Winter Quarter 1981

DEPARTMENT OF MECHANICAL ENGINEERING

Finite Element Methods, ME 235A,B,C

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The course will be taught over a three term period. Each succeeding term will build upon previous work. A tentative outline is as follows:

First term: Emphasis on fundamental concepts and techniques of "primal" finite element methods. Method of weighted residuals, Galerkin's method, and variational equations. Linear elliptic boundary value problems in one, two, and three space dimensions; applications in structural, solid, and fluid mechanics, and heat transfer. Properties of standard element families, numerically integrated elements including reduced integration. Penalty and generalized displacement methods for application to constrained field theories such as classical plate theory, incompressible elasticity, Stokes flow, etc. Thick and thin beams, plates and shells. "Upwind" finite elements for convective/diffusive transport phenomena.

Implementation of the finite element method. A simple finite element computer code for static problems. Compacted column equation solver, assembly of equations and element routines. Comparison of finite element results with exact solutions.

Mathematical theory of finite elements. Approximation properties of finite element spaces; stability, consistency, convergence, and accuracy. Standard error estimates, Aubin-Nitsche theory, patch test, superconvergence and Barlow stress points. (Useful results will be emphasized, the treatment will be primarily expository and thus no serious functional analysis background will be required.

Brief mention of hybrid, mixed, and equilibrium models.

Second Term: Finite element methods for linear dynamic analysis. Eigenvalue, parabolic and hyperbolic problems. Mathematical properties of semi-discrete (t-continuous) Galerkin approximations. Modal decomposition and direct spectral truncation techniques. Stability, consistency, convergence, and accuracy of ordinary differential equation solvers. Asymptotic stability, "overshoot" and conservation laws for discrete algorithms. Mass reduction. Applications in heat convection/conduction, structural vibrations and elastic wave propagation. Computer implementation of finite element methods in linear dynamics. Implicit, explicit, and "implicit-explicit" algorithms and code architectures. Various operator splitting techniques.

Third term: Nonlinear finite element analysis. Brief survey of nonlinear continuum mechanics. Eulerian and Lagrangian formulations. Galerkin formulation of nonlinear elliptic, eigenvalue, parabolic and hyperbolic problems. Explicit, implicit, and "implicit-explicit" algorithms in nonlinear transient analysis. Stability of ordinary differential equation solvers for nonlinear problem classes; "energy-conserving algorithms." Methods of solving nonlinear algebraic systems. Newton-type methods and quasi-Newton updates.

Architecture of computer codes for nonlinear finite element analysis. Applications from structural, solid, and fluid mechanics, e.g. nonlinear elasticity, plasticity, viscoplasticity, nonlinear structural models, material and geometric nonlinearities, postbuckling, Navier-Stokes equations, etc. Special topics of interest to the class: for example, nonlinear heat conduction, contact-impact problems, large problem computer code architecture, blocking, overlay structures, multilevel meshes, etc.

Prerequisites: A serious desire to learn the material will suffice in place of any hard-line prerequisties. However, desirable background would include a working knowledge of standard FORTRAN IV, some elasticity and/or structural analysis, linear algebra, an understanding of ordinary differential equations, and the basic properties of the fundamental classes of partial differential equations (i.e. elliptic, parabolic, and hyperbolic).

ESMB 113 ME 235A 1/6/81

Finite Element Methods Hughes 283 Durand x 7-2040

Office Hrs Tu-Th Afternoons

No text required - Class Notes will be self contained

Grades: 1 Hr. Midterm (In Class) 25% x } (1-ε)

3 Hr. Final (In Class) 75% x

HW's ε %

Course Outline : Handouts 100%

1/8/81

Part I Linear Static Problems

Chapt 1 - Fund. Concepts presented in context of simple 1-D Bdy Value Prob

§0. Prelims: 2 main constituents of F.E.M.

(i) Variational or weak formulation of BVP

(ii) approx. Soln of weak formulations via F.E. fns.

Ex: $u_{,xx} + f(x) = 0$ f is given & defined on $x \in [0,1]$ & is real valued $f:[0,1] \to \mathbb{R}$

we also assume f(x) is smooth

this is representation of heat conduction or transverse displ of string under tension or longitudinal displacement of beam.

§1. Strong or Classical Statement of the BVP

BC's suppose we want $u(1) = g$ const & $u_{,x}(0) = h$ const.

this leads to 2-PT BVP

(S) The strong form of problem will be stated as: give $f:[0,1] \to \mathbb{R}$, constants g, h,

Find $u:\underset{\bar{\Omega}}{[0,1]} \to \mathbb{R}$ ∋ $u_{,xx} + f = 0$ on $\Omega =]0,1[$ (or (0,1))

and $u(1) = g$ $u_{,x}(0) = h$

Aside

exact soln $u(x) = g + (x-1)h + \int_x^1 \int_0^y f(z)\,dz\,dy$

§2. Weak, variational Counterpart, form of BVP beginnings of F.E.

Introduce 2 collections of functions

$\mathcal{S}$ = collection of trial solns = $\{ u \mid u(1) = g,$ require u to have 1st deriv that are square integrable $\int_0^1 (u_{,x})^2 dx < \infty$ ie u is in H^1 (ie $u \in H^1$) & $u:\bar{\Omega} \to \mathbb{R}$ $\}$

$\mathcal{V}$ = collection of weighting functions (variations)

$= \{ w \mid w(1) = 0$ (homogeneous counterpart), $w \in H^1 \}$

(W) The weak statement of the problem is stated as:

Given f, g, h, as before, find $u \in \mathcal{S}$ $\ni$ $\forall\, w \in \mathcal{V}$

$$(*) \quad \int_0^1 w_{,x} u_{,x}\, dx = \int_0^1 w f\, dx - w(0) h$$

— Variational Equns.

also known as virtual work equns.

— principle of virtual displ (these are the w's)

Remark: 1. $\exists$ other weak statements. But this is most useful for our appl.

2. (W) is the basis for F.E. approx.

§3. (S) $\Longleftrightarrow$ (W) equivalence ie the soln u of (S) $\equiv$ soln u of (W)

soln is unique for each problem.

"formal proof" but will not be rigorous. We will use "smoothness" but result true in general.

$$\int_0^1 w(u_{,xx} + f)\, dx = 0 \qquad \int_0^1 w u_{,xx}\, dx = w u_{,x}\Big|_0^1 - \int_0^1 w_{,x} u_{,x}\, dx$$

$$= -w(0) h - \int_0^1 w_{,x} u_{,x}\, dx$$

$$\Rightarrow \int_0^1 w f\, dx - w(0) h - \int_0^1 w_{,x} u_{,x}\, dx = 0 \quad \text{Result follows.}$$

Proposition: a. Let u be a soln of (S) $\Rightarrow$ then u is a soln of (W)

b. Let u be a soln of (W) $\Rightarrow$ then u is a soln of (S)

"Formal Proof" a (Trivial) Assume u soln of (S) $\begin{bmatrix} u_{,xx} + f = 0 \\ u(1) = g \\ u_{,x}(0) = h \end{bmatrix}$ must show u is a sol of (W)

since satisfies $u(1) = g \Rightarrow u \in \mathcal{S}$

now mult by $w \in \mathcal{V}$ & integrate over $[0,1]$ $\therefore$ $0 = \int_0^1 w\left[u_{,xx} + f\right]$

integrate by part $\quad = -\int_0^1 w_{,x} u_{,x}\, dx + (w\, u_{,x})\Big|_0^1 + \int_0^1 w f\, dx$

since $w \in \mathcal{V}$ & $w(1) = 0$ then $w(u_{,x})\Big|_0^1 = -w(0)\, u_{,x}(0)$

since $u \in \mathcal{S}$ then $u_{,x}(0) = h$ $\therefore$

$$0 = -\int_0^1 w_{,x} u_{,x}\, dx - w(0) h + \int_0^1 w f\, dx \quad \text{Result follows}$$

Since this was satisfied $\forall\, w \in \mathcal{V}$ hence we've shown a.

To show b must use fundamental lemma of Calculus of Variations n.b. u is a sol. of (W) $\left[\begin{array}{l} u \in \mathcal{S} \\ u(1)=g \\ \text{and } u \text{ satisfies variational eqn} \end{array}\right]$ must show that u satisfies $\left[\begin{array}{l} u_{xx}+f=0 \\ u(1)=g \\ u_{,x}(0)=h \end{array}\right]$

Start with $\int_0^1 w_{,x}u_{,x}\,dx = \int_0^1 wf\,dx - w(0)h \quad \forall w \in \mathcal{V}$

Integrate by parts

$$-\int_0^1 w\,(u_{,xx}+f)\,dx + wu_{,x}\Big|_0^1 = -w(0)h$$

Since $w \in \mathcal{V}$ $w(1)=0$ thus we can write

$$0 = \int_0^1 w\,(u_{,xx}+f)\,dx + w(0)[u_{,x}(0)-h] \qquad (**)$$

Must show $u_{,xx}+f=0$ & $u_{,x}(0)=h$. To do this define $w = \phi(u_{,xx}+f)$ where $\phi(x)>0$ $x \in (0,1)$ smooth & $\phi(x)=0$ on $x=0,1$ ie $w(0)=w(1)=0$

Now substitute into above (**) thus $0 = \int_0^1 \phi\,(u_{,xx}+f)^2\,dx + 0$; $(u_{xx}+f)^2 \geq 0$ and $\phi(x)>0 \Rightarrow u_{,xx}+f=0 \quad \forall x \in [0,1]$

Now use this to establish $u_{,x}(0)=h$ as follows: since $u_{,xx}+f=0$ $\forall x \in [0,1]$ for any w then $0 = 0 + w(0)[u_{,x}(0)-h]$; now $w \in \mathcal{V}$ hence $w(0)$ need not be zero thus $u_{,x}(0)-h=0$ and b is proved.

<u>Terminology assoc. with types of bdy cond.</u>

1. BC's which are built into the dfns. of the collections of fns. (ie $u(1)=g$ or $w(1)=0$) are called essential bdy conditions
2. BC's for which there is no explicit statement in (W), but which are implied by satisfaction of (*) are called <u>natural bdy. conditions</u> (ie $u_{,x}(0)=h$)

Summary (W) ⟺ (S) FE's BEGINS here

New Notation introduced to simplify writing and to provide important understanding of mathematical propositions of F.E.M.'s. Also it applies to diverse problems.

Define $a(w,u) = \int_0^1 w_{,x}u_{,x}\,dx$ $\quad a(\cdot,\cdot)$ } symmetric bilinear forms

$(w,f) = \int_0^1 wf\,dx$ $\quad (\cdot,\cdot)$

Symmetric bilinear form. $a(w,u) = a(u,w)$, $(w,f) = (f,w)$ } Symmetry property

Bilinearity – linearity in each slot ie if $c_1, c_2 \in \mathbb{R}$ u, v, w fns.

$$a(c_1 u + c_2 v, w) = a(c_1 u, w) + a(c_2 v, w) = c_1\, a(u,w) + c_2\, a(v,w)$$

also $(c_1 u + c_2 v, w) = c_1 (u,w) + c_2 (v,w)$

Thus (*) can be written as $a(w,u) = (w,f) - w(0)h$

§ 4. Galerkin's Approximation Method – uses symmetric forms very well

1/13/81

Final sched 7-10PM Friday; exam will be given 7-10 PM Thurs 12 March

§ 4. Galerkin's Approximation Method.

(abstract version) we will approx the weak form.

We will introduce more collections of fns.

1. $\mathcal{S}^h$ is an approx. of $\mathcal{S}$ h – grid size in our FE mesh. (characteristic length)

$\mathcal{S}^h$ = all linear combinations of a finite set of fns. This will be given

$\mathcal{S}^h \subset \mathcal{S}$ (means if $u^h \in \mathcal{S}^h \Rightarrow u^h \in \mathcal{S} \Rightarrow u^h(1) = g$ + square integrable cond.

2. $\mathcal{V}^h$ is an approx. of $\mathcal{V}$ (finite dimensional subset of $\mathcal{V}$)

(means if $w^h \in \mathcal{V}^h \Rightarrow w^h \in \mathcal{V} \Rightarrow w^h(1) = 0$ + square integrable conditions

(Bubnov) Galerkin Method. – Assume that $\mathcal{V}^h$ is given; construct $\mathcal{S}^h$ as follows: Take $v^h \in \mathcal{V}$ define $u^h = v^h + g^h$

$(\in \mathcal{S}^h)$ $(\in \mathcal{S} \ni g^h(1) = g)$ g^h is given

unknown part of trial sol.

$\therefore u^h(1) = v^h(1) + g^h(1) = 0 + g$

Thus $\mathcal{S}^h$ = all fns of the form $u^h = v^h + g^h$

Now put $\binom{u^h}{w^h}$ into $a(w,u) = (f,w) - w(0)h$

$$a(w^h, v^h + g^h) = (w^h, f) - w^h(0)h$$
$$a(w^h, v^h) + \underbrace{a(w^h, g^h)}_{\text{known}} = \underbrace{(w^h, f)}_{\text{known}} - w^h(0)h$$

only v^h is unknown. The above implies the following problem

(G) $\begin{cases} \text{given } f, g, h \text{ find } u^h \in \mathcal{S}^h \ni \forall w^h \in \mathcal{V}^h \text{ with the condition} \\ a(w^h, v^h) = (w^h, f) - w^h(0)h - a(w^h, g^h) \end{cases}$

(w^h, v^h $\in \mathcal{V}^h$; right side given)

Aside

(Petrov) Galerkin Method

Formalism is the same as above <u>but</u> we don't require $v^h \in \mathcal{V}^h$
but we can say that $v^h \in \mathcal{S}_0^h$ ie satisfies homog. b.c.

§ 5. MATRIX EQN'S, (STIFFNESS MATRIX $\underset{\sim}{K}$).

- We will adopt Bubnov version

$\mathcal{V}^h$: linear combinations of N_A's $A = 1, 2, \ldots, n$ $\ni$. $N_A \in [0,1]$
with $N_A(1) = 0$

if $w^h \in \mathcal{V}^h$ then $w^h = \sum_{A=1}^{n} c_A N_A$ c_A's are constants

N_A's called shape fns, interpolation fns, basis fns.

$\mathcal{S}^h$: $\underset{\in \mathcal{V}^h}{u^h = v^h} + g^h$ we can write $u^h = \sum_{A=1}^{n} d_A N_A + g^h$
where $g^h = g N_{n+1}$ $\ni$. $N_{n+1}(1) = 1$
where $N_{n+1} \notin \mathcal{V}^h$ (otherwise $N_{n+1}(1) = 0$).

$\{N_A\}_1^{n+1}$ are normally given. The unknown in u^h is the d_A's. Put into (G) next

$$\therefore \quad a\left(\sum_{A=1}^{n} c_A N_A, \sum_{B=1}^{n} d_B N_B\right) = \left(\sum_{A=1}^{n} c_A N_A, f\right) + \left(\sum_{A=1}^{n} c_A N_A(0)\right) h + a\left(\sum_{A=1}^{n} c_A N_A, g N_{n+1}\right)$$

now we use bilinearity of $a(\cdot,\cdot)$, $(\cdot,\cdot)$ to get

$$\therefore \quad \sum_{A=1}^{n} c_A \left\{ \sum_{B=1}^{n} d_B a(N_A, N_B) - (N_A, f) + N_A(0) h + a(N_A, N_{n+1}) g \right\} = 0$$

this must hold $\forall\, w^h \in \mathcal{V}^h$ ie $\forall\, c_A$

$$\sum_{A=1}^{n} c_A \{\cdots\}_A = 0 \Rightarrow \{\cdots\}_A = 0 \quad \forall A = 1, \ldots, n$$

or

$$\sum_{B=1}^{n} d_B\, a(N_A, N_B) = (N_A, f) - N_A(0) h - a(N_A, N_{n+1}) g$$

These are n linear eqns in n unknowns (d_B). Define $K_{AB} = a(N_A, N_B)$

define $F_A = (N_A, f) - N_A(0) h - a(N_A, N_{n+1}) g$

$$\therefore \quad \sum_{B=1}^{n} K_{AB} d_B = F_A \qquad \text{Thus } [K]\underset{\sim}{d} = \underset{\sim}{F}$$

$$K = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & \ddots & & \vdots \\ \vdots & & & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix}$$

Now define the following matrix problem

(M) $\begin{cases} \text{given } \underset{\sim}{K}, \underset{\sim}{F} \text{ find } \underset{\sim}{d} \\ \ni. \ K\underset{\sim}{d} = \underset{\sim}{F} \end{cases}$

$\underset{\sim}{d}$ is akin to displacement matrix
$\underset{\sim}{F}$ " " " force "

$\therefore \quad \underset{\sim}{d} = \underset{\sim}{K}^{-1} \underset{\sim}{F}$ as long as $\underset{\sim}{K}^{-1}$ exists.

Remarks 1. (G) is a special case of a so-called weighted residual method. Standard reference Finlayson [F3] [F4]

2. $\underset{\sim}{K}$ is symmetric ($\underset{\sim}{K} = \underset{\sim}{K}^T$)

Summary

Pattern of Development of Element Technique

Start with (S) $\Longleftrightarrow$ (W) $\approx$ (G) $\Longleftrightarrow$ (M)

infinite dim \ finite dim — only approx being between (W) & (G)

§6. Examples involving 1 & 2 degrees of freedom.

1. 1-Degree of freedom (n=1) $\mathcal{V}^h$ is 1-Dim ie if $w^h \in \mathcal{V}^h \Rightarrow w^h = c_1 N_1$

what is $u^h = v^h + g N_2 = d_1 N_1 + g N_2$

$N_1(1) = 0 \quad N_2(1) = 1$ pick N_1, N_2 $\quad N_1(x) = 1 - x \quad N_2(x) = x$

Now $\underset{\sim}{K} = K_{11} \quad \underset{\sim}{F} = F_1 \quad \underset{\sim}{d} = d_1$ ie $K_{11} = \int_0^1 (N_{1,x})^2 dx = 1$

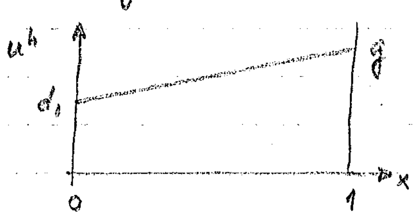
$\therefore \quad d_1 = F_1$

$= (N_1, f) - N_1(0) h - a(N_1, N_2) g$

$d_1 = \int_0^1 (1-x) f(x) dx - h + g \qquad a(N_1, N_2) = \int_0^1 N_{1,x} N_{2,x} dx = -1$

$u^h = d_1 N_1 + N_2 g = (1-x)\left[\int_0^1 (1-x) f(x) dx - h + g\right] + xg = g - (1-x)h + (1-x)\int_0^1 (1-x') f(x') dx'$

$\int_x^1 \int_0^y (1-x') f(x') dx' \, dy$



recall exact sol. $u(x) = g + (x-1) h + \int_x^1 \left\{ \int_0^y f(z) dz \right\} dy.$

Case (i) $f(x) = 0 \quad \therefore \quad u(x) = g + (x-1) h$
$u^h(x) = g + (x-1) h$ $\quad \Big\} \; u = u^h$

1/15/81

Beginning Thurs Jan 22 - Office Hrs 1:30-2:30 Tu & Th.

LAST TIME: (G) $a(w^h, v^h) = (w^h, f) - w^h(0) h - a(w^h, g^h)$

Find $v^h \in \mathcal{V}^h \ni \; \forall \; w^h \in \mathcal{V}^h$ the above eqn holds

Matrix Eqns $\quad w^h = \sum_{A=1}^{n} c_A N_A \quad$ w/ $\quad N_A(1) = 0 \qquad A = 1, 2, \dots, n$

$$u^h = v^h + g^h = \sum_{A=1}^{n} d_A N_A + g N_{n+1} \qquad N_{n+1}(1) = 1$$

$\Rightarrow$ (M) $\quad K\underset{\sim}{d} = \underset{\sim}{F}$

$$K_{AB} = a(N_A, N_B)$$

$$F_A = \dots \ (N_A, f) - N_A(0)h - a(N_A, N_{n+1})\, g$$

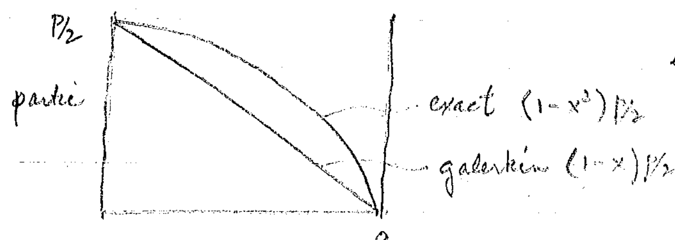
Ex's $\quad$ 1 - DOF

$N_2(1) = 1$
$N_1(1) = 0$

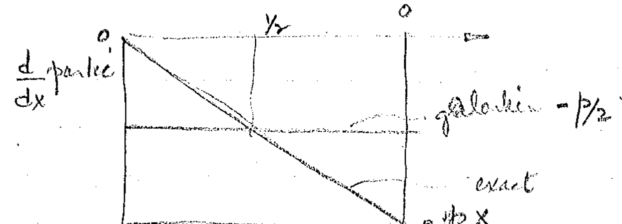
assuming $f = 0 \qquad u^h = u = g + (x-1)h$

Assume $f = p \qquad u = g + (x-1)h + p/2\,(1-x^2)$
it turns out $\quad u^h = g + (x-1)h + p/2\,(1-x)$ } particular
homogeneous part is exact.

the particular part will be where the approx. comes in



exact at two end points but no where else

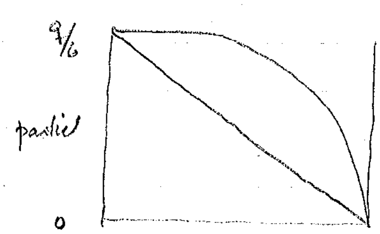


exact at one point

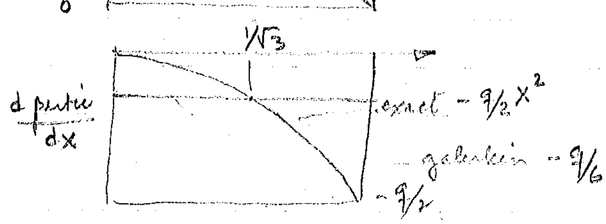
we will note the same pattern

Assume $f = qx \qquad q = $ const.

$u(x) = $ homogeneous $+ q/6\,(1-x^3)$
$u^h(x) = \quad$ " $\quad + q/6\,(1-x)$ } particular



exact at two points

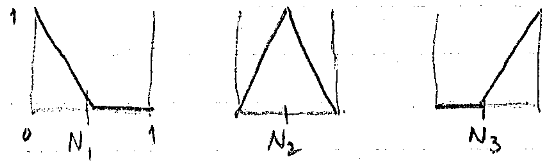


exact at one point

Example 2-DOF ($n=2$)

$w^h = c_1 N_1 + c_2 N_2$

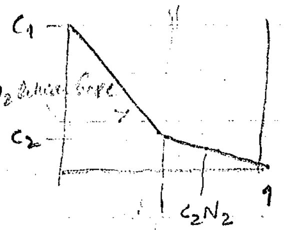
$u^h = d_1 N_1 + d_2 N_2 + g N_3$



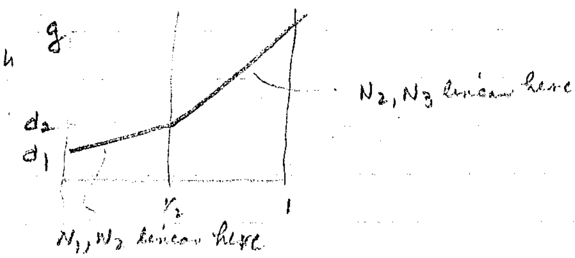
pick midpt as pt of slope discont.

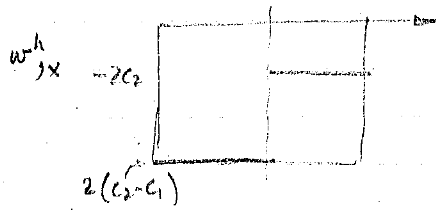
$N_1(1) = N_2(1) = 0 \quad N_3(1) = 1$

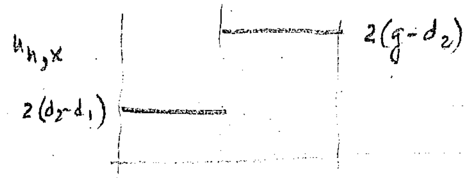
typical w^h



typical u^h







$$
\begin{aligned}
N_1 &= 2(1/2 - x) && 0 \le x \le 1/2 && N_{1,x} = -2 \\
&\quad 0 && 1 \ge x \ge 1/2 && \quad = 0 \\
N_2 &= 2x && 0 \le x \le 1/2 && N_{2,x} = 2 \\
&\quad 2(1-x) && 1/2 \le x \le 1 && \quad = -2 \\
N_3 &= 0 && 0 \le x \le 1/2 && N_{3,x} = 0 \\
&= 2x - 1 && 1/2 \le x \le 1 && \quad = 2
\end{aligned}
$$

$$
K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \qquad \underset{\sim}{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}
$$

$$
K_{AB} = \int_0^1 N_{A,x} N_{B,x}\, dx = \left(\int_0^{1/2} + \int_{1/2}^1 \right) N_{A,x} N_{B,x}\, dx
$$

thus if we plug in & work out proble

$$
K_{AB} = 2 \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}
$$

$$
F_1 = (N_1, f) - N_1(0) h - a(N_1, N_3) g
$$

$$
= \int_0^1 N_1 f\, dx - \ldots h - \int_0^1 N_{1,x} N_{3,x}\, g\, dx
$$

since $N_1(0) = 1$; $= 0$ since $N_{1,x}$ & $N_{3,x} = 0$ in each domain

$$
F_2 = \int_0^1 N_2 f\, dx - N_2(0) h - \int_0^1 N_{2,x} N_{3,x}\, g\, dx
$$

$= 0$ since $N_2(0) = 0$

$$= 2\int_0^{1/2} x f\,dx + 2\int_{1/2}^{1} (1-x) f(x)\,dx + 2g$$

Case 1 $f = 0$

$$\underset{\sim}{F} = \begin{Bmatrix} -h \\ 2g \end{Bmatrix}$$

$$\underset{\sim}{d} = K^{-1}\underset{\sim}{F} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{pmatrix} -h \\ 2g \end{pmatrix}$$

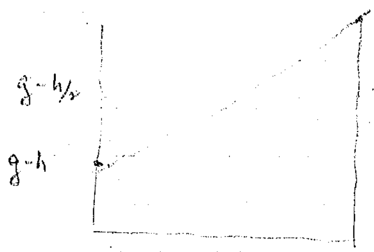
$$\underset{\sim}{d} = \begin{pmatrix} g-h \\ g-h/2 \end{pmatrix}$$

$$\therefore \quad u^h = (g-h)N_1 + (g-h/2)N_2 + gN_3$$

$$(g-h)(1-2x) + (g-h/2)(2x) \qquad x \le 1/2$$

$$(g-h/2)2(1-x) + g(2x-1) \qquad x \ge 1/2$$

$$= h(x-1) + g \equiv u$$



Case 2 if $f = p = \text{const}$

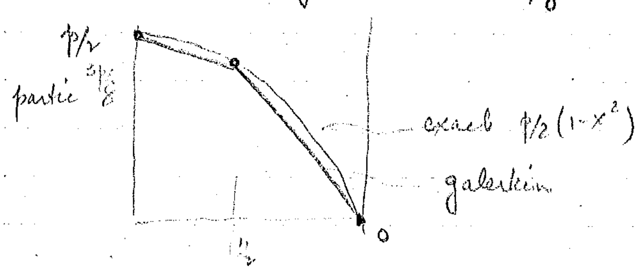
$$F_1 = p/4 - h$$

$$F_2 = p/2 + g$$

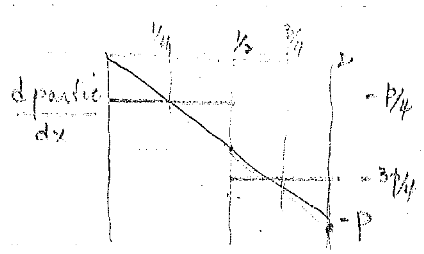
$$\underset{\sim}{d} = \begin{Bmatrix} p/2 + g-h \\ 3p/8 + g-h/2 \end{Bmatrix}$$

(partic) (homog.)

$$u_h = \text{homog} + p/2\,N_1 + 3p/8\,N_2$$



exact at 3 points



2 points coincide at $x = 1/4,\ 3/4$

$\therefore$ for n degrees of freedom
u^h is exact at $n+1$ pts (2 pts are ends)
$\frac{du^h}{dx}$ " " at n pts

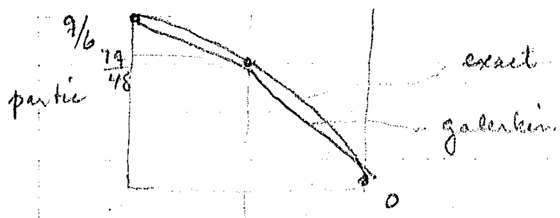
Case 3 when $f = qx$

$$F_1 = q/24 - h$$

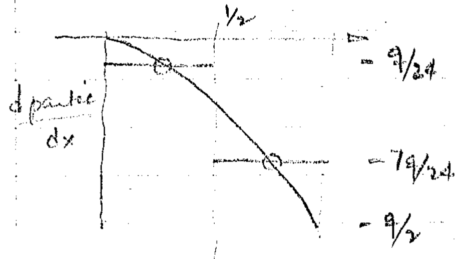
$$F_2 = q/4 + 2g$$

(homog)

$$\underset{\sim}{d} = \begin{pmatrix} q/6 + \text{homog} \\ 7q/48 + \text{homog} \end{pmatrix}$$



u^h is exact at 3 points



derivative has 2 pts exact.

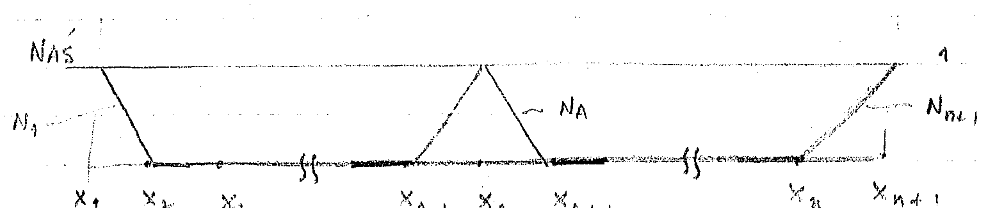
§ 7. PIECEWISE LINEAR F.E. Space

Generalizing to N-dimensional space U^h

partition [0,1] into N subintervals

$$0 = x_1 < x_2 \ldots < x_n < x_{n+1} = 1.$$

x_A's are called nodes, joints, knots $\qquad h_A = x_{A+1} - x_A \neq$ constant.



for interior nodes - look at typical fn N_A

for x_1 use N_1

for x_{n+1} use N_{n+1}

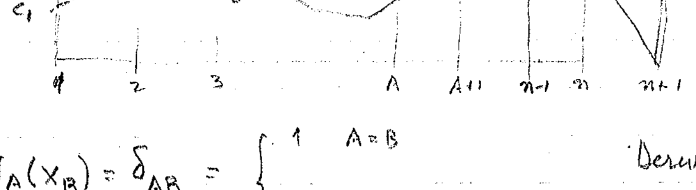
$$N_A = \begin{cases} (x - x_{A-1})/h_{A-1} & x_{A-1} \le x \le x_A \\ (x_{A+1} - x)/h_A & x_A \le x \le x_{A+1} \\ 0 & \text{elsewhere} \end{cases}$$

$$N_1(x) = \begin{cases} (x_2 - x)/h_1 & x_1 \le x \le x_2 \\ 0 & \text{elsewhere} \end{cases}$$

$$N_{n+1}(x) = \begin{cases} (x - x_n)/h_n & x_n \le x \le x_{n+1} \\ 0 & \text{elsewhere} \end{cases}$$

$$w^h = \sum_{A=1}^{n} c_A N_A \in U^h$$ is piecewise linear fn.

w/ values c_A at node x_A

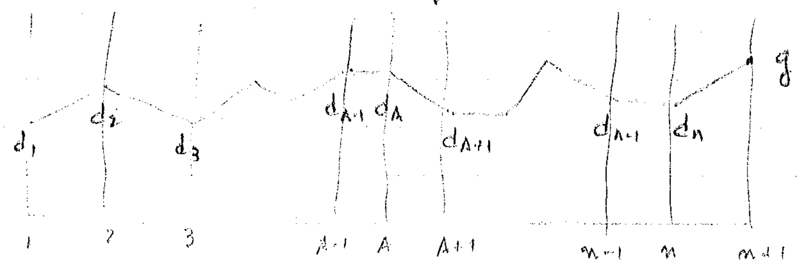


Remark $N_A(x_B) = \delta_{AB} = \begin{cases} 1 & A = B \\ 0 & A \neq B \end{cases}$

Derivatives are piecewise constant

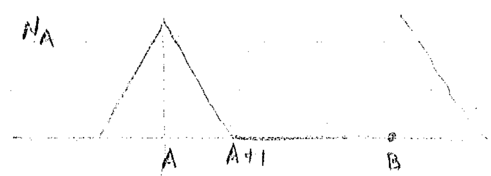
$h \overset{\text{def}}{=} \max_{A=1,\dots,n} (h_A)$ if h_A were equal $h_A = h = 1/n$

for $u^h \in \mathcal{S}^h$ $\quad u^h = \sum_{A=1}^{n} d_A N_A + g N_{n+1}$



§ 8 Properties of $\underset{\sim}{K}$

focus our attention on node A



if $B > A+1$, $B < A-1$ $\quad K_{AB} = \int_0^1 N_{A,x} N_{B,x}\, dx \equiv 0$

thus

$$K = \begin{bmatrix} K_{11} & K_{12} & 0 & \cdots & & 0 \\ K_{12} & K_{22} & K_{23} & 0 & \cdots & 0 \\ 0 & K_{32} & K_{33} & K_{34} & 0 \cdots & 0 \\ \vdots & & & & & \\ & & & & & K_{n-1\,n} \\ 0 & 0 & \cdots & & K_{n\,n-1} & K_{nn} \end{bmatrix}$$

since $\underset{\sim}{K}$ is sym. then

bandwidth = 3 elements

1/2 bandwidth = 2 elements

Now defn. $n \times n$ matrix $\underset{\sim}{A}$ is +-ive definite if

1. $\underset{\sim}{c}^T A \underset{\sim}{c} \geq 0 \quad \forall \underset{\sim}{c}$
2. $\underset{\sim}{c}^T A \underset{\sim}{c} = 0$ if $\underset{\sim}{c} = \underset{\sim}{0}$

$\underset{\sim}{A}$ is always invertible if $\underset{\sim}{A}$ is positive definite

Proposition $\underset{\sim}{K}$ is positive definite

Pf: $K_{AB} = a(N_A, N_B)$ $\quad$ let $\{c_A\}$ define an n-vector $\underset{\sim}{c} = \{c_A\}$

Define also $w^h = \sum_{A=1}^{n} c_A N_A \quad w^h \in \mathcal{V}^h$

then $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = \sum_{A,B=1}^{n} c_A K_{AB} c_B = c_A\, a(N_A, N_B) c_B = a\left(\sum_A c_A N_A, \sum_B c_B N_B\right)$

$= a(w^h, w^h) = \int_0^1 (w^h_{,x})^2 dx \geq 0$

Assume $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = 0$ then by part 1 $\Rightarrow \int_0^1 (w^h_{,x})^2 dx = 0$

$\Rightarrow w^h_{,x} \equiv 0 \Rightarrow w^h =$ piecewise constant but $w^h \in \mathcal{V}^h \Rightarrow w^h(1) = 0$

$\therefore w^h_n = 0 \Rightarrow w^h_{n-1} = 0 \cdots w^h_1 = 0 \therefore w^h \equiv 0$ everywhere $\Rightarrow \sum c_A N_A \equiv 0$

but N_A form a basis & span $\Rightarrow c_A \equiv 0 \quad \forall A$.

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Section 8 continued

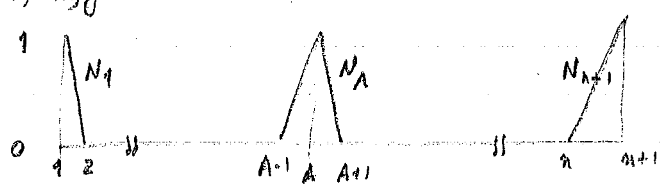
we note that for 1 and 2 degrees of freedom

1. u^h is exact at nodes
2. $u^h_{,x}$ " " 1 point / element
3. $u^h_{\text{homogeneous}}$ is exact

$\underset{\sim}{K}$ (i) Symmetry
(ii) Banded bandwidth = 3
(iii) Positive definite

$w^h = \sum^n c_A N_A \in \mathcal{V}^h \quad u^h = \sum^n d_A N_A + g N_{n+1} \in \mathcal{S}^h$

c_A, d_A, g are the nodal values



§ 9 Mathematical Analysis

Recall the original BVP

(S) $u_{xx} + f = 0$ w/ $u_{,x}(0) = h \quad u(1) = g$

Now what is the green's fn formulation. To do this let $g = h = 0$

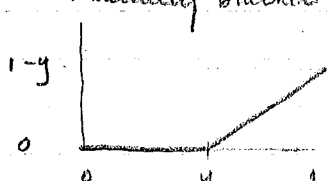
and let $f = \delta_y$ the dirac delta fn at $y \in (0,1)$

Call the soln $\mathcal{G}$. We must now find $\mathcal{G}_{,xx} + \delta_y = 0 \quad \mathcal{G}_{,x}(0) = 0$
$\mathcal{G}(1) = 0$

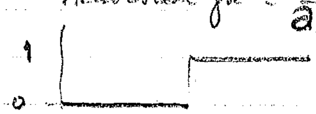
a delta fn $\int_0^1 \delta_y w \, dx = w(y)$ for w cont about y.

<u>Generalized fn's</u>

Macauley Bracket $\langle x-y \rangle = \begin{cases} 0 & x \le y \\ x-y & x > y \end{cases}$



Heaviside fn $= \frac{\partial}{\partial x} \langle x-y \rangle = H(x-y)$



$\delta(x-y) = \frac{\partial}{\partial x} H(x-y)$

Now pick $\phi(0) = \phi(1) = 0$ where ϕ is a nice smooth fn.

$\int_0^1 H\phi_{,x}\, dx = -\int_0^1 H_{,x}\phi\, dx$ IBP

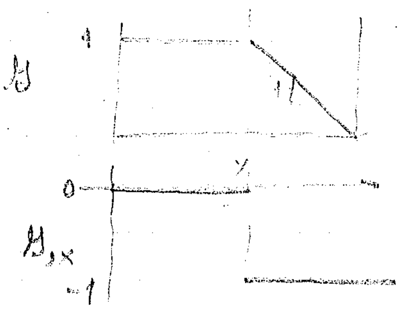
$\int_y^1 \phi_{,x}\, dx = \phi(1)^{\nearrow 0} - \phi(y) = -\int_0^1 \delta_y \phi\, dx \quad \therefore \quad \delta_y = H_{,x}$.

Take $G_{,xx} + \delta_y = 0$ & integrate over 0,1

$G_{,x} + H_y + C_1 = 0$ using bc $G_{,x}(0)^{\nearrow 0} + H(0-y)^{\nearrow 0} + C_1 = 0 \Rightarrow C_1 = 0$

$G_{,} + \langle x-y\rangle + C_2 = 0$ using bc $G(1)^{\nearrow 0} + \langle 1-y\rangle + C_2 = 0 \quad C_2 = -\langle 1-y\rangle$

$\therefore \quad G(x) = -C_2 - \langle x-y\rangle = \underset{\text{const}}{\langle 1-y\rangle} - \langle x-y\rangle$ y = const here



the fn suffers a kink just like our N_A's. N_A's suffer kinks at nodes

Looking at a weak formulation of the green's fn.

(W) = $a(w, u) = (w, f) - w(0)h$

w/ $u = G$ $f = \delta_y$ $h = 0$

(*) weak formulation $a(w, G) = (w, \delta_y)$ $\forall w \in \mathcal{V}$ and $\int_0^1 (w_{,x})^2 dx < \infty$

A theorem exists saying that if $\int_0^1 (w_{,x})^2 dx < \infty$ in 1D $\Longrightarrow$ w is continuous

Theorem : in Piecewise linear F.E. space then $u^h(x_A) = u(x_A)$ $\forall A$

the proof will be done only for $A = 2, \ldots, N$

Lemma 1 $\quad a(w^h, u - u^h) = 0 \quad \forall w^h \in \mathcal{V}^h$

Proof

Take (W) for u : $a(w, u) = (w, f) - w(0) h \quad \forall w \in \mathcal{V}$

remark $w^h \in \mathcal{V}^h \subset \mathcal{V}$ $\therefore$ $w^h \in \mathcal{V}$ $\therefore$ replace w by w^h

$a(w^h, u) = (w^h, f) - w^h(0) h$

for u^h : $a(w^h, u^h) = (w^h, f) - w^h(0) h$. since the rhs's are same

$a(w^h, u) - a(w^h, u^h) = 0$; by bilinearity

$a(w^h, u - u^h) = 0$ QED

this tells you about the error in the discretization : Optimality to be discussed

Lemma 2 $\quad u(y) - u^h(y) = a(u-u^h, G-w^h) \quad \forall w^h \in V^h$

Proof: $u(y) - u^h(y) \triangleq \int_0^1 (u-u^h)\,\delta(y)\,dx = (u-u^h, \delta_y)$

since $u^h \in S^h \subset S \therefore u^h \in S \quad \& \; u \in S$

now $u(1) = u^h(1) = g \quad \therefore \quad u(1) - u^h(1) = 0 \quad \therefore u - u^h \in V$ (and only V is $\ni w(1)=0$)

using $(*)$ replace $u-u^h$ for w

$\therefore$

$$a(u-u^h, G) = (u-u^h, \delta_y)$$

by linearity $\quad a(u-u^h, G-w^h) = a(u-u^h, G) - a(u-u^h, w^h)$

but $a(u-u^h, w^h) = a(w^h, u-u^h) = 0$ as proven by Lemma 1

$\therefore$

$$a(u-u^h, G-w^h) = a(u-u^h, G) = (u-u^h, \delta_y) = u(y) - u^h(y)$$

QED

Pf. of our theorem: if $y = x_A$ then $G \in V^h$

Assume $y = x_A$ and substitute into Lemma 2

$\therefore u(x_A) - u^h(x_A) = a(u-u^h, G-w^h) \quad \forall w^h \in V^h$ using Lemma 1

since $G \;\&\; w^h \in V^h$, so is the difference, then by Lemma 1

$a(u-u^h, G-w^h) = 0 \quad \therefore \quad u(x_A) \equiv u^h(x_A)$

Analysis of Derivatives

Prelims: we will use exactness at nodes & uses Taylor's remainder form

consider fns, f, that are C^k on $[0,1]$ i.e. cont & cont deriv for $0 \le l \le k$

for $y \;\&\; z \in [0,1]$ then $\exists$ a pt $c \ni \quad c \in (y,z)$

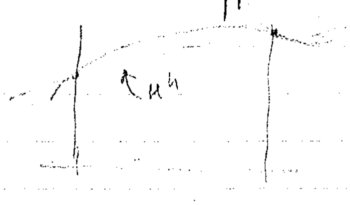
$$\ni \; f(z) = f(y) + (z-y) f'(y) + \cdots + \frac{(z-y)^k}{k!} f^{(k)}(c)$$

Mean value theorem: if $f \in C^1 \Rightarrow f(z) = f(y) + (z-y) f_{,x}(c)$

Proposition: assume $u \in C^1$ (u is contin & differentiable) then $\exists$ a point in each interval or element $\ni u^h_{,x}$ is exact.

Proof: look at a typical element since u^h is linear $\Rightarrow u^h(x) = u^h(x_A) + \frac{(x-x_A)}{h_A}[u^h(x_{A+1}) - u^h(x_A)]$

then



$$u^h_{,x}(x) = \frac{u^h(x_{A+1}) - u^h(x_A)}{h_A}$$

$x \in (x_{A+1}, x_A)$

and since we are at nodes then $u^h(-) = u(-)$ $\Rightarrow u^h_{,x}(x) = \frac{u(x_{A+1}) - u(x_A)}{h_A}$ = constant (**)

by mean value theorem (replace u for f, x_A for y, x_{A+1} for z) ∋ a point

$$u_{,x}(c) \overset{\text{by mean value theorem}}{=} \frac{u(x_{A+1}) - u(x_A)}{h_A} \overset{\text{by }(**)}{=} u^h_{,x} = \text{constant}$$

∴ $u_{,x}$ & $u^h_{,x}$ are exact at one point in the interval QED.

To make this optimal we assert that midpoint is exceptional wrt the derivative in some required context (what that is we will define).

Define the error in (x_A, x_{A+1}) to be $e_{,x}(\alpha) = u^h_{,x} - u_{,x}(\alpha)$ $\alpha \in (x_A, x_{A+1})$

<u>First lemma</u> - assume $u \in C^3$ then for $\bar{x} = (x_A + x_{A+1})/2$

$$e_{,x}(\alpha) = (\bar{x} - \alpha)\, u_{,xx}(\alpha) + \frac{1}{3!\,h_A}\left\{(x_{A+1}-\alpha)^3 u_{,xxx}(c_1) - (x_A - \alpha)^3 u_{,xxx}(c_2)\right\}$$

w/ c_1 & $c_2 \in (x_{A+1}, x_A)$

<u>Proof</u>: $e_{,x}(\alpha) = u^h_{,x} - u_{,x}(\alpha) = \frac{u(x_{A+1}) - u(x_A)}{h_A} - u_{,x}(\alpha)$

expand $u(x_{A+1})$ & $u(x_A)$ by taylors series; since $u^h(x_A) = u^h(x_A)$ & $u^h \in \mathcal{S}^h \subset \mathcal{S}$, $u \in \mathcal{S}$ then replace $u(x_A)$

lagits Taylors Series (1) $u(x_{A+1}) = u(\alpha) + u_{,x}(\alpha)(x_{A+1} - \alpha) + u_{,xx}(\alpha)\frac{(x_{A+1}-\alpha)^2}{2!} + u_{,xxx}(c_1)\frac{(x_{A+1}-\alpha)^3}{3!}$

(2) do same for $u(x_A)$ about α.

(3) then by rearranging we get the required result

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Last time
$$\left.\begin{aligned} g_{,xx} + \delta_y &= 0 \\ g_{,x}(0) &= 0 \\ g(1) &= 0 \end{aligned}\right\} \text{used to show linear FE on (S)} \quad u^h(x_A) = u(x_A)$$

also ∃ a pt $c \in (x_A, x_{A+1})$ ∋ $u^h_{,x} = u_{,x}(c)$

Accuracy of Deriv for $\alpha \in (x_A, x_{A+1})$ w/ $c_1, c_2 \in (x_A, x_{A+1})$ $\bar{x} = \frac{x_A + x_{A+1}}{2}$

$$(u^h_x - u_x)\Big|_\alpha = (\bar{x} - \alpha)\, u_{,xx}(\alpha) + \frac{1}{3!\,h_A}\left[(x_{A+1}-\alpha)^3 u_{,xxx}(c_1) - (x_A - \alpha)^3 u_{,xxx}(c_2)\right]$$

<u>Remark</u> $f(x)$ is said to be $O(x^k)$ if $\lim_{x\to 0} \frac{f(x)}{x^k}$ = constant.

also recall $h = \max_A (h_A)$. We define the following: if $|e_{,x}(\alpha)| \le O(h^k)$ then $u^h_{,x}(\alpha)$ is k^{th} order accurate. Or k is the order of convergence

As mesh size ↓ $h^k \ll h^{k-1}$ ∴ the larger the k the smaller the error. Thus we desire k as large as possible.

For a point $\alpha \to x_A$ $\quad e_{,x}(\alpha) = \left(\frac{x_A + x_{A+1}}{2} - x_A\right) u_{,xx}(x_A) + \frac{1}{3!\,h_A}\left[(x_{A+1}-x_A)^3 u_{,xxx}(c_1)\right]$

$$= \frac{h_A}{2} u_{,xx}(x_A) + \frac{1}{3!} h^2 u_{,xxx}(c_1)$$

$$|e_{,x}(\alpha)| \leq \frac{h_A}{2} |u_{,xx}(x_A)| + \frac{1}{3!} h_A^2 |u_{,xxx}(c_1)|$$

$$\leq \frac{h}{2} \max_{x\in[0,1]} |u_{,xx}| + \frac{1}{3!} h^2 \max_{x\in[0,1]} |u_{,xxx}| \leq O(h)$$

$|u_{,xx}|$ $|u_{,xxx}|$ are fixed as $h \to 0$

thus at nodal points $\boxed{\lim_{\alpha \to \text{nodal values}} |e_{,x}(\alpha)| \text{ is } O(h)}$

Let $\alpha = \bar{x}$ then $|e_{,x}(\bar{x})| = \frac{1}{3!\,h_A}\left[(x_{A+1}-\bar{x})^3 u_{,xxx}(c_1) + (x_A - \bar{x})^3 u_{,xxx}(c_2)\right]$

$$\leq \frac{1}{3!\,h_A}\left[\left|\frac{h_A^3}{2^3}\right| |u_{,xxx}(c_1)| + \left|\frac{h_A^3}{2^3}\right| |u_{,xxx}(c_2)|\right]$$

$$\leq \frac{1}{3!\cdot 8} h^2 \cdot 2 \max_{x\in[0,1]} |u_{,xxx}|$$

$$\leq \frac{h^2}{24} \max_{x\in[0,1]} |u_{,xxx}|$$

$\boxed{\lim_{\alpha \to \bar{x}} |e_{,x}(\alpha)| \text{ is } O(h^2)}$ thus $\bar{x}$ is exceptionally accurate w.r.t. derivatives.

and $\bar{x}$ is the optimal sampling point for derivative. Barlow was the first to deduce these exceptional points and are sometimes called "Barlow Stress points" Stresses being derivatives hence stress points.

<u>Another Remark</u> suppose u is quadratic ie $u = Ax^2 + Bx + C$
Then $u^h_{,x}(\bar{x})$ is exact since $|u_{,xxx}| = 0$ $\therefore$ $|e_{,x}(\bar{x})| \leq 0 \Rightarrow e_{,x}(\bar{x}) = 0$

Remembering for $n=1,2$ $f = \text{const} = p$ we observed this fact — which follows from the fact that $u_{,xxx} = 0$.

Summary
1. $u^h(x_A) = u(x_A)$
2. $\exists$ a pt $c \in (x_A, x_{A+1})$ $\ni$ $u^h_{,x}(c) = u_{,x}(c)$
3. $u^h_{,x}(\bar{x})$ is $O(h^2)$ & exact if u is quadratic, ie $f = \text{const}$.

§ 10. Gauss Elimination will be handed out next time.
An Exercise will be provided at the end.

§ 11. The "Element Point of View."

A. So far – we used Galerkin's method w/ a particular choice of fns (trial & weighting fns) to solve (S). This is a Global Point of View which lends itself to mathematical analysis of the FE method.

B. However if we want to look at the local point of view we look at the Element point of view. In engineering this is good for computer implementation

Now to begin with we must define what we mean by what is an element? The global description (w.r.t. the linear finite element space) (g_1) is the domain $[x_A, x_{A+1}]$; (g_2) nodes $\{x_A, x_{A+1}\}$; the degrees of freedom DOF $\{d_A, d_{A+1}\}$ (g_3) value at the nodes

(with the Galerkin method)

(g_4) the shape fns $\{N_A, N_{A+1}\}$

; (g_5) interpolation fn, if restricted to El. Domain

$u^h(x) = d_A N_A(x) + d_{A+1} N_{A+1}(x)$, $x \in h_A$

the local point of view: (l_1) Domain $[\xi_1, \xi_2] = [-1, 1]$ ie domain is rescaled since we need to have element subroutines that generate the elements use standard element & rescale all subintervals; (l_2) nodes $\{\xi_1, \xi_2\}$

(l_3) DOF $\{d_1^e, d_2^e\}$ $(\ )^e$ associated value for element "e"

1, 2 are the local ordering of each node. For curved elements, will have more than 2.

(l_4) Shape fns $\{N_1, N_2\}$

(l_5) The interpolation fn $u^h(\xi) = N_1(\xi)d_1^e + N_2(\xi)d_2^e$, $\xi \in [\xi_1, \xi_2]$

this is a dual of the global except for interpolation formulas.

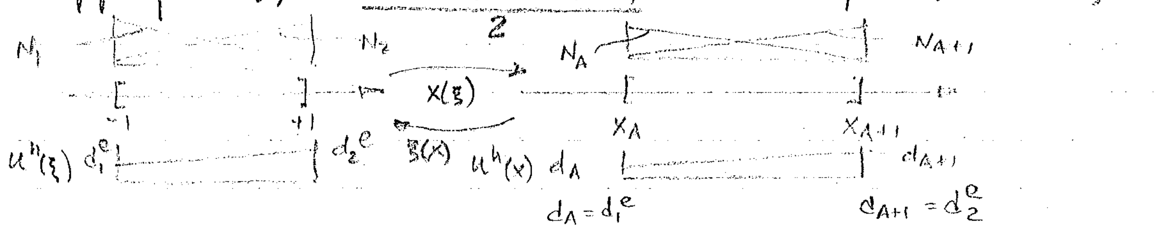
Define $\xi : [x_A, x_{A+1}] \to [\xi_1, \xi_2] = [-1, 1]$

Assume $\xi = c_1 + c_2 x$ so that $\xi(x_A) = \xi_1 = -1$ and $\xi(x_{A+1}) = \xi_2 = 1$

$$\left.\begin{array}{l} -1 = c_1 + c_2 x_A \\ 1 = c_1 + c_2 x_{A+1} \end{array}\right\} \Rightarrow \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & x_A \\ 1 & x_{A+1} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

$$\xi = \frac{2x - (x_A + x_{A+1})}{h_A}$$

The inverse mapping $x(\xi) = \dfrac{\xi h_A + (x_A + x_{A+1})}{2}$ which maps $[-1, 1] \to [x_A, x_{A+1}]$



$h_A = h^e$ here.

$N_a(\xi) = N_a(x(\xi)) = \frac{1}{2}(1+\xi_a \xi)$ — are not fns of x_A's

$(-1)^a = \xi_a$

Now $x(\xi) = (h_A \xi + x_A + x_{A+1})/2$

$= \sum_{a=1}^{2} N_a(\xi)\, x_a^e$

$u^h(\xi) = \sum_{a=1}^{2} N_a(\xi)\, d_a^e$

$(\ )_a$ is local index

global domain can be ~~made~~ represented by the same shape fns as displacements. This has non-trivial consequences. This is also known as isoparametricity

For future reference

$N_{a,\xi} = \frac{\xi_a}{2} = \frac{(-1)^a}{2}$ — the derivatives are a constant & don't depend on (x_A, x_{A+1})

$x_{,\xi} = \sum \frac{\partial N_a}{\partial \xi} x_a^e = \sum_{a=1}^{2} \frac{(-1)^a}{2} x_a^e = h^e/2 = h_A/2$

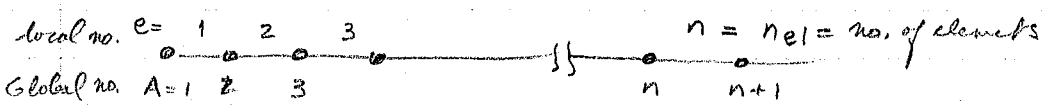
$\xi_{,x} = 2/h_A = 2/h^e = (x_{,\xi})^{-1}$

1/27/81

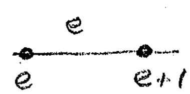
Final Exam will be Thurs March 12 7-10PM in our Classroom

§10. Gauss-Elimin - see Handout

§12. Element Stiffness and Force Vector



For a general element e



Recalling Global Defns of $\underset{\sim}{K}, \underset{\sim}{F}$

$K_{AB} = a(N_A, N_B)$

$= \int_0^1 (N_{A,x})(N_{B,x})\,dx$

$F_A = (N_A, f) - N_A(0)h - a(N_A, N_{n+1})\,g$

$= \int_0^1 N_A f\,dx - N_A(0)h - \int_0^1 N_{A,x} N_{n+1,x}\,dx \cdot g$

now by discretizing for each e subelement ie

$\underset{\sim}{K} = \sum_{e=1}^{nel} \underset{\sim}{K}^e; \quad \underset{\sim}{K}^e = [K_{AB}^e]_{n \times n}$

$\underset{\sim}{F} = \sum_{e=1}^{nel} \underset{\sim}{F}^e; \quad \underset{\sim}{F}^e = \{F_A^e\}$

§10. Interlude: Gauss Elimination: Hand-Calculation Version

It is important for anyone who wishes to do finite element analysis to become familiar with the efficient and sophisticated computer schemes which arise in the finite element method. It is felt that the best way to do this is to begin with the simplest scheme, perform some hand calculations, and gradually increase the sophistication as time goes on.

To do some of the problems, one will need a fairly efficient method of solving matrix equations by hand. The following scheme is applicable to systems of equations $\underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F}$ in which no pivoting (i.e. reordering) is necessary. For example, symmetric, positive-definite coefficient matrices never require pivoting. The procedure is as follows:

Gauss elimination

- Solve eq. 1 for d_1 and eliminate d_1 from the remaining $n-1$ equations
- Solve eq. 2 for d_2 and eliminate d_2 from remaining $n-2$ equations

  ⋮

- Solve eq. $n-1$ for d_{n-1} and eliminate d_{n-1} from eq. n
- Solve eq. n for d_n

The preceding steps are called forward reduction. The original matrix is reduced to upper triangular form. For example, suppose we began with a system of four equations as follows:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

The augmented matrix corresponding to this system is:

$$\underbrace{\left[\begin{matrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{matrix}\right.}_{\underset{\sim}{K}} \; \underbrace{\left|\begin{matrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{matrix}\right]}_{\underset{\sim}{F}}$$

After the forward reduction, the augmented matrix becomes

$$\underbrace{\left[\begin{matrix} 1 & K'_{12} & K'_{13} & K'_{14} \\ 0 & 1 & K'_{23} & K'_{24} \\ 0 & 0 & 1 & K'_{34} \\ 0 & 0 & 0 & 1 \end{matrix}\right.}_{\underset{\sim}{U}} \; \underbrace{\left|\begin{matrix} F'_1 \\ F'_2 \\ F'_3 \\ d_4 \end{matrix}\right]}_{\underset{\sim}{F}'} \qquad (1)$$

corresponding to the upper triangular system $\underset{\sim}{U}\,\underset{\sim}{d} = \underset{\sim}{F}'$. It is a simply

*"Primes" will be used to denote intermediate quantities throughout this section.

verified fact that if $\underset{\sim}{K}$ is banded, then so will be $\underset{\sim}{U}$.

Employing the reduced augmented matrix, proceed as follows:

- Eliminate d_n from equations n-1, n-2, . . . , 1.
- Eliminate d_{n-1} from equations n-2, n-3, . . . , 1.

  .
  .
  .

- Eliminate d_2 from eq. 1.

This procedure is called back substitution. For example, in the case above, after back substitution, we obtain

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\underset{\sim}{I}} \left| \underbrace{\begin{matrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{matrix}}_{\underset{\sim}{d}} \right] \qquad (2)$$

corresponding to the identity $\underset{\sim}{I}\underset{\sim}{d} = \underset{\sim}{d}$. The solution winds up in the last column.

Hand calculation algorithm

In a hand calculation, Gauss elimination can be performed on the augmented matrix as follows:

Forward reduction

- Divide row 1 by K_{11}.

- Subtract K_{21} × row 1 from row 2.
- Subtract K_{31} × row 1 from row 3.

  ⋮

- Subtract K_{n1} × row 1 from row n.

Consider the example of four equations. The preceding steps reduce the first column to the form

$$\left[\begin{array}{cccc|c} 1 & K'_{12} & K'_{13} & K'_{14} & F'_1 \\ 0 & K''_{22} & K''_{23} & K''_{24} & F''_2 \\ 0 & K'_{32} & K''_{33} & K''_{34} & F''_3 \\ 0 & K''_{42} & K''_{43} & K''_{44} & F''_4 \end{array}\right].$$

Note that if $K_{A1} = 0$ then the computations for the A^{th} row can be ignored.

Now proceed to reduce the 2nd column

- Divide row 2 by K''_{22}.
- Subtract K''_{32} × row 2 from row 3.
- Subtract K''_{42} × row 2 from row 4.

  ⋮

- Subtract K''_{n2} × row 2 from row n.

The result for the example will look like

$$\left[\begin{array}{cccc|c} 1 & K'_{12} & K'_{13} & K'_{14} & F'_1 \\ \hdashline 0 & 1 & K''_{23} & K''_{24} & F''_2 \\ 0 & 0 & K'''_{33} & K'''_{34} & F'''_3 \\ 0 & 0 & K'''_{43} & K'''_{44} & F'''_4 \end{array}\right]$$

Note that only the submatrix enclosed in dashed lines is affected in this procedure.

Repeat as above until columns 1 to n are reduced and the upper triangular form (1) is obtained.

<u>Back substitution</u>

- Subtract $K'_{n-1,\,n}$ × row n from row n-1.
- Subtract $K'_{n-2,\,n}$ × row n from row n-2.

  ⋮

- Subtract $K'_{1,\,n}$ × row n from row 1.

After these steps the augmented matrix, for the example, will look like

$$\left[\begin{array}{cccc|c} 1 & K'_{12} & K'_{13} & 0 & F''''_1 \\ 0 & 1 & K'_{23} & 0 & F'''_2 \\ 0 & 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 1 & d_4 \end{array}\right]$$

[illegible]

[illegible]

a. [illegible]

a. [illegible]

a. [illegible]

[illegible]

[illegible] result is [illegible]

Remarks

1. [illegible] which we shall [illegible] later. [illegible] computer program [illegible] simult- [illegible] to retain [illegible] and [illegible] change. [illegible] small change in the [illegible] phase [illegible] in the [illegible]

2. The numerical example with which we close this section illustrates the preceding elimination scheme. Note that the band is maintained (i.e. the zero in the upper right-hand corner of the coefficient matrix remains zero throughout the calculations). The reader is urged to perform the calculations for him/herself.

Example of Gauss elimination:*

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

Augmented matrix

$$\left[\begin{array}{cccc|c} 5 & -4 & 1 & 0 & 0 \\ -4 & 6 & -4 & 1 & 1 \\ 1 & -4 & 6 & -4 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

Forward reduction

$$\left[\begin{array}{cccc|c} 1 & -4/5 & 1/5 & 0 & 0 \\ 0 & 1 & -8/7 & 5/14 & 5/14 \\ 0 & -16/5 & 29/5 & -4 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

*From K. J. Bathe and E. L. Wilson, Numerical Methods in Finite Element Analysis, Prentice-Hall, Englewood Cliffs, N.J. (1976).

$$\left[\begin{array}{cccc|c} 1 & -4/5 & 1/5 & 0 & 0 \\ 0 & 1 & -8/7 & 5/14 & 5/14 \\ 0 & 0 & 1 & -4/3 & 8/15 \\ 0 & 0 & -20/7 & 65/14 & -5/14 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -4/5 & 1/5 & 0 & 0 \\ 0 & 1 & -8/7 & 5/14 & 5/14 \\ 0 & 0 & 1 & -4/3 & 8/15 \\ 0 & 0 & 0 & 1 & 7/5 \end{array}\right]$$

<u>Back substitution</u>

$$\left[\begin{array}{cccc|c} 1 & -4/5 & 1/5 & 0 & 0 \\ 0 & 1 & -8/7 & 0 & -1/7 \\ 0 & 0 & 1 & 0 & 12/5 \\ 0 & 0 & 0 & 1 & 7/5 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -4/5 & 0 & 0 & -12/25 \\ 0 & 1 & 0 & 0 & 13/5 \\ 0 & 0 & 1 & 0 & 12/5 \\ 0 & 0 & 0 & 1 & 7/5 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 8/5 \\ 0 & 1 & 0 & 0 & 13/5 \\ 0 & 0 & 1 & 0 & 12/5 \\ 0 & 0 & 0 & 1 & 7/5 \end{array}\right]$$

$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} 8/5 \\ 13/5 \\ 12/5 \\ 7/5 \end{Bmatrix}$$

Exercise

Consider the boundary-value problem discussed in class:

$$u_{,xx}(x) + f(x) = 0 \qquad x \in \,]0,1[$$

$$u(1) = g$$

$$u_{,x}(0) = h$$

$u_{xx} = -qx$
$u_x = -qx^2/2 + C$
$u_x(0) = 0 \Rightarrow C = 0$
$u = -qx^3/6 + D$
$u(1) = 0 \quad D = q/6 \qquad u = q/6\,(1 - x^3)$ Exact

Assume $f = q\,x$, where q is constant, and $g = h = 0$.

(a) Employing the linear finite element space with equally spaced nodes, set up and solve the Galerkin--finite element equations for $n = 4$ (h = mesh parameter = 1/4). Recall that in class we carried this out for $n = 1$ and $n = 2$ ($h = 1$ and $h = 1/2$, respectively). Do not invert the stiffness matrix $\underset{\sim}{K}$; use Gauss elimination to solve $\underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F}$ or a more sophisticated direct factorization scheme if you know one. Show all calculations. You can check your answers since they must be exact at the nodes.

(b) Let $re_{,x} = |u^h_{,x} - u_{,x}|/(q/2)$, the relative error in $u_{,x}$. Compute $re_{,x}$ at the midpoints of the 4 elements. They should all be equal. (This was also the case for $n = 2$ in class.)

(c) Employing the data presented in class for h = 1 and 1/2, plot ln $re_{,x}$ versus ln h.

(d) Using the error analysis for $re_{,x}$ at the midpoints presented in class, answer the following questions:

- What is the significance of the slope of the graph in part (c)? $\dfrac{\ln(re_{,x_2}/re_{,x_1})}{\ln(h_2/h_1)}$ gives the order of the error

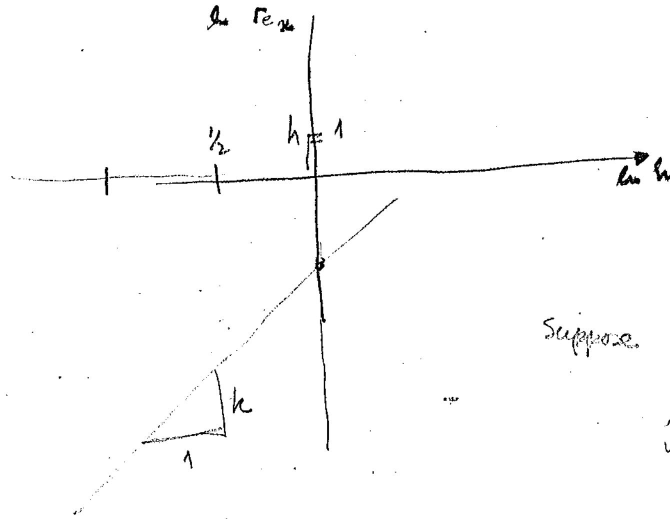
- What is the significance of the y-intercept?

ie at midpoint

$re_{,x} \sim \frac{h^2}{24} \max|u_{xxx}|$

$\therefore re_{x_2}/re_{x_1} = h_2^2/h_1^2$

$\therefore \ln(\ ) = 2\ln(h_2/h_1)$



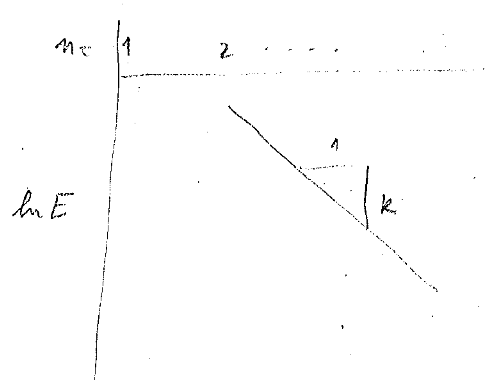
Suppose $E = (re)_{,x} \cong c h^k$

$\underbrace{\ln E}_{y} = \ln c + k \underbrace{\ln h}_{x}$

slope gives order of convergence

y = 0 gives y intercept

if $h = 1/n$ $\ln h = -\ln n$



$$K^e_{AB} = a(N_A, N_B)^e = \int_{\Omega^e = [x_1^e, x_2^e] = [x_A, x_{A+1}]} N_{A,x} N_{B,x}\, dx$$

are same in meaning since N_A is related to $[x_A, x_{A+1}]$

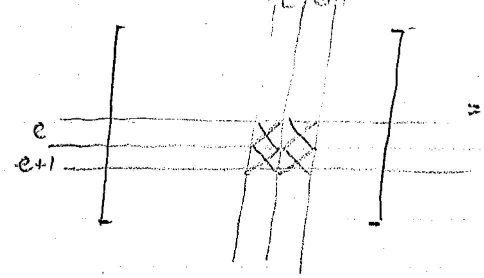
$$F_A^e = (N_A, f)^e - N_A(0)^e h - a(N_A, N_{n+1})^e g$$

$$= \int_{x_1^e}^{x_2^e} N_A f\, dx - N_A^e(0) h - \int_{x_1^e}^{x_2^e} N_{A,x} N_{n+1,x}\, dx\; g$$

This is known as the direct stiffness method ie direct sum of element by element contributions

$K^e_{AB} = 0$ if $A \neq e$, or $e+1$ and $B \neq e$, or $e+1$

we note that $F_A^e = 0$ if $A \neq e$ or $e+1$



$= \underset{\sim}{K}^e$ X - nonzero elements, zero everywhere

$$\underset{\sim}{F}^e = \begin{Bmatrix} \\ \times \\ \times \\ \end{Bmatrix} \begin{matrix} \\ e \\ e+1 \\ \end{matrix}$$

Focusing on the nonzero elements by defining

$\underset{\sim}{k}^e$, $\underset{\sim}{f}^e$ to be the element stiffness & force. We will use $1 \le a, b \le 2$ to be node nos. in local order

$$\underset{\sim}{k}^e = \left[k^e_{ab} \right]_{2\times 2} \qquad \underset{\sim}{f}^e = \left\{ f^e_a \right\}_{2\times 1}$$

where

$$k^e_{ab} = a(N_a, N_b)^e = \int_{x_1^e}^{x_2^e} N_{a,x} N_{b,x}\, dx$$

$$f_a^e = \int_{x_1^e}^{x_2^e} N_a f\, dx + \begin{cases} -\delta_{a1} h & e = 1 \\ 0 & e = 2, n_{el}-1 \\ -k^e_{a2}\, g & e = n_{el} \end{cases}$$

$\delta_{a1} = 1$ $a=1$; $= 0$ $a \neq 1$

since $g \int_{x_1^e}^{x_2^e} N_1^e(x) N_2^e(x)\, dx$ only

thus $\underset{\sim}{k}^e \Longrightarrow$ put into $\underset{\sim}{K}$ $\underset{\sim}{f}^e \Rightarrow$ put into $\underset{\sim}{F}$ will be done by book keep'g array

§ 13 Assembly of $\underset{\sim}{K}$, $\underset{\sim}{F}$: LM array (Location Matrix = LM)

The function of an "element subroutine" is to construct element arrays (ie $\underset{\sim}{k}^e$, $\underset{\sim}{f}^e$)
Then the fn of an "assembly algorithm" is to locate the entries of $\underset{\sim}{k}^e$, $\underset{\sim}{f}^e$ into $\underset{\sim}{K}$, $\underset{\sim}{f}$

$[LM] = n_{el\,eqs} \times n_{el}$

2 = no. of local eqn × no of elmts

$LM(a, e)$ = global eqn. no. (e = element no.; a = local node, or eq. no.)

thus for $1 \le e \le n_{el} - 1$ $\quad LM(a,e) = \begin{cases} e & a=1 \\ e+1 & a=2 \end{cases}$ (global eqn. no.)

for $e = n_{el}$ $\quad LM(a,e) = \begin{cases} e & a=1 \\ 0 & a=2 \end{cases}$

(0 means that ∃ a globally specified bdry cond. – ie means g-type or essential bdry cond.)

ie there is no global DOF corresponding to this local DOF.

Consequently the terms of $k_{12}^{n_{el}} = k_{21}^{n_{el}}$, $k_{22}^{n_{el}}$ are not assembled into $\underset{\sim}{K}$; likewise $f_2^{n_{el}}$ is not assembled into $\underset{\sim}{F}$

Example of the above.

Consider e^{th} element contributions to $\underset{\sim}{K}, \underset{\sim}{f}$. This is in the DO LOOP context; assume $\underset{\sim}{K}, \underset{\sim}{f}$ are set to zero initially. For $(e < n_{el})$

$K_{(e=LM(a,e),\ e=LM(b,e))} = K_{ee} + k_{ab}^e$

$F_{e=LM(a,e)} = F_e + f_a^e$

From LM (e global = 1 local eqn no.; e+1 global = 2 local)

$K_{ee} \xleftarrow{\text{replaced by}} K_{ee} + k_{11}^e$ $\qquad F_e \leftarrow F_e + f_1^e$ $\qquad$ (e global = 1 local; e+1 " = 2 ")

$K_{e,e+1} \longleftarrow K_{e,e+1} + k_{12}^e$ $\qquad F_{e+1} \leftarrow F_{(e+1)} + f_{(2)}^e$

$K_{e+1,e} \longleftarrow K_{e+1,e} + k_{21}^e$ $\qquad$ symmetric so this is not necessary to actually do.

$K_{e+1,e+1} \longleftarrow K_{e+1,e+1} + k_{22}^e$

at $e = n_{el} = n$

$K_{nn} \longleftarrow K_{nn} + k_{11}^{n_{el}}$ $\qquad F_n \leftarrow F_n + f_1^{n_{el}}$

rest is ignored by virtue of the LM array.

Thus $\underset{n\times n}{\underset{\sim}{K}} = \overset{n_{el}}{\underset{e=1}{A}} (\underset{2\times 2}{\underset{\sim}{k}^e})$ $\qquad \underset{\sim}{F} = \overset{n_{el}}{\underset{e=1}{A}} (\underset{\sim}{f}^e)$

(A: assembly operation (coded in "LM"))

for a larger no. of degrees of freedom must define $LM(a, i, e)$ (i = local degree of freedom)

in our case $i = 1$ DOF for each node.

1/29/81

We will explicitly calculate element stiffness matrix

§ 14. Explicit Calculation of $\underset{\sim}{k}^e$ and $\underset{\sim}{f}^e$

Prelims — Change of variables formula

$$\text{if } f : [x_A, x_{A+1}] \to \mathbb{R} \text{ and } f \in C^0 \text{ (continuous)}$$ f can be the integrand of the stiffness array for example.

Define a fn $x = x(\xi) : [\xi_1, \xi_2] \to [x_A, x_{A+1}]$, with $x \in C^1$ contin & differentiable

Now $$\int_{x_A}^{x_{A+1}} f(x)\,dx = \int_{\xi_1}^{\xi_2} f(x(\xi))\,\frac{dx(\xi)}{d\xi}\,d\xi.$$

Recall also chain rule : if g is C^1 and $g(x(\xi)) = g(x)$ then $\frac{\partial}{\partial \xi} g = \frac{\partial}{\partial x} g(x) \frac{\partial x}{\partial \xi}$

Remember that $\underset{\sim}{k}^e$ is representatively given by $k^e_{ab} = \int_{x_1^e}^{x_2^e} N_{a,x}(x)\, N_{b,x}(x)\,dx$

with $x_1^e = x_A$, $x_2^e = x_{A+1}$. If we also remember that $\xi_a = (-1)^a$ also $x(\xi) = \sum_{a=1}^{2} N_a x_a^e$

and $x_{,\xi} = h^e/2$; $h^e = x_2^e - x_1^e$ then by applying the change of variables & chain rule ⇒

$$= \int_{-1}^{1} N_{a,x}(x(\xi))\, N_{b,x}(x(\xi))\,d\xi \cdot \frac{h^e}{2} = \int_{-1}^{1} N_{a,\xi}(\xi)\, N_{b,\xi}(\xi)\,d\xi \cdot \frac{h^e}{2} \cdot \left(\frac{\partial \xi}{\partial x}\right)^2$$

$$= \int_{-1}^{1} N_{a,\xi}(\xi)\, N_{b,\xi}(\xi)\,d\xi \cdot \frac{2}{h^e} \quad \text{since } \frac{\partial \xi}{\partial x} = 1/\partial x/\partial \xi$$

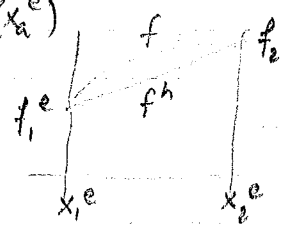
since $N_a = \frac{1}{2}(1 + (-1)^a \xi)$ $\quad N_{a,\xi} = \frac{1}{2}\xi_a$ ∴

$$= \int_{-1}^{1} \frac{1}{4}(-1)^{a+b} \cdot \frac{2}{h^e}\,d\xi = \frac{2 \cdot 2}{4 h^e}(-1)^{a+b} = \frac{(-1)^{a+b}}{h^e}$$

$$\therefore \quad \underset{\sim}{k}^e = \frac{1}{h^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{for linear } N_a(\xi)$$

what about source term $f(x)$? approximate f by f^h and define $f^h(\xi) = \sum_{a=1}^{2} N_a(\xi) f_a^e$ (*)

with $f_a^e = f(x_a^e)$



thus we've made the piecewise linear interpol of the the fn f

thus $f_a^e = \int_{x_1^e}^{x_2^e} N_a(x) f(x)\,dx \cong \int_{x_1^e}^{x_2^e} N_a(x) f^h(x)\,dx$ by changing variables & substitute (*)

$$= \int^{1} N_a(\xi)\,\frac{dx}{d\xi}\,d\xi \left(\sum^{2} N_b(\xi) f_b^e\right) = \frac{h^e}{2}\sum^{2} f_b^e \int^{1} N_a(\xi) N_b(\xi)\,d\xi$$

Now $N_a = \frac{1}{2}(1+(-1)^a\xi)$ $N_a N_b = \frac{1}{4}\left(1+(-1)^a\xi\right)\left(1+(-1)^b\xi\right)$
$= \frac{1}{4}\left(1+[(-1)^a+(-1)^b]\xi + (-1)^{a+b}\xi^2\right)$

$$\int_{-1}^{1} N_a N_b \, d\xi = \frac{1}{4}\left(\xi + [(-1)^a+(-1)^b]\frac{\xi^2}{2} + (-1)^{a+b}\frac{\xi^3}{3}\right)\Big|_{-1}^{1} = \frac{1}{4}\left[2+(-1)^{a+b}\frac{2}{3}\right]$$

(=0 because of symmetry of interval) $= \frac{1}{2}\left[1+(-1)^{a+b}\cdot\frac{1}{3}\right]$

thus $f_a^e = \frac{h^e}{4}\sum_{b=1}^{2} f_b^e \left[1+(-1)^{a+b}\cdot\frac{1}{3}\right]$

$$= \quad f^e = \frac{h^e}{4}\begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix}\begin{bmatrix} f_1^e \\ f_2^e \end{bmatrix} = \frac{h^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} f_1^e \\ f_2^e \end{Bmatrix}$$

Now this becomes more complicated with different elements

Chap 2 Formulation of 2/3 - DIM Probs

§ 1. Prelims

n_{sd} = no. of space dimensions (2 or 3)

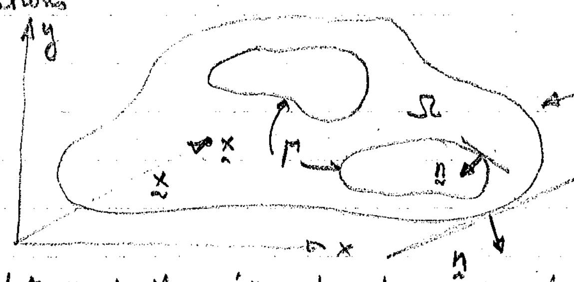
define

Ω = Domain ("open domain" without Bdry)

we will assume that the boundary is piecewise smooth

Γ = Boundary of $\Omega = \partial\Omega$

Definitions



a general point $\underset{\sim}{x} \in \mathbb{R}^{n_{sd}}$

in 2-D

$\underset{\sim}{x} = \{x_i\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix}$

in 3-D $\underset{\sim}{x} = \{x_1, x_2, x_3\}^T = \ldots$

Let $\underset{\sim}{n}$ be the unit outward normal vector $\underset{\sim}{n} = \{n_i\} = \{n_1, n_2\}^T = \{n_x, n_y, n_z\}^T$

The spacial indices will always be denoted by i, j, k, l (almost always) $1 \le i,j,k,l \le n_{sd}$

The BVP will normally be given by a PDE + Bdy data on Γ. Roughly speaking we will have g-type & h-type bdy conditions
(Essential) (Natural)

g-type are Dirichlet (displacement / temp) type ; h-type are Neumann (traction / heat flux) type

we will assume $\Gamma = \Gamma_g \cup \Gamma_h$ $\cup$ - union Γ_a - the part of Γ where we have a-type bdry conditions we will include pbs where Γ_g & Γ_h overlap are zero.

$$\Gamma_g \cap \Gamma_h = \phi \text{ empty} \qquad \cap \text{ intersection}$$

Assume $\Gamma_g \neq \phi$ to give us nonsingular matrices as well as unique solutions but Γ_h may be empty. These assumptions allow us to have mixed bdy value problem

Notation $u_{,i} = u_{,x_i} = \frac{\partial u}{\partial x_i}$. We will also use Einstein summation convention

Ex

$$\sum_{i=1}^{n_{sd}} u_{,ii} = u_{,ii} = u_{,11} + \dots + u_{,n_{sd}n_{sd}} = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_{n_{sd}}^2}$$

if we have more than two indices ≡ assume convention is off

Recall The divergence theorem if $f : \bar{\Omega} \to \mathbb{R}$ where $\bar{\Omega} = \Omega \cup \Gamma$

then $\int_\Omega f_{,i}\, d\Omega = \int_\Gamma f n_i\, d\Gamma$ where n_i is the unit outward normal vector

Also we will be using integration by parts. Thus let $f, g \in C^1$ then

$$\int_\Omega f_{,i} g\, d\Omega = -\int_\Omega f g_{,i}\, d\Omega + \int_\Omega (fg)_{,i}\, d\Omega = -\int_\Omega f g_{,i}\, d\Omega + \int_\Gamma (fg) n_i\, d\Gamma$$

§2. Classical Linear Heat Conduction

Definition of Generalized Fourier Law $\quad q_i = -\kappa_{ij} u_{,j} = -\kappa_{i1} u_{,1} - \dots - \kappa_{in_{sd}} u_{,n_{sd}}$

where u is the temperature; $\underset{\sim}{q}$ is the heat flux vector; κ_{ij} are conductivities & are given.

This law is the constitutive equation. $\kappa_{ij} = \kappa_{ji}$ (symmetric); they may be fns of the spatial coordinates. If they are not then the body is homogeneous. Also we assume $\underset{\sim}{\kappa}$ is positive definite (ie $\underset{\sim}{c}^T \underset{\sim}{\kappa}\, \underset{\sim}{c} \geq 0$ where $\underset{\sim}{c}$ is a vector of length n_{sd}. Also $\underset{\sim}{c}^T \underset{\sim}{\kappa}\, \underset{\sim}{c} = 0 \Longrightarrow \underset{\sim}{c} = \underset{\sim}{0}$.)

Strong Statement of BVP

(S)

$\nabla \cdot \underset{\sim}{q} = q_{i,i} = f$ where f is prescribed: $\Omega \to \mathbb{R}$ (f = source term) — This is a 2nd order PDE, unknown is u.

also $u = g$ on Γ_g ie $u(\underset{\sim}{x}) = g(\underset{\sim}{x})$ for $\underset{\sim}{x} \in \Gamma_g$ $\qquad g : \Gamma_g \to \mathbb{R}$

also $-q_i n_i = h$ on Γ_h ie $-q_i(\underset{\sim}{x}) n_i(\underset{\sim}{x}) = h(\underset{\sim}{x})$ for $\underset{\sim}{x} \in \Gamma_h$ $\qquad h : \Gamma_h \to \mathbb{R}$

Now given (S) we must find $u : \overline{\Omega} \to \mathbb{R}$

more terminology if $\kappa_{ij}(\underset{\sim}{x}) = \underset{\sim}{\kappa}(\underset{\sim}{x})\,\delta_{ij}$ the body is isotropic
kronecker delta.

$$\underset{\sim}{\kappa} = \begin{bmatrix} \kappa_{11} & \kappa_{21} \\ \kappa_{12} & \kappa_{22} \end{bmatrix} = \kappa \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Suppose we assume isotropy & look at the transverse displ of a membrane
Let u = disp f = transverse force $g = 0$ (no bdy displ) & $\Gamma = \Gamma_g$
Then κ represents tension in the membrane.

2/3/81

Midterm 2/12/81 1 hr inclass open-notes.
Last-time : heat conduction problem - given f, g, h and the following PDE & BC

$$(S) \quad \left.\begin{array}{l} q_{i,i} = f \quad \text{in } \Omega \\ u = g \quad \text{on } \Gamma_g \\ -q_i n_i = h \quad \text{on } \Gamma_h \end{array}\right\} \quad \text{w/ the dfn } q_i = -\kappa_{ij} u_{,j} \quad (+) \qquad \kappa_{ij} = \kappa_{ji}$$

For the weak form we will define our spaces

$$\mathcal{S} = \{ u \mid u = g \text{ for } u \text{ on } \Gamma_g \}$$

$$\mathcal{V} = \{ w \mid w = 0 \text{ for } w \text{ on } \Gamma_g \}$$

we want to find $u \ni \ \forall\, w \in \mathcal{V}$

$$-\int_\Omega w_{,i} q_i \, d\Omega = \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma \qquad \text{where } -q_i n_i = h \text{ on } \Gamma_h$$

Sketching the equivalence : assuming everything is smooth enough
(S) and (W) are equivalent and possess unique soln.

Part 1

(S) $\Longrightarrow$ (W). First assume u satisfies (S)

$$0 = q_{i,i} - f$$

$$= \int_\Omega w(q_{i,i} - f)\, d\Omega = \int_\Omega \left\{ \left[(w q_i)_{,i} - (w_{,i} q_i) \right] - f w \right\} d\Omega = 0$$

$$\therefore \; -\int_\Omega (w_{,i} q_i)\, d\Omega - \int_\Omega f w \, d\Omega = -\int_\Omega (w q_i)_{,i}\, d\Omega = -\int_\Gamma w q_i n_i \, d\Gamma$$

but $w = 0$ on Γ_g $\forall\, w \in \mathcal{V}$ $\therefore \; -\int_\Gamma w q_i n_i \, d\Gamma = -\int_{\Gamma_h} w q_i n_i \, d\Gamma = \int_{\Gamma_h} w h \, d\Gamma$

by (+)

$\therefore \; -\int_\Omega w_{,i} q_i \, d\Omega = \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma$ also since $u = g$ $\therefore$ $u \in \mathcal{S} \Rightarrow u$ is a soln of (W)

Part 2

$(W) \Rightarrow (S)$ Assuming u satisfies (W) $\therefore$ $u \in \mathcal{S}$ ($u = g$ on Γ_g) and u satisfies variational eqns.

$$0 = \int_\Omega w_{,i} q_i \, d\Omega + \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma$$

integrate by parts (*)

$$= \int_\Omega w(-q_{i,i} + f) d\Omega + \int_\Gamma w \, (q_i n_i + h) d\Gamma$$ since $w \in \mathcal{V}$ $w = 0$ on Γ_g

now we must show that $\underbrace{q_{i,i} - f}_{\alpha} = 0$ on Ω $\underbrace{q_i n_i + h}_{\beta} = 0$ on Γ

suppose $\alpha \neq 0$: then define $w = \alpha \phi$ w/ (i) $\phi = 0$ on Γ but (ii) $\phi > 0$ on Ω also (iii) assume ϕ is nice. Thus this $w \in \mathcal{V}$

using this into (*). since w vanishes on Γ bdy term disappears

$$\therefore \quad 0 = \int_\Omega \alpha^2 \phi \, d\Omega \quad \text{but } \phi > 0 \text{ in } \Omega \Rightarrow \alpha \equiv 0.$$

$\therefore$ $q_{i,i} = f$. now to show $\beta = 0$ on Γ_h pick $w = \beta \psi$. (i') Pick $\psi = 0$ on Γ_g where ψ is defined on $\bar{\Omega}$

also (ii') $\psi > 0$ on Γ_h and (iii') ψ is nice enough. Put into (*)

$$\therefore \quad 0 = 0 + \int_\Gamma \beta^2 \psi \, d\Gamma$$ thus since $\psi \neq 0$ everywhere $\Rightarrow \beta \equiv 0$

hence $-q_i n_i = h$ QED

× × × × × ×

We now will link up with the abstract notation. As before we write

$$a(w,u) = \int_\Omega w_{,i} \kappa_{ij} u_{,j} \, d\Omega$$

$$(w,f) = \int_\Omega w f \, d\Omega$$

$$(w,h)_\Gamma = \int_{\Gamma_h} w h \, d\Gamma$$

Remembering $a(\cdot,\cdot)$; $(\cdot,\cdot)$; $(\cdot,\cdot)_\Gamma$ are symmetric, bilinear forms

Hence

$$a(w,u) = (w,f) + (w,h)_\Gamma$$

now the indices are going to be a problem; to suppress the spatial indices is our goal (ie $ijkl$ indices). To do this we define

$$\underset{\sim}{\nabla} u = \{u_{,i}\}, \quad \underset{\sim}{\kappa} = [\kappa_{ij}] \quad \underset{\sim}{\nabla} w = \{w_{,i}\}$$

for the isotropic case $\underset{\sim}{\kappa} = [\kappa \delta_{ij}] = [\kappa_{ij}] = \kappa \underset{\sim}{I}$

thus $w_{,i} \kappa_{ij} u_{,j} = (\nabla w)^T \kappa (\nabla u) = (\kappa (\nabla w)^T (\nabla u)$ if $\kappa = \kappa I$

then $a(w,u) = \int_\Omega (\nabla w)^T \kappa (\nabla u) d\Omega$

§ 3. Galerkin For Heat Conduction

we define $\delta^h \approx \delta$ and $\mathcal{V}^h \approx \mathcal{V}$ as before (approximation spaces)

Assume members of $\mathcal{V}^h$ are zero (or approximately zero) on Γ_g

as $h \to 0$ $w(\text{on } \Gamma_g) \to 0$

also assume members of $\delta^h = g$ (" " $= g$) on Γ_g

again as $h \to 0$ $u(\text{on } \Gamma_g) \to g$

Assume $u^h \in \delta^h$ can be written as $u^h = v^h + g^h \in \delta^h$

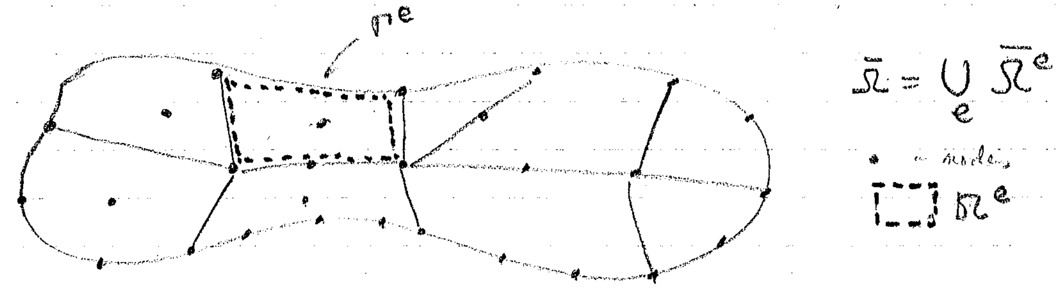
$v^h \in \mathcal{V}^h$, $g^h \in \delta^h$ accounting for g-BC

We can then define

(G) $\begin{cases} \text{Given } g, h \text{ etc,} \\ \text{find } u^h \in \delta^h \ni \forall w^h \in \mathcal{V}^h \\ a(w^h, v^h) = (w^h, f) + (w^h, h)_\Gamma - a(w^h, g^h) \end{cases}$

$$\sum_{A \in \eta - \eta_g} c_A \left[\sum_{B \in \eta - \eta_g} d_B a(N_A, N_B) = (N_A, f) + (N_A, h)_\Gamma - \sum_{B \in \eta_g} a(N_A, N_c) g_c \right]$$

We must develop a data processing technique. Cannot be same as 1-D problem since we now have 2 space variables.



let $\eta = \{1, 2, \ldots, n_{np}\}$ np - no. of nodal points all points

suppose g-node is a node "A" $\ni$ $u^h = g$ $(u^h(x_A) = g(x_A))$

let $\eta_g \subset \eta$ be the set of g-nodes in our 1-D problem this was one node namely n+1 node

Let $\eta - \eta_g$ = "complement" of g nodes i.e. the set of nodes at which u^h is to be determined (i.e. $\mathcal{V}^h$)

The no. of nodes in $\eta - \eta_g$ is n_{eq} (no. of eqns).

Looking at the typical members of

$\mathcal{V}^h$

$$w^h(\underset{\sim}{x}) = \sum_{A \in \eta - \eta_g} N_A(\underset{\sim}{x}) c_A$$

$$v^h(\underset{\sim}{x}) = \sum_{A \in \eta - \eta_g} N_A(\underset{\sim}{x}) d_A \quad \text{where } d_A \text{ are the unknown nodal value}$$

$\mathcal{S}^h$

$$g^h(\underset{\sim}{x}) = \sum_{A \in \eta_g} N_A(\underset{\sim}{x}) g_A \quad \text{where } g_A = g(\underset{\sim}{x}_A)$$

Note that the approximation of g is nodal interpolation by way of the shape fns.

2/5/81

Heat conduction Problem from last time

$$(G) \begin{cases} a(w^h, v^h) = (w^h, f) + (w^h, h)_\Gamma - a(w^h, g^h) \quad w^h \in \mathcal{V}^h \underset{\subset}{\approx} \mathcal{V} \\ \text{w/ } u^h = v^h + g^h \in \mathcal{S}^h \underset{\subset}{\approx} \mathcal{S} \end{cases}$$

We defined $\eta = \{1, 2, \dots, n_{np}\}$ $\quad$ np – no. of nodal points

η_g = "g-nodes" $\quad$ $\eta - \eta_g$ = no. of non prescribed nodal points

$$w^h = \sum_{A \in \eta - \eta_g} N_A c_A \qquad v^h = \sum_{A \in \eta - \eta_g} N_A d_A \qquad g^h = \sum_{A \in N_g} N_A g_A \quad \text{w/ } g_A = g(\underset{\sim}{x}_A)$$

to go to the matrix problem we substitute into (G) formulation and argue as before (i.e. § 1.15)

$$\sum_{B \in \eta - \eta_g} a(N_A, N_B) d_B = (N_A, f) + (N_A, h)_\Gamma - \sum_{B \in \eta_g} a(N_A, N_B) g_B$$

because of these two sums being complimentary we define another matrix ID that maps node nos. into eqn nos. ID is a destination array.

$$ID : \eta \longrightarrow \text{"eqn. nos."}$$

$$\underset{\text{global node no.}}{ID(A)} = \begin{cases} P & \text{– global eqn. no.} \quad \text{if } A \in \eta - \eta_g \\ 0 & \quad \text{if } A \in \eta_g \end{cases}$$

$1 \le P \le n_{eq}$ (no. of global eqns) $\qquad$ n_{eq} = no. of non zero nodes

This leads to the matrix equivalents $\quad \underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F} \qquad \underset{\sim}{K} = [K_{PQ}] \quad 1 \le P, Q \le n_{eq}$

$$\underset{\sim}{d} = \{d_Q\} \qquad \underset{\sim}{F} = \{F_P\}$$

$$K_{PQ} = a(N_A, N_B) \quad \text{w/ } \begin{matrix} P = ID(A) \\ Q = ID(B) \end{matrix} \qquad F = (N_A, f) + (N_A, h)_\Gamma - \sum_{B \in \eta_g} a(N_A, N_B) g_B$$

Properties of $\underset{\sim}{K}$ for the heat conduction problem.

1. $\underset{\sim}{K}$ is symmetric
2. $\underset{\sim}{K}$ is positive def.

Proof of 1. $K_{PQ} = a(N_A, N_B) = a(N_B, N_A) = K_{QP}$ "symm bilinear form" - follows from symmetry of κ_{ij}'s

Proof of 2. Recall if (i) $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} \geq 0 \quad \forall c$, & (ii) $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = 0 \Rightarrow \underset{\sim}{c} = 0$

Proof of (i) $\underset{\sim}{c} = [c_P]_{n_{eq} \times 1}$ to it we can associate A $w^h \in \mathcal{V}^h$ by $w^h = \sum_{A \in \eta - \eta_g} N_A \bar{c}_A$ where $\bar{c}_A = c_P$, $P = ID(A)$

evaluation $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = \sum_{P,Q=1}^{n_{eq}} c_P K_{PQ} c_Q = \sum_{A,B \in \eta - \eta_g} \bar{c}_A a(N_A, N_B) \bar{c}_B = a\left(\sum_{A \in \eta - \eta_g} \bar{c}_A N_A, \sum_{B \in \eta - \eta_g} \bar{c}_B N_B\right)$

$$= a(w^h, w^h) = \int_\Omega w^h_{,i} \kappa_{ij} w^h_{,j} \, d\Omega$$

we hypothesized that κ_{ij} are positive definite $\Rightarrow \int_\Omega \quad d\Omega \geq 0$

Proof of (ii) : assume $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = 0 \Rightarrow \int_\Omega w^h_{,i} \kappa_{ij} w^h_{,j} \, d\Omega = 0 \Rightarrow w^h_{,i} \kappa_{ij} w^h_{,j} = 0$

if κ_{ij} is pos def $\Rightarrow \nabla w^h = 0 \quad \therefore \quad w^h =$ const. But $w^h \in \mathcal{V}^h \Rightarrow w^h = 0$ on $\Gamma_g \Rightarrow w^h \equiv 0$ everywhere $\Rightarrow c_A = 0 \quad A \in \eta - \eta_g$

§4. Element Arrays for the Heat Conduction Problem

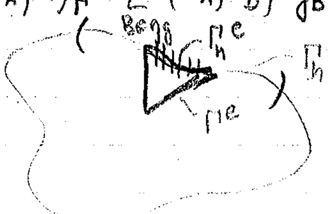
Define
$$\underset{\sim}{K} = \sum_{e=1}^{n_{el}} \underset{\sim}{K}^e \qquad \underset{\sim}{K}^e = [K^e_{PQ}]$$
$$\underset{\sim}{F} = \sum_{e=1}^{n_{el}} \underset{\sim}{F}^e \qquad \underset{\sim}{F}^e = \{F^e_P\}$$

$K^e_{PQ} = a(N_A, N_B)^e = \int_{\Omega^e} \underset{\sim}{\nabla} N_A^T \, \underset{\sim}{\kappa} \, \underset{\sim}{\nabla} N_B \, d\Omega$ ← localized (has $n_{en} \times n_{en}$ elements)

$F^e_P = (N_A, f)^e + (N_A, h)^e_{\Gamma} - \sum_{B \in \eta_g} a(N_A, N_B)^e g_B = \int_{\Omega^e} N_A f \, d\Omega + \int_{\Gamma^e_h} N_A h \, d\Gamma - \sum_{B \in \eta_g} a(N_A, N_B)^e g_B$

η_g is really $\eta^e_g = \eta^e \cap \eta_g$

$\Gamma^e_h = \Gamma_h \cap \Gamma^e$



also $P = ID(A) \quad Q = ID(B)$

Element Stiffness in local ordering etc. a, b - local notation

$\underset{\sim}{k}^e = [k^e_{ab}] \qquad \underset{\sim}{f}^e = \{f^e_a\} \qquad 1 \leq a, b \leq n_{en}$ no. of elemental nodes
3 - triangle
4 - quadril
2 - line in 1-D

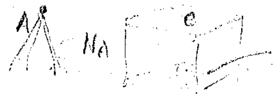
$$k^e_{ab} = a(N_A, N_B)^e = \int_{\Omega^e} \nabla^T N_a \, \underset{\sim}{k} \, \nabla N_b \, d\Omega$$

$$f^e_a = \int_{\Omega^e} N_a f \, d\Omega + \int_{\Gamma^e_h} N_a h \, d\Gamma - \sum_{b=1}^{n_{en}} k^e_{ab} \, g^e_b$$

some of these will be zero since $n_{en} = n_g + n_{eng}$, also $n_{eq} = n_{en} \times n_{el} - n_g$

where $g^e_b = g(\underset{\sim}{x}^e_b)$ if g is prescribed at node b
$= 0$ otherwise

we implicitly will assume that if node A is not attached to element e then $N_A(\underset{\sim}{x}) = 0$ if $\underset{\sim}{x} \in \bar{\Omega}^e$



Standard Notation (Good for Programming)

$$\underset{\sim}{k}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \, d\Omega \qquad \underset{\sim}{D} = \underset{n_{sd} \times n_{sd}}{\underset{\sim}{k}} \quad \begin{matrix} 2\times 2 \text{ in 2-D} \\ 3\times 3 \text{ in 3-D} \end{matrix}$$

$$\underset{n_{sd} \times n_{en}}{\underset{\sim}{B}} = [\underset{\sim}{B}_1, \underset{\sim}{B}_2, \ldots, \underset{\sim}{B}_{n_{en}}] \qquad \underset{\sim}{B}_a = \nabla N_a \underset{\text{in 2-D}}{\triangleq} \begin{Bmatrix} N_{a,1} \\ N_{a,2} \end{Bmatrix} \underset{\text{in 3d}}{\triangleq} \begin{Bmatrix} N_{a,1} \\ N_{a,2} \\ N_{a,3} \end{Bmatrix}$$

$$\text{so} \quad k^e_{ab} = \int_{\Omega^e} \underset{\sim}{B}^T_a \underset{\sim}{D} \underset{\sim}{B}_b \, d\Omega \qquad \begin{Bmatrix} N_{a,1} \\ N_{a,2} \end{Bmatrix}^T \begin{bmatrix} \nu & 0 \\ 0 & \nu \end{bmatrix} \begin{Bmatrix} N_{a,1} \\ N_{a,2} \end{Bmatrix}$$

__exercise__: Let $\underset{\sim}{d}^e$ – element temperature vector at nodes $= \underset{n_{el}\times 1}{\{d^e_a\}} = \begin{Bmatrix} d^e_1 \\ \vdots \\ d^e_{n_{el}} \end{Bmatrix}$

$d^e_a = u^h(\underset{\sim}{x}^e_a)$ show that for $\underset{\sim}{x} \in \Omega^e$

$$\underset{\sim}{q}(\underset{\sim}{x}) = -\underset{\sim}{D}(\underset{\sim}{x}) \underset{\sim}{B}(\underset{\sim}{x}) \underset{\sim}{d}^e = -\underset{\sim}{D}(\underset{\sim}{x}) \sum_{a=1}^{n_{el}} \underset{\sim}{B}_a d^e_a$$

$$\underset{\sim}{q} = -\underset{\sim}{k} \cdot \nabla T = -\underset{\sim}{D} \cdot (\nabla T) \qquad \underset{\sim}{q} = -\underset{\sim}{D} \, \underset{\sim}{B} \, \underset{\sim}{d}$$

$$T = \sum d_a N_a \qquad \nabla T = \sum d_a \nabla N_a \qquad \nabla T = \sum d_a \underset{\sim}{B}_a = \underset{\sim}{B} \underset{\sim}{d}$$

§5. Heat Conduction Processing Arrays

ID, IEN, LM

IEN (element–node) links local node no. to global node no. $\text{IEN}(a,e) = A$ – global node no. (a local node no., e element no.)

ID, IEN: Set up from input data

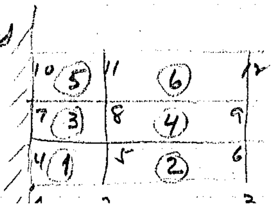
IEN takes local element node to global node. Take this & go to ID

ID takes global node to global eqn. Now LM is a composite of ID, IEN

Thus $\text{LM}(a,e) = \text{ID}(\text{IEN}(a,e))$ is really redundant but is used anyway to cut down recomputing the composition

__Ex__ 1 Consider mesh of 4 node elements

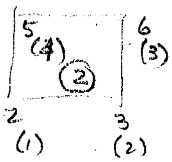
g specified ie at nodes 1, 4, 7, 10



$\eta = \{1, 2, \ldots, 12\} \qquad \eta_g = \{1, 4, 7, 10\}$

element nos. ⊙

Assume local node nos (1, 2, 3, 4) are read in counter-clockwise fashion starting w/ lower left hand corner of each ele example



(2) – element #
5, 6, 2, 3 – global node no.
(1), (2), (3) .. – local node no.

what is ID array: global node nos. n_{np}

	global node A	1	2	3	4	5	6	7	8	9	10	11	12
0 – for g type	equo. P	0	1	2	0	3	4	0	5	6	0	7	8

$n_{eq} = \eta - \eta_g = 12 - 4 = 8$ we pick the ordering, we assign these

$P = ID(A)$ if $A = 6$ $P = 4$

IEN we have 6 elements & 4 nodes / elmt.

no. of elements (e)

no. of local nodes (a)	1	2	3	4	5	6
1	1	2	4	5	7	8
2	2	3	5	6	8	9
3	5	6	8	9	11	12
4	4	5	7	8	10	11

$IEN(a,e) \overset{eq.}{=} IEN(4,3) = 7$
this gives global node no.

LM(a,e)	1	2	3	4	5	6
1	0	1	0	3	0	5
2	1	2	3	4	5	6
3	3	4	5	6	7	8
4	0	3	0	5	0	7

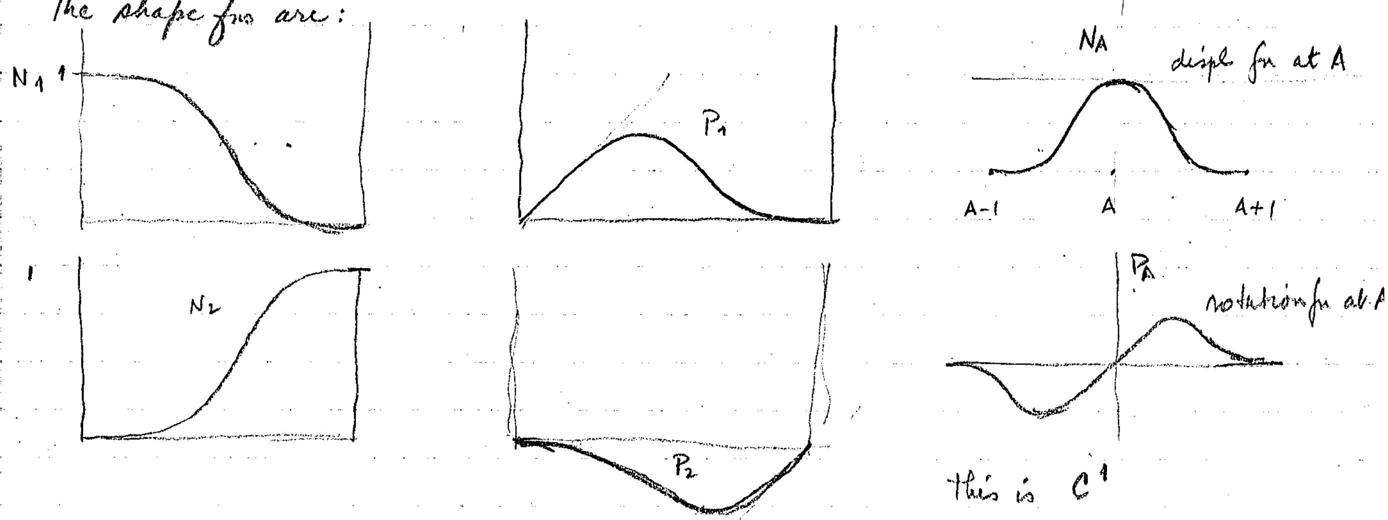
$P = LM(a,e) = ID(\underbrace{IEN(a,e)}_{A})$

this gives global eqn no.
$P = ID(7) = 0$

10 Feb 81

Natural BC's for HW #2 $u_{,xx}(0) = M$ $u_{,xxx}(0) = Q$

The shape fns are:



this is C^1

Element Stiffness is a 4x4 matrix $\underset{\sim}{k} = [k^e_{ab}] \Rightarrow \int_0^1 N_{1,xx} N_{2,xx} dx ; \int N_{1,xx}^2 ;$ etc $\int N_{1,xx} P_{1,xx} ; \int P_{1,xx}^2 ; \int P_{1,xx} P_{2,xx}$

Do the beam problem in its entirety for clarity

Recall the g-data contribution to $\underset{\sim}{f}^e$. There was a term

$$f_a^e = (f + h \text{ terms}) \quad - \sum_{b=1}^{n_{en}} k_{ab}^e \, g_b^e$$

(n_{en} = no. of elemental nodes) — adjustment of force due to g type bdy cond.

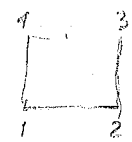
How do we use g_a^e via LM & IEN?

$$\Rightarrow \quad g_a^e = \begin{cases} 0 & \text{if } LM(a,e) \neq 0 \quad \text{ie no g-type b.c. at that node } (a,e) \\ g_A & \text{if } LM(a,e) = 0 \quad \text{ie this is a g-type node } A = IEN(a,e) \end{cases}$$

An example is given to show how these ideas are used

<u>Example</u> Consider a typical 4-node element "e"

Assume
$LM(1,e) = 5$
$LM(2,e) = 0$ } g type
$LM(3,e) = 0$
$LM(4,e) = 9$



Aside: $p = ned(a-1) + i$ (element degree of freedom, $1 \le i \le ned$); $ned = 1$, $i = 1$
$\therefore \; p = (1)(1-1)+1 = 1$
$p = (1)(4-1)+1 = 4$

for $\underset{\sim}{K}$: $K(P,Q)$ where $P = LM(a,e) \neq 0$; $Q = LM(b,e) \neq 0$ $\therefore$

$K_{55} \Leftarrow K_{55} + k_{11}^e$
$K_{59} \Leftarrow K_{59} + k_{14}^e$
$K_{95} \Leftarrow K_{95} + k_{41}^e$) symmetric
$K_{99} \Leftarrow K_{99} + k_{44}^e$

$K(P,Q) = K(P,Q) + k_{ab}^e$ where a, b are connected to P, Q by LM
$K(ID(IEN(a,e)), ID(IEN(b,e)))$

$$f_a^e = (f + h \text{ terms}) \quad -\sum_{b=1}^{n_{en}} k_{ab}^e \, g_b^e$$

Now $\underset{\sim}{f}$: $F(P) = F(P) + f(a)^e = (f + h \text{ terms}) - \sum k_{ab}^e g_b^e$ where $LM(b,e) = 0$

where $P = LM(a,e)$

$F_5 \Leftarrow F_5 + f_1^e$ where (*) $f_1^e = (f \text{ and } h \text{ terms}) - [k_{12}^e g_2^e + k_{13}^e g_3^e]$

$F_9 \Leftarrow F_9 + f_4^e$ $\qquad f_4^e = (\qquad\qquad) - [k_{42}^e g_2^e + k_{43}^e g_3^e]$

$$\underset{4\times4}{\underset{\sim}{k}^e} = \begin{bmatrix} k_{11}^e & k_{12}^e & k_{13}^e & k_{14}^e \\ k_{21}^e & k_{22}^e & k_{23}^e & k_{24}^e \\ k_{31}^e & k_{32}^e & k_{33}^e & k_{34}^e \\ k_{41}^e & k_{42}^e & k_{43}^e & k_{44}^e \end{bmatrix}$$

○ contribute to $\underset{\sim}{K}$

□ contribute to $\underset{\sim}{f}^e$ via (*)

$$\underset{\sim}{f}^e = \begin{Bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{Bmatrix}$$

◇ contribute to $\underset{\sim}{F}$

Because of the fact that g-type conditions don't appear in large nos. in the elements it would <u>not</u> behoove us to code specific things like not calculating the 2 middle rows of $\underset{\sim}{k}^e$. Just let your arrays do work

§6. Classical Linear Elastostatics : Strong and Weak Forms

References : Physically Oriented → Timoshenko & Goodier
Sokolnikoff
Math increasing in ... Gurtin ... in Handbuch der Physik Vol. 1a/2

Duvaut - Lions   Inequalities in Mechanics & Physics
Fichera (Existence theorems) - Handbuch der Physik

Notes to be aware of:

Spatial indices for this problem is i, j, k, l. Summation will be implied on repeated indices. $1 \le i, j, k, l \le n_{sd}$ (no. of space dimensions)

Define $\boldsymbol{\sigma} = [\sigma_{ij}]$ = Cauchy Stress Tensor   equiv to heat flux vector
$\mathbf{u} = u_i$ = Displacement vector (unknown).   equiv to Temp
$\mathbf{f} = f_i$ = body force vector/per unit volume (Given).
$\boldsymbol{\varepsilon} = \varepsilon_{ij}$ = (infinitesimal) strain tensor

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2 = u_{(i,j)}$$ symmetric part of displacement grad.   (strain-displ eq) equiv to temp grad.

Constitutive Eqn is the generalized Hooke's Law

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$$ C_{ijkl} - equiv to conductivities (elastic coeffs)

if $C_{ijkl}(\underset{\sim}{x})$ = constants then Ω is homogeneous. (Ω - volume of the body)

Symmetry $C_{ijkl} = C_{klij}$ this is the major symmetry needed for galerkin method that lets $\underset{\sim}{K}$ be symmetric

also since σ_{ij} is symmetric $C_{ijkl} = C_{jikl}$
" " ε_{kl} " " $C_{ijkl} = C_{ijlk}$

Since we also need positive definiteness we must have that

① $C_{ijkl}\, \psi_{ij}\psi_{kl} \ge 0$
and ② if $= 0$ then $\psi_{ij} = 0$

This must hold $\forall$ pts in body and $\psi_{ij} = \psi_{ji}$

This is related to the strain energy of the body. This condition leads to but does not directly imply that $\underset{\sim}{K}$ is positive definite

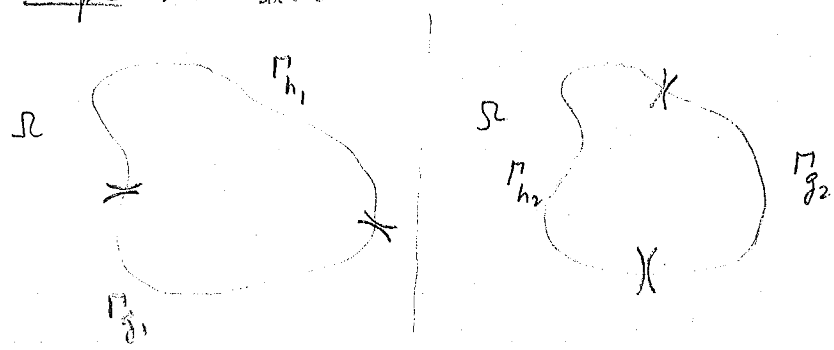
Last time $\overline{\Gamma = \Gamma_g \cup \Gamma_h}$ for heat conduction; for this problem it's even more complicated

$\Gamma = \Gamma_{g_i} \cup \Gamma_{h_i}$ where g_i - prescribed bdy displacement components
h_i - " bdy tractions

and $\overline{\quad}$ above $\Rightarrow$ all other pts not picked up by $\Gamma_{g_i} \cup \Gamma_{h_i}$

also $\phi = \Gamma_{g_i} \cap \Gamma_{h_i}$

Example let $n_{sd} = 2$



The strong statement of the Problem

(S) Given $f_i : \Omega \to \mathbb{R}$; $g_i : \Gamma_{g_i} \to \mathbb{R}$; $h_i : \Gamma_{h_i} \to \mathbb{R}$

Find $u_i : \bar{\Omega} \to \mathbb{R}$ $\ni$. $\sigma_{ij,j} + f_i = 0$ (Equilib eqn) $\forall \underset{\sim}{x} \in \Omega$

and $u_i = g_i$ on Γ_{g_i} (essential bc.) and $\sigma_{ij} n_j = h_i$ on Γ_{h_i} (traction) (natural bc)

The weak statement :

Let S_i = trial solution space ; if $u_i \in S_i$ then $u_i = g_i$ on Γ_{g_i}

V_i = variation or weighting fn space if $w_i \in V_i$ if $w_i = 0$ on Γ_{g_i}

Now

(W) Given $f_i ; \Omega \to \mathbb{R}$; $g_i : \Gamma_{g_i} \to \mathbb{R}$; $h_i : \Gamma_{h_i} \to \mathbb{R}$

Find $u_i \in S_i$ $\ni$. $\forall$ $w_i \in V_i$

$$\int_\Omega w_{(i,j)} \sigma_{ij} \, d\Omega = \int_\Omega w_i f_i \, d\Omega + \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i \, d\Gamma \right)$$

Remark (W) in mechanics states the principle of virtual work or (virtual displacemen

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Elastostatics - review

(S)
$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma_g \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{h} & \text{on } \Gamma_h \\ \boldsymbol{\sigma} = \mathbb{C} \boldsymbol{\epsilon} \quad \boldsymbol{\epsilon} = (\mathbf{u}\nabla + \nabla \mathbf{u})/2. \end{cases}$$

$\Rightarrow$ (W)

$$\int_\Omega w_{(i,j)} \sigma_{ij} \, d\Omega = \int_\Omega w_i f_i \, d\Omega + \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i \right.$$

$V_i : w_i = 0$ on Γ_{g_i}

$S_i : u_i = g_i$ on Γ_{g_i}

$i = n_{sd}$

Exam (i) $f = \text{const}$

Eq 1

$$f_a^e = f \int_{x_1^e}^{x_2^e} N_a(x) \, dx = f \int_{-1}^{1} N_a(\xi) \, \frac{h^e}{2} \, d\xi = f \frac{h^e}{2}$$

$$\underset{\sim}{f}^e = f \frac{h^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

(ii) $f_a^e = \int_{x_1^e}^{x_2^e} N_a(x)\,\delta(x-\bar{x})\,dx = N_a(\bar{x})$ for $x_1^e < \bar{x} < x_2^e$

Cases $\bar{x} = x_b^e$ $\quad f_a^e = N_a(x_b^e) = \delta_{ab}$ $\qquad \underset{\sim}{f}^e = \begin{Bmatrix} \delta_{1b} \\ \delta_{2b} \end{Bmatrix}$

$\bar{x} = x_a^e$ $\quad f_a^e = N_a(x_a^e) = 1$

$\bar{x} = x_a^e + x_b^e/2$ $\quad f_a^e = N_a(x_a^e + x_b^e/2) = 1/2$ $\qquad \underset{\sim}{f}^e = \frac{1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

Problem 2 you finally get $-\int_\Omega w_{,i}\, q_i\, d\Omega = \int_\Omega w f\, d\Omega + \int_{\Gamma_h} w(h - \lambda u)\, d\Gamma$

the additional cont. is
total: $\int_\Gamma w \lambda u\, d\Gamma$; $\int_{\Gamma_h^e} N_a \lambda N_b\, d\Gamma$ by element.
where $\Gamma_h^e = \Gamma^e \cap \Gamma_h$

(iii) $\underset{\sim}{c}^T \underline{K}\, \underset{\sim}{c} = \int_\Omega w_{,i}^h\, \kappa_{ij}\, w_{,j}^h\, d\Omega + \int_{\Gamma_h} \lambda (w^h)^2 d\Gamma \geq 0$
$\lambda \geq 0$

if $\underline{c}^T \underline{K}\, \underline{c} = 0$ each term is zero $\Rightarrow w^h = 0$

Problem 3 is quite trivial & are basically obtainable

Proposition that we make is that (S) ⟺ (W). We will prove this but we must have the following preliminaries:

1. Euclidean Decomp. of Second-Rank Tensor
Let S_{ij} be a non symmetric tensor $(S_{ij} \neq S_{ji})$
Then we can always write $S_{ij} = S_{(ij)} + S_{[ij]}$ where $S_{(ij)}$ is symmetric, $S_{[ij]}$ " skew symmetric

THUS
Define $S_{(ij)} = (S_{ij} + S_{ji})/2$.
$S_{[ij]} = (S_{ij} - S_{ji})/2$.

2. Let S_{ij} be nonsymmetric & t_{ij} be symmetric, then
$S_{ij} t_{ij} = S_{(ij)} t_{ij}$

Proof $S_{ij} t_{ij} = (S_{(ij)} + S_{[ij]}) t_{ij} = S_{(ij)} t_{ij} +$ ⏤ $S_{[ij]} t_{ij} = -S_{[ji]} t_{ij}$ — skew sym of $S_{[ij]}$
$= -S_{[ji]} t_{ji}$ — symm of t
$= -S_{[ij]} t_{ij}$ — arbitrariness of indices

+ 0 $\quad$ Sum = −Sum ⇒ Sum = 0

Now to tackle the formal proof

Pf if (S) ⟹ (W) : Assume u satisfies (S)

then $0 = \int \omega_i (\sigma_{ij,j} + f_i)\, d\Omega$ with $\omega_i \in \mathcal{V}_i$

integrate by parts

$$= -\int_\Omega \omega_{i,j}\sigma_{ij}\, d\Omega + \int_\Omega \omega_i f_i\, d\Omega + \int_\Gamma \omega_i \sigma_{ij} n_j\, d\Gamma = \int_\Omega (\omega_i \sigma_{ij})_{,j}\, d\Omega$$

$$= -\int_\Omega \omega_{(i,j)}\sigma_{ij}\, d\Omega + \int_\Gamma \omega_1 \sigma_{1j} n_j\, d\Gamma + \int_\Gamma \omega_2 \sigma_{2j} n_j\, d\Gamma + \ldots + \int_\Omega \omega_i f_i\, d\Omega$$

by preliminary #2

since $\omega_i = 0$ on Γ_{g_i} then e.g. $\int_\Gamma \omega_1 \sigma_{1j} n_j\, d\Gamma = \int_{\Gamma_{h_1}} \omega_1 \sigma_{1j} n_j\, d\Gamma = \int_{\Gamma_{h_1}} \omega_1 h_1\, d\Gamma$ since $\sigma_{ij} n_j = h_i$ on Γ_{h_i}

hence $0 = -\int_\Omega \omega_{(i,j)}\sigma_{ij}\, d\Omega + \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} \omega_i h_i\, d\Gamma + \int_\Omega \omega_i f_i\, d\Omega$ Result follows.

Second Part: assume $u_i \in \mathcal{S}_i$ & satisfies (W) we want to show (W) ⟹ (S)

$$0 = (W) = -\int_\Omega \omega_{(i,j)}\sigma_{ij}\, d\Omega + \int_\Omega \omega_i f_i\, d\Omega + \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} \omega_i h_i\, d\Gamma$$

Using part 2 of preliminaries $\int_\Omega \omega_{(i,j)}\sigma_{ij}\, d\Omega = \int_\Omega \omega_{i,j}\sigma_{ij}\, d\Omega$

Now we integrate by parts to get

$$0 = \int_\Omega \omega_i \sigma_{ij,j}\, d\Omega + \int_\Omega \omega_i f_i\, d\Omega - \int_\Gamma \omega_i \sigma_{ij} n_j\, d\Gamma + \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} \omega_i h_i\, d\Gamma$$

now since $\omega_i \in \mathcal{V}_i$ then $\int_\Gamma \omega_i \sigma_{ij} n_j\, d\Gamma = \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} \omega_i \sigma_{ij} n_j\, d\Gamma$

$$\therefore \quad 0 = \int_\Omega \omega_i \underbrace{(\sigma_{ij,j} + f_i)}_{①}\, d\Omega - \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} \omega_i \underbrace{(\sigma_{ij} n_j - h_i)}_{②}\, d\Gamma$$

natural bdy cond.

let $\sigma_{ij,j} + f_i = \alpha_i$ and $(h_i - \sigma_{ij} n_j) = \beta_i$

Now we proceed as before

let $\alpha_i \phi = \omega_i$ w/ $\phi > 0$ on Ω ; $\phi = 0$ on Γ ; ϕ is nice.

since $\phi = 0$ on Γ $\omega_i \in \mathcal{V}_i$

then $0 = \int_\Omega \alpha_i^2 \phi\, d\Omega - \overset{\text{bdry}}{0} = \int_\Omega \phi(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)\, d\Omega \geq 0$ (>0)

since $\phi > 0$ then for $\int_\Omega$ to be $= 0$ ⟹ each $\alpha_i = 0$ ⟹ $\sigma_{ij,j} + f_i = 0$

for the boundary term let $w_i = \delta_{i1} \beta_1 \psi$ w/ $\psi > 0$ on Γ_{h_1}, $\psi = 0$ on Γ_{g_1}, ψ nice

now substitute into our (W) then

$$0 = 0 + \underbrace{\int_{\Gamma_{h_1}} \beta_1^2 \psi \, d\Gamma}_{\geq 0,\ >0} + (\text{rest} = 0 \text{ of bdry terms}) \Rightarrow \beta_1 = 0 \text{ for } \int_{\Gamma_{h_1}} = 0$$

we can do this for the other two terms $\beta_2, \beta_3, \ldots$ $\therefore$ $\beta_i = h_i - \sigma_{ij} n_j = 0$

thus result follows and $(S) \Leftrightarrow (W)$

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last time:

(W) $\int_\Omega w_{(i,j)} \sigma_{ij} \, d\Omega = \int_\Omega w_i f_i \, d\Omega + \sum_{i=1}^{n_{sd}} \int_{\Gamma_{h_i}} w_i h_i \, d\Gamma$

$(S) \Leftrightarrow (W)$ proven

HW #3 a) work in $[-1, 1]$ space $\Rightarrow$ Barlow Stress Points $\pm 1/\sqrt{3}$ (Gauss Points)

converting to $[-h/2, h/2] \Rightarrow \pm \frac{h}{2\sqrt{3}}$

b) $O(h^3)$

c) $u^{IV} = 0 \Rightarrow$ all pts are exact.

Because of indices we will develop the symbology to take care of (W)

$$a(\underset{\sim}{w}, \underset{\sim}{u}) = \int_\Omega w_{(i,j)} \, C_{ijkl} \, u_{(k,l)} \, d\Omega \qquad \text{letting } \sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \ \varepsilon_{kl} = u_{(k,l)}$$

$$(\underset{\sim}{w}, \underset{\sim}{f}) = \int_\Omega w_i f_i \, d\Omega$$

$$(\underset{\sim}{w}, \underset{\sim}{h})_\Gamma = \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i \, d\Gamma \right)$$

$$\therefore (W) \Rightarrow \quad a(\underset{\sim}{w}, \underset{\sim}{u}) = (\underset{\sim}{w}, \underset{\sim}{f}) + (\underset{\sim}{w}, \underset{\sim}{h})_\Gamma$$

Now $a(\underset{\sim}{w}, \underset{\sim}{u})$, $(\underset{\sim}{w}, \underset{\sim}{f})$, $(\underset{\sim}{w}, \underset{\sim}{h})_\Gamma$ are symmetric bilinear forms as shown previously

Major symmetry of $a(\underset{\sim}{w}, \underset{\sim}{u})$ comes from $C_{ijkl} = C_{klij}$

We now use the weak form in the galerkin method:

(W) The problem statement becomes given $\underset{\sim}{f}, g, \underset{\sim}{h}$ find $\underset{\sim}{u} \in \underset{\sim}{S} \ni \forall \underset{\sim}{w} \in \underset{\sim}{V}$

$$a(\underset{\sim}{w}, \underset{\sim}{u}) = (\underset{\sim}{w}, \underset{\sim}{f}) + (\underset{\sim}{w}, \underset{\sim}{h})_\Gamma$$

We now convert to matrix form.

Assume $n_{sd} = 2$ $\quad 1 \le i, j, k, l \le 2$

Let $\underset{\sim}{\epsilon}(\underset{\sim}{u})$ be a "strain vector" $= (u_{1,1}\ ;\ u_{2,2}\ ;\ u_{1,2} + u_{2,1})^T$

↑ Engineering shear strain

$\underset{\sim}{\epsilon}(\underset{\sim}{w})$ is same as $\underset{\sim}{\epsilon}(\underset{\sim}{u})$ except we replace $\underset{\sim}{u}$ by $\underset{\sim}{w}$

Define $\quad \underset{\sim}{D} = [D_{IJ}] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ \text{Sym.} & & D_{33} \end{bmatrix}$

I \ J	i \ k	j \ l
1	1	1
3	1	2
3	2	1
2	2	2

where D_{IJ} is related to C_{ijkl}

and I depends on ij

J depends on kl

C_{1211} $\quad 12 \Rightarrow I = 3 \quad \therefore C_{1211} = D_{31}$

$\quad 11 \Rightarrow J = 1$

Thus using all this we get

$$w_{(i,j)}\, C_{ijkl}\, u_{(k,l)} = \underset{\sim}{\epsilon}(\underset{\sim}{w})^T \underset{\sim}{D}\, \underset{\sim}{\epsilon}(\underset{\sim}{u}) \qquad \text{Verify this}$$

$w_{(i,j)} = w(I)$; $C_{ijkl} = D(I,J)$; $u_{(k,l)} = u(J)$ $\quad \therefore w_{(i,j)} C_{ijkl} u_{(k,l)} = \sum_J \sum_I w^T D u = \underset{\sim}{\epsilon}(\underset{\sim}{w})^T \underset{\sim}{D}\, \underset{\sim}{\epsilon}(\underset{\sim}{u})$

For 3-D set up the analog of the I/J table by choosing the $\underset{\sim}{\epsilon}(\underset{\sim}{u})$ to be

$$(u_{1,1};\ u_{2,2};\ u_{3,3};\ u_{2,3} + u_{3,2};\ u_{3,1} + u_{1,3};\ u_{1,2} + u_{2,1})^T$$

thus $I = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

etc.

I \ J	i \ k	j \ l
1	1	1
6	1	2
5	1	3
6	2	1
2	2	2
4	2	3
5	3	1
4	3	2
3	3	3

We can also define a stress "vector" $\underset{\sim}{\sigma}$ where

$$\underset{\sim}{\sigma} \overset{n_{sd}=2}{=} (\sigma_{11} \quad \sigma_{22} \quad \sigma_{12})^T$$

and

$$\underset{\sim}{\sigma} \overset{n_{sd}=3}{=} (\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{23} \quad \sigma_{31} \quad \sigma_{12})^T$$

and finally $\underset{\sim}{\sigma} = \underset{\sim}{D}\, \underset{\sim}{\epsilon}(\underset{\sim}{u})$ which is the analog $\sigma_{ij} = C_{ijkl}\, \epsilon_{kl}$

$\underset{\sim}{\sigma} = C_{ijkl}\, u_{(k,l)} = \underset{\sim}{D}\, \underset{\sim}{\epsilon}(\underset{\sim}{u})$

A body is said to be isotropic if

$$C_{ijkl}(\underset{\sim}{x}) = \mu(\underset{\sim}{x})\,[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda(\underset{\sim}{x})\,\delta_{ij}\delta_{kl}$$

μ & λ are the Lamé parameters $\quad \mu = G$ the shear modulus

λ and μ can be related with E and ν where $\lambda = \nu E / ((1+\nu)(1-2\nu))$

$\mu = E / (2(1+\nu))$

<u>Exercise</u> For the isotropic case - define D matrix

$n_{sd} = 2$

$i = 1 = j = k = l$; $I = J = 1$ $\quad D(1,1) = 2\lambda + \mu$
$i = j = 1$, $k = 1, l = 2$; $\quad D(1,3) = 0$
$i = j = 1$; $k = 2, l = 2$ $\quad D(1,2) = \lambda$
$i = j = k = l = 2$ $\quad D(2,2) = D(1,1)$
$\Big\} \longrightarrow$

$$\underset{\sim}{D} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

the $n_{sd} = 2$ case we have defined the plane strain hypothesis.

for 3-D case
$n_{sd} = 3$

$$\underset{\sim}{D} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

plane stress case is basically plane strain with some basic items redefined

i.e. isotropic plane stress replace λ with $\bar{\lambda} = 2\mu\lambda/(\lambda+2\mu)$

if D is coded in terms of E and ν & we want to convert from plane strain to plane stress replace E by $\bar{E} = E/(1-\nu^2)$
ν by $\bar{\nu} = \nu/(1-\nu)$

§ 7. Galerkin form of (W)

Define as previously $\underset{\sim}{\mathcal{V}}^h \approx \mathcal{V}$ assume if $\underset{\sim}{w}^h \in \underset{\sim}{\mathcal{V}}^h$ then $w_i^h = 0$ on Γ_{g_i}

$\underset{\sim}{\mathcal{S}}^h \approx \mathcal{S}$ assume if $\underset{\sim}{u}^h \in \underset{\sim}{\mathcal{S}}^h$ then $u_i^h = g_i$ " Γ_{g_i}

As before if $\underset{\sim}{u}^h \in \mathcal{S}^h$ $\underset{\sim}{u}^h = \underset{\sim}{v}^h + \underset{\sim}{g}^h$ where $\underset{\sim}{v}^h \in \underset{\sim}{\mathcal{V}}^h$

(G) $\Bigg\{$ Thus we now define (G) : using the givens of (W)
Find $\underset{\sim}{u}^h = (\underset{\sim}{v}^h + \underset{\sim}{g}^h) \in \underset{\sim}{\mathcal{S}}^h$ ∋ $\forall \underset{\sim}{w}^h \in \underset{\sim}{\mathcal{V}}^h$

$$a(\underset{\sim}{w}^h, \underset{\sim}{v}^h) = (\underset{\sim}{w}^h, \underset{\sim}{f}) + (\underset{\sim}{w}^h, \underset{\sim}{h})_\Gamma - a(\underset{\sim}{w}^h, \underset{\sim}{g}^h)$$

We now begin the data processing aspects – different because of the no. of degrees of freedom

∴ $[ID]_{n_{dof} \times n_{np}}$ where n_{dof} = no. of degrees of freedom/no.
n_{np} – no. of nodal points

we now define $\eta = \{1, 2, \ldots, n_{np}\}$
η_{g_i} = subset of η where g_i^h is prescribed
$\eta - \eta_{g_i}$ = where u_i^h is unknown.

$$\therefore \quad ID(i, A) = \begin{cases} P & A \in \eta - \eta_{g_i} \\ 0 & A \in \eta_{g_i} \end{cases}$$

P – global eqn. no.

(i – no. of Dof; A – global node no.)

for one degree of freedom $i = 1$

We now define $v_i^h = \sum_{A \in \eta - \eta_{g_i}} N_A d_{iA}$ (node no.; DOF no.) $d_{iA} = d_P$

$$g_i^h = \sum_{A \in \eta_{g_i}} N_A g_{iA}$$

Now we can also let $\underset{\sim}{v}^h = v_i^h \underset{\sim}{e}_i$ where $e_i = \begin{Bmatrix} \delta_{1i} \\ \delta_{2i} \end{Bmatrix}$ if $i \le 2$

Similarly $\underset{\sim}{g}^h = g_i^h \underset{\sim}{e}_i$ and $\underset{\sim}{w}^h = w_i^h \underset{\sim}{e}_i$ $w_i^h = \sum N_A c_{iA}$

therefore $\underset{\sim}{w} = w_i^h \underset{\sim}{e}_i = N_A c_{iA} \underset{\sim}{e}_i = N_A \underset{\sim}{e}_i c_{iA}$

$$\sum_P c_P \left[\sum_Q d_Q a(N_A \underset{\sim}{e}_i, N_B \underset{\sim}{e}_j) = (N_A \underset{\sim}{e}_i, \underset{\sim}{f}) + (N_A \underset{\sim}{e}_i, \underset{\sim}{h})_\Gamma - \sum_{Q^{comp}} a(N_A \underset{\sim}{e}_i, N_B \underset{\sim}{e}_j) g_{jB} \right]$$

Thus substitut'g into (G) and argue'g as before to get rid of c_{iA}'s then

$$\underset{\sim}{K} \underset{\sim}{d} = \underset{\sim}{F} \quad \text{where} \quad \underset{\sim}{K} = [K_{PQ}]_{neq \times neq}$$

$$\underset{\sim}{F} = \{F_P\}$$

$$\underset{\sim}{d} = \{d_Q\}$$

As previous $K_{PQ} = a(N_A \underset{\sim}{e}_i, N_B \underset{\sim}{e}_j)$ $\quad P = ID(i,A)$, $Q = ID(j,B)$

and $F_P = (N_A \underset{\sim}{e}_i, \underset{\sim}{f}) + (N_A \underset{\sim}{e}_i, \underset{\sim}{h})_\Gamma - \sum_{j=1}^{n_{dof}} \left(\sum_{B \in \eta_{gj}} a(N_A \underset{\sim}{e}_i, N_B \underset{\sim}{e}_j) g_{jB} \right)$

(this is summed first)

now $\underset{\sim}{\varepsilon}(N_A \underset{\sim}{e}_i) = \underset{\sim}{B}_A \underset{\sim}{e}_i$ — these are the strain displ — for $n_{sd} = 2$ $\quad \underset{\sim}{B}_A = \begin{bmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{bmatrix}$

for $n_{sd} = 3$

$$\underset{\sim}{B}_A = \begin{bmatrix} N_{A,1} & 0 & 0 \\ 0 & N_{A,2} & 0 \\ 0 & 0 & N_{A,3} \\ 0 & N_{A,3} & N_{A,2} \\ N_{A,3} & 0 & N_{A,1} \\ N_{A,2} & N_{A,1} & 0 \end{bmatrix}$$

to show this remember

$$\underset{\sim}{\varepsilon}(\underset{\sim}{w}) = \begin{Bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} + w_{2,1} \end{Bmatrix} \xrightarrow{\text{to get}} \begin{Bmatrix} N_{A,1}\delta_{1i} \\ N_{A,2}\delta_{2i} \\ N_{A,2}\delta_{1i} + N_{A,1}\delta_{2i} \end{Bmatrix} = \begin{Bmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{Bmatrix} \begin{Bmatrix} \delta_{1i} \\ \delta_{2i} \end{Bmatrix}$$

where $\underset{\sim}{w} = N_A \underset{\sim}{e}_i \rightarrow w_1 = N_A \delta_{1i}$, $w_2 = N_A \delta_{2i}$ (stick into)

voilà! $= \underset{\sim}{B}_A \underset{\sim}{e}_i$

Similarly (comes for w_i) $(N_A \underset{\sim}{e}_i, \underset{\sim}{f}) = \int_\Omega N_A f_i \, d\Omega$ $\quad$ since $\underset{\sim}{f} = f_j \underset{\sim}{e}_j$ & $\underset{\sim}{e}_j^T \underset{\sim}{e}_i = \delta_{ij}$ $\therefore \underset{\sim}{f} \underset{\sim}{e}_i = f_j \delta_{ij} = f_i$

likewise we can do this for $(\;,\;)_\Gamma = \int_\Gamma N_A h_i \, d\Gamma$ $\quad$ check above

For $K_{PQ} = \underset{\sim}{e}_i^T \int_\Omega \underset{\sim}{B}_A^T \underset{\sim}{D} \underset{\sim}{B}_B \, d\Omega \, \underset{\sim}{e}_j$ $\quad$ to get this we work with $\int_\Omega \underset{\sim}{\varepsilon}^T(\underset{\sim}{w}^h) \underset{\sim}{D} \underset{\sim}{\varepsilon}(\underset{\sim}{u}^h) d\Omega$

So far we have made no assumptions about rigid body motion. For unique soln we must define them:

Analytical versions of rigid body modes. We must take them out for system to be non singular

IF $w_{(i,j)} = \underset{2\times1}{0}$ ("zero strains") Then w can be written as

$\tilde{w}(\tilde{x}) = \tilde{c} + c_3(x_1 e_2 - x_2 e_1)$ $n_{sd} = 2$

$\tilde{w}(x) \quad \underset{3\times1}{\tilde{c}_1} + \tilde{c}_2 \otimes \underset{3\times1}{\tilde{x}}$ $\otimes$ - cross product $n_{sd} = 3$

These are general representation of rigid body motions

$\tilde{c}, \tilde{c}_1$ are translational motions ; second term are rotational motions

- Some points about $\tilde{K}$
  - $\tilde{K}$ is symmetric
  - $\hat{K}$ is positive def. if conditions built into the definition of $\tilde{v}^h$ preclude non-trivial rigid body motions

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Continuation of discussion of $\tilde{K}$

Proof of Symmetry & positive definiteness

1.[st] pt: symmetry of $\hat{K}$

Proof

$$K_{PQ} = \tilde{e}_i^T \int_\Omega \hat{B}_A^T \tilde{D} \hat{B}_B \, d\Omega \, \tilde{e}_j = \int_\Omega \tilde{\varepsilon}(N_A e_i)^T \tilde{D} \, \varepsilon(N_B e_j) d\Omega$$

$$= \tilde{e}_j^T \int_\Omega \tilde{B}_B^T \tilde{D}^T \hat{B}_A \, d\Omega \, \hat{e}_i \quad \text{since } K_{PQ} \text{ is a number.}$$

Now $\tilde{D} = [D_{IJ}] = c_{ijkl} = c_{klij} = [D_{JI}] = \tilde{D}^T$ from major symmetry of $\tilde{C}$

put into above to get.

$$= \tilde{e}_j^T \int_\Omega \tilde{B}_B^T \hat{D} \hat{B}_A \, d\Omega \, \hat{e}_i = K_{QP}$$

2nd pt. : Positive Definiteness

Must show

(i) $\tilde{c}^T \tilde{K} \tilde{c} \geq 0 \quad \forall \tilde{c}$ this is trivial

(ii) if $\tilde{c}^T \tilde{K} \tilde{c} = 0$ then $\tilde{c} = 0$

To do the above define: $w_i^h = \sum_{A \in \eta - \eta_{g_i}} N_A \bar{c}_{iA} \in \mathcal{V}_i^h \quad 1 \leq i \leq n_{sd}$

now define $c_P = \bar{c}_{iA}$ where $P = ID(i, A)$ $1 \leq P \leq$ no. of eqns.

$\therefore \quad w_i^h = \sum_{A \in \eta - \eta_{g_i}} N_A c_P$ and $\tilde{c} = [c_P]$

(i) $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = \sum_{P,Q=1}^{n_{eq}} c_P K_{PQ} c_Q$

$$= \sum_{\substack{i=1 \\ j=1}}^{n_{dof}} \left\{ \sum_{\substack{A \in \eta - \eta_{g_i} \\ B \in \eta - \eta_{g_j}}} \left[\bar{c}_{iA} \quad a(N_A \underset{\sim}{e}_i, N_B \underset{\sim}{e}_j) \quad \bar{c}_{jB} \right] \right\}$$

do this first

Now if we use the bilinearity

$$= a\left(\sum_{i=1}^{n_{dof}} \left\{ \sum_{A \in \eta - \eta_{g_i}} \bar{c}_{iA} N_A \underset{\sim}{e}_i \right\}, \sum_{j=1}^{n_{dof}} \left\{ \sum_{B \in \eta - \eta_{g_j}} \bar{c}_{jB} N_B \underset{\sim}{e}_j \right\} \right)$$

Now $\underset{\sim}{w}^h = w_i^h \underset{\sim}{e}_i = \sum_{i=1}^{n_{dof}} \sum_{A \in \eta - \eta_{g_i}} \bar{c}_{iA} N_A \underset{\sim}{e}_i$ $\therefore$ the above $= a(\underset{\sim}{w}^h, \underset{\sim}{w}^h)$

$$= a(\underset{\sim}{w}^h, \underset{\sim}{w}^h) = \int_\Omega w^h_{(i,j)} c_{ijkl} w^h_{(k,l)} d\Omega.$$

the integrand is ≥ 0 at each pt. since we had defined C_{ijkl} to be positive definite $\therefore$ since integrand ≥ 0 so must the integral be. $\therefore$ $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} \geq 0$

(ii) Assume $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = 0 \overset{\text{from (i)}}{\Rightarrow} w^h_{(i,j)} C_{ijkl} w^h_{(k,l)} = 0$ since $\int_\Omega (\ ) = 0 \Rightarrow (\ ) = 0$

since $C_{ijkl} > 0$ $\forall$ pts. in $\Omega \Rightarrow w^h_{(i,j)} = 0$ this defines a rigid body motion.

We also said that the B.C. must preclude rigid motion $\Rightarrow w^h_i = 0$

$\Rightarrow \underset{\sim}{w}^h = 0$ $\forall$ pts $\eta - \eta_g$ since $\underset{\sim}{w}^h = w_i^h \underset{\sim}{e}_i$. But $w^h_i = 0 \Rightarrow c_P = 0 \Rightarrow \underset{\sim}{c} = 0$

§ Elastostatics, Element Stiffness, Element Forms.

We will skip the definitions of $\underset{\sim}{K}^e$ and $\underset{\sim}{F}^e$ and we will focus on $\underset{\sim}{k}^e, \underset{\sim}{f}^e$

$\underset{\sim}{k}^e = [k^e_{pq}]$ $\quad$ $\underset{\sim}{f}^e = \{f^e_p\}$ $\quad$ $1 \leq p,q \leq n_{eq} = n_{\text{element nodes}} \times n_{\text{dof/node}} = n_{en} \times n_{ed}$

$n_{ed} \leq (n_{dof}$ this is global version of the n_{ed} parameter)

$$k^e_{pq} = \underset{\sim}{e}_i^T \int_{\Omega^e} \underset{\sim}{B}_a^T \underset{\sim}{D} \underset{\sim}{B}_b \, d\Omega \, \underset{\sim}{e}_j$$

a, b are the local nodal indices

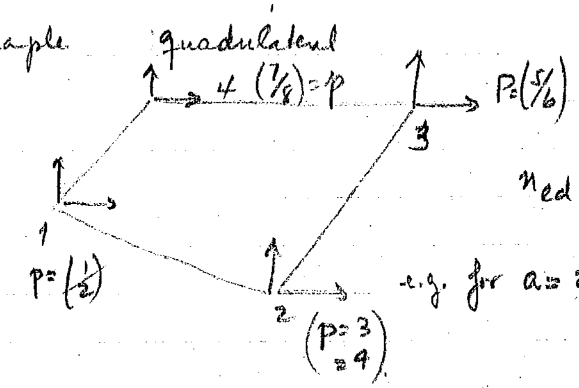
if we define we don't have to work w/3-D matrices

$p = n_{ed}(a-1) + i$

$q = n_{ed}(b-1) + j$

if $a=1$, $p = 1, 2, 3$ (no. of dof) $\quad$ if $a=2$ & $n_{ed}=3 \Rightarrow p = 4, 5, 6$

example quadrilateral



suppose that we have $n_{sd} = 2$ & $n_{ed} = 2$; thus $1 \le i \le 2$

e.g. for $a = 2$ $\quad p = 2(2-1) + i$
$= 2 + 1 = 3$ & $2 + 2 = 4$

for ($n_{sd} = 2$)

$$\underset{\sim}{B}_a = \begin{bmatrix} N_{a,1} & 0 \\ 0 & N_{a,2} \\ N_{a,2} & N_{a,1} \end{bmatrix}$$

for ($n_{sd} = 3$)

$$\underset{\sim}{B}_a = \begin{bmatrix} N_{a,1} & 0 & 0 \\ 0 & N_{a,2} & 0 \\ 0 & 0 & N_{a,3} \\ 0 & N_{a,3} & N_{a,2} \\ N_{a,3} & 0 & N_{a,1} \\ N_{a,2} & N_{a,1} & 0 \end{bmatrix}$$

$1 \le a, b \le n_{ed}$

$$f_p^e = \int_{\Omega^e} N_a f_i \, d\Omega + \int_{\Gamma^e_{h_i}} N_a h_i \, d\Gamma - \sum_{q=1}^{n_{eq}} k_{pq}^e g_q^e \qquad \text{where } p = n_{ed}(a-1) + i$$

we define as previously
$$g_q^e = g_{jb}^e = g_j(\underset{\sim}{x}_b^e) \quad \text{if } g_j \text{ is given at } \underset{\sim}{x}_b^e$$
$$= 0 \quad \text{if } g_j \text{ is } \underline{\text{not}} \text{ give}$$
and $q = n_{ed}(b-1) + j$

we will now define $\int_{\Omega^e} \underset{\sim}{B}_a^T \underset{\sim}{D} \underset{\sim}{B}_b \, d\Omega = \underset{\sim}{k}_{ab}^e$ where $[k_{ab}^e] = n_{ed} \times n_{ed}$

then $k_{pq}^e = \underset{\sim}{e}_i^T \, \underset{\sim}{k}_{ab}^e \, \underset{\sim}{e}_j$ which is useful for coding purposes.

we also have $\underset{\sim}{k}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \, d\Omega$ where $\underset{\sim}{B} = [\underset{\sim}{B}_1, \underset{\sim}{B}_2, \ldots, \underset{\sim}{B}_{n_{en}}]$

Using the 4 node quadrilateral - 2 DOF/node $\therefore$ stiffness matrix 8x8 thus

$$\underset{\sim}{k}^e = \begin{bmatrix} \underset{\sim}{k}^e_{11} & \underset{\sim}{k}^e_{12} & \underset{\sim}{k}^e_{13} & \underset{\sim}{k}^e_{14} \\ & \underset{\sim}{k}^e_{22} & \underset{\sim}{k}^e_{23} & \underset{\sim}{k}^e_{24} \\ \text{Symmetric} & & \underset{\sim}{k}^e_{33} & \underset{\sim}{k}^e_{34} \\ & & & \underset{\sim}{k}^e_{44} \end{bmatrix}$$

where $\underset{\sim}{k}^e_{ab}$ are 2x2 matrices

How do we assemble? Same as before. $K^e_{PQ} = K^e_{PQ} + k^e_{pq}$ where $P = LM(i,a,e)$, $Q = LM(j,b,e)$

Suppose we define the element displacement vector

$$\underset{n_{ee} \times 1}{\underset{\sim}{d}^e} = \left\{ \underset{\sim}{d}^e_a \right\} = \begin{Bmatrix} \underset{\sim}{d}_1^e \\ \underset{\sim}{d}_2^e \\ \vdots \\ \underset{\sim}{d}^e_{n_{en}} \end{Bmatrix}$$

no. of element eqns. ; $n_{ed} \times n_{en}$

$F^e_p = F^e_p + f^e_p$

$f^e_p = (f+q) - \sum k^e_{pq} g^e_q$ where $LM(j,b,e)=0$, $g^e_q \neq 0$

n. ~~element/node~~ node/element × n. element dof = # element eq

Example

$(n_{ed} = 2)$ $\quad \underset{\sim}{d}^e_a = \begin{Bmatrix} d^e_{1a} \\ d_{2a} \end{Bmatrix}$ $\quad$ if $(n_{ed} = 3)$ $\quad d^e_a = \begin{Bmatrix} d^e_{1a} \\ d^e_{2a} \\ d^e_{3a} \end{Bmatrix}$

How do these fit into our overall picture

$u^h_i(\underset{\sim}{x}^e_a) = d^e_{ia}$ {includes unknown part & g_i part}

<u>Exercise</u>: show that $\underset{\sim}{\sigma}(\underset{\sim}{x}) = \underset{\sim}{D}(\underset{\sim}{x})\, \underset{\sim}{B}(\underset{\sim}{x})\, \underset{\sim}{d}^e \quad \underset{\sim}{x} \in \Omega^e$

$$= \underset{\sim}{D}(\underset{\sim}{x}) \sum_{A=1}^{n_{en}} B_a(\underset{\sim}{x})\, \underset{\sim}{d}^e_a$$

Since $\sigma_{ij} = C_{ijkl}\, u_{(k,l)}$ where $u_{(k,l)} = \epsilon_{kl} = \sum_{A \in \eta - \eta_g} (N_{A,l}\, d_{kA} + N_{A,k}\, d_{lA})/2$.

now $\underset{\sim}{u}_{(k,l)} = \sum_{A \in \eta - \eta_g} \begin{pmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{pmatrix} \begin{pmatrix} d_{1A} \\ d_{2A} \end{pmatrix} = \sum_A B_A(\underset{\sim}{x})\, d_{iA} = \sum_{a=1}^{n_{en}} B_a(\underset{\sim}{x})\, \underset{\sim}{d}^e_a = \underset{\sim}{B}(\underset{\sim}{x})\, \underset{\sim}{d}^e$

Now $C_{ijkl} = \underset{\sim}{D}(x)$ when $\sigma_{ij} = \underset{\sim}{\sigma}(\underset{\sim}{x})$ $\quad \therefore \quad \underset{\sim}{\sigma}(\underset{\sim}{x}) = \underset{\sim}{D}(x)\, \underset{\sim}{B}(\underset{\sim}{x})\, \underset{\sim}{d}^e$

§ 9. Data Processing: ID, IEN, LM

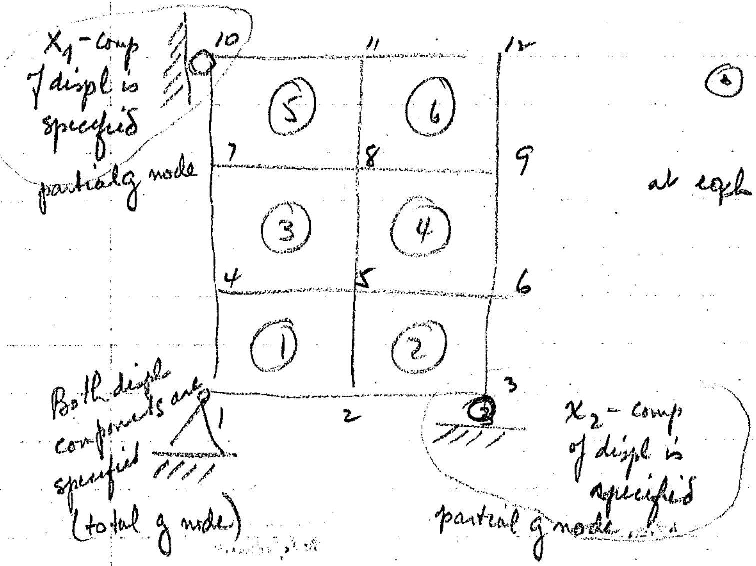
ID was defined $P = ID(i, A)$ (global eq, ndof, global no.) and was generalized heat conduction

IEN was relation between local & global node no. same as for heat conduction

Now $LM(i, a, e) = P = ID(i, A = IEN(a, e))$

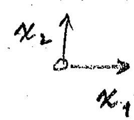
(i: ndof, a: local node no., e: element no.; P: global eqn.; i: ndof, A: global node, a: local node, e: element no.)

We will consider the 4 node quad elements as an example with local ordering being counterclockwise beginning lower-left hand node



(circled number) – element nos.

at each node 2 degrees of freedom.



see handout for the full works similar to the heat transfer problem

2/26/80

Chapter 3 Isoparametric elements and calculation of element arrays

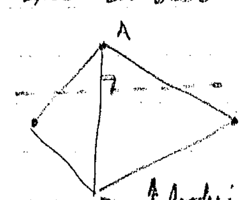
§ 1. Preliminary concepts; continuity and completeness

Defn of shape fns require fns to provide us with a "convergent formulation"
The following "basic convergence requirements" are not necessary & sufficient for N_A
These are not hard & fast rules & "rule-breaking" will be looked at later.

(C1) Fns must be smooth on element interiors ie (C^1 - fns.)
(C2) Require continuity across element boundaries (e.g. Γ^e)
(C3) Complete set must exist.

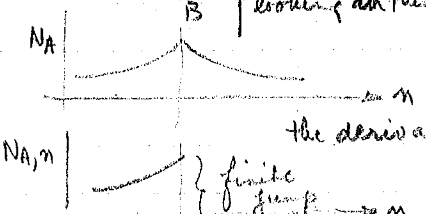
Let us look at (C1)-(C2): we are looking only at C^1 requirements here
if not (C2) $\Rightarrow$ δ fns will exist in our integration schemes in general.

Look at 2 triangles w/ interface



take cut & look at shape fn. & plot

looking in this direction then



the derivative

these are C^0 fns but the discont are at the bdrys & integrations will be OK.

For the problem done in class

ID Array:

global node numbers (A)

(fixed node: node 1; fixed in x_1 dir: node 10; fixed in x_2 dir: node 3)

degree-of-freedom numbers (i)	1	2	3	4	5	6	7	8	9	10	11	12
1	0*	1	3	4	6	8	10	12	14	0	17	19
2	0	2	0	5	7	9	11	13	15	16	18	20

$(n_{np} = 12)$

$(n_{ed} = 2)$ (dof) $P = ID(i, A)$ $(n_{eq} = 20)$

$n_{eq} = n_{dof} \times n_{np} - n_g$

IEN Array:

element numbers (e)

local node numbers (a)	1	2	3	4	5	6
1	1	2	4	5	7	8
2	2	3	5	6	8	9
3	5	6	8	9	~~[illegible]~~ 11	12
4	4	5	7	8	~~[illegible]~~ 10	11

$(n_e = 6)$ $(n_{en} = 4)$

this is based on picture we had in class

$A = IEN(a, e)$

LM Array:

$LM(i,a,e)$

element numbers (e) $(n_e = 6)$

$n_{ed}(a-1)+i = p = f(a, i)$

local equation numbers (p)	local node numbers (a)	degree-of-freedom numbers (i)	1	2	3	4	5	6
1	1	1	0*	1	4	6	10	12
2		2	0	2	5	7	11	13
3	2	1	1	3	6	8	12	14
4		2	2	0	7	9	13	15
5	3	1	6	8	12	14	17	19
6		2	7	9	13	15	18	20
7	4	1	4	6	10	12	0	17
8		2	5	7	11	13	16	18

$(n_{ee} = 8)$

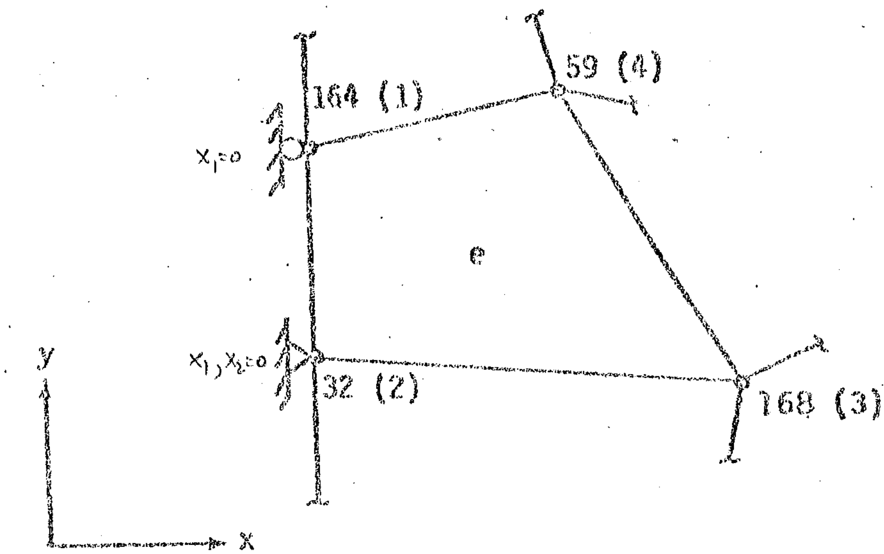
degree-of-freedom numbers (i) [illegible]
local node numbers (a)
local equation numbers $(p = n_{ed}(a-1) + i)$

$P = LM(p, e) = LM(i, a, e) = ID(i, IEN(a, e))$

* displacement boundary conditions denoted by zeros

Figure 2. ID, IEN and LM arrays for the mesh of Figure 1.

[illegible] n_{ed} [illegible] $n_{ed} = 2$



( ) - local node numbers

Figure 3. Typical 4-node elasticity element; global and local node numbers.

This is shown for element 4, which is typical. Four displacement (i.e., "g-type") boundary conditions are specified; namely the horizontal displacement is specified at nodes 1 and 10, and the vertical displacement is specified at nodes 1 and 3. Since $n_{np} = 12$, $n_{ed} = 2$, and four displacement degrees-of-freedom are specified, we have $n_{eq} = 20$. As is usual, we adopt the convention that the global equation numbers run in ascending order with respect to the ascending order of global node numbers.* The ID, IEN and LM arrays are given in Figure 2. The reader is urged to verify the results. ∎

In terms of the IEN and LM arrays, a precise definition of the g_p^e's may be given (see Eq. 5, §B):

$$g_p^e = g_{ia}^e = \begin{cases} 0 , & \text{if LM}(i, a, e) \neq 0 \\ g_{iA} , & \text{if LM}(i, a, e) = 0, \text{ where } A = \text{IEN}(a, e) \end{cases} \tag{3}$$

This definition may be easily programmed.

<u>Example 2.</u> As a final example, we consider a typical 4-node, elasticity element in some large mesh, see Figure 3. We assume the pertinent entries of the ID array are given as follows:

---

* In practice, equations numbers are often renumbered internally to minimize the band width of K and thus decrease storage and solution effort. This is especially important in large-scale analyzing equations involving tens of thousands of equations. An algorithm for renumbering bandwidth is presented in [ ].

$$
\left.\begin{aligned}
ID(1, 32) &= 0 \\
ID(2, 32) &= 0 \\
ID(1, 59) &= 115 \\
ID(2, 59) &= 116 \\
ID(1, 164) &= 0 \\
ID(2, 164) &= 325 \\
ID(1, 168) &= 332 \\
ID(2, 168) &= 333
\end{aligned}\right\} \qquad (3)
$$

The entries of IEN follow from Figure 3:

$$
\left.\begin{aligned}
IEN(1, e) &= 164 \\
IEN(2, e) &= 32 \\
IEN(3, e) &= 168 \\
IEN(4, e) &= 59
\end{aligned}\right\} \qquad (4)
$$

Combining (3) and (4), by way of (1), yields entries of the LM array:

$$
\left.\begin{aligned}
LM(1, 1, e) &= 0 \\
LM(2, 1, e) &= 325 \\
LM(1, 2, e) &= 0 \\
LM(2, 2, e) &= 0 \\
LM(1, 3, e) &= 332 \\
LM(2, 3, e) &= 333 \\
LM(1, 4, e) &= 115 \\
LM(2, 4, e) &= 116
\end{aligned}\right\} \qquad (5)
$$

The contribution to the global arrays may be deduced from LM:

stiffness (due to symmetry, only the upper triangular portion need be assembled.)

corresponding to LM array location

$P = ID(i, IEN(a,e)) = LM(i,a,e)$

$115 = LM(1,4,e)$

$p = ned(a-1)+i$

$p = 2(3)+1 = 7$

$$K_{115,115} \leftarrow K_{115,115} + k^e_{77}$$
$$K_{115,116} \leftarrow K_{115,116} + k^e_{78}$$
$$K_{115,325} \leftarrow K_{115,325} + k^e_{72}$$
$$K_{115,332} \leftarrow K_{115,332} + k^e_{75}$$
$$K_{115,333} \leftarrow K_{115,333} + k^e_{76}$$

$$K_{116,116} \leftarrow K_{116,116} + k^e_{88}$$
$$K_{116,325} \leftarrow K_{116,325} + k^e_{82}$$
$$K_{116,332} \leftarrow K_{116,332} + k^e_{85}$$
$$K_{116,333} \leftarrow K_{116,333} + k^e_{86}$$

$$K_{325,325} \leftarrow K_{325,325} + k^e_{22}$$
$$K_{325,332} \leftarrow K_{325,332} + k^e_{25}$$
$$K_{325,333} \leftarrow K_{325,333} + k^e_{26}$$

$$K_{332,332} \leftarrow K_{332,332} + k^e_{55}$$
$$K_{332,333} \leftarrow K_{332,333} + k^e_{56}$$

$$K_{333,333} \leftarrow K_{333,333} + k^e_{66} \qquad (6)$$

force

$P = LM(i,a,e)$ $\quad p = ned(a-1)+i \quad ned = 2$

$p = ned(a-1)+i$ where $P = ID(i, IEN(a,e))$

$$F_{115} \leftarrow F_{115} + f^e_7 \quad (i=1\ a=4)$$
$$F_{116} \leftarrow F_{116} + f^e_8 \quad (i=2\ a=4)$$
$$F_{325} \leftarrow F_{325} + f^e_2 \quad (i=2\ a=1)$$
$$F_{332} \leftarrow F_{332} + f^e_5 \quad (i=1\ a=3)$$
$$F_{333} \leftarrow F_{333} + f^e_6 \quad (i=2\ a=3) \qquad (7)$$

where

$$f_p^e = \cdots - \sum_{q=1}^{n_{ee}} k_{pq}^e g_q^e \qquad (8)$$

(We have omitted the first two terms in the right-hand side of (5), §8, in writing (8).) In the present example, only g_1^e, g_3^e and g_4^e may be nonzero. Therefore (8) may be simplified to

$q = n_{ed}(b-1)+j$ where $LM(j,b,e)=0$

$$f_p^e = \cdots - k_{p1}^e g_1^e - k_{p3}^e g_3^e - k_{p4}^e g_4^e \qquad (9)$$

contrib j	b	q
1	1	1
2	2	3
		4

The multiplications indicated in (9) are only performed in practice if the g_p^e's are nonzero. A schematic representation of the contributions of $\underset{\sim}{k}^e$ and $\underset{\sim}{f}^e$ to $\underset{\sim}{K}$ and $\underset{\sim}{F}$ is shown in Figure 4.

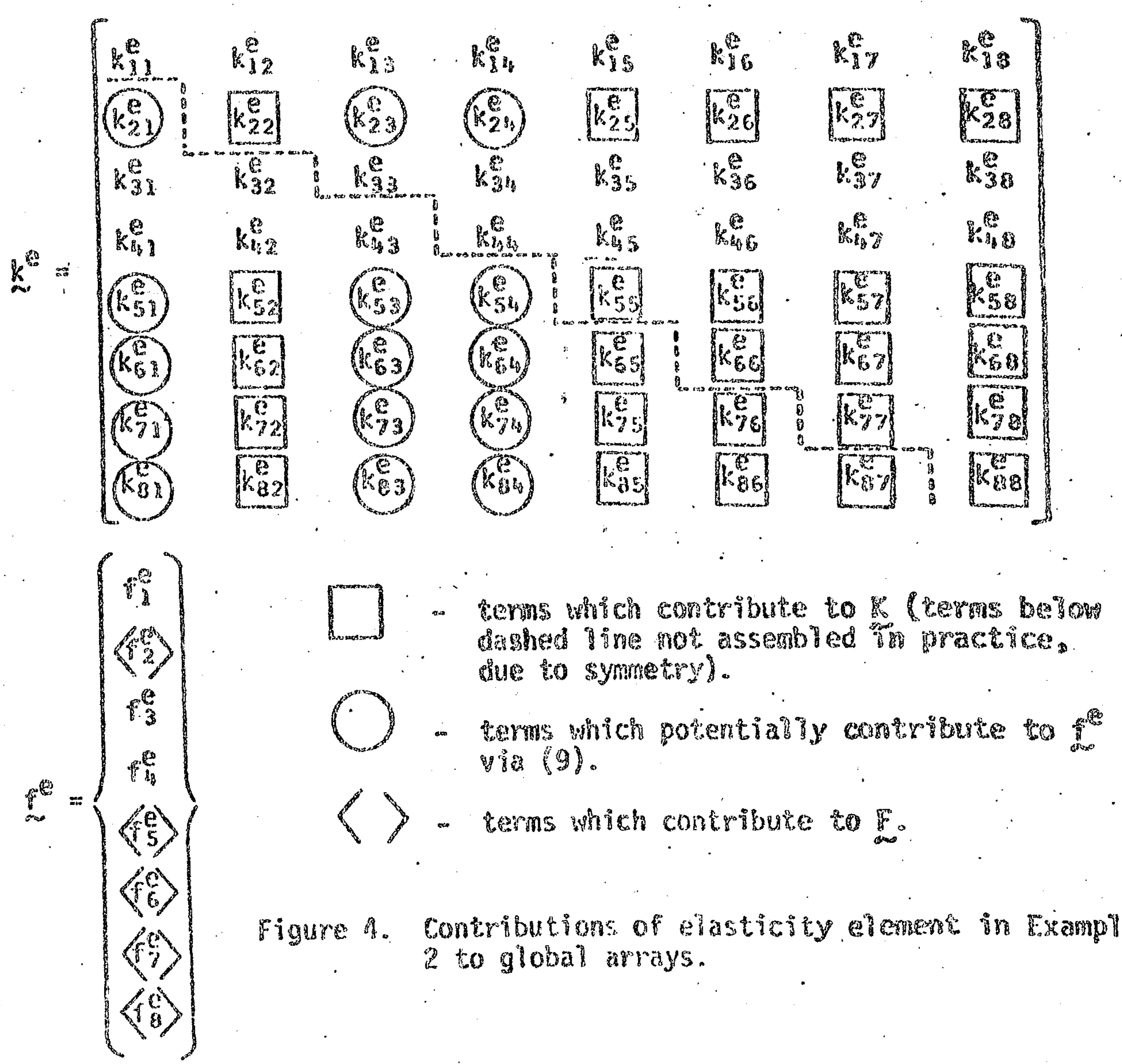


Figure 4. Contributions of elasticity element in Example 2 to global arrays.

for Bernoulli - Euler Beam theory modify (C1) to C^2 fns & (C2) to C^1 across Γ^e

In a theory involving squares of mth derivatives in the stiffness integrand, then (C1) becomes C^m fns in the interiors C^m on Ω^e
(C2) " C^{m-1} fn at the bdy C^{m-1} on Γ^e
[ie the PDE will have terms like $\partial^{2m} u / \partial x^{2m} + \dots$

$\exists$ non-conforming or incompatible elements which violate the continuity condition (C2)

If N_A satisfies (C1)-(C2) so does u^h

(C3) Completeness – using heuristic argument we will illustrate on heat conduction problem. Look at u^h restricted to element e – ⓘ needed if talking about elasticity problem

$$u^h_{(i)} = \sum_{a=1}^{n_{en}} N_a d^e_{a(i)} \qquad \text{where } d^e_{a(i)} = u^h(x^e_{a(i)})$$

for $n_{sd} = 3$, say we will say the shape fns are complete if

$$d^e_{a(i)} = c_0 + c_1 x^e_a + c_2 y^e_a + c_3 z^e_a$$

implies that

$$u^h_{(i)}(x) = c_0 + c_1 x + c_2 y + c_3 z$$

where c's are arbitrary constants $x = (x, y, z)^T$ and $x^e_a = (x^e_a, y^e_a, z^e_a)^T$ are nodal coordinates

for elasticity ⓘ must be used

This completeness will not allow us to lose "rigid body motions".

For C^{m-1} type elements (squares of m^{th} deriv) completeness involves m^{th} order polynomials

for euler-bernoulli $m = 2$ $\therefore$ order of poly must be at least 2. But we found $N_1(x), N_2(x), \dots$ to be 3rd order hence we can represent all quadratics w/o problems

Example 1: Piecewise linear f.e.'s
Completeness requires that $d^e_a = c_0 + c_1 x^e_a$, then $u^h(x) = c_0 + c_1 x$

$$u^h(x) = \sum_{a=1}^{2} N_a(x) d^e_a = \sum_{a=1}^{2} N_a (c_0 + c_1 x^e_a)$$

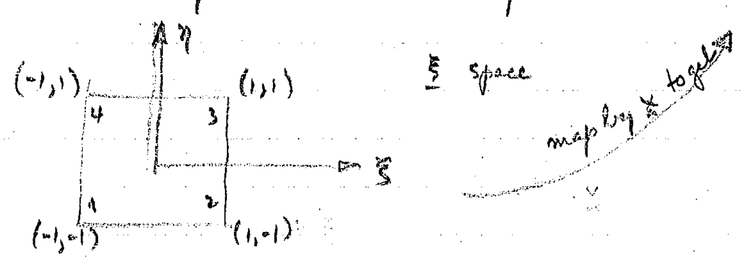
$$= c_0 \underbrace{\left(\sum N_a\right)}_{1} + c_1 \underbrace{\left(\sum N_a x^e_a\right)}_{x} \Rightarrow \text{looking back to section §1.11}$$

Reminder: $N_a(\xi) = \frac{1}{2}(1+\xi_a\xi)$ $\xi_a = (-1)^a$; Hence we have shown that we can represent a linear polynomial exactly

§2. Bilinear, Quadrilateral (TAIG) Argyris used a rectangle.

$1 \le a \le e;\ \underset{\sim}{x}_a^e$

we must remap into a biunit square. We assume the mappings exist. The Biunit square is called the Parent domain



we will use
$$x(\xi,\eta) = \sum_{a=1}^{4} N_a(\xi,\eta)\, x_a^e$$
$$y(\xi,\eta) = \sum_{a=1}^{4} N_a(\xi,\eta)\, y_a^e$$
$$\Bigg\}\ \underset{\sim}{x}(\xi,\eta) = \sum_{a=1}^{4} N_a\, \underset{\sim}{x}_a^e$$

Assume bilinear expansion
$$x(\xi,\eta) = \alpha_0 + \alpha_1\xi + \alpha_2\eta + \alpha_3\xi\eta \qquad \alpha_i\text{'s are param to be defined}$$
similarly
$$y(\xi,\eta) = \beta_0 + \beta_1\xi + \beta_2\eta + \beta_3\xi\eta \qquad \beta_i\text{'s} \quad " \quad "$$

we want $x(\xi_a,\eta_a) = x_a^e$ & $y(\xi_a,\eta_a) = y_a^e$

where

a	ξ_a	η_a
1	-1	-1
2	1	-1
3	1	1
4	-1	1

now $x_b^e = x(\xi_b,\eta_b) = \sum_{a=1}^{4} N_a(\xi_b,\eta_b)\, x_a^e$ $\quad \therefore N_a(\xi_b,\eta_b) = \delta_{ab} \begin{cases} 1 & a=b \\ 0 & a \ne b \end{cases}$

This is the interpolation property now by applying the above we find

$$\begin{pmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{pmatrix} = \underset{\sim}{x}^e = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \underset{\sim}{\alpha} \quad \text{thus } \underset{\sim}{y}^e = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \underset{\sim}{\beta}$$

Thus $N_a(\underset{\sim}{\xi}) = N_a(\xi,\eta) = \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta)$ = product of 1D shape fns. (?)

$= \frac{1}{4}(1+\xi_{[a/2+1]}\xi)(1+\eta_{[\frac{a+1}{2}]}\eta)$ where $[x]$ means smallest integer

Thus we can use the classical Lagrange interpolation techniques once geometry is straightened out. Hence

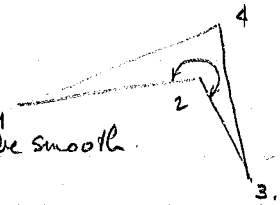
$$\underset{\sim}{x}(\underset{\sim}{\xi}) = \sum_{a=1}^{4} N_a(\underset{\sim}{\xi})\, \underset{\sim}{x}_a^e$$

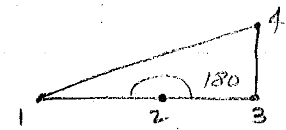
$$\Longrightarrow \quad u^h_{(i)}(\underset{\sim}{\xi}) = \sum_{a=1}^{4} N_a(\xi)\, \underset{\sim}{d}^e_{a(i)}$$ (i) if we use the elasticity problem.

To look at convergence criterion

(C1)? N_a to be smooth as a fn of $\underset{\sim}{x}$. N_a was smooth as a fn of ξ.

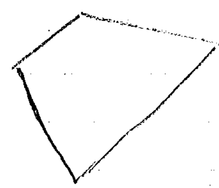
if we take quadrilateral & make interior angles $\geq 180°$ N_a fails to be smooth





if we take quadrilateral & make interior angles $< 180°$ N_a is smooth.

ie. this quadril is acceptable



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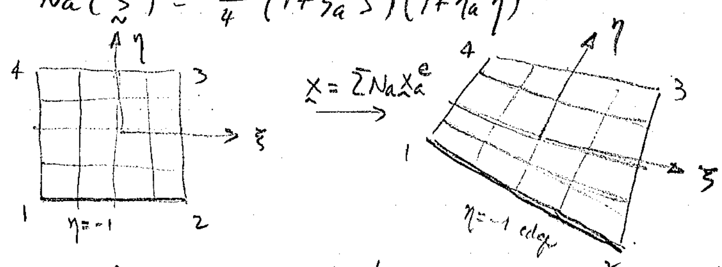
Continuation of last time.

From the bilinear quad

$$N_a(\underset{\sim}{\xi}) = \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta)$$



$$\begin{pmatrix} x \\ y \end{pmatrix} = \underset{\sim}{x} = \sum N_a \underset{\sim}{x}_a^e = \sum N_a \begin{pmatrix} x_a^e \\ y_a^e \end{pmatrix}$$

$$u^h_{(i)} = \sum N_a\, d^e_{(i)a}$$

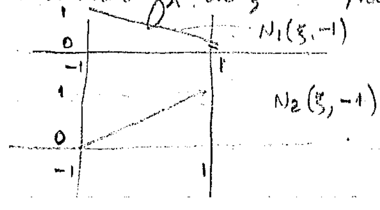
We assumed the mapping didn't distort quadril too much.

The other item (C2) continuity across Γ^e: look at N_a at an edge.

N_a across edge $\overline{12}$ equivalent to $\eta=-1$

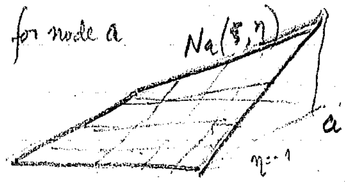
$N_a(\xi,\eta)$ $\quad N_a(\xi,-1) = \frac{1}{2}(1+\xi_a\xi)$ for $a=1,2$

this is a linear fn in ξ. This is the 1-D piecewise shape fn hence we find that



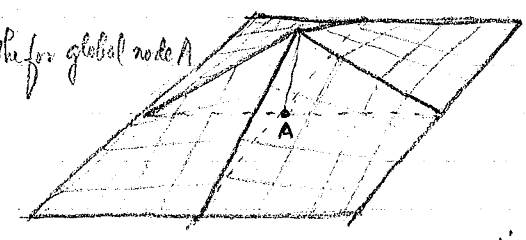
now $N_a = 0$ for $a=3,4$ $\therefore$ $\underset{\sim}{x} = \sum_{1}^{2} N_a(\xi,-1)\underset{\sim}{x}_a$

Thus for node a



for 4 elements the fn looks like

The fn global node A



tent like in nature but 1st derivs dis cont. along $\widehat{12}$ since $N_a(\underset{\sim}{\xi})$ is linear in $\underset{\sim}{\xi}$

$$u^h(\xi,-1) = \sum_{a=1}^{2} N_a(\xi,-1)\, d_a^e$$

hence fn is linear along the edge thus the fn. is continuous across the boundaries

(C3) Completeness: if we can interpolate a linear polynomial by u^h then we've demonstrated it. $u^h = \sum_{a=1}^{4} N_a d_a^e$ $\quad (c_0 + c_1 x_a^e + c_2 y_a^e)$ $\quad c_i$ are constants.

$$= c_0 \sum_{a=1}^{4} N_a + c_1 \sum N_a x_a^e + c_2 \sum N_a y_a^e$$

$$c_0\left(\sum \tfrac{1}{4}\cdot 4\right) + c_1 x + c_2 y$$

$$= c_0 + c_1 x + c_2 y.$$

§3. Isoparametric Elements TAIG 1961 looked at it, IRONS completes 1966

Let $\square$ be the parent domain in $\underset{\sim}{\xi}$ - space

Define: $\underset{\sim}{x} : \square \longrightarrow \bar{\Omega}^e$ ie $\underset{\sim}{x}(\xi) = \sum_{a=1}^{n_{en}} N_a(\xi)\, \underset{\sim}{x}_a^e$

if u^h is of same form $u^h(\xi) = \sum_{a=1}^{n_{en}} N_a(\xi)\, d_a^e$ then the element is isoparametric

if we can show this then (C1)-(C3) are virtually satisfied.

We will need the Jacobian determinant of the mapping

in $n_{sd}=2$ $\quad j = \det \begin{pmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{pmatrix} \Rightarrow \overset{j}{\left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}\right)} d\xi\, d\eta = dx\,dy$

in $n_{sd}=3$

$$j = \det \begin{pmatrix} x_{,\xi} & x_{,\eta} & x_{,\zeta} \\ y_{,\xi} & y_{,\eta} & y_{,\zeta} \\ \ldots \end{pmatrix}$$

we now give a statement for (C1)

Fact: if $\underset{\sim}{x}$: is
(i) 1-1
(ii) onto (every point in □ is in ◇)
(iii) C^k ; $k \geq 1$
so that integrals are of correct sign (iv) $j(\underset{\sim}{\xi}) > 0 \quad \forall \underset{\sim}{\xi} \in$ □ then
$\underset{\sim}{\xi} = \underset{\sim}{x}^{-1} : \bar{\Omega}^e \to$ □ is C^k and exists

} $\Rightarrow Na(\underset{\sim}{x})$ is smooth (in C^k sense)
$\therefore u^h(\underset{\sim}{x})$ " "

$\underset{\sim}{\xi}(\underset{\sim}{x})$ is a measure of the distortion of mapping hence if $\underset{\sim}{\xi}$ is smooth, & $\underset{\sim}{x}$ is smooth u^h & $Na(\underset{\sim}{x})$ are smooth. problem is not $Na(\underset{\sim}{\xi})$ but $\underset{\sim}{\xi}(\underset{\sim}{x})$

We will skip (C2) & go on to (C3) completeness.

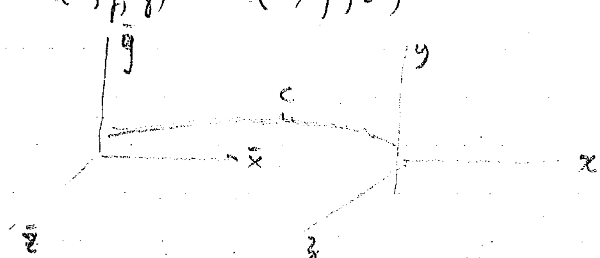
$u^h = \sum_{a=1}^{n_{en}} N_a d_a^e$ & define $d_a^e = c_0 + c_1 x_a^e + c_2 y_a^e + c_3 z_a^e$)

$\therefore \quad = c_0 \sum N_a + c_1 \underbrace{\sum N_a x_a^e}_{x} + c_2 \underbrace{\sum N_a y_a^e}_{y} + c_3 \underbrace{\sum N_a z_a^e}_{z}$

by virtue of isoparametricity

now $\sum N_a = 1$ must proved. Consider 2 rectangular coord. system

$(x, y, z) \qquad (\bar{x}, \bar{y}, \bar{z})$



let $\bar{x} = x + c$
$\bar{y} = y$
$\bar{z} = z$

define $\bar{x}(\xi) = \sum N_a(\xi)\, \bar{x}_a^e$
$x(\xi) = \sum N_a(\xi)\, x_a^e$

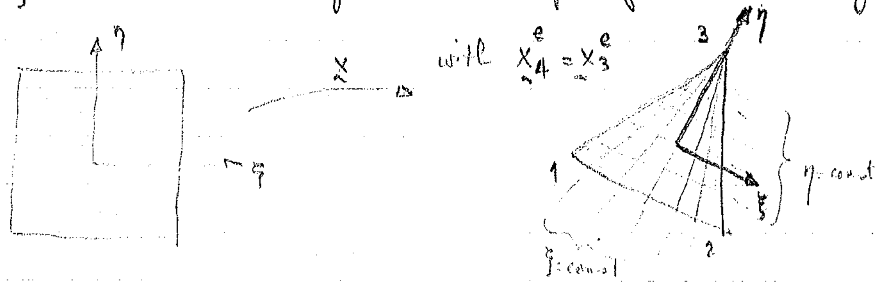
since $x(\xi) + c = \bar{x}(\xi)$ substitute the above.

$= \sum N_a(\xi)\, \bar{x}_a^e$
$= \sum N_a(\xi)\, (x_a^e + c)$

$x(\xi) + c = x(\xi) + c \sum N_a(\xi) \quad \Rightarrow \sum N_a(\xi) = 1.$

We look at isoparametric elements because linear poly is approx. by a linear poly. A linear poly for elasticity represents rigid body modes. Hence use of the elements allows us not to lose r. b. modes.

§4. Linear Triangle - an example of element "degeneration"



$$\underset{\sim}{x} = \sum_{a=1}^{4} N_a \underset{\sim}{x}_a^e \qquad \text{where } N_a = \text{are the bilinear shape fun as before}$$

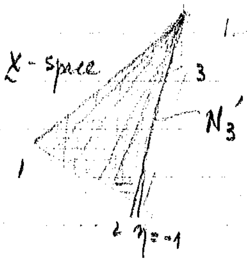
$$= N_1 \underset{\sim}{x}_1^e + N_2 \underset{\sim}{x}_2^e + (N_3 + N_4)\underset{\sim}{x}_3^e$$

$$= \sum_{a=1}^{3} N_a' \underset{\sim}{x}_a^e \qquad \text{where} \quad N_a' = N_a \quad a = 1,2 \qquad \tfrac{1}{4}(1+(-1)^a \xi)(1-\eta)$$

$$N_a' = N_a + N_{a+1}, \quad a = 3 \qquad \tfrac{1}{2}(1+\eta)$$

$a=1$: $\frac{1}{4}(1-\xi)(1-\eta)$
$a=2$: $\frac{1}{4}(1+\xi)(1-\eta)$
$a=3$: $\frac{1}{2}(1+\eta)$.

in $\underset{\sim}{X}$-space



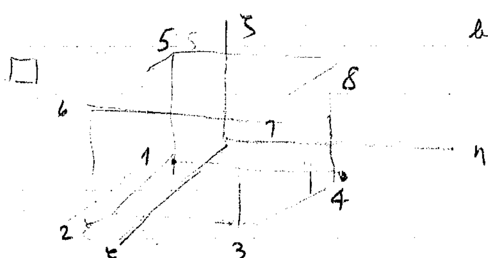
likewise for N_1', N_2'

since N_a are planes, then continuity is satisfied by linear like behavior

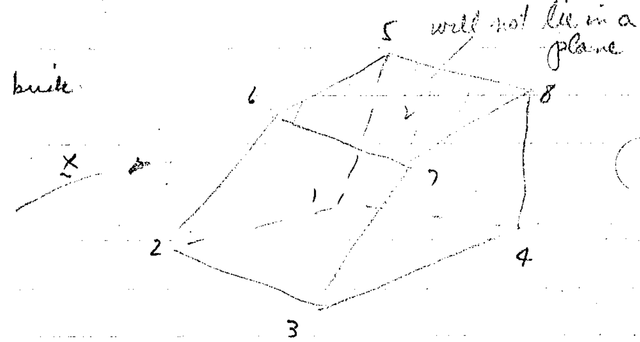
now $u^h = \sum_1^3 N_a' d_a^e$; at each edge each fn satisfies continuity hence (C2) is satisfied

we will skip (C3) even though its easy to prove as before

§ 5. Trilinear hexahedron



biunit brick



We will assume: $x(\underset{\sim}{\xi}) = \alpha_0 + \alpha_1 \xi + \alpha_2 \eta + \alpha_3 \zeta + \alpha_4 \xi\eta + \alpha_5 \eta\zeta + \alpha_6 \xi\zeta$
(also y & z has same form), $+ \alpha_7 \xi\eta\zeta$

Require: $x(\underset{\sim}{\xi}_a) = x_a^e$ likewise for y, z

if we write $\quad x(\underset{\sim}{\xi}) = \sum_{a=1}^{8} N_a(\underset{\sim}{\xi}) \underset{\sim}{x}_a^e \qquad$ where $\quad N_a(\underset{\sim}{\xi}) = \frac{1}{8}(1+\xi_a \xi)(1+\eta_a \eta)(1+\zeta_a \zeta)$

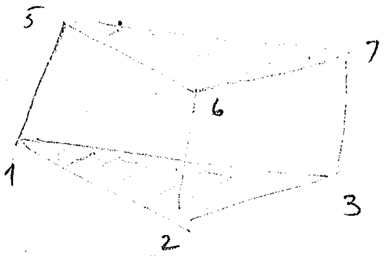
$$= \frac{1}{8}\left(1 + \xi_{[\frac{a}{2}+1]}\xi\right)\left(1 + \eta_{[\frac{a+1}{2}]}\eta\right)\left(1 + \zeta_{[\frac{a}{5}+1]}\zeta\right)$$

$[x]$ being smallest integer

a	ξ_a	η_a	ζ_a
1	-1	-1	-1
2	1	-1	-1
3	1	1	-1
4	-1	1	-1
5	-1	-1	1
6	1	-1	1
7	1	1	1
8	-1	1	1

$N_1(\underset{\sim}{\xi}) = \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)$
$N_2 = \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)$
$N_3 = \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)$
$N_4 = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)$
$N_5 = \frac{1}{8}(1-\xi)(1-\eta)(1+\zeta)$
$N_6 = \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta)$
$N_7 = \frac{1}{8}(1+\xi)(1+\eta)(1+\zeta)$
$N_8 = \frac{1}{8}(1-\xi)(1+\eta)(1+\zeta)$

Wedge element (collapsing the brick)



we will require

$$\underset{\sim}{x}_4^e = \underset{\sim}{x}_3^e \qquad \underset{\sim}{x}_7^e = \underset{\sim}{x}_8^e$$

hence $\underset{\sim}{x} = \sum_{a=1}^{3} N_a' \underset{\sim}{x}_a^e + \sum_{a=5}^{7} N_a' \underset{\sim}{x}_a^e$

$N_a' = N_a \quad a = 1,2,5,6$

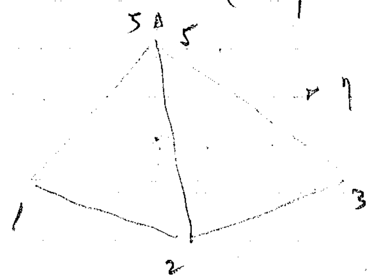
$N_a' = N_a + N_{a+1} \quad a = 3,7$

$N_3' = \frac{1}{4}(1+\eta)(1-\zeta)$

$N_7' = \frac{1}{4}(1+\eta)(1+\zeta)$

has linear variat in triangles

Tetrahedron (very useless) coalesce $\underset{\sim}{x}_7^e = \underset{\sim}{x}_5^e = \underset{\sim}{x}_6^e$



$$\underset{\sim}{x} = \sum_{a=1}^{3} N_a' \underset{\sim}{x}_a^e + N_5'' \underset{\sim}{x}_5^e \qquad N_5'' = N_5' + N_6' + N_7'$$

$$N_5'' = \tfrac{1}{4}(1-\eta)(1+\zeta) + \tfrac{1}{4}(1+\eta)(1+\zeta) = \tfrac{1}{2}(1+\zeta)$$

(the first term is $N_5' + N_6'$, the second N_7')

$$N_3' = \tfrac{1}{4}(1+\eta)(1-\zeta)$$

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For all isoparametric elements we will have $\underset{\sim}{x} = \sum N_a \underset{\sim}{x}_a^e$

$\underset{\sim}{u}^h = \sum N_a \underset{\sim}{d}_a^e$

§ 6. Higher-Order Elements.

Involves the use of Lagrange Polynomials in 1-dimension having n_{en} nodes

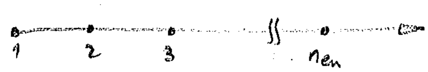
$$l_a^{n_{en}-1}(\xi) = \prod_{\substack{b=1\\ b\neq a}}^{n_{en}} (\xi - \xi_b) \Big/ \prod_{\substack{b=1\\ b\neq a}}^{n_{en}} (\xi_a - \xi_b)$$

($n_{en}-1$: order of poly; a: node; ξ_b: location of nodes in ξ-space)

denominator normalizes things

now on each elem.: $\underset{\sim}{x}(\xi) = \sum_{a=1}^{n_{en}} l_a^{n_{en}-1}(\xi)\, \underset{\sim}{x}_a^e$

$\underset{\sim}{u}(\xi) = \sum_{a=1}^{n_{en}} l_a^{n_{en}-1}(\xi)\, \underset{\sim}{d}_a^e$



we want $l_a(\xi_a) = 1 \qquad l_a(\xi_c) = 0$ where $c \neq a \qquad \therefore\ l_a(\xi_c) = \delta_{ac}$

thus $N_a(\xi) = l_a^{n_{en}-1}(\xi)$ is the definition

Example: let $n_{en} = 2$ for linear shape fns.

$$N_1(\xi) = l_1^1(\xi) = (\xi - \xi_2)/(\xi_1 - \xi_2) = (\xi - 1)/(-2) = (1-\xi)/2$$

$$N_2(\xi) = l_2^1(\xi) = (\xi - \xi_1)/(\xi_2 - \xi_1) = (\xi + 1)/2 = (1+\xi)/2$$

$\Big\}\ (1 + \xi_a \xi)/2$

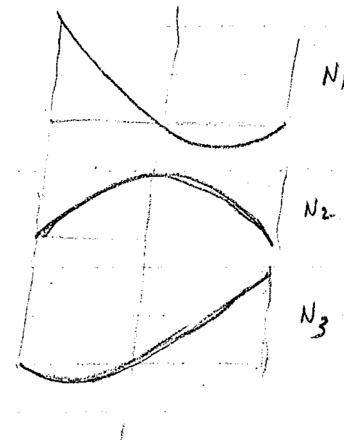
Now $\xi_1 = -1 \quad \xi_2 = 1$ here and these are our linear shape fns.

Example 2 — a 3 noded element: nodes at -1, 0, 1 (ξ_1, ξ_2, ξ_3)

$$N_1(\xi) = l_1^2(\xi) = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} = \frac{\xi(\xi-1)}{(-1)(-2)} = \frac{\xi(\xi-1)}{2}$$

$$N_2(\xi) = l_2^2(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} = \frac{(\xi^2-1)}{1\cdot(-1)} = 1-\xi^2$$

$$N_3(\xi) = l_3^2(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} = \frac{\xi(\xi+1)}{2}$$



HW Exercise $n_{en} = 4$: nodes at -1, $-1/3$, $1/3$, 1

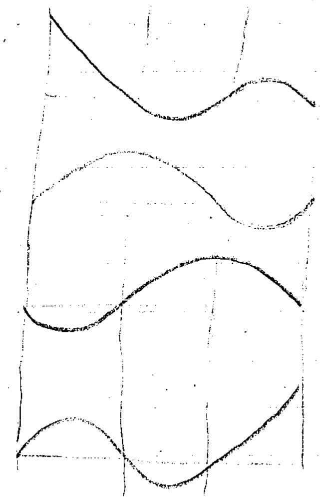
define the N_A's

$$N_1(\xi) = l_1^3(\xi) = \frac{(\xi+1/3)(\xi-1/3)(\xi-1)}{(-2/3)(-4/3)(-2)} = \frac{-9(\xi^2-1/9)(\xi-1)}{16}$$

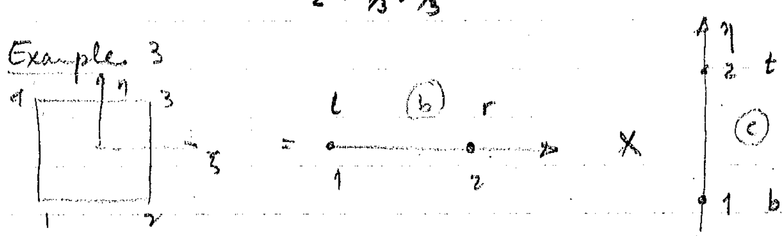
$$N_2(\xi) = l_2^3(\xi) = \frac{(\xi+1)(\xi-1/3)(\xi-1)}{(2/3)(-2/3)(-4/3)} = \frac{27}{16}(\xi^2-1)(\xi-1/3)$$

$$N_3(\xi) = l_3^3(\xi) = \frac{(\xi+1)(\xi+1/3)(\xi-1)}{(4/3)(2/3)(-2/3)} = \frac{-27}{16}(\xi^2-1)(\xi+1/3)$$

$$N_4(\xi) = l_4^3(\xi) = \frac{(\xi+1)(\xi^2-1/9)}{2\cdot 4/3\cdot 2/3} = \frac{9}{16}(\xi+1)(\xi^2-1/9)$$



Example 3



we noted that this was how we got the formulas; thus we can do same again

$$N_a(\xi,\eta) = l_b^1(\xi)\, l_c^1(\eta) \quad \text{where}$$

	a	b	c
lb	1	1	1
rb	2	2	1
rt	3	2	2
lt	4	1	2

now

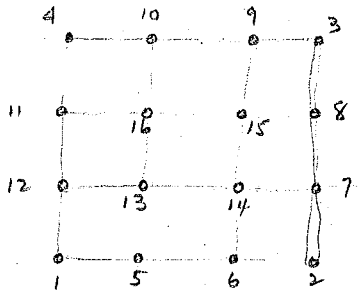
$$N_1(\xi,\eta) = l_1^1(\xi)\, l_1^1(\eta) = \tfrac{1}{4}(1-\xi)(1-\eta)$$

2	2	1	$= \tfrac{1}{4}(1+\xi)(1-\eta)$
3	2	2	$= \tfrac{1}{4}(1+\xi)(1+\eta)$
4	1	2	$= \tfrac{1}{4}(1-\xi)(1+\eta)$

a, b, c from

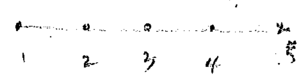
4×4

Ⓐ



a	b	c
1	1	1
2	4	1
3	4	4
4	1	4
5	2	1
6	3	1

Ⓑ



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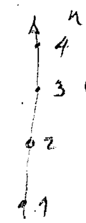


Ⓒ

a	b	c	a	b	c
7	4	2	11	1	3
8	4	3	12	1	2
9	3	4	13	2	2
10	2	4	14	3	2

a	b	c
15	3	3
16	2	3

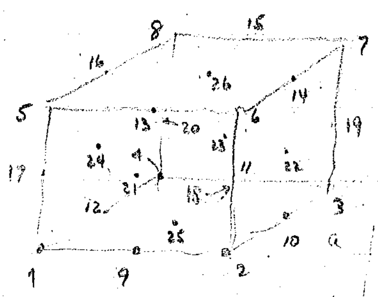
$$N_a(\xi,\eta) = \ell_b^3(\xi)\,\ell_c^3(\eta)$$

$$N_1(\xi,\eta) = \ell_1^3(\xi)\,\ell_1^3(\eta)$$

$$= \left(\tfrac{9}{16}\right)^2 (\xi^2 - 1/9)(\xi-1)(\eta^2-1/9)(\eta-1)$$

$$N_5(\xi,\eta) = \ell_2^3(\xi)\,\ell_1^3(\eta) = \frac{-9\cdot 27}{(16)^2}(\xi^2-1)(\xi-1/3)(\eta^2-1/9)(\eta-1)$$

$$N_{13}(\xi,\eta) = \ell_2^3(\xi)\,\ell_2^3(\eta) = \left(\frac{27}{16}\right)^2(\xi^2-1)(\xi-1/3)(\eta^2-1)(\eta-1/3)$$



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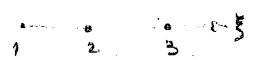

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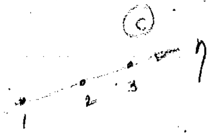

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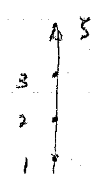


Ⓓ

a	b	c	d	a	b	c	d	a	b	c	d
1	1	1	1	8	1	3	3	15	2	3	3
2	3	1	1	9	2	1	1	16	1	2	3
3	3	3	1	10	3	2	1	17	1	1	2
4	1	3	1	11	2	3	1	18	3	1	2
5	1	1	3	12	1	2	1	19	3	3	2
6	3	1	3	13	2	1	3	20	1	3	2
7	3	3	3	14	3	2	3	21	2	1	2

add corners

corners 1-8
midside 9-20
faces 21-26
center 27

22	3	2	2
23	2	3	2
24	1	2	2
25	2	2	1
26	2	2	3
27	2	2	2

$N_4(\xi,\eta,\zeta) = \frac{\xi(\xi-1)}{2}\,\frac{\eta(\eta+1)}{2}\,\frac{\zeta(\zeta-1)}{2} = l_1^2\; l_3^2\; l_1^2$

$$N_a(\underset{\sim}{\xi}) = \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta)$$

$$\underset{\sim}{X}(\underset{\sim}{\xi}) = \sum_{a=1}^{4} N_a(\underset{\sim}{\xi})\,\underset{\sim}{x}_a^e$$

$$\frac{\partial x}{\partial \xi} = \sum \frac{1}{4}\xi_a(1+\eta_a\eta)\,x_a^e \qquad \frac{\partial y}{\partial \xi} = \sum \frac{1}{4}\xi_a(1+\eta_a\eta)\,y_a^e$$

$$\frac{\partial x}{\partial \eta} = \sum \frac{1}{4}\eta_a(1+\xi_a\xi)\,x_a \qquad \frac{\partial y}{\partial \eta} = \sum \frac{1}{4}\eta_a(1+\xi_a\xi)\,y_a$$

$$\frac{\partial x}{\partial \xi} = \frac{1}{4}(-1)(1-\eta)x_1 + \frac{1}{4}(1-\eta)x_2 + \frac{1}{4}(1+\eta)x_3 - \frac{1}{4}(1+\eta)x_4$$

$$\frac{\partial y}{\partial \eta} = \frac{1}{4}(-1)(1-\xi)y_1 + \frac{1}{4}(-1)(1+\xi)y_2 + \frac{1}{4}(1)(1+\xi)y_3 + \frac{1}{4}(1-\xi)y_4$$

$$\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} = \frac{1}{16}(1-\eta)(1-\xi)x_1y_1 + \frac{1}{16}(1-\xi)(1-\eta)x_2y_1 + \cdots$$

$$\frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} = \frac{1}{16}(1-\eta)(1-\xi)x_1y_1 - \frac{1}{16}(1-\xi)(1-\eta)y_2x_1 + \cdots$$

$$= \frac{1}{16}(1-\xi)(1-\eta)[x_2y_1+y_2x_1] + \frac{1}{16}(1-\xi)(1+\eta)(y_1x_3-x_1y_3) + \frac{1}{16}(1+\eta)(1-\xi)(x_4y_1-x_1y_4)$$

$1-\xi+\eta-\xi\eta$ $\quad$ $1+\eta-\xi-\xi\eta$

$$+\frac{1}{16}(1+\xi)(1-\eta)(y_2x_1-y_1x_2) + \frac{1}{16}(1+\xi)(1-\eta)(y_3x_2-y_2x_3) + \frac{1}{16}(1+\xi)(1-\eta)(y_1x_3-y_3x_1)$$

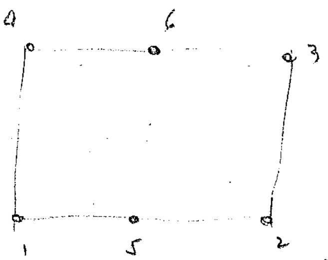
$1+\xi-\eta+\xi\eta$ $\quad$ $1-\xi-\eta+\xi\eta$

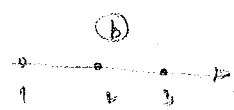
$$+\frac{1}{16}(1-\xi)(1-\eta)(y_4x_3-y_4x_1)$$

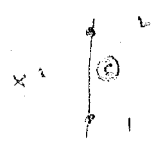
Jacobian

$$\frac{-1}{8}\xi(1-\eta)(y_2x_1-y_1x_2) + \frac{1}{8}(-\xi+\eta)(y_1x_3-y_3x_1) + \frac{\eta}{8}(1-\xi)(y_1x_4-y_4x_1)$$

3x2







3x2

a	b	c
1	1	1
2	3	1
3	3	2
4	1	2
5	2	1
6	2	2

$$N_a(\xi,\eta) = l_b^2(\xi)\; l_c^1(\eta)$$

$N_1 = \frac{(\xi-1)\xi}{2}\;\frac{(1-\eta)}{2} = \xi\frac{(\xi-1)(1-\eta)}{4}$

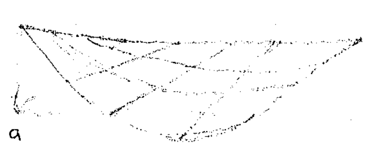
$N_2 = \frac{\xi(\xi+1)}{2}\;\frac{(1-\eta)}{2} = \xi\frac{(\xi+1)(1-\eta)}{4}$

$\frac{\xi(\xi+1)}{2}\;\frac{(1+\eta)}{2} = \frac{1}{4}\xi(\xi+1)(1+\eta)$

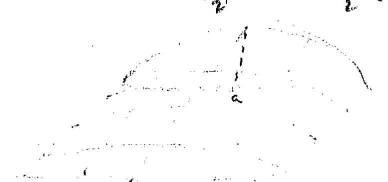
$\frac{(\xi-1)\xi}{2}\;\frac{1+\eta}{2} = \frac{1}{4}\xi(\xi-1)(1+\eta)$

$1-\xi^2\;\;\frac{1-\eta}{2} = \frac{1}{2}(1-\xi^2)(1-\eta)$

$1-\xi^2\;\;\frac{1+\eta}{2} = \frac{1}{2}(1-\xi^2)(1+\eta)$



corner



mid

Now $f(\xi,\eta,\zeta)\,j(\xi,\eta,\zeta)$ can be viewed as a fn $g(\underset{\sim}{\xi})$ $\quad \underset{\sim}{\xi} = (\xi,\eta,\zeta)$

In 1-D : $\displaystyle\int_{-1}^{1} g(\xi)\,d\xi = \sum_{l=1}^{n_{int}} g(\tilde{\xi}_l)\,W_l + R$

(n_{int} = # integration pts; $\tilde{\xi}_l$ = location at integration pts; W_l = weights; R = remainder)

$$\cong \boxed{\sum_{l=1}^{n_{int}} g(\tilde{\xi}_l)\,W_l}$$

must define weights, n_{int}, $\tilde{\xi}_l$ in order to see how this comes close to the actual integral

trapezoidal rule $\quad n_{int} = 2 \quad \tilde{\xi}_1 = -1 \quad \tilde{\xi}_2 = 1 \quad W_l = 1 \quad l = 1,2 \qquad R = -\tfrac{2}{3}\, g_{,\xi\xi}(\bar{\xi})$

exact integ of const & linear fns $\quad W_l = h/2 = \tfrac{2}{2} = 1.$

and has 2nd order or quadratic accuracy $\quad -1 \le \bar{\xi} \le 1$

simpson's rule : $\quad n_{int} = 3 \quad \tilde{\xi}_1 = -1 \quad \tilde{\xi}_2 = 0 \quad \tilde{\xi}_3 = +1 \quad W_l = \tfrac{1}{3}\; l = 1,3 \quad W_2 = \tfrac{4}{3}$

$R = -g^{IV}(\bar{\xi})/90$ $\quad$ will integrate all cubics $\therefore$ accuracy is quartic or 4th order

$W_l = \tfrac{h}{6}$ for $l = 1,3$ $\quad \tfrac{4h}{6}$ for $l = 2$

Gauss rules: give highest order accuracy for fewest # of points & optimal in 1-D

Ex 1 & 2 are "inefficient" by these rules

Order of accuracy is $2 \cdot n_{int}$ for n_{int} points ie $n_{int} = 1 \Rightarrow$ 2nd order accur.

integrate linear exactly (1) $\quad n_{int} = 1 \quad \tilde{\xi}_1 = 0 \quad W_1 = 2 \quad R = g''(\bar{\xi})/3 \qquad$ ie $\int_{-1}^{1} g(\xi)\,d\xi = 2g(0)$

" cubic " (2) $\quad n_{int} = 2 \quad \tilde{\xi}_1 = -1/\sqrt{3} \quad \tilde{\xi}_2 = 1/\sqrt{3} \quad W_l = 1 \quad l = 1,2 \qquad R = g^{IV}(\bar{\xi})/135$ Gauss pts

" quintic " (3) $\quad n_{int} = 3 \quad \tilde{\xi}_1 = -\sqrt{3/5}\,, \quad \tilde{\xi}_2 = 0\,, \quad \tilde{\xi}_3 = \sqrt{3/5} \quad W_l = 5/9 \;\; 1,3 = l \quad W_l = 8/9 \;\; l = 2$

will integrate 5th order polyn. $\qquad R = g^{VI}(\bar{\xi})/15750$

Exercise - show that poly of degree $2n_{int}-1$ is exact for each rule.

for $n_{int} = 1$ $\quad p(x) = c_0 + c_1 x \quad R = p''(\bar{\xi})/3 = 0$ for any $|\bar{\xi}| \le 1$

$n_{int} = 2$ $\quad p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \quad R = p^{IV}(\bar{\xi})/135 = 0$

} note that for n_{int} order of deriv. is $2n_{int}$ but poly is of degree $2n_{int}-1$

Example : we will derive 2 point rules - we can exactly integrate cubic since $g(\xi) \sim O(\xi^{2n_{int}-1})$

$\therefore$ pick $\quad g(\xi) = \alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3 \qquad \alpha_3$ are parameters.

Assumpt: assume $\tilde{\xi}_1 = -\tilde{\xi}_2 \quad W_1 = W_2$ (always must be symmetric)

$$\int_{-1}^{1} g(\xi)\,d\xi = 2\alpha_0 + \frac{2\alpha_2}{3} \text{ (exact)} = \sum_{l=1}^{2} g(\tilde{\xi}_l)\,W_l$$

$$= W_2\left[\, g(-\tilde{\xi}_2) + g(\tilde{\xi}_2) \,\right]$$

Now subst. $g(\bar{\xi}_2)$ & $g(-\bar{\xi}_2)$

$$W_2\left[\alpha_0 - \alpha_1\bar{\xi}_2 + \alpha_2\bar{\xi}_2^2 - \alpha_3\bar{\xi}_2^3 + \dots\right] = \left[2\alpha_0 + 2\alpha_2\bar{\xi}_2^2\right]W_2$$

Now $2\alpha_0 + \frac{2}{3}\alpha_2 = W_2\left[2\alpha_0 + 2\alpha_2\bar{\xi}_2^2\right]$; this must be true $\forall\, \alpha_0, \alpha_2$

$\Rightarrow W_2 = 1$ & $\frac{2}{3} = 2\bar{\xi}_2^2$ $\therefore$ $\bar{\xi}_2 = \pm\sqrt{\frac{1}{3}}$ choose $+\sqrt{\frac{1}{3}}$

$\Rightarrow W_1 = W_2 = 1$ $\bar{\xi}_2 = \sqrt{\frac{1}{3}}$ $\bar{\xi}_1 = -\sqrt{\frac{1}{3}}$

Exercise: do for 3 pt. rule. Assume $\begin{cases}\bar{\xi}_1 = -\bar{\xi}_3 & \bar{\xi}_2 = 0 \\ W_1 = W_3\end{cases}$ symmetric cond.

For 4th order $\int_{-1}^{1} g(\xi)d\xi = 2\alpha_0 + \frac{2}{3}\alpha_2 + \frac{2}{5}\alpha_4 = \left[\left(2\alpha_0 + 2\alpha_2\bar{\xi}_3^2 + 2\alpha_4\bar{\xi}_3^4\right)W_1 + W_2\cdot\alpha_0\right]$

$\therefore$ $2W_1 + W_2 = 2$ $\frac{2}{3} = 2\bar{\xi}_3^2 W_1$ $\frac{2}{5} = 2\bar{\xi}_3^4 W_1$ $\frac{1}{3} = \bar{\xi}_3^2 W_1$ $\frac{1}{5} = \bar{\xi}_3^4 W_1 = \frac{1}{3}\bar{\xi}_3^2 \Rightarrow \bar{\xi}_3 = \pm\sqrt{\frac{3}{5}}$

$W_2 = 8/9 \Leftarrow$ substitute; $W_1 = 5/9 \Leftarrow$ substitute

for $n_{sd} = 2$ we iterate 1-D rule

$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\,d\xi\,d\eta \cong \int_{-1}^{1}\left\{\sum_{\ell^{(1)}=1}^{n_{int}^{(1)}} g(\xi_{\ell^{(1)}},\eta)\,W_{(1)}^{(1)}\right\}d\eta \qquad \eta \text{ fixed}$$

(① and ② mark the index no. of ξ and η)

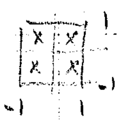
rule # 1 used in ξ-direction.

$$\cong \sum_{\ell^{(1)}=1}^{n_{int}^{(1)}} \sum_{\ell^{(2)}=1}^{n_{int}^{(2)}} g(\tilde{\xi}_{\ell^{(1)}}^{(1)}, \eta_{\ell^{(2)}}^{(2)})\,W_{(1)}^{(1)}\,W_{(2)}^{(2)} \qquad (*)$$

Example 4 1 pt rule in 2 dim

$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\,d\xi\,d\eta \cong 4\,g(0,0)$$

2pt × 2pt rule $a = 1/\sqrt{3}$



$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\,d\xi\,d\eta = g(-a,-a) + g(a,-a) + g(-a,a) + g(a,a) \qquad \text{all } W = 1$$

Alternative form is $\int_{-1}^{1}\int_{-1}^{1} \cong \sum_{\ell=1}^{n_{int}} g(\tilde{\xi}_\ell, \tilde{\eta}_\ell)\,W_\ell$ where $n_{int} = n_{int}^{(1)} \times n_{int}^{(2)}$

$\tilde{\xi}_\ell = \tilde{\xi}_{\ell^{(1)}}^{(1)}$ $\tilde{\eta}_\ell = \tilde{\eta}_{\ell^{(2)}}^{(2)}$

$W_\ell = W_{\ell^{(1)}}^{(1)}\,W_{\ell^{(2)}}^{(2)}$

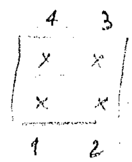
A table is now required which relates $\ell^{(1)}, \ell^{(2)}$ to ℓ

Example 1 1×1 pt rule $n_{int} = 1 \times 1 = 1$ $\xi_1 = 0$ $\eta_1 = 0$

$W_1 = W_1 \cdot W_1 = 4$

Example 2 2×2 rule

l	$l^{(1)}$	$l^{(2)}$
1	1	1
2	2	1
3	2	2
4	1	2



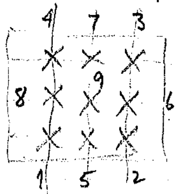
$n_{int} = 2 \times 2 = 4$ $\quad a = \sqrt{1/3}$

$\tilde{\xi}_1 = -a \quad \tilde{\xi}_2 = a \quad \tilde{\xi}_3 = a \quad \tilde{\xi}_4 = -a$

$\tilde{\eta}_1 = -a \quad \tilde{\eta}_2 = -a \quad \tilde{\eta}_3 = a \quad \tilde{\eta}_4 = a$

$W_l = 1 \ \forall$ pts.

Exercise set up table & derive $\tilde{\eta}_l, \tilde{\xi}_l, \tilde{W}_l$ for the 3×3 rule.



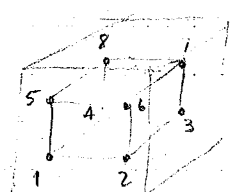
$\tilde{W}_l$	l	$l^{(1)}$	$l^{(2)}$	$\tilde{\xi}_l$	$\tilde{\eta}_l$	l	$l^{(1)}$	$l^{(2)}$	$\tilde{\xi}_l$	$\tilde{\eta}_l$	
25/81	1	1	1	$-\surd$	$-\surd$	6	3	2	$\surd$	0	40/81
25/81	2	3	1	$\surd$	$-\surd$	7	2	3	0	$\surd$	"
"	3	3	3	$\surd$	$\surd$	8	1	2	$-\surd$	0	"
"	4	1	3	$-\surd$	$\surd$	9	2	2	0	0	64/81
40/81	5	2	1	0	$-\surd$						

$\surd = \sqrt{3/5}$

Similarly

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta \cong \sum_{l^{(1)}=1}^{n_{int}^{(1)}} \sum_{l^{(2)}=1}^{n_{int}^{(2)}} \sum_{l^{(3)}=1}^{n_{int}^{(3)}} g(\tilde{\xi}^{(1)}_{l^{(1)}}, \tilde{\eta}^{(2)}_{l^{(2)}}, \tilde{\zeta}^{(3)}_{l^{(3)}})\, W^{(1)}_{l^{(1)}} W^{(2)}_{l^{(2)}} W^{(3)}_{l^{(3)}}$$

$$\sum_{l=1}^{n_{int}} g(\tilde{\xi}_l, \tilde{\eta}_l, \tilde{\zeta}_l)\, W_l$$



Exercise: determine $\tilde{\xi}_{l^{(1)}}$, $\tilde{\eta}_{l^{(2)}}$, $\tilde{\zeta}_l$, W_l for 1 pts rule $W_1 = 8$ $\tilde{\xi}_l = \eta_l = \zeta_l = 0$

l	$\tilde{\xi}$	$\tilde{\eta}$	$\tilde{\zeta}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	l	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$\tilde{\xi}$	$\tilde{\eta}$	$\tilde{\zeta}$
1	$-1/\surd$	$-1/\surd$	$-1/\surd$	1	1	1	5	1	1	2	$-1/\surd$	$-1/\surd$	$1/\surd$
2	$1/\surd$	$-1/\surd$	$-1/\surd$	2	1	1	6	2	1	2	$1/\surd$	$-1/\surd$	$1/\surd$
3	$1/\surd$	$1/\surd$	$-1/\surd$	2	2	1	7	2	2	2	$1/\surd$	$1/\surd$	$1/\surd$
4	$-1/\surd$	$1/\surd$	$-1/\surd$	1	2	1	8	1	2	2	$-1/\surd$	$1/\surd$	$1/\surd$

2×2×2 rule all $W_l = 1$

$1/\surd = 1/\sqrt{3}$

2×2×2 is not optimal: 8 points must be eval. However we can get a 6 pt rule to do the job ie a 3×2×1

wrt order of accuracy (ie the ability to exactly integrate complete polynomials)

(Complete polynomials: in $n_{sd} = 2$ linear poly: $\alpha_0 + \alpha_1\xi + \alpha_2\eta$ all terms must be present

quadratic: $\alpha_0 + \alpha_1\xi + \alpha_2\eta + \alpha_3\xi^2 + \alpha_4\xi\eta + \alpha_5\eta^2$)

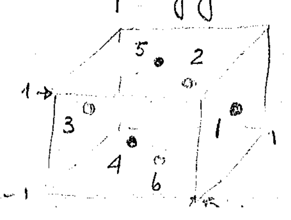
Gaussian rules are not necessarily optimal (efficient) in multi dim

Example $n_{sd} = 3$ 6-pt rule achieves 4th order accuracy.

l	$\tilde{\xi}_l$	$\tilde{\eta}_l$	$\tilde{\zeta}_l$	W_l
1	1	0	0	4/3
2	−1	0	0	4/3
3	0	1	0	4/3
4	0	−1	0	4/3

l	$\tilde{\xi}_l$	$\tilde{\eta}_l$	$\tilde{\zeta}_l$	W_l
5	0	0	1	4/3
6	0	0	−1	4/3

mid pts of faces



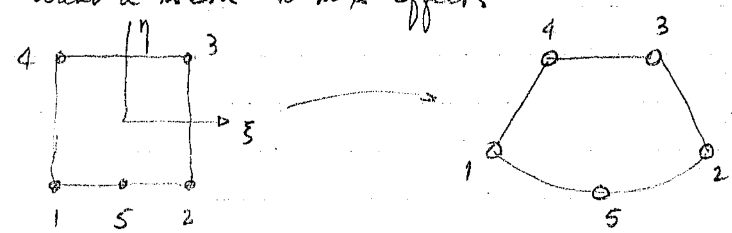
14 pt rule 6^{th} order accurate in 3d   compares favorably w/ 3×3×3 (27 pts)
8 corners + 6 faces.

3/12/81

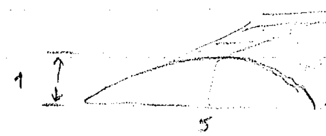
No calculators - Open notes: everything is on exam
Bring paper: 300 pt final ≈ 3 problems (?)

Back to section 7.

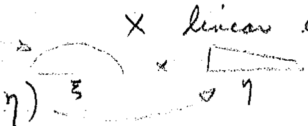
§ 7. Variable no. of nodes element.
Want a mesh to mix effects



a candidate fn is



use lagrange for 3 node element
× linear element.



$$\therefore \quad N_5(\xi,\eta) = \ell_2^2(\xi)\, \ell_1^1(\eta)$$

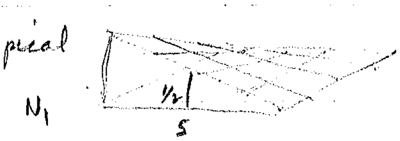
$$N_5(\xi,\eta) = \tfrac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_5(\xi_a,\eta_a) = \delta_{5a} \qquad a = 1,2,\ldots,5$$

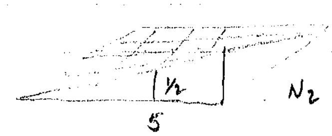
now we look at the other pts. Original were bilinear elements.

$$N_a(\xi,\eta) = \tfrac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta) \qquad a=1,2,\ldots,4 \quad (*)$$

look at typical



this doesn't satisfy cond that $N_a\big|_{\text{node }5} = 0$



now N_3, N_4 do satisfy since N_3, N_4 at $\eta=-1$ $= 0$
Thus must modify N_1, N_2 to satisfy the interpol.
Thus if we add a multiple of N_5 to N_1 & N_2 since N_5 vanishes at all other elements nodes

then define $N_1' = N_1 - \frac{1}{2} N_5$ $\quad$ $\frac{1}{2}$ = value of N_1 at node 5

$N_2' = N_2 - \frac{1}{2} N_5$

(**) $N_a'(\xi_b, \eta_b) = \delta_{ab} \quad a=1,2 \quad b=1,\dots,5$

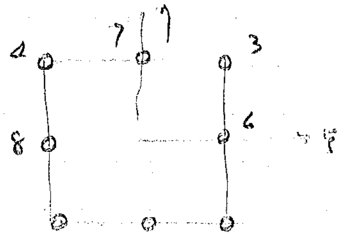
Shape fns. for 5 node elemets, are defined by

$$N_a = \begin{cases} N_a & (*) \quad a=3,4 \\ N_a' & (**) \quad a=1,2 \\ N_a & \quad a=5 \end{cases}$$

& $N_a(\underset{\sim}{\xi}_b) = \delta_{ab} \quad 1 \le a, b \le 5$

this then will change the variation on edge

For



for the addition of node 6, 7, 8

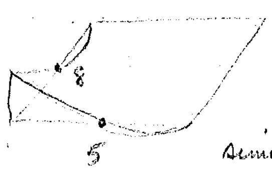
$$N_6 = \tfrac{1}{2}(1-\eta^2)(1+\xi)$$
$$N_7 = \tfrac{1}{2}(1-\xi^2)(1+\eta)$$
$$N_8 = \tfrac{1}{2}(1-\eta^2)(1-\xi)$$

now these all affect the others e.g. N_8 & N_5 affect N_1

why $N_1|_5$ & $N_1|_8 = \frac{1}{2}$ but $N_5|_5 = 1 = N_8|_8$ but zero everywhere else.

$\therefore$ replace N_1 by $N_1 - \frac{1}{2}(N_5 + N_8)$

$\therefore$



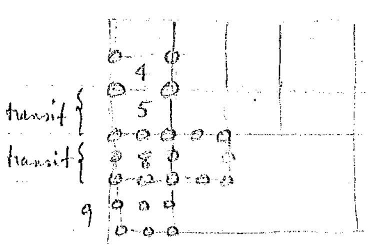
thus if a is absent $N_a = 0$

similarly N_2 by $N_2 - \frac{1}{2}(N_5 + N_6)$

N_3 by $N_3 - \frac{1}{2}(N_7 + N_6)$

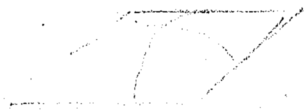
N_4 by $N_4 - \frac{1}{2}(N_7 + N_8)$ $\quad$ (***)

where do we use these type elements - transition



thus we have gone from 4 to 8 node

adding node nine at center.



candidate for node nine using Lagrange

$$N_9(\xi,\eta) = (1-\xi^2)(1-\eta^2)$$

to adjust for this start w/ 4 nodes element. & add node nine

N_9 vanishes on bdry ∴

stage 1 take $N_a \leftarrow N_a - \frac{1}{4} N_9$ since $N_a(0,0) = \frac{1}{4}$ $a = 1,2,3,4$ (N_a on the right: bilinear)

Now the above have not been affected on bdry.

stage 2 for $a = 5,6,7,8$

$$N_a = \begin{cases} N_a - \frac{1}{2} N_9 & \text{if } a \text{ is present} \quad \text{since } N_a(0,0) = \frac{1}{2} \\ 0 & \text{if } a \text{ is not} \end{cases}$$

stage 3 use (***) using the N_a's of stage 1 & stage 2.

thus

N_9 is $(1-\xi^2)(1-\eta^2)$

$a=5,6,7,8$ $\quad N_a = N_a - \frac{1}{2} N_9$ (N_a on the right: bilinear)

$a=1,2,3,4$ $\quad N_1 = \underbrace{\left(N_1 - \frac{1}{4} N_9\right)}_{N_1'} - \frac{1}{2} \left(\underbrace{N_5 - \frac{1}{2} N_9}_{N_5'} + \underbrace{N_8 - \frac{1}{2} N_9}_{N_8'}\right)$

$\quad = N_1 - \frac{1}{2}(N_5 + N_8) - \frac{1}{4} N_9$

so to go from 4 to 9

(start with) → $N_a = \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta)$, $a = 1,2,3,4$ | bilinear 1-4 corner

↓

Is node 9 present? — Yes → $N_9 = (1-\xi^2)(1-\eta^2)$; $N_a \leftarrow N_a - \frac{1}{4} N_9$, $a=1,2,3,4$ | add central bubble & modify your bilinear set of corners

↓ No

$N_9 = 0$

↓

define mid side pts taking account of center node:

$$N_5 = \begin{cases} \frac{1}{2}\left[(1-\xi^2)(1-\eta)\right] - \frac{1}{2} N_9 & \text{if 5 is present} \\ 0 & \text{if 5 is absent} \end{cases}$$

$\left.\begin{matrix} N_6 \\ N_7 \\ N_8 \end{matrix}\right\}$ = likewise

↓

finally modify corner nodes due to addition of midside nodes:

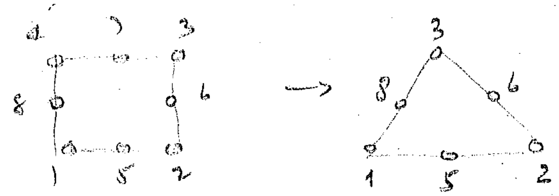
$N_1 \leftarrow N_1 - \frac{1}{2}(N_5 + N_8)$

(***)

stop

exercise 1 use flow chart to define shape fn for 8 node element.

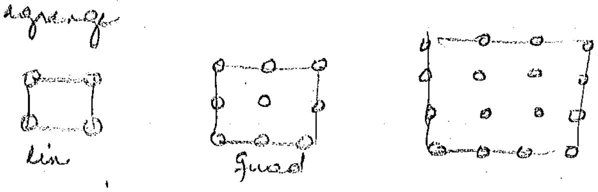
2. Generalize flow chart to allow degeneration of 8 node to 6 node triangle



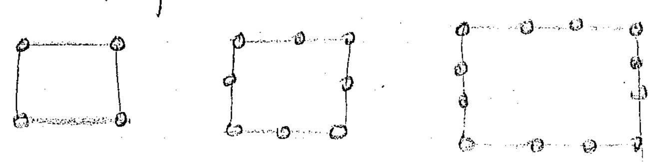
let $N_3' = N_4 + N_7 + N_3$

Elements Summarized

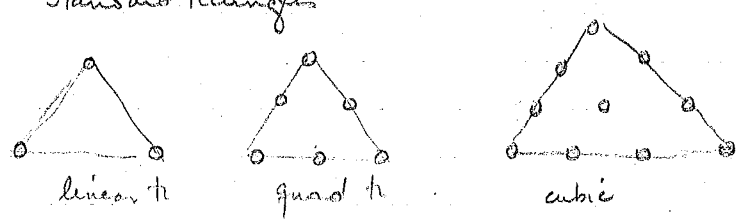
Lagrange



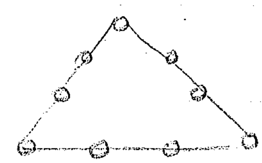
Serendipity



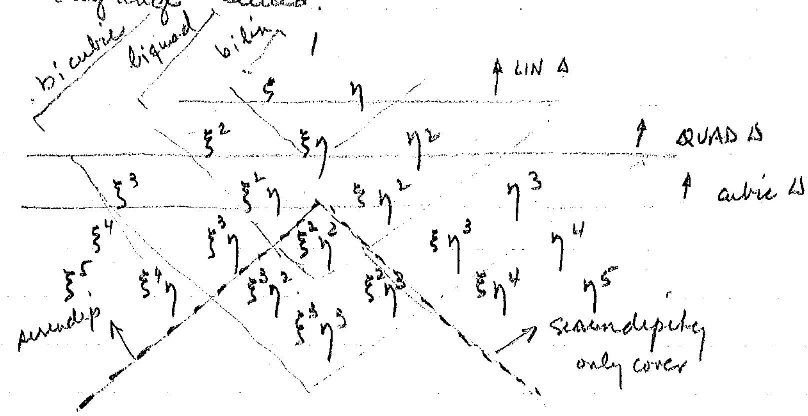
Standard triangles



Serendip



Lagrange Quad.



for serendip of 8 node start w/ lagrange biquad & drop terms inside of ... this gives the terms in the polyn. to be used.

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Some Current Trends in Finite Element Research

THOMAS J. R. HUGHES*

Introduction

In the 1960's the finite element method was under rapid development in the field of structural mechanics. The essential features of the methodology had been identified and it was clear to the research community what additional extensions were needed to achieve general capabilities for the bulk of structural mechanics problems which had traditionally faced analysts. The main thrusts of research at the time were directed towards the development of curved two- and three-dimensional higher-order continuum elements, and the development of effective shell elements.

Straight edged higher-order triangular and tetrahedral continuum elements were available early on, but were inconvenient in many situations, and did not permit accurate geometrical modeling of curved domains. The importance of elements of general shape, particularly curvilinear triangles, quadrilaterals, wedges and brick-shaped elements, was acknowledged by researchers as a key step in the development of the finite element method for it would enable convenient modeling of intricately shaped, real-world engineering designs. All classical analytical techniques, and even competing numerical techniques of the time, were severely limited due to the inability to handle complex geometries. One may view the "geometry problem" as one which has beleaguered the history of analysis. Essentially with the development of curved "isoparametric" elements [E1-E3, I1, I2, T1, Z2] the problem was solved once and for all for most practical purposes.

The ability to solve general shell configurations was of considerable interest to the developers of finite element methods in the 1960's due to its obvious importance throughout structural analysis, and particularly because most researchers were actively engaged in aerospace projects, a focal point of engineering endeavors at the time. By the latter part of the decade a wide variety of shell elements had been developed which were adequate for most linear analyses [A7, B24, C2, C7, C8, G17, P7, S6, Z8].

With the main problems of finite element/structural mechanics research (as perceived at the time) essentially solved, and the occurrence of a simultaneous decline in the aerospace industry, which had provided the major support for finite element developments, one would have expected a decrease in finite element research. In fact, to some extent this did occur. In aerospace, interest shifted to computer-program development employing, by and large, the existing methodology. Basic research activities dwindled. Evidence which indicates the reduced amount of finite element research may be garnered from the number of papers published per year. From 1960 onward, the number increased "exponentially" through 1969. For example, ten papers were published in 1961 and five hundred and thirty-one[1] were published in 1969. The first (and only) year that there was a decrease in the number of papers was in 1970 in which five hundred and ten appeared. Considering the usual delay between performing research and getting it published of about two years, it may be seen that the decline in number of papers may be correlated with the publication of major works on curved elements and shells, and the general decline in aerospace activities.

The predictable decline cited above was immediately reversed in 1971 in which eight hundred and forty-four papers on finite elements appeared. The subsequent year to year increases have been dramatic. In 1974, the last year for which complete data is available, thirteen hundred and seventy-seven papers were published.[2] What happened was that a counter trend of greater magnitude had begun and overwhelmed the decline associated with aerospace structural research: the finite element method had spread to other areas of engineering and analysis. The techniques that had proven so successful in structural analysis were seen to be more general and were being applied elsewhere. The seeds for this development had already been planted in the 1960's [W9, V1, Z4, Z6] and considerable momentum had been created by the early 1970's. This trend continues unabated today.

Within each field to which finite elements have been applied, success on simple problem classes has encouraged bolder applications, typically nonlinear and time-dependent.

[1] All data quoted are from [N4].

[2] As of the writing of this manuscript, thirty-two texts on finite elements had been published. See [H14] for a compilation.

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(Author's full title and affiliation given at paper's end.)

These have in turn created new problems of methodology and so research has continued to grow. Structural mechanics is a case in point. The emphasis in recent research has been oriented more to nonlinear dynamic phenomena involving, for example, finite deformations of inelastic materials, contact-impact, etc., and this has led to fundamentally new problems which continue to be grappled with.

In the remainder of this review I will try to identify some of the main contemporary trends in finite element research and their significance. The present work builds upon Zienkiewicz's earlier review in these pages [Z1], and complements several other feature articles dealing with other aspects of this subject, namely, Finlayson and Scriven's review of the method of weighted residuals [F4], Argyris and Patton's discourse on the role of the computer in research [A12] and, most recently, Oden and Bathe's [O3] commentary on the state of computational mechanics.[3]

The selection of topics discussed is necessarily a personal one shaped by my experience. The theme is essentially positive and optimistic as this is my overall impression of activity in the field. Due to the enormity of the literature on finite elements, no attempt has been made to mention all important recent works. Nevertheless, a large number of references are cited which should prove useful to readers wishing to delve further into particular areas.

Fluid Mechanics

The vast majority of problems in solid and structural mechanics involve symmetric, positive, spatial differential operators. The finite element method, as is most commonly practiced, is taken to be the *Galerkin* finite element method, a special member of the family of so called weighted residual methods [F3] in which both trial solutions and weighting functions are selected from the *same* class of functions. This canonical symmetry of the approximation method turns out to exploit the structure of symmetric operators. It is a simple exercise to show that a "best approximation" property is achieved in the natural energy norm defined by the operator [S4]. This optimality is the underlying reason for the high accuracy consistently attained by Galerkin/finite element methods in solid and structural mechanics. (Certain elements, incapable of convergence in the limit, have actually exhibited good accuracy in practical situations. See [B9].)

The basic problems of fluid mechanics involve nonsymmetric "convection" operators. In most physical problems of engineering interest convection is dominant. The Galerkin/finite element method loses the best approximation property under these circumstances and there are many situations in which the solution may be shown to be very poor indeed [H6, H32]. This issue was raised in [S4] and the potential usefulness of Galerkin/finite element methods in simulating flows was cast in doubt. It turns out that, despite the noted deficiencies, the basic Galerkin/finite element method is capable of high accuracy in *many* flow problems, even convection dominated ones, and a great deal of success and progress has been achieved in this mode on a wide range of fluid mechanics problems [G4-G8, G18, T2].

Nevertheless, it is still correct to point out that the basic Galerkin/finite element method exhibits spurious behavior for *some* convection dominated situations. These problems have been attributed to downstream essential, or "hard," boundary conditions [G9, Z3], but are still not fully understood.

This deficiency in the basic "recipe" has caused the realization that the degree of generality and reliability achieved by Galerkin/finite element methods in symmetric problem classes can never be achieved in fluid mechanics without a fundamental breakthrough in the analytical treatment of nonsymmetric operators. This necessarily will take us beyond the Galerkin/finite element method as classically used, but to what is the question.

The research problem is thus clearly set: To develop efficient finite element schemes with all the usual attributes vis-à-vis geometry, etc., which retain a best approximation property in an appropriate sense. Presently, there is considerable activity on this topic and it is being pursued in diverse areas of engineering and the physical sciences. A variety of schemes have been proposed and are under investigation [B31, G19, H6-H8, H18, H23, H24, H32].[4] A name which has been used to describe some of these techniques is "upwind finite element methods," since some of the initial efforts were very similar in intent to classical upwind finite difference methods [R2]. The name has stuck, which may be unfortunate, because some of the new finite element methods are very different from upwind finite difference techniques and are not subject to the well-known defects of the latter. An Applied Mechanics Division Symposium at the 1979 Winter Annual Meeting attempts to assess the state-of-the-art in this area [H19].

If a methodology emerges from these endeavors which attains the reliability of Galerkin/finite element methods on symmetric problems, then an era of development of "general purpose" computer programs in fluid mechanics may be anticipated. In the meantime we can at least expect many special purpose finite element programs with (hopefully) accompanying admonitions that in some situations the performance may be inadequate.

The whole effort to develop new finite element methods for fluid mechanics problems has been criticized by some investigators because of the current ability to accurately solve *particular* problems with Galerkin/finite element methods. This begs the central *numerical* problem of the general class of nonsymmetric operators which needs to be satisfactorily resolved to have significant effect on engineering practice.

Prior to any serious hope of attempting a numerical study of complicated physical phenomena such as turbulence, a much greater degree of reliability and *a priori* optimality must be achieved. The use of techniques which work well "sometimes" is simply not good enough.

The types of flow problems to which finite element methods have been applied are by now too numerous to describe in an article of this scope. The interested reader

[3] An excellent early review and history of matrix methods of structural analysis may be found in J. H. Argyris, "On the Analysis of Complex Elastic Structures," *Applied Mechanics Reviews,* Vol. 11, 331-338 (1958).

[4] Morton and colleagues at the University of Reading have made several recent important contributions. See K. W. Morton and J. W. Barrett, "Optimal Finite Element Methods for Diffusion–Convection Problems," pp. 134-148 in *Proceedings of the Conference on Boundary and Interior Layers--Computational and Asymptotic Methods,* J. J. H. Miller (Ed.), Boole Press, Dublin, 1980, and references therein.

may profitably consult [G4-G6, T2], and references cited therein, for exposure to the literature.

Fluid-Structure Interaction

Traditionally, fluid-structure interaction analysis has been an area in which special techniques have been developed to exploit the structure of specific problems. With advances being made in finite element methods for fluid mechanics, it becomes clear that a merging with the now well established discipline of finite element structural analysis will lead to greater analytical ability in this area of significant contemporary interest. Given basically sound numerical capabilities for fluids and structures, some way of "interfacing" the fluid and structural domains is, of course, required. This is complicated by the fact that different kinematical descriptions are generally favored for each. For example, it is usually convenient in modeling structures to adopt the classical Lagrangian description in which the finite element mesh moves with the material particles. On the other hand, the Eulerian description, in which the mesh is fixed in space, is generally favored for fluids due to typically large fluid motions. Clearly, some compromise in description must be adopted to model the fluid interface region between the purely Eulerian and Lagrangian subdomains. More flexible descriptions are also required to model free-surface flows. Several techniques have been developed with these ends in mind [B13, D3, D4, H33, W2]. The terminology generally applied to these techniques is "mixed Lagrangian-Eulerian finite element methods." They possess the ability to continuously interpolate between the classical descriptions, and/or directionally split descriptions at each point. Generalizations along these lines [H15] and further applications of techniques such as these promise to considerably extend problem solving capabilities in ensuring years. Areas other than fluid-structure interaction will no doubt also benefit from these developments (e.g., metal forming).

Transient Analysis

Until a few years ago, classical "off-the-rack" techniques were used to time-integrate the large systems of ordinary differential equations generated by finite element spatial discretizations. These schemes could be segregated into two distinct classes: *explicit* and *implicit*.

Explicit algorithms involve no matrix equations, consequently computer storage requirements and cost per time step are relatively small. The shortcoming is that numerical stability considerations dictate the use of small time steps, often smaller than necessary for accuracy. Explicit algorithms are generally felt to be cost effective for analysis of wave-propagation phenomena in continua.

Implicit algorithms, on the other hand, require solution of matrix equations during each time step which engenders a considerable increase in storage and number of operations per step over explicit algorithms. The attribute is that numerical stability is improved, and in many cases implicit algorithms can be shown to be unconditionally stable (i.e., no restriction, aside from accuracy, is imposed on the time step). In cases in which the response is dominated by the low modes, the larger time step permitted often makes implicit schemes more cost-effective.

It has been concluded that neither explicit nor implicit algorithms are superior for all problem classes. Some classes, such as structure-continuous media interaction, suggest that an optimal scheme might employ both implicit and explicit concepts within one algorithm. Finite element applications of this idea were first introduced in [B14, B15]. More recent works [H29, H30, H34, H35] have shown how to develop such schemes within the context of the standard finite element "assembly" algorithm. This procedure may be implemented in many existing implicit computer programs with only minor modifications. Furthermore, the basic structure of the schemes has suggested many other generalizations [F2, H20, H34, H35, P3]. It now appears that many different algorithmic concepts such as implicit-explicit schemes, alternating direction methods [T8], staggering schemes [P4], etc., may be deducible from a general coherent theory, and the implementation may be facilitated in similar fashion. The ideas are general in the sense that they pertain to both linear and nonlinear analysis in the context of any field theory. This represents a considerable generalization and consolidation of ideas and is a significant step toward optimally efficient transient analysis.

Various other improvements in transient algorithms have also been recently achieved. Examples which may be mentioned are refined damping characteristics which enable filtering of spurious higher modes, without adverse effect on the accuracy in the lower modes [H11-H13], and algorithms which rigorously achieve unconditional stability in nonlinear elastodynamics [H5, H25].

Two areas which are now being given increased attention, for their obvious importance in lowering computational cost, are automatic time stepping strategies based upon accuracy considerations [H10, P5, U3] and subcycling techniques in which subdomains of the mesh are integrated at different time steps [B18, W10]. Although progress has been made, neither of these areas seems to have reached full fruition yet, and thus it may be anticipated that considerable further activity will be seen.

Synthesis of Theoretical Concepts

At one point in the development of finite element methods, a bewildering array of approaches confronted the potential finite element developer for even the simplest class of problems. The so called displacement, force, mixed and hybrid formulations may be mentioned as examples. In linear elastostatics, displacement and force models are based upon the variational theorems of minimum potential and complementary energy, respectively, whereas mixed and hybrid models are based upon particular forms of mixed-field variational theorems such as those of Hellinger-Reissner and Hu-Washizu. There was a period when much activity took place in developing elements based on the various formulations, and the relative strengths and weaknesses were strongly argued. This has for the most part subsided. The majority of work is presently being performed with the simplest formulation. (This is the displacement method in elastostatics.) The reason for this is that success can generally best be achieved with the simplest formulation.

The trend in recent years has been to establish the equivalences or similarities between the various approaches rather than emphasize their differences [B19, pp. 276-280 in G1, H17, H32, L2, M2-M4]. These efforts have to some extent lessened interest in the more exotic approaches since equivalent results have been achieved with much simpler procedures. A case in point has been the increased use of reduced integration and allied techniques in constrained media applications (e.g., incompressible solids and fluids, and beam, plate and shell models based upon theories which account for transverse shear deformations; see [F6, G14, H21, H26-H28, H32, H36, H37, K1, K2, M1, N1-N3, P10, Z5, Z9] and references therein). In the same spirit, penalty-function methodology is increasingly being used as an alternative to Lagrange multiplier formulations [B19, H32, H37, M3].

Equivalences between some finite element and finite difference techniques have also been established recently [J2, K10]. One consequence is that dominant areas of finite-difference methodology, such as the so called "Lagrangian hydrocodes" [G13], are now being subsumed by competing finite element codes [B10, G15, H1-H4, K4] which are often faster and always much more versatile with respect to mesh topology.

Additional Topics

Inelastic Analysis. The numerical ability to solve large-scale nonlinear inelastic problems has advanced beyond the ability to characterize nonlinear materials. This fact has provided increased motivation for the development and study of new constitutive models which more faithfully model the response of materials, especially geological ones such as soils and rocks [C6, M11, P8, P9]. New models of this type often represent a considerable increase in complexity, and their practical usefulness depends upon efficient and reliable implementation in finite element computer programs. This has lead to the study of "stress point algorithms," the methodology concerned with the integration of constitutive equations. Considerable progress and understanding have been achieved in relatively simple settings [K7-K9, S1] and much additional work may be foreseen along these lines.

Forming Processes. Metal forming processes (e.g., extrusion, stamping, bending, etc.) have been based for the most part upon empiricism since the large-deformation, inelastic behavior characteristic of these phenomena was in the past outside the realm of analytical capabilities. The situation is now changing due to the increased ability to solve large-deformation problems by finite element methods. Considerable attention to forming processes has been given recently [A13, K5, Z7] and this may eventually have considerable beneficial consequences to the efficient design of forming tools and machinery.

Nonlinear Equation Solving. The solution of large-scale finite element problems is contingent upon solution of the matrix-equation systems developed. Although numerous studies have been undertaken to develop effective iterative solution procedures, in hope of lessening storage requirements and calculations associated with direct elimination schemes, very little significant progress has been made to date. Indeed, direct procedures are still almost universally relied upon for finite element equation systems, and are included in all major codes available to the general public. Presently, "compacted column" [B7, F1, M9, M10, T3, W7, W8] and "frontal" [I3, J1] techniques have replaced "band solvers" in most major codes. These newer procedures are virtually identical in speed and storage requirements, assuming each is optimally coded. The compacted column technique is often favored due to its conceptual simplicity and greater efficiency with respect to re-solutions [T5].

Many attempts are being made to improve the efficiency of direct nonlinear equation-solving algorithms [M5]. In these efforts, use is being made of direct, iterative and combined concepts. Of particular note lately is the adoption of conjugate gradient and quasi-Newton updates [B6, B30, C10, C11, G12, I4, M6, S5] which have been employed extensively in other fields, such as optimization. It is anticipated that in the future increased use will be made of transient analysis methodology in static and quasi-static situations [U1] due to the higher-level of understanding of transient algorithms and variety of approaches now available.

Contact-Impact. Many important engineering problems involve contact, impact and/or frictional sliding between two or more bodies (e.g., problems emanating from weapons technology, bearing design, vehicle safety and crash-worthiness, etc.). A completely general theoretical framework, suitable as a basis for the development of finite element methods, does not yet exist for the entire class of problems of this type, although variational inequalities may be used to characterize a subclass [K6]. Despite this, considerable recent progress has been made in the practical resolution of many difficult problems [B16, F5, H1, H10, H38]. It is hoped that the gap between theory and practice will close in the coming years.

Fracture Mechanics. The technological importance of fracture mechanics has motivated considerable finite element research. Several approaches have been proposed to model the singularities which are present in problems of this type [A2, A7, B3, B4, B22, H9, S3, T7]. Perhaps the most general approach thus far has been to include special singular functions, determined from analytical procedures, amongst finite element basis functions. This insures full rate-of-convergence of the method in the presence of the singularity. Considerable success has been achieved for two-dimensional cases and current attention is focused on three-dimensional cases, surface cracks, crack propagation, and nonlinear and dynamic situations [G3, P6].

Special Elements. As in the case of fracture mechanics, the special features of particular problem classes, such as singularities, may be embedded in a finite element formulation if sufficient analytical results are available. This marriage of classical and numerical concepts has proven quite successful in several areas of interest [A3]. Recent applications include boundary-layer elements [H39] for viscous flow calculations, and infinite elements [B21, G10] for modeling unbounded domains. A concise computer algorithm has been developed for producing finite element interpolation schemes from mixed classes of functions, necessary for achieving "special" element properties [H22]. Techniques are now available for developing elements with an arbitrary number of boundary surfaces, each of which takes on a prescribed analytical form [W1]. For most practical purposes, such generality is unnecessary, as isoparametric elements suffice. However, special circumstances will no doubt be encountered in the future when elements of this more general kind will prove useful.

Boundary Element Methods. Finite element discretizations of boundary integral formulations – termed "boundary element methods" [B12] – have gained increased attention in recent years. The popularity of this procedure stems from the fact that a reduction in dimensionality can often be achieved which may result in a significant decrease in computational effort. For example, a problem of three-dimensional elastostatics may require only a discretization of its boundary, which is two-dimensional. The complex boundaries of three-dimensional engineering geometries are conveniently discretized using isoparametric finite elements [L1]. Although some nonlinear and time-dependent applications of the boundary element method have been given [G11, M7, U2], the major area of success so far has been linear static problems [B26, C12]. Boundary element procedures are also well suited for the modeling of infinite domains [U2], such as occur in the analysis of soil-structure interaction.

Electromagnetic Field Theory. One of the many areas that has recently been impacted upon by finite element methodology is electromagnetism [A5, A6, C3, Z4]. As is often the case, the geometric complexity of electromagnetic devices favors the use of finite element procedures over competing numerical techniques. Typically, problems in electromagnetic field theory involve infinite domains. A particularly interesting recursive procedure has been developed for this purpose which does not entail use of special elements or analytical results [S2].

Mesh Optimization. Some recent progress has been made in the development of mesh optimization algorithms and adaptive mesh refinement strategies [B1, C1]. Nevertheless, it is fair to say that procedures of this type are to date little used in practical problem solving. Based on current developments, one would anticipate that limited purpose computer codes, which employ optimization/refinement techniques, will be available in the near future for certain problem classes.

Recently, an alternative procedure to mesh refinement has been proposed in which the mesh is fixed once and for all, but additional functions are added to the finite element basis. It has been shown that implementation may be expeditiously carried out and that exceptional convergence characteristics are achieved in practice [S7]. The terminology "p-convergence" has been applied to these methods due to the fact that convergence rate has been shown to be proportional to the order of the polynomial, p, contained in the finite element basis.

Finite Deformation Shell Analysis. Much recent progress has been made in the large deformation analysis of three-dimensional shell structures by finite elements [A8-A11, B5, B11, B12, B17, B23, B29, G2, H31, I5, P1, P2, R1, W3]. Many of these works employ the "degeneration technique" [A1] in which shell element properties are derived directly from general three-dimensional nonlinear continuum concepts. This avoids some of the limitations and inconveniences inherent in the use of shell theories. Difficult nonlinear post-buckling phenomena exhibiting imperfection sensitivity have been shown to be computable in certain cases by high precision elements [B29, I5]. The primary drawback to these developments is the computer cost associated with the formulation of elements. Consequently, simpler, lower-order alternatives are under investigation. These necessarily involve more sophisticated concepts and methodology since the low-order functions employed are incapable in themselves of exhibiting good bending behavior (see [A11, B8, B20, H16, H31, H36, K1-K3, M1, R3, T4] for contributions along these lines). It is sometimes frustrating to the non-specialist that, at this stage of the development of finite element methods, new elements, such as those cited, are still being proposed. These efforts, however, are important as it is generally acknowledged that optimally accurate and efficient shell analysis is far from a reality.

Mathematical Theory. In the early years of development of the finite element method there was little concern for, or awareness of, its mathematical basis. The situation is now completely changed. Since the late 1960's, the mathematical theory of finite elements has steadily grown [A14, C5, D1, M8, O1, S4, W4, W5]. Many of the basic questions regarding stability, order-of-accuracy, and convergence were established in the early 1970's for simple classes of problems. To some extent the recent thrust is associated with developing mathematical theories of the governing classes of nonlinear differential equations themselves (e.g., the Navier-Stokes equations [T6], first order hyperbolics [B2], the equations of nonlinear elasticity [O2], etc.), since it is recognized that without a complete mathematical theory of the underlying problem, there is no rigorous basis for a mathematical theory of approximation via finite elements.

The development of mathematical theories has tended to lag behind the realm of feasible computations and thus the emphasis in computer model validation has been on comparisons with experimental results. For example, despite a paucity of mathematical results, a degree of confidence has been attained in elastic-plastic computations.

Computers

It may be deduced from the preceding remarks that considerable advancement has been made in recent years in the development and understanding of finite element methods for engineering analysis. If digital computer capabilities were stagnant, one could still look forward to continual progress in the years to come based upon the current rate of improvements in algorithms and methodology. As almost everyone is aware, however, digital computer capabilities are far from stagnant. We are living in an era which has been aptly described as the "integrated circuit revolution." Digital computer capabilities are increasing in speed and capacity, while simultaneously decreasing in cost. The rate at which these events have been taking place is staggering, and the limits are nowhere in sight. Orders-of-magnitude improvements are still technically feasible and, in fact, are anticipated in the ensuring years.

This is of great significance to finite element research and application since digital computers are the technological foundation upon which finite element analysis rests. Every increase in the performance/cost ratio enhances the ability of existing finite element methods to solve engineering problems effectively.

Improved computer graphics hardware/software also contributes greatly to the use of element procedures in that time and cost involved in preparing data and assimilating results are considerably decreased. Finally, new types of hardware (e.g., microcomputers) are becoming available which promise to enlarge the scope of finite element computer analysis in the coming years [W6].

Concluding Remarks

The finite element method represents one of the most significant developments in the history of engineering analysis. What was viewed as analytically inconceivable as little as twenty years ago is often commonplace today. The primary reasons for the success of the method are its generality, ability to handle arbitrarily complex geometries and its consistent treatment of difficult differential-type boundary conditions. These strengths are often decisive when compared against competing techniques. In addition, finite element techniques may be brought to bear on any problem which can be stated in terms of differential, integral, or integro-differential equations. Once the basic procedures are mastered, the door is open for the practitioner to apply his skills to a wide range of physical problems. Successful application of these skills must, of course, be buttressed by a sound physical understanding of the problem area in which the analyst works. We are presently seeing the development of finite element sub-disciplines, within several fields of engineering and science [B27, B28, C4, C9, D2, G16, G20, L3, L4, O4]), that seek to thoroughly combine both the physical and computational aspects of the problem. This trend will no doubt continue since proper application of complex element methodology cannot be performed independently of physical insight.

Perhaps the most salient feature of current finite element research is the variety of areas in which developments are being undertaken. A sampling of the more prominent disciplines has been discussed in the body of this review. The most important area of endeavor appears to be fluid mechanics. Successful resolution of the treatment of nonsymmetric operators, which dominate much fluid mechanical phenomena, will at once enhance the numerical calculation of flows and have significant spin-off effects on other disciplines.

Financial support for finite element development should continue to be good due to the area's "track record," its continued potential, widespread use in the industrial sector, and an increasing demand for sophisticated and accurate analyses of evermore complicated engineering systems.

The continual improvements in methodology and computers augers an era in which the development and use of finite element methods should continue to rapidly grow.

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Historical Sketch of Research Activity in Finite Elements

Prehistory: 1943

- Lattice analogies of Hrennikoff [1] and McHenry [2]
- Courant's solution of the torsion problem [3]
- Prager and Synge's function space method for problems of elasticity [4]

First decade: 1954-6 to 1965

- Argyris and Kelsey's series of papers in Aircraft Engineering [5]
- Turner, Clough, Martin and Topp's paper on the Direct Stiffness Method [6]
- The terminology "Finite Elements" coined by Clough [7]
- Formulations for linear static structural and elasticity problems [8]
- Virtually no mathematical work

Second decade: 1966-1975

- Applications to diverse linear and nonlinear engineering problems [9]
- Mathematical foundations established [10]
- Large scale linear static and dynamic structural analysis programs become readily available [11-13]

Third decade: 1976-1985

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Chapter 6 contains a useful computer program which can solve the problems presented in the three previous chapters.

In Chapter 7, as a means of introducing the idea of higher order finite element approximations, beams are considered. One-dimensional problems in mass-transport (diffusion–convection), overland flow due to rainfall and wave propagation are covered in Chapters 8, 9 and 10 respectively.

Chapters 11–14 deal with two-dimensional finite element analysis including torsion, flow and stress analysis problems.

A final chapter on advanced study and applications and four Appendices complete the book.

One of the admirable features of this book is the amount of space given to one-dimensional problems which, if worked through in detail by the reader, will provide an excellent basis for understanding the finite element method.

As Desai notes it is not easy to write an elementary finite element textbook with so many auxiliary disciplines. However, this book is likely to become a classic introductory text in the field of finite elements. Hopefully a cheaper, paperback, student edition will become available soon.

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ME 235A
Fall 1981

HW # 1

1. Consider the BVP discussed in class:

$$u_{,xx}(x) + f(x) = 0 \qquad x \in (0,1)$$
$$u(1) = g$$
$$u_{,x}(0) = h.$$

Now assume $f(x) = qx$ q being a constant, and $g = h = 0$.

(a) Employing the linear FE space with equally spaced nodes, set up and solve the Galerkin - finite element equations for $n = 4$ $(h = 1/4)$.

- The exact solution is $u = \frac{q}{6}(1 - x^3)$
- We first define $w^h(x) = \sum_{i=1}^{4} c_i N_i(x)$ and $u^h(x) = \sum_{i=1}^{4} d_i N_i(x) + g N_5(x)$ where $N_i(1) = 0$ for $i = 1, \ldots, 4$ and $N_5(1) = 1$

$N_1(x) = 1 - 4x \qquad 0 \le x \le 1/4$
$\qquad = 0 \qquad x \ge 1/4$

$N_2(x) = 4x \qquad 0 \le x \le 1/4$
$\qquad = 2 - 4x \qquad 1/4 \le x \le 1/2$
$\qquad = 0 \qquad x \ge 1/2$

$N_3(x) = 4x - 1 \qquad 1/4 \le x \le 1/2$
$\qquad = 3 - 4x \qquad 1/2 \le x \le 3/4$
$\qquad = 0 \qquad x \ge 3/4$ & $x \le 1/4$

$N_4(x) = 4x - 2 \qquad 1/2 \le x \le 3/4$
$\qquad = 4 - 4x \qquad 3/4 \le x \le 1$
$\qquad = 0 \qquad x \le 1/2$

$N_5(x) = 0 \qquad x \le 3/4$
$\qquad = 4x - 3 \qquad 3/4 \le x \le 1$

$f_i(x) = f(x_i)$ thus $f_1 = 0$ $f_2 = q/4$ $f_3 = q/2$ $f_4 = 3q/4$ $f_5 = q$

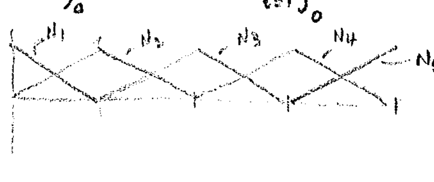
$K_{AB} = \int_0^1 (N_{A,x} N_{B,x})\,dx$. Using the above formulas

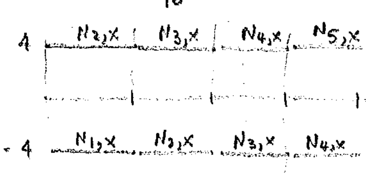
$$\underset{\sim}{K} = 4\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

. Since h and g are equal to zero then $F_A = (N_A, f)$ only

$$F_A = \int_0^1 N_A f\,dx = \sum_{i=1}^{5} \int_0^1 N_A N_i f_i\,dx \Rightarrow \underset{\sim}{F} = \frac{q}{96}(1, 6, 12, 18)^T$$

Note:





Thus we have that $\underset{\sim}{K}\,\underset{\sim}{d} = \underset{\sim}{F}$ or forming the augmented matrix and defining $4\hat{K} = \underset{\sim}{K}$ $\frac{9}{96}\hat{F} = \underset{\sim}{F}$ then $\hat{d} = \frac{384}{9}\underset{\sim}{d}$ and $\hat{K}\hat{d} = \hat{F}$

or

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 6 \\ 0 & -1 & 2 & -1 & 12 \\ 0 & 0 & -1 & 2 & 18 \end{array}\right] \xrightarrow[\text{replace 2}]{\text{add 1 to 2}} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 7 \\ 0 & -1 & 2 & -1 & 12 \\ 0 & 0 & -1 & 2 & 18 \end{array}\right] \xrightarrow[\text{replace 3}]{\text{add 2 to 3}}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 7 \\ 0 & 0 & 1 & -1 & 19 \\ 0 & 0 & -1 & 2 & 18 \end{array}\right] \xrightarrow[\text{repl 4}]{\text{add 3 to 4}} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 7 \\ 0 & 0 & 1 & -1 & 19 \\ 0 & 0 & 0 & 1 & 37 \end{array}\right]$$

Thus $\hat{d}_4 = 37$ and $\hat{d}_3 = 19 + \hat{d}_4 = 56$ and $\hat{d}_2 = 7 + \hat{d}_3 = 63$ and $\hat{d}_1 = 1 + \hat{d}_2 = 64$

but $\underset{\sim}{d} = \frac{9}{384}\hat{d}$ $\therefore$ $d_1 = 9/6$; $d_2 = 9/6\left(\frac{63}{64}\right)$; $d_3 = 9/6\left(7/8\right)$; $d_4 = 9/6\left(\frac{37}{64}\right)$

Using the exact solution at $x = 0, 1/4, 1/2, 3/4$ we get the same results as above.

2. Let $re_{,x} = |u^h_{,x} - u_{,x}| / (9/2)$ the relative error in $u_{,x}$. Compute $re_{,x}$ at the midpoint of the 4 elements (ie @ $x = 1/8, 3/8, 5/8, 7/8$)

$u_{,x} = -9x^2/2$

$u^h_{,x} = \sum_{i=1}^{4} d_i N_{i,x}$ since N_i, N_{i+1} are non zero in interval i then only 2 contributions come in

Thus

$u^h_{,x}/(9/2) =$	$u_{,x}/(9/2) =$	@	$re_{,x} =$
$-4/3\,(1/64)$	$-1/64$	$x = 1/8$	$2/384$
$-4/3\,(7/64)$	$-9/64$	" $= 3/8$	"
$-4/3\,(19/64)$	$-25/64$	" $= 5/8$	"
$-4/3\,(37/64)$	$-49/64$	" $= 7/8$	"

3. Using the results for $n=1$, $n=2$, $n=4$ plot $\ln re_{,x}$ vs. $\ln h$.

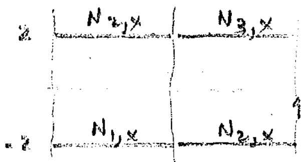
From class and $n=1$ $\quad u^h = \frac{9}{6}(1-x) \quad u_{,x} = -\frac{9}{2}(x^2)$

$u^h_{,x} = -\frac{9}{2}\left(\frac{1}{3}\right) \quad u_{,x}\Big|_{x=1/2} = -\frac{9}{2}\left(\frac{1}{4}\right)$

$\therefore \quad re_{,x} = \frac{1}{12}$

For $n=2$ $\quad u^h = \frac{9}{6} N_1 + \frac{79}{48} N_2$

$u^h_{,x} = \frac{9}{6} N_{1,x} + \frac{79}{48} N_{2,x}$ where



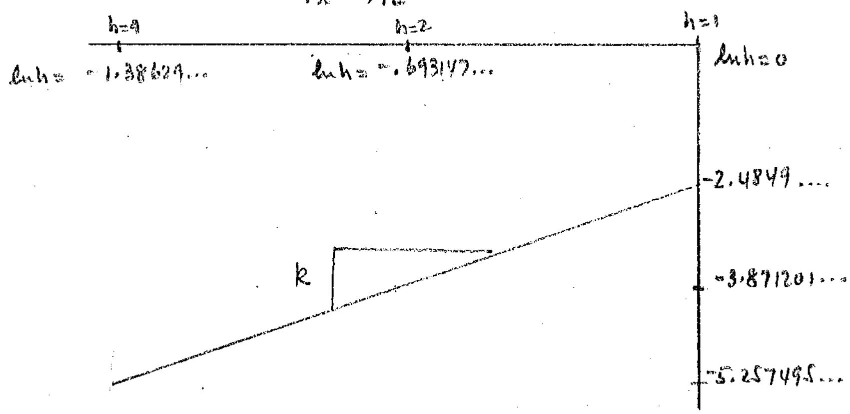
@ $x = \frac{1}{4}$ $\quad u^h_{,x} = \frac{9}{6}(-2) + \frac{79}{48}(2) = \frac{-2}{48}9 = -\frac{9}{2}\left(\frac{1}{12}\right) \qquad u_{,x}\Big|_{x=1/4} = -\frac{9}{2}\left(\frac{1}{16}\right) \qquad re_{,x} = \frac{1}{48}$

@ $x = \frac{3}{4}$ $\quad u^h_{,x} = \frac{79}{48}(-2) = \frac{-14}{24} \cdot \frac{9}{2} \qquad u_{,x}\Big|_{x=3/4} = -\frac{9}{2}\left(\frac{9}{16}\right)$

thus for $h = \frac{1}{4}$ $\quad re_{,x} = \frac{2}{384} \quad \ln h = -1.38629 \quad \ln\left(\frac{2}{384}\right) = -5.257495\ldots$

$h = \frac{1}{2}$ $\quad re_{,x} = \frac{1}{48} \quad \ln h = -.693147\ldots \quad \ln\left(\frac{1}{48}\right) = -3.871201\ldots$

$h = 1$ $\quad re_{,x} = \frac{1}{12} \quad \ln h = 0 \quad \ln\left(\frac{1}{12}\right) = -2.4849\ldots$



$\ln re_{,x}$

4. Using the error analysis for $re_{,x}$ at the midpoints presented in class, answer the following:

a. What is the significance of the slope of the graph in question (3)?

It represents the order of the error if we wrote $\ln re_{,x} = \ln C + k \ln h$ (ie $re_{,x} = Ch$

b. What is the significance of the y intercept?

It represents the coefficient C (or really the $\ln C$) in $re_{,x} = Ch^k$

[illegible]

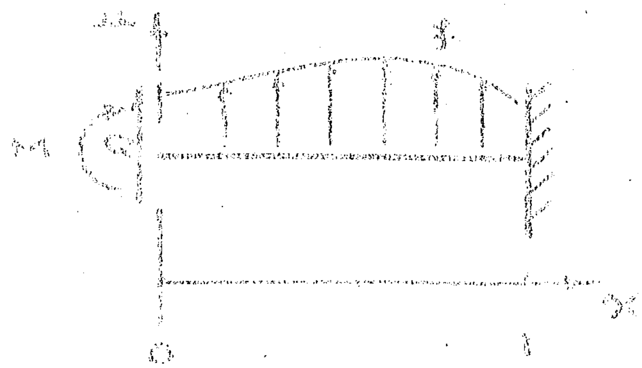
[illegible] thin [illegible]

[illegible]

[illegible] ($\Omega =]0,1[$, $\bar{\Omega} = [0,1]$).

$$
(S)\quad
\begin{cases}
\text{Given } f : \bar{\Omega} \to \mathbb{R} \text{ [illegible]}, M, Q \text{ [illegible]} \\
u_{xxxx} = f \quad \text{in } \Omega \quad \text{([illegible])} \\
u(1) = 0 \quad \text{([illegible])} \\
u_x(1) = 0 \quad \text{([illegible])} \\
-u_{xx}(0) = M \quad \text{([illegible])} \\
-u_{xxx}(0) = Q \quad \text{([illegible])}
\end{cases}
$$

[illegible]:



Let $\mathcal{S} = \mathcal{V} = \{ w \mid w \in H^2(\Omega)^*, w(1) = w_{,x}(1) = 0 \}$.
Then a corresponding weak form of the problem is:

(W) Given f, M and Q, find $u \in \mathcal{S}$ such that for all $w \in \mathcal{V}$

$$a(u,w) = (f,w) - w_{,x}(0)\,M + w(0)\,Q$$

where

$$a(u,w) = \int_0^1 u_{,xx}\, w_{,xx}\, dx$$

$$(f,w) = \int_0^1 f w \, dx$$

The collection of functions, $\mathcal{S}$, may be thought of as the space of finite strain-energy configurations of the beam, satisfying the kinematic (essential) boundary conditions at $x = 1$. It is a consequence of Sobolev's Theorem that each $w \in \mathcal{S}$ is in $C^1(\bar{\Omega})$. For reasonable f, the above problem possesses unique solutions.

Let $\mathcal{S}^h = \mathcal{V}^h$ be a finite-dimensional approximation of $\mathcal{S}$. In particular, each member $w^h \in \mathcal{S}^h$ satisfies $w^h(1) = w^h_{,x}(1) = 0$.

The Galerkin statement of the problem goes as follows:

(G) Given f, M and Q, find $u^h \in \mathcal{S}^h$ such that for all $w^h \in \mathcal{V}^h$

$$a(u^h, w^h) = (f, w^h) - w^h_{,x}(0)\,M + w^h(0)\,Q$$

* This means essentially that $w_{,xx}$ is "square-integrable" (i.e., $\int (w_{,xx})^2\, dx < \infty$)

(a) Assuming the functions are piecewise smooth, show that the solutions of (S) and (W) are identical. What are the natural boundary conditions?

(b) Assume $0 = x_1 < x_2 < \ldots < x_{n+1} = 1$, and

$$\mathcal{S}^h = \{ w^h \mid w^h \in C^1(\bar{\Omega}),\ w^h(1) = w^h_{,x}(1) = 0,$$

and w^h restricted to $[x_A, x_{A+1}]$ is a cubic polynomial (i.e. consists of a linear combination of $1, x, x^2, x^3$) $\}$

This is a space of piecewise cubic Hermite shape functions. Observe that $w^h \in \mathcal{S}^h$ must have continuous first derivatives at the nodes.

On a typical element, show that w^h may be uniquely written

$$w^h(x) = N_1(x)\, w^h(x_1) + N_2(x)\, w^h(x_2) + P_1(x)\, w^h_{,x}(x_1) + P_2(x)\, w^h_{,x}(x_2)$$

where

$N_1(x_1) = 1$ $\quad N_{1,x}(x_1) = 0$ $\quad P_1(x_1) = 0$

$N_1(x_2) = 0$ $\quad P_{1,x}(x_1) = 1$

$N_{1,x}(x_2) = 0$ $\quad P_1(x_2) = 0$ $\quad A(x-x_2)^2(x-x_1)$

$P_{1,x}(x_2) = 0$

$$N_1(x) = -(x-x_2)^2\,[-h + 2(x_1 - x)]/h^3$$

$$N_2(x) = (x-x_1)^2\,[h + 2(x_2 - x)]/h^3$$

$$P_1(x) = (x-x_1)(x-x_2)^2/h^2$$

$$P_2(x) = (x-x_1)^2(x-x_2)/h^2$$

(Hint: Let $w^h(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$, where the c's are constants. Determine them by requiring the following four conditions hold:

$w^h(x_1) = c_1 + c_2 x_1 + c_3 x_1^2 + c_4 x_1^3$

$w^h(x_2) = c_1 + c_2 x_2 + c_3 x_2^2 + c_4 x_2^3$

$w^h_{,x}(x_1) = c_2 + 2c_3 x_1 + 3c_4 x_1^2$

$w^h_{,x}(x_2) = c_2 + 2c_3 x_2 + 3c_4 x_2^2$)

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Winter 1981

HW #2

1. If the strong formulation of the problem is given by

$$(S)\begin{cases} \text{Given } f:\Omega\to\mathbb{R} \text{ and constants } M \text{ and } Q\text{; find } u:\bar\Omega\to\mathbb{R} \ni. \\ u_{,xxxx}-f=0 \text{ on } \Omega \quad (*) \\ u(1)=0\ ;\ u_{,x}(1)=0 \quad \text{built-in end} \\ u_{,xx}(0)=M\ ;\ u_{,xxx}(0)=Q \quad \text{prescribed conditions at the other end}\end{cases}$$

and the weak formulation of the problem is given by

$$(W)\begin{cases} \text{Let } \mathcal{S}=\mathcal{V}=\{w\mid w\in H^2(\Omega)\ ;\ w(1)=w_{,x}(1)=0\} \\ \text{Given } f, M \text{ and } Q\text{, find } u\in\mathcal{S}\ \ni.\ \forall w\in\mathcal{V} \\ \int_0^1 u_{,xx}w_{,xx}\,dx=\int_0^1 fw\,dx-w_{,x}(0)M+w(0)Q \quad (**)\end{cases}$$

Show that (S) and (W) are equivalent and that the solution to (S) is also a solution to (W). What are the natural bdry conditions.

a. Assume u is a soln of (S). Since u satisfies $u(1)=u_{,x}(1)=0$ then $u\in\mathcal{S}=\mathcal{V}$. Now multiply the equation (*) by a $w\in\mathcal{V}$ and integrate over $\bar\Omega$

$$\int_0^1 w(u_{,xxxx}-f)\,dx=0=\int_0^1 w_{,xx}u_{,xx}\,dx+w\,u_{,xxx}\Big|_0^1-w_{,x}u_{,xx}\Big|_0^1-\int_0^1 wf\,dx$$

since $w\in\mathcal{V}$ $w(1)=w_{,x}(1)=0$ also $u_{,xx}(0)=M$ and $u_{,xxx}(0)=Q$ since $u\in\mathcal{S}$

$$0=\int_0^1 w_{,xx}u_{,xx}\,dx-w(0)Q+w_{,x}(0)M-\int_0^1 wf\,dx.$$ The result follows after transposing the last three terms and noting that this is true $\forall w\in\mathcal{V}$.

b. Assume u is a soln of (W). Since $u\in\mathcal{S}$ then $u(1)=u_{,x}(1)=0$. Starting with (**) we integrate by parts to get

$$\int_0^1 w\,u_{,xxxx}\,dx+u_{,xx}w_{,x}\Big|_0^1-u_{,xxx}w\Big|_0^1=\int_0^1 fw\,dx-w_{,x}(0)M+w(0)Q$$

since $w\in\mathcal{V}$ $w(1)=w_{,x}(1)=0$. Thus by using this and transposing to one side of equation

$$\int_0^1 w(u_{,xxxx}-f)\,dx+w_{,x}(0)[M-u_{,xx}(0)]-w(0)[Q-u_{,xxx}(0)]=0 \quad (***)$$

Now define $w=\phi(x)(u_{,xxxx}-f)$ $\ni.$ $\phi(0)=\phi(1)=\phi_{,x}(0)=\phi_{,x}(1)=0$ and $\phi(x)>0$ $x\in(0,1)$ then by substituting into (***) we obtain that

$\int_0^1 \phi(u_{,xxxx}-f)^2dx=0$; since the integrand is ≥ 0 in the interval & the integral $=0$ $\Rightarrow$ integrand must be zero. But $\phi(x)>0$ in the interval $\Rightarrow u_{,xxxx}-f\equiv 0$ in the interval. Using this result in (***) $\Rightarrow$

$$w_{,x}(0)[M-u_{,xx}(0)]-w(0)[Q-u_{,xxx}(0)]=0$$

in general for any $w\in\mathcal{V}$ $w_{,x}(0)$ & $w(0)\neq 0$ simultaneously $\Rightarrow M-u_{,xx}(0)=0$ and $Q-u_{,xxx}(0)=0$

Thus we have shown that $(S)\Leftrightarrow(W)$ and soln of $(S)\equiv$ soln of (W)

(b) Assuming $0 = x_1 < x_2 < \dots < x_{n_{np}} = 1$ and

$$A^h = \left\{ w^h \mid w^h \in C^1(\bar{\Omega}),\ w^h(1) = w^h_{,x}(1) = 0, \text{ and } w^h(x) = \sum_{i=0}^{3} c_i x^i \text{ on } [x_A, x_{A+1}] \right\}$$

show that for each subinterval that w^h may be uniquely written

$$w^h(x) = N_1(x)\, w^h(x_1) + N_2(x)\, w^h(x_2) + P_1(x)\, w^h_{,x}(x_1) + P_2(x)\, w^h_{,x}(x_2)$$

where

$$N_1(x) = +\,(x-x_2)^2\,[h - 2(x_1-x)]/h^3$$

$$N_2(x) = (x-x_1)^2\,[h + 2(x_2-x)]/h^3$$

$$P_1(x) = (x-x_1)(x-x_2)^2/h^2$$

$$P_2(x) = (x-x_1)^2(x-x_2)/h^2$$

To do this look at

$$w^h(x_1) = N_1(x_1)\, w^h(x_1) + N_2(x_1)\, w^h(x_2) + P_1(x_1)\, w^h_{,x}(x_1) + P_2(x_1)\, w^h_{,x}(x_2) \quad (\dagger)$$

(a) let us pick $N_1(x_1) = 1$ $N_2(x_1) = 0$ $P_1(x_1) = 0$ $P_2(x_1) = 0$

likewise for our conditions for $w^h(x_2)$:

(b) let us pick $N_1(x_2) = 0$ $N_2(x_2) = 1$ $P_1(x_2) = 0$ $P_2(x_2) = 0$

likewise for our conditions for $w^h_{,x}(x_1)$:

(c) let us pick $N_{1,x}(x_1) = 0$ $N_{2,x}(x_1) = 0$ $P_{1,x}(x_1) = 1$ $P_{2,x}(x_1) = 0$

likewise for our conditions for $w^h_{,x}(x_2)$:

(d) let us pick $N_{1,x}(x_2) = 0$ $N_{2,x}(x_2) = 0$ $P_{1,x}(x_2) = 0$ $P_{2,x}(x_2) = 1$

<u>For example</u> let us look at the conditions for $N_1(x)$

By the fundamental theorem of algebra (b) and (d) state that $N_1(x) = (x-x_2)^2 R(x)$ where $R(x)$ is a polynomial. The degree of $R(x)$ must be no higher than 1 by the restriction on $w^h(x)$ $\therefore$ $R(x) = (a + bx)$.

Now $N_1(x_1) = (x_1-x_2)^2\,[a + bx_1] = h^2\,[a + bx_1] = 1$ $\therefore$ $a + bx_1 = 1/h^2$ (*)

also $N_{1,x}(x_1) = 2(x_1-x_2)\,[a+bx_1] + h^2 \cdot b = -2h \cdot 1/h^2 + h^2 b = 0$ $\therefore$ $b = 2/h^3$ (**)

Use of (*) and (**) yields that $a = 1/h^2 - bx_1 = 1/h^2 - \frac{2x_1}{h^3}$

and $R(x) = \frac{1}{h^3}\,[h - 2x_1 + 2x] = 1/h^3\,[h - 2(x_1 - x)]$

or $N_1(x) = (x-x_2)^2\,[h - 2(x_1-x)]/h^3$. We can follow this same argument to find $N_2(x)$

Let us look at $P_1(x)$

By the fundamental theorem of algebra (b) and (d) state that $P_1(x) = (x-x_2)^2 Q(x)$ where $Q(x)$ is no higher than degree 1 by the restriction on $w^h(x)$. By (a)

$Q(x) = A(x-x_1)$ $\therefore$ $P_1(x) = A(x-x_1)(x-x_2)^2$. Now $P_{1,x}(x_1) = A(x_1-x_2)^2 + 2A(x_1-x_2)\cdot 0$

$\therefore$ $P_{1,x}(x_1) = 1 = A \cdot h^2$ $\therefore$ $A = 1/h^2$ $\Rightarrow$ $P_1(x) = (x-x_1)(x-x_2)^2/h^2$.

we can follow this same argument to find $P_2(x)$

To show uniqueness suppose $\exists$ $\hat{N}_1$, $\hat{N}_2$, $\hat{P}_1$, $\hat{P}_2$ $\ni$

$$w^h(x) = \hat{N}_1 w^h(x_1) + \hat{N}_2 w^h(x_2) + \hat{P}_1 w^h_{,x}(x_1) + \hat{P}_2 w^h_{,x}(x_2) \qquad (\dagger\dagger)$$

then by subtracting ($\dagger$) from ($\dagger\dagger$) we obtain

$$0 = (N_1 - \hat{N}_1) w^h(x_1) + (N_2 - \hat{N}_2) w^h(x_2) + (P_1 - \hat{P}_1) w^h_{,x}(x_1) + (P_2 - \hat{P}_2) w^h_{,x}(x_2)$$

in the most general case $w^h(x_1)$, $w^h(x_2)$, $w^h_{,x}(x_1)$ and $w^h_{,x}(x_2)$ will not be zero

$\Rightarrow$ $(\quad) = 0$ $\quad \therefore \quad N_1 \equiv \hat{N}_1$, $N_2 \equiv \hat{N}_2$, $P_1 \equiv \hat{P}_1$, $P_2 \equiv \hat{P}_2$

QED

Homework Assignment No. 3

This is a continuation of Homework Assignment No. 2.

The finite element space described in part (b) results in exact nodal displacements and slopes (first derivatives); analogous to the case presented in class. In problems of beam bending we are generally interested in curvatures (second derivatives) for bending moment calculations.

[-1, 1]

(c) Locate the optimal curvature points in the sense of Barlow. Barlow pts @ $\pm 1/\sqrt{3}$

(d) What is the rate of convergence of curvature at these points? $O(h^3)$.

(e) If the segment of the beam $[x_A, x_{A+1}]$ is unloaded (i.e. $u_{,xxxx} = 0$, where u is the exact solution), which points are optimal?

$e_{,xx}(\alpha) = u^h_{,xx} - u_{,xx} =$

HW # 3.

1. given $u^h(x) = N_1(x)\, u^h(x_1) + N_2(x)\, u^h(x_2) + P_1(x)\, u^h_{,x}(x_1) + P_2(x)\, u^h_{,x}(x_2)$

$$\therefore \quad u^h_{,xx}(x) = \left(\tfrac{6}{h^2} - \tfrac{12b}{h^3}\right) u^h_1 + \left(\tfrac{6}{h^2} + \tfrac{12a}{h^3}\right) u^h_2 + \left(\tfrac{2}{h} - \tfrac{6b}{h^2}\right) u^h_{,x}\Big|_1 + \left(\tfrac{2}{h} + \tfrac{6a}{h^2}\right) u^h_{,x}\Big|_2 \qquad (1)$$

where $b = x_2 - x \quad a = x_1 - x \quad u^h_1 = u^h(x_1)$ etc.

Since $u_1^h = u_1 \quad u_2^h = u_2 \quad u^h_{,x}\big|_1 = u_{,x}\big|_1 \quad u^h_{,x}\big|_2 = u_{,x}\big|_2$ and $u^h \in \mathcal{S}^h \subset \mathcal{S}$ then we can replace u^h and $u^h_{,x}$ with continuous fns. and let us write

$$u(x_1) = \sum_{n=0}^{\infty} u^{(n)}(x)\frac{a^n}{n!} \qquad u(x_2) = \sum_{n=0}^{\infty} u^{(n)}(x)\frac{b^n}{n!} \qquad u'(x_1) = \sum_{n=1}^{\infty} u^{(n)}(x)\frac{a^{n-1}}{(n-1)!} \quad \text{etc.}$$

Then by plugging into (1) we find that

$$u^h_{,xx}(x) = 0\cdot u(x) + 0\cdot u'(x) + u_{,xx}(x) + 0\cdot u_{,xxx}(x) - [h^2 + 6ab]\, u^{IV}(x) + u^{V}(x)\frac{A(a,b)}{60h^2}$$

now since $b = h + a$ then for optimality we need to make $h^2 + 6ah + 6a^2 = 0 \quad \therefore$

$$a = \frac{-6h \pm 2h\sqrt{3}}{12} \quad \text{or} \quad \boxed{x = x_1 + h/2 \pm \frac{h\sqrt{3}}{6} \quad \text{Barlow Points}}$$

2. $A(a,b)$ is $O(a^5, b^5)$ or $O(h^5)$ $\therefore$ $A(a,b)/h^2$ is $O(h^3)$ hence $u^h_{,xx} - u_{,xx}$ is $O(h^3)$ at the Barlow points Note $A(a,b) = -2\left[a^5 + b^5 + 5(a^4 b + b^4 a) - 3(a^3 b + a^2 b^2 + a b^3)\right]$

3. If $u^{IV} = 0 \quad \forall x \in [x_A, x_{A+1}]$ then $u^{(n)} = 0 \quad \forall n \geq 4$ hence $e_{xx} = u^h_{,xx} - u_{,xx} = 0$ and all the points are optimal

Homework Assignment No. 4

This is a continuation of Homework Assignment No. 3.

(f) Assume $n_{el}=1$ (i.e. one element) and $f(x)=c=$ constant. Set up and solve the Galerkin finite element equations. Plot u^h and u; $u^h_{,x}$ and $u_{,x}$; and $u^h_{,xx}$ and $u_{,xx}$. Indicate the locations of the Barlow stress points. (See Homework Assignment No. 3.)

(g) <u>Optional</u> Prove that

$$u^h(x_p) = u(x_p)$$
$$u^h_{,x}(x_p) = u_{,x}(x_p)$$

where x_p is a typical node (i.e. prove <u>the displacements and slopes are exact at the nodes</u>). To do the second part you will have to be familiar with the dipole, $\delta_{,x}(x-x_p)$, which is the generalized derivative of the delta function.

(h) Show that the Barlow stress points are exact when $f(x)=c=$ constant (cf. Homework Assignment No. 3).

(i) Why do we require that the functions in $\mathcal{S}^h$ have continuous first derivatives?

from energy $\int_\Omega w_{,xx} u_{,xx} \, d\Omega$ for integrals to make sense must up the continuity of derivatives in Ω.

$$\int_0^1 u_{,xx}\, w_{,xx}\, dx = \int_0^1 f w\, dx - w_{,x}(0) M + w(0) Q$$

for any element $\quad u^h(x) = N_1(x)\, u^h(x_1) + N_2(x)\, u^h(x_2) + P_1(x)\, u_{,x}^h(x_1) + P_2(x)\, u_{,x}^h(x_2)$

$$\therefore \quad u^h(\xi) = u^h(\xi(x)) = N_1(\xi)\, u^h(x_1^e) + N_2(\xi)\, u^h(x_2^e) + P_1(\xi)\, u_{,x}^h(x_1^e) + P_2(\xi)\, u_{,x}^h(x_2^e)$$

where $\quad \xi(x) = \dfrac{2x - (x_2^e + x_1^e)}{h_A} \qquad$ let $h_A^e = x_2^e - x_1^e$

thus for any element

$$\int_{x_1}^{x_2} u_{,xx}\, w_{,xx}\, dx = \int_{x_1^e}^{x_2^e} u_{,\xi\xi}\, w_{,\xi\xi}\, d\xi \cdot \frac{dx}{d\xi}\left(\frac{d\xi}{dx}\right)^4 \qquad \text{since } \frac{d^2\xi}{dx^2} = 0.$$

$$\int_{x_1}^{x_2} f w\, dx = \int_{x_1^e}^{x_2^e} f w\, d\xi\, \frac{dx}{d\xi}$$

$$w_{,x}(0) M = w_{,\xi}(\xi = -1)\, M \cdot \delta_{1e} \cdot \frac{d\xi}{dx} \qquad \text{where } \delta_{1e} = \begin{cases} 1 & \text{if } e = 1 \\ 0 & e \neq 1 \end{cases}$$

$$w(0) Q = w(\xi = -1)\, Q \cdot \delta_{1e}$$

the N_2, P_2 terms are not there by virtues of the g, h type bc

Now if nel = 1 $\quad x_1 = 0 \quad x_2 = 1 \quad \therefore \quad b = 1 - x \quad a = -x \quad h = 1 \quad \therefore \quad w(0) = w_1^h \quad w_{,x}(0) =$

$$u_{,xx}^h(x) = \left(\tfrac{6}{h^2} - \tfrac{12b}{h^3}\right) u_1^h + \left(\tfrac{6}{h^2} + \tfrac{12a}{h^3}\right) u_2^h + \left(\tfrac{2}{h} - \tfrac{6b}{h^2}\right) u_{1,x}^h - \left(\tfrac{2}{h} + \tfrac{6a}{h^2}\right) u_{2,x}^h$$

$6 - 12(1-x)$

$12x - 6$

$$w_{,x}^h(x) = \frac{6ab}{h^3} w_1^h - \frac{6ab}{h^3} w_2^h + \frac{3b^2 - 2bh}{h^2} w_{1,x}^h - \left(\frac{3a^2 + 2ah}{h^2}\right) w_{2,x}^h$$

$2 - 6(1-x)$

$-4 + 6x$

$6x - 4$

$$u_{,xx}(x) = (12x - 6) u_1^h + (6 - 12x) u_2^h + (6x - 4) u_{1,x}^h - (2 - 6x) u_{2,x}^h$$

$$w_{,x}(x) = -6(1-x)x\, w_1^h + 6(1-x)x\, w_2^h + (1-x)(1-3x)\, w_{1,x}^h - (3x-2)x\, w_{2,x}^h$$

$\dfrac{(12x-6)^3}{36}\Big|_0^1$

$6 + 6 = 12$

$$M w_{,x}(0) = w_{1,x}^h M \qquad w(0) Q = w_1^h Q$$

$72x^2 - 84x + 24 \qquad +12 - 60x + 72x^2$

$$\int_0^1 (12x-6)^2 \left(u_1^h w_1^h + u_2^h w_2^h\right) dx - \int_0^1 dx\, (12x-6)^2 \left(u_2^h w_1^h + u_1^h w_2^h\right) + \int_0^1 (6x-4)(12x-6)\, u_{1,x}^h w_1^h - (w_2^h u_{1,x}^h) + (w_{1,x}^h u_1^h) + \int (-2+6x)(12x-6) \left(u_{2,x}^h w_1^h - u_{2,x}^h w_2^h + w_{2,x}^h u_1^h - w_{2,x}^h u_2^h\right) dx$$

$\dfrac{(12x-6)^3}{36}\Big|_0^1 = 12 \qquad = 12 \qquad 24x^3 - 42x^2 + 24x = 6 \qquad 24x^3/3 - 30x^2/2 + 12x \quad 6$

$36x^2 - 36x + 8$

$$+ \int_0^1 (6x-4)^2\, u_{1,x}^h w_{1,x}^h\, dx + \int_0^1 (-2+6x)(6x-4)\, w_{1,x}^h u_{2,x}^h\; w_{2,x}^h u_{1,x}^h - (w_{1,x}^h u_2^h)$$

$\dfrac{(6x-4)^3}{18}\Big|_0^1 = \dfrac{8}{18} + \dfrac{64}{18} = 4 \qquad 36x^3/3 - 18x^2 + 8x \quad 2$

$$+ \int_0^1 (2-6x)^2\, w_{2,x}^h u_{2,x}^h\, dx$$

$\dfrac{(2-6x)^3}{-18}\Big|_0^1 = \dfrac{+64}{18} + \dfrac{8}{18} = 4$

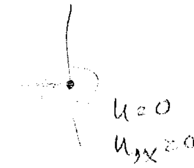
$$12(u_1 w_1 + u_2 w_2) - 12(u_2 w_1 + u_1 w_2) + 6(u_{1,x} w_1 - w_2 u_{1x} + w_{1x} u_1 - w_{1x} u_2) + 6(u_{2x} w_1 - u_{2x} w_2 + w_{2x} u_1 - w_{2x} \ldots$$

$$+ 4 u_{1x} w_{1x} + 2(w_{1x} u_{2x} + w_{2x} u_{1x}) + 4 u_{2x} w_{2x} = C \int \left[(1-x)^2(2x+1) w_1 + x^2[3-2x] w_2 + x(1-x)^2 w_{1x} - x^2(1-x) w_{2x}\right]$$

$x^2 - \frac{4x^3}{3} + \frac{x^4}{2} + \frac{(1-x)^3}{-3} \qquad x^3 - \frac{x^4}{2} \qquad \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \qquad -\left(\frac{x^3}{3} - \frac{x^4}{4}\right)$

$\frac{x^2}{2} - \frac{4x^3}{3} + \frac{x^4}{2} + \frac{(1-x)^3}{-3} \qquad \frac{1}{6} + \frac{1}{3} = \frac{1}{2} w_1 + \frac{1}{2} w_2 \qquad \frac{1}{12} w_{1x} \quad -\frac{1}{12} w_{2,x}$

$\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$



$$\therefore\ w_1\left[12u_1 - 12u_2 + 6u_{1,x} + 6u_{2,x}\right] + w_2\left[12u_2 - 12u_1 - 4u_{1,x} - 6u_{2,x}\right] + w_{1,x}\left[6u_1 + 4u_{1,x} + 2u_{2x} - 4u_2\right]$$

$$+ w_{2,x}\left[6u_1 - 6u_2 + 2u_{1,x} + 4u_{2x}\right] = c\left\{\tfrac{1}{2}w_1 + \tfrac{1}{2}w_2 + \tfrac{1}{12}w_{1x} - \tfrac{1}{12}w_{2x}\right\} - w_{1x}M + w_1 Q$$

$$12u_1 - 12u_2 + 6u_{1,x} + 6u_{2,x} = \frac{c}{2} + Q$$

$$-12u_1 + 12u_2 - 4u_{1,x} - 6u_{2,x} = \frac{c}{2}$$

$$6u_1 - 4u_2 + 4u_{1x} + 2u_{2x} = \frac{c}{12} - M$$

$$6u_1 - 6u_2 + 2u_{1x} + 4u_{2x} = -\frac{c}{12}$$

$u_2 = 0$, $u_{2x} = 0$ } built in

$$\begin{bmatrix} 12 & -12 & 6 & 6 \\ -12 & 12 & -4 & -6 \\ 6 & -4 & 4 & 2 \\ 6 & -6 & 2 & 4 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_{1,x} \\ u_{2x}\end{pmatrix} = \begin{pmatrix} A \\ B \\ C \\ D\end{pmatrix}$$

$-12 \quad +12 \quad -4 \quad -8$

$$\begin{pmatrix} 12 & 6 \\ 6 & 4\end{pmatrix}\begin{pmatrix} u_1 \\ u_{1,x}\end{pmatrix} = \begin{pmatrix} A \\ C\end{pmatrix} = \begin{pmatrix} c/2 + Q \\ c/12 - M\end{pmatrix}$$

$$\begin{bmatrix} 12 & -12 & 4 & 6 & A \\ 0 & 0 & 0 & 0 & B+A \\ 0 & 0 & -8 & 0 & A-3C \\ 0 & 0 & 0 & -2 & A-2D \end{bmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix} \ \\ \ \end{pmatrix} = \begin{pmatrix} A \\ C\end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 0 & -2\end{pmatrix}\begin{pmatrix} A/4 \\ -3C/4 + A/4\end{pmatrix}$$

$$u_1 \quad \frac{\left[c/2 + Q - c/12 + M\right]}{8} = \frac{5}{96} + \frac{Q}{8} + \frac{M}{8}$$

$$\frac{1}{8}\left[3C - A\right] = \frac{1}{8}\left[\frac{c}{4} - 3M - \frac{c}{2} - Q\right]$$

$$= \frac{18c}{96} + \frac{3Q}{8} \ldots \frac{5}{96} + \frac{M}{8}$$

$$\left[-\frac{c}{32} - \frac{3M}{8} - \frac{Q}{8}\right]$$

$$u_{1,x} = \frac{17}{96}c + \frac{3Q}{8} + \frac{M}{8}$$

$$-\left[\frac{c}{32} + \frac{3M}{8} + \frac{Q}{8}\right]$$

$$\therefore\ u_{1,x} = -\frac{1}{2}\left\{-\frac{3}{4}C + \frac{A}{4}\right\} \qquad u_1 = \frac{1}{3}\left\{\frac{A}{4} - u_{1,x}\right\}$$

$$= -\frac{1}{2}\left\{-\frac{3}{4}\left(\frac{c}{12} - M\right) + \left(\frac{c}{2} + Q\right)\frac{1}{4}\right\}$$

$$= -\frac{1}{2}\left\{-\frac{c}{16} + \frac{3}{4}M + \frac{Q}{4} + \frac{c}{8}\right\}$$

$$u_{1,x} = -\frac{c}{32} - \frac{3}{8}M - \frac{Q}{8} \qquad u_1 = \frac{1}{3}\left\{\frac{c}{8} + \frac{Q}{4} + \frac{c}{32} + \frac{3}{8}M + \frac{Q}{8}\right\}$$

$$= \frac{5}{96}c + \frac{Q}{8} + \frac{M}{8}$$

$$\therefore\ u_1^h = \left(\frac{5}{96}c + \frac{Q}{8} + \frac{M}{8}\right)N_1(x) - \left(\frac{c}{32} + \frac{3}{8}M + \frac{Q}{8}\right)P_1(x)$$

if f = const $\qquad u^{IV} = f \qquad u''' = fx + c_1 \qquad u'''(0) = c_1 = Q \quad \Rightarrow \quad u''' = fx + Q$

$$u'' = f\frac{x^2}{2} + Qx + c_2 \qquad u''(0) = c_2 = M \quad \therefore\ u'' = f\frac{x^2}{2} + Qx + M$$

$$u' = f\frac{x^3}{6} + Q\frac{x^2}{2} + Mx + d \qquad u'(1) = 0 \quad \therefore -\left(\frac{f}{6} + \frac{Q}{2} + M\right) = d$$

$$\therefore\ u = f\frac{x^4}{24} + Q\frac{x^3}{6} + M\frac{x^2}{2} + dx + e \qquad u(1) = 0$$

$$= \frac{f}{24} + \frac{Q}{6} + \frac{M}{2} - \frac{4f}{24} - \frac{3Q}{6} - \frac{2M}{2} + e = 0$$

$$e = +\frac{3f}{24} + \frac{2Q}{6} + \frac{M}{2} = \frac{f}{8} + \frac{Q}{3} + \frac{M}{2}$$

$\det = 48-36 = 12 \qquad u_1 = \frac{4A-6C}{12} = \frac{A}{3} - \frac{C}{2} = \frac{C}{6} + \frac{Q}{3} - \frac{C}{24} + \frac{M}{2}$

$$u_1 = \frac{C}{8} + \frac{Q}{3} + \frac{M}{2}$$

$$u_{1,x} = \frac{12C - 6A}{12} = C - A/2 = \frac{C}{12} - M - C/4 - \frac{Q}{2} = -\frac{C}{6} - M - \frac{Q}{2}$$

$$\therefore \quad u^h = \left(\frac{C}{8} + \frac{Q}{3} + \frac{M}{2}\right) N_1(x) - \left(\frac{C}{6} + \frac{Q}{2} + M\right) P_1(x)$$

$$u = \frac{Cx^4}{24} + Qx^3/6 + Mx^2/2 - \left(\frac{C}{6} + \frac{Q}{2} + M\right)x + \left(\frac{C}{8} + \frac{Q}{3} + \frac{M}{2}\right)$$

$$u^h(0) = u(0) \qquad u^h_{,x}(0) = u_{,x}(0)$$

$$u^h_{,x} = \left(\frac{C}{8} + \frac{Q}{3} + \frac{M}{2}\right)(-6x(1-x)) - \left(C/6 + \frac{Q}{2} + M\right)(1-x)(1-3x)$$

$$u_{,x} = \frac{Cx^3}{6} + \frac{Qx^2}{2} + Mx - \left(C/6 + \frac{Q}{2} + M\right).$$

$$u^h_{,xx} = (12x-6)\left(\frac{C}{8} + \frac{Q}{3} + \frac{M}{2}\right) - (6x-4)\left(C/6 + \frac{Q}{2} + M\right)$$

$$u_{,xx} = \frac{Cx^2}{2} + Qx + M$$

barlow pts are $x = (x_1 + h/2) \pm \frac{h\sqrt{3}}{6} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$ (with $x_1 = 0$, $h = 1$)

@ $x = \frac{3 \pm \sqrt{3}}{6}$

$$u^h_{,xx} = \pm 2\sqrt{3}\left(C/8 + Q/3 + \frac{M}{2}\right) - (3 \pm \sqrt{3} - 4)\left(C/6 + Q/2 + M\right)$$

$$= C/6 + Q/2 + M \pm \sqrt{3}\left(C/4 + \frac{2Q}{3} + M - C/6 - Q/2 - M\right)$$

$$= C/6 + Q/2 + M \pm \sqrt{3}\left(C/12 + Q/6\right) \checkmark$$

$$u_{,xx} = \frac{C}{2}\left[\frac{1}{3} \pm \frac{\sqrt{3}}{6}\right] + Q\,\frac{3 \pm \sqrt{3}}{6} + M$$

$$= \left(\frac{C}{6} + \frac{Q}{2} + M\right) \pm \sqrt{3}\left(\frac{C}{12} + \frac{Q}{6}\right) \checkmark\checkmark$$

h) if $u^{IV} = \text{const} \Rightarrow u^V = 0$ and $u^{(n)} = 0 \ \forall n \geq 5 \ \therefore$

$$u^h_{,xx} - u_{,xx} = -u^{IV}(x)\left[h^2 + 6ah + 6a^2\right]$$

at the barlow points $u^h_{,xx} - u_{,xx} \equiv 0 \quad \therefore$ barlow pts are exact.

i) from the formulation of (W) we must evaluate $\int_\Omega w_{,xx} u_{,xx} \, dx$

if $u_{,x}, w_{,x} \in C^0$ then by the time we get to $u_{,xx}, w_{,xx}$ we have discontinuities of 2nd order ie derivatives of delta fns. which are not nice to deal with

$(w) \Rightarrow \quad a(w,u) = (w,f) - w_{,x}(0)M + w(0)Q$

if $u = G$ $\quad f = \delta$ $\quad \ni \quad G^{IV} = \delta(y)$ w/ $\quad G_{,x}(1) = G(1) = 0$

then $a(w,G) = (w,\delta)$ $\quad (*)$ $\quad G_{,xx}(0) = G_{,xxx}(0) = 0$

Now $\quad a(w,u) = (w,f) - w_{,x}(0)M + w(0)Q$

since $\quad w^h \in \mathcal{V}^h \subset \mathcal{V} \therefore w^h \in \mathcal{V}$

$a(w^h,u) = (w^h,f) - w^h_{,x}(0)M + w^h(0)Q$

for u^h $\quad a(w^h,u^h) = (w^h,f) - w^h_{,x}(0)M + w^h(0)Q$ $\quad$ since $u^h \in \mathcal{S}^h \subset \mathcal{S} \therefore u^h \in \mathcal{S} = \mathcal{V}$

$\therefore a(w^h,u) - a(w^h,u^h) = 0 \quad \therefore \quad a(w^h, u-u^h) = 0$ by bilinearity $(**)$

now $\quad u(y) - u^h(y) = \int_0^1 (u-u^h)\,\delta\,dx = (u-u^h,\delta)$

since $u \in \mathcal{S}$ $\quad u^h \in \mathcal{S}$ $\quad u-u^h \in \mathcal{S} = \mathcal{V} \therefore$ replace $u-u^h$ for w in $(*)$

$\therefore \quad a(u-u^h, G) = (u-u^h,\delta) = u(y) - u^h(y)$

but since $a(w^h,u-u^h) = a(u-u^h,w^h) = 0$ then

$a(u-u^h, G-w^h) = u(y) - u^h(y)$

when $y = x_A \Rightarrow G \in \mathcal{V}^h \therefore G - w^h \in \mathcal{V}^h \subset \mathcal{V} \Rightarrow a(u-u^h, G-w^h) = 0$ by $(**)$

$\therefore \quad u(x_A) - u^h(x_A) = 0.$

$u_{,x}(y) - u^h_{,x}(y) = \int_0^1 (u_{,x} - u^h_{,x})\,\delta\,dx = -\int_0^1 (u-u^h)\,\delta_{,x}\,dx$

$= (u-u^h, -\delta_{,x})$ $\quad$ but since $\quad (u_{,x} - u^h_{,x}, \delta) = (u-u^h, -\delta_{,x})$

if $\exists\; G \ni \quad G^{IV} = -\delta_x(y)$ w/ $G_{,x}(1) = G(1) = 0$ $\quad G_{,xx}(0) = G_{,xxx}(0) = 0$

then $a(w,G) = (w,-\delta_x) = (w_{,x},\delta)$

ME 235A
Finite Element Methods
Winter Quarter 1981

Stanford University
Instructor: T. Hughes

Cesar Levy

Midterm Exam

Time = 1 hour
Points = 100

Instructions: Open book, notes and homework allowed. Be concise!

1. 0

2. 30

3. 20

50

1. (20 points) Consider the one-dimensional model problem described in class. Obtain exact expressions for $\underset{\sim}{f} = \{f_a^e\}$, $a = 1, 2$, for the following cases (ignore g and h contributions):

(i) f = constant.

(ii) $f = \delta(x - \bar{x})$ the "delta function", where $x_1^e \le \bar{x} \le x_2^e$. Specialize for the cases $\bar{x} = x_1^e$ and $\bar{x} = (x_1^e + x_2^e)/2$.

$f_1 = (N_1, f) - N_1(0)h - a(N_1, N_2)g$

$= f\int_0^1 (1-x)\,dx - h - 1\cdot g$

$= -f\left.\frac{(1-x)^2}{2}\right|_0^1 - h - g = \boxed{f/2} - h - g$

$f_2 = (N_2, f) - N_2(0)h - a(N_2, N_1)g$

$= f\int_0^1 x\,dx - 0 - g\int_0^1 dx$

$= f\left.\frac{x^2}{2}\right|_0^1 - 0 - g = f/2 - 0 - g = \boxed{f/2}$

PLEASE DON'T DO THINGS LIKE THIS. WHAT IS THE POINT?

$f_1 = \int_0^1 (1-x)(\delta(x-\bar{x}))\,dx - h - \int$

$f_1 = 1 - \bar{x}$

$f_2 = \int_0^1 x\,\delta(x-\bar{x})\,dx = \bar{x}$

$\bar{x} = x_1^e$	$\bar{x} = x_2^e$	$\bar{x} = (x_1^e + x_2^e)/2$
$f_1 = 1$	$f_1 = 0$	$f_1 = 1/2$
$f_2 = 0$	$f_2 = 1$	$f_2 = 1/2$

ACCIDENT

$N_1(x) \neq 1 - x$
$N_2(x) \neq x$

YOUR EXECUTION IS OK, STARTING FROM

$N_1 = 1 - x$
$N_2 = x$

BUT THESE ARE NOT THE SHAPE FTNS IN THIS CASE.

30

2. (60 points) Consider the strong statement of the boundary-value problem in classical linear heat conduction in which the h-type boundary condition is replaced by the following expression:

$$\lambda u - q_i n_i = h \quad \text{on } \Gamma_h \qquad (1)$$

where λ is a given non-negative function of $x \in \Gamma_h$.

(i) Generalize the weak formulation to include (1) as a natural boundary condition.

(ii) Obtain an expression for the additional contribution to k^e_{ab} arising from (1).

(iii) Show that K is positive-definite.

Remark: The boundary condition is often called Newton's law of heat transfer; λ is called the coefficient of heat transfer. This boundary condition applies to the case in which the heat flux is proportional to the difference of the surface temperatures of the body and surrounding medium, the latter formally represented by h/λ in (1).

(S) $\nabla \cdot q = f$ in Ω

$u = g$ on Γ_g

$\lambda u - q_i n_i = h$ on Γ_h

20

(i) define $\mathcal{V} = \{ w \,/\, w = 0 \text{ on } \Gamma_g \}$

$\mathcal{S} = \{ u \,/\, u = g \text{ on } \Gamma_g \}$

$$\int_\Omega w(\nabla \cdot q - f)\, d\Omega = \int w \,\nabla \cdot q \, d\Omega - \int w f \, d\Omega = \int_\Omega \nabla \cdot (w q)\, d\Omega - \int_\Omega (q \cdot \nabla w)\, d\Omega - \int_\Omega w f \, d\Omega = 0$$

$$\int_\Omega \nabla \cdot (w q)\, d\Omega = \int_\Gamma w\, q \cdot n \, d\Gamma = \int_\Omega (q \cdot \nabla w)\, d\Omega + \int_\Omega w f \, d\Omega$$

since $w = 0$ on Γ_g & $q \cdot n = \lambda u - h$ on Γ_h then

✓ $$\int_{\Gamma_h} w(\lambda u - h)\, d\Gamma = \int_\Omega (q \cdot \nabla w)\, d\Omega + \int_\Omega w f \, d\Omega$$

$$\lambda \int_\Gamma w u \, d\Gamma - \int_\Omega q_i \, w_{,i} \, d\Omega = +\int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma$$

$$\lambda \int_{\Gamma_h^e} w u \, d\Gamma + \int_{\Omega^e} K_{ij} u_{,j} w_{,i} \, d\Omega = k^e_{ab}$$

~~$\lambda (w^h, v^h)_\Gamma + a(w^h, v^h) = (w^h, f) + (w^h, h)_\Gamma - a(w^h, g^h) - \lambda (w^h, g^h)_\Gamma$~~

~~$\sum_{B \in \eta_h} d_B \lambda (N_A, N_B)_\Gamma + \sum_{B \in \eta - \eta_g} d_B\, a(N_A, N_B) = \dots$ η_h being the nodes of Γ_h~~

~~the addition contrib is~~ $\sum_{B \in \eta_h} \lambda (N_a, N_b)^e d^e_B$ $\quad \eta^e_h$ is $\eta^e_h \in \Gamma_h \cap \Gamma^e$

and 0 for all other $\eta - \eta_g \notin \eta_h$ over

$\underset{\sim}{K} = K_{(1)} + \lambda K_{(2)}$ $\quad K_{(1)} = a(w,w) = \int_{\Omega} \nabla w^T \underset{\sim}{\kappa} \nabla w \, d\Omega$

let $w^h \in \mathcal{V}^h$ s.t. $w = \sum N_A c_A$ $\quad K_{(2)} = (w,w)_\Gamma$ ~~$\int_\Gamma \nabla w^T \nabla w \, d\Gamma$~~

$\bar{c}_A = c_P \quad P = ID(A)$

$\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = \sum c_Q K^{(1)}_{PQ} c_P \quad \lambda \sum_{P,Q} c_Q K^{(2)}_{PQ} c_P$

$\underset{\sim}{\kappa}$ - is conductance positive definite hypothesis

proven ≥ 0 in class ✓

$+ \lambda \int_\Gamma \nabla w^T \nabla w$

$\int_\Gamma |\nabla w|^2 d\Gamma \geq 0$

$\therefore$ ~~$\underset{\sim}{K} = K_{(1)} + \lambda K_{(2)} \geq 0$ since $\lambda \not< 0$~~ OK

assume $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = 0 \Rightarrow \int_\Omega \nabla w^T \underset{\sim}{\kappa} \nabla w \, d\Omega + \lambda \int \nabla w^T \nabla w \, d\Gamma = 0$

since λ is $\not< 0$ & $\underset{\sim}{\kappa}$ is positive definite

$\Rightarrow \nabla w^T \underset{\sim}{\kappa} \nabla w = 0$ & $\nabla w^T \nabla w = 0 \Rightarrow \nabla w = 0$

from here, proof is same as in class & $\Rightarrow c_A = 0$ ✓

IDEA RIGHT BUT....

(iii)

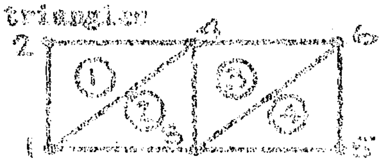
~~YOU DON'T HAVE PART (ii)~~
~~IN FACT, WHAT YOU DO HAVE FOR ADDITIONAL IS b.2.b IS TOTALLY WRONG~~

−20 (ii) 0

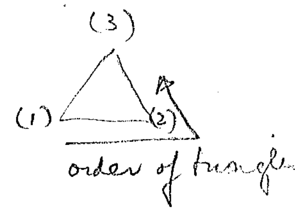
QUESTION: (FOR ME)
HOW MUCH CREDIT ?

(iii) + 10

3. (20 points) Set up the ID, IEN and LM arrays for the following mesh of three-node triangles



Assume the temperature is specified at nodes 2, 4 and 6 ("g-data"). Furthermore, assume the lower left-hand node of each triangle is number one in the local ordering.



20

ID (global node no. / global eqn no.)

1	2	3	4	5	6
1	0	2	0	3	0

IEN (local element no. across; local node no. down; entries: global node no.)

local node no.	1	2	3	4
1	1	1	3	3
2	4	3	6	5
3	2	4	4	6

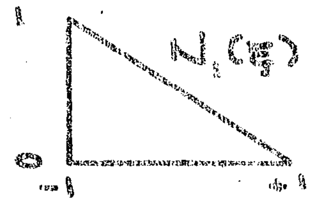
LM (local element across; local node no. down; entries: global eqn.)

local node no.	1	2	3	4
1	1	1	2	2
2	0	2	0	3
3	0	0	0	0

$$f_a^e = \int_{x_1^e}^{x_2^e} N_a(x)\, f(x)\, dx$$

(ii) $$f_a^e = f \int_{x_1^e}^{x_2^e} N_a(x)\, dx = \frac{f h^e}{2} \underbrace{\int_{-1}^{+1} N_a(\xi)\, d\xi}_{1}$$

$$\underset{\sim}{f}^e = \frac{f h^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$



(iii) $$f_a^e = \int_{x_1^e}^{x_2^e} N_a(x)\, \delta(x - \bar{x})\, dx = N_a(\bar{x})$$

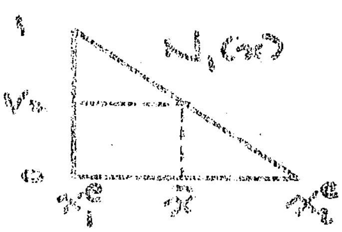
[illegible]: $\bar{x} = x_b^e$

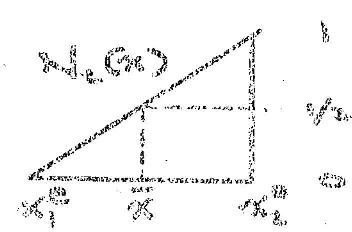
$$f_a^e = N_a(\bar{x}) = N_a(x_b^e) = \delta_{ab} \quad \begin{pmatrix} \text{KRONECKER} \\ \text{DELTA} \end{pmatrix}$$

$$\underset{\sim}{f}^e = \begin{Bmatrix} \delta_{1b} \\ \delta_{2b} \end{Bmatrix}$$

: $\bar{x} = (x_1^e + x_2^e)/2$

$$f_a^e = N_a(\bar{x}) = N_a\left(\frac{x_1^e + x_2^e}{2}\right) = 1/2$$





2. (i) $-\int_{\Omega} w_{,i} q_i \, d\Omega = \int_{\Omega} w f \, d\Omega +$

$$+ \int_{\Gamma_h} w (h - \lambda u) \, d\Gamma$$

INTEGRATION BY PARTS, AND THE USUAL ARGUMENT, ESTABLISHES THAT $\lambda u - q_i n_i = h$ ON Γ_h IS A NATURAL BOUNDARY CONDITION.

(ii) $k^e_{ab} = \underbrace{\int_{\Gamma^e_h} N_a \lambda N_b \, d\Gamma}_{\text{ADDITIONAL CONTRIBUTION}} + \dots$

WHERE $\Gamma^e_h = \Gamma_h \cap \Gamma^e$

(iii) PROCEED AS IN CLASS NOTES TO ARRIVE AT:

(*) $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = \underbrace{\int_{\Omega} w^h_{,i} \kappa_{ij} w^h_{,j} \, d\Omega}_{\geq 0} + \underbrace{\int_{\Gamma_h} \lambda \underbrace{(w^h)^2}_{\geq 0} d\Gamma}_{\geq 0}$

≥ 0 (FIRST PART)

SECOND PART: SHOW $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = 0 \Rightarrow \underset{\sim}{c} = \underset{\sim}{0}$

2. (CONT'D) BY (**), $\underset{\sim}{\zeta}^T \underset{\sim}{K} \underset{\sim}{\zeta} = 0 \Rightarrow$

$$0 = \int_{\Omega} u^h_{,i} \, \kappa_{ij} \, u^h_{,j} \, d\Omega \qquad (***)$$

and $$0 = \int_{\Gamma_h} \lambda (u^h)^2 \, d\Gamma$$

CONDITION (***) AND THE ARGUMENT IN CLASS NOTES $\Rightarrow \underset{\sim}{\zeta} = \underset{\sim}{0}$.

3. ID:

(a)	1	2	3	4	5	6
	1	0	2	0	3	0

IEN:

(a) \ (e)	1	2	3	4
1	1	1	3	3
2	4	3	6	5
3	2	4	4	6

LM:

(a) \ (e)	1	2	3	4
1	1	1	2	2
2	0	2	0	3
3	0	0	0	0

ME 235a
Finite Element Methods
Winter Quarter 1981

Stanford University
Instructor: T. J. R. Hughes

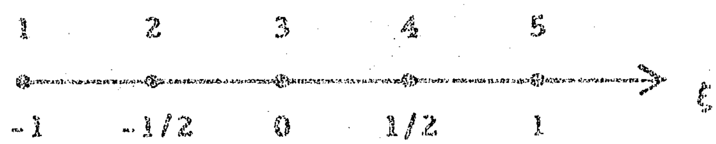
LEVY
work is in enclosed sheets

Final Exam

Time = 3 Hours
Total Points = 300

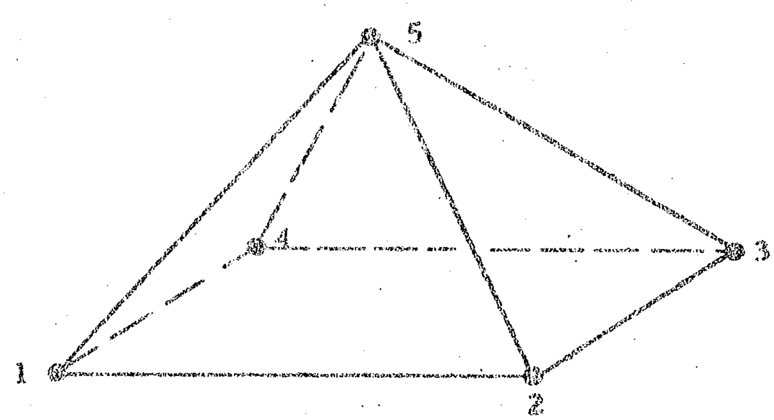
Instructions: Open notes and homework are allowed. You may use all results previously obtained in the course lectures and homework, so be as brief as possible.

1. (25 points) Determine the shape functions $N_a(\xi)$, a = 1, 2, 3 of the 5-node element shown below:

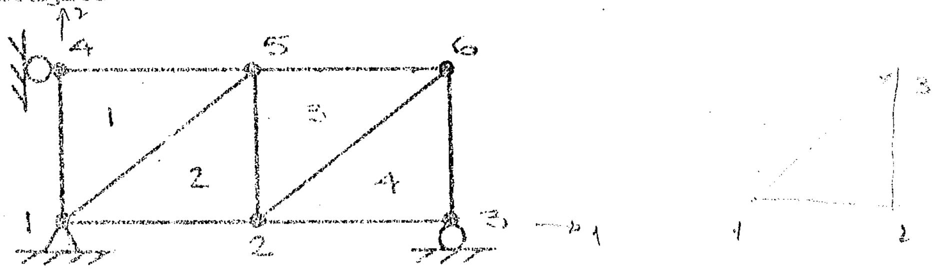


2. (25 points) Derive the weights W_ℓ and quadrature points $\tilde{\xi}_\ell$ of the 4-point Gauss quadrature rule in one-dimension.
(Hint: $\tilde{\xi}_1$ = .86113 and W_1 = .34785.)

3. (25 points) Degenerate the shape functions of the trilinear hexahedral element to determine shape functions for the pyramidal element shown below.

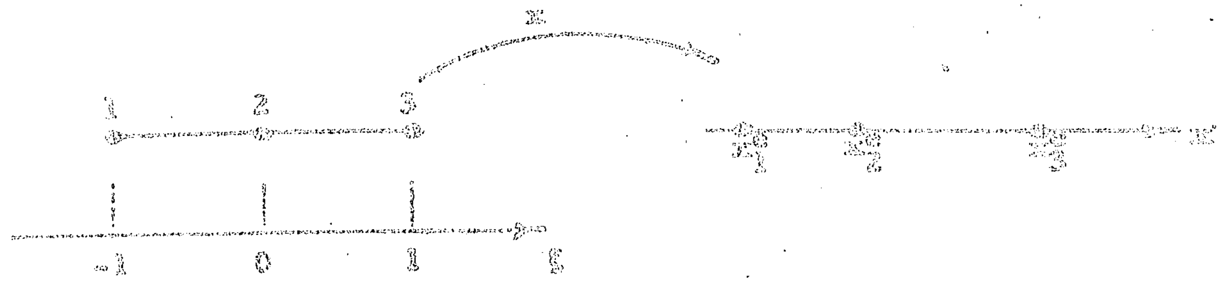


4. (25 points) Consider a two-dimensional elastostatic boundary-value problem. Set up the ID, IEN, and LM arrays for the following mesh of three-node triangles:



Assume the lower left-hand node of each triangle is number one in the local ordering.

5. Consider a 1-dimensional, quadratic, 3-node element for which the following relations hold:



$$x(\xi) = \sum_{a=1}^{3} N_a(\xi) x_a^e \qquad h^e = x_3^e - x_1^e$$

$$u^h(\xi) = \sum_{a=1}^{3} N_a(\xi) d_a^e \qquad \underset{\sim}{f}^e = \{f_a^e\}$$

$$N_1(\xi) = \tfrac{1}{2}\xi(\xi-1) \qquad f_a^e = \int_{x_1^e}^{x_3^e} f N_a \, dx$$

$$N_2(\xi) = 1 - \xi^2$$

$$N_3(\xi) = \tfrac{1}{2}\xi(\xi+1) \qquad \underset{\sim}{k}^e = [k_{ab}^e]$$

$$k_{ab}^e = \int_{x_1^e}^{x_3^e} N_{a,x} N_{b,x} \, dx , \qquad 1 \le a,b \le 3$$

Given a "loading" , the elements of the consistently derived "force" vector $\underset{\sim}{f}^e$ are sometimes surprising, especially for higher-order elements. This problem establishes some basic results concerning the calculation of $\underset{\sim}{f}^e$, appropriate quadrature formulas, and Barlow stress points for the 3-node element.

5. continued

(a) (25 points) Assume f = const. and let $x_2^e = (x_1^e + x_3^e)/2$.

Determine exact expressions for f_a^e, $a = 1,2,3$.

(b) (25 points Assume $f = \delta(x-\bar{x})$, $x_1^e \leq \bar{x} \leq x_3^e$, (the "delta function") and let $x_2^e = (x_1^e + x_3^e)/2$. Obtain an exact expression for f_a^e, $a = 1,2,3$, and specialize for the cases $\bar{x} = x_1^e$ and $\bar{x} = x_2^e$.

(c) (25 points) Assume f = const., but make no assumption on the location of x_2^e other than $x_1^e < x_2^e < x_3^e$. Determine the lowest-order Gaussian quadrature formula (n_{int} = ?) which exactly integrates f^e. Justify your answer.

(d) (25 points) Assume $x_2^e = (x_1^e + x_3^e)/2$. Determine the lowest-order Gaussian quadrature formula (n_{int} = ?) which exactly integrates k^e. Justify your answer.

(e) (25 points) An analysis of the quadratic element is being performed to locate the Barlow stress points (i.e. the points at which $e_{,x} = u^h_{,x} - u_{,x}$ optimally converges). It is assumed in the analysis that $x_2^e = (x_1^e + x_3^e)/2$ and the nodal values $d_a^e = u(x_a)$. Determine the Barlow stress points.

(f) (25 points) Let

$$x_2^e = x_1^e + h^e/4 \qquad \text{("quarter point")}$$

$$r = (x - x_1^e)/h^e$$

$$h^e = x_3^e - x_1^e$$

Determine an expression for $u^h_{,r}(r)$ and indicate the order of the singularity at $r = 0$ [i.e. determine α, where $u^h_{,r} = 0(r^{-\alpha})$]. (This result is useful for the construction of "crack elements.")

6. (50 points) In practice, it is often useful to generalize the constitutive equation of classical elasticity to the form

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon^{o}_{kl}) + \sigma^{o}_{ij} \qquad (1)$$

where ε^{o}_{ij} and σ^{o}_{ij} are the initial strain and initial stress, both given functions of $\underset{\sim}{x}$. The initial strain term may be used to represent thermal expansion effects by way of

$$\varepsilon^{o}_{kl} = -\theta c_{kl}$$

where θ is the temperature and the c_{kl}'s are the thermal expansion coefficients (both given functions). Clearly, (1) will in no way change the stiffness matrix. However, there will be additional contributions to f^{e}_{p}. Generalize the definition of f^{e}_{p} to account for these additional terms.

$$u^h_{,r}(\xi) = \frac{2}{\xi+1}\left[(\xi-\tfrac{1}{2})u_1 - 2\xi u_2 + (\xi+\tfrac{1}{2})u_3\right]$$

$$= \frac{2}{\xi+1}\left[\xi(u_1-2u_2+u_3)+\tfrac{1}{2}(u_3-u_1)\right] \qquad \xi = \frac{-\frac{he}{2}+\sqrt{\left(\frac{he}{2}\right)^2 - he(x_2-x)}}{he/2}$$

let $u_1 = u + (x_1-x)u_{,x} + \dots$

$u_2 = u + (x_2-x)u_{,x} + \dots$

$u_3 = u + (x_3-x)u_{,x} + \dots$

$$= \frac{2}{\xi+1}\left[\xi(x_1+x_3-2x_1)u_{,x} + \tfrac{1}{2}(x_3-x_1)u_{,x} + O(u_{,xx})\right)$$

$$\left[\xi\,\tfrac{he}{2}u_{,x} + \tfrac{1}{2}\tfrac{he}{2}u_{,x}\right.$$

$$= he\,u_{,x}$$

$$u^h_{,r}(\xi) = u^h_{,\xi}\,\xi_r = u^h_{,\xi}\,\xi_x\cdot x_r = he\,u^h_{,\xi}\frac{+he}{he/2\sqrt{\left(\frac{he^2}{2}\right)-he(x_2-x)}}$$

$$= \frac{2u^h_{,\xi}\,he}{\sqrt{\frac{he^2}{4}-he\left(x_1-x+\frac{he}{4}\right)}} = \frac{2u^h_{,\xi}\,he}{\sqrt{he^2 r}}$$

$$= \frac{2u^h_{,\xi}}{\sqrt{r}}$$

6. from our formulation we had that $\int_\Omega \omega_{(i,j)}\sigma_{ij}\,d\Omega = \int_\Omega w_i f_i\,d\Omega + \int_{\Gamma_{h_i}} w_i\sigma_{ij}n_j\,d\Gamma$

now $\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \overset{\circ}{\varepsilon}_{kl}) + \overset{\circ}{\sigma}_{ij}$ where $\overset{\circ}{\varepsilon}_{kl}$ & $\overset{\circ}{\sigma}_{ij}$

$$\Rightarrow \int_\Omega \omega_{(i,j)}C_{ijkl}\,u_{(k,l)}\,d\Omega - \int_\Omega \omega_{(i,j)}C_{ijkl}\,\overset{\circ}{\varepsilon}_{kl}\,d\Omega + \int_\Omega \omega_{(i,j)}\overset{\circ}{\sigma}_{ij}\,d\Omega$$

hence we can rewrite

$$\int_\Omega \omega_{(i,j)}C_{ijkl}\,u_{(k,l)}\,d\Omega = \int_\Omega w_i f_i\,d\Omega + \int_\Gamma w_i\sigma_{ij}n_j\,d\Gamma + \int_\Omega \omega_{(i,j)}C_{ijkl}\,\overset{\circ}{\varepsilon}_{kl}\,d\Omega - \int_\Omega \omega_{(i,j)}\overset{\circ}{\sigma}_{ij}\,d\Omega \quad - \int \omega_{(i,j)}C_{ijkl}\ldots$$

now $f_p^e = \int_{\Omega e} N_a f_i\,d\Omega + \int_{\Gamma^e_{h_i}} N_a h_i\,d\Gamma + \int_{\Omega e}\omega_{(i,j)}C_{ijkl}\,\overset{\circ}{u}_{(k,l)} - \int_\Omega \ldots$

$u_{,x} = u_{,\xi}\, \xi_{,x}$ where $\xi_{,x} = 2/h_e$

3 pt element

$$0 = \xi^2 \left(\frac{x_1 - 2x_2 + x_3}{2}\right) + \xi \left(\frac{x_3 - x_1}{2}\right) + x_2 - x$$

$$x = \frac{\xi^2 - \xi}{2} x_1 + (1-\xi^2) x_2 + \frac{\xi^2 + \xi}{2} x_3$$

$$u^h(\xi) = \sum_{a=1}^{3} N_a(\xi)\, u^h(x_a) \quad \text{in each}$$

$$x = \sum N_a(\xi) x_a$$

$$\xi = \frac{-\frac{(x_3 - x_1)}{2} + \sqrt{\frac{(x_3 - x_1)^2}{4} - 2(x_1 - 2x_2 + x_3)(x_2 - x)}}{x_1 - 2x_2 + x_3}$$

$$= \frac{\xi(\xi-1)}{2} u_1 + (1-\xi^2) u_2 + \frac{\xi(\xi+1)}{2} u_3$$

$N_{1,\xi}$

$$\xi = \frac{-\frac{(h_2+h_1)}{2} + \sqrt{\frac{(h_2+h_1)^2}{4} - 2(h_2 - h_1)(x_2 - x)}}{h_2 - h_1}$$

$$u_{,x} = u_{,\xi}\, \xi_{,x} = \left[\sum N_{a,\xi}\, u^h(x_a)\right] \xi_{,x}$$

$$h_2^2 + 2h_2h_1 + h_1^2 - 2h_2h_1 \ldots$$

$$\xi_{,x} = \frac{1}{h_2 - h_1} \cdot \frac{1}{2} \cdot \frac{+2(h_2-h_1)\,1}{\sqrt{\frac{(h_2+h_1)^2}{4} - 2(h_2-h_1)(x_2 - x)}}$$

$$\frac{h_2^2}{4} - \frac{6}{4} h_2 h_1 + \frac{9}{4} h_1^2$$

$$x_{,\xi} = \frac{2\xi - 1}{2} x_1 - 2\xi x_2 + \frac{2\xi + 1}{2} x_3$$

$$\frac{1}{4}(3h_1 - h_2)^2$$

$$x_{,\xi} = \xi(x_1 - 2x_2 + x_3) + \frac{x_3 - x_1}{2}$$

$$= \frac{1}{2}(3h_1 - h_2) - \frac{1}{2}\left(\frac{h_2 + h_1}{1}\right)$$

$$= \frac{2h_1 - 2h_2}{2} = \frac{h_1 - h_2}{h_2 - h_1} = -1$$

$$\xi_{,x} = \frac{1}{x_{,\xi}} = \frac{1}{\xi(h_2 - h_1) + \frac{h_2 + h_1}{2}} \quad ✓$$

$$u^h_{,x} = \left[\frac{2\xi - 1}{2} u_1 - 2\xi u_2 + \frac{2\xi + 1}{2} u_3\right]$$

$$= \left[(u_1 - 2u_2 + u_3)\xi + \left(\frac{u_3 - u_1}{2}\right)\right] \cdot \frac{1}{\xi(h_2 - h_1) + \frac{h_2 + h_1}{2}}$$

$$= (u_1 - 2u_2 + u_3)\left[\frac{-\frac{(h_2 + h_1)}{2} \pm \overset{A}{\sqrt{\quad}}}{h_2 - h_1}\right] + \left(\frac{u_3 - u_1}{2}\right)$$

$$\sqrt{\quad}$$

$$u_1 = u + (x_1 - x) u_{,x} + \frac{(x_1 - x)^2}{2} u_{,xx} + \ldots$$

$$u_1 - 2u_2 + u_3 = 0 \cdot u + \left[(x_1 - x) - 2(x_2 - x) + (x_3 - x)\right] u_{,x}$$
$$(h_2 - h_1)$$

$$u_2 = u + (x_2 - x) u_{,x} + \ldots$$

$u_3 = \ldots$

$$(x_1 - x)^2 - (x_2 - x)^2 = (x_1 - x_2)(x_1 + x_2 - 2x)$$

$$\frac{u_3 - u_1}{2} = 0 \cdot u + \left(\frac{h_2 + h_1}{2}\right) u_{,x} + \frac{1}{2}\left[\frac{(x_3 - x)^2}{2!} - \frac{(x_1 - x)^2}{2!}\right] u_{,xx} + \ldots$$

$$= \left\{(h_2 - h_1) u_{,x} + \frac{u_{,xx}}{2}\left[(x_1 - x)^2 - 2(x_2 - x)^2 + (x_3 - x)^2\right] + u_{,xxx}\left[\frac{(x_1 - x)^3}{3!} - 2\frac{(x_2 - x)^3}{3!} + \frac{(x_3 - x)^3}{3!}\right] + \ldots\right\}$$

$$a^2 - b^2 = (a - b)(a^2 + ab + b^2)$$

$$[\qquad]$$

$$+ \frac{h_2 + h_1}{2} u_{,x} + \frac{1}{2}\left[\frac{(x_3 - x)^2}{2!} - \frac{(x_1 - x)^2}{2!}\right] u_{,xx} + \frac{1}{2}\left[\frac{(x_3 - x)^3}{3!} - \frac{(x_1 - x)^3}{3!}\right] u_{,xxx} + \ldots$$

$$\sqrt{\quad}$$

$$= u_{,x} + \frac{u_{,xx}}{2}\left\{\frac{\left[-(x_2 - x_1)(x_1 + x_2 - 2x)\right] A}{\sqrt{\quad}} + \frac{(x_3 - x_1)(x_3 + x_1 - 2x)}{2\sqrt{\quad}}\right\} + \frac{u_{,xxx}}{3!}\left[\right.$$

$$= 0$$

$$\frac{u^h_{,x} - u_{,x}}{x_3 - x_1} = \quad \text{if} \quad \sqrt{\quad} = \frac{(x_3 - x_1)(x_3 + x_1 - 2x)(x_1 + x_3 - 2x_2)}{4(x_2 - x_1)(x_2 + x_1 - 2x)} + \frac{2(x_3 - x_1)(x_2 - x_1)(x_2 + x_1 - 2 \ldots)}{4(x_2 - x_1)(x_2 + x_1 - 2x)}$$

$$\left[4(x_2 - x_1)(x_2 + x_1 - 2x)\right]^2 \left[\frac{(x_3 - x_1)^2}{4} - 2(x_1 - 2x_2 + x_3)(x_2 - x)\right] = f(x^?)$$

∴ 3 pt. looks like will be

$$A\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho A v) = 0 \qquad \text{let } \rho A = u$$

$$A\frac{\partial}{\partial t}(\rho V) + \frac{\partial}{\partial x}(\rho A v^2) + A\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uv) = 0$$

$$\frac{\partial p}{\partial x} - c^2\frac{\partial \rho}{\partial x} = 0 \qquad \frac{\partial}{\partial t} \qquad \frac{\partial(uv)}{\partial t} + \frac{\partial}{\partial x}(uv^2) + c^2 A\frac{\partial \rho}{\partial x}$$

$$\frac{\partial u}{\partial x} - c^2\rho\frac{\partial A}{\partial x}$$

let $t = t(\xi, \eta)$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$c_1 \quad A\xi_t(\rho)_{,\xi} + \eta_t A(\rho)_{,\eta} + (\rho A v)_{,\xi}\xi_x + (\rho A v)_{,\eta}\eta_x = 0 \qquad Ac^2[\rho_\xi \xi_x + \rho_\eta \eta_x] =$$

$$c_2 \quad A\xi_t(\rho v)_{,\xi} + \eta_t A(\rho v)_{,\eta} + (\rho A v^2)_{,\xi}\xi_x + (\rho A v^2)_{,\eta}\eta_x + A p_{,\xi}\xi_x + A p_{,\eta}\eta_x = 0$$

$$\sout{c_3 \quad p_\xi \xi_x + p_\eta \eta_x - c^2(\rho_\xi \xi_x + \rho_\eta \eta_x) = 0}$$

$$dt\,\xi_t + \xi_{,x}dx = 0 \qquad d\eta = \eta_t\,dt + \eta_x\,dx$$

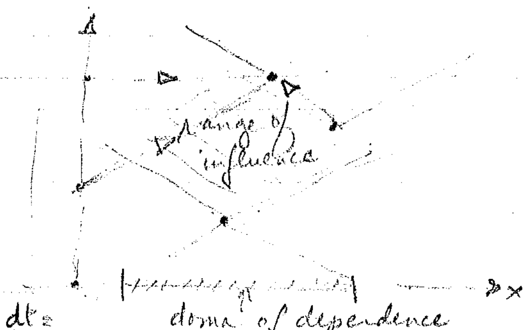
$$\xi_t\left(c_1[\rho_\xi A] + c_2[\rho_\xi A v + \rho v_\xi A]\right) + \eta_t\left(c_1 A\rho_\eta + c_2(A\rho_\eta v + A\rho v_\eta)\right) + \xi_x\left[c_1(\rho_\xi A v + \rho A_\xi v + \rho A v_\xi) + c_2(\rho_\xi A v^2 + \rho A_\xi v^2 + 2\rho A v v_\xi) + c_3(p_\xi - c^2\rho_\xi)\right] + \ldots$$

$$= 0$$

$$\begin{pmatrix} (c_1 A + c_2 A v) & (A\rho c_2) \\ (\quad) & (\quad) \end{pmatrix}\begin{pmatrix}\rho_\eta \\ v_\eta\end{pmatrix} = 0 \qquad \eta_x$$

$$\begin{pmatrix} A\rho_\eta & A\rho_\eta v + A\rho v_\eta & 0 \\ (\rho A v)_{,\eta} & (\rho A v^2)_{,\eta} & \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\frac{dt}{dx} \qquad \frac{dx}{dt} = \infty \qquad dx - c_1\,dt = \qquad dt - \frac{1}{c_1}dx = 0$$



$$Ac_2c^2\xi_x$$

$$\rho_\xi\left[c_1 A\xi_t + c_1 A v\xi_x + c_2 A\xi_t v + c_2 A v^2\xi_x \; \text{[illegible]}\right] + v_\xi\left[c_1\rho A\xi_x + c_2 A\xi_t\rho + 2\rho A v\xi_x c_2\right]$$

$$+ p_\xi\left[c_2 A\xi_x + c_3\xi_x\right] + \ldots = 0$$

$$x_n - x_0 = dx$$

$$(c_1 A + c_2 A v) \qquad (c_1 A v + c_2 A v^2 + c_2 c^2) \qquad \frac{dx}{dt} \quad dx = (v \pm c)\,dt$$

$$(A\xi_t + Av\xi_x) \qquad (Av\xi_t + Av^2\xi_x + Ac^2\xi_x)$$

$$x_t = \frac{2v \pm \sqrt{4v^2 - 4v^2 + 4c^2}}{2}$$

$$(\rho A\xi_x) \qquad A\xi_t\rho + 2\rho A v\xi_x \qquad x_t = \frac{2v \pm 2c}{2}$$

$$x_t^2\rho - x_t\rho v - \rho v x_t + \rho v^2 - \rho c^2 = 0$$

$$\sout{A^2\xi_t^2\rho + A^2 v\xi_x\rho\xi_t + 2}$$

$$x_t^2\xi_x^2\rho - x_t\xi_x^2\rho v - \rho v x_t\xi_x^2 + 2\rho v^2\xi_x^2 - \rho v^2\xi_x^2 - \rho c^2\xi_x^2 = 0 \qquad x_t^2 - x_t(2v) + (v^2 - c^2) = 0$$

$$\xi_t^2\rho + \xi_t\xi_x\rho v + \rho v\xi_t\xi_x + 2\rho v^2\xi_x^2 - \rho\xi_x v\xi_t - \rho v^2\xi_x^2 - \rho c^2\xi_x^2 = 0 \qquad \sout{\frac{+(1+v) \pm \sqrt{(1+v)^2 - 4(v^2 - c^2)}}{2}}$$

$$\xi_t\,dt + \xi_x\,dx = 0 \qquad \xi_t = -\xi_x x_t$$

$$N_1(x) = (x-x_2)^2\,[h-2(x_1-x)]/h^3$$
$$N_2(x) = (x-x_1)^2\,[h+2(x_2-x)]/h^3$$
$$P_1(x) = (x-x_1)(x-x_2)^2/h^2$$
$$P_2(x) = (x-x_1)^2(x-x_2)/h^2$$

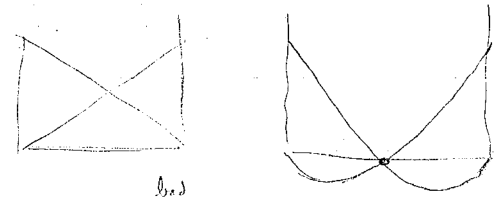
$$u_e^h(x) = \sum_i N_i^e(x)\, c_i^e$$
$$u^h(x) = \sum_e u_e^h(x)$$

$$\xi = [2x-(x_1+x_2)]/h$$
$$[\xi h + (x_1+x_2)]/2 = x$$

$$x - x_2 = \frac{\xi h + (x_1-x_2)}{2} = \frac{h(\xi-1)}{2}$$
$$x - x_1 = \frac{\xi h + (x_2-x_1)}{2} = \frac{h(\xi+1)}{2}$$

$$N_1(\xi) = \frac{(\xi-1)^2}{4}[\xi+2]$$
$$N_2(\xi) = \frac{(\xi+1)^2}{4}[2-\xi]$$
$$P_1(\xi) = h\,\frac{\xi+1}{2}\,\frac{(\xi-1)^2}{4} = \frac{h}{8}(\xi^2-1)(\xi-1)$$
$$P_2(\xi) = \frac{h}{8}(\xi+1)(\xi^2-1)$$

$$P_1'(\xi) = \frac{h}{8}(2\xi)(\xi-1) + \frac{h}{8}(\xi^2-1)$$
$$P_1'(x) = P_1'(\xi)\frac{d\xi}{dx} = \frac{2}{h}\left[\frac{h}{8}\cdot 4\right] = 1 \checkmark$$



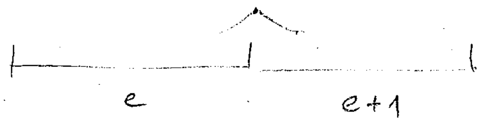
Now $u^h(\xi) = N_i(\xi)\, c_i$

$$\frac{d}{dx}u^h(x) = \frac{d}{d\xi}u^h(\xi)\frac{d\xi}{dx} = \frac{2}{h}\,u^{h\prime}(\xi)$$
$$\frac{d^2}{dx^2}u^h(x) = u^{h\prime\prime}(\xi)\,\frac{4}{h^2}$$

$$\int_\Omega \left(\sum c_j N_{j,x}\; \kappa_{ji}\left(\sum d_j N_{i,x} + \right)\right)$$
$$\sum c_j \sum d_i \int \left(N_{j,x}\,\kappa_{ji}\,N_{i,x}\right)$$
$$\int (\nabla N)^T \underset{\sim}{\kappa}\, (\nabla N)$$

$$x = [(x_{A+1}-x_A)\,\xi + (x_A + x_{A+1})]/2.$$
$$= x_{A+1}\left[\frac{\xi+1}{2}\right] + x_A\left[\frac{1-\xi}{2}\right]$$
$$x_2^e\, N_1(\xi) + x_1^e\, N_2(\xi) = \sum N_i(\xi)\, x_i^e$$

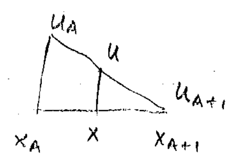
$$\sigma_{ij} = \lambda\,\varepsilon_{kk}\delta_{ij} + 2\mu\,\varepsilon_{ij}$$
$$i \neq j \quad \sigma_{ij} = 2\mu\,\varepsilon_{ij}$$
$$= \frac{u_{i,j}+u_{j,i}}{2} = \mu(u_{i,j}+u_{j,i})$$
$$\sigma_{ij} = C_{ijkl}\,(u_{k,l}+u_{l,k})$$



$$u^{h^e}(\xi) = \sum_{a=1}^{4} N_a(\xi)\, u^h(x_a)$$

$$u^h(\underset{\sim}{x}) = \sum_{a=1}^{4} N_a\left(\hat{\xi}(\underset{\sim}{x})\right) u^h(x_a)$$

$$u^h_{,x}(\underset{\sim}{x}) = u^h_{,\xi} \cdot \xi_{,x} = \sum N_a'(\xi)\, u^h(x_a)\, \xi_{,x}$$

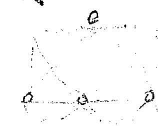


$$\frac{u_{A+1} - u_A}{x_{A+1} - x_A} = \frac{u - u_A}{x - x_A}$$

$$\frac{u_{A+1} - u_A}{h_A}(x - x_A) + u_A = u$$

questions 1. bea problem w/ Lagrange polyn. cannot satisfy cont. of deriv across bdry.

$1 \le b, a \le 3$



this is for $u_{xx} + f = 0$ using 3 node elements.

$$\int_{x_1^e}^{x_3^e} N_{a,x}(x)\, N_{b,x}(x)\, dx$$

$$\int_{-1}^{1} N_{a,\xi}\, N_{b,\xi}\, d\xi (\xi_x)^2 \cdot x_\xi = N_{a,\xi} N_{b,\xi}\, \xi_x\, d\xi$$

Now $x = \sum N_a(\xi) x_a^e$ $\qquad N_1 = \frac{\xi(\xi-1)}{2}$

$$x_{,\xi} = \left[\frac{2\xi-1}{2} x_1^e + -2\xi x_2^e + \frac{2\xi+1}{2} x_3^e \right] \qquad N_2 = -\xi^2+1 \qquad N_3 = \frac{\xi(\xi+1)}{2}$$

$$= \xi\left(x_3^e + x_1^e - 2x_2^e\right) + \frac{x_3^e - x_1^e}{2} \qquad x_{,\xi} = 1/\xi_{,x}$$

$$\xi_{,x} = \frac{1}{\xi(h_2-h_1) + \left(\frac{h_2+h_1}{2}\right)} \qquad \xi = \frac{-\left(\frac{h_2+h_1}{2}\right) + \sqrt{\left(\frac{h_2+h_1}{2}\right)^2 - 2(h_2-h_1)(x_2-x)}}{h_2-h_1}$$

$$k_{11}^e = \int_{-1}^{1} \left(\frac{2\xi-1}{2}\right)^2 \frac{d\xi}{\xi\left(x_3^e+x_1^e-2x_2^e\right) + \frac{(x_3^e-x_1^e)}{2}}$$

if error is $g''(\xi)$ ie 1pt int $\left(\frac{-1}{2}\right)^2 \frac{2\cdot 2}{h^e} = \frac{1}{h_e}$

$$k_{12}^e = \int_{-1}^{1} \frac{2\xi-1}{2} \cdot \frac{-2\xi \, d\xi}{\xi(a)+b} \qquad \frac{-1}{2}\cdot\frac{0}{b} = 0$$

$$k_{13}^e = \int_{-1}^{1} \frac{2\xi-1}{2}\cdot\frac{2\xi+1}{2}\, \frac{d\xi}{\xi(a)+b} \qquad -\frac{1}{2}\cdot\frac{1}{2}\,\frac{2\cdot 2}{h_e} = -\frac{1}{h_e}$$

$$k_{23}^e = \int_{-1}^{1} -2\xi \cdot \frac{2\xi+1}{2}\, \frac{d\xi}{\xi(a+b)} \qquad 0$$

$$k_{33}^e = \int_{-1}^{1} \left(\frac{2\xi+1}{2}\right)^2 \frac{d\xi}{\xi a + b} \qquad \left(\frac{1}{2}\right)^2 \cdot \frac{2\cdot 2}{h_e} = \frac{1}{h_e}$$

$$k_{22}^e = \int_{-1}^{1} 4\xi^2 \frac{d\xi}{\xi a b} \qquad = 0$$

$$\underset{\sim}{k}^e = \frac{1}{h^e}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

integration element: -1 0 1, x x x

not good look at 2 pts g^{iv}

for 2 elem.

$$\begin{pmatrix} 1 & 0 & -1 & & \\ 0 & 0 & 0 & & \\ -1 & 0 & 2 & 0 & -1 \\ & & 0 & 0 & 0 \\ & & -1 & 0 & 1 \end{pmatrix}$$

$$\left(\frac{-3/\sqrt{3}-1}{2}\right)^2 \frac{1}{-\frac{1}{\sqrt{3}}a+b} + \left(\frac{\frac{3}{\sqrt{3}}-1}{2}\right)^2 \frac{1}{\frac{1}{\sqrt{3}}a+b}$$

$$+\frac{(2+\sqrt{3})^2}{4\sqrt{3}}\,\frac{1}{-a+\sqrt{3}b} + \frac{(2-\sqrt{3})^2}{4\sqrt{3}}\,\frac{1}{a+b\sqrt{3}}$$

$$\frac{(7+4\sqrt{3})(a+b\sqrt{3}) + (7-4\sqrt{3})(b\sqrt{3}-a)}{4\sqrt{3}(3b^2-a^2)}$$

$$14b\sqrt{3} + 4\sqrt{3}(2a) = \frac{14b\sqrt{3}+8\sqrt{3}a}{4\sqrt{3}} = \frac{7b+4a}{2(3b^2-a^2)}$$

if $h_2 = h_1$ $\quad k_{11}^e = \int_{-1}^{1} (\xi-1/2)^2 \frac{2^2}{h} d\xi \qquad h = x_3^e - x_1^e$

$$= \frac{2}{3h}(\xi-1/2)^3 \Big|_{-1}^{1} = \frac{2}{3h}\left\{\frac{1}{8} + \frac{27}{8}\right\} = \frac{7}{3h}$$

$$k_{22} = \int_{-1}^{1} 4\xi^2 \cdot \frac{2}{h} = \frac{8}{3}\frac{\xi^3}{h} = \frac{16}{3h}$$

$$k_{12}^e = \int_{-1}^{1} (\xi-1/2)\cdot -2\xi\cdot\frac{2}{h}\,d\xi = \int_{-1}^{1} -\frac{4\xi^2}{h}d\xi = -\frac{4}{3}\frac{\xi^3}{h}\Big|_{-1}^{1} = -\frac{8}{3h}$$

$$k_{23} = \int_{-1}^{1} -2\xi^2 - 2\xi \cdot \frac{2}{h} = \int_{-1}^{1} -\frac{4\xi^2}{h} d\xi = -\frac{4}{3} = -\frac{8}{3h}$$

$$k_{13}^e = \int_{-1}^{1} \frac{4\xi^2-1}{4}\,\frac{2}{h}\,d\xi = \frac{1}{h}\left(\frac{4\xi^3}{3}-\xi\right)\Big|_{-1}^{1} = \frac{1}{3h}\left(\frac{2}{3}+\right) = \frac{1}{3h}$$

$k_{33} = \int (\xi + \tfrac{1}{2})^2 \tfrac{2}{h} = \frac{2}{3h}(\xi + \tfrac{1}{2})^3 \Big|_{-1}^{1} = \frac{2}{3h}\left(\frac{27}{8} + \frac{1}{8}\right) = \frac{7}{3h}$

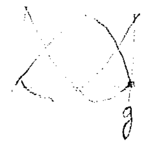
$$\underset{\sim}{k}^e = \frac{1}{3h}\begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix} \quad \text{if}$$

e=1	e=2
o o o	o o

3 node element.

$$f_a^e = \int_{x_1^e}^{x_3^e} N_a f + N_a(0) - g \int_{-1}^{1} N_{a,\xi}\, N_{3,\xi}\, \xi_x \, d\xi - \delta_{a1} h \leftarrow \text{for } e=1$$

$c_1 N_1 + c_2 N_2$



$$f^h = \sum_{b=1}^{3} N_b f_b^e$$

$$= \sum_{b=1}^{3} \int_{-1}^{1} N_a^e N_b^e \, d\xi \, x_{,\xi} \, f_b^e - \delta_{a,1} h - g \int_{-1}^{1}$$

$$f_1^e = \sum_{b=1}^{3} \int \frac{\xi(\xi-1)}{2} [\ N_b\ ](\xi a + b)\, f_b^e \quad - h \quad \text{for } e=1$$

$$f_2^1 = \sum \int 1-\xi^2 [\ N_b\ ](\xi a + b)\, f_b^e$$

$$f_3^1 = \sum \int \frac{\xi(\xi+1)}{2} [\ N_b\ ](\xi a + b) f_b^e$$

for $e \neq 1$ or n_{el}

$$f_1^{nel} = \sum_{b=1}^{2} \int_{-1}^{1} \frac{\xi(\xi-1)}{2} [\ N_b\ ](\xi a + b) f_b^e - g \int_{-1}^{1} \frac{2\xi - 1}{2} \cdot \frac{2\xi + 1}{2} \cdot \frac{d\xi}{\xi a + b}$$

$$f_2^{nel} = \sum_{b=1}^{2} \int_{-1}^{1} (1-\xi^2)[\ N_b\ ](\xi a + b) - g \int_{-1}^{1} (1-\xi^2) \frac{2\xi+1}{2} \frac{d\xi}{\xi a + b}$$

only need 2 since last element is not needed in assembly

$$\underset{\sim}{f} = \begin{pmatrix} \times & & \times \\ & 3\times 3 & \\ \times & & \times \end{pmatrix} \begin{pmatrix} f_1^e \\ f_2^e \\ f_3^e \end{pmatrix}$$

2e-1	2e	2e+1				
1	1	1	2	1	3	1
o	o	o	o	o	o	o
				5	6	7

$a = 1, 2, 3$

now

$F_1 = F_1 + f_1^1$

$F_2 = F_2 + f_2^1$

$F_3 = F_3 + f_3^1 + f_1^2$

$F_4 = F_4 + f_2^2$

$F_5 = F_5 + f_3^2 + f_1^3$

$F_6 = \ldots$

$\vdots$

$F_{2nel-1} = F_{2nel-1} + f_3^{nel-1} + f_1^{nel}$

$F_{2nel} = F_{2nel} + f_2^{nel}$

$F_p = F_p + f_a^e \quad LM(a,e) = \begin{cases} 2e-1 \\ 2e \\ 2e+1 \end{cases} = p$

$F_3 = F_3 + f_3^1 + f_1^2$

$F_4 = F_4 + f_2^2$

$F_5 = F_5 + f_3^2 + f_1^3 \quad LM(a, n_{el}) = \begin{cases} 2n_{el}-1 \\ 2n_{el} \\ 2n_{el}+1 \end{cases}$

$\vdots$

ME 235a
FINITE ELEMENT METHODS
WINTER QUARTER 1981

STANFORD UNIVERSITY
INSTRUCTOR: T.J.R. HUGHES

Final Exam – Solutions

1. Lagrange polynomial formula:

$$N_1 = \frac{(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)(\xi-\xi_5)}{(\xi_1-\xi_2)(\xi_1-\xi_3)(\xi_1-\xi_4)(\xi_1-\xi_5)}$$

$$= \frac{(\xi+1/2)(\xi-0)(\xi-1/2)(\xi-1)}{\underbrace{(-1+1/2)}_{-1/2}\underbrace{(-1-0)}_{-1}\underbrace{(-1-1/2)}_{-3/2}\underbrace{(-1-1)}_{-2}}$$

$$= \frac{2}{3}\xi(\xi+1/2)(\xi-1/2)(\xi-1)$$

$$N_2 = \frac{(\xi-\xi_1)(\xi-\xi_3)(\xi-\xi_4)(\xi-\xi_5)}{(\xi_2-\xi_1)(\xi_2-\xi_3)(\xi_2-\xi_4)(\xi_2-\xi_5)}$$

$$= \frac{(\xi+1)(\xi-0)(\xi-1/2)(\xi-1)}{\underbrace{(-1/2+1)}_{+1/2}\underbrace{(-1/2-0)}_{-1/2}\underbrace{(-1/2-1/2)}_{-1}\underbrace{(-1/2-1)}_{-3/2}}$$

$$= -\frac{8}{3}\xi(\xi+1)(\xi-1/2)(\xi-1)$$

$$N_3 = \frac{(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_4)(\xi-\xi_5)}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)(\xi_3-\xi_5)}$$

$$= \frac{(\xi+1)(\xi+1/2)(\xi-1/2)(\xi-1)}{(+1)(+1/2)(-1/2)(-1)}$$

$$= 4(\xi+1)(\xi+1/2)(\xi-1/2)(\xi-1)$$

2. $\displaystyle\int_{-1}^{+1} \xi^{i}\, d\xi = \begin{cases} \frac{2}{i+1} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$

Assume: $W_4 = W_1$, $W_3 = W_2$
$\xi_4 = -\xi_1$, $\xi_3 = -\xi_2$

$$\sum_{l=1}^{4} (\xi_l)^i W_l = \begin{cases} 2(W_1 \xi_1^i + W_2 \xi_2^i) & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

$$\therefore \frac{1}{i+1} = W_1 \xi_1^i + W_2 \xi_2^i, \quad i = 0, 2, 4, 6,$$

$$\left.\begin{aligned} 1 &= W_1 + W_2 \\ 1/3 &= W_1 \xi_1^2 + W_2 \xi_2^2 \\ 1/5 &= W_1 \xi_1^4 + W_2 \xi_2^4 \\ 1/7 &= W_1 \xi_1^6 + W_2 \xi_2^6 \end{aligned}\right\} \text{4 eqs / 4 unknowns}$$

$\xi_1 = .86113$

$W_1 = .34785$

$W_2 = 1 - W_1 = .65214$

$\xi_2 = [(1/3 - W_1 \xi_1^2)/W_2]^{1/2} = .33998$

3. Shape functions for the trilinear hexahedron:

$$N_a = (1+\xi_a\xi)(1+\eta_a\eta)(1+\zeta_a\zeta)/8 \quad , \quad a=1,2,\ldots,8$$

Shape functions for the pyramid:

$$N_a' = N_a \qquad a = 1,2,3,4$$

$$N_5' = N_5 + N_6 + N_7 + N_8$$

4. ID

dof # \ global node #	1	2	3	4	5	6
1	0	1	3	0	5	7
2	0	2	0	4	6	8

IEN

local node # \ element #	1	2	3	4
1	1	1	2	2
2	5	2	6	3
3	4	5	5	6

LM

local eq. #	local node #	dof #	1	2	3	4
1	1	1	0	0	1	1
2		2	0	0	2	2
3	2	1	5	1	7	3
4		2	6	2	8	0
5	3	1	0	5	5	7
6		2	4	6	6	8

5. a) $x(\xi) = \sum_{a=1}^{3} N_a(\xi) x_a^e$

$$= N_1 x_1^e + N_2 (x_1^e + x_3^e)/2 + N_3 x_3^e$$

$$= \frac{1}{2}(1-\xi) x_1^e + \frac{1}{2}(1+\xi) x_3^e$$

$$x_{,\xi} = h^e/2$$

$$f_a^e = \int_{x_1^e}^{x_3^e} N_a \mathcal{f} \, dx = \mathcal{f} \int_{-1}^{+1} N_a(\xi) x_{,\xi} \, d\xi$$

$$= \frac{\mathcal{f} h^e}{2} \int_{-1}^{+1} N_a(\xi) \, d\xi$$

$$\underset{\sim}{f}^e = \{ f_a^e \} = \frac{\mathcal{f} h^e}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

b) $f_a^e = \int_{x_1^e}^{x_3^e} N_a(x) \, \delta(x - \bar{x}) \, dx = N_a(\bar{x})$

if $\bar{x} = x_1^e$, $\quad f_a^e = N_a(x_1^e) = \delta_{a1}$

if $\bar{x} = x_2^e$, $\quad f_a^e = N_a(x_2^e) = \delta_{a2}$

c) $f_a^e = \mathcal{f} \int_{-1}^{+1} \underbrace{\underbrace{N_a(\xi)}_{\text{quad.}} \underbrace{x_{,\xi}(\xi)}_{\text{linear}}}_{\text{cubic}} \, d\xi$

$\therefore n_{int} = 2.$

5. cont'd

d) $$k^e_{ab} = \int_{x_1^e}^{x_3^e} N_{a,x} N_{b,x}\, dx$$

$$= \int_{-1}^{+1} N_{a,\xi} N_{b,\xi} (\xi_{,x})^2 x_{,\xi}\, d\xi$$

$$= \underbrace{\frac{2}{h^e}}_{\text{part a}} \int_{-1}^{+1} \underbrace{\underbrace{N_{a,\xi}(\xi)}_{\text{linear}} \underbrace{N_{b,\xi}(\xi)}_{\text{linear}}}_{\text{quad.}} d\xi$$

$\therefore\ n_{int} = 2.$

e) $e = u^h - u$

$e_{,x} = e_{,\xi}\, \xi_{,x} = e_{,\xi}\, 2/h^e$ (part a)

<u>simple sol.:</u> observe if $u(\xi)$ is a quadratic poly, then $u^h(\xi) = u(\xi)$. Therefore, take $u(\xi) = \text{const.}\ \xi^3$.

$e_{,\xi} = u^h_{,\xi} - u_{,\xi} = \left(-\cancel{\xi} + \tfrac{1}{2} + \cancel{\xi} + \tfrac{1}{2} - 3\xi^2\right)\text{const}$

$e_{,\xi} = 0$ at $\xi = \pm 1/\sqrt{3}$.

or else grind it out; coefficient of $u_{,\xi\xi\xi}$ term (obviously) leads to same result

6. $$\int_\Omega w_{(i,j)}\,\sigma_{ij}\,d\Omega = \int_\Omega w_i f_i\,d\Omega + \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i\,d\Gamma \right)$$

$$\sigma_{ij} = c_{ijkl}\,(\varepsilon_{kl} - \varepsilon^0_{kl}) + \sigma^0_{ij}$$

$$\varepsilon^0_{kl} = \Theta\, c_{kl}$$

$$\int_\Omega w_{(i,j)}\, c_{ijkl}\, \varepsilon_{kl}\, d\Omega =$$

$$\int_\Omega w_i f_i\,d\Omega + \sum_{i=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_i h_i\,d\Gamma \right)$$

↑ as before

↓ new contrib'n to rhs

$$+ \int_\Omega w_{(i,j)}\, c_{ijkl}\, \varepsilon^0_{kl}\, d\Omega$$

$$- \int_\Omega w_{(i,j)}\, \sigma^0_{ij}\, d\Omega$$

$w_{3,1}\sigma_{31} + w_{3,2}\sigma_{32}$

$$f^e_p = \dots + e_i^T \int_{\Omega^e} B_a^T\, D\, \Theta\, c\, d\Omega - e_i^T \int_{\Omega^e} B_a^T\, \sigma^0\, d\Omega$$

↑ any function inside integral

e.g. $c = \begin{Bmatrix} c_{11} \\ c_{22} \\ 2c_{12} \end{Bmatrix}$; $\sigma^0 = \begin{Bmatrix} \sigma_{11}^0 \\ \sigma_{22}^0 \\ \sigma_{12}^0 \end{Bmatrix}$; etc.

without loss of generality, may assume symm., i.e. $c_{12} + c_{21} = 2c_{12}$

1	25	c	25
2	25	d	25
3	25	e	0
4	25	f	25
5a	25	6	35
b	25		260

COURSE A

1. **25** $l_1^4 = \dfrac{\Pi(\xi+\frac{1}{2})(\xi)(\xi-\frac{1}{2})(\xi-1)}{\Pi(-1+\frac{1}{2})(-1-0)(-1-\frac{1}{2})(-1-1)} = \dfrac{(\xi^2-\frac{1}{4})(\xi-1)\xi}{(-\frac{1}{2})(-1)(-\frac{3}{2})(-2)} = \boxed{\frac{2}{3}\xi(\xi-1)(\xi^2-\frac{1}{4}) = l_1^4}$

$l_2^4 = \dfrac{(\xi+1)(\xi)(\xi-\frac{1}{2})(\xi-1)}{(-\frac{1}{2}+1)(-\frac{1}{2}-0)(-\frac{1}{2}-\frac{1}{2})(-\frac{1}{2}-1)} = \boxed{\dfrac{-8(\xi^2-1)(\xi)(\xi-\frac{1}{2})}{3} = l_2^4}$

$\frac{1}{2} \quad -\frac{1}{2} \quad -1 \quad -\frac{3}{2}$

$l_3^4 = \dfrac{(\xi+1)(\xi+\frac{1}{2})(\xi-\frac{1}{2})(\xi-1)}{(0+1)(0+\frac{1}{2})(0-\frac{1}{2})(0-1)} = \boxed{4(\xi^2-1)(\xi^2-\frac{1}{4}) = l_3^4}$

2. **25** $\bar\xi_1 \quad \bar\xi_2 \quad \bar\xi_3 \quad \bar\xi_4$ (on -1 to 1)

$\int_{-1}^{1} g\,d\xi = \int_{-1}^{1}(\alpha_0+\alpha_1\xi+\alpha_2\xi^2+\alpha_3\xi^3+\alpha_4\xi^4+\alpha_5\xi^5+\alpha_6\xi^6+\alpha_7\xi^7)\,d\xi$ ✓

let $\bar\xi_1 = -\bar\xi_4 \quad \bar W_1 = \bar W_4$

$\bar\xi_2 = -\bar\xi_3 \quad \bar W_2 = \bar W_3$

$= 2\alpha_0 + \frac{2\alpha_2}{3} + \frac{2}{5}\alpha_4 + \frac{2}{7}\alpha_6$

$\Sigma g(\xi)W = (2\alpha_0 + 2\alpha_2\bar\xi_1^2 + 2\alpha_4\bar\xi_1^4 + 2\alpha_6\bar\xi_1^6)W_1 + (2\alpha_0 + 2\alpha_2\bar\xi_3^2 + 2\alpha_4\bar\xi_3^4 + 2\alpha_6\bar\xi_3^6)W_2$

equating the two $\Rightarrow$ ① $W_1+W_2=1$ ✓ ; ② $\frac{1}{3} = \bar\xi_1^2 W_1 + \bar\xi_3^2 W_2$ ③ $\frac{1}{5} = \bar\xi_1^4 W_1 + \bar\xi_3^4 W_2$

④ $\frac{1}{7} = \bar\xi_1^6 W_1 + \bar\xi_3^6 W_2$

$\boxed{W_2 = 1 - .34785 = .65215}$ ✓

Haven't got the time but using W_2 & the remaining 3 eqns get ξ_1 in terms of ξ_3 ?

? ONLY NEED ONE

25

3. $N_1 = \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)$

$N_2 = \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)$

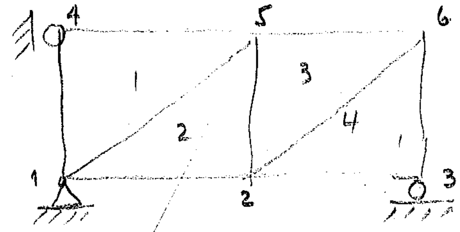
$N_3 = \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)$

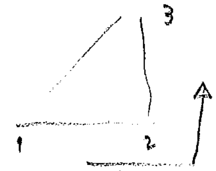
$N_4 = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)$

$N_5' = N_6+N_5+N_7+N_8 = \frac{1}{4}(1-\eta)(1+\zeta) + \frac{1}{4}(1+\eta)(1+\zeta)$

$N_5' = \frac{1}{4}(1+\zeta)\cdot 2 = \boxed{\frac{1+\zeta}{2} = N_5'}$ ✓

25





4.

ID

	1	2	3	4	5	6
1	0	1	3	0	5	7
2	0	2	0	4	6	8

IEN

a \ e	1	2	3	4
1	1	1	2	2
2	5	2	6	3
3	4	5	5	6

LM

p	a	i	1	2	3	4
1	1	1	0	0	1	1
2		2	0	0	2	2
3	2	1	5	1	7	3
4		2	6	2	8	0
5	3	1	0	5	5	7
6		2	4	6	6	8

5. $f_a^e = \int_{x_1^e}^{x_3^e} f N_a \, dx = f \int_{-1}^{1} N_a(\xi)\, d\xi \; x_{,\xi}$; $x_{,\xi} = \xi\,(x_3^e + x_1^e - 2x_2^e) + \frac{x_3^e - x_1^e}{2}$

a. 25 if $x_2^e = (x_1^e + x_3^e)/2$ then $x_{,\xi} = \frac{h^e}{2} = \frac{x_3^e - x_1^e}{2}$ ✓

$\Rightarrow \quad f_a^e = \frac{f h^e}{2} \int_{-1}^{1} N_a(\xi)\, d\xi$

$f_1^e = \frac{f h^e}{4} \int_{-1}^{1} (\xi^2 - \xi)\, d\xi = \frac{2 f h^e}{3 \cdot 4} = \boxed{\frac{f h^e}{6} = f_1^e}$ ✓

$f_2^e = \frac{f h^e}{2} \int_{-1}^{1} (1 - \xi^2)\, d\xi = \frac{f h^e}{2} \left(\xi - \frac{\xi^3}{3}\right)\Big|_{-1}^{1} = \frac{f h^e}{2}\left[\frac{2}{3} - \left(-\frac{2}{3}\right)\right] = \boxed{\frac{2}{3} f h^e = f_2^e}$ ✓

$f_3^e = \frac{f h^e}{4} \int_{-1}^{1} (\xi^2 + \xi)\, d\xi = \frac{f h^e}{4} \left(\frac{\xi^3}{3} + \frac{\xi^2}{2}\right)\Big|_{-1}^{1} = \frac{f h^e}{4} \cdot \frac{2}{3} = \boxed{\frac{f h^e}{6} = f_3^e}$ ✓

b. 25 $f_a^e = \int_{x_1^e}^{x_3^e} \delta(x - \bar{x})\, N_a(x)\, dx = N_a(\bar{x})$ ✓

when $\bar{x} = x_1^e$	$N_a(x_1^e) = \delta_{a1}$	$= f_a^e$
when $\bar{x} = x_2^e$	$N_a(x_2^e) = \delta_{a2}$	$= f_a^e$

✓ let $x_3 - x_2 = h_2$ $x_2 - x_1 = h_1$

c. 25 $f_a^e = f \int_{x_1^e}^{x_3^e} N_a\, dx = f \int_{-1}^{1} N_a(\xi) \left[\xi (h_2 - h_1) + \frac{h_2 + h_1}{2}\right] d\xi$. when $x_3 + x_1 - 2x_2 \neq 0$ $h_2 - h_1 \neq 0$

$= f \int_{-1}^{1} g(\xi)\, d\xi$; since $g(\xi)$ is cubic then using $n_{int} = 2$ integrates ~~interpolates~~ $g(\xi)$

which is $O(x^{2 int - 1})$ exactly

d. 25 $\underset{\sim}{k}^e = \int_{x_1^e}^{x_e^3} N_{a,x}(x)\, N_{b,x}(x)\, dx = \int_{-1}^{1} N_{a,\xi}\, N_{b,\xi}\, \xi_{,x}^2\, d\xi \cdot x_{,\xi}$

$= \int_{-1}^{1} N_{a,\xi}\, N_{b,\xi}\, \xi_{,x}\, d\xi \qquad \xi_{,x} = \frac{2}{h_e}$ for $h_2 - h_1 = 0$

now $N_{a,\xi}$ is linear $\therefore$ $g(\xi) = N_{a,\xi} N_{b,\xi} \xi_{,x}$ is quadratic

thus lowest order must be for $n_{int} = 2$ for the same reason as c

e. $u^h(\xi) = \sum N_a(\xi)\, u^h(x_a)$

$$u^h_{,x} = u^h_{,\xi}\,\xi_{,x} = u^h_{,\xi}\frac{2}{h_e} = \sum N_{a,\xi}(\xi)\, u^h(x_a)\cdot\frac{2}{h_e}$$

$$u^h_{,x} = \frac{2}{h_e}\left[\frac{2\xi-1}{2}u^h_1 - 2\xi u^h_2 + \frac{2\xi+1}{2}u^h_3\right]$$ with $u_i = u(x_i^e)$; since $u^h_i \in \mathcal{S}^h \subset \mathcal{S}$ then

$u^h_i = u_i$ at nodes $\Rightarrow$ $u_i = u + (x_i - x)\,u_{,x} + \frac{(x_i-x)^2}{2}\,u_{,xx} + \dots$

now since $x_1 + x_3 - 2x_2 = 0 \Rightarrow \xi = 2(x-x_2)/h_e$

need to look at $u_{,xxx}$

thus

$$u^h_{,x} = \frac{2}{h_e}\Big[\overset{N_{1,\xi}}{(\xi-\tfrac12)}u_1 - \overset{N_{2,\xi}}{2\xi}\,u_2 + \overset{N_{3,\xi}}{(\xi+\tfrac12)}u_3\Big] = \frac{2}{h_e}\Big[\xi(u_1-2u_2+u_3) + \tfrac12(u_3-u_1)\Big]$$

now $u_1 - 2u_2 + u_3 = \Big[(x_1-x) - 2(x_2-x) + (x_3-x)\Big]u_{,x} + \frac12\Big[(x_1-x)^2 - 2(x_2-x)^2 + (x_3-x)^2\Big]u_{,xx} + \dots$ (first bracket $\to 0$)

$u_3 - u_1 = (x_3-x_1)\,u_{,x} + \frac{(x_3-x)^2 - (x_1-x)^2}{2}\,u_{,xx} + \dots$

plugging into $u^h_{,x}$

$$= \frac{2}{h_e}\left[\frac{2}{h_e}\frac{x-x_2}{2}\left[(x_1-x)^2 - 2(x_2-x)^2 + (x_3-x)^2\right]u_{,xx} + \dots + \tfrac12(x_3-x_1)\,u_{,x} + \tfrac12\left[\;\;\right] + \text{h.o.t}\right]$$

collecting

$$= u_x + \left[\frac{2}{h_e^2}(x-x_2)\left[(x_1-x)^2 - 2(x_2-x)^2 + (x_3-x)^2\right]u_{,xx} + \frac{4}{h_e^2}\left[(x_3-x)^2 - (x_1-x)^2\right]u_{,xx}\right]$$

Now $u^h_{,x} - u_x = \frac{2}{h_e^2}\Big\{(x-x_2)\Big[(x_1-x)^2 - 2(x_2-x)^2 + (x_3-x)^2\Big] + (x_3-x_1)\Big[(x_3-x)^2 - (x_1-x)^2\Big] + \dots\Big\}$

$= u_{,xx}\frac{2}{h_e^2}\Big\{(x-x_2)\Big[(x_1-x_2)(x_1+x_2-2x) + (x_3-x_2)(x_3+x_2-2x)\Big] + (x_3-x_1)\Big[(x_3-x_1)(x_3+x_1-2x)\Big]$

$= \frac{2u_{,xx}}{h_e^2}\Big[(x-x_2)\Big[(x_1^2-x_2^2) + (x_3^2-x_2^2)\Big] - 2x(x_1+x_3-2x_2) + (x_3-x_1)^2(x_3+x_1-2x)\Big]$ ($x_1+x_3-2x_2 \to 0$)

✗

thus $e_{,x}^h$ has one one barlow point namely $x = x_2$?

coef. of $u_{,xx} \equiv 0$

f. 25 $u^h(\xi) = \sum N_a(\xi)\, u^h(x_a)$

$u^h_{,r}(\xi) = u^h_{,\xi}(\xi)\,\xi_{,x}\,x_{,r}$ for the givens $\xi_{,x} = \frac{2}{h_e\sqrt{r}}$ ✗ $\therefore x_{,r} = h_e$ ✓

$= u^h_{,\xi}(\xi)\cdot\frac{2}{\sqrt{r}}$ ✗ ✓ $\therefore O(r^{-\alpha}) \Rightarrow$ $\boxed{\alpha = 1/2}$ ✓

$\xi = \xi(r) = 2\sqrt{r} - 1$

35

6. from our formulation we had that $\int_\Omega w_{(i,j)}\sigma_{ij}\,d\Omega = \int_\Omega w_i f_i\,d\Omega + \int_\Gamma w_i \sigma_{ij} n_j\,d\Gamma$ [$\sigma_{ij} n_j$ crossed out, h_i written above]

now $\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon^0_{kl}) + \sigma^0_{ij}$ where ε^0_{kl} & σ^0_{ij} are given

$\Rightarrow$

$$\int_\Omega u_{(i,j)} C_{ijkl} u_{(k,l)}\,d\Omega = \int_\Omega w_{(i,j)} C_{ijkl}\varepsilon^0_{kl}\,d\Omega - \int_\Omega w_{(i,j)}\sigma^0_{ij}\,d\Omega + \int_\Omega w_i f_i\,d\Omega + \int_\Gamma w_i h_i\,d\Gamma$$

+ 25

---

let $\sigma^0_{ij} \times \sigma^0(\underset{\sim}{x}) = \underset{\sim}{D}(\underset{\sim}{x})\,\underset{\sim}{B}(\underset{\sim}{x})\,\underset{\sim}{d}^{e^0}$ ← you are given σ^0_{ij}

$\varepsilon^0_{kl} = -\Theta c_{kl}$ ✗ $-\Theta\,\underset{\sim}{\epsilon}(\underset{\sim}{u}^0)$ where is like the "strain vector" discussed in class

↑ given

then

$$f^e_p = -\underset{\sim}{e}_i^T \int_{\Omega^e} \underset{\sim}{B}_a^T \Theta \underset{\sim}{D} \underset{\sim}{c}^0\,d\Omega - \underset{\sim}{e}_i^T \int_{\Omega^e} \underset{\sim}{B}_a^T (\underset{\sim}{D}\,\underset{\sim}{B}\,\underset{\sim}{d}^{e^0})\,d\Omega + \int_{\Omega^e} w_i f_i\,d\Omega + \int_{\Gamma^e_{h_i}} w_i h_i\,d\Gamma$$

[$\underset{\sim}{B}\,\underset{\sim}{d}^e$ circled and crossed out, labelled $\underset{\sim}{\varepsilon}^0$; $\underset{\sim}{D}\,\underset{\sim}{B}\,\underset{\sim}{d}^{e^0}$ labelled $\underset{\sim}{\sigma}^0$]

10

-15

+ g term

$$\underset{\sim}{e}_i^T \int \underset{\sim}{B}_a^T \underset{\sim}{D}\,\Theta\,\underset{\sim}{C}^0\,d\Omega$$

there may not exist compatible disp. fields corresponding to $\underset{\sim}{\varepsilon}^0, \underset{\sim}{\sigma}^0$.