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SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

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Note: Figures are positioned at the end of each chapter.

Chapter 1

THE ORIGIN OF PARTIAL DIFFERENTIAL EQUATIONS

1.1 Fields and Partial Derivatives

Partial differential equations (PDE's) arise in the mathematical formulation of physical problems involving quantities that vary in more than one space dimension, or in both space and time. Such quantities are called field variables; the temperature field in the atmosphere and the wave-height field on the ocean are familiar examples.

In most problems the field variables are the dependent variables of the problem, and the space coordinates and time are the independent variables. What one seeks as the solution of the PDE is a representation of the dependent variables as functions of the independent variables. Thus, if $f(x,y,t)$ represents the field variable of interest, here a function of the two space coordinates x,y and of time, the pertinent first-order partial derivatives are

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x,y,t) - f(x,y,t)}{\Delta x}\right] \tag{1.1.1.a}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \left[\frac{f(x,y+\Delta y,t) - f(x,y,t)}{\Delta y}\right] \tag{1.1.1b}$$

$$\frac{\partial f}{\partial t} = \lim_{\Delta t \to 0} \left[\frac{f(x,y,t+\Delta t) - f(x,y,t)}{\Delta t}\right] \tag{1.1.1.c}$$

Thus, partial derivatives can be thought of as ordinary derivaties with respect to one variable, with the other variables held constant. They represent the slopes of lines along which all the independent variables but one are held constant (Fig. 1.1.1).

In writing partial derivatives, it is important to remember which variables are held constant. In most of the literature this is left to the reader to infer. Thermodynamics literature often employs subscripts to indicate which variables are held fixed. For example, a thermodynamicist might at one time be thinking of the entropy as a function of temperature and pressure, $S = S(T,P)$ and at another time work with entropy as a function of temperature and volume,

$S = S(T,V)$. Rather than write $\partial S/\partial T$, which does not indicate the fixed variable, one could write

$$\left.\frac{\partial S}{\partial T}\right|_P \quad \text{or} \quad \left.\frac{\partial S}{\partial T}\right|_V \tag{1.1.2}$$

Higher-order partial derivatives are simply partial derivatives of partial derivatives. For example,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \tag{1.1.3}$$

If there are no discontinuities in f and its first two derivatives, then repeated application of the basic definition shows that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{1.1.4}$$

That is, the order of partial differentiation is unimportant.

If one makes a small change in one <u>independent</u> variable, the change in the <u>dependent</u> variable is, to first approximation, related just to the first-order partial derivative. For example, from (1.1.1a),

$$f(x+\Delta x,y,t) = f(x,y,t) + \frac{\partial f}{\partial x}\Delta x + O(\Delta x^2) \tag{1.1.5}$$

where the notation $O(\Delta x^2)$ means that the correction will involve powers of Δx of 2 and higher, making them negligible compared to the first-order term as $\Delta x \to 0$. Similar expressions hold for the changes with respect to the other independent variables, so that

$$f(x+\Delta x, y+\Delta y, t+\Delta t) = f(x,y,t) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial t}\Delta t + O(\Delta x^2) + O(\Delta y^2) + O(\Delta t^2) \tag{1.1.6}$$

The student may recognize this as the beginnings of a Taylor's series. This series is especially useful in the derivation of PDE's governing physical problems. Note that the partial derivatives are understood to be evaluated at the point (x,y,t).

If all of the changes are infinitesimal, it is usual to denote them by dx, dy, dt, etc., and to represent the total change in f by df = f(x+dx,y+dy,t+dt) - f(x,y,t). Then, since the higher-order corrections are infinitesimally smaller than the first-order terms, in the limit

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial t} dt \qquad (1.1.7)$$

Here df is called the total differential of f. Eqn. (1.1.7) is also useful in deriving PDE's for physical problems.

Example 1.1.1

Suppose $f(x,t) = e^{-\lambda^2 x} \sin(\omega t)$, where λ and ω are constants. Find $\partial f/\partial x$ and $\partial f/\partial t$:

$$\frac{\partial f}{\partial x} = -\lambda^2 e^{-\lambda^2 x} \sin \omega t \quad ; \quad \frac{\partial f}{\partial t} = \omega e^{-\lambda^2 x} \cos \omega t$$

1.2 Changes of Variables

Often the solution of PDE's are simplified by changes of variables. For example, one might ultimately be interested in f(x,y) but find it easier to solve the problem in other independent variables (ξ,η), where

$$\xi = \xi(x,y) \quad , \quad \eta = \eta(x,y) \qquad (1.2.1)$$

To do this requires expression of the PDE in terms of the new independent variables (ξ,η). This is accomplished by expressing the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ in terms of $\partial f/\partial \xi$ and $\partial f/\partial \eta$. These linkings may be derived by first equating the total differentials in both sets of variables,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \eta} d\eta \qquad (1.2.2)$$

Next we express $d\xi$ and $d\eta$ in terms of dx and dy by taking the total differentials of (1.2.1),

$$d\xi = \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy \quad , \quad d\eta = \frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy \qquad (1.2.3)$$

Substituting in (1.2.2) and regrouping terms,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}\right) dx + \left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}\right) dy \qquad (1.2.4)$$

Comparing the right- and left-hand sides, we find

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} \qquad (1.2.5a)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} \qquad (1.2.5b)$$

Eqns. (1.2.5) are called the <u>chain rule</u>; they tell us how to express the derivatives with respect to the original independent variables in terms of derivatives with respect to the new independent variables. The student should master the use of the chain rule, as we shall use it frequently.

<u>Example 1.2.1</u>

Suppose $\xi = x + y$, $\eta = x - y$. Transform $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x^2$, and $\partial^2 f/\partial y^2$.

$$\frac{\partial \xi}{\partial x} = 1 \; , \qquad \frac{\partial \xi}{\partial y} = 1 \; , \qquad \frac{\partial \eta}{\partial x} = 1 \; , \qquad \frac{\partial \eta}{\partial y} = -1$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \eta} \; , \qquad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi} - \frac{\partial f}{\partial \eta}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial f}{\partial x}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial f}{\partial x}\right)\frac{\partial \eta}{\partial x} \\ &= \left(\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \xi \partial \eta}\right) + \left(\frac{\partial^2 f}{\partial \eta \partial \xi} + \frac{\partial^2 f}{\partial \eta^2}\right) \\ &= \frac{\partial^2 f}{\partial \xi^2} + 2\frac{\partial^2 f}{\partial \xi \partial \eta} + \frac{\partial^2 f}{\partial \eta^2}\end{aligned}$$

Similarly,

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial \xi^2} - 2\frac{\partial^2 f}{\partial \xi \partial \eta} + \frac{\partial^2 f}{\partial \eta^2}$$

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3. Derivation of Partial Differential Equations

Partial differential equations are usually derived by applying a set of basic physical principles to a region in space (a control volume) which is infinitesimally small in at least one dimension. Phenomenological relationships or equations of state frequently are involved.

We shall illustrate the approach with an example. Consider the problem of transient heat conduction within and convection from a thin plate subjected to internal heating. The control volume used is the infinitesimal piece of the plate shown in Fig. (1.3.1). The six heat conduction rates $\dot{Q}_1 \ldots \dot{Q}_6$ move thermal energy in and out of the control volume as shown. In addition, there is a source of thermal energy (perhaps chemical or nuclear energy release) within the plate, occurring at the rate of s (W/m^3). The energy balance on the control volume is

$$\dot{Q}_1 - \dot{Q}_2 + \dot{Q}_3 - \dot{Q}_4 - \dot{Q}_5 - \dot{Q}_6 + sV = \frac{dE}{dt} \tag{1.3.1}$$

where V is the volume of the control volume and E is the thermal energy contained therein.

Having applied the pertinent basic principles, we next bring in the appropriate phenomenological relationships, in this case the Fourier heat conduction law, which relates the conductive heat flows to the local temperature gradients,

$$q_x = -k\frac{\partial T}{\partial x}, \qquad q_y = -k\frac{\partial T}{\partial y} \tag{1.3.2}$$

where q_x and q_y are the heat conduction rates per unit area (W/m^2) at any point (x,y) in the plate, and k is the thermal conductivity, W/(m•K). Note that we have implicitly assumed that the plate is sufficiently thin and the material conductivity k sufficiently high that the temperature does not vary in the direction perpendicular to the plate. We can regard $q_x(x,y,t)$ and $q_y(x,y,t)$ as field variables.

To evaluate the heat flows we shall determine the values of the fluxes at the center of each face of the control volume, and then multiply by the area. Thus,

$$\dot{Q}_1 = q_x(x,y+dy/2,t)dy \cdot \delta$$

$$\dot{Q}_2 = q_x(x+dx,y+dy/2,t)dy \cdot \delta$$

$$\dot{Q}_3 = q_y(x+dx/2,y,t)dx \cdot \delta \qquad (1.3.3)$$

$$\dot{Q}_4 = q_y(x+dx/2,y+dy,t)dx \cdot \delta$$

Next we use a Taylor's series;

$$q_x(x,y+dy/2,t) = q_x(x,y,t) + \frac{\partial q_x}{\partial y}\frac{dy}{2} + \ldots$$

$$q_x(x+dx,y+dy/2,t) = q_x(x,y,t) + \frac{\partial q_x}{\partial x}dx + \frac{\partial q_x}{\partial y}\frac{dy}{2} + \ldots \qquad (1.3.4)$$

It is helpful to notice that the difference in these quantities is what is needed, because then only one term remains. Hence,

$$\dot{Q}_1 - \dot{Q}_2 = -\frac{\partial q_x}{\partial x}dx \cdot dy \cdot \delta = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dxdy\delta \qquad (1.3.5)$$

Similarly,

$$\dot{Q}_3 - \dot{Q}_4 = -\frac{\partial q_y}{\partial y}dy\,dx\delta = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)dxdy\delta \qquad (1.3.6)$$

We also need phenomenological equations for the convective heat transfer terms $\dot{Q}_5$ and $\dot{Q}_6$. We shall assume that the convective heat transfer rate per unit area (W/m^2) on each side at any point is given by the field variable $q_c(x,y,t) = h \cdot (T-T_o)$, where $T(x,y,T)$ is the local plate temperature, T_o is the temperature in the surrounding fluid, and h is the convective heat transfer coefficient, $W/(m^2 \cdot K)$. Then we represent $\dot{Q}_5 + \dot{Q}_6$ as the value of q_c at the center of the sides of the control volume times the area of the side,

$$\dot{Q}_5 + \dot{Q}_6 = 2q_c(x+dx/2,y+dy/2,t) \cdot dx \cdot dy \qquad (1.3.7)$$

Next we expand the field variable q_c in a Taylor's series,

$$q_c(x+dx/2,y+dy/2,t) = q_c(x,y,t) + O(dx) + O(dy) \qquad (1.3.8)$$

$$= h\left[T(x,y,t) - T_o\right] + O(dx) + O(dy) \quad .$$

where the $O(dx)$ and $O(dy)$ are to remind us that the neglected terms have factors of dx and dy, respectively; we shall not need these, since when we multiply them by $dx \cdot dy$ (see (1.3.7)) these terms will be infinitesimal in comparison to the other terms in the equation, which are $O(dx \cdot dy)$.

The source term sV is given by determining the value of the field variable s at the center of the control volume, $s(x+dx/2, y+dy/2, t)$, multiplied by the volume. Using the Taylor's series for s,

$$sV = s(x,y,t)dxdy\delta + O(dxdxdy) + O(dxdydy) \tag{1.3.9}$$

Note that again the higher-order terms need not be included.

We now have all of the terms on the left in (1.3.1), expressed in terms of the temperature field $T(x,y,t)$. To get the time-rate of change of thermal energy we use the <u>equation of state</u> for the material*, $e = e(T)$, where e is the energy per unit <u>mass</u> (J/kg). Since T is a field variable, e is also a field variable. We evaluate the total energy in the control volume at any instant as the value of e at the <u>center</u> of the control volume times the density ρ (kg/m^3), times the volume.

$$\begin{aligned} E &= \rho e(x+dx/2, y+dy/2, t)dxdy\delta \\ &= \rho e(x,y,t)dxdy\delta + O(dxdydx) + O(dxdydy) \end{aligned} \tag{1.3.10}$$

Again the higher-order terms are infinitesimal in comparison to those retained, and may be neglected. Hence

$$\frac{dE}{dt} = \rho \frac{\partial e}{\partial t} dxdy\delta \tag{1.3.11}$$

Finally, we use the <u>equation of state</u> relating thermal energy to temperature, $e = e(T)$, which, differentiated, gives $de = cdT$, where c is the <u>specific heat</u> (J/kg-K) of the plate material. Using the chain rule,

$$\frac{\partial e}{\partial t} = \frac{de}{dT}\frac{\partial T}{\partial t} = c\frac{\partial T}{\partial t} \tag{1.3.12}$$

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* Thermodynamicists will recognize that we are treating the plate as an incompressible medium.

Hence,

$$\frac{dE}{dt} = \rho c \frac{\partial T}{\partial t} dxdy\delta \tag{1.3.13}$$

We now substitute (1.3.5), (1.3.6), (1.3.7), (1.3.9), and (1.3.13) in the basic energy balance (1.3.1), and drop all terms smaller than $dx \cdot dy$. The result is

$$\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right] dxdy\delta \quad + \quad 2h(T-T_o)dxdy + s\ dxdy\delta = \rho c \frac{\partial T}{\partial t} dxdy\delta \tag{1.3.14}$$

Note that each term has $dxdy$ as a factor. Finally, we divide by $dxdy\delta$ to obtain the PDE for the temperature field,

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{2h}{\delta}(T-T_o) + s = \rho c \frac{\partial T}{\partial t} \tag{1.3.15}$$

To summarize, the process for deriving PDE's is as follows:

- Select an appropriate elemental control volume.
- Apply basic principles (conservation of energy, momentum, mass, etc.).
- Express the flow or force <u>differences</u> in terms of the desired field variables using phenomenological equations and Taylor's series expansions.
- Bring in equations of state as necessary to express the problem completely in terms of the desired field variables.

With some experience, the student will know in advance which of the Taylor's series terms will not appear, and can leave them out of the development. Another trick is useful in dealing with products of field variables $(f \cdot g \cdot h)$. Instead of doing separate Taylor's series expansions for each, simply do the expansion for the product, producing terms like $\partial(f \cdot g \cdot h)/\partial x$. The student should select some physical problems from his own area of specialty and derive the associated PDE's. Exercises from a selection of fields are included at the end of this chapter.

Equation (1.3.15) is a <u>second-order</u> PDE for T, because it involves derivatives no higher than second. If k, h, ρ, and c are <u>independent</u> of T, the equation is <u>linear</u> in T.

Many physical problems give rise to second-order PDE's or coupled second-order PDE's. Therefore, we shall spend considerable time studying methods for solution of such equations, particularly those that are linear in the dependent variables.

1.4 Boundary Conditions, Initial Conditions, and Well-Posedness

In addition to the PDE's, one must also have an appropriate set of boundary and/or initial conditions. For example, the initial conditions for the temperature PDE developed in the previous section would be the temperature field at some starting time t_o. We can think of the initial conditions as the specification

$$T(x,y,t_o) = f(x,y) \tag{1.4.1}$$

In addition, we would require boundary conditions constraining the temperature field all around the edge of the plate. One such possibility is to prescribe the edge temperature,

$$T(x,y,t) = g(x,y) \quad \text{on} \quad C \tag{1.4.2}$$

where C is the outer edge shown in Fig. 1.3.1. The initial conditions, the boundary conditions, the PDE, and values for the parameters in the equations define the problem to be solved.

If a portion of the needed boundary or initial condition information is missing, the problem is said to be incompletely posed. If too much information, or the wrong type of information, is prescribed, the problem is said to be ill-posed. For example, the problem defined by Eqns. (1.3.15), (1.4.1), and (1.4.2) would be ill-posed if in addition we attempted to prescribe any of the following:

- The rate of change of temperature, $\partial T/\partial t$, at time zero.
- The temperature field at some later time, $T(x,y,t_1)$.
- The heat transfer rates into the plate along C, $(-k\partial T/\partial n)$.

It is important that the student learn to identify when a problem is not well-posed. A good way to do this is to use physical intuition. There are certain mathematical rules, for some simple types of problems, and we shall discuss these later in the text. Unfortunately, they are applicable only to the simplest of problems, and in dealing with the problems that typically arise in engineering or applied science one has no choice but to rely on experience and intuition.

A case in point can be illustrated by considering the steady-state heat transfer problem formed by dropping the right-hand side of (1.3.15). For simplicity, let's also drop the convective term, treat the conductivity as constant, and write the equation as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + s/k = 0 \tag{1.4.3}$$

This problem clearly has a solution if one prescribes the boundary condition $T = g(x,y)$ on C; physically, a plate with internal thermal energy sources, held at a prescribed edge temperature, can indeed reach a steady-state solution. But suppose we instead insulate the edges of the plate, corresponding to a boundary condition

$$\frac{\partial T}{\partial n} = 0 \quad \text{on} \quad C \tag{1.4.4}$$

where n is the outward normal direction. Clearly this problem does not have a solution; there is no way for the internally generated thermal energy to get out, and consequently the temperature will go up and up and the system will never reach a steady state. Hence (1.4.3) and (1.4.4) form an ill-posed problem.

Suppose we insulate only a portion of the boundary, prescribing

$$\left.\begin{aligned} \frac{\partial T}{\partial n} &= 0 && \text{on part of } C \\ T &= g(x,y) && \text{on the remainder} \end{aligned}\right\} \tag{1.4.5}$$

This *is* a well-posed problem; the energy can escape at the points where T is specified, and thus a steady-state solution can be reached.

Suppose now we remove the source term and have

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1.4.6}$$

This is called Laplace's equation; it is one of the few equations for which theorems about well-posedness are known. Clearly it is proper to specify T on the boundary, or to specify T and some combination of $\partial T/\partial n$ on the boundary, as we have discussed. However, can we specify only the heat flux, i.e.,

$\partial T/\partial n$, all around the boundary? On physical grounds we can argue that a steady-state temperature field will be obtained only if the net energy input is zero, i.e., in this case only if

$$\int \frac{\partial T}{\partial n}\, ds = 0 \quad \text{on} \quad C \tag{1.4.7}$$

Thus, the Laplace equation with $\partial T/\partial n$ specified on the boundary forms a well-posed problem only if (1.4.7) is satisfied. But then the solution is not unique, as one may add a constant to any solution and produce another!

Boundary conditions in which the function is specified on the boundary are called Dirichlet conditions, while boundary conditions specifying the normal derivative are called Neumann conditions. Problems involving combinations of these are called mixed or Churchill conditions. Theorems giving the conditions for well-posedness of simple equations (e.g., the Laplace equation) for these conditions may be found in more advanced books on the theory of PDE's.

In addition to the Laplace equation, there are two other simple equations which form useful models for deciding upon the well-posedness of problems. These are the wave equation,

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2} = 0 \tag{1.4.8}$$

and the (so-called) heat equation,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t} \tag{1.4.9}$$

The wave equation arises in simple problems of linearized vibration, one-dimensional acoustics, shallow-water wave theory, and other simple wave propagation problems. The heat equation (a special case of (1.3.15)) arises in simple one-dimensional diffusion problems in heat transfer, fluid mechanics, and other fields. Most problems of interest involve more complex equations, often nonlinear. However, an understanding of these three simple equations and their solutions is very helpful when one has to solve more difficult problems, analytically or numerically, and so we shall give them due attention in this book. For each there are boundary and initial conditions which render the problem well-posed, and we shall investigate these matters in a later chapter.

1.5 Other Notations

So far we have used the most common notation for partial differentiation, $\partial f/\partial x$. Other notations in use include

$$\partial_x f \quad \text{to mean} \quad \frac{\partial f}{\partial x}$$

$$f_x \quad \text{to mean} \quad \frac{\partial f}{\partial x}$$

$$f_{xy} \quad \text{to mean} \quad \frac{\partial^2 f}{\partial x \partial y}$$

$$\nabla^2 f \quad \text{to mean} \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{(in Cartesian coordinates)}$$

In Cartesian coordinate systems the subscript convention is frequently used. The independent variables are written as x_i (meaning x_1, x_2, or x_3), and the x derivatives are then denoted by subscripts after commas,

$$f_{,i} \quad \text{to mean} \quad \frac{\partial f}{\partial x_i}$$

$$f_{,ij} \quad \text{to mean} \quad \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Alternatively, one sometimes sees

$$\partial_i f \quad \text{to mean} \quad \frac{\partial f}{\partial x_i}$$

In this convention the appearance of a repeated subscript means that the term represents a sum in which the repeated subscript in turn receives each of its permitted values. For example,

$$f_{,ii} = \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \tag{1.5.1}$$

$$x_i x_i = x_1^2 + x_2^2 + x_3^2 \tag{1.5.2}$$

Often the dependent variables are also vector quantities, such as velocity, and their components are denoted by subscripts. For example, the three-component velocity vector (u_1,u_2,u_3) is represented by u_i. Then the continuity equation, which expresses the principle of conservation of mass for an incompressible medium, is simply

$$u_{i,i} = 0 \quad \text{meaning} \quad \frac{\partial u_i}{\partial x_i} = 0$$

$$\text{or} \quad \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \tag{1.5.3}$$

When using the comma approach to spatial derivatives, time-derivatives are denoted by overdots:

$$\dot{u}_i \quad \text{to mean} \quad \frac{\partial u_i}{\partial t}$$

The subscript and comma notations have the great advantage of being very compact; with some experience you can become quite comfortable with them. You should practice writing the equations of your own special field in these different notations.

Exercises

1.1 Express the first and second derivatives of $f = \sinh(\lambda x)\cos(\lambda y)$. Show that f satisfies Laplace's equation (1.4.6).

1.2 Show that $f = e^{-\lambda^2 t}\cos(\lambda x)$ satisfies the heat equation (1.4.9).

1.3 Show that $f = \sin(\lambda x)\cos(\lambda t)$ satisfies the wave equation (1.4.8).

1.4 Show that $e^{it}e^{ix}$ satisfies the wave equation (1.4.8).

1.5 Show that $e^{ix}e^{y}$ satisfies the Laplace equation (1.4.6).

1.6 Let $x = r\cos\theta$ and $y = r\sin\theta$. Show that the Laplace equation (1.4.6) for $f(x,y)$ transforms for $f(r,\theta)$ to

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0$$

1.7 Let $\xi = x + t$, $\eta = x - t$. Show that the wave equation (1.4.8) transforms to $\partial^2 f/\partial\xi\partial\eta = 0$. Then develop the general solution to this equation by direct integration.

1.8 Let $\xi = x + iy$, $\eta = x - iy$. Show that this transforms the Laplace equation (1.4.6) to $\partial^2 f/\partial\xi\partial\eta = 0$. Then develop the general solution to this equation by direct integration.

1.9 (For fluid mechanicians) Derive the PDE's governing unsteady, one-dimensional, inviscid compressible flow of a perfect gas. Assuming isentropic flow, obtain a single differential equation for the pressure field. Show that if the pressure changes are small you obtain the acoustic approximation (c^2 is the sound speed)

$$\Box^2 p \equiv \nabla^2 p - c^2 \frac{\partial^2 p}{\partial t^2} = 0$$

1.10 (For heat transfer persons) Derive the PDE governing the temperature field in an incompressible, inviscid fluid flowing with a prescribed velocity field $u_i(x_i)$. Show that if the thermal conductivity is zero the temperature of each fluid particle remains constant.

1.11 (For heat transfer/fluid mechanics persons) Derive the set of PDE's governing the temperature and velocity fields in a porous medium heated by internal sources and cooled by the interstitial flow of a fluid. Use Darcy's law for the fluid mechanics. Write some boundary and initial conditions that you think are well-posed.

1.12 (For elasticians) Derive the PDE governing the vibrations of an elastic bar. Assume linear stress laws and small amplitude deflections, and consider only bending distortions.

1.13 (For oceanographers) Derive the PDE governing the wave height for long waves in shallow water. Assume small-amplitude inviscid motion. You should obtain a form of the wave equation. Write some boundary and initial conditions appropriate to a tidal wave entering the San Francisco Bay. Under what circumstances would your analysis apply to a water bed?

1.14 (For guitarists) Derive the PDE governing the vibrations of a taut string. Assume small deflections and uniform string properties. You should obtain a form of the wave equation. Write the boundary and initial conditions for your favorite plucking mode.

1.15 (For drummers) Derive the PDE governing the vibrations of a circular drum. Assume small deflections and uniform properties. Write the initial and boundary conditions for your favorite whomp.

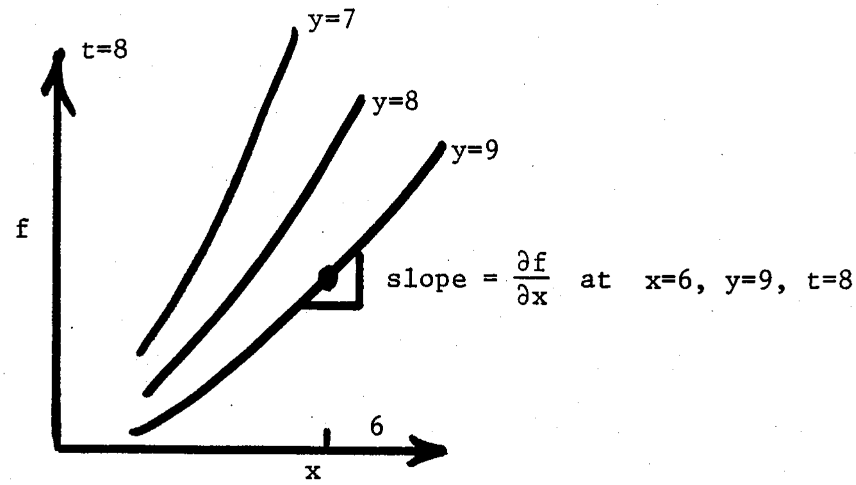

Fig. 1.1.1 Interpretation of Partial Derivatives

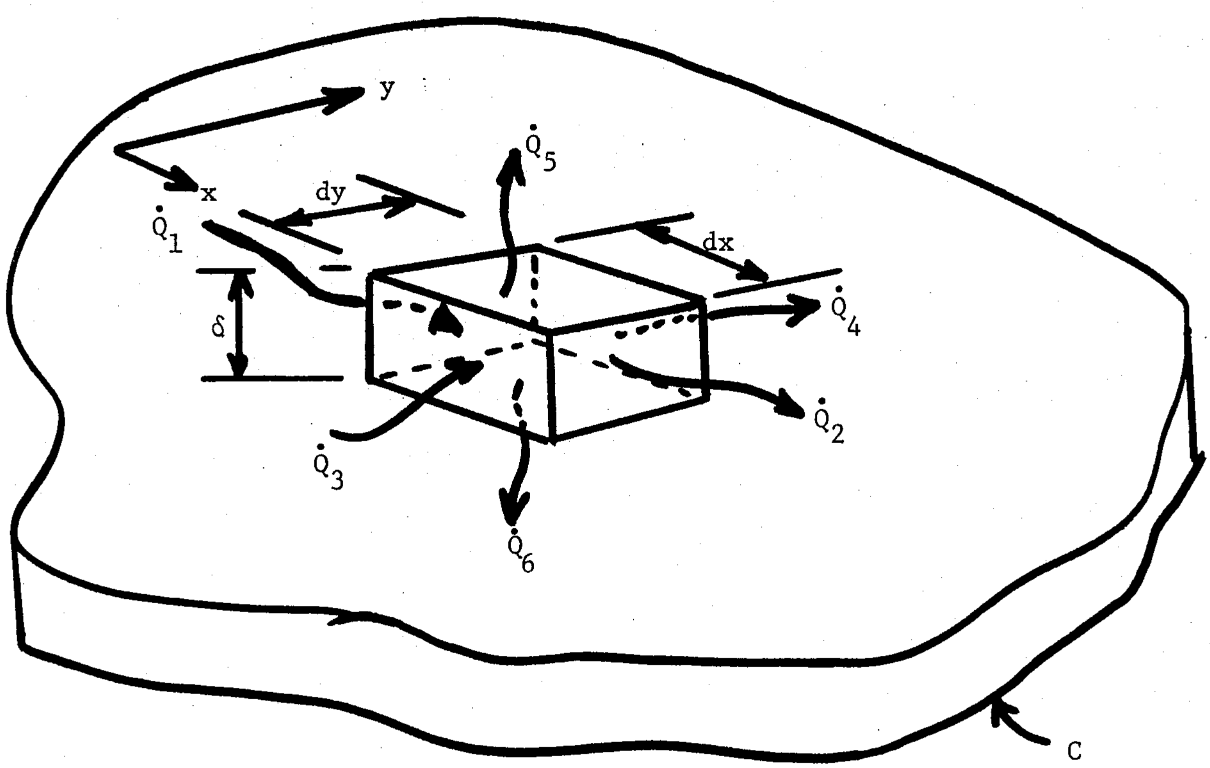

Fig. 1.3.1 Control Volume for Derivation of the PDE

Chapter 2

SELF-SIMILAR SOLUTIONS

2.1 Characteristic Scales; Scale-Similar Problems

It is often convenient to present the solution to a PDE problem in non-dimensional form. This makes the results independent of the size of the system for which the solution was obtained as well as independent of any choice of dimensional system. Non-dimensionalization is usually accomplished by choosing some length and time scales characterizing the problem, and then defining non-dimensional independent variables based on these scales. For example, the solution for fluid flow in a rotating sphere might be expressed non-dimensionally in terms of the dimensionless radius, $R = r/r_o$, where r_o is the radius of the sphere. Here r_o is the characteristic length scale of the problem. If the fluid is initially at rest, and at time zero it is put into rotation at angular velocity ω, then the period of rotation is $\tau = 2\pi/\omega$, and τ would be the characteristic time scale. Then a suitable dimensionless time would be $T = t/\tau$. Note that one of the characteristic scales for the independent variables (r_o) came from the geometry of the system, and the other (τ) from the boundary conditions.

The dependent variables also can be represented non-dimensionally. For example, in the rotating sphere problem the equatorial velocity is $u_o = \omega r_o$ and may be used as a characteristic velocity in the dimensionless velocity $\underline{U} = \underline{u}/u$.

The problem may also contain some parameters, such as the kinematic viscosity ν. The parameters also can be reduced to non-dimensional form, and in the case of viscosity it is customary to use a reciprocal dimensionless viscosity called the Reynolds number, $Re = u_o r_o/\nu$.

The solution for the velocity within the rotating sphere could then be expressed non-dimensionally as

$$\underline{U} = \underline{U}(R,T;Re)$$

This says that the dimensionless velocity (a vector) $\underline{U}$ will be a function of the dimensionless radial coordinate R, the dimensionless time coordinate T, and the parameter Re. It might also happen that the flow depends upon the polar angular coordinates ϕ and θ, which are additional non-dimensional independent variables.

Problems which have natural characteristic scales for the <u>independent</u> variables (here r_o and τ) are called <u>scale-similar</u>. Scale-similar solutions for systems of different size will have the same non-dimensional solution, provided that the two problems also have the same values of the dimensionless parameters and dimensionless boundary and initial conditions.

2.2 Self-Similarity

There are a few very interesting and important PDE problems for which no natural characteristic scales for the independent variables exist in the problem formulation. For example, consider the case of heat conduction in a semi-infinite slab initially at uniform temperature, subjected to a step increase in the surface temperature at time zero (Fig. 2.2.1). The appropriate PDE is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.2.1)$$

where α is a constant parameter called the thermal diffusivity of the medium. The initial condition is

$$T(x,0) = T_i \qquad x > 0 \qquad (2.2.2)$$

The boundary condition at the surface is

$$T(0,t) = T_s \qquad (2.2.3)$$

The temperature field must fall off to the initial temperature T_i as $x \to \infty$, giving a second boundary condition

$$T(x,t) \to T_i \qquad \text{as} \quad x \to \infty \qquad (2.2.4)$$

There are no characteristic scales for either length or time in this problem. This fact is the clue that a self-similar solution must exist. Since the solution to all physical problems must be expressible in dimensionless form (nature is unaware of the length of a meter), there must be some way to non-dimensionalize the solution to this problem. The only possible way is for the variables to appear together in a non-dimensional group. Looking at the denominators in (2.2.1), it is readily apparent that x^2 and αt have the same dimensions, and therefore the quantity $x^2/(\alpha t)$ is dimensionless. Somehow the solution must be expressible in terms of this quantity, in order to have dimensionless form. Solutions made non-dimensional by combinations of the independent variables, rather than by characteristic scales imposed by the geometry, boundary, or initial conditions, are called self-similar solutions.

There is a characteristic temperature for this problem, namely the step increase in temperature $T_s - T_i$. Therefore, one might guess that the non-dimensional form of the solution is

$$\frac{T - T_i}{T_s - T_i} = f\left(\frac{x^2}{\alpha t}\right) \tag{2.2.5}$$

As we shall see, this guess is correct. In a moment we shall develop a systematic way of discovering the forms of self-similar solutions.

If (2.2.5) is indeed correct, then another fully equivalent form would be

$$\frac{T - T_i}{T_s - T_i} = g(x/\sqrt{\alpha t}) \tag{2.2.6}$$

and another would be

$$\frac{T - T_i}{T_s - T_i} = \frac{x}{\sqrt{\alpha t}}\, h(x/\sqrt{\alpha t}) \tag{2.2.7}$$

All of these solutions would really be the same, but the functions f, g, and h would be different.

In terms of the similarity variable, $\eta = x/\sqrt{\alpha t}$, the family of temperature profiles existing at different times will collapse to a single curve (Fig. 2.2.1b). This is the essence of self-similarity; the solution does not scale on the size of the system, instead it scales on itself.

At first glance, it may appear disadvantageous to seek a solution in terms of the non-linear combination of variables $\eta = x/\sqrt{\alpha t}$. However, note that a single function $g(\eta)$ would be involved, and therefore one would only have to deal with an ordinary differential equation (ODE). This is the practical advantage of a self-similar problem in two independent variables. The existence of self-similarity will always reduce the number of independent variables by one.

To summarize, self-similar solutions exist when a problem is not scale-similar, i.e. when characteristic scales for the independent variables do not exist in the problem formulation. In problems with two independent variables, self-similar solutions represent a collapse of the family of solutions as functions of the two variables to a single function of the similarity variable. The governing PDE is thereby reduced to an ODE, which may be solved by some appropriate analytical or numerical method. The proper form of the transformation depends upon the equation, the initial conditions, and the boundary conditions. The transformation can be discovered systematically, as we shall now illustrate by some examples.

2.3 Example with Constant Boundary Conditions

Consider the transient heat transfer problem discussed in section 2.2 The differential equation, boundary conditions, and initial conditions are (2.2.1)-(2.2.4). The solution must be expressible in terms of some similarity variable, which must be non-dimensional. Let's assume that the similarity variable is of the form

$$\eta = Ax/t^n \qquad (2.3.1)$$

where A and n are constants to be chosen in a manner that reduces the PDE problem to an ODE problem. Now, suppose we assume that the dimensionless solution has the form

$$\frac{T - T_i}{T_s - T_i} = f(\eta) \qquad (2.3.2)$$

This is suggested by the observation that the significant aspect is the difference between the temperature at any point $T(x,y)$ and the initial temperature T_i.* The form of η is suggested by the fact that the solution for $t=0$ and $x=\infty$ must give the same value of T, and hence must correspond to the same value of f, and hence to the same value of η. Now, we could have taken $\eta = Ax^m/t^n$ but this is no more general than the (2.3.1), since this η is just a power of the other η. Also, we could have taken $\eta = At/x^n$, which also is no more general. However, we will have to differentiate twice with respect to x, and only once with respect to t, and we will find our work easier if we keep the x-dependence of η as simple as possible. For this reason, we make η linear in x, and then divide by t to a power (to be chosen later).

The next step is to transform the PDE. Using the chain rule,

$$\frac{\partial T}{\partial x} = (T_s-T_i)\frac{df}{d\eta}\frac{\partial \eta}{\partial x} = (T_s-T_i)\, f' \cdot \frac{A}{t^n} \tag{2.3.3a}$$

$$\frac{\partial^2 T}{\partial x^2} = (T_s-T_i)\frac{A}{t^n}\frac{df'}{d\eta}\frac{\partial \eta}{\partial x} = (T_s-T_i)\frac{A}{t^n}\, f'' \cdot \frac{A}{t^n} \tag{2.3.3b}$$

$$\frac{\partial T}{\partial t} = (T_s-T_i)\frac{df}{d\eta}\frac{\partial \eta}{\partial t} = (T_s-T_i)\, f' \cdot \left(-\frac{Anx}{t^{n+1}}\right) \tag{2.3.3c}$$

Then, substituting in (2.2.1), we obtain

$$(T_s-T_i)\frac{A^2}{t^{2n}}\, f'' = -\frac{1}{\alpha}(T_s-T_i)\frac{Anx}{t^{n+1}}\, f'$$

---

*We could instead take

$$\frac{T}{T_s-T_i} = g(\eta\ ;\ T_s/T_i) \tag{2.3.2x}$$

The student should work through the problem with this starting assumption to verify that the same solution is obtained.

which simplifies to

$$f'' + \frac{1}{\alpha A^2} Anx\, t^{n-1} f' = 0 \qquad (2.3.4)$$

Now, this is supposed to be an ODE for $f(\eta)$. Therefore, it can only contain f , f' , f'' , and η ; somehow we must make x and t disappear. To do this, we first replace x using (2.3.1), $x = t^n \eta / A$, and find

$$f'' + \frac{n}{\alpha A^2} t^{2n-1} \eta f' = 0 \qquad (2.3.5)$$

Next, we can select the proper value of n as that which drops out t , namely $n = 1/2$. With this choice, (2.3.5) reduces to

$$f'' + \frac{1}{2\alpha A^2} \eta f' = 0 \qquad (2.3.6)$$

This is an ODE, as desired. We still are free to choose A any way we like. To make (2.3.6) as simple as possible, let's pick

$$A = 1/\sqrt{2\alpha} \qquad (2.3.7)$$

which reduces our ODE to

$$f'' + \eta f' = 0 \qquad (2.3.8)$$

Note that η is a dimensionless variable. Now we have

$$\eta = x/\sqrt{2\alpha t} \qquad (2.3.9)$$

We must also be able to express the boundary and initial conditions in terms of $f(\eta)$ in order to complete the self-similar transformation. Eqs. (2.2.2) and (2.2.4) both require

$$f(\eta) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty \qquad (2.3.10)$$

And, (2.2.3) requires

$$f(0) = 1 \qquad (2.3.11)$$

Eqs. (2.3.8), (2.3.10), and (2.3.11) define the ODE problem that we must solve.

Eqn. (2.3.8) can be written as

$$\frac{df'}{f'} = -\eta d\eta \tag{2.3.12}$$

Integrating,

$$\ell n\ f' = -\frac{\eta^2}{2} + C_0$$

or,

$$f' = C_1\ e^{-\eta^2/2} \tag{2.3.13}$$

Integrating again,

$$f = C_1 \int_{\infty}^{\eta} e^{-\sigma^2/2}\ d\sigma + C_2 \tag{2.3.14}$$

The lower limit is arbitrary, and ∞ is a good choice. We must be careful not to confuse the limit of integration (η) with the variable of integration, and therefore have introduced σ as the "dummy variable" of integration.

The boundary condition (2.3.10) requires $C_2 = 0$. The boundary condition (2.3.11) requires

$$1 = C_1 \int_{\infty}^{0} e^{-\sigma^2/2}\ d\sigma \tag{2.3.15}$$

Hence, we can write the solution as

$$f = \int_{\eta}^{\infty} e^{-\sigma^2/2}\ d\sigma \Bigg/ \int_{0}^{\infty} e^{-\sigma^2/2}\ d\sigma \tag{2.3.16}$$

We can express the solution in terms of known special functions by letting $z = \sigma/\sqrt{2}$. Then, $d\sigma = \sqrt{2}\ dz$, and

$$f = \int_{\eta/\sqrt{2}}^{\infty} e^{-z^2}\ dz \Bigg/ \int_{0}^{\infty} e^{-z^2}\ dz \tag{2.3.17}$$

The denominator has the value $\sqrt{\pi}/2$. The numerator is $\sqrt{\pi}/2 \;\text{erfc}\;(\eta/\sqrt{2})$, where erfc is the complementary error function.* Hence, the solution is

$$\frac{T - T_i}{T_s - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{2.3.18}$$

2.4 Example with Variable Boundary Conditions

The motion of a viscous fluid, initially at rest, over an infinite plate that is set into motion at time zero is described by (Fig. 2.4.1)

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \tag{2.4.1}$$

where u is the velocity tangential to the plate, and ν is the (constant) kinematic viscosity. Suppose the boundary condition at the plate $y=0$, is

$$u(0,t) = at^b \tag{2.4.2}$$

where a and b are fixed parameters. The other boundary condition is

$$u(y,t) \to 0 \qquad \text{as} \quad y \to \infty \tag{2.4.3}$$

The initial condition is

$$u(y,0) = 0 \tag{2.4.4}$$

There are no characteristic length or time scales in either the domain or boundary conditions of this problem, hence, we expect a self-similar solution. Suppose we assume

$$u = A\, f(\eta) \quad , \quad \eta = By/t^n \tag{2.4.5}$$

where A, B, and n are parameters that we will try to select to produce an ODE problem. The form of η is suggested by (2.4.3) and (2.4.4), which require

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*See HMF, Section 7.1.

that the solution have the same behavior for large y as for small t. However, when we try to fit the boundary condition (2.4.2) with this form, we get

$$A\, f(0) = a t^b \tag{2.4.6}$$

Since A and $f(0)$ will be constants, (2.4.6) can't be true except for the special case $b=0$ (which reduces this example to the previous one). Hence, (2.4.5) will not work.

We need to allow additional freedom. If we expect the curves of Fig. (2.4.1a) to collapse on a single non-dimensional curve, the value of the fluid velocity must somehow scale on the instantaneous wall velocity. This suggests that we try

$$u = A\, t^m f(\eta) \qquad \eta = By/t^n \tag{2.4.7}$$

Where now A, m, B, and n may be chosen to give us the desired self-similar solution.*

We can immediately determine m using (2.4.2),

$$u(0,t) = A\, t^m f(0) = a t^b \tag{2.4.8}$$

Hence, we <u>must</u> choose $m=b$. We <u>may</u> choose A any way we like. If we choose $A=a$, then we <u>must</u> impose the boundary condition

$$f(0) = 1 \tag{2.4.9}$$

Now, we have

$$u = a\, t^b f(\eta) \qquad \eta = By/t^n \tag{2.4.10}$$

which <u>will</u> fit the boundary conditions.

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*We could have used $u = A\, y^k t^m g(\eta)$, or $u = A\, y^m h(\eta)$. These forms are equivalent to (2.4.7), with different functions f, g, and h. Eq. (2.4.7) is the simplest, since we must take two y derivatives and only one t derivative.

Next, we substitute (2.4.10) in the differential equation (2.4.1), and find ($f' = df/d\eta$, $f'' = d^2f/d\eta^2$)

$$\nu a B^2 t^{b-2n} f'' = abt^{b-1} f - at^{b-n-1} nByf' \qquad (2.4.11)$$

As an ODE in $f(\eta)$, this may contain only f and its derivatives, η , and constants; y and t may not appear. So, we will replace y by

$$y = t^n \eta / B \qquad (2.4.12)$$

Then, (2.4.11) reduces to

$$\nu a B^2 t^{b-2n} f'' = abt^{b-1} f - at^{b-1} n\eta f' \qquad (2.4.13)$$

In order that t drop out, we must choose n such that

$$b-2n = b-1 \quad \text{or} \quad n = 1/2$$

With this choice, our ODE becomes

$$\nu B^2 f'' = bf - \frac{1}{2} \eta f' \qquad (2.4.14)$$

Let's choose B such that $\nu B^2 = \frac{1}{2}$, or $B = 1/\sqrt{2\nu}$. Then we have

$$f'' + \eta f' - 2bf = 0 \qquad (2.4.15)$$

and our similarity variable η is

$$\eta = y/\sqrt{2\nu t} \qquad (2.4.16)$$

The boundary conditions on (2.4.15) are, from (2.4.9),

$$f(0) = 1 \qquad (2.4.17a)$$

and, from (2.4.3),

$$f(\eta) \rightarrow 0 \qquad \text{as} \quad \eta \rightarrow \infty \tag{2.4.17b}$$

To complete the problem, we must solve (2.4.15) subject to (2.4.17). This will provide a good review of some ODE solution methods and will introduce us to some special functions.

In order to solve (2.4.15), one must be specific about the value of b. Let's first take $b = 1/2$, for which (2.4.15) becomes

$$f'' + \eta f' - f = 0 \tag{2.4.18}$$

The general solution will be of the form

$$f = C_1 f_1 + C_2 f_2 \tag{2.4.19}$$

where f_1 and f_2 are two linearly-independent solutions. For this case, $f_1 = \eta$ is one obvious solution; when the first solution to a second-order linear ODE is known, the second can always be constructed by setting

$$f_2(\eta) = f_1(\eta) \cdot g(\eta) \tag{2.4.20}$$

So, we assume

$$f_2(\eta) = \eta\, g(\eta)$$

Differentiating, and substituting in (2.4.18), we find

$$\eta g'' + 2g' + \eta(\eta g' + g) - \eta g = 0 \tag{2.4.21}$$

The zero-derivative terms cancel, which is why this methods works. So, we have

$$\eta g'' + (2+\eta^2) g' = 0 \tag{2.4.22}$$

which is really a <u>first</u>-order ODE for g'; separating the variables,

$$\frac{dg'}{g'} = -\left(\frac{2}{\eta} + \eta\right) d\eta \tag{2.4.23}$$

Integrating, and taking the exponential,*

$$g' = \exp\left(-2 \ln \eta - \frac{\eta^2}{2}\right) = \frac{1}{\eta^2} e^{-\eta^2/2} \tag{2.4.24}$$

Integrating again,

$$g(\eta) = \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\sigma^2/2} d\sigma \tag{2.4.25}$$

The lower limit choice is arbitrary, except that zero will cause problems; infinity is an "artistic" choice. So, we now have the <u>general</u> solution to (2.4.18) as

$$f = C_1\eta + C_2\eta \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\sigma^2/2} d\sigma \tag{2.4.26}$$

Note that again we were careful not to confuse the limit of integration (η) with the variable of integration (σ).

We now apply the boundary condition (2.4.17b), which will require $C_1 = 0$ <u>if</u> we can show that the second solution f_2 is bounded as $\eta \to \infty$. We have

$$f_2(\eta) = \eta \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\sigma^2/2} d\sigma < \eta \int_{\infty}^{\eta} \frac{1}{\eta} e^{-\sigma^2/2} d\sigma = \int_{\infty}^{\eta} e^{-\sigma^2/2} d\sigma$$

$$(\text{for } \eta > 1) \tag{2.4.27}$$

So, clearly $f_2(\eta) \to 0$ as $\eta \to \infty$. Therefore, C_1 is indeed zero.

---

*We choose the constant of integration to be 0. Any $g(\eta)$ will do since we can use any second solution.

The behavior of f_2 at $\eta = 0$ can be clarified through use of one of the most powerful tools of analysis-integration by parts.* With it, f_2 can be rewritten as

$$f_2 = \eta\left[-\frac{1}{\sigma}\, e^{-\sigma^2/2}\,\Big|_{\infty}^{\eta} - \int_{\infty}^{\eta}\left(-\frac{1}{\sigma}\right)(-\sigma)\, e^{-\sigma^2/2}\, d\sigma\right]$$

$$= -e^{-\eta^2/2} - \eta\int_{\infty}^{\eta} e^{-\sigma^2/2}\, d\sigma \qquad (2.4.28)$$

Now it is clear that $f_2(0) = -1$. Since (2.4.17a) requires that $f(0) = 1$, $C_2 = -1$. Therefore, the final solution is

$$f(\eta) = e^{-\eta^2/2} + \eta\int_{\infty}^{\eta} e^{-\sigma^2/2}\, d\sigma \qquad (2.4.29)$$

Using the charge of variables, $z = \sigma/\sqrt{2}$, this can be written as

$$f(\eta) = e^{-\eta^2/2} - \eta\sqrt{\frac{\pi}{2}}\, \mathrm{erfc}(\eta/\sqrt{2}) \qquad (2.4.30)$$

(for $b = 1/2$)

Next, let's consider the case $b = n/2$, where n is an integer. Eqn. (2.4.15) is then

$$f'' + \eta f' - nf = 0 \qquad (2.4.31)$$

If we let $z = \eta/\sqrt{2}$, then (2.4.31) becomes

$$\frac{d^2f}{dz^2} + 2z\,\frac{df}{dz} - 2nf = 0 \qquad (2.4.32)$$

---

*Recall that $\int u\,dv = uv - \int v\,du$; this is called integration by parts; become adept at doing it, because it is tremendously useful and important.

The two linearly independent solutions of this equation are repeated integrals of the error function,*

$$f = C_1 i^n \mathrm{erfc}(z) + C_2 i^n \mathrm{erfc}(-z) \tag{2.4.33}$$

where the function $i^n \mathrm{erfc}(x)$ is**

$$i^n \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{(t-x)^n}{n!} e^{-t^2} dt \tag{2.4.34}$$

Hence, our solution is

$$f = C_1 i^n \mathrm{erfc}(\eta/\sqrt{2}) + C_2 i^n \mathrm{erfc}(-\eta/\sqrt{2}) \tag{2.4.35}$$

The boundary condition $f(\infty) = 0$ requires $C_2 = 0$, since $i^n \mathrm{erfc}(-\infty)$ is a constant. The boundary condition $f(0) = 1$ fixes C_1 as***

$$C_1 = \frac{1}{i^n \mathrm{erfc}(0)} = 2^n \, \Gamma(\tfrac{n}{2}+1) \tag{2.4.36}$$

where $\Gamma(x)$ is the Gamma function,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{2.4.37}$$

Hence, the solution is

$$f(\eta) = 2^n \, \Gamma(\tfrac{n}{2}+1) \, i^n \mathrm{erfc}(\eta/\sqrt{2}) \tag{2.4.38}$$

(for $b = n/2$)

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*HMF Section 7.2.2.

**The student should verify (2.4.33) by substitution in (2.4.32). Integration by parts will be required.

***See HMF Section 7.2.7.

2.5 Example with Integral Constraint

Consider the problem of diffusion of a contaminant deposited at time zero at the surface of a semi-infinite slab (Fig. 2.5.1). The diffusion process is described by

$$\alpha \frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t} \tag{2.5.1}$$

where $c(x,t)$ is the concentration per unit volume, and α is the diffusion coefficient for the contaminant. The initial condition is

$$c(x,0) = 0 \qquad x > 0 \tag{2.5.2}$$

The boundary condition for large x is

$$c(x,t) \to 0 \qquad \text{as} \quad x \to \infty \tag{2.5.3}$$

The total amount of contaminant contained in the slab is fixed. This gives an integral constraint,

$$\int_0^\infty c\,dx = Q \tag{2.5.4}$$

This problem has no natural characteristic length or time scales and, hence, we expect a self-similar solution.

Let's try to construct the solution in the form*

$$c = At^n f(\eta) \qquad \eta = Bx/t^m \tag{2.5.5}$$

where A, n, B, and m are constants to be chosen. The integral constraint (2.5.4) immediately tells us something about n

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*Again, the similar boundary condition (2.5.3) and initial condition (2.5.2) suggest the form of η.

$$Q = \int_0^\infty At^n f(\eta)\,dx = \int_0^\infty At^n f(\eta) \cdot \frac{t^m}{B}\,d\eta$$

$$= \frac{A}{B}\,t^{n+m} \int_0^\infty f(\eta)\,d\eta = \text{constant} \qquad (2.5.6)$$

The integral will be some number. Therefore, for Q to be constant $n = -m$. We will later use (2.5.6) to help determine other constraints.

Next, we substitute (2.5.5), with $n = -m$, in (2.5.1), and obtain

$$\alpha AB^2 t^{-3m} f'' = -mt^{-m-1} Af - m\eta At^{-m-1} f' \qquad (2.5.7)$$

Note we have already eliminated x in favor of η. For this to be an ODE, t must drop out, hence $-3m = -m-1$, or $m = 1/2$. We pick $\alpha B^2 = 1/2$, $B = 1/\sqrt{2\alpha}$, and then our ODE becomes

$$f'' + \eta f' + f = 0 \qquad (2.5.8)$$

The boundary condition (2.5.3) requires

$$f(\eta) = 0 \quad \text{as} \quad \eta \to \infty \qquad (2.5.9)$$

We have the freedom to match this integral constraint with the choice of A. Hence, let's choose

$$f(0) = 1 \qquad (2.5.10)$$

Eqs. (2.5.8)-(2.5.10) define the ODE problem to be solved.

Our task becomes easy when we recognize that (2.5.8) is expressible as

$$(f')' + (\eta f)' = 0 \qquad (2.5.11)$$

Integrating,

$$f' + \eta f = C_1 \qquad (2.5.12)$$

Since the boundary condition (2.5.9) requires $f(\eta) \to 0$ as $\eta \to \infty$, $f'(\eta) \to 0$ as $\eta \to \infty$, and hence C_1 will have to be zero unless $\eta f \to$ constant as $\eta \to \infty$. Let's assume (subject to later verification) that $\eta f \to 0$ as $\eta \to \infty$, and hence that $C_1 = 0$. Separating the variables and integrating again,

$$f = C_2 e^{-\eta^2/2} \tag{2.5.13}$$

Note that indeed $\eta f \to 0$ as $\eta \to \infty$, as assumed. Our choice $f(0) = 1$ requires $C_2 = 1$. Hence,

$$f(\eta) = e^{-\eta^2/2} \tag{2.5.14}$$

To complete the solution we need*

$$\int_0^\infty f(\eta)d\eta = \int_0^\infty e^{-\eta^2/2}\, d\eta = \int_0^\infty e^{-\sigma^2}\sqrt{2}\, d\sigma = \sqrt{\frac{\pi}{2}} \tag{2.5.15}$$

Using this in (2.5.6), we find

$$A = \frac{Q}{\sqrt{\pi\alpha}} \tag{2.5.16}$$

Hence, the final solution is

$$c = \frac{Q}{\sqrt{\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right) \tag{2.5.17}$$

Note that the concentration at $x = 0$ is <u>infinite</u> at $t = 0$. This reflects a modest deficiency in the model, namely we assumed that we could place a finite amount of contaminant in a zero thickness layer at time zero. Thus, the solution is not useful for very small times. Fig. 2.5.1 shows the form of this solution.

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*See HMF Section 7.1.

2.6 A Non-Linear Problem

The laminar boundary layer over a flat plate is described by

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy} \tag{2.6.1}$$

where ν is the kinematic viscosity and $\psi(x,y)$ is the stream function, which must satisfy the boundary conditions

$$\psi_x = 0 \quad \text{at} \quad y = 0 \tag{2.6.2a}$$

$$\psi_y = 0 \quad \text{at} \quad y = 0 \tag{2.6.2b}$$

$$\psi_y \to U_o \quad \text{as} \quad x \to 0 \tag{2.6.3a}$$

$$\Psi_y \to U_o \quad \text{as} \quad y \to \infty \tag{2.6.3b}$$

Students of fluid mechanics should look up the derivation of this problem; others may treat it simply as a mathematical example.

Since there are no characteristic scales in the problem, we look for a self-similar solution of the form

$$\psi(x,y) = Ax^n f(\eta) \quad , \quad \eta = By/x^m \tag{2.6.4}$$

Note that we will need three y derivatives, and only one x derivative, so we chose a form that keeps the y dependence simple.

Substituting (2.6.4) in (2.6.3),

$$\psi_y = ABx^{n-m} f'(\eta) \to U_o \quad \text{as} \quad (\eta \to \infty) \tag{2.6.5}$$

Now, $f'(\infty)$ will be a number; hence, for this to be constant, $m = n$. We will make the arbitrary choice $f'(\infty) = 1$. Then, we will have to choose A and B such that $AB = U_o$. With these choices, (2.6.5) will be satisfied for all x .

Next, we substitute (2.6.4) in (2.6.1), using $m = n$. This produces

$$-AB\,f' \cdot ABx^{-1}m\eta\,f'' - Bx^{m-1}m(f-\eta f')BA^2x^{-m}\,f'' = \nu B^3Ax^{-2m}\,f''' \tag{2.6.6}$$

Note that we have already replaced y by $\eta x^m/B$. For x to drop out, $-2m = -1$, or $m = 1/2$. Then, if we pick $A^2B^2 = \nu B^3A$, the equation reduces to

$$f''' + \frac{1}{2} f\, f'' = 0 \tag{2.6.7}$$

We have already chosen $f'(\infty) = 1$, which led us to $AB = U_o$. Hence,

$$A = \sqrt{U_o/\nu} \qquad B = \sqrt{U_o\nu} \tag{2.6.8}$$

The boundary conditions are, from (2.6.2a)

$$f(0) = 0 \tag{2.6.9a}$$

and from (2.6.2b)

$$f'(0) = 0 \tag{2.6.9b}$$

Eqs. (2.6.3) will be safisfied by our choice of constants if

$$f'(\eta) \to 1 \qquad \text{as } \eta \to \infty \tag{2.6.9c}$$

Eqn. (2.6.7) must now be solved, subject to the boundary conditions (2.6.9). The solution will introduce you to two useful ideas; rescaling, and numerical solution as an initial value problem.

In problems of this sort, it is often possible to use a "rescaling technique" to convert the two-point boundary value problem to a one-point initial value problem. The advantage of this is that the initial value problem can be solved numerically with a single-pass technique. To rescale, we let

$$z = C\eta \qquad f(\eta) = C^n g(z) \tag{2.6.10}$$

The idea is to pick an n such that we can solve the g equation without knowing the value of the constant C, which will be determined after the g equation has been solved. Substituting (2.6.10) in (2.6.7), one finds ($g' = dg/dz$, etc) .

$$C^{n+3} g''' + \frac{1}{2} C^{2n+2} g g'' = 0$$

Now, if we pick $n+3 = 2n+2$, i.e. $n = 1$, the g equation is

$$g''' + \frac{1}{2} g g'' = 0 \tag{2.6.11}$$

The boundary conditions on g are, from (2.6.9a and b),

$$g(0) = 0 \tag{2.6.12a}$$
$$g'(0) = 0 \tag{2.6.12b}$$

We replace the outer boundary condition by a third condition at $z = 0$. Let's use

$$g''(0) = 1 \tag{2.6.12c}$$

If we can solve (2.6.11), subject to (2.6.12), we can choose C to produce an f satisfying (2.6.9c), and the solution will be complete.

So now we go to the local computer center, and use a program that solves systems of first order ODE's by a marching method. These methods deal with systems of the form

$$\frac{dy_i}{dx} = A_i(x,\underline{y}) \tag{2.6.13}$$

with the "initial" $(x = x_o)$ values of the solution vector y_i prescribed. We define the three variables as

$$y_1 = g \qquad (2.6.14a)$$

$$y_2 = g' \qquad (2.6.14b)$$

$$y_3 = g'' \qquad (2.6.14c)$$

Then, (2.6.11) is the <u>first</u> order equation

$$y_3' = -\frac{1}{2} y_1 y_3 \qquad (2.6.15a)$$

The other two equations are, from the definitions,

$$y_1' = y_2 \qquad (2.6.15b)$$

$$y_2' = y_3 \qquad (2.6.15c)$$

The initial conditions are

$$y_1(0) = 0 \qquad (2.6.16a)$$

$$y_2(0) = 0 \qquad (2.6.16b)$$

$$y_3(0) = 1 \qquad (2.6.16c)$$

It takes only a few lines of program to tell the general purpose program that we want it to solve (2.6.15), subject to (2.6.16), over a range from $x = 0$ to some large x (perhaps 20). We execute, and print y_1, y_2, and y_3 at different values of x. If all has gone well, at large x y_1 will be growing linearly, $y_2 = g'$ will be constant, and $y_3 = g''$ will be very small.

Knowing the value of $g'(z)$ as $z \to \infty$, we go back to the rescaling transformation (2.6.10) and the outer boundary condition (2.6.9c)

$$f'(\infty) = C^2 g'(\infty) = 1$$

Hence, $C = 1/\sqrt{g'(\infty)}$. We can now calculate and plot $f(\eta)$ for $0 \leq \eta < \infty$, and the problem is finished.

2.7 An Example in More Dimensions

The transient heat conduction in a quarter-infinite block (Fig. 2.7.1) is described by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.7.1)$$

where the quantities are as defined in §2.3. Suppose that the initial condition is

$$T(x,y,0) = T_i \quad \text{for} \quad x > 0\,, \; y > 0 \qquad (2.7.2)$$

the boundary conditions are

$$T(x,0,t) = T_s \qquad (2.7.3a)$$

$$T(0,y,t) = T_s \qquad (2.7.3b)$$

Let's seek a self-similar solution in terms of two similarity variables,*

$$\xi = Ax/t^n \qquad \eta = Ay/t^n \qquad (2.7.4)$$

Following the example in §2.3, we assume

$$\frac{T - T_i}{T_s - T_i} = F(\xi,\eta) \qquad (2.7.5)$$

Substituting in (2.7.1),

$$A^2 t^{-2n}(F_{\xi\xi}+F_{\eta\eta}) = -\frac{nt^{-1}}{\alpha}(\xi F_\xi + \eta F_\eta) \qquad (2.7.6)$$

We choose $n = 1/2$ to reduce (2.7.6) to a PDE in just ξ and η. Then, with $A = 1/\sqrt{\alpha}$,

---

*Because the problem is symmetric in x and y, we have no reason to use different powers or coefficients in the two similarity variables.

$$F_{\xi\xi} + F_{\eta\eta} + \xi F_\xi + \eta F_\eta = 0 \qquad (2.7.7)$$

Note that the self-similar transformation has reduced the number of independent variables by one.

The boundary and initial conditions produce

$$F(0,\eta) = 1 \qquad (2.7.8a)$$

$$F(\xi,0) = 1 \qquad (2.7.8b)$$

Now, as $\xi \to \infty$, the solution should approach that of the semi-infinite solid (see § 2.3), so

$$F \to \hat{g}(\eta) = \text{erfc}(\eta) \quad \text{as} \quad \xi \to \infty \qquad (2.7.8c)$$

Similarly,

$$F \to \hat{f}(\xi) = \text{erfc}(\xi) \quad \text{as} \quad \eta \to \infty \qquad (2.7.8d)$$

The PDE for F can be solved by the method of separation of variables, discussed in the next three chapters. Following the approach to be presented there, we assume

$$F(\xi,\eta) = \hat{f}(\xi) + \hat{g}(\eta) + H(\xi,\eta) \qquad (2.7.9)$$

Since $\hat{f}'' + \xi\hat{f}' = 0$ and $\hat{g}'' + \eta\hat{g}' = 0$ (see 2.3.8), H also satisfies (2.7.7). Now, we assume a separable solution for H

$$H(\xi,\eta) = f(\xi) \cdot g(\eta) \qquad (2.7.10)$$

Substituting (2.7.9) in (2.7.7), and dividing by H, one finds

$$\frac{f'' + \xi f'}{f} = -\left(\frac{g'' + \eta g'}{g}\right) \qquad (2.7.11)$$

Since the left-hand side is independent of η, and the right-hand side is independent of ξ, both must be constant, and

$$\frac{f'' + \xi f'}{f} = C$$

$$\frac{g'' + \eta g'}{g} = -C$$

or,

$$f'' + \xi f' = Cf \tag{2.7.12a}$$

$$g'' + \eta g' = -Cg \tag{2.7.12b}$$

The boundary conditions on H are, from (2.7.8),

$$H \to 0 \quad \text{as} \quad \xi \to \infty \tag{2.7.13a}$$

$$H \to 0 \quad \text{as} \quad \eta \to \infty \tag{2.7.13b}$$

$$H(0,\eta) = -\hat{g}(\eta) \tag{2.7.13c}$$

$$H(\xi,0) = -\hat{f}(\xi) \tag{2.7.13d}$$

By symmetry, f and g must be the same function, hence C = 0. So if we take

$$C = 0 \qquad f(\xi) = i\hat{f}(\xi) \qquad g(\eta) = i\hat{g}(\eta) \tag{2.7.14}$$

(2.7.12) will be satisfied, and the boundary conditions (2.7.13) are all satisfied. Hence, the solution is

$$F(\xi,\eta) = -\mathrm{erfc}(\xi/\sqrt{2})\ \mathrm{erfc}(\eta/\sqrt{2}) + \mathrm{erfc}(\xi/\sqrt{2}) + \mathrm{erfc}(\eta/\sqrt{2}) \tag{2.7.15}$$

2.8 Summary

We have seen that self-similar solutions arise when there are no natural characteristic scales for the independent variables in the problem formulation. The self-similar transformation will always reduce the number of independent variables by one, so that in a problem with two independent variables the PDE will become an ODE. The steps used to systematically develop the self-similar solution are as follows:

(1) Assume a general form for the transformation, guided by the initial and boundary conditions. Use a form in which the variable that appears in the most complex way in the equations appears as simply as possible in the transformation.

(2) Express the boundary and initial conditions in terms of the similarity transformation, and verify that they can be satisfied by the

assumed transformation. If they can not, add additional degrees of freedom.

(3) Remove one (or more) of the independent variables using the similarity variable. Then, determine the parameters of the transformation necessary to reduce the PDE order by one.

(4) Express the boundary and initial conditions for the reduced problem, and solve by appropriate methods.

In all of the examples worked here, the similarity variable involved forms like $y/\sqrt{x}$. The square-root behavior occurs frequently, but not exclusively. Some of the problems at the end of this chapter will require other powers in the similarity variable.

For Further Reading on Similarity Solutions

- Kline, S. J., Similitude and Approximation Theory, McGraw-Hill Book Co., New York, 1965.
- Hansen, A. G., Similarity Analysis of Boundary Value Problems in Engineering, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.
- Sedov, L. I., Similarity and Dimensional Methods in Mechanics, Academic Press, New York, 1959.

Exercises:

2.1 The temperature field $T(x,t)$ in a semi-infinite slab with a constant heat flux is described by

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \quad ; \quad T(x,0) = T_i$$

$$T(x,t) \rightarrow T_i \text{ as } x \rightarrow \infty \quad ; \quad -k\frac{\partial T}{\partial x} = q \text{ at } x = 0$$

Solve for the temperature field for $x \geq 0$, $t \geq 0$.

2.2 The temperature field in the thermal boundary layer that grows within a hydrodynamic boundary layer at a step in wall temperature is described by

$$\alpha \frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \quad ; \quad T(0,y) = T_\infty \quad y > 0$$

$$T(x,y) \rightarrow T_\infty \text{ as } y \rightarrow \infty \quad ; \quad T(x,0) = T_w \; ;$$

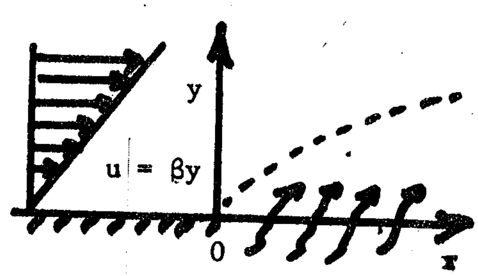
Solve for the temperature field for $x \geq 0$, $y \geq 0$.

2.3 A device for measuring the velocity gradient in flows is shown in the figure. It consists of a heated plate at the wall, over which a thermal boundary layer grows. As long as the thermal boundary layer is confined to the region where the flow velocity u is linear $(u = \beta y)$, the problem is described by

$$\alpha \frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \quad ; \quad T(0,y) = T_\infty \quad y > 0$$

$$T(x,y) \rightarrow T_\infty \text{ as } y \rightarrow \infty \quad ; \quad -k\frac{\partial T}{\partial y} = q \text{ at } y = 0$$

Derive an expression relating the <u>local</u> wall temperature, $T_w(x)$, to the flow parameters and x . Evaluate any constants in this expression. Hint: Γ .



2.4 The diffusion of a contaminant deposited along a line within an infinite medium is described by

$$\alpha \frac{\partial}{\partial r} \left(r \frac{\partial c}{\partial r}\right) = r \frac{\partial c}{\partial t} \quad ; \quad c(r,t) \to 0 \quad \text{as} \quad r \to \infty$$

$$c(r,0) = 0 \quad r > 0 \quad ; \quad 2\pi \int_0^\infty c r\, dr = Q$$

Solve this problem, and give an expression for $c(0,t)$.

2.5 The diffusion of a contaminant deposited at a point in an infinite medium is described by

$$\alpha \frac{\partial}{\partial r} \left(r^2 \frac{\partial c}{\partial r}\right) = r^2 \frac{\partial c}{\partial t} \quad ; \quad c(r,t) \to 0 \quad \text{as} \quad r \to \infty$$

$$c(r,0) = 0 \quad r > 0 \quad ; \quad 4\pi \int_0^\infty c r^2\, dr = Q$$

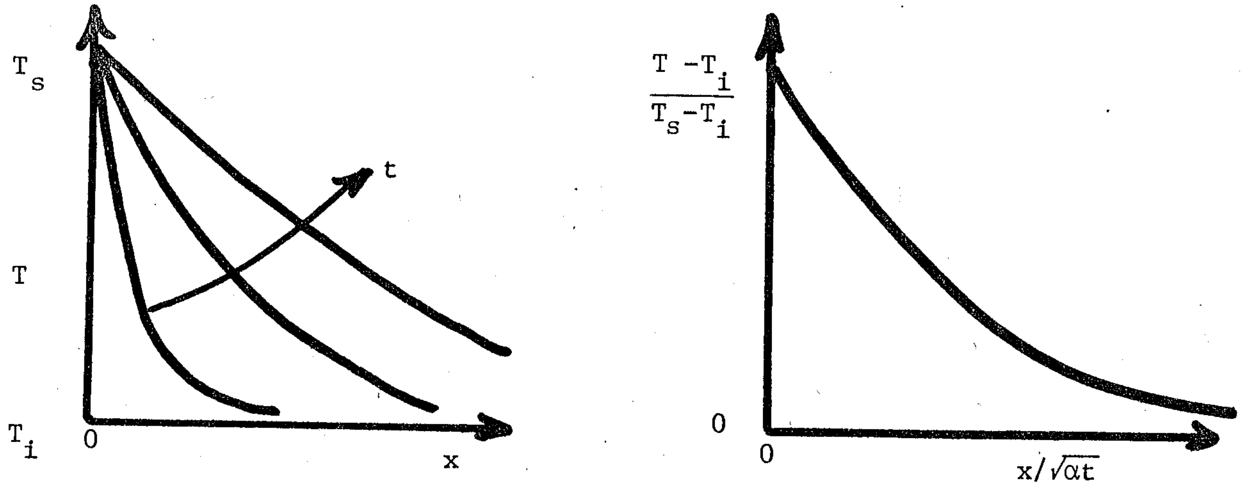
Solve this problem, and give an expression for $c(0,t)$.

2.6 Consider a non-linear diffusion problem described by

$$\frac{\partial}{\partial x} \left[\alpha(1+\beta c) \frac{\partial c}{\partial x}\right] = \frac{\partial c}{\partial t} \quad ; \quad c(x,0) = 0 \quad x > 0$$

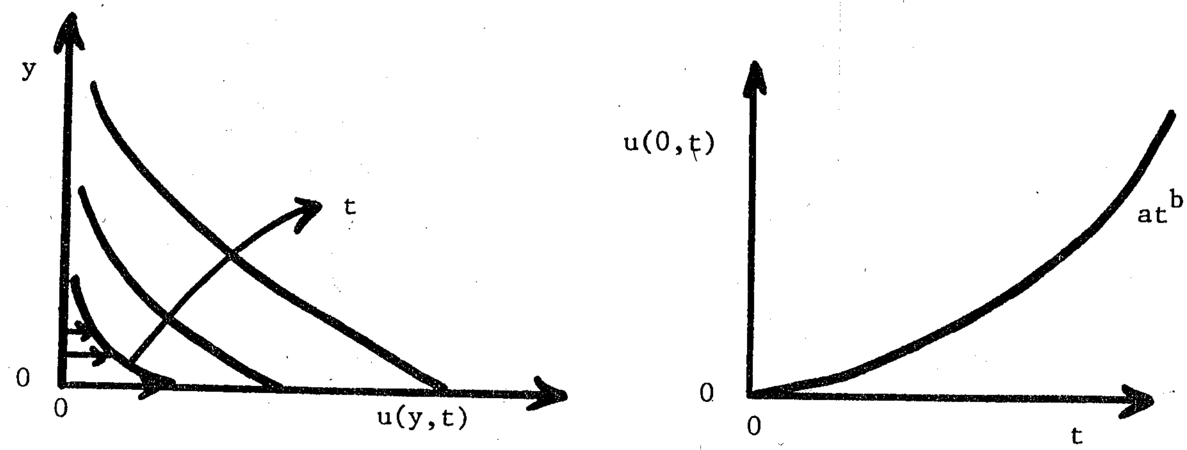
$$c(0,t) = 1 \quad ; \quad c(x,t) \to 0 \quad \text{as} \quad x \to \infty$$

Derive the similarity transform and associated ODE. Solve the problem numerically for $\beta = -0.5$, 0, and 0.5. Use the $\beta = 0$ case to check the numerical solution against the exact solution, and to guide the starting and direction of numerical marching.



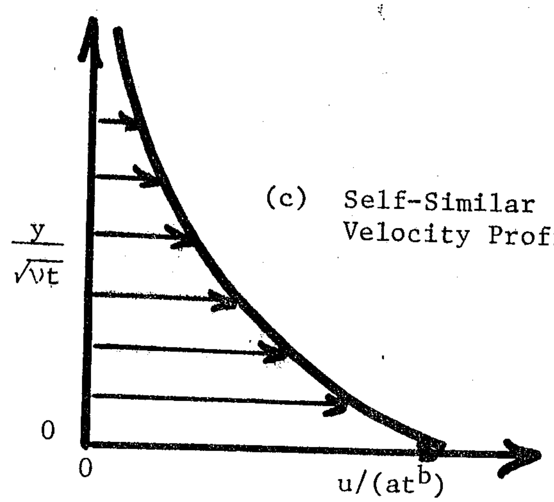
(a) (b)

Fig. 2.2.1 Temperature Field in a Semi-Infinite Slab



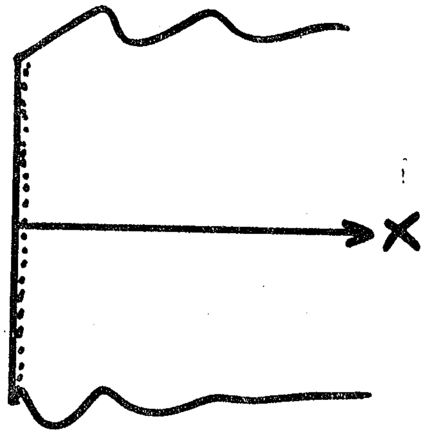
(a) Velocity Field

(b) Plate Velocity

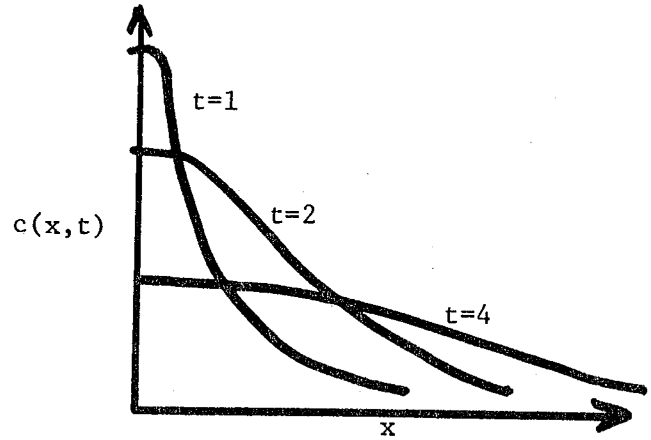


(c) Self-Similar Velocity Profile

Fig. 2.4.1 Velocity Field in Viscous Flow over A Moving Plate

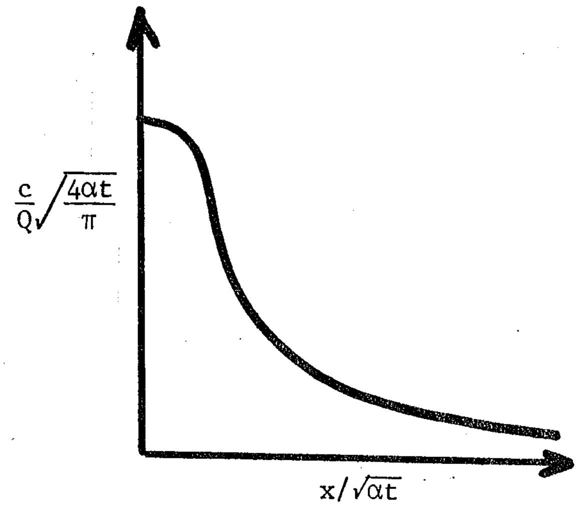


(a) The System



(b) Concentration Profiles

Fig. 2.5.1.



(c) Self-Similar Profile

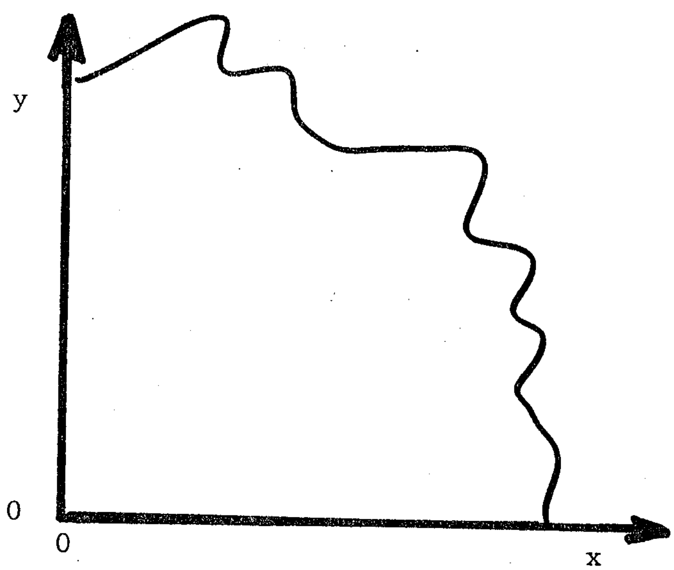


Fig. 2.7.1. Geometry for Analysis of Heating of a Corner

Chapter 3

SOLUTION OF EIGENVALUE PROBLEMS BY SEPARATION OF VARIABLES

3.1 Introduction

Solutions to linear, homogeneous partial differential equations, or coupled systems of such equations, can usually be obtained by the method of separation of variables (SOV). In some problems, such as vibration analysis, the SOV solutions are of great physical interest, as they represent the natural modes or eigenmodes of vibration. In other problems the SOV solutions are of little interest by themselves, but they can be used as building blocks to construct solutions that are of interest. In this chapter we shall discuss the general idea behind the development of solutions by SOV, and present a number of examples in which the SOV method is used to construct interesting eigensolutions. In subsequent chapters we will examine the role of these solutions in constructing more complicated solutions.

With very few exceptions, the SOV method is only useful in linear, homogeneous equations. A linear equation is one in which the dependent variable and its partial derivatives appear only to the first power, and never in products. In a homogeneous equation one may multiply the dependent variable (or variables) by a constant, with the result that the constant drops out of the equation. Examples of linear, non-linear, homogeneous, and inhomogeneous PDE's are given in Table 3.1.1.

Eigenvalue problems arise in linear homogeneous PDE's for which the boundary conditions are also linear and homogeneous. In such cases it is obvious that one solution to the equation and boundary conditions is that the

Table 3.1.1

EXAMPLES OF PDE's OF DIFFERENT TYPES

<u>Linear, homogeneous</u>

(1) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + au = 0$$

(2) $$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + v = 0 \\ \dfrac{\partial^2 v}{\partial x^2} + u = 0 \end{cases}$$

<u>Linear, inhomogeneous</u>

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

(4) $$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + v = a \\ \dfrac{\partial^2 v}{\partial x^2} + u = b \end{cases}$$

<u>Non-linear</u>

(5) $$u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} = 0$$

(6) $$\frac{\partial^2 u}{\partial x^2} + u^2 = 0$$

Table 3.1.2

EXAMPLES OF BOUNDARY CONDITION TYPES

<u>Linear, homogeneous</u>

(1) $$u(0,t) = 0$$
$$u(L,t) = 0$$

(2) $$\begin{cases} u + \dfrac{\partial v}{\partial x} = 0 & \text{at } x = 0 \\ v + \dfrac{\partial u}{\partial t} = 0 & \text{at } x = L \end{cases}$$

<u>Linear, inhomogeneous</u>

(3) $$\begin{cases} u(0,t) = a \\ u(L,t) = b \end{cases}$$

(4) $$\begin{cases} u + \dfrac{\partial v}{\partial x} = a & \text{at } x = 0 \\ u + v = b & \text{at } x = L \end{cases}$$

<u>Non-linear</u>

(5) $$\begin{cases} \dfrac{\partial u}{\partial x} = au^3 & \text{at } x = 0 \\ \dfrac{\partial u}{\partial x} + u^2 = 0 & \text{at } x = L \end{cases}$$

the dependent variable is zero; this is called the trivial solution. If nontrivial solutions also exist, they are called eigensolutions; each is usually associated with a particular value of a parameter, called the eigenvalue. In vibration problems the eigenvalues are the natural frequencies of vibration; in nuclear reactors the eigenvalue is the critical mass. Table 3.1.2 gives some examples of homogeneous and inhomogeneous boundary conditions.

The general idea of SOV is to assume that the solution for the dependent variable (or variables) exists as a product of functions, each of which is a function of only one of the independent variables. For example, one would assume

$$u(x,y,t) = X(x) \cdot Y(y) \cdot T(t) \tag{3.1.1}$$

(The use of capital letters for the functions, and small letters for their arguments, is customary). Then, one manipulates with the equation to separate terms that depend upon each of the variable from one another. This leads to a situation where the equation requires that a function (not those above) of one independent variable must be equal to a function of another independent variable, for any arbitrary values of the two independent variables. The only way that this can be true is for the functions to be constants. This in turn leads one to ordinary differential equations for the assumed functions $X(x)$, $Y(y)$, etc., in which the separation constant appears. The separation constant, or eigenvalue, is then determined from the linear homogeneous boundary conditions. We will now illustrate this methodology by several examples.

3.2 Vibration of a String

The equation describing the small-amplitude motion of a taut string is the wave equation,

$$a^2 u_{xx} - u_{tt} = 0 \qquad (3.2.1)$$

Here u is the transverse deflection of the string (Figure 3.2.1), x is the coordinate along the string, t is time, and a^2 is a physical constant that depends on the string tension and mass per unit length.*

Let us suppose that the ends of the string are fixed, so that the boundary conditions are

$$u(0,t) = 0; \qquad u(L,t) = 0 \qquad (3.2.2a,b)$$

We will seek the solutions to (3.2.1) and (3.2.2) that can be obtained by SOV.

We assume

$$u(x,t) = X(x) \cdot T(t) \qquad (3.2.3)$$

Substituting in (3.2.1),

$$a^2 X''T - XT'' = 0 \qquad (3.2.4)$$

Here primes denote the derivatives of the functions with respect to their own arguments, i.e. $X'' = d^2X/dx^2$, $T'' = d^2T/dt^2$. The variables are separated by dividing by $X \cdot T$, which yields

$$a^2 \frac{X''}{X} = \frac{T''}{T} \qquad (3.2.5)$$

Since the left-hand side is a function of x alone, and the right-hand side depends upon t only, and x and t are independent variables that may have any values, (3.2.4) can only hold true if each side is (the same)

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*Students of mechanics should be able to derive this equation.

constant. We may call this constant anything we like (e.g. C, -C, C^2, α, $-\omega^2$, etc.) The "artistic" choice is $-\omega^2$, for reasons that will be clear very shortly. So we have

$$a^2 \frac{X''}{X} = \frac{T''}{T} = -\omega^2 \qquad (3.2.6)$$

Therefore, the SOV solution requires that the functions X and T satisfy the two ordinary differential equations

$$T'' + \omega^2 T = 0 \; ; \qquad a^2 X'' + \omega^2 X = 0 \qquad (3.2.7a,b)$$

The T equation (3.2.7a) has two linearly independent solutions sin (ωt) and cos (ωt). Thus, if ω turns out to be a real number, the solution will oscillate with frequency ω (rad/s). The expectation of this behavior is what prompted the choice of $-\omega^2$ for the separation constant. The general solution to the T equation is

$$T = A_1 \sin \omega t + A_2 \cos \omega t \qquad (3.2.8)$$

which, using trigonometric identities, may be recast as

$$T = A_3 \cos(\omega t - \phi) \qquad (3.2.9)$$

where ϕ is a phase angle. At this point A_3 , ϕ and ω are all unknown.

The X equation (3.2.7b) may be rewritten as

$$X'' + \lambda^2 X = 0 \qquad (3.2.10)$$

$$\lambda^2 = \omega^2/a^2$$

The general solution is

$$X = B_1 \sin(\lambda x) + B_2 \cos(\lambda x) \qquad (3.2.11)$$

Therefore, the SOV process has produced the following solution to the PDE:

$$u(x,t) = A_3 \cos(\omega t - \phi) [B_1 \sin(\lambda x) + B_2 \cos(\lambda x)] \tag{3.2.12}$$

Any set of values for A_3 , ϕ , B_1 , B_2 , and ω will produce a solution to the PDE. However, the boundary conditions limit the values which give solutions meeting the boundary conditions. Eq. (3.2.2a) requires that u be zero at $x = 0$ for all t. This can only be true if

$$X(0) = 0 \tag{3.2.13}$$

This in turn requires that $B_2 = 0$. Likewise, the condition on u at $x = L$ requires

$$X(L) = 0 \tag{3.2.14}$$

This requires that

$$B_1 \sin(\lambda L) = 0 \tag{3.2.15}$$

One possibility is $B_1 = 0$, but this will produce a trivial solution. The other possibility is

$$\sin(\lambda L) = 0 \tag{3.2.16}$$

This will be satisfied if λL is assigned any of the following values:

$$\lambda L = \pi, 2\pi, 3\pi, \ldots, n\pi \tag{3.2.17}$$

Associated with each of these possible choices, or eigenvalues, is an eigenfunction

$$X_n(x) = B_{1n} \sin(\lambda_n x)$$
$$\lambda_n = n\pi/L \tag{3.2.18}$$

Since $\omega = a\lambda$, for each n there is a corresponding frequency

$$\omega_n = a\lambda_n = n\pi a/L \tag{3.2.19}$$

We now have a complete description of the normal modes or eigenmodes for this problem. They are described by

$$\begin{aligned} u_n(x,t) &= A \sin(\lambda_n x) \cos(\omega_n t - \phi) \\ \omega_n &= n\pi a/L\, ; \quad \lambda_n = n\pi/L \end{aligned} \tag{3.2.20}$$

Note that the amplitude A cannot be determined; this is a consequence of the homogeneous feature problem. In addition, the phase ϕ is also undetermined. All we can determine are the <u>mode shape</u> $\sin(\lambda_n x)$ and the vibration frequency ω_n. The shapes of the first few modes are shown in Figure 3.2.1b. Note that the higher modes oscillate at higher frequencies, and have <u>nodal points</u> at which the string remains motionless.

Suppose that a guitarist had sufficient dexterity to pluck his strings in one of the normal mode shapes shown in Figure 3.2.1. At time zero the deflection would then exactly match the x-dependence of the eigensolution, and the string velocity at the moment of release $\partial u/\partial t$, would be zero. The condition $\partial u/\partial t = 0$ requires $\phi = 0$, and the amplitude of the pluck would determine A. Thus, if we add to the PDE and boundary conditions the <u>initial conditions</u>

$$u(x,0) = A \sin(\lambda_n x) \tag{3.2.21a}$$

$$\frac{\partial u}{\partial t} = 0 \quad \text{at} \quad t = 0 \tag{3.2.21b}$$

then the solution is fully determined as

$$\begin{aligned} u &= A \sin(\lambda_n x) \cos(\omega_n t) \\ \omega_n &= a\lambda_n \end{aligned} \tag{3.2.22}$$

3.3 One-Dimensional Acoustic Vibrations

The deviation of the fluid pressure from ambient during one-dimensional acoustic vibrations is also described by the wave equation,

$$c^2 \frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2} = 0 \tag{3.3.1}$$

where c is the speed of sound in the fluid. The lateral velocity u associated with acoustic motions is related to the pressure field by*

$$\rho \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial x} \tag{3.3.2}$$

where ρ is the fluid density.

At an open-end of a tube or duct the boundary condition $p = 0$ is a good approximation if the wavelength is long compared to the duct diameter. At a closed-end of the duct the fluid velocity u is zero, and hence the boundary condition $\partial p/\partial x = 0$ is appropriate. Figure 3.3.1 shows the boundary conditions of interest in a number of simple one-dimensional acoustic vibration problems.

As an example, let's take the case of a tube closed at $x = 0$ and open at $x = L$, and seek the normal modes of acoustic vibration for this case.** Thus, the boundary conditions are

$$\frac{\partial p}{\partial x} = 0 \quad \text{at} \quad x = 0 \tag{3.3.3a}$$

$$p = 0 \quad \text{at} \quad x = L \tag{3.3.3b}$$

---

*Acoustic waves correspond to motions of the fluid in the same direction as the wave propagation (lateral waves), while in the string vibration previously considered the string motion is perpendicular to the direction of the wave propagation (the string).

**These modes can be excited in a tall-slender bottle by blowing across the mouth.

Note that these are both linear homogeneous conditions. We assume

$$P = X(x) \cdot T(t) \tag{3.3.3}$$

Substituting in (3.3.1), and separating the variables, we find

$$c^2 \frac{X''}{X} = \frac{T''}{T} = -\omega^2 \tag{3.3.4}$$

where again we chose to call the separation constant $-\omega^2$ for the same reasons as in the previous example. Hence,

$$T'' + \omega^2 T = 0$$

$$X'' + \lambda^2 X = 0$$

where

$$\lambda = \omega/c \tag{3.3.5}$$

The general solution of the X equation is

$$X = C_1 \sin(\lambda x) + C_2 \cos(\lambda x) \tag{3.3.6}$$

The boundary conditions, which must be satisfied for all t, require that

$$X'(0) = 0 \qquad X(L) = 0 \tag{3.3.7a,b}$$

The first tells us that $C_1 = 0$. The second requires

$$C_2 \cos(\lambda L) = 0 \tag{3.3.8}$$

$C_2 = 0$ is one possibility, but this produces a trivial solution. So we conclude that λ must be such that

$$\cos(\lambda L) = 0 \tag{3.3.9}$$

which will be true if λL has any of the following values:

$$\lambda L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, (2n-1)\frac{\pi}{2}, \ldots \tag{3.3.10}$$

Hence, for the nth mode, $\lambda_n = (2n-1)\pi/(2L)$ and $X_n = C_2 \cos(\lambda_n x)$. The T equation has the general solution

$$T = A_1 \cos(\omega t) + A_2 \sin(\omega t) = A_3 \cos(\omega t - \phi) \quad (3.3.11)$$

So, the normal modes of acoustic oscillation are described by

$$\begin{aligned} p(x,t) &= A \cos(\lambda_n x) \cos(\omega_n t - \phi) \\ \lambda_n &= (2n-1)\pi/(2L) \\ \omega_n &= (2n-1)\pi c/(2L) \end{aligned} \quad (3.3.12)$$

Again, we find that the amplitude A and phase ϕ of the acoustic pressure field cannot be determined (without the addition of initial conditions). This is always a characteristic of normal-mode vibration problems.

We can now calculate the velocity field associated with each eigenmode, using (3.3.2);

$$\rho \frac{\partial u}{\partial t} = A\lambda_n \sin(\lambda_n x) \cos(\omega_n t - \phi) \quad (3.3.13)$$

Integrating and using the fact that $\omega_n = \lambda_n c$,

$$u(x,t) = \frac{A}{\rho c} \sin(\lambda_n x) \sin(\omega_n t - \phi) \quad (3.3.14)$$

(The constant of integration must be zero for the fluid to remain motionless at the closed end.) Note that the <u>nodes</u> of p, where the pressure fluctuations are always zero, are the <u>antinodes</u> of u (points where the u field has maximum amplitude).

Let's look briefly at a very simple application of this theory. Suppose you are trying to reduce the noise present in a long room in which the acoustic motions of this type occur to the bother of the occupants. One solution is to damp the motions by providing a fine fibrous material that will oppose the fluid motion through viscosity. Obviously the best place to locate this material is where the fluid velocity is greatest (a velocity antinode); the material would have little effect if placed at a velocity node*. This means that the material to quiet the room should be placed at a node in the pressure field, i.e., where the sound you are trying to kill can not be heard!

3.4 Membrane Vibration

The vibration of a taut membrane is described by

$$a^2\nabla^2 u - u_{tt} = 0 \tag{3.4.1}$$

In Cartesian coordinates the Laplace operator ∇^2 is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{3.4.2}$$

while in cylindrical coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{3.4.3}$$

The parameter a depends upon the membrane tension and density. The boundary condition is that the displacement u must be zero all around the edge of the membrane.

To study the normal modes, we assume

$$u(x,y,t) = F(x,y) \cdot T(t) \tag{3.4.4}$$

---

* In actual situations the acoustic field is much more complicated. However, it is in general true that placing the acoustic damping material away from solid walls, out in the room where the air can move through it, is most effective.

We have chosen not to split F as yet in order that the analysis apply to a variety of membrane geometries. Substituting in (3.4.1) and separating the variables,

$$a^2 \frac{\nabla^2 F}{F} = \frac{T''}{T} = -\omega^2 \qquad (3.4.5)$$

The choice of $-\omega^2$ as the separation constant was again dictated by our expectation that the time-solutions would be periodic. Thus, we have

$$T'' + \omega^2 T = 0 \qquad (3.4.6)$$

$$\nabla^2 F + \lambda^2 F = 0 \qquad (3.4.7)$$

$$\lambda^2 = \omega^2/a^2$$

The time solution again has the general form

$$T = A \cos(\omega t - \phi)$$

The eigenvalues (ω or λ) are determined by the solutions to the eigenvalue problem formed by (3.4.7), with the boundary condition $F = 0$ on C, where C denotes the outer edge of the membrane.

For a rectangular or circular membrane we can solve the F problem by SOV in appropriate coordinates. For other shapes the SOV method will not produce a useful solution, but in such cases it is possible to obtain the solution to the F problem by suitable numerical means.

Let's now consider the case of a square membrane (Fig. 3.4.1b). Our F equation is then

$$F_{xx} + F_{yy} + \lambda^2 F = 0 \qquad (3.4.8)$$

We assume

$$F = X(x) \cdot Y(y) \qquad (3.4.9)$$

Substituting in (3.4.8) and separating the variables,

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda^2 = -\alpha^2$$

We could have placed the λ^2 term either on the x-side or on the y-side; the same solution would be obtained in the end in either case. We chose to call the separation constant $-\alpha^2$ because this will produce an X-equation of

$$X'' + \alpha^2 X = 0 \qquad (3.4.10)$$

and hence X will be periodic in x and can go from zero at one side of the membrane to zero on the other, as the boundary conditions require.*

The Y-equation may be written as

$$Y'' + \beta^2 Y = 0 \quad \text{where} \quad \beta^2 = \lambda^2 - \alpha^2 \qquad (3.4.11)$$

Equation (3.4.10) has the general solution

$$X = C_1 \sin(\alpha x) + C_2 \cos(\alpha x)$$

The boundary conditions $F(0,y) = 0$, $F(L,y) = 0$ require

$$C_2 = 0 \ ; \quad C_1 \sin(\alpha L) = 0 \qquad (3.4.12a,b)$$

Hence the eigenvalues α_n are

$$\alpha_n = n\pi/L \qquad (3.4.13)$$

Similarly, the Y solution is

$$Y = C_3 \sin(\beta y) + C_4 \cos(\beta y) \qquad (3.4.14)$$

and the boundary conditions $F(x,0) = 0$, $F(L,0) = 0$ require

$$C_4 = 0 \ ; \quad C_3 \sin(\beta L) = 0 \qquad (3.4.15a,b)$$

---

* If instead we had chosen $+\alpha^2$, we would have ended up with an imaginary value for α, which is "inartistic".

Hence, there is another set of eigenvalues β_m,

$$\beta_m = m\pi/L \tag{3.4.16}$$

Thus, it takes <u>two</u> indices (n and m) to identify each eigensolution of the F problem. The solutions are

$$F_{nm} = A_{nm}\sin\left(\frac{n\pi x}{L}\right)\,\sin\left(\frac{m\pi y}{L}\right) \tag{3.4.17}$$

and the associated eigenvalues are

$$\lambda_{nm}^2 = \alpha_n^2 + \beta_m^2 = (n^2+m^2)\pi^2/L^2 \tag{3.4.18}$$

We can now express the frequency of each mode of oscillation using the relationship between ω and λ,

$$\omega_{nm} = a\sqrt{(n^2+m^2)}\;\pi/L \tag{3.4.19}$$

Note that the higher modes (larger n and m) oscillate at higher frequencies, and that modes having the same values of $n^2 + m^2$ will oscillate at the same frequency (but with different mode shapes). The <u>lowest</u> frequency of vibration, called the fundamental, is associated with the (1,1) mode, and is $\sqrt{2}\pi a/L$.

Figure 3.4.2 shows the first few mode shapes. Note that all except the fundamental mode have <u>nodal lines</u> along which the membrane remains motionless. These lines divide the membrane into regions within which, at any instant of time, the membrane is moving in the same direction, and across which the direction of motion changes.

3.5 Circular Membrane -- Bessel Functions

Let's now examine the case of the vibration of a circular membrane (Fig. 3.5.1), which will serve to introduce us to some new special functions. In cylindrical coordinates (3.4.7) is

$$F_{rr} + \frac{1}{r} F_r + \frac{1}{r^2} F_{\theta\theta} + \lambda^2 F = 0 \tag{3.5.1}$$

We assume

$$F = R(r) \cdot \Theta(\theta) \tag{3.5.2}$$

Substituting in (3.5.1) and separating the variables,

$$\frac{r^2 R'' + rR' + \lambda^2 r^2 R}{R} = -\frac{\Theta''}{\Theta} = \alpha^2 \tag{3.5.3}$$

Note that we had to multiply by $r^2/R\Theta$ to effect the separation. The decision to call the separation constant $+\alpha^2$ in this case was dictated by the realization that the solution must be periodic in θ. Thus we have

$$\Theta'' + \alpha^2 \Theta = 0 \tag{3.5.4}$$

$$r^2 R'' + rR' + (\lambda^2 r^2 - \alpha^2) R = 0 \tag{3.5.5}$$

The Θ solution will be written as

$$\Theta = B \cos(\alpha\theta - \psi)$$

where ψ is a phase angle constant. Again, we can not determine B or ψ without initial conditions. However, we can determine the possible values of α. The argument used in the solution can not be double-valued, i.e.,

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$

This can only be true if α is an integer, so

$$\alpha_n = n \tag{3.5.6}$$

The R-equation (3.5.5) has solutions in terms of <u>Bessel functions</u> (see HMF 9.1). If we let $z = \lambda r$, (3.5.5) becomes

$$z^2 \frac{d^2R}{dz^2} + z \frac{dR}{dz} + (z^2 - \alpha^2) R = 0 \tag{3.5.7}$$

The general solution consists of two linearly independent Bessel functions and may be expressed as

$$R = C_1 J_\alpha(z) + C_2 Y_\alpha(z) \qquad (3.5.8)$$

where the functions $J_\alpha(z)$ and $Y_\alpha(z)$ are called the Bessel functions of the first and second kinds, respectively, of order α. These functions are no different in concept from the $\sin(z)$ and $\cos(z)$ functions that satisfy the ODE $d^2R/dz^2 + R = 0$. Power series can be found for the Bessel functions using standard methods for developing series solutions of ODE's, just as one can for the trigonometric functions. If you have not encountered them before, think of them as slightly complicated functions which can be computed, tabulated, looked up, or called in FORTRAN programs, just as those functions with which you are already familiar.

Just as one takes linear combinations of e^z and e^{-z} to define new functions $\sinh(z)$ and $\cosh(z)$, one can define other functions as linear combinations of J and Y. Therefore, in the literature one may encounter the Hankel functions of the first and second kinds, $H_\alpha^{(1)}(z)$ and $H_\alpha^{(2)}(z)$, which are particular linear combinations of J and Y. Hence, other valid forms of the solution to (3.5.7) are

$$R = C_3 H_\alpha^{(1)}(z) + C_4 H_\alpha^{(2)}(z) \qquad (3.5.9a)$$

$$R = C_5 J_\alpha(z) + C_6 H_\alpha^{(1)}(z) \qquad (3.5.9b)$$

etc. In some literature Y_α is denoted by N_α.

Note that (3.5.7) is <u>even</u> in the parameter α. If one replaces α by $-\alpha$ in the series expansion for $J_\alpha(z)$, one obtains another Bessel function, $J_{-\alpha}(z)$. If α is <u>not</u> an integer, then $J_\alpha(z)$ and $J_{-\alpha}(z)$ are linearly independent, and an acceptable general solution to (3.5.7) is

$$R = C_7 J_\alpha(z) + C_8 J_{-\alpha}(z) \quad (\alpha \text{ non-integer}) \qquad (3.5.9c)$$

However, if α is an integer, then $J_\alpha(z)$ and $J_{-\alpha}(z)$ are the same function (except for a constant factor) and hence are not linearly independent and (3.5.9c) is not the general solution for integer α. Eqn. (3.5.8) is the usual representation of the general solution in the modern literature, and we shall use it for the remainder of the analysis. Fig. 3.5.2 shows the form of $J_\alpha(z)$ and $Y_\alpha(z)$.

The Bessel functions $Y_\alpha(z)$, $J_{-\alpha}(z)$, and $H_\alpha^{(2)}(z)$ have one important property in common: they all are infinite at $z = 0$. Therefore, these functions cannot appear in the description of the deflection of a continuous membrane that includes the point $z = 0$ (although they would appear in the solution for an annular membrane). The functions $J_\alpha(z)$ and $H_\alpha^{(1)}(z)$ are well-behaved at $z = 0$ and present no problem. Hence, for our circular membrane we must take $C_2 = 0$, and with $\alpha = n$ our solution reduces to

$$R = C_1 J_n(\lambda r) \qquad (3.5.10)$$

The remaining boundary condition will determine the eigenvalue λ. Since the deflection must vanish at $r = r_o$ for all θ, R must be zero at $r = r_o$.

$$C_1 J_n(\lambda r_o) = 0 \qquad (3.5.11)$$

Looking at Fig. 3.5.2, we see that there are indeed points at which the Bessel function $J_n(z)$ is zero. These points are given in HMF Table 9.5, where $j_{n,m}$ is the value of z at which $J_n(z)$ has its m^{th} zero. Hence, it again takes two indices to identify the eigenfunction and eigenvalue,

$$\lambda_{nm} = j_{n,m}/r_o \qquad (3.5.12)$$

Recalling the relationship between λ and the frequency (see 3.4.7)), the frequency of vibration of the n,m mode is seen to be

$$\omega_{nm} = a j_{n,m}/r_o \qquad (3.5.13)$$

Table 3.5.1 gives the first few values of these frequencies in dimensionless form. The solution for the membrane displacement u_{nm} in the vibration eigenmode n,m is then

$$u_{nm}(r,\theta,t) = A_{nm} J_n(\lambda_{nm} r) \cos(n\theta - \psi) \cos(\omega_{nm} t - \phi)$$

$$\lambda_{nm} = j_{n,m}/r_o \qquad \omega_{nm} = \lambda_{nm} a \tag{3.5.14}$$

The phase angles ϕ and ψ, and the amplitude A remain undetermined.

The lowest frequency occurs for the 0,1 mode. Note that for $n = 0$ the motion is axisymmetric, and has no nodes. The next higher frequency occurs for the 1,1 mode. This mode has one diametral node along which the

TABLE 3.5.1

DIMENSIONLESS MEMBRANE FREQUENCIES

n	m	$j_{n,m} = \omega_{nm} r_o / a$
0	1	2.40483
1	1	3.83171
2	1	5.13562
0	2	5.52008
3	1	6.38016
1	2	7.01559
4	1	7.58834

membrane does not move. (The phase angle of this node cannot be determined without initial conditions). The third mode is the 2,1 mode, which has two diametral nodes, and the fourth is the 0,2 mode, with one circular node at the point where $J_0(\lambda_{02} r) = 0$, i.e. at $\lambda_{02} r = j_{0,1} = 2.40483$. Figure 3.5.3 shows the nodal lines for the first several modes.

3.6 Nuclear Reactor Criticality

A very simple but conceptually useful model for the neutron density in a nuclear reactor is that the neutron density ϕ is described by

$$\nabla^2\phi + \mu^2\phi = 0 \tag{3.6.1a}$$

where

$$\nabla^2\phi = \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz} \tag{3.6.1b}$$

The parameter μ^2 depends upon the reactor size and design and the position of the control rods. For a cylindrical reactor, the boundary conditions are (Fig. 3.6.1)

$$\phi_z = 0 \quad \text{at} \quad z = 0 \tag{3.6.2}$$

$$-\phi_r = \beta\phi \quad \text{at} \quad r = r_0 \tag{3.6.3}$$

$$-\phi_z = \beta\phi \quad \text{at} \quad z = L/2 \tag{3.6.4}$$

Equation (3.6.2) is a symmetry condition; (3.6.3) and (3.6.4) equate the neutron diffusive flux at the outer surface to the diffusive loss through shielding. This model is far too simple to be useful in reactor design. Nevertheless, it does display many of the features of more complex models that are solved by any heavy numerical analysis in actual reactor design. The example will also serve to introduce some aspects of Bessel functions and graphical solutions of transcendental algebraic equations. Equations (3.6.2) - (3.6.4) are <u>linear</u> and <u>homogeneous</u>. It is clear that $\phi = 0$ is one solution. For small μ (small reactor volume) it is the only solution, but as μ is increased the point is reached at which a non-trivial solution becomes possible. This non-trivial solution, an <u>eigensolution</u> of the linear homogeneous problem, represents the neutron density field for steady-state reactor operation. The lowest value of μ

(lowest eigenvalue) determines the <u>critical mass</u> of the reactor.* The objective of our analysis is therefore to calculate μ.

We will seek axisymmetric eigensolutions of the form

$$\phi(r,z) = R(r) \cdot Z(z) \tag{3.6.5}$$

Substituting in (3.6.1), and separating the variables,

$$\frac{R'' + \frac{1}{r}R'}{R} = -\left(\frac{Z''}{Z} + \mu^2\right) = -\alpha^2 \tag{3.6.6}$$

The separation constant was called $-\alpha^2$ because this will lead to positive, real α. So,

$$Z'' + \gamma^2 Z = 0 \tag{3.6.7}$$

where

$$\gamma^2 = \mu^2 - \alpha^2 \tag{3.6.8}$$

and

$$R'' + \frac{1}{r}R' + \alpha^2 R = 0 \tag{3.6.9}$$

The Z solution is

$$Z = C_1 \sin(\gamma Z) + C_2 \cos(\gamma Z) \tag{3.6.10}$$

But $C_1 = 0$ by (3.6.2), then since (3.6.4) requires

$$Z' + \beta Z = 0 \quad \text{at} \quad z = L/2 \tag{3.6.11}$$

---

*The transient behavior would be governed by

$$\alpha \frac{\partial \phi}{\partial t} = \nabla^2 \phi + \mu^2 \phi \tag{3.6.X1}$$

If $\mu < \mu_c$, $\frac{\partial \phi}{\partial t} < 0$ and the reactor shuts down. If $\mu > \mu_c$, $\frac{\partial \phi}{\partial t} > 0$ and the neutron population builds up. The reactor control system increases μ to slightly above μ_c, then allows the neutron population to build up to the desired operating level (reactor power), and then resets μ to μ_c to hold a steady-state critical condition.

it follows that

$$-\gamma \sin(\gamma L/2) + \beta \cos(\gamma L/2) = 0 \qquad (3.6.12)$$

or

$$\tan(\gamma L/2) = \left(\frac{\beta L}{2}\right)\left(\frac{2}{\gamma L}\right) \qquad (3.6.13)$$

This defines an eigenvalue problem for γ. Figure 3.6.2 shows how the eigenvalue γ can be determined graphically. This plot would be very useful in helping one structure a computer program to calculate the eigenvalue(s).

The R equation general solution is

$$R = B_1 J_0(\alpha r) + B_2 Y_0(\alpha r) \qquad (3.6.14)$$

But $B_2 = 0$ since $Y_0(0) = -\infty$. Then, (3.6.3) require

$$R'(r_0) + \beta R(r_0) = 0 \qquad (3.6.15)$$

Now (see HMF 9.1.28)

$$J_0'(x) = -J_1(x) \qquad (3.6.16)$$

So (3.6.14) gives

$$B_1[-J_1(\alpha r_0) \cdot \alpha + \beta J_0(\alpha r_0)] = 0 \qquad (3.6.17)$$

$B_1 = 0$ produces a trivial solution. Therefore, we require the term in brackets to be zero. This determines the eigenvalue α. HMF Table 9.7 gives the roots of

$$-\lambda J_0(x) + xJ_1(x) = 0 \qquad (3.6.18)$$

Hence, if we multiply (3.6.17) by r_0 and set $\beta r_0 = \lambda$ and $\alpha r_0 = x$, we can use HMF Table 9.7 to determine αr_0. For example, if $\beta r_0 = 0.2$, $\alpha r_0 = 0.6170$. Finally, we know that $\mu^2 = \alpha^2 + \gamma^2$, and hence can calculate μ from the known values of α and γ.

The neutron density distribution is then

$$\phi = A \cos(\gamma z) J_0(\alpha r)$$

The amplitude A cannot be determined. It depends on the reactor thermal power output, and will increase as the reactor power increases.

3.7 Summary

We have seen that the eigensolution to linear, homogeneous PDE's subjected to linear, homogeneous boundary conditions, can be obtained by SOV. The general approach is as follows:

1. assume the solution in SOV form;
2. separate the variables and define the separation constants,
3. invoke the boundary conditions, first to simplify the solution and then finally to determine the separation constants (eigenvalues).

The amplitude of the eigenfunctions cannot be determined, because of the homogeneity of the boundary conditions.

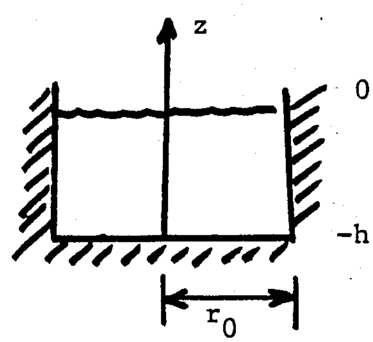
c. Show that there are no modes where

$p = p(\theta,t)$.

d. Find the eigenmodes and frequencies for the general case where

$p = p(r,\theta,z,t)$

3.4 In the analysis of seismic loading on nuclear reactors, oil storage tanks and other large fluid containers, one needs to know the natural frequencies of sloshing motions. This problem will acquaint you with the typical analysis.



Consider a circular geometry, with vertical walls at $r = r_0$, and the bottom at $z = -h$. The equations governing the sloshing are

(1) $\nabla^2\phi = \Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta} + \Phi_{zz} = 0$

(2) $\frac{\partial\Phi}{\partial t} + g\eta = 0$ on $z = 0$

(3) $\frac{\partial\eta}{\partial t} - \frac{\partial\Phi}{\partial z} = 0$ on $z = 0$

(4) $\frac{\partial\Phi}{\partial r} = 0$ at $r = r_0$

(5) $\frac{\partial\Phi}{\partial z} = 0$ at $z = -h$

$\Phi(r,\theta,z,t)$ is the velocity potential; the fluid velocity is the gradient of Φ; $\eta(r,\theta,t)$ is the surface displacement. g is the acceleration of gravity, $g = 9.8\text{m/sec}^2$. Equation (1) is the continuity equation for irrotational flow, (2) is the Bernouli equation applied on the free surface, (3) is a kinematic condition relating surface motion to velocity, and (4) and (5) are boundary conditions that the flow cannot penetrate the wall. Students with expertise in fluid mechanics should derive (1) - (5).

(a) Using the method of separation of variables, derive an expression for the natural frequencies. Express them non-dimensionally as

$$(6) \quad \Omega^2 \equiv \omega^2 r_0/g = f(h/r_0)$$

Express the solution for the surface deflection $\eta(r,\theta,t)$ in the non-dimensional form

$$(7) \quad \frac{\eta}{\eta_a} = F\left(\frac{r}{r_0}\right) G(\omega_{mn}t)\ H(m\theta)$$

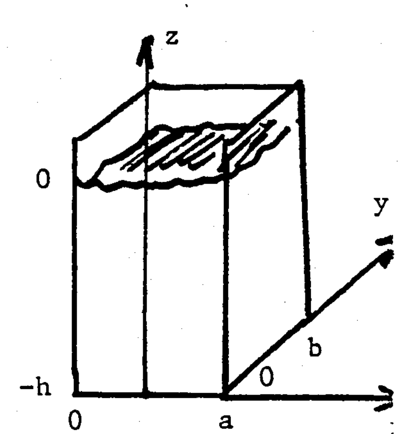
where η_a is the maximum deflection at $r = r_0$ (the sloshing amplitude)

(b) For the special case $h/r_0 = \infty$, calculate the values of Ω^2 for the modes having the five lowest natural frequencies, and sketch the node-lines in the surface displacement $\eta(r,\theta,t)$ for each of these modes. Check-point: the fundamental has $\Omega^2 = 1.841$. HINT: See HMF 9.1.1, 9.1.11, Table 9.5.

(c) Consider a large oil tank 30m in diameter, filled to a depth of 10m. Calculate the lowest natural frequency of vibration (hz).

(d) Find a coffee cup, jar, or other circular container. Fill with water to a selected depth, and manually excite the first mode by moving the container sideways. Compare the "measured" frequency (hz) with the value predicted by the analysis. Visualize the radial node-lines of part (d) in your cup by banging it (gently!) on the table.

3.5 Consider the sloshing of a fluid in a rectangular tank. The motion is described by the equations of Problem 3.4, except that



$$\nabla^2\phi = \phi_{xx} + \phi_{yy} + \phi_{zz}$$

and (4) is replaced by

$$\phi_x = 0 \quad \text{at} \quad x = 0,a$$

$$\phi_y = 0 \quad \text{at} \quad y = 0,b$$

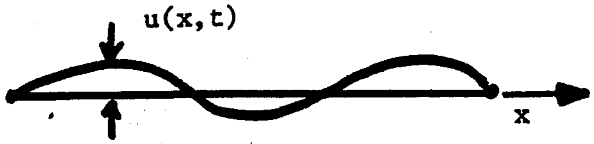
(a) Calculate the natural frequencies of fluid sloshing in the tank. Show that they are given by

$$\omega^2_{nm} = gk \tanh(kh) \qquad k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

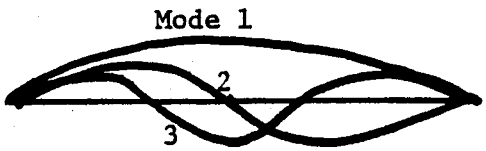
(b) Give the expression for $\eta_{nm}(x,y,t)$, apart from an undetermined phase and amplitude.

(c) Find a bathtub, wash-basin, or kitchen sink, fill with water to a reasonable depth. Manually excite the fundamental sloshing frequency and compare the theoretical value with an "eyeball" experimental measurement (hz).

FIGURE 3.2.1 VIBRATING STRING



(a) System sketch



(b) Mode Shapes

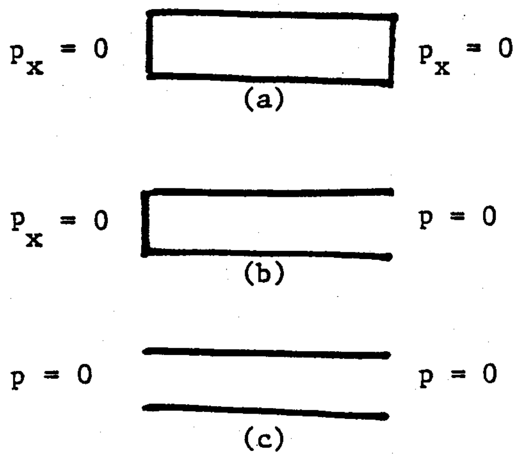
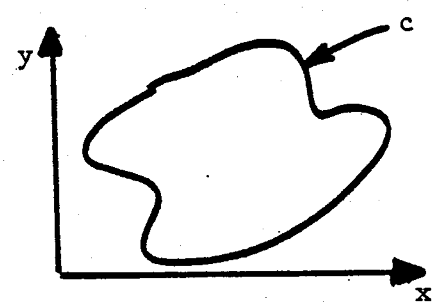


FIGURE 3.3.1 END BOUNDARY CONDITIONS

FIGURE 3.4.1a GENERAL MEMBRANE



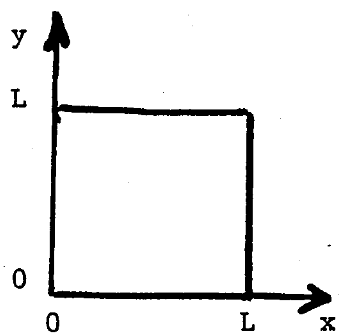


FIGURE 3.4.1b
SQUARE MEMBRANE

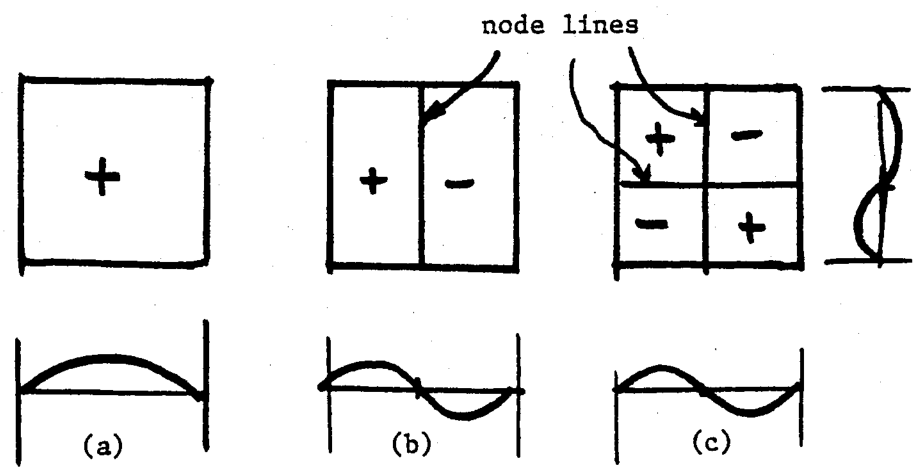


FIGURE 3.4.2 MODE SHAPES FOR THE SQUARE MEMBRANE

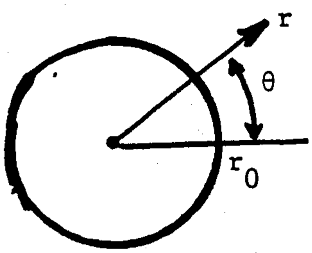


FIGURE 3.5.1 COORDINATES FOR CIRCULAR MEMBRANE ANALYSIS

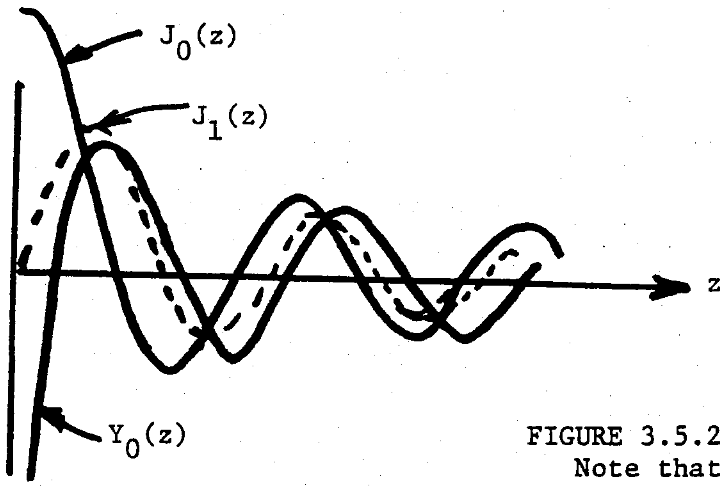


FIGURE 3.5.2 BESSEL FUNCTIONS
Note that $|Y_0|$ is infinite at $z = 0$

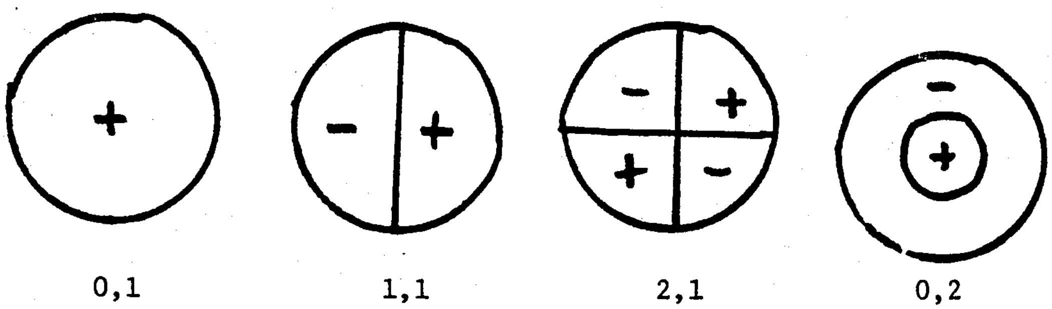


FIGURE 3.5.3 MODE SHAPES

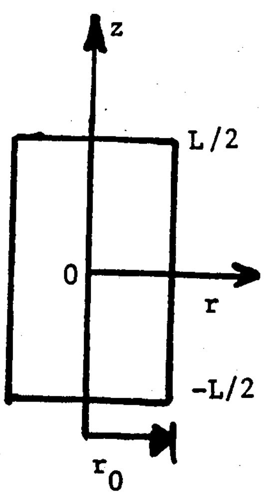


FIGURE 3.6.1

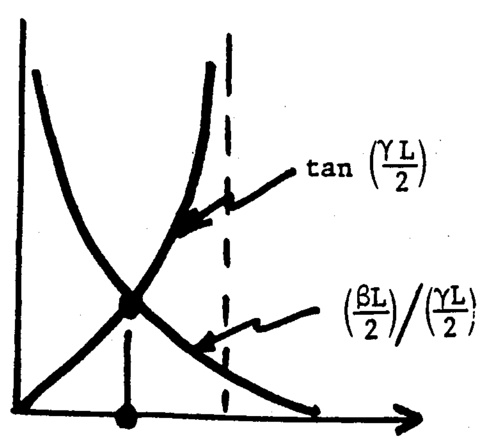


FIGURE 3.6.2 $\frac{\gamma L}{2}$

Chapter 4

EIGENFUNCTION EXPANSIONS IN LINEAR PROBLEMS

4.1 Introduction

In the previous chapter, we saw how linear, homogeneous problems led to eigenvalue problems when attacked by the method of SOV. For the problems examined there, the initial condition information needed to complete the problem formulation was missing. In this chapter, we will see that it is possible to combine the various eigensolutions of the "partial problem" formed by the homogeneous PDE and homogeneous BC's to generate the solutions to the complete problem.

Suppose a PDE is a linear, homogeneous equation; let's denote this equation by $L(u) = 0$. Then, if u_1 and u_2 are two functions that satisfy the equation, $L(u_1) = 0$ and $L(u_2) = 0$. It follows from the linearity and homogeneity that $u_3 = Au_1 + Bu_2$ also satisfies the equation, since

$$L(u_3) = L(Au_1 + Bu_2) = L(Au_1) + L(Bu_2)$$

$$= AL(u_1) + BL(u_2) = 0$$

So, we can take arbitrary linear combinations of functions satisfying the same linear homogeneous equation and thereby construct new functions satisfying the same equation. This is the property of linear homogeneous equations that enables us to use the eigensolutions as building blocks for more complex problems.

Let's illustrate the idea with a simple example. In the vibrating string problem (Section 3.2), we had the linear homogeneous problem

$$a^2 u_{xx} - u_{tt} = 0 \qquad (4.1.1a)$$

$$u(0,t) = 0 \qquad (4.1.1b)$$

$$u(L,t) = 0 \qquad (4.1.1c)$$

We can group the eigensolutions that we found into two classes:

$$u_n^{(1)} = \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a}{L} t\right) \tag{4.1.2a}$$

$$u_n^{(2)} = \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi a}{L} t\right) \tag{4.1.2b}$$

Eqn. (4.1.2a) represents those solutions that have $\partial u/\partial t = 0$ at $t = 0$, while (4.1.2b) gives the eigensolutions with $u(x,0) = 0$. Now, suppose that we pose a complete problem by adding to (4.1.1) the initial conditions

$$\partial u/\partial t = 0 \qquad \text{at} \quad t = 0 \tag{4.1.3a}$$

$$u(x,0) = 3 \sin\left(\frac{\pi x}{L}\right) - 2 \sin\left(\frac{2\pi x}{L}\right) \tag{4.1.3b}$$

Since all solutions (4.1.2a) satisfy both (4.1.1) and (4.1.3a), and since each of these equations is linear and homogeneous, the linear combination

$$u = 3u_1^{(1)} - 2u_2^{(1)}$$

$$= 3 \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi a}{L} t\right) - 2 \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi a}{L} t\right) \tag{4.1.4}$$

also satisfies (4.1.1) and (4.1.3a). Moreover, at $t = 0$ it matches precisely the inhomogeneous initial condition (4.1.3b). Hence, it is the solution to the complete problem formed by (4.1.1) and (4.1.3).

This example had such a simple initial condition that the proper mix of the eigensolutions could be found by inspection. But we would like to be able to deal with more complex situations, for example, the case where (4.1.3b) is replaced by

$$u(x,0) = f(x) \tag{4.1.5}$$

where $f(x)$ is <u>any</u> continuous function consistent with the end boundary conditions (4.1.1b) and (4.1.1c). You might correctly guess that this general case would require a mix of <u>all</u> of the eigensolutions $u_n^{(1)}$, such that

$$u(x,t) = \sum_{n=1}^{\infty} A_n u_n^{(1)}(x,t) \tag{4.1.6}$$

The question then is, how can the coefficients A_n be evaluated? Students who have encountered Fourier series will recognize that in this case the coefficients A_n are just the Fourier coefficients in the sine series for $f(x)$, since

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \tag{4.1.7}$$

However, suppose the eigenfunctions were not sinusoidal in x (as would be the case for a non-uniform string). Recalling that

$$u_n(x,t) = X_n(x) \cdot T_n(t) \tag{4.1.8}$$

since we can always normalize the eigensolution* such that $T_n(0) = 1$, the inhomogeneous initial condition would take the form

$$u(x,0) = \sum_{n=1}^{\infty} A_n X_n(x) = f(x) \tag{4.1.9}$$

The function X_n might be sines, or Bessel Functions, Legendre polynomials, or other special functions, depending upon the problem. The need to determine the coefficients A_n still remains, as we shall see, a property of the eigenfunctions $X_n(x)$, the orthogonality property, allows the A_n to be determined.

Thus, the solution to the complete problem can indeed be constructed as a linear combination of the solutions to the linear homogeneous partial problem.

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*Meaning those for which $T_n(0) \neq 0$, i.e., the $u_n^{(1)}$ functions in (4.1.2a).

4.2 The Sturm Liouville Problem

The key to determining the A_n's in an expansion solution is the eigenfunction orthogonality property, which can be established for each particular eigenvalue problem by mathematical analysis of the ODE's governing the eigenfunctions. Second-order ODE's arise in many problems; a reasonably general second order problem is the Sturm Liouville problem.

The Sturm-Liouville problem is the eigenvalue problem described in the linear homogeneous ODE

$$\frac{d}{dx}\left(S(x)\,\frac{dy}{dx}\right) + \left[Q(x) + \lambda^2\,P(x)\right]y = 0 \tag{4.2.1}$$

and the linear homogeneous boundary conditions

$$\alpha y + \beta y' = 0 \qquad \text{at} \quad x = a \tag{4.2.2a}$$

$$\gamma y + \delta y' = 0 \qquad \text{at} \quad x = b \tag{4.2.2b}$$

All of the second-order ODE's that appeared in Ch. 3 can be placed in this form; indeed, any second-order linear homogeneous ODE can be transformed to the form of (4.2.1).

In this section, we are going to explore the nature of the solutions to problems of the Sturm-Liouville class, and develop the orthogonality property of the Sturm-Liouville eigenfunctions. We will discuss questions of convergence of eigenfunction expansions, but will use heuristic arguments rather than formal mathematical proofs. Exacting proofs are available in books on advanced theory of ordinary differential equations.*

Let's begin by looking at a simplified form of (4.2.1),

$$y'' + \lambda^2\,P(x)y = 0 \tag{4.2.3}$$

$$P(x) > 0$$

with the boundary conditions

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*See, for example, Ince, E. L., Ordinary Differential Equations, Dover, New York, 1956.

$$y(0) = 0 \qquad y(L) = 0 \tag{4.2.4a,b}$$

If we momentarily replace (4.2.4b) by the inhomogeneous condition

$$y'(0) = 1 \tag{4.2.5}$$

then (4.2.3), (4.2.4a), and (4.2.5) precisely defines a function $y(x;\lambda)$ for any λ. Fig. 4.2.1 shows what these functions might look like for $P(x) > 0$ over $0 \leq x \leq L$. Note that y'' will change sign when $y = 0$. Hence, the solutions will oscillate as shown in the figure. The larger the value of λ, the larger will be y'' and for any given y, and hence large λ solutions will oscillate most rapidly.

Now, in this case the equation solutions $y(x;\lambda)$ do not necessarily satisfy the boundary condition (4.2.4b). However, for certain values of λ this boundary condition will be satisfied. The eigenfunction $y_n(x)$ is then

$$y_n = y(x;\lambda_n) \tag{4.2.6}$$

It is clear that for this case there is an infinite set of eigenvalues and no two eigenvalues are the same. Hence, we can think of ordering them such that

$$\lambda_1^2 < \lambda_2^2 < \lambda_3^2 < \lambda_4^2 \ \ldots \tag{4.2.7a}$$

$$\text{and} \quad \lambda_n^2 \to \infty \qquad \text{as} \quad n \to \infty \tag{4.2.7b}$$

In general, for (4.2.1) subject to (4.2.2) it may be shown that, if $S(x) > 0$ and $P(x) > 0$ over the problem domain $(a \leq x \leq b)$, then the eigenvalues are distinct and may be ordered as in (4.2.7).

We are now ready to develop the orthogonality property. Let y_n and y_m be two eigenfunctions associated with eigenvalues λ_n and λ_m, respectively. Then, y_n and y_m satisfy slightly different ODE's.

$$(Sy_n')' + \left[Q + \lambda_n^2 P\right] y_n = 0 \tag{4.2.8a}$$

$$(Sy_m')' + \left[Q + \lambda_m^2 P\right] y_m = 0 \tag{4.2.8b}$$

Now, multiplying (4.2.8a) by y_m and (4.2.8b) by y_n, subtracting the second equation from the first, and integrating, one obtains,

$$\int_a^b \left[y_m(Sy_n')' - y_n(Sy_m')'\right]dx = (\lambda_m^2 - \lambda_n^2) \int_a^b Py_ny_m dx \qquad (4.2.9)$$

Integrating the integral on the left by parts, the left-hand side becomes

$$y_mSy_n' - y_nSy_m' \Big|_a^b - \int_a^b (sy_n'y_m' - sy_m'y_n')dx$$

But the integrand is zero, hence (4.2.9) is

$$s(y_my_n' - y_ny_m')\Big|_a^b = (\lambda_m^2 - \lambda_n^2) \int_a^b Py_ny_m dx \qquad (4.2.10)$$

The boundary conditions at $x = a$ (4.2.2a) require

$$\left.\begin{aligned} \alpha y_n + \beta y_n' &= 0 \\ \alpha y_m + \beta y_m' &= 0 \end{aligned}\right\} \text{ at } x = a \qquad (4.2.11)$$

Thinking of this as a pair of linear homogeneous equations for α and β, which must have the problem values as a non-trivial solution, it follows that the determinant of the coefficients must vanish, or

$$(y_ny_m' - y_my_n') = 0 \quad \text{at } x = a \qquad (4.2.12)$$

A similar result is found at $x = b$. Hence, the left-hand side of (4.2.10) is exactly zero. Hence,

$$(\lambda_m^2 - \lambda_n^2) \int_a^b Py_ny_m dx = 0 \qquad (4.2.13)$$

Thus, if $\lambda_n^2 \neq \lambda_m^2$, as will usually be the case if $n \neq m$

$$\int_a^b Py_n y_m dx = 0 \qquad n \neq m \tag{4.2.14}$$

Eqn. (4.2.14) is the orthogonality property of the eigenfunctions. The eigenfunctions are said to be orthogonal with respect to the weight function $P(x)$.

Now, suppose that, in the course of trying to construct the solution to a PDE as a linear combination of eigensolutions of the linear, homogeneous partial problem, we are led to the point where we wish to determine the coefficients in an eigenfunction expansion,

$$f(x) = \sum_{n=1}^{\infty} A_n y_n(x) \tag{4.2.15}$$

where the y_n are eigensolutions of a Sturm-Liouville problem. Multiplying (4.2.15) by Py_m, and integrating over the problem domain,

$$\int_a^b fPy_m dx = \sum_{n=1}^{\infty} A_n \int_a^b Py_n y_m dx \tag{4.2.16}$$

But, because of the orthogonality property (4.2.14), all of the integrals on the right will drop out, except the one where $n = m$. Hence, we can immediately solve for A_m,

$$A_m = \frac{\int_a^b fPy_m dx}{\int_a^b Py_m^2 dx} \tag{4.2.17}$$

The infinite series (4.2.15) will be useless if it fails to converge to $f(x)$. In specific problems where one calculates the A_n it is easy to perform the standard tests for series convergence. It is somewhat more difficult to prove convergence in general. However, if f is square-integrable, i.e., if

$$\int_a^b Pf^2 dx \quad \text{is finite}$$

then the series converges in the sense that*

$$\lim_{N\to\infty} \int_a^b \left| f(x) - \sum_{n=1}^{N} A_n y_n(x) \right|^2 dx \rightarrow 0$$

This means that, if f is continuous over the interval $a \leq x \leq b$, the series converges uniformly (at all x). However, if f is discontinuous at some point, then the series will give a value at that point that is the average of the values of f at points infinitesimally above and below the point of discontinuity.

There are many problems of interest involving higher order system of linear homogeneous equations. In these cases, there are no theorems or general proofs of convergence of the eigenfunction expansions. One has to proceed by examining each case separately. However, problems arising from well-thought through physical formulations rarely, if ever, give rise to non-convergent expansions, so the analyst is usually safe in going ahead, assuming convergence, and then verifying it after the fact by ratio tests, numerical calculations, or other appropriate means.

4.3 Example - Vibrating String

For the vibrating string problem discussed in §4.1, the solution is given by (4.1.6). The coefficients A_n must be chosen such that (4.1.9) is satisfied. The eigenfunctions X_n are eigensolutions of

$$X_n'' + \lambda_n^2 X_n = 0 \tag{4.3.1}$$

and hence, from Sturm-Liouville theory, have the orthogonality property

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*See, for example, Ince, Ordinary Differential Equations, Dover, New York, 1956.

$$\int_0^L X_n X_m dx = 0 \qquad n \neq m \tag{4.3.2}$$

Recalling that

$$X_n = \sin\left(\frac{n\pi x}{L}\right) \tag{4.3.3}$$

we see that (4.3.2) is equivalent to

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0 \qquad n \neq m$$

which is indeed correct. So, to determine the A_n we multiply (4.1.9) by X_m and integrate,

$$A_m = \frac{\int_0^L f X_m dx}{\int_0^L X_m^2 dx} = \frac{\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{m\pi x}{L}\right) dx} \tag{4.3.4}$$

The integral in the denominator has the value $L/2$, for all m . Hence

$$A_m = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx \tag{4.3.5}$$

To be more specific, let's suppose that

$$f(x) = \begin{cases} 2\varepsilon\ x/L & x \leq L/2 \\ 2\varepsilon(1-x/L) & x \geq L/2 \end{cases} \tag{4.3.6}$$

This corresponds to an initial pluck in the center. Integrating, one finds

$$A_m = \begin{cases} \dfrac{4\varepsilon}{(m\pi)^2}(-1)^{(m+1)/2} & m \text{ odd} \\ 0 & m \text{ even} \end{cases} \tag{4.3.7}$$

Hence, the complete solution for (4.3.6) is

$$u(x,t) = \sum_{\substack{n=1 \\ (n\ \text{odd})}}^{\infty} \frac{4\varepsilon}{(n\pi)^2}(-1)^{(n+1)/2} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a}{L}\,t\right) \tag{4.3.8}$$

Noting that $A_n \sim 1/n^2$, we see that (4.3.8) is <u>absolutely convergent</u> for all x and t.

A real string would exhibit damping, not present in the mathematical model used here. Damping would cause the higher frequency modes to decay faster than the lower frequency modes, with the result that the fundamental (lowest frequency) mode would dominate after a period of time.

4.4 Example - Quenched Sphere

A metal sphere is heated to a uniform temperature T_o, then quenched by dunking in water. If we assume that the surface temperature is instantly dropped to $T = 0$ by the dunking, the temperature history in the sphere is described by the linear, homogeneous PDE

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{r^2}{\alpha}\frac{\partial T}{\partial t} \tag{4.4.1}$$

and the linear, homogeneous boundary condition

$$T(r_o,t) = 0 \tag{4.4.2}$$

and the linear, <u>in</u>homogeneous initial condition

$$T(r,0) = T_o \qquad r > 0 \tag{4.4.3}$$

This problem can be solved by the methods just described. The first step is to construct the eigensolutions of the homogeneous problem formed by the linear, homogeneous PDE and boundary condition (4.4.1) and (4.4.2). Then, a linear combination of these solutions will be taken to satisfy the inhomogeneous initial condition (4.4.3).

For the homogeneous problem, we look for eigensolutions in the form

$$T_n(r,t) = R(r) \cdot F(t) \tag{4.4.4}$$

Separating the variables,

$$\frac{(r^2R')'}{r^2R} = \frac{1}{\alpha}\frac{F'}{F} = -\lambda^2 \tag{4.4.5}$$

The decision to name the separation constant $-\lambda^2$ was dictated by the fact that the F equation then becomes

$$F' + \lambda^2\alpha F = 0 \tag{4.4.6}$$

which has the solution

$$F = C_1 e^{-\lambda^2\alpha t} \tag{4.4.7}$$

Thus, each of the eigensolutions will decay in time. The R equation is

$$(r^2R')' + \lambda^2r^2R = 0 \tag{4.4.8}$$

This equation has solutions in terms of the <u>spherical Bessel Functions</u> (HMF 10.1.1); the general solution is

$$R = C_2\, j_o(\lambda r) + C_3\, y_o(\lambda r) \qquad (4.4.9)$$

The functions $j_o(z)$ and $y_o(z)$ are related to the Bessel functions of order 1/2 , as shown in HMF §10.1. Now, $y_o(0)$ is infinite (see HMF 10.1.5), so $C_3 = 0$ is required. Since we are going to multiply the eigensolutions by expansion coefficients A_n when we construct the complete solution, we lose nothing by setting $C_1 \cdot C_2 = 1$, and hence take the eigensolutions $T_n(r,t)$ as

$$T_n(r,t) = j_o(\lambda_n r)\, e^{-\lambda_n^2 \alpha t} \qquad (4.4.10)$$

The eigenvalues λ_n are determined by the homogeneous boundary condition (4.4.2), which requires $R_n(r_o) = 0$. Hence, the condition

$$j_o(\lambda_n r_o) = 0 \qquad (4.4.11)$$

fixes the λ_n . HMF Table 10.6 gives the roots of this equation,

Each of the eigensolutions (4.4.10) satisfies the PDE and boundary condition, both of which are linear and homogeneous. Therefore, any arbitrary sum of the eigensolutions will also satisfy the PDE and boundary condition. So, we take

$$T(r,t) = \sum_{n=1}^{\infty} A_n\, T_n(r,t) \qquad (4.4.12)$$

and will try to find expansion coefficients A_n such that the infinite sum also satisfies the initial condition (4.4.3). Thus, the initial condition takes the form

$$T_o = \sum_{n=1}^{\infty} A_n\, j_o(\lambda_n r) = \sum_{n=1}^{\infty} A_n\, R_n(r) \qquad (4.4.13)$$

The orthogonality property, used to evaluate the A_n , may be developed by manipulations with the ODE (4.4.8). One multiplies the equation for R_n by R_m , the equation for R_m by R_n , subtracts, and integrates by parts. Alternatively, the result may be taken directly from Sturm-Liouville theory. In either approach one finds

$$\int_0^{r_o} r^2 R_n R_m \, dr = 0 \qquad \text{if} \quad n \neq m \tag{4.4.14}$$

So, multiplying (4.4.13) by $r^2 R_m$, and integrating, one finds

$$T_o \int_0^{r_o} r^2 R_m dr = \sum_{n=1}^{\infty} A_n \int_0^{r_o} r^2 R_n R_m dr = A_m \int_0^{r_o} r^2 R_m^2 dr \tag{4.4.15}$$

Note that the orthogonality property drops out all of the integrals except the one with $n = m$. Hence,

$$A_m = \frac{T_o \int_0^{r_o} r^2 R_m dr}{\int_0^{r_o} r^2 R_m^2 dr} = \frac{T_o I_1}{I_2} \tag{4.4.16}$$

Thus, once these integrals have been calculated, the solution will be completely known.

The integrals I_1 and I_2 can be evaluated using the differential equation (4.4.8). This avoids the need for explicit integration. Integrating (4.4.8), one finds

$$I_1 = \int_0^{r_o} r^2 R_n dr = -\frac{1}{\lambda_n^2} \int_0^{r_o} (r^2 R_n')' dr = -\frac{1}{\lambda_n^2} r_o^2 R_n'(r_o) \tag{4.4.17}$$

Thus, the integral can be evaluated in terms of the derivative of the eigenfunction at the boundary.

In order to evaluate I_2, a special trick is helpful. Let $R(r,\lambda)$ denote any solution to (4.4.8) that is finite at the origin (here $R = j_o(\lambda r)$). Note that $R(r,\lambda)$ satisfies (4.4.8) for all λ. Differentiating with respect to λ, we obtain

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial^2 R}{\partial r \partial \lambda}\right) + \lambda^2 r^2 \frac{\partial R}{\partial \lambda} + 2\lambda r^2 R = 0 \qquad (4.4.18)$$

We multiply (4.4.18) by R_n and integrate over the problem range, obtaining

$$\int_0^{r_o} R_n \left\{\frac{\partial}{\partial r}\left(r^2 \frac{\partial^2 R}{\partial r \partial \lambda}\right) + \lambda^2 r^2 \frac{\partial R}{\partial \lambda} + 2\lambda r^2 R \right\} dr = 0 \qquad (4.4.19)$$

Integrating the first integral by parts,

$$R_n r^2 \frac{\partial^2 R}{\partial r \partial \lambda}\Bigg|_0^{r_o} - \int_0^{r_o} r^2 \frac{\partial^2 R}{\partial r \partial \lambda} R_n' \, dr + \ldots = 0 \qquad (4.4.20)$$

The boundary terms drops at $r = 0$, and also at $r = r_o$ because $R_n(r_o) = 0$. Hence, only the integral remains. Integrating it again by parts, (4.4.20) becomes

$$- r^2 \frac{\partial R}{\partial \lambda} R_n' \Bigg|_0^{r_o} + \int_0^{r_o} \frac{\partial R}{\partial \lambda} \left\{ (r^2 R_n')' + \lambda^2 r^2 R_n \right\} dr + 2\lambda \int_0^{r_o} r^2 R \, R_n dr = 0$$

(4.4.21)

Now, the integrand of the first integral is zero everywhere (it contains the equation for R_n), and hence this term drops out. With $\lambda = \lambda_n$, $R(r,\lambda_n) = R_n(r)$, and hence the second integral is a constant times I_2; thus

$$I_2 = \int_0^{r_o} r^2 R_n^2 dr = \frac{1}{2\lambda_n} r_0^2 R_n'(r_o) \frac{\partial R}{\partial \lambda} \Bigg|_{\substack{r = r_o \\ \lambda = \lambda_n}} \tag{4.4.22}$$

Thus, we are able to evaluate the integral I_2 in terms of properties of the solution at the boundary points. This is very useful, especially in problems where we must resort to numerical solution of the eigenvalue ODE. Hence,

$$A_n = \frac{- T_o r_o^2 R_n'(r_o)}{\lambda_n^2} \cdot \frac{2\lambda_n}{r_o^2 R_n'(r_o)(\partial R/\partial\lambda)} \Bigg|_{\substack{\lambda = \lambda_n \\ r = r_o}}$$

$$= - \frac{2T_o}{\lambda_n(\partial R/\partial\lambda)} \Bigg|_{\substack{\lambda = \lambda_n \\ r = r_o}} \tag{4.4.23}$$

For the problem at hand, $j_o(z) = \sin(z)/z$ (see HMF 10.1.25). Hence, the eigenvalue-defining condition $R_n(r_o) = 0$ becomes

$$\frac{\sin(\lambda_n r_o)}{\lambda_n r_o} = 0 \tag{4.4.24}$$

and we see that the eigenvalues are given by the roots of $\sin(\lambda_n r_o)$,

$$\lambda_n r_o = n\pi \tag{4.4.25}$$

Moreover,

$$R(r,\lambda) = \frac{\sin(\lambda r)}{\lambda r} \tag{4.4.26}$$

$$\left.\frac{\partial R}{\partial \lambda}\right)_{\substack{\lambda = \lambda_n \\ r = r_o}} = \frac{1}{\lambda_n} \cos(\lambda_n r_o) \tag{4.4.27}$$

Noting that $\cos(\lambda_n r_o) = (-1)^n$, (4.2.23) gives

$$A_n = 2(-1)^{n+1} \; T_o \tag{4.4.28}$$

So, our final solution is, from (4.4.12),

$$T(r,t) = 2T_o \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n\pi r/r_o)}{n\pi r/r_o} e^{-n^2\pi^2\alpha t/r_o^2} \tag{4.4.29}$$

Note that the series converges for all t . The series for $\partial T/\partial r$, developed from (4.4.29) by differentiation, will converge for all $t > 0$ because of the exponential, but does not converge at $t = 0$. But this is not a serious limitation. As t increases the series converges more rapidly, and at large t the solution is given (approximately) by just the first term,

$$T \simeq 2T_o \frac{\sin(\pi r/r_o)}{\pi r/r_o} e^{-\pi^2\alpha t/r_o^2} \tag{4.4.30}$$

4.5 Sturm-Liouville Denominator Integral

In analyses, leading to the Sturm-Liouville problems, the orthogonality property will produce (4.2.17). The denominator integral may be expressed in terms of quantities evaluated at the boundary using a generalization of the trick employed in the previous example. Let $y(x,\lambda)$ be a solution to (4.2.1) not necessarily satisfying the boundary conditions (4.2.2). Then, $y(x,\lambda_n)$

will be an eigensolution satisfying the boundary conditions. We differentiate (4.2.1) with respect to λ , obtaining

$$\frac{\partial}{\partial x}\left(S\,\frac{\partial^2 y}{\partial x \partial \lambda}\right) + \left[Q + \lambda^2 P\right]\frac{\partial y}{\partial \lambda} + 2\lambda P y = 0 \qquad (4.5.1)$$

Next, we multiply (4.5.1) by y_n and integrate over the problem range,

$$\int_a^b y_n \left\{\frac{\partial}{\partial x}\left(S\,\frac{\partial^2 y}{\partial x \partial \lambda}\right) + \left[Q + \lambda^2 P\right]\frac{\partial y}{\partial \lambda} + 2\lambda P y\right\} dx = 0 \qquad (4.5.2)$$

The first integral is integrated twice by parts, and (4.5.2) becomes

$$y_n\, S\, \frac{\partial^2 y}{\partial x \partial \lambda}\Bigg|_a^b - \frac{\partial y}{\partial \lambda}\, y_n' S\Bigg|_a^b$$

$$+ \int_a^b \frac{\partial y}{\partial \lambda}\left\{(S y_n')' + \left[Q + \lambda^2 P\right] y_n\right\} dx + 2\lambda \int_a^b P y y_n\, dx = 0 \qquad (4.5.3)$$

Now, if we set $\lambda = \lambda_n$, the first integral drops out (because the integrand contains the y_n equation), and hence

$$\int_a^b P y_n^2 dx = \frac{1}{2\lambda_n}\left\{ y_n' S\, \frac{\partial y}{\partial \lambda}\Bigg|_a^b - y_n S\, \frac{\partial^2 y}{\partial x \partial \lambda}\Bigg|_a^b \right\}_{\lambda = \lambda_n} \qquad (4.5.4)$$

Thus, the denominator in A_n can be evaluated without recourse to integration.

4.6 Removal of Inhomogeneities in the PDE and BCs

In the previous problem, the PDE and BCs were homogeneous, and therefore eigensolutions of this homogeneous problem could be found. By taking a linear combination of these eigensolutions, we were able to construct a solution which satisfies an inhomogeneous initial condition. In many problems, the PDE and BCs also are inhomogeneous. In such cases, one must "remove" the inhomogeneities to form a homogeneous problem which can be attacked successfully by the methods presented above.

To illustrate the problem, suppose we are interested in the time-history of the diffusion of a contaminant in an annular system in which the contaminant is continually produced (Fig. 4.6.1). The governing PDE is

$$\frac{\partial}{\partial r}\left(r\,\frac{\partial c}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial c}{\partial t} - rs \tag{4.6.1}$$

Here c is the contaminant concentration (kg/m^3), α is the (constant) diffusivity of the contaminant, and s is a "source" term. Let's suppose that the outward diffusion is blocked at $r = r_o$ by a barrier, so that the boundary condition at $r = r_o$ is

$$\frac{\partial c}{\partial r} = 0 \qquad \text{at} \quad r = r_o \tag{4.6.2}$$

And, let's suppose that the contaminant is removed convectively at the inner radius r_i, so that the inner boundary condition is

$$h(c-c_\infty) = \mathcal{D}\,\frac{\partial c}{\partial r} \qquad \text{at} \quad r = r_i \tag{4.6.3}$$

Here h is the convective transport coefficient, c_∞ is the concentration (fixed) in the fluid passing through the annular hole, and $\mathcal{D}$ is the diffusion coefficient for c in the solid. Finally, let's suppose that initially $c = c_o$ throughout the solid,

$$c(r,0) = c_o \tag{4.6.4}$$

Eqs. (4.6.1) - (4.6.4) define the problem to be solved. (4.6.1) contains the inhomogeneous term rs ; (4.6.2) is homogeneous; (4.6.3) contains the inhomogeneous term hc_∞. If we could somehow remove these inhomogeneities, the

PDE and BCs would be linear and homogeneous, and we could find the eigensolutions and then combine them to satisfy the initial condition.

In transient problems such as this one, the inhomogeneities can usually be removed by use of the steady-state solution. Let $\psi(r)$ be a solution of (4.6.1), (4.6.2), and (4.6.3) that is independent of time (t). Hence, the steady-state solution $\psi(r)$ will satisfy

$$\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = -rs \tag{4.6.5}$$

$$\frac{\partial \psi}{\partial r} = 0 \quad \text{at} \quad r = r_o \tag{4.6.6}$$

$$h(\psi - c_\infty) = \mathcal{D}\frac{\partial \psi}{\partial r} \quad \text{at} \quad r = r_i \tag{4.6.7}$$

Now, if we put

$$c = \psi(r) + \phi(r,t) \tag{4.6.8}$$

then the transient function $\phi(r,t)$ will have to satisfy

$$\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial \phi}{\partial t} \tag{4.6.9}$$

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{at} \quad r = r_o \tag{4.6.10}$$

$$h\phi = \mathcal{D}\frac{\partial \phi}{\partial r} \quad \text{at} \quad r = r_i \tag{4.6.11}$$

Note that (4.6.9) - (4.6.11) are all linear and homogeneous in ϕ ; the inhomogeneities that appeared in the equations for c have been "removed". Hence, the ϕ problem can be attacked by separation of variables, and its eigensolutions found. The initial condition for the ϕ problem is then

$$\phi(r,0) = c(r,0) - \psi(r) = c_o - \psi(r) \tag{4.6.12}$$

Thus, the structure of the ϕ problem is the same as in the problems previously studied: homogeneous PDE and BCs, inhomogeneous initial condition.

The ψ problem is solved simply by integrating; the first integration yields

$$r\psi' = -\frac{r^2}{2}s + c_1 \tag{4.6.13}$$

but (4.6.6) requires that $c_1 = r_o^2 s/2$. Thus, the second integration yields

$$\psi = \frac{r_o^2 s}{2} \ln r - \frac{sr^2}{4} + c_2 \tag{4.6.14}$$

c_2 is found using (4.6.7) . The result is

$$\psi = c_\infty + \frac{s}{2}\left\{ r_o^2 \ln\left(\frac{r}{r_i}\right) + \frac{(r_i^2 - r^2)}{2} + \frac{\mathcal{D}}{h}\left(\frac{r_0^2}{r_i} - r_i\right)\right\} \tag{4.6.15}$$

Next, we attack the ϕ problem. We look for eigensolutions of the form

$$\phi_i = R(r) \cdot T(t) \tag{4.6.16}$$

Eqn. (4.6.9) produces

$$\frac{(rR')'}{rR} = \frac{1}{\alpha}\frac{T'}{T} = -\lambda^2 \tag{4.6.17}$$

Hence, the T equation is

$$T' + \lambda^2 \alpha T = 0 \tag{4.6.18}$$

and

$$T = \exp(-\lambda^2 \alpha t) \tag{4.6.19}$$

Since we expect the eigensolutions to decay with t (leaving the steady-state solution), the choice of $-\lambda^2$ as the separation constant was appropriate. The R equation is

$$(rR')' + \lambda^2 rR = 0 \qquad (4.6.20)$$

which has the solution (see HMF 9.1.1).

$$R = C_1 J_o(\lambda r) + C_2 Y_o(\lambda r) \qquad (4.6.21)$$

The boundary conditions (4.6.10) and (4.6.11) require

$$C_1 \lambda J_o'(\lambda r_o) + C_2 \lambda Y_o'(\lambda r_o) = 0 \qquad (4.6.22a)$$

$$C_1 \left[h J_o(\lambda r_i) - \mathcal{D}\lambda J_o'(\lambda r_i) \right] + C_2 \left[h Y_o(\lambda r_i) - \mathcal{D}\lambda Y_o'(\lambda r_i) \right] = 0 \qquad (4.6.22b)$$

Eqs. (4.6.22) are a pair of linear, homogeneous algebraic equations for C_1 and C_2. Non-trivial solutions can be obtained only if the determinant of the coefficients is zero,

$$D(\lambda) = \begin{vmatrix} \lambda J_o'(\lambda r_o) & \lambda Y_o'(\lambda r_o) \\ h J_o(\lambda r_i) - \mathcal{D}\lambda J_o'(\lambda r_i) & h Y_o(\lambda r_i) - \mathcal{D}\lambda Y_o'(\lambda r_i) \end{vmatrix} = 0 \qquad (4.6.23)$$

$D(\lambda)$ is called the "characteristic determinant" of the problem. The zeros of $D(\lambda)$ defines the eigenvalues λ_n. They could be determined by a suitable graphical or numerical routine, using Fig. 4.6.2 as a guide.

Now, the amplitude of the eigenfunctions can be anything, since they satisfy homogeneous equations. Therefore, we can arbitrarily scale the eigenfunctions in any way we like. The choice $C_1 = 1$ is convenient; with $C_{1n} = 1$, either of (4.6.22) will produce

$$C_{2n} = - J_o'(\lambda_n r_o) \Big/ Y_o'(\lambda_n r_o) \tag{4.6.24}$$

The eigenfunctions $R_n(r)$ are now completely known, and hence the $R_n(r)$ can be calculated.

Finally, we expand the solution for ϕ in terms of the eigensolutions,

$$\phi = \sum_{n=1}^{\infty} A_n \phi_n(r,t) = \sum_{n=1}^{\infty} A_n R_n(r) e^{-\lambda_n^2 \alpha t} \tag{4.6.25}$$

Evaluation of the A_n requires the orthogonality property. It is derived by multiplying the R_n equation by R_m, the R_m equation by R_n, subtracting the two equations, and integrating. After integrating by parts, using the boundary conditions, one finds

$$\int_{r_i}^{r_o} r R_n R_m \, dr = 0 \qquad n \neq m \tag{4.6.26}$$

We could have taken this directly from Sturm-Liouville Theory (4.2.14). Hence multiplying (4.6.25) by rRm, and integrating, one finds, for $t = 0$,

$$A_m = \int_{r_i}^{r_o} \phi(r,0) r R_m dr \Big/ \int_{r_i}^{r_o} r R_m^2 \, dr \tag{4.6.27}$$

Using the initial condition on ϕ, (4.6.12), the A_m can now be evaluated.

If we were doing this analysis completely, we would write up a computer program to evaluate the A_m and graph the solution for a range of r and t. Computer center libraries have routines that generate Bessel functions with the same ease as exponentials, sines, and other functions, so this would be a very easy task. The hardest part would be solving $D(\lambda) = 0$ for a large number of λ's. One would start by plotting $D(\lambda)$ vs. λ, which would give an idea of the structure of the problem, after which an "automatic" root-finder could be constructed.

We note that the denominator integral in (4.6.27) could be evaluated using the approach of the previous section. The numerator integral would involve

$$I_1 = \int_{r_i}^{r_o} rR_n dr \qquad I_2 = \int_{r_i}^{r_o} r^3 R_n dr \qquad I_3 = \int_{r_i}^{r_o} r \, \ell n(r) R_n dr$$

(4.6.28a,b,c)

I_1 can be found, in terms of boundary quantities, directly by integrating (4.6.20). I_2 can be found by multiplying (4.6.20) by r^2 and integrating:

$$\int_{r_i}^{r_o} r^2 (rR_n')' dr + \lambda_n^2 I_2 = 0 \qquad (4.6.29)$$

Integrating by parts, the first integral is

$$r^2 rR_n' \Big|_{r_i}^{r_o} - \int_{r_i}^{r_o} 2r \cdot rR_n' dr = r^3 R_n' \Big|_{r_i}^{r_o} - 2\left[r^2 R_n \Big|_{r_i}^{r_o} - 2 \int_{r_i}^{r_o} rR_n dr\right]$$

So, we can evaluate I_2 in terms of I_1 and boundary quantities. I_3 can also be found by multiplying (4.6.20) by $\ell n(r)$ and integrating,

$$\int_{r_i}^{r_o} \ell n(r)(rR_n')' dr + \lambda_n^2 I_3 = 0 \qquad (4.6.30)$$

Integrating by parts, the first integral is

$$\ell n(r) \cdot rR_n' \Big|_{r_i}^{r_o} - \int_{r_i}^{r_o} rR_n' \cdot \frac{1}{r} dr = r \, \ell n(r) \cdot R_n' \Big|_{r_i}^{r_o} - R_n \Big|_{r_i}^{r_o}$$

and hence I_3 can also be evaluated in terms of boundary quantities. Thus, the A_n can be found without recourse to any numerical integration! This is often the case; the key is always integration by parts.

4.7 Splitting

We have seen that problems with linear PDEs and BCs can be solved by constructing linear combinations of the eigensolutions for appropriate homogeneous partial problems. We also saw that in transient problems the inhomogeneities can be "removed" by "splitting" the solution into steady-state and transient parts. The concept of problem splitting can also be used to "remove inhomogeneities" in other problems.

To illustrate the idea, consider the problem shown in Fig. 4.7.1. The PDE is the inhomogeneous Laplace equation,

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = h(x,y) \qquad (4.7.1)$$

The domain is the rectangle shown, and the boundary conditions specify ϕ around the boundary, in terms of the functions shown. Note that all of these boundary conditions are inhomogeneous.

To use the methods developed in this chapter, we can "split" the problem into the five problems shown in Fig. 4.7.1. Problem (p) will take care of the inhomogeneity in the PDE. The solution $\phi^{(p)}$ is any particular solution of the PDE, without regard for boundary conditions. It will yield the values of $\phi^{(p)}$ on the boundaries denoted by the functions $g_1 - g_4$. We shall discuss means for finding the particular solution shortly. The four problems $\phi^{(1)} - \phi^{(4)}$ involve homogeneous PDEs and nearly completely homogeneous boundary conditions. Therefore, for each the eigensolutions of the homogeneous partial problem can be found, and then a linear combination of these eigenfunctions taken to construct a solution satisfying the remaining inhomogeneous boundary condition. Note that the sum

$$\phi = \phi^{(p)} + \sum_{k=1}^{4} \phi^{(k)} \qquad (4.7.2)$$

satisfies the inhomogeneous PDE and inhomogeneous boundary conditions. This type of splitting can, of course, only be done in linear problems.

Let's presume that we have the particular solution $\phi^{(p)}$, and are ready to solve problems $\phi^{(1)} - \phi^{(4)}$. We will do the $\phi^{(1)}$ problem; the other three are done in the same way.

The $\phi^{(1)}$ PDE is, dropping the superscript (1),

$$\phi_{xx} + \phi_{yy} = 0 \tag{4.7.3}$$

and the boundary conditions are

$$\phi = 0 \quad \text{on} \quad y = 0 \tag{4.7.4}$$

$$\phi = 0 \quad \text{on} \quad x = 0 \tag{4.7.5}$$

$$\phi = 0 \quad \text{on} \quad x = a \tag{4.7.6}$$

$$\phi = f_1(x) - g_1(x) = q(x) \quad \text{on} \quad y = b \tag{4.7.7}$$

We look for eigensolutions to the homogeneous partial problem (4.7.3) - (4.7.6) in the form

$$\phi = X(x)\, Y(y) \tag{4.7.8}$$

and, from (4.7.3), find

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2 \tag{4.7.9}$$

Hence,

$$X'' + \lambda^2 X = 0 \tag{4.7.10}$$

$$Y'' - \lambda^2 Y = 0 \tag{4.7.11}$$

The decision to name the separation constant $-\lambda^2$ was dictated by the recognition that the X-solutions must oscillate in X in order to match the boundary conditions. The X solution is

$$X = C_1 \sin(\lambda x) + C_2 \cos(\lambda x) \tag{4.7.12}$$

The BC (4.7.5) gives $C_2 = 0$. Then, the BC (4.7.6) requires $\sin(\lambda a) = 0$. Hence,

$$\lambda_n a = n\pi \tag{4.7.13}$$

The Y equation solution is

$$Y = C_3 \sinh(\lambda y) + C_4 \cosh(\lambda y) \tag{4.7.14}$$

The BC (4.7.4) requires $C_4 = 0$. Hence, the eigensolutions are (apart from a scaling constant)

$$\phi_n(x,y) = \sin(n\pi x/a) \sinh(n\pi y/a) \tag{4.7.15}$$

Finally, we seek the solution satisfying the inhomogeneous condition (4.7.7) as an expansion in the eigenfunctions,

$$\phi = \sum_{n=1}^{\infty} A_n \phi_n \tag{4.7.16}$$

Thus, at $y = b$,

$$\phi(b,x) = q(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh(n\pi b/a) \tag{4.7.17}$$

The orthogonality property for the X_n eigenfunctions is*

$$\int_0^a X_n X_m dx = 0 \qquad n \neq m \tag{4.7.18}$$

*Developed in the usual way.

So, multiplying (4.7.17) by $\sin(m\pi x/a)$, and integrating

$$A_m = \frac{\int_0^a q(x)\ \sin(m\pi x/a)}{\sinh(m\pi b/a)\int_0^a \sin^2(m\pi x/a)dx} \tag{4.7.19}$$

Given $q(x)$, we could compute the A_n. Hence, the $\phi^{(1)}$ solution is completely known.

The $\phi^{(2)}$, $\phi^{(3)}$, and $\phi^{(4)}$ problems could be handled in much the same way. In the $\phi^{(3)}$ problem, the Y equations would again be (4.7.11), and $Y(b) = 0$. Hence, rather than (4.7.14), a "more artistic" form of the Y solution is

$$Y = C_5 \sinh[\lambda(y-b)] + C_6 \cosh[\lambda(y-b)] \tag{4.7.20}$$

because C_6 will have to be zero for $Y(b) = 0$.

Let's now discuss the particular solution. If h depends upon only one of the independent variables, say x, the particular solution may be developed by assuming

$$\phi^{(p)} = F(x) \tag{4.7.21}$$

The inhomogeneous PDE is then

$$F'' = h(x) \tag{4.7.22}$$

which has the solution (by double integration)

$$F = \int_0^x \int_0^\xi h(\sigma)\ d\sigma\ d\xi \tag{4.7.23}$$

If $h = h(x,y)$, the particular solution can be obtained by expanding h in a Fourier series in either x or y. If we choose to do it in x, we would write

$$h(x,y) = \sum_{n=o}^{\infty} a_n(y)\cos(2n\pi x/a) + \sum_{n=1}^{\infty} b_n(y)\sin(2n\pi x/a) \tag{4.7.24}$$

The coefficients a_n and b_n are determined using the orthogonality property of the sine and cosine functions;

$$a_o = \frac{1}{a}\int_0^a h\,dx \tag{4.7.25a}$$

$$a_m = \frac{2}{a}\int_0^a h\cos(2m\pi x/a)dx \tag{4.7.25b}$$

$$b_m = \frac{2}{a}\int_0^a h\sin(2m\pi x/a)dx \tag{4.7.25c}$$

Next, one would look for a particular solution in the form

$$\phi^{(p)} = \sum_{n=o}^{\infty} F_n(y)\cos(2n\pi x/a) + \sum_{n=1}^{\infty} G_n(y)\sin(2n\pi x/a) \tag{4.7.26}$$

Substituting into the PDE, and equating coefficients of the sines and cosines, one finds

$$F_o'' = a_o \tag{4.7.27a}$$

$$F_n'' - \left(\frac{2n\pi}{a}\right)^2 F_n = a_n \tag{4.7.27b}$$

$$G_n'' - \left(\frac{2n\pi}{a}\right)^2 G_n = b_n \tag{4.7.27c}$$

Particular solutions to these three ODEs can be obtained by standard methods (e.g., the method of separation of variables).

In this problem, the corners would be singular points. The series solutions would converge everywhere, except at the corners, where the solutions $\phi^{(1)} - \phi^{(4)}$ would all be zero because of the method of solution.

4.8 Some Generalizations

While some problems fall into the Sturm-Liouville form, others do not. However, the same general ideas can be used with the help of a new concept, adjoint operations.

Suppose that the SOV process in a linear, homogeneous PDE problem produces the ODE

$$Lu = Mu + \lambda Nu = 0 \tag{4.8.1}$$

where L, M, and N are linear operators. Suppose that the linear, homogeneous boundary conditions are a set of equations of the form

$$\{B_i u = 0\} \quad \text{at} \quad x = a \text{ or } b \tag{4.8.2}$$

where the B_i are also linear operators. The eigenvalues λ are those values for which non-trivial solutions to (4.8.1), and (4.8.2) exist. The adjoint operators L^*, M^*, N^*, and B_i^* are defined by the requirement that

$$\int_a^b vLudx = \int_a^b uL^*vdx \tag{4.8.3}$$

where $a \leq x \leq b$ is the range of the problem. Thus, the adjoint operators are identified by integration by parts. The adjoint ODE, $L^*v = 0$, will be uniquely determined, but the adjoint boundary condition set $\{B_i^* v = 0\}$ is not always unique.

For example, suppose that the ODE is

$$Lu = u'''' + f(x)u'' + \lambda^2 g(x)u = 0 \tag{4.8.4}$$

and the boundary conditions are

$$u(a) = u'(a) = u(b) = u'(b) = 0 \tag{4.8.5}$$

To identify the adjoint operators, we multiply (4.8.4) by v, and integrate

$$\int_a^b vLudx = \int_a^b v\left[u'''' + fu'' + \lambda^2 gu\right] dx = 0 \tag{4.8.6}$$

Integrating by parts (several times) to transfer the differentiation from u to v, one finds

$$\int_a^b vLudx = (vu''' - v'u'' + v''u' - v'''u)\Big|_a^b$$

$$+ \left[vfu' - u(vf)'\right]_a^b + \int_a^b u\left[v'''' + (fv)'' + \lambda^2 gv\right] dx = 0$$

(4.7.7)

Now, (4.8.5) drops out almost all of the boundary terms. The remaining boundary terms will drop if we choose

$$v(a) = v'(a) = v(b) = v'(b) = 0 \tag{4.7.8}$$

These, then, are the adjoint boundary conditions. In this problem, they are the same as the basic boundary conditions (4.8.5). The adjoint ODE is then

$$L^*v = v'''' + (fv)'' + \lambda^2 gv = 0 \tag{4.8.9}$$

Note that, if f is not constant, (4.8.9) is different than the basic ODE (4.8.4).

If the adjoint ODE and adjoint boundary conditions are the same as those of the basic problem, the problem is called self-adjoint. The Sturm-Liouville problem is self-adjoint, which is why we did not require the concept of the adjoint in our earlier examples.

The adjoint problem is also linear and homogeneous. It will have solutions only for particular values of λ .

It is easy to argue that the eigenvalues λ of the adjoint problem are identical with those of the basic problem. Let $v(x,\lambda)$ be a solution of the adjoint equation $L^*v = 0$ that satisfies all but one of the adjoint boundary conditions $\{B_i^*v = 0\}$. Suppose that λ is an eigenvalue of the basic problem, associated with eigenfunction u . Hence, integration by parts gives

$$\int_a^b vLudx = \int_a^b uL^*vdx + \text{boundary terms} \tag{4.8.10}$$

But, the integrals are both zero, and therefore the boundary terms must vanish. Hence, v must also satisfy the one remaining boundary condition; therefore v is an eigensolution of the adjoint problem, with eigenvalue λ .

The orthogonality property of the eigenfunctions is derived by multiplying the equation for u_n by v_m , the equation for v_m by u_n , subtracting and integrating (by parts, of course!). Thus,

$$\int_a^b \left\{ v_m(Mu_n + \lambda_n Nu_n) - u_n(M^*v_m + \lambda_m N^*v_m) \right\} dx = 0 \tag{4.8.11}$$

Integrating the first terms by parts, this becomes

$$\int_a^b \left\{ u_n(M^*v_m + \lambda_n N^*v_m) - u_n(M^*v_m + \lambda_m N^*v_m) \right\} dx = 0$$

or

$$(\lambda_n - \lambda_m) \int_a^b u_n N^* v_m \, dx = 0 \qquad (4.8.12)$$

Hence, if $\lambda_n \neq \lambda_m$,

$$\int_a^b u_n N^* v_m \, dx = 0 \qquad (4.8.13a)$$

Alternatively, if $\lambda_n \neq \lambda_m$,

$$\int_a^b v_m N u_n \, dx = 0 \qquad (4.8.13b)$$

Eqs. (4.8.13) are the <u>orthogonality property</u> of the eigenfunctions (and their adjoints).

In PDE problems, one may want to expand in terms of the eigenfunctions. The orthogonality property allows the expansion coefficients to be evaluated. For example, if we set

$$h(x) = \sum_{n=1}^{\infty} A_n \, u_n(x) \qquad (4.8.14)$$

then

$$A_m = \frac{\int_a^b h(x) N^* v_m \, dx}{\int_a^b u_m N^* v_m \, dx} \tag{4.8.15}$$

Many PDE problems involve more than one dependent variable. If we denote the vector of variables in a coupled set of ODEs arising in a SOV analysis by $u_i(x)$, then the coupled ODEs will be of the form

$$M_{ij} \, u_j + \lambda N_{ij} \, u_j = 0 \tag{4.8.16}$$

where here we use the subscript summation convention (see §1.5), the sums to be carried out over the k variables in the solution vector. The adjoint equations

$$M^*_{ij} v_j + \lambda N^*_{ij} \, v_j = 0 \tag{4.8.17}$$

are identified by integration by parts in the scalar equation

$$\int_a^b v_i L_{ij} u_j dx = \int_a^b u_j L^*_{ji} v_i dx \tag{4.8.18}$$

The boundary condition will be of the form

$$\{B_{ij} \, u_j = 0\} \tag{4.8.19}$$

The adjoint boundary conditions

$$\{B^*_{ij} \, v_j = 0\} \tag{4.8.20}$$

are also identified in the integration by parts operation. The orthogonality property is the scalar equation

$$\int_a^b u_j^{(n)} N_{ij}^* v_j^{(m)} dx = 0 \quad . \quad n \neq m \qquad (4.8.21)$$

where here we use superscripts to denote the (vector) eigenfunctions.

In complicated problems, the ODEs will have to be solved numerically to calculate the basic and adjoint eigenfunctions, and the integrals for the expansion coefficients (4.8.15) will have to be done numerically. There are now numerous problems in the literature that have been solved in this way.

4.9 Summary

In problems described by linear equations and linear boundary conditions, in domains of simple shape, solutions can be obtained by the following process;

1. Split the problem into a number of parts, each of which takes care of some of the inhomogeneities. A particular solution of the PDE will take care of the PDE inhomogeneity. Where boundary conditions are to be specified, the split problems should involve homogeneous PDEs and boundary conditions that are homogeneous in at least one of the coordinate directions.

2. Solve the homogeneous partial problems associated with the split problems by SOV. Take a linear combination of these eigensolutions to form the complete solution to the split problem, using the eigenfunction orthogonality property to evaluate the expansion coefficients.

3. Assemble the full solution.

Exercises

4.1 The temperature field in a slab, initially at uniform temperature, subjected to a sudden increase in the temperature of one face, is described by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad T(x,0) = T_o$$

$$T(0,t) = T_o \qquad T(L,t) = T_1$$

Develop the solution to this problem, giving expressions for any integrals involved in the solution. Does your (series) solution converge?

4.2 The temperature field in a slab, initially at uniform temperature, subjected to a step input in heat flux at one surface, is described by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad T(x,0) = T_o$$

$$T(0,t) = T_o \qquad k\left.\frac{\partial T}{\partial x}\right|_{x=L} = q''$$

Solve this problem, giving expressions for any integrals involved in the solution. Does your (series) solution converge?

4.3 The azimuthal velocity field in a cylinder of radius a filled with fluid initially at rest, subject to a sudden rotation of the cylinder is described by

$$\nu\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r}\right] = r\frac{\partial u}{\partial t} \qquad u(r,0) = 0$$

$$u(a,t) = u_o$$

Solve this problem, giving expressions for any integrals involved in the solution. Hint: The steady-state solution is solid body rotation.

4.4 The motion of the fluid in an annular cylinder, set into motion by the sudden rotation of the outer surface, is described by the PDE and initial condition of exercise 4.3, and the boundary conditions

$$u(r_i,t) = 0 \qquad u(r_o,t) = u_o$$

where r_i and r_o are the inner and outer radii, respectively. Solve this problem. Express any integrals involved in terms of functions evaluated at r_i and r_o.

4.5 The concentration of a contaminant in a hollow sphere, initially "clean", subjected to a step jump in the concentration at the inner radius r_i, is described by

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right) = \frac{r^2}{\alpha}\frac{\partial c}{\partial t} \qquad c(r,0) = 0$$

$$c(r_i,t) = c_o \qquad c(r_o,t) = 0$$

Solve this problem, developing expressions for any integrals involved in terms of functions evaluated at r_i and r_o. This problem has application in the geological diffusion of nuclear wastes.

4.6 The transient temperature of a circular fin is described by

$$k\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \beta^2 r(T-T_o) = \frac{r}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial r} = 0 \quad \text{at } r = r_o \qquad T(r_i,t) = 0$$

$$T(r,0) = T_o$$

Solve this problem, developing expressions for any integrals in terms of functions evaluated at r_i and r_o.

4.7 The steady potential field in a circular object, with potential specified around the perimeter $(r = a)$, is described by

$$\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$$\phi(a,\theta) = f(\theta)$$

Develop the solution to this problem, expressing the result in terms of appropriate integrals.

4.8 The temperature field in a quarter-circular sector plate cooled by convection is described by

$$\left\{\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right) + \frac{1}{r}\,\frac{\partial^2 T}{\partial \theta^2}\right\} - \beta^2 r(T-T_o) = 0$$

$$T(r_i,\theta) = T_1 \qquad T(r_o,\theta) = a + b\theta$$

$$\frac{\partial T}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0,\ \pi/2$$

Solve this problem, evaluating any integrals that appear in terms of functions evaluated at r_i or r_o. What happens at the corners?

4.9 Study the Sturm-Liouville problem (4.2.1). Show that, if P is real, the eigenvalues are all real. Hint: Let them be complex; consider the conjugate equations. Use our favorite tool, integration by parts.

4.10 Consider Bessel's equation and boundary condition

$$r^2R'' + rR' + \lambda^2 r^2 R = 0$$

$$R(a) = 0$$

Find the adjoint equation. If the eigenfunction is $J_o(\lambda r)$, what is the adjoint eigenfunction, and what is the orthogonality property? Is this the same as obtained from the Sturm Liouville form of Bessel's equation?

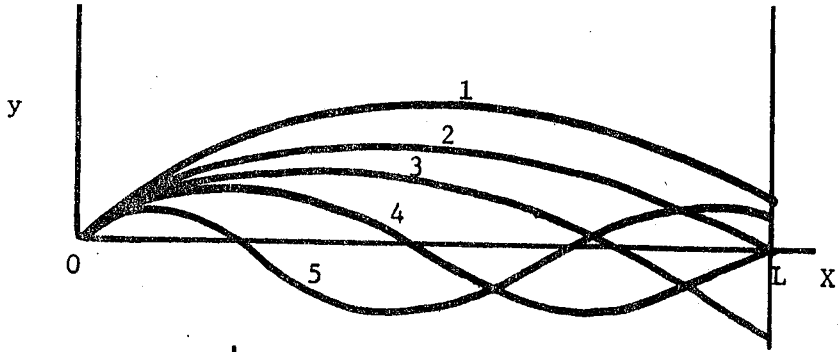

FIGURE 4.2.1

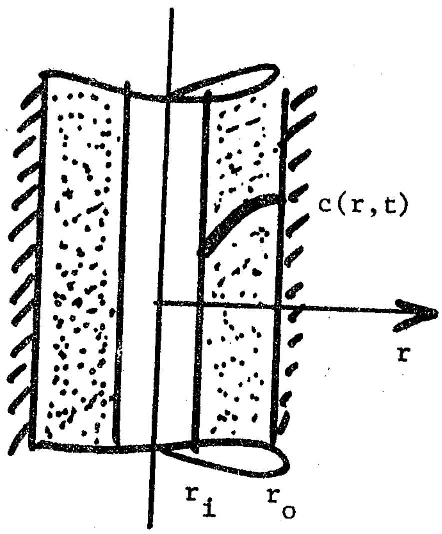

Fig. 4.6.1

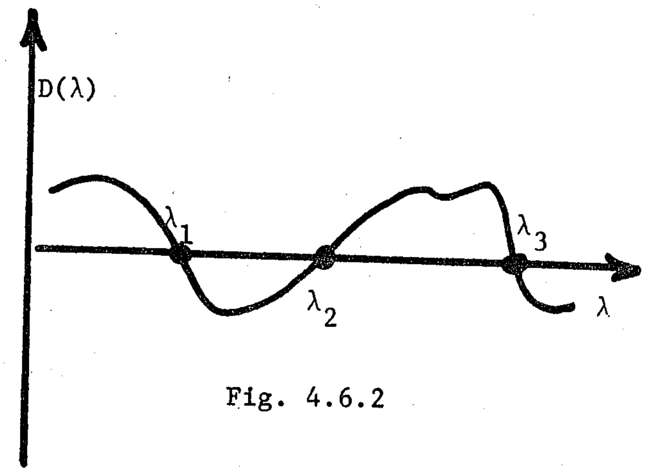

Fig. 4.6.2

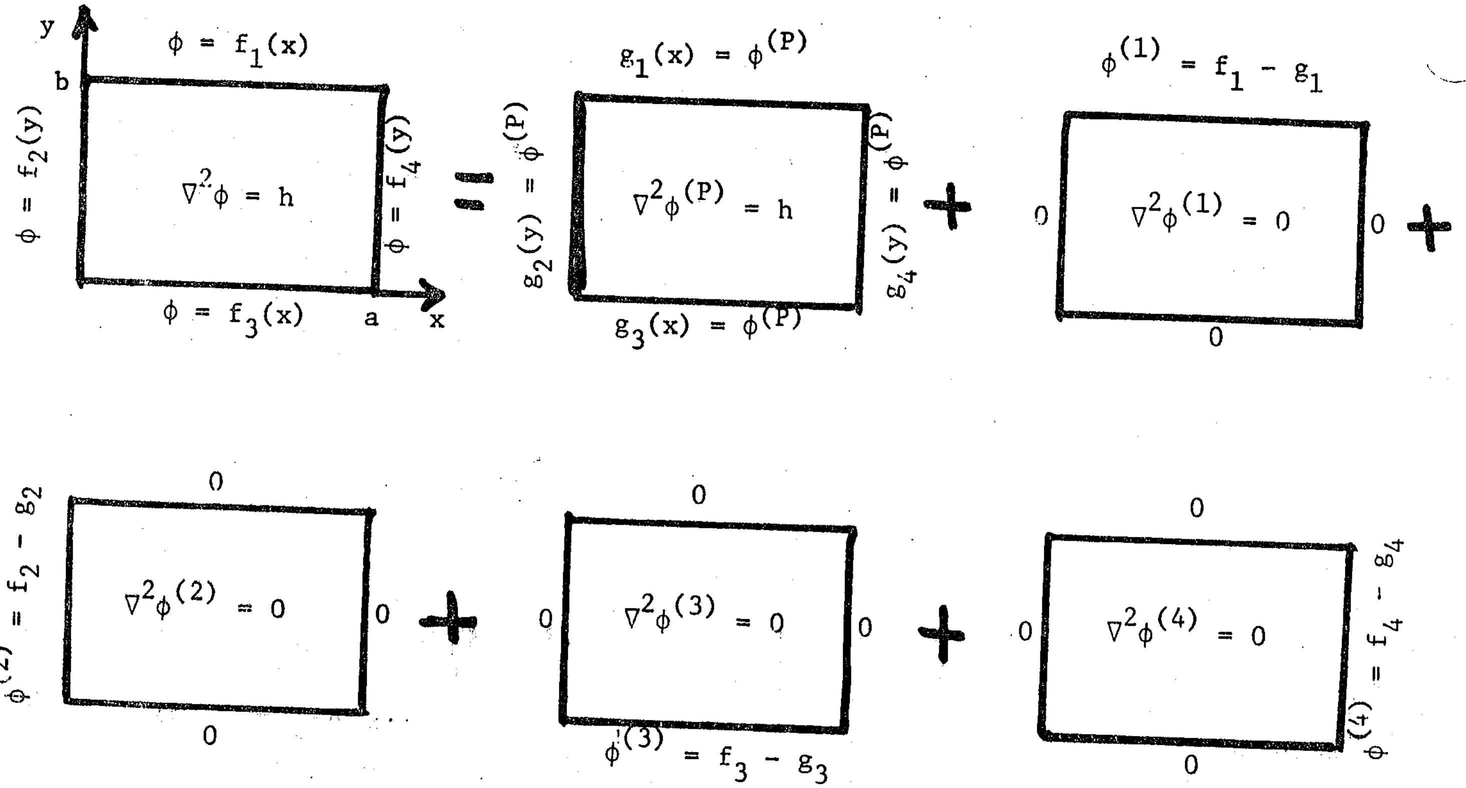

Fig. 4.7.1. Splitting

Notes on Problems 4.1 & 4.4

4.1 Convergence: most of you found

$$T(x,t) = -2(T_0-T_1)\sum_{n=1}^{\infty} e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}\frac{\cos n\pi}{n\pi}\cdot\sin\frac{n\pi x}{L} + T_0 + (T_1-T_0)\frac{x}{L}$$

Here $A_n = e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}\frac{\cos n\pi}{n\pi}\sin\frac{n\pi x}{L}$. For any x,t $A_n \to 0$ as $n\to\infty$. For series to converge,

$$\left|\frac{A_{n+1}}{A_n}\right| = \left|\frac{e^{-\frac{\alpha[n+1]^2\pi^2}{L^2}t}\ \frac{\cos(n+1)\pi}{n+1\ \pi}\ \sin\frac{(n+1)\pi x}{L}}{e^{-\frac{\alpha n^2\pi^2 t}{L^2}}\ \frac{\cos n\pi}{n\pi}\ \sin\frac{n\pi x}{L}}\right| = \underbrace{\left|e^{-\frac{\alpha(2n+1)\pi^2}{L^2}t}\right|}_{①}\underbrace{\left|\frac{\cos(n+1)\pi}{\cos n\pi}\right|}_{②}\underbrace{\left|\frac{n\pi}{(n+1)\pi}\right|}_{③}\underbrace{\left|\frac{\sin\frac{(n+1)\pi x}{L}}{\sin\frac{n\pi x}{L}}\right|}_{④}$$

lim $n\to\infty$ of ① is 0, any t; ② is 1; ③ is 1; ④ is 1, any x } product is 0, thus series converges for any x,t

4.4 $\nu\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r}\right] = r\frac{\partial u}{\partial t}$ $\quad u(r_o,t)=u_o,\ u(r_i,t)=0\quad u(r,t=0)=0$

let $u = v + \psi(r)$

$$\therefore\ \nu\left[r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} - \frac{v}{r} + r\frac{\partial^2\psi}{\partial r^2} + \frac{\partial\psi}{\partial r} - \frac{\psi}{r}\right] = r\cancelto{0}{\frac{\partial\psi}{\partial t}} + r\frac{\partial v}{\partial t}$$

if we choose $\psi(r)$ so that $r\psi'' + \psi' - \frac{\psi}{r} = 0$ (this is Euler Equidimensional Eqn), $\psi = \frac{C_1}{r} + C_2 r$

try $\psi(r) = Cr^\alpha$ & show $\alpha = 1,-1$. This means $\nu\left[\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{v}{r}\right] = r\frac{\partial v}{\partial t}$ HOMOG.

now since $u(r_o,t) = u_o = v(r_o,t) + \psi(r_o)$ choose $\psi(r_o) = u_o \Rightarrow v(r_o,t) = 0$ HOMOG.

$u(r_i,t) = 0 = v(r_i,t) + \psi(r_i)$ choose $\psi(r_i) = 0 \Rightarrow v(r_i,t) = 0$ "

$u(r,t=0) = 0 = v(r,t=0) + \psi(r) \Rightarrow v(r,t=0) = -\psi(r)$

$\psi(r) = \frac{C_1}{r} + C_2 r\quad \therefore\quad \psi(r_o) = u_o = \frac{C_1}{r_o} + C_2 r_o\ ;\quad \psi(r_i) = 0 = \frac{C_1}{r_i} + C_2 r_i \Rightarrow C_1 = -C_2 r_i^2$

$\therefore\ u_o r_o = C_1[1] + C_2 r_o^2 = +C_2[r_o^2 - r_i^2]\quad \therefore C_2 = \frac{u_o r_o}{r_o^2 - r_i^2}$ & $C_1 = -\frac{u_o r_o\cdot r_i^2}{r_o^2 - r_i^2}$

Now: $\nu\left[\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{v}{r}\right] = r\frac{\partial v}{\partial t}$ with $v(r_o,t)=0\quad v(r_i,t)=0$ & $v(r,t=0) = -\psi(r)$

all of you did this part.

If you chose $\psi(r) = \frac{u_o(r-r_i)}{r_o - r_i}$ to satisfy the BC's, $\psi''=0$ but $r\psi'' + \psi' - \frac{\psi}{r} \neq 0$

so v would have to solve $\nu\left[\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{v}{r}\right] - r\frac{\partial v}{\partial t} = G(x,t) = -\nu\left[\psi' - \frac{\psi}{r}\right]$

with $v(r_o,t) = v(r_i,t) = 0$ & $v(r,t=0) = -\psi(r)$. You'd need to solve an inhomog. PDE

7.6 General Solution of the Wave Equation

The general solution of the wave equation can be found if we use both of the characteristics as coordinates. We take

$$\eta = x - at \quad , \quad \xi = x + at \tag{7.6.1}$$

Here, η corresponds to the "+" characteristics and ξ to the "-" characteristic.* Transforming (7.5.1) to these coordinates,

$$u_x = u_\xi + u_\eta \quad , \quad u_t = a(u_\xi - u_\eta)$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \quad , \quad u_{tt} = a^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$

So the wave equation (7.5.1) becomes

$$u_{\xi\eta} = 0 \tag{7.6.2}$$

We can solve this exactly. Integrating with respect to ξ,

$$u_\eta = g(\eta) \tag{7.6.2a}$$

Now, integrating with respect to η,

$$u = \int g(\eta')\, d\eta' + F(\xi) \tag{7.6.2b}$$

or,

$$u = F(\xi) + G(\eta) \tag{7.6.3}$$

---

*The use of both characteristics as the coordinates for the equation is functional only if there are two characteristics; it is not a useful approach in a two-dimensional problem with more than two characteristics, such as occurs in certain types of turbulent boundary layer models.

Hence, the general solution of the wave equation is

$$u = F(x+at) + G(x-at) \tag{7.6.4}$$

The general solution can be used to solve some problems, but it is a cumbersome approach for others (those handled better by separation of variables). For example, let's consider the problem where the initial conditions are specified, for $-\infty < x \leq +\infty$, as

$$u(x,0) = p(x) \tag{7.6.5a}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=o} = 0 \tag{7.6.5b}$$

Applying (7.6.5b) to (7.6.4),

$$a[F'(x) - G'(x)] = 0 \tag{7.6.6}$$

Therefore,

$$G(x) = F(x) + C_1 \tag{7.6.7}$$

Now (7.6.5a) requires

$$F(x) + G(x) = p(x) \tag{7.6.8}$$

Combining with (7.6.7),

$$F(x) = \frac{1}{2}\,p(x) - \frac{1}{2}\,C_1 \tag{7.6.9a}$$

$$G(x) = \frac{1}{2}\,p(x) + \frac{1}{2}\,C_1 \tag{7.6.9b}$$

So the solution satisfying (7.6.5) is

$$u(x,t) = \frac{1}{2}\,p(x+at) + \frac{1}{2}\,p(x-at) \tag{7.6.10}$$

At point (x,t), the quantity $p(x+at)$ will have a value determined by the intercept of the "–" characteristic passing through point (x,t) with the line $t = 0$ (Fig. 7.6.1); similarly, the quantity $p(x-at)$ is constant along the "+" characteristic passing through (x,t). Therefore, for this problem the value of the solution at point 3 in Fig. 7.6.1 depends only upon the values of the initial data at points 1 and 2! The solution at point 3 is merely the average of the initial values at points 1 and 2.

For example, suppose that the initial distribution is a Gaussian pulse

$$u(x,0) = \exp(-x^2) \tag{7.6.11}$$

Then the solution at later times will be

$$u(x,t) = \frac{1}{2}\exp[-(x+at)^2] + \frac{1}{2}\exp[-(x-at)^2] \tag{7.6.12}$$

The solution says that the initial pulse splits into two parts, one which propagates to the left, the other to the right. The center of each pulse moves out along a characteristic line, so each pulse travels at the speed a.

7.7 Imaging in Wave Equation Solutions

Suppose we are interested in the reflection of a wave from a boundary. Eqn. (7.5.1) and the initial conditions (7.6.5) again govern the problem, but now we add the boundary condition

$$\left.\frac{\partial u}{\partial x}\right|_{x=o} = 0 \tag{7.7.1}$$

and restrict our interest to the domain $0 \leq x \leq \infty$. This problem can be solved by the general solution. We set

$$u = F(x+at) + G(x-at) \tag{7.7.2}$$

The initial conditions (7.6.5) require

$$F(x) + G(x) = p(x) \qquad x \geq 0 \qquad (7.7.3a)$$

$$F'(x) - G'(x) = 0 \qquad x \geq 0 \qquad (7.7.3b)$$

So (7.6.4) again give F and G, but only for positive arguments! Note that now the functions F and G are not defined for $x < 0$ by the initial conditions. Instead, we have, from (7.7.1),

$$F'(at) + G'(-at) = 0 \qquad (7.7.4)$$

This must hold at all times. Therefore, for negative arguments the function G must be such that its derivative is the negative of the derivative of the function F for the same value of positive argument; i.e.,

$$G'(-\sigma) = -F'(\sigma) \qquad (7.7.5)$$

This will be the case when G is the mirror image of F (Fig. 7.7.1). In mathematical terms,

$$G(-x) = F(x) = p(x)/2 \qquad x > 0 \qquad (7.7.6a)$$

$$F(-x) = G(x) = p(x)/2 \qquad x > 0 \qquad (7.7.6b)$$

Therefore, since the c_1 terms cancel, we can take

$$\begin{aligned}
(x+at) \geq 0 \quad & F(x+at) = \frac{1}{2}\, p(x+at) \\
(x+at) < 0 \quad & F(x+at) = \frac{1}{2}\, p[-(x+at)] \\
(x-at) \geq 0 \quad & G(x-at) = \frac{1}{2}\, p(x-at) \\
(x-at) < 0 \quad & G(x-at) = \frac{1}{2}\, p[-(x-at)]
\end{aligned} \qquad (7.7.7)$$

The solution (7.7.2) therefore can be thought of as a combination of four wave packets, as shown in Fig. 7.7.1. The first is half of the $p(x)$

wave, which moves to the right away from the reflecting boundary. The second is the other half of this wave, which moves to the left and passes through the boundary to negative x. The third is the mirror image of the $p(x)$ wave, which starts to the left of the reflecting wall (outside of the real problem) and travels to the right, entering the wall as its "mate" passes through going left. This image wave then appears in the domain of interest as a reflected wave. The fourth wave is the other half of the image $p(x)$ wave, which travels to the left and never enters the domain of interest.

Wave-equation solutions obtained by these imaging methods must be represented segmentally. If there are only one or two segments, this is not too difficult and is a convenient way to get the solution. However, if there are many reflections, such as would be the case for the solution of standing acoustic waves in a duct or the long-term vibration of a finite string, the approach becomes very cumbersome and the separation of variables technique usually is easier to execute and present.

7.8 Characteristics for the Laplace Equation

For the Laplace equation,

$$u_{xx} + u_{yy} = 0 \tag{7.8.1}$$

the characteristic slopes are $y' = \pm i$, so the characteristics are given by

$$x + iy = \text{const.} \quad \text{and} \quad x - iy = \text{const.} \tag{7.8.2}$$

On the surface this does not appear too useful, because the characteristics are not lines in the real x-y plane. However, we can learn something by transforming the equation to new variables ξ, η such that

$$\xi = x + iy \ , \qquad \eta = x - iy$$

$$u_x = u_\xi + u_\eta \ , \qquad u_y = i(u_\xi - u_\eta)$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \quad , \qquad u_{yy} = - u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta}$$

So (7.8.1) becomes

$$u_{\xi\eta} = 0 \tag{7.8.3}$$

Integrating as before, we have the general solution of Laplace's equation as

$$u = F(\xi) + G(\eta)$$

or

$$u(x,y) = F(x+iy) + G(x-iy) \tag{7.8.4}$$

This equation forms the basis for solution of the Laplace equation by the method of complex variables. These methods are beyond the scope of the present text, but are discussed in depth in courses on the applications of complex variables.

Exercises

7.1. The fluid temperature field in a nuclear reactor is described by the simple model

$$\frac{\partial T}{\partial t} + V \frac{\partial T}{\partial x} = A \sin(\pi x/L)$$

$$T(0,t) = 0 , \qquad T(x,0) = 0$$

where A and V are constants. Develop the solution to this problem using the method of characteristics. At what time is the steady-state solution achieved?

7.2. The Bradshaw-Ferris turbulent boundary layer model is described by

$$uu_x + vu_y = e_y + h$$

$$ue_x + ve_y = aeu_y + g$$

$$u_x + v_y = 0$$

where h, g, and a are functions of the dependent variables u, v, and e. u and v are velocity components, and e is the turbulent kinetic energy per unit mass. Find the slopes of the three characteristics for this problem, and write the three quasi-ODEs that apply on these characteristics.

7.3. The equations for time-dependent, one-dimensional compressible flow are

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(A\rho V) = 0$$

$$A \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(A\rho V^2) + \frac{\partial}{\partial t}(A\rho V) = 0$$

$$\frac{\partial p}{\partial x} - c^2 \frac{\partial \rho}{\partial x} = 0$$

where $A(x)$ is the prescribed duct cross-section area, and where $C^2 = kp/\rho$ is the isentropic sound speed. The independent variables are the velocity V, pressure p, and density ρ.

Develop the expressions for the slopes of the characteristics, and write pseudo-IDEs that apply on each characteristic. Organize an approximate numerical algorithm to solve this problem marching forward in time, using the method of characteristics.

7.4. Consider the wave equation $u_{xx} - u_{tt} = 0$, with the initial conditions $u(x,0) = 0$, $u_t(x,0) = \exp(-x^2)$ in $-\infty \leq x \leq +\infty$. Derive an expression for the solution using the general solution of the wave equation.

7.5. Consider the wave equation $u_{xx} - u_{tt} = 0$, with the initial and boundary conditions

$$u(x,0) = xe^{-x} \qquad 0 \leq x \leq \infty$$

$$u_t(x,0) = 0$$

$$u(0,t) = 0$$

Develop (segmental) expressions for the solution to this problem in $0 \leq x \leq \infty$, and give an expression for the solution at $t = 1$. Interpret in terms of right- and left-running waves, using a sketch.

7.6. Consider the wave equation $u_{xx} - u_{tt} = 0$, with the initial and boundary conditions

$$u(x,0) = 0 \qquad 0 \leq x \leq 1$$

$$u_t(x,0) = \begin{cases} 1 & 0 \leq x \leq 1/2 \\ 0 & x \geq 1/2 \end{cases}$$

$$u(0,t) = 0$$

$$u(1,t) = 0$$

Develop this solution by the method of characteristics and by separation of variables, and compare.

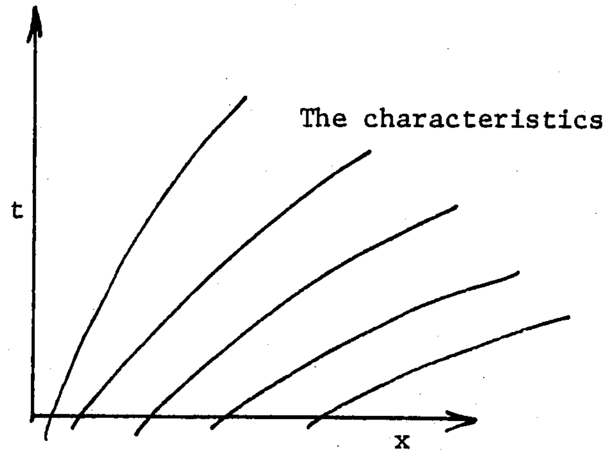

Fig. 7.2.1

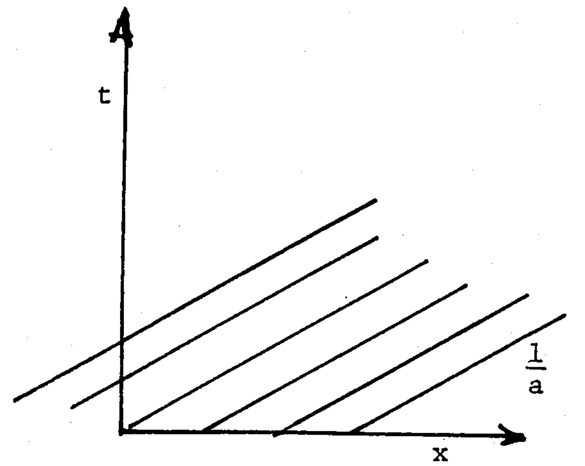

Fig. 7.2.2

The characteristics are straight

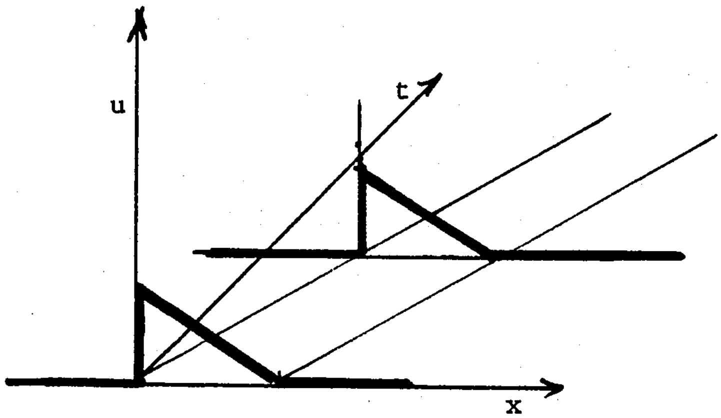

Fig. 7.2.3

Wave form propagation on characteristics

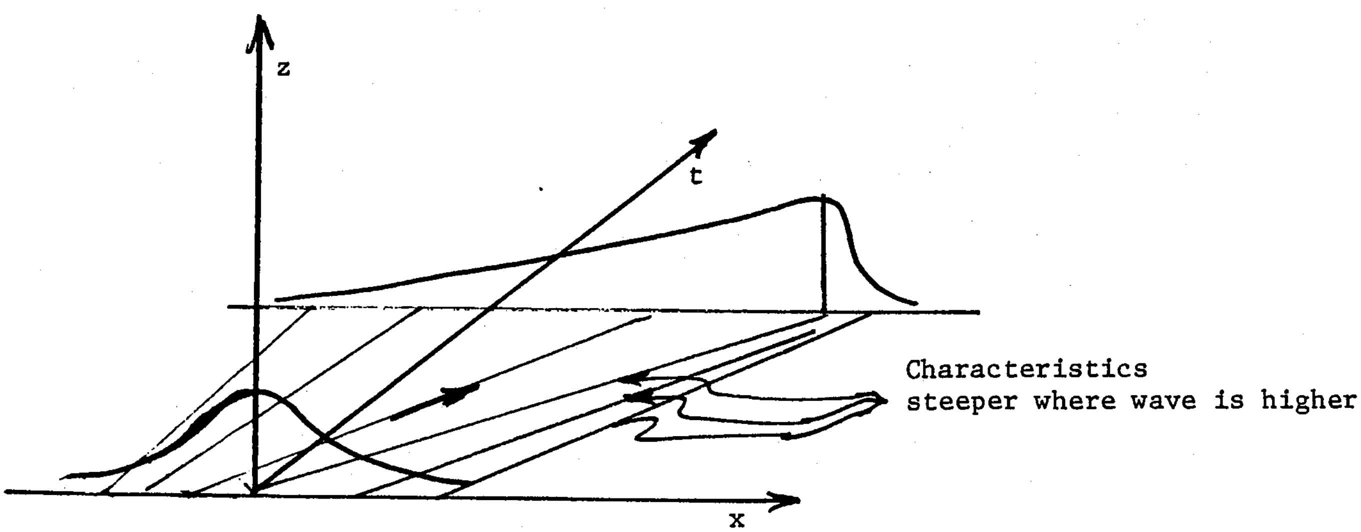

Fig. 7.3.1

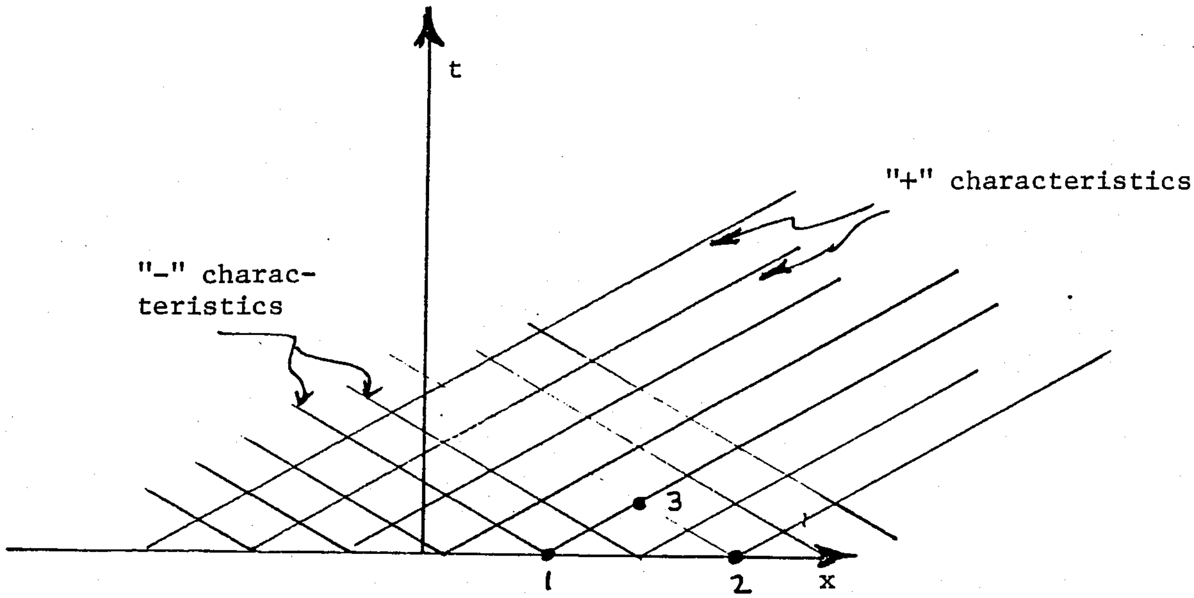

Fig. 7.5.1 The two sets of characteristics

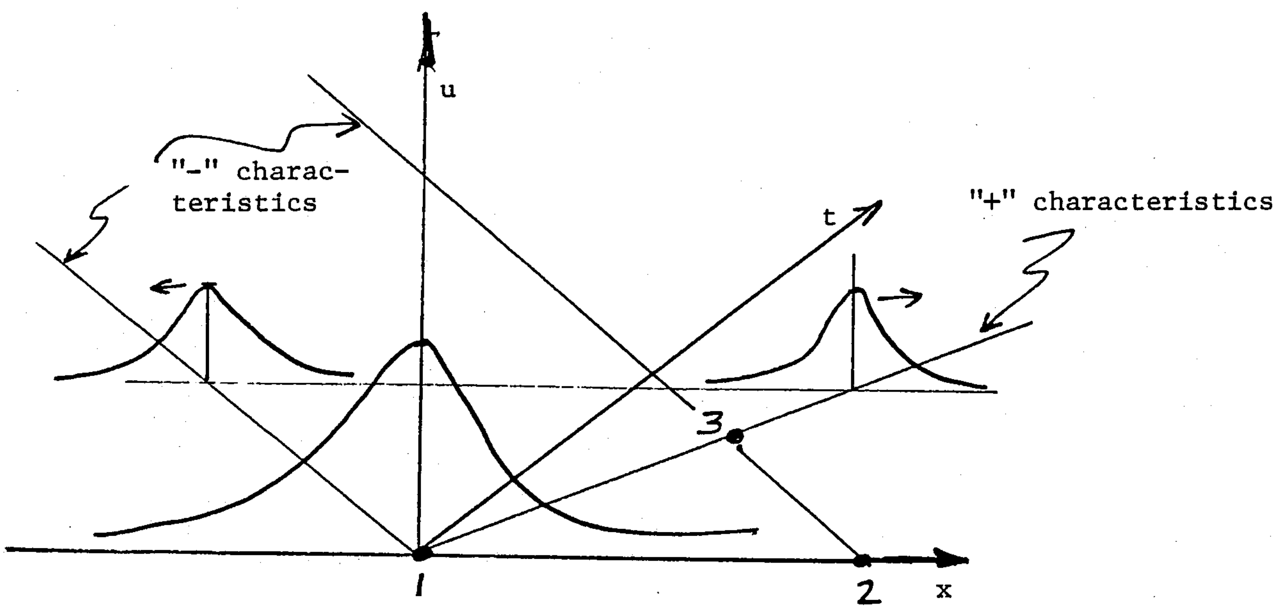

Fig. 7.6.1. Solution for an initial pulse

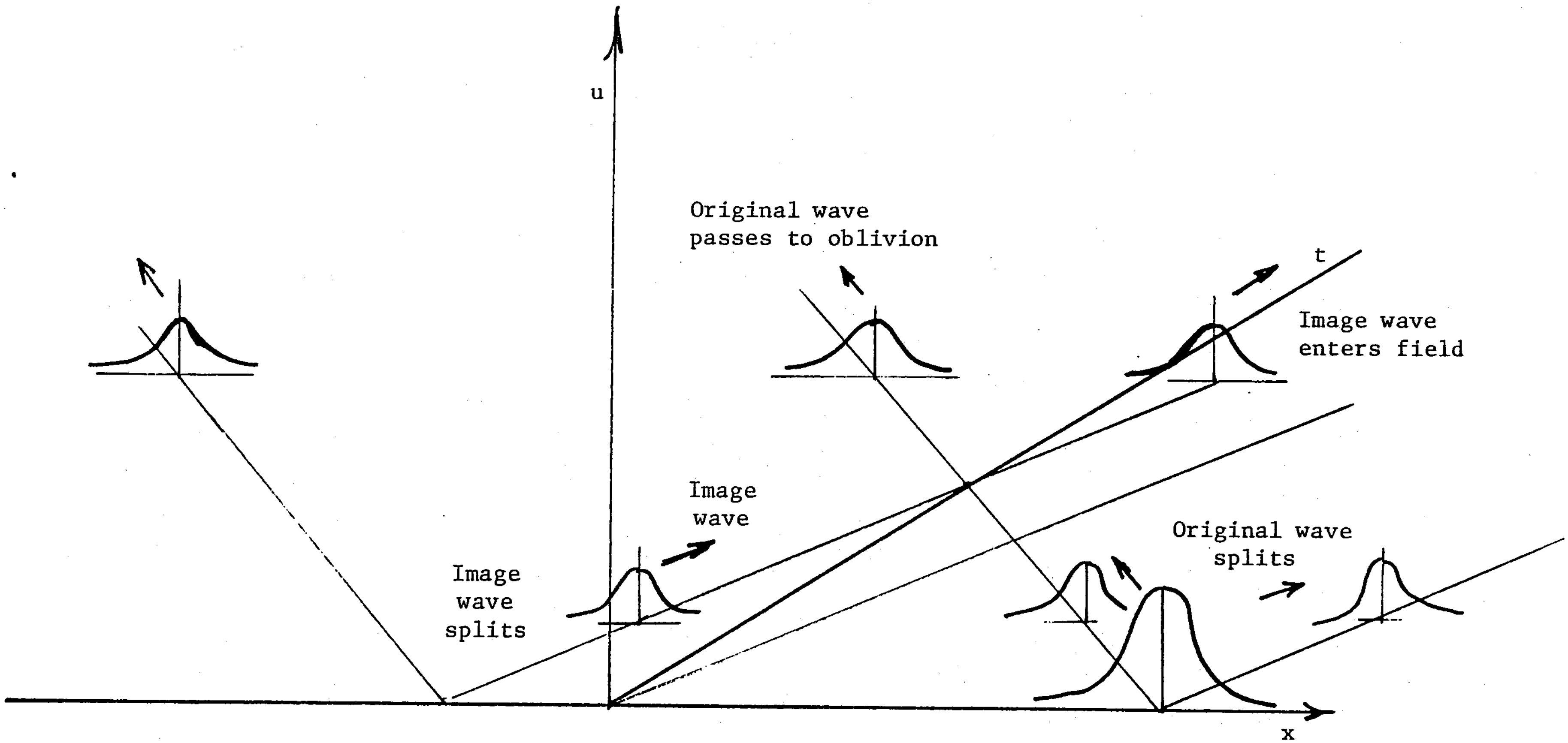

Fig. 7.7.1. Solution decomposition