ME 201. APPLICATIONS OF COMPLEX VARIABLES

Autumn 1979-80, MWF 11:00, Room 550 D

Instructor	Milton Van Dyke, 381 Durand Bldg., Tel. 497-2443. Office hours Tuesday & Friday 1:30-3:00 (and other times when in)
Textbooks	Recommended: Carrier, Krook & Pearson: Functions of a Complex Variable, Theory and Technique. McGraw-Hill, 1966, $22 or Churchill, Brown & Verhey: Complex Variables and Applications. Mc-Graw-Hill, 1976, $21 Both on reserve in Engineering Library, and available in Bookstore.
Exercises	Given out each Monday (and exceptionally at first meeting Wednesday 26 September). Marked and returned promptly if handed in by following Monday.
Examinations	Midterm, Monday 5 November, open-book; Final, Tuesday 11 December, 8:30-11:30, open-book
Honor Code	An undertaking of the students individually and collectively ... that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading ...

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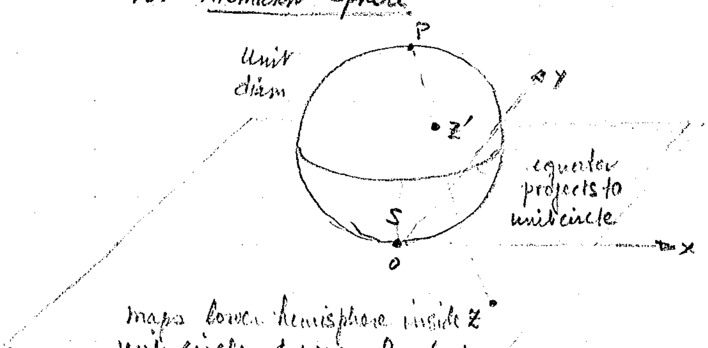
7/28/79

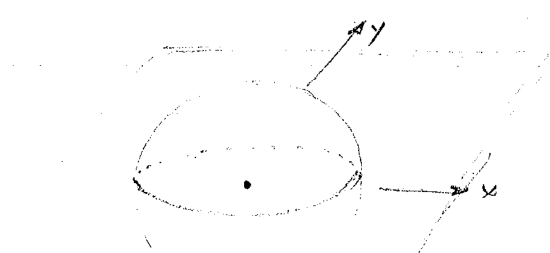
Applied Math - Seminar 3:15 PM

Complex Nos.

1. Dfn: $z = (x, y)$ ordered pair $z = x + iy$ Equality, associ, commut, distrib law
2. Algebraic properties equal, assoc., commut, distrib
3. Complex Plane: Argand Diagram
4. Vector Interpretation mod z vector z $|z| = (x^2+y^2)^{1/2}$
   arg z $\theta = \tan y/x$
5. Triangle Inequality $|z_1+z_2| \le |z_1|+|z_2|$
6. Polar Coords $z = re^{i\theta}$ $0 \le \theta < 2\pi$, $-\pi < \theta \le \pi$ $r = |z|$ $\theta = \tan \frac{\operatorname{Im} z}{\operatorname{Re} z}$
7. Integral power $z^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta) = r^n(\cos\theta + i\sin\theta)^n$
   DeMoivre's Thm.
8. Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$ take Taylor Series of e^x w/ $x = i\theta$
9. Complex Conj $\bar{z} = (x, -y)$ $\bar{z} = x - iy$ $z^*, \bar{z}$ same. $\bar{z} = re^{-i\theta}$
10. Roots $z^{1/n}$ has n distinct roots
11. Point at ∞ Entire or extended complex plane contains point at ∞
    to examine the point at ∞, set: $w = \frac{1}{z}$ and examine neighborhood of $w = 0$
12. Riemann Sphere



maps lower hemisphere inside z unit circle & upper hemisphere outside circle



maps upper hemisphere outside unit circle
maps lower hemisphere inside unit circle

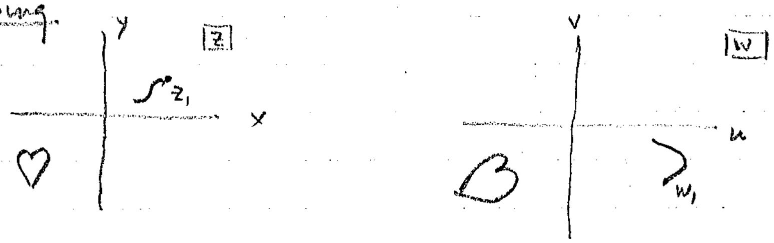
Functions & Mapping

13. Functions of a complex variable $w = f(z)$ may single or multiple valued

Analytic Functions have derivatives

Inverse fn: $z = f^{-1}(w) = g(w)$

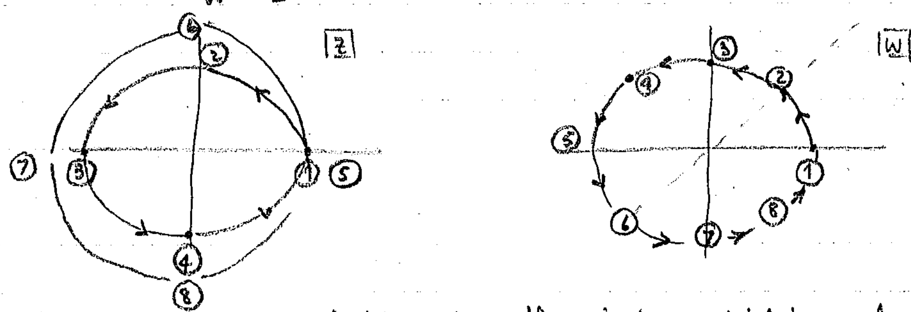
14. Mapping



w_1 is image of z_1

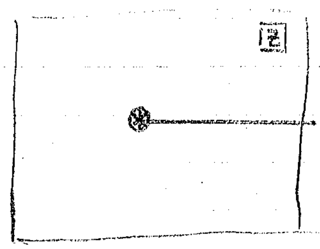
15. Branch Point

$w = z^{1/2}$



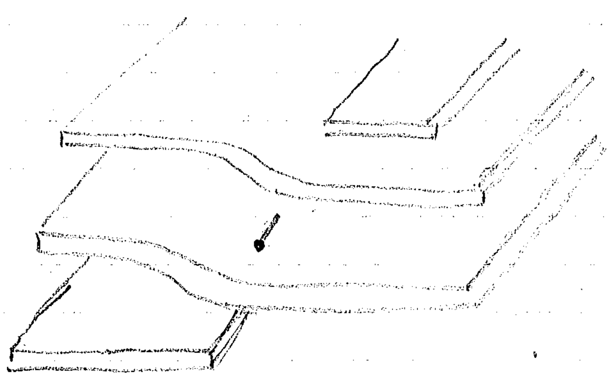
circuit encloses the origin which is a branch point

16. Branch Cut



Branch cut must be between branch points
$\therefore$ $z = \infty$ is a Branch point.
Branch cut need not be a ray but any curve
simple curve

10/1 17. Riemann Surface for $z^{1/3} = w$



18. Elementary Functions

a. Polynomials $w = P(z) = \sum_{i=0}^{n} a_i z^i$

b. Rational functions $\quad w = \dfrac{P(z)}{Q(z)} = \dfrac{\sum_{i=0}^{n} a_i z^i}{\sum_{i=0}^{m} b_i z^i}$

(1) Bilinear Transformation: $\quad \underline{w = az+b / cz+d}$

maps circles $\Rightarrow$ circles and preserves angles

(2) Euler Transformation: $\quad \underline{w = \dfrac{z}{z+d}}$ preserves the origin

c. Exponential $\quad e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!} \qquad e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

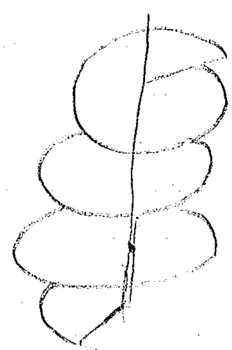
d. Trig fn. $\quad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

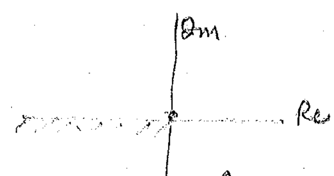
e. Logarithmic $\quad$ if $z = e^w$ then $w = \ln z$ (C, M & P)

$w = \log z$ (Churchill)

$= \ln(re^{i\theta}) = \ln r + i\theta$

multivalued





principal value $-\pi < \theta \le \pi$: denoted by $\ln_p(z)$, $\mathrm{Log}(z)$

$\sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right)$

Complex exponent $\quad z^\alpha = \left(e^{\ln z}\right)^\alpha = e^{\alpha \ln z}$ multivalued

but $\underline{e^z \text{ is single valued.}}$

f. Infinite series - power series / Taylors Series

$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad$ Converges in a circle about z_0 extending to nearest singularity

<u>Tests for Convergence</u>

(1) Dalembert Ratio test $\quad$ radius of convergence R of Taylor series $\sum a_n z^n$ is $\ \lim \left| \dfrac{a_{n+1}}{} \right| \qquad R = \dfrac{1}{}$

ii Cauchy Root test $R = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}}$ $L = \frac{1}{R} = |a^n|^{1/n}$

Analytic fns
Requires $f(z)$ has a derivative $\ni$.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$ independent of angle of approach.

consider
$$f(z) = x^2 + i y$$
$$f(z) = Re(z) = x$$
$$f(z) = \bar{z} = x - iy$$
none are analytic

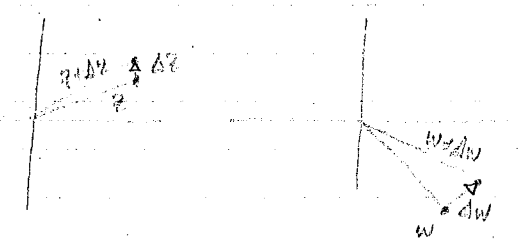
Proof.
take $f(z) = \bar{z}$ $f(z+\Delta z) = \overline{z+\Delta z} = \bar{z} + \rho e^{-i\phi}$
take $\Delta z = \rho e^{i\phi}$

$$\lim_{\rho\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\rho e^{-i\phi}}{\rho e^{i\phi}} = e^{-2i\phi}$$

and depends on ϕ $\therefore$ $f(z) = \bar{z}$ is not analytic

- Cauchy-Riemann Eqns
For that satisfies CR Eqns is analytic & has all derivs

$$W(z) = u(x,y) + i V(x,y)$$



$$\frac{dW}{dz} = W'(z) = \lim_{\Delta z \to 0} \frac{W(z+\Delta z) - W(z)}{\Delta z}$$

$$= \frac{u(x+\Delta x, y+\Delta y) + iV(x+\Delta x, y+\Delta y) - u(x,y) - iV(x,y)}{\Delta x + i\Delta y}$$

for a deriv to exist we must have same limit as $\Delta x, \Delta y \to 0$ in any manner
1) take $\Delta y = 0$, $\Delta x \to 0$

$$\Rightarrow \frac{u(x+\Delta x, y) + iV(x+\Delta x, y) - u(x,y) - iV(x,y)}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

2) take $\Delta x = 0$ $\Delta y \to 0$

$$\Rightarrow u(x, y+\Delta y) + iV(x, y+\Delta y) - u(x,y) - iV(x,y) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

thus if limit exists $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$ CR Eqns

or if ~~second~~ first cross deriv exists $\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}$ or $\Delta u = 0 \quad (\nabla^2 u = 0)$, $\Delta v = 0 \quad (\nabla^2 v = 0)$

Necessary & sufficient condition for existence of derivatives

in plane polar coords $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Check $f(z) = z^2$: $u = x^2 - y^2 \quad v = 2xy$
$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = 2x$ ✓; $\frac{\partial u}{\partial y} = -2y \quad -\frac{\partial v}{\partial x} = -2y$ ✓

Check $f(z) = \bar{z} = x - iy \qquad \frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = -1; \quad \frac{\partial u}{\partial y} = 0 \quad \frac{\partial v}{\partial x} = 0$

$f(z) = |z| = u \quad v = 0$

∴ u & v are conjugate fns.

Given: $u = y^3 - 3x^2y$ Find v: $\frac{\partial u}{\partial x} = -6xy = \frac{\partial v}{\partial y} \qquad v = -3xy^2 + f(x)$
$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -3y^2 + 3x^2 = -3y^2 + f'$
$\therefore f' = 3x^2$ or $f = x^3 + c$

$\therefore \underline{v = -3xy^2 + x^3 + \tilde{c}} \qquad \therefore f(z) = iz^3 + C$

For fluids $u_x + v_y = 0$ Continuity $\nabla \cdot vel = 0$ no dilatation
$v_x - u_y = 0$ Irrotationality $\nabla \times vel = 0$ no rotation.

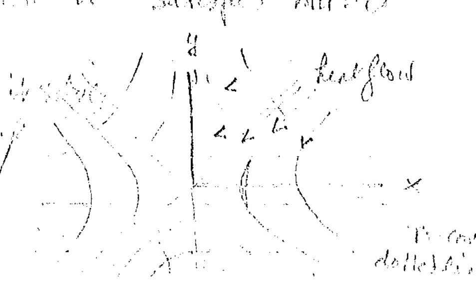
Real & imaginary part of analytic functs are (plane) harmonics.

Example Steady heat conduction: Temperature u satisfies $\Delta u = 0$

Inverse solution given a fn. infer problem it solves

$w = f(z) = z^2 \quad u = x^2 - y^2$

$v = 2xy$



solid lines are lines of constant heat flux

Integration in the complex plane 10/5/??

Analytic continuation : unique

1. Suppose f defined only on real axis   Example: $\sin x = f(x)$

2. Analytic cont. $\Rightarrow f(z) = \sin z$

3. $f(z)$ is defined by a power series   Example: $f(z) = 1 + z + z^2 + \dots + z^n$
converge for $|z| < 1$
now we can define $f(z) = \frac{1}{1-z}$ which is the analytic cont. of the power series that converge everywhere except at $z = 1$

Singular Points
is a point where $f(z)$ ceases to be analytic ie derivative not defined

A fn which has no singular points $\Rightarrow f'(z) = \text{const} \Rightarrow f(z) = $ linear fn of z

Ex: $f(z) = \sqrt{z}$ $f'(z) = \frac{1}{2\sqrt{z}}$ $z = 0$ is singular point

Classify Branch Point   Ex: $\sqrt{z}$, $\log z$ (origin & ∞)

Poles   Ex: $\frac{1}{z}$ is a simple pole at $z = 0$

$\frac{1}{(z-1)^2}$ is a double pole

Removable Singularity   Ex: $\frac{\sin z}{z}$ define $f(0) = 1$

Essential Singularity - any singularity other than the above

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \dots + \frac{1}{n!z^n} + \dots \quad (\text{pole of } \infty \text{ order})$$

Corresponding Classification of functions
function is Analytic in a domain if it has at most finite singular points

fn is Regular in a domain if it has no singularities

Entire (Integral) no singularities in finite plane   Ex: e^z (essent. sing. at ∞)
~~has essential singularity~~ $A + Bz + Cz^2$ (has pole at ∞)

Integration in Complex Plane.

Real Variable : defined as limit of sum
depends only on values of end points
integral can be interpreted as area under curve

Integral has no meaning $\int_{\boxed{z}}$ unless we define the contour

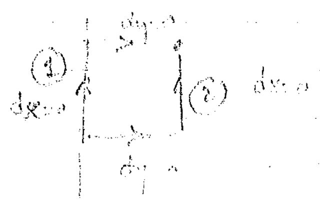
as long as there is no singularity within the contour

$$\int_C f(z)\,dz = \lim \sum_{i=1}^{n} f(\xi_i)\,\Delta z_i$$

$$\int_C f(z)\,dz = \int (u+iv)(dx+idy) = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

Result depends on choice of contour if $f(z)$ is not analytic

$f(z) = \bar{z}$ integrate from $(0,0)$, $(1,i)$ $u = x$, $v = -y$



path (1) gives $\int_C f\,dz = \frac{1}{2} + \frac{1}{2} + i = 1+i$

path (2) gives $\int_C f\,dz = \frac{1}{2} + \frac{1}{2} - i = 1-i$

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Cauchy's Integral Theorem (Cauchy Goursat)

$\int_C f(z)\,dz = 0$ if $f(z)$ is regular in and on C. ie no singularities in or on bound.

ccw direction

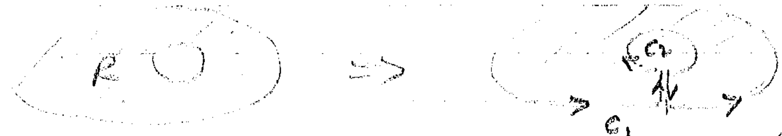
The result allows us to deform the contour

Integration is reverse of differentiation if Cauchy Integral Theorem is satisfied

$\therefore$ for branch cuts $\int z^s\,dz = \frac{z^{s+1}}{s+1}$ $(s \neq -1)$ must stay on same branch for z^s, z^{s}

Converse is Morera's theorem : if $\oint f(z)\,dz = 0$ for every C then $f(z)$ is regular

Multiply Connected Region can be made singly connected by cut



Then Cauchy Theorem can be used as

$$\oint_{C_1} + \oint_{C_2} = 0$$

if  then $\int_A^B$ may differ by a cyclic constant

Proof depends on
1. decomposing integral into real - variable integrals
2. apply Green / Divergence 2-D thm th
3. use C-R eqns

Div th. given any $P(x,y)$ $Q(x,y)$ are C^2

$$\oint_C P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

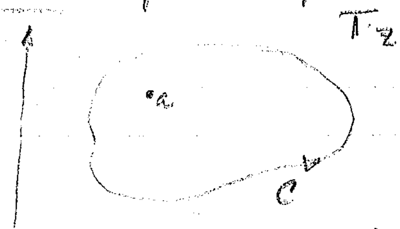
$$\int f(z)\,dz = \int (u+iv)(dx+i\,dy) = \int (u\,dx - v\,dy) + i\int (v\,dx + u\,dy)$$

1. Let $u = P$, $-v = Q$ $\Rightarrow$ $\iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = 0$

but $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ by C-R eq $\Rightarrow$ $\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$

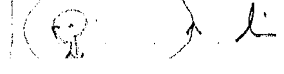
2. Let $v = P$ $Q = u$ $\Rightarrow$ $\iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0$ by C-R eq

Cauchy's Integral Formula expresses f at any interior point in terms of Boundary values



$$\boxed{f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz}$$

Proof: apply Cauchy's theorem to $\frac{f(z)}{z-a} = g(z)$ $\quad \oint_C g(z)\,dz = 0$ if a is outside

 $\lim \int \frac{f(a+\rho e^{i\phi})}{\ldots}(i\,d\phi\,\rho e^{i\phi}) = i2\pi f(a) = \oint \frac{f(z)}{z-a}\,dz \Rightarrow$ desired

Cauchy Principal Value

$$CPV\int_{-1}^{1}\frac{dx}{x} = \lim_{R\to 0}\int_{-1}^{R}\frac{dx}{x} + \lim_{R\to 0}\int_{R}^{+1}\frac{dx}{R} = \ln x\Big|_{-1}^{R} + \ln x\Big|_{R}^{1} = -\ln x\Big|_{-1} + \ln x\Big|_{1}$$

ident contour inside since we know fcn is analytic in region except at a_1

$\int \frac{f(z)}{z-a}dz = 0$ if a is outside

$\int \frac{f(z)}{z-a}dz = \frac{2\pi i f(a)}{2}$ if a is on smooth bdy

$\int \frac{f(z)}{z-a}dz = 2\pi i f(a)$ if a is inside

derivatives can be found by formal differentiation of integral formula

$$f'(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^2}dz$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}dz$$

since $f'(a) = \lim_{\Delta z\to 0}\frac{f(a+\Delta z)-f(a)}{\Delta z}$

$= \lim_{\Delta z\to 0}\frac{1}{2\pi i\,\Delta z}\oint\left[\frac{f(z)}{z-a-\Delta z} - \frac{f(z)}{z-a}\right]dz$

$= \lim_{\Delta z\to 0}\frac{1}{2\pi i\,\Delta z}\oint_c \frac{f(z)\,\Delta z}{(z-a-\Delta z)(z-a)}dz$

$= \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^2}dz$

- Thus an analytic fn has derivatives of all order
- This is also true for u, v.

10/10/79

Branch Points - center coords around suspected point & go around once to see if change occurs

Models: $\sqrt{z}$ $\log z$ Branch Points at 0 & ∞

$\frac{d}{dz}\sqrt{z} = \frac{1}{2\sqrt{z}}$ $\frac{d}{dz}\log z = \frac{1}{z}$

Algebraic Branch Points - $\sqrt[3]{z}$ branch pt at $0, \infty$ Order 2

$\sqrt[n]{z}$ - $(n-1)$ order

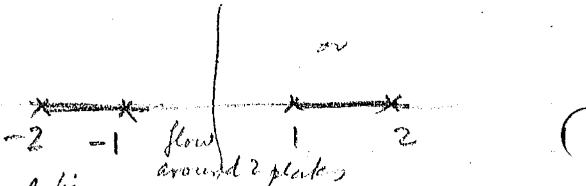
$z^{m/n}$ - $(n-1)$ order $z^{1/\pi} = \infty$ order

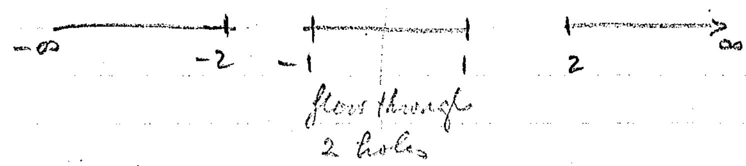
Logarithmic ∞ order

Root of $P(z)$ $\sqrt[n]{P(z)} = \sqrt[n]{(z-z_0)^{N_0}\cdots(z-z_m)^{N_m}}$ depends on pts inside contour & order of zero.

$f(z) = \sqrt{(1-z^2)(2-z^2)}$ has branch points

∞ not br pt since $f(z) \sim \sqrt{z^4} = z^2$ which is analytic





$f(z) = \cos^{-1} z = -i \log\left(z + \sqrt{z^2-1}\right)$ has $z = \pm 1$ & $z = \infty$ branch points

Recap

Cauchy $\oint_C F(z)\,dz = 0$ if $F(z)$ is reg in & on C

Cauchy Integral form $\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\zeta}\,dz = f(\zeta)$

Derivs Exists if $f'(z)$ exists $\Rightarrow f^{(n)}(z)$ exists $\forall n$

$$f^{(n)}(\zeta) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-\zeta)^{n+1}}\,dz$$

Mean Value theorem

Analytic fn = average value on any circle

if $z - \zeta = \rho e^{i\phi}$

$$\frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(\rho e^{i\phi})}{\rho e^{i\phi}}\,\rho i e^{i\phi}\,d\phi = \qquad f(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta + \rho e^{i\phi})\,d\phi$$

Average on Circle

This is also true for the real or imaginary parts.

f (or u, v) can't attain max in interior & Max. always on bdy.
Also min [if $f(z)$ doesn't vanish] is on bdy.

Cauchy's Inequal let $\zeta + re^{i\theta} = z$

$$\left|f^{(n)}(\zeta)\right| = \left|\frac{n!}{2\pi \cancel{i}}\oint_C \frac{f(\zeta + re^{i\theta})\,\cancel{i} r e^{i\theta}\,d\theta}{(re^{i\theta})^{n+1}}\right| \le \frac{n!}{2\pi r^n} M \cdot 2\pi \le \frac{n!\,M}{r^n}$$

where $M = \max_{z \in C} f(z)$ $\Rightarrow$ Higher derivatives die off faster than lower deriv

Liouville's Thm: Entire fn: (no singularity in finite plane) $\Rightarrow$ let fn be bounded & entire $\Rightarrow$ $f = $ constant ie $|f^{(n)}(\zeta)| = 0 \Rightarrow f(z) = f(\zeta)$ $\forall z \in \mathbb{C}$

<u>O-R</u>

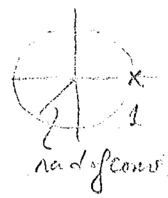
A non constant entire fn assumes arbitrarily large values ie e^{z}

TAYLOR SERIES

$$f(z) = a_0 + a_1 (z - z_0) + \dots + a_n (z - z_0)^n + \dots$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$$\sqrt{1-z} = 1 - \tfrac{1}{2} z + \dots$$



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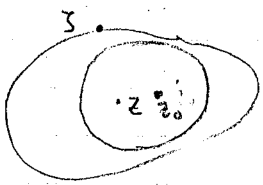
Taylor series From Cauchy Integral formula

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} d\zeta \qquad \& \quad f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{(\zeta - z_0)\left[1 - \frac{z - z_0}{\zeta - z_0}\right]} = \frac{1}{\zeta - z_0}\left[1 + \frac{z - z_0}{\zeta - z_0} + \dots\right]$$

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \dots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0)$$



$|z - z_0| <$ radius of convergence $\Rightarrow$ a power series exists

Since a series converges w/in some radius of convergence $\Rightarrow$ I can differentiate & integrate term by term

$$\oint \frac{1}{(1+z)} dz = \underbrace{\log(1+z)}_{\text{multiple valued}} = \underbrace{z - \frac{z^2}{2} + \frac{z^3}{3!} - \dots}_{\text{single valued}}$$

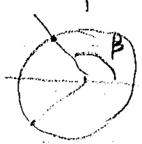
thus series represents soln on one of their Riemann sheets

Pattern of Signs of Taylor Series - The ultimate pattern of signs corresponds to nearest singularities

1. Fixed signs $\frac{1}{1-z} = 1 + z + z^2 + \dots$ Nearest sing is on + ive real axis

2. Alternating Signs $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots$ Nearest sing is on - ive real axis

3. Irregular patterns - if coeffs are real nearest sing are complex conj pair



if β is rational $\frac{M\pi}{N}$ pattern repeats in N or 2N signs

Example $(1+z)^{5/2} = 1 + \frac{5}{2}z + \frac{15}{8}z^2 + \frac{5}{16}z^3 - \frac{5}{128}z^4 + \underbrace{- + - + - + - \ldots}_{\text{ultimate pattern}}$

$\frac{1}{1+z} + \frac{6}{2-z} = 4 + \frac{1}{2}z + \frac{7}{4}z^2 - \frac{5}{8}z^3 + - + - + -$

The stronger the singularity $\frac{1}{1-z}$ beats $\sqrt{1+z}$

Bigger coeff $\frac{4}{1-z}$ beats $\frac{2}{1+z}$

} to determine ultimate pattern.

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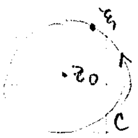
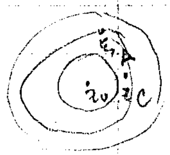


Taylor Series: $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$; $a_n = \frac{1}{2\pi i}\oint_C \frac{f(\xi)\,d\xi}{(\xi - z_0)^{n+1}} = \frac{1}{n!} f^{(n)}(z_0)$

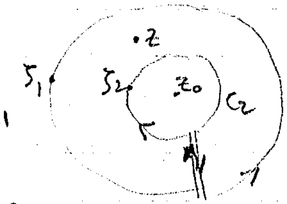
Laurent Series: fn is analytic in an annulus



$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n ; \quad a_n = \frac{1}{2\pi i}\oint_C \frac{f(\xi)\,d\xi}{(\xi - z_0)^{n+1}}$$

we cannot relate a_n to $f^{(n)}(z_0)$ as we did before

$$= \underbrace{\ldots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)}}_{\text{Principal part}} + a_0 + a_1(z-z_0) + \ldots$$



$C = C_1 + C_2$

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) + (z_0 - z)} = \frac{1}{(\zeta - z_0)}\left[1 - \frac{z-z_0}{\zeta - z_0}\right]^{-1}$$

for $\zeta \in C_1$ since $\left|\frac{z-z_0}{\zeta-z_0}\right| < 1$

for $\zeta \in C_2$ $\frac{1}{\zeta - z} = \frac{1}{-(z-z_0)}\left[1 - \frac{\zeta - z_0}{z - z_0}\right]^{-1}$ since $\left|\frac{\zeta - z_0}{z-z_0}\right| < 1$

reduces to taylor series if $f(z)$ regular inside C_1

$\frac{e^z}{z^2} = \frac{1}{z^2}\left[1 + z + \frac{z^2}{2!} + \ldots\right] = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{1}{6}z + \ldots$ for expansion about $z=0$

- fn has double pole at $z=0$ since highest power of $\left(\frac{1}{z}\right)$ is -2

- Residue is a_{-1}

Isolated singular point -

If $f(z)$ is regular & single-valued in some nbd of z_0; but not $z = z_0$

example: e^z/z^2 has $z=0$ is an isolated singular point.

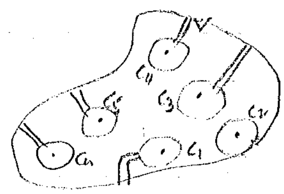
Classify singularity as principal part has

1. no terms (removable singularity: $\frac{\sin z}{z}$)
2. a finite # of terms, say N. (N^{th} order pole)
3. an ∞ # of terms, essential singularity (example $e^{\frac{1}{z}}$)

for point at ∞ let $\zeta = \frac{1}{z}$ & check what happens when $\zeta \to 0$.

Residue theorem: If $f(z)$ analytic inside & single valued w/ finite (infinite) # of isolated singular

$$\oint_C f(z)\,dz = 2\pi i\left[\text{Sum of residues in } C\right]$$



$$\oint_{C-[C_1+C_2+\cdots+C_n]} f(z)\,dz = 0 \Rightarrow \oint_C f(z)\,dz = -\oint_{C_1+C_2+C_3+C_4+\cdots C_n} f(z)\,dz$$

$$= \oint_{C_1+\cdots+C_n} f(z)\,dz = 2\pi i\left[a_{-1}^{(1)} + a_{-1}^{(2)} + \cdots + a_{-1}^{(n)}\right]$$

Evaluation of Definite Integrals By Contour Integration

Classification of technique

1. $\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$ = [f rational or not] E.g. Dwight § 859.2 $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}$ $|a|<1$
2. $\int_{-\infty}^{\infty} f(x)\,dx$ f rational $\times$ times sine/cosine or E.g. Dwight § 858.5 $\int_{-\infty}^{\infty} \frac{\sin mx}{x}\,dx = \operatorname{sgn}(m)\cdot$
3. $\int_0^{\infty} x^{\alpha-1} f(x)\,dx$ 856.3 $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}} = \pi$

Trivial Example $\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$

① $I = \int_0^{2\pi} \cos\theta\,d\theta$ let $z = e^{i\theta}$ & integrate around unit circle

$\operatorname{Re}(z) = \cos\theta$ $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$ie^{i\theta}d\theta = dz$ $\therefore d\theta = \frac{dz}{iz}$

$$\oint_C \frac{1}{2}\left(z+\frac{1}{z}\right)\frac{dz}{iz} = \frac{1}{2i}\int dz + \frac{dz}{z^2} = 0 \quad \text{using Cauchy Residue Th}$$

Only singularity inside unit circle: $z=0$ but coeff of $\frac{1}{z}$ in Laurent exp $=$ Residue $= 0$.

(2) $\displaystyle\int_0^{2\pi}\cos^2\theta\, d\theta \stackrel{?}{=} \pi$ let $z=e^{i\theta}$ $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$ $d\theta = \frac{dz}{iz}$

then $\displaystyle\frac{1}{i}\oint_{|z|=1}\frac{1}{4}(z+1/z)^2\frac{dz}{z} = \frac{1}{4i}\oint\left(z+\frac{2}{z}+\frac{1}{z^3}\right)dz = \frac{1}{2i}\oint\frac{dz}{z} = \frac{\log(z)}{2i} = \frac{2\pi i}{2i} = \pi$

triple pole.

(3)

$$I = \int_0^{2\pi}\frac{d\theta}{1-2a\cos\theta+a^2} = \oint\frac{dz}{iz}\,\frac{1}{1-\frac{2a}{2}(z+1/z)+a^2} = \frac{1}{i}\oint\frac{dz}{z-az^2-a+a^2z}$$

$= \displaystyle\frac{1}{i}\oint\frac{dz}{(z-a)(1-az)} =$ if $|a|<1$ then a simple pole at $a=z$ inside circle
& simple pole at $z=1/a$ outside circle

$= \displaystyle\frac{1}{i}2\pi i\left(\frac{1}{1-a^2}\right) = \frac{2\pi}{1-a^2}$

if $|a|>1$ then $z=a$ is outside; $z=1/a$ is inside

$\displaystyle\frac{1}{i}2\pi i\left(\frac{-1}{1-a^2}\right) = \frac{2\pi}{a^2-1}$ $\qquad I = \displaystyle\oint -\frac{dz}{ai(z-a)(z-1/a)}$

$$I = \begin{cases} 2\pi/1-a^2 & |a|<1 \\ \infty & |a|=1 \\ \dfrac{2\pi}{a^2-1} & |a|>1 \end{cases} \quad \text{or} \quad \frac{2\pi}{|1-a^2|} \quad \text{if } a \text{ is real.}$$

(4) $\displaystyle I = \int_0^{2\pi}\frac{\cos^2\theta\, d\theta}{1-2a\cos\theta+a^2} \quad \frac{1}{4i}\oint\frac{(1+z^2)^2}{-az^2(z-a)(z-1/a)}$ poles at $z=0, z=a, z=1/a$

if $|a|<1$ Residue at $z=a = \dfrac{(1+a^2)^2}{a^2(1-a^2)}$

Double pole at $z=0$

Expand these with Laurent series $\displaystyle\frac{[1+2z^2+\dots]}{-a^2z^2(1-z/a)(z-1/a)} = \frac{[1+2z^2](1+z/a+z^2/a^2+\dots)}{-a^2z^2(z-1/a)}$

$\displaystyle\frac{1+z/a+O(z^2)}{+az^2(1-az)} = \left[\frac{1}{az^2}+\frac{1}{a^2z}+\dots\right]\left(\frac{1}{1-az}\right)$ $\therefore$ res at $z=$

Computing residues

(1) decompose into partial fractions if $f(z)$ is rational

$$\frac{1+3z}{z^2-1} = \frac{2}{z-1} + \frac{1}{z-1}$$

Simple pole at $z=a$: residue @ $z=a$ is $\lim_{z\to a} (z-a)f(z)$

If it's a rational fraction result is immediate

" " transcendental use $\lim_{z\to a}(z-a)f(z)$ ie $\lim_{z\to 0} \cot z = \frac{z\cos z}{\sin z} = \frac{1}{1} = 1$

If a multiple pole of order n: mult. by $(z-a)^n$ & differentiate $n-1$ times and divide by $(N-1)!$

$$\frac{1}{(N-1)!}\left\{\frac{d^{N-1}}{dz^{N-1}}(z-a)^N f(z)\right\}\Bigg|_{z=a} = a_{-1}$$

Essential singularity write Laurent series & pick out $\frac{1}{z}$ term.

Variations on $\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$

A. Exploit Symmetry

$$\int_0^{\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{1-2a\cos\theta+a^2} = \begin{cases} \frac{\pi}{1-a^2} & |a|<1 \\ \frac{\pi}{a^2-1} & |a|>1 \end{cases}$$

B. Take real (or imaginary) part

$$I = \int_0^{2\pi}\frac{\cos\theta\,d\theta}{1-2a\cos\theta+a^2} = \frac{1}{2i}\oint\frac{1+z^2}{z(z-a)(1-az)}dz$$ pole at $z=0$ & $z=a$ or $\frac{1}{a}$

Alternatively, rewrite as (if a is real)

$$I = \text{Re}\int_0^{2\pi}\frac{e^{i\theta}\,d\theta}{1-2a\cos\theta+a^2} = \text{Re}\,\frac{1}{i}\oint\frac{z\,dz}{(z-a)(1-az)}$$ only 1 pole at $z=a$, $z=1/a$

if $|a|<1$ $\quad \text{Re}\,\frac{1}{i}\left(\frac{a}{1-a^2}\right)\cdot 2\pi i = \frac{2\pi a}{1-a^2}$

if $|a|>1$ $\quad \text{Re}\,\frac{1}{i}\,\frac{1}{a(a^2-1)}\cdot 2\pi i = \frac{2\pi}{a(a^2-1)}$

Bonus

$$\text{Im}\int_0^{2\pi}\frac{\sin\theta\,d\theta}{1-2a\cos\theta+a^2} = 0$$ also obvious by symmetry.

$$\int_0^{2\pi} \frac{\cos^2\theta \, d\theta}{1-2a\cos\theta+a^2} \neq \text{Re}\int \frac{e^{2i\theta}\,d\theta}{1-2a\cos\theta+a^2} = \int \frac{\cos^2\theta-\sin^2\theta}{1-2a\cos\theta+a^2}\,d\theta$$

$$= \text{Re}\int \frac{\cos\theta\, e^{i\theta}\,d\theta}{1-2a\cos\theta+a^2}$$

if $f(\sin\theta,\cos\theta)$ not a rational function

$$I = \int_0^{2\pi} \ln\left(1-2a\cos\theta+a^2\right)d\theta$$

this has a branch cut between $z=a$ & $z=\frac{1}{a}$ & doesn't have isolated singularity. Look at $I=I(a)$ & take $\frac{dI}{da}$

$$I(a) = \int_0^{2\pi} \ln(1-2a\cos\theta+a^2)d\theta \Rightarrow \frac{dI}{da} = \int_0^{2\pi} \frac{1}{1-2a\cos\theta+a^2}(2a-2\cos\theta)\,d\theta$$

thus $\displaystyle\frac{dI}{da} = \int_0^{2\pi}\frac{2a\,d\theta}{1-2a\cos\theta+a^2} - \int_0^{2\pi}\frac{2\cos\theta\,d\theta}{1-2a\cos\theta+a^2}$

$= 2a\cdot\frac{2\pi}{1-a^2} - 2\cdot\frac{2\pi a}{1-a^2} \qquad |a|<1$

$= 2a\cdot\frac{2\pi}{a^2-1} - 2\cdot\frac{2\pi}{a(a^2-1)} \qquad |a|>1$

$= 0 \qquad |a|<1$

$= \frac{4\pi}{a} \qquad |a|>1$

$\therefore\ I = 0 + \text{const} \qquad |a|<1 \qquad$ to get const let $a=0 \Rightarrow \text{const}=0$

$I(0) = \int_0^{2\pi} \ln 1\, d\theta = 0$

$I = 4\pi\ln a + \text{const} \qquad |a|>1 \qquad$ to get const let $a\to 1 \Rightarrow \text{const}=0$

$I(1) = \int_0^{2\pi} \ln 4\left(\frac{1-\cos\theta}{2}\right)d\theta$ ($\sin^2\frac{\theta}{2}$)

Not rational fn

$$I = \int_0^{2\pi}\frac{\theta\sin\theta\,d\theta}{(1-2a\cos\theta+a^2)^2}$$

integrate by parts $u=\theta \quad dv = \frac{\sin\theta\,d\theta}{(1-2a\cos\theta+a^2)^2}$

$du = d\theta$

$v = -\frac{1}{2a}\frac{1}{1-2a\cos\theta+a^2}$

$$\left[\frac{\cdot\,\theta}{(1-2a\cos\theta+a^2)^2}\cdot\frac{1}{2a}\right]_0^{2\pi} + \frac{1}{2a}\int_0^{2\pi}\frac{d\theta}{1-2a\cos\theta+a^2}$$

$$\frac{-2\pi}{2a(1-a)^2} + \frac{1}{2a}\begin{cases} 2\pi/1-a^2 & |a|<1 \\ \dfrac{2\pi}{a^2-1} & |a|>1 \end{cases}$$

$$\frac{-2\pi\,(1+a)}{2a(1-a^2)(1-a)} + \frac{1}{2a}\frac{(2\pi)(1-a)}{(1-a^2)(1-a)} = \frac{-2\pi}{(1-a^2)(1-a)} \qquad |a|<1$$

$$\frac{-2\pi\,(a+1)}{2a(1-a)^2(a+1)} + \frac{1}{2a}\frac{2\pi\,(a-1)}{(a^2-1)(a-1)} = \frac{-2\pi}{a(a+1)(a-1)^2} \qquad |a|>1$$

Evaluate
must use 3 tricks $\displaystyle\int_0^{2\pi}\frac{x\sin x\,dx}{1-2\alpha\cos x+\alpha^2} = \frac{1}{2}\int_{-\pi}^{\pi}$

Since not rational integrate by parts to get rid of x

$\Rightarrow \frac{1}{2}\int_0^{2\pi}[\qquad] - \int \ln[\qquad]$

$\Rightarrow \int \ln[\qquad]$ must use $I(a)$ trick

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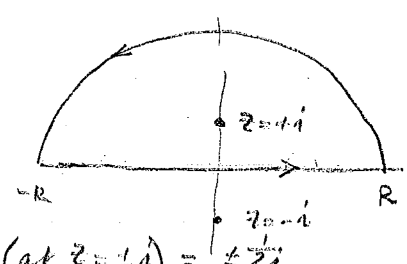
Evaluation of $\int_{-\infty}^{\infty} f(x)\,dx$, f rational or times $\sin ax$, $\cos ax$

Device to evaluate the integral: replace x by z and consider $\lim_{R\to\infty}\int_{-R}^{R} f(z)\,dz$
We must now do the following:
Complete the contour in the upper (or lower) plane, usually by a semi circle or rectangle. Show that the extra integral vanishes, is known, or is related to the desired integral.

Example $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = 2\int_0^{\infty}\frac{dx}{1+x^2} = \pi$

Proof: Consider the integral $I_1 = \oint \frac{dz}{1+z^2}$



$\dfrac{1}{1+z^2} = \dfrac{1}{(z-i)(z+i)}$

residues (at $z=\pm i$) $= \pm\frac{1}{2i}$

$I_1 = \displaystyle\int_{-R}^{R}\frac{dx}{1+x^2} + \int^{\pi}\frac{Re^{i\theta}i\,d\theta}{1+R^2e^{2i\theta}} \le \int^{R}\frac{dx}{1+x^2} + \left|\frac{i}{R}\int^{\pi}\frac{d\theta}{\cdot\theta}\right|$ but $\oint f(z)\,dz = 2\pi i\, f(+i)$

$\therefore$ as $R\to\infty$ $\left|\int \frac{d\theta}{e^{i\theta}}\right| \to$ const as $\lim_{R\to\infty} \left|\frac{i}{R}\right| \left|\int \frac{d\theta}{e^{i\theta}}\right| < \lim_{R\to\infty} \frac{1}{R} = 0$

$$\therefore \quad I_1 = \pi = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \quad \underline{\text{but}} \quad \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \arctan\Big|_{-\infty}^{\infty} = \pi/2 - (-\pi/2) = \pi \checkmark$$

<u>Example</u> $I = \displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx$ if $x = z$ $\cos x = \cos z = \frac{1}{2}\left(z + 1/z\right)$
this is not good since $\int$ doesn't vanish

Instead let $Re(e^{iz}) = \cos z$

$$I_1 = \oint \frac{e^{iz}\,dz}{1+z^2}$$ along $z = x$ we get the integral we want residues at $z = \pm i$ $\pm \frac{e^{-1}}{2i}$

$$\frac{\pi}{e} = \frac{e^{-1}}{2i} \cdot 2\pi i = \int_{-R}^{R} \frac{e^{ix}dx}{1+x^2} + \left|\int \frac{e^{i(x+iy)} i d\theta\, z}{1+z^2}\right| < \int_{0}^{2\pi} \frac{e^{ix-y} z i d\theta}{1+z^2} + \int_{-R}^{R} \frac{e^{ix}dx}{1+x^2}$$

$\to 0$

$$\therefore \quad \int_{-R}^{R} \frac{e^{ix}dx}{1+x^2} = \int \frac{\cos x\,dx}{1+x^2} + i\int \frac{\sin x\,dy}{1+x^2} \implies \int_{-\infty}^{\infty} \frac{\sin x\,dx}{1+x^2} = 0$$

$$\int \frac{\cos x\,dx}{1+x^2} = \pi/e$$

<u>Note</u>: $\displaystyle\int \frac{\sin x\,dx}{A + Bx^2 + Cx + \dots} \sim \frac{\sin x\,dx}{x^2}$ for large x. Integral over large circle vanishes in the limit if integrand behaves like $\frac{1}{z^{\alpha}}$ for large z, $\alpha >$

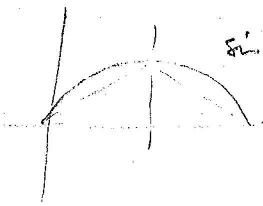
otherwise we may appeal to Jordan's Lemma

<u>Example</u> $I = \displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2}\,dx \sim \frac{1}{x}$ for large x – but from above $\Rightarrow$ we get non convergence. However the oscillatory factor is good – it makes the series conditionally convergent.

$I = \oint \frac{z e^{iz}\,dz}{1+z^2}$. Residue (at $z = \pm i$) $= \pm\frac{i e^{-1}}{2i}$

$= 2\pi i \cdot \frac{i e^{-1}}{\ } = \frac{\pi i}{\ } = \int^{R} \frac{x e^{ix}\,dx}{\ } + \int^{\pi} R e^{i\theta} e^{i(R\cos\theta + iR\sin\theta)} \frac{i\,d\theta\, e^{i\theta} R}{\ }$

$$\leq \int_{-R}^{R} \frac{x e^{ix} dx}{1+x^2} + \left| \int_0^{\pi} \frac{R^2}{1+R^2} e^{-R\sin\theta} d\theta \right| \qquad \left|\int_0^{\pi} e^{-R\sin\theta} d\theta\right| < \int_0^{\pi} e^{-R\dots} d\dots$$



$$\sin\theta > \frac{2}{\pi}\theta \qquad \therefore \quad -R\sin\theta < -\frac{2}{\pi}\theta R \qquad \& \quad e^{-R\sin\theta} < e^{-R\frac{2\theta}{\pi}}$$

$$\left|\int e^{-R\sin\theta} d\theta\right| < 2\int_0^{\pi/2} e^{-R2\theta/\pi} d\theta = -\frac{\pi}{2R}\left[e^{-2\theta/\pi R}\right]_0^{\pi/2} \Rightarrow \frac{e^{-R}}{2R}$$

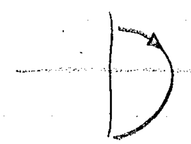
this $\to 0$ as $R \to \infty$

Jordan's Lemma. Suppose that on a circular arc C_R with radius R and center at the origin, $f(z) \to 0$ uniformly as $R \to \infty$, then

$$\lim_{R\to\infty} \int_{C_R} f(z) e^{imz} dz = 0 \qquad (m>0)$$

$$\int_{C_R} f(z) e^{-imz} dz = 0 \qquad (m>0)$$

$$\int_{C_R} f(z) e^{-mz} dz = 0 \qquad (m>0)$$



$$\int_{C_R} f(z) e^{mz} dz = 0 \qquad (m>0)$$

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Indented Contour: (Cauchy's Principal Value)

Example $\displaystyle\int_{-a}^{b} \frac{dx}{x} = \int_{-a}^{0} \frac{dx}{x} + \int_{0^+}^{b} \frac{dx}{x}$ $\qquad \displaystyle PV\int_{-a}^{b} \frac{dx}{x} = \lim_{\epsilon\to 0} \int_{-a}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{b} \frac{dx}{x}$

Normally $\displaystyle\int_{-a}^{b} \frac{dx}{x}$ doesn't exist but $\displaystyle PV\int_{-a}^{b} \frac{dx}{x} = \ln\frac{b}{a}$

$$= \lim_{\epsilon\to 0} \ln x \Big|_{-a}^{-\epsilon} + \ln x \Big|_{\epsilon}^{b}$$

$$\lim_{\epsilon\to 0} \left[\ln\frac{-\epsilon}{-a} + \ln b/\epsilon\right] = \ln\frac{b}{a}$$

$$PV\int_{-a}^{b} \frac{dx}{x^3} = \frac{1}{2}\left[\frac{1}{a^2} - \frac{1}{b^2}\right]$$

if $f(x) \uparrow \infty$ when $x = c$ is in the interval of integral then

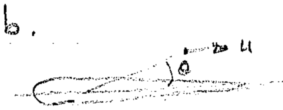
$$PV\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^{b} f(x)\,dx$$

also $PV\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$

Ex'ple $\int_{-\infty}^{\infty} x\,dx$ doesn't exist but $\lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty}\left[\frac{x^2}{2}\Big|_{-R}^{R}\right] = 0.$

Distribution of plane potential sources (of local stren $f(x)$) from $-a$ to $+b$.

Horiz velocity



$$u = \int_{-a}^{b} \frac{(x-\xi)}{(x-\xi)^2+y^2} f(\xi)d\xi$$

for u as $y \to 0^+$ $\quad u_{surf.} = PV\int_{-a}^{b}\frac{f(\xi)d\xi}{(x-\xi)}$ but it has a principal value

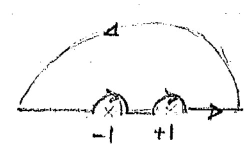
<u>Contour integration to evaluate</u> PV

$\int_{-\infty}^{\infty} \frac{\cos x}{1-x^2}dx \nexists$ since fn blows up at $x=\pm 1$

if we instead assume $PV\int_{-\infty}^{\infty}\frac{\cos x}{1-x^2}dx = \lim_{\substack{R\to\infty\\ \epsilon\to 0}} \int_{1+\epsilon}^{R} + \int_{-1+\epsilon}^{1-\epsilon} + \int_{-R}^{-1-\epsilon}$

Consider integration over some contour

$$\oint_C \frac{e^{iz}dz}{1-z^2} = 0$$



$$\int_{-R}^{-1-\epsilon} + \int_{-1+\epsilon}^{1-\epsilon} + \int_{+1+\epsilon}^{R} \frac{e^{ix}dx}{1-x^2} + I_R + I_1 + I_{-1} = 0$$ since no singularities within contour

$I_1 = -\pi i\,\frac{e^i}{-2} = \frac{\pi i e^i}{2}$ $\qquad I_{-1} = -\pi i\,\frac{e^{-i}}{2}$

integration in CW sense ½ contour

Proof for I_1 $\quad$ let $z = \epsilon e^{i\theta}+1$

$$\int_{\pi}^{0} \frac{\exp(i\epsilon e^{i\theta}+i)\,\epsilon e^{i\theta}\cdot i d\theta}{-\epsilon e^{i\theta}(2+\epsilon e^{i\theta})} \to \int \frac{\exp(i)\cdot i d\theta}{-2} = \frac{e^i\cdot i}{-2}(-\pi)$$
$$= \pi i e^i$$

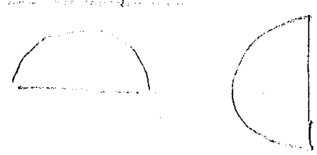
$$\therefore \quad I_1 + I_{-1} = -\frac{\pi}{2} i \left[\frac{e^{i} - e^{-i}}{2i}\right] = \frac{\pi i}{2}\left[e^{i} - e^{-i}\right]$$

$$\therefore \quad PV \int \quad = -(I_1 + I_2) = +\pi \sin 1$$

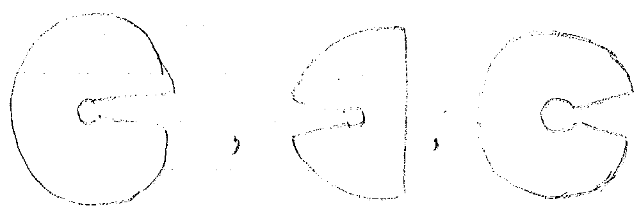
$$\left| \int_{+R}^{-R} \frac{e^{iz} dz}{1-z^2} \right| \le \int_{R}^{-R} \frac{dz}{|1-z^2|} \to 0 \quad \text{as } R \to \infty$$

$$PV \int_{-\infty}^{\infty} \frac{\cos x}{1-x^2}\, dx = \pi \sin 1$$

Integral Transforms

Use  if integrand is single-valued

or



for factors w/ branch cuts.

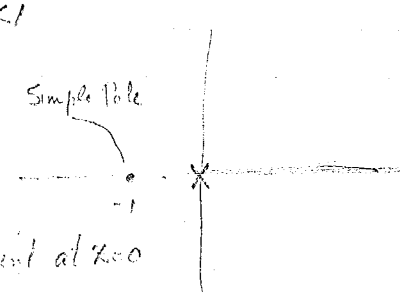
3. Evaluate $\int_0^\infty x^{a-1} f(x)\,dx$ & other multiple-valued fns ($a \neq$ integer)

Standard example

$$I = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\, dx \qquad (0 < a < 1$$

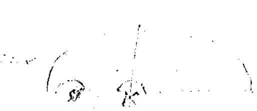
Consider

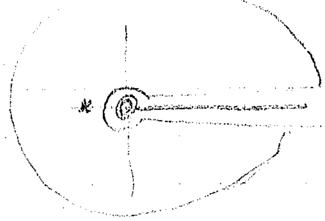
$$\int \frac{z^{a-1}}{1+z}\, dz$$



$\text{Res}\big|_{z=-1} = (-1)^{a-1}$; since $\exists$ a branch point at $z=0$

then $(-1)^{a-1} \Rightarrow$ must use principal value of $-1 = e^{i\pi}$

Choice of Contour  Large semicircle $\to 0$ but negative axis gives complex multiple $PV \int_0^\infty \frac{r^{a-1}}{1-r}\, dr$

Choose 

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_\epsilon^R \frac{x^{a-1}}{1+x}\,dx + \underbrace{\int_0^{2\pi}}_{\text{as } R\to\infty}{}^{=0} + \int_R^\epsilon \frac{(e^{2\pi i}x)^{a-1}e^{2\pi i}\,dx}{1+e^{2\pi i}x}$$

$$+ \underbrace{\int_{2\pi}^{0} \frac{(\epsilon e^{i\theta})^{a-1}}{1+\epsilon e^{i\theta}}\, i\,d\theta\, \epsilon e^{i\theta}}_{\to 0 \text{ as } \epsilon\to 0}$$

$$\therefore \int_0^\infty \frac{x^{a-1}}{1+x}\,dx + \int_\infty^0 \frac{e^{(2\pi i)(a-1)}\cdot e^{2\pi i}\, x^{a-1}\,dx}{1+e^{2\pi i}x} = 2\pi i\,(e^{i\pi})^{a-1}$$

$$\left[1-e^{2\pi i a}\right]\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = 2\pi i\,(e^{i\pi})^{a-1}$$

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{2\pi i\,(e^{i\pi})^{a-1}}{1-e^{2\pi i a}} = \frac{2\pi i\, e^{i\pi a}\cdot e^{-i\pi}}{e^{\pi i a}\left[e^{-\pi i a}-e^{\pi i a}\right]}$$

$$= \frac{\pi}{\sin \pi a}$$

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$$\frac{(1-z)(1+3z)}{1-z^3} = (1+2z-3z^2)(1+z^3+z^6+z^9+\dots)$$

Ratio Test doesn't work $\Rightarrow \frac{a_n}{a_{n+1}} = \frac{1}{2}, -\frac{2}{3}, 3$ we can't say anything

Root Tests work $R = \lim \frac{1}{|a_n|^{1/n}} \to 1$

What is "residue at ∞

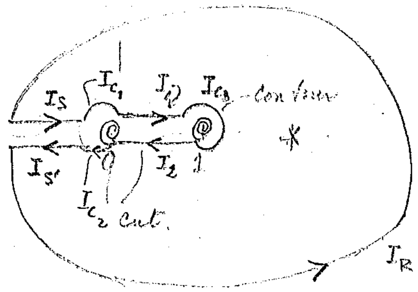
$$f(z) = \dots \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \dots$$

$$= a_{-2}\zeta^2 + a_{-1}\zeta + a_0 + a_1/\zeta + a_2/\zeta^2 + \dots$$

Residue should be a_{+1} – But according to Dennis ("Taylor Series" pg 238) Residue is $-a_1$. Markushevich (p. 53, II) Residue is $-a_{-1}$

Look at $\displaystyle\int_0^1 \frac{dx}{(q-px)\sqrt{x(1-x)}} = \frac{\pi}{\sqrt{q(q-p)}}$ $\quad 0<p<q$

in complex plane Consider $\int \frac{dz}{(q-pz)\sqrt{z(1-z)}}$ Choose branch so that integral is + & real on upper part



$I_R \to 0$ $I_S + I_S' = 0$

$I_{c_1} + I_{c_2} = 0$ $I_{c_3} = 0$

$I_1 + I_2 = 2\pi i \,\text{Res}\,(z = q/p)$

but $\int_0^1 F_1(z)\,dz = I_1$ $I_2 = \int_1^0 F_2(z)\,dz$

but branch in lower part of cut = − value in upper cut $\therefore F_2 = -F_1$ $\therefore I_1 + I_2 = 2I_1$

$$\therefore I_1 = \frac{1}{2}\cdot 2\pi i \frac{i \text{ from def of branch}}{-p\sqrt{q/p(1-q/p)}} = \frac{\pi}{\sqrt{q(1-q/p)}}$$

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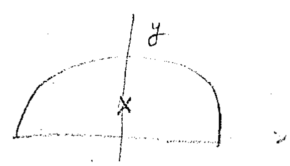
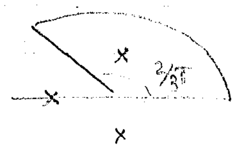
Midterm monday 11/5/79

① Tricks $\int_0^\infty \frac{dx}{1+x^2}$



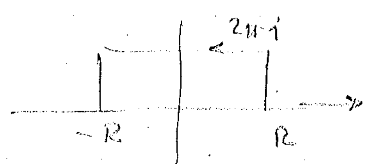
$\int_0^\infty \frac{dx}{1+x^3}$



Exercise #9 pg 84 $\int_0^\infty \frac{dx}{1+x^{300}}$

② Rectangular Contour

$$\int_{-\infty}^{\infty} e^{ax} f(e^x)\,dx$$

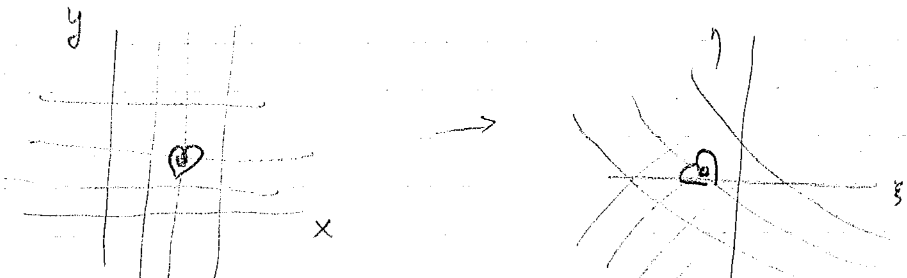


$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx \quad (0<a<1) = \int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a}$$

③ $\int_0^\infty \frac{dx}{x^2+3x+2}$ no way to complete contour $x = -2, -1$ Consider instead $\int \frac{\ln z}{z^2+3z+2}$

Conformal mapping & Laplace's Eqn in Plane

$\xi = \xi(x,y)$ $\eta = \eta(x,y)$

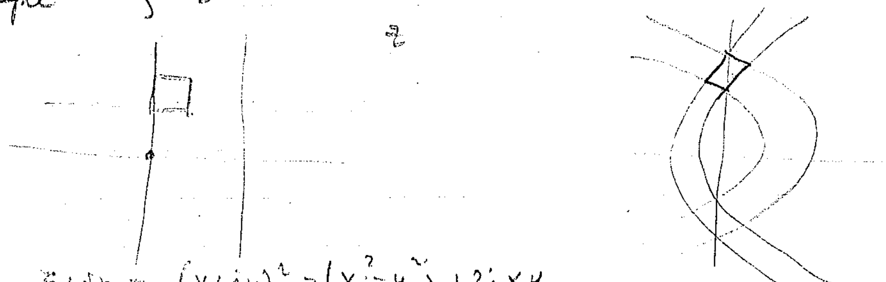


Conformal mapping given by

$$\zeta = F(z)$$
$$\zeta = \xi + i\eta$$
$$z = x + iy$$

orientation & angle preserving

Example $\zeta = z^2$



$$\xi + i\eta = (x+iy)^2 = (x^2-y^2) + 2i xy$$

$2xy = \eta \qquad \xi = x^2 - y^2$

This Example

orthogonal – local angles are preserved

conformal – small squares map into small squares

Polar Coords are not conformal – not give by an analytic complex fn.

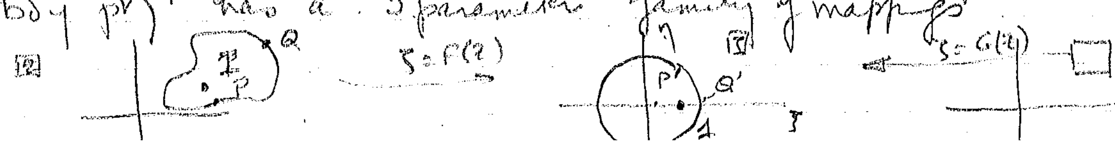
$$F' = \frac{d\zeta}{dz} \simeq \frac{\Delta\zeta}{\Delta z} \qquad \& \qquad \Delta\zeta \simeq F'\Delta z$$

– line element is shifted from z to ζ by rotating through $\arg F'(z)$ and magnified by $|F'|$

– area element is magnified by $\left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right| = |F'|^2$

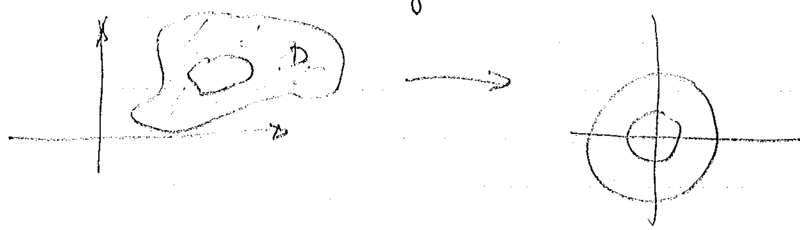
Riemann Mapping Theorem

A simply connected domain D in z plane (having more than 1 bdy pt) has a 3 parameter family of mappings



1. Transform $z=a$ to the origin and point on bdy to pt on bdy
2. $z=a$ to $0'$ and direction at $z=a$ to the North axis
3. Points on bdy to 3 pts on circle in same order

What about annular region



what is radius of circle?
whatever we get.

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$\zeta = F(z)$ Conformal mapping : $z = f(\zeta)$

1. Retain shape but are translated, rotated or magnified
2. Except at critical points where $f'(\zeta) = 0$

   Example $z = \zeta^2$ (angles are doubled at $\zeta = 0$ since $f'(0) = 0$

3. Riemann mapping theorem. given $z = z_0$ then $\zeta = z - z_0$

(Potential)
2-D Laplace Eqn is invariant under conformal mapping

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \equiv \nabla^2_{(x,y)} \equiv \nabla^2_{\xi,\eta} \times |F(z)|^2 \qquad \text{where } \frac{d\zeta}{dz} = F'(z)$$

$$\equiv \frac{1}{|f'(\zeta)|^2} \nabla^2_{(\xi,\eta)}$$

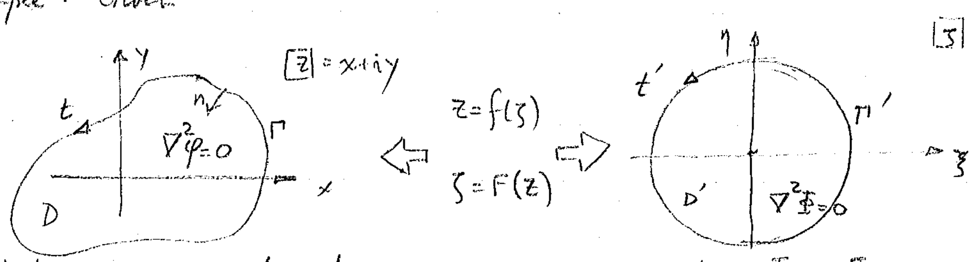
if $\nabla^2_{(x,y)} = 0$ & if $|F'(z)|^2 \neq 0 \Rightarrow \nabla^2_{(\xi,\eta)} = 0$

Harmonic fns are transformed into Harmonic fns in ξ, η
In contrast: ∇^4 is not harmonic — not conf.

Poisson eqn : $\nabla^2_{x,y}\varphi = G(x,y)$ $\quad \nabla^2_{\xi,\eta}\varphi = |f'(\zeta)|^2 G(\xi,\eta)$ — not conf.

Helmholtz eqs : $\nabla^2_{x,y}\varphi + k\varphi = 0 \Rightarrow \nabla^2_{\xi,\eta}\varphi + k|f'(\xi)|^2\varphi = 0$ — not conf.

Example : Given



Complex Potential : $\Omega(z) = \phi + i\psi$ $\qquad \chi(\zeta) = \Phi + i\Psi$

where ψ & ϕ satisfy C.R. eqn. $\qquad \chi(\zeta) = \Omega[f(\zeta)]$

if $\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi = 0 \Rightarrow \nabla^2\Phi = \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\Phi = 0$

w/ BC: $A(t)\,\phi + B(t)\,\frac{\partial\varphi}{\partial n} = C(t)$ on $\Gamma \Rightarrow A'(t')\,\Phi + B'(t')\,\frac{\partial\Phi}{\partial n'} = C'(t')$ on Γ'

$\Omega'(z)$: complex intensity
complex velocity in fluid mech.

1. Steady state heat cond. ϕ = temp
2. Electrostatic ϕ = potential
3. Potential fluid flow ϕ = veloc poten (velocity = grad ϕ)
ψ = stream f.

4. Solid Mech . for shaft warping fn. ϕ
$$BC \quad \left(\frac{\partial\phi}{\partial x} - y\right)\frac{dx}{dt} + \left(\frac{\partial\phi}{\partial y} - x\right)\frac{dy}{dt} = 0 \text{ on } \Gamma$$

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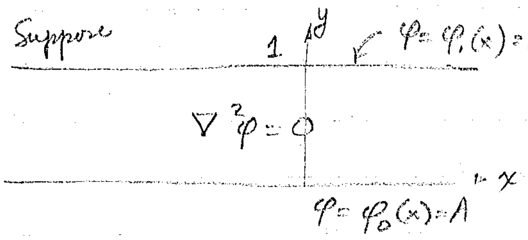
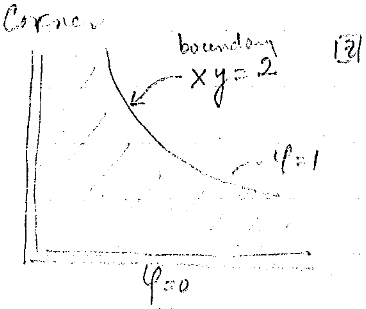
(*) Suppose



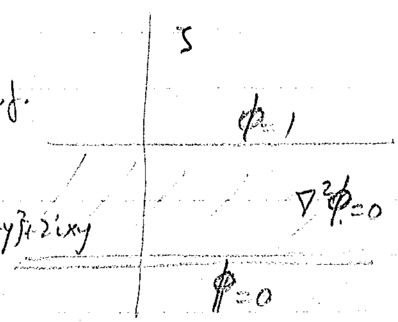
Obviously
$$\phi(x,y) = A + y(B-A)$$
$$\Omega = \phi + i\psi = A - i(B-A)z$$

Simple mappings of harmonic fn.

(A) Corner



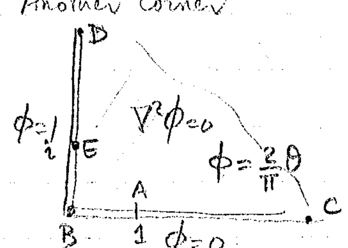
underhamf. $\Rightarrow$
$\zeta = z^2$
$= (x^2-y^2) + 2ixy$



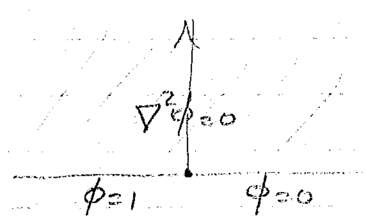
$\chi = \Phi + i\Psi = -i\zeta$

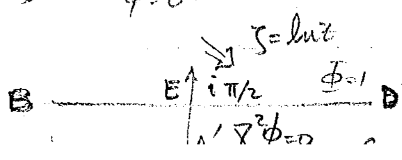
$$\Omega = \phi + i\psi = -iz^2$$
$$\phi = 2xy \qquad \psi = -(x^2-y^2)$$
we can now take the soln for (*)

(B) Another Corner

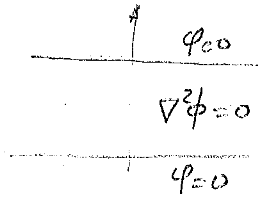


$\zeta = z^2$
$\Longrightarrow$





Eigensoln.

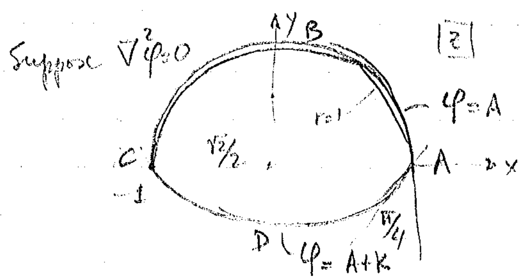


Can add soln of trivial proble to all particular soln.
Trivial sol is $\text{Im}\, C_n e^{i(2n-1)\pi z}$

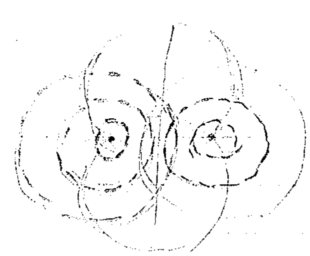
Transformations
Fractional power
Logarithmic
Bilinear transformation $\quad \zeta = \dfrac{az+b}{cz+d} \qquad ad - cb \neq 0$
map circles into circles

Example of transformation p127 CKP

Suppose $\nabla^2\varphi = 0$



Bipolar Coords - lenticular region



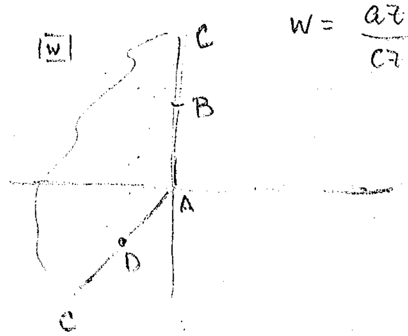
Using Bilinear Transformation to map one of corners at ∞
map $z = -1$ to ∞ to get a right corner at ∞
Denom. should thus be mult. of $z+1$. next for simplicity map right corner to origin: ie let num be mult of $z-1$

Let

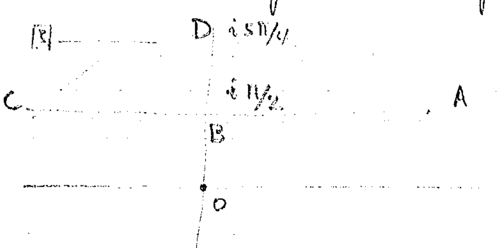
$$w = \frac{az+b}{cz+d} = \text{const}\frac{(z-1)}{(z+1)}$$

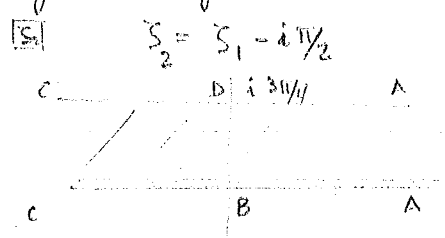
take as simple choice $w = \dfrac{z-1}{z+1}$

near $z = 1 \quad w = \frac{1}{2}(z-1) + O(z-1)^2$



next map into a strip using log $\quad \zeta_1 = \log w$

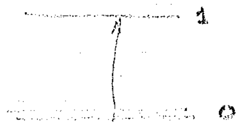


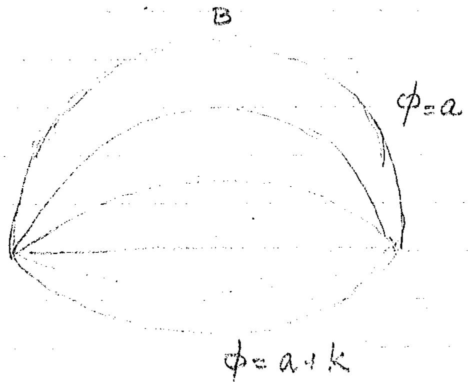


$\zeta = \zeta_2 \cdot \frac{4}{3}$ to magnify the strip to unit strip

$$\zeta = \frac{4}{3}\zeta_2 = \frac{4}{3}\left[\zeta_1 - i\pi/2\right] = \frac{4}{3}\left[\ln w - i\pi/2\right] = \frac{4}{3}\left[\ln\left(\frac{z-1}{z+1}\right) - i\pi/2\right]$$

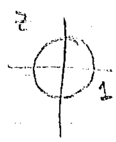
Thus ~~ϕ~~ ϕ



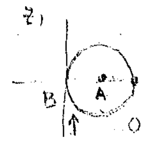


Connection between "Standard" regions

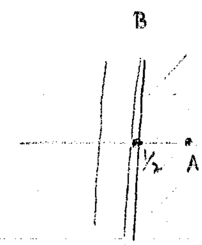
Unit circle $\Rightarrow$ half plane



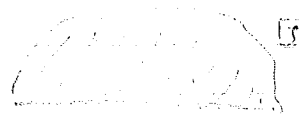
let $z_1 = z+1$



invert in unit circle $w = \frac{1}{z_1}$



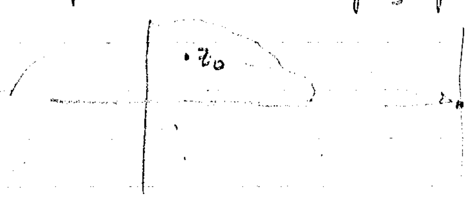
$\rightarrow$



$$\zeta = (w - 1/2)\,e^{i\pi} = \left(\frac{1}{z+1} - \frac{1}{2}\right)e^{i\pi} = \frac{1-z}{2(z+1)}\,e^{i\pi}$$

Most general mapping of unit circle in z-plane onto upper half ζ-plane

$$\zeta = e^{i\theta_0}\,\frac{z - z_0}{z - \bar{z}_0}$$



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notes on Midterm

If a fn has isolated singularities, the sum of the res at all the singularities must equal zero.

Thus res of $\frac{A}{z}$ at $z=0$ is A

res at " at $z=\infty$ is $-A$

Analytic Cont.

(1) Classic method of approach is by Taylor's Thm

(2) Another method is to sum series. If $f(z) = \sum_{n=0}^{\infty} (-1)^n z^n$ converges $|z|<1$ then $f(z) = \frac{1}{1+z}$ converges $\forall z \neq -1$

(3) method for midterm problem is inspection

for a fn given on real axis: $x \to z$

" " " " " imag " : $y \to -iz$

" " " " " unit circle: $\theta \to -i\ln z$ $\quad z = e^{i\theta}$

(4) Transform variable (ie conformal map). Transformed series may have a different rad of convergence. A particularly useful transf. is the Euler transf. (special case of bilinear trans. $\zeta = \frac{az+b}{cz+d}$, with $b=0$

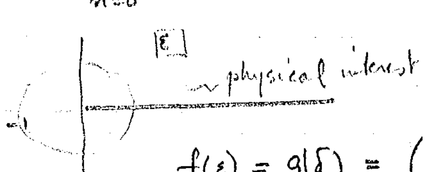
$z=0 \Rightarrow \zeta = 0$ (truncated series) Supposing $f(\varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - + \dots - \frac{1}{13}\varepsilon^{13}$

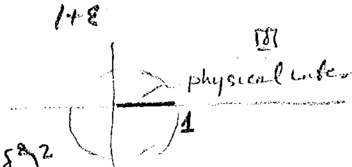
Alternating signs - nearest sing is on negative real axis. The ratio test gives rad of convergence is 1

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = 1$$

To allow us to use this representation along the entire positive axis (the region of physical interest) use: $\delta \frac{\varepsilon}{1+\varepsilon}$

$$f(\varepsilon) = -\sum_{n=0}^{\infty} \varepsilon^n(-1)^n + 1 = -\frac{1}{1+\varepsilon} + 1 = \frac{\varepsilon}{1+\varepsilon} = \delta \qquad \varepsilon = \frac{\delta}{1-\delta}$$





$$f(\varepsilon) = g(\delta) = \left(\frac{\delta}{1-\delta}\right) - \frac{1}{2}\left(\frac{\delta}{1-\delta}\right)^2 + \dots$$
$$= (\delta + \delta^2 + \delta^3 + \dots) - \frac{1}{2}(\delta + \delta^2 + \delta^3 + \dots)^2 + \dots = \delta + \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3 + \dots$$

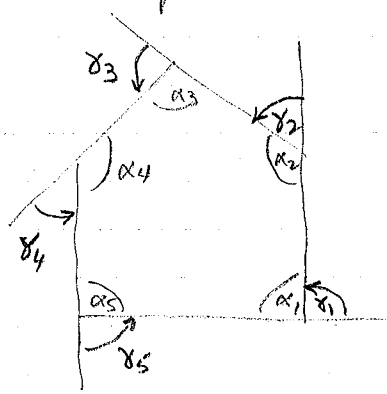
Converges for $|\delta| < 1$?

Conformal Mapping

"Strange angles" – described by C K & P

Sum of interior angles of n-gon $= (n-2)\pi$

Sum of exterior angles $= 2\pi$



exterior angles: measured from the extended line (which forms one side of n-gon) to the adjacent side.

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Singular Solutions of Plane Laplace Eqn.

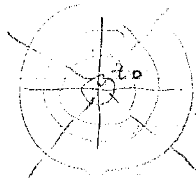
$$\nabla^2 = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\varphi = 0$$

Axysim requires solution of $\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\varphi = 0$ $\qquad \varphi = A \ln r + B$

$\Omega = A \ln z + B = \phi + i\psi \qquad \psi = \theta$

Examples:

1. $\Omega = q \ln(z - z_0)$ represents a source at $z = z_0$ of strength q (or $2\pi q$)



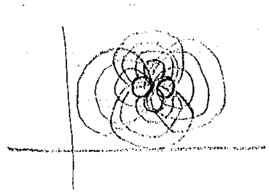
– streamlines

– equipot (veloc pot)

2. Vortex at $z = z_0$ Strength of p (or $2\pi p$) $\quad \Omega = i p \ln(z - z_0)$

reverses streamlines & equipot.

3. $\Omega = i\psi + \varphi = \frac{m}{z - z_0}$ – Dipole or doublet at $z = z_0$ strength m is obtained from source by differentiation



Further differentiation of the dipole gives higher order poles

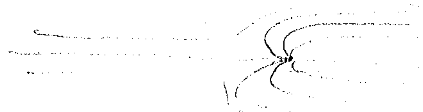
quadrapoles $\frac{1}{(z-z_0)^2}$

octapoles $\frac{1}{(z-z_0)^3}$

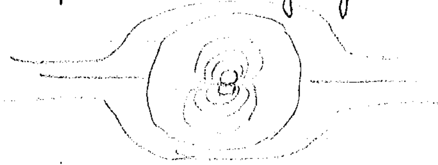
1. $\Omega = z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ represents uniform flow
$\psi = y \quad \phi = x$ in $+x$ direction

Inverse soln.

(1) Source in a uniform field
$$\Omega = Uz + C\ln z$$

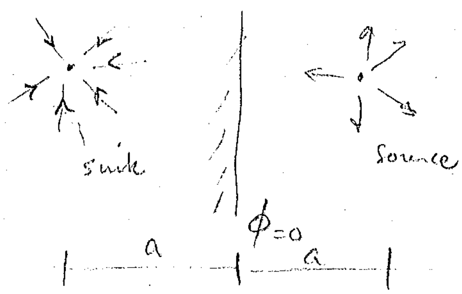


(2) Dipole in a uniform field



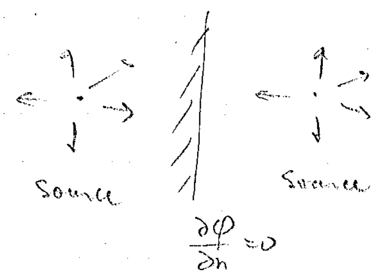
$$\Omega = Uz + \frac{m}{z - z_0}$$

Method of Images.



Put imaginary sink behind wall

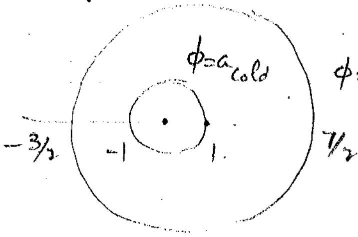
In heat transfer this ⏚ wall is isotherm



Put imaginary source behind wall

heat transfer – wall is insulator

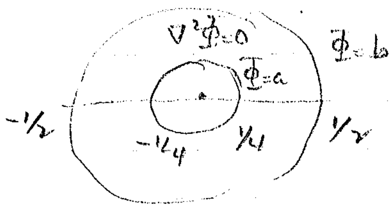
Steady heat conduction between concentric circles (CK&P Ex 6, Pg 129)



Use bilinear to map into concentric circles
$$\zeta = \frac{az+b}{cz+d}$$

$$\frac{az+b}{cz+d} \longrightarrow \frac{z+b}{z+d} \longrightarrow \frac{z+b}{z+1/b} \longrightarrow \frac{z+1/4}{z+4}$$

no magnif — maintain unit circle symm.



$F(\zeta) = \Phi + i\Psi = A \ln \zeta + B$ imposing
bc on z circles give A & B

$$\Omega(z) = \varphi + i\psi = \underbrace{(2b-a)}_{B} + \underbrace{\frac{b-a}{\ln 2}}_{A} \ln\left(\frac{z-1/4}{z+1/4}\right).$$

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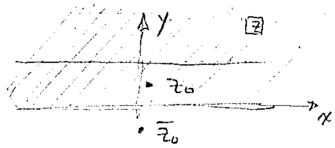
Mapping a half plane into circle of unit rad.



$$\boxed{w = \frac{z - z_0}{z - \bar{z}_0}\, e^{i\alpha}}$$

General bilinear $\frac{az+b}{cz+d}$

for our case we have 3 unknowns by Riemann mapping. However 2 of them are fixed, when we pick z_0 (ie x_0, y_0).

we want z_0 into origin : $a=1$ $b = \bar{z}_0$

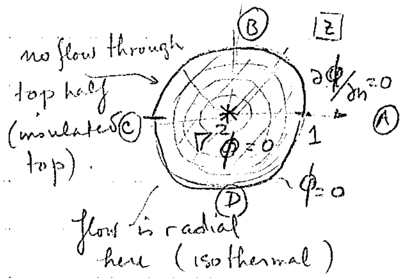
we want $z = \infty$ to map into unit circle : $\left|\frac{1}{c}\right| = 1$ since $w \sim \frac{1}{c}$ take $c=1$

we want $z=0$ into unit circle $\left|\frac{-z_0}{-z_\phi}\right| = 1$ $\left(w = \frac{z-z_0}{z-z_1}\right)$ $\therefore z_1 = \bar{z}_0$

finally, multiply by $e^{i\alpha}$ to rotate circle in any direction.

Inverse $z = \frac{e^{i\alpha} - w}{e^{i\alpha} + w}\, z_0$

CK&P Source a center of circle with mixed bc.



Source at origin $\Omega = \phi + i\psi \sim q \ln z$

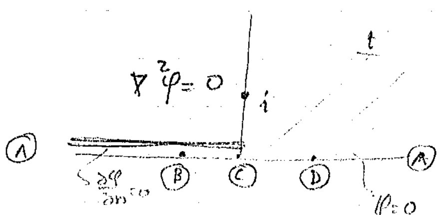
Suppose we don't know how to solve this

Map into upper half plane $w = e^{i\alpha}\frac{z - z_0}{z - \bar{z}_0}$

$z = e^{i\alpha}\frac{t - t_0}{t - t_0^*}$

map origin to point i

	z	t
A	1	∞
B	i	-1
C	1	0
D	$-i$	1



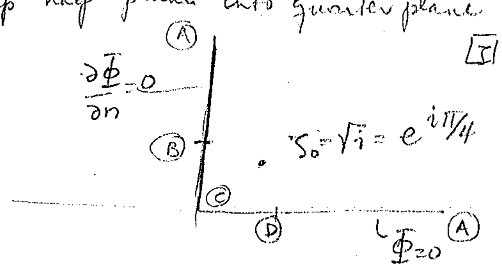
top half mapped to Ⓐ - Ⓒ

take $t_0 \to i$

$\alpha = 0$

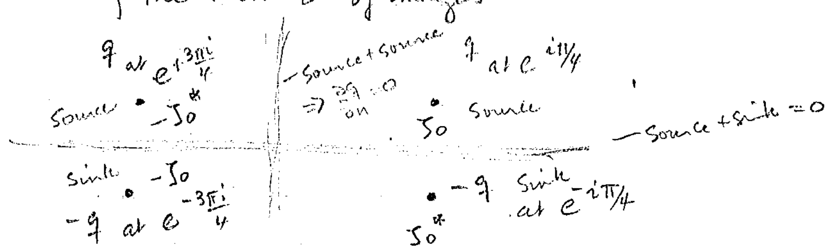
$$\boxed{z = \frac{t-i}{t+i} \qquad t = i\,\frac{1+z}{1-z}}$$

map half-plane into quarter plane



$\chi = \Phi + i\Psi$

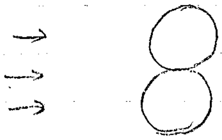
Using the method of images



thus

$$\chi = \ln\frac{(\zeta - \zeta_0)(\zeta + \zeta_0^*)}{(\zeta - \zeta_0^*)(\zeta + \zeta_0)}$$

flow past a figure. 8



Flux & circulation

$$\text{Flux} = \int_C d\psi$$

$$\text{Circulation} \quad \Gamma = \int_C d\phi$$

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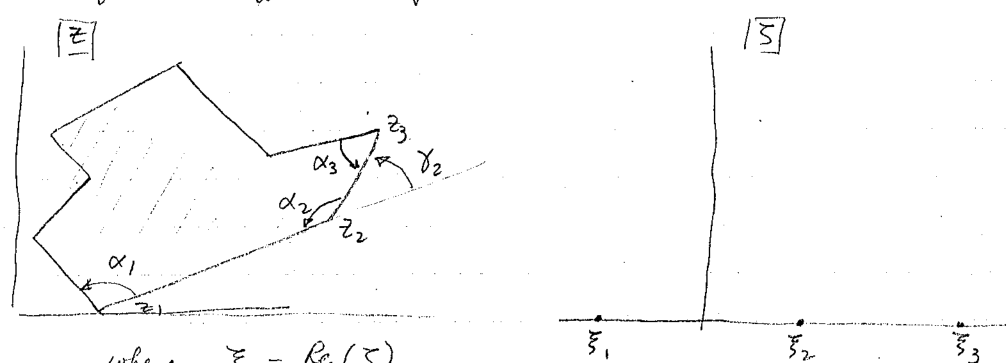
Schwarz - Christoffel Transformation

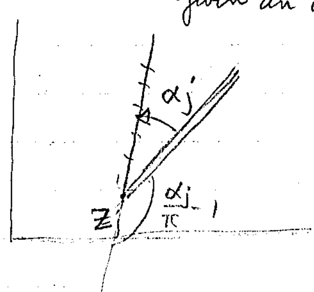
mapping



where $\xi_1 = Re(\zeta_1)$
and $\zeta_1 = f(z_1)$

Usually map one vertex to $\zeta = \infty$. Since 3 pts fix the body in z plane
$\Rightarrow$ we can map any 3 pts to whatever we want. Rest fall wherever they may

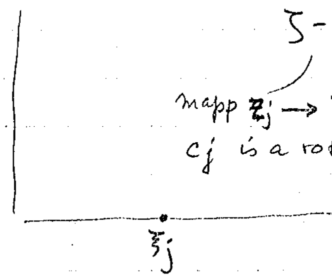
Given an arbitrary corner



$$\zeta - \xi_j = c_j (z - z_j)^{\pi/\alpha_j}$$

mapp $z_j \rightarrow \xi_j$
c_j is a rotation

opens angle to $180°$



$$z - z_j = A_j (\zeta - \xi_j)^{\alpha_j/\pi}$$

We however only need differential form.

$$dz/d\zeta = a_j (\zeta - \xi_j)^{\alpha_j/\pi - 1}$$

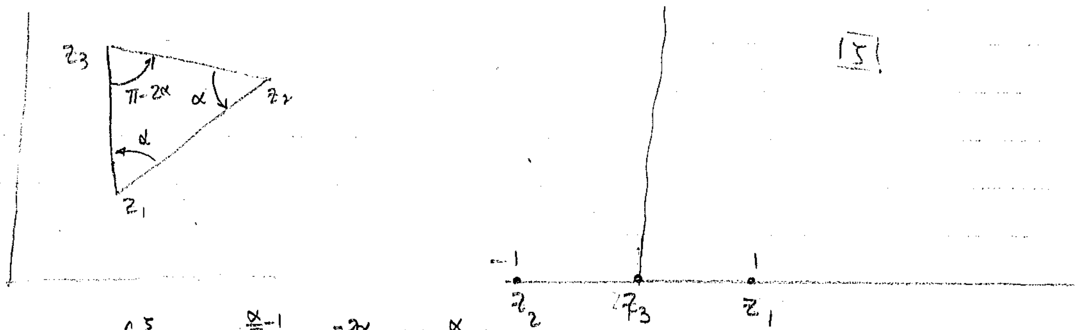
for several corners $\quad dz/d\zeta$ = products of $\frac{dz}{d\zeta}$ for one corner

$$dz/d\zeta = a(\zeta - \xi_1)^{\alpha_1/\pi - 1}(\zeta - \xi_2)^{\alpha_2/\pi - 1} \cdots$$

thus $$\boxed{z = \int_0^{\zeta} a \prod_{j=1}^{n} (\zeta' - \xi_j)^{\frac{\alpha_j}{\pi} - 1} d\zeta' + b}$$

b shifts locatn. in z plane
a magnifies & rotates

Exaple
Isoc. Triangle.



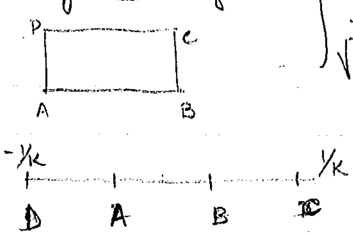
$$z = a \int^{\zeta} (\zeta'+1)^{\frac{\alpha}{\pi}-1} (\zeta'-0)^{-\frac{2\alpha}{\pi}} (\zeta'-1)^{\frac{\alpha}{\pi}-1} d\zeta' + b$$

$$= a \int^{\zeta} \frac{d\zeta'}{(\zeta'-0)^{\frac{2\alpha}{\pi}} (\zeta'^2 - 1)^{1-\alpha/\pi}} + b$$

for equil $\alpha = \pi/3$

integrals are simple in terms of elem. fns. only if exponent $\frac{\alpha_j}{\pi} - 1$ are integers or half-integers and not more than 2 of the latter.

for rectangle



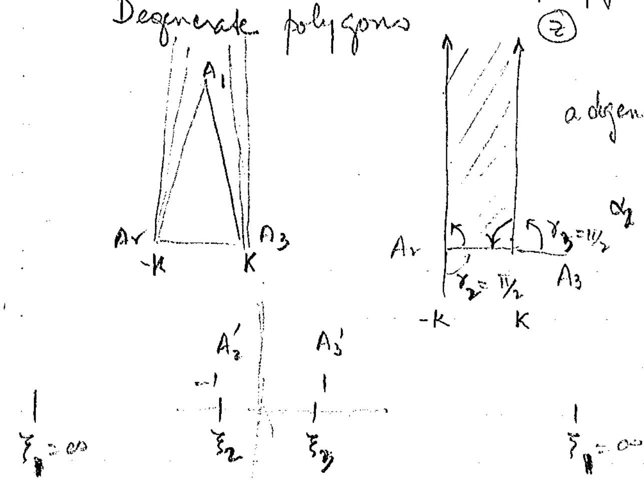
$$\int \frac{d\zeta}{\sqrt{1-\zeta^2}\sqrt{1-k^2\zeta^2}}$$

this is an elliptic integral.

So this is tough to do since most polygons don't satisfy the above.

We can look at degenerate polygons as closed polygons with on point mapped to ∞

Degenerate polygons



a degenerate isosc triangle with vertex at ∞

$\alpha_2 = \pi/2 \quad \alpha_3 = \pi/2 \quad \alpha_1 = 0$ at ∞

$\gamma_1 = \pi \Rightarrow \alpha_1 + \gamma_1 = \pi$ or $\alpha_1 = 0$

mapping $A_2 \to \xi_2 \quad A_3 \to \xi_3 \quad A_1 \to \infty$

for corners at ∞

$$a(\zeta-\xi_j)^{\frac{\alpha_j}{\pi}-1} \sim \underbrace{a(-\xi_j)^{\frac{\alpha_j}{\pi}-1}}_{\substack{\text{keep this fixed} \\ \text{as } \xi_j \to \infty}} \underbrace{\left(1-\zeta/\xi_j\right)^{\frac{\alpha_j}{\pi}-1}}_{\hookrightarrow \text{ goes to } 1}$$

thus for this problem

$$z = a\int_0^{\zeta} (\zeta'-1)^{-1/2}(\zeta'+1)^{-1/2}\,d\zeta' + b$$

$$a\int_0^{\zeta} \frac{d\zeta'}{\sqrt{\zeta'^2-1}} + b = a\cosh^{-1}\zeta + b.$$

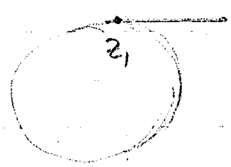
or $\quad a_2\int_0^{\zeta}\frac{d\zeta'}{\sqrt{1-\zeta'^2}} + b = a_2\sin^{-1}\zeta + b.$

$\zeta = 1 \Rightarrow a_2\cdot\pi/2 + b = k$
$\zeta = -1 \Rightarrow -a_2\pi/2 + b = -k$ $\Rightarrow b=0$, $a_2 = \frac{k\cdot 2}{\pi}$

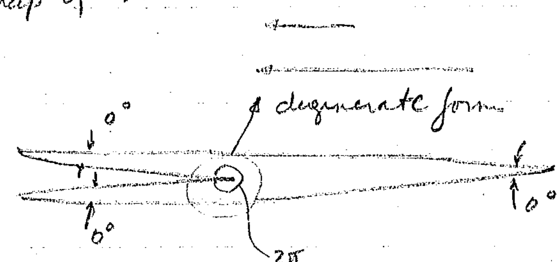
now locate original half strip as shown (by symmetry) $b=0$ & $a_2 = \frac{2k}{\pi}$

Most examples are degenerate polygons

Example



map $z_1 \to \infty$



gives rise to linear ~~[illegible]~~ factors

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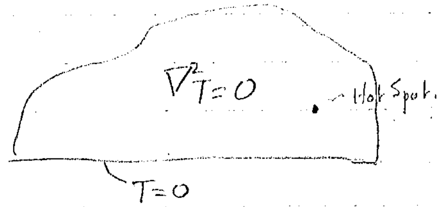


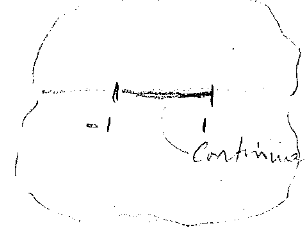
We can analytically continue into negative real region by use of method of images or by Schwarz Reflection principle.

Reflection Principle

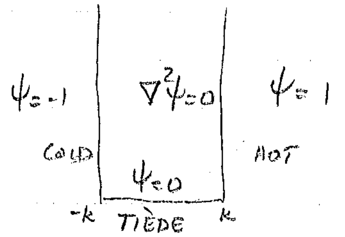
if $f(z)$ is analytic in region of upper half plane & real on a segment of real axis, Then analytic continuation into lower half plane is $\quad f^*(z^*) = \overline{f(\bar{z})} = f(z)$

if $f(z) = u(x,y) + i v(x,y)$ $\qquad \overline{f(\bar{z})} = u(x,-y) - i v(x,-y)$
$f(\bar{z}) = u(x,-y) + i v(x,-y)$

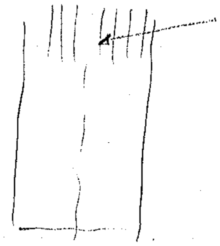
ie $f(z) = \sqrt{1-z^2}$



Now look back to problem of last time for the half strip.

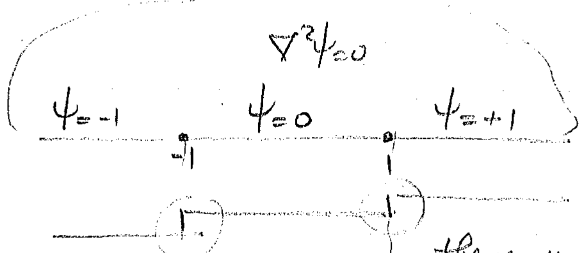


can add any mult. of
$\psi = \cos\frac{\pi x}{2k} \sinh\frac{\pi y}{2k}$; $\psi(x = \pm k) = 0$; $\psi(y=0) = 0$
which grows exponentially upward. This ψ is an eigensoln.

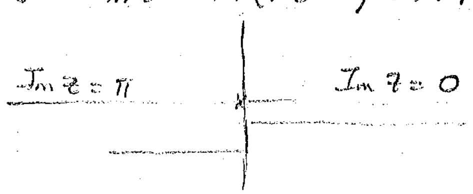


$\psi \sim \frac{x}{k}$ very far from lower boundary – Lower boundaries effects die out very quickly & "steady state" in the spatial variable is $\psi \sim \frac{x}{k}$.

But applying the schwarz christoffel



these jumps can be taken care of by logarithms $\ln z = \ln(re^{i\theta}) = \ln r + i\theta$



obviously $\chi = \Phi + i\Psi$

for $\zeta > 1$ $\quad = i \quad (\Psi = 1)$

for $|\zeta| < 1$ $\quad = -\frac{1}{\pi}\ln(\zeta - 1) \qquad \Psi = 0 = \Psi_{\zeta>1} - \text{jump}$

for $\zeta < -1$ $\quad = -\frac{1}{\pi}\ln(\zeta + 1) \qquad \Psi = -1 = \Psi_{\zeta>-1} - \text{jump}$

thus $\chi = \Phi + i\Psi = i - \frac{1}{\pi}\ln(\zeta^2 - 1)$

but from last time $z = \frac{2k}{\pi}\sin^{-1}\zeta$ or $\zeta = \sin\frac{z\pi}{2k}$

$\therefore \quad \Omega(z) = \chi[\zeta(z)] = i - \frac{1}{\pi}\ln\left[\sin^2\frac{z\pi}{2k} - 1\right]$

thus $\Omega' = \frac{1}{k}\tan\frac{\pi z}{2k}$ as $z \uparrow$ along the imag axis $\tan\frac{\pi z}{2k} = \tanh\frac{\pi z}{2k}$

as $|z| \uparrow \infty$ then $\tanh\frac{\pi z}{2k} \to 1$ $\therefore \Omega' \to \frac{1}{k}$ which is what

we get when we note that for large z $\varphi \sim \frac{x}{k}$ $\frac{d\Omega}{dz} = \frac{d\varphi}{dx} \sim \frac{1}{k}$

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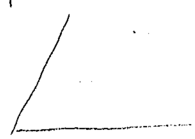
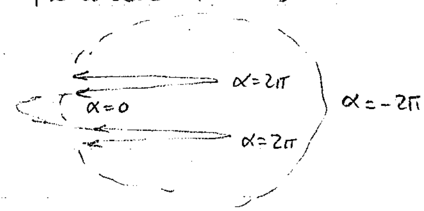
Polygons from Mapping by Elementary Functions

A. Only one vertex, arbitrary angle

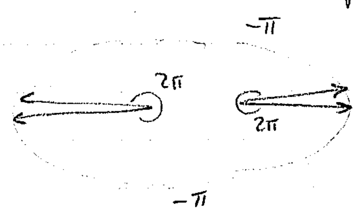


B. Any no. of vertices all angles mult of π C, K & P Ex 8

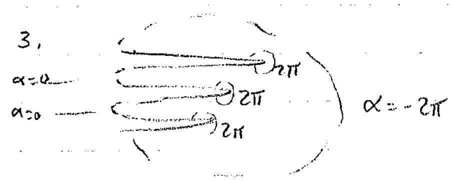
1. Parallel Plates



2. Flow ~~in slit~~ between plates C K & P Ex 9.

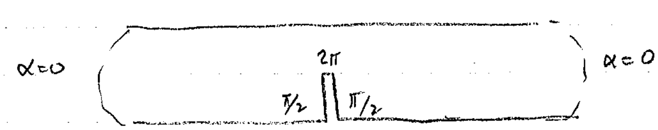


3. C K & P Ex 10

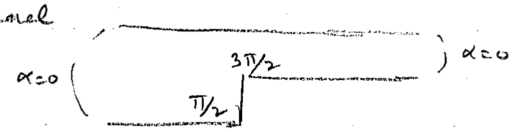


C. Any number of vertices, all angles mult of π except two that are mult of $\pi/2$

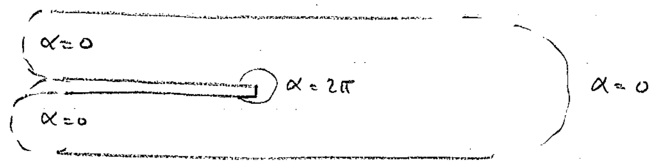
1. Infinite strip with vertical plate ex. 11



2. Step in channel Ex. 13

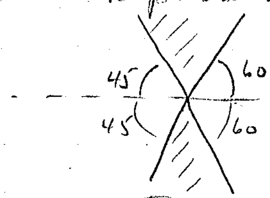


3. Slab midway between walls Ex. 15



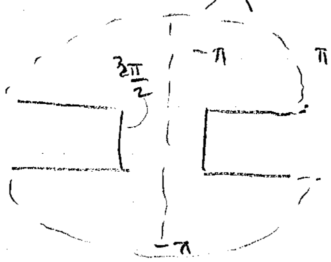
D. Reducible to previous cases A or B by symmetry

1.



Ex 18

2.

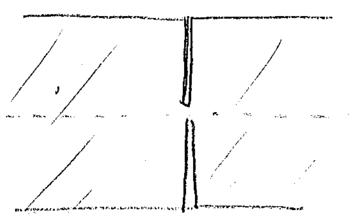


exploit symmetry about central vertical line

Ex 19

3. Flow through Constriction

Ex 12

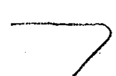


<u>Variations of Schwarz-Christoffel Mapping</u>

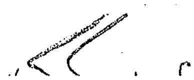
1. Standard: mapping interior of polygon into upper half plane
2. Interior of poly into interior of unit circle
3. Exterior of poly into upper half plane CK&P pg 152
4. " " " " exterior of unit circle CK&P ex 25

<u>Methods of rounding off corners</u>

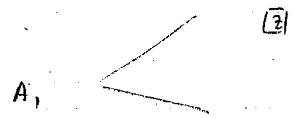
1. Add a small circular fillet

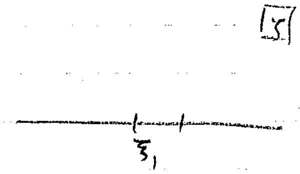


2. Choose a nearly equipot line as corner



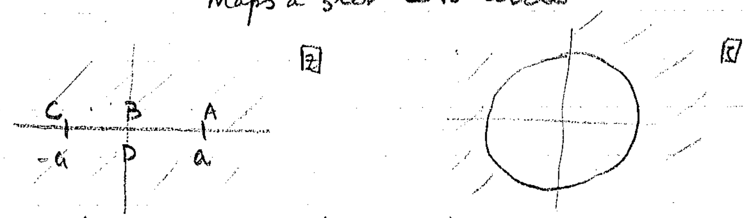
3. Add a pair of Schwarz Christoffel factors. not really explained in C,K&P





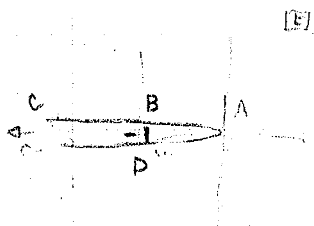
Joukowsky Transform.

Maps a slit into circle



This mapping can be done by

1. mapping exterior of poly (slit) to exterior of circle shown in CP/K
2. A chain of 2 bilinear mappings separated by a squareroot mapping

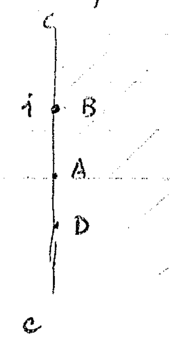


$\frac{z-a}{z+a}$

Semi infinite Strip

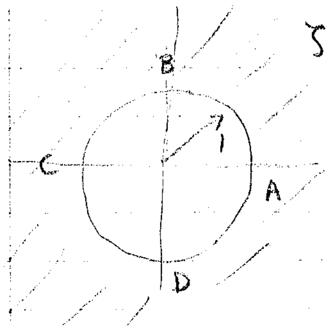
$z = a\left(\frac{1+t}{1-t}\right)$

$\Rightarrow$



$\boxed{w} = \sqrt{t}$

right half plane



$\zeta = \frac{1+w}{1-w}$

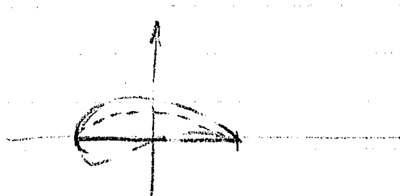
$$\frac{z-a}{z+a} = \left(\frac{\zeta-1}{\zeta+1}\right)^2 \quad \text{or}$$

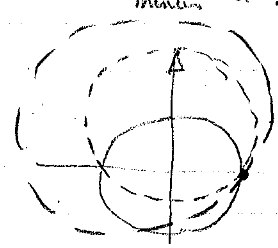
$$z = a/2\left(\zeta + 1/\zeta\right)$$

$$\zeta = 1/a\left(z \pm \sqrt{z^2-a^2}\right)$$

Plus sign : exterior of circle for large z $\zeta \sim \frac{z\pm}{a}$

Minus " : interior

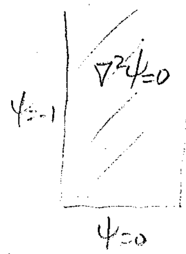


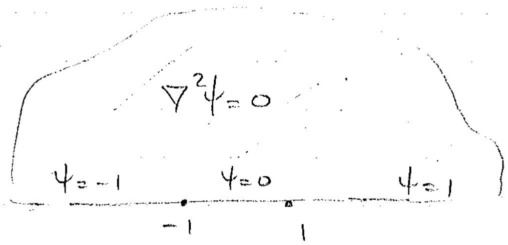


11/21/79

Positive unit source at origin : $\Omega = \phi + i\psi = \ln z$ see 11/16/79 notes

" " dipole " " : $\frac{d}{dz}(\ln z) = \frac{1}{z}$

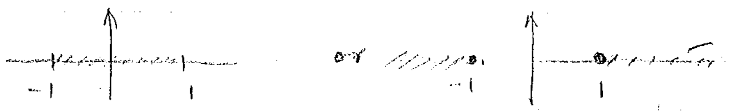




map into

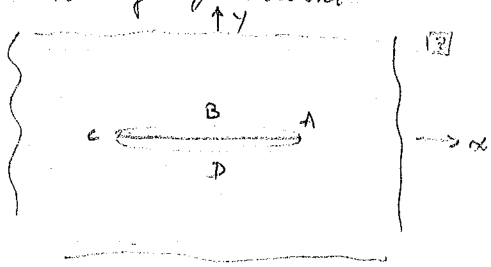
$$\chi = i\,c_n - \frac{1}{\pi}\sum (c_{j-1} - c_j)\ln(\zeta - \xi_j)$$ most general due to branch cuts

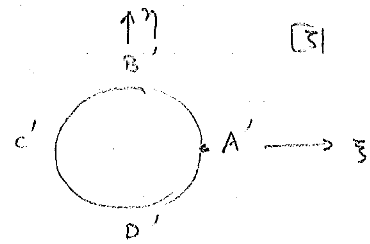
$$\chi = \Phi + i\psi = i - \frac{1}{\pi}\ln(\zeta - 1) - \frac{1}{\pi}\ln(\zeta + 1) = i - \frac{1}{\pi}\ln(\zeta^2 - 1)$$



branch cuts.

Application of Joukowski





$z = \frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right) \implies \zeta = \frac{1}{a}\left(z + \sqrt{z^2 - a^2}\right)$ maps to exterior of circle

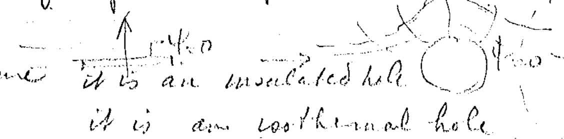
any transformation leaving distant field unchanged in form $z \sim \text{const} \times \zeta$ ~~form~~ has laurent expansion

$$z = a_1\zeta + a_0 + \frac{a_{-1}}{\zeta} + \dots$$

Joukowski is simplest non trivial case of this kind

Soln of laplace Eq. for this case are:

① Uniform horiz field in z plane $\Omega = \phi + i\psi = Uz$ $\phi = Ux$ $\psi = Uy$

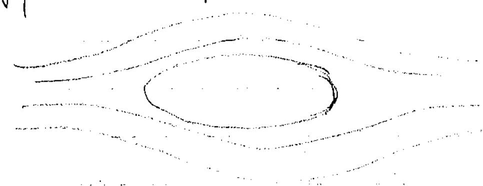


if ϕ is temperature it is an insulated hole

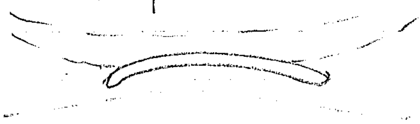
if ψ " " it is an isothermal hole

② if we rotate circle $\pi/2$ & map back flow about plate

③ if we magnify circle & map back

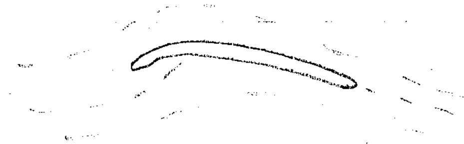


④ if we raise & magnify circle to pass through $\zeta = \pm 1$ & map back

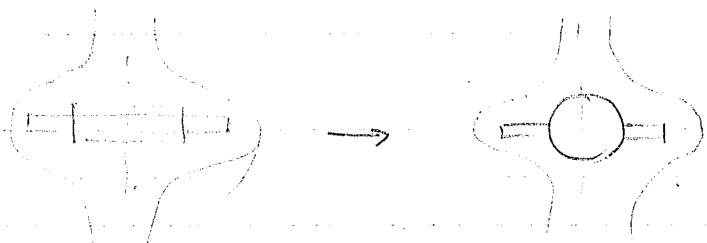


flow around trailing edge results in ∞ veloc & vortices, must apply Kutta cond. in ζ space add circul. to circle – equiv to raising & magnifying to pass beyond front point

⑤ Raise & magnify circle to pass thru $\zeta = +1$, but outside Elsewhere



⑥

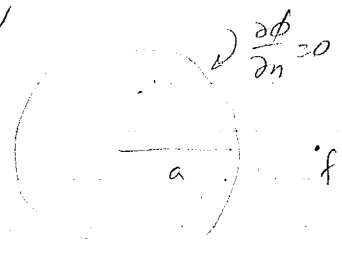


Integral Transforms
Fourier (exp, cos, sine)
Laplace
Hankel
Mellin

"good book"
C J Tranter Integral transform in Math Phys.

11/26/79

HW



Using Milne-Thompson Circle theorem
$$\Omega = q\ln(z-f) + q\ln\left(z-\frac{a^2}{f}\right) - q\ln z$$

must exist since, on inside without it, we would have a source of unit strength & would violate continuity

Transfer on transforms - we temporarily exclude one of the indep. variable to reduce dependency on the number of indep variables by 1.

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0 \quad \text{Try } X(t) = e^{kt} \Rightarrow (k^2 + ak + b)e^{kt} = 0$$

reduce ODE to algebraic eqn.

integral transform: $\bar{F}$ of $f(t)$

$$F(s) = \int_a^b \underbrace{K(s,t)}_{\text{known Kernel}} f(t)\,dt$$

Laplace Transform $\quad F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Fourier Complex $\quad F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{i\lambda t} f(t)\,dt$

Fourier Sine / Cos $\quad F(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \begin{matrix} \sin(st) \\ \cos(st) \end{matrix} f(t)\,dt$

Procedure

1. Eqn should be linear PDE w/ linear BC
   a. Select appropriate transformation, guided by examples of experience
2. mult diff eq by kernel & integrate wrt variable to be excluded.
3. Integrate by parts to eliminate deriv, using initial & bc.
4. solve the transform problem
5. Invert the transformation

Erdélyi: Table of Integral Transforms, Vol 1: McGraw-Hill 1954

Show table of transforms

$f(t)$	$F(s)$
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\,dt$
1	$1/s$
t	$1/s^2$
t^n	$n!/s^{n+1}$

$f(t)$ $\qquad$ $F(s)$

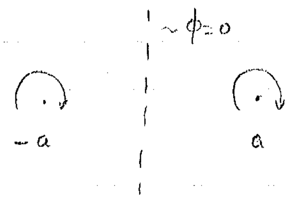
$e^{-\alpha t}$ $\qquad$ $\dfrac{1}{s+\alpha}$

let $\alpha = i\alpha_1$

$\cos\alpha_1 t + i\sin\alpha_1 t \qquad \dfrac{1}{s+i\alpha_1} = \dfrac{s}{s^2+\alpha_1^2} + i\,\dfrac{\alpha}{s^2+\alpha_1^2}$

$\cancel{\dfrac{e^{-a/4t}}{t}}$ $\quad$ $\cancel{\text{erf} t}$ $\quad$ $\text{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$ $\quad$ $\cancel{\text{erf}\left(\frac{a}{2}\right.}$ $\quad$ $\cancel{\dfrac{e^{-a\sqrt{s}}}{s}}$ $\quad$ $\dfrac{e^{-a\sqrt{s}}}{s}$

11/28/79



for vortex

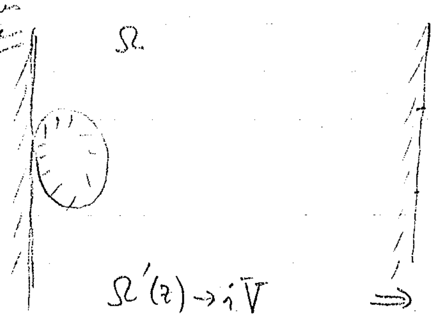
$\Omega = i\ln(z-a) + i\ln(z+a) =$

$\phi \neq 0$ @ $x = 0$ but $\phi = \pi$

$\therefore$ if we add a constant $= -\pi$

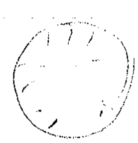
$\Omega = i\ln(z-a) + i\ln(z+a) - \pi$

For this problem



$\Omega'(z) \to iV \quad \Longrightarrow \quad \chi' \to \text{const}$ as $\zeta \to \infty$

for flow outside circle



$\Omega = Qz + \dfrac{i\gamma}{2\pi}\ln z/a + Q^* \dfrac{a^2}{z}$ only good for multiply connected region

<u>Laplace transform continued</u>

$f(t) \Longrightarrow F(s)$ or $\bar{f}(s)$ or $\mathcal{L}\{f(t)\}$

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ $\qquad$ Used in <u>linear problems</u> only

Trivial $\quad \dfrac{df}{dt} + \alpha f = 0 \qquad f(0) = 1 \qquad f = e^{-\alpha t}$

Do by Laplace Transform

Exclude t by multiplying by Kernel

$$\int_0^\infty e^{-st} f'(t)\,dt + \alpha \int_0^\infty e^{-st} f(t)\,dt = 0$$

$$sF(s) - f(0+) \qquad \alpha F(s) \quad = 0 \quad \Rightarrow \quad (s+\alpha)F(s) = 1 \quad \text{or} \quad F(s) = \frac{1}{s+\alpha}$$

integrate by parts & assume $f(t)\,e^{-st} \to 0$ as $t \to \infty$

Now

$$f(t) = \frac{1}{2\pi i}\int_{c+i\infty}^{c+i\infty} e^{+st} F(s)\,ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s+\alpha}\,ds \qquad \underset{-\alpha}{*}$$

$C \qquad c > -\alpha$

Laplace transform analytic in some right-half s-plane

To invert: use tables that are there.

Two. - extend tables $\quad \cos\alpha t \Longrightarrow \dfrac{s}{s^2+\alpha^2}$

$$\frac{d}{d\alpha}\cos\alpha t = -t\sin\alpha t \Longleftrightarrow \frac{-2\alpha s}{(s^2+\alpha^2)^2}$$

Three: Use partial fractions to express $F(s)$

Linear Ode, const coeff W/RHS being exp, sin, cos

then $F(s)$ = Rational fn. $\dfrac{a_0 + a_1 s + \dots a_{n-1} s^{n-1}}{1 + b_1 s + \dots + b_n s^n}$

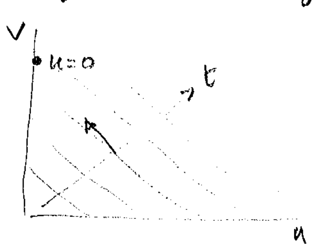
$$= \frac{A}{c_1 - s} + \frac{A_1}{c_2 - s} + \dots$$

$$F(s) = \frac{1}{(s+\alpha)(s+\beta)} = \frac{1}{\beta-\alpha}\left(\frac{1}{s+\alpha} - \frac{1}{s+\beta}\right) \Longleftrightarrow \frac{1}{\beta-\alpha}\left(e^{-\alpha t} - e^{-\beta t}\right)$$

Four: Convolution ("Faltung") integral

if $\quad F(s) = G(s)\cdot H(s) \Longleftrightarrow f(t) = \int_0^t g(\tau)\,h(t-\tau)\,d\tau = \int_0^t g(t-\tau)\,h(\tau)\,d\tau$

$$G\cdot H = \int_0^\infty e^{-su} g(u)\,du \int_0^\infty e^{-sv} h(v)\,dv = \int_0^\infty\int_0^\infty e^{-s(u+v)} g(u)\,h(v)\,du\,dv$$



let $t = u+v \qquad \displaystyle\int_0^\infty e^{-st}\int_{u=0}^{t} g(u)\,h(t-u)\,du\,dt$

Eliminate $v = t-u \qquad$ now $\quad u=0 \quad v=t$
$\qquad\qquad\qquad\qquad\qquad\qquad u=v \quad t=0$

(5) Large & small t or s $\qquad$ small t ⟺ large s ; large t ⟺ small s

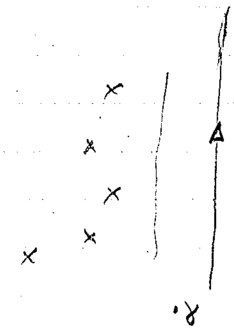
$F(s)\ \frac{1}{s+\alpha}$ for small time large s

$$\Rightarrow \frac{1}{s(1+\alpha/s)} = \frac{1}{s}\left[1-\frac{\alpha}{s}+\frac{\alpha^2}{s^2}+\dots\right]$$
$$= 1-\alpha t+\frac{\alpha^2 t^2}{2!}\dots = e^{-\alpha t}$$

May be purely asymptotic

(6) use inversion integral

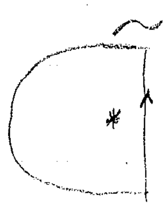
$$\left.\begin{array}{ll}\text{for } t>0 & f(t)\\ t<0 & 0\end{array}\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds$$



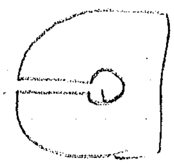
γ

$\int \to 0$ by Jordan's lemma

For simple pole



For branch



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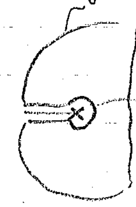
from last time

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}}{s+\alpha}\,ds = e^{-\alpha t}$$

Pole at $s=-\alpha$ residue is $e^{-\alpha t}$

Suppose $F(s) = \frac{1}{(s+\alpha)^{1/2}}$ (branch point at $s=-\alpha$



vanishes by Jordan lem like e (Re $\alpha<0$)

vanishes like $\sqrt{\varepsilon}$

on top of cut $s = e^{i\pi}\cdot R = -R$ $\quad ds = -dR$

$$(s+\alpha)^{1/2} = (e^{i\pi}R+\alpha)^{1/2} = \left[e^{i\pi}(R-\alpha)\right]^{1/2} = i(R-\alpha)^{1/2}$$

$$\text{Contribution} = \frac{1}{2\pi i}\int_{r=\alpha}^{R\to\infty}\frac{e^{-rt}}{i(r-\alpha)^{1/2}}\cdot -dr = \frac{1}{2\pi}\int_\alpha^\infty \frac{e^{-rt}}{(r-\alpha)^{1/2}}\,dr$$

let $r-\alpha = u$ $\quad\therefore\quad u+\alpha = r \quad\therefore\quad e^{-rt} = e^{-\alpha t}e^{-ut}$

$dr = du$

$$\therefore \frac{e^{-\alpha t}}{2\pi}\int_0^\infty e^{-ut}u^{-1/2}\,du \qquad \text{now let } ut = x^2 \quad du = \frac{2x\,dx}{t} \quad \sqrt{u} = \frac{x}{\sqrt{t}}$$

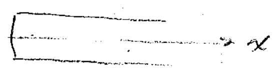
$$\frac{e^{-\alpha t}}{\ }\int^\infty e^{-x^2}\frac{\sqrt{t}}{\ }\cdot 2x\,dx = \frac{2e^{-\alpha t}}{\ }\int^\infty e^{-x^2}dx = \frac{e^{-\alpha t}}{\pi\sqrt{t}}\frac{\sqrt{\pi}}{2}$$

or $\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{1}{(s+\alpha)^{1/2}}\,ds = \frac{e^{-\alpha t}}{2\sqrt{\pi t}}\cdot 2$ since

from bottom cut same thing

Exaple: applications to Heat conduction in bar impulsively heated at end.

Initially @ $T=T_0$ @ $t=0$ end is heated to T_1



$T_{,t} = \alpha T_{,xx}$ α : coeff of thermal diffusivity

Set $T = T_0 + (T_1 - T_0) f(t)$

$\therefore$ put into DE to get $\dfrac{\partial f}{\partial t} = \alpha \dfrac{\partial f}{\partial x^2}$

Initial cond. $f = 0$ @ $t = 0$

BC $f = 1$ @ $x = 0$

Apply the Transform $sF = \alpha F''$ $\therefore$ $F'' - \frac{s}{\alpha}F = 0$ or

$F = A(s)\,e^{\sqrt{\frac{s}{\alpha}}x} + B(s)\,e^{-\sqrt{\frac{s}{\alpha}}x}$ for bounded sol. $A(s) = 0$

$\therefore$ $F = B(s)\,e^{-\sqrt{s/\alpha}\,x}$

Now BC converts to $\frac{1}{s}$ ie $f(0,t) \Rightarrow \frac{1}{s} = F(0,s)$

$\therefore$ $B(s) = \frac{1}{s}$ thus

$F = \frac{1}{s}e^{-\sqrt{s/\alpha}\,x}$

observe that $F_1(s) = e^{-a\sqrt{s}} \Longleftrightarrow f_1(t) = \dfrac{a}{2\sqrt{\pi}}\,\dfrac{e^{-a^2/4t}}{t^{3/2}}$ $a = \frac{x}{\sqrt{\alpha}}$

$\therefore$ convolve w/ $G(s) = \frac{1}{s} \Longleftrightarrow g(t) = 1$ to get

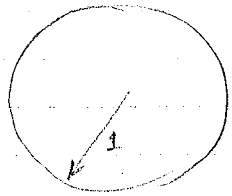
$$f = f_1 * g = \int_0^t f_1(\tau)\,g(t-\tau)\,d\tau = \frac{a}{2\sqrt{\pi}}\int_0^t \frac{e^{-a^2/4\tau}}{\tau^{3/2}}\cdot 1\,d\tau$$

if $a^2/4\tau = y^2 \Rightarrow -\frac{a^2}{4\tau^2}d\tau = 2y\,dy$
$= 2\frac{a}{2\sqrt{\tau}}dy$ $\therefore$ $\dfrac{a}{2\sqrt{\pi}}\displaystyle\int_{\infty}^{a/2\sqrt{t}} e^{-y^2}\cdot -\frac{4}{a}\,dy$

$$\therefore\ f(t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

12/3/79

Impulsively heated sphere



Init Temp $f=0$ at $t=0$
Surface Temp $f=1$ at $r=1$ $(t>0)$
Temperature is bounded inside

Governing PDE Diffusion heat eqn. $\frac{\partial f}{\partial t} = \alpha \nabla^2 f$ — 3D but w/axial sym.

$$= \alpha\left(f_{,rr} + \frac{2}{r} f_{,r}\right)$$

Apply Laplace Transformation

$$s\bar{F} - \underbrace{f(x,0)}_{\to 0 \text{ by initial cond.}} = \alpha\left(\bar{F}_{,rr} + \frac{2}{r}\bar{F}_{,r}\right)$$

$F = \frac{1}{s}$ @ $r=1$ and F bounded in interior

$$\therefore\ \frac{d^2F}{dr^2} + \frac{2}{r}\frac{dF}{dr} - \frac{s}{\alpha}F = 0$$

Multiply by r this eqn is transformed into 1d eqn for rF

$$\frac{d^2}{dr^2}(rF) - \frac{s}{\alpha}(rF) = 0 \qquad \text{General sol. is}$$

$$rF = \tilde{C}_1 e^{+\sqrt{s/\alpha}\cdot r} + \tilde{C}_2 e^{-\sqrt{s/\alpha}\, r} = C_1 \sinh\sqrt{s/\alpha}\, r + C_2 \cosh\sqrt{s/\alpha}\, r$$

$\therefore$

$$F(r,s) = \frac{C_1}{r}\sinh\sqrt{s/\alpha}\, r + \frac{C_2}{r}\cosh\sqrt{s/\alpha}\, r$$

by bddness of soln at center & $\frac{\sinh}{r} \xrightarrow[r\to 0]{} \text{const.}$ $\quad \frac{\cosh}{r} \to \infty \quad \therefore C_2 = 0$

$$F(r,s) = \frac{C_1}{r}\sinh\sqrt{s/\alpha}\, r \ ; \quad \text{since } F(1,s) = \frac{1}{s} \implies C_1 = \frac{1}{s}\frac{1}{\sinh\sqrt{\alpha/s}}$$

$$\therefore\ F(r,s) = \frac{1}{r}\,\frac{1}{s}\,\frac{\sinh\sqrt{s/\alpha}\, r}{\sinh\sqrt{s/\alpha}}$$

Use inversion $$f(t,r) = \frac{1}{2\pi i}\frac{1}{r}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{s}\,\frac{\sinh\sqrt{s/\alpha}\, r}{\sinh\sqrt{s/\alpha}}\, ds$$

Where are the singularities in order to place γ

1. Simple pole at $s=0$ Res $= r$
2. Essential sing @ ∞
3. Simple poles at zeroes of $\sinh\sqrt{s/\alpha}$

Remember
$\sinh ix = i\sin x$
$\cosh ix = i\cos x$

$\implies \pm i n\pi = \sqrt{s/\alpha}$

$$\therefore\ s = -\alpha N^2\pi^2$$

$\therefore \mathrm{Res}_{s=-\alpha N^2\pi^2} = \lim_{s\to -\alpha N^2\pi^2} (s+\alpha N^2\pi^2)\frac{1}{s}\frac{\sinh\sqrt{s/\alpha}\,r}{\sinh\sqrt{s/\alpha}}e^{st} = \frac{2(-1)^N}{N\pi}\sin(N\pi r)e^{-\alpha N\pi^2 t}$

$$\therefore f(r,t) = \frac{1}{r}\left[r - \sum_{N=1}^{\infty}\frac{2}{N\pi}(-1)^{N-1}\sin(N\pi r)e^{-\alpha N^2\pi^2 t}\right]$$

$$= 1 - \frac{2}{\pi r}\sum_{N=1}^{\infty}\frac{(-1)^{N-1}}{N}\sin(N\pi r)e^{-\alpha N^2\pi^2 t}$$

as $t\to\infty$ $\quad f(r,t)\to 1$

at center $\quad f(0,t) = 1 - 2\sum_{N=1}^{\infty}(-1)^{N-1}e^{-\alpha N^2\pi^2 t}$

approx for small time: expand $F(s,r)$ for large s & invert term by term

Final: Handed out Wednesday due Wed 12/12

$$rF(r,s) = \frac{1}{s}\frac{\sinh\sqrt{s/\alpha}\,r}{\sinh\sqrt{s/\alpha}} = \frac{1}{s}\,\frac{e^{\sqrt{s/\alpha}\,r}-e^{-\sqrt{s/\alpha}\,r}}{e^{\sqrt{s/\alpha}}-e^{-\sqrt{s/\alpha}}}$$

for $s\to\infty$ the exp $e^{-\sqrt{s/\alpha}\,r}$ $e^{-\sqrt{s/\alpha}}\to 0$

$\therefore rF(r,s) \sim \frac{1}{s}e^{-\sqrt{s/\alpha}\,(1-r)} + \dots$ as $s\to\infty$

remember $\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}\right) = \frac{e^{-a\sqrt{s}}}{s}$ $\quad\therefore rf(r,t)\sim \mathrm{erfc}\,\frac{(1-r)}{2\sqrt{\alpha t}}$

Fourier Transformation – normally applied to space coords 12/5/79

Laplace " – " " " timelike "

$$F(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\lambda\tau}f(\tau)\,d\tau \Longleftrightarrow f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\lambda t}F(\lambda)\,d\lambda$$

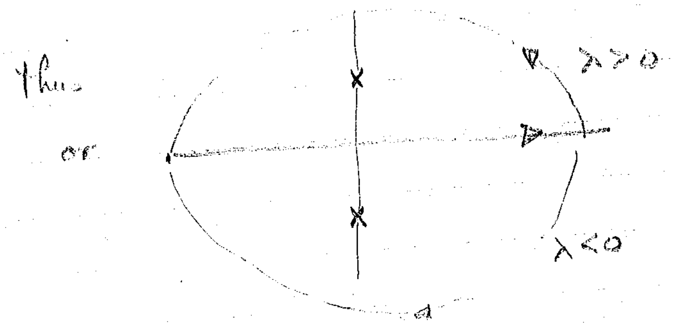
beware of other notation Erdélyi $g(y) = \int_{-\infty}^{\infty}f(x)e^{-iyx}dx$

Lighthill: $g(y) = \int_{-\infty}^{\infty}e^{-2\pi ixy}f(x)\,dx$ $\quad\int |F(\lambda)|\,d\lambda \to 0$ as $\lambda\to\infty$

Example of Fourier Transf. CKP, pg 303 Ex 1 part b Exponential

$f(t) = \frac{1}{a^2+t^2}$ $\quad F(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{i\lambda\tau}}{a^2+\tau^2}d\tau$

roots : none poles at $\tau = \pm ai$ Res $\dfrac{e^{\mp\lambda a}}{\pm 2ai}$



Complete contour use res.

$$\therefore \sqrt{\tfrac{\pi}{2}}\ \frac{e^{-\lambda a}}{a} \qquad \lambda > 0 \qquad \text{or } \sqrt{\tfrac{\pi}{2}}\ \frac{e^{-|\lambda| a}}{a} \quad \lambda \text{ real.}$$

$$\sqrt{\tfrac{\pi}{2}}\ \frac{e^{+\lambda a}}{a} \qquad \lambda < 0$$

Fourier sine & cosine approp for symmetry.

if $f(t)$ is an even fn. Fourier Cosine Trans.

$$F_c(\lambda) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \cos\lambda\tau\, f(\tau)\,d\tau = \sqrt{\tfrac{2}{\pi}} \int_0^\infty \cos(\lambda\tau) f(\tau)\,d\tau$$

$$f(t) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty \cos(\lambda t)\, F(\lambda)\,d\lambda$$

if $f(t)$ is odd in t Fourier Sin Trans.

$$F_s(\lambda) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty \sin(\lambda\tau)\, f(\tau)\,d\tau.$$

$$f(t) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty \sin(\lambda t)\, f(\lambda)\,d\lambda$$

Transform of derivatives : Integrate by parts.

(1) Exponential : Transform of $\frac{df}{dt}$ = $-i\lambda$ times the transform of $f(t)$

Transform of $\frac{d^n f}{dt^n}$ = $(-i\lambda)^n$ " " " "

(2) Cosine Transform : Transform of $\frac{df(t)}{dt}$ = $\lambda \times$ sine transform of $f(t) - \sqrt{\tfrac{2}{\pi}}$

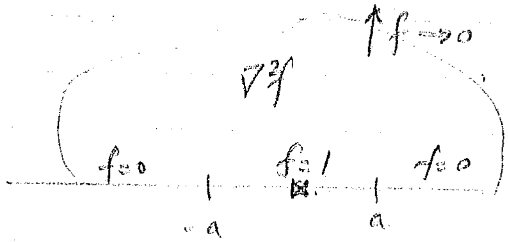
2nd deriv : $\frac{d^2f}{dt^2}$ transform $= -\lambda^2$ cos transf of $f(t) - \sqrt{\frac{2}{\pi}}\, f'(0)$

It is more approp to use these sine & cosine transf if the PDE involve even ~~no~~ # of deriv

only good if $\int_{-\infty}^{\infty} |f(t)|\,dt$ is bounded ie. $|f(t)| \sim \frac{1}{t^\alpha}$ $\alpha > 1$.

M. J. Lighthill : Fourier Analysis & Generalized Functions
Tranter : Ex 1 pg 44

find the steady temp in the semi inf solid $y > 0$ with $f > 0$ on $y = 0$ except temp $= 1$ for $-a < x < a$



use result that $\int_0^\infty e^{-sx} x^{-1} \sin(rx)\,dx = \arctan(r/s)$ $\quad r>0, s>0$

$f(x,y)$ is symmetric in x BC at $y=0$

apply fourier transf wrt x & cosine trans.

use $F_c(y;\lambda) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \cos(\lambda x)\, f(x,y)\,dx$

transform $\frac{\partial^2 f}{\partial x^2} = -\lambda^2 F_c - \sqrt{2/\pi}\, f'(0,y)$ $\qquad f'(0,y) = 0$ by sym.

$\therefore$ we get $-\lambda^2 F_c + F_{c\,yy} = 0 \Rightarrow$

$$F_c = C_2(\lambda)\, e^{\lambda y} + C_1(\lambda)\, e^{-\lambda y} \quad (\to 0)$$

bc. $f(x,0) = \begin{cases} 1 & |x|<a \\ 0 & |x|>a \end{cases} \Rightarrow F(0,\lambda) = \sqrt{\frac{2}{\pi}} \int^a \cos\lambda x \cdot 1\,dx = \sqrt{\frac{2}{\pi}}\, \frac{\sin\lambda}{\lambda}$

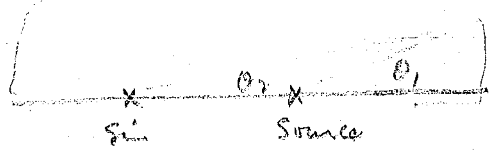
$$\therefore \quad F(\lambda; y) = \sqrt{\tfrac{2}{\pi}}\, \frac{A \sin \lambda a}{\lambda}\, e^{-\lambda y}$$

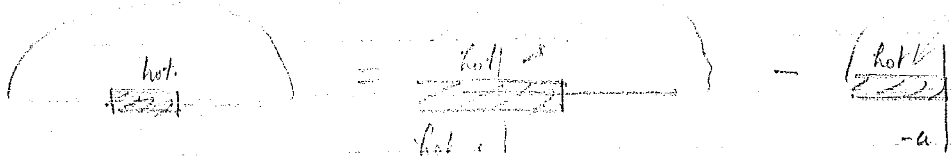
$$f(x,y) = \sqrt{2/\pi} \int_0^\infty \cos \lambda x\, F(y;\lambda)\, d\lambda = \frac{2}{\pi}\int_0^\infty \cos \lambda x\, \frac{\sin \lambda a}{\lambda}\, e^{-\lambda y}\, d\lambda$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin \lambda a \cos \lambda x = \frac{1}{2}\left[\sin \lambda(a+x) + \sin \lambda(a-x)\right]$$

$$\Rightarrow \frac{1}{\pi}\left[\tan^{-1}\frac{a+x}{y} + \tan\frac{a-x}{y}\right] = \frac{1}{\pi}\left[\tan^{-1}\frac{y}{x-a} - \tan^{-1}\frac{y}{x+a}\right]$$

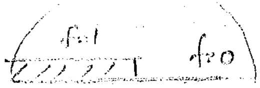




$$F(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda t} f(t)\, dt \quad \text{has certain restrictions}$$

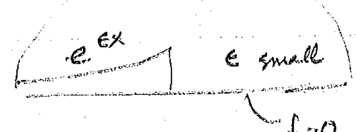
$$F(x,y) = \frac{1}{\pi}\left[\tan^{-1}\frac{y}{x-a} \mp \tan^{-1}\frac{y}{x+a}\right]$$

each is a self similar; if $a \neq 0$, all we need to do is to shift the origin



the problem cannot be fourier transformed since $f \not\to 0$ as $z \to \infty$ on the axis

Add exponential damping: solve instead



we always assume integration is along real axis. If $f(t) \not\to 0$ as $t \to \infty$ change contour $\ni$ we integrate over $t = x + i\epsilon$ $\quad i\lambda t = i\lambda x - \lambda\epsilon$

$$\text{if } f(x) = \begin{cases} x^{1/2} & x > 0 \\ 0 & x < 0 \end{cases}$$

C, K & P using change contour. $F(\lambda) = \frac{1}{2\sqrt{2}} e^{i\frac{3}{4}\pi}\lambda^{-3/2}$

$Im\,\lambda > 0$

INTRODUCTION TO FOURIER ANALYSIS AND GENERALISED FUNCTIONS

BY

M. J. LIGHTHILL, F.R.S.

Beyer Professor of Applied Mathematics in the University of Manchester

CAMBRIDGE
AT THE UNIVERSITY PRESS
1958

Table 1

	1	$\operatorname{sgn} x$	$H(x)$
$\lvert x\rvert^{\alpha}$	$\{2\cos\frac{1}{2}\pi(\alpha+1)\}\,\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}$	$\{-2i\sin\frac{1}{2}\pi(\alpha+1)\}\,\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}\operatorname{sgn} y$	$\{e^{-\frac{1}{2}\pi i(\alpha+1)\operatorname{sgn} y}\}\,\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}$
x^{n}	$(-2\pi i)^{-n}\,\delta^{(n)}(y)$	$2(n!)\,(2\pi iy)^{-n-1}$	$(-2\pi i)^{-n}\left\{\frac{1}{2}\delta^{(n)}(y)+\frac{(-1)^{n}\,n!}{2\pi iy^{n+1}}\right\}$
x^{-m}	$-\pi i\,\frac{(-2\pi iy)^{m-1}}{(m-1)!}\operatorname{sgn} y$	$-2\,\frac{(-2\pi iy)^{m-1}}{(m-1)!}\,(\log\lvert y\rvert+C)$	$-\frac{(-2\pi iy)^{m-1}}{(m-1)!}\times\{\frac{1}{2}\pi i\operatorname{sgn} y+\log\lvert y\rvert+C\}$
$\lvert x\rvert^{\alpha}\log\lvert x\rvert$	$\{2\cos\frac{1}{2}\pi(\alpha+1)\}\,\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}\times\{-\log(2\pi\lvert y\rvert)+\psi(\alpha)-\frac{1}{2}\pi\tan\frac{1}{2}\pi(\alpha+1)\}$	$\{-2i\sin\frac{1}{2}\pi(\alpha+1)\}\,\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}\operatorname{sgn} y\times\{-\log(2\pi\lvert y\rvert)+\psi(\alpha)+\frac{1}{2}\pi\cot\frac{1}{2}\pi(\alpha+1)\}$	$\{e^{-\frac{1}{2}\pi i(\alpha+1)\operatorname{sgn} y}\}\,\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}\times\{-\log(2\pi\lvert y\rvert)+\psi(\alpha)-\frac{1}{2}\pi i\operatorname{sgn} y\}$
$x^{n}\log\lvert x\rvert$	$-\pi i\,\frac{n!}{(2\pi iy)^{n+1}}\operatorname{sgn} y$	$-2\,\frac{n!}{(2\pi iy)^{n+1}}\{\log(2\pi\lvert y\rvert)-\psi(n)\}$	$-\frac{n!}{(2\pi iy)^{n+1}}\times\{\frac{1}{2}\pi i\operatorname{sgn} y+\log(2\pi\lvert y\rvert)-\psi(n)\}$
$x^{-m}\log\lvert x\rvert$	$\pi i\,\frac{(-2\pi iy)^{m-1}}{(m-1)!}\operatorname{sgn} y\times\{\log(2\pi\lvert y\rvert)-\psi(m-1)\}$	$\frac{(-2\pi iy)^{m-1}}{(m-1)!}\times[\{\log(2\pi\lvert y\rvert)-\psi(m-1)\}^{2}+C]$	$\frac{(-2\pi iy)^{m-1}}{(m-1)!}\,[\frac{1}{2}\{\frac{1}{2}\pi i\operatorname{sgn} y+\log(2\pi\lvert y\rvert)-\psi(m-1)\}^{2}+C]$

Table 1. Each entry is the Fourier transform of the product of the expression at the head of its row with the expression at the head of its column. Thus, the F.T. of $x^{n}\operatorname{sgn} x$ is $2(n!)\,(2\pi iy)^{-n-1}$. In table 1, as well as throughout chapter 3, α stands for any real number not an integer, n for any integer $\geqslant 0$, m for any integer > 0, and C for an arbitrary constant which is present because $x^{-m}\operatorname{sgn} x$ and $x^{-m}\log\lvert x\rvert\operatorname{sgn} x$ are both indeterminate to the extent of an arbitrary multiple of $\delta^{(m-1)}(x)$. The table is specially valuable when used together with the fact that, if the F.T. of $f(x)$ is $g(y)$, then the F.T. of $f(ax+b)$ is $\frac{1}{\lvert a\rvert}\,e^{2\pi iby/a}\,g\left(\frac{y}{a}\right)$. $\left(\text{Conversely, the F.T. of } e^{ikx}f(x) \text{ is } g\left(y-\frac{k}{2\pi}\right).\right)$

if $g(y) = \int_{-\infty}^{\infty} e^{-2\pi i xy} f(x)\,dx$; $f(x) = \int_{-\infty}^{\infty} e^{2\pi i xy} g(y)\,dy$

Lighthill

Use his table on pg 43, First w/ $\alpha = 1/2$ $\quad f(x) = x^{1/2} H(x)$

$$g(y) = \left\{ e^{-\frac{1}{2}\pi i (3/2)\,\mathrm{sgn}\,y} \right\} \tfrac{1}{2}!\ (2\pi |y|)^{-3/2}$$

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ $\quad \alpha! = \Gamma(\alpha+1)$ $\quad \Gamma(1/2) = \sqrt{\pi}$ $\quad \tfrac{1}{2}! = \Gamma(1+1/2) = \tfrac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$

$$\therefore \quad g(y) = \frac{1}{\sqrt{\pi}} e^{-\frac{3}{4}\pi i\,\mathrm{sgn}\,y} (2\pi |y|)^{-3/2}$$

Lighthill: Good fns: integrable, differentiable & die off faster than x^n as $x \to \infty$: e^{-x}
Fairly Good fns: " " " " " at same rate as x^n " : poly

Generalized fn: A sequence of good fns that are regular" in sense that $f(x)$ is a fairly good fn. $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, F(x)\,dx$ exists

Dirac δ fn. $\quad \int_{-\infty}^{\infty} \delta(x)\, F(x)\,dx = F(0)$

Define δ by sequence $\lim_{n\to\infty} e^{-nx^2} (n/\pi)^{1/2}$

Unit fn. $I(x) = 1$.
defined by $\lim_{n\to\infty} e^{-x^2/n^2}$ or e^{-x^2/n^4}

thus Fourier Transform of $\delta(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-nt^2} (n/\pi)^{1/2} e^{i\lambda t}\,dt$

Solutions to Final Examination
ME 201. APPLICATIONS OF COMPLEX VARIABLES
December 1979

1. A model of the Weiss-Fogh mechanism. The flow region can be regarded as a degenerate quadrilateral with angles $\pi-\beta$, 2π, β, and a "strange" angle of $-\pi$ at infinity. If we map these vertices respectively to -1, 0, k (to be found), and ∞ in the ζ-plane, the transformation is

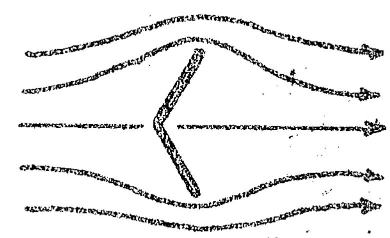
$$z = a \int_{\zeta_0}^{\zeta} (\zeta+1)^{-\beta/\pi} \zeta (\zeta-k)^{\beta/\pi-1} \, d\zeta + b .$$

When $\beta = \pi/2$, it is clear from symmetry that $k = 1$, and we may take

$$z = \int \frac{\zeta \, d\zeta}{\sqrt{\zeta^2-1}} = \sqrt{\zeta^2-1} , \qquad \zeta = \sqrt{z^2+1} .$$

For $\psi = 0$ on the wall and flap, and growing like Uy far away, the corresponding complex potential in the ζ-plane is simply $\chi(\zeta) = U\zeta$, which gives in the physical plane

$$\phi + i\psi = \Omega(z) = U\sqrt{z^2+1} .$$



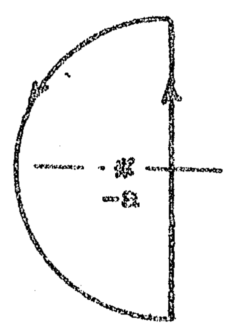
If we continue this solution in the upper half-plane analytically into the lower half-plane, we simply reflect the geometry and streamline pattern as shown. (This is the interpretation used for the two wings of the wasp.)

2. Solving an ordinary differential equation by Laplace transformation. Applying the Laplace transformation with respect to t reduces the problem to

$$sF + aF = \frac{1}{s+a} , \qquad F(s) = \frac{1}{(s+a)^2} ,$$

Then the inversion theorem gives the solution as

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{(s+a)^2} \, ds , \qquad \gamma > -a .$$



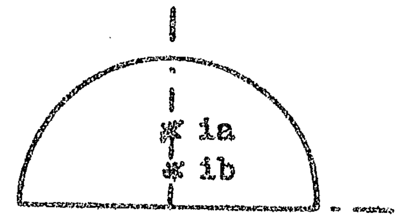
We complete the contour with a large semicircle to the right, as shown, whose contribution vanishes in the limit by Jordan's lemma. The double pole at $s = -a$ has residue te^{-at}. Hence the residue theorem gives

$$f(t) = te^{-at}$$

which can be verified by direct substitution.

3. An infinite integral. Suppose that $m > 0$, as well as $a, b > 0$. Then consider the contour integral

$$\oint \frac{e^{imz}\,dz}{(z^2+a^2)(z^2+b^2)}$$



about the semi-circular contour in the upper half-plane. There are simple poles inside the contour at

$$z = ia: \quad \text{Residue} \quad \frac{e^{-ma}}{2ia(b^2-a^2)}$$

$$z = ib: \quad \text{Residue} \quad \frac{e^{-mb}}{2ib(b^2-a^2)}$$

The integral over the semi-circle vanishes, so the residue theorem gives

$$\int_{-\infty}^{\infty} \frac{e^{imx}\,dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^{\infty} \frac{\cos mx\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-mb}}{b} - \frac{e^{-ma}}{a}\right).$$

The desired integral is, by symmetry, just half of this. For $m < 0$, the contour must be completed in the lower half plane. The result is not the same, because although the cosine is an even function of its argument, the exponential is not; the signs of the exponents in the exponentials must be reversed. The special case $a = b$ may be treated either by starting again and evaluating the residue at the double pole, or by letting $a \to b$ and using l'Hospital's rule or series expansion to evaluate the indeterminate form.

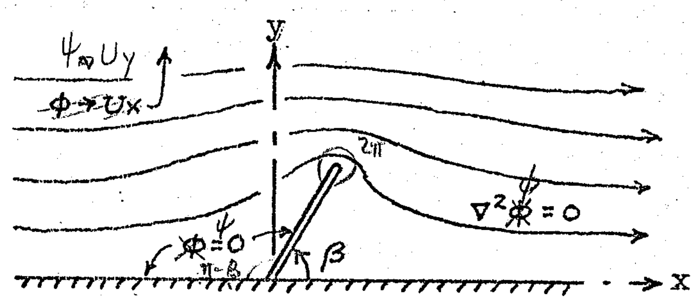
Final Examination

ME 201. APPLICATIONS OF COMPLEX VARIABLES

December 1979

This is an open-book take-home examination conducted in accord with the Stanford Honor Code. Write on any convenient paper. Spend not more than five hours -- in several sessions if desired -- and record the total time spent. Return before Wednesday evening December 12 to 'Van Dyke' mailbox in Applied Mechanics office on second floor of Durand building, or by interdepartmental mail.

1. A model of the Weiss-Fogh mechanism. (40 points) In analyzing the "clap-fling" method of producing lift that Weiss-Fogh discovered is employed by the wasp *Encarsia formosa*, M. J. Lighthill found it useful to calculate the plane potential flow along a wall with a thin flap protruding from it at an angle β.



a. Express in terms of an integral the conformal mapping of this flow region into the upper half of another plane.

b. Show that in the special case when the flap is normal to the wall ($\beta = \pi/2$) the transformation can be given in simple closed form.

c. In that case find the solution ϕ of Laplace's equation that is zero on the wall and on the flap, and approaches a uniform field Ux far away.

d. Suppose that the wall at $y = 0$ is removed (leaving the flap) and the solution continued analytically into the lower half plane. For an arbitrary value of β, as above, sketch the complete pattern of streamlines for negative as well as positive y.

2. Solving an ordinary differential equation by Laplace transformation. (30 points) Treat the simple problem

$$\frac{df}{dt} + af = e^{-at}, \qquad f(0) = 0$$

by theory of ODE $f(t) = te^{-at}$

by applying the Laplace transformation. Then, ignoring the fact that the inversion can be found in any table of Laplace transforms, complete the solution by using directly the inversion theorem together with contour integration.

(Of course such relatively sophisticated techniques are by no means necessary in this simple problem; and you might want to check your solution by some easier method.)

✓ 3. <u>An infinite integral</u>. (30 points) Evaluate the definite integral

$$I = \int_0^\infty \frac{\cos mx \, dx}{(x^2+a^2)(x^2+b^2)}$$

for real m, a, b. Then specialize your result to the case a = b, and so check it against the result of Spiegel (p. 196, no. 54) that if $m > 0$

$$\int_0^\infty \frac{\cos mx}{(x^2+1)^2}\, dx = \frac{\pi e^{-m}(1+m)}{4} .$$

$\int_{-\infty}^{\infty} = 2\int_0^\infty \frac{\cos mx}{(x^2+1)^2}\, dx$

look at $\frac{e^{imz}}{(z^2+a^2)(z^2+b^2)}$

A grade

Please Return to C. LEVY
Dept. of MECH. ENG.
Applied Mech. Div (2nd FLOOR DURAND BUILDING

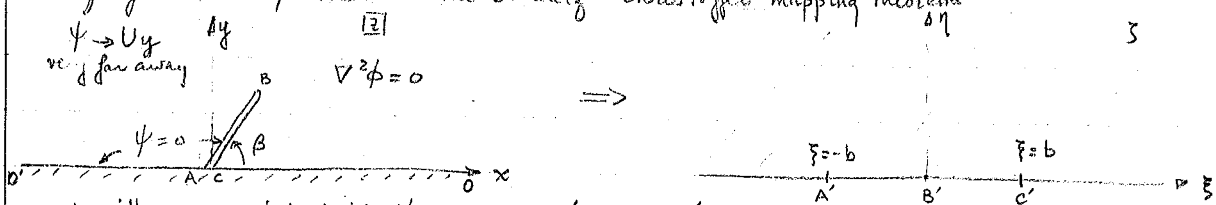
Time: $3\frac{1}{2}$ hrs

Cesar Levy
ME 201 Fall 79
Complex Variables
Prof. Van Dyke

Final Exam

1. M.J. Lighthill calculated the plane potential flow along a wall with a thin flap protuding at some angle β.

a. Express in terms of an integral the conformal mapping of this flow region into the upper half of another plane. Use the schwarz - Christoffel mapping theorem



we will map point A into A', B into B', C in C' assuming that the length of the flap is b.
The angle for $O'AB = \pi - \beta$; the angle for $ABC = 2\pi$; angle for $BCO = \beta$; angle for $O'\infty O$ is π. The only factors that will play a part are the first three since the 4^{th} one will give $\pi/\pi - 1 = 0$ $\therefore$

No, $-\pi$

Hence would naturally chose to map "strange" angle at $z=\infty$ to $\zeta=\infty$; but then cannot choose all 3 remaining points (according to Riemann).

$$\frac{dz}{d\zeta} = a(\zeta+b)^{-\beta/\pi}(\zeta-b)^{\beta/\pi - 1}\zeta^{1} \quad \text{or} \quad z = \tilde{a}\int_0^{\zeta} \frac{\zeta'}{\zeta'-b}\left(\frac{\zeta'-b}{\zeta'+b}\right)^{\beta/\pi} d\zeta' + \tilde{b}$$

when $\zeta = 0$ $z = be^{i\beta}$ $\therefore$ $\tilde{b} = be^{i\beta}$

b. Show that if $\beta = \pi/2$ we can get a simple transformation

If $\beta = \pi/2$ $\quad \beta/\pi = \frac{1}{2}$ then $\dfrac{dz}{d\zeta} = a(\zeta+b)^{-1/2}(\zeta-b)^{-1/2}\zeta$ or

$$z = \tilde{a}\int^{\zeta} \frac{\zeta'}{(\zeta'^2 - b^2)^{1/2}}\, d\zeta' + \tilde{b} \quad \text{and} \quad z = \tilde{a}(\zeta^2 - b^2)^{1/2} + \tilde{b} \quad ✓$$

since $z = 0^-$ maps into $-b$, $z = 0^+$ maps into b and $z = ib$ maps into 0 from the first two $\Rightarrow$ $\tilde{b} = 0$ from the 3^{rd} $\tilde{a} = 1$.

$$\therefore \quad z = (\zeta^2 - b^2)^{1/2} \quad \text{thus} \quad \zeta = (z^2 + b^2)^{1/2} \quad ✓$$

c. In that case find the solution ψ of $\nabla^2\psi = 0$ so that $\psi = 0$ on the wall & flap and approaches Uy as $|z| \to \infty$

Since $\Omega(z) = \chi(\zeta)$ then $\nabla^2_{xy}\psi = \nabla^2_{\xi,\eta}\Psi(\xi,\eta)\left|\dfrac{d\zeta}{dz}\right|^2 = 0 \Rightarrow \nabla^2\Psi = 0$ if $\dfrac{d\zeta}{dz} \neq 0$
and $\psi = 0$ on the wall and flap $\Rightarrow$ $\Psi = 0$ on $\eta = 0$

for very large z then $\zeta \sim z$ $\therefore$ $\psi \to Uy$ for large z $\Rightarrow$ $\Psi \to U\eta$ for large ζ

If I take $\chi(\zeta) = U\zeta + C$ then $\chi(\zeta = \xi + i0) = U\xi + i\cdot 0 + C$ and $\Psi = 0$ on $\eta = 0$
and $\Psi \to U\eta$ as ζ becomes very large if C is a real constant

$$\therefore \quad \Omega(z) = \chi(\zeta[z]) = U(z^2 + b^2)^{1/2} \quad \text{and} \quad \psi = \mathrm{Im}\, U(z^2+b^2)^{1/2} \quad ✓$$

d. If we analytically continue our results we find that the boundary is a mirror by Schwarz's lemma see other side ✓

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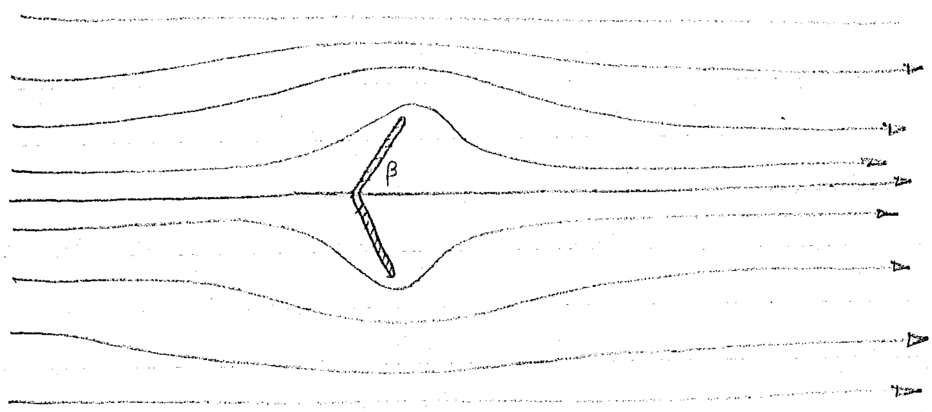

β

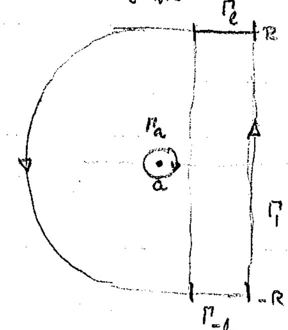
2. Solve by Laplace transform $f' + af = e^{-at}$, $f(0)=0$; Let $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

∴ Applying Laplace gives $sF(s) - f(0) + aF(s) = (s+a)^{-1}$ for $s > -a$

∴ $F(s) = \frac{1}{(s+a)^2}$ after applying initial conditions

to invert we must look at $f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\,ds$ with $\gamma > -a$

thus look at contour integration over



∴ $\int_{\Gamma_1} + \int_{\Gamma_\ell} + \int_{\Gamma_c} + \int_{\Gamma_{-\ell}} + \oint_{\Gamma_a} = 0$ when argument of each integral is $e^{st}F(s)$

∴ we find that $\int_{\Gamma_\ell}$ & $\int_{\Gamma_{-\ell}} \to 0$ as $R \to \infty$ since integration is with respect to x (which is bounded) and integration will be independent of R; $|e^{st}| = |e^{xt+iRt}| \le |e^{xt}|$ and $\left|\frac{ds}{(s+a)^2}\right| = \left|\frac{dx}{(iR+x+a)^2}\right| \le \frac{dx}{R^2}$

∴ $\left|\int_{\Gamma_\ell}\right| \le \frac{1}{R^2}\int_0^\gamma e^{xt}\,dx$ and $\lim_{R\to\infty}$ of integral is 0

also $\lim_{R\to\infty}\int_{\Gamma_c} e^{st}F(s)\,ds \to 0$: $s = Re^{i\theta}$ ∴ $e^{st} = e^{Rt\cos\theta}\cdot e^{iRt\sin\theta}$

$ds = Rie^{i\theta}d\theta$ and $\frac{1}{(s+a)^2} = \frac{1}{R^2(e^{i\theta}+a/R)^2}$

$$\therefore \left|\int_{\Gamma_c} e^{st}F(s)\,ds\right| = \left|\frac{Ri}{R^2}\int_{\pi/2}^{3\pi/2}\frac{e^{Rt\cos\theta}\,e^{i\theta}\,e^{iRt\sin\theta}}{(e^{i\theta}+a/R)^2}\,d\theta\right| \le \frac{1}{R}\int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\,d\theta$$

$$\le \frac{2}{R}\left(\int_{\pi/2}^{\pi} e^{-\frac{Rt2\theta}{\pi}}d\theta\right) = \frac{\pi}{R^2 t}e^{-\frac{2Rt\theta}{\pi}}d\theta\Big|_{\pi/2}^{\pi} \; ; \; \lim_{R\to\infty} \text{ of this is } 0$$

$$\therefore \int_{\Gamma_1} = \oint_{\Gamma_a} = 2\pi i \,\mathrm{Res}\Big|_{s=-a} \qquad \therefore \frac{1}{2\pi i}\int_{\Gamma_1} e^{st}F(s)\,ds = f(t)$$

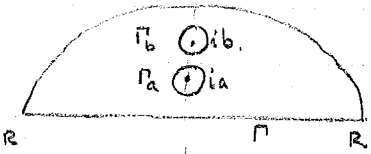
$$\therefore f(t) = \mathrm{Res}\Big|_{s=-a} = \frac{d}{ds}e^{+st}\Big|_{s=-a} = +te^{st}\Big|_{s=-a} = te^{-at}$$

CHECK: The solution to the homogeneous ODE is $Ae^{-at} = f_h(t)$. Since the forcing function is of same form then let $f(t) = (Bt+A)e^{-at}$ by placing this into DE we find $B=1$, (no info on A) and by using initial condition $A=0$. ∴ $f(t) = te^{-at}$

3. Evaluate

$$I = \int_0^\infty \frac{\cos mx \, dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos mx \, dx}{(x^2+a^2)(x^2+b^2)}$$

Look at $\oint_\Gamma \frac{e^{imz}dz}{(z^2+a^2)(z^2+b^2)}$ over



with $\underline{m>0}$

$$\oint_\Gamma + \oint_{\Gamma_a} + \oint_{\Gamma_B} = 0 \quad \text{where} \quad \oint_\Gamma = \int_{-R}^{R} \frac{e^{imx}dx}{(x^2+a^2)(x^2+b^2)} + \int \frac{e^{imz}dz}{(z^2+a^2)(z^2+b^2)}$$

$\therefore \int \frac{e^{imz}dz}{(z^2+a^2)(z^2+b^2)} \to 0$ like $\frac{1}{R^3}$ as $R \to \infty$: for $m>0$ & $R \to \infty$ $\quad |e^{imz}| < e^{-Rm\sin\theta} < e^{-mR} \to 0$; $\frac{dz}{(z^2+a^2)(z^2+b^2)} \sim \frac{R}{R^4}$ or $\frac{1}{R^3}$

$$\therefore \left| \int \frac{e^{imz}dz}{(z^2+a^2)(z^2+b^2)} \right| \le \frac{1}{R^3}\int_0^\pi d\theta \; ; \; \lim_{R\to\infty} \frac{1}{R^3}\int_0^\pi d\theta \to 0.$$

$$\therefore \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{imx}dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i \left\{ \mathrm{Res}\Big|_{z=ai} + \mathrm{Res}\Big|_{z=bi} \right\} = 2\pi i \left\{ \frac{e^{-ma}}{2ai(b^2-a^2)} + \frac{e^{-mb}}{2bi(a^2-b^2)} \right\}$$

$$= \frac{\pi}{(b^2-a^2)} \left\{ \frac{e^{-ma}}{a} - \frac{e^{-mb}}{b} \right\}$$

$$\therefore \quad I = \mathrm{Re}\, \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{imx}dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(b^2-a^2)} \left\{ \frac{e^{-ma}}{a} - \frac{e^{-mb}}{b} \right\} \quad \text{for } m>0$$

if $a=b$ then we would have a double pole at $z=\pm ai$, then all proof remaining same

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{imx}}{(x^2+a^2)^2}dx = 2\pi i \, \mathrm{Res}\Big|_{z=ai} = 2\pi i \, \frac{d}{dz}\left(\frac{e^{imz}}{(z+ai)^2}\right)\Bigg|_{z=ai} = 2\pi i \left\{ \frac{ime^{imz}}{(z+ai)^2} - \frac{2e^{imz}}{(z+ai)^3} \right\}_{z}$$

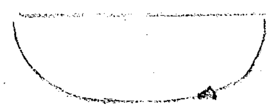
$$= 2\pi i \left\{ \frac{ime^{-ma}}{(2ai)^2} - \frac{2e^{-ma}}{(2ai)^3} \right\} = 2\pi i \left\{ \frac{-ime^{-ma}}{4a^2} - \frac{2ie^{-ma}}{8a^3} \right\} = 2\pi i e^{-ma}\left\{\frac{-im}{4a^2} \ldots\right.$$

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)^2}dx = \frac{2\pi}{4a^3}e^{-ma}\{1+am\}$$

$$I = \int_0^\infty \frac{\cos mx \, dx}{(x^2+a^2)^2} = \frac{1}{2}\mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{imx}dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}e^{-ma}\{1+am\}$$

for $a=1$ $\quad I = \frac{\pi}{4}e^{-m}(1+m) \quad m>0$

over

if $m<0$ choose  so that $\sin\theta$ is negative and evaluate the residues at $z=-bi, -ai$ ✓

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Exercise Set 0

ME 201. APPLICATIONS OF COMPLEX VARIABLES

Due Monday 1 October 1979

1. Put $\frac{1+2i}{3-4i}+\frac{2-i}{5i}$ and $(1-i)^4$ into the standard form $(x+iy)$.

2. Find one value of $\arg z$ when $z = \frac{-2}{1+i\sqrt{3}}$. $\frac{-2(1-i\sqrt{3})}{4} = -\frac{(1-i\sqrt{3})}{2}$ $\arg z = \tan^{-1}($ $\theta = 2\pi/3 \pm 2$

3. Show that the hyperbola $x^2-y^2=1$ can be written as $Az^2+B\bar{z}^2=C$, finding the constants A, B, C. (Adapted from Churchill, Brown & Verhey) $A(x^2+2xyi-y^2)+B(x^2-2xyi-y^2)=1$ $A+B=1$ $A-B=0$ $A=\frac{1}{2}$ $B=\frac{1}{2}$ $C=1$ $\frac{1}{2}(z^2+\bar{z}^2)=1$

4. Use the polar form to calculate $(-1+i)^7$. $z=re^{i\theta}=\sqrt{2}e^{i3\pi/4}$ $8\sqrt{2}e^{i\frac{21\pi}{4}=(5+\frac{1}{4}}$ $-\frac{\sqrt{2}}{2}-i$ $-8(1+i)$

5. Use the equation $z^n = |z|^n(\cos n\theta + i\sin n\theta)$ and the binomial expansion to obtain a formula for $\cos n\theta$ in terms of $\cos\theta$ and $\sin\theta$, and show that the result is a polynomial in $\cos\theta$. (CK&P p. 5) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
$\cos^n\theta + \dots$

6. Find the radii of convergence of the power series whose coefficients are $1/n$, $(-1)^n n^4$, $\ln n$, $i^n/n!$, $(-1)^n n!$ (CK&P p. 9, modified) $|a_{n+1}/a_n| = \left|r\left(\frac{n}{n+1}\right)\right|$ $r<1$; $\left|(-1)r\left(\frac{n+1}{n}\right)^4\right|$ $r<1$; $r\left|\frac{\ln(n+1)}{\ln n}\right|$ $r<$

7. Show that the net result of two consecutive bilinear transformations is again a bilinear transformation. (CK&P p. 20) $\frac{i}{(n+1)}$ $-i(n+1)$

8. Where are the branch points of $(z^2-1)^{1/2}$? $z=\pm 1$
Is the point at infinity a branch point?
How many different values does this multi-valued function have for each value of z?
Sketch in the complex plane the branch cuts that you would introduce to make this function single-valued.
Sketch them also on the Riemann sphere.
Alternatively, sketch the Riemann surface, with a separate sheet for each of the multiple values, and showing how they are connected.

$(\cos^2\theta + \sin^2\theta) + 2i\cos\theta\sin\theta$

$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$\cos^3\theta - 3\cos\theta\sin^2\theta$

Cesar Levy
ME 201
Prof Van Dyke

Exercise Set 0

1. $\frac{1+2i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{-5+10i}{25} = \frac{-1+2i}{5} = -\frac{1}{5} + \frac{2}{5}i$

$\frac{2-i}{5i} \cdot \frac{i}{i} = \frac{2i+1}{-5} = \frac{-1-2i}{5} = -\frac{1}{5} - \frac{2i}{5}$

$+ \quad = -\frac{2}{5}$ ✓

$(1-i)^4 = (\sqrt{2}\, e^{-\pi i/4})^4 = 4e^{-\pi i} = -4$ ✓

2. $z = \frac{-2}{1+i\sqrt{3}} \cdot \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{-2+i\,2\sqrt{3}}{4} = -\frac{1}{2}(1-i\sqrt{3}) \qquad \tan\theta = \frac{+\sqrt{3}}{-1} \qquad \theta = \frac{2\pi}{3} \pm 2\pi k$ ✓

3. $Az^2 + B\bar{z}^2 = C \qquad z = x+iy \qquad \bar{z} = x-iy \qquad z^2 = x^2-y^2+2xyi \qquad \bar{z}^2 = x^2-y^2-2xyi$ ✓

$A(x^2-y^2+2xyi) + B(x^2-y^2-2xyi) = C \equiv x^2-y^2 = 1 \quad \Rightarrow \quad A+B=1 \;\&\; A-B=0 : C=1, \; A=B=\frac{1}{2}$

4. $(-1+i)^7 = (\sqrt{2}e^{i3\pi/4})^7 = 8\sqrt{2}\, e^{\frac{21\pi}{4}i} \qquad e^{21\pi i/4} = e^{5\pi i} \cdot e^{i\pi/4} = -\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}(1+i)$

$\therefore \quad (-1+i)^7 = -8\sqrt{2}\left[\frac{\sqrt{2}}{2}(1+i)\right] = -8(1+i)$ ✓

5. $z^n = |z|^n(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

using the binomial theorem and taking only the real terms (which occur when n is even)

$\cos n\theta = \sum_{j=1}^{n} \binom{\ }{\ } \cos^{2j+1}\theta\,(-1)^{j+1} \sin^{n-2j+1}\theta$ for n even ✓

$\sum_{j=2}^{(n+1)/2}$ for n odd

6. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{n+1} \cdot \frac{n}{1} \cdot r\right| \Rightarrow \lim_{n\to\infty} \quad |r| < 1 \qquad$ if $a_n = \frac{1}{n}$ ✓

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)(n+1)^4}{n^4} r\right| \Rightarrow \lim_{n\to\infty} \quad |r| < 1 \qquad$ if $a_n = (-1)^n n^4$ ✓

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\left(\frac{\ln(n+1)}{\ln n}\right) r\right| \Rightarrow \lim_{n\to\infty} \quad |r| < 1 \qquad$ if $a_n = \ln n$ ✓

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-i)\; n!}{(n+1)!} r\right| \Rightarrow \lim_{n\to\infty} \left|\frac{r}{n+1}\right| = 0 \Rightarrow |r| < \infty \qquad$ if $a_n = \frac{i^n}{n!}$ ✓

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)(n+1)!}{n!} r\right| = |(-1)(n+1)r| \Rightarrow \lim_{n\to\infty} = \infty \Rightarrow |r|=0$ for $a_n = (-1)^n n!$

7. let $z' = \dfrac{a'w+b'}{c'w+d'}$, $a'd'-c'b' \neq 0$ and $w = \dfrac{az+b}{cz+d}$, $ad-cb \neq 0$ then $z' = \dfrac{a'\left(\frac{az+b}{cz+d}\right)+b'}{c'\left(\frac{az+b}{cz+d}\right)+d'}$

$$z' = \frac{a'(az+b)+b'(cz+d)}{c'(az+b)+d'(cz+d)} = \frac{(a'a+b'c)z+(a'b+b'd)}{(c'a+d'c)z+(c'b+d'd)} = \frac{Az+B}{Cz+D}$$

now $AD-CB = (a'a+b'c)(c'b+d'd) - (c'a+d'c)(a'b+b'd)$

$\cancel{a'c'ab} + a'ad'd + b'bcc' + \cancel{b'd'cd} - [\cancel{c'a'ab} + c'b'ad + d'ca'b + \cancel{d'b'cd}]$

$= ad(a'd'-c'b') + bc(b'c'-d'a') = (ad-bc)(a'd'-c'b') \neq 0$ ∴

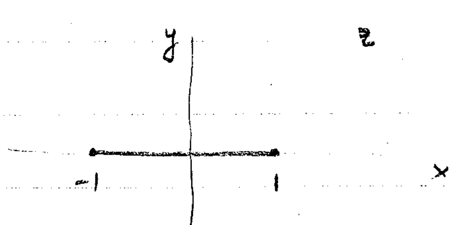
two consecutive bilinear transformations $\Rightarrow$ a bilinear transformation.

8. a. The branch points of $(z^2-1)^{1/2}$ are at $z=\pm 1$

b. ~~yes~~ no.

c. There are two values of the funct for each value of z

d.



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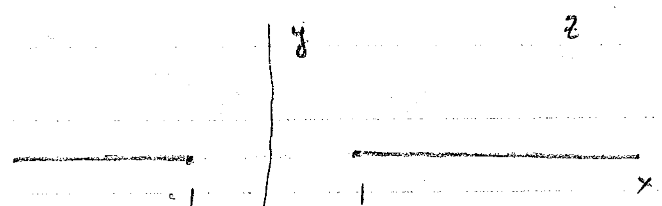
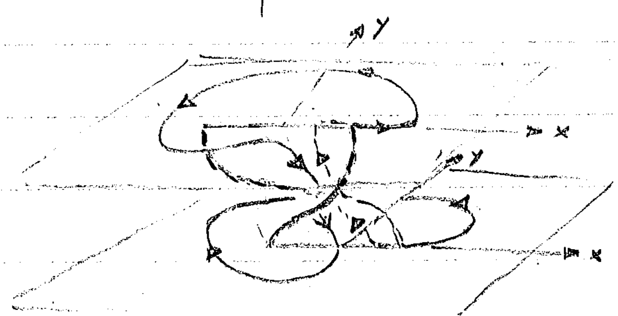
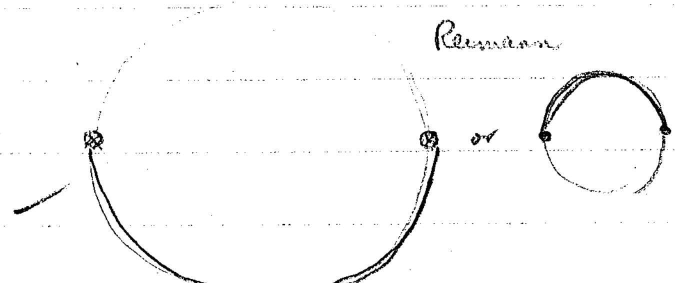


e.





A

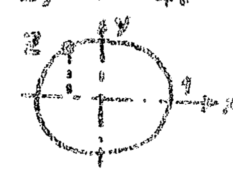
Solutions to Exercise Set 0
ME 201. APPLICATIONS OF COMPLEX VARIABLES
5 October 1979

1. Multiplying the numerator as well as the denominator of the first term by the complex conjugate $3+4i$, and similarly with $5i$ for the second term, gives

$$\frac{1+2i}{3-4i}\,\frac{3+4i}{3-4i} + \frac{2-i}{5i}\,\frac{-5i}{-5i} = \frac{-5+10i}{9+16} + \frac{-5-10i}{25} = -\frac{2}{5}$$

Writing $1-i$ in polar form as $2^{\frac{1}{2}}e^{-i\pi/4}$ and taking the fourth power, or simply applying the binomial formula gives $(1-i)^4 = -4$.

2. Multiplying numerator and denominator by $1-i\sqrt{3}$ gives $z = -1/2 + i\sqrt{3}/2$. Hence $\arg z = \tan^{-1}(-\sqrt{3})$, and recognizing that z lies in the second (not the fourth!) quadrant gives $\arg z = 2\pi/3 = 120^\circ$



3. Since $z = x+iy$ and $\bar{z} = x-iy$, $x = (z+\bar{z})/2$ and $iy = (z-\bar{z})/2$, and substituting into $x^2-y^2 = 1$ gives $z^2+\bar{z}^2 = 2$.

4. In polar form $-1+i = 2^{\frac{1}{2}}e^{i3\pi/4}$ so $(-1+i)^7 = 2^{7/2}e^{i21\pi/4}$, and reducing the argument by multiples of 2π gives $2^{7/2}e^{i5\pi/4} = -8(1+i)$.

5. With $r = 1$ for simplicity we have

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n = (\cos\theta)^n + in(\cos\theta)^{n-1}\sin\theta - \frac{n(n-1)}{2}(\cos\theta)^{n-2}\sin^2\theta + \dots$$

and equating the real parts, and using $\sin^2\theta = 1-\cos^2\theta$, gives the desired polynomial in $\cos\theta$:

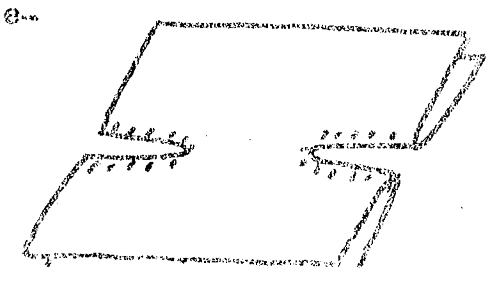
$$\cos n\theta = (\cos\theta)^n - \frac{n(n-1)}{2}(\cos\theta)^{n-2}(1-\cos^2\theta) + \dots$$
$$= 2^{n-1}(\cos\theta)^n - n2^{n-3}(\cos\theta)^{n-2} + \dots$$

6. Applying d'Alembert's ratio test, and using l'Hospital's rule in the third case, gives the radii of convergence as 1, 1, 1, ∞, and 0. In the last case the series is called semi-convergent or (purely) asymptotic -- see chapter 6 of Carrier, Krook & Pearson.

7. If $w = \frac{az+b}{cz+d}$ and, in turn, $t = \frac{Aw+B}{Cw+D}$, then simply eliminating the intermediate variable w by substitution shows that t is again a bilinear transformation of z:

$$t = \frac{A\frac{az+b}{cz+d}+B}{C\frac{az+b}{cz+d}+D} = \frac{(Aa+Bc)z+(Ab+Bd)}{(Ca+Dc)z+(Cb+Dd)}.$$

8. Factoring gives $[(z-1)(z+1)]^{\frac{1}{2}}$, which shows that this function has square-root branch points at $z = 1$ and $z = -1$. It is therefore double-valued at every other point. The derivative $z/(z^2-1)^{\frac{1}{2}}$ is finite and single-valued elsewhere, so there are no other singular points. In particular, setting $w = 1/z$ gives $(1-w^2)^{\frac{1}{2}}/w$. This behaves like $1/w$ near $w = 0$; this is not a branch point. The generally most useful branch cut runs along the real axis, for either $|x| \leq 1$ or $|x| \geq 1$. One connection of the two Riemann sheets ...



Exercise Set 1

ME 201. APPLICATIONS OF COMPLEX VARIABLES

Due Monday 8 October 1979

1. The bilinear transformation. A useful mapping in many applications is the bilinear (or linear fractional, or Möbius) transformation

$$w = \frac{az+b}{cz+d}$$

from the z- to the w-plane.

(a) Explain why one requires that $ad - bc \neq 0$.

(b) Show that this can be decomposed into a sequence of transformations that represent translation in the z-plane $(w = z + A)$, rotation and magnification $(w = Bz)$, and inversion $(w = 1/z)$.

(c) Show that the inverse mapping, from the w- to the z-plane, is also bilinear.

(d) Show that the net result of two consecutive bilinear transformations is again a bilinear transformation.

(e) An important special case is the Euler transformation

$$w = \frac{z}{z+d},$$

which leaves the origin unchanged. Sketch the image in the w-plane of the circle $z = |d|$.

(f) Using the d'Alembert ratio test or the Cauchy root test, show that the radius of convergence of the power series

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots + (-1)^{n+1}\frac{z^n}{n} + \ldots$$

is limited to a circle in the z-plane that passes through the nearest singularity. If we recast the series as an expansion in powers of $w = z/(1+z)$, for what real values of z does it converge?

2. Branch points and the Riemann surface.

(a) Consider the function $f(z) = (z-1)^{\frac{1}{2}}$. Show that the point $z = 1$ is a branch point. [Suggestion: Set $z = 1 + \rho e^{i\phi}$, choose a sign for $f(z)$ when $\phi = 0$, and show that the function does not return to its original value when ϕ is increased by 2π.]

(b) A function cannot have a single branch point. Show that $(z-1)^{\frac{1}{2}}$ has another branch point at $z = \infty$. [Set $\zeta = 1/z$, and consider how the function varies as ζ traverses a circle about $\zeta = 0$.]

(c) Now consider the function $f(z) = [(z-1)/(z+1)]^{\frac{1}{2}}$. What are its branch points? Is $z = \infty$ a branch point? Sketch several choices for a branch cut that will make this function single-valued. Include the shortest possible cut, and an infinitely long one.

(d) Consider the alternative way of making this double-valued function single-valued, by regarding it as a single-valued function on a two-sheeted Riemann surface. Try to sketch that surface — or to build a paper model. Then starting at the point $z = 1.500$, on the sheet for which $f = +0.57735\ldots$, imagine encircling the point $z = 1$ in the counterclockwise direction at a distance of 1/2, and give the numerical values of $f(z)$ after one, two, three, and four half-revolutions.

(e) As in Alice's Looking-Glass World, a function may have quite different behavior on its different Riemann surfaces. You can appreciate this by examining Exercise 13 on page 24 of Carrier, Krook & Pearson:

> Verify that a suitably defined branch of
>
> $$f(z) = \ln\left(5 + \sqrt{\frac{z+1}{z-1}}\right)$$
>
> is single-valued in the z-plane outside of a line joining the points $z = 1$ and $z = -1$. Show, however, that if one enters another sheet of the Riemann surface by crossing this line, there will be a branch point at $z = 13/12$.

ME 201
Cesar Levy
Fall 1978
SIT-IN

HW Set #1

1. a. if $ad-bc=0 \Rightarrow$ denom is a mult of numerator $\Rightarrow$ $w=$ const & hence there is no transformation (ie all pts map into one point). ✓

1b. let $w'=z+c_1$ then $w''=c_2w'=c_2z+c_1c_2$ then $w'''=\frac{1}{w''}=\frac{1}{c_2z+c_1c_2}$
then $w^{iv}=w'''+c_3=\frac{1}{c_2z+c_1c_2}+c_3=\frac{c_3z+c_1c_2c_3+1}{c_2z+c_1c_2}$

let $c_3=a$ $\quad c_1c_2c_3+1=b$ $\quad c_2=c$ $\quad c_1c_2=d$ ✓

1c. Solving for z

$wcz+dw=az+b \Rightarrow (a-cw)z=dw-b$ or $z=\frac{dw-b}{a-cw}$ ✓

let $A'=d$ $B'=-b$ $C'=-c$ $D'=a$ $\Rightarrow z=(A'w+B')/(C'w+D')$ which is a bilinear tran

1d. Shown in HW #0 : let $z'=\frac{a'w+b'}{c'w+d'}$ $\quad w=\frac{az+b}{cz+d}$ $\therefore$ $a'd'-c'b'\neq 0$ & $ad-cb\neq 0$

then $z'=\frac{Az+B}{Cz+D}$ where $A=a'a+b'c$ $\quad B=a'b+b'd$ $\quad C=c'a+d'c$ ✓
$D=c'b+d'd$ & $AD-CB=(ad-bc)(a'd'-b'c')\neq 0$.

1e. $w=\frac{z}{z+d}$ of a circle $\ni$ $|z|=d$ $\therefore$ $z=de^{i\phi}$ or $w=\frac{e^{i\phi}}{e^{i\phi}+1}=\frac{e^{i\phi}(e^{-i\phi}+1)}{2(1+\cos\phi)}$

$=\frac{1+e^{i\phi}}{2(1+\cos\phi)}=\frac{1}{2}+i\frac{\sin\phi}{2(1+\cos\phi)}$; as ϕ ranges from $\phi=0$ to 2π the

transformation will be a line whose $u=\frac{1}{2}$ and $v\geq 0$ for $0\leq\varphi<\pi$; $v\leq 0$ for $\pi<\varphi$ ✓

1f. D'Alembert's ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(-1)^{n+2}z^{n+1}/n+1}{(-1)^{n+1}z^n/n}\right|=|z|\lim_{n\to\infty}\frac{n}{n+1}=|z|$

Thus for $|z|<1$ the series converges ✓ $|z|=1$ we find that for $z=-1$ $\ln(1+z)=-\sum\frac{1}{n}$ ✓

if $w=z/_{1+z}$ $\quad z=\frac{w}{1-w}$ then $\ln(1+z)=-\ln(1-w)$ $= w+\frac{w^2}{2}+\frac{w^3}{3!}+\dots+\frac{w^n}{n!}+\dots$ then by the D'Alembert Test

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|z|=\left|\frac{w}{1-w}\right|<1$ or $-1<\frac{w}{1-w}<1$?

thus if $1-w>0$ $\therefore$ $|w|<1/2$ if $1-w<0$ there are no sol

2a. $f(z) = (z-1)^{1/2}$ let $z = 1+\rho e^{i\phi}$ $\therefore$ $z-1 = \rho e^{i\phi}$ $\therefore$ $f(z) = f(\phi) = \rho^{1/2} e^{i\phi/2}$

for $0 \le \phi \le 2\pi$ let $f(0) = +\rho^{1/2}$ $f(2\pi) = -\rho^{1/2} \neq f(0)$

hence we don't return to same values when ϕ makes one rotation $\therefore$ $z=1$ is a branch point

2b. to show $z=\infty$ is a branch point let $z = \frac{1}{w}$: when $z=\infty$ $w=0$

$\therefore$ $\left(\frac{1}{w}-1\right)^{1/2} = f(w) = \left(\frac{1-w}{w}\right)^{1/2} \Rightarrow$ branch points are $w=1$ & $w=0$,

(but $w=0 \Rightarrow z=\infty$) because if we let $w = \rho e^{i\phi}$, $\Rightarrow$ $f(w) = f(\phi) = \left(\frac{1-\rho e^{i\phi}}{\rho e^{i\phi}}\right)^{1/2}$

where $0 \le \phi \le 2\pi$ $f(0)$ will not be $= f(2\pi)$. ✓

2c.(1) $f(z) = \left(\frac{z-1}{z+1}\right)^{1/2}$ if $z = 1+\rho e^{i\phi}$ $\Rightarrow$ $\left(\frac{\rho e^{i\phi}}{2+\rho e^{i\phi}}\right)^{1/2}$: as $0 \le \phi \le 2\pi$

$f(\phi=0) \neq f(\phi=2\pi)$ $\Rightarrow$ <u>$z=1$ is a branch point</u>. if $z = -1+\rho e^{i\phi} \Rightarrow \left(\frac{-2+\rho e^{i\phi}}{\rho e^{i\phi}}\right)^{1/2}$

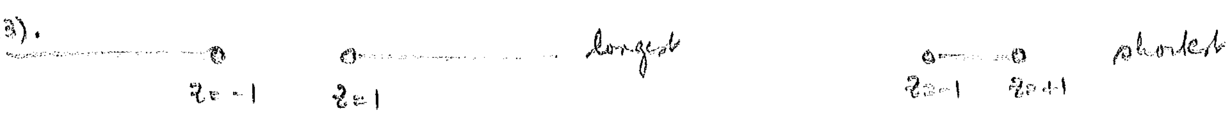
$f(\phi=0) \neq f(\phi=2\pi)$ $\Rightarrow$ <u>$z=-1$ is a branch point.</u> ✓

2c(2). $z=\infty$ is not a branch point since for $z = 1/w$ $f(w) = \left(\frac{1-w}{1+w}\right)^{1/2} \Rightarrow f(0)=1$

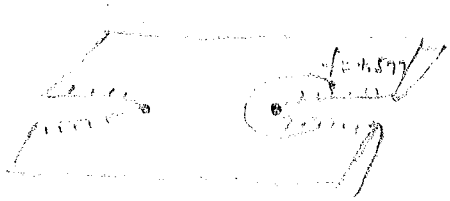
or if $w = \rho e^{i\phi}$ [with $\phi=0$ & $f(0)=+1$] when $\phi = 2\pi$, $f(0) = +1$ $\therefore$ same sign

& no branch point ✓

2c(3).



2d.



$\frac{1}{2}$ rev $f = -.57735\ldots$ **times i**

1 rev $f = .57735\ldots$ **~~times~~ NO, $-.4772$**

$\frac{3}{2}$ rev $f = -.57735$ **times i**

2 rev $f = .~~57735~~\ldots$ **$.4772$ (my fault!)**

See my remark about the shortcoming of this picture

2e. $f(z) = \ln\left(5 + \sqrt{\frac{z+1}{z-1}}\right) = \ln 5 \cdot \ln\left(1 + \frac{1}{5}\sqrt{\frac{z+1}{z-1}}\right)$

using the previous results of f, with $u = \frac{1}{5}\sqrt{\frac{z+1}{z-1}}$. $\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\dfrac{\left((-1)^{n+1}\frac{1}{5^{n+1}}\left(\frac{z+1}{z-1}\right)^{\frac{n+1}{2}}\right)\Big/ (n+1)}{\left((-1)^{n}\frac{1}{5^{n}}\left(\frac{z+1}{z-1}\right)^{n/2}\right)\Big/ n}\right|$

Why have you envisioned expanding in series?

to the nearest sing: $\frac{1}{5}\left|\frac{z+1}{z-1}\right|^{1/2} = 1 \Rightarrow \frac{z+1}{z-1} = 25 \qquad z+1 = 25z - 25 \qquad z = \frac{26}{24} = \frac{13}{12}$

thus $z = \frac{13}{12}$ is a value which will cause the series to diverge

But there are branch points at $z = \pm 1$!?

for series to converge $\frac{1}{5}\left|\frac{z+1}{z-1}\right|^{1/2} < 1$ $\qquad z = +1$ doesn't satisfy the inequality.

using the result $z = \pm 1 + \rho e^{i\phi}$ then $\sqrt{\frac{z+1}{z-1}}$ changes signs at $\pm 1 = z$ unless they are branch points

(B)

Solutions to Exercise Set 1

ME 201. APPLICATIONS OF COMPLEX VARIABLES

12 October 1979

1. The bilinear transformation. (a) If $ad - bc = 0$, the transformation degenerates to $x = a/c$, which maps the entire z-plane into a single point. (b) Unless $c = 0$,

$$w = \frac{a}{c} + \frac{bc-ad}{c(cz+d)}$$

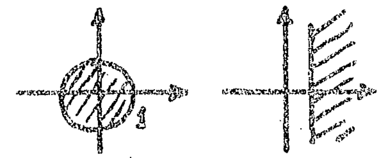
which is clearly a rotation-and-magnification followed by a translation followed by an inversion followed by a second translation.

(c) Solving for z gives $z = (-dw+b)/(cw-a)$, which is bilinear.

(d) If $w = (az+b)/(cz+d)$ and $s = (Aw+B)/(Cw+D)$, then

$$s = \frac{(Aa+Bc)z + (Ab+Bd)}{(Ca+Dc)z + (Cb+Dd)},$$ which is again bilinear.

(e) Setting $z = de^{i\theta}$ gives $w = \frac{1}{2}(1 + i\tan\theta/2)$, which is the infinite vertical line through $w = 1/2$. The interior of the circle in the z-plane maps to the right of that line.



(f) The derivative $(d/dz)\, n(1+z) = 1/(1+z)$ is not defined at $z = -1$, which is the nearest singular point. The ratio test gives the radius of convergence

thus one maps z=-1 into ∞

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{n+1}{n} = 1.$$

Recasting the function $\ln(1+z)$ as a series in powers of $w = z/(1+z)$ gives

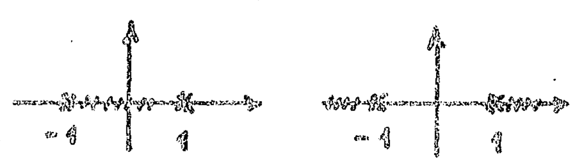
$$\ln(1+z) = -\ln(1-w) = w + \frac{w^2}{2} + \frac{w^3}{3!} + \ldots \frac{w^n}{n!} + \ldots \quad \frac{w^{n+1}}{(n+1)n!}\ \frac{n!}{w^n}$$

This likewise has unit radius of convergence; but $|w| < 1$ corresponds to $\mathrm{Re}\, z > -1/2$, so the recast series converges for all finite real z greater than (or in fact equal to) $-1/2$.

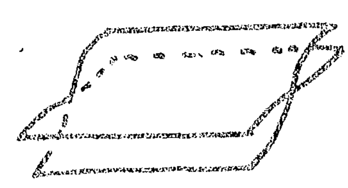
2. Branch points and the Riemann surface. (a) Since $f'(z) = \frac{1}{2}(z-1)^{-\frac{1}{2}}$, the derivative is undefined at $z = 1$, which is therefore a singular point. Setting $z = 1 + \rho e^{i\varphi}$ gives $f(z) = \rho^{\frac{1}{2}} e^{i\varphi/2}$, and this changes sign when φ is increased by 2π, so the singularity is a branch point (of order 1).

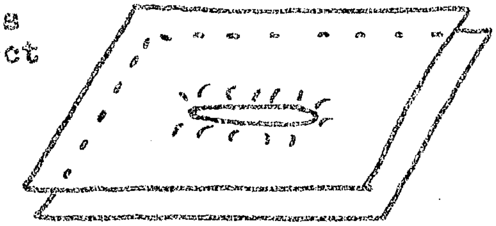
(b) Setting $\zeta = 1/z$ gives $f(z) = (1-\zeta)^{\frac{1}{2}}\zeta^{-\frac{1}{2}}$. The origin is evidently a singular point, and setting $\zeta = \rho e^{i\varphi}$ shows as above that it is a branch point.

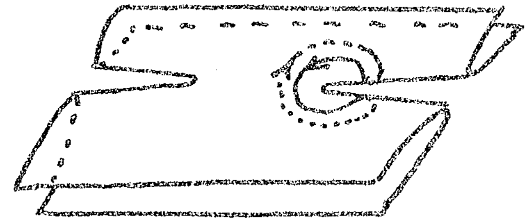
(c) Evidently $f(z) = \left[(z-1)/(z+1)\right]^{\frac{1}{2}}$ has branch points at $z = \pm 1$. However, although each factor $(z-1)^{\frac{1}{2}}$ and $(z+1)^{\frac{1}{2}}$ has a branch point at infinity, it is easy to verify that their ratio does not. In applications, one would often use one or the other of the two branch cuts sketched.



(d) The corresponding two-sheeted Riemann surface cannot be properly constructed without one sheet passing through the other (without intersecting!). That defect can be avoided by constructing the surface in either of the ways shown below.

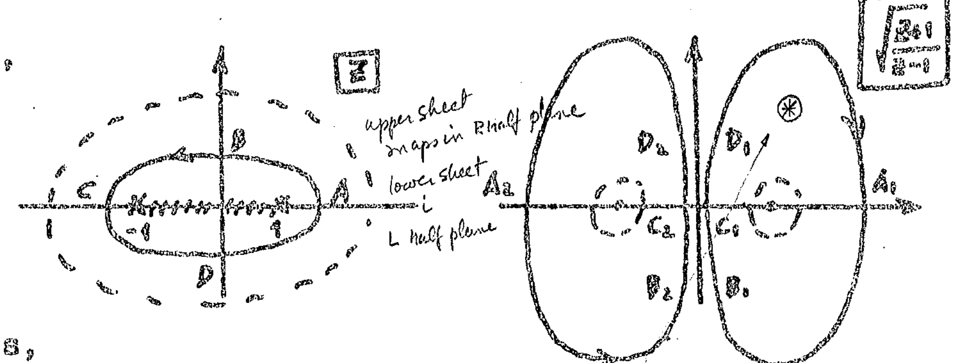






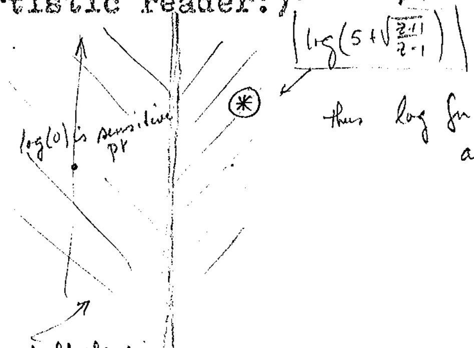
However, this has the defect that in encircling either singular point the conventional positive counterclockwise sense is reversed on the lower sheet. (Of course that is only as viewed from the front; a man making the trip would see the lower sheet from our rear, and regard himself as still going counterclockwise.) On the circle $z-1 = 1/2$, the values of f after zero, one, two, three, and four half-revolutions are $(5)^{-\frac{1}{2}}$ $= 0.4472$ (NOT 0.57735 as stated!), $(3)^{-\frac{1}{2}}i$, $-(5)^{-\frac{1}{2}}$, $-(3)^{-\frac{1}{2}}i$, and again $(5)^{-\frac{1}{2}}$.

(e) The function $(z+1)/(z-1)$ has branch points only at $z = 1$ and $z = -1$, and so remains single-valued if we do not cross a cut joining them. However, that does not mean that the logarithm is single-valued. (For instance, z is single-valued, but its logarithm is not.) However, traversing various contours around the cut in the z-plane we find ourselves describing contours (in the opposite sense) in either the right or left half of the plane of $\sqrt{(z+1)/(z-1)}$.



We choose the right half-plane (so that, for example, $z = 13/12$ maps into 5 rather than -5). Then the logarithm is confined to one of its infinitely many single-valued sheets, because the angle is confined between $-\pi$ and π — for example, to the principal branch, on which the logarithm of a positive real number is real. However, if we cross the branch cut in the z-plane, we enter the other Riemann sheet, which maps into the left half of the plane of $\sqrt{(z+1)/(z-1)}$. Then $z = 13/2$ maps instead into -5, so the argument of the logarithm can vanish, giving a logarithmic branch point on the second sheet. (The task of connecting those denumerably infinite sheets with the other two sheets is left to the artistic reader!)

$\log\left(5 + \sqrt{\frac{z+1}{z-1}}\right)$



thus log fn in this part has an argument of $|\theta| \le \pi/2$

left half plane mapped here & 2nd sheet will contain pt where $z = 13/12$ ie $\log(\quad) = \log(0)$

Exercise Set 2

ME 201. APPLICATIONS OF COMPLEX VARIABLES

Due Monday 15 October 1979

2.1. Singular points. Find and classify (e.g., branch point, simple pole, Nth-order pole, removable, essential) all the singularities of the function

$$\frac{(z^2+1)^{\frac{1}{2}}}{z(1-z)\sin z}$$

$z = \pm i$ are branch pts

~~$z=0$~~, $z=1$ are simple poles

$z=0$ is a double pole $z = \pm n\pi$ are simple poles, $n=1,2,\ldots$

2.2. Heating a half-space. Explain how the function

$$\log \frac{z-1}{z+1}$$

can be regarded as a single-valued function in the upper half plane, having zero imaginary part at (for example) $z = 2$. How does this function behave for large z; does it have a singularity there?

Convince yourself that the imaginary part of this function satisfies the two-dimensional Laplace equation. Thus show that it gives the solution for steady plane heat conduction in a semi-infinite medium at zero temperature far away, which is heated along a portion of its boundary. Sketch roughly the isotherms.

2.3. Integration around a branch point. If we make the function $z^{\frac{1}{2}}$ single-valued with a cut, we cannot complete a contour, and are therefore unable to consider, say, its integral around the unit circle. If, however, we instead make it single-valued by introducing a two-sheeted Riemann surface, we can then integrate around a closed contour that consists of the unit circle traversed twice — once on each sheet.

Calculate in that way what might be denoted by

$$\oint\!\!\!\oint z^{\frac{1}{2}}\,dz$$

Then draw a general conclusion about the applicability of Cauchy's theorem to functions with branch points.

Cesar Levy
ME 201
Prof Van Dyke Fall 79
AUDIT

Problem Set #2

2.1: $\dfrac{(z^2+1)^{1/2}}{z(1-z)\sin z}$ has $z=\pm i$ as branch points ; $z=0$ as a double pole ; $z=1$ as a simple pole and $z=\pm n\pi$ $n=1,2,3,\ldots$ as simple poles

The point at ∞: $\dfrac{(z^2+1)^{1/2}}{z(1-z)\sin z} \sim \dfrac{z}{-z^2\sin z} = \dfrac{-1}{z\sin z} = \dfrac{-1}{z^2(z^2-\pi^2)(z^2-4\pi^2)\cdots(z^2-n^2\pi^2)\cdots}$?

if $z=\frac{1}{w}$ then the function $\sim \dfrac{-w^{2(n+1)}}{1\cdot(1-\pi^2w^2)\cdot(1-4\pi^2w^2)\cdots(1-n^2\pi^2w^2)} \to 0$ as $w\to 0$ or as $z\to\infty$

thus $z=\infty$ is an analytic point (pt where fn is analytic)

Setting $z=\frac{1}{w}$: $\dfrac{w}{\sin 1/w}$
and $w=0$ is point of accumulation of poles — an essential singularity.

2.2: $\log\dfrac{z-1}{z+1} = \log(z-1)-\log(z+1) = \frac{1}{2}\log((x-1)^2+y^2) - \frac{1}{2}\log((x+1)^2+y^2) + i\left[\tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x+1}\right)\right]$

if when $z=2$ $(x=2, y=0)$ $\Im\left(\log\frac{z-1}{z+1}\right)=0$ we ~~must~~ confine the angle $0\le\theta<2\pi$ & put branch cut between $z=\pm1$. ~~[illegible]~~ let $z-1=\rho_1e^{i\theta_1}$, $z+1=\rho_2e^{i\theta_2}$ $\Rightarrow$ $\theta_1-\theta_2=0$ at $z=2$, or $\theta_1=\theta_2$

– for large z $\log\dfrac{z-1}{z+1} = \log\dfrac{z(1-1/z)}{z(1+1/z)} = \log\left([1-1/z][1-1/z+1/z^2+\ldots]\right) \sim \log[1-2/z] \to 0^{+i0}$ as $z\to\infty$

thus there is no singularity if we restrict $0\le\theta<2\pi$

take only $\log(z-1)$ for example $\log(z-1) = \log((x-1)^2+y^2) + i\tan^{-1}\left(\frac{y}{x-1}\right)$ $\therefore$ $v_1=\tan^{-1}(y/x-1)$

$$\frac{\partial^2 v_1}{\partial x^2} = \frac{2y(x-1)}{[(x-1)^2+y^2]^2} \qquad \frac{\partial^2 v_1}{\partial y^2} = -\frac{2y(x-1)}{[(x-1)^2+y^2]^2} \qquad \therefore \quad \Delta v_1 \equiv 0$$

Now we can do same w/ $\log(z+1)$, just change $-$ sign in $(x-1)$ factors to $+1$. But since $\log\left(\frac{z-1}{z+1}\right)$ $=\log(z-1)-\log(z+1)$ $\Im\left(\log\frac{z-1}{z+1}\right) = \Im\log(z-1) - \Im\log(z+1) = \tan^{-1}(y/x-1) - \tan^{-1}(y/x+1) = v$

and $\Delta(v_1-v_2) = \Delta v_1 - \Delta v_2 = 0$

for steady state heat condition with $T=T_0$ for $y=0, |x|\le1$ and $T=0$ $\forall$ $|x|>1$ & $T=0$ as $|y|\to\infty$

then $\Delta T=0 \Rightarrow \dfrac{\partial^2T}{\partial x^2}+\dfrac{\partial^2T}{\partial y^2}=0$ using the fourier integral $T(x,y) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty e^{-\lambda y}f(\xi)\cos\lambda(\xi-x)d\xi d\lambda$

where $f(\xi)=T_0$ for $|\xi|\le1$ $f(\xi)=0$ for $|\xi|>1$

but $\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty e^{-\lambda y}f(\xi)\cos\lambda(\xi-x)d\xi d\lambda = \frac{1}{\pi}\int_{-\infty}^\infty \frac{y\,f(\xi)\,d\xi}{y^2+(\xi-x)^2} = \frac{1}{\pi}\int_{-1}^1\frac{y\,T_0\,d\xi}{y^2+(\xi-x)^2}$

$$-\frac{T_0}{\pi}\tan^{-1}\left(\frac{y}{\xi-x}\right)\Bigg|_{\xi=-1}^{\xi=1} = \frac{T_0}{\pi}\left\{\tan^{-1}\left(\frac{y}{x+1}\right) - \tan^{-1}\left(y/x-1\right)\right\}$$

which, up to a multiplicative constant of $-\frac{T_0}{\pi}$ this is the imag part of $\log\frac{z-1}{z+1}$

2.3: $\oint z^{1/2} dz$ let $\rho e^{i\varphi} = z$ & $dz = i d\varphi \rho e^{i\varphi}$ for a circular path & for a unit circle ρ

$$\oint e^{i\varphi/2} i e^{i\varphi} d\varphi = i \int_0^{4\pi} e^{i3\varphi/2} d\varphi = \frac{2}{3} e^{i3\varphi/2}\Big|_0^{4\pi} = \frac{2}{3}\left[e^{6\pi i} - e^{0}\right] = 0$$

If you integrate so that you make one full revolution in the transformed plane then you will satisfy Cauchy's thm that $\oint_C f(z)dz = 0$ if $f(z)$ is analytic in C

If you integrate so that you make one full revolution in the z plane then $\oint_C f(z) dz \neq 0$

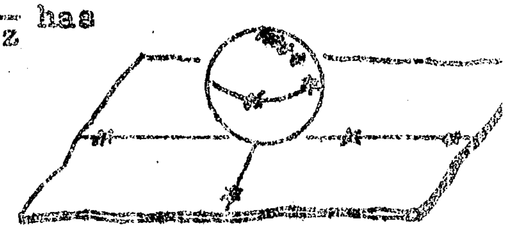
A

Solutions to Exercise Set 2

ME 201. APPLICATIONS OF COMPLEX VARIABLES

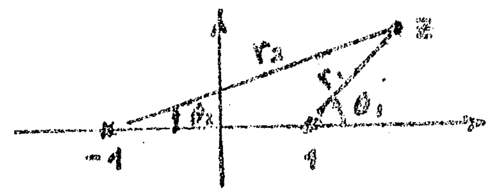
17 October 1979

2.1. Singular points. The function $\frac{(z^2+1)^{\frac{1}{2}}}{z(1-z)\sin z}$ has



branch points at $z = \pm i$, of order 1;
simple pole at $z = 1$;
double pole at $z = 0$ (where $z, \sin z = 0$)
simple poles at $z = \pm N\pi$, $N = 1,2,3,\ldots$
essential singularity at $z = \infty$ which is a point of accumulation (limit point) of poles.

2.2 Heating a half-space. The function $\log (z-1)/(z+1)$ has logarithmic branch points at $z = \pm 1$. It is single-valued if the z-plane is cut between those points, and hence single-valued in the upper half-plane. We choose the branch on which the function is real for $x > 1$. The derivative $f'(z) = 2/(z^2-1)$ vanishes as $z \to \infty$, which is a regular point.



We know that the imaginary part of any analytic function satisfies Laplace's equation. Alternatively, writing the Laplace equation in polar coordinates as

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

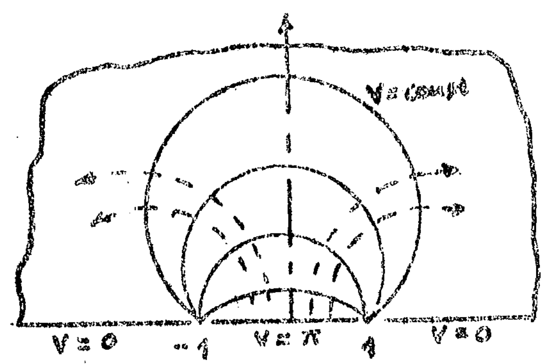
and calculating the imaginary part of our function:

$$f(z) = \log(z-1) - \log(z+1) = \log\frac{r_1}{r_2} + i(\theta_1 - \theta_2),$$

we see that θ_1 and θ_2 are solutions, and hence so is their difference. (The corresponding verification in Cartesian coordinates is more laborious!) Our harmonic function

$$v = \theta_1 - \theta_2 = \tan^{-1}\frac{2y}{x^2+y^2-1}$$

is constant on the circles $2y/(x^2+y^2-1) =$ constant. It vanishes at infinity and on the boundary outsize $-1 < x < 1$, where it equals π. Hence it solves the problem of plane steady heat conduction in a half-space with a length of the boundary heated uniformly.



2.3. Integration around a branch point. Introducing polar coordinates according to $z = re^{i\theta}$ and integrating from $\theta = 0$ to 4π shows that the integral around both Riemann sheets vanishes.

Exercise Set 3

ME 201. APPLICATIONS OF COMPLEX VARIABLES

Due Monday 22 October 1979

3.1. Taylor series and pattern of signs. Classify the function

$$f(z) = \frac{(1-z)(1+3z)}{1-z^3} = \frac{1+3z}{1+z+z^2} = \frac{1+3z}{z(\frac{1}{z}+1+z)} \cdot \frac{1+3z}{1+z(1+z)}\left[1-z(1+z)+z^2(1+z)^2+\right.$$

$$z^2\left[\tfrac{1}{z^2}+\tfrac{1}{z}+1\right]$$

according to its analytic structure (number, location, and nature of zeroes and singularities in the complex plane, including the point at infinity; multi-valuedness; etc.). Calculate its Taylor series expansion about the origin ("Maclaurin series"). Is the radius of convergence given by the ratio test? The root test? Determine the pattern of signs, and explain what produces it. Can you invent a comparably simple model function whose Maclaurin series has a sign pattern with period five?

3.2. Computation of Laurent series. Find the Laurent expansions of

$$\frac{e^{(z-1)}}{(z-1)^3}$$

@ z=1 $\frac{1}{(z-1)^3} + \frac{1}{(z-1)^2} + \frac{1}{2!}\frac{1}{z-1} + \frac{1}{3!} + \frac{(z-1)}{4!} + \dots + \frac{(z-1)^{n-3}}{n!} + \dots$ 3rd order pole, res $= \frac{1}{2}$, R

about the points $z = 0$, 1, and ∞. In each case classify the singularity, calculate the residue, and give the radius of convergence.

3.3. Computation of residues. Find the residue of each of the following functions at its singular point nearest to the origin:

$$\frac{1}{z^2+z-z}, \qquad \frac{\sin z}{z^3}, \qquad z^3\, e^{1/(z-1)}.$$

$(z+2)(z-1)$

Residue if $= a_{-1}$ of Laurent series about singular point

$\frac{-1 \pm \sqrt{5}}{2}$ nearest is $\frac{-1+\sqrt{5}}{2}$ a. $f\left(\frac{-1+\sqrt{5}}{2}\right)$

b. $f(0)$

c. $f(z=1)$

$-\frac{1}{e}\left\{1+z+z^2/2!+\dots\right\}\left\{1+z+z^2/2+\dots\right\}^3$

$1+3z+6z^2+10z^3+15z^4+\dots$

$\left(\frac{21}{0!}+\frac{15}{1!}+\frac{10}{2!}+\frac{6}{3!}+\frac{3}{4!}+\frac{1}{5!}\right)z^6$

z^4

z^3

$(j-n+3)\cdot(j-n+2)$

$\left(\frac{15}{0!}+\frac{10}{1!}+\frac{6}{2!}+\frac{3}{3!}+\frac{1}{4!}\right)z^4$

$(j-n+1)(j-n+2)\ z^j \quad j \geq 2$

$1 - z - \cancel{z^2} + \cancel{z^2} + 2z^3 + z^4 - z^3 - 3z^4 - 3z^5 - \cancel{z^6} + z^4 + 4z^5 + 6z^6 + \cancel{4z^7} + z^8 - z^5 - 5z^6 + 10z^7 - 10z^8 - 5z^9 + \cancel{z^6} + 6z^7$

$-z^{10}$

$1 - z + z^3 - z^4 + z^6 + 2z^7$

$-z^7$

Solutions to Exercise Set 3

ME 201. APPLICATIONS OF COMPLEX VARIABLES

Wednesday 24 October 1979

3.1. Taylor series and pattern of signs. This is a single-valued function with a complex conjugate pair of simple poles at $z = e^{i2\pi/3}$ and a simple zero at $z = -1/3$. We may say that the first form given also has a removable singularity at $z = 1$. Setting $w = 1/z$ gives

$$\frac{w(3+w)}{1+w+w^2}$$

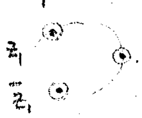
and this has a simple zero at the origin, so we say that $f(z)$ has a simple zero at infinity.

The Taylor series about the origin is most easily found from the first form given, by expanding the denominator by the binomial theorem:

$$\frac{1+3z}{1+z+z^2} = (1+2z-3z^2)(1+z^3+z^6+\dots) = 1+2z-3z^2+z^3+2z^4-3z^5+\dots$$

depends on $\frac{1}{1-z^3}$

Thus the coefficients are a repetition of 1, 2, -3. Hence the ratios a_n/a_{n+1} are a repetition of $1/2$, $-2/3$, 3; these have no limit, so the ratio test fails. On the other hand, the nth root of 1, 2, or 3 approaches unity, so Cauchy's root test shows that the radius of convergence is unity — the distance to the nearest singularities.



The signs are $+ + -$ repeated. The period of three is set by the pair of poles on the unit circle, spaced at multiples of 1/3 of a full circle; and the particular pattern is due to the signs of the numerator. A pattern of signs with period five would result from $1-z^5$ in the denominator.

if you null $\frac{1}{1-z^3}(z-z_1)$ you get rid of 1 sign

3.2. Computation of Laurent series. The origin is a regular point, so the Laurent expansion is a Taylor (Maclaurin) series:

$$\frac{e^{z-1}}{(z-1)^3} = \frac{-1}{e}\left(1+z+\frac{z^2}{2!}+\dots\right)(1+3z+6z^2+\dots) = \frac{-1}{e}\left(1+4z+\frac{19}{2}z^2+\dots\right)$$

The coefficient of z^n is $\frac{-1}{2e}\sum_{k=0}^{n}\frac{(k+1)(k+2)}{(n-k)!}$. The radius of convergence is set by the pole at $z=1$.

The Laurent expansion about $z=1$ has a triple pole:

$$\frac{1}{(z-1)^3}+\frac{1}{(z-1)^2}+\frac{1}{2(z-1)}+\frac{1}{6}+\dots+\frac{1}{(n+3)!}(z-1)^n+\dots$$

The residue is seen to be 1/2; and this converges for all $|z| > 0$, because there are no other singularities in the finite plane.

The Laurent expansion about the essential singularity at $z=\infty$ will converge for all $|z| > 1$. Setting $z = 1/\zeta$ and expanding for small ζ yields

$$\frac{1}{e}\left(\zeta^3+\zeta^2+\frac{1}{2}\zeta+\frac{1}{6}+\dots+\frac{\zeta^{3-n}}{n!}+\dots\right)\left(1+3\zeta+6\zeta^2+\dots+\frac{(n+1)(n+2)}{2}\zeta^n+\dots\right)$$

The coefficient of the terms in $1/\zeta$, and hence of z, is

$$\frac{1}{e}\left(\frac{1}{4!}+\frac{3}{5!}+\frac{6}{6!}+\dots+\frac{(n-3)(n-2)}{2n!}+\dots\right) = \frac{3}{2}-\frac{4}{e} = 0.02848\dots$$

and any other coefficient can likewise be found using $e = 1+\sum_{n=1}^{\infty}\frac{1}{n!}$.

3.3. Computation of residues. These functions have respectively simple poles at $z = 1$ and -2, a double pole at $z = 0$, and an essential singularity at $z = 1$.

$$\frac{1}{z^2+z-2} = \frac{1}{(z-1)(z+2)}$$ evidently has residue 1/3 at $z = 1$.

$$\frac{\sin z}{z^3} = \frac{z - z^3/6 + \cdots}{z^3}$$ has residue 0 at $z = 0$.

$$z^3 e^{1/(z-1)} = \left[1 + (z-1)\right]^3 e^{1/(z-1)} =$$

$$\left[(z-1)^3 + 3(z-1)^2 + (z-1) + 1\right]\left[1 + \frac{1}{z-1} + \frac{1}{2!(z-1)^2} + \frac{1}{3!(z-1)^3} + \cdots\right]$$

$$= \cdots + \left(\frac{1}{4!} + \frac{3}{3!} + \frac{3}{2!} + 1\right)\frac{1}{z-1} + \cdots$$ so residue is 73/24.

ME201
Fall 1979
Prof Van Dyke
Cesar Levy AUDIT

Problem Set # 3

3.3: a. $\frac{1}{z^2+z-2} = \frac{1}{(z+2)(z-1)}$ find $f(1)$

-2 1

thus $\frac{1}{(z+2)(z-1)} = \frac{-1/3}{z+2} + \frac{1/3}{z-1}$ since $\frac{1}{z+2}$ is analytic at $z_0=1$ then $\frac{-1/3}{z+2} = \frac{-1/3}{3\left[1+\frac{z-1}{3}\right]} = -\frac{1}{9}(z-1)^n ...$

and contributes nothing to residue $\therefore$ residue is coeff of $\frac{1}{z-1} = \frac{1}{3}$ ✓

b. $\frac{\sin z}{z^3} = \frac{1}{z^3}\prod_{n=-\infty}^{\infty}(z-n\pi) = \frac{1}{z^2}\prod_{n=1}^{\infty}(z^2-n^2\pi^2) = \frac{z-\frac{1}{3}z^3+\frac{1}{5}z^5+\dots}{z^3} = \frac{1}{z^2}-\frac{1}{3!}+\frac{1}{5!}z^2+\dots$
and has no $\frac{1}{z}$ term. $\therefore$ residue $=0$ ✓; since singular point is at the origin

c. $z^3e^{\frac{1}{z-1}}$: singular point is at $z=1$ $\therefore$ $z^3e^{\frac{1}{z-1}} = (z-1)^3+4(z-1)^2+\frac{13}{2}(z-1)+\frac{34}{6}$
$+\frac{73}{24}\frac{1}{z-1}+\dots$

thus @ $z_0=1$ residue is $\frac{73}{24}$. $e^{\frac{1}{z-1}}z^3 = \left[(z-1)^3+1\right]^3 e^{\frac{1}{z-1}}$ take all the $\frac{1}{z-1}$ terms to get $\frac{73}{24}$ ✓

3.2. $\frac{e^{z-1}}{(z-1)^3} = \frac{1}{(z-1)^3}+\frac{1}{(z-1)^2}+\frac{1}{2!(z-1)}+\frac{1}{3!}+\frac{(z-1)}{4!}+\dots+\frac{(z-1)^n}{(n+3)!}+\dots$ ✓

- about $z_0=1$: It has a pole of order 3 and a residue of $\frac{1}{2}$ and has a radius of convergence 0 since by limit test $L=\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+4}|z-1| \to 0$ as $n\to\infty$; but $R=\frac{1}{L} \Rightarrow$ thus the result for the region of convergence is $|z-1|<\infty$ ✓
- about $z_0=0$: since the singularity $z=1$ is a problem; we must use the fact the for a circle of $|z|<1$ the fn is analytic and has no principal value $\therefore$ its residue is 0 & has a radius of convergence of 1 (radius from $z_0=0$ to nearest singularity) ✓

$$f(z) = -\frac{1}{e}\left\{1+z+\frac{z^2}{2!}+\dots\right\}^3\left\{1+z+z^2+\dots\right\}^3 = \sum_{j=0}^{\infty}a_j z^j \quad \text{where } a_j = \sum_{n=1}^{j+1}\frac{-1}{e}\frac{(j-n+3)(j-n+\dots)}{2(n-1)!}$$ ✓

about $z_0=0$ and $|z|>1$ $\frac{e^{z-1}}{(z-1)^3} = +\frac{1}{e}\frac{1}{z^3}\frac{e^z}{(1-1/z)^3} = +\frac{1}{e}\left\{\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right\}^3\left\{1+z+\frac{z^2}{2!}+\dots\right\}$

now its region of convergence is $1<|z|<\infty$. It has an essential singularity at ∞ ✓

the residue $\dfrac{e^{z-1}}{(z-1)^3} = \sum_{n=1}^{\infty} \dfrac{(n-1)(n-2)}{2e\,(n-1)!}$ if $f(z) = \sum_{n=\infty}^{\infty} a_j z^j$

$a_j = \sum_{n=1}^{\infty} \dfrac{n(n+1)}{2e\,(n+2+j)} \quad j>0$

$a_j = \sum_{n=1} \dfrac{1}{e} \dfrac{(n-j-2)(n-j-3)}{2(n-1)!} \quad j<0$

$a_0 = \sum \dfrac{1}{2e} \dfrac{(n-1)(n-2)}{(n-1)!} \quad j=0$

about $z_0 = \infty$ let $z = \frac{1}{\omega}$ $\quad \dfrac{e^{\frac{1}{\omega}-1}}{(\frac{1}{\omega}-1)^3} = \dfrac{\omega^3}{e} \cdot \dfrac{e^{\frac{1}{\omega}}}{(1-\omega)^3}$

$= \dfrac{\omega^3}{e}\left\{1+\omega+\omega^2 \dots\right\}^3\left\{1 + \frac{1}{\omega} + \frac{1}{2!\,\omega^2} + \dots\right\}$ this has an essential singularity at $\omega = 0$ ie $z=\infty$ since the 1 from the first bracket will mult the entire 2nd bracket which has a pole of ∞ order (thus an essential singularity). Region of convergence is $0<|\omega|<1$

$a_{-1} = \sum_{n=1}^{\infty} \frac{1}{e} \dfrac{n(n+1)}{2(n+3)!} = \dfrac{1}{2e} \sum_{n=1}^{\infty} \dfrac{1}{(n+2)(n+1)(n-1)!}$; $= \dfrac{3}{2} - \dfrac{4}{e}$

if $f(z) = \sum_{j=-\infty}^{\infty} a_j \omega^j$ then

$a_j = \sum_{n=1}^{\infty} \frac{1}{e} \dfrac{(n+j-3)(n+j-2)}{2(n-1)!} \quad j>0$

$a_j = \sum_{n=1}^{\infty} \dfrac{n(n+1)}{2e(n+2-j)!} \quad j<0$

$a_0 = \sum_{n=1}^{\infty} \dfrac{1}{2e} \dfrac{n(n+1)}{(n+2)!} \quad j=0$

3.1. $f(z) = \dfrac{(1-z)(1+3z)}{(1-z^3)} = \dfrac{1+3z}{(1+z)+z^2}$ when $z \neq 1$ $\quad f(z) = \dfrac{1+3z}{(z-w)(z-\bar{w})}$ where $w = \dfrac{-1+\sqrt{3}i}{2}$

has a zero at $z=-\frac{1}{3}$; has a removable singularity at $z=1$ and a pole at $z = w$ & $\bar{w}$

at $z=\infty$ let $z = \frac{1}{v}$ $\quad f(z) = \dfrac{1+3/v}{1+1/v+1/v^2} = \dfrac{(v+3)v}{v^2+v+1}$ $\quad f(v) = 0$ at $v=0 \Rightarrow z=\infty$ is not a point of singularity.

Much simpler to use first form given.

To find the Taylor series about $z_0 = 0$ & $|z|<1$: $f(z) = \dfrac{1+3z}{w\bar{w}(1-z/w)(1-z/\bar{w})} = \dfrac{\frac{3}{2}+i\frac{\sqrt{3}}{6}}{z-w} + \dfrac{\frac{3}{2}-i\frac{\sqrt{3}}{6}}{z-\bar{w}}$

$\therefore \dfrac{\frac{3}{2}+i\frac{\sqrt{3}}{6}}{-w(1-z/w)} + \dfrac{\frac{3}{2}-i\frac{\sqrt{3}}{6}}{-\bar{w}(1-z/\bar{w})} = \dfrac{\frac{1}{2}+\frac{5}{6}\sqrt{3}i}{1-z/w} + \dfrac{\frac{1}{2}-\frac{5}{6}\sqrt{3}i}{1-z/\bar{w}} = \dfrac{A}{1-z/w} + \dfrac{\bar{A}}{1-z/\bar{w}}$

$\dfrac{1}{(1-z/w)} = 1 + z/w + z^2/w^2 + z^3/w^3 + \dots \qquad \dfrac{1}{1-z/\bar{w}} = 1 + z/\bar{w} + (z/\bar{w})^2 + \dots$

$\therefore f(z) = A\left[1 + z/w + z^2/w^2 + z^3/w^3 + \dots\right] + \bar{A}\left[1 + z/\bar{w} + z^2/\bar{w}^2 + \dots\right]$

$\therefore f(z) = (A+\bar{A}) + z\left(A/w + \bar{A}/\bar{w}\right) + z^2\left(\dfrac{A}{w^2} + \dfrac{\bar{A}}{\bar{w}^2}\right) + z^3\left(\dfrac{A}{w^3} + \dfrac{\bar{A}}{\bar{w}^3}\right) + \dots$

$= 1 + 2\mathrm{Re}\left(\dfrac{A}{w}\right)z + 2\mathrm{Re}\left(\dfrac{A}{w^2}\right)z^2 + \dots + 2\mathrm{Re}\left(\dfrac{A}{w^n}\right)z^n + \dots = 1 + \sum_{j=1}^{\infty} 2\mathrm{Re}\left(A\bar{w}^j\right)z^j$

and radius of convergence is $\left| \frac{2\mathrm{Re}(A\bar{w}^{j+1})z^{j+1}}{2\mathrm{Re}(A\bar{w}^{j})z^{j}} \right|^{-1}$

$$\frac{\mathrm{Re}(A\bar{w}^{j+1})}{\mathrm{Re}(A\bar{w}^{j})} = \frac{\mathrm{Re}A\,\mathrm{Re}(\bar{w}^{j+1}) - \mathrm{Im}A\,\mathrm{Im}(\bar{w}^{j+1})}{\mathrm{Re}A\,\mathrm{Re}(\bar{w}^{j}) - \mathrm{Im}A\,\mathrm{Im}(\bar{w}^{j})}$$

but $\mathrm{Re}(\bar{w}^{j+1}) = \mathrm{Re}\,\bar{w}^{j}\,\mathrm{Re}\,\bar{w} - \mathrm{Im}(\bar{w}^{j})\,\mathrm{Im}(\bar{w})$

$\mathrm{Im}(\bar{w}^{j+1}) = \mathrm{Re}(\bar{w}^{j})\,\mathrm{Im}\,\bar{w} + \mathrm{Re}\,\bar{w}\,\mathrm{Im}(\bar{w}^{j})$

$$\therefore \quad = \frac{\mathrm{Re}(\bar{w}^{j})\mathrm{Re}\bar{w}\,\mathrm{Re}A - \mathrm{Re}A\,\mathrm{Im}\,\bar{w}^{j}\,\mathrm{Im}\,\bar{w} - \mathrm{Im}A\,\mathrm{Re}\,\bar{w}^{j}\,\mathrm{Im}\,\bar{w} - \mathrm{Im}A\,\mathrm{Re}\,\bar{w}\,\mathrm{Im}\,\bar{w}^{j}}{\mathrm{Re}A\,\mathrm{Re}\,\bar{w}^{j} - \mathrm{Im}A\,\mathrm{Im}\,\bar{w}^{j}}$$

$$= \mathrm{Re}\,\bar{w} - \frac{\mathrm{Im}\,\bar{w}\,\mathrm{Im}(A\bar{w}^{j})}{\mathrm{Re}(A\bar{w}^{j})} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\mathrm{Im}(A\bar{w}^{j})}{\mathrm{Re}(A\bar{w}^{j})} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\tan(\phi - j\theta)$$

where $\bar{w} = re^{-i\theta}$ and $A = pe^{i\phi}$; now $\tan(\phi - j\theta) = \frac{\tan\phi - \tan j\theta}{1 + \tan\phi\tan j\theta}$ where $\theta = \frac{4\pi}{3}$.

$\therefore$ $\tan j\theta$ will never blow up and $\phi = \tan^{-1} 5/\sqrt{3}$, which doesn't complement $\tan j\theta$ and cause denom to blow up.

thus $\left| \frac{2\mathrm{Re}(A\bar{w}^{j+1})}{2\mathrm{Re}(A\bar{w}^{j})} z \right| < \left| -\frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{5/\sqrt{3} + \sqrt{3}}{1 - 5/\sqrt{3}\,\sqrt{3}} \right) \right| |z| < \frac{3|z|}{2} < 1$

The ratio has no limit, so ratio test is inapplicable.

or $|z| < 2/3 \Rightarrow$ radius of convergence is $3/2$. Here we've taken $j\theta = \frac{8\pi}{3}$ to minimize the $\tan(\phi - j\theta)$. If $j\theta = 0$ then $\tan(\phi - j\theta) = \frac{5\sqrt{3}}{3}$ and we get $|2z| < 1$ & $|z| < \frac{1}{2}$ and radius of convergence is 2. If $j\theta = \frac{4\pi}{3}$ $|z/3| < 1$ or $|z| < 3$ radius of convergence is $\frac{1}{3}$

thus for assured convergence take $\min(1/3, 2/3, 2) = 1/3$

Series is $1 + 2z - 3z^2 + z^3 + 2z^4 - 3z^5 + \dots$

if $j = 1, 4, \dots$ $\frac{a^{j+1}}{a^j} = -1/3$? so if $a^{(1)} > 0$ $a^2 < 0$

$j = 2, 5, \dots$ $\frac{a^{j+1}}{a^j} = -1.5$ if $a^2 < 0$ $a^{(3)} > 0$

$j = 3, 6, \dots$ $\frac{a^{j+1}}{a^j} = 2$ $a^{(3)} > 0$ $a^{(4)} > 0$

$+ \; - \; + \; + \; - \; + \; + \; - \; + \; +$

This gives the sign pattern. $+ \; + \; - \; + \; + \; -$

$f(z) = \sum_{j=1}^{\infty} \mathrm{Re}(e^{2\pi j/5}) z^j$ should give a pattern of 5

For $|z| > 1$ $f(z) = \frac{S}{z(1 - w/z)} + \frac{\bar{S}}{z(1 - \bar{w}/z)} = \frac{S + \bar{S}}{z} + \frac{2\mathrm{Re}(Sw)}{z^2} + \frac{2\mathrm{Re}(Sw^2)}{z^3} + \dots$ where $S = \frac{A}{-\bar{w}}$

$f(z) = \sum_{j=1}^{\infty} \frac{2\mathrm{Re}(Sw^{j-1})}{z^j}$ — again we would go through same type of argument as above

A −

Exercise Set 4

ME 201. APPLICATIONS OF COMPLEX VARIABLES

Due Monday 29 October 1979

4.1. Integral of a rational function of $\cos\theta$. Derive the result (Gradstein & Ryzhik 3.613.4)

$\cos n\theta = z^n$ with $|z|=1$

$$\int_0^{\pi} \frac{\cos n\theta \cos\theta \, d\theta}{1-2a\cos\theta+a^2} = \frac{\pi}{2}\,\frac{1+a^2}{1-a^2}\,a^{n-1}, \qquad |a| < 1.$$

Suggestion: The computation can be simplified by first regarding a as real, and writing part of the integrand as the real part of an exponential.

4.2. A non-rational function. Verify Gradstein & Ryzhik's 3.716.2:

$$\int_0^{\pi/2} \cos(a\tan\theta)\, d\theta = \frac{\pi}{2} e^{-a}.$$

4.3. An integral not in the tables. Evaluate the following definite integral (which I can't find in any tables) for both $|a| < 1$ and $|a| > 1$. Discuss the situation for $a = \pm 1$ and $|a| = 1$.

$$I = \int_0^{2\pi} \cos\theta \,\ln(1-2a\cos\theta+a^2)\, d\theta$$

$I = I(\psi) = \int_0^{2\pi} \cos\theta \ln\left(1 - a\left[e^{i(\theta+\psi)} + e^{-i(\theta-\psi)}\right] + e^{2i\psi}\right) d\theta$

$\frac{dI}{d\psi}$

$1 - 2a\cos\theta$

$1 - ze^{i\theta}\left[e^{i\psi} + e^{-i\psi}\right] + e^{2i\psi}$

$1 - e^{i(\theta+\psi)} - e^{i(\theta-\psi)} + e^{2i\psi}$

Wherever convenient in the above, you may use the results derived in class:

$$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = \begin{cases} \dfrac{2\pi}{1-a^2}, & |a| < 1 \\[2mm] \dfrac{2\pi}{a^2-1}, & |a| > 1 \end{cases}$$

$$\int_0^{2\pi} \frac{\cos\theta \, d\theta}{1-2a\cos\theta+a^2} = \begin{cases} \dfrac{2\pi a}{1-a^2}, & |a| < 1 \\[2mm] \dfrac{2\pi}{a(a^2-1)}, & |a| > 1 \end{cases}$$

$$\int_0^{2\pi} \frac{\cos^2\theta \, d\theta}{1-2a\cos\theta+a^2} = \begin{cases} \pi\dfrac{1+a^2}{1-a^2}, & |a| < 1 \\[2mm] \dfrac{\pi}{a^2}\,\dfrac{a^2+1}{a^2-1}, & |a| > 1 \end{cases}$$

Solutions to Exercise Set 4

ME 201. APPLICATIONS OF COMPLEX VARIABLES

Friday 2 November 1979

4.1. Integral of a rational function of $\cos\theta$. By symmetry, the integral is half the integral from 0 to 2π. Hence for real a it can be written

$$I = \frac{1}{2}\int_0^{2\pi} \frac{e^{in\theta}\cos\theta\, d\theta}{1-2a\cos\theta+a^2}.$$

Then setting $z = e^{i\theta}$ gives a contour integral around the unit circle:

$$I = \frac{1}{4i}\int_C \frac{(1+z^2)z^{n-1}}{(z-a)(1-az)}dz.$$

This has simple poles at $z = a$ and $z = 1/a$. For $|a| < 1$ the first of these lies inside the contour. Its residue is

$$\frac{(1+a^2)a^{n-1}}{1-a^2}.$$

Hence by Cauchy's residue theorem $I = \frac{\pi}{2}\frac{1+a^2}{1-a^2}a^{n-1}$. By appeal to the principle of analytic continuation we see that this holds for complex a with $|a| < 1$.

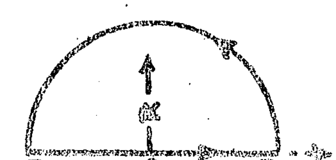
Alternatively, one can use $\cos n\theta\cos\theta = \frac{1}{2}\cos(n+1)\theta + \cos(n-1)\theta$.

4.2. A non-rational function. Proceeding just as in the previous example gives

$$I = \frac{1}{4}\mathrm{Re}\int_0^{2\pi} e^{ia\tan\theta}d\theta = \frac{1}{4i}\mathrm{Re}\int_C \frac{1}{z}e^{a(z^2-1)/(z^2+1)}dz.$$

However, in addition to the simple pole at the origin, there are essential singularities on the contour of integration at $z = \pm i$. Rather than compute their residues, it is simpler to make the change of variable $\tan\theta = x$. Then for real a

$$I = \frac{1}{2}\mathrm{Re}\int_{\infty}^{\infty}\frac{e^{iax}}{1+x^2}dx.$$



If we integrate $e^{iaz}/(1+z^2)$ around the indicated contour, the contribution of the large semicircle vanishes in the limit $R \to \infty$ provided that a is positive. The pole at $z = i$ has residue $e^{-a}/2i$; so we find

$$I = \frac{\pi}{2}e^{-a}, \qquad a > 0.$$

However, if a is negative, the contour must be completed with a semicircle in the lower half plane in order that its contribution vanish, and the result is instead

$$I = \frac{\pi}{2}e^{a}, \qquad a < 0.$$

That the result quoted in the exercise cannot be correct for both positive and negative a is evident from the fact that the original integral is an even function of a, whereas e^{-a} is not.

4.3. <u>An integral not in the tables.</u> Integration by parts gives

$$I = \Big[\sin\theta \;\; \ln(1-2a\cos\theta + a^2)\Big]_0^{2\pi} - \int_0^{2\pi} \frac{2a \sin^2\theta \, d\theta}{1 - 2a\cos\theta + a^2}$$

The term in brackets vanishes, and setting $\sin^2\theta = 1 - \cos\theta^2$ and using the results derived previously gives

$$I = \begin{cases} -2\pi a, & |a| < 1 \\ -2\pi/a, & |a| > 1 \end{cases}$$

Alternatively, differentiating with respect to a gives

$$\frac{\partial I}{\partial a} = 2\int_0^{2\pi} \frac{(a - \cos\theta)}{1 - 2a\cos\theta + a^2} \, d\theta = \begin{cases} -2\pi, & |a| < 1 \\ 2\pi/a^2, & |a| > 1 \end{cases}$$

and then integrating gives

$$I = \begin{cases} -2\pi a + C_+ , & |a| < 1 \\ -2\pi/a + C_- , & |a| > 1 \end{cases}$$

When $a = 0$ the logarithm and hence the integrand and the integral are zero, so $C_+ = 0$. The second constant C_- can likewise be shown to be zero, for example by expanding the integrand term by term for large a.

For $a = \pm 1$ the logarithm is infinite at $\theta = \pm\pi/2$, but the cosine is zero, and l'Hospital's rule shows that the product is zero, so the integral converges. Thus the integral is continuous on the real axis of a, but with discontinuous derivative. However, this cancellation does not occur for other values of a on the unit circle, so the integral diverges. It does, however, have a Cauchy principal value that is found to be just the average of the values approached from the inside and the outside.

Cesar Levy
MA 106
Prof. Van Dyke

Problem Set #4

1. $\int_0^{\pi} \frac{\cos n\theta \cos\theta \, d\theta}{1-2a\cos\theta+a^2} = \frac{\pi}{2}\frac{1+a^2}{1-a^2} a^{n-1} \qquad |a|<1$ Prove it

Let $z = e^{i\theta}$ then $\cos\theta = \frac{1}{2}(z+1/z)$; $\cos n\theta = Re\,(z^n)$ $\quad 1-2a\cos\theta + a^2 = (1-az)(z-a)/z$

$d\theta = dz/iz$. we note also that $\int_0^{\pi} = \frac{1}{2}\int_0^{2\pi}$ ✓

$\therefore \int_0^{\pi} \frac{\cos n\theta \cos\theta \, d\theta}{1-2a\cos\theta+a^2} = Re \frac{2\pi i}{4i}\oint \frac{(z^n)(z^2+1)\,dz}{z(1-az)(z-a)}$. We note that at $z=0$ we have a removable singularity since a is real, Since $|a|<1$ there is only one residue within unit circle at $z=a$ $\therefore$ $Res\Big|_{z=a} = Re\frac{2\pi i}{4i} a^{n-1}\frac{a^2+1}{1-a^2}$ and $\int_0^{\pi}\frac{\cos n\theta\cos\theta\, d\theta}{1-2a\cos\theta+a^2} = \frac{\pi}{2}\frac{a^2+1}{1-a^2} a^{n-1}$ ✓ $|a|<1$ n≥

3. $I = \int_0^{2\pi} \cos\theta \cdot f(a,\theta)\, d\theta$ where $f(a,\theta) = \ln(1-2a\cos\theta+a^2)\,d\theta$

Let $I = I(a)$ $\therefore$ $dI/da = \int_0^{2\pi} \frac{2\cos\theta\,[a-\cos\theta]}{1-2a\cos\theta+a^2}\,d\theta = 2a\int_0^{2\pi}\frac{\cos\theta\, d\theta}{1-2a\cos\theta+a^2} - 2\int_0^{2\pi}\frac{\cos^2\theta\, d\theta}{1-2a\cos\theta+a^2}$

if $|a|<1 \Rightarrow \frac{dI}{da} = 2a\cdot\frac{2\pi a}{1-a^2} - 2\pi\frac{(1+a^2)}{1-a^2} = \frac{2\pi(a^2-1)}{1-a^2} = -2\pi$

thus $I = -2\pi a + \text{const.}$ if $a=0$ $I(0)=0 \Rightarrow \text{const}=0$ ✓ $\therefore I = -2\pi a$ ✓

if $|a|>1 \Rightarrow \frac{dI}{da} = 2a\cdot\frac{2\pi}{a(a^2-1)} - 2\frac{\pi}{a^2}\frac{a^2+1}{a^2-1} = \frac{2\pi(a^2-1)}{a^2(a^2-1)} = \frac{2\pi}{a^2}$ ✓

thus $I = -\frac{2\pi}{a} + \text{const}$ we now show that if $b = 1/a$ ($\Rightarrow |b|<1$) that $I(b=0)=0$

$\Rightarrow$ const $=0$.

$$I = \int_0^{2\pi}\cos\theta\ln\left(1-\frac{2}{b}\cos\theta+\frac{1}{b^2}\right)d\theta = \underbrace{\int_0^{2\pi}\cos\theta\ln b\, d\theta}_{=0 \text{ since } b=\text{const}} + \underbrace{\int_0^{2\pi}\cos\theta\ln(b^2-2b\cos\theta+1)\,d\theta}_{=2\pi b \text{ since } |b|<1}$$

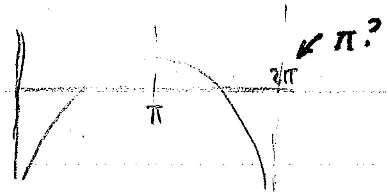
but $I = -\frac{2\pi}{a} + \text{const} = -2\pi b + \text{const} \equiv -2\pi b \Rightarrow \text{const}=0$.

Thus $I = \begin{cases} -2\pi a & |a|<1 \\ -2\pi/a & |a|>1 \end{cases}$ ✓

if $a=+1$ $I = \underbrace{\int_0^{2\pi}\cos\theta\ln 2\, d\theta}_{=0} + \int_0^{2\pi}\cos\theta\ln(1-\cos\theta)\,d\theta$

But $\cos\theta$ times this vanishes where the log is ∞.

if we plot $f(\theta) = \ln(1-\cos\theta)$ vs. θ



we note that $\int_0^{2\pi} f(\theta)\,d\theta$ does not exist but that $\lim_{\epsilon\to 0}\int_\epsilon^{2\pi-\epsilon} f(\theta)\,d\theta$ does

so we can use the principal value to get an answer

$$PV\int_0^{2\pi}\cos\theta\ln(1-\cos\theta)\,d\theta = \lim_{\epsilon\to 0}\sin\theta\ln(1-\cos\theta)\Big|_\epsilon^{2\pi-\epsilon} - \lim_{\epsilon\to 0}\int_\epsilon^{2\pi-\epsilon}\frac{\sin^2\theta}{1-\cos\theta}\,d\theta$$

$$= -\sin\epsilon\ln(1-\cos\epsilon) - \sin\epsilon\ln(1-\cos\epsilon) - \lim_{\epsilon\to 0}\int_\epsilon^{2\pi-\epsilon}\frac{(1+\cos\theta)\cancel{(1-\cos\theta)}}{\cancel{1-\cos\theta}}\,d\theta$$

$$-2\sin\epsilon\ln(1-\cos\epsilon) \qquad - \lim_{\epsilon\to 0}\Big[\theta+\sin\theta\Big]_\epsilon^{2\pi-\epsilon}$$

$$-2\sin\epsilon\ln(2\cdot\sin^2\tfrac{\epsilon}{2}) \qquad -2\pi$$

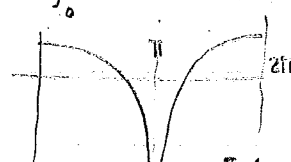
since $\epsilon\to 0$ $\sin\epsilon\sim\epsilon$ and $-2\sin\epsilon\cdot\ln 2 - 2\sin\epsilon\cdot 2\ln\sin\epsilon/2 = -2\sin\epsilon\ln(2\sin^2\epsilon/2)$ (*)

the first term of (*) $\to 0$ as $\epsilon\to 0$ $\sin\epsilon\ln\sin\epsilon/2 \sim \epsilon\ln\epsilon/2$ as $\epsilon\to 0$ but $\epsilon\ln\epsilon/2\to 0$

$\therefore$ $PV\int_0^{2\pi}\cos\theta\ln(1-\cos\theta)\,d\theta = -2\pi$ which is what one obtains if we take the limit as $a\to 1$ of the result $I = \begin{cases} -2\pi a & |a|<1 \\ -2\pi/a & |a|>1\end{cases}$

for $a=-1$ $\quad I = \int_0^{2\pi}\cos\theta\underbrace{\ln 2}_{=0}\,d\theta + \int_0^{2\pi}\cos\theta\ln(1+\cos\theta)\,d\theta$

if we plot $f(\theta) = \ln(1+\cos\theta)$ vs. θ



; again the same arguments can be used & we can define I

$$I = PV\int_0^{2\pi}\cos\theta\ln(1+\cos\theta)\,d\theta = \lim_{\epsilon\to 0}\sin\theta\ln(1+\cos\theta)\Big|_0^{\pi-\epsilon} + \Big|_{\pi+\epsilon}^{2\pi} + \lim_{\epsilon\to 0}\left[\int_0^{\pi-\epsilon} + \int_{\pi+\epsilon}^{2\pi}\frac{\sin^2\theta}{1+\cos\theta}\,d\theta\right]$$

$$= \cos\epsilon\ln\epsilon - \overset{=0}{\cos\epsilon\ln\epsilon} + \lim_{\epsilon\to 0}\left[\int_0^{\pi-\epsilon} + \int_{\pi+\epsilon}^{2\pi}(1-\cos\theta)\,d\theta\right] = \lim_{\epsilon\to 0}\Big\{\theta-\sin\theta\Big\}_0^{\pi+\epsilon} + \Big\{\theta-\sin\theta\Big\}_{\pi+\epsilon}^{2\pi} = 2\pi$$

which is exactly what we obtain if we take the limit when $a\to -1$ or

$$I = \begin{cases} -2\pi a & |a|<1 \\ -2\pi/a & |a|>1\end{cases}$$

for $|a|=1$ then let $a = e^{i\psi}$ $\therefore$ $I(\psi) = \int_0^{2\pi}\cos\theta\ln\left(1 - e^{i(\psi+\theta)} - e^{i(\psi-\theta)} + e^{2i\psi}\right)d\theta$

if one looks at the ( ) $= 2\cos\psi(\cos\psi-\cos\theta) + i\,2\sin\psi(\cos\psi-\cos\theta)$

then $\ln(\quad) = \ln(0)$ when $\theta = \arg(a)$. If we rewrite the bracket as $r e^{i\varphi}$

$r = 2(\cos\psi-\cos\theta)$ $\quad\varphi = \psi$ $\quad$ then $\ln(\quad) = \ln 2 + \ln(\cos\psi-\cos\theta) + i\psi$

and $\int_0^{2\pi}\cos\theta\ln(\quad)\,d\theta = \int_0^{2\pi}\ln(\cos\psi-\cos\theta)\,d\theta$

now take $\dfrac{dI}{d\psi} = \displaystyle\int_0^{2\pi} \frac{-\sin\psi\cos\theta}{\cos\psi-\cos\theta}\,d\theta = -\sin\psi\int_0^{2\pi}\frac{\cos\theta\,d\theta}{\cos\psi-\cos\theta} = -\sin\psi\left[\int_0^{2\pi} -d\theta + \frac{\cos\psi\,d\theta}{\cos\psi-\cos\theta}\right]$

$$= -\sin\psi\,\{-2\pi\} - \sin\psi\cos\psi\left[\int_0^{\psi-\epsilon} + \int_{\psi+\epsilon}^{2\pi}\frac{d\theta}{\cos\psi-\cos\theta}\right]$$

using the principal value theorem.

Using Dwight $\displaystyle\int\frac{dx}{a+b\cos x}$ $\left(\text{if } a^2<b^2\right) = \dfrac{1}{\sqrt{b^2-a^2}}\log\left|\dfrac{(b-a)\tan x/2+\sqrt{b^2-a^2}}{(b-a)\tan x/2-\sqrt{b^2-a^2}}\right|$ $\quad a=\cos\psi \quad b=1$

thus $\displaystyle\int\frac{d\theta}{\cos\psi-\cos\theta} = \frac{1}{\sin\psi}\log\left|\frac{(1-\cos\psi)\tan\theta/2+\sin\psi}{(1-\cos\psi)\tan\theta/2-\sin\psi}\right|$

Now invoking P.V. $\displaystyle\int_0^{2\pi}\frac{d\theta}{\cos\psi-\cos\theta} = 0$ hence $\dfrac{dI}{d\psi} = 2\pi\sin\psi$ and $I = -2\pi\cos\psi + \text{const.}$

Using the result that at $a=1$ $(\psi=0)$ $I=-2\pi \Rightarrow \text{const}=0$

hence

$$\text{PV}\int_0^{2\pi}\cos\theta\,\ln(1-2a\cos\theta+a^2)\,d\theta = -2\pi\cos\psi \quad \text{where } a=e^{i\psi}$$

4.2

$I = \displaystyle\int_0^{\pi/2}\cos(a\tan\theta)\,d\theta = \pi/2\; e^{-a}$ prove it.

let $I = \text{Re}\displaystyle\int_0^{\pi/2} e^{i(a\tan\theta)}\,d\theta = I(a)$ & take $\dfrac{d^2}{da^2}$

$$\frac{d^2I}{da^2} = \text{Re}\left(-\int_0^{\pi/2}\tan^2\theta\, e^{ia\tan\theta}d\theta\right) = \text{Re}\Big[-\int_0^{\pi/2}\sec^2\theta\, e^{ia\tan\theta}d\theta + \underbrace{\int_0^{\pi/2} e^{ia\tan\theta}d\theta}_{I}\Big]$$

or

$$\frac{d^2I}{da^2} - I = \text{Re}\left(-\frac{1}{ia}\int_0^{\pi/2} d(ia\tan\theta)\, e^{ia\tan\theta}\right) = \text{Re}\left(-\frac{1}{ia}\, e^{ia\tan\theta}\right)\Big|_0^{\pi/2} = \text{Re}\,\frac{i}{a}\left[e^{ia\infty}-1\right]$$

Now I get stuck here trying to justify that the real part $=0$, but I will assume it.

The solution to $I''-I=0$ is $I(a) = c_1e^{-a}+c_2e^{a}$, However as $a\to\infty$

$|\cos(a\tan\theta)|<1 \Rightarrow I$ must be bounded as $a\to\infty$ or $c_2=0$. When $a=0$

$\cos(a\tan\theta)=1$ and $\displaystyle\int_0^{\pi/2}1\,d\theta = \pi/2 \Rightarrow c_1=\pi/2$

THUS $\displaystyle\int_0^{\pi/2}\cos(a\tan\theta)\,d\theta = \pi/2\; e^{-a}$

This is evidently correct only for $a>0$. The original integral is an even function of a, whereas e^{-a} is not.

Question	Score
1	
2	
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7	
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Total	

STANFORD UNIVERSITY

OFFICIAL EXAMINATION BOOK

24 Page Ruled

Name of student CESAR LEVY

Date of examination Nov 5, 79

Course ME201 Complex Var.

THE STANFORD UNIVERSITY HONOR CODE

A. The Honor Code is an undertaking of the students, individually and collectively:

(1) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

(2) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

I acknowledge and accept the Honor Code.

(Signed) Cesar Levy

Interpretations and applications of the Honor Code appear on the back cover of this examination book.

$$\frac{e^{1/z}}{1-z} = \left(1+z+z^2+\dots\right)\left(1+\frac{1}{z}+\frac{1}{2}\frac{1}{z^2}+\dots\right) =$$

$$= \dots \quad + \frac{1}{z}\left(1+\frac{1}{2}+\frac{1}{6}+\dots\right)$$

Revision of First Midterm Problem

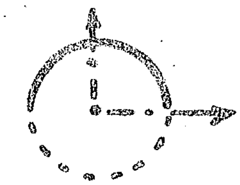
ME 201. APPLICATIONS OF COMPLEX VARIABLES

14 November 1979

<u>Original version:</u>

A single-valued function is defined on the upper half of the unit circle in the complex plane by

$$\frac{i\theta}{2\cos\theta}, \qquad 0 < \theta < \pi$$

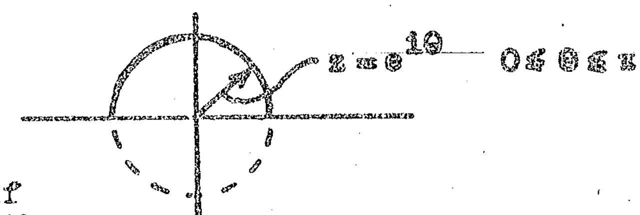


Find its analytic continuation into as much as possible of the complex plane, including perhaps onto Riemann sheets. Then describe its analytic structure by characterizing in detail each of its singular points.

<u>Revised version:</u>

The complex function

$$f(r,\theta) = \frac{i\theta}{2\cos\theta}$$



is single-valued on the upper half plane $0 < \theta < \pi$. It represents the upper semi-circle of a unit circle in the z-plane. Obtain the function $f(z)$ which represents the analytic continuation of this function over as much of the complex z-plane as possible. Do not attempt to derive this continuation via the Taylor series approximation. The multivaluedness might perhaps warrant the inclusion of several Riemann sheets. Also, classify the singular points for the function $f(z)$ that you obtain.

Solutions to Midterm Examination

ME 201. APPLICATIONS OF COMPLEX VARIABLES

Wednesday 7 November 1979

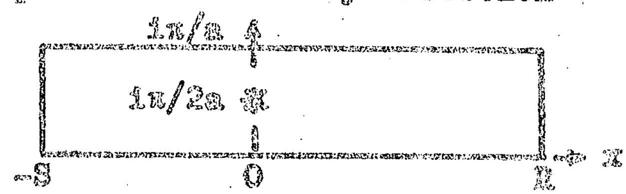
1. On the unit circle $z = e^{i\theta}$, so $i\theta = \ln z$ and $2\cos\theta = z + 1/z$. Making these substitutions gives the analytic continuation into the entire complex plane, and on a denumerably infinite number of Riemann sheets:
$$f(z) = \frac{z \ln z}{1+z^2}.$$
This has simple poles at $z = \pm i$, with residues $i(\pm\frac{\pi}{4} + n\pi)$ on the nth sheet of the Riemann surface, and logarithmic branch points at $z = 0$ and $z = \infty$.

2. Expanding about $z = 0$ gives
$$\frac{e^{1/z}}{1-z} = (1 + z + z^2 + \ldots + z^n + \ldots)(1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \ldots + \frac{1}{n!}\frac{1}{z^n} + \ldots)$$
$$= \ldots + (1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots)\frac{1}{z} + \ldots$$
Hence we see that the residue at the essential singularity at the origin is $e-1$, and the integral about $|z| = 1/2$ is $2\pi i(e-1)$ by the residue theorem. The residue at the simple pole at $z = 1$ is easily seen to be $-e$, so the integral about $|z| = 2$ is $-2\pi i$.

* A shortcut for finding the residue at the origin is to compute the residue at infinity according to
$$\lim_{z\to\infty} \left[-zf(z)\right]$$
which is easily seen to be 1, and then use the fact that

  if $f(z)$ is analytic except at isolated singular points, the sum of all the residues including that at infinity is zero.

3. We extend the integration from $-\infty$ to ∞ by symmetry. Since $\cosh(aiy) = \cos(ay)$, the integrand is periodic in the y-direction with period $2\pi/a$; so we complete the contour along $y = \pi/a$. The integrals over the ends of the rectangle vanish exponentially as R and $S \to \infty$ separately. For $a > 0$ the only singularity within the contour is a simple pole at $z = i\pi/2a$, with residue



$$\lim_{z\to i\pi/2a} \frac{z - i\pi/2a}{\cosh az} = \lim_{z\to i\pi/2a} \frac{1}{a \sinh az} \quad \text{by l'Hospital's rule}$$
$$= 1/ia.$$
Hence in the limit
$$4\int_0^\infty \frac{dx}{\cosh ax} = 2\pi i(1/ia) = \frac{2\pi}{a}, \quad a > 0.$$
For $a < 0$, the only singularity within the contour lies at $z = -i\pi/2a$, and the residue — and hence the integral — reverses sign.

Exa.

1. $f(e^{i\theta}) = \dfrac{i\theta}{2\cos\theta}$ $\quad 0<\theta<\pi \quad$ analytically continue

$e^{i\theta} = z \qquad i\theta = \ln z \qquad \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) \qquad \therefore \quad f(z) = \dfrac{z\ln z}{z^2+1}$

has poles at $z = \pm i$; is analytic at $z=\infty$; since $f(z) \to \dfrac{\ln z}{z} \to 0$ as $z\to\infty$

at $z=0$ you have a removable singularity since $z\ln z = z\ln r + zi\theta$

as $z\to 0$ in whatever direction $(r\ln r)e^{i\theta} + zi\theta = (r\ln r\cos\theta - r\theta\sin\theta) + i\, r\ln r\sin\theta + r\theta\cos$

$\to 0$ as $|z|\to 0$ $\quad r\ln r \to 0,\ r\to 0. \qquad \therefore$ no problems there

2. what is $\displaystyle\oint_c \frac{e^{1/z}}{1-z}\,dz$ when $|z| = 1/2 \qquad |z| = 2$

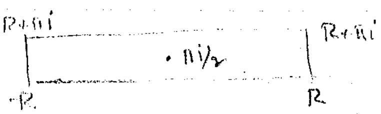
$e^{1/z}$ has pole about $z=0$ (essential sing) $\quad \dfrac{e^{1/z}}{1-z} = \left\{1+\frac{1}{z}+\frac{1}{z^2 2!}+\cdots\right\}\left\{1+z+z^2+\cdots\right\}$

look at $1/z$ coeff $= 1+\frac{1}{2!}+\frac{1}{3!}+\cdots = e-1 \quad \Rightarrow \displaystyle\oint \frac{e^{1/z}}{1-z}\,dz = 2\pi i \,\mathrm{Res}\Big|_{z=0} = 2\pi i\,(e-1)$

$\therefore \oint_c$ [illegible]

when $|z| = 2 \qquad \displaystyle\oint \frac{e^{1/z}\,dz}{1-z} = 2\pi i\left(\mathrm{Res}\Big|_{z=0} + \mathrm{Res}\Big|_{z=1}\right) = 2\pi i\,(e-1) + 2\pi i\,\mathrm{Res}\Big|_{z=1}$

$2\pi i\,\mathrm{Res}\Big|_{z=1} = -\displaystyle\oint \frac{e^{1/z}}{z-1}\,dz = -e\,2\pi i \qquad \therefore \oint dz = 2\pi i\,(e-1-e) = -2\pi i$

3. $\displaystyle\int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2a}$ $\quad$ prove it $\quad$ let $\displaystyle\oint \frac{d\zeta}{\cosh\zeta}$  $\quad \zeta = az$

$$\oint_c \frac{d\zeta}{\cosh\zeta} = \int_{-R}^{R}\frac{d\xi}{\cosh\xi} + \int_0^\pi \frac{i\,d\eta}{\cosh(R+i\eta)} + \int_R^{-R}\frac{d\xi}{\cosh(\xi+\pi i)} + \int_\pi^0 \frac{i\,d\eta}{\cosh(-R+i\eta)} = 2\pi i\,\mathrm{Res}\Big|_{\zeta=\pi i/2}$$

$|\cosh(R+i\eta)| \geq \frac{1}{4}e^{R} \quad \therefore \quad 2^{nd}$ & 4^{th} integrals $\to 0$

$$\int_{-\infty}^{\infty}\frac{d\xi}{\cosh\xi} + \int_{\infty}^{-\infty}\frac{d\xi}{\cosh(\xi+\pi i)} = \frac{2\pi i}{\sinh i\pi/2} = \frac{2\pi}{\sin\pi/2} = 2\pi$$

$$\Downarrow$$

$$\int_{-\infty}^{\infty}\frac{d\xi}{\cosh\xi} - \int_{\infty}^{-\infty}\frac{d\xi}{\cosh\xi} = 4\int_0^\infty \frac{d\xi}{\cosh\xi} = 2\pi \qquad \text{but } \frac{d\xi}{\cosh\xi} = \frac{a\,dx}{\cosh ax}$$

$\int dx = \pi/$

if $a < 0$ then Real $az < 0$ for $\operatorname{Re} z > 0$

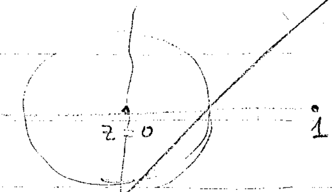
remember $\cosh(x) = \cosh(-x)$ $\therefore$ fn. will still be ok.

and $\displaystyle\int_0^\infty \frac{dx}{\cosh ax} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{\cosh ax}$ if $a = -b$, $b > 0$ then $\displaystyle\int_{-\infty}^\infty \frac{dx}{\cosh(-bx)} = \int \frac{dx}{\cosh bx}$ $\underline{\underline{b > 0}}$

but $\displaystyle\int_0^\infty \frac{dx}{\cosh(bx)} = \frac{\pi}{2b} = \frac{\pi}{-2a}$; thus if we use $|a|$,

$$\int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2|a|}$$

2. $\frac{e^{1/z}}{1-z}$ ~~$= f(z)$ has an essential sing at $z=0$, $z=1$ a simple pole~~



$e^{1/z} = 1 + 1/z + 1/z^2 2! + \cdots \quad \frac{1}{z^n n!} + \cdots$

$\therefore$ at $z = 1/2$ $\frac{1}{1-z}$ is analytic

$$\therefore \oint f(z)\,dz = \oint \frac{1}{1-z}\left[1 + \frac{1}{z} + \frac{1}{z^2 2!} + \cdots\right]dz = 2\pi i \operatorname{Res}\Big|_{z=0} = 2\pi i \cdot a_{-1}$$

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots$$

$$\frac{e^{1/z}}{1-z} = \begin{array}{l} 1 + 1/z + \cdots \\ z + 1 + 1/2z + \cdots \\ z^2 + z + 1/2 + 1/3!z + \cdots \end{array}$$

$$a_{-1} = \sum_{n=1}^{\infty} \frac{1}{n!} = e - 1$$

$\therefore$ the only nonzero integral will be the $\frac{1}{z}$ integral

Thus $\oint_{|z|=1/2} \frac{e^{1/z}}{1-z}\,dz = 2\pi i\,(e-1)$

$$\frac{e^{1/z}}{1-z} \quad \text{for } |z| = 2 \qquad -z\left[1 - 1/z\right] = 1 - z$$

$$\therefore \frac{e^{1/z}}{1-z} = -\frac{1}{z}\left[\frac{e^{1/z}}{1-1/z}\right] = -1/z\left[1 + 1/z + \frac{1}{z^2 2!} + \frac{1}{3! z^3} \cdots\right]\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right]$$

$$-\frac{1}{z}\left[\begin{array}{l} 1 + 1/z + 1/z^2 + 1/z^3 + \cdots \\ 1/z + 1/z^2 \cdots \\ 1/z^2 + \end{array}\right.$$

for $|z| = 1/2$: $\quad 1 - z$ = analytic $\therefore$ only contrib is from $\frac{1}{z}$ portion of $e^{1/z}$

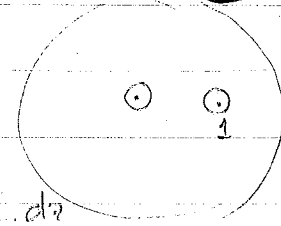
No $\leftarrow$

$$\Rightarrow \int_C f(z)\,dz = \oint_{|z|=1/2} \frac{1}{z} \cdot \frac{1}{1-z}\,dz = \frac{2\pi i}{1/2} = 4\pi i \qquad \frac{(e-1)}{2}$$

?

for $|z| = 2$:

$$\oint_C f(z)\,dz = 0$$



$$\oint_C f(z)\,dz = \int_{|z|=\varepsilon} \frac{e^{1/z}}{1-z}\,dz + \int_{|z-1|=\varepsilon} \frac{e^{1/z}}{1-z}\,dz$$

Residue at $z = 1$ is <u>minus</u> e

Why not same as for $|z| = 1/2$

$$= \frac{2\pi i}{-1} + 2\pi i\, e^{1} = \cancel{2\pi i} + 2\pi i\,(e - 1)$$

?

6

Use C-R eqns.

1. Let $\frac{\theta}{2\cos\theta} = v \quad 0 < \theta \le \pi$ since $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ (2)

Let $f(z) = u + iv$

$$\frac{\partial v}{\partial \theta} = \frac{2\cos\theta \cdot 1 - \theta(-2\sin\theta)}{4\cos^2\theta} = \frac{2(\cos\theta + \theta\sin\theta)}{4\cos^2\theta} = r\frac{\partial u}{\partial r}$$

$$\frac{1}{r}\frac{2(\cos\theta + \theta\sin\theta)}{4\cos^2\theta} = \frac{\partial u}{\partial r} \Rightarrow u = \ln r\left[\frac{\cos\theta + \theta\sin\theta}{2\cos^2\theta}\right] + f(\theta)$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\ln r}{r}\left[\frac{2\cos^2\theta\{-\sin\theta + \theta\cos\theta + \sin\theta\} + 4\cos\theta\sin\theta(\cos\theta + \theta\sin\theta)}{4\cos^4\theta}\right] - \frac{f'(\theta)}{r}$$

use CR eqns to get u & v then form $f(z) = u + iv$
this is the analytic continuation

$$\frac{\theta}{2\cos\theta} = v \qquad \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{2\cos\theta \cdot 1 + 2\theta\sin\theta}{4\cos^2\theta} = \frac{1}{r}\frac{\cos\theta + \theta\sin\theta}{2\cos^2\theta} = \frac{\partial u}{\partial r}$$

$$u = \ln r\left[\frac{\cos\theta + \theta\sin\theta}{2\cos^2\theta}\right] + f(\theta)$$

$4\cos^2\theta\sin\theta + 2\cos\theta \cdot \theta\sin^2\theta$

$$\frac{\partial u}{\partial \theta} = \ln r\left[\frac{2\cos^2\theta\{\theta\cos\theta\} + (\cos\theta + \theta\sin\theta)4\cos\theta\sin\theta}{4\cos^4\theta}\right] + f'(\theta)$$

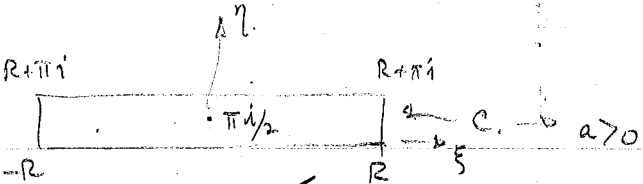
$$= \frac{\ln r}{r}\left[\frac{2\theta\cos\theta + 2\theta\cos\theta\sin^2\theta + 4\cos^2\theta\sin\theta}{4\cos^4\theta}\right] + \frac{f'(\theta)}{r}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \theta = \cos^{-1}\left(\frac{1}{2}z + \frac{1}{2z}\right)$$

$$\therefore f(z) = \frac{i\cos^{-1}\left(\frac{1}{2}z + \frac{1}{2z}\right)}{\frac{1}{2}\left(z + \frac{1}{z}\right)}$$

$$\frac{2iz\cos^{-1}\left(\frac{1}{2}z + \frac{1}{2z}\right)}{z^2 + 1} \qquad z \pm i = \text{poles}$$

3. Look at $\oint_C \frac{d\xi}{\cosh \xi}$ over



$\zeta = az$ ✓

poles of $\cosh \xi$ when $\xi = (n+\tfrac{1}{2})\pi i = az$

$$\cosh az = \tfrac{1}{2}\left(e^{a(x+iy)} + e^{-a(x+iy)}\right)$$

$$\int_{-R}^{R} \frac{d\xi}{\cosh \xi} + \int_0^{\pi} \frac{i\,d\eta}{\cosh(R+i\eta)} + \int_R^{-R} \frac{d\xi}{\cosh(\xi+\pi i)} + \int_{\pi}^{0} \frac{i\,d\eta}{\cosh(-R+i\eta)}$$

$$= 2\pi i \operatorname{Res}\Big|_{\xi=\pi i/2} = 2\pi i \lim_{\xi\to\pi i/2} \left(\xi - \frac{\pi i}{2}\right)\cdot\frac{1}{\cosh \xi} = \frac{2\pi i}{\sinh \frac{\pi i}{2}} =$$

~~$\cosh(ax+a\pi i) = \cosh ax \cos a$~~

$$\int_0^{\pi} \frac{i\,d\eta}{\cosh(R+i\eta)} \qquad \cosh(R+i\eta) = \cosh R \cos\eta + i \sinh R \sin\eta$$

$$|\cosh(R+i\eta)| = \left|\frac{e^{(\ )}+e^{-(\ )}}{2}\right| \geq \tfrac{1}{2}\{|\ \ | - |\ \ |\} = \tfrac{1}{2}(e^{R}-e^{-R})$$

$$\geq \tfrac{1}{4}e^{R}$$

$\therefore$ 2nd & 4th integrals go to 0 as $R\to\infty$ ✓

$$\therefore \int_{-\infty}^{\infty} \frac{d\xi}{\cosh \xi} + \int_{\infty}^{-\infty} \frac{d\xi}{\cosh(\xi+\pi i)} = \frac{2\pi i}{i \sin \pi/2} = 2\pi \ ✓$$

$$2\int_0^{\infty} \frac{d\xi}{\cosh \xi} + 2\int_0^{\infty} \frac{d\xi}{\cosh \xi} = 4\int_0^{\infty} \frac{d\xi}{\cosh \xi} = 2\pi \quad \text{or}$$

$$\int_0^{\infty} \frac{d\xi}{\cosh \xi} = \frac{\pi}{2} \ ✓$$

but $\displaystyle\int_0^{\infty} \frac{a\,dx}{\cosh ax} = \int_0^{\infty} \frac{d\xi}{\cosh \xi}$

$$\int_0^{\infty} \frac{a\,dx}{\cosh ax} = \int_0^{\infty} \frac{d\xi}{\cosh \xi} \quad \therefore \quad \boxed{\int_0^{\infty} \frac{dx}{\cosh ax} = \frac{\pi}{2a}}$$

For $a>0$! Sign reversed for $a<0$
(since $\cosh ax$ even fn. of a)

Exercise Set 5

ME 201. APPLICATIONS OF COMPLEX VARIABLES

Due Monday 5 November 1979

5.1. Checking Dwight's tables. One cannot trust tables implicitly; they occasionally contain mistakes. Suppose that you were using Dwight's 856.8

$$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}$$

$\frac{1}{2i(b^2-a^2)}\left[\frac{b}{a}-\frac{a}{b}\right]$

in an important analysis. Check this result by contour integration. Can a and b be complex? Does the result hold if a has a negative real part?

5.2. Alternative treatment of integral with branch point. The integral

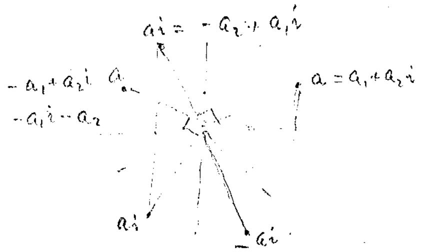
$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin(\pi a)}, \qquad 0 < a < 1$$

was treated in class by replacing x by $z = x+iy$. Show that it can alternatively be treated by setting $x = e^{\zeta}$ and considering an integral in the plane of the variable $\zeta = \xi + i\eta$ taken about the rectangle with vertices at R, $R+2\pi i$, $-S+2\pi i$, $-S$, where R and S are real and positive.

5.3. Checking Schaum's Outline. Just as in exercise 5.1 above, one should not use the two definite integrals involving logarithms that are given on pages 197 and 198 of Spiegel's Schaum's Outline,

$$\int_0^\infty \frac{\ln x}{x^2+a^2}\,dx = \frac{\pi \ln 2}{2a}, \qquad \int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx = \frac{1}{4}\pi \ln 2$$

in an important analysis without checking them. Are they both right?



$\frac{e^{a\pi i}}{e^{\pi i}}$ $\quad 1 + e^{(\zeta-\pi i)}$ $\quad e^{\pi i}$ $\quad \zeta - \pi i$ $\quad \frac{1}{1+e^{\zeta}}$

$e^{-\pi i} + e^{(\zeta-\pi i)}$

$e^{(a-1)\pi i}$

$1 + e^{\pi i}$

$e^{\zeta-\pi i}$ $\quad \frac{1}{1+e^{\zeta}}$ $\quad \frac{1}{e^{\zeta}}$ $\quad \frac{e^{\pi i a}}{e^{\pi i}}$

$= e^{i\pi(a-1)}$

$= \cos(a-1)\pi$

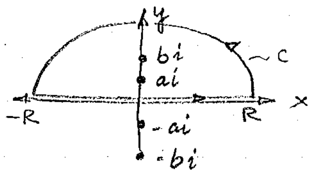
Cesar Levy
ME 301 | Fall 79
Prof Van Dyke
AUDIT

Problem Set #5

5.1 Check $\displaystyle\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}$

To do this look at the following integral over the contour shown

$$\oint f(z)dz = \oint \frac{dz}{(a^2+z^2)(b^2+z^2)}$$



$$\oint_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_0^\pi \frac{Re^{i\theta} i\,d\theta}{(a^2+R^2e^{2i\theta})(b^2+R^2e^{2i\theta})} + \oint_{|z-ai|=\varepsilon_1} f(z)\,dz + \oint_{|z-bi|=\varepsilon_2} f(z)\,dz = 0$$

one mouse, 2 mice; one radius, 2 radii.

where ε_1 & ε_2 are radiis made as small as we wish

Now since $f(x) = f(-x) \Rightarrow \int_{-R}^{R} = 2\int_0^R$; $\lim_{R\to\infty} \int_0^\pi = \lim_{R\to\infty} \frac{i}{R^3}\int e^{-3i\theta}d\theta = 0$

$$\therefore \lim_{R\to\infty}\int_0^R f(x)\,dx = \frac{1}{2}\left\{ \oint_{|z-ai|=\varepsilon_1} f(z)\,dz + \oint_{|z-bi|=\varepsilon_2} f(z)\,dz \right\} = \frac{1}{2}\left\{ \mathrm{Res}\Big|_{z=ai} + \mathrm{Res}\Big|_{z=bi} \right\}\cdot 2\pi i$$ ✓

$$\mathrm{Res}\Big|_{z=ai} = \frac{1}{2ai}\cdot\frac{1}{b^2-a^2} \qquad \mathrm{Res}\Big|_{z=bi} = \frac{1}{2bi}\cdot\frac{1}{a^2-b^2} \qquad \therefore \int_0^\infty f(x)\,dx = \frac{\pi}{2ab(a+b)}$$

now if a and b are complex: then $(z^2-a^2) = (z-ai)(z+ai)$ or the poles are shifted by by $90°$ counterclockwise. If a & b are in the 1^{st} quadrant, then ai & bi will be in the 2^{nd} quadrant but still in the upper half plane. Hence result will be same. If a & b are in 2^{nd} quadrant the ai, bi will be in 3^{rd} quadrant but $-ai$, $-bi$ will be in the first quadrant. Thus

$$\oint_c f(z)dz = \int_{-R}^{R} f(x)\,dx + \int_0^\pi \frac{Rie^{i\theta}d\theta}{(a^2+z^2)(b^2+z^2)} + \oint_{|z+ai|=\varepsilon_1} + \oint_{|z+bi|=\varepsilon_2} = 0$$

or $$\int_0^\infty f(x)\,dx = \frac{1}{2}\left[\mathrm{Res}\Big|_{z=-ai} + \mathrm{Res}\Big|_{z=-bi} \right]\cdot 2\pi i$$

$$\mathrm{Res}\Big|_{z=-ai} = \frac{1}{-2ai}\,\frac{1}{b^2-a^2} \qquad \mathrm{Res}\Big|_{z=-bi} = \frac{1}{-2bi}\,\frac{1}{a^2-b^2} \qquad \therefore \int_0^\infty f(x)\,dx = -\frac{\pi}{2ab(a+b)}$$

Real $a, b < 0$

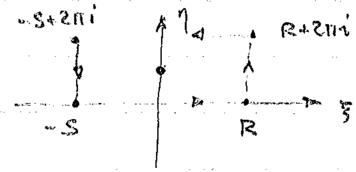
If only one of them is in the 2^{nd} quadrant: ai in 3^{rd} quad, $-ai$ in 1^{st} quad

$$\int_0^\infty f(x)\,dx = \frac{1}{2}\left[\mathrm{Res}\Big|_{z=-ai} + \mathrm{Res}\Big|_{z=bi} \right]\cdot 2\pi i = \frac{\pi}{2ab(a-b)} \qquad \text{Real } a < 0$$ ✓

and if viseversa: $\displaystyle\int_0^\infty f(x)\,dx = \frac{-\pi}{2ab(a-b)}$ Real $b < 0$

5.2 if we let $x = e^{\xi}$ then $dx = e^{\xi} d\xi$ and $\displaystyle\int_{-\infty}^{\infty} \frac{e^{(a\xi)}}{1+e^{\xi}} d\xi = \int_0^{\infty} \frac{x^{a-1}}{1+x^2} dx$ ✓

Look at $\displaystyle\oint_c \frac{e^{a\zeta} d\zeta}{1+e^{\zeta}} = f(\zeta) d\zeta$



denominator $1+e^{\xi}$ has simple poles at $\zeta = \pm\pi i, \pm 3\pi i, \pm 5\pi i$. The only pole in the contour C is at $\zeta = \pi i$. Thus ✓

$$\oint_c f(\zeta) d\zeta = \int_{-S}^{R} \frac{e^{a\xi}}{1+e^{\xi}} d\xi + \frac{e^{aR}}{e^{R}} \int_0^{2\pi} \frac{i e^{ai\eta}}{1 e^{-R} \cdot e^{i\eta}} d\eta + \int_R^{-S} \frac{e^{a2\pi i} \cdot e^{a\xi}}{1 + e^{2\pi i} \cdot e^{\xi}} d\xi$$

$$+ e^{-(a-1)S} \int_{2\pi}^{0} \frac{i e^{ai\eta}}{e^{iS} + e^{i\eta}} d\eta + \oint_{|\zeta - \pi i| = \varepsilon_1} f(\zeta) d\zeta = 0$$

we take limit as $R \to \infty$ & $S \to \infty$

$$\therefore \lim_{\substack{R\to\infty \\ S\to\infty}} \int_{-S}^{R} \frac{e^{a\xi}}{1+e^{\xi}} d\xi = \int_{-\infty}^{\infty} \frac{e^{a\xi} d\xi}{1+e^{\xi}} \quad ; \quad \lim_{R\to\infty} e^{(a-1)R} i \int_0^{2\pi} \frac{e^{ai\eta}}{e^{-R} + e^{i\eta}} d\eta = 0 \cdot i \int_0^{2\pi} e^{(a-1)i\eta} d\eta = 0$$

$$\therefore \lim_{\substack{R\to\infty \\ S\to\infty}} \int_R^{-S} \frac{e^{a2\pi i} e^{a\xi}}{1 + e^{2\pi i} e^{\xi}} d\xi = e^{(a-1)2\pi i} \int_{\infty}^{-\infty} \frac{e^{a\xi}}{e^{-2\pi i} + e^{\xi}} d\xi = -e^{(a-1)2\pi i} \int_{-\infty}^{\infty} \frac{e^{-a\xi}}{e^{-2\pi i} + e^{-\xi}} d\xi$$

$$= -e^{a2\pi i} \int_{-\infty}^{\infty} \frac{e^{-a\xi}}{1+e^{-\xi}} d\xi = -e^{2\pi i a} \int_{-\infty}^{\infty} \frac{e^{a\xi}}{1+e^{\xi}} dx, \text{ since } \int_{-\infty}^{\infty} = -\int_{\infty}^{-\infty}$$

$$\therefore e^{-(a-1)S} i \int_{2\pi}^{0} \frac{e^{ai\eta}}{e^{S} + e^{i\eta}} d\eta \to 0 \text{ as } S \to \infty \quad ; \quad \oint f(\zeta) d\xi = 2\pi i \operatorname{Res}\Big|_{\zeta = \pi i} = 2\pi i \cdot e^{i\pi(a-1)}$$

using L'Hôpital's rule on $\lim_{\zeta \to \pi i} (\zeta - \pi i) \cdot \frac{1}{1+e^{\zeta}}$.

$$\text{Now} \Rightarrow \left[\int_{-\infty}^{\infty} \frac{e^{a\xi}}{1+e^{\xi}} d\xi\right] \frac{1 - e^{2\pi i a}}{1} = \frac{2\pi i \cdot e^{i\pi(a-1)}}{1}$$

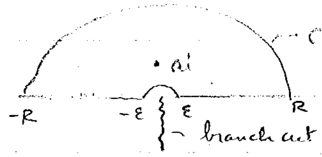
$$\therefore \int_{-\infty}^{\infty} \frac{e^{a\xi}}{1+e^{\xi}} dx = \frac{2\pi i \, e^{i\pi(a-1)}}{1 - e^{2\pi i a}} = \frac{-2\pi i \, e^{i\pi a}}{(e^{-i\pi a} - e^{\pi i a}) e^{i\pi a}} = \frac{\pi}{\sin(\pi a)}$$

since $\dfrac{e^{\pi a i} - e^{-\pi i a}}{2i} = \sin(\pi a)$

$$\therefore \int_0^{\infty} \frac{x^{a-1}}{1+x^2} dx = \frac{\pi}{\sin(\pi a)}$$ ✓

5.3a $\displaystyle\int_0^\infty \frac{\ln x}{x^2+a^2}\,dx = \frac{\pi \ln 2}{2a}$ prove this

look at $\displaystyle\oint_C \frac{\ln(z)}{z^2+a^2}\,dz$ over



the integrand only has a simple pole in the upper half plane thus

$$\oint_C f(z)\,dz = \int_{-R}^{-\varepsilon} \frac{\ln x\,dx}{x^2+a^2} + \int_\pi^0 \frac{\ln(\varepsilon e^{i\theta})\,\varepsilon i e^{i\theta}\,d\theta}{\varepsilon^2 e^{2i\theta}+a^2} + \int_\varepsilon^R \frac{\ln x\,dx}{x^2+a^2}$$

$$+ \int_0^\pi \frac{\ln(Re^{i\theta})\,iRe^{i\theta}\,d\theta}{R^2e^{2i\theta}+a^2} + \oint_{|z-ai|=\varepsilon_1} f(z)\,dz = 0$$

Now: in 1st integral let $x' = -x$ $\displaystyle\int_R^\varepsilon \frac{-\ln(-x')\,dx'}{x'^2+a^2} = \int_{-R}^{-\varepsilon}\frac{\ln x\,dx}{x^2+a^2} = \int_\varepsilon^R \frac{\ln x' + i\pi}{x'^2+a^2}\,dx$

the 2nd & 4th integrals $\to 0$ as $\varepsilon \to 0$ & $R \to \infty$

$$\therefore \quad 2\int_\varepsilon^R \frac{\ln x\,dx}{x^2+a^2} + i\pi\int_\varepsilon^R \frac{dx}{x^2+a^2} = \oint_{|z-ai|=\varepsilon_1} f(z)\,dz = 2\pi i \,\mathrm{Res}\Big|_{z=ai} = 2\pi i\,\frac{\ln(ai)}{2ai}$$

as $\varepsilon \to 0$ & $R \to \infty$

$$2\int_0^\infty \frac{\ln x\,dx}{x^2+a^2} + i\pi\int_0^\infty \frac{dx}{x^2+a^2} = 2\pi i\left[\frac{\ln a + i\pi/2}{2ai}\right] = \frac{\pi\ln a}{a} + \frac{i\pi^2}{2a}$$

Now equating real & imaginary parts $\displaystyle 2\int_0^\infty \frac{\ln x\,dx}{x^2+a^2} = \frac{\pi \ln a}{a}$; $\displaystyle \pi\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi^2}{2a}$

$\therefore \displaystyle\int_0^\infty \frac{\ln x\,dx}{x^2+a^2} = \frac{\pi\ln a}{2a}$ ✓ & $\displaystyle\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$; $\displaystyle\int_0^\infty \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\Big|_0^\infty = \frac{\pi}{2a}$

thus the integral $\displaystyle\int_0^\infty \frac{\ln x\,dx}{x^2+a^2} = \frac{\pi\ln a}{2a}$ <u>not</u> $\displaystyle\frac{\pi \ln 2}{2a}$

we note that $\displaystyle\int_0^\infty \frac{\ln x\,dx}{x^2+1} = 0$ which is what we get when $a = 1$

5.3b $\displaystyle\int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx = \frac{1}{4}\pi\ln 2$ prove it

again look at $\displaystyle\oint_C \frac{\ln z}{(z^2+1)^2}\,dz$ over the same contour as above

Then

$$\oint_C f(z)dz = \int_{-R}^{-\varepsilon} \frac{\ln x}{(x^2+1)^2}dx + \int_{\pi}^{0} \frac{\varepsilon i \ln(\varepsilon e^{i\theta}) e^{i\theta} d\theta}{(\varepsilon^2 e^{2i\theta}+1)^2} + \int_{\varepsilon}^{R} \frac{\ln x\, dx}{(x^2+1)^2} + \int_0^{\pi} \frac{\ln(Re^{i\theta}) Rie^{i\theta} d\theta}{(R^2e^{2i\theta}+1)^2}$$

$$+ \oint_{|z-i|=\varepsilon_1} f(z)dz = 0$$

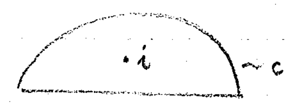
or performing the same trick as before (ie $x' = -x$ in first integral)

$$-\int_R^{\varepsilon} \frac{\ln x' + \pi i}{(x'^2+1)^2}dx' = \int_{\varepsilon}^{R} \frac{(\ln x + \pi i)}{(x^2+1)^2}dx$$

then we have $\quad 2\int_0^{\infty} \frac{\ln x}{(x^2+1)^2}dx + \pi i \int_0^{\infty} \frac{dx}{(x^2+1)^2} = \oint_{|z-i|=\varepsilon_1} f(z)dz = 2\pi i \operatorname{Res}\Big|_{z=i}$ (1)

$$\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$$

Now to find $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$ look at $\oint_{C'} \frac{dz}{(z^2+1)^2}$ over



$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = 2\pi i \operatorname{Res}\Big|_{z=i} = 2\pi i \frac{d}{dz}(z-i)^2 \frac{1}{(z-i)^2(z+i)^2}\Bigg|_{z=i} = 2\pi i \frac{-2}{(z+i)^3}\Bigg|_{z=i} = 2\pi i\left(\frac{+1}{4i}\right.$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2} \quad ; \quad \pi i \int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi i}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi^2}{4}i$$

$\therefore$ from (1) $\quad 2\int_0^{\infty} \frac{\ln x}{(x^2+1)^2}dx + \frac{\pi^2}{4}i = 2\pi i \frac{d}{dz}\left[(z-i)^2 \frac{\ln z}{(z-i)^2(z+i)^2}\right]_{z=i} = 2\pi i \frac{(z+i)^2/z - 2\ln z\,(z+}{(z+i)^4}\Bigg|_{z}$

$$= 2\pi i \left\{\frac{(2i)^2/i - 2(\ln i)(2i)}{(2i)^4}\right\} = 2\pi i\left\{-\frac{1}{4i} + \frac{\pi}{8}\right\}$$

thus $\quad 2\int_0^{\infty} \frac{\ln x}{(x^2+1)^2}dx = -\frac{\pi}{2} \qquad \therefore \boxed{\int_0^{\infty} \frac{\ln x\, dx}{(x^2+1)^2} = -\frac{\pi}{4}}$ ✓

hence the result given in the book is incorrect.

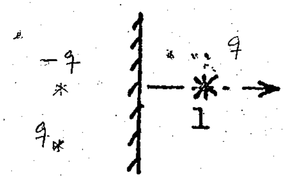
A

Exercise Set 6

ME 201. APPLICATIONS OF COMPLEX VARIABLES

Due 19 November 1979

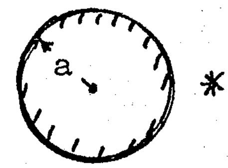
6.1. Image systems for circle. Use the method of images to find the complex potential $\Omega = \phi + i\psi$ for each of the following potential problems in the half-plane $\operatorname{Re} z > 0$:



a. Source located at (without loss of generality) $z = 1$, with $\phi = 0$ on boundary $x = 0$.

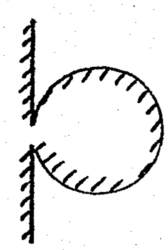
b. Same with $\partial\phi/\partial n = 0$ on boundary.

c. Same for vortex and for dipole at $z = 1$.



Then map the half-plane into the exterior of a circle of radius a, and show that the corresponding solutions for that boundary can be interpreted as the result of adding to the singularity outside the circle an "image" system of singularities inside. List for each case the locations and relative strengths of the images.

6.2. Flow past a circular bump on a wall. Exercise 2 on page 134 of Carrier, Krook & Pearson asks for



> the complex potential $\Omega(z) = \phi + i\psi$ in the domain D defined by the inequalities $\operatorname{Re} z > 0$ and $|z-1| > 1$, subject to the conditions that (1) the boundary forms a streamline say $\psi = 0$ and (2) $\Omega'(z) \to U > 0$ as $z \to \infty$ in D.

Map the domain into some standard one, say the upper half-plane. Then explain (physically, mathematically, or both) why no solution exists to this problem. Find the solution with the distant boundary condition changed to a requirement that the complex intensity $\Omega'(z)$ approach an imaginary constant iV $(V > 0)$. Likewise find the solution with $\Omega'(z) \to Uz$ as $z \to \infty$ (which is probably what the authors had in mind).

$(z-1)(\bar{z}-1) = z\bar{z} - z - \bar{z} + 1 = x^2+y^2-2x+1$

ib ζ' $\phi = 0$ ζ $-b$ b

let $\zeta' = -i\zeta$, $\zeta = i\zeta'$, $\zeta^2 = -\zeta'^2$

$z = a\sqrt{\zeta^2+b^2}$

$z = \sqrt{\zeta^2-b^2}$

$z = i\sqrt{\zeta'^2+b^2}$

as $z \to \infty$, $z \sim i\zeta'$

$\phi = U\eta$

Let $\Phi = Ux$, $\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$

$\chi \sim U\zeta + \frac{B}{\zeta} + \dots$

$\chi = \phi + i\psi = U\zeta + \frac{B}{\zeta}$

$\Phi_x + i\psi_x = -i\Phi_y + \psi_y$

$\Phi_x = \psi_y$, $-\Phi_y = \psi_x$

$U = \psi_y$

$\psi = Uy + g(x)$

$\psi_x = g' = 0$

$\therefore \chi \sim U\zeta + C$

as $z \to \infty$, $\Phi = Ux$

$Uz \sim Ui\zeta'$

$Ux \sim U\eta$

Solutions to Exercise Set 6

ME 201. APPLICATIONS OF COMPLEX VARIABLES

Monday 26 November 1979

6.1. Image systems for circle. (a) A point source of unit strength located at $z = 1$ has (CK&P p. 117) complex potential $\Omega = \ln(z-1)$. We can cancel its real part ϕ on the line $x = 0$ by adding a source of opposite strength (a unit "sink") at the image point $z = -1$. Then the complex potential is

$$\Omega(z) = \ln(z-1) - \ln(z+1) = \ln\frac{z-1}{z+1}.$$

(b) Similarly, adding a source of equal strength at the image point gives

$$\Omega(z) = \ln(z-1) + \ln(z+1) = \ln(z^2-1),$$

and this is purely real at $z = iy$. Hence ψ vanishes on the line $x = 0$, so that $\partial\psi/\partial y = 0$ there, and hence $\partial\phi/\partial x = 0$ by the Cauchy-Riemann equations.

(c) A unit vortex at $z = 1$ has complex potential $\Omega = i\ln(z-1)$, so the roles of real and imaginary parts are reversed from the two cases above: for $\phi = 0$ on the line $x = 0$ the image is an equal vortex, so that

$$\Omega(z) = i\ln(z^2-1),$$

and for $\partial\phi/\partial n = 0$ it is an opposite vortex, so that

$$\Omega(z) = i\ln\frac{z-1}{z+1}.$$

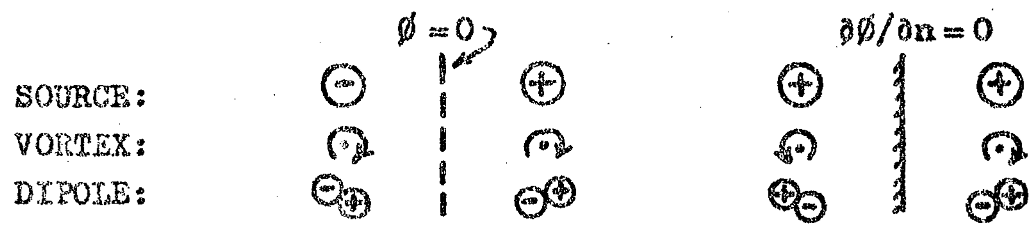
A dipole with axis inclined at angle α is obtained by letting a source and sink approach each other along a line with that inclination, their strengths increasing inversely as their distance apart. This gives for a unit dipole located at $z = 1$ the complex potential $\Omega = e^{i\alpha}/(z-1)$. Then using the results of part (a) above shows that for $\phi = 0$ on $x = 0$ the potential is

$$\Omega(z) = e^{i\alpha}/(z-1) + e^{-i\alpha}/(z+1),$$

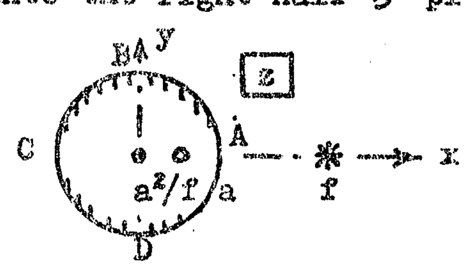
and using the results of part (b) shows that for $\partial\phi/\partial n = 0$ on $x = 0$

$$\Omega(z) = e^{i\alpha}/(z-1) - e^{-i\alpha}/(z+1).$$

All these results may be indicated graphically as follows:

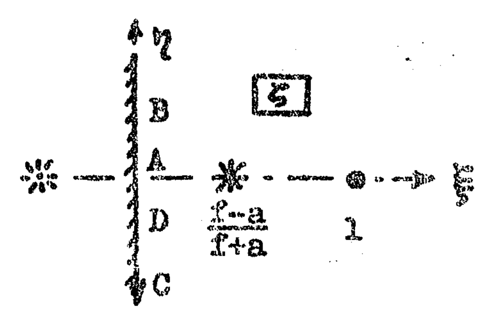


The circle of radius a about the origin in the z-plane is mapped into the right half ζ-plane by



$$z = a\,\frac{1+\zeta}{1-\zeta}$$

$$\zeta = \frac{z-a}{z+a}$$



A unit source outside the circle on the positive real axis at $z = f$ is mapped into a unit source in the ζ-plane at $\zeta = (f-a)/(f+a)$. For $\phi = 0$ on the boundary, the solution in the ζ-plane is given by the method of images (as in part a above) by

$$\chi(\zeta) \equiv \Phi + i\Psi = \ln\left(\zeta - \frac{f-a}{f+a}\right) - \ln\left(\zeta + \frac{f-a}{f+a}\right) = \ln \frac{\zeta - (f-a)/(f+a)}{\zeta + (f-a)/(f+a)} .$$

Then mapping back gives the solution in the original z-plane as

$$\Omega(z) \equiv \phi + i\psi = \ln \frac{a(z-f)}{fz - a^2} = \ln(z-f) - \ln\left(z - \frac{a^2}{f}\right) + \text{const.}$$

This can be interpreted as the source at $z = f$ outside the circle plus an image source of opposite strength (a "sink") inside the circle at the inverse point $z = a^2/f$.

For $\partial\phi/\partial n = 0$ on the boundary we might suppose that the method of images gives similarly (as in part b above)

$$\chi(\zeta) = \ln\left(\zeta - \frac{f-a}{f+a}\right) + \ln\left(\zeta + \frac{f-a}{f+a}\right) = \ln\left[\zeta^2 - \left(\frac{f-a}{f+a}\right)^2\right] .$$

However, mapping back to the original z-plane would give

$$\Omega(z) = \ln \frac{4a(z-f)(zf-a\ )}{(f+z)\ (z+a)} = \ln(z-f) + \ln\left(f - \frac{a^2}{z}\right) - 2\ln(z+a) + \text{const.}$$

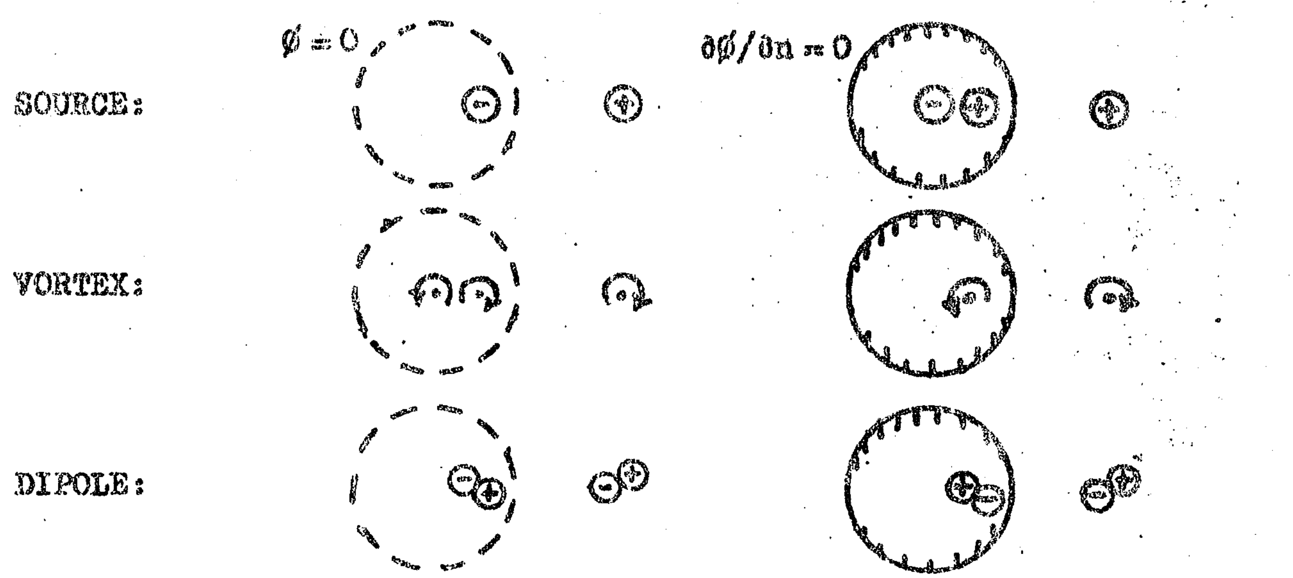
and this shows, in addition to an image source at the inverse point, a double sink on the boundary at $z = -a$, which spoils the solution there. We can correct this defect by recognizing that we can add a source of any strength at the point $\zeta = 1$, together with its image at $\zeta = -1$, because these correspond to $z = \infty$ and $z = 0$. Choosing its strength to cancel the sink at $z = -a$ gives

$$\chi(\zeta) = \ln\left[\zeta^2 - \left(\frac{f-a}{f+a}\right)^2\right] - \ln(1 - \zeta^2)$$

and

$$\Omega(z) = \ln \frac{(z-f)(zf-a^2)}{(f+a)^2\, z} = \ln(z-f) + \ln\left(z - \frac{a^2}{f}\right) - \ln z + \text{const.}$$

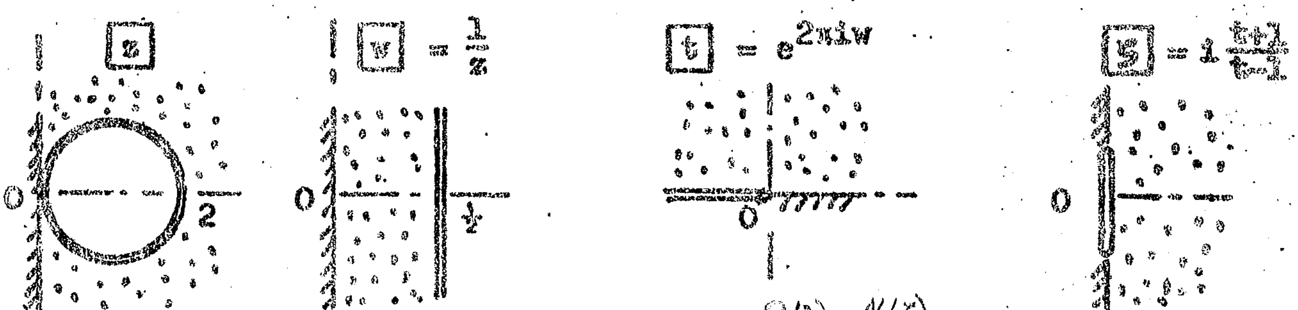
Thus the correct image system is an equal source at the inverse point plus an equal sink at the center of the circle. Proceeding in the same way for the other cases yields the results summarized graphically below:



In the case of a dipole, the image inside the circle is weaker than the dipole outside by the factor $(a/f)^2$.

For the cases with $\partial\phi/\partial n = 0$ on the circle, these results are obtained more quickly by applying the circle theorem of Milne-Thomson (Theoretical Hydrodynamics, p. 154, Macmillan, NY 1960): if the solution without the circular boundary is $\Omega = f(z)$, then that with the boundary is $\Omega = f(z) + f^*(a^2/z)$. [Here if $f(z) = 6z + 3iz^2$, for example, then $f^*(z) = 6z - 3iz^2$.] Carrier, Krook & Pearson ask for the proof of this theorem in their Exercise 10, p. 119; but they quote it incorrectly.

6.2. Flow past a circular bump on a wall. The given domain is simply connected (so that there is no circulation to be found), and must therefore be mapped into another simply connected domain, such as a half plane. Of the many possible sequences of mapping, the following is perhaps the simplest:



$\Omega(z) = \chi(\zeta)$

$\Omega' = \chi' \frac{d\zeta}{dz} = -\frac{1}{\pi}\chi' = U$ $\quad -\frac{1}{\pi}dz \sim d\zeta$

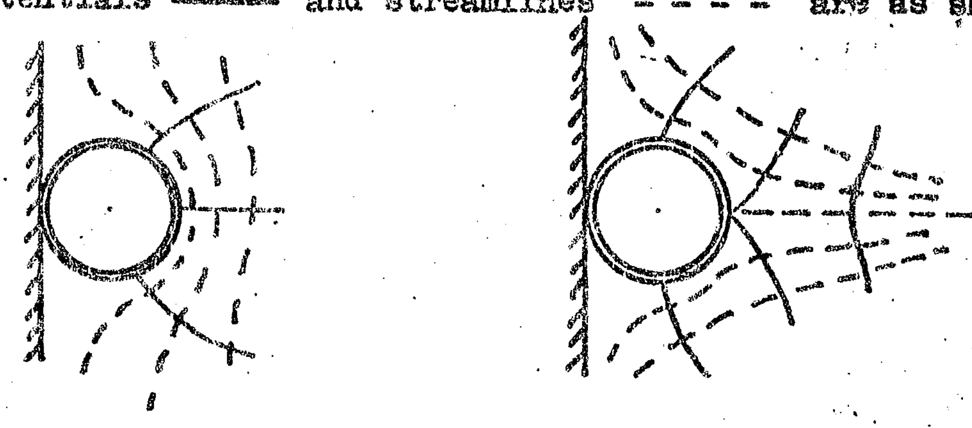
Contracting this chain gives the mapping as $\zeta = \cot(\pi/z)$. The point $z = \infty$ is mapped into $\zeta = \infty$, with $z \sim \pi\zeta$ at great distances. The problem in the ζ-plane is therefore to find an analytic function whose imaginary part vanishes on the imaginary axis and whose derivative approaches U/π far away. Evidently no such function exists — mathematically because the two requirements are contradictory far up the imaginary axis, physically because this would give uniform flow normal far from an infinite wall.

$\chi' = U\pi$ $\Rightarrow \chi \sim U\pi\zeta$ $\quad \chi = \phi + i\psi = U\pi\xi + U\pi i\eta$

With the distant boundary condition changed to $\Omega'(z) \to iV$ or $\Omega'(z) \to Uz$ the solution in the ζ-plane is evidently just

$$\chi = i\pi V\zeta \qquad \text{or} \qquad \chi = \tfrac{1}{2}\pi^2 U\zeta^2$$

and then substituting $\zeta = \cot(\pi/z)$ gives the required solutions. The equipotentials ——— and streamlines - - - - are as shown below:



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Problem Set #6

6.1 Use the method of images to find the complex potential Ω for each of the following:

a. a source located at $z=1$ with $\phi=0$ on the boundary $x=0$

Ω' for a source with strength $2\pi q$ is $\Omega' = q\ln(z-1)$. now

$\Omega = \frac{q}{2}\ln[(x-1)^2+y^2] + q\theta i$; thus along $x=0$ $\phi = \frac{q}{2}\ln(y^2+1)$

thus we need to place a sink at $z=-1$ of strength $-q$

$\Omega'' = +q''\ln(z+1) = +\frac{q''}{2}\ln[(x+1)^2+y^2] + iq''\theta$; along $x=0$ $\phi'' = +\frac{q''}{2}\ln(y^2+1)$

thus for $\phi=0$ $q''=-q$ or

$\boxed{\Omega = \Omega'+\Omega'' = q\ln\left[\frac{z-1}{z+1}\right]}$ note that for large $\Im z$ $\Omega \sim q\ln 1 = 0$ as a check.

b. As before $\Omega' = q\ln(z-1)$ $\phi' = \frac{q}{2}\ln[(x-1)^2+y^2]$. Since we want $\frac{\partial\phi}{\partial n}=0$ thus $\frac{\partial\phi}{\partial x}=0$; $\frac{\partial\phi'}{\partial x} = \frac{q}{2}\frac{2(x-1)}{[(x-1)^2+y^2]} = \frac{-q}{[y^2+1]}$ @ $x=0$

if we let $\Omega'' = q''\ln(z+1)$ $\phi'' = \frac{q''}{2}\ln[(x+1)^2+y^2]$ and $\frac{\partial\phi''}{\partial x} = \frac{q''}{2}\frac{2[x+1]}{[(x+1)^2+y^2]}$

thus at $x=0$ $\frac{\partial\phi''}{\partial x} = \frac{q''}{[1+y^2]}$. for $\phi=0$ let $q''=q$

thus $\boxed{\Omega = \Omega'+\Omega'' = q\ln[z^2-1]}$. for large $\Im z$ then $\Omega \sim 2\ln(iy)$ and $\frac{d\Omega}{dx} = \frac{d}{dx}\ldots = 0$ as a check.

c. for a vortex: $\Omega' = ip\ln(z-1)$ and $\phi' = -\theta p$ along $x=0$ $\theta = \tan^{-1}\frac{y}{-1}$

in order for $\phi=0$ superpose $\Omega'' = +ip''\ln(z+1) \Rightarrow \phi'' = -\theta p''$ along $x=0$ $\theta = \tan^{-1}y$

$\phi' = p\tan^{-1}y$ $\phi'' = -p''\tan^{-1}y$ take $p''=p$ $\therefore$ $\boxed{\Omega = \Omega'+\Omega'' = ip\ln(z^2-1)}$ for $\phi=0$ at $x=0$

With usual definition of principal value of logarithm, this actually gives $\Phi=\pi$ on boundary; but that is easily remedied by subtracting π from Ω.

and for $\frac{\partial\phi}{\partial n}=0$ following the same lines as above $\phi' = -\theta p = -p\tan^{-1}\frac{y}{x-1}$

$\frac{\partial\phi'}{\partial n} = \frac{\partial\phi'}{\partial x} = \frac{-p\,\,{}^{-y}/(x-1)^2}{\left(\frac{y}{x-1}\right)^2+1} = \frac{+yp}{y^2+(x-1)^2}$; at $x=0$ $\frac{\partial\phi'}{\partial x} = \frac{+yp}{y^2+1}$

if $\Omega'' = ip''\ln(z+1)$ $\phi'' = -\theta p''$ $\theta = \tan^{-1}\frac{y}{x+1}$ $\frac{\partial\phi''}{\partial x} = -p''\frac{-y/(x+1)^2}{(y/x+1)^2+1} = \frac{p''y}{y^2+(x+1)^2}$

$\left.\frac{\partial\phi''}{\partial x}\right|_{x=0} = \frac{p''y}{y^2+1}$ for $\frac{\partial\phi}{\partial x}=0$ take $p''=-p$ $\therefore$ $\Omega'' = -ip\ln(z+1)$

$\therefore$ $\Omega = \Omega'+\Omega'' = \boxed{ip\ln\left(\frac{z-1}{z+1}\right) = \Omega \text{ for } \frac{\partial\phi}{\partial n}=0 \text{ at } x=0}$

for a dipole: $\Omega' = \frac{m}{z-1}$ then $\Omega' = \frac{m(\bar{z}-1)}{(x-1)^2+y^2}$ and $\phi' = \frac{m(x-1)}{(x-1)^2+y^2}$ (@ $x=0$ $\phi' = \frac{-m}{y^2}$...

let $\Omega'' = \frac{m''}{z+1} = \frac{m''(\bar{z}+1)}{(x+1)^2+y^2}$ $\phi'' = \frac{m''(x+1)}{(x+1)^2+y^2}$ @ $x=0$ $\frac{m''}{y^2+1} = \phi''$ take $m''=m$

thus $\boxed{\Omega = \frac{m}{z-1} + \frac{m}{z+1} = \frac{2mz}{z^2-1}}$ for large $\Im z$ then $\Omega \sim \frac{2mz}{z^2} \sim \frac{i2m\,\Im z}{-(\Im z)^2}$
and $\operatorname{Re}\Omega = \phi = 0$ as a check.

for $\frac{\partial\phi}{\partial n} = \frac{\partial\phi}{\partial x} = 0$ as before $\frac{\partial\phi'}{\partial x} = \frac{\partial}{\partial x}\frac{m(x-1)}{(x-1)^2+y^2} = m\left[\frac{(x-1)^2+y^2-2(x-1)^2}{[(x-1)^2+y^2]^2}\right]$

@ $x=0$ $\frac{\partial\phi'}{\partial x} = m\frac{y^2-1}{(y^2+1)^2}$

Similarly $\frac{\partial\phi''}{\partial x} = m''\frac{[(x+1)^2+y^2-2(x+1)^2]}{[(x+1)^2+y^2]^2}$; @ $x=0$ $\frac{\partial\phi''}{\partial x} = m''\frac{y^2-1}{(y^2+1)^2}$

take $m''=-m$ then $\Omega = \Omega''+\Omega' = \boxed{\frac{m}{z-1} - \frac{m}{z+1} = \frac{2m}{z^2-1} = \Omega}$

Mapping the half plane into the exterior of a circle of radius a:
$w' = \frac{z-2}{z+2}$ maps the half plane into the unit circle; $w'' = aw'$ map the unit circle into a circle of radius a; and $w = \frac{a^2}{w''}$ maps the circle into the exterior of itself $\therefore$ $w = \frac{a(z+2)}{(z-2)}$ $\Rightarrow$ $z = 2\frac{a+w}{w-a}$

because the mapping is conformal, the boundary conditions $\phi=0$ & $\frac{\partial\phi}{\partial n}=0$ go into $\Phi=0\big|_{r=a}$ and $\frac{\partial\Phi}{\partial r}=0\big|_{r=a}$ since the Jacobian of the transformation is not zero on the boundary.

Now since at $z=1$ $\Rightarrow$ $w=-3a$ the source is there for $\Phi=0$ and $z=-1$ $\Rightarrow$ $w=-a/3$ is inside the circle and the sink is there.

proof: $z-1 = \frac{2a+2w-w+a}{w-a} = \frac{3a+w}{w-a}$; $z+1 = \frac{2a+2w+w-a}{w-a} = \frac{a+3w}{w-a}$

$\mathcal{X}(w) = q\ln\left(\frac{z-1}{z+1}\right) = q\ln\left(\frac{3a+w}{3w+a}\right)$ thus $\mathcal{X}(w)$ is made up of a source at $w=-3a$ & a sink at $w=-a/3$, each of strength q and $-q\ln 3$ resp. **No!**

for $\frac{\partial\Phi}{\partial r}=0$ $\mathcal{X}(w) = q\ln(z^2-1) = q\ln\left[\frac{(3a+w)(a+3w)}{(w-a)^2}\right]$

thus we have a source at $w=-a/3$ and $-3a$ of strength $q\ln 3$ & q resp. and a sink at $w=a$ of strength $2q$. **As explained in solution, this sink is unacceptable.**

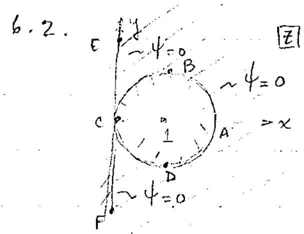
$\Phi=0$: for the vortex $\mathcal{X}(w) = ip\ln\left[\frac{(3a+w)(a+3w)}{(w-a)^2}\right]$. Thus we have vortices at $w=-3a$ and $-a/3$ of strength p and a vortex at $w=a$ of strength $-2p$

for $\frac{\partial\Phi}{\partial r}=0$: for the vortex $X(w)= ip\ln\left[\frac{3a+w}{a+3w}\right]$. This situates a vortex at $w=-3a$ and one at $-a/3$ of strength p and $-p\ln 3$ respect.

for $\Phi=0$ and the dipole case: $X(w)=\frac{m(w-a)}{(3a+w)}+\frac{m(w-a)}{(a+3w)}$

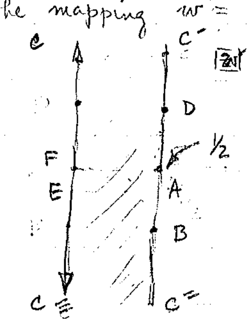
Consists of dipoles of strength $-4ma$ at $w=-3a$ and $-\frac{4ma}{9}$ at $w=-a/3$ and a constant potential of $\frac{4m}{3}$ superposed

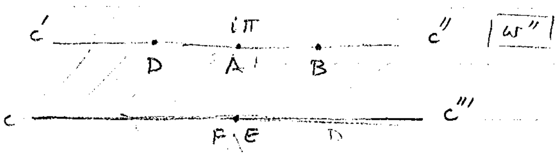
6.2.



and $\Omega'(z)\to U$ as $z\to\infty$ in D.

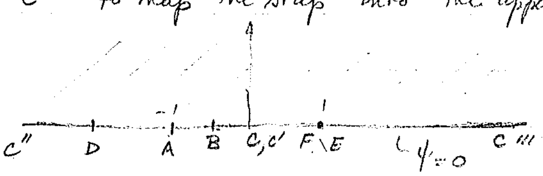
by the mapping $w'=\frac{1}{z}$ we map the z plane into





let $w''=2\pi i w'$ to map w' into

and finally $w=e^{w''}$ to map the strip into the upper half plane



∴ $w=e^{2\pi i/z}$ ⇒ $\ln w = 2\pi i/z$ ∴ $z=\frac{2\pi i}{\ln w}$

given $\psi=0$ on the bdy and $\Omega'(z)\to U$ as $z\to\infty$ then this translates to $\Psi=0$ on the bdy and $X'(w)\frac{dw}{dz}=X'(w)\cdot\left[-\frac{w(\ln w)^2}{2\pi i}\right]\to U$ as $w\to 1$

but $w(\ln w)^2\to 0$ as $w\to1$ ⇒ $X'(w)$ must $\to -\frac{2\pi i U}{w(\ln w)^2}$ as $w\to 1$

thus $X(w)\sim\frac{2\pi i U}{\ln w}+(C_1+iC_2)$ But we also want $\Psi=0$ along $\operatorname{Im} w=0$

this is impossible since $\Psi=\operatorname{Im}X = \frac{2\pi i U}{\ln u}+C_1+iC_2$ ⇒ $C_2=-\frac{2\pi U}{\ln u}$ + const

Thus we have no solution

If $\Omega'(z)\to iV$ as $z\to\infty$ ⇒ $X'(w)\left[-\frac{w(\ln w)^2}{2\pi i}\right]\to Vi$ as $w\to1$

∴ $X'(w)\to\frac{-Vi\cdot 2\pi i}{w(\ln w)^2}=\frac{2\pi V}{w(\ln w)^2}$ as $w\to1$ ∴ $X(w)\sim-\frac{2\pi V}{\ln w}$ + const

Thus let $\chi(w) = \frac{-2\pi V}{\ln w} + \text{const} = -\frac{2\pi V}{\ln w} + c_1 + i c_2$

and along $\Im m\, w = u$ then $\Im m\, \chi = \Psi = c_2 = 0 \Rightarrow c_2 = 0 \quad \therefore \chi(w) = -\frac{2\pi V}{\ln w} + c$

let $c_1 = 0$ for simplicity $\therefore$ $\boxed{\chi(w) = \Omega(z) = -iVz}$ **This obviously doesn't satisfy boundary conditions on circular bump**

If $\Omega'(z) \to Vz$ as $z \to \infty$ $\quad \chi'(w)\left[-\frac{w(\ln w)^2}{2\pi i}\right] \to \frac{V \cdot 2\pi i}{\ln w}$ as $w \to 1$

$\Rightarrow \chi'(w) \to \frac{V 4\pi^2}{w(\ln w)^3}$ as $w \to 1 \quad \therefore \quad \chi(w) \sim -\frac{2\pi^2 V}{(\ln w)^2} + \text{const.}$

let $\chi(w) = -\frac{2\pi^2 V}{(\ln w)^2} + c_1 + i c_2 \qquad \Psi = \Im m\, \chi(u) = 0 = c_2 \quad \therefore \chi(w) = -\frac{2\pi^2 V}{(\ln w)^2}$

(for simplicity we let $c_1 = 0$) $\quad \therefore \quad \boxed{\Omega(z) = \frac{-2\pi^2 z^2 V}{-4\pi^2} = Vz^2}$ **Nor this.**

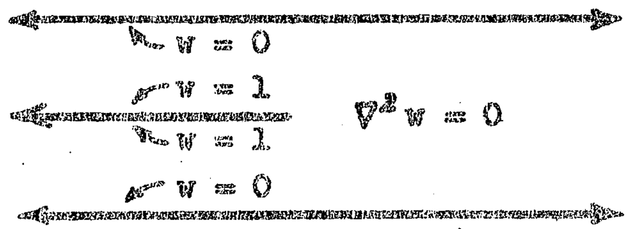
As indicated in solution, it is much simpler to map once more with $\zeta = i\,\frac{w+1}{w-1}$, giving a trivial problem to solve in the ζ-plane

Exercise Set 7

ME 201. APPLICATIONS OF COMPLEX VARIABLES

Due Monday 3 December 1979

7.1. Maxwell's edge correction for disk viscometer. James Clerk Maxwell measured the viscosity of air by observing the damping of a circular disk oscillating in its own plane between two closely spaced parallel fixed plates. To calculate approximately a correction for the edge of the disk, his friend William Thomson showed him how to solve the plane potential problem indicated in the sketch: the velocity w normal to the plane satisfies Laplace's equation, vanishes on two parallel infinite plates, and is equal to unity on a semi-infinite plate midway between them.



(a) Show how Thomson mapped the region between the plates conformally into the upper half of another plane.

(b) Translate the original boundary-value problem into that auxiliary plane, and solve it there.

(c) Thus find the solution for w in the original plane of the sketch.

7.2. Laplace inversion by contour integration. The solution of a certain vibration problem (Carrier, Krook & Pearson's Exercise 11, p. 354) requires the inversion of the Laplace transform

$$F(s) = \frac{s}{(s^2+a^2)(s^2+b^2)} \qquad (a,\ b,\ \text{real})$$

Invert this directly by evaluating the inversion formula

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,F(s)\,ds,$$

where γ is any value large enough that the line of integration lies to the right of all the singularities of the integrand. Use contour integration, completing the contour in the left half-plane. Consider the special case $a = b$. Check your result by decomposing $F(s)$ into two terms of the form $ks/(s^2+c^2)$ and using known results.

ME 201
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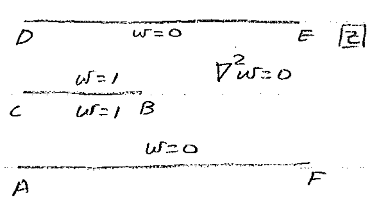
Problem Set #7

1.

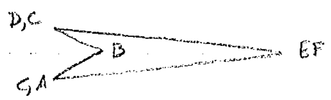


a. Show how this maps into upper half plane

b. Translate the original bvp into the new bvp

c. Find the sol of w in the original plane.

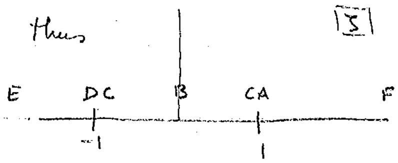
a. Consider the degenerate form.



$\measuredangle BAF = \measuredangle FDB = \measuredangle DEA = 0 \qquad \measuredangle CBA = 2\pi$ ✓

map CA into $\zeta = 1$ map DC into $\zeta = -1$ map B in $\zeta = 0 \Rightarrow \frac{dz}{d\zeta} = a\frac{\zeta}{\zeta^2-1}$ ✓

Thus



$\therefore \quad z = \frac{a}{2}\ln(\zeta^2-1) + b$ ✓

$\zeta = 0 \Rightarrow \quad 0 = \frac{a}{2}\cdot\pi i + b \quad$ or $\quad b = -\frac{a\pi i}{2}$. Now lets assume that CB is on the negative x axis and DE, AF map into the line $y = \pm\frac{\pi i}{2}$ respectively. Now to find a look at a circle of radius ρ about $\zeta = 1 \Rightarrow \zeta - 1 = \rho e^{i\theta}$

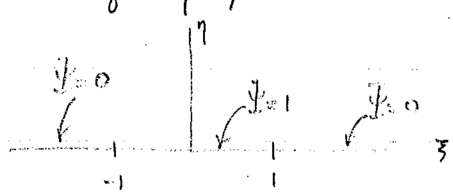
then $dz = a\frac{\zeta\, d\zeta}{\zeta^2-1} = a\frac{(1+\rho e^{i\theta})\rho i e^{i\theta}d\theta}{(2+\rho e^{i\theta})\rho e^{i\theta}}$ now let $\rho \to 0$ ie $\zeta \to 1$ $dz \sim \frac{a}{2}i\,d\theta$

now $z\Big|_A^C = \frac{ai}{2}\theta\Big|_0^\pi$ since along line BC in ζ plane $\theta = \pi$ & line AF in ζ plane $\theta = 0$

but $z\Big|^C = -\infty \quad z\Big|^A = -\infty - i\pi/2 \quad \therefore \quad i\pi/2 = \frac{ai\pi}{2} \quad a = 1 \Rightarrow b = -\frac{\pi i}{2}$

Thus $\boxed{z = \frac{1}{2}\ln(\zeta^2-1) - i\pi/2}$ ✓. Note that $\zeta = \pm a + 0i \Rightarrow \mathrm{Im}\, z = \mp i\pi/2$

b. We can now use logs for the χ function in the ζ plane since it produces jumps in its imaginary part. The original problem translates to



for $\chi \sim \ln(\zeta-\zeta_0)$ and

$\eta = 0 \quad \xi > 1$ then let $\chi = 0$

$|\xi| < 1$ then let $\chi = \frac{1}{\pi}\ln(\zeta-1) = \frac{1}{2\pi}\ln[(\xi-1)^2+\eta^2] + i \Rightarrow \Psi = 1$

$\xi < 1$ then let $\chi = -\frac{1}{\pi}\ln(\zeta+1) = -\frac{1}{2\pi}\ln[(\xi+1)^2+\eta^2]\cdot i \Rightarrow \Psi = 0$ ✓

Thus let $\boxed{\chi = \frac{1}{\pi}\ln\left(\frac{\zeta-1}{\zeta+1}\right)}$; as $|\zeta| \to \infty \quad \mathrm{Re}\,\chi = 0$ and $\mathrm{Im}\,\chi = \Psi = 0$

$|\xi| < 1 \Rightarrow -\infty \le \frac{\xi-1}{\xi+1} \le 0$ on real axis $\therefore \mathrm{Im}\,\frac{1}{\pi}\ln\left(\frac{\zeta-1}{\zeta+1}\right) = 1 = \Psi$

now from previous $(2z + i\pi) = \ln(\zeta^2-1)$ or $\left[-e^{2z}+1\right]^{1/2} = \zeta$

thus $\mathcal{K}[\zeta(z)] = \Omega(z) = \frac{1}{\pi} \ln\left[\frac{(1-e^{2z})^{1/2}-1}{(1-e^{2z})^{1/2}+1}\right]$ thus $\boxed{w = \operatorname{Im}\Omega(z)}$

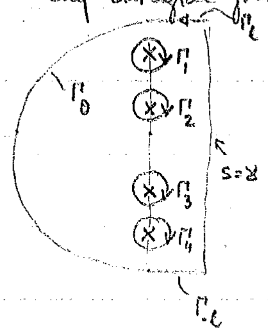
or $\boxed{w = \operatorname{Im} \frac{1}{\pi} \ln\left\{\frac{(1-e^{2z})^{1/2}-1}{(1-e^{2z})^{1/2}+1}\right\}}$ ✓

2. If $F(s) = \frac{s}{(s^2+a^2)(s^2+b^2)}$ simple poles exist at $s = \pm ia$ $s = \pm ib$

Using the Laplace inverse transform to find $f(t)$

Since $f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds$

We use the Cauchy integral formula



thus $\int_c e^{st}F(s)ds = \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds + \int_{\Gamma_\ell} + \int_{\Gamma_0} + \int_{\Gamma_{-\ell}} + \sum_{i=1}^{4}\oint_{\Gamma_i} e^{st}F(s)ds = 0$

since real s over Γ_0 is $< 0 \Rightarrow e^{st}F(s)ds \sim \frac{e^{(\mathrm{Re}\,s)t}}{s^2} \to 0$ as $s \to$

now over Γ_ℓ & $\Gamma_{-\ell}$ the integrals go to zero since integration is over x and the interval of integration is bounded but the integrals $\sim \frac{1}{y^3}$. As $y \to \infty$ the integral must vanish ✓ $\therefore$

$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds = \sum \text{Res}$ at $s = \pm ai,\ s = \pm bi$; $\text{Res}\big|_{s=ai} = \frac{e^{ait}\,ai}{2ai\,(b^2-a^2)} = \frac{e^{ait}}{2(b^2-a^2)}$

$\text{Res}\big|_{s=-ai} = \frac{e^{-ait}(-ai)}{-2ai\,(b^2-a^2)} = +\frac{e^{-ait}}{2(b^2-a^2)}$, $\text{Res}\big|_{s=bi} = \frac{e^{bit}\,bi}{(a^2-b^2)\cdot 2bi} = \frac{e^{bit}}{2(a^2-b^2)}$ ✓

$\text{Res}\big|_{s=-bi} = \frac{e^{-bit}(-bi)}{-2bi\,(a^2-b^2)} = \frac{e^{-bit}}{2(a^2-b^2)}$ ✓

$\therefore f(t) = \frac{e^{ait}+e^{-ait}}{2(b^2-a^2)} + \frac{e^{bit}+e^{-bit}}{2(a^2-b^2)} = \frac{\cos at - \cos bt}{(b^2-a^2)}$ ✓

we may also write $\frac{s}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$ $\mathcal{L}^{-1}\left(\frac{s}{s^2+c^2}\right) = \cos ct$

$\therefore \frac{1}{b^2-a^2}\left[\cos at - \cos bt\right] = f(t)$ QED

if $b = a$ then $f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}s}{(s^2+a^2)^2}ds = \sum \text{Res}\big|_{s=\pm ai}$

$$\text{Res}\Big|_{s=ai} = \frac{d}{ds}\left(\frac{e^{st}s}{(s+ai)^2}\right)\Big|_{s=ai} = \left[\frac{tse^{st}}{(s+ai)^2} + \frac{e^{st}}{(s+ai)^2} - \frac{2e^{st}s}{(s+ai)^3}\right]_{s=ai} = \frac{taie^{ait}}{-4a^2} + \frac{e^{ait}}{-4a^2} - \frac{2aie^{ait}}{-8a^3i}$$

$$\text{Res}\Big|_{s=-ai} = \frac{d}{ds}\left(\frac{e^{st}s}{(s-ai)^2}\right)\Big|_{s=-ai} = \left[\frac{tse^{st}}{(s-ai)^2} + \frac{e^{st}}{(s-ai)^2} - \frac{2se^{st}}{(s-ai)^3}\right]_{s=-ai} = \frac{-aite^{-ait}}{-4a^2} + \frac{e^{-ait}}{-4a^2} + \frac{2aie^{-ait}}{8a^3i}$$

$$\therefore f(t) = \frac{t}{2a}\sin at \checkmark$$

Now $\frac{s}{(s^2+a^2)^2} = -\frac{1}{2a}\frac{d}{da}\left(\frac{s}{s^2+a^2}\right) \Leftrightarrow -\frac{1}{2a}\frac{d}{da}\cos at = -\frac{1}{2a}(-t\sin at) = \frac{t\sin at}{2a} \checkmark$

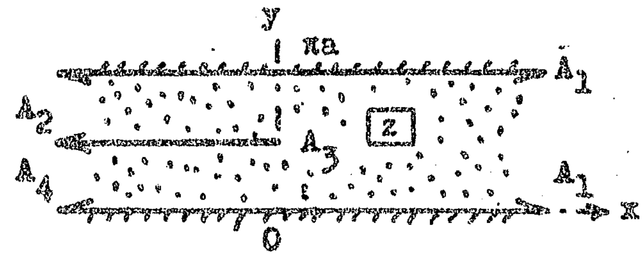
A

Solutions to Exercise Set 7

ME 201. APPLICATIONS OF COMPLEX VARIABLES

5 December 1979

7.1. Maxwell's edge correction for disk viscometer. We may regard the three plates as the degenerate limit of the symmetric quadrilateral of the second sketch. If we map vertex A_1 to $\zeta = \infty$, A_2 to $\zeta = -1$, and A_3 to $\zeta = 0$, then by symmetry A_4 maps to $\zeta = 1$ (as is easily verified). Then the Schwarz-Christoffel formula gives

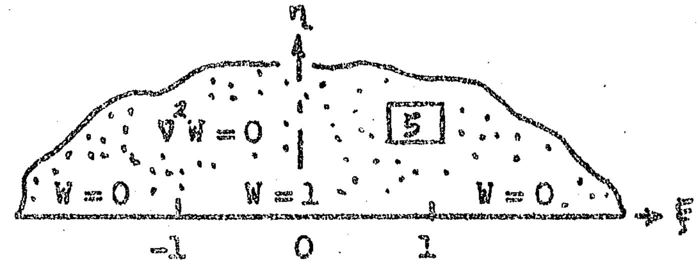


$$\frac{dz}{d\zeta} = a(\zeta+1)^{-1}(\zeta-0)^{1}(\zeta-1)^{-1}.$$

If we take the constant of integration as zero, this gives

$$z = \frac{a}{2}\ln(\zeta^2-1), \quad \zeta^2 = 1+e^{2z/a},$$

and the orientation and scale of the geometry in the original z-plane is as shown in the first sketch. Alternatively, with $z = (a/2)\ln(1-\zeta^2)$ the origin of the z-plane is located at the end of the middle plate.



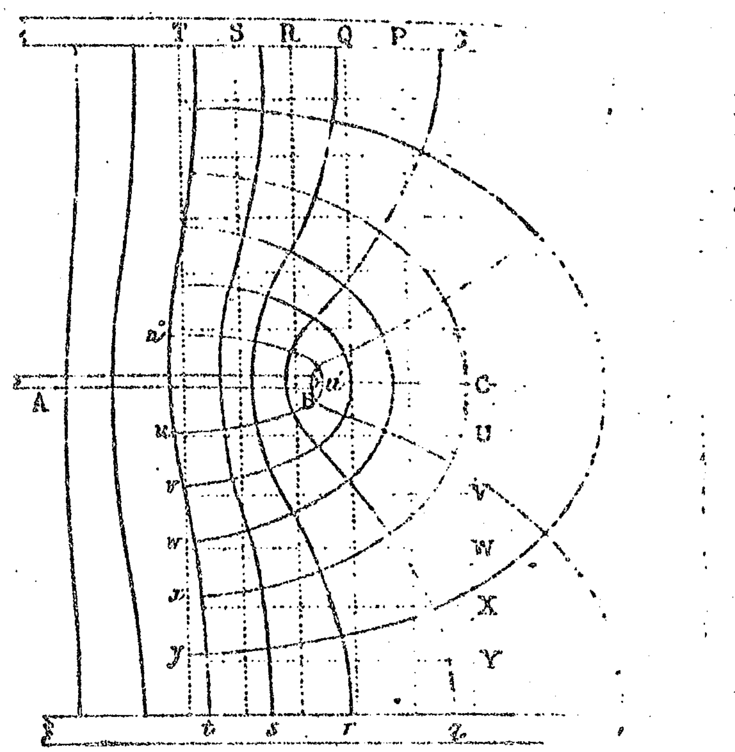
The transformed boundary-value problem is indicated in the third sketch. Using the fact that the imaginary part of $\ln(z-z_0)$ changes from 0 for $z > z_0$ to $i\pi$ for $z < z_0$ we find the solution as

$$W = \mathrm{Im}\frac{1}{\pi}\left[\ln(\zeta-1) - \ln(\zeta+1)\right] = \frac{1}{\pi}\,\mathrm{Im}\ln\frac{\zeta-1}{\zeta+1} = -\frac{2}{\pi}\,\mathrm{Im}\,\mathrm{ctnh}^{-1}\zeta.$$

Hence in the original physical plane

$$w = -\frac{2}{\pi}\,\mathrm{Im}\,\mathrm{ctnh}^{-1}(1+e^{2z/a}).$$

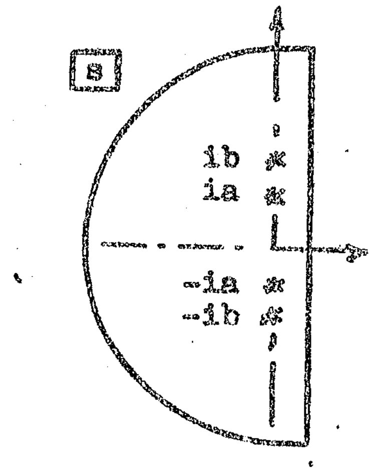
At the right is Maxwell's figure showing lines of constant speed w, taken from his 1866 Bakerian lecture "On the viscosity or internal friction of air and other gases," Philos. Trans. Roy. Soc. 156, 249-268.



7.2. <u>Laplace inversion by contour integration.</u> The inversion theorem gives

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}\, s\, ds}{(s^2+a^2)(s^2+b^2)}$$

where we must take γ greater than a and b. If we complete the contour to the left, the integral along the extra path vanishes by Jordan's lemma, and we enclose simple poles at



$$s = \pm ia \quad \text{with residues} \quad \frac{e^{\pm iat}}{2(b^2-a^2)},$$

$$s = \pm ib \quad \text{with residues} \quad \frac{e^{\pm ibt}}{2(b^2-a^2)}.$$

Then the residue theorem gives

$$f(t) = \frac{\cos at - \cos bt}{b^2 - a^2}.$$

To check this, we decompose

$$F(s) = \frac{s}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\,\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

and use the known result that the transform of $\cos ct$ is $s/(s^2+c^2)$.

In the special case $b = a$ there are double poles at $s = \pm ia$ with residues

$$\pm \frac{t\, e^{iat}}{4ia}$$

and the residue theorem gives

$$f(t) = \frac{1}{2a}\, t \sin t.$$

Alternatively, we may extract this result from the previous one by letting $b \to a$ and using l'Hospital's rule.