4/4/79

ME 207

Books

Ref: Cole, Pert. Methods in Appl. Mech 1968 } Reserved @ Eng. Lib.
Nayfeh, Pert. Methods 1973.

Midterm, Final + HW (8-9 sets of HW due weekly)

Office Hr: MWF 11-12 F 2-3 + By Appt.

Perturbation Methods are normally done using parameter expansion ($\varepsilon \to 0$ etc) or coordinate expansion (ie $t \to 0$ Remember expansion of $a(t)$ as $t \to 0$ in Prof. Herrmann's 3 Engineering exercises for ME291).

When we need an approximation in a systematic method that will allow us to obtain higher order approx. then it is known as a rational approximation. If we cannot, it is known as an irrational approximation, (ie. ad hoc approx.)

What we will cover:

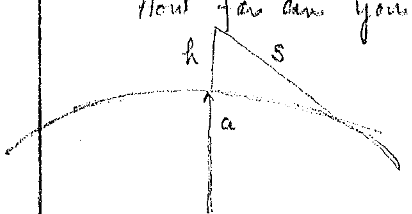
1. Regular Perturbation (single expansion is valid throughout region of interest)

most of the time we find we must use singular

2. Slow variation is a method of transition between Regular Pert. & Singular Pert.
3. Singular pert. (method of matched asymptotic expansions) Variations in one of the scales is matched at some point ie $y \sim O(\delta)$ $x \sim O(L)$ in b.L. & $y \sim O(L)$ outside b.L.

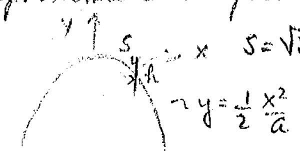
REGULAR PERTURBATIONS

How far can you see from the mast of a ship



Exactly $S = a \cos^{-1} \frac{a}{a+h}$

Approximate the problem by a earth by parabolic surface



$S = \sqrt{2ah}$

$y = \frac{1}{2}\frac{x^2}{a}$

Approximate answer by $S = a \cos^{-1} \frac{1}{1 + a/h} = \sqrt{2ah}\left[1 - \frac{5}{12}\frac{h}{a} + O\left(\frac{h^2}{a^2}\right)\right]$

4/6/79

Categories

Rational or Irrational

Coordinate & parameter perturb

Regular or Singular

Convergent or divergent series (Asymptotic)

Example

Exponential integral $E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt = -\ln x - \gamma - \sum \frac{(-1)^n x^n}{n\, n!}$, $x < \infty$

$\gamma = .57721\ldots$

need 15 terms for 4 digit accuracy this is a series expansion

using integration by parts we can get $\frac{e^{-x}}{x}\left[1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots\right]$, $x \to \infty$

this is the asymptotic expansion it is "semi convergent"; terms decrease up to a point and then start increasing

a better approximation is $\frac{e^{-x}}{x}\left[\frac{x^2 + a_1 x + a_2}{x^2 + b_1 x + b_2}\right] + \varepsilon(x)$, $1 < x < \infty$ is a rational approximation where $\varepsilon(x)$ can not be better than 5×10^{-5}

using the following notation $\sim$ means asymptotic

$\approx$ approximately equal to

if $f(\varepsilon)$ is $O(\varepsilon^n)$ that means $\lim_{\varepsilon\to 0} \frac{f(\varepsilon)}{\varepsilon^n} = A$ where $A < \infty$

Parameter Perturbation

Iteration or Assume series   HOW TO get an expansion

Quasi linear ODE (Nayfeh's Ex 1.10)

$\frac{dx}{dt} + x = \varepsilon x^2$ $x(0) = 1$

Approximate for small ε: $\varepsilon \ll 1$. [Formally, we let $\varepsilon \to 0$]

1st Approx let $\varepsilon = 0$ $\frac{dx^I}{dt} + x^I = 0$ $x^I(0) = 1$ $x^I = e^{-t}$

2nd Approx, use iteration or series

Iteration: use 1st approx to get 2nd approx

$\frac{dx^{II}}{dt} + x^{II} = \varepsilon x^{I2} = \varepsilon e^{-2t}$ $x^{II}(0) = 1$

$$\therefore \quad X^{II} = A\overset{\text{homog}}{e^{-t}} - \overset{\text{Partic}}{\varepsilon e^{-2t}} \qquad x(0) = 1 = A - \varepsilon \qquad \therefore \ A = 1+\varepsilon$$

$$\therefore \quad e^{-t} + \varepsilon(e^{-t} - e^{-2t}) = X^{II}$$

$\therefore$ we can use this to get $\frac{dX^{III}}{dt} + X^{III} = \varepsilon X^{II^2} = \varepsilon\left[(1+\varepsilon)e^{-t} - \varepsilon e^{-2t}\right]^2$ w/ $X^{III}(0) = 1$

it is inefficient since each time we solve complete problem over.

$$\frac{dX^{III}}{dt} + X^{III} = \varepsilon e^{-2t} + \varepsilon^2[\quad] + [\quad]$$

(first bracket) significant since they remain unchanged w/ next approx.

(second bracket) meaningless since they will be changed in next approximation

Alternatively, substitute assumed expansion (for small ε)

① Iteration suggest $x(t;\varepsilon) \sim x_1(t) + \varepsilon x_2(t) + \varepsilon^2 x_3(t) + \dots$

② $x(0;\varepsilon) \sim x_1(0) + \varepsilon x_2(0) + \varepsilon^2 x_3(0) + \dots \Rightarrow$ $x_1(0) = 1$, $x_2(0) = x_3(0) = 0$ etc

③ substitute expansion into DE

$$\left(\frac{dx_1}{dt} + \varepsilon\frac{dx_2}{dt} + \varepsilon^2\frac{dx_3}{dt} + \dots\right) + \left(x_1 + \varepsilon x_2 + \varepsilon^2 x_3 + \dots\right) = \varepsilon\left[x_1^2 + 2\varepsilon x_1 x_2 + \dots\right]$$

④ equate like powers of ε

$$\varepsilon^0: \ \frac{dx_1}{dt} + x_1 = 0 \quad x_1(0) = 1; \quad \varepsilon^1: \ \frac{dx_2}{dt} + x_2 = x_1^2 \quad x_2(0) = 0$$

$$\varepsilon^2: \ \frac{dx_3}{dt} + x_3 = 2x_1x_2 \quad x_3(0) = 0 \quad \text{etc.}$$

now use bc for corresponding power of ε

This gives infinite set of d.e. $\quad x_1 = e^{-t} \quad x_2 = e^{-t}(1 - e^{-t}) \quad x_3 = e^{-t}[1 - e^{-t}]^2$

thus $x(t;\varepsilon) = \dfrac{e^{-t}}{1 - \varepsilon(1 - e^{-t})}$

4/9/79

showed mostly examples of expansions ie around cylinders Jantzen - Rayleigh M^2 expansion for subsonic flow ; oseen flow - drag of sphere at low Reynolds no.

Also put down picard method as it becomes cumbersome & it is simpler to assume $x(t;\varepsilon) = \sum_{n=0}^{\infty} t^n X_n(\varepsilon)$

1st bal

look at $\varepsilon x^2 + x - c = 0$

2nd bal

4/11/79

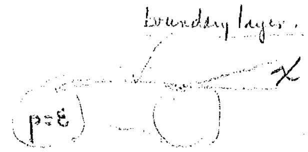
if ε was small we approximated $x - c = 0$
if we want 2nd root we can assume that
$$X = \varepsilon^m x \Rightarrow \frac{\varepsilon}{\varepsilon^{2m}} X^2 + \frac{X}{\varepsilon} - c = 0$$
$$\Rightarrow m = 1$$

$$\cos\sqrt{t}\,\frac{dx}{dt} - \frac{x}{t} = 1 \quad \Rightarrow \quad t\cos\sqrt{t}\,\frac{dx}{dt} - x = t$$
$$t\left(1 - \tfrac{1}{2}t + \frac{t^2}{24} + \cdots\right)\frac{dx}{dt} - x = t$$

Prof. Van Dyke's method
$$\frac{dx_n}{dt} - \frac{x_n}{t} = 1 + \left(1 - \cos\sqrt{t}\right)\frac{dx_{n-1}}{dt} \qquad x_0 = 0$$

Typical perturbation Series : expect the following

0. Integral powers (as for regular perturb)
1. Fractional " (in singular perturb)
   ie Prandtl Boundary Layer $\delta \sim Re^{-1/2}$
   pressurized toroidal shell $x = c_1 \varepsilon^{3/4} + \ldots$



2. Logarithm w/ powers
   C_D over sphere $\quad C_D = \frac{6}{R}\left[1 + \frac{3}{8}R + c_1 R^2 \log R + c_2 R^2 + \ldots\right]$

3. Logarithms in denominator
   flow on circular cylinder gives C_D
   electrostatic capac. of long slender rod
$$\frac{c_1}{\ln\varepsilon} + \frac{c_2}{(\ln\varepsilon)^2} + \frac{c_3}{(\ln\varepsilon)^3} + \ldots + O\left(\frac{\varepsilon}{\ln\varepsilon}\right) + \ldots$$

4-13-79

$\sin\varepsilon = o(\cos\varepsilon)$ means $\frac{\sin\varepsilon}{\cos\varepsilon} \to 0$ as $\varepsilon \to 0$

$\sin\varepsilon = O(\varepsilon)$ means $\frac{\sin\varepsilon}{\varepsilon} \to$ finite no as $\varepsilon \to 0$

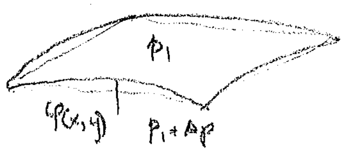
an asymptotic series will converge at some point & start diverging afterwards
we must pick cut off point since error in asymptotic series is O(next term)

Regular Perturbations

Slight deflection of pressurized membrane

Torsion of cylinder

Poiseuille Flow: Laminar flow through cylinder



DE $= -\Delta p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ where R_1, R_2 are radii of curvature

$w/ \frac{1}{R} = \frac{-y''}{(1+y'^2)^{3/2}}$ & slight deflection $y' \ll 1$

$\therefore \frac{1}{R} \sim -y''$ $\quad\therefore\quad -\Delta p = -\sigma(\varphi_{,xx} + \varphi_{,yy})$ or non dim: $\nabla^2\varphi = -1$

Torsion of cylinder let $\tau_{xz} = \partial\varphi/\partial y$ $\quad \tau_{yz} = \frac{\partial\varphi}{\partial x}$ Prandtl stress function

$\nabla^2\varphi = -2G\alpha$ $\quad\alpha$ is the twist

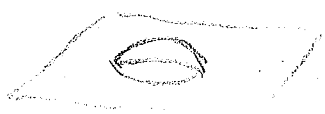
for a circle

basic soln $\phi = 1/4\,(1-r^2)$ for $\phi = 0$ on bdy of circle $w/\, r=1$

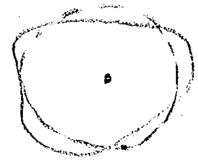
Infinite strip $\phi = 1/2\,(1-x^2)$

Perturbation may come into play in any of 3 ways

1. Perturbed DE
2. Perturbed BC
3. " Location of bc (this is hardest)



BC.



Location

Perturbed DE. Torsion of a slightly non isotropic shaft. (aelotropic

Using principal axes $A'\phi_{xx} + B'\phi_{yy} = 1$

for slight non isotropy on a dowel

$$(1-\varepsilon)\phi_{xx} + \phi_{yy} = -1 \qquad \phi = 0 \text{ on unit circle}$$

when $\varepsilon = 0$ $\quad \phi_0 = 1/4\,(1-r^2) = 1/4\,(1-x^2-y^2)$

Second approx: either write

$$\phi^{(1)}_{xx} + \phi^{(1)}_{yy} = 1 + \varepsilon\phi^{(0)}_{xx}$$

or let $\Phi(x,y;\varepsilon) = \phi_0(x,y) + \varepsilon\phi_1(x,y)$

where appx upon

$$\phi_{1,xx} + \phi_{1,yy} = -1 \qquad \varphi_1 = 0 \text{ on circle} \qquad \varphi_1 = \tfrac{1}{2}(1-r^2)$$

$$\phi_{2,xx} + \phi_{2,yy} = -1 + \varphi_{1,xx} = -\tfrac{1}{2} \qquad \varphi_2 = 0 \text{ on circle}$$

$$\phi_2 = \tfrac{1}{2}\phi_1 \qquad \therefore \qquad \phi = \tfrac{1}{4}(1-x^2-y^2)\left(1 + \tfrac{\varepsilon}{2} + \dots\right)$$

we can show that $\phi_{i,xx} + \phi_{i,yy} = -1 + \varphi_{i-1,xx} = -\left(\frac{1}{2}\right)^{i-1}$

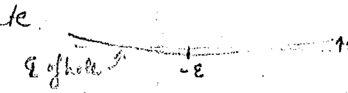
$$\therefore \quad \phi = \frac{1}{4}\,\frac{1}{1-\frac{1}{2}\varepsilon}\,(1-r^2)$$

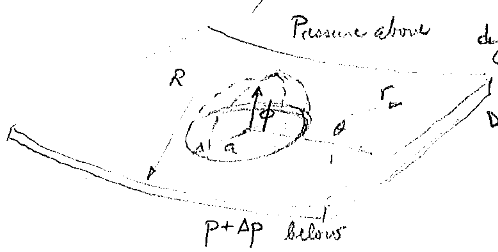
4/16/79

Perturbation of Boundary Conditions

a) BC is modified in its value § 2.4.5 of notes

Soap bubble spanning circular hole in slightly flexed plate





define $\epsilon = a/R$

DE: $\nabla^2\phi = -1$ where $a=1$; w/slight curvature of plate then

$$\phi = \varepsilon\cos 2\theta + O(\varepsilon^2) \quad \text{at } r = 1 + O(\varepsilon^2) \text{ on bdy}$$

Basic soln $\varepsilon = 0$: $\phi_0 = \frac{1}{4}(1-r^2)$

Assume $\phi(r,\theta;\varepsilon) = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots$

Substitute & equate ε $\qquad \nabla^2\phi = \nabla^2\phi_0 + \varepsilon\nabla^2\phi_1 + \dots = -1 + 0\cdot\varepsilon + 0\cdot\varepsilon^2$

$$\therefore \quad \nabla^2\phi_1 = 0 \qquad \therefore \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\phi = 0$$

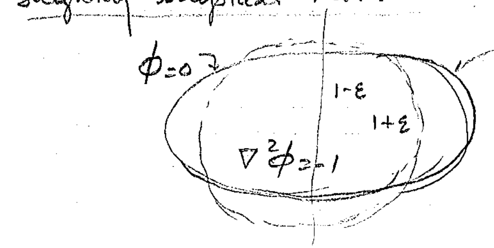
B.C.: $\phi = \varepsilon\cos 2\theta = \phi_0 + \varepsilon\phi_1 + \dots$ but $\phi_0 = 0 \;\therefore\; \phi_1 = \cos 2\theta$

Soln is $\phi_1 = r^2\cos 2\theta$ since $r^{\pm n}\begin{matrix}\sin(n\theta)\\ \cos(n\theta)\end{matrix}$ is harmonic

$$\therefore \text{ deflection } \phi = \tfrac{1}{4}(1-r^2) + \varepsilon r^2\cos 2\theta + O(\varepsilon^2)$$

b) BC is modified in location in § 2.5
Transfer of bound. cond. when position is perturbed

Slightly ellyptical section



$r = 1 + \varepsilon \cos 2\theta + \dots$

$$\frac{x^2}{(1+\varepsilon)^2} + \frac{y^2}{(1-\varepsilon)^2} = 1$$

$$x^2(1 - 2\varepsilon + \dots) + y^2(1 + 2\varepsilon + \dots) = 1$$

$$x^2 + y^2 = 1 + 2\varepsilon(x^2 - y^2) + \text{h.o.t.}$$

$$r^2 = 1 + 2\varepsilon r^2 \cos 2\theta \qquad \text{since } r \sim = 1 \text{ then}$$

$$r \simeq \sqrt{1 + 2\varepsilon\cos 2\theta} \sim 1 + \varepsilon\cos 2\theta$$

Assume a regular perturb expansion

$$\phi(r,\theta;\varepsilon) = \phi_0 + \varepsilon\phi_1(r,\theta) + O(\varepsilon^2) \qquad \phi_0 = \tfrac{1}{4}(1-r^2) \text{ for } \nabla^2\phi_0 = -1$$

Subst. $\nabla^2\phi = \nabla^2\phi_0 + \varepsilon\nabla^2\phi_1 + \dots = 0$

$$\Rightarrow \nabla^2\phi_1 = 0 \qquad \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\theta = 0$$

BC. $\phi = 0$ at $r = 1 + \varepsilon\cos 2\theta + \dots$

$$\phi = \tfrac{1}{4}\left[1 - (\underbrace{1 + \varepsilon\cos 2\theta}_{r^2})^2\right] + \varepsilon\phi_1(1 + \varepsilon\cos 2\theta + \dots, \theta) + \dots = 0$$

Must extract ε explicitly in ϕ_1

Assume ϕ_1 is analytic in r then expand in taylors series about $r=1$

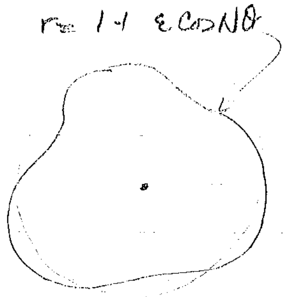
bc @ $r = 1 + \varepsilon\cos 2\theta$: $\phi = \frac{-1}{4}\left[(2 + \varepsilon\cos 2\theta)\varepsilon\cos 2\theta\right] + \varepsilon\left\{\phi_1(1,\theta) + \varepsilon\cos 2\theta\, \phi_{1,r}(1,\theta) + \dots\right\} = 0$

$\therefore \quad \phi_0 = 0 \qquad 0 = -\frac{1}{2}\cos 2\theta + \phi_1(1,\theta)$; etc $O(\varepsilon^2)$

$\therefore \quad \phi_1(1,\theta) = +\frac{1}{2}\cos 2\theta \qquad \therefore \quad \phi_1(r,\theta) = +\frac{1}{2}r^2\cos 2\theta$ using S.O.V.

$\therefore \quad \phi = \frac{1}{4}(1 - r^2) + \frac{\varepsilon}{2}r^2\cos 2\theta + O(\varepsilon^2)$

for $r = 1 + \varepsilon\cos N\theta$



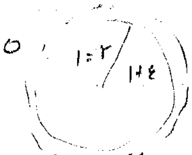
$$\phi = \tfrac{1}{4}(1 - r^2) + \tfrac{1}{2}\varepsilon r^N\cos N\theta + O(\varepsilon^2)$$

$N = 0 \quad \phi = \frac{1}{4}(1 - r^2) + \frac{1}{2}\varepsilon$

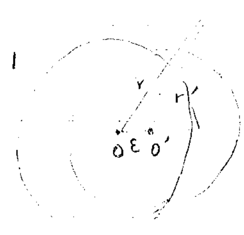
Exact soln $\quad r = 1 + \varepsilon \quad \therefore \phi = \frac{1}{4}\left[(1+\varepsilon)^2 - r^2\right]$

$\sim \frac{1}{4}(1 + 2\varepsilon - r^2) + \dots$

for N=0



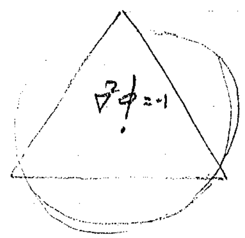
for N=1



$r'^2 = r^2 + 2\varepsilon r\cos\theta + \varepsilon^2$

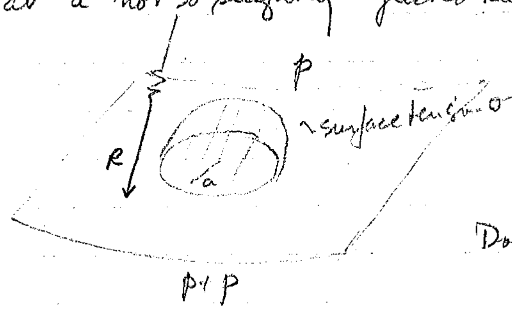
Exact Sol $= \frac{1}{4}(1 - r'^2) = \frac{1}{4}(r^2 + 2\varepsilon r\cos\theta + 1)$

use for Δ shape ∋. area of circle = area of Δ



Φ = product of each of eq of 3 sides

Look at a not so slightly deflected bubble w/slightly deflected plate



Bubble slope not very small

Expand in powers of $\frac{Pa}{\sigma} = \varepsilon_1$

" " " of $a/R = \varepsilon_2$

Double expansion

$$\phi(r,\theta;\varepsilon_1,\varepsilon_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon_1^m \varepsilon_2^n \phi_{mn}(r,\theta)$$

Exercise 2.1

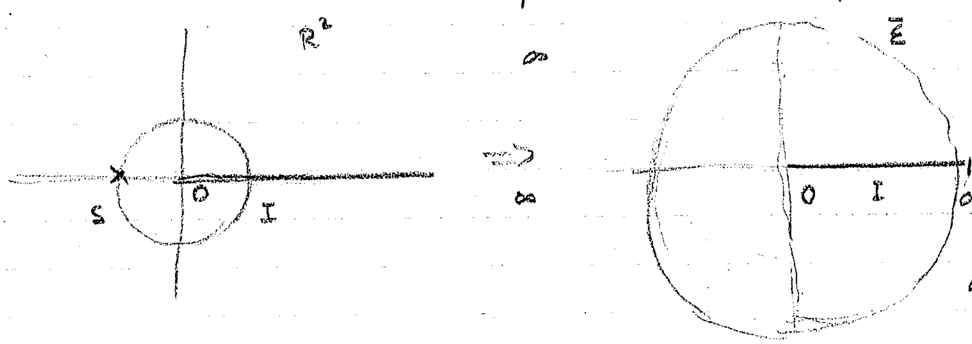
4/18/79

"Extended Stokes Series: Heated Horiz Pipe"
Thurs 19 Apr 4PM Rm 501A

Remember Problem Set #2 : 2nd problem

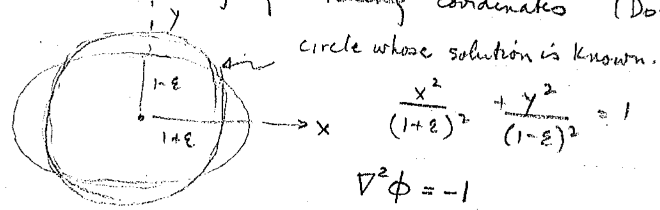
To extend a series analytically use Euler transf. $\bar{E} = \frac{R^2}{R^2+105}$

singularity



convergence rad is no greater than 1 but we have covered all physical R^2

§2.6 in notes, Slightly Stretching coordinates (Domain Perturbation) 4/20/79



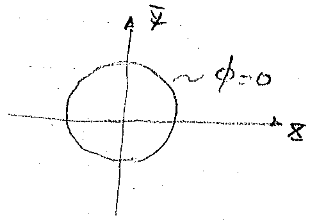
$$\frac{x^2}{(1+\varepsilon)^2} + \frac{y^2}{(1-\varepsilon)^2} = 1$$

$$\nabla^2\phi = -1$$

$$\phi = 0 \text{ at } \frac{x^2}{(1+\varepsilon)^2} + \frac{y^2}{(1-\varepsilon)^2} = 1$$

where $\varepsilon = 0$ $\phi_0 = \frac{1}{4}(1-r^2)$ where $\phi = \phi_0 + \varepsilon\phi_1 + \dots$

Stretched coords let $\bar{X} = \frac{x}{1+\varepsilon}$ $\bar{Y} = \frac{y}{1-\varepsilon}$ this takes the problem out of the location of the bound & puts it into DE.



$$(1+\varepsilon^2)(\phi_{\bar{x}\bar{x}} + \phi_{\bar{y}\bar{y}}) = -(1-\varepsilon^2)^2 + 2\varepsilon(\Phi_{\bar{x}\bar{x}} - \phi_{\bar{y}\bar{y}})$$

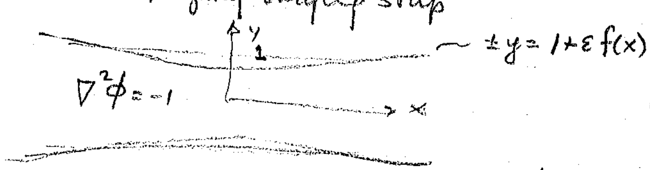
$$\Phi(\bar{X},\bar{Y}) = 0 \quad \text{or} \quad \bar{X}^2 + \bar{Y}^2 = 1$$

Assume $\Phi = \frac{1}{4}(1 - \bar{X}^2 - \bar{Y}^2) + \varepsilon\,\Phi_2(\bar{X},\bar{Y})$

ε^0 : $\nabla^2\phi_0 = -1$

ε^1 : $\nabla^2\phi_1 = 2(\phi_{0\bar{x}\bar{x}} - \phi_{0\bar{y}\bar{y}}) = 2\left(\frac{-1}{2} + \frac{1}{2}\right) = 0$ $\nabla^2\phi_2 = 0$ $\phi_2 = 0$ at $\bar{X}^2 + \bar{Y}^2 = 1$ $\phi_2 = 0$

look at a slightly varying strip

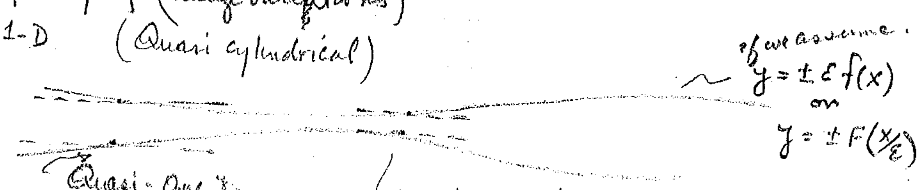


$$\phi = \frac{1}{2}(1-y^2)$$

$$\phi = \frac{1}{2}(1-y^2) + \varepsilon\,\phi_2(x,y)$$

accurate only for small ε.

Slowly varying (large variations)
Quasi - 1-D (Quasi cylindrical)



$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = -1$$

$\phi = 0$ at $y = \pm\varepsilon f(x)$

Obviously for smooth f and small ε : φ varies faster in y than x-dir

1st approx
$$\phi^{(1)}_{yy} = -1 \quad - \phi_{xx} \text{ small}$$
$$\phi^{(1)} = 0 \quad \text{on } y = \pm \varepsilon f(x)$$

General soln $\phi^{(1)} = -\frac{1}{2}y^2 + A(x) + B(x)y$ using bc $B(x) = 0$ since $\phi = 0$ $y = \pm\varepsilon f$

$\therefore \; A(x) - \frac{1}{2}\varepsilon^2 f^2(x) = 0$

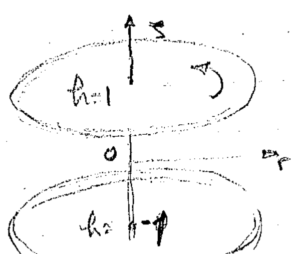
$\phi^{(1)} = \frac{1}{2}\left[\varepsilon^2 f^2(x) - y^2\right]$ good for any large excursions of y

for 2nd app
$$\phi^{(2)}_{yy} = -1 - \tfrac{1}{2}\varepsilon^2 \left[f^2\right]''$$
$$\phi^{(2)} = A_2(x) + B_2(x)\,y - \left[1 + \tfrac{1}{2}\varepsilon^2 (f^2)''\right] \frac{y^2}{2}$$

4/23/79

Exercise Set #4
Due Monday 30 April
Exercise 2.4, 3.1 of notes

Coaxial Perturbation



$R = \frac{\Omega h^2}{\nu}$

$$g'' = (hg' - gh')R^2$$
$$h^{iv} = 4gg' + R^2 hh'''$$
g, h were odd fn of ζ
$g = 1,\; h = h' = 0$ @ $\zeta = 1$

Assume $g = A\zeta + B\zeta^3 + C\zeta^5 + \dots$
$h = a\zeta + b\zeta^3 + c\zeta^5 + \dots$

Substitute into DE
$$6B\zeta + 20C\zeta^3 + \dots = 2R^2(aB + bA)\zeta^3 + \dots$$
$$120c\zeta + 840d\zeta^3 + \dots = (4A^2 + 6abR^2)\zeta + \dots$$

Equate like powers of ζ & write in terms of a, b, A
since 1) odd fn property was taken into account by dropping even power terms
2) $g' \neq 0$ at $\zeta = 0$ since g is odd $\Rightarrow A \neq 0$
3) h' and h''' are $\neq 0$ at $\zeta = 0$ since h is odd

Now $g = A\zeta - \frac{R^2}{10}Ab\zeta^5 - \frac{R^2}{315}Aa\left(3R^2 b + A\right)\zeta^7 + \dots$

$h = a\zeta + b\zeta^3 + \frac{A^2}{30}\zeta^5 + \dots$

Impose BC
$$g(1) = A - \frac{R^2}{10}Ab - \frac{R^2}{315}Aa(3R^2 b + A) + \dots$$
$h(0) = \dots$

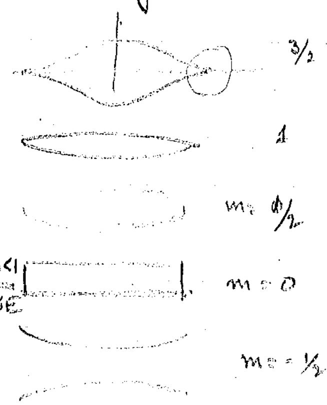
if you truncate at $g = A\zeta$ $h = a\zeta + b\zeta^3$ $\Rightarrow A = 1,\ a = b = 0$
if you truncate at the next terms you get a cubic in A.

this entire approx. about small ζ

SYMPTOMS OF NONUNIFORMITY for singular perturb

1. infinities in region of interest

torsion of beam $\nabla^2\phi = -1$ for a thin section where $y = \pm\varepsilon f(x)$ (Pg 29 notes)
let $f(x) = (1-x^2)^m$ for $|x| \le 1$



~~Small~~ Second approx (2.32) for small ε contains

$$[f^2]'' = -2m\left[1-(4m-1)x^2\right](1-x^2)^{2m-2}$$

∞ for $x = \pm 1$ for $m<1$ except $m = 1/2$ ELLIPSE

first approx was $\phi_I = \frac{1}{2}\left[\varepsilon^2 f^2 - y^2\right]$
second " "

$$\phi_{II} = \frac{1}{2}\left[\varepsilon^2 f^2 - y^2\right]\left\{1 + \frac{1}{2}\varepsilon^2 [f^2]''\right\}$$

2. Ratios of successive terms not small

$\frac{1}{2}\varepsilon^2[f^2]'' + 1$
2nd term 1st term

$$\frac{\text{2nd term}}{\text{1st term}} = 4m(2m-1)\left[\frac{\varepsilon}{(1-x^2)^{1-m}}\right]^2 + \dots$$

if $m \geq 1$ we're ok
infinite as $x \to \pm 1$ for $m<1$
except $m = 1/2$

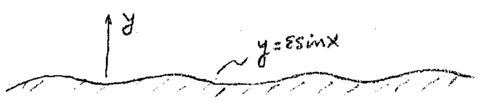
we can approx how good (how big our region is)
2nd term should be ε^2 times the first but $\frac{\varepsilon}{2(1-x)^{1-m}} \neq O(1)$

where $1-x = O(\varepsilon^{\frac{1}{1-m}})$

thus solution breaks down within $O(\varepsilon^{\frac{1}{1-m}})$ of the points $x = \pm 1$
this implies that near the tips the char length is not $O(1)$ but order of width near singular point. Thus must expand $\frac{x}{\varepsilon}$ where ε is order of width near singular term.

3. Unsatisfiable boundary conditions

4/25/79



$u = U = 1$, $v = 0$ } $\psi \sim y$ as $y \to \infty$, $\lim_{y\to\infty} \frac{\psi}{y} = 1$ || $\psi = y + o(y)$ as $y \to \infty$

Examples

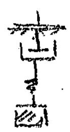
Plane Potential flow.

$\psi \sim y + \frac{\text{const}}{y}$

$\psi \sim y + \text{const} \log y$

Symptoms of Non Uniformity

1. Infinite value of solution (including at ∞ distance)
2. Ratios of successive terms is not small.
3. Secularity: non periodic terms crop up



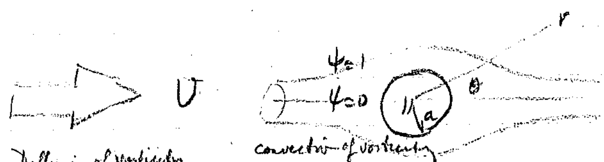
$m\ddot{x} + \varepsilon\dot{x} + kx = 0$ $\quad \frac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4mk}}{2m} = \omega \quad -\frac{\varepsilon}{2m} \pm \sqrt{\frac{\varepsilon^2}{4m^2} - \frac{k}{m}} = -\frac{\varepsilon}{2m} \pm i\sqrt{\frac{K}{m}}$

soln is $e^{-\varepsilon t} \sin t$ where $m = 1 = K$

for small ε then $\sin t - \underbrace{\varepsilon t \sin t}_{\text{secular}}$

4. Nonexistence of sol. Stokes paradox + whitehead.

Laminar flow past circle on sphere slow motion assume quasi static



$\nabla^4\psi = R\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y}\right)\nabla^2\psi$: $D^4\psi = R[\text{non lin}]$ for sphere

$R = \frac{Ua}{\nu}$

$\nabla^2\psi = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi = 0$ for $R \ll 1$ Stokes approx

upstream $\psi \sim y = r\sin\theta$

Surface $\psi = \text{const} = 0$ $(\psi_\theta = 0)$

no slip $\psi_r = 0$

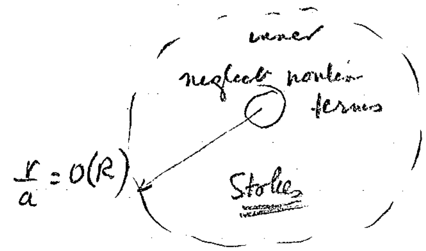
ψ for cylinder $\psi = (r^3, r\log r, r, \frac{1}{r})\sin\theta$

ψ for sphere $\psi = (r^4, r^2, r, \frac{1}{r})\sin^2\theta$

blowup at ∞ — friction — stokeslet — dipole

Now Oseen made the next approx ⇒ $\nabla^4\psi = RU\frac{\partial}{\partial x}\nabla^2\psi$ in 2-Dimension

∴ Stokes $\nabla^4\psi = 0$ Singular Part
Oseen $\nabla^4\psi = RU\frac{\partial}{\partial x}\nabla^2\psi$ Regular Part.



outer
vorticity convected by
free stream.
Oseen.

viscous Reference length ν/U

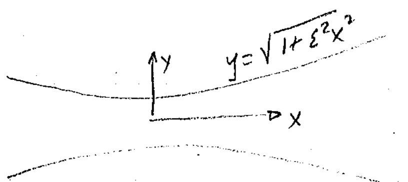
Outer reference length ν/U
Inner " " a } Ratio $\frac{a}{\nu/U} = \frac{Ua}{\nu} = R$

R = ratio of 2 reference lengths.

4/27/79

Methods of removing singularities
1. Slow variation
2. Matched Asympt.
3. Methods of ~~Strained~~ multiple Scales.

<u>Slow variation</u>



flow through a slowly varying duct.

Potential flow $\nabla^2\psi = 0$
$\psi(x, \sqrt{1+\varepsilon^2x^2}) = 1$
$\psi(x,0) = 0$
} Let $\psi(x,y;\varepsilon) = \psi_0(x,y) + \varepsilon^2\psi_2(x,y) + \varepsilon^4\psi_4(x,y) + \ldots$

Now $\nabla^2\psi_0 = 0$
$\psi_0(x,1) = 1$
$\psi_0(x,0) = 0$
} soln. $\psi_0 = y$

$\therefore \psi = y + \varepsilon^2\psi_2 + O(\varepsilon^4) + \ldots$

$\psi\big|_{y=\sqrt{1+\varepsilon^2x^2}} = \sqrt{1+\varepsilon^2x^2} + \varepsilon^2[\psi_2(x,1) + O(\varepsilon^2)] + O(\varepsilon^4) + \ldots = 1$

$\psi = 1 + \frac{\varepsilon^2x^2}{2} + O(\varepsilon^4) + \varepsilon^2\psi_2(x,1) + O(\varepsilon^4) = 1$

$\therefore \psi_2(x,1) = -\frac{x^2}{2}$

$\therefore\ \psi_2(x,1) = -\frac{1}{2}x^2$

$\psi_2(x,0) = 0 \qquad \psi_2 = -\frac{1}{6}\left[3x^2y - y^3 + y\right] \qquad$ using $\nabla^2 = \frac{\partial}{\partial z \partial \bar{z}}$

$\nabla^2 \psi_2 = 0$

$$\therefore \quad \psi = y - \frac{\varepsilon^2}{6}\left(3x^2y - y^3 + y\right) + O(\varepsilon^4)$$

but this will go to ∞ for $x \to \infty$ ie $x = O\left(\frac{1}{\varepsilon}\right)$ this shows that at $x = \infty$ the channel is no longer parabolic but forms a wedge $\therefore$ $y \sim \varepsilon x$

to make problem tractable: you must make x, y scale the same then you can contract x or magnify y

We will contract x by $X = \varepsilon x$

Put into DE $\quad 0 = \nabla^2\psi \Rightarrow \quad \frac{\partial^2 \psi}{\partial y^2} + \varepsilon^2 \frac{\partial^2 \psi}{\partial X^2} = 0$

BC $\quad \Psi(X, \sqrt{1+X^2}) = 1$

$\Psi(X, 0) = 0$

take $\quad \Psi = \Psi_0 + \varepsilon^2 \Psi_2 + \varepsilon^4 \Psi_4 + \dots$

for 1st order $\quad \frac{\partial^2 \Psi_0}{\partial y^2} = \nabla^2 \Psi_0 = 0 \qquad \Psi_0(X, \sqrt{1+X^2}) = 1 \qquad \Psi_0(X, 0) = 0$

ie variations are much large in y direction than x direction

we can then take $\quad \Psi_0 = y / \sqrt{1+X^2}$

using iteration: $\quad \frac{\partial^2 \Psi_2}{\partial y^2} = -\frac{\partial^2 \Psi_0}{\partial X^2} \qquad$ w/ $\Psi_2(X, \sqrt{1+X^2}) = 0$; $\Psi_2(X, 0) = 0$

$$\frac{\partial^2 \Psi_0}{\partial X^2} = \left(\frac{y}{\sqrt{1+X^2}}\right)'' = y\left(\frac{1}{\sqrt{1+X^2}}\right)''$$

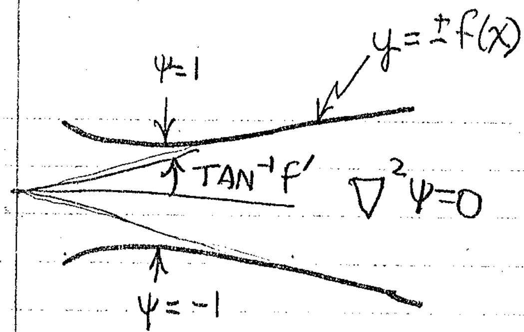
$\therefore \quad \Psi_2 = -y^3/6 \left(\frac{1}{\sqrt{1+X^2}}\right)'' + c_1 y + c_2 \qquad c_2 = 0$ using

$= $ or $\quad \frac{1}{6}\left(\frac{y}{\sqrt{1+X^2}}\right)''\left[(1+X^2) - y^2\right]$ using first bc

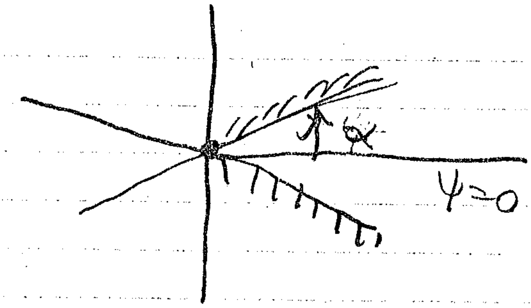
$$\therefore \quad \Psi = \frac{y}{\sqrt{1+X^2}} + \frac{\varepsilon^2}{6} y \left(\frac{1}{\sqrt{1+X^2}}\right)''\left[(1+X^2) - y^2\right] + O(\varepsilon^4)$$

for large X we must recover flow through a wedge

$$\psi = y/x - \frac{\varepsilon^2}{6}\frac{y}{(1+x^2)^{5/2}}\left[-1 + 2x^2\right](1 + x^2 - y^2) + O(\varepsilon^4)$$



FOR SLOWLY VARYING CURVATURE,



ISOLATED SOURCE

FITTING WEDGE FLOWS

$$\psi \approx \frac{\tan^{-1}\left[y\, f'(x)/f(x)\right]}{\tan^{-1}\left[f'(x)\right]}$$

EXAMPLE:

HYPERBOLIC CHANNEL

$$y = \pm\sqrt{1+\varepsilon^2 x^2}$$

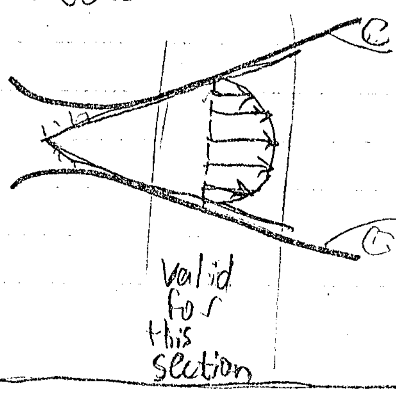
$$\psi \approx \frac{\tan^{-1}\left(\frac{\varepsilon^2 x y}{1+\varepsilon^2 x^2}\right)}{\tan^{-1}\left(\frac{\varepsilon^2 x}{\sqrt{1+\varepsilon^2 x^2}}\right)} \sim \frac{y}{\sqrt{1+\varepsilon^2 x^2}}\left[1 + \frac{\varepsilon^4 x^2}{3}\left(1 - \frac{y^2}{1+\varepsilon^2 x^2}\right) + \ldots\right]$$

Eq'n (3.12) of notes:

THIS IS MISSING FROM THIS

$$x \to \varepsilon x \quad \psi \sim \frac{y}{\sqrt{1+\varepsilon x^2}}\left[1 - \varepsilon^2 \frac{(1-)\,2\varepsilon^2 x^2}{6\,(1+\varepsilon^2 x^2)}\left(1 - \frac{y^2}{1+\varepsilon^2 x^2}\right) + \ldots\right]$$

Viscous Flow

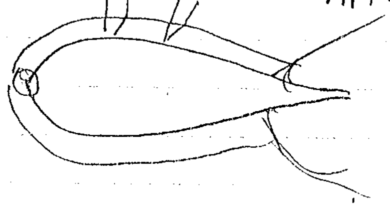


JEFFERY - HAMEL FLOW

ELLIPTIC FUNCTIONS

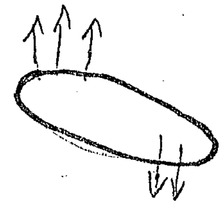
FIT PIECEWISE "WEDGE SOLUTIONS" TO OBT[AIN] SECOND APPROXIMATION

FALKNER - SKAN FAMILY OF SOL'NS OF LAMINAR BOUNDAR[Y] LAYER EQ'NS - APPLY PIECEWISE SOL'N TO EQUATIONS IN LAMINAR

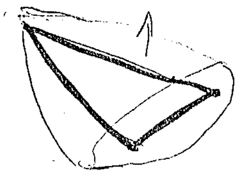


4-30

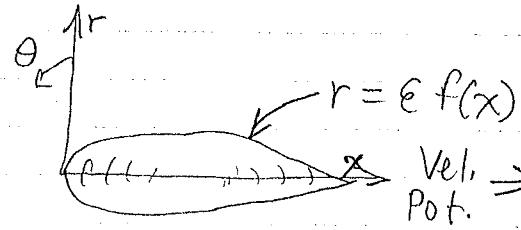
SLENDER AIRSHIP THEORY



MUNK AIRSHIP THEORY
JONES SLENDER-WING THEORY



INCLINED BODY OF REVOLUTION, INCOMPRESSIBLE FLOW



Vel. Pot. $\Rightarrow \nabla^2\phi = \phi_{rr} + \frac{\phi_r}{r} + \frac{\phi_{\theta\theta}}{r^2} = 0$

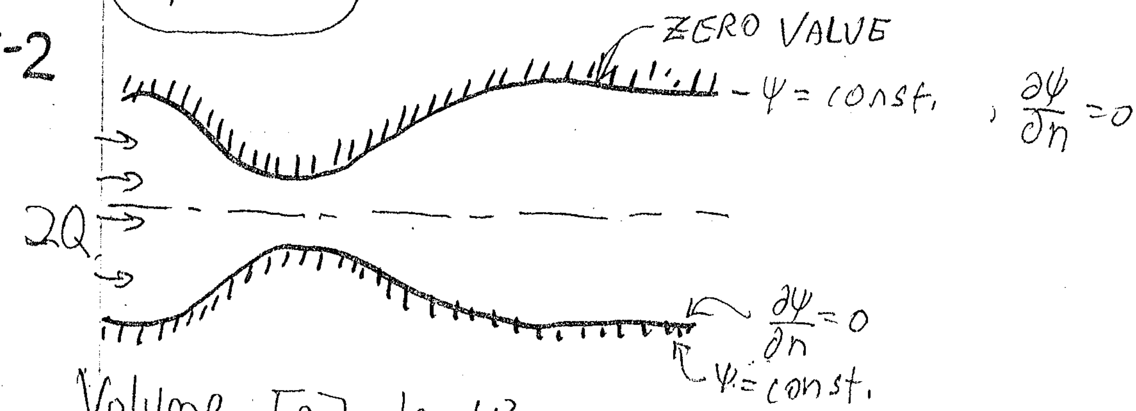
Infinity $\phi = U[x\cos\alpha + r\cos\theta\sin\alpha]$

velocity $\tilde{q}$ Surface: Tangency [illegible]
Surface $F(x,y,z)=0$ (NO FLOW) $\phi_r = \epsilon f'(x)\phi_x$ @ $r = \epsilon f(x)$
no flow $\Rightarrow \tilde{q}\cdot \text{grad}\, F = 0$

We can neglect variations in one directions

5-2

(EXER. 3.5)



Volume FLUX $[Q] = \frac{\text{length}^2}{\text{time}}$

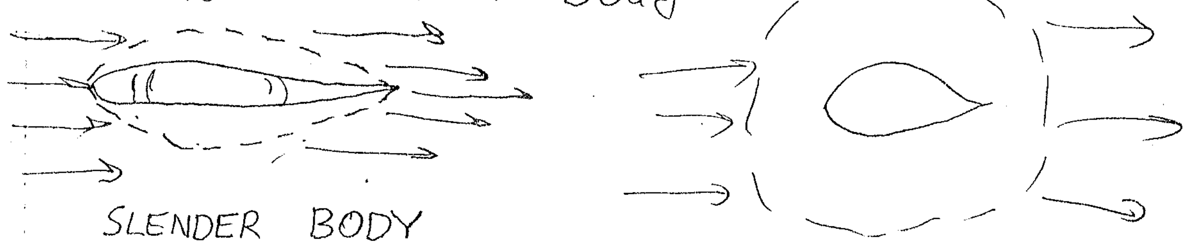
$$\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y}\right)\nabla^2\psi = \nu\nabla^4\psi \qquad R = \frac{Q}{\nu} \to \frac{1}{\nu}$$

($\nu \to 1/R$)

$\frac{\partial \psi}{\partial x} = 0$ on $y=0$, $\frac{\partial^2 \psi}{\partial y^2} = 0$ also

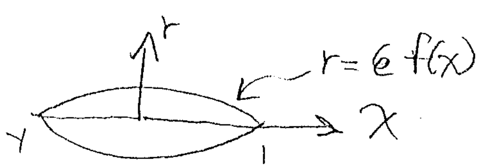
---

"Slender" vs. "Thin" Body

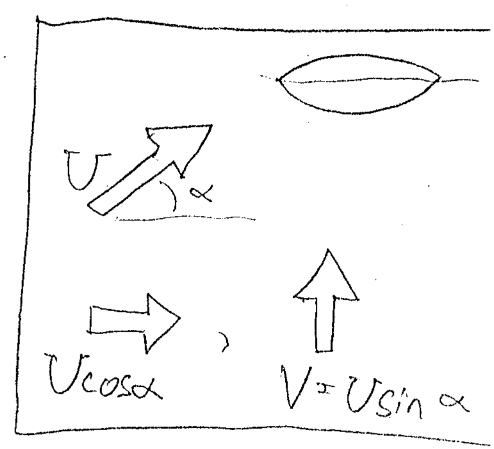


SLENDER BODY

CROSS - FLOW PAST SLENDER BODY



$V = U \sin\alpha = 1$



FULL PROBLEM

$$\nabla^2\phi \equiv \phi_{xx} + \phi_{rr} + \frac{\phi_r}{r} + \frac{\phi_{\theta\theta}}{r^2} = 0$$

surface $\phi_r = \varepsilon f'(x)\,\phi_x$ @ $r = \varepsilon f(x)$

distant $\phi \sim r\cos\theta$ as $r \to \infty$

Stretched problem: $\phi(x,r,\theta;\epsilon) = \epsilon\,\Phi(R = r/\epsilon)$

$r = \epsilon f(x)$, $f(x) = r/\epsilon$

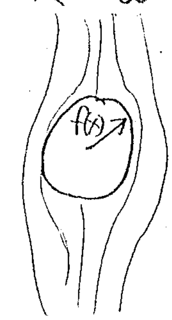
$$\epsilon\,\Phi_{xx} + \frac{1}{\epsilon}\underbrace{\left[\Phi_{RR} + \frac{\Phi_R}{R} + \frac{\Phi_{\theta\theta}}{R^2}\right]}=0$$

2 Dim'l Laplace operator in cross flow

$\Phi_R = \epsilon^2 f'(x)\Phi_x$ on $R = f(x)$

$\Phi \sim R\cos\theta$ as $R \to \infty$

1ST APPROX.



$$\Phi_1 = \left[R + \frac{f^2(x)}{R}\right]$$

Iterate for 2nd approx.

$$\Phi = \Phi_1 + \epsilon^2\Phi_2 + \dots$$

$$\Phi_{2,RR} + \frac{\Phi_{2,R}}{R} + \frac{\Phi_{2,\theta\theta}}{R^2} = -\Phi_{1,xx} = -\frac{\cos\theta}{R}\left[f^2(x)\right]''$$

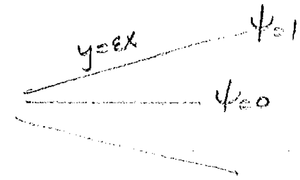
$\Phi_{2,R} = f'(x)\cdot\Phi_{1,x}$ ON $R = f(x)$

$\Phi_2 \to 0$, or @ any rate, $o(R)$ as $R \to \infty$

Particular Integral - Multiple of $R \log R \cos\theta$

Velocity $\sim \log R$ as $R \to \infty$

expanding wedge sol. $\psi = \dfrac{\tan^{-1}\left(\frac{\epsilon y}{x}\right)}{\tan^{-1}(\epsilon)}$



we find the result is the same.

in general asymptotic expansion

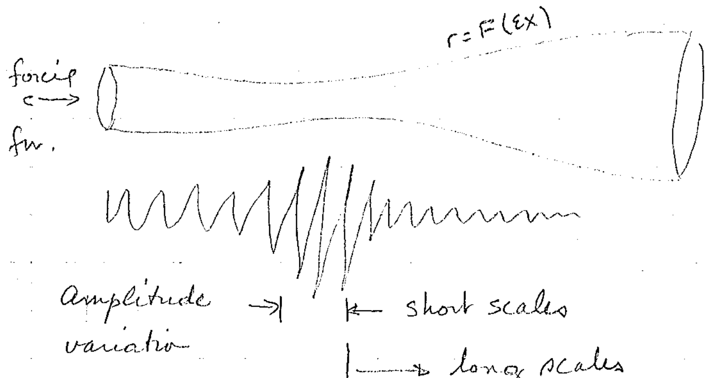
$$f(x,y;\epsilon) = \sum \delta_n(\epsilon)\, f_n(x,y;\epsilon)$$

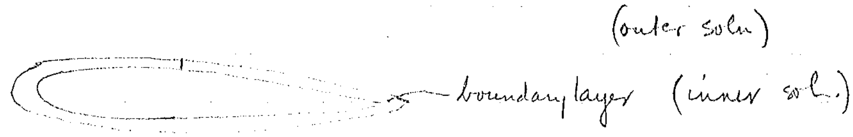
when $\delta_n(\epsilon)$ is the gage fn.

See book inserts for 4-30-79
5-2-79

5/4/79

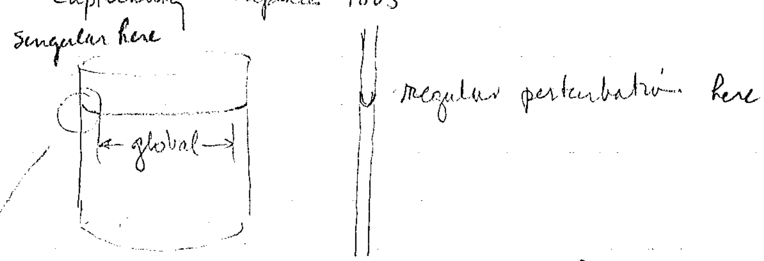
Midterm - Monday 14 May
Openbook/notes



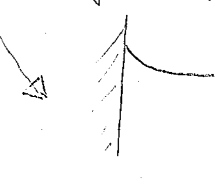


Method of Matched asymptotics

Capillarity - Laplace 1805

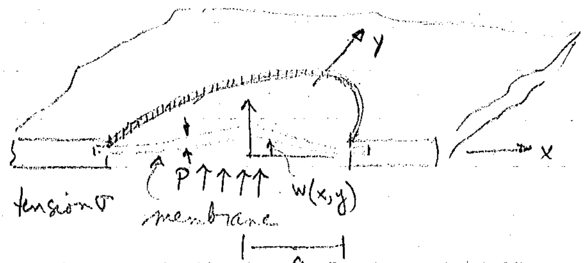


Small slope over global region - In boundary region slopes are large and the glass appears to be a plane wall on the small scale



Rule: as you approach edge of outer solution – you must see result when you go far away from inner solution.

Deflection of slightly stiff membrane (ie has some flexural rigidity)



Small deflections (tension constant)

Governing PDE $K\nabla^4 w - \sigma \nabla^2 w = p$

flexural rigidity — surface tension — pressure

Assumed small away from wall – $K\nabla^4 w$ is large in bl near wall

$K = \dfrac{Et^3}{12(1-\nu^2)}$

introduce dimensionless variables: refer x, y to a. deflection w to pa^2/σ

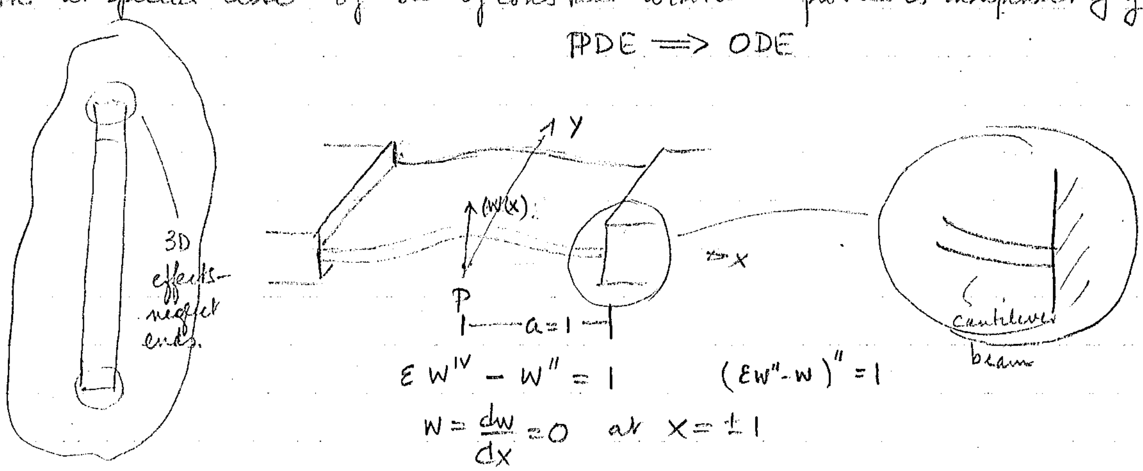
then PDE becomes $\varepsilon \nabla^4 w - \nabla^2 w = 1$ $\qquad \varepsilon = \dfrac{K}{\sigma a^2} = \dfrac{Et^3}{12(1-\nu^2)\sigma a^2}$

this is a ratio of 2 lengths: we should expect singularity

clamped bc $\quad w = 0 \quad \dfrac{\partial w}{\partial n} = 0$ $\qquad$ soln is bessel eqn operator

look at special case of slit of constant width – problem is independent of y

PDE ⟹ ODE



$\varepsilon w^{IV} - w'' = 1 \qquad (\varepsilon w'' - w)'' = 1$

$w = \dfrac{dw}{dx} = 0$ at $x = \pm 1$

General Sol $\qquad -\frac{1}{2}x^2 + Ax + B + Ce^{x/\sqrt{\varepsilon}} + De^{-x/\sqrt{\varepsilon}}$

impose BC $\qquad w = \frac{1}{2}(1-x^2) - \dfrac{\sqrt{\varepsilon}}{\sinh 1/\sqrt{\varepsilon}}\left(\cosh\dfrac{1}{\sqrt{\varepsilon}} - \cosh\dfrac{x}{\sqrt{\varepsilon}}\right)$

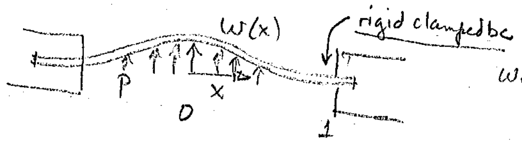
$w(0) = \frac{1}{2} - \dfrac{\sqrt{\varepsilon}}{\sinh 1/\sqrt{\varepsilon}}\left(\cosh\dfrac{1}{\sqrt{\varepsilon}} - 1\right)$

$= \dfrac{1}{2} - \sqrt{\varepsilon}\coth\dfrac{1}{\sqrt{\varepsilon}}$ + exponentially small terms

$\approx \frac{1}{2} - \sqrt{\varepsilon} + \dots$

Cylindrical Deflection of Pressurized Slightly Rigid Membrane
Continuation of last time

5/7/79



we found last time that $\varepsilon w^{IV} - w'' = 1$

$w = \frac{dw}{dx} = 0$ @ $x = \pm 1$

$\varepsilon = \frac{\text{rigidity factor}}{\text{tension factor}}$

Straightforward first approx: let $\varepsilon = 0$

$\frac{d^2 w}{dx^2} = -1$ general symmetric solution $w = C - \frac{1}{2}x^2$

if $\varepsilon = 0$ we can only satisfy $w = 0$ @ $x = \pm 1$

1^{st} approx $\therefore$ is $w_1 = \frac{1}{2}(1 - x^2)$ and thus we only lose $\frac{dw}{dx}$

$\frac{dw}{dx} = -x$

2^{nd} app: Straight forward expansion try $w = w_1 + \varepsilon w_2(x) + \ldots$

put into DE $\Rightarrow$ $\frac{d^2 w_2}{dx^2} = 0$ then the general symm. sol. $w_2 = \text{Const.}$

$w_2 = 0$ and $\frac{dw_2}{dx} = \pm\frac{1}{\varepsilon}$ @ $x = \pm 1$!!?? can't impose any bc.

We note that because of these problems we have to look at singular perturbations

- We begin at first approx. above: we call it the global solutions, since it is normally the solution over the majority of the ~~plate~~ membrane
- We need consider two cases 1. is it a boundary layer problem
  2. is it a multiple scales problem.

Here we've got a boundary layer problem. Move coordinate to point near bc and magnify the scales and the function w. But since w is linear it will drop out

thus we can define $X = x/\varepsilon^n$ $W = w/\varepsilon^m$

nonuniformity near $x = -1$ (and $x = +1$)

Introduce "magnified" local variables $W(X)$

try powers let $w(x) = \varepsilon^a W(X)$

$X = (1 + x)/\varepsilon^b$

Put into DE

$$\varepsilon^{1+a-4b} \frac{d^4 W}{dX^4} - \varepsilon^{a-2b} \frac{d^2 W}{dX^2} = 1$$

bc $w = \frac{dw}{dx} = 0$ @ $x = \pm 1$ $\Rightarrow$ $W = \frac{dW}{dX} = 0$ $X = 0$ only: not will ~~need~~ have enough bc to solve problem but it's ok since we must match local & global

Rigidity — tension — Pressure

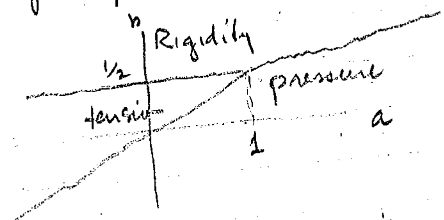
$$\varepsilon^{1+a-4b} W^{IV} - \varepsilon^{a-2b} W'' = 1$$

a is physical interpretation near x=±1 ⇒ ① if rigidity balances tension $1+a-4b = a-2b \implies b = \tfrac{1}{2} \quad a = \text{anything}$

② " " " " $1+a-4b = 0 \quad a = 4b-1$

③ " all 3 balance $1+a-4b = a-2b = 0 \implies b = \tfrac12 \quad a = 1$

③ gives you full problem again but you lose looking at small ε results



as a physical argument rigidity must balance tension : as $\bar{X} \to 0$ $p \to 0$ but tension remains constant & rigidity effects are at $\bar{X} = 0$

⇒ Since a = anything let a = 0.

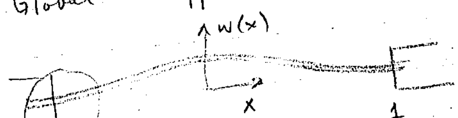
5/9/79

Returning to our problem.

$\varepsilon W^{IV} - W'' = 1$ $\quad W = W' = 0$ @ $x = \pm 1$

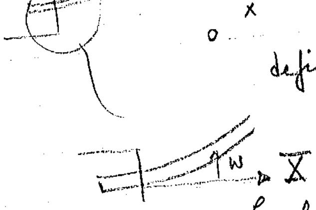
rigidity tension pressure

Global 1st approx $W_1 = \tfrac12 (1 - x^2)$

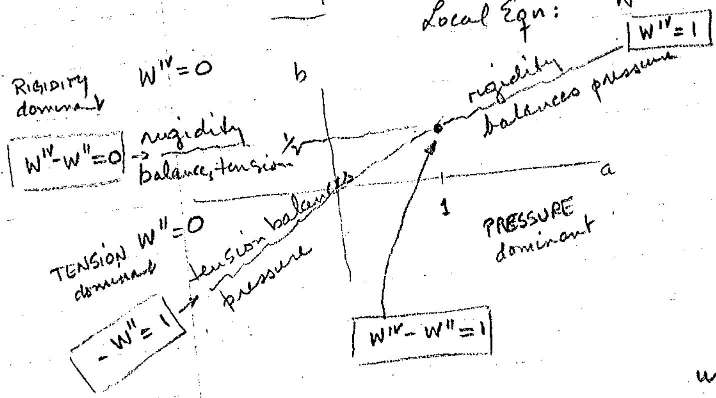


defining: Local variables $w = \varepsilon^a W(\bar{X})$

$$\bar{X} = \frac{1+x}{\varepsilon^b}$$



Local Eqn: $W^{IV} - \varepsilon^{2b-1} W'' = \varepsilon^{4b-a-1}$ w/ $W(0) = 0$ $W'(0) = 0$



– $W^{IV} = 1$ has soln which will not match the global solution hence don't consider

– look at rigidity balanced by tension $W^{IV} - W'' = \varepsilon$

Consider $b = \tfrac12$, a arbitrary $(a < 1)$

1st approx: $W_1^{IV} - W_1'' = 0$

$W_1 = A e^{\bar{X}} + B e^{-\bar{X}} + C\bar{X} + D$

using BC at left end $W_1(0) = A + B + D = 0$,

$W_1'(0) = A - B + C = 0$

keep A, C ⇒ $B = A + C \quad D = -2A - C$

$$\therefore W_1 = A e^{\bar{X}} + (A+C) e^{-\bar{X}} + C\bar{X} - (2A + C)$$

$$= A(e^{\bar{X}} + e^{-\bar{X}} - 2) + C(e^{-\bar{X}} + \bar{X} - 1)$$

... global → outer expansion of local

Global soln near $x = -1$ expand in terms of $x+1$

$$\varepsilon^a W_1 = w_1 = \tfrac{1}{2}(1+x)(1-x) = \tfrac{1}{2}(1+x)\left[2-(1+x)\right] \quad \text{for small } 1+x$$

$$= (1+x) - \tfrac{1}{2}(1+x)^2 \sim (1+x) + \dots$$

Local soln for large $X \Rightarrow \varepsilon \to 0$

$$W_1 \sim C(X - 1 + \exp(\to 0)) + A(e^X + \dots \text{ smaller terms})$$

since e^X is larger than $X-1$ then $\Rightarrow e^X$ (local) must match $1+x$ (global) $\Rightarrow$ impossible (exp ≠) $\Rightarrow A = 0$

$\Rightarrow C(X - 1 + \dots)$ must match $1+x$; but subst. for X (subst global coord into local,

$C\left(\frac{1+x}{\varepsilon^{1/2}} - 1 + \dots\right)$ must match $\frac{1+x}{\varepsilon^a}$ $\Rightarrow$ $C = \varepsilon^{1/2 - a}$. Let $a = 0$

$$W \sim \varepsilon^a W_1 = \varepsilon^a\left[A\left(e^X + e^{-X} - 2\right) + C\left(X - 1 + e^{-X}\right)\right] \quad \text{1st local}$$

$$w_* \sim \tfrac{1}{2}\left[1 - x^2\right] \quad \text{1st global}$$

2. Intermediate limits

define Global: $\varepsilon \to 0$ w/ $\frac{1+x}{\varepsilon^0}$ fixed

define local: $\varepsilon \to 0$ w/ $X = \frac{1+x}{\varepsilon^{1/2}}$ fixed

define Intermediate: $\varepsilon \to 0$ w/ $\tilde{X} = \frac{1+x}{\varepsilon^\alpha}$ fixed pick ε^α: can be any thing between $\varepsilon^{1/2}$ & $\varepsilon^0 = 1$

pick $0 < \alpha < \tfrac{1}{2}$ take $\alpha = \tfrac{1}{4}$ $\therefore \tilde{X} = \frac{1+x}{\varepsilon^{1/4}}$

Rewrite global in intermediate variables

$$w = \tilde{X}\varepsilon^{1/4} - \tfrac{1}{2}\tilde{X}^2\varepsilon^{1/2}$$

then expand for small ε $\quad w \sim \varepsilon^{1/4}\tilde{X}$

Rewrite local in intermed. variables

$$W \sim \varepsilon^a\left[A\left(e^{\tilde{X}/\varepsilon^{1/4}} + e^{-\tilde{X}/\varepsilon^{1/4}} - 2\right) + C\left(\tilde{X}/\varepsilon^{1/4} - 1 + e^{-\tilde{X}/\varepsilon^{1/4}}\right)\right]$$

expand for small ε

$$W \sim \varepsilon^a\left[Ae^{\tilde{X}/\varepsilon^{1/4}} \text{ or (if } A=0) \; C\left(\tilde{X}/\varepsilon^{1/4}\right)\right]$$

$\Rightarrow$ must now equate $\Rightarrow A = 0$ $\quad \varepsilon^a C/\varepsilon^{1/4} = \varepsilon^{1/4}$ $\quad C = \varepsilon^{1/2 - a}$ $\quad$ Let $a = 0$

HW Assignment #6.1. 5/11/79

1. Check axisymmetric case
2. Local Scale. "Obviously" ε for x, y Not clear for w

$$w = W(R) \to \nabla^2 W = 0$$

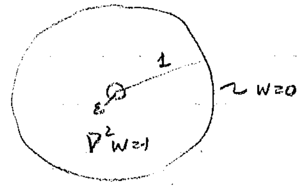
or

$$w = \varepsilon^2 W(R) \to \nabla^2 W = -1$$

3. Purely log case: expansion in powers of $\frac{1}{\log(1/\varepsilon)}$ due to cylindrical geometry

Also transcendentally small $\frac{\varepsilon}{(\log 1/\varepsilon)^n}$

Circular Case



$$w = A \log r + B - \tfrac{1}{4} r^2$$

$$w(1) = B - \tfrac{1}{4} = 0 \qquad B = \tfrac{1}{4}$$

$$w(\varepsilon) = A \log \varepsilon + B - \tfrac{1}{4}\varepsilon^2 = 0 \qquad A = -\tfrac{1}{4}\left(\frac{1-\varepsilon^2}{\log \varepsilon}\right)$$

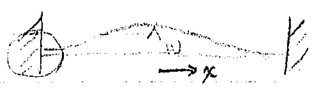
$$w = \frac{1}{4}(1 - r^2) - \frac{1}{4}\,\frac{1-\varepsilon^2}{\log \varepsilon}\log r$$

Matching Principles

1. Intuitive –
2. Intermediate limits
3. Asymptotic Matching principle

5/10/79

Review: Slightly rigid membrane



Global approx $\quad w = \tfrac{1}{2}(1 - x^2)$

rigid is balanced by tension $\quad$ Local approx $\quad w = \left\{ A\left(e^{X} + e^{-X} - 2\right) + C\left(X - 1 + e^{-X}\right)\right.$

rigid is " " pressure $\quad = BX^2 + AX^3 - \tfrac{1}{24}X^4$

where $X = \dfrac{1+x}{\varepsilon^{1/2}}$

1 term local of (1 – term global) = 1-term global of (1 – term local)

1 term global $\quad w = \tfrac{1}{2}(1 - x^2)$

rewrite in local variable $\quad = \sqrt{\varepsilon}X\left(1 - \tfrac{1}{2}\sqrt{\varepsilon}X\right)$

local expansion $\quad = \sqrt{\varepsilon}X - \tfrac{1}{2}\varepsilon X$

1 term local $\quad = \sqrt{\varepsilon}X$

1 term local $\quad W = A\left(e^{X} + e^{-X} - 2\right) + C\left(X - 1 + e^{-X}\right)$

rewrite in global variables

$$W = A\left(e^{\frac{1+x}{\sqrt{\varepsilon}}} + e^{-\frac{1+x}{\sqrt{\varepsilon}}} - 2\right) + C\left(\frac{1+x}{\sqrt{\varepsilon}} - 1 + e^{-\frac{1+x}{\sqrt{\varepsilon}}}\right)$$

global expansion ~~either~~ $A\left(e^{\frac{1+x}{\sqrt{\varepsilon}}} - 2 + \dots\right) + C\left(\frac{1+x}{\sqrt{\varepsilon}} - 1 + \dots\right)$

either $\quad A e^{\frac{1+x}{\sqrt{\varepsilon}}}$ or (if $A=0$) $C\left(\frac{1+x}{\sqrt{\varepsilon}}\right)$

the exponential will not match $A = 0$

Rewrite in global variables only to compare $\quad \sqrt{\varepsilon}X = CX \qquad C = \sqrt{\varepsilon}$

If however it is balanced by pressure (or nothing)

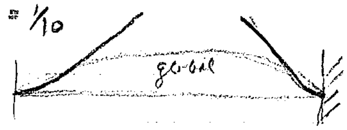
1 – term local $\quad w = BX^2 + AX^3\left(-\tfrac{1}{24}X^4\right)$

rewrite in global $\quad = B\dfrac{(1+x)^2}{\varepsilon} + A\dfrac{(1+x)^3}{\varepsilon^{3/2}} - \dfrac{1}{24}\dfrac{(1+x)^4}{\varepsilon^2}$

1-term global $= \begin{cases} -\frac{1}{24}\frac{(1+x)^4}{\varepsilon^2} & \text{if } A, B = O(1) \quad = -\frac{1}{2}X^4 \\ A\left(\frac{1+x}{\varepsilon^{1/2}}\right)^3 & \quad = AX^3 \\ B\,\frac{(1+x)^2}{\varepsilon} & \quad = BX^2 \end{cases}$ would only exist if rigidity is balanced by nothing

however we can't match ⇒ rigid cannot balance pressure

if $\varepsilon = 1/10$



1st global $\quad w = \frac{1}{2}(1-x^2)$

1st local $\quad w = \sqrt{\varepsilon}\,(X - 1 + e^{-X})$

$\sim \sqrt{\varepsilon}\,(\frac{1}{2}X^2 + \dots)$ small X

$\sim \sqrt{\varepsilon}\,(X)$ fo large X

this must now be "composited"

Composite = local + global − Common part

Common part: local of global = global of local

$$w \sim \frac{1}{2}(1-x^2) + \sqrt{\varepsilon}\left[\frac{1+x}{\sqrt{\varepsilon}} - 1 + e^{-\frac{1+x}{\sqrt{\varepsilon}}}\right] - \sqrt{\varepsilon}\left(\frac{1+x}{\sqrt{\varepsilon}}\right)$$

$$\sim \frac{1}{2}(1-x^2) + \sqrt{\varepsilon}\left[-1 + e^{-(1+x)/\sqrt{\varepsilon}}\right] \qquad \text{good to } x=0 \quad -1 \le x \le 0$$

(the $\sqrt{\varepsilon}[\dots]$ term: correction boundary layer)

for other end (replace x by $-x$)

$$\frac{1}{2}(1-x^2) + \sqrt{\varepsilon}\left[-1 + e^{\frac{-1+x}{\sqrt{\varepsilon}}}\right]$$

thus composite of composite: $\quad w = \frac{1}{2}(1-x^2) + \sqrt{\varepsilon}\left[-1 + e^{-\left(\frac{1+x}{\sqrt{\varepsilon}}\right)}\right] + \sqrt{\varepsilon}\left[-1 + e^{-\left(\frac{1-x}{\sqrt{\varepsilon}}\right)}\right]$

w left + w right − common part

Composite gives deflection

at left end: $(x=-1)$ $\quad w \approx -\sqrt{\varepsilon}\left(1 - e^{-2/\sqrt{\varepsilon}}\right) \cong -\sqrt{\varepsilon}$ $\quad$ Should be zero

at middle: $x = 0$ $\quad w \sim \frac{1}{2} + \left[-2\sqrt{\varepsilon} + 2e^{-1/\sqrt{\varepsilon}}\right] \approx \frac{1}{2} - 2\sqrt{\varepsilon}$ (not significant)

Composite of composite really should be

$$w = \frac{1}{2}(1-x^2) + \sqrt{\varepsilon}\left[-1 + e^{-\frac{(1+x)}{\sqrt{\varepsilon}}} + e^{-\left(\frac{1-x}{\sqrt{\varepsilon}}\right)}\right]$$

composite left + composite right − common part.

$w = \frac{1}{2} - \sqrt{\varepsilon}$ at center which happens to be correct to 2nd order

to second order take 2nd global $= \frac{1}{2}(1-x^2) + \text{const}\sqrt{\varepsilon}$ is what we found

expand in local & keep 1 term

take 1st local

expand in global & keep 2 terms

the two results must match ⇒ const = −1

We can also define to multiplicative composite

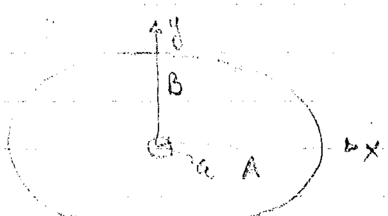
$$\text{Multiplicative} = \frac{\text{Global} \times \text{local}}{\text{Global of local} = \text{local of global}}$$

5/18/79

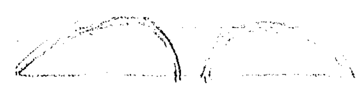
Composite = Global + local − Global (of local)

Local (Comp) = local (Global) + loc (local) − loc (Glob (local))
local (Global) + local(local) − loc (Glob)
= local

We go over the homework problem #6.1
- Forms of matched expansions
  1. Fractional powers
  2. powers & logs of ε
  3. Inverse powers of log. $\sum \frac{a_n}{(\log 1/\varepsilon)^n}$



assume $a/A \ll 1$ $\quad a/A = \varepsilon$



Refer x, y to global scale ie $x = \frac{\bar{x}}{A}$ $y = \frac{\bar{y}}{A}$ original eqn $\nabla^2 w = -\left\{\begin{matrix}(-dp/dx)/\mu \\ \Delta p/\sigma\end{matrix}\right.$ laminar flow / soap bubble

$\nabla^2 w = -1$

$w = 0$ on $\begin{cases} x^2 + y^2/b^2 = 1 \\ x^2 + y^2 = \varepsilon^2 \end{cases}$

For 1st Global Expression $\varepsilon = 0$ $\quad w_1 = \frac{b^2}{2(1+b^2)}\left(1 - x^2 - y^2/b^2\right)$

Try regular perturbation and set $w = w_1 + \varepsilon^2 w_2 + \dots$

$\Rightarrow \nabla^2 w_2 = 0$

$w_2 = 0$ on ellipse

$w_2 = \frac{-1}{\varepsilon^2} \frac{b^2}{2(1+b^2)}\left(1 - \varepsilon^2 - \varepsilon^2/b^2\right)$ at $x^2 + y^2 = \varepsilon^2$!!

obviously refer x, y to local scale $a = \varepsilon A$ $\quad X = \frac{x}{\varepsilon}$ $\quad Y = y/\varepsilon$ if $\bar{W} = w$

$\Rightarrow \bar{\nabla}^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$ $\quad \therefore \frac{1}{\varepsilon^2}\bar{\nabla}^2 W = -1$ or $\bar{\nabla}^2 W = -\varepsilon^2$ (pressure is negligible) (no scaling)

[if we want to keep the pressure $\varepsilon^2 \tilde{W} = w$
$\Rightarrow \tilde{\nabla}^2 W = -1$]

BC $\quad W = 0$ at $X^2 + Y^2 = 1 = R^2$

solving $\tilde{\nabla}^2 W = 0$ w/ $W = 0$ on $R^2 = 1$ for $\varepsilon = 0$

$$W = A \log \sqrt{X^2 + Y^2}$$

1st term global

$$w = \frac{b^2}{2(1+b^2)} \left[1 - x^2 - y^2/b^2 \right]$$

rewrite in local var

$$\tilde{W} = w = \frac{b^2}{2(1+b^2)} \left[1 - \varepsilon^2 \left[X^2 + Y^2/b^2 \right] \right]$$

expand & take 1 term local ie as $\varepsilon \to 0$

$$\tilde{W} = \frac{b^2}{2(1+b^2)}$$

1st term local

$$W = \tilde{A} \log (X^2 + Y^2) \qquad \tilde{A} = \frac{A}{2}$$

rewrite in global var

$$w = A \log (\sqrt{x^2 + y^2}) + A \log \varepsilon$$

expand in small ε & take 1st term global as $x, y \to 0$

$$w = -\tilde{A} \log \varepsilon^2$$

1 term global $\quad w = -A \log \varepsilon$

Equating $\quad \dfrac{b^2}{2(1+b^2)} = -A \log \varepsilon \qquad A = \dfrac{1}{2 \log 1/\varepsilon} \dfrac{b^2}{(b^2+1)}$

if we look at $\tilde{\nabla}^2 W = -1$ w/ $w = \varepsilon^2 W$ $\quad W = 0$ on $X^2 + Y^2 = 1$

$$\tilde{W} = \frac{1}{4}(1 - R^2) + A \log R + C\left(R^2 - \frac{1}{R^2}\right) \cos 2\theta$$

1st term local $\quad w = \varepsilon^2 W = \varepsilon^2 \left[A \log R + \frac{1}{4}(1 - R^2) + C\left(R^2 - \frac{1}{R^2}\right) \cos 2\theta \right]$

rewrite in local var

$$= \varepsilon^2 \left[A \log r/\varepsilon + \frac{1}{4}\left(1 - r^2/\varepsilon^2\right) + C\left(\frac{r^2}{\varepsilon^2} - \frac{\varepsilon^2}{r^2}\right) \cos 2\theta \right]$$

expand for small ε:

$$A\varepsilon^2 \log \frac{1}{\varepsilon} + \varepsilon^2 A \log r + \frac{1}{4}\varepsilon^2 - \frac{1}{4} r^2 + C r^2 \cos 2\theta - \varepsilon^4 \frac{C}{r^2} \cos 2\theta$$

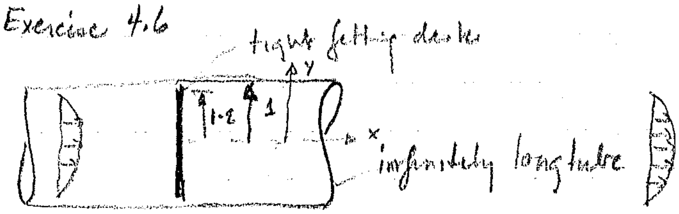
1 - term global
leading term is

if $A = C = O(1)$: $\quad -\dfrac{1}{4} r^2 + C r^2 \cos 2\theta \qquad$ can't match then if $A = O\left(\frac{1}{\varepsilon^2}\right)$ leading term is $\varepsilon^2 A \log\frac{1}{\varepsilon}$ $\Rightarrow A = \dfrac{b^2}{\varepsilon^2\, 2(1+b^2)\log\frac{1}{\varepsilon}}$

$$\therefore \quad \omega = \varepsilon^2 W = \frac{b^2}{2(1+b^2)\log\frac{1}{\varepsilon}} \quad \text{as previous}$$

5/21/79

Exercise 4.6



let Rad = A
δ = distance of gap

w/either disks moving though tube or keep disk fixed & move fluid through.
Ask what is drag as $\varepsilon \to 0$

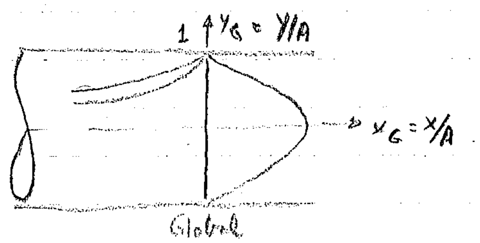
We can then pick global coordinates normalized on radius of tube

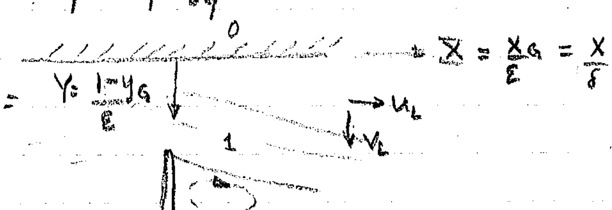
fluid is incompressible
plane or axisymmetric case
$R_N = \frac{U \cdot A}{\nu}$ A: radius of tube

Navier Stokes Eqn. - Steady
Continuity for plane case $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

x - mom $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu(\nabla^2 u)$

y - mom $u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu(\nabla^2 u)$





local problem: will appear as plane wall

$u_G = U/U_\infty$ $v_G = V/U_\infty$

$p_G = P/\rho U_\infty^2$

$u_G = \frac{1}{\varepsilon} U_L$

$v_G = \frac{1}{\varepsilon} V_L$

horiz velo = $\frac{1}{\varepsilon}$ times as big in the gap. u
vertic " = $\frac{1}{\varepsilon}$ " " " " " "
pressure $(\rho u^2) \sim \frac{1}{\varepsilon^2}$ if balancing inertia
$(\rho\nu \frac{du}{dx}) \sim \frac{1}{\varepsilon^2}$ " " viscous

pressure difference across the plate rises to $O(\frac{1}{\varepsilon^2})$ in distance of size of gap.

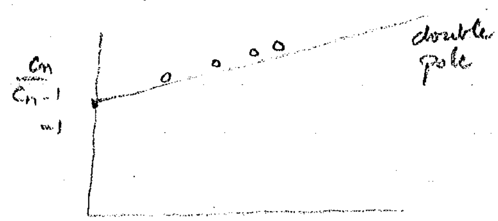
$$D = \int p \, dy = O(\tfrac{1}{\varepsilon^2})$$

when we through this into eqn we still get full Navier Stokes eqns in both local

Shail & Norton – circular disk of radius a moving slowly along axis of circular tube of unit radius – expanding in powers of a (radius of disk to tube radii)

$$\frac{D}{16\mu Ua} = 1 + 1.7684a + 3.11912a^2 + 4.5728a^3 + 6.1578a^4 + 7.6425a^5 + 9.1900a^6 + \dots$$

since the signs are fixed: nearest singularity on positive axis like $[(1-a)^2]^{-1}$



Drag $\sim \frac{1}{(1-a)^2}$

see insert

3/28/79

1851 Stokes: Sphere $D = 6\pi\mu Ua$

Circle $\nabla^4\psi = 0$ $\psi = \sin\theta\,(\frac{1}{r}, r, r\log r, r^3)$ doesn't satisfy bc at ∞

No slip: $\psi = c\,(r\log r - \frac{1}{2}r + \frac{1}{2}\frac{1}{r})$ Blows up at infinity

1910 Oseen: perturbs sphere results by introducing perturbation of free stream.

$$D = 6\pi\mu Ua\left[1 + \frac{3}{8}R\right]$$

1911 Lamb: for circle $D' = 4\pi\mu U \dfrac{1}{\log 4/R - \gamma + \frac{1}{2}}$ $R = \frac{Ua}{\nu} = \frac{a}{\nu/U} = \frac{\text{geometric}}{\text{viscous}}$

1957: Proudman & Pearson for sphere $D = 6\pi\mu Ua\left[1 + \frac{3}{8}R + \frac{9}{48}R^2\log R + O(R^2)\right]$

and circle $D' = 4\pi\mu U\left[\dfrac{1}{\log(1/R)} - \dfrac{\log 4 - \gamma + \frac{1}{2}}{(\log 1/R)^2} + O\left(\dfrac{1}{\log\frac{1}{R}}\right)^3\right]$

"purely log case"

1957: Kaplun for circle $D' = 4\pi\mu U\left[\dfrac{1}{\log(1/R)} - \dfrac{\log 4 - \gamma + \frac{1}{2}}{[\log(1/R)]^2} + .87\left(\dfrac{1}{\log\frac{1}{R}}\right)^3\right]$

Kaplun telescoped this result to get $4\pi\mu U\left[\dfrac{1}{\log 4/R - \gamma + 1/2} - \dfrac{.87}{(\log 4/R - \gamma + 1/2)^3} + O(\right.$

matching for circle $r = \frac{\text{radius}}{a}$ $\rho = \frac{\text{radius}}{\nu/a}$

1st global approx $\psi = y$ (dimensionless) ψ = streamfn$/Va$ $= \frac{Uy}{Ua}$

$\psi = y = (r\sin\theta) = \rho/R \sin\theta$

rewrite in local var $= r\sin\theta$

expand for small R $= r\sin\theta$

1 - term local $= r\sin\theta$

1 term local sol $\psi = c\,(r\log r - \frac{1}{2}r + \frac{1}{2}r)\sin\theta$

rewrite in global var $= C\,(\rho/R \log \rho/R - \frac{1}{2}\rho/R + \frac{R}{2\rho})\sin\theta$

expand for small R $= C\left(\frac{1}{R}\log\frac{1}{R}\cdot\rho + \frac{1}{R}(\rho\log\rho - \frac{1}{2}) + \frac{1}{2}R\frac{1}{\rho}\right)\sin\theta$

1-term global $= C\left(\frac{1}{R}\log\frac{1}{R}\cdot\rho\sin\theta\right)$

$$r\sin\theta = \rho/R\sin\theta = C\rho/R\log\frac{1}{R}\sin\theta \quad \therefore \quad C = +\frac{1}{\log 1/R}$$

C is not a pure constant

Drag/unit length of circle $D' = 4\pi\mu U\left[\Delta - .87\Delta^3 + O(\Delta^4)\right] + 4\pi\mu\left[-\frac{1}{32}R^2(1 - \frac{1}{2}\Delta + \ldots)\right]$

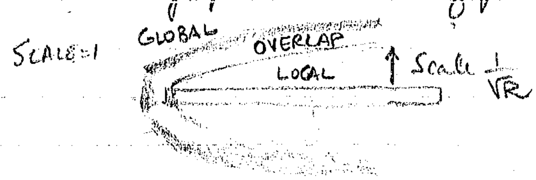
where $\Delta = \frac{1}{\log\frac{4}{R} - \gamma + \frac{1}{2}}$

which will give the ballpark soln to separation point

5/25/79

3 Results of Frankel

1. "Asymptotic matching principle" vs. matching by intermediate limits



in Asymptotic matching principle - counting of terms can be ambiguous ∴ go out to a particular gage function & not a particular term ie go to ε, or ε² or ε log ε instead of 2nd or 4th term.

2. Asymptotic matching principle may ~~fail~~ give incorrect results if there are cuts between logarithms.

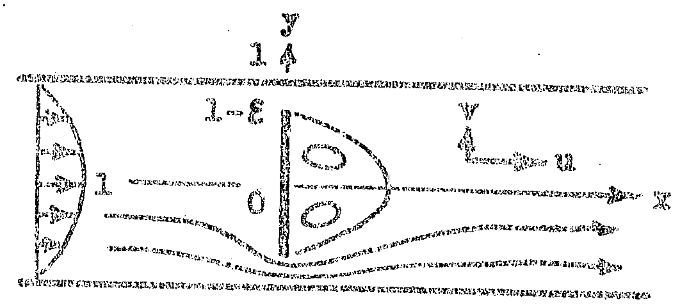
3. Asymptotic matching may fail in "forbidden regions" in "purely logarithmic case".

Solution to Exercise 4.6 of Notes

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

25 May 1979

4.6. Qualitative matching for viscous flow past barrier. The pattern of streamlines to be expected is as sketched. Here it is natural to refer lengths to the half-width h of the channel, velocities to the maximum upstream speed U, and pressure to ρU^2. Then the dimensionless Navier-Stokes equations are

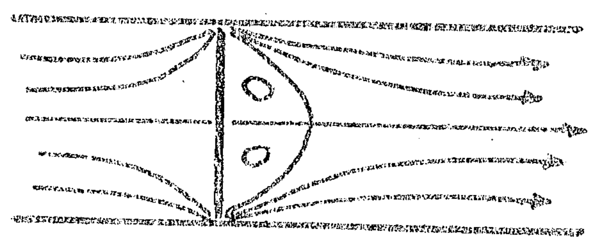


FULL DIMENSIONLESS PROBLEM

$$u_x + v_y = 0$$
$$uu_x + vu_y + p_x = (u_{xx} + u_{yy})/R$$
$$uv_x + vv_y + p_y = (v_{xx} + v_{yy})/R$$

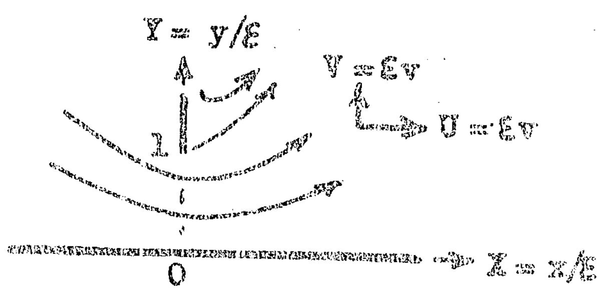
The Reynolds number $R = Uh/\nu$ may be small or large (but not so large that the flow becomes unsteady and turbulent).

If we approximate for small gap ratio $\varepsilon = d/h$, the first approximation (and all higher ones) will evidently be singular at the gap. We therefore supplement this global picture with a local one, in which the origin is shifted to (say) the bottom wall, and the coordinates magnified by a factor $1/\varepsilon$ (so that the gap remains of unit length in the limit $\varepsilon \to 0$).


GLOBAL APPROXIMATION

The global velocity components u and v will be of order unity as $\varepsilon \to 0$ except near the gap. There continuity of flow shows that u must be large like $1/\varepsilon$; and the continuity equations indicates that the same is true of v. Hence we can introduce local velocities of order unity near the gap by setting

$$u(x,y) = \frac{1}{\varepsilon}U(X,Y), \quad v(x,y) = \frac{1}{\varepsilon}V(X,Y)$$



LOCAL APPROXIMATION

Then substituting into the x-momentum equation shows that

$$p_x \sim \frac{1}{\varepsilon^3}UU_X \quad \text{or} \quad \frac{1}{\varepsilon^3}\frac{1}{R}U_{XX}$$

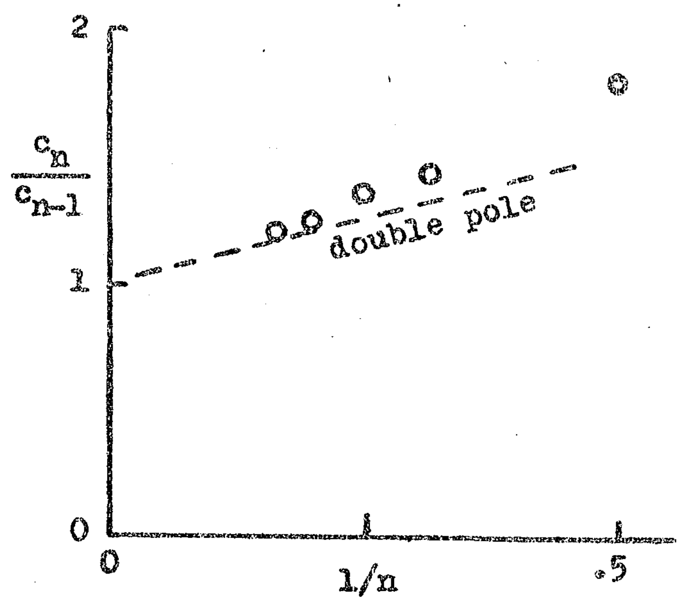
according as the pressure gradient is balanced by inertia at high Reynolds number or viscosity at low Reynolds number. In any case, this means that pressure in the gap is large like $1/\varepsilon^2$, so we can set

$$p(x,y) = \frac{1}{\varepsilon^2}P(X,Y).$$

Thus in terms of global variables the pressure difference across the plate will rise from zero at the edge to some multiple of $1/\varepsilon^2$ in a distance of order $1/\varepsilon$. Thereafter it will change only slightly, since p_y is of order unity in the global region, until it falls rapidly to zero again at the other edge. Hence the drag (referred to $\rho U^2 h$ or μU) will be proportional to $1/\varepsilon^2$ for small ε.

This conclusion evidently applies equally well to a circular disk in a circular tube (or even an elliptic disk in an elliptic tube), and to a disk moving through the tube as well as one fixed in it. We can therefore check this conclusion at low Reynolds numbers, because Shail and Norton (1969: Proc. Camb. Phil. Soc. 65, 793) have calculated approximately the drag of a circular disk of radius a that slowly moves broadside along the axis of a circular tube of radius unity. If the disk is very small, its drag is unaffected by the tube, and is known to be equal to $16\pi\mu U a$. For a larger disk, this value is increased by a factor that Shail and Norton compute as a regular perturbation expansion to seven terms:

$$1 + 1.786\,a + 3.1912\,a^2 + 4.5728\,a^3 + 6.1578\,a^4 + 7.6425\,a^5 + 9.1900\,a^6 + \dots$$

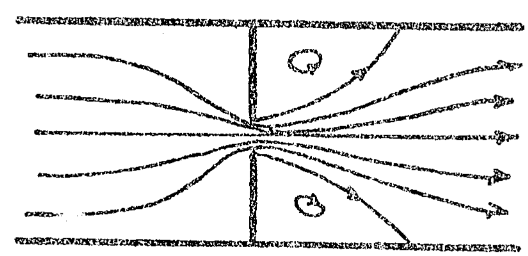


This series converges slowly for a approaching unity as the gap disappears. However, we can apply d'Alembert's ratio test in the graphical form devised by Domb and Sykes. The ratios of successive coefficients c_n/c_{n-1}, plotted versus $1/n$, clearly extrapolate linearly to unit radius of convergence — indicating a singularity at $a = 1$ — and to a double pole. If we extract that singularity multiplicatively, the series becomes

$$\frac{1}{(1-a)^2}\left[1 - .2136\,a + .6184\,a^2 - .0232\,a^3 + .2034\,a^4 - .1003\,a^5 + .0628\,a^6 + \dots\right]$$

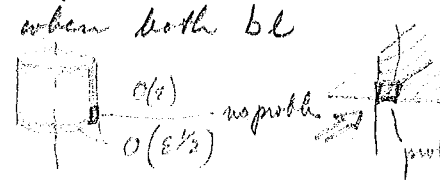
Here the last coefficients are one per cent of the corresponding values before extraction of the double pole, and decreasing rapidly. This confirms that the drag is singular like $1/(1-a)^2$ or $1/\varepsilon^2$.

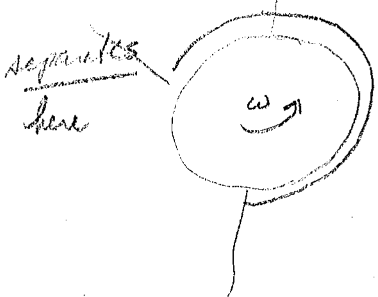
The same conclusion applies to a plate with a small slit through which viscous fluid is forced. In this case, for plane flow at low Reynolds number, the local approximation is known in closed form. As Morse and Feshbach show (Methods of Theoretical Physics, p. 1197) the slow viscous flow through a slit is found by separating variables in elliptic coordinates. (Note, however, that in applying that result to flow in the ear, Prof. Steele has found factors of 4 missing in several equations.)



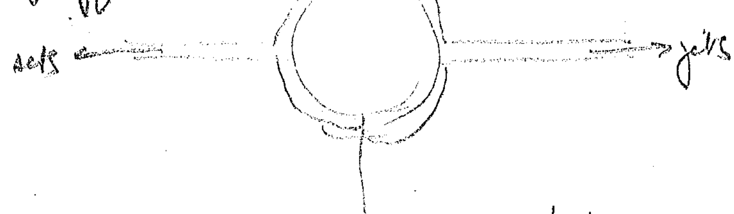
Multiple Boundary Layers

1. Concentric – Layering just match each layer to its neighbor
2 Intersecting or Overlapping. – difficulty is when both bl are of same Order (thickness)
3. Colliding b.l.





but since there are two they collide and go off



5/30/79

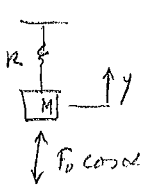
Check Nayfeh's book out - if it's good buy it. Perturbation Methods 1973.

Multiple scales

Slowly modulated oscillations involving 2 time scales which are of equal importance through region. Unlike matched asympt. exp. where either of the scales are dominant in their appropriate region

Near Resonant excitation of linear oscillator pg 90 J notes

Natural freq $\omega = \sqrt{\frac{k}{m}}$



$$m\ddot{y} + ky = F_0 \cos\alpha t \qquad y(0) = \dot{y}(0) = 0$$

$$\frac{1}{\omega^2}\ddot{y} + ky = \frac{F_0}{k}\cos\alpha t$$

Refer time to reciprocal of natural freq $\frac{1}{\omega} = \sqrt{m/k}$ & y to static deflect

let $\frac{\partial}{\partial \tau} = \frac{1}{\omega}\frac{\partial}{\partial t}$ $\quad \bar{Y} = \frac{k}{F_0} y \quad \Rightarrow \quad \bar{y}'' + \bar{y} = \cos\alpha t = \cos(1-\varepsilon)\tau$ if $\alpha = \omega(1-\varepsilon)$

w/ IC $\bar{y}(0) = \bar{y}'(0) = 0$

Exact sol. $\bar{y} = \frac{1}{\varepsilon(2-\varepsilon)}\left[(\cos\varepsilon t - 1)\sin t + \sin\varepsilon t \cos t\right]$

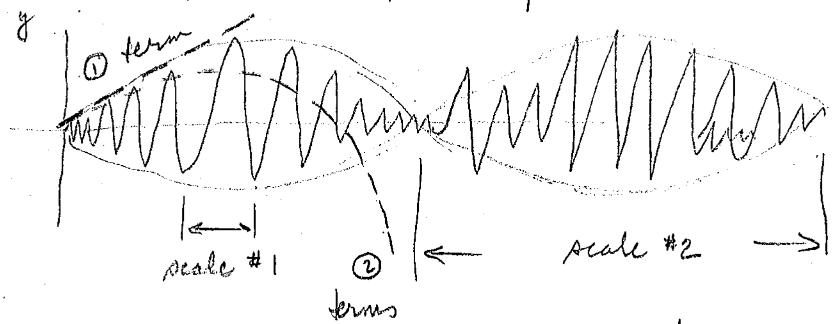
Regular perturb $\cos(1-\varepsilon)\tau = \cos\tau\cos\varepsilon\tau + \sin\tau\sin\varepsilon\tau \approx \cos\tau + \varepsilon\tau\sin\tau$

we now find that $\bar{y} = \overset{①}{t\sin\tau} - \frac{1}{4}\varepsilon(\overset{②}{t^2\cos\tau - \tau\sin\tau}) + O(\varepsilon^2 t^3)$

$\frac{2^{nd}\text{ term}}{1^{st}\text{ term}} = -\frac{1}{4}\varepsilon\left(t\frac{\cos\tau}{\sin\tau} - 1\right)$ small unless $t = O(\frac{1}{\varepsilon})$

"Secular term"

this is not correct since we lost periodicity and boundedness of original problem



$\Rightarrow$ boundary layer at ∞ if we let $T = \varepsilon t$ and $\bar{y} = \frac{1}{\varepsilon} Y(\tau)$

then the DE $\Rightarrow$ $\varepsilon^2 Y'' + Y = \varepsilon \cos\left(\frac{T}{\varepsilon} - T\right)$ since ε^0 prob $Y = 0$

which gives as a first approx $Y = 0$ which doesn't help; this really not bl.

look at two scales let fast scale be τ

let slow scale be $T = \varepsilon \tau$

we tried expanding $y = y_1(\tau) + \varepsilon y_2(\tau) + \varepsilon^2 y_3(\tau) + \cdots$ failed at $t = O(1/\varepsilon)$

now try two scales

$$\bar{y} = y_1(\tau, T) + \varepsilon y_2(\tau, T) + \varepsilon^2 \ldots$$

this will show that $Y = \bar{y}/\varepsilon$ for problem scaling

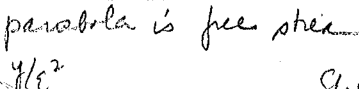
Calculate derivatives by chain rule:

$$\dot{(\ )} = \frac{\partial}{\partial \tau} + \varepsilon \frac{\partial}{\partial T}$$

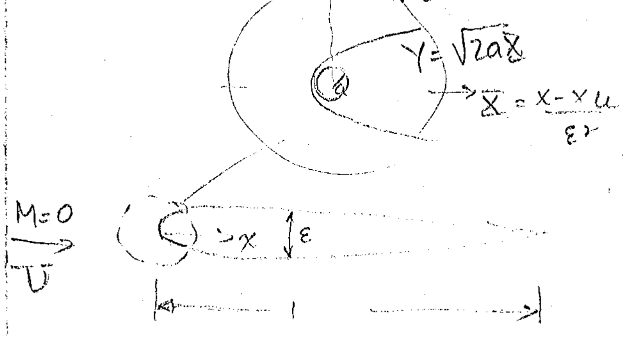
6/1/79

Exam handed out Wed 6 June due 11 June

Munk's rule - Surface speed on an ellipsoid in flow along principal axis proj. of max.



Surface speed on parabola is free stream times cosine of angle



Slope = tangent of angle $= \frac{dY}{dX} = \sqrt{\frac{a}{2X}}$

$$\cos\theta = \frac{\sqrt{2X}}{\sqrt{a + 2X}}$$

$$\text{Surface speed} = U\sqrt{\frac{2X}{a + 2X}} = \frac{U}{\sqrt{1 + \frac{a}{2X}}} = \frac{U}{\sqrt{1 + \frac{\varepsilon^2 a}{2x}}}$$

$$\approx U\left[1 - \frac{\varepsilon^2 a}{4x} + \ldots\right]$$

Near Resonant Excitation of linear oscillator – continued

Dimensionless $\ddot{y} + y = \cos(1-\varepsilon)t$, $y(0) = \dot{y}(0) = 0$

we saw regular perturb fails when $t = O(1/\varepsilon)$

We also saw no boundary layer approx would help

Multiple scale $\tau = t$ (fast) $\quad T = \varepsilon t$ (slow)

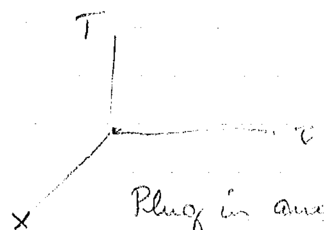
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \varepsilon\frac{\partial}{\partial T} \qquad \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \tau^2} + 2\varepsilon\frac{\partial^2}{\partial\tau\partial T} + \varepsilon^2\frac{\partial^2}{\partial T^2}$$

Full problem

$$\left(\frac{\partial^2 y}{\partial \tau^2} + 2\varepsilon\frac{\partial^2 y}{\partial\tau\partial T} + \varepsilon^2\frac{\partial^2 y}{\partial T^2}\right) + y = \cos(\tau - T)$$

w/ ic $y = 0$ at $\tau = T = 0$

$0 = \dot{y}(0) \Rightarrow \frac{\partial y}{\partial\tau} + \varepsilon\frac{\partial y}{\partial T}$ at $\tau = T = 0$



Expanding Let $y = \frac{1}{\varepsilon}y_1(\tau, T) + y_2(\tau, T) + \dots$

Plug in and look at ε^0 term

$\frac{1}{\varepsilon}$: $\quad \partial^2 y_1/\partial\tau^2 + y_1 = 0 \qquad$ w/bc $\quad y(0) = 0 \Rightarrow y_1(0) = 0$

$\dot{y}(0) = 0 \Rightarrow \frac{\partial y_1}{\partial \tau}(0) = 0$

ε^0: $\quad \frac{\partial^2 y_2}{\partial\tau^2} + 2\frac{\partial^2 y_1}{\partial\tau\partial T} + y_2 = \cos(\tau - T) \qquad$ w/ ic $\quad y_2(0) = 0$

$\frac{\partial y_2}{\partial \tau} + \frac{\partial y_1}{\partial T} = 0$ @ $t = 0, T = 0$

Classical asymptotic expansion (Poincaré) $f(x;\varepsilon) \sim \sum_{n=0} \varepsilon^n f_n(x)$ as $\varepsilon \downarrow 0$

Generalized (Erdélyi) $f(x;\varepsilon) \sim \sum \varepsilon^n f_n(x;\varepsilon)$

Soln to 1st Problem if $y_1 = A(T)\cos\tau + B(T)\sin\tau$

$y_1(0,0) = 0 \Rightarrow y_1 = A(0) + B(0)\cdot 0 \Rightarrow \boxed{A(0) = 0}$

$dy_1/d\tau\,(0,0) = 0 \Rightarrow dy_1/d\tau = 0 \Rightarrow \boxed{B(0) = 0}$

Soln to 2nd Problem will (should) give the fns $A(T)$; $B(T)$

$$\frac{\partial^2 y_2}{\partial\tau^2} + y_2 = -2(B'\cos\tau - A'\sin\tau) + \cos\tau\cos T + \sin\tau\sin T$$

$$= \cos\tau\left[-2B' + \cos T\right] + \sin\tau\left[2A' + \sin T\right]$$

choose A, B to avoid nonuniformity

best way is to annihilate rhs

$A' = -\tfrac{1}{2}\sin T \qquad B' = \frac{\cos T}{2}$

$A = +\tfrac{1}{2}\cos T + A_0 \qquad B = \frac{\sin T}{2} + B_0$

using $A(0) = 0$ $\quad B(0) = 0$ $\quad A_0 = -\tfrac{1}{2}$ $\quad B_0 = 0$

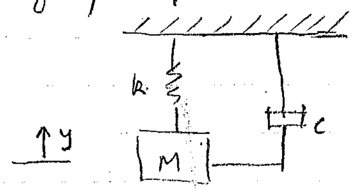
thus we have a uniformly valid first order approx

To get 2nd order approx solve ε^0 problem with BC and examine

ε^1 equation as we did before to find the functions A, B of the slow/fast variable so that we can make non uniformity disappear

5/4/79

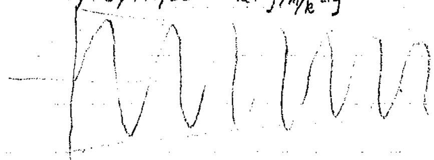
Slightly damped Oscillator



$\ddot{y} + 2\varepsilon\dot{y} + y = 0 \Rightarrow y = Ce^{-\varepsilon t}\cos(\sqrt{1-\varepsilon^2}\,t + p)$

$m\ddot{y} + c\dot{y} + k\hat{y} = 0$ let $\hat{y}/m/k = y$

slow (the $e^{-\varepsilon t}$ factor), fast (the $\sqrt{1-\varepsilon^2}\,t$ argument)



→ try regular perturb $\quad y/c = \cos(t+p) - \varepsilon t\cos(t+p) + \varepsilon^2\left[\frac{1}{2}t^2\cos(t+p) + \frac{1}{2}t\sin(t+p)\right] + \ldots$

note secular term cannot be neglected if $t = O(1/\varepsilon)$

→ if we use 2 simple scale $t = \tau \quad T = \varepsilon t$

then $e^{-T}\cos(\sqrt{1-\varepsilon^2}\,\tau + p) \sim e^{-T}\left[\cos(\tau+p) + \frac{1}{2}\varepsilon^2 t\sin(\tau+p) + O(\varepsilon^4\tau^3)\right]$

invalid when $\tau = t = O(1/\varepsilon^2)$

If we want to go further we can

① define $T_2 = \varepsilon^2 t$ a much slower time & will give results that are valid for $t < O(1/\varepsilon^3)$

etc let $T_3 = \varepsilon^3 t$, $T_4 = \varepsilon^4 t$ ⟶ this gives us an infinite number of successive slower scales

we must then define $\frac{d}{dt} = (\dot{\ }) = \frac{\partial}{\partial\tau} + \varepsilon\frac{\partial}{\partial T} + \varepsilon^2\frac{\partial}{\partial T_2} + \varepsilon^3\frac{\partial}{\partial T_3} + \ldots + \varepsilon^n\frac{\partial}{\partial T_n}$

"Derivative - Expansion Method"

Versions of multiple scales

1. Two simple scales $\quad t, \varepsilon t$
2. Infinite no. $\quad t, \varepsilon t, \varepsilon^2 t, \ldots$
3. Two more difficult scales ie stretch fast scales $\quad \varepsilon t, \ \tau = t(1 + a\varepsilon + b\varepsilon^2 + c\varepsilon^3 + \ldots)$

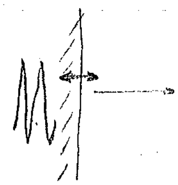
   a can be taken as zero since εt already appears in the slower scale

   if $\tau = t$ then let $T = \varepsilon t(1 + A\varepsilon + B\varepsilon^2 + \ldots)$ note here since the linear scale is already covered in τ don't need it in T. ⇒ start at εt

4. Et Cetera:
   - Stretch both scales
   - More than two scales
   - "Strained" Scales $\qquad \tau = t + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \ldots$

Plane waves with viscous damping

w/o damp $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow f(x-ct) = u$



w/damping $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \frac{4\nu}{3} \frac{\partial^3 u}{\partial x^2 \partial t}$ if $\frac{4\nu\omega}{3c^2} \sim \varepsilon$

Solution $u = e^{\alpha x} \cos(t - \beta x)$ see pg 97

$u = U \cos\omega t$ @ $x=0$

$\alpha = \frac{1}{2}\varepsilon - \frac{5}{16}\varepsilon^3 + \frac{63}{128}\varepsilon^5 + \dots$

$\beta = 1 - \frac{3}{8}\varepsilon^2 + \frac{35}{128}\varepsilon^4 + \dots$

} stretching both scales.

If regular perturb fails when $t = O(\frac{1}{\varepsilon})$ then in general, multiple scales will only give results valid to $t = O(\frac{1}{\varepsilon})$. In linear problems you can indefinitely extend soln but in non linear case you cannot extend it indefinitely

6/6/79

A Aging Spring H Cheng & T.T. Wu SIAM 46 (1970) 183–185

"Regular" pert fails to $t = O(1/\varepsilon)$

" Multiple scales OK to $t = O(1/\varepsilon)$

not generally true to $t = O\left(\frac{1}{\varepsilon^{3/2}}\right)$

Linear harmonic oscill $\ddot{x} + e^{-\varepsilon t} x = 0$ $\left.\begin{matrix} x = 0 \\ \dot{x} = 1 \end{matrix}\right\}$ @ $t = 0$

Exact soln $x = \frac{\pi}{\varepsilon}\left[Y_0\left(\frac{2}{\varepsilon}\right) J_0\left(\frac{2}{\varepsilon} e^{-\varepsilon t/2}\right) - J_0\left(\frac{2}{\varepsilon}\right) Y_0\left(\frac{2}{\varepsilon} e^{-\varepsilon t/2}\right)\right]$

Uniform approx for small ε $x \sim \sqrt{\frac{\pi}{\varepsilon}}\left[\sin\left(\frac{2}{\varepsilon} - \frac{\pi}{4}\right) J_0(\;) - \cos\left(\frac{2}{\varepsilon} - \frac{\pi}{4}\right) Y_0(\;)\right]$

can't touch those factors where $\frac{2}{\varepsilon} e^{-\varepsilon t/2}$ since at $t \to \infty$ variable → 0

Straightforward expansion

$x = \sin t + \frac{1}{4}\varepsilon\,(t^2 \cos t - t \sin t) + O(\varepsilon^2)$ fails at $t = O(\frac{1}{\varepsilon})$

Multiple scales

$x \sim e^{\varepsilon t/4} \sin\left[\frac{2}{\varepsilon}(1 - e^{-\varepsilon t/2})\right]$ valid only for $\varepsilon e^{\varepsilon t/2} \ll 1$

Not discussed

1. WKB (linear problems)
2. Ray Methods – Geometrical Optics
3. Method of Strained Coordinates PLK (Poincaré – Lighthill – Kuo) unreliable

Exercise Set 1

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

Due ~~Monday 10 April 1979~~
Wed 12 Apr 79

1.1. Solution of quadratic equation: a parameter perturbation. When $\varepsilon = 0$, the equation

$$\varepsilon x^2 + x - c = 0$$

has the solution $x = c$. Find a better approximation to that root when ε is small but not zero, to at least $O(\varepsilon^2)$, by (a) iterating on the approximation $x = c$, and (b) substituting an assumed expansion in powers of ε. Which of these processes is the more systematic? The simpler in higher approximations? Check your result by expanding the exact solution for small ε. For what values of ε does the series converge? What happened to the other root in this process? Devise a procedure for finding an analogous expansion for it, and calculate two or three terms.

1.2. Solution of an ordinary differential equation: a coordinate perturbation. Suppose that we are interested in the general solution of

$$\cos\sqrt{t}\,\frac{dx}{dt} - \frac{x}{t} = 1$$

for small time. The familiar expression (involving an integrating factor) for the solution of any first-order linear differential equation leads in this case to unfamiliar integrals that cannot be expressed in simple terms. Find a first approximation for small t by replacing the coefficient $\cos\sqrt{t}$ by its value at $t = 0$ and solving. Then improve on this in a systematic way to find the second approximation for small time.

Cesar Levy
ME 207
Prof. Van Dyke

Problem Set # 1

1.1a. Iteration Method

Let $x_{n+1} = c - \epsilon x_n^2$ with $x_0 = c$

$x_1 = c - \epsilon x_0^2 = c - \epsilon c^2 = c(1-\epsilon c)$ ✓

$x_2 = c - \epsilon c^2(1-\epsilon c)^2 = c - \epsilon c^2(1-2\epsilon c+\epsilon^2 c^2) = c - \epsilon c^2 + 2\epsilon^2 c^3 - \epsilon^3 c^4 = c\left[1-\epsilon c(1-\epsilon c)^2\right]$

$x_3 = c - \epsilon x_2^2 = c - \epsilon c^2\left[1-\epsilon c(1-\epsilon c)^2\right]^2 = c - \epsilon c^2 + 2\epsilon^2 c^3 + O(\epsilon^3)$ ✓

b. Expansion Method

Let $x = \sum_{i=0}^{\infty} a_i \epsilon^i$; then $\epsilon x^2 + x - c = \sum_{i=1}^{\infty}\left[\sum_{j=0}^{i-1} a_j a_{i-1-j} - a_i\right]\epsilon^i + a_0 - c = 0$

by equating like powers we can obtain the a_i's — **would you believe 14 ?**

$\therefore \quad x = c - \epsilon c^2 + 2c^3\epsilon^2 - 5c^4\epsilon^4 - 42c^6\epsilon^5 + - + - \ldots$

- The iteration method is more systematic and much easier to handle for higher approximations

The exact soln is $x_{1,2} = \dfrac{-1 \pm \sqrt{1+4\epsilon c}}{2\epsilon}$

expanding $\sqrt{1+4\epsilon c}$ for small ϵ, taking the plus sign before the $\sqrt{\ }$ we obtain

$$x = \frac{-1 + \left(1 + 2c\epsilon - 2c^2\epsilon^2 + 4c^3\epsilon^3 + O(\epsilon^3)\right)}{2\epsilon} = c - c^2\epsilon + 2\epsilon^2 c^3 + O(\epsilon^3)$$ ✓

from the above we can get the n^{th} term $\dfrac{2^{n-1}c^n\epsilon^{n-1}(-1)^{n+1}\prod_{k=0}^{n}(2k-1)}{n!}$

using the ratio test we obtain that for $|4c\epsilon| < 1$ we get convergence for fixed c, ϵ

$\therefore \quad |\epsilon| < \dfrac{1}{|4c|}$ for convergence of series ✓

To obtain the other root we note that $x_1 \cdot x_2 = -c$. The second root should begin with $-\frac{1}{\epsilon} + \ldots$

so that I assume $x = \sum_{i=0}^{\infty} a_i \epsilon^{i-1}$

putting this into the eqn we get

$$\epsilon x^2 + x - c = \left(\frac{a_0^2 + a_0}{\epsilon}\right) + (2a_0a_1 + a_1 - c) + (a_1^2 + 2a_0a_2 + a_2)\epsilon + (2a_0a_3 + 2a_1a_2 + a_3)\epsilon^2 + \ldots$$

from the first term for nontrivial soln $a_0 = -1$; this leads to $a_1 = -c$, $a_2 = c^2$, $a_3 = -2c^3$, etc

hence $x = -\frac{1}{\epsilon} - c + c^2\epsilon - 2c^3\epsilon^2 + O(\epsilon^2)$ ✓

which converges again for $|\epsilon| < \dfrac{1}{4c}$

the actual soln using $\frac{-1-\sqrt{1+4\epsilon c}}{2\epsilon} = \frac{-2-2c\epsilon+2c^2\epsilon^2-4\epsilon^3c^3+O(\epsilon^3)}{2\epsilon}$

$$x = -\frac{1}{\epsilon} - c + c^2\epsilon - 2\epsilon^2c^3 + O(\epsilon^2)$$

as to what happened to the other solution – nothing; when we put in the expansion we are only solving for one root at a time. We could obtain the second root if we remember that

$$(x-x_1)(x-x_2) = x^2-(x_1+x_2)x + x_1x_2 = x^2+\frac{x}{\epsilon}-\frac{c}{\epsilon}$$ and that if $-x_1-x_2=\frac{1}{\epsilon}$

then $x_2 = -\frac{1}{\epsilon}-x_1$ (or more difficult if $x_1x_2 = -\frac{c}{\epsilon}$ then $x_2 = -\frac{c}{x_1\epsilon}$)

If we try the iteration method on this problem it becomes exceedingly difficult to obtain a solution

1.2 @ small t $\cos\sqrt{t} \approx 1$ and DE is $\frac{dx}{dt}-\frac{x}{t}=1$ which leads to $d(x/t) = \frac{dt}{t}$

hence $\frac{x}{t} = \ln t + c$ or $\underline{x = t\ln t + ct}$ ✓

Now $\cos\sqrt{t} = 1-\frac{1}{2}t+\frac{1}{4!}t^2-\frac{1}{6!}t^3 + \ldots$

taking $\cos\sqrt{t} = 1-\frac{1}{2}t$ then

$$(2-t)\frac{dx}{dt} - \frac{2x}{t} = 2$$ ✓ using the integrating factor $\mu(t) = e^{-\int\frac{2}{t(2-t)}dt} = t^2-2t$

Yes No!

then

$$x = \frac{1}{t^2-2t}\int_0^t \frac{2}{2-t'}\cdot t'(t'-2)\,dt' + \frac{c}{t^2-2t} = \frac{t}{2-t} + \frac{c}{t^2-2t}$$

But this is a proper method.

④

1.2: @ $t=0$ $\cos\sqrt{t}=1$ and DE is $\frac{dx}{dt}-\frac{x}{t}=1$ or $\frac{t\,dx-x\,dt}{t^2}=\frac{t\,dt}{t^2}$

$d\left(\frac{x}{t}\right)=\frac{dt}{t}$

$\cos u = 1-\frac{u^2}{2!}+\frac{u^4}{4!}+\dots$

$\frac{x}{t}=\ln t + c$ or $x = t\ln t + ct$

$\cos\sqrt{t} = 1-\frac{1}{2}t+O(t)$

$(2-t)\frac{dx}{dt}-\frac{2x}{t}=2$: $\frac{dx}{dt}-\frac{2}{t(2-t)}x=\frac{2}{2-t}$

use integrating factor $\mu(t)=e^{-\int\frac{2}{t(2-t)}dt}$ $\quad \int\frac{2}{t(2-t)}dt=\int\frac{dt}{t}-\frac{dt}{2-t}=\ln t+\ln(2-t)=\ln(2t-t^2)$

$\mu(t) = -2t+t^2$

$x=\frac{1}{-2t+t^2}\int\frac{2}{(2-t)}t(t-2)dt+\frac{c}{t^2-2t}=\frac{t^2}{(2-t)t}+\frac{c}{t^2-2t}=\frac{t}{2-t}+\frac{c}{t^2-2t}$

✓

1.1b let $x=\sum_{i=0}^{\infty}a_i\epsilon^i$ then $\epsilon x^2+x-c=\sum_{i=1}^{\infty}\left[\left(\sum_{j=0}^{i-1}a_j a_{i-1-j}\right)-a_i\right]\epsilon^i+a_0-c=0$

$\therefore\quad x = c-c^2\epsilon+2c^3\epsilon^2-5c^4\epsilon^3+14c^5\epsilon^4-42c^6\epsilon^5+-+-\dots$

this is easier for prediction

for series to converge $\frac{\left|\sum_{j=0}^{i}(a_j a_{i-j})\right|\epsilon^{i+1}}{\left|\sum_{j=0}^{i-1}(a_j a_{i-1-j})\right|\epsilon^i}=\left|\frac{Nc^{i+2}\epsilon^{i+1}}{Mc^{i+1}\epsilon^i}\right|<1$ or

$\frac{N}{M}c\epsilon<1$ or $\epsilon<\frac{1}{c}\frac{M}{N}$

<u>Exact</u> ~~actual~~ is $\frac{-1\pm\sqrt{1+4c\epsilon}}{2\epsilon}$ one root is $\frac{-1+\sqrt{1+4c\epsilon}}{2\epsilon}=\frac{-1+(1+2c\epsilon-2c^2\epsilon^2+4c^3\epsilon^3+O(\epsilon^3)}{2\epsilon}$

$= c-c^2\epsilon+2c^3\epsilon^2+O(\epsilon^2)$ we can get the n^{th} term $\frac{2^{n-1}c^n\epsilon^{n-1}(-1)^{n+1}\prod_{k=0}^{n}(2k-1)}{n!}$

converges for $|4c\epsilon|<1$ or $|\epsilon|<\frac{1}{4c}$

the other root is $\frac{-1-\sqrt{\ }}{2\epsilon}=\frac{-1-(1+2c\epsilon-2c^2\epsilon^2+4c^3\epsilon^3+O(\epsilon^3))}{2\epsilon}$

$=\frac{-2-2c\epsilon+2\epsilon^2c^2-4c^3\epsilon^3+O(\epsilon^3)}{2\epsilon}$

$=-\frac{1}{\epsilon}-c+c^2\epsilon-2c^3\epsilon^2+O(\epsilon^2)$ n^{th} ~~order~~ term is same as above

1.1b. since product of roots must be $\frac{-c}{\epsilon}$ then second root should begin with $-\frac{1}{\epsilon}+\dots$

so for this I will assume $x = \frac{a_0}{\epsilon} + a_1 + a_2\epsilon + a_3\epsilon^2 + \dots$ ✓

then we find that $x^2 = \frac{a_0^2}{\epsilon^2} + \frac{2a_0a_1}{\epsilon} + (a_1^2 + 2a_0a_2) + (2a_0a_3 + 2a_1a_2)\epsilon + \dots$

then $\epsilon x^2 + x - c = \left(\frac{a_0^2}{\epsilon} + \frac{a_0}{\epsilon}\right) + (2a_0a_1 + a_1 - c) + (a_1^2 + 2a_0a_2 + a_2)\epsilon + (2a_0a_3 + 2a_1a_2 + a_3)\epsilon^2$

$c^2 - a_2$

$(a_0^2 + a_0) = 0 \Rightarrow a_0 = -1$ or $a_0 = 0$ not what we want

$-2a_1 + a_1 - c = 0 \Rightarrow a_1 = -c$; $a_2 = +c^2$; $a_3 = -2c^3$

1.1a ✓

iteration is $x_{n+1} = c - \epsilon x_n^2$ if $x_0 = c$ $x_1 = c - \epsilon c^2$

$x_2 = c - \epsilon[c - \epsilon c^2]^2 = c - \epsilon c^2(1 - \epsilon c)^2$

$= c - \epsilon c^2 + 2\epsilon^2 c^3 - ~~\epsilon^3 c^4~~ O(\epsilon^2)$

the other iteration for $\epsilon x^2 + x - c = 0$ is

a way to the root ~~#2~~ is to remember that the sum of the roots $= -\frac{1}{\epsilon}$

~~and the product is $\frac{c}{\epsilon}$~~

since ~~$x_1 = c - c^2\epsilon + 2c^3\epsilon^2 + O(\epsilon^2)$~~ $(x_1)_0 = c$

and $x_{1_n} + x_{2_n} = -\frac{1}{\epsilon}$ then $(x_2)_0 = -\frac{1}{\epsilon} - c$

~~$x_2 = -\frac{1}{\epsilon} - x_1 = -\frac{1}{\epsilon} - c + c^2\epsilon - 2c^3\epsilon^2 + O(\epsilon^2)$~~

$(x_1)_1 = c - \epsilon c^2 + 2\epsilon^2 c^3 - \epsilon^3 c^4$

$\therefore (x_2)_1 = -\frac{1}{\epsilon} - (x_1) = -\frac{1}{\epsilon} - c + c^2\epsilon - 2c^3\epsilon^2 + ~~\epsilon^3 c^4~~ O(\epsilon^2)$

another way is to remember that $x_1 x_2 = \frac{c}{-\epsilon}$ and $(x_1)_0 \cong c$ $(x_2)_0 \cong -\frac{1}{\epsilon}$

then $(\epsilon x + 1)x = c$ or $x_{n+1} = \frac{c}{x_n\epsilon} - \frac{1}{\epsilon}$

if $x_0 = -\frac{1}{\epsilon}$ then $x_1 = -c - \frac{1}{\epsilon}$

$x_2 = \frac{c}{(-c - \frac{1}{\epsilon})\epsilon} - \frac{1}{\epsilon} = \frac{c}{-c\epsilon - 1} - \frac{1}{\epsilon} = \frac{-c}{1 + c\epsilon} - \frac{1}{\epsilon}$

since $c\epsilon$ is small then $(1 + c\epsilon)^{-1} = 1 - c\epsilon + c^2\epsilon^2 - c^3\epsilon^3 + \dots$

$x_2 = -\frac{1}{\epsilon} - c(1 - c\epsilon + c^2\epsilon^2 - c^3\epsilon^3 + \dots) = -\frac{1}{\epsilon} - c + c^2\epsilon - c^3\epsilon^2 + c^4\epsilon^3$

$-\sin\sqrt{t} \quad \frac{1}{2}\frac{1}{\sqrt{t}} = -\frac{1}{2}$

$-\frac{\cos\sqrt{t}}{2\sqrt{t}} \quad \frac{1}{2\sqrt{t}} \quad + \frac{\sin\sqrt{t}}{2\cdot 2\sqrt{t}\cdot t}$

$\frac{-\cos\sqrt{t}}{4t} + \frac{\sin\sqrt{t}}{4t\sqrt{t}}$

$-(1-t$

Solutions to Exercise Set 1

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

11 April 1979

1.1. Solution of quadratic equation: a parameter perturbation. If we first neglect the quadratic term because it is multiplied by ε, and then evaluate it from the preceding approximation, we obtain successively

$$x = c - \varepsilon c^2$$
$$c - \varepsilon c^2 + 2\varepsilon^2 c^3 \left[- \varepsilon^3 c^4 \right. \leftarrow \text{meaningless terms}$$
$$c - \varepsilon c^2 + 2\varepsilon^2 c^3 - 5\varepsilon^3 c^4 \left. \right] + 6\varepsilon^4 c^5 - 6\varepsilon^5 c^6 + 4\varepsilon^6 c^7 - \varepsilon^7 c^8$$

Alternatively, substituting $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$ and equating like powers of ε gives

$$x = c - \varepsilon c^2 + 2\varepsilon^2 c^3 - 5\varepsilon^3 c^4 + 14\varepsilon^4 c^5 - \ldots$$

The second method is at the same time simpler and more systematic, in that at each stage it produces only meaningful terms. The quadratic formula gives the exact solution as

$$x = \frac{-1 \pm \sqrt{1 + 4\varepsilon c}}{2\varepsilon} = \frac{-1 \pm (1 + 2\varepsilon c - 2\varepsilon^2 c^2 + \ldots)}{2\varepsilon}$$

and taking the upper sign reproduces the above result. Here the expansion by the binomial theorem converges for $|4\varepsilon c| \leq 1$. Taking the lower sign gives the other root as

$$x = -\frac{1}{\varepsilon}(1 + \varepsilon c - \varepsilon^2 c^2 + \ldots)$$

1.2. Solution of an ordinary differential equation: a coordinate perturbation. Expanding the cosine in powers of t gives

$$\left(1 - \frac{1}{2}t + \frac{1}{24}t^2 - \ldots\right)\frac{dx}{dt} - \frac{x}{t} = 1.$$

Replacing the parenthesis by unity and solving yields a first approximation for small time:

$$\frac{dx_I}{dt} - \frac{x_I}{t} = 1 : \qquad x_I = t(\log t + c).$$

Next we can use this approximation to evaluate the first neglected term, which gives for the second approximation

$$\frac{dx_{II}}{dt} - \frac{x_{II}}{t} = 1 + \tfrac{1}{2}t\frac{dx_I}{dt} = 1 + \tfrac{1}{2}t(\log t + 1 + c),$$

with solution

$$x_{II} = t(\log t + c) + \frac{1}{2}t^2(\log t + c)$$

This suggests the expansion for small time

$$x = \log t\,(At + Bt^2 + Ct^3 + \ldots) + ct + Dt^2 + Et^3 + \ldots$$

and substituting and equating like powers of t and $\log t$ yields algebraic equations that give

$$x = t(\log t + c) + \frac{1}{2}t^2(\log t + c) + \frac{1}{48}t^3\left[11(\log t + c) - \frac{1}{2}\right] + O(t^4 \log t)$$

Solutions to Exercise Set 1

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

13 April 1979

<u>1.1. Solution of quadratic equation: a parameter perturbation.</u> If we first neglect the quadratic term because it is multiplied by ε, and then evaluate it from the preceding approximation, we obtain successively

$$x = c - \varepsilon c^2$$
$$c - \varepsilon c^2 + 2\varepsilon^2 c^3 \left[- \varepsilon^3 c^4 \right. \quad \text{meaningless terms}$$
$$c - \varepsilon c^2 + 2\varepsilon^2 c^3 - 5\varepsilon^3 c^4 \left. \right] + 6\varepsilon^4 c^5 - 6\varepsilon^5 c^6 + 4\varepsilon^6 c^7 - \varepsilon^7 c^8$$

Alternatively, substituting $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$ and equating like powers of ε gives

$$x = c - \varepsilon c^2 + 2\varepsilon^2 c^3 - 5\varepsilon^3 c^4 + 14\varepsilon^4 c^5 - \ldots$$

The second method is at the same time simpler and more systematic, in that at each stage it produces only meaningful terms. The quadratic formula gives the exact solution as

$$x = \frac{-1 \pm \sqrt{1 + 4\varepsilon c}}{2\varepsilon} = \frac{-1 \pm (1 + 2\varepsilon c - 2\varepsilon^2 c^2 + \ldots)}{2\varepsilon}$$

and taking the upper sign reproduces the above result. Here the expansion by the binomial theorem converges for $|4\varepsilon c| \leq 1$. Taking the lower sign gives the other root as

$$x = -\frac{1}{\varepsilon}(1 + \varepsilon c - \varepsilon^2 c^2 + \ldots)$$

Mr. Zick found this root by setting $x = 1/y$ to give the equation

$$y = -\varepsilon - cy^2,$$

then solving by iteration, starting with the last term neglected, and finally taking the reciprocal. Another viewpoint is to stretch x by introducing $X = \varepsilon^m x$ and seek an m for which some pair of terms in the equation other than the last two are dominant. This occurs only for $m = 1$, when

$$X = -1 + \frac{\varepsilon c}{X};$$

and this can be solved by iteration, starting with the last term neglected.

<u>1.2. Solution of an ordinary differential equation: a coordinate perturbation.</u> Expanding the cosine in powers of t gives

$$\left(1 - \frac{1}{2}t + \frac{1}{24}t^2 - \ldots\right)\frac{dx}{dt} - \frac{x}{t} = 1.$$

Replacing the parenthesis by unity and solving yields a first approximation for small time:

$$\frac{dx_1}{dt} - \frac{x_1}{t} = 1 \quad \text{or} \quad \frac{d}{dt}\left(\frac{x_1}{t}\right) = \frac{1}{t}: \qquad x_1 = t(\log t + c).$$

Similarly, a second approximation can be found by keeping two terms in the expansion of $\cos\sqrt{t}$:

$$(1-\tfrac{1}{2}t)\frac{dx_2}{dt} - \frac{x_2}{t} = 1 \quad \text{or} \quad \frac{d}{dt}\left[(1-\tfrac{1}{2}t)\frac{x_2}{t}\right] = \frac{1}{t}; \qquad x_2 = \frac{t(\log t + c)}{1 - \frac{1}{2}t}.$$

Higher approximations can in principle be found by continuing this procedure, but it rapidly becomes unwieldy.

A simpler alternative is to use the previous approximation to evaluate the new term, giving

$$\frac{dx_2}{dt} - \frac{x_2}{t} = 1 + \tfrac{1}{2}t\frac{dx_1}{dt} = 1 + \tfrac{1}{2}t(\log t + 1 + c),$$

with solution

$$x_2 = t(\log t + c) + \tfrac{1}{2}t^2(\log t + c).$$

Expanding the previous result for small t gives this to second order. Here it would appear that the second c could be different from the first, but they must be the same because the solution of a first-order differential equation contains only one arbitrary constant.

It is not permissible, however, to use the previous approximation to evaluate the entire derivative term:

$$(1-\tfrac{1}{2}t)\frac{dx_1}{dt} - \frac{x_2}{t} = 1: \qquad x_2 = t(1-\tfrac{1}{2}t)(\log\ t + c + 1)$$

This result is wrong, because x_1 was not found by the first step of any iteration scheme of which this is the second. Neither is it permissible to use the previous approximation to evaluate the term $-x/t$.

These results suggest the expansion for small time

$$x = (At + Bt^2 + Ct^3 + \ldots)\log t + ct + Dt^2 + Et^3 + \ldots$$

and substituting and equating like powers of t and $\log t$ yields algebraic equations that give

$$x = t(\log t + c) + \frac{1}{2}t^2(\log t + c) + \frac{1}{48}t^3\left[11(\log t + c) - \frac{1}{2}\right] + O(t^4 \log t).$$

<u>A nonlinear version.</u> In this solution $\log t$ appears only to the first power because the differential equation is linear. By modifying it to be weakly nonlinear, we introduce the compounding of powers of the logarithm that is encountered in many singular perturbations, such as the expansion (1.14) for the drag of a sphere. For example, the equation

$$\frac{dx}{dt} - \frac{x}{t} = 1 + x + x^2$$

has the solution

$$x = t(\log t + c) + t^2(\log t + c - 1) + \tfrac{1}{2}t^3(\log^2 t + 2c\log t + 2c^2 + c - \tfrac{1}{2}) + O(t^4\log^2 t).$$

Because little is known about the properties of series of this form, it might be worhtwhile to study this and similar model equations.

Exercise Set 2

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

Due Monday 17 April 1979

2.1. Laminar flow between counterrotating disks. Von Kármán showed in 1921 that the Navier-Stokes equations in cylindrical polar coordinates with axial symmetry,

$$\text{Cont.} \qquad (ru)_r + (rw)_z = 0$$

$$\text{mom } r \qquad uu_r + wu_z - v^2/r = -p_r/\rho + \nu[u_{rr} + (u/r)_r + u_{zz}]$$

$$\text{mom } \theta \qquad uv_r + wv_z + uv/r = \nu[v_{rr} + (v/r)_r + v_{zz}]$$

$$\text{mom } z \qquad uw_r + ww_z = -p_z/\rho + \nu[w_{rr} + w_r/r + w_{zz}]$$

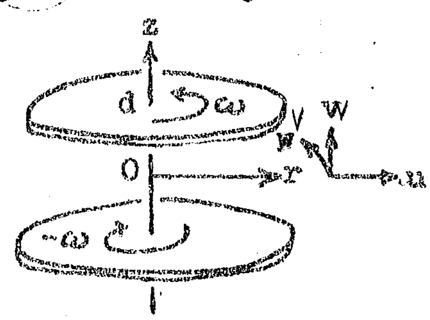
can be reduced to ordinary differential equations by setting

$$u = -\tfrac{1}{2}\frac{\omega^2 d^2}{\nu} r\, h'(\xi), \quad w = \frac{\omega^2 d^3}{\nu} h(\xi), \quad v = \omega r\, g(\xi), \quad \xi = z/d,$$

where d and ω are a reference length and angular velocity. The result is

$$g'' = R^2(hg' - gh')$$

$$h'''' = 4gg' + R^2 hh'''$$



where $R = \omega d^2/\nu$ is a Reynolds number. In 1953 Stewartson considered the motion between infinite disks located at $z = \pm d$ and rotating with angular velocities $\pm\omega$, so that the boundary conditions become

$$g = \pm 1, \quad h = h' = 0 \quad \text{at} \quad \xi = \pm 1.$$

He expanded the solution in powers of the Reynolds number R, and calculated the first few terms.

Repeat that calculation to find the second approximation for v and the first approximation for u and w. How many terms to you estimate you could find correctly in a reasonable time (say, a week)? How long would it take you to write a Fortran program to compute the general term?

2.2. Convergence of series for torque. The torque out to any finite radius on either of the disks sketched above is proportional to the integral of $\partial v/\partial z$ at the surface, and hence to g' at $\xi = 1$, which has the expansion for small Reynolds number R

$$g'(1;\, R) = g_1'(1) + R^2 g_2'(1) + R^4 g_3'(1) + \ldots$$

The first 56 of these coefficients have been computed by machine. The first ten and last ten of those values are tabulated on the next page.

n	$g_n'(1)$	n	$g_n'(1)$
1	1.000000000000000 E-00	47	-4.107684982261942 E-97
2	7.619047619047619 E-03	48	3.609679121423434 E-99
3	-2.918281013519109 E-05	49	-3.174243163078715 E-101
4	1.546574653478234 E-07	50	2.793183452793601 E-103
5	-9.400799362804898 E-10	51	-2.459431340463040 E-105
6	6.192329427740572 E-12	52	2.166880295150269 E-107
7	-4.303220771930771 E-14	53	-1.910248012430022 E-109
8	3.107423645892505 E-16	54	1.684959435555610 E-111
9	-2.309690557906093 E-18	55	-1.487047321124373 E-113
10	1.755873317078919 E-20	56	1.313067557769390 E-115

Estimate as accurately as you can the largest value of R for which this series converges (for example, by adapting one of the standard tests of d'Alembert, Cauchy, etc., for the convergence of a Taylor series). Give your estimate only to as many significant figures as you believe are accurate. Where does the nearest singularity, which limits convergence, lie in the plane of Reynolds number? Can you devise some method of estimating the nature of that singularity, for example, by comparing with the binomial-series expansion of $(1-x)^{\alpha}$?

thus if $S_n = a_0 + a_1 x + \dots + a_n x^n$

$xS_{n+1} = a_0 x + a_1 x^2 + \dots a_n x^{n+1}$

$S_n (1-x) = \frac{a_0 - a_n x^{n+1}}{1-x}$

bin coeff

since this series

$1 - \alpha R^2 + \alpha(\alpha-1) R^4 + \alpha(\alpha-1)(\alpha-2) R^6 + 1 + (-1)^n \prod_{i=1}^{n} (\alpha - i + 1) R^{2n}$

55 52 $\alpha - n + 1$ R^2

$\frac{1}{1+\alpha}$

$(\alpha - n + 1) R^2$

$(\alpha - n + 1) R^2 \sim$

$R^2 < \frac{1}{\alpha - n + 1} = 113$

$\frac{1}{\alpha - 54}$

$\frac{1}{\alpha - 55 + 1}$

54....

S_n $1 + x + x^2 + x^3 + \dots$

xS_n $x - x^2 + x^3 + x^4 \dots$

$\frac{1 + (-1)^n x^{n+1}}{1+x}$

$\frac{1}{\alpha - 54}$

$\alpha = + 54.888$

54.0088

Pick one.) Hi Cesy! , ~~Hi ...!~~ , Hi Cesar!,

~~30 × 8 = 240~~
~~100 × 8 = 800~~
~~1140~~

$$g(\zeta;R) = g_1(\zeta) + Rg_2(\zeta) + R^2 g_3(\zeta) + R^3 g_4(\zeta) + \dots$$
$$h(\zeta;R) = h_1(\zeta) + R h_2(\zeta) + R^2 h_3(\zeta) + R^3 h_4(\zeta) + \dots$$
$$h'(\zeta;R) = h_1'(\zeta) + R h_2'(\zeta) + R^2 h_3'(\zeta) + R^3 h_4'(\zeta) + \dots$$

~~$\therefore\ g_1''(\zeta) = R^2$~~

~~$$\sum_{i=0}^{\infty} R^i g_i(\zeta) = R^2 \left\{ \sum_{i=0}^{\infty} h_i R^i \sum_{j=0}^{\infty} g_j' R^j - \sum_{j=0}^{\infty} g_j R^j \sum_{i=0}^{\infty} h_i' R^i \right\}$$~~

~~$$\sum_{i=0} R^i h_i^{iv}(\zeta) = 4$$~~

$R=0 \Rightarrow \quad g_0''(\zeta) = 0 \quad \Rightarrow \quad g_0 = C_1\zeta + C_2 \qquad g_0' = C_1$

$h_0^{iv}(\zeta) = 4 g_0(\zeta) g_0'(\zeta) \Rightarrow h_0^{iv} = 4(C_1\zeta + C_2)C_1 \Rightarrow$

$$h_0''' = 2C_1^2\zeta^2 + 4C_2C_3\zeta + D_1$$
$$h_0'' = \frac{2C_1^2}{3}\zeta^3 + 2C_2C_3\zeta^2 + D_1\zeta + D_2$$
$$h_0' = \frac{C_1^2}{6}\zeta^4 + 2C_2C_3\frac{\zeta^3}{3} + D_1\frac{\zeta^2}{2} + D_2\zeta + D_3$$
$$h_0 = \frac{C_1^2\zeta^5}{30} + \frac{C_2C_3\zeta^4}{6} + \frac{D_1\zeta^3}{6} + \frac{D_2\zeta^2}{2} + D_3\zeta + D_4$$

$g(1) = 1 \quad g(-1) = -1 \quad \Rightarrow \quad C_1 + C_2 = 1, \quad +C_1 + C_2 = +1 \qquad C_1 = 1 \quad C_2 = 0 \qquad \therefore\ g_0(\zeta) = \zeta$

$h_0^{iv} = 4\zeta \quad \Rightarrow \quad h_0''' = 2\zeta^2 + D_1 \quad h_0'' = \frac{2\zeta^3}{3} + D_1\zeta + D_2 \qquad h_0' = \frac{2\zeta^4}{6} + D_1\frac{\zeta^2}{2} + D_2\zeta + D_3$

~~$2\zeta^2 - 2/5$~~

$h_0 = \frac{\zeta^5}{30} + D_1\frac{\zeta^3}{6} + D_2\frac{\zeta^2}{2} + D_3\zeta + D_4$

$\left.\begin{array}{l} h_0 = 0 \\ h_0' = 0 \end{array}\right\}\ \zeta = \pm 1 \qquad 2\zeta^2 - \frac{2}{5}$

$$\frac{1}{6} + \frac{D_1}{2} + D_2 + D_3 = 0$$
$$\frac{1}{30} + \frac{D_1}{6} + \frac{D_2}{2} + D_3 + D_4 = 0$$
$$\frac{1}{6} + \frac{D_1}{2} - D_2 + D_3 = 0$$
$$-\frac{1}{30} - \frac{1}{6}D_1 + \frac{D_2}{2} - D_3 + D_4 = 0$$

$$\frac{5}{15} + D_1 + 2D_3 = 0$$
$$\frac{1}{15} + \frac{D_1}{3} + 2D_3 = 0$$

~~$\frac{1}{60} + \frac{2D_1}{3} = 0 \Rightarrow D_1 = -\frac{3}{2\cdot 60} = -\frac{1}{40}$~~

$\frac{4}{15} + \frac{2}{3}D_1 = 0 \qquad D_1 = -\frac{3}{2}\cdot\frac{4}{15} \qquad \boxed{D_1 = -2/5}$

$\frac{1}{15} - \frac{2}{15} + 2D_3 = 0 \qquad -\frac{1}{15} = -2D_3 \qquad \boxed{D_3 = \frac{1}{30}}$

$\boxed{D_2 = -\frac{5}{30} + \frac{6}{30} - \frac{1}{30} \ldots = 0}$

~~$\frac{5}{30} - \frac{6}{30} + \frac{1}{30}$~~

$\boxed{D_4 = -\frac{1}{30} + \frac{1}{15} + 0 - \frac{1}{30} = 0}$

$-\frac{1}{30} + \frac{1}{15} + 0 - \frac{1}{30} + D_4 = 0$

$\boxed{h_0 = \frac{\zeta^5}{30} - \frac{2}{30}\zeta^3 + \frac{1}{30}\zeta}$

$\boxed{g_0 = \zeta}$

$h_0' = \frac{\zeta^4}{6} - \frac{1}{5}\zeta^2 + \frac{1}{30}$

$$g'' = R^2(h_0 g_0' - g_0 h_0')$$
$$= R^2\left(\frac{\zeta^5}{30} - \frac{2}{15}\zeta^3 + \frac{1}{30}\zeta - \frac{5\zeta^5}{30} + \frac{3\zeta^3}{15} - \frac{\zeta}{30}\right)$$
$$= R^2\left(-\frac{4\zeta^5}{30} + \frac{2}{30}\zeta^3\right) = 2R^2(\zeta^3 - \zeta^5) \ldots$$

$\frac{8}{15}\left(\frac{\zeta^5}{4} - \frac{\zeta^7}{6} - \frac{115}{420}\right) + \frac{8}{15}\left(\frac{\zeta^5}{20} - \frac{\zeta^7}{42} - \frac{11}{420}\zeta\right)$

$(2\zeta^2 - 7/5)\ \frac{1}{30}(\zeta^5 - 2\zeta^3 + \zeta)$

$$g_1'' = \frac{2}{15}(\zeta^3 - \zeta^5)$$

$$g_1' = \frac{2}{15}\left(\frac{\zeta^4}{4} - \frac{\zeta^6}{6} + C\right)$$

$$g_1 = \frac{2}{15}\left(\frac{\zeta^5}{20} - \frac{\zeta^7}{42} + C\zeta + D\right)$$

$g_1 = 0 \quad \pm 1 \qquad \left(\frac{1}{20} - \frac{1}{42} + C + D\right) = 0$

$\left(\frac{-1}{20} + \frac{1}{42} - C + D\right) = 0$

$\frac{3 \cdot 1 \cdot 7}{3 \cdot 2 \cdot 2 \cdot 5 \cdot 7} - \frac{2 \cdot 1 \cdot 5}{2 \cdot 2 \cdot 3 \cdot 7 \cdot 5} + 2D = 0$

$\frac{21 - 10}{12 \cdot 35} = \boxed{\frac{-11}{12 \cdot 35} = C}$

$4 \cdot 3 \cdot 7 \cdot 5$

$$g_1 = \frac{2}{15}\left(\frac{\zeta^5}{20} - \frac{\zeta^7}{42} - \frac{11}{20 \cdot 21}\zeta\right)$$

$$g = g_0 + R^2 g_1$$

					R		$< R^2$
48	$\frac{a_{n+1}}{a_n}$ =	.8787624 $\times 10^{-2}$	R^2	< 1	R^2		1.137964
49		.8793699 "	R^2				
50		.8799526 "	R^2				
51		.8805119 "	R^2				
52	.00052	.8810493 "	R^2				1.1350103
53	.00050	.881546 "	R^2		10.650563		1.134345
54	.00048	.8820632 "	R^2		10.64756		1.1337056
55	.00046	.8825419 "	"		10.644673		1.1330907
56		.8830032 "	"		10.64189	R^2 =	1.1324987×10^2

$R^2 \sim 113$

R

$R^2 <$

$R < 10.64189$

Dear Mr D

sorta of an unusual chap

CL + CK

KK

she's a cutie pie!

yes CL likes her - he probably hugs her every night. Not hugs

A

He hugs his calculator instead!

Solutions to Exercise Set 2

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

Wednesday 19 April 1979

2.1. Laminar flow between counterrotating disks. For $R = 0$ the problem for g becomes

$$g'' = 0, \quad g(\pm 1) = 1.$$

The solution is $g = \zeta$, and then the problem for h becomes

$$h'''' = 4\zeta, \qquad h(\pm 1) = h'(\pm 1) = 0,$$

with solution

$$h = \frac{1}{30}\zeta(1-\zeta^2)^2.$$

To find the first effects of finite Reynolds number, we can substitute these results into the equation for g to obtain the iteration equation

$$g'' = \frac{2}{15}R^2\zeta^3(1-\zeta^2), \qquad g(\pm 1) = 0,$$

with solution

$$g = \zeta - \frac{1}{3150}R^2\zeta(11 - 21\zeta^4 + 10\zeta^6).$$

This gives for the (dimensionless) torque on the disks

$$g'(1) = 1 + \frac{4}{525}R^2$$

and this checks with the first two tabulated values. To proceed further it is convenient to expand in powers of R; evidently only even powers are required, since the Reynolds number appears only as R^2, and this is a regular perturbation:

$$g(\zeta; R) = g_1(\zeta) + R^2 g_2 + R^4 g_3 + \dots, \qquad h = h_1 + R^2 h_2 + R^4 h_3 + \dots$$

It is not hard to calculate

$$h_2 = \frac{1}{1,134,000}\zeta(-143 + 407\zeta^2 - 390\zeta^4 + 126\zeta^6 + 5\zeta^8 - 5\zeta^{10})$$

and even g_3; but thereafter hand calculation becomes tedious and untrustworthy.

It is clear from induction that g_n and h_n are polynomials in odd powers of ζ of degrees $6n-5$ and $6n-1$. Hence substituting a double expansion in powers of R^2 and odd powers of ζ yields recursion relations for the coefficients that can be solved in succession. A Fortran program running in quadruple precision on our IBM machine computed 56 terms in 9 minutes, at a cost of \$32. Because the essential computation involves DO loops nested four deep, the time increases as the fourth power of the number of terms.

2.2. Convergence of series for torque. D'Alembert's ratio test gives the radius of convergence of the Taylor series $\sum c_n \varepsilon^n$ as

$$\varepsilon < \lim_{n\to\infty} |c_{n-1}/c_n|.$$

For $n = 52$ through 56 these radios are

$$|c_{n-1}/c_n| = 113.50, \quad 113.43, \quad 113.37, \quad 113.31, \quad 113.25$$

It seems safe to estimate that our series for the torque will converge for $R^2 < 100$, or $R < 10$. Mr. Parnell plotted these ratios versus n and extrapolated (to infinity!) to get the better value 10.5. Messrs. Hajrudin and Nathman obtained the same value more easily by extrapolating versus $1/n$.

As discussed on page 246 of <u>Perturbation Methods in Fluid Mechanics</u>, it is better still to plot the reciprocal ratio $|c_n/c_{n-1}|$ versus $1/n$. The reason is that for the simple function

$$f(\varepsilon) = \text{const}\,(\varepsilon_0 \pm \varepsilon)^{\alpha} = \sum c_n \varepsilon^n, \qquad \alpha \neq 0, 1, 2, \ldots$$

which has a singularity at $\varepsilon = \mp\varepsilon_0$, the binomial theorem gives

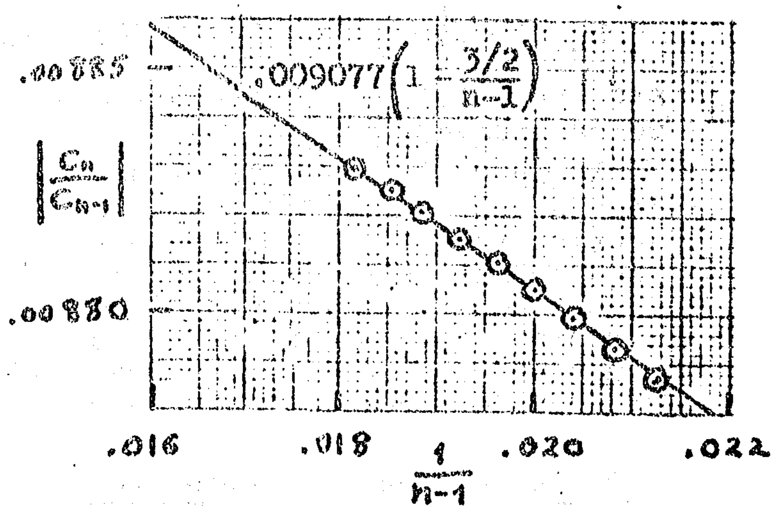
$$\frac{c_n}{c_{n-1}} = \pm\frac{1}{\varepsilon_0}\left(1 - \frac{1+\alpha}{n}\right).$$

Hence a plot versus $1/n$ is linear. A further advantage is that the intercept on the axis of abscissas gives the exponent α of the singularity. (If it corresponds to $\alpha = 0, 1, 2, \ldots$, the singularity is logarithmic rather than algebraic.) This simple graphical ratio test was devised by the British physicists C. Domb and M. F. Sykes in 1957 for estimating the critical points of crystal lattices from series expansions of their thermodynamic properties in powers of temperature.*

For a more complicated function, this linear relation will be approached for large n, with α corresponding to the nearest singularity (or the strongest, if several are equally distant). Then

$$\frac{c_n}{c_{n-1}} \sim \mp\frac{1}{\varepsilon_0}\left[1 - \frac{1+\alpha}{n} + O\left(\frac{1}{n^2}\right)\right].$$

Because the term in $1/n^2$ is unknown, it is permissible to try to straighten the plot by shifting n. For our problem, the plot is remarkably straight when plotted versus $1/(n-1)$. The straight line shown corresponds to a square-root singularity located at $R^2 = (10.496)^2$. This value can be further refined by fitting polynomials in $1/n$, which is accomplished efficiently by forming a so-called <u>Neville table</u>. For details, see the reference to Gaunt & Guttmann 1974 in <u>PMFM</u>.



*And re-invented for this problem by Messrs. Chaderjian and Sangani.

ME 207
Cesar Levy
Prof. M. Van Dyke

Problem Set #2

Evidently even powers of R suffice.

2.1 if we assume $g(\zeta;R) = \sum_{i=0}^{\infty} g_i(\zeta) R^i$ $\quad g'(\zeta;R) = \sum_{i=0}^{\infty} g_i'(\zeta) R^i$

$h(\zeta;R) = \sum_{i=0}^{\infty} h_i(\zeta) R^i$ $\quad h'(\zeta;R) = \sum_{i=0}^{\infty} h_i'(\zeta) R^i$ $\quad h''(\zeta;R) = \sum_{i=0}^{\infty} h_i''(\zeta) R^i$ $\quad h'''(\zeta;R) = \sum_{i=0}^{\infty} h_i'''(\zeta)$

$h^{IV}(\zeta;R) = \sum_{i=0}^{\infty} h_i^{IV}(\zeta) R^i$

with $g_0(\pm1) = \pm1$ $\quad g_i(\pm1) = 0 \quad \forall i > 1$ and $h_i(\pm1) = h_i'(\pm1) = 0 \quad \forall i$ if we accept a priori that each series converges and that the derivative functions converge within the radius of convergence of the g and h series respectively.

Then for

$i=0$ we obtain $g_0'' = 0$ w/ $g_0(\pm1) = \pm1$ $\quad \boxed{g_0(\zeta) = \zeta}$ ✓

$h_0^{IV} = 4 g_0 g_0'$ w/ $h_0(\pm1) = h_0'(\pm1) = 0$ $\quad \boxed{h_0(\zeta) = \frac{1}{30}[\zeta^5 - 2\zeta^3 + \zeta]}$ ✓

$i=1$ we obtain $g_1'' = 0$ w/ $g_1(\pm1) = 0$ $\quad \boxed{g_1(\zeta) = 0}$ ✓

$h_1^{IV} = 4 g_0 g_1' + 4 g_0' g_1 = 0$ w/ $h_1(\pm1) = h_1'(\pm1) = 0$ $\quad \boxed{h_1(\zeta) = h_1'(\zeta) = 0}$ ✓

$i=2$ we obtain $g_2'' = (h_0 g_0' - g_0 h_0') = \frac{2}{15}(\zeta^3 - \zeta^5)$ w/ $g_2(\pm1) = 0$ $\quad \boxed{g_2(\zeta) = \frac{2}{15}\left(\frac{\zeta^5}{20} - \frac{\zeta^7}{42} - \frac{11}{420}\zeta\right)}$ ✓

$h_2'' = 4(g_0 g_2' + g_2 g_0' + 2 g_1 g_1') + h_0 h_0'''$

Now $g(\zeta;R) \sim g_0(\zeta) + R^2 g_2(\zeta) + O(R^2)$

$h(\zeta;R) \sim h_0(\zeta) + o(R)$

We note that $g_3(\zeta) = g_5(\zeta) = \dots = g_m(\zeta) = 0$ $\quad$ m odd

$h_3(\zeta) = h_5(\zeta) = \dots = h_m(\zeta) = 0$ $\quad$ m odd

With each non zero g and h we need to evaluate 6 constants. The higher the value of i the hairier the coefficients are to evaluate. From preliminary results it appears as if the only nonzero coeff of ζ^m of the $g_i(\zeta)$'s and $h_i(\zeta)$'s are for m=odd. This would reduce the number of constants down to 3. It would then take me a week to compute about 10 terms and once I had found the trend it would take me about another 2/3 days to program **Sounds reasonable.**

2.2 based on these results I find that

$\left|\frac{a_{48}}{a_{47}}\right| = .8787624 \times 10^{-2} R^2$ $\quad \left|\frac{a_{50}}{a_{49}}\right| = .8799526 \times 10^{-2} R^2$ $\quad \left|\frac{a_{52}}{a_{51}}\right| = .8810493 \times 10^{-2} R^2$ $\quad \left|\frac{a_{54}}{a_{53}}\right| = .8820632 \times 10^{-2} R^2$

$\left|\frac{a_{49}}{a_{48}}\right| = .8793699 \times 10^{-2} R^2$ $\quad \left|\frac{a_{51}}{a_{50}}\right| = .8805119 \times 10^{-2} R^2$ $\quad \left|\frac{a_{53}}{a_{52}}\right| = .881566 \times 10^{-2} R^2$ $\quad \left|\frac{a_{55}}{a_{54}}\right| = .8825419 \times 10^{-2} R^2$

$\left|\frac{a_{56}}{a_{55}}\right| = .8830032 \times 10^{-2} R^2$

for convergence $\left|\frac{a_{n+1}}{a_n}\right| < 1$ then $R < 10.6$, $R^2 < 113$ 3 figure accuracy. **No, 2.**

If we look at divided differences it <u>looks</u> as if the $\left|\frac{a_{n+1}}{a_n}\right| \to 0$ in about 23 terms ? this will lead to a value of $R^2 < 112.6$. Hence we can look at the $fn = \frac{1}{1+x}$ since terms alternate in sign. $\frac{1}{1+x} = \frac{1}{1 + .8880632 \times 10^{-3} R^2}$ or $R^2 = -112.6$ will give a singularity in complex R plane or $R = 10.6i$ is the singularity estimated to 3 places.

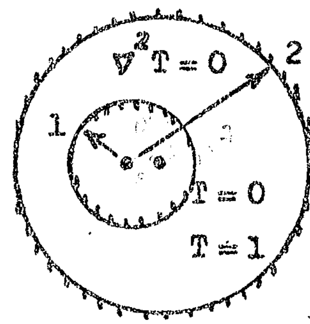
(A)

Exercise Set 3

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

Due Monday 23 April 1979

3.1. Temperature field between slightly eccentric circles. (From last year's midterm examination) Find the steady-state temperature distribution (a solution of the two-dimensional Laplace equation) between an inner circle of radius 1 maintained at zero temperature and an outer circle of radius 2 maintained at unit temperature, if their centers are displaced by a small distance ε, so that the outer circle has radius $2 + \varepsilon \cos\theta + O(\varepsilon^2)$ measured from the center of the inner circle. Simplify by neglecting terms of order ε^2.



(Alternatively, you may interpret this as the problem of finding the electrostatic potential between circles maintained at potentials 0 and 1, or the small deflection of a soap film that spans two rings lying in different parallel planes.)

3.2. Flow past wavy wall. (Exercise 2.2 of notes with last sentence clarified) A simple problem that illustrates some of the features of water waves and of thin-ship theory is that of potential flow past an infinite sinusoidal wall with a uniform stream far from the wall. If continuity is satisfied by introducing a stream function according to $u = \psi_y$, $v = -\psi_x$ the dimensionless problem is

$$\psi_{xx} + \psi_{yy} = 0, \qquad \psi = 0 \text{ on the wall}, \qquad \psi \sim y \text{ far from the wall.}$$

No solution of this problem is known in closed form. Show that the first approximation for small ε is

$$\psi \sim y - \varepsilon \sin x \, e^{-y}$$

and carry this to the next approximation. Use your result to calculate the maximum and minimum speed in the field (which occur on the wall at the tops and bottoms of the sinusoidal bumps), putting them in the form $1 + a\varepsilon + b\varepsilon^2$. The necessity for transferring the surface boundary condition can be avoided -- at the expense of complicating the differential equation -- by using coordinates that conform to the surface. A simple possibility is to replace the ordinate by its value measured from the surface, introducing the new independent variables

$$X = x, \qquad Y = y - \varepsilon \sin x .$$

Carry out the solution on this basis to order ε, and show by comparison with the previous result that it happens to give the maximum and minimum speeds correct to order ε^2.

Cesar Levy
ME 207
Prof. Van Dyke

Problem Set #3

3.1 Given $\nabla^2 T = 0$ w/ $T=0$ on $r=1$ and $T=1$ on $r = 2+\varepsilon\cos\theta$

let $T(r,\theta;\varepsilon) = T_0(r,\theta) + \varepsilon T_1(r,\theta) + \varepsilon^2 T_2(r,\theta) + \dots$

(1) $\nabla^2 T = \nabla^2 T_0 + \varepsilon\nabla^2 T_1 + \varepsilon^2 \nabla^2 T_2 + \dots = 0 \Rightarrow \nabla^2 T_i = 0 \quad i=0,1,2,\dots$ ✓

(2) $T(1,\theta) = T_0(1,\theta) + \varepsilon T_1(1,\theta) + \varepsilon^2 T_2(1,\theta) + \dots = 0 \Rightarrow T_i(1,\theta) = 0 \quad i=0,1,2,\dots$ ✓

(3) $T(2+\varepsilon\cos\theta,\theta) = T_0(2+\varepsilon\cos\theta,\theta) + \varepsilon T_1(2+\varepsilon\cos\theta,\theta) + \varepsilon^2 T_2(2+\varepsilon\cos\theta,\theta) + \dots = 1$

$$= T_0(2,\theta) + T_{0,r}(2,\theta)\varepsilon\cos\theta + T_{0,rr}(2,\theta)\frac{\varepsilon^2\cos^2\theta}{2} + \varepsilon\left[T_1(2,\theta) + T_{1,r}(2,\theta)\varepsilon\cos\theta + T_{1,rr}(2,\theta)\frac{\varepsilon^2\cos^2\theta}{2}\right]$$
$$+ \varepsilon^2\left[T_2(2,\theta) + T_{2,r}(2,\theta)\varepsilon\cos\theta + T_{2,rr}(2,\theta)\frac{\varepsilon^2\cos^2\theta}{2}\right] + \dots = +1$$ ✓

From (3) $T_0(2,\theta) = 1$; $T_{0,r}(2,\theta)\cos\theta + T_1(2,\theta) = 0$; $T_{0,rr}(2,\theta)\frac{\cos^2\theta}{2} + T_{1,r}(2,\theta)\cos\theta + T_2(2,\theta) = 0$; etc.

(1) Must Solve $\nabla^2 T_0 = 0$ w/ $T_0(1,\theta)=0$ and $T_0(2,\theta)=1$ since this is axisymmetric bc soln is only fn of r ∴ after the algebra $\boxed{T_0 = \ln r/\ln 2}$ ✓

(2) Must Solve $\nabla^2 T_1 = 0$ w/ $T_1(1,\theta)=0$ and $T_1(2,\theta) = -\frac{\cos\theta}{2\ln 2}$ the soln to $\nabla^2 T_1 = 0$ is

$$T_{1n} = (A_n r^n + B_n r^{-n})(C_n\cos n\theta + D_n \sin n\theta) \quad \text{and} \quad T_1(r,\theta) = \sum_{n=1}^{\infty} T_{1n}(r,\theta)$$

n must be integer to satisfy periodicity condition and since $T_1(2,\theta)$ involves only $\cos\theta \Rightarrow n=1$.

by applying bc we find $T_2(r,\theta) = -\frac{1}{3\ln 2}\left(r - \frac{1}{r}\right)\cos\theta$

$$\therefore \boxed{T(r,\theta) = \frac{\ln r}{\ln 2} + \frac{\varepsilon}{3\ln 2}\left(\frac{1}{r} - r\right)\cos\theta + O(\varepsilon^2)}$$ ✓

3.2. given $\psi_{xx} + \psi_{yy} = 0$ and $\psi = 0$ on $y = \varepsilon\sin x$ with $\psi \sim y$ for large y

let $\psi(x,y;\varepsilon) = \psi_0(x,y) + \varepsilon\psi_1(x,y) + \varepsilon^2\psi_2(x,y) + \dots$

b.c. (1) $\psi(x, y=\varepsilon\sin x) = \psi_0(x,\varepsilon\sin x) + \varepsilon\psi_1(x,\varepsilon\sin x) + \varepsilon^2\psi_2(x,\varepsilon\sin x) + \dots = 0$

$$= \psi_0(x,0) + \psi_{0,y}(x,0)\varepsilon\sin x + \psi_{0,yy}(x,0)\frac{\varepsilon^2\sin^2 x}{2} + \varepsilon\left[\psi_1(x,0) + \psi_{1,y}(x,0)\varepsilon\sin x + \psi_{1,yy}(x,0)\frac{\varepsilon^2\sin^2 x}{2}\right] +$$
$$\varepsilon^2\left[\psi_2(x,0) + \psi_{2,y}(x,0)\varepsilon\sin x + \psi_{2,yy}(x,0)\frac{\varepsilon^2\sin^2 x}{2}\right] + \dots = 0$$

$\Rightarrow \psi_0(x,0) = 0$; $\psi_{0,y}(x,0)\sin x + \psi_1(x,0) = 0$; $\psi_{0,yy}(x,0)\frac{\sin^2 x}{2} + \psi_{1,y}(x,0)\sin x + \psi_2(x,0) = 0$ ✓

(2) $\psi(x,y) = \psi_0(x,y) + \varepsilon\psi_1(x,y) + \varepsilon^2\psi_2(x,y) + \dots \sim y$

here we will take $\psi_0(x,y) = y$; $\psi_1(x,y) = 0$; $\psi_2(x,y) = 0$ etc. (will modify later)

(3) also $\nabla^2\psi = 0 = \nabla^2\psi_0 + \varepsilon\nabla^2\psi_1 + \varepsilon^2\nabla^2\psi_2 + \dots = 0 \Rightarrow \nabla^2\psi_i = 0 \quad i = 0,1,2,\dots$

Thus using (1), (2), (3)

ε^0: $\nabla^2\psi_0 = 0$ w/ $\psi_0(x,y) = y$ as $y \to \infty$ and $\psi_0(x,0) = 0$

Solution to this problem since ψ_0 must be periodic in x (using separation of variables) gives for $n \neq 0$ $\psi_{0n}(x,y) = (A_n e^{+ny} + B_n e^{-ny})(C_n \sin nx + D_n \cos nx)$; this solution will not solve the bc at ∞

$\therefore$ for $n=0$ $\psi_0 = Axy + Bx + Cy + D$ using bc's we find $\psi_0(x,y) = y$ (which we could have picked by observation.

Thus:

ε^1: $\nabla^2\psi_1 = 0$ w/ $\psi_1(x,0) = -\sin x$ $\psi_1(x,y) = 0$

Since the bc involves a sine fn. $\Rightarrow$ $\psi_{1n}(x,y) = (A_n e^{+ny} + B_n e^{-ny})(C_n \cos nx + D_n \sin nx)$

$\psi_1(x,y) = \sum_{n=1}^{\infty} \psi_{1n}$ since we want fn to go to 0 as $y \to \infty$ $A_n = 0$

$\psi_1(x,0) = B_n(C_n \cos nx + D_n \sin nx) = -\sin x \Rightarrow C_n = 0$ and $B_n D_n = 0 \ \forall n > 1$ $B_1 D_1 = -1$

$\therefore \quad \psi_1(x,y) = -e^{-y}\sin x$ ✓

thus to $O(\varepsilon)$: $\boxed{\psi(x,y) = y - \varepsilon e^{-y}\sin x}$

ε^2: $\nabla^2\psi_2 = 0$ using $\psi_{0,yy}(x,0) = 0$ $\psi_{1,y}(x,0) = \sin x \Rightarrow \psi_2(x,0) = -\sin^2 x = -\left[\frac{1-\cos 2x}{2}\right]$ ✓

also $\psi_2(x,y) = 0$ as $y \to \infty$

Now since $\psi_2(x,0) = -\frac{1}{2} + \frac{\cos 2x}{2}$ we note that the constant term will really screw up any attempt to find a solution, We can remedy this by remembering that since $\psi(x,y) \sim y$ we can let $\psi(x,y) = y + \varepsilon^2 \cdot c$ where c is a constant to be picked. Let me pick $c = -\frac{1}{2}$ to make the solution simple. Let $\psi_2(x,y) = \psi_{2p}(x,y) + \psi_{2c}(x,y) \Rightarrow \psi_{2p}(x,y) = \psi_{2p}(x,0) = -\frac{1}{2}$ and $\nabla^2\psi_{2p} = 0$ thus $\boxed{\psi_{2p}(x,y) = -\frac{1}{2}}$. We thus can take $\nabla^2\psi_{2c} = 0$, $\psi_{2c}(x,0) = \frac{\cos 2x}{2}$ and $\nabla^2\psi_{2c}(x,y) = 0$ as $y \to \infty$.

Again solution to this is $\psi_{2c_n}(x,y) = (A_n e^{ny} + B_n e^{-ny})(C_n \cos nx + D_n \sin nx)$ same reasons as before as to why n must be integer. For soln $\psi_{2c}(x,y) = \sum_{n=1}^{\infty} \psi_{2c_n}(x,y)$ using bc we find that $\psi_{2c}(x,y) = e^{-2y}\frac{\cos 2x}{2}$

$\therefore$ to $O(\varepsilon^3)$ we get $\boxed{\psi(x,y) = y - \varepsilon e^{-y}\sin x + \varepsilon^2\left(e^{-2y}\frac{\cos 2x}{2} - \frac{1}{2}\right)}$ ✓

now $\frac{\partial\psi}{\partial y} = u = 1 + \varepsilon e^{-y}\sin x + \varepsilon^2\left(-2e^{-2y}\frac{\cos 2x}{2}\right) + O(\varepsilon^3)$

at wall : $y = \varepsilon\sin x$ then $u = 1 + \varepsilon\sin x\left[1 - \varepsilon\sin x + \varepsilon^2\frac{\sin^2 x}{2}\right] - 2\varepsilon^2\frac{\cos 2x}{2}\left[1 - 2\varepsilon\sin x + 4\varepsilon^2\frac{\sin^2 x}{2}\right] + \dots$

$= 1 + \varepsilon\sin x - \varepsilon^2\left(\sin^2 x + 2\frac{\cos 2x}{2}\right) + \varepsilon^3\left(\frac{\sin^3 x}{2} + 4\frac{\cos 2x\sin x}{2}\right) + \dots$

Note caution: ε^3 terms are not exactly correct since terms of ε^2 will come in from the ψ_3 soln. **True.**

thus $u = 1 + \varepsilon\sin x - \varepsilon^2(\sin^2 x + \cos 2x) + O(\varepsilon^3) = 1 + \varepsilon\sin x - \cos^2 x\,\varepsilon^2 + O(\varepsilon^3)$

$-\frac{\partial\psi}{\partial x} = v = \varepsilon e^{-y}\cos x + \varepsilon^2 e^{-2y}\sin 2x + O(\varepsilon^3)$

at wall: $y = \varepsilon\sin x$ then $v = \varepsilon\cos x\left[1 - \varepsilon\sin x + \varepsilon^2\frac{\sin^2 x}{2}\right] + \varepsilon^2\sin 2x\left[1 - 2\varepsilon\sin x + 4\varepsilon^2\frac{\sin^2 x}{2}\right] + \dots$

$= \varepsilon\cos x + \varepsilon^2\left[-\frac{\sin 2x}{2} + \sin 2x\right] + \varepsilon^3\left[\frac{\cos x\sin^2 x}{2} - 2\sin x\sin x\right] + \dots$

again the same caution about the ε^3 terms

Since u is $O(1)$ and v is $O(\varepsilon)$ we can then say that for the speed $U = \sqrt{u^2 + v^2} \sim u$

for ε small $u = 1 + \varepsilon\sin x - \varepsilon^2\cos^2 x$

@ $x = \pi/2$ $u = 1 + \varepsilon$ max ; $x = -\pi/2$ $u = 1 - \varepsilon$ min ✓

Now if $\bar{X} = x$ and $\bar{Y} = y - \varepsilon\sin x$

$\frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}}\cdot 1 + \frac{\partial}{\partial \bar{y}}(-\varepsilon\cos x) = -\varepsilon\cos x\frac{\partial}{\partial \bar{y}} + \frac{\partial}{\partial \bar{x}}$; $\frac{\partial}{\partial y} = \frac{\partial}{\partial \bar{y}}$

$\frac{\partial^2}{\partial x^2} = +\varepsilon\sin x\frac{\partial}{\partial \bar{y}} + \varepsilon^2\cos^2 x\frac{\partial^2}{\partial \bar{y}^2} - 2\varepsilon\cos x\frac{\partial^2}{\partial \bar{y}\partial \bar{x}} + \frac{\partial^2}{\partial \bar{x}^2}$; $\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \bar{y}^2}$

$\therefore\ \nabla^2\psi = \bar{\nabla}^2\hat{\psi} + \varepsilon\left(\sin\bar{x}\frac{\partial\hat{\psi}}{\partial\bar{y}} - 2\cos\bar{x}\frac{\partial^2\hat{\psi}}{\partial\bar{y}\partial\bar{x}}\right) + \varepsilon^2\cos^2\bar{x}\frac{\partial^2\hat{\psi}}{\partial\bar{y}^2} = 0$ ✓

(1) Let $\psi = \hat{\psi}_0 + \varepsilon\hat{\psi}_1 + \varepsilon^2\hat{\psi}_2 + \dots$ then

ε^0: $\bar{\nabla}^2\hat{\psi}_0 = 0$ ε^1: $\bar{\nabla}^2\hat{\psi}_1 + \left(\sin\bar{x}\frac{\partial\hat{\psi}_0}{\partial\bar{y}} - 2\cos\bar{x}\frac{\partial^2\hat{\psi}_0}{\partial\bar{y}\partial\bar{x}}\right) = 0$

ε^2: $\bar{\nabla}^2\hat{\psi}_2 + \left(\sin\bar{x}\frac{\partial\hat{\psi}_1}{\partial\bar{y}} - 2\cos\bar{x}\frac{\partial^2\hat{\psi}_1}{\partial\bar{x}\partial\bar{x}}\right) + \cos^2\bar{x}\frac{\partial^2\hat{\psi}_0}{\partial\bar{y}^2} = 0$ etc

(2) BC $\psi(x, \varepsilon\sin x) = 0$ but $\varepsilon\sin x = y \Rightarrow \bar{y} = 0$ $\therefore$ $\hat{\psi}(\bar{x}, \bar{y} = 0) = 0$

or $\hat{\psi}_0(\bar{x}, 0) = 0$; $\hat{\psi}_1(\bar{x}, 0) = 0$; $\hat{\psi}_2(\bar{x}, 0) = 0$ etc

(3) $\psi(x,y) \sim y$ as $y \to \infty$ ie $\hat{\psi}(\bar{x}, \bar{y} + \varepsilon\sin\bar{x}) \sim \bar{y} + \varepsilon\sin\bar{x}$ for $\bar{y} \to \infty$

thus $\hat{\psi}_0(\bar{x},\bar{y}+\varepsilon\sin\bar{x}) + \varepsilon\hat{\psi}_1(\bar{x},\bar{y}+\varepsilon\sin\bar{x}) + \varepsilon^2\hat{\psi}_2(\bar{x},\bar{y}+\varepsilon\sin\bar{x}) + \dots \sim \bar{y}+\varepsilon\sin\bar{x}$

or $\left[\hat{\psi}_0(\bar{x},\bar{y}) + \hat{\psi}_{0,\bar{y}}(\bar{x},\bar{y})\varepsilon\sin\bar{x} + \hat{\psi}_{0,\bar{y}\bar{y}}(\bar{x},\bar{y})\varepsilon^2\frac{\sin^2\bar{x}}{2}\right] + \varepsilon\left[\hat{\psi}_1(\bar{x},\bar{y}) + \hat{\psi}_{1,\bar{y}}(\bar{x},\bar{y})\varepsilon\sin\bar{x} + \hat{\psi}_{1,\bar{y}\bar{y}}(\bar{x},\bar{y})\varepsilon^2\frac{\sin^2\bar{x}}{2}\right] +$

$\varepsilon^2\left[\hat{\psi}_2(\bar{x},\bar{y}) + \hat{\psi}_{2,\bar{y}}(\bar{x},\bar{y})\varepsilon\sin\bar{x} + \hat{\psi}_{2,\bar{y}\bar{y}}(\bar{x},\bar{y})\varepsilon^2\frac{\sin^2\bar{x}}{2}\right] + \dots \sim \bar{y}+\varepsilon\sin\bar{x}$

thus: $\hat{\psi}_0(\bar{x},\bar{y}) \sim \bar{y}$ for $\bar{y}\to\infty$ and $\hat{\psi}_1(\bar{x},\bar{y}) + \hat{\psi}_{0,\bar{y}}(\bar{x},\bar{y})\sin\bar{x} \sim \sin\bar{x}$

since $\hat{\psi}_0(\bar{x},\bar{y})\sim\bar{y}$ for $\bar{y}\to\infty$ then $\hat{\psi}_{0,\bar{y}}(\bar{x},\bar{y})\sim 1 \Rightarrow \hat{\psi}_1(\bar{x},\bar{y})\sim 0$ for $\bar{y}\to\infty$

also $\hat{\psi}_{0,\bar{y}\bar{y}}(\bar{x},\bar{y})\sim 0$ and $\hat{\psi}_{1,\bar{y}}(\bar{x},\bar{y})\sim 0 \Rightarrow \hat{\psi}_2(\bar{x},\bar{y})\sim 0$ for $\bar{y}\to\infty$

$\therefore$ (1) $\bar{\nabla}\hat{\psi}_0 = 0$ w/ $\psi_0(\bar{x},0)=0$ and $\psi_0(\bar{x},\bar{y})=\bar{y}$ for large $\bar{y}$

Thus by doing separation of variables we find that $\hat{\psi}_0 = F(\bar{x})G(\bar{y}) \Rightarrow F(\bar{x})$ is linear & so in $G(\bar{y})$

applying bc gives us $\hat{\psi}_0(\bar{x},\bar{y}) = \bar{y}$ (which we could have gotten by inspection, also) ✓

(2) $\bar{\nabla}^2\hat{\psi}_1 = -\sin\bar{x}$ w/ $\hat{\psi}_1(\bar{x},0)=0$ and $\hat{\psi}_1(\bar{x},\bar{y})=0$ for $\bar{y}\to\infty$

to get rid of the inhomogeneity take $f(x) = \hat{\psi}_{1p}$ thus $f(x) = \sin x + Ax + B$. Also the solution to $\bar{\nabla}^2\psi_{1c}=0$ is $\sum_{n=1}^{\infty}(A_n e^{n\bar{y}} + B_n e^{-n\bar{y}})\cos n\bar{x} + \sum_{n=1}(C_n e^{n\bar{y}} + D_n e^{-n\bar{y}})\sin n\bar{x}$

thus $\hat{\psi}_1 = +\sin x + Ax + B \sum_{n=1}(A_n \quad)\cos n\bar{x} + \sum_{n=1}(\quad)\sin n\bar{x}$

applying BC at $\bar{y}=0$ $\forall\bar{x} \Rightarrow A=0$ $B=0$ $A_n+B_n=0$ $\forall n$ $C_n+D_n=0$ $\forall n>1$ $C_1+D_1=-1$

by causality $\psi_1(\bar{x},\bar{y})\to 0$ as $\bar{y}\to\infty \Rightarrow A_n=0$ $C_n=0$ $\forall n \Rightarrow B_n=0$ $D_n=0$ except $D_1=-1$

thus $\hat{\psi}_1 = -e^{-\bar{y}}\sin\bar{x} + \sin\bar{x}$

thus $\hat{\psi}(\bar{x},\bar{y}) = \bar{y} + \varepsilon\sin x(e^{-\bar{y}}-1) + O(\varepsilon^2) = y - \varepsilon\sin x - \varepsilon\sin x\left(e^{-y}e^{+\varepsilon\sin x} - 1\right)$ ✓

$= y - \varepsilon\sin x\, e^{\varepsilon\sin x}e^{-y}$

$\frac{\partial\psi}{\partial y} = u = 1 + \varepsilon\sin x\, e^{-y}e^{\varepsilon\sin x}$; at wall $y=\varepsilon\sin x$ $\therefore$ $u = 1+\varepsilon\sin x + O(\varepsilon^2)$

$-\frac{\partial\psi}{\partial x} = v = \varepsilon\cos x\, e^{\varepsilon\sin x}e^{-y} + \varepsilon^2\sin x\cos x\, e^{\varepsilon\sin x}e^{-y} + O(\varepsilon^2)$; at wall $y=\varepsilon\sin x$ $\therefore$ $v = \varepsilon\cos x + \varepsilon^2\frac{\sin 2x}{2}$

$U = \sqrt{u^2+v^2} \sim u = 1+\varepsilon\sin x$

u_{max} (occurs at $x=+\pi/2$) $= 1+\varepsilon$ $\qquad$ u_{min} (occurs at $x=-\frac{\pi}{2}$) $= 1-\varepsilon$

at $x=+\pi/2$ $\bar{y} = y-\varepsilon = 0$ $y=+\varepsilon$ $\qquad$ at $x=-\pi/2$ $\bar{y}=y+\varepsilon=0$ $y=-\varepsilon$

again we note that max & min u for order ε solution is same as that for order ε^2 solution previously found. ✓

A

$\nabla^2 T = 0$ w/ $T=0$ on $r=1$

$T=1$ on $r = 2+\varepsilon\cos\theta$

$\nabla^2 T = \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}\right) T = 0$ use sep of var $T(r,\theta) = F(r)G(\theta)$

$G \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial F}{\partial r} + \frac{F}{r^2}\frac{\partial^2 G}{\partial \theta^2} = 0$ or $-\frac{G''}{G} = \frac{r^2}{F}\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial F}{\partial r}\right] = \lambda^2$

$r\frac{\partial F}{\partial r} + r^2 F'' - \lambda^2 F = 0$

$r^2F'' + r\frac{\partial F}{\partial r} - \lambda^2 F = 0$ let $r^{\pm n}$

$n(n-1) + n - \lambda^2 = 0$

$-n(-n-1) - n - \lambda^2 = 0$

$n(n+1) - n - \lambda^2 = 0$

$n^2 - \lambda^2 = 0 \quad n = \pm\lambda$

$(Ar^{+n} + Br^{-n})$

$G = A\cos\lambda\theta + B\sin\lambda\theta$ λ must be integer

Let $T(r,\theta;\varepsilon) = T_0(r,\theta) + \varepsilon T_1(r,\theta) + \varepsilon^2 T_2(r,\theta) + \dots$

$\nabla^2 T = \nabla^2 T_0 + \varepsilon \nabla^2 T_1 + \varepsilon^2 \nabla^2 T_2 = 0$

$T(1,\theta) = T_0(1,\theta) + \varepsilon T_1(1,\theta) + \varepsilon^2 T_2(1,\theta) + \dots = 0$

$T(2+\varepsilon\cos\theta,\theta) = T_0(2+\varepsilon\cos\theta,\theta) + \varepsilon T_1(2+\varepsilon\cos\theta,\theta) + \varepsilon^2 T_2(2+\varepsilon\cos\theta,\theta) + \dots = 1$

$= T_0(2,\theta) + T_{0,r}(2,\theta)\cdot\varepsilon\cos\theta + \varepsilon\left[T_1(2,\theta) + T_{1,r}(2,\theta)\varepsilon\cos\theta\right] + \varepsilon^2\left[T_2(2,\theta) + T_{2,r}\right.$

$= T_0(2,\theta) + \varepsilon\left[T_1(2,\theta) + T_{0,r}\cos\theta\right] + \varepsilon^2\left[T_2(2,\theta) + T_{1,r}(2,\theta)\cos\theta + T_{0,rr}\frac{\cos^2\theta}{2}\right.$

$= 1 + \varepsilon\cdot 0 + \varepsilon^2\cdot 0$

$\therefore$ $\nabla^2 T_0 = 0$, $T_0(1,\theta) = 0$, $T_0(2,\theta) = 1$

$\nabla^2 T_1 = 0$, $T_1(1,\theta) = 0$, $T_1(2,\theta) = -T_{0,r}(2,\theta)\cos\theta$

$\nabla^2 T_2 = 0$, $T_2(1,\theta) = 0$, $T_2(2,\theta) = -T_{1,r}(2,\theta)\cos\theta - T_{0,rr}\frac{\cos^2\theta}{2}$

Soln $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}T_0\right) = 0$

$r\frac{\partial}{\partial r}T_0 = C_1$

$T_0 = C_1 \ln r + C_2$

$0 = C_1\cdot 0 + C_2 \Rightarrow C_2 = 0$

$1 = C_1 \ln 2 \Rightarrow C_1 = \frac{1}{\ln 2}$

$\therefore \boxed{T_0 = \ln r/\ln 2}$

$T_{0,r}(2,\theta) = \frac{1}{2\ln 2}$ $\therefore T_1(2,\theta) = -\frac{1}{2\ln 2}\cos\theta$

let $F_1(r)G_1(\theta)$

$F_1(r) = Ar^n + Br^{-n}$

$G_1(\theta) = C\cos n\theta + D\sin n\theta$

$T_1(1,\theta) = 0 \Rightarrow A + B = 0$

$T_1(2,\theta) = \frac{-1}{2\ln 2}\cos\theta$

$T_1(r,\theta) = \sum_n (A_n r^n + B_n r^{-n})(C_n\cos n\theta + D_n\sin n\theta)$

$T_1(1,\theta) = 0 = \sum_{n=1}^{\infty}(A_n + B_n)(C_n\cos n\theta + D_n\sin n\theta)$ $\forall\theta \Rightarrow A_n + B_n = 0$

$T_1(2,\theta) = \frac{-1}{2\ln 2}\cos\theta = \sum_{n=1}^{\infty}(A_n 2^n + B_n 2^{-n})(C_n\cos n\theta + D_n\sin n\theta)$ $D_n = 0$

$A_n 2^n + 2^{-n}B_n = 0 \quad \forall\, n>1$

$\Rightarrow A_n, B_n\ (\forall n>1) = 0$

$A_1 2 + \frac{1}{2}B_1 = \frac{-1}{2\ln 2}$

$A_1 + B_1 = 0$

$A_1(2 - 1/2) = \frac{-1}{2\ln 2}$

$A_1 = \frac{-1}{3\ln 2}$

$B_1 = +\frac{1}{3\ln 2}$

$\boxed{T_2(r,\theta) = \frac{-1}{3\ln 2}\left(r - \frac{1}{r}\right)\cos\theta}$

$\therefore \quad T(r,\theta) = \frac{\ln r}{\ln 2} + \frac{\varepsilon}{3\ln 2}\left(\frac{1}{r} - r\right)\cos\theta + O(\varepsilon^2)$

<u>3.2</u> given $\psi_{xx} + \psi_{yy} = 0$ and $\psi = 0$ on $y = \varepsilon \sin x$

$\psi \sim y$ for very far y

then if $\psi(x,y;\varepsilon) = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \dots$

$\psi(x, y = \varepsilon\sin x) = \psi_0(x, \varepsilon\sin x) + \varepsilon\psi_1(x, \varepsilon\sin x) + \varepsilon^2\psi_2(x, \varepsilon\sin x) + \dots = 0$

$\psi(x,y) = y \qquad \psi_0(x,0) + \psi_{0,y}(x,0)\,\varepsilon\sin x + \frac{\varepsilon^2\sin^2 x}{2}\psi_{0,yy} + \varepsilon\left[\psi_1(x,0) + \varepsilon\sin x\,\psi_{1,y}(x,0) + \frac{\varepsilon^2\sin^2 x}{2}\psi_{1,yy}\right] + \varepsilon^2\left[\psi_2(x,0) + \varepsilon\sin x\,\psi_2'(x,0)\right]$

(1) $\Rightarrow \nabla^2\psi_0 = 0$

$\psi_0(x,0) = 0$

$\psi_0(x,y) \Rightarrow y \quad$ as $y \to \infty$

$\therefore \quad F''G + G''F = 0$ or $\frac{F''(x)}{F} = -\frac{G''(y)}{G} = -\lambda^2 \qquad F'' + \lambda^2 F = 0 \qquad F(x) = C\sin\lambda x + D\cos\lambda x$

$G'' - \lambda^2 G = 0 \qquad G(y) = Ae^{\lambda y} + Be^{-\lambda y}$

a particular solution to this is $\psi_0(x,y) = y$

(2) $\nabla^2\psi_1 = 0$

$\psi_{0,y}(x,0)\sin x + \psi_1(x,0) = 0 \qquad \psi_{0,y}(x,0) = 1 \qquad \therefore \psi_1(x,0) = -\sin x$

$\psi_1(x,y) \Rightarrow 0$ as $y \to \infty$

$\therefore \psi(x,y) = \sum^{\infty} (\cancel{Ae^{y}} + Be^{-y})(C\sin\lambda x + D\cos\lambda x)$

$\psi(x,0) = -\sin x \quad \Rightarrow \quad \psi(x,y) = -e^{-y}\sin x$

$\therefore \quad \psi_0 + \varepsilon\psi_1 \sim y - \varepsilon e^{-y}\sin x$

(3) $\nabla^2\psi_2 = 0$

$\psi_{0,yy}\frac{\varepsilon^2\sin^2 x}{2} + \sin x\,\varepsilon^2\,\psi_{1,y}(x,0) + \psi_2(x,0)\,\varepsilon^2 = 0$

$\psi_2(x,y) \to 0$ as $y \to \infty$

$\psi_{0,yy}(x,0) = 0 \qquad \psi_{1,y}(x,0) = +e^{-y}\sin x\Big|_{y=0} = \sin x$

ψ_{2x}

$\lambda = 0 = \cancel{Axy + Bx + Cy + D}$

$\therefore -\sin^2 x = \psi_2(x,0) = -\left(\frac{1-\cos 2x}{2}\right) = \frac{\cos 2x}{2} - \frac{1}{2} \qquad \psi_{2,2} = \frac{1}{2}e^{-2y}\cos 2x$

$Bx + D = 2$

$Bx + D = 0$

$\psi_2(x,y) \to 0 \qquad y \to \infty$

$\psi_2(x,y) = \sum C_n \cancel{e^{\lambda y}\sin\lambda_n x} + D_n \cancel{e^{\lambda_n y}\cos\lambda x} + E_n e^{-\lambda_n y}\sin\lambda_n x + F_n e^{-\lambda_n y}\cos\lambda_n x$

$\psi(x,0) = -k \qquad \psi(x,y) \to 0$ as $y \to \infty \qquad f''(y) = e^{-y}(y - y_2) \qquad \cancel{\dots}\ ye^{-y}$

Let $\psi_2(x,y) \to -\frac{1}{2}$ as $y \to \infty$ This is similar to changing the bc
$\psi(x,y) \sim y$ to $\psi(x,y) \to y + \frac{1}{2}\varepsilon^2$
thus $\psi_2(x,y) = e^{-y}\frac{\cos 2x}{2} - \frac{1}{2}$ or ~~$\frac{\cos 2x}{2} - (\frac{1}{2} - \sin^2 x)$~~

$$\psi(x,y;\varepsilon) = y - \varepsilon e^{-y}\sin x + \varepsilon^2\left(e^{-y}\frac{\cos 2x}{2} - \frac{1}{2}\right) + O(\varepsilon^2)$$
$$= y - \varepsilon e^{-y}\sin x - \varepsilon^2\left(\frac{1}{2} - e^{-y}\frac{\cos 2x}{2}\right) + O(\varepsilon^2)$$
$$\frac{\partial\psi}{\partial y} = u = 1 + \varepsilon e^{-y}\sin x + \varepsilon^2\left(-e^{-y}\frac{\cos 2x}{2}\right) + O(\varepsilon^2) \quad ; \quad \text{on } y = \varepsilon\sin x$$
$$= 1 + \varepsilon\left[1 - \varepsilon\sin x + \varepsilon^2\frac{\sin^2 x}{2} + \dots\right] - \varepsilon^2\left[1 - \varepsilon\sin x + \varepsilon^2\frac{\sin^2 x}{2} + \dots\right]\frac{\cos 2x}{2} + O(\varepsilon^2)$$
$$= 1 + \varepsilon - \varepsilon^2\left(\sin x + \frac{1}{2}\cos 2x\right) + O(\varepsilon^3)$$
$$\left[1 + \varepsilon - \varepsilon^2\left(\sin x + \frac{\cos 2x}{2}\right)\right]^2 \left[1 + \varepsilon - \varepsilon^2\left(\sin x + \frac{\cos 2x}{2}\right)\right]$$

$$-\frac{\partial\psi}{\partial x} = +v = +\varepsilon e^{-y}\cos x + \varepsilon^2\left(+e^{-y}\sin 2x\right) + O(\varepsilon^2) \quad ; \quad \text{on } y = \varepsilon\sin x$$
$$= +\varepsilon\left[1 - \varepsilon\sin x + \varepsilon^2\frac{\sin^2 x}{2} + \dots\right]\cos x + \varepsilon^2\left[1 - \varepsilon\sin x + \varepsilon^2\frac{\sin^2 x}{2} + \dots\right]\sin 2x$$
$$= \varepsilon\cos x - \varepsilon^2\left(\frac{\sin 2x}{2} - 1\right) + O(\varepsilon^2)$$

$\left(1 + \varepsilon\sin x - \frac{\varepsilon^2}{2}\right)\left(1 + \varepsilon\sin x - \frac{\varepsilon^2}{2}\right)$

$1 + 2\varepsilon\sin x - \frac{\varepsilon^2}{2} + \varepsilon^2\sin^2 x - 2\varepsilon^3\frac{\sin x}{2}$

$1 + 2\varepsilon\sin x + (\sin^2 x - 1)\varepsilon^2$

$$u^2 = 1 + 2\varepsilon + \varepsilon^2\left(1 - 2\sin x - \frac{2\cos 2x}{2}\right) + [$$
$$v^2 = \varepsilon^2\cos^2 x \qquad (1-\sin x)^2 - \cos^2 x \quad -\cos^2 x + \sin^2 x\ \cos^2 x$$
$$u^2 + v^2 = 1 + 2\varepsilon + (1 - 2\sin x - \cos 2x)\varepsilon^2$$
$$(1 - 2\sin x + \sin^2 x)\varepsilon^2$$
$$1 + 2\varepsilon \qquad [\varepsilon(1-\sin x)]^2$$
$$-2\varepsilon(1-\sin x)\cos x = 0$$

$x = n\pi \qquad u^2 + v^2 = 1 + 2\varepsilon + \varepsilon^2$
$x = \frac{2n+1}{2}\pi \qquad u^2 + v^2 = 1 + 2\varepsilon$

Max: $u^2 + v^2 = (1 + 2\varepsilon + \varepsilon^2)$ on peaks $\quad u = 1 + \varepsilon$
min: $u^2 + v^2 = 1 + 2\varepsilon$ in trough

$\cos x = \frac{2n+1}{2}\pi$ min
$\sin x = n\pi$ max

$X = x \qquad Y = Y(y,x)$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial}{\partial X}\frac{\partial X}{\partial x} = -\varepsilon\cos x\frac{\partial}{\partial Y} + \frac{\partial}{\partial X}$$

$-\varepsilon\cos x \qquad 1$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial}{\partial Y}\frac{\partial Y}{\partial x}\right)\frac{\partial}{\partial x}$$

$$\frac{\partial^2}{\partial x^2} = \left[+\varepsilon\sin x\frac{\partial}{\partial Y} - \varepsilon\cos x\frac{\partial}{\partial Y}\left[-\varepsilon\cos x\frac{\partial}{\partial Y} + \frac{\partial}{\partial X}\right]\right] = \varepsilon\sin x\frac{\partial}{\partial Y} + \varepsilon^2\cos^2 x\frac{\partial^2}{\partial Y^2} - 2\varepsilon\cos x\frac{\partial^2}{\partial Y\partial X} + \frac{\partial^2}{\partial X^2}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial Y}\frac{\partial Y}{\partial y} = \frac{\partial}{\partial Y}$$

$$\nabla^2\psi = \varepsilon\sin X\frac{\partial\psi}{\partial Y} + \varepsilon^2\cos^2 X\frac{\partial^2\psi}{\partial Y^2} - 2\varepsilon\cos X\frac{\partial^2\psi}{\partial Y\partial X} + \frac{\partial^2\psi}{\partial Y^2} + \frac{\partial^2\psi}{\partial X^2} = 0$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial Y^2}$$

let $\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \dots$

$\partial^2\psi_0(x,y) = 0$ w/ $\psi_0(x,0) = 0$ and $\psi_0(x,Y)$

solution $\psi_0(\bar{x},\bar{Y}) = c_0\bar{Y} + c_1$

$\psi_0(\bar{x},\bar{Y}) = c_0\bar{Y} + c_1$

$\psi_0(\bar{x},\bar{Y}) = \bar{Y}$

$\psi_0(\bar{x},0) + \varepsilon\psi_1(\bar{x},0) + \varepsilon^2\psi_2(\bar{x},0) + \dots = 0$

$\psi_0(\bar{x},\bar{Y}) + \varepsilon\psi_1(\bar{x},\bar{Y}) + \varepsilon^2\psi_2(\bar{x},\bar{Y}) + \dots = \bar{Y} + \varepsilon\sin\bar{x}$ for large $\bar{Y} \Rightarrow \bar{Y}$

$\varepsilon\sin\bar{x}\,\frac{\partial\psi_0}{\partial\bar{y}} - \varepsilon\cos\bar{x}\,\frac{\partial^2\psi_0}{\partial\bar{y}\,\partial\bar{x}} + \varepsilon\,\frac{\partial^2\psi_1}{\partial\bar{y}^2} = 0$

w/ $\psi_1(\bar{x},0) = 0$

$\psi_1(\bar{x},\bar{Y}) = \sin\bar{x}$ for large $\bar{Y}$

$\varepsilon\sin\bar{x} - \varepsilon\cos\bar{x}\cdot 0 + \varepsilon\,\frac{\partial^2\psi_1}{\partial\bar{y}^2} = 0$

$\frac{\partial^2\psi_1}{\partial\bar{y}^2} = -\sin\bar{x} \quad \therefore \quad \frac{\partial\psi_1}{\partial\bar{y}} = -\bar{y}\sin\bar{x} + f(\bar{x})$

$\psi_1 = -\bar{y}^2\frac{\sin\bar{x}}{2} + \bar{y}f(\bar{x}) + g(\bar{x})$

$\psi_1(\bar{x},0) = \Rightarrow g(\bar{x}) = 0 \quad \forall x \Rightarrow g(\bar{x}) = 0$

$\psi_1(\bar{x},\bar{Y}) = -\bar{Y}^2\frac{\sin\bar{x}}{2} + \bar{Y}f(\bar{x}) = \sin\bar{x}$

$\psi_1 = F(x)G(y)$

$G''F = -\sin\bar{x}$

$G'' = -\frac{\sin\bar{x}}{F} = \text{const } c_1 \quad \therefore \quad G = c_1\frac{y^2}{2} + c_2 y + c_3$

$F = -\frac{1}{c_1}\sin\bar{x}$

$\psi_1 = FG = -\frac{y^2}{2}\sin\bar{x} - \frac{c_2}{c_1}y\sin\bar{x} - \sin\bar{x}\,\frac{c_3}{c_1}$

$\psi_1(\bar{x},0) = -\sin\bar{x}\,\frac{c_3}{c_1} = 0 \qquad c_3 = 0 \quad \forall x$

$\psi_1(\bar{x},\bar{Y}) \Rightarrow \sin\bar{x}$ for large $\bar{Y}$

$\frac{\partial u}{\partial x} = 0 = \varepsilon\cos x + 2\cos x\sin x\,\varepsilon^2$

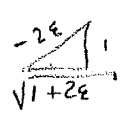
~~$\varepsilon\cos x(1 - 2\varepsilon\cos x)$~~ $\qquad \cos x = \pi/2,\ 3\pi/2$

$\cos x = \frac{1}{2\varepsilon}$

$\varepsilon\cos x\,[1 + 2\varepsilon\sin x] \qquad \cos x = 0 \Rightarrow x = \pi/2$

$\sin x = -\frac{1}{2\varepsilon}$

$\cos x$



$f(x) = -\sin x + Ax + B$

$\psi_1(\bar{x},\bar{y}) = \sum (A_n e^{n\bar{y}} + B_n e^{-n\bar{y}}) \cos n\bar{x} + \sum (C_n e^{n\bar{y}} + D_n e^{-n\bar{y}}) \sin n\bar{x} - \sin\bar{x} + A\bar{x} + B$

$\psi_1(\bar{x},0) = \sum_{n=0}^{\infty} (A_n + B_n) \cos nx - \sin\bar{x} + A\bar{x} + B = 0$

$\Rightarrow \quad (C_n + D_n) \sin nx$

$C_1 + D_1 = +1 \qquad A=0 \quad B=0 \qquad A_n + B_n = 0 \ \forall n \qquad A_n = -B_n$

$C_n + D_n = 0 \ \forall n>1 \qquad C_n = -D_n$

since $\psi_1(\bar{x},\bar{y}) \to 0$ for $\bar{y} \to \infty$

$\Rightarrow e^{ny}$ cannot exist as sol. $\Rightarrow$ $A_n = 0,\ C_n = 0 \quad \forall n \Rightarrow B_n = 0$

$D_n(>1) = 0$

$C_1 = 0$

$\therefore \quad \psi_1(\bar{x},\bar{y}) = -\sin x + e^{-y}\sin x$

$\therefore \quad \psi(\bar{x},\bar{y}) = \bar{y} - \varepsilon \sin\bar{x}(1 - e^{-\bar{y}}) = ~~y + 2\sin x - \varepsilon\sin x~~$

$= y - \varepsilon\sin x - \varepsilon\sin x + \varepsilon \sin x\, e^{-\bar{y}}$

$= y - 2\varepsilon\sin x + \varepsilon\sin x\, e^{-y + \varepsilon\sin x}$

$\psi(x,y) = ~~y - 2\varepsilon\sin x + \varepsilon e^{-y}\sin x~~ \left[1 + \varepsilon\sin x + \varepsilon^2 \frac{\sin^2 x}{2}\right]$

$u = \frac{\partial}{\partial y}\psi = \frac{\partial}{\partial \bar{y}} = 1 - \varepsilon\sin\bar{x}(+e^{-\bar{y}}) = 1 - \varepsilon\sin\bar{x}$

$v = -\frac{\partial}{\partial x}\psi = +\varepsilon\cos x \frac{\partial}{\partial \bar{y}} - \frac{\partial}{\partial \bar{x}} = \varepsilon\cos\bar{x}\left[1 - \varepsilon\sin\bar{x}\, e^{-\bar{y}}\right] - \varepsilon\cos\bar{x}(1 - e^{-\bar{y}})$

$= -\varepsilon^2 \cos\bar{x}\sin\bar{x}\, e^{-\bar{y}} + \varepsilon\cos\bar{x}\, e^{-\bar{y}}$

$v = -\varepsilon^2\cos\bar{x}\sin\bar{x}\left[1 - \bar{y} + \bar{y}^2/2 + \dots\right] + \varepsilon\cos\bar{x}\left[1 - \bar{y} + \bar{y}^2/2 + \dots\right.$

$= -\varepsilon^2 \frac{\sin 2x}{2} + \varepsilon\cos x$

$v^2 + u^2 = (1 - \varepsilon\sin x)^2 + \varepsilon^2\cos^2 x$

$v^2 + u^2 = (1 - 2\varepsilon\sin x + \varepsilon^2)$ $\qquad$ max is when $x = \pi/2$

$x =$

$$\therefore \quad \nabla^2\psi = \varepsilon \sin\bar{x}\,\frac{\partial \psi}{\partial \bar{y}} + \varepsilon^2\cos^2\bar{x}\,\frac{\partial^2\psi}{\partial \bar{y}^2} - \varepsilon 2\cos\bar{x}\,\frac{\partial^2\psi}{\partial\bar{y}\,\partial\bar{x}} + \bar{\nabla}^2\psi = 0$$

for $\sum_{i=0}^{\infty} \psi_i(\bar{x},\bar{y})\,\varepsilon^i$

then for ε^0: $\quad \bar{\nabla}^2\psi_0 = 0$

ε^1: $\quad \sin\bar{x}\,\frac{\partial\psi_0}{\partial\bar{y}} - 2\cos\bar{x}\,\frac{\partial^2\psi_0}{\partial\bar{y}\,\partial\bar{x}} + \bar{\nabla}^2\psi_1 = 0$

ε^2: $\quad \sin\bar{x}\,\frac{\partial\psi_1}{\partial\bar{y}} + \cos^2\bar{x}\,\frac{\partial^2\psi_0}{\partial\bar{y}^2} - 2\cos\bar{x}\,\frac{\partial^2\psi_0}{\partial\bar{y}\,\partial\bar{x}} + \bar{\nabla}^2\psi_2 = 0$

B.C. $\psi(x, \varepsilon\sin x) = 0$ but $\varepsilon\sin x = y \Rightarrow \bar{y} = 0 \quad \therefore \quad \psi(\bar{x}, \bar{y}=0) = 0$

$\therefore \quad \psi_0(\bar{x},0) = 0 \qquad \psi_1(\bar{x},0) = 0 \qquad \psi_2(\bar{x},0) = 0 \quad \ldots$

B.C. $\psi(x,y) \sim y$ for $y \to \infty$ ie $\psi(\bar{x}, \bar{y}+\varepsilon\sin\bar{x}) \sim \bar{y} + \varepsilon\sin\bar{x}$ for $\bar{y} \to \infty$

$$\psi_0(\bar{x}, \bar{y}+\varepsilon\sin\bar{x}) + \varepsilon\,\psi_1(\bar{x}, \bar{y}+\varepsilon\sin\bar{x}) + \varepsilon^2\psi_2(\bar{x}, \bar{y}+\varepsilon\sin\bar{x}) + \ldots \sim \bar{y} + \varepsilon\sin\bar{x}$$

$$\left[\psi_0(\bar{x},\bar{y}) + \psi_{0,\bar{y}}(\bar{x},\bar{y})\,\varepsilon\sin\bar{x} + \psi_{0,\bar{y}\bar{y}}(\bar{x},\bar{y})\,\frac{\varepsilon^2\sin^2\bar{x}}{2}\right] + \varepsilon\left[\psi_1(\bar{x},\bar{y}) + \psi_{1,\bar{y}}(\bar{x},\bar{y})\,\varepsilon\sin\bar{x} + \psi_{1,\bar{y}\bar{y}}(\bar{x},\bar{y})\,\frac{\varepsilon^2\sin^2\bar{x}}{2} + \ldots\right]$$

$$+ \varepsilon^2\left[\psi_2(\bar{x},\bar{y}) + \psi_{2,\bar{y}}(\bar{x},\bar{y})\,\varepsilon\sin\bar{x} + \psi_{2,\bar{y}\bar{y}}(\bar{x},\bar{y})\,\frac{\varepsilon^2\sin^2\bar{x}}{2} + \ldots\right] \sim \bar{y} + \varepsilon\sin\bar{x}$$

ε^0: $\quad \psi_0(\bar{x},\bar{y}) = \bar{y}$

ε^1: $\quad \psi_{0,\bar{y}}(\bar{x},\bar{y})\sin\bar{x} + \psi_1(\bar{x},\bar{y}) = \sin\bar{x} \quad \Rightarrow \quad \psi_1(\bar{x},\bar{y}) = 0$

ε^2: $\quad \psi_{0,\bar{y}\bar{y}}(\bar{x},\bar{y})\frac{\sin^2\bar{x}}{2} + \psi_{1,\bar{y}}(\bar{x},\bar{y})\sin\bar{x} + \psi_2(\bar{x},\bar{y}) = 0 \quad \Rightarrow \quad \psi_2(\bar{x},\bar{y}) = 0$

$\therefore \quad \bar{\nabla}^2\psi_0 = 0 \quad$ w/ $\quad \psi_0(\bar{x},0) = 0 \qquad \psi_0(\bar{x},\bar{y}) = \bar{y}$ for large $\bar{y}$

$\psi_0 = F(\bar{x})G(\bar{y}) \qquad F''G + G''F = 0 \quad \Rightarrow \quad \frac{F''}{F} = -\frac{G''}{G} = -\lambda^2$

gives sin, cos, sinh, cosh except for $\lambda = 0$; note $\lambda^2 \neq 0$ since y sol. will not satisfy far field b.c.

$F''(\bar{x}) = 0 \quad \Rightarrow \quad \left.\begin{array}{l} F = \hat{A}\bar{x} + \hat{B} \\ G = \hat{C}\bar{y} + \hat{D} \end{array}\right\} \quad \psi_0 = A\bar{x}\bar{y} + B\bar{x} + C\bar{y} + D$

$\psi_0(x,0) = B\bar{x} + D = 0 \quad \forall x \Rightarrow B, D = 0$

$\psi_0(x,y) = \bar{y} = A\bar{x}\bar{y} + C\bar{y} \quad \forall\bar{x} \Rightarrow A = 0 \quad C = 1$

$\bar{\nabla}^2\psi_1 = -\sin\bar{x} \quad$ w/ $\quad \psi_1(\bar{x},\bar{y}) = 0 \quad \bar{y} \to \infty$

Let ψ_1 parti $= f(x)$ ~~$\psi_h = e^{-\bar{y}}\sin\bar{x}$~~ $\psi_1(\bar{x},0) = 0$

$f''(\bar{x}) = -\sin\bar{x}$

$f'(\bar{x}) = +\cos\bar{x} + A \qquad$ choose periodicity

$$= \sum_{n=1}^{\infty}\left(A_n e^{-n\bar{y}} + B_n e^{+n\bar{y}}\right)(\cos nx) + \left(C_n e^{-n\bar{y}} + D_n e^{+n\bar{y}}\right)\sin nx$$

$f(x) = -\sin x$ ~~$+ Ax + B$~~

$\sim -\sin x$ ~~$+ Ax + B$~~ $+ \sum (A_n + B_n)(\cos nx) +$

Solutions to Exercise Set 4

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

Wednesday 2 May 1979

2.4 of Notes. Plane waves traveling through slowly changing environment. With $F(x) = 1 + \epsilon x$ the problem becomes

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \epsilon\left[\frac{\partial u}{\partial x} - x\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right)\right], \qquad u(0,t) = \cos t\,,$$

together with the "radiation condition" that waves travel only in the positive x-direction. For $\epsilon = 0$ we have the classical wave equation, with general solution $f(t-x) + g(t+x)$, and imposing the boundary conditions gives

$$u_I = \cos(t-x)\,.$$

Substituting this into the right-hand side gives the iteration equation

$$\frac{\partial^2 u_{II}}{\partial t^2} - \frac{\partial^2 u_{II}}{\partial x^2} = \epsilon \sin(t-x)\,.$$

A particular integral is $-\frac{1}{2}\epsilon x \cos(t-x)$, and adding a solution of the homogeneous equation leads to

$$u_{II} = \cos(t-x) - \tfrac{1}{2}\epsilon x \cos(t-x)\,.$$

This approximation evidently breaks down when x is as large as $1/\epsilon$. Substituting this again into the right-hand side gives the new iteration equation

$$\frac{\partial^2 u_{III}}{\partial t^2} - \frac{\partial^2 u_{III}}{\partial x^2} = \epsilon \sin(t-x) - \tfrac{1}{2}\epsilon^2\left[\cos(t-x) + 3x\sin(t-x)\right].$$

The last term on the right will contribute to the particular integral a term

$$\tfrac{3}{8}\epsilon^2 x^2 \cos(t-x)$$

which shows that the difficulty when x is as large as $1/\epsilon$ is compounded in higher approximations.

Substituting the proposed approximation into the equation gives

$$\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) - F\frac{\partial^2 u}{\partial t^2} = \frac{\cos(x-t)}{4F(x)^{3/2}}\left(F'^2 - 2FF''\right).$$

Hence it is an exact solution if $F'^2 = 2FF''$. Writing this as $F'/F = 2(F''/F')$ we can integrate it to find that $F(x) = (a+bx)^2$ or, since $u = \cos t$ at $x = 0$, $F(x) = (1+bx)^2$. For $F(x) = 1+\epsilon x$ the equation is

$$\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) - F\frac{\partial^2 u}{\partial t^2} = \epsilon^2\,\frac{\cos(x-t)}{4(1+\epsilon x)^{3/2}}\,.$$

This shows that the correction is only of order ϵ^2, so that the difficulty at large x now appears in the term of order ϵ^2 rather than order ϵ.

To this order we could equally well take as our first approximation

$$u \approx \frac{\cos(x-t)}{1+\frac{1}{2}\epsilon x} \approx (1-\tfrac{1}{2}\epsilon x)\cos(x-t)\,.$$

This suggests expanding the amplitude further in a power series in ϵ with coefficients depending on x. However, it is clear that the phase as well as the amplitude of the wave must be allowed to vary with x. Hence we try

$$u = \left[1+\epsilon A(x)+\epsilon^2 B(x)+\ldots\right]\cos\left[t-x-\epsilon a(x)-\epsilon^2 b(x)-\ldots\right].$$

The equation becomes

$$\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right)-F\frac{\partial^2 u}{\partial t^2} = \left[(A''-2a')\epsilon+(B''-2Aa'+A''x+A'-2b'-2a'x-a'^2)\epsilon^2+\ldots\right]\cos$$
$$+\left[(1+2A'+a'')\epsilon+(2B'+2A'a+Aa''+2A'X+A+b''+a''x+a')\epsilon^2\right]\sin.$$

We can annihilate the terms of order ϵ by choosing the free functions A and a according to

$$\left.\begin{array}{r}A''-2a'=0\\ 1+2A'+a''=0\end{array}\right\}\quad \begin{array}{l}A'''+4A'+2=0\\ a'''+4a'=0\end{array}\quad \text{of which the simplest solution is}\quad \left\{\begin{array}{l}A=-\frac{1}{2}x\\ a=0.\end{array}\right.$$

Similarly, we can annihilate the terms of order ϵ^2 by setting

$$\left.\begin{array}{r}B''-\frac{1}{2}-2b'=0\\ 2B'-\frac{3}{2}x+b''=0\end{array}\right\}\quad \begin{array}{l}B'''+4B'=3x\\ b'''+4b'=\frac{1}{2}\end{array}\quad \text{of which the simplest solution is}\quad \left\{\begin{array}{l}B=\frac{3}{8}x^2\\ b=\frac{1}{8}x.\end{array}\right.$$

Hence to this order

$$u \approx \left[1-\tfrac{1}{2}\epsilon x+\tfrac{3}{8}\epsilon^2 x^2+\ldots\right]\cos\left(t-x-\tfrac{1}{8}\epsilon^2 x+\ldots\right)$$

(Notice that to this order the amplitude is the expansion of the square root of the previous approximation.) Evidently this process can be continued indefinitely, using each successive pair of free functions C,c; D,d; ... to postpone the difficulty at large x indefinitely.

This approximation is discussed for acoustic waves in a trumpet, and in air of slowly varying density, by Rayleigh in section 266 of his Theory of Sound. For longitudinal waves in a bar of cross-sectional area $F = 1+\epsilon x^n$ it has been deduced using the method of multiple scales (without invoking the energy idea) by Wingate and Davis (1970: J. Acoust. Soc. Amer. 47, 1334-1337).

3.1. of Notes. Torsion of circular shaft of varying radius. For constant radius the solution is independent of z, so the differential equation becomes $\phi_{rr}-3\phi_r/r=0$ or $(\phi_r/r^3)_r=0$. The solution is a multiple of r^4, and imposing the surface boundary condition gives $\phi=(r/a)^4$.

For the hyperboloidal shaft we set

$$\phi = r^4+\epsilon^2\phi_2(r,z)+\ldots$$

Substitution shows that ϕ_2 satisfies the full differential equation and condition at $r = 0$; and transferring the surface condition by Taylor series gives

$$\phi(1, z) = -2z^2.$$

Trying a polynomial in r^2 and z^2 leads to

$$\phi_2 = -2r^4z^2 - \frac{1}{3}r^4(1-r^2).$$

Contracting the abscissa to $Z = \epsilon z$ reduces the differential equation to

$$\phi_{rr} - 3\frac{\phi_r}{r} \equiv r^3(\phi_r/r^3)_r = -\epsilon^2\phi_{ZZ} \cdot \quad = -\varepsilon^2 r^4 \left[\frac{1}{f^4(Z)}\right]''$$

Then for the shaft described by $r = f(\epsilon z) = f(Z)$ the quasi-one-dimensional solution is given by quadratures as $\phi = [r/f(Z)]^4$; and iterating or expanding in powers of ϵ^2 leads to the second approximation

$$\phi = \frac{r^4}{f^4(Z)} + \epsilon^2 \frac{r^4[f^2(Z) - r^2]}{12} \left[\frac{1}{f^4(Z)}\right]'' + O(\epsilon^4)$$

Setting $f^2(Z) = 1 + \epsilon^2Z^2$ gives an approximation for the hyperboloidal shaft that is uniformly valid. Restoring $z = Z/\epsilon$ and expanding for small ϵ reproduces the previous result, which is not valid when $z = O(1/\epsilon)$. The result for the cone of semi-vertex angle $\tan^{-1}\epsilon$ can be extracted from that for the hyperboloid simply by letting $Z \to \infty$;

$$\phi = \frac{r^4}{Z^4} + \frac{5}{3}\epsilon^2\left(\frac{r^4}{Z^4} - \frac{r^6}{Z^6}\right) + \dots$$

and expanding Föppl's exact solution for small r/z and $\alpha = \tan^{-1}\epsilon$ gives this same result.

Numerator = $\frac{3}{4}\left(\frac{r}{z}\right)^4 - \frac{5}{4}\left(\frac{r}{z}\right)^6 + .$

Denominator = $\frac{3}{4}\varepsilon^4 - \frac{5}{4}\varepsilon^6 + .$

Cesar Levy
ME 207
Prof. Van Dyke

Problem Set #4

2.4 a Given $\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) - F\frac{\partial^2 u}{\partial t^2} = 0$ w/ $u(0,t) = \cos t$ and $F = 1+\varepsilon x$

Put the $F(x)$ into DE to obtain $\varepsilon\frac{\partial u}{\partial x} + (1+\varepsilon x)\Box^2 u = 0$

if $u(x,t;\varepsilon) = u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \dots$ then $u(0,t;\varepsilon) = \cos t \Rightarrow u_0(0,t) = \cos t$

and $u_i(0,t) = 0 \quad \forall i > 0$.

for ε^0: $\Box^2 u_0 = 0$ w/ $u_0(0,t) = \cos t$ $\boxed{u_0(x,t) = \cos(x-t)}$ ✓

for ε^1: $\varepsilon\left[\frac{\partial u_0}{\partial x} + x\Box^2 u_0\right] + \varepsilon\Box^2 u_1 = 0$ but $\Box^2 u_0 = 0$ and $\frac{\partial u_0}{\partial x} = -\sin(x-t)$

thus $\sin(x-t) = \Box^2 u_1$ w/ $u_1(0,t) = 0$

we note that the general solution to $\Box^2 u_{1_h} = 0$ is $f(x-t) + g(x+t)$. With the IC $u_1(0,t) = 0$ we find that $u_{1_h} = \sin(x-t) + \sin(x+t)$ indeed solves the homogeneous problem. For the particular problem to have a solution trial fns of the form $x[A\sin(x-t) + B\cos(x-t)]$. This indicates that for large values of x, u_1 results in a secular term which is singular at ∞.

if $\xi = x-t$ $\eta = x+t$ then $\Box^2 u_1(x,t) = \sin(x-t) \Rightarrow 4\frac{\partial^2 u_1}{\partial\xi\partial\eta} = \sin\xi$

this has solution $u_1(\xi,\eta) = -\frac{1}{4}\eta\cos\xi + f(\eta) + g(\xi)$ and for $+x$ direction propagation of waves, take $f(\eta) = 0$ $\therefore$ $u_1(x,t) = -\frac{1}{4}(x+t)\cos(x-t) + g(x-t)$. Using b.c. $u_1(0,t) = -\frac{1}{4}t\cos(-t) + g(-t)$:

then $g(\sigma) = -\frac{1}{4}\sigma\cos\sigma$ or $u_1(x,t) = -\frac{1}{4}(\eta+\xi)\cos(x-t) = \boxed{-\frac{1}{2}x\cos(x-t) = u_1(x,t)}$ ✓

As expected we get the secular term $-\frac{1}{2}x\cos(x-t)$

for ε^2: $\varepsilon^2\left[\frac{\partial u_1}{\partial x} + x\Box^2 u_1\right] + \varepsilon^2\Box^2 u_2 = 0$ but $\Box^2 u_1 = \sin(x-t)$ and $\frac{\partial u_1}{\partial x} = -\frac{1}{2}\cos(x-t) + \frac{1}{2}x\sin(x-t)$

thus $\Box^2 u_2 = \frac{1}{2}\cos(x-t) - \frac{3}{2}x\sin(x-t)$ w/ $u_2(0,t) = 0$

as before the functions on the right hand side are composed of the solutions to the homogeneous problems and hence u_2 would have to be some power of x and/or t times these functions, which tends to propagate the secular term. We find by converting to ξ,η coordinates and integrating, then returning to x,t coordinates and plugging in the initial condition that

$\boxed{u_2(x,t) = \frac{1}{16}x\sin(x-t) - \frac{9}{16}xt\cos(x-t) + \frac{3}{16}x^2\cos(x-t)}$. Hence the secularity is reinforced in x and introduced in t. ✓

2.4b Let $u = \frac{\cos(x-t)}{\sqrt{F}} + \varepsilon f_1(x,t) + \varepsilon^2 f_2(x,t) + \dots$

where $f_1(0,t) = 0$ and $f_2(0,t) = 0$ etc

then $\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) - F\frac{\partial^2 u}{\partial t^2} = [\cos(x-t)]_{,xx}\sqrt{F} - \frac{1}{2}\cos(x-t)\frac{\cdot 0}{\sqrt{F}} - \frac{1}{4}\cos(x-t)\frac{\varepsilon^2}{\sqrt{F^3}} - \sqrt{F}\,[\cos(x-t)]_{,tt} +$
$\varepsilon\left[\frac{\partial}{\partial x}\left(F\frac{\partial f_1}{\partial x}\right) + F\frac{\partial^2 f_1}{\partial t^2}\right] + \varepsilon^2\left[\frac{\partial}{\partial x}\left(F\frac{\partial f_2}{\partial x}\right) - F\frac{\partial^2 f_2}{\partial t^2}\right] + \ldots = 0$

then for ε^0: we identical satisfy the DE

then for ε^1: $\Box^2 f_1 = 0$ with $f_1(0,t) = 0 \Rightarrow f_1 = 0$

then for ε^2: $-\frac{1}{4}\cos(x-t) + \frac{\partial f_1}{\partial x} + \Box^2 f_1 + \Box^2 f_2 = 0$ w/ $f_2(0,t) = 0$ or $\Box^2 f_2 = \frac{1}{4}\cos(x-t)$ w/ $f_2(0,t) = 0$

thus the problem is postponed to the ε^2 approximation since $\Box^2 f_2 = 0$ w/ $f_2(0,t) = 0$ has solutions of $\cos(x-t) - \cos(x+t)$, and since the rhs contains one of the solutions $\Rightarrow f_2$ must include secular terms. ✓

2.4c if $u = \frac{\cos(x-t)}{\sqrt{F(x)}}$ then $\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) - F\frac{\partial^2 u}{\partial t^2} = \left(\frac{1}{4}\frac{(F')^2}{\sqrt{F^3}} - \frac{1}{2}\frac{F''}{\sqrt{F}}\right)\cos(x-t) = 0$ ✓

thus if $F(x) \neq 0$ then $\Rightarrow (F')^2 - 2FF'' = 0$. This has the solution $\boxed{F(x) = \left(\frac{x}{c_1} + c_2\right)^2}$

using the fact that $(F')^2 - 2FF'' = 0 \Rightarrow \frac{F'}{F} = \frac{2F''}{F'} \Rightarrow \ln F = 2\ln F' + \ln C$

$\Rightarrow F = CF'^2$ or $\sqrt{\frac{F}{C}} = \frac{dF}{dx}$ or $\frac{x}{\sqrt{C}} + \hat{C}_2 = \int\frac{dF}{\sqrt{F}} = 2\sqrt{F} \Rightarrow \left(\frac{x}{2\sqrt{C}} + \frac{\hat{C}_2}{2}\right)^2 = F(x)$

now let $2\sqrt{C} = c_1$, $c_2 = \hat{c}_2/2$. QED. ✓

2.4d Scales are needed in order the make the order of the equations for large x = order of eqns. in t.

5/5

3.1a A simple solution for a shaft or radius a for $\phi_{rr} - \frac{3\phi_r}{r} + \phi_{zz} = 0$ w/ $\phi(0,z) = 0$ and $\phi(a,z) = 1$:

Since ϕ on the boundary is only a fn of r then we can assume $\phi_{rr} - \frac{3\phi_r}{r} = 0$ or

$\phi = \frac{c r^4}{4} + c_2$ using BC. we get $c_2 = 0$ and $c = \frac{4}{a^4}$

$\therefore \phi = (r/a)^4$ ✓

3.1 b let $\phi(r,z;\epsilon) = \phi_0(r,z) + \varepsilon\phi_1(r,z) + \varepsilon^2\phi_2(r,z) + \ldots$

w/ $\phi(0,z;\varepsilon) = 0 \Rightarrow \phi_0(0,z) = \phi_1(0,z) = \ldots = \phi_n(0,z) = \ldots = 0$

and $\phi(\sqrt{1+\epsilon^2 z^2}, z;\epsilon) = \phi_0(\sqrt{1+\epsilon^2 z^2}, z) + \epsilon\phi_1(\sqrt{1+\epsilon^2 z^2}, z) + \epsilon^2\phi_2(\sqrt{1+\epsilon^2 z^2}, z) + \ldots$

$= \phi_0(1,z) + \epsilon\phi_1(1,z) + \epsilon^2\left[\frac{z^2}{2}\phi_{0,r}(1,z) + \phi_2(1,z)\right] + \ldots = 1$

then $\phi_0(1,z)=1$; $\phi_1(1,z)=0$; $\frac{z^2}{2}\phi_{0,r}(1,z) = -\phi_2(1,z)$ etc.

Now

$$\varepsilon^0: \ \phi_{0,rr} - 3\frac{\phi_{0,r}}{r} + \phi_{0,zz} = 0 \quad w/\ \phi_0(0,z)=0 \quad \phi_0(1,z)=1 \quad \Rightarrow \boxed{\phi_0(r,z) = r^4}$$ ✓

$$\varepsilon^1: \ \phi_{1,rr} - 3\frac{\phi_{1,r}}{r} + \phi_{1,zz} = 0 \quad w/\ \phi_1(0,z) = \phi_1(1,z) = 0 \quad \Rightarrow \boxed{\phi_1(r,z) = 0}$$ ✓

$$\varepsilon^2: \ \phi_{2,rr} - 3\frac{\phi_{2,r}}{r} + \phi_{2,zz} = 0 \quad w/\ \phi_2(0,z)=0, \quad \phi_2(1,z) = -\frac{z^2}{2}\phi_{0,r}(1,z) = -2z^2$$

from the B.C. we can infer again that $\phi_2(r,z) = f(r)z^2$ where $f(0)=0$ $f(1)=-2$

Now let us assume $\phi_2(r,z) = \tilde{\phi}_2(r,z) + \hat{\phi}_2(r,z)$ where $\tilde{\phi}_2(r,z) = -2r^4z^2$. $\tilde{\phi}_2$ satisfies BC's but not DE. We will pick $\hat{\phi}_2$ $\ni$ DE$[\tilde{\phi}_2 + \hat{\phi}_2] = 0$ and $\hat{\phi}_2(0,z) = \hat{\phi}_2(1,z) = 0$.

Now DE$[\tilde{\phi}_2] = -4r^4$ $\therefore$ DE$[\hat{\phi}_2] = 4r^4$. Since DE$[\hat{\phi}_2]$ = fn. of r assume $\hat{\phi}_2$ is only a fn. of r $\therefore$ $\hat{\phi}_{2,rr} - 3\frac{\hat{\phi}_{2,r}}{r} + \hat{\phi}_{2,zz} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\hat{\phi}_2}{dr}\right) - 4\frac{\hat{\phi}_{2,r}}{r} = 4r^4$

A solution to this, using the integrating factor of $\frac{1}{r^4}$ is

$$\hat{\phi}_2 = \frac{r^6}{3} - \frac{1}{4}\mathcal{B} + \mathcal{C}r^4$$

Applying BC $\Rightarrow$ $\mathcal{B} = 0$ $\mathcal{C} = -1/3$ $\therefore$ $\hat{\phi}_2 = 1/3[r^6 - r^4]$

or $\boxed{\phi_2 = -2r^4z^2 + \frac{1}{3}[r^6 - r^4]}$ ✓

$$\therefore \quad \phi = r^4 + \varepsilon^2\left[-2r^4z^2 + \frac{1}{3}r^6 - \frac{1}{3}r^4\right] + O(\varepsilon^4)$$ ✓

3.1c. if $Z = \epsilon z$ then $\phi_{,rr} - 3\frac{\phi_{,r}}{r} + \phi_{,zz} = \phi_{,rr} - 3\frac{\phi_{,r}}{r} + \epsilon^2\phi_{,ZZ} = 0$

w/ BC. $\phi(0,Z)=0$ and $\phi(R,Z)=1$ where $R^2 = 1+Z^2$

Now if $\phi(r,Z;\epsilon) = \phi_0(r,Z) + \epsilon^2\phi_2(r,Z) + \epsilon^4\phi_4(r,Z) + \dots$ since DE contains an ϵ^2 term.

$\Rightarrow$ $\phi(0,Z;\epsilon) = \phi_0(0,Z) + \epsilon^2\phi_2(0,Z) + \epsilon^4\phi_4(0,Z) + \dots = 0$ $\Rightarrow$ $\phi_{2i}(0,Z) = 0 \ \forall\ i \geq 0$

$\Rightarrow$ $\phi(R,Z;\epsilon) = 1 = \phi_0(R,Z) + \epsilon^2\phi_2(R,Z) + \epsilon^4\phi_4(R,Z) + \dots$ $\Rightarrow$ $\phi_0(R,Z) = 1$ and $\phi_{2i}(R,Z) = 0 \ \forall\ i \geq$

$$\varepsilon^0, \ \phi_{0,rr} - 3\frac{\phi_{0,r}}{r} = 0 \quad w/\ \phi_0(R,Z) = 1 \text{ and } \phi_0(0,Z) = 0 \quad \Rightarrow \quad \phi_0 = (r/R)^4$$ ✓

$$\varepsilon^2: \ \phi_{2,rr} - 3\frac{\phi_{2,r}}{r} + \phi_{0,ZZ} = 0 \quad w/\ \phi_{0,ZZ} = r^4(R^{-4})'' \text{ and } \phi_2(0,Z) = 0 = \phi_2(R,Z) = 0$$

Choose $\phi_2 = f(r)(R^{-4})''$ $\Rightarrow$ $f'' - 3\frac{f'}{r} = -r^4$ w/ $f(0)=0$ and $f(R)=0$

As before we obtain, using $1/r^4$ integrating factor,

$$f = -\frac{r^6}{12} - \frac{1}{4}\mathscr{C} + \mathbb{E} r^4$$

using the BC $\Rightarrow$ $\mathscr{C}=0$ and $\mathbb{E} = +\frac{R^2}{12}$ $\therefore$ $f = \frac{-1}{12}(r^6 - R^2 r^4)$

thus $\phi_2 = \left\{-\frac{r^6}{12} + \frac{R^2 r^4}{12}\right\}[R^{-4}]''$ ✓

$$\therefore \quad \phi = (r/R)^4 + \varepsilon^2\left[-\frac{r^6}{12} + \frac{R^2 r^4}{12}\right](R^{-4})'' + O(\varepsilon^4) \qquad (R^{-4})'' = \frac{-4(1-5z^2)}{[1+z^2]^4}$$ ✓

3.1d for very small Z:

$$\phi = r^4(1-2z^2+\dots) + \varepsilon^2\left[-\frac{r^6}{12} + \frac{r^4}{12}(1+z^2)\right](-4)(1-5z^2)(1-4z^2+\dots) + O(\varepsilon^4)$$

$$= r^4 - 2r^4\varepsilon^2 Z^2 + \varepsilon^2\left[\frac{r^6}{3} - \frac{r^4}{3}(1+\varepsilon^2 Z^2)\right](1-9z^2+\dots) + O(\varepsilon^4)$$

$$= r^4 + \varepsilon^2\left[-2r^4 Z^2 + \frac{r^6}{3} - \frac{r^4}{3}\right] + O(\varepsilon^4)$$ which matches the results of (b) ✓

for very large Z ✓

$$\phi \sim (r/Z)^4 + \varepsilon^2\left[\frac{-r^6}{12} + \frac{r^4 Z^2}{12}\right]\left(\frac{20}{Z^6}\right) + O(\varepsilon^4) \sim (r/Z)^4 + \varepsilon^2\left[-\tfrac{5}{3}(r/Z)^6 + \tfrac{5}{3}(r/Z)^4\right] + O(\varepsilon^4)$$

Given $\phi = \dfrac{2 - 3z(r^2+z^2)^{-1/2} + z^3(r^2+z^2)^{-3/2}}{2 - 3\cos\alpha + \cos^3\alpha}$; for a conical angle α $\tan\alpha = \frac{r}{z} = \frac{\epsilon Z}{Z} = \epsilon$

$\cos\alpha = \dfrac{1}{(1+\epsilon^2)^{1/2}}$ & $\cos^3\alpha = \dfrac{1}{(1+\epsilon^2)^{3/2}}$

$$\therefore \quad 2 - 3\cos\alpha + \cos^3\alpha = 2 - 3\left(1 - \frac{\epsilon^2}{2} + \frac{3\epsilon^4}{8} - \frac{15}{48}\epsilon^6 + \dots\right) + \left(1 - \frac{3}{2}\epsilon^2 + \frac{15}{8}\epsilon^4 - \frac{105}{48}\epsilon^6 + \dots\right) = \frac{6}{8}\epsilon^4 - \frac{5}{4}\epsilon^6 + \dots$$

Now $$\frac{z}{(r^2+z^2)^{+1/2}} = \frac{1}{\varepsilon}\cdot\frac{Z}{(r^2 + Z^2/\epsilon^2)^{1/2}} = \frac{1}{\left(1 + \frac{\epsilon^2 r^2}{Z^2}\right)^{1/2}} = 1 - \frac{1}{2}\frac{\epsilon^2 r^2}{Z^2} + \frac{3}{8}\frac{\epsilon^4 r^4}{Z^4} - \frac{15}{48}\frac{\epsilon^6 r^6}{Z^6} + O(\epsilon^8)$$

$$\frac{z}{(r^2+z^2)^{3/2}} = \frac{1}{\varepsilon}\frac{Z}{(r^2+Z^2/\epsilon^2)^{3/2}} = \frac{1}{(1+\epsilon^2 r^2/Z^2)^{3/2}} = 1 - \frac{3}{2}\frac{\epsilon^2 r^2}{Z^2} + \frac{15}{8}\frac{\epsilon^4 r^4}{Z^4} - \frac{105}{48}\frac{\epsilon^6 r^6}{Z^6} + O(\epsilon^8)$$

thus $$\phi = \frac{\frac{6}{8}\epsilon^4\frac{r^4}{Z^4} - \frac{5}{4}\epsilon^6\frac{r^6}{Z^6} + O(\epsilon^8)}{\frac{6}{8}\epsilon^4(1 - \frac{5}{3}\epsilon^2) + O(\epsilon^8)} = \frac{r^4/Z^4 - \frac{5}{3}\epsilon^2 r^6/Z^6 + O(\epsilon^4)}{(1-\frac{5}{3}\epsilon^2) + O(\epsilon^8)}$$

$$= \left[r^4/Z^4 - \frac{5}{3}\epsilon^2 r^6/Z^6 + O(\epsilon^4)\right]\left(1 + \frac{5}{3}\epsilon^2 + O(\epsilon^4)\right)$$

$$= r^4/Z^4 + \frac{5}{3}\epsilon^2\left(r^4/Z^4 - r^6/Z^6\right) + O(\varepsilon^4)$$

Solution matches the results for very large Z. ✓

A

2.4a if $F = \text{const}$ the $\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) - F\frac{\partial^2 u}{\partial t^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$

$\Rightarrow u = g(x+t) + f(x-t)$; for a wave prop. in +x-dir $u = f(x-t)$

also $u(0,t) = \cos t = f(-t) = \cos(-t) \Rightarrow f(\sigma) = \cos(\sigma) \quad \therefore \quad u = \cos(x-t)$

if $F = 1 + \varepsilon x$ then we get

$$\varepsilon\frac{\partial u}{\partial x} + (1+\varepsilon x)\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2}\right] = 0$$

if $u(x,t;\varepsilon) = u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \dots$ w/ $u(0,t;\varepsilon) = \cos t$

for ε^0: $\Rightarrow \Box^2 u_0 = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)u_0 = 0$ w/ $u_0(0,t) = \cos t$

Thus as above $u_0(x,t) = \cos(x-t)$

for ε^1: $\varepsilon\left[\frac{\partial u_0}{\partial x} + x\left[\Box^2 u_0\right]\right] + \varepsilon\Box^2 u_1 = 0 \Rightarrow \sin(x-t) = \Box^2 u_1$ w/ $u_1(0,t) = 0$

since $\frac{\partial u_0}{\partial x} = -\sin(x-t)$, $\Box^2 u_0 = 0$

but w/ $\xi = x - t \quad \eta = x + t \qquad \Box^2 u_1 = 4\frac{\partial^2 u_1}{\partial\xi\partial\eta} = \sin\xi$

this has a solution of $u_1(\xi,\eta) = -\frac{1}{4}\eta\cos\xi + f(\eta) + g(\xi)$

for +X prop waves $f(\eta) = 0 \quad \therefore \quad u_1(x,t) = -\frac{1}{4}(x+t)\cos(x-t) + g(x-t)$

$u_1(0,t) = -\frac{1}{4}t\cos(-t) + g(-t) = 0 \quad \therefore \quad -g(\sigma) = \frac{1}{4}\sigma\cos\sigma$ or $g(\sigma) = -\frac{1}{4}\sigma\cos\sigma$

$\therefore \quad u_1(x,t) = -\frac{1}{4}(x+t)\cos(x-t) - \frac{1}{4}(x-t)\cos(x-t) = -\frac{1}{2}x\cos(x-t)$

Note that this gives rise to secular term which goes to ∞ for large x

for ε^2: $\varepsilon^2\left[\frac{\partial u_1}{\partial x} + x\left[\Box^2 u_1\right]\right] + \varepsilon^2\Box^2 u_2 = 0 \qquad \frac{\partial u_1}{\partial x} = -\frac{1}{2}\cos(x-t) + \frac{1}{2}x\sin(x-t)$

$\Box^2 u_1 = \sin(x-t)$

$\therefore \quad \Box^2 u_2 = \frac{1}{2}\cos(x-t) - \frac{1}{2}x\sin(x-t) - x\sin(x-t) = \frac{1}{2}\cos(x-t) - \frac{3}{2}x\sin(x-t)$ w/ $u_2(0,t) = 0$

$4\frac{\partial^2 u_2}{\partial\xi\partial\eta} = \frac{1}{2}\cos\xi - \frac{3}{4}(\xi+\eta)\sin\xi$ integrate w/t η $4u_{2,\xi} = \frac{1}{2}\eta\cos\xi - \frac{3}{4}\xi\eta\sin\xi - \frac{3}{8}\eta^2\sin\xi + \hat{g}'(\xi)$

or $4u_2 = \frac{1}{2}\eta\sin\xi - \frac{3}{8}\eta\left[-\xi\cos\xi + \sin\xi\right] + \frac{3}{8}\eta^2\cos\xi + \hat{g}(\xi) + \hat{f}(\eta)$

or $u_2(\xi,\eta) = \frac{1}{8}\eta\sin\xi + \frac{3}{32}\eta\left[\xi\cos\xi - \sin\xi\right] + \frac{3}{32}\eta^2\cos\xi + g(\xi) + f(\eta)$

for +X prop waves $f(\eta) = 0$

$u_2(x=0,t) = \frac{1}{8}t\sin(-t) + \frac{3}{32}t\left[-t\cos(-t) - \sin(-t)\right] + \frac{3}{32}t^2\cos(-t) + g(-t) \quad 0$

$\therefore \quad g(-t) = -\frac{1}{8}t\sin(-t) - \frac{3}{32}t\left[-t\cos(-t) - \sin(-t)\right] - \frac{3}{32}(-t)^2\cos(-t)$

$g(\sigma) = \frac{1}{8}\sigma\sin\sigma + \frac{3}{32}\sigma\left[\sigma\cos\sigma - \sin\sigma\right] - \frac{3}{32}\sigma^2\cos\sigma$

$\therefore u_2(\xi,\eta) = \frac{1}{8}\sin\xi\left[\eta+\xi\right] + \frac{3}{32}\left[\eta+\xi\right]\left\{\xi\cos\xi - \sin\xi\right\} + \frac{3}{32}\left[\eta^2 - \xi^2\right]\cos\xi$

$= \frac{1}{4}x\sin(x-t) + \frac{3}{16}x\left[(x-t)\cos(x-t) - \sin(x-t)\right] - \frac{3}{8}xt\cos(x-t)$

$= \frac{1}{16}x\sin(x-t) + \frac{3}{16}x(x-t)\cos(x-t) - \frac{3}{8}xt\cos(x-t)$

again we start with the

again we start and we let $U = \frac{u}{\sqrt{F}}$ $U\sqrt{F} = u$ for $\sqrt{F} = \sqrt{1+\varepsilon x} \cong 1 + \frac{\varepsilon x}{2}$

then $\frac{\partial u}{\partial x} = \frac{\partial U}{\partial x}\sqrt{F} + \frac{1}{2}\frac{F'}{\sqrt{F}}U$

$F\frac{\partial u}{\partial x} = \frac{\partial U}{\partial x}(F^{3/2}) + \frac{1}{2}F^{1/2}F'U$

$$\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) = \frac{\partial^2 U}{\partial x^2}F^{3/2} + \frac{3}{2}F^{1/2}F'\frac{\partial U}{\partial x} + \frac{\partial U}{\partial x}\frac{1}{2}F^{1/2}F' + \frac{1}{4}\frac{(F')^2}{F^{1/2}}U + \frac{1}{2}F^{1/2}F''U$$

for $F = 1+\varepsilon x$ $F' = \varepsilon$; $F'' = 0$

$$\therefore \quad \frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) = U_{,xx}(1+\tfrac{3}{2}\varepsilon x + \tfrac{3}{8}\varepsilon^2 x^2) + 2U_{,x}\varepsilon\left(1+\frac{\varepsilon}{2}x\right) + \frac{1}{4}\varepsilon^2\left(1-\tfrac{1}{2}\varepsilon x + \tfrac{3}{8}\varepsilon^2 x^2 + \ldots\right)U$$

$$-F\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 U}{\partial t^2}F^{3/2} = -\frac{\partial^2 U}{\partial t}\left(1 + \tfrac{3}{2}\varepsilon x + \tfrac{3}{8}\varepsilon^2 x^2 + \ldots\right)$$

$U(x,t) = u\left(1 - \tfrac{1}{2}\varepsilon x + \tfrac{3}{8}\varepsilon^2 x^2 + \ldots\right)$ $\quad U(0,t) = u(0,t) \Rightarrow U_0(0,t) = \cos t$, $U_i(0,t) = 0 \;\; \forall i > 0$

ε^0: $\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial t^2} = 0$ $\quad \therefore \quad U_0 = f(x-t) + \cancel{g(x+t)}$ $\quad U_0 = \cos(x-t)$

ε^1: $\frac{3}{2}xU_{0,xx} + U_{1,xx} + 2U_{0,x} - U_{1,tt} - \frac{3}{2}xU_{0,tt} = 0$ $\quad \Box^2 U_1 = -2U_{0,x} = +2\sin(x-t)$

2.4 b Let $u = \frac{U}{\sqrt{F}}$ $\quad F\frac{\partial u}{\partial x} = \frac{\partial U}{\partial x}\frac{F}{\sqrt{F}} - \frac{1}{2}\frac{UFF'}{\sqrt{F^3}}$ $\quad F\frac{\partial u}{\partial x} = \frac{\partial U}{\partial x}\sqrt{F} - \frac{1}{2}U\frac{F'}{\sqrt{F}}$

and $U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots$

$$\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) = \frac{\partial^2 U}{\partial x^2}\sqrt{F} + \frac{1}{2}\frac{\partial U}{\partial x}\frac{F'}{\sqrt{F}} - \frac{1}{2}\frac{\partial U}{\partial x}\frac{F'}{\sqrt{F}} - \frac{1}{2}U\frac{F''}{\sqrt{F}} - \frac{1}{4}U\frac{(F')^2}{\sqrt{F^3}}$$

then $\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) - F\frac{\partial^2 u}{\partial t^2} = U_{,xx}\sqrt{F} - \frac{1}{2}U\frac{F''}{\sqrt{F}} - \frac{1}{4}U\frac{(F')^2}{\sqrt{F^3}} - \sqrt{F}\frac{\partial^2 U}{\partial t^2} = 0$

$F = 1+\varepsilon x$ $\quad F' = \varepsilon$ $\quad F'' = 0$

$\sqrt{F}\,\Box^2 U - \frac{1}{2}U\frac{F''}{\sqrt{F}} - \frac{1}{4}U\frac{(F')^2}{\sqrt{F^3}} = 0$ (with $F''=0$) $\quad$ or $\quad \Box^2 U - \frac{1}{4}U\varepsilon^2(1+\varepsilon x)^{-2} = 0$ $\quad$ as long as $x \neq -\frac{1}{\varepsilon}$

or $\Box^2 U - \frac{1}{4}U\varepsilon^2\left(1 - 2\varepsilon x - 3\varepsilon^2 x^2 + \ldots\right) = 0$ $\quad$ w/ $U(x=0,t) = \cos t$

ε^0: $\Box^2 U_0 = 0$ $\quad U_0(0,t) = \cos t$

ε^1: $\Box^2 U_1 = 0$ $\quad U_1(0,t) = 0$ $\Rightarrow U_1(x,t) = 0$ postponement of problem to ε^2 term.

ε^2: $\Box^2 U_2 - \frac{1}{4}U_0 = 0$

ε^3: $\Box^2 U_3 - \frac{1}{4}U_1 + \frac{1}{2}U_0 x = 0$ (with $U_1 = 0$)

$u = \cos(x-t)\left[1 - \tfrac{1}{2}\varepsilon x + \tfrac{3}{8}\varepsilon^2 x^2\right]$

use $u = \frac{\cos(x-t)}{\sqrt{F}} + \varepsilon f_1(x,t) + \varepsilon^2 f_2(x,t) + \ldots$

w/ $f_i(0,t) = 0 \quad \forall i \geq 1$

and get same results as above

$$\varepsilon\left[F'\frac{\partial f_1}{\partial x} + F\left(\frac{\partial^2 f_1}{\partial x^2} - \frac{\partial^2 f_1}{\partial t^2}\right)\right]$$

$$\varepsilon\left[\varepsilon\frac{\partial f_1}{\partial x} + \left(1+\tfrac{1}{2}\varepsilon x+\dots\right)\Box^2 f_1\right] + \varepsilon^2\left[\varepsilon\frac{\partial f_2}{\partial x} + \left(1+\tfrac{1}{2}\varepsilon x+\dots\right)\Box^2 f_2\right.$$

$f(-t) = 0 \quad \therefore \ \forall\, \sigma \ f(\sigma) = 0$

2.4c

$$\text{if } u = \frac{\cos(x-t)}{\sqrt{F(x)}} = \cos(x-t)\,F(x)^{-1/2} = \cos(x-t)\{1+\varepsilon x\}^{-1/2}$$

$$= \cos(x-t) + \cos(x-t)\left(-\tfrac{1}{2}\varepsilon x\right) + \cos(x-t)\frac{\left(+\tfrac{1}{2}\cdot\tfrac{3}{2}\right)}{2!}\varepsilon^2 x^2$$

$$+\cos(x-t)\left(-\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\right)\frac{\varepsilon^3 x^3}{3!}+\dots$$

$$\therefore \quad \frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) - F\frac{\partial^2 u}{\partial t^2} =$$

$$F'\frac{\partial u}{\partial x} + F\,\Box^2 u = 0 \quad \therefore \quad F'\left\{\frac{-\sqrt{F}\sin(x-t)}{F} - \cos(x-t)\frac{1}{2}\frac{F'}{\sqrt{F}\,F}\right\} + F\,\Box^2 u = 0$$

$$\frac{\partial u}{\partial t} = \frac{\sin(x-t)}{\sqrt{F}} \qquad \frac{\partial^2 u}{\partial t^2} = \frac{-\cos(x-t)}{\sqrt{F}} \qquad \Box^2 u$$

$$\frac{\partial u}{\partial x} = \frac{-\sin(x-t)}{\sqrt{F}} - \frac{1}{2}\frac{\cos(x-t)F'}{\sqrt{F}\,F} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{-\cos(x-t)}{\sqrt{F}} + \frac{1}{2}\frac{\sin(x-t)F'}{\sqrt{F^3}} + \frac{1}{2}\frac{\sin(x-t)F'}{\sqrt{F^3}}$$

$$-\frac{1}{2}\frac{\cos(x-t)F''}{\sqrt{F^3}} + \frac{3}{4}\frac{\cos(x-t)(F')^2}{\sqrt{F^5}}$$

$$\therefore \text{ DE becomes } F'\left\{\frac{-\sin(x-t)}{\sqrt{F}} - \frac{1}{2}\cos(x-t)\frac{F'}{\sqrt{F^3}}\right\} + \left\{+\frac{1}{2}\frac{\sin(x-t)F'}{\sqrt{F}} + \frac{1}{2}\frac{\sin(x-t)F'}{\sqrt{F}} - \frac{1}{2}\frac{\cos(x-t)F''}{\sqrt{F}}\right.$$

$$\left.+\frac{3}{4}\frac{\cos(x-t)(F')^2}{\sqrt{F^3}}\right\} = 0$$

$$\therefore \quad \frac{(F')^2}{\sqrt{F^3}}\left[-\frac{1}{2}+\frac{3}{4}\right] - \frac{1}{2}\frac{F''}{\sqrt{F}} = 0 \qquad \text{if } F(x) \neq 0 \text{ then ~~divide~~ mult By } 4\sqrt{F}$$

$$\tfrac{1}{4} \qquad (F')^2 - 2FF'' = 0 \quad \text{or}$$

$$\text{or} \quad \frac{(F')}{F} = 2\frac{F''}{F'} \qquad \ln F = 2\ln F' + \ln c \ \text{~~+ const~~} \quad \text{or}$$

$$\frac{d}{dx}\left(\frac{F}{F'}\right) \quad \frac{FF''}{(F')^2} \qquad \ln\frac{F}{(F')^2} = \ln c_1 \qquad \therefore \quad F = c_1(F')^2$$

$$\text{or} \pm\sqrt{\frac{F}{c_1}} = \frac{dF}{dx} \quad \text{or}$$

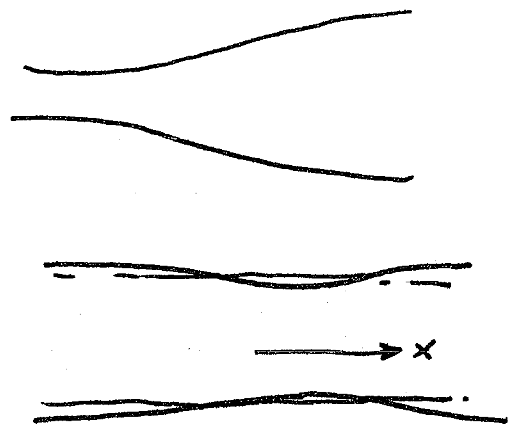
$$\frac{dx}{\pm\sqrt{c_1}} = \frac{dF}{\pm\sqrt{F}} \quad \text{or} \quad \frac{x}{\pm\sqrt{c_1}} + c_2 = \pm 2\sqrt{F}$$

$$\text{or} \quad \left(\frac{x}{c_1} + c_2\right)^2 = F(x)$$

$$\text{or} \quad F(x) = \left(\frac{x}{\bar{c}_1} + \bar{c}_2\right)^2 \qquad \text{let} \quad F(x) = (c_1 x + c_2)^2 \qquad \text{as long as } x \neq -\frac{c_2}{c_1}$$

then for $F(x) = (c_1 x + c_2)^2$ we get exact sol

~~with proper scaling on can delay~~ we need to redefine



$$\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right) - F\frac{\partial^2 u}{\partial t^2} = 0$$

if F were const $\Rightarrow$ $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$ of $u = f(x-t) + g(\cancel{x+t})$

when $x=0$ $u = \cos t$ $\quad u = f(-t) = \cos t = \cos(-t)$

$\therefore$ $f(-t) = \cos(-t)$ or $f(\sigma) = \cos\sigma$ $\quad u = \cos(x-t)$

if $F = 1+\varepsilon x$

$$\varepsilon\frac{\partial u}{\partial x} + (1+\varepsilon x)\frac{\partial^2 u}{\partial x^2} - (1+\varepsilon x)\frac{\partial^2 u}{\partial t^2} = 0$$

if $u = U_0(x,t) + \varepsilon U_1(x,t) + \cdots$

if $\varepsilon = 0$ then $\forall x$ $U_0(x) \Rightarrow$ soln to $\Box^2 U_0 = 0$ w/ $x=0$ $U = \cos t$

$\therefore$ $U_0 = \cos(x-t)$

$$\therefore\ \varepsilon\left(\frac{\partial U_0}{\partial x} + x\frac{\partial^2 U_0}{\partial x^2} - x\frac{\partial^2 U_0}{\partial t^2}\right) + \varepsilon\left[\frac{\partial^2 U_1}{\partial x^2} - \frac{\partial^2 U_1}{\partial t^2}\right] = 0$$

$$\frac{\partial U_0}{\partial x} + x\left[\Box^2 U_0 = 0\right] + \Box^2 U_1 = 0 \quad \text{w/} \quad U_1(0,t) = 0$$

$$+\sin(x-t) = \Box^2 U_1$$

$$\Box^2 U_1 = \sin(x-t)$$

if $\quad 4u_{\xi\eta} = \sin\xi$

$$u_\eta = -\frac{1}{4}\cos\xi + f'(\eta)$$

$$u = -\tfrac{1}{4}\eta\cos\xi + f(\eta) + g(\xi)$$

$u_1(0,t) = 0 = -\tfrac{1}{4}t\cos(-t) + g(-t) = 0 \quad \therefore \quad -g(\sigma) = \frac{1}{4}\sigma\cos\sigma$

$$+g(\sigma) = -\frac{1}{4}(\sigma)\cos(\sigma)$$

$$u_1(x,t) = -\tfrac{1}{4}(x+t)\cos(x-t) + g(x-t)$$

a. simple soln for a shaft of radius a

a. $\phi_{rr} - \frac{3\phi_r}{r} + \phi_{zz} = 0$ $\phi = 0$ on $r=0$, $\phi = 1$ on $r = a$.

since ϕ only is a fn of r $\forall z$ then let us solve $\phi_{rr} - \frac{3\phi_r}{r} = 0$

this gives $\phi = c\frac{r^4}{4} + c_2$

using $\phi(r=0) = 0 \Rightarrow c_2 = 0$ $\phi(r=a) = 1$ $c = \frac{4}{a^4}$ $\therefore$ $\phi = \left(\frac{r}{a}\right)^4$

b. for $a=1$ let $\phi(r,z;\varepsilon) = \phi_0(r,z) + \varepsilon\phi_1(r,z) + \varepsilon^2\phi_2(r,z) + \dots$

$\phi(0,z;\varepsilon) = \phi_0(0,z) + \dots \Rightarrow \phi_i(0,z) = 0 \quad i \geq 0$

$\phi(\sqrt{1+\varepsilon^2 z^2}, z;\varepsilon) = \phi_0(\sqrt{1+\varepsilon^2 z^2}, z) + \varepsilon\phi_1(\sqrt{1+\varepsilon^2 z^2}, z) + \varepsilon^2\phi_2(\sqrt{1+\varepsilon^2 z^2}, z) + \dots$

$= \phi_0(1,z) + \frac{\varepsilon^2}{2}z^2\phi_{0,r}(1,z) + \dots O(\varepsilon^4) + \varepsilon\left[\phi_1(1,z) + \frac{\varepsilon^2 z^2}{2}\phi_{1,r}(1,z) + \dots O(\varepsilon^4)\right]$

$+ \varepsilon^2\left[\phi_2(1,z) + \frac{1}{2}\varepsilon^2 z^2\phi_{2,r}(1,z) + \dots O(\varepsilon^4)\right]$

$= 1$

$\Rightarrow \phi_0(1,z) = 1$; $\phi_1(1,z) = 0$; $\frac{z^2}{2}\phi_{0,r}(1,z) + \phi_2(1,z) = 0$ etc.

Now $\boxed{\phi_0 = r^4}$

$\phi_1(0,z) = 0$

ε^1: $\phi_{rr} - \frac{3\phi_r}{r} + \phi_{zz} = 0$ $\phi_1(1,z) = 0 \Rightarrow \phi_1 \equiv 0$

ε^2: $\phi_{rr} - \frac{3\phi_r}{r} + \phi_{zz} = 0$ $2z^2 + \phi_2(1,z) = 0$ $\therefore$ $\phi_2(1,z) = -2z^2$

$\phi_2(0,z) = 0$

~~let $\phi = Ar^4 z^2$ $\phi_{rr} = A\cdot 12r^2z^2$~~

$-3\frac{\phi_r}{r} = -12Ar^2z^2$

$\phi_{zz} = 2Ar^4$

$\therefore$ $DE(\phi) = 2Ar^4$

if $\phi = Br^4$ $DE\phi = 0$

$\phi = Cr^6$ $DE\phi = 30Cr^4 - 18Cr^4 = 12Cr^4$

$\therefore$ let ~~C = A~~

$-2A = 12C$

$-\frac{1}{6}A = C$

$\therefore$ $\phi = A(r^4z^2 - \frac{r^6}{6}) + Br^4$

$\phi(0,z) = 0$ $\phi(1,z) = -2z^2$

$\therefore$ $\phi = A(z^2 - \frac{1}{6}) + B$ $\therefore$ ~~A(z² ... B...)~~

$\therefore$ $\phi_2 = -2(r^4z^2 - r^6) - 2r^4$ $A = -2$ $B = -\frac{1}{3}$

$\therefore$ $\phi = -2(r^4z^2 - \frac{r^6}{6}) - \frac{1}{3}r^4$ ✓

$e^{-\int \frac{4dr}{r}} = e^{-4\ln r} = \frac{1}{r^4}$

let $\phi_{rr} - 3\frac{\phi_r}{r} = -Ar^4$

~~r~~ $\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi\right) - 4\phi_r = -Ar^5$

~~r~~ $\frac{\partial\phi}{\partial r} - \frac{4}{r}\phi = -6Ar^5$

let $\phi = Br^6$

$6Br^5 - 4Br^5 =$

$2Br^5 = -6Ar$

$\therefore B = -3A$

$\left(\frac{1}{r^4}\phi\right)' = -6Ar$

$\phi = -3Ar^6 + Cr^4$

$\phi_{2,rr} - 3\frac{\phi_{2,r}}{r} + \phi_{2,zz} = 0 \qquad \phi_2(0,z) = 0 \qquad \phi_2(1,z) = -2z^2$

let $\hat{\phi}_2(r,z) = -2r^2z^2 \ni \hat{\phi}_2(0,z) = 0$ and $\hat{\phi}_2(1,z) = -2z^2$

Then $-4z^2 - 3(-4z^2) - 4r^2 = DE[\hat{\phi}_2]$

$DE[\hat{\phi}_2] = 8z^2 - 4r^2$ Now we need a fn. $\hat{\hat{\phi}}_2 \therefore DE[\hat{\hat{\phi}}_2] = 4r^2 - 8z^2$

and $[\hat{\hat{\phi}}_2(0,z) = 0 \quad \hat{\hat{\phi}}_2(1,z) = 0$

$\frac{1}{r}\frac{d}{dr}(\phi)$

$\frac{1}{r}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)$

$r^2\frac{d}{dr}\left(r^{-2}\frac{d\phi}{dr}\right) = \frac{d^2\phi}{dr^2} - \frac{2r^2}{r^3}\frac{d\phi}{dr}$

since r^4 solves $\phi_{rr} - 3\frac{\phi_r}{r} = 0$ then Ar^4z^2 also solves $\phi_{rr} - 3\frac{\phi_r}{r} = 0$

$\therefore DE[\phi] = f(r)$ only $\therefore$

$\tilde{\phi}_2 = Ar^4z^2$ solves $\tilde{\phi}(0,z) = 0$ and $\tilde{\phi}_2(1,z) = -2z^2 = Az^2 \Rightarrow A = -2$

$\therefore \tilde{\phi}_2 = -2r^4z^2$

$DE[\tilde{\phi}_2] = -4r^4$

Now we must find a $\hat{\phi}_2 \ni DE[\hat{\phi}_2] = 4r^4$ & $\hat{\phi}_2(0,z) = \hat{\phi}_2(1,z) = 0$ then $\phi_2 = \tilde{\phi}_2 + \hat{\phi}_2$

but since $DE[\hat{\phi}_2] = f(r)$ only then choose a particular sol. $\Rightarrow$

$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\hat{\phi}}{\partial r}\right) - \frac{4\hat{\phi}_{,r}}{r} = \hat{\phi}_{2,rr} - \frac{3\hat{\phi}_{2,r}}{r} = 4r^4$ ~~choose $\hat{\phi}_2$~~

$\therefore \left(r\frac{\partial\hat{\phi}}{\partial r}\right)' - 4\hat{\phi}_{,r} = 4r^5$

$\left(\frac{\partial\hat{\phi}}{\partial r}\right) - \frac{4\hat{\phi}}{r} = \frac{4}{6}r^5 + \frac{c}{r}$ using integrating factor $\frac{1}{r^4}$

$\hat{\phi}/r^4 \quad -4\hat{\phi}/r^5$

$\left(\frac{\hat{\phi}}{r^4}\right)' = \frac{2}{3}r + \frac{c}{r^5}$

$\hat{\phi}/r^4 = \frac{r^2}{3} + \frac{1}{4}\frac{c}{r^4} + C$

$\hat{\phi} = \frac{r^6}{3} - \frac{1}{4}c + Cr^4$

$\hat{\phi}(0,z) = 0 \Rightarrow c = 0$

$\hat{\phi}(1,z) = 0 \Rightarrow \frac{1}{3} + C = 0$

$C = -1/3$

$\therefore \hat{\phi} = \frac{1}{3}\left[r^6 - r^4\right]$

$\therefore \phi_2 = -2r^4z^2 + \frac{1}{3}\left[r^6 - r^4\right]$

$DE[\hat{\phi}_2] = \frac{1}{3}\left[30r^4 - 12r^2\right] - \frac{3}{3}\left[6r^4 - 4r^2\right]$
$= 10r^4 - 4r^2 - 6r^4 + 4r^2 = 4r^4$

$\therefore \phi = r^4 + \epsilon^2\left[-2r^4z^2 + \frac{1}{3}r^6 - \frac{1}{3}r^4\right] + O(\epsilon^4)$

$\phi_{rr} - 3\frac{\phi_r}{r} + \phi_{zz} = 0 \qquad \phi = 0 \text{ at } r = 0 \qquad \phi = 1 \text{ @ } r = r(z)$

if $Z = \epsilon z$ then $\phi_{rr} - 3\frac{\phi_r}{r} + \epsilon^2 \phi_{ZZ} = 0$ with $\phi(0,Z) = 0 \quad \phi(R,Z) = 1$

Now if $\phi(r, Z; \varepsilon) = \phi_0(r,Z) + \varepsilon^2 \phi_2(r,Z) + \varepsilon^4 \phi_4(r,Z) + \dots$

$\Rightarrow \phi_0(0,Z) = 0 \; ; \; \phi_2(0,Z) = 0; \dots$

also $\phi_0(R, Z) = 1 \; ; \; \phi_2(R, Z) = 0 \; ; \; \phi_4(R, Z) = 0 \dots$

for ε^0: $\phi_{rr} - \frac{3\phi_r}{r} = 0$ w/ $\phi_0(0,Z) = 0$ and $\phi_0(R, Z) = 1$ where $R = R(Z)$

as before $\phi_0 = \left(r/R\right)^4$

for ε^2: also $\phi_{2,rr} - 3\frac{\phi_{2,r}}{r} = -\phi_{0,ZZ} \qquad \phi_{0,ZZ} = r^4 (R^{-4})'' \qquad$ and $\phi_2(0,Z) = 0$, $\phi_2(R,Z) = 0$

$= -r^4 [R^{-4}]'' \qquad$ Let $\phi_2 = f(r)[R^{-4}]''$

then $f'' - \frac{3f'}{r} = -r^4 \quad \Rightarrow \quad \frac{1}{r}(rf')' - \frac{4f'}{r} = -r^4$

or $(rf')' - 4f' = -r^5$

$rf' - 4f = -\frac{r^6}{6} + c \Rightarrow f' - \frac{4}{r}f = -\frac{r^5}{6} + \frac{c}{r}$

$\therefore$ integ factor is $1/r^4 \quad \left(\frac{f}{r^4}\right)' = -\frac{r}{6} + \frac{c}{r^5}$ or $\frac{f}{r^4} = -\frac{r^2}{12} - \frac{1}{4}\frac{c}{r^4} + C$

$\therefore \quad f = -\frac{r^6}{12} - \frac{c}{4} + Cr^4$

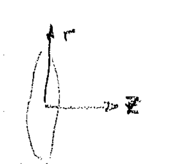
$\therefore \quad f(0) = 0 \Rightarrow c = 0 \qquad f(R) = 0 \Rightarrow -\frac{R^6}{12} + CR^4 = 0 \qquad \therefore C = \frac{R^2}{12}$

$\therefore \quad \phi_2 = \left\{-\frac{r^6}{12} + \frac{R^2}{12} r^4\right\}[R^{-4}]''$

$\therefore \quad \phi = \left(r/R\right)^4 + \varepsilon^2\left[-\frac{r^6}{12} + \frac{R^2 r^4}{12}\right](R^{-4})'' + O(\varepsilon^4) \qquad$ where $R = R(Z)$

$(R^{-4})' = \left[\frac{1}{(1+Z^2)^2}\right]' = \frac{-2 \cdot 2Z}{(1+Z^2)^{+3}} = \frac{-4Z}{(1+Z^2)^3}$

for a conical



$(R^{-4})'' = \left[\frac{-4Z}{(1+Z^2)^3}\right]' = \frac{-4}{[1+Z^2]^3} + \frac{+12Z \cdot 2Z}{(1+Z^2)^4}$

$= \frac{-4 - 4Z^2 + 24Z^2}{(1+Z^2)^4} = \frac{-4(1-5Z^2)}{[1+Z^2]^4}$

for $R^2(Z) = 1 + Z^2$ then $\phi = \frac{r^4}{(1+Z^2)^2} + \varepsilon^2\left[\frac{-r^6}{12} + \frac{(1+Z^2)r^4}{12}\right]\left[\frac{-4(1-5Z^2)}{[1+Z^2]^4}\right] + O(\varepsilon^4)$

for very small Z: $\phi = r^4\left[1 - 2Z^2 + \dots\right] + \varepsilon^2\left[\frac{-r^6}{12} + \frac{r^4}{12}(1+Z^2)\right](-4)(1-5Z^2)(1-4Z^2 + \dots) + O(\varepsilon^4)$

$= r^4 - 2r^4\varepsilon^2 z^2 + \varepsilon^2\left[\frac{r^6}{3} - \frac{r^4}{3}(1+\varepsilon^2 z^2)\right](1 - 9Z^2 + \dots) + O(\varepsilon^4)$

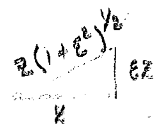
$= r^4 + \varepsilon^2\left[-2r^4 z^2 + \frac{r^6}{3} - \frac{r^4}{3}\right] + O(\varepsilon^4)$ which matches results of (b)

for large z $\phi \sim \frac{r^4}{z^4} + \varepsilon^2\left[+\frac{r^6}{3}\left(\frac{-5}{z^6}\right) - \frac{1}{3}r^4\left(\frac{-5}{z^4}\right)\right] + O(\varepsilon^4)$

$\phi \sim (r/z)^4 + \varepsilon^2\left[-\frac{5}{3}(r/z)^6 + \frac{5}{3}(r/z)^4\right] + O(\varepsilon^4)$

$\phi = \frac{1}{M}\left[2 - \frac{3z}{(r^2+z^2)^{1/2}} + \frac{z^3}{(r^2+z^2)^{+3/2}}\right] = \frac{1}{M}\left[2 - \frac{3}{(1+\varepsilon^2)^{1/2}} + \frac{1}{(1+\varepsilon^2)^{3/2}}\right]$ w/ $r = \varepsilon z$

$= \frac{1}{M}\left[2 - 3\left(1 - \frac{1}{2}\varepsilon^2 + \dots\right) + \left(1 - \frac{3}{2}\varepsilon^2 + \dots\right)\right]$

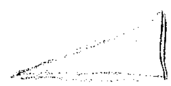


$\cos\alpha = \frac{1}{(1+\varepsilon^2)^{1/2}}$ $\cos^3\alpha = \frac{1}{(1+\varepsilon^2)^{3/2}} = \left(1 - \frac{3}{2}\varepsilon^2 + \frac{15}{8}\varepsilon^4\right.$

$= 1 - \frac{1}{2}\varepsilon^2 + \frac{3}{8}\varepsilon^4 + \dots$

$+ \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{3!}\epsilon^6$ $-\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{1}{3!}$

~~$2 - 3 + \frac{3}{2}\varepsilon^2 - \frac{9}{8}\varepsilon^4 + 1 - \frac{3}{2}\varepsilon^2 + \frac{15}{8}\varepsilon^4 + \dots$~~ $\frac{6}{8}\varepsilon^4 - \frac{5/4}{8}\epsilon^6$



$\frac{45}{8\cdot 3!} - \frac{105}{8\cdot 3!} = \frac{60}{8\cdot 3!} = 5/4$ $\frac{6}{8}\varepsilon^4\left(1 - \frac{5}{3}\epsilon^2\right)$

$\frac{d\tan^{-1}\varepsilon}{d\varepsilon} = \frac{1}{1+\varepsilon^2} = 1 - \varepsilon^2 + \varepsilon^4 + \dots$

$\tan^{-1} = \varepsilon - \frac{\varepsilon^3}{3} + \frac{\varepsilon^5}{5} + \dots$

$\frac{\tan^{-1}(\varepsilon y/x)}{\varepsilon\left(1 - \varepsilon^2/3 + \dots\right)} = \frac{\varepsilon y/x - \frac{\varepsilon^3}{3}y^3/x^3 + \frac{\varepsilon^5}{5}y^5/x^5}{} = \frac{y}{x} - \frac{\varepsilon^2}{3}\frac{y^3}{x^3} + \dots$

$\left(1 + \varepsilon^2/3 + \dots\right.$

$y/x + \varepsilon^2\left(\frac{1}{3}y/x - \frac{y^3}{3x^3}\right) + O$

$\frac{2 - 3z/\varepsilon\left(r^2 + z^2/\varepsilon^2\right)^{-1/2} + z^3/\epsilon^3\left(r^2 + z^2/\epsilon^2\right)^{-3/2}}{2 - 3\cos\alpha + \cos^3\alpha}$ $\therefore \left(r^2 + z^2/\epsilon^2\right) = \left(\frac{\epsilon^2 r^2}{z^2} + 1\right)z^2/\epsilon^2$

$\frac{2 - 3\,z/\epsilon\left(\frac{\epsilon^2 r^2}{z^2} + 1\right)^{-1/2}\cdot\frac{\epsilon}{z} + \left(\frac{\epsilon^2 r^2}{z^2} + 1\right)^{-3/2}}{2 - 3\cos\alpha + \cos^3\alpha}$

$\frac{2 - 3\left[1 - \frac{1}{2}\frac{\epsilon^2 r^2}{z^2} + \frac{3}{8}\frac{\varepsilon^4 r^4}{z^4}\right] + \left[1 - \frac{3}{2}\left(\frac{\varepsilon^2 r^2}{z^2}\right) + \frac{15}{8}\frac{\varepsilon^4 r^4}{z^4}\right]}{2 - 3\cos\alpha + \cos^3\alpha}$ ~~$2 - 3\left(1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{4!} + \dots\right) + \left(1 - \frac{3}{2}\alpha^6 + \dots\right)$~~

$\frac{\frac{6}{8}\varepsilon^4\left(\frac{r^4}{z^4}\right) - \frac{5}{4}\varepsilon^6\left(\frac{r^6}{z^6}\right)}{\frac{6}{8}\epsilon^4\left(1 - \frac{5}{3}\epsilon^2\right)} = \left[\left(\frac{r^4}{z^4}\right) - \frac{5}{3}\varepsilon^2\left(\frac{r^6}{z^6}\right)\right]\left[1 + \frac{5}{3}\epsilon^2 + \dots\right.$

$\left(r^4/z^4\right) + \frac{5}{3}\epsilon^2\left[\left(r^4/z^4\right) - \left(r^6/z^6\right)\right] + O(\varepsilon^4)$ very good.

$f(\delta) = (1+\delta)^{-1/2} \Rightarrow f(0)=1$

$f'(\delta) = -\frac{1}{2}(1+\delta)^{-3/2} \Rightarrow f'(0) = -1/2$

$f''(\delta) = \frac{3}{4}(1+\delta)^{-5/2} \Rightarrow f''(0) = 3/4$

$f'''(\delta) = -\frac{15}{8}(1+\delta)^{-7/2} \Rightarrow f''' = -15/8$

$f(\delta) = (1+\delta)^{-3/2} \Rightarrow f(0)=1$

$f'(\delta) = -\frac{3}{2}(1+\delta)^{-5/2} \Rightarrow f(0) = -3/2$

$f''(\delta) = \frac{15}{4}(1+\delta)^{-7/2} \Rightarrow f(0) = 15/4$

$f'''(\delta) = \frac{-105}{8}(1+\delta)^{-9/2} \Rightarrow f(0) = -\frac{105}{8}$

$f(\delta) = f(0) + \delta f'(0) + \frac{\delta^2}{2!} f''(0) + \frac{\delta^3}{3!} f'''(0) + \ldots$

$3 \cdot 2 \cdot 1 = 6$

$f(\delta) = 1 - \frac{3}{2}\delta + \frac{15}{8}\delta^2 - \frac{105}{48}\delta^3$

$= 1 - \frac{1}{2}\delta + \delta^2 \frac{3}{8} - \delta^3 \frac{15}{48}$

$-3 \quad + 3/2 \quad - 9/8 \quad + \frac{45}{48}$

$f(\delta) = \frac{1}{1+\delta} \Rightarrow f(0) = 1$

$f'(\delta) = -(1+\delta)^{-2} \Rightarrow f'(0) = -1$

$f''(\delta) = 2(1+\delta)^{-3} \Rightarrow f''(0) = 2$

$f'''(\delta) = -6(1+\delta)^{-4} \Rightarrow f'''(0) = -6$

$f^{iv}(\delta) = 24(1+\delta)^{-5} \Rightarrow f^{iv}(0) = 24$

$1 - \delta + \delta^2 - \delta^3 + \delta^4 + \ldots$

$f(\delta) = \frac{1}{1+\delta} \cdot \frac{1}{1-\delta} = \frac{1}{1-\delta^2} \overset{?}{=} (1 + \delta^2 + \delta^4 + \ldots$

$= (1 + \delta + \delta^2 + \delta^3 + \ldots)(1 - \delta + \delta^2 - \delta^3 + \ldots)$

$= 1 + \delta^2$

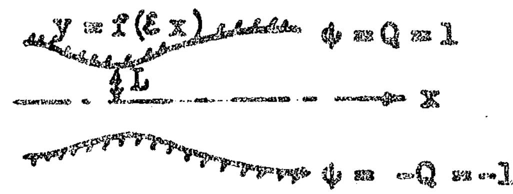
$\frac{60}{48} = \frac{5}{4}$

Solutions to Exercise Set 5

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

11 May 1979

3.5 of notes. Plane laminar flow in slowly varying channel. We make the variables dimensionless by referring the coordinates x and y to some characteristic half-width L of the channel, and the stream function ψ to the flux Q through half the channel. Then with the walls described by $y = \pm f(\varepsilon x)$, the problem is



$$\nabla^4 \psi = R\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y}\right)\nabla^2 \psi, \qquad \left.\begin{matrix}\psi = \pm 1 \\ \psi_y = 0\end{matrix}\right\} \text{ at } y = \pm f(\varepsilon x).$$

Here the Reynolds number is $R = Q/\nu$. Introducing the contracted abscissa $X = \varepsilon x$ transforms this to

$$\psi_{yyyy} = \varepsilon R\left(\psi_y \frac{\partial}{\partial X} - \psi_X \frac{\partial}{\partial y}\right)\left(\psi_{yy} + \varepsilon^2 \psi_{XX}\right) - 2\varepsilon^2 \psi_{XXyy} - \varepsilon^4 \psi_{XXXX},$$

$$\psi = \pm 1, \quad \psi_y = 0 \quad \text{at} \quad y = \pm f(X).$$

For $\varepsilon = 0$ the first approximation satisfies $\psi_{yyyy} = 0$, and quadratures give the quasi-cylindrical result, with parabolic velocity profile:

$$\psi = \frac{1}{2}\left(3\frac{y}{f(X)} - \frac{y^3}{f^3}\right), \qquad u = \psi_y = \frac{3}{2}\frac{1}{f(X)}\left(1 - \frac{y^2}{f^2}\right).$$

The skin friction is proportional to the vorticity at the wall, or to u_y in this approximation, and hence to $1/f^2$. Iterating on the neglected terms, or seeking a regular expansion in powers of ε, shows that the second approximation satisfies

$$\psi_{yyyy} = 9\varepsilon R\frac{f'}{f^5}\left(y - y^3/f^2\right).$$

Hence quadratures lead to

$$\psi = \frac{1}{2}\left(3\frac{y}{f} - \frac{y^3}{f^3}\right) + \frac{3}{280}\varepsilon R f'\left(5\frac{y}{f} - 11\frac{y^3}{f^3} + 7\frac{y^5}{f^5} - \frac{y^7}{f^7}\right) + O(\varepsilon^2, \varepsilon^2 R^2)$$

To this order the skin friction is still proportional to $u_y = \psi_{yy}$ at the wall, and therefore contains a factor

$$1 - \frac{4}{35}\varepsilon R f'(X)$$

Hence for a linearly growing channel (f' constant) the skin friction vanishes for a slope of $35/4R = 8.75 R^{-1}$. This result was given by Blasius (1910). Abramowitz (1949) calculated the next approximation, refining this to $6.16 R^{-1}$ at high Reynolds number. The exact value is $6.87 R^{-1}$. Lucas (1972) computed 25 terms of our series, and found that at large R the radius of convergence is 20 per cent greater than this value.

BLASIUS, H. 1910 Laminare Strömung in Kanälen wechselnder Breite. Zeit. f. Math. u. Physik 58, 225-233

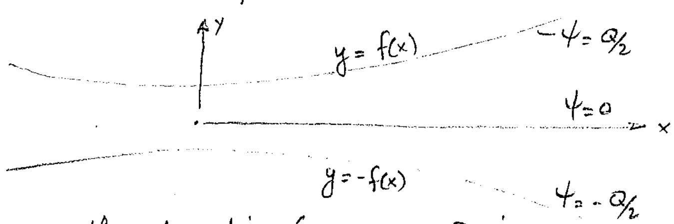
ABRAMOWITZ, M. 1949 On backflow of a viscous fluid in a diverging channel. J. Math. & Phys. 28, 1-21

LUCAS, R. D. 1972 A perturbation solution for viscous incompressible flow in channels. Ph.D. diss., Stanford Univ.

Cesar Ling(?)
ME 207
Prof. Van Dyke

Problem Set #5

3.5: Given $(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y})\nabla^2\psi = \nu\nabla^4\psi$ where $u = \psi_y$, $v = -\psi_x$. Show



1. That ψ_0 yields a parabolic velocity profile
2. the skin friction is inversely proportional to the channel width
3. Find ψ_1
4. if y = linear function in x find the value of the wall slope as a fn of R_N where $\tau = 0$

if we remember the volumetric flow rate Q is proportional to $\Delta\psi$. If the line $y=0$ is tagged $\psi=0$ then the lines $y = \pm f(x)$ can be tagged $\psi = \pm Q/2$. Since for large x we would have non uniformities, we contract the x coordinate to write $X = \epsilon x$; then

$$(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y})\nabla^2\psi = \nu\nabla^4\psi \Rightarrow \epsilon(\psi_y\psi_{Xyy} - \psi_X\psi_{yyy}) + \epsilon^3(\psi_y\psi_{XXX} - \psi_X\psi_{yXX}) = \nu\epsilon^4\psi_{XXXX} + 2\nu\epsilon^2\psi_{XXyy} + \nu\psi_{yyyy}$$ ✓

the b.c. are that $\frac{\partial\psi}{\partial n} = 0$ on $y = \pm f(x)$; $\psi = \pm Q/2$ on $y = \pm f(x)$. (normal velocity = 0 on wall is first bc)

We can replace these due to symmetry about $y=0$ by

(1) $\frac{\partial\psi}{\partial n} = 0$ on $y = +f(x) = F(X)$ (2) $\psi = Q/2$ on $y = +f(x) = F(X)$ (3) $\psi = 0$ on $y = 0$

and finally $\frac{\partial^2\psi}{\partial y^2} = 0$ on $y=0$ (this condition says the change in u in y direction is 0 across center line - just a symmetry condition)

Now. (1) $\frac{\partial\psi}{\partial n} = \underline{n}\cdot\nabla\psi$ if $y = F(X)$ then $g(X,y) = y - F(X) = 0$. Thus $\underline{n} = \pm\frac{\nabla g}{|\nabla g|} = \pm\frac{-\epsilon F'(X)\,\underline{i} + \underline{j}}{\sqrt{1 + F'(X)^2\epsilon^2}}$

now $\nabla\psi = \epsilon\frac{\partial\psi}{\partial X}\underline{i} + \frac{\partial\psi}{\partial y}\underline{j}$ thus $\frac{\partial\psi}{\partial n} = \pm\frac{-F'(X)\epsilon^2\,\partial\psi/\partial X + \partial\psi/\partial y}{\sqrt{1 + F'(X)^2\epsilon^2}}$ ✓ since $f'(x) = \frac{d}{dx}f(x) = \epsilon\frac{d}{dX}F(X) = \epsilon F'$

if $\psi(X,y;\epsilon) = \psi_0(X,y) + \epsilon\psi_1(X,y) + \epsilon^2\psi_2(X,y) + \ldots$

BC $\Rightarrow$ $\psi(X, F(X);\epsilon) = Q/2 = \psi_0(X,F(X)) + \epsilon\psi_1(X,F(X)) + \epsilon^2\psi_2(X,F(X)) + \ldots \Rightarrow \psi_0(X,F(X)) = Q/2;\ \psi_i(X,F(X)) = 0\ \forall i>0$

$\psi(X,0;\epsilon) = 0 = \psi_0(X,0) + \epsilon\psi_1(X,0) + \epsilon^2\psi_2(X,0) + \ldots \Rightarrow \psi_i(X,0) = 0 \quad \forall i \geq 0$

$\frac{\partial^2\psi}{\partial y^2}(X,0;\epsilon) = 0 = \frac{\partial^2\psi_0}{\partial y^2}(X,0) + \epsilon\frac{\partial^2\psi_1}{\partial y^2}(X,0) + \epsilon^2\frac{\partial^2\psi_2}{\partial y^2}(X,0) + \ldots \Rightarrow \frac{\partial^2\psi_i}{\partial y^2}(X,0) = 0 \quad \forall i \geq 0$

for the outward normal take + sign

$$\frac{\partial\psi}{\partial n}(X,F(X);\epsilon) = 0 = \left[\frac{\partial\psi_0}{\partial y}(X,F(X)) + \epsilon\frac{\partial\psi_1}{\partial y}(X,F(X)) + \epsilon^2\frac{\partial\psi_2}{\partial y}(X,F(X)) + \ldots - \epsilon^2 F'(X)\left\{\frac{\partial\psi_0(X,F(X))}{\partial X} + \epsilon\frac{\partial\psi_1}{\partial X}(X,F(X)) + \ldots\right\}\right]$$

$\Rightarrow \psi_{0,y}(X,F(X)) = 0$ and $\psi_{i,y}(X,F(X)) - F'(X)\,\psi_{i-2,X}(X,F(X)) = 0 \quad \forall i > 1$.

$\psi_{1,y}(X,F(X)) = 0$

thus the ε^0 problem reduces to

1. $$\psi_{0,yyyy} = 0 \quad w/\ \psi_0(X,0)=0\ ;\ \frac{\partial^2\psi_0}{\partial y^2}(X,0)=0\ ;\ \psi_0(X,F(X)) = Q/2\ ;\ \psi_{0,y}(X,F(X))=0 \qquad (1-4)$$

the DE gives $\psi_0 = Ay^3 + By^2 + Cy + D$

applying BC (1 & 2) $\Rightarrow$ $D, B = 0$ applying BC (3 & 4) gives $A = -\frac{Q}{4F(X)^3}$ $C = \frac{3Q}{4F(X)}$

$$\therefore \quad \boxed{\psi_0 = \frac{Q}{4}\left(\frac{y}{F(X)}\right)\left[3 - \left(\frac{y}{F(X)}\right)^2\right]} \ ✓$$

Now

$$u = \frac{\partial\psi_0}{\partial y} = -\frac{3Q}{4F(X)}\left[\frac{y^2}{F(X)^2} - 1\right]\ ;$$ for any value of X, $F(X)$ is fixed and we note that $u \propto y^2$ which is an equation of a parabola

2. Now for non parallel laminar flow $\tau = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]\Big|_{wall} = \mu\left[\varepsilon\frac{\partial v}{\partial X} + \frac{\partial u}{\partial y}\right] = \mu\left[-\varepsilon^2\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial y^2}\right]\Big|_{wall}$

to a first approximation

$$\tau = \mu\frac{\partial^2\psi_0}{\partial y^2}\Big|_{y=F(X)} = \mu\left[-\frac{6Q}{4}\frac{y}{F(X)^3}\right]\Big|_{y=F(X)} = -6\mu Q\cdot\frac{1}{4F(X)^2}$$

Since the channel width is $2F(X)$ then the skin friction is $\propto \frac{1}{\text{width}^2}$ as required.

3. for the ε^1 problem

$$\psi_{0,y}\,\psi_{0,Xyy} - \psi_{0,X}\,\psi_{0,yyy} = \nu\,\psi_{1,yyyy}$$

$$w/\ \psi_1(X,F(X)) = 0\ ;\ \psi_1(X,0)=0\ ;\ \frac{\partial^2\psi_1}{\partial y^2}(X,0)=0\ ;\ \psi_{1,y}(X,F(X))=0$$

Now $\psi_{0,y} = -\frac{3Q}{4F^3}y^2 + \frac{3}{4}\frac{Q}{F}$; $\psi_{0,yy} = -\frac{6Q}{4F^3}y$ $\quad \psi_{0,yyy} = -\frac{6Q}{4F^3}$

$\psi_{0,Xyy} = \frac{18Qy}{4F^4}F'$ $\quad \psi_{0,X} = +\frac{3Q}{4F^4}y^3F' - \frac{3Q}{4F^2}yF'$

Put into DE

$$\frac{18}{4}\frac{QyF'}{F^4}\left\{\frac{3Q}{4F}\left[1-(y/F)^2\right]\right\} - \frac{3QF'}{4F}\left\{\left(\frac{y}{F}\right)^3 - y/F\right\}\left\{\frac{-6Q}{4F^3}\right\} = \frac{54Q^2yF'}{16F^4F}\left\{1-(y/F)^2\right\} + \frac{18Q^2yF'}{16F^4F}\left\{(y/F)^2 - 1\right\} =$$

$$\frac{36}{16}\frac{Q^2yF'}{F^4F}\left\{1-(y/F)^2\right\} = \nu\frac{\partial^4\psi_1}{\partial y^4} \quad \text{or} \quad \frac{36Q^2F'}{16\nu}\left[Y - Y^3\right] = \frac{\partial^4\psi_1}{\partial Y^4} \quad \text{where } Y = y/F(X)$$

Let $P = \frac{36Q^2F'}{16\nu}$ then $P\left[\frac{Y^2}{2} - \frac{Y^4}{4} + C_1\right] = \psi_{1,YYY}$; $P\left[\frac{Y^3}{6} - \frac{Y^5}{20} + C_1Y + C_2\right] = \psi_{1,YY}$

$$P\left[\frac{Y^4}{24} - \frac{Y^6}{120} + C_1\frac{Y^2}{2} + C_2Y + C_3\right] = \psi_{1,Y} \quad ; \quad P\left[\frac{Y^5}{120} - \frac{Y^7}{840} + C_1\frac{Y^3}{6} + C_2\frac{Y^2}{2} + C_3Y + C_4\right] = \psi_1$$

w/ BC now converted to $\psi_1(X,1) = 0$; $\psi_1(X,0) = 0$; $\psi_{1,YY}(X,0) = 0$ and $\psi_{1,Y}(X,1) = 0$

from 2nd BC $C_4 = 0$ $\quad$ 3rd BC $C_2 = 0$

from 1st BC $\frac{c_1}{6} + c_3 = -1/140$; $\frac{c_1}{2} + c_3 = -\frac{1}{30}$ $\Rightarrow$ $c_1 = \frac{-5.5}{70}$ $c_3 = \frac{2.5}{420}$

thus

$$\psi_1 = \frac{36\, Q^2 F'(X)}{840 \cdot 16\nu}\left[7\left(\frac{y}{F(X)}\right)^5 - \left(\frac{y}{F(X)}\right)^7 - 11\left(\frac{y}{F(X)}\right)^3 + 5\left(\frac{y}{F(X)}\right) \right]$$

thus

$$\psi = \left[\frac{Q}{4}\left(3Y - Y^3\right)\right] + \varepsilon\left[\frac{3\, Q^2 F'(X)}{70 \cdot 16\nu}\left(7Y^5 - Y^7 - 11Y^3 + 5Y\right)\right] + O(\varepsilon^2) \checkmark$$

(4) if $F(X) = AX$ then $F'(X) = A$ then

$$\psi = \frac{Q}{4}\left(3Y - Y^3\right) + \varepsilon\left[\frac{3AQ^2}{70 \cdot 16\nu}\left(7Y^5 - Y^7 - 11Y^3 + 5Y\right)\right] + O(\varepsilon^2)$$

for skin friction to a second approximation $\tau = \mu \frac{\partial^2\psi_0}{\partial y^2}\Big|_{wall} + \varepsilon\mu \frac{\partial^2\psi_1}{\partial y^2}\Big|_{wall} = \frac{\mu}{F^2}\frac{\partial^2\psi_0}{\partial Y^2}\Big|_{wall} + \frac{\varepsilon\mu}{F^2}\frac{\partial^2\psi_1}{\partial Y^2}\Big|_{wall}$

$$\psi_{,YY} = \frac{Q}{4}\left(-6Y\right) + \frac{3\varepsilon AQ^2}{70 \cdot 16\nu}\left[140Y^3 - 42Y^5 - 66Y\right];$$ at the wall $$\psi_{,YY} = \frac{-6Q}{4} + \frac{3\varepsilon AQ^2}{70 \cdot 16\nu}\left[32\right]$$

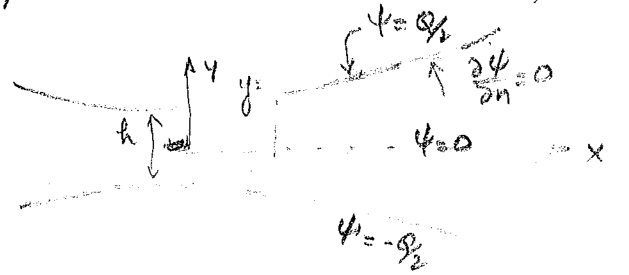
for a linear wall $\tau = \frac{\mu}{A^2X^2}\left[\frac{-6Q}{4} + \frac{6\varepsilon AQ^2}{70 \cdot \nu}\right]$; if $\tau = 0 \Rightarrow +\frac{6Q}{4} = \frac{6\varepsilon AQ^2}{70 \cdot \nu} \Rightarrow A = \frac{70\nu}{4\varepsilon Q}$

since $\frac{\nu}{\varepsilon Q} = \frac{1}{R}$ then $\boxed{A = \frac{70}{4R}}$ $\checkmark$ where $R = \frac{\text{veloc}\, AX}{\nu}$ $Q = \text{veloc} \cdot Ax = \text{veloc} \cdot \frac{AX}{\varepsilon}$

$$\frac{df(x)}{dx} = \varepsilon \frac{dF(X)}{dX} = \varepsilon F'$$

$$\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y}\right) \nabla^2 \psi = \nu \nabla^4 \psi \qquad \text{where } u = \psi_y \quad v = -\psi_x$$

if $\Delta\psi = Q$



1. Show that ψ_0 yields a parabolic veloc. profile
2. skin friction $\propto \frac{1}{4 h^2}$
3. find ψ_1
4. if $y = h + AX$ find the slope $A = A(R)$ ∋ skin friction = 0

let $X = \varepsilon x \Rightarrow$

$$\left(\psi_y \varepsilon \frac{\partial}{\partial X} - \varepsilon \psi_X \frac{\partial}{\partial y}\right)\left[\psi_{yy} + \varepsilon^2 \psi_{XX}\right] = \nu\left\{\varepsilon^4 \psi_{XXXX} + 2\varepsilon^2 \psi_{XXyy} + \psi_{yyyy}\right\}$$

let $\psi = \psi_0 + \varepsilon \psi_1(X, y) + \varepsilon^2 \psi_2(X, y) + \dots$

$$\left(\varepsilon \psi_y \psi_{Xyy} + \varepsilon^3 \psi_y \psi_{XXX} - \varepsilon \psi_X \psi_{yyy} - \varepsilon^3 \psi_X \psi_{yXX}\right) = \nu \varepsilon^4 \psi_{XXXX} + 2\nu\varepsilon^2 \psi_{XXyy} + \nu \psi_{yyyy}$$

ε^0: is $\quad \nu \psi_{0\,yyyy} = 0 \quad$ or $\quad \psi_0 = ay^3 + by^2 + cy + d$

BC. $\psi = 0$ on $y = 0 \Rightarrow \psi_i(X, 0) = 0 \;\forall i$
$\psi = Q/2$ on $y = f(x) \Rightarrow \psi_0 = Q/2 \quad \psi_i(X, f(X/\varepsilon))$ $i > 0$

BC: when $y = 0 \quad \psi_0 = 0 \Rightarrow d = 0$

BC: $\psi = Q/2$ on $y = f(x) \qquad y - f(x) = g(x, y) = 0 \qquad \underline{n} = \frac{\nabla g}{|\nabla g|} = \frac{-f'(x)\,\underline{i} + \underline{j}}{\sqrt{1 + f'(x)^2}}$

BC: and $\frac{\partial \psi}{\partial n} = \underline{n} \cdot \nabla \psi = \dfrac{-f'(x)\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}}{\sqrt{1 + f'(x)^2}} = 0$ on $y = f(x)$

$$\therefore \frac{-f'(x)\varepsilon \frac{\partial \psi}{\partial X} + \frac{\partial \psi}{\partial y}}{\sqrt{1 + f'(x)^2}} = 0 \Rightarrow \frac{\partial \psi}{\partial y} = 0 \text{ on } y = f(x)$$

BC: and $\frac{\partial^2 \psi}{\partial y^2}$ on $y = 0$ (since u velocity is symmetric)

$$\Rightarrow \frac{\partial^2 \psi_0}{\partial y^2} = 0 \quad \text{and} \quad \partial^2 \psi_i / \partial y^2 = 0 \text{ on } y = 0 \;\; \forall i$$

$\therefore \quad \psi_0 = ay^3 + by^2 + cy + d \quad$ w/ (1) $\psi_0 = 0$ on $y = 0$ | (3) $\psi_0 = Q/2$ on $y = f(x)$
(2) $\frac{\partial^2 \psi_0}{\partial y^2} = 0$ on $y = 0$ | (4) $\frac{\partial \psi_0}{\partial y} = 0$ on $y = f(x)$

$\therefore$ (1) $\Rightarrow d = 0 \qquad$ (3) $\Rightarrow Q/2 = a f(x)^3 + c f(x)$

(2) $\Rightarrow b = 0 \qquad$ (4) $\Rightarrow 0 = 3a f(x)^2 + c \qquad c = -3a f(x)^2$

$$\Rightarrow Q/2 = -2a f(x)^3 \quad \therefore \quad a = -\frac{Q}{4 f(x)^3} \quad c = +\frac{3Q}{4 f(x)}$$

$$\therefore \psi_0 = -\frac{Q y^3}{4 f(x)^3} + \frac{3Q}{4 f(x)} y = \frac{Q}{4}\left(\frac{y}{f(x)}\right)\left[3 - \left(\frac{y}{f(x)}\right)^2\right]$$

$$\frac{\partial \psi_0}{\partial y} = u = -\frac{3Q}{4 f(x)}\left[\frac{y^2}{f(x)^2} - 1\right]$$

$$\frac{\partial \psi_0}{\partial x} = v = 0$$

for any value of x $f(x)$ is fixed thus $-\frac{3Q}{4f} = \text{const}$ $\left(\frac{y^2}{f^2} - 1\right)$ is a parabola since $u \sim (y^2)$

$\frac{\partial v}{\partial n} = \underline{n} \cdot \nabla(\underline{v})$ $\left[\frac{\partial \psi}{\partial y} i - \varepsilon \frac{\partial \psi}{\partial X} j\right]$

$\frac{\partial}{\partial n} \sqrt{u^2+v^2} = \frac{\partial}{\partial y}\frac{\partial y}{\partial n} + \frac{\partial}{\partial X}$

$\frac{\partial v}{\partial n}$ $V n' = u$, $V \cos = v$

$\therefore V\left(\frac{f'}{\sqrt{1+f'^2}}\right) = u$

$V\left(\frac{1}{\sqrt{1+f'^2}}\right) v$

$\frac{\partial}{\partial n} = \frac{\partial}{\partial y}$

$$\tau = \mu \frac{\partial u}{\partial y} = \mu \frac{\partial^2 \psi_0}{\partial y^2}\Big|_{y=f(x)}$$

since $\psi_0 = -\frac{Q}{4}\left(\frac{y}{f}\right)^3 + \frac{3}{4} Q \frac{y}{f}$

then $\frac{\partial^2 \psi_0}{\partial y^2} = -\frac{Q}{4} \cdot \frac{6y}{f^3}\Big|_{y=f(x)}$

$\tau = -\frac{\mu Q}{4} \cdot \frac{6y}{f^3}$ but $y = f(x)$ on the wall $\therefore$ $\tau = -\frac{\mu Q}{4} \cdot 6 \frac{1}{f(x)^2}$

~~$y = f(x)$, $dy = f'(x)dx$~~

$\therefore \tau \sim \frac{1}{f(X)^2}$

~~$F = \int \tau \, dx$~~

$= \int$ ~~$\frac{-\mu Q}{4} \cdot 6 \frac{f(x)dx}{f(X)}$~~

~~2nd approx if $y = w + X$~~

~~$\varepsilon \psi_{0,y} \psi_{0,Xyy} - \varepsilon \psi_{0,X} \psi_{0,yyy} = \nu \psi_{1,yyyy}$~~

w/ ~~$\psi_1(X,0) = 0$ & $\psi_1(X, w+X) = 0$~~

$\psi_{0,y} \psi_{0,Xyy} - \psi_{0,X} \psi_{0,yyy} = \nu \psi_{1,yyyy}$

(1) ✓ $\psi_1(X,0) = 0$ and $\psi_1(X, f(X)) = 0$ (2)

(3) $$\frac{-f'(X)\frac{\partial \psi_0}{\partial X} + \frac{\partial \psi_1}{\partial y}}{\sqrt{1+f'(X)^2}} = 0 \text{ on } y = f(X)$$

(4) $\frac{\partial^2 \psi_1}{\partial y^2} = 0$ on $y = 0$ ✓

$\psi_0 = -\frac{Q}{4}\left[\frac{y}{f}\right]^3 + \frac{3}{4} Q \frac{y}{f}$

$\psi_{0,y} = -\frac{3Q}{4f^3} y^2 + \frac{3}{4}\frac{Q}{f}$

$\psi_{0,yy} = -\frac{6Q}{4f^3} y$ $\psi_{0,yyy} = -\frac{6Q}{4f^3}$

$$\frac{3Q}{4f}\left[1 - \left(\frac{y}{f}\right)^2\right]\left\{+\frac{6Qy}{4f^4}(+3)f'\right\} - \left\{-\frac{6Q}{4f^3}\right\}\left\{+\frac{Q}{4}\frac{y^3}{f^4}(+3)f' + \frac{3Q}{4}\frac{y}{f^2}(-1)f'\right\} = \nu \psi_{1,yyyy}$$

$$\frac{54Q^2 f'}{16 f^5} y - \frac{54 Q^2 f' y^3}{16 f^7} - \left\{-\frac{18 Q^2 y^3 f'}{16 f^7} + \frac{18 Q^2}{16 f^5} y f'\right\} = \nu \frac{\partial^4 \psi_1}{\partial y^4}$$

$$\frac{36 Q^2 f'}{16 f^5} y - \frac{36 Q^2 f'}{16 f^7} y^3 = \frac{36 Q^2 f'}{16 f^5}\left(y - \frac{y^3}{f^2}\right) = \nu \frac{\partial^4 \psi_1}{\partial y^4}$$

$= P\nu$

$\therefore P\left(\frac{y^2}{2} - \frac{y^4}{4f^2} + c_1\right) = {}^3\psi_1$

$P\left(\frac{y^3}{6} - \frac{y^5}{20f^2} + c_1 y + c_2\right) = {}^2\psi_1$

$P\left(\frac{y^4}{24} - \frac{y^6}{120f^2} + c_1\frac{y^2}{2} + c_2 y + c_3\right) = {}^1\psi_1$

$P\left(\frac{y^5}{120} - \frac{y^7}{840f^2} + c_1\frac{y^3}{6} + c_2\frac{y^2}{2} + c_3 y + c_4\right) = \psi_1$ by (1) and (4)

$\therefore \psi_1 = \frac{36 Q^2 f'}{16 f^5 \nu}\left[\frac{y^5}{120} - \frac{y^7}{840 f^2} - \frac{16.5 y^3 f^2}{1260} + \frac{15.5 y}{420}\right]$

$c_3 = \frac{15.5 f^4}{420}$

$c_1 = \frac{-16.5 f^2}{210}$

$\uparrow$

$\frac{7f^5}{840} - \frac{f^5}{840} + c_1\frac{f^3}{6} + c_3 f = 0 \Rightarrow$ $c_1\frac{f^3}{6} + c_3 f = \frac{-f^5}{140}$

$c_1\frac{f^3}{2} + c_3 f = \frac{-f^5}{30}$

$c_1\frac{f^3}{3} = -\frac{f^4}{30} + \frac{f^5}{140}$

$= -\frac{7f^5}{210} + \frac{1.5 f^5}{210}$

$\frac{\partial \psi_1}{\partial y} = P\left[\frac{f^4}{24} + c_1\frac{f^2}{2} + c_3\right] = f'\left\{\frac{3Q}{4f} - \frac{3Q}{4f}\right\} = 0$

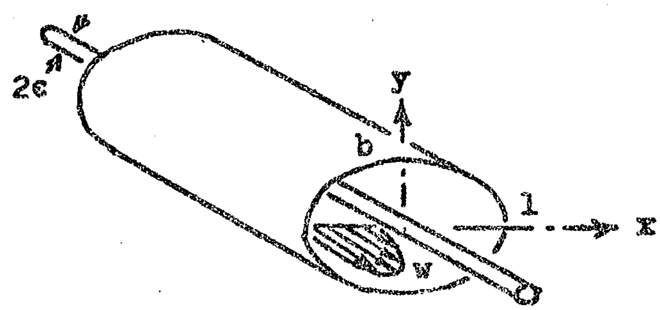
Exercise Set 6

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

Due Monday 14 May 1979

6.1. Effect of central wire on laminar flow through elliptic pipe.

A slender obstruction has a surprisingly large effect on laminar flow through a pipe. To show this, consider the fully developed ("Poiseuille") flow through an elliptic pipe with a wire stretched along its center-line. In appropriate dimensionless variables, the axial velocity across any longitudinal section satisfies



$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -1,$$

$$w = 0 \quad \text{at} \quad \begin{cases} x^2+y^2/b^2 = 1 \\ x^2+y^2 = \epsilon^2 \end{cases}$$

local scale: ϵ for x, y $\quad W = w/\epsilon^\alpha$ $\quad$ solve $\nabla^2 W = 0$ in local or $\nabla^2 W = -1$

[Using the membrane analogy, we may interpret this instead as the problem of the deflection, under a small pressure differential, of a soap film that spans the annular gap between an ellipse and a small circle. However, we cannot use the other analogy with torsion of a shaft, because the stress function for a hollow shaft has a different value on each boundary.]

a. Show that when the wire is absent ($\epsilon = 0$) the solution for w is a multiple of the simplest function that vanishes on the boundary, namely $1 - x^2 - y^2/b^2$.

b. Introduce local coordinates X and Y appropriate to the neighborhood of the wire. Rewrite the differential equation and local boundary condition in those coordinates, and find the first local approximation for small ϵ, to within an undetermined multiplicative constant.

$W = A \ln R + \frac{1}{4}(1-R^2)$ $\quad R = r/\epsilon$ $\quad W = w/\epsilon^2$ $\quad$ Axisymmetric

$W =$

c. Complete the local approximation by matching with the global approximation of part a.

d. Construct a first approximation that is valid everywhere across the section by forming the additive composite.

e. Use the composite approximation to calculate the rate of volume flux (or area under the soap film) $\iint w\,dx\,dy$ in the special case of a circular pipe. How much is the flux reduced when the wire has 1/100 the cross-sectional area of the pipe?

Of course for the circular pipe the exact solution is easily found; and you may want to use it as a check on your solution. For the elliptic pipe, however, no closed-form solution seems possible.

$\nabla^2 W = -1$

w/ $W = 0$ on $x^2+y^2 = 1$, $x^2+y^2 = \epsilon^2$

$$\frac{r \frac{\partial}{\partial r}\left(R \frac{\partial f}{\partial r}\right)}{-f} = \frac{g''}{g} = -\lambda$$
$-\lambda^2$

a. $\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -1$ w/ $w(x,y) = 0$ on $x^2 + y^2/b^2 = 1$

if we pick $w(x,y) = A\left(1 - x^2 - y^2/b^2\right)$ then $w\left(x, y = \pm b\sqrt{1-x^2}\right) = 0$

$\nabla^2 w = A\left[-2 - \frac{2}{b^2}\right] = -1$ take $A = \frac{1}{2}\frac{b^2}{b^2+1}$ then

$$w(x,y) = \frac{1}{2(b^2+1)}\left[b^2 - b^2x^2 - y^2\right]$$

let $X = \frac{x}{\varepsilon}$ $Y = \frac{y}{\varepsilon}$ let $W = \frac{w}{\varepsilon^2}$ since $w \sim O(\text{coordinate squared})$

$\nabla^2 w = \varepsilon^2 \bar\nabla^2 W = \varepsilon^2\left[\frac{\partial^2 W}{\partial X^2}\left(\frac{\partial X}{\partial x}\right)^2 + \frac{\partial^2 W}{\partial Y^2}\left(\frac{\partial Y}{\partial y}\right)^2\right] = \bar\nabla^2 W = -1$

w/ BC $\varepsilon^2 X^2 + \varepsilon^2 \frac{Y^2}{b^2} = 1$ cannot satisfy the BC

and $\varepsilon^2 X^2 + \varepsilon^2 Y^2 = \varepsilon^2$ or $X^2 + Y^2 = 1$ $\Rightarrow$ on $R = 1 = \sqrt{X^2+Y^2}$

$\bar\nabla^2 W_0 = -1$ or $\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}W_0\right) + \frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2} = -1$ and $W_0(1,\theta) = 0$

this is axisymmetric $\therefore$

$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W_0}{\partial R}\right) = -1$ $\frac{\partial W_0}{\partial R} = -\frac{R}{2} + \frac{C_1}{R}$

$W_0 = -\frac{R^2}{4} + C_1 \ln R + C_2$

@ $R = 1$ $W_0(1,\theta) = 0$ $\therefore$ $W_0 = -1/4 + C_1 \cdot 0 + C_2 = 0$ $C_2 = 1/4$

$\therefore$ $W_0(R,\theta) = 1/4\left(1 - R^2\right) + C_1 \ln R$

Let $W = W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + \ldots$

$\bar\nabla^2 W = \bar\nabla^2 W_0 + \varepsilon \bar\nabla^2 W_1 + \varepsilon^2 \bar\nabla W_2 + \ldots = -1$ $\therefore$ $\bar\nabla^2 W_0 = -1$; $\bar\nabla^2 W_i = 0$ $i \geq 1$

$W(1,\theta) = 0 = W_0(1,\theta) + \varepsilon W_1(1,\theta) + \ldots = 0$ $\Rightarrow$ $W_i(1,\theta) = 0$ $i \geq 0$ $R=1$

$\therefore$ $W_0(R,\theta) = 1/4\left(1 - R^2\right) + C_1 \ln R$

$\nabla^2 W_1 = 0$ and $W_1(1,\theta) = 0$ $\Rightarrow \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W_1}{\partial R}\right) = 0$ $\frac{\partial W_1}{\partial R} = \frac{\hat C_1}{R}$ $W_1 = \hat C_1 \ln R + \hat C_2$

$W_1(1,\theta) = 0 \Rightarrow \hat C_2 = 0$ $\therefore$

$W(R,\theta) = W_0(R,\theta) + \varepsilon W_1(R,\theta) + O(\varepsilon^2) = \frac{1}{4}\left(1 - R^2\right) + \left(C_1 + \varepsilon \hat C_1\right)\ln R + O(\varepsilon^2)$

1. global is $w(x,y) = \frac{1}{2}\frac{b^2}{b^2+1}\left[1 - x^2 - y^2/b^2\right]$

rewrite in local variables

$$\varepsilon^2 W = \frac{1}{2}\frac{b^2}{b^2+1}\left[1 - \varepsilon^2 X^2 - \varepsilon^2 \frac{Y^2}{b^2}\right] = \frac{1}{2}\frac{b^2}{b^2+1}\left[1 - \varepsilon^2 R^2 - \varepsilon^2 Y^2 \left(\frac{b^2+1}{b^2}\right)\right]$$

local expansion is $W = \frac{1}{2}\frac{b^2}{b^2+1}\left[\frac{1}{\varepsilon^2} - X^2 - \frac{Y^2}{b^2}\right] = \frac{1}{2}\frac{b^2}{b^2+1}\left[\frac{1}{\varepsilon^2} - R^2 - Y^2\left(\frac{b^2+1}{b^2}\right)\right]$

1st term ~~local~~ global in local is $W = -\frac{1}{2}\frac{b^2}{b^2+1}\left[X^2 + \frac{Y^2}{b^2}\right]$

2. local is $W = \frac{1}{4}(1 - R^2) + c_1 \ln R$

rewrite in global var

$$W = \frac{w}{\varepsilon^2} = \frac{1}{4}\left(1 - \frac{x^2+y^2}{\varepsilon^2}\right) + \frac{c_1}{2}\ln\frac{x^2+y^2}{\varepsilon^2}$$

$$w(x,y) = \frac{1}{4}\left(\varepsilon^2 - [x^2+y^2]\right) + \frac{c_1 \varepsilon^2}{2}\ln\frac{x^2+y^2}{\varepsilon^2}$$

~~1st~~ 2 terms global expansion

$$w(x,y) = -\frac{1}{4}(x^2+y^2) + \frac{1}{4}\varepsilon^2 + \frac{c_1}{2}\varepsilon^2 \ln(x^2+y^2) - c_1 \varepsilon^2 \ln\varepsilon$$

1st term of local in global expansion

cannot match

$\therefore$ go back and let $w(x,y) = W(X,Y)$

$$\therefore \nabla^2 w = \frac{1}{\varepsilon^2}\bar{\nabla}^2 W = -1 \Rightarrow \bar{\nabla}^2 W = -\varepsilon^2$$

let $W = W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + \dots$

$$\Rightarrow \bar{\nabla}^2 W_0 = 0 \quad \bar{\nabla}^2 W_1 = 0 \quad \bar{\nabla}^2 W_2 = -1 \quad \bar{\nabla}^2 W_i = 0 \quad i \geq 3$$

BC is at $X^2+Y^2=1$ ~~$W(X,Y)=0 \Rightarrow W_0(1,\theta)$~~

$$\therefore W(X,Y) = W(1,\theta) = 0 \Rightarrow W_i(1,\theta) = 0 \quad i \geq 0$$

since problem is sym.

$$\Rightarrow \bar{\nabla}^2 W_0 = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}W_0\right) = 0 \Rightarrow \frac{C_1}{R} = \frac{\partial W_0}{\partial R} \Rightarrow C_1 \ln R + C_2 = W$$

$W_0(1,\theta) = 0 \Rightarrow C_2 = 0$ or $W_0 = C_1 \ln R$

$$= C_1\left[(R-1) - \frac{1}{2!}(R-1)^2 + \frac{2}{3!}(R-1)^3 + \dots\right.$$

$$R-1 - \left[\frac{1}{2}R^2 - \frac{2R}{2} + \frac{1}{2}\right]$$

$W_1(1,\theta) = 0$ & $\bar{\nabla}^2 W_1 = 0 \Rightarrow C_2 \ln R \Rightarrow (C_1 + \varepsilon C_2)\ln R = W(X,Y) = W(R,\theta)$

for the global $W = w = \frac{1}{2}\frac{b^2}{b^2+1}\left[1 - \varepsilon^2 X^2 - \varepsilon^2 Y^2/b^2\right]$

1st global ter in local coords $W = \frac{1}{2}\frac{b^2}{b^2+1}$

local $c_1 \ln(X^2+Y^2) = -c_1 \ln \varepsilon^2 + c_1 \ln(x^2+y^2) = w$

$\therefore$ matching $-c_1 \ln \varepsilon^2 = \frac{1}{2}\frac{b^2}{b^2+1}$ $\therefore$ $c_1 = -\frac{1}{2}\frac{b^2}{b^2+1}\frac{1}{\ln \varepsilon^2}$

$\therefore$ $\frac{1}{2}\frac{b^2}{b^2+1}\left[1 - x^2 - y^2/b^2\right] + -\frac{1}{2}\frac{b^2}{b^2+1}\frac{1}{\ln \varepsilon^2}\ln R$

Solution to Exercise Set 6

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

Wednesday 23 May 1979

6.1. Effect of central wire on laminar flow through elliptic pipe.
The solution in the absence of the wire is easily found as

$$w = \frac{b^2}{2(1+b^2)}(1-x^2-y^2/b^2).$$

It is clear physically that the appropriate magnified local coordinates near the wire are $X = x/\varepsilon$ and $Y = y/\varepsilon$. However, it is not clear how w is to be magnified. If we leave it unaltered, we are tacitly assuming that the pressure gradient is negligible near the wire -- which turns out to be correct. Then the local problem is

$$\frac{\partial^2 w}{\partial X^2} + \frac{\partial^2 w}{\partial Y^2} = -\varepsilon^2, \quad w = 0 \text{ at } X^2+Y^2 = 1.$$

The first approximation (for $\varepsilon = 0$) can be assumed axisymmetric, and so has the form $A \log(X^2+Y^2)^{\frac{1}{2}} + B$. Imposing the surface condition shows that $B = 0$, and we find A by matching:

1-term global approximation: $w = \frac{b^2}{2(1+b^2)}(1-x^2-y^2/b^2)$

rewritten in local variables: $= \frac{b^2}{2(1+b^2)}(1-\varepsilon^2X^2-\varepsilon^2Y^2/b^2)$

1-term local expansion: $= \frac{b^2}{2(1+b^2)}$

1-term local approximation: $w = A \log(X^2+Y^2)^{\frac{1}{2}}$

rewritten in global variables: $= A \log\frac{(x^2+y^2)^{\frac{1}{2}}}{\varepsilon}$

expanded for small ε: $= A \log\frac{1}{\varepsilon} + A\log(x^2+y^2)^{\frac{1}{2}}$

1-term global expansion: $= A\log\frac{1}{\varepsilon}$

Matching requires that these two expressions be identical. Thus we discover that A is not a pure constant, but depends weakly on ε:

$$A = \frac{1}{\log 1/\varepsilon}\frac{b^2}{2(1+b^2)}, \quad \text{so} \quad w = \frac{1}{\log 1/\varepsilon}\frac{b^2}{2(1+b^2)}\log(X^2+Y^2)^{\frac{1}{2}}$$

This shows that w should have been magnified slightly, by the factor $(\log 1/\varepsilon)^{-1}$. Forming the additive composite gives

$$w \approx \frac{b^2}{2(1+b^2)}\left[1-x^2-\frac{y^2}{b^2}+\frac{\log(x^2+y^2)^{\frac{1}{2}}}{\log 1/\varepsilon}\right].$$

For a circular pipe ($b = 1$) this is

$$w \approx \frac{1}{4}\left(1-r^2+\frac{\log r}{\log 1/\varepsilon}\right)$$

whereas the exact solution is easily found to be

$$w = \frac{1}{4}\left(1 - r^2 + \frac{1-\varepsilon^2}{\log 1/\varepsilon} \log r\right).$$

Hence the relative error is only $O(\varepsilon^2)$ in this special case. For the general elliptic pipe, however, considering the next global approximation shows that the relative error is of order $(\log 1/\varepsilon)^{-1}$.

Integrating across a circular section shows that for $\varepsilon = 1/10$, corresponding to a wire that fills only one per cent of the cross section, the flux is reduced by 43 per cent.

If we play safe by setting $w = \varepsilon^2 W(X,Y)$ in order to maintain the pressure gradient in the local approximation (in case it is important), the matching is more complicated. The local problem is

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = -1, \qquad W = 0 \quad \text{at} \quad X^2 + Y^2 = 1.$$

The appropriate axisymmetric solution is

$$W = \frac{1}{4}(1 - R^2) + A \log R ;$$

and now the second half of our matching scheme becomes

<u>1-term local approximation:</u> $w = \varepsilon^2\left[\frac{1}{4}(1-X^2-Y^2) + A \log (X^2+Y^2)^{\frac{1}{2}}\right]$

rewritten in global variables: $\varepsilon^2\left[\frac{1}{4}\left(1 - \frac{x^2+y^2}{\varepsilon^2}\right) + A \log \frac{(x^2+y^2)^{\frac{1}{2}}}{\varepsilon}\right]$

expanded for small ε: $= -\frac{x^2+y^2}{4} + \varepsilon^2 A \log \frac{1}{\varepsilon} + \varepsilon^2 A \log (x^2+y^2) + \varepsilon^2/4$

It would appear that the first term is the dominant one for small ε, and then the matching fails. However, we have already seen that the constant A may depend on ε. If it is as large as $1/\varepsilon^2$, the second term is the dominant one, and then matching gives

$$A = \frac{1}{\varepsilon^2 \log 1/\varepsilon} \; \frac{b^2}{2(1+b^2)}.$$

Here the extra factor of ε^2 in the denominator simply serves to cancel that introduced with W, so that the local approximation is the same as before, aside from an additive term that is meaningless because it is of relative order ε^2.

The matching turned out to be marginal in this case, hinging on terms of order unity being neglected compared with terms of order $(\log 1/\varepsilon)^{-1}$. As a consequence, higher approximations will proceed as a series in powers of $(\log 1/\varepsilon)^{-1}$. This is what Fraenkel has called the <u>purely logarithmic case</u>; and he warns that matching may fail at certain stages. Perhaps his <u>forbidden region</u> would appear in this problem in higher approximations.

Cesar Levy
ME207
Prof Van Dyke

Problem Set # 6

6.1 a. given $\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -1$ w/ $w(x,y) = 0$ on $x^2 + y^2/b^2 = 1$

$w(x,y) = 0$ on $x^2 + y^2 = \varepsilon^2$

We must abandon the inner bc to get the global soln. ✓

Thus we need to solve $\nabla^2 w = -1$ w/ $w = 0$ $x^2 + y^2/b^2 = 1$. If we pick $w(x,y) = A(1 - x^2 - y^2/b^2)$ and substitute into DE we obtain that $A = \frac{1}{2}\frac{b^2}{b^2+1}$

Thus our global soln is $w(x,y) = \frac{1}{2(b^2+1)}\left[b^2 - b^2x^2 - y^2\right]$ (1) ✓

b. To obtain or local solution since the principal length is ε let $X = \frac{x}{\varepsilon}$, $Y = \frac{y}{\varepsilon}$ and since $w(x,y) \sim O(\text{coordinate}^2$ in the global) then let $W = \frac{w}{\varepsilon^2}$

then the DE becomes $\varepsilon^2 \nabla^2 W = \frac{\varepsilon^2}{\varepsilon^2}\bar{\nabla}^2 W = -1$ and $W(X,Y) = 0$ when $x^2 + y^2 = \varepsilon^2$ or $(X^2 + Y^2 = 1)$ where we now abandon the global boundary condition

Thus we must solve $\bar{\nabla}^2 W = -1$ and $W(R,\theta) = 0$ on $R^2 = 1$ where now $R^2 = X^2 + Y^2$ and $R^2 = r^2/\varepsilon^2$ $(r^2 = x^2 + y^2)$

if $W(R,\theta;\varepsilon) = \sum_{i=0}^{\infty} W_{2i}(R,\theta)\,\varepsilon^{2i}$ then $\bar{\nabla}^2 W_0 = -1$ and BC is $W_0 = 0$ on $R^2 = 1$

The axisymmetric solution is

$$W_0(R,\theta) = C_0 \ln R + \frac{1}{4}(1 - R^2)$$ ✓

and $\bar{\nabla}^2 W_{2i} = 0$ and BC is $W_{2i} = 0$ on $R^2 = 1$ => $W_{2i} = C_{2i}\ln R$ ✓

$$\therefore\ W(R,\theta) = \sum_{i=0}^{\infty} c_i\,\varepsilon^{2i}\ln R + \frac{1}{4}(1 - R^2) \qquad (2)$$

if however we pick $w = W$ (no magnification) then putting into DE

$\nabla^2 W = \frac{1}{\varepsilon^2}\bar{\nabla}^2 W = -1$ or $\bar{\nabla}^2 W = -\varepsilon^2$ and $W(X,Y) = 0$ on $X^2 + Y^2 = 1$

again reverting $W(R,\theta) = \sum_{i=0}^{\infty} W_{2i}(R,\theta)\,\varepsilon^{2i}$

then $\bar{\nabla}^2 W_0 = 0$ $\bar{\nabla}^2 W_2 = -1$ $\bar{\nabla}^2 W_{2i} = 0$ $i \geq 2$ and BC again is

$W(X,Y) = 0$ on $R^2 = 1$ => $W_{2i}(R,\theta) = 0$ on $R^2 = 1$

solving $\bar{\nabla}^2 W_0 = 0$ w/ $W_0(1,\theta) = 0$ => $W_0(R,\theta) = C_0 \ln R$ ✓

$\bar{\nabla}^2 W_2 = -1$ w/ $W_2(1,\theta) = 0$ => $W_2(R,\theta) = \frac{1}{4}(1 - R^2) + C_2 \ln R$

etc $\therefore$ $W(R,\theta) = \sum C_{2i}\,\varepsilon^{2i}\ln R + \frac{\varepsilon^2}{4}(1 - R^2)$ (3)

c. lets begin matching by taking the one term global

$\Rightarrow$ 1 term global $\quad w(x,y) = \frac{1}{2}\frac{b^2}{b^2+1}\left[1 - x^2 - y^2/b^2\right]$

rewrite in local variables

if $w = \varepsilon^2 W$ $\quad \varepsilon^2 W = \frac{1}{2}\left[\frac{b^2}{b^2+1}\right]\left[1 - \varepsilon^2 X^2 - \varepsilon^2 Y^2/b^2\right]$

$$W = \frac{1}{2}\left[\frac{b^2}{b^2+1}\right]\left[\frac{1}{\varepsilon^2} - X^2 - Y^2/b^2\right]$$

Now let $\varepsilon \to 0$ and look for leading term

$$W \quad \frac{1}{2}\left[\frac{b^2}{b^2+1}\right]\left\{-\left[X^2 + Y^2/b^2\right] + \frac{1}{\varepsilon^2}\right\}$$

1 term local expansion

$$W = -\frac{1}{2}\left[\frac{b^2}{b^2+1}\right]\left(X^2 + Y^2/b^2\right) \qquad ? \qquad \varepsilon \to 0 \; ? \quad (4)$$

$\Rightarrow$ 1 term local expansion

$$W = \frac{1}{4}\left(1 - X^2 - Y^2\right) + C_0 \ln\left(\sqrt{X^2+Y^2}\right)$$

rewritten in global variables $\quad \frac{w}{\varepsilon^2} = \frac{1}{4}\left(1 - \frac{x^2+y^2}{\varepsilon^2}\right) + C_0 \ln \frac{\sqrt{x^2+y^2}}{\varepsilon}$

or $\quad w = \frac{1}{4}\left(\varepsilon^2 - [x^2+y^2]\right) + C_0 \varepsilon^2 \ln \frac{\sqrt{x^2+y^2}}{\varepsilon}$

Now let $\varepsilon \to 0$ and look for leading term

$$w = -\frac{1}{4}r^2 + \frac{1}{4}\varepsilon^2 + C_0 \varepsilon^2 \ln \frac{\sqrt{x^2+y^2}}{\varepsilon}$$

rewrite 1 term expansion in global $\quad \varepsilon^2 W = -\frac{1}{4}\varepsilon^2 R^2 \Rightarrow W = -\frac{1}{4}(X^2+Y^2) \quad (5)$

there's no way we can match (4) & (5) by using $\varepsilon^2 W = w$; at first look we cannot match. C_0 cannot be $O(\frac{1}{\varepsilon^2})$

Now we try the solution with $w = W$

rewrite global in local variables

$$W = \frac{1}{2}\left[\frac{b^2}{b^2+1}\right]\left(1 - \varepsilon^2\left[X^2 + Y^2/b^2\right]\right)$$

expanded in terms of ε and look for leading term

$$W = \frac{1}{2}\left[\frac{b^2}{b^2+1}\right] - \frac{\varepsilon^2 b^2}{2(b^2+1)}\left[X^2 + Y^2/b^2\right]$$

taking the 1 term local expansion

$$W = \frac{1}{2}\left[\frac{b^2}{b^2+1}\right] \quad ✓ \qquad (6)$$

rewrite local in global variables taking one term

$$W = w = C_0 \ln\left(\frac{\sqrt{x^2+y^2}}{\varepsilon}\right)$$

now let $\varepsilon \to 0$ and look for leading term

$$W = -c_0 \ln \varepsilon + \frac{c_0}{2} \ln (x^2+y^2)$$

hence we can define $-c_0 \ln \varepsilon = \frac{1}{2}\left[\frac{b^2}{b^2+1}\right] \Rightarrow c_0 = \frac{-1}{2\ln\varepsilon}\frac{b^2}{b^2+1}$ ✓

d. $\therefore$ we can define the composite : local + global − local of global

$$W = -\frac{1}{2\ln\varepsilon}\frac{b^2}{b^2+1}\ln R + \frac{1}{2}\left[\frac{b^2}{b^2+1}\right]\left(1-\varepsilon^2\left[X^2+Y^2/b^2\right]\right) - \frac{1}{2}\left[\frac{b^2}{b^2+1}\right]$$

$$w = W = -\frac{1}{2\ln\varepsilon}\frac{b^2}{b^2+1}\ln\frac{\sqrt{x^2+y^2}}{\varepsilon} - \frac{1}{2}\left[\frac{b^2}{b^2+1}\right]\left(x^2+y^2/b^2\right)$$ ✓

e. for a circular pipe $(b=1)$ $\quad w = -\frac{1}{4\ln\varepsilon}\ln\left(\frac{\sqrt{x^2+y^2}}{\varepsilon}\right) - \frac{1}{4}(x^2+y^2)$

$$= -\frac{1}{4\ln\varepsilon}\ln\frac{r}{\varepsilon} - \frac{1}{4}r^2$$

for a circular pipe $(b=1)$ without the wire

$$w = \frac{1}{4}(1-r^2)$$

for pipe w/wire $\quad \iint w\,dx\,dy = \iint w\,r\,dr\,d\theta = \int_0^1\left[\int_0^{2\pi} d\theta\right]\left\{-\frac{r\ln r}{4\ln\varepsilon} + \frac{1}{4}r - \frac{1}{4}r^3\right\}dr$

$$= 2\pi\left\{\frac{1}{8}r^2 - \frac{1}{16}r^3 - \frac{1}{4\ln\varepsilon}\left[\frac{r^2}{2}\ln r - \frac{r^2}{4}\right]\right\}_0^1 = 2\pi\left\{\frac{1}{16} + \frac{1}{16\ln\varepsilon}\right\}$$

for pipe w/o wire

$$\iint w\,dx\,dy = \iint w\,r\,dr\,d\theta = \int_0^1\int_0^{2\pi} d\theta \left\{\frac{1}{4}(r-r^3)\,dr\right\}$$

$$= 2\pi\left[\frac{1}{8}r^2 - \frac{1}{16}r^4\right] = \frac{2\pi}{16}$$

$\therefore$ for crossectional area of pipe $= \pi$ then crossection of wire $= \pi/100$ or $\varepsilon = \frac{1}{10}$ ✓

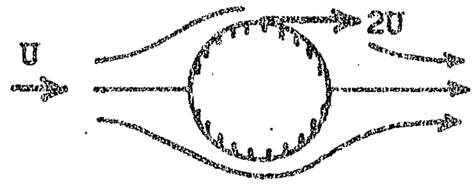
thus flow reduction $= \frac{2\pi}{16\ln\varepsilon} = -.170547$ or $\frac{2\pi}{16\ln\varepsilon}\Big/\frac{2\pi}{16} = \frac{1}{\ln\varepsilon} \Rightarrow 43.43\%$

Exercise Set 7

ME 207. PERTURBATION METHODS IN FLUID MECHANICS

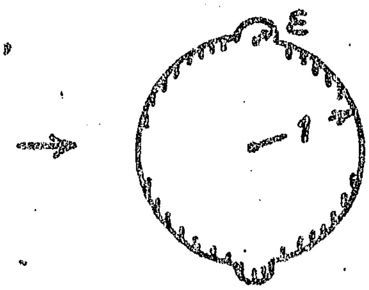
Due Wednesday 30 May 1979

7.1. Potential flow past circle with small semi-circular bumps. Every book on fluid mechanics derives the symmetric plane potential flow past a circle in a uniform stream. An easily remembered result is that the maximum speed, which occurs on the sides of the circle, is twice the free-stream speed.



Now suppose that to each side of a circle of unit radius is added a small semi-circular bump of radius ε. The flow past this shape cannot be calculated exactly, so we approximate for small ε. Consider the first approximation for the flow everywhere, and compute the new maximum flow speed.

Try to do this as intuitively as possible, using only words if you can, but resorting to equations if you must.



Alternatively, if you feel more comfortable with elasticity than fluid mechanics, consider the corresponding problem of a circular hole in a large plate submitted to uniform tension S. The maximum stress, which occurs at the ends of the diameter perpendicular to the direction of tension, is 3S. What is the maximum stress if small semi-circular holes are added at the ends of that diameter?

7.2. Matching in the purely logarithmic case: sliding rod. Solve Exercise 4.2 on pages 84-86 of the Notes. However, please correct the second line on page 86 to read "n = 0, 1." (That is, you aren't expected to calculate the second term in the Oseen expansion.)

Stokes' $w = 1 + \left[\frac{a}{\log 1/\varepsilon} + \frac{b}{(\log 1/\varepsilon)^2} + \frac{c}{(\log 1/\varepsilon)^3} + \dots\right] \log R$

Oseen $w = \frac{1}{\log 1/\varepsilon} \cdot \frac{1}{2}\left\{-(\log \rho^2 + \gamma) + O(\rho^2)\right\} = \frac{1}{\log 1/\varepsilon}\cdot\frac{1}{2}\left\{-2\log\rho - \gamma + O(\rho^2)\right\}$

$= \frac{1}{\log 1/\varepsilon}\cdot\left\{\log 1/\varepsilon - \log R - \gamma/2 + \dots\right\}$

Stokes $n=0$ Oseen $m=0$

Stokes $n=0 \Rightarrow w=1$ Expanded in Oseen variables $w=1$; take 1st order terms $(n=0) \Rightarrow w=1$

Oseen $m=0 \Rightarrow w=0$ Expanded in Stokes var $w=0$; take 1st order terms $(m=0) \Rightarrow w=0$

This shows we __don't__ match

Stokes $n=0$ Oseen $m=1$

Stokes $n=0 \Rightarrow w=1$ Exp in Oseen var $w=1$; take 2 term $(m=1) \Rightarrow w=1$

Oseen $m=1 \Rightarrow w = \frac{-\frac{1}{2}(\log\rho^2+\gamma)}{\log 1/\varepsilon}$ Expand in Stokes var $w = 1 - \frac{(\log R + \gamma/2)}{\log 1/\varepsilon}$ take 1 term $(n=0) \Rightarrow w=1$

This shows that we match

Stokes $n=1$ Oseen $m=0$

Stokes $n=1 \Rightarrow w = 1 + \frac{a}{\log 1/\varepsilon}\log R$ Expand in Oseen var $w = 1 + \frac{a\log R}{\log 1/\varepsilon} = 1 + a\frac{\log \rho/\varepsilon}{\log 1/\varepsilon} = 1 + a + \frac{a\log\rho}{\log 1/\varepsilon}$

take $m=0$ order $\Rightarrow w = 1+a$

Oseen $m=0 \Rightarrow w=0$ Expand in Stokes var $w=0$; take 2nd term $(m=1) \Rightarrow w=0$

$\Rightarrow$ for $w = 0 = 1+a$ $\boxed{a=-1}$ for matching

Stokes $n=1$ Oseen $m=1$

Stokes $n=1 \Rightarrow w = 1 + \frac{a\log R}{\log 1/\varepsilon}$ Expand in Oseen var $w = 1 + a + \frac{a\log\rho}{\log 1/\varepsilon}$ take $m=1$ order

$\Rightarrow w = 1 + a + \frac{a\log\rho}{\log 1/\varepsilon}$

Oseen $m=1 \Rightarrow w = \frac{-\frac{1}{2}(\log\rho^2+\gamma)}{\log 1/\varepsilon}$ Expand in Stokes var $w = 1 - \frac{(\log R + \gamma/2)}{\log 1/\varepsilon}$ take $m=1$ term

$w = 1 - \frac{(\log R + \gamma/2)}{\log 1/\varepsilon} = 1 - \frac{(\log\rho + \log 1/\varepsilon + \gamma/2)}{\log 1/\varepsilon} = -\frac{\log\rho + \gamma/2}{\log 1/\varepsilon}$

$\Rightarrow$ we __cannot__ match since $\Rightarrow a=-1$ and $0 = \frac{\gamma/2}{\log 1/\varepsilon}$

Stokes $n=2$ Oseen $m=0$

Stokes $n=2 \Rightarrow w = 1 + \left[\frac{a}{\log 1/\varepsilon} + \frac{b}{(\log 1/\varepsilon)^2}\right]\log R$ Expand in Oseen var $= 1 + \left[\frac{a}{\log 1/\varepsilon} + \frac{b}{(\log 1/\varepsilon)^2}\right](\log\rho + \log 1/\varepsilon$

$w = (1+a) + \frac{(a\log\rho + b)}{\log 1/\varepsilon} + \frac{b\log\rho}{(\log 1/\varepsilon)^2}$ take $m=0 \Rightarrow w = 1+a$

Oseen $m=0 \Rightarrow w=0$ Expand in Stokes var $w=0$; take 3rd order $(n=2)$ term $w=0$

matching $\boxed{a=-1}$

Stokes $n=2$ Oseen $m=1$

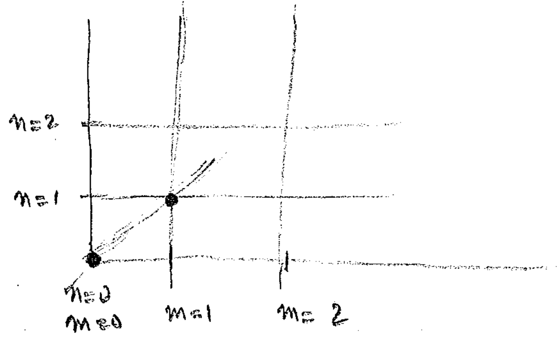
Stokes $n=2 \Rightarrow W = 1 + \left[\frac{a}{\log 1/\epsilon} + \frac{b}{(\log 1/\epsilon)^2}\right]\log R$ expand in oseen var $W = (1+a) + \frac{a\log\rho + b}{\log 1/\epsilon} + \frac{b\log\rho}{(\log 1/\epsilon)}$

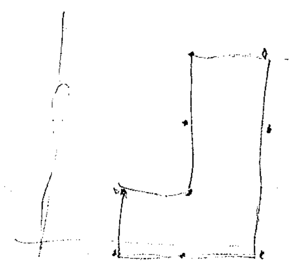
take 2 terms $(m=1)$

$$W = (1+a) + \frac{a\log\rho + b}{\log 1/\epsilon}$$

Oseen $m=1$ $\quad W = \frac{-\frac{1}{2}(\log\rho^2 + \gamma)}{\log 1/\epsilon}$ write in Stokes vars $W = 1 - \frac{(\log R + \gamma/2)}{\log 1/\epsilon} = \frac{\log\rho + \gamma/2}{\log 1/\epsilon}$

$\Rightarrow$ $\boxed{a = -1 \quad b = -\gamma/2}$ for matching





Since the Stokes expansion is

$$W = 1 + \left[\frac{a}{\log 1/\epsilon} + \frac{b}{(\log 1/\epsilon)^2} + \frac{c}{(\log 1/\epsilon)^3}\right]\log R$$

$$= 1 + \left[\frac{a}{\log K/\epsilon} + \frac{c'}{(\log K/\epsilon)^3} + \cdots\right]\log R$$

$$\left[\frac{a}{\left(\frac{\log K}{\log 1/\epsilon} + 1\right)\log 1/\epsilon} + \frac{c}{(\log 1/\epsilon)^3\left(\frac{\log K}{\log 1/\epsilon} + 1\right)^3}\right]$$

for $\epsilon \to 0$ then $\quad \frac{a}{\log 1/\epsilon}\left[1 - \frac{\log K}{\log 1/\epsilon} + \left(\frac{\log K}{\log 1/\epsilon}\right)^2 + \cdots\right] + \frac{c}{(\log 1/\epsilon)^3}\left(1 - \frac{3\log K}{\log 1/\epsilon} + \cdots\right)$

$\frac{a(\log K)^3}{(\log 1/\epsilon)^4} - \frac{3c\log K}{(\log 1/\epsilon)^4}$ $\quad \Rightarrow \frac{a}{\log 1/\epsilon} - \frac{a\log K}{(\log 1/\epsilon)^2} + \frac{a(\log K)^2 + C}{(\log 1/\epsilon)^3}$ $\quad + C = C$

$\Rightarrow -a\log K = b \qquad c' + a(\log K)^2 = C \quad \Rightarrow c' - b\log K = c$

$c' = c + b\log K$

$\log K/\epsilon - \log K = \log 1/\epsilon$

$$\frac{a}{\log K/\epsilon\left[1 - \frac{\log K}{\log K/\epsilon}\right]} + \frac{b}{\quad} + \frac{c}{(\log K/\epsilon)^3\left(1 - \frac{\log K}{\log K/\epsilon}\right)^3}$$

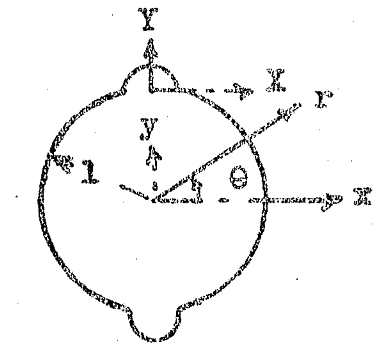
$$\frac{a}{\log K/\epsilon}\left[1 + \frac{\log K}{\log K/\epsilon} + \left(\frac{\log K}{\log K/\epsilon}\right)^2 + \cdots\right] + \frac{b}{(\log K/\epsilon)^2} + \frac{c}{(\log K/\epsilon)^3}\left[1 + \frac{3\log K}{\log K/\epsilon} + \cdots\right.$$

Solutions to Exercise Set 7

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

Friday 1 June 1979

7.1. Potential flow past circle with small semi-circular bumps. This is clearly a singular perturbation problem, and one that can be solved by the method of matched asymptotic expansions.



The global coordinates are the r and θ of the problem — or their Cartesian counterparts, which are easier to use; the local coordinates are (for the upper bump) centered at the top of the big circle and magnified by the ratio $1/\varepsilon$ of large to small radii:

$$X = \frac{x}{\varepsilon}, \qquad Y = \frac{y-1}{\varepsilon}.$$

We can calculate the maximum speed without any computation. We know that the global solution gives a horizontal velocity of speed $2U$ near the top. It is clear that the small bump sees this as a uniform stream far away. On its scale the big circle is flat, so that by symmetry the local problem is — like the global one — just that of a circle in a uniform stream. The only differences are in the scale, and a doubled speed far away. Then just as the big circle doubles the free-stream speed of U to a maximum of $2U$ at its top, so the little circle doubles its effective free-stream speed of $2U$ to a maximum of $4U$ at its top. More formally, we have

1-term global expansion: $\phi = U\left(r - \frac{1}{r}\right)\sin\theta = U\left(y - \frac{y}{x^2+y^2}\right)$

rewritten in local variables: $= U\left[(1+\varepsilon Y) - \frac{1+\varepsilon Y}{(1+\varepsilon Y)^2 + \varepsilon^2 X^2}\right]$

1-term local expansion: $= 2\varepsilon UY$

1-term local expansion: $\phi = C\left(Y - \frac{Y}{X^2+Y^2}\right)$

rewritten in global vars: $= C\left(\frac{y-1}{\varepsilon} - \varepsilon\frac{y-1}{x^2+(y-1)^2}\right)$

1-term global expansion: $= C\,\frac{y-1}{\varepsilon}$

rewritten in local variables for comparison $= CY$

Matching gives $C = 2\varepsilon U$; and then the maximum speed is found to be $4U$, in accord with our intuitive argument.

Mr. Human points out that we could put a still much smaller bump on top of the first to double the free-stream speed again to $8U$, and so on. In the limit we would have a sharp convex edge, which is known to give infinite local speed in potential flow.

Likewise, in the elasticity problem the big hole triples the stress to $3S$, and the small semi-circular holes triple it again to $9S$.

7.2. Matching in the purely logarithmic case: sliding rod. Rewriting the Oseen expansion in Stokes variables, expanding for small τ, and collecting terms gives

$$\text{Stokes of (Oseen)} = \begin{array}{c|ccc|c} & m=0 & 1 & 2 & \\ \hline & 0 & & & n=0 \\ & 1 & -\dfrac{\log R+\gamma/2}{\log 1/\tau} & +\dfrac{0}{(\log 1/\tau)^2} & n=1 \end{array}$$

Performing the reverse operation on the Stokes expansion gives

$$\text{Oseen of (Stokes)} = \begin{array}{c|cc|c} & n=0 & 1 & \\ \hline & 1 & +\ 0 & m=0 \\ & (1+a) & +\dfrac{a\log\rho}{\log 1/\tau} & m=1 \\ & (1+a) & +\dfrac{a\log\rho+b}{\log 1/\tau} & m=2 \end{array}$$

Hence the asymptotic matching principle gives, with everything written in Stokes variables for comparison

m	n	Matching	Result
0	0	$0 = 1$	False: forbidden region
0	1	$1 = 1$	True: but trivial
1	0	$0 = 1+a$	True: useful, gives $a = -1$
1	1	$1-\dfrac{\log R+\gamma/2}{\log 1/\tau} = 1+\dfrac{a\log R}{\log 1/\tau}$	False: forbidden region
2	0	$0 = 1+a$	True: but not useful at this stage
2	1	$1-\dfrac{\log R+\gamma/2}{\log 1/\tau} = 1+\dfrac{a\log R+b}{\log 1/\tau}$	True: useful, gives $b = -\gamma/2$

Thus the forbidden region appears to be the diagonal line $m = n$.

We can telescope three terms of the Stokes expansion into two, as Kaplun did for the drag of a circle at low Reynolds number:

$$w = 1-\left[\frac{1}{\log 1/\tau}-\frac{\gamma/2}{(\log 1/\tau)^2}+O\left\{\frac{1}{(\log 1/\tau)^3}\right\}\right]\log R$$

$$1-\left[\frac{1}{\log 1/\tau-\gamma/2}+O\left\{\frac{1}{(\log 1/\tau-\gamma/2)^3}\right\}\right]\log R\,.$$

We cannot form a composite expansion with $m = n$, because then the Stokes and Oseen expansions do not match. Forming the additive composite using our maximum information, with $m = 2$ and $n = 1$, gives the Stokes approximation plus a single higher-order correction:

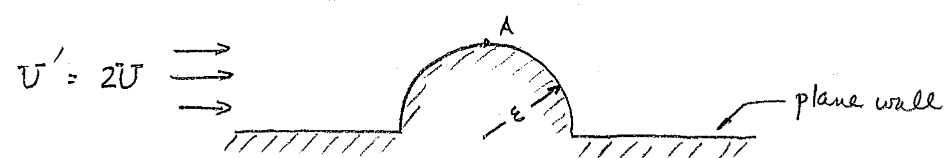
$$w \sim \frac{1}{\log 1/\tau}\,\frac{1}{2}E_1(\rho^2)+\frac{\gamma}{2}\,\frac{\log\rho}{(\log 1/\tau)^2}$$

Cesar Levy
ME 207
Prof Van Dyke

Problem Set #7

7.1 Intuitively find the maximum value of the velocity on the surface of a circle with small semicircular bumps if the upstream velocity is U.

It is well known that maximum velocity occurs on the north and south poles of a circle if the flow travels in the east-west direction and has a value of $2U$. If there are small bumps at the north and south poles I expect that at their north/south poles a velocity of $4U$ will be observed. Here's why: In the global point, to a first approx, the velocity near the bumps will be on the order of $2U$. On the local scale the flow will look like this:



Since at the wall the no slip condition is not satisfied, we can look at the analogous problem of a circle in a fluid with incoming velocity U'. The maximum velocity on the circle occurs at A and has a value of $2U' = 4U$. Thus at A the velocity is $4U$ ✓

For the case of a plate under uniform tension we could use the same argument and show that near the bump the plate experiences a uniform tension of $S' = 3S$ and hence at point A one would expect a maximum $\sigma_{\theta\theta} = 3S' = 9S$. ✓

7.2 Given the Stokes and Oseen expansions for a suddenly accelerated rod in a viscous fluid perform the matching for orders of $\log 1/\epsilon$: $n = 0,1$ (Oseen) and $m = 0,1,2$ (Stokes)

Stokes expansion is $w = 1 + \left[\frac{a}{\log 1/\epsilon} + \frac{b}{(\log 1/\epsilon)^2} + \frac{c}{(\log 1/\epsilon)^3} + \cdots\right] \log R$

Oseen expansion is $w = \frac{1}{\log 1/\epsilon} \cdot \frac{1}{2}\left\{-(\log \rho^2 + \gamma) + O(\rho^2)\right\} = 1 - \frac{\log R + \gamma/2}{\log 1/\epsilon} + O\left(\frac{1}{\log 1/\epsilon}\right)^2$ in Stokes coordinates

Stokes $n=0$ Oseen $m=0$

Stokes for $n=0$	$w=1$		Oseen $m=0$	$w=0$
Expanded in Oseen var	$w=1$		Expanded in Stokes var	$w=0$
take $m=0$ terms	$w=1$		take $n=0$	$w=0$

Rewrite in Oseen var $w=1$ Here we see no matching ✓

Stokes $n=0$ Oseen $m=1$

Stokes for $n=0$ $\quad w=1$

Expanded in Oseen var $\quad w=1$

take $m=1$ terms $\quad w=1$

Oseen $m=1$ $\quad w = \frac{-\frac{1}{2}(\log \rho^2 + \gamma)}{\log 1/\varepsilon}$

Expand in Stokes var $\quad w = 1 - (\log R + \gamma/2)/\log 1/\varepsilon$

take $n=0$ terms $\quad w=1$

Rewrite in Oseen var $\quad w=1$ $\quad$ Here we see matching ✓

Stokes $n=1$ Oseen $m=0$

Stokes for $n=1$ $\quad w = 1 + \frac{a \log R}{\log 1/\varepsilon}$

Expand in Oseen var $\quad w = (1+a) + a\log\rho/\log 1/\varepsilon$

take $m=0$ terms $\quad w = (1+a)$

Oseen $m=0$ $\quad w=0$

Expand in Stokes var $\quad w=0$

take $n=1$ $\quad w=0$

Rewrite in Oseen var $\quad w = 1+a$ $\quad$ Here matching occurs when $a=-1$ ✓

Stokes $n=1$ Oseen $m=1$

Stokes $n=1$ $\quad w = 1 + a\log R/\log 1/\varepsilon$

Expand in Oseen var $\quad w = (1+a) + a\log\rho/\log 1/\varepsilon$

take $m=1$ terms $\quad w = (1+a) + a\log\rho/\log 1/\varepsilon$

Oseen $m=1$ $\quad w = -\frac{1}{2}(\log\rho^2+\gamma)/\log 1/\varepsilon$

Expand in Stokes var $\quad w = 1 - \left[1 - \frac{\log\rho + \gamma/2}{\log 1/\varepsilon}\right]$

take $n=1$ terms $\quad w = -(\log\rho + \gamma/2)/\log 1/\varepsilon$

Rewrite in Oseen var $\Rightarrow$ $1+a=0$ and $\frac{\gamma/2}{\log 1/\varepsilon} = 0$ impossible to match ✓

Stokes $n=2$ Oseen $m=0$

Stokes $n=2$ $\quad w = 1 + \left[\frac{a}{\log 1/\varepsilon} + \frac{b}{(\log 1/\varepsilon)^2}\right]\log R$

Expand in Oseen var $\quad w = 1 + \left[\frac{a}{\log 1/\varepsilon} + \frac{b}{(\log 1/\varepsilon)^2}\right](\log\rho + \log 1/\varepsilon)$

take $m=0$ $\quad w = 1+a$

Oseen $m=0$ $\quad w=0$

Expand in Stokes var $\quad w=0$

take $n=2$ terms $\quad w=0$

Rewrite in Oseen var $\Rightarrow$ $1+a=0$ or $a=-1$ ✓

Stokes $n=2$ Oseen $m=1$

Stokes $n=2$ $\quad w = 1 + \left[\frac{a}{\log 1/\varepsilon} + \frac{b}{(\log 1/\varepsilon)^2}\right]\log R$

Expand in Oseen var $\quad w = (1+a) + \frac{a\log\rho + b}{\log 1/\varepsilon} + \dots$

takes $m=1$ $\quad w = (1+a) + \frac{a\log\rho + b}{\log 1/\varepsilon}$

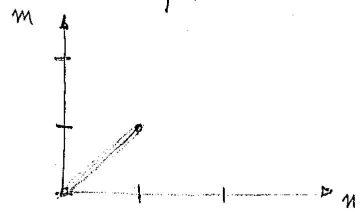
Oseen $m=1$ $\quad w = -\frac{1}{2}(\log\rho^2+\gamma)/\log 1/\varepsilon$

write in Stokes var $\quad w = -\frac{\log\rho + \gamma/2}{\log 1/\varepsilon}$

take $n=2$ $\quad w = -(\log\rho + \gamma/2)/\log 1/\varepsilon$

Rewrite in Oseen var $\Rightarrow$ $a=-1$ $\quad b=-\gamma/2$ for matching ✓

the two non matching points are $n=0, m=0$ $m=1, n=1$ and the forbidden region is as shown



✓

b. Since the Stokes expansion is

$$W = 1 + \left[\frac{a}{\log 1/\epsilon} + \frac{b}{(\log 1/\epsilon)^2} + \frac{c}{(\log 1/\epsilon)^3}\right]\log R = 1 + \left[\frac{a'}{\log K/\epsilon} + \frac{c'}{(\log K/\epsilon)^3} + \cdots\right]\log R$$

$$= 1 + \left\{\frac{a'}{\log 1/\epsilon}\left[1 - \frac{\log K}{\log 1/\epsilon} + \left(\frac{\log K}{\log 1/\epsilon}\right)^2 \cdots\right] + \frac{c'}{(\log 1/\epsilon)^3}\left[1 - \frac{3\log K}{\log 1/\epsilon} + \cdots\right]\right\}\log R$$

using the power series expansion

thus $a' = a$ and $-a\log K = b$ also $c' + a(\log K)^2 = c$; therefore

$a' = a = -1$ $K = e^{-b/a} = e^{-\gamma/2}$ $c' = c + (-\gamma/2)^2$ ✓

c. In the $n=2, m=1$ case the composite = stokes expansion and we thus lose the oseen solution far away from the rod. Because of the non matchability of two of the values this case might be one that follows Frankel's first point.

A

ME 207 / HW # 8

5.1 $F(\varepsilon x)\, u_{xx} + F_x(\varepsilon x)\, u_x - F(\varepsilon x)\, u_{tt} = 0$

i.e. $u_{xx} - u_{tt} + \frac{F_x(\varepsilon x)}{F(\varepsilon x)} u_x = 0$ (1)

B.C. $u(0,t) = \cos t$ (2)

Let short space scale be $\xi = x$
& long " " " $X = \varepsilon x$ } $\Rightarrow u(x,t) = u(X,\xi,t)$ ✓

$$\therefore \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \varepsilon \frac{\partial}{\partial X}$$ ✓

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2\varepsilon \frac{\partial^2}{\partial \xi \partial X} + \varepsilon^2 \frac{\partial^2}{\partial X^2}$$ ✓

(1) $\Rightarrow$

$$u_{\xi\xi} + 2\varepsilon u_{\xi X} + \varepsilon^2 u_{XX} - u_{tt} + \frac{\varepsilon F'}{F}(u_\xi + \varepsilon u_X) = 0 \; ; \quad F' \equiv F_X(X)$$

i.e. $u_{\xi\xi} - u_{tt} + \varepsilon\left(2u_{\xi X} + \frac{F'}{F}u_\xi\right) + \varepsilon^2\left(u_{XX} + \frac{F'}{F}u_X\right) = 0$ (3) ✓

Let $u(X,\xi,t) = u^1(X,\xi,t) + \varepsilon u^2(X,\xi,t) + \varepsilon^2 u^3(X,\xi,t) + \ldots$ (4)

(4) in (3) $\Rightarrow$

$$u^1_{\xi\xi} - u^1_{tt} + \varepsilon(u^2_{\xi\xi} - u^2_{tt}) + \varepsilon^2(u^3_{\xi\xi} - u^3_{tt}) + \varepsilon\left[2u^1_{\xi X} + \frac{F'}{F}u^1_\xi + \varepsilon\left(2u^2_{\xi X} + \frac{F'}{F}u^2_\xi\right)\right.$$

$$\left. + \varepsilon^2\left(2u^3_{\xi X} + \frac{F'}{F}u^3_\xi\right)\right] + \varepsilon^2\left[u^1_{XX} + \frac{F'}{F}u^1_X + \varepsilon\left(u^2_{XX} + \frac{F'}{F}u^2_X\right) + \varepsilon^2\left(u^3_{XX} + \frac{F'}{F}u^3_X\right)\right] = 0$$

Equating like powers of ε,

ε^0: $u^1_{\xi\xi} - u^1_{tt} = 0$
(2,4) $\Rightarrow u^1(0,0,t) = \cos t$ } (5)

ε: $u^2_{\xi\xi} - u^2_{tt} = -\left[2u^1_{\xi X} + \frac{F'}{F}u^1_\xi\right]$
(2,4) $\Rightarrow u^2(0,0,t) = 0$ } (6)

ε^2: $u^3_{\xi\xi} - u^3_{tt} = -\left[2u^2_{\xi X} + \frac{F'}{F}u^2_\xi + u^1_{XX} + \frac{F'}{F}u^1_X\right]$
(2,4) $\Rightarrow u^3(0,0,t) = 0$ } (7)

First Approximation

(5) $\Rightarrow u^1(X,\xi,t) = A(X)\cos(t-\xi)$ (8)

where $A(0) = 1$ and A is obtained by requiring the R.H.S. of (6) to vanish to insure no secular terms in the second approximation

i.e. $2u^1_{\xi X} + \frac{F'}{F}u^1_\xi = 0$

so, $2A' + \frac{F'}{F}A = 0$ or $\frac{F'}{F} = -2\frac{A'}{A} \Rightarrow \log F = -2\log A + \text{Const.}$ ✓ 1

or in general $u'(x,\xi,t) = \sqrt{\frac{F(0)}{F(\xi)}} \cos(t-\xi)$

$$\therefore A(X) = \frac{1}{\sqrt{F(X)}} \underset{(8)}{\Rightarrow} u'(X,\xi,t) = \frac{\cos(t-\xi)}{\sqrt{F(X)}} \qquad (9)$$

This is the approximation based on energy conservation. ✓

Second Approximation

$(6) \Rightarrow \quad u^2(X,\xi,t) = B(X) f(t-\xi)$

where f is periodic (sine or cosine), $B(0)=0$, and $B(X)$ is obtained by requiring, in (7), that:

$$2u^2_{\xi X} + \frac{F'}{F}u^2_{\xi} + u^1_{XX} + \frac{F'}{F}u^1_X = 0 \qquad (10)$$

i.e. $-2B'f' - \frac{F'}{F}Bf' + \left[\left(\frac{1}{\sqrt{F}}\right)'' + \frac{F'}{F}\left(\frac{1}{\sqrt{F}}\right)'\right]\cos(t-\xi) = 0$

$$\left[\frac{F'}{F}B + 2B'\right]f'(t-\xi) = \left[\frac{F'}{F}\left(\frac{1}{\sqrt{F}}\right)' + \left(\frac{1}{\sqrt{F}}\right)''\right]\cos(t-\xi)$$

By comparison,

$$f(t-\xi) = \sin(t-\xi)$$

and for $F = 1+X$,

$$\frac{1}{1+X}B + 2B' = \frac{1}{4}\frac{1}{(1+X)^{5/2}} \quad \text{i.e.} \quad B' + \frac{1}{2(1+X)}B = \frac{1}{8(1+X)^{5/2}} \qquad (11)$$

Integrating factor is $e^{\frac{1}{2}\int\frac{1}{1+X}dX} = (1+X)^{\frac{1}{2}}$

$$\therefore \left(B(1+X)^{\frac{1}{2}}\right)' = \frac{1}{8(1+X)^2}$$

$$B(1+X)^{\frac{1}{2}} = -\frac{1}{8(1+X)} + C, \quad \text{where } B(0)=0 \Rightarrow C = \frac{1}{8}$$

$$B(1+X)^{\frac{1}{2}} = \frac{1}{8}\left[1 - \frac{1}{1+X}\right] = \frac{X}{8(1+X)}$$

$$B(X) = \frac{X}{8(1+X)^{\frac{3}{2}}}$$

So, $u^2 = \dfrac{X\sin(t-\xi)}{8(1+X)^{\frac{3}{2}}}$.

and $$u = \frac{1}{(1+X)^{\frac{1}{2}}}\left\{\cos(\xi-t) - \varepsilon\frac{X\sin(\xi-t)}{8(1+X)}\right\} + O(\varepsilon^2)$$

$$= \frac{1}{(1+\varepsilon x)^{\frac{1}{2}}}\left\{\cos(x-t) - \varepsilon^2\frac{x\sin(x-t)}{8(1+\varepsilon x)}\right\} + O(\varepsilon^2)$$ ✓

(*) a rather contrived choice, justifiable in that it seems to give the required result

Using Nayfeh's scheme, $\begin{cases} \xi = x + \varepsilon g(\varepsilon x) + O(\varepsilon^2) \stackrel{?}{=} x + \varepsilon g(X) + O(\varepsilon^2) \\ X = \varepsilon x \end{cases}$

$$\frac{\partial}{\partial x} = (1+\varepsilon^2 g')\frac{\partial}{\partial \xi} + \varepsilon\frac{\partial}{\partial X} \quad ✓$$

$$\frac{\partial^2}{\partial x^2} = (1+\varepsilon^2 g')^2\frac{\partial^2}{\partial \xi^2} + 2\varepsilon(1+\varepsilon^2 g')\frac{\partial^2}{\partial\xi\partial X} + \varepsilon^2\frac{\partial^2}{\partial X^2}$$

note that $g' = g'(X)$ implying ξ, X not independent ?

(1) ⇒

$$(1+\varepsilon^2 g')^2 u_{\xi\xi} + 2\varepsilon(1+\varepsilon^2 g')u_{\xi X} + \varepsilon^2 u_{XX} - u_{tt} + \frac{\varepsilon F'}{F}\left[(1+\varepsilon^2 g')u_\xi + \varepsilon u_X\right] = 0$$

i.e.

$$u_{\xi\xi} - u_{tt} + \varepsilon\left[2u_{\xi X} + \frac{F'}{F}u_\xi\right] + \varepsilon^2\left[2g'u_{\xi\xi} + u_{XX} + \frac{F'}{F}u_X\right] + O(\varepsilon^3) = 0 \qquad (3a)$$

Substituting (4) in (3a),

$$u^1_{\xi\xi} - u^1_{tt} + \varepsilon(u^2_{\xi\xi} - u^2_{tt}) + \varepsilon^2(u^3_{\xi\xi} - u^3_{tt}) + \varepsilon\left[2u^1_{\xi X} + \frac{F'}{F}u^1_\xi + \varepsilon\left(2u^2_{\xi X} + \frac{F'}{F}u^2_\xi\right) + \ldots\right]$$
$$+ \varepsilon^2\left[2g'u^1_{\xi\xi} + u^1_{XX} + \frac{F'}{F}u^1_X + \ldots\right] + O(\varepsilon^3) = 0$$

i.e.

$$u^1_{\xi\xi} - u^1_{tt} = 0$$
$$u^2_{\xi\xi} - u^2_{tt} = -\left[2u^1_{\xi X} + \frac{F'}{F}u^1_\xi\right] \quad ✓$$
$$u^3_{\xi\xi} - u^3_{tt} = -\left[2u^2_{\xi X} + \frac{F'}{F}u^2_\xi + \underbrace{2g'u^1_{\xi\xi}}_{\text{not in (10)}} + u^1_{XX} + \frac{F'}{F}u^1_X\right] \quad ✓$$

As before, $u^1 = \frac{\cos(t-\xi)}{\sqrt{F(X)}}$.

And assuming $u^2 = B(X)\sin(t-\xi)$, and $F = 1+X$, we get, corresponding to (11):

$$B' + \frac{1}{2(1+X)}B = \frac{1}{8(1+X)^{5/2}} - \frac{g'}{(1+X)^{\frac{1}{2}}} \qquad (12)$$

(*)

← Require that R.H.S. vanishes, in order to insure a decaying solution for B

$$\therefore\ g' = \frac{1}{8}\frac{1}{(1+X)^2} \Rightarrow \boxed{g = \frac{X}{8(1+X)}} \qquad (13)$$

$$(12)\ \&\ (13) \Rightarrow B' = -\frac{1}{2(1+X)}B \Rightarrow B = \frac{1}{(1+X)^2} + C$$

$$B(0) = 0 \Rightarrow C = -1$$

$$B = \frac{1}{(1+X)^2} - 1 = -\frac{X(2+X)}{(1+X)^2} = -X(2+X)(1-2X+$$
$$= -X(2-3X+\ldots) = -2X + O(X^2)$$

$$\therefore\ u^2 = -\left[2X + O(X^2)\right]\sin(t-\xi) = \left[2X + O(X^2)\right]\sin(\xi - t)$$

$$u = \frac{\cos(\xi - t)}{(1+X)^{\frac{1}{2}}}\left\{1 + \varepsilon\left[2X + O(X^2)\right]\tan(\xi - t)\right\}$$

$$= \frac{\cos(\xi - t)}{(1+X)^{\frac{1}{2}}}\left\{1 + O(\varepsilon^2)\right\}$$

✓

$$= \frac{1}{(1+\varepsilon x)^{\frac{1}{2}}}\cos\left[x + \frac{\varepsilon^2 x}{8(1+\varepsilon x)} - t + O(\varepsilon^2)\right] + O(\varepsilon^2)$$

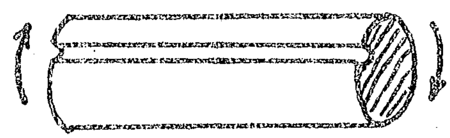
Final Examination

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

6 - 11 June 1979

Instructions: This is an open-book examination, conducted under the Honor Code. Use any convenient paper. Spend not more than six hours (in several sessions if desired), and record your total time. Return to "Van Dyke" mailbox on second floor of Durand building, not later than Monday afternoon June 11. Give campus address for return of examination.

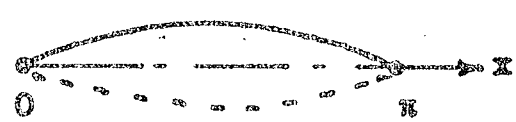
1. Torsion of shaft with keyway. A long circular shaft in torsion has a semi-circular keyway cut along its length. On the assumption that the radius of the keyway is very small compared with the radius of the shaft, calculate a first approximation for the stress-concentration factor (the ratio of the maximum stress with and without the keyway).



(Alternatively, this is the problem of finding how the maximum slope changes for a soap film stretched across a hole having the cross-section of the shaft, and subjected to a slight pressure difference; or the problem of how the skin friction changes for steady laminar flow through a long cylindrical pipe having the cross-section of the shaft.)

2. Torsion of slightly damped string. In dimensionless form, the vibration of a string stretched between $x = 0$ and $x = \pi$ is represented by a nontrivial solution of the one-dimensional wave equation

$$\frac{d^2y}{dt^2} - \frac{d^2y}{dx^2} = 0$$



subject to the boundary conditions

$$y = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \pi$$

and, say, the initial condition

$$y = 0 \quad \text{at} \quad t = 0.$$

Separation of variables shows that the fundamental mode (the first eigensolution), which is sketched, has unit frequency, being any multiple of

$$y = \sin x \sin t .$$

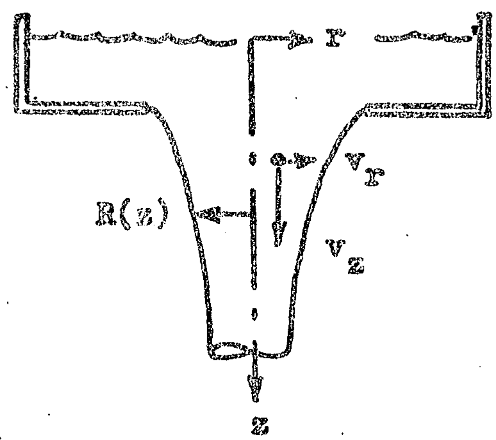
Now consider the situation when the string is immersed in a viscous fluid, so that its motion is governed by the damped wave equation

$$\frac{d^2y}{dt^2} + 2\varepsilon\frac{dy}{dt} - \frac{d^2y}{dx^2} = 0.$$

First find a straightforward correction to the above solution on the assumption that the damping is so slight that the square of ε is negligible. Second, observing that your result breaks down for large times, calculate a refined first correction for the damping that remains valid at those times.

3. Falling jet of water. Water flows under gravity at rate Q through a circular hole in the bottom of a large vessel, forming a steady axisymmetric jet. It is convenient to describe the downward and radial velocity components v_z and v_r in terms of the Stokes stream function according to

$$v_z = \frac{\phi_r}{r}, \quad \sout{v_r = \frac{-\phi_r}{r}} \quad v_r = -\frac{\phi_z}{r}$$



Then ϕ satisfies the differential equation

$$\phi_{rr} - \frac{\phi_r}{r} + \phi_{zz} = 0.$$

The local radius $R(z)$ of the jet will be determined by the pair of conditions that the volumetric flow rate across any horizontal section is constant; if we take $\phi = 0$ at $r = 0$:

$$\int_{r=0}^{R(z)} 2\pi r v_z \, dr = 2\pi \int_{r=0}^{R(z)} \phi_r \, dr = 2\pi \phi\Big|_{r=R(z)} = Q;$$

and that the pressure p, as calculated from Bernoulli's equation in terms of the density ρ and speed, is equal to the atmospheric pressure p_a at the surface of the jet:

$$\frac{p - p_a}{\rho} = gz - \tfrac{1}{2}(v_z^2 + v_r^2) = 0 \quad \text{at} \quad r = R(z).$$

This nonlinear free-boundary problem is far too difficult to solve exactly. However, we can find a simple first approximation at great distances z below the surface of the water in the vessel by recognizing that there the radius R is decreasing so slowly that the velocity is practically vertical, and hence uniform across the jet. Calculate, according to this first approximation, the stream function ϕ and the radius R of the jet, showing that $R(z)$ varies inversely as the fourth root of z. Improve on that result in a systematic way to find a second approximation for ϕ and $R(z)$ at large z.

7 June 1979

CORRECTION TO

Final Examination

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

In problem 3, the first pair of equations, the last term should have subscript z rather than r :

$$v_r = \frac{-\psi_z}{r}$$

(as in exercise 3.2 on page 53 of the notes). I'm very sorry!

Prof Van Dyke
ME 207
Cesar Levy
$5\frac{1}{2}$ - 6 hrs

Final Exam

1. If we define $\hat{\tau}_{\hat{z}\hat{x}} = \frac{\partial\hat{\phi}}{\partial\hat{y}}$ and $\tau_{\hat{y}\hat{z}} = -\frac{\partial\hat{\phi}}{\partial\hat{x}}$ then we find that from compatability that

$$\hat{\nabla}^2\hat{\phi} = -2G\alpha \quad \text{where} \quad \hat{\nabla}^2 = \frac{\partial^2}{\partial\hat{x}_i\partial\hat{x}_i} \quad i = 1, 2$$

where $\hat{x}, \hat{y}, \hat{z}$ are the physical coordinates.

The boundary conditions are that $\hat{\phi} = 0$ on the surface of the bar. If we let $x = \hat{x}/a$, $y = \hat{y}/a$, $\hat{z} = z$ and $\phi = 2G\alpha a^2 \hat{\phi}$ then the problem is non dimensionalized (with a = shaft radius) to

$$\nabla^2\phi = -1 \quad \text{and} \quad \phi = 0 \text{ on the surface}$$

The surface is defined as $x^2+y^2 = 1$ except at the key where the surface is defined by $x^2+(y-1)^2 = \varepsilon^2$. We pick the center of the keyway arbitrarily at $(0,1)$

If we solve the problem for the first global approximation ϕ_0 then $\nabla^2\phi_0 = -1$ and $\phi_0 = 0$ on $x^2+y^2 = 1$. This has the solution

$$\boxed{\phi_0 = \tfrac{1}{4}(1 - x^2 - y^2)} \checkmark \quad \text{if we assume } \phi_0 \text{ is axisymmetric}$$

If we solve the problem for the 2nd global approx ϕ_1 then

$\nabla^2\phi_1 = 0$ and $\phi_1 = 0$ on $x^2+y^2 = 1$. This has the axisymmetric solution $\boxed{\phi_1 = \text{constant}}$. Thus

$$\phi_{global} = \tfrac{1}{4}(1 - x^2 - y^2) + \varepsilon C$$

We note that this solution does not satisfy $\phi = 0$ on $x^2+(y-1)^2 = \varepsilon^2$

We now define local coordinates $X = \frac{x}{\varepsilon}$ $Y = (y-1)/\varepsilon$ ✓ but we will not magnify ϕ thus $\phi = \Phi$

Put this into PDE remembering that $\frac{\partial}{\partial x} = \frac{1}{\varepsilon}\frac{\partial}{\partial X}$; $\frac{\partial}{\partial y} = \frac{1}{\varepsilon}\frac{\partial}{\partial Y}$

thus we obtain defining $\bar{\nabla}^2 = \frac{\partial^2}{\partial X_i \partial X_i}$

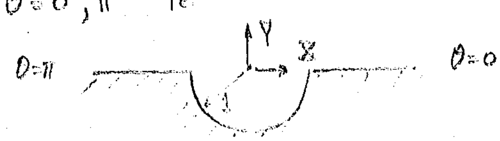
$\frac{1}{\varepsilon^2}\bar{\nabla}^2\Phi = -1$ or $(\bar{\nabla}^2\Phi = -\varepsilon^2)$ with bc that if $X^2+Y^2 = 1$ then $\Phi = 0$ $-1 \le Y \le 0$

defining $\Phi = \Phi_0(X,Y) + \varepsilon^2\Phi_1(X,Y) + O(\varepsilon^4)$

the first local approximation is found by solving $\nabla^2\Phi_0 = 0$ and $\Phi_0 = 0$ on the surface. The most general solution is (for n integer)

$$\Phi_0 = \sum_{n=1}^{\infty} C_n\left(R^n - \frac{1}{R^n}\right)\sin n\theta + D_n\left(R^n - \frac{1}{R^n}\right)\cos n\theta + A_0 R + B_0 + C_0\theta + D_0$$

we note that in the local approximation the circle $x^2+y^2=1$ appears as a flat surface with $\theta = 0, \pi$ ie



This effectively gives that $D_n = 0$ $\forall n$ and $A_0 = B_0 = C_0 = D_0 = 0$

thus $\Phi \sim c_1 \left(R - \frac{1}{R}\right) \sin\theta$ since the far boundary condition is $\sim 2\varepsilon Y$ ✓

Now we begin by matching

Global $\quad \phi = \frac{1}{4}(1 - x^2 - y^2)$

Rewritten in local terms $\quad \Phi = \frac{1}{4}\left(1 - \varepsilon^2 X^2 - (1+\varepsilon Y)^2\right) = -\frac{\varepsilon Y}{2} - \varepsilon^2 (X^2 + Y^2)$

take 1 term expansion $\quad \Phi \sim -\frac{\varepsilon Y}{2}$

Local $\quad \Phi = c_1\left(R - \frac{1}{R}\right)\sin\theta$

Rewritten in global terms $\quad \phi = c_1\left[\frac{(y-1)}{\varepsilon} - \frac{(y-1)\varepsilon}{x^2 + (y-1)^2}\right]$

take 1 term expansion $\quad \phi = c_1\left(\frac{y-1}{\varepsilon}\right)$

rewrite in local $\quad \Phi_0 = c_1 Y \quad \Rightarrow \quad \boxed{c_1 = -\frac{\varepsilon}{2}}$ ✓

The composite is the additive composite

$$\phi_{comp} \sim \frac{1}{4}(1 - x^2 - y^2) - \frac{\varepsilon}{2}\left(\frac{y-1}{\varepsilon} - \frac{\varepsilon(y-1)}{x^2 + (y-1)^2}\right) + \frac{\varepsilon}{2}\left(\frac{y-1}{\varepsilon}\right) + O(\varepsilon^2)$$

$$\frac{1}{4}(1 - x^2 - y^2) + \frac{\varepsilon^2}{2}\frac{y-1}{x^2 + (y-1)^2}$$

To find the stress intensity factor we must look at $\tau_{zx} = \frac{\partial\phi}{\partial y} = \frac{2G\alpha a^2}{a}\frac{\partial\hat{\phi}}{\partial\hat{y}} = 2G\alpha a \frac{\partial\hat{\phi}}{\partial\hat{y}}$

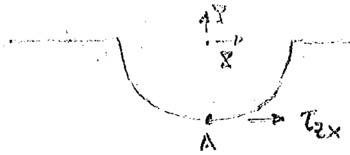
thus we can define $\tau_{zx} = \hat{\tau}_{\hat{z}\hat{x}}(2G\alpha a)$ and the ratio of τ_{zx} with keyway to τ_{zx} without keyway $= \hat{\tau}_{\hat{z}\hat{x}}$ with keyway $/\ \hat{\tau}_{\hat{z}\hat{x}}$ without keyway $= K$

Now τ_{zx} without keyway $= \frac{\partial\phi_{glob}}{\partial y} = -\frac{2}{4}y = -\frac{1}{2}y = -\frac{1}{2}[1+\varepsilon Y]$

τ_{zx} with keyway $= \frac{\partial\phi_{comp}}{\partial y} = -\frac{y}{2} + \frac{\varepsilon^2}{2}\left[\frac{1}{x^2+(y-1)^2} - \frac{2(y-1)^2}{[x^2+(y-1)^2]^2}\right]$

$$= -\frac{y}{2} + \frac{\varepsilon^2}{2}\left[\frac{x^2-(y-1)^2}{[x^2+(y-1)^2]^2}\right] = -\frac{1}{2}[1+\varepsilon Y] + \frac{1}{2}\left[\frac{X^2 - Y^2}{[X^2+Y^2]^2}\right]$$

We find that max. stresses can occur A i.e. $Y = -1 \quad X = 0$



then $\frac{\partial\phi}{\partial y}$ comp $= -\frac{1}{2}[1-\varepsilon] - \frac{1}{2} = -1 + \frac{\varepsilon}{2}$ $\quad$ and $\frac{\partial\phi}{\partial y}$ global $= -\frac{1}{2}[1-\varepsilon] = \frac{-1+\varepsilon}{2}$

Hence $K = \frac{\partial\phi}{\partial y}\text{comp} \Big/ \frac{\partial\phi}{\partial y}\text{global} = \frac{-2+\varepsilon}{-1+\varepsilon} = 2 + \varepsilon + O(\varepsilon^2)$ ✓

10

2. For $\varepsilon = 0$ the PDE results in $\frac{\partial^2 y_0}{\partial t^2} - \frac{\partial^2 y_0}{\partial x^2} = 0$ where y_0 is the 1st order approximation.

The boundary and initial conditions translate to $y_0(0,t) = y_0(\pi,t) = 0$; $y_0(x,0) = 0$

The solution to this problem is

$$y_0 = \sin x \sin t$$

to get the 2nd order approximation if we let $y(x,t;\varepsilon) = y_0 + \varepsilon y_1 + O(\varepsilon^2)$ into the pde and bc/ic's we get that

$$\frac{\partial^2 y_1}{\partial t^2} - \frac{\partial^2 y_1}{\partial x^2} = -2\frac{\partial y_0}{\partial t} = -2\cos t \sin x = -\left[\sin(x+t) + \sin(x-t)\right]$$

Letting $\xi = x-t$, $\eta = x+t$ then

$$\Box^2 y_1 = 4\frac{\partial^2 y_1}{\partial \xi \partial \eta} = -\sin\xi - \sin\eta$$

the general solution to this is

$$y_1 = \frac{1}{4}\left[\eta\cos\xi + \xi\cos\eta + f(\xi) + g(\eta)\right]$$

applying the bc

$y_1(0,t) = 0 \Rightarrow f(-t) + g(t) = 0$

$y_1(x,0) = 0 \Rightarrow f(x) + g(x) = -2x\cos x$

$y_1(\pi,t) = 0 \Rightarrow f(\pi-t) + g(\pi+t) = 2\pi\cos t$

we find that $f(x-t) = -(x-t)\cos(x-t)$

$g(x+t) = -(x+t)\cos(x+t)$

thus $y_1 = \frac{t}{2}\left[\cos(x-t) - \cos(x+t)\right]$ this hence brings in a secular term since

$$y = \sin x \sin t + \frac{\varepsilon t}{2}\left[\cos(x-t) - \cos(x+t)\right]$$ will blow up when $t \geq O\left(\frac{1}{\varepsilon}\right)$

Let us use multiple scales with $t_1 = t$ $t_2 = \varepsilon t$ then $y = y(t_1, t_2, x; \varepsilon)$

then $\frac{d}{dt} = \frac{d}{dt_1} + \varepsilon\frac{d}{dt_2} \Rightarrow \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t_1^2} + 2\varepsilon\frac{\partial^2}{\partial t_1 \partial t_2} + \varepsilon^2\frac{\partial^2}{\partial t_2^2}$

Now plug this into damped PDE

$$\therefore \frac{\partial^2 y}{\partial t_1^2} + 2\varepsilon\left[\frac{\partial^2 y}{\partial t_1 \partial t_2} + \frac{\partial y}{\partial t_1}\right] + \varepsilon^2\left[\frac{\partial^2 y}{\partial t_2^2} + 2\frac{\partial y}{\partial t_2}\right] - \frac{\partial^2 y}{\partial x^2} = 0$$

the bc/ic convert to $y(0,0,x;\varepsilon) = 0$ $y(t_1,t_2,0;\varepsilon) = 0$ $y(t_1,t_2,\pi;\varepsilon) = 0$

Let $y = y_0(t_1,t_2,x) + \varepsilon y_1(t_1,t_2,x) + O(\varepsilon^2)$ and put into PDE/BC/ic to get

$$\frac{\partial^2 y_0}{\partial t_1^2} - \frac{\partial^2 y_0}{\partial x^2} = 0 \qquad y_0(0,0,x) = y_0(t_1,t_2,0) = y_0(t_1,t_2,\pi) = 0$$

Letting $y_0(t_1, t_2, x) = T_0(t_1, t_2)\, X_0(x)$ and putting into PDE and applying BC we obtain

$$X_0 = \sin x \; ; \quad T_0 = A(t_2)\sin t_1 + B(t_2)\cos t_1$$

with $B(0) = 0$, $A(t_2)$ has no restrictions

To pin down $B(t_2)$ and $A(t_2)$ look at solution to the second order problem

ie $$\frac{\partial^2 y_1}{\partial t_1^2} - \frac{\partial^2 y_1}{\partial x^2} = -2\left[\frac{\partial^2 y_0}{\partial t_1 \partial t_2} + \frac{\partial y_0}{\partial t_1}\right] = -2\left[(A+A')\cos t_1 + (B+B')\sin t_1\right]\sin x$$

for the results to remain bounded (ie no secular terms) and to be valid for all x

then $A + A' = 0$ or $A = Ce^{-t_2}$

$B + B' = 0$ or $B = De^{-t_2}$

If we apply the restriction $B(0) = 0 \Rightarrow D = 0$ and if we normalize A so that $A(0) = 1$ then we find that

$$y_0 = e^{-t_2}\sin x \sin t_1 \quad \text{or} \quad y \sim e^{-\varepsilon t}\sin x \sin t$$

which is the 'refined' first correction

What is ε?

3. We will use the method of slow variation with ψ varying much faster in the r direction than in the z direction. Since for large z we expect nonuniformities we will reduce the z coordinate by $Z = \varepsilon z$. Putting this into the D.E.,

We now have to solve

$$\psi_{,rr} - \psi_{,r}/r + \varepsilon^2 \frac{\partial^2 \psi}{\partial Z^2} = 0 \quad \text{with} \quad \psi(R(z)) = \psi(\tilde{R}(Z)) = \frac{Q}{2\pi}$$

and $\psi = 0$ at $r = 0$

We now assume $\psi = \psi(r, Z; \varepsilon) \doteq \psi_0(r, Z) + \varepsilon^2 \psi_1(r, Z) + \dots$ and plug into PDE and BC's to get

ε^0: $\psi_{0,rr} - \psi_{0,r}/r = 0$; $\psi_0(R, Z) = \frac{Q}{2\pi}$; $\psi_0(0, Z) = 0$

ε^2: $\psi_{1,rr} - \psi_{1,r}/r + \psi_{0,ZZ} = 0$; $\psi_1(R, Z) = 0$; $\psi_1(0, Z) = 0$

ε^{2n}: $\psi_{n,rr} - \psi_{n,r}/r + \psi_{n-1,ZZ} = 0$; $\psi_n(R, Z) = 0$; $\psi_n(0, Z) = 0$ $\quad n \geq 1$

Solving the ε^0 problem we obtain that $\psi_0(r, Z) = c_1 r^2 + c_2$. Applying BC gives $c_1 = \frac{Q}{2\pi R^2}$ $\quad c_2 = 0$

$$\boxed{\psi_0(r, Z) = \frac{Q r^2}{2\pi R^2}}$$ ✓

now $v_r = -\psi_{0,z}/r = -\varepsilon \psi_{0,Z}/r = \frac{\varepsilon Q r R'(z)}{2\pi R^3}$ $\qquad v_r\Big|_{r=R} = \frac{\varepsilon Q R'}{2\pi R^2}$

$v_z = \psi_{0,r}/r = \frac{Q}{\pi R^2}$ $\qquad v_z\Big|_{r=R} = \frac{Q}{\pi R^2}$

plugging into the bernoulli equation and noting that $v_z \gg v_r$ especially for large z then $v_z^2 + v_r^2 \sim v_z^2$ ie for the ε^0 problem neglect any terms involving ε's. at $r = R$

$$0 = gz - \tfrac{1}{2}(v_z^2 + v_r^2) = gz - \tfrac{1}{2}\left(\frac{Q^2}{\pi^2 R^4} + \frac{\varepsilon^2 Q^2 R'^2}{4\pi^2 R^4}\right) \sim gz - \frac{Q^2}{2\pi^2 R^4}$$

or $\quad R \sim R_0(z) = \left[Q^2/2\pi^2 g z\right]^{1/4}$ as required ✓

Now to get the next step solve the ε^2 problem, ie:

$$\psi_{1,rr} - \psi_{1,r}/r = -\psi_{0,ZZ} = -\frac{Q r^2}{2\pi}(R_0^{-2})''$$ ✓

with $\psi_1(R, Z) = 0$ and $\psi_1(0, Z) = 0$

Letting $\psi_1 = \Psi(R) F(Z)$ we find that $F(Z) = -\frac{Q}{2\pi}(R_0^{-2})''$

and $\Psi = r^4/8 + c_3 r^2/2 + c_4$. Putting in BC $c_4 = 0$ $\quad c_3 = -R^2/4$

or $\Psi = \frac{r^2}{8}(r^2 - R^2)$

or $\psi_1 = \Psi \cdot F = -\frac{Q}{16\pi}(R_0^{-2})'' r^2(r^2 - R^2)$

and hence

$$\psi \approx \psi_0 + \varepsilon^2 \psi_1 = \frac{Q}{2\pi} r^2 \left\{ 1 - \frac{\varepsilon^2 (R_0^{-2})''}{8}(r^2 - R^2) \right\}$$

↑ R^2

thus $V_z = \psi_{,r}/r = \frac{Q}{\pi}\left\{ \frac{1}{R^2} - \frac{\varepsilon^2}{8}(R_0^{-2})''(2r^2 - R^2) \right\}$ ✓

$$V_r = -\psi_{,z}/r = -\varepsilon \psi_{,Z}/r = \frac{\varepsilon Q}{\pi}\left\{ \frac{rR'}{R^3} + \frac{\varepsilon^2}{16}\left[(R_0)''' r(r^2 - R^2) - (R_0^{-2})'' 2rRR' \right] \right\}$$

evaluating at $r = R$

$$V_z = \frac{Q}{\pi R^2} - \frac{Q\varepsilon^2}{8\pi}(R_0^{-2})'' \cdot R^2$$

$$V_r = \frac{\varepsilon Q}{\pi}\left\{ \frac{R'}{R^2} - \frac{\varepsilon^2}{8}(R_0^{-2})'' R^2 R' \right\}$$

and $V_z^2 + V_r^2 = \frac{Q^2}{\pi^2 R^4} - \frac{Q^2 \varepsilon^2}{4\pi^2}(R_0^{-2})'' + \frac{\varepsilon^2 Q^2}{\pi^2}\frac{R'^2}{R^4} + O(\varepsilon^4)$

Since $R_0 = \left(\frac{Q^2}{2\pi^2 g z}\right)^{1/4}$ then $\varepsilon^2 (R_0^{-2})'' = -\frac{1}{4}\left(\frac{2\pi^2 g}{Q^2 z^3}\right)^{1/2}$

or $V_z^2 + V_r^2 = \frac{Q^2}{\pi^2 R^4} + \frac{Q^2}{16\pi^2}\left(\frac{2\pi^2 g}{Q^2 z^3}\right)^{1/2} + \frac{\varepsilon^2 Q^2 R'^2}{\pi^2 R^4}$

for very large z R' is small and the ε^2 term is much smaller than the rest so we can drop it in comparison to the other two

thus $gz - \frac{1}{2}(V_z^2 + V_r^2) \cong gz - \frac{Q^2}{2\pi^2 R^4} - \frac{Q^2}{32\pi^2}\cdot\frac{\pi}{Q}\left(\frac{2g}{z^3}\right)^{1/2} = 0$

thus $\left[gz - \frac{Q}{32\pi}\left(\frac{2g}{z^3}\right)^{1/2}\right] \cdot \frac{2\pi^2}{Q^2} \cong R^{-4}$

or $R = R_1 = \left[\frac{Q^2}{2\pi^2} \frac{1}{gz - \frac{Q}{32\pi}\left(\frac{2g}{z^3}\right)^{1/2}} \right]^{1/4}$ (can expand out; I then get twice your second term.)

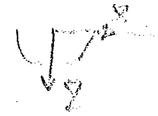
to continue: go to the ε^4 problem and use R_1 in the $\psi_{1,zz}$ term solve as we did and look at the terms in the Bernoulli equation keeping only the largest terms when looking at large values of z.

10

10 + 10 + 10 = 30/30 (A)

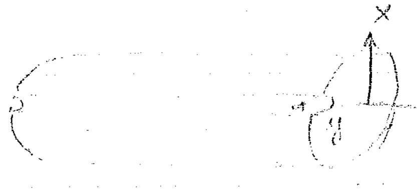
$\varepsilon^2 X^2 + (\varepsilon Y + 1)^2 = 1$

$\varepsilon^2 (X^2 + Y^2) + 2\varepsilon Y = 0$



$\frac{\partial \hat\phi}{\partial \hat x^2} + \frac{\partial \hat\phi}{\partial \hat y^2} = -2G\alpha$

$\frac{1}{a^2}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\hat\phi = -2G\alpha$



If we define the stress fn $\tau_{zx} = \frac{\partial\phi}{\partial y}$; $\tau_{yz} = -\frac{\partial\phi}{\partial x}$ which satisfies $\nabla^2\hat\phi = -2G\alpha$ then by redefining $y = \hat y/a$, $x = \hat x/a$, $\hat\phi = 2G\alpha a^2 \phi$

Then we can redefine $\nabla^2\phi = -2G\alpha \Rightarrow \boxed{\nabla^2\phi = -1}$ $r = \hat r/a$

In the same manner $\hat\phi = 0$ on $\hat r = a \Rightarrow \phi = 0$ on $x^2+y^2=1$ for the most part; and for convenience lets take $\phi = 0$ for $x^2 + (y-1)^2 = \varepsilon^2$

The first global approx is $\phi_0 = \frac{1}{4}(1 - x^2 - y^2)$ it satisfies $\nabla^2\phi_0 = -1$ and $\phi_0 = 0$ on $r = 1$

The second global approx must satisfy $\nabla^2\phi_1 = 0$ and $\phi_1 = 0$ on $r=1$

for an axisymmetric solution is $\phi_1 = c$

$\therefore \quad \phi \approx \frac{1}{4}(1 - x^2 - y^2) + \varepsilon c$

to obtain a local solution — let $X = \frac{x}{\varepsilon}$, $Y = \frac{1-y}{\varepsilon}$

and we must satisfy on $X^2 + Y^2 = 1$ $\phi = 0$ we will not magnify ϕ but $\phi = \Phi$

This put into de $\Rightarrow \frac{d}{dx} = \frac{d}{dX}\cdot\frac{+1}{\varepsilon}$, $\frac{d}{dy} = \frac{d}{dY}\cdot\frac{-1}{\varepsilon}$

Thus $\nabla^2\phi = \frac{1}{\varepsilon^2}\nabla^2\Phi = -1$ or $\nabla^2\Phi = -\varepsilon^2$ with $\Phi = 0$ on $(X^2+Y^2=1)$

let $\Phi = \Phi_0 + \varepsilon^2\Phi_1 + \varepsilon^4\Phi_2 + \dots$

then $\nabla^2\Phi = -\varepsilon^2 \Rightarrow \nabla^2\Phi_0 = 0 \quad \nabla^2\Phi_1 = -1$ etc

$\Phi = 0$ on $X^2+Y^2=1 \Rightarrow \Phi_0 = 0 \quad \Phi_1 = 0$ etc

let $R^2 = X^2 + Y^2$ thus $\nabla^2\Phi_0 = \frac{1}{R}\frac{d}{dR}\left(R\frac{d\Phi}{dR}\right) + \frac{1}{R^2}\frac{d^2\Phi}{d\Theta^2} = 0$

$\Phi_n = C_n\left(R^n - \frac{1}{R^n}\right)\sin n\theta + D_n\left(R^n - \frac{1}{R^n}\right)\cos n\theta$

$\Rightarrow \Phi = C\left(R - \frac{1}{R}\right)\sin\theta$ since $\Phi = 0$ along the lines $\theta = 0, \pi$

~~then we pick~~ most general is R^n we pick n cause it matches best

Match'g also as $Y \to \infty$ it approaches $-y$

Global $\phi = \frac{1}{4}(1 - x^2 - y^2)$

Rewrite in terms of local $= \frac{1}{4}\left(1 - \varepsilon^2 X^2 - (1 - \varepsilon Y)^2\right) = +\frac{\varepsilon Y}{2} - \frac{\varepsilon^2}{4}(X^2+Y^2)$

take 1 term expansion : $+\frac{\varepsilon Y}{2}$

Local $\Phi = C_1\left(R - \frac{1}{R}\right)\sin\theta = CY - \frac{C\sin\theta}{R} = C\left[Y - \frac{Y}{R^2}\right]$

Rewrite in global $= C_1\left(\frac{1-y}{\varepsilon}\right) - \frac{C\sin\theta\,\varepsilon}{\sqrt{x^2+(1-y)^2}}$

take 1 term : $= C_1\left(\frac{1-y}{\varepsilon}\right)$

Rewrite in local $C_1 Y$. let $C_1 = +\frac{\varepsilon}{2}$

comp = local + global − local of global

$$= \frac{1}{4}(1-x^2-y^2) + \frac{\varepsilon}{2}\left[\frac{(1-y)}{\varepsilon} - \frac{\varepsilon^2(1-y)}{\varepsilon(x^2+(1-y)^2)}\right] - \frac{\varepsilon}{2}\left(\frac{1-y}{\varepsilon}\right)$$

$$= \frac{1}{4}(1-x^2-y^2) - \frac{\varepsilon^2}{2}\frac{(1-y)}{x^2+(1-y)^2}$$

With the key way

$$\hat{\tau}_{zx} = \frac{\partial\phi}{\partial y}\Big| = -\frac{y}{2} - \frac{\varepsilon^2}{2}\left[\frac{\overset{+}{\ominus}\,1}{x^2+(1-y)^2} + \frac{2(1-y)^2}{(x^2+(1-y)^2)^2}\right]$$

$$= -\frac{y}{2} - \frac{\varepsilon^2}{2}\left[\frac{-x^2+(1-y)^2}{(x^2+(1-y)^2)^2}\right] = \frac{\varepsilon Y-1}{2} - \frac{\varepsilon^2}{2}\left[\frac{-X^2+Y^2}{(X^2+Y^2)^2\,\varepsilon^2}\right]$$

$$= \frac{\varepsilon Y-1}{2} - \frac{1}{2}\left[\frac{-X^2+Y^2}{(X^2+Y^2)^2}\right]$$

Without the key way

$$\hat{\tau}_{zx} = \frac{\partial\phi}{\partial y}\Big|_{1-\varepsilon} = -\frac{y}{2}\Big|_{1-\varepsilon} = \frac{\varepsilon Y-1}{2}\Big|_{Y=1} = \frac{-1+\varepsilon}{2}$$

With the keyway

$$-\hat{\tau}_{zy} = \frac{\partial\phi}{\partial x} = -\frac{x}{2} + \frac{\varepsilon^2}{2}\left[\frac{-2(1-y)x}{(x^2+(1-y)^2)^2}\right] = -\frac{\varepsilon X}{2} + \left[\frac{YX}{(X^2+Y^2)^2}\right]$$

~~With the keyway~~

$$\sout{-\hat{\tau}_{zy} = \frac{\partial\phi}{\partial x} = -\frac{x}{2}} = -\frac{\varepsilon X}{2}$$

$$\hat{\tau}_{zx} = \frac{\partial\phi}{\partial y}\Big|_{\substack{1-\varepsilon\\ x=0}} \text{ or } \frac{\partial\phi}{\partial y}\Big|_{\substack{Y=1\\ X=0}} = \frac{-1+\varepsilon}{2} - \frac{1}{2}\left[1\right] = \frac{-2+\varepsilon}{2}$$

$$\frac{\hat{\tau}_{zx}\text{ w/keway}}{\hat{\tau}_{zx}\text{ w/o keway}} = \frac{-2+\varepsilon}{-1+\varepsilon} = (+2-\varepsilon)(1+\varepsilon+\varepsilon^2+\dots)$$

$$= 2+\varepsilon+O(\varepsilon^2)$$ ✓✓

The negative signs occur since we defined Y in the opposite sense to y-axis hence shear will change sign ⇒ $\tau_{zx} < 0$ is a positive shear

2a

for $\varepsilon = 0$ then $\frac{d^2 y_0}{dt^2} - \frac{d^2 y_0}{dx^2} = 0$ using $y = y_0 + \varepsilon y_1 + \ldots$

then $y_i(x=0, x=\pi) = 0$

and $y_i(t=0) = 0$

if

$y_0 = \sin x \sin t$

and $\frac{d^2 y_1}{dt^2} - \frac{d^2 y_1}{dx^2} = -2 \cos t \sin x = -2 \frac{dy_0}{dt}$

$= -[\sin(x+t) + \sin(x-t)] =$

$y_1 = f(x-t) + g(x+t)$

$\Box^2 y_1 = 4 \frac{\partial^2 y_1}{\partial\xi \partial\eta} = -\sin\xi - \sin\eta$ $\xi = x-t$ $\eta = x+t$

$4 \frac{\partial y_1}{\partial \xi} = -\eta \sin\xi + \cos\eta + f'(\xi)$

$4 y_1 = +\eta \cos\xi + \xi \cos\eta + f(\xi) + g(\eta)$

$y_1 = \frac{1}{4}[\eta \cos\xi + \xi \cos\eta + f(\xi) + g(\eta)]$

$y_1 = \frac{1}{4}\{(x+t)\cos(x-t) + (x-t)\cos(x+t) + f(x-t) + g(x+t)\}$

$f(-t) + g(t) = 0$

$y_1(x=0) = 0$ ~~$t\cos t - t\cos t$~~ $+ f(-t) + g(t) = 0$ ~~$g(t) = f(-t)$~~

$y_1(t=0) = 0$ $x\cos x + x\cos x + f(x) + g(x) = 0$ $f(x) + g(x) = 2\cos x$

$y_1(x=\pi) = 0$ $-(\pi+t)\cos t - (\pi-t)\cos t + f(\pi-t) + g(\pi+t) = 0$

$2\pi\cos t = f(\pi-t) + g(\pi+t)$

$f(-\sigma) + g(\sigma) = 0$ $g(\sigma) = -f(-\sigma)$

$2\sigma\cos\sigma + f(\sigma) + g(\sigma) = 0$ $f(\sigma) - f(-\sigma) = -2\sigma\cos\sigma$ $f(\sigma) = -\sigma\cos\sigma$

$f(\pi - t) = -(\pi - t)\cos(\pi - t) = +(\pi - t)\cos t$ $g(\sigma) = -\sigma\cos\sigma$

$g(\pi + t) = -(\pi + t)\cos(\pi + t) = +(\pi + t)\cos t$

$\therefore f(x-t) = -(x-t)\cos(x-t)$

$g(x+t) = -(x+t)\cos(x+t)$

$y_1 = \frac{1}{4}\{2t\cos(x-t) - 2t\cos(x+t)\}$ $\therefore y = y_0 + \varepsilon y_1 + O(\varepsilon^2)$

$\therefore$ when $t = O(\frac{1}{\varepsilon})$ this breaks down

Let

Let $T = \varepsilon t$ then

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T}\frac{\partial T}{\partial t} = \varepsilon\frac{\partial}{\partial T} \qquad \frac{\partial^2}{\partial t^2} = \varepsilon^2\frac{\partial^2}{\partial T^2}$$

$\therefore$

$$\varepsilon^2\frac{\partial y}{\partial T^2} + 2\varepsilon^2\frac{\partial y}{\partial T} - \frac{\partial^2 y}{\partial x^2} = 0 \qquad w/\ Y(T=0)=0,\quad Y(x,T)=0\ \ x=0,\pi$$

$\therefore\ Y_0 = F(T)(Ax+B)$ and $B=0$

2b. Use multiple scales with no scaling of y

let $t = t_1 \quad \varepsilon t = t_2$

$$\frac{d}{dt} = \frac{d}{dt_1} + \varepsilon\frac{d}{dt_2}$$

$$\frac{d^2}{dt^2} = \left(\frac{d}{dt_1} + \varepsilon\frac{d}{dt_2}\right)\left(\frac{d}{dt_1} + \varepsilon\frac{d}{dt_2}\right) = \frac{d^2}{dt_1^2} + 2\varepsilon\frac{d^2}{dt_1dt_2} + \varepsilon^2\frac{d^2}{dt_2^2}$$

$$\therefore\ \frac{d^2y}{dt_1^2} + 2\varepsilon\frac{d^2 y}{dt_1dt_2} + \varepsilon^2\frac{d^2y}{dt_2^2} + 2\varepsilon\left(\frac{dy}{dt_1} + \varepsilon\frac{dy}{dt_2}\right) - \frac{d^2y}{dx^2} = 0$$

if $y = y_0(t_1,t_2,x) + \varepsilon y_1(t_1,t_2,x) + \dots$ $\qquad y_0(0,0,x)=0 \quad y_0(t_1,t_2,0)=y_0(t_1,t_2,\pi)=0$

$$\frac{d^2y_0}{dt_1^2} - \frac{d^2y_0}{dx^2} = 0 \qquad y_0 = Y_0(t_1,t_2)X(x)$$

$$Y_0''X - Y_0X'' = 0 \qquad \frac{Y_0''}{Y_0} = -\lambda^2 = \frac{X''}{X} \Rightarrow X'' + \lambda^2 X = 0$$

$X = \sin x$

$$Y_0 = A(t_2)\sin t_1 + B(t_2)\cos t_1$$

$\Rightarrow B(0) = 0 \qquad A(t_2) =$ anything

w/ y_1

$$\frac{d^2y_1}{dt_1^2} + \frac{2d^2y_0}{dt_1dt_2} + 2\frac{dy_0}{dt_1} - \frac{d^2y_1}{dx^2} = 0$$

$\frac{dy_0}{dt_1} = (A\cos t_1 - B\sin t_1)\sin$

$\frac{d^2y_0}{dt_1dt_2} = (A'\cos t_1 - B'\sin t_1)\sin$

$$\frac{d^2y_1}{dt_1^2} - \frac{d^2y_1}{dX^2} = -2\frac{dy_0}{dt_1} - 2\frac{d^2y_0}{dt_1dt_2}$$

$$= \sin x\left\{\left[-2A\cos t_1 + 2B\sin t_1\right] + \left[-2A'\cos t_1 + 2B'\sin t_1\right]\right\}$$

$\Rightarrow -2A - 2A' = 0 \qquad 2B + 2B' = 0 \qquad B = Ce^{-t_2}$

$A = c_0e^{-t_2} \qquad B(0)=0 \Rightarrow C=0$

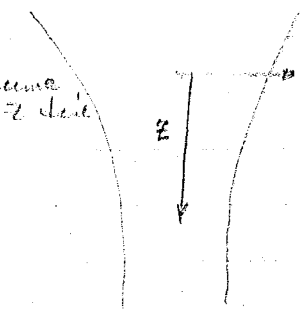
thus $y_0 = c e^{-t_2} \sin t_1 \sin x$ let $c = 1$ or $y \sim e^{-\varepsilon t} \sin t \sin x$ $(1+O($

$= 2 [\sin(x+t) + \sin(x-t)]$

$\sim (1 - \varepsilon t) \sin t \sin x$

given $\psi_{rr} - \frac{\psi_r}{r} + \psi_{zz} = 0$

Use slow variation approx – we assume ψ variation in r direct >> ψ variation in z direction since for large z we have non



$r = R(z) = \tilde{R}(z)$

let $Z = \varepsilon z$

then $\frac{\partial}{\partial z} = \frac{\partial}{\partial Z}\frac{\partial Z}{\partial z} = \varepsilon \frac{\partial}{\partial Z}$

then $\psi_{rr} - \frac{\psi_r}{r} + \varepsilon^2 \frac{\partial^2 \psi}{\partial Z^2} = 0$ with $2\pi\,\psi(R) = Q = 2\pi\,\psi(\tilde{R})$

and $\psi = 0$ at $r = 0$

then to first approx if $\psi = \psi(r, Z; \varepsilon) = \psi_0(r, Z) + \varepsilon^2 \psi_1(r, Z) + \dots \varepsilon^{2n}\psi_n(r, Z)$

then $\psi_{0,rr} - \psi_{0,r}/r = 0$

$\psi_{1,rr} - \psi_{1,r}/r + \psi_{0,ZZ} = 0$

$\vdots$ $\psi_{i,rr} - \psi_{i,r}/r + \psi_{i-1,ZZ} = 0$ $\quad \forall i \geq 1$

also $\psi(\tilde{R}, Z) = \frac{Q}{2\pi} \Rightarrow \psi_0(\tilde{R}, Z) = \frac{Q}{2\pi}$, $\psi_i(\tilde{R}, Z) = 0 \quad \forall i \geq 1$

also $\psi(0, Z) = 0 \Rightarrow \psi_i(0, Z) = 0 \quad \forall i$

then take ε^0 problem $\psi_{0,rr} - \frac{\psi_{0,r}}{r} = 0$ or $\frac{\psi_{0,rr}}{\psi_{0,r}} = \frac{1}{r}$ or

$\frac{d(\psi_{0,r})}{\psi_{0,r}} = \frac{dr}{r}$ $\qquad \ln \psi_{0,r} = \ln r + \ln c_1$

$\psi_{0,r} = r c_1$ and $\psi_0 = \frac{r^2 c_1}{2} + c_2$

$\psi_0(\tilde{R}, Z) = \frac{Q}{2\pi} = \frac{\tilde{R}^2 c_1}{2} + c_2$ $\qquad \psi_0(0, Z) = 0 = c_2$

$\therefore$ ~~R~~ $c_1 = \frac{Q}{\pi \tilde{R}^2}$ $\qquad \therefore \psi_0 = \frac{Q}{2\tilde{R}^2\pi} r^2 = \frac{Q}{2\pi R^2(Z)} r^2$

Thus $\psi_{0,r} = \frac{Qr}{\pi R^2(Z)}$ $\qquad \frac{\psi_{0,r}}{r} = V_z = \frac{Q}{\pi R^2(Z)}$

$\psi_{0,z} = \psi_{0,Z}\frac{dZ}{dz} = \varepsilon\,\psi_{0,Z} = \frac{-2\varepsilon Q r^2}{2\pi R^3} \cdot R'(Z)$

and $V_r = -\frac{\psi_{0,z}}{r} = \frac{+2\varepsilon Q r}{2\pi R^3} R'(Z) = \frac{\varepsilon Q r}{\pi R^3} R'$

Now for ε^0 probl. $V_r = 0$ $V_z = \frac{Q}{\pi R^2(z)}$

thus $gz - \frac{1}{2}(V_z^2 + V_r^2)\Big|_{r=R} = 0$ $\therefore$ $gz - \frac{1}{2}\left[\frac{Q^2}{\pi^2 R^4}\right]\Big|_{r=R} = 0$

thus $\frac{2\pi^2 gz}{Q^2} = \frac{1}{R^4}$ thus $R_0 = \left(\frac{Q^2}{2\pi^2 gz}\right)^{1/4}$ as required

the 2nd approx. is

$$\psi_{1,rr} - \psi_{1,r}/r = -\psi_{0,zz} = -\frac{Qr^2}{2\pi}(R_0^{-2})''$$

with $\psi_1(\tilde{R}, z) = 0$ and $\psi_1(0, z) = 0$

Let $\psi_1 = \Psi(r)F(z)$ then $\Psi'' F - \frac{1}{r}\Psi' F = -\frac{Qr^2}{2\pi}(R_0^{-2})''$

Let $F(z) = \frac{-Q(R_0^{-2})''}{2\pi}$ and $\Psi'' - \Psi'/r = r^2$

$r\left(\frac{1}{r}\Psi'\right)' = r^2$ $\therefore$ $\left(\frac{1}{r}\Psi'\right)' = r$

$\left(\frac{1}{r}\Psi'\right) = \frac{r^2}{2} + c_3$ $\Psi' = \frac{r^3}{2} + c_3 r$

$\Psi = \frac{r^4}{8} + c_3\frac{r^2}{2} + c_4$

$\Psi(0, z) = 0$ $c_4 = 0$

$\Psi(\tilde{R}, z) = \tilde{R}^4/8 + c_3 \tilde{R}^2/2 = 0$ $\therefore$ $c_3 = -\tilde{R}^2/4$

$\Psi = \frac{r^2}{8}(r^2 - \tilde{R}^2)$ thus $\psi_1 = (R_0^{-2})''\frac{r^2}{8}(r^2 - R^2)\cdot\frac{-Q}{2\pi}$

Now $\psi = \frac{Qr^2}{2\pi R^2} - \varepsilon^2\frac{r^2}{8}(r^2 - R^2)(R_0^{-2})''\frac{Q}{2\pi}$

$\psi_{,r} = \frac{Qr}{\pi R^2} - \frac{Q\varepsilon^2}{2\pi 8}(R_0^{-2})''\{2r(r^2 - R^2) + 2r^3\}$

$V_z = \psi_{,r}/r = \frac{Q}{\pi R^2} - \frac{Q\varepsilon^2}{2\pi\, 4}(R_0^{-2})''(2r^2 - R^2) = \frac{Q}{\pi R^2} - \frac{\varepsilon^2 Q (R_0^{-2})''(2r^2 - R^2)}{8\pi}$

$\psi_{,z} = \varepsilon\frac{\partial\psi}{\partial z} = \varepsilon\left[-\frac{Qr^2}{\pi R^3}R' - \frac{Q\varepsilon^2}{16\pi}\left[(R_0^{-2})''' r^2(r^2 - R^2) - (R_0^{-2})'' r^2 2RR'\right]\right]$

$V_r = +\frac{\varepsilon Q r}{\pi R^3}R' + \frac{Q\varepsilon^3}{16\pi}\left[(R_0^{-2})''' r(r^2 - R^2) - (R_0^{-2})'' 2rRR'\right]$

$$V_z\Big|_{r=R} = \frac{Q}{\pi R^2} + \frac{Q}{2\pi}\frac{\varepsilon^2 R^2}{4}\left(R_0^{-2}\right)''$$

$$V_r\Big|_{r=R} = \frac{\varepsilon Q R'}{\pi R^2} - \frac{\varepsilon^3}{8}\left\{\left(R_0^{-2}\right)'''\cdot 2R^2 - \left(R_0^{-2}\right)''\cdot 4RR'\right\}$$

$$V_z^2 + V_R^2 = \frac{Q^2}{\pi^2 R^4} + \frac{Q}{2\pi}\varepsilon^2\frac{Q}{2\pi}\left(R_0^{-2}\right)'' + \frac{\varepsilon^2 Q^2 R'^2}{\pi^2 R^4} + O(\varepsilon^4)$$ take the up to ε^2 term

$$R_0 = \left(\frac{Q^2}{2\pi^2 g Z}\varepsilon\right)^{1/4} \quad \therefore \quad R_0^{-2} = \left(\frac{2\pi^2 g Z}{Q^2\varepsilon}\right)^{1/2} \quad \text{and} \left(R_0^{-2}\right)'' = -\frac{1}{4}\left(\frac{2\pi^2 g}{Q^2\varepsilon Z^3}\right)^{1/2} = -\frac{1}{4\varepsilon^2}\left(\frac{2\pi^2 g}{Q^2 z}\right)$$

$$\therefore V_z^2 + V_R^2 = \frac{Q^2}{\pi^2 R^4} + \frac{1}{4\cdot 4\pi^2}Q^2\left(\frac{2\pi^2 g}{Q^2 z^3}\right)^{1/2} + \frac{\varepsilon^2 Q^2 R'^2}{\pi^2 R^4} + O(\varepsilon^4)$$

$$= \frac{Q^2}{\pi^2 R^4}\left(1+\varepsilon^2 R'^2\right) + \frac{Q}{16\pi}\left(\frac{g}{2z^3}\right)^{1/2}$$

$$\therefore gz - \frac{1}{2}\left(V_z^2+V_R^2\right) = gz + \frac{Q}{16\pi}\left(\frac{g}{2z^3}\right)^{1/2} - \frac{Q^2}{2\pi^2 R^4}\left(1+\varepsilon^2 R'^2\right) = 0$$

~~for large z the above is asymptotic to~~

$$gz\frac{\pi^2 R^4}{Q^2} = 1 + \varepsilon^2 R'^2$$ take − sign since R ↓ as z ↑

thus $-\sqrt{\frac{gz\pi^2}{Q^2}\frac{R^4}{\varepsilon^2} - \frac{1}{\varepsilon^2}} = R'$ thus we assume R slowly decreases and z >> large ⇒

that the first term >> $\frac{1}{\varepsilon^2}$ ∴

$$-dz\sqrt{z}\left(\sqrt{\frac{g\pi^2}{Q^2\varepsilon^2}}\right) = \frac{dR}{R^2} \quad \text{and} \quad -R^{-1} = -\frac{2}{3}z^{3/2}\sqrt{\frac{g\pi^2}{Q^2\varepsilon^2}}$$

$$\text{or} \quad R^{-1} = \frac{2\pi}{3Q\varepsilon}z\sqrt{gz} \quad \text{or} \quad R = \frac{3Q\varepsilon}{2\pi z}\sqrt{\frac{1}{gz}}$$

$$\frac{\pi^2 R^4}{Q^2}\left[gz + \frac{1}{8}\left(\frac{g}{2z^3}\right)^{1/2}\right] = 1+\varepsilon^2 R'^2$$ if ε is small ⇒

$$R = \left[\frac{Q^2}{\pi^2}\left(gz + \frac{1}{8}\left(\frac{g}{2z^3}\right)^{1/2}\right)\right]^{1/4}$$

$$= \left[\frac{Q^2}{2\pi^2}\frac{1}{gz + \frac{Q}{16\pi}\left(\frac{2g}{8z^3}\right)^{1/2}}\right]^{1/4}$$

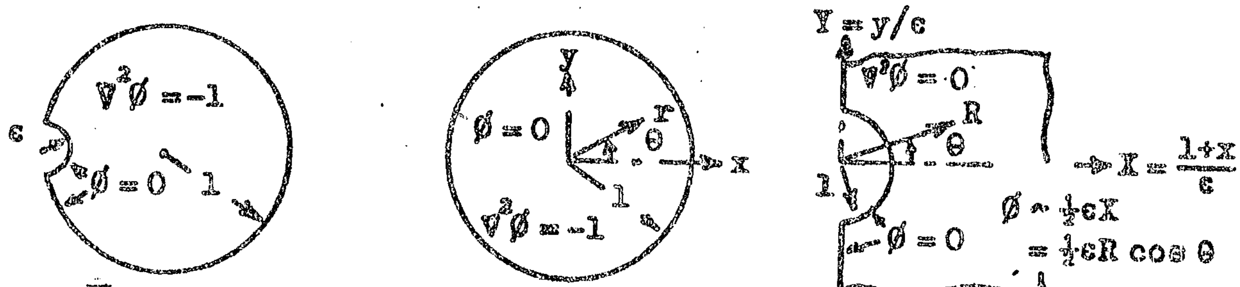
Solutions to Final Examination

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

13 June 1979

1. Torsion of shaft with keyway. The exact solution can be found using bipolar coordinates. It was first derived in clumsy fashion by Gronwall (1919: Amer. Math. Soc. Trans. 20, 234-244), and then more simply by Weber (1921: VDI-Forsch.-Heft 249, Berlin). Gronwall gives the stress-concentration factor as $2-(4/\pi)\epsilon+\ldots$, where ϵ is the ratio of the radius of the keyway to that of the shaft; but Weber finds $2-\epsilon$ (see Timoshenko & Goodier p. 268 or Sokolnikoff p. 126). The perturbation problem for small ϵ is clearly singular, the maximum stress being twice as great for a very small keyway as for none at all. The approximate solution is readily found by the method of matched asymptotic expansions, as indicated below for the dimensionless form of Prandtl's stress function:

FULL PROBLEM | GLOBAL APPROXIMATION | LOCAL APPROXIMATION



$$\phi = \tfrac{1}{4}(1-r^2)+\ldots \qquad \phi = \tfrac{1}{2}\epsilon(R-1/R)\cos\theta+\ldots$$

$$|\partial\phi/\partial r|_{max} = \tfrac{1}{2}+\ldots \qquad |\partial\phi/\partial R|_{max} = \epsilon+\ldots$$

The first approximation can be deduced almost without any computation. The global solution for the stress function without the keyway is a paraboloid, with slope 1/2 at the boundary. To this must be matched a solution of Laplace's equation that vanishes on the half-circle of unit radius (in the magnified local coordinates) and on the extended diameter (or, by reflection, simply on the entire unit circle), and behaves far away like $(1+x)/2 = \tfrac{1}{2}R\cos\theta$ in order to match. This is simply the potential, familiar to a fluid dynamicist as that for flow past a circle, obtained by adding a dipole at the origin; and this is known to double the free stream at its side.

[In this way we can now easily generalize to an elliptic keyway, using the result that in potential flow along the major axis of an ellipse the maximum speed is $1+b/a$ times the free-stream speed. It follows that the stress concentration is $1+b/a$. Essentially this intuitive matching argument is used by Neuber (1946: Theory of Notch Stresses, Edwards Bros., Ann Arbor), who gives the equivalent result $1+\sqrt{t/\rho}$, where t is the depth of a semi-elliptic notch and ρ the radius at its end.]

2. Vibration of slightly damped string. Straightforward iteration on the solution for $\epsilon = 0$ gives the problem

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = -2\epsilon\frac{\partial y}{\partial t} = -2\epsilon\sin x\cos t, \quad y = 0 \text{ at } x = 0, \pi, \quad t = 0.$$

A particular integral is $-\epsilon t \sin x \sin t$, and since this satisfies the boundary and initial conditions, we can simply add it to the basic solution, giving

$$y = (1-\epsilon t)\sin x \sin t + O(\epsilon^2)$$

This evidently breaks down at large time, for $t = O(1/\epsilon)$, when the second term becomes as large as the first. This suggests introducing $T = \epsilon t$ as a second time in addition to $\tau = t$, and then expanding

$$y = y_1(\tau,T) + \epsilon y_2(\tau,T) + \ldots$$

Substituting gives for the first approximation

$$\frac{\partial^2 y_1}{\partial\tau^2} - \frac{\partial^2 y_1}{\partial x^2} = 0, \qquad y_1 = 0 \text{ at } x = 0, \pi \text{ and at } \tau = T = 0.$$

The general solution is

$$y_1 = [A(T)\sin\tau + B(T)\cos\tau]\sin x, \quad \text{where } B(0) = 0 \text{ and, say, } A(0) = 1.$$

Then the equation for the second approximation becomes

$$\frac{\partial^2 y_2}{\partial\tau^2} - \frac{\partial^2 y_2}{\partial x^2} = -2\epsilon\left(\frac{\partial y_1}{\partial\tau} + \frac{\partial y_1}{\partial\tau\partial T}\right) = -2\epsilon\,[(A'+A)\cos\tau - (B'+B)\sin\tau]\sin x .$$

Now the nonuniformity at large time can be avoided by choosing $A(T)$ and $B(T)$ so that this right-hand side vanishes. This gives $B = 0$ and $A = e^{-T}$, so that our uniform first approximation is

$$y = \sin x\; e^{-\epsilon t}\sin t .$$

For comparison, the exact solution is

$$y = \sin x\; e^{-\epsilon t}\sin(\sqrt{1-\epsilon^2}\,t) .$$

3. Falling jet of water. In the plumber's approximation we assume that the radial velocity v_r is negligible and the vertical velocity v_z constant at each elevation:

$$v_z = (\psi_r/r) = \text{const.} = U_1(z).$$

Quadrature gives the stream function as $\frac{1}{2}Ur^2$, since we normalize it to zero on the axis of the jet. Then the conditions of atmospheric pressure on the outside of the jet and prescribed volumetric flux rate Q become

$$gz - \tfrac{1}{2}U_1^2(z) = 0,$$

$$\pi U_1(z)\, R_1^2(z) = Q .$$

Solving these shows that the speed of the jet is, in the first approximation, that of water in free fall:

$$U_1(z) = \sqrt{2gz},$$

and then the stream function and jet radius are

$$\psi_1 = \sqrt{\frac{gz}{2}}\, r^2 = \frac{Q}{2\pi}\frac{r^2}{R_1(z)^2}, \qquad R_1(z) = \left(\frac{Q^2}{2\pi^2 gz}\right)^{1/4} .$$

This is evidently a "slowly-varying" approximation. To improve it systematically, we might want to contract z (or expand r) using a small parameter ϵ; but it is not clear what to use. Under these circumstances it is best to iterate on the differential equating, using the previous approximation to evaluate the neglected term:

$$\psi_{2_{rr}} - \frac{\psi_{2_r}}{r} = r(\psi_{2_r})_r = -\psi_{1_{zz}} = \frac{1}{8}\sqrt{2g}\,\frac{r^2}{z^{3/2}}$$

Hence

$$\psi_2 = \tfrac{1}{2}U_2(z)\,r^2 + \frac{1}{64}\sqrt{2g}\,\frac{r^4}{z^{3/2}}$$

where U_2 is the vertical velocity on the axis of the jet. We now substitute into the two conditions, simplifying consistently wherever possible — for example, using the known ψ_1 rather than the unknown ψ_2 to evaluate the second-order term $\frac{1}{2}v_r^2$ in the Bernoulli equation, using U_1 instead of U_2 in the second-order cross-product term $\frac{1}{2}{v_z}^2$, etc. This gives

$$gz - \tfrac{1}{2}U_2^{\,2}(z) - \frac{Q}{16\pi}\sqrt{2g}\,z^{-3/2} - \frac{Q}{32\pi}\sqrt{2g}\,z^{-3/2} = 0$$

$$2\pi\left[\tfrac{1}{2}U_2(z)R_2^{\,2}(z) + \frac{Q^2}{128\pi^2}\sqrt{2/g}\,z^{-5/2}\right] = Q.$$

Solving these, again approximating consistently, gives

$$U_2(z) = \sqrt{2gz}\left[1 - \frac{3}{32}\left(\frac{Q^2}{2\pi^2 gz^5}\right)^{1/2}\right]$$

$$R_2(z) = \left(\frac{Q^2}{2\pi^2 gz}\right)^{1/4}\left[1 + \frac{1}{32}\left(\frac{Q^2}{2\pi^2 gz^5}\right)^{1/2}\right]$$

We now see clearly what sort of approximation this is: it is not a parameter expansion, but a coordinate expansion for z large compared with the characteristic reference length $(Q^2/g)^{1/5}$, which is the only length that can be formed from the parameters in the problem; and the expansion proceeds in inverse 5/2-powers of z. The stream function is

$$\psi_2 = \sqrt{\frac{gz}{2}}\left[1 - \frac{3}{32}\left(\frac{Q^2}{2\pi^2 gz^5}\right)^{1/2}\right]r^2 + \frac{1}{64}\sqrt{2g}\,\frac{r^4}{z^{3/2}}$$

$$= \frac{Q}{2\pi}\,\frac{r^2}{R_2^{\,2}(z)}\left[1 - \frac{3}{32}\left(\frac{Q^2}{2\pi^2 gz^5}\right)^{1/2}\left(1 - \frac{r^2}{R_2^{\,2}(z)}\right)\right]$$

This problem, including also the effects of surface tension and viscosity, was treated by

N. S. CLARKE 1969 The asymptotic effects of surface tension and viscosity on an axially-symmetric free jet of liquid under gravity. Quart. J. Mech. Appl. Math. 22, 247-256

He used the stream function and velocity potential as independent variables, and introduced an "artificial parameter" ϵ.

Solution to Midterm Examination

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

16 May 1979

1. Torsion of inhomogeneous shaft. The first approximation, for $\mathcal{E} = 0$, satisfies

$$\nabla^2 \phi \equiv \phi_{rr} + \phi_r/r = \frac{1}{r}(r\phi_r)_r = -1, \quad \phi = 0 \text{ at } r = 1,$$

and we know from eq. (2.7) of the notes that the solution is

$$\phi_I = \frac{1}{4}(1 - r^2).$$

Iterating, we substitute this into the neglected term to find

$$\nabla^2 \phi_{II} \equiv \frac{1}{r}(r\phi_{II_r}) = -1 + \mathcal{E} r^2 \nabla^2 \phi_I = -1 - \mathcal{E} r^2,$$

and integrating this subject to the boundary condition $\phi_{II} = 0$ at $r = 1$ gives

$$\phi_{II} = \frac{1}{4}(1 - r^2) + \frac{1}{16}\mathcal{E}(1 - r^4).$$

Alternatively, the same result is obtained by substituting an assumed expansion $\phi = \phi_1 + \varepsilon\ \phi_2 + \ldots$

- Note that if we try to solve this exactly, we can integrate once to find $\phi_r = \frac{1}{2\mathcal{E}} \frac{\log(1 - \mathcal{E} r^2)}{r} + \frac{C}{r}$;

  but this cannot be integrated again in closed form in terms of elementary functions. However, Mr. Spalart expanded and integrated term by term to find

  $$\phi = \sum_{n=0}^{\infty} \frac{1 - r^{2n+2}}{4(n+1)^2} \mathcal{E}^n$$

  and this converges up to $\mathcal{E} = 1$ in the range $0 \leq r \leq 1$.

2. Temperature in corrugated plate. This is clearly a problem of slow variation, with z varying much more rapidly than x or y. We accordingly magnify z by introducing $Z = z/\mathcal{E}$, where $\mathcal{E} = 0.04$. Then the problem becomes

$$\frac{\partial^2 T}{\partial Z^2} = -\mathcal{E}^2\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \qquad \begin{aligned} T &= 0 \quad \text{at} \quad Z = 0, \\ T &= 100 \quad \text{at} \quad Z = 3 + 2\cos x \cos y. \end{aligned}$$

Neglecting the terms in $\mathcal{E}^2$ leads to the first approximation

$$T_1 = 100 \frac{z}{0.04(3 + 2\cos x \cos y)}.$$

In the next approximation we would expand in powers of $\mathcal{E}^2$, or iterate, solving

$$\frac{\partial^2 T}{\partial Z^2} = -100\,\mathcal{E}^2 z\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{1}{3 + 2\cos x \cos y},$$

and quadrature yields a cubic in Z, with coefficients depending on x and y.

The thickness of the plate varies by a factor of five between its thinnest and thickest points. It would therefore evidently be a very poor approximation to assume that the thickness is nearly constant, with slight variations. (The analysis is also more complicated, because the boundary condition must be transferred by Taylor series expansion.) This assumption of slight rather than slow variations would give

$$T = 100\left[\frac{z}{0.12} - \frac{2}{3}\,\frac{\sinh\sqrt{2}\,z}{\sinh\sqrt{2}\,(.12)}\,\cos x \cos y + \ldots\right];$$

and since z is so small, this is close to

$$T = 100\,\frac{z}{0.12}\left(1 - \frac{2}{3}\cos x \cos y + \ldots\right).$$

It is clear that this is the result of expanding the denominator of our previous first approximation, assuming that 2/3 is very small compared with 1.

Our slow-variation approximation gives maximum and minimum values of $\partial T/\partial z$ of 2500 and 500 at the thinnest and thickest parts of the plate, and these are correct to order ε^2, and hence to within a few per cent. By contrast, the assumption of slight variations gives a constant value of 833 in the first approximation, and maxima and minima of 1389 and 278 in the second, so that many terms would be needed for accurate results.

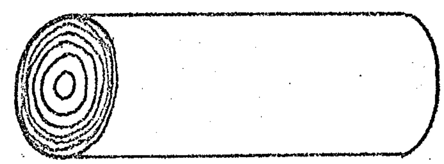
Midterm Examination

ME 207. PERTURBATION METHODS IN ENGINEERING MECHANICS

14 May 1979

INSTRUCTIONS: Spend 50 minutes (10:00 to 10:50)
Open book and notes
Write in bluebooks or any convenient paper
Be sure your name appears

1. Torsion of inhomogeneous shaft. A circular wooden shaft has been made from an oak tree that grew up during a long period of gradually decreasing rainfall.
As a consequence, the growth rings are spaced more closely near the surface than at the axis, so the modulus of elasticity in shear increases with radius. Consequently the problem for Prandtl's stress function, when suitably normalized, is found to be

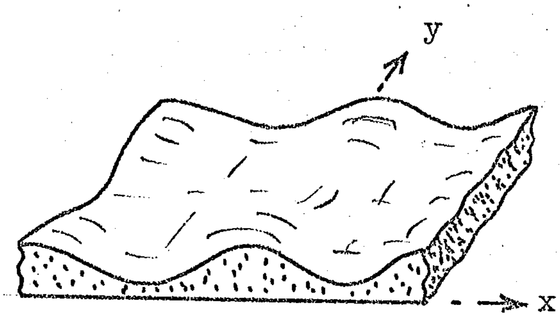


$$(1 - \varepsilon r^2) \nabla^2 \phi = -1, \qquad \phi = 0 \quad \text{at} \quad r = 1.$$

Find, for small ε, the first correction to the solution for a homogeneous shaft (with $\varepsilon = 0$).

2. Temperature in corrugated plate. In a heat exchanger, boiling water is separated from ice water by a thin corrugated plate, which is flat on the cold side, and whose thickness (in feet) is

$$0.04\,(3 + 2\cos x \cos y). \quad = t(x,y)$$



Calculate (in degrees Celsius) the steady-state temperature distribution within the plate as a solution of the 3-dimensional Laplace equation. Find a first approximation on the basis that 0.04 is a small number. Indicate -- without carrying out the details -- how you would improve on it to find a closer approximation.

1. $(1-\varepsilon r^2)\nabla^2\phi = -1 \qquad \phi = 0$ at $r=1$

for small ε Let $\phi = \phi_0 + \varepsilon\phi_1 + \ldots$

$\Rightarrow \phi(1,\theta) = \phi_0(1,\theta) + \varepsilon\phi_1(1,\theta) + \ldots = 0 \Rightarrow \phi_1(1,\theta) = 0$ ✓

$\therefore$ for ε^0

PDE $\Rightarrow \nabla^2\phi_0 = -1 \qquad \phi_0(1,\theta) = 0 \qquad \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi\right) + \frac{1}{r^2}\frac{\partial\phi}{\partial\theta}$

let proble be symmetric $\therefore \quad \phi_0 = \frac{1}{4}(1-r^2)$ ✓

for ε^1

No!

then $\nabla^2\phi_1 - r^2\nabla^2\phi_0 = -1 \qquad \phi_1(1,\theta) = 0$

but $\nabla^2\phi_0 = -1 \quad \therefore \quad \nabla^2\phi_1 + r^2 = -1$ or $\nabla^2\phi_1 = -1 - r^2$

take symmetric results $\therefore \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi_1\right) = -1 - r^2$

$\frac{\partial}{\partial r}(r\phi_1') = -r - r^3$

$r\phi_1' = -r^2/2 - r^4/4 + C$

$\phi_1' = -r/2 - r^3/4 + C \qquad \phi = -r^2/4 - r^4/16 + Cr + D$

$\phi(1) = 0$ at $\Rightarrow \phi = -4/16 - 1/16 + D = 0 \qquad D = 5/16$

$\phi_1(r,\theta) = 5/16 - r^2/4 - r^4/16$

$\therefore \quad \phi(r,\theta) = 1/4(1-r^2) + \varepsilon\left(5/16 - r^2/4 - r^4/16\right)$

8/10

2. $\nabla^2 T(x,y,z) = 0$

$$T(z = t(x,y),\ x, y) = 100°C$$

$$T(0, x, y) = 0°C$$

thickness fn. $t(x,y) = \varepsilon(3 + 2\cos x \cos y) = \varepsilon f(x,y)$ ✔

let $T(x,y,z) = T_0(x,y,z) + \varepsilon T_1(x,y,z) + \ldots$

$$T(x,y,0) = 0 \Rightarrow T_i(x,y,0) = 0 \quad \forall i$$

$$T(x,y,\varepsilon f(x,y)) = T_0(x,y,\varepsilon f(x,y)) + \varepsilon T_1(x,y,\varepsilon f(x,y)) + \cdots$$

$$= T_0(x,y,0) + \varepsilon f(x,y)\, T_{0,z}(x,y,0) + \ldots + \varepsilon\left[T_1(x,y,0) + \varepsilon f(x,y) T_{1,z}(x,y,0) \ldots\right]$$

$\therefore T_0(x,y,0) = 100°C$

$T_{0,z}(x,y,0)\, f(x,y) + T_1(x,y,0) = 0$ etc.

$\therefore$

ε^0: PDE is $\nabla^2 T_0 = 0 \quad T_0(x,y,0) = 0 \quad T_0(x,y,0) = 100°C$

let $T_0(x,y,z) = g(z)$

$\therefore \quad g'' = 0$ or $g = Az + B$ impossible to satisfy bc.

---

let $Z = z - t(x,y) \quad \bar{X} = x \quad \bar{y} = y$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial x} + \frac{\partial}{\partial Z}\frac{\partial Z}{\partial x} = \frac{\partial}{\partial X} + \frac{\partial}{\partial Z}\left[-\varepsilon(-2\sin x \cos y)\right]$$

$$= \frac{\partial}{\partial X} + \frac{\partial}{\partial Z}(2\varepsilon \sin x \cos y)$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial X} + \frac{\partial}{\partial Z}(2\varepsilon \sin x \cos y)\right] = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}(4\varepsilon^2 \sin^2 x \cos^2 y) + \frac{\partial}{\partial Z}\cdot 2\varepsilon \cos x \cos y$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial}{\partial \bar{Y}}\cdot\frac{\partial \bar{Y}}{\partial y} + \frac{\partial}{\partial Z}\frac{\partial Z}{\partial y} = \frac{\partial}{\partial \bar{Y}} + \frac{\partial}{\partial Z}\left[+2\varepsilon \cos x \sin y\right]$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial Z}\,(2\varepsilon \cos x \sin y)\right)$$

$$= \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial Z^2}\,(4\varepsilon^2 \cos^2 x \sin^2 y) + \frac{\partial}{\partial Z}\,(2\varepsilon \cos x \cos y)$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial Z} \quad \& \quad \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial Z^2}$$

$\therefore$ $\nabla^2 T = \bar{\nabla}^2 T + \varepsilon \frac{\partial}{\partial Z}$ terms $+ \varepsilon^2 \frac{\partial}{\partial Z^2}$ terms

w/ bc when $Z = -\varepsilon f(x,y)$ $T(x,y,-\varepsilon f(x,y)) = 0$

when $Z = 0$ $T(x,y,0) = 100\,°C$

let $T(x,y,Z,\varepsilon) = T_0(x,y,Z) + \varepsilon T_1(x,y,Z) + \dots$

$\nabla^2 T_0 = 0$

$T(x,y,0$

again impossible: we can't satisfy BC – singular perturb

---

let $Z = z/\varepsilon$ then **Yes!**

$$\nabla^2 T = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\varepsilon^2}\frac{\partial}{\partial Z^2}\right) T = 0 \Rightarrow \left(\varepsilon^2 \nabla_1^2 + \frac{\partial}{\partial Z^2}\right) T = 0$$

$\nabla_1^2 =$ 2-D laplace

and $T(x,y,Z) = 0$ on $z = Z = 0$

$T(x,y,Z) = 100$ on $Z = \hat{Z} = f(x,y)$

using $\sum_{i=0}^{\infty} T_i(x,y,Z)\,\varepsilon^i = T(x,y,Z;\varepsilon)$

$\therefore$ $\nabla^2 T_0 = 0$ $\Rightarrow \frac{\partial}{\partial Z^2} = 0$ with $T(x,y,0) = 0$

with $T(x,y,\hat{Z}) = 100$

Let $T_0(x,y,\hat{Z}) = AZ + B$ w/ BC $B = 0$ & $A = 100/\hat{Z}$

$\therefore$ $T_0(x,y,Z) = 100\,Z/\hat{Z}$ ✓

using the above I would systematically do the following

solve $\frac{\partial^2}{\partial z^2} T_2 = -\nabla_1^2 T_0$ (since DE is in ε^2 terms)

w/ BC $T(x,y,0) = 0$ and $T(x,y,\hat{z}) = 0$

etc. ✓

10/10

8 + 10 = 18/20

(A)