10/1/80

Integral Equations - Levine MA224

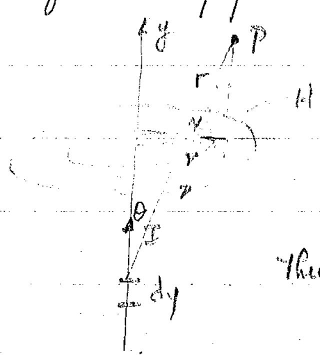
Durand 358

We will work with linear integral equations, Weiner - Hopf Technique

Hochstadt - Integral Equations

Example in Electrostatics leading to Volterra Eqn.

Infinitely Long wire carrying current $i$ gives rise to magnetic field $H$. Given a point $P$




we know that $H = \frac{2I}{x}$; we want to find magnetic field as a superposition of $H$ due to current filament.

$dH = dy \cdot I \cdot \sin\theta \, f(r)$ $\quad$ $\sin\theta$ since $H$ only has component $\perp$ to current

Thus $\displaystyle\int_{-\infty}^{\infty} dH = \frac{2I}{x} = \int_{-\infty}^{\infty} I \sin\theta \, f(r)\, dy$

$$= \int_{-\infty}^{\infty} I \frac{x}{r} f(r)\, dy \qquad r^2 = x^2 + y^2$$

$$= 2\int_{0}^{\infty} I \frac{x}{r} f(r)\, dy \qquad \text{since } r \text{ is an eve fn of } x \,\&\, y.$$

now $2r\,dr = \underbrace{2x\,dx}_{x \text{ fixed}} + 2y\,dy$ $\quad\therefore\quad$ $\displaystyle = 2\int_{x}^{\infty} I \frac{r\,dr}{\sqrt{r^2 - x^2}} \frac{x}{r} f(r)$

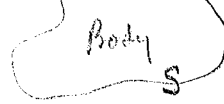
and $y = \sqrt{r^2 - x^2}$

$$\frac{1}{x^2} = \int_{x}^{\infty} \frac{f(r)\,dr}{\sqrt{r^2 - x^2}} \qquad x > 0$$

$f(r) = \frac{1}{r^2}$ is solution : to show this let $r = x\cosh\varphi$

$$\cancel{\frac{1}{x^2}} = \int_0^{\infty} \frac{1}{x^2\cosh^2\varphi}\, d\varphi = \int_0^{\infty} \frac{1}{\cosh^2\varphi}\, d\varphi$$

Another example




with static charge density

$\Phi = 1$ (electrostatic potential)

$\phi(\underline{r})$ = Electrostatic potential satisfies

$$\nabla^2 \Phi(\underline{r}) = 0 \text{ outside } S$$

$$\Phi \to 0 \text{ as } \underline{r} \to \infty$$

converting this to an eqn of integral form.

define $\Phi(\underline{r}) = \dfrac{1}{|\underline{r} - \underline{r}'|}$ . potential due to a point charge at $\underline{r}'$

Suppose the charge on the body is $\sigma(\underline{r})$ not uniform

thus $\phi(\underline{r}) = \displaystyle\int_S \frac{\sigma(\underline{r}')\,dS'}{|\underline{r} - \underline{r}'|}$ is a convolution of charge density . dirac delta f. over the surface

thus $\nabla^2\phi = 0$ and as $\underline{r} \to \infty$ $\phi(\underline{r}) \to 0$

and now to satisfy $\phi = 1$ at $\underline{r}$ on S

$$1 = \int_S \frac{\sigma(\underline{r}')\,dS'}{|\underline{r} - \underline{r}'|} \quad \text{for } \underline{r} \text{ on } S$$ this is an integral on $\sigma$

this is Fredholm type (singular kernel) of first kind since the unknown doesn't appear on both sides of eqn. Note this doesn't care what the surface form looks like.

Classification of equations Fredholm

$1^{st}$ kind $\quad g(x) = \displaystyle\int_a^b K(x,y)\, f(y)\, dy$ this smooths out the f

known parameter

$2^{nd}$ kind $\quad f(x) = g(x) + \lambda \displaystyle\int_a^b K(x,y)\, f(y)\, dy \qquad a < x < b$

if either of limits is replaced by variable become volterra types.

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Fredholm Type Eqns.

$$f(x) = g(x) + \lambda \int_a^b K(x,y)\, f(y)\, dy \qquad \text{of } 2^{nd} \text{ Kind}$$

Inhomogeneous since $g(x) \neq 0$. In most problems $g(x), \lambda, K(x,y)$ are known.
Fredholm Alternative: If $\lambda$ is some general value then $\exists!$ a solution for arbitrary function of $g(x)$
or $\lambda$ is an eigenvalue of the associated homogeneous problem, which gives rise to a solution of

$$f(x) = \lambda \int_a^b K(x,y) f(y)\, dy$$

with this eqn $\exists$ solutions $\varphi_1(x), \varphi_2(x), \ldots \varphi_n(x)$ associated with $\lambda_1$
ie

$$\varphi_j(x) = \lambda_1 \int_a^b K(x,y)\, \varphi_j(y)\, dy$$

and $\exists$ functions $\psi_1(x), \ldots, \psi_n(x)$ associated with $\lambda_1$ so that

$$\psi_j(x) = \lambda_1 \int_a^b K(y,x)\, \psi_j(y)\, dy \qquad K(y,x) \neq K(x,y) \text{ in general}$$

$$\int_a^b g(x)\, \psi_j(x)\, dx = 0$$ A condition for the existence of a soln. if $\lambda$ is an eigenvalue

proof $\int_a^b f(x)\psi_j(x)\,dx = \int_a^b g(x)\psi_j(x)\,dx + \lambda \int_a^b \psi_j(x) \int_a^b K(x,y) f(y)\, dy\, dx = \int g\psi_j\, dx + \lambda \iint_a^b \psi_j(x) K(x,y)\, dx\; f(y)\, dy$
$= \int g\psi_j\, dx + \int_a^b \psi_j(y) f(y)\, dy \Rightarrow \int g\psi_j$

Fredholm Alternative is easy to prove when $K(x,y)$ can be separated as $\sum k_i(x) l_i(y)$.
This gives rise to a linear system

$$f(x) = g(x) + \lambda \sum A_i k_i(x) \qquad \text{where } A_i = \int_a^b dy\; l_i(y) f(y)$$

For the $1^{st}$ Kind (Fredholm Eq): Not generally the case that if $0 = \int_a^b K(x,y) f(y)\, dy$ has only the trivial soln $f(y) = 0$ then

$$\int_a^b K(x,y)\, f(y)\, dy = g(x) \text{ has a (unique) soln.}$$

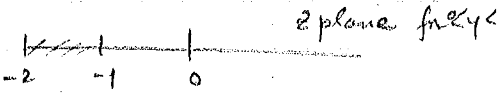
Proof by Counterexample to Fredholm Alternative $K(x,y) = \dfrac{1}{x+y+1}$, $0 \le x, y \le 1$

Consider $\displaystyle\int_0^1 \frac{f(y)}{x+y+1}\, dy = 0 \Rightarrow f(y) \equiv 0$ for continuous fns. for $0 \le x \le 1$

Suppose $\int_0^1 \frac{f(y)\,dy}{x+y+1} = 1$ , $0<x<1$

Define a fn.

$$F(z) \triangleq \int_0^1 \frac{f(y)\,dy}{z+(y+1)} \qquad z \text{ complex}$$




Now $F(z)$ is regular (analytic) throughout the cut $z$-plane cut on $(-1,-2)$

now let $z = -1-\xi \pm i\eta$ where $\eta \to 0^+$ $\xi \in (0,1)$

$$\text{thus } F(z) = \int_0^1 \frac{f(y)\,dy}{-\xi \pm i\eta + y} = \int_0^1 \frac{f(y)\cdot(y-\xi \mp i\eta)\,dy}{(y-\xi)^2+\eta^2}$$

$$\to PV\int_0^1 \frac{f(y)\,dy}{y-\xi} \mp i \int_0^1 \lim_{\eta\to 0} \frac{\eta}{(y-\xi)^2+\eta^2} f(y)\,dy \qquad \text{as } \eta \to 0$$

$$\int_0^{\xi-\epsilon} + \int_{\xi+\epsilon}^1 \qquad \mp i \int_0^1 \pi\,\delta(y-\xi)\, f(y)\,dy$$

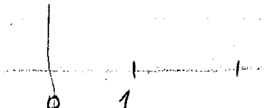
$$F(z) = ⨍_0^1 \frac{f(y)\,dy}{y-\xi} \qquad \mp i\pi f(\xi)$$

thus $F(z)$ has a discontinuity $\propto f(\xi)$ on crossing the cut at the coordinate $\xi$.

What does this mean? The condition $F(x)=0$, $0<x<1$

corresponding to homogeneous I. Eqn.

$$\int_0^1 \frac{f(y)\,dy}{x+y+1} = 0, \quad 0<x<1$$




$\Rightarrow$ $F(x) = 0$ in cut plane by analytic continuation

Thus $F(x)=0$ by approaching from both sides $\Rightarrow f(x) = 0$ on the cut

thus we have shown that soln to homog eq is the trivial sol.

now $F(x) = \int_0^1 \frac{f(y)\,dy}{x+y+1} = 1$ look at $F(z)-1$ : this analytic fn which vanishes on 0,1

Then $F(z)-1 = 0$ in cut plane

but when $z\to\infty$ $F(z)\to 0$ $\therefore$ $F(z)-1 \to -1$ as $z\to\infty$ and the

fn $f(y)$ is not unique.

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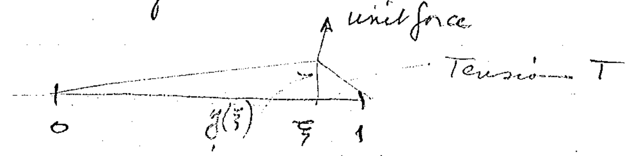
Example

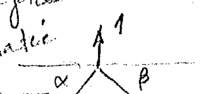
Consider a string $l = 1$




what is the static profile of string $\quad y(x) = y(\xi)\left[\frac{x}{\xi}\right] \qquad 0 \le x < \xi$

$$y(x) = y(\xi)\left[\frac{1-x}{1-\xi}\right] \qquad \xi < x \le 1$$

Resolve force from static equil




$\Rightarrow \quad T\cos\alpha = T\cos\beta \quad$ Horiz Eq $\quad$ (Trivial if $\alpha, \beta$ are small)

$T(\sin\alpha + \sin\beta) = 1 \quad$ Vertical Eq $\Rightarrow \quad T(\alpha + \beta) = 1$

$\alpha \cong \tan\alpha = y(\xi)/\xi$

$\beta \cong \tan\beta = y(\xi)/1-\xi$

$\therefore \quad \dfrac{y(\xi)}{\xi(1-\xi)} T = 1 \qquad$ from $\quad T(\alpha+\beta) = 1$

Thus $\quad y(\xi) = \dfrac{1}{T}\xi(1-\xi) \qquad$ for small deflections

for load P $\quad y(\xi) = P/_T(1-\xi)\xi$

thus $\quad y(\xi)/_\xi = \dfrac{1-\xi}{T} \quad$ and $\quad y(x) = \dfrac{x(1-\xi)}{T} \qquad 0 \le x \le \xi$

$$y(x) = \frac{(1-x)\xi}{T} \qquad \xi < x < 1$$

if we had forces $P_i$ at $\xi_i$ then $\quad y(x) = \displaystyle\sum_{i=1}^{N} \frac{x P_i(1-\xi_i)}{T}$

$y(x) = \displaystyle\sum_{i=1}^{N} P_i\, y(x, \xi_i) \qquad = \sum_{i=1}^{N} \frac{(1-x)P_i\xi_i}{T}$

Thus $\quad y(x) = \displaystyle\int_0^1 f(\xi)\, V(x,\xi)\, d\xi$

continuously distributed load or force

where $\quad V(x,\xi) = \dfrac{x(1-\xi)}{T} \qquad 0 \le \xi < x$

$$\frac{(1-x)\xi}{T} \qquad x < \xi \le 1$$

Now look at Free vibrations of the taut string

$$y(x,t) = \varphi(x)\cos\omega t$$

Then $\quad \rho(x)$ - local density $\qquad \underbrace{\rho\dfrac{\partial^2 y}{\partial t^2}}_{\text{Newtonian force}} = -\rho\omega^2\varphi(x)\cos\omega t$

thus $\rho \frac{\partial^2 y}{\partial t^2} = -f(\xi)$ by D'Alembert principle

$= \omega^2 \rho(x) \varphi(x) \cos\omega t$

thus $y(x) = \int_0^1 f(x') V(x,x') dx'$

$$\varphi(x)\cancel{\cos\omega t} = \int_0^1 \omega^2 \rho(x') \varphi(x') \cancel{\cos\omega t} V(x,x') dx'$$

thus $\varphi(x) = \int_0^1 \omega^2 \rho(x') V(x,x') \varphi(x') dx'$

$$\updownarrow$$

$$V(x',x)$$

This is homogeneous integral equation for $\varphi(x)$ with $V(x,x') = \frac{x(1-x')}{T}$ $x<x'$

$\frac{x'(1-x)}{T}$ $x'<x<1$

this has a Non symmetric kernel since $\rho(x)$ is not const.

Since $\rho(x)$ is non negative then

$$\varphi(x) = \int_0^1 \omega^2 \sqrt{\rho(x')}\, V(x,x') \sqrt{\rho(x')}\, \varphi(x') dx'$$

now mult by $\sqrt{\rho(x)}$ on both sides

$$\sqrt{\rho(x)}\, \varphi(x) = \int_0^1 \omega^2 \underbrace{\sqrt{\rho(x)}\, V(x,x') \sqrt{\rho(x')}}_{\text{symmetric kernel } K(x,x')} \underbrace{\sqrt{\rho(x')}\, \varphi(x')}_{\psi(x')} dx'$$

$$\psi(x) = \lambda \int_0^1 K(x,x')\, \psi(x') dx'$$

also $\lambda = \omega_1^2, \omega_2^2, \dots, \omega_n^2, \dots$

and $\exists!$ $\psi_1, \psi_2, \dots, \psi_n$ associated with the $\omega$'s

and that $\omega_n^2$ has one & only one $\psi_n$

Consider the forced motion of the string

$F(x,t) = q(x) \cos\Omega t$

external force acting on string. The synchronous string displacement is the

$y(x,t) = \varphi(x) \cos\Omega t$

thus looks at $g(x)\cos\Omega t$ $\quad -\rho(x)\dfrac{\partial^2 y}{\partial t^2} = g(x)\cos\Omega t + \Omega^2\rho\,\varphi\cos\Omega t$

this is now $f(x')$ in our integral eqn. thus

$$\varphi(x) = \int_0^1 \{ g(x') + \Omega^2\rho(x')\,\varphi(x') \} V(x,x')\,dx'$$

$$= \int_0^1 g(x')\,V(x,x')\,dx' + \Omega^2 \int_0^1 V(x,x')\,\rho(x')\,\varphi(x')\,dx'$$

This is Fredholm eq of the 2nd kind inhomogeneous

then let $\psi(x)$ be the same as before

$$\psi(x) = \underbrace{\int_0^1 \sqrt{\rho(x)}\,g(x')\,V(x,x')\,dx'}_{\mathcal{F}(x)} + \Omega^2 \int_0^1 \underbrace{K(x,x')}_{\text{symmetric}}\,\psi(x')\,dx'$$

if $\Omega \neq \omega$ then $\exists!$ a soln for the given $g(x')$

if $\Omega = \omega$ we do not have a unique soln but have resonance and will require the existence conditions described by Fredholm' Alternative

if $\Omega = \omega$ let $\psi(x) = \sum_{n=1}^{\infty} \alpha_n \psi_n(x)$ where $\psi_n$ are the EF of homog prob

then $\sum \alpha_n \psi_n(x) = \mathcal{F}(x) + \Omega^2 \int_0^1 K(x,x') \sum_{n=1}^{\infty} \alpha_n \psi_n(x')\,dx'$

suppose $\Sigma$ converges uniformly then $\int \Sigma = \Sigma \int$

$$= \mathcal{F}(x) + \Omega^2 \sum_{n=1}^{\infty} \alpha_n \int_0^1 K(x,x')\,\psi_n(x')\,dx'$$

$$+ \Omega^2 \sum_{n=1}^{\infty} \alpha_n \frac{\psi_n(x)}{\omega_n^2}$$

then $\displaystyle\sum_{n=1}^{\infty} \alpha_n \left(1 - \frac{\Omega^2}{\omega_n^2}\right) \psi_n(x) = \mathcal{F}(x)$

$\psi_n$ also satisfies $\int_0^1 \psi_n \psi_m\,dx = 0$ then $\int_0^1 (\Sigma \quad) \psi_m(x) = \int_0^1 \mathcal{F}(x)\,\psi_m(x)$

and $\alpha_m\left(1 - \dfrac{\Omega^2}{\omega_m^2}\right) = \dfrac{\int_0^1 \mathcal{F}(x')\,\psi_m(x')\,dx'}{\int_0^1 \psi_m^2\,dx}$ and $\alpha_m$ is obtained iff $\Omega \neq \omega_m$

if $\Omega = \omega_m$ : for a soln, $\int_0^1 \mathcal{F}\,\psi_m\,dx \equiv 0$

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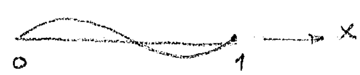
$f(x) = g(x) + \lambda \int_a^b K(x,x') f(x') dx'$ ; another form was $f(x,\lambda) = g(x) + \int_a^b R(x,x';\lambda) g(x') dx'$

where $R$ is the Resolvent kernel $= \dfrac{\hat{D}(x,\hat{x},\lambda)}{D(\lambda)}$

Fredholm argued that $\hat{D}$ & $D$ are entire fns (analytic w/zeroes) and that $D(\lambda)=0$ meant that $\lambda$ were EV's and that solutions existed if $\int_a^b g\psi\, dx = 0$

where $\psi$ is a soln of $f = \int_a^b R(x,x';\lambda) g(x') dx'$

Example Look at a string held taught



Let $y(x,t) = \varphi(x) \cos \omega t$ $\quad \varphi(0) = \varphi(1) = 0$

$\rho(x)\ddot{y} = T y''$ where $T$ is tension

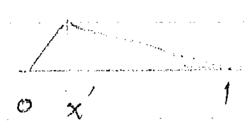
$\therefore \quad T y'' \equiv T\varphi'' \cos\omega t = -\rho\omega^2 \varphi \cos\omega t \equiv \rho\ddot{y}$ thus $\varphi'' = -\dfrac{\omega^2\rho(x)}{T}\varphi$

a solution is $\varphi \equiv 0$ or if $\dfrac{\omega^2\rho}{T}$ is an EV then $\varphi$ has non trivial soln.

Defining a fn $V(x,x') = \begin{cases} x(1-x') & 0 \le x \le x' \\ x'(1-x) & x' < x \le 1 \end{cases}$

this satisfies $\dfrac{d^2V}{dx^2} = 0 \quad x \ne x'$

$V(0,x') = V(1,x') = 0$



$\left.\dfrac{dV}{dx}\right|_{x'-0}^{x'+0} = -x' - (1-x') = -1$

if we use this fn then

$\varphi(x) = \dfrac{\omega^2}{T}\int_0^1 V(x,x')\varphi(x')dx' \cdot \rho(x')$ which satisfies $\varphi'' = -\dfrac{\omega^2\rho}{T}\varphi$

$\varphi(0) = \varphi(1) = 0$

This is a homogeneous integral eqn Fredholm Eq of 1st kind.

Normally has trivial soln. It is not symmetric kernel

$\sqrt{\rho(x)}\,\varphi(x) = \lambda \int_0^1 K(x,x') \sqrt{\rho(x')}\,\varphi(x')\, dx'$

$\psi(x) = \lambda \int_0^1 K(x,x')\psi(x')dx' \qquad K = \sqrt{\rho(x)}\, V(x,x')\sqrt{\rho(x')}$

Suppose an external force $q(x)\cos\Omega t$ acts on string

$$\rho(x)\,\ddot{y} = T\frac{\partial^2 y}{\partial x^2} + q(x)\cos\Omega t$$

if $y = y(\Omega; x, t)$ then $y(x) = \varphi(x)\cos\Omega t$

$$-\rho(x)\,\varphi(x)\,\Omega^2\cos\Omega t = T\,\varphi''(x)\cos\Omega t + q(x)\cos\Omega t$$

thus $\varphi'' = -\frac{\rho(x)\Omega^2}{T}\varphi(x) - \frac{q(x)}{T}$

thus $\varphi(x) = g(x) + \frac{\Omega^2}{T}\int_0^1 V(x,x')\,\rho(x')\,\varphi(x')\,dx' \quad (*)$

where $g(x) = \frac{1}{T}\int_0^1 V(x,x')\,q(x')\,dx' \longrightarrow$ a known fn. static displ profile

now mult $\sqrt{\rho(x)}$ times $(*)$ then if $\psi = \sqrt{\rho}\,\varphi$

$$\psi(x) = \sqrt{\rho(x)}\,g(x) + \bar{\lambda}\int_0^1 K(x,x')\,\psi(x')\,dx'$$

either $\bar{\lambda}$ is an EV or it isn't. If it isn't then can find a unique soln $\psi(x)$. If it is, then a non unique soln exists

Assuming that $\psi(x) = \sum^{\infty} \alpha_n \psi_n(x)$ for purposes of solving the unknown I.E. of the forced motion where

$$\psi_n(x) = \lambda_n\int_0^1 K(x,x')\,\psi(x')\,dx' \quad \text{where } n = 1,\ldots$$

also $\int_0^1 \psi_n\psi_m\,dx = 0$ for $n \neq m$. Now put $\psi(x)$ into diff eq.

$$\sum\alpha_n\,\psi_n(x) = \psi(x) = \bar{\lambda}\int_0^1 K(x,x')\sum\alpha_n\psi_n(x')\,dx' + G(x)$$

then $\int_0^1 K(x,x')\,\psi_n(x)\,dx' = \frac{\psi_n(x')}{\lambda_n}$ and

$$\sum\alpha_n\psi_n = \bar{\lambda}\sum\alpha_n\frac{\psi_n}{\lambda_n} + G(x)$$

thus $\sum\alpha_n\psi_n\left(1 - \bar{\lambda}/\lambda_n\right) = G(x)$

where $\alpha_n\left(1 - \bar{\lambda}/\lambda_n\right) = \int_0^1 G(x)\,\psi_n(x)\,dx$

thus $\psi(x) = \sum_{n=1}^{\infty}\alpha_n\psi_n = \sum_{n=1}^{\infty}\frac{\beta_n\psi_n}{1 - \bar{\lambda}/\lambda_n}$ where $\beta_n = \int_0^1 G(x)\,\psi_n(x)\,dx$

if $\bar{\lambda} \neq \lambda_n$ then we are ok if $\bar{\lambda} = \lambda_n \Rightarrow \beta_n = 0$ ie $\int_0^1 G(x)\psi_n(x)dx$

Another method of finding $f(x)$ by letting $f(x') = g(x')$ in

$$f_1(x) = g(x) + \lambda \int_a^b K(x,x')\, g(x')\, dx'$$

Now $f_2(x) = g(x) + \lambda \int_a^b K(x,x') \left[ g(x') + \lambda \int_a^b K(x',x'')\, g(x'')\, dx'' \right] dx'$

etc $\therefore f(x) = \lim_{n\to\infty} f_n(x) = g(x) + \sum_{n=1}^{\infty} \lambda^n g_n(x)$

$g_1(x) = \int_a^b K(x,x')\, g(x')\, dx'$ $\quad g_2(x) = \int_a^b K_2(x,x')\, g(x')\, dx'$

1st iterated kernel.

Sufficient condition for convergence is that

$$\lambda < \frac{1}{M(b-a)} \qquad M = \max \text{ of } K$$

Refinement by Schmidt

$$\lambda < \frac{1}{\sqrt{\int_a^b \int_a^b K^2(x,x')\, dx\, dx'}}$$

This is the 3rd method of representation

1st is Fredholm

2nd is Hilbert Schmidt

3rd is Neumann

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3 representation for soln of Fredholm's IE of 2nd kind

1. Fredholm showed $f(x) = \dfrac{\hat{D}(\lambda, x)}{D(\lambda)} = \dfrac{d_0 f_0(x) + d_1 \lambda f_1(x) + d_2 \lambda^2 f_2(x) + \cdots}{1 + c_1\lambda + c_2\lambda^2 + \cdots}$

2 power series in $\lambda$. Power series converges $\forall \lambda$ ($\lambda$ being complex); these fns are entire fns of $\lambda$. Where $f_0(x)$ & $f_1(x)$ .... are found by evaluation of kernel at particular values. $D(\lambda) = 0$ when $\lambda$ are EV of $f(x) = \lambda \int K f\, dx$

2. Liouville - Neumann $f(x) = g(x) + \lambda f_1(x) + \lambda^2 f(x) + \cdots$

formed by iteration. And is normally restricted to $\lambda < \dfrac{1}{M(b-a)}$

thus by taking $\frac{1}{D(\lambda)}$ and write it as $1 + e_1\lambda + e_2\lambda^2 + \ldots$ for $\lambda <$ ___
and mult by $\hat{D}(\lambda, x)$ would give L-N soln.

thus Fredholm form is continuation (complex) of L-N soln.

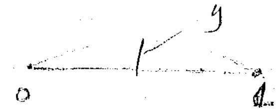
3. Hilbert Schmidt showed that

$$f(x) = g(x) + \sum_{n} \frac{c_n \psi_n(x)}{\lambda - \lambda_n} \qquad \text{where } \psi_n(x) = \lambda_n \int_a^b K(x,x')\psi_n(x')\,dx'$$

if $\lambda = \lambda_n$ then $\quad c_n = \int_a^b g_n \psi_n(x)\,dx = 0$

Example look at buckling of beam

$$\frac{d}{dx}\left(EI\frac{dy}{dx}\right) + Py = 0$$




w/ $y(0) = y(1) = 1$ for $0 < x < 1$

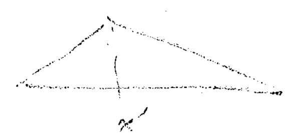
leads $\quad y(x) = \lambda \int_0^1 K(x,x')\,y(x')\,dx'$. This normally has trivial soln. unless $P/EI$ is a critical EV.

how to turn it into integral eqn. if $EI \neq f(x)$

$$y'' = -\frac{Py}{EI} \qquad \underline{\text{recast}} \qquad \frac{d^2K(x,x')}{dx^2} = 0, \quad x \neq x'$$

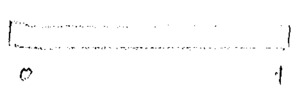
$$K(0,x') = K(1,x') = 0$$

$$\left.\frac{dK}{dx}\right|_{x'-0}^{x'+0} = -1$$




where $K = \begin{matrix} x(1-x') \\ x'(1-x) \end{matrix}$

Example look at diffusion eqn.




$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x)\cos\omega t$$

heat source etc.

temp.
$u(0,t) = u(1,t) = 0$
$u(x,t) = $ Temp

Convert this to an integral eqn.
if we had two derivatives normally, let $u(x,t) = \varphi(x)\cos\omega t$
but since we only one deriv in t let $u = \varphi(x)\cos\omega t + \psi(x)\sin\omega t$
put into diff eq

$\therefore \quad -\omega\varphi\sin\omega t + \omega\psi\cos\omega t = \varphi''\cos\omega t + \psi''\sin\omega t + g\cos\omega t$
equating like coeff of cos & sin $\omega t$

then $\varphi'' - \omega\psi = -g(x)$ $\quad\psi'' + \omega\varphi = 0$ $\quad$ Coupled eqns. ODE

let $\varphi'' = \omega\psi - g(x) \quad\Rightarrow\quad \varphi(x) = -\omega\int_0^1 K(x,x')\,\psi(x')\,dx' + f(x)$

where $f(x) = \int_0^1 K(x,x')\,g(x')\,dx'$

where $K(x,x') = \begin{matrix} x(1-x') & 0<x<x' \\ x'(1-x) & x'<x<1 \end{matrix}$

also $\psi'' = -\omega\varphi \quad\Rightarrow\quad \psi(x) = \omega\int_0^1 K(x,x')\,\varphi(x')\,dx'$

now put this into 1st eq

$$\varphi(x) = -\omega^2\int_0^1 K_2(x,x')\,\varphi(x')\,dx' + f(x)$$

where $K_2$ is first iterate to the Kernel.

$$K_2(x,x') = \int_0^1 K(x,y)\,K(y,x')\,dy$$

for $\psi$ put 1st eq into 2nd

$$\psi(x) = -\omega^2\int_0^1 K_2(x,x')\,\psi(x')\,dx' + g(x)$$

where $g(x) = \omega\int_0^1 K_2(x,x')\,g(x')\,dx'$

now we use Fredholm: eqns have unique sol if corresponding homog. sol. has no EV which must be positive. EV of $K$ are $+$ & EV of $K_2 = (\text{EV of } K)^2 > 0$

therefore sol. exists no matter what $\omega$ is

$$\varphi(x) = f(x) - \omega^2\sum_n \frac{c_n\varphi_n(x)}{\omega_n^2+\omega^2} \qquad c_n = \int_0^1 f(x)\,\varphi_n(x)\,dx$$

note there is no resonance

and $\varphi_n(x) = \omega_n^2\int_0^1 K_2(x,x')\,\varphi_n(x')\,dx'$

$$\psi(x) = g(x) - \omega^2\sum \frac{d_n\psi_n(x)}{\omega_n^2+\omega^2} \qquad d_n = \int_0^1 g(x)\,\psi_n(x)\,dx$$

and $\psi_n(x) = \omega_n^2\int_0^1 K_2(x,x')\,\varphi_n(x')\,dx'$

Another example

$$\frac{d^2\varphi}{dx^2} - g\ln x \cdot \varphi = 0 \qquad 0 < x < \infty \quad \varphi(0) = 1 \quad \varphi(\infty) = 0$$

Desired $\varphi'(0)$

if $g$ is smaller we can write eqn in fredholm type.

let $x = \tau y \Rightarrow \frac{d^2\varphi(y)}{dy^2} - g\tau^2\ln\tau\left[1 + \frac{\ln y}{\ln\tau}\right]\varphi = 0$

choose $\tau \Rightarrow g\tau^2\ln\tau = 1$ if $g$ is small $\tau \uparrow$

$\frac{\ln y}{\ln\tau} \sim \epsilon \ln y$

$\therefore \quad \varphi''(y) - (1 + \epsilon\ln y)\varphi = 0 \qquad \underline{\epsilon = \ln\tau}$

by choosing $K(y,y') = \frac{1}{2}e^{-|y-y'|}$

10/13

HW Determine & Characterize $\vec{EF}$ & EV

$$\varphi(x) = \lambda\int_0^1 e^{-|x-x'|}\varphi(x')\,dx' \qquad \text{what are the } \varphi(x) \text{ \& } \lambda$$

Hint: convert to DE

$$\varphi(x) = \lambda\int_0^{2\pi}\sin(x+x')\,\varphi(x')\,dx' \qquad \text{expand sin; degenerate kernel}$$

Characterize the $\vec{EF}$ & EV of the kernel $K(x,x') = \begin{cases} x' & 0 \le x' \le x < 1 \\ x & 0 \le x \le x' \le 1 \end{cases}$

ie $\varphi(x) = \lambda\int_0^1 K(x,x')\,\varphi(x')\,dx'$

$= \lambda\int_0^x x'\varphi(x')\,dx' + \lambda x\int_x^1 \varphi(x')\,dx$

continuation of last time

$\varphi'' - g\ln x\,\varphi = 0 \qquad 0 < x < \infty$ since we have ln.

w/ $\varphi(0) = 1, \quad \varphi(\infty) = 0 \qquad$ want $\varphi'(0)$

for $g$ small : let $x = \tau y$ then $\varphi'' - g\tau^2[\ln\tau + \ln y]\varphi(y) = 0$

with $g\tau^2\ln\tau = 1$ & $\epsilon = (\ln\tau)^{-1}$ $\quad \varphi'' - (1 + \epsilon\ln y)\varphi = 0$ for $0 < y < \infty$

Consider now a fn (Green's fn)

$$\left(\frac{d^2}{dy^2} - 1\right)G = 0 \qquad y \neq y'$$

$G = e^{y}, e^{-y}$ & also $\left.\frac{d}{dy}G\right|_{y'-0}^{y'+0} = 1$

and $G \to 0$ as $y \to \infty$

to find $G$ $\quad 0<y<y'$ the soln is $G = e^{y} + e^{-y}$

$y'<y<\infty$ the soln is $G = e^{-y}$

then piece together s.t. $\frac{dG}{dy}$ is satisfied

thus $$G(y;y') = \tfrac{1}{2} e^{-|y-y'|}$$

The fn. $$\Phi_1(y) \overset{\text{defn.}}{=} \frac{\epsilon}{2}\int_0^\infty e^{-|y-y'|} \ln y' \varphi(y') dy'$$

satisfies $$\left(\frac{d^2}{dy^2} - 1\right)\Phi_1 = -\epsilon \ln \varphi(y)$$

to show this take $\int_0^\infty = \int_0^y + \int_y^\infty$

or recognize $$\left(\frac{d^2}{dy^2} - 1\right) G = -\delta(y-y'); \quad -\int_{y'-\eta}^{y'+\eta} \delta(y-y')\, dy = -1 \quad \text{and } \eta > 0$$

$$\left.\frac{dG}{dy}\right|_{y'-0}^{y'+0} = \int^{y'+\eta}$$

$$\therefore \quad \text{Let } \left(\frac{d^2}{dy^2} - 1\right)\Phi_1 = -\frac{\epsilon}{2}\left(\frac{d^2}{dy^2} - 1\right)\int_0^\infty \delta(y-y') \ln y' \varphi(y') dy'$$

$$= -\epsilon/2 \frac{d^2}{dy^2}\int + \epsilon/2 \ln y\, \varphi(y) = -\epsilon/2 \frac{d}{dy}\left[-\frac{d}{dy} \ln y' \varphi(y')\Big|_{y'=y}\right] + \epsilon/2\, l$$

$$= -\epsilon \ln y\; \varphi(y)$$

Now define $$\Phi_2(y) = \varphi(y) + \Phi_1(y)$$

then $$\left(\frac{d^2}{dy^2} - 1\right)\Phi_2(y) = \left(\frac{d^2}{dy^2} - 1\right)\varphi(y) + \left(\frac{d^2}{dy^2} - 1\right)\Phi_1(y)$$

$$= \underbrace{\epsilon \ln y\, \varphi(y)}_{\text{from original d.e.}} - \underbrace{\epsilon \ln y\, \varphi(y)}_{\text{by defn of } \Phi_1} = 0$$

to satisfy that $\varphi \to 0$ as $y \to \infty$ then soln is $\Phi_2(y) = Ce^{-y}$

$$\varphi(y) = Ce^{-y} - \Phi_1(y) = Ce^{-y} - \frac{\epsilon}{2}\int_0^\infty e^{-|y-y'|} \ln y' \varphi(y') dy' \qquad 0<y<\infty$$

This recommends itself to iteration; $\varphi(0) = C - \epsilon/2 \int_0^\infty e^{-y'} \ln y' \varphi(y') dy' = 1$

$$\Rightarrow -\epsilon/2 \int_0^\infty e^{-y'} \ln y' \varphi(y') dy' = -C + 1$$

what is $\left.\frac{d\varphi}{dy}\right|_{y=0} = \tau A$ since $\dot\varphi(0) = A$

$$A = \frac{1}{\tau}\left.\frac{d\varphi}{dy}\right|_{y=0} = -C - \epsilon/2\left[\int_0^y + \int_y^\infty\right]$$

$$= -C - \frac{\epsilon}{2}\frac{d}{dy}\int_0^y e^{-(y'-y)} \ln y' \varphi(y') dy'$$

thus

$$= -C - \epsilon/2 \int^\infty e^{-y'} \ln y' \varphi(y') dy' = -2C + 1$$

now to solve by iteration

$$\varphi(y) = \underbrace{Ce^{-y}}_{} - \frac{\epsilon}{2}\int_0^{\infty} e^{-|y-y'|}\ln y'\,\varphi(y')\,dy'$$

to get a value of $C$ (put into $1 = C - \epsilon/2 \int \ldots$ to get $C$ then) put into $A = \frac{1}{\epsilon}\left[-2C+1\right]$ to get $A$.

Thus we can find

$$A = -g^{1/2}\epsilon^{-1/2}\left\{1 - \tfrac{1}{2}\epsilon\,(\ln 2+\gamma) - \tfrac{1}{8}\epsilon^2\left[\tfrac{1}{3}\pi^2 + (\ln 2+\gamma-1)^2 - 3\right] + O(\epsilon^3)\right\}$$

$$\gamma = \lim_{n\to\infty}\left(\textstyle\sum_{k=1}^{n} \tfrac{1}{k} - \ln n\right) = .577 \quad \text{Euler constant.}$$

if $\varphi(y') = Ce^{-y'}$ what is $\int_0^{\infty} e^{-2y'}\ln y'\,dy'$

let $I(\alpha) = \int_0^{\infty} e^{-\alpha y}\ln y\,dy \qquad \alpha > 0$

$$\frac{dI}{d\alpha} = -\int_0^{\infty} e^{-\alpha y}\, y\ln y\,dy$$

$$= \int_0^{\infty}\left(\frac{d}{dy}\left(\frac{e^{-\alpha y}}{\alpha}\right)dy\right) y\ln y = -\int_0^{\infty}\frac{e^{-\alpha y}}{\alpha}\left[1+\ln y\right]dy \quad \text{int by parts}$$

$dv$, $u$, $v = \frac{e^{-\alpha y}}{\alpha}$, $du = (1+\ln y)\,dy$

$$\frac{dI}{d\alpha} = -\frac{1}{\alpha}I - \frac{1}{\alpha}\int_0^{\infty} e^{-\alpha y}dy$$

$$\frac{dI}{d\alpha} = -\frac{1}{\alpha}I - \frac{1}{\alpha^2} \quad \text{or} \quad \alpha I'(\alpha) + I = -\frac{1}{\alpha}$$

$$\frac{d}{d\alpha}(\alpha I) = -\frac{1}{\alpha}$$

$\alpha I = -\ln\alpha + C.$

or $\quad I(\alpha) = -\frac{1}{\alpha}\ln\alpha + \frac{C}{\alpha}$

what is $C$? $C = I(1)$

Proof $\qquad I(2) = \int_0^{\infty} e^{-2y}\ln y\,dy = \int_0^{\infty} e^{-s}\ln(s/2)\,\frac{ds}{2} = \frac{I(1)}{2} - \frac{1}{2}\ln 2$

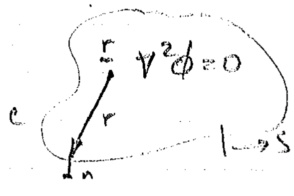
$\therefore\ I(2) = -\frac{\ln 2}{2} + \frac{C}{2}$ from $I(\alpha) = -\frac{1}{\alpha}\ln\alpha + C/\alpha$

also

$I(2) = -\frac{\ln 2}{2} + \frac{I(1)}{2} \qquad \therefore\ C = I(1)$

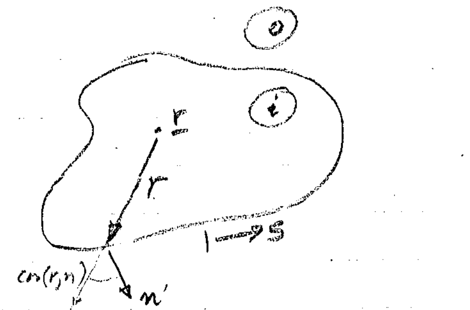
Fredholm Alternative for 1st BVP of potential theory

i.e. to find a harmonic function within a smooth plane curve which takes on prescribed values over the curve ie Dirichlet problem




$\phi = f(s)$ on $C$

for a point $\underline{r}$ in interior




$$\phi(\underline{r}) = \int_C \mu(s) \frac{\partial}{\partial n} \ln\left(\frac{1}{r}\right) ds$$

with $\mu(s)$ being double layer density

This is the double layer potential

Single layer potential $\quad \phi(\underline{r}) = \int_C \nu(s) \underbrace{\ln\left(\frac{1}{r}\right)}_{\text{Green's fn.}} ds \qquad \nu$ is single layer density

can let $\underline{r} \to$ bdy & this single layer potential eqn gives Fredholm of 1st kind: not too good. By using double layer potential gives rise to Fredholm of 2nd kind

10/15/80

Aside // Going back to evaluation of $I(1) = \int_0^\infty e^{-\zeta} \ln \zeta \, d\zeta$ from last time

now to evaluate $\int_0^\infty e^{-\zeta} \ln \zeta \, d\zeta$. look at $I(\alpha) = \int_0^\infty e^{-\zeta} \zeta^\alpha d\zeta$

$$= \int_0^\infty e^{-\zeta} e^{\alpha \ln \zeta} d\zeta \quad \& \quad \frac{dI}{d\alpha} = \int_0^\infty e^{-\zeta} \ln \zeta \, e^{\alpha \ln \zeta} d\zeta$$

thus when $\alpha = 0 \quad \frac{dI}{d\alpha} = I(\alpha)\Big|_{\alpha=0}$

Now $\quad \Gamma(z) = \int_0^\infty e^{-\zeta} \zeta^{z-1} d\zeta \qquad \Gamma(1) = 1 \;\; \Gamma(2) = 1 \;\; \Gamma(3) = 2$

for us $I(\alpha) = \int_0^\infty e^{-\zeta} \zeta^\alpha d\zeta = \Gamma(1+\alpha)$

$$\therefore \frac{dI}{d\alpha} = \frac{d}{d\alpha} \Gamma(1+\alpha)\Big|_{\alpha=0}$$

$\ln \Gamma(1+z) = -\gamma z + \sum_{n=2}^\infty (\quad) z^n \quad$ for $|z|<1 \qquad \gamma$ = Euler Const.

$\frac{d}{dz} \ln \Gamma(1+z) = -\gamma + \sum_{n=2} n(\quad) z^{n-1} \qquad$ now $\left.\dfrac{\frac{d\Gamma(1+z)}{dz}}{\Gamma(1)}\right|_{z=0} = -\dfrac{\gamma}{1} = -\gamma$

$\dfrac{\frac{d}{dz}\Gamma(1+z)}{\Gamma(1+z)}$

$$\therefore I(\alpha) = -\gamma \qquad \therefore I(1) = \int_0^\infty e^{-z} \ln \zeta \, d\zeta = -\gamma$$

$$\therefore I(2) = -\tfrac{1}{2} \ln 2 - \frac{\gamma}{2} \quad \text{and} \quad I(\alpha) = -\tfrac{1}{\alpha} \ln \alpha - \frac{\gamma}{\alpha} \quad //$$

Now returning to the BVP (Dirichlet Problem)

we remember from last time that

$$\phi(\underline{r}) = \int_C \mu(s') \frac{\partial}{\partial n'} \ln r \, ds'$$

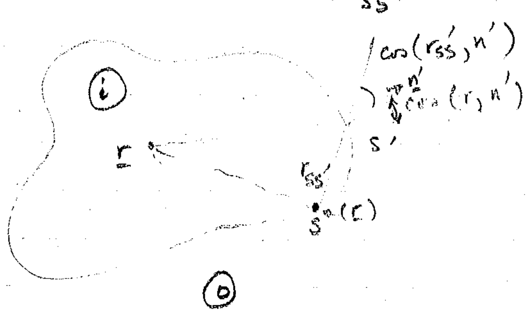
$r = \sqrt{(x-x')^2 + (y-y')^2}$ and $\nabla^2 \ln r = 0$ for $x \neq x'$, $y = y'$

thus $\phi(\underline{r}) = \int_c \mu(s') \frac{\cos(r, n')}{r}\, ds'$

suppose $\underline{r}$ moves onto the curve $C$

$$\phi_i(\underline{r})\Big|_{at\, C} = f(s) = \pi\mu(s) + \int_C \mu(s') \frac{\cos(r, n')}{r_{ss'}}\, ds'$$

where $r = r_{ss'}$




$$\mu(s) = \frac{f(s)}{\pi} - \frac{1}{\pi}\int_C \mu(s') \frac{\cos(r_{ss'}, n')}{r_{ss'}}\, ds'$$

when $s \to s'$ $\cos(r_{ss'}, n') \to \cos(90°) = 0$

$r_{ss'} \to 0$

$\therefore \frac{\cos(r_{ss'}, n')}{r_{ss'}}$ is well behaved & is $\propto$ to curvature.

by fredholm alternative : unique sol unless $\exists$ a non trivial sol. to assoc homog eqn.

Assoc Homog Eq is $\pi\bar{\mu}(s) + \int_C \bar{\mu}(s') \frac{\cos(r_{ss'}, n')}{r_{ss'}}\, ds' = 0$

Is there a non trivial sol $\bar{\mu}$ ??

let $\bar{\Phi} = \int_C \bar{\mu}(s') \frac{\cos(r, n')}{r}\, ds'$ now $\bar{\phi}_i$ at $C = 0$ by virtue of passing from the interior, integral eqn for $\bar{\mu}$

but since $\bar{\phi}$ satisfies $\nabla^2\bar{\phi} = 0$ w $\bar{\phi} = 0$ on $C$ $\Rightarrow$ $\bar{\phi}_i \equiv 0$ in D

thus $\frac{\partial \bar{\phi}_i}{\partial n} = 0$ at $C$

& there is a unique sol.

$\bar{\phi}_o(r)$ at $C$ $= -\pi\mu + \int_C \mu(s') \frac{\cos(r_{ss'}, n')}{r_{ss'}}\, ds'$

$\mu(s) = \frac{1}{2\pi}(\phi_i - \phi_o)$ at $C$

but $\phi$ = discont at $C$ but the normal deriv is continuous there

$\Rightarrow \frac{\partial}{\partial n}\bar{\phi}_o = 0$ at $C$

Consider now for the exterior problem

$$\int_{C_1} (\nabla\bar{\phi})^2\, dx\, dy = \int_{C_1} \left(\frac{\partial\bar{\phi}}{\partial x}\right)^2 + \left(\frac{\partial\bar{\phi}}{\partial y}\right)^2 dx\, dy$$

$$= \int_{Ext} \left\{ \nabla\cdot(\bar\phi \nabla\bar\phi) - \cancel{\bar\phi \nabla^2 \bar\phi} \right\} dx\,dy$$

since $\nabla^2 \bar\phi = 0$

$$= \int_C \underset{\sim}{n}\cdot(\bar\phi \nabla\bar\phi)\, ds + \int_{C_\infty} \underset{\sim}{n}\cdot(\bar\phi\nabla\bar\phi)\, dS$$

$$= \int_C \Big(\bar\phi \frac{\partial \bar\phi}{\partial n}\Big) ds + \int_{C_\infty} \bar\phi \frac{\partial\bar\phi}{\partial n}\, dS \quad = 0$$

$= 0$ on $C$ and $\bar\phi \to 0$ at $\infty$

$\therefore\ \bar\phi = \text{const}$ but $\bar\phi \to 0$ at $\infty$ $\therefore$ const $= 0$

$\therefore\ \bar\phi_i$ at $C = 0 = \bar\phi_0 \implies \bar\mu(s) = \frac{1}{2\pi}(\bar\phi_i - \bar\phi_0)$ at $C$

$\implies \bar\mu(s) \equiv 0$

Thus we have no nontriv sol to the Assoc Homog eqn $\therefore\ \exists!$ a

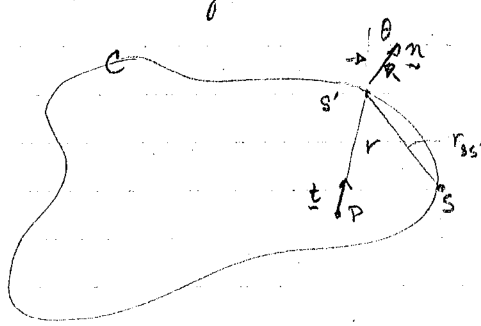
sol of $\mu(s)$

thus $f(s) = \pi\mu(s) + \int_C \mu(s') \frac{\cos(r_{ss'}, n')}{r_{ss'}}\, ds'$ can be solved

discretely for $\mu(s)$. Now put $\mu(s)$ in $\phi(\underset{\sim}{r}) = \int_C \mu(s) \frac{\partial}{\partial n} \ln r\, ds$ to get $\phi(\underset{\sim}{r})$ everywhere in $D$

10/17/80

Returning to last lecture




$\nabla^2\phi(\underset{\sim}{r}) = 0$ in $D$ and $\phi(s) = f(s)$ on $C$

we found $\phi(\underset{\sim}{r}) = \int_C \mu(s') \frac{\partial}{\partial n} \ln r\, ds'$

$\underset{\sim}{r} = (x,y) \qquad r = \sqrt{(x-x')^2 + (y-y')^2}$

$$\frac{\partial \ln r}{\partial n}\ \underset{\sim}{n}\cdot\nabla' \ln r$$

$$= \underset{\sim}{n}\cdot\left\{ \underset{\sim}{i}\frac{\partial}{\partial x'} + \underset{\sim}{j}\frac{\partial}{\partial y'} \right\} \ln\sqrt{(x-x')^2 + (y-y')^2}$$

$$= \underset{\sim}{n}\cdot\left\{ \underset{\sim}{i}\,\frac{(x'-x)}{r} + \underset{\sim}{j}\,\frac{(y'-y)}{r} \right\}$$

$$= \underset{\sim}{n}\cdot\underset{\sim}{t} = \cos\theta \qquad |\underset{\sim}{t}| = 1$$

if P → to point on boundary

$$f(s) = \pi\mu(s) + \int_C \mu(s') \frac{\cos(r_{ss'}, n)}{r_{ss'}}\, ds'$$

to show $\int_C \frac{\partial}{\partial n} \ln r \, ds' = \pi$


$$\int_C \frac{\partial}{\partial n} \ln r \, ds' = \int_D \nabla^2 \ln r \, dA$$

$$\int_D \left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right) \ln \sqrt{(x-x')^2+(y-y')^2}\, dx'dy'$$

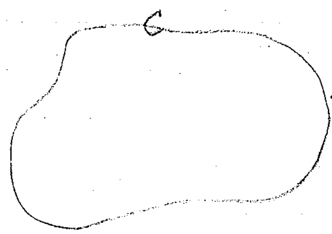
$$\nabla^2 \ln r = 0 \quad \text{if } x,y \neq x',y'$$

$$\int_D \nabla'^2 \ln \tfrac{1}{r} dA' = -2\pi\, \delta(x-x')\,\delta(y-y')$$

$$\int_D \nabla^2 \ln r \, dA' = 2\pi \quad \text{if } x,y \text{ in } D$$

$\therefore$ if $x,y$ ~~in~~ on $C \Rightarrow \int_C \nabla^2 \ln r \, dA' = \pi$

<u>Dirichlet Exterior Problem</u> $\quad \phi \not\to 0$ at $\infty$ $\quad$ thus $\phi(\underline{r}) = \int_C \mu(s') \frac{\partial}{\partial n} \ln r \, ds' + \hat{C}$




$\phi = f(s)$

$\hat{C} = $ const.

on the bdy $\quad f(s) = -\pi\mu(s) + \int_C \mu(s') \frac{\cos(r_{ss'}, n)}{r_{ss'}} ds + \hat{C}$

$$\therefore \quad \mu(s) = \tfrac{1}{\pi}\left[\hat{C} - f(s)\right] + \frac{1}{\pi}\int_C \frac{\cos(r_{ss'}, n')}{r_{ss'}} \mu(s')\, ds'$$

here $K(s,s') = \dfrac{\cos(r_{ss'}, n)}{r_{ss'}}$

The soln exists if $\quad \bar{\mu}(s) = \frac{1}{\pi}\int_C \dfrac{\cos(r_{ss'}, n')}{r_{ss'}} \bar{\mu}(s')\, ds' \quad$ Ⓐ

the nontrivial sol is $\bar{\mu}(s) = 1$ since $\int_C \dfrac{\cos(r_{ss'}, n)}{r_{ss'}} ds' = \pi$

infact $\bar{\mu}(s) = $ const is also a sol.

since the problem has a nontrivial sol. then $\bar{\nu}(s) = \frac{1}{\pi}\int_C K(s',s)\,\bar{\nu}(s')\, ds'$ has a nontriv sol. Since this is true then

$$\mu(s) = \tfrac{1}{\pi}\left[\hat{C} - f(s)\right] + \frac{1}{\pi}\int K(s,s')\,\mu(s')\, ds'$$

has a sol if $\quad \int_C \tfrac{1}{\pi}\left[\hat{C} - f(s)\right] \bar{\nu}(s)\, ds' = 0$

$$\hat{C} = \frac{\int_C f(s)\,\bar{\nu}(s')\, ds'}{\int_C \bar{\nu}(s')\, ds'}$$

thus $\mu(s) = $ sol. to eqn + const times homog associated eqn * now $\phi(\underline{r}) = \ln \left\{ \dfrac{1}{\sqrt{(x-x_s)^2 + (y-y_s)^2}} \right\} + \int \sigma(x') \ln \dfrac{1}{\sqrt{(x-x')^2 + y^2}}\, dx'$

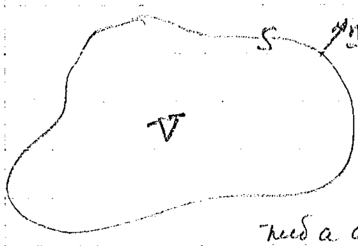
for the following: $\cdot (x_s, y_s)$ loc of a line source

screen — with $\phi = 0$

$$I E \ (1^{st} \text{ kind}) = \ln \frac{1}{\sqrt{(x-x_s)^2 + y_s^2}} = \int_{x'=0}^{\infty} \ln |x - x'| \, \sigma(x') \, dx', \quad x > 0$$

10/20/80

Let us now look at Neumann Problems



$\nabla^2 \phi(\underline{r}) = -4\pi \rho(\underline{r})$ in $V$ Poisson Eqn.

w/ $\dfrac{\partial \phi}{\partial n} = 0$ $\forall$ $\underline{r}$ on $S = \partial V$

need a constraint because

$$\int_V \nabla^2 \phi \, dV = \int_S \frac{\partial \phi}{\partial n} \, dS = 0 \quad \text{since } \frac{\partial \phi}{\partial n} = 0 \text{ on } S$$

$$\therefore \Rightarrow \int_V \rho(\underline{r}) \, dV = 0 = \int \nabla^2 \phi \, dV \quad \text{this is a constraint}$$

This turns out to be the orthog condition in the Fredholm Alternative

let us define $\psi(\underline{r}) \stackrel{def}{=} \int_V \dfrac{\rho(\underline{r}')}{|\underline{r} - \underline{r}'|} dV' \Rightarrow \nabla^2 \psi = -4\pi \rho(\underline{r}) \quad \underline{r} \neq \underline{r}'$

now let $\phi = \psi(\underline{r}) + \chi(\underline{r}) \Rightarrow \nabla^2 \chi(\underline{r}) = 0$ since $\nabla^2 \phi = \nabla^2 \psi = -4\pi \rho(\underline{r})$

and $\dfrac{\partial \chi}{\partial n} = -\dfrac{\partial \psi}{\partial n}$ at $\underline{r}$ on $S$.

Suppose we define $\chi(\underline{r}) \stackrel{def}{=} \int_S \dfrac{\sigma(\underline{r}')}{|\underline{r} - \underline{r}'|} dS'$ with $\sigma$ being a simple source distrib

now what is $\dfrac{\partial \chi}{\partial n}$ as $\underline{r} \to S$ from the interior, must now say something about $\sigma$

$$\frac{\partial \chi}{\partial n} = 2\pi \sigma(\underline{r}) + \int_S \frac{\partial}{\partial n} \left( \left| \frac{1}{\underline{r} - \underline{r}'} \right| \right) \sigma(\underline{r}') \, dS' = -\frac{\partial \psi}{\partial n} \ ; \ \underline{r} \text{ on } S$$

Fredholm I.E. for $\sigma$ since

$$\sigma(\underline{r}') + \frac{1}{2\pi} \int_S K(\underline{r}, \underline{r}') \, \sigma(\underline{r}') \, dS' = \frac{-1}{2\pi} \frac{\partial \psi}{\partial n} \quad \underline{r} \text{ on } S$$

This is an IE w/ a polar Kernel.

for the exterior problem $\frac{\partial \chi}{\partial n} = -2\pi\sigma + \dots$

for a soln we would need to find sol. to $\sigma(\underline{r}) = -\frac{1}{2\pi}\int_S K(\underline{r},\underline{r}')\,\sigma(\underline{r}')\,dS'$ w/ Eigenvalue $-\frac{1}{2\pi}$

Now let's derive a second integral eq using green's theorem.

$$\int_V (\Phi \nabla'^2 \psi - \psi \nabla'^2 \Phi)\,dV' = \int_S \left(\Phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial}{\partial n'}\Phi\right) dS'$$

let $\Phi = \phi(\underline{r})$ and $\psi = \frac{1}{|\underline{r}-\underline{r}'|}$

then
$$\int_S \left(\Phi \frac{\partial\psi}{\partial n'} - \psi\frac{\partial}{\partial n'}\Phi\right) dS = \int_S \left[\phi(\underline{r}')\frac{\partial}{\partial n'}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right) - \frac{1}{|\underline{r}-\underline{r}'|}\overbrace{\frac{\partial \phi(\underline{r}')}{\partial n'}}^{0 \text{ by b.c.}}\right] dS'$$
$$= \int_V \left[\phi(\underline{r}')\underbrace{\nabla'^2 \frac{1}{|\underline{r}-\underline{r}'|}}_{(-4\pi\delta(\underline{r}-\underline{r}'))} - \frac{1}{|\underline{r}-\underline{r}'|}\underbrace{\nabla'^2\phi(\underline{r}')}_{(-4\pi\rho(\underline{r}'))}\right] V$$
$$= 4\pi\int_V \frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}\,dV' + \begin{cases} 0 & \text{if } \underline{r} \text{ is outside } V \\ -4\pi\phi(\underline{r}) & \text{if } \underline{r} \text{ is inside } V \\ -2\pi\phi(\underline{r}) & \text{if } \underline{r} \text{ is on } S \end{cases}$$

thus
$$\int_S \phi(\underline{r}')\underbrace{\frac{\partial}{\partial n'}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)}_{K(\underline{r}',\underline{r})} dS' = 4\pi\int_V \frac{\rho(\underline{r}')\,dV'}{|\underline{r}-\underline{r}'|} - 2\pi\,\phi(\underline{r})$$

This is the kernel

The assoc. homog. eq is $\int_S \phi(\underline{r}')K(\underline{r}',\underline{r})\,dS' = -2\pi\,\phi(\underline{r})$

There is a non triv $\phi_0(\underline{r}') = 1$ on a const. since $\int_S K(\underline{r}',\underline{r})\,dS' = -2\pi$ $\quad \lambda = -\frac{1}{2\pi}$

$\Rightarrow$ There exists a non-triv sol. to $2\pi\,\sigma(\underline{r}) + \int_S K(\underline{r},\underline{r}')\,\sigma(\underline{r}')\,dS' = 0$

The existence of a sol. to $2\pi\sigma + \int_S \quad = -\frac{\partial\psi}{\partial n}$ means $\int_S \frac{\partial\psi}{\partial n}\phi_0(\underline{r})\,dS = 0$

or
$$\int_S \frac{\partial\psi}{\partial n}\,dS = \int_S dS\int_V \rho(\underline{r}')\frac{\partial}{\partial n}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)dV' = 0$$
$$= \int_V \rho(\underline{r}')\,dV'\underbrace{\int_S \frac{\partial}{\partial n}\frac{1}{|\underline{r}-\underline{r}'|}\,dS}_{-4\pi \text{ since } \underline{r} \text{ in } V} = 0 \Rightarrow \int_V \rho(\underline{r}')\,dV' = $$

is exactly the result we had before.

now the condition for soln of $\int_S \phi\, K(\underline{r}', r')\, dS' = -2\pi\phi + 4\pi \int_V \frac{\rho(r')}{|\underline{r}-\underline{r}'|}\, dS$

is that $4\pi \int_V \frac{\rho(r')\,dS}{|\underline{r}-\underline{r}'|}$ must be orthog to soln of $2\pi\bar{\sigma}(\underline{r}) + \int_S \frac{\partial}{\partial n}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)\bar{\sigma}(\ldots$

thus $\int_S \bar{\sigma}(\underline{r}) \left[\int_V \frac{\rho(\underline{r}')\,dV'}{|\underline{r}-\underline{r}'|}\right] dS = 0$

No class 10/21

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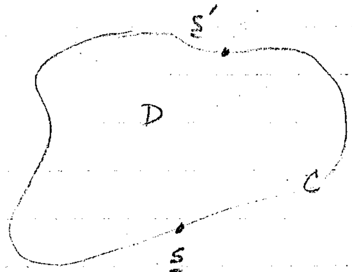
Looking at Eqns of First Kind

Given



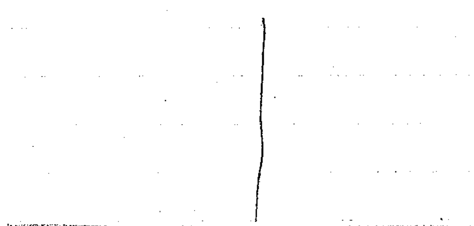

$\underline{s}', \underline{s}$ are points on C. With each pt. we assign normals $\underline{n}_i, \underline{n}_e$ (interior & exterior) and let s define a running parameter

⊛ then $\phi(\underline{s}) = \int_C \ln|\underline{s}-\underline{s}'|\, \sigma(\underline{s}')\, ds'$

ie $\nabla^2\phi = 0$ in D & $\phi = \phi(\underline{c})$ on C

Now $\int_C \frac{\partial}{\partial n_e'} \ln|\underline{s}-\underline{s}'|\, ds' = \pi = -\int_C \frac{\partial}{\partial n_i'} \ln|s-s'|\, ds'$

Another such problem
look at soln of $\nabla^2\phi = 0$ for $y > 0$



$$\phi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\,\phi(x',0)}{y^2 + (x-x')^2}\, dx'$$

$$= \frac{1}{\pi}\int_0^{\infty} y\,\phi(x',0)\left\{\frac{1}{(x-x')^2+y^2} + \frac{1}{(x+x')^2+y^2}\right.$$

now let $x = 0$ $\quad = \frac{2}{\pi}\int_0^{\infty} \frac{y\,\phi(x',0)\,dx'}{x'^2+y^2} = \phi(0,y)$

$$\therefore \frac{\phi(0,y)}{y} = \frac{2}{\pi}\int_0^{\infty} \frac{\phi(x',0)}{x'^2+y^2}\, dx'$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{x'}{x'^2+y^2}\, \frac{\phi(x',0)}{x'}\, dx'$$

Going back to beginning

If (*) has a soln it generates an exterior potential

for $\underline{r} \notin D$ $\qquad \phi_e(\underline{r}) = \int_c \ln|\underline{r}-\underline{s}'|\sigma(\underline{s}')ds'$ and an interior potential

$\underline{r} \in D$ $\qquad \phi_i(\underline{r}) = \int_c \ln|\underline{r}-\underline{s}'|\sigma(\underline{s}')ds'$

Now $\frac{\partial}{\partial n_i}\phi_i(\underline{r}) = \int_c \frac{\partial}{\partial n_i}\ln|\underline{r}-\underline{s}'|\sigma(\underline{s}')ds' + \pi\sigma(\underline{r})$

$\frac{\partial}{\partial n_e}\phi_e(\underline{r}) = \int \frac{\partial}{\partial n_e} \quad \ldots \quad + \pi\sigma(\underline{r})$

$\Rightarrow \quad \frac{\partial}{\partial n_i}\phi_i + \frac{\partial}{\partial n_e}\phi_e = 2\pi\sigma(\underline{s}')$ or discont. in normal derivative of $\phi$

is proportional to $\sigma(\underline{r})$.

Mushkelishvili: $\int_c \ln|\underline{s}-\underline{s}'|\lambda(\underline{s}')ds' = 1 \qquad$ (**)

showed that a non $\int_c \ln|\underline{s}-\underline{s}'|\lambda(\underline{s}')ds' = 0 \qquad$ (***)

triv soln exists for either

of these if C is smooth

Consider $\int_c \frac{\partial}{\partial n_e'}\ln|\underline{s}-\underline{s}'|\mu(\underline{s}')ds' - \pi\mu(\underline{s}) = 0$ ; it has a nontriv soln $\mu(\underline{s}') =$

note that differentiation is wrt $n_e'$

By fredholm $\Rightarrow$ adjoint eqn also has nonzero soln.

$\therefore \quad \int_c \frac{\partial}{\partial n_e}\ln|\underline{s}-\underline{s}'|\lambda(\underline{s}')ds' - \pi\lambda(\underline{s}) = 0$ has a soln.

now $\frac{\partial}{\partial n_e} = -\frac{\partial}{\partial n_i} \quad \therefore \Rightarrow \int_c \frac{\partial}{\partial n_i}\ln|\quad|\lambda(\underline{s}')ds' - \pi\lambda(\underline{s}) = 0 = \frac{\partial}{\partial n_i}\phi_i(\underline{s})$

thus $\left.\begin{array}{l}\frac{\partial}{\partial n_i}\phi_i(\underline{s}) = 0 \text{ at } C \\ \& \ \nabla^2\phi_i = 0\end{array}\right\} \Rightarrow \phi_i(\underline{s}) = \text{const.} = \int_c \ln|\underline{s}-\underline{s}'|\lambda(\underline{s}')ds'$

by neumann problem. $\phi_i$ is an equipot.

But RHS of (**) & (***) are constant hence Mushkelishvili conjecture is proven.

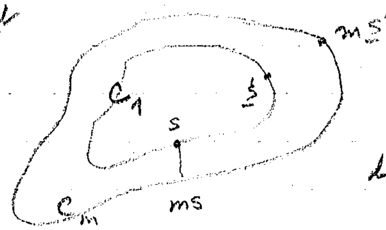
Now if C is an equipot then $\frac{\partial}{\partial n_e}\phi_e$ maintains its sign

where $\phi_e(\underline{r}) = \int_c \ln|\underline{r}-\underline{s}'|\sigma(\underline{s}')ds'$ & $\frac{\partial}{\partial n_e}\phi_e(\underline{s}) = \int_c \frac{\partial}{\partial n_e}\ln|\underline{s}-\underline{s}'|\sigma(\underline{s}')ds' + \pi\sigma$

& $\int_c \frac{\partial}{\partial n_e}\ln|\underline{s}-\underline{s}'|\lambda(\underline{s}')ds' - \pi\lambda(\underline{s}) = 0$

$\Rightarrow \quad 2\pi\lambda(s) = \frac{\partial}{\partial n_e}\phi_e$

Look at




$C_1 : f(x,y) = 0$

$C_m : f\left(\frac{x}{m}, \frac{y}{m}\right) = 0$

define as $\lambda(\underline{s})$ for $C_1$ & $\Lambda(m\underline{s})$ for $C_m$.

We will distinguish between (**) & (***)

If $\lambda(\underline{s}')$ satisfies $\int_{C_1} \ln|s - \underline{s}'|\,\lambda(\underline{s}')\,ds' = 1 \Rightarrow \Lambda(m\underline{s}')/m(1+\alpha\ln m)$

satisfies same eqn on $C_m$.

Proof

$|m\underline{s} - m\underline{s}'| = m|\underline{s} - \underline{s}'| \qquad \int_{C_m} \Lambda(ms')\,d(ms') = m\int_{C_1} \lambda(s')ds' = m\alpha$

what is $\int_{C_m} \Lambda(m\underline{s}')\ln|m\underline{s} - m\underline{s}'|\,d(ms')$

$$= \int_{C_m} \Lambda(m\underline{s}')\left[\ln m + \ln|\underline{s} - \underline{s}'|\right] m\,ds'$$

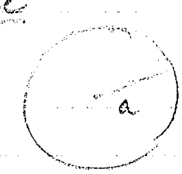
$$= m\ln m \cdot \alpha + m\int_{C_1} \lambda(s')\ln|s - s'|\,ds'$$

$$m\ln m\,\alpha + m$$

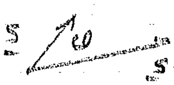
thus $\int_{C_m} \Lambda(ms')\ln|ms - ms'|\,dms' = m(1+\alpha\ln m)$

$\Rightarrow \exists$ a unique $C_m$ where $m(1+\alpha\ln m) = 0$ and $m > 0 \quad m = e^{-1/\alpha}$

Example




$r = a \quad$ (**) gives $\lambda = \frac{1}{2\pi a\ln a}$




now $\ln|\underline{s} - \underline{s}'| = \ln s_< + \sum \frac{s_<^n\cos\theta}{s_>^{n+1}} \qquad \underline{s}_> = \max(\underline{s}, \underline{s}')$

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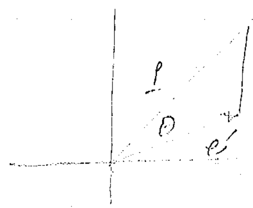
Eqns of 1st & 2nd kind are equally attractive when discretizing

$$\int_C \ln|\underline{s}-\underline{s}'|\, \sigma(\underline{s}')\, ds' = \phi(\underline{s})$$

for a difference kernel as here we can expand $\ln|s-s'|$ in a series




$$|\underline{\rho}-\underline{\rho}'| = [\rho^2+\rho'^2-2\rho\rho'\cos\theta]^{1/2}$$

$-\frac{1}{2\pi}\ln|\underline{\rho}-\underline{\rho}'|$ is a regular harmonic fn of $\underline{\rho}$ when not $=$ to $\underline{\rho}'$.

now $-\frac{1}{2\pi}\oint \frac{\partial}{\partial n}\ln|\underline{\rho}-\underline{\rho}'|\,ds' = -1$ since $\underbrace{\left(\partial_{\rho\rho}+\frac{1}{\rho}\partial_{\rho}+\frac{1}{\rho^2}\partial_{\theta\theta}\right)}_{\nabla^2_{(\rho,\theta)}} G(\underline{\rho},\underline{\rho}') = -\frac{\delta(\rho,\rho')}{\rho}\,\delta(\theta$

to construct $G$: $G(\underline{\rho},\underline{\rho}') = \sum_{n=1}^{\infty} f_n(\rho,\rho')\cos n(\theta-\theta') + f_0(\rho,\rho')$

$$\Rightarrow \nabla^2 G = \sum_{n=1}\left[\partial_{\rho\rho}+\frac{1}{\rho}\partial_{\rho}-\frac{n^2}{\rho^2}\right] f_n(\rho,\rho')\cos n(\theta-\theta') + \left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)f_0 = -\frac{1}{\rho}\delta(\rho-\rho')\delta($$

now $\int_0^{2\pi}$ of $\nabla^2 G = 2\pi\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)f_0 = -\frac{\delta(\rho-\rho')}{\rho}$

now $\int_0^{2\pi}\cos m(\theta-\theta')\,d\theta = \pi\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{n^2}{\rho^2}\right)f_n(\rho,\rho') = -\frac{1}{\rho}\delta(\rho-\rho')$

for a soln of homog is $\rho^n, \rho^{-n}$ use only $\rho^{-n}$ for bdd sol.
for $f_0$ sol is $\ln|\underline{\rho}-\underline{\rho}'|$

then $-\frac{1}{2\pi}\ln|\underline{\rho}-\underline{\rho}'| = -\frac{1}{2\pi}\ln\rho_> - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho_<}{\rho_>}\right)^n\cos n(\theta-\theta')$

$\rho_> = \max(\rho,\rho')$.

$$\ln|\underline{\rho}-\underline{\rho}'| = \ln a + \sum_{n=1}^{\infty}\frac{1}{n}\cos n(\theta-\theta')$$

now expand $\sum_0^{\infty} c_n\cos n\theta' = \sigma(s')$ & let $\sum_0^{\infty} d_n\cos n\theta = \phi(s)$ which is known

put in and integrate & use orthog.

Consider $\int_0^{\infty}\frac{\xi f(\xi)\,d\xi}{x^2+\xi^2} = g(x)$ $\quad x>0$ this was obtained from poisson integral formula

Assume $f(\xi)$ has integrable singularities & is s.t. $\int_N^\infty \frac{f(\xi)\,d\xi}{\xi}$ exists for all large $N$

i.e. when $N$ is fixed $\quad \xi > N \Rightarrow \xi^2 > N^2$ & $\int_0^\infty \frac{\xi f\,d\xi}{x^2+\xi^2} \Longrightarrow \int_0^\infty \frac{f}{\xi}\,d\xi$ is integrable

because of this we use complex theory to get that $g(x)$ is analytic for $x>0$

now analytically continue $g(x)$ : $\tilde{g}(x) = g(x) \quad x>0$

$= g(-x), \; x<0$

also let us continue $\bar{f}(\xi) = f(\xi) \quad \xi > 0$

$\qquad - f(-\xi) \quad \xi < 0$

since we want $\xi f(\xi)$ to be even

thus by continuation $2g(x) = \int_{-\infty}^{\infty} \frac{\xi \bar{f}(\xi)}{x^2+\xi^2}\,d\xi = \int_0^\infty \frac{\xi f(\xi)}{x^2+\xi^2}\,d\xi + \int_{-\infty}^{0} \frac{\xi \bar{f}(\xi)}{x^2+\xi^2}\,d\xi$

$\int_0^\infty \frac{\xi f(\xi)}{x^2+\xi^2}\,d\xi + \int_{-\infty}^{0} \frac{(-\xi)\left(-f(-\xi)\right)}{x^2+\xi^2}\,(-d\xi) = 2\int_0^\infty \frac{\xi f(\xi)\,d\xi}{x^2+\xi^2}$

Now $\int_{-\infty}^{\infty} \frac{\xi \bar{f}(\xi)}{x^2+\xi^2} = 2g(x)$ ; now consider $\int_{-\infty}^{\infty} \frac{\xi}{z^2+\xi^2}\bar{f}(\xi)\,d\xi = 2G(z)$

Now this has discont. on imag axis in $z$ plane

$\frac{\xi}{z^2+\xi^2} = \frac{1}{2i}\left(\frac{1}{z-i\xi} - \frac{1}{z+i\xi}\right)$

$\therefore \frac{1}{2i}\int_{-\infty}^{\infty}\left(\frac{1}{z-i\xi} - \frac{1}{z+i\xi}\right)\bar{f}(\xi)\,d\xi = 2G(z)$

$= 2G(z) = \frac{1}{i}\int_{-\infty}^{\infty} \frac{\bar{f}(\xi)}{z-i\xi}\,d\xi = \int_{-\infty}^{\infty} \frac{\bar{f}(\xi)\,d\xi}{\xi+iz}$

$\int_{-\infty}^{\infty} \frac{1}{z+i\xi}\bar{f}(\xi)\,d\xi = \int_{\infty}^{-\infty} \frac{1}{z-i\xi}\bar{f}(-\xi)(-d\xi)$

$= (-1)^2\int_{\infty}^{-\infty} \frac{1}{z-i\xi}\bar{f}(\xi)\,d\xi$

$= -\int_{-\infty}^{\infty} \frac{\bar{f}(\xi)\,d\xi}{z-i\xi}$

Since $\bar{f}(-\xi) = -\bar{f}(\xi)$

by symmetry of $\bar{f}(\xi)$

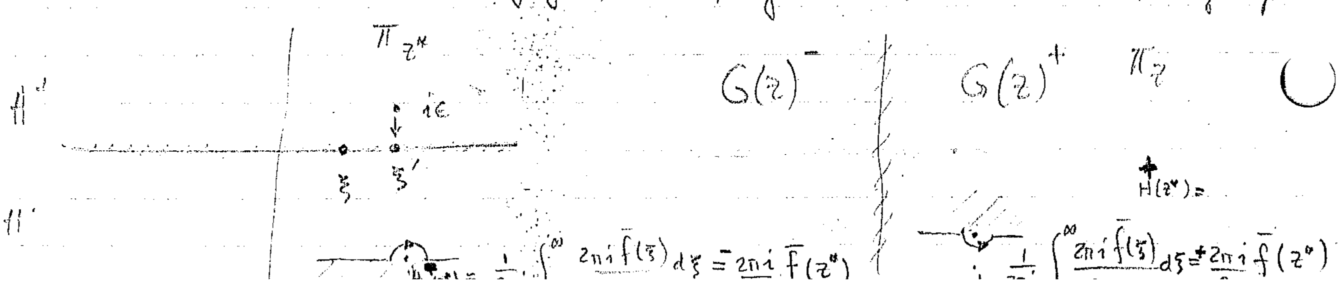
write $z^* = -iz$ then $2G(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{2\pi i\,\bar{f}(\xi)\,d\xi}{\xi - z^*} = H(z^*)$

sectionally holomorphic

analytic contin of $g(x)$ $\quad$ this fn has discont on real axis of $z^*$ plane

$G^+ \xrightarrow{z^*} H^-$

$G^- \xrightarrow{z^*} H^+$



$\Pi_{z^*}$ $\quad H^+$ $\quad i\epsilon$ $\quad \xi \quad \xi'$ $\quad H^-$

$G(z)^- \quad G(z)^+ \quad \Pi_z$

$\frac{1}{2\pi i}\int^{\infty} \frac{2\pi i\,\bar{f}(\xi)}{} d\xi = \frac{2\pi i}{2}\bar{f}(z^*)$

$H^+(z^*) = \frac{1}{2\pi i}\int^{\infty} \frac{2\pi i\,\bar{f}(\xi)}{}\,d\xi = \pm\frac{2\pi i}{2}\bar{f}(z^*)$

Now $\lim_{\epsilon\to 0} H^{+}(z^{+})_{z^{+}\to\xi'+i\epsilon} - \lim_{\epsilon\to 0} H^{-}(z^{+})_{z^{+}\to\xi'-i\epsilon} = 2\pi i f(\xi')$

Proposition obtained from Plemelj formula.

Now replace $2G(z)$ for $H$ and use the symmetry condition to get and eqn of $g^{+}$

since $z=x$ then $\left\{ -\lim_{\epsilon\to 0} 2G^{-}(z)_{(i\xi-\epsilon)} + \lim_{\epsilon\to 0} 2G^{+}(z)_{(i\xi+\epsilon)} \right\} \overset{?}{=} +2\pi i f(\xi') = -\frac{2\pi}{i} f(\xi')$

$$\boxed{f(x) = \frac{1}{\pi} \lim_{\epsilon\to 0} \left[ g(ix+\epsilon) - g(-ix+\epsilon) \right]} \quad \text{where } x, \epsilon > 0$$

Since $\lim_{\epsilon\to 0} G^{-}(-(\epsilon-i\xi)) - \lim_{\epsilon\to 0} G^{+}(\epsilon+i\xi) = -\pi/i\, f(\xi') \Rightarrow \frac{i}{\pi} \lim_{\epsilon\to 0}\left[ G^{+}(\epsilon+i\xi) - G^{-}(-(\epsilon-i\xi))\right] = f(\xi)$ ; by $G(-x) = G(x)$ then $f(x) = \frac{i}{\pi} \lim_{\epsilon\to 0} [g(\epsilon+ix) - g(\epsilon-ix)]$ QED

Example $f(\xi) = \sin\xi$, no singularities if $\int_N^{\infty} \frac{\sin\xi}{\xi}\, d\xi$ integral exists

now $\int_0^{\infty} \frac{\xi \sin\xi}{x^2+\xi^2}\, d\xi = -\frac{d}{d\alpha} \int_0^{\infty} \frac{\cos\alpha\xi}{x^2+\xi^2}\, d\xi \Big|_{\alpha=1} = -\frac{1}{2}\frac{d}{d\alpha} \int_{-\infty}^{\infty} \frac{\cos\alpha\xi}{x^2+\xi^2}\, d\xi \Big|_{\alpha=1}$

$$= -\frac{1}{2}\frac{d}{d\alpha} \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{i\alpha\xi}}{\xi^2+x^2}\, d\xi = -\frac{1}{2}\frac{d}{d\alpha} \mathrm{Re}\left[2\pi i \frac{e}{2ix}\right.$$

$\frac{e^{+i\alpha\xi}}{(\xi+ix)(\xi-ix)} = \frac{-1}{2ix} e^{+i\alpha\xi} \left[ \frac{1}{\xi+ix} - \frac{1}{\xi-ix} \right]$

thus residue $\frac{2\pi i}{2ix} e^{i\alpha\xi}\Big|_{\xi=ix}$ $\frac{\pi}{x} e^{-\alpha x}$ ; $\frac{d}{d\alpha}(\ ) = -\pi e^{-\alpha x}$

$$= -\frac{1}{2} \frac{d}{d\alpha}\left[ -2\pi \frac{e^{-\alpha x}}{2x} \right]_{\alpha=1} = \frac{\pi e^{-x}}{2}$$

$\therefore\ g(x) = \frac{\pi}{2} e^{-x}$ $\quad \therefore\ g(z) = \pi/2\, e^{-z}$ by cont.

now $f(x) = \frac{1}{\pi} \lim_{\epsilon\to 0} \left( g(ix+\epsilon) - g(-ix+\epsilon) \right)$

$$= \frac{1}{\pi} \lim_{\epsilon\to 0} \left[ \pi/2\, e^{-(ix+\epsilon)} - \pi/2\, e^{-(-i\pi+\epsilon)} \right]$$

$$= \frac{i}{2}\left( e^{-ix} - e^{ix} \right) = \sin x \ \checkmark$$

$\frac{i}{2}(-2i\sin x) = \sin x$

10/29/80

$$\int_0^{\infty} \frac{\xi f(\xi)}{x^2+\xi^2}\, d\xi = g(x), \quad x > 0$$

$f$ has integrable singularities if $\int_N^{\infty} \frac{f(\xi)}{\xi} d\xi < \infty$

this converts to $\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{2\pi i f(\xi)}{\xi - iz} d\xi = 2G(z)$

Suppose $F(z) \overset{def}{=} \frac{1}{2\pi i} \int_{\text{arc}} \frac{f(s)\,ds}{s-z}$ Harmonic fn of $z \to z$ is not on arc or curve.

here $F(z)$ can be identified as $\frac{G(z)}{\pi i}$ curve from above

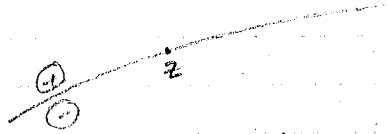
$f(s)$ being Hölder continuous ie $|f(s) - f(s_1)| \le A|s-s_1|^{n}$ $\quad n > 0$

$f(s)$ is also discont as we approach arc (curve) from $\oplus$ or neg side

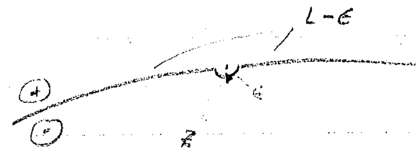
PLEMELJ formula shows answer.

if $z$ is on curve then we must bend arc away from the region which is harmonic

Suppose $f(s)$ is harmonic & approach curve from ⊕ side

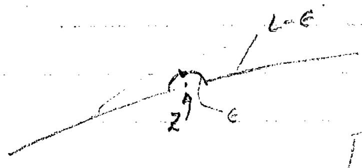



$\longrightarrow$




$$\text{thus} \quad \int_{L-\epsilon} + \int_{\epsilon} = \frac{1}{2} f(z) + \frac{1}{2\pi i} \int_{L-\epsilon} \frac{f(s)\,ds}{s-z} = F^+(z)$$

if $f(s)$ is harmonic & approach curve from ⊖ side then



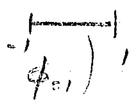

$$\int_{L-\epsilon} + \int_{\epsilon} = -\frac{1}{2} f(z) + \frac{1}{2\pi i} \int_{L-\epsilon} \frac{f(s)\,ds}{s-z} = F^-(z)$$

$$\text{then} \quad \begin{aligned} f(z) &= F^+(z) - F^-(z) \\ g(z) &= F^+(z) + F^-(z) = \frac{1}{\pi i} \int_{L-\epsilon} \frac{f(s)\,ds}{s-z} \end{aligned} \qquad \text{PLEMELJ}$$

Carleman has used these to solve Cauchy type problems.

Example

strip with potential




$$\text{thus} \quad \phi(x,y) = \int_{-1}^{1} \sigma(x') \ln \sqrt{(x-x')^2 + y^2}\, dx'$$

also $\nabla^2 \phi = 0$ on strip. $\quad \sigma(x) = \left[\frac{\partial \phi}{\partial n}\right]$

$$\text{now} \quad \phi(x,0) = \int_{-1}^{1} \sigma(x') \ln \sqrt{(x-x')}\, dx' = 1 \qquad \frac{d}{dx} \ln \sqrt{\ } = \frac{1}{\sqrt{x-x'}} \cdot \frac{1}{2}\frac{1}{\sqrt{x-x'}}$$

$$\text{now} \quad \frac{\partial \phi}{\partial x} = 0 = \int_{-1}^{1} \frac{\sigma(x')}{x-x'}\, dx' \qquad -1 \le x \le 1$$

homog. i.e. with a Cauchy kernal thus $\int_{-1}^{1} \frac{\sigma(x')}{x'-x}\, dx' = 0$

by letting $\sigma$ replace $f$ and $x$ by $z$ then by PLEMELJ

$$F^+(z) + F^-(z) = 0$$

$$\text{let} \quad \bar{g}(x) = \begin{cases} g(x), & x > 0 \\ g(-x), & x < 0 \end{cases} \qquad \text{and} \quad \bar{f}(\xi) = \begin{cases} f(\xi) & \xi > 0 \\ -f(-\xi) & \xi < 0 \end{cases}$$

remember by doing this $\displaystyle\int_0^{\infty} \frac{\xi f(\xi)}{x^2+\xi^2} d\xi = g(x) \Rightarrow 2\bar{g}(x) = \int_{-\infty}^{\infty} \frac{\xi \bar{f}(\xi)}{x^2+\xi^2} d\xi.$

then $\displaystyle\int_{-\infty}^{\infty} \frac{\xi \bar{f}(\xi)}{z^2+\xi^2} = 2G(z)$ where $G(z) = G(-z)$

the Dis cont. location for $G(z)$ is the imag axis $z = i\tau$.

$G(z)$ is harmonic outside the axis

now $\displaystyle\frac{\xi}{z^2+\xi^2} = \frac{1}{2i}\left[\frac{1}{z-i\xi} - \frac{1}{z+i\xi}\right]$

$\therefore\ \displaystyle 2G(z) = \frac{1}{2i}\int_{-\infty}^{\infty} \bar{f}(\xi)\left[\frac{1}{z-i\xi} - \frac{1}{z+i\xi}\right] d\xi$ now if $\xi' = -\xi$ then

$$\int_{-\infty}^{\infty} f(\xi')\left[\frac{1}{z+i\xi'} - \frac{1}{z-i\xi'}\right](-d\xi')$$

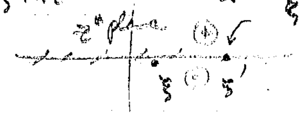
$$= +\int_{+\infty}^{-\infty} f(-\xi')\left[\frac{1}{z-i\xi'} - \frac{1}{z+i\xi'}\right] d\xi'$$

but $f(-\xi') = -f(\xi')$ $\quad\therefore\ \displaystyle -\int_{\infty}^{-\infty} f(\xi')\left[\quad\right] d\xi' = \int_{-\infty}^{\infty} f(\xi')\left[\quad\right]$

$$\therefore\ \int_{-\infty}^{\infty} f(\xi')\left[\quad\right] d\xi = \frac{2}{2i}\int_{-\infty}^{\infty} \frac{\bar{f}(\xi)}{z-i\xi} d\xi = \int_{-\infty}^{\infty} \frac{\bar{f}(\xi)\, d\xi}{\xi+iz}$$

now let $z^* = -iz$

$$2G(z) = \int_{-\infty}^{\infty} \frac{\bar{f}(\xi)}{\xi+iz} d\xi = \int_{-\infty}^{\infty} \frac{\bar{f}(\xi)\, d\xi}{\xi - z^*} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\left(2\pi i\, \bar{f}(\xi)\right) d\xi}{\xi - z^*} = H(z^*)$$




⊕ $z$ plane $\qquad H^+ \to G^-$

$$\lim_{\substack{\epsilon\to 0\\ z^* = \xi'+i\epsilon}} H^+(z^*) - \lim_{\substack{\epsilon\to 0\\ z^* = \xi'-i\epsilon}} H^-(z^*) = 2\pi i\, \bar{f}(\xi')$$ by plemelj formula

now $z = iz^*$ $\quad$ ⊕ ⊕ $z$ plane $\quad$ rotation by $+\pi/2$

thus $H^-(z^*) \to 2G^+(z)$

$$\lim_{\substack{\epsilon\to 0\\ z=-\epsilon+i\xi'}} 2G^-(z) - \lim_{\substack{\epsilon\to 0\\ z=\epsilon+i\xi'}} 2G^+(z) = 2\pi i\, \bar{f}(\xi')$$

now since $G$ has an even character $\qquad G(z) = G(-z)$

$$\lim_{\substack{\epsilon \to 0 \\ z = \epsilon - i\xi'}} G^{+}(z) - \lim_{\substack{\epsilon \to 0 \\ z = \epsilon + i\xi'}} G^{+}(z) = \pi i \bar{f}(\xi')$$

if $\xi' > 0 \quad \bar{f} = f$

$$\therefore \lim_{\epsilon \to 0} \left\{ g(\epsilon - i\xi) - g(\epsilon + i\xi) \right\} = \pi i f(\xi) \qquad \epsilon, \xi > 0$$
$$= \frac{-\pi}{i} f(\xi)$$

$$\Rightarrow \lim_{\epsilon \to 0} \tfrac{i}{\pi} \left\{ g(\epsilon + i\xi) - g(\epsilon - i\xi) \right\}$$

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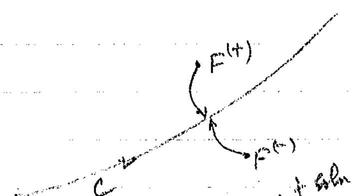
As shown before

$$F^{(+)}(\zeta) - F^{(-)}(\zeta) = f(\zeta)$$

$$F^{(+)}(\zeta) + F^{(-)}(\zeta) = \frac{1}{\pi i} \int \frac{f(\xi)\,d\xi}{\xi - \zeta}$$ where $f(\zeta)$ is a harmonic fn except along the arc $C$




Suppose we want soln to $\displaystyle \frac{1}{\pi i} \int_{-1}^{1} \frac{f(\zeta)}{\zeta - \xi}\, d\zeta = g(\xi) \quad -1 < \xi < 1$ $\qquad$ IE w/ Cauchy Kernel

what if $\displaystyle F(z) \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{-1}^{1} \frac{f(\zeta)\,d\zeta}{\zeta - z}$, $\lim_{z \to \infty} F(z) = O\left(\frac{1}{z}\right)$

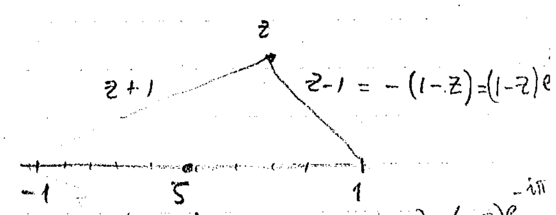
I.E. become (by Plemelj Formula)

I $\quad F^{(+)}(\zeta) + F^{(-)}(\zeta) = g(\zeta)$ known on $-1 < \zeta < 1$

II $\quad F^{(+)}(\zeta) - F^{(-)}(\zeta) = f(\zeta)$ unknown. — to find it: to do this we want to find a fn $\hat{F}(z) \ni \hat{F}^{+} + \hat{F}^{-} = 0$ such a fn is $\hat{F}(z) = \frac{1}{\sqrt{z^2 - 1}}$

now given $\displaystyle \hat{F}(z) = \frac{1}{\sqrt{z^2 - 1}}$ this has a branch cut between $-1$ & $1$

if we pick $\hat{F}(x)$ real for $x > 1$ and imaginary if $x < 1$




Now $\displaystyle \hat{F}^{+}(\zeta) = \frac{1}{i\sqrt{(1+\zeta)(1-\zeta)}} = \frac{-i}{\sqrt{1-\zeta^2}}$

since $\sqrt{z-1} = \sqrt{1-\zeta}\, e^{i\pi/2}$ $\qquad \arg(z \pm 1) = 0$ $\qquad z - 1 = -(1-z) = (1-z)e^{-i\pi}$

$\therefore \sqrt{z-1} = -i\sqrt{1-z}$

$\displaystyle F^{(-)}(\zeta) = \frac{i}{\sqrt{1-\zeta^2}}$ $\qquad$ since $\sqrt{z-1} = \sqrt{1-\zeta}\; e^{-i\pi/2}$

so that $\hat{F}^{(+)}(\zeta) + \hat{F}^{(-)}(\zeta) = 0$ on $-1 < \zeta < 1$ $\qquad$ this is the homog counterpart to II

Rewrite I in the form: $F^{+}(\zeta) + F^{(-)}(\zeta) = \dfrac{g(\zeta)}{\hat{F}^{+}(\zeta)}$ and $\hat{F}^{(+)}(\zeta) = -\hat{F}^{(-)}(\zeta)$

$$\frac{F^{(+)}(\zeta)}{\hat{F}^{(+)}(\zeta)} - \frac{F^{(-)}(\zeta)}{\hat{F}^{(-)}(\zeta)} = \left(\frac{g(\zeta)}{\hat{F}^{(+)}(\zeta)}\right)$$ known,

note that this is similar to PLEMELJ II if $F/\hat{F}$ is ident w/ F & $g/\hat{F}^{+}$ is ident w/ f

so that
$$\frac{F(z)}{\hat{F}(z)} = \frac{1}{2\pi i}\int_{-1}^{1} \frac{g(\xi)\,d\xi}{\hat{F}^{(+)}(\xi)(\xi - z)} + A \qquad \text{A being const}$$

but since $F = \frac{1}{2\pi i}\int \frac{f}{\xi - z}\,d\xi$ then $F/\hat{F} = \frac{1}{2\pi i}\int \frac{g}{\hat{F}^{+}(\xi - z)}\,d\xi$; but actually $F/\hat{F}$ is

but $F^{+} - F^{-} = f(\zeta)$, $\Longrightarrow$ $\hat{F}^{+} - \hat{F}^{-} = -2i/\sqrt{1-\zeta^2}$ &

$$\boxed{f(\zeta) = \frac{1}{2\pi i}\frac{-2i}{(\sqrt{1-\zeta^2})}\int_{-1}^{1} \frac{g(\xi)\sqrt{1-\xi^2}}{-i(\xi - \zeta)}\,d\xi + \frac{A}{\sqrt{1-\zeta^2}}}$$ is the soln.

$\hat{F}(\zeta) = \frac{1}{\sqrt{\zeta^2 - 1}}$ $\qquad$ $\frac{-i}{\sqrt{1-\xi^2}} = \hat{F}^{+}(\xi)$

hence $\frac{A}{\sqrt{1-\zeta^2}}$ satisfies $\int_{-1}^{1} \frac{f\,d\xi}{\xi - \zeta} = 0$

If f is substituted in, there are two princip integrals. When the integrals are interchanged it brings in the extra term. Consider $F/\hat{F} + \frac{1}{2\pi i}\int = J(z)$, $F \to O(\frac{1}{z})$ $\hat{F} \to O(\frac{1}{z})$ and $\int \to O(\frac{1}{z})$ as $z \to \infty$ thus $J \to$ const as $z \to \infty$. By Liouville J=const which we called A, $\therefore$ $F/\hat{F} = A + \frac{1}{2\pi i}\int$

For the strip at const. potential

$$0 = \int_a^b \frac{\sigma(x')\,dx'}{x' - x} \qquad a < x < b \quad ; \text{ since } g = 0, \text{ then}$$

$$\sigma(x) = \frac{A}{\sqrt{(b-x)(x-a)}}$$ if given total charge on strip $Q = \int_a^b \sigma(x)\,dx = A\pi$

Another example

$\varphi = 0$ $\qquad$ $\varphi = 1$

a $\qquad$ b

$$1 = -\frac{1}{2\pi}\int_a^b \left[\ln|x - x'| - \ln|x + x'|\right]\sigma(x')\,dx'$$

differentiate w r t x

$$0 = \int_a^b \frac{\sigma(x')\,x'\,dx'}{x'^2 - x^2} \quad ; \quad a < x < b$$

let $\xi = x^2$ $\quad$ $\zeta = x'^2$

$$0 = \int_{a^2}^{b^2} \frac{\sigma(\zeta)\,d\zeta}{\zeta - \xi}$$ back to cauchy form.

So $$\sigma(x) = \frac{A}{\sqrt{(b^2 - x^2)(x^2 - a^2)}}$$ since $\sigma(\xi) = \dfrac{1}{\sqrt{(b^2 - \xi)(\xi - a^2)}}$

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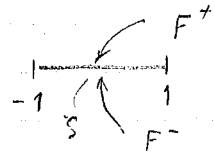
we had considered

$$\int_{-1}^{1} \frac{f(x')dx'}{x'-x} = g(x) \qquad -1<x<1$$

as well as eqns like $h(x)f(x) + \int \qquad = g(x)$

can be solved by using fns that are sectionally holomorphic

ie define $2\pi i F(z) = \int_{-1}^{1} \frac{f(x')dx'}{x'-z}$




now $F^+(\zeta) - F^{(-)}(\zeta) = f(\zeta)$

problems of volterra can be solved, if kernel is a difference fn, by use of Fourier or Laplace transform.

<u>HW</u> consider

$$\frac{1}{\pi i}\int_{-1}^{1} \Big\{ \underbrace{\frac{1}{x'-x}}_{\text{Cauchy part}} + \underbrace{k(x,x')}_{\text{bounded}} \Big\} f(x')dx' = g(x)$$

Cauchy part + bdd part

Solve it by considering

$\frac{1}{\pi i}\int \frac{1}{x'-x} f(x')dx' = g(x)$ first

then write eqn as an integral of 2nd kind & use iteration to get sol.

Suppose $\frac{1}{\pi i}\int_{-1}^{1} \frac{f(\zeta)d\zeta}{\zeta-\xi} = g(\xi) \qquad -1<\xi<1$

we said $f(\zeta) = \frac{1}{\pi i}\frac{1}{\sqrt{1-\zeta^2}}\int_{-1}^{1} \frac{g(\xi)\sqrt{1-\xi^2}}{\xi-\zeta}d\xi + \frac{A}{\sqrt{1-\zeta^2}}$ arbitrary const

to verify put into integral, use rule about inverting and we will get

$$I = \int_{-1}^{1} \frac{d\zeta}{\sqrt{1-\zeta^2}}\,\frac{1}{\zeta-\xi} = 0 \implies \frac{A}{\sqrt{1-\zeta^2}} \text{ solves } \int \frac{f(\zeta)d\zeta}{\sqrt{1-\zeta^2}} = 0$$

To prove this let $\zeta = \cos\theta$ $\quad \xi = \cos\Theta$ $\quad$ then $I = \int_0^\pi \frac{\sin\theta\, d\theta}{\sin\theta \cdot \cos\theta - \cos\Theta}$

$$I = \int_0^\pi \frac{d\theta}{\cos\theta - \cos\Theta} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{\cos\theta - \cos\Theta} \quad \text{by evenness of } \cos\theta$$

now let $z = e^{i\theta}$ $\Rightarrow$ $= \frac{1}{2}\oint \frac{dz/iz}{\frac{1}{2}(z + 1/z) - \cos\Theta} = \frac{1}{i}\oint \frac{dz}{(z - e^{i\Theta})(z - e^{-i\Theta})}$

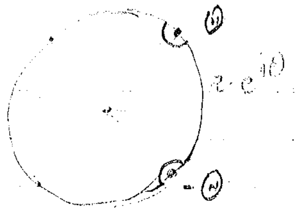
$\frac{1}{i}\oint \frac{1}{z^2 + 1 - 2z\cos\Theta}$ ; $2\cos\Theta = e^{i\Theta} + e^{-i\Theta}$

$\frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{\cos\theta + \cos\Theta}$

$\frac{dz/iz}{\frac{1}{2}(z + 1/z) + \frac{1}{2}(e^{i\Theta} + e^{-i\Theta})}$

$\frac{1}{i}\int \frac{dz}{z^2 + (e^{i\Theta} + e^{-i\Theta})z + 1}$

$(z + e^{i\Theta})(z + e^{-i\Theta})$




$\oint$ means circle w/ $z = e^{i\Theta}$, $z = e^{-i\Theta}$ deleted

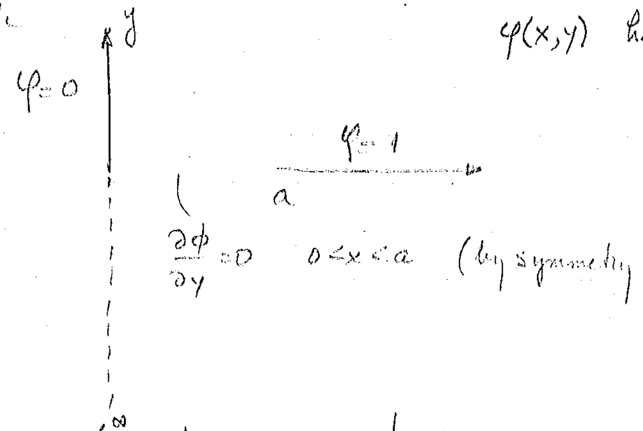
but since this means fn is regular then circle

$\oint = 0$

Note that $f(\zeta) = \frac{A}{\sqrt{1-\zeta^2}}$ represents $\varphi = \text{const}$ on $-1$ to $1$ $\Rightarrow$ $\frac{1}{\pi i}\int_{-1}^{1} f(\zeta)\ln|\zeta - \xi|\, d\zeta = A$

then $\frac{\partial}{\partial\xi} = \frac{1}{\pi i}\int_{-1}^{1} \frac{f(\zeta)}{(\zeta - \xi)}\, d\zeta = 0$

For problem $\qquad\qquad\qquad\qquad$ $\varphi(x,y)$ harmonic in $x, y > 0$




$\phi(x,y) = \int_0^\infty \phi(x',0)\, \frac{\partial}{\partial y'} G\Big|_{y'=0}\, dx'$ $\quad$ now $\frac{\partial\phi}{\partial y} = \int_0^\infty \phi \frac{\partial^2}{\partial y^2} G \; dx' = \int_0^\infty \phi \frac{\partial^2}{\partial x^2} G$

hence $\quad 0 = \frac{\partial\phi}{\partial y}\Big|_{y=0} = \frac{\partial^2}{\partial x^2}\int_0^\infty \varphi(x',0)\Big(\underbrace{-\frac{1}{\pi}\ln\left|\frac{x - x'}{x + x'}\right|}_{G}\Big)\, dx'$ , $0 < x < a$

now take $\frac{\partial}{\partial x}$

$$0 = \frac{\partial}{\partial x}\frac{1}{\pi}\int_0^\infty \varphi(x',0)\left\{-\frac{1}{x - x'} + \frac{1}{x + x'}\right\} dx'$$

$$= \frac{\partial}{\partial x}\int_0^\infty \frac{\varphi(x',0)\, x'\, dx'}{\ldots} = 0$$

now since $\varphi = 1$ for $x > a$

$\Rightarrow \int_0^a \frac{\varphi x'\, dx'}{\ldots} + \frac{1}{2}\int_a^\infty \frac{2x'\, dx'}{\ldots} = \frac{\partial}{\partial x}\frac{1}{2}\ln(x'^2 - x^2)\Big|_a^\infty + \ldots$

$$\frac{\partial}{\partial x}\int_0^a \frac{\phi(x',0)\,x'dx'}{x'^2-x^2} = \lim_{\bar{X}\to\infty}\frac{\partial}{\partial x}\int_a^{\bar{X}} \frac{x'dx'}{x'^2-x^2} = -\frac{\partial}{\partial x}\left\{\frac{1}{2}\ln\frac{\bar{X}^2-x^2}{a^2-x^2}\right\}$$

$$= -\frac{\bar{X}}{\bar{X}^2-x^2} - \frac{x}{a^2-x^2} \longrightarrow -\frac{x}{a^2-x^2} \quad \text{as } \bar{X}\to\infty$$

$$\therefore \quad \frac{\partial}{\partial x}\int_0^a \frac{\phi(x',0)\,x'dx'}{x'^2-x^2} = -\frac{x}{a^2-x^2} \qquad 0 \le x < a$$

now $\displaystyle\int_0^a \frac{\phi(x',0)\,x'dx'}{x'^2-x^2} = \frac{1}{2}\ln(a^2-x^2) + \text{const.} \qquad 0 \le x < a$

now let $x^2 = \xi \quad x'^2 = \zeta$

$$\frac{1}{2}\int_0^{a^2} \frac{\phi(\zeta)\,d\zeta}{\zeta-\xi} = \underbrace{\frac{1}{2}\ln(a^2-\xi) + \text{const.}}_{g(\xi)} \qquad 0 \le \xi < a^2$$

using Carleman for. $\Rightarrow$ $\displaystyle\phi(\xi) = -\frac{1}{\pi^2}\frac{1}{\sqrt{\xi(a^2-\xi)}}\int_0^{a^2} \frac{\ln(a^2-\xi) + C}{(\xi-\zeta)}\sqrt{\xi(a^2-\xi)}\,d\xi + \frac{A}{\sqrt{\zeta(a^2-\zeta)}}$

by general eqn of Carlemann

singularities at $\zeta=0, a^2$

now $\phi(0) = 0 \quad \& \quad \phi(a^2) = 1$

is enough to give $A, C$. After much manipulation answer is obtained.

thus $\displaystyle\phi(x,0) = \frac{2}{\pi}\sin^{-1}(x/a)$

now $\displaystyle\left.\frac{\partial\phi}{\partial x}\right|_{y=0}$ = electric field intensity $\displaystyle= \frac{2}{\pi}\frac{1}{\sqrt{a^2-x^2}}$

recall $\displaystyle\frac{1}{\pi i}\int_{-1}^{1}\frac{f(\xi)}{\xi-\zeta} = g(\zeta) \qquad -1 \le \zeta < 1$

was replaced by

$$F^{(+)}(\zeta) + F^{(-)}(\zeta) = g(\zeta)$$

where $\displaystyle F(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{f(\zeta)\,d\zeta}{\zeta-z}$

plemmelj formulas gave

and $F^{(+)}(\zeta) - F^{(-)}(\zeta) = f(\zeta)$

this is useless

we needed to find sol. to $F^{+} + F^{-} = 0$ which we called $\hat{F}$

# SOME INTEGRAL EQUATIONS

The solution of a singular integral equation of the first kind with a Cauchy kernel, namely

I.E. $$\frac{1}{\pi}\int_{-1}^{1}\frac{f(\xi)\,d\xi}{\xi - x} = g(x), \quad -1<x<1 \tag{1}$$

can be expressed in the form (due to Carleman)

solution $$f(x) = -\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-\xi^2}{1-x^2}}\,\frac{g(\xi)}{\xi - x}\,d\xi + \frac{C}{\sqrt{1-x^2}} \tag{2}$$

with a principal value integral and an arbitrary constant $C$. To adapt this result for the corresponding equation on a general range,

I.E. $$\frac{1}{\pi}\int_{a}^{b}\frac{f(\xi)\,d\xi}{\xi - x} = g(x), \quad a<x<b \tag{3}$$

the changes of variable

$$x' = \frac{2(x-a)}{b-a} - 1, \quad \xi' = \frac{2(\xi - a)}{b-a} - 1$$

$1-\xi' = 2 - \frac{2(\xi-a)}{b-a} = \frac{2b-2a-2\xi+2a}{b-a} = \frac{2(b-\xi)}{b-a}$

are appropriate and it turns out that

$1+\xi' = \frac{2(\xi-a)}{b-a}$ $\quad 1-\xi'^2 = \frac{4(b-\xi)(\xi-a)}{(b-a)^2}$

$\therefore 1-x'^2 = \frac{4(b-x)(x-a)}{(b-a)^2}$

solution $$f(x) = -\frac{1}{\pi}\int_{a}^{b}\sqrt{\frac{(\xi-a)(b-\xi)}{(x-a)(b-x)}}\,\frac{g(\xi)}{\xi - x}\,d\xi + \frac{C}{\sqrt{(x-a)(b-x)}}.$$

$\therefore \sqrt{\frac{1-\xi'^2}{1-x'^2}} = \sqrt{\frac{(b-\xi)(\xi-a)}{(b-x)(x-a)}}$

$d\xi' = \frac{2\,d\xi}{b-a}$ $\quad \xi'-x' = \frac{2(\xi-x)}{b-a}$

Carleman also considered the singular integral equation

I.E. $$\frac{1}{\pi}\int_{0}^{1}\ln|x-\xi|\,f(\xi)\,d\xi = g(x), \quad 0<x<1$$

if we take $\frac{\partial}{\partial x}$ of eq (5) then solve we get $f(\xi) = \frac{1}{\pi}\int_0^1 \frac{\sqrt{\xi(1-\xi)}}{\sqrt{x(1-x)}}\,g'(\xi)\,d\xi + \frac{\Lambda}{\sqrt{x(1-x)}}$

and gave its solution in the form

solution $$f(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}\int_0^1 \frac{\sqrt{\xi(1-\xi)}}{\xi - x}\,g'(\xi)\,d\xi - \frac{1}{2\pi\ln 2\,\sqrt{x(1-x)}}\int_0^1\frac{g(\xi)\,d\xi}{\sqrt{\xi(1-\xi)}}. \tag{6}$$

If we convert equation (5) by the transformation

$$x = \frac{X-a}{b-a}, \quad \xi = \frac{\zeta - a}{b-a}, \quad (b-a)\hat{f}(\zeta) = f\left(\frac{\zeta-a}{b-a}\right), \quad \hat{g}(X) = g\left(\frac{X-a}{b-a}\right)$$

then the solution of

$$\frac{1}{\pi}\int_a^b \left\{\ln|X-\zeta| - \ln(b-a)\right\}\hat{f}(\zeta)\,d\zeta = \hat{g}(X), \quad a<X<b \tag{7}$$

is $\hat{f}(X) = \frac{1}{\pi}\int_a^b \sqrt{\frac{(\xi-a)(b-\xi)}{(X-a)(b-X)}}\,\frac{\hat{g}'(\xi)}{\xi - X}\,d\xi - \frac{1}{2\pi\ln 2}\,\frac{I}{\sqrt{(X-a)(b-X)}}$

where
$$I = \int_a^b \frac{\hat{g}(X)\,dX}{\sqrt{(X-a)(b-X)}} \qquad (8)$$

The additional term $\ln(b-a)$ which appears in the integrand of (7) needs to be recast for purposes of generalizing the interval in (5) and this may be achieved with use of the result

$$\int_a^b \frac{\ln|\xi - x|}{\sqrt{(\xi-a)(b-\xi)}}\,d\xi = \pi\left[\ln(b-a) - 2\ln 2\right], \quad a<x<b; \qquad (9)$$

then, after multiplying (7) by $1/\sqrt{(X-a)(b-X)}$ and integrating over $a<X<b$ we obtain

$$\int_a^b \hat{f}(\xi)\{\ln(b-a) - 2\ln 2\}\,d\xi - \ln(b-a)\int_a^b \hat{f}(\xi)\,d\xi = I$$

or
$$\int_a^b \hat{f}(\xi)\,d\xi = -\frac{I}{2\ln 2}.$$

Hence (7) becomes

$$\int_a^b \hat{f}(\xi)\ln|X-\xi|\,d\xi + \frac{\ln(b-a)}{2\ln 2}\,I = \pi\hat{g}(X)$$

and, utilizing (9) once again,

$$\int_a^b \ln|X-\xi|\left\{\hat{f}(\xi) + \frac{1}{\sqrt{(\xi-a)(b-\xi)}}\,\frac{\ln(b-a)}{2\ln 2}\,\frac{I}{\pi\{\ln(b-a)-\ln 2\}}\right\}d\xi = \pi\hat{g}(X)$$

which means that the solution of

integral eqn. $\frac{1}{\pi}\int_a^b \ln|x-\xi|\,f(\xi)\,d\xi = g(x), \quad a<x<b$ (10)

is represented by

$$f(x) = \frac{1}{\pi}\int_a^b \sqrt{\frac{(\xi-a)(b-\xi)}{(x-a)(b-x)}}\,\frac{g'(\xi)\,d\xi}{\xi - x} + \frac{1}{\pi}\,\frac{1}{\sqrt{(x-a)(b-x)}}\,\frac{1}{\ln(b-a)-2\ln 2}\int_a^b \frac{g(\xi)\,d\xi}{\sqrt{(\xi-a)(b-\xi)}}.$$

soln. (11)

The exceptional (or distinguished) case occurs if $b-a=4$ and then the last term of (11) has the form

$$c\left((x-a)(b-x)\right)^{-1/2}$$

with arbitrary $c$, this being a null solution of (10).

Consider next the integral equation

$$\frac{1}{\pi}\int_0^1 \ln\left|\frac{x+\xi}{x-\xi}\right| f(\xi)\,d\xi = g(x), \qquad 0<x<1\;; \tag{12}$$

here an integral representation for the kernel,

$$\ln\left|\frac{x+\xi}{x-\xi}\right| = 2\int_0^{\min(x,\xi)} \frac{\lambda\,d\lambda}{\sqrt{(x^2-\lambda^2)(\xi^2-\lambda^2)}}, \qquad 0<x,\xi<1 \tag{13}$$

affords a means of solution. To establish (13) we examine the integral

$$I(x,\xi) = \int_0^x \frac{\lambda\,d\lambda}{\sqrt{(x^2-\lambda^2)(\xi^2-\lambda^2)}} \qquad \text{when } 0<x<\xi<1\;;$$

let $\lambda = x\sin\vartheta$ and

$$I = \int_0^{\pi/2} \frac{x\sin\vartheta\,d\vartheta}{\sqrt{(\xi^2-x^2+x^2\cos^2\vartheta)}} = \ln\frac{x+\xi}{\sqrt{\xi^2-x^2}} = \frac{1}{2}\ln\frac{x+\xi}{\xi-x}$$

If $0<\xi<x<1$ we find, analogously, $I=\frac{1}{2}\ln\frac{x+\xi}{x-\xi}$ and hence (13) follows.

Substitute (13) into (12) and observe that

$$2\int_0^x f(\xi)\,d\xi\int_0^\xi \frac{\lambda\,d\lambda}{\sqrt{(x^2-\lambda^2)(\xi^2-\lambda^2)}} + 2\int_x^1 f(\xi)\,d\xi\int_0^x \frac{\lambda\,d\lambda}{\sqrt{(x^2-\lambda^2)(\xi^2-\lambda^2)}} = \pi g(x)$$

whence, inverting the orders of integration, which gives $\int_0^x \frac{d\lambda}{\sqrt{x^2-\lambda^2}}\int_\lambda^1 \frac{f(\xi)\,d\xi}{\sqrt{\xi^2-\lambda^2}}$ ; add'g the two

$$\pi g(x) = 2\int_0^x \frac{\lambda\,d\lambda}{\sqrt{x^2-\lambda^2}}\int_\lambda^1 \frac{f(\xi)\,d\xi}{\sqrt{\xi^2-\lambda^2}} = 2\int_0^x \frac{\lambda F(\lambda)\,d\lambda}{\sqrt{x^2-\lambda^2}} \tag{14}$$

with

$$F(\lambda) = \int_\lambda^1 \frac{f(\xi)\,d\xi}{\sqrt{\xi^2-\lambda^2}} \tag{15}$$

interchange is OK since integrands are integrable [illegible]

The latter is an ABEL equation whose solution takes the form

$$f(\xi) = -\frac{2}{\pi}\frac{d}{d\xi}\int_\xi^1 \frac{\lambda F(\lambda)\,d\lambda}{\sqrt{\lambda^2-\xi^2}} \tag{16}$$

Likewise, the solution of (14) is expressed by

$$F(\lambda) = \frac{1}{\lambda}\frac{d}{d\lambda}\int_0^\lambda \frac{\xi\, g(\xi)\, d\xi}{\sqrt{\lambda^2-\xi^2}} = \frac{g(0+)}{\lambda} + \int_0^\lambda \frac{g'(\xi)\, d\xi}{(\lambda^2-\xi^2)^{1/2}} \tag{17}$$

and, taken in conjunction with (16), we then arrive at the solution of (12), viz.,

$$f(x) = -\frac{2}{\pi}\frac{d}{dx}\int_x^1 \frac{\lambda\, G(\lambda)\, d\lambda}{\sqrt{\lambda^2-x^2}} + \frac{2}{\pi}\frac{g(0)}{x\sqrt{1-x^2}} \tag{18}$$

where

$$G(\lambda) = \int_0^\lambda \frac{g'(\xi)\, d\xi}{(\lambda^2-\xi^2)^{1/2}}. \tag{19}$$

An alternative representation of the solution is established by substituting (19) into (18) and inverting order of integration; this yields

$$f(x) = -\frac{2}{\pi}\frac{x}{\sqrt{1-x^2}}\int_0^1 \frac{\sqrt{1-\xi^2}\, g'(\xi)\, d\xi}{\xi^2-x^2} + \frac{2}{\pi}\frac{g(0)}{x\sqrt{1-x^2}} \qquad 0<x<1 \tag{20}$$

or the so-called Carleman form of solution.

More generally $F^{(+)}(\zeta) - G(\zeta)F^{(-)}(\zeta) = g(\zeta)$

For Plemelj $G(\zeta) = -1$

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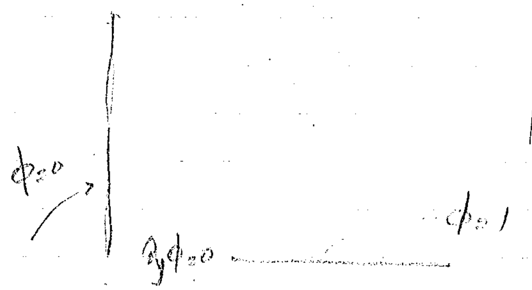
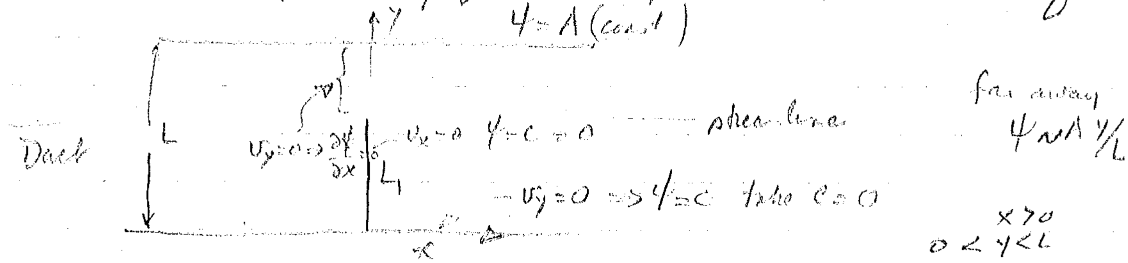
Handout given on different types of singular integral eqns



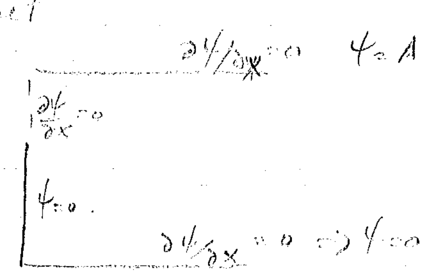

do this problem using the handout HW
using (12) on handout

Look at problem of steady flow through duct w/a transverse fin




Stream fn $\psi(x,y)$ now $\nabla^2\psi = 0$ $v_x = \frac{\partial\psi}{\partial y}$, $v_y = -\frac{\partial\psi}{\partial x}$

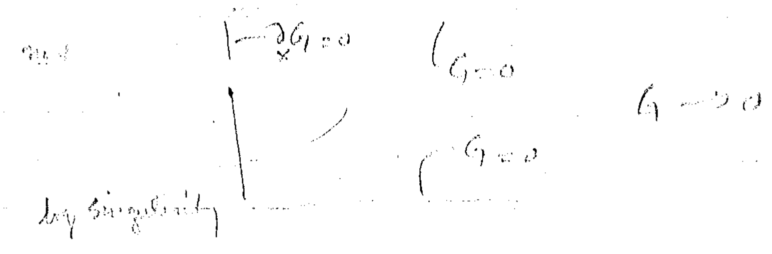
Then look only at half the duct




Thus we can say

$$\psi(x,y) = A\frac{y}{L} + \int_0^{L_1} G(x,y,0,y')\,V(y')\,dy' \qquad V(y') = v_y(0,y)$$

$$G(x,y,x',y') = \frac{1}{\pi}\sum \frac{1}{n}\sin\frac{n\pi y}{L}\sin\frac{n\pi y'}{L}\left(e^{-\frac{n\pi}{L}|x-x'|} + e^{-\frac{n\pi}{L}(x+x')}\right)$$



Integ Eq $\psi = 0$, $x = 0$, $0 < y < L_1$

$$\int_0^{L_1} K(y,y')\, V(y')\,dy' = -A\,y/L \qquad 0 < y < L_1$$

since
$$K(y,y') = G(0,y,0,y') = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{n\pi y}{L}\sin\frac{n\pi y'}{L}$$
$$= \frac{1}{\pi}\sum \frac{1}{n}\left\{\cos\frac{n\pi}{L}(y-y') - \cos\frac{n\pi}{L}(y+y')\right\}$$
$$= \frac{1}{\pi}\ln\left\{\frac{\sin\frac{\pi}{2L}(y+y')}{\sin\frac{\pi}{2L}|y-y'|}\right\}$$

using $\sum \frac{1}{n}\cos n\alpha = -\ln\left(2\left|\sin\frac{\alpha}{2}\right|\right)$ for $|\alpha| < \pi$

now
$$\frac{\partial K}{\partial y}(y,y') = \frac{1}{L}\,\frac{\sin \pi y'/L}{\cos \pi y/L - \cos \pi y'/L}$$

Put into integral

$$\int_0^{L_1} \frac{\sin \pi y'/L\; V(y')}{\cos \pi y/L - \cos \pi y'/L}\,dy' = -A, \quad 0 < y < L_1$$

let $Y = \sin \pi y/2L \qquad Y' = \sin \pi y'/2L \qquad dY' = \pi/2L \cos\frac{\pi y'}{2L}\,dy'$

$$\cos \pi y/L = 1 - 2Y^2$$

then
$$\int_0^{\sin\frac{\pi L_1}{2L}} \frac{Y' V(Y')}{Y'^2 - Y^2}\,dY' = -\frac{\pi A}{2L}$$

now $V(-Y') = -V(Y')$

$$\int_{-L_1}^{L_1}$$

Since $V(Y)$ is an odd fn, we use this to show that first $\frac{1}{2}\int_{-\sin\frac{\pi L_1}{2L}}^{\sin\frac{\pi L_1}{2L}} = \int_0^{\sin\frac{\pi L_1}{2L}}$; and then $\frac{1}{Y'+Y} + \frac{1}{Y'-Y} = \frac{2Y'}{Y'^2 - Y^2}$; and that $\int_{-a}^{a}\frac{V(Y')}{Y'+Y}\,dY' = \int_{-a}^{a}\frac{V(Y')}{Y'-Y}\,dY'$; hence combining all effects then we get

$$\int_{-\sin\frac{\pi L_1}{2L}}^{\sin\frac{\pi L_1}{2L}} \frac{V(Y')\,dY'}{Y' - Y} = -\frac{\pi A}{L}$$

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Suppose
$$\frac{1}{\pi}\int_{-1}^{1}\frac{df(y)}{dy}\,\frac{dy}{x-y}=g(x)\qquad 0<x<1$$
Lifting Line theory w/ $f(\pm 1)=0$

$f(y)$ is the circulation of the flow around a wing.

Now the principal value integral $\frac{1}{\pi}\int_0^{\pi}\frac{\cos(n+1)\delta}{\cos\delta-\cos\theta}\,d\delta=\frac{\sin(n+1)\theta}{\sin\theta}\qquad n=0,1,2,\ldots$

$\int_0^{\pi}\frac{d}{d\delta}\left(\sin(n+1)\delta\right)\cdot\frac{d\delta}{\cos\delta-\cos\theta}=\frac{\pi(n+1)\sin(n+1)\theta}{\sin\theta}$

we use this PVI and $x=\cos\theta$ $y=\cos\delta$

to get $\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{df(\delta)}{d\delta}\,\frac{d\delta}{\cos\delta-\cos\theta}=g(\theta)$

now if $f(\theta)=\sum_{n=0}^{N}\frac{c_n}{n+1}\sin(n+1)\theta$ & $g(\theta)\sin\theta=\sum_{n=0}^{N}c_n\sin(n+1)\theta$

we can use the above PVI & a superposition to find the solution

thus $f(x)=\frac{1}{2\pi}\int_{-1}^{1}\ln\frac{1-xy+\sqrt{(1-x^2)(1-y^2)}}{1-xy-\sqrt{(1-x^2)(1-y^2)}}\,g(y)\,dy$ thus $f(\pm 1)=0$ ✓

thus $f(\theta)=\frac{1}{2\pi}\int_0^{\pi}\ln\frac{1-\cos(\theta+\delta)}{1-\cos(\theta-\delta)}\,g(\delta)\sin\delta\,d\delta$ using $\binom{x}{y}=\cos\binom{\theta}{\delta}$

this is a source type representation of the fn $f$

Also $F(x)=\int_a^b K(x,y)\,G(y)\,dy$ is a source representation with source density $G(y)$

lets write it as $\sum_{n=1}^{\infty}c_n\phi_n(x)$ where $\phi_n(x)$ are EF of the kernel $K(x,y)$

and $c_n=\int_a^b\frac{G(y)}{\lambda_n}\phi_n(y)\,dy$

with $\int_a^b K(x,y)\,\phi_n(y)\,dy=\frac{1}{\lambda_n}\phi_n(x)$

Now $\sum c_n\phi_n(x)$ is UConv absolutely conv when $\int_a^b\left(K(x,y)\right)^2dy<\infty$ (Suff cond for $x\in[a$

range can be extended to an infinite limit

suppose $K(\theta,\delta)=\frac{1}{2\pi}\ln\left[\frac{1-\cos(\theta+\delta)}{1-\cos(\theta-\delta)}\right]$ does this $\Rightarrow\int[K(\theta,\delta)]^2d\delta<\infty$

what are the EF & EV? Start with $\frac{1}{\pi}\frac{\sin n\theta}{\sin\theta}=\int_0^{\pi}\frac{\cos n\delta}{\cos\delta-\cos\theta}\,d\delta$

now $\lim_{\epsilon\to 0}\int_0^{\theta-\epsilon}\frac{\cos n\delta}{\cos\delta-\cos\theta}\,d\delta+\int_{\theta+\epsilon}^{\pi}\frac{\cos n\delta}{\cos\delta-\cos\theta}\,d\delta$

we will integrate by parts using $\frac{d\delta}{\cos\delta - \cos\theta} = dv$

$$\lim_{\epsilon\to 0}\left\{ \frac{\cos n\delta}{\sin\theta} \ln\left(\frac{\sin\left(\frac{\theta+\delta}{2}\right)}{\sin\left(\frac{\theta-\delta}{2}\right)}\right) + \int_0^{\theta-\epsilon} \frac{n\sin n\delta}{\sin\theta} \ln \frac{\sin\left(\frac{\theta+\delta}{2}\right)}{\sin\left(\frac{\theta-\delta}{2}\right)} d\delta + \theta+\epsilon \right\} + \int_{\theta+\epsilon}^{\pi} \dots$$

This shows that $\sin(n+1)\theta = \text{EF}$ , $\text{EV} = n+1$

Note that $K(\theta,\delta) = K(\delta,\theta)$

if we define A bilinear transform for $K(x,y)$

$$K(x,y) = \sum_{n=0}^{\infty} \frac{\hat{\phi}_n(x)\hat{\phi}_n(y)}{\lambda_n} \qquad \int \hat{\phi}_n^2 dx = 1$$

for us. $\overline{EF} = \frac{\sin(n+1)\theta}{\sqrt{\pi/2}}$ since $\int_0^\pi \overline{EF}^2 dx = 1$

thus

$$K(\theta,\delta) = \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\sin(n+1)\theta \sin(n+1)\delta}{n+1} = \frac{1}{\pi}\sum \frac{\cos(n+1)(\theta-\delta) - \cos(n+1)(\theta+\delta)}{n+1}$$

converges for $\theta \neq \delta$ but diverges when $\theta = \delta$

for the mth iterat $K^{(m)}(x,y) = \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\sin(n+1)\theta \sin(n+1)\delta}{(\lambda_{n+1})^m}$

ie $K^{(2)}(\theta,\delta) = \int_0^\pi K(\theta,\xi) K(\xi,\delta)\, d\xi \Rightarrow K^{(2)}(\theta,\theta) = \int_0^\pi K^2(\theta,\xi) d\xi$ by sym of $K$

$K^m(\theta,\delta) = \int_0^\pi K^{(m-1)}(\theta,\xi) K(\xi,\delta)\, d\xi$

$$K^{(2)} = \frac{2}{\pi}\sum_{n=0} \frac{\sin^2(n+1)\theta}{(n+1)^2} = \frac{2}{\pi}\sum_{n=1}\frac{\sin^2 n\theta}{n^2} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos 2n\theta}{n^2}$$

Thus if $F(x) = \int_a^b \sum_n \frac{\hat{\phi}(x)\phi_n(y)}{\lambda_n} G(y)\, dy = \int_a^b K(x,y)\, G(y) dy$

$$= \sum a_n \hat{\phi}_n(x)$$

thus back to our original problem

$\frac{1}{\pi} \int_{-1}^{1} \frac{df(y)}{dy} \frac{dy}{x-y} = g(x)$ is an eqn of first kind

an eqn of 2nd kind is $f(\theta) + \underbrace{p(\theta)}_{\text{dependent of wing shape}} \int_0^\pi \ln \frac{1-\cos(\theta+\delta)}{1+\cos(\theta-\delta)} f(\delta) d\delta = g(\theta)$

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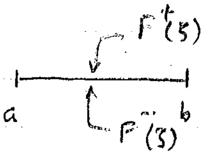
looking at
$$a(\zeta) f(\zeta) + \int_a^b \frac{b(\xi) f(\xi)}{\xi - \zeta} d\xi = g(\zeta) \qquad a < \zeta < b \quad a \leq \xi \leq b$$
Fredholm eqn of $2^{nd}$ kind w/ Cauchy Kernel.

Using Carleman procedure based on $F(z) \triangleq \frac{1}{2\pi i} \int_a^b \frac{b(\xi) f(\xi)}{\xi - z} d\xi$

This fn has line of descent on $[a, b]$

Now replace I.E. by relation of limit values of $F$ from above & below.



From plemelj $F^+(\zeta) = \frac{1}{2\pi i} \int_a^b \frac{b(\xi) f(\xi) d\xi}{\xi - z} + \frac{1}{2} f(\zeta) b(\zeta)$

$F^-(\zeta) = \qquad | \qquad - \qquad \downarrow$

thus $F^+ - F^- = f(\zeta) b(\zeta) = f(\zeta)$ if $b(\zeta) = 1$

also $F^+ + F^- = \frac{1}{\pi i} \int_a^b \frac{f(\xi) d\xi}{\xi - \zeta}$ if $b(\xi) = 1$

now go back to original (plug into them) to get $a(\zeta)\left[F^+ - F^-\right] + \pi i \left[F^+ + F^-\right] = g(\zeta)$

Now $[a(\zeta) + \pi i] F^+ - [a(\zeta) - \pi i] F^- = g(\zeta)$ after regroup.

Suppose that $a(\zeta) \pm \pi i \neq 0$ on $[a, b]$ then
$$\frac{[a(\zeta) + \pi i]}{a(\zeta) - \pi i} F^+ - F^- = \frac{g(\zeta)}{a(\zeta) - \pi i} \quad \text{or} \quad K(\zeta) F^+ - F^- = \frac{g(\zeta)}{a(\zeta) - \pi i} \quad (*)$$

if $a$ is a real fn. Then $K(\zeta) = \frac{a(\zeta) + \pi i}{a(\zeta) - \pi i} \triangleq e^{2i\psi(\zeta)}$ ; if $a(\zeta) = 0$ $K(\zeta) = -1$ $\psi(\zeta) = \frac{\pi}{2}$

why? $\overset{\nu(\zeta) =}{a(\zeta) \pm \pi i} = \frac{1}{\sqrt{a^2 + \pi^2}} e^{\pm i\psi(\zeta)}$ $\therefore \frac{\nu(\zeta)}{\bar{\nu}(\zeta)} = e^{2i\psi(\zeta)}$

The eqn (*) constitutes an inhomog. hilbert problem on the arc $c$.

Now we need to find $F$? Can we find now $K(\zeta)$ so that $K(\zeta) = \frac{K^+(\zeta)}{K^-(\zeta)}$

now we seek $K(z)$ which is analytic in the cut plane w/o zeros.

$K(z) = \frac{K^+(z)}{K^-(z)} \Rightarrow \ln(K^+) - \ln(K^-) = 2i\psi(\zeta)$

using plemelj II $\Rightarrow \ln K(z) = \frac{1}{2\pi i} \int_a^b \frac{2i\psi(\xi) d\xi}{\xi - z}$

for $K(\xi) = -1$ $\quad \psi(\xi) = \pi/2$

$$\ln K = \tfrac12 \int_a^b \frac{d\xi}{\xi - z} = \frac12 \ln\left|\frac{b-z}{a-z}\right| = \frac12 \ln\left|\frac{b-z}{z-a}\right| + \text{const.}$$

$$\therefore \quad K(z) = \sqrt{\frac{b-z}{z-a}}$$

known fn.

$$\Rightarrow \frac{K^+}{K^-} F^+ - F^- = \frac{g(\xi)}{a(\xi)-\pi i} \qquad \text{or} \qquad K^+F^+ - F^-K^- = \frac{g(\xi)}{a(\xi)-\pi i}\, K^-$$

now $K(\xi)F(\xi) = \dfrac{1}{2\pi i} \displaystyle\int_a^b \frac{g(\xi)K^-(\xi)}{(a(\xi)-\pi i)(\xi-\zeta)}\, d\xi + A$ from plemelj II

and in the factoriz we are looking for $K\hat F^+ - \hat F^- = 0$ (homog counterpart of plemelj I
& $K = \hat F^-/\hat F^+$

now $F(z) = \dfrac{1}{2\pi i K(z)} \displaystyle\int_a^b \frac{g(\xi)K^-(\xi)}{(a(\xi)-\pi i)(\xi-z)}\, d\xi$ using this and

$f(z) = \left\{\frac{1}{2\pi i}\int \quad\right\}\left\{\frac{K^- - K^+}{K^-K^+}\right\} = \left\{ \quad \right\}\frac{-2\pi i}{a^2(\xi)+\pi^2} = -\frac{1}{a^2(z)+\pi^2}\int \frac{g(\xi)}{\xi-\zeta}\,d\xi$

$f(z) = F^+ - F^-$ to get soln

Now we note that $K(z) = \exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_a^b \frac{2i\psi(\xi)\,d\xi}{\xi-z}\right\}$

I can mult $K(z)$ by $P(z)$ a polynomial since $K(z) = \dfrac{K^+}{K^-} = \dfrac{\exp\{\frac1\pi\int\quad\}^+ P(z)}{\exp\{\frac1\pi\int\quad\}^- P(z)}$

hence $K(z) = \exp\left\{\dfrac1\pi \displaystyle\int_a^b \frac{\psi(\xi)\,d\xi}{\xi-z}\right\} P(z)$

Remember $F(z) = \dfrac{1}{2\pi i}\displaystyle\int_a^b \frac{f(\xi)\,d\xi}{\xi-z}$ $\quad$ if $z\to\infty$ $\quad F(z) \sim O\left(\frac1z\right)$

what about $K_0(z)$ (when $P(z)=1$). if $z\to\infty$ $\quad K_0(z) \to \exp\left\{O\left(\frac1z\right)\right\} \to 1$

if the fns are integrable. what is behavior as $z\to a$ of $K_0(z)$?

$$K_0(z) \doteq \exp\left\{\frac1\pi\int_a^b \frac{\psi(\xi)\,d\xi}{\xi-z}\right\} \doteq \exp\left\{\frac1\pi\psi(a)\int_a^b\frac{d\xi}{\xi-z} + \cdots\right\}$$

$$\doteq \exp\left\{-\frac1\pi\psi(a)\ln(a-z) \underbrace{+\cdots}_{\text{regular at } z=a}\right\}$$

when $z\to b$ $\quad K_0(z) \doteq \exp\left\{\frac1\pi\psi(b)\ln(b-z) \underbrace{+\cdots}_{\text{regular at } z=b}\right\}$

$$\doteq (b-z)^{\psi(b)/\pi}$$

$z\to a$ $\quad K_0(z) \doteq (a-z)^{-\psi(a)/\pi}$

if $\psi = \pi/2$ $\quad K_0(z) = \exp\left\{ \frac{1}{2}\int_{-1}^{1} \frac{d\xi}{\xi - z} \right\}$ $\quad$ if $a=-1,\ b=1$

$$= \exp\left\{ \tfrac{1}{2} \ln \frac{1-z}{-1-z} \right\} = \exp\left\{ \ln \sqrt{\frac{z-1}{z+1}} \right\} = \sqrt{\frac{z-1}{z+1}}$$

note that $K^+ + K^- = 0 \Rightarrow \frac{K^+}{K^-} = K(\xi) = -1 \Leftarrow a(\xi) = 0$ also $K$ is real outside $|z|=1$

note that we have a zero at $z=-1$ but at $z=+1$ we only have a branch point thus to remove the zero mult by polynomial (since doesn't affect answer). Hence multiply $K_0(z)$ by $z+1$

$\therefore K_1(z) = \sqrt{(z-1)(z+1)} = \sqrt{z^2-1}$ this has branch points only; call this $K(z)$

Suppose we look at $\quad J(z) = K(z)F(z) - \frac{1}{2\pi i}\int_a^b \frac{g(\xi)K^-(\xi)}{(a(\xi)-\pi i)}\frac{d\xi}{\xi - z}$

$J(z)$ is analytic in full plane

behavior as $z \to \infty$

$$J(z) = O(z)\,O\left(\tfrac{1}{z}\right) - O\left(\tfrac{1}{z}\right) \Rightarrow \text{const}$$

$\therefore$ since analytic in full plane & $\Rightarrow$ const at $\infty$ must be (by Liouville Thm.) a constant

thus $\quad F(z) = \frac{\text{const}}{K(z)} + \frac{1}{2\pi i}\int_a^b \quad$ as Carlemann showed.

11/14/80

Poincaré-Bertrand Formula to invert PVI

$$\int \frac{d\xi}{\xi - \xi_0} \int \frac{f(\xi,\zeta)}{\zeta - \xi} d\zeta = -\pi^2 f(\xi_0,\xi_0) \pm \int d\zeta \int \frac{f(\xi,\zeta)}{(\xi - \xi_0)(\zeta - \xi)} d\xi$$

some parameter

let us look at

$$\frac{1}{\pi}\int_{-1}^{1} \frac{f(\xi)}{\xi - \zeta} d\xi = g(\zeta)$$

to solve it we define a fn $\quad F(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{f(\xi)}{\xi - z} d\xi = O\left(\frac{1}{z}\right)$ as $z \to \infty$

denom never vanishes for $z \notin$ on $[-1,1]$. Thus line $[-1,1]$ is a dis cont. having introduced $F(z)$ then

$$f(\zeta) = F^+(\zeta) - F^-(\zeta)$$
$$\frac{1}{\pi i}\int_{-1}^{1} \frac{f(\xi)}{\xi - \zeta} d\xi = F^+(\zeta) + F^-(\zeta)$$

also $\quad \dfrac{1}{\xi - z} = \dfrac{1}{-z(1-\xi/z)} = -\dfrac{1}{z}\left[1 + \dfrac{\xi}{z} + \cdots\right] \sim O\left(\dfrac{1}{z}\right)$

from the eqn $\dfrac{1}{\pi}\displaystyle\int_{-1}^{1} \dfrac{f(\xi)}{\xi - \zeta}\,d\xi = g(\zeta)$ then $-i\,g(\zeta) = F^{+}(\zeta) + F^{-}(\zeta)$ by 2nd Plemelj

suppose we can construct a fn $X(z)$ such that $X^{+}/X^{-} = -1 = e^{i\pi}$

(i.e. looking at homog eq of $F^{+} + F^{-} = 0$) now

$$\ln X^{+} - \ln X^{-} = i\pi$$

by 1st plemelj formula $\Rightarrow \ln X(z) = \dfrac{1}{2\pi i}\displaystyle\int_{-1}^{1} \dfrac{i\pi\, d\xi}{\xi - z}$

$$\therefore \quad X(z) = e^{1/2\int_{-1}^{1}\frac{d\xi}{\xi - z}} \qquad X(z) \to 1 \text{ as } z \to \infty$$

now look at $F^{+} + F^{-} = -i\,g(\zeta)$ divide by $X^{+}$

$$\therefore \quad \frac{F^{+}}{X^{+}} + \left(\frac{F^{-}}{X^{+}} = -\frac{F^{-}}{X^{-}}\right) = \left(-\frac{i\,g(\zeta)}{X^{+}} = \frac{i\,g(\zeta)}{X^{-}}\right)$$

$$\therefore \quad F/X = \frac{1}{2\pi i}\int_{-1}^{1} \frac{i\,g(\xi)\, d\xi}{X^{-}(\xi)(\xi - z)}$$

thus $F(z) = \dfrac{X(z)}{2\pi}\displaystyle\int_{-1}^{1} \dfrac{g(\xi)\, d\xi}{X^{-}(\xi)(\xi - z)}$

for any point $z$ then $X(z) = e^{1/2 \ln(\xi - z)\big|_{-1}^{1}} = e^{1/2\{\ln(1-z) - \ln(-1-z)\}} = e^{1/2 \ln\left(\frac{1-z}{-1-z}\right)}$

$= e^{1/2 \ln\left(\frac{z-1}{z+1}\right)} = \sqrt{\dfrac{z-1}{z+1}}$ but this $\to 1$ as $z \to \infty$; $z = \pm 1$ are branch points but $z = -1$ is also a pt of singularity

now $f(\zeta) = F^{+} - F^{-} = \dfrac{X^{+}(\zeta)}{2\pi}\displaystyle\int_{-1}^{1} \dfrac{g(\xi)\, d\xi}{X^{-}(\xi)(\xi - \zeta)} - \dfrac{X^{-}(\zeta)}{2\pi}\displaystyle\int_{-1}^{1} \dfrac{g(\xi)\, d\xi}{X^{-}(\xi)(\xi - \zeta)}$

but $X^{+} - X^{-} = 2X^{+} \Rightarrow \quad = \dfrac{X^{+}(\zeta)}{\pi}\displaystyle\int_{-1}^{1} \dfrac{g(\xi)}{X^{-}(\xi)}\dfrac{d\xi}{\xi - \zeta}$

$\sqrt{z-1} = \sqrt{1-\xi}\, e^{i\pi/2}$

$\sqrt{z-1} = \sqrt{1-\xi}\, e^{-i\pi/2}$

$$= \frac{1}{\pi} e^{i\pi/2}\sqrt{\frac{1-\zeta}{1+\zeta}} \int_{-1}^{1} e^{i\pi/2}\sqrt{\frac{1+\xi}{1-\xi}}\, \frac{g(\xi)\, d\xi}{\xi - \zeta}$$

$$= -\frac{1}{\pi}\sqrt{\frac{1-\zeta}{1+\zeta}} \int_{-1}^{1} \sqrt{\frac{1+\xi}{1-\xi}}\, \frac{g(\xi)\, d\xi}{\xi - \zeta}$$

Special case: what if $g(\zeta) = 1$ then $f(\zeta) = -\dfrac{1}{\pi}\sqrt{\dfrac{1-\zeta}{1+\zeta}}\displaystyle\int_{-1}^{1}\sqrt{\dfrac{1+\xi}{1-\xi}}\,\dfrac{d\xi}{\xi - \zeta}$

$$f(\zeta) = -\frac{1}{\pi}\sqrt{\frac{1-\zeta}{1+\zeta}} \int_{-1}^{1} \frac{1+\xi}{\sqrt{1-\xi^2}} \; \frac{d\xi}{\xi-\zeta}$$

now let $\xi = \cos\theta$, $\zeta = \cos\psi$ then $f(\zeta) = -\frac{1}{\pi}\sqrt{\frac{1-\zeta}{1+\zeta}} \int_0^{\pi} \frac{1+\cos\theta}{\cos\theta-\cos\psi}\, d\theta$

Now $\int_0^{\pi} \frac{\cos n\theta}{\cos\theta - \cos\psi}\, d\theta = \frac{\pi \sin n\psi}{\sin\psi}$ $\quad n = 0, 1, 2, \ldots$

$= \frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos n\theta \, d\theta}{\cos\theta - \cos\psi}$ since $\cos\theta$ is even

now $\int_0^{\pi} \frac{d\theta}{\cos\theta-\cos\psi} = 0$ since this is $n=0$ case

$\int_0^{\pi} \frac{\cos\theta \, d\theta}{\cos\theta-\cos\psi} = \pi$ since this is $n=1$ case

hence

$$f(\zeta) = -\frac{1}{\pi}\sqrt{\frac{1-\zeta}{1+\zeta}}\cdot\pi = -\sqrt{\frac{1-\zeta}{1+\zeta}}$$

now to check

$$\frac{1}{\pi}\int_{-1}^{1} -\sqrt{\frac{1-\xi}{1+\xi}} \; \frac{d\xi}{\xi-\zeta} = -\frac{1}{\pi}\int_{-1}^{1} \frac{1-\xi}{\sqrt{1-\xi^2}} \; \frac{d\xi}{\xi-\zeta}$$

now let $\xi = \cos\theta$ $\quad \zeta = \cos\psi$

$$-\frac{1}{\pi}\int_{\pi}^{0} \frac{1-\cos\theta}{\sin\theta}\left(\frac{-\sin\theta\, d\theta}{\cos\theta-\cos\psi}\right) = -\frac{1}{\pi}\cdot -\pi = 1$$

$$\int_0^{\pi} \frac{1-\cos\theta}{\cos\theta-\cos\psi}\, d\theta = -\pi$$

now $X(z)$ can be mult by any rational fn of $z$ which has no singularities (zeroes) away from the cut $(-1,1)$

Thus $\sqrt{z+1}\, X(z) = \sqrt{z^2-1} = \hat{X}(z)$ $\qquad z-1 =$

put into $f(\zeta) = 2F^+(\zeta) = \frac{1}{\pi} X^+(\zeta) \int_{-1}^{1} \frac{g(\xi)\, d\xi}{X^+(\xi)(\xi-\zeta)} = \frac{1}{\pi} e^{i\pi/2}\sqrt{1-\zeta^2}\int_{-1}^{1} \frac{g(\xi)\, d\xi}{e^{-i\pi/2}\sqrt{1-\xi^2}(\xi-\zeta)}$

$$= -\frac{1}{\pi}\sqrt{1-\zeta^2}\int_{-1}^{1} \frac{g(\xi)\, d\xi}{\sqrt{1-\xi^2}(\xi-\zeta)}$$

if $g = $ const $\quad \int_{-1}^{1} \frac{d\xi}{\sqrt{1-\xi^2}(\xi-\zeta)} = 0 \quad$ how come?

note that $F(z) \sim X^+ \cdot O(\frac{1}{z})$ but $X^+ \sim O(z)$

hence $F(z) \not\to O(1/z)$ but $\to$ const. $\quad z \to \infty$

Suppose we mult $X(z)\cdot \frac{1}{z-1} = \frac{1}{\sqrt{z^2-1}} = \hat{X}$

now $\hat{f}(\zeta) = \frac{1}{\pi}\hat{X}^{+} \int_{-1}^{1} \frac{g(\xi)\,d\xi}{\hat{X}^{-}\,(\xi-\zeta)} = \frac{1}{\pi}\, e^{i\pi/2}\frac{1}{\sqrt{1-\zeta^2}} \int_{-1}^{1} \frac{g(\xi)\,d\xi}{e^{-i\pi/2}\frac{1}{\sqrt{1-\xi^2}}\,\xi-\zeta}$

$$= -\frac{1}{\pi}\frac{1}{\sqrt{1-\zeta^2}} \int_{-1}^{1} \frac{\sqrt{1-\xi^2}\; g(\xi)\,d\xi}{\xi-\zeta}$$

now if $g(\xi) = \text{const}$ $\quad f(\xi) = -\frac{1}{\pi}\frac{1}{\sqrt{1-\xi^2}} \int_0^{\pi} \frac{\sin^2\theta\, d\theta}{\cos\theta - \cos\psi}$ by change as before

$$\sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta$$

$$\hat{f}(\zeta) = -\frac{1}{\pi\sqrt{1-\zeta^2}} \int_0^{\pi} \frac{\frac{1}{2} - \frac{1}{2}\cos 2\theta}{\cos\theta - \cos\psi}\, d\theta$$

$$-\tfrac{1}{2}\cdot 2\pi\zeta$$

$$= \frac{\zeta}{\sqrt{1-\zeta^2}}$$

now $f(\zeta) = -\sqrt{\frac{1-\zeta}{1+\zeta}} = \frac{\zeta}{\sqrt{1-\zeta^2}} - \frac{1}{\sqrt{1-\zeta^2}}$

hence $f, \hat{f}$ are different by $\frac{1}{\sqrt{1-\zeta^2}}$ which is sol. of $\frac{1}{\pi}\int_{-1}^{1}\frac{f(\xi)\,d\xi}{\xi-\zeta} = 0$

Thus must put conditions on $X$ : at end pts must not have poles

in general $X(z) = \exp\left\{\int_{-1}^{1} z\,\frac{\psi(\xi)\,d\xi}{\xi - z}\right\}$ if $\frac{X^+}{X^-} = e^{2\pi i\psi(\zeta)} = K$

then our plemelj formula is $KF^+ - F^- =$ known fn.

now when $z \to 1$ $\quad X(z) \to \exp\left\{\frac{1}{\pi}\psi(1)\ln(z-1)\right\} = (z-1)^{\psi(1)/\pi}$

now to get a proper $X$ must mult by $(z-1)^m$ s.t. $0 < m + \frac{\psi(1)}{\pi} < 1$

for $z \to -1$ $\quad X(z) \to \exp\left\{\frac{1}{\pi}\psi(-1)\,\ln(z+1)\right\} = (z+1)^{-\psi(-1)/\pi}$

must mult by $(z+1)^n$ s.t. $0 < n - \frac{\psi(-1)}{\pi} < 1$

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given
$$\frac{1}{\pi i}\int_{-1}^{1}\frac{f(\xi)d\xi}{\xi-\zeta}=g(\zeta)$$

we found $\quad f(\zeta)=\frac{1}{\pi i}\int_{-1}^{1}\frac{\sqrt{1-\xi^2}}{\sqrt{1-\zeta^2}}\cdot\frac{g(\xi)}{\xi-\zeta}d\xi+\frac{A}{\sqrt{1-\zeta^2}}$

to check this we must use the Poincaré-Bertrand formula

$$\left(\frac{1}{\pi i}\right)^2\int_{-1}^{1}\frac{d\xi}{\xi-\zeta}\left\{\frac{1}{\sqrt{1-\xi^2}}\int_{-1}^{1}\frac{g(\bar\xi)\sqrt{1-\bar\xi^2}}{\bar\xi-\xi}d\bar\xi+\frac{A}{\sqrt{1-\xi^2}}\right\}\overset{?}{=}g(\zeta)$$

now $\left(\frac{1}{\pi i}\right)^2\left\{-\pi^2 g(\zeta)+\int_{-1}^{1}d\bar\xi\sqrt{1-\bar\xi^2}\,g(\bar\xi)\int_{-1}^{1}\frac{d\xi}{\sqrt{1-\xi^2}}\frac{1}{(\xi-\zeta)(\bar\xi-\xi)}+\int_{-1}^{1}\frac{A\,d\xi}{\sqrt{1-\xi^2}(\xi-\zeta)}\right\}$

if $\xi=\cos\theta$ $\zeta=\cos\varphi$ $\quad\int_{-1}^{1}\frac{d\xi}{\sqrt{1-\xi^2}(\xi-\zeta)}=\frac{1}{2}\int_{-\pi}^{\pi}\frac{d\theta}{\cos\theta-\cos\varphi}=0$ for $n=0$ case as before

$$\int_{-1}^{1}\frac{d\xi}{\sqrt{1-\xi^2}}\frac{1}{(\xi-\zeta)(\bar\xi-\xi)}=\int_0^{\pi}\frac{d\theta}{(\cos\theta-\cos\varphi)(\cos\hat\varphi-\cos\theta)}=\int_0^{\pi}\frac{A\,d\theta}{\cos\theta-\cos\varphi}+\frac{B\,d\theta}{\cos\theta-\cos\hat\varphi}=0$$

$$\therefore\quad\left(\frac{1}{\pi i}\right)^2\left\{-\pi^2 g(\zeta)+\int_{-1}^{1}\cdots\right\}=g(\zeta)$$

look at $\frac{1}{\pi}\int_a^b\frac{f(\xi)d\xi}{\xi-\zeta}+a(\zeta)f(\zeta)=g(\zeta)$

define $F(z)=\frac{1}{2\pi i}\int_a^b\frac{f(\xi)d\xi}{\xi-z}\quad\ni.\quad O\left(\frac{1}{z}\right)$ as $z\to\infty$ $\quad\int_a^b f(\xi)d\xi\neq 0$

$\therefore$ but using plemelj then

$$\underbrace{i\left[F^+(\zeta)+F^-(\zeta)\right]}_{\text{integral}}+a(\zeta)\underbrace{\left[F^+(\zeta)-F^-(\zeta)\right]}_{f(\zeta)}=g(\zeta)$$

thus $\quad(a(\zeta)+i)F^+-(a(\zeta)-i)F^-=g(\zeta)\quad$ then $\quad\frac{(a(\zeta)+i)}{(a(\zeta)-i)}F^+-F^-=\frac{g(\zeta)}{a(\zeta)-i}$

define $X(\zeta)\ni.\quad\frac{a(\zeta)+i}{a(\zeta)-i}=\frac{X^+}{X^-}$

and hence $\quad X^+F^+-X^-F^-=\frac{X^+g(\zeta)}{a(\zeta)-i}$

thus $\quad XF=\frac{1}{2\pi i}\int_a^b\frac{X^+(\xi)\,g(\xi)}{(a(\xi)-i)\,(\xi-z)}d\xi$

now $F(z)=\frac{1}{2\pi i\,X(z)}\int_a^b\frac{X^+(\xi)g(\xi)\,d\xi}{(a(\xi)-i)\,(\xi-z)}\qquad$ hence $f(z)=F^+-F^-$

we thus also defined $\ln X^+ - \ln X^- = \ln \dfrac{a(\xi)+i}{a(\xi)-i}$

hence by plemelj $\ln X(z) = \dfrac{1}{2\pi i}\displaystyle\int_a^b \ln \dfrac{a(\xi)+i}{a(\xi)-i} \dfrac{d\xi}{\xi - z}$

$$\text{hence} \quad X = \exp\left\{ \frac{1}{2\pi i} \int_a^b \downarrow \right\}$$

$m(a,b)$

we can also mult by a rational fn. How does mult by this fn affect sol

even then $X P(z) = \dfrac{X^+ P}{X^- P} = \dfrac{X^+}{X^-}$

if $\dfrac{a(\xi)+i}{a(\xi)-i} = e^{2i\psi(\xi)}$ then

$$X(z) = \exp\left\{ \frac{1}{2\pi i} \int_a^b \frac{2i\psi(\xi)}{\xi - z} d\xi \right\}$$

now as $z \to b$ $\quad X(z) = \exp\{\frac{1}{\pi}\psi(b) \ln(z-b)\} = (z-b)^{\psi(b)/\pi} + h.o.t.$

$z \to a$ $\quad X(z) = \exp\{-\frac{1}{\pi}\psi(a)\ln(z-a)\} = (z-a)^{-\psi(a)/\pi} + h.o.t.$

choose $M$ & $N$ $\ni$. $\quad (z-b)^M X(z)$ has no zeroes or pole at $b$

thus $\quad 0 \le M + \dfrac{\psi(b)}{\pi} < 1 \quad$ & $\quad 0 \le N - \dfrac{\psi(a)}{\pi} < 1$

thus choose the rational fn $(z-b)^M (z-a)^N$

define the fundamental $X$ to be $X_0$ $\quad \therefore \quad X = X_0(z)(z-b)^M(z-a)^N$

now as $z \to \infty$ the asymptotic behavior of $X_0(z) \to 1$ & $(z-b)^M(z-a)^N \to z^{M+N}$

Consider $G(z) = X(z)F(z) - \dfrac{1}{2\pi i}\displaystyle\int_a^b \dfrac{g(\xi)X^-(\xi)}{a(\xi)-i}\dfrac{d\xi}{\xi - z}$

this fn is continuous throughout finite $z$ plane ie analytic everywhere.

$\cong z^{M+N-1} + O(1/z)$

$F(z) \sim O(\frac{1}{z})$

we have following possibilities

suppose $M+N=0$ $(M=N=0)$ $\quad G(z) \sim O(1/z)$ as $z \to \infty$ $\to$ $G(z) = 0$ $\forall z$ by Liouville

$$\Rightarrow F(z) = \frac{1}{2\pi i X_0(z)} \int_a^b \frac{g(\xi) X_0^-(\xi)}{a(\xi)-i} \frac{d\xi}{\xi - z} \quad \text{since } X = X_0$$

if $M+N > 0$ then $G(z) \cong z^{M+N-1}$ $\Rightarrow$ $G(z)$ is poly of degree $M+N-1$

hence $F(z) = \dfrac{1}{2\pi i X(z)}\displaystyle\int_a^b \dfrac{g(\xi)X^-(\xi)\,d\xi}{(a(\xi)-i)(\xi - z)} + \dfrac{P_{M+N-1}(z)}{X(z)}$ $\quad$ now unique if $M+N<0$ then $G(z) \cong O\left(\frac{1}{z}\right)$ and $G \equiv 0$ by Liouville

thus $F(z) = \frac{1}{2\pi i\, X(z)} \int_a^b \frac{g(\xi) X^-(\xi)}{a(\xi)-i} \frac{d\xi}{\xi - z} \cong z^{-(M+N)} \int_a^b H(\xi)\left\{\frac{1}{z} - \frac{\xi}{z^2} \cdots\right\} d\xi$

now first term is $z^{-(M+N+1)}$ hence this power exceeds $O\left(\frac{1}{z}\right)$

thus we must restrict $g(\xi)$ s.t.

$$\int_a^b \frac{g(\xi)\ X^-(\xi)}{a(\xi)-i}\ \xi^{\text{power}}\, d\xi = 0 \quad \text{for power } 1, 2, \ldots (M+N)$$

Examples $\frac{1}{\pi} \int_0^1 \frac{\sin \pi \xi}{\xi - \zeta} f(\xi) d\xi + \cos \pi \zeta\, f(\zeta) = g(\zeta)$

using our results $i\left[F^+(\zeta) + F^-(\zeta)\right] + \cos \pi \zeta \left[\frac{F^+(\zeta) - F^-(\zeta)}{\sin \pi \zeta}\right] = g(\zeta)$

now $\underbrace{F^+\left[\cos \pi \zeta + i \sin \pi \zeta\right]}_{F^+ e^{i\pi\zeta}} + \underbrace{F^-\left[\cos \pi \zeta - i \sin \pi \zeta\right]}_{-F^- e^{-i\pi\zeta}} = g(\zeta) \sin \pi \zeta$ $f(\zeta)$

now $X = \frac{e^{i\pi\zeta}}{e^{-i\pi\zeta}} = e^{2\pi i \zeta}$

$\therefore\ F^+ e^{2\pi i \zeta} - F^- = g(\xi) \sin \pi \xi\, e^{i\pi\xi}$

now $X_0(z) \stackrel{\text{def}}{=} \exp\left\{\frac{1}{2\pi i} \int_0^1 \frac{2\pi i \zeta}{\zeta - z} d\zeta\right\} = \exp\left\{\int_0^1 \frac{\zeta - z + z}{\zeta - z}\right) d\zeta$

$\exp\left\{1 + \cancel{\exp} \left(-z \int_0^1 \frac{d\zeta}{z-\zeta}\right)\right.$

$= \exp\left(1 + z \ln \frac{z-1}{z}\right)$

as $z \to 0$ $X_0(z) \to e \exp\left[\underbrace{z \ln(z-1) - z \ln z}_{\to 0}\right] = e \exp(0) = e \Rightarrow N = 0$

Const ∴ since no pole or zero $N=0$

as $z \to 1$ $X_0(z) \to (z-1)^z$ this has a zero at 1 hence pick $M = -1$

$\Rightarrow\ X(z) = \frac{1}{z-1} X_0$

hence

$$F(z) = \frac{1}{2\pi i\, X(z)} \int_0^1 \frac{\sin \pi \xi\, e^{i\pi\xi} g(\xi)}{(\xi - z)} d\xi$$

$\underset{z \to \infty}{\cong} O(z-1) \cdot O\left(\frac{1}{z}\right) = \text{const}$, This is a soln if $0 = \int_0^1 g(\xi) e^{\pi i \xi} \sin \pi \xi$

examples of 3rd case

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$$\int_{-1}^{1} \frac{f(\xi)}{\xi-\zeta}\, d\xi = g(\zeta) \quad \Rightarrow \quad \int_{-1}^{1} \underbrace{K(\xi-\zeta)}_{\text{convolution Kernel}} f(\xi)\, d\xi = g(\zeta)$$

Carleman procedure requires $K$ to be a special type. A sol can be found by defining a fn which is sectionally holomorphic discont on a line or curve or closed loop.

Weiner – Hopf Technique

W.H eqn. $\int_0^\infty K(x-x')f(x')\,dx' = g(x) \qquad x>0$

Uses fact that $K$ is a convol kernel.
Now apply fourier transfor.

let $\int_0^\infty K(x-x')\, f(x')\,dx' = F(x)$. what is $\int_{-\infty}^{\infty} e^{-i\zeta x} F(x)\,dx \overset{?}{=} \int_{-\infty} e^{-i\zeta x}\int_0^\infty K f$

One characteristic of WH is to extend the fn for $x<0$ also

ie let $\hat{f}(x) = \begin{cases} f(x) & \text{for } x>0 \\ 0 & \text{for } x<0 \end{cases}$

hence $\int_{-\infty}^{\infty} e^{-i\zeta x} \hat{F}(x)\,dx = \int_{-\infty}^{\infty} e^{-i\zeta x}\left(\int_{-\infty}^{\infty} K(x-x')\hat{f}(x')\,dx'\right)dx$

$$= \int_{-\infty}^{\infty} e^{-i\zeta(x-x')} K(x-x')\,dx \int_{-\infty}^{\infty} \hat{f}(x') e^{-i\zeta x'}\,dx'$$

hence let $x-x'=\bar{x}$ ($x'$ being a parameter) then $dx = d\bar{x}$ and

$\int_{-\infty}^{\infty} e^{-i\zeta x}\hat{F}(x)\,dx = K(\zeta)\hat{f}(\zeta)$ or $K(\zeta) f(\zeta)$ since $\int_{-\infty}^{0} e^{-i\zeta x'}\hat{f}(x')\,dx' = 0$

we can redefine the original eqn in the $(\infty,-\infty)$ line as
i.e. $-\infty < x < \infty$ : look at

$$\int_{-\infty}^{\infty} K(x-x')f(x')\,dx' = \begin{Bmatrix} g(x) \\ 0 \end{Bmatrix}^{\text{Known}} + \begin{Bmatrix} h(x) \end{Bmatrix}^{\text{Unknown}}$$

$f=0, \; x<0$
$h=0 \quad x>0$

$h\neq 0 \qquad f\neq 0$

$x$

we are not finding 2 unknowns but $h$ & $f$ are complimentary fns so that hence $K(\zeta)\ f(\zeta) = g(\zeta) + h(\zeta)$ $\qquad$ $K$ & $g$ are known.

$\int_0^\infty e^{-i\zeta x} f(x)\,dx$ $\qquad$ $\int_{-\infty}^0 e^{-i\zeta x} h(x)\,dx$

$\int_{-\infty}^{\infty} e^{-i\zeta \tilde{x}} K(\tilde{x})\,d\tilde{x}$

Now $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta x} f(\zeta)\,d\zeta$

But since $f(\zeta) = \int_0^\infty e^{-i\zeta x} f(x)\,dx$ $\qquad$ if $\zeta = \xi + i\eta$ then $f(\zeta) = \int_0^\infty e^{-i\xi x} e^{\eta x} f(x)\,dx$

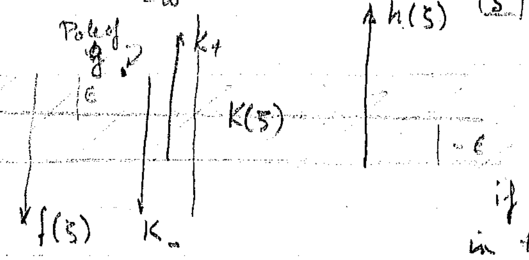
if $\eta < 0$ then $f(\zeta)$ will not grow if $f$ is an integrable fn. Hence

if $f(\zeta) \sim e^{-\epsilon x}$ then $\eta < \epsilon$ for $f$ to be integrable

for $h(x)$ $\qquad$ $h(\zeta) = \int_{-\infty}^0 e^{-i\xi x} e^{\eta x} h(x)\,dx$ : if it is integrable & bounded $\Rightarrow \eta >$

Now ~~Suppose~~ K is defined in a strip which may be for $(-\omega, \omega)$. Now look for overlap

K is defined in strip due to 2 sided integral




hence soln is defined in the joint analyticity region, ie the shaded strip.

if $\epsilon \to 0$ we have a hilbert problem in the $\zeta$ plane

example: if $g(\zeta) = 0$ then $Kf_- = h_+$ $\qquad$ $f_-$ regular in lower half, $h_+$ " " upper half

now factor $K = \frac{K_-}{K_+}$ $\Rightarrow K_- f_- = K_+ h_+ = I$

this defines a fn that is entire in whole plane since they agree in the strip & $K_- f_-$ is analytic in lower plane & $K_+ h_+$ is analytic in upper plane. $I$ is entire & must have asymptotic representation consistent w/ $f_- K_-$ & $h_+ K_+$ as $|\zeta| \to \infty$

hence $f_-(\zeta) = \frac{I(\zeta)}{K_-(\zeta)}$

now $K_-(\zeta)$ as $|\zeta| \to \infty$ can be found presumably from $K$

also $f_-(\zeta) \overset{\text{def}}{=} \int_0^\infty e^{-i\zeta x} f(x)\,dx$

the behavior of $f$ as $\eta \to -\infty$ depends only near $x = 0$ since $e^{\eta x} f(x)$ is largest there.

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Eqns of 1st Kind

$$\int_a^b K(x,y)f(y)\,dy = g(x) \qquad \text{suppose } \int K^2(x,y)\,dy < \infty \text{ \& } \int g^2\,dx < \infty$$

we want to find a fn $f(x)$ (also $\int f^2\,dx < \infty$).

Case I: if $K(x,y) = K(y,x) \Rightarrow \exists$ at least 1 eigenfn $\ni$.

$$\varphi_n(x) = \lambda_n \int K(x,y)\varphi_n(y)\,dy$$

assume $\exists$ $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ are the eigenfns &

$\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ ev

and they satisfy

$$\int \phi_n \phi_m\,dx = \delta_{mn}$$

Suppose we take $\int \phi_n(x) g_n(x)\,dx = c_n$ [Fourier Coeff of $g(x)$ relative to the orthog. system $\{\phi_n\}$]

if $g$ is sq integrable then $\sum_{i=1}^{\infty} c_i^2 < \infty$.

If this is so then $\sum \lambda_n^2 c_n^2 < \infty$ if $\int f^2\,dy < \infty$ this is necess & sufficient condition

$$\text{Now} \quad \int g(x)\phi_n(x)\,dx = c_n = \int K(x,y)\, f(y)\, \phi_n(x)\,dx\,dy$$

$$= \int f(y) \left( K(x,y)\phi_n(x)\,dx \right) dy$$

$$= \int f(y) \left( K(y,x)\phi_n(x)\,dx \right) dy$$

$$\int f(y)\, \frac{\phi_n(y)}{\lambda_n}\, dy$$

thus

$$c_n \lambda_n = \int f(y)\phi_n(y)\,dy = d_n$$

now if $\int f^2 dy < \infty \Rightarrow \sum d_n^2 < \infty \Longrightarrow \sum \lambda_n^2 c_n^2 < \infty$ QED.

Complete set of fn $\Rightarrow \not\exists$ a nontrivial fn $h \ni \int h\phi_n\,dx = 0$ $n=1,2,\ldots$

now $\sum \lambda_n^2 c_n^2 < \infty$ (is sufficient for a soln) if $\phi_n$'s form a complete set.

if $K(x,y) \neq K(y,x)$ define $\phi(x) = \lambda \int K(x,y)\psi(y)\,dy$

$$\psi(x) = \lambda \int K(y,x)\phi(y)\,dy$$

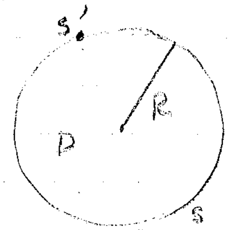
if non triv sol exist we can define $\phi_1, \phi_2, \ldots$

$\psi_1, \psi_2, \ldots$

& ev $\lambda_1, \lambda_2, \ldots$

this will then make the fn $\phi$ or $\psi$ ef of a symmetric kernel.

Suppose we have a circle



$\nabla^2 u = 0$ in the doma

$u = g(s)$ on C

$\therefore\ u(\underline{r}) = \int_c \ln \frac{1}{|\underline{r} - \underline{s}'|} f(s')\,ds'$

$g(s) = \int_c \ln \frac{1}{|s - s'|} f(s')\,ds'$ on C

thus $\phi_n(s) = \lambda_n \int_c \ln \frac{1}{|\underline{s} - \underline{s}'|} \phi_n(s')\,ds'$

HW $\boxed{\lambda_n \sim \frac{n\pi}{R} \qquad \lambda_0 \sim \ln \frac{1}{R}}$ — Homework — show this

then from us $\sum c_n^2 \lambda_n^2 \sim \sum n^2 \lambda_n^2 < \infty$ hence $\lambda_n \sim n^{-p}$ $p > 1$

let $g(s)$ be represented by $g(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^{\alpha}}$ $\quad \frac{3}{2} > \alpha > 1$

hence $\lambda_n \sim n^{-\alpha}$ hence $\sum n^2 \frac{1}{n^{2\alpha}}$ diverges for $\frac{3}{2} > \alpha > 1$

$\ln(z' - z) = \ln z' + \ln(1 - z/z')$

Going back to Wiener Hopf Technique

Suppose

$$\int_0^{\infty} e^{-\alpha|x-x'|} f(x')\,dx' = e^{-\beta x} \qquad x > 0$$

extended eq.

$$\int_{-\infty}^{\infty} e^{-\alpha|x-x'|} f(x')\,dx' = \begin{Bmatrix} e^{-\beta x} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ g(x) \end{Bmatrix} \qquad \begin{matrix} x > 0 \\ x < 0 \end{matrix}$$

$\Rightarrow\ f(x) = 0$ for $x < 0$

hence $g(x) = \int_0^{\infty} e^{-\alpha|x-x'|} f(x')\,dx' \qquad x < 0$

multiply by $e^{i s x}$ & integrate $(-\infty, \infty)$

hence

$$K(\xi)\, f(\xi) = \int_0^\infty e^{-\beta x + i\xi x}\, dx + g(\xi)$$

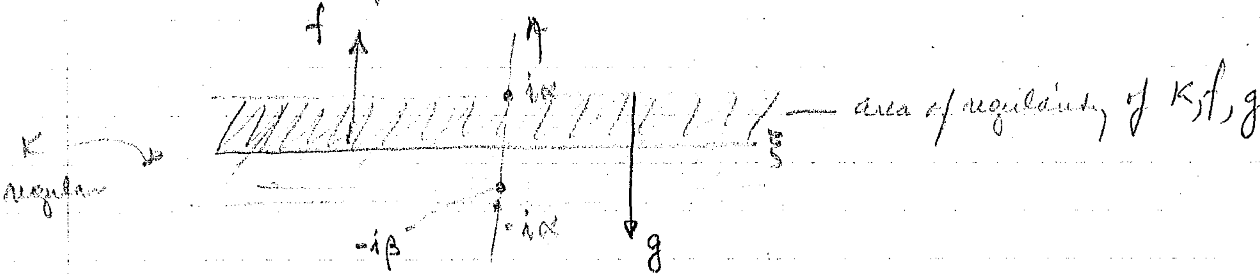
$$= \frac{1}{\beta - i\xi} + g(\xi)$$

$$K(\xi) = \int_{-\infty}^{\infty} e^{i\xi z} e^{-\alpha|z|}\, dz \qquad \text{since } e^{-\alpha|z|} \text{ is even use only cosine part}$$

$$2\int_0^\infty \cos \xi z\, e^{-\alpha z}\, dz = \frac{2\alpha}{\xi^2+\alpha^2}$$

now $$f(\xi)\cdot \frac{2\alpha}{\xi^2+\alpha^2} = \frac{i}{\xi + i\beta} + g(\xi)$$

has poles at $\pm i\alpha$




now $$f(\xi) = \int_0^\infty e^{i\xi x} f(x)\, dx \qquad f(\xi) \text{ is defined for } \eta > 0$$

now $$g(\xi) = \int_{-\infty}^{0} e^{i\xi x} g(x)\, dx \quad \text{and} \quad g(x) \sim e^{\alpha x}\int_0^\infty e^{-\alpha x'} f(x')\, dx$$

hence $$g(\xi) \sim \int_{-\infty}^{0} e^{i\xi x} e^{\alpha x}$$

$\therefore (-\eta+\alpha)x$

hence $-\eta+\alpha > 0$

$\eta < \alpha$

and $g(\xi)$ is defined for $\eta < \alpha$

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Last time we looked at

$$\int_0^\infty e^{-\alpha|x-x'|} f(x')\,dx' = e^{-\beta x} \quad \alpha > 0$$

. Using the wiener-Hopf technique: we first extend eqn. for $x \in (-\infty, \infty)$ so that

$$\int_{-\infty}^{\infty} e^{-\alpha|x-x'|} f(x')\,dx' = \begin{Bmatrix} e^{-\beta x} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ g(x) \end{Bmatrix} \qquad x \begin{matrix} > 0 \\ < 0 \end{matrix}$$

and $f(x) = 0$ for $x < 0$ hence $g(x) = \int_0^\infty e^{-\alpha|x-x'|} f(x')\,dx'$

Now def $F(\zeta) = \int_0^\infty e^{i\zeta x} f(x)\,dx \qquad G(\zeta) = \int_{-\infty}^0 e^{i\zeta x} g(x)\,dx$

~~then~~ and $K(\zeta) = \int_{-\infty}^\infty e^{i\zeta x} e^{-\alpha|x|}\,dx \quad + \quad H(\zeta) = \int_0^\infty e^{i\zeta x - \beta x}\,dx$

by convolution

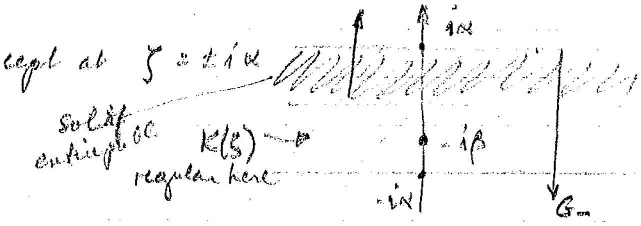
$$K(\zeta)F(\zeta) = H(\zeta) + G(\zeta)$$

✓ – known fns.

This is a Hilbert problem since F & G have ~~[illegible]~~ defined in different part of ζ plane

$$F(\zeta) \to 0 \text{ as } \eta > 0 \qquad G(\zeta) \to 0 \text{ as } \eta < 0$$

now $K(\zeta) = \frac{2\alpha}{\zeta^2 + \alpha^2}$ analytic except at $\zeta = \pm i\alpha$




now $H(\zeta) = \frac{1}{\beta - i\zeta} = \frac{i}{\zeta + i\beta}$ has singularity at $\zeta = -i\beta$

now $g(x) = e^{\alpha x}\int_0^\infty e^{-\alpha x'} f(x')\,dx'$ since $g$ is def for $x < 0$ & $f$ for $x > 0$

now $G(\zeta) = \int_{-\infty}^0 e^{i\zeta x} e^{\alpha x} \cdot \text{const}\,dx \quad \therefore G(\zeta) \sim e^{(-\eta+\alpha)x} \quad \therefore -\eta + \alpha > 0$

$\therefore \alpha < \eta$

hence $G$ is defined for $i\eta < i\alpha$

now $K(\zeta) = \frac{2\alpha}{(\zeta - i\alpha)(\zeta + i\alpha)}$

and $\frac{2\alpha F_+}{\zeta + i\alpha} = H(\zeta)\cdot(\zeta - i\alpha) + G_-(\zeta)(\zeta - i\alpha)$

regular in lower half since $G$ is regular for $\eta < \alpha$ & so is $\zeta - i\alpha$ (−)

regular in upper half since $F_+$ regular for $\eta > 0$ & $\zeta + i\alpha$ is regular for $\eta > 0$ (+)

$= i\,\frac{\zeta - i\alpha}{\zeta + i\beta}$

regular in upper half since $\frac{1}{\zeta + i\beta}$ has a pole in lower half plane & $\zeta - i\alpha$ has [illegible] upper half plane (+)

$\frac{2\alpha F_+}{\zeta + i\alpha} - i\left(\frac{\zeta - i\alpha}{\zeta + i\beta}\right) = G_-(\zeta)(\zeta - i\alpha)$ = entire fn of $\zeta$ in the strip

now $G_-(\zeta)$ has its largest contrib near $x = 0$ now $g(x) \to$ const

now $G_-(\zeta) \underset{\eta \to -\infty}{\sim} \text{const} \int_{-\infty}^0 e^{i\zeta x}\,dx \sim \frac{e^{i\zeta x}}{i\zeta} \sim O\!\left(\frac{1}{\zeta}\right)$

hence $G_-(\zeta)(\zeta - i\alpha)$

$\to O(1)$

$$\Rightarrow \quad F_+(\xi) \underset{\eta\to\infty}{\sim} \int_0^{\infty} e^{i\xi x} f(x)\,dx \qquad \text{let } f(x) \sim x^{\gamma} \quad -1 < \gamma \le 0$$

ie $f(x)$ has an integrable singularity

then let $\xi x = \tau$

$$\int_0^{\infty} \frac{e^{i\tau}}{\xi} f\!\left(\frac{\tau}{\xi}\right) d\tau = \int_0^{\infty} \frac{e^{i\tau}}{\xi}\left(\frac{\tau}{\xi}\right)^{\gamma} d\tau = \xi^{-(1+\gamma)} \int_0^{\infty} e^{i\tau}\tau^{\gamma}\,d\tau$$

hence $F_+(\xi) \sim O\left(\frac{1}{\xi^{\gamma+1}}\right)$ thus $\frac{(2\alpha)F_+(\xi)}{\xi + i\alpha} \to 0$ like $\frac{1}{\xi^{\gamma+2}}$

thus the entire fn is a constant

$$\text{thus } F_+(\xi) = \frac{1}{2\alpha}(\xi + i\alpha)\left[C + i\,\frac{\xi + i\alpha}{\xi + i\beta}\right]$$

$$\text{hence} \quad F_+(\xi) = \frac{C}{2\alpha}(\xi + i\alpha) + \frac{i}{2\alpha}\,\frac{(\xi - i\alpha)(\xi + i\alpha)}{\xi + i\beta}$$

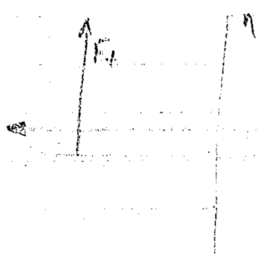
$$\text{now } \frac{\xi + i\alpha}{\xi + i\beta} = 1 + \frac{i(\alpha - \beta)}{\xi + i\beta} \quad \text{and } \frac{\xi - i\alpha}{\xi + i\beta} = \frac{1 - i(\alpha - \beta)}{\xi + i\beta}$$

using these gives

not correct

$$F_+(\xi) = \underbrace{C(\xi + i\alpha)}_{O(\xi)} + \frac{i}{2\alpha}\left\{\underbrace{1 + i(\alpha - \beta)}_{\text{Const}} + \underbrace{\frac{\alpha^2 - \beta^2}{\xi + i\beta}}_{O(1/\xi)}\right\}$$

for the last term $\quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_+(\xi)\, e^{-i\xi x}\, d\xi \implies F_+(\xi) = \int_0^{\infty} e^{i\xi x} f(x)\,dx$



$\to$ integral for $F_+(\xi)$ $\quad \therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i}{2\alpha}(\alpha^2 - \beta^2)\,\frac{e^{-i\xi x}}{\xi + i\beta}\,d\xi \quad x > 0$

use residues close of contour in lower half plane

$$\therefore \frac{i}{2\pi}\,\frac{(-2\pi i)}{2\alpha}(\alpha^2 - \beta^2)\, e^{-i(-i\beta)x} = \frac{\alpha^2 - \beta^2}{2\alpha}\, e^{-\beta x}$$

$$\frac{i}{2\alpha}\left(1 + i(\alpha - \beta)\right) \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\,d\xi}_{\delta(x)} \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\xi + i\alpha)\, e^{-i\xi x}\,d\xi \sim \delta'(x)$$

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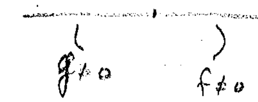
Continuing from last time

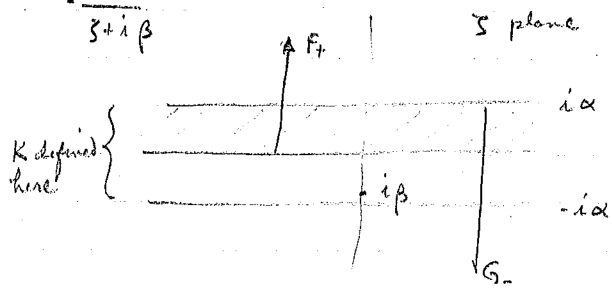
$$\int_{-\infty}^{\infty} e^{-\alpha|x-x'|} f(x')\,dx' = \begin{Bmatrix} e^{-\beta x} & 0 \\ 0 & g(x) \end{Bmatrix} \quad \begin{matrix} x>0 \\ x<0 \end{matrix}$$




apply fourier transform $\Rightarrow K(\zeta)\, F_+(\zeta) = H(\zeta) + G_-(\zeta)$

w/ $\int_0^{\infty} e^{i\zeta x} f(x)\,dx$ $\qquad \int_0^{\infty} e^{i\zeta x} e^{-\beta x}\,dx = \frac{i}{\zeta + i\beta}$ $\qquad \int_{-\infty}^{0} e^{i\zeta x} g(x)\,dx$

$$K(\zeta) = \int_{-\infty}^{\infty} e^{i\zeta x} e^{-\alpha|x|}\,dx = \frac{2\alpha}{\zeta^2 + \alpha^2}$$




Now rewriting

$$\frac{2\alpha F_+(\zeta)}{\zeta + i\alpha} - i\,\frac{\zeta - i\alpha}{\zeta + i\beta} = (\zeta - i\alpha)\, G_-(\zeta)$$

defines an entire fn. $=$ const. $= A$

$G_-(\zeta) \to \frac{1}{\zeta}$ as $\zeta \to \infty$

(+) regular in $\eta > 0$ $\qquad$ (−) regular in $\eta > 0$

Now $F_+(\zeta) = \frac{1}{2\alpha}\left(A + i\,\frac{\zeta - i\alpha}{\zeta + i\beta}\right)\zeta + i\alpha$ $\qquad$ now $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\zeta x} F_+(\zeta)\,d\zeta$

contour must be in domain of regularity & must lie above $-i\beta$

$$F_+(\zeta) = \frac{1}{2\alpha}\left[A(\zeta + i\alpha) + i(\zeta + i\alpha)\left(\frac{\zeta + i\beta - i(\alpha+\beta)}{\zeta + i\beta}\right)\right]$$

$$= \frac{1}{2\alpha}\left[A(\zeta + i\alpha) + i(\zeta + i\alpha) + (\alpha+\beta)\left(\frac{\zeta + i\beta + i(\alpha-\beta)}{\zeta + i\beta}\right)\right]$$

$$= \frac{1}{2\alpha}\left[A(\zeta + i\alpha) + i(\zeta + i\alpha) + (\alpha+\beta) + i\left(\frac{\alpha^2 - \beta^2}{\zeta + i\beta}\right)\right]$$

Now must close contour in lower half plane & use residue theorem.

$$\therefore f(x) = \frac{1}{2\alpha \cdot 2\pi} \cdot i \cdot \underbrace{-2\pi i}_{\text{contour is clockwise in lower half plane}} (\alpha^2 - \beta^2)\, e^{-i(-i\beta)x} + \ldots$$

$$= \frac{1}{2\alpha}\Big\{(\alpha^2 - \beta^2)\, e^{-\beta x} + \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\zeta x}\,d\zeta\,(\alpha+\beta)}_{(\alpha+\beta)\,\delta(x)} - \frac{\alpha}{2\pi}\int e^{-i\zeta x}\,d\zeta + \frac{i}{2\pi}\int \zeta\, e^{-i\zeta x}\,d\zeta +$$

Now $\frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta e^{-i\zeta x}\,d\zeta = i\,\delta'(x)$ & $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\zeta x}\,d\zeta = \delta(x)$ using this gives

$$f(x) = \frac{1}{2\alpha}\left\{(\alpha^2 - \beta^2)e^{-\beta x} + (\alpha+\beta)\,\delta(x) - \alpha\,\delta(x) - \delta'(x) + Ai\left[\delta'(x) + \alpha\delta(x)\right]\right\}$$

Sol. of corresponding homog.

eqn. $\int^{\infty} e^{-\alpha|x-x'|}$ [illegible] thus $\frac{1}{2\alpha}\left\{ (\alpha^2-\beta^2)e^{-\beta x} + \beta\,\delta(x) - \delta'(x) \right\} = f(x) \quad x>0$

$\qquad\qquad\qquad\qquad 0 \qquad\qquad\qquad\qquad = f(x) \quad x<0$

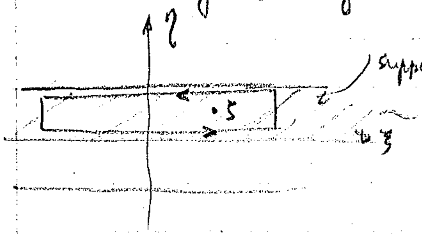
Verify this is true also show that $A\,i[\delta' + \alpha\delta] \Rightarrow \int_0^\infty e^{-\alpha|x-x'|} f(x')\,dx' = 0$

Complicatedness in Wiener Hopf is how $K(\zeta)$ is split

Recipe is give by Cauchy Formula

ie suppose we had a strip of regularity and we want to split $K(\zeta) = \frac{K_+(\zeta)}{K_-(\zeta)}$

$\log K(\zeta) = \log K_+(\zeta) - \log K_-(\zeta)$

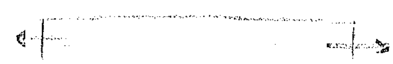


suppose $K(\zeta) \neq 0$ here

look at rectangular contour

if $F(\zeta')$ is analytic in the strip except at $\zeta = \zeta'$ $\qquad F(\zeta) = \frac{1}{2\pi i}\int \frac{F(\xi)}{\xi - \zeta}\,d\xi$

let the Contour extend



then $\frac{F(\xi)}{\xi-\zeta} \to 0$ since $\xi \to \infty$ & $F(\xi)$ is bounded

hence sides give no contrib

$$\therefore F(\zeta) = \frac{1}{2\pi i}\int \frac{F(\xi)}{\xi-\zeta}\,d\xi + \frac{1}{2\pi i}\int \frac{F(\xi)}{\xi-\zeta}\,d\xi$$

above contour — define this by $F_+(\zeta)$ — below contour — define this as $F_-(\zeta)$

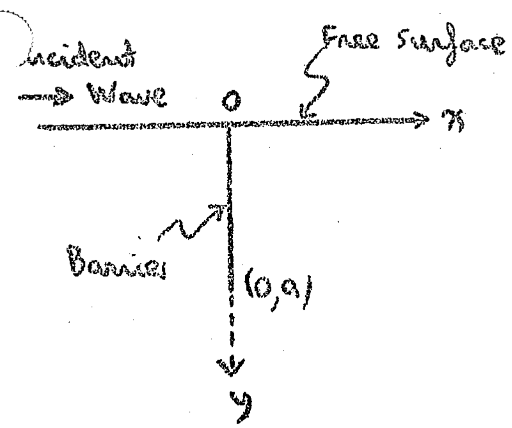
$$\text{thus} \quad \ln K(\zeta) = \frac{1}{2\pi i}\int \frac{\ln K(\xi)}{\xi-\zeta}\,d\xi - \frac{1}{2\pi i}\int \frac{\ln K(\xi)}{\xi-\zeta}\,d\xi$$

Now $\underset{\text{regular in upper}}{K_+F_+} = \underset{\text{? what is it regular}}{K_-H} + \underset{\text{regular i lower}}{K_-G_-}$

must break it up into 2 additive fns that are defined in upper & lower half plane thus $K_-H = \frac{1}{2\pi i}\int \frac{K_-H}{\xi-\zeta}\,d\xi + \frac{1}{2\pi i}\int \frac{K_-H}{\xi-\zeta}\,d\xi$

using the Cauchy formula.

# SCATTERING OF A SURFACE WAVE BY A VERTICAL BARRIER IN A FLUID




Velocity potential for plane incompressible and non-vortical motion, $\phi(x,y)e^{-i\omega t}$, satisfies

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = 0 \quad \text{in the fluid } (y>0)$$

constant pressure surface condition

$$\left(\frac{\partial}{\partial y}+K\right)\phi = 0 \quad \text{at the (equilibrium) free surface } y=0,\ -\infty<x<\infty$$

$\frac{\omega^2}{g}$

where
$$K=\frac{\omega^2}{g}=\frac{(\text{angular frequency of oscillatory motion})^2}{\text{acceleration of gravity}}>0$$

and
$$\frac{\partial\phi}{\partial x}=0,\quad x=0,\ 0<y<a,\quad \text{on the rigid barrier}$$

The incident surface wave is described by (the real part of)

$$\phi^{inc}(x,y)\,e^{-i\omega t} = \alpha\, e^{iKx-Ky-i\omega t}$$

Representation for the complete velocity potential in the presence of the barrier

discont. in little $\phi$ across the barrier

$$\phi(x,y)=\phi^{inc}(x,y)-\int_0^a \Phi(y')\frac{\partial}{\partial x'}G(x,y;0,y')\,dy'$$

$$\Phi(y)\overset{\text{def}}{=}\phi(-0,y)-\phi(+0,y) \quad \text{(the discontinuity in } \phi \text{ across the barrier)};$$

$\Phi(a)=0$ since no discont. there

here
$$G(x,y;x',y')=-\frac{1}{2\pi}\ln\left(\frac{(x-x')^2+(y-y')^2}{(x-x')^2+(y+y')^2}\right)^{1/2}+\frac{1}{\pi}\int_0^\infty\frac{dk}{k-K}\cos k(x-x')\,e^{-k(y+y')},$$

with an integral that passes below the pole at $k=K$, is characterized by

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)G=-\delta(x-x')\,\delta(y-y')$$

and
$$\left(\frac{\partial}{\partial y}+K\right)G=-\frac{1}{2\pi}\left(\frac{y-y'}{(x-x')^2+(y-y')^2}-\frac{y+y'}{(x-x')^2+(y+y')^2}\right)-\frac{1}{\pi}\int_0^\infty\frac{k-K+K}{k-K}\cos k(x-x')\,e^{-k(y+y')}\,dk$$
$$+KG$$

$$= -\frac{1}{2\pi}\left(\frac{y-y'}{(x-x')^2+(y-y')^2} + \frac{y+y'}{(x-x')^2+(y+y')^2}\right) - \frac{K}{2\pi}\ln\left(\frac{(x-x')^2+(y-y')^2}{(x-x')^2+(y+y')^2}\right)^{1/2}$$

whence

$$\left(\frac{\partial}{\partial y}+K\right)G(x,0;x',y') = 0, \qquad -\infty < x < \infty$$

Boundary condition (zero normal velocity) on the barrier

$$\frac{\partial}{\partial x}\phi^{inc}\Big|_{x=0} = \lim_{x\to 0}\frac{\partial}{\partial x}\int_0^a \Phi(y')\frac{\partial}{\partial x'}(G(x,y;x',y'))_{x'=0}\,dy'$$

i.e.,

$$i\alpha K e^{-Ky} = \frac{d^2}{dy^2}\int_0^a \Phi(y')\,G(0,y;0,y')\,dy', \quad 0<y<a \longleftarrow$$

Integrating the latter relation twice in succession,

$$\frac{d}{dy}\int_0^a \Phi(y')G(0,y;0,y')\,dy' = -i\alpha e^{-Ky} + A$$

and

$$\int_0^a \Phi(y')\,G(0,y;0,y')\,dy' = \frac{i\alpha}{K}e^{-Ky} + Ay + B\ ;$$

thus

$$\int_0^a \Phi(y')\left[KG(0,y;0,y') + \frac{dG(0,y;0,y')}{dy}\right]dy' = A(1+Ky) + KB$$

which implies

$$\int_0^a \Phi(y')\left[K\ln\left|\frac{y-y'}{y+y'}\right| + \frac{1}{y-y'} + \frac{1}{y+y'}\right]dy' = C_1 y + C_2, \quad 0<y<a \longleftarrow$$

where $C_1 = -2\pi K A$, $C_2 = -2\pi A - 2\pi K B$.

Define

$$F(y) = \int_y^a \Phi(y')\,dy', \quad \frac{dF}{dy} = -\Phi(y), \quad F(a) = 0$$

and then

$$\int_0^a \Phi(y')\ln\left|\frac{y-y'}{y+y'}\right|dy' = -\int_0^a \ln\left|\frac{y-y'}{y+y'}\right|dF(y') = +\int_0^a F(y')\frac{d}{dy'}\ln\left|\frac{y-y'}{y+y'}\right|dy'$$

$$= -\int_0^a F(y')\left(\frac{1}{y-y'}+\frac{1}{y+y'}\right)dy'\ ;$$

accordingly,

$$\int_0^a [\Phi(y')-K F(y')] \left(\frac{1}{y-y'}+\frac{1}{y+y'}\right)dy' = C_1 y + C_2, \quad 0<y<a.$$

Setting $y=0$ it may be concluded that $C_2=0$, whence

$$\int_0^a \mathcal{F}(y') \frac{dy'}{y^2-y'^2} = \frac{1}{2}C_1, \quad 0<y<a$$

with

$$\mathcal{F}(y) = \Phi(y) - K F(y)$$

The solution of this integral equation takes the form

$$\mathcal{F}(y)\sqrt{1-\frac{y^2}{a^2}} = C - \frac{C_1}{2\pi}\left(2\frac{y^2}{a^2}-1\right)$$

where $C$ is an arbitrary constant; inasmuch as the left-hand side vanishes at $y=a$ it follows that $C = C_1/2\pi$ and so

$$\mathcal{F}(y) = \frac{C_1}{\pi}\sqrt{1-\frac{y^2}{a^2}}$$

or

$$\frac{d\Phi(y)}{dy} + K\Phi(y) = -\frac{C_1}{\pi a^2}\frac{y}{\sqrt{1-\frac{y^2}{a^2}}}.$$

Thus

$$\Phi(y) = \frac{C_1}{\pi a^2} e^{-Ky}\int_y^a \frac{\xi e^{K\xi}}{\sqrt{1-\frac{\xi^2}{a^2}}}d\xi$$

in keeping with the condition $\Phi(a)=0$.

The remaining constant, $C_1$, can be related to $\alpha$ and $Ka$ through the integro-differential equation

$$\frac{d^2}{dy^2}\int_0^a \Phi(y')\, G(0,y;0,y')\,dy' = iK\alpha\, e^{-Ky}, \quad 0<y<a$$

employing, among other results, the fact that

$$\int_0^a \Phi(y)\, e^{-Ky}\, dy = \frac{\hat{C} a^2}{K}\cdot\frac{\pi}{2} I_1(Ka), \quad \hat{C} = C_1/Ka^2$$

where $I_1(z)$ designates a cylinder function ($J_1(iz) = i I_1(z)$), one finds

$$\hat{C} = \frac{2\alpha}{a^2}\,\frac{1}{\pi I_1(Ka) + i K_1(Ka)},$$

in terms of another cylinder function $K_1(z)$.

Since

$$\frac{\partial}{\partial x'} G(x,y;x',y')\Big|_{x'=0} = \frac{1}{2\pi}\left(\frac{x}{x^2+(y-y')^2} + \frac{x}{x^2+(y+y')^2}\right) + \frac{K}{\pi}\int_0^\infty \frac{dk}{k-K}\sin kx\, e^{-k(y+y')}$$

$$= \frac{1}{2\pi}\left(\frac{x}{x^2+(y-y')^2} + \frac{x}{x^2+(y+y')^2}\right) + \frac{\pi}{2} e^{iKx} e^{-K(y+y')}\cdot\frac{K}{\pi}$$

$$- \frac{K}{\pi}\int_0^\infty \frac{dk}{k^2+K^2}\, e^{-kx}\left(K\cos k(y+y') + k\sin k(y+y')\right)$$

$$= \frac{K}{2} e^{iKx} e^{-K(y+y')} + \frac{1}{\pi}\int_0^\infty \frac{dk}{k^2+K^2}\, e^{-kx}\,(k\cos ky - K\sin ky)(k\cos ky' - K\sin ky')$$

$x > 0$

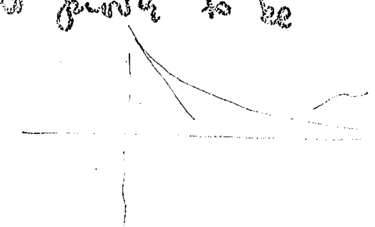
$$\sim \frac{K}{2} e^{iKx} e^{-K(y+y')}, \quad x \to +\infty$$

it follows that

$$\phi(x,y) \sim \phi^{inc}(x,y) - K e^{iKx-Ky}\int_0^a \Phi(y) e^{-Ky}\, dy$$

and the transmission coefficient proves to be

$$T = 1 - \frac{K}{\alpha}\int_0^a \Phi(y) e^{-Ky}\, dy$$



$K_1$, $I_1$ are Bessel fns of first & second kind.

$$= \frac{i K_1(Ka)}{\pi I_1(Ka) + i K_1(Ka)}$$

$T$ is exponentially small when $Ka \gg 1$ and approaches unity in the limit $Ka \to 0$.

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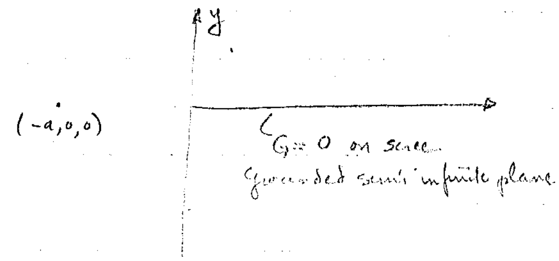
Given a semi-infinite screen edge in $z$ axis for $x>0$
Put a pt charge at $(-a,0,0)$




from electrostatic
$$\nabla^2_{xyz} G = -\delta(x+a)\,\delta(y)\,\delta(z)$$
cannot use conformal mapping since pt charge

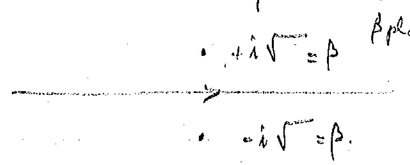
Since sol is symmetric wrt $z$ : Fourier transform wrt $z$ then $\bar{G} = \int_{-\infty}^{\infty} e^{i\zeta z} G\, dz$

thus $\left(\nabla^2_{xy} - \zeta^2\right)\bar{G} = -\delta(x+a)\delta(y)$ w/ $G$ & $\nabla G \to 0$ as $z\to\infty$
also $\bar{G} = 0$ for $x>0, y=0$

Suppose we look at $\left(\nabla^2_{xy} - \zeta^2\right) g = -\delta(x-x')\,\delta(y-y')$ this is a fundamental sol for a line source

$$g = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{i\alpha(x-x') + i\beta(y-y')}}{\alpha^2+\beta^2+\zeta^2}\, d\alpha\, d\beta$$
thus $\left(\nabla^2_{xy} - \zeta^2\right) g = \frac{-1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\alpha(x-x') + i\beta(y-y')}\, d\alpha\, d\beta$
$= -\delta(x-x')\,\delta(y-y')$

now factor denominator $\beta^2 + (\alpha^2+\zeta^2) = \left(\beta - i\sqrt{\alpha^2+\zeta^2}\right)\left(\beta + i\sqrt{\alpha^2+\zeta^2}\right)$




if $y-y'>0$ close in upper half
$y-y'<0$ " " lower half

thus by doing this $g = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{i\alpha(x-x') - \sqrt{\alpha^2+\zeta^2}\,|y-y'|}}{\sqrt{\alpha^2+\zeta^2}}\, d\alpha$ by integrating wrt $\beta$.

now this fn is $\propto K_0$ ie $g = \frac{1}{2\pi} K_0\left(|\zeta|\sqrt{(x-x')^2+(y-y')^2}\right) \sim \ln\left(|\zeta|\sqrt{\quad}\right) + \dots e^{(\quad)}$

Now to find: $\bar{G} = \underset{\text{original source}}{g(x,y,-a,0)} + \int_0^{\infty} g(x,y,x',0)\,\sigma(x')\,dx'$ sources distributed on screen

now $\left(\nabla^2_{xy} - \zeta^2\right)\bar{G} = -\delta(x+a)\,\delta(y) + \left(\nabla^2(x,y) - \zeta^2\right)\int_0^{\infty} g(x,y,x',0)\,\sigma(x')\,dx'$
$-\delta(x+x')\,\delta(y)\,\sigma(x')\,dx'$

What is $\sigma(x')$ but $x'>0$ $y>0$ the $\delta = 0$
this is single layer potential and $\sigma(x) = -\frac{\partial}{\partial y}\bar{G}\Big|_{y=0-}^{y=0+}$ $x>0$

$\longrightarrow$ Verify by using $g = \frac{1}{4\pi}\int \dots$ & $y'=0$ thus let $y\to 0^+$ & $y \to 0^-$

by using b.c.

$$0 = g(x,0,-a,0) + \int_0^\infty g(x,0,x',0)\,\sigma(x')\,dx' \qquad x>0$$

integral eq of second kind for $\sigma(x')$
this is a wiener-hopf eqn & extend interval

extend by defining $\quad h(x) = g(x,0,-a,0) + \int_{-\infty}^{\infty} g(x,0,x',0)\,\sigma(x')\,dx' \qquad -\infty<x<\infty$

$\sigma(x) = 0 \quad x<0$
$h(x) = 0 \quad x>0$

g doesn't need to be defined as zero for $x<0$ since it $\to 0$ as $|x|\to\infty$

define complex fourier transforms
$$\sigma(\mu) = \int_0^\infty e^{i\mu x}\sigma(x)\,dx \qquad h(\mu) = \int_{-\infty}^0 e^{i\mu x} h(x)\,dx$$

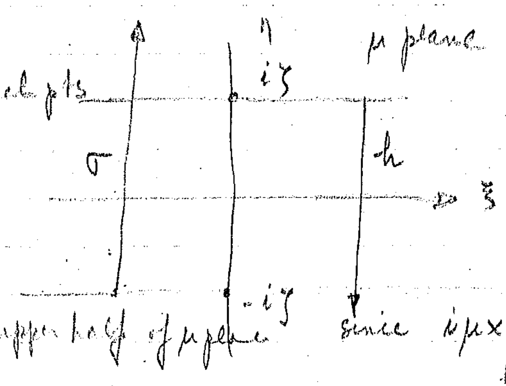
transform eqn is
$$h(\mu) = \frac{e^{-i\mu a}}{\sqrt{\mu^2+\zeta^2}} + \frac{1}{2}\frac{\sigma(\mu)}{\sqrt{\mu^2+\zeta^2}}$$

since $\quad g(x,0,-a,0) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{e^{i\alpha(x+a)}}{\sqrt{\alpha^2+\zeta^2}}\,d\alpha \Rightarrow g(\mu) = \int_{-\infty}^{\infty} e^{i\mu x} g(x,0,-a,0)\,dx = \frac{1}{4\pi}\iint_{-\infty}^{\infty}\frac{d\alpha\,dx}{\sqrt{\alpha^2+\zeta^2}}\,e^{i\mu x+i\alpha(x+a)}$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\alpha}{\sqrt{\alpha^2+\zeta^2}}\,\delta(\mu+\alpha)\,e^{i\alpha a}$$

and $\quad \iint_{-\infty}^{\infty} g(x-x',0-0)\,\sigma(x')\,dx'\,e^{i\mu x}\,dx = \frac{e^{i\mu(0)}}{2\sqrt{\mu^2+\zeta^2}} = \frac{1}{2\sqrt{\mu^2+\zeta^2}} \qquad = \frac{e^{-i\mu a}}{2\sqrt{\mu^2+\zeta^2}}$

$\sqrt{\mu^2+\zeta^2}$ has branch pts




$\sigma(x)$ is regular in upper half of μ plane since $i\mu x = i(\xi+i\eta)x = -\eta x + i\xi x$
hence since $x>0$ η must be $>0$

also the charges on the grid must be such as to balance that at $(-a,0,0)$
ie $\int \sigma(x')\,dx'\,dy'\,dz' = -1 \Rightarrow \sigma(x')$ must decrease exponentially hence
we can extend regular part of $\sigma(x)$ to $-i\zeta$
also since $g \to 0$ exponentially since $\propto$ to $K_0$ then we can
extend $h(x)$ to the $+i\zeta$ line

not ⊕ since $\sqrt{-i\mu a}>0$ in upper half plane

hence $\quad h_- = \frac{1}{2}\frac{e^{-i\mu a}}{\sqrt{\mu^2+\zeta^2}} + \frac{1}{2}\frac{\sigma_+(\mu)}{\sqrt{\mu^2+\zeta^2}} \Rightarrow \underbrace{2\sqrt{\mu-i\zeta}\,h_-(\mu)}_{\ominus f} = \underbrace{\frac{\sigma_+(\mu)}{\sqrt{\mu+i\zeta}}}_{\oplus f} + \frac{e^{-i\mu a}}{\sqrt{\mu+i\zeta}}$

12/3/80

Continuation of last lecture

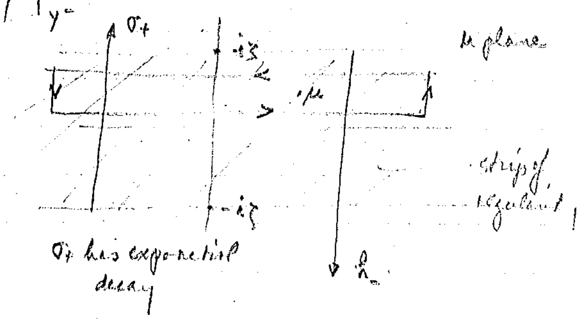
We had written an extended integral eqn

$$h(x) = g(x,0,-a,0) + \int_{-\infty}^{\infty} g(x,0,x',0)\,\sigma(x')\,dx' \qquad -\infty < x < \infty$$

represents potential in the continuation of the extended plane with $-\frac{\partial G}{\partial y}\Big|_{y^-}^{y^+} = \sigma$

Thus by fourier transforming

$$h(\mu) = \tfrac{1}{2}\frac{e^{-i\mu a}}{\sqrt{\mu^2+a^2}} + \tfrac{1}{2}\frac{\sigma(\mu)}{\sqrt{\mu^2+\zeta^2}}$$




Then $\dfrac{\sigma_+(\mu)}{\sqrt{\mu+i\zeta}} + \dfrac{e^{-i\mu a}}{\sqrt{\mu+i\zeta}} = 2\sqrt{\mu-i\zeta}\,h_-(\mu)$

(+) fn regular in upper half plane — $\ominus$ fn

$$\Phi(\mu,a,\zeta) = \frac{e^{-i\mu a}}{\sqrt{\mu+i\zeta}} = \Phi_+ - \Phi_-$$

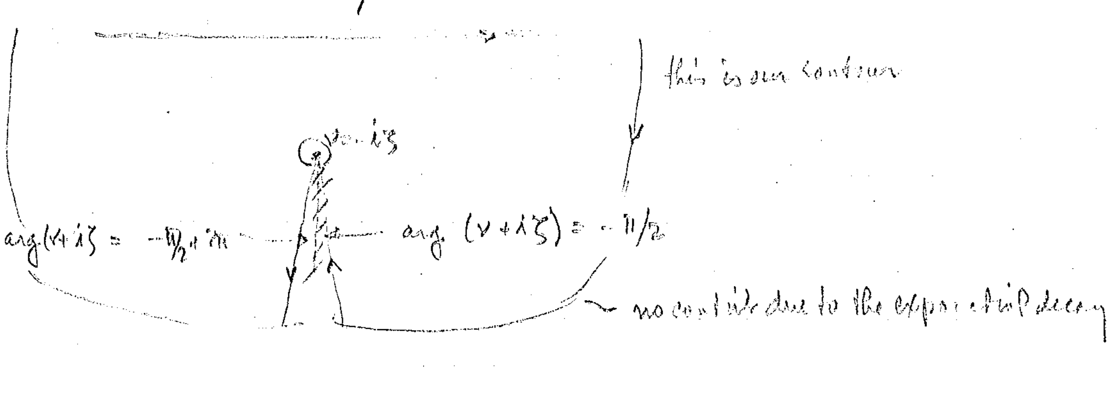
now $F(\mu) = \dfrac{1}{2\pi i}\displaystyle\oint \frac{F(z)\,dz}{z-\mu}$ by cauchy's theorem $= \dfrac{1}{2\pi i}\displaystyle\int \frac{F(z)\,dz}{z-\mu} + \frac{1}{2\pi i}\int \frac{F(z)\,dz}{z-\mu}$

$$\Phi_+ = \frac{1}{2\pi i}\int \frac{e^{-i\nu a}}{\sqrt{\nu+i\zeta}}\,\frac{d\nu}{\nu-\mu}$$

$\cdot\mu$

($\nu$) integration → to close contour $-i\nu a = -i\xi a + \eta a$ if $a>0 \Rightarrow \eta<0$ for no contrib from closing ∴ close in bottom contour

since closing below, only singularity contributing is the branch cut since $\nu-\mu$ lies outside contour.



thus $\int_{\longrightarrow} + \int_{\uparrow} + \int_{\downarrow} = 0 \Rightarrow \int_{\longrightarrow} = -\left(\int_{\uparrow} + \int_{\downarrow}\right)$

$$\Phi_+ = \frac{e^{-i\pi/4}}{\pi} \int_{\zeta}^{\infty} \frac{e^{-\tau a}}{\sqrt{\tau - \zeta}} \frac{d\tau}{\tau - i\mu}$$

let $\tau - \zeta = -i(\mu + i\zeta)\eta$ thus $\Phi_+ = \frac{1}{\pi} \frac{e^{-\zeta a}}{\sqrt{\mu + i\zeta}} \int_0^{\infty} \frac{e^{i(\mu + i\zeta)\eta a}}{\sqrt{\eta}\,(1+\eta)} d\eta$

since $\eta$, $a$ are positive $\Longrightarrow \Phi_+$ is regular in upper half plane since $\Phi_+ \sim e^{-\operatorname{Im}\mu} \to 0$

$$\therefore \frac{\sigma_+(\mu)}{\sqrt{\mu + i\zeta}} + \frac{1}{\pi} \frac{e^{-\zeta a}}{\sqrt{\mu + i\zeta}} \int_0^{\infty} \frac{e^{i(\mu + i\zeta)\eta a}}{\sqrt{\eta}\,(1+\eta)} d\eta = \text{entire fn.}$$

but $\frac{\sigma_+}{\sqrt{\mu}} \to 0$ as $\mu \to \infty$ $\therefore$ entire fn $= 0$

$$\therefore \sigma_+(\mu) = -\frac{1}{\pi} e^{-\zeta a} \int_0^{\infty} \frac{e^{i(\mu + i\zeta)\eta a}}{\sqrt{\eta}\,(1+\eta)} d\eta$$

$$\sigma(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} \sigma_+(\mu)\, d\mu = -\frac{1}{2\pi^2} e^{-\zeta a} \int_{-\infty}^{\infty}\int_0^{\infty} \frac{e^{-i\mu x} e^{i\mu\eta a} e^{-\zeta\eta a}}{\sqrt{\eta}\,(1+\eta)} d\eta\, d\mu$$

$$(\quad) \cdot \int_0^{\infty} \frac{d\eta}{\sqrt{\eta}\,(1+\eta)} \underbrace{\int_{-\infty}^{\infty} e^{-i\mu x} e^{i(\mu + i\zeta)\eta a} d\mu}_{2\pi \cdot \delta(\eta a - x)\, e^{-\zeta a \eta}}$$

$$(\quad) \cdot 2\pi \int_0^{\infty} \frac{d\eta}{\sqrt{\eta}\,(1+\eta)} \delta(\eta a - x)\, e^{-\zeta a \eta}$$

$$\left(-\frac{1}{2\pi^2} e^{-\zeta a}\right) \cdot \frac{2\pi}{a\sqrt{x/a}\,(1 + x/a)} e^{-\zeta x} \overset{\eta = x/a}{=} -\frac{1}{\pi} \frac{e^{-\zeta(a+x)}}{\sqrt{x/a}\,(x+a)} \qquad x > 0$$

if one forms

$$Q = -\int_{-\infty}^{\infty} dz \; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\zeta z} d\zeta \int_0^{\infty} \frac{dx}{\pi} \frac{e^{-|\zeta|(x+a)}}{\sqrt{x/a}\,(x+a)} = -1$$

since this must equal the charge at $(-a,0,0)$

<u>Consider</u> the homog eqn $f(x) - \lambda \int_0^{\infty} e^{-|x-x'|} f(x')\, dx' = 0$, $x > 0$

Consider sol. s. $f(0+)$ is bounded & $f(x) \sim O(e^{\mu x})$ $x \to \infty$ $\mu < 1$

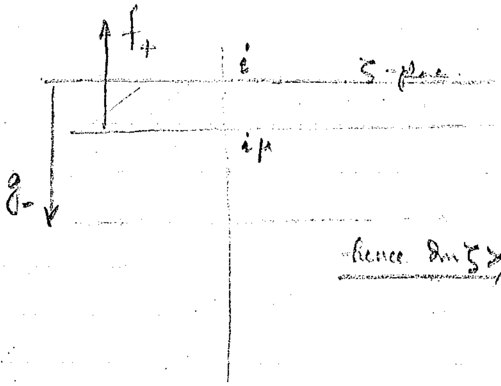
Now extended eqn leads to $f(x) - \lambda \int_{-\infty}^{\infty} e^{-|x-x'|} f(x')\, dx' = g(x)$

$g(x) = 0 \quad x > 0 \qquad f(x) = 0 \quad x < 0$

by transforming $\quad f(\zeta) - \frac{2\lambda}{\zeta^2+1} f(\zeta) = g(\zeta) \quad$ using $\int_{-\infty}^{\infty} e^{i\zeta x} e^{-|x|} dx = 2\int_0^{\infty} \cos \zeta x \, e^{-x} dx = \frac{2}{\zeta^2+1}$

$\therefore \left(1 - \frac{2\lambda}{\zeta^2+1}\right) f(\zeta) = g(\zeta) \quad$ looking in $\zeta$ plane

$\left(\frac{\zeta^2+1-2\lambda}{\zeta^2+1}\right) f(\zeta) = g(\zeta)$




now $f(\zeta) = \int_0^{\infty} e^{i\zeta x} f(x) dx$

$e^{-\mathrm{Im}\,\zeta x} O(e^{\mu x}) = e^{(-\mathrm{Im}\,\zeta + \mu)x}$

this must decrease for $x > 0 \quad \therefore -\mathrm{Im}\,\zeta + \mu < 0$

hence $\mathrm{Im}\,\zeta > \mu$ for $f$. now $g(x) \overset{\text{def}}{=} -\lambda \int_0^{\infty} e^{-|x-x'|} f(x') dx'$ from eqn.

$= -\lambda e^{x} \underbrace{\int_0^{\infty} e^{-x'} f(x') dx}_{\text{number}} \quad \downarrow$ as $x < 0$

$\left(\frac{\zeta^2+1-2\lambda}{\zeta^2+1}\right) f_+ = g_-(\zeta)$

$g$ is regular in lower half since $g = \int_{-\infty}^{0} e^{i\zeta x} (\sim e^{x}) dx$

$\therefore -\mathrm{Im}\,\zeta + 1 > 0$ since $x < 0 \quad \underline{\mathrm{Im}\,\zeta < 1 \text{ for } g}$

$\underset{(+)}{\left(\frac{\zeta^2+1-2\lambda}{\zeta+i}\right) f_+} = \underset{(-)}{g_-(\zeta)(\zeta - i)} \quad$ (dependent on $\lambda$) = entire f.

$g_-(\zeta)$ as $\zeta \to \infty$ is same as $g(x)$ as $x \to 0$ ie const.

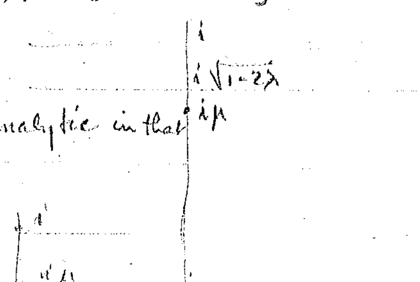
$\therefore g(\zeta) \sim O\left(\frac{1}{\zeta}\right)$ as $\zeta \to \infty$ from $\int e^{i\zeta x} \cdot \text{const}$

$\therefore (\zeta - i) g_-(\zeta) = \text{const.}$

$\therefore \quad f_+(\zeta) = C \frac{\zeta+i}{\zeta^2 - 2\lambda + 1}$ ; if $\lambda > \frac{1}{2}$ $\quad \zeta^2 - a^2$ (denom.) $\therefore \zeta = \pm a \quad a = \sqrt{2\lambda - 1}$

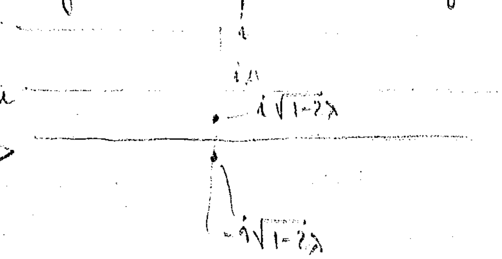
if $0 < \lambda < \frac{1}{2} \quad \zeta^2 + b^2 \quad \therefore \zeta = \pm ib \quad b = \sqrt{1-2\lambda}$

if $\sqrt{1-2\lambda} > \mu$ must begin strip after $\mu \Rightarrow$

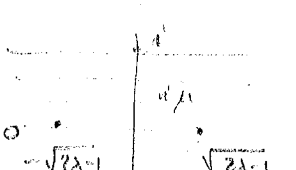



so that $f_+$ is analytic in that region

If $\sqrt{1-2\lambda} < \mu \Longrightarrow$

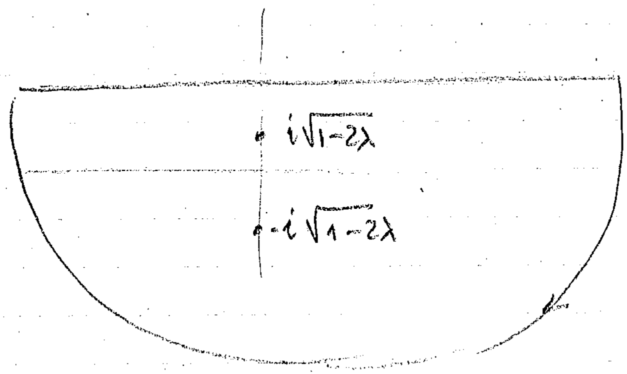



if $\sqrt{2\lambda - 1} > 0$ no problems




hence because of this we get a continuum of sol. dependent on $0 < \lambda < \frac{1}{2}$

if $\lambda < 0 \quad \sqrt{1-2\lambda} > 1$ and points lie outside strip now since $f_+(\zeta) = \dfrac{c\ \ \zeta+i}{\zeta^2+1-2\lambda}$ and $f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-i\zeta x}\, c\, \dfrac{\zeta+i}{\zeta^2+1-2\lambda}$




close of $\ni$ $-i\zeta = i\eta + x$
for vanishing exponential close of in negative region

then

$$\frac{1}{\zeta^2+1-2\lambda} = \left(\frac{1}{\zeta+i\sqrt{\ }} - \frac{1}{\zeta-i\sqrt{\ }}\right)\frac{1}{2i\sqrt{\ }}$$

$$\therefore\quad f(x) = \frac{1}{2\pi\cdot 2i\sqrt{1-2\lambda}}\int_{-\infty}^{\infty} e^{-i\zeta x}\, c\cdot(\zeta+i)\left(\frac{1}{\zeta+i\sqrt{\ }} - \frac{1}{\zeta-i\sqrt{\ }}\right)dx$$

$$f(x) = \frac{-2\pi i}{2\pi\cdot 2i\sqrt{\ }}\, ic\,(1+\sqrt{\ })\, e^{-\sqrt{\ }x} \quad + \frac{2\pi i}{2\pi\cdot 2i\sqrt{\ }}\, ic\,(1-\sqrt{\ })\, e^{\sqrt{\ }x}$$

if we go back to IE & differentiate we get $f'' + (2\lambda-1)f = 0$

or $\left(\dfrac{d^2}{dx^2} - 1\right)e^{-|x-x'|} = -2\delta(x-x')$ where $e^{-|x-x'|}$ is the green's fn.

$$f(x) = A\, e^{\sqrt{\ }x} + B e^{-\sqrt{\ }x} \qquad \sqrt{\ } = \sqrt{1-2\lambda}$$

from IE $f(0) = \lambda\displaystyle\int_0^{\infty} e^{-x} f(x)\,dx$ gives relation betw. A & B

& as shown above $f(x) = \underset{\text{const}}{\text{arbitrary}}\cdot\left(e^{\sqrt{\ }x}\,\dfrac{1+\sqrt{\ }}{\sqrt{\ }} + e^{-\sqrt{\ }x}\,\dfrac{\sqrt{\ }-1}{\sqrt{\ }}\right)$

ie only one unknown const.

Given a new eqn $\displaystyle\int_0^{\infty} K_0|x-x'|\, f(x')\,dx' = 1 \qquad x>0$

w/ $\quad K_0(\alpha) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} e^{i\zeta\alpha}\,\dfrac{d\zeta}{\sqrt{\zeta^2+1}}$

Extend interval

$$\int_{-\infty}^{\infty} K_0|x-x'|\ f(x')\,dx' = \begin{matrix}1 \\ g(x) & x<0\end{matrix}$$

Now apply transform.

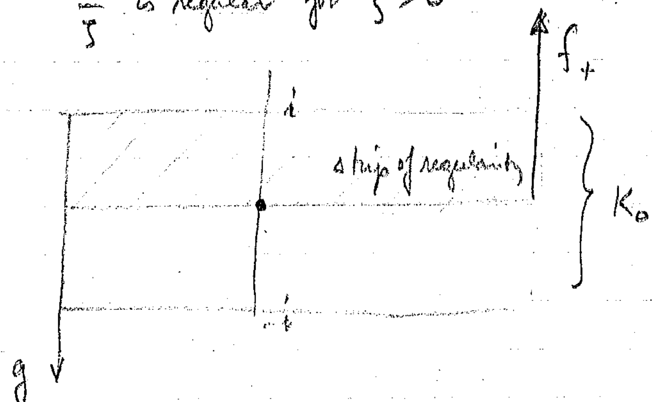
Using wiener hopf $\mathcal{F}\{K_0\}\, F = \dfrac{\pi}{\sqrt{\zeta^2+1}} f(\zeta) = \dfrac{i}{\zeta} + g(\zeta)$

NOT USED

NOT USED

transform of Kernel singular at $\zeta = \pm i$
since $g(\zeta) = \int_{-\infty}^{0}$ is regular in lower half plane but for $x<0$ $K_0$ is exponentially small
$\therefore$ $g$ is regular below $\zeta = i$

$\frac{i}{\zeta}$ is regular for $\zeta > 0$




$\therefore$ from our transform eqn.

$$\underset{\oplus}{\frac{\pi}{\sqrt{\zeta+i}} f_+(\zeta)} = \underbrace{\frac{i}{\zeta}\sqrt{\zeta-i}}_{\text{not regular below } \zeta = i} \underset{\ominus}{+ (\zeta-i)^{1/2} g_-(\zeta)}$$

use additive decomp formula $\frac{i}{\zeta}(\sqrt{\zeta-i}) = \frac{i}{\zeta}\underset{\ominus}{\left[\sqrt{\zeta-i} - \sqrt{-i}\,\right]} + \underset{\oplus}{\frac{i\sqrt{-i}}{\zeta}}$ regular in upper.

removes pole

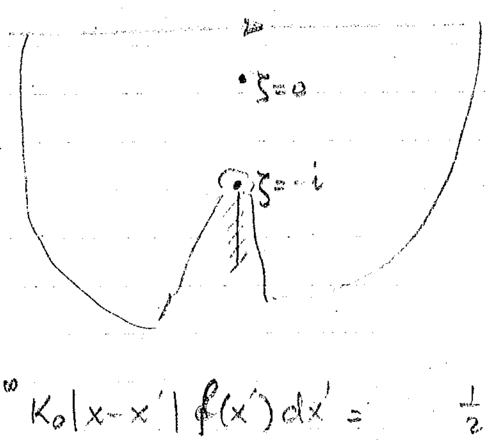
Thus $-\frac{i}{\zeta}\sqrt{-i} + \frac{\pi}{\sqrt{\zeta+i}} f_+ = \sqrt{\zeta-i}\; g_-(\zeta) + \frac{i}{\zeta}\left[\sqrt{\zeta-i} - \sqrt{-i}\right] =$ Entire fn.

entire fn. $= 0$ since $f_+$ is exponential small

$$\therefore \quad f_+ = \frac{i\sqrt{-i}}{\zeta}\frac{\sqrt{\zeta+i}}{\pi} \qquad \text{define } \sqrt{-i} = e^{-i\pi/4}$$

$$f = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\zeta x}\,\frac{1}{\pi\zeta}\,e^{-i\pi/4}\sqrt{\zeta+i}\;d\zeta$$

$f$ is $0$ for $x<0$ as required since $\Im\zeta > 0$




$f \neq 0$ for $x>0$ $\therefore$ close off in lower half since $\Im\zeta < 0$

$$\int_{-\infty}^{\infty} K_0|x-x'|\, f(x')\,dx' = \frac{1}{2}\int_{-\infty}^{\infty} e^{-i\zeta x}\,\frac{f_+(\zeta)}{\sqrt{\zeta^2+1}}\,d\zeta$$

substituting $K_0$ represent. & note that we have transform off

now replace for $f_+$ $\therefore$

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{-i\zeta x}\,\frac{i\sqrt{-i}\sqrt{\zeta+i}}{\zeta\pi\sqrt{\zeta^2+1}}\,d\zeta = \frac{1}{2}\int_{-\infty}^{\infty} e^{-i\zeta x}\,\frac{e^{i\pi/4}}{\pi\zeta\sqrt{\zeta-i}}\,d\zeta$$

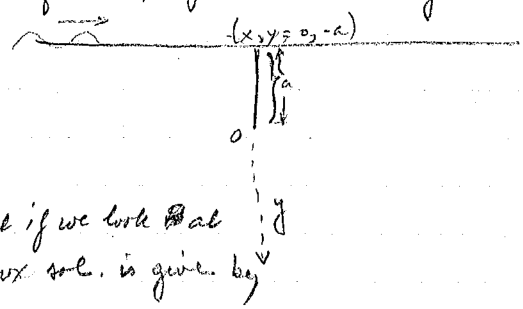
close contour in lower half and only residue is at $\zeta = 0$

12/8/80

Wiener Hopf cannot be used for a system of eqns.

Returning to a surface wave traveling over a barrier

We have 2 parameters to worry about: wavelength & barrier length
if $\lambda \ll a$ barrier appears to be $\infty$




so that if we look at
the approx sol. is given by

$$\tfrac{1}{2} e^{-K(y+a)} = \int_0^\infty f(y')\,\frac{i}{4}\,H_0^{(1)}(i\epsilon|y-y'|)\,dy', \quad y>0$$

Henkel fn of zeroth order of 1st kind $\sim e^{-y}$ as $y \to \infty$

Thus if a wave $e^{iKx-Ky}$ hits barrier it returns as a $Re^{-iKx-Ky}$ where R is reflection coeff.

$$R = 1 - 2i\int_0^\infty f(y)\,e^{-K(y+a)}\,dy$$

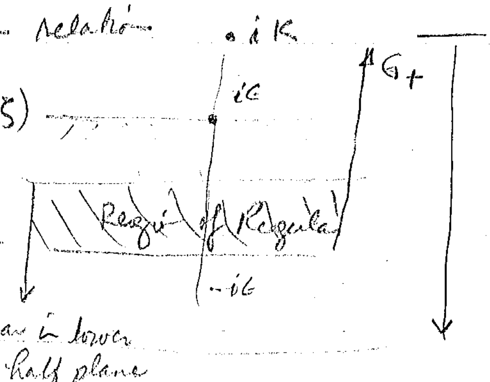
thus if we have $\infty$ barrier $R = 1$ since all of it will be reflected

$$\text{thus}\quad R = 1 - 2i\,e^{-Ka}\,F(-iK) \quad w/\ F(\zeta) = \int_0^\infty e^{-i\zeta x} f(x)\,dx$$

thus if we know $F(\zeta)$ we can plug into $F(-iK)$ to get R without inverting

from previous we had found the transform relation

$$\frac{-2e^{-Ka}}{\zeta - iK} = \frac{2F(\zeta)}{i\sqrt{\zeta^2+\epsilon^2}} + G(\zeta)$$




$$\therefore\quad \frac{1}{i}\frac{F(\zeta)}{\sqrt{\zeta - i\epsilon}} = e^{-Ka}\frac{\sqrt{\zeta+i\epsilon}}{\zeta - iK} + \frac{1}{2}\sqrt{\zeta+i\epsilon}\,G_+(\zeta)$$

(−) (+)

$$e^{-Ka}\frac{\left[\sqrt{\zeta+i\epsilon} - \sqrt{i(\epsilon+K)}\right]}{\zeta - iK} + e^{-Ka}\frac{\sqrt{i(K+\epsilon)}}{\zeta - i\epsilon}$$

(+) no pole at iK (−) regular below $\zeta = i\epsilon$

now equating (+) & (−) and noting that $f$ is integrable, that as $\zeta \to \infty$ each side of eqn tends to zero and $I = 0$

$\therefore$ take (+) side & solve for $F$

$$F(\zeta) = -i\, e^{-Ka} \frac{\sqrt{\zeta - i\epsilon}\,\sqrt{i(K+\epsilon)}}{\zeta - iK}$$ Now if we take

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta x}\, \frac{-i e^{-Ka}\sqrt{\zeta - i\epsilon}\,\sqrt{i(K+\epsilon)}}{\zeta - iK}\, d\zeta$$ this is approx soln for surface wave $\lambda \ll a$.

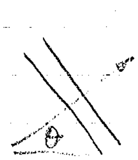
now $$F(-iK) = -i\, e^{-Ka}\frac{\sqrt{-i(K+\epsilon)}\,\sqrt{i(K+\epsilon)}}{-2iK} = -i e^{-Ka}\frac{K+\epsilon}{-2iK}$$
$$= +\tfrac{1}{2} e^{-Ka} + \dots \quad \text{as } \epsilon \to 0$$

this is valid when $Ka \gg 1$. thus $R = 1 - 2iF(-iK)\,e^{-Ka}$
$$= 1 - i e^{-2Ka}$$

The exact soln $$R = 1 - \frac{i K_1(Ka)}{\pi I_1(Ka) + i K_1(Ka)}$$ for large $Ka$ $K_1 \sim e^{-Ka}$, $I_1 \sim e^{Ka}$

$$R \cong 1 - \frac{i}{\pi}\cdot \text{Const}\; e^{-2Ka}$$

Another problem solved by W-H Sommerfeld half plane problem
Given a rigid plane for $x > 0$
Given sound wave incident on plane, $\phi$ is velocity potential
& compressible medium



$$\tilde{\phi}^{inc} = e^{-ik(x\cos\theta + y\sin\theta)}, e^{-i\omega t} \quad 0 \le \theta \le \pi/2 \qquad K = \frac{\omega}{c}$$
looks at $\tilde{\phi} = \phi e^{-i\omega t}$ ~~define~~ $\phi^{inc}$

incident wave

if plane is rigid $\phi^{reflect} = e^{ik(x\cos\theta - y\sin\theta)}$
and $\frac{\partial}{\partial y}(\phi^{ref} + \phi^{inc}) = 0$ at $y = 0$

We will solve by W.H. we must define $\phi^{sc}$ ($\phi$ scatter) $\Rightarrow (\nabla^2 + k^2)\phi^{sc} = 0$
and $\frac{\partial \phi^{sc}}{\partial y} = -ik\sin\theta\, e^{ikx\cos\theta}$, $x > 0$ $y = 0$

another condition $\Rightarrow$ Radiation (outgoing wave) now for stability of soln $k = \frac{\omega}{c} + ik_2$ where $k_2 \to 0$

for 2-D $\phi^{sc} \cong f(\theta)\frac{e^{ikr}}{\sqrt{r}}$

12/10/80

Meeting in 383W at 2PM ~~at~~ on 12/15/80 Have all homeworks

Look at HW problem $f(x) - \frac{\sin \pi\lambda}{2\pi}\int_0^\infty \frac{f(x')}{\cosh\frac{x-x'}{2}}\,dx' = 0 \quad , \quad x>0 \qquad 0<\lambda<\frac{1}{2}$

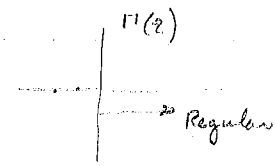
use W.H. Fourier transform extended eqn. find transform of $\operatorname{sech}\frac{x}{2}$ or $\operatorname{sech} ax$

$$F\left(1 - \frac{\sin \pi\lambda}{2\pi\cosh\zeta}\right) = G$$

find strip, define upper & lower fn also use $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$

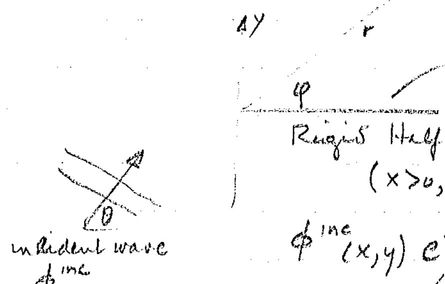
$\Gamma(z)$ has poles at $z = 0, -1, -2, -3, \quad \therefore \Gamma(z)$ is regular for $\operatorname{Re} z > 0$

$\Gamma(1-z)$ " " " " $< 0$




now break up eqn to define entire fn. & solve for $f(x)$.

Back to Somerfeld problem




$(x>0, y=0, -\infty<z<\infty)$

$\phi^{inc}(x,y)\,e^{-i\omega t} = e^{ik(x\cos\theta + y\sin\theta)} \quad k = \frac{\omega}{c}$

suppress

$c$ is $\sqrt{\frac{\lambda+2\mu}{\rho}}$

$\therefore (\nabla^2_{xy} + k^2)\,\phi^{inc} = 0 \qquad \phi^{TOT} = \phi^{inc} + \phi^{sc}$

also $(\nabla^2_{xy} + k^2)\,\phi^{TOT} = 0$ and $\frac{\partial\phi^{TOT}}{\partial y} = 0$ at $x>0 \; y=0 \; |z|<\infty$

thus $\left.\frac{\partial\phi^{sc}}{\partial y}\right|_{\substack{y=0\\x>0}} = -ik\sin\theta\, e^{ikx\cos\theta} = -\left.\frac{\partial\phi^{inc}}{\partial y}\right|_{\substack{y=0\\x>0}}$

and $\phi^{sc}$ is outgoing wave $\quad \therefore \phi^{sc} \sim A(r)\,e^{ik(x\cos\theta - y\sin\theta)} \sim \frac{e^{ikr}}{\sqrt{kr}} f(\varphi), \quad r\to\infty$ in 2 dim

what is $\phi^{sc}(x,0) \sim A e^{-k_2 x\cos\theta} \; x\to\infty$ if $k = k_1 + ik_2 \quad k_2>0$

what is $\phi^{sc}(x,0) \sim \bar{A} e^{k_2 x} \quad x\to-\infty$

$\phi^{TOT} = \phi^{inc} + \int_0^\infty \frac{\partial}{\partial y'} G(x,y,x',0)\,P(x')\,dx' \qquad G \; (\nabla^2 + k^2)G = -\delta(x-x')\delta(y-y')$

$G = \frac{i}{4} H_0^{(1)}\left(k\sqrt{(x-x')^2+(y-y')^2}\right)$

$H_0^{(1)}(kr) \sim \left(\ln r + \frac{e^{ikr}}{\sqrt{r}}\right) \therefore \frac{i}{4\pi}\int_{-\infty}^\infty \frac{e^{i\zeta(x-x') + i\sqrt{k^2-\zeta^2}\,|y-y'|}}{\sqrt{k^2-\zeta^2}}\,d\zeta = G(x,y,x',y')$

note no difference betwee $x, y$ could have written $\int e^{i\sqrt{\;}\,|x-x'| + i\zeta|y-y'|}$

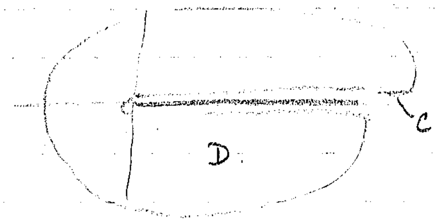
but for convenience we do it the way we have it so that integration is simple

what is $P(x') = \phi^{sc}(x,y)\Big|_{y=0^-}^{y=0^+}$ , this sol is a double layer potential sol.

look at $\int(\phi\nabla^2\psi - \psi\nabla^2\phi)\,dA = \int_D \left(\phi(\nabla^2+k^2)\psi - \psi(\nabla^2+k^2)\phi\right)dA = \int_C \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds$

$\phi^{TOT}$ $\quad G \quad$ $\phi^{TOT}$ on C $\quad$ 0 on C

but $\phi^{inc}$ $\quad$ & $\phi\to 0$ as $r\to\infty$



now using $\frac{\partial\phi^{TOT}}{\partial y} = 0 \Rightarrow 0 = ik\sin\theta\, e^{ikx\cos\theta} + \lim_{y\to 0}\frac{\partial}{\partial y}\int_0^\infty \frac{\partial}{\partial y'}G(x,y,x',0)P(x')dx' \quad x>0$

now $G$ is dependent on $y-y'$ $\quad\therefore\ \frac{\partial G}{\partial y} = -\frac{\partial G}{\partial y'}$

$\quad\text{''}\quad + \lim_{y\to 0}\frac{\partial}{\partial y^2}\int_0^\infty G(x,y,x',0)P(x')dx'$

$\quad\text{''}\quad \left(\frac{\partial^2}{\partial x^2}+k^2\right)\int_0^\infty G(x,0,x',0)\,P(x')\,dx'$

since for $y\neq 0$ $\quad \delta(y-y')=0 \quad\therefore\ (\nabla^2+k^2)G = 0 \qquad \therefore \lim_{y\to 0}\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2}+k^2$

now $G \sim \ln r$ $\qquad$ hence we've defined an integro-differential eqn.

12/12/80

Continuing Sommerfeld problem.

Now using Weiner Hopf we extend

define $\begin{matrix} 0 & x>0 \\ 0 & x<0 \end{matrix} \begin{matrix} = \\ \neq \end{matrix} V(x) = \left\{\begin{matrix} ik\sin e^{ikx\cos\theta} & x>0 \\ 0 & x<0\end{matrix}\right\} + \left(\frac{\partial^2}{\partial x^2}+k^2\right)\int_{-\infty}^{\infty} G(x-x')\ P(x')\,dx'$

$P(x) = 0 \quad x<0$

V is the normal y deriv of the scattered potential

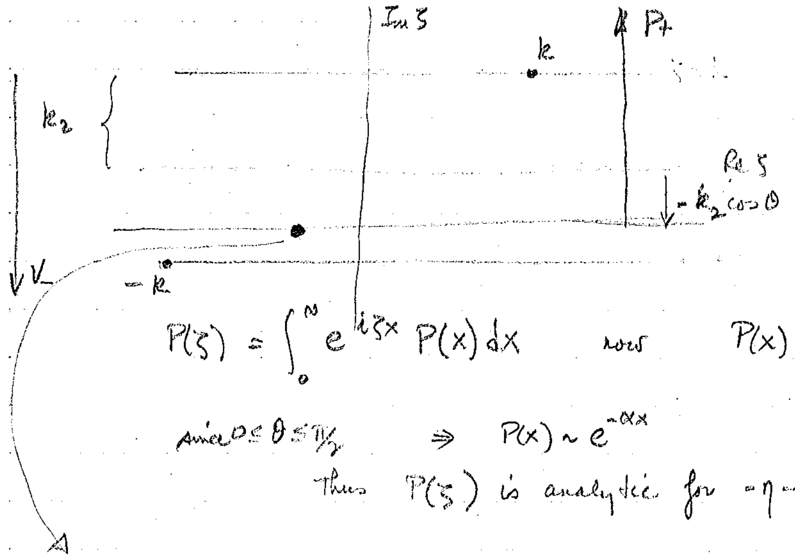
$V(x) - \frac{\partial\phi^{inc}}{\partial y}(x,0)$ - normal veloc. when $x<0$

we will fourier transform if $k = \frac{\omega}{c} = k_R + ik_I \quad 0<k_I \ll 1$

$\Rightarrow \quad V(\zeta) = \frac{-k\sin\theta}{\zeta + k\cos\theta} + \frac{i}{2}\frac{(k^2-\zeta^2)}{\sqrt{k^2-\zeta^2}}P(\zeta) \quad ; \quad k^2-\zeta^2 = \mathcal{F}\left\{\frac{\partial^2}{\partial x^2}+k^2\right\}$

since $\int_{-\infty}^{\infty} e^{i\zeta x}G(x)\,dx = \frac{i}{4\pi}\int_{-\infty}^{\infty} e^{i\zeta x}dx\int_{-\infty}^{\infty}\frac{e^{i\bar\zeta(x)}}{\sqrt{k^2-\bar\zeta^2}}\,d\bar\zeta = \frac{i}{4\pi}\cdot 2\pi\int_{-\infty}^{\infty}\frac{\delta(\zeta+\bar\zeta)}{\sqrt{k^2-\bar\zeta^2}}\,d\bar\zeta$

$$= \frac{i}{2} \frac{1}{\sqrt{k^2-\zeta^2}} \qquad \therefore \qquad V(\zeta) = -\frac{k\sin\theta}{\zeta + k\cos\theta} + \frac{i}{2}\sqrt{k^2-\zeta^2}\,P(\zeta)$$




$$\zeta = \pm k = \pm(k_R + i k_2)$$

$$P(\zeta) = \int_0^\infty e^{i\zeta x} P(x)\,dx \qquad \text{now} \qquad P(x) = \phi^{sc}\Big|_{y<0^-}^{y=0^+} \; ; \quad \text{as } x \to +\infty$$

since $0 \le \theta \le \pi/2 \quad \Rightarrow \quad P(x) \sim e^{-\alpha x} \qquad \sim e^{ikx\cos\theta} \sim e^{-k_2\cos\theta\, x}$

Thus $P(\zeta)$ is analytic for $-\eta - k_2\cos\theta > 0 \quad \therefore \quad \eta \geq k_2\cos\theta$

now $\zeta = -k\cos\theta$ is a pole of the incident wave transform.

$$V(x) \sim \frac{e^{ikr}}{\sqrt{r}} \quad \text{as } r \to \infty \qquad \frac{\partial V}{\partial y} \cong \frac{ik}{\sqrt{r}} e^{ikr}\frac{\partial r}{\partial y} \quad \text{as } r \to \infty$$

$$\cong e^{ikr}\frac{iky}{r^{3/2}} \quad \sim y/r$$

$$\cong e^{i(k_R + ik_I)x} = e^{-k_2 x} \quad \text{at } y=0 \quad r = \sqrt{|x|}$$

$$\therefore \quad V(\zeta) = \int_{-\infty}^0 e^{i\zeta x} V(x)\,dx \cong e^{(-\eta - k_2)x} \quad \therefore \text{ for } x \to -\infty$$

$$-\eta - k_2 > 0 \qquad \eta < -k_2$$

$$\underbrace{\frac{V_-(\zeta)}{\sqrt{k-\zeta}}}_{\ominus} = \frac{-k\sin\theta}{\sqrt{k-\zeta}\,(\zeta + k\cos\theta)} + \underbrace{\frac{i}{2}\sqrt{k+\zeta}\;P_+(\zeta)}_{\oplus}$$

$$= \frac{-k\sin\theta}{\zeta + k\cos\theta}\underbrace{\left[\frac{1}{\sqrt{k-\zeta}} - \frac{1}{\sqrt{k(1+\cos\theta)}}\right]}_{\ominus} - \underbrace{\frac{k\sin\theta}{\zeta + k\cos\theta}\,\frac{1}{\sqrt{k(1+\cos\theta)}}}_{\oplus}$$

removes singularity at $\zeta = -k\cos\theta$

now as $|\zeta| \to \infty$ all the $\ominus$ parts the to zero since $V_-(\zeta) \to 0$
and entire fn $= 0$

$P(x) \doteq Cx^{1/2}, \quad x \to 0+$

$P(\zeta) = O(\zeta^{-3/2}) \qquad |\zeta| \to \infty \qquad \text{Im}\,\zeta > 0$

$V(x) \doteq D(-x)^{-1/2} \qquad x \to 0-$

$V(\zeta) = O(\zeta^{-1/2}) \qquad \text{Im}\,\zeta < 0$

$$\therefore \quad P_+(\zeta) = \frac{k \sin\theta}{\zeta + k\cos\theta} \; \frac{i}{\sqrt{k(1+\cos\theta)}\,2\sqrt{k+\zeta}}$$

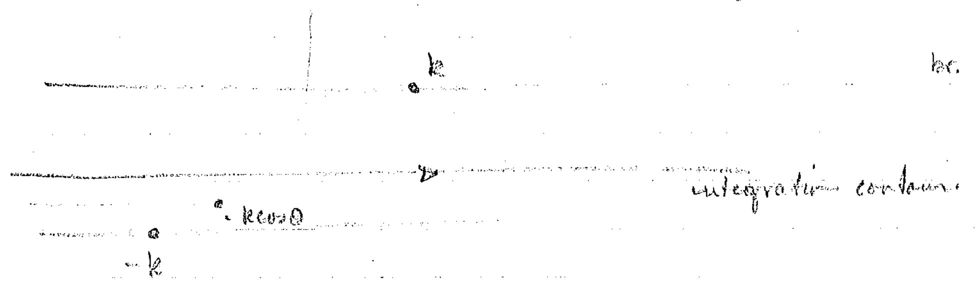
$$\therefore \quad \phi^{sc}(x,y) = \frac{\operatorname{sgn} y}{4\pi} \int_{-\infty}^{\infty} e^{-i\zeta x}\, e^{i\sqrt{k^2-\zeta^2}\,|y|}\, P_+(\zeta)\, d\zeta$$

since $\phi^{sc} = \int_0^\infty \frac{\partial}{\partial y'} G(x,y,x',0)\, P(x')\, dx'$

then $\phi^{sc} = -\frac{\partial}{\partial y} \int_0^\infty G \quad P(x')\, dx'$ ; put in $G = \frac{i}{4\pi} \int_{-\infty}^{\infty} \frac{e^{i\zeta x} \cdots}{\sqrt{k^2-\zeta^2}}$

Thus put in $P_+$ to get $\phi^{sc} = \frac{1}{2\pi i} \frac{\sqrt{k}\,\operatorname{sgn} y}{(1+\cos\theta)^{1/2}} \sin\theta \int_{-\infty}^{\infty} \frac{e^{-i\zeta x + i\sqrt{k^2-\zeta^2}\,|y|}}{\sqrt{k+\zeta}\;(\zeta + k\cos\theta)}\, d\zeta$

br. point    pole




can be solved by means of steepest descent.      Neil - 324 0563

Cesar Levy
MA 224
Fall 80 Prof Lewine

Find the solution to $f(x) - \frac{\sin \pi\lambda}{2\pi} \int_0^\infty f(x') \operatorname{sech}\left(\frac{x-x'}{2}\right) dx' = 0 \qquad 0 < \lambda < 1/2$

To use the Weiner-Hopf technique we extend the integral by defining $\tilde{f}(x) = \begin{cases} f(x) & x>0 \\ 0 & x<0 \end{cases}$

and $\tilde{f}(x) - \frac{\sin \pi\lambda}{2\pi} \int_{-\infty}^\infty \tilde{f}(x') \operatorname{sech}\left(\frac{x-x'}{2}\right) dx' = g(x)$ where $g(x) = \begin{cases} 0 & x>0 \\ \neq 0 & x<0 \end{cases}$

Now applying the fourier transform to this extended equation we get

$$F(\zeta) - \sin\pi\lambda \cosh\pi\zeta\, F(\zeta) = G(\zeta) \qquad \text{where} \quad F(\zeta) = \int_{-\infty}^\infty e^{i\zeta x} \tilde{f}(x)dx = \int_0^\infty e^{i\zeta x} f(x)dx$$

and $K(\zeta) = \int_{-\infty}^\infty e^{i\zeta x} \operatorname{sech}\frac{x}{2}\, dx = 2\pi \operatorname{sech}\pi\zeta$. We do this since the integral becomes a convolution integral. Also

$$\int_{-\infty}^\infty e^{i\zeta x} g(x)dx = \int_{-\infty}^0 e^{i\zeta x} g(x)\, dx = G(\zeta)$$

Now $K(\zeta)$ is defined in the strip $|\operatorname{Im}\zeta| < 1/2$ at $|\operatorname{Im}\zeta| = 1/2$ $\operatorname{sech}\pi\zeta$ has poles at $\zeta = \frac{2k+1}{2}$

for small $\zeta$ (large $x$) we need to know the character of $F(\zeta)$ and $G(\zeta)$

First for large $x>0$ $\quad f(x) = \frac{\sin\pi\lambda}{2\pi} \int_0^\infty \frac{2f(x')\,dx'}{e^{\frac{(x-x')}{2}} + e^{-\frac{(x-x')}{2}}} \leq \frac{\sin\pi\lambda}{2\pi} e^{-\frac{x}{2}} \int_0^\infty 2f(x') e^{x'/2} dx'$

or $f(x)$ is $O(e^{-x/2})$ $\quad \therefore \quad F(\zeta) \sim c_1 \int_0^\infty e^{i\zeta x} e^{-x/2} dx \sim c_1 \int_0^\infty e^{i(\xi+i\eta)x} e^{-x/2} dx$

$\therefore$ for convergence $c_1 \int_0^\infty e^{i\xi x} e^{x(-\eta - 1/2)} dx < \infty$ and $-\eta - 1/2 < 0$ for $x>0$

or $\eta > -1/2$

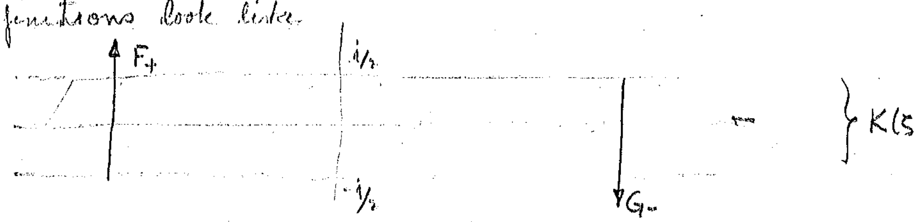
for $g(x) = \frac{\sin\pi\lambda}{2\pi} \int_0^\infty \frac{2f(x')\,dx'}{e^{\frac{x-x'}{2}} + e^{-\frac{(x-x')}{2}}} \leq \frac{\sin\pi\lambda}{2\pi} e^{x/2} \int_0^\infty 2f(x') e^{-x'/2} dx'$ for large $x<0$

or $g(x)$ is $O(e^{x/2})$ $\quad \therefore \quad G(\zeta) = c_2 \int_{-\infty}^0 e^{i\zeta x} e^{x/2} dx \cong c_2 \int_{-\infty}^0 e^{i(\xi+i\eta)x} e^{x/2} dx$

$\therefore$ for convergence

$-\eta + 1/2 > 0$ or $\eta < 1/2$

Hence our definitions look like




Our transformed equation then is $F_+ \left(\frac{\cosh\pi\zeta - \sin\pi\lambda}{\cosh\pi\zeta}\right) = G_-$

Now for large $\zeta$ (small $x$) $\quad \frac{1}{\cosh\left(\frac{x-x'}{2}\right)} \leq \frac{1}{e^{x/2}(1+x/2)} \leq e^{-x/2}\left(1 - \frac{x}{2}\right)$

thus $f(x) \cong c(1 - x/2)$ and $F_+ \cong c\int e^{i\zeta x}(1 - x/2)dx \sim O\left(\frac{c}{i\zeta}\right)$

similarly $G_- \cong O\left(\frac{c}{i\zeta}\right)$

Now let's look at $\frac{\cosh\pi\zeta - \sin\pi\lambda}{\cosh\pi\zeta}$

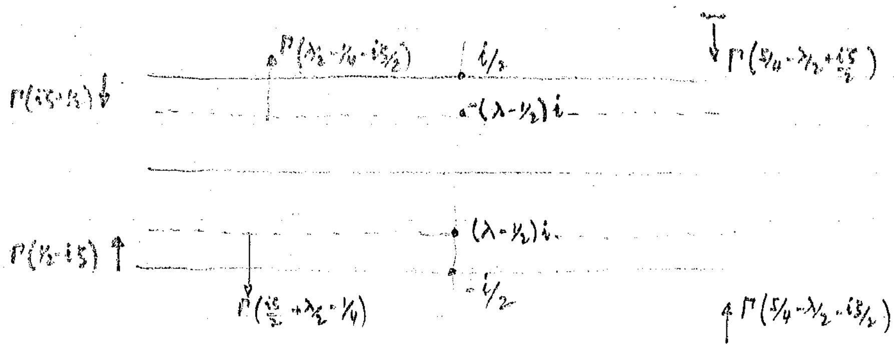
Using $\cosh \pi\zeta = \cos i\pi\zeta$ ; $\sin\pi\lambda = \cos(\pi\lambda - \pi/2)$ ; and $\cos\alpha - \cos\beta = 2\sin(\alpha+\beta)\sin(\beta$
and $\sin \pi\zeta = \dfrac{\pi}{\Gamma(\zeta)\Gamma(1-\zeta)}$ then

$$\cosh\pi\zeta - \sin\pi\lambda = \frac{2\pi^2}{\Gamma\left(\frac{i\zeta}{2}+\frac{\lambda-1/2}{2}\right)\Gamma\left(1-\frac{i\zeta+\lambda-1/2}{2}\right)\Gamma\left(\frac{\lambda-1/2}{2}-\frac{i\zeta}{2}\right)\Gamma\left(1-\frac{\lambda-1/2}{2}+\frac{i\zeta}{2}\right)}$$

$$\cosh\pi\zeta = \cos i\pi\zeta = \sin(\pi i\zeta + \pi/2) = \frac{\pi}{\Gamma(1/2+i\zeta)\Gamma(1/2-i\zeta)}$$

$$\therefore \frac{\cosh\pi\zeta - \sin\pi\lambda}{\cosh\pi\zeta} = \frac{2\pi\,\Gamma(i\zeta+1/2)\,\Gamma(1/2-i\zeta)}{\Gamma\left(\frac{i\zeta}{2}+\lambda/2-1/4\right)\Gamma\left(5/4-\lambda/2-i\zeta/2\right)\Gamma\left(\lambda/2-1/4-i\zeta/2\right)\Gamma\left(5/4-\lambda/2+i\zeta/2\right)}$$

Roots $\Gamma(i\zeta+1/2)$ has poles at $(n+1/2)i$ $n=0,1,2,\ldots$ and is regular below $\zeta = 1/2 i$
$\Gamma(1/2-i\zeta)$ " " at $-(n+1/2)i$ " " " " above $\zeta = -1/2 i$
$\Gamma\left(\frac{i\zeta}{2}+\lambda/2-1/4\right)$ " " at $(2n+\lambda-1/2)i$ " and is regular below $\zeta = (\lambda-1/2)i$
$\Gamma\left(5/4-\lambda/2-i\zeta/2\right)$ " " at $-(2n+5/2-\lambda)i$ " " " " above $\zeta = (\lambda-5/2)i$
$\Gamma\left(\lambda/2-1/4-i\zeta/2\right)$ " " at $-(2n+\lambda-1/2)i$ " and is regular above $\zeta = -(\lambda-1/2)i$
$\Gamma\left(5/4-\lambda/2+i\zeta/2\right)$ " " at $(2n+5/2-\lambda)i$ " " " " below $\zeta = -(\lambda-5/2)i$




Thus the problem is with $\Gamma(\lambda/2-1/4-i\zeta/2)$ or $\Gamma\left(\frac{i\zeta}{2}+\lambda/2-1/4\right)$ since they don't have overlapping regions. Let us use $z\Gamma(z) = \Gamma(1+z)$ & let $\Gamma(z) = \Gamma\left(\frac{i\zeta}{2}+\lambda/2-1/4\right)$

$$\therefore \quad \Gamma\left(\frac{i\zeta}{2}+\frac{\lambda}{2}+3/4\right)\Big/\left(\frac{i\zeta}{2}+\frac{\lambda}{2}-1/4\right) = \Gamma\left(\frac{i\zeta}{2}+\lambda/2-1/4\right)$$

$\Gamma\left(\frac{i\zeta}{2}+\frac{\lambda}{2}+3/4\right)$ has poles at $(2n+3/2+\lambda)i$ and is regular below $(\lambda+3/2)i$
and $\frac{i\zeta}{2}+\frac{\lambda}{2}-1/4$ is a pole at $-(1/2-\lambda)i$. Thus the strip of regularity for all the $\Gamma$ fns $F_+$ and $G_-$ is the strip $\mathrm{Im}\,\zeta \in (1/2-\lambda,\ 1/2)$

$$\text{thus}\quad \frac{\cosh\pi\zeta - \sin\pi\lambda}{\cosh\pi\zeta} = \frac{H_+}{H_-} = \frac{\pi\,\Gamma(1/2-i\zeta)(i\zeta+\lambda-1/2)\big/\Gamma(5/4-\lambda/2-i\zeta/2)\Gamma(\lambda/2-1/4-i\zeta/2)}{\Gamma(i\zeta/2+\lambda/2+3/4)\,\Gamma(5/4-\lambda/2+i\zeta/2)\big/\Gamma(i\zeta+1/2)}$$

Thus

$$F_+H_+ = G_-H_- \quad ; \quad F_+H_+ = F_+\,\pi\,\frac{\Gamma(1/2-i\zeta)(i\zeta+\lambda-1/2)}{\Gamma(5/4-\lambda/2-i\zeta/2)\Gamma(\lambda/2-1/4-i\zeta/2)}$$ is an entire function of $\zeta$

for large $\zeta$ $\quad \Gamma(a-i\zeta) \sim \sqrt{2\pi(a-i\zeta)}\,(a-i\zeta)^{(a-i\zeta)}e^{-(a-i\zeta)}$ $\quad$ Thus $H_+ \sim 2^{-i\zeta}(i\zeta)\cdot\text{const.}$

and $\ln F_+H_+ \sim -i\zeta\ln 2 + \text{const}$

for the entire fn to be constant we must multiply $F_+H_+$ and $G_-H_-$ by $e^{P(\zeta)}$
Thus $\ln(F_+H_+e^{P(\zeta)}) \sim P(\zeta) - i\zeta\ln 2 + \ln\text{const}$; so take $P(\zeta) = i\zeta\ln 2$.
hence by multiplying $F_+H_+$ by $2^{i\zeta}$

$$F_+H_+\cdot 2^{i\zeta} = \text{Const.} \quad \therefore \quad F_+ = \frac{C}{H_+2^{i\zeta}} = \frac{C\cosh\pi\zeta\cdot 2^{-i\zeta}}{(\cosh\pi\zeta - \sin\pi\lambda)H_-}$$

and $f(x) = \frac{C}{2\pi}\int_{\infty i\tau}^{\infty+i\tau}\frac{e^{-i\zeta x}\cosh\pi\zeta\, 2^{-i\zeta}}{(\cosh\pi\zeta - \sin\pi\lambda)H_-}d\zeta$ note that $H_-$ is regular in the lower half plane and for convergence we must close our contour in the negative imaginary $\zeta$ region.

where $-(\lambda-\frac{1}{2}) < \tau < \frac{1}{2}$. The zeros of the denominator are at $\cosh\pi\zeta - \sin\pi\lambda = 0$
Thus $\cosh\pi\zeta - \sin\pi\lambda = 2\sin\pi\left(\frac{i\zeta+\lambda-\frac{1}{2}}{2}\right)\sin\pi\left(\frac{\lambda-\frac{1}{2}-i\zeta}{2}\right)$
and the zeros are when $\frac{i\zeta+\lambda-\frac{1}{2}}{2} = \pm n$ or $\frac{-i\zeta+\lambda-\frac{1}{2}}{2} = \pm n$ $\quad n=0,1,\dots$

$\therefore \pm 2n - (\lambda-\frac{1}{2}) = i\zeta$ or $-i\zeta = (\lambda-\frac{1}{2}) \mp 2n$ and $-i\zeta = \pm 2n - (\lambda-\frac{1}{2})$

for those poles in the lower half plane take the $-$ sign before the $2n$ hence $-i\zeta = -2n \pm (\lambda-\frac{1}{2})$

Thus $H_-\left(\frac{i\zeta}{2} = n + \frac{(\lambda-\frac{1}{2})}{2}\right) = \Gamma(n+\lambda+\frac{1}{2})\Gamma(1+n)/\Gamma(2n+\lambda)$

$H_-\left(\frac{i\zeta}{2} = n - \frac{(\lambda-\frac{1}{2})}{2}\right) = \Gamma(n+1)\Gamma(n-\lambda+\frac{3}{2})/\Gamma(2n-\lambda+1)$

$\cosh\pi\zeta = \cos\pi i\zeta = \cos\pi(2n+(\lambda-\frac{1}{2})) = \cos\pi(\lambda-\frac{1}{2})$ $\qquad i\zeta = 2n+(\lambda-\frac{1}{2})$

$\cosh\pi\zeta = \cos\pi i\zeta = \cos\pi(2n-(\lambda-\frac{1}{2})) = \cos\pi(\lambda-\frac{1}{2})$ $\qquad i\zeta = 2n-(\lambda-\frac{1}{2})$

for $i\zeta = 2n-(\lambda-\frac{1}{2})$ $\quad i\zeta+(\lambda-\frac{1}{2}) = 2n$ ; $\quad i\zeta-(\lambda-\frac{1}{2}) = 2n$ for $i\zeta = 2n+(\lambda-\frac{1}{2})$
for $-i\zeta = -2n+(\lambda-\frac{1}{2})$ $\qquad$ for $-i\zeta = -2n-(\lambda-\frac{1}{2})$

$$\lim_{i\zeta\to 2n+\lambda-\frac{1}{2}}\frac{\cosh\pi\zeta - \sin\pi\lambda}{i\zeta-2n-(\lambda-\frac{1}{2})} = -2\sin\pi\left(\frac{i\zeta+\lambda-\frac{1}{2}}{2}\right)\frac{\sin\frac{\pi}{2}(i\zeta-(\lambda-\frac{1}{2})-2n)(\cos\pi n)\frac{\pi}{2}}{\frac{\pi}{2}(i\zeta-2n-(\lambda-\frac{1}{2}))}$$

$$= -\pi\sin\pi(n+(\lambda-\tfrac{1}{2}))\cos\pi n = -\pi\sin\pi(\lambda-\tfrac{1}{2}) \quad \text{for } i\zeta = 2n+\lambda-\tfrac{1}{2}$$

$$\lim_{i\zeta\to 2n-(\lambda-\frac{1}{2})}\frac{\cos\pi\zeta - \sin\pi\lambda}{i\zeta-2n+(\lambda-\frac{1}{2})} = \frac{2\sin\frac{\pi}{2}(i\zeta+(\lambda-\frac{1}{2})-2n)}{\frac{\pi}{2}(i\zeta-2n+(\lambda-\frac{1}{2}))}\sin\frac{\pi}{2}(\lambda-\tfrac{1}{2}-i\zeta)\cos\pi n\cdot\frac{\pi}{2}$$

$$= \pi\cos\pi n\sin\pi(\lambda-\tfrac{1}{2}-n) = \pi\sin(\lambda-\tfrac{1}{2})$$

Now since we close it in the lower half plane we must take $-2\pi i \times$ Residues at the poles

$$\therefore f(x) = \frac{C}{2\pi}(-2\pi i)\sum_{n=0}^{\infty}\left[\frac{1}{i}\frac{(2e^x)^{-(2n+\lambda-\frac{1}{2})}\cos\pi(\lambda-\frac{1}{2})\Gamma(2n+\lambda)}{-\pi\sin\pi(\lambda-\frac{1}{2})\Gamma(n+\lambda+\frac{1}{2})n!} + \frac{1}{i}\frac{(2e^x)^{-(2n-(\lambda-\frac{1}{2}))}\cos\pi(\lambda-\frac{1}{2})\Gamma(2n-\lambda+1)}{\pi\sin(\lambda-\frac{1}{2})\Gamma(n-\lambda+\frac{3}{2})n!}\right]$$

$$= \frac{C\cot\pi(\lambda-\frac{1}{2})}{\pi}\sum_{n=0}^{\infty}\left(-\frac{(2e^x)^{-(2n-(\lambda-\frac{1}{2}))}\Gamma(2n-\lambda+1)}{\Gamma(n-\lambda+\frac{3}{2})n!} + \frac{(2e^x)^{-(2n+(\lambda-\frac{1}{2}))}\Gamma(2n+\lambda)}{\Gamma(n+\lambda+\frac{1}{2})n!}\right)$$

$$f(x) - \frac{\sin \pi\lambda}{2\pi} \int_0^\infty \frac{f(x')}{\cosh\left(\frac{x-x'}{2}\right)} dx' = 0 \qquad \text{for } x>0 \qquad 0<\lambda<\tfrac{1}{2}$$

extend the integral by defining

$$\tilde{f}(x) - \frac{\sin\pi\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{f}(x')}{\cosh\left(\frac{x-x'}{2}\right)} dx' = g(x) \qquad \text{where} \quad \tilde{f}(x) = \begin{cases} f(x) & x>0 \\ 0 & x<0 \end{cases}$$

$$\hat{f}(x) - \frac{\sin\pi\lambda}{2\pi} \int_{-\infty}^{\infty} K(x-x')\,\hat{f}(x')\,dx' = g(x) \qquad g(x) = \begin{cases} 0 & x>0 \\ \neq 0 & x<0 \end{cases}$$

Now $K(x-x') = \operatorname{sech}\left(\frac{x-x'}{2}\right)$ and $K(x) = K(-x)$

Now fourier transform the extended eqn. w/ $F(\zeta) = \int_{-\infty}^{\infty} e^{i\zeta x}\tilde{f}(x)\,dx = \int_0^\infty e^{i\zeta x} f(x)\,dx$

$$G(\zeta) = \int_{-\infty}^{\infty} e^{i\zeta x} g(x)\,dx = \int_{-\infty}^{0} e^{i\zeta x} g(x)\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} e^{i\zeta x}\,dx \int_{-\infty}^{\infty} K(x-x')\,\tilde{f}(x')\,dx' = K(\zeta)F(\zeta)$$

where $K(\zeta) = \int_{-\infty}^{\infty} e^{i\zeta x} K(x)\,dx$

Now in Bateman $\int_{-\infty}^{\infty} e^{-i\zeta x} K(\zeta)\,d\zeta = K(x)$ and $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta x} K(x)\,dx = K(\zeta)$

since $K(x) = K(-x)$

$$\int_{-\infty}^{\infty} e^{+i\zeta x} K(\zeta)\,d\zeta = K(-x)$$

let $\zeta' = -\zeta$

$$\int_{-\infty}^{\infty} e^{-i\zeta' x} K(-\zeta')\,d\zeta' = K(-x) = \int_{-\infty}^{\infty} e^{-i\zeta x} K(\zeta)\,d\zeta$$

$$= K(x) \Rightarrow K(\zeta) = K(-\zeta)$$

$\therefore$ the transform is also even. We want to find a $K(\zeta)$ $\ni$ $\mathcal{F}\{K(\zeta)\} = 2\pi\operatorname{sech}\frac{x}{2}$

because the transform is even

$$\int_{-\infty}^{\infty} e^{-i\zeta x} K(\zeta)\,d\zeta = 2\int_0^\infty K(\zeta)\cos\zeta x\,d\zeta$$

if $\hat{K}(\zeta) = \operatorname{sech}(a\zeta)$ then $\int_0^\infty \hat{K}(\zeta)\cos\zeta x\,d\zeta = \frac{1}{2}\int_{-\infty}^{\infty} K(\zeta) e^{-i\zeta x}\,d\zeta = \frac{1}{2}\frac{\pi}{a}\operatorname{sech}\left(\frac{\pi x}{2a}\right)$

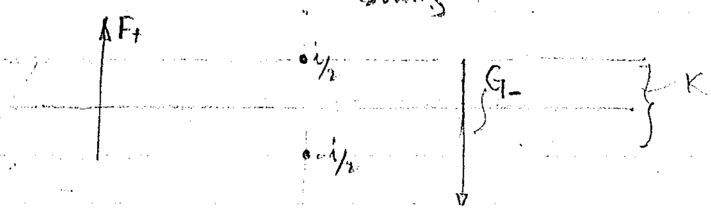
$\therefore \int_{-\infty}^{\infty} \hat{K}(\zeta) e^{-i\zeta x}\,d\zeta = \frac{\pi}{a}\operatorname{sech}\left(\frac{\pi x}{2a}\right)$ if $\hat{K}(\zeta) = \operatorname{sech} a\zeta$ ; if $a=\pi \Rightarrow \operatorname{sech}\pi\zeta \longleftrightarrow \operatorname{sech}\frac{x}{2}$

Now $2\pi\operatorname{sech}\pi\zeta \longleftrightarrow 2\pi\operatorname{sech}\frac{x}{2}$ ie $\int_{-\infty}^{\infty} e^{-i\zeta x}\cdot 2\pi\operatorname{sech}\pi\zeta\,d\zeta = 2\pi\operatorname{sech}\frac{x}{2}$

and $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta x}\cdot 2\pi\operatorname{sech}\frac{x}{2}\,dx = 2\pi\operatorname{sech}\pi\zeta$ thus $\int_{-\infty}^{\infty} e^{i\zeta x}\operatorname{sech}\frac{x}{2}\,dx = 2\pi\operatorname{sech}\pi\zeta = K(\zeta)$

thus the transform eqn is $F(\zeta) - \frac{\sin\pi\lambda}{2\pi}\cdot 2\pi\operatorname{sech}\pi\zeta\, F(\zeta) = G(\zeta)$

$\therefore F(\zeta)\left[1 - \frac{\sin\pi\lambda}{\cosh\pi\zeta}\right] = G(\zeta)$




$\cosh\pi\zeta$ has poles at $\zeta = \frac{2k+1}{2}i$ $\quad k = 0, \pm1, \ldots$

since $\cosh\pi i\frac{2k+1}{2} = \cos\frac{2k+1}{2}\pi$.

Thus the strip is analytic for $|\operatorname{Im}\zeta| < \frac{1}{2}$

now $F(\zeta) = \int e^{i(\xi+i\eta)x} f(x)\,dx$

$e^{i\xi x}\cdot e^{-\eta x} f(x)$ for $x>0$

unless $f(x) \sim e^{-\frac{1}{2}x}$ then $\eta > -\frac{1}{2}$ $\quad \eta > 0$

$$\left|\int_0^\infty \frac{2f(x')\,dx'}{e^{\frac{(x-x')}{2}}+e^{-\frac{(x-x')}{2}}}\right| \quad \text{for large } x,\ \text{small } \zeta \quad \le \left|e^{-\frac{x}{2}}\int_0^\infty 2f(x')e^{x'/2}\,dx'\right| \sim e^{-x/2}\cdot c.$$

const.

$\therefore\ F_+(\zeta)$ is defined from $\operatorname{Im}\zeta > -\tfrac12$

now $\quad G(\zeta) \sim \int e^{i(\xi+i\eta)x} g(x)\,dx \quad$ for $x<0$

$\int e^{i\xi x}e^{-\eta x}\,g(x)\,dx \quad \Rightarrow \eta<0$

for $x<0$ $\quad \int_0^\infty \frac{2f(x')}{e^{\frac{x-x'}{2}}+e^{-\frac{(x-x')}{2}}} \cong g(x) \qquad e^{x/2}\int_0^\infty 2f(x)e^{-x/2}dx' \cong g(x) = O(e^{+x/2}$ or $e^{-x/2}$ $x<0$

$\int e^{-\eta x}e^{x/2}\,dx \qquad x<0 \qquad -\eta+\tfrac12>0 \quad \eta<\tfrac12$

$\therefore \quad F_+\left[1-\frac{\sin\pi\lambda}{\cosh\pi\zeta}\right] = G_-(\zeta) \qquad$ extend $\sin\pi\lambda \ni \lambda$ is complex

$\therefore \quad F_+\left[\frac{\cosh\pi\zeta-\sin\pi\lambda}{\cosh\pi\zeta}\right] = G_-(\zeta).$

$\cosh\pi\zeta = \cos\pi i\zeta \quad \therefore\ \sin\pi\lambda = \cos(\pi\lambda-\pi/2) \quad \therefore\ \cosh\pi\zeta - \sin\pi\lambda = \cos\pi i\zeta - \cos(\pi\lambda-\pi/2)$

but $\cos(\alpha)-\cos(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right) = 2\sin\tfrac12(\pi i\zeta+\pi\lambda-\pi/2)\sin\tfrac12(\pi\lambda-\pi/2-\pi i\zeta)$

$= 2\sin\frac{\pi}{2}\left(i\zeta+\lambda-\tfrac12\right)\sin\pi\left(\frac{\lambda-\frac12-i\zeta}{2}\right)$

but $\sin\pi(z) = \frac{\pi}{\Gamma(z)\Gamma(1-z)} \quad \Rightarrow \quad \cosh\pi\zeta-\sin\pi\lambda = 2\pi^2\dfrac{1}{\Gamma\left(\frac{i\zeta}{2}+\frac{\lambda-\frac12}{2}\right)\Gamma\left(1-\frac{i\zeta+\lambda-\frac12}{2}\right)\Gamma\left(\frac{\lambda-\frac12-i\zeta}{2}\right)\Gamma\left(1-\frac{\lambda-\frac12-i\zeta}{2}\right)}$

now $\cosh\pi\zeta = \cos i\pi\zeta = \sin(i\pi\zeta+\pi/2) = \sin\pi(i\zeta+\tfrac12) = \dfrac{\pi}{\Gamma(i\zeta+\frac12)\Gamma(\frac12-i\zeta)}$

$$\therefore\ \frac{\cosh\pi\zeta-\sin\pi\lambda}{\cosh\pi\zeta} = \frac{\Gamma(i\zeta+\frac12)\Gamma(\frac12-i\zeta)\cdot 2\pi}{\Gamma(\frac{i\zeta}{2}+\frac{\lambda}{2}-\frac14)\Gamma(\frac54-\frac{\lambda}{2}-\frac{i\zeta}{2})\Gamma(\frac{\lambda}{2}-\frac14-\frac{i\zeta}{2})\Gamma(\frac54-\frac{\lambda}{2}+\frac{i\zeta}{2})}$$

non analytic

Now $\Gamma_-(i\zeta+\tfrac12)$ has poles when $i\zeta+\tfrac12 = 0,-1,-2,\ldots$ ie $\zeta = +\tfrac12 i, \tfrac32 i, \tfrac52 i,\ldots$ upper

$\Gamma_+(\tfrac12-i\zeta)$ " " " $\tfrac12-i\zeta =$ " ie $\zeta = -\tfrac12 i, -\tfrac32 i, -\tfrac52 i,\ldots$ lower

$\Gamma_-(\tfrac{i\zeta}{2}+\tfrac{\lambda}{2}-\tfrac14)$ " " " $\tfrac{i\zeta}{2}+\tfrac{\lambda}{2}-\tfrac14 = -n \quad n>0$ ie $\zeta = (2n+\lambda+\tfrac12)i\,;\ (\lambda-\tfrac12)i, (\lambda+\tfrac32)i,$ u/l

$\Gamma_+(\tfrac54-\tfrac{\lambda}{2}-\tfrac{i\zeta}{2})$ " " " $-\tfrac{i\zeta}{2}+\tfrac54-\tfrac{\lambda}{2} = -n$ ie $\zeta = (-2n-\tfrac52+\lambda)i\,;\ (\lambda-\tfrac52)i, (\lambda-\tfrac92)i,\ldots$

$\Gamma_+(\tfrac{\lambda}{2}-\tfrac14-\tfrac{i\zeta}{2})$ " " " $-\tfrac{i\zeta}{2}+\tfrac{\lambda}{2}-\tfrac14 = -n$ ie $\zeta = (-2n-\lambda+\tfrac12)i\,;\ -(\lambda-\tfrac12)i, -(\lambda+\tfrac32)i,$

$\Gamma_-(\tfrac54-\tfrac{\lambda}{2}+\tfrac{i\zeta}{2})$ " " " $\tfrac{i\zeta}{2}+\tfrac54-\tfrac{\lambda}{2} = -n$ ie $\zeta = -(2n-\tfrac52+\lambda)i\,;\ -(\lambda-\tfrac52)i, -(\lambda-\tfrac92)i,$

now have problems since one of the $\Gamma$'s is in the other part

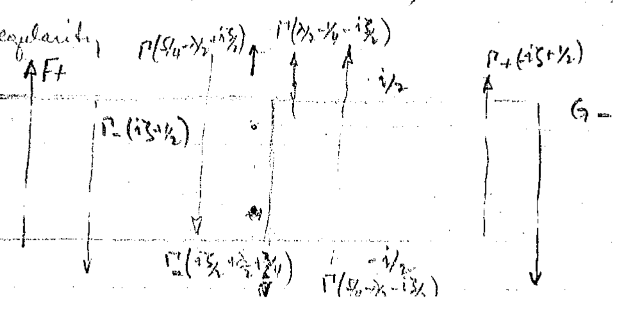
$\therefore$ rewrite as $\Gamma(\tfrac{i\zeta}{2}+\tfrac{\lambda}{2}-\tfrac14) = \dfrac{\Gamma_-(\frac{i\zeta}{2}+\frac{\lambda}{2}+\frac34)}{(\frac{i\zeta}{2}+\frac{\lambda}{2}-\frac14)_+}$

and note that $\dfrac{\Gamma_+(\frac{\lambda}{2}-\frac14-\frac{i\zeta}{2})}{(\frac{i\zeta}{2}+\frac{\lambda}{2}-\frac14)_+}$

thus

$$= \frac{\Gamma_-(i\zeta+\frac12)\,\Gamma_+(\frac12-i\zeta)\,(\frac{i\zeta}{2}+\frac{\lambda}{2}-\frac14)}{\Gamma_-(\frac{i\zeta}{2}+\frac{\lambda}{2}+\frac34)\,\Gamma_+(\frac54-\frac{\lambda}{2}-\frac{i\zeta}{2})\,\Gamma_+(\frac{\lambda}{2}-\frac14-\frac{i\zeta}{2})\,\Gamma_-(\frac54-\frac{\lambda}{2}+\frac{i\zeta}{2})}$$

where $\pm$ denotes region of regularity



$$\Gamma(a+i\varsigma) = \sqrt{2\pi(a+i\varsigma)}\ (a+i\varsigma)^{(a+i\varsigma)}\ e^{-(a+i\varsigma)}$$

$$= \sqrt{2\pi i\varsigma}\left(1+\frac{\pi a}{i\varsigma}\right)\cancel{(i\varsigma)^{a+i\varsigma}}\left[1+a/i\varsigma\right]^{(a+i\varsigma)} e^{-a} e^{-i\varsigma} (i\varsigma)^{i\varsigma} \cdot i\varsigma^{a}$$

$$\sqrt{2\pi i\varsigma}\ (i\varsigma)^{i\varsigma} e^{-i\varsigma}\left(1+\frac{\pi a}{i\varsigma}\right)\cancel{\left(1+a/i\varsigma\right)^{i\varsigma}}\ (i\varsigma+a)^{a}\cancel{(i\varsigma)^{i\varsigma}}\ \cancel{e^{-a}\cdot e^{a}}$$

$$\Gamma(a+i\varsigma/2) = \sqrt{2\pi(a+i\varsigma/2)}\ (a+i\varsigma/2)^{(a+i\varsigma/2)}\ e^{-(a+i\varsigma/2)} \quad e^{a} \quad (i\varsigma+a)^{a}$$

$$= \sqrt{\pi i\varsigma\left(\frac{2a}{i\varsigma}+1\right)}\ (a+i\varsigma/2)^{a}\left(\frac{i\varsigma}{2}\right)^{i\varsigma/2}\left(\frac{2a}{i\varsigma}+1\right)^{i\varsigma/2} e^{-a}\ e^{-i\varsigma/2}$$

$$= \sqrt{\pi i\varsigma}\ \left(1+\pi a/i\varsigma\right)(a+i\varsigma/2)^{a}\ (i\varsigma/2)^{i\varsigma/2}\ \cancel{e^{a}\cdot e^{-a}}\ e^{-i\varsigma/2} \qquad \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}$$

$$\ln\Gamma(a+i\varsigma) = \tfrac{1}{2}\ln\pi i\varsigma + \ln\left(1+\pi a/i\varsigma\right) + a\ln(a+i\varsigma) + i\varsigma\ln(i\varsigma) + i\varsigma$$

$$\ln(b+i\varsigma)$$

$$\ln\Gamma(c-i\varsigma/2) = \tfrac{1}{2}\ln -i\varsigma/2$$

thus $$2\pi F_+ \cdot \frac{\Gamma_+(\tfrac12 - i\zeta)}{\Gamma_+(\tfrac54-\tfrac{\lambda}{2}-\tfrac{i\zeta}{2})\Gamma_+(\tfrac{\lambda}{2}-\tfrac14-\tfrac{i\zeta}{2})} = G_- \frac{\Gamma_-(\tfrac{i\zeta}{2}+\tfrac{\lambda}{2}-\tfrac14)\Gamma_-(\tfrac54-\tfrac{\lambda}{2}+\tfrac{i\zeta}{2})}{\Gamma_-(i\zeta+\tfrac12)}$$

$$\pi F_+ \frac{\Gamma_+(\tfrac12-i\zeta)(\lambda-\tfrac12+i\zeta)}{\Gamma_+(\tfrac{\lambda}{2}-\tfrac14-\tfrac{i\zeta}{2})\Gamma_+(\tfrac54-\tfrac{\lambda}{2}-\tfrac{i\zeta}{2})} = G_- \frac{\Gamma_-(\tfrac34+\tfrac{\lambda}{2}+\tfrac{i\zeta}{2})\Gamma_-(\tfrac54-\tfrac{\lambda}{2}+\tfrac{i\zeta}{2})}{\Gamma(\tfrac12+i\zeta)} =$$

$z\Gamma(z) = \Gamma(1+z)$

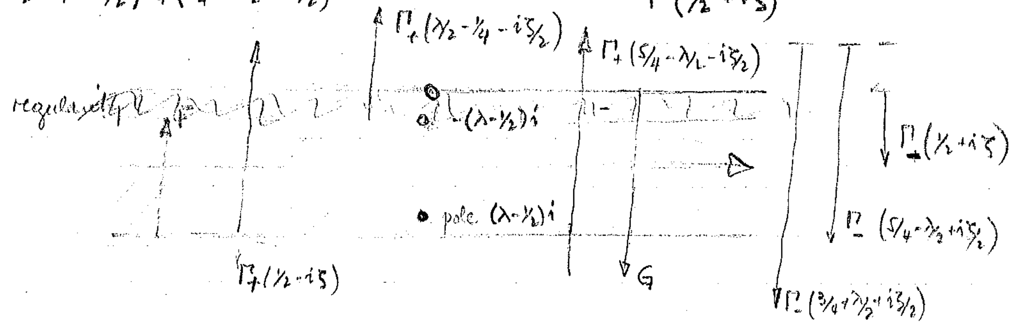
$(z-1)\Gamma(z-1) = \Gamma(z)$

$\lambda-\tfrac12+i\zeta=0 \Rightarrow (\lambda-\tfrac12)i$ has a zero

$\lambda-\tfrac12+i\zeta$ has a zero at $\zeta=(\lambda-\tfrac12)i$

$-2n\ \ 2n(\tfrac32-2\lambda)i$

$(+2n+\tfrac32+\lambda)i=\zeta$ for $\Gamma_-(\tfrac34+\tfrac{\lambda}{2}+\tfrac{i\zeta}{2})$




now for large $\zeta$ $\quad \Gamma(z) \sim \sqrt{2\pi z}\, z^z e^{-z}$ $\therefore$

$$\therefore \quad \underset{|\zeta|\to\infty}{H_+} = \frac{\Gamma(-i\zeta)(i\zeta)}{\Gamma(-\tfrac{i\zeta}{2})\Gamma(-\tfrac{i\zeta}{2})} \sim \frac{\sqrt{-2\pi i\zeta}\,(-i\zeta)^{(-i\zeta)}e^{-i\zeta}(i\zeta)}{(\sqrt{-\pi i\zeta})^2\cdot(-\tfrac{i\zeta}{2})^{(-i\zeta)}e^{-i\zeta}} \sim_{\text{Const}} \frac{2^{-i\zeta}\, i\zeta}{\sqrt{i\zeta}} \text{ or } \frac{2^{-i\zeta}}{\sqrt{i\zeta}}\cdot\text{Const}$$

$$H_- = \frac{(\sqrt{i\zeta})(i\zeta)^{i\zeta}\cdot 2^{-i\zeta}e^{-i\zeta}}{\sqrt{i\zeta}\,(i\zeta)^{i\zeta}e^{-i\zeta}} = \sqrt{i\zeta}\,2^{-i\zeta}$$

$\therefore \quad \ln H_+ \sim -i\zeta\ln 2 + \tfrac12\ln i\zeta + \text{const}$

Now $F_+$ (for small $x$ and large $\zeta$) looks like $\quad f(x) = C\int_0^\infty \frac{f(x')}{\cosh\frac{x}{2}+x\sinh\frac{x}{2}} = \frac{\text{Const}}{1+x}$

$f(x) \cong C(1-x) \cong$ constant for small $x$

$\therefore \quad F_+ \sim \frac{1}{i\zeta} \quad \therefore \quad \ln(F_+H_+) \approx \frac{2^{-i\zeta}}{\sqrt{i\zeta}}\text{const} = -i\zeta\ln 2 - \tfrac12\ln i\zeta + \text{const}$

if $I$ is entire of polynomial nature $\ln I \sim \ln i\zeta \quad \therefore$ Must mult. $H_+\cdot e^{P(\zeta)}$ $\therefore \ln e^{P(\zeta)} = i\zeta\ln 2 =$

and $\ln(H_+F_+e^{P(\zeta)}) \cong \ln i\zeta$ as required $\therefore$

$P(\zeta) = i\zeta\ln 2$ or $e^{P(\zeta)} = e^{i\zeta\ln 2}$

hence $I(\zeta) = \text{Constant} = C$ and $\quad F_+ = \frac{C}{H_+} = C\frac{\Gamma(\tfrac{\lambda}{2}-\tfrac14-\tfrac{i\zeta}{2})\Gamma(\tfrac54-\tfrac{\lambda}{2}-\tfrac{i\zeta}{2})}{\pi\,\Gamma(\tfrac12-i\zeta)(\lambda-\tfrac12+i\zeta)}e^{i\zeta\ln 2}$

$$\frac{H_+}{H_-} = \frac{\cosh\pi\zeta-\sin\pi\lambda}{\cosh\pi\zeta} \quad \therefore \quad H_+ = H_-\left(\frac{\cosh\pi\zeta-\sin\pi\lambda}{\cosh\pi\zeta}\right)$$

$$F_+ = \frac{C}{H_-}\,\frac{\cosh\pi\zeta\, e^{i\zeta\ln 2}}{\cosh\pi\zeta-\sin\pi\lambda}$$

and $$f(x) = \frac{C}{2\pi}\int_{-\infty}^{\infty}\frac{\cosh\pi\zeta\, e^{-i\zeta x}e^{i\zeta\ln 2}}{\cosh\pi\zeta-\sin\pi\lambda}\cdot\frac{d\zeta}{H_-(\zeta)} \qquad H_- = \frac{\Gamma_-(\tfrac34+\tfrac{\lambda}{2}+\tfrac{i\zeta}{2})\Gamma_-(\tfrac54-\tfrac{\lambda}{2}+\tfrac{i\zeta}{2})}{\Gamma(\tfrac12+i\zeta)}$$

for $x>0$ $\;-i\zeta x = -i(\operatorname{Re}\zeta + i\operatorname{Im}\zeta)x = \operatorname{Im}\zeta x - i\operatorname{Re}\zeta x$ $\quad$ for convergence close in lower half plane

where $H_-(\zeta)$ is also regular: The only poles in lower half plane are at $-ik = -2n \pm (\lambda-\tfrac12)$

$k = -2n i(\lambda-\tfrac12)i$

$$\therefore \quad f(x) = \frac{C}{2\pi}\left[\frac{\cot(\pi(\lambda-\tfrac12))}{\pi}\right]\sum_{n=0}\frac{\Gamma(\tfrac12+\alpha+2n)}{\Gamma(1+\alpha+n)}\frac{(2e^x)^{-2n+\alpha}}{n!} - \sum_{n=0}\frac{\Gamma(\tfrac12-\alpha+2n)}{\Gamma(1-\alpha+n)}\frac{(2e^x)^{-(2n-\alpha)}}{n!}$$

$\alpha = \lambda - \tfrac12$

Probl. #1

$$\int_0^\infty \overset{u}{e^{-px}}\ \overset{dv}{\sinh ax} = e^{-px}\frac{\cosh ax}{a}\Big|_0^\infty + p\int e^{-px}\frac{\cosh ax}{a}$$

$$-pe^{-px}\,du \quad \frac{\cosh ax}{a} = v \quad = \frac{1}{2}\left[\frac{e^{x(a-p)}}{a} + e^{-x(a+p)}\right]\Big|_0^\infty$$

$p > a$

$$-\frac{1}{a} + \frac{p}{a}\left[e^{-px}\cosh px\Big|_0^\infty + \frac{p}{a} I\right.$$

$$-\frac{1}{a} + \cancel{\frac{p}{a}} - \frac{p^2}{a^2} I$$

$$I\left(1 + \frac{p^2}{a^2}\right) = -\frac{1}{a}$$

$$a^2 - p^2 = -a$$

$$I = \frac{a}{p^2 - a^2} = s$$

for $\quad \varphi(x) = \lambda \int_0^1 e^{-|x-x'|}\varphi(x')\,dx'$

if $\quad x - x' > 0 \qquad |x-x'| = x - x'$

$\quad\;\; x - x' < 0 \qquad |x - x'| = x' - x$

if $x \geq 1 \qquad (x - x') \geq 0 \quad$ for $0 \leq x' \leq 1 \quad \therefore \quad |x-x'| = x - x'$

$$\varphi(x) = \lambda \int_0^1 e^{-(x-x')}\varphi(x')\,dx'$$

$$= \lambda e^{-x}\int_0^1 e^{+x'}\varphi(x')\,dx' \; ; \quad \text{let } \int_0^1 e^{x'}\varphi(x')\,dx' = c$$

$$\varphi(x) = \lambda e^{-x}\cdot c$$

$$\therefore \quad \lambda e^{-x}\cdot c = \lambda e^{-x}\int_0^1 e^{x'}\cdot \lambda e^{-x'}\cdot c\,dx'$$

$$\lambda e^{-x}\cdot c = \lambda^2 e^{-x}\cdot c$$

thus $\lambda e^{-x} c(1-\lambda) = 0 \quad$ either $\lambda = 1$ or $c = 0$ or $\lambda = 0$

if $\lambda = 0 \implies \varphi(x) = 0 \quad x > 1$

if $c = 0 \implies \varphi(x) = 0 \quad x > 1 \qquad \therefore$ for non trivial soln $\lambda = 1$

$$\therefore \quad \underline{\varphi(x) = ce^{-x}} \quad \text{for } \lambda = 1 \quad \text{and} \quad x \geq 1$$

if $x \leq 0 \qquad (x - x') \leq 0 \quad$ for $0 \leq x' \leq 1 \quad \therefore \quad |x - x'| = x' - x$

$$\varphi(x) = \lambda \int_0^1 e^{-(x'-x)}\varphi(x')\,dx'$$

$$= \lambda e^{x}\int_0^1 e^{-x'}\varphi(x')\,dx' \; ; \quad \text{let } \int_0^1 e^{-x'}\varphi(x')\,dx' = c$$

$$\varphi(x) = \lambda e^{x} c$$

$$\therefore \quad \lambda e^{x} c = \lambda e^{x}\int_0^1 e^{-x'}\cdot \lambda e^{x'}\cdot c\,dx$$

$$\lambda e^{x} c = \lambda^2 e^{x} c$$

$$\therefore \quad \lambda e^{x} c(1-\lambda) = 0 \qquad \therefore \lambda = 0,\ c = 0 \text{ or } \lambda = 1.$$

if $\lambda = 0 \implies \varphi(x) = 0 \quad \forall\, x \leq 0$

if $\quad c = 0 \implies \varphi(x) = 0 \quad \forall\, x \leq 0$

if $\lambda = 1 \implies \varphi(x) = ce^{x}$ for $\lambda = 1$ for $\quad x \leq 0$

$$\varphi(x) = \lambda\int_0^x e^{-(x-x')}\varphi(x')\,dx' + \lambda e^{x}\int_0^1 e^{-x'}\varphi(x')\,dx' - \lambda e^{x}\int_0^x e^{-x'}\varphi(x')\,dx'$$
$$= \lambda\int_0^x \left\{e^{-(x-x')} - e^{(x-x')}\right\}\varphi(x')\,dx' + \lambda e^{x}\int_0^1 e^{-x'}\varphi(x')\,dx'$$
$$\Phi(p) = \lambda\left[\mathcal{L}\{e^{-x}\} - \mathcal{L}\{e^{x}\}\right]\Phi(p) + C\lambda\mathcal{L}\{e^{x}\} \quad C$$

$$\int_0^\infty e^{-px}\cdot e^{-x}\,dx = \int_0^\infty e^{-x(p+1)}\,dx = \left.\frac{e^{-x(p+1)}}{-(p+1)}\right|_0^\infty = \frac{1}{p+1}$$
$$\int_0^\infty e^{-px}e^{x}\,dx = \frac{1}{p-1}$$

$$= \lambda\left[\frac{1}{p+1} - \frac{1}{p-1}\right]\Phi(p) + C\lambda\cdot\frac{1}{p-1}$$
$$= \frac{-2\lambda}{p^2-1}\Phi(p) + \frac{C\lambda}{p-1}$$
$$\Phi(p)\left\{1 + \frac{2\lambda}{p^2-1}\right\} = \frac{C\lambda}{p-1} \quad \therefore \quad \Phi(p)\left(\frac{p^2-1+2\lambda}{\cancel{p^2-1}}\right) = \frac{C\lambda(p+1)}{\cancel{p^2-1}}$$

$$= \frac{C\lambda(p+1)}{p^2-1+2\lambda}$$

$$\frac{C\lambda(p+1)}{p^2-1+2\lambda} = C\lambda\left\{\frac{p}{p^2+(2\lambda-1)} + \frac{\sqrt{2\lambda-1}}{(\sqrt{2\lambda-1})\,p^2+(2\lambda-1)}\right\}$$

$$= p^2+(2\lambda-1)$$

$$= C\lambda\left[\frac{p}{p^2+(2\lambda-1)} + \frac{1}{p^2+(2\lambda-1)}\right]$$

$$p^2-(1-2\lambda) \qquad p^2-(1-2\lambda)$$

if $2\lambda-1>0 \Rightarrow \lambda>\frac{1}{2}$

$$\varphi(x) = C\lambda\left\{\cos\sqrt{2\lambda-1}\,x + \frac{1}{\sqrt{2\lambda-1}}\sin\sqrt{2\lambda-1}\,x\right\}$$

if $\lambda = \frac{1}{2}$ $\qquad \dfrac{C}{2}\dfrac{(p+1)}{p^2} = \dfrac{C}{2}\left[\dfrac{1}{p} + \dfrac{1}{p^2}\right] \Rightarrow \dfrac{C}{2}\left[1+x\right] = \varphi(x)$

if $2\lambda-1<0 \Rightarrow \lambda<\frac{1}{2}$

$$\Phi(p) = C\lambda\left\{\frac{p}{p^2-(1-2\lambda)} + \frac{\sqrt{1-2\lambda}}{\sqrt{1-2\lambda}\,(p^2-(1-2\lambda))}\right\}$$

$$\varphi(x) = C\lambda\left\{\cosh\sqrt{1-2\lambda}\,x + \frac{1}{\sqrt{1-2\lambda}}\sinh\sqrt{1-2\lambda}\,x\right\}$$

$$C = \int_0^1 e^{-x}\cdot C\lambda\left\{\cos\sqrt{2\lambda-1}\,x + \frac{1}{\sqrt{2\lambda-1}}\sin\sqrt{2\lambda-1}\,x\right\}dx$$
$$1 = \lambda\int_0^1 e^{-x}\cos\sqrt{2\lambda-1}\,x + \frac{\lambda}{\sqrt{2\lambda-1}}\int_0^1 e^{-x}\sin\sqrt{2\lambda-1}\,x$$

$$\int_0^1 e^{-x}\cos px\,dx = e^{-x}\frac{\sin px}{p}\Big|_0^1 + \frac{1}{p}\int_0^1 e^{-x}\sin px\,dx$$

$du = -e^{-x}$ $\quad v = \frac{\sin px}{p}$

$$\int_0^1 e^{-x}\sin px\,dx = -\frac{\cos px}{p}e^{-x}\Big|_0^1 - \frac{1}{p}\int_0^1 e^{-x}\frac{\cos px}{p}\,dx$$

$du = -e^{-x}$ $\quad v = -\frac{\cos px}{p}$

$$\left(1+\frac{1}{p^2}\right)\int_0^1 \quad = e^{-x}\frac{\sin px}{p}\Big|_0^1 - \frac{1}{p^2}e^{-x}\cos px\Big|_0^1$$

$$\frac{p^2+1}{p^2}\int \quad = e^{-1}\frac{\sin p}{p} - \frac{1}{p^2}e^{-1}\cos p + \frac{1}{p^2}$$

$$= \frac{e^{-1}\left(\frac{p\sin p}{p^2} - \frac{\cos p}{p^2}\right) + \frac{1}{p^2}}{\frac{p^2+1}{p^2}}$$

$$\boxed{\int_0^1 e^{-x}\cos px\,dx = \frac{e^{-1}(p\sin p - \cos p) + 1}{p^2+1}}$$

$$\left(\frac{p^2+1}{p^2}\right)\int_0^1 e^{-x}\sin px\,dx = -\frac{e^{-x}\cos px}{p}\Big|_0^1 - \frac{1}{p}e^{-x}\frac{\sin px}{p}\Big|_0^1$$

$$= -\frac{p}{p^2}\left[e^{-1}\cos p - 1\right] - \frac{1}{p^2}e^{-1}\sin p$$

$$\boxed{\int_0^1 e^{-x}\sin px\,dx = \frac{p - e^{-1}(p\cos p + \sin p)}{p^2+1}}$$

$p = \sqrt{2\lambda - 1}$

$p^2 = 2\lambda - 1 \qquad p^2 + 1 = 2\lambda$

$$1 = \lambda\left[\frac{e^{-1}(p\sin p - \cos p) + 1}{2\lambda}\right] + \frac{\lambda}{p}\left[\frac{p - e^{-1}(p\cos p + \sin p)}{2\lambda}\right]$$

$$= \frac{1}{2}e^{-1}(p\sin p - \cos p) + \frac{1}{2} + \frac{1}{2} - e^{-1}\frac{\cos p}{2} - e^{-1}\frac{\sin p}{2p}$$

$$1 = 1 - e^{-1}\cos p + \frac{e^{-1}}{2p}(p^2-1)\sin p$$

$\cos p$ $\qquad \frac{2(\lambda-1)}{2\sqrt{2\lambda-1}}$

$$\cos p = \frac{p^2-1}{2p}\sin p \qquad \therefore \quad \frac{2p}{p^2-1} = \tan p$$

$= 0$

char eqn. $\boxed{\frac{2\sqrt{2\lambda-1}}{2(\lambda-1)} = \tan\sqrt{2\lambda-1}}$ $\quad \lambda > 1$

~~$\varphi(x) = c\lambda\cos\sqrt{2\lambda-1}\,x\left\{1 + \frac{1}{(\lambda-1)}\right\}$~~

$\tan(+) = \frac{(+)}{(+)}$ ✓

$$\varphi(x) = c\lambda\left\{\cos\left(\sqrt{2\lambda-1}\right)x + \frac{1}{\sqrt{2\lambda-1}}\sin\sqrt{2\lambda-1}\,x\right\}$$

$$c\lambda\cos\sqrt{2\lambda-1}\left\{1 + \frac{1}{\lambda-1}\right\}$$

$$\frac{\lambda}{\lambda-1}$$

$\varphi(x) = \frac{\xi}{2}(1+x)$

$$c = \frac{c}{2}\int_0^1 e^{-x}(1+x)\,dx = \frac{c}{2}\left[-(1+x)e^{-x}\Big|_0^1 + e^{-x}\Big|_0^1\right]$$

$$2 = \int_0^1 e^{-x} + \int_0^1 x e^{-x}\,dx \qquad \frac{c}{2}\left(-2e^{-1}+1 - e^{-1}+1\right) = \frac{c}{2}(-3e^{-1}+2)$$

$$c = c\left(-\tfrac{3}{2}e^{-1}+1\right)$$

$$= \frac{e^{-x}}{-1}\Big|_0^1 + -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx$$

$$-2e^{-x}\Big|_0^1 - xe^{-x}\Big|_0^1 \qquad \frac{e^{-x}}{-1}\Big|_0^1$$

$$-2(e^{-1}-1) - (e^{-1})$$

2

$\therefore \lambda \neq \frac{1}{2}$

$$\int_0^1 e^{-x}\cosh qx\,dx = \frac{1}{q}\sinh qx\, e^{-x}\Big|_0^1 + \frac{1}{q}\int_0^1 e^{-x}\sinh qx\,dx$$

$du = -e^{-x}$ $\quad v = \frac{\sinh qx}{q}$

$$\int_0^1 e^{-x}\sinh qx\,dx = e^{-x}\frac{\cosh qx}{q}\Big|_0^1 + \frac{1}{q}\int_0^1 e^{-x}\cosh qx\,dx$$

$du = -e^{-x}$ $\quad v = \frac{\cosh qx}{q}$

$$\left(\frac{q^2-1}{q^2}\right)\int_0^1 e^{-x}\cosh qx\,dx = \frac{1}{q}\sinh qx\, e^{-x}\Big|_0^1 + \frac{1}{q^2}e^{-x}\cosh qx\Big|_0^1$$

$$= \frac{q}{q^2}\sinh q\, e^{-1} + \frac{1}{q^2}\left(e^{-1}\cosh q - 1\right)$$

$$\boxed{\int_0^1 e^{-x}\cosh q = \frac{e^{-1}(\cosh q + q\sinh q) - 1}{q^2-1}}$$

$$\left(\frac{q^2-1}{q^2}\right)\int_0^1 e^{-x}\sinh qx\,dx = q e^{-x}\frac{\cosh qx}{q^2}\Big|_0^1 + \frac{1}{q^2}\sinh qx\, e^{-x}\Big|_0^1$$

$$= q(e^{-1}\cosh q - 1) + e^{-1}\sinh q$$

$$\boxed{\int_0^1 e^{-x}\sinh qx\,dx = \frac{e^{-1}(q\cosh q + \sinh q) - q}{q^2-1}}$$

$1-2\lambda-1 = -2\lambda$

$$C = \int_0^1 e^{-x}\varphi(x)\,dx = C\lambda\int_0^1 e^{-x}\cosh qx + \frac{C\lambda}{q}\int_0^1 e^{-x}\sinh qx$$

$$= C\lambda\left[\frac{e^{-1}(\cosh q + q\sinh q) - 1}{q^2-1}\right] + \frac{C\lambda}{q}\left[\frac{e^{-1}(q\cosh q+\sinh q) - q}{q^2-1}\right]$$

$$1 = -\tfrac12\left[e^{-1}(\cosh q + q\sinh q) - 1\right] + -\tfrac12\left[e^{-1}\left(\cosh q + \frac{\sinh q}{q}\right) - 1\right]$$

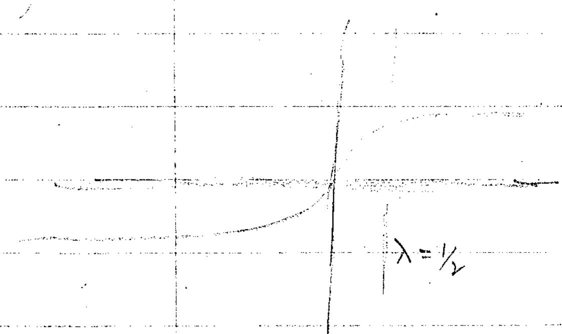
$$\cancel{1} = \cancel{1} - e^{-1}\cosh q - \cancel{\quad} - \tfrac12 e^{-1}\frac{\sinh q}{q}$$
$$-\tfrac12 e^{-1} q\sinh q$$

$$0 = -e^{-1}\cosh q \quad - \tfrac12 e^{-1}\left(\frac{q^2+1}{q}\right)\sinh q$$

$$\cosh q = -\frac{q^2+1}{2q}\sinh q$$

$$-\frac{2q}{q^2+1} = \tanh q$$

$$\frac{2\sqrt{1-2\lambda}}{2(1-\lambda)} = \tanh\sqrt{1-2\lambda} \qquad \lambda<\tfrac12$$



$\lambda=\frac12$

$$\boxed{\frac{\sqrt{1-2\lambda}}{\lambda-1} = \tanh\sqrt{1-2\lambda}}$$
$$(-) = \frac{(+)}{-} = \tanh(+)$$

not correct.
$\Rightarrow \tanh(+) = (-)$
not possible

$$\Phi(p)\left\{1+\frac{2}{p^2-1}\right\} = \frac{C}{p-1}$$

$$\left\{\frac{p^2+1}{\cancel{p^2-1}}\right\} = \frac{C(p+1)}{\cancel{p^2-1}}$$

$$\Phi(p) = C\left(\frac{p+1}{p^2+1}\right)$$
$$= C\left[\frac{p}{p^2+1}+\frac{1}{p^2+1}\right]$$
$$\varphi(x) = C\left[\cos x + \sin x\right]$$

$$1 = 1\int_0^1 e^{-x}\cos x\,dx + \int_0^1 e^{-x}\sin x\,dx$$

$$0 \;\cancel{1} = \frac{e^{-1}}{2}(\cancel{\sin 1}-\cos 1) + \cancel{\tfrac12} + \cancel{\tfrac12} - \frac{e^{-1}}{2}(\cos 1+\cancel{\sin 1})$$

$$0 \overset{?}{=} -\frac{e^{-1}}{2}\cos 1 \quad \text{no!}$$

$$\Phi(p) = -2\lambda\ \mathcal{L}\{\sinh x\}\ \Phi(p) + c\lambda\ \mathcal{L}\{e^x\}$$

$$\left[1 + 2\lambda \frac{1}{p^2-1}\right]\Phi(p) = \frac{c\lambda}{p-1}$$

$$[p^2-1+2\lambda]\ \Phi(p) = c\lambda(p+1)$$

$$\Phi(p) = c\lambda \frac{p+1}{p^2+(2\lambda-1)}$$

$$\varphi(x) = c\lambda\left[\frac{p}{p^2+(\sqrt{2\lambda-1})^2} + \frac{1}{\sqrt{2\lambda-1}}\frac{\sqrt{2\lambda-1}}{p^2+(\sqrt{2\lambda-1})^2}\right]$$

$$= c\lambda\left[\cos\sqrt{2\lambda-1}\,x + \frac{1}{\sqrt{2\lambda-1}}\sin\sqrt{2\lambda-1}\,x\right]$$

Now let $p = \sqrt{2\lambda-1}$ or $p^2+1 = 2\lambda$

$$\varphi(x) = c\lambda\left[\cos px + \frac{1}{p}\sin px\right] = \lambda\cdot\left[\int_0^x e^{-(x-x')} c\lambda\left(\cos px' + \frac{1}{p}\sin px'\right)dx' + c\lambda\int_x^1 e^{-(x'-x)}\left(\cos px' + \frac{1}{p}\sin\right.\right.$$

$$\overbrace{\int_0^x \underbrace{e^{(x'-x)}}_{u}\underbrace{\left(\cos px' + \frac{1}{p}\sin px'\right)dx'}_{dv}}^{I} = e^{(x'-x)}\left[\frac{\sin px'}{p} - \frac{\cos px'}{p^2}\right]\Bigg|_0^x - \int_0^x e^{(x'-x)}\left(\frac{\sin px'}{p} - \frac{\cos px'}{p^2}\right)d$$

$$= \left[\frac{\sin px}{p} - \frac{\cos px}{p^2}\right] + \frac{e^{-x}}{p^2} + \left\{e^{(x'-x)}\left(\frac{\cos px'}{p^2} + \frac{\sin px'}{p^3}\right)\Bigg|_0^x - \frac{1}{p^2}I\right\}$$

$e^{(x'-x)}dx' = du \qquad v = +\frac{\sin px}{p} - \frac{\cos px}{p^2}$

$$\int_0^x e^{(x'-x)}\left(\frac{\sin px'}{p} - \frac{\cos px'}{p^2}\right)dx'$$

$e^{(x'-x)}dx' = du \qquad v = -\frac{\cos px}{p^2} - \frac{\sin px}{p^3}$

$$\left(\frac{p^2+1}{p^2}\right)I = \frac{\sin px}{p} - \cancel{\frac{\cos px}{p^2}} + \cancel{\frac{e^{-x}}{p^2}} + \cancel{\frac{\cos px}{p^2}} + \frac{\sin px}{p^3} - \cancel{\frac{e^{-x}}{p^2}}$$

$$= \frac{p^2+1}{p^3}(\sin px)$$

$$I = \frac{\sin px}{p}$$

$$\int_x^1 \underbrace{e^{-(x'-x)}}_{u}\underbrace{\left(\cos px' + \frac{1}{p}\sin px'\right)dx'}_{dv} = \left\{e^{-(x'-x)}\left[\frac{\sin px}{p} - \frac{\cos px}{p^2}\right]\right\}\Bigg|_x^1 + \int_x^1 e^{-(x'-x)}\left[\frac{\sin px'}{p} - \frac{\cos px'}{p^2}\right]dx'$$

$du = -e^{-(x'-x)}dx \qquad v = \frac{\sin px}{p} - \frac{\cos px}{p^2}$

$$\int_x^1 \underbrace{e^{-(x'-x)}}_{u}\underbrace{\left(\frac{\sin px'}{p} - \frac{\cos px'}{p^2}\right)dx'}_{dv} = -e^{-(x'-x)}\left(\frac{\cos px}{p^2} + \frac{\sin px}{p^3}\right)\Bigg|_x^1 - \int \frac{e^{-(x'-x)}}{p^2}\left[\cos + \frac{1}{p}\sin\right]$$

$$-\frac{1}{p^2}I_p$$

$du = e^{(x'-x)} \qquad v = -\frac{\cos px}{p^2} - \frac{\sin px}{p^3}$

$$\varphi(x) = \lambda A\left[\frac{\cos px - e^{-x} + \cancel{p\sin px}}{1+p^2} + \frac{-e^{-1+x}\cos p + \cos px + pe^{-1+x}\sin p - \cancel{p\sin px}}{1+p^2}\right]$$

$$+\lambda B\left[\frac{\sin px - \cancel{p\cos px} + pe^{-x}}{1+p^2} + \frac{-e^{-1+x}\sin p + \sin px - pe^{-1+x}\cos p + \cancel{p\cos px}}{1+p^2}\right]$$

$$= \lambda A\left[\frac{2}{1+p^2}\cos px + e^{-1+x}[p\sin p - \cos p] - \frac{e^{-x}}{1+p^2}\right]$$

$$+\lambda B\left[\frac{2\sin px + \cancel{2p\cos px}}{1+p^2} + \frac{pe^{-x} - e^{-1+x}(\sin p + p\cos p)}{1+p^2}\right]$$

$$= \frac{\lambda 2}{1+p^2}\left[A\cos px + B\sin px\right] + \frac{\lambda e^{-x}}{1+p^2}\left[-A + pB\right] + \frac{\lambda e^{-1+x}}{1+p^2}\left[A(p\sin p - \cos p) - B(\sin p + p\cos p)\right]$$

if $\frac{2\lambda}{1+p^2} = 1$ & $pB - A = 0$ and. $A(p\sin p - \cos p) - B(\sin p + p\cos p) = 0$

$p^2+1 = 2\lambda$ ✓ $\quad A = pB \quad B\left[(p^2-1)\sin p - 2p\cos p\right] = 0$

$$\frac{e^{-1+x}}{2}\left[\tan p = \frac{2p}{p^2-1} = \frac{2\sqrt{2\lambda-1}}{2(\lambda-1)}\right]$$

$$\tan p = \frac{\sqrt{2\lambda-1}}{\lambda-1}$$

$$\sin p\left[p^2 - p - 1\right] = p\cos p \qquad \tan\sqrt{2\lambda-1} = \frac{\sqrt{2\lambda-1}}{\lambda-1}$$

$$(p^2-p-1) \quad \frac{p}{p^2-p-1} = \tan p \qquad 2\lambda - 2$$

$$\frac{\sqrt{2\lambda-1}}{2(\lambda-1) - \sqrt{2\lambda-1}}$$

$$\varphi(x) = \lambda\int_0^x e^{-(x-x')}\left[A\cosh gx' + B\sinh gx'\right]dx' + \lambda\int_x^1 e^{-(x'-x)}\left[A\cosh gx' + B\sinh gx'\right]dx'$$

$$= \lambda e^{-x}\left\{\int_0^x Ae^{x'}\cosh gx' + Be^{x'}\sinh gx'\right\} + \lambda e^{x}\left\{\int_x^1 Ae^{-x'}\cosh + Be^{-x'}\sinh gx'\right\}$$

$$\therefore \quad I_p\left(\frac{p^2+1}{p^2}\right) = \left\{ e^{-(x'-x)}\left(\frac{\sin px'}{p} - \frac{\cos px'}{p^2}\right)\Bigg|_x^1 - e^{-(x'-x)}\left(\frac{\cos px'}{p^2} + \frac{\sin px'}{p^3}\right)\Bigg|_x^1 \right\}$$

$$= e^{x-1}\left(\frac{\sin p}{p} - \frac{\cos p}{p^2}\right) - \left(\frac{\sin px}{p} - \frac{\cos px}{p^2}\right) - e^{x-1}\left(\frac{\cos p}{p^2} + \frac{\sin p}{p^3}\right)$$

$$+ \left(\frac{\cos px}{p^2} + \frac{\sin px}{p^3}\right)$$

$$I_p\left(\frac{p^2+1}{p^2}\right) = \text{or} \quad \cancel{\frac{2\cos px}{p^2} - \frac{\sin px}{p}\left(\frac{p^2-1}{p^2}\right) + e^{x-1}\left\{-\frac{2\cos p}{p^2} + \frac{\sin p}{p}\left(\frac{p^2-1}{p^2}\right)\right\}}$$

$$(1 - 1/p^2)$$

$\varphi(x)$

$$c\lambda\left[\cos px + \frac{1}{p}\sin px\right] = c\lambda\left(\cancel{\frac{\sin px}{p}(1 - 1/p^2) - \frac{2\cos px}{p^2} + e^{x-1}\left\{-\frac{2\cos p}{p^2} + \frac{\sin p}{p}\left(\frac{p^2-1}{p^2}\right)\right.}\right.$$

$$= \cancel{\frac{c\lambda}{p^2}\left(\frac{\sin px}{p} - 2\cos px\right.}$$

$$\int_0^x$$

$$= c\lambda\left(\frac{\sin px}{p} + \frac{2\cos px}{p^2+1} - \frac{\sin px}{p}\left(\frac{p^2-1}{p^2+1}\right)\right) + e^{x-1}\left\{\frac{-2\cos p}{p^2+1} + \frac{\sin p}{p}\left(\frac{p^2-1}{p^2+1}\right.\right.$$

$$= c\lambda\left(\left[\frac{\sin px}{p}\left[1 - \frac{p^2-1}{p^2+1}\right] + \frac{2\cos px}{p^2+1}\right]\right) + e^{x-1}$$

$$\frac{\sin px}{p}\left[1 - \frac{p^2-1}{p^2+1}\right. \qquad \frac{p^2+1-p^2+1}{p^2+1} \qquad \frac{p^2+1-p^2+1}{p^2+1} = \frac{2}{p^2+1} \qquad \frac{-2\cos p}{p^2+1}$$

$$1 \qquad \frac{2}{p^2+1}$$

$$= c\lambda\left[\frac{2\lambda}{p^2+1}\left[\frac{\sin px}{p} + \cos px\right] + \lambda e^{x-1}\left\{\frac{-2\cos p}{p^2+1} + \frac{\sin p}{p}\left(\frac{p^2-1}{p^2+1}\right)\right\}\right.$$

$$\varphi(x)\left[1 - \frac{2}{p^2+1}\right] = e^{x-1}\cdot\frac{c\lambda}{p^2+1}\left\{-2\cos p + \frac{\sin p}{p}\left(p^2-1\right)\right\}$$

$$\text{and } \frac{2\lambda}{p^2+1} = 1 = \frac{2}{2\lambda} = \frac{1}{\lambda}$$

$$\frac{p^2-1}{p^2+1} \qquad p = \sqrt{2\lambda-1} \Rightarrow p^2-1 = 2(\lambda-1)$$

$$p^2+1 = 2\lambda$$

$$\varphi(x)\left[\frac{\lambda-1}{\lambda}\right] = e^{x-1}\cdot c\left\{\frac{2\lambda}{2\lambda}\cos p + \frac{\sin p}{p}\cdot\frac{\lambda}{2\lambda}\cdot 2(\lambda-1)\right\} \qquad \frac{-2\cos p}{p^2+1} \qquad 2(\lambda-1)$$

$$-\cos p + \frac{\sin p}{2p}\cdot(p^2-1) = 0$$

$$\lambda = 1$$

$$= 0 \quad -\cos p + \frac{\sin p}{p}\,(\cancel{\lambda-1}) = 0 \qquad \frac{\cos p}{\sqrt{2\lambda-1}} = \sin p\,\frac{\lambda-1}{\sqrt{2\lambda-1}}$$

$$\frac{2}{p^2+1} = 1 \quad \&$$

$$\frac{2}{2\lambda} = 1 \qquad \therefore \quad \frac{\sin p}{\cos p} = \frac{\sqrt{2\lambda-1}}{\lambda-1} = \frac{\lambda-1}{\tan\sqrt{2\lambda-1}}$$

ok.

$$\tan\sqrt{2\lambda-1} =$$

$$\tan 1 = \frac{1}{0} = \infty \quad \therefore \lambda \neq 1$$

hence only sol is trivial sol.

$$\cot 1 = \frac{0}{1} = 0$$

for the 2 to be equal $\Rightarrow \dfrac{2}{p^2+1} = 1 \;\&\; \dfrac{2}{p^2+1}\left[\cos p + \dfrac{\sin p}{2p}(p^2-1)\right] = 0$ $\quad \lambda=1$

$$\frac{2c\lambda}{p^2+1} \qquad \frac{2}{2\lambda} = 1 \quad \lambda = 1 \qquad \cos p = \frac{\sin p}{2\sqrt{2\lambda-1}}\,2(\lambda-1) \qquad \cot\sqrt{2\lambda-1} = \frac{(\lambda-1)}{\sqrt{2\lambda-1}}$$

$$\cotan(\sqrt{2\lambda-1})$$

$$\varphi'(x) = \lambda\varphi(x) - \lambda\int_0^x e^{-(x-x')}\varphi(x')dx' - \lambda\varphi(x) + \lambda\int_x^1 e^{-(x'-x)}\varphi(x')dx'$$

$$\varphi''(x) = -\lambda\varphi(x) + \lambda\int_0^x e^{-(x-x')}\varphi(x')dx' - \lambda\varphi(x) + \lambda\int_x^1 e^{-(x'-x)}\varphi(x')dx'$$

$$= (-2\lambda+1)\varphi(x)$$

$$\varphi''(x) + (2\lambda-1)\varphi(x) = 0 \qquad \varphi = Ax+B \quad \text{if } \lambda = \tfrac{1}{2}$$

$$\varphi = A\cos\sqrt{2\lambda-1}\,x + B\sin\sqrt{2\lambda-1}\,x$$

$$\varphi = A\cosh\sqrt{1-2\lambda}\,x + B\sinh\sqrt{1-2\lambda}\,x$$

$$\varphi(x) = \lambda\int_0^x e^{-(x-x')}[A\cos\sqrt{2\lambda-1}\,x' + B\sin\sqrt{2\lambda-1}\,x']dx' + \lambda\int e^{(x-x')}[A\cos\sqrt{2\lambda-1}\,x + B\sin$$

$$= \lambda e^{-x}A\int_0^x e^{x'}\cos\sqrt{\ }\,x' + \lambda e^{-x}B\int_0^x e^{x'}\sin\sqrt{\ } + \lambda A e^{x}\int_x^1 e^{-x'}\cos\sqrt{\ }\,x' + \lambda B e^{x}\int_x^1 e^{-x'}\sin\sqrt{\ }$$

dv u

$$\int_0^x e^{x'}\cos\sqrt{p}\,x' = e^{x'}\cos px'\Big|_0^x + p\int_0^x e^{x}\sin px\,dx = e^{x'}\cos px'\Big|_0^x + pe^{x}\sin px\Big|_0^x - p^2\int_0^x e^{x}\cos px\,dx$$

$e^{x'} = v$ $du = -p\sin px$

dv u

$$\int_0^x e^{x'}\sin px = e^{x'}\sin px\Big|_0^x - p\int_0^x e^{x'}\cos px\,dx = e^{x'}\sin px\Big|_0^x - pe^{x'}\cos px'\Big|_0^x - p^2\int_0^x e^{x}\sin$$

$e^{x'} = v$ $p\cos px = du$

$$\cancel{(1+p^2)}\int_0^x e^{x}\sin px = e^{x'}\sin px\Big|_0^x - pe^{x'}\cos px\Big|_0^x = \frac{e^x\sin px - pe^x\cos px + p}{1+p^2}$$

$$\int_0^x e^{x}\cos px = \frac{e^{x}\cos px\Big|_0^x + pe^x\sin px\Big|_0^x}{1+p^2} = \frac{e^x\cos px - 1 + pe^x\sin px}{1+p^2}$$

dv u

$$\int_x^1 e^{-x'}\cos px'dx' = -e^{-x'}\cos px'\Big|_x^1 - p\int_x^1 e^{-x'}\sin px'dx' = -e^{-x'}\cos px'\Big|_x^1 + pe^{-x'}\sin px'\Big|_x^1 - p^2 I$$

$v = -e^{-x}$ $du = -p\sin px$

dv u

$$\int_x^1 e^{-x}\sin px'dx' = -e^{-x'}\sin px'\Big|_x^1 + p\int_x^1 e^{-x}\cos px'dx' = -e^{-x'}\sin px'\Big|_x^1 - pe^{-x'}\cos px'\Big|_x^1 - p^2\int_x^1 e^{-x'}\sin px'dx'$$

$v = e^{-x}$ $p\cos px'dx' = du$

$$\therefore \cancel{(1+p^2)}\int_x^1 = \frac{-e^{-1}\sin p + e^{-x}\sin px - pe^{-1}\cos p + pe^{-x}\cos px}{1+p^2} = \int_x^1 e^{-x}\sin px$$

$$\cancel{1+p^2}\int e^{-x}\cos p = \frac{-e^{-1}\cos p + e^{-x}\cos px + pe^{-1}\sin p - pe^{-x}\sin px}{1+p^2}$$

Proble. #2

$$\varphi(x) = \lambda \int_0^{2\pi} (\sin x \cos x' + \cos x \sin x')\varphi(x')\,dx'$$

$$= \lambda \sin x \int_0^{2\pi} \cos x' \varphi(x')\,dx' + \lambda \cos x \int_0 \sin x' \varphi(x')\,dx'$$

$$= \lambda (A \sin x + B \cos x) \quad A \qquad B$$

$$\lambda A \sin x + \lambda B \cos x = \lambda \sin x \int_0^{2\pi} \cos x' [\lambda A \sin x' + \lambda B \cos x']\,dx' + \lambda \cos x \int_0^{2\pi} \sin x' [\lambda A \sin x' + \lambda B \cos x']$$

$$= \lambda \sin x \left\{ \lambda A \int_0^{2\pi} \cos x' \sin x'\,dx' + \lambda B \int_0^{2\pi} \cos x'^2\,dx' \right.$$

$$\left. + \lambda \cos x \left\{ \lambda A \int_0^{2\pi} \sin x'^2\,dx' + \lambda B \int_0^{2\pi} \sin x' \cos x'\,dx' \right\} \right.$$

$$-\frac{\cos 2x}{4}\Big|_0^{2\pi} \qquad \int_0^{2\pi} \frac{1+\cos 2x}{2} \qquad \frac{x}{2} + \frac{\sin 2x}{4}\Big|_0^{2\pi} = \pi$$

$$\lambda A \sin x + \lambda B \cos x = (\lambda^2 B \sin x)\pi + (A\lambda^2 \cos x)\cdot\pi$$

$$\sin x (\lambda A - \pi\lambda^2 B) + \cos x (\lambda B - \pi A \lambda^2) = 0$$

$$\lambda A - \pi\lambda^2 B = 0$$
$$\lambda B - \pi A \lambda^2 = 0$$

$$\begin{pmatrix} \lambda & -\pi\lambda^2 \\ -\pi\lambda^2 & \lambda \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0 \qquad \text{for nontrivial } A \,\&\, B \quad \det(\;) = 0$$

$$\lambda^2 - \pi^2\lambda^4 = 0 \qquad \lambda = 0, \lambda = 0 \text{ or } \lambda = \pm 1/\pi$$

$$\lambda^2(1 - \pi^2\lambda^2) = 0 \qquad \lambda = \pm 1/\pi$$

$$\frac{\lambda A}{\pi\lambda^2} = B = \frac{A}{\pi\lambda}$$

$\therefore$ for $\lambda = 0 \quad \varphi = 0$

for $\lambda = \pm 1/\pi \quad \varphi(x) = \pm 1/\pi \,[A \sin x + B \cos x]$

if $\lambda = 1/\pi \quad B = A$

$\lambda = -1/\pi$

if $\lambda = -1/\pi \Rightarrow \quad -1/\pi A - 1/\pi B = 0$

$$-1/\pi B - 1/\pi A = 0$$

$$u \quad dv$$

if $\varphi(x) = \lambda \int_0^x x' \varphi(x')\,dx' + \lambda \int_x^1 x \varphi(x')\,dx'$

$du \quad v = \int^x \varphi$

$$\frac{d\varphi}{dx} = \lambda[x\varphi(x)] - \lambda[x\varphi(x)] = 0 \qquad \varphi = \text{const}$$

$$\varphi(x) = \text{const} = \lambda C \int_0^x x^2 + \lambda C x \int_x^1 dx$$

$$\lambda C[\;] + \lambda C[x \cdot x^2]$$

$$\varphi(x) = \lambda \left[ x' \int_0^{x'} \varphi(v)\,dv \Big|_0^x - \int_0^x \int_0^{x'} \varphi(v)\,dv \right.$$

let $\varphi_0(x) = 1 \qquad \varphi(x) = \lambda \int_0^x x'\,dx' + \lambda x \int_x^1 dx'$

$$\lambda(x^2/2 + x - x^2) = \lambda(x - x^2/2)$$

Proble #3

$$\varphi(x) = \lambda \int_0^x x'\varphi(x')\,dx' + \lambda \int_x^1 x\varphi(x')\,dx'$$

$$= \lambda \int_0^x x'\varphi(x')\,dx' - \lambda x \int_0^x \varphi(x')\,dx' + \lambda x \int_0^x \varphi(x')\,dx' + \lambda x \int_x^1 \varphi(x')\,dx'$$

$$= -\lambda \int_0^x (x-x')\varphi(x')\,dx' + \lambda x \int_0^1 \varphi(x')\,dx'$$

$\cancel{f(x') = x'}$ $\cancel{f(x-x') = x-x'}$ $\quad c$

$$F(p) = \int_0^\infty e^{-px} x\,dx = \frac{x}{-p}e^{-px}\Big|_0^\infty + \frac{1}{p}\int_0^\infty e^{-px}dx = -\frac{1}{p^2}e^{-px}\Big|_0^\infty = \frac{1}{p^2}$$

$$\int_0^\infty e^{-px}\varphi(x)\,dx = \Phi(p) = -\lambda F(p)\,\Phi(p) + \lambda F(p)\cdot c$$

$$\Phi(p)\,[1+\lambda F(p)] = \lambda F(p)\,c$$

$$\Phi(p) = \frac{\lambda F(p) c}{1+\lambda F(p)} = \frac{c\lambda/p^2}{1+\lambda/p^2} = \frac{c\lambda}{p^2+\lambda}$$

and $$\varphi(x) = \frac{c}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\frac{\lambda F(p)}{1+\lambda F(p)}\,dp = \frac{c}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\frac{\lambda}{(p-\sqrt{\lambda}i)(p+\sqrt{\lambda}i)}\,d\lambda$$

$$\frac{c\lambda}{2\pi i}\int \frac{e^{px}\lambda}{p^2+\lambda}\,dp$$

$$\frac{c}{2\pi i}\lambda\left[\frac{e^{\sqrt{\lambda}ix}}{2\sqrt{\lambda}i} + \frac{e^{-\sqrt{\lambda}ix}}{-2\sqrt{\lambda}i}\right]$$

$$\Phi(p) = c\sqrt{\lambda}\,\frac{\sqrt{\lambda}}{p^2+(\sqrt{\lambda})^2} \qquad \frac{\lambda}{p^2+\lambda}$$

$$\frac{c\lambda}{-4\pi\sqrt{\lambda}}\left[\frac{e^{\sqrt{\lambda}ix} - e^{-\sqrt{\lambda}ix}}{c+is - (c-is)}\right] \quad 2is$$

$$\underline{\varphi(x) = c\sqrt{\lambda}\sin\sqrt{\lambda}x}$$

$c = \int_0^1 \varphi(x')\,dx'$ $\quad \therefore\ \varphi(x) = -\frac{c\sqrt{\lambda}}{2\pi}i\sin\sqrt{\lambda}x$

$= -\frac{c}{2\pi}\sqrt{\lambda}i\int_0^1 \sin\sqrt{\lambda}x\,dx$

$$\varphi(x) = \lambda\int_0^x x'\cdot K\sin\sqrt{\lambda}x'\,dx' + \lambda x\int_x^1 K\sin\sqrt{\lambda}x'\,dx'$$

$\left(-\frac{c}{2\pi}\right)\frac{-\cos\sqrt{\lambda}x}{\sqrt{\lambda}}\Big|_0^1$ $\quad \lambda K \quad v = -\frac{\cos\sqrt{\lambda}x'}{\sqrt{\lambda}} \quad + \lambda x K \frac{-\cos\sqrt{\lambda}x'}{\sqrt{\lambda}}\Big|_x^1$

$c = \frac{ci}{2\pi}(\cos\sqrt{\lambda}-1)$ $\quad \lambda K\left[\frac{-x'}{\sqrt{\lambda}}\cos\sqrt{\lambda}x'\Big|_0^x + \int_0^x \frac{\cos\sqrt{\lambda}x'}{\sqrt{\lambda}}dx'\right]$

$1 = \frac{i}{2\pi}(\cos\sqrt{\lambda}-1)$

$\cos\sqrt{\lambda}-1 = -2\pi i$ $\quad \lambda K\left[\frac{-x'\cos\sqrt{\lambda}x'}{\sqrt{\lambda}}\Big|_0^x + \frac{\sin\sqrt{\lambda}x'}{\lambda}\Big|_0^x\right] - \lambda K x \frac{\cos\sqrt{\lambda}x'}{\sqrt{\lambda}}\Big|_x^1$

$\cos\sqrt{\lambda} = 1-2\pi i$ $\quad \lambda K\left[\cancel{\frac{-x\cos\sqrt{\lambda}x}{\sqrt{\lambda}}} + \frac{\sin\sqrt{\lambda}x}{\lambda}\right] - \frac{\lambda K x\cos\sqrt{\lambda}}{\sqrt{\lambda}} + \cancel{\frac{\lambda K x\cos\sqrt{\lambda}x}{\sqrt{\lambda}}}$

$$K\sin\sqrt{\lambda}x = K\sin\sqrt{\lambda}x - \sqrt{\lambda}Kx\cos\sqrt{\lambda}$$

pick $\sqrt{\lambda} = \pi/2$ or any odd multiple of $\pi/2$

$$\varphi(x) = \lambda\int_0^x x' c\sqrt{\lambda}\sin\sqrt{\lambda}x'\,dx' + \lambda x c\sqrt{\lambda}\int_x^1 \sin\sqrt{\lambda}x'\,dx'$$

$$= \lambda c\sqrt{\lambda}\left[\frac{-x'\cos\sqrt{\lambda}x'}{\sqrt{\lambda}}\Big|_0^x + \frac{\sin\sqrt{\lambda}x'}{\lambda}\Big|_0^x\right] + \left(\lambda x c\sqrt{\lambda}\right)\left(-\frac{\cos\sqrt{\lambda}x'}{\sqrt{\lambda}}\right)\Big|_x^1$$

$$\lambda c\sqrt{\lambda}\left[\frac{-x\cos\sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{\sin\sqrt{\lambda}x}{\lambda}\right] + \lambda c x\sqrt{\lambda}\left[-\frac{\cos\sqrt{\lambda}}{\sqrt{\lambda}} + \frac{\cos\sqrt{\lambda}x}{\sqrt{\lambda}}\right]$$

$$-\lambda c x \cancel{\cos\sqrt{\lambda}x} + c\sqrt{\lambda}\sin\sqrt{\lambda}x - \lambda c x\cos\sqrt{\lambda} + \cancel{\lambda c x\cos\sqrt{\lambda}x}$$

$$c\sqrt{\lambda}\sin\sqrt{\lambda}x = c\sqrt{\lambda}\sin\sqrt{\lambda}x - \lambda c x\cos\sqrt{\lambda}$$

$\sqrt{\lambda} = \pi/2 \quad \lambda_1 = \pi^2/4$

$\sqrt{\lambda} = 3\pi/2 \quad \lambda_2 = 9\pi^2/4$

$\vdots$

$$c = \int_0^1 c\sqrt{\lambda}\sin\sqrt{\lambda}x'\,dx'$$

$$1 = \sqrt{\lambda}\left(\frac{-\cos\sqrt{\lambda}x'}{\sqrt{\lambda}}\right)_0^1 = -\cos\sqrt{\lambda}x\Big|_0^1$$

if $\sqrt{\lambda} = \pi/2$

$$-\cos \tfrac{\pi}{2}x\Big|_0^1 = 1$$

$$\therefore \quad \sqrt{\lambda} = \pi/2,\ 3\pi/2,\ 5\pi/2,\ \dots,\ \frac{2n+1}{2}\pi \qquad n = 0, 1, 2, \dots$$

$$\therefore \quad \varphi_n(x) = C\left(\frac{2n+1}{2}\pi\right)\sin\left(\frac{2n+1}{2}\pi x\right)$$

$$f(x)$$

$$\lambda\int_0^1 x'\varphi(x')dx' + \lambda\int_x^1 (x-x')\varphi(x')dx'$$

let $x - x' = u$ when $x' = x \quad u = 0$

then $\qquad x' = 1 \quad u = x-1$

$x - u = x' \qquad dx' = -du$

$$-\lambda\int_0^{x-1} u\,\varphi(x-u)\,du = -\lambda\int_0^{x-1} u\varphi(x-u)\,du - \lambda\int_{x-1}^{x} \quad + \lambda\int_x$$

$\varphi(x) = f(x) + \lambda\int_0^1 x'\varphi(x')dx'$

$\varphi(x) = \lambda c$

$\lambda c = \lambda^2 c\int_0^1 x'dx' = \frac{\lambda^2 c}{2}$

$\lambda c(1 - \lambda/2) = 0 \quad \lambda = 0,\ \lambda = 2,\ c = 0$

$$\lambda\int_0^1 (x'-x)\varphi(x')dx' \qquad -\lambda\int_0^x \qquad + \lambda\int_{x-1}^{x}(x-u)\varphi(u)$$

$$+\lambda\int_0^1 x\varphi(x')dx'$$

$$\lambda\int_0^x \varphi + \lambda\int_x^1 x\varphi$$

$$\lambda x\int_0^1 \varphi(x')dx' + \lambda\int_x^1 (x-x')\varphi$$

Integral Equations
MA 224 Prof Levine
Cesar Levy Fall 80

HW # 1

Find the solution to $\varphi(x) = \lambda \int_0^1 e^{-|x-x'|}\varphi(x')dx'$

$$\varphi(x) = \lambda \int_0^x e^{-(x-x')}\varphi(x')dx' + \lambda \int_x^1 e^{-(x'-x)}\varphi(x')dx' \qquad (**)$$

$$= \lambda \int_0^x e^{-(x-x')}\varphi(x')dx + \lambda \int_0^1 e^{-(x'-x)}\varphi(x')dx' - \lambda \int_0^x e^{-(x'-x)}\varphi(x')dx'$$

$$= -2\lambda \int_0^x \sinh(x-x')\varphi(x')dx' + \lambda \int_0^1 e^{-(x'-x)}\varphi(x')dx'$$

$$\varphi(x) = -2\lambda \sinh(x) * \varphi(x) + \lambda e^x \int_0^1 e^{-x'}\varphi(x')dx' \quad \text{a Volterra IE.}$$

by taking the Laplace transform of both sides and also defining $\int_0^1 e^{-x'}\varphi(x')dx' = c$, then

$$\Phi(p) = -2\lambda\,\Phi(p)\cdot\frac{1}{p^2-1} + c\lambda\frac{1}{p-1} \quad \text{thus}$$

$$\Phi(p)\left[1+\frac{2\lambda}{p^2-1}\right] = \frac{c\lambda}{p-1} \quad \text{or} \quad \frac{c\lambda(p+1)}{p^2-1+2\lambda} = \Phi(p)$$

Now by taking the inverse transform then

$$\varphi(x) = c\lambda\left[\cos\sqrt{2\lambda-1}\,x + \frac{1}{\sqrt{2\lambda-1}}\sin\sqrt{2\lambda-1}\,x\right] \quad \text{if } \lambda > 1/2$$

$$\varphi(x) = \frac{c}{2}[1+x] \quad \text{for } \lambda = 1/2$$

$$\varphi(x) = c\lambda\left[\cosh\sqrt{1-2\lambda}\,x + \frac{1}{\sqrt{1-2\lambda}}\sinh\sqrt{1-2\lambda}\,x\right] \quad \text{for } \lambda < 1/2$$

since $\int_0^1 e^{-x'}\varphi(x')dx' = c$ let us look at $\varphi(x) = \frac{c}{2}[1+x]$

$$\frac{c}{2}\int_0^1 e^{-x'}[1+x']dx' = \frac{c}{2}\left\{-e^{-x'}(1+x')\Big|_0^1 + \int_0^1 e^{-x'}dx'\right\} = \frac{c}{2}\left\{-e^{-x'}(2+x')\Big|_0^1\right\}$$

$$= \frac{c}{2}\left\{2-3e^{-1}\right\} \neq c \quad \therefore \lambda \neq 1/2$$

looking at $\varphi(x) = c\lambda\left[\cos px + \frac{1}{p}\sin px\right]$ where $p = \sqrt{2\lambda-1}$ $\quad \lambda > 1/2$

$$K(x) = \lambda\int_0^x e^{-(x-x')}\varphi(x')dx' + \lambda\int_x^1 e^{-(x'-x)}\varphi(x')dx'$$

$$= c\lambda\left[\frac{2\lambda}{p^2+1}\left[\frac{\sin px}{p} + \cos px\right] + \frac{\lambda e^{-1+x}}{p^2+1}\left\{-2\cos p + \sin p\left(\frac{p^2-1}{p}\right)\right\}\right] \quad \text{after substitution}$$

but since $K(x) = \varphi(x)$ and $\frac{2\lambda}{p^2+1} = 1 \Rightarrow -2\cos p + \sin p\left(\frac{p^2-1}{p}\right) = 0$

or $\tan\sqrt{2\lambda-1} = \frac{\sqrt{2\lambda-1}}{\lambda-1}$

this is precisely the condition we get when we check

$$c = \int_0^1 e^{-x'}\varphi(x')dx' = c - e^{-1}\left[\cos p - \frac{p^2-1}{2p}\sin p\right]$$

thus $\tan\sqrt{2\lambda-1} = \frac{\sqrt{2\lambda-1}}{\lambda-1}$ is our characteristic equation and $\varphi(x,\lambda) = c\lambda\left[\cos\sqrt{2\lambda-1}\,x + \frac{1}{\sqrt{2\lambda-1}}\sin\sqrt{2\lambda-1}\right.$

is the eigenfunction for $\lambda \geq 1/2$.

for $\lambda < 1/2$ $\qquad \varphi(x) = C\lambda \left[\cosh qx + \frac{1}{q}\sinh qx\right]$ where $q = \sqrt{1-2\lambda}$

$$C = \int_0^1 e^{-x'}\varphi(x')dx'$$

$$= C\lambda\left[\frac{e^{-1}(\cosh q + q\sinh q) - 1}{q^2-1}\right] + \frac{C\lambda}{q}\left[\frac{e^{-1}(q\cosh q + \sinh q) - q}{q^2-1}\right]$$

$$= C - Ce^{-1}\left\{\cosh q + \frac{q^2+1}{2q}\sinh q\right\}$$

this means $\tanh q = -\frac{2q}{q^2+1} = \frac{-\sqrt{1-2\lambda}}{1-\lambda}$. Looking at $\tanh\sqrt{1-2\lambda}$ we note that the argument being positive means that the tanh is positive. But $-\frac{\sqrt{1-2\lambda}}{1-\lambda}$ is negative for $\lambda < 1/2$; hence no solutions exist in this domain. Thus

$\tan\sqrt{2\lambda_n - 1} = \frac{\sqrt{2\lambda_n-1}}{\lambda_n - 1}$ is the eigenvalue equation and

$\varphi_n(x) = C\lambda_n\left[\cos\sqrt{2\lambda_n-1}\,x + \frac{1}{\sqrt{2\lambda_n-1}}\sin\sqrt{2\lambda_n-1}\,x\right]$ is the corresponding eigenfunction.

Note: another method of solution is to differentiate equation (**) with respect to x to obtain

$\varphi''(x) + (2\lambda - 1)\varphi(x) = 0$. Suppose $\lambda < 1/2 \Rightarrow \varphi(x) = A\cosh qx + B\sinh qx \quad q = \sqrt{1-2\lambda}$

By putting this back into (**) and evaluating we get.

$$\varphi(x) = \frac{2\lambda}{1-q^2}\cdot\varphi(x) - \frac{\lambda e^{-x}}{1-q^2}[A - Bq] + \frac{\lambda e^{-1+x}}{1-q^2}\left[-\cosh q(A+Bq) - \sinh q(Aq+B)\right]; \text{ but } 1-q^2 = 2\lambda$$

$$\therefore \quad \varphi(x) = \varphi(x) - \frac{1}{2}e^{-x}[A - Bq] - \frac{1}{2}e^{x-1}\left[\cosh q(A+Bq) + \sinh q(Aq+B)\right]$$

since this is true $\forall x \Rightarrow A = Bq$ and $\cosh q(A+Bq) + \sinh q(Aq+B) = 0$

which yields the results shown above, i.e. that no value of $\lambda < 1/2$ can be an E-value.

2. Find the solution to $\varphi(x) = \lambda \int_0^{2\pi} \sin(x+x')\, \varphi(x')\, dx'$

$$\varphi(x) = \lambda \int_0^{2\pi} (\sin x \cos x' + \cos x \sin x')\, \varphi(x')\, dx'$$

$$= \lambda \left[ A \sin x + B \cos x \right] \quad \text{where} \quad \begin{pmatrix} A \\ B \end{pmatrix} = \int_0^{2\pi} \begin{pmatrix} \cos x' \\ \sin x' \end{pmatrix} \varphi(x')\, dx'$$

Now put this into $\varphi(x)$ under the integration, then

$$\lambda \left[ A \sin x + B \cos x \right] = \lambda^2 \int_0^{2\pi} (\sin x \cos x' + \cos x \sin x') \left[ A \sin x' + B \cos x' \right] dx'$$

$$= \lambda^2 \sin x \int_0^{2\pi} (A \sin x' \cos x' + B \cos x'^2)\, dx' + \lambda^2 \cos x \int_0^{2\pi} (A \sin x'^2 + B \sin x' \cos x')$$

$$= \lambda^2 B \sin x\, \pi \quad + \lambda^2 A \cos x\, \pi$$

$\therefore \quad \lambda A = \lambda^2 B \pi \quad \& \quad \lambda B = \lambda^2 A \pi \;;$ solving for $A$ & $B$

$$\begin{pmatrix} \lambda & -\lambda^2 \pi \\ -\lambda^2 \pi & \lambda \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \;; \text{ if } A, B \text{ are to be non trivial} \Rightarrow \lambda^2 - (\lambda^4 \pi^2) = 0$$

or $\lambda^2 (1 - \lambda^2 \pi^2) = 0$. if $\lambda = 0 \Rightarrow \varphi(x) \equiv 0 \quad \therefore \quad \lambda = \pm 1/\pi$

$\therefore \quad$ if $\lambda = 1/\pi \qquad A - B = 0 \quad \therefore \quad B = A$ and $\varphi(x) = \frac{1}{\pi} A (\sin x + \cos x)$

if $\lambda = -1/\pi \qquad -A - B = 0 \quad \therefore \quad B = -A$ and $\varphi(x) = -1/\pi\, A (\sin x - \cos x)$

finally

$$\int_0^{2\pi} \cos x' \left[ \pm 1/\pi\, A \left( \sin x' \pm \cos x' \right) \right] dx' = \int_0^{2\pi} \cos x'\, \varphi(x')\, dx'$$

$$= \pm 1/\pi\, A \int_0^{2\pi} (\cos x' \sin x' \pm \cos x'^2)\, dx' = \pm 1/\pi\, A \int_0^{2\pi} \pm \cos x'^2\, dx' = \frac{+A}{\pi} \cdot \pi = A \checkmark$$

similarly one can show that $\int_0^{2\pi} \sin x'\, \varphi(x')\, dx' = B$

And if one puts $\varphi(x')$ in the RHS

$$\pm 1/\pi \int_0^{2\pi} (\sin x \cos x' + \cos x \sin x') (\pm 1/\pi\, A) (\sin x' \pm \cos x')\, dx'$$

$$= 1/\pi^2 A \int_0^{2\pi} \left[ \sin x (\cos x' \sin x' \pm \cos x'^2) + \cos x (\sin x'^2 \pm \sin x' \cos x') \right] dx'$$

$$= 1/\pi^2 A \left[ \sin x (\pm \pi) + \cos x \cdot \pi \right] = \pm 1/\pi\, A \left[ \sin x \pm \cos x \right] = \varphi(x) \quad \text{QED}$$

3. Find the solution to $\varphi(x) = \lambda\int_0^1 K(x,x')\,\varphi(x')\,dx'$ where $K(x,x') = \begin{cases} x' & 0 \le x' < x \\ x & 0 < x < x' \le 1 \end{cases}$

$$\therefore \quad \varphi(x) = \lambda\int_0^x x'\varphi(x')\,dx' + \lambda\int_x^1 x\,\varphi(x')\,dx'$$

$$= \lambda\int_0^x (x'-x)\varphi(x')\,dx' + \lambda\int_0^1 x\,\varphi(x')\,dx'$$

$$= -\lambda\int_0^x (x-x')\,\varphi(x')\,dx' + \lambda\int_0^1 x\,\varphi(x')\,dx'$$ ; this is a volterra type equation.

The first integral is a convolution integral $\int_0^x h(x-x')\,\varphi(x')\,dx'$ where $h(x) = x$

The second integral is equal to $\lambda x C$ where $C = \int_0^1 \varphi(x')\,dx'$. Now taking the laplace transform of both sides gives

$$\Phi(p) = -\lambda\, H(p)\,\Phi(p) + \lambda C\, H(p) \quad \text{where } H(p) = \int_0^\infty e^{-px}\cdot x\,dx = \frac{1}{p^2}$$

$$\therefore \quad \Phi(p) = \frac{\lambda C/p^2}{1+\lambda/p^2} = \frac{\lambda C}{p^2+\lambda}$$ . Now we just need to find the inverse transform of $\Phi(p)$

$$\therefore \quad \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{px}\cdot\frac{\lambda C}{p^2+\lambda}\,dp = \varphi(x) = C\sqrt{\lambda}\,\sin\sqrt{\lambda}\,x \quad \text{for } \lambda > 0$$

if $\lambda < 0$ then let $\mu = -\lambda$ and $\varphi(x) = -C\sqrt{\mu}\,\sinh\sqrt{\mu}\,x$

look at $\lambda > 0$ case

$$\int_0^1 \varphi(x')\,dx' = C\sqrt{\lambda}\int_0^1 \sin\sqrt{\lambda}\,x'\,dx' = C\sqrt{\lambda}\left(-\frac{\cos\sqrt{\lambda}\,x'}{\sqrt{\lambda}}\right)\Big|_0^1 = C\left\{-\cos\sqrt{\lambda}+1\right\};$$

but this must be equal to $C$ by definition. Therefore $\sqrt{\lambda} = \frac{2n+1}{2}\pi \quad n = 0,1,2,\ldots$

and $\lambda = \left(\frac{2n+1}{2}\pi\right)^2$

$$\text{Now} \quad \lambda\int_0^1 K(x,x')\,\varphi(x')\,dx' = \lambda\int_0^x x'\,C\sqrt{\lambda}\,\sin\sqrt{\lambda}\,x'\,dx' + \lambda x\int_x^1 C\sqrt{\lambda}\,\sin\sqrt{\lambda}\,x'\,dx'$$

$$= C\lambda\sqrt{\lambda}\left\{-x'\frac{\cos\sqrt{\lambda}\,x'}{\sqrt{\lambda}}\Big|_0^x + \frac{\sin\sqrt{\lambda}\,x'}{\lambda}\Big|_0^x\right\} + \lambda\sqrt{\lambda}\,Cx\left\{-\frac{\cos\sqrt{\lambda}\,x'}{\sqrt{\lambda}}\Big|_x^1\right\}$$

$$= \cancel{-C\lambda x\cos\sqrt{\lambda}\,x} + C\sqrt{\lambda}\,\sin\sqrt{\lambda}\,x + \cancel{C\lambda x\cos\sqrt{\lambda}\,x} - \lambda Cx\cos\sqrt{\lambda}$$

$$= C\sqrt{\lambda}\,\sin\sqrt{\lambda}\,x - \lambda Cx\cos\sqrt{\lambda}$$

since this must be equal to $\varphi(x) \Longrightarrow \cos\sqrt{\lambda} = 0$, for non trivial solution, as before

thus $\varphi_n(x) = C\left(\frac{2n+1}{2}\pi\right)\sin\left(\frac{2n+1}{2}\pi x\right) \quad$ for $n = 0,1,2,\ldots$

and the most general $\varphi(x) = \sum_{n=0}^{\infty} a_n\varphi_n(x)$

for the case $\lambda < 0$ (ie $\mu > 0$)

$$\int_0^1 \varphi(x')\,dx' = -C\sqrt{\mu}\int_0^1 \sinh\sqrt{\mu}\,x'\,dx' = -C\sqrt{\mu}\left(\frac{\cosh\sqrt{\mu}\,x'}{\sqrt{\mu}}\right)\Bigg|_0^1 = -C\left(\cosh\sqrt{\mu}-1\right)$$

but since this must equal $C$ by definition that implies that $\cosh\sqrt{\mu} = 0$ which is impossible

hence no solutions exist for $\lambda < 0$.

Hence our final result is $\varphi_n(x) = C\left(\frac{2n+1}{2}\pi\right)\sin\left(\frac{2n+1}{2}\pi x\right)$ gives the eigenfunctions,

and the eigenvalues are given by $\lambda = \left(\frac{2n+1}{2}\pi\right)^2$ for $n \geq 0$

Integral Eqns
MA 224 Profla
Cesar Levy Fall8

HW #2

Suppose

$$\frac{1}{\pi i}\int_{-1}^{1}\left\{\frac{1}{x'-x}+k(x,x')\right\}f(x')\,dx' = g(x)$$ . Find $f(x)$ is $k(x,x')$ is small

define $$g(x) - \frac{1}{\pi i}\int_{-1}^{1} k(x,x')\,f(x')\,dx' = G(x) \qquad (1)$$

then $$\frac{1}{\pi i}\int_{-1}^{1}\frac{f(x')}{x'-x}\,dx' = G(x)$$

and $$f(x) = \frac{1}{\pi i}\frac{1}{\sqrt{1-x^2}}\int_{-1}^{1}\frac{G(\xi)\sqrt{1-\xi^2}}{\xi - x}\,d\xi + \frac{A}{\sqrt{1-x^2}} \qquad (2)$$

if $k(x,x')$ is small then $g(x) = G_0(x)$ in (1)

$$\therefore\ f_0(x) = \frac{1}{\pi i}\frac{1}{\sqrt{1-x^2}}\int_{-1}^{1}\frac{g(\xi)\sqrt{1-\xi^2}}{\xi-x}\,d\xi + \frac{A}{\sqrt{1-x^2}}$$

now put this into $$g(x) - \frac{1}{\pi i}\int_{-1}^{1} k(x,x')\,f_0(x')\,dx' = G_1(x)$$

then $$f_1(x) = \frac{1}{\pi i}\frac{1}{\sqrt{1-x^2}}\int_{-1}^{1}\frac{G_1(\xi)\sqrt{1-\xi^2}}{\xi-x}\,d\xi + \frac{A}{\sqrt{1-x^2}}$$

then put this into (1) to get $G_2(x)$ which when placed into (2) gives $f_2(x)$ . Thus

if $$G_n(x) = g(x) - \frac{1}{\pi i}\int_{-1}^{1} k(x,x')\,f_{n-1}(x')\,dx' \quad \& \quad G_0(x) = g(x)$$

then $$f_n(x) = \frac{1}{\pi i}\int\sqrt{\frac{1-\xi^2}{1-x^2}}\,\frac{G_n(\xi)}{\xi-x}\,d\xi + \frac{A}{\sqrt{1-x^2}}$$

and $$f(x) = \lim_{n\to\infty} f_n(x)$$

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MA 224
Integral Eqns
Fall 1980

HW # 3

Show that for $x>0$ $f(x) = \delta'(x) + \alpha\delta(x)$ is a soln to $\int_0^\infty e^{-\alpha|x-x'|} f(x')dx' = 0$

Extend the interval to $(-\infty,\infty)$ and define $f(x)=0$ for $x<0$. Then for $x<0$ the equation is identically satisfied, thus

$$\int_{-\infty}^{\infty} e^{-\alpha|x-x'|}(\delta'(x')+\alpha\delta(x'))dx' = \int_{-\infty}^{x} e^{-\alpha(x-x')}(\delta'(x')+\alpha\delta(x'))dx' + \int_x^\infty e^{-\alpha(x'-x)} f(x')dx'$$

since $\delta'(x)$ & $\delta(x)$ are not zero at $x=0$ then for $x>0$ the only contribution is from the first integral thus

$$\int_{-\infty}^{x} e^{-\alpha(x-x')}[\delta'(x')+\alpha\delta(x')]dx' = -\alpha e^{-\alpha x} + \alpha e^{-\alpha x} = 0 \qquad \text{QED}$$

Using the fact that $\int_{-\infty}^{\infty} e^{p(x')}\delta'(x')dx' = -p'(x')e^{p(x')}\Big|_{x'=0}$ & $\int_{-\infty}^{\infty} e^{p(x')}\delta(x')dx' = e^{p(x')}\Big|_{x'=0}$

Show that for $x>0$ that $\frac{1}{2\alpha}\left\{(\alpha^2-\beta^2)e^{-\beta x} + \beta\delta(x) - \delta'(x)\right\}$ solves $\int_0^\infty e^{-\alpha|x-x'|}f(x')dx' = e^{-\beta x}$

let use extend the interval and define $f(x)=0$ for $x<0$ $\therefore$

$$\int_{-\infty}^{x} e^{-\alpha(x-x')} f(x')dx' + \int_x^\infty e^{-\alpha(x'-x)} f(x')dx'$$

Now for $x \geqslant 0^+$ the $\delta(x)$ & $\delta'(x)$ contrib comes from the first integral only.
Hence

$$\int_{-\infty}^{x} e^{-\alpha(x-x')}[\beta\delta(x')-\delta'(x')]dx' = \beta e^{-\alpha x} + \alpha e^{-\alpha x} = (\beta+\alpha)e^{-\alpha x}$$

also

$$\int_{-\infty}^{x} e^{-\alpha(x-x')}e^{-\beta x'}dx' + \int_x^\infty e^{-\alpha(x'-x)}e^{-\beta x'}dx' = \int_0^x e^{-\alpha(x-x')-\beta x'}dx' + \int_x^\infty e^{-\alpha(x'-x)-\beta x'}dx'$$

$$= \frac{2\alpha}{\alpha^2-\beta^2}e^{-\beta x} - \frac{e^{-\alpha x}}{\alpha-\beta} \qquad \text{for } \alpha>\beta$$

hence $\frac{1}{2\alpha}\left\{(\alpha^2-\beta^2)e^{-\beta x} + \beta\delta(x) - \delta'(x)\right\}$ gives $e^{-\beta x} + \frac{1}{2\alpha}\left\{-(\alpha+\beta)e^{-\alpha x} + (\beta+\alpha)e^{-\alpha x}\right\} = e^{-\beta x}$

as required

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HW #4

$$\bar{G} = g(x,y,-a,0) + \int_0^\infty g(x,y,x',0)\,\sigma(x')\,dx' \quad w/ \quad g(x,y,x',y') = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{i\alpha(x-x') - \sqrt{\alpha^2+\zeta^2}|y-y'|}}{\sqrt{\alpha^2+\zeta^2}}\,d\alpha$$

$$\frac{\partial \bar{G}}{\partial y} = \frac{\partial g}{\partial y} + \int_0^\infty \frac{\partial g}{\partial y}(x,y,x',0)\,\sigma(x')\,dx'$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{i\alpha(x+a) - \sqrt{\alpha^2+\zeta^2}|y|}\,d\alpha + \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_0^\infty e^{i\alpha(x-x') - \sqrt{\alpha^2+\zeta^2}|y|}\,d\alpha\,\sigma(x')\,dx'$$

Now $$\left.\frac{\partial \bar{G}}{\partial y}\right|^{y\to 0^+} = -\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{i\alpha(x+a) - \sqrt{\alpha^2+\zeta^2}\,y}\,d\alpha - \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_0^\infty e^{i\alpha(x-x') - \sqrt{\alpha^2+\zeta^2}\,y}\,dx'\,\sigma(x')\,d\alpha$$

$$\left.\frac{\partial \bar{G}}{\partial y}\right|^{y\to 0^-} = \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{i\alpha(x+a) + \sqrt{\alpha^2+\zeta^2}\,y}\,d\alpha + \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_0^\infty e^{i\alpha(x-x') + \sqrt{\alpha^2+\zeta^2}\,y}\,dx'\,\sigma(x')\,d\alpha$$

now $$\left.\frac{\partial \bar{G}}{\partial y}\right|^{y\to 0^-} - \left.\frac{\partial \bar{G}}{\partial y}\right|^{y\to 0^+} = -\left.\frac{\partial \bar{G}}{\partial y}\right|_{y=0^-}^{y=0^+} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha(x+a)}\,d\alpha + \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_0^\infty e^{i\alpha(x-x')}\,2\cosh\sqrt{\phantom{x}}\,y\,\sigma(x')\,dx'\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha(x+a)}\,d\alpha + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^\infty e^{i\alpha(x-x')}\,\sigma(x')\,dx'\,d\alpha$$

$$= \delta(x+a) + \int_0^\infty \delta(x-x')\,\sigma(x')\,dx'$$

$$= \delta(x+a) + \sigma(x) \quad \text{since } x>0 \ \& \ 0\le x'<\infty$$

Since $x>0$ then $\delta(x+a)=0$ and

then $$-\left.\frac{\partial \bar{G}}{\partial y}\right|_{y=0^-}^{y=0^+} = \sigma(x) \qquad \text{QED}$$

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HW #5

Show that the soln to $0 = \frac{\partial^2}{\partial x^2}\int_0^\infty \varphi(x',0)\left(-\frac{1}{\pi}\ln\left|\frac{x-x'}{x+x'}\right|\right)dx'$ is $\varphi(x,0) = \frac{2}{\pi}\sin^{-1}\left(\frac{x}{a}\right)$ $0<x<a$

if $\varphi(x,0) = 1$ for $x>a$

$$\therefore \quad 0 = \frac{\partial^2}{\partial x^2}\int_0^a \varphi(x',0)\left(-\frac{1}{\pi}\ln\left|\frac{x-x'}{x+x'}\right|\right)dx' + \frac{\partial^2}{\partial x^2}\int_a^\infty -\frac{1}{\pi}\ln\left|\frac{x-x'}{x+x'}\right|dx' \qquad 0<x<a$$

$$= \frac{\partial^2}{\partial x^2}\int_0^a \varphi(x',0)\left(-\frac{1}{\pi}\ln\left|\frac{x-x'}{x+x'}\right|\right)dx' + \frac{2x}{\pi}\int_a^\infty \frac{2x'\,dx'}{(x^2-x'^2)^2}$$

$$\| \qquad + \left(\frac{2x}{\pi}\cdot\frac{1}{x^2-x'^2}\Big|_a^\infty = -\frac{2x}{\pi(x^2-a^2)}\right)$$

or $$-\frac{2x}{\pi(a^2-x^2)} = \frac{\partial^2}{\partial x^2}\int_0^a \varphi(x',0)\left(-\frac{1}{\pi}\ln\left|\frac{x-x'}{x+x'}\right|\right)dx'$$

then integrate once to get $\frac{1}{\pi}\ln(a^2-x^2) + c_1 = \frac{\partial}{\partial x}\int_0^a \varphi(x',0)\left(-\frac{1}{\pi}\ln\left|\frac{x-x'}{x+x'}\right|\right)dx'$.

using $\int \ln(a^2-x^2)\,dx = x\ln(a^2-x^2) - 2x + a\ln\left|\frac{a+x}{a-x}\right|$ we get

$$\frac{1}{\pi}\left[x\ln(a^2-x^2) - 2x + a\ln\left|\frac{a+x}{a-x}\right|\right] + c_1x + c_2 = \frac{1}{\pi}\int_0^a \varphi(x',0)\ln\left|\frac{x+x'}{x-x'}\right|dx'$$

now let $x'' = x'/a$ $\quad \bar{x} = x/a$ $\quad$ then $\varphi(x',0)$ is replaced by $\bar{\varphi}(x'',0)$ and

$$\frac{1}{\pi}\left[a\bar{x}\ln a^2(1-\bar{x}^2) - 2a\bar{x} + a\ln\left|\frac{1+\bar{x}}{1-\bar{x}}\right|\right] + c_1a\bar{x} + c_2 = \frac{a}{\pi}\int_0^1 \bar{\varphi}(x'',0)\ln\left|\frac{\bar{x}+x''}{\bar{x}-x''}\right|dx''$$

$$\therefore \quad \frac{1}{\pi}\left[\bar{x}\ln a^2 + \bar{x}\ln(1-\bar{x}^2) - 2\bar{x} + \ln\left|\frac{1+\bar{x}}{1-\bar{x}}\right|\right] + c_1\bar{x} + \frac{c_2}{a} = \frac{1}{\pi}\int_0^1 \bar{\varphi}(x'',0)\ln\left|\frac{\bar{x}+x''}{\bar{x}-x''}\right|dx''$$

hence since

$$\frac{1}{\pi}\int_0^1 \ln\left|\frac{\bar{x}+\xi}{\bar{x}-\xi}\right| f(\xi)\,d\xi = g(\bar{x}) \quad \text{has a soln} \quad f(\bar{x}) = -\frac{2}{\pi}\frac{\bar{x}}{\sqrt{1-\bar{x}^2}}\int_0^1 \frac{\sqrt{1-\xi^2}}{\xi^2-\bar{x}^2}g'(\xi)\,d\xi + \frac{2}{\pi}\frac{g(0)}{\bar{x}\sqrt{1-\bar{x}^2}}$$

then

by defining $\frac{1}{\pi}\left[\bar{x}\ln a^2 + \bar{x}\ln(1-\bar{x}^2) - 2\bar{x} + \ln\left|\frac{1+\bar{x}}{1-\bar{x}}\right|\right] + c_1\bar{x} + \frac{c_2}{a} = g(\bar{x})$

$$g(0) = \frac{c_2}{a} \qquad g'(\bar{x}) = \frac{1}{\pi}\left[\ln a^2 + \ln(1-\bar{x}^2) - 2 - \frac{2\bar{x}^2+2}{1-\bar{x}^2}\right] + c_1 = \frac{1}{\pi}\left[\ln a^2 + \ln(1-\bar{x}^2)\right] + c_1$$

$$= \frac{1}{\pi}\ln(1-\bar{x}^2) + \mathscr{C}_1 \quad \text{where} \quad \mathscr{C}_1 = \frac{1}{\pi}\left[\ln a^2\right] + c_1$$

$$\therefore \varphi(x,0) = \bar{\varphi}(\bar{x},0) = -\frac{2}{\pi}\frac{\bar{x}}{\sqrt{1-\bar{x}^2}}\int_0^1 \frac{\sqrt{1-\xi^2}}{\xi^2-\bar{x}^2}\left\{\frac{1}{\pi}\ln(1-\xi^2) + \mathscr{C}_1\right\}d\xi + \frac{2}{\pi}\frac{c_2}{a}\frac{1}{\bar{x}\sqrt{1-\bar{x}^2}}$$

now by replacing $\bar{x}$ by $\frac{x}{a}$ and $x^2$ by $\zeta$ also $\xi$ by $\frac{y}{a}$ and $y^2$ by $\mu$

$$\varphi(\zeta,0) = -\frac{1}{\pi}\frac{\sqrt{\zeta}}{\sqrt{a^2-\zeta}}\int_0^{a^2}\frac{\sqrt{a^2-\mu}}{\mu-\zeta}\left\{\frac{1}{\pi}\ln(a^2-\mu) + C_1\right\}\frac{d\mu}{\sqrt{\mu}} + \frac{2}{\pi}\frac{c_2 a}{\sqrt{\zeta(a^2-\zeta)}}$$

originally $\varphi(\zeta=0,0) = 0 \Rightarrow -\frac{1}{\pi}\zeta\int_0^{a^2}\frac{\sqrt{a^2-\mu}}{\sqrt{\mu}}\left\{\frac{1}{\pi}\ln(a^2-\mu) + C_1\right\}d\mu + \frac{2a}{\pi}C_2 = 0 \Rightarrow C_2 = 0$

$\varphi(\zeta=a^2,0) = 1 \Rightarrow 0 = +\frac{1}{\pi}\sqrt{\zeta}\int_0^{a^2}\frac{\sqrt{a^2-\mu}}{a^2-\mu}\left\{\frac{1}{\pi}\ln(a^2-\mu) + C_1\right\}\frac{d\mu}{\sqrt{\mu}}$ $\zeta=0$

we know that when $\bar{x}=0$ $\varphi(0,0)=0$ and that when $\bar{x}=1$ $\varphi(1,0)=1$

thus $$\bar{x}\sqrt{1-\bar{x}^2}\;\bar{\varphi}(\bar{x},0) = -\frac{2}{\pi^2}\bar{x}^2\int_0^1 \frac{\sqrt{1-\xi^2}}{\xi^2-\bar{x}^2}\ln(1-\xi^2)\,d\xi - \frac{2}{\pi}\bar{x}^2\int_0^1 \frac{\sqrt{1-\xi^2}}{\xi^2-\bar{x}^2}\left(\frac{1}{\pi}\ln a^2 + C_1\right)d\xi + \frac{2C_2}{\pi a}.$$

let's look at
$$\int_0^1 \frac{\sqrt{1-\xi^2}}{\xi^2-\bar{x}^2}\,d\xi = \frac{1}{2}\int_{-1}^{1}\frac{\sqrt{1-\xi^2}}{\xi^2-\bar{x}^2}\,d\xi \quad \text{and let } \xi=\cos\theta \quad \bar{x}=\cos\delta \text{ ; then}$$

$$= \frac{1}{2}\int_\pi^0 \frac{-\sin^2\theta\,d\theta}{\cos^2\theta-\cos^2\delta} = \frac{1}{2}\int_0^\pi \frac{\sin^2\theta\,d\theta}{\cos^2\theta-\cos^2\delta} = \frac{1}{2}\,\frac{(1-\cos^2\delta)}{2\cos\delta}\int_0^\pi\left(\frac{d\theta}{\cos\theta-\cos\delta}-\frac{d\theta}{\cos\theta+\cos\delta}\right)-\frac{\pi}{2}$$

$$= \left(\frac{1}{2}\,\frac{1-\cos^2\delta}{2\cos\delta}\right)\frac{1}{2}\int_{-\pi}^{\pi}\left(\frac{d\theta}{\cos\theta-\cos\delta}-\frac{d\theta}{\cos\theta+\cos\delta}\right)-\frac{\pi}{2} = \frac{1}{2}\left(\frac{1-\bar{x}^2}{2\bar{x}}\right)$$

by using complex representation & integrating in the complex plane then
$$\frac{1}{2}\int_{-\pi}^{\pi}\frac{d\theta}{\cos^2\theta-\cos^2\delta} = \frac{1}{i}\oint_{|z|=1}\left[\frac{dz}{(z-e^{i\delta})(z-e^{-i\delta})} - \frac{dz}{(z+e^{i\delta})(z+e^{-i\delta})}\right] = -\frac{1}{2\sin\delta}\cdot\left[(\pi i-\pi i)+(\pi i-\pi i)\right]=0$$

hence $\displaystyle\int_0^1 \frac{\sqrt{1-\xi^2}}{\xi^2-\bar{x}^2}\,d\xi = -\pi/2$ for $\bar{x}\neq 0$ ; $\displaystyle\int_0^1\frac{\sqrt{1-\xi^2}}{\xi^2-\bar{x}^2}\ln(1-\xi^2)\,d\xi$ for $\bar{x}=0$ is bounded

hence the two integrals are finite and $\therefore C_2=0$ when $\bar{x}=0$

at $\bar{x}=1$ then $$\bar{x}\sqrt{1-\bar{x}^2}\;\bar{\varphi}(\bar{x},0) = 1\cdot 0\cdot 1 = 0 = +\frac{2}{\pi^2}\int_0^1\frac{\ln(1-\xi^2)\,d\xi}{\sqrt{1-\xi^2}} + \frac{2}{\pi}\int_0^1\frac{d\xi}{\sqrt{1-\xi^2}}\left(\frac{1}{\pi}\ln a^2+C_1\right)$$

now if $1-\xi^2=u^2$ $\quad -2\xi\,d\xi = 2u\,du \Rightarrow d\xi = \dfrac{u\,du}{-\sqrt{1-u^2}}$ $\quad\therefore \displaystyle\int_0^1\frac{\ln(1-\xi^2)\,d\xi}{\sqrt{1-\xi^2}} = -\int_1^0\frac{2\ln u\cdot u\,du}{u\sqrt{1-u^2}}$

$$= 2\int_0^1\frac{\ln u}{\sqrt{1-u^2}}\,du = \pi\log 2 \qquad \text{and} \int_0^1\frac{d\xi}{\sqrt{1-\xi^2}} = \arcsin\xi\Big|_0^1 = \pi/2$$

$$\therefore \quad 0 = \frac{2}{\pi^2}\cdot\pi\log 2 + \frac{2}{\pi}\cdot\frac{\pi}{2}\left(\frac{1}{\pi}\ln a^2+C_1\right) = \frac{2}{\pi}\log 2 + \frac{1}{\pi}\ln a^2 + C_1$$

$$\therefore \quad C_1 = -\frac{1}{\pi}\ln 4a^2 \qquad \text{and } C_1 + \frac{1}{\pi}\ln a^2 = -\frac{2}{\pi}\log 2$$

and $$\bar{x}\sqrt{1-\bar{x}^2}\;\bar{\varphi}(\bar{x},0) = -\frac{2}{\pi^2}\bar{x}^2\int_0^1\frac{\sqrt{1-\xi^2}}{\xi^2-\bar{x}^2}\ln(1-\xi^2)\,d\xi + \frac{4}{\pi^2}\log 2\,\bar{x}^2\int_0^1\frac{\sqrt{1-\xi^2}}{\xi^2-\bar{x}^2}\,d\xi$$

$$= -\frac{2}{\pi^2}\bar{x}^2\int_0^1\frac{\sqrt{1-\xi^2}}{\xi^2-\bar{x}^2}\ln(1-\xi^2)\,d\xi \qquad \text{since } \int_0^1\frac{\sqrt{1-\xi^2}}{\xi^2-\bar{x}^2}\,d\xi = 0$$

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# SOME AXIALLY SYMMETRIC POTENTIAL PROBLEMS

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**Abstract**

Some axially symmetric boundary value problems of potential theory are formulated as integral equations of the first kind. In each case the kernel admits an expansion, for small values of a parameter of the problem, that leads to an approximate integral equation whose solution provides a direct asymptotic estimate for the physical quantity of primary interest. A manipulation of the original and modified integral equations provides an efficient formula for calculating higher order terms in the asymptotic expansion.

## 1. Introduction

A variety of methods is available for the analysis of boundary value problems in axisymmetric potential theory (cf. Sneddon (**10**)), though explicit solutions seldom result and practical objectives are met by establishing the initial terms of series expansions for individual cases. If more than one characteristic length is involved in the geometrical description of a problem, expansions for an integrated quantity such as the total charge or flux take on distinctive forms according as these lengths are comparable in magnitude or not. Expansions of an irregular type, involving powers and logarithms of a small parameter (which is related to the difference of a pair of lengths, say) are more difficult to specify and relatively few representatives from this category are known with any degree of completeness.

It is our intention in this paper to show that details of irregular or singular perturbation expansions for linear boundary value problems of potential theory are furnished without elaborate calculation if a suitable integral equation formulation is devised at the outset; a stream function is sometimes appropriate for the latter purpose and integral equations on infinite rather than finite intervals may be used to advantage.

The particular integral equations which form the basis for our analysis are all of the first kind and, despite the unavailability of exact solutions, prove to be advantageous insofar as

(1) the quantity of interest for each problem, which is related to an integral of the solution, appears explicitly and linearly therein, and

(2) the kernels admit expansion for small values of the relevant parameter so that analytic approximation can be obtained directly or the quantity of interest estimated by comparison with a fully determinate integral equation of independent origin.

When a first approximation to the solution of a basic integral equation is known, with a concomitant evaluation for the quantity of interest, an efficient representation of the difference between this estimate and the exact value, obtained by manipulation of the original and modified integral equation, facilitates the derivation of additional or higher order terms in the sought-for expansion.

The technique is applied to several boundary value problems in what follows, corroborating or extending results obtained by Spence (**11, 12**) through intricate transform analysis for two planar configurations that involve an annular disc or gap, confirming a prediction by Kirchhoff (**5**) for the thickness effect in a circular disc condenser and complementing a calculation by Fock (**3**) which pertains to the conductivity of a circular aperture in a transverse section of a cylindrical tube.

## 2. The electrified annular disc

An annular disc of inner and outer radii $a$ and $b$ is held at unit potential, and it is required to find its capacity when $a$ and $b$ are nearly equal. If polar coordinates $(r, \theta, z)$ are chosen so that the disc occupies the region $z = 0$, $a \leqq r \leqq b$, then the symmetry of the configuration clearly implies that the potential $\phi(r, z)$ is independent of $\theta$, and is an even function of $z$. Thus we may confine our attention to the half-space $z \geqq 0$, with $\phi(r, z)$ specified as the function that vanishes at infinity and satisfies the conditions

$$\left.\begin{aligned} &\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\phi = 0, \quad z \geqq 0 \\ &\phi = 1, \quad z = 0,\ a<r<b \\ &\phi_z = 0, \quad z = 0,\ r<a \text{ or } r>b, \end{aligned}\right\} \tag{2.1}$$

where the notation $\phi_z$ means $\partial\phi/\partial z$.

The problem can readily be reduced to that of an integral equation of the first kind, by applying Green's theorem to the potential $\phi(r, z)$ and a Green's function $G(\mathbf{r}; \mathbf{r}_1)$ chosen to have vanishing normal derivative on the plane $z = 0$. Thus

$$\phi(r_1, z_1) = \int_0^{2\pi} d\theta \int_a^b rG(r, \theta, 0;\ r_1, \theta_1, z_1)\phi_z(r, 0)dr, \quad z_1>0 \tag{2.2}$$

where

$$G = -\frac{1}{4\pi R_+} - \frac{1}{4\pi R_-};\ R_\pm = (r^2+r_1^2-2rr_1\cos(\theta-\theta_1)+(z\pm z_1)^2)^{\frac{1}{2}}. \tag{2.3}$$

Formula (2.2) gives the potential at any point $r_1$ in terms of the unknown charge density

$$\sigma(r) = -\frac{1}{4\pi}\phi_z(r, 0) \tag{2.4}$$

on the upper surface. In particular, if $r_1$ lies on this surface then $\phi$ assumes its prescribed value unity, and we have the integral equation

$$1 = \int_a^b K(r, r_1) r\sigma(r) dr, \quad a<r_1<b, \tag{2.5}$$

for the charge $\sigma$, where the kernel $K(r, r_1)$ is given by

$$\begin{aligned} K(r, r_1) &= 2\int_0^{2\pi} (r^2+r_1^2-2rr_1\cos\theta)^{-\frac{1}{2}} d\theta \\ &= \frac{8}{(r+r_1)}\,\mathbf{K}\left(\frac{2r^{\frac{1}{2}}r_1^{\frac{1}{2}}}{r+r_1}\right) \end{aligned} \tag{2.6}$$

in terms of the elliptic integral

$$\mathbf{K}(\alpha) = \int_0^{\pi/2} (1-\alpha^2\sin^2\theta)^{-\frac{1}{2}} d\theta. \tag{2.7}$$

The capacity $C$ of the annulus is equal to the total charge divided by the potential; thus

$$C = 2\int_a^b 2\pi r\sigma(r) dr. \tag{2.8}$$

Now the integral equation (2.5), which is valid for all values of the parameters $a$ and $b$, is particularly well suited to the case of interest here, with the two radii nearly equal. To exploit the fact that the pair of variables $r$ and $r_1$ appearing in (2.5) differ little from $a$, we define the small dimensionless parameter

$$\varepsilon = (b/a)-1, \tag{2.9}$$

and rescale the variables by the transformation

$$r = a(1+\varepsilon x), \quad r_1 = a(1+\varepsilon x_1), \tag{2.10}$$

so that $x$ and $x_1$ vary between 0 and 1. It is also convenient to normalise the charge density according to the formula

$$V(x) = \frac{4\pi a\varepsilon}{C} r\sigma(r) = \frac{4\pi a\varepsilon}{C}(a+a\varepsilon x)\sigma(a+a\varepsilon x). \tag{2.11}$$

In terms of the function $V(x)$, the integral equation (2.5) and the expression (2.8) take the form

$$\frac{4\pi}{C} = \int_0^1 K(x, x_1)V(x) dx, \quad 0<x_1<1, \tag{2.12}$$

and

$$1 = \int_0^1 V(x)dx, \tag{2.13}$$

where $K(x, x_1)$ denotes $K(r, r_1)$ with $r = 1+\varepsilon x$ and $r_1 = 1+\varepsilon x_1$.

The kernel $K$, given by (2.6), can readily be expanded for small values of $\varepsilon$, using the known expansion (cf. Gradstein and Rhysik (9)) for the elliptic integral of order close to unity; an expansion of this type is carried out in a similar problem by Grinberg and Kuritsyn (4). It is found that, as $\varepsilon \to 0$,

$$K(x, x_1) \sim -\frac{4}{a}\left\{\log \varepsilon + \log \frac{|x-x_1|}{8} - \tfrac{1}{2}(x+x_1)\varepsilon \log \varepsilon - \tfrac{1}{2}\varepsilon \log \frac{|x-x_1|\, e}{8}\right\} + o(\varepsilon). \tag{2.14}$$

A first approximation $V_0(x)$ for the unknown function $V(x)$, and a corresponding estimate $C_0$ for the capacity, is now obtained from the governing equations (2.12, 2.13) on replacing the exact kernel $K$ by the first pair of terms in the uniformly valid expansion (2.14). If we define

$$K_0(x, x_1) = -\frac{4}{a}\left\{\log \varepsilon + \log \frac{|x-x_1|}{8}\right\}, \tag{2.15}$$

then $V_0(x)$ and $C_0$ satisfy the equation

$$\frac{4\pi}{C_0} = \int_0^1 K_0(x, x_1)V_0(x)dx, \quad 0<x_1<1, \tag{2.16}$$

i.e.

$$\frac{a}{C_0} + \frac{1}{\pi}\log\frac{\varepsilon}{8} = -\frac{1}{\pi}\int_0^1 V_0(x)\log|x-x_1|\,dx, \tag{2.17}$$

if $V_0(x)$ is normalised to satisfy the scaling condition (2.13).

Such an integral equation, with a logarithmic kernel, can readily be inverted: its solution, given by several authors (cf. Cooke (2)) is

$$V_0(x) = \frac{1}{\log 4}\left\{\frac{a}{C_0} + \frac{1}{\pi}\log\frac{\varepsilon}{8}\right\} x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}, \tag{2.18}$$

whence integration between 0 and 1, and use of the condition (2.13), leads to the results

$$C_0^{-1} = (\pi a)^{-1}\log(32/\varepsilon), \tag{2.19}$$

and

$$V_0(x) = \pi^{-1}x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}. \tag{2.20}$$

An improved estimate for the function $V(x)$, and hence for the capacity $C$ can in principle be obtained from the governing equations (2.12, 2.13), on the basis of an iteration scheme using $V_0(x)$ as a first approximation. In particular, since the kernel $K$ is seen from (2.14) to differ from $K_0$ by a term of order $\varepsilon \log \varepsilon$, such a scheme readily provides the information that the next term in the expansion of $V(x)$ is also of the order $\varepsilon \log \varepsilon$. Higher order terms are obtained along similar lines in a related low frequency wave problem by Grinberg and Kuritsyn (**4**).

A more efficient procedure for gaining a direct improvement on our estimate (2.19) for $C$ is now shown to be possible, using only the first approximation $V_0(x)$ together with the exact equations (2.12), (2.16) satisfied by $V(x)$ and $V_0(x)$. For if (2.16) is multiplied by $V(x_1)$ and integrated from 0 to 1, then

$$\frac{4\pi}{C_0} = \int_0^1 \int_0^1 K_0(x, x_1)V_0(x)V(x_1)dxdx_1,$$

on account of the scaling condition (2.13). Similarly, if (2.12) is multiplied by $V_0(x_1)$,

$$\frac{4\pi}{C} = \int_0^1 \int_0^1 K(x, x_1)V(x)V_0(x_1)dxdx_1,$$

whence, on subtraction, the symmetry of $K(x, x_1)$ yields the formula

$$\frac{1}{C} - \frac{1}{C_0} = \frac{1}{4\pi}\int_0^1 \int_0^1 V_0(x)V(x_1)\{K(x, x_1)-K_0(x, x_1)\}dxdx_1; \quad (2.21)$$

this is an *exact* expression for the difference between $C$ and its estimate $C_0$, given by (2.19).

A first approximation for the error (2.21) is obviously obtained by replacing $V(x_1)$ by $V_0(x_1)$, and using the expansion (2.14) to estimate $K-K_0$. Further, since $V$ differs from $V_0$ by a term of order $\varepsilon \log \varepsilon$, the form of the expansion (2.14) for $K(x, x_1)$ shows that the first *two* terms of the integral (2.21) are obtained by replacing $V$ by $V_0$ and using the two leading terms in the expansion for $K-K_0$. Since $C_0^{-1}$ already contains two terms in the expansion for the capacity, the formula (2.21) is seen to be very efficient in providing the *four* leading terms for $C^{-1}$ in terms of only the *first* approximation $V_0(x)$ for $V$.

Thus from (2.20), and (2.14, 2.15), we get

$$\frac{1}{C} - \frac{1}{C_0} = \frac{1}{2a\pi^3}\int_0^1 \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}x_1^{-\frac{1}{2}}(1-x_1)^{-\frac{1}{2}}\left\{(x+x_1)\varepsilon \log \varepsilon \right.$$

$$\left. +\varepsilon \log \frac{|x-x_1|\, e}{8} + o(\varepsilon)\right\} dxdx_1 = \frac{1}{2a\pi^3}\{\pi^2\varepsilon \log \varepsilon + \varepsilon\pi^2 \log (e/32)\} + o(\varepsilon).$$

Finally, from (2.19),

$$\frac{1}{C} = \frac{1}{2\pi a}\left\{2 \log \frac{32}{\varepsilon} + \varepsilon \log \varepsilon + \varepsilon \log \frac{e}{32}\right\} + o(\varepsilon), \quad (2.22)$$

which is in agreement with the result due to Spence (**11**), who expresses his result in non-dimensional form by multiplying by the known capacity $C^* = 2b/\pi = 2a(1+\varepsilon)/\pi$ for the disc of radius $b$. Thus

$$C^*/C = \pi^{-2}\{-2 \log \varepsilon + 2 \log 32 - \varepsilon \log \varepsilon + \varepsilon \log (32e) + o(\varepsilon)\}. \quad (2.23)$$

**3. An electrified circular disc with an infinite coplanar screen**

A complementary problem is that posed by Spence (**12**), in which a disc of radius $a$, held at unit potential, is separated by an annular gap of width $b-a$ from an earthed screen that extends from $r = b$ to $r = \infty$. The symmetry about the plane $z = 0$ containing the plates leads to a boundary value problem for the axisymmetric potential function $\phi(r, z)$ in the half-space $z \geqq 0$. Thus

$$\left.\begin{aligned} \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\phi = 0, &\quad z \geqq 0 \\ \phi = 1, &\quad z = 0,\ r<a, \\ \phi_z = 0, &\quad z = 0,\ a<r<b, \\ \phi = 0, &\quad z = 0,\ r>b, \end{aligned}\right\} \tag{3.1}$$

and we are again concerned with finding the capacity of the system when the gap is small, i.e. for small values of the parameter $\varepsilon = (b/a)-1$.

Our aim is to reduce the problem (3.1) to an integral equation of the first kind, over the region $a<r<b$, so that an approximate solution may be obtained by the procedure of the previous section. Such an equation is most readily derived by introducing a suitable *stream function* for the problem, in the manner described in (**7**) and (**8**). This function $\psi(r, z)$ is defined by the formulae

$$r\phi_r = r\psi_z \quad \text{and} \quad r\phi_z = -\frac{\partial}{\partial r}(r\psi), \tag{3.2}$$

with $\psi = 0$ when $r = 0$, $z = 0$, this implying that $\psi(0, z)$ is zero for all positive $z$.

The boundary value problem for $\psi(r, z)$ resulting from (3.1) and (3.2) has the specifications

$$\left.\begin{aligned} \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}-\frac{1}{r^2}\right)\psi = 0, &\quad z \geqq 0 \\ \psi_z = 0, &\quad z = 0,\ r<a, \\ \psi = C/r, &\quad z = 0,\ a<r<b, \\ \psi_z = 0, &\quad z = 0,\ r>b; \end{aligned}\right\} \tag{3.3}$$

the constant $C$ is an unknown of the problem, to be found from the scaling condition that the potential assumes the constant value unity on the inner disc, whence

$$-1 = \int_a^b \phi_r(r, 0)dr = \int_a^b \psi_z(r, 0)dr. \tag{3.4}$$

An essential feature of this formulation is that the constant $C$ appearing in the specifications (3.3) is the numerical value of the capacity of the system. For the capacity, which is the total charge on the disc divided by the potential difference, is therefore equal to twice the charge on the upper surface $r<a$, $z=0$, whence

$$\text{capacity} = -\frac{2}{4\pi}\int_0^a \phi_z(r, 0)2\pi r dr = \int_0^a \frac{\partial}{\partial r}(r\psi)dr = C. \tag{3.5}$$

The stream function $\psi(r, z)$ at any point $z \geqq 0$ can be represented in terms of the normal derivative $\psi_z(r, 0) = \phi_r(r, 0)$, $a<r<b$, through the use of Green's theorem applied to $\psi(r, z)$ and a Green's function $G(\mathbf{r};\ \mathbf{r}_1)$ that has zero normal derivative across the plane $z=0$.
Thus

$$\psi(r_1, z_1) = \int_0^{2\pi} d\theta \int_a^b G(r, \theta, 0;\ r_1, \theta_1, z_1)\psi_z(r, 0)rdr, \quad z_1>0$$

$$= \int_a^b g(r, 0;\ r_1, z_1)\psi_z(r, 0)rdr, \qquad z_1>0,$$

where $g(r, z;\ r_1, z_1)$ is specified by the conditions

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}-\frac{1}{r^2}\right)g = \frac{1}{r}\delta(r-r_1)\delta(z-z_1), \quad g_z(r, 0;\ r_1, z_1) = 0. \tag{3.6}$$

In particular, if $a<r_1<b$ and $z_1\to 0$, then $\psi$ is seen from (3.3) to take the value $C/r_1$, and the radial electric field in the gap, $\psi_z(r, 0) = \phi'(r)$, satisfies the integral equation

$$C = \int_a^b K(r, r_1)\phi'(r)dr, \quad a<r_1<b. \tag{3.7}$$

The symmetric kernel $K(r, r_1) = rr_1 g(r, 0;\ r_1, 0)$ is given in (7) by the formula

$$K(r, r_1) = \frac{2}{\pi}r_> \left\{\mathbf{E}\left(\frac{r_<}{r_>}\right) - \mathbf{K}\left(\frac{r_<}{r_>}\right)\right\}, \tag{3.8}$$

where $r_<$ and $r_>$ denote the lesser and greater of the variables $r$ and $r_1$; $\mathbf{E}$ and $\mathbf{K}$ are complete elliptic integrals, $\mathbf{K}$ being given by (2.7) and

$$\mathbf{E}(\alpha) = \int_0^{\pi/2} (1-\alpha^2\sin^2\theta)^{\frac{1}{2}}d\theta. \tag{3.9}$$

Equation (3.7) and the scaling condition (3.4) are of the same type as the corresponding pair (2.12, 2.13) discussed in section 2, and can be dealt with by exactly the same procedure for small values of $\varepsilon = (b/a)-1$. Thus if the rescaled variables $(x, x_1)$ of (2.10) are used in place of $(r, r_1)$, and if we define

$$U(x) = -a\varepsilon\phi'(a+a\varepsilon x), \tag{3.10}$$

then the governing equations (3.7, 3.4) become

$$-C = \int_0^1 K(x, x_1)U(x)dx, \quad 0<x_1<1, \tag{3.11}$$

with

$$1 = \int_0^1 U(x)dx. \tag{3.12}$$

The kernel $K(x, x_1)$, which means $K(r, r_1)$ written in terms of $(x, x_1)$, can easily be expanded for small values of $\varepsilon$, making use of the known results (cf. Gradstein and Rhysik (**9**)) for the elliptic integrals.

It is found that

$$K = K_0 + \frac{a}{2\pi}\left\{(x+x_1)\varepsilon \log \varepsilon + \varepsilon(x+x_1)\log\frac{|x-x_1|e}{8} + o(\varepsilon)\right\}, \tag{3.13}$$

where

$$K_0 = \frac{a}{\pi}\log\frac{\varepsilon e^2}{8} + \frac{a}{\pi}\log|x-x_1|. \tag{3.14}$$

Proceeding as in section 2, a first approximation $U_0(x)$, and $C_0$, is obtained by replacing $K$ by $K_0$ in the integral equation (3.11). Thus

$$\left.\begin{aligned} -C_0 &= \int_0^1 K_0(x, x_1)U_0(x)dx \\ \text{and} \qquad 1 &= \int_0^1 U_0(x)dx, \end{aligned}\right\} \tag{3.15}$$

with the solution

$$U_0(x) = \frac{1}{\log 4}\left\{\frac{C_0}{a} + \frac{1}{\pi}\log\frac{\varepsilon e^2}{8}\right\}x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} = \frac{1}{\pi}x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}, \tag{3.16}$$

and

$$C_0 = \frac{a}{\pi}\left\{-\log\varepsilon + \log\frac{32}{e^2}\right\}. \tag{3.17}$$

An improved estimate follows, as in section 2, from the exact formula

$$C - C_0 = -\int_0^1\int_0^1 (K-K_0)U(x_1)U_0(x)dx dx_1, \tag{3.18}$$

whence from (3.13) and (3.16),

$$C - C_0 = -\frac{a}{2\pi^3}\int_0^1\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}x_1^{-\frac{1}{2}}(1-x_1)^{-\frac{1}{2}}$$
$$\left\{(x+x_1)\varepsilon\log\varepsilon + \varepsilon(x+x_1)\log\frac{|x-x_1|e}{8} + o(\varepsilon)\right\}dx dx_1. \tag{3.19}$$

Finally then, from (3.17) and (3.19), we get

$$C = \frac{a}{\pi}\left\{-\log\varepsilon + \log\frac{32}{e^2} - \tfrac{1}{2}\varepsilon\log\varepsilon - \tfrac{1}{2}\varepsilon\log\frac{e}{32} + o(\varepsilon)\right\} \tag{3.20}$$

in which the first two terms are those obtained by Spence (**12**).

**4. A condenser with parallel circular discs of finite thickness**

Two identical circular discs, each of unit radius and thickness $h$, are held at equal and opposite potentials $+1$ and $-1$, and form a condenser whose capacity $C$ is to be found when the separation $2\varepsilon$ and thickness $h$ are small compared with the radius.

An earlier analysis (7), dealing with the case of zero thickness, $h = 0$, has provided the first few terms of the asymptotic expansion for $C$, in agreement with results obtained by Kirchhoff (5) and others. The approach developed in (7), which is now extended to include finite thickness, is to exploit the fact that the potential $\phi$ near the end faces closely resembles that of the corresponding two-dimensional problem involving a pair of semi-infinite plates. An exact integral equation formulation, to determine the potential, is derived and compared with the corresponding equations for the analogous two-dimensional problem; a direct comparison of the respective equations is sufficient to determine the capacity to a high degree of accuracy.

The symmetry of the problem implies that the potential $\phi$ is zero over the plane midway between the discs, so that we may confine our attention to the half-space above this plane. A polar coordinate system $(r, \theta, z)$ is chosen so that the symmetry plane is $z = -\varepsilon$, with the disc at potential $+1$ occupying the region $r \leqq 1$, $0 \leqq z \leqq h$. The boundary value problem to determine the potential $\phi(r, z)$ is now reduced to that of a pair of coupled integral equations over the domains $z = 0$, $r > 1$ and $z = h$, $r > 1$, again by introducing a suitable stream function for the problem.

Working in the half-space $z \geqq -\varepsilon$, it is convenient to subdivide into the three regions $z \geqq h$, $0 \leqq z \leqq h$, and $-\varepsilon \leqq z \leqq 0$, these being denoted respectively by the subscripts 1, 2 and 3 (Fig. 1). The *stream function* $\psi(r, z)$ is defined by the formulae

$$r\phi_r = r\psi_z, \quad r\phi_z = -\frac{\partial}{\partial r}(r\psi), \tag{4.1}$$

with constants of integration chosen so that

$$\psi_1(0, h) = 0, \quad \psi_3(0, 0) = 0, \quad \text{and} \quad \lim_{r\to\infty} r\psi_2(r, z) = 0. \tag{4.2}$$

The boundary value problem for $\psi(r, z)$ is given by

$$\left.\begin{aligned} &\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)\psi = 0, \\ &\quad \psi_z = 0,\ r \leqq 1,\ z = 0 \text{ or } z = h, \\ &\quad \psi_z = 0,\ z = -\varepsilon, \\ &\frac{\partial}{\partial r}(r\psi) = 0,\ r = 1,\ 0 \leqq z \leqq h. \end{aligned}\right\} \tag{4.3}$$

The choice of constants of integration defined by (4.2) implies that $r\psi$ is discontinuous across the planes $z = h$ and $z = 0$,

i.e. $$\psi_1-\psi_2 = A/r, \quad z = h, \quad r \geqq 1, \tag{4.4}$$

and $$\psi_2-\psi_3 = B/r, \quad z = 0, \quad r \geqq 1. \tag{4.5}$$

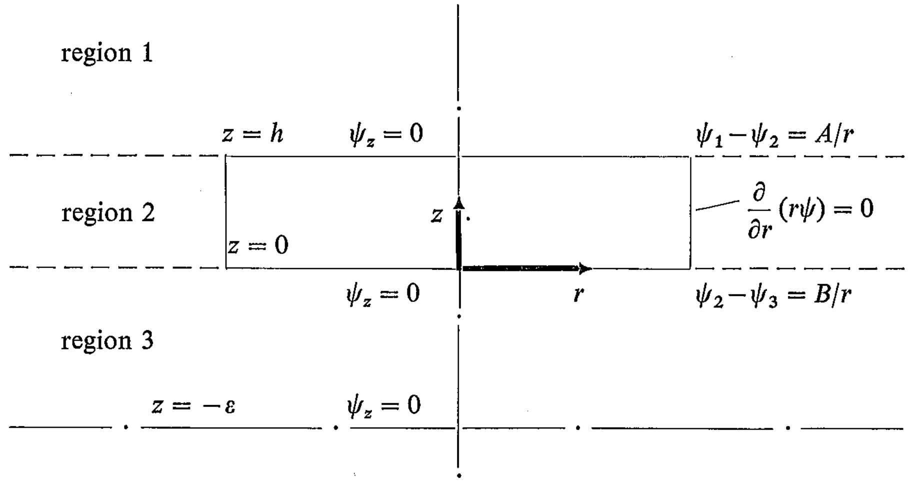


FIG. 1

The constants $A$ and $B$ are to be found, and are related to the capacity $C$ of the system. For the total charge on the upper surface is

$$\Sigma_1 = -\int_0^1 \frac{1}{4\pi}\frac{\partial\phi}{\partial z}(r, h)2\pi r dr = \tfrac{1}{2}\int_0^1 \frac{\partial}{\partial r}(r\psi_1)dr = \tfrac{1}{2}\psi_1(1, h),$$

and the charge on the lower surface is given similarly as

$$\Sigma_3 = -\tfrac{1}{2}\psi_3(1, 0),$$

while the curved surface, $r = 1$, $0 \leqq z \leqq h$, carries a charge

$$\Sigma_2 = -2\pi\int_0^h \frac{1}{4\pi}\frac{\partial\phi}{\partial r}(1, z)dz = -\tfrac{1}{2}\{\psi_2(1, h)-\psi_2(1, 0)\}.$$

It follows that the capacity $C = \frac{1}{2}(\Sigma_1+\Sigma_2+\Sigma_3)$ is related to the constants $A$ and $B$ by the formula

$$C = \tfrac{1}{4}(A+B). \tag{4.6}$$

*Integral equation formulation.* A representation for the stream function $\psi_1(r_1, z_1)$, for any point in region 1, $(z_1 \geqq h)$ is obtained by choosing a Green's function $g_1(r, z;\ r_1, z_1)$ that vanishes at infinity and has zero normal derivative on the plane $z = h$. Specifically,

$$\psi_1(r_1, z_1) = \int_1^\infty g_1(r, h;\ r_1, z_1)r\phi_r(r, h)dr, \quad z_1 \geqq h, \tag{4.7}$$

where $g_1$ is such that

$$g_1(r, h;\ r_1, h) = \frac{2}{\pi r_<}\left\{\mathbf{E}\left(\frac{r_<}{r_>}\right) - \mathbf{K}\left(\frac{r_<}{r_>}\right)\right\}, \tag{4.8}$$

with details given in (7).

A similar analysis for region 3 ($-\varepsilon \leqq z_1 \leqq 0$) requires a Green's function that has vanishing normal derivative on each of the planes $z = 0$ and $z = -\varepsilon$, whence

$$\psi_3(r_1, z_1) = -\int_1^\infty g_3(r, 0;\ r_1, z_1) r\phi_r(r, 0)dr, \quad -\varepsilon \leqq z_1 \leqq 0, \tag{4.9}$$

where the Green's function,

$$g_3(r, z;\ r_1, z_1) = -\frac{1}{2\varepsilon}\frac{r_<}{r_>} - \frac{2}{\varepsilon}\sum_1^\infty \cos\frac{n\pi z}{\varepsilon}\cos\frac{n\pi z_1}{\varepsilon} I_1\left(\frac{n\pi r_<}{\varepsilon}\right) K_1\left(\frac{n\pi r_>}{\varepsilon}\right), \tag{4.10}$$

expressed in terms of the modified Bessel functions $I_1$ and $K_1$, is the same as that appearing in the analysis (7) for discs of zero thickness.

The corresponding formula for the region 2 requires a Green's function $g_2(r, z;\ r_1, z_1)$ specified by the conditions

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right) g_2 = \frac{1}{r}\delta(r-r_1)\delta(z-z_1),$$

with $(g_2)_z = 0$ when $r \geqq 1$, $z = 0$ or $h$, and $r(g_2)_r + g_2 = 0$ when $r = 1$, $0 \leqq z \leqq h$. Thus we find, for $0 \leqq z_1 \leqq h$,

$$\psi_2(r_1, z_1) = -\int_1^\infty g_2(r, h;\ r_1, z_1) r\phi_r(r, h)dr + \int_1^\infty g_2(r, 0;\ r_1, z_1) r\phi_r(r, 0)dr, \tag{4.11}$$

with $g_2$ given by

$$g_2 = -\frac{1}{2h}\frac{r_<}{r_>} - \frac{2}{h}\sum_1^\infty \cos\frac{n\pi z}{h}\cos\frac{n\pi z_1}{h} I_1\left(\frac{n\pi r_<}{h}\right) K_1\left(\frac{n\pi r_>}{h}\right) + \frac{1}{2h}\frac{r}{r_1}$$
$$+\frac{2}{h}\sum_1^\infty \cos\frac{n\pi z}{h}\cos\frac{n\pi z_1}{h}\frac{I_1\left(\frac{n\pi}{h}\right) + \frac{n\pi}{h} I_1'\left(\frac{n\pi}{h}\right)}{K_1\left(\frac{n\pi}{h}\right) + \frac{n\pi}{h} K_1'\left(\frac{n\pi}{h}\right)} K_1\left(\frac{n\pi r}{h}\right) K_1\left(\frac{n\pi r_1}{h}\right). \tag{4.12}$$

Finally, the discontinuity conditions (4.4, 4.5) are applied, using the three formulae (4.7), (4.9) and (4.11), and the problem is reduced to the following pair of coupled integral equations for the radial fields $\phi_r(r, 0)$ and $\phi_r(r, h)$:

$$\int_1^\infty \phi_r(r, h) rr_1(g_1+g_2)(r, h;\ r_1, h)dr - \int_1^\infty \phi_r(r, 0) rr_1 g_2(r, 0;\ r_1, h)dr = A \tag{4.13}$$

and

$$-\int_1^\infty \phi_r(r,\, h)rr_1 g_2(r,\, h;\; r_1,\, 0)dr + \int_1^\infty \phi_r(r,\, 0)rr_1(g_2+g_3)(r,\, 0;\; r_1,\, 0)dr = B, \tag{4.14}$$

for $r_1 \geqq 1$, with the constants $A$ and $B$ determined from the requirement that $\phi = 1$ on the disc, whence

$$\int_1^\infty \phi_r(r,\, h)dr = \int_1^\infty \phi_r(r,\, 0)dr = -1. \tag{4.15}$$

*Approximation near the edges.* The coupled equations (4.13) and (4.14), which are, in principle, sufficient to determine the solution for any values of the parameters $\varepsilon$ and $h$, are now shown to be particularly useful for the case of interest here, when $\varepsilon$ and $h$ are small. For although their exact solution is not possible, some information regarding the desired constants $A$ and $B$ can be obtained by simply comparing (4.13) and (4.14) with the corresponding equations for a pair of *semi-infinite* plates of the same thickness and separation. The argument is that, for points sufficiently close to the end faces (i.e. $r_1$ close to unity), the potential will be insensitive to the small curvature of the discs, as was found to be the case in the earlier work (7) on the problem of discs with zero thickness.

Thus we write

$$r = 1+x, \quad r_1 = 1+x_1, \tag{4.16}$$

and expand the kernels of the equations (4.13) and (4.14) for small values of $x$ and $x_1$. This is a simple procedure since the Bessel functions $I_1$ and $K_1$, that appear in $g_2$ and $g_3$, have uniformly large arguments and may be replaced by their asymptotic forms in terms of exponentials, and the sums can readily be performed. Denoting the approximate solution so obtained by the suffix 0, the integral equations (4.13) and (4.14) reduce to the form

$$\begin{aligned}\int_0^\infty \phi_{0x}(x,\, h)&\left\{\frac{(x-x_<)}{h} + \frac{1}{\pi}\log\left[|x-x_1|(1-e^{-\pi|x-x_1|/h})(1-e^{-\pi|x+x_1|/h})\right]\right\}dx \\ &- \int_0^\infty \phi_{0x}(x,\, 0)\left\{\frac{(x-x_<)}{h} + \frac{1}{\pi}\log\left[(1+e^{-\pi|x-x_1|/h})(1+e^{-\pi|x+x_1|/h})\right]\right\}dx \\ &= A_0 + \frac{1}{\pi}\log\frac{e^2}{8},\end{aligned} \tag{4.17}$$

$$\begin{aligned}-\int_0^\infty \phi_{0x}(x,\, h)&\left\{\frac{(x-x_<)}{h} + \frac{1}{\pi}\log\left[(1+e^{-\pi|x-x_1|/h})(1+e^{-\pi|x+x_1|/h})\right]\right\}dx \\ &+ \int_0^\infty \phi_{0x}(x,\, 0)\left\{\frac{(x-x_<)}{h} + \frac{1}{\pi}\log\left[(1-e^{-\pi|x-x_1|/h})(1-e^{-\pi|x+x_1|/h})\right.\right. \\ &\left.\left.(1-e^{-\pi|x-x_1|/\varepsilon})\right] - \frac{x_<}{\varepsilon}\right\}dx = B_0 - \frac{1}{2\varepsilon},\end{aligned} \tag{4.18}$$

for $x_1 > 0$, where use has been made of the additional scaling conditions that

$$\int_0^{\infty} \phi_{0x}(x, h)dx = \int_0^{\infty} \phi_{0x}(x, 0)dx = -1. \tag{4.19}$$

Although the approximate equations (4.17, 4.18) still look difficult to deal with, it is to be noted that they are derived only for comparison with those appropriate to the corresponding two-dimensional configuration, for which a solution can be obtained by conformal mapping. It will now be shown that the two-dimensional equations differ from (4.17, 4.18) only in that definite constants appear on the right-hand sides: a direct comparison is therefore sufficient to determine $A_0$ and $B_0$ without the need to solve the equations.

*Two-dimensional problem.* The corresponding two-dimensional problem, whose solution leads to a first approximation $A_0$, $B_0$ for the constants $A$, $B$, is that of a pair of semi-infinite plates of thickness $h$ and separation $2\varepsilon$, maintained at potentials $\pm 1$. Exploiting the symmetry about the mid-plane, we have to find a harmonic function $\Phi(x, y)$ that vanishes on the plane $y = -\varepsilon$, and takes the value unity on the three surfaces $(x \leqq 0, y = h)$, $(x \leqq 0, y = 0)$ and $(x = 0, 0 \leqq y \leqq h)$.

This problem is readily solved by means of the following conformal transformation from the complex plane $z = x+iy$ to $\zeta = \xi+i\eta$:

$$z = \zeta^{\frac{1}{2}}(\zeta+\alpha)^{\frac{1}{2}}+(\beta+\tfrac{1}{2}\alpha)\cosh^{-1}\left(\frac{2\zeta}{\alpha}+1\right)-\beta^{\frac{1}{2}}(\alpha+\beta)^{\frac{1}{2}}\cosh^{-1}\frac{(1+2\beta/\alpha)\zeta+\beta}{\zeta-\beta}$$
$$-i\pi\beta^{\frac{1}{2}}(\alpha+\beta)^{\frac{1}{2}}, \tag{4.20}$$

(cf. Kober (6)), where the constants $\alpha$ and $\beta$ are given in terms of $\varepsilon$ and $h$ by the formulae

$$\pi\beta^{\frac{1}{2}}(\alpha+\beta)^{\frac{1}{2}} = \varepsilon, \quad \pi(\beta+\tfrac{1}{2}\alpha) = \varepsilon+h. \tag{4.21}$$

The transformation (4.20) is found to map the plate $ABCD$ and the line $EF$ on to the real $\xi$-axis (Fig. 2).

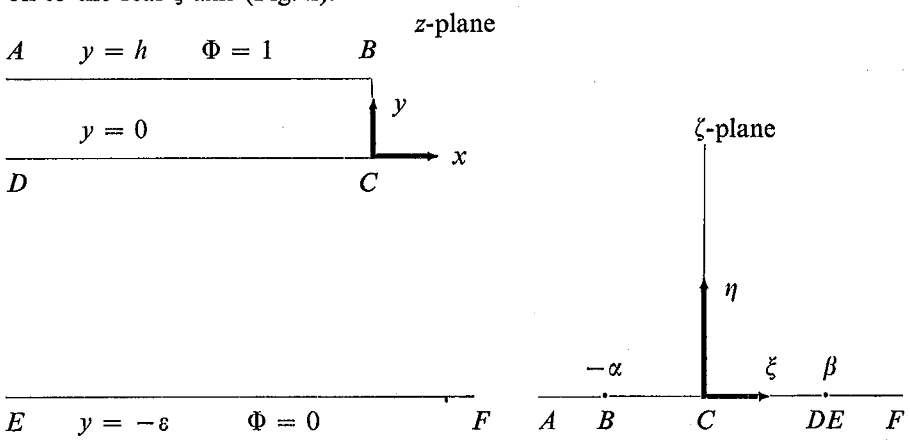


FIG. 2

The required complex potential $\Omega = \Phi + i\Psi$ is easily calculated in terms of $\zeta$, and is found to be

$$\Omega(\zeta) = -\frac{i}{\pi}\log\left(\frac{\zeta-\beta}{\beta}\right). \tag{4.22}$$

An explicit solution in terms of $x$ and $y$ is not possible, since the transformation (4.20) cannot be inverted in a simple form; but the solution (4.22), together with (4.20), can be used to infer the limiting form of the solution at large distance from the origin, for points both between and outside the plates. For points between the plates, it is found that

$$\Psi(x, y) \sim -x/\varepsilon + P \text{ as } |z| \to \infty, \text{ through } DE, \tag{4.23}$$

where

$$P = \frac{1}{\pi}\left\{1 + \frac{\beta+\frac{1}{2}\alpha}{\beta^{\frac{1}{2}}(\alpha+\beta)^{\frac{1}{2}}}\cosh^{-1}\left(\frac{2\beta}{\alpha}+1\right) - \log\frac{4}{\alpha}(\alpha+\beta)\right\}. \tag{4.24}$$

For points outside the plates, on the other hand,

$$\Psi(x, y) \sim -\frac{1}{\pi}\log(x^2+y^2)^{\frac{1}{2}} + \frac{1}{\pi}\log\beta \text{ as } |z| \to \infty, \text{ through } AF. \tag{4.25}$$

A pair of coupled integral equations for $\Phi_x(x, 0)$ and $\Phi_x(x, h)$, corresponding to (4.13, 4.14) for the discs, can readily be derived by a similar procedure. Using the asymptotics (4.23, 4.25) to calculate the contributions from arcs at infinity, it is found that the equations for $\Phi_x(x, 0)$ and $\Phi_x(x, h)$ are exactly like (4.17) and (4.18), with the constants on the right replaced by

$$\left.\begin{aligned} &(1-\log\beta)/\pi \quad \text{instead of} \quad A_0 + \frac{1}{\pi}\log\frac{e^2}{8} \\ &\text{and} \\ &P - 1/\pi \qquad \text{instead of} \quad B_0 - 1/2\varepsilon \end{aligned}\right\}. \tag{4.26}$$

*Approximate evaluation of the capacity.* Since the governing equations for $\phi_0$ and $\Phi$ differ only by the scaling constants on the right, and since each function has the same scaling condition (4.19), the pair must be *identically equal.* Thus the constant terms of their integral equations can be equated, and the constants $A_0$ and $B_0$ are seen from (4.26) to have the values

$$\left.\begin{aligned} A_0 &= -\frac{1}{\pi}\log\frac{e^2}{8} + \frac{1}{\pi} - \frac{1}{\pi}\log\beta \\ &\text{and} \\ B_0 &= 1/2\varepsilon + P - 1/\pi \end{aligned}\right\}. \tag{4.27}$$

Finally, the capacity $C = \frac{1}{4}(A+B)$ takes the approximate form

$$\begin{aligned} C \sim C_0 &= \frac{1}{8\varepsilon} - \frac{1}{4\pi}\log\left(\frac{e^2\beta}{8}\right) + \frac{1}{4}P \\ &= \frac{1}{8\varepsilon} + \frac{1}{4\pi}\log\left\{\frac{8\pi}{e\varepsilon}\left(1+\frac{h}{2\varepsilon}\right)\right\} + \frac{1}{8\pi}\frac{h}{\varepsilon}\log\left(1+\frac{2\varepsilon}{h}\right), \end{aligned} \tag{4.28}$$

from (4.24), and using (4.21) to express the answer in terms of $\varepsilon$ and $h$. The result (4.28) is precisely that proposed by Kirchhoff (**5**).

The infinite range of integration in the exact formulae (4.13, 4.14) implies that the asymptotic approximation of the kernels leading to (4.17, 4.18) is not uniform, therefore not strictly valid. The justification for this step is that for values of $r$ significantly greater than unity, for which the asymptotics are not valid, the functions $\phi_r(r, h)$ and $\phi_r(r, 0)$ are very small on account of the dipole nature of the field away from the discs. This point is discussed more fully in the analysis (**7**) for discs of zero thickness, for which an improved estimate for the capacity is derived. It is likely that the next term in the expansion for $C$ could be obtained similarly here, by manipulating the exact equations (4.13, 4.14) and the approximate equations (4.17, 4.18) to provide efficient formulae for $A$ and $B$, analogous to (2.21) and (3.18) in the earlier sections.

## 5. Steady flow through a constricted circular tube

A problem investigated by Fock (**3**) concerns the steady flow of current down a circular tube, of radius $b$, across which is placed a plane screen containing a circular aperture, of radius $a$, whose centre is on the axis of the tube. Fock deals with the case when the aperture radius $a$ is small compared with $b$, while the present investigation concerns the complementary problem when the two radii are nearly equal.

Cylindrical polar coordinates $(r, \theta, z)$ are chosen so that the tube is given by $r = b$, $-\infty < z < \infty$, with the screen occupying the region $a < r < b$, $z = 0$. The scalar potential $\phi(r, z)$ is a harmonic function whose asymptotic form $\phi \sim Az$, for large $|z|$, corresponds to the free flow down the tube with no constriction, where $A$ is a constant proportional to the total current $I$. The symmetry of the problem implies that $\phi(r, z)$ is an odd function of $z$, so that we may consider only the region $z \geqq 0$, in which $\phi$ is specified by the conditions

$$\left.\begin{aligned} &\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\phi = 0, \quad z \geqq 0 \\ &\phi = 0, \qquad z = 0,\ r < a, \\ &\phi_z = 0, \qquad z = 0,\ a < r < b, \\ &\phi_r = 0, \qquad z \geqq 0,\ r = b \\ &\phi \sim Az + B \quad \text{as } z \to \infty, \end{aligned}\right\} \tag{5.1}$$

where $A$ is given and $B$ is an unknown constant to be found.

In order to obtain an integral equation representation over the region $z = 0$, $a \leqq r \leqq b$, we follow the lead given in sections 3 and 4 by defining the *stream function* $\psi(r, z)$:

$$r\phi_r = r\psi_z \quad \text{and} \quad r\phi_z = -\frac{\partial}{\partial r}(r\psi), \tag{5.2}$$

with the condition that $\psi(0, 0) = 0$. This implies that $\psi(0, z) = 0$ along the whole $z$-axis, whence on taking the limit $r \to 0$ in the second of formulae (5.2) we have

$$\phi_z(0, z) = -2\psi_r(0, z), \quad z \geqq 0. \tag{5.3}$$

The boundary value problem for $\psi(r, z)$ is found from (5.1) and (5.2) to be:

$$\left.\begin{array}{ll} \left(\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} - \dfrac{1}{r^2} + \dfrac{\partial^2}{\partial z^2}\right)\psi = 0, & z \geqq 0, \\ \psi_z = 0, & z = 0,\ r < a \\ \psi = -\tfrac{1}{2}Ab^2/r, & z = 0,\ a < r < b \\ \psi = -\tfrac{1}{2}Ab, & z \geqq 0,\ r = b \\ \psi \sim -\tfrac{1}{2}Ar, & \text{as } z \to \infty. \end{array}\right\} \tag{5.4}$$

The total current $I$ flowing down the tube is given from conditions at large $| z |$ as

$$I = \sigma \int_0^b \frac{\partial \phi}{\partial z} 2\pi r dr = \sigma \pi b^2 A, \tag{5.5}$$

where $\sigma$ is the conductivity of the medium. Fock (**3**) defines the conductivity of the aperture as

$$C = I/2\sigma B, \tag{5.6}$$

where $B$ is the unknown constant appearing in the specifications (5.1), and our aim is to calculate this number, hence the conductivity $C$, in the limit when $\delta = 1-(a/b)$ is small.

*Integral equation formulation.* The problem (5.4) is reduced to an integral equation, through the use of a Green's function $g(r, z;\ r_1, z_1)$, described in the appendix, that is chosen to vanish when $r = b$ and to have zero normal derivative when $z = 0$. It is found that $\psi(r_1, z_1)$ has the representation

$$\psi(r_1, z_1) = \int_a^b g(r, 0;\ r_1, z_1)\psi_z(r, 0) r dr - \tfrac{1}{2}Ab \int_0^\infty \frac{\partial g}{\partial r}(b, z;\ r_1, z_1) b dz,$$

i.e.

$$\psi(r_1, z_1) + \tfrac{1}{2}Ar_1 = \int_a^b g(r, 0;\ r_1, z_1)\psi_z(r, 0) r dr, \tag{5.7}$$

using equation ($A7$) of the appendix. In particular, if $z_1 \to 0$ and $a < r_1 < b$, then (5.4) shows that $\psi$ assumes the value $-Ab^2/2r_1$, and we are led to the integral equation

$$-\tfrac{1}{2}Ab\left(\frac{b}{r_1} - \frac{r_1}{b}\right) = \int_a^b g(r, 0;\ r_1, 0)\psi_z(r, 0) r dr, \quad a < r_1 < b, \tag{5.8}$$

for the function $\psi_z(r, 0)$.

It remains to express the constant $B$ in terms of the function $\psi_z(r, 0)$, and this is achieved as follows. Since $\phi \sim Az+B$ for large $z$, we have

$$B = \int_0^\infty \{\phi_{z_1}(0, z_1)-A\}dz_1$$

$$= -2\int_0^\infty \{\psi_{r_1}(0, z_1)+\tfrac{1}{2}A\}dz_1, \text{ from (5.3)},$$

$$= -2\int_a^b r\psi_z(r, 0)dr \int_0^\infty g_{r_1}(r, 0;\ 0, z_1)dz_1, \text{ from (5.7)},$$

i.e.

$$B = \frac{1}{b}\int_a^b \left(\frac{b}{r}-\frac{r}{b}\right) r\psi_z(r, 0)dr, \tag{5.9}$$

using formula ($A$8) of the appendix. Thus if we define

$$W_1(r) = \frac{1}{bB}\left(\frac{b}{r}-\frac{r}{b}\right) r\psi_z(r, 0),$$

and

$$K(r, r_1) = \left(\frac{b}{r}-\frac{r}{b}\right)^{-1}\left(\frac{b}{r_1}-\frac{r_1}{b}\right)^{-1} g(r, 0;\ r_1, 0), \tag{5.10}$$

then equations (5.8) and (5.9) take the normalised form

$$\int_a^b K(r, r_1)W_1(r)dr = -C/\pi b^2, \quad a<r_1<b, \tag{5.11}$$

with

$$\int_a^b W_1(r)dr = 1. \tag{5.12}$$

The equations (5.11, 5.12), which together determine the potential field, and in particular the conductivity $C$, are of exactly the same type as have appeared in the earlier sections 2 and 3, and can be solved by the same procedure.

*Approximate solution.* An approximate solution for $W_1(r)$, and the constant $C$, is sought for small values of the parameter

$$\delta = 1-(a/b). \tag{5.13}$$

The variables $r$, $r_1$ and $W_1(r)$ are rescaled according to the transformations

$$r = b(1-\delta x), \quad r_1 = b(1-\delta x_1), \tag{5.14}$$

and

$$W(x) = b\delta W_1(r) = b\delta W_1(b-b\delta x). \tag{5.15}$$

The governing equations (5.11, 5.12) then become

$$\int_0^1 K(x, x_1)W(x)dx = -C/\pi b^2, \quad 0<x_1<1, \tag{5.16}$$

$$\int_0^1 W(x)dx = 1, \tag{5.17}$$

and are solved approximately by expanding the kernel $K(x, x_1)$ for small $\delta$. It is shown in the appendix that

$$K(x, x_1) \sim K_0(x, x_1) = \frac{1}{4\pi b\delta^2}\frac{1}{xx_1}\log\left|\frac{x-x_1}{x+x_1}\right| - \frac{3}{16\pi b}\log\delta, \tag{5.18}$$

whence a first approximation $W_0(x)$, and $C_0$, is defined by the integral equation

$$\int_0^1 K_0(x, x_1)W_0(x)dx = -C_0/\pi b^2, \quad 0<x_1<1, \tag{5.19}$$

i.e.

$$-\frac{1}{\pi}\int_0^1 \log\left|\frac{x-x_1}{x+x_1}\right|\frac{W_0(x)}{x}dx = x_1\left\{\frac{4C_0}{\pi b}\delta^2 - \frac{3}{4\pi}\delta^2\log\delta\right\}, \quad 0<x_1<1, \tag{5.20}$$

where use has been made of the scaling condition

$$\int_0^1 W_0(x)dx = 1. \tag{5.21}$$

The integral equation (5.20) is a familiar one whose solution is known (cf. Cooke (**2**)). It can also be solved from first principles, since the approximate formula (5.19) corresponds to the two-dimensional problem of a steady flow past a plane with a strip of unit length protruding at right angles; this can easily be solved by conformal transformation and leads to the solution

$$\frac{W_0(x)}{x} = \frac{x}{(1-x^2)^{\frac{1}{2}}}\left\{\frac{4C_0}{\pi b}\delta^2 - \frac{3}{4\pi}\delta^2\log\delta\right\}.$$

The constant $C_0$ is then found from condition (5.20), whence

$$W_0(x) = \frac{4x^2}{\pi(1-x^2)^{\frac{1}{2}}}, \tag{5.22}$$

and

$$C_0 = b(1+\tfrac{3}{16}\delta^2\log\delta)/\delta^2. \tag{5.23}$$

To improve this estimate, the exact formula

$$C - C_0 = -\pi b^2\int_0^1\int_0^1 (K-K_0)W(x_1)W_0(x)dxdx_1 \tag{5.24}$$

$$\sim -\pi b^2\int_0^1\int_0^1 (K-K_0)W_0(x_1)W_0(x)dxdx_1 \tag{5.25}$$

is obtained from (5.16) and (5.19). An estimate for (5.25) requires a further expansion of the kernel $K(x, x_1)$, and is not pursued here since the next term, of order $\delta^2$, is found to have a coefficient that is expressed only in the form of a complicated definite integral involving modified Bessel functions.

Formula (5.23) complements the results of Fock (**3**) who deals with the regular limit of small $a/b$, and obtains a power series expansion for $C^{-1}$ up to terms of order $(a/b)^{12}$. The value obtained from (5.23) agrees well with Fock's prediction when $a/b = 9/10$.

## Appendix. Properties of a Green's function

The Green's function $g(r, z;\ r_1, z_1)$ of section 5 is specified by

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\partial^2}{\partial z^2}\right)g=\frac{1}{r}\delta(r-r_1)\delta(z-z_1),\quad r_1<b,\ z_1>0, \tag{A1}$$

with

$$g_z=0 \text{ when } z=0,\ g=0 \text{ when } r=b. \tag{A2}$$

Writing $g$ in the form of a Fourier integral,

$$g(r, z;\ r_1, z_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(r, r_1;\ s)\{e^{-is(z-z_1)}+e^{-is(z+z_1)}\}ds, \tag{A3}$$

the transform function $\hat{g}$ is such that

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}-\delta^2\right)\hat{g}=\frac{1}{r}\delta(r-r_1),$$

with $\hat{g}=0$ when $r=b$, and has the solution

$$\hat{g}=-K_1(|s|r_>)I_1(|s|r_<)+\frac{K_1(b|s|)}{I_1(b|s|)}I_1(|s|r)I_1(|s|r_1); \tag{A4}$$

the functions $I_1$ and $K_1$ are modified Bessel functions. In particular then, when $z=z_1=0$, we have

$$\begin{aligned} g(r, 0;\ r_1, 0) = &-\frac{2}{\pi b}\int_0^{\infty}K_1\left(\frac{tr_>}{b}\right)I_1\left(\frac{tr_<}{b}\right)dt \\ &+\frac{2}{\pi b}\int_0^{\infty}\frac{K_1(t)}{I_1(t)}I_1\left(\frac{tr}{b}\right)I_1\left(\frac{tr_1}{b}\right)dt. \end{aligned} \tag{A5}$$

A useful related function is that obtained by integrating $g(r, z;\ r_1, z_1)$ from $z=0$ to $z=\infty$, and is most easily calculated directly from equation ($A$1). Thus

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\int_0^{\infty}g\,dz=\frac{1}{r}\delta(r-r_1),$$

with $\int_0^\infty g dz = 0$ when $r = b$, and has the solution

$$\int_0^\infty g(r, z;\ r_1, z_1)dz = \int_0^\infty g(r, z_1;\ r_1, z)dz = \frac{rr_1}{2b^2} - \frac{r_<}{2r_>}. \tag{A6}$$

In particular,

$$\int_0^\infty \frac{\partial g}{\partial r}(b, z;\ r_1, z_1)dz = r_1/b^2 \tag{A7}$$

and

$$\int_0^\infty \frac{\partial g}{\partial r_1}(r, z_1;\ 0, z)dz = -\frac{1}{2b}\left(\frac{b}{r} - \frac{r}{b}\right). \tag{A8}$$

*Asymptotics for small* $\delta$. We require the asymptotic form of the function $g(r, 0;\ r_1, 0)$ when $r = b(1-\delta x)$ and $r_1 = b(1-\delta x_1)$ with $\delta$ small.

The first integral of ($A5$) is readily dealt with, since it can be expressed (**9**) in the form of a hypergeometric function,

$$-\frac{2}{\pi b}\int_0^\infty K_1\left(\frac{tr_>}{b}\right)I_1\left(\frac{tr_<}{b}\right)dt = -\frac{2}{b}F(\tfrac{1}{2}, \tfrac{1}{2};\ 1;\ r_<^2/r_>^2), \tag{A9}$$

whose known expansion (cf. (**1**)), with argument close to unity, leads to the result that

$$-\frac{2}{\pi b}\int_0^\infty K_1(t-t\delta x_>)I(t-t\delta x_<)dt$$

$$\sim -\frac{1}{\pi b}\Big\{-\log\delta - \log\frac{|x-x_1|e^2}{8} - \tfrac{1}{2}\delta(x+x_1)\log\delta$$

$$-\tfrac{1}{2}\delta(x+x_1)\log\frac{|x-x_1|e^3}{8} - \tfrac{1}{16}\delta^2(9x^2-2xx_1+9x_1^2)\log\delta + O(\delta^2)\Big\}. \tag{A10}$$

To deal with the remaining integral of ($A5$) we note that, for small $\delta$, the factor $I_1(t-t\delta x)I_1(t-t\delta x_1)$ appearing in the integrand is nearly the same as $I_1(t)I_1(t-t\delta(x+x_1))$. Thus

$$\frac{2}{\pi b}\int_0^\infty \frac{K_1(t)}{I_1(t)}I_1(t-t\delta x)I_1(t-t\delta x_1)dt = \frac{2}{\pi b}\int_0^\infty K_1(t)I_1(t-t\delta(x+x_1))dt + R, \tag{A11}$$

where

$$R = \frac{2}{\pi b}\int_0^\infty \frac{K_1(t)}{I_1(t)}\{I_1(t-t\delta x)I_1(t-t\delta x_1) - I_1(t)I_1(t-t\delta(x+x_1))\}dt \tag{A12}$$

is small when $\delta$ is small.

The first term of $(A11)$, which gives the main contribution when $\delta$ is small, can be written as a hypergeometric function, whence

$$\frac{2}{\pi b}\int_0^\infty K_1(t)I_1(t-t\delta(x+x_1))dt = \frac{2}{b}F(\tfrac{1}{2}, \tfrac{1}{2};\ 1;\ [1-\delta(x+x_1)]^2)$$

$$\sim \frac{1}{\pi b}\left\{-\log\delta-\log\frac{(x+x_1)e^2}{8}-\tfrac{1}{2}\delta(x+x_1)\log\delta-\tfrac{1}{2}\delta(x+x_1)\log\frac{(x+x_1)e^3}{8}\right.$$
$$\left.-\tfrac{9}{16}\delta^2(x+x_1)^2\log\delta+O(\delta^2)\right\}. \qquad (A13)$$

It remains to deal with the correction term $R$ of formula $(A11)$. This can be estimated by dividing the range of integration at $t = N$, where $N$ is a large number such that

$$N\to\infty,\ N\delta\to 0 \quad \text{as } \delta\to 0. \qquad (A14)$$

Over the interval $0<t<N$, we have $t\delta \ll 1$ and the modified Bessel functions may be replaced by their Taylor series approximations about $t$; for $t>N$, on the other hand, the Bessel functions have uniformly large arguments and may therefore be replaced by their appropriate asymptotic forms. This procedure leads to the estimate

$$R\sim\frac{xx_1}{2\pi b}\delta^2\log\delta+O(\delta^2), \qquad (A15)$$

whence from $(A10)$, $(A13)$ and $(A15)$ we get

$$g(r,0;\ r_1,0)\sim\frac{1}{\pi b}\left\{\log\left|\frac{x-x_1}{x+x_1}\right|+\tfrac{1}{2}\delta(x+x_1)\log\left|\frac{x-x_1}{x+x_1}\right|\right.$$
$$\left.-\tfrac{3}{4}xx_1\delta^2\log\delta+O(\delta^2)\right\}. \qquad (A16)$$

Finally, the kernel of equation (5.16) is

$$K(x,x_1)=\frac{1}{4\delta^2 xx_1}\frac{(1-\delta x)(1-\delta x_1)}{(1-\frac{1}{2}\delta x)(1-\frac{1}{2}\delta x_1)}g$$

and is found from $(A16)$ to take the asymptotic form

$$K(x,x_1)\sim\frac{1}{4\pi b\delta^2 xx_1}\left\{\log\left|\frac{x-x_1}{x+x_1}\right|-\tfrac{3}{4}xx_1\delta^2\log\delta+O(\delta^2)\right\}. \qquad (A17)$$

## REFERENCES

(1) M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, 1965).

(2) J. C. Cooke, The solution of some integral equations and their connection with dual integral equations and series, *Glasgow Math. J.* **11** (1970), 9-20.

(3) V. A. Fock, A theoretical investigation of the acoustical conductivity of a circular aperture in a wall put across a tube, *C.R.* (*Doklady*) *Acad. Sci. URSS* (*N.S.*) **31** (1941), 875-878.

(4) G. A. Grinberg and V. N. Kuritsyn, Diffraction of a plane electromagnetic wave by an ideally conducting plane ring and the electrostatic problem for such a ring, *Soviet Physics Tech. Phys.* **6** (1962), 743-749.

(5a) G. R. Kirchhoff, Zur Theorie des Kondensators, *Monatsber. König. Preuss. Akad. Wiss. Berlin* (1877), 144-162.

(5b) G. R. Kirchhoff, *Vorlesungen über mathematische Physik* 3. bd (Leipzig, 1891).

(6) H. Kober, *Dictionary of Conformal Representations* (Dover, 1952).

(7) F. G. Leppington and H. Levine, On the capacity of the circular disc condenser at small separation, *Proc. Cambridge Philos. Soc.* **68** (1970), 235-254.

(8) F. G. Leppington and H. Levine, On the problem of closely separated circular discs at equal potential, *Q.J. Mech. Appl. Math.* **25** (1972), 225-245.

(9) I. M. Rhysik and I. S. Gradstein, *Tables of Series, Products and Integrals* (Berlin, 1963).

(10) I. N. Sneddon, *Mixed Boundary Value Problems in Potential Theory* (North-Holland, 1966).

(11) D. A. Spence, *Math. Res. Centre Tech. Report* **1039** (University of Wisconsin, 1970).

(12) D. A. Spence, A Wiener-Hopf solution to the triple integral equations for the electrified disc in a coplanar gap, *Proc. Cambridge Philos. Soc.* **68** (1970), 529-545.

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