Siu MA 106 (Complex Var) Office 383J 9:30–11 am MWF

Text: Levison & Redheffer

Ch 1–4

one to two tests ; one HW assign/wk

Complex No: Review

A complex no. is a pair of real numbers (a, b)

$a+bi \qquad i = \sqrt{-1}$

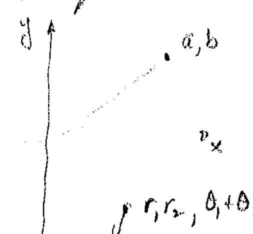
$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

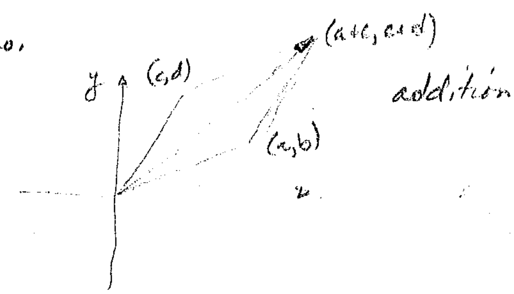
$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac-bd)}{c^2+d^2} + \frac{(ad+bc)}{c^2+d^2} i$$

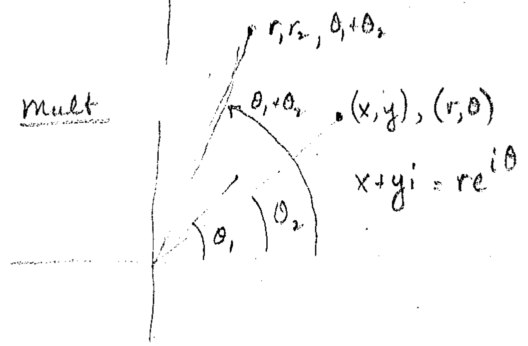
complex conjugate of (a, b) is $(a, -b)$

Geometric Representation of Complex No.





Mult



$$(x_1, y_1)\cdot(x_2, y_2) = (r_1, \theta_1)(r_2, \theta_2) = (r_1 r_2, \theta_1+\theta_2)$$

Find nth root of a or $z^n = a \Rightarrow z_k = \sqrt[n]{a}\, e^{i\left(\frac{\arg a + 2k\pi}{n}\right)} \quad k = 0, 1, 2, \dots, n-1$

Express $\sin 5\theta$, $\cos 5\theta$ in terms of $\sin\theta$, $\cos\theta$

Sol: $\cos 5\theta + i\sin 5\theta = e^{i5\theta} = (\cos\theta + i\sin\theta)^5$ use pascal triangle to get $\sin 5\theta$, $\cos 5\theta$ equate real & imaginary parts

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

Absolute value of a complex no. $a+bi$

$$|a+bi| = \sqrt{a^2+b^2}$$ or the distance to origin

$$|a+bi|^2 = (a+bi)(a-bi)$$

Example: z, w are complex no.

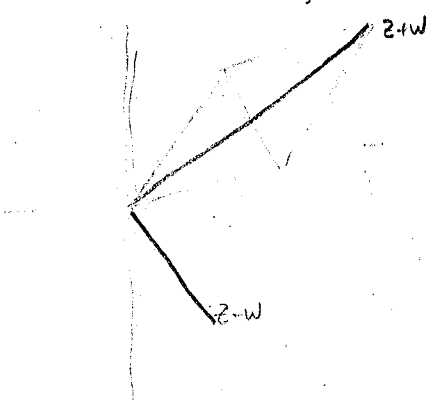
$$2|z|^2 + 2|w|^2 = |z-w|^2 + |z+w|^2$$

sum of squares of sides — sum of squares of diag

Pf

$$|z-w|^2 + |z+w|^2$$

$$(z-w)(\bar{z}-\bar{w}) + (z+w)(\bar{z}+\bar{w}) = 2z\bar{z} + 2w\bar{w}$$ after mult & canceling



Example: if $z = a+bi$ $\quad a = \mathrm{Re}\, z$, $b = \mathrm{Im}\, z$

$$|\mathrm{Re}\, z| \le |z|$$

Ex Δ inequality

$$|z+w| \le |z| + |w|$$

aside $\overline{(\bar{z})} = z$

$z + \bar{z} = 2\,\mathrm{Re}\, z$

$z - \bar{z}\, i = 2i\, \mathrm{Im}\, z$

Proof: $|z+w|^2 = (z+w)\overline{(z+w)}$

$z\bar{z} + w\bar{z} + \bar{w}z + w\bar{w}$

$z\bar{z} + 2\mathrm{Re}(w\bar{z}) + w\bar{w} \le z\bar{z} + 2|zw| + w\bar{w}$

or $\qquad \le |z|^2 + 2|z||w| + |w|^2$ or $(|z|+|w|)^2$

and take $\sqrt{}$ of both sides

9/29/30

Let A = subset of $\mathbb{C}$

Def 1. $z_0 \in A$ is an interior point of A if for some $\epsilon > 0$ the disc of radius ϵ centered at z_0 is contained in A.

2. A is open if every point of A is an interior point

3. a point a is called a boundary point of A if every ϵ-disc ($\epsilon > 0$) centered a a contains both pts in A & points not in A.

4. the set of boundary points of A is the boundary of A

5. A set is closed if it contains its boundary points.

A set is closed iff its complement is open

6. An open set is called connected if any pair of points by a broken path inside the set.

7. A domain is a connected open set.

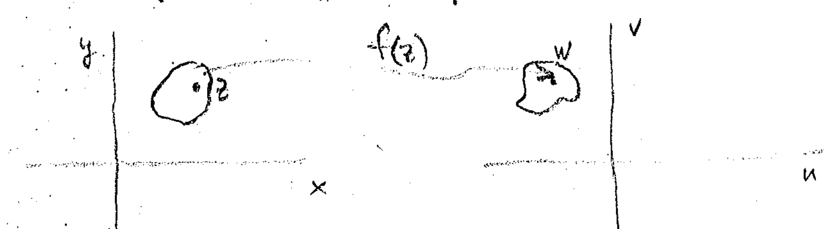
8. An ϵ-neighborhood of a pt z is an open disc of radius ϵ centered at z ($\epsilon > 0$).

9. Region a domain plus some or all bdy pts.

Continuity Limit

$f(z)$ = complex valued function of a complex variable.

$z \longrightarrow f(z)$



Def 1. $\lim_{z \to a} f(z) = L$ - given an ϵ-neighborhood W of L $\exists$. $\delta > 0$ such that f maps the deleted δ neighborhood into W.

2. A deleted ϵ-neighborhood of a = a ϵ-nbhood of a less a.

3. $f(z)$ is continuous at a if $\lim_{z \to a} f(z) = f(a)$

Deriv Review $y = f(x)$ $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$f(x) = f(a) + L(x-a) + \epsilon(x)(x-a)$, $\lim_{x \to a} \epsilon(x) = 0$ $L = f'(x)$

Suppose $f(z)$ is a complex valued funct. of a complex value.

L is called the complex deriv of f at a denoted by $f'(a)$ if

$$L = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

let $\epsilon(z) = \frac{f(z) - f(a)}{z - a} - L$ or $f(z) = f(a) + L(z-a) + \epsilon(z)(z-a)$

- V a vector space over $\mathbb{C}$

$v_1, v_2 \to v_1 + v_2$

$\lambda \in \mathbb{C}, v \to \lambda v$ Regard V as over $\mathbb{R}$

if the complex deriv of $f'(a)$ of $f\big|_a$ exists then $\left.\frac{\partial f}{\partial x}\right|_a = f'(a)$

$$\left.\frac{\partial f}{\partial y}\right|_a = i f'(a)$$

<u>OR</u> $i f_{,x} = f_{,y}$ $\therefore$ $i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$

$$\boxed{\text{or} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0}$$

CAUCHY-RIEMANN EQU.

$f(z) = (x^2+y^2) + ix$ $\quad u_{,x} = 2x \quad v_{,y} = 0 \Rightarrow x = 0$

$u = x^2+y^2 \quad v = x$ $\quad u_{,y} = 2y \quad v_{,x} = 1 \Rightarrow y = -\tfrac{1}{2}$ $\Big\}$ $f'(z)$ exists

Geometric interpretation is Jacobian

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

conformal mapping which preserves-orientation

C.R. equations say total differentiability of real & imag. parts $\Rightarrow \exists f'$

harmonic function is a real valued function of 2 real variables $\ni$ $\Delta u = 0$ (Laplace)

Example: $w = f(z)$ $w = u + iv$; $z = x + iy$

Suppose f' exists at every point then u & v are harmonic

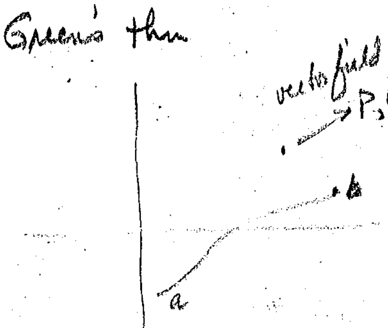
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right)$$

$$+$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right) = u_{xx} + u_{yy} = 0 = \Delta u$$

now diff wrt $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right) = v_{xx} + v_{yy} = 0 = \Delta v$

Green's thm

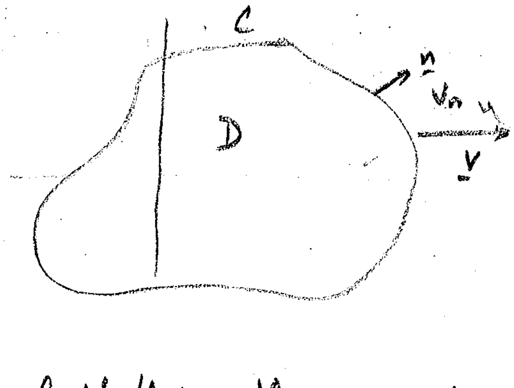


P, Q are fns of x, y & $x = x(t)$, $y = y(t)$ $a \le t \le b$

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b \Big(P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\Big)\,dt$$

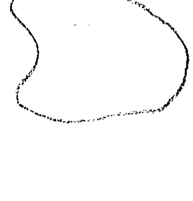
Greens theorem says for $C = \partial D$ D = domain

$$\int_C P\,dx + Q\,dy = \int_{\partial D}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$



$\underset{\sim}{v}\cdot\underset{\sim}{n} = v_n$ $\qquad \int_C \underset{\sim}{v}\cdot\underset{\sim}{n}\,dS = \int_D \nabla\cdot\underset{\sim}{v}\,dU$

fluid flow problems, ρ = const, source/sink free, steady, irrotational



$\int_C v_n\,dS = 0$ continuity $\int_D \nabla\cdot\underset{\sim}{v}\,dU$

$\int_C v_t \, ds = 0$ (irrotational)

unit tangent is $\left(\frac{dx}{ds}, \frac{dy}{ds}\right)$ then $v_t = \underline{v} \cdot \left(\frac{dx}{ds} + \frac{dy}{ds}\right) = A\frac{dx}{ds} + B\frac{dy}{ds}$

$$\int_{s=a}^{\beta} \left(A\frac{dx}{ds} + B\frac{dy}{ds}\right) ds = \iint \left(\frac{\partial B}{\partial y} - \frac{\partial A}{\partial x}\right) dy\, dx$$

Fractional linear transformation (FLT)

$w = \frac{az+b}{cz+d}$ $a, b, c, d \in \mathbb{C}$ & $ad - bc \neq 0$ so that $w \neq$ const.

Theorem FLT maps lines & circles to lines & circles

$w = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}$

$z \longrightarrow cz = \zeta_1$ rotation

$\zeta_1 \longrightarrow \zeta_1 + d = \zeta_2$ translation

$\zeta_2 \longrightarrow \frac{1}{\zeta_2} = \zeta_3$ inversion

$\zeta_3 \longrightarrow \left(b - \frac{ad}{c}\right)\zeta_3 = \zeta_4$ rotation

$\zeta_4 \longrightarrow \frac{a}{c} + \zeta_4$ = translation

10/6/98

FLT maps lines & circles to lines & circles

Case 1 $c = 0$ $w = \frac{a}{d} z + \frac{b}{d}$

$z \longrightarrow \zeta_1 = \frac{a}{d} z$ stretch/contract

$\zeta_1 \longrightarrow \zeta_1 + \frac{b}{d}$ = translation

Case 2 $c \neq 0$ $w = \frac{a}{c} + \frac{b - \frac{da}{c}}{cz+d}$

$z \to \zeta_1 = cz$

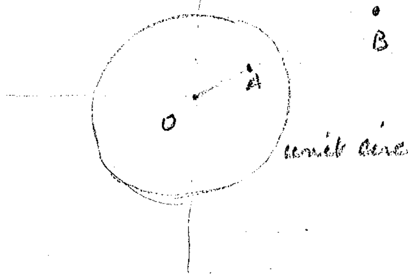
$\zeta_1 \to \zeta_1 + d = \zeta_2$ translation

$\zeta_2 \to \frac{1}{\zeta_2} = \zeta_3$ inversion + reflect

$\zeta_3 \to \left(b - \frac{ad}{c}\right)\zeta_3 = \zeta_4$

$\zeta_4 \to \frac{a}{c} + \zeta_4$ = translation $= w$

$w = \frac{1}{\zeta}$ (inversion w/reflection)



$OA \cdot OB = 1$

unit circle   for circle radius r $OA \cdot OB = r^2$

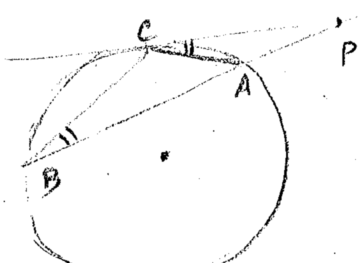
$w = \frac{1}{\bar{z}}$ is the inversion wrt the unit circle

$z = re^{i\theta}$ $w = \frac{1}{re^{-i\theta}} = \frac{1}{r}e^{i\theta}$

note that w, z are colinear since $\arg w = \arg z$ note that $w\bar{z} = 1$

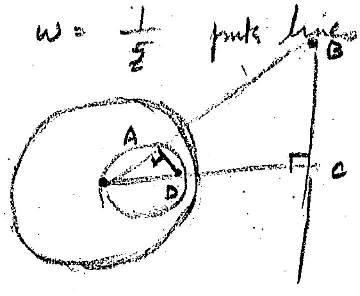
$|w|\,|\bar{z}| = 1$

thus $\frac{1}{\bar{z}} = w$ is inversion

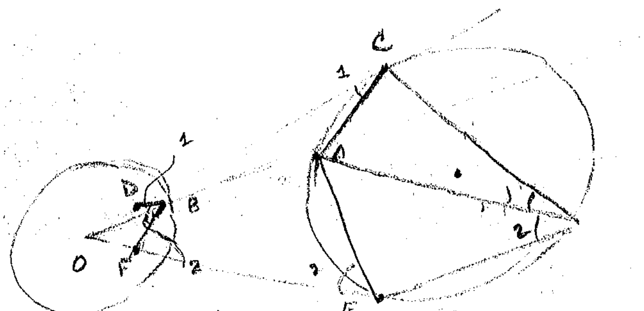


$\frac{PC}{PB} = \frac{PA}{PC}$ $\Delta PCB \sim \Delta PAC$

to show $w = \frac{1}{\bar{z}}$ puts lines into lines   draw line $\perp$



$OA \cdot OB = OD \cdot OC$



$OD \cdot OC = 1$

$OB \cdot OA = 1$

$\dfrac{OD}{OB} = \dfrac{OA}{OC}$

show $\measuredangle DBF = $ const since $\frac{1}{z}$ is a conformal map

a circle w/ center a radius r $\quad |z-a| = r$ or $(z-a)(\bar z - \bar a) = r^2$

$$z\bar z - a\bar z - \bar a z + a\bar a = r^2$$

a circle of form $z\bar z - a\bar z - \bar a z + \rho = 0$ if $\rho < |a|$

if $z = \frac{1}{w}$ then $\dfrac{1}{w\bar w} - \dfrac{a}{\bar w} - \dfrac{\bar a}{w} + \rho = 0$

or $1 - a w - \bar a \bar w + \rho w \bar w = 0$ $\quad$ if $\rho = 0$

$1 - aw - \overline{aw} = 0$ is a line

since $1 - 2\,\mathrm{Re}(aw) = 1 - 2(\alpha u - \beta v) = 0$

$\alpha, \beta = $ const

if $\rho \neq 0$ $\quad w\bar w - \dfrac{\overline{aw}}{\rho} - \dfrac{aw}{\rho} + \dfrac{1}{\rho} = 0$ $\quad$ if $\dfrac{1}{\rho} < |a|^2$ true for circle

Cross ratio $\quad \dfrac{z_3 - z_1}{z_3 - z_2} \Big/ \dfrac{z_4 - z_1}{z_4 - z_2}$ is invariant under FLT

if $w = \dfrac{az+b}{cz+d}$ $\qquad w_3 - w_1 = \dfrac{az_3+b}{cz_3+d} - \dfrac{az_1+b}{cz_1+d} = \dfrac{(ad-bc)(z_3-z_1)}{(cz_3+d)(cz_1+d)}$

$$w_3 - w_2 = \frac{(ad-bc)(z_3-z_2)}{(cz_3+d)(cz_1+d)}$$

$$\frac{w_3 - w_1}{w_3 - w_2} = \frac{z_3 - z_1}{z_3 - z_2}\,\frac{cz_2+d}{cz_1+d}$$

and $\quad \dfrac{w_4 - w_1}{w_4 - w_2} = \dfrac{z_4 - z_1}{z_4 - z_2}\,\dfrac{cz_2-d}{cz_1+d}$

$$\left.\right\} \quad \frac{\dfrac{w_3 - w_1}{w_3 - w_2}}{\dfrac{w_4 - w_1}{w_4 - w_2}} = \text{cross ratio}$$

$a \cdot \quad \overset{b}{\cdot} \quad \overset{c}{\cdot} \longrightarrow \quad \overset{\cdot}{\alpha} \quad \overset{\cdot}{\beta} \quad \overset{\cdot}{\gamma}$

$\overset{\cdot}{z} \qquad\qquad \overset{\cdot}{w}$

given 3 pts in z plane that map into 3 pts in w plane there is only one FLT that maps $a, b, c \to \gamma, \beta, \delta$

$$\frac{c-a}{c-b} \Big/ \frac{z-a}{z-b} = \frac{\gamma-\alpha}{\gamma-\beta} \Big/ \frac{w-\alpha}{w-\beta}$$

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$f'(a) = \lim_{z \to a} \frac{f(z)-f(a)}{z-a}$ Cauchy Riemann Eqs. $\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}$ $\left(\begin{array}{l} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{array} \right)$ orientation preserving conformal map

Examples of Harmonic Functions

fluid flow incompressible, source free, irrotational steady

$f = u - iv$ is analytic

Def: a function f is analytic if f' exists

$\Delta u = 0, \quad \Delta v = 0$ harmonic function

Def if $f = u + iv$ is analytic, then v is called a conjugate harmonic fn of u

Problem Given $u = x^2 - y^2$ Find a function v s.t. $u + iv$ is analytic

Necessary cond — $\Delta u = 0$ $\quad u_{xx} = 2, \ u_{yy} = -2 \quad \therefore \Delta u = 0$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

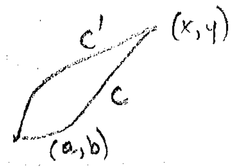
$2x = v_{,y} \Rightarrow v = 2xy + g(x)$ diff this $\frac{\partial v}{\partial x} = 2y + g'(x)$

$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = 2y$

$\Downarrow g'(x) = 0$ or $g =$ con

$\therefore v = 2xy \quad \therefore f = (x^2 - y^2) + i\,2xy + \text{const} = (x + iy)^2 + \text{const}$

$\left\{ \begin{array}{l} \frac{\partial v}{\partial x} = P(x,y) \\ \frac{\partial v}{\partial y} = Q(x,y) \end{array} \right.$ A necess cond & suff cond is $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in simply connected domain sufficiency proof



$$V(x,y) = \int_c P(x,y)\,dx + Q(x,y)\,dy$$

for it to be well chosen line integral ~~must~~ be indep of Path

$$0 = \int_{c-c'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

if true for all $c-c'$ by green's theorem true ~~by~~ $\forall D \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$

<u>Define</u> $e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$ $\quad z = x+iy$

1) e^z is analytic everywhere since it is differentiable

$u = e^x \cos y \quad V = e^x \sin y$

$u_x = e^x \cos y \quad V_y = + e^x \cos y$

$u_y = -e^x \sin y \quad V_x = e^x \sin y$

$u_x = V_y \quad u_y = -V_x$

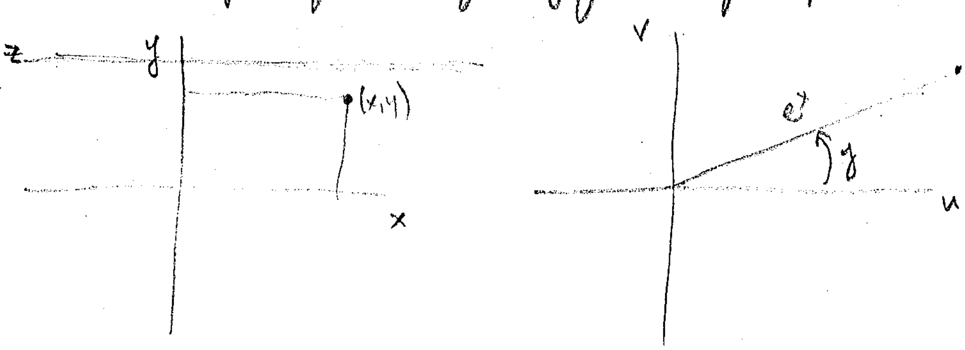
2) e^z satisfies exponential law $e^{z_1+z_2} = e^{z_1} e^{z_2}$

$$e^{z_1} \cdot e^{z_2} = e^{x_1}(\cos y_1 + i \sin y_1)\, e^{x_2}(\cos y_2 + i \sin y_2) = e^{x_1+x_2}[\cos(y_1+y_2) + i \sin(y_1+y_2)$$
$$= e^{z_1+z_2}$$

$$\left.\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \cosh(i\theta) \\ i \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2} = \sinh(i\theta) \end{aligned}\right\} \Rightarrow \begin{matrix} i \sin z \\ \cos z \end{matrix} = \left\{ \frac{e^{i\theta} \mp e^{-i\theta}}{2} \right.$$

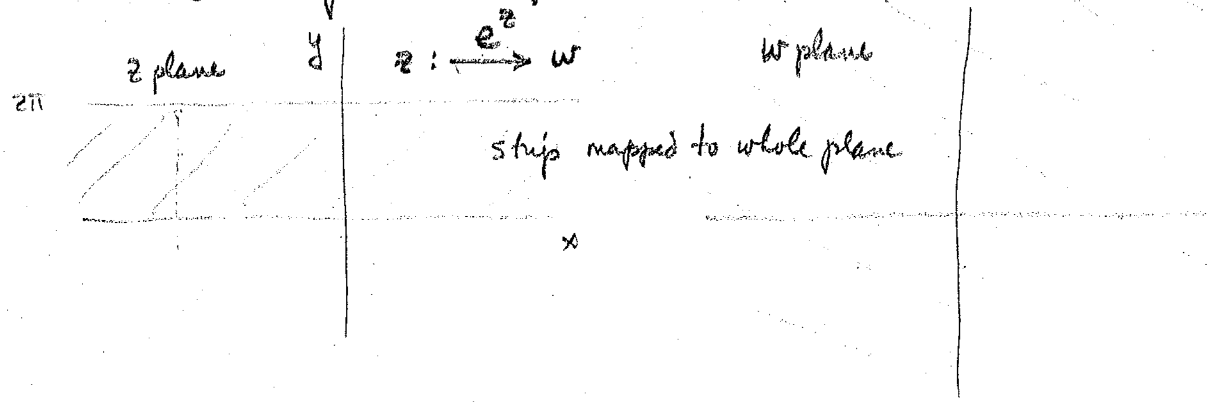
this allows for definition of trig function of complex variable

$w = e^z$



for $y_{new} = y + k2\pi$ the function for constant x will map $z = (x + i(y + 2\pi k))$

into the same point in w plane



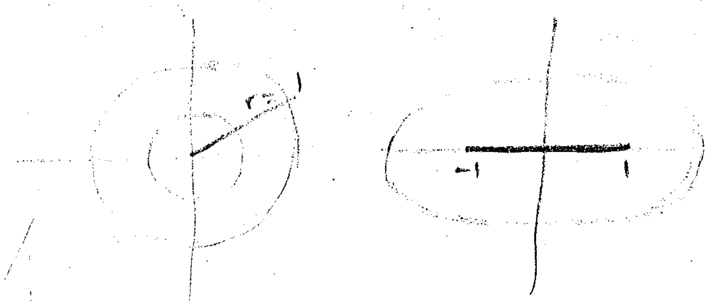
$$w = \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

let $\zeta = iz$ (rotation by 90°) ; let $\alpha = e^{\zeta}$; let $w = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right)$

what is $\frac{1}{2}\left(z + \frac{1}{z}\right)$ mapping $\quad u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta$, $v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta$

image of the circle of radius r is an ellipse w/ semi major axis $\frac{1}{2}\left(r + \frac{1}{r}\right)$

as $r \to \infty \quad u, v \to \frac{1}{2} r \cos\theta, \sin\theta$



as $r \to 1$ semi minor axis $\to 0$, semi maj is line

maps whole circle $r \geq 1$ onto whole plane less slit

map whole circle $r < 1$ onto whole plane less slit

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Aside

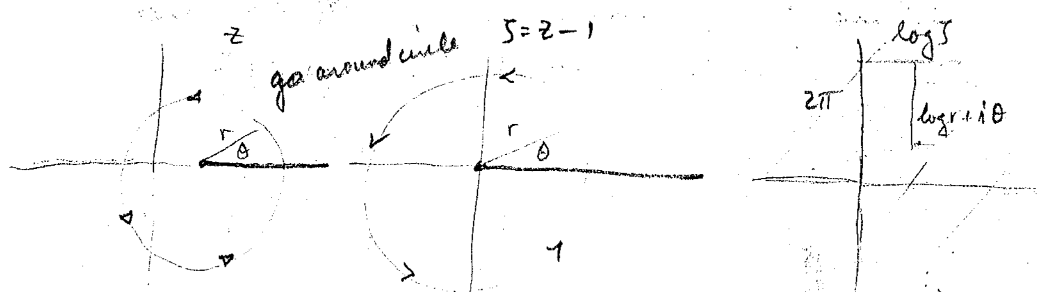
$$h = \frac{Btu}{lbm} \cdot 32.2\frac{lbm}{slug} = h \cdot 32.2 = \frac{Btu}{slug} \qquad \frac{V^2}{2J} = \frac{ft^2}{sec^2}\frac{Btu}{ft\text{-}lb} = \frac{Btu}{\frac{lb\text{-}sec^2}{ft}} = \frac{Btu}{slug}$$

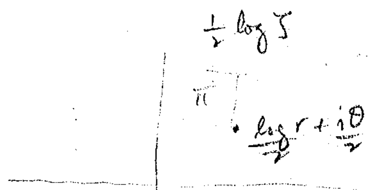
$$\frac{ft\text{-}lb}{lb_m\text{-}{}^\circ R} \times \frac{V^2}{2} = \frac{\gamma}{\gamma-1}R(T-T_0) = h - h_0 = \frac{V^2}{2g_cJ} \qquad h - h_0 = \frac{V^2}{2g_cJ} \text{ if } v = \text{ft/sec},\ h = \frac{Btu}{\text{\# mass}}$$

$$\frac{ft\text{-}lb}{lbm} \times g_c\frac{lbm}{slug} = \frac{ft\text{-}lb}{\frac{lb\text{-}sec^2}{ft}}$$

log function $\quad \mathbb{C} - 0 \quad$ has a branch cut we can take it to be $+x$ axis

$$f(z) = \sqrt{z^2-1} = e^{\frac{1}{2}\log(z^2-1)} = e^{\frac{1}{2}\log(z-1)} \cdot e^{\frac{1}{2}\log(z+1)} = \sqrt{z-1}\sqrt{z+1}$$





note that $f(z) = (\pm\sqrt{z-1})(\pm\sqrt{z+1})$ which shows that each fn can take on 2 values ($\pm$) but $f(z)$ restricts them to being the same (+ or −)

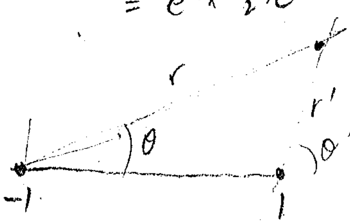
note that using the log argument we can get the branch cut of $z \pm 1$ to be from $z = \mp 1$ to ∞. However note that for $z_1 = 1 + \epsilon e^{i\theta}$, $f(z)$ = well define, continuous, single valued on whole plane less -1 —— $+1$; hence branch is from $[-1, 1]$

Question find the value of the branch of (?) defined on $\mathbb{C} - [-1,1]$ at $z = 1+i$ if the branch assumes the value $-\sqrt{3}$ at $z = 2$

$$f(z) = e^{\frac{1}{2}\log(z-1)} \cdot e^{\frac{1}{2}\log(z+1)}$$
$$= e^{\frac{1}{2}\log r + i\frac{\theta}{2}} \cdot e^{\frac{1}{2}\log r' + i\frac{\theta'}{2}} = e^{\frac{1}{2}(\log r + \log r') + i\left(\frac{\theta+\theta'}{2}\right)}$$

$f(z) = \sqrt{3}\, e^{i\pi}$

$\Rightarrow \frac{\theta+\theta'}{2} = \pi$ or $\theta + \theta' = 2\pi$



since $z = 2$ is where $f(z) = \sqrt{3}$ it cannot lie on a branch cut $\therefore$ must change our cut $\therefore$ take the cut on the negative axis

10/11/78

$z = \log w$ is not unique

$w = u + iv = e^{x+iy} = e^x \cos y + e^x i \sin y$

$e^x = |w| \qquad y = \arg w$

$x = \log|w| \qquad y = \arg w$ as an angle it is unique but not as a no.

$z = \log w = \log|w| + i \arg w$

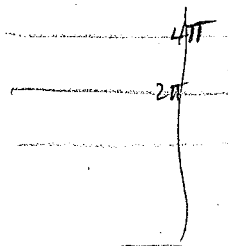
if select fixed strip 2π wide (well defined) choose points only in strip

$0 \le \arg w < 2\pi$ is discontinuous $\therefore$ take $0 < \arg w < 2\pi$ this will give a continuous fn but leaves out the entire line from $w=0$ to ∞

if like all branches - take an infinite no. of w planes (since $\arg w$ is same for $\arg w + 2\pi n$) ; For every w plane delete + axis

$\therefore$ on 0^{th} w plane, 0^{th} branch defined

on 1^{st} w plane, 1^{st} branch is defined etc.



In this way we define the Riemann spiral surface of the log function which is single valued and well defined.

rest of 10/13/78

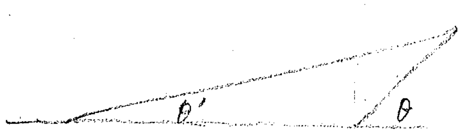
for θ take $-\pi < \theta < \pi$ if

for θ' take $2\pi < \theta' < 4\pi$

θ continuously varies for 0 to $\pi/2$

θ' continuously varies for 0 to $\tan^{-1}\frac{1}{2}$

total increment $\frac{\pi}{2} + \tan^{-1}\frac{1}{2}$



since @ $z=2$ $f(z) = e^{i\pi/3} \Rightarrow e^{i\pi} = -1$

$\therefore (-1)\, e^{i(\pi/2 + \tan^{-1}\frac{1}{2})}$

find the value of the branch of $\log(1-z^2)$ at point $2+3i$ which is defined on the right half plane less line segment $[0,1]$ and takes on value $3\pi i$ @ $\sqrt{2}=z$

$$1-z^2 = 1-(2+3i)^2 = 1-(4+12i-3) = 6-12i$$

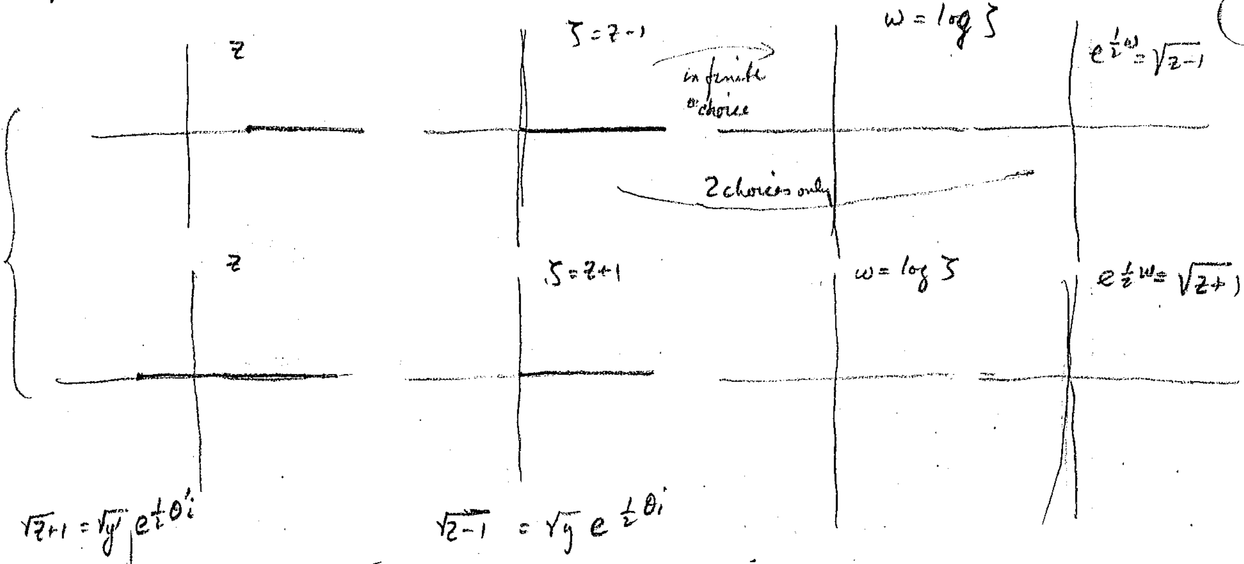
$$\log(6-12i) = \log|6-12i| + i\arg(6-12i)$$
$$= 6\sqrt{5} + i\tan^{-1}\left(\frac{-12}{6}\right)$$

Since $\arg$ of $\log(1-z^2)|_{z=\sqrt{2}} = 3\pi i$ then $\tan^{-1}-2$ must be between 2π & 4π or $2\pi < \tan^{-1}(-2) < 4\pi$

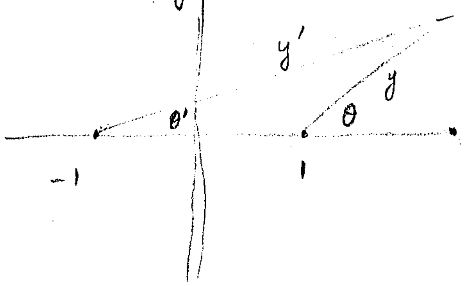
10/16/78

$$w = \sqrt{z^2-1} = \sqrt{z-1}\ \sqrt{z+1}$$

product will only give you choices

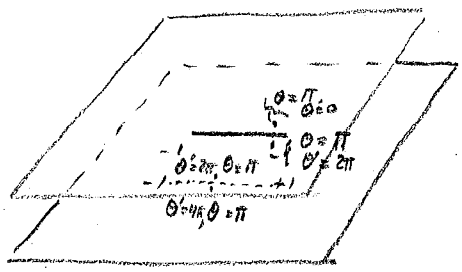


$\sqrt{z+1} = \sqrt{r'}\, e^{\frac{1}{2}\theta' i}$ $\sqrt{z-1} = \sqrt{r}\, e^{\frac{1}{2}\theta i}$



$$\sqrt{z^2-1} = \sqrt{rr'}\, e^{\frac{1}{2}(\theta+\theta')i}$$

points outside slit as θ increases by 2π θ' also increases by 2π



take $0 < \theta < 2\pi$ $\quad 0 < \theta' < 2\pi$ one branch
$0 < \theta < 2\pi$ $\quad 2\pi < \theta' < 4\pi$ the other br

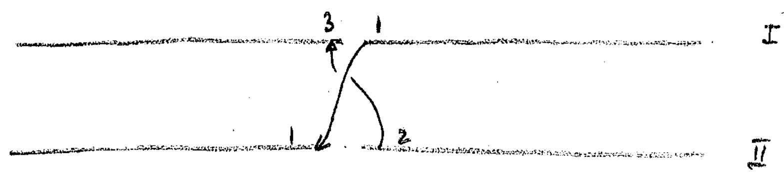
$2\pi < \theta < 4\pi$ and $2\pi < \theta' < 4\pi$ one branch
$2\pi < \theta < 4\pi$ & $4\pi < \theta' < 6\pi$ the other br.

at 1st sheet from above $f(z) = \sqrt{rr'}\, e^{i(\pi + 0)\frac{1}{2}} = i\sqrt{rr'}$

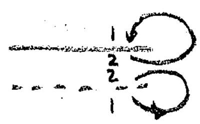
below $f(z) = \sqrt{rr'}\, e^{i(3\pi)\frac{1}{2}} = -i\sqrt{rr'}$

on 2nd sheet from above $f(z) = \sqrt{rr'}\, e^{i\left(\frac{3\pi}{2}\right)} = -i\sqrt{rr'}$

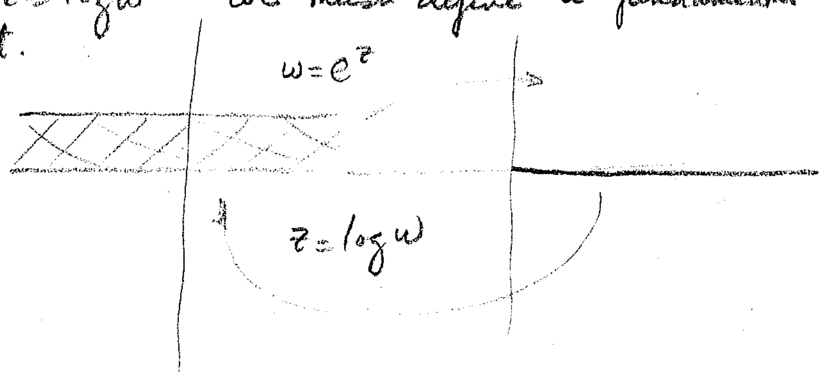
$f(z) = \sqrt{rr'}\, e^{i(5\pi/2)} = i\sqrt{rr'}$



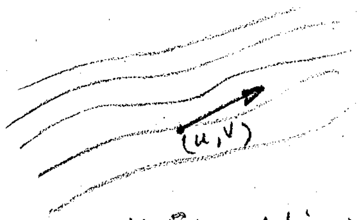
cross section



for $w = e^z$ this is not a 1-1 map: in order to define an inverse such as $z = \log w$ we must define a fundamental region after making a cut.



2-D, Incomp, irrotational, steady, source-free



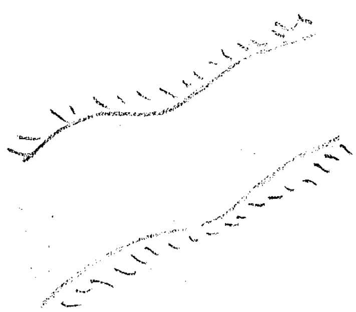
define $f = u - iv$ suppose $f = F'$ for some analytic function F

if F is analytic $\Rightarrow F' = \frac{dF}{dz}$ exists $= \frac{\partial F}{\partial x} \Rightarrow \begin{matrix} U_x = V_y \\ -V_x = U_y \end{matrix}$ and for $F' = U_x + iV_x$
$\frac{\partial F'}{\partial x} = \begin{cases} U_{xx} = V_{xy} \\ U_{xy} = -V_{xx} \end{cases}$

$u = U_x$, $v = -V_x = -U_y \Rightarrow (u,v) = \text{grad}\, U$ U is the velocity potential
$U =$ const define equipotential lines.
grad $V \perp$ grad U since
$\nabla V \cdot \nabla U = U_x V_x + (-V_x)(U_x) = 0$ $\therefore$ $\nabla V \perp \nabla U$ if $U, V \neq 0$

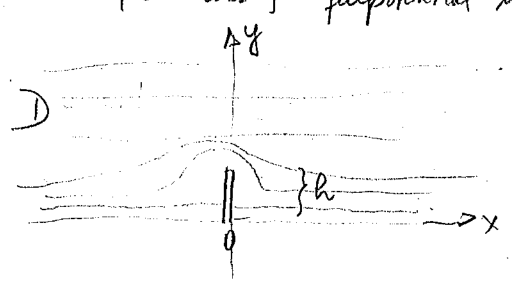
V are the streamlines.

boundaries in fluid flow problem is also a stream line
if $f = u - iv$
$\text{Im}\, F =$ const. since $\text{Im}\, F$ represents streamline



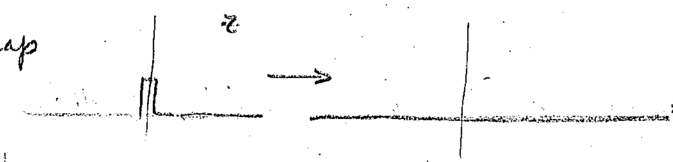
10/18/78

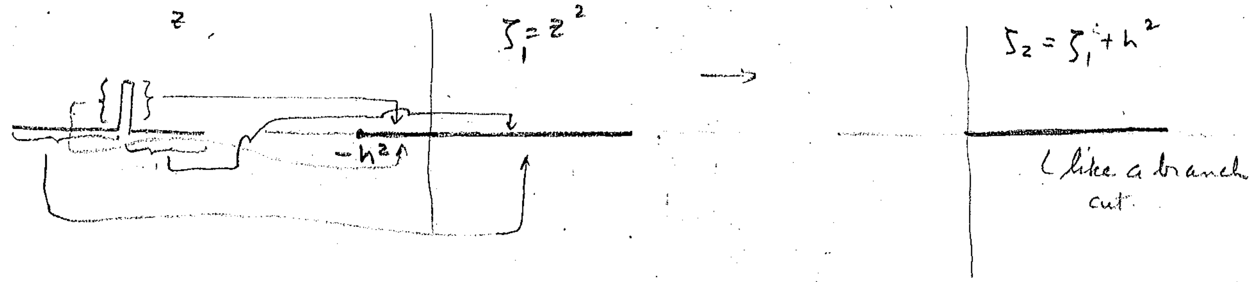
Given flow and velocity vector (u,v) define $f = u - iv$ which is analytic
so that $F' = f$. Then $\exists$ an $F = U + iV$ s. $\nabla U = (u,v)$
and $\{U = \text{const}\}$ equipotential lines $\{V = \text{const}\}$ streamlines



Find an analytic fn F on the domain D $\ni$ $\operatorname{Im} F = \text{const}$ on $y=0$ and $\{x=0, 0 \le y \le h\}$

We want to map



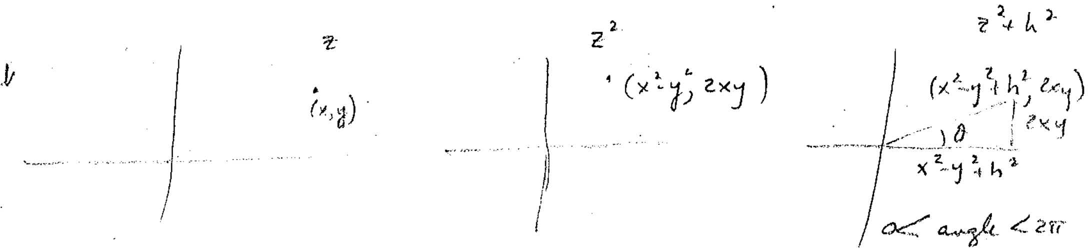


let $W = \sqrt{\zeta_2}$

1 possible solution $F(z) = \sqrt{z^2+h^2}$ $\quad \operatorname{Im} F > 0$ when $z < 0$

$$F'(z) = \frac{-z}{\sqrt{z^2+h^2}} = u - iv \qquad u = \operatorname{Re}\left(\frac{z}{\sqrt{z^2+h^2}}\right) \qquad v = -\operatorname{Im}\left(\frac{z}{\sqrt{z^2+h^2}}\right)$$

take a point



$$\frac{1}{2}\operatorname{atan}\frac{2xy}{x^2-y^2+h^2} = \frac{\theta}{2} \qquad r = \sqrt{(x^2-y^2+h^2)^2+(2xy)^2}$$

$$\sqrt{z^2+h^2} = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \sqrt{r}\left[\sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{\frac{1-\cos\theta}{2}}\right]$$

$$\sqrt{\ } = \sqrt{r}\left\{\sqrt{\frac{1+\frac{x^2-y^2+h^2}{r}}{2}} + i\sqrt{\frac{1-\frac{x^2-y^2+h^2}{r}}{2}}\right\}$$

$$\text{or } \frac{z}{\sqrt{z^2+h^2}} = \frac{z\left(\overline{(z^2+h^2)^{1/2}}\right)}{|z^2+h^2|^{2}} = \frac{(x+iy)\sqrt{r}\,(A - iB)}{(\sqrt{r})^2}$$

$$= \frac{xA+yB}{\sqrt{r}} \quad \therefore \quad u = \frac{x}{\sqrt{r}}\sqrt{\frac{1+\frac{x^2-y^2+h^2}{r}}{2}} + \frac{y}{\sqrt{r}}\sqrt{\frac{1-\frac{x^2-y^2+h^2}{r}}{2}}$$

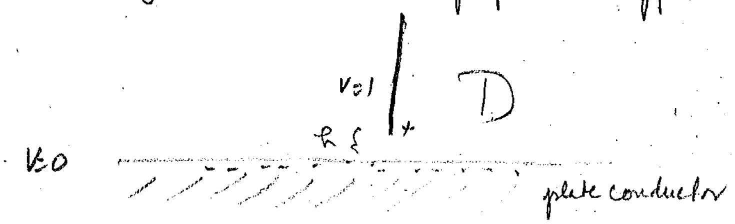
Electrostatics $\underline{E} = (E_1, E_2) = -\operatorname{grad} V$

$\operatorname{div} \underline{E} = c\rho$

no charge inside $\rho = 0$ source free, sink free $\operatorname{div} \operatorname{grad} V = 0$

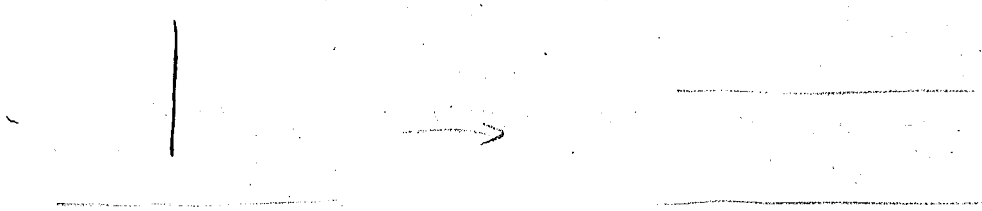
$\Delta V = 0$

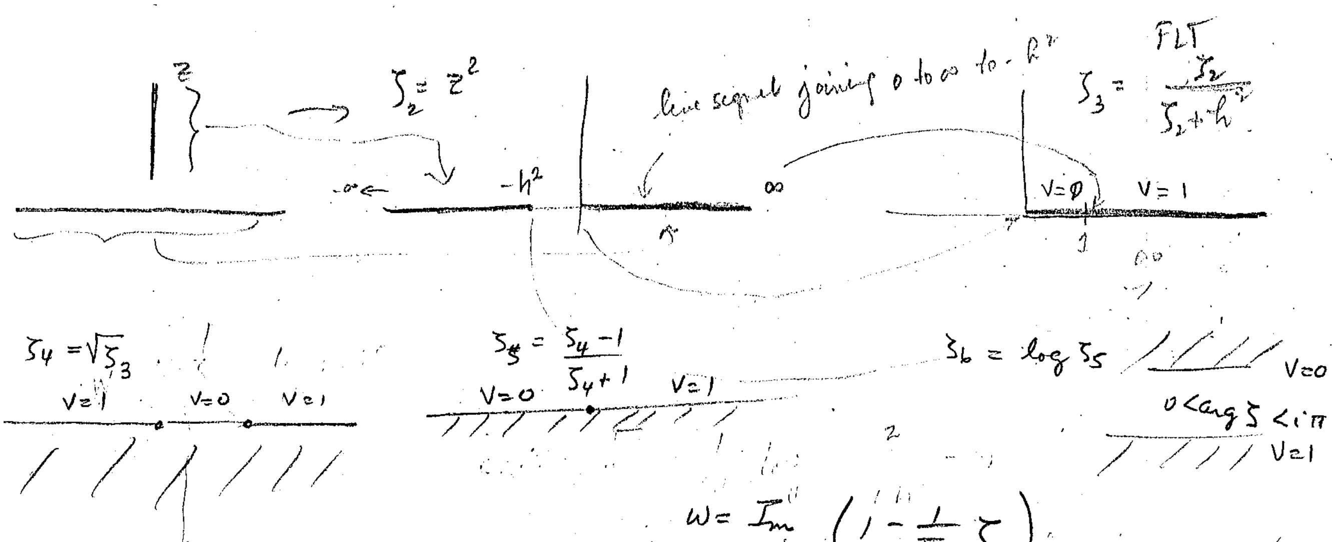
determine field between 2 cond if potential diff is $= 1$



Find an analytic fn F in D $\rightarrow$ $\operatorname{im} F = 0$ on $y = 0$

$\operatorname{im} F = 1$ on $x = 0$ $y \geq h$





$w = \operatorname{Im}\left(1 - \frac{1}{\pi}\zeta_6\right)$

Im w is the potential for $\frac{1}{2\pi} \arg\left(\frac{z^2}{z^2+h^2}\right)$ 10/20/78

given $F = U + iV$

V = potential

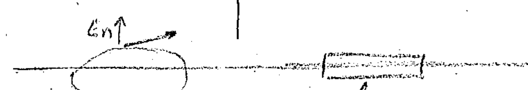
$$\bar{E} \text{ field} = -\nabla V = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}\right) = \left(-\frac{\partial V}{\partial x}, -\frac{\partial U}{\partial x}\right) = -\frac{\partial V}{\partial x} - i\frac{\partial U}{\partial x}$$

$$= -i\left(\frac{\partial U}{\partial x} - i\frac{\partial V}{\partial x}\right) = -i\overline{F'}$$

$$\|\bar{E}\| = |F'|$$

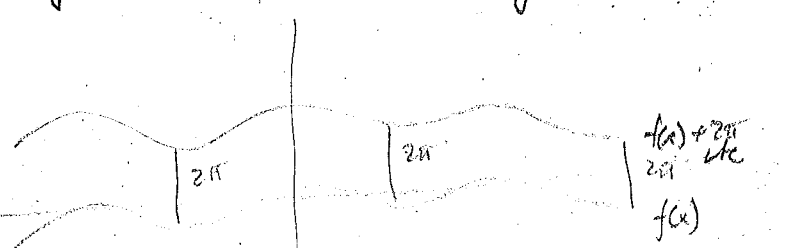
div field $= \frac{1}{\epsilon_0}$ charge density $\nabla \cdot \bar{E} = \frac{\rho}{\epsilon_0}$

$$\int_{\partial D} E_n \, ds = \int_D \nabla \cdot E \, dV = \int_D \frac{\rho}{\epsilon_0} \, dV$$



since field is $\perp$ to surface $l\|\bar{E}\| = \frac{l\rho}{\epsilon_0} \Rightarrow \|\bar{E}\| = \frac{\rho}{\epsilon_0}$

You can map $e^z = w$ however the inverse is not singlevalued so you can define a particular strip of $z = \log w$ for the inverse map but you can also use a curved function which should satisfy



if $e^z = e^{x+iy} \Rightarrow r = e^x, \; y = \theta$

$\Rightarrow$ since $0 < \theta < 2\pi \Rightarrow f(x) < y < f(x) + 2\pi$

where $f(x)$ must be defined for $-\infty, \infty$

$\therefore f(\log r) < \theta < f(\log r) + 2\pi$

$\therefore$ we can show that this curved strip will be mapped into a curved line. This cut need not be a straight line but a curve, Remember

- $\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ $\left\{ \begin{array}{l} \text{div th. } \int_{\partial D} \underline{E}\cdot\underline{n}\,ds = \iint_D \nabla\cdot\underline{E} \\ \int_{\partial D} \underline{E}_{\text{tangential}} = \iint_D \nabla\times\underline{E} \end{array}\right.$

- $\int \nabla f \cdot dr = f(b) - f(a)$

$= \int df$

Recall Given a curve C $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $a \le t \le b$

Given a vector field (or a pair of functions $P(x,y)$ $Q(x,y)$)

$$\int_C P\,dx + Q\,dy \overset{\text{def}}{=} \int \left[P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\right] dt$$

P, Q should be piecewise continuous

$$\int_a^w f(z)\,dz \text{ in complex plane} \implies \lim \sum_i f(z_i)\,(z_{i+1} - z_i)$$

where z_i's are points on a curve which is in the domain of f for which f is defined

let $f = u + iv$ $z = x + iy$

$$\sum_k f(z_k)(z_{k+1} - z_k) = \sum (u_\ell + i v_\ell)\left[(x_{\ell+1} + i y_{\ell+1}) - (x_\ell + i y_\ell)\right]$$

$$= \sum_\ell \left[u_\ell (x_{\ell+1} - x_\ell) - v_\ell (y_{\ell+1} - y_\ell)\right] + i \sum \left[v_\ell (x_{\ell+1} - x_\ell) + u_\ell (y_{\ell+1} - y_\ell)\right]$$

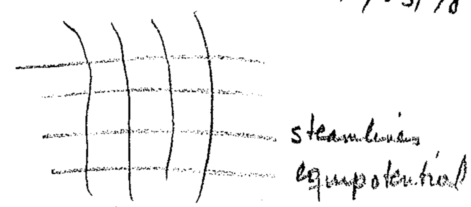
$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i \int_C v\,dx + u\,dy$$

10/23/78

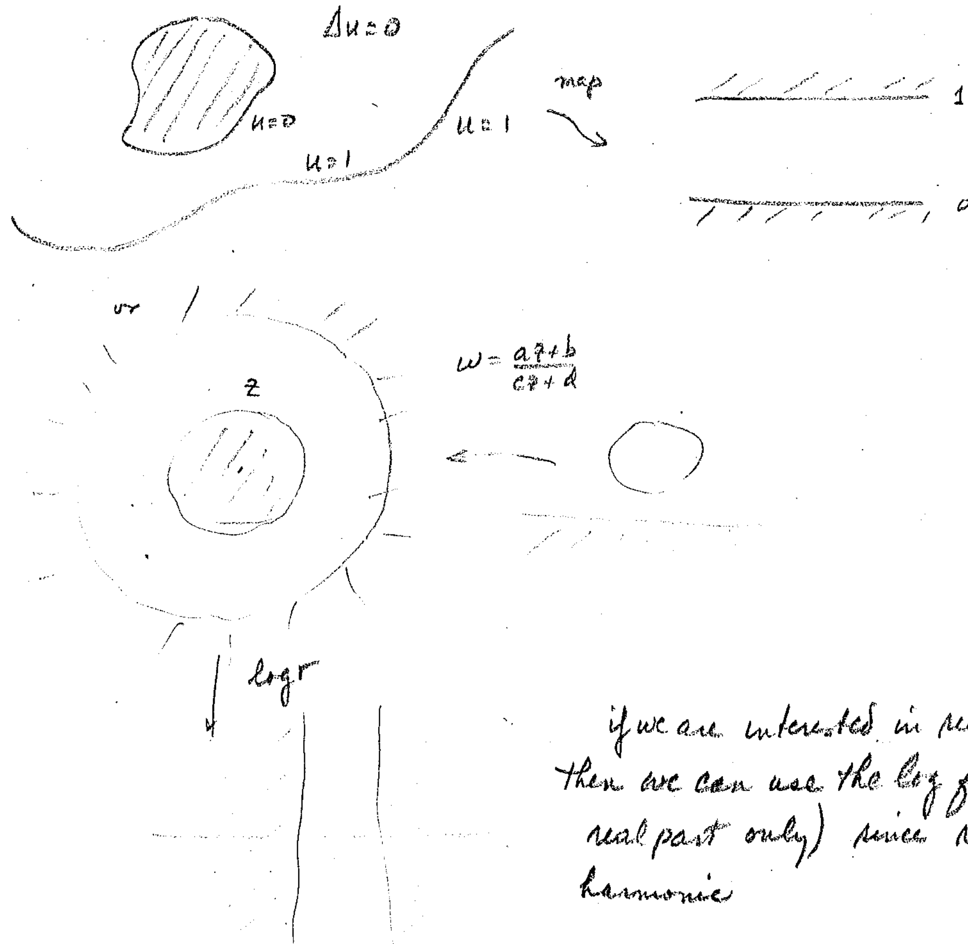
Conjugate flow $f = u - iv$

$F' = f$ $\quad U = \text{const}$

$V = \text{const}$



if $F \to iF$

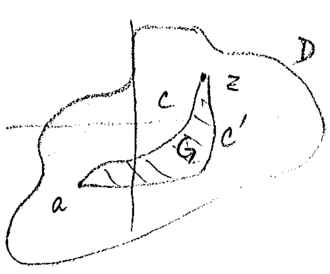


if we are interested in real part only then we can use the log function (its real part only) since real part is also harmonic

Motivating

Given f analytic on D Find an analytic function F such that $F' = f$

$$\int_c f dz \overset{def}{=} \int_c (u+iv)(dx+idy) \overset{def}{=} \int_c (u\,dx - v\,dy) + i\int_c v\,dx - u\,dy$$



investigate $\int_c u\,dx - v\,dy \overset{?}{=} \int_{c'} u\,dx - v\,dy$

look at 2 curves

is $\int_{c'-c} u\,dx - v\,dy = 0$ & is $\int_{c'-c} v\,dx - u\,dy = 0$

we look at green's theorem if u, v are continuously differentiable

$$\iint_G \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \int_{C'C} u\,dx - v\,dy$$

however analytic functions only need to have derivatives existing
if we take the stronger condition that u, v have cont. first deriv

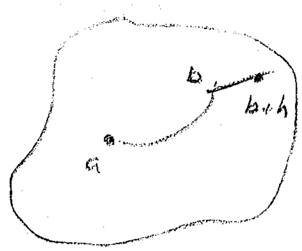
$$\therefore \iint_G (\qquad) dx\,dy = \int_{CC'} u\,dx - v\,dy = 0$$

but $(\quad) = 0$ by Cauchy-Riemann so therefore $\int f(z)\,dz$
exists & is independent of path.
If the region D has a hole in it Green's theorem doesn't work
$\therefore$ we must restrict D to a simply connected region

$\therefore$ if D is simply connected and u, v have continuous partial deriv
of 1st order and f is analytic on D

Then $F(b) = \int_C f(z)\,dz$ C = any curve joining a to b

Now we claim $F'(z) = f(z)$ to prove



$$\lim_{h\to 0} \frac{F(b+h) - F(b)}{h}$$

$$\lim_{h\to 0} \frac{F(b+h) - F(b)}{h} = \lim_{h\to 0} \frac{1}{h} \int_{[b,\,b+h]} f(z)\,dz$$

$$\frac{F(b+h) - F(b)}{h} - f(b) = \frac{1}{h} \int_{[b,\,b+h]} f(z)\,dz - f(b)$$

$$= \frac{1}{h} \int_{[b,\,b+h]} \{f(z) - f(b)\}\,dz$$

assuming $\left| \int_C f(z)\,dz \right| \leq ML$ where $M = \sup_{z \in C} |f(z)|$ L = length of C

we look at right hand side

$$\left| \frac{1}{h} \int \{f(z) - f(b)\}\,dz \right| \leq \frac{1}{|h|} \sup_{z \in [b, b+h]} |f(z) - f(b)| \cdot |h| \to 0 \text{ as } h \to 0$$

$$\Rightarrow \lim \frac{F(b+h) - F(b)}{h} = f(b) \quad \text{QED}$$

let $C: \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \quad 0 \leq t \leq L$ we will prove our assumption by parametrizing C wrt arclength

$$\int_C f\,dz = \int_C (u\,dx - v\,dy) + i \int_C v\,dx + u\,dy$$

$$= \int_0^L \left(u \frac{dx}{dx} - v \frac{dy}{dt} \right) dt + i \int_0^L \left(v \frac{dx}{dt} + u \frac{dy}{dt} \right) dt$$

if we take $\left| \int_C f\,dz \right|^2 = \left(\int_0^L \left(u \frac{dx}{dt} - v \frac{dy}{dt} \right) dt \right)^2 + \left(\int_0^L \left(v \frac{dy}{dt} + u \frac{dy}{dt} \right) dt \right)^2$

and we use the schwarz inequality $\left| \int_a^b g(t) h(t)\,dt \right|^2 \leq \int_a^b |g(t)|^2 dt \int_a^b |h(t)|^2 dt$
we will prove later

$$\left| \int_C f\,dz \right|^2 \leq \int_0^L 1\,dt \cdot \int_0^L \left(u \frac{dx}{dt} - v \frac{dy}{dt} \right)^2 dt + \int_0^L 1 \cdot dt \int_0^L \left(v \frac{dx}{dt} + u \frac{dy}{dt} \right)^2 dt$$

$$\leq L \int_0^L (u^2 + v^2) \left(\left[\frac{dx}{dt} \right]^2 + \left[\frac{dy}{dt} \right]^2 \right) dt + L \int_0^L (u^2 + v^2) \left(\left[\frac{dx}{dt} \right]^2 + \left[\frac{dy}{dx} \right]^2 \right) dt$$

$$\leq 2L^2 \sup (u^2 + v^2)$$

$$\leq 2L^2 |f|^2$$

10/25/78

$F' = f$

D is simply connected domain

f analytic $f = u + iv$, u, v continuously differentiable

$F(b) = \int f(z)\,dz$ (any curve joining a, b) $F'(z) = f(z)$

Lemma $\left|\int_c f(z)\,dz\right| \le ML$ where M is the max of $|f|$ on C
L = length of C

use Schwarz ineq $\left|\int_{t=a}^{b} fg\,dt\right|^2 \le \left(\int_a^b |f|^2 dt\right)\left(\int_a^b |g|^2 dt\right)$

Parametrize C by arc length t from $t=0$ to $t=L$

$$\left|\int_c f(z)\,dz\right|^2 = \left|\int_0^L (u+iv)\left(\frac{dx}{dt}+i\frac{dy}{dt}\right)dt\right| \le \left(\int_0^l |f|^2 dt\right)\left(\int_0^L \left(\frac{dx}{dt}+i\frac{dy}{dt}\right)^2 dt\right)$$

since we are using arclength $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=1$ since $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$

arclength in terms of any parameter t

if $t=s$ the result follows

$$\le \left(\int_0^l |f|^2 dt\right)\left(\int_0^l dt\right) \le \left(\int_0^L M^2 dt\right) L = M^2L^2$$

$$\therefore \left|\int_c f(z)\,dz\right| \le ML$$

look at Schwarz inequality especially $0 \le \int |g+\lambda\bar{h}|^2 dt \quad \forall \lambda$

$$= \int (g+\lambda\bar{h})(\bar{g}+\bar{\lambda}h)\,dt = \int |g|^2 dt + \lambda\int \bar{h}\bar{g}\,dt + \bar{\lambda}\int gh\,dt + \lambda\bar{\lambda}\int |h|^2 dt$$

now $\left|\int g(t)h(t)\,dt\right|^2 = \left(\int gh\,dt\right)\left(\int \bar{g}\bar{h}\right)dt$

if we let $\lambda = -\left(\int gh\,dt\right) / \int |h|^2 dt$

$$0 \le \int |g|^2 dt - \frac{\int gh\,dt}{\int |h|^2 dt}\int \bar{h}\bar{g}\,dt - \frac{\int \bar{g}\bar{h}\,dt}{\cancel{\int |h|^2 dt}}\int gh\,dt + \frac{\cancel{\left(\int gh\,dt\right)}\left(\int \bar{g}\bar{h}\right.}{\int |h|^2 dt}$$

$$\therefore \left|\int gh\,dt\right|^2 \le \left(\int |g|^2 dt\right)\left(\int |h|^2 dt\right)$$

geometric proof of $\left|\int_C f(z)\,dz\right| \le ML$ take n pts on C $\exists$ a max value of $f(z)$ on the closed arc

$$\therefore \left|\int_c f(z)\,dz\right| \le \sum_i f(z_i)\,\Delta z_i \le M \sum \Delta z_i$$

$\lim \|\Delta z\| \to 0$ says $\le ML$

Cauchy theorem

f is analytic in D, C is piecewise smooth, simple, closed curve in D $\ni$. D contained the domain enclosed by C $\Rightarrow \int_C f(z)\,dz$

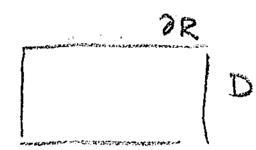
Proof: parametrized C with $x = x(t)$, $y = y(t)$ $\quad a \le t \le b$

$$\int_C f(z)\,dz = \int_a^b (u x' - v y')\,dt + i \int_a^b (v x' - u y')\,dt$$

(To define this u, v must be cont; $\frac{dx}{dt}, \frac{dy}{dt}$ must also continuous C must be piecewise cont.)

(Simple curve means it doesn't cross itself)

Look a simple case where D = rectangle $C = \partial R$



$$\int_{\partial R} f(z)\,dz = \sum_{\nu=1}^{4} \int_{\partial R_\nu} f(z)\,dz$$

Assume $\left| \int_{\partial R} f(z)\,dz \right| = c > 0$ and least one of $\int_{\partial R_\nu} f(z)\,dz$

must be $\ge \frac{c}{4}$ in absolute value and take it and cut it into another rectangle. We get another similar result. If we repeat it we get a nest rectangles $R \supset R^{(1)} \supset R^{(2)} \supset R^{(3)} \supset \ldots \supset R^{(n)}$. Now $\exists!$ a point a belong to all closed rectangles, $R, R^{(1)}, \ldots R^{(n)}$ until

$$\left| \int_{\partial R^{(N)}} f(z)\,dz \right| \ge \frac{c}{4^n}$$

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Test Nov 6 Covers up to Cauchy Thm

Continuation of proof:

Call point a $\quad f'(a)$ exists (since we are analytic in D) and

$f(z) = f(a) + f'(a)(z-a) + \epsilon(z)(z-a)$ where $\epsilon(z) = 0$ when $\lim_{z \to a}$

$$\int_{R^{(N)}} f(z)\,dz = \int_{R^{(N)}} (f(a) + f'(a)(z-a))\,dz + \int_{R^{(N)}} \epsilon(z)(z-a)\,dz$$

Since $f(a) + f'(a)(z-a)$ is a continuous linear funct then by Greens theorem $\int (\quad)\,dz = 0$

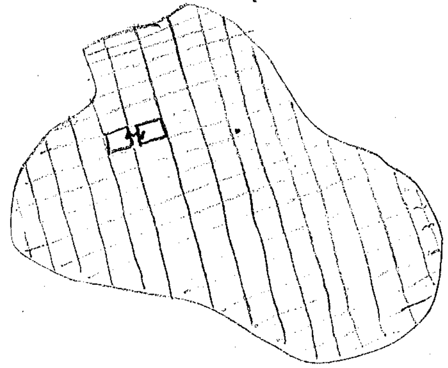
$\therefore \quad \left| \int_{R^N} f(z)dz \right| = \left(\int_{R^N} \epsilon(z)\,(z-a)\,dz \right) \le \max \left|\epsilon(z)(z-a)\right| \frac{L}{2N}$

$|z-a|$ in $R^N < \frac{L}{2 \cdot 2N}$ for sure

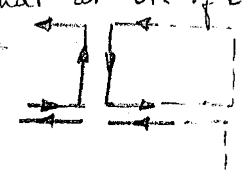
$\therefore \le \max|\epsilon(z)| \; \frac{L^2}{2} \; \frac{1}{4N}$ when z is sufficiently large.

$\max|\epsilon(z)|$ can be made as small as we please $\therefore$ make $\max|\epsilon(z)| \cdot \frac{L^2}{2} < e$

<u>General case</u> - approximate by a finite # of rectangles



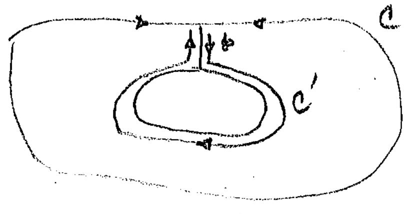
break up - integration over each square. note that at ∂R of each square the components from to adjoining squares will cancel out



Cauchy's formula says for the same condition as the theorem. and for a point a in D enclosed by C

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz$$

We don't care that $f(z)/(z-a)$ is <u>not</u> defined at a because we integrate around C not across a.



Suppose we have a hole - Make a slit and

$$\int_C f(z)\,dz + \int_{C'} f(z)dz + \underbrace{\int_P f(z)\,dz + \int_{-P} f(z)dz}_{=0 \text{ since } \int_{-P} = -\int_P} = 0$$

define $g(z) = \frac{f(z)}{z-a} \implies \int g(z)dz \ne 0$ since $g(z)$ is not analytic at a. Thus take a small circular around a.

$$\int_C g(z)\,dz = \int_{|z-a|=r} g(z)\,dz \quad \forall r$$

let $z = a + re^{i\theta}$ $0 \le \theta < 2\pi$

Look at the special case $f(z) = 1$

$$\int_C \frac{1}{z-a}\,dz = \int \frac{1}{re^{i\theta}}\, ire^{i\theta}\,d\theta = \int i\,d\theta = 2\pi i$$

$$\int_{|z-a|=r} \frac{f(z)\,dz}{z-a} = \int_{|z-a|=r} \frac{f(z)-f(a)}{z-a}\,dz + \int_{|z-a|=r} \frac{f(a)\,dz}{z-a}$$

$$= \int \frac{f(z)-f(a)}{z-a}\,dz + f(a)\cdot 2\pi i$$

$$\left| f(a)\cdot 2\pi i - \int \frac{f(z)\,dz}{z-a} \right| = \left| \int \frac{f(z)-f(a)}{z-a}\,dz \right| \le \max \left| \frac{f(z)-f(a)}{z-a} \right| \cdot 2\pi r$$

since $f(z)$ has a derivative at a; then as $z \to a \Rightarrow r \to 0$

or

$$\lim_{z\to a} \left| f(a)\cdot 2\pi i - \int \frac{f(z)\,dz}{z-a} \right| = 0 \quad \text{or} \quad \boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz}$$ Cauchy Kernel!

but $f(z) = \dfrac{1}{2\pi i}\displaystyle\int \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i} \lim_{\|\Delta\zeta_\nu\|\to 0} \sum \frac{f(\zeta_\nu)\,\Delta\zeta_\nu}{\zeta_\nu - z}$

thus any analytic function is a linear sum of functions of the form $\frac{1}{a-z}$

From the Cauchy Kernel: given boundary values of a function on a circle (or curve) we know the function at every point inside the circle (or curve).

Why because Cauchy formula relates points on the boundary to points inside the domain.

10/31/78

Grader 381-B

Solutions to problems will be posted at his door by the end of this week.

Problem: Given boundary values on C. Does there always exist an analytic function f on domain D enclosed by C with given bdry values?

Dirichlet problem same as the one stated above except that "analytic" replaced by "harmonic"

analytic function $\Rightarrow$ $Im f$, $Re f$ satisfy $\Delta(Im f) = \Delta(Re f) = 0$ with $(Im f)_{\partial D}$ with $(Im f)_{\partial D}$

but an analytic problem must also be $\ni$. f satisfies C.R. equation

not all functions will satisfy this compatability

Example $f = u + iV$ on $C = \partial D$

Special case $V = 0$ on C

$u \not\equiv$ const in C

suppose we have a solution $f = u + iV$ on D

$V \equiv 0$ on D.

by cauchy riemann $\Rightarrow u \equiv$ const $\therefore$ we have $\rightarrow\leftarrow$ contradiction

$\therefore$ Dirichlet problem is a lot weaker since the function is <u>not</u> constrained by some compatability equations (such as C.R. eqns)

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z} \qquad z \in D \quad , \quad C = \partial D$$

<u>Thm</u>: Cauchy's integral formula for derivatives

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}$$

lets look at $g_\zeta(z) = \dfrac{1}{\zeta - z}$ $\quad g'_\zeta(z) = \dfrac{+1}{(\zeta - z)^2}$ $\quad g''_\zeta(z) = \dfrac{2}{(\zeta - z)^3}$ $\quad \cdots \quad g^{(n)}_\zeta(z) = \dfrac{n!}{(\zeta - z)^{n+1}}$

thus we note that the burden of differentiation is shifted from f to the kernel and f is infinitely differentiable since g is infinitely differentiable this integral formula shows that integration is equivalent to differentiation

Note

2 complex variable use 1. evaluation of definite integrals by method of residues 2. Conformal mapping of Poisson type problems

$$f(z+h) = \frac{1}{2\pi i}\int_{\zeta\in C} \frac{f(\zeta)\,d\zeta}{\zeta-(z+h)} \qquad f(z) = \frac{1}{2\pi i}\int_{\zeta\in C} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\int_{\zeta\in C} \frac{1}{h}\left(\frac{1}{\zeta-(z+h)} - \frac{1}{\zeta-z}\right) f(\zeta)\,d\zeta$$

$$= \frac{1}{2\pi i}\int_{\zeta\in C} \frac{f(\zeta)}{[\zeta-(z+h)][\zeta-z]}\,d\zeta$$

$$\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_{\zeta\in C}\frac{f(\zeta)\,d\zeta}{(\zeta-z)^2} = \frac{1}{2\pi i}\int_{\zeta\in C}\left[\frac{1}{(\zeta-(z+h))(\zeta-z)} - \frac{1}{(\zeta-z)^2}\right] f(\zeta)\,d\zeta$$

$$= \frac{h}{2\pi i}\int \frac{f(\zeta)\,d\zeta}{(\zeta-z+h)(\zeta-z)^2}$$

In general $f_n \to f \not\Rightarrow \int_a^b f_n(x)dx \to \int_a^b f(x)dx$

Lebesgue convergence thm states that $f_n \to f \Rightarrow \int_a^b f_n(x)dx \to \int_a^b f(x)dx$

if $\exists$ a $g(x) \ni$ for $|f_n| \le g \ \forall_n$ & $\int g\,dx < \infty$

take $|\quad|$ of both sides then $\frac{h}{2\pi}\max|f(\zeta)|\frac{L}{|\zeta-z|^3} \geq |$

lim $h\to 0$ says that $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} \to f^{(2)}(z)$ QED to prove that if u is harmonic in D & $u = 0$ on ∂D $\Rightarrow$ u = zero everywhere.

<u>Maximum modulus principle</u>:

Suppose D is a bounded domain in $\mathbb{C}$. Suppose f is an analytic fn on D and is continuous on $\bar{D}$. Then $|f(z)|$ assumes its maximum at some point of ∂D, and it assumes its maximum at some point of D iff $f \equiv$ const

$\bar{D} = D \cup \partial D$ = closure of D (smallest closed set containing D)

11/1/78

Let's look at uniqueness of Dirichlet probl

Thm Suppose $u = \mathrm{Re} f$. By max mod principle $\Rightarrow$ u assumes its max at some pt of ∂D. (u assumes its min at some pt of ∂D). This is max principle for harmonic functions. Moreover if u assumes its max (min) at some interior point of D, then $u \equiv$ const.

<u>Proof</u>

let $g = e^f \Rightarrow |g| = e^u$. $|g|$ assumes its max precisely at pt where u assumes its max (since e^u is an increasing fn.)

In particular if $u \equiv 0$ @ ∂D then $u = 0$ in D

Reason $\max_{\bar{D}} u \le \max_{\partial D} u = 0$

also $\min_{\bar{D}} u \ge \min_{\partial D} u = 0$

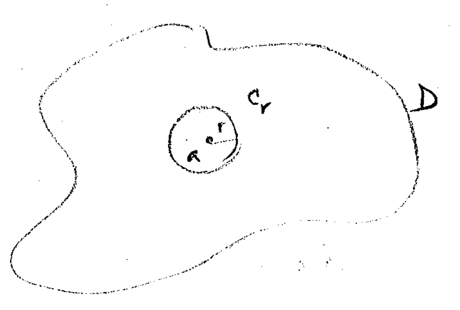
$$0 = \min_{\partial D} u \le \min_{\bar{D}} u \le \max_{\bar{D}} u < \max_{\partial D} u = 0$$

$\Rightarrow u = 0$ QED

To prove max mod theory $|f|$ is cont on $\bar{D}$

Assume $|f|$ assumes its max at some interior pt. $a \in D$ Have to prove that $f \equiv$ const.

for a pt a lets find $f(a) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)\,dz}{z-a}$



let $z = a + re^{i\theta}$ on C_r $\Rightarrow$

$$f(a) = \frac{1}{2\pi i}\int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})\, rie^{i\theta}}{re^{i\theta}}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

– This formula gives that the value of an analytic fn. at the center of a circle equals the average of its values in the circumference. (mean value property)

– this property also holds for the harmonic fn.

$$|f(a)| \leq \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} |f(a+re^{i\theta})|\, d\theta = \max f(a+re^{i\theta}) \cdot \frac{2\pi}{2\pi}$$

we know that $|f(a)| \geq |f(a+re^{i\theta})| \quad \forall\theta$

$\therefore$ for this holds $|f(a)| = f(a+re^{i\theta})|$

but $f(a+re^{i\theta}) = |f(a)|e^{i\varphi(\theta)}$

and $f(a) = |f(a)|\, e^{i\alpha}$ now to prove $e^{i\alpha} = e^{i\varphi(\theta)}$

$\therefore$ since $f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\, d\theta = |f(a)|e^{i\alpha} = \frac{1}{2\pi}\int_0^{2\pi} |f(a)|\, e^{i\varphi}\, d\theta$

if we ~~try~~ divide by $e^{i\alpha}$ & know that $|f(a)| = $ const.

$\therefore$
$$1 = \frac{1}{2\pi}\int_0^{2\pi} e^{i\varphi-\alpha}\, d\theta$$

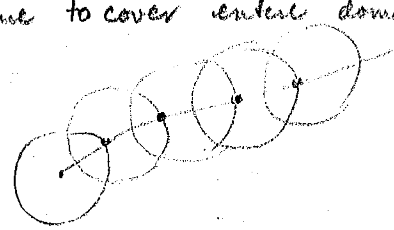
Take the real part $1 = \frac{1}{2\pi}\int_0^{2\pi} \cos(\varphi(\theta) - \alpha)\, d\theta$

since $|\cos(\varphi - \alpha)| \leq 1 \quad \therefore \quad 1 \leq \frac{1}{2\pi}\int 1 \; |d\theta$

we have that $1 \leq 1$ only true if $\cos(\varphi - \alpha) = 1$

or we then say that $\varphi = \alpha$ & $\therefore$ $f(a) = f(a+re^{i\theta})$

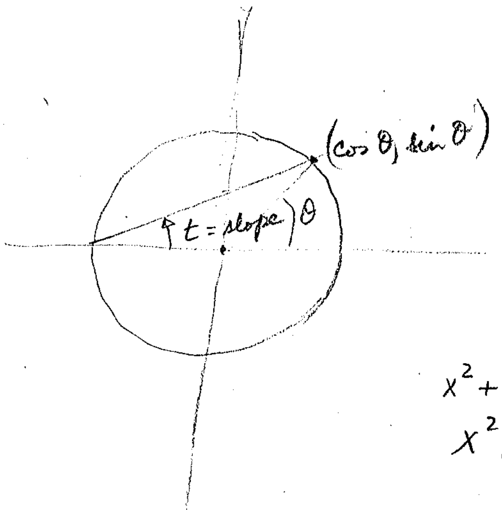
or that the value of the fn is same at bdy & at center, we can then analytically continue to cover entire domain hence $f \equiv$ Const everywhere



11/3/78

Test Monday 6 Nov up to Cauchy

$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$ In calculus courses it is evaluated by rationalization of the trig fns.



$$y - y_0 = m(x - x_0)$$
$$y = t(x+1)$$
along $x^2 + y^2 = 1$
$$x^2 + t^2(x+1)^2 = 1$$
$$x^2(t^2+1) + 2t^2x + (t^2-1) = 0$$

$x_1 = -1$ now $x_2 + x_1 = \frac{-2t^2}{t^2+1}$ $\therefore\ x_2 = \frac{1-t^2}{t^2+1}$

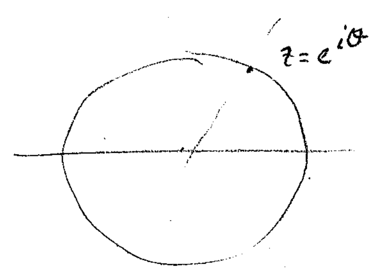
$\therefore\ y = \frac{2t}{1+t^2} \implies \therefore\ (x,y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$

since $t = \tan\frac{\theta}{2}$ $\quad d\theta = \frac{2\,dt}{1+t^2}$ $\quad \cos\theta = \frac{1-t^2}{1+t^2}$ $\quad \sin\theta = \frac{2t}{1+t^2}$

normally for adv calc the integration could be done over indefinite limits

$\therefore \int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta$ ← but this integrand in the complex plane must be integrated over $(2\pi, 0)$ because or as a consequence of cauchy's theorem

now we can integrate over unit circle



$$\left.\begin{matrix}\cos\theta \\ i\sin\theta\end{matrix}\right\} = \frac{e^{i\theta} \pm e^{-i\theta}}{2} = \frac{1}{2}\left(z - \frac{1}{z}\right)$$
$$dz = i e^{i\theta}\,d\theta = iz\,d\theta$$

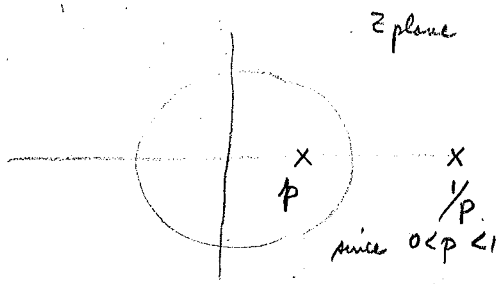
$$\int_{|z|=1} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{dz}{iz}$$

Example $\int_{\theta=0}^{2\pi} \frac{d\theta}{1-2p\cos\theta+p^2}$ using $z = e^{i\theta}$ $\quad \int_{|z|=1} \frac{\frac{dz}{iz}}{1 - p\left(z+\frac{1}{z}\right) + p^2}$

$$\int_{|z|=1} \frac{-i\,dz}{z - p(z^2+1) + p^2 z} = -i\int \frac{dz}{pz^2 - (p^2+1)z + p}$$

$$pz^2 - (1+p^2)z + p = (pz-1)(z-p)$$

$$\therefore = i\int_{|z=1|} \frac{dz}{p(z-1/p)(z-p)} = \int_{|z=1|} \frac{\frac{i}{p(z-1/p)}}{z-p}\,dz$$



now $\dfrac{i}{p(z-1/p)}$ is analytic in the region $|z=1|$

$\therefore$ by cauchy's integral formula

$$2\pi i\left[\frac{i}{p(p-1/p)}\right] = \int_{|z=1|} \frac{i\,dz}{p(z-1/p)(z-p)}$$

$$\frac{-2\pi}{p^2-1} = \frac{2\pi}{1-p^2}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{1-2p\cos\theta+p^2} = \frac{2\pi}{1-p^2}$$

Method of solution

$$\int_{\theta=0}^{2\pi} R(\cos\theta, \sin\theta)\,d\theta = \int_{|z|} \frac{P(z)}{Q(z)}\,dz = \int_{|z=1|} \frac{P(z)\,dz}{c(z-a_1)^{m_1}\cdots(z-a_k)^{m_k}}$$

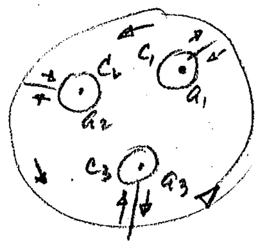
next define which poles are inside/outside

$$\int_{|z=1|} \frac{\dfrac{P(z)}{c(z-a_{l+1})^{m_{l+1}}\cdots(z-a_k)^{m_k}}}{(z-a_1)^{m_1}\cdots(z-a_l)^{m_l}}\,dz$$

where a_i $i=1,\dots,l$ are pts inside
a_i $i=l+1,\dots,k$ are pts outside

$$\therefore f(z) = \frac{P(z)}{c(z-a_{l+1})^{m_{l+1}}\cdots(z-a_k)^{m_k}}$$

let $\int$ over $z = e^{i\theta}$ for each a_i



$$\sum_{i}^{l} \int_{C_i} \frac{f(z)\,dz}{(z-a_1)^{m_1} \cdots (z-a_l)^{m_l}}$$

$$= \sum_i \int_{C_i} \frac{\overbrace{\dfrac{f(z)\,dz}{(z-a_1)^{m_1} \cdots (z-a_{i-1})^{m_{i-1}} (z-a_{i+1})^{m_{i+1}} \cdots}}^{g(z)}}{(z-a_i)^{m_i}}$$

$g(z)$ is analytic on C_i and inside the domain

$$\sum_{i=1}^{l} \frac{2\pi i}{(m_i-1)!} \frac{d^{m_i-1}}{dz^{m_i-1}} \left(g(z) \right) \Big|_{z=a_i}$$

$$\int_{|z|=1} \frac{(z^2+1)\,dz}{\left(z-\frac{1}{2}\right)^2 \left(z-\frac{1}{3}\right)} = \int_{C_1} \frac{\dfrac{z^2+1}{\left(z-\frac{1}{2}\right)^2}\,dz}{z-\frac{1}{3}} + \int_{C_2} \frac{\dfrac{z^2+1}{z-\frac{1}{3}}}{\left(z-\frac{1}{2}\right)^2}\,dz$$

$$= 2\pi i \left[\frac{z^2+1}{(z-\frac{1}{2})^2} \right] \Big|_{z=1/3} + 2\pi i \frac{d}{dz} \left(\frac{z^2+1}{z-\frac{1}{3}} \right) \Big|_{z=1/2}$$

$$\int_{\theta=0}^{\pi} \frac{d\theta}{2+\cos\theta} = \int_{\theta=0}^{-\pi} \frac{-d\theta}{2+\cos\theta} \text{ by symmetry } \therefore \int_0^{\pi} \frac{d\theta}{2+\cos\theta} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta}{2+\cos\theta}$$

11/8/78

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta.$$

Power series expansions. (Taylor Series)

Def: $p(z) = \sum a_n (z-c)^n$

Look at special case $\sum a_n z^n$

1st problem find the set E consisting of all z $\ni$ $\sum a_n z^n$ converges

Thm set E must be = the open disk of radius r centered at 0 together w some bdy points

Equivalence form $\sum a_n z^n$ converges at $z = b \Rightarrow \sum a_n z^n$ converges $\forall z \ni |z| < |b|$

let $r = \sup_{b \in E} |b|$ then $|z| < r \Leftrightarrow |z| < |b|$ for some $b \in E \Rightarrow z \in E$

$\therefore$ $E \supset \{ |z| < r \}$ $\xrightarrow{\text{to end proof}}$ Show if $|z| > r$, then $z \notin E$ otherwise $\sup_{b \in E} |b| > r$

Suppose $\sum a_n b^n$ converges $\{$ recall convergence of sequence $s_n \to L \Rightarrow$ For every ϵ $\exists$ $N = N(\epsilon)$ $\ni$ $|s_n - L| < \epsilon$ for $n > N \}$

we can define

$s_n = \sum_0^n c_k$ as a sequence of partial sums. $\left. \begin{array}{l} s_{n+1} \to L \\ s_n \to L \end{array} \right\} \Rightarrow s_{n+1} - s_n \to 0$ or $a_{n+1} \to 0$

$\lim_{n \to \infty} a_n \to 0$ is a necessary but not suff cond.

$\therefore$ $a_n b^n \to 0 \Rightarrow \exists M$ st $|a_n b^n| \le M$ $\forall n$

Divergence

Cauchy sequence $\Rightarrow \epsilon > 0$ $\exists N \ni$ $|s_n - s_m| < \epsilon$ for $n, m \geq N$

a convergence seq is a cauchy seq

a cauchy seq need not be convergent

a bounded cauchy sequence is a convergent sequence.

Bounded sequence $|S_n| \leq R$ for all n

Given $\epsilon > 0$ $\exists N$ s.t. $|S_n - S_m| < \frac{\epsilon}{2}$ for $n, m \geq N$

$\exists\, p \geq N \ni |p - L| < \frac{\epsilon}{2}$

take $n \geq N$ $|S_n - L| \leq |S_n - S_p| + |S_p - L| < \epsilon$

Back to absolute convergence

Want to show $S_n = \sum c_n$ are bdd & cauchy

since $|S_n| \leq \sum |c_n| < \infty$ ∴ bounded.

to show cauchy

for $\varepsilon > 0$ $\exists N \ni |S_n - S_m| < \varepsilon$ for $m, n \geq N$

ie $\left|\sum_{k=n+1}^{m} c_k\right| < \varepsilon$ $(*)$

Corresponding statement for the series $\sum_{n=0}^{\infty} |c_k|$

Given $\epsilon > 0$ $\exists N \ni$ $\sum_{k=n+1}^{m} |c_k| < \varepsilon$ $(**)$

but this$^{(**)}$ is much stronger than $(*)$ ∴ $\sum c_k$ is convergent

To prove (finally) that $\sum a_n z^n$ converges if $\sum a_n b^n$ converges for $|z| < |b|$

$$\sum |a_n z^n| = \sum \left|a_n b^n \left(\frac{z}{b}\right)^n\right| \leq M \sum \left|\frac{z}{b}\right|^n = \frac{M}{1 - |z/b|}$$

if $|z/b| < 1$ or $|z| < |b|$

Radius of Convergence $= R$

Series conv for $|z| < R$

not conv $|z| > R$

unknown $|z| = R$

To find radius of conv.

Ratio test if $\sup_n \frac{|a_n z^{n+1}|}{|a_n z^n|} < 1$ series conv

if $\inf \left|a_{n+1} z^{n+1}\right| > 1$ series dive sup shorthand = least upper bound = lub

ie for a sequence $|c_n| \le M$ for some M. $\exists$ a set E of M_i's $\ni$ $|c_n| \le M_i$ the smallest $M_i \ni |c_n| \le M_i$ is lub

inf = infimum glb

prove ratio test take $\alpha \ni \sup_n \left|\frac{c_{n+1}}{c_n}\right| = \alpha \quad \therefore \quad \left|\frac{c_{n+1}}{c_n}\right| \le \alpha \quad \forall n$

$$\therefore |c_{n+1}| \le \alpha |c_n| \le \alpha^2 |c_{n-1}| \ldots\ldots \le \alpha^{n+1} c_0$$

$$\sum |c_{n+1}| \le \sum \alpha^n |c_0| = \frac{|c_0|}{1-\alpha} \text{ which conv.}$$

same for case where $\left|\frac{c_{n+1}}{c_n}\right| > 1$

Exam Review 11/11/78

express $\sin 5\theta$, $\cos 5\theta$ in terms, $\cos\theta$

$$\cos 5\theta + i \sin 5\theta = (\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta\,(i\sin\theta)$$
$$+ 10\cos^3\theta\,(i\sin\theta)^2 + 10\cos^2\theta\,(i\sin\theta)^3 + 5\cos\theta\,(i\sin\theta)^4 + (i\sin\theta)^5$$

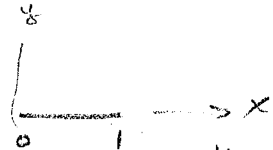
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

$V = x^3 + axy^2 \qquad V_x = 3x^2 + ay^2 \qquad V_y = 2axy$

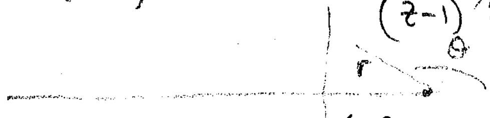
$V_{xx} = 6x \qquad V_{yy} = 2ax \qquad a = -3$ since $\Delta V = 0$

$\therefore \quad V = x^3 - 3y^2x \qquad V_x =$

Show for $\sqrt[4]{(z-1)^3 z}$ branch cut is



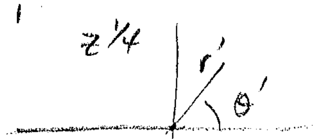
$(z-1)^{3/4}$



$-\pi < \theta < \pi$

$-\pi + 2k\pi < \theta < \pi + 2k\pi$ general branch

$z^{1/4}$



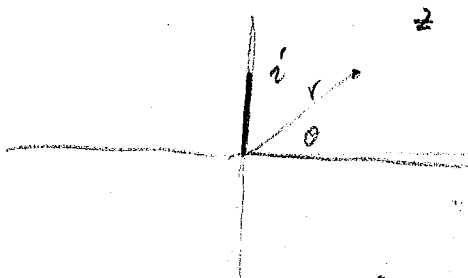
$-\pi < \theta' < \pi$

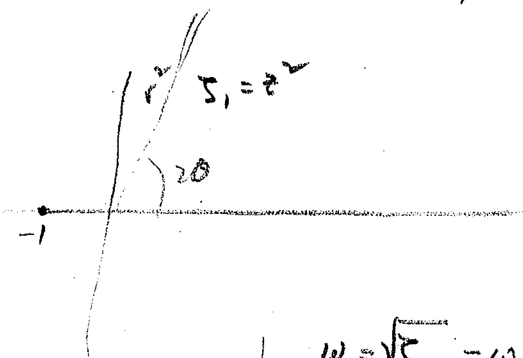
$-\pi + 2k\pi < \theta' < \pi + 2k\pi$ general branch

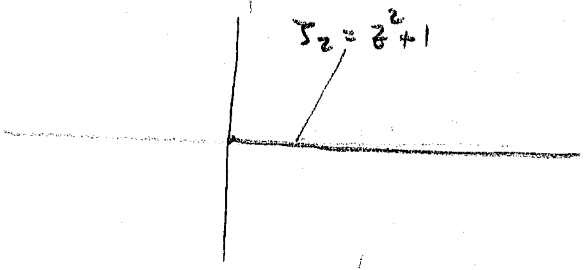
$f(z) = r^{3/4} r'^{1/4} e^{i\left(\frac{3\theta}{4} + \frac{\theta'}{4}\right)}$ take $|\theta|, |\theta'| < \pi$ & define branch

for $z = i$ $\theta = \frac{3\pi}{4}$ $\theta' = \frac{\pi}{2}$

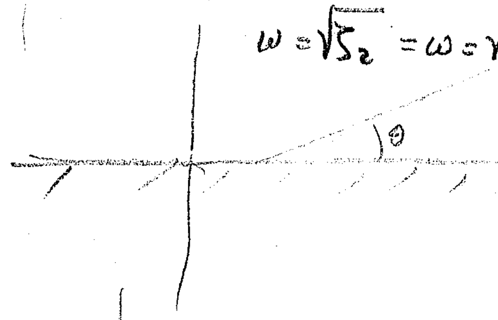
at $z=2$ For $z = 2$ $\sqrt{2}^{3/4}\left(e^{i\left(\frac{3}{4}\cdot\frac{3\pi}{4} + \frac{1}{4}\cdot\frac{\pi}{2}\right)}\right) = 2^{3/8} e^{i\frac{11\pi}{16}}$

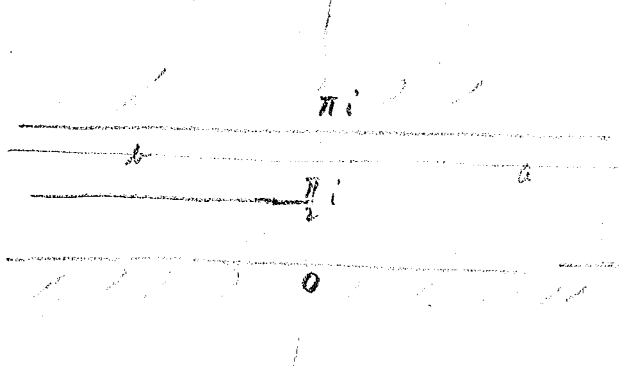


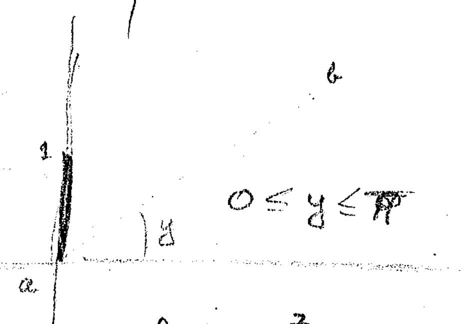




$w = \sqrt{\zeta_2} = w = \sqrt{z^2+1}$







$0 \le y \le \pi$

$f(z) = e^z$

$\therefore \sqrt{e^{2z}+1} = f(z)$

Math 106 TEST I Fall 1978

1. Use the multiplication of complex numbers to derive a formula expressing

$e^{5i\theta} = \cos 5\theta + i \sin 5\theta = (\cos\theta + i \sin\theta)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

a) $\sin 5\theta$ in terms of $\sin\theta$ and $\cos\theta$

b) $\cos 5\theta$ in terms of $\sin\theta$ and $\cos\theta$

c) $\tan 5\theta$ in terms of $\tan\theta$.

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta$

$\sin 5\theta = 5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta$

$\tan 5\theta = \tan\theta \left(\frac{5\cos^4\theta - 10\cos^2\theta \sin^2\theta + \sin^4\theta}{\cos^4\theta - 10\cos^2\theta \sin^2\theta + 5\sin^4\theta} \right)$

2. Determine the constant a so that the function $v = x^3 + axy^2$ is the imaginary part of an analytic function f and find f.

$f = u + iv$

$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ $\therefore u_{,x} = v_{,y}$, $v_{,x} = -u_{,y}$

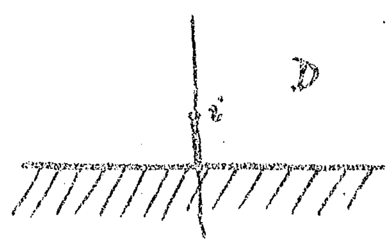
$v_{,y} = 2axy = u_{,x} \Rightarrow ax^2y + f(y) = u$

$v_{,x} = 3x^2 + ay^2$ $ax^2 + f' = u_{,y} = -v_{,x}$ $\Rightarrow a = 3$ $f(y) = -y^3 + C$ $\therefore f = (3x^2y + y^3) + i(x^3 + 3xy^2)$ or $(y + ix)^3$

3. Show that we can define a branch of the function $\sqrt{(z-1)^3 z}$ on the plane minus the line segment $0 \le x \le 1$, $y = 0$. Find the value at the point $z = i$ of the branch which has a positive value at the point $z = 2$.

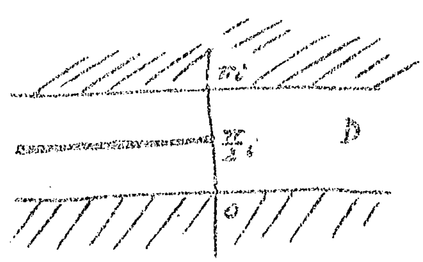
4. Find a conformal map (i.e. a one-to-one analytic function) which maps the following domain D onto the upper half-plane.

a) D = the upper half-plane minus the line segment joining 0 to i



b) $D = \{0 < y < \pi\} - \{y = \frac{\pi}{2}, x \le 0\}$

(Hint: for b) use first the exponential map)



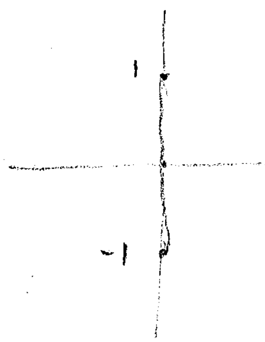
$x + iy$, $(x^2 + y^2)\,dx + i\,dy$

5. Evaluate a) $\int_C |z|\,dz$, when C is the line segment joining $-i$ to i

b) $\int_{|z|=3} \frac{2z-1}{z(z-1)}\,dz$ (Hint: take small circles around 0 and 1 and use Cauchy's integral formula)

$2\pi i \cdot 1 = \int \frac{(2z-1)/z}{z-1}\,dz + \int \frac{(2z-1)/(z-1)}{z}\,dz = 4\pi i$ $\quad 2\pi i \cdot 1$

$\int_C |z|\,dz$ c line seg joining $-i$ to i



let $z = it$

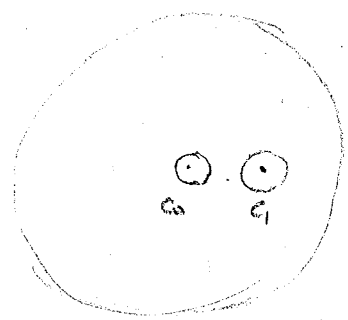
$dz = i\,dt$

$|z| = |it| = |t|$

$$\int |z|\,dz = i\int_{-1}^{1} |t|\,dt = i\int_0^1 t\,dt + i\int_{-1}^{0} -t\,dt = i$$

$$\tfrac{1}{2} \qquad + \tfrac{1}{2}$$

$$\int_{|z|=3} \frac{(2z-1)}{z(z-1)}\,dz$$



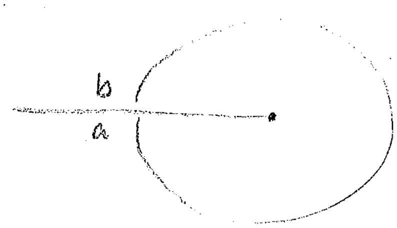
$$\int_{C_0} \frac{2z-1}{z-1}\,\frac{dz}{z} = 2\pi i$$

$$\int_{C_1} \frac{2z-1}{z}\,\frac{dz}{z-1} = 2\pi i$$

$$\therefore \int_C = \int_{C_0} + \int_{C_1} = 4\pi i$$

Another way

$\int \frac{dz}{z} = \log z$ if we define the cut & take proper branch



$\log z = \log|z| + i\arg z$

$\log b = \log 3 + i\pi$ if branch is $-\pi < \theta < \pi$

$\log a = \log 3 - i\pi$

$\therefore \log b - \log a = \log z\Big|_a^b =$ in the limit $2\pi i$

Convergence of series $\overset{def}{\Longleftrightarrow}$ convergence of the sequence of partial sums
$\Rightarrow$ n^{th} term $\to 0$

Convergence sequence $\Leftrightarrow$ cauchy sequence $|S_n - S_m| < 1$ for $n > N$

$\sum_{n=0}^{\infty} c_n$ $\quad \sup_n \left|\frac{c_{n+1}}{c_n}\right| < 1 \Rightarrow$ conv.

$\inf_n \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow$ no conv

Ratio test

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} < 1 & \text{conv} \\ > 1 & \text{no conv} \end{cases}$

$\sup_n |c_n|^{\frac{1}{n}} < 1 \Rightarrow$ conv

$\inf_n |c_n|^{\frac{1}{n}} > 1 \Rightarrow$ no conv

let $\alpha = \sup_n |c_n|^{\frac{1}{n}}$ & $\alpha < 1$

$|c_n| \leq \alpha^n \qquad \sum |c_n| \leq \sum \alpha^n = \frac{1}{1-\alpha}$

11/13/78

Exerc VI pg 138 1–6 pg 145 1–4, 6, 11, 12 due 20 Nov

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}$$

$C =$ circle of radius R centered at z

$$|f^{(n)}(z)| = \frac{n!}{2\pi} \left| \int_{|\zeta - z| = R} \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}} \right| \leq \frac{n!}{2\pi} \max_{|z-\zeta|=R} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} \cdot 2\pi R$$

$$= \frac{n!}{R^n} \max_{|\zeta - z| = R} |f(\zeta)|$$

Cauchy's inequal $\quad |f^{(n)}(z)| \leq \frac{n!}{R^n} \max_{|\zeta - z| = R} |f(\zeta)|$

Liouville's theorem - analytic on whole plane (entire fns)

if f analytic on the whole plane (entire) and f is bounded $|f(z)| \le M$ $\forall z$

$\Rightarrow f \equiv$ constant

Pf cauchy's ineq with $n=1$ $\quad |f'(z)| \le \frac{1}{R} M$ $\quad$ true $\forall$ all z + all R

Fix z. Let $R \to \infty$ $\quad f'(z) = 0 \quad \Rightarrow f(z) = \text{const}$

Fundamental theorem of algebra

Every poly of degree $n \geq 1$ has at least one root in the complex number field (ie the complex number field is algebraically closed)

Given $P(z) = \sum_{i=0}^{n} a_i z^i$ show it has at least 1 root

Suppose P has no root $\quad$ let $f(z) = \frac{1}{P(z)}$ $\quad$ f is entire since $P(z) \neq 0$

$\therefore f \neq \text{const}$ $(P^{(n)} = n! a_n \neq 0)$

$$|f(z)| = \frac{1}{|\sum a_n z^n|} = \frac{1}{\sum |a_n z^n|} \cdot \frac{1}{\left|1 + \frac{a_{n-1}}{a_n z} + \dots \frac{a_0}{a_n z^n}\right|} \to \frac{1}{a_n z^n} \cdot 1$$

as $z \to \infty$, $|f(z)| \to 0$ $\quad$ f .: is bounded $\rightarrow\leftarrow$ $\therefore$ P has at least 1 root.

A series $\sum_{n} f_n(z)$ ~~converges~~ unif to a fn. $L(z)$ on the set S if $\forall \varepsilon > 0$ there $\exists$ a number $N = N(\varepsilon)$ indep of z and that

$$\left| \sum_{k=0}^{n} f_k(z) - L(z) \right| < \varepsilon \quad \text{for } n \geq N \text{ and } \underline{\forall z \in S}.$$

In the case of UC the series can be integrated term by term.

the limit of the series is a continuous function if each term is cont.

$f_n(x) \to f(x)$ unif of $[a,b]$

$$\Rightarrow \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Given $\epsilon > 0$ want to find N ∋.

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| < \epsilon \quad \forall n \geq N$$

we know that $\exists\ N$ ∋. $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for $\forall n \geq N$ & $\forall\ x \in [a,b]$

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| = \left| \int_a^b (f_n(x) - f(x))\,dx \right| \leq \int_a^b |f_n - f|\,dx$$

$$\leq \int_a^b \frac{\epsilon\,dx}{b-a} = \epsilon \quad \text{QED}$$

11/15/78

Test II <u>Nov 22</u>

$\sum a_n z^n$: Asolute & unf conv Root & Ratio Test

$f_n(t) \to f(t)$ uniformly on $[a,b] \Rightarrow$ 1 $\int_a^b f_n(t)\,dt \to \int_a^b f(t)\,dt$

2 f_n continuous $\Rightarrow f$ continuous (use 3ϵ proof)

1. f cont at $c \Leftrightarrow$ given $\epsilon > 0$ $\exists$ $\delta = \delta(\epsilon) > 0$ ∋. $|f(t) - f(c)| < \epsilon$ whenever $|t-c| < \delta$
2. $f_n \to f$ unif $\Rightarrow$ given any $\epsilon > 0$ $\exists N$ ∋. $|f_n(t) - f(t)| < \epsilon$ for $n > N$ & $\forall t$
3. f_n cont $\Rightarrow$ give $\epsilon > 0$ $\exists$ $\delta = \delta(\epsilon) > 0$ ∋. $|f_n(t) - f_n(c)| < \epsilon$ when $|t-c| < \delta$

Take $n \geq N$ and $\delta = \delta_n(\epsilon)$ $\quad |f(t) - f(c)| \leq |f(t) - f_n(t)| + |f_n(t) - f_n(c)| + |f_n(c) - f(c)|$

when $|t-c| < \delta$

1st term $< \epsilon$ by 2

2nd term $< \epsilon$ by 3.

3th term $< \epsilon$ by 2.

$\therefore \quad |f(t) - f(c)| \leq 3\epsilon$

Analytic fns.

Example: polynomials, rational fns, exponential, related fn.

$P(z) = \sum_j^n a_j z^j$ when expanded $f(z) = \sum_j^\infty a_j z^j$ and converges on $|z|<R$ radius of converg.

<u>Thm</u> $f(z)$ is analytic on $|z|<R$

we must prove that $f'(z)$ exists but this involves $\lim_{n\to\infty} \sum^n \lim_{\Delta z \to 0} \frac{f(z)-f(z_0)}{z-z_0}$ which may not be commutative

We will prove that for unif + abso convergence then $\lim_{n\to\infty} \lim_{\Delta z\to 0} = \lim_{\Delta z} \lim_{n}$

i) $\sum_{n=0}^\infty a_n z^n$ conv unif & absolutely on $|z| \le R_1 < R$

Take b s.t. $R_1 \le |b| \le R$ $\quad \sum a_n b^n \to 0 \quad |a_n b^n| \le M \ \forall n$

For $|z| \le R_1$ $\quad \sum_{n=0}^\infty |a_n z^n| = \sum_{n=0}^\infty \left|a_n b^n \left(\frac{z}{b}\right)^n\right| \le M \sum_{n=0}^\infty \left(\frac{|z|}{b}\right)^n < M \sum_{n=0}^\infty \left(\frac{|R_1|}{|b|}^n\right) < \infty$

ii) $\frac{\partial}{\partial x} f(z) = \sum_{n=0}^\infty \frac{\partial}{\partial x}(a_n z^n)$ $\quad \left(\text{Likewise } \frac{\partial}{\partial y} f(z) = \sum_{n=0}^\infty \frac{\partial}{\partial y}(a_n z^n)\right)$

$\sum_{n=0}^\infty \frac{\partial}{\partial x}(a_n z^n) = \sum_{n=0}^\infty n a_n z^{n-1}$ this series conv. unif + absol on $|z| \le 1$

<u>Proof</u> $\sum_{n=0}^\infty |n a_n z^{n-1}| = \sum_{n=0}^\infty \left| n \underset{\le M}{a_n b^n} \frac{1}{b}\left(\frac{z}{b}\right)^{n-1}\right| \le \frac{M}{|b|} \sum_{n=0}^\infty n \left(\frac{R_1}{|b|}\right)^{n-1}$

ratio test $\quad \dfrac{\frac{n+1}{n}\left(\frac{R_1}{|b|}\right)^n}{\left(\frac{R_1}{|b|}\right)^{n-1}} \to \frac{R_1}{|b|} < 1$

$$\int_0^{x_0} \sum_{n=0}^\infty \frac{\partial}{\partial x}(a_n z^n)\,dx = \sum_{n=0}^\infty \int_{x=0}^{x_0} \frac{\partial}{\partial x}(a_n z^n) = \int_0 \left(\sum n a_n z^{n-1}\right) dx$$

$$\sum_{n=0}^\infty \left(a_n z_0^n - a_n (iy_0)^n\right) = f(z_0) - f(iy_0) =$$

now take $\frac{\partial}{\partial x} f(z_0) = \sum_{n=0}^\infty n a_n z_0^{n-1}$ by F. T. of Calculus.

now since $i\frac{\partial}{\partial x} f = \frac{\partial}{\partial y} f$ then this shows that $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z}$

$= \frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}$

$\therefore \quad f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} \quad \text{if } f(z) = \sum a_n z^n$

Thm: $f(z)$ analytic on $|z| < R$

$$\Rightarrow f(z) = \sum a_n z^n \text{ whose radius of convergence is } \geq R$$

this series is unique

$$\frac{1}{a-z} = \frac{1}{a}\left(\frac{1}{1 - z/a}\right) = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^n \quad \text{where } \left|\frac{z}{a}\right| < 1$$

Pf that $f'(z) = \sum n a_n z^{n-1}$ is unique wrt $f(z) = \sum_0^{\infty} a_n z^n$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad f(0) = a_0$$

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad f'(0) = a_1$$

$$f^{(k)}(z) = \sum_{n=1}^{\infty} k! a_n z^{n-k} \qquad f^{(k)}(0) = a_k \cdot k!$$

Proof Exists

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta| = R_1} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\,\frac{1}{1 - \frac{z}{\zeta}} = \frac{1}{\zeta}\sum\left(\frac{z}{\zeta}\right)^n \qquad |z| < R_1 = |\zeta|$$

now $\dfrac{f(\zeta)}{\zeta - z} = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{\zeta^n} f(\zeta)$

$$\frac{1}{\zeta}\int \frac{f(\zeta)}{\zeta - z}\,d\zeta = \sum \int \frac{z^n f(\zeta)}{\zeta^{n+1}}\,d\zeta = \sum \underbrace{\left(\int \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta\right)}_{\text{coeff of power series}} z^n$$

Math 106
Prof. Sica
H.W. #4

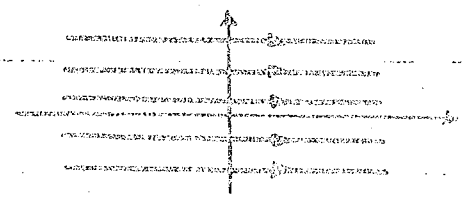
Text, p. 93 (1-7)

(1) Given complex potential $f(z)$ for ideal fluid flow, find: φ = velocity potential, ψ = stream function, V = complex velocity, $|V|$ = speed and plot stream lines.

$f(z) = z$

$\varphi = x$, $\psi = y$

$V = 1 = |V|$



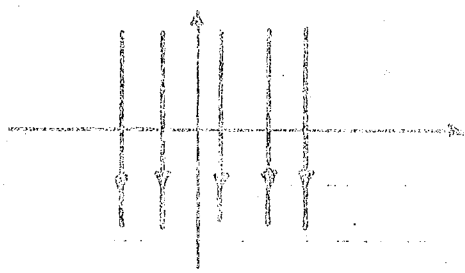
streamlines $y = \psi = \text{const}$

$f(z) = iz$

$\varphi = -y$, $\psi = x$

$V = \bar{i} = -i$

$|V| = 1$



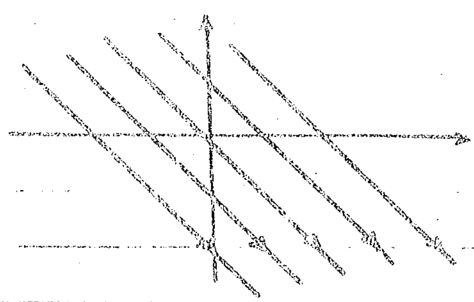
streamlines $x = \text{const}$.

$f(z) = (1+i)z$

$\varphi = x-y$, $\psi = x+y$

$V = \overline{1+i} = 1-i$

$|V| = \sqrt{2}$



$x+y = \text{const}$.

$f(z) = \frac{1}{z}$

$\varphi = x/x^2+y^2$, $\psi = -y/x^2+y^2$

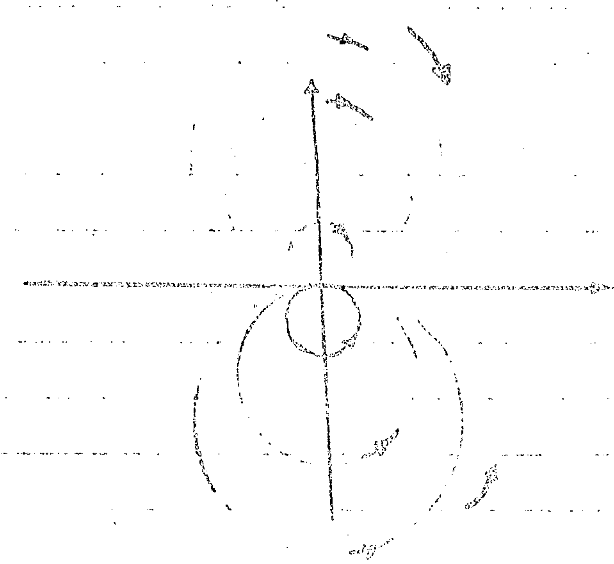
$V = \frac{-1}{\bar{z}^2} = \frac{-z^2}{|z|^4} = \frac{1}{|z|^4}(y^2-x^2-2xyi)$

$|V| = \frac{1}{|z^2|}$

streamlines:

$\frac{y}{x^2+y^2} = \frac{1}{2c}$;

$x^2+(y-c)^2 = c^2$

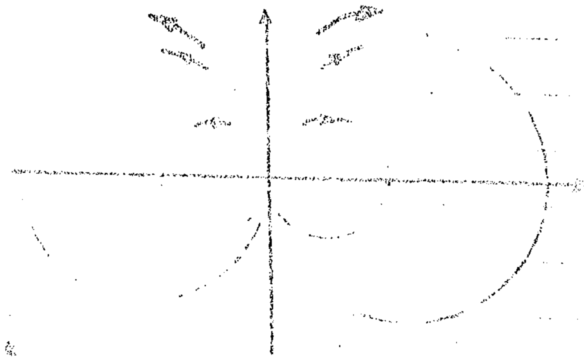


$f(z) = i/z$

$\varphi = y/(x^2+y^2)\ , \quad \psi = x/(x^2+y^2)$

$v = \overline{\frac{-i}{z^2}} = i\,\frac{z^2}{|z^4|} = (-2xy + (x^2-y^2)i)/|z|^4$

$|v| = 1/|z^2|$



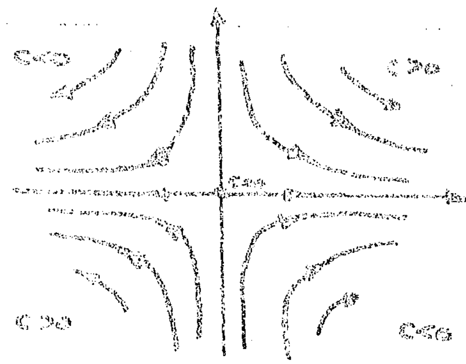
streamlines $(x-c)^2+y^2 = c^2$

$f(z) = z^2$

$\varphi = x^2-y^2\ , \quad \psi = 2xy$

$v = \overline{2z} = 2x - 2yi$

$|v| = 2|z|$



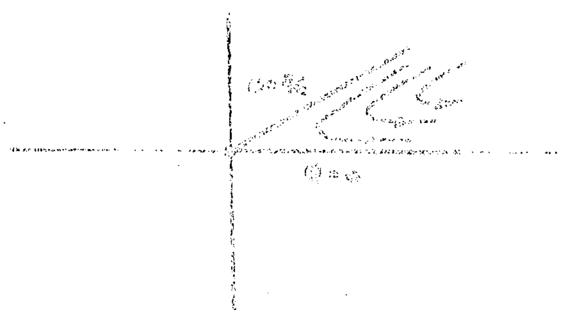
streamlines $xy = c$

(2) $f(z) = z^c \qquad c > 1/2$

$z = re^{i\theta}\ , \quad 0 \le \theta < 2\pi$

$f(z) = r^c e^{ic\theta} = r^c(\cos(c\theta) + i\sin c\theta) = \varphi + i\psi$

When $\theta = 0$ or π/c, $\psi = \text{const} = 0$, so these are streamlines (fluid behaves as if there was a barrier on these lines); note that $0 < \pi/c < 2\pi$

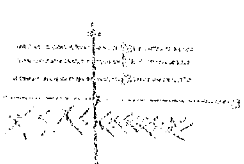


$f'(z) = c z^{c-1}$

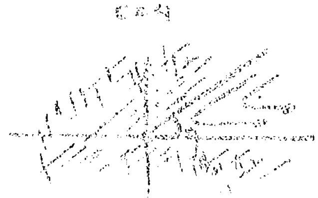
speed $c\, r^{c-1}$

So we have

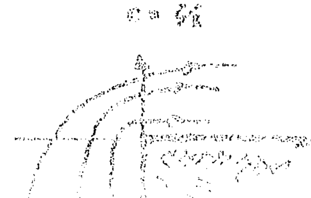
$c = 1$



$c = 2$



$c = 3/2$



(3) (cf. example 6.2)

(a) $f(z) = c \log(z^2 - a^2) + 2\pi k c i \quad k \in \mathbb{Z} \quad (a > 0)$

At wall $(x = 0)$, $V = 2cyi/y^2+a^2$ and the speed $|V| = 2cy/y^2+a^2$ (set $c = 1$ for convenience) which is seen to have a maximum for $y = a$:

max value $= 1/a$

On the other hand, without the wall, $f(z) = \log(z-a)$; velocity $\overline{f'(z)} = \overline{z-a}/|z-a|^2$ and at the wall, the speed is $1/\sqrt{y^2+a^2}$ which has a maximum at $y=0$ — namely $1/a$. So the two coincide.

(b) Put a source of strength $2\pi c$ $(c \neq 0)$ at $a > 0$; we have seen that

$$\text{speed with wall} = \frac{|2cz|}{|z^2-a^2|} = s_1 \quad \text{and}$$

$$\text{speed without wall} = \frac{|c|}{|z-a|} = s_2$$

Want to determine those z for which $s_1 > s_2$ i.e.

$$2|z| > |z+a|$$

After a short calculation this gives

$(x - \frac{1}{3}a)^2 + y^2 > \frac{4}{9}a^2$ which is the exterior of the circle centred at $a/3$ and radius $2a/3$.

(4) $f(z) = \alpha \log z \quad \alpha \neq 0$

$V = \bar{\alpha}/\bar{z} = \frac{\bar{\alpha} z}{|z|^2}$. The angle between this and the vector z is (remember previous HW) the real part of $\frac{\bar{\alpha} z}{|z|^2} \cdot \bar{z} = \bar{\alpha}$. Hence it is constant

(the streamlines are therefore logarithmic spirals)

(4) cont'd... : Streamlines ...

$$f(z) = \alpha \log z = a \log r - b\theta + i\,(b \log r + a\theta) = \phi + i\psi$$

which gives equation

$$\text{const. } k = b \log r + a\theta \;; \qquad r = \exp\left(\frac{k - a\theta}{b}\right).$$

(5) routine ...

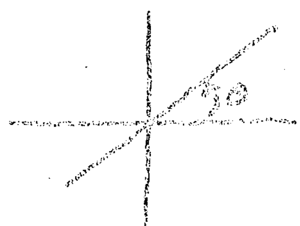
(6) $f(z) = \log\left(\frac{z-1}{z+1}\right)$ $\left\{\begin{array}{l}\text{is a potential due to source at} \\ z=1 \text{ and sink at } z=-1\end{array}\right.$

Note that (the bilinear) $w = \frac{z-1}{z+1}$ maps the right half-plane onto the unit disk, and the left half-plane onto the exterior of the unit disk (proof ?). In particular, it preserves angles (except at $z=-1$, the pole) and maps "circles" (in broad sense: lines or circles) onto "circles".

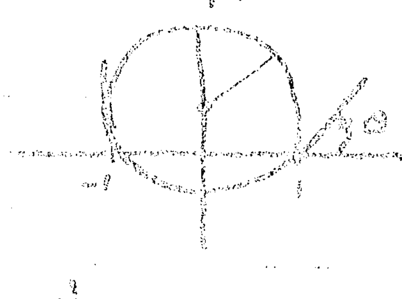
Now, for the imaginary part of $f(z)$ to be constant $= \theta$; we have

$$\arg(w) = \theta$$

which says that w lies in a line throug O and therefore z must lie in a "circle" through the pole -1 making same angle θ at $z=1$ with the real line (which is mapped onto itself !)



plane w



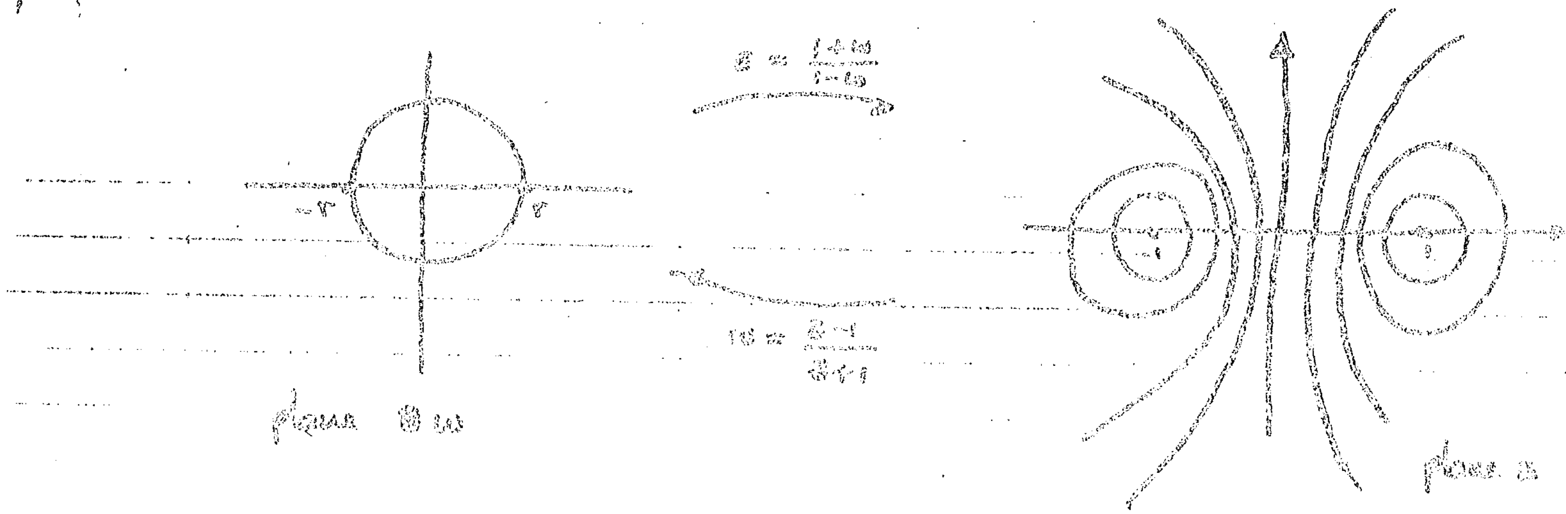
plane z

which says that the stream line is a circle as shown !

(6) cont'd... By a trivial computation, $|V| = 2/|z^2-1|$

(7) The streamlines of the conjugate flow are — same notation — the loci where w has constant modulus; namely, circles centered at 0 with radius $r =$ log (const). These are orthogonal to real line in the points $-r$, r and, with the exception of $r=1$ (which passes through pole), the preimages in plane z are circles with center <u>on real line</u> through the points

$$\frac{1+r}{1-r} \text{ and } \frac{1-r}{1+r} \quad (= w^{-1}(-r),\ w^{-1}(r))$$



The case $r=1$ is seen to degenerate to imaginary axis.

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A power series represents an analytic fn inside the disc of convergence

We look at converse Th if $f(z)$ is analytic on $|z|<R$

$\Rightarrow f(z) = \sum a_n z^n$ whose radius of convergence is $\geq R$

a_n is unique $= \frac{1}{n!} f^{(n)}(0)$

Pf $0<R_1<R$ $\qquad f(z) = \frac{1}{2\pi i} \int_{|\zeta|=R_1} \frac{f(\zeta)\,d\zeta}{\zeta - z} \qquad |z|<R_1$

$$= \frac{1}{2\pi i} \int \frac{f(\zeta)}{\zeta(1-z/\zeta)} = \frac{1}{2\pi i} \int \frac{1}{\zeta}\left(\sum^{\infty} \left(\frac{z}{\zeta}\right)^n\right) f(\zeta)\,d\zeta$$

we can integrate term by term since we have $\underline{\underline{UC}}$ $\quad = \frac{1}{2\pi i} \sum \int \frac{z^n}{\zeta^{n+1}} f(\zeta)\,d\zeta$

$$= \frac{1}{2\pi i} \sum a_n z^n \text{ where}$$

since $a_n = \frac{1}{2\pi i} \int \frac{f(\zeta)\,d\zeta}{\zeta^{n+1}} = \frac{1}{n!} f^{(n)}(0)$. now since this series convergence is independent of R since a_n dependent on $f^{(n)}(0)$ $\Rightarrow$ series converges up to at least R

Find power series expansion of $f = \frac{1}{(z-1)(z-2)}$ at i & find radius of convergence

Radius of convergence is $\min(|i-1|, |i-2|)$ since power series must be analytic in a disk up to R if $R > \min |i-1| \Rightarrow$ a singularity in disk $\therefore$ f is not analytic there $\quad \therefore R = \sqrt{2}$

$$\frac{1}{z-1} = \frac{1}{(z-i)-(1-i)} = -\frac{1}{1-i} \cdot \frac{1}{1-\frac{z-i}{1-i}} = \frac{-1}{1-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i}\right)^m$$

$$\frac{1}{z-2} = \frac{1}{(z-i)-(2-i)} = -\frac{1}{2-i} \cdot \frac{1}{1-\frac{z-i}{2-i}} = \frac{-1}{2-i} \sum_{0}^{\infty} \left(\frac{z-i}{2-i}\right)^n$$

$$\frac{1}{z-1} \cdot \frac{1}{z-2} = \frac{1}{(1-i)(2-i)} \sum\sum \left(\frac{z-i}{1-i}\right)^m \left(\frac{z-i}{2-i}\right)^n$$

if $m+n=p$ $\quad \sum_n \sum_m = \frac{1}{(1-i)(2-i)} \sum_{p=0}^{\infty} (z-i)^p \sum_{\substack{m+n=p \\ m,n \geq 0}} \frac{1}{(1-i)^m (2-i)^n}$

now $(a^q - b^q) = (a-b)\left(a^{q-1} + a^{q-2}b + \dots + a^0 b^{q-1}\right)$

$$\sum_{\substack{m+n=p \\ m,n\geq 0}} \frac{1}{(1-i)^m (2-i)^n} = \frac{1}{(1-i)(2-i)} \sum_{p=0}^{\infty} \left(\frac{\frac{1}{(1-i)^{p+1}} - \frac{1}{(2-i)^{p+1}}}{\frac{1}{1-i} - \frac{1}{2-i}} \right) (z-i)^p$$

Other method is partial fractions

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

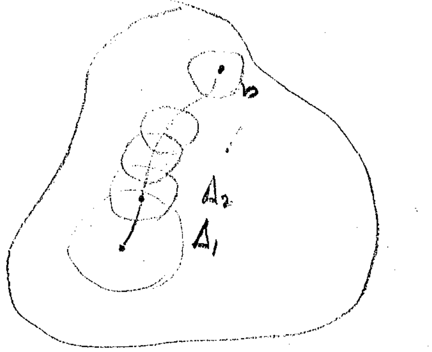
$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

mult both sides by $z-1$ & set $z=1$ gives $\frac{1}{1-2} = A + \frac{B\cancel{(z-1)}^{\,0}}{z-2} \Rightarrow \underline{A=-1}$

another way of getting $\underline{A, B}$

Uniqueness (identity) theorem for analytic fns.

$f(z)$ is analytic on a domain D
$a \in D \quad f^{(n)}(a) = 0 \quad \forall n \geq 0$
$\Longrightarrow f \equiv 0$ on D.
In partic if $f=0$ on some (nonempty) open subset of D, then $f=0$ on D.



f = power series at a on Δ_1
$\therefore \sum a_n (z-a)^n$ but all $a_n = 0 \Rightarrow f=0$
since center $\Delta_2 \in \Delta_1 \Rightarrow f^{(n)} = 0$ at center Δ_2 $\forall n$
f = power series at center Δ_2 on Δ_2
etc. we can thus argue to the other point b that $f=0$.

we need center of Δ_ν on the path, $\Delta_\nu \subset D$,
center of $\Delta_{\nu+1}$ is inside Δ_ν, center of $\Delta_1 = a$, b is in Δ_m

Example $f(z)$ cont on $|z| \leq 1$

$f(z)$ analytic on $|z| < 1$

$f(z) = 0$ on $\{ e^{i\theta} \mid 0 \leq \theta \leq \alpha \} \Rightarrow f \equiv 0.$

11/20/78

HW #8

P 152: 1, 6, 7, 8, 9
P 161 1, 4, 5, 6, 9, 10, 11, 12
P 169 1, 2, 4, 5

due the 29th
Test 29 Nov.

Power Series

fn analytic on a disc $\longleftrightarrow$ power series convergent on the disc.

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} (x^i)$$

radius of convergence $= 1$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$$ defined $\forall x$ by ratio test gives $R = 1$

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 \ldots$$ analytic except at $z = \pm i$ $\therefore$ R is 1

Analytic fn is defined as solution to PDE Cauchy Riemann eq
Analytic fn is defined as limit of power series Weierstrass
uniqueness theorem if $f^{(n)}(a) = 0$ $n = 0, 1, 2, \ldots$ then $f(z) \equiv 0$

f cont on $|z| \leq 1$, analytic on $|z| < 1$

$f(e^{i\theta}) = 0$ for $0 \leq \theta \leq \alpha$ (where α is some $+$ #)

$\Rightarrow f \equiv 0.$

Take $|z| < 1$ & look at $f(z) = \frac{1}{2\pi i} \int_{|\zeta| = 1} \frac{f(\zeta)\, d\zeta}{\zeta - z}$ true for $|z| < r < 1$
let $r \to 1^-$ since f is cont

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)\, d\zeta}{\zeta - z} \underset{\zeta = e^{i\theta}}{=} \frac{1}{2\pi i} \int_{\alpha \leq \theta \leq 2\pi} \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

Consider $\frac{1}{2\pi i} \int_{\{\zeta = e^{i\theta} \mid \alpha \le \theta \le 2\pi\}} \frac{f(\zeta)\,d\zeta}{\zeta - z} = g(z)$ for z not necessarily having absolute value < 1

well defined

$$\frac{f(\zeta)}{\zeta - z} = h(z, \zeta) \quad \text{well defined } \forall z \neq \zeta \text{ and } |\zeta| = 1$$

$g(z) = \int_{\{\zeta = e^{i\theta} \mid \alpha \le \theta \le 2\pi\}} h(z,\zeta)\,d\zeta$ continuous [illegible] for $\forall$ z not $= \zeta$'s used $\Rightarrow$ analytic for $z \notin \{\zeta = e^{i\theta} \mid \alpha < \theta \le 2\pi\}$

$g(z) \therefore$ is analytic on whole plane less $\{\zeta = e^{i\theta} \mid \alpha \le \theta \le 2\pi\}$

$g(z) = f(z)$ for $|z| < 1$ $\quad$ g is cont. with circle & for g $0 \le \theta \le \alpha$

$\therefore g(z) = f(z)$ for $z = e^{i\theta}$ $0 \le \theta \le \alpha$

but $f(z) = 0$ for $z = e^{i\theta}$, $0 \le \theta \le \alpha$

$\therefore g(z) = 0$ for $z = e^{i\theta}$, $0 < \theta < \alpha$

$\therefore g(z) = 0$ for $z = e^{i\theta}$, $0 < \theta < \alpha$ $\Rightarrow \frac{d^n}{dz^n} g(z) = 0$ $\forall z = e^{i\theta}$, $0 < \theta < \alpha$ for all n $\Rightarrow g \equiv 0$

Th the zero-set of an analytic fn is isolated (discrete)

if f is analytic on a domain D & $f \not\equiv 0$ and if $f(a) = 0$ for some a in D, then $\exists$ a disk of radius R centered at a $\ni$. f has no zeros inside the disc other than a.

Proof $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ $\quad c_0 = 0$

$\exists$ a $c_n \neq 0$ otherwise $f \equiv 0$

let c_k be the first non vanishing one

$f(z) = c_k (z-a)^k + c_{k+1}(z-a)^{k+1} + \ldots$

$f(z) = (z-a)^k \left[c_k + c_{k+1}(z-a) + c_{k+2}(z-a)^2 + \ldots \right]$

$= (z-a)^k \, Q(z)$

$\forall$ $g(z)$ is analytic in the disc where series conv. $\quad g(a) = c_k \neq 0$

$\therefore$ $g(z)$ is nowhere zero on $|z-a| < r$ for some $r > 0$.

$\therefore$ $f(z)$ is nowhere zero on $0 < |z-a| < r$

Suppose we have an analytic fn on $|z| \leq 1$

Restrict f to the bdry ie $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$

$$f(z) = \sum c_n z^n$$
$$f(e^{i\theta}) = \sum_{n=0}^{\infty} c_n e^{in\theta}$$

Fourier series

$$g(\theta) = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$
$$= \sum_{n=0}^{\infty} \left(a_n \frac{e^{in\theta} + e^{-in\theta}}{2} + b_n \frac{e^{in\theta} - e^{-in\theta}}{2i} \right)$$
$$= \sum_{n=0}^{\infty} \left(\left(\frac{a_n + b_n/i}{2} \right) e^{in\theta} + \left(\frac{a_n - b_n/i}{2} \right) e^{-in\theta} \right) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

$$g(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} = \sum_{-\infty}^{\infty} c_n z^n \quad \text{LAURENT SERIES}$$

$$g = \sum_{-\infty}^{\infty} c_n (z-a)^n$$ is Laurent series at $z = a$.

11/20/78

<u>Test II: next Wed</u>

Laurent Series $\quad \sum_{n=-\infty}^{\infty} c_n z^n$

$$S_{m,n} = \sum_{k=-m}^{n} c_k z^k$$

Say that $S_{m,n} \to L$ as $m,n \to \infty$ if for any $\epsilon > 0$ $\exists$ $N = N(\epsilon)$ s.t.
$|S_{m,n} - L| < \epsilon$ for $m,n \geq N$

Principal value of series $\lim_{n\to\infty} \sum_{k=-n}^{n} c_k z^k$ (for integral $\int_{-\infty}^{\infty} = \lim_{R\to\infty} \int_{-R}^{R}$)

Suppose $\sum_{n=-\infty}^{\infty} c_n a^n$ exists for some a then

	Principal value
$S_{m,n} - S_{m,n-1} \to 0$	$c_n a^n \to 0$ as $n \to \infty$
$S_{m,n} - S_{m-1,n} \to 0$	$c_{-n} a^{-n} \to 0$ as $n \to \infty$

$$\sum_{-\infty}^{\infty} = \underbrace{\sum_{0}^{\infty} c_n z^n}_{\substack{\text{convergence inside} \\ \text{circle } |z|<|a|}} + \underbrace{\sum_{1}^{\infty} c_{-n} \left(\frac{1}{z}\right)^n}_{\substack{\text{convergence outside} \\ \text{circle } |z|>|a|}} = f_1 + f_2$$

convergence is dependent on the radii of convergence of f_1 & f_2

Consider absolute convergence

$$\sum_{k=-\infty}^{\infty} |c_k z^k| < \infty \iff \begin{array}{ll} \sum |c_k z^k| < \infty & \text{Domain: disk \& some bdry pts} \\ \sum |c_{-k} z^{-k}| < \infty, & \text{Domain: disk \& some bdry pts.} \end{array}$$

Such pts form an annulus together with some boundary points

$$\sum_{k=-\infty}^{\infty} c_k z^k = \sum_{0}^{\infty} c_k z^k + \sum_{1}^{\infty} c_{-k} z^{-k}$$

$\sum c_k z^k$ has radius of convergence R_2

$\sum c_{-k} z^{-k}$ " " " " for $\frac{1}{z}$ say $\frac{1}{R_1}$ (ie conv for $\frac{1}{|z|} < \frac{1}{R_1}$)

Laurent series convergence is for $R_1 < |z| < R_2$ (absolute convergence in annulus)

$\sum_{-\infty}^{\infty} c_k z^k$ defines an analytic fn on $R_1 < |z| < R_2$

<u>Thm</u> (LAURENT) every analytic fn. $f(z)$ on $R_1 < |z| < R_2$ can be uniquely written as $\sum_{-\infty}^{\infty} a_n z^n$

Power series $f(z) = \sum a_n z^n \Rightarrow a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)\,d\zeta}{\zeta^{n+1}}$

Cannot use this
since principal part is not analytic there

$$\therefore\ c_n = \int_{|\zeta|=R} \frac{f(\zeta)\,d\zeta}{\zeta^{n+1}}$$

can be replaced by any C simple curve)

the convergence is uniform & absolute on any annulus $R_1 < R_1' \le |z| \le R_2' < R_2$

Pf Uniqueness suppose $f(z) = \sum_{-\infty}^{\infty} c_n z^n$ Since the conv is uniform, we can integrate term by term

$$\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)\,d\zeta}{\zeta^{n+1}} = \sum_{k=-\infty}^{\infty} c_k \int \zeta^{k-n-1}\,d\zeta = \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi i}\int_{\theta=0}^{2\pi} R^{k-n-1} e^{i(k-n-1)\theta}\, iRe^{i\theta}$$

$$= \sum_{k=-\infty}^{\infty} \frac{c_k}{2\pi}\int_0^{2\pi} R^{k-n} e^{i(k-n)\theta}\,d\theta$$

When $k=n \Rightarrow \int_0^{2\pi} e^0\,d\theta = 2\pi$

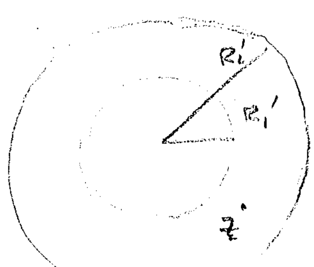
$k \ne n \Rightarrow \int_0^{2\pi} e^{i(k-n)\theta}\,d\theta = \frac{1}{i(k-n)} e^{i(k-n)\theta} = 0$

$$\therefore\ \sum_{k=-\infty}^{\infty} \frac{c_k}{2\pi}\int_0^{2\pi} R^{k-n} e^{i(k-n)\theta}\,d\theta = \frac{c_n}{2\pi}\int_0^{2\pi} e^0\,d\theta = c_n$$

hence we show uniqueness since $c_n = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)\,d\zeta}{\zeta^{n+1}}$

to prove Existence working in $R_1 < R_1' < R_2' < R_2$ take $R_1' < |z| < R_2'$

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R_2'} \frac{f(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{|\zeta|=R_1'} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$



$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\,\frac{1}{1-\frac{z}{\zeta}} = \frac{1}{\zeta}\sum_0^{\infty}\left(\frac{z}{\zeta}\right)^n = \sum_0^{\infty}\frac{z^n}{\zeta^{n+1}}$$

for $1^{st} \int$, $R_2' = |\zeta| > |z|$

for $2^{nd} \int$, $R_1' = |\zeta| < |z|$ we must rewrite

$$\frac{1}{\zeta - z} = -\frac{1}{z}\,\frac{1}{1-\zeta/z} = -\frac{1}{z}\sum\left(\frac{\zeta}{z}\right)^n = -\sum\frac{\zeta^n}{z^{n+1}}$$

since we have UC we can integrate term by term

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|\zeta|=R_2'} \frac{f(\zeta)\,d\zeta}{\zeta^{n+1}} \right) z^n - \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|\zeta|=R_1'} f(\zeta)(-\zeta^n)\,d\zeta \right) \frac{1}{z^{n+1}}$$

$$-\zeta^n = -\frac{1}{\zeta^{-n}} = \frac{-1}{\zeta^{(-n-1)+1}}$$

$$= \sum_0^{\infty} c_n z^n + \sum c_{-n-1} z^{-n-1}$$

$$\sum_0^{\infty} c_n z^n + \sum_{k=-1}^{-\infty} c_k z^k = \sum_{-\infty}^{\infty} c_n z^n \qquad \text{QED}$$

Isolated singularities

Suppose D is a domain in $\mathbb{C}$ & $a \in D$

Suppose $f(z)$ is analytic on $D - \{a\}$ then a is called an isolated singularity of $f(z)$. If f is analytic at a then the singularity is called removable.

let us assume that $a = 0$ f is analytic in $0 < |z| < R$

Laurent series expansion $f(z) = \sum_{-\infty}^{\infty} c_n z^n$ on $0 < |z| < R$

if f is analytic also at 0 then $f(z) = \sum_{n=0}^{\infty} a_n z^n$

Since Laurent series is unique $\Longrightarrow$ $\sum_{-\infty}^{\infty} c_n z^n = \sum_0^{\infty} a_n z^n$ or $c_n = 0 \ \forall n < 0$

hence principal part $= 0$

Classification of isolated sing

1) The principal part $= 0$ — removal

2) " " " has a finite # terms — pole of order k $\sum_{n=-k}^{-1} c_n z^n$ with k...

3) " " " has infinite " " " " ∞ — essential singular...

11/27/78

Laurent Series

$f(z)$ analytic $R_1 < |z| < R_2 \implies f(z) = \sum_{-\infty}^{\infty} c_n z^n$

$$c_n = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)\,d\zeta}{\zeta^{n+1}} \qquad R_1 < R < R_2$$

Isolated sing

f analytic on $0 < |z| < R$

$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ with principal part $\sum_{-\infty}^{-1} c_n z^n$

i) Removable sing $\sum_{-\infty}^{-1} c_n z^n = 0$

ii) pole of order k $\quad \sum_{-k}^{-1} c_n z^n$ with $c_{-k} \neq 0$

iii) essential sing (infinite # of terms in the principal part)

Th (Riemann removable sing thm)

f is bdd $\implies$ 0 is a removable sing

($|f| \le M$ for $0 < |z| < R$)

Pf we need to show principal part $= 0$

$\therefore n < 0 \qquad c_n = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)\,d\zeta}{\zeta^{n+1}}$

$$|c_n| \le \frac{1}{2\pi} \cdot \frac{|f(\zeta)|}{r^{n+1}} \cdot 2\pi r \le \frac{1}{2\pi} \frac{M\, 2\pi r}{r^{n+1}} \le M r^{-n}$$

let $r \to 0 \implies c_n = 0$

Thm f has a pole of order k iff $\lim_{z \to 0} z^{k+1} f(z) = 0$ and k is the smallest integer with this property.

Pf: $f(z) = \frac{c_{-k}}{z^k} + \frac{c_{-k+1}}{z^{k-1}} + \dots + c_0 + c_1 z + \dots$

$z^{k+1} f(z) = c_{-k} z + c_{-k+1} z^2 + \dots + c_0 z^{k+1} + c_1 z^{k+2} + \dots$

this is a power series $\therefore$ $z^{k+1}f$ is cont $\therefore$ @ $z=0$ $z^{k+1}f=0$

Assume $l<k$ if $\lim_{z\to 0} z^{l+1}f(z)=0$, then $\lim z^k f(z) = \underbrace{z^{k-(l+1)}}_{z^{k-(l+1)}=0} \cdot z^{l+1}f(z)=0$

but $z^{k-(l+1)} = z^m$ $m>0$ but

$z^k f(z) = C_{-k} + C_{-k+1}z + \dots$ and $\lim_{z\to 0} z^k f(z) \neq 0$ $\therefore$ $\Rightarrow\Leftarrow$

Proof if $\lim z^{k+1}f(z)=0 \Rightarrow f$ has a pole of order k

$z^{k+1}f(z)$ is anal on $0<|z|<R$ is bdd near 0 by Riemann's theorem it's analytic on whole disk & is a power series since at $z=0$) $=0$ $\Rightarrow a_0=0$

$z^{k+1}f(z) = a_0 + a_1 z + a_2 z^2 + \dots$

$\therefore z^{k+1}f(z) = a_1 z + a_2 z^2 + \dots$

$f(z) = \frac{a_1}{z^k} + \dots$

<u>Th</u> f has a pole $\Longleftrightarrow$ $\lim_{z\to 0} |f(z)| = \infty$

Pf $\Rightarrow$ $f(z) = \frac{C_{-k}}{z^k} + \dots$ power series

$z^{k-1}f = \frac{C_{-k}}{z} + \underbrace{C_{-k+1}}$

$\lim_{\text{as } z\to 0}$ of this is C_{-k+1}

$\lim_{z\to 0} z^{k-1}f = \lim_{z\to 0}\frac{C_{-k}}{z} + C_{-k+1} = \infty$

$\lim_{z\to 0} f = \lim_{z\to 0} z^{k-1}f \cdot \lim_{z\to 0}\frac{1}{z^{k-1}} = \infty \cdot \begin{cases} 1 & k=1 \\ \infty & k\neq 1 \end{cases} = \infty$

$\Leftarrow$ $f(z)$ is analy on $0<|z|<R$

near "0" $\exists\, 0<R_1<R$ $\ni$ $|f(z)|\geq 1$ on $0<|z|<R_1$

$\frac{1}{f}$ is analytic on punctured disk. But $\lim_{z\to 0}\frac{1}{f(z)}=0$

but since near 0 it is bdd ∴ 0 is a removable sing and is analy over entire disk & has power series

$$\frac{1}{f(z)} = a_0 + a_1 z + \cdots \qquad w/\ a_0 = 0 \text{ since } \frac{1}{f(0)} = 0$$

let first nonvanish term be $a_k z^k$

$$\therefore \frac{1}{f(z)} = z^k \left(a_k + a_{k+1} z + a_{k+2} z^2 + \cdots\right) = z^k g(z)$$

w/ $g(z)$ is analy on disk. $g(z)\Big|_{z=0} \neq 0$

$\therefore g(z) \neq 0$ on a disk $|z| < R_2$ $(0 < R_2 < R_1)$

since $g(z) \neq 0 \Rightarrow \frac{1}{g(z)}$ is anal & can be written as a power series

$$\therefore f(z) = \frac{1}{z^k}\frac{1}{g(z)} = \frac{1}{z^n}\left[b_0 + b_1 z + \cdots\right] = \text{Laurent series for } f.$$

finite no. of terms of Principal part and since it is not a removable singularity $\Rightarrow$ f has a pole.

<u>Thm</u> f analy on $0 < z < R$ and 0 is an essential sing $\Rightarrow$ the image of f is everywhere dense, ie given any complex # a & any positive number ε $\exists$ some $0 < |z| < R$ $\ni$ $|f(z) - a| < \varepsilon$

Example $e^z = \sum \frac{z^n}{n!}$ $\qquad e^{1/z} = \sum \frac{1}{n!\, z^n}$ analyt on $0 < |z| < \infty$
Principal has infinite no. of terms

<u>Pf</u> assume $\exists$ an a, ε $\ni$ $|f(z) - a| \geq \varepsilon$ $\forall z$
Consider $g(z) = \frac{1}{f(z) - a}$ on $0 < |z| < R$ and is bdd by $\frac{1}{\varepsilon}$

∴ has a removable singularity ∴ analytic in $|z| < R$

$g(z) = \frac{1}{f - a} = b_0 + b_1 z + b_2 z^2 + \cdots$

since it is not ident $= 0 \Rightarrow \frac{1}{f - a} = z^k\left(b_k + b_{k+1} z + \cdots\right)$

$= z^k h(z)$

since $h(z) \neq 0$ near 0 $\therefore$ $h(z) = \sum_0^\infty c_n z^n$

now we invert $f - a = \frac{1}{z^n} \frac{1}{h(z)} = \frac{1}{z^n} \sum c_n z^n = \frac{c_0}{z^n} + \cdots$

this shows that f has a finite no. of terms in this principal part $\rightarrow\leftarrow$ since we assumed that f had an essential singularity

Schwarz Lemma

f analytic on $|z| \leq 1$, $f(0) = 0$, $|f(z)| < 1$ for $|z| < 1$

$\Rightarrow |f(z)| \leq |z|$

Pf $f(z) = \underset{\text{since anal}}{a_0 + a_1 z + \cdots} = \underset{\text{since } f(0)=0}{a_1 z + \cdots}$

let $\underline{\frac{g(z)}{z}} = a_1 + a_2 z + \cdots$ is analytic on $|z| < 1$ since power series

For $|z| < r < 1$ by max mod theorem $|g(z)| \leq \max_{|\zeta| = r} |g(\zeta)|$

$$= \frac{\max_{|\zeta|=r} |f(\zeta)|}{r} < \frac{1}{r}$$

now let $r \to 1$ (fix z & let $r \to 1$)

$\therefore$ $|g(z)| \leq 1$ and $|f(z)| \leq |z|$

12/1/78

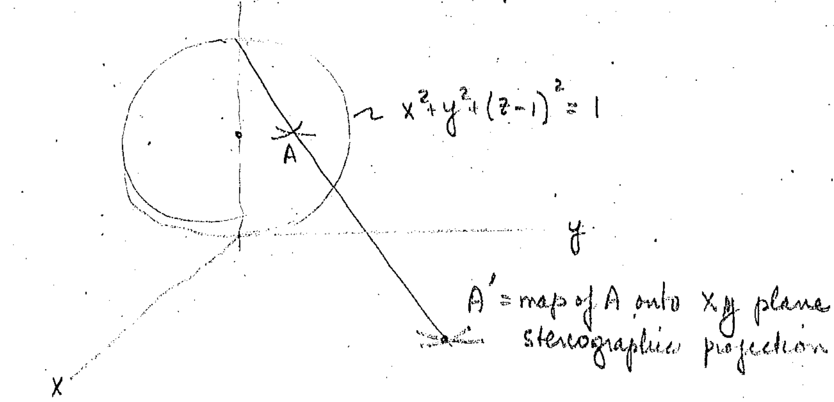
HW Due 12/8/78

Pg 195 (1-6) 202 (1-3, 7, 8) 209 (2, 3) 215 (1, 2, 4-6, 8)

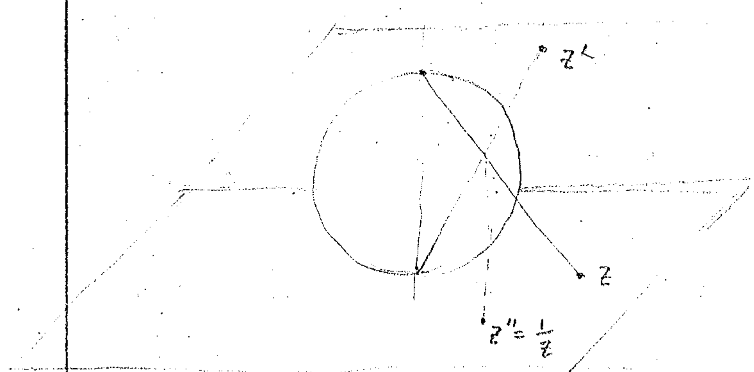
- for a point at ∞: where $f(z)$ being analytic on $|z| > R$ we can define a $g(w) = f(\frac{1}{w})$ which is analytic on the punctured disk $0 < |w| < \frac{1}{R}$

- Hence singularity of f at ∞ is defined as the singularity for g at $w = 0$

- Dfn: the extended z plane is the z plane + ∞
  This is called the Riemann sphere.

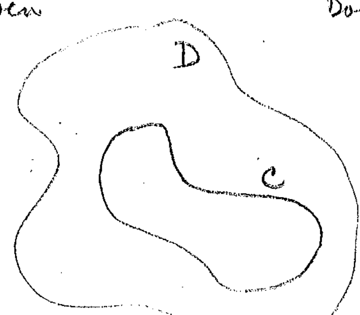


This is a conformal map because angle between 2 curves on the sphere = angle between the two curves mapped on the plane.



Evaluation of definite integrals using residues

Given Domain D w/ curve C



If $f(z)$ is analytic in D the $\int_C f(z)\,dz = 0$ (Cauchy Theorem)

If $f(z)$ is not analytic then $\int_C f(z)\,dz$ = residue

Defn: if f has an isolated singularity at $\underline{a}$ then we define

$$\text{Residue of } f \text{ at } a = \operatorname{Res}(f,a) = \frac{1}{2\pi i}\int_{|z-a|=r} f(z)\,dz = \operatorname{Res}_a f$$

r is small

Thm: if a is an isolated singularity of f and the Laurent series of f is

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k$$

Res $(f,a) = \mathrm{Res}_a f =$ coeff of 1^{st} neg power term $= c_{-1}$

Prof $\frac{1}{2\pi i}\int_{|z-a|=r} f(z)\,dz$ let $z = re^{i\theta}+a \therefore z-a = re^{i\theta}$

$$\therefore \frac{1}{2\pi i}\int f(z)\,dz = \frac{1}{2\pi i}\sum c_k \int_{\theta=0}^{2\pi} r^k e^{ki\theta} \cdot ri\,e^{i\theta}\,d\theta = \sum \frac{r^{k+1}}{2\pi i} c_k \int e^{(1+k)i\theta}\,d\theta$$

$$= \sum_k c_k \frac{r^{k+1}}{2\pi(k+1)i} e^{i\theta(k+1)}\Big|_0^{2\pi} = 0 \text{ if } k+1 \neq 0 \text{ and if } k+1=0 \text{ then}$$

$$= \frac{c_{-1}}{2\pi}\int_0^{2\pi} d\theta = c_{-1}$$

if Res at $a=0$ doesn't necessarily mean fn is analytic at a

ex: $e^{\frac{1}{z^2}}$ singular at $z=0$

$$e^{\frac{1}{z^2}} = 1 + \frac{1}{z^2} + \frac{1}{z^4} + \cdots$$

note that c_{-1} (residue at $z=0$) $= 0$

Hence if a fn has a residue $=0$ at a and if the fn is continuous at a, then fn is analytic at a.

Thm: (Residue theorem)

Suppose f is analytic on $D - \{a_1, \dots, a_k\}$ and C is any piecewise smooth simple closed curve such that the domain enclosed by C is inside D and C contains $\{a_1, \dots, a_k\}$

then $\frac{1}{2\pi i}\int f(z)\,dz = \sum_{\nu=1}^{k} \mathrm{Res}_{a_\nu} f$

This follows directly from Cauchy's theorem by taking integrals around small circles enclosing each a_ν

<u>Definite integrals</u>

Type I : $\int_{x=-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$ $\quad$ P, Q polynomials

Aside

- if limits are a, b try to use Fractional linear transformation such that $a \to 0$ and $b \to \infty$ $\quad x_1 = \frac{x-a}{x-b}$
- if fn is symmetric then $\int_0^\infty = \frac{1}{2}\int_{-\infty}^{\infty}$

Assume $Q(x) \neq 0$ when $x \in \mathbb{R}$ or $\int_{-\infty}^{\infty} \frac{P}{Q}\,dx$ is undefined

Convergence $\quad$ degree of Q > degree of P + 1

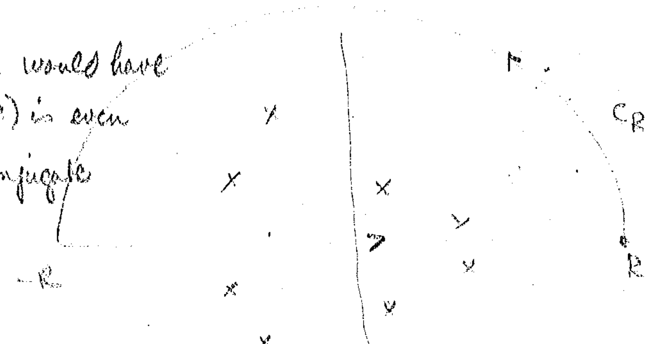
or

degree of Q $\geq$ degree of P + 2

define $\quad f(z) = P(z)/Q(z)$

Although Q has no zeros when $z \in \mathbb{R}$ it must have n zeros when $z \in \mathbb{C}$ by fundamental theorem of algebra <u>if Q has degree = n</u>

Q(z) must not be odd $\Rightarrow$ Q would have a real root. Thus Q(z) is even $\Rightarrow$ Q(z) would have conjugate pairs of roots.



$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum_\nu \mathrm{Res}_{a_\nu} f$ $\quad$ a_ν inside semicircle note that $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$ reason $f(z) \sim \frac{1}{z^n}$ (since degree $Q >$ degree P) where $n > 1$ $\therefore \int f(z)\,dz = \int_0^{\pi} \frac{1}{R^n e^{in\theta}} R \cdot e^{i\theta} i\,d\theta$

$$= \frac{i}{R^{n-1}} \int_0^{\pi} e^{i(1-n)\theta}\,d\theta = \frac{1}{R^{n-1}(1-n)} \cdot 0$$

or if $n = 1$ $\int_{C_R} f(z)\,dz = \pi i \cdot \text{const.}$

Hence second integral will disappear when we take limit as $R \to \infty$

Thus

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \text{ (Sum of all residues of } P/Q \text{ at roots of } Q(z) \text{ in upper half plane)}$$

Must prove that $\int_{C_R} f(z)\,dz = 0$ as $R \to \infty$

$$\left| \int_{C_R} \frac{P(z)}{Q(z)}\,dz \right| = \left| \int_0^{\pi} \frac{P(Re^{i\theta})}{Q(Re^{i\theta})} Rie^{i\theta}\,d\theta \right| \le \sup_{0 \le \theta < \pi} \left| \frac{P(Re^{i\theta})}{Q(Re^{i\theta})} \right| R\pi$$

$$\text{but } \sup_{0 \le \theta < \pi} \left| \frac{P(Re^{i\theta})/R^{n-2}}{Q(Re^{i\theta})/R^{n}} \right| R\pi = \frac{1}{R}\pi$$

degree of num must be at least 2 less than degree of poly in denom.

limit of denom as $R \to \infty$ is a finite no. numerator $\to 0$ or finite no. as $R \to \infty$

thus $\int \frac{P(z)}{Q(z)}\,dz = 0$

Math 106 Test II Nov. 29, 1978

1. Find the radius of convergence of the following power series.

a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{z}{n}\right)^n$

b) $\displaystyle\sum_{n=1}^{\infty} n^{\log n} z^n$

2. Prove that the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} z^{(n!)}$ is 1 (note that $\displaystyle\sum_{n=1}^{\infty} z^{(n!)}$ means the power series $\displaystyle\sum_{k=1}^{\infty} a_k z^k$ with $a_k = \begin{cases} 1 & \text{if } k \text{ is of the form } n! \\ 0 & \text{if } k \text{ is not of the form } n! \end{cases}$).

use $\sqrt{\ }$ test

3. Find the power series or Laurent series expansion of the function $\dfrac{1}{(z-1)(z-2)}$ in the following disc and annuli.

a) $|z| < 1$, b) $1 < |z| < 2$, c) $2 < |z| < \infty$

4. Find the Laurent series for $e^{\frac{1}{z}}$ in $0 < |z| < \infty$ and then show that for $n = 0, 1, 2, 3, \ldots$

$$\frac{1}{\pi}\int_0^{\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{1}{n!}$$

5. Suppose $f(z)$ is an analytic function on $0 < |z| < 1$ and there is a positive number M such that $|f(z)| \le \dfrac{M}{|z|^{\frac{1}{2}}}$ for all $0 < |z| < 1$.

Prove that 0 is a removable singularity of f.

12/4/78

1) $\sum \left(\frac{z}{n}\right)^n$ Root test $\sqrt[n]{\left|\frac{z}{n}\right|^n} = \left|\frac{z}{n}\right| \to 0 < 1 \quad \forall z$ radius $= \infty$

Ratio test $\dfrac{\left|\frac{z}{n+1}\right|^{n+1}}{\left|\frac{z}{n}\right|^n} = z\left|\frac{n}{n+1}\right|^n \cdot \frac{1}{n+1} \to z \cdot \frac{1}{e} \cdot 0 = 0 \quad \forall z$

$\Rightarrow$ radius $= \infty$

$\sum n^{\log n} z^n$ Root test $\sqrt[n]{n^{\log n} z^n} = n^{\frac{1}{n}\log n} z = e^{\frac{1}{n}(\log n)^2}|z| \to |z|$

$\therefore$ radius $|z| \leq 1$ since $\frac{1}{n} \to 0$ faster than $(\log n)^2 \to \infty$

ratio test $\dfrac{(n+1)^{\log (n+1)}}{n^{\log n}} \dfrac{|z|^{n+1}}{|z|^n} = \dfrac{e^{(\log(n+1))^2}}{e^{(\log n)^2}}|z|$

$= e^{\left((\log(n+1))^2 - (\log n)^2\right)}|z|$

but $(\log(n+1))^2 - (\log n)^2 = (\log(n+1) - \log n) \cdot (\log(n+1) + \log n)$

$= n\log\left(1+\frac{1}{n}\right) \cdot \dfrac{\log(n(n+1))}{n} = 1 \cdot 0 = 0$

2) $\sum z^{n!}$, $\left|\dfrac{z^{(n+1)!}}{z^{n!}}\right| = \left|\dfrac{z^{n \cdot n!}}{z^{n!}}\right| \to 0$ as $|z| < 1$

for $|z| \geq 1$ $|z^{n!}| \geq 1$ $\therefore$ rad of conv $= 1$

3. $\dfrac{1}{(z-1)(z-2)} = f(z)$ $f(z) = \dfrac{-1}{z-1} + \dfrac{1}{z-2}$

$|z| < 1$ $\dfrac{-1}{z-1} = \dfrac{1}{1-z} = 1 + z + z^2 + \cdots$

$\dfrac{1}{z-2} = -\dfrac{1}{2}\left(\dfrac{1}{1-z/2}\right) = -\dfrac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right]$

b) $1 < |z| < 2$ $\dfrac{-1}{z-1} = \dfrac{1}{1-z} = \dfrac{-1}{z} \dfrac{1}{1-1/z} = \dfrac{-1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right]$

$\dfrac{1}{z-2} = -\dfrac{1}{2}\left(\dfrac{1}{1-z/2}\right) = -\dfrac{1}{2}\left[1 + \frac{z}{2} + \cdots\right]$

c) $|z|>2$ $\quad \frac{1}{z-2} = \frac{1}{z}\left[\frac{1}{1-2/z}\right] = \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^{2}+\cdots\right]$

$$\frac{1}{z-1} = -\frac{1}{z}\,\frac{1}{1-1/z} = -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

4) $e^{\frac{1}{z}} = 1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\cdots\frac{1}{n!\,z^n}+\cdots$

$$\frac{1}{n!} = c_{-n} = \frac{1}{2\pi i}\int_{|z|=1}\frac{e^{1/z}\,dz}{z^{-n+1}} = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{e^{\left(\frac{1}{e^{i\theta}}\right)}\,ie^{i\theta}d\theta}{(e^{i\theta})^{-n+1}}$$

$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{(\cos\theta - i\sin\theta)}\,e^{in\theta}d\theta = \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{\cos\theta - i\sin\theta}(\cos n\theta + i\sin n\theta)$$

$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{\cos\theta - i(\sin\theta - n\theta)}d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\cos\theta}\underbrace{\cos(\sin\theta - n\theta)}_{\text{even fn.}} - i\,e^{\cos\theta}\underbrace{\sin(\sin\theta - n\theta)}_{\text{odd f.}}$$

by $\frac{1}{2}\int_{-\pi}^{\pi} = \int_0^{\pi}$ hence result $\frac{1}{\pi}\int_0^{\pi} e \qquad d\theta = \frac{1}{n!}$

5.

$f(z) = \sum c_n z^n \qquad c_n = 0$ for $n<0$

$$c_n = \frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi i}\int f(re^{i\theta})\,r^{-n-1}e^{-i(n+1)\theta}\,ire^{i\theta}d\theta$$

$$|c_n| \le \frac{1}{2\pi}\,\frac{M}{r^{1/2}}\,r^{-n}\,2\pi \longrightarrow 0 \quad \text{as } r\to 0$$

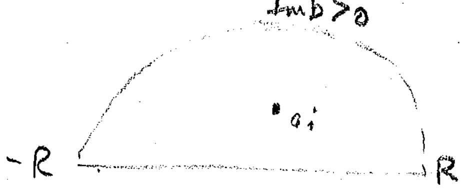
$g(z) = zf(z) \to 0$ as $z\to 0$

g has a removable sing at $z=0$ since g is bounded

$\therefore\ g(z) = \underset{\swarrow 0}{a_0} + a_1 z + a_2 z^2 + \cdots$ but $\Rightarrow f =$ power series

Continue

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} = 2\pi i \left(\operatorname*{Res}_{\substack{b \\ \operatorname{Im} b > 0}} f(z) \right)$$



$f(z) = \frac{1}{(z^2+a^2)^3}$ $\quad z = \pm ai$ since $\operatorname{Im} b > 0 \Rightarrow z = ai \quad a > 0$

$$\frac{1}{(z^2+a^2)^3} = \frac{1}{(z+ai)^3} \frac{1}{(z-ai)^3} = \frac{1}{((z-ai)+2ai)^3} \cdot \frac{1}{(z-ai)^3}$$

$$= \frac{1}{(2ai)^3} \cdot \frac{1}{\left[1 - \left(\frac{z-ai}{-2ai}\right)\right]^3} \cdot \frac{1}{(z-ai)^3}$$

$$\frac{1}{1-\left(\frac{z-ai}{-2ai}\right)} = 1 + \left(\frac{z-ai}{-2ai}\right) + \left(\frac{z-ai}{-2ai}\right)^2 + \dots$$

now take cube $\quad \frac{1}{\left[1-\left(\frac{z-ai}{-2ai}\right)\right]^3} = 1 + \left[\frac{z-ai}{-2ai} \cdot \frac{z-ai}{-2ai} \cdot 1\right.$

$$+ 2 \cdot 1 \left(\frac{z-ai}{-2ai}\right)^2 + \frac{z-ai}{-2ai} \cdot 2 \cdot \frac{z-ai}{-2ai}$$

$$\left. + \left(\frac{z-ai}{-2ai}\right)^2 \right] + \dots$$

from the theorem we know that the result occurs from 1^{st} negative power term

$$\frac{c_{-1}}{z-ai} = \frac{\therefore \operatorname{Res}_{ai} f}{z-ai} = \frac{1}{(2ai)^3} \cdot \frac{6}{(-2ai)^2} = \frac{1}{(z-ai)^{\cancel{2}}} \cdot \frac{1}{(2ai)^3} \cdot 6 \left(\frac{z-ai}{-2ai}\right)^2 = \frac{6}{(2ai)^5} \frac{1}{z-ai}$$

$$\therefore \quad c_{-1} = \operatorname{Res}_{ai} f = \frac{6}{(2ai)^5} \qquad \therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} = 2\pi i \cdot \frac{6}{(2ai)^5} = \frac{3\pi}{8a^5}$$

analytic at $z = ai$

$$\frac{1}{(z^2+a^2)^3} = \frac{1}{(z-ai)^3} \frac{1}{(z+ai)^3} = \frac{1}{(z-ai)^3} \left[c_0 + c_1 (z-ai) + c_2 (z-ai)^2 + \dots \right.$$

$$\frac{1}{2\pi i} \int \frac{dz}{(\ )^3} = \cancel{\frac{1}{2\pi i} \int \frac{dz\, c_0}{(\ )^3}}^{0} + \cancel{\frac{1}{2\pi i} \int \frac{c_1\, dz}{(\ )^2}}^{0} + \frac{1}{2\pi i} \int \frac{c_2\, dz}{(z-ai)} + \cancel{\frac{1}{2\pi i} \oint \sum_{n=0}^{\infty} (z-ai)^n dz}^{0}$$

$$\therefore = c_2 \quad \text{but} \quad c_2 = \frac{1}{2!} \frac{d^2}{dz^2} \frac{(z-ai)^3}{(z^2+a^2)^3}$$

$$c_2 = \frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{(z-ai)^3}{(z^2+a^2)^3}\right)\Bigg|_{z=ai} = \frac{1}{2!}\frac{d^2}{dz^2}\frac{1}{(z+ai)^3} = \frac{12}{2!}\frac{1}{(z+ai)^5}$$
$$= \frac{12}{2!}\frac{1}{(2ai)^5}$$

if f has a pole of order k at a then $\text{Res}_a f = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}} f(z)(z-a)^k$

12/6/78

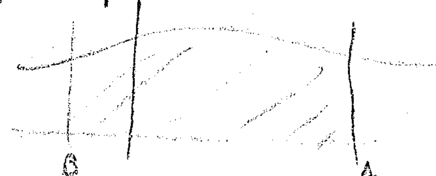
Now Type I $\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{\text{Im}\,a} \text{Res}_a \frac{P(z)}{Q(z)}$

where degree $Q \geq \deg P + 2$

Type II $\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)} \begin{matrix}\sin x\\ \cos x\end{matrix}\,dx$ $\quad \deg Q \geq \deg P + 1$ and $Q(x) = 0$ where $\begin{matrix}\sin x\\ \cos x\end{matrix} = 0$

Integrals are to be interpreted as principal values

in general $\int_{-\infty}^{\infty} = \lim_{\substack{A\to\infty\\ B\to-\infty}}\int_B^A$



Principal value $\int_{-\infty}^{\infty} = \lim_{R\to\infty}\int_{-R}^{R}$

example $\int_{-\infty}^{\infty}\frac{dx}{x^{1/3}}$ $\because x^{1/3}=0$ at $x=0$ but $\int\frac{dx}{x^{1/3}} = x^{2/3}\Big|_{-\infty}^{\infty} = 0$

for type II extend the fn in z plane by defining $f(z) = \frac{P(z)}{Q(z)}e^{iz}$

and look at real & im parts since z (on real line) $= x$

$\therefore f(z) = \frac{P(x)}{Q(x)}e^{ix} = \frac{P(x)}{Q(x)}(\cos x + i\sin x)$

now we look at convergence of $\int \left|\frac{P}{Q} e^{iz}\right| dz \sim \pi R \cdot \frac{1}{R} \sim \pi$ in absolute val

$\therefore$ look at $\frac{d}{dz}\left(\frac{P}{Q}\right) = \frac{P'Q - Q'P}{Q^2}$ degree $Q^2 \geq 2$ deg $(P'Q - Q'P)$

now $$\int \frac{P(z)}{Q(z)} e^{iz} dz = \frac{P(z)}{Q(z)} \frac{e^{iz}}{i} \Bigg|_R^{-R} - \int \frac{d}{dz}\left(\frac{P}{Q}\right) \frac{e^{iz}}{i} dz$$

$$= \underbrace{\frac{P(-R)}{Q(-R)} \frac{e^{-iR}}{i} - \frac{P(R) e^{iR}}{Q(R)} \frac{}{i}}_{\to 0 \text{ since PV}} - \underbrace{\int_{C_R} \left(\frac{d}{dz}\frac{P}{Q}\right) \frac{e^{iz}}{i} dz}_{\to 0 \text{ as proven above}}$$

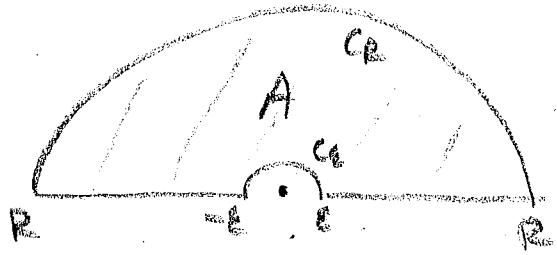
$$\therefore \int_{-\infty}^{\infty} \frac{P}{Q} \sin x \, dx = \operatorname{Im} 2\pi i \sum_{\operatorname{Im} a > 0} \operatorname{Res}_a \frac{P(z)}{Q(z)} e^{iz}$$

<u>example</u> $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \operatorname{Re}\left\{ 2\pi i \operatorname{Res}_{ai} \frac{e^{iz}}{z^2 + a^2} \right\}$$
$$= \operatorname{Re} 2\pi i \left[\frac{e^{-a}}{2ai} \right] = \frac{\pi e^{-a}}{a}$$

<u>Example</u>

for $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$ define $f(z) = \frac{e^{iz}}{z} \sim \operatorname{Im} 2\pi i \int \frac{e^{iz}}{z} dz$

singularity exists due to real part even though $\frac{\sin x}{x} \to 0$ as $x \to \infty$



$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x} dx + \int_{C_\varepsilon} \frac{e^{iz}}{z} dz + \int_{\varepsilon}^{R} \frac{e^{ix}}{x} dx$$
$$+ \int_{C_R} \frac{e^{iz}}{z} dz = 0 \text{ by Cauchy Th.}$$

since in A no poles and hence fn is analytic

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}dz}{z} \to 0$$

analytic fn.

now $\frac{e^{iz}}{z} = \frac{1}{z} + i + \left(\frac{iz}{2!z}\right)^2 + \dots = \frac{1}{z} + g(z)$

$$\int \frac{e^{iz}}{z} dz = -\frac{1}{2}\oint \frac{1}{z} dz - \int g(z) dz^{\to 0} = -\pi i$$

since $\left|\int g(z)\,dz\right| \le \lim_{\varepsilon\to 0} |g(0)| \cdot \varepsilon = 0$

Half Residue thm.

if $f(z)$ has a pole of order 1 (simple pole) and C_ε is a half circle of radius ε centered at a.

$$\lim_{\varepsilon\to 0} \int f(z)\,dz = 2\pi i \cdot \frac{1}{2} \mathrm{Res}_a f$$

Proof let $\mathrm{Res}_a f = C_{-1}$ $\quad f(z) = \frac{C_{-1}}{z-a} + g(z)$

now let $z = a + \varepsilon e^{i\theta}$ $\quad \alpha \le \theta \le \pi + \alpha$

$$\int f(z)\,dz = \int \frac{C_{-1}\,dz}{z-a} + \int g(z)\,dz$$

$$= \left(\int C_{-1} \frac{\varepsilon i e^{i\theta}}{\varepsilon e^{i\theta}} d\theta = C_{-1} \cdot \pi i\right) + \int g(a + \varepsilon e^{i\theta})\, \varepsilon i e^{i\theta} d\theta$$

$\le \max g(\quad) \cdot \varepsilon \cdot \pi$

$\lim_{\varepsilon\to 0} g(a) \cdot \varepsilon\pi = 0$

$$\int_0^\infty \frac{x^\alpha}{(1+x^2)}\,dx \qquad 0<\alpha<1$$

— we must take a branch cut since x^α is an α root of x for; when we extend $f(x) \to f(z) = \frac{z^\alpha}{1+z^2}$

then we must be careful & take branch cut and we do this by take $0 \le \theta \le 2\pi$

12/8/76

Final 12 – 3 Rm 3809

ME251 Get from Skey

ME 200 (ask John to give)

· ME250 (ask Rich to get one)

\+ ME255 Monday I do.

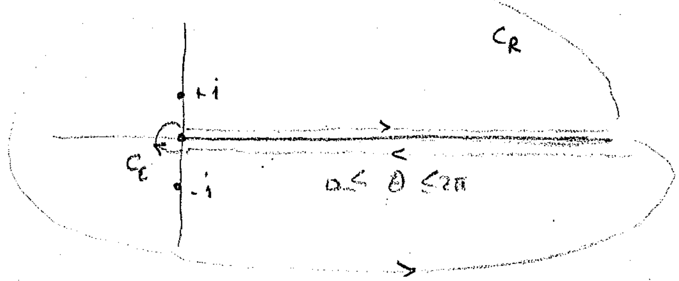
\+ ME238 I must do

MA106 Dec 15 (12^15 – 3^15) ask guy in Rm 340

ME 111 (sunday 1-4 go pick up or AC)

$$\int_0^\infty \frac{x^\alpha\,dx}{x^2+1} \qquad 0<\alpha<1$$

$$f(z) = \frac{z^\alpha}{1+z^2}$$



$$\int_\varepsilon^R \frac{x^\alpha}{1+x^2}\,dx + \int_R^\varepsilon \frac{x^\alpha e^{i\alpha 2\pi}}{1+x^2}\,dx + \oint_{C_R} f(z)\,dz + \oint_{C_\varepsilon} f(z)\,dz = 2\pi i \operatorname{Res}_i f + 2\pi i \operatorname{Res}_{-i} f$$

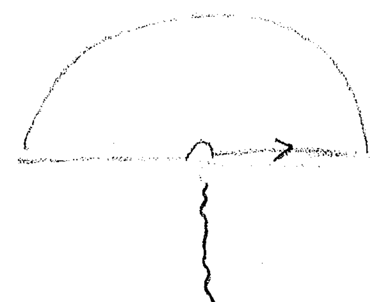
$$\lim_{\varepsilon\to 0} \frac{\varepsilon^\alpha\,\varepsilon}{1+\varepsilon^2} = \lim_{\varepsilon\to 0} \varepsilon^{1+\alpha} \to 0$$

$$\lim_{R\to\infty} \frac{R^\alpha}{1+R^2}\cdot R = R^{-1+\alpha} \to 0$$

$$= 2\pi i \left[\frac{e^{\frac{\alpha\pi i}{2}}}{2i} + \frac{e^{\frac{3\alpha\pi i}{2}}}{-2i}\right]$$

$$(1-e^{i\alpha 2\pi})\int_0^\infty \frac{x^\alpha}{1+x^2}\,dx = \frac{2\pi i}{2i}\left[\frac{e^{\frac{\alpha\pi i}{2}} - e^{\frac{3\alpha\pi i}{2}}}{1}\right] \quad \text{or} \quad \int_0^\infty \frac{x^\alpha}{1+x^2}\,dx = \pi\left[\frac{e^{\frac{\alpha\pi i}{2}} - e^{\frac{3\alpha\pi i}{2}}}{1-e^{i\alpha 2\pi}}\right]$$

$$\int_0^\infty \frac{\log x\,dx}{1+x^2}\ ; \quad \frac{\log z}{1+z^2} = f(z)$$



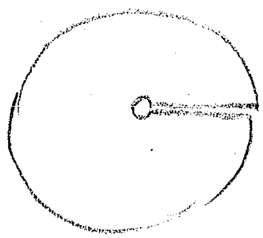
$$\frac{(\log r + i\theta r)\,r}{r^2} \text{ as } r\to\infty \longrightarrow 0$$

$$\int_\varepsilon^R \frac{\log x\,dx}{1+x^2} + \oint_{CR} \frac{\log z\,dz}{1+z^2} + \int_{-R}^{-\varepsilon} \frac{\log x + \pi i}{1+x^2}\,dx + \oint_{-\varepsilon}^{\varepsilon} \frac{\log z}{1+z^2}\,dz = 2\pi i \operatorname{Res}_i$$

$\operatorname{Res}_i f = \left.\frac{\log z}{z+i}\right|_{z=i} = \frac{i\pi/2}{2i} = \frac{\pi}{4}$

$\therefore \int_0^\infty \frac{\log x\,dx}{1+x^2} + \int_0^\infty \frac{(\log x + \pi i)\,dx}{1+x^2} = \frac{2\pi i \pi}{4}$

$\therefore \operatorname{Re}\ 2\int_0^\infty \frac{\log x\,dx}{1+x^2} = 0$ &   gives $\int_0^\infty \frac{\pi\,dx}{1+x^2} = \frac{2\pi^2}{4}$

for  use $f(z) = \frac{(\log z)^2}{1+z^2}$

Course

Analytic fns.

1. Complex Deriv $\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$ $\Rightarrow$ C.R. or $\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}$ $\Rightarrow$ $u_y = -v_x$, $u_x = v_y$

||

Cauchy's Thm.

2. Power series

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)\, d\zeta}{\zeta - z} \quad ; \quad c_k = \frac{k!}{2\pi i} \int \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{k+1}}$$

Laurent, power

Conformal maps $\Delta u = \Delta v = 0$ f L Transf log fn.

powers

exponential maps

Residues

Max mod principle

Uniqueness

Schwarz inequalities

Ch #1-3 $\frac{1}{2}$ Ch 4 § 1-3 Ch 5.

Math 106 Exercise I Oct. 2, 1978

1. Given $|z_1| = |z_2| = |z_3|$ and $z_1 + z_2 + z_3 = 0$, prove that the points z_1, z_2, z_3 lie at the vertices of an equilateral triangle.

2. Use $1 + z + \cdots + z^n = \dfrac{z^{n+1} - 1}{z - 1}$ to prove that for $0 < \theta < 2\pi$,

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[(n+\frac{1}{2})\theta\right]}{2\sin\frac{\theta}{2}}$$

3. Let z, w be two complex numbers and that $\bar{z}w \neq 1$. Prove that

$$\left|\frac{z-w}{1-\bar{z}w}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

$$\left|\frac{z-w}{1-\bar{z}w}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

4. Prove that $\operatorname{Im} \dfrac{az+b}{cz+d} > 0$ if $ad - bc = 1$ and $\operatorname{Im} z > 0$.

5. Find $\dfrac{3+4i}{1-2i}$, $(4+3i)^{\frac{1}{2}}$, $(-64)^{\frac{1}{6}}$, $(i-1)^{\frac{1}{3}}$ (the answers should be in the form $a+bi$ and contain all possible roots).

6. Prove that $z\bar{z} + (1+i)z + (1-i)\bar{z} - 1 = 0$ represents a circle. Find its center and radius.

7. Let θ be the angle between two (non zero) vectors in $\mathbb{R}^2$ represented by 2 complex numbers α, β. Show that

i) $\operatorname{Re} \alpha\bar{\beta} = |\alpha|\,|\beta| \cos\theta$

ii) $\operatorname{Im} \alpha\bar{\beta} = \pm |\alpha|\,|\beta| \sin\theta$

iii) $\alpha \parallel \beta$ iff $\operatorname{Im} \alpha\bar{\beta} = 0$

iv) $\alpha \perp \beta$ iff $\operatorname{Re} \alpha\bar{\beta} = 0$

v) the area of the $\triangle$ with α, β as two sides is $\frac{1}{2} |\operatorname{Im} \alpha\bar{\beta}|$.

vi) the area of a polygon with vertices $z_1, z_2, \dots, z_{n-1}, z_n = z_1$ arranged in a counterclockwise order is

$$\frac{1}{2} \operatorname{Im}\left(z_1 \bar{z}_2 + z_2 \bar{z}_3 + \cdots + z_{n-1} \bar{z}_n\right).$$

8. Textbook p. 23, # 8.

9. Textbook p. 23, # 9.

10. Textbook p. 43, # 4.1

11. Textbook p. 43, # 4.3

1. $|z_1| = |z_2| = |z_3|$ says that if $z_j = \rho_j e^{i\theta_i}$ ρ_i are equal thus from the origin z_1, z_2, z_3 are points on a circle of radius ρ.

Define a circle of radius ρ with points $z_j = \rho_j e^{i\theta_j}$

and let $\theta_1 = \theta$, $\theta_2 = \theta + \alpha$ $\theta_3 = \theta + \beta$

then $z_1 + z_2 + z_3 = 0 \Rightarrow \cos\theta + \cos(\theta+\alpha) + \cos(\theta+\beta) = 0$ & $\sin\theta + \sin(\theta+\alpha) + \sin(\theta+\beta) = 0$

Thus $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = \cos\theta(1 + \cos\alpha + \cos\beta) + \sin\theta(-\sin\alpha - \sin\beta) = 0$

and $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = \sin\theta(1 + \cos\alpha + \cos\beta) + \cos\theta(\sin\alpha + \sin\beta) = 0$

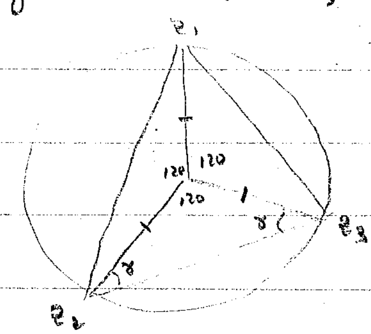
or $(1 + \cos\alpha + \cos\beta) = 0$ and $(\sin\alpha + \sin\beta) = 0$

$\therefore \cos\beta = -1 - \cos\alpha$ $\sin\beta = -\sin\alpha$ or $\beta = -\alpha$

if $\beta = -\alpha \Rightarrow \cos\beta = -1 - \cos\alpha$ becomes $2\cos\beta = -1$ or $\beta = \frac{2\pi}{3}$

thus the angles between θ_1 & θ_2, θ_1 & θ_3, θ_2 & θ_3 are $120°$ $\Rightarrow$ from diag $\gamma = 30°$

or the $\measuredangle$ of vertices $= 60°$ or equil triangle



2. $\sum_{j=0}^{n} z^j = \frac{z^{n+1} - 1}{z - 1}$ prove $\sum_{j=0}^{n} \cos j\theta = \frac{1}{2} + \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}}$

if $z = e^{i\theta}$ then $z^j = e^{ji\theta} = \cos j\theta + i\sin j\theta$ thus $Re(z^j) = \cos j\theta$

then $Re\left(\frac{z^{n+1} - 1}{z - 1}\right) = Re\left(\frac{e^{(n+1)i\theta} - 1}{e^{i\theta} - 1}\right) = Re\left(\frac{\cos([n+1]\theta) + i\sin([n+1]\theta) - 1}{(\cos\theta - 1) + i\sin\theta}\right)$

mult by the conj of denom gives

$$\frac{\cos(n+1)\theta\cos\theta - [\cos(n+1)\theta + \cos\theta] + 1 + \sin(n+1)\theta\sin\theta}{2(1 - \cos\theta)}$$

4. Prove $\operatorname{Im}\left(\frac{az+b}{cz+d}\right) > 0$ if $ad-bc=1$ & $\operatorname{Im} z > 0$

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\cdot\frac{c\bar{z}+d}{c\bar{z}+d}\right) = \operatorname{Im}\left(\frac{acz\bar{z}+azd+bc\bar{z}+bd}{c^2z\bar{z}+cd(z+\bar{z})+d^2}\right)$$

note $z+\bar{z} = 2\operatorname{Re} z$ and $z\bar{z} = |z|^2$ thus denom is real & since $w\cdot\bar{w} = |w|^2$, denom >0

$$acz\bar{z} + azd + bc\bar{z} + bd = ac|z|^2 + bd + (ad+bc)\operatorname{Re} z + (ad-bc)\operatorname{Im} z$$

now $\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{ad-bc \operatorname{Im} z}{w\bar{w}}$ where $\bar{w} = cz+d$ thus

if $ad-bc=1$ and $\operatorname{Im} z > 0 \implies \operatorname{Im}\left(\frac{az+b}{cz+d}\right) > 0$

5. $\frac{3+4i}{1-2i} = \frac{5e^{i\,\operatorname{atan} 3/4}}{\sqrt{5}\,e^{i\,\operatorname{atan} -1/2}} = \sqrt{5}\,e^{i(\operatorname{atan} 3/4 + \operatorname{atan} \frac{1}{2})}$

$$(4+3i)^{1/2} = \left(5e^{i\,\operatorname{atan} 3/4}\right)^{1/2} = \sqrt{5}\left[\cos\left(\operatorname{atan}\tfrac{3}{4}\right) + i\sin\left(\operatorname{atan}\tfrac{3}{4}\right)\right]$$
$$\sqrt{5}\left[\cos\left(\operatorname{atan}\tfrac{3}{4}+\pi\right) + i\sin\left(\operatorname{atan}\tfrac{3}{4}+\pi\right)\right]$$

$$(-64)^{\frac{1}{6}} = \left(64e^{i(\pi+2\pi k)}\right)^{1/6} = 2e^{i\pi/6 + i\frac{k\pi}{3}} = 2\left[\cos\left(\frac{\pi}{6}+\frac{k\pi}{3}\right) + i\sin\left(\frac{\pi}{6}+\frac{k\pi}{3}\right)\right]$$
$k = 0,1,2,..5$

$$[i-1]^{1/3} = \left(\sqrt{2}\,e^{i\frac{3\pi}{2}}\right)^{1/3} = \sqrt[6]{2}\;e^{i\frac{\pi}{2}+i\frac{2\pi k}{3}} = \sqrt[6]{2}\left(\cos\left(\frac{\pi}{2}+\frac{2\pi k}{3}\right) + i\sin\left(\frac{\pi}{2}+\frac{2\pi k}{3}\right)\right)$$
$k = 0,1,2$

6. $z\bar{z} + (1+i)z + (1-i)\bar{z} - 1 = 0$ prove it is a representation for a circle

let $|z-a| = r$ then $|z-a||\bar{z}-\bar{a}| = r^2$ or $z\bar{z} - a\bar{z} + a\bar{a} - z\bar{a} = r^2$

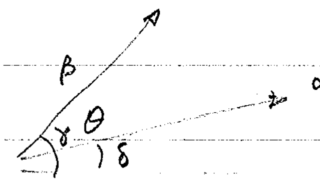
since $|z-a| = r$ represents a circle w/ center @ a & radius of r then

$z\bar{z} - \bar{a}z - a\bar{z} + a\bar{a} - r^2 = 0$ note that the equations are alike if

$a = (-1+i)$ & $-1 = a\bar{a} - r^2$; thus $a\bar{a} = 2$ & hence $r^2 = 3$ or $r = \sqrt{3}$.

thus we have a circle w/ radius $r = \sqrt{3}$ and centered at $z = (-1, 1)$

7.



$\alpha = re^{i\delta}$ & $\beta = \rho e^{i\gamma}$, $\bar{\beta} = \rho e^{-i\gamma}$

Prove $Re(\alpha\bar{\beta}) = |\alpha||\beta| \cos\theta$

$$\alpha\bar{\beta} = r\rho e^{i(\delta-\gamma)}$$

(a.) $Re(\alpha\bar{\beta}) = r\rho \cos(\delta-\gamma) = |\alpha||\beta| \cos(\delta-\gamma)$; note that $\cos(\delta-\gamma) = \cos(\gamma-\delta) = \cos\theta$ $\therefore$ $Re(\alpha\bar{\beta}) = |\alpha||\beta|\cos\theta$

(b) $Im(\alpha\bar{\beta}) = r\rho\sin(\delta-\gamma) = |\alpha||\beta| \sin(\delta-\gamma) = \begin{cases} +|\alpha||\beta|\sin\theta & \text{if } \delta > \gamma \\ -|\alpha||\beta|\sin\theta & \text{if } \gamma > \delta \end{cases}$

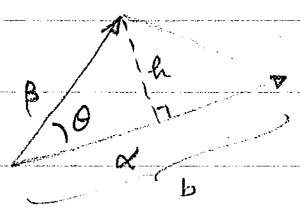
(c) if $\alpha \parallel \beta \Rightarrow \delta = \gamma$ or $Im(\alpha\bar{\beta}) = |\alpha||\beta|\sin 0 = 0$

if $Im(\alpha\bar{\beta}) = 0 \Rightarrow \sin\theta = 0 \Leftrightarrow \theta = 0 \Leftrightarrow \delta = \gamma \Leftrightarrow \alpha \parallel \beta$ other case is trivial α or $\beta = 0$

(d) if $\alpha \perp \beta \Rightarrow \delta = \gamma \pm 90°$ or $Re(\alpha\bar{\beta}) = |\alpha||\beta| \cos 90° = |\alpha||\beta| \cdot 0 = 0$

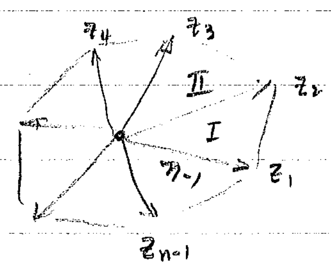
if $Re(\alpha\bar{\beta}) = 0 \Rightarrow \cos\theta = 0 \Leftrightarrow \theta = 90°$ or $270° \Leftrightarrow \alpha \perp \beta$

(e)



Area $= \frac{1}{2} bh = \frac{1}{2}|\alpha| \sin\theta \cdot |\beta| = \frac{1}{2}|\alpha||\beta|\sin\theta = \frac{1}{2} Im(\alpha$

(f.)



area of $\Delta I = \frac{1}{2} Im(z_1 \bar{z}_2)$

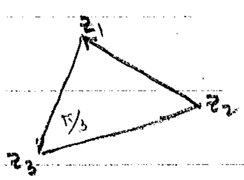
$\Delta II = \frac{1}{2} Im(z_2 \bar{z}_3)$

$\vdots$

$\Delta n\text{-}1 = \frac{1}{2} Im(z_{n-1} \bar{z}_n)$

$\therefore$ Total area $= \frac{1}{2} \sum_{k=2}^{n} (z_{k-1} \bar{z}_k)$

\# 8 Pg. 23 Problem 8



note $z_1 - z_3 = e^{\frac{i\pi}{3}}(z_2 - z_3)$

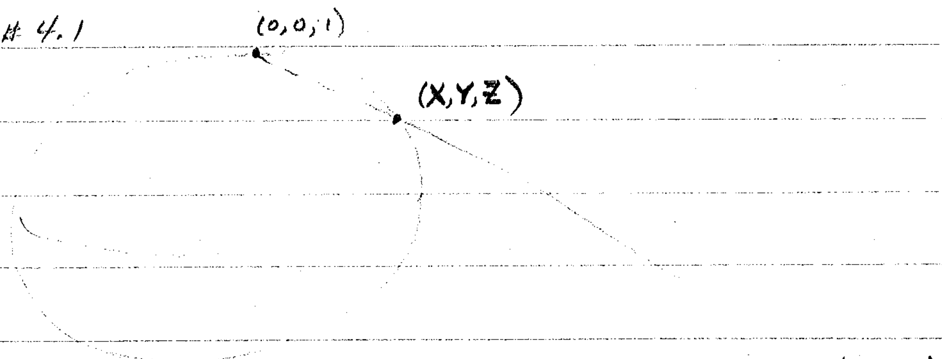
$\overline{z_3 - z_2} = e^{\frac{i\pi}{3}}(z_1 - z_2)$

$z_1^2 - z_1 z_3 - z_1 z_2 - z_2 z_3 = z_3 z_1 - z_3^2 - z_2^2 + \cancel{z_2 z_3}$

result follows.

Pg #23/9 $a \equiv b \pmod{2\pi}$ $c \equiv d \pmod{2\pi}$ show that $a+c \equiv b+d \mod 2\pi$

Pg 43 # 4.1



P43 # 4.3

Math 106 — Exercise II — Oct. 9, 1978

1. Suppose f is analytic on a domain D and $\operatorname{Re} f \equiv \text{const}$. Prove that $f \equiv \text{const}$.

   if $f = u + iv$ $\quad \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \Rightarrow \frac{\partial u}{\partial x} = 0$ since f is analytic & satisfies CR $\Rightarrow \frac{\partial v}{\partial y} = 0$ & $\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \Rightarrow \frac{\partial u}{\partial y} = 0 \Rightarrow -\frac{\partial v}{\partial x} = 0$ $\therefore$ since $v \neq v(x)$, $v \neq v(y)$ $\}$ $v = \text{const}$ $\therefore \operatorname{Im} f$ & $\operatorname{Re} f$ ...

2. Suppose f and $\bar{f}$ are both analytic on a domain D. Prove that $f \equiv \text{const}$.

   $f = \text{analytic} = u + iv \Rightarrow \begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix}$ $\quad$ $\bar{f} = g + ih$ $\quad \begin{matrix} g_x = h_y \\ g_y = -h_x \end{matrix}$ but $\begin{matrix} g = u \\ h = -v \end{matrix}$ $\therefore g_x = h_y \Rightarrow \begin{matrix} u_x = -v_y \\ u_y = v_x \end{matrix}$

   since $u_x = v_y$ & $-v_y \Rightarrow u_x = 0$; $u_y = -v_x$ & $v_x \Rightarrow u_y = 0$ $\}$ $\operatorname{Re} f = \text{const}$ $\Rightarrow$ $\operatorname{Im} f = \text{const}$ $\Rightarrow$ $f = \text{const}$

3. Suppose f is analytic on a domain D and f maps D one-one onto another domain D'. Prove that the area of D' is given by $\iint_D |f'|^2 \, dx\,dy$. (Hint: compute $\frac{\partial(u,v)}{\partial(x,y)}$ by using the Cauchy-Riemann equations.)

   $\iint_D dx\,dy = \iint_{D'} du\,dv$

   now $dx\,dy = J\begin{pmatrix} u & v \\ x & y \end{pmatrix} du\,dv$

   $\therefore \iint J \, dx\,dy = \iint du\,dv$ $\quad$ but $J = |f'|^2$

4. For the following functions $\underline{\underline{u}}$ find a function v such that $u + iv$ is analytic.

   a) $3x^2y + 2x^2 - y^3 - 2y^2$

   b) $xe^x \cos y - ye^x \sin y$

   c) $e^{-2xy} \sin(x^2 - y^2)$.

5. Find the fractional linear transformation which maps the triple of points $(2,\ i+\sqrt{2},\ 3-i)$ to the triple of points $(1-i,\ 0,\ 3+i)$.

6. Find the centre and radius of the circle which is the image of the circle with centre $2i$ and radius 3 under the fractional linear transformation $w = \dfrac{5z + 3 + i}{iz - 2}$.

7. #2, p.52.

8. #9, p.53.

9. #8, p.61

10. #9, p.61

11. #10, p.61

12. #13, p.62

13. #1, p.69

14. #3, p.69

15. #4, p.69

16. #6, p.69

17. #7, p.69.

Correction: In Exercise I #4, a,b,c,d are real

Math 106 # Exercise III Oct. 16, 1998

1. Show that we can separate a branch of the function $\sqrt[3]{(1-z)z^2}$ in the domain exterior to the segment $0 \le x \le 1$, $y = 0$. Find the value at the point $z = i$ of the branch which has a negative value at the point $z = 2$.

2. Show that we can separate a branch of $\log(1-z^2)$ in the domain formed from the z-plane by deleting the segments $[-1, i]$, $[1, i]$ and the ray $x = 0$, $y \ge 1$. Find the value at $z = 2$ of the branch which has the value 0 at $z = 0$.

3. Show that we can separate a branch of $\log(z + \sqrt{z^2-1})$ in the domain formed from the z-plane by deleting the rays $-\infty \le x \le -1$, $1 \le x \le \infty$, $y = 0$. Find the value at $z = i$ of the branch which has the value $\frac{\pi}{2} i$ at $z = 0$.

4. p. 76, # 1

5. p. 76, # 2.

6. p. 82, # 2

7. p. 82, # 3

8. p. 82, # 4

9. p. 82, # 6.

Math 105 Exercise IV Oct. 28, 1958

Textbook p. 93, Problems 1 — 7.

8. The potential is V_0 along the ray $x = 0$, $y > 1$ and is zero along the x-axis. Find the density of the charge distribution along the x-axis. (The density of charge at a point on a conductor is $\sigma = \pm \frac{E}{4\pi}$, where E is the magnitude of the field intensity at this point.)

9. The cylindrical conductor represented by the circle $|z - 2i| = 1$ bears a charge 2π per unit height. Find the distribution of this charge on the cylinder if the conducting sheet represented by the x-axis is earthed.

10. Find the electrostatic field in the space between two conducting cylinders which are perpendicular to the z-plane and intersect this plane in the circles $|z| = 1$ and $|z - 1| = \frac{5}{2}$. The potential difference between the cylinders is unity. Find the least and greatest values for the density of charge in the distribution on the cylinders.

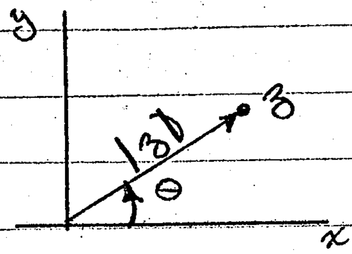
$$\boxed{\text{Students not interested in the application of complex analysis to electrostatics can ignore Problems 8, 9, 10}}$$

Math 106 Exercise V Oct. 30, 1978

1. p. 106, #5
2. p. 116, #1
3. p. 116, #2
4. p. 116, #3
5. p. 116, #5
6. p. 126, #2
7. p. 126, #3
8. p. 126, #4
9. p. 126, #5
10. p. 132, #3

Complex Variable Review

Standard form: $\quad z = x + iy$



modulus $= |z| = r = \sqrt{x^2+y^2}$

argument $= \theta = \arg(z) = \tan^{-1} y/x$

Polar form: $\quad z = r(\cos\theta + i\sin\theta)$

De Moivre's Theorem: $\quad z^n = r^n(\cos n\theta + i\sin n\theta)$

Complex Conjugate $\quad \bar{z} = z^* = x - iy$

Taylor series for $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots$

if $z = i\theta$

$$\Rightarrow e^{i\theta} = 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} + \ldots$$

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\right)$$

$$\Rightarrow \boxed{e^{i\theta} = \cos\theta + i\sin\theta} \qquad \text{Exponential form}$$

$$\Rightarrow e^z = e^x e^{iy} = e^x(\cos y + i\sin y)$$

Branch point: arises from multivalued nature of a function
(e.g. $z^{1/2} = r^{1/2} e^{i\theta/2}$)

Branch cut: imaginary cut made in complex plane so as to render the function single-valued

<u>Analytic</u> (holomorphic) <u>functions</u>:

consider $f(z) = u(x,y) + i\,v(x,y)$

to be analytic, $f'(z)$ must exist

(for the derivative to exist, it must be independent of direction of approach in the limit, this implies the Cauchy-Riemann equations)

$$\frac{df}{dz} = \frac{df}{dx} = \frac{df}{i\,dy}$$

<u>Cauchy-Riemann Equations</u>

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \Rightarrow \Delta u = \Delta v = 0$$

<u>Some Properties of Analytic Functions</u>

1. possess derivatives of all orders
2. both real and imaginary parts are harmonic (i.e. satisfies Laplace's equation)
3. can be specified by its singularities (to within an additive constant)
4. can be continued into other regions (analytic continuation)

Cauchy's Integral Theorem

let $f(z)$ be analytic in a region R and on its boundary

then $$\oint_C f(z)\,dz = 0$$

some consequences of Cauchy's theorem are:

if a and z are two points in R

then 1. $\int_a^z f(z)\,dz$ is path independent in R

and

2. $G(z) = \int_a^z f(z)\,dz$

then $G(z)$ is analytic and $G'(z) = f(z)$

Morera's theorem

let $f(z)$ be continuous in a simply-connected region R and suppose that

$$\oint_C f(z)\,dz = 0$$

around every simple closed curve C in R

then $\boxed{f(z) \text{ is analytic in } R}$

Cauchy's integral formulas

if $f(z)$ is analytic in and on a simple closed curve C and a is any point inside C then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

$$f^{n}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 1, 2, 3,$$

$\int \nabla^2 u\, dV = 0 = \int_S \nabla u \cdot n\, dS = \int_S \frac{\partial u}{\partial n}\, dS = 0$

Useful tests for convergence of a series

1. D'Alembert's Ratio Test

if $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = L$ then Σu_n converges absolutely if $L<1$ and diverges if $L>1$, test fails if $L=1$

2. Cauchy's Root Test

if $\lim_{n\to\infty} \sqrt[n]{|u_n|} = L$ then Σu_n converges absolutely if $L<1$, and diverges if $L>$ test fails if $L=1$

Power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

a. power series can be differentiated term by term in any region which lies entirely inside its circle of convergence

b. Power series can be integrated term by term along any curve C which lies entirely inside its circle of convergence.

Taylor's theorem

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(z-a)^n$$

A Taylor series converges to its nearest singularity i.e. $|\bar{z}-a| = R$ where $\bar{z}$ is the location of the singularity nearest a

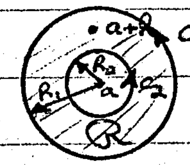
For $|r|<1$, a geometric series gives

$$\frac{1}{1-r} = 1 + r + r^2 + \cdots \quad \text{which converges for } |r|<1$$

Binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Laurent's Theorem



$z = a + h$

suppose $f(z)$ is single valued and analytic on C_1 and C_2 and in the annulus between these two curves.

Let $a+h$ be any point in R, then:

$$f(a+h) = \overbrace{a_0 + a_1 h + a_2 h^2 + \cdots}^{\text{analytic part}}$$

$$+ \underbrace{\frac{a_{-1}}{h} + \frac{a_{-2}}{h^2} + \frac{a_{-3}}{h^3} + \cdots}_{\text{principal part}}$$

where
$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \qquad n = 0, \pm 1, \pm 2, \pm 3,$$

• if the principal part is zero, the Laurent series reduces to a Taylor series

Ex. of Laurent Series

Find Laurent series for $\dfrac{1}{z^2(z-3)^2}$ about $z=3$

singularities at $z=0$, and $z=3$ (both poles of order 2)

let $u = z-3$ (want ~~the exp~~ series valid about $z=3$)

then
$$\frac{1}{z^2(z-3)^2} = \frac{1}{(u+3)^2 u^2} = \frac{1}{9u^2\left(1+u/3\right)^2}$$

from Binomial Theorem (top p. 5)

for $a = 1$

$$\Rightarrow (1+b)^n = 1 + nb + \frac{n(n-1)}{2!} b^2 + \dots + \frac{n(n-1)\dots(n-p+1)}{p!} b^n + \dots$$

$$\Rightarrow \left(1+\frac{u}{3}\right)^{-2} = 1 - 2\left(\frac{u}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3 + \dots$$

$$\frac{\left(1+\frac{u}{3}\right)^{-2}}{9u^2} = \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \dots$$

$$\boxed{\frac{1}{z^2(z-3)^2} = \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4}{243}(z-3) + \dots}$$

Desired Laurent series

radius of convergence = 3 (to singularity at $z=0$)

residue $= a_{-1} = -2/27$

function analytic $\forall z$ except $z=\infty$ are entire fns. entire fns have ∞ radii of convergence

Residues and Residue Theorem

residue is the value of the coefficient a_{-1} of the Laurent series expansion about the singularity in question

However, if the singularity is a pole of order k,

$$a_{-1} = \lim_{z\to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left\{ (z-a)^k f(z) \right\}$$

and if $k=1$ (simple pole)

$$a_{-1} = \lim_{z\to a} (z-a) f(z)$$

Residue theorem

(Cauchy's theorem and Cauchy's Integral Formulas are special cases of this)

$$\oint_C f(z)\,dz = 2\pi i \left(\sum_{n=1}^{N} a_{-1}^{(n)} \right)$$

where, $a_{-1}^{(n)}$ is the residue of the n^{th} singularity within the contour C

Conformal Mapping

General Transformations

1. Translation $\quad w = z + \beta$
2. Rotation $\quad w = e^{i\theta_0} z$
3. Stretching $\quad w = az$
4. Inversion $\quad w = 1/z$

Linear transformation

$w = \alpha z + \beta$ (translation, rotation, stretch

Bilinear (fractional) transformation

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}$$

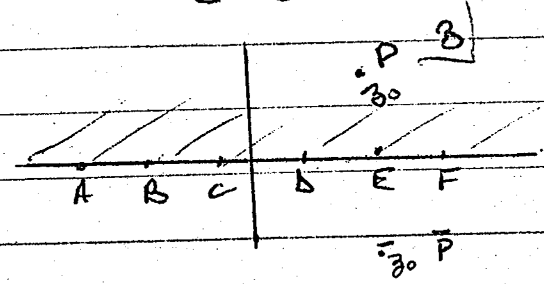
(translation, rotation, stretching, inversion)

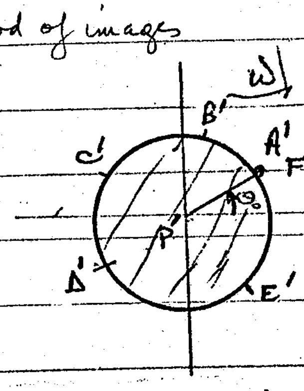
maps circles into circles, where circles includes straight lines (circle of infinite radius)

ex.

$$w = e^{i\theta_0}\left(\frac{z - z_0}{z - \bar{z}_0}\right)$$

Using method of images



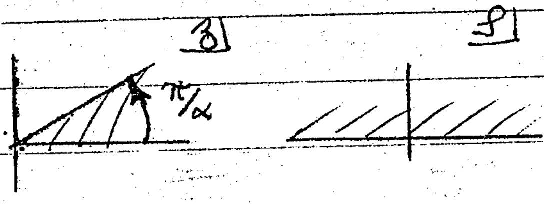


Conformal mapping of an analytic function, preserves (except at singular points)

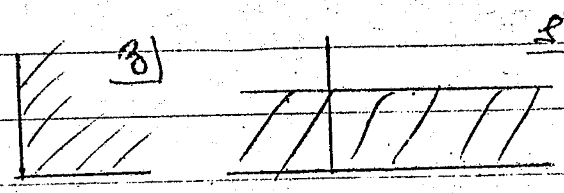
1. Angles, relative lengths
2. Laplace operator
3. Form of boundary conditions

Other useful mappings

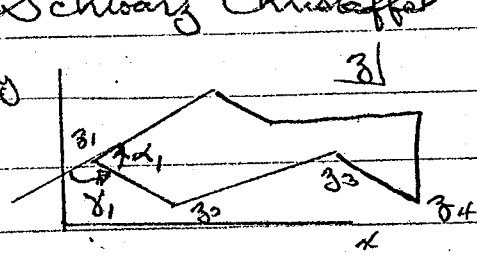
1. Power $\zeta = z^{\alpha}$

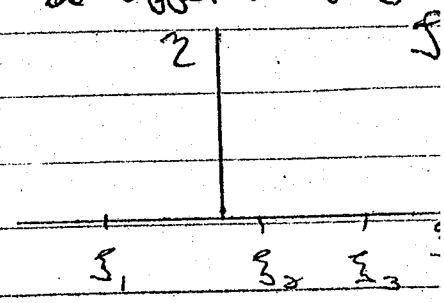


2. logarithm (or exp.) $\zeta = \ln$



3. Schwarz Christoffel Transformation (maps any polygon to the upper half-pla





locally $\quad \zeta - \xi_1 = c(z - z_1)^{\pi/\alpha_1}$

inverting $\quad z - z_1 = A(\zeta - \xi_1)^{\alpha_1/\pi}$

$$\frac{dz}{d\zeta} = a(\zeta - \xi_1)^{\alpha_1/\pi - 1} \Big\} \to 0 \quad \text{at } \zeta =$$

in general

$$\frac{dz}{d\zeta} = \bar{A} \prod_{j=1}^{N} (\zeta - \xi_j)^{\frac{\alpha_j}{\pi} - 1}$$

and

$$\boxed{z = \bar{a} \int_{\zeta_0}^{\zeta} \prod_{j=1}^{N} (\zeta - \xi_j)^{\frac{\alpha_j}{\pi} - 1} \, d\zeta + B}$$

for any polygon

$$\begin{cases} \sum_{i=1}^{N} \alpha_i = (N-2)\pi \\ \sum_{i=1}^{N} \gamma_i = 2\pi \end{cases}$$

To be able to ~~__~~ evaluate the transformation in terms of elementary functions:

1. angles must be multiples of π or $\pi/2$ and no more than 3 vertices

$\Longrightarrow$ degenerate polygons

<u>Complex Variables</u>

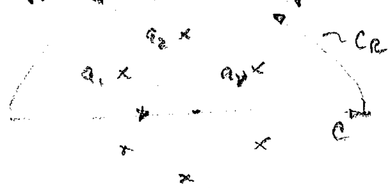
Definite integrals

A. Type I : $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$ $\quad P(x), Q(x)$ are polynomials

- if $Q(x) \neq 0$ when $x \in \mathbb{R}$: for convergence require $\deg Q > \text{degree of } P+1$
  or $\deg Q \geq \text{degree of } P+2$

if so define $f(z) = \frac{P(z)}{Q(z)}$

- altho $Q(x)$ has no real roots $Q(z)$ must have n zeroes when $z \in \mathbb{C}$
- $Q(x)$ must not be odd $\Rightarrow Q$ has a real root. Thus since $Q(x)$ is even $\Rightarrow Q(z)$ would have conjugate roots since coeffs of of powers of z are real.



$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum_\nu \text{Res} f(a_\nu)$$

$\int_{C_R} f(z)\,dz \to 0$ since $f(z) \sim \frac{1}{z^n}$ w/ $n>1$ $\quad \therefore f(z)\,dz \approx \frac{1}{z^{n-1}} \to 0$ as $z \to \infty$ since $n-1>0$

$$\therefore \int_C f(z)\,dz = 2\pi i \sum_\nu \text{Res} f(a_\nu) = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$

B. Type II : $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \begin{Bmatrix} \sin x \\ \text{or} \\ \cos x \end{Bmatrix} dx$ $\quad$ for convergence we require $\deg Q \geq \deg P+1$
<u>and</u> $Q(x) = 0$ when $\sin x, \cos x = 0$

define $f(z) = \frac{P(z)}{Q(z)} e^{iz}$ and look for real or im part

Integ part $\int_{C_R} \frac{P(z)}{Q(z)} e^{iz}\,dz = \cancel{\frac{P(z)}{Q(z)} \frac{e^{iz}}{i} \Big|_{-R}^{R}} - \int \frac{d}{dz}\left(\frac{P}{Q}\right) \frac{e^{iz}}{i}\,dz$

Cauchy PV $\Rightarrow \to 0$ $\quad \to 0$ if $\deg Q^2 \geq 2 \deg(QP' - PQ')$

Aside · if limits are a, b try to use fractional linear transformation
So that $a \to 0$ & $b \to \infty$ $\quad x_1 = \frac{x-a}{x-b}$

· if fn is symmetric then $\int_0^\infty = \frac{1}{2}\int_{-\infty}^{\infty}$

conformal maps are orthog. Analytic fns map conformally

if $f(z)$ is anal on a circle $C_n^{(z=a)}$ of rad r then $|f^{(n)}(a)| \le M\frac{n!}{r^n}$ where $M = \max_{z\in C}|f(z)|$

use $f^{(n)}(a) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}\,dz \Rightarrow |f^{(n)}(a)| \le \frac{n!}{2\pi}\,\frac{M}{r^{n+1}} \oint |dz| = \frac{n!}{2\pi} M \cdot \frac{2\pi r}{r^{n+1}}$

Liouville: if $f(z)$ is ~~entire~~ analytic $\forall z$ & bounded $\Rightarrow f(z) = \text{const}$.

$f(z)$ is analy inside on on C $f(z) \ne \text{const} \Rightarrow |f(z)|$ is max on C & a min on C.

Math 106 FINAL EXAMINATION December 15, 1978
12:15 p.m. - 3:15 p.m.

Each problem carries 20 points except Problems 2 and 9, each of which carries 30 points. Total points = 200.

1. a) Find the real and imaginary parts of $(1+i\sqrt{3})^{-10}$.

b) Derive the formulas

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{1}{2} + \frac{\sin[(n+\frac{1}{2})\theta]}{2\sin\frac{\theta}{2}}$$

$$\sin\theta + \sin 2\theta + \ldots + \sin n\theta = \frac{1}{2}\cot\frac{\theta}{2} - \frac{\cos[(n+\frac{1}{2})\theta]}{2\sin\frac{\theta}{2}}$$

from the identity $1 + z + z^2 + \ldots + z^n = \dfrac{1-z^{n+1}}{1-z}$. $= \dfrac{1-e^{(n+1)i\theta}}{1-e^{i\theta}} \cdot \dfrac{1-e^{-i\theta}}{1-e^{-i\theta}}$

let $z = e^{i\theta} = \cos\theta + i\sin\theta$ $(1-c)-is$ $-e^{-i\theta}-e^{i\theta}$ $(1-c)+is$

2. Suppose $f(z) = u(x,y) + iv(x,y)$ is a function on the open unit disc $D = \{z = x+iy;\ |z| < 1\}$ and both u and v are continuously differentiable. By using the definition of $\frac{df}{dz}$ and Green's theorem, prove that the following three statements are equivalent

a) $\frac{df}{dz}$ exists at every point of D.

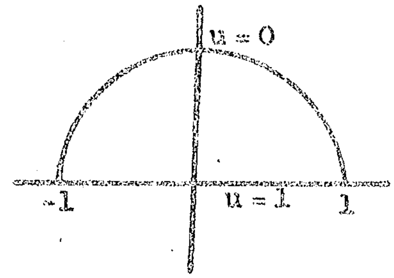
b) $$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \quad \text{on } D \end{cases}$$

c) $\int_C f(z)dz = 0$ for any piecewise smooth simple closed curve C in D.

3. Find an analytic function whose real part is $e^{-y}\cos x + e^{x}\sin y$.

$u = e^{-y}\cos x + e^{y}\sin y$

4. Find a harmonic function u on the upper half of the unit disc $\{|z| < 1,\ \operatorname{Im} z > 0\}$ which has the boundary value 1 on the diameter $\{-1 < \operatorname{Re} z < 1,\ \operatorname{Im} z = 0\}$ and the boundary value 0 on the semicircle $\{|z| = 1,\ \operatorname{Im} z > 0\}$.



5. Suppose $f(z)$ is analytic on $\{|z| \le 1\}$. Prove that

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{\overline{f(z)}\,dz}{z-a} = \begin{cases} \overline{f(0)} & \text{if } |a| < 1 \\ \overline{f(0)} - \overline{f(\frac{1}{\bar a})} & \text{if } |a| > 1 . \end{cases}$$

(Hint: Take conjugate of the integral and use $z\bar z = 1$ and $d\bar z = -\frac{dz}{z^2}$).

6. Suppose $f(z)$ is analytic on the whole plane and $|f(z)| \le 100 + |z|^{11/2}$ for all z. Prove that f is a polynomial of degree ≤ 5.

$\Longrightarrow f(z) = \sum a_i z^i$ $\qquad \int f(z)\,dz = \int \sum a_i z^i\,dz = \sum a_i \int z^i\,dz$

7. Find the Laurent series expansions of the function $f(z) = \frac{1}{z^2(1-z)}$ in the following annuli.

$\left|\int f(z)\,dz\right| < \int |f(z)$

a) $0 < |z| < 1$

b) $1 < |z| < \infty$

c) $0 < |z-1| < 1$

d) $1 < |z-1| < \infty$.

8. Find the radii of convergence of the following power series

a) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!} z^n$

b) $\sum_{n=2}^{\infty} (\log n)^{\sqrt{n} \log n} z^n$.

9. Evaluate the following integrals

a) $\int_0^{2\pi} \frac{d\theta}{1-2a \cos \theta + a^2}$ $(0 < a < 1)$

b) $\int_0^{\infty} \frac{\cos x \, dx}{x^2 + 1}$

c) $\int_0^{\infty} \frac{\log x \, dx}{(x^2 + 1)^2}$.