). For this purpose it is convenient to rephrase (12) as

$$0 = \mathscr{G}(\underset{\sim}{d}) = \underset{\sim}{1}_Q^T\, \underset{\sim}{d} - \mathscr{g} \tag{17}$$

where

$$\underset{\sim Q}{1}^T = \langle 0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0 \rangle \quad \text{(}\uparrow\text{ Qth term)} \tag{18}$$

The brackets $\langle \ldots \rangle$ signify a row vector. Clearly (17) is equivalent to (12).

Lagrange-multiplier method

Consider the following function:

$$\mathscr{H}(\underset{\sim}{d}, m) = \mathscr{F}(\underset{\sim}{d}) + m\,\mathscr{G}(\underset{\sim}{d}) \tag{19}$$

where m is a scalar parameter called the Lagrange multiplier. Rendering (19) "stationary" is equivalent to satisfaction of the constrained problem (i.e. eqs. 11 and 17). The condition of stationarity is that

$$0 = \left(\frac{d}{d\varepsilon}\mathscr{H}(\underset{\sim}{d} + \varepsilon\,\underset{\sim}{c}\ ,\ m + \varepsilon\,\ell)\right)_{\varepsilon=0} \tag{20}$$

for all values of the vector $\underset{\sim}{c}$ and scalar ℓ. The multiplier m plays the role of the force which maintains the constraint (12). It is an additional unknown corresponding to the additional equation which must be satisfied, namely (12). Let us carry out the calculation indicated in (20):

$$\begin{aligned} 0 &= \left(\frac{d}{d\varepsilon}\left(\mathscr{F}(\underset{\sim}{d} + \varepsilon\,\underset{\sim}{c}) + (m + \varepsilon\,\ell)\mathscr{G}(\underset{\sim}{d} + \varepsilon\,\underset{\sim}{c})\right)\right)_{\varepsilon=0} \\ &= \underset{\sim}{c}^T(\underset{\sim}{K}\,\underset{\sim}{d} - \underset{\sim}{F}) + \ell\,\mathscr{G}(\underset{\sim}{d}) + m\,\underset{\sim Q}{1}^T\,\underset{\sim}{c} \\ &= \underset{\sim}{c}^T(\underset{\sim}{K}\,\underset{\sim}{d} + m\,\underset{\sim Q}{1} - \underset{\sim}{F}) + \ell(\underset{\sim Q}{1}^T\,\underset{\sim}{d} - g) \end{aligned} \tag{21}$$

Due to the arbitrariness of $\underset{\sim}{c}$ and ℓ, (21) implies

$$\underset{\sim}{K}\,\underset{\sim}{d} + m\,\underset{\sim}{1}_Q = \underset{\sim}{F} \tag{22}$$

and

$$\underset{\sim}{1}_Q^T \underset{\sim}{d} = \mathscr{g} \tag{23}$$

or, equivalently,

$$\begin{bmatrix} \underset{\sim}{K} & \underset{\sim}{1}_Q \\ \underset{\sim}{1}_Q^T & 0 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{d} \\ m \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{F} \\ \mathscr{g} \end{Bmatrix} \tag{24}$$

This is the equation system which determines $\underset{\sim}{d}$ and m, the solution of the modified problem. Note that to account for the constraint, the original system, (11), needed to be modified (see eq. 22). This is physically reasonable.

The equation system (24) is a prototype of a <u>mixed method</u>, that is, one in which there are both displacements and forces as unknowns. Its importance in regard to problems of incompressibility—a constraint—is that we have from the outset both displacements and pressures—the force-like variable—as unknowns. Consequently, we may ultimately anticipate algebraic systems having features in common with (24). Note that the coefficient matrix in (24) is symmetric, but not positive-definite. Symmetry follows from the definition of a transposed partitioned matrix:

$$\begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{C} & \underset{\sim}{D} \end{bmatrix}^T = \begin{bmatrix} \underset{\sim}{A}^T & \underset{\sim}{C}^T \\ \underset{\sim}{B}^T & \underset{\sim}{D}^T \end{bmatrix} \tag{25}$$

Failure of the positive-definiteness condition (see (ii) of the definition in §8, Chapter 1) follows from:

$$\begin{Bmatrix} \underset{\sim}{0} \\ 1 \end{Bmatrix}^T \begin{bmatrix} \underset{\sim}{K} & \underset{\sim}{1}_Q \\ \underset{\sim}{1}_Q^T & 0 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{0} \\ 1 \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{0} \\ 1 \end{Bmatrix}^T \begin{Bmatrix} \underset{\sim}{1}_Q \\ 0 \end{Bmatrix} = 0 \tag{26}$$

<u>penalty method</u>

The penalty method of formulating the constrained problem may be viewed as an approximation to the Lagrange-multiplier method. In the penalty formulation the Lagrange multiplier is approximated as follows:

$$\lambda \approx k\mathscr{G}(\underset{\sim}{d}) \tag{27}$$

where k is a large positive number having the physical interpretation of a stiff spring constant. Note that k is <u>not</u> an unknown. With this approximation we may define a new function

$$\mathscr{I}(\underset{\sim}{d}) = \mathscr{F}(\underset{\sim}{d}) + \frac{k}{2}\, \mathscr{G}(\underset{\sim}{d})^2 \tag{28}$$

whose minimum defines an approximate solution to the constrained problem. Calculating, as before,

$$\begin{aligned} 0 &= \left(\frac{d}{d\varepsilon}\, \mathscr{I}(\underset{\sim}{d} + \varepsilon\, \underset{\sim}{c})\right)_{\varepsilon=0} \\ &= \left(\frac{d}{d\varepsilon}\left(\mathscr{F}(\underset{\sim}{d} + \varepsilon\, \underset{\sim}{c}) + \frac{k}{2}\mathscr{G}(\underset{\sim}{d} + \varepsilon\, \underset{\sim}{c})^2\right)\right)_{\varepsilon=0} \\ &= \underset{\sim}{c}^T(\underset{\sim}{K}\, \underset{\sim}{d} - \underset{\sim}{F}) + k\, \mathscr{G}(\underset{\sim}{d})\underset{\sim}{1}_Q^T\underset{\sim}{c} \\ &= \underset{\sim}{c}^T\left((\underset{\sim}{K} + k\, \underset{\sim}{1}_Q\underset{\sim}{1}_Q^T)\underset{\sim}{d} - (\underset{\sim}{F} + k\, \mathscr{g}\, \underset{\sim}{1}_Q)\right) \end{aligned} \tag{29}$$

which implies

$$(\underset{\sim}{K} + k\,\underset{\sim}{1}_Q \underset{\sim}{1}_Q^T)\underset{\sim}{d} = \underset{\sim}{F} + k g\,\underset{\sim}{1}_Q \tag{30}$$

Explicating (30) yields

$$\left(\underset{\sim}{K} + \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & k & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\right)\underset{\sim}{d} = \underset{\sim}{F} + \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ kg \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{31}$$

k appears in the Qth diagonal entry; kg appears in the Qth row

Thus it is clear the as $k \to \infty$, $d_Q \to g$ (i.e. the constraint is satisfied) and thus (27) is an approximation to the constraining force (Lagrange multiplier). The larger k, the better the approximation. This formulation is suggestive of general ways of approximating constrained problems. In the context of elasticity, one would interpret ideas like this as approximating the incompressible case by a slightly compressible formulation. That is one for which the ratio B/μ, or equivalently λ/μ, is very large (see eqs. 1 and 3).

References

1. I.S. Sokolnikoff, Mathematical Theory of Elasticity, Second Edition, McGraw-Hill, New York, 1956.

§3 A Mixed Formulation of Compressible Elasticity Capable of Representing the Incompressible Limit

It is desirable to have a formulation of isotropic elasticity which is valid for both compressible and incompressible behavior. To this end we may introduce a pair of constitutive equations

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu\, u_{(i,j)} \tag{1}$$

$$0 = u_{i,i} + p/\lambda \tag{2}$$

where the pressure parameter, p, is viewed as an independent unknown. If $\nu = \frac{1}{2}$, (2) becomes the incompressibility condition and p is the hydrostatic pressure as in the previous section. If $\nu < \frac{1}{2}$, p may be eliminated from (1) by way of (2) to obtain the constitutive equation of the compressible case, namely

$$\sigma_{ij} = \lambda\, u_{k,k}\,\delta_{ij} + 2\mu\, u_{(i,j)} \tag{3}$$

Thus we see that (1) and (2) are valid in both the compressible and incompressible cases.

Remark

Note that p may be interpreted as the hydrostatic pressure only in the incompressible case. In general, the hydrostatic pressure is $-\sigma_{ii}/3$. Thus in the compressible case, (3) yields

$$-\sigma_{ii}/3 = -\underbrace{(\lambda + 2\mu/3)}_{B}\, u_{i,i} \tag{4}$$

whereas (2) gives

$$p = -\lambda\, u_{i,i} \tag{5}$$

If $\mu \ll \lambda$ (nearly incompressible case), (5) is a good approximation to (4). On the other hand, in the incompressible limit

$$p = -\sigma_{ii}/3 \tag{6}$$

follows directly from (1).

§3.1 Strong form

(5) With f_i, g_i and h_i given as before, we wish to find u_i and p such that

$$\left.\begin{aligned} \sigma_{ij,j} + f_i &= 0 \quad (7) \\ u_{i,i} + p/\lambda &= 0 \quad (8) \end{aligned}\right\} \text{ in } \Omega$$

$$u_i = g_i \text{ on } \Gamma_{g_i} \tag{9}$$

$$\sigma_{ij} n_j = h_i \text{ on } \Gamma_{h_i} \tag{10}$$

where σ_{ij} is defined by (1).

Remark

The formulation presented in this section was first proposed by Herrmann [1]. Generalizations to anisotropic cases were proposed by Taylor, Pister and Herrmann [2] and Key [3]. The subject of anisotropy and incompressibility is taken up in §6.

§3.2 Weak form

The weak formulation of the problem is similar to the one for compressible elasticity (see Chapter 2, §6, eq. 11) except we need to introduce a term which implies satisfaction of (8). In addition to the displacement weighting and trial solution spaces ($\mathcal{V}_i$ and $\mathcal{S}_i$, respectively) <u>a space of pressures</u>, $\mathcal{P}$, is required. The functions in $\mathcal{P}$ are required to be square-integrable (i.e., "L^2-functions," cf. Appendix I, Chapter 1). Because there are no explicit boundary conditions on the pressures, $\mathcal{P}$ suffices as both a trial solution space and a weighting function space.

The weak formulation may then be stated as:

(W) Given f_i, g_i and h_i, as before, find $u_i \in \mathcal{S}_i$ and $p \in \mathcal{P}$, such that for all $w_i \in \mathcal{V}_i$ and $q \in \mathcal{P}$ (pressure weighting function)

$$\int_\Omega w_{(i,j)}\sigma_{ij}\, d\Omega - \int_\Omega q(u_{i,i} + p/\lambda)\, d\Omega =$$

$$\int_\Omega w_i f_i\, d\Omega + \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i h_i\, d\Gamma \tag{11}$$

where σ_{ij} is given by (1).

To see what equations are implied by satisfaction of (11), we need to integrate (11) by parts (we assume all functions are smooth):

$$0 = \int_\Omega w_i(\underbrace{\sigma_{ij,j} + f_i}_{\text{equilibrium}})d\Omega$$

$$+ \int_\Omega q(\underbrace{u_{i,i} + p/\lambda}_{\text{eq. (8)}})d\Omega$$

$$+ \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i(\underbrace{h_i - \sigma_{ij}n_j}_{\text{traction boundary condition}})d\Gamma \qquad (12)$$

The usual arguments enable us to establish that the terms in parentheses in (12) vanish identically on their respective domains of definition (see Chapter 2, §6, for detailed arguments of this type).

The variational equation (11) may be written in the abstract form:

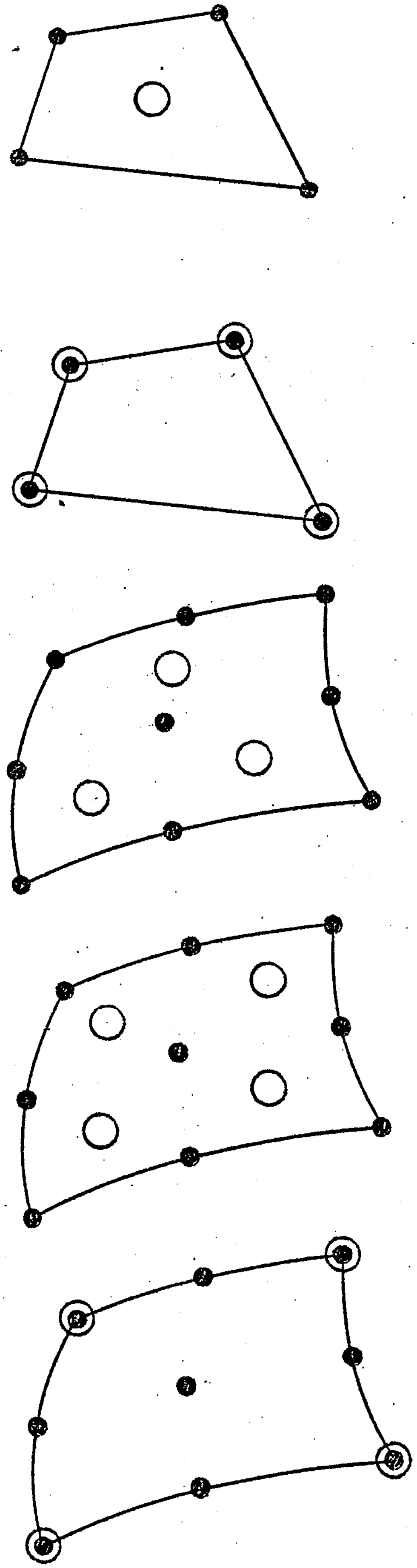
$$\bar{a}(\tilde{w}, \tilde{u}) - (\text{div}\, \tilde{w}, p) - (q, \text{div}\, \tilde{u} + p/\lambda)$$
$$= (\tilde{w}, \tilde{f}) + (\tilde{w}, \tilde{h})_\Gamma \qquad (13)$$

where $\bar{a}(\cdot, \cdot)$ is the symmetric bilinear form defined by

$$\bar{a}(\tilde{w}, \tilde{u}) = \int_\Omega w_{(i,j)} \bar{c}_{ijk\ell}\, u_{(k,\ell)} d\Omega \qquad (14)$$

in which

$$\bar{c}_{ijk\ell} = \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \qquad (15)$$



continuous bilinear displacements and discontinuous constant pressure

continuous bilinear displacements and pressure

continuous biquadratic displacements and discontinuous linear pressure

continuous biquadratic displacements and discontinuous bilinear pressure

continuous biquadratic displacements and continuous bilinear pressure

Key: ● displacements

○ pressure

Figure 1. Examples of possible displacement and pressure interpolation in two dimensions. Warning: Not all are effective in practice.

Exercises

1. Verify that (11) and (13) are equivalent.

2. Show that $\overline{a}(\cdot, \cdot)$ satisfies the definition of a symmetric bilinear form (see Chapter 1, §3, eqs. 11 and 13).

3. Show that if $\mu > 0$, then

$$\overline{c}_{ijk\ell} \, \Psi_{ij} \, \Psi_{k\ell} \geq 0 \tag{16}$$

for all symmetric Ψ_{ij} and furthermore

$$\overline{c}_{ijk\ell} \, \Psi_{ij} \, \Psi_{k\ell} = 0 \tag{17}$$

if and only if $\Psi_{ij} = 0$ (positive definiteness, see Chapter 2, §6, eqs. 5 and 6).

§3.3 Galerkin formulation

Recall that $\underset{\sim}{\mathcal{S}}^h$ and $\underset{\sim}{\mathcal{V}}^h$ are the finite-dimensional approximations to $\underset{\sim}{\mathcal{S}}$ and $\underset{\sim}{\mathcal{V}}$, respectively. Likewise, let $\mathcal{P}^h$ be the finite-dimensional space which approximates $\mathcal{P}$. The Galerkin formulation may then be stated as:

(G) Given $\underset{\sim}{f}$, $\underset{\sim}{g}$ and $\underset{\sim}{h}$, as in (W), find $\underset{\sim}{u}^h = \underset{\sim}{v}^h + \underset{\sim}{g}^h \in \underset{\sim}{\mathcal{S}}^h$ and $p^h \in \mathcal{P}^h$ such that for all $\underset{\sim}{w}^h \in \underset{\sim}{\mathcal{V}}^h$ and $q^h \in \mathcal{P}^h$

$$\bar{a}(\underset{\sim}{w}^h, \underset{\sim}{v}^h) - (\text{div}\, \underset{\sim}{w}^h, p^h) - (q^h, \text{div}\, \underset{\sim}{v}^h + p^h/\lambda) = (\underset{\sim}{w}^h, \underset{\sim}{f})_\Gamma$$

$$+ (\underset{\sim}{w}^h, \underset{\sim}{h})_\Gamma - \bar{a}(\underset{\sim}{w}^h, \underset{\sim}{g}^h) + (q^h, \text{div}\, \underset{\sim}{g}^h) \qquad (18)$$

§3.4 Matrix Problem

In order to develop the matrix form of the problem we need to introduce interpolatory expansions for p^h. Since p^h just needs to be square-integrable, it may be discontinuous across element boundaries. (Note that no derivatives of p^h, or q^h, appear in the variational equation; see eq. 15.) Thus a wider range of interpolations are permissible for pressure than for displacements. Possible combinations are illustrated in Figure 1. It must however be emphasized that arbitrary combinations of interpolations may lead to poor numerical performance and even nonconvergence. It is difficult to give intuitive guidelines because seemingly "natural" combinations may fail in practice. The subject of appropriate combinations is dealt with in §5.

Let us denote the pressure interpolation by

$$p^h(\underset{\sim}{x}) = \sum_{\tilde{A} \in \tilde{\eta}} \tilde{N}_{\tilde{A}}(\underset{\sim}{x}) p_{\tilde{A}} \tag{19}$$

where $\tilde{\eta}$ is the set of pressure node numbers, $\tilde{N}_{\tilde{A}}$ is the pressure shape function associated with pressure node number $\tilde{A}$, and $p_{\tilde{A}}$ is the value of pressure at node number $\tilde{A}$.

Similarly, the pressure weighting function may be expressed as

$$q^h(\underset{\sim}{x}) = \sum_{\tilde{A} \in \tilde{\eta}} \tilde{N}_{\tilde{A}}(\underset{\sim}{x}) q_{\tilde{A}} \tag{20}$$

Substitution of (19) and (20), along with the expressions for $\underset{\sim}{v}^h$, $\underset{\sim}{w}^h$ and $\underset{\sim}{g}^h$ (see eqs. 4-10, §7, Chapter 2), into (18) lead to the global matrix equation which can be written in the following partitioned form:

<u>segregated $\underset{\sim}{d}$, $\underset{\sim}{p}$-form of the matrix equation</u>

$$\begin{bmatrix} \underset{\sim}{\overline{K}} & \underset{\sim}{G} \\ \underset{\sim}{G}^T & \underset{\sim}{M} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{d} \\ \underset{\sim}{p} \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{\overline{F}} \\ \underset{\sim}{H} \end{Bmatrix} \tag{21}$$

The arrays in (21) and corresponding terms in the variational equations are identified in Table 1.

Table 1

Global array	Term in Galerkin equation from which global array emanates
$\bar{\mathbf{K}}$	$\bar{a}(\mathbf{w}^h, \mathbf{v}^h)$
$\mathbf{G}$	$-(\mathrm{div}\ \mathbf{w}^h, p^h)$
$\mathbf{G}^T$	$-(q^h, \mathrm{div}\ \mathbf{v}^h)$
$\mathbf{M}$	$-(q^h, p^h/\lambda)$
$\bar{\mathbf{F}}$	$(\mathbf{w}^h, \boldsymbol{f}) + (\mathbf{w}^h, \boldsymbol{h})_\Gamma - \bar{a}(\mathbf{w}^h, \boldsymbol{g}^h)$
$\mathbf{H}$	$(q^h, \mathrm{div}\ \boldsymbol{g}^h)$

The matrix $\mathbf{G}$ is the discrete gradient operator, $\mathbf{G}^T$ is the discrete divergence operator, $\bar{\mathbf{K}}$ and $\mathbf{M}$ are both symmetric, $\bar{\mathbf{K}}$ is positive definite, whereas $\mathbf{M}$ is negative definite, except in the case $\nu = ½$ in which $\mathbf{M} = \mathbf{0}$.

There are several possible ways of solving equation (21). Some of these procedures are described below:

<u>Procedure 1</u>

If $\mathbf{M} \neq \mathbf{0}$ (compressible case) then $\mathbf{p}$ can be eliminated by expanding (21),

$$\bar{\mathbf{K}}\,\mathbf{d} + \mathbf{G}\,\mathbf{p} = \bar{\mathbf{F}} \tag{22}$$

$$\mathbf{G}^T\mathbf{d} + \mathbf{M}\,\mathbf{p} = \mathbf{H}\ , \tag{23}$$

solving (23) for $\mathbf{p}$ and substituting into (22):

$$\underbrace{(\bar{\underset{\sim}{K}} - \underset{\sim}{G}\,\underset{\sim}{M}^{-1}\,\underset{\sim}{G}^T)}_{\underset{\sim}{K}}\underset{\sim}{d} = \underbrace{\bar{\underset{\sim}{F}} - \underset{\sim}{G}\,\underset{\sim}{M}^{-1}\,\underset{\sim}{H}}_{\underset{\sim}{F}} \qquad (25)$$

Equation (25) has the usual format. Observe that $\underset{\sim}{K}$ is symmetric and positive definite. After solving (25) for $\underset{\sim}{d}$, $\underset{\sim}{p}$ may be calculated from (24).

Procedure 2

If $\underset{\sim}{M} = \underset{\sim}{0}$ (incompressible case) the preceding elimination cannot be performed. However, the following procedure may be employed. Solve (22) for $\underset{\sim}{d}$, premultiply by $\underset{\sim}{G}^T$, and employ (23) to obtain the discrete Poisson equation for pressure:

$$\underbrace{(\underset{\sim}{G}^T\,\bar{\underset{\sim}{K}}^{-1}\,\underset{\sim}{G})}_{\hat{\underset{\sim}{K}}}\underset{\sim}{p} = \underset{\sim}{G}^T\underbrace{(\bar{\underset{\sim}{K}}^{-1}\,\bar{\underset{\sim}{F}} - \underset{\sim}{H})}_{\hat{\underset{\sim}{F}}} \qquad (26)$$

After $\underset{\sim}{p}$ is obtained by solving (26), (22) may be used to obtain $\underset{\sim}{d}$.

Exercise 4.

Generalize (26) to the case in which $\underset{\sim}{M} \neq \underset{\sim}{0}$.

Procedure 3

Both of the above solution procedures are global in nature and are valid whether p^h is continuous or discontinuous. If pressure is discontinuous between elements then, in the compressible case, the pressure degrees-of-freedom may be eliminated on the element level. The element arrays which correspond to the global arrays are notationally defined in Table 2.

Table 2

Global array	Corresponding element array
$\bar{K}$	$\bar{k}^e$
G	g^e
M	m^e
$\bar{F}$	$\bar{f}^e$
H	h^e

In this case we may write the global system in the usual way as an assembly of element arrays, that is

$$K\, d = F .$$

where

$$K = \mathop{\mathbf{A}}_{e=1}^{n_{e\ell}} (k^e) \tag{28}$$

$$F = \mathop{\mathbf{A}}_{e=1}^{n_{e\ell}} (f^e) \tag{29}$$

and

$$k^e = \bar{k}^e - g^e (m^e)^{-1} (g^e)^T \tag{30}$$

$$f^e = \bar{f}^e - g^e (m^e)^{-1} h^e \tag{31}$$

Note that (30) and (31) are the element analogs of the arrays which appear in (25). The above although equivalent to Procedure 1 is much more convenient from a practical standpoint in that only operations on the element level are involved. Likewise, the element vector of nodal pressures, $\tilde{p}^e$, can be calculated from the element nodal displacements once the latter are determined. This can be written as

$$\tilde{p}^e = -(\tilde{m}^e)^{-1} (\tilde{g}^e)^T \tilde{d}^e \tag{32}$$

where $\tilde{d}^e$ is the element displacement vector. Recall that $\tilde{d}^e$ includes specified displacement degrees-of-freedom (see Chapter 2, §8, eqs. 10-13). This is the reason why no $\tilde{h}^e$-term appears (cf. the global counterpart of (32), i.e., (23)). By consulting Table 1, the rationale behind (32) may be verified.

§3.5 Definition of Element Arrays

The components of the element arrays introduced in the previous section are given in this section:

$$\bar{\underset{\sim}{k}}^e = [\bar{k}^e_{pq}] \quad , \quad 1 \le p,\ q \le n_{ee} \tag{33}$$

$$\bar{k}^e_{pq} = \underset{\sim}{e}_i^T\ \bar{\underset{\sim}{k}}^e_{ab}\ \underset{\sim}{e}_j \quad , \quad 1 \le a,\ b \le n_{en} \tag{34}$$

$$p = n_{ed}(a - 1) + i \quad , \quad q = n_{ed}\ (b - 1) + j \tag{35}$$

$$\bar{\underset{\sim}{k}}^e_{ab} = \int_{\Omega^e} \underset{\sim}{B}_a^T\ \bar{\underset{\sim}{D}}\ \underset{\sim}{B}_b\ d\Omega \tag{36}$$

$$\bar{\underset{\sim}{f}}^e = \{\bar{f}^e_p\} \tag{37}$$

$$\bar{\underset{\sim}{f}}^e_p = \int_{\Omega^e} N_a\ \ell_i\ d\Omega + \int_{\Gamma_{h_i}} N_a\ h_i\ d\Gamma - \sum_{q=1}^{n_{ee}} \bar{k}^e_{pq}\ g^e_q \tag{38}$$

The preceding formulas are close analogs of those given for elasticity in Chapter 2. The only difference involves the appearance of the matrix $\bar{\underset{\sim}{D}}$ instead of $\underset{\sim}{D}$. From (15), it may be concluded that $\bar{\underset{\sim}{D}}$ is the part of $\underset{\sim}{D}$ in which λ-terms are omitted. Therefore we have (see e.g., eq. 34 of §6, Chapter 2):

three dimensions

$$\bar{\underset{\sim}{D}} = \mu \begin{bmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \tag{39}$$

plane strain

$$\bar{\underset{\sim}{D}} = \mu \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix} \tag{40}$$

axisymmetry

$$\bar{\underset{\sim}{D}} = \mu \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 1 & \\ & & & 2 \end{bmatrix} \tag{41}$$

All omitted terms in (39)-(41) are zero.

In (33)-(38) the indexing pertains to displacement degree-of-freedom only. We assume that the element in question possesses $\tilde{n}_{en}$ pressure nodes and that $1 \leq \tilde{a}, \tilde{b} \leq \tilde{n}_{en}$, where $\tilde{a}$ and $\tilde{b}$ are element pressure node numbers. With these we may write

$$\underset{\sim}{m}^e = [m^e_{\tilde{a}\tilde{b}}] \tag{42}$$

$$m^e_{\tilde{a}\tilde{b}} = \int_{\Omega^e} \frac{1}{\lambda} \tilde{N}_{\tilde{a}} \tilde{N}_{\tilde{b}} \, d\Omega \tag{43}$$

$$\underset{\sim}{g}^e = [g^e_{p\tilde{a}}] \tag{44}$$

$$g^e_{p\tilde{a}} = - \int_{\Omega^e} \text{div} \, (N_a \underset{\sim}{e}_i) \, \tilde{N}_{\tilde{a}} \, d\Omega \tag{45}$$

$$\underset{\sim}{h}^e = \{h^e_{\tilde{a}}\} \tag{46}$$

$$h^e_{\tilde{a}} = - \sum_{p=1}^{n_{ee}} g^e_{p\tilde{a}} \mathscr{g}^e_p \tag{47}$$

In (12), $\operatorname{div}(N_a \underset{\sim}{e}_i)$ is given by:

<u>three dimensions and plane strain</u>

$$\operatorname{div}(N_a \underset{\sim}{e}_i) = N_{a,i} \tag{48}$$

<u>axisymmetry</u> (see §11, Chapter 2)

$$\operatorname{div}(N_a \underset{\sim}{e}_i) = \begin{cases} N_{a,1} + N_a/r & i = 1 \\ N_{a,2} & i = 2 \end{cases} \tag{49}$$

The stress "vector" in an element may be computed from the formula (cf. eq. 14, §8, Chapter 2):

$$\underset{\sim}{\sigma}(\underset{\sim}{x}) = -\left(\sum_{\tilde{a}=1}^{\tilde{n}_{en}} \tilde{N}_{\tilde{a}}(\underset{\sim}{x}) p^e_{\tilde{a}}\right)\underset{\sim}{V} + \underset{\sim}{\bar{D}}(\underset{\sim}{x}) \sum_{a=1}^{n_{en}} \underset{\sim}{B}_a(\underset{\sim}{x}) \underset{\sim}{d}^e_a \tag{50}$$

where $\underset{\sim}{V}$ is defined by

$$\underset{\sim}{V} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{three dimensions} \tag{51}$$

$$\underset{\sim}{V} = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \quad \text{plane strain} \tag{52}$$

$$\underset{\sim}{V} = \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{Bmatrix} \qquad \underline{\text{axisymmetry}} \tag{53}$$

Remark

In practical computing, the segregated $\underset{\sim}{d}$, $\underset{\sim}{p}$-form of the global matrix equation, (21), is rarely employed. The reason for this is that the band-profile structure of the coefficient matrix is lost unless the displacement and pressure degrees-of-freedom associated with an element are grouped together in the overall equation number ordering. The following exercises are useful in understanding the data processing aspects of the preceding formulation.

Exercises

5. Consider the mesh shown in Figure 1 of Chapter 2, §9. Assume that the present mixed formulation of elasticity is being employed and that both displacement and pressure are interpolated in continuous bilinear fashion over each element. Thus there are three degrees-of-freedom per node, less displacement boundary conditions. Assume the pressure degree-of-freedom at a node directly follows the displacement degrees-of-freedom. Set up the ID, IEN and LM arrays.

Sketch the band-profile structure of the global coefficient matrix. Calculate the half-band width (see Figure 1, §8, Chapter 1). Reorder the rows and columns of the coefficient matrix so that the pressure degrees-of-freedom come last. This ordering puts the coefficient matrix into the segregated $\underset{\sim}{d}$, $\underset{\sim}{p}$-form, equation (21). Calculate the new half-band width and compare the result with the previous ordering.

6. Repeat the preceding exercise, but assume that pressure is piecewise constant on each element. Again assume three degrees-of-freedom per node and associate the single element pressure degree-of-freedom with the last node of the element in the local ordering. The third degree-of-freedom at each of the first three element nodes is a dummy degree-of-freedom and may be eliminated as if it were "prescribed" (i.e., set a zero in the appropriate position of the ID array).

Remark

It is important to realize that the matrix equation of the present formulation is somewhat different than the form considered heretofore. The present coefficient matrix is symmetric, but not positive definite. It possesses both positive and negative eigenvalues. In fact, in the incompressible case, improper interpolatory combinations may also lead to spurious zero eigenvalues in which case the coefficient matrix is rendered singular and solution is impossible. These are frequently referred to as "pressure modes" in the literature. (Equal-order interpolations, such as in Exercise 5, generally create this pathology. The interpolations of Exercise 6 do too under certain circumstances! See §5 for elaboration.)

In well-set cases, in which there are no spurious pressure modes, typical symmetric band-profile equation solvers are also capable of solving systems of the present type. However, some precautions must be taken. For example, an equation with a zero diagonal element must not appear first in the global ordering. Because the global ordering is arbitrary this can always be accomplished. Due to the fact that some eigenvalues will be negative, equation solving techniques which take square roots are inapplicable (e.g. the Cholesky decomposition). Alternatives which do not take square roots, such as the Crout algorithm, are however acceptable (see R. L. Taylor's chapter on computing in [4]).

§3.6 Illustration of a Fundamental Difficulty

We have already given some forewarning that arbitrary combinations of displacement and pressure interpolations may prove ineffective in incompressible cases. An example of one of the difficulties is illustrated by the mesh in Figure 2. Suppose linear displacement--constant pressure triangular elements are being employed. Furthermore, assume the left-hand side and bottom edges of the mesh are fixed (i.e., the displacements are identically zero).

It may be concluded from the Galerkin equation, (18), that the condition of incompressibility is

$$(q^h, \operatorname{div} \underset{\sim}{u}^h) = 0 \tag{54}$$

Because

$$(q^h, \operatorname{div} \underset{\sim}{u}^h) = \sum_{e=1}^{n_{e\ell}} \int_{\Omega^e} q^h \operatorname{div} \underset{\sim}{u}^h \, d\Omega \tag{55}$$

and q^h is an arbitrary constant on each triangle, we infer from (54) and (55) that incompressibility is satisfied in the mean, that is

$$\int_{\Omega^e} \operatorname{div} \underset{\sim}{u}^h \, d\Omega = 0 \quad , \quad 1 \leq e \leq n_{e\ell} \tag{56}$$

Thus the area of each triangle must necessarily remain constant. (Due to the fact that $\underset{\sim}{u}^h$ is a linear polynomial over each triangle and thus $\operatorname{div} \underset{\sim}{u}^h$ is constant, (56) actually implies the stronger point-wise condition

$$\operatorname{div} \underset{\sim}{u}^h = 0 \tag{57}$$

on each Ω^e. However, this result is not needed to illustrate the present

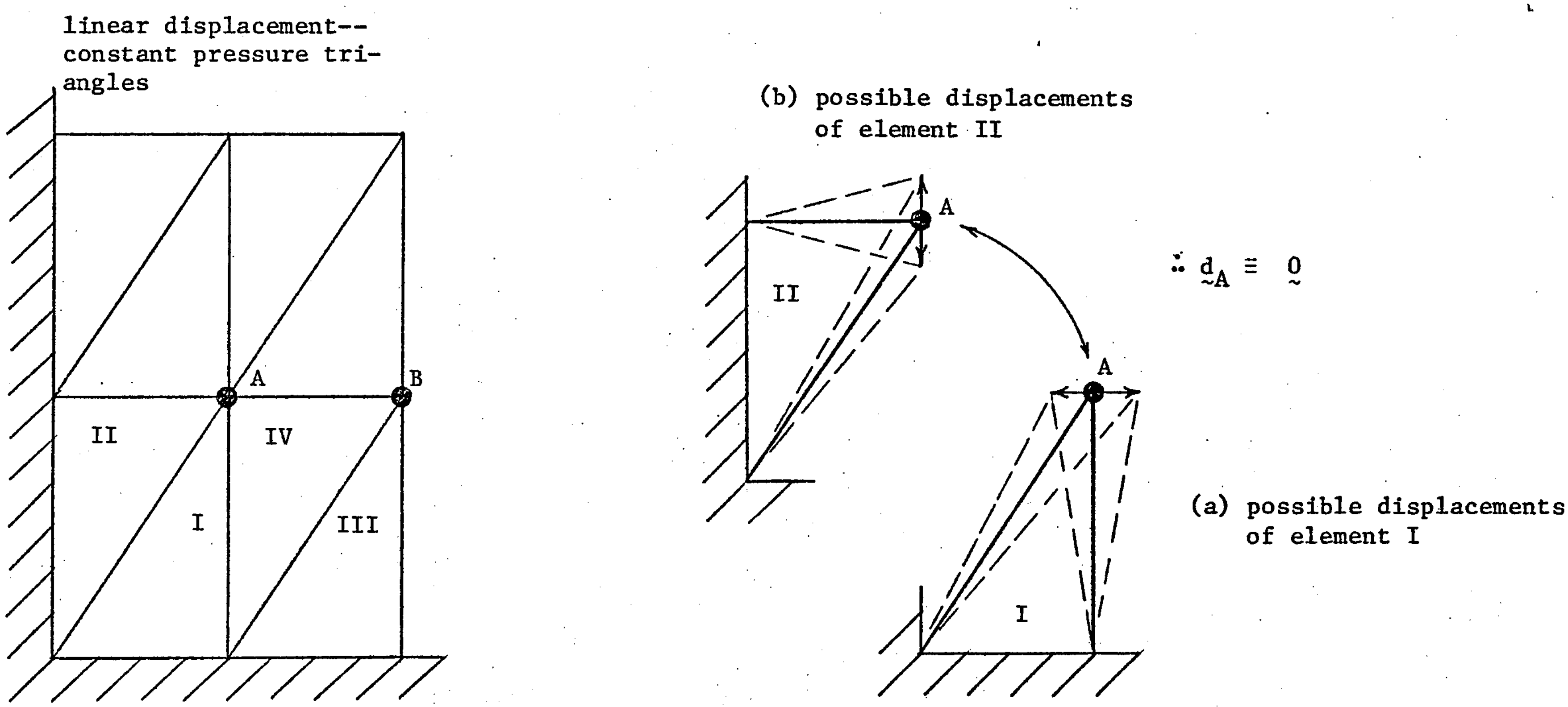


Figure 2. Mesh for which incompressibility dictates zero displacements.

difficulty.)

Let us examine what conditions equation (55) enforces on the kinematics of the mesh in Figure 2. Consider element I. We see from Figure 2a that constant volume prevents the displacement at node A , $\underset{\sim}{d}_A$, from having a nonzero vertical component. Now consider element II. For this element the constant volume condition precludes horizontal motion of node A (see Figure 2b). Taken together, $\underset{\sim}{d}_A$ must be identically zero. We can now repeat the argument for elements III and IV to conclude $\underset{\sim}{d}_B$ must also be zero. In fact, identical reasoning may be used to conclude that every node in the entire mesh must have zero displacement. Thus the only possible incompressible displacement is $\underset{\sim}{u}^h \equiv \underset{\sim}{0}$. This result holds no matter how many elements are present in each direction. Clearly, this type of mesh offers no approximation power whatsoever. This phenomenon is often referred to as mesh "locking". It is but one of the difficulties afflicting problems of incompressibility.

In the nearly-incompressible case, the same phenomenon occurs only this time $\underset{\sim}{u}^h \approx \underset{\sim}{0}$. Thus introducing slight incompressibility does not make the problem go away. To varying degrees, a tendency to lock afflicts many standard elements.

It is desirable to have a simple procedure for assessing whether or not an element will lock. For this purpose the method of constraint counting proves quite effective [5-7].

<u>Constraint counts</u>

This method is an heuristic approach for determining the ability of an element to perform well in incompressible and nearly incompressible applications. It should be emphasized that this is not a precise mathematical method for assessing elements, but rather a quick and simple tool for obtaining an indication of element potential. However, it does seem to be able to predict a propensity for locking. There are, of course, other issues which need to be considered in an overall evaluation of element performance.

Let us introduce a standard mesh which for two-dimensional problems is illustrated in Figure 3. Let n_{eq} represent the total number of displacement equations after boundary conditions have been imposed (i.e. the length of the vector $\underset{\sim}{d}$ in eq. 21) and let n_c represent the total number of incompressibility constraints. As long as the pressure equations are linearly independent, n_c will equal $\tilde{n}_{eq}$, the number of pressure equations (i.e. length of the vector $\underset{\sim}{p}$ is eq. 21). We shall define the constraint ratio, r, by

$$r = n_{eq}/n_c \tag{58}$$

We are interested in values of r as the number of elements per side, n_{es}, approaches infinity. The conjecture is that r should mimic the behavior of the number of equilibrium equations divided by the number of incompressibility conditions for the governing system of partial differential equations. These are n_{sd}, the number of space dimensions, and 1, respectively. So in two dimensions, the ideal value of r would be 2. A value of r less than 2 would indicate a tendency to lock. If $r \leq 1$ there are more constraints on $\underset{\sim}{d}$ than there are displacement degrees-of-freedom available and thus severe locking would be anticipated, such as

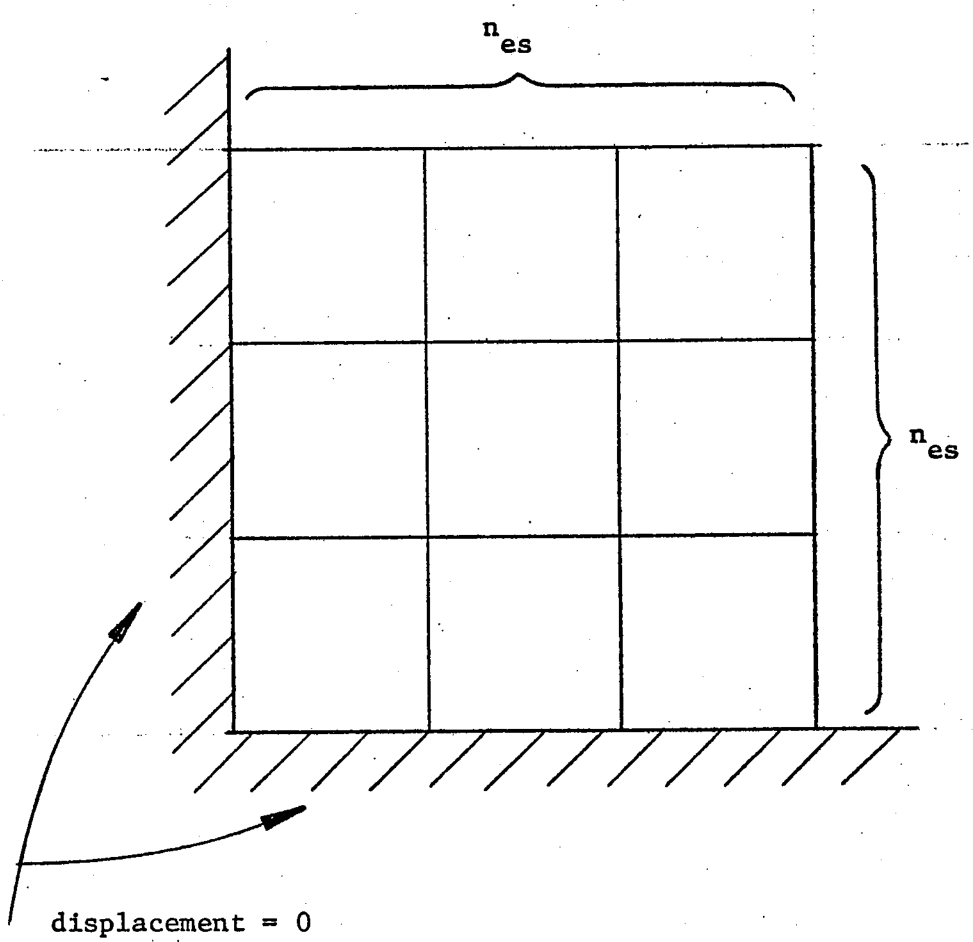


Figure 3. Standard mesh

was seen for the linear displacements—constant pressure triangle. A value of r much greater than 2 indicates that not enough incompressibility conditions are present, so the incompressibility condition may be poorly approximated in some problems. An imprecise summary of these ideas follow:

$r > 2$, too few incompressibility constraints

$r = 2$, optimal

$r < 2$, too many incompressibility constraints

$r \leq 1$, locking

We shall begin by calculating r for some two-dimensional elements in which pressure is discontinuous. In this case r is constant as a function of n_{es} so r may be determined by considering a single element ($n_{es} = 1$) . (The reader may wish to verify this statement for some of the cases considered).

Example 1

Consider the case of the bilinear displacements—constant pressure triangle. See Figure 4a. In this case

$$r = n_{eq}/n_c = n_{eq}/\tilde{n}_{eq} = 2/2 = 1$$

which is consistent with the behavior previously deduced. This element was first studied by Hughes and Allik [8].

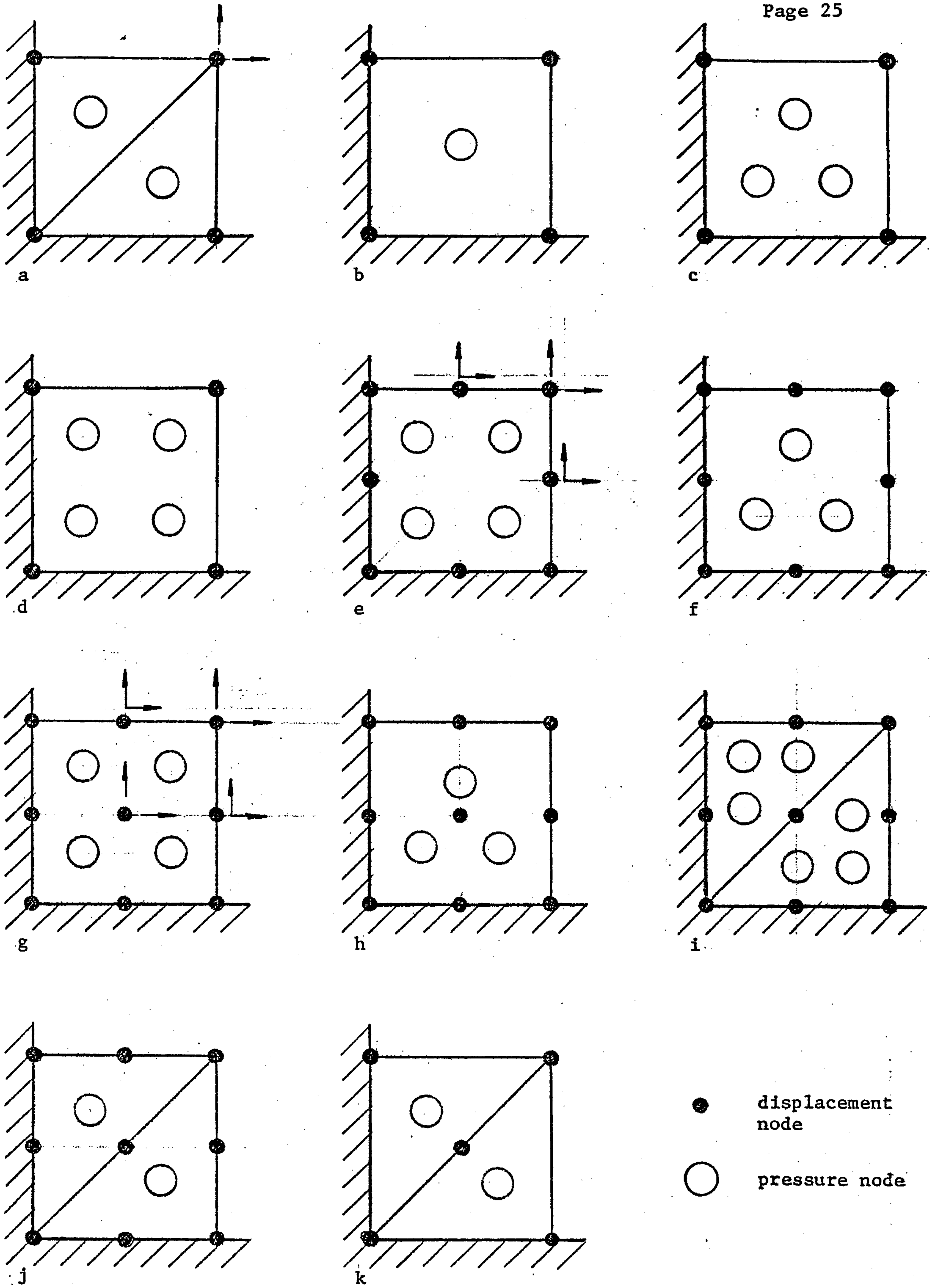


Figure 4. Discontinuous pressure-field elements.

Example 2

Consider the bilinear displacements--constant pressure element (see Figure 4b). For this element, incompressibility is achieved in the mean, that is

$$\int_{\Omega^e} \operatorname{div} \underset{\sim}{u}^h \, d\Omega = 0 \tag{59}$$

This may be shown by way of the same reasoning which lead to (56). In this case we have

$$r = n_{eq}/n_c = n_{eq}/\tilde{n}_{eq} = 2/1 = 2$$

Optimal behavior is indicated and this element is widely used. This element was first proposed by Hughes and Allik [8].

From Exercise 7 of Chapter 3, §11, we recall that the mean value of div $\underset{\sim}{u}^h$ occurs at the origin of the isoparametric coordinate system (i.e. $\xi = \eta = 0$). This is the only point in the element at which incompressibility is identically satisfied. The mean-value point shifts somewhat for the case of the axisymmetric version of this element.

Example 3

Consider the bilinear displacements--linear pressure element shown in Figure 4c. In this case,

$$r = n_{eq}/n_c = n_{eq}/\tilde{n}_{eq} = 2/3$$

which indicates severe locking.

Although no improvement could be expected by increasing the pressure interpolation to bilinear, we wish to consider this element (see Figure 4d) since it illustrates a point. In this case*

$$\int_{\Omega^e} \underbrace{q^h}_{\text{bilinear}} \underbrace{\operatorname{div} \underset{\sim}{u}^h}_{\text{linear}} d\Omega = 0 \tag{60}$$

Thus there are more pressure weighting functions, and consequently incompressibility conditions, than there are independent monomials in $\operatorname{div} \underset{\sim}{u}^h$. As a result the four incompressibility conditions are linearly dependent and so

$$r = n_{eq}/n_c = n_{eq}/(\tilde{n}_{eq} - 1) = 2/(4-1) = 2/3$$

as for the preceding case. Increasing the order of pressure interpolation further does not change r, but increases the order of singularity of the matrix system. Clearly an approximation of this kind is useless since the global matrix could not be inverted in the incompressible case.

Singularities of this type affect many elements in incompressible applications. It is not always obvious that this can occur for an element. For example, the bilinear displacements--constant pressure elements exhibits a singularity in the global pressure equations for the mesh shown in Figure 5a as long as n_{es} is even. This pathology is referred to as the "checkerboard mode" since the pressure degrees-of-freedom of the eigenvector of the

---

*For the standard mesh we can write $\xi = \xi(x)$ and $\eta = \eta(y)$ where each of these functions is linear. Thus a bilinear expansion in ξ and η can also be written as a bilinear expansion in x and y. For example, $u_i^h = \alpha_{0i} + \alpha_{1i}\,\xi + \alpha_{2i}\,\eta + \alpha_{3i}\,\xi\eta = \beta_{0i} + \beta_{1i}\,x + \beta_{2i}\,y + \beta_{3i}\,xy$ where the α's and β's are parameters which depend on the nodal values of u_i^h. The β's also depend on the lengths of the element edges. Clearly, $\operatorname{div} \underset{\sim}{u}^h = (\beta_{11} + \beta_{22}) + \beta_{32}\,x + \beta_{31}\,y$ which is identically zero pointwise if and only if $0 = \beta_{11} + \beta_{22} = \beta_{32} = \beta_{31}$, that is, there are three independent constraints on $\underset{\sim}{u}^h$.

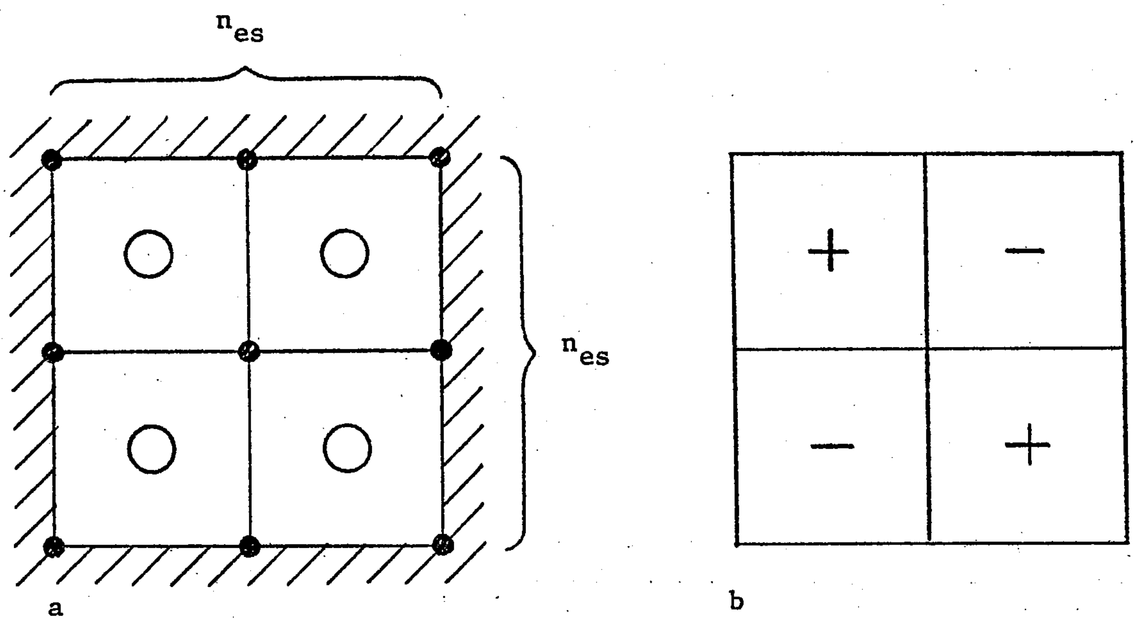


Figure 5. Checkerboard pressure mode

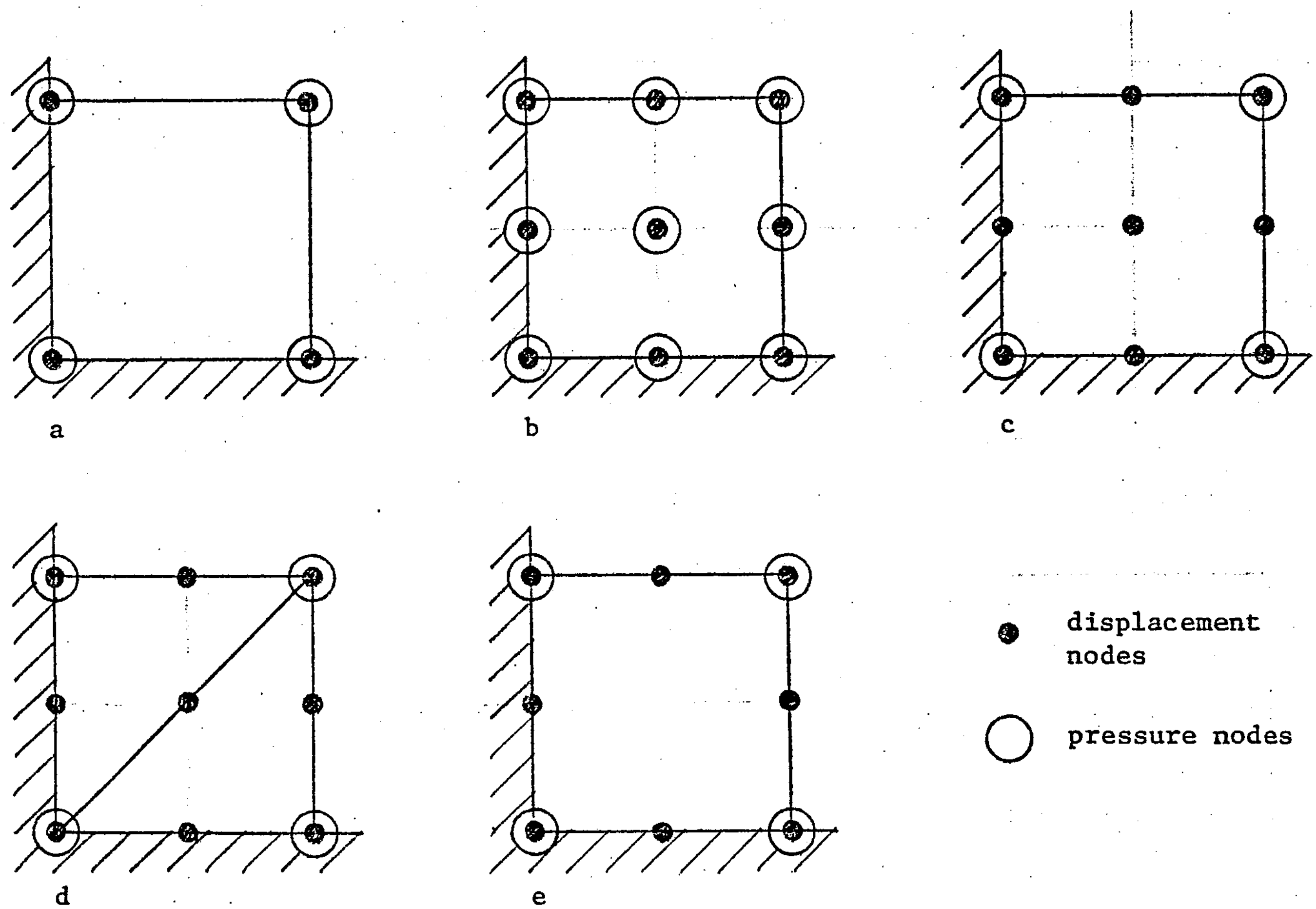


Figure 6. Continuous pressure - field elements.

global equations corresponding to the zero eigenvalue takes the form +1 on the red squares and -1 on the black squares (Figure 5b). When we come to the "penalty method", we will show that this mode can be removed resulting in an effective formulation for this element.

In a sense all elements are subject to the problem of singularity in the incompressible limit. This can easily be seen from (21). If enough displacement degree-of-freedom have been specified such that $n_{eq} < \tilde{n}_{eq}$, then the matrix <u>must</u> be singular at least to degree $\tilde{n}_{eq} - n_{eq}$.

<u>Example 4</u>

Consider the 8-node serendipity displacement--bilinear pressure quadrilateral shown in Figure 4e. A calculation of $\operatorname{div} \tilde{u}^h$ reveals that it can be expressed as a full quadratic polynomial in x and y and thus involves six independent coefficients which must vanish for incompressibility to be satisfied pointwise. For a bilinear pressure expansion, we have the incompressibility condition takes the form

$$\int_{\Omega^e} \underbrace{q^h}_{\text{bilinear}} \; \underbrace{\operatorname{div} \tilde{u}^h}_{\text{quadratic}} \, d\Omega = 0 \tag{61}$$

and thus four conditions emanate from (61). The constraint ratio is therefore

$$r = n_{eq}/n_c = n_{eq}/\tilde{n}_{eq} = 6/4 = 3/2$$

This indicates that their tends to be too many incompressibility constraints.

Some ostensible improvement can be made by reducing the pressure interpolation to linear, see Figure 4f. In this case

$$r = n_{eq}/\tilde{n}_{eq} = 6/3 = 2 \quad ,$$

the optimal value. However, neither of these elements (i.e. Figures 4e and f) is currently favored.

Example 5

Consider the biquadratic displacements--bilinear pressure quadrilateral element shown in Figure 4g for which

$$r = n_{eq}/n_c = n_{eq}/\tilde{n}_{eq} = 8/4 = 2 \quad .$$

Thus from a constraint ratio point of view this element appears ideal. However, it also gives rise to a pressure mode such as that for the bilinear displacements--constant pressure element.

Reducing the pressure interpolation to linear (Fig. 4h) removes this pressure mode and the constraint ratio increases to $r = 8/3 = 2^2/_3$. This element is currently felt to be one of the most effective elements for incompressible analysis.

Example 6

Consider the quadratic displacements--linear pressure triangle shown in Figure 4i. The constraint ratio is

$$r = n_{eq}/n_c = n_{eq}/\tilde{n}_{eq} = 8/6 = 4/3 \quad .$$

Thus this element possesses too many incompressibility constraints. (Observe that pointwise satisfaction of the incompressibility constraint is attained.)

To reduce the number of incompressibility constraints, constant pressure may be employed (see Fig. 4j) resulting in incompressibility in the mean and a relatively high constraint ratio of $r = 8/2 = 4$. This element is nevertheless favored by some analysts [9] because it does not possess pressure modes of the kind described previously. The drawback, however, is the crude approximation of incompressibility (i.e. piecewise constant) relative to displacements (i.e. piecewise quadratic) which results in sub-optimal convergence. (For optimal rate-of-convergence, the complete polynomial in the pressure field should be one order lower than the complete polynomial in the displacements.)

A more balanced approximation employing constant pressure is shown in Figure 4k [10]. Each quadrilateral "macro-element" is composed of linear displacement--constant pressure triangles in which quadratic displacement modes are added along the diagonal edge. The constraint ratio is $r = 4/2 = 2$, which is optimal. Additionally, this element exhibits no spurious pressure modes.

The remaining examples involve continuous pressure fields. The first to study continous pressure-field elements were Hughes and Allik [8]. In these cases r varies with n_{es} . However, it may be argued that

$$\lim_{n_{es} \to \infty} r$$

may be obtained by again considering the single corner element of the standard mesh and ignoring all degrees-of-freedom--pressure in addition to displacement--on the left and bottom boundaries. (The reader may wish to verify this statement as an exercise for some of the following elements.)

Example 7

Consider the bilinear displacements-(continuous!) bilinear pressure quadrilateral shown in Figure 6a. The constraint ratio is

$$\lim_{n_{es} \to \infty} r = 2$$

which is optimal. However, the convergence is from below and this element may exhibit spurious pressure modes. This appears to be a general fact for typical elements possessing identical displacement and pressure interpolations (see e.g. Fig. 6b).

Example 8

The deficiencies noted in the previous example can be corrected by lowering the pressure interpolation. Figures 6c and d depict elements of this type. For both these elements

$$\lim_{n_{es} \to \infty} r = 8$$

which is very high (i.e. there are too few incompressibility constraints). Although no pressure modes are exhibited by these elements, and the convergence rate is theoretically optimal, very poor approximations of the incomressibility condition are frequently noted. Nevertheless these elements are widely used in incompressible analysis.

The constraint ratio can be reduced somewhat by removing internal displacement degrees-of-freedom as, for example, in Figure 6e. For this element,

$$\lim_{n_{es} \to \infty} r = 6 \quad ,$$

which is still rather high. This element has also been widely used in incompressible analysis.

Remark

It is an empirical observation that all elements which are "effective" in incompressible analysis possess constraint ratios greater than or equal to n_{sd}. We wish to reiterate, however, that these elements may give rise to so-called "pressure modes" (i.e. singularities in the global equations). Thus the analysis of incompressible media is a delicate matter and care must be exercised. We consider the matter further in subsequent sections.

References

1. L.R. Herrmann, "Elasticity Equations for Nearly Incompressible Elasticity," AIAA J., Vol. 3, 1896-1900 (1965).

2. R.L. Taylor, K.S. Pister and L.R. Herrmann, "On a Variational Theorem for Incompressible and Nearly-incompressible Orthotropic Elasticity, Int. J. Solids Structures, Vol. 4, 875-883 (1968).

3. S.W. Key, "A Variational Principle for Incompressible and Nearly-incompressible Anisotropic Elasticity," Int. J. Solids Structures, Vol. 5, 951-964 (1969).

4. O.C. Zienkiewicz, The Finite Element Method, McGraw-Hill, London, 1977.

5. J.C. Nagtegaal, D.M. Parks and J.R. Rice, "On Numerically Accurate Finite Element Solutions in the Fully Plastic Range," Computer Methods in Applied Mechanics and Engineering, 4, 153-178 (1974).

6. J.H. Argyris, P.C. Dunne, T. Angelopoulos and B. Bichat, "Large Natural Strains and Some Special Difficulties Due to Nonlinearity and Incompressibility in Finite Elements," Computer Methods in Applied Mechanics and Engineering, 4, 219-278 (1974).

7. D.S. Malkus and T.J.R. Hughes, "Mixed Finite Element Methods--Reduced and Selective Integration Techniques: A Unification of Concepts," Computer Methods in Applied Mechanics and Engineering, Vol. 15, No.1, 68-81 (1978).

8. T.J.R. Hughes and H. Allik, "Finite Elements for Compressible and Incompressible Continua," Proc. Symp. Civil Eng., Vanderbilt University, Nashville, Tenn., 27-62 (1969).

9. Thompson

10. R. Glowinski

§4 Penalty Formulation: Reduced and Selective Integration Techniques; Equivalence with Mixed Methods

Let us recall the formulations we introduced previously for modelling a single constraint (§2). These were the Lagrange multiplier method and the penalty function formulation. In the context of the incompressibility constraint, the mixed formulation of §3 is a Lagrange multiplier type of formulation in which the pressure field plays the role of the Lagrange multiplier.

We also recall from §2 that in the penalty formulation we simply approximated the single constraint by introducing a stiff elastic spring. In the context of incompressibility this amounts to allowing for slight compressibility. That is, λ is taken finite, but large with respect to μ:

nearly-incompressible case

$$\lambda/\mu \gg 1 \tag{1}$$

This can be easily done within the framework of §3 and has some advantages. For example, in the case of discontinuous pressure fields, the pressure degrees-of-freedom can be eliminated on the element level. The idea is to select λ sufficiently large so that compressibility errors are negligible, but not so large that numerical problems arise. The ratio λ/μ thus depends on the floating-point word length of the computer being utilized. In our experience with words of length 60-64 bits, we have found that the range

$$10^7 \leq \lambda/\mu \leq 10^9 \tag{2}$$

is effective. As noted in §3, allowing for slight compressibility does not change the situation with regard to elements. One must employ only those elements which are also effective in the incompressible limit as slight compressibility does not eliminate the fundamental difficulties.

Once we are willing to introduce some compressibility, the theory of Chapter 2 is also applicable. The question naturally arises then as to the performance of the standard "displacement - only" elements of Chapter 3 in these circumstances.* The answer, as might be anticipated, is that these elements tend to perform poorly in nearly - incompressible applications, frequently exhibiting a tendency to "lock". However, a slight modification of the usual formulation enables the construction of elements which are identical to many of the discontinuous pressure field elements generated by the mixed formulation. The basic tools in this process are reduced and selective integration procedures which are described as follows:

Reduced and selective integration

The expression for element stiffness in the displacement formulation is given by (we omit the element number superscript, e , for simplicity):

$$\underset{\sim}{k} = [k_{pq}] \quad , \qquad 1 \leq p \; , \; q \leq n_{ee} \tag{3}$$

$$k_{pq} = \underset{\sim}{e}_i^T \, \underset{\sim}{k}_{ab} \, \underset{\sim}{e}_j \tag{4}$$

$$p = n_{ed}(a - 1) + i \quad , \qquad q = n_{ed}(b - 1) + j \tag{5}$$

$$\underset{\sim}{k}_{ab} = \int_{\Omega^e} \underset{\sim}{B}_a^T \, \underset{\sim}{D} \, \underset{\sim}{B}_b \, d\Omega \tag{6}$$

*The finite element methodology of Chapters 2 and 3 is generally referred to as the "displacement formulation".

The material properties matrix $\underset{\sim}{D}$ may be written as

$$\underset{\sim}{D} = \underset{\sim}{\overline{D}} + \underset{\sim}{\overline{\overline{D}}} \qquad (7)$$

where $\underset{\sim}{\overline{D}}$ is the "μ - part" of $\underset{\sim}{D}$, defined in §3, and $\underset{\sim}{\overline{\overline{D}}}$, the remainder, is the "λ - part". The reader may easily verify the following formulas:

plane strain

$$\underset{\sim}{\overline{\overline{D}}} = \lambda \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ \text{symm.} & & 0 \end{bmatrix} \qquad (8)$$

axisymmetry

$$\underset{\sim}{\overline{\overline{D}}} = \lambda \begin{bmatrix} 1 & 1 & 0 & 1 \\ & 1 & 0 & 1 \\ \text{symm.} & & 0 & 0 \\ & & & 1 \end{bmatrix} \qquad (9)$$

three dimensions

$$\underset{\sim}{\overline{\overline{D}}} = \lambda \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \text{symm.} & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \qquad (10)$$

Employing (7) in (6) enables us to write

$$\underset{\sim}{k}_{ab} = \overline{\underset{\sim}{k}}_{ab} + \overline{\overline{k}}_{ab} \tag{11}$$

where

$$\overline{\underset{\sim}{k}}_{ab} = \int_{\Omega^e} \underset{\sim}{B}_a^T \overline{\underset{\sim}{D}} \underset{\sim}{B}_b \, d\Omega \tag{12}$$

$$\overline{\overline{\underset{\sim}{k}}}_{ab} = \int_{\Omega^e} \underset{\sim}{B}_a^T \overline{\overline{\underset{\sim}{D}}} B_b \, d\Omega \tag{13}$$

Note that $\overline{\underset{\sim}{k}}_{ab}$ is the part of the stiffness which also appears in the mixed formulation (see §3). Due to the fact that $\lambda/\mu \gg 1$, and $\overline{\overline{\underset{\sim}{k}}}$ is proportional to λ whereas $\overline{\underset{\sim}{k}}$ is proportional to μ, the numerical value of terms in $\overline{\overline{\underset{\sim}{k}}}$ tend to be very large compared with those of $\overline{\underset{\sim}{k}}$. The $\overline{\overline{\underset{\sim}{k}}}$ term is the part of the stiffness which attempts to maintain the volumetrically stiff behavior. Because typical finite elements tend to lock (i.e. there are proportionally too many incompressibility-type conditions), special treatment of $\overline{\overline{\underset{\sim}{k}}}$ is required to alleviate this tendency. One simple and practically important way of going about this is to reduce the order of numerical quadrature employed to evaluate $\overline{\overline{\underset{\sim}{k}}}$ below that which is "normally" used. The basic idea is illustrated in the following example:

Example 1

Consider the 4-node bilinear displacement element in plane strain. "Normal" quadrature for this element is considered to be the 2 × 2 Gauss rule. The stiffness matrix turns out to be identical to that obtained in the mixed formulation for bilinear displacements and (discontinuous) linear pressures in which the pressure degrees-of-freedom are eliminated on the element level as indicated in (30) of §3. This fact is known from an "equivalence" theorem due to Malkus and Hughes [1]. The constraint ratio of this element was calculated to be 2/3 in §3 and thus locking-type behavior would be anticipated.

Recall also from §3 that by employing constant pressure over each element, the bilinear element attains an optimal constraint ratio of 2. The equivalent displacement model [1], may be obtained by reducing the quadrature on the $\bar{\bar{\underset{\sim}{k}}}$-term to 1-point Gauss. Thus, as $\lambda/\mu \to \infty$, incompressibility in the mean is attained.

The performance of this element is illustrated by the following numerical example.

Consider the equations of two-dimensional linear isotropic elasticity theory on the domain illustrated in Figure 1. The boundary conditions are given as follows:

(displacement)

$$u_1(0,0) = u_2(0,0) = 0$$

$$u_1(0, \pm c) = 0$$

(traction)

$$h_1(x_1, \pm c) = h_2(x_1, \pm c) = 0 \quad , \quad x_1 \in]0,L[$$

$$\left.\begin{aligned} h_1(L,x_2) &= 0 \\ h_2(L,x_2) &= \frac{P}{2I}(c^2 - x_2^2) \end{aligned}\right\} \quad x_2 \in]-c,c[$$

$$\left.\begin{aligned} h_1(0,x_2) &= \frac{PLx_2}{I} \\ h_2(0,x_2) &= -\frac{P}{2I}(c^2 - x_2^2) \end{aligned}\right\} \quad x_2 \;]-c,0[\cup]0,c[$$

where P is a given constant and $I = 2c^3/3$.

The traction boundary conditions are those encountered in simple bending theory for a cantilever beam with root section at $x = 0$; that is, parabolically varying end shear and linearly varying bending stress at the root. The displacement boundary conditions allow the root section to warp.

The following data were employed in the numerical calculations:

$$L = 16 , \quad c = 2 .$$

The mesh used is depicted in Figure 2. (Only half the domain need be modelled since the x_1-axis is a line of antisymmetry.)

Plain strain conditions were assumed and 4-node quadrilateral elements were employed.

Vertical tip displacements [i.e., $u_2(16,0)$] are compared in Table 1. Both the "standard" 2 × 2 Gauss quadrature element and the selective/reduced element provide adequate results for $\nu = 0.3$. However, for the nearly-incompressible case, the standard quadrilateral degenerates, whereas the selective/reduced element retains accuracy.

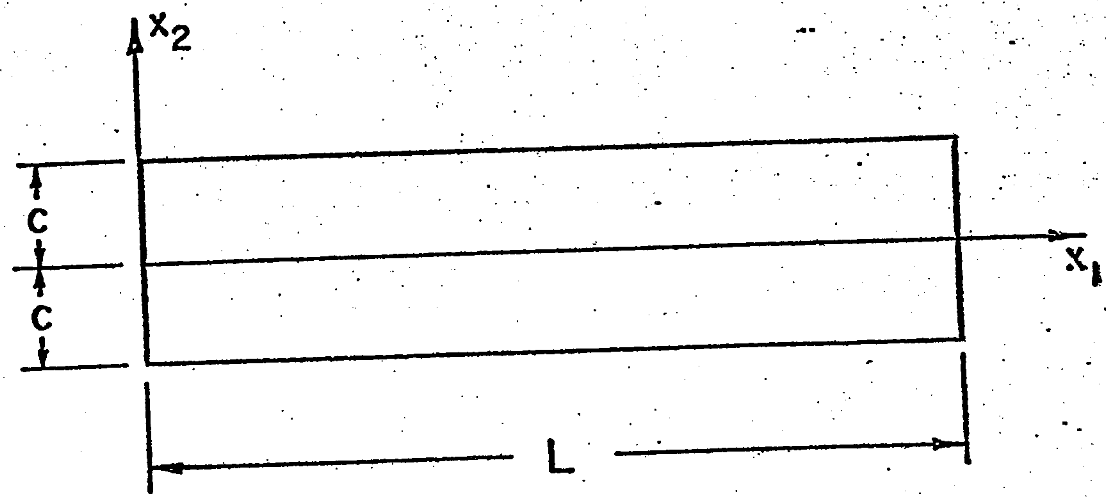


Figure 1. Domain for plane strain elasticity problem.

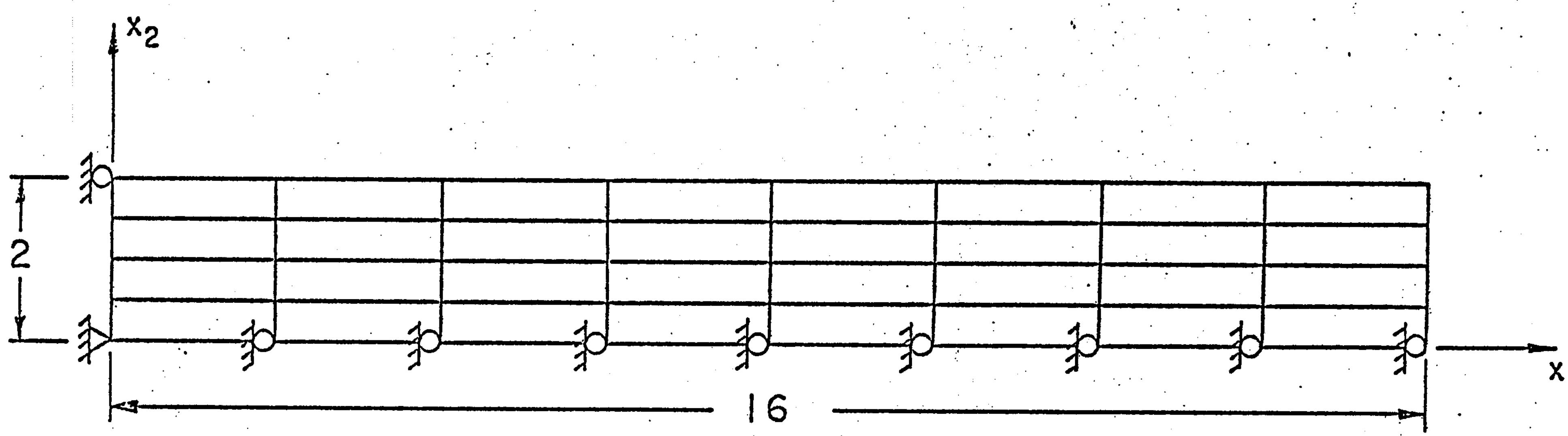


Figure 2. Element mesh and boundary conditions for plane strain elasticity problem.

Table 1

Normalized vertical tip displacements of
plain strain beam

ν	2 × 2 Gauss	1 × 1 Gauss volume term 2 × 2 Gauss remainder
.3	.904	.912
.499	.334	.937

Definitions

Selective reduced integration refers to the case in which reduced integration is used only on the λ-term, whereas normal integration is used on the μ-term. Uniform reduced integration refers to the case when both terms are integrated with a reduced rule.

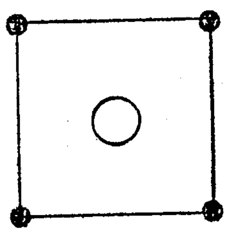
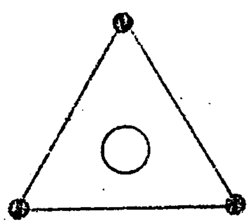
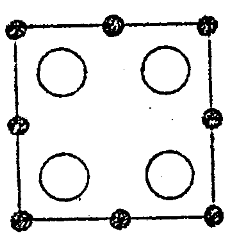
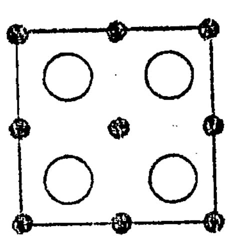
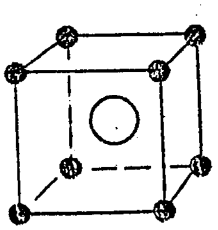
The ostensible advantage of uniform reduced integration is the economy of element formulation. The disadvantage is that the rank of the element stiffness may be reduced, resulting in singularity of the global matrix. Selective integration retains the correct rank of the element stiffness and therefore the global stiffness also possesses correct rank. This follows from the fact that $\bar{a}(\cdot, \cdot)$ is positive definite. (The $\bar{\bar{a}}(\cdot, \cdot)$ term may be entirely ignored without affecting rank.)

Equivalence theorem

The general theorem presented in Malkus and Hughes [1] established the equivalence of many mixed and reduced/selective integration elements. Some examples are presented in Figure 3. It may thus be concluded that the reduced/selective integration procedure is a very simple way of attaining the performance of the mixed formulation without engendering the additional complications.

The equivalence theorem states that the element stiffnesses, and consequently the displacements, are identical. At the Gauss points of the reduced integration rule,

$$p^h = -\lambda \operatorname{div} \underset{\sim}{u}^h \qquad (14)$$

	mixed element		selective integration element	
	displacement interpolation	pressure interpolation	normal Gaussian quadrature* $(\bar{\underset{\sim}{k}})$	reduced Gaussian quadrature $(\bar{\bar{\underset{\sim}{k}}})$
	bilinear	constant	2 × 2	1 – point
	linear	constant	1 – point	1 – point
	serendipity	bilinear	3 × 3	2 × 2
	biquadratic	bilinear	3 × 3	2 × 2
	trilinear	constant	2 × 2 × 2	1 – point

*also used on all terms of mixed formulation

Figure 3. Some equivalent elements

agrees with the pressure field of the mixed method. Elsewhere in the element, the pressure may be determined from the displacements by interpolating the values of (14) with the pressure shape functions. The equivalence theorem thus provides the interpretation rule by which the pressure should be defined within the reduced/selective integration approach.

For example, in the case of the selectively integrated 4-node bilinear element, the constant element pressure of the mixed method agrees with the value computed from (14) at the origin of isoparametric coordinates (i.e., $\xi = \eta = 0$). As a second example, consider the selectively integrated 9-node element. In this case, the values of (14) computed at the 2 × 2 Gauss points need to be interpolated via bilinear shape functions. The resulting formula is

$$p^h(\xi, \eta) = \sum_{\tilde{a}=1}^{4} \tilde{N}_{\tilde{a}}(\xi, \eta) p_{\tilde{a}} \qquad (15)$$

where

$$p_{\tilde{a}} = -\lambda(\text{div } \underset{\sim}{u}^h)(\tilde{\xi}_{\tilde{a}}, \tilde{\eta}_{\tilde{a}}) \qquad (16)$$

and $\tilde{\xi}_{\tilde{a}}$, $\tilde{\eta}_{\tilde{a}}$ are the coordinates of the $\tilde{a}^{th}$ Gauss point. The shape functions in (15) are defined by

$$\tilde{N}_{\tilde{a}}(\xi, \eta) = 9(1 + \tilde{\xi}_{\tilde{a}}\, \xi)(1 + \tilde{\eta}_{\tilde{a}}\, \xi)/16 \qquad (17)$$

Exercises

1. Verify that (17) satisfies the interpolation property at the reduced Gauss points, that is

$$\tilde{N}_{\tilde{a}}(\tilde{\xi}_{\tilde{b}}, \tilde{\eta}_{\tilde{b}}) = \delta_{\tilde{a}\tilde{b}} \tag{18}$$

2. Describe how to program the reduced/selective integration procedure.

3. Show that for the isotropic case, the symmetric bilinear form $a(\cdot, \cdot)$ can be written as

$$a(\underset{\sim}{w}, \underset{\sim}{u}) = \bar{a}(\underset{\sim}{w}, \underset{\sim}{u}) + \bar{\bar{a}}(\underset{\sim}{w}, \underset{\sim}{u}) \tag{19}$$

where

$$\bar{\bar{a}}(\underset{\sim}{w}, \underset{\sim}{u}) = (\text{div } \underset{\sim}{w}, \lambda \text{ div } \underset{\sim}{u}) \tag{20}$$

This result should reinforce the assertion that it is the $\bar{\bar{\underset{\sim}{k}}}$-stiffness which is responsible for enforcing the volumetrically stiff behavior.

Remarks

1. Constraint ratios may be determined for reduced/selective integration elements as follows: Consider the expression which leads to $\bar{\bar{\underset{\sim}{k}}}$, namely (20). The maximum number of constraints possible is given by the number of independent monomials in div $\underset{\sim}{u}^h$. Likewise, the maximum can be no greater than the number of quadrature points used to evaluate (20). Thus

n_c = min {number of independent monomials present in div $\underset{\sim}{u}^h$; number of quadrature points used to evaluate (20)}

As an example, consider the 4-node bilinear element. Normal quadrature results in $n_c = 3$ (i.e., the number of independent monomials in div $\underset{\sim}{u}^h$)

and so $r = 2/3$. On the other hand, if reduced 1-point quadrature is used on (20), $n_c = 1$ (i.e., number of quadrature points) and so $r = 2$. These values of the constraint ratio agree with those computed for the equivalent mixed elements.

2. Fried [3] argued in a somewhat different way in favor of reducing the integration rule of the λ-term. Consider the case in which λ and μ are constants and let

$$\bar{\underset{\sim}{K}} = \mu \underset{\sim}{K}_1 \tag{21}$$

$$\bar{\bar{\underset{\sim}{K}}} = \lambda \underset{\sim}{K}_2 \tag{22}$$

Then

$$\begin{aligned} \underset{\sim}{F} &= \underset{\sim}{K}\,\underset{\sim}{d} \\ &= (\bar{\underset{\sim}{K}} + \bar{\bar{\underset{\sim}{K}}})\underset{\sim}{d} \\ &= (\mu \underset{\sim}{K}_1 + \lambda \underset{\sim}{K}_2)\underset{\sim}{d} \end{aligned} \tag{23}$$

Fried noted that for typical elements $\underset{\sim}{K}_2$ tended to have too great a rank (i.e., too many incompressibility constraints). In fact, in some situations $\underset{\sim}{K}_2$ is non-singular and therefore for $\lambda/\mu >> 1$

$$\begin{aligned} \underset{\sim}{d} &= (\mu \underset{\sim}{K}_1 + \lambda \underset{\sim}{K}_2)^{-1}\underset{\sim}{F} \\ &\cong \frac{1}{\lambda}\,\underset{\sim}{K}_2^{-1}\,\underset{\sim}{F} \end{aligned} \tag{24}$$

From (24), it can be seen that as $\lambda \to \infty$, $\underset{\sim}{d} \to \underset{\sim}{0}$. This situation may be seen to be equivalent to the locking phenomenon noted in the example of

Figure 2, §3. Thus Fried argued that for a formulation to be successful in nearly-incompressible applications, $\underset{\sim}{K}_2$ must be singular so that (24) does not hold. Fried further argued that one way of achieving this end was to reduce the order of quadrature on the element contributions to $\underset{\sim}{\overline{K}}$.

3. The linear triangular displacement element, which is equivalent to the linear displacement -- constant pressure triangular mixed element (Hughes and Allik [4]), exhibits pathological locking on the standard mesh (Figure 2, §3). However, Nagtegaal et al. [5] found that in the cross-diagonal pattern (see Figure 4a) this element improves. Ostensibly, the constraint ratio for this pattern is still 1. However, what occurs is that the incompressibility conditions exhibit a linear dependency and thus there are only three incompressibility constraints per quadrilateral macro-element. Thus $r = 4/3$.

Mercier [6] provided an elegant argument which established similar improvement for quadratic triangles in the cross-diagonal pattern (see Figure 4b).

At this point, however, there appear to be several disadvantages in adopting the macro-element approach. First of all, since there is a necessity of assembling into quadrilateral macro-elements, no additional flexibility is gained by the use of the triangular elements over standard quadrilateral elements. Second, when compared to the selectively integrated Lagrange quadrilaterals of equal interpolatory order, less accuracy is attained by the triangular macro-elements despite the fact that approximately two times as many unknowns are involved.

On the other hand, a potential advantage of the triangular macro-elements is that in the plane strain case, if the edges are straight, pointwise incompressibility may be obtained. Incompressibility, in general, only occurs at

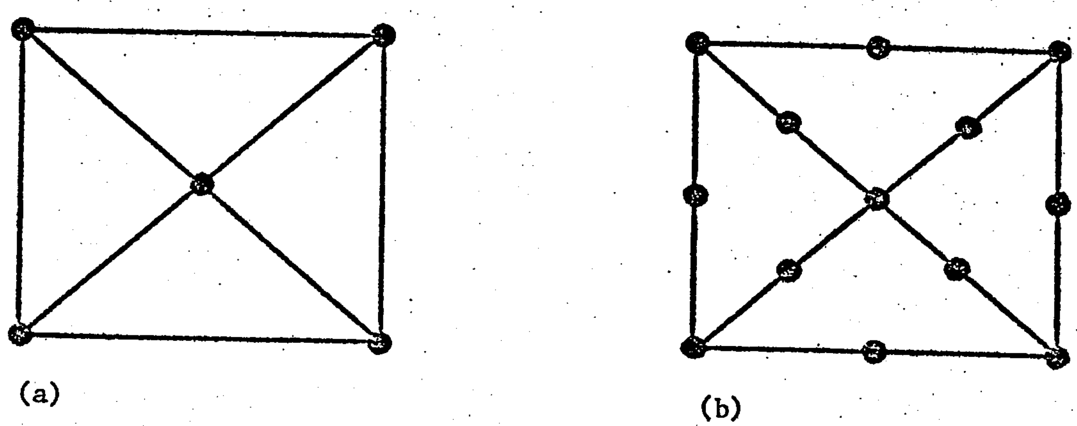


Figure 4. Quadrilateral macro-elements formed from triangles in the cross-diagonal pattern.

the quadrature points of the reduced integration rule for the selectively integrated Lagrange elements.

Exercise 4

Consider the Lagrange family of quadrilaterals and bricks in which normal integration is used for the entire stiffness. Determine expressions for the constraint ratio as functions of the number of nodes per element. Deduce that the constraint ratio is poorer for lower-order members of the family than for higher-order members. Show that as $n_{en} \to \infty$, $r \to n_{sd}$ (optimal). (This result is consistent with the observation that fully integrated higher-order elements are more successful in nearly-incompressible applications than fully integrated lower-order elements.)

Some historical remarks on mixed and reduced/selective integration methods.

Mixed finite element formulations were first discussed by Fraeijs de Veubeke [7] and Herrmann [8]. Herrmann developed a reduced form of Reissner's variational principle particularly suited to problems of incompressible and nearly-incompressible elasticity and, based upon this principle, established the first effective finite elements for such cases. This is the formulation given in §3. Prior to this development many displacement models were applied to these problems and poor behavior was typically observed. The reasons for this were not understood at the time. Certain elements derived from Herrmann's formulation also failed. Hughes and Allik [4] traced this failure to a correspondence between mixed and displacement models, contained within Fraeijs de Veubeke's "limitation principle" [7].

The first example of a uniform reduced integration element was apparently the plate/shell element presented by Zienkiewicz, Taylor and Too [9]. This element, among others, is discussed in Chapter 5. The same concept was employed in other areas by Zienkiewicz and colleagues. In particular, Naylor [10] and Zienkiewicz and Godbole [11] advocated the use of the 8-node serendipity element in problems involving incompressibility. The procedure, however, was viewed by many as more a "trick" than a method and subsequently some bad experiences were noted for the serendipity element.

The concept of selective integration was first employed by Doherty, Wilson and Taylor [12] to obtain improved bending behavior in simple 4-node elasticity elements. One-point Gauss quadrature was used on the shear-strain term, and 2×2 Gauss quadrature was used to integrate the remaining terms. Although improved behavior was noted in some configurations, lack of invariance opened the approach to criticism.

Studies performed by Fried [3], Nagtegaal et al. [5], and Argyris et al. [13] provided fresh insights into why the displacement approach failed in constrained problems. Malkus [14, 15] proved the equivalence of a class of mixed models with reduced/selective integration single-field elements in linear elasticity theory. The equivalence results of Malkus and Hughes [1] elevated the reduced and selective integration approach from the realm of "tricks" to a legitimate methodology. Considerable research on the behavior of mixed and reduced/selective integration elements has taken place in recent years. A summary of more recent developments is contained in the following sections.

References

1. D.S. Malkus and T.J.R. Hughes, "Mixed Finite Element Methods - Reduced and Selective Integration Techniques: A Unification of Concepts," Computer Methods in Applied Mechanics and Engineering, Vol. 15, No. 1, 63-81 (1978).

2. T.J.R. Hughes, "Equivalence of Finite Elements for Nearly-Incompressible Elasticity," Journal of Applied Mechanics, 44, 181-183 (1977).

3. I. Fried, "Finite Element Analysis of Incompressible Material by Residual Energy Balancing," Int. J. Solids Structures, 10, 993-1002 (1974).

4. T.J.R. Hughes and H. Allik, "Finite Elements for Compressible and Incompressible Continua," Proc. Symp. Civil Eng., Vanderbilt University, Nashville, Tenn., 27-62 (1969).

5. J.C. Nagtegaal, D.M. Parks and J.R. Rice, "On Numerically Accurate Finite Element Solutions in the Fully Plastic Range," Computer Methods in Applied Mechanics and Engineering, Vol. 4, 153-178 (1974).

6. B. Mercier, "A Conforming Finite Element Method for Two-Dimensional Incompressible Elasticity," International Journal for Numerical Methods in Engineering, Vol. 14, No. 6, 942-945 (1979).

7. B. Fraeijs de Veubeke, "Displacement and Equilibrium Models in the Finite Element Method," in Stress Analysis, (eds. O.C. Zienkiewicz and G.S. Holister), John Wiley, London, 1965.

8. L.R. Herrmann, "Elasticity Equations for Incompressible and Nearly Incompressible Materials by a Variational Theorem," AIAA J., 3, 1896-1900 (1965).

9. O.C. Zienkiewicz, R.L. Taylor and J.M. Too, "Reduced Integration Technique in General Analysis of Plates and Shells," Int. J. Num. Meth. Engng., 3, 275-290 (1971).

10. D.J. Naylor, "Stresses in Nearly Incompressible Materials by Finite Elements with Application to the Calculation of Excess Pore Pressures," Int. J. Num. Meth. Engng., 8, 443-460 (1974).

11. O.C. Zienkiewicz and P.N. Godbole, "Viscous Incompressible Flow with Special Reference to Non-Newtonian (Plastic) Fluids," in Finite Element Methods in Fluids, Vol. 1, J. Wiley and Sons, London, 1975.

12. W.P. Doherty, E.L. Wilson and R.L. Taylor, "Stress Analysis of Axisymmetric Solids Utilizing Higher Order Quadrilateral Finite Elements," SESM Report No. 69-3, Department of Civil Engineering, University of California, Berkeley, 1969.

13. J.H. Argyris, P.C. Dunne, T. Angelopoulus and B. Bichat, "Large Natural Strains and Some Special Difficulties Due to Nonlinearity and Incompressibility in Finite Elements," Computer Methods in Applied Mechanics and Engineering, Vol. 4, 219-278 (1978).

14. D.S. Malkus, "Finite Element Analysis of Incompressible Solids," Ph.D. Thesis, Boston University, Boston (1975).

15. D.S. Malkus, "A Finite Element Displacement Model Valid for Any Value of the Compressibility," International Journal of Solids and Structures, Vol. 12, 731-738 (1976).

Question	Score
1	8/30
2	25/35
3	10/35
4	
5	
6	
7	
8	
Total	~~4~~3/100

43/100

Itzhak Levitt

STANFORD UNIVERSITY

OFFICIAL EXAMINATION BOOK

24 Page Ruled

Name of student LEVY

Date of examination

Course ME 235C

THE STANFORD UNIVERSITY HONOR CODE

A. The Honor Code is an undertaking of the students, individually and collectively:

(1) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

(2) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

I acknowledge and accept the Honor Code.

(Signed) ____________________

Interpretations and applications of the Honor Code appear on the back cover of this examination book.

ME 235C
Nonlinear Finite Element Analysis
Fall Quarter 1981

Stanford University
Instructor: T. Hughes

Midterm Exam

Time = 1 Hour

Total Points = 100

Open notes and homeworks allowed

CESAR LEVY

Name

1. (30 points) Consider the following eigenvalue problem

$$(\underset{\sim}{K} - \lambda \underset{\sim}{M})\underset{\sim}{d} = \underset{\sim}{0} \quad ; \qquad \lambda = \omega^2,$$

where

$$\underset{\sim}{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$\underset{\sim}{K} = \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$\underset{\sim}{d} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

Assume $k_1 = k_2 = 1$, $m_1 = 2$, $m_2 = 1$.

(a) (5 points) Calculate the frequencies (i.e., ω_1, ω_2) and mode shapes (i.e., $\underset{\sim}{d}_{(1)}, \underset{\sim}{d}_{(2)}$).

(b) (5 points) Assuming that the fundamental mode load pattern is given approximately by

$$\underset{\sim}{P} = \begin{Bmatrix} m_1/(k_1+k_2) \\ m_2/k_2 \end{Bmatrix},$$

use the discrete Ritz reduction procedure to obtain an estimate of the fundamental frequency and mode shape.

(c) (10 points) Use the Irons-Guyan procedure to reduce the problem to one degree of freedom. Pick the degree of freedom to be retained according to the criterion presented in class. Determine the approximate fundamental frequency and mode shape.

(d) (10 points) Use the subspace iteration procedure to calculate the fundamental frequency and mode shape. Initialize the computations with the load pattern

$$\underset{\sim}{P} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Employ 2 iterations.

2. (35 points) Consider the following nonlinear equation:

$$\frac{d}{dx}\left(\frac{du}{dx}\left(1 + \left(\frac{du}{dx}\right)^2\right)\right) + \ell = 0 \quad \text{on} \quad \Omega =]0, 1[. \qquad (1)$$

A boundary-value problem for (1) consists of finding a function $u : \Omega \to \mathbb{R}$ satisfying (1) and the boundary conditions

$$u(0) = g \qquad (= \text{a given constant})$$

$$u(1) = 0$$

(i) (15 points) Set up a weak formulation of the problem.

(ii) (20 points) Use the result of part (i) to derive the elemental contributions of a Newton-Raphson solution algorithm:

$$D\underset{\sim}{N}(\underset{\sim}{d}) \cdot \Delta \underset{\sim}{d} = \underset{\sim}{F} - \underset{\sim}{N}(\underset{\sim}{d})$$

(that is, define $\underset{\sim}{f}^e$, $\underset{\sim}{n}^e(\underset{\sim}{d}^e)$, and $D\underset{\sim}{n}^e(\underset{\sim}{d}^e)$).

3. (35 points) Consider the first order system of linear ordinary differential equations

$$\dot{\underset{\sim}{y}} = \underset{\sim}{G}\,\underset{\sim}{y} + \underset{\sim}{H}(t) \tag{1}$$

and the following two-step LMS method ("leap-frog method")

$$\underset{\sim}{y}_{n+1} - \underset{\sim}{y}_{n-1} = 2\Delta t(\underset{\sim}{G}\,\underset{\sim}{y}_n + H_n) \tag{2}$$

Assume that (1) and (2) may be reduced to SDOF model equations.

(a) (15 points) Determine an expression for the local truncation error. What is the order-of-accuracy of this method.

(b) (10 points) Assuming (1) represents the semi-discrete heat equation, describe the stability characteristics of (2).

(c) (10 points) Assuming (1) represents the undamped equation of motion, describe the stability characteristics of (2).

ME 235 C MIDTERM

1. a)

$$\underset{\sim}{K} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad ; \quad \underset{\sim}{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(\underset{\sim}{K} - \lambda \underset{\sim}{M}) = 0 \Rightarrow 2\lambda^2 - 4\lambda + 1 = 0$$

$$\lambda_1 = 1 - \frac{1}{\sqrt{2}} \quad ; \quad \lambda_2 = 1 + \frac{1}{\sqrt{2}}$$

$$\omega_1 = \sqrt{\lambda_1} = .5412 \quad ; \quad \omega_2 = 1.3066$$

$$\underset{\sim}{\psi}^{(\lambda)} = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix} \qquad \text{Let } \psi_1 = 1 \Rightarrow \psi_2 = 2(1-\lambda)$$

$$\underset{\sim}{\psi}^{(1)} = \begin{Bmatrix} 1 \\ \sqrt{2} \end{Bmatrix} \quad ; \quad \underset{\sim}{\psi}^{(2)} = \begin{Bmatrix} 1 \\ -\sqrt{2} \end{Bmatrix}$$

b)

$$\underset{\sim}{P} = \begin{Bmatrix} 2/2 \\ 1/1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\underset{\sim}{K}\,\underset{\sim}{R} = \underset{\sim}{P} \Rightarrow \underset{\sim}{R} = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$$

$$\underset{\sim}{K}^* = \underset{\sim}{R}^T \underset{\sim}{K}\,\underset{\sim}{R} = \underset{\sim}{R}^T \underset{\sim}{P} = 5$$

$$\underset{\sim}{M}^* = \underset{\sim}{R}^T \underset{\sim}{M}\,\underset{\sim}{R} = 17$$

$$(\underset{\sim}{K}^* - \lambda^* \underset{\sim}{m}^*)\,\underset{\sim}{d}^* = \underset{\sim}{0} \Rightarrow \lambda^*_{(1)} = \frac{5}{17} \; ; \; d^*_{(1)} = 1$$

2\. a) Find $u \in \mathcal{S}$ s.t. $\forall\, w \in \mathcal{V}$:

$$\int_{\Omega} \{ w \underbrace{[u_{,x} (1 + u_{,x}^2)]}_{\equiv\, v(x)}{}_{,x} + w f \} \, d\Omega = 0$$

$$\int_0^1 w \cdot v_{,x} \, dx = \underbrace{\int_0^1 [w v]_{,x} \, dx}_{\Downarrow \; w v|_0^1 \to 0} - \int_0^1 w_{,x} v(x) \, dx$$

since $w \in \mathcal{V} \Rightarrow w(0) = w(1) = 0$

So :

$$\underbrace{\int_0^1 w_{,x} [u_{,x} (1 + u_{,x}^2)] \, dx}_{\eta(w, u)} = \underbrace{\int_0^1 w f \, dx}_{(w, f)}$$

(W) $\begin{cases} \text{Find } u \in \mathcal{S} \text{ s.t. } \forall\, w \in \mathcal{V} \\ \eta(w, u) = (w, f) \end{cases}$

b) $\underset{\sim}{f}^e = \{ f_a^e \}$; $\underset{\sim}{F} = \overset{n_{el}}{\underset{e=1}{A}} \underset{\sim}{f}^e$

$$f_a^e = \int_0^1 N_a f(x)\, dx$$

$\underset{\sim}{n}^e(\underset{\sim}{d}^e) = \{ n_a^e(\underset{\sim}{d}^e) \}$; $\underset{\sim}{N}(\underset{\sim}{d}) = \overset{n_{el}}{\underset{e=1}{A}} \underset{\sim}{n}^e(\underset{\sim}{d}^e)$

$$n_a^e(\underset{\sim}{d}^e) = \int_0^1 N_{a,x} \left[\sum_{b=1}^{n_{en}} N_{b,x} d_b^e \left(1 + \left(\sum_{c=1}^{n_{en}} N_{c,x} d_c^e \right)^2 \right) \right] \cdot dx$$

$$D\underset{\sim}{N}(\underset{\sim}{d}) = \overset{n_{el}}{\underset{e=1}{A}} D\underset{\sim}{n}^e(\underset{\sim}{d}^e)$$

$$D\underset{\sim}{n}^e(\underset{\sim}{d}^e) = [Dn_{ab}^e(\underset{\sim}{d}^e)]$$

$$Dn_{ab}^e(\underset{\sim}{d}^e) = \partial n_a^e(\underset{\sim}{d}^e) / \partial d_b^e =$$

$$= \int_0^1 N_{a,x} N_{b,x} \left(1 + 3 \left(\sum_{c=1}^{n_{en}} N_{c,x} d_c^e \right)^2 \right) dx$$

3 a) s.d.o.f. model

$$\begin{cases} y_{n+1} - y_{n-1} = 2\Delta t \lambda y_n + h_n 2\Delta t & (2) \\ \dot{y} = \lambda y + h & (1) \end{cases}$$

$$\tau(t_n) = y(t_{n+1}) - y(t_{n-1}) - 2\Delta t \lambda y(t_n) - h(t_n) \times 2\Delta t \quad (3)$$

where

$y(t_n)$ is a sol'n of (1) at $t = t_n = n\Delta t$

Taylor expansion with remainder

$$y(t_{n+1}) = y(t_n) + \Delta t \dot{y}(t_n) + \frac{\Delta t^2}{2} \ddot{y}(t_n) + \frac{\Delta t^3}{6} \dddot{y}(c_1) \quad (4)$$

$c_1 \in [t_n, t_{n+1}]$

$$y(t_{n-1}) = y(t_n) - \Delta t \dot{y}(t_n) + \frac{\Delta t^2}{2} \ddot{y}(t_n) - \frac{\Delta t^3}{6} \dddot{y}(c_2) \quad (5)$$

$c_2 \in [t_{n-1}, t_n]$

~~subst~~ substitute (4) & (5) into (3) to get

$$\tau(t_n) = 2\Delta t (\lambda y(t_n) + h(t_n)) - 2\Delta t \dot{y}(t_n) + \frac{\Delta t^3}{6} [\dddot{y}(c_1) + \dddot{y}(c_2)] \quad (6)$$

Subst. $\dot{y}(t_n)$ from (1) into (6) to get

$$\tau(t_n) = \frac{\Delta t^3}{6}\left[\dddot{y}(c_1) + \dddot{y}(c_2)\right] = C\,\Delta t^3 = C\,\Delta t^{k+1}$$

$\Rightarrow$ $k = 2$ $\therefore$ second order accurate

b) Assume $y_{n+1} = \eta\, y_n$ (7)

Subst in the S.D.O.F eqtn (2) (without h) to get

$$(\eta^2 - 2\Delta t \lambda \eta - 1)\, d_{n-1} = 0 \qquad (8)$$

For stability require:

$$|\eta| \leq 1$$

(8) $\Rightarrow$

$$\eta_{1,2} = \Delta t \lambda \pm \sqrt{(\Delta t \lambda)^2 + 1} \qquad (9)$$

For heat conduction $\lambda \in R$

Thus for $\Delta t \lambda < 0$ (physical pb.)

(9) $\Rightarrow$ $\eta_2 < -1$ $\Rightarrow$ uncond. unstable

(Remark: if either of the roots of (8) is corresponding to unstability the method is unstable)

c) For undamped equation of motion

$$\lambda = i\omega \quad ; \quad \text{Let } \Omega = \omega \Delta t$$

subst. in (9) to get:

$$\eta_{1,2} = i\Omega \pm \sqrt{1-\Omega^2}$$

i) For $\alpha\Omega^2 < 1$

$$|\eta_{1,2}| = \left(\Omega^2 + 1 - \Omega^2\right)^{1/2} = 1$$

$$\Rightarrow \Omega \leq 1 \qquad \left(\Delta t \leq \frac{1}{\omega}\right)$$

stable

ii) $\Omega^2 > 1$

$$\eta_{1,2} = i\Omega \pm i\sqrt{\Omega^2 - 1}$$

$$|\eta_1| = \Omega + \sqrt{\Omega^2-1} > 1 \Rightarrow \text{unstable}$$

$\therefore$ conditionally stable method

$$\Delta t_{crit} = \frac{1}{\omega_{max}}$$

$$\omega_{(1)} \cong \omega^*_{(1)} = \sqrt{5/17} = .5423$$

$$\underset{\sim}{d}_{(1)} \cong \underset{\sim}{R}\,\underset{\sim}{d}^* = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.5 \end{Bmatrix}$$

c) $m_{11}/K_{11} = m_{22}/K_{22} = 1$ can retain either d.o.f., say 1

$$\underset{\sim}{R} = \begin{Bmatrix} 1 \\ -K_{22}^{-1} K_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\left.\begin{aligned} \underset{\sim}{K}^* &= \underset{\sim}{R}^T \underset{\sim}{K}\,\underset{\sim}{R} = 1 \\ \underset{\sim}{M}^* &= \underset{\sim}{R}^T \underset{\sim}{M}\,\underset{\sim}{R} = 3 \end{aligned}\right\} \Rightarrow \lambda^* = 1/3 \;,\; d^* = 1$$

$$\underset{\sim}{d}_{(1)} \cong \underset{\sim}{R}\,\underset{\sim}{d}^* = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\omega_{(1)} \cong \omega^* = \sqrt{1/3} = .5774$$

d) i) $\underset{\sim}{P}' = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$; $\underset{\sim}{K}\,\underset{\sim}{R}' = \underset{\sim}{P} \longrightarrow \underset{\sim}{R}' = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

same as $\underline{c}$ $\Rightarrow$ $\begin{cases} \underset{\sim}{d}'_{(1)} \cong \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \\ \omega'_{(1)} \cong .5774 \end{cases}$

ii) $\underset{\sim}{P} = \underset{\sim}{M}\,\underset{\sim}{d}'_{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

same as $\underline{b}$ $\Rightarrow$ $\underset{\sim}{d}^2_{(1)} \cong \begin{Bmatrix} 1 \\ 1.5 \end{Bmatrix}$; $\omega^2_{(1)} \cong .5423$

1. $\left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right)\left(\underset{\sim}{d}\right) = \underset{\sim}{0}$ ✓

$\det\begin{bmatrix} 2-2\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} = 2(1-\lambda)^2 - 1 = 0$

$2(1-2\lambda+\lambda^2) - 1 = 0$

$2 - 4\lambda + 2\lambda^2 - 1 = 0 \qquad 2\lambda^2 - 4\lambda + 1 = 0$ ✓

$\lambda = \frac{4 \pm \sqrt{16 - 4 \cdot 2}}{2} = \frac{4 \pm \sqrt{8}}{2}$ ~4

$= \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$

$2 + 1.414 = 3.414 = \lambda_1 \qquad \lambda_2 = 2 - 1.414 = .586$ ✗ ✗

$\omega_1 = 1.8478 \qquad \omega_2 = .7654$

$\lambda_1 = 3.414 \qquad \begin{pmatrix} 2(1-3.414) & -1 \\ -1 & -2.414 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 0$

3/5

$\lambda_2 = .586 \qquad \begin{pmatrix} 2 - 2(.586) & -1 \\ -1 & 1 - .586 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 0$ ✗ [above: $2(1-.586)$]

$\begin{pmatrix} 2a & -1 \\ -2a & 2a^2 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 0$

b. $KR = P$

$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\left(\underset{\sim}{R}\right) = \begin{pmatrix} m_1/(k_1+k_2) \\ m_2/k_2 \end{pmatrix}$

$\underset{\sim}{R} = K^{-1}\underline{P}$

Subst. numbers for $\underset{\sim}{P}$! +1

$R_1 = \frac{\begin{bmatrix} m_1/k_1+k_2 & -1 \\ m_2/k_2 & -1 \end{bmatrix}}{1} = -m_1/k_1+k_2 + m_2/k_2$

$\begin{matrix} R_1 \\ R_2 \end{matrix} = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} m_1/k_1+k_2 \\ m_2/k_2 \end{pmatrix}$

$R_2 = \begin{bmatrix} 2 & m_1/k_1+k_2 \\ -1 & m_2/k_2 \end{bmatrix} = 2m_2/k_2 + m_1/k_1+k_2$

now ~~$P^T K^T K K^{-1} P$~~ $P^T I K^{-1} P$ $-\lambda$ ~~$[P^T K^{-T} M K^{-1} P$~~

$$\underset{\sim}{R} = \begin{pmatrix} \overset{+1}{\textcircled{-1}} & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \underset{\sim}{P} \end{pmatrix}$$

0/5

$$R^T = P^T \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

$\underset{\sim}{P}$ is known use it !

$$R^T K R = \underline{P}^T \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -\phi \\ -\phi & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \underline{P}$$

$$\begin{pmatrix} -2 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} = \underline{P}^T \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix} \underset{\sim}{P} = K^*$$

X

$$R^T \underline{M} R = P^T \begin{pmatrix} \textcircled{-1} & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \textcircled{-1} & 1 \\ 1 & 2 \end{pmatrix} P$$

$$= P^T \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 1 & 2 \end{pmatrix} P = P^T \begin{pmatrix} 3 & 3 \\ 0 & 6 \end{pmatrix} P$$

$$(K^* - \lambda^* M^*)$$

$$\left\{ \underline{P}^T \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix} \underline{P} - \lambda \underline{P}^T \begin{pmatrix} 3 & 3 \\ 0 & 6 \end{pmatrix} \underline{P} \right\} \underline{d}^* = 0$$

solution of

X

$$\det \left\{ \underline{P}^T \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix} \underline{P} - \lambda \underline{P}^T \begin{pmatrix} 3 & 3 \\ 0 & 6 \end{pmatrix} \underline{P} \right\} = 0 \quad \text{gives } \lambda$$

and for use smallest λ to get $\underset{\sim}{d}_1^*$

then $\underset{\sim}{d} \cong R \underset{\sim}{d}^*$

~~or $\underline{R} = \begin{bmatrix} 1 \\ -1 \cdot -1 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$~~

~~now~~

c. $\underset{\sim}{R} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $k_{11} = 2$ $k_{12} = -1$ $k_{21} = -1$ $k_{22} = 1$

$m_{11} = 2$ $m_{22} = 1$

$\underset{\sim}{K}^* = 2 - (-1) \cdot 1 \cdot (-1) = 2 - 1 = 1$ ✓

~~$M^* = (1\ 1)\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1\ 1)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = (3)$~~

$\therefore \underset{\sim}{M}^* = 2 - 0 - (-1) \cdot 1 \, (0 - 1 \cdot 1 \cdot (-1)) = 3.$ ✓

3/10

$(1 - \lambda^* 3)\underset{\sim}{d}^* = 0$ $\lambda^* = 1/3$ $d^* = 0$ ✗ $d^* = \text{const}$ (say 1)

$-1 d_1 + -1 d_2$ $d_1 = -d_2$ ✗

$m_{11}/k_{11} = 1$ $m_{22}/k_{22} = 1$

$\underset{\sim}{d} \sim \underset{\sim}{R}\,\underset{\sim}{d}^* = \underset{\sim}{R} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

d. $\begin{bmatrix} 2 & -1 \\ -1 & +1 \end{bmatrix}\begin{pmatrix} R \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow R \begin{pmatrix} +1 \\ +2 \end{pmatrix}$ ✓

2/10

2. $\frac{d}{dx}\left(u_{,x}\left(1+\left(\frac{du}{dx}\right)^2\right)\right)+f=0$ on f

$u_{,xx}+\frac{d}{dx}\left(u_{,x}\cdot u_{,x}^2\right)$

$\frac{d}{dx}(u_{,x})^3$

g type B.C

let at $w\in\mathcal{V}$ exists $w(1)=0$ $\quad w(0)=0$

$\Rightarrow$

$$\int w\frac{d}{dx}\left(u_{,x}(1+(u_x)^2)\right)dx+\int fw\,dx=0$$

$$\int\frac{d}{dx}\left[w\left(u_{,x}(1+(u_x)^2)\right)\right]dx-\int_0^1 w_{,x}u_{,x}(1+(u_{,x})^2)\,dx=-\int fw\,dx=0$$

$w\,u_{,x}(1+u_{,x}^2)\Big|_0^1$

$$-w(0)u_{,x}(0)(1+u_{,x}^2(0))-\int_0^1 w_{,x}(u_{,x}+u_{,x}^3)dx$$

10/15

$$\therefore\int_0^1 w_{,x}(u_{,x}+u_{,x}^3)dx=\int_0^1 w\tilde{f}dx+w(0)\left[u_{,x}(0)+u_{,x}^2(0)\right]\quad w\in\mathcal{V}\Rightarrow w(0)=0$$

if $w=\sum N_c d_c$ $\quad u=\sum N_a d_a$.

$$\int_0^1 w_{,x}(N_{a,x}+N_{a,x}^3)dx=\sum_a\int N_c N_a\,dx+\sum d_a N_a(0)\left[N_{a,x}(0)+N_{a,x}^3(0)\right]$$

$\kappa(u,u_{,x};x)=(1+u_{,x}^2)$.

$=\tilde{f}^e+w(0)(u_{,x}(0)+u_{,x}^3(0))=F^{ext}$

15/20

$$n_c^e(\tilde{d}^e)=\int_0^1 N_{c,x}(1+N_{c,x}^2)N_{a,x}\,dx=\int_0^1 N_{c,x}\left(\sum N_{b,x}d_b+\sum N_{b,x}^3 d_b\right)$$

$u_{,x}\,\kappa(u_{,x})\,u_{,x}$

$$\frac{\partial n_a^e}{\partial d_b}=\int_0^1 N_{c,x}\frac{\partial\kappa}{\partial u_{,x}}\cdot\frac{\partial u_{,x}}{\partial d_b}u_{,x}\,dx+\int N_{c,x}\kappa(u_{,x})\frac{\partial u_{,x}}{\partial d_b}dx$$

$$=\int N_{c,x}\,2u_{,x}^2 N_{b,x}\,dx+\int N_{c,x}\kappa(u_{,x})N_{b,x}\,dx$$

$$Dn(d)=\frac{\partial n_a^e}{\partial d_b}=\int N_{c,x}(1+3u_{,x}^2)N_{b,x}\,dx$$

$\left(\sum_{c=1}^{n_{en}}N_{c,x}d_c\right)^2$

$f_a^e=?$

3. $\dot{y} = G y + H(t)$

$$y_{n+1} - y_n = 2\Delta t\,(G y_n + H_n)$$

Subst
$$\begin{cases} y_{n+1} = z^2 y_{n-1} \\ y_n = z\, y_{n-1} \end{cases}$$

$$\sum \alpha_i\, y_{n+1-i} + \Delta t\, \beta_i f(\quad) = 0$$

$$\alpha_0 y_{n+1} + \beta_0 \Delta t\, f(y_{n+1}, t_{n+1}) + \alpha_1 y_n + \Delta t\, \beta_1 f(y_n, t)$$

$$\alpha_0 = 1 \quad \alpha_1 = 0 \quad \beta_0\, G y_{n+1} + H(t_{n+1}) \quad + \quad \Delta t\, \beta_1 (G y_n + H(t_n)).$$

$$\alpha_2 = -1 \qquad \beta_0 = 0 \quad \beta_1 = -2. \quad \beta_2 = 0.$$

(a) (midpoint) here $\alpha = 1/2$ $\quad e(t) \sim C\,\Delta t^2$ in my notes of last quarter $t_m\, C\,\Delta t^2$.

(b) $\dot{y} = G y$ for heat eqn is uncond stable since $\alpha = 1/2$.

$$\dot{y} = \begin{pmatrix} \dot{d} \\ \ddot{d} \end{pmatrix} = G \begin{pmatrix} d \\ \dot{d} \end{pmatrix}$$ the system is stable but the periodic criterion governs the numerical stability ?

(10/35)

$$K = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \qquad M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(K - \lambda M)\underline{d} = 0 \qquad \begin{bmatrix} 2-2\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} \underset{\sim}{d} = 0$$

$$\det = 2(1-\lambda)^2 - 1 = 2 - 4\lambda - 1 + 2\lambda^2 = 2\lambda^2 - 4\lambda + 1 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4\cdot 2\cdot 1}}{2\cdot 2} = \frac{4 \pm \sqrt{8}}{4} = 1 \pm \frac{1}{\sqrt{2}}$$

$\lambda_1 = 1.7071 \qquad \lambda_2 = .2929$

$\omega_1 = 1.30656 \qquad \omega_2 = .5412$

now
$$\begin{pmatrix} 2 - 2(1.7071) & -1 \\ \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 0$$

$$2 - 2\left(1 + \frac{1}{\sqrt{2}}\right) \qquad -1$$

$$\begin{pmatrix} -\sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 0 \qquad \psi_1 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 2 - 2(.29..) & -1 \\ 2 - 2\left(1 - \frac{1}{\sqrt{2}}\right) & -1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 0 \qquad \psi_2 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

$$\underline{K}\,\underline{R} = \underline{P} \qquad \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \underline{P} = \begin{pmatrix} m_1/k_1+k_2 \\ m_2/k_2 \end{pmatrix} = \begin{pmatrix} 2/2 \\ 1/1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now $\det = 1 \therefore \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = R_1 = 2$

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = R_2 = 3 \qquad \therefore \quad \underset{\sim}{R} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$K^* = \underset{R^TP}{R^TKR} = \begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 5$$

$$M^* = R^T M R = \begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 17$$

form. $(K^* - \lambda M^*)d^* = 0$

$$\therefore (5 - \lambda\cdot 17)\,d^* = 0 \qquad \lambda = 5/17 = .2941\ldots \qquad d^* = 1 \qquad \underset{\sim}{d} = \underset{\sim}{R}\,\underset{\sim}{d}^* = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}$$

$$\omega_1 = \sqrt{\lambda} = .5423$$

(c) in non-guyan ignore mass term in 2nd eqn.

$\therefore \; K_{21} \underline{d}_1 + K_{22} \underline{d}_2 = 0 \qquad -d_1 + d_2 = 0 \; ; \; d_1 = d_2$

let $\underline{R} = \begin{bmatrix} \underline{I} \\ -K_{22}^{-1} K_{21} \end{bmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{1} \cdot (-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$K^* = R^T K R = 5$ as before $\qquad M^* = R^T M R = 17 \qquad (K^* - \lambda^* M^*) d^* = 0$

$\Rightarrow \lambda^* = 5/17 \qquad d^* = 1$ as before.

(d) Subspace iter

start $\quad \underline{P} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ solve for $\underline{R}$ in $\underline{K}\underline{R} = \underline{P}$

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \underline{R} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{R} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

now find $\quad K^* = R^T \underbrace{KR}_{P} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2$

and $\quad R^T M R = M^* \quad \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 6$

form $\quad \therefore \; (K^* - \lambda^* M^*) \underline{d}^* = 0 \qquad (2 - \lambda^* \cdot 6) d^* = 0 \qquad \lambda^* = 1/3 \qquad d^* = 1$

first approx $\quad d = \begin{pmatrix} 1 \\ 2 \end{pmatrix} d^* = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \lambda^* = 1/3 \qquad \omega^{(1)} = \frac{1}{\sqrt{3}}$

form new approx $\quad \underline{P} = \underline{M}\,\underline{d} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

2nd iter

now solve for R $\quad \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \underline{R} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ det=1 $\quad \underline{R} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

form $\quad K^* = R^T K R = \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 20$

form $\quad M^* = R^T M R = \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} = 68$

find $(20 - \lambda^* \cdot 68) d^* = 0$ $\lambda = \frac{20}{68} = .2941\ldots$ $d^* = 1$

new approx $d = R d^* = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \cdot 1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}$

2. $\int_0^1 w\left(u_{,x}(1+u_{,x}^2)\right)_{,x} dx + \int_0^1 wf\,dx = 0$

$+ \int_0^1 \left[w(u_{,x}(1+u_{,x}^2))\right]_{,x} dx - \int_0^1 w_{,x}(u_{,x}(1+u_{,x}^2))dx + \int_0^1 wf\,dx = 0$

$+ w(u_{,x}(1+u_{,x}^2))\Big|_0^1 - \int_0^1 \cdots$

$u(0)=g \Rightarrow w(0)=0$
$u(1)=0 \Rightarrow w(1)=0$
$\therefore \int_0^1 w_{,x}(u_{,x} + u_{,x}^3)dx = \int_0^1 wf\,dx$

~~let $\underset{\sim}{f} = \int_0^1 w\,dx$~~ $\int_0^1 w_{,x}(u_{,x}+u_{,x}^3)dx = N$

let $w = \sum N_a c_a$ $u = \sum N_a d_a$ $x = \sum N_a x_a$ $f = \sum N_a f_a$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}$ $\frac{\partial \xi}{\partial x} = 1/\frac{\partial x}{\partial \xi} = 1/\frac{x_2-x_1}{2} = \frac{2}{x_2-x_1}$

$\int_0^1 \_\_\_\_ dx = \int_{-1}^1 w_{,\xi}\, \xi_{,x} \left(u_{,\xi}\,\xi_{,x} + (u_{,\xi}\,\xi_{,x})^3\right)\frac{dx}{d\xi}\, d\xi$

$= \sum f_b \int_{-1}^1 N_a N_b \frac{2}{h} d\xi = \int_{-1}^1 N_{a,\xi}\left(\sum N_{b,\xi} d_b \cdot \frac{2}{h} + \left(\sum N_{b,\xi} d_b \cdot 2/h\right)^3\right) d\xi$

for each element

$\underset{\sim}{f}^e = \sum f_b^e \int_{-1}^1 N_a N_b \cdot \frac{2}{h^e} d\xi$ $\underset{\sim}{f} = \overset{n_{el}}{\underset{e=1}{A}} \underset{\sim}{f}^e$

$\hat{n}^e(\underset{\sim}{d}^e) = \int_{-1}^1 N_{a,\xi}\left(\sum_{d=1}^{n_{en}} N_{b,\xi} d_b^e \cdot \frac{2}{h^e} + \left(\sum N_{b,\xi} d_b^e \cdot 2/h^e\right)^3\right) d\xi$ $N(d) = A N($

$Dn = \int_{-1}^1 N_{a,\xi}\left(\sum N_{b,\xi}\,\delta_{bc}\, 2/h^e + 3\left(\sum N_{b,\xi} d_b^e \cdot 2/h^e\right)^2 \cdot \sum N_{b,\xi}\,\delta_{bc}\, 2/h^e\right) d$

$$DN^e = \int_{-1}^{1} N_{a,\xi} \left(N_{c,\xi} \, 2/h^e + 3\left(\sum_{b=1}^{2} N_{b,\xi}\, d_b^e \cdot 2/h^e\right)^2 N_{c,\xi} \cdot 2/h^e \right) d\xi$$

$$DN = A(DN^e)$$

3. $\dot{y} = Gy + H(t)$

$$\frac{y_{n+1} - y_{n-1}}{2\Delta t} = G y_n + H \qquad \text{Thus } \dot{y} \text{ uses centered difference}$$

$|\tau_n| \le C \Delta t^{k+1}$ where k = order of accuracy

$k = 2$ for $\alpha = 1/2$ centered diff / midpoint rule

$$y_{n+1} = y(t_n) + \Delta t\, y'(t_n) + \Delta t^2/2\, y''(t_n) + \Delta t^3/6\, y'''(\bar{t}_{n_1})$$
$$y_{n-1} = y(t_n) - \Delta t\, y'(t_n) + \Delta t^2/2\, y''(t_n) - \Delta t^3/6\, y'''(\bar{t}_{n_2}) + \ldots$$
$$\frac{y_{n+1} - y_{n-1}}{2\Delta t} = y'(t_n) + \frac{\Delta t^2}{6} y'''(t_n)$$

now $\tau(t_n) = y(t_{n+1}) - y(t_{n-1}) - 2\Delta t G\, y(t_n) - 2\Delta t\, H(t_n)$

$$= 2\Delta t\, y'(t_n) + \Delta t^3/6 \left[y'''(\bar{t}_{n_1}) + y'''(\bar{t}_{n_2}) \right] - 2\Delta t \left[G y(t_n) + H(t_n) \right]$$

now $y'(t_n) = G\, y(t_n) + H(t_n)$

$$\tau(t_n) = 2\Delta t \left[G\, y(t_n) + H(t_n) \right] - 2\Delta t \left[G y(t_n) + H(t_n) \right] + \Delta t^3/3 \cdot [\quad]$$

$\therefore \tau(t_n) = C \Delta t^3 \quad \therefore \tau(t_n) \sim \Delta t^3$ or $k = 2$ 2nd order accurate

for (b) heat equation $k = 2 \Rightarrow \alpha = 1/2 \quad \therefore$ it is unconditional stable

Proof ~~let~~ ~~let~~ ~~$y_{n+1} = \lambda^{n+1}$~~ ~~$y_{n+1} = \lambda^{n-1}$~~

STANFORD UNIVERSITY

OFFICIAL EXAMINATION BOOK

24 Page Ruled

Question	Score
1	60/60
2	85/90
3	80/90
4	60/60
5	
6	
7	
8	
Total	285/300

course A

Name of student CESAR LEVY

Date of examination 18 December 81

Course ME 235C

THE STANFORD UNIVERSITY HONOR CODE

A. The Honor Code is an undertaking of the students, individually and collectively:

(1) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

(2) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

I acknowledge and accept the Honor Code.

(Signed) Cesar Levy

Interpretations and applications of the Honor Code appear on the back cover of this examination book.

ME 235C
Finite Element Methods
Fall Quarter 1981

Stanford University
Instructor: T.J.R. Hughes

Final Exam

Time = 3 hours

Total Points = 300

Open notes and homework allowed

CESAR LEVY

Name

1. (60 points) Consider the following ordinary differential equation:

pg 96 in 1st handout
Ω_{crit} given on pg 85

$$\ddot{d} + \dot{d} + (1 + d^2)d = 0 \qquad (1)$$

A predictor-corrector algorithm for (1) is given by

$$a_{n+1} + \tilde{v}_{n+1} + (1 + \tilde{d}^2_{n+1})\tilde{d}_{n+1} = 0$$

and equations (2)-(5) on pages 80 and 81 of Handout #1.

Determine an expression for the critical time step by performing a <u>linearized</u> stability analysis of the algorithm.

midterm

2. (90 points) Consider the following nonlinear equation ✓

$$\frac{\partial}{\partial x}\left(\left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial u}{\partial x}\right) + f = \rho \frac{\partial^2 u}{\partial t^2} \qquad (2)$$

An initial/boundary-value problem for (2) consists of finding a function u satisfying (2) and the initial and boundary conditions:

$$u(x,0) = u_o(x) \qquad \text{(IC)}$$

$$\left.\begin{aligned} u(o,t) &= g(t) \\ u(l,t) &= 0 \end{aligned}\right\} \text{(BC's)}$$

(i) Set up a weak formulation of the problem.

(ii) Define the elemental contributions to the arrays in the equation

$$\underset{\sim}{M}^* \Delta \underset{\sim}{a} = \underset{\sim}{R}^{(i)}$$

where

$$\underset{\sim}{M}^* = \underset{\sim}{M} + \beta\Delta t^2 \, D\underset{\sim}{N}(\underset{\sim}{d})$$

$$\underset{\sim}{R}^{(i)} = \underset{\sim}{F} - \underset{\sim}{M}\,\underset{\sim}{a}^{(i)} - \underset{\sim}{N}(\underset{\sim}{d}^{(i)})$$

(That is define $\underset{\sim}{m}^e$, $\underset{\sim}{n}^e(\underset{\sim}{d}^{(i)})$, $\underset{\sim}{f}^e$ and $D\underset{\sim}{n}^e(\underset{\sim}{d}^{(i)})$.)

pg 117

3. (90 points) Consider the central difference method for the second-order wave equation:

$$u_{,tt} = c^2 u_{,xx}$$

$$u_{n-1}(m) - 2u_n(m) + u_{n+1}(m)$$

$$= Cr^2 \left[u_n(m-1) - 2u_n(m) + u_n(m+1)\right] \tag{3}$$

where $Cr = c\Delta t/h$ is the Courant number, $h = x_{m+1} - x_m = x_m - x_{m-1}$ and $u_n(m) \approx u(x_m, t_n)$, etc. Perform a von Neumann stability analysis of (3). Compare the stability condition obtained with that obtained by other means (see dynamics notes, eq. (129), p. 54).

Consider the generalized trapezoidal method for the first-order wave equation in which central differencing is used for the space derivative:

$$u_{,t} = c\, u_{,x}$$

$$u_{n+1}(m) - u_n(m) = \frac{Cr}{2} \Big\{(1 - \gamma) \left[u_n(m + 1) - u_n(m - 1)\right]$$

$$+ \gamma \left[u_{n+1}(m + 1) - u_{n+1}(m - 1)\right]\Big\} \tag{4}$$

Perform a von Neumann stability analysis of (4). Show that (a) if $\gamma = 0$, the algorithm is unconditionally unstable; (b) if $\gamma \geq 1/2$, the algorithm is unconditionally stable.

energy conserving algo pg 104

4. (60 points) Solution of the matrix problem

$$\underset{\sim}{K}\,\underset{\sim}{d} = \underset{\sim}{F}$$

where $\underset{\sim}{K}$ is positive definite, may be viewed equivalently as a minimization problem; namely, find $\underset{\sim}{c}$ such that

$$\mathscr{F}(\underset{\sim}{c}) = \tfrac{1}{2}\,\underset{\sim}{c}^T\,\underset{\sim}{K}\,\underset{\sim}{c} - \underset{\sim}{c}^T\,\underset{\sim}{F}$$

is minimized. Suppose we wish to impose the following linear constraint:

$$a\,d_P + b\,d_Q = c \qquad (5)$$

where a, b and c are given constants, and $P, Q \in \{1, 2, \ldots, n_{eq}\}$. Use the Lagrange multiplier method to obtain a modified matrix problem in which (5) is satisfied.

1. $\delta\left(\ddot{d} + \dot{d} + (1+d^2)d = 0\right)$ here $M = 1$

$\delta\ddot{d} + \delta\dot{d} + \delta d(1+3d^2) = 0$

$\delta d_{n+1} = \delta \tilde{d}_{n+1}$

$\delta a_{n+1} + \delta \tilde{v}_{n+1} + \delta \tilde{d}_{n+1}(1 + 3\tilde{d}_{n+1}^2) = 0$ ✓

$M = 1 \quad C_T^I = 0 \quad K_T^I = 0 \quad C_T^E = 1 \quad K_T^E = (1 + 3\tilde{d}_{n+1}^2)$ ✓

$\delta d_{n+1} = \delta\tilde{d}_{n+1} + \Delta t^2 \beta\, \delta a_{n+1}$ ✓

$\delta v_{n+1} = \delta\tilde{v}_{n+1} + \Delta t \gamma\, \delta a_{n+1}$ ✓

$\delta\tilde{d}_{n+1} = \delta d_n + \Delta t\, \delta v_n + \frac{\Delta t^2}{2}(1-2\beta)\, \delta a_n$ ✓

$\delta\tilde{v}_{n+1} = \delta v_n + \Delta t(1-\gamma)\, \delta a_n$ ✓

60/60

this leads to (14) pg 98 pick $M^I = 0$

$B^I = 0 + \Delta t(\gamma - 1/2)\cdot 0 + \Delta t^2(\beta - \gamma/2)\cdot 0 = 0$ ✓

$A^I = 1 + \Delta t(\gamma - 1/2)\cdot 0 = 1$

$\bar{B}^E = B^E - \Delta t\gamma C_T^E - \Delta t^2 \beta K_T^E = B^E - \Delta t\gamma - \Delta t^2\beta(1+3\tilde{d}_{n+1}^2)$ ✓

pick $M^E = 1$ $\quad B^E = 1 + \Delta t(\gamma - 1/2)\cdot 1 + \Delta t^2(\beta - \gamma/2)(1+3\tilde{d}_{n+1}^2)$ ✓

$\therefore \quad \bar{B}^E = 1 - 1/2\,\Delta t - \gamma/2\,\Delta t^2(1+3\tilde{d}_{n+1}^2)$

$\bar{A}^E = \bar{B}^E + \Delta t(\gamma - 1/2)\cdot 1 = \Delta t(\gamma - 1) + \gamma/2\,\Delta t^2(1+3\hat{d}_{n+1}^2)$

Using the fact that the time step restriction is sufficient for the explicit elements and the stability class is determined by conditions that render $\bar{B}$ positive then $\bar{B}^E \geq 0$

$\Delta t^2 \frac{\gamma}{2}(1+3d_{n+1}^2) + 1/2\,\Delta t - 1 \geq 0$ ✓

$$\Delta t = \frac{-1/2 \pm \sqrt{1/4 + \frac{4}{2}\gamma/2(1+3d_{n+1}^2)}}{\gamma(1+3d_{n+1}^2)}$$ ✓

$$= \frac{-1/2 \pm \sqrt{(1/2)^2 + 2\gamma(1+3d_{n+1}^2)}}{\gamma(1+3d_{n+1}^2)}$$ ✓

$$\Delta t_{crit} = \frac{-1/2 + \sqrt{(1/4) + 2\gamma(1+3d_{n+1}^2)}}{\gamma(1+3d_{n+1}^2)}$$ and $\gamma \geq 1/2$ ✓

2. We can write this as

$$\int_0^1 w\left\{\frac{\partial}{\partial x}\left[(1+u_{,x}^2)u_{,x}\right]+f-\rho\ddot{u}\right\}dx=0 \quad \text{on} \quad x\in(0,1)$$ ✓

or

$$-\int_0^1 w_{,x}(1+u_{,x}^2)u_{,x}dx+\int_0^1\left[w(1+u_{,x}^2)u_{,x}\right]_{,x}dx+\int_0^1 wf\,dx-\int_0^1\rho\ddot{u}w\,dx=0$$ ✓

$$w(1+u_{,x}^2)u_{,x}\Big|_0^1 = 0$$ ✓

40/45

since $u(0,t)=g(t)$ $w(0,t)=0$
$u'(1,t)=0$ $w(1,t)=0$

$$\therefore \begin{cases}\int_0^1 w_{,x}(1+u_{,x}^2)u_{,x}dx+\int_0^1 w\rho\ddot{u}\,dx=\int_0^1 wf\,dx & (1)\ ✓\\ \int_0^1 w[u(x,0)-u_0(x)]dx=0 & \text{I.C's}\end{cases}$$

must find $u\in\mathcal{S}$ $\ni$. $\forall w\in\mathcal{V}$ (1) holds

or

$$\int_0^1 N_{A,x}(1+u_{,x}^2)u_{,x}dx+\int_0^1 N_A\rho\ddot{u}\,dx=\int_0^1 N_A f\,dx \Rightarrow \underline{F}=M\ddot{\underline{d}}+N(d,t)$$

ie $F_{n+1}=M\ddot{d}_{n+1}^{(i)}+N(d_{n+1}^{(i+1)},t_{n+1})=M\ddot{d}_{n+1}+\frac{\partial N}{\partial d}(d_{n+1}^{(i)},t_{n+1})\Delta d+N(d_n,t_n)$

$R^{(i)}=F_{n+1}-F_n=F_{n+1}-M\ddot{d}_n^{(i)}-N(d_n,t_n)=M\Delta\ddot{d}+DN\Delta d$; $\Delta d=\beta\Delta t^2\Delta\ddot{d}$ from previous work.

Now let $u=\sum_{B=1}^{nen}N_B d(t)_B$ put into (1):

$$\int_0^1 N_{A,x}\left(1+\left(\sum_C^{nen}N_{C,x}d_C\right)^2\right)\sum_B^{nen}N_{B,x}d_B\,dx+\int_0^1\rho\sum_B N_B\ddot{d}_B N_A\,dx=\int_0^1 N_A f\,dx$$

$$N(d,t)+M\ddot{d}=F$$

$$m^e=[m_{AB}^e]=\int_{\Omega^e}\rho N_A N_B\,dx$$ ✓

$$\underset{\sim}{n}^e(\underset{\sim}{d}^{(i)})\Rightarrow[n_A^e]=\int_{\Omega^e}N_{A,x}\left(1+\left[\sum_C N_{C,x}d_C^{e(i)}\right]^2\right)\sum_B N_{B,x}d_B^{e(i)}\,dx$$ ✓

$$Dn_A^e(d^{(i)})\Rightarrow[D\underset{\sim}{n}_A^e(d^{(i)})]=\int_{\Omega^e}N_{A,x}N_{B,x}\left(1+3\left(\sum_C N_{C,x}d_C^{e(i)}\right)^2\right)dx$$ ✓

$$f_A^e=\int_{\Omega^e}N_A f\,dx \quad ✓ \quad \underset{\sim}{f}^e=\{f_A^e\}\ ;\quad \underset{\sim}{F}=\underset{e=1}{\overset{nel}{\mathrm{A}}}\underset{\sim}{f}^e$$

45/45

now $\underset{\sim}{n}(\underset{\sim}{d}^{(i)})=\{n_A(\underset{\sim}{d}^{(i)})\}$; $N(\underset{\sim}{d}^{(i)})=\underset{e=1}{\overset{nel}{\mathrm{A}}}n(\underset{\sim}{d}^{(i)})$

$D\underset{\sim}{n}(\underset{\sim}{d}^{(i)})=\{Dn_A(\underset{\sim}{d}^{(i)})\}$; $DN=\mathrm{A}\,Dn(d^{(i)})$

no way
follow yo
soln and
of your
is wrong 8

3.a) $u_{n-1}(m) - 2u_m(m) + u_{n+1}(m) = Cr^2 \left[u_n(m-1) - 2u_m(m) + u_n(m+1)\right]$

the error induced by small perturb leads to

$e_{n-1}(m) - 2e_n(m) + e_{n+1}(m) = Cr^2\left[e_n(m-1) - 2e_n(m) + e_n(m+1)\right]$ ✓

if $e_n(m) = \lambda^n e^{im\xi}$ then

$e^{im\xi}\left[\lambda^{n-1} - 2\lambda^n + \lambda^{n+1}\right] = Cr^2 \lambda^n \left[e^{im\xi}\left\{e^{-i\xi} - 2 + e^{i\xi}\right\}\right]$ ✓

$\left[1 - 2\lambda + \lambda^2\right] = Cr^2\lambda \left\{2(\underbrace{\cos\xi - 1}_{-2\sin^2\frac{\xi}{2}})\right\}$ ✓

$\therefore \quad \lambda^2 - \lambda\left(2 + 2Cr^2(\cos\xi - 1)\right) + 1 = 0$

$\lambda = \dfrac{2\left(1 + Cr^2(\cos\xi - 1)\right) \pm \sqrt{4\left(1 + Cr^2(\cos\xi-1)\right)^2 - 4}}{2}$

$\lambda = 1 + Cr^2(\cos\xi - 1) \pm \sqrt{\left(1 + Cr^2(\cos\xi - 1)\right)^2 - 1}$

if $\forall \xi$ $\cos\xi - 1$ goes from -2 to 0 ?

$|\lambda| \leq 1 \qquad -1 \leq \lambda \leq 1$

$\lambda_1 = (\quad) + \sqrt{\quad}$ $\quad -1 < \lambda_1 < 1$

$\left(Cr^2(\cos\xi - 1) + \sqrt{\quad}\right) \leq 0$

$\therefore$ λ_2 is critical; $\lambda_2 < 1$ for $\Rightarrow Cr^2(\cos\xi - 1) - \sqrt{\quad} \leq 0$ true

$-1 \leq \lambda_2 \Rightarrow -2 \leq Cr^2(\cos\xi - 1) - \sqrt{(1 + Cr^2(\cos\xi - 1))^2 - 1}$

$2 + Cr^2(\cos\xi - 1) \geq \sqrt{\quad}$

$\left(2 + Cr^2(\cos\xi - 1)\right)^2 \geq \left(1 + Cr^2(\cos\xi - 1)\right)^2 - 1$

$4 + 4Cr^2(\cos\xi - 1) + Cr^4(\cos\xi - 1)^2 \geq +2Cr^2(\cos\xi - 1) + Cr^4(\quad)^2$

$4 \geq -2Cr^2(\cos\xi - 1)$ since $-2 \leq \cos\xi - 1 \leq 0$

35/45

$\lambda_2 \Rightarrow Cr \leq 1 \quad \forall\xi$
$\lambda_1 \Rightarrow Cr \leq 1 \quad \forall\xi$

✓ ie $\Delta t \leq \frac{h}{c}$ same as given in notes.

to ... in some algebra

for $-1 \leq \lambda_1 \Rightarrow 0 \leq \overbrace{2 + Cr^2(\cos\xi - 1)}^{A} + \sqrt{A \cdot Cr^2(\cos\xi - 1)}$

$\Rightarrow (-)A \leq \sqrt{\quad}$

$\Rightarrow (+)A^2 \leq A\,Cr^2(\cos\xi - 1)$ if $A > 0 \Rightarrow A \leq Cr^2(\cos\xi - 1)$?

$2 + Cr^2(\quad) \leq Cr^2(\quad)$?! no.

$\Rightarrow A < 0$

$A \geq Cr^2(\quad) \Rightarrow 2 + Cr^2(\cos\xi - 1) \leq 0$

$\Rightarrow Cr^2 \leq \frac{2}{1 - \cos\xi}$, $Cr \leq 1$

$\lambda_1 \leq 1$

$Cr^2(\quad) + \sqrt{A\,Cr^2(\quad)} \leq 0$

$\sqrt{\quad} \leq -Cr^2(\quad)$

$A\,Cr^2(\quad) \leq Cr^4(\quad)^2$

$A \geq Cr^2(\quad)^2$

$2 \geq 0$ ✓

3b for the second case again error in perturb satisfies

$$e_{n+1}(m) - e_n(m) = \frac{Cr}{2}\left\{(1-\gamma)\left[e_n(m+1) - e_n(m-1)\right] + \gamma\left[e_{n+1}(m+1) - e_{n+1}(m-1)\right]\right\}$$ ✓

if $e_n(m) = \lambda^n e^{im\xi}$ then.

$$e^{im\xi}(\lambda^{n+1} - \lambda^n) = \frac{Cr}{2}\left\{(1-\gamma)\left[\lambda^n e^{im\xi}(e^{i\xi} - e^{-i\xi})\right] + \gamma\left[\lambda^{n+1} e^{im\xi}(e^{i\xi} - e^{-i\xi})\right]\right\}$$ ✓

$$(\lambda - 1) = \frac{Cr}{2}\left\{(1-\gamma)\cdot 2i\sin\xi + \gamma\lambda\cdot 2i\sin\xi\right\}$$ ✓

$$\lambda - 1 = i\, Cr \sin\xi\left\{(1-\gamma) + \gamma\lambda\right\}$$ ✓

if $\gamma = 0$ $\quad \lambda - 1 = i\, Cr\sin\xi$

$$\therefore \lambda = 1 + i\,Cr\sin\xi \qquad |\lambda| = \sqrt{1 + Cr^2\sin^2\xi} \geq 1 \quad \forall\xi.$$

$\therefore$ system is uncond. unstable error grows in amplitude ✓

if $\gamma = 1/2$ $\quad (\lambda - 1) = i\frac{Cr\sin\xi}{2}(1+\lambda)$

$$\therefore \lambda = \frac{1 + i\frac{Cr}{2}\sin\xi}{1 - i\frac{Cr}{2}\sin\xi} = \frac{\left(1 + i\frac{Cr}{2}\sin\xi\right)^2}{1 + \frac{Cr^2}{4}\sin^2\xi} = \frac{\left(1 - \frac{Cr^2\sin^2\xi}{4}\right) + iCr\sin\xi}{1 + \frac{Cr^2\sin^2\xi}{4}}$$

Note that $|\lambda| = \dfrac{\sqrt{\left[1 - \frac{Cr^2\sin^2\xi}{4}\right]^2 + Cr^2\sin^2\xi}}{1 + \frac{Cr^2\sin^2\xi}{4}} = 1.$ hence no growth at $\gamma = 1/2$ ✓

for any γ $\quad \lambda\left[1 - \gamma i Cr\sin\xi\right] = 1 + i\,Cr\sin\xi\,(1-\gamma)$

$$\lambda = \frac{\left(1 + i\,Cr(1-\gamma)\sin\xi\right)\left(1 + i\gamma Cr\sin\xi\right)}{1 + \gamma^2 Cr^2\sin^2\xi}$$

$$= \frac{1 - \gamma(1-\gamma)Cr^2\sin^2\xi + i\,Cr\sin\xi}{1 + \gamma^2 Cr^2\sin^2\xi}$$ ✓

$$|\lambda| = \frac{\sqrt{\left[1 - \gamma(1-\gamma)Cr^2\sin^2\xi\right]^2 + Cr^2\sin^2\xi}}{1 + \gamma^2 Cr^2\sin^2\xi}$$

$$= \frac{\sqrt{1 - 2\gamma(1-\gamma)Cr^2\sin^2\xi + \gamma^2(1-\gamma)^2 Cr^4\sin^4\xi + Cr^2\sin^2\xi}}{1 + \gamma^2 Cr^2\sin^2\xi = \text{denominator}}$$

$$\frac{\sqrt{1 + Cr^2\sin^2\xi\,(2\gamma^2 - 2\gamma + 1) + \gamma^2(1-\gamma)^2 Cr^4\sin^4\xi}}{\text{denom}}$$

for $\gamma = 1/2$, $2\gamma^2 - 2\gamma + 1 = 2/4 - 1 + 1 = 1/2$ $\gamma^2(1-\gamma)^2 = 1/4(1/4) = \frac{1}{16}$

$$\sqrt{1 + \tfrac{1}{2} Cr^2 \sin^2\xi + \tfrac{1}{16} Cr^4 \sin^4\xi} = 1 + \tfrac{1}{4} Cr^2 \sin^2\xi$$

$|\lambda| = 1$ the solution doesn't grow in amplitude $\forall \xi$.

for $\gamma > 1/2$

$$\text{numerator} = (1 + 2\gamma^2 Cr^2 \sin^2\xi + \gamma^4 Cr^4 \sin^4\xi) + Cr^2 \sin^2\xi (1-2\gamma) + Cr^4 \sin^4\xi (\gamma^2 - 2\gamma^3)$$
$$\gamma^2(1-2\gamma)$$

Thus we can write $|\lambda| = \sqrt{1 + \dfrac{(1-2\gamma) Cr^2 \sin^2\xi (1 + \gamma^2 Cr^4 \sin^4\xi)}{(1 + \gamma^2 Cr^2 \sin^2\xi)^2}}$

this term is negative for $\gamma > 1/2$ $\therefore |\lambda| \le 1$ $\forall \xi$.

hence system is unconditionally stable.

45/45

1. $\underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F}$ if $N(\underset{\sim}{d}) = \underset{\sim}{K}\underset{\sim}{d} = \frac{\partial U}{\partial \underset{\sim}{d}}$

then $U = \frac{1}{2} \underset{\sim}{d}^T \underset{\sim}{K} \underset{\sim}{d}$ put into eqn 43 of pg 104 w/ $M = 0$

then

$$\mathcal{F}(\underset{\sim}{d}_{n+1}) = U(\underset{\sim}{d}_{n+1}) - \underset{\sim}{d}_{n+1}^T \underset{\sim}{F}_{n+1}$$

or $$\mathcal{F}(\underset{\sim}{d}) = \frac{1}{2} \underset{\sim}{d}^T \underset{\sim}{K} \underset{\sim}{d} - \underset{\sim}{d}^T \underset{\sim}{F}$$

thus let $\underset{\sim}{c} = \underset{\sim}{d}$ and $\mathcal{F}(\underset{\sim}{c})$ is minimized ie $\underset{\sim}{c}$ is the vector that satisfies $\underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F}$ ✓

~~Define $\underset{\sim}{b} = \{e, d_p, \ldots d_Q\}$ and $\underset{\sim}{P} = (0, 0, \ldots, c, 0, 0, \ldots)$ very long~~

60/60

~~$R\underset{\sim}{b} = \underset{\sim}{P}$ then~~

let $G(\underset{\sim}{d}) = c - a d_p - b d_Q = 0$ ✓

form $H(\underset{\sim}{d}) = \mathcal{F}(\underset{\sim}{d}) + \lambda G(\underset{\sim}{d})$ ✓

then $$\frac{\partial H}{\partial \underset{\sim}{d}}(\delta \underset{\sim}{d}) = \frac{\partial \mathcal{F}}{\partial \underset{\sim}{d}} \delta \underset{\sim}{d} + \lambda \frac{\partial G}{\partial \underset{\sim}{d}} \delta \underset{\sim}{d} + \delta\lambda\, G(\underset{\sim}{d}) = 0$$ ✓

$\forall$ variations $\therefore \Rightarrow G(\underset{\sim}{d}) = 0$ constraint eqn. ✓

and $$\left\{ \frac{\partial \mathcal{F}}{\partial \underset{\sim}{d}} + \lambda \frac{\partial G}{\partial \underset{\sim}{d}} \right\} = 0 \quad \text{or.}$$

$$\cancel{\tfrac{1}{2}}\boxed{\underset{\sim}{K}\underset{\sim}{d}} + \cancel{\tfrac{1}{2} \underset{\sim}{d}^T \underset{\sim}{K}} - \underset{\sim}{F} + \lambda\{-a\,\delta_{\ell p} - b\,\delta_{\ell Q}\} = 0$$ ✓

ie for the p^{th} eqn $K_{p\ell}\, d_\ell - F_p - a\lambda = 0$

for the q^{th} eq $K_{Q\ell}\, d_\ell - F_Q - b\lambda = 0$

for all others $K_{R\ell}\, d_\ell - F_R = 0$

✓ $$\left[\ \underset{\sim}{K}\ \right]\begin{pmatrix} \underset{\sim}{d} \end{pmatrix} = \begin{pmatrix} F \end{pmatrix} + a\lambda \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + b\lambda \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(1 in p^{th} spot; 1 in q^{th} spot)

find $(20 - \lambda^* \cdot 68) d^* = 0 \quad \lambda = \frac{20}{68} = .2941\ldots \quad d^* = 1$

new approx $\quad d = R d^* = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \cdot 1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}$

2. $\int_0^1 w\left(u_{,x}(1+u_{,x}^2)\right)_{,x} dx + \int_0^1 wf\,dx = 0$

$+ \int_0^1 \left[w\left(u_{,x}(1+u_{,x}^2)\right)\right]_{,x} dx - \int_0^1 w_{,x}\left(u_{,x}(1+u_{,x}^2)\right) dx + \int_0^1 wf\,dx = 0$

$+ \; w\left(u_{,x}(1+u_{,x}^2)\right)\Big|_0^1 - \int_0^1 \cdots$

$u(0) = g \Rightarrow w(0) = 0$
$u(1) = 0 \Rightarrow w(1) = 0$
$\quad \therefore \int_0^1 w_{,x}(u_{,x} + u_{,x}^3)\,dx = \int_0^1 wf\,dx$

~~let $\underset{\sim}{f} = \int_0^1 w\,dx \quad \int_0^1 w_{,x}(u_{,x} + u_{,x}^3)\,dx = N$~~

let $\quad w = \sum N_a c_a \quad u = \sum N_a d_a \quad x = \sum N_a x_a \quad f = \sum N_a f_a$

$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} \qquad \frac{\partial \xi}{\partial x} = 1\Big/\frac{\partial x}{\partial \xi} = 1\Big/\frac{x_2 - x_1}{2} = \frac{2}{x_2 - x_1}$

$\int_0^1 \_\_\_\_\_\_ \, dx = \int_{-1}^1 w_{,\xi}\, \xi_{,x} \left(u_{,\xi}\,\xi_{,x} + (u_{,\xi}\,\xi_{,x})^3\right) \frac{dx}{d\xi}\, d\xi$

$\Rightarrow \sum f_b \int_{-1}^1 N_a N_b \frac{2}{h} d\xi = \int_{-1}^1 N_{a,\xi}\left(\sum N_{b,\xi} d_b \cdot \frac{2}{h} + \left(\sum N_{b,\xi} d_b \cdot 2/h\right)^3\right) d\xi$

for each element

$\underset{\sim}{f}^e = \sum f_b^e \int_{-1}^1 N_a N_b \cdot \frac{2}{h^e} d\xi \qquad \underset{\sim}{f} = \overset{nel}{\underset{e=1}{A}} \underset{\sim}{f}^e$

$\hat{n}^e(\underset{\sim}{d}^e) = \int_{-1}^1 N_{a,\xi}\left(\sum_{d=1}^{n_{en}} N_{b,\xi} d_b^e \cdot \frac{2}{h^e} + \left(\sum N_{b,\xi} d_b^e \cdot 2/h^e\right)^3\right) d\xi \qquad N(d) = A\,N($

$Dn = \int_{-1}^1 N_{a,\xi}\left(\sum N_{b,\xi}\,\delta_{bc}\, 2/h^e + 3\left(\sum N_{b,\xi} d_b^e \cdot 2/h^e\right)^2 \cdot \sum N_{b,\xi}\,\delta_{bc}\, 2/h^e\right) d\xi$

$$DN^e = \int_{-1}^{1} N_{a,\xi}\left(N_{c,\xi}\, ^2/_{h^e} + 3\left(\sum_{b=1}^{2} N_{b,\xi}\, d_b^e \cdot {}^2/_{h^e}\right)^2 N_{c,\xi} \cdot {}^2/_{h^e}\right) d\xi$$

$$DN = A(DN^e)$$

3. $\dot{y} = Gy + H(t)$

$$\frac{y_{n+1} - y_{n-1}}{2\Delta t} = G y_n + H \qquad \text{Thus } \dot{y} \text{ uses centered difference}$$

$|\tau_n| \le C \Delta t^{k+1}$ where k = order of accuracy

$k = 2$ for $\alpha = 1/2$ centered diff / midpoint rule

$$y_{n+1} = y(t_n) + \Delta t\, y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + \frac{\Delta t^3}{6} y'''(\overline{t_{n_1}})$$
$$y_{n-1} = y(t_n) - \Delta t\, y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) - \frac{\Delta t^3}{6} y'''(\overline{t_{n_2}}) + \ldots$$
$$~~\frac{y_{n+1} - y_{n-1}}{2\Delta t} = y'(t_n) + \frac{\Delta t^2}{6} y'''(t_n)~~$$

now $\tau(t_n) = y(t_{n+1}) - y(t_{n-1}) - 2\Delta t G\, y(t_n) - 2\Delta t\, H(t_n)$

$$= 2\Delta t\, y'(t_n) + \frac{\Delta t^3}{6}\left[y'''(\overline{t_{n_1}}) + y'''(\overline{t_{n_2}})\right] - 2\Delta t\left[G y(t_n) + H(t_n)\right]$$

now $y'(t_n) = G\, y(t_n) + H(t_n)$

$$\tau(t_n) = 2\Delta t\left[G\, y(t_n) + H(t_n)\right] - 2\Delta t\left[G y(t_n) + H(t_n)\right] + \frac{\Delta t^3}{3} \cdot [\quad]$$

$\therefore \tau(t_n) = C \Delta t^3 \qquad \therefore \tau(t_n) \sim \Delta t^3$ or $k = 2$ 2nd order accurate

for (b) heat equation $k = 2 \Rightarrow \alpha = 1/2 \quad \therefore$ it is unconditional stable

Proof ~~let~~ let ~~$y_{n+1} = \lambda^{n+1} \quad y_{n-1} = \lambda^{n-1}$~~

then $\lambda^{n-1}\left[\lambda^2 - 1 - 2\Delta t G \lambda\right] = 0$

$$\therefore \quad \therefore \lambda = \frac{+2\Delta t G \pm \sqrt{4\Delta t^2 G^2 + 4}}{2}$$

$$= +\Delta t G \pm \sqrt{\Delta t^2 G^2 + 1}$$

for stability $|\lambda| < 1$

$$-1 < \Delta t G \pm \sqrt{} < 1$$

$$\pm\sqrt{} < (1 - \Delta t G)$$

$$\Delta t^2 G^2 + 1 < 1 - \Delta t G$$

---

for the undamped eqn of motion. ~~this corresponds to central difference method which is conditionally stable.~~

~~and $\Delta t = \frac{2}{\omega_{max}}$~~

$$M\dot{v} + Kd = F$$

$$\dot{v} + M^{-1}Kd = M^{-1}F$$

$$\begin{pmatrix} \dot{d} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & 0 \end{pmatrix}\begin{pmatrix} d \\ v \end{pmatrix}$$

G

$$\dot{y} = G y$$

$$\begin{pmatrix} d_{n+1} \\ v_{n+1} \end{pmatrix} - \begin{pmatrix} d_{n-1} \\ v_{n-1} \end{pmatrix} = 2\Delta t G \begin{pmatrix} d_n \\ v_n \end{pmatrix}$$

$$y_{n+1} = y_n +$$

$$\left(\right)_{n+1} = \left(\right)_n + \Delta t \left(\dot{}\right)_n + \frac{\Delta t}{2}\begin{pmatrix} \dot{v}_n \Delta t \\ \dot{v}_{n+1} \end{pmatrix}$$

~~$\left(\right)_n = \left(\right)_{n-1} + \Delta t \left(\dot{}\right)$~~

$$d_{n+1} = d_n + \Delta t v_n + \frac{\Delta t^2}{2} a_n \qquad \left(\right)_{n+1} = \left(\right)_n +$$

$$v_{n+1} = v_n + \frac{\Delta t}{2}(a_n + a_{n+1})$$

$$v_{n+1} - \frac{\Delta t}{2} a_{n+1} = v_n + \frac{\Delta t}{2} a_n \; ; \; v_{n+1} + \frac{\Delta t}{2} a_{n-1} = v_n + \frac{\Delta t}{2} a_n$$

$$\overset{1}{\alpha_0}\, \underline{y}_{n+1} + \overset{0}{\alpha_1}\, \underline{y}_n + \overset{-1}{\alpha_2}\, \underline{y}_{n-1} + \Delta t\, \overset{0}{\beta_0}\, \underline{f}(\underline{y}_{n+1}, t_{n+1})$$

$$+ \Delta t\, \beta_1\, \underline{f}(\underline{y}_n + \ldots)$$

$$\beta_1 = -1$$

$$\dot{\begin{pmatrix} d \\ v \end{pmatrix}}_n = G \begin{pmatrix} d \\ v \end{pmatrix}_n$$

$$\begin{pmatrix} d \\ v \end{pmatrix}_{n+1} = \begin{pmatrix} d \\ v \end{pmatrix}_n + \Delta t \dot{\begin{pmatrix} d \\ v \end{pmatrix}}_n + \frac{\Delta t^2}{2} \underset{\sim}{a}_n$$

w/ $\gamma = 0$ $\beta = 0$

$\Rightarrow$

$$\begin{pmatrix} d \\ v \end{pmatrix}_{n-1} = \begin{pmatrix} d \\ v \end{pmatrix}_n - \Delta t \dot{\begin{pmatrix} d \\ v \end{pmatrix}}_n + \frac{\Delta t^2}{2} \underset{\sim}{a}_n$$

diff of the two leads to

$$\left(\ \right)_{n+1} - \left(\ \right)_{n} = 2\Delta t \dot{\left(\ \right)}_n$$

Ω_{crit} $(\xi = 0,\ \beta = 0,\ \gamma = 0) \Rightarrow$ unconditional unstability

$$\dot{\begin{pmatrix} Y \\ v \end{pmatrix}} = G \qquad M\ddot{d} + Kd = 0$$

let $e^{i\omega t} = d$

$$\underset{\sim}{d}\,(-\omega^2 M + K) = 0$$

$$(-\omega^2 I + G)\,\underset{\sim}{d}$$