ME 200B
Winter '80
WCR

MATHEMATICAL METHODS IN MECHANICAL ENGINEERING

Course Plan

Numbers in brackets refer to chapters in the notes.

M	W	F	Workshops 500U	Homework
	Jan. 9 Introduction pre-test	Jan. 11 Self-similar solutions	-----	Problem 1: Self-similar solutions (LOTS-ODE) Due Jan. 21
Jan. 14 Self-similar solutions [2]	Jan. 16	Jan. 18 Linear Prob. [2]	Self-similar transformations	
Jan. 21 Linear eigenvalue problems and special functions [3]	Jan. 23	Jan. 25	ODE Review	Problem 2: An eigenvalue problem (Optional LOTS) Due Feb. 4
Jan. 28 Linear eigenvalue problems and special functions [3]	Jan. 30	Feb. 1	Linear PDE solution; SOV idea	
Feb. 4 Superposition solutions in linear PDE problems [4]	Feb. 6	Feb. 8	Special functions	Problem 3: Inhomogeneous problem solution by superposition (LOTS evaluation) Due Feb. 25
Feb. 11 Superposition solutions in linear PDE problems [4]	Feb. 13	Feb. 15 MIDTERM	Linear boundary value problems	
Feb. 18 HOLIDAY	Feb. 20 Inhomogeneous problems [4,5]	Feb. 22	Inhomogeneous problems	
Feb. 25 Periodic and wave solutions [5]	Feb. 27	Mar. 1	Wave solutions	Problem 4: Wave propagation problems by SOV and characteristics
Mar. 3 Characteristics [7]	Mar. 5	Mar. 7	Characteristics	
Mar. 10 Characteristics	Mar. 12	Mar. 14 Review [7]	Open review	

FINAL EXAM, Thursday, March 20, 8:30-11:30a.m., Terman Auditorium.

All TV students in the Bay Area must come to campus for the exams.

WCR
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ME 200B - MATHEMATICAL METHODS IN MECHANICAL ENGINEERING

INSTRUCTOR: William C. Reynolds; office, 500C, 497-4021; home, 948-2952

TEACHING ASSISTANTS:

Robert La Roche, Room 501U, 497-4039
Ramash Jayaraman, Room 501U, 497-4039
Cesar Levy, Room 264 (Durand), 497-2189

TUTORS: Several tutors will be selected from the class enrollment, based on pre-examination. Tutors will be expected to be available to assist other students for two hours each week, in return for which they will be exempted from turning in the class homework. The tutors will meet with the instructor and teaching assistants weekly for coordination. Tutor Room: 500U The principal role of the tutors will be to assist other students as they encounter difficulties in the homework.

TEXT: We have experimented with several texts and have not found a good one that fits this course. Therefore, Professor Reynolds is writing a new book, Solution of Partial Differential Equations. A preliminary edition of this book will be sold through the Bookstore.

It is important that each student have access to the material included in Abramowicz and Stegun, Handbook of Mathematical Functions, NBS publication 15 (Dover Press), which is available in the Bookstore.

CLASS OBJECTIVES: The primary objective of this class is to help you develop skill in solving partial differential equations. Therefore, emphasis on theory of PDE's will be minimal. If you desire a more theoretical background, there are courses for this purpose in our math department.

CLASS FORMAT: The lectures will cover the basic material. The primary teaching methods will be by example. The notes contain additional examples and exercises that may be worked for practice.

The teaching assistants will conduct small-group workshops each week. These will be opportunities to practice the methods presented in lecture in a close-feedback situation. In addition, some supplementary and remedial material will be presented.

There will be four major analysis problems which must be done professionally as homework. These will involve the use of LOTS to a small degree.

TV TAPES: Videotapes will be available for replay in the Engineering Library for approximately one week following live presentation.

GRADING: Grading will be done by the instructor and other helpers. Rough break-down: homework, 40%; midterm exam, 25%; final exam 35%. The homework scores are usually high, and therefore, the chief penalty will come if they are not done.

TA's and Tutors will meet one hour each week to discuss course progress and to tutor the Tutors.

Handouts which students miss receiving in class will be available in metal boxes outside of Room 500C. Problems will be returned there also.

TA Office Hours

Robert La Roche, Room 501U, 497-4039
10:00 - 12:00 MW

Ramesh Jayaraman, Room 501U, 497-4039
2:15 - 3:00 MF
9:00 - 11:00 T

Tutoring Hours (Lounge)

MONDAY
1:15 - 2:05 Pat Shea

TUESDAY
10:00 - 10:50 Steve Pronchik
~~4:15~~ 3:15 - ~~5:05~~ 4:05 Dave Goodwin

WEDNESDAY
1:15 - 2:05 Pat Shea
4:15 - 5:05 Steve Pronchik

THURSDAY
1:15 - 2:05 Currie Munce
~~4:15~~ 1:30 - ~~5:05~~ 2:30 Dave Goodwin

FRIDAY
1:15 - 3:05 Pat Lowery
3:15 - 4:05 Currie Munce

Workshops (501U)

MONDAY
2:15 - Robert La Roche
3:15 - Cesar Levy
4:15 - Ramesh Jayaraman

TUESDAY
11:00 - Ramesh Jayaraman
1:15 - Robert La Roche
2:15 - Ramesh Jayaraman
3:15 - Robert La Roche

WEDNESDAY
2:15 - Robert La Roche
3:15 - Ramesh Jayaraman

THURSDAY
10:00 - Ramesh Jayaraman
11:00 - Cesar Levy
2:15 - Ramesh Jayaraman
3:15 - Robert La Roche

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1. MOND 3 PM
2. TUES 1 PM
3. Th 11 AM

W1-1

have students do: I By Parts, Sol. of ODE, CHAIN RULE

WORKSHOP # 1

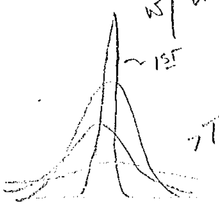
DIFFUSION IN SPHERICAL COORDS. from a point source into an infinite medium. SELF SIMILAR

NO TIME, LENGTH SCALE ∴ USE COMB. OF INDEP. VARIABLE TO FOR NON DIMENS. GROUP

diffusion eqn w/ integral const.

$$\alpha \frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right) = r^2 \frac{\partial c}{\partial t}$$

$c = \infty$ at $r=0,\ t=0$
$= 0$ at $r>0$ for $t=0$
also $c(r,t) \to 0$ as $r \to \infty$



$c = 0$ at $t = 0$, except at $r = \infty$

use integral const first

$$\int_0^\infty 4\pi r^2 c\, dr = Q$$

CONSTANT QUANTITY
Total amount of chemical

$c \to 0$ as $r \to \infty$

what is temp. due to this

self-similar soln collapse all these curves to one

$$c = A t^m f(\eta) \qquad \eta = \frac{Br}{t^n}$$

$$d\eta = \frac{B}{t^n} dr \Big|_{t=\text{CONST}}$$

$$4\pi \int_0^\infty \left(\frac{t^n \eta}{B}\right)^2 A t^m f(\eta) \cdot \frac{t^n}{B} d\eta = Q$$

Now Get rid of time dependence

$$2n + m + n = 0 \qquad 3n + m = 0 \qquad m = -3n$$

$$\frac{4\pi A}{B^3}\int_0^\infty \eta^2 f(\eta)\, d\eta = Q$$

WILL USE TO SET A or B.

CHAIN RULE
TAKE 5 min

c is a fn of η & t but η is also a fn of t

$\therefore \frac{\partial c}{\partial t} = \frac{dc}{dt} + \frac{\partial c}{\partial \eta}\frac{d\eta}{dt}$

$$\frac{\partial c}{\partial t}\Big|_{\text{const } r} = A m t^{m-1} f + A t^m f' \cdot \left(-\frac{Brn}{t^{n+1}}\right)$$

$$= A t^{m-1}\left[-3nf - n\eta f'\right]$$

$$\frac{\partial c}{\partial r}\Big|_{\text{const } t} = A t^m f' \cdot \frac{B}{t^n} \qquad r^2 \frac{\partial c}{\partial r} = ABt^{m-n} r^2 f'(\eta)$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right)_t = ABt^{m-n} \frac{d}{d\eta}\left[\left(\frac{\eta^2 t^{2n}}{B^2}\right) f'\right]_{t=\text{CONST}} \cdot \frac{B}{t^n}$$

$$= A \cdot t^{m} \frac{d}{d\eta}(\eta^2 f')$$

So, $\alpha (r^2 c')' = r^2 \dot{c}$

$$\alpha A t^m \frac{d}{d\eta}(\eta^2 f') = \frac{\eta^2 t^{2n}}{B^2} \cdot A t^{m-1}\left[-3nf - n\eta f'\right]$$

Get rid of ...

$\cancel{m} = 2n + \cancel{m} - 1 \qquad \underline{n = 1/2} \qquad m = -3/2$

$$\alpha B^2 \frac{d}{d\eta}(\eta^2 f') + \eta^2\left[\tfrac{1}{2}\eta f' + \tfrac{3}{2} f\right] = 0$$

why? with experience you learn to pick groups of coeffs to simplify eqns. - Normally we will tell you how.

PICK $\alpha B^2 = 1/2 \qquad \eta = \frac{r}{\sqrt{2\alpha t}} \qquad B = \frac{1}{\sqrt{2\alpha}}$

$$\frac{d}{d\eta}(\eta^2 f') + \underbrace{\left[\eta^3 f' + 3\eta^2 f\right]}_{[\eta^3 f]'} = 0$$

WANT $\boxed{f(\infty) \to 0.}$

$\frac{\partial c}{\partial r} = \frac{A}{(2\alpha t)^{3/2}} f' \qquad \frac{d}{d\eta}\left[\eta^2 f' + \eta^3 f\right] = 0$

$\eta^2 f' + \eta^3 f = \text{const}$

physically concentration $\to 0$ as $r \to \infty$ independent of time $\therefore f(\infty) \to 0$ also $f'(\infty) \to 0$

$$\eta^2 f' + \eta^3 f = \text{CONST} = 0$$

SOLUTION OF ODE

TAKE 5 min $\quad f' + \eta f = 0$

$$\frac{df/d\eta}{f} = \frac{f'}{f} = -\eta \qquad \ln f = -\eta^2/2 + C$$

$$f = C e^{-\eta^2/2}$$

find that you will always have a free constant

since f is defined for c & c has an arbitrary const we can Pick $f(0) = 1 \quad \therefore C = 1$

$$f = e^{-\eta^2/2} = \exp\left\{-\frac{1}{2}\frac{r^2}{2\alpha t}\right\} = \exp\left\{-\frac{r^2}{4\alpha t}\right\}$$

GO BACK TO integral CONSTRAINT

$$Q = 4\pi A (2\alpha)^{3/2} \int_0^\infty \eta^2 f(\eta)\, d\eta$$

$$= 4\pi A (2\alpha)^{3/2} \int_0^\infty \eta^2 e^{-\eta^2/2}\, d\eta$$

let $x = \eta^2/2 \qquad \eta = \sqrt{2x} \qquad d\eta = \sqrt{2}\,\frac{1}{2} x^{-1/2}\, dx$

GAMMA FUNCTION

$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1}\, dx \qquad z > 0$

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(z+1) = z\,\Gamma(z)$

$$Q = 4\pi A (2\alpha)^{3/2} \int_0^\infty 2x \cdot \frac{1}{\sqrt{2}} x^{-1/2} e^{-x}\, dx =$$

$$= 4\pi A (2\alpha)^{3/2} \sqrt{2} \underbrace{\int_0^\infty x^{1/2} e^{-x}\, dx}_{\Gamma(3/2) = \frac{\sqrt{\pi}}{2}} = \cancel{4}^{2} A \cdot (2\alpha\pi)^{3/2} \frac{\sqrt{2}}{\cancel{2}}$$

$$\boxed{A = \frac{Q}{8(\alpha\pi)^{3/2}}}$$

$$c = \frac{Q}{8(\alpha\pi t)^{3/2}} e^{-\frac{r^2}{4\alpha t}}$$

WORKSHOP #2 ODE REVIEW

Linearity: dep. variable to first power only.
Homogeneity: replace $y \to ky$ gives same eqn.

Linear ODE: n^{th} order
if homogeneous:
gen'l solution is $\sum C_n y_n$
where y_n are n linearly independent solutions

if inhomogeneous:
gen'l solution is $\sum C_n y_{n_{homo}} + \underset{\text{not unique}}{y_{particular}}$

Lin ODEW | Constant Coefficients: always give exponential (emphasis) solutions to homogeneous problems.

eg.
$$y''' + ay'' + by' + cy = 0$$

assume $y = e^{px}$
$$p^3e^{px} + ap^2e^{px} + bpe^{px} + ce^{px} = 0$$
$$p^3 + ap^2 + bp + c = 0$$
The 3 roots of this equation give 3 linearly independent solutions.

EXAMPLES

$$y'' - a^2y = 0 \quad \leftarrow \text{got to know}$$
$$p^2 - a^2 = 0 \qquad p = \pm a$$
$$y = C_1e^{ax} + C_2e^{-ax} \qquad \text{let } C_1 = \frac{A_1 + A_2}{2} \quad C_2 = \frac{-(A_1 + A_2)}{2}$$
$$\left(A_1\frac{e^{ax} - A_1e^{-ax}}{2}\right) + \left(A_2\frac{e^{ax} + A_2e^{-ax}}{2}\right)$$
or $y = A_1 \sinh(ax) + A_2 \cosh(ax)$

what happens if you don't have ~~…~~ indep roots

if 2 roots $y_h = (c_1 + c_2x)e^{rx}$
if 3 roots $y_h = (c_1 + c_2x + c_3x^2)e^{rx}$
if n roots $y_h = \left(\sum^{n-1} c_i x^i\right) e^{rx}$

$$y'' + 2ay' + a^2y = 0$$
$$(p^2 + 2ap + a^2)y = 0$$
$$(p+a)^2 y = 0$$

$$y'' + a^2 y = 0$$ must know

$$p^2 + a^2 = 0 \qquad p^2 = -a^2 \qquad p = \pm i a$$

$$y = c_1 e^{iax} + c_2 e^{-iax}$$

$c_1(\cos ax + i\sin ax) + c_2(\cos ax - i\sin ax) = \underbrace{(c_1+c_2)}_{A_1}\cos ax + \underbrace{(c_1-c_2)i}_{A_2}\sin ax$

or

$$y = A_1 \sin(ax) + A_2 \cos(ax)$$

INHOMOG L. ODE

Variation of parameters for construction of particular solutions

$$y'' + ay' + by = f(x)$$

SOLVE: Homogeneous problem $\quad y_h'' + a y_h' + b y_h = 0$

$$y_h = c_1 y_1 + c_2 y_2 \qquad \Rightarrow y_1, y_2$$

$$y_p = g_1(x) y_1 + g_2(x) y_2$$

$$y_p' = \underbrace{g_1' y_1 + g_2' y_2}_{\text{pick} = 0} + g_1 y_1' + g_2 y_2'$$

$$y_p'' = g_1' y_1' + g_2' y_2' + g_1 y_1'' + g_2 y_2''$$

subst in eqn. $y_p'' + a y_p' + b y_p$

$$g_1\cancel{[y_1'' + a y_1' + b y_1]}^{0} + g_2\cancel{[y_2'' + a y_2' + b y_2]}^{0}$$
$$+ g_1' y_1'' + g_2' y_2'' = f(x)$$

So,
$$g_1' y_1' + g_2' y_2' = f(x)$$
$$g_1' y_1 + g_2' y_2 = 0$$

Solve for g_1

$$g_1' = \frac{\begin{vmatrix} f(x) & y_2' \\ 0 & y_2 \end{vmatrix}}{\begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix}} \qquad g_2' = \frac{\begin{vmatrix} y_1' & f \\ y_1 & 0 \end{vmatrix}}{\begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix}}$$

$\begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} = W(y_1, y_2) \neq 0$

$$\begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} = W \qquad = \text{Wronskian}$$

$W \neq 0$ is condition of linear independence

$$g_1(x) = \int^x \frac{y_2(x')}{W(x')} f(x')\,dx'$$

$$g_2(x) = \int^x \frac{-y_1(x')f(x')\,dx'}{W(x')}$$

So, we can carry out the integrations to get the solutions.

This works at all orders n, with constant or variable coefficients. for higher orders more restriction

Non-constant coefficients

$$y'' + \frac{1}{x}y' + y = 0$$

$y = gf$
$y' = g'f + gf'$
$y'' = g''f + 2g'f' + gf''$
$= g''f + 2g'f' + gf'' + g'f/x + gf'/x + gf = 0$
$+ f(g'' + g'/x + g)$

Series solution $\quad y = x^r \sum_{n=0}^{\infty} a_n x^n$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad \sum_{n=0}^{\infty} a_n x^{n+r} + \cdots$$

$$y' = a_1 + 2a_2 x + \cdots = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n-1+r}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n-2+r} \qquad \text{let } n-2=m,\ n=m+2$$

$$y'' + \frac{1}{x}y' + y = \sum_{n=0}^{\infty} \underbrace{(n+r)(n+r-1)}_{} a_n x^{n-2+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n-2+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$m = n-2$
$n = 2+m$ $\qquad m = n-2$

$$\sum_{m=-2}^{\infty} (2+m+r)(1+m+r)\, a_{m+2} x^{m+r} + \sum_{m=-2}^{\infty} (m+2+r)\, a_{m+2} x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$\underline{m=-2}$ $\qquad a_0$ arbitrary $\qquad 0 = \sum 0 \cdot x^{-}$

$$\underbrace{[r(r-1)+r]}a_0x^{r-2} + [(r+1)r+(r+1)]a_1x^{r-1} + \sum_{m=0}^{\infty}\left\{\left[(2+m+r)(m+1+r)+(m+2+r)\right]a_{m+2} + a_m\right\}x^{m+r} = 0$$

$$\underbrace{r^2}a_0 + (r+1)^2a_1 + \sum\left\{(m+2+r)^2a_{m+2} + a_m\right\} = 0$$

indicial equation — solution of which are the roots r_1, r_2

$m = -1$

$$1\,a_1 = 0 \qquad \therefore \quad a_1 = 0$$

$r^2 = 0$ or $a_0 = 0$

if $r = 0 \Rightarrow a_0 \neq 0 \Rightarrow a_1 = 0$

$m = 0$

$$2a_2 + 2a_2 + a_0 = 0$$

for the nonzero: $a_{m+2} = -\dfrac{a_m}{(m+2)^2}$

if $a_0 = 0 \Rightarrow r = -1$ or $a_1 = 0$

$$a_2 = -\frac{a_0}{4}$$

$a_2 = -\dfrac{a_0}{2^2}$

$a_4 = -\dfrac{a_2}{4^2} = \dfrac{a_0}{2^2\cdot 4^2} = \dfrac{a_0}{2^4(2!)^2}$

$m > 0$

$$a_{m+2} = -\frac{a_m}{(2m+1)(m+1) + (m+2)}$$

$a_6 = -\dfrac{a_4}{6^2} = \dfrac{-a_0}{6^2 2^2 4^2} \cdots$

$= \dfrac{-a_0}{2^6\, 3!\, 3!}$

Recurrence equations

let $a_{2(l+1)} = -\dfrac{a_{2l}}{2^2(l+1)}$

$= -\dfrac{a_{2(l-1)}}{2^2\cdot 2^2(l+1)l}$

$= -\dfrac{a_{2(l-2)}}{2^6(l+1)^2(l)^2(l$

$a_{2(l+1)} = \dfrac{(-1)^l a_0}{2^{2l}(l+1)!^2}$

We have a_0, a_1, and can get all coefficients by this process.

What about the second linearly independent solution?

Could construct by the method taught in class:

$\therefore \displaystyle\sum_{l=0}^{\infty}\frac{(-1)^l\,x^{2l}}{2^{2l}(l!)^2}$

$$y_2 = g(x)\sum_{n=0}^{\infty} a_n x^n$$

$$\cancel{y_2' = g'\sum_{n=0}^{\infty} + g\sum_{n=0}^{\infty} n a_{n-1} x^{n-1}}$$

$$\cancel{y_2'' = g''\sum_{n=0}^{\infty} a_n x^n + 2g'\sum_{n=0}^{\infty} n a_{n-1} x^{n-1} + g\sum_{n=0}^{\infty} n(n-1)x^{n-2}}$$

~~Subst into eqn the terms out g as coefficient all cancel out.~~

$$\cancel{g''\sum_{n=0}^{\infty} a_n x^n + 2g'\sum_{n=0}^{\infty} n a_{n-1} x^{n-1} + g'\sum_{n=0}^{\infty} a_n x^n = 0}$$

$$\cancel{\frac{g''}{g'} =}$$

Works easier to assume

$$y_2 = \ln x \cdot \sum_{n=0}^{\infty} b_n x^n + \sum_{n=0}^{\infty} C_n x^n$$

and a second solution can be developed this way.

The <u>Bessel</u> <u>functions</u> are developed in this manner. Review this in your ODE Text

if $r_1 - r_2$ is not an integer

$$y_1(x) = |x|^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n x^n\right] \qquad a_n(r_1) = a_n$$

$$y_2(x) = |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} \bar{a}_n x^n\right] \qquad \bar{a}_n = \bar{a}_n(r_2)$$

if $r_1 = r_2$ then

$$y_1(x) = |x|^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n x^n\right] \qquad a_n = a_n(r_1)$$

$$y_2(x) = y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n(r_1) x^n$$

if $r_1 - r_2 = N$ a positive integer

$$y_1(x) = |x|^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1) x^n\right]$$

$$y_2(x) = a\, y_1(x)\ln|x| + |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n(r_2) x^n\right]$$

where $a_n(r_1)$, $\bar{a}_n(r_2)$, $b_n(r_1)$, $c_n(r_2)$ and the constant a can be determined by substituting the form of the series solution for y into the differential equation. The constant a may turn out to be zero.

if we have an equation of the form

15 min **WORKSHOP #3 SPECIAL FUNCTIONS**

OBJECTIVE: ACQUAINT THEM WITH THE FUNCTIONS

GUIDE THEM TO SOURCES OF INFO IN HMF

1. GAMMA FUNCTION HMF 6.1.1

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

INTEGRATE BY PARTS

$$\Gamma(z) = \cancel{-t^{z-1} e^{-t}\Big|_0^\infty} - \int_0^\infty (z-1)\, t^{z-2}(-e^{-t})\, dt$$

$$= (z-1)\int_0^\infty t^{(z-1)-1} e^{-t}\, dt = (z-1)\,\Gamma(z-1)$$

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1$$

$$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1\, \Gamma(1) = 2$$

$$\Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1 \cdot \Gamma(1) = 3!$$

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$$

$$\Gamma(n) = (n-1)! \qquad n! = \Gamma(n+1)$$

Sometimes you will see $(\tfrac{1}{2})!$ meaning $\Gamma(3/2)$

35 min 2. DIFFERENTIAL EQS LEADING TO BESSEL FUNC.

HMF 9.1.1 9.1.49

EXAMPLE $y'' + \lambda^2 x y = 0$

let $y = x^{1/2} g(x)$

$y' = \frac{1}{2x^{1/2}} g + x^{1/2} g'$

$y'' = -\frac{1}{4x^{3/2}} g + \frac{1}{x^{1/2}} g' + x^{1/2} g''$

$y'' + \lambda^2 x y = -\frac{1}{4x^{3/2}} g + \frac{1}{x^{1/2}} g' + x^{1/2} g'' + \lambda^2 x^{3/2} g$

FROM HMF 9.1.51 $\quad P = 3$

$$y = x^{1/2}\, \mathscr{C}_{1/3}\left(\frac{2}{3}\lambda x^{3/2}\right)$$

$\mathscr{C}$ stands for any linear combination of the Bessel functions (9.1.27)

SO,

$$y = c_1 x^{1/2} J_{\frac{1}{3}}\left(\frac{2}{3}\lambda x^{3/2}\right) + c_2 x^{1/2} J_{-\frac{1}{3}}\left(\frac{2}{3}\lambda x^{3/2}\right)$$

since last time indicial eqn gave $r^2=0$ or $r=0$ & we defined $J_0(x)$

here the indicial eqn leads to $r = 1/3$, $r = -1/3$ & $r_1 - r_2 \neq$ integer then solns are

OR

$$y = c_1 x^{\frac{1}{2}} J_{\frac{1}{3}}\left(\frac{2}{3}\lambda x^{3/2}\right) + c_2 x^{1/2} Y_{\frac{1}{3}}\left(\frac{2}{3}\lambda x^{3/2}\right)$$

SEE Paragraph 9.1 HMF for discussion of the linearly independent solutions.

How about

$$y'' - \lambda^2 x y = 0 \;?$$

let $z = -x$

$$\frac{d}{dx} = \frac{d}{dz}\cdot(-1) \qquad y'' = \frac{d^2}{dz^2}$$

$$-xy = zy$$

$$\frac{d^2y}{dz^2} + \lambda^2 z y = 0 \qquad \text{so } y = \cancel{z}^{1/2}\, \mathcal{C}_{\frac{1}{3}}\left(-\tfrac{2}{3}\lambda \cancel{z}^{3/2}\right)$$

(HMF 9.1.51)

soln $\cancel{z}^{1/2}\, \mathcal{C}_{1/3}\left(\frac{2\lambda}{3} z^{3/2}\right)$

sol. in $i x^{1/2}\, \mathcal{C}_{1/3}\left(\cancel{+} i \tfrac{2}{3}\lambda x^{3/2}\right)$

$y = \cancel{x}^{1/2}\, \mathcal{J}_{1/3}\left(\tfrac{2}{3}\lambda x^{1/2}\right)$ F 9.6.26

ALSO SEE HMF 10.4

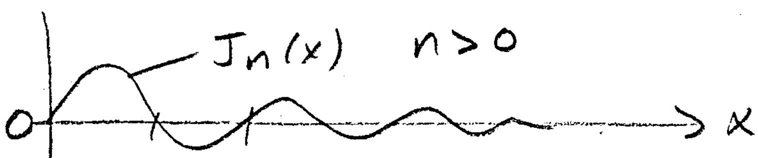
$$y = C_1 A_i(z) + C_2 B_i(z)$$

$$x^2 y'' + x y' - (x^2 + \nu^2)\, y = 0 \qquad \boxed{\text{HMF } 9.6.1}$$

DISCUSS I & K FUNCTIONS

POINT OUT THAT THESE FUNCTIONS ARE DEVELOPED BY SERIES METHODS.

ROOTS OF BESSEL FUNCTIONS

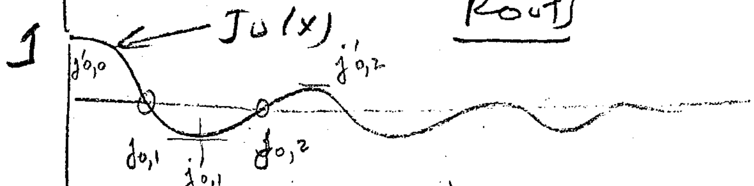


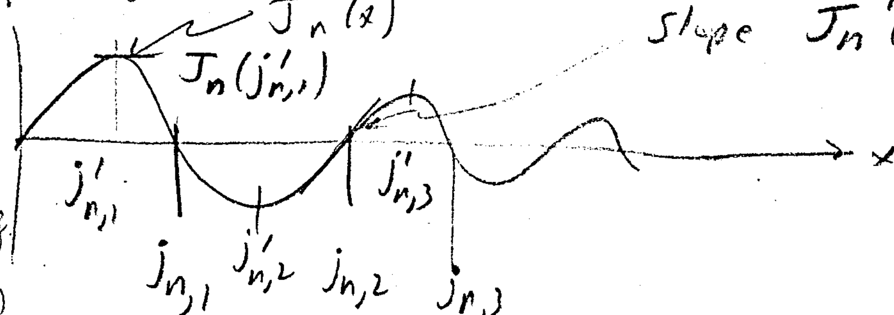
EXPLAIN TABLES IN HMF pp 409, 411

for large x

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \alpha_n)$$

$$\alpha_n = (2n+1)\pi/4$$





$J_n'(x)$ = slope of $J_n(x)$

$j_{n,s}$ = s^{th} zero of $J_n(x)$

$J_n'(j_{n,s})$ = slope of $J_n(x)$ at the s^{th} zero of $J_n(x)$

$j'_{n,s}$ = s^{th} zero of $J_n'(x)$

$J_n(j_{n,s})$ = value of $J_n(x)$ where s^{th} zero of $J'_n(x)$

DERIVATIVE RELATIONSHIPS

HMF 9.1.27, 9.1.28

$$J_0' = -J_1$$

prove by differentiation of series or by working ODE

$$J_2' = J_1 - \frac{2}{z} J_2$$

;

Problem SET #2 ; 3.5/3.4 NOTES

$\frac{\partial \phi}{\partial x_i} = \dot{x}_i$

3.5 (1) $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$ Contin eqn

(2) $\phi_t + g\eta = 0$ unsteady Bernoulli surface bdy cnd @ $z = 0$ $\phi(x,y,z,t)$

(3) $\eta_t - \phi_z = 0$ @ $z = 0$ $\eta(x,y,t)$ surface movement eqn. defined at $z=0$

(4a) $\phi_x = 0$ at $x = 0, a$

(4b) $\phi_y = 0$ at $y = 0, b$

(5) $\phi_z = 0$ @ $z = -h$ no velocity through tank.

(6) Assume $\phi = T(t) \cdot X(x) \cdot Y(y) \cdot Z(z)$

$(1) \Rightarrow \underbrace{\frac{X''}{X}}_{-\alpha^2} + \underbrace{\frac{Y''}{Y}}_{-\beta^2} + \underbrace{\frac{Z''}{Z}}_{+\gamma^2} = 0$ each term must be constant. $\gamma^2 = \alpha^2 + \beta^2$

(7) $X'' + \alpha^2 X = 0$ (8) $Y'' + \beta^2 Y = 0$ (9) $Z'' - \gamma^2 Z = 0$

(10) $X = A_1 \cos\alpha x + A_2 \sin\alpha x$ $(4a) \Rightarrow A_2 = 0, \sin(\alpha a) = 0$ $\alpha_m a = m\pi$

(11) $X_m = A_m \cos\left(\frac{m\pi x}{a}\right)$

(12) $Y = B_1 \cos\beta y + B_2 \sin\beta y$ $(4b) \Rightarrow B_2 = 0, \sin(\beta b) = 0$ $\beta_n b = n\pi$

(13) $Y_n = B_n \cos\left(\frac{n\pi}{b} y\right)$

(14) $Z = C_1 \sinh(\gamma z) + C_2 \cosh(\gamma z) = C_3 \cosh[\gamma(z+h)] + C_4 \sinh[\gamma(z+h)]$ to get rid of bound cond @ $z=-h$

$(5) \Rightarrow Z'(-h) = 0 \Rightarrow C_4 = 0$

(15) So $\cancel{Z_3} = C_3 \cosh[\gamma(z+h)]$ (16) $\gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2} = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$

(17) $\phi = A \cos\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{b}\right] \cosh[\gamma(z+h)] \times T(t)$

This potential satisfies all of the wall boundary conditions.

$(2)_t \Rightarrow \phi_{tt} + g\eta_t = 0$, with $(3) \Rightarrow \phi_{tt} + g\phi_z = 0$ at $z = 0$

using (17) $T'' \cosh[\gamma h] + g\gamma \sinh[\gamma h] T = 0$

$T'' + \underbrace{g\gamma \tanh(\gamma h)}_{\omega^2} \cdot T = 0$

$T'' + \omega^2 T = 0 \Rightarrow T = \cos(\omega t - \psi)$

$\omega^2 = g\gamma \tanh(\gamma h)$ $\gamma^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$

go back to (2) $\eta = -\frac{1}{g}\phi_t = A\frac{\omega}{g} \sin(\omega t - \psi) \cdot \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \cosh(\gamma h)$

3.4 $\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz} = 0$

(2) $\phi_t + g\eta = 0$
(3) $\eta_t - \phi_z = 0$ } @ $z = 0$

(4) $\phi_r = 0$ at $r = r_0$
(5) $\phi_z = 0$ at $z = -h$

Assume

$$\phi = T(t) \cdot R(r) \cdot \Theta(\theta) \cdot Z(z)$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \underbrace{\frac{\Theta''}{\Theta}}_{-\beta^2} + r^2 \underbrace{\frac{Z''}{Z}}_{\gamma^2} = 0$$

$Z'' - \gamma^2 Z = 0$

$\Theta'' + \beta^2\Theta = 0$

$$r^2R'' + rR' + (\gamma^2r^2 - \beta^2)R = 0$$

$$R'' + \frac{1}{r}R' + \left(\gamma^2 - \frac{m^2}{r^2}\right)R = 0$$

$\Theta = C_1 \cos(\beta\theta - \mu)$ phase angle

$\beta = m$ (integer) for continuity in θ.

$R = B_1 J_m(\gamma r) + B_2 Y_m(\gamma r) \Rightarrow B_2 = 0$ for finite solution at $r = 0$

$\phi_r = 0$ at $r = r_0 \Rightarrow R'(r_0) = 0$

$R_1' = B_1 \gamma J_m'(\gamma r)$ $\qquad J_m'(x) = 0$ at x $\qquad Z = \cosh[\gamma(z+h)]$ as in prob. 3.5

HMF PG411

m	$x_m = \gamma_{nm} r_0$	(ORDER)
0	3.83171	3
	7.01558	
	10.17346	
1	1.84118	1
	5.33144	5
	8.5362	
2	3.05424	2
	6.7061	
3.	4.20119	4
4	5.31755	

So, we now know the γ_{nm} values, corresponding to the γ_{nm} values in prob 3.5.

(2) & (3) $\Rightarrow$ $\phi_{tt} + g\phi_z = 0$ on $z = 0$

$$T'' \cosh[\gamma z] + g\gamma T \sinh(\gamma z) = 0$$

$$T'' + \underbrace{g\gamma \tanh(\gamma z)}_{\omega^2} T = 0 \qquad \underbrace{\omega^2 = g\gamma \tanh(\gamma h)}_{\text{frequency}}$$

$$T = A\cos(\omega t - \psi)$$

$$\Omega_{nm}^2 = \frac{\omega_{nm}^2 r_0}{g} = \gamma_{nm} r_0 \tanh(\gamma_{nm} r_0)$$

$$\Omega^2 = \frac{\omega^2 r_0}{g} = \gamma r_0 \tanh\left(\gamma r_0 \frac{h}{r_0}\right)$$

$$\Omega^2_{nm} = j'_{m,n} \tanh\left[j'_{m,n} \frac{h}{r_0}\right]$$

for $h/r_0 \to \infty$ $\quad \Omega^2_{nm} = j'_{nm}$

For mode n, m:

$$\eta = -\frac{1}{g}\phi_t = A\frac{\omega}{g}\cos(\omega t - \psi)\cos(m\theta - \mu)\, J_m(\gamma_{nm} r)$$

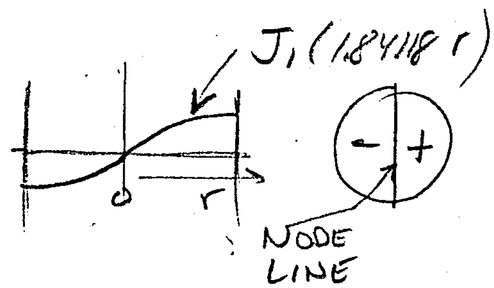
$$\frac{\eta}{\eta_a} = \frac{J_m(\gamma_{nm} r)}{J_m(\gamma_{nm} r_0)}$$

MODE SHAPES

① LOWEST MODE $\qquad \gamma r_0 = 1.84118$

$m = 1$, first root

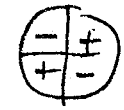
$\eta \sim \cos\theta \cdot J_1\left(1.84118 \frac{r}{r_0}\right)$



② SECOND MODE

$m = 2$, first root

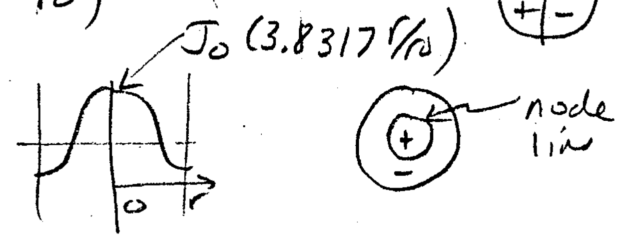
$\eta \sim \cos 2\theta \cdot J_2\left(3.05424 \frac{r}{r_0}\right)$



③ THIRD MODE

$m = 0$, first root

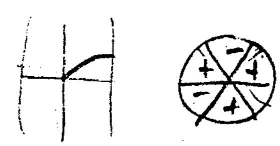
$\eta \sim J_0\left(3.8317 \frac{r}{r_0}\right)$



④ FOURTH MODE

$m = 3$, First root

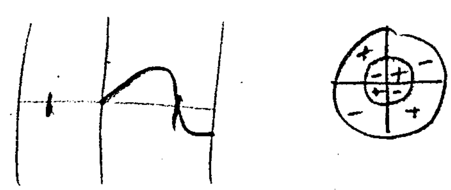
$\eta \sim \cos 3\theta\; J_3\left(4.202 \frac{r}{r_0}\right)$



⑤ FIFTH MODE

$m = 1$, second root

$\eta \sim \cos 2\theta \cdot J_1\left(5.3314 \frac{r}{r_0}\right)$



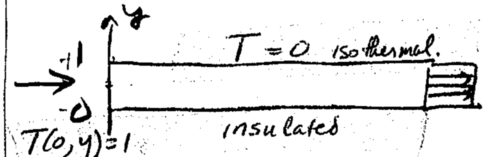
DO #1 & #3 WH-1

If time due #2 / if not give it to them to go home & do.

SLUG FLOW BETWEEN TWO WALLS

Reason

SOV



recognizing homogeneous proble

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial T}{\partial x} \quad (1) \qquad (2)\ T(x,y): \quad T(0,y) = 1 \quad (IC)$$

$$(3)\ T(x,1) = 0; \quad \left.\frac{\partial T}{\partial y}\right)_{y=0} = 0 \quad (4) \Big\} (BC)$$

what is $T(x,y)$

SOV. (HAVE THEM WORK THROUGH HOMOGENEOUS PROBLEM) $\underline{(1) + (3)}$ + (4)

Let them solve

what is homogeneous probl (3) & (4) w PDE

$$T = X(x) \cdot Y(y) \qquad \frac{Y''}{Y} = \frac{X'}{X} = -\lambda^2$$

why $-\lambda^2$?

$$Y'' + \lambda^2 Y = 0 \qquad X' + \lambda^2 X = 0$$

$$X = e^{-\lambda^2 x}$$

$$Y = C_1 \sin(\lambda y) + C_2 \cos(\lambda y)$$

$$(4) \Rightarrow Y'(0) = 0 \Rightarrow C_1 = 0 \qquad Y = C\cos(\lambda y)$$

$$(3) \Rightarrow Y(1) = 0 \Rightarrow \cos(\lambda) = 0 \qquad \lambda = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots (2n-1)\frac{\pi}{2}$$

$$T_n = \cos\left[(2n-1)\frac{\pi y}{2}\right] e^{-(2n-1)^2 \pi^2/4\, x}$$

NOTE - eigensolutions decay as $x \to \infty$

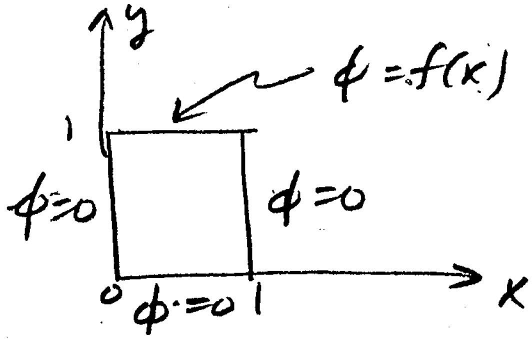
$$T_n(0,y) = \cos\left[(2n-1)\frac{\pi}{2} y\right]$$

We would satisfy initial conditions by

$$T(x,y) = \sum_{n=1}^{\infty} C_n T_n(x,y) = \sum_{n=1}^{\infty} C_n \cos\left[(2n-1)\frac{\pi}{2} y\right] e^{-\left[(2n-1)\frac{\pi}{2}\right]^2 x}$$

$$T(0,y) = 1 = \sum_{n=1}^{\infty} C_n \cos\left[(2n-1)\frac{\pi}{2} y\right]$$

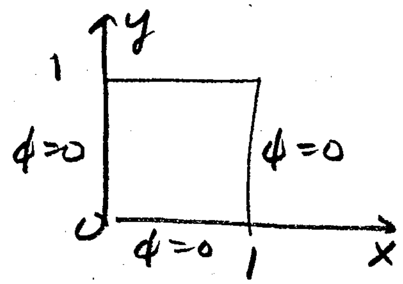
ORTHOGONALITY WOULD BE USED TO GET C_n. (will discuss next week).



POTENTIAL PROBLEM

$$\nabla^2 \phi = 0 = \phi_{xx} + \phi_{yy}$$

HOMO PROBLEM



$$\phi = X(x) \cdot Y(y)$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \underbrace{-\lambda^2}_{\text{why?}}$$

$$X'' + \lambda^2 X = 0 \qquad Y'' - \lambda^2 Y = 0$$

$$X = C_1 \sin \lambda x + \cancel{C_2 \cos \lambda x} \qquad Y = A_1 \sinh \lambda y + \cancel{A_2 \cosh \lambda y}$$

$C_2 = 0$ for $X(0) = 0$ $\qquad$ $A_2 = 0$ FOR $Y(0) = 0$

$X(1) = 0 \Rightarrow \sin(\lambda \cdot 1) = 0$

$\lambda = n\pi$

$$\phi_n = \sin(n\pi x) \sinh(n\pi y)$$

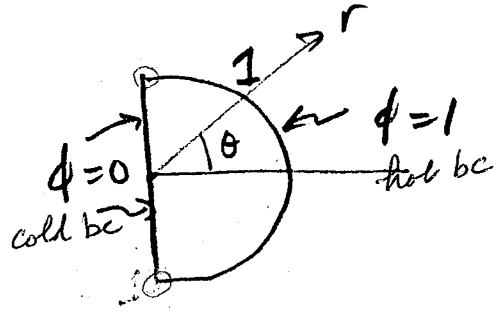
soln to homogeneous problem

For inhomogeneous problem, we would take

$$\phi = \sum_{n=1}^{\infty} A_n \phi_n = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh(n\pi y)$$

$$\phi(x,1) = f(x) = \sum_{n=1}^{\infty} A_n \sinh(n\pi) \cdot \sin(n\pi x)$$

ORTHOGONALITY WOULD BE USED TO GET A_n.



Steady State heat conduction
ϕ - non dimensional temperature
r - non dimensional distance
$-\frac{\pi}{2} \le \theta \le \pi/2$

$$\nabla^2 \phi = 0$$
$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = 0$$
$$\phi(r, -\tfrac{\pi}{2}) = \phi(r, \tfrac{\pi}{2}) = 0$$
$$\phi(1, \theta) = 1$$

what is the **HOMO PROBLEM**

$$\nabla^2 \phi = 0 \qquad \phi(r, -\tfrac{\pi}{2}) = \phi(r, \tfrac{\pi}{2}) = 0$$

SOLUTIONS:

Solve by SOV

$$\phi = R(r) \cdot \Theta(\theta)$$

choice of λ^2 driven by BC.

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2$$

$$\Theta'' + \lambda^2 \Theta = 0 \rightarrow \Theta = C_1 \sin(\lambda\theta) + C_2 \cos(\lambda\theta)$$

$$\Theta(-\tfrac{\pi}{2}) = 0 \Rightarrow \quad -C_1 \sin(\tfrac{\pi}{2}\lambda) + C_2 \cos(\lambda\tfrac{\pi}{2}) = 0$$

$$\Theta(+\tfrac{\pi}{2}) = 0 \Rightarrow \quad C_1 \sin(\tfrac{\pi}{2}\lambda) + C_2 \cos(\tfrac{\lambda\pi}{2}) = 0$$

what is soln to this set of Linear Hom. algebraic eqn

$$\begin{vmatrix} -\sin(\frac{\lambda\pi}{2}) & \cos(\frac{\lambda\pi}{2}) \\ \sin(\frac{\lambda\pi}{2}) & \cos(\frac{\lambda\pi}{2}) \end{vmatrix} = 0$$

CHARACTERISTIC DETERMINANT

$$-2\sin(\tfrac{\lambda\pi}{2})\cos(\tfrac{\lambda\pi}{2}) = 0 = -\sin(\lambda\pi)$$

$$\lambda\pi = n\pi \qquad \text{so } \underline{\lambda = n}. \text{ integer}$$

What is C_1 & C_2
only 1 eqn is L.I.

$$-C_1 \sin(\tfrac{n\pi}{2}) + C_2 \cos(\tfrac{n\pi}{2}) = 0$$

2 families of Soln.

for $n = 1, \quad C_1 = 0$
$n = 2 \quad C_2 = 0$

$$\Theta = \begin{cases} \sin(n\theta) & \text{even } n \\ \cos(n\theta) & \text{odd } n \end{cases}$$

$$r^2R'' + rR' - n^2R = 0$$

NOT Bessels eqn. It is equidimensional (homogeneous in $\underline{\underline{r}}$)

$$r^{2}\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R = 0$$

let $\tilde{r} = cr$ then $\tilde{r}^2\frac{d^2R}{d\tilde{r}^2} + \tilde{r}\frac{dR}{d\tilde{r}} - n^2R = \frac{c^2r^2}{c^2}\frac{d^2R}{dr^2} + \frac{cr}{c}\frac{dR}{dr} - n^2R = 0$

Assume $R = r^p$

$$p(p-1)\cancel{r^p} + p\cancel{r^p} - n^2\cancel{r^p} = 0$$

$$p^2 - \cancel{p} + \cancel{p} - n^2 = 0$$

so, $p = \pm n$ remember n is integer

$$R = B_1 r^n + \underbrace{B_2 r^{-n}}$$

Blows up at $r = 0$ so $B_2 = 0$

$$\phi_n = r^n \begin{Bmatrix} \sin(n\theta) & \text{even } n \\ \cos(n\theta) & \text{odd } n \end{Bmatrix}$$ Homogeneous problem solutions

$$\phi = \sum_{n=1}^{\infty} A_n \phi_n(r,\theta)$$

The A_n would be determined by orthogonality

even though problem is inhomog the sol will always involve homog.

fractional orders will always be L.I.

Suggestions - use J_ν, Y_ν

I_ν, K_ν

Ramesh - speak very quickly, find you hard to understand

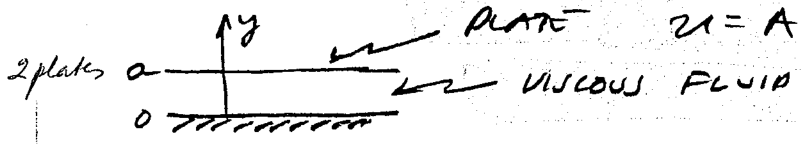
Some would like a "feel" for problem

$t = .003$ $dt = .005$

$t = .003$ $dt = .007$

$t = .015$ $dt = .007$

Periodic AND WAVE SOLUTIONS



$u = A\cos\omega t$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\nu = \mu/\rho$$

COMPLEX IMBED

Assume complex $\underline{u}$ — good for linear problems.

$$u_t = \nu u_{yy} \qquad u(0,t) = 0 \qquad u(a,t) = \underline{A}\, e^{i\omega t}$$

↑ since this const doesn't depend on t

assume $u(y,t) = F(y)\, e^{i\omega t}$

$$i\omega F = \nu F''$$

$$F'' - \frac{i\omega}{\nu} F = 0 \qquad F(0) = 0 \quad F(a) = A$$

$$F = C_1 \sinh\left(\sqrt{\frac{i\omega}{\nu}}\, y\right) + C_2 \cosh\left(\sqrt{\frac{i\omega}{\nu}}\, y\right)$$

since 0 b.c.

$\sinh(0) = 0 \quad \cosh(0) = 1 \quad$ so $\quad C_2 = 1$ by $F(0) = 0$.

$$F = C \sinh\left(\sqrt{\frac{i\omega}{\nu}}\, y\right)$$

$$F(a) = A = C \sinh\left(\sqrt{\frac{i\omega}{\nu}}\, a\right) \qquad C = \frac{A}{\sinh\left(\sqrt{\frac{i\omega}{\nu}}\, a\right)}$$

$$F = \frac{A \sinh\left(\sqrt{\frac{i\omega}{\nu}}\, y\right)}{\sinh\left(\sqrt{\frac{i\omega}{\nu}}\, a\right)}$$

best way to get sol from this point - use computer.

Shear stress at upper surface: $\displaystyle \tau = \mu \left.\frac{\partial u}{\partial y}\right)_{y=a}$

COMPLEX IMBED:

$$\tau = \mu e^{i\omega t} \cdot F'(a) = \mu e^{i\omega t} \frac{\sqrt{\frac{i\omega}{\nu}} \cosh\left(\sqrt{\frac{i\omega}{\nu}}\, a\right)}{\sinh\left(\sqrt{\frac{i\omega}{\nu}}\, a\right)}$$

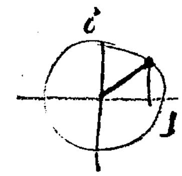
$$\tau = \sqrt{i\omega\mu\rho}\, e^{i\omega t} \frac{1}{\tanh\left(\sqrt{\frac{i\omega}{\nu}}\, a\right)}$$

HOW TO EVALUATE THE REAL PART?

Need to define i $\quad i = e^{i\pi/2}$ (UNIQUE: NO $n\pi$)

$$\sqrt{i} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$$

$$\tau = \sqrt{\omega\mu\rho} \times e^{i(\pi/4 + \omega t)} \; \underbrace{\frac{e^{\sqrt{\frac{i\omega}{\nu}}a} + e^{-\sqrt{\frac{i\omega}{\nu}}a}}{e^{\sqrt{\frac{i\omega}{\nu}}a} - e^{-\sqrt{\frac{i\omega}{\nu}}a}}}_{B}$$

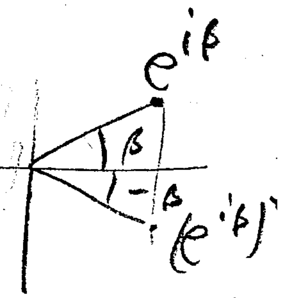


$$e^{\sqrt{\frac{i\omega}{\nu}}a} = e^{\sqrt{\frac{\omega}{\nu}}a \cdot e^{i\pi/4}} = e^{\sqrt{\frac{\omega}{\nu}}a\,(1+i)/\sqrt{2}}$$

$$= e^{\sqrt{\frac{\omega}{2\nu}}a}\, e^{i\sqrt{\frac{\omega}{2\nu}}a} = e^{\beta} e^{i\beta} \qquad \beta = \sqrt{\frac{\omega}{2\nu}}\, a$$

$$B = \frac{e^{\beta}e^{i\beta} + e^{-\beta}e^{-i\beta}}{e^{\beta}e^{i\beta} - e^{-\beta}e^{-i\beta}} \times$$

Rationalize denom



let $B = \frac{a+bi}{c+di}\,\frac{(c-di)}{c-di} = \frac{}{c^2+d^2}$

$$\left(e^{i\beta}\right)^{*} = e^{-i\beta} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$B = \frac{e^{\beta}e^{i\beta} + e^{-\beta}e^{-i\beta}}{e^{\beta}e^{i\beta} - e^{-\beta}e^{-i\beta}} \times \left(\frac{e^{\beta}e^{-i\beta} - e^{-\beta}e^{+i\beta}}{e^{\beta}e^{-i\beta} - e^{-\beta}e^{+i\beta}}\right)$$

$$= \frac{e^{2\beta} - e^{2i\beta} + e^{-2i\beta} - e^{-2\beta}}{e^{2\beta} - e^{2i\beta} - e^{-2i\beta} + e^{-2\beta}} = \frac{\sinh(2\beta) - i\sin(2\beta)}{\cosh(2\beta) - \cos(2\beta)}$$

$$\underbrace{\cos(\omega t+\pi/4) + i\sin(\omega t+\pi/4)}$$

So,

$$\tau = \frac{A\sqrt{\omega\rho\mu}\; e^{i(\underbrace{\pi/4}_{\text{from }\sqrt{i}}+\omega t)}}{\cosh(2\beta)-\cos(2\beta)}\left[\sinh(2\beta) - i\sin(2\beta)\right]$$

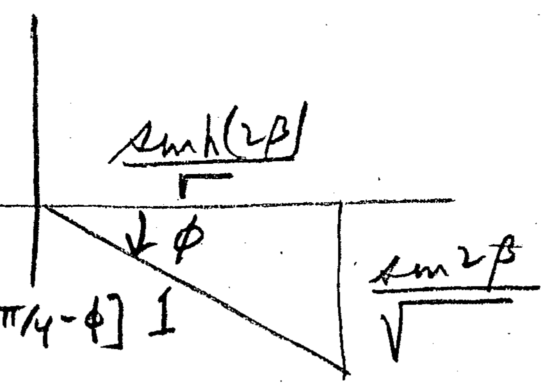
rewrite $A+Bi = \frac{A+Bi}{\sqrt{A^2+B^2}}\sqrt{A^2+B^2} = e^{i\phi}\sqrt{A^2+B^2}$

the real part is then

$$\boxed{\tau = \frac{A\sqrt{\omega\rho\mu}}{\cosh(2\beta)-\cos(2\beta)}\left[\cos\left(\omega t+\frac{\pi}{4}\right)\sinh(2\beta) + \sin\left(\omega t+\frac{\pi}{4}\right)\sin(2\beta)\right]}$$

$$\tau = \frac{A\sqrt{\omega\rho\mu}}{\cosh(2\beta)-\cos(2\beta)}\overbrace{\left[\frac{\sinh(2\beta) - i\sin(2\beta)}{\sqrt{\sinh^2(2\beta)+\sin^2(2\beta)}}\right]}^{e^{i\phi}}\sqrt{\sinh^2(2\beta)+\sin^2(2\beta)}$$

$$\times\, e^{i(\pi/4+\omega t)}$$

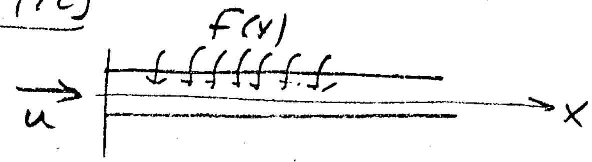


$$\tau = \frac{A\sqrt{\omega\rho\mu\left[\sinh^2(2\beta)+\sin^2(2\beta)\right]}}{\cosh(2\beta)-\cos(2\beta)}\underbrace{e^{i[\omega t+\pi/4-\phi]}}_{\cos\,+\,i\sin}$$

$$\downarrow$$

$$\tau_{\text{real}} = \qquad\qquad \cdot\cos\left[\omega t+\frac{\pi}{4}-\phi\right]$$

WORKSHOP ON CHARACTERISTICS



(1) $c_t + \bar{u}\, c_x = X$

(2) $c(0,t) = 1 \quad t \geq 0 \quad$ B.C.

(3) $c(x,0) = 0 \quad x > 0 \quad$ I.C.

TA Explain the idea — want to transform to ODE on lines. Need to find the lines.

SOLVE BY CHARACTERISTICS

$\eta = \eta(x,t) \qquad \xi = \xi(x,t)$

$c_\eta \eta_t + c_\xi \xi_t + \bar{u}\,(c_\eta \eta_x + c_\xi \xi_x) = X$

(4) $c_\eta(\eta_t + \bar{u}\,\eta_x) + c_\xi(\xi_t + \bar{u}\,\xi_x) = X$

Have them do this. (take 5 min if required)

Pick η, ξ such that this term is zero on a line of constant η. Then, on a line of constant η,

$$\left.\frac{\partial c}{\partial \xi}\right)_{\eta} = \left[\frac{X}{\xi_t + \bar{u}\,\xi_x}\right] = f(\xi, \eta)$$

CAN BE INTEGRATED AT CONSTANT η.

TA

HOMO LIN. EQNS FOR η_t, η_x.

$\eta_t + \bar{u}\,\eta_x = 0$ on a characteristic

$d\eta = \eta_t\, dt + \eta_x\, dx = 0$ on a line of constant η

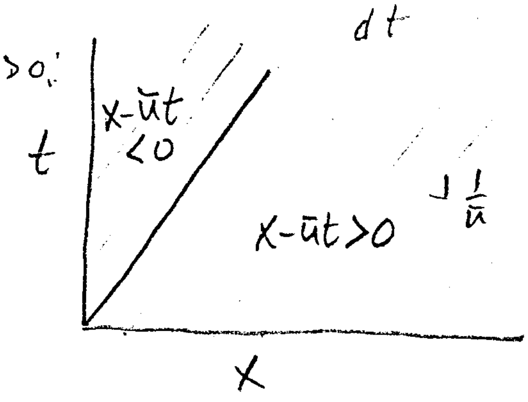
GET THEM TO REALIZE THIS

$$\begin{vmatrix} 1 & \bar{u} \\ dt & dx \end{vmatrix} = 0$$

on a line of constant η (a "characteristic" line)

$dx = \bar{u}\, dt \qquad \dfrac{dx}{dt} = \bar{u}$ defines slopes of the characteristics

if $\bar{u} = \text{const} > 0$:



TA GET THEM TO DRAW THIS

($\bar{u}$ = const)

on characteristic, $\frac{dx}{dt} = \bar{u}$ = const, so $x = \bar{u}t$ + const.

This is a line of constant η. So lets choose

$$\boxed{\eta = x - \bar{u}t}$$

We need to choose ξ that intersects with this.
So, lets choose

$\xi = x$ since the rhs of our eqn is just x

Then, (4) →

$$c_\eta[-\bar{u} + \bar{u}] + c_\xi[0 + \bar{u}] = \xi$$

and our eqn is

$$\frac{\partial c}{\partial \xi} = \frac{\xi}{\bar{u}}$$

(5) $$c = \frac{\xi^2}{2\bar{u}} + g(\eta) = \frac{x^2}{2\bar{u}} + g(x - \bar{u}t)$$

SOLUTION.

Discuss how c is carried along the characteristics by g.
Now we must fit the BC and IC.

(a), for $x - ut < 0$, characteristics intercept $x = 0$ line

(2) ⇒ $1 = 0 + g(-\bar{u}t)$ ⇒ $g(\xi) = 1$

So, along these characteristics,

$$\boxed{c = \frac{x^2}{2\bar{u}} + 1 \quad \text{for } t > \frac{x}{\bar{u}} \quad \text{steady state}}$$

(b) for $x - ut > 0$ characteristics intercept $t = 0$ line

(3) → $$0 = \frac{x^2}{2\bar{u}} + g(x - 0)$$

$$g(x) = -\frac{x^2}{2\bar{u}} \quad \text{so} \quad g(\xi) = -\frac{\xi^2}{2\bar{u}}$$

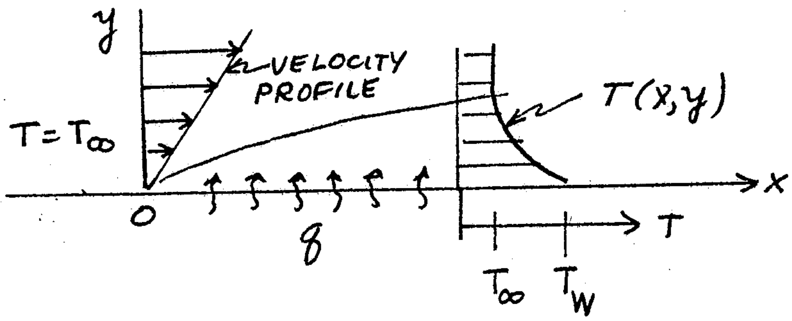
$$\boxed{c = \frac{x^2}{2\bar{u}} - \frac{(x - \bar{u}t)^2}{2\bar{u}} \quad \text{for } t < \frac{x}{\bar{u}} \quad \text{transient}}$$

Note that a discontinuity (c jumps from 0 to 1) moves on $x - \bar{u}t = 0$

ME 200B
Winter '80
WCR

Problem 1 -- Due January 21

A heat transfer problem of interest near the entrance of ducts, and in hot-film gages for wall shear stress measurement, is described mathematically by



$$\alpha\frac{\partial^2 T}{\partial y^2} = \beta y\frac{\partial T}{\partial x} \qquad (1)$$

$$T \to T_\infty \quad \text{as} \quad y \to \infty \qquad (2a)$$

$$T \to T_\infty \quad \text{as} \quad x \to 0 \qquad (2b)$$

$$-k\frac{\partial T}{\partial y} = q \quad \text{at} \quad y = 0 \quad \text{for} \quad x \geq 0 \qquad (2c)$$

Here β is the velocity gradient at the wall, q is the local wall heat flux, α is the thermal diffusivity in the fluid, and k is the thermal conductivity. These parameters are all constant.

(a) Develop the self-similar solution to this problem. Choose your variables such that the ODE is

$f''' + 2\eta f' + \eta^2 f'' - \eta f' - f = 0$

$f'' = \eta f - \eta^2 f'$
$= \eta(f - \eta f')$

$$f'' + \eta^2 f' - \eta f = 0$$

$\eta = e^z \qquad z = \ln\eta$

$\frac{d}{d\eta} = \frac{d}{dz}\cdot\frac{1}{\eta}$

and

$f''' + \eta f'(1-\eta^3) + (\eta^3 - 1)f = 0$

$$f'(0) = -1, \qquad f(\infty) = 0$$

(b) Solve the ODE problem analytically. Use the fact that one solution of the ODE is $f = \eta$. Express the second solution in terms of integrals. Use integration by parts to get this integral into a manageable form.

(c) Develop an expression for $T_w - T_\infty$ as a function of x, where $T_w = T(x,0)$. Evaluate the constant in this expression, first in terms of gamma functions, and then give its numerical value.

(d) Using the ODE equation routine at LOTS, or any other routine for solving a system of simultaneous first-order ODEs, solve the ODE problem numerically. To do this, generate a solution to the ODE satisfying $f'(0) = 0$, $f(0) = 1$. Then, take a linear combination of this solution and the solution η to construct a solution satisfying the boundary conditions. Check the value of $f(0)$ determined numerically with the value from your analytical solution.

(e) Plot the self-similar solution in a form that renders it as a single curve (something v.s. η). This curve would be useful in determining the degree to which the thermal boundary layer penetrates the flow.

$A f_1(\eta) + B f_2(\eta) \qquad f(\eta)$

$f(\infty)$

$$\frac{d}{d\eta}\left(\frac{f}{\eta}\right) = \frac{\eta f' - f}{\eta^2}$$

$$\eta^2 (f/\eta)' = \eta f' - f$$

$$\eta^3 (f/\eta)' = \eta^2 f' - \eta f$$

$$f'' + \eta^3 (f/\eta)' \underset{=0}{=} f'' + \eta^2 f' - \eta f = 0$$

$$\frac{f''}{\eta^3} = -(f/\eta)'$$

$f'' d\eta = dv \quad \frac{1}{\eta^3} = u$

$$\int \frac{f''}{\eta^3} d\eta = -f/\eta + c$$

$v = f' \quad -3\eta^{-4} d\eta = du$

$$f'/\eta^3 \Big|_0^\eta - \int f'$$

Feel free — Right or wrong, don't ~~help them~~ do proofs for them.

CONFIDENTIAL - TUTORS/TA'S ONLY!

Problem 1 Solution

$$\alpha \frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \quad (1) \qquad T \to T_\infty \text{ as } y \to \infty$$

$$T \to T_\infty \text{ as } x \to 0$$

$$-k \frac{\partial T}{\partial x} = q \text{ @ } y = 0$$

assume

$$T = T_\infty + Bx^m f(\eta) \qquad \eta = Ay/x^n$$

$$\frac{\partial T}{\partial x} = Bmx^{m-1} f + Bx^m f' \left(-n\frac{Ay}{x^{n+1}}\right) = Bx^{m-1}\left[mf - n\eta f'\right]$$

$$\frac{\partial T}{\partial y} = Bx^m f' \frac{A}{x^n}$$

since $\left.\frac{\partial T}{\partial y}\right|_{y=0} = \text{CONST}$, $m = n$ ($f'(0)$ = NUMBER)

$$\frac{\partial^2 T}{\partial y^2} = BAf'' \frac{A}{x^n}$$

$$(1) \Rightarrow \alpha BAf'' \frac{A}{x^n} = \beta y \cdot Bx^{n-1} n\left[f - \eta f'\right]$$

$$y = \eta x^n / A$$

$$\alpha BAf''A = x^n \beta\left(\eta \frac{x^n}{A}\right) Bnx^{n-1}\left[f - \eta f'\right]$$

$3n - 1 = 0$ for x to drop out $n = 1/3$

$$\frac{\alpha A^3}{\beta} f'' + \frac{1}{3}\left[\eta^2 f' - \eta f\right] = 0$$

Pick

$$\frac{\alpha A^3}{\beta} = 1/3 \qquad A = \sqrt[3]{\frac{\beta}{3\alpha}}$$

$$f'' + \eta^2 f' - \eta f = 0 \qquad f(\infty) = 0$$

$$q = -kABf'(0) \qquad \text{pick } f'(0) = -1$$

$$B = \frac{q}{kA} = \frac{q}{k}\sqrt[3]{\frac{3\alpha}{\beta}}$$

So, the form of the self-similar solution is

$$\boxed{\begin{aligned} &T = T_\infty + \frac{q}{k}\sqrt[3]{\frac{3\alpha x}{\beta}}\, f(\eta) \\ &f'' + \eta^2 f' - \eta f = 0 \qquad f(\infty) = 0 \qquad f'(0) = -1 \end{aligned}}$$

$f'' = \eta f - \eta^2 f'$
$\phi_2' = \eta\phi_1 - \eta^2\phi_2$
$\phi_1' = \phi_2$

SOLVE ODE: ONE SOL'N IS $f_1 = \eta$

$f_2 = \eta g(\eta) \qquad f_2' = g + \eta g' \qquad f_2'' = 2g' + \eta g''$

$$2g' + \eta g'' + (\cancel{\eta^2 g} + \eta^3 g') - \cancel{\eta^2 g} = 0$$

$$\frac{g''}{g'} = -\left(\eta^2 + \frac{2}{\eta}\right) \qquad \ln g' = -\frac{\eta^3}{3} - 2\ln\eta$$

$$g' = \frac{1}{\eta^2} e^{-\eta^3/3}$$

so, $g = \displaystyle\int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^3/3}\, d\sigma$

and $f_2 = \eta \displaystyle\int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^3/3}\, d\sigma$ is a second solution

Integral blows up as $\eta \to 0$. To fix this, we integrate by parts

$$f_2 = \eta\left[-\frac{1}{\sigma} e^{-\sigma^3/3}\Big|_\infty^\eta - \int_\infty^\eta \left(-\frac{1}{\sigma}\right)\left(-\sigma^2 e^{-\sigma^3/3}\right) d\sigma\right]$$

$$= -e^{-\eta^3/3} - \eta\int_\infty^\eta \sigma\, e^{-\sigma^3/3}\, d\sigma$$

Now, $f = C_1 f_1 + C_2 f_2 \qquad C_1 = 0$ for $f \to 0$ as $\eta \to \infty$

$$f = C_2\left[\eta\int_\eta^\infty \sigma e^{-\sigma^3/3}\, d\sigma - e^{-\eta^3/3}\right]$$

$f'(0) = -1$

$$f'(0) = C_2\left[\int_0^\infty \sigma e^{-\sigma^3/3}\, d\sigma - \cancel{\eta \cdot \eta e^{-\eta^3/3}}\Big|_{\eta \to 0} - \left(\cancel{-\eta^2 e^{-\eta^3/3}}\right)_{\eta = 0}\right.$$

TO EVALUATE THE INTEGRAL LET $t = \sigma^3/3 \qquad \sigma = \sqrt[3]{3t} \qquad d\sigma = \sqrt[3]{3}\,\frac{1}{3} t^{-2/3}\, dt$

$$\int_0^\infty \sigma e^{-\sigma^3/3}\,d\sigma = \int_0^\infty 3^{1/3} t^{1/3} e^{-t} \cdot 3^{-2/3} t^{-2/3}\,dt$$

$$= 3^{-1/3}\int_0^\infty t^{-1/3} e^{-t}\,dt = 3^{-1/3}\,\Gamma\left(\tfrac{2}{3}\right) \quad \text{(HMF 6.1.1)}$$

$$= 3^{-1/3} \times 1.3541179394 \quad \text{(HMF 6.1.13)}$$

$$= 0.938893$$

so,

$$-1 = C_2 \times 0.938893 \qquad C_2 = -1.06508 = -\frac{1}{3^{-1/3}\,\Gamma(2/3)}$$

$$f = \frac{3^{1/3}}{\Gamma(\frac{2}{3})}\left[e^{-\eta^3/3} - \eta\int_\eta^\infty \sigma e^{-\sigma^3/3}\,d\sigma\right]$$

WE CAN EXPRESS THIS AS

$$f(\eta) = \frac{3^{1/3}}{\Gamma(2/3)}\left[e^{-\eta^3/3} - \eta\, 3^{-1/3}\int_{\eta^3/3}^\infty t^{-1/3} e^{-t}\,dt\right]$$

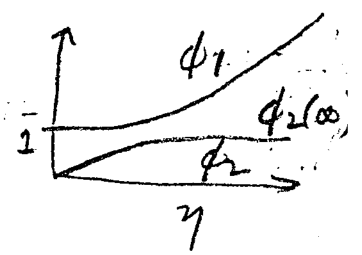
$$= \frac{3^{1/3}}{\Gamma(2/3)}\left[e^{-\eta^3/3} - \eta\, 3^{-1/3}\,\Gamma\left(\tfrac{1}{3}, \eta^3/3\right)\right] \quad \text{(HMF 6.5.3)}$$

$$f(0) = \frac{3^{1/3}}{\Gamma(2/3)} = 1.06508$$

$$T_w - T_\infty = \frac{q}{k}\sqrt[3]{\frac{\alpha x}{\beta}} \times \frac{3^{2/3}}{\Gamma(2/3)} = \underline{1.536\,\frac{q}{k}\left(\frac{\alpha x}{\beta}\right)^{1/3}}$$

<u>NUMERICAL SOLUTION FOR f</u>

$$\phi_1 = f_3 \qquad \phi_2 = f_3'$$



$$\boxed{\begin{array}{ll} \phi_2' = -\eta^2\phi_2 + \eta\,\phi_1 & \phi_1(0) = 1 \\ \phi_1' = \phi_2 & \phi_1'(0) = 0 \end{array}}$$

THEN, TAKE $f(\eta) = C_3 f_3 + C_4\,\eta$

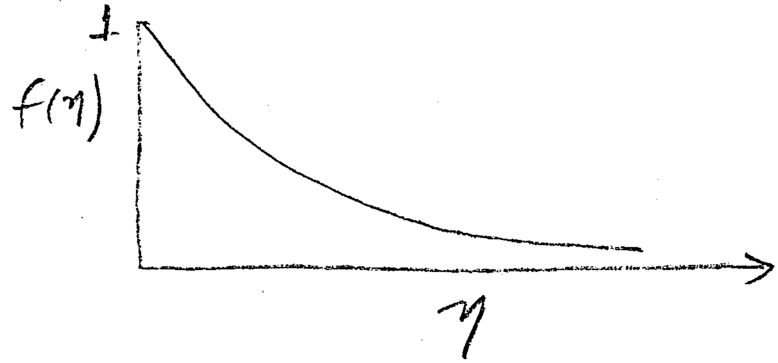
$f'(0) = -1 \rightarrow \quad c_4 = -1$

$f'(\infty) \rightarrow 0 \rightarrow \quad c_3\,\phi_2(\infty) + c_4 = 0 \qquad c_3 = 1/\phi_2(\infty)$

Then, $\boxed{f = \dfrac{1}{\phi_2(\infty)}\,\phi_1 - \eta}$

makes no sense
since $\phi_2(\infty) = f'(\infty) = 0$

PLOT $f(\eta)$



use tolerances 1×10^{-6}, 1×10^{-6}

ME ~~270~~ 200B
WCR
Winter 1980

Problem #2, due: February 4

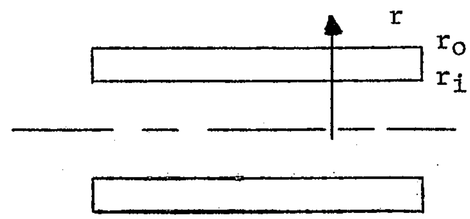
Work Problem 3.5 in the notes. You may skip Part C. Sketch the mode shapes and node lines for a square tank for the first five modes.

Now work Problem 3.4 in the notes. Do the experiment requested in Part D, using a ruler and a watch as the basic measuring instruments.

ME 200B
Winter 1980
WCR

Problem Set 3, Due: February 25

Aluminum is to be cast in the form of long annular billets, shown below:



The billets will be cooled by natural convections to surrounding air. The transient temperature history in the billet is described by:

$$\alpha \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) = r \frac{\partial T}{\partial t} \qquad (1)$$

where r is the radial coordinate, t is time, T is temperature, and α is the (constant) thermal diffusivity of the aluminum.

The boundary conductions are

$$\frac{\partial T}{\partial r} = 0 \quad \text{at} \quad r = r_i \qquad (2)$$

(negligible heat loss to the inside air)

$$-k \frac{\partial T}{\partial r} = h(T - T_\infty) \quad \text{at} \quad r = r_0 \qquad (2)$$

(convective energy balance on the outside surface)

The initial condition is

$$T(r,0) = T_o \qquad (3)$$

Your job is to develop the analytical solution to this problem, and to prepare a non-dimensional graph describing the surface temperature $T_s = T(r_o,t)$ and the inside temperature $T_i = T(r_i,t)$ as functions of time, for use in engineering analysis.

1. Note that $T = T_\infty$ is a particular solution of (1) that also satisfies (2). This suggests that the solution should be saught in the form

$$T(r,t) = \phi(r,t) + T_\infty$$

Develop the ϕ problem. State the PDE, BC's, and IC for ϕ.

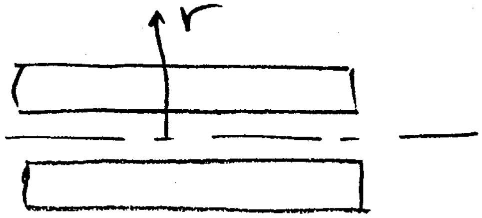
2. Consider next the homogeneous sub-problem for ϕ, formed by the PDE, and homogeneous boundary conditions. Develop the eigensolutions for this problem using SOV, with $\phi_n = R_n(r) \cdot F_n(t)$. Choose your constants so that R satisfies $(rR')' + \lambda^2 rR = 0$. Derive the characteristic determinant $D(\lambda r_o;\ r_o/r_i)$ from which the eigenvalues will be determined. This will involve both J and Y Bessel functions.
   HINT: See HMF 9.1.28.

3. Denoting the eigenfunctions as R_n, derive the orthogonality property of the eigenfunctions needed to determine the coefficients in the eigenfunction expansion of the problem solution. Then, construct the solution to the full problem, including the initial condition. Express the constants in this expansion in terms of the R_n. (It is easier not to write R_n in terms of Bessel functions at this point.) Derive an expression for the denominator integral in the expansion coefficient using the methods developed in the notes (and class).

4. For the case $r_o/r_i = 2$, $B = (r_o h/k) = 1$, calculate and plot $D(\lambda r_o)$ v.s. λr_o and find at least the first two roots of this equation. You can do this by hand, using HMF, or you can do it on LOTS using my Bessel function routine and root finding routines. If you use LOTS, you might as well find at least five roots. Then, determine the constants multiplying J and Y in the solution. Normalize the eigenfunctions so that $R_n(r_i) = 1$. Plot the first two eigenfunctions v.s. r/r_o for this case, using a few points.

5. Finally, for the case of part 4 above, derive expressions for $(T_i - T_\infty)/(T_o - T_\infty)$ and $(T_s - T_\infty)/(T_o - T_\infty)$ as functions of $t/(\alpha r_o^2)$. Plot these quantities on log - log paper, showing both the one and two-term approximations to the solutions. Graphs such as these, with B and r_o/r_i as parameters, would be the results of most utility in engineering analysis, and would be what you would produce if doing this problem for ALCOA.

HW



$$\alpha \frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = r\frac{\partial T}{\partial t} \quad (1)$$

$$\frac{\partial T}{\partial r} = 0 \ @\ r = r_i \quad (2)$$

$$-k\frac{\partial T}{\partial r} = h(T - T_\infty) \text{ at } r = r_o \quad (3)$$

$T = \phi(r,t) + T_\infty$ $\alpha\frac{\partial}{\partial r}(r\phi_r) = r\phi_t$ (4)

(5) $\phi_r = 0$ @ $r = r_i$ (6) $k\phi_r + h\phi = 0$ at $r = r$

(6) $\phi = R(r)\cdot F(t)$

(7) $$\frac{(rR')'}{rR} = \frac{T'}{\alpha T} = -\lambda^2$$

(8) $T' + \lambda^2\alpha T = 0$ $(rR')' + \lambda^2 rR = 0$ (9)

(10) $T = e^{-\alpha\lambda^2 t}$ $R = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$ (11)

(5) $\Rightarrow$ $c_1\lambda J_0'(\lambda r_i) + c_2\lambda Y_0'(\lambda r_i) = 0$ (12)

(6) $\Rightarrow$ $c_1\left[\lambda J_0'(\lambda r_o) + \frac{h}{k}J_0(\lambda r_o)\right] + c_2\left[\lambda Y_0'(\lambda r_o) + \frac{h}{k}Y_0(\lambda r_o)\right]$ (13)

For nontrivial c_i, $B = hr_o/k$

(14) $$\underbrace{\begin{vmatrix} J_0'[\lambda r_o/(r_o/r_i)] & Y_0'[\lambda r_o/(r_o/r_i)] \\ (\lambda r_o)J_0'(\lambda r_o) + BJ_0(\lambda r_o) & (\lambda r_o)Y_0'(\lambda r_o) + BY_0(\lambda r_o) \end{vmatrix}}_{D(\lambda r_o;\ r_o/r_i)} = 0$$

$X = \lambda r_o$ $a = r_i/r_o$ $B = hr_o/k$ $y = aX$

(15) $$D = J_0'(y)\left[XY_0'(X) + BY_0(X)\right] - Y_0'(y)\left[XJ_0'(X) + BJ_0(X)\right] = 0$$

for fixed a, B, roots where $X = 0$ $\Rightarrow$ the eigenvalues λ.

Roots found by WCR: 1.51197, 6.68660, 12.78031, 18.99395, 25.24152

HINT: IN FORTRAN,
REAL J0, J0P, ..

Now, after we know the λ_j from BC,

(16) $$\frac{C_2}{C_1} = -\frac{J_0'(\lambda r_i)}{Y_0'(\lambda r_i)}$$ ← only indep sol.

So,

(17) $$R_n = C_1\left[J_0(\lambda_n r) - \frac{J_0'(\lambda_n r_i)}{J_0'(\lambda_n r_i)} \times Y_0(\lambda_n r)\right]$$

Pick C_1 to make $R_n(r_i) = 1$

(18) $$1 = C_1\left[J_0(\lambda_n r_i) - \frac{J_0'(\lambda_n r_i)}{J_0'(\lambda_n r_i)} Y_0(\lambda_n r_i)\right] \Rightarrow C_1$$

Then $\phi_n = R_n \times \exp\{-\alpha\lambda_n^2 t\}$

(19) $$T = T_\infty + \sum_{n=\infty}^{\infty} B_n\phi_n = T_\infty + \sum_{n=1}^{\infty} B_n R_n e^{-\alpha\lambda_n^2 t}$$

(20) $$T_0 - T_\infty = \sum_{n=1} B_n R_n(r)\cdot 1 \quad \text{at } t=0$$

ORTHOGONALITY PROPERTY:

(21a) $$(rR_n')' + \lambda^2 r R_n = 0 \qquad \times R_m$$

(21b) $$(rR_m')' + \lambda_m^2 r R_m = 0 \qquad \times R_n$$

(22) $$\int_{r_i}^{r_0}\left\{R_m\left[(rR_n')'\right] - R_n\left[(rR_m')'\right]\right\}dr + (\lambda_n^2 - \lambda_m^2)\int_{r_i}^{r_0} rR_nR_m\,dr = 0$$

IBP,

(23) $$R_m(rR_n')\Big|_{r_i}^{r_0} - R_n(rR_m')\Big|_{r_i}^{r_0} + (\lambda_n^2-\lambda_m^2)\int_{r_i}^{r_0} rR_nR_m\,dr = 0$$

$R_n' = R_m' = 0$ at $r = r_i$ FROM (5)

$kR_n' + hR_n = 0$ at $r = r_0$ FROM (6)

$R_n' = -\frac{h}{k}R_n$ at $r = r_0$. So,

(24) $$-\frac{h}{k} r \cancel{R_m R_n}\Big|^{r_0} + \frac{h}{k} r \cancel{R_n R_m}\Big|^{r_0} + (\lambda_n^2-\lambda_m^2)\int_{r_i}^{r_0} rR_nR_m\,dr = 0$$

(25) $\int_{r_i}^{r_o} r R_n R_m \, dr = 0$ if $\lambda_n \neq \lambda_m$ ORTHOG

using (20),

(26) $$B_m = \frac{(T_o - T_\infty)\int_{r_i}^{r_o} r R_m \, dr}{\int_{r_i}^{r_o} r R_m^2 \, dr} = (T_o - T_\infty)\cdot\frac{N}{D}$$

(27) $$N = \int_{r_i}^{r_o} r R_m \, dr = -\int_{r_i}^{r_o} (r R_m')' \frac{1}{\lambda_m^2} dr = -\frac{1}{\lambda_m^2}(r R_m')\Big|_{r_i}^{r_o}$$

$$= -\frac{r_o R_m'(r_o)}{\lambda_m^2} \qquad \text{SINCE } R_m'(r_i) = 0$$

(28) $$D = \int_{r_i}^{r_o} r R_m^2 \, dr.$$

TO GET D, CONSIDER $R(r,\lambda)$ as solution of

(29a) $$(rR')' + \lambda^2 r R = 0 \qquad \frac{\partial R}{\partial r}(r_i) = 0 \quad (29b)$$

(30) $$\int_{r_i}^{r_o} R_m \cdot \frac{\partial}{\partial\lambda}(29)\, dr \Rightarrow$$

(31) $$\int_{r_i}^{r_o} R_m\left\{\frac{\partial}{\partial r}\left(r\frac{\partial^2 R}{\partial r\partial\lambda}\right) + \lambda^2 r\frac{\partial R}{\partial\lambda} + 2\lambda r R\right\} dr = 0$$

IBP.

(32) $$R_m\left(r\frac{\partial^2 R}{\partial r\partial\lambda}\right)\Big|_{r_i}^{r_o} - \int_{r_i}^{r_o} r\frac{\partial^2 R}{\partial r\partial\lambda} R_m' \, dr + \lambda^2\int_{r_i}^{r_o} r\frac{\partial R}{\partial\lambda} R_m \, dr + 2\lambda\int_{r_i}^{r_o} r R R_m \, dr = 0$$

(33) $\frac{\partial R}{\partial r} = 0$ at $r = r_i$ for all λ

Hence $\frac{\partial^2 R}{\partial r\partial\lambda} = 0$ @ $r = r_i$

(34) $$R_m r\frac{\partial^2 R}{\partial r\partial\lambda}\Big|^{r_o} - \frac{\partial R}{\partial\lambda}\cdot(r R_m')\Big|_{r_i} + \int_{r_i}^{r_o}\frac{\partial R}{\partial\lambda}\left[\cancelto{0}{(rR_m')' + \lambda^2 r R_m}\right] dr + 2\lambda\int_{r_i}^{r_o} r R R_m \, dr = 0$$

(35) $$R_m(r_o)\cdot r_o \left.\frac{\partial^2 R}{\partial r \partial \lambda}\right)_{\substack{r=r_o\\ \lambda=\lambda_m}} - \left.\frac{\partial R}{\partial \lambda}\right|_{\substack{r=r_o\\ \lambda=\lambda_m}} r_o R_m'(r_o) + 2\lambda_m \int_{r_i}^{r_o} r R_m^2 dr = 0$$

So,

(36) $$\int_{r_i}^{r_o} r R_m^2 dr = \frac{1}{2\lambda_m}\left\{ r_o R_m'(r_o) \left.\frac{\partial R}{\partial \lambda}\right)_{\substack{r=r_o\\ \lambda=\lambda_m}} - R_m(r_o) r_o \left.\frac{\partial^2 R}{\partial r \partial \lambda}\right)_{\substack{r=\\ \lambda=\lambda}}\right.$$

we have

(37) $$R = A_1 J_0(\lambda r) + A_2 Y_0(\lambda r)$$

(38) $$R'(r_i) = 0 \Rightarrow A_1 J_0'(\lambda r_i) + A_2 Y_0'(\lambda r_i) = 0$$

(39) $$A_2 = -A_1 \frac{J_0'(\lambda r_i)}{Y_0'(\lambda r_i)}$$

So,

(40) $$R = A_1\left[J_0(\lambda r) - \frac{J_0'(\lambda r_i)}{Y_0'(\lambda r_i)} \times Y_0(\lambda r)\right]$$

---

use A_1 from (18) as a fixed constant for each λ_n, independent of λ. This R (40) satisfies (29a, b)

Since $J_0' = -J_1$, $Y_0' = -Y_1$,

(41) $$R = A_1\left[J_0(\lambda r) - \frac{J_1(\lambda r_i)}{Y_1(\lambda r_i)} Y_0(\lambda r)\right]$$

$$\frac{1}{A_1}\frac{\partial R}{\partial \lambda} = r\left[J_0'(\lambda r) - \frac{J_1(\lambda r_i)}{Y_1(\lambda r_i)} Y_0'(\lambda r)\right]$$
$$+ r_i Y_0(\lambda r)\left[\frac{J_1'(\lambda r_i)}{Y_1(\lambda r_i)} - \frac{J_1'(\lambda r_i)}{Y_1^2(\lambda r_i)} Y_1'(\lambda r_i)\right]$$

COPY <W.WCR> BESJYN.FØR given $J_n(x)$, $J_n'(x)$, $Y_n(x)$, $Y_n'(x)$

declare J_n's to be real JN, JNP if values like 10^{10} not declaring real

<W.WCR> ROOTS.FØR

cmts, example.

main prog

EXTERNAL FUNC, ØSUB

CALL ROOTS (FUNC, XMIN, DX, XMAX, OSUB)

output subroutine

FUNCTION FUNC (x)

pick coeff of J to be = 1

ME 200B
FINAL EXAM SOLN
WINTER 80

$\boxed{/50}$

1. $T_{yy} = \beta y T_x$ (1) $T_y = 0$ $y \to 0$ (2) $T \to T_\infty$ as $y \to \infty$ and $x \to 0$ for $y > 0$ (3)

(4) $\int_0^\infty y(T-T_\infty)\,dy = E$

$T - T_\infty = A x^n f(\eta)$ $\eta = B y / x^m$ (5) $\boxed{/15}$ form $y = x^m \eta / B$

(1) $\Rightarrow$ $A x^n \left(\frac{B}{x^m}\right)^2 f'' = \beta y \cancel{A}\left[n x^{n-1} f - x^n \frac{m B y}{x^{m+1}} f'\right]$

$\frac{B^2}{\beta} f'' = x^{2m-n} \cdot \eta \frac{x^m}{B}\left[n x^{n-1} f - m x^{n-1} \eta f'\right]$ $\boxed{/10}$ use of eqn to find m

(4) $\Rightarrow$ $2m - \cancel{n} + m + \cancel{n} - 1 = 0$ $m = 1/3$.

$\int_0^\infty \frac{\eta x^m}{B} A x^n f(\eta) \cdot \frac{x^m}{B}\,d\eta = x^{2m+n} \frac{A}{B^2} \int_0^\infty \eta f(\eta)\,d\eta = E$

so $2m + n = 0$ $n = -2m = -2/3$. $\boxed{/10}$ use of integral in n determination

PICK $B^3/\beta = 1$ $\underline{f'' + \frac{1}{3}\eta^2 f' + \frac{2}{3}\eta f = 0}$ $f(\infty) = 0$ by (3)
$f'(0) = 0$ by (2)

SOLVE ODE;
CALCULATE $I = \int_0^\infty \eta f(\eta)\,d\eta$ $\boxed{/5}$ completion (A) $\boxed{/10}$ ode problem

CALC $A = E B^2 / I = E \beta^{2/3} / I$

NOTE; Need one BC on f not homogeneous.
pick $f(0) = 1$ to normalize

Here is the solution

$$3f'' + \frac{d}{d\eta}(\eta^2 f) = 0 \qquad 3f' + \eta^2 f = c_1$$
$c_1 = 0$ by $f'(0) = 0$

$$\frac{f'}{f} = -\frac{\eta^2}{3} \qquad f = c_2 e^{-\eta^3/9}$$

$c_2 = 1$ For normalization $f = e^{-\eta^3/9}$

2. $\phi_{xx} + \phi_{yy} + \phi_{zz} + \mu^2 \phi = 0$ (1)

$$\left.\begin{array}{l} \phi = 0 \text{ at } x=1, y=1 \\ \phi_z = 0 \text{ at } z = 1 \end{array}\right\} (2) \qquad \left.\begin{array}{ll} \phi_x = 0 & x=0 \\ \phi_y = 0 & y=0 \\ \phi_z = 0 & z = 0 \end{array}\right\} (3)$$

$\phi = X(x) \cdot Y(y) Z(z)$

$$\underbrace{\frac{X''}{X}}_{-\alpha^2} + \underbrace{\frac{Y''}{Y}}_{-\beta^2} + \underbrace{\frac{Z''}{Z}}_{-\gamma^2} + \mu^2 = 0 \qquad \mu^2 = \alpha^2 + \beta^2 + \gamma^2 \quad \boxed{\ /10} \text{ SOV}$$

$X'' + \alpha^2 X = 0 \qquad Y'' + \beta^2 Y = 0 \qquad Z'' + \gamma^2 Z = 0$

With (3),

$X = \cos(\alpha x) \qquad Y = \cos(\beta y) \qquad Z = \cos(\gamma z)$.

$\cos(\alpha) = 0 \qquad \cos(\beta) = 0 \qquad -\gamma \sin\gamma = 0$

$\alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \ \boxed{\ /10}$ EV $\quad \beta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \ \boxed{\ /10}$ EV $\quad \gamma = 0, \pi, 2\pi, \ldots \ \boxed{\ /10}$ EV

Smallest μ is for $\alpha = \pi/2$, $\beta = \pi/2$, $\gamma = 0$

$$\mu = \sqrt{\left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2}\right)^2 + 0} = \sqrt{2}\,\frac{\pi}{2} = \underline{\pi/\sqrt{2}}$$

eigenfunction $\quad \boxed{\ /10}$ first mode

$$\phi = \cos\left(\frac{\pi}{2} x\right) \cos\left(\frac{\pi}{2} y\right) \cdot 1$$

(No z dependence!)

$\boxed{/50}$ 3. $a^2 u_{xx} - u_{tt} = 0$ (1) $\quad u(x,0) = 0$ (2) $\quad u_t(x,0) = 0 \quad x < L$ (3) $\quad u(0,t) = 0$ (4)

$u(L,t) = B\sin(\Omega t)$ (5).

Particular solution to take care of (5)

(6) $u_p = F(x)\sin(\Omega t)$

(1) $\Rightarrow a^2 F'' + \Omega^2 F = 0 \qquad F'' + \Lambda^2 F = 0 \qquad \Lambda = \Omega/a$

(7) $F = C_1 \sin(\Lambda x) + C_2 \cos(\Lambda x)$ pick $F(0) = 0$ (see (4))

$F(1) = B$ (see (5))

Then $C_1 = \dfrac{B}{\sin(\Lambda L)} \qquad C_2 = 0$

$\boxed{/10}$ PART, SOLN (8) $u_p = \dfrac{B\sin(\Lambda x)}{\sin(\Lambda L)}\sin(\Omega t) \qquad u_p(x,0) = 0 \qquad u_{p_t}(x,0) = \dfrac{-B\Omega\sin(\Lambda x)}{\sin(\Lambda L)}$

(9) $u = u_p + \phi \qquad a^2\phi_{xx} - \phi_{tt} = 0$ (10) $\quad \phi(x,0) = 0$ (11)

(12) $\phi_t(x,0) = -u_{p_t}(x,0) = -B\Omega\dfrac{\sin(\Lambda x)}{\sin(\Lambda L)}$ $\boxed{/10}$ ϕ problem

SOV

(13) $\phi = X(x)\,T(t) \qquad a^2\dfrac{X''}{X} = \dfrac{T''}{T} = -\omega^2 \qquad X'' + \lambda^2 X = 0$ (14)

$T'' + \omega^2 T = 0$ (15)

$\lambda = \omega/a$

(16) $X = A_1\sin(\lambda x) + A_2\cos(\lambda x) \qquad A_2 = 0 ; \ \sin(\lambda L) = 0$

$\lambda L = n\pi \qquad \lambda_n = n\pi/L \qquad n = 1, 2, \ldots$

(17) $X_n = \sin(n\pi x/L)$ $\boxed{/10}$ ϕ prob. soln $\qquad T = B_1\sin(\omega t) + B_2\cos(\omega t)$

$T(0) = 0 \Rightarrow B_2 = 0$

So

(18) $\phi = \sum_{n=1}^{\infty} C_n X_n(x)\sin(\omega_n t)$

(19) $\phi_t(x,0) = \sum_{n=1}^{\infty} C_n \omega_n X_n(x) = -B\Omega\dfrac{\sin(\Lambda x)}{\sin(\Lambda L)}$

ORTHOGONALITY!

$\int_0^L X_n(x) X_m(x)\,dx = 0 \quad n \neq m$

proof:

$0 = \int_0^L \{X_n(X_m'' + \lambda_m^2 X_m) - X_m(X_n'' + \lambda_n^2 X_n)\}\,dx$

$= (X_n X_m' - X_m X_n')\Big|_0^L - \int_0^L (X_n' X_m' - X_m' X_n')\,dx + (\lambda_m^2 - \lambda_n^2)\int_0^L X_n X_m\,dx = 0$ $\boxed{/10}$

so,

$C_n = \dfrac{-B\Omega\sin(\Lambda x)}{\omega_n\sin(\Lambda L)}\,\dfrac{\int_0^L \sin(\Lambda x)\sin(\lambda_n x)\,dx}{\int_0^L \sin^2(\lambda_n x)\,dx}$

$$u = -\frac{B\sin(\Lambda x)}{\sin(\Lambda L)}\left\{\sin(\Omega t) - \sum_{n=1}^{\infty}\frac{\Omega}{\omega_n}\,\frac{\int_0^L \sin(\Lambda x)\sin(\lambda_n x)\,dx}{\int_0^L \sin^2(\lambda_n x)\,dx}\cdot\sin\left(n\pi\frac{x}{L}\right)\right\}$$

$\boxed{/10}$ completed solution

$\boxed{/50}$ 4. $\quad C_t + At\, C_x = 1 \qquad (1)$

$$\xi = \xi(x,t) \qquad \eta = \eta(x,t)$$

$(1) \Rightarrow$

$$C_\xi \xi_t + C_\eta \eta_t + At\,(C_\xi \xi_x + C_\eta \eta_x) = 1 \qquad (2)$$

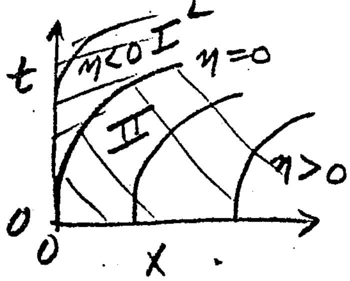
$$C_\xi\,[\xi_t + At\,\xi_x] + C_\eta\,[\underbrace{\eta_t + At\,\eta_x}_{\text{pick } = 0}] = 1$$

(3) $\quad \eta_t + At\,\eta_x = 0$

(4) $\quad d\eta = \eta_t\,dt + \eta_x\,dx = 0$ $\qquad \begin{vmatrix} 1 & At \\ dt & dx \end{vmatrix} = 0 \quad (5)$ $\quad \boxed{/10}$ slopes

(6) $\quad \dfrac{dx}{dt} = At$ (slopes of characteristics)

$x = \dfrac{At^2}{2} + \text{const}$ $\qquad$ pick $\eta = x - At^2/2 \quad (7)$
$\qquad \xi = t$



$\boxed{/10}$ chars.

$$\frac{\partial c}{\partial \xi}\,[1\ ] = 1$$

$$c = \xi + f(\eta)$$ $\boxed{/10}$ Soln. eqn on char

$c(x,0) = 0 \Rightarrow \quad 0 = 0 + f(\eta) \qquad f(\eta) = 0$ for $\eta > 0$

$c(0,t) = 1 \Rightarrow \quad 1 = t + f\left(-\frac{At^2}{2}\right)$

let $\sigma = -At^2/2 \qquad t = \sqrt{-2\sigma/A}$

$f(\sigma) = 1 - \sqrt{\frac{-2\sigma}{A}} \qquad \sigma < 0$

So, $f(\eta) = 1 - \sqrt{\frac{2}{A}\left(\frac{At^2}{2} - x\right)}$ for $\eta < 0$

$c(x,t) = t + 1 - \sqrt{\frac{2}{A}\left(\frac{At^2}{2} - x\right)}$ $\qquad$ Region ~~III~~ I $\boxed{/10}$

$c(x,t) = t$ $\qquad$ Region II $\boxed{/10}$

SUPPORT THE HONOR CODE!

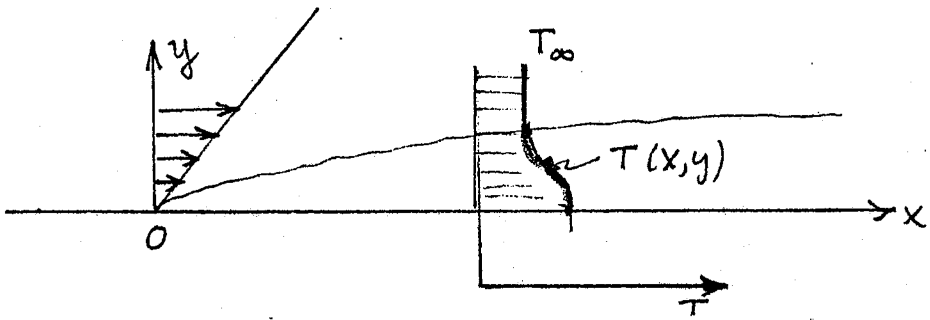
ME 200B
Winter 1980
WCR

FINAL EXAMINATION -- Closed Books and Notes

Part I is optional. If you are satisfied with your performance on the self-similar problem on the midterm, you do not need to work the self-similar problem in Part I. If you are satisfied with your performance on the eigenvalue problem on the midterm, you do not need to work the eigenvalue problem in Part I. If you work either of these problems, the higher of your midterm/final scores on each type of problem will be used in determining your course grade.

PART I -- Optional; see above.

(50) 1. The equation describing the temperature field in the boundary layer of a fluid downstream of a heated region is



$$\frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \tag{1}$$

$$\frac{\partial T}{\partial y} = 0 \quad \text{at} \quad y = 0 \tag{2}$$

$$T \to T_\infty \quad \text{as} \quad y \to \infty \quad \text{and as } x \to 0 \text{ for } y > 0 \tag{3}$$

$$\int_0^\infty y(T - T_\infty)\,dy = E \tag{4}$$

Here $T(x,y)$ is the temperature, T_∞ is the constant free-stream temperature, β is a constant, and E is a constant related to the energy deposited into the layer over the heated portion.

Develop the self-similar form of the solution to this problem, in the form

$$T - T_\infty = \ldots.$$

This will involve a function $f(\eta)$. Give the ODE and boundary conditions for $f(\eta)$, BUT DO NOT SOLVE THIS EQUATION. Explain how you would evaluate all constants you introduce in forming the solution.

(50) 2. The neutron flux in a cubical nuclear reactor is described by

$$\phi_{xx} + \phi_{yy} + \phi_{zz} + \mu^2\phi = 0$$

$$\begin{array}{llll} \phi = 0 & \text{at} \quad x = 1 & \phi_x = 0 & \text{at} \quad x = 0 \\ \phi = 0 & \text{at} \quad y = 1 & \phi_y = 0 & \text{at} \quad y = 0 \\ \phi_z = 0 & \text{at} \quad z = 1 & \phi_z = 0 & \text{at} \quad z = 0 \end{array}$$

The parameter μ is an eigenvalue associated with the control rod placement. Derive an expression for the smallest eigenvalue of this problem. Express the eigensolution $\phi(x,y,z)$ for this problem.

(50) 3. A high-voltage wire stretched between two towers is put into oscillation by an earthquake. The displacement $u(x,t)$ of the wire is described by

$$a^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \qquad (1)$$

$$u(x,0) = 0 \qquad (2) \qquad\qquad u(0,t) = 0 \qquad (4)$$

$$u_t(x,0) = 0 \qquad (3) \qquad\qquad u(L,t) = B \sin(\Omega t) \qquad (5)$$

Here a and B are constants, and Ω is the frequency of oscillation at the tower where $x = L$.

Develop the solution to this problem. Prove any orthogonality relationships involved. Give expressions for all constants that appear in the solution. You may leave constants expressed in terms of definite integrals.
HINT: Start with a particular solution.

(50) 4. The one-dimensional convection of a scalar contaminant in an <u>unsteady</u> flow is (neglecting diffusion) described by

$$c_t + Atc_x = 1 \qquad (1)$$

Here At is the velocity of the convecting flow, which you will note increases linearly in time.

(a) Determine the characteristics for this equation. Express x as a function of t for the characteristics, and sketch their positions in the $x - t$ plane.

(b) Develop the solution c to (1) satisfying the following conditions:

$$c(x,0) = 0 \qquad (2)$$

$$c(0,t) = 1 \qquad (3)$$

This will involve two regions in the $x - t$ plane. Express the solution in each region as a function of x and t.

NOTE: The solution is desired only for $t \geq 0$, $x \geq 0$.

let $T - T_\infty = S$ then $\frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \Rightarrow \frac{\partial^2 S}{\partial y^2} = \beta y \frac{\partial S}{\partial x}$

$\frac{\partial T}{\partial y} = 0$ at $y = 0 \Rightarrow \frac{\partial S}{\partial y} = 0$ at $y = 0$

$T \to T_\infty$ as $y \to \infty$ & as $x \to 0$ for $y > 0 \Rightarrow S \to 0$ as $y \to \infty$ & as $x \to 0$ for $y > 0$

$\therefore$ $\int_0^\infty y(T - T_\infty)\,dy = E \Rightarrow \int_0^\infty y(S)\,dy = E$

let $S = Ax^n f(\eta)$ $\quad \eta = \frac{By}{x^m}$

when $y \to \infty$ $\eta \to \infty \Rightarrow S \to 0$ when $\underline{f(\eta) \to 0 \text{ for } \eta \to \infty}$

$x \to 0$ for $y > 0 \Rightarrow \eta \to \infty$

$$\frac{\partial S}{\partial y} = \frac{\partial S}{\partial \eta}\frac{\partial \eta}{\partial y} = Ax^n f'(\eta) \cdot \frac{B}{x^m} = Ax^{n-m} f'(\eta) = 0 \text{ for } y = 0 \Rightarrow \underline{f'(0) = 0}$$

$$\int_0^\infty y S\,dy = E \Rightarrow \int_0^\infty \frac{\eta x^m}{B} \frac{d\eta\, x^m}{B} \cdot Ax^n f(\eta) = E \Rightarrow x^{2m+n}\frac{A}{B^2}\int_0^\infty \eta f\,d\eta = E$$

$\therefore$ $2m + n = 0$ or $\underline{n = -2m}$.

Now
$$\frac{\partial S}{\partial y} = \frac{\partial S}{\partial \eta}\frac{\partial \eta}{\partial y} = ABx^{n-m} f'(\eta)$$

$$\frac{\partial^2 S}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial S}{\partial y} = \frac{\partial}{\partial \eta}\left(\frac{\partial S}{\partial y}\right)\frac{\partial \eta}{\partial y} = ABx^{n-m} f'' \cdot \frac{B}{x^m} = AB^2 x^{n-2m} f''$$

$y = \eta \frac{x^m}{B}$ and $\frac{\partial S}{\partial x} = Anx^{n-1} f + Ax^n f'\frac{(-m)}{x}\eta$

$$= Ax^{n-1}[nf - mf'\eta]$$

$\therefore$ DE $\Rightarrow$ $\cancel{A}B^2 x^{n-2m} f'' = \beta \eta \frac{x^m}{B} \cancel{A} x^{n-1}[nf - mf'\eta]$

$\Rightarrow n - 2m = m + n - 1$

$-4m = -m - 1$ $\quad \underline{m = 1/3}$ $\quad \underline{n = -2/3}$

$$\frac{B^3}{\beta} f'' = \eta\left[-\tfrac{2}{3}f - \tfrac{1}{3}\eta f'\right] \quad \text{or} \quad \frac{3B^3}{\beta}f'' + 2\eta f + \eta^2 f' = 0$$

$$\frac{3B^3}{\beta}f'' + (\eta^2 f)' = 0$$

choose $B = \left(\frac{\beta}{3}\right)^{1/3}$ $\quad \therefore f'' + (\eta^2 f)' = 0$ or

$(f' + \eta^2 f)' = 0$ or $f' + \eta^2 f = \text{Const}$

if $f(\eta) \to 0$ as $\eta \to \infty$ $\quad f'(\eta) \to 0$ as $\eta \to \infty$ $\quad \therefore$ Const $= 0$

$\therefore$ $\frac{df}{f} = -\eta^2 \cancel{f}\, d\eta$ $\quad \ln f = -\eta^3/3 + \ln \text{Const}_2$

$f = C_2 e^{-\eta^3/3}$

Now $f'(\eta)\big|_{\eta=0} = -C_2\eta^2 e^{-\eta^3/3}\big|_{\eta=0} = 0$ $\quad \therefore$ choose $C_2 = 1$ for simplicity

$\therefore$ $f = e^{-\eta^3/3}$

also $\dfrac{A}{\beta^2}\int_0^\infty \eta f\, d\eta = E \Rightarrow \left(\frac{3}{\beta}\right)^{2/3} A \int_0^\infty \eta e^{-\eta^3/3} d\eta = E$

let $\eta^3/3 = x \quad \therefore \sqrt[3]{3x} = \eta \qquad \eta^2 d\eta = dx$

$d\eta = \dfrac{dx}{(3x)^{2/3}} \qquad \therefore \dfrac{(3x)^{1/3} dx}{(3x)^{2/3}} = \eta\, d\eta$

$\therefore \left(\frac{3}{\beta}\right)^{2/3} A \cdot \left(\frac{1}{3}\right)^{1/3} \int_0^\infty x^{-1/3} e^{-x} dx = E$

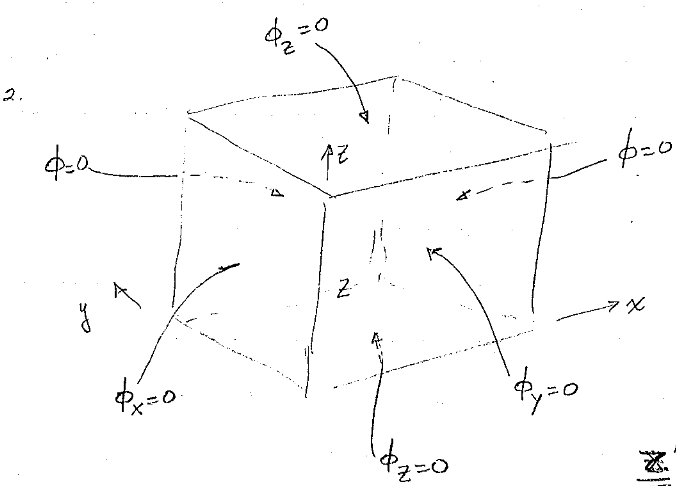
$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} ds$

$\therefore A = \dfrac{E \beta^{2/3} \; 3^{1/3}}{\Gamma(2/3)}$

$\therefore S = \dfrac{E \beta^{2/3}}{3^{1/3}\Gamma(2/3)} \dfrac{1}{x^{2/3}} e^{-\eta^3/3} \qquad \eta = y\left(\frac{\beta}{3x}\right)^{1/3}$

$S = \dfrac{E \beta^{2/3}}{3^{1/3} x^{2/3} \Gamma(2/3)} e^{-\frac{y^3 \beta}{9x}}$

2.



Let $\Phi(x,y,z) = X(x)Y(y)Z(z)$

$X''YZ + XY''Z + XYZ'' + \mu^2 XYZ = 0$

$\dfrac{X''}{X} + \dfrac{Y''}{Y} + \dfrac{Z''}{Z} + \mu^2 = 0$

$\dfrac{Z''}{Z} = -\dfrac{Y''}{Y} - \dfrac{X''}{X} - \mu^2 = -c_1^2$

$\dfrac{Z''}{Z} = -c_1^2 \; ; \quad \dfrac{Y''}{Y} = c_1^2 - \mu^2 - \dfrac{X''}{X} = -c_2^2$

$Z'' + c_1^2 Z = 0 \qquad Z = A_z\cos c_1 z + \underline{B_z \sin c_1 z}\; ; \; -A c_1 \sin c_1 z = Z' \quad c_1 = n\pi \quad \therefore \underline{Z = A_z \cos n\pi z} \quad n = 0, \pm 1, \ldots$

$Y'' + c_2^2 Y = 0 \qquad Y = A_y \cos c_2 y + \underline{B_y \sin c_2 y}\; ; \quad Y' = -A_y c_2 \sin c_2 y + C_2 B_y \cos c_2 y \qquad c_2 = (2m+1)\dfrac{\pi}{2}$

$m = 0, \pm 1, \ldots$

if $c_1 = 0 \quad Z'' = 0 \quad \therefore Z = A_z z + B_z \qquad \underline{Z = \text{Const}}$

if $c_2 = 0 \quad Y'' = 0 \qquad Y = A_y y + B_y \Rightarrow Y = 0 \qquad \therefore C_2 \neq 0.$

$\therefore$ if $c_1 = 0$ & $c_2 \neq 0 \qquad \dfrac{X''}{X} = c_2^2 - \mu^2 \Rightarrow X'' + (\mu^2 - c_2^2)X = 0 \quad \therefore X = A_x \sin\sqrt{\mu^2 - c_2^2}\,x + B_x \cos\sqrt{\mu^2 - c_2^2}\,x$

$X'(0) = 0 \Rightarrow A_x = 0 \qquad \sqrt{\mu^2 - c_2^2} = \dfrac{2k+1}{2}\pi \qquad \therefore \mu^2 = \dfrac{\pi^2}{4}\left[(2k+1)^2 + (2m+1)^2\right]$

$k = 0, \pm 1, \ldots \qquad \mu^2 = \dfrac{\pi^2}{4}[2] = \dfrac{\pi^2}{2} \qquad \underline{\mu = \dfrac{\pi}{\sqrt{2}}}$

if $c_1 \neq 0$ then $\dfrac{X''}{X} = c_1^2 - \mu^2 + c_2^2 = \left[n^2\pi^2 - \mu^2 + (2m+1)^2\dfrac{\pi^2}{4}\right] = -\alpha^2$

$-\left[\mu^2 - n^2\pi^2 - (2m+1)^2\dfrac{\pi^2}{4}\right]$

$X = A_x \cos\alpha x + B_x \sin\alpha x \qquad B_x = 0$ when $X'(0) = 0 \qquad \alpha = \dfrac{2k+1}{2}\pi \quad \therefore \alpha^2 = (2k+1)^2\dfrac{\pi^2}{4}$

$$\therefore \mu^2 = \alpha^2 + n^2\pi^2 + (2m+1)^2 \frac{\pi^2}{4} = \left[(2k+1)^2 + (2m+1)^2\right]\frac{\pi^2}{4} + n^2\pi^2$$

let $n=0 \quad k=m=1 \qquad \mu^2 = \pi^2/2 \qquad \mu = \pm \pi/\sqrt{2}$

4. $c_t + At\,c_x = 1 \qquad$ let $\eta(x,t)$

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial c}{\partial \xi}\frac{\partial \xi}{\partial t} \qquad \frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial c}{\partial \xi}\frac{\partial \xi}{\partial x}$$

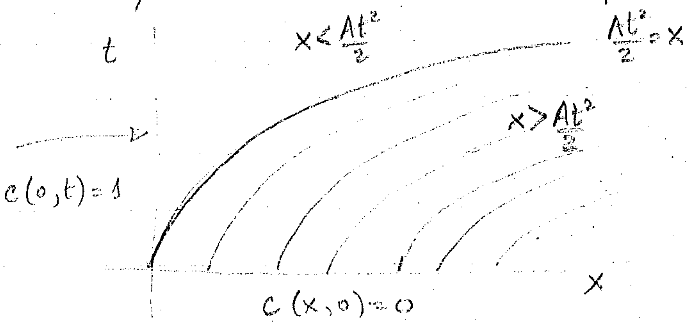
$$\frac{\partial c}{\partial \eta}\underbrace{\left(\frac{\partial \eta}{\partial t} + At\frac{\partial \eta}{\partial x}\right)}_{=0} + \frac{\partial c}{\partial \xi}\left(\frac{\partial \xi}{\partial t} + At\frac{\partial \xi}{\partial x}\right) = 1$$

$$\frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial t}dt = 0 \qquad \begin{vmatrix} dt & dx \\ 1 & At \end{vmatrix}\begin{pmatrix} \frac{\partial \eta}{\partial t} \\ \frac{\partial \eta}{\partial x} \end{pmatrix} = 0$$

$dt \cdot At - dx = 0$

$At^2/2 - x = \hat{c} = \eta \qquad$ let $\xi = t$

$$\frac{\partial c}{\partial \xi}(1) = 1 \qquad \therefore \quad c = \xi + f(\eta)$$



if $x=0,\ t=t \quad c(0,t) = 1 = c(\eta,\xi) = c\left(A\xi^2/2 - \eta, \xi\right) = \xi + f(\eta)\ ;\ \xi = \sqrt{\frac{2\eta}{A}} \quad 1 = \sqrt{\frac{2\eta}{A}} + f(\eta) \quad \therefore f(\eta) = 1 - \sqrt{\frac{2\eta}{A}}$

$\therefore \quad c(\xi,\eta) = \xi + 1 - \sqrt{\frac{2\eta}{A}} = (t+1) - \sqrt{t^2 - \frac{2x}{A}} = c(x,t)$ for $\quad x < At^2/2$

if $(x \neq 0,\ t=0)$

$c(x,0) = c(-x=\eta, \xi=0) = 0 + f(\eta) = 0 \qquad f(\eta) = 0 \qquad \therefore \quad c = \xi \quad$ or $\quad c = t$ for $x > \frac{At^2}{2}$

$c(x,t) = t$

when $\quad At^2/2 - x = 0 \qquad$ then $\qquad c_1(\xi,\eta) = \xi + 1 \qquad c_1 - c_2 = 1$

$c_2(\xi,\eta) = \xi.$

3. $$a^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \qquad (1)$$

$u(x,0)=0 \quad (2)$ $\qquad u(0,t)=0 \quad (4)$

$\frac{\partial u}{\partial t}(x,0)=0 \quad (3)$ $\qquad u(L,t) = B\sin\Omega t \quad (5)$

let $u(x,t) = u_0(x,t) + u_1(x,t)$

let $u_1(x,t) = \frac{Bx}{L}\sin\Omega t$ then $a^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial t^2} + \frac{Bx\Omega^2}{L}\sin\Omega t = 0$

$$\therefore \quad a^2\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial t^2} = -\frac{Bx\Omega^2}{L}\sin\Omega t$$

Also

$u(x,0) = u_0(x,0) + 0 = 0$ $\qquad u(0,t) = u_0(0,t) + 0 = 0$

$u_t(x,0) = u_{0,t}(x,0) + \frac{B\Omega x}{L} = 0$ $\qquad u(L,t) = u_0(L,t) + B\sin\Omega t = B\sin\Omega t$

$\therefore u_0(L,t) = 0$

$$\Rightarrow \quad u_0(x,0)=0$$
$$u_{0,t}(x,0) = -\frac{B\Omega x}{L}$$
$$u_0(0,t)=0$$
$$u_0(L,t)=0$$

let $u_0(x,t) = u_0^{I}(x,t) + u_0^{II}(x,t)$ $\ni$.

$u_0^{I}(x,t)$ satisfies

$$a^2\frac{\partial^2 u_0^{I}}{\partial x^2} + \frac{\partial^2 u_0^{I}}{\partial t^2} = -\frac{Bx\,\Omega^2}{L}\sin\Omega t$$

& $u_0^{I}(x,0)=0$

$u_{0,t}^{I}(x,0)=0$

$u_0^{I}(0,t)=0$

$u_0^{I}(L,t)=0$

let $u_0^{I}(x,t) = \sum T_n(t)\sin\frac{n\pi x}{L}$

and let $-\frac{Bx}{L}\Omega^2\sin\Omega t = \sum F_n(t)\sin\frac{n\pi x}{L}$

then $\sum\left(-a^2\frac{n^2\pi^2}{L^2}T_n - T_n''\right)\sin\frac{n\pi x}{L} = \sum F_n(t)\sin\frac{n\pi x}{L}$

$\therefore \quad -a^2\frac{n^2\pi^2}{L^2}T_n - T_n'' = F_n(t)$

where $$F_n(t) = -\frac{B\Omega^2\sin\Omega t}{L}\,\frac{\int_0^L x\sin\frac{n\pi x}{L}\,dx}{\int_0^L \sin^2\frac{n\pi x}{L}\,dx}$$

$\therefore$ if $\Omega \neq \left(\frac{n\pi a}{L}\right)^2$ then we may use var. of param

using $T_n = A(t)\sin\frac{n\pi a}{L}t + B(t)\cos\frac{n\pi a}{L}t$

$T_n' = \underbrace{A's + B'c}_{=0} + \frac{n\pi a}{L}\left[Ac - Bs\right]$

pick

$T_n'' = \frac{n\pi a}{L}\left[A'c - B's\right] + \frac{n^2\pi^2a^2}{L^2}\left[-As - Bc\right]$

$u_0^{II}(x,t)$ satisfies

$$a^2\frac{\partial^2 u_0^{II}}{\partial x^2} - \frac{\partial^2 u_0^{II}}{\partial t^2} = 0$$

& $u_0^{II}(x,0)=0$

$u_{0,t}^{II}(x,0) = -\frac{B\Omega x}{L}$

$u_0(0,t)=0$

$u_0(L,t)=0$

let $u_0^{II}(x,t) = F(x)T(t)$

$a^2F''T - FT'' = 0$ or $a^2\frac{F''}{F} = \frac{T''}{T} = -\omega^2$

then $F = A_x\cos\frac{\omega}{a}x + B_x\sin\frac{\omega}{a}x$

applying BC $\frac{\omega L}{a} = n\pi$ or $\frac{\omega}{a} = \frac{n\pi}{L}$

$F(x) = (B_x)_n\sin\frac{n\pi x}{L}$

$T(x) = B_t\sin\frac{n\pi a}{L}t + A_t\cos\frac{n\pi a}{L}t$

Using IC $\Rightarrow A_t = 0$ &

$\sum B_n\frac{n\pi a}{L}\sin\frac{n\pi x}{L} = -\frac{B\Omega x}{L}$

$\therefore B_n\frac{n\pi a}{L}\int_0^L s\,s = -\frac{B\Omega}{L}\int_0^L x\sin\frac{n\pi x}{L}$

or $B_n = -\frac{B\Omega}{L}\cdot\frac{L}{n\pi a}\,\frac{\int x\sin}{\int\sin^2\frac{n\pi x}{L}}$

$\therefore$ put into DE to get $-\frac{a^2n^2\pi^2}{L^2}\left[As+Bc\right] - \frac{n\pi a}{L}\left[A'c-B's\right] + \frac{n^2\pi^2a^2}{L^2}\left[As+Bc\right] = F_n$

$\therefore \quad A'c - B's = -\frac{L}{n\pi a}F_n \qquad A' = -\frac{L}{n\pi a}F_n \cos\frac{n\pi a}{L}t$

$A's + B'c = 0 \qquad B' = \frac{L}{n\pi a}F_n \sin\frac{n\pi a}{L}t$

$\therefore \quad A_n = -\frac{L}{n\pi a}\int^t F_n(\tau)\cos\frac{n\pi a}{L}\tau\, d\tau \qquad B_n = +\frac{L}{n\pi a}\int_0^t F_n(\tau)\sin\frac{n\pi a}{L}\tau\, d\tau.$

put back into dfn^{o} of T_n

$T_n = -\frac{L}{n\pi a}\int_0^t F_n(\tau)\left[\cos\frac{n\pi a}{L}\tau \sin\frac{n\pi a}{L}t - \sin\frac{n\pi a}{L}\tau \cos\frac{n\pi a}{L}t\right]d\tau$

$= -\frac{L}{n\pi a}\int_0^t F_n(\tau)\sin\frac{n\pi a}{L}(t-\tau)\,d\tau$; put dfn of $F_n(\tau)$ back in

$T_n = \frac{B\Omega^2}{L}\int_0^t \sin\Omega\tau \sin\frac{n\pi a}{L}(t-\tau)d\tau \quad \left(\frac{\int_0^L x\sin\frac{n\pi x}{L}dx}{\int_0^L \sin^2\frac{n\pi x}{L}dx}\right) = L/2$

$u_0^{I}(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi x}{L}$

$u_0^{II}(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$ where $B_n = -\frac{B\Omega}{n\pi a}\frac{\int_0^L x\sin\frac{n\pi x}{L}dx}{\int_0^L \sin^2\frac{n\pi x}{L}dx}$

$u_1(x,t) = \frac{Bx}{L}\sin\Omega t$

$u(x,t) = u_1 + u_0^{I} + u_0^{II}$

near room 548

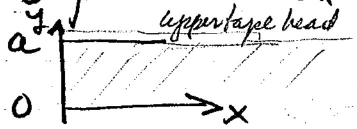
MIDTERM SOLUTION

Self similar & EV. problem.

Analysis of temperature between magnetic tape & read head

1, $$\frac{\partial^2 T}{\partial y^2} = \alpha y \frac{\partial T}{\partial x} \quad (1)$$

$T(0,x) = 0$ (2); $T(a,x) = 0$ (3); $T \to 0$ as $x \to \infty$ (4);



Homogeneous problem: ∴ Eigenvalue problem

$$T = X(x)\,Y(y) \Rightarrow \frac{Y''}{yY} = \alpha \frac{X'}{X} = -\lambda$$

$$X' + \frac{\lambda}{\alpha} X = 0 \quad (5) \Rightarrow X = \exp\left(-\frac{\lambda}{\alpha} x\right) \quad (6)$$

$$Y'' + \lambda y Y = 0 \quad (7)$$

NOTE! MANY FOULED UP HERE. LEARN THIS APPROACH!

$y = C_1 Ai(x) + C_2 Bi(x)$

Given a handout work on EV part of HW!

Looks almost like $y'' + xy = 0$

let $z = cy \Rightarrow \frac{d^2Y}{dy^2} = c^2 \frac{d^2Y}{dz^2}$

SO, THE DE IS

$$c^2 \frac{d^2Y}{dz^2} + \lambda \frac{z}{c} Y = 0$$

so, if we choose $c^3 = \lambda$, $\Rightarrow d^2Y/dz^2 + zY = 0$

So,

$$Y = C_1\, ai(\lambda^{1/3} y) + C_2\, bi(\lambda^{1/3} y)$$ IS THE SOLUTION

$bi(0) = \sqrt{3}\, ai(0)$

$$Y(0) = 0 \Rightarrow ai(0) \cdot C_1 + \sqrt{3}\, ai(0) \cdot C_2 = 0$$

$$Y(a) = 0 \Rightarrow ai(\lambda^{1/3} a)\, C_1 + \sqrt{3}\, bi(\lambda^{1/3} a)\, C_2 = 0$$

$$D(\lambda) = \begin{vmatrix} 1 & \sqrt{3} \\ ai(\lambda^{1/3} a) & bi(\lambda^{1/3} a) \end{vmatrix} = 0$$ FIXES $\underline{\lambda_n}$ eigenvalues

$$C_1 = -\sqrt{3}\, C_2$$

$$Y_n = A\left[\sqrt{3}\, ai(\lambda_n^{1/3} y) - bi(\lambda_n^{1/3} y)\right]$$ eigenfunction for $\lambda = \lambda_n$.

ORTHOGONALITY PROPERTY! $\int_0^a y\, Y_n Y_m\, dy = 0$ prove in usual way;

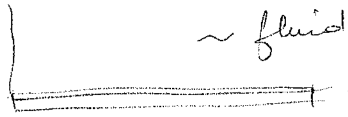


plate moves at $t = t_0$ what is u

viscous fluid initially motionless about flat plate at $t=0$ fluid moves to right w/ linearly increasing velo.

2. $$\frac{\partial^2 u}{\partial y^2} = \alpha \frac{\partial u}{\partial t} \quad (1) \qquad u(0,t) = \beta t \quad (2) \qquad u(y,0) = 0 \quad (3)$$

$$u(y,t) \to 0 \text{ as } y \to \infty \quad (4)$$

Self-similar solution – no characteristic scales!

assume $u = A t^n f(\eta)$ $\quad \eta = By/t^m$

But $n = 1$ by (2)! Pick $A = \beta$.

So $\underline{u = \beta t F(\eta) \qquad F(0) = 1 \qquad F(\infty) \to 0}$

$$\beta t \frac{B^2}{t^{2m}} F'' = \alpha\left[\beta F - \beta t\, m \frac{By}{t^{m+1}} F'\right]$$

$$\frac{t}{t^{2m}} B^2 F'' = \alpha\left[F - m\eta F'\right] \qquad \underline{2m = 1 \quad m = 1/2}$$

$$\frac{B^2}{\alpha} F'' + \frac{1}{2}\eta F' - F = 0$$

Pick $B^2/\alpha = 1/2$ Then $\eta = y\sqrt{\dfrac{\alpha}{2t}}$

$$\frac{1}{2} F'' + \frac{1}{2}\eta F' - F = 0 \qquad F'' + \eta F' - 2F = 0$$

NOTE! if you pick $B^2/\alpha = 1$, $\quad F'' + \frac{1}{2}\eta F' - F = 0$

Now let $z = c\eta \qquad c^2 \dfrac{d^2F}{dz^2} + \dfrac{1}{2} z \dfrac{dF}{dz} - F = 0$

Pick $c^2 = \frac{1}{2} \qquad c = 1/\sqrt{2} \Rightarrow \dfrac{d^2F}{dz^2} + z\dfrac{dF}{dz} - 2F = 0$

So,
$$F = C_1 G_2\left(y\sqrt{\frac{\alpha}{2t}}\right) + C_2 H_2\left(y\sqrt{\frac{\alpha}{2t}}\right)$$

$C_1 = 1$ for $F(0) = 1 \qquad C_2 = 0$ for $F(\infty) \to 0$.

$$\underline{u = \beta t\, G_2\left(y\sqrt{\frac{\alpha}{2t}}\right)}$$ IS THE SOLUTION

1st order Given $y' + p(x)y = g(x)$ if $p(x) \neq$ const, $\exists$ a factor $\mu(x)$ $\ni$ $(\mu y)' = \mu g$ 1

$\therefore \mu p y = \mu' y$ or $\underline{\mu = e^{\int p\,dx}}$ INTEGRATING FACTOR

$$\therefore \quad \underline{y = \frac{1}{\mu}\int^x g\, e^{\int^{x''} p\,dx'}\,dx'' + \frac{c}{\mu}}$$ SOLN

Bernoulli Eqn. $y' + p(x)y = q(x)\,y^n$ by subst. $v = y^{1-n}$ we can reduce eqn. $v' = (1-n)y^{-n}y'$

Seperable Solutions

1. if $\frac{dy}{dx} = f(x,y)$ & we can write $y' - f = 0 \Rightarrow M\,dx + N\,dy = 0 = d\psi$

where $M(x,y)$, $N(y,x)$ are functions of x, y then $\exists$ a $\psi(x,y)$ $\ni$ the solution is

$\psi(x,y) = c = \int^x M\,dx' + \int^y N\,dy'$ if $M_y = N_x = \psi_{xy}$

2. if $M\,dx + N\,dy = 0$ cannot be written as $d\psi = 0$ $\exists$ a $\mu = \mu(x,y)$

$\ni$ $\mu M = \psi_x$ $\mu N = \psi_y$ and $(\mu M)_y = (\mu N)_x \Rightarrow \mu M_y = \mu N_x + \mu_x N$ } if $\mu = \mu(x)$
$\mu(M_y - N_x) = \mu_x N$

if $\mu = \mu(x)$ only then $\therefore \frac{d\mu}{\mu} = \frac{M_y - N_x}{N}\,dx$; if $\mu = \mu(y)$ $\frac{d\mu}{\mu} = \frac{N_x - M_y}{M}\,dy$

3. if $y' = f(x,y) = F(y/x)$ then if $y = vx$, where $v = v(x)$
the equation leads to $\frac{dx}{x} = \frac{dv}{F(v) - v}$

4. if $y' = \frac{ax + by + m}{cx + dy + n}$ let $x = X - h$ $y = Y - k$ $\ni$ $\frac{dY}{dX} = \frac{aX + bY}{cX + dY}$ and use 3. above.

which will then be of the form $Y' = F(Y/X)$

5. Method of successive approx. given $y' = f(x,y)$ with $y(0) = 0$ define
$\phi_{k+1}(x) = \int_0^x f(t, \phi_k(t))\,dt$ then if the sequence $\phi_0, \phi_1, \phi_2, \ldots \phi_n \to \phi$ then ϕ is the solution

2nd order / ~~higher orders~~ where $y^{(i)} = \frac{d^i y}{dx^i}$ and c_i are constant coeff

if $\sum_{i=0}^{n} c_i\, y^{(i)} = f(x)$

the homogeneous solution y_h $\ni$ $\sum_{i=0}^{n} c_i\, y_h^{(i)} = 0$ depends on the solution to the

algebraic eqn $\sum_{i=0}^{n} c_i m^i = 0$ where $y_h = e^{mx}$ is assumed.

LINEAR EQ one in which the highest degree of the fn (or its derivatives) is 1 and the equation does not have mixed products of deriv w/ deriv or deriv w/ fn itself

degree power to which ~~last~~ highest term is raised $(y'')^5 = 5$th degree

order - the highest derivative in the eqn.

Reduction of order is we know the y_1 is a solution to the ODE then $y_2 = v\,y_1$
will ~~reduce the order of the ode~~ to give you an order ODE for v which is of red

if $y'' + py' + qy = 0$ then if $y = vy_1 \Rightarrow v'' + (p + 2\frac{y_1'}{y_1})v' = 0 \Rightarrow v' = c\, e^{-\int (p + 2y_1'/y_1)}$

hence $v = c\int^{x} e^{-\int^{x'} (p + 2y_1'/y_1) ds}\, dx' + c_1$

- for 2nd order Hom $ay'' + by' + cy = 0$ $a, b, c \neq 0$ then if $y = e^{mx}$ char eq is $(am^2 + bm + c) = 0$

if $b^2 - 4ac > 0$ then $y_H = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ $m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$b^2 - 4ac = 0$ then $y_H = (c_1 + c_2 x) e^{mx}$ $m = -\frac{b}{2a}$ use reduction of order w/ $y_1 = e^{mx}$

$b^2 - 4ac < 0$ then if $m_{1,2} = \mu \pm i\lambda$ $\mu = -\frac{b}{2a}$ $\lambda = \frac{\sqrt{4ac - b^2}}{2a}$

$$y_H = e^{\mu x}[A \sin\lambda x + B\cos\lambda x]$$

- For non homogeneous case use the method of undetermined coeffs.

if $y'' + py' + qy = g$ and $y_h = c_1 y_1 + c_2 y_2$ then $y_p = u_1(x) y_1 + u_2(x) y_2$

$$\therefore\ u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g$$

- Euler Eq $x^2 y'' + \alpha x y' + \beta y = 0$ $\alpha, \beta = $ const $y = x^r$ solve problem & give char eq

$(r^2 - r + \alpha r + \beta) = 0$ $\therefore$ $r^2 + (\alpha - 1) r + \beta = 0$ or $r_{1,2} = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$

if $\sqrt{(\alpha-1)^2 - 4\beta} > 0$ $y = c_1 x^{r_1} + c_2 x^{r_2}$

$= 0$ $y = (c_1 + c_2 \ln x) x^r$ $r = \frac{1-\alpha}{2}$

< 0 then let $\frac{1-\alpha}{2} = \lambda$ $-\mu^2 = (\alpha - 1)^2 - 4\beta$ $y = x^\lambda (c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x))$

if we define $x = e^z$ we can transform the diff eq to $d^2y/dz^2 + (\alpha - 1)\frac{dy}{dz} + \beta y = 0$

Series solns given that $P(x) y'' + Q(x) y' + R(x) y = 0$ — Solutions near a regular singular point

and if $\lim_{x\to 0} x \frac{Q}{P} < \infty$ $\lim_{x\to 0} x^2 \frac{R}{P} < \infty$: let $x\frac{Q}{P} = \sum_{n=0} a_n' x^n$ $x^2 \frac{R}{P} = \sum_{n=0} b_n' x^n$

let us recast $Py'' + Qy' + Ry = 0 \Rightarrow x^2 y'' + x\left[x\frac{Q}{P}\right]y' + \left[x^2 \frac{R}{P}\right] y = 0$

this is Euler eq with power series coeff $\Rightarrow$ assume $y = x^r \sum_{n=0}^{\infty} a_n x^n$

notice that for $x \to 0$ $x\frac{Q}{P} \to a_0'$ $x^2 \frac{R}{P} \to b_0'$ and $Py'' + Qy' + Ry = 0 \Rightarrow x^2 y'' + a_0' x y' + b_0' y = 0$ | The Euler Eq.

Method of Frobenius : get indicial equation, roots are the exponents of singularity at regular singular point

① if $r_1 > r_2$ 2 distinct solutions, if $r_1 - r_2 \neq$ integer

$r_1 = r_2$

r_1, r_2 are imag 2 distinct solutions

radius of convergence of series is at least the distance from $(x = 0)$ to the nearest zero of $P(x)$.

- if $r_1 - r_2 \neq$ integer & $r_1 \neq r_2$

$$y_i = |x|^{r_i} \sum_{n=0}^{\infty} a_n(r_i) x^n \quad \text{where } a_n(r_i) \text{ are coefficients evaluated based on } r_i$$

if $r_1 = r_2$ then

$$y_1 = |x|^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n$$

$$y_2 = y_1 \ln|x| + |x|^{r_1} \sum_{n=1}^{\infty} b_n(r_1) x^n$$

if $r_1 - r_2 = N$ then $\qquad a > 0$

$$y_1 = |x|^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n$$

$$y_2 = a\, y_1 \ln|x| + |x|^{r_2} \sum_{n=0}^{\infty} c_n(r_2) x^n$$

a can be determined by putting series back into diff eq.

Higher order Lin ODE

$\sum_{i=0}^{n} a_i y^{(i)} = g(x)$ has solutions (for const coeffs) of $e^{m_i x}$ & $y_h = \sum c_i e^{m_i x}$

m_i depends on solution of $\sum_{m=0}^{n} a_i m^i = 0$

if a_i's are all real and $\exists$ complex roots: roots come in conjugate pairs

if $g(x)$ is of the same form as one of the solutions to homogeneous problem Let $y_p = x^s g(s)$ where s is s.t. $x^s g(s)$ is different from y_h

Var param: for n^{th} degree let y_i be a sol to $Ly = 0$ s.t. $y_h = \sum c_i y_i$

define $y_p = \sum u_i y_i$ if $\sum_{i=1}^{n} u_i' y_i^{(n-1)} = g$, $\sum_{i=1}^{n} u_i' y_i^{(j)} = 0$ for $j \leq n-2$

then
$$\begin{bmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_i' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g \end{bmatrix}$$
& we can solve for u_i' integrate & get u_i

Laplace Transform
$$F(s) = \int_0^{\infty} f(t) e^{-st} dt = \mathcal{L}\{f(t)\}$$

if $f^{(n)}(t)$ $\quad \mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L} f - \left[\sum_{i=1}^{n} s^{(n-i)} f^{(i-1)}(0)\right]$

Fourier series if a function $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$ $\quad \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \begin{pmatrix} \cos \frac{n\pi x}{\ell} \\ \sin \frac{n\pi x}{\ell} \end{pmatrix} dx$

Separation of variables requires a simple domain

Heat conduction equation $\alpha^2 u_{xx} = u_t$ Parabolic Eq. requires IC, BC

Wave Equation $a^2 u_{xx} = u_{tt}$ (1-D) hyperbolic "

$a^2(u_{xx} + u_{yy}) = u_{tt}$ (2-D) "

Laplace Eqn. $a^2(u_{xx} + u_{yy}) = 0$ Elliptic BC

Some general information on Laplace's eqn

Any fn governed by $\nabla^2\phi = 0$ will attain its max & min on the boundary

Given $P(x,y)\dfrac{\partial z}{\partial x} + Q(x,y)\dfrac{\partial z}{\partial y} = R_1(x,y)z + R_2(x,y) = R$ linear equation of 1st order

has solution $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ if $P = 0 \Rightarrow dx = 0$, $Q = 0 \Rightarrow dy = 0$, $R = 0 \Rightarrow dz = 0$ will give 2 curves $u_1(x,y,z) = c_1$, $u_2(x,y,z) = c_2$ and $u_2 = f(u_1)$

if $a z_{,xx} + b z_{,xy} + c z_{,yy} = 0$ assume $z = f(y+mx)$ then m is a soln of $am^2 + bm + c = 0$

if $m_1 \neq m_2$ $z = f(y+m_1x) + g(y+m_2x)$

$m_1 = m_2$ $z = f(y+m_1x) + x\,g(y+m_1x)$

- An equation is hyperbolic if $\exists$ a transformation $\xi(x,y),\ \eta(x,y) \Rightarrow$ PDE reduces to $\dfrac{\partial^2 u}{\partial\xi\,\partial\eta} = 0$
  or if $\alpha = (\eta+\xi)/2$ $\beta = (\eta-\xi)/2$ then $u_{\alpha\alpha} - u_{\beta\beta} = 0$
- An equation is parabolic if $\exists$ a transformation " " $\Rightarrow$ PDE reduces to $\dfrac{\partial^2 u}{\partial\eta^2} = 0$
- " " " elliptic " " " " " " $\Rightarrow$ PDE " to $u_{\alpha\alpha} + u_{\beta\beta} = 0$

if PDE $= a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy}$

then $\dfrac{dy}{dx} = \dfrac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{12}}}{a_{11}}$ for parabolic take $\dfrac{dy}{dx} = \dfrac{a_{12}}{a_{11}}$

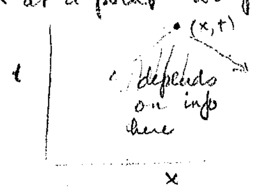
after you solve you will get $c_1 = f_1(x,y) = \xi$ $c_2 = f_2(x,y) = \eta$

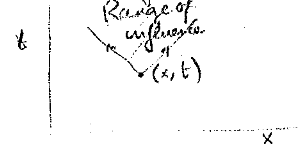
- if a_{11}, a_{12}, a_{22} are constants

  then $y = \dfrac{a_{12} \pm \sqrt{}}{a_{11}}\,x + c$ $\therefore\ \xi = y - \left[\dfrac{a_{12} + \sqrt{}}{a_{11}}\,x\right]$

  $\eta = y - \left[\dfrac{a_{12} - \sqrt{}}{a_{11}}\,x\right]$

the wave eqn 1-D   solution at a point is dependent on all info from characteristics





Heat Equation   fn assumes max at $u(x,0)$ initially or @ $u(0,t)$ & $u(l,t)$ or on boundary

if $u_1 \le u_2$ at bdy & everywhere for $t=0$ then $u_1 \le u_2$ $\forall\,(x,t)$

if $\underline{u} \le u \le \bar{u}$ on $t=0, x=0, x=l$ then $\underline{u} \le u \le \bar{u}$ $\forall\, x,t$ in interval

if $|u_1 - u_2| \le \varepsilon$ initially then $|u_1 - u_2| \le \varepsilon$ $\forall\, x,t$

1. Max occurs on boundary / min occurs on bdy
2. value of fn at an interior point is determined by value of fn & value of deriv (normal to ~~surface~~ bdy) at bdy
3. if $u \le U$ on bdy then $u \le U$ everywhere in domain
4. Solution to Dirichlet problem is unique
5. Solution to Neumann " is unique to an arbitrary constant.

if $F(x,y,u,u_x,u_y)=0$ then if $p = u_x$ $q = u_y$
to solve equation look at

$$dp/dt = -F_{,x} - F_{,u}\, p$$
$$dq/dt = -F_{,y} - F_{,u}\, q$$
$$\frac{dx}{dt} = F_{,p} \quad \frac{dy}{dt} = F_{,q} \quad \frac{du}{dt} = pF_{,p} + qF_{,q}$$

in hildebrand $u \Rightarrow z$

$$u(M) = +\frac{1}{4\pi}\int_{\Sigma}\left[\frac{1}{r_{MP}}\frac{\partial u}{\partial n} - u(P)\frac{\partial}{\partial n}\left(\frac{1}{r_{MP}}\right)\right]dS - \frac{1}{4\pi}\int_V \frac{\Delta u}{r}\, dV$$ for non harmonic fn,

$$= \frac{1}{4\pi}\int_{\Sigma}\left[\frac{1}{r_{MP}}\frac{\partial u}{\partial n} - u(P)\frac{\partial}{\partial n}\left(\frac{1}{r_{MP}}\right)\right]dS$$ for harmonic using $\int [u\Delta v - v\Delta u]dV = \int\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS$

if v is harmonic $\int \Delta v\, dV = \int_S \frac{\partial v}{\partial n}\, dS = 0$

PDE's

- if no char ~~sehale~~ scale exists define a new param η $\ni$ $\eta(x,y)$ is of the form that whatever variable is differentiated the most will appear in symplest way & the one which is differentiated the least appears in the most complicated way ie $\eta = Ax/y^n$ if $\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial x^2}$
  define the dependent variable $u = Bf(\eta)$ if the bc are of the form $u(x,t) =$ const, no, fn.
  if b.c. are derivative or integral the let $u = By^m f(\eta)$. The 2 coeffs A, B are free to be chosen to satisfy BC & simplify diff eq.
  The self scimilar soln reduces the no. of independent variables by one. If we know one soln then the 2nd soln is $f_2 = f_1(\eta)\, g(\eta)$ where we get a new de for $g(\eta)$.
  LINEAR if coeffs of then fn. f are not ~~fns~~ products of derivatives or the fn. itself
  HOMOG if when you replace f by af & get same eqns. or if $\eta \to a\eta$ gives same eqn.

- SEP. OF. VAR - lead to eigenfunct & evalues for homogeneous b.c. & diff eq. Normally used in simple geometries $\nabla^2\phi = \frac{\partial^2\phi}{\partial x_i \partial x_i}$ or $\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$
  IF PDE is linear a linear comb of solns are also solutions.
  E.V. and E. functions $\Rightarrow$ orthogonality conditions
- ADJOINT PROBLEM if $Lu = (su')' + (Q+\lambda^2 P)u = 0$ w/BC then using IBP on $\int_0^a v\left[(su')' + (Q+\lambda^2P)u\right]dx = 0$ will give you adjoint L^*v & we can pick bc on v $\ni$ $L^*v = 0$; if $L^* = L$ we have selfadj problem.
  Solution to $Lu = h$ exists if v is the soln to adjoint homog. $L^*v = 0$; then $\int_a^b vLu = \int_a^b hv = \int u L^* v = 0$
  ie $\int_a^b h(x)\, v_n\, dx = 0$ or h is orthog to the adjoint soln.
- Complex imbedding solution and taking real part: makes things simpler
- For inhomog problem get partic solution w/o care to solve bc, then use homog. soln w/ changed b.c. & solve.
- Method of char $\exists$ char lines $\ni$ PDE will be reduced to ODE take ξ, η coordinates and put into PDE set coeff of $u_\eta = 0$ $\left[(\ )\eta_x + (\ )\eta_y\right]$ & for const η along const η lines, $d\eta = 0 = \eta_x dx + \eta_y dy$ use these two eqns to get characteristics. Now coeff of $u_\xi \neq 0$ integrate u_ξ along const η to get $u = f(\eta)$

- Systems of Higher order - ① reduce to first order by ~~into~~ intro. of new variables w, v $\ni$ $v = u_x$ $w = u_y$, $v_y - w_x = 0$ is new eq
  ~~take~~ ② transform eqs to ξ, η and ③ take linear combo of eqs. ④ Choose coeff of v_n $w_n = 0$ & solve for

(5) also $d\eta = \eta_x dx + \eta_y dy = 0 \Rightarrow y' = -\eta_x/\eta_y$ put result into eq from (4) to get y' identify eqn solve & get $y = \int^x dx' + c$ $\quad c = y - \int^x dx' = \eta$ take ξ to be any coord that crosses η only once

for wave eqn $u_{tt} - a^2 u_{xx} = 0$ $\quad u(x,t) = F(x+at) + G(x-at)$ $\quad$ if $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$ then

$$u(x,t) = \frac{f(x+at) + f(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)\,d\xi$$

b.c. will produce reflections

Characteristics

one of 3 ways to reduce PDE to ODEs

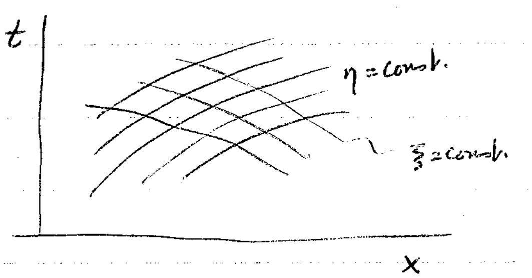
Self Sim

Sep of Var

Character

continuity $\frac{\partial \phi}{\partial t} + \bar{u}\frac{\partial \phi}{\partial x} = f(x)$ source term with $\phi(x,0)=0$ IC, $\phi(0,t)=0$ BC.

Look for lines in x,t space $\ni$. PDE $\rightarrow$ ODE



let PDE $\Rightarrow \frac{\partial}{\partial \xi}$ on lines of const η.
can then integrate along characteristic

$$(x,y) \Rightarrow (\xi,\eta) \quad \therefore \xi(x,y) \quad \eta(x,y)$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial t}$$

$$\frac{\partial \phi}{\partial x} = \phi_\xi \xi_x + \phi_\eta \eta_x$$

put into PDE $\quad \phi_\xi(\xi_t + \bar{u}\xi_x) + \phi_\eta(\eta_t + \bar{u}\eta_x) = f(\xi,\eta)$.

want derivs wrt η disappear $\quad\therefore$ pick $\eta,\xi \ni.\ \eta_t + \bar{u}\eta_x = 0$ along const η

$$d\eta = \eta_x dx + \eta_t dt = 0 \text{ along const } \eta.$$

have 2 equs for η_x, η_t

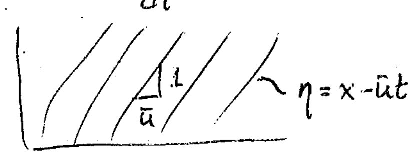
$$\begin{aligned} \eta_t + \bar{u}\eta_x &= 0 \\ dt\,\eta_t + dx\,\eta_x &= 0 \end{aligned} \Rightarrow \det\begin{vmatrix} 1 & \bar{u} \\ dt & dx \end{vmatrix} = 0$$

$dx - \bar{u}\,dt = 0$ on const η lines or $\frac{dx}{dt} = \bar{u}$

$\bar{u}$ is or can be variable or const

let $\bar{u}$ = const $\quad\therefore\quad x = \bar{u}t + c$ $\qquad \frac{dx}{dt} = \bar{u} \leftarrow$ velocity $\therefore x - \bar{u}t = c$ path of part

& particle carries ϕ along the path.



can take ξ to be any fn that intersects η $\quad$ take $\xi = t$ or $\xi = x$

best to take $\xi = x \Rightarrow f(x) = f(\xi)$

put results into PDE   to show that $\phi_t + \bar{u}\phi_x = \bar{u}\phi_\xi = f(\xi)$

$$\phi_\xi(0 + \bar{u}\cdot 1) = f(\xi) \qquad [\phi_\eta(-\bar{u}) + \phi_\xi\cdot 0] + \bar{u}[\phi_\eta\cdot 1 + \phi_\xi\cdot 1]$$

$$\bar{u}\phi_\xi = f(\xi)$$

$$\therefore \quad \phi_\xi = \frac{1}{\bar{u}} f(\xi) \qquad \phi = \int_{\xi=0}^{\xi} \frac{1}{\bar{u}} f(\xi')\, d\xi' + g(\eta)$$

~~when t = 0 ξ = a φ(x,0) = 0~~

$$\phi(x,t) = \phi(\xi,\eta)$$

IC. $0 = \phi(x,0) \Rightarrow \phi(\xi, \eta = x = \xi) = 0$

$$\text{then} \quad \underbrace{\bar{u}\,\phi(\xi,\xi)}_{=0} + \int_{\xi=0}^{\xi} f(\xi')\,d\xi' + g(\eta = x = \xi)$$

$$\therefore \quad g(\xi) = -\int_0^{\xi} f(\xi')\,d\xi' \quad \Rightarrow \quad g(\eta) = -\int_0^{\eta} f(\xi')\,d\xi'$$

B $$\bar{u}\,\phi(\xi,\eta) = \int_0^{\xi} f(\xi')\,d\xi' - \int_0^{\eta} f(\xi')\,d\xi' = \int_{\eta}^{\xi} f(\xi')\,d\xi'$$

these only ~~after~~ affected by IC.

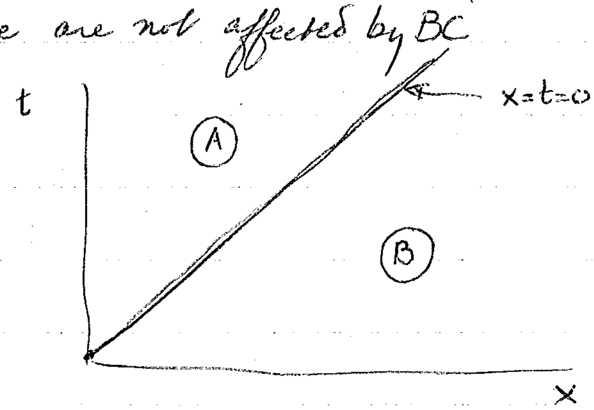
BC.

$$\bar{u}\,\phi(0,\eta) = g(\eta) \quad \Rightarrow \quad \phi(0,t) = 0 \qquad x = 0,\ \xi = 0 \quad \eta = -\bar{u}t$$

A $$\bar{u}\,\phi(0,\eta) = \cancel{\int_0^{0} f(\xi')\,d\xi'} + g(\eta) \qquad \therefore \quad g(\eta) = \bar{u}\,\phi(0,\eta)$$

$$\therefore \quad \phi(\xi,\eta) = \frac{1}{\bar{u}}\int_0^{\xi} f(\xi')\,d\xi' + \phi(0,\eta)$$

these are not affected by BC



$x = t = 0 \Rightarrow \eta = 0$ & $\phi(0,0) = 0 \Rightarrow \phi(\xi,\phi) = \frac{1}{\bar{u}}\int_0$

$\phi(\xi,0) = \frac{1}{\bar{u}}\int$

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_m \\ y_1' & y_2' & \cdots & y_m' \\ \vdots & & & \\ y_1^{(m-1)} & y_2^{(m-1)} & \cdots & y_m^{(m-1)} \end{bmatrix} \begin{bmatrix} c_1' \\ \vdots \\ \vdots \\ c_m' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ f(x) \end{bmatrix}$$

$$y'' + y = f(x) \qquad y_h = \hat{c}_1 \sin x + \hat{c}_2 \cos x$$

take $\quad y_p = c_1(x) \sin x + c_2(x) \cos x$

$$y_p' = c_1(x)\cos x - c_2(x) \sin x + \boxed{c_1'(x) \sin x + c_2'(x) \cos x = 0}$$

$$y_p'' = -c_1(x) \sin x - c_2(x) \cos x + \boxed{c_1'(x) \cos x - c_2'(x) \sin x = f(x)}$$

$$y_p'' + y_p = c_1'(x) \cos x - c_2'(x) \sin x = f(x)$$

$$\therefore \quad \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

$$c_1' = + f(x) \cos x \qquad \therefore \quad c_1(x) = \int f(x) \cos x \, dx$$

$$c_2' = - f(x) \sin x \qquad c_2(x) = -\int f(x) \sin x \, dx$$

Example 1

$$c_1(x) = \int \cos x \, dx = \sin x$$

$$c_2(x) = -\int \sin x \, dx = \cos x$$

$$y_p = \sum c_i(x)\, y_i(x) = 1$$

Review of ODE

given $\left.\begin{array}{l} f'' + a^2 g = 0 \\ g'' + a^2 f = 0 \end{array}\right\}$ $f^{IV} + a^4 f = 0 \Rightarrow m = \pm ai, \pm a$

or let $f = \hat{f} e^{mx}$ $\quad g = \hat{g} e^{mx}$ $\quad$ if we had assumed $g = \hat{g} e^{nx} \Rightarrow m = n$

$$\therefore \quad (\hat{f} m^2 + a^2 \hat{g}) e^{mx} = 0$$

$$(\hat{g} m^2 + a^2 \hat{f}) e^{mx} = 0$$

$$A \underset{\sim}{x} = \therefore \begin{pmatrix} m^2 & a^2 \\ a^2 & m^2 \end{pmatrix} \begin{pmatrix} \hat{f} \\ \hat{g} \end{pmatrix} = 0 \Rightarrow \text{if } \hat{f}, \hat{g} \neq 0$$

$\det A = 0 \Rightarrow m = \pm ai, \pm a$

if $y'' + a^2 y = f(x)$ $\qquad f(x) = \begin{cases} 1 \\ x \\ \sin \omega x \end{cases}$

take $y = \frac{1}{a^2}$ for 1st

$y = cx + d \Rightarrow y'' = 0$ ODE: $a^2(cx + d) = x \Rightarrow c = \frac{1}{a^2}$ $\; d = 0$

take $y = A \sin \omega x$ $\; -\omega^2 A (\sin \omega x) + A a^2 \sin \omega x = \sin \omega x$

$$A(a^2 - \omega^2) = 1 \quad \therefore \quad A = \frac{1}{a^2 - \omega^2} \quad \therefore \quad y = \frac{\sin \omega x}{a^2 - \omega^2}$$

Variation of param $\quad$ must know homogeneous solutions $\{y_i(x)$ are homog sol$\}$

$$y_p = \sum c_i(x) \{ y_i(x) \}$$

$$y_p' = \sum c_i y_i' + \sum c_i' y_i \quad \text{set this} = 0$$

$$y_p'' = \sum c_i y_i'' + \sum c_i' y_i' \quad \text{set this} = 0$$

$$\vdots$$

$$y_p^{(n)} = \sum c_i y_i^{(n)} + \boxed{\sum c_i' y_i^{(n-1)} = f(x)}$$

if $f(x) = x$ $\quad c_1 = \int x\cos x\,dx = x\sin x + \cos x$

$$c_2 = \int -x\sin x\,dx = x\cos x - \sin x$$

if $f(x) = \sin x$

$$c_1 = \int \sin x\cos x\,dx = \frac{1}{2}\int \sin 2x\,dx = \frac{-\cos 2x}{4}$$

$$c_2 \int -\sin^2 x\,dx = -\int \frac{1-\cos 2x}{2}\,dx = -\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]$$

$\therefore$

$$y_p = -\frac{\cos 2x}{4}\sin x - \frac{x}{2}\cos x + \frac{\cos x \sin 2x}{4}$$

$$= \frac{\sin x^{\,2x-x}}{4} - \frac{x}{2}\cos x$$

ODE - Bessel's Eqn.

$y'' + a(x)y' - b(x)y = 0$ use series solution iff a, b are or can be expressed as power series

$$y = \sum_{k=0}^{\infty} A_k r^{k+s}$$

Bessel's fn. (1) $x^2y'' + xy' + (x^2 - p^2)y = 0$ $\quad c_1 J_0(x) + c_2 Y_0(x)$

if $p = 0$ $\quad x^2y'' + xy' + (x^2 y) = 0$

$$y = \sum_{k=0}^{\infty} A_k x^{k+s} \qquad y' = \sum_{k=1}^{\infty} A_k\, k(k+s)\, x^{k+s-1}$$
$$y'' = \sum_{k=2} A_k\, k(k+s)(k+s-1)\, x^{k+s-2}$$

put into (1)

$$\sum A_k (k+s)(k+s-1)\, x^{k+s} + A_k\, k(k+s)\, x^{k+s} + A_k k\, x^{k+s+2} = 0$$

assume $A_n = 0$ for $n < 0$

look for coeff of x^s: $\sum_{k=0}^{\infty}$ $\quad \rightarrow \quad A_0\, s(s-1) + A_0 s = 0 \qquad s^2 = 0$

1) $s_1 = s_2$ / solution in series form
2) $s_1 \neq s_2$ $\quad s_1 - s_2 \neq n$ / 2 soln in series form
3) $s_1 \neq s_2$ $\quad s_1 - s_2 =$ integer $\Rightarrow s_1 = c\, s_2$

to develop series $x^{s+1}\left(A_1 (1+s)(1+s-1) + A_1 (1+s)\right) = 0 \Rightarrow A_1 (1+s)^2 = 0$
$\Rightarrow$ if $1+s \neq 0$ $\quad A_1 = 0$

x^{s+2}: $A_2 (2+s)(2+s-1) + A_2 (2+s) + A_0 = 0$ or $A_2 = \dfrac{-A_0}{(s+2)^2}$

x^{s+3}: $A_3 (s+3)^2 + A_1 = 0 \Rightarrow A_3 = 0$

$A_{2m+1} = 0 \quad m = 0, 1, 2, \ldots$

$A_{2m} = \dfrac{-A_{2m-2}}{(s+2m)^2} \qquad A_{2m} = \dfrac{(-1)^m A_0}{(s+2m)^2 (s-2m-2)^2 \cdots (s+2)^2}$

$$\therefore \; y = \sum_{k=0}^{\infty} A_k x^{k+s} = A_0 \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{(m!)^2} = A_0 J_0(x)$$

Fröbenius Technique

$$y = \sum_{k=0}^{\infty} A_{2k} x^{2k+s} \qquad y = y(s,x) \qquad A_{2k} = A_{2k}(s)$$

$$= \sum y_k$$

$y_1 = A_0 x^s$

$y_1' = A_0 s x^{s-1}$ $\qquad$ let $\mathcal{L} = x^2(\ )'' + x(\ )' + x^2(\ )$

$y_1'' = A_0 s(s-1) x^{s-2}$

$$\mathcal{L}y = 0 \Rightarrow \quad A_0 s(s-1)x^s + A_0 s x^s + A_0 x^{s+2}$$

$$\mathcal{L}y_2 = A_2 (s+2)(s+1) x^{s+2} + A_2 (s+2) x^{s+2} + A_4 x^{s+4}$$

$y_2 = A_2 x^{s+2}$

$y_2' = A_2 (s+2) x^{s+1}$

$y_2'' = A_2 (s+2)(s+1) x^s$

$\mathcal{L}y_2 = A_2(s+2)(s+1) x^{s+2} + A_2(s+2) x^{s+2} + A_2 x^{s+4}$

$$\therefore \mathcal{L}y_1 + \mathcal{L}y_2 = A_0 s^2 x^s + x^{s+2}\{\underbrace{A_0 + A_2(s+2)^2}_{0}\} + x^{s+4} A_2$$

but $A_2 = -\dfrac{A_0}{(s+2)^2}$

$$\therefore A_0 s^2 x^s + x^{s+4} A_2 = 0$$

now $\mathcal{L}y_1 + \mathcal{L}y_2 + \mathcal{L}y_3 = x^{s+2}\{\underbrace{A_2 + A_4(s+4)^2}_{0}\} + A_4 x^{s+4} +$

$\therefore \mathcal{L}y = A_0 x^s s^2$ $\qquad$ where $y = y_1 + y_2 + y_3 + \cdots + y_n$

$$y(x,s)\Big|_{s=0} = y(0,x) = J_0(x) \qquad \frac{\partial}{\partial s}[\mathcal{L}y]\Big|_{s=0} = A_0 2s x^s + A_0 s^2 x^s \log x = 0$$

@ $s=0$ $\quad$ 1st term $\to 0$

$$\frac{\partial}{\partial s}[\mathcal{L}(\ )] = \mathcal{L}\left[\frac{\partial}{\partial s}(\ )\right] = \mathcal{L}\left[\frac{\partial y}{\partial s}\right]\Bigg|_{s=0} = 0 \qquad \text{2nd sol. is}$$

$\therefore$ 1st sol obtained when $s_1 = s_2 = 0$

2nd sol. deriv of 1st soln @ $s=0$

$$\left.\frac{\partial y}{\partial s}\right|_{s=0} = \sum \left.\frac{\partial A_{2k}}{\partial s}\right|_{s=0} x^{2k} + \underbrace{\sum_{k=0}^{\infty} A_{2k}(0)\, x^{2k}}_{J_0(x)\log x} \log x$$

$$Y_0(x) = J_0(x)\log x + \sum (-1)^{k+1} \frac{\varphi(k)}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$$

$$\varphi(k) = \sum_{m=1}^{k} \frac{1}{m} \qquad k \geq 1$$

$$\varphi(0) = 0$$

$$Y_0(x) = \frac{2}{\pi}\left[Y^0(x) + (\gamma - \log 2)\, J_0(x) \right] \qquad \text{where } \gamma = \lim_{k\to\infty}\left[\varphi(k) - \log k\right] = .5772157$$

$$y = c_2\, y_0(x) + c_1 J_0(x)$$

if $p \neq 0$ $\quad s^2 - p^2 = 0 \qquad s_1 = p,\ s_2 = -p \quad$ thus $s_1 = s_2$ iff $p = 0$

$\varepsilon = \text{const}$ $\quad \varepsilon \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t}$ $\quad$ let $T = \varepsilon t$ $\quad \frac{\partial}{\partial T} = \frac{1}{\varepsilon}\frac{\partial}{\partial t}$

S.S. sol.
since no time, scale charact.

let $\quad \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial T}$

Let $u(z,T) = Z(z)\,\bar{T}(T)$

$$\frac{Z''}{Z} = \frac{\bar{T}'}{\bar{T}} = -\lambda^2 \quad \text{then}$$

$$Z'' + \lambda^2 Z = 0 \qquad Z = A\cos\lambda z + B\sin\lambda z$$

$$\bar{T}' + \lambda^2 \bar{T} = 0 \qquad \text{then} \quad \bar{T} = e^{-\lambda^2 T}$$

$\cancel{B}$ $\quad$ ic $\quad u(z,0) = 0 \quad \Rightarrow \quad A\cos\lambda z + B\sin\lambda z = 0 \quad \forall z$

$$\varepsilon \frac{\partial u}{\partial z}\Big|_{z=0} = \tau \qquad \varepsilon\left[-A\lambda\sin\lambda z + B\lambda\cos\lambda z\right]\bar{T}\Big|_{z=0} = \tau$$

$$B = 0$$

$\therefore \quad u(-\infty, t) = 0$

let $u = f(mz + \sqrt{t})$ $\qquad f(z + \sqrt{T}) + g(z - \sqrt{T})$

$-\frac{z}{\sqrt{T}} \qquad \cancel{+\frac{z}{\sqrt{T}}}$

$u = f(0)$ $\qquad f'' + \eta f' = 0 \quad \eta f'$

$\lim_{\eta \to 0}$

$u(0,t) = \frac{\sqrt{2t}\,\tau}{\sqrt{\lambda\pi}}$ $\qquad u(0,t)$ $\qquad f = c_1\eta + c_2\left[e^{-\eta^2/2} + \eta\int_{-\infty}^{\eta} e^{-\theta^2/2}\,d\theta\right]$

$$f'' + \eta f' - f = 0$$

let $\varepsilon = \left(\kappa l^2 \frac{\partial u}{\partial z} + \nu\right)$ $\qquad \kappa l^2\left(\frac{\partial u}{\partial z}\right)_1^2 + \nu\left(\frac{\partial u}{\partial z}\right) = \tau_0$

$u = A g(\eta) \qquad \eta = \frac{Cz}{t^m}$

ME 200 B Prof Reynolds 1/5/79

His notes (6.95) + Hand book (10.)

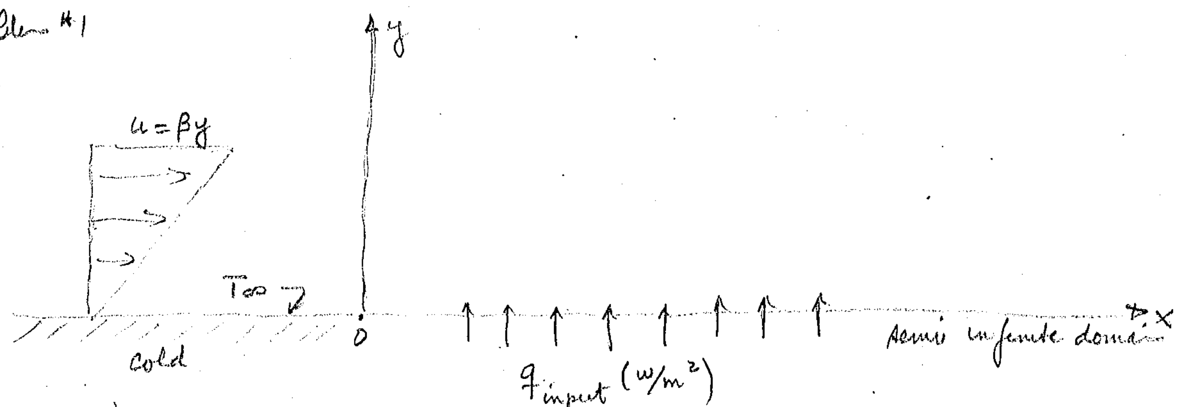
1/9/79

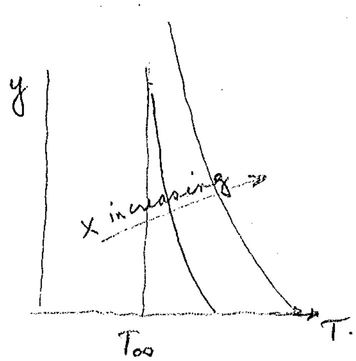
Wednesday 3:15 PM <u>Tutorials</u>

Self - Similar Solution

1. whenever no characteristic scales the problem will lead to a self-similar solution ie the variables combine to form some non dimensionless group.

Problem #1





problem is such that q & β are related and allows up to find out veloc profile in b.l. based on how much q is inputted.

$$\alpha \frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \qquad \frac{\alpha T}{L^2} = \beta \frac{LT}{L} \Rightarrow \frac{\alpha}{\beta} = L^2$$

BC @ $x = 0$ $t = T_\infty$

$y \to \infty \Rightarrow t \to T_\infty$

at $y = 0$ $q = -k \frac{\partial T}{\partial y} = \text{const.}$

let $\Delta T = T - T_\infty$

$$\alpha \frac{\partial^2 \Delta T}{\partial y^2} = \beta y \frac{\partial \Delta T}{\partial x} \qquad (1)$$

@ $x = 0$ $\Delta T = 0$ (2)

$y \to \infty$ $\Delta T \to 0$ (3)

@ $y = 0$ $-k \frac{\partial \Delta T}{\partial y} = q$ (4)

(2) and (3) suggest that we try

$\eta = By/x^n$ since for $y \to \infty$ or $x \to 0$ η has same eff we want $\Delta T = \overset{\text{const}}{A} f(\eta)$ $\rightarrow$ all curves should collapse onto 1

$$\frac{\partial \Delta T}{\partial x} = A f'(\eta) \frac{\partial \eta}{\partial x} = A f' \frac{-n\,By}{x^{n+1}}$$

$$\frac{\partial \Delta T}{\partial y} = A f' \frac{\partial \eta}{\partial y} = A f' \cdot \frac{B}{x^n} \qquad \frac{\partial^2 \Delta T}{\partial y^2} = A f'' \frac{B^2}{x^{2n}}$$

putting into PDE $\alpha A f'' \frac{B^2}{x^{2n}} = \beta y \cdot \left[-A \frac{n\,By}{x^{n+1}}\right]$ this must be an ODE in $f, f', f'', \eta, \eta^2 \ldots$

in an ode we cannot have y's or x's if ODE is only in η

solve for y as a fn of η and x

$$\therefore \quad \alpha A f'' \frac{B^2}{x^{2n}} = \beta\, \eta \frac{x^n}{B} \left[-\frac{n}{x^{n+1}} B\, \eta \frac{x^n}{B} f' \right]$$

since this cant be a fn of x $\therefore$ the powers of x on both sides will be the same $\therefore$

$$-2n = 2n - n - 1 = n - 1 \quad \therefore \quad n = \tfrac{1}{3} \quad \text{hence}$$

$$\frac{\alpha B^3}{\beta} f'' + \frac{1}{3}\eta^2 f' = 0 \qquad \text{could pick } \frac{3\alpha B^3}{\beta} = 1$$

however this <u>doesn't satisfy</u> the B.C. (eq (4)) since $\frac{\partial \Delta T}{\partial y} = \text{const}$

and from the derivation: $\frac{\partial \Delta T}{\partial y} = \text{fn of } x$ at $x = 0$.

therefore as a guess assume

$$\Delta T = A x^m f(\eta) \quad \text{with} \quad \eta = By/x^n$$

look at BC. $\frac{\partial \Delta T}{\partial y} = A x^m f'(\eta) \frac{B}{x^n}$

$$\text{const} = q = -k \left.\frac{\partial \Delta T}{\partial y}\right|_{y=0} = -k A x^m f'(0) \frac{B}{x^n} \Rightarrow m = n$$

$\therefore \quad \Delta T = A x^n f(\eta) \qquad (*)$

now check other BC @ $x=0$ $\Delta T \equiv 0$

@ $y \to \infty$ $\Delta T \to 0$ $\Rightarrow$ $f(\eta)$ must decay exponential

Put $(*)$ in PDE

$$\alpha A x^n f'' \frac{B^2}{x^{2n}} = \beta y \left[A n x^{n-1} f - n A x^n f' \frac{By}{x^{n+1}} \right]$$

replace $y = \eta \frac{x^n}{B} \Rightarrow \quad = \beta y \left[A n x^{n-1} f - n A \eta f' x^{n-1} \right]$

$$\alpha \cancel{A} x^{-n} f'' B^2 = \beta \eta \frac{x^n}{B} x^{n-1} \cancel{A} \left[n f - n \eta f' \right]$$

$\therefore \quad -n = 2n-1 \qquad n = \frac{1}{3}$

$$\therefore \quad \left(\frac{3 \alpha B^3}{\beta} \right) f'' + \eta^2 f' - \eta f = 0$$

choose A based on BC (eq(4)) choose B $\ni$ 1st coef $= 1$

$\therefore \quad B = \sqrt[3]{\beta / 3\alpha}$

BC $\quad f(\infty) = 0 \Leftarrow x = 0 \quad \Delta t = 0$

BC $\quad f'(0) = -1 \quad$ choose $A \ni f'(0) = -1 \therefore \frac{1}{kAB} = 1$

1/10/78

$\Delta T(x,y) = A x^{1/3} f(\eta)$

$$\eta = y \left(\frac{\beta}{3 \alpha x} \right)^{1/3} \qquad f'' + \eta^2 f' - \eta f = 0$$

w/ BC $\quad f(\infty) = 0 \ ; \ f'(0) = -1$

Remember $f'(0) = -1$ is obtained from @ $y=0$ $q = -k \frac{\partial (\Delta T)}{\partial y}$

$q = -kA x^{1/3} f'(0) \left(\frac{\beta}{3 \alpha x} \right)^{1/3}$ if we take $f'(0) = -1$ then $A = \dfrac{q}{k \left(\frac{\beta}{3\alpha} \right)^{1/3}}$

self-similar solution is one where there is no char length. For example even in BC a char length can exist ie (oscillating bc)

$$f'' + \eta^2 f' - \eta f = 0$$

$f(\infty) = 0 \quad f'(0) = -1 \qquad\qquad f = c_1 f_1(\eta) + c_2 f_2(\eta)$

O.D.E.

linear (since coeff of f, f' etc do not involve derivs)

homogeneous in f ie if we replace f by af

not " in η " " " " η by $c\eta$

$f_1(\eta) = \eta$ is a solution

Construct $f_2 = g(\eta) \cdot f_1(\eta)$ it always works.

$\therefore \; f_2 = \eta \cdot g(\eta)$

$$f_2' = g + \eta g'$$

$$f_2'' = g' + g' + \eta g'' = 2g' + \eta g''$$

$$(\eta g'' + 2g') + \eta^2(g + \eta g') - \eta(\eta g) = 0$$

$$\eta g'' + 2g' + \eta^3 g' = 0 \quad \therefore \quad g'' = \left(-\frac{2}{\eta} - \eta^2\right) g'$$

$$g''/g' = -\frac{2}{\eta} - \eta^2$$

$$\log g' = -2 \ln \eta - \eta^3/2 + \cancel{c}_0$$ pick this

$$g' = \frac{1}{\eta^2} e^{-\eta^3/3}$$

$$g = \int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^3/3} \, d\sigma$$

choose

reason since $f_1(\infty) = \infty \Rightarrow C_1 = 0$ since $f(\infty) = 0$

$\Rightarrow f_2 = \eta \int_c^\eta \frac{1}{\sigma^2} e^{-\sigma^2/3} d\sigma \longrightarrow 0$ when $\eta \to \infty \Rightarrow c = \infty$

$$f_2(\eta) = \eta \int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^3/3} d\sigma$$

for large $\eta > 0$ for $\eta > 1$

$$f_2(\eta) = \eta \int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^3/3} d\sigma < \left|\eta \int_\infty^\eta 1 \; e^{-\sigma^3/3} d\sigma\right| < \left|\eta \int_\infty^\eta 1 \cdot e^{-\sigma^3/3} d\sigma\right|$$

$\therefore f(\eta) \to 0$ as $\eta \to \infty$ since $\frac{1}{\eta^2} < 1$ and $e^{-\eta^3/3} < e^{-\eta^3}$ for $\eta > 1$

$$f(\eta) = c_2 \eta \int_\infty^\eta \frac{e^{-\sigma^3/3}}{\sigma^2} d\sigma \qquad c_1 = 0 \text{ when } f(\infty) = 0$$

INTEGRATION by parts

$$\int u\,dv = uv - \int v\,du$$

let $dv = \sigma^{-2}$ $v = -\frac{1}{\sigma}$
$u = e^{-\sigma^3/3}$ $du = -\sigma^2 e^{-\sigma^3/3}$ $\Big\}$ $\int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^3/3} d\sigma = -\frac{1}{\sigma} e^{-\frac{\sigma^3}{3}}\Big|_\infty^\eta - \int_\infty^\eta \sigma e^{-\frac{\sigma^3}{3}} d\sigma$

$$\therefore f(\eta) = c_2 \eta \left[-\frac{1}{\eta} e^{-\eta^3/3} - \int_\infty^\eta \sigma e^{-\sigma^3/3} d\sigma\right] = -c_2 e^{-\eta^3/3} - c_2 \eta \int_\infty^\eta \sigma e^{-\sigma^3/3} d\sigma$$

$f(0) = -c_2$ to get c_2 must get $f'(\eta)\big|_{\eta=0} = -1$

$$f'(\eta) = c_2 \left\{\eta^2 e^{-\eta^3/3} - \int_\infty^\eta \sigma e^{-\sigma^3/3} d\sigma - \eta \left[d(\text{integral})\right]\right\}$$

$d(\text{integral}) = \text{integrand}\Big|_\eta = \eta e^{-\eta^3/3}$

$$f'(0) = c_2 \left[\cancel{\eta^2 e^{-\eta^3/3}} - \int_\infty^0 \sigma e^{-\sigma^3/3} d\sigma - \cancel{\eta^2 e^{-\eta^3/3}}\right]$$

let $\sigma^3/3 = t$ $\sigma = (3t)^{1/3}$ $d\sigma = \frac{1}{\cancel{3}}(3t)^{-2/3} \cdot \cancel{3} dt$ $\sigma d\sigma = (3t)^{1/3}(3t)^{-2/3} dt = (3t)^{-1/3}$

$$f'(0) = c_2 \int_0^\infty \sigma e^{-\sigma^3/3} d\sigma = c_2 3^{-1/3} \int_0^\infty t^{-1/3} e^{-t} dt = c_2 3^{-1/3} \Gamma(2/3) \qquad c_2 = \frac{-1}{\Gamma(2/3) 3^{-1/3}}$$

1/12/79

$f'' + \eta^2 f' - \eta f = 0$

$f \to 0$ as $\eta \to \infty$

$f'(0) = -1$

Numerical ODE solvers solve systems like

$y_i' = F_i(y_1, y_2, y_3, \cdots, y_n, x)$

w/IC $y_i(x_0) =$ given $\quad i = 1, \cdots, n$

With our solution we will have problem using these solvers since

1) Problem is a BVP not IVP

2) 2nd order ODE not a system of first order

Recast system as follows

$x = \eta$

$y_1 = f$

$y_2 = f'$

$\therefore \; y_1' = y_2$

$y_2' = f'' = \eta f - \eta^2 f' = x(y_1 - x y_2)$

w/IC $\quad f'(0) = -1 = y_2(0)$

need $\quad y_1(0) = f_1(0)$

1 method: we can guess $\quad y_1(0) =$ number & see if $f \to 0$ as $\eta \to \infty$

Generate 2 linearly indep solns (since 2nd order ODE)

first solution is

$y_1(0) = 0 \qquad y_2(0) = -1 \qquad : f_1$

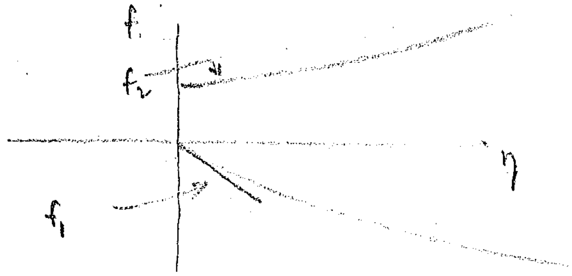
2nd solution is

$y_2(0) = 0 \qquad y_1(0) = 1 \qquad : f_2$

Then $f = C_1 f_1 + C_2 f_2$ will be the solution if C_1 & C_2 are picked correctly

But $\quad f'(0) = C_1 \underbrace{f_1'}_{y_2(0) \text{ for } f_1} + C_2 \underbrace{f_2'}_{y_2(0) \text{ for } f_2} = C_1(-1) + C_2 \cdot 0 = -C_1 = -1 \Rightarrow C_1 = 1$

Now pick C_2 ∋ $f \to 0$ as $\eta \to \infty$



Thus if we know one solution and problem is linear one can find the 2nd by use of the first $\quad y_2(x) = y_1(x) \cdot g(x)$

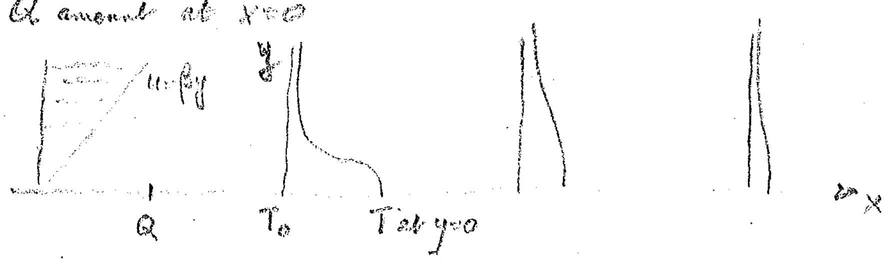
$y'' - y = 0 \quad$ has solution $\quad f_1 = \sinh x \quad f_2 = \cosh x$

$$f_1(0) = 0 \qquad f_2(0) = 1$$
$$f_1'(0) = 1 \qquad f_2'(0) = 0$$

trouble is for large $x \quad f_1, f_2 \to \dfrac{e^x}{2}$ we must therefore use filtering in order to remove this effect.

Integral constraint problem - self similar soln

Consider flow over flat plate ∋ $u = \beta y$ above it and heat source inputs Q amount at $x = 0$



define $\Delta T = T - T_0$ then governing PDE is $\quad \dfrac{\partial}{\partial y}\left(K \dfrac{\partial(\Delta T)}{\partial y}\right) = \rho c_p\, u \dfrac{\partial(\Delta T)}{\partial x}$

ie the energy balance of heat conduction in y direction is the heat carried away in x dir

if $\alpha = \dfrac{K}{\rho c_p}$ & $K \neq K(y)$ then

$$\alpha \frac{\partial^2 \Delta T}{\partial y^2} = \beta y \frac{\partial \Delta T}{\partial x}$$

Since no other sources at $y = 0 \quad \dfrac{\partial(\Delta T)}{\partial y} = 0$

and as $y \to \infty \quad \Delta T \to 0 \quad$ return to ambient condition

Fixed energy flow given by Q is

$$\int_0^\infty \text{enthalpy} \cdot \text{veloc. } dA = Q \qquad \int_0^\infty \rho c_p \Delta T \cdot \beta y \, dy$$

profile has no characteristic distance problem is self similar in ΔT

take $\eta = \frac{By}{x^n}$ since 2 diff wrt y and only 1 wrt x

define $\Delta T = Ax^m f(\eta)$ as in problem #1 pg #1

integral constraint is $\int_0^\infty \rho c_p \beta \, \Delta T y \, dy = Q$

now $y = \eta \frac{x^n}{B}$ and $dy = d\eta \frac{x^n}{B}$ $\therefore$ $\rho c_p \beta \int_0^\infty A x^m f(\eta) \cdot \eta \frac{x^n}{B} \cdot \frac{x^n}{B} d\eta = Q$

$\Rightarrow$ $m + 2n = 0$ or $m = -2n$ and constraint is $\rho c_p \beta \frac{A}{B^2} \int_0^\infty \eta f \, d\eta = Q$

this will fix A once B & f are fixed (from DE & B/C)

1/15/79

$$\alpha \frac{\partial^2 \Delta T}{\partial y^2} = \beta y \frac{\partial \Delta T}{\partial x} \quad (1)$$ Temperature of fluid field moving w/ velocity $u = \beta y$

w/ $\int_0^\infty \rho c_p \beta y \Delta T \, dy = Q$ (2)

$\frac{\partial \Delta T}{\partial y} = 0$ @ $y = 0$ $\Delta T \to 0$ as $y \to \infty$

Assume $\Delta T = A x^m f(\eta)$ $\eta = By/x^n$

from (2) $\Rightarrow$ $m = -2n$

$$(2) \Rightarrow \quad \frac{\rho c_p \beta A}{B^2} \int_0^\infty \eta f(\eta) \, d\eta = Q$$

$$\alpha \left[A x^m \cdot f'' \cdot \frac{B^2}{x^{2n}} \right] = \beta \eta \frac{x^n}{B} \left[A m x^{m-1} f + A x^m f' \left(-\frac{n}{x} \eta \right) \right]$$

order of x $\quad (x^{m-2n}) = x^n [x^{m-1}] \Rightarrow x^{-4n} = x^{n+(-2n-1)} = x^{-n-1}$

$\therefore$ $n = 1/3$ $m = -2/3$

$$\therefore \text{DE} \rightarrow \alpha \cancel{A} f'' B^2 = \frac{\beta \eta}{B} \left[-\frac{2A}{3} f - \frac{1}{3} A f' \eta \right] = -\frac{\beta \eta \cancel{A}}{3B} \left[2f + \eta f' \right]$$

or $\frac{3 \alpha B^3}{\beta} f'' = -\eta [2f + \eta f']$ pick $\frac{3\alpha B^3}{\beta} = 1$ $B = \left(\frac{\beta}{3\alpha} \right)^{1/3}$

$f'' + \underbrace{\eta^2 f' + 2\eta f}_{} = 0$

$f'' + (\eta^2 f)' = 0 \qquad \therefore \quad f' + \eta^2 f = c_1$

now since $\frac{\partial \Delta T}{\partial y} = 0$ @ $y = 0 \Rightarrow Ax^m f' \cdot \frac{B}{x^n} = \frac{\partial \Delta T}{\partial y} \quad \therefore f'(0) = 0$

$\Delta T \to 0$ as $y \to \infty \Rightarrow f(\eta) \to 0$ as $\eta \to \infty$

can pick $f(0) = 1$ since b.c. are homogeneous $\therefore c_1 = 0$

hence $f' + \eta^2 f = 0$ or $\frac{df}{f} = -\eta^2 d\eta$ or $f = c_2 e^{-\eta^3/3}$

but since we picked $f(0) = 1 \Rightarrow c_2 = 1 \qquad \underline{f = e^{-\eta^3/3}}$

de is homogeneous since if we replace f by Kf and the value K will dropout ($Kf'' + \eta^2 Kf = 0 \Rightarrow f'' + \eta^2 f = 0$) $\Rightarrow$ we can pick $f(0)$

$I = \int_0^\infty \eta f \, d\eta = \int_0^\infty \eta e^{-\eta^3/3} d\eta \qquad$ let $\sigma = \eta^3/3 \quad \eta = (3\sigma)^{1/3}$

$d\eta = \frac{1}{3}(3\sigma)^{-2/3} \cdot 3 = \sigma^{-2/3} d\sigma$

$I = \int_0^\infty (3\sigma)^{1/3} e^{-\sigma} \cdot \sigma^{-2/3} d\sigma = 3^{1/3} \int_0^\infty \sigma^{-1/3} e^{-\sigma} d\sigma = 3^{1/3}\, \Gamma(2/3)$

$\therefore \quad \frac{\rho c_p \beta A}{B^2} I = Q \qquad A = \frac{QB^2}{I \rho c_p \beta}$

$\therefore$

$$\Delta T = \left(\frac{\beta}{3\alpha x}\right)^{2/3} \frac{Q}{\rho c_p \beta} \cdot \frac{3}{9^{2/3}\, \Gamma(2/3)} \cdot \exp\left(-\frac{y^3 \beta}{27\alpha x}\right)$$

Self Similar – no char scale exists

Need to scale ΔT w/ X otherwise we can't satisfy integral constraint.

<u>Method of separation of variables</u>

EV problems solve BC only not I/C

Linear EV problems are
linear, homogeneous DE's and BC
Separation of Variables only works in simple domains

These will be building block problems.

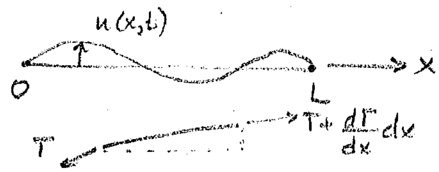
1/17/79

TYPE <NA-LIB> ODE.DOC

Vibrating String

Eqn of motion

$$a^2 \underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{restoring force}} - \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{acceleration}} = 0 \quad (1)$$



B.C. $u=0$ @ $x=0$ (2)

$u=0$ @ $x=L$ (3)

Separation of Variables - try to find solutions of form $u(x,t) = X(x)\,T(t)$ w/BC.

(1), (2), (3) are homogeneous in u ∴ Ku is also a solution
homogeneous problems have $u=0$ as a trivial solution. Other problems are eigensolutions

$$\therefore \quad a^2 \frac{X''T}{XT} - \frac{XT''}{XT} = 0 \qquad \text{from (1)}$$

$$\therefore \quad \underbrace{a^2 \frac{X''}{X}}_{\text{fn of } x} = \underbrace{\frac{T''}{T}}_{\text{fn of } t} = -\omega^2 \implies \text{constant}$$

$$T'' + \omega^2 T = 0 \implies T = c_1 \sin\omega t + c_2 \cos\omega t = \overbrace{c_3}^{\text{amplitude}} \cos(\omega t + \overbrace{\phi}^{\text{phase shift}})$$

$$a^2 X'' + \omega^2 X = 0 \implies \text{let } \lambda = \omega/a \quad \therefore \quad X'' + \lambda^2 X = 0 \implies$$

$$X = c_4 \sin\lambda x + c_5 \cos\lambda x$$

B.C. $\forall t$ $u=0$ @ $x=0$

$u=0$ @ $x=L$

$$u(0,t) = X(0)\,T(t) = 0 \quad \Rightarrow \quad \forall t \quad X(0) = 0 \quad \Rightarrow \quad C_3 = 0$$

$$u(L,t) = X(L)\,T(t) = 0 \quad \Rightarrow \quad \forall t \quad X(L) = 0 \quad \Rightarrow \quad C_4 \sin \lambda L = 0$$

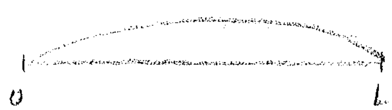
$$\therefore \quad \lambda L = n\pi \quad \text{or} \quad \lambda_n = \frac{n\pi}{L} = \frac{\omega_n}{a} \qquad \lambda_n \text{ are the EV's}$$

$$\therefore \quad X(x) = C_n \sin \frac{n\pi x}{L} \quad \overrightarrow{EV} \quad \therefore \quad \frac{n\pi a}{L} = \omega_n$$

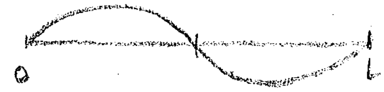
$$\text{thus} \quad u(x,t) = X(x)\,T(t) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi x}{L} \left(\cos\left[\frac{n\pi a t}{L} - \phi_n\right]\right)$$

A_n, ϕ_n are not determined since we have not given IC.

$$u_1(x,t) = A_1 \sin\frac{\pi x}{L} \cos\left(\frac{\pi a t}{L} - \phi_1\right)$$

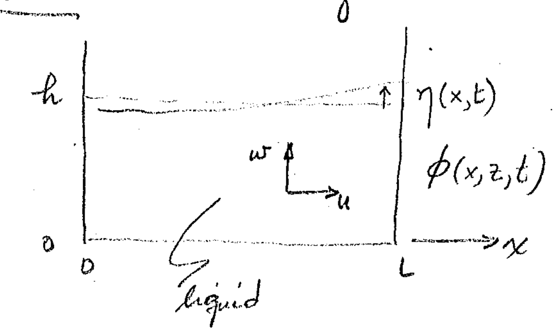


$$u_2(x,t) = A_2 \sin\frac{2\pi x}{L} \cos\left(\frac{2\pi a t}{L} - \phi_2\right)$$



EV problems arise because of the homogeneities of the problem or the b.c.

<u>Problem</u> Sloshing



potential function ϕ $\ni$ $\dfrac{\partial \phi}{\partial x} = u$, $\dfrac{\partial \phi}{\partial z} = w$

DE: Continuity $\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = \phi_{xx} + \phi_{zz} = 0$ (1)

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at } z = h \qquad (2)$$

g – accel of gravity; Bernoulli Eq (momentum Eq)

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial z} = 0 \quad @\ z = h \text{ on surface} \qquad \text{Kinematics (}$$

$\frac{\partial \phi}{\partial x} = u = 0$ @ $x=0, L$ and $\frac{\partial \phi}{\partial z} = 0 = w$ @ $z=0$ (4) Impermeable wall bc. fluid can't go through wall

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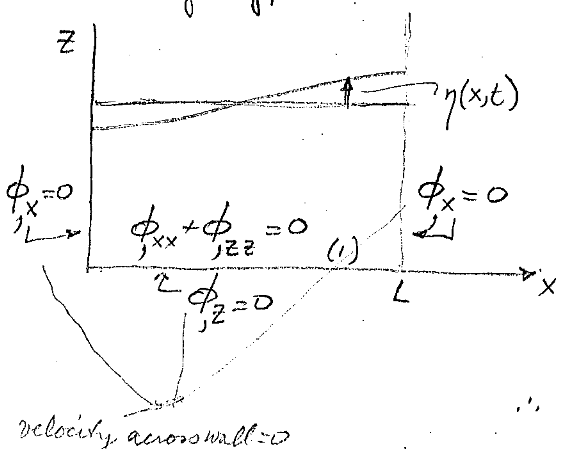
Sloshing Phys/Math



on $z=h$

$\phi_{,t} + g\eta = 0$ (2) $\quad \phi : \phi(x,z,t)$ velocity pot.

$\eta_{,t} - \phi_{,z} = 0$ (3)

Assume $\phi = X(x)Z(z)T'(t)$

$\eta = F(x)G(t)$

$\therefore \quad XZT' + gFG = 0 \quad (2)$

$X''ZT + XZ''T = 0 \quad (1) \Rightarrow \frac{X''}{X} = \frac{-Z''}{Z} = -\lambda^2$

For (1):

$X'' + \lambda^2 X = 0 \; ; \quad Z'' - \lambda^2 Z = 0 \Rightarrow X = c_1 \sin\lambda x + c_2 \cos\lambda x \; ; \; Z = c_3 \sinh\lambda z + c_4 \cosh\lambda z$

BC $\left.\phi_{,x}\right|_{x=0} = 0 \Rightarrow X'(0) = 0 \Rightarrow c_1 = 0$

$\left.\phi_{,x}\right|_{x=L} = 0 \Rightarrow X'(L) = 0 \Rightarrow \lambda L = n\pi$

$\longrightarrow \quad X = c_2 \cos\frac{n\pi x}{L}$

$\left.\phi_{,z}\right|_{z=0} \Rightarrow Z'(0) = 0 \Rightarrow c_3 = 0 \longrightarrow Z = c_4 \cosh\lambda z$

combine (2) & (3) take $\frac{\partial}{\partial t}(2) - g(3) \Rightarrow \phi_{,tt} + g\phi_{,z} = 0$ or

on $z=h$ $\quad XZT'' + gXZ'T' = 0$ thus $T'' + g\frac{Z'(h)}{Z(h)}T' = 0$

now $\frac{Z'(h)}{Z(h)} = \frac{+\lambda c_4 \sinh(\lambda h)}{c_4 \cosh(\lambda h)} = \lambda \tanh(\lambda h)$ $\quad$ let $\frac{\omega^2}{g} = \lambda \tanh\lambda h$

$\therefore \quad T = c_5 \sin\omega t + c_6 \cos\omega t = \underbrace{a}_{\text{amplitude}} \cos(\omega t - \underbrace{\psi}_{\text{phase shift}})$

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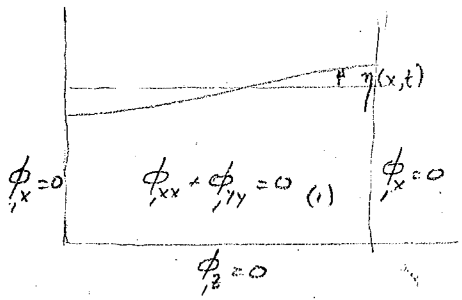


$\phi = XZT$

$\eta = FG$

$\phi = A\cos\lambda x \cosh\lambda z \cos(\omega t-\psi)$

$\lambda_n = n\pi/L$

$\omega_n^2 = g\lambda_n \tanh\lambda_n h$

$T = a\cos(\omega t-\psi)$

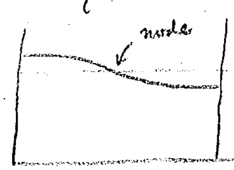
on $z=h$ $\quad \phi_{,t} + g\eta = 0$ (2)

$\eta_{,t} - \phi_{,z} = 0$ (3)

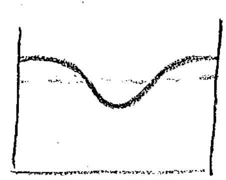
for $h\to\infty$ $\quad \tanh(\lambda_n h) \to 1$ $\quad \omega_n^2 = g\lambda_n = \frac{n\pi g}{L}$ $\quad \omega = \sqrt{\frac{n\pi g}{L}}$ $\quad f = \frac{\omega}{2\pi}$

Use equation (2) $\quad \eta = -\phi_{,t}/g = \boxed{\frac{A\omega}{g}\cos\lambda x \cosh\lambda h \sin(\omega t-\psi) = \eta(x,t)}$

First mode $n=1$ $\quad \eta(x,t) = \text{const}$ $\quad \text{const} \times \cos\left(\frac{\pi x}{L}\right)\sin(\omega t-\psi)$



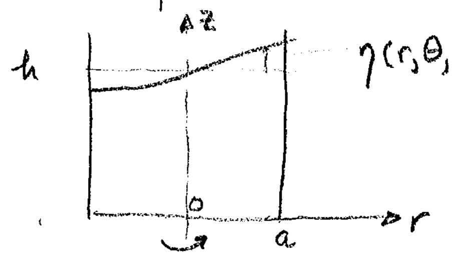
$n=2$



When do we use separation of var.

If a linear homogeneous d.e. in a simple geometry w/ zero b.c. give eigensolution

Circular Cylindrical coordinate system.



$\eta(r,\theta,t)$

define a velocity potential ϕ

Continuity $\quad \nabla^2\phi = 0 = \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz}$

B.C. $\left.\begin{array}{l}\phi_{,z} = 0 \text{ @ } z=0 \\ \phi_{,r} = 0 \text{ @ } r=a\end{array}\right\}$ solid wall bc

$\left.\underset{\text{Dynamics}}{\phi_t + g\eta = 0 \ (2)} \quad \underset{\text{Kinematics}}{\eta_t - \phi_z = 0 \ (3)}\right\}$ at $z=h$

assume $\phi = R(r)\,\Theta(\theta)\,Z(z)\,T(t)$

$$\underbrace{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}}_{\text{independent of } z} + \underbrace{\frac{Z''}{Z}}_{\text{fn of } z} = 0$$

$$\therefore \quad \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\frac{Z''}{Z} = -\lambda^2 \quad \Rightarrow \quad Z'' - \lambda^2 Z = 0$$

$$r^2\left[\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right] = -r^2\lambda^2 \quad \Rightarrow \quad r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda^2 = -\frac{\Theta''}{\Theta} = \beta^2$$

$$\Rightarrow \Theta'' + \beta^2\Theta = 0$$

and $r^2R'' + rR' + (r^2\lambda^2 - \beta^2)R = 0$

$\Theta = c_1\sin\beta\theta + c_2\cos\beta\theta$ solutions must be periodic in θ $\Rightarrow$ $\beta = \text{integer} = n$

AMF 9.1.1: $z^2\dfrac{d^2w}{dz^2} + z\dfrac{dw}{dz} + (z^2 - \nu^2)w = 0$ let $\lambda r = z$ $w = R$ $\nu = n$ defines a bessel

$$R = c_1 J_n(\lambda R) + c_2 Y_n(\lambda R)$$

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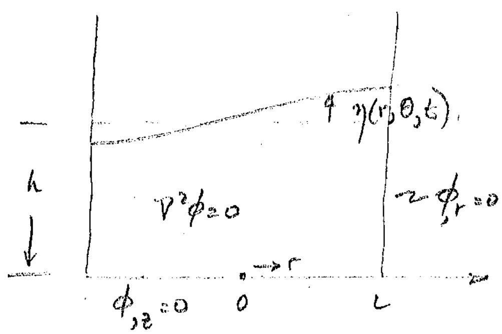


$$\left.\begin{array}{ll} \phi_t + g\eta = 0 & (2) \\ \eta_t - \phi_z = 0 & (3) \end{array}\right\} @\ z = h$$

$\phi = R\Theta Z T$

$\Theta'' + n^2\Theta = 0$

$r^2R'' + rR' + (r^2\lambda^2 - n^2)R = 0$

$Z'' - \lambda^2 Z = 0$

(2) & (3) $\Rightarrow$ $\phi_{tt} + g\phi_z = 0$ @ $z = h$

$T''Z + gZ'T = 0$ or $\dfrac{T''}{T} = -g\dfrac{Z'(h)}{Z(h)} = -\omega^2$ $\therefore$ let $g Z'(h)/Z(h) = \omega^2$

time: $\therefore$ $T'' + \omega^2 T = 0 \Rightarrow T = A_1\sin\omega t + A_2\cos\omega t$

height: $Z'' - \lambda^2 Z = 0 \Rightarrow Z = B_1\sinh\lambda z + B_2\cosh\lambda z$

$\phi_{,z} = 0 \Rightarrow Z'(0) = 0 \Rightarrow B_1 = 0$ $\therefore$ $\boxed{Z = B_2\cosh\lambda z}$

hence $\Rightarrow$ $Z'/Z\Big|_{z=h} = \lambda\tanh\lambda h \Rightarrow \boxed{\omega^2 = g\lambda\tanh\lambda h}$

ω^2 determined by λ : @ $h \to \infty$ $\omega^2 \to g\lambda$

- Radial problem : $\phi_{,r} = 0 \;\forall$ points on wall $\quad \therefore R'(a) = 0$

2nd BC : R is finite everywhere

$$r^2R'' + rR' + (\lambda^2 r^2 - n^2)R = 0$$

Linearly independent soln : pick any two

$J_n(\lambda r) \qquad J_{-n}(\lambda r) \qquad$ are l.i. if $n \neq$ integer

$Y_n(\lambda r)\,,\quad H_n^{(1)}(\lambda r)\,,\quad H_n^{(2)}(\lambda r)\,,\quad N_N(\lambda r)$

let $R(r) = C_1 J_n(\lambda r) + C_2 Y_n(\lambda r) \qquad Y_n(z) \to \infty$ at $z = 0 \quad \therefore C_2 = 0$

for finite soln.

$\therefore \; R(r) = C_1 J_n(\lambda r)$

$R'(a) = C_1 \lambda J_n'(\lambda a) = 0 \qquad$ So on pg 409 of the HMF $\quad j_{0,s} = J_0(X_s) = 0$

$j'_{0,s} = J_0'(X_s) = 0$

n	s	λa	mode #
0	1	3.831	3
0	2	7.0156	
1	1	1.841	1
1	2	5.331	
2	1	3.054	2

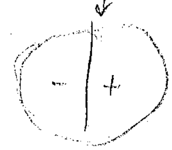
from mode ① : $R_{1,1} = C_1 J_1(\lambda_{1,1} r) \qquad \lambda_{1,1} = \frac{1.841}{a}$

for infinite depth $\quad \omega_{1,1}^2 = g\lambda_{1,1} = g\frac{1.841}{a}$

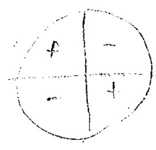
now mode 1 $\Rightarrow$ n = 1 $\qquad \Theta'' + \Theta = 0 \;\Rightarrow\; \Theta = A\cos\theta + B\sin\theta$

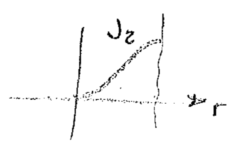
$$\phi_{1,1} = A\cos\theta \cosh\left(\frac{1.841z}{a}\right) J_1\left(\frac{1.841r}{a}\right)\cos\left(\sqrt{\frac{1.841g}{a}}\,t\right)$$

node line

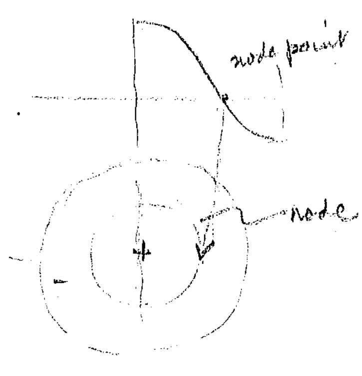


for mode ② :





for ③ :



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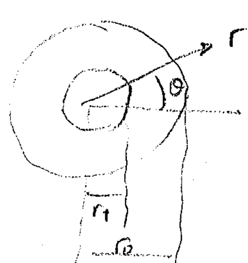
Annular Membrane



$c^2\nabla^2 u - u_{tt} = 0$ u is deflection c^2 = membrane param.

$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$

BC $u=0$ @ $r = r_i, r_o$

Linear homogeneous w/ char lengths: use separation of variable

Try $u = R\Theta T$

$$\therefore \quad \underbrace{c^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right)}_{\text{fn of } r,\theta} = \underbrace{\frac{T''}{T}}_{\text{fn of } t} = \text{const} = -\omega^2$$

$$T'' + \omega^2 T = 0 \Rightarrow T = c_1\cos\omega t + c_2\sin\omega t = B\cos(\omega t - \phi)$$

$$r^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right) + \frac{\omega^2 r^2}{c^2} = -\frac{\Theta''}{\Theta} = \beta^2$$

$\Theta'' + \beta^2\Theta = 0 \Rightarrow \Theta = \hat{c}_1\sin\beta\theta + \hat{c}_2\cos\beta\theta$ for continuity in θ $\beta \equiv$ integer $= n$

$r^2R'' + rR' - (\beta^2 - \alpha^2r^2)R = 0$ where $\alpha^2 = \frac{\omega^2}{c^2}$

$r^2R'' + rR' + (\alpha^2r^2 - n^2)R = 0$ $R = A_1J_n(\alpha r) + A_2Y_n(\alpha r)$

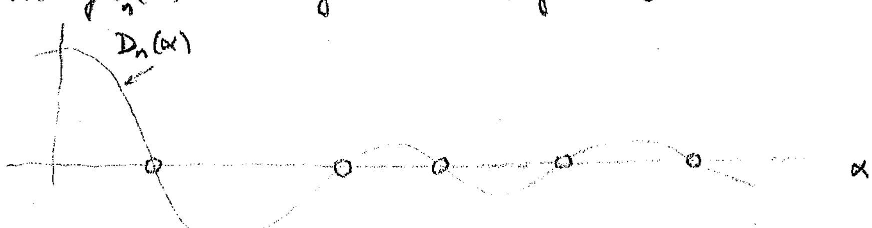
BC. $u = 0$ $r = r_i, r_o$ $R(r_i) = 0$ or $R(r_o) = 0$

$$A_1J_n(\alpha r_i) + A_2Y_n(\alpha r_i) = 0$$
$$A_1J_n(\alpha r_o) + A_2Y_n(\alpha r_o) = 0$$

For non trivial A_1, A_2

$$\therefore \quad \det\left[J_n(\alpha r_i)Y_n(\alpha r_o) - J_n(\alpha r_o)Y_n(\alpha r_i)\right] = 0 = D_n(\alpha)$$

thus the roots of $D_n(\alpha) = 0$ give the eigenvalues α



let $x = \alpha r_i$ $\lambda = r_o/r_i$ $\det\left(J_n(x)Y_n(\lambda x) - J_n(\lambda x)\cdot Y_n(x)\right) = 0$

How to get $R_{nm}(r) \Rightarrow R_{nm}(r) : J_n(\ \ )$ m^{th} root of D.

go back to eqns for BC pick first eqn: $\frac{A_1}{A_2} J_n(\alpha r_i) + Y_n(\alpha r_i) = 0$
this gives ratio of A_1/A_2 only this gives shape

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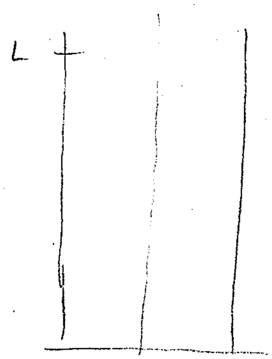


Acoustics in a tube

p = pressure fluctuation

Governing: $c^2 p_{,xx} - p_{,tt} = 0$ $\quad$ c = speed of sound (1)

@ $x = 0$ $\quad \frac{\partial p}{\partial x} = 0$ (2)

We will force press at open end i.e. @ $x = L$ $\quad p = a\cos\omega t$ (3)

Still there is no initial conditions but we seek periodic soln.

Soln: this is a separable problem and we observe that in a linear homogeneous PDE where the coeffs are independent of a variable (say t) then solutions proportional to e^{at} are possible.

Let $P = X(x)\cos\omega t$

$$c^2 p_{xx} - p_{tt} = 0 \Rightarrow c^2 X'' \cos\omega t + \omega^2 X \cos\omega t = 0$$

$$\therefore \quad X'' + \lambda^2 X = 0 \qquad \lambda = \frac{\omega}{c}$$

w/ $\frac{\partial P}{\partial x} = 0$ @ $x = 0$ $\Rightarrow$ $X'(0) = 0$

@ $x = L$ $\qquad X(L) = a$ $\qquad$ since $P(L,t) = a\cos\omega t$

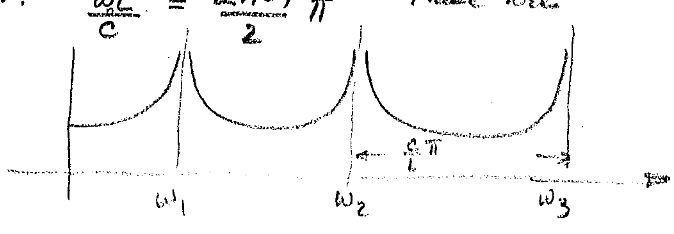
$\therefore$ let $X = C_1 \sin\lambda x + C_2 \cos\lambda x \Rightarrow C_1 = 0 \quad C_2 = \frac{a}{\cos\lambda L}$

$$\therefore \quad P(x,t) = \frac{a}{\cos\lambda L}\cos\lambda x \cos\omega t. = \frac{a}{\cos\frac{\omega}{c}L}\cos\frac{\omega x}{c}\cos\omega t$$

For $\cos\frac{\omega L}{c} = 0$ and a being small $\Rightarrow p \to \infty$: resonant condition

$\therefore$ It turns out that $\cos\frac{\omega L}{c}$ represents the natural modes of the system.

$\therefore \quad \frac{\omega L}{c} = \frac{2n-1}{2}\pi$ $\quad$ these are the normal mode



Now let's increase the complexity at the $x = L$ bc
and let $p(L,t) = a\cos\omega t - \frac{1}{2}a(\cos 2\omega t)$ (3a)

again assume $P = XF(t)$ $\therefore$ $c^2X''F - XF'' = 0$
by our normal arguments $c^2\frac{X''}{X} = \frac{F''}{F} \stackrel{?}{=}$ constant but is $\frac{F''}{F}$ a const..

$$F' = -a\omega\sin\omega t + a\omega\sin 2\omega t$$
$$F'' = -a\omega^2\cos\omega t + 2a\omega^2\cos 2\omega t \qquad F''/F \neq \text{const.}$$

Thus we assumed an incorrect solution
We have to do it by partial solns.

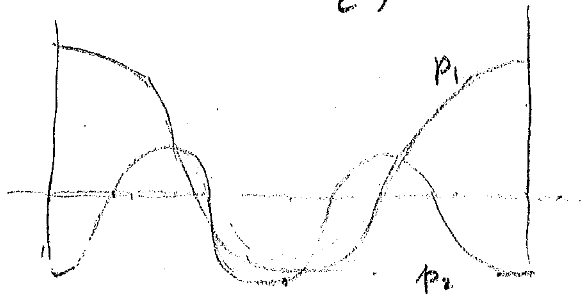
1) Solve (1), (2) and (3) to get $p_1(x,t)$

2) Solve (1), (2) and (3a) w/bc $p_2(L,t) = -\frac{1}{2}a\cos 2\omega t$

$\therefore$ $P = p_1 + p_2$. Since p_1, p_2 solve the DE & the bc @ $x=0$
and $p(L,t) = p_1(L,t) + p_2(L,t) =$ (3a) then p_1 is the solution

$$\therefore \; p_2(x,t) = \frac{a}{\cos\left(\frac{2\omega L}{c}\right)}\cos\left(\frac{2\omega x}{L}\right)\cos(2\omega t)$$

$\therefore$



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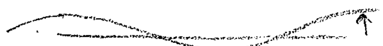
Vibrating rope



(1) $$c^2 u_{xx} - u_{tt} = 0 \qquad (1)$$

$$w/\ u(x=0,t) = 0 \qquad (2)$$

$$w/\ u(x=L,t) = \sum_{n=1} a_n \cos(n\omega t) + b_n \sin(n\omega t) \qquad (3)$$

This is an incomplete problem since IC not specified

– define partial solutions $\sum u_n(x,t) = u(x,t)$

Let $u_n = X(x)\begin{Bmatrix}\sin(n\omega t)\\ \cos(n\omega t)\end{Bmatrix}$ since $u(0,t)=0 \Rightarrow X(x) = \dfrac{A \sin\frac{n\omega x}{c}}{\sin\frac{n\omega L}{c}}$

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} \frac{a_n \sin\frac{n\omega x}{c}}{\sin\frac{n\omega L}{c}} \cos n\omega t + \sum_{n=1}^{\infty} \frac{b_n \sin\frac{n\omega x}{c}}{\sin\left(\frac{n\omega L}{c}\right)} \sin n\omega t$$

this $\neq F(X)G(t)$ since the problem has an inhomogeneous bc

Suppose $a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) = f(t)$

in Fourier series (1) $f(t)$ is periodic in T with period $2\pi/\omega$

(2) $f(t)$ bounded (finite)

(3) need not be continuous $f(x) = \dfrac{f(x^+) + f(x^-)}{2}$

if f is periodic, bounded, and C^1. The series $\sum_{n=0}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$ converges to $f(t)$ $\forall t$.

if f is periodic, bounded and only C^0, the series converges but not at discont.
at discontinuity x_i $f(x_i) = \dfrac{f(x_i^+) + f(x_i^-)}{2}$

a_n and b_n are determined by the orthogonality property of sines and cosines

$$\frac{1}{\pi}\int_0^{2\pi} \cos n\theta \cos m\theta\, d\theta = \begin{matrix}0\\1\end{matrix}\Bigg\}\ \begin{matrix}n\neq m\\ n=m\end{matrix} \qquad \frac{1}{\pi}\int_0 \cos n\theta \sin m\theta\, d\theta = 0 \quad \forall n m$$

$$\frac{1}{\pi}\int_0^{2\pi} \sin n\theta \sin m\theta\, d\theta = \begin{matrix}0\\1\end{matrix}\Bigg\}\ \begin{matrix}n\neq m\\ n=m\end{matrix}$$

$$f(t) = \sum_{n=0}^{\infty} a_n \cos n\omega t + \sum_{n=0}^{\infty} b_n \sin(n\omega t)$$

mult by $\sin m\omega t$ over period 2π

$$\therefore \int_0 f(t) \sin m\omega t\, dt = \frac{\pi b_n}{\omega} \qquad \therefore \quad b_n = \frac{\omega}{\pi} \int_0^{2\pi} f(t) \sin n\omega t\, dt$$

if $T = \frac{2\pi}{\omega}$ then in general $\frac{2}{T} \int_0^{T} f(t) \sin n\theta\, d\theta = b_n$

$$\int_0^{T} f(t) \cos(m\omega t)\, dt = a_m \frac{\pi}{\omega} \qquad a_m = \frac{\omega}{\pi} \int_0^{2\pi} f(t) \cos m\omega t\, dt$$

$$\int_0^{T} f(t)\, dt = a_0 T \qquad a_0 = \frac{\omega}{2\pi} \int_0^{2\pi} f(t)\, dt$$

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$c^2 u_{xx} - u_{tt} = 0$

BC $u = 0$ @ $x = 0, L$ (1)

IC $u(x,0) = f(x)$; $u_t(x,0) = g(x)$ (2)

1. get partial solns that satisfy homo conditions
2. Partial soln by SOV

$$u_n(x,t) = X_n(x) T(t)$$

(1.) & (2) $\Rightarrow$ $T'' + \omega^2 T = 0$

$X'' + \lambda^2 X = 0$ $\quad \lambda = \frac{\omega}{c} \Rightarrow X_n = \sin(\lambda_x X) \quad \lambda_n L = n\pi$

$T_n = \sin \omega_n t$ or $\cos \omega_n t$ but $\cos(\omega_n t)$ has $T_n'(0) = 0$

$\sin(\omega_n t)$ has $T_n(0) = 0$

$\therefore \cos(\omega_n t)$ would be assoc with $u_n(x,0) = f(x)$

$\sin(\omega_n t)$ " " " " $u_t(x,0) = g(x)$

All satisfy (1) and (2)

Now construct

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n u_n^{(1)} + B_n u_n^{(2)} \right) \text{ when } \begin{array}{ll} u_n^{(1)} = \sin \lambda_n x \cos \omega_n t & (3) \\ u_n^{(2)} = \sin \lambda_n x \sin \omega_n t & (4) \end{array}$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \lambda_n x \qquad (5)$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} A_n \omega_n \cos \omega_n t \qquad (6)$$

$$\int_0^L X_n(x) X_m(x)\, dx = 0 \quad n \neq m \quad \text{orthog of EV } X_n$$

ASIDE. PROOF:

Using the diff eq

(a) $\quad X_n'' + \lambda_n^2 X = 0 \quad$ also $\quad X_m'' + \lambda_m^2 X_m = 0 \quad$ (b)

Mult (a) by X_m and (b) by X_n

$$\therefore \int_0^L [(a) X_m - (b) X_n]\, dx = \int_0^L (X_m X_n'' - X_n X_m'')\, dX + (\lambda_n^2 - \lambda_m^2) \int_0^L X_n X_m\, dx = 0$$

Integrate by parts

$$X_m X_n' \Big|_0^L - \int_0^L X_n' X_m'\, dx - X_n X_m' \Big|_0^L + \int_0^L X_m' X_n'\, dx + (\lambda_n^2 - \lambda_m^2) \int_0^L X_n X_m\, dx = 0$$

from b.c.: $X_m, X_n = 0$ @ $x = 0, L \quad \therefore$ if $\lambda_n^2 - \lambda_m^2 \neq 0 \quad \int_0^L X_n X_m\, dx \equiv 0$

Partial solutions must satisfy b.c. that are homogeneous.

since $\quad f(x) = \sum_{n=1}^{\infty} A_n X_n(x) \qquad \int_0^L f(x) \sin \lambda_n x\, dx = A_n \cdot \frac{L}{2}$

$$\therefore A_n = \frac{2}{L} \int_0^L f(x) \sin \lambda_n x\, dx = \frac{2}{n\pi} \int_0^L f(x) \sin \lambda_n x\, dx$$

since $\quad g(x) = \sum_{n=1}^{\infty} B_n \sin \lambda_n x \cdot \omega_n\, dx \implies \int_0^L g(x) \sin \lambda_n x\, dx = B_n \omega_n \frac{L}{2}$

$$\therefore B_n = \frac{2}{\omega_n L} \int_0^L g(x) \sin \lambda_n x\, dx$$

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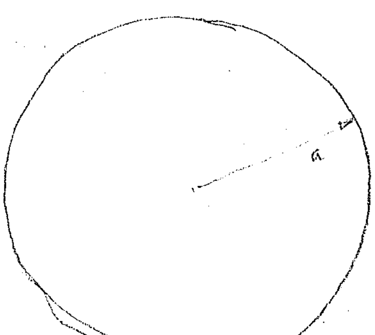

circular membrane

$u(r,t)$ displacement

(1) $\quad c^2 \Delta u - u_{tt} = f \quad w/ \ \Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right)$

where $f = \bar{a}$ const

(3,4) $u(r,0) = 0 \quad \frac{\partial u}{\partial t}(r,0) = 0 \quad$ are the initial

(2) BC: $u(a,t) = 0$

(1) u is inhomogeneous $\qquad$ (2-4) u is homogeneous

Want to find particular solution $u_p(r,t)$ which satisfies PDE

let $u_p(r,t) = F(r)\cos\omega t$

$$\therefore \quad c^2 \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u_p\right)\right) + \omega^2 F = \bar{a} \qquad \text{let } F = \frac{\bar{a}}{\omega^2} + c_1 J_0\left(\frac{\omega r}{c}\right) + c_2 Y_0\left(\frac{\omega}{c}r\right)$$

if $F = \frac{a}{\omega^2}$ only this doesn't satisfy bc $\therefore$ look at entire soln. now $c_2 = 0$ since $r=0$ is in the domain.

Pick c_1 $\ni$. $F(a) = \frac{\bar{a}}{\omega^2} + c_1 J_0\left(\frac{\omega}{c}a\right) = 0$

since $u(a,t) = 0$

$$\therefore u_p = \frac{\bar{a}}{\omega^2}\left[1 - J_0\left(\frac{\omega r}{c}\right)/J_0\left(\frac{\omega a}{c}\right)\right]\cos\omega t$$

$J_0\left(\frac{\omega a}{c}\right) = 0 \Rightarrow \omega_n$ (normal modes)

Now look at u_h $\ni$. $c^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_h}{\partial r}\right)\right) - u_{h tt} = 0$ (5)

BC. $u(a,t) = u_p(a,t) + u_h(a,t) = 0 + u_h(a,t) = 0 \Rightarrow u_h(a,t) = 0$ (6)

IC. $u(r,0) = u_p(r,0) + u_h(r,0) = \frac{\bar{a}}{\omega^2}\left[1 - \frac{J_0\left(\frac{\omega r}{c}\right)}{J_0\left(\frac{\omega a}{c}\right)}\right] + u_h(r,0) = 0$

$$\therefore u_h(r,0) = \frac{\bar{a}}{\omega^2}\left[\frac{J_0\left(\frac{\omega r}{c}\right)}{J_0\left(\frac{\omega a}{c}\right)} - 1\right] \qquad (7)$$

$$u_t(r,0) = u_{p,t}(r,0) + u_{h,t}(r,0) = 0 + u_{h,t}(r,0) = 0$$

$$\therefore u_{h,t}(r,0) = 0 \qquad (8)$$

(5), (6), (7) can be solved by sep of variables $\Rightarrow u_{hn}(r,t)$ eigensolns

homogeneous define $u_h = \sum_{n=1}^{\infty} c_n u_{hn}(r,t)$ find c_n $\ni$. (8) is satisfied

let $u_{hn} = R(r)T(t)$

$$\Rightarrow c^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right) = \frac{T''}{T} = -\gamma^2$$

$T = A\cos\gamma t + \underbrace{B\sin\gamma t}_{\to 0}$ since $u_t(r,0) = 0$

$r^2R'' + rR' + \left(\frac{\gamma^2}{c^2}r^2\right)R = 0 \Rightarrow R = c_1 J_0\left(\frac{\gamma}{c}r\right) + c_2 \cancel{Y_0}\left(\frac{\gamma}{c}r\right)$

$\to 0$ since $r=0$ is in domain

$$\therefore R = c_1 J_0\left(\frac{\gamma}{c}r\right)$$

$u_h(a,t) = 0 \Rightarrow R(a) = 0 \quad \therefore \quad a = \text{a zero of } J_0\left(\frac{\gamma a}{c}\right) = 0$

$u_{nn} = R_n(r)\cos\gamma_n t \qquad R_n(r) = J_0(\lambda_n r) \; : \; \lambda_n = \frac{\gamma_n}{c}$

$\therefore \; u_n = \sum c_n R_n(r)\cos\gamma_n t$

Since each R_n satisfies the ODE then

$$rR_n'' + R_n' + \lambda_n^2 r R_n = 0$$

$$(rR_n')' + \lambda_n^2 r R_n = 0 \qquad \text{This is the Sturm-Liouville form.}$$

Now we will prove $\int_0^a rR_m R_n \, dr = 0 \quad n \neq m.$

$$\left.\begin{aligned} &\int_0^a R_m (rR_n')' dr + \int_0^a \lambda_n^2 r R_n R_m \, dr = 0 \\ &\int_0^a R_n (rR_m')' dr + \int_0^a \lambda_m^2 r R_m R_n \, dr = 0 \end{aligned}\right\} \int_0^a \left[R_m (rR_n')' dr - R_n (rR_m')' dr\right. + (\lambda_n^2 - \lambda_m^2)\int_0^a rR_n R_m \, dr = 0$$

now integ by parts

$$\Big[R_m(R_n' r) - R_n(rR_m')\Big]_0^a - \int_0^a \Big[(rR_n')\cdot R_m' - (rR_m')R_n'\Big] dr + \ldots$$

at $r=0 = 0$ @ $r=a$ $R_m(a) = R_n(a) = 0$ (BC)

since R is bdd on $r=0$

$$\therefore \int_0^a rR_n R_m \, dr = 0 \quad \text{iff } \lambda_n^2 - \lambda_m^2 \neq 0$$

but $\quad u_n(r,0) = \sum_{n=1}^{\infty} c_n R_n(r)$

$$\frac{a}{\omega^2}\left[\frac{J_0\left(\frac{\omega r}{c}\right)}{J_0\left(\frac{\omega a}{c}\right)} - 1\right] = \sum_{n=1}^{\infty} c_n R_n(r) = f(r)$$

use orthog prop to get c_n

$$\int_0^a r R_m f(r)\, dr = c_n \sum \int_0^a r R_n R_m \, dr = c_m \left[\int_0^a r R_m^2 \, dr\right]$$

$$\therefore \; c_m = \frac{\int_0^a r R_m(r) f(r)\, dr}{\int_0^a r R_m^2 \, dr}$$

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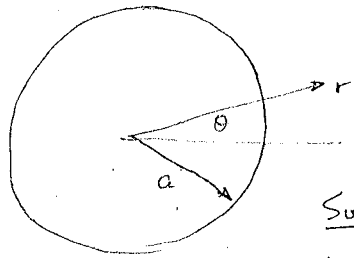

$$c^2\left(u_{rr}+\frac{1}{r}u_r\right)-u_{tt}=\bar{a}\cos\omega t$$

IC $u(r,0)=u_t(r,0)=0$

BC $u(a,t)=0$

Soln

$u=u_p+v(r,t)$

$u_p=F(r)\cos\omega t \implies (rF')'+\left(\frac{\omega^2}{c^2}\right)rF=r\bar{a} \qquad F(a)=0$

$v_{tt}-c^2\nabla^2 v=0$

$v(0,t)=0 \qquad v_t(r,0)=0 \qquad v(r,0)=-u_p=-F(r)=\sum_{n=1}^{\infty}c_nR_n(r)$

$F=\ldots\ldots$

$v=\sum_{n=1}^{\infty}c_nR_n(r)\cos\gamma_nt \implies (rR_n')'+\lambda_n^2rR_n=0 \quad ; R_n(a)=0$

$\gamma_n=\lambda_nc$

Orthogonality: $\int_0^a rR_nR_m\,dr=0 \quad n\neq m$

we found $c_m=\dfrac{-\int_0^aF(r)\,rR_m\,dr}{\int_0^a rR_m^2\,dr}=\dfrac{I_1}{I_2}$

consider $\int_0^a rF(r)R_m(r)\,dr$ use the DE: since $rR_m=-\frac{1}{\lambda_m^2}(rR_m')'$

then $\int_0^a-\frac{F(r)}{\lambda_m^2}(rR_m')'\,dr=-\frac{1}{\lambda_m^2}F(rR_m')\Big|_0^a+\frac{1}{\lambda_m^2}\int_0^a(rR_m')F'(r)\,dr$

since $F(a)=0$ & $r=0$ @ $r=0$ the first term $=0$

$$=\frac{1}{\lambda_m^2}\int_0^aR_m'(rF'(r))\,dr=\frac{1}{\lambda_m^2}\left[R_m(rF')\Big|_0^a-\int_0^aR_m(rF')'\,dr\right]$$

$R_m(a)=0$ & $r=0$ @ $r=0$ $\therefore$ 1st term again drops out

now

$$\int_0^a rFR_m\,dr=-\frac{1}{\lambda_m^2}\int_0^aR_m(rF')'\,dr=-\frac{1}{\lambda_m^2}\left[\int_0^aR_m\left[r\bar{a}-r\left(\frac{\omega^2}{c^2}\right)F\right]dr\right],$$

using the diff eq on F.

$$\therefore\int_0^a\left(rFR_m-\frac{1}{\lambda_m^2}\left(\frac{\omega^2}{c^2}\right)rFR_m\right)dr=-\frac{1}{\lambda_m^2}\int_0^a\bar{a}\,rR_m\,dr$$

$$=\left[1-\frac{1}{\lambda_m^2}\left(\frac{\omega^2}{c^2}\right)\right]\int_0^arFR_m\,dr=-\frac{\bar{a}}{\lambda_m^2}\int_0^arR_m\,dr$$

$$\text{or} \quad \int_0^a r F R_m \, dr = \frac{-\dfrac{\bar{a}}{\lambda_m^2}\displaystyle\int_0^a r R_m \, dr}{\left(1 - \dfrac{\omega^2}{\gamma_m^2}\right)}$$

$$\int_0^a r R_m dr = -\frac{1}{\lambda_m^2}\int_0^a (r R_m')' dr = -\frac{1}{\lambda_m^2} a R_m'(a)$$

$$\therefore \quad C_m = \frac{I_1}{I_2}$$

$$I_1 = -\frac{\bar{a}\, a}{\lambda_m^4} \frac{R_m'(a)}{\left(1 - \dfrac{\omega^2}{\gamma_m^2}\right)}$$

defining I_2: Consider $R(r,\lambda)$ satisfying $(rR')' + \lambda^2 r R = 0$ * $R' = \frac{\partial R}{\partial r}$
with $R(0)$ finite satisfying the BC @ $r=0$
in general $R(a,\lambda) \neq 0$
ie we define a functional $R(r,\lambda)$ $\ni$ $R(a,\lambda_n) = 0$ define $R(\lambda_n, r) = R_n$
(Shades of Calc of Variations) differentiate wrt λ since R's de is as above*

① $(r R_\lambda')' + 2\lambda r R + \lambda^2 r R_\lambda = 0$ with $R_\lambda = \frac{\partial R}{\partial \lambda}$

② mult by R_n

$$\int_0^a \left[(r R_\lambda')' + 2\lambda r R + \lambda^2 r R_\lambda\right] R_n \, dr = 0$$

I. By Parts $\rightarrow$ $\cancel{(r R_\lambda') R_n \Big|_0^a}$ $+ \int_0^a -(r R_\lambda') R_n' + (2\lambda r R + \lambda^2 r R_\lambda) R_n \, dr = 0$

0 since $R_n(a,\lambda) = 0$, $r=0$

$$- R_\lambda (r R_n')\Big|_0^a + \int_0^a R_\lambda \cdot (r R_n')' + R_\lambda \lambda^2 r R_n + 2\lambda r R R_n$$

$$- R_\lambda(a,\lambda)\, a\, R_n'(a) \qquad \int_0^a R_\lambda \Big[\underbrace{(r R_n')' + \lambda^2 r R_n}_{=0 \text{ when } R_\lambda = R_{\lambda_n}}\Big] + 2\lambda r R R_n \, dr$$

$R(r,\lambda_n) = R_n$.

$$\therefore \; -\frac{dR}{d\lambda}(a,\lambda)\, a\, R_n'(a) + \int_0^a 2\lambda_n r R_n^2 dr = 0 \quad \therefore \quad \int_0^a r R_n^2 dr = \frac{a}{2\lambda_n} R_n'(a) \frac{\partial R}{\partial \lambda}\Big|_{\substack{\lambda=\lambda_n \\ r=a}}$$

$$\text{now } \frac{\partial R}{\partial \lambda} = \frac{\partial J_0(\lambda r)}{\partial \lambda} = r J_0'(\lambda r)\Big|_{\substack{\lambda=\lambda_n \\ r=a}} a \cdot \lambda_n J_0'(\lambda_n a)$$

$$\therefore \; \frac{a}{2\lambda_n}\left[a^2 \lambda_n J_0'^2\right] = \frac{a^3}{2} J_0'(a) = \frac{a^3}{2} J_1^2$$

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Look at the Sturm-Liouville problem

$$\frac{d}{dx}\left(S\frac{dy}{dx}\right) + (Q+\lambda^2 P)y = 0 \quad \text{where } S, Q, P \text{ are fns of } x \qquad (1)$$

Any 2nd order eqn can be put in Sturm Liouville form.

$A(x)y'' + B(x)y' + C(x)y = 0$ (2) to make (2) look like (1) we multiply by some function $f(x)$

$Afy'' + Bfy' + Cfy = 0 = S\frac{d^2y}{dx^2} + S'\frac{dy}{dx} + (Q+\lambda^2P)y = 0$ To find f match coeffs.

$S = Af$

$S' = Af' + A'f$ but $S' = Bf$ $\therefore$ $f' = \frac{B-A'}{A}f$ or $\frac{f'}{f} = \frac{B-A'}{A}$

$\therefore \ln f = \int \frac{B-A'}{A}dx$

Now general Sturm-Liouville B.C. is $\alpha y + \beta y' = 0$ at $x=a$

$\gamma y + \delta y' = 0$ at $x = b$

The orthogonality condition is $\int_a^b P(x)\, y_n y_m\, dx = 0$ if $\lambda_n^2 \neq \lambda_m^2$

Any general linear ODE $Ly = 0 \Rightarrow My + \lambda Ny = 0$ where λ is the e.v.

L, M, N are linear diff operators

We can define a linear adjoint operator L^* $\ni$ $\int_a^b v\,Lu\,dx =$ boundary value terms (something like $\alpha UV|_a^b$) $+ \int_a^b u\,L^*v\,dx$

$L^*v = 0$ is the adjoint equation a self adjoint operator is when $L = L^*$

$$\int_a^b v\,Lu\,dx = \int_a^b v\left[(Su')' + (Q+\lambda^2P)u\right]dx = v(Su')\Big|_a^b - \int_a^b (Su')v'dx + \int_a^b v(Q+\lambda^2P)u\,dx$$

$$= \underbrace{v(Su')\Big|_a^b - u(Sv')\Big|_a^b}_{\text{boundary terms}} + \int_a^b u\underbrace{\left[(Sv')' + (Q+\lambda^2P)v\right]}_{L^*v = Lu}dx$$

Any 2nd order problem (linear DE) is self adjoint

Suppose $u=0$ @ $x = a, b$ and $Lu = 0$ then u is an eigenfunction if we pick $v = 0$ at the b.c., the b.c. are self adjoint and $L^*v = 0$.

Suppose $Lu = 0$ and BC's are such that some of the boundary conditions drop out - Consider a function v $\ni$ $L^*v = 0$

Fact: if u is an eigensolution to $Lu=0$ w/ ($B_i u=0$ diff operator on bdy) then $L^* v=0$ is also an eigensoln if we pick all but one bc on v to be satisfied.

Example $y^{IV}+xy''+\lambda y=0$ $\quad \overset{A}{y(0)}=\overset{B}{y'(0)}=\overset{C}{y(1)}=\overset{D}{y'(1)}=0$

to find adj problem. $\int_0^1 v\,(y^{IV}+xy''+\lambda y)=0$
integrate by parts

$$v y'''\Big|_0^1 - v'y''\Big|_0^1 + \underset{B,D}{v''y'\Big|_0^1} - \underset{A,C}{v'''y\Big|_0^1} + \underset{B,D}{vxy'\Big|_0^1} - \underset{A,C}{y\,(vx')\Big|_0^1} +$$

$$\int_0^1 y\,(\underbrace{v^{IV}+(vx)''+\lambda v}_{L^*v})\,dx \qquad L\neq L^* \quad \text{problem not self adj}$$

Adj problem $\quad v^{IV}+(xv)''+\lambda v=0 \quad$ Chose $v(0)=v(1)=v'(0)=v'(1)=0$

Claim when λ is $\ni$. the main problem (for y) has non-trivial soln. So does the adjoint problem.

Orthog

$Lu=Mu+\lambda Nu=0 \quad$ for the n_{th} eigensoln $Mu_n+\lambda_n Nu_n=0$

now

$(M^* u_m+\lambda_m N^* v_m)=0$ is the adjoint $m^{\underline{th}}$ soln.

$\Rightarrow \int_a^b v_m(Mu_n+\lambda_n Nu_n)\,dx = \int_a^b u_n(M^*v_m+\lambda_m N^* v_m)\,dx=0$

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General Linear Homogeneous ODE Problem

$Lu=0 \quad$ w/ $\{B_i u=0$ set of homogeneous bc$\}$

Adjoint operator L^* w/ adjoint BC: $\{B_i^* v=0\}$

$\ni. \quad L^*v=0 \quad$ & L^*, B_i^* are chosen so that $\int_a^b v Lu\,dx=\int_a^b u L^* v\,dx \quad (1)$

if u & v are fns that satisfy $\{B_i u=0\}$ & $\{B_i^* v=0\}$ (1) will be true even if $Lu\neq 0, L^*v\neq 0$.

if $Lu=0$, & $\{B_i u=0\}$ then an eigensolution to main problem. If there is an eigensoln then the adjoint problem also has an eigensoln. Both problems have the same EV.

Orthog Prop: Now let $L=M+\lambda N$

Since $Lu_n = (M+\lambda_n N)u_n = 0$
also $L^* v_m = (M+\lambda_m N)v_m = 0$ } now form $\int_a^b (v_m L u_n - u_n L^* v_m)\,dx = 0$

Now $\int_a^b (v_m L u_n - u_n L^* v_m)\,dx = \int_a^b [(\cancel{v_m M u_n - u_n M^* v_m}) + (\lambda_n v_m N u_n - \lambda_m u_n N^* v_m)]\,dx = 0$

$+ \int_a^b (\lambda_n - \lambda_m) u_n N^* v_m\,dx = 0$

boundary terms will drop out since M^*, N^* are pick such that $\int v_m N u_n = \int u_n N^* v_m + \underbrace{\{\text{boundary terms}\}}_{=0}$

$\therefore$ if $\lambda_n - \lambda_m \neq 0$ then $\int_a^b u_n N^* v_m\,dx = 0$

Orthog property: If $\lambda_n - \lambda_m \neq 0$ $\quad \int_a^b u_n N^* v_m\,dx = 0$ or $\int v_m N u_n\,dx = 0$

System of Eqns

$$\sum_{j=1}^{n} L_{ij} u_j = 0 \qquad i=1,\dots,n \qquad \text{ODE}$$

$\{B_{ij}^{(k)} u_j = 0\}$ set of B.C.

$$\int_a^b \sum_{i=1}^{n} v_i \sum_{j=1}^{n} L_{ij} u_j\,dx = \int_a^b \sum_{i=1}^{n} u_i \sum_{j=1}^{n} L^*_{ij} v_j\,dx$$

Use

$Mu_n + \lambda_n N u_n = 0$ w/ BC etc.

we want to expand

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x) \qquad \lambda_n \neq \lambda_m$$

$$\int_a^b f(x) N^* v_m\,dx = \sum_{n=1}^{\infty} c_n \int_a^b u_n N^* v_m\,dx = 0 \qquad \text{for } n \neq m$$

$$\therefore \quad c_m = \frac{\int_a^b f N^* v_m\,dx}{\int_a^b u_m N^* v_m\,dx}$$

Inhomogeneous Eqn, homogeneous B.C. $\quad Lu = h(x)$ w/ $\{B_i u = 0\}$

Solutions are possible if $Lu = 0$ $\{B_i u = 0\}$ has no non trivial soln.

But if $Lu = 0$ $\{B_i u = 0\}$ does have a solution then you cannot necessary find a solution for arbitrary $h(x)$. But if $h(x)$ is orthog to the adjoint eigensolution then a solution is possible (Fredholm's Alternative)

Proof $\quad Lu = h \qquad$ Let v_n^* = adjoint eigenfunction, ie $L^* v_n = 0$

$$\int_a^b v_n^* L u \, dx = \int_a^b h v_n^* \, dx$$

$\downarrow$ IBP

$$\int_a^b u \underbrace{L^* v_n^*}_{0} \, dx = \int_a^b h v_n^* dx = 0 \qquad \text{a solvability condition}$$

LIN ALG	ODE
$A_{ij} x_j = 0$	$Lu = 0$
$A_{ij} = A_{ji}$ sym	if $L^* = L$ self adj
if $A_{ij} \neq A_{ji}$	
x_i, x_i^T (transpose EV)	u, v adjoint EV

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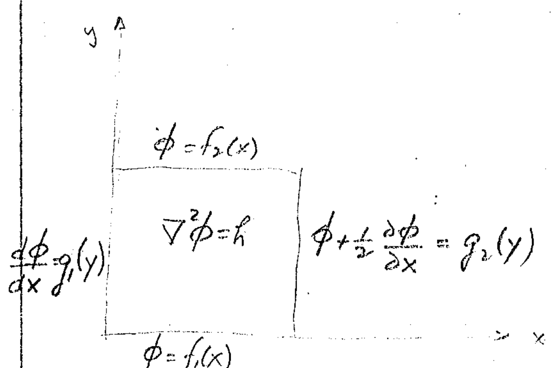
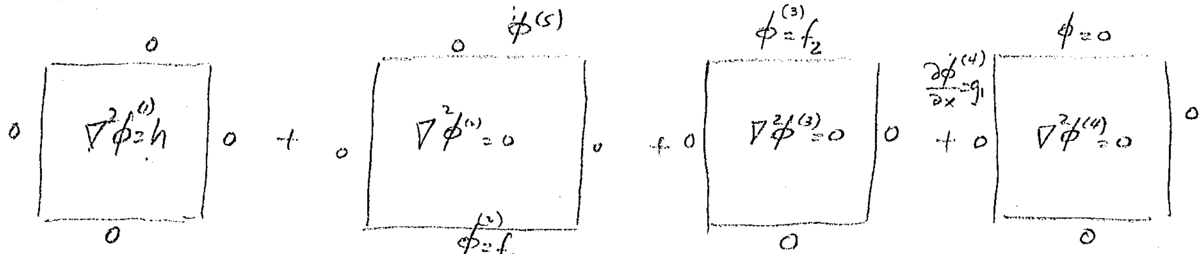


Splitting $\quad \phi = \sum_{i=1}^{5} \phi^{(i)} \quad \ni \quad \phi^{(1)}$ satisfies $\nabla^2 \phi = h \ni BC = 0$

$\phi^{(2)}$ " $\nabla^2 \phi = 0 \ni BC\ (y=0)\ \phi = f_1(x)$

$\vdots$

$\phi^{(5)}$



$\frac{\partial \phi^{(5)}}{\partial x} = -\left(\frac{\partial \phi^{(1)}}{\partial x} + \frac{\partial \phi^{(2)}}{\partial x} + \frac{\partial \phi^{(3)}}{\partial x}\right) \quad \nabla^2 \phi^{(5)} = 0.$

this is not best way to solve

Thus we can split this as

$\phi = 0$

$\frac{\partial \phi}{\partial x} = 0$ $\nabla^2 \phi = h$ $\phi + \frac{1}{2}\frac{\partial \phi}{\partial x} = 0$

$\phi = 0$

(1)

$+$

0

$\frac{\partial \phi}{\partial x} = 0$ $\nabla^2 \phi = 0$ $\frac{1}{2}\frac{\partial \phi}{\partial x} + \phi = 0$

$\phi = f_1$

(2)

$+$

$\phi = f_2$

$\phi_x = 0$ $\nabla^2 \phi = 0$ $\phi + \frac{1}{2}\frac{\partial \phi}{\partial x} = 0$

0

$+$

0

$\frac{\partial \phi}{\partial x} = g(y)$ $\nabla^2 \phi = 0$ $\phi + \frac{1}{2}\frac{\partial \phi}{\partial x} = 0$

0

$+$

0

$\frac{\partial \phi}{\partial x} = 0$ $\nabla^2 \phi = 0$ $\phi + \frac{1}{2}\frac{\partial \phi}{\partial x} = g_2(x)$

0

Toughest problem is #1. However we could have done it even better. to do this pick any fn that satisfies $\nabla^2 \phi_p = h$. and don't worry about bc thus.

$F_2(x)$

$G_1(y)$ $\nabla^2 \phi_p = h$ $G_2(y)$

$F_1(x)$

where $G_1(y)$, $G_2(y)$, $F_2(x)$, $F_1(x)$ is the bc generated by the solution to the DE.

then solve.

$\phi = f_2 - F_2$

$+$

$\phi_x = g_1 - G_1$ $\nabla^2 \phi = 0$ $g_2 - G_2 = \phi + \frac{1}{2}\phi_x$

$\phi = f_1 - F_1$

$=$

$\phi = f_2$

g_1 $\nabla^2 \phi = h$ g_2

f_1

Now look at one solution block only

0

$\phi_x = 0$ $\nabla^2 \phi = 0$ $\phi + \frac{1}{2}\phi(x) = G(y)$

$\phi = 0$

Soln is $X(x)Y(y) = \phi$

or $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2$

$\therefore$ $Y'' + \lambda^2 Y = 0$

$X'' - \lambda^2 X = 0$

$\therefore$ using bc $\quad Y = A \sin \lambda y \quad$ with $\lambda_n = n\pi \qquad$ where height $= 1 = y$

using bc $\quad X = \cosh \lambda x$

$$\phi = \sum_{n=0}^{\infty} c_n \sin \lambda_n y \cosh \lambda_n x$$

bc $\quad$ at $x = 1 \qquad \phi + \tfrac{1}{2}\phi_x = g(y)$

$$\phi + \tfrac{1}{2}\phi_x = \sum_{n=0}^{\infty} c_n \underbrace{\sin(\lambda_n y)\left\{\cosh \lambda_n + \tfrac{1}{2}\lambda_n \sinh \lambda_n\right\}}_{h_m(y)} = G(y)$$

By use of orthog. we can get c_n

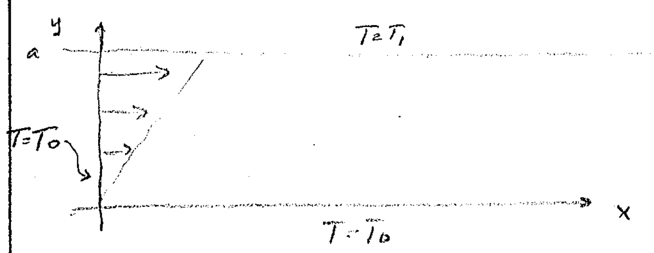
$$\therefore \int_0^1 Y_n Y_m \, dx = 0 \qquad Y_n = \sin \lambda_n y$$

$$\boxed{c_m = \frac{\int_0^1 G(y) \sin(\lambda_m) y \, dy}{\int_0^1 h_m(y) \sin \lambda_m y \, dy}}$$

for $n=0 \quad \lambda = 0 \quad \therefore \quad X = ax + b \quad Y = ay + b \quad \Rightarrow Y = 0 \qquad \begin{matrix} \phi = 0 & y = 0 \\ \phi = 0 & y = 1 \end{matrix}$

$$\therefore \quad c_0 = 0$$

2/22/79



$$\frac{\partial^2 T}{\partial y^2} = \alpha y \frac{\partial T}{\partial x} \qquad (1)$$

$$T(x,0) = T_0 \qquad (2,3)$$
$$T(x,a) = T_1$$

$$T(0,y) = T_0 \qquad (4)$$

Homogeneous BC $\quad$ Homo DE. $\qquad$ Not self similar

General approach is to get partic soln to take care of the inhom + partial solns to take care of the homog DE.

Particular Soln. by the physics we expect $T_p \neq f(x)$ "steadystate"

ie let $T_p = f(y)$

From (1) $f''(y) = 0$ $f = c_1 + c_2 y$ let $f = T_0 + (T_1 - T_0)(y/a) = T_p(y)$

soln of PDE that satisfies the inhomog bc (2,3)

we must find a $\phi(x,y)$ ∋. $T = T_p + \phi$ w/ $\phi_{,yy} = \alpha y \, \phi_{,x}$

$\phi(x,0) = 0$ from (2)

$\phi(x,a) = 0$ " (3)

$\phi(0,y) = T_0 - \{T_0 + (T_1 - T_0) y/a\}$

$= (T_0 - T_1) y/a$

Let $\phi = XY$ $\Rightarrow PDE \Rightarrow$ $Y''X = \alpha y X'Y \Rightarrow \alpha \frac{X'}{X} = \frac{Y''}{yY} = -\lambda^2$

$X' + \frac{\lambda^2}{\alpha} X = 0$ $\Rightarrow$ $X = C e^{-\frac{\lambda^2 x}{\alpha}}$ as $x \to \infty$ X decays & we reach steadystate

$Y'' + \lambda^2 y Y = 0$ Let $z = by$ $\Rightarrow$ $b^2 Y''(z) + \lambda^2 \frac{z}{b} Y(z) = 0$

$\therefore b^2 = \frac{\lambda^2}{b}$ $\therefore$ $\underline{b = (\lambda^{2/3})}$ $\Rightarrow$ $Y'' + zY = 0$

in HMF

$Y = c_1 A_i(-z) + c_2 B_i(-z)$ $\Rightarrow$ $Y(y) = c_1 A_i(-\lambda^{2/3} y) + c_2 B_i(-\lambda^{2/3} y)$

now $\phi(x,0) = Y(0) = 0$ $\Rightarrow c_1 A_i(0) + c_2 B_i(0) = 0$

$\phi(x,a) = Y(a) = 0$ $\Rightarrow c_1 A_i(-\lambda^{2/3} a) + c_2 B_i(-\lambda^{2/3} a) = 0$

$\Rightarrow \det\left[A_i(0) \cdot B_i(-\lambda^{2/3} a) - B_i(0) \cdot A_i(-\lambda^{2/3} a)\right] = 0$

This is the char eqn for the Efns & EV λ

pick $c_1 = 1$ $\Rightarrow$ c_2/c_1 can be obtained from either of

$\therefore$ Define $Y_n(y)$, λ_n now known.

a constant is not a soln to ϕ since to satisfy BC: $\phi(x,0) = 0$, $\phi(x,a) = 0 \Rightarrow \phi = const = 0$

$$\phi = \sum c_n Y_n(y) e^{-\lambda^2 x/\alpha}$$

now $Y_n'' + \lambda_n^2 y Y_n = 0$ (a) $Y_m'' + \lambda_m^2 y Y_m = 0$ (b)

use this form to get orthog. $\int_0^a [Y_m(a) - Y_n(b)] dy = 0$

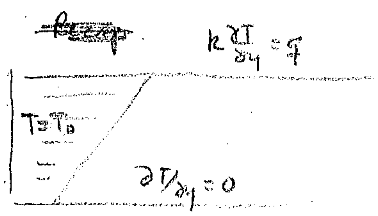
$$\int_0^a \left[\cancel{(Y_m Y_n'' - Y_m'' Y_n)}^{\to 0} + (\lambda_n^2 - \lambda_m^2)\, y\, Y_n Y_m \right] dy = 0 \Rightarrow \int y\, Y_n Y_m\, dy = 0$$

now since $\phi(0,y) = (T_0 - T_1)\, y/a = \sum_{n=1}^{\infty} C_n Y_n(y)$

$$\int_0^a \phi\, y\, Y_m\, dy = \sum C_n \int_0^a y\, Y_n Y_m\, dy$$

$$\int_0^a y\, \phi\, Y_n\, dy = C_n \int y\, Y_n^2\, dy \qquad \therefore \quad C_n = \frac{\int_0^a y\, \phi\, Y_n\, dy}{\int y\, Y_n^2\, dy}$$

25 Feb 79



w/ PDE : $\frac{\partial^2 T}{\partial y^2} = \alpha y \frac{\partial T}{\partial y}$

Physics suggest that $T_p(x,y) = f(y) + Ax$

① Find particular soln

② Get all partial solns to homog problem.

③ Let $T = T_p + \sum C_n$ (eigenfunctions)

Adjust $C_n \ni T(0,y) = T_0$

using $T_p = f(y) + Ax \Rightarrow$ PDE $\Rightarrow \quad f'' = \alpha y A \qquad$ w/ BC $f'(0) = 0 \quad f'(a) = q/k$

then

$$f' = \frac{A\alpha y^2}{2} + B_1 \Rightarrow B_1 = 0 \quad (\text{using } f'(0) = 0) \qquad A = \frac{q}{k} \cdot \frac{2}{\alpha a^2} = \frac{2q}{k a^2 \alpha}$$

$$\therefore \quad f = \frac{2q}{k a^2 \alpha} \cdot \frac{\alpha y^3}{6} + B_2 = \frac{q y^3}{3 k a^2} + B_2 \qquad \text{pick } B_2 = 0$$

$$\therefore \quad T_p(x,y) = \frac{2q}{k \alpha a^2} \left[x + \frac{\alpha y^3}{6} \right]$$

now $T = T_p(x,y) + \phi(x,y)$

$\Rightarrow \phi_{,yy} = \alpha y\, \phi_{,x} \qquad$ w $\qquad \phi_{,y} = 0 \quad @ \quad y = 0$

$\phi_{,y} = 0 \quad @ \quad y = a$

and $T_0 = \frac{q y^3}{3 k a^2} + \phi(0,y) \qquad$ or $\quad \phi(0,y) = T_0 - \frac{q y^3}{3 k a^2}$

as before we found $\phi = YX$

$$Y = C_1 A_i(-\lambda^{2/3} y) + C_2 B_i(-\lambda^{2/3} y) \quad \text{and} \quad X = e^{-\lambda^2 x/\alpha}$$

using BC $y'(0) = y'(a) = 0$

$\Rightarrow$

$$\begin{bmatrix} -\lambda^{2/3} Ai'(0) & -\lambda^{2/3} Bi'(0) \\ -\lambda^{2/3} Ai'(a\lambda^{2/3}) & -\lambda^{2/3} Bi'(a\lambda^{2/3}) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$D(\lambda) = 0$ give the λ_n's

when you have deriv BC the $\lambda = 0$ is a root & a const can be a solution

$$\phi = \sum c_n Y_n(y)\, e^{-\lambda_n^{2/3} x/\alpha} \qquad n = 0;\ \lambda_0 = 0 \qquad Y_0 = 1$$

IC $\quad T = T_0$ @ $x = 0$

$$T_0 = \frac{a_y y^3}{3a^2 K} + \phi(0, y) \qquad \phi(0,y) = T_0 - \frac{a_y y^3}{3Ka^2} = g(y)$$

$$g(y) = \sum_{n=0}^{\infty} c_n Y_n(y) \quad \text{n thog prop.} \quad \int_0^a y\, Y_n Y_m\, dy = 0 \quad n \neq m.$$

for the first coeff take $\int_0^a y\, Y_m\, dy = 0$

$$\therefore \int_0^a g(y)\, y\, Y_m(y)\, dy = c_m \int_0^a y\, Y_m^2\, dy \qquad m > 1$$

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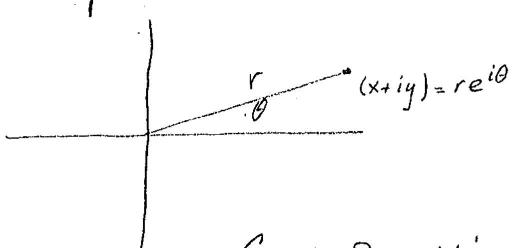
Complex Variables



$$\nu \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = r \frac{\partial u}{\partial t} \qquad (1)$$

@ $r = r_0 \quad u = 0 \qquad (2)$

$r = r_1 \quad u = A \cos \omega t \qquad (3)$

Complex Imbedding - let u be complex

Replace (3) by $A e^{i\omega t}$ Then u would be the real part of u

Assume $u = AR(r)\, e^{i\omega t}$ Seek periodic steady state solution. Put u into PDE

$$\nu (r R')' = r R\, i\omega$$

$$(rR')' - rR \frac{i\omega}{\nu} = 0 \implies r^2 R'' + rR' + \left(-\frac{i\omega}{\nu}\right) r^2 R = 0$$

let $z = \beta r \qquad \beta = \sqrt{\frac{-i\omega}{\nu}} = e^{3\pi i/4} \sqrt{\frac{\omega}{\nu}}$

$$R = c_1 J_0\left(\sqrt{\tfrac{-i\omega}{\nu}}\, r\right) + c_2 Y_0\left(\sqrt{\tfrac{-i\omega}{\nu}}\, r\right) = c_1 J_0\left(e^{\frac{3\pi i}{4}}\sqrt{\tfrac{\omega}{\nu}}\, r\right) + c_2 Y_0\left(e^{\frac{3\pi i}{4}}\sqrt{\tfrac{\omega}{\nu}}\, r\right)$$

HMF 9.9.3 $\quad x^2\omega'' + x\omega' - (ix^2+\nu^2)\omega = 0 \qquad$ Let $z = \sqrt{\frac{\omega}{\nu}}\, x$ in original fn.

$$\omega = c_1\left[\mathrm{ber}_\nu(x) + i\,\mathrm{bei}_\nu(x)\right] + c_2\left[\mathrm{ker}_\nu(x) + i\,\mathrm{kei}_\nu(x)\right]$$

$$\mathrm{ber}_\nu(x) + i\,\mathrm{bei}_\nu(x) = J_\nu(xe^{3\pi i/4})$$
$$\mathrm{Ker}_\nu(x) + i\,\mathrm{Kei}_\nu(x) = K_\nu(xe^{3\pi i/4})$$

Kelvin function. "THIS IS TEDIOUS METHOD" JV REYNOLDS

INSTEAD define complex $z = e^{3\pi i/4}\sqrt{\frac{\omega}{\nu}}\, r$ then $R(z) = C_1 J_0(z) + C_2 Y_0(z)$

apply bc @ r_i $\quad R(r_i) = 1$

@ r_o $\quad R(r_o) = 0$

$$\begin{bmatrix} J_0(z_i) & Y_0(z_i) \\ J_0(z_o) & Y_0(z_o) \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution give c_1, c_2

To get plots of $u(r,t;\omega)$ $\qquad$ CALL CBESJYN (z,...)

CALL CLINSYS $\Rightarrow c_1, c_2$

construct $R(z)$ for desired r, ω.

$u = AR(z)e^{i\omega t} \qquad u_R = \text{REAL}(u)$

We now want to add initial conditions $\quad u(r,0) = 0$

$u = u_{ss}(r,t) + \phi(r,t) \qquad u_{ss} = A R(z) e^{i\omega t}$

periodic SS just obtained

ϕ will satisfy $\quad \nu \frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = r\frac{\partial\phi}{\partial t} \quad (4)$

$\phi(r_i) = 0$

$\phi(r_o) = 0 \qquad (5\text{-}7)$

I.C. $\quad \phi(r,0) = -u_{ss}(r,0)$

4-5-6 give partial solution with $\phi = F(r)\,G(t) \Rightarrow \frac{\nu(rF')'}{rF} = \frac{G'}{G} = -\nu\lambda^2$

$\therefore G' + \nu\lambda^2 G = 0 \Rightarrow G = e^{-\lambda^2\nu t}$

$(rF')' + \lambda^2 r F = 0 \Rightarrow F = B_1 J_0(\lambda r) + B_2 Y_0(\lambda r)$

using bc. $\quad F(r_i) = F(r_o) = 0 \Rightarrow \det\left[J_0(\lambda r_o)Y_0(\lambda r_i) - Y_0(\lambda r_i)J_0(\lambda r_o)\right] = 0$

This gives λ_n and ratio of B_2/B_1 $\quad$ pick $B_1 = 1 \Rightarrow F_n(r)$ known

$\therefore$ orthog cond. is $\displaystyle\int_{r_i}^{r_o} r F_n(r) F_m(r)\, dr = 0$

let $\phi = \sum A c_n F_n(r) e^{-i\nu\lambda_n^2 t}$

$$\phi(r,0) = -u(r,0) = \sum A c_n F_n(r) = -A R(r)$$

use orthog

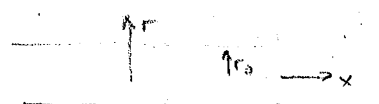
$$\int_{r_i}^{r_0} r F_m \phi \, dr = \sum A c_n \int_{r_i}^{r_0} r F_m F_n \, dr = -A \int_{r_i}^{r_0} r R F_m \, dr$$

$$c_m = \frac{-\int r F_m R \, dr}{\int r F_m^2 \, dr} = \frac{I_1}{I_r} \quad \text{prove} \quad \frac{r F_n'(r_i) R(r_i)}{(\lambda_n^2 + i\omega)} = I_1$$

2/28/79

1-D Wave prop. Governing Eq $c^2 u_{xx} - u_{tt} = 0$

Solutions to this are of form $u = G(x-at)$ for a forward propagating wave w/ a as wave speed if $a=c$ then G can be any function

2-D  Pressure fluctuations in a tube are governed by

$$c^2 \nabla^2 p - p_{tt} = 0 \quad \text{w/ } \frac{\partial p}{\partial r} = 0 \text{ @ } r = r_0$$

in polar this results in $c^2\left(P_{rr} + \frac{1}{r}P_r + \frac{1}{r^2}P_{\theta\theta} + P_{xx}\right) - P_{tt} = 0$

What are solutions of form $F(r,\theta)\, G(x-at) = P$

Plug in to get $c^2\left(F_{rr} + \frac{1}{r}F_r + \frac{1}{r^2}F_{\theta\theta}\right)G + c^2 F G'' - a^2 F G'' = 0$

we can separate $f(r,\theta) = g(x,t)$

$\therefore \quad c^2\left(F_{rr} + \frac{1}{r}F_r + \frac{1}{r^2}F_{\theta\theta}\right)/F = (a^2 - c^2)\, G''/G$

$\boxed{\text{Case 1} \quad a = c}$ then G can be anything $\therefore$

$F_{rr} + \frac{1}{r}F_r + \frac{1}{r^2}F_{\theta\theta} = 0 \quad$ try $F = R(r)\Theta(\theta)$

then $r^2\left(R'' + \frac{1}{r}R'\right)/R = -\Theta''/\Theta = \beta^2$

This gives $\Theta'' + \beta^2\Theta = 0$ or $\Theta = \cos(\beta\theta - \phi)$ $(\beta = 0, 1, 2, \dots n)$ for single valuedness

The R equation $r^2R'' + rR' - n^2R = 0$, w/ $R'(r_0) = 0$ $R(0)$ is finite

Solution to this Euler eq. is $R(r) = r^\alpha$ Put into DE

$\therefore \quad \alpha(\alpha - 1) + \alpha - n^2 = 0$ or $\alpha = \pm n$ throw out negative for bounded r

$\therefore \quad R = c_1 r^n$ to satisfy $R'(r_0) = 0 \Rightarrow c_1 = 0 \quad \therefore R = 0$ for $n > 0$

$\therefore$ Back to DE $r^2R'' + rR' = 0$ $\quad R = \text{const}$ solves DE & BC

also $\frac{R''}{R'} = -\frac{1}{r}$ or $R' = c_1 \ln r + c_1$ $\quad R = -c_1(r \ln r - r) + c_2$

$c_1 = 0$ for $R'(r_0) = 0$

$\therefore$ for $a = c$ the only possibilities are $n=0$ $\Theta = \text{const}$ $R = \text{const}$ (ie no radial or angular dependence, and $G(x-at)$ can be anything

Case II if $a \neq c$

if $a < c$ then we will get decaying waves

if $a > c$

$$c^2\left(F_{rr} + \tfrac{1}{r}F_r + \tfrac{1}{r^2}F_{\theta\theta}\right)/F = (a^2-c^2)\frac{G''}{G} = -\mu^2$$

$$G'' + \frac{\mu^2}{a^2-c^2}G = 0 \qquad G = \cos\left[\sqrt{\frac{\mu^2}{a^2-c^2}}\,(x-at) - \phi\right]$$ waves must be sinusoidal

from $c^2\left(F_{rr} + \frac{1}{r}F_r + \frac{1}{r^2}F_{\theta\theta}\right) + \mu^2 F = 0$

take $F = R(r)\,\Theta(\theta)$

$$\therefore\ c^2\left(\frac{r^2R'' + rR'}{R}\right) + \mu^2 r^2 = -\frac{\Theta''}{\Theta}c^2 = \underbrace{\beta^2 c^2}_{\text{const}}$$

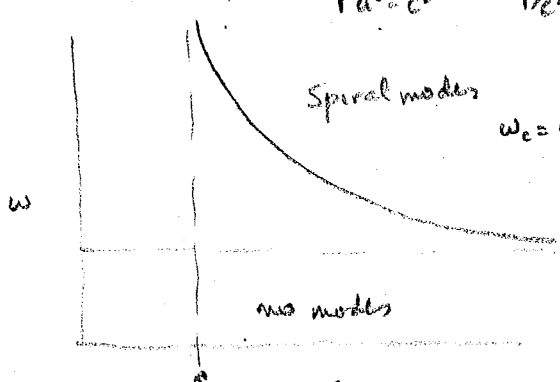
then $\Theta'' + \beta^2\Theta = 0$ $\beta = n$ an integer and $\Theta = A_1\cos(n\theta - \phi)$ for single valued reas

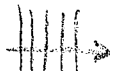
for the R eq we get $r^2R'' + rR' + \left(\frac{\mu^2}{c^2}r^2 - n^2\right)R = 0$ let $\mu/c = \gamma$

for finite $R(0)$ then $R = C_1 J_n(\gamma r)$ and $R'(r_0) = J_n'(\gamma r_0) = 0$ gives the EV γ_{mn}

define $\omega = a\sqrt{\frac{\mu^2}{a^2-c^2}} \Rightarrow a\sqrt{\frac{\gamma_{mn}^2 c^2}{a^2-c^2}} = \omega_{mn}$ for given n, m γ_{mn} are fixed

now $\omega = \text{fn of } a = \sqrt{\frac{a^2c^2\gamma^2}{a^2-c^2}} = \sqrt{\frac{\gamma^2}{1/c^2 - 1/a^2}}$



plane waves  $\to c$

Spiral wave  i $\nearrow c$ $\to V$ but propagate info faster in this direction

$p = \frac{\sin i}{c} = \frac{1}{V}$ $\therefore V = \frac{c}{\sin i} > c$

3/5/79

Characteristics: Basically lines along which we can reduce our indep variables so that PDE is now ODE

$$A(x,y,u)\,u_x + B(x,y,u)\,u_y = C(x,y)$$

To create ODE we find new coords (chars) ξ, η $\ni$ when we integrate ODE wrt, let's say, ξ over ~~let's say~~ $\eta = \text{const}$, we can get u.

Let $\xi = \xi(x,y)$ $\quad \eta = \eta(x,y)$ $\therefore$ $u(x,y) \Rightarrow u(\xi,\eta)$ then

$$A\left[u_\xi \xi_x + u_\eta \eta_x\right] + B\left[u_\xi \xi_y + u_\eta \eta_y\right] = C$$

or

$$(*) \qquad (A\xi_x + B\xi_y)u_\xi + (A\eta_x + B\eta_y)u_\eta = C(x,y)$$

since we want to integrate u_ξ along constant η $\quad \left(\frac{\partial u}{\partial \xi}\right)_{\eta=\text{const}} = \frac{du}{d\xi}$ then must take

$A\eta_x + B\eta_y = 0$ $\quad$ but if $\eta = \text{const}$ $\quad d\eta = 0 = \eta_x dx + \eta_y dy$ $\quad \therefore \begin{pmatrix} A & B \\ dx & dy \end{pmatrix}\begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} = 0$

only nontriv soln is if det $= 0$ or $\frac{dy}{dx} = \frac{B}{A}$ on constant η lines

Specific cases

I: $\quad B/A = f(x) = \frac{B(x)}{A(x)}$ $\quad$ then $y = \int_a^x f(x')dx' + C$ $\quad$ define $\eta = C = y - \int_a^x f(x')dx'$

then take ξ to be anything that intersects η $\quad$ pick $\xi = x$

Example $B = x^3$, $A = 1$, $a = 0$ $\quad$ then $y = x^4/4 + C$ $\quad$ or $\eta = y - x^4/4$ $\quad \xi = x$

put into (*) to get

$$\left[1\cdot 1 + x^3 \cdot 0\right]u_\xi + \left[1\cdot\cancel{(-x^3)} + x^3\cdot 1\right]u_\eta = C(x,y)$$

or $\quad u_\xi\Big)_{\eta=\text{const}} = C(x,y)$

if $C = 0$ then $\left(\frac{\partial u}{\partial \xi}\right)_\eta = 0$ $\quad$ or $\quad u = f(\eta) = f(y - x^4/4)$

this is general soln to $u_x + x^3 u_y = 0$

Another case $\quad u_t + v u_x = 0 \Rightarrow (u_\xi \xi_t + u_\eta \eta_t) + v(u_\xi \xi_x + u_\eta \eta_x) = 0$

or $\quad (\xi_t + v\xi_x)u_\xi + \underbrace{(\eta_t + v\eta_x)}_{=0 \text{ on constant } \eta \text{ lines}} u_\eta = 0 \qquad (1)$

also $d\eta = 0 = \eta_x dx + \eta_t dt \qquad (2)$ $\qquad$ (1) & (2) give $dx/dt = v \Rightarrow u_\xi = 0$ or $u = f(\eta) = f(x - v$

3/7/79

Recall

$\phi_{,t} + V\frac{\partial\phi}{\partial x} = 0$ slope of char is $\frac{dx}{dt} = V$ even if $V = V(x,t,\phi)$

Case a $V = e^{-x}$ at $t=0$ $\phi(x,0) = g(x)$

$x=0$ $\phi(0,t) = 1$

to get $\frac{dx}{dt} = V$ define $\xi(x,t)$ $\eta(x,t)$ characteristics ∋ we reduce no. of terms

$\therefore \phi_{,t} = \phi_\xi \xi_t + \phi_\eta \eta_t \qquad \phi_\xi \xi_x + \phi_\eta \eta_x = \phi_{,x}$

$\therefore \phi_\xi(\xi_t + V\xi_x) + \phi_\eta(\eta_t + V\eta_x) = 0$ pick one of the coeffs to be zero: let $= 0$

make the requirement that $\eta = \text{const}$ ∋ we can integrate ϕ_ξ along constant η

but $d\eta = 0 = \eta_x dx + \eta_t dt$
$0 = \eta_x \cdot V + \eta_t \cdot 1$) soln exists if $\begin{vmatrix} dx & dt \\ V & 1 \end{vmatrix} = 0$ or $\frac{dx}{dt} = V$ on line of const η

return to $\frac{dx}{dt} = V = e^{-x}$ $\quad dt = \frac{dx}{e^{-x}}$ $\quad \therefore t = e^x + c \quad \therefore t - e^x = c$

Let $\eta = t - e^x$, and pick ξ so that they intersect with η choose $\xi = x$

now put these into diff eq

$\phi_t = \phi_\eta \cdot 1 + \phi_\xi \cdot 0 \qquad \phi_x = \phi_\eta(-e^x) + \phi_\xi \cdot 1$

So eqn becomes $e^{-x}\frac{\partial\phi}{\partial\xi}\Big)_\eta = 0$ or $\frac{\partial\phi}{\partial\xi}\Big)_\eta = 0$

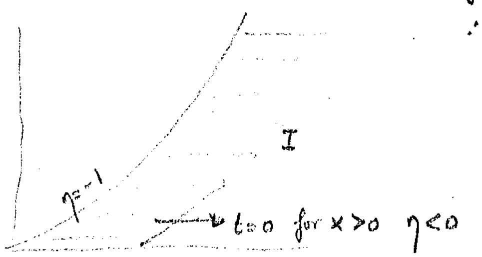
$\therefore \phi = f(\eta) = F(t - e^x)$

Now apply bc & IC

$t=0$ $\phi(x,0) = g(x) = F(-e^x)$ let $\sigma = e^x$ $\quad x = \ln\sigma \quad x \geq 0 \;\; \sigma > 1$

$\therefore g(\ln\sigma) = F(-\sigma) \quad \sigma > 1$ // $F(z) = g[\ln(-z)]$ $z < -1$

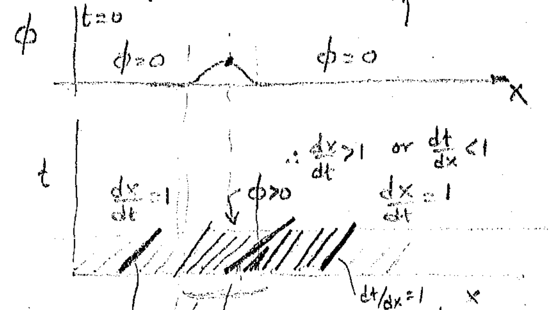
Region I soln



$x=0$ $\phi(0,t) = 1 = F(t-1)$ $\forall\, t > 0$

So $F(\sigma) = 1$ if $\sigma \geq -1$ Region II

Case b: $V = 1 + \frac{3}{2}\phi$ $\qquad \frac{dx}{dt} = V = 1 + \frac{3}{2}\phi$ long waves model the eqn along the char ( ) $\left.\frac{\partial\phi}{\partial\xi}\right)_\eta = 0$ so again $\phi = $ const on char.

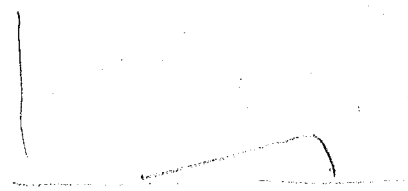


wave steepens

along this line $\phi =$ const $\therefore$ V = const $\therefore$ $\frac{dx}{dt}$ = const of chars are linear

Slope goes from 1 to < 1 to 1

as time $t > 0$ the front of wave steepens since the chars tend to intersect in front and disperse at rear thus at $t = t_1 > 0$



3/9/79

Higher order — Systems of 1st order

1. Reduce to first order system
2. Transform the eqns $\Rightarrow \xi, \eta$
3. Take linear comb of eqns.
4. Pick Characteristics to drop out derive w/t one variable ie η
5. Transformed eqns can now be integrated along const η

Example

$$A u_{xx} + B u_{xy} + C u_{yy} + D = 0 \qquad (1)$$

A, B, C, D are fns of (x, y, u, u_x, u_y)

only restriction Quasi linear - linear in highest deriv

① Let $v = u_x$ $\quad w = u_y$

We will reduce 1st eqn to coupled 1st order system.

$$\therefore \quad A v_x + B v_y + C w_y + D = 0 \qquad (2a)$$

$$v_y - w_x = 0 \qquad (2b)$$

② take $\xi = \xi(x,y) \qquad \eta = \eta(x,y)$

$$A\left(v_\xi \xi_x + v_\eta \eta_x\right) + B\left(v_\xi \xi_y + v_\eta \eta_y\right) + C\left(w_\xi \xi_y + w_\eta \eta_y\right) + D = 0 \qquad (3a)$$
$$\left(v_\xi \xi_y + v_\eta \eta_y\right) - \left(w_\xi \xi_x + w_\eta \eta_x\right) = 0 \qquad (3b)$$

3. take $C_1 \cdot (3a) + C_2 (3b) = 0$

$$v_\xi\left(C_1 A \xi_x + C_1 B \xi_y + C_2 \xi_y\right) + v_\eta\left(C_1 A \eta_x + C_1 B \eta_y + C_2 \eta_y\right) + w_\xi\left(C_1 C \xi_y - C_2 \xi_x\right)$$
$$w_\eta\left(C_1 C \eta_y - C_2 \eta_x\right) + C_1 D = 0$$

4. Choose $C_1 A \eta_x + C_1 B \eta_y + C_2 \eta_y = 0$ (4a) and $C_1 C \eta_y - C_2 \eta_x = 0$ (4b) so we can integrate along constant η

Solutions have nontrivial C_1, C_2 if $\det(4a, 4b) = 0$

$$\therefore \quad \begin{vmatrix} A\eta_x + B\eta_y & \eta_y \\ C\eta_y & -\eta_x \end{vmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \underline{0} \qquad (5)$$

$$\det = -A\eta_x^2 - B\eta_x\eta_y - C\eta_y^2 = 0 \qquad (6)$$

must be true on char: Also η must be const on characteristics, so that $\left.\frac{\partial v}{\partial \xi}\right)_{\eta = \text{const}}$ can be integrated: $d\eta = \eta_x dx + \eta_y dy = 0$ along const η

$$\therefore \quad \frac{dy}{dx} = -\frac{\eta_x}{\eta_y} \qquad (7)$$

put (7) in (6) $\quad \eta_x = -\eta_y \dfrac{dy}{dx} \Rightarrow$

on characteristic (8) $\quad A y'^2 \eta_y^2 - B y' \eta_y^2 + C\eta_y^2 = 0 \Rightarrow \quad \eta_y = 0$ or $Ay'^2 - By' + C = 0$

$$\therefore \quad y' = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

if $B^2 - 4AC < 0$ there are no characteristics ELLIPTIC
if $B^2 - 4AC > 0$ 2 real characteristics HYPERBOLIC
if $B^2 - 4AC = 0$ 1 real characteristics PARABOLIC

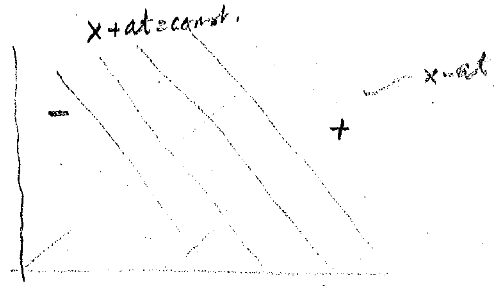
Example.

$$A u_{xx} + B u_{xy} + C u_{yy} + D = 0$$

$a^2 u_{xx} - u_{tt} = 0 \qquad A = a^2 \quad B = 0 \quad C = -1 \qquad$ w/ a = const

$$\frac{dt}{dx} = \frac{\pm\sqrt{4a^2}}{2a^2} = \pm\frac{1}{a} \qquad t = \pm\frac{1}{a}x + \text{const}$$

$$\boxed{x \pm at = \text{Const}}$$



oney in a 2nd order systems can we transform the eqns as PDE's wrt the characteristics

let $\eta_+ = x-at$ $\quad \xi = t$ $\quad$ on + characteristics

$$v = u_x \quad w = u_t \qquad a^2 v_x - w_t = 0$$
$$v_t - w_x = 0$$

$$a^2\left[v_\xi \cdot 0 + v_{\eta_+}\cdot 1\right] - \left[w_\xi \cdot 1 + w_{\eta_+}(-a)\right] = a^2 v_{\eta_+} - w_\xi + a w_{\eta_+} = 0 \quad (9a)$$

$$v_\xi + v_{\eta_+}(-a) - w_{\eta_+} = 0 \qquad (9b)$$

mult $(9b)\cdot a + (9a) = \quad -w_\xi + a v_\xi = 0$

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$a^2 u_{xx} - u_{tt} = 0$ (1) Let $v = u_x$ $w = u_t$ $\Rightarrow$ $\begin{array}{l} a^2 v_x - w_t = 0 \\ -w_x + v_t = 0 \end{array}$ (2a,b)

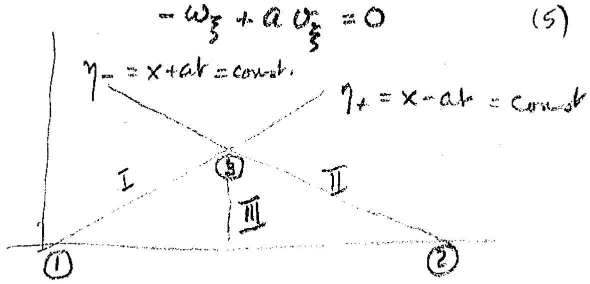
$\dfrac{dx}{dt} = \pm a$ Characteristics (3)

<u>+ char</u>: $\quad \eta_+ = x - at \qquad \xi = t$

$$a^2 v_{\eta_+} - w_\xi + a w_{\eta_+} = 0$$
$$v_\xi - a v_{\eta_+} - w_{\eta_+} = 0 \qquad (4a,b)$$

take (1) × 4a + a · 4b

$$-w_\xi + a v_\xi = 0 \qquad (5)$$



since info at ③ depends on info at ① and ② we need 2 eqns one good on ①-③ and one for ②-③.

$\int_1^3 (5)$ on line I $= -[\omega_3 - \omega_1] + a[v_3 - v_1] = 0$ since it's a definite integral

in general $\int_1^3 dw = \int_1^3 \frac{\partial w}{\partial \xi}\Big)_{\eta_+} d\xi + \frac{\partial w}{\partial \eta_+}\Big)_{\xi} d\eta_+$ along the line $d\eta_+ = 0$ since $\eta_+ =$ const

$\therefore \quad -\omega_3 + av_3 = -\omega_1 + a v_1 \qquad (7a)$

on the $\frac{dx}{dt} = -a$ then $\eta_- = x + at \qquad \xi = t$

$(2a) \Rightarrow \quad a^2 v_{\eta_-} - \omega_{\eta_-} a - \omega_{\xi} \cdot 1 = 0$

$(2b) \Rightarrow \quad v_{\eta_-} \cdot a + v_{\xi} \cdot 1 - \omega_{\eta_-} \cdot 1 = 0$ $\qquad (5.5a, b)$

$(1) \cdot (5.5a) + a(5.5b) = -\omega_\xi - a v_\xi = 0 \quad (6)$ integrate along (2/3)

Integrate along 2/3

$\therefore \quad -\left[\omega_3 - \omega_2 + a(v_3 - v_2)\right] = 0 \qquad (7b)$

take 7a, 7b and solve for v_3, ω_3

add them

$-2\omega_3 + \omega_1 + \omega_2 + a(-v_1 + v_2) = 0$

$$\boxed{\omega_3 = \frac{\omega_1 + \omega_2}{2} + \frac{a}{2}(v_2 - v_1)}$$

subtract the

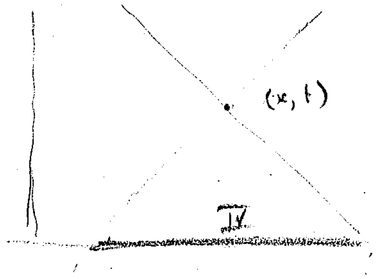
$\omega_1 - \omega_2 + 2a v_3 - a(v_1 + v_2) = 0$

$$\boxed{v_3 = \frac{a}{2a}(v_1 + v_2) + \frac{\omega_2 - \omega_1}{2a} = \frac{1}{2a}\left[(\omega_2 - \omega_1) + a(v_1 + v_2)\right]}$$

Solve for v, w at point 3 depends <u>only</u> on "initial" data at point 1 & 2 exact for v, w

since we want u then $\frac{\partial u}{\partial t}\Big)_x = \omega \qquad u_3 = u_{1.5} + \int_{1.5}^3 w(x,t)\,dt$ integrate along line III

but Solution of u at 3 depends only on data for u, u_t between (1) and (2)



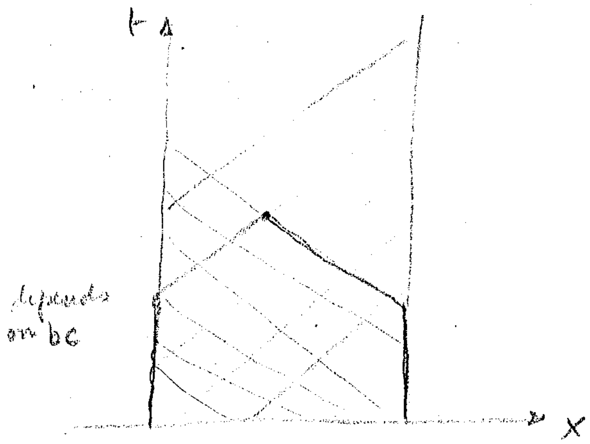
$a^2 u_{xx} - u_{tt} = 0$

$u(x,t)$ depends only upon u, u_t initial data on $\underline{IV}$

Boundary Cond.

$u(0,t) = u(L,t) = 0$
$u(x,0) = f(x)$
$u_t(x,0) = g(x)$

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi$$



there are two chars because there are two dependent variables

Use char as coords $\eta = x+at$ $\xi = x-at$

$u_x = u_\xi + u_\eta$
$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$
$u_t = u_\xi \cdot (-a) + u_\eta (a)$
$u_{tt} = a^2(u_{\xi\xi} + u_{\xi\eta}(-1) + u_{\eta\eta} a^2 - u_{\xi\eta})$

we transform $a^2 u_{xx} - u_{tt} = 4a^2 u_{\xi\eta} = 0$

$u = \int f(\xi')\,d\xi' + G(\eta)$
$u = F(\xi) + G(\eta)$
general soln

$u = F(x+at) + G(x+at)$

3/14/79

Wave Eqns.

(1) $a^2 u_{xx} - u_{tt} = 0$

$\xi = x - at \qquad \eta = x + at$

(1) $\Rightarrow \quad u_{\xi\eta} = 0$

(2) $u = F(x-at) + G(x+at)$

General solution of (1)

Case 1 $\qquad u(x,0) = f(x)$

$u_t(x,0) = 0 \qquad -\infty \le x \le \infty$

Normally will contain char lengths within the wave form given by $f(x)$ $\therefore$ ~~char~~ cannot use similarity & since $|x| \le \infty$ then cannot use S.O.V.

$\therefore$ use general solution $\quad u(x,0) = f(x) = G(x) + F(x)$ (*) where F, G are determined by I.C./B.C.

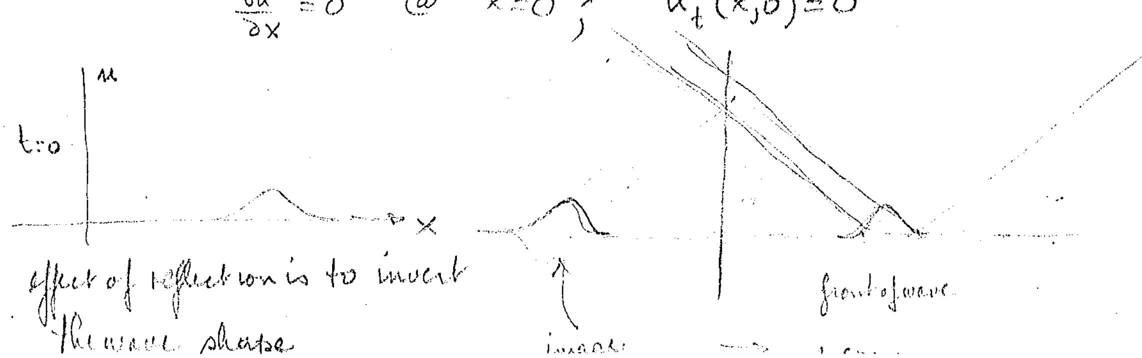
$$u_t(x,0) = 0 = -aF'(x) + aG'(x) = 0$$
$$\Rightarrow F'(x) = G'(x) \Rightarrow$$
$$F(x) = G(x) + C_0 \quad (**)$$

(*), (**) given : $\Rightarrow$

$$G(\sigma) = \tfrac{1}{2} f(\sigma) - C_0/2$$
$$F(\sigma) = \tfrac{1}{2} f(\sigma) + C_0/2$$
$$\therefore \; u(x,t) = \tfrac{1}{2} f(x-at) + \cancel{\tfrac{1}{2}C_0} + \tfrac{1}{2} f(x+at) - \cancel{\tfrac{1}{2}C_0}$$
$$\boxed{u(x,t) = \tfrac{1}{2}\left[f(x-at) + f(x+at)\right]}$$

Case (2) $\qquad u(x,0) = f(x)$

$\dfrac{\partial u}{\partial x} = 0 \quad @ \quad x = 0; \qquad u_t(x,0) = 0$



effect of reflection is to invert the wave shape

use general soln w/ 1 refl or no reflect
use S.O.V for more than 1 reflect

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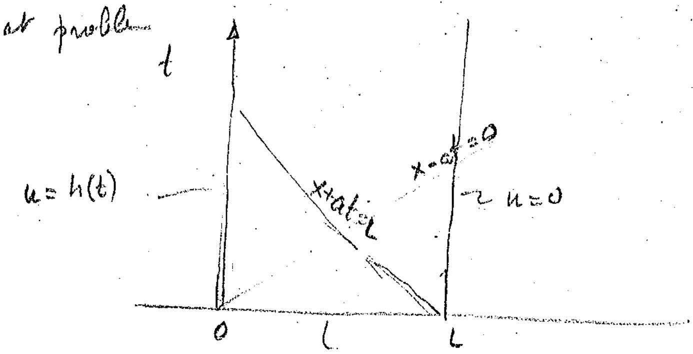
PDE $a^2 u_{xx} - u_{tt} = 0$ has general soln $u = F(x-at) + G(x+at) = F(\xi) + G(\eta)$

Look at problem



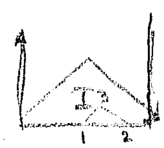
$x + at = \xi = \text{const} = \xi\Big|_{\substack{x=L\\t=0}} = L$

$x + at = \eta = \text{const} = \eta\Big|_{\substack{x=0\\t=0}} = 0$

$\left.\begin{matrix} u = f(x) \\ u_t = 0 \end{matrix}\right\}$ I.C. $\quad 0 \le x \le L$

Graphical approach

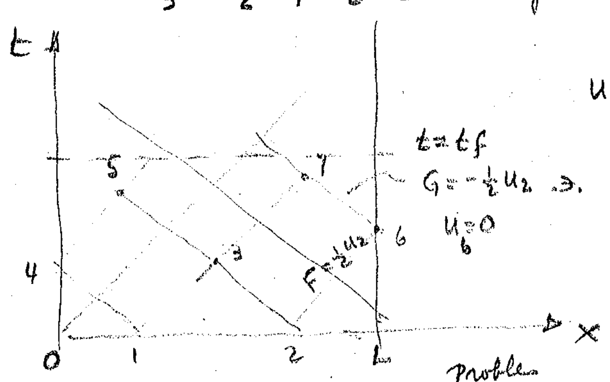
Region I: $\xi \geq 0 \quad \eta \le L$



do not depend on b.c. but only I.C.

$\therefore \quad u(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right] = \frac{1}{2}\left[f(\xi) + g(\eta)\right]$ from previous

$u_3 = \frac{1}{2}u_1 + \frac{1}{2}u_2$ depends only



$u_7 = \frac{1}{2}u_1 - \frac{1}{2}u_2$

$G = -\frac{1}{2}u_2 \Rightarrow F_6 + G_6 = 0$

$u_6 = 0$

It would be an ill posed ~~space~~ problem if we ~~define~~ prescribe to solution at $t = t_f$

$u_4 = h(4) = \frac{1}{2}u_1 + F(4) \quad \therefore \quad F(4) = h(4) - \frac{1}{2}u_1 = F(5)$

$G(5) = G(2)$

$\therefore \quad u_5 = h(4) - \frac{1}{2}u_1 + \frac{1}{2}u_2$

elliptic char is not real: info in region is obtained from boundary

WCR
ME 200B
Winter 79

ME 200B - MATHEMATICAL METHODS IN MECHANICAL ENGINEERING

INSTRUCTOR: William C. Reynolds; office, 500C, 497-4021; home, 948-2952

TEACHING ASSISTANTS:

Alan Cain: Office, 501J, 497-3220
Ranga Jayaraman: Office, 501T, 497-4039
Cathy Koshland: Office, 520 Loft, 497-0393

TUTORS: Several tutors will be selected from the class enrollment, based on pre-examination. Tutors will be expected to be available to assist other students for two hours each week, in return for which they will be exempted from turning in the class homework. The tutors will meet with the instructor and teaching assistants weekly for coordination. Tutor Room: 500U The principal role of the tutors will be to assist other students as they encounter difficulties in the homework.

TEXT: We have experimented with several texts and have not found a good one that fits this course. Therefore, Professor Reynolds is writing a new book, Solution of Partial Differential Equations. A preliminary edition of this book will be sold through the Bookstore.

It is important that each student have access to the material included in Abramowicz and Stegun, Handbook of Mathematical Functions, NBS publication SI (Dover Press), which is available in the Bookstore.

CLASS OBJECTIVES: The primary objective of this class is to help you develop skill in solving partial differential equations. Therefore, emphasis on theory of PDE's will be minimal. If you desire a more theoretical background, there are courses for this purpose in our math department.

CLASS FORMAT: The lectures will cover the basic material. The primary teaching methods will be by example. The notes contain additional examples and exercises that may be worked for practice.

The teaching assistants will conduct small-group workshops each week. These will be opportunities to practice the methods presented in lecture in a close-feedback situation. In addition, some supplementary and remedial material will be presented.

There will be four major analysis problems which must be done professionally as homework. These will involve the use of LOTS to a small degree.

TV TAPES: Videotapes will be available for replay in the Engineering Library for approximately one week following live presentation.

GRADING: Grading will be done by the instructor and other helpers. Rough breakdown: homework, 40%; midterm exam, 25%; final exam 35%. The homework scores are usually high, and therefore, the chief penalty will come if they are not done.

TA's and Tutors will meet one hour each week to discuss course progress and to tutor the Tutors.

Handouts which students miss receiving in class will be available in metal boxes outside of Room 500C.

ME 200B
Winter '79
WCR

MATHEMATICAL METHODS IN MECHANICAL ENGINEERING

Course Plan

M	W	F	Workshops	Homework
		Jan. 5 Introduction, pre-test	-----	Problem 1: Self-similar Solutions
Jan. 8 Self-similar solutions	Jan. 10	Jan. 12	ODE Review	
Jan. 15 Linear eigenvalue problems and special functions	Jan. 17	Jan. 19	Self-similar transformations	Due Jan. 22
Jan. 22 Linear eigenvalue problems and special functions	Jan. 24	Jan. 26	Linear PDE solution	Problem 2: An eigenvalue problem Due Feb. 5
Jan. 29 Superposition solutions in linear PDE problems	Jan. 31	Feb. 2	Special functions	
Feb. 5 Superposition solutions in linear PDE problems	Feb. 7	Feb. 9	Linear boundary value problems	Problem 3: Boundary value problem solution by superposition, due Feb. 26
Feb. 12 ~~Green's functions~~	Feb. 14 ODE GENERALIZ.	Feb. 16 MIDTERM	Open review	
Feb. 19 HOLIDAY	Feb. 21 ~~Characteristics~~ INHOM. PROB.	Feb. 23	None INHOMOG. PROB.	
Feb. 26 ~~Use of characteristics in problem solution~~ Periodic & wave soln.	Feb. 28	Mar. 2	Wave Soln ~~Characteristics~~	Problem 4: Wave prop probl. by SOV & char ~~Solution by characteristics, Due Mar. 12.~~
Mar. 5 ~~Introduction to numerical solution.~~ Characteristics	Mar. 7	Mar. 9	~~Numerical solutions~~ Characteristic	
Mar. 12 ~~Mixed analytical/numerical solution~~ characteristics	Mar. 14	Mar. 16	None Open Review	None

FINAL EXAM, Friday, March 20, 8:30-11:30 a.m.

ME 200B
Winter '79
WCR

MATHEMATICAL METHODS IN MECHANICAL ENGINEERING
(revised)

Course Plan

Numbers in brackets refer to chapters in the notes.

M	W	F	Workshops	Homework
		Jan. 5 Introduction, pre-test	-----	Problem 1: Self-similar Solutions
Jan. 8 Self-similar solutions	Jan. 10	Jan. 12 [2]	ODE Review	
Jan. 15 Linear eigenvalue problems and special functions	Jan. 17	Jan. 19 [3]	Self-similar transformations	Due Jan. 22
Jan. 22 Linear eigenvalue problems and special functions	Jan. 24	Jan. 26 [3]	Linear PDE solu- tion; SOV idea	Problem 2: An eigenvalue problem
Jan. 29 Superposition solutions in linear PDE problems	Jan. 31	Feb. 2 [4]	Special functions	Due Feb. 5
Feb. 5 Superposition solutions in linear PDE problems	Feb. 7	Feb. 9 [4]	Linear boundary value problems	
Feb. 12 ODE Generalizations	Feb. 14 [4]	Feb. 16 MIDTERM	Open review	Problem 3: Inho- mogeneous problem solution by super- position, due Feb. 26
Feb. 19 HOLIDAY	Feb. 21 Inhomogeneous problems	Feb. 23 [4,5]	Inhomogeneous problems	
Feb. 26 Periodic and wave solutions	Feb. 28	Mar. 2 [5]	Wave solutions	Problem 4: Wave propagation prob- lems by SOV and characteristics Due Mar. 16
Mar. 5 Characteristics	Mar. 7	Mar. 9 [7]	Characteristics	
Mar. 12 Characteristics	Mar. 14	Mar. 16 [7]	Open review	

FINAL EXAM, Tuesday, March 20, 8:30-11:30 a.m.

ME 200B
Winter 1979
W. C. Reynolds

PROBLEM 1 - Self-Similar Solutions

This problem involves the solution of two self-similar problems associated with undersea disposal. Write this problem up in a readable manner, as you would if doing it as an analyst for the Ocean Dumping Enterprise (ODE).

1. G kg of gunk is deposited at time zero on the ocean floor. Neglecting currents, the equation describing the concentration g of the gunk is

$$\alpha \frac{\partial}{\partial r}\left(r^2 \frac{\partial g}{\partial r}\right) = r^2 \frac{\partial g}{\partial t} \qquad (1.1)$$

where α is the (constant) diffusivity for gunk in the sea. The boundary condition far from the spot is

$$g(r,t) \to 0 \quad \text{as} \quad r \to \infty \qquad (1.2)$$

The fact that the total amount of gunk is fixed is conveyed by the constraint

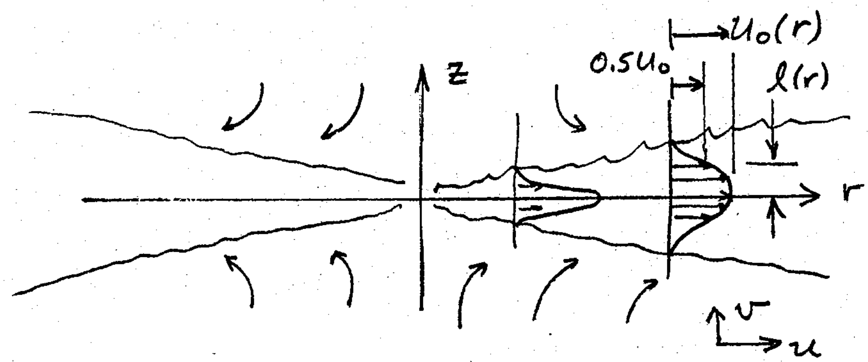
$$2\pi \int_0^\infty r^2\, g(r,t)\, dr = G = \text{constant} \qquad (1.3)$$

a) Solve the problem and give an expression for $g(r,t)$. Choose your similarity variable in the form Br/t^n and normalize your similarity function such that $f(o) = 1$.

Hint: What is $(\eta^3 f)'$? See also HMF §6.1.

b) Let $\tau = \alpha t/r^2$. Calculate (LOTS suggested) and plot gr^3/G vs. τ. This is a curve that an engineer might use to estimate the maximum concentration that would develop (and when) as a function of distance away from the deposit.

2. A steady stream of fluid emerges from a pipe submerged under the sea. This forms a "radial slot jet" as shown in the sketch.



In a simple model of the turbulent diffusion processes involved, the equations for the radial and vertical velocity components, u and v, are

$$u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial z} = \varepsilon \frac{\partial^2 u}{\partial z^2} \qquad (2.1)$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0 \qquad (2.2)$$

(2.1) is the radial momentum equation and (2.2) is the equation of mass conservation. The turbulent diffusivity ε is modeled as

$$\varepsilon = aq\ell \qquad (2.3)$$

where a is a constnat, q is a reference velocity scale for the turbulence, and ℓ is a reference length scale for the turbulence. Experience with other jets suggests that we use the jet centerline velocity for q,

$$q = u(r,0)$$

This velocity will decrease with r as the jet fans out. ℓ must be a measure of the layer thickness, which increases as the jet fans out. We will use as ℓ the distance from the jet centerline to the point where $u(r,\ell) = 0.5u(r,0)$, as shown in the sketch.

The strength of the flow is fixed by the radial momentum flux, which may be shown to be constant for this problem,

$$\int_0^\infty ru^2 dz = M = \text{constant} \qquad (2.4)$$

The boundary conditions are as follows:

$$u \to 0 \quad \text{as} \quad z \to \infty \tag{2.5a}$$

$$v = 0 \quad \text{at} \quad z = 0 \tag{2.5b}$$

$$\frac{\partial u}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{2.5c}$$

With these assumptions, a self-similar solution can be obtained. This solution represents the "far field" flow away from the jet origin. We can express ε in terms of the similarity function for u, namely $f(\eta)$, and in terms of η_o, where $f(\eta_o) = 0.5f(0)$.

a) Assuming that $u = Ar^n f(\eta)$, $v = Br^m g(\eta)$, $\eta = \beta z/r^\alpha$, find the values of n, m, and α which yield a self-similar solution. Choose B and β such that the differential equations for f and g are

$$g' = \eta f' \tag{2.6a}$$

$$f(0)f'' + f'(\eta f - g) + f^2 = 0 \tag{2.6b}$$

Note that, with these choices, $\ell = \eta_o r^\alpha/\beta$ and $q = Ar^n f(0)$. Give the boundary conditions on f and g. Give an expression for A as determined by the integral constraint.

b) Solve the problem numerically using a standard routine for systems of first-order ODE's (see notes pp. 220-221). You will do this as an initial value problem, starting at $\eta = 0$. Since (2.5) and (2.6) are homogeneous in f and g, if f and g are solutions then kf and kg are solutions. Hence, you can choose $f(0) = 1$ without loss of generality. Then, you can use the normalization condition for f as a third condition at $\eta = 0$. (2.5a) will be satisfied automatically.

The integral condition involves $I(\infty)$, where

$$I(\eta) = \int_0^\eta f^2(\sigma)\, d\sigma \tag{2.7}$$

Thus, by adding a fourth variable $I(\eta)$ to your equation set, governed by the equation

$$I' = f^2 \tag{2.8}$$

you will be able to compute the necessary integral quite directly.

Carry the solution out to $\eta = 8$. Produce a solution in which each variable is accurate to four digits.

Challenge: The two-dimensional jet has an exact analytical solution. Can you find the exact solution here? Hint: try using $\eta f - g$ as a variable.

c) Plot f and g as functions of η , and determine the value of η_o . $g(\infty)$ should be negative, indicating entrainment by the jet.

d) The value of a would have to be determined by experiments. Develop an expression for $\ell(x)$ in terms of a , and discuss how velocity profile measurements might be used to determine a .

e) Based on knowledge of other similar flows, a value of $a \simeq 0.04$ is expected. Using this value, express the jet centerline velocity, half-width ℓ , and induced upwelling velocity $v(r,-\infty)$ as functions of r and M . These are the "engineering results" that would be useful in designing an undersea disposal system incorporating the slot jet.

let $\eta = \frac{Br}{t^m}$ $\quad g = At^n f(\eta)$ $\quad \frac{d\eta}{dt} = -m\frac{Br}{t^{m+1}} = -\frac{m\eta}{t}$ $\quad \frac{d\eta}{dr} = \frac{B}{t^m}$

$$2\pi\int_0^\infty r^2 g(r,t)\,dr = G \Rightarrow 2\pi\int_0^\infty \eta^2\frac{t^{2m}}{B^2}\cdot At^n f(\eta)\cdot\frac{t^m}{B}\,d\eta = t^{3m+n}\frac{A}{B^3}2\pi\int_0^\infty \eta^2 f(\eta)\,d\eta = \text{const}$$

for fixed t

$\therefore\ 3m+n=0 \quad n=-3m \quad \therefore\ \eta=\frac{Br}{t^m} \quad g = At^{-3m}f(\eta)$

$$\frac{\partial g}{\partial r} = At^n\frac{\partial f}{\partial r} = At^n\frac{\partial f}{\partial\eta}\cdot\frac{\partial\eta}{\partial r} = At^n\cdot\frac{B}{t^m}f' = BAt^{-4m}f'$$

$$\frac{\partial g}{\partial t} = Ant^{n-1}f(\eta) + At^n\frac{\partial f}{\partial t} = nAt^{n-1}f(\eta) + At^n\frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial t} = nAt^{n-1}f(\eta) - Amt^{n-1}\eta f'$$

$\cancel{3m}$ $\quad At^{n-1}[nf - m\eta f']$

$-At^{n-1}[3mf + m\eta f']$

$$r^2\frac{\partial g}{\partial r} = \eta^2\frac{t^{2m}}{B^2}\cdot BAt^{-4m}f'$$

and $\frac{\partial()}{\partial r} = \frac{\partial()}{\partial\eta}\frac{\partial\eta}{\partial r} = \frac{B}{t^m}\frac{\partial}{\partial\eta}\left[\eta^2\frac{t^{2m}}{B^{\cancel{2}}}\cdot BAt^{-4m}f'\right] = At^{-3m}\frac{\partial}{\partial\eta}[\eta^2 f'] = \cancel{At^{-3m-1}(-3mf - m\eta f')}$

$$\Rightarrow \alpha\frac{\partial}{\partial r}\left[r^2\frac{\partial g}{\partial r}\right] = \alpha At^{-3m}\frac{\partial}{\partial\eta}(\eta^2 f')$$

$$r^2\frac{\partial g}{\partial t} = \eta^2\frac{t^{2m}}{B^2}At^{-3m-1}\left[-3mf - m\eta f'\right]$$

$\left(-\frac{3}{2}f - \frac{1}{2}\eta f'\right)$

$$\alpha\frac{\partial}{\partial\eta}(\eta^2 f') = -\frac{m}{B^2}\eta^2[3f + \eta f']$$

$$\frac{\partial}{\partial\eta}(\eta^2 f') = \frac{1}{2\alpha B^2}\left(-\eta^2[3f + \eta f']\right)$$

$\therefore\ t^{-3m} = t^{-m-1} \quad \therefore\ \boxed{m=\frac{1}{2} \quad n=-\frac{3}{2}} \quad \therefore\ \eta = \frac{Br}{\sqrt{t}} \quad g = At^{-3/2}f(\eta)$

choose $2\alpha B^2 = 1 \quad \therefore\ \boxed{B = \frac{1}{\sqrt{2\alpha}}}$ $\quad \eta = \frac{r}{\sqrt{2\alpha t}}$

or $\frac{\partial}{\partial\eta}(\eta^2 f') = -3\eta^2 f - \eta^3 f' = -(\eta^3 f)'$

or $\frac{\partial}{\partial\eta}\left[\eta^2 f' + \eta^3 f\right] = 0 \quad \therefore\ \eta^2 f' + \eta^3 f = c_1$ or $f' + \eta f = \frac{c_1}{\eta^2}$

$\cancel{f' + \eta f = \frac{c_1}{\eta^2}}$

BC $g(r,t)\to 0$ as $r\to\infty \Rightarrow f(\eta)\to 0$ as $\eta\to\infty$

also pick $A \ni f(0)=1$

since $f(\eta)\to 0$ as $\eta\to\infty$ we again impose that $f'(\eta)\to 0$ as $\eta\to\infty$ to be checked later

$\therefore\ \eta^2[f' + \eta f] = 0$ and $\frac{df}{f} = -\eta\,d\eta$ or $f = c_2 e^{-\eta^2/2}$

again $\eta^3 f \to 0$ since exponential $\to 0$ faster than $\eta^3\to\infty$

Proof $\eta^3 e^{-\eta^2/2} = \frac{\eta^3}{e^{+\eta^2/2}}$ L'hopitals' rule $= \frac{3\eta^2}{\eta e^{\eta^2/2}} = \frac{3\eta}{e^{\eta^2/2}}$ & again $\frac{3}{\eta e^{\eta^2/2}} \to 0$ as $\infty \to \infty$.

Proof that $\eta^2 f' \to 0$ as $\eta \to \infty$ $\eta^2 f' = -\eta^3 e^{-\eta^2/2} = -\frac{\eta^3}{e^{+\eta^2/2}}$ as before this ~~are~~ goes to 0.

when $\eta = 0$ $f(0) = 1 \Rightarrow C_1 = 1$

$\therefore f(\eta) = e^{-\eta^2/2}$

~~now $g(r,t) = 2At^{-3/2} f(\eta)$~~

~~and $r^2 g = 2A \frac{r^2}{t^{3/2}} 2 f(\eta)$~~

we had that $\frac{A}{B^3} 2\pi \int_0^\infty \eta^2 f(\eta)\, d\eta = G \Rightarrow \frac{A(2\alpha)^{3/2}}{\cancel{2\alpha}} 2\pi \int_0^\infty \eta^2 e^{-\frac{\eta^2}{2}} d\eta = G$

let $\frac{\eta^2}{2} = \zeta$ $\therefore \eta^2 = 2\zeta$ $\frac{2\eta\, d\eta}{2} = d\zeta$

$\therefore d\eta = \frac{d\zeta}{\eta} = \frac{d\zeta}{\sqrt{2\zeta}}$

$A \cdot (2\alpha)^{3/2} 2\pi \int_0^\infty 2\zeta\, e^{-\zeta} \frac{d\zeta}{\sqrt{2\zeta}} = G$

$A(2\alpha)^{3/2} \cdot 2\pi \frac{2}{\sqrt{2}} \underbrace{\int_0 \zeta^{1/2} e^{-\zeta} d\zeta}_{\Gamma(3/2)} = G$ $\quad A \cdot \frac{(2\alpha)^{3/2} 2\pi}{\sqrt{2}} \underbrace{\int_0^\infty \zeta^{1/2} e^{-\zeta} d\zeta}_{\Gamma(3/2)} = G$

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

$\Gamma(n+1) = n\Gamma(n)$

$A(2\alpha)^{3/2} \cdot 2\pi \cdot \frac{2}{\sqrt{2}} \frac{1}{2} \sqrt{\pi} = A(2\alpha)^{3/2} \cdot \frac{\sqrt{2\pi}\, 2\pi}{2}$

$\frac{A(2\alpha \cdot 2\pi)^{3/2}}{2}$

$2A\left(\frac{2\alpha\pi}{4}\right)^{3/2}$

$\Gamma(\tfrac{1}{2} + 1) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi}$

$A =$

$A \cdot \frac{(2\alpha)^{3/2} 2\pi}{\sqrt{2}} \cdot \frac{1}{2}\sqrt{\pi} = G \quad \therefore A = \cancel{\frac{G}{8\alpha}\left(\frac{2}{\pi}\right)^{-3/2}} = \frac{1}{2}\left(\frac{1}{\alpha\pi}\right)^{3/2}$

$\therefore g(r,t) = \frac{1}{2(\alpha\pi)^{3/2}} \cancel{\frac{G}{8\alpha}\left(\frac{2}{\pi}\right)^{3/2}} t^{-3/2} e^{-\eta^2/2}$ $\quad \eta = \frac{r}{\sqrt{2\alpha t}}$ $\quad \frac{\eta^2}{2} = \frac{r^2}{4\alpha t} = \frac{\tau}{4}$

$\left(\frac{r}{\sqrt{\alpha t}}\right)^3 = \tau^{3/2}$

$\frac{g r^3}{G} = \frac{1}{2} \frac{r^3}{(\alpha\pi t)^{3/2}} e^{-\eta^2/2}$ $\quad \eta^2 = \frac{\tau}{2}$

$g r^3/G = \frac{1}{2(\pi)^{3/2}} \tau^{3/2} e^{-\tau/4}$

$$c_1 + \ln\left(\frac{1}{2B}\sqrt{\frac{\pi}{\alpha}}\right) = 0 \qquad \therefore\; c_1 = -\ln\left(\frac{1}{2B}\sqrt{\frac{\pi}{\alpha}}\right)$$

$$\therefore\; e^{c_1} = e^{-\ln\left(\frac{1}{B}\sqrt{\frac{\pi}{\alpha}}\right)} \quad e^{\ln\left(2B\sqrt{\alpha/\pi}\right)} = 2B\sqrt{\alpha/\pi}$$

and $\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\,e^{c_1} = B \qquad 2B\sqrt{\alpha} = \sqrt{\pi}\,e^{c_1}$

$$\therefore\quad f(\chi) = \frac{1}{2\sqrt{\pi}}\left[\int_0^{\sqrt{\pi}\chi' e^{c_1}} \chi^{-3/2} e^{-\chi}\,d\chi + 2\sqrt{\pi}\right]$$

$$2\pi\int_0^\infty r^2 g(r,t)\,dr = G$$

$$(\eta^3 f)' = 3\eta^2 f + \eta^3 f'$$

$$B = \frac{1}{\sqrt{2\alpha}} \quad\therefore\quad B^3 = \left(\frac{1}{2\alpha}\right)^{3/2} \text{ or } \frac{1}{B^3} = (2\alpha)^{3/2}$$

$$\frac{A}{B^3}\cdot 2\pi\int_0^\infty \eta^2 f\,d\eta = G \quad\Rightarrow\quad A(2\alpha)^{3/2}\cdot 2\pi\int_0^\infty \eta^2 e^{-\eta^2/2}\,d\eta = G$$

let $\eta^2 = 2\zeta \quad\therefore\quad \cancel{2}\eta\,d\eta = \cancel{2}\,d\zeta$ or $\eta\,d\eta = \sqrt{2\zeta}\,d\eta = d\zeta \Rightarrow d\eta = \frac{d\zeta}{\sqrt{2\zeta}}$

$$\int_0^\infty \eta^2 e^{-\eta^2/2}\,d\eta = \int_0^\infty 2\zeta\, e^{-\zeta}\frac{d\zeta}{\sqrt{2\zeta}} = \frac{2}{\sqrt{2}}\int_0^\infty \zeta^{1/2} e^{-\zeta}\,d\zeta$$

$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-x}x^{-1/2}\,dx$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ since $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$

$$\Gamma(3/2) = \frac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$$

$$\therefore \int_0^\infty \eta^2 e^{-\eta^2/2}\,d\eta = \frac{2}{\sqrt{2}}\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{2}}$$

and $A\,(2\alpha)^{3/2}\cdot 2\pi\sqrt{\frac{\pi}{2}} = G \qquad A = \frac{G}{\sqrt{2}\,(2\pi\alpha)^{3/2}}$

$\therefore\; g = At^n f = \frac{G}{\sqrt{2}(2\pi\alpha)^{3/2}}\,t^{-3/2}\exp\left(-\frac{r^2}{4\alpha t}\right)$ if $\frac{r^2}{\alpha t} = \frac{1}{\tau}$ then

$g = \frac{G}{\sqrt{2}}\frac{1}{(2\pi\alpha t)^{3/2}}\exp\left(-\frac{1}{4\tau}\right) \qquad gr^3/G = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\left(\frac{r}{\sqrt{\alpha t}}\right)^3 \exp\left(-\frac{1}{4\tau}\right)$ $\frac{1}{\tau^{1/2}}$

$$gr^3/G = \frac{1}{4}\left(\frac{1}{\pi^{3/2}}\right)\tau^{-3/2}\exp\left(-\frac{1}{4\tau}\right)$$

$u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z} = \varepsilon\frac{\partial^2 u}{\partial z^2}$

$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0$

$\varepsilon = aql \qquad q = u(r,0) = Ar^n f(0)$

$q\downarrow$ as $r\to\infty \qquad l\uparrow$ as $r\to\infty$

$u(r,l) = .5\,u(r,0) \qquad f(\eta_0) = .5 f(0)$

$\varepsilon = aq\frac{\eta_0}{\beta} r^\alpha = aAr^n \frac{\eta_0}{\beta} r^\alpha f(0)$

$\int_0^\infty r u^2 dz = M = \text{const}$

BC

$u = 0$ at $z = \infty \qquad \frac{\partial u}{\partial z} = 0$ at $z = 0$

$v = 0$ @ $z = 0$

$\eta_0 = \beta l / r^\alpha \qquad l = \eta_0 \frac{r^\alpha}{\beta}$

if $u = Ar^n f(\eta) \quad v = Br^m g(\eta)$ and $\eta = \beta z / r^\alpha \qquad \frac{\partial \eta}{\partial z} = \frac{\beta}{r^\alpha} \quad \frac{\partial \eta}{\partial r} = \frac{-\alpha\beta z}{r^{\alpha+1}}$

then $\frac{\partial u}{\partial r} = Anr^{n-1} f(\eta) + Ar^n f' \cdot \frac{\partial \eta}{\partial r}$

$= Anr^{n-1} f(\eta) - \frac{\alpha}{r}\eta\, Ar^n f'(\eta)$

$\frac{\partial u}{\partial z} = Ar^n f' \frac{d\eta}{dz} = Ar^n \beta \frac{1}{r^\alpha} f'(\eta)$

$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial \eta}\cdot\frac{\partial \eta}{\partial z}\cdot\left(\frac{\partial u}{\partial z}\right) = \frac{\beta}{r^\alpha}\cdot Ar^n \beta \frac{1}{r^\alpha} f''(\eta) = \beta^2 A r^{n-2\alpha} f''(\eta)$

$\therefore \quad \underset{u}{Ar^n f}\,[\underset{du/dr}{Anr^{n-1} f - Ar^{n-1}\alpha\eta f'}] + \underset{v}{Br^m g(\eta)}\,\underset{du/dz}{Ar^{n-\alpha}\beta f'(\eta)} = \varepsilon\, \beta^2 A r^{n-2\alpha} f''$

$= aAr^n f(0)\ \beta^2 Ar^{n-}f$

$$u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z} = \varepsilon\frac{\partial^2 u}{\partial z^2} \qquad \varepsilon = \overset{\text{Const}}{a}\, q\, l \qquad q = u(r,0) \qquad u(r,l) = .5\,u(r,0)$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0$$

w/ $\int r u^2 dz = M$

$u \to 0$ as $z \to \infty$

$v = 0$ at $z = 0$

$\frac{\partial u}{\partial z} = 0$ at $z = 0$

let $u = Ar^n f(\eta)$ $\quad v = Br^m g(\eta)$ $\quad \eta = \beta z / r^\alpha$

$u(r,0) = Ar^n f(0) = q$ and $u(r,l) = .5\,u(r,0) \Rightarrow$

$Ar^n f(\eta_0) = Ar^n \cdot .5 f(0)$ where $\eta_0 = \beta l / r^\alpha$

$$\therefore \boxed{l = \frac{r^\alpha}{\beta}\eta_0}$$

$$\boxed{\varepsilon = a \cdot Ar^n f(0) \cdot \frac{r^\alpha}{\beta}\eta_0}$$

$$\frac{\partial u}{\partial r} = nAr^{n-1}f + Ar^n f'\frac{d\eta}{dr} \qquad \frac{\partial v}{\partial z} = Br^m\frac{d\eta}{dz}g' \qquad \frac{d\eta}{dr} = -\frac{\alpha\beta z}{r^{\alpha+1}} = -\frac{\alpha}{r}\eta$$

$$\frac{\partial u}{\partial z} = Ar^n f'\frac{d\eta}{dz} \qquad \frac{\partial^2 u}{\partial z^2} = Ar^n f''\left(\frac{d\eta}{dz}\right)^2 \qquad \frac{d\eta}{dz} = \frac{\beta}{r^\alpha}$$

$\therefore$

Mom.

$$\cancel{A}r^n f\left[n\cancel{A}r^{n-1}f + Ar^n\left(-\frac{\alpha}{r}\eta\right)f'\right] + B\cancel{r}^m g\left[\cancel{A}r^n\left(\frac{\beta}{r^\alpha}\right)f'\right] = \cancel{A}a\frac{r^{n+\alpha}}{\beta}\eta_0 f(0)\left[\cancel{A}r^n f''\left(\frac{\beta^2}{r^{2\alpha}}\right)\right]$$

$$r^n[r^{n-1}] + r^m[r^{n-\alpha}] = r^{n+\alpha}[r^{n-2\alpha}]$$

or r^{2n-1} $\quad r^{m+n-\alpha}$ $\quad r^{2n-\alpha}$ $\qquad \therefore 2n-1 = m+n-\alpha = 2n-\alpha$ $\quad m=n$ $\quad \boxed{\alpha = 1}$

Cont

$$n\cancel{A}r^{n-1}f + Ar^n\left(-\frac{\alpha\eta}{r}\right)f' + \cancel{A}r^{n-1}f + B\cancel{r}^m\frac{\beta}{r^\alpha}g' = 0 \Rightarrow r^{n-1} + r^{m-\alpha}$$

or $n-1 = m-\alpha \;\therefore$

from integral const $\quad \int_0^\infty r u^2 dz = \int r A^2 r^{2n} f^2 \frac{r^\alpha}{\beta}\, d\eta = \text{const}$

$\Rightarrow r^{2n+\alpha+1}$ must vanish $\quad \therefore 2n+\alpha+1 = 0$

or $\frac{A^2}{\beta}\int_0^\infty f^2(\eta)\,d\eta = M$ $\qquad \therefore 2n = -(1+\alpha)$

$\boxed{n = -1 \quad m = -1}$

$\therefore$ $f[nAf + A(-\alpha\eta)f'] + Bg\beta f' = \frac{a}{\beta}\eta_0 f(0)Af''\beta^2 = a\beta A\eta_0 f(0) f''$

$$\frac{A}{B\beta}\left[-f + \eta f'\right] + g f' = a\frac{A}{B}\eta_0 f(0) f''$$

$$\left(a\frac{A}{B}\eta_0\right) f(0) f'' + f'\left(\frac{A}{B\beta}\eta f - g\right) + \frac{A}{B\beta} f^2 = 0 \qquad \text{pick } \boxed{\frac{A}{B\beta} = 1} \text{ and } \boxed{\frac{aA}{B}\eta_0 = 1}$$

Cont $\quad -\cancel{Af} + A\eta f' + \cancel{Af} + B\beta g' = 0 \quad$ or $\frac{A}{B\beta}\eta f' = g'$

$$\therefore \quad f(0) f'' + f'(\eta f - g) + f^2 = 0 \qquad \therefore \frac{A}{B} = \beta \quad a\beta\eta_0 = 1 \quad \therefore \boxed{\begin{array}{l}\beta = \frac{1}{a\eta_0} \\ B = Aa\eta_0\end{array}}$$

$$\eta f' = g'$$

$u = 0$ as $z \to \infty \Rightarrow f(\eta) \to 0$ as $\eta \to \infty$

$v = 0$ at $z = 0 \Rightarrow g(0) = 0$

$\frac{\partial u}{\partial z} = 0$ at $z = 0 \Rightarrow \frac{A}{r^2}\beta f' = 0$ at $z = 0$ or $f'(0) = 0$ at $\eta = 0$

from 2nd eq $g'(0) = 0$ and from 1st eq $f(0) f''(0) + f^2(0) = 0 \Rightarrow$ if $f(0) \neq 0$ $f''(0) + f^2(0) = 0$

let $f(0) = 1 \Rightarrow f''(0) = -1$

let $p = f'$ $\qquad f(0)p' + p(\eta f - g) + f^2 = 0 \Rightarrow p' = -\left[f^2 + p(\eta f - g)\right]\frac{1}{f(0)}$

g' $\qquad p = f' \qquad f' = p$

$\eta p = g' \qquad g' = \eta p$

$I' = f^2$

w/IC $\quad I'(0) = 1$

$g'(0) = 0$

$f'(0) = 0$

$p'(0) = -1$

if $\eta f - g = x$ $\qquad \eta f - g\Big|_{\eta=0} = 0 \Rightarrow x(0) = 0$

$\underbrace{f + \eta f' - g'}_{0} = x' = f \qquad f(0) = 1 \Rightarrow x'(0) = 1$

$x'' = f' \qquad f'(0) = 0 \Rightarrow x''(0) = 0$

$x''' = f''$

$$x''' + x''(x) + x'^2 = 0$$

$$x''' + (xx')' = 0$$

or $x''' + (xx')' = 0 \Rightarrow x'' + xx' = C_1 \qquad x(0) = 0 \quad x''(0) = 0$

$\Rightarrow C_1 = 0$

$\therefore x'' + (xx') = 0 \qquad \therefore \left(x' + \frac{x^2}{2}\right)' = 0 \qquad \therefore x' + \frac{x^2}{2} = C_2$

@ $\eta = 0 \quad x'(0) = 1 \quad x^2(0) = 0 \quad \therefore C_2 = 1$

$$x' + \frac{1}{2}x^2 = 1$$

$\therefore x' = 1 - \frac{1}{2}x^2$ and $\frac{dx}{1 - \frac{1}{2}x^2} = d\eta \qquad \frac{dx}{\frac{1}{2}[2 - x^2]} = d\eta$

$\frac{dx}{2 - x^2} = \frac{1}{2}d\eta \qquad \therefore dx\left[\frac{A}{\sqrt{2} - x} + \frac{B}{\sqrt{2} + x}\right] = \frac{1}{2}d\eta$

$$\frac{\frac{1}{2\sqrt{2}}}{\sqrt{2} - x} + \frac{\frac{1}{2\sqrt{2}}}{\sqrt{2} + x}$$

$$\frac{1}{2\sqrt{2}}dx\left[\frac{1}{\sqrt{2} - x} + \frac{1}{\sqrt{2} + x}\right] = \frac{1}{2}d\eta$$

$$\frac{\sqrt{2} - \cancel{x} + \sqrt{2} + \cancel{x}}{2\sqrt{2}}$$

$\therefore \frac{1}{\sqrt{2}}dx\left[\frac{1}{\sqrt{2} - x} + \frac{1}{\sqrt{2} + x}\right] = d\eta$

$\ln\frac{\sqrt{2} + x}{\sqrt{2} - x} = \sqrt{2}\eta \qquad \therefore \frac{\sqrt{2} + x}{\sqrt{2} - x} = e^{\sqrt{2}\eta} \quad \therefore x = \frac{e^{\sqrt{2}\eta} - 1}{e^{\sqrt{2}\eta} + 1}\sqrt{2}$

but $x' = f = 1 - \frac{1}{2}x^2 = 1 - \frac{1}{2}\cdot 2\left[\frac{e^{\sqrt{2}\eta} - 1}{e^{\sqrt{2}\eta} + 1}\right]^2 = \frac{[e^{\sqrt{2}\eta} + 1]^2 - [e^{\sqrt{2}\eta} - 1]^2}{(e^{\sqrt{2}\eta} + 1)^2}$

$\therefore \frac{2e^{\sqrt{2}\eta}\cdot 2}{(e^{\sqrt{2}\eta} + 1)^2} = \boxed{\frac{4e^{\sqrt{2}\eta}}{(e^{\sqrt{2}\eta} + 1)^2} = f}$

$g = \eta f - x = \frac{4\eta e^{\sqrt{2}\eta}}{(e^{\sqrt{2}\eta} + 1)^2} - \sqrt{2}\frac{(e^{\sqrt{2}\eta} - 1)}{(e^{\sqrt{2}\eta} + 1)} = \boxed{\frac{4\eta e^{\sqrt{2}\eta} - \sqrt{2}(e^{2\sqrt{2}\eta} - 1)}{(e^{\sqrt{2}\eta} + 1)^2} = g}$

Now from integral $\frac{A^2}{\beta}\int_0^\infty \frac{16\, e^{2\sqrt{2}\eta}\cdot\, d\eta}{(e^{2\sqrt{2}\eta}+2e^{\sqrt{2}\eta}+1)^2} = M$

let $e^{\sqrt{2}\eta} = x$ $\quad \eta=0 \quad x=1 \quad \eta=\infty \quad x=\infty$ $\qquad \ln x = \sqrt{2}\eta$

$$\frac{16A^2}{\beta}\int_1^\infty \frac{x^2}{(x+1)^4}\cdot\frac{dx}{\sqrt{2}x} = \frac{16}{\sqrt{2}}\frac{A^2}{\beta}\int_1^\infty \frac{x\,dx}{(x+1)^4}$$ and $\frac{dx}{x} = \sqrt{2}\,d\eta \;\therefore\; d\eta = \frac{dx}{\sqrt{2}x}$

now $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x+1)^4}$

$$A(x+1)^3 + B(x+1)^2 + C(x+1) + D = x$$

$$\cancel{Ax^3} + \cancel{A\cdot 3x^2} + \cancel{A3x} + \cancel{A} + \cancel{Bx^2} + \cancel{2Bx} + \cancel{B} + Cx + D + C$$

$D+C=0 \qquad A=0$ (no x^3) $\quad B=0$ (no x^2) $\quad C=1 \qquad D=-1$

$$\int_1^\infty \frac{dx}{(x+1)^3} - \int_1^\infty \frac{dx}{(x+1)^4}$$

$$-\frac{1}{2}\frac{1}{(x+1)^2}\Big|_1^\infty + \frac{1}{3}\frac{1}{(x+1)^3}\Big|_1^\infty \quad +\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 - \frac{1}{3}\left(\frac{1}{2}\right)^3 = \frac{2}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$$

$$\left(\tfrac{1}{2}\right)^3\left[1-\tfrac{1}{3}\right]$$

$$\therefore\ \frac{\cancel{4}\,16}{3\,\sqrt{2}}\frac{A^2}{\beta}\cdot\frac{1}{\cancel{12}} = M \qquad \boxed{A^2 = \frac{3\sqrt{2}}{4}\frac{M}{\beta}}$$

$f(0)=1 \qquad \therefore \qquad \frac{4e^{\sqrt{2}\eta_0}}{(e^{\sqrt{2}\eta_0}+1)^2} = .5$

$\therefore\ \frac{4x}{(x+1)^2} = .5 \qquad \therefore\ 8x = x^2+2x+1 \qquad \therefore\ x^2-6x+1=0$

$$x = \frac{6\pm\sqrt{36-4}}{2} = \frac{6\pm 4\sqrt{2}}{2} = 3\pm 2\sqrt{2}$$

$x_1 = 3+2.828 = 5.828$ and $x_2 = .172$

check to see which is to be used x_1 since this should represent far field & only physically significant one

$\therefore\ \therefore\ e^{\sqrt{2}\eta_0} = x_1 \qquad \sqrt{2}\eta_0 = \ln x_1 \qquad \eta_0 = \frac{1}{\sqrt{2}}\ln(3+2\sqrt{2})$

thus $\beta = \frac{\sqrt{2}}{a\ln(3+2\sqrt{2})}$ and $A^2 = \frac{3\sqrt{2}\ M\,a\ln(3+2\sqrt{2})}{4\ \sqrt{2}} = \frac{3Ma\ln(3+2\sqrt{2})}{4}$

and $A = \left(\frac{3Ma\ln(3+2\sqrt{2})}{4}\right)^{1/2} \qquad B = \left(\frac{3Ma\ln(3+2\sqrt{2})}{4}\right)^{1/2}\frac{a}{\sqrt{2}}\ln(3+2\sqrt{2})$

$$l = \frac{r\alpha}{\beta}\eta_0 \quad \text{and} \quad \beta = \frac{1}{a\eta_0} \quad \therefore \quad l = ra\,\eta_0^2 = \frac{ra}{2}\left(\ln(3+2\sqrt{2})\right)^2$$

$$l = \frac{X}{\cos\theta}\,\frac{a}{2}\left[\ln(3+2\sqrt{2})\right]^2$$

$$u = \frac{A}{r}f(\eta) \qquad u(r,0) = \frac{A}{r}f(0) = \frac{A}{r} = \frac{1}{r}\left[\frac{3Ma\ln(3+2\sqrt{2})}{4}\right]^{1/2}$$

$$l = \frac{ra}{2}\left(\ln(3+2\sqrt{2})\right)^2$$

$$v(r,z) = \frac{B}{r}g(\eta) \qquad v(r,-\infty) = \frac{B}{r}g(-\infty) = \frac{B}{r}\left[0 - \sqrt{2}\,\frac{(0-1)}{(0+1)}\right] = \frac{+\sqrt{2}B}{r}$$

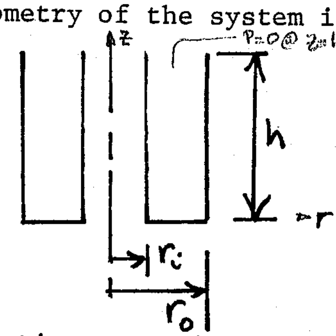
$$v(r,-\infty) = \frac{1}{r}\left(\frac{3Ma\ln(3+2\sqrt{2})}{4}\right)^{1/2} a\ln(3+2\sqrt{2}) = \frac{1}{r}\left[\frac{3M}{4}\right]^{1/2}\left[a\ln(3+2\sqrt{2})\right]^{3/2}$$

ME 200B
Winter 1979
W.C. Reynolds

PROBLEM 2 - Due Monday, February 5, 1979

This problem deals with the acoustic resonance of rectangular and annular cavities in emergency steam relief valves in nuclear power stations. These valves are designed to open when the system pressure exceeds a set value. The design includes some annular chambers in which the pressure builds up to snap the valve open (or closed). These valves leak slightly, allowing hot steam to escape. This is no hazzard, as the steam is captured and condensed in the system, but the loss of hot steam, if excessive, does degrade the system performance. The manufacturer desires to install a leak detection system (retroactive to construction), and the idea is to use a non-intrusive acoustic scheme. It is believed that the leaking steam blows over the mouth of an annular cavity, exciting acoustic resonances much as blowing over a bottle makes sound. This sound is louder than the background noise at the resonant frequency; then by measuring the strength of the sound at the resonant frequencies of the system the magnitude of the leak can be determined (using laboratory calibrations of sound generation versus leak rate). In this problem you will determine the resonant frequencies of an annular geometry that models that in the valve.

The geometry of the system is shown below:



BC $r=r_i$, $r=r_o$ $\frac{\partial p}{\partial r}=0$

$z=0$ $\frac{\partial p}{\partial z}=0$

$h = 15$ cm

$r_i = 4$ cm

$r_o = 10$ cm

The equation governing the acoustic pressure field in the annular chamber is

$$\nabla^2 p - \frac{1}{c^2} p_{tt} = 0$$

where $\nabla^2 p$ is the Laplace operator; in cylindrical coordinates,

$$\nabla^2 p = p_{rr} + \frac{1}{r} p_r + \frac{1}{r^2} p_{\theta\theta} + p_{zz}$$

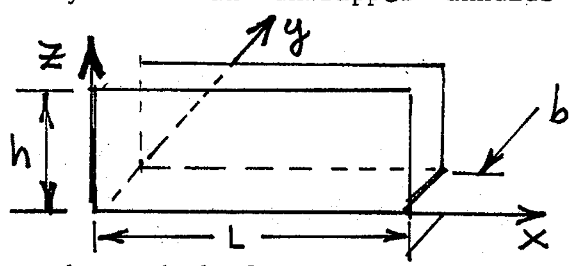
while in cartesian coordinates

$$\nabla^2 p = p_{xx} + p_{yy} + p_{zz}$$

c is the sound speed, which you may take as 500 m/s for this problem.

The boundary condition at a solid wall is $\partial p/\partial n = 0$, where n is the normal direction. This is equivalent to the requirement that the fluid not move at the solid boundary. At an open boundary (the plane $z=h$ in this problem) the boundary condition is $p=0$ (corrections are sometimes made at open ends, but we will use this simple condition here).

1. In order to get a feeling for the analysis, begin by doing an analysis for an "unwrapped" annulus below:



$\frac{\partial p}{\partial y} = 0$ @ $y=0, y=b$,

$p=0$ @ $z=h$

$p(x=L) = p(x=0)$

$\frac{\partial p}{\partial z} = 0$ @ $z=0$

$h = 15$ cm
$b = 6$ cm
$L = 14\pi$ cm

Using the method of separation of variables, determine the eigenmodes for this problem, using the following notation:

$$P_{nmj} = X_n(x)\, Y_m(y)\, Z_j(z) \cos(\omega_{nmj} t)$$

$$\omega_{nmj} = 2\pi f_{nmj}$$

Here f_{nmj} is the frequency (hz) of the n,m,j mode, and n,m, and j are integers identifying the modes. You should find that n and m can both be zero, indicating solutions independent of X and Y, and that j starts at 1. Note that the solutions must be periodic in x with period L. Express the dimensionless quantity $F_{nmj} \equiv f_{nmj}h/c$ as a function of n,m, and j and the ratios h/L and h/b.

Then, calculate F_{nmj} for the given geometry and sound speed, and order the first six modes by increasing frequency in a table as follows:

now $Z'(0)=0 \quad Z(h)=0 \quad \therefore \quad A\cos\gamma z + B\sin\gamma z = Z$

$\therefore A\cos\gamma h + B\sin\gamma h = 0$

and $-A\gamma\sin\gamma z + B\gamma\cos\gamma z = Z' \qquad Z'(0) \Rightarrow B=0$ or $\gamma=0$

if $B=0 \Rightarrow Z = A\cos\gamma z$ but $Z(h)=0 \Rightarrow \cos\gamma h = 0$

or $\gamma h = (2j-1)\frac{\pi}{2}$ where $j=1,2,3,\ldots \quad \therefore \gamma_j = (2j-1)\frac{\pi}{2h}$

thus we define γ; since the soln to Bessel's eq gave $\beta_{i\mu}$ & this defines γ_j then $\beta_{i\mu}^2 - \gamma_j^2 = \frac{\omega^2}{c^2}{}_{ij\mu} \Rightarrow \omega_{ij\mu}$

if $\mu=0$ then $P = RZT$ | if $\gamma=0 \Rightarrow Z=0$ to satisfy BC. Proof $\gamma=0 \Rightarrow Z''=0 \quad Z=az+b \quad \frac{\partial P}{\partial z}=0=a$

$\therefore Z=b$ but $P=0$ @ $z=h \Rightarrow b=0$ or $Z=0$ trivial case

define $P = XYZT$

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} - \frac{1}{c^2}\frac{T''}{T} = 0$$

$$\therefore \quad c^2\left[\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right] = \frac{T''}{T} = -\omega^2 \qquad \therefore T'' + \omega^2 T = 0 \qquad T = C_T\cos(\omega t + \phi)$$

$$\text{and } \frac{X''}{X} + \frac{Y''}{Y} = -\frac{Z''}{Z} - \frac{\omega^2}{c^2} = -\beta^2 \Rightarrow Z'' + \left(\frac{\omega^2}{c^2} - \beta^2\right)Z = 0 \qquad \text{let } \frac{\omega^2}{c^2} - \beta^2 = \gamma^2$$

$\therefore Z'' + \gamma^2 Z = 0$ or

$Z = A_z\cos\gamma z + B_\mu\sin\gamma z$

w/BC $\Rightarrow B=0$ or $\gamma=0$

for non trivial case take $B=0$

$\Rightarrow \gamma h = (2j-1)\frac{\pi}{2} \quad j \geq 1$

$\gamma_j = (2j-1)\frac{\pi}{2h}$

now $\frac{X''}{X} = -\beta^2 - \frac{Y''}{Y} = -\mu^2$ since X is periodic

$\therefore \quad X'' + \mu^2 X = 0 \quad \therefore \quad X = A_x\cos\mu x + B_x\sin\mu x \quad ; \mu L = n\pi \quad \therefore \mu = n\pi/L$

$[P(x=0) = P(x=L)] \Rightarrow \mu$ is integer

if $\mu \neq 0$ then for $X(0) = X(L)$

if $\mu=0 \Rightarrow X = ax+b$ for $X(0)=X(L) \Rightarrow a=0$ & $X=$ const or $P = YZT$

$\beta^2 + Y''/Y = \mu^2 \quad \therefore \quad Y'' + (\beta^2 - \mu^2)Y = 0$ w/ $\frac{\partial P}{\partial y}=0$ @ $y=0$ $y=b$ to satisfy BC $\beta^2 - \mu^2 \geq 0$

if $\beta^2 - \mu^2 = 0 \quad Y = ay+b \quad \frac{\partial P}{\partial y} = \frac{\partial Y}{\partial y} = a = 0 \Rightarrow Y =$ const.

$\therefore$ if $\mu \neq 0 \Rightarrow P = XZT$

if $\mu=\beta=0 \Rightarrow P = ZT$ if $\beta^2 - \mu^2 = \lambda^2 > 0$ then $Y = A_\lambda\cos\lambda y + B_\lambda\sin\lambda y$

$\frac{\partial P}{\partial y} = \frac{\partial Y}{\partial y} = \lambda\left[-A_\lambda\sin\lambda y + B_\lambda\cos\lambda y\right] \qquad \frac{\partial P}{\partial y} = 0\Big|_{y=0,y=b} \Rightarrow B_\lambda = 0 \quad /y=0$

$\lambda b = m\pi$

$\lambda = \frac{m\pi}{b}$

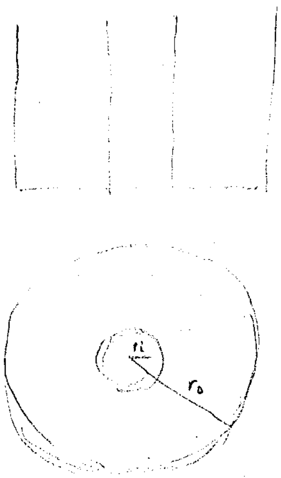
$\therefore \lambda^2 = \frac{m^2\pi^2}{b^2} = \beta^2 - \frac{n^2\pi^2}{L^2} \quad \therefore \quad \beta^2 = \frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{L^2}$

and $\gamma_j^2 = (2j-1)^2\frac{\pi^2}{4h^2}$

$$\therefore \frac{\omega^2}{c^2} = \frac{(2j-1)^2}{4}\frac{\pi^2}{h^2} + \frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{L^2} \qquad \therefore \frac{h^2 f_{nmj}^2}{c^2} = F_{nmj}^2 = \frac{1}{4}\left\{\left(\frac{2j-1}{2}\right)^2 + \frac{m^2h^2}{b^2} + n^2\left(\frac{h}{L}\right)^2\right\}$$

$$\frac{4\pi^2 f_{nmj}^2}{c^2} = \frac{\pi^2}{h^2}\left\{\left(\frac{2j-1}{2}\right)^2 + \frac{m^2h^2}{b^2} + \frac{n^2h^2}{L^2}\right\} \qquad F_{nmj} = \frac{1}{2}\sqrt{\frac{1}{4}(2j-1)^2 + m^2\left(\frac{h}{b}\right)^2 + n^2\left(\frac{h}{L}\right)^2}$$

2.



$$\nabla^2 p - \frac{1}{c^2} p_{tt} = 0$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

Assume $P = R\Theta Z T$

$$\therefore \quad \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} - \frac{1}{c^2}\frac{T''}{T} = 0$$

$$\therefore c^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z}\right) = \frac{T''}{T} = -\omega^2$$

$$\therefore \quad T'' + \omega^2 T = 0 \qquad T = A_t \cos\omega t + B_t \sin\omega t = C_t \cos(\omega_{ij\mu} t - \phi)$$

now $\quad \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\frac{Z''}{Z} - \frac{\omega^2}{c^2} = -\beta^2$

$$\therefore \quad \frac{Z''}{Z} = \left(\beta^2 - \frac{\omega^2}{c^2}\right) = -\gamma^2 \quad \therefore \quad Z'' + \gamma^2 Z = 0 \quad \therefore \; Z = A_z \cos\gamma z + B_z \sin\gamma z$$

now $\; r^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right) + \beta^2 r^2 = -\frac{\Theta''}{\Theta} = \mu^2$

$$\therefore \quad \Theta'' + \mu^2\Theta = 0 \quad \text{or} \quad \Theta = A_\theta \sin\mu\theta + B_\theta \cos\mu\theta$$

$$\therefore \; r^2 R'' + rR' + (\beta^2 r^2 - \mu^2)R = 0$$

$$R = A_\mu J_\mu(\beta r) + B_\mu Y_\mu(\beta r)$$

BC. $\; \frac{\partial p}{\partial r} = 0 \;@\; r = r_i,\, r = r_o \qquad \therefore \; R'(r_i) = 0 \qquad R'(r_o) = 0$

$\frac{\partial p}{\partial z} = 0 \;@\; z = 0 \qquad \therefore \; Z'(0) = 0 \qquad \& \; p = 0 \;@\; z = h$

also fn must be continuous $\Rightarrow \; P(r, z, t, 0 \text{ rad}) = P(r, z, t, 2\pi \text{ rad}) \Rightarrow \Theta(0) = \Theta(2\pi) \Rightarrow \mu$ is integer

if $\mu = 0 \Rightarrow \Theta$ = const solution $\neq f(\theta)$

from $R'(r_i) = 0 \quad R'(r_o) = 0$

$\Rightarrow \; A_\mu J_\mu'(\beta r_i) + B_\mu Y_\mu'(\beta r_i) = 0$

$\Rightarrow \; A_\mu J_\mu'(\beta r_o) + B_\mu Y_\mu'(\beta r_o) = 0$

if $\mu = 0 \quad J_0'(\beta r) = -\beta J_1(\beta r)$
$\qquad\qquad Y_0'(\beta r) = -\beta Y_1(\beta r)$

$\therefore \; A_\mu J_1(\beta r_i) + B_\mu Y_1(\beta r_i) = 0$
$\quad A_\mu J_1(\beta r_o) + B_\mu Y_1(\beta r_o) = 0$

$\therefore \; \left| J_1(\beta r_i) Y_1(\beta r_o) - J_1(\beta r_o) Y_1(\beta r_i) \right| = 0$

but for $\mu = 1$: $\; J_1'(\beta r) = \beta J_0(\beta r) - \frac{1}{r} J_1(\beta r)$
$\qquad Y_1'(\beta r) = \beta Y_0(\beta r) - \frac{1}{r} Y_1(\beta r)$

also $\; A_\mu\left[r_i\beta J_0(\beta r_i) - J_1(\beta r_i)\right] + B_\mu\left[r_i\beta Y_0(\beta r_i) - Y_1(\beta r_i)\right] = 0$

for $\mu = 1$: $\; A_\mu\left[r_o\beta J_0(\beta r_o) - J_1(\beta r_o)\right] + B_\mu\left[r_o\beta Y_0(\beta r_o) - Y_1(\beta r_o)\right] = 0$

but from $\mu = 0$ we got that the second terms to drop

$\therefore \quad A_\mu J_0 + B_\mu Y_0 = 0 \Rightarrow \left| J_0(\beta r_i) Y_0(\beta r_o) - J_0(\beta r_o) Y_0(\beta r_i) \right| = 0$
$\quad\; A_\mu J_0 + B_\mu Y_0 = 0$

thus from these solutions we get $\beta_{i\mu}$

n	m	j	F_{nmj}	f_{nmj}, hz
0	0	1	0.25	-
-	-	-	-	-
-	-	-	-	-
-	-	-	-	-
-	-	-	-	-

2. Repeat the analysis in the cylindrical geometry, using the following notation:

$$P_{nmj} = \Theta_n(\theta)\ R_m(r)\ Z_j(z)\ \cos(\omega_{nmj} t)$$

$$\gamma_j = (2j-1)\frac{\pi}{2h} \qquad\qquad \frac{\omega^2}{c^2} - \gamma^2 = \beta^2$$

Your differential equation for $R(r)$ will involve integer-order Bessel functions (HMF 9.1.1) and the functions $J_n(\beta r)$ and $Y_n(\beta r)$. The boundary conditions on R will lead you to a determinant that must vanish. See HMF 9.1.28. For one value of n the Tables on HMF, page 415, will give you the required roots.

For the given geometry, calculate (at least) the six smallest values of $\beta_{nm} r_i$. To aid you in this we have placed two computer programs on LOTS as follows:

BESJYN (N,X,JN,YN,JPN,YPN,ICN)

calculates

$J_n(x), Y_n(x), J_n'(x), Y_n'(x)$; ICN is an error return index.

ROOTS (FUNC, XMIN, DX, XMAX, OSUB)

calculates the roots of a user-defined oscillating function F=FUNC(X). In the interval XMIN$\leq$ X $\leq$XMAX. At intervals of DX the user-defined subroutine OSUB(X,F) is called so that the user can print F if desired. Roots are given when the OSUB is called F = 0.

FORTRAN decks for these two routines will be made available to organizations having registered students in 200B. On LOTS these are in directory <W. WCR> , files BESJYN and ROOTS.

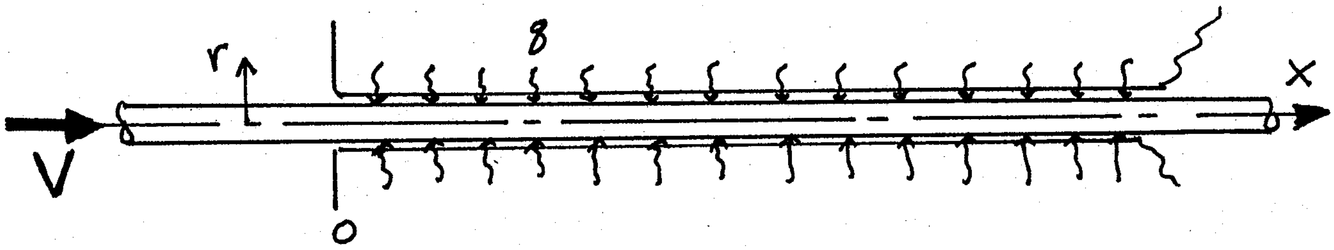
To use these routines, you need only to write a small FUNCTION program defining F. Func will call BESJYN. Have your OSUB routine write F(X) as well as the roots. You can verify that everyting is working properly by checking against the case for which you can look up the roots in HMF p. 415.

Once you have calculated the βr_1 values indicated above, you will be able to make a table similar to that in part 1. Give the first six modes, ordered by increasing frequency. For each sketch the radial variation R(r). Feel free to use BESJYN again to get this information if necessary.

END

PROBLEM 3: DUE MONDAY, FEBRUARY 26

In the extrusion of metal from rod, it is first necessary to heat the metal rod up. A continuous metal wire moves steadily at velocity V through an electrical heater, as shown below:



The equation describing the temperature field in the wire is* (neglecting axial heat conduction)

$$k\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) = r\rho c V\frac{\partial t}{\partial x} \tag{1}$$

$K \nabla^2 T = \rho c_p \frac{DE}{Dt}$ steady state $\underset{\sim}{u} = (V,0)$, no pressure

where k, ρ, and c are the thermal conductivity $W/(m\cdot K)$, density kg/m^3, and specific heat $J/(kg\cdot K)$, respectively. The boundary condition is, for constant heat flux,

$$k\frac{\partial t}{\partial r} = q \quad \text{at} \quad r = a \tag{2}$$

where q is the heat flux W/m^2 delivered by the heater. The "initial" condition is

$$t = t_o \quad \text{at} \quad x = 0 \tag{3}$$

The purpose of this problem is to solve for the temperature in the wire $t(r,x)$, and then to prepare some graphs useful in engineering design.

The first step in the analysis is to find a solution that takes care of the inhomogeneity in the boundary conditions. This is the far-downwire solution; a little physical thinking suggests that far downwire the temperature at any point should be rising linearly in x, and that the radial temperature

*Thermosciences students should derive this equation. Use a stationary control volume.

profile should be invariant with x. This suggests that you might find the desired particular solution in the form

$$t_p = Ax + F(r)$$

where A is a constant determined by the boundary condition.

1. Develop the particular solution; find the value of A (in terms of the problem variables) and the function $F(r)$. Choose the constant of integration in F such that $t_p(0,0) = t_o$ (an artistic choice).

2. Now you are ready to carry out the "entry region" part of the solution. Assume $t = t_p(x,r) + \phi(x,r)$ and set up the problem for ϕ. Note that ϕ will have homogeneous boundary conditions, and a more complicated initial condition.

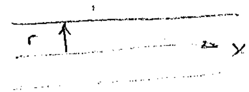
3. Using separation of variables, develop the partial solutions to the homogeneous part of the ϕ problem. Choose your separation constants such that your R equation is $(rR')' + \lambda^2 rR = 0$, and denote $\beta = k/(\rho cV)$. Normalize your eigenfunctions such that $R_n(0) = 1$, and $X_n(0) = 1$, and give the expressions for the eigenfunctions $R_n(r)$ and the first three values of the eigenvalues $\lambda_n a$ (see HMF Sec. 9).

4. Taking ϕ as the sum of an infinite number of the partial solutions,

$$\phi = \sum_{n=1}^{\infty} C_n R_n(r) X_n(x) + C_o$$

Evaluate the C_n using the orthogonality property of the R eigenfunctions. These will involve integrals, <u>all</u> of which may be integrated in closed form and expressed in terms of known functions (including those calculable from BESJYN). <u>Work out</u> these integrals using one of the most important tools ibp of applied mathematics. See notes, Sect. 4.5. CHECK: $C_n k/(qa)$ is a <u>very</u> simple function of $J_o(\lambda_n a)$ and $\lambda_n a$.

$k\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) = r\rho c V\frac{\partial t}{\partial x}$ let $\rho c V = \alpha$



w/ $k\frac{\partial t}{\partial r} = q$ @ $r=a$

$t = t_0$ @ $x=0$

take $t_p = Ax + F(r)$ $k\frac{\partial t_p}{\partial r} = kF'(r) = q$

$k\frac{\partial}{\partial r}\left(r\frac{\partial t_p}{\partial r}\right) = r\alpha A$

$k\frac{\partial}{\partial r}\left(rF'\right) = r\alpha A$

$k\left(F' + rF''\right) = r\alpha A$

$(rF')' = r\frac{\alpha A}{k}$

$rF' = \frac{r^2}{2}\frac{\alpha A}{k} + c_1$

$F' = \frac{r}{2}\frac{\alpha A}{k} + \frac{c_1}{r}$

$F = \frac{r^2}{4}\frac{\alpha A}{k} + c_1 \ln r + c_2$ $\therefore$ $t_p = Ax + F = Ax + \frac{r^2}{4}\frac{\alpha A}{k} + c_1\ln r + c_2$

@ $r=0, x=0$ $t_p = t_0$ $\Rightarrow c_1 = 0$ for $|t_p|<\infty$ $c_2 = t_0$

$\therefore$ $t_p = Ax + \frac{r^2}{4}\frac{\alpha A}{k} + t_0$ (1)

now $k\frac{\partial t_p}{\partial r} = kF'(r) = \cancel{k}\frac{a}{2}\frac{\alpha A}{\cancel{k}} = q$ $A = \frac{2q}{a\alpha}$

$t_p = \frac{2q}{a\alpha}\left[x + \frac{r^2\alpha}{4k}\right] + t_0 = \frac{2q}{ak}\beta\,[$

now $t(x,r) = t_p(x,r) + \phi(x,r)$ $\therefore$ $k\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = r\alpha\frac{\partial\phi}{\partial x}$

$k\frac{\partial\phi}{\partial r} = 0$ @ $r=a$

@ $x=0$ $t=t_0$

$t(0,r) = t_p(0,r) + \phi(0,r) = t_0$

$= \frac{2q}{a\cancel{\alpha}}\frac{r^2\cancel{\alpha}}{4k} + t_0 + \phi(0,r) = t_0$

$\therefore \phi(0,r) = -\frac{2qr^2}{4ak} = -$

$$k\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)X = r\alpha X'R$$ now let $\frac{k}{\alpha}=\beta$ 2

$$\therefore \frac{k(rR')'}{\alpha r R} = \frac{1}{\beta}\frac{X'}{X} = \quad ; \quad \frac{(rR')'}{rR} = -\lambda^2$$

then $\underbrace{X' = -\lambda^2\beta X}$ and $(rR')' = -\lambda^2 r R \Rightarrow (rR')' + \lambda^2 r R = 0$

$$\underline{X} = Ae^{-\lambda^2\beta x} \Rightarrow X = e^{-\lambda^2\beta x}$$

w/ $R(r) = -\frac{qr^2}{2aK}$ at $x=0$

$$R'(a) = 0$$

$$r^2R'' + rR' + \lambda^2 r^2 R = 0$$

let $\xi = \lambda r$ $\quad d\xi = \lambda dr$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \lambda^2r^2R \qquad d^2\xi = \lambda d^2r \qquad d\xi^2 = \lambda^2 dr^2$$

$$\lambda^2r^2\frac{d^2R}{\lambda^2dr^2} + \lambda r\frac{dR}{\lambda dr} + \lambda^2r^2R$$

$$\xi^2R''(\xi) + \xi R'(\xi) + \xi^2R(\xi) = 0$$

Solution is $C_1J_0(\lambda r) + C_2\cancel{Y_0}(\lambda r) = R(\lambda r)$ $\quad\therefore J_0(\xi)$ satisfies or $J_0(\lambda_n r)$ gets

Y_0 since it $\Rightarrow \infty$ @ $r=0$

$\therefore$ now since $R(0) = C_1J_0(0) = C_1 = 1$ $\quad\therefore\quad R(\lambda r) = J_0(\lambda r)$

$R'(\lambda_n a) = J_0'(\lambda_n a) = 0$ as the $\underline{\underline{EF}}$ eqn.

$-\lambda_n J_1(\lambda_n a) = 0$

Now let

$$\phi_n(x,r) = e^{-\lambda_n^2\beta x}J_0(\lambda_n r)$$

Now $\phi = \sum C_n e^{-\lambda_n\beta x}J_0(\lambda_n r) + C_0$

now ϕ must satisfy $\phi(0,r) = -\frac{qr^2}{2aK}$ and

$$\cancel{\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = \sum C_n e^{-\lambda_n\beta x}(-\lambda_n)J_1(\lambda_n a) = 0}$$

$$\phi(0,r) = -\frac{qr^2}{2aK} = \sum C_n J_0(\lambda_n r) + C_0$$

$$\cancel{\int_0^a J_0(\lambda_m r)\phi\,dr = \int_0^a -\frac{qr^2}{2aK}J_0(\lambda_m r)\,dr = \sum C_n\int_0^a J_0(\lambda_m r)J_0(\lambda_n r)\,dr + C_0\int_0^a J_0(\lambda_m r)\,dr}$$

since $\int_0^a r R\,dr = 0$ then to find c_0

$$\int_0^a \phi(0,r)\,r\,dr = \int_0^a -\frac{qr^3\,dr}{2ak} = \int_0^a \sum c_n\, r \cancel{J_0}\,dr + c_0 \int_0^a r\,dr$$

$\to 0$ by the above

$$= -\frac{qr^4}{8ak}\Big|_0^a = c_0 \frac{r^2}{2}\Big|_0^a \Rightarrow c_0 = -\frac{qa}{4k}$$

the orthog prop for bessel fn is $\int_a^b r R_m R_n\,dr = 0 \quad m \neq n$

$$\therefore \int_0^a r\,\phi(0,r) R_m\,dr = \sum c_n \int_0^a r R_n R_m\,dr + c_0 \int_0^a \cancel{r R_m\,dr}$$

$$\int_0^a -\frac{qr^3}{2ak} R_m\,dr = c_m \int_0^a r R_m^2\,dr \qquad \therefore \quad c_m = \frac{\int_0^a -\frac{qr^3}{2ak} R_m\,dr}{\int_0^a r R_m^2\,dr}$$

$$\therefore \quad c_m = \frac{\int_0^a \frac{-qr^2}{2ak}(rJ_0)\,dr}{\int_0^a r J_0^2\,dr} = \frac{\int_0^a \frac{qr^2}{2ak}\frac{(rJ_0')'}{\lambda_m^2}\,dr}{} = \frac{\frac{q}{2a\lambda_m^2 k}\left[r^2(J_0' r)\Big|_0^a - \int 2r^2 J_0'\,dr\right]}{}$$

0 bc

use $r^2\,dv = J_0'\,dr$, $du = 2r$, $v = \frac{J_0}{\lambda_m}$

$$-2\int r^2 J_0'\,dr = -2\left[r^2 \frac{J_0}{\lambda_m}\Big|_0^a - \int \frac{2r \cancel{J_0}}{\lambda_m}\,dr\right] = -2r^2\frac{J_0}{\lambda_m}\Big|_0^a$$

0 by bc

$$\therefore \text{numerator} = \frac{-q a^2 J_0(\lambda_m a)}{\lambda_m \cdot 2a\lambda_m^2 k}\cdot q = -\frac{a J_0(\lambda_m a)q}{\lambda_m^3 k}$$

$$\int_0^a r J_0^2\,dr = \cancel{\int_0^a -\frac{(rJ_0')'}{\lambda_m^2} J_0\,dr} = \cancel{-\frac{1}{\lambda_m^2}\int_0^a (rJ_0')' J_0\,dr}$$

$$\cancel{\left[J_0(rJ_0') - \lambda_m \int r J_0' J_0'\right]}$$

$$= \frac{1}{2\lambda_m}\left\{J_0' r^2 \cancel{J_0'}\Big|_0^a - J_0 r (rJ_0')'\right\} = \frac{1}{2\lambda_m} J_0 r^2 \lambda_m^2 J_0\Big|_0^a = \frac{\lambda_m}{2} a^3 J_0^2(\lambda_m a)$$

0

$$\therefore \quad c_m = \frac{-a J_0(\lambda_m a) q}{\lambda_m^3 k}\cdot\frac{2}{\lambda_m a^3 J_0^2(\lambda_m a)} = -\frac{2q}{ka}\,\frac{1}{J_0(\lambda_m a)\,\lambda_m^4}$$

$$\frac{c_m k}{qa} = \frac{-2}{a^2 J_0(\lambda_m a)\lambda_m^4} \overset{?}{=} \frac{2}{\xi_n^2 J_0(\xi a)} \quad ?$$

$$\therefore \quad t = t_p + \phi$$

$$\frac{k}{qa}(t_c - t_0) = \left(Ax + \sum c_n e^{-\lambda^2\beta x} + c_0\right)\frac{k}{qa} = -\frac{1}{4} + \sum \frac{c_m k}{qa} e^{-\lambda^2\beta x} + \frac{2}{a^2}\beta x$$

$$-\frac{1}{4} + \sum \frac{-2}{J_0(\lambda_m a)(\lambda_m a)^4} e^{-\lambda_m^2 a^2 \xi} \qquad 2\xi$$

$$t = t_p + \phi$$

$$t_s(a,x) = t_p(a,x) + \phi(a,x) \Rightarrow t_s - t_0 = \frac{2q\beta}{ak}\left[x + \frac{a^2}{4\beta}\right] + \sum c_n e^{-\lambda^2\beta x} J_0(\lambda_m a) + c_0$$

$$(t_s - t_0)\frac{k}{qa} = \frac{k}{qa}\left[-\frac{1}{4} + 2\xi + \quad + \sum \frac{-2}{(\lambda_m a)^2} e^{-(\lambda_m a)^2 \xi}\right]$$

5. From the extruder's point of view, the critical problem is getting the center temperature high enough to allow extrusion, and keeping the surface temperatures from getting too high. The centerline temperature $t_c(x) = t(0,x)$ and the surface temperature $t_s(x) = t(a,x)$ can be expressed non-dimensionally as

$$T_c = \frac{(t_c - t_o)\,k}{qa} = f(\xi)$$

$$T_s = \frac{(t_s - t_o)\,k}{qa} = g(\xi)$$

where

$$\xi = \frac{kx}{\rho c V a^2} = \frac{k}{\alpha}\frac{x}{a^2} = \frac{\beta x}{a^2} \qquad \therefore \beta x = \xi a^2$$

Using the results from above, give expressions for the functions f and g, and plot these functions vs ξ. In making this plot you will have to decide how many terms to use in the series and will need more terms for small ξ. Make this a single plot on linear paper covering the range.

PROBLEM 4: DUE FRIDAY, MARCH 16

This problem deals with the motion of long water waves moving over shallow water of depth $h(x,y)$. The governing equation for the free-surface displacement $z(x,y,t)$ is*

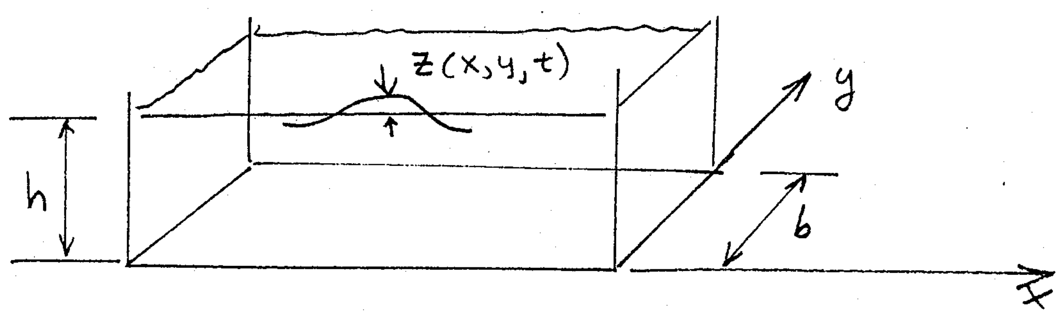
$$\frac{\partial^2 z}{\partial t^2} - g\left[\frac{\partial}{\partial x}\left(h\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial z}{\partial y}\right)\right] = 0$$

1. Consider first the case h = constant. Investigate traveling wave solutions in a canal of width b in the y direction, assuming

$$z = F(y)\, G(x - at)$$

At the sides of the canal the boundary condition is

$$z_y = 0 \quad \text{at} \quad y = 0, b$$

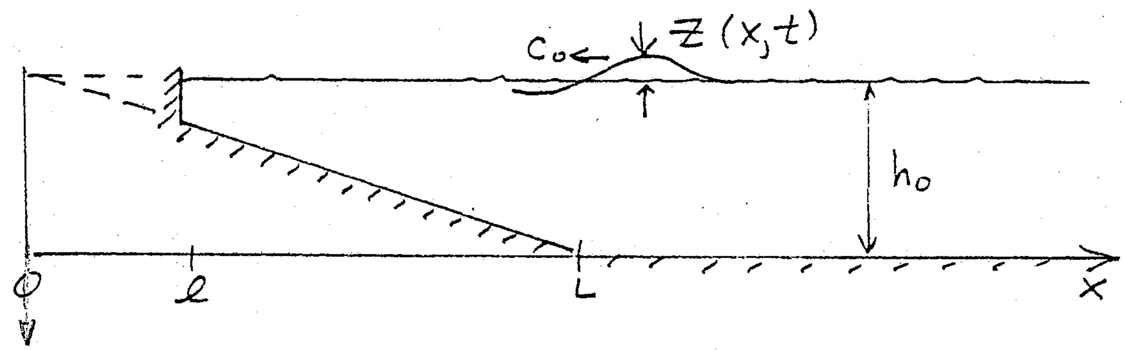


a) First show that waves independent of y, having any shape G, will propagate at velocity $a = \sqrt{gh}$. Denote $c = \sqrt{gh}$ here and subsequently. Express the frequency of sinusoidal waves as a function of their length λ.

*To derive, assume uniform tangential velocity profiles in the x and y directions, hydrostatic variation of the pressure with depth, and neglect non-linear terms in velocity and velocity-displacement products. See tutors for complete derivations.

1\. b) Next, derive expressions for the wave-length (in the x direction) and frequency of waves which have a y dependence. Sketch the wave speed a and frequency ω as functions of the axial wavelength λ for the "first" y-dependent wave, displaying the cut-off frequency ω_c below which the waves would not propagate in the channel. A channel of width b will propagate only one-dimensional waves at frequencies below the cut-off frequency.

2\. Next, consider one-dimensional waves $z(x,t)$ approaching a coastal area with the bottom contour as shown below:



$$h = ax \qquad \ell \le x \le L$$

$$h = h_o = aL \qquad L \le x \le \infty$$

A steady sinusoidal train of waves approaches the coast, with waves described by

$$z = A\cos\left[\alpha(x + c_o t)\right] \qquad x \ge L \tag{1}$$

where $c_o = \sqrt{gh_o}$.

a) Verify that (1) is a solution to the governing equation for $x \ge L$.

b) Develop the solution for $\ell \le x \le L$ using complex imbedding. Match z and z_x of this solution to (1) at the point $x = L$. Assume $z = e^{i\omega t}F(x)$, and solve the problem for F . HINT: See HMF 9.1.53. Then, write F as $B(f_r + i\, f_i)$, where $f_r^2 + f_i^2 = 1$, and express the (real) solution in the beach area in the form

$$\eta = C\cos\left[\omega t + \theta(x)\right]$$

which reveals the traveling wave character of the solution. Give the expression for C and $\theta(x)$. NOTE: This analysis assumes that the wave energy is dissipated in a breaking wave at the seawall.

2. c) For the following values, calculate the ratio of the wave height at the seawall to the wave height in the open sea:

$$g = 9.8 \text{ m/s}^2 \qquad \ell = 20\text{m} \qquad L = 1000\text{m}$$

$$h_o = 10\text{m} \qquad \lambda = 2\pi/\alpha = 20\text{m}$$

Are the waves shorter or longer at the seawall? See HMF 9.2.1, 9.2.2.

3. Next, consider one-dimensional waves $z(x,t)$ where $h = h(x)$.

   a) Derive the equation for the slopes of the characteristics, using the method presented in the notes (and in class). Denote $c(x) = \sqrt{gh}$.

   b) Transform the governing equation to characteristic coordinates, using the following independent variables $(c = \sqrt{gh})$

$$\xi = t + \int_o^x \frac{dx'}{c(x')}$$

$$\eta = t - \int_o^x \frac{dx'}{c(x')}$$

   For the case of constant c the transformed equation may be solved in closed form. Is the same true here?

   c) For the case of constant depth, find a solution that satisfies the following initial conditions:

$$z(x,0) = Ae^{-ax^2}$$

$$z_t(x,0) = Be^{-bx^2}$$

****

CONGRATULATIONS!!! YOU ARE FINISHED WITH ME 200B HOMEWORK!!!!!

$z = F(y)G(x-at)$

1. $z_{,tt} - gh\left[z_{,xx} + z_{,yy}\right] = 0$ w/ $z_{,y} = 0$ @ $y = 0, b$

if $z = G(x-at)$ only put into DE $\Rightarrow$ $a^2 G'' - gh\left[G'' + 0\right] = (a^2 - gh)G'' = 0$

for any wave shape take $a^2 - gh = 0$ then $a = \sqrt{gh} = c$.

IF $z = G(x-at) = A\cos(\alpha(x-ct))$ $\omega = \alpha c$ and $\lambda = 2\pi/\alpha$

2. if $z = f(y)G(x-at)$ then $(a^2 - gh)\frac{G''}{G} = gh\frac{F''}{F}$ or $\left[\left(\frac{a}{c}\right)^2 - 1\right]\frac{G''}{G} = \frac{F''}{F} = -\beta^2$

$a \neq c$

or $F = A\cos\beta y + B\sin\beta y$ for F to solve bc. $\Rightarrow$ $B = 0$ and $\beta b = n\pi$ or $\beta = \frac{n\pi}{b}$

if $\beta = 0$ then $F'' = 0$ or $F' = c_1$ $F = c_1 y + c_2$; to satisfy bc $c_1 = 0$ $\therefore$ $F = c_2$ and no y depend.

if $\beta = 0$ $\Rightarrow$ $G'' = 0$ or $G' = c_3$ $\hat{G} = c_3(x-at) + \hat{c}_4$ $\therefore$ $z = c_3(x-at) + c_4$ and no y dependence

if $\beta \neq 0$ then

$$\frac{a^2 - c^2}{c^2}\frac{G''}{G} = -\frac{n^2\pi^2}{b^2} \quad \text{or} \quad \frac{a^2 - c^2}{c^2}G'' + \frac{n^2\pi^2}{b^2}G = 0 \quad \text{or} \quad G'' + \frac{c^2 n^2 \pi^2}{b^2(a^2 - c^2)}G = 0$$

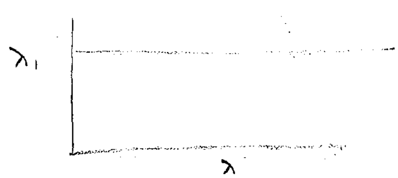
or

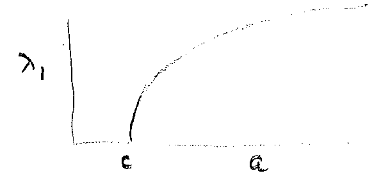
$$G = D\sin\frac{cn\pi}{b(a^2-c^2)^{1/2}}(x-at) + E\cos\frac{cn\pi(x-at)}{b(a^2-c^2)^{1/2}} \qquad \alpha = \frac{cn\pi}{b(a^2-c^2)^{1/2}} \qquad \lambda = \frac{2\pi b(a^2-c^2)^{1/2}}{cn\pi}$$

$$\lambda = \frac{2b}{cn}(a^2 - c^2)^{1/2}$$

$$\omega = \frac{cn\pi a}{b(a^2-c^2)^{1/2}} = \frac{ca\,n\pi}{bac\left(\frac{1}{c^2} - \frac{1}{a^2}\right)^{1/2}} = \frac{n\pi}{b\left(\frac{1}{c^2} - \frac{1}{a^2}\right)^{1/2}}$$

now $$\lambda/\lambda_1 = \frac{\frac{2b(a^2-c^2)^{1/2}}{cn}}{\frac{2b(a^2-c^2)^{1/2}}{c}} = \frac{1}{n} \qquad \lambda_1 = \frac{2b}{c}(a^2 - c^2)^{1/2} \text{ or } \sqrt{\left(\frac{\lambda_1 c}{2b}\right)^2 + c^2} = a$$





$$\omega_{\text{cut off}} = \frac{n\pi c}{b}$$

2a. for a 1 dimensional wave $z_{xx} - \frac{1}{c_0^2}z_{tt} = 0$ where $c_0 = \sqrt{gh_0}$ $z = A\cos\left[\alpha(x + c_0 t)\right]$

or $gh_0 z_{xx} - z_{tt} = 0$

we also that $z_{xx} - \frac{1}{a^2}z_{tt} = 0$ is DE for $l \leq x \leq L$ with $h = ax$

$gh z_{xx} - z_{tt} = 0$ for $l \leq x \leq L$ where $h = h(x) = ax$ for $l \leq x \leq L$

$aL = h_0$ $x = L$

then $gax F'' e^{i\omega t} + \omega^2 F e^{i\omega t} = 0$ or $gax F'' + \omega^2 F = 0$

ME 200B
WCR
Winter 79

MIDTERM EXAMINATION: CLOSED BOOKS AND NOTES

Please occupy alternate seats if possible, and refrain from any conversation with your neighbors.

Please help us enforce the Honor Code!

The exam consists of two problems. For each, reduce the problem to one or more ordinary differential equations. Define any new variables, constants, or functions that you introduce. Find the solution to the ordinary differential equation(s) in terms of special functions given on the attached two sheets, and give the expression(s) for evaluating eigenvalue(s) that appear in this solution (if any), in terms of these functions and their derivatives. If you can evaluate the amplitude(s) of the solution(s), do so; if you cannot, explain why not. If eigenfunctions are involved, DEMONSTRATE their orthogonality property.

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(50) 1. In analysis of the temperature field between a magnetic tape and the read head, the following differential equation problem arises.

$$\frac{\partial^2 T}{\partial y^2} = \alpha y \frac{\partial T}{\partial x}$$

$$T(0,x) = 0 \qquad T(a,x) = 0 \qquad T \to 0 \quad \text{as} \quad x \to \infty$$

Solve as indicated above.

(50) 2. A viscous fluid is initially motionless above a flat plate. At time zero, the plate begins moving to the right with a linearly increasing velocity. The velocity field $u(y,t)$ in the fluid is described by

$$\frac{\partial^2 u}{\partial y^2} = \alpha \frac{\partial u}{\partial t}$$

$$u(0,t) = \beta t$$

$$u(y,0) = 0$$

$$u(y,t) \to 0 \quad \text{as} \quad y \to \infty$$

Solve as indicated above.

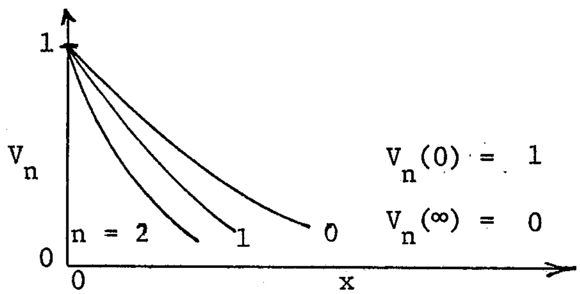
SOME MATERIAL FROM HMF

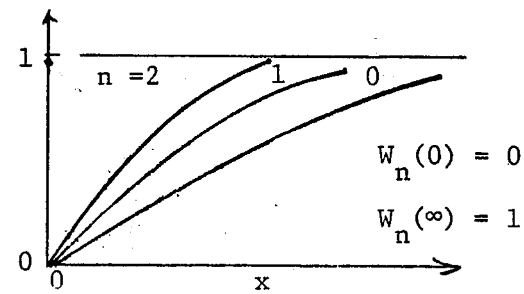
(Slightly distorted for simplicity and good humor)

1. $y'' + xy' + ny = 0$

$$y = C_1 V_n(x) + C_2 W_n(x)$$

where V_n and W_n are the n^{th} order Wight functions,

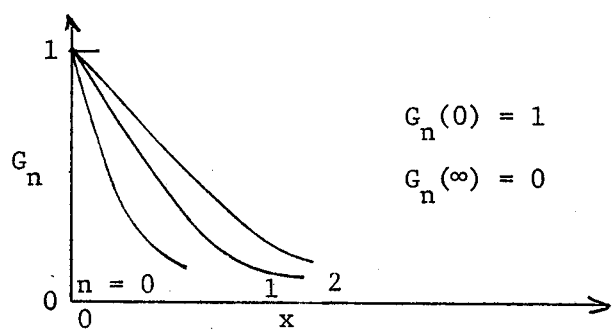


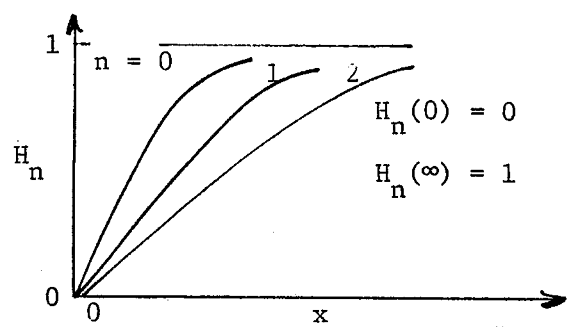


2. $y'' + xy' - ny = 0$

$$y = C_1 G_n(x) + C_2 H_n(x)$$

where G_n and H_n are the n^{th} order Wong functions,





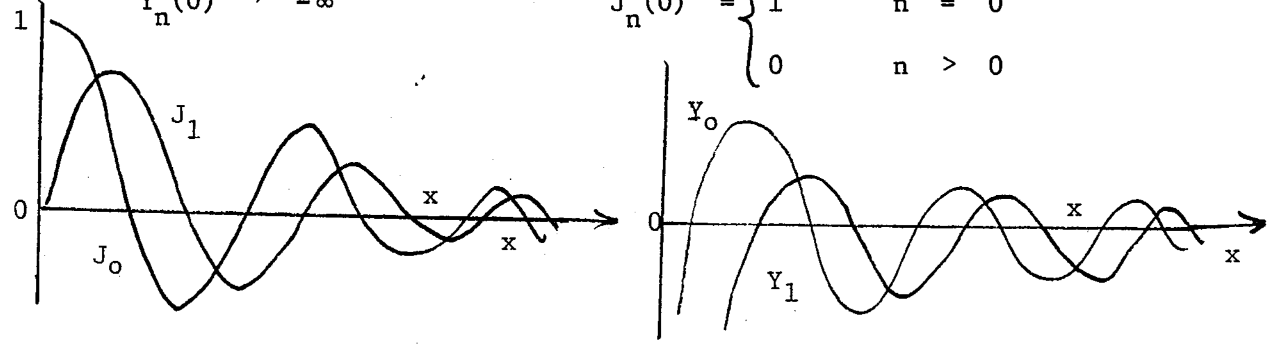
3. $x^2y'' + xy' + (x^2 - n^2)y = 0$

$$y = C_1J_n(x) + C_2Y_n(x)$$

where $J_n(x)$ and $Y_n(x)$ are the n^{th} order Bessel functions,

$Y_n(0) \to -\infty$

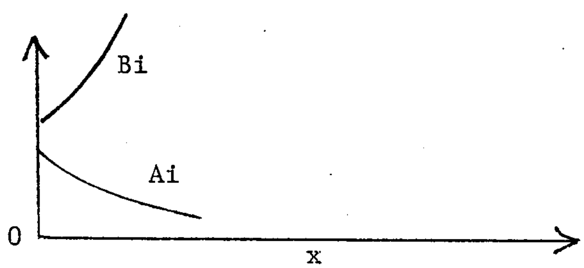
$$J_n(0) = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$$



4. $y'' - xy = 0$

$$y = C_1Ai(x) + C_2Bi(x)$$

where Ai(x) and Bi(x) are the Airy Functions,



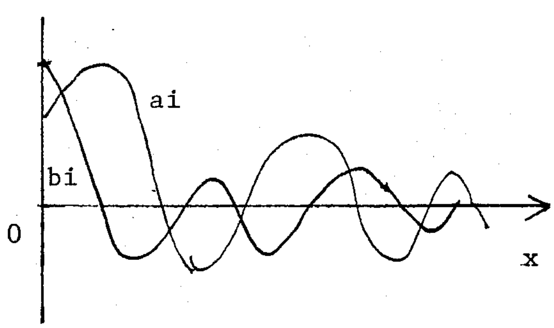
$Ai(0) = 0.355 \qquad Bi(0) = \sqrt{3}Ai(0)$

$Ai(\infty) = 0 \qquad Bi(\infty) = \infty$

5. $y'' + xy = 0$

$$y = C_1ai(x) + C_2bi(x)$$

where ai(x) and bi(x) are the Nairy functions,

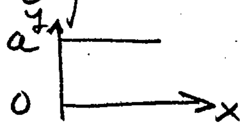


$ai(0) = 0.355 \qquad bi(0) = \sqrt{3}ai(0)$

$ai(\infty) = bi(\infty) = 0$

MIDTERM SOLUTION

1. $\dfrac{\partial^2 T}{\partial y^2} = \alpha y \dfrac{\partial T}{\partial x}$ (1) $T(0,x) = 0$ (2); $T(a,x)=0$ (3);
$T \to 0$ as $x \to \infty$ (4);



Homogeneous problem:
∴ Eigenvalue problem

$$T = X(x)\,Y(y) \Rightarrow \frac{Y''}{yY} = \alpha \frac{X'}{X} = -\lambda$$

$$X' + \frac{\lambda}{\alpha} X = 0 \quad (5) \Rightarrow X = \exp\left(-\frac{\lambda}{\alpha} x\right) \quad (6)$$

$$Y'' + \lambda y Y = 0 \quad (7)$$

NOTE: MANY FOULED UP HERE.

LEARN THIS APPROACH!

Looks almost like $y'' + xy = 0$

let $z = cy \Rightarrow \dfrac{d^2Y}{dy^2} = c^2 \dfrac{d^2Y}{dz^2}$

SO, THE DE IS

$c^2 \dfrac{d^2Y}{dz^2} + \lambda \dfrac{z}{c} Y = 0$ so, if we choose $c^3 = \lambda$,
$\Rightarrow d^2Y/dz^2 + zY = 0$

So,

$Y = C_1\, ai(\lambda^{1/3} y) + C_2\, bi(\lambda^{1/3} y)$ IS THE SOLUTION

$Y(0) = 0 \quad \Rightarrow \quad ai(0) \cdot C_1 + \sqrt{3}\, ai(0) \cdot C_2 = 0$

$Y(a) = 0 \quad \Rightarrow \quad ai(\lambda^{1/3} a)\, C_1 + \sqrt{3}\, bi(\lambda^{1/3} a)\, C_2 = 0$

$$D(\lambda) = \begin{vmatrix} 1 & \sqrt{3} \\ ai(\lambda^{1/3} a) & bi(\lambda^{1/3} a) \end{vmatrix} = 0$$ FIXES $\underline{\lambda_n}$ eigenvalues

$$C_1 = -\sqrt{3}\, C_2$$

$Y_n = A\left[\sqrt{3}\, ai(\lambda_n^{1/3} y) - bi(\lambda_n^{1/3} y)\right]$ eigenfunction for $\lambda = \lambda_n$.

ORTHOGONALITY PROPERTY: $\displaystyle\int_0^a y\, Y_n Y_m\, dy = 0$ prove in usual way;

2. $\dfrac{\partial^2 u}{\partial y^2} = \alpha \dfrac{\partial u}{\partial t}$ (1) $\quad u(0,t) = \beta t$ (2)
$u(y,0) = 0$ (3)

$u(y,t) \to 0$ as $y \to \infty$ (4)

Self-similar solution – no characteristic scales:

assume $u = A t^n f(\eta) \qquad \eta = By/t^m$

But $n = 1$ by (2)! Pick $A = \beta$.

So $\underline{u = \beta t F(\eta) \qquad F(0) = 1 \qquad F(\infty) \to 0}$

$$\beta t \frac{B^2}{t^{2m}} F'' = \alpha\left[\beta F - \beta t \frac{mBy}{t^{m+1}} F'\right]$$

$$\frac{t}{t^{2m}} B^2 F'' = \alpha\left[F - m\eta F'\right] \qquad \underline{2m = 1 \quad m = 1/2}$$

$$\frac{B^2}{\alpha} F'' + \frac{1}{2}\eta F' - F = 0$$

Pick $B^2/\alpha = 1/2$ Then $\eta = y\sqrt{\dfrac{\alpha}{2t}}$

$$\frac{1}{2}F'' + \frac{1}{2}\eta F' - F = 0 \qquad F'' + \eta F' - 2F = 0$$

Note! if you pick $B^2/\alpha = 1$, $F'' + \frac{1}{2}\eta F' - F = 0$

Now let $z = c\eta \qquad c^2 \dfrac{d^2F}{dz^2} + \dfrac{1}{2} z \dfrac{dF}{dz} - F = 0$

pick $c^2 = \frac{1}{2} \qquad c = 1/\sqrt{2} \Rightarrow \dfrac{d^2F}{dz^2} + z\dfrac{dF}{dz} - 2F = 0$

So, $F = C_1 G_2\left(y\sqrt{\dfrac{\alpha}{2t}}\right) + C_2 H_2\left(y\sqrt{\dfrac{\alpha}{2t}}\right)$

$C_1 = 1$ for $F(0) = 1 \qquad C_2 = 0$ for $F(\infty) \to 0$

$\underline{u = \beta t\, G_2\left(y\sqrt{\dfrac{\alpha}{2t}}\right)}$ is the solution

Let $u = Af(\eta)$ $\quad \eta = By/t^n \quad \frac{\partial \eta}{\partial t} = \frac{-nBy}{t\,t^n} = \frac{-\eta n}{t}$

$\frac{\partial u}{\partial t} = Af'\left(\frac{-nB\eta}{t}\right) \quad \frac{\partial u}{\partial y} = Af' \cdot \frac{B}{t^n} \quad \frac{\partial^2 u}{\partial y^2} = Af'' \frac{B^2}{t^{2n}}$

$\therefore \quad u = At^m f(\eta) \quad \eta = By/t^n$

$$\frac{\partial u}{\partial t} = Amt^{m-1}f + At^m f' \cdot \frac{-n\eta}{t} = At^{m-1}\left[mf - n\eta f'\right]$$

$$\frac{\partial u}{\partial y} = At^m f' \cdot \frac{B}{t^n} \qquad \frac{\partial^2 u}{\partial y^2} = At^m f'' \cdot \frac{B^2}{t^{2n}}$$

BC $\quad u(0,t) = At^m f(0) = \beta t \quad$ pick $f(0) = 1 \quad \boxed{m=1} \therefore \boxed{A = \beta}$

PDE $\quad \cancel{A}t f'' \frac{B^2}{t^{2n}} = \alpha\left[\cancel{A}f + \cancel{A}t f' \cdot \frac{-n\eta}{t}\right]$

$$t^{-2n+1} = t^0 \quad \therefore \boxed{n = \tfrac{1}{2}}$$

$$f'' B^2 = \alpha\left[f - \tfrac{1}{2}\eta f'\right] \quad \therefore$$

$$f'' \frac{2B^2}{\alpha} + \eta f' - 2f = 0 \qquad \text{let } \frac{2B^2}{\alpha} = 1 \quad \therefore \boxed{B = \sqrt{\frac{\alpha}{2}}}$$

$$f'' + \eta f' - 2f = 0$$

$u(y,0) = 0 \implies u = Atf\left(\frac{By}{\sqrt{t}}\right) \quad tf = 0 \quad \therefore f \sim \frac{1}{\sqrt{t}}$

$u(y,t) \to 0$ as $y \to \infty \implies f(\eta) \to 0$ as $y \to \infty$

$\therefore \quad f = C_1 G_2(\eta) + C_2 H_2(\eta)$

$f(\eta) \to 0$ as $\eta \to \infty \quad \therefore C_2 = 0$

$f = C_1 G_2(\eta) \qquad f(0) = 1 \quad \therefore C_1 = 1$

$$\therefore \boxed{f = G_2(\eta) \qquad \therefore u = \beta t G_2(\eta) \qquad \eta = y\sqrt{\frac{\alpha}{2t}}}$$

let $T = XY$

$$Y''X = \alpha y X'Y$$

$$\frac{1}{\alpha y}\frac{Y''}{Y} = \frac{X'}{X} = -\lambda^2 \quad \therefore$$

$$\therefore \quad X = e^{-\lambda^2 x} \qquad (1)$$

$$Y'' = -\lambda^2 \alpha y Y$$

$$\text{or } Y'' + \lambda^2 \alpha y Y = 0 \qquad (2)$$

$$T(0,x) = 0 \implies Y(0) = 0$$

$$T(a,x) = 0 \qquad Y(a) = 0$$

$$T \to 0 \text{ as } x \to \infty \implies X \to 0 \text{ as } x \to \infty$$

let ~~let~~ $cz = y$

$\therefore T$

$$\therefore Y' = \frac{dY}{dz}\frac{1}{c}$$

$$Y'' = \frac{d^2Y}{dz^2}\frac{1}{c^2}$$

$$Y = Y$$

$$\frac{1}{c^2} Y'' + \lambda^2 \alpha c z Y = 0$$

let $\frac{1}{c^2} = \lambda^2 \alpha c$

$$\therefore \boxed{c = \sqrt[3]{\frac{1}{\lambda^2 \alpha}}}$$

$$Y = c_1\, ai(z) + c_2\, bi(z)$$

$$Y(0) = 0$$

$$Y\left(\frac{a}{c}\right) = 0$$

$$c_1 (.355) + c_2 (.355)\sqrt{3} = 0$$

$$c_1\, ai(a/c) + c_2\, bi(a/c) = 0$$

$$\therefore \implies bi(a/c) - \sqrt{3}\, ai(a/c) = 0 \quad \text{or} \quad \boxed{bi(a/c) = \sqrt{3}\, ai(a/c)} \quad \text{EV eqn}$$

$\therefore \quad Y'' + zY = 0$ $\quad$ have Efns

$$Y_m Y_n'' + z_n Y_m Y_n - Y_n Y_m'' - z_m Y_n Y_m = 0$$

$$\therefore \quad T = e^{-\lambda^2 x}\left[c_1\, ai(y/c) + c_2\, bi(y/c)\right]$$

$$z_n - z_m \int (Y_n Y_m)\, dz$$

Let $T = A f(\eta)$ $\qquad \eta = \dfrac{By}{X^m}$ $\qquad \dfrac{d\eta}{dx} = By(-m x^{-m-1}) = \dfrac{By(-m)}{X^{m+1}} = \dfrac{By(-m)}{X^m}\dfrac{\cdot}{X} = \eta \dfrac{m}{X}$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y} = Af' \cdot \frac{B}{X^m} \qquad \frac{\partial T}{\partial x} = \frac{\partial T}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = -Af' \cdot By\frac{m}{X^{m+1}} = -Af'\frac{m\eta}{X}$$

$$\therefore \frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial \eta}\left(Af'\frac{B}{X^m}\right)\cdot\frac{B}{X^m} = AB^2\frac{f''}{X^{2m}}$$

$$T_{yy} = \alpha y T_x \qquad \frac{B^2}{X^{2m}} A f'' = \alpha \frac{X^m}{B}\eta \left(-Af'\frac{m\eta}{X}\right)$$

or $X^{-2m} = X^{m-1}$ $\quad \therefore \quad -2m = m-1$ or $\underline{\underline{m = 1/3}}$

BC. $\quad T(0,x) = 0 \Rightarrow f(0) = 0$

$T(a,x) = 0 \Rightarrow f(a) = 0$

$T \to 0$ as $x \to \infty$ $\quad f(\eta) \to 0$ as $\eta \to 0$

$$B^2 f'' = \frac{\alpha}{B}\eta\left(-f'\frac{1}{3}\eta\right) = -\frac{\alpha\eta^2}{3B}f'$$

$$\frac{3B^3}{\alpha}f'' + \eta^2 f' = 0 \qquad \text{let } \frac{3B^3}{\alpha} = 1 \quad \therefore \quad \boxed{B = \sqrt[3]{\frac{\alpha}{3}}}$$

$f'' + \eta^2 f' = 0$ has soln. $\quad \dfrac{f''}{f'} = -\eta^2$ or $\dfrac{df'}{f'} = -\eta^2 d\eta$

or $\log f' = -\dfrac{\eta^3}{3}$

$f' = e^{-\frac{\eta^3}{3}} + C$

$\therefore \quad f = C_1 \displaystyle\int^{\eta} e^{-\frac{\hat\eta^3}{3}} d\hat\eta + C_2$

when $\eta = 0$ $\quad f(\eta) = 0 \Rightarrow C_2 = 0$

$$\eta = a \qquad 0 = C_1 \int_0^a e^{-\hat\eta^3/3} d\hat\eta \quad \Rightarrow \quad \int_0^a e^{-\hat\eta^3/3} d\eta = 0$$

$$T = A\int_0^{\eta} e^{-\frac{\hat\eta^3}{3}} d\hat\eta \qquad \eta = \sqrt[3]{\frac{\alpha y^3}{3x}}$$

ME 200B
Winter 79
WCR

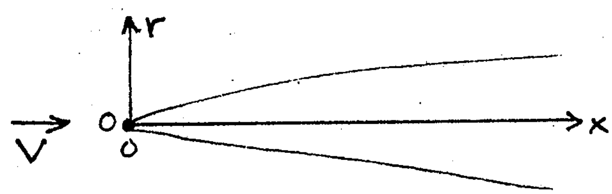
FINAL EXAMINATION - Closed Books and Notes

PLEASE SUPPORT THE HONOR CODE!

Part I is optional. If you are satisfied with your performance on the self-similar problem on the midterm, you do not need to work the self-similar problem in Part I. If you are satisfied with your performance on the eigenvalue problem on the midterm, you do not need to work the eigenvalue problem in Part I. If you work either of these problems, the higher of your midterm/final scores on each type of problem will be used in determining your course grade.

PART I - Optional; see above.

1. The equation describing the concentration c of a pollutant downstream from a point source is



$$rV\frac{\partial c}{\partial x} = \alpha\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) = \alpha\left(\frac{\partial c}{\partial r} + r\frac{\partial^2 c}{\partial r^2}\right) \tag{1}$$

The total pollutant flow is fixed, so,

$$\int_0^\infty rVc(r,x)\,dr = Q \tag{2}$$

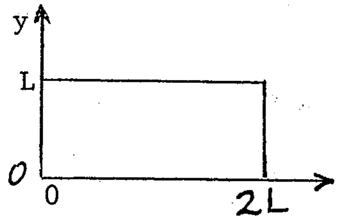
V, α, and Q are constants. Also,

$$c(r,x) \to 0 \quad \text{as} \quad r \to \infty \tag{3}$$

a) Develop the solution to this problem, and express $c(r,x)$ in terms of r, x, and the constant parameters V, α, and Q.
HINT: $(x^2y)' = 2xy + x^2y'$.

PART I (continued)

2. Consider the vibration of a rectangular membrane.



$$a^2(u_{xx} + u_{yy}) - u_{tt} = 0 \qquad (1)$$

$u = 0$ on all bdry

a) Derive an expression for the natural frequencies of vibration of this membrane. Express

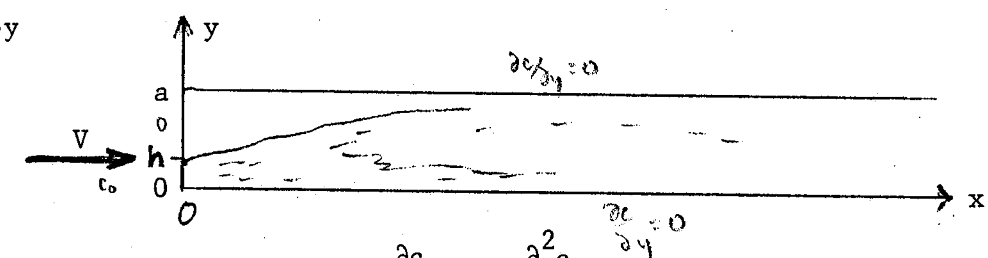
$$\Omega_{nm} = \frac{\omega_{nm} L}{a} = f(n,m)$$

where n and m are integers associated with the x and y behavior, respectively.

b) Give the values of Ω_{nm} for the lowest two frequencies of oscillation, and sketch the node lines for these modes.

PART II - To be Done by All Participants!

3. The diffusion of a pollutant from a point source in a river is described by



$$V\frac{\partial c}{\partial x} = \alpha\frac{\partial^2 c}{\partial y^2} \qquad (1)$$

with the boundary conditions

$$\left.\frac{\partial c}{\partial y}\right| = 0 \quad @ \quad y = 0,a \qquad (2)$$

At the dump point, the distribution of c is

$$c(0,y) = \begin{cases} c_0 & 0 \leq y \leq h \\ 0 & h < y \leq a \end{cases} \qquad (3)$$

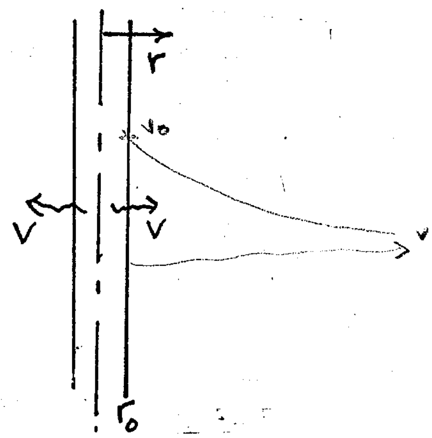
PART II

3. continued

a) Derive the solution for $c(x,y)$. Express c in terms of x, y, and the constant parameters V, α, h, ~~and~~ a and c_o.

b) <u>Prove</u> the orthogonality property of any eigenfunctions you encounter in this problem, <u>without</u> using any trigometric identities.

4. The convection of a pollutant from a cylindrical waste disposal well is described by



$$r\frac{\partial c}{\partial t} + \frac{\partial}{\partial r}(rVc) = 0 \qquad (1)$$

$$r \geq r_o$$

where the radial flow velocity is

$$V = V_o \frac{r_o}{r} \qquad (2)$$

The initial condition is

$$C(r,0) = \left\{ \begin{array}{lll} 0 & r > r_o \\ c_o & r = r_o \end{array} \right\} \qquad (3)$$

Because the outflow depletes the concentration in the well, the boundary condition at $r = r_o$ is

$$r_o^2 \frac{\partial c}{\partial t} = -V_o r_o c \quad \text{at} \quad r = r_o \qquad (4)$$

a) Find the characteristics for this problem.

b) How long does it take for the first element of pollutant to reach the point r ?

c) In the region where $c \neq 0$ express the solution $c(r,t)$ as a function of r, t, and the constant parameters V_o, r_o, and c_o.

1. $rV\frac{\partial c}{\partial x} = \alpha \frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right)$ (1)

$\int_0^\infty rVc\,dr = Q$ (2) $\qquad c(r,x) \to 0$ as $r \to \infty$

$c = Ax^n f(\eta) \qquad \eta = Br/x^m$

(2) $\Rightarrow V\int_0^\infty \left(\frac{\eta x^m}{B}\right)\cdot Ax^n f(\eta)\cdot\left(\frac{x^m}{B}d\eta\right) = Q \qquad 2m+n=0 \qquad \underline{\underline{n=-2m}}$

(1) $\Rightarrow \left(\frac{\eta x^m}{B}\right) VA\left[nx^{n-1}f - mx^{n-1}\eta f'\right] = \alpha\frac{d}{d\eta}\left[\left(\frac{\eta x^m}{B}\right)Ax^n f'\right]\frac{B^2}{x^{2m}}$

$m+n-1 = -m+n \qquad \underline{m = 1/2 \quad \therefore \quad n=-1}$

$$\left(\frac{V}{\alpha B^2}\right)\left[\eta f + \tfrac{1}{2}\eta^2 f'\right] + \frac{d}{d\eta}\left[\eta f'\right] = 0$$

Pick $\underline{\frac{V}{\alpha B^2} = 2}$ $\qquad \frac{d}{d\eta}\left[\eta f'\right] + \left[2\eta f + \eta^2 f'\right] = 0$

$\eta = r\sqrt{\frac{V}{2\alpha x}} \qquad \eta f' + (\eta^2 f) = C_1 = 0$ by BC.

$\frac{f'}{f} = -\eta \qquad f = C_2 e^{-\eta^2/2}$

Pick $C_2 = 1$
$f = e^{-\eta^2/2}$

(2) $\Rightarrow \frac{VA}{B^2}\int_0^\infty \eta e^{-\eta^2/2}d\eta = Q = \frac{VA}{B^2}\cdot\int_0^\infty e^{-\sigma}d\sigma = \frac{VA}{B^2} \qquad \sigma = \eta^2/2$

$A = \frac{QB^2}{V} = \frac{Q\,V}{2\alpha V} = \frac{Q}{2\alpha}$

So,

$$\underline{c = \frac{Q}{2\alpha x}e^{-r^2V/(4\alpha x)}}$$

2.

$$a^2(u_{xx}+u_{yy})-u_{tt}=0$$

$$u = X(x)\cdot Y(y)\, T(t)$$

$$a^2\Big(\underbrace{\frac{X''}{X}}_{-\alpha^2}+\underbrace{\frac{Y''}{Y}}_{-\beta^2}\Big)=\frac{T''}{T}=-\omega^2$$

$$\omega^2=a^2(\alpha^2+\beta^2)$$

$$X''+\alpha^2 X=0 \qquad Y''+\beta^2 Y=0 \qquad \omega=a\sqrt{\left(\frac{n\pi}{2L}\right)^2+\left(\frac{m\pi}{L}\right)^2}$$

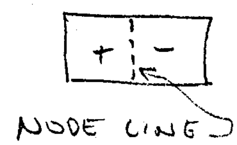
$$X(0)=X(2L)=0 \qquad Y(0)=Y(L)=0$$

$$X=\sin(\alpha x) \qquad Y=\sin(\beta y) \qquad \frac{\omega_{nm}L}{a}=\Omega_{nm}=\sqrt{\frac{n^2}{4}+m^2}\cdot\pi$$

$$\alpha=\frac{n\pi}{2L} \qquad \beta=\frac{m\pi}{L}$$

lowest $\quad n,m=1 \qquad \Omega_{nm}=\pi\sqrt{1+\frac{1}{4}}=\pi\sqrt{5}/2$ (NO NODES)

Next lowest $\quad m=1, n=2 \qquad \Omega_{21}=\pi\sqrt{1+\frac{4}{4}}=\pi\sqrt{2}$



3.

$$V\frac{\partial c}{\partial x}=\alpha\frac{\partial^2 c}{\partial y^2} \qquad \frac{\partial c}{\partial y}=0 \quad @\ y=0, a$$

eigensolutions $\quad c=X(x)\cdot Y(y) \qquad \frac{V}{\alpha}\frac{X'}{X}=\frac{Y''}{Y}=-\lambda^2$

$$\left.\begin{aligned}&Y''+\lambda^2 Y=0\\&Y'(0)=Y'(a)=0\end{aligned}\right\}\Rightarrow Y_n=\cos(\lambda_n y) \qquad \lambda_n=\frac{n\pi}{a}$$

$n=0,1,2,\ldots$

$$X_n=\exp\left(-\lambda_n^2\frac{\alpha}{V}x\right)=\exp\left(-\frac{n^2\pi^2}{a^2}\frac{\alpha}{V}x\right)$$

Now, put

$$c=\sum_{n=0}^{\infty}A_n\cos\left(\frac{n\pi y}{a}\right)\exp\left(-\frac{n^2\pi^2}{a^2}\frac{\alpha}{V}x\right)$$

$$c(0,y)=g(y)=\sum_{n=0}^{\infty}A_n\cos\left(\frac{n\pi y}{a}\right)$$

$$A_m=\frac{\int_0^a g(y)\cdot\cos\left(\frac{m\pi y}{a}\right)dy}{\int_0^a\cos^2\left(\frac{m\pi y}{a}\right)dy}=\frac{\frac{aC_0}{m\pi}\sin\left(\frac{m\pi h}{a}\right)}{\frac{a}{2}} \qquad m>0$$

For $m=0$,

$$aA_0 = \int_0^a g(y)\,dy = h$$

So,

$$c = c_0 \frac{h}{a} + \sum_{n=1}^{\infty} \frac{2c_0}{n\pi} \sin\left(\frac{n\pi h}{a}\right) \cos\left(\frac{n\pi y}{a}\right) \exp\left(-\frac{n^2\pi^2}{a^2}\frac{\alpha}{V}x\right)$$

---

b) $\int_0^a Y_n Y_m \, dy = 0 \quad n \neq m.$

Proof: $\int_0^a \left\{ (Y_n'' + \lambda_n^2 Y_n) Y_m - (Y_m'' + \lambda_m^2 Y_m) Y_n \right\} dx$

$$= 0 = \left(Y_n' Y_m - Y_m' Y_n\right)\Big|_0^a - \int_0^a \left(Y_n' Y_m' - \cancel{Y_m' Y_n'}\right) dx$$

$$+ \underline{(\lambda_n^2 - \lambda_m^2) \int_0^a Y_n Y_m \, dx = 0} \qquad \text{QED}$$

4. $$r\frac{\partial c}{\partial t} + V_0 r_0 \frac{\partial c}{\partial r} = 0 \qquad (1)$$

$\xi = \xi(t,r) \qquad \eta = \eta(t,r)$

$r(c_\xi \xi_t + c_\eta \eta_t) + V_0 r_0 (c_\xi \xi_r + c_\eta \eta_r) = 0$

$(r\xi_t + V_0 r_0 \xi_r) c_\xi + \underbrace{(r\eta_t + V_0 r_0 \eta_r)}_{=0} c_\eta = 0$

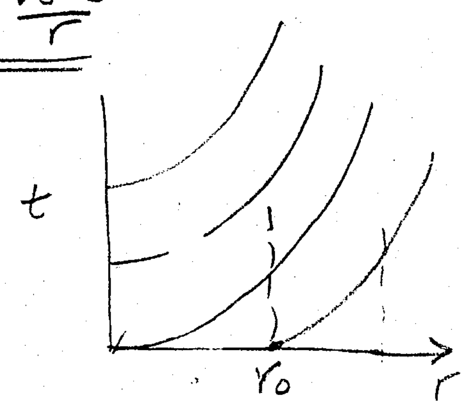
$d\eta = \eta_t dt + \eta_r dr = 0 \qquad \eta_t = -\eta_r \frac{dr}{dt}$ on characteristics

$-r\eta_r \frac{dr}{dt} + V_0 r_0 \eta_r = 0 \qquad \underline{\underline{\frac{dr}{dt} = \frac{V_0 r_0}{r}}}$

$\boxed{\frac{r^2}{2} = V_0 r_0 t + \text{constant} \quad \underline{\text{on chars.}}}$

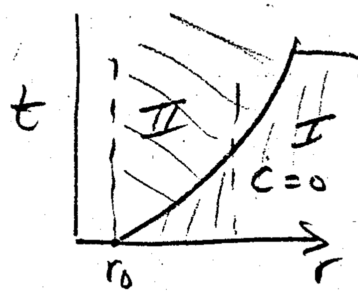
let $\eta = \frac{r^2}{2} - V_0 r_0 t \qquad \xi = t$

c



$$r[c_\eta(-\cancel{V_0 r_0}) + c_\xi \cdot 1] + V_0 r_0 r \cancel{c_\eta} = 0$$

$c_\xi = 0$ $\qquad c = f(\eta)$ $\qquad$ <u>c constant on chars.</u>



$$\frac{r^2}{2} = \frac{r_0^2}{2} + V_0 r_0 t$$

So, $\boxed{t = \dfrac{r^2 - r_0^2}{2 V_0 r_0}}$ time for first crud to reach r.

Now, for region II, B.c. $\Rightarrow$

$$r_0^2 \frac{\partial c}{\partial t} = -V_0 r_0 c$$

$$r_0^2 f'\left(\frac{r_0^2}{2} - V_0 r_0 t\right) \cdot (-\cancel{V_0 r_0}) = -\cancel{V_0 r_0} f\left(\frac{r_0^2}{2} - V_0 r_0 t\right)$$

$$r_0^2 f'(\sigma) = f(\sigma) \qquad f(\sigma) = A e^{\sigma/r_0^2}$$

$$c = A \exp\left[\left(\frac{r^2}{2} - V_0 r_0 t\right)/r_0^2\right]$$

$$c_0 = A \exp\left[\left(\frac{r_0^2}{2} - 0\right)/r_0^2\right] = A \exp\left(\frac{1}{2}\right)$$

So,

$$c = c_0 \exp\left[\left(\frac{r^2}{2} - V_0 r_0 t\right)/r_0^2 - \frac{1}{2}\right]$$

3

a) $v\frac{\partial c}{\partial x} = \alpha \frac{\partial^2 c}{\partial y^2}$

w/BC $\frac{\partial c}{\partial y} = 0$ @ $y=0,\ y=a$

$c(0,y) = \begin{cases} c_0 & 0 \le y \le h \\ 0 & h < y \le a \end{cases}$

Let $c = X(x)Y(y)$

$vX'Y = \alpha XY''$ $\therefore$ $\frac{Y''}{Y} = \frac{vX'}{\alpha X} = -\lambda^2$

$\therefore$ $Y'' + \lambda^2 Y = 0$ $\quad Y = A\sin\lambda y + B\cos\lambda y$

$X' + \lambda^2\frac{\alpha}{v}X = 0$ $\quad X = Ce^{-\frac{\alpha}{v}\lambda^2 x}$

$\therefore$ w/BC $\frac{\partial c}{\partial y} = 0$ @ $y=0,\ y=a\ \forall x \Rightarrow Y'(0) = Y'(a) = 0$

$\therefore$ $A\lambda\cos(\lambda y) - B\lambda\sin(\lambda y) = Y'$

$Y'(0) \Rightarrow A=0 \quad Y'(a) \Rightarrow \lambda a = n\pi \quad \therefore \boxed{\lambda_n = \frac{n\pi}{a}}$ ✓

$\therefore$ $c = \sum_{n=1}^{\infty} D_n e^{-\frac{\alpha\lambda_n^2}{v}x}\cos\frac{n\pi}{a}y$

if $\lambda^2=0 \Rightarrow Y''=0$ or $Y = \hat{A}y + \hat{B}$ $\quad Y'(0)=Y'(a)=0 \Rightarrow \hat{A}=0 \quad \hat{B}=Y$ or $c=$ const

$\therefore$ $c = \sum_{n=0}^{\infty} D_n e^{-\frac{\alpha n^2\pi^2}{va^2}x}\cos\frac{n\pi}{a}y$ ✓

since $c(0,y) = \begin{cases} c_0 & 0 \le y \le h \\ 0 & h < y \le a \end{cases} \Rightarrow c(0,y) = \sum_{n=0}^{\infty} D_n\cos\frac{n\pi}{a}y$

$\int_0^a c(0,y')\cos\frac{m\pi}{a}y'\,dy' = D_m\int_0^a \cos\frac{m\pi}{a}y'\cos\frac{n\pi}{a}y'\,dy' = D_m\int_0^a \cos^2\frac{m^2\pi^2}{a^2}y'\,dy'$

$\int_0^h c_0\cos\frac{m\pi}{a}y'\,dy' \qquad = D_m \int \frac{1+\cos\frac{2m\pi}{a}y'}{2}dy'$

$\qquad \frac{y'}{2} + \frac{\sin\frac{2m\pi}{a}y'}{4\frac{m\pi}{a}}\Big|_0^a$

$\frac{ac_0}{m\pi}\sin\frac{m\pi}{a}y'\Big|_0^h = \qquad D_m\frac{a}{2}$

$\frac{ac_0}{m\pi}\sin\frac{m\pi h}{a} = D_m\frac{a}{2} \quad \therefore \quad D_m = \frac{2c_0}{m\pi}\sin\frac{m\pi h}{a}$

$c_0\int_0^h dy = D_0\int_0^a dy' : c_0 h = D_0 a \quad \therefore \quad D_0 = \frac{c_0 h}{a}$

$\therefore$ $c(x,y) = \sum_{m=0} e^{-\frac{\alpha m^2\pi^2}{a^2 v}x}\left[\frac{c_0 h}{a} + \frac{2c_0}{m\pi}\sin\frac{m\pi h}{a}\cos\frac{m\pi}{a}y\right]$?

$= \frac{c_0 h}{a} + \frac{2c_0}{\pi}\sum_{m=1}^{\infty} e^{-\frac{\alpha m^2\pi^2}{a^2 v}x}\frac{1}{m}\sin\frac{m\pi h}{a}\cos\frac{m\pi}{a}y$

since $y'' + \lambda^2 y = 0$ is eq we can write this as $(Sy')' + (Q + \lambda^2 P y) = 0$ Sturm-Liouville form.

$Q = 0 \quad P = 1 \quad S = 1$

now to solve $\int_0^a P y_n^2 dy = \frac{1}{2\lambda_n}\left\{ y_n' S \frac{\partial y}{\partial \lambda}\Big|_0^a - y_n S \frac{\partial^2 y}{\partial y \partial \lambda}\Big|_0^a \right\}_{\lambda=\lambda_n}$

now we know that $y = \cos\lambda y$ then $\frac{\partial y}{\partial \lambda} = -y \sin\lambda y \quad \frac{\partial^2 y}{\partial\lambda \partial y} = -\sin\lambda y - y\lambda\cos\lambda y$

$y_n' = -\sin\lambda_n y \cdot \lambda_n$

$\therefore \frac{1}{2\lambda_n}\left\{ +\lambda_n y \sin\lambda_n y \sin\lambda y \Big|_0^a + \cos\lambda_n y(\sin\lambda y + y\lambda\cos\lambda y)\Big|_0^a \right\}_{\lambda=\lambda_n}$

$= \frac{1}{2\lambda_n}\left\{ \cos\lambda_n a(\sin\lambda_n a + a\lambda\cos\lambda a)\Big|_{\lambda=\lambda_n} \right\} = \frac{1}{2\lambda_n}\left\{ +a\cos^2\lambda_n a \lambda_n \right\} = +\frac{a}{2}$ ✓

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4. $r\frac{\partial c}{\partial t} + \frac{\partial}{\partial r}(rVc) = 0$ Continuity $\quad r \geq r_0 \quad V = V_0\frac{r_0}{r}$

$c(r,0) = \begin{cases} 0 & r > r_0 \\ c_0 & r = r_0 \end{cases}$

$r_0^2 \frac{\partial c}{\partial t} = -V_0 r_0 c \quad @\ r = r_0$

let $c = c(\xi, \eta)$ then $\frac{\partial c}{\partial t} = \frac{\partial c}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial t}$

$\frac{\partial c}{\partial r} = \frac{\partial c}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial r}$

$r\left(\frac{\partial c}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial t}\right) + r\frac{r_0 V_0}{r}\left(\frac{\partial c}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial r}\right) = 0$

$\therefore r\frac{\partial c}{\partial \xi}\left[\xi_t + \frac{r_0 V_0}{r}\xi_r\right] + r\frac{\partial c}{\partial \eta}\left[\eta_t + \frac{r_0 V_0}{r}\eta_r\right] = 0 \quad (1)$

along constant η then $d\eta = 0 = \eta_t\, dt + \eta_r\, dr$

$\therefore \eta_t + \frac{r_0 V_0}{r}\eta_r = 0$ along constant η ✓

$\eta_t\, dt + \eta_r\, dr = 0 \quad \therefore \begin{vmatrix} 1 & \frac{r_0 V_0}{r} \\ dt & dr \end{vmatrix} = 0$ or $dr = \frac{r_0 V_0}{r} dt$ $\cancel{\frac{dr}{dt} = \frac{r_0 V_0}{r} = V}$

~~or since r_0, V_0 are const~~ $\cancel{t = \frac{r_0 V_0}{r}\eta + \text{const}}$ $\therefore \cancel{t - (r_0 V_0) r = \eta}$ $\frac{dr}{dt} = V$

~~$\therefore$ along constant η (1) reduces to~~ let $t = \xi$

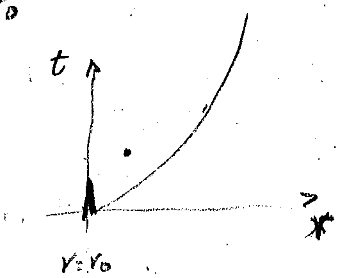
$\cancel{c_\xi\left[1 + r_0 V_0 \cdot 0\right] = 0 \text{ or } c_\xi = 0 \quad c = f(\eta) = f(t - r_0 V_0 r)}$

$dt\frac{r_0 V_0}{r} = dr \quad dt(r_0 V_0) = r\,dr \quad \therefore (r_0 V_0)t = \frac{r^2}{2} + c \quad \therefore \quad \eta = r_0 V_0 t - \frac{r^2}{2}$

$t = \xi$

b) when $t=0$ $r=r_0$ $\therefore$ $\eta = -\frac{r_0^2}{2}$ $\therefore$ $-\frac{r_0^2}{2} = r_0 v_0 t - \frac{r^2}{2}$

$\therefore$ when ~~the~~ we reach r then $\boxed{\frac{r^2 - r_0^2}{2 r_0 v_0} = t}$ ✓ $r \geq r_0$



along constant η (1) reduces to

$r c_\xi \left[1 - \frac{r_0 v_0}{r} \cdot 0\right] = 0$ or $c_\xi = 0$ for $r \geq r_0$

$c = f(\eta) = f\left(r_0 v_0 t - \frac{r^2}{2}\right) = c(r,t)$

$\cancel{c(r,0) = f\left(-\frac{r^2}{2}\right) = c_0 = f(\sigma)}$

as a bc $c(r_0,t) = f\left(r_0 v_0 t - \frac{r_0^2}{2}\right)$

$\left.\frac{\partial c}{\partial t}\right|_{r=r_0} = f' r_0 v_0$

$r_0^2 \cancel{v_0} f' = -\cancel{v_0} \cancel{r_0} f$

$r_0^2 f' = -f$ or $f'(\sigma) = -\frac{1}{r_0^2} f(\sigma)$ or $f(\sigma) = A e^{-\frac{1}{r_0^2}\sigma}$ ✓

$\therefore$ $f\left(r_0 v_0 t - \frac{r_0^2}{2}\right) = A e^{-\frac{1}{r_0^2}\left(r_0 v_0 t - \frac{r_0^2}{2}\right)}$ $A = c_0 e^{-1/2}$

$c(r,0) = c_0 = f\left(-\frac{r_0^2}{2}\right)$ $r = r_0$ $\to$ $c_0 = f\left(r_0 v_0 t - \frac{r^2}{2}\right)$

$= 0 = f\left(-\frac{r^2}{2}\right)$ $r > r_0$

when $c = f\left(-\frac{r^2}{2}\right) = f(\sigma)$

$f(\sigma) = 0 \quad \forall\, \sigma < \sigma_0$

$f(\sigma_0) = c_0\, \delta(\sigma_0)$ where $\delta(x) \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

$\sigma < \sigma_0 < 0$

$f(\sigma) = c_0\, \delta\left(\sigma - \left(-\frac{r_0^2}{2}\right)\right)$

$f\left(r_0 v_0 t - \frac{r^2}{2}\right) = c_0\, \delta\left(r_0 v_0 t - \frac{r^2}{2} + \frac{r_0^2}{2}\right)$

now $r_0^2 \frac{\partial c}{\partial t} = -v_0 r_0 c$

$$c = f(\eta) = f(r_0 v_0 t - r^2/2)$$

$$r_0^2 \frac{\partial c}{\partial t} = -v_0 r_0 c \quad \text{at } r = r_0$$

$$r_0^2 \,[\cancel{r_0 v_0}]\, f' = -\cancel{v_0 r_0} f$$

$$\therefore \; f'(\sigma) = -f(\sigma)/r_0^2 \qquad \therefore \; f = A e^{-\sigma/r_0^2}$$

$$f(\sigma) = A e^{-\sigma/r_0^2}$$

$$f(r_0 v_0 0 - r_0^2/2) = A e^{-r_0^2/2r_0^2} = c_0$$

$$f(-r_0^2/2) = c_0 = A e^{-1/2}$$

$$\therefore \; A = c_0 e^{1/2}$$

$$f \qquad \therefore \; f(\sigma) = c_0 e^{1/2} e^{-\sigma/r_0^2}$$

$$= c_0$$

1c. Let $\eta = \dfrac{Br}{x^m}$ $\quad c = Ax^n f(\eta)$

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial x} = \cancel{Ax^n f'\left(\frac{-mBr}{x^{m+1}}\right)} = Ax^{n-1}f'(-m\eta) = Anx^{n-1}f + Ax^{n-1}f'(-m\eta)$$

$$\frac{\partial c}{\partial r} = Ax^n f'\frac{\partial \eta}{\partial r} = Ax^n f'\frac{B}{x^m} \qquad \frac{\partial^2 c}{\partial r^2} = Ax^n f''\frac{B^2}{x^{2m}}$$

$$= Ax^n f'\frac{B}{x^m}$$

$$\eta\frac{x^m}{B}v\left[Anx^{n-1}f + Ax^{n-1}m\eta f'\right] = \alpha\left[Ax^{n-m}f'B + AB^2 f'' x^{n-2m}\cdot\eta\frac{\cancel{x^m}}{\cancel{B}}\right]$$

$$\frac{A\eta x^{m+n-1}}{B}v\left[nf - m\eta f'\right] = \alpha AB x^{n-m}\left[f' + f''\eta\right]$$

$m+n-1 = n-m$ $\quad \boxed{m = 1/2}$

$$\frac{Av}{B}\eta\left[nf - \tfrac{1}{2}\eta f'\right] = \alpha AB\left[f' + f''\eta\right] \qquad (1)$$

$$\int_0^\infty rvc\,dr = \int_0^\infty \eta\frac{x^m}{B}v\,Ax^n f\,\frac{x^m}{B}\,d\eta = Q$$

$$= x^{2m+n}\int_0^\infty\left[\frac{Av}{B^2}\eta f\,d\eta\right] = Q \qquad \Rightarrow 2m+n=0 \quad \boxed{n=-1}$$

$$\text{or} \quad \boxed{\int_0^\infty \eta f\,d\eta = \frac{QB^2}{Av}} \qquad \therefore \quad A = \frac{QB^2}{v}\frac{1}{\int_0^\infty \eta f\,d\eta}$$

let $\boxed{B = \sqrt{\tfrac{v}{2\alpha}}}$ $\quad \therefore \dfrac{v}{2\alpha B^2} = 1$

$$(1)\quad \frac{v}{2\alpha B^2}\eta\left[-2f - \eta f'\right] = 2f' + 2f''\eta$$

$$\cancel{\left[f + \tfrac{1}{2}\eta f' - f' - f''\eta\right]} = 0$$

$$\cancel{\left[2f + \eta f' - 2f' - 2f''\eta\right]}$$

$$\cancel{(\eta-1)f' - f''\eta}$$

$$\cancel{\eta f = \tfrac{1}{2}\eta^2 f' - f\eta - f''\eta}$$

$$\tfrac{1}{2}\left[(\eta^2 f)' - 2\eta f\right]$$

$$\cancel{\eta f = \tfrac{1}{2}(\eta^2 f)' - \eta f}$$

$$\cancel{2\eta f = \tfrac{1}{2}(\eta^2 f)' - \eta\left[f' + f''\right]}$$

$$-2\eta f - \eta^2 f' = 2f' + 2f''\eta$$

$$-2\eta f - (\eta^2 f)' + 2\eta f = 2\left[f'\eta\right]'$$

$$\text{or } -\eta^2 f = 2f'\eta + c \qquad \text{or} \quad -\eta f = 2f' \qquad \therefore -\eta\,d\eta = \frac{df}{f} \quad \text{or} \quad -\frac{\eta^2}{4} = \ln f + c$$

since we expect

let $\eta = \frac{Br}{x^m}$ $\quad c = Ar^n f(\eta)$

then $\frac{\partial c}{\partial x} = Anx^{n-1} f(\eta) + Ax^n f' \frac{\partial \eta}{\partial x}$ $\quad \frac{\partial \eta}{\partial x} = Br\,(-m x^{-m-1})$

$\frac{\partial c}{\partial r} = Ax^n f' \frac{\partial \eta}{\partial r}$ $\quad \frac{\partial \eta}{\partial r} = \frac{B}{x^m}$ $\quad \therefore \frac{\partial c}{\partial r} = Ax^n f' \frac{B}{x^m}$

$\frac{\partial^2 c}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial c}{\partial r}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial c}{\partial r}\right)\frac{\partial \eta}{\partial r} = Ax^n f'' \frac{B^2}{x^{2m}}$

$\therefore \quad rV\left[Anx^{n-1} f - Ax^n f' \frac{Brm}{x^{m+1}}\right] = \alpha\left[Ax^n f' \frac{B}{x^m} + rAx^n f'' \frac{B^2}{x^{2m}}\right]$

$\left[\qquad \right] \quad \alpha\left[Ax^n f' \eta/r + Ax^n f'' \eta^2/r\right]$

let $\eta = \frac{Br}{x^m}$ $\quad c = Ar^n f(\eta)$

$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial x} = Ar^n f' Br\left(\frac{-m}{x^{m+1}}\right) = -Ar^n f' \frac{Brm}{x^{m+1}} = -Ar^n f' \eta \frac{m}{x}$

$\frac{\partial c}{\partial r} = nAr^{n-1} f + Ar^n f' \frac{B}{x^m}$

$r\frac{\partial c}{\partial r} = nAr^n f + Ar^{n+1} f' \frac{B}{x^m}$

$\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) = n^2 Ar^{n-1} f + nAr^n f' \frac{B}{x^m} + A(n+1) r^n f' \frac{B}{x^m} + Ar^{n+1} f'' \frac{B^2}{x^{2m}}$

$n^2 Ar^{n-1} f + nAr^{n-1} f' \eta + A(n+1) r^{n-1} f' \eta + Ar^{n-1} f'' \eta^2$

$\therefore \quad rV\frac{\partial c}{\partial x} = \alpha \frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right)$

$= \alpha\left[nAr^{n-1} f' \eta + A(n+1) r^{n-1} f' \eta + Ar^{n-1} f'' \eta^2 + n^2 Ar^{n-1} f\right]$

let $\eta = Brx^m$ $\quad c = Ar^n f(\eta)$

then $\int_0^\infty rVc\,dr = Q = \int \eta$

$\frac{\eta x^m}{B} = r$ $\quad dr = \frac{d\eta\, x^m}{B}$ $\quad c = Ax^n f(\eta)$

$\therefore \quad \int_0^\infty d\eta\, \eta \frac{x^{2m}}{B^2} Ax^n f(\eta) = dQ$ $\quad 2m + n = 0$

$\therefore \quad f = C e^{-\eta^2/4}$ $\quad$ choose $f = 1$ @ $\eta = 0$ $\quad \therefore C_1 = 1$

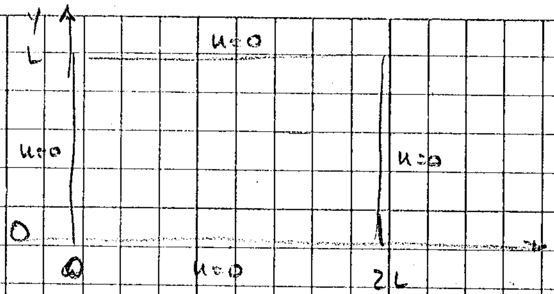
$\int_0^\infty \eta f\,d\eta = \int_0^\infty \eta e^{-\eta^2/4} d\eta = \frac{QB^2}{AV} \Rightarrow \quad 2 = \frac{QB^2}{AV} \quad \therefore \quad A = \frac{QB^2}{2V} = \frac{QV}{2 \cdot 2\alpha} = \frac{QV}{4\alpha}$

let $\eta^2/4 = \xi$ $\quad \eta\, d\eta = 2d\xi$ $\quad 2\int_0^\infty e^{-\xi} d\xi = -2e^{-\xi}\Big|_0^\infty = 2$

next pg

$\eta = \sqrt{\frac{v}{\alpha \alpha x}}\, r \qquad c = \frac{Q v}{4 \alpha X}\, e^{-\eta^2/4}$

2



$$a^2\nabla^2 u - u_{tt} = 0$$

then

let $u = F(x,y)e^{i\omega t}$ then

$$a^2\nabla^2 F e^{i\omega t} + \omega^2 e^{i\omega t}F = 0 \quad \therefore \text{ since } e^{i\omega t}\neq 0 \text{ then}$$

$a^2\nabla^2 F + \omega^2 F = 0$ or

$$\nabla^2 F = -\frac{\omega^2}{a^2}F \qquad \text{if } F = X(x)\,Y(y) \text{ then}$$

$$X''Y + Y''X = -\frac{\omega^2}{a^2}XY \text{ or } \qquad \frac{X''}{X} + \frac{Y''}{Y} = -\frac{\omega^2}{a^2}$$

$$\frac{X''}{X} = -\frac{\omega^2}{a^2} - \frac{Y''}{Y} = -\lambda^2 \qquad \therefore \qquad X'' + \lambda^2 X = 0 \text{ or}$$

$$X = A\cos\lambda x + B\sin\lambda x$$

$X(0) = X(2L) = 0 \quad \therefore \quad A = 0 \quad 2\lambda L = n\pi$ or $\lambda = \frac{n\pi}{2L}$

if $\lambda^2 = 0 \quad \therefore \quad X = ax + b \qquad X(0) = 0 = X(2L) = 0 \Rightarrow X = 0$ no non-triv sol.

$$\therefore \quad \frac{Y''}{Y} = \lambda^2 - \frac{\omega^2}{a^2} \qquad \therefore \qquad Y'' + \underbrace{\left(\frac{\omega^2}{a^2} - \lambda^2\right)}_{\nu^2} Y = 0 \qquad \nu^2 = \frac{\omega^2}{a^2} - \lambda^2$$

$$\therefore \quad Y = C\cos\nu y + D\sin\nu y = 0$$

$Y(0) = Y(L) = 0 \Rightarrow C = 0 \quad \nu L = m\pi \quad \therefore \quad m\neq 0,\ n\neq 0$

$$\nu^2 L^2 = m^2\pi^2$$

$$\left(\frac{\omega^2}{a^2} - \frac{n^2\pi^2}{4L^2}\right)L^2 = m^2\pi^2$$

$$\text{or} \quad \frac{\omega^2 L^2}{a^2} = m^2\pi^2 + \frac{n^2\pi^2}{4}$$

$$\frac{\omega^2 L^2}{a^2} = \pi^2\left[m^2 + \frac{n^2}{4}\right]$$

$$\Omega_{nm} = \frac{\omega_{mn}L}{a} = \frac{\pi}{2}\sqrt{4m^2 + n^2}$$

$m=1 \quad n=1 \quad \Omega_{mn} = \sqrt{5}\,\pi/2$

$m=1 \quad n=2 \quad \Omega_{mn} = \pi\sqrt{2}$

$\Omega_{mn} = 0$

$\Omega_{mn} = \sqrt{3}$