cubic

$y^3 + py^2 + qy + r = 0$

can be reduced to $x^3 + ax + b = 0$ by letting $y = x - p/3$

$$a = \tfrac{1}{3}(3q - p^2) \qquad b = \frac{1}{27}(2p^3 - 9pq + 27r)$$

let

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \qquad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

$$x = A+B, \quad -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}, \quad -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3}$$

if $\frac{b^2}{4} + \frac{a^3}{27}$ > 0 1 real 2 im

$= 0$ 3 real 2 eq

< 0 3 real & uneq

quartic

$x^4 + ax^3 + bx^2 + cx + d = 0$ can be rewritten

$y^3 - by^2 + (ac - 4d)y - a^2d + 4bd - c^2 = 0$

$R = \sqrt{\frac{a^2}{4} - b + y}$ where y

if $R \neq 0$

$$D = \sqrt{\frac{3a^2}{4} - R^2 - 2b + \frac{4ab - 8c - a^3}{4R}}$$

$$E = \sqrt{\frac{3a^2}{4} - R^2 - 2b - \frac{4ab - 8c - a^3}{4R}}$$

$R = 0$

$$D = \sqrt{\frac{3a^2}{4} - 2b + 2\sqrt{y^2 - 4d}}$$

$$E = \sqrt{\frac{3a^2}{4} - 2b - 2\sqrt{y^2 - 4d}}$$

then 4 roots

$$x = -\frac{a}{4} + \frac{R}{2} \pm \frac{D}{2}$$

$$x = -\frac{a}{4} - \frac{R}{2} \pm \frac{E}{2}$$

$$\frac{dv}{dt} + \frac{k}{m}v + g = 0$$

$$\frac{dv}{dt} + \frac{k}{m}v = -g \qquad v = \frac{mg}{k} \qquad v = -\frac{mg}{k}$$

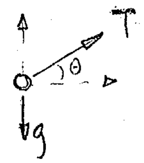
$$(x + \tfrac{k}{m})v = 0 \qquad e^{-kt/m} = v$$

$$-\frac{k}{m}e^{-kt/m}$$

(1)

$\ddot{x}$

$$m = m_0 - \dot{m}t = m_0 + \beta t$$



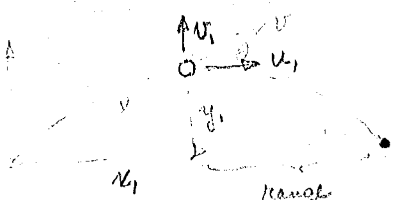
$$\dot{u} = \bar{a}\cos\theta$$
$$\dot{v} = \bar{a}\sin\theta - g$$
$$\dot{x} = u$$
$$\dot{y} = v$$

$$\bar{a} = \frac{T}{m} = \frac{c\beta}{m}$$

also $x = y = u = v = 0 \quad t_0 = 0 \quad t_1 = T$

$$\bar{a} = -\frac{c\,dM}{M\,dt} = +\frac{c}{M}\beta \qquad \beta \text{ is already known}$$

no drag uniform field



$$u_1 = V_1\cos\theta_1$$
$$v_1 = V_1\sin\theta_1 - gt$$

$$\tan\theta_1 = \frac{v_1}{u_1} = \frac{v_1}{u_1}$$

$$\sin\theta_1 = \frac{v_1}{\sqrt{u_1^2 + v_1^2}} = \frac{v_1}{V}$$

$$x_R = (V\cos\theta_1)t_1$$
$$y_R = (V\sin\theta_1)t_1 - \tfrac{1}{2}gt_1^2 = y_1$$
$$\tfrac{1}{2}g\,t_1^2 - V\sin\theta_1 t_1 - y_1 = 0$$
$$t_1 = \frac{V\sin\theta_1 \pm \sqrt{V^2\sin^2\theta_1 + 2gy_1}}{g} = \qquad t = \frac{v_1}{g} + \frac{\sqrt{v_1^2 + 2gy_1}}{g}$$

$$x_r = V^2\sin\theta_1\cos\theta_1 \pm \sqrt{}$$

$$x_r = \frac{v_1 u_1}{g} + \frac{u_1}{g}\sqrt{v_1^2 + 2gy_1} = \frac{u_1}{g}\left[v_1 + \sqrt{v_1^2 + 2gy_1}\right]$$

$$R = x_1 + \frac{u_1}{g}\left[v_1 + \sqrt{v_1^2 + 2gy_1}\right]$$

$$dR = dx_1 + \frac{du_1}{g}\left[v_1 + \sqrt{}\right] + \frac{u_1}{g}\left[dv_1 + \frac{v_1 dv_1 + g\,dy_1}{r}\right]$$

u, v, x, y, θ

$$F = \lambda_u[\dot{u} - a\cos\theta] + \lambda_v[\dot{v} - a\sin\theta + g] + \lambda_x[\dot{x} - u] + \lambda_y[\dot{y} - v]$$

$$-\lambda_x - \dot{\lambda}_u = 0 \;\; u \qquad -\lambda_y - \dot{\lambda}_v = 0 \;\; v \qquad -\dot{\lambda}_x = 0 \;\; x \qquad -\dot{\lambda}_y = 0 \;\; y$$

$$+ a(\lambda_u\sin\theta - \lambda_v\cos\theta) = 0 \qquad \theta$$

$$\lambda_x = c_1 \qquad \lambda_y = c_2 \qquad \lambda_u = -c_1 t + c_{11} \qquad \lambda_v = -c_2 t + c_{22}$$

~~from transversality~~ FI $\quad \lambda_u\dot{u} + \lambda_v\dot{v} + \lambda_x\dot{x} + \lambda_y\dot{y} = C$

Transversality - dx $\quad -C\,dt + \lambda_u\,du + \lambda_v\,dv + \lambda_x\,dx + \lambda_y\,dy\Big|_i^f = 0$

$$C = 0 \qquad \lambda_{u_f} = \qquad \lambda_{v_f} = 0 \qquad \lambda_{x_f} = -1 \qquad \lambda_{y_f} = 0$$

$$\cancel{\lambda_u = -c_1(t - T)} \qquad \lambda_v = -c_2(t - T)$$

$$\lambda_x = -1 \qquad \lambda_y =$$

transv $\quad -C\,dt + \left[\lambda u + \frac{v_1+r}{g}\right] du + \left[\lambda v + \frac{u_1(r+v_1)}{gr}\right] dv + \left[\lambda y + \frac{u_1}{r}\right] dy + [\lambda x + 1]\, dx \Big|_1^f = 0$

$C = 0 \qquad \lambda u_f = -\frac{v_1+r}{g} \qquad \lambda v_f = -u_1\frac{(r+v_1)}{gr} \qquad \lambda y_f = -\frac{u_1}{r} \qquad \lambda x_f = -1$

$r = \sqrt{v_1^2 + 2gy_1}$

$C_1 = \lambda x = -1$ everywhere $\quad C_2 = \lambda y = -\frac{u_1}{r}$ everywhere

$\tan\theta = \frac{\lambda v}{\lambda u} = \frac{u_1}{r}$, constant angle to horiz

$\lambda u_f = -C_1(T) + C_{11}$

$-\frac{(v_1+r)}{g} = +T + C_{11} \qquad C_{11} = -\left[T + \frac{(v_1+r)}{g}\right]$

$\lambda v_f = -C_2 T + C_{22} \qquad -\frac{u_1(r+v_1)}{gr} = +\frac{u_1}{r}T + C_{22} \qquad C_{22} = -\frac{u_1}{r}\left[T + \frac{(r+v_1)}{g}\right]$

$\therefore \quad \lambda u = (t-T) - \left(\frac{v_1+r}{g}\right) \qquad \lambda v = \frac{u_1}{r}\left[(t-T) - \left(\frac{r+v_1}{g}\right)\right]$

$\lambda x = -1 \qquad \lambda y = -\frac{u_1}{r}$

$\cos\theta = \frac{1}{\sqrt{1 + u_1^2/r^2}} \qquad \sin\theta = \frac{u_1/r}{\sqrt{1+u_1^2/r^2}}$

total flight time $T + \left(\frac{v_1+r}{g}\right)$

Weierstrass Condition $\quad \lambda u\,\mathring{u} + \lambda v\,\mathring{v} + \lambda x\,\mathring{x} + \lambda y\,\mathring{y} \le \lambda u\,\mathring{U} + \lambda v\,\mathring{V} + \lambda x\,\mathring{X} + \lambda y\,\mathring{Y}$

where $\mathring{U} = \bar{a}\cos\theta^*$

$\mathring{V} = \bar{a}\sin\theta^* - g$

$\mathring{X} = u = \mathring{x} \qquad \therefore \quad \lambda u\,\mathring{u} + \lambda v\,\mathring{v} \le \lambda u\,\mathring{U} + \lambda v\,\mathring{V}$

$\mathring{Y} = v = \mathring{y} \qquad \bar{a}(\lambda u\cos\theta + \lambda v\sin\theta) \le \bar{a}(\lambda u\cos\theta^* + \lambda u\sin\theta^*)$

$-\frac{\bar{a}}{g}(v_1+r)\left[\cos\theta + \frac{u_1}{r}\sin\theta\right] \le -\frac{\bar{a}(v_1+r)}{g}\left[\cos\theta^* + \frac{u_1}{r}\sin\theta^*\right]$

$\cos\theta + \frac{u_1}{r}\sin\theta \ge \cos\theta^* + \frac{u_1}{r}\sin\theta^*$ $\quad$ satisfied $\forall\,\theta^*$ if θ makes LHS max $\quad u_1 > 0$

$\therefore\ a(t)$ is irrelevant in the weierstrass condition, θ is acute angle

from FI $\quad \lambda u\,\mathring{u} + \lambda v\,\mathring{v} + \lambda x\,\mathring{x} + \lambda y\,\mathring{y} = C$ $\quad$ from corner cond $\quad \lambda u_+ = \lambda u_-$

θ is not discont since it is a solution to $\tan\theta = \frac{u_1}{r}$

$\lambda v_+ = \lambda v_-$

$\lambda x_+ = \lambda x_-$

$\lambda y_+ = \lambda y_-$

$C_+ = C_- = 0$

$T \cdot \dot m$

$\frac{c\beta}{m}\sin\theta - gt$

$c\sin\theta$

$\dot m = -\beta$

if $\dot m = \text{const}$ $\quad \int du = -c\dot m \int \frac{\cos\theta}{m}\,dt = -c\cos\theta \int_0^t \frac{\dot m\,dt}{m_1 + \dot m t} = c\cos\theta \ln(m_1 + \dot m t)\Big|_0^t$

$m = m_1 + \dot m t$

$$u = -c\cos\theta \ln(m_1 + \dot m t) + c\cos\theta \ln(m_1)$$

$\longrightarrow$ $$u = c\cos\theta \ln\left(\frac{m_1}{m_1 + \dot m t}\right) = c\cos\theta \ln\left(\frac{m_1}{m_1 - \beta t}\right)$$

$\longrightarrow$ $$v = \int -\frac{c\dot m}{m}\sin\theta\,dt - gt = +c\sin\theta \ln\left(\frac{m_0}{m_0 - \beta t}\right) - gt$$

$\longrightarrow$ $$x = \int u\,dt = \int c\cos\theta \ln\left(\frac{m_1}{m_1 - \beta t}\right)dt = [c\cos\theta \ln(m_1)]t - c\cos\theta\left[t\ln(m_1 - \beta t) - t - \frac{m_1}{\beta}\ln\left(\frac{m_1 - \beta t}{m_1}\right)\right]$$

$\longrightarrow$ $$y = \int_0^t v\,dt = \int c\sin\theta \ln\left(\frac{m_1}{m_1 - \beta t}\right)dt - g\frac{t^2}{2}$$

$$y = c\sin\theta \ln m_1 t - c\sin\theta\left[t\ln(m_1 - \beta t) - t - \frac{m_1}{\beta}\ln\left(\frac{m_1 - \beta t}{m_1}\right)\right] - g\frac{t^2}{2}$$

$$v_1 = c\ln\left(\frac{m_0}{m_0 - \beta T}\right)\sin\theta - gT, \quad u_1 = c\cos\theta \ln\left(\frac{m_1}{m_1 - \beta T}\right) \qquad \frac{m_1}{m_1 - \beta T} = B$$

$$y_1 = c\sin\theta\left[T\left\{\ln\left(\frac{m_1}{m_1 - \beta T}\right) + 1\right\} - \frac{m_1}{\beta}\ln\left(\frac{m_1}{m_1 - \beta T}\right)\right] - g\frac{T^2}{2}$$

Change

$$x_1 = c\cos\theta\left[T\left\{\ln\left(\frac{m_1}{m_1 - \beta T}\right) + 1\right\} - m_1 \ln\left(\frac{m_1}{m_1 - \beta T}\right)\right]$$

$$r = \sqrt{v_1^2 + 2gy_1} = \sqrt{c^2 \underbrace{\ln^2[B]}_{v_1^2} \sin^2\theta + g^2T^2 - 2gcT\ln B^{\sin\theta} + 2gc\sin\theta\left[T\{\ln B + 1\} - \frac{m_1}{\beta}\ln B\right]}$$

$$u_1 = c\cos\theta \ln\left(\frac{m_0}{m_0 - \beta T}\right) = c\ln B\cos\theta$$

$A = (c\ln B)$

$D = 2gcT$

$E = -\frac{2gm_1}{\beta}$ Change

$B = \frac{m_0}{m_1}$

$$\tan\theta = \frac{\cos\theta}{\sqrt{\sin^2\theta + \left(\frac{D + EA}{A^2}\right)\sin\theta}}$$

where θ is acute

$$c^2 \ln^2 B \sin^2\theta + \cancel{g^2T^2} - 2gTc \ln B \sin\theta + 2gc\sin\theta\left[T\left\{\ln B + 1\right\} + \frac{m_1}{\beta}\right]$$
$$- \cancel{g^2T^2}$$

$$c^2 \ln^2 B \sin^2\theta \cancel{- 2gTc\ln B \sin\theta} \cancel{+ 2gcT\sin\theta \ln B}$$
$$+ 2gcT\sin\theta + 2g\frac{cm_1}{\beta}\ln B \sin\theta$$

$$\tan\theta = \frac{c \ln B \cos\theta}{\sqrt{c^2 \ln^2 B \sin^2\theta + 2gcT\sin\theta + 2g\frac{cm_1}{\beta}\ln B \sin\theta}}$$

$$= \frac{\cos\theta}{\sqrt{\sin^2\theta + \frac{2gcT}{c^2\ln^2 B}\sin\theta + \frac{2gm_1}{\beta}\left[\frac{1}{c\ln B}\right]\sin\theta}}$$

$A = c\ln B$
$D = 2gcT$
$E = \frac{2gm_1}{\beta}$

$$= \frac{\cos\theta}{\sqrt{\sin^2\theta + \left(\frac{D + EA}{A^2}\right)\sin\theta}}$$

$$\frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta}{\left(\sin^2\theta + \left(\frac{D+EA}{A^2}\right)\sin\theta\right)}$$

$$\cancel{\sin^4\theta} + \sin^3\theta\left[\frac{D+EA}{A^2}\right] = \left(1 - 2\sin^2\theta \cancel{+ \sin^4\theta}\right)$$

$$\left[\frac{D+EA}{A^2}\right]\sin^3\theta + 2\sin^2\theta - 1 = 0$$

$$\sin^3\theta + \frac{2A^2}{D+EA}\sin^2\theta - \frac{A^2}{D+EA} = 0$$

$q = 0 \qquad r = -\frac{A^2}{D+EA} \qquad p = \frac{2A^2}{D+EA}$

$a = -\frac{1}{3}\left(\frac{4A^4}{D^2 + 2DEA + E^2A^2}\right) \qquad b = \frac{1}{27}$

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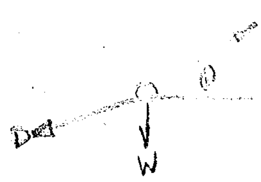
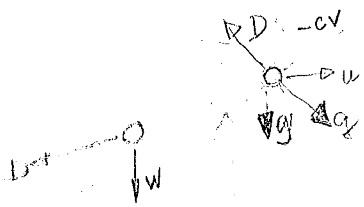


$D = kq$

$\dot{u} = a\cos\theta - \frac{kq}{m}\cos\theta_D \qquad \cos\theta_D = \frac{u}{q}$

$\dot{v} = a\sin\theta - \frac{kq}{m}\sin\theta_D - g \qquad \sin\theta_D = \frac{v}{q}$



$a_x = -\frac{D}{m_T}\cos\theta_{D_1} = -\frac{ku}{m_T} \qquad -\frac{cv}{m_T} - g$

$a_y = -\frac{D}{m_T}\sin\theta_{D_1} - g = -\frac{kv}{m_T} - g$

$\frac{du}{dt} = -\frac{ku}{m_T} \qquad \frac{du}{u} = -\frac{k}{m_T}\,dt \qquad \ln u = -\frac{kt}{m_1} \quad u = u_1 e^{-\frac{kt}{m_T}}$ ✓

$u_1\left(-\frac{k}{m_T}\right)e^{-\frac{kt}{m_T}} = -\frac{k}{m_T}u_1 e^{-\frac{kt}{m_T}}$

$\frac{dv}{dt} = -\frac{kv}{m_T} - g \qquad \frac{dv}{dt} = \frac{-kv - m_T g}{m_T} \qquad dt = \frac{m_T\,dv}{-kv - m_T g}$

$t = m_T \int \frac{dv}{-kv - m_T g} = -\frac{m_1}{k}\ln\left[kv + m_T g\right] + C$

$t = 0 \quad v = v_1 \qquad \frac{m_1}{k}\ln\left[kv_1 + m_T g\right] = C$

$t = +\frac{m_T}{k}\left[\ln\left(\frac{kv_1 + m_T g}{kv + m_T g}\right)\right] \qquad e^{\frac{kt}{m_1}} = \frac{kv_1 + m_T g}{kv + m_T g}$

$e^{-\frac{kt}{m_1}}\left(v_1 + \frac{m_T g}{k}\right) = v + \frac{m_T g}{k} \qquad e^{-n_1 t} = \frac{kv + m_T g}{kv_1 + m_T g}$

$\frac{(kv_1 + m_T g)}{k}e^{-n_1 t} - \frac{m_T g}{k} = v$

$-\frac{m_T g}{k} + \left(v_1 + \frac{m_T g}{k}\right)e^{-\frac{kt}{m_T}} = v$ ✓

$x_r = \int u_1 e^{-\frac{kt}{m_T}}\,dt = u_1\left(-\frac{m_T}{k}\right)e^{-\frac{kt}{m_T}} + C \qquad x_r = u_1\frac{m_1}{k}$

$x = \frac{u_1}{n_T}\left[1 - e^{-\frac{kt}{m_T}}\right]$? at $t=0$

$y = -\frac{m_T g}{k}t - \frac{m_1}{k}\left(v_1 + \frac{m_1 g}{k}\right)e^{-\frac{kt}{m_1}} + \cancel{\frac{m_1}{k}\left(v_1 + \frac{m_1 g}{k}\right)} \qquad \text{let } \frac{k}{m_1} = n_1$

$y_1 = -\frac{m_1}{k}\left(v_1 + \frac{m_1 g}{k}\right) + C \qquad \frac{k}{m} = n$

let $t_1 = f(y_1, v_1, u_1)$

$$dR = dx_1 + \frac{du_1}{k} m_1 \left[1 - e^{-\frac{kt}{m_1}}\right] + \frac{u_1 m_1}{k}\left[\left(+e^{-\frac{kt}{m_1}}\right)\left(+\frac{k}{m_1}\,dt_1\right)\right]$$

$$0 = y_1 - \frac{m_1 g}{k} t_1 + \frac{m_1}{k}\left(v_1 + \frac{m_1 g}{k}\right)\left[1 - e^{-kt/m_1}\right]$$

$$0 = dy_1 - \frac{m_1 g}{k}\,dt_1 + \cancel{\frac{m_1}{k}\left(dv_1 + \frac{m_1 g}{k}\right)\left[1 - e^{-kt/m_1}\right]}$$

$$\left(\frac{m_1 v_1}{k}\right) + \left(\frac{m_1^2 g}{k^2}\right)\left[\quad\right]$$

$$\frac{m_1}{k}\left[\quad\right]dv_1 + \frac{m_1 v_1}{k}\left[d \quad\right] + \frac{m_1^2 g}{k^2}\left[d \quad\right]$$

$$0 = dy_1 - \frac{m_1 g}{k}\,dt_1 + \frac{m_1}{k}\left[1 - e^{-kt/m_1}\right]dv_1 + \frac{m_1}{k}\left[v_1 + \frac{m_1 g}{k}\right]\left[+e^{-\frac{kt_1}{m_1}}\left(+\frac{k}{m_1}dt_1\right)\right.$$

$$\frac{m_1 g}{k}\,dt_1 - \left[v_1 + \frac{m_1}{k} g\right]e^{-kt_1/m_1}\,dt_1 = dy_1 + \frac{m_1}{k}\left[1 - e^{-kt/m_1}\right]dv_1$$

$$dt_1 = \frac{dy_1 + \frac{m_1}{k}\left[1 - e^{-\frac{kt_1}{m_1}}\right]dv_1}{\frac{m_1 g}{k} - \left[v_1 + \frac{m_1}{k} g\right]e^{-k/m_1 t_1}}$$

$$dR = dx_1 + \frac{du_1}{k} m_1\left[1 - e^{-\frac{kt_1}{m_1}}\right] + u_1 e^{-\frac{kt_1}{m_1}}\left[\frac{dy_1 + \frac{m_1}{k}\left[1 - e^{-\frac{kt_1}{m_1}}\right]dv_1}{\frac{m_1 g}{k} - \left[v_1 + \frac{m_1 g}{k}\right]e^{-\frac{kt_1}{m_1}}}\right]$$

$$\lambda_x = -1 \qquad \lambda_{u_f} = -\frac{m_1}{k}\left[1 - e^{-\frac{kt_1}{m_1}}\right]$$

$$\lambda_{y_f} = \frac{u_1 e^{-\frac{kt_1}{m_1}}}{\left[v_1 + \frac{m_1 g}{k}\right]e^{-\frac{kt_1}{m_1}} - \frac{m_1 g}{k}} \qquad \lambda_{v_f} = \frac{\frac{m_1 u_1}{k_1} e^{-\frac{kt_1}{m_1}}\left[1 - e^{-\frac{kt_1}{m_1}}\right]}{\left[v_1 + \frac{m_1 g}{k}\right]e^{-\frac{kt_1}{m_1}} - \frac{m_1 g}{k}}$$

$$\lambda_u = -\frac{m}{K+\beta} + C_{11}\, m^{-K/\beta}$$

$$\lambda_v = \frac{K u_1 e^{-\frac{kt_1}{m_1}}}{\left[kv_1 + m_1 g\right] e^{-\frac{kt_1}{m_1}} - m_1 g} \left(\frac{m}{K+\beta}\right) + C_{22}\, m^{-K/\beta}$$

$$\lambda_{u_f} = -\frac{m_1}{K+\beta} + C_{11}\, m_1^{-K/\beta} = -\frac{m_1}{K}\left[1 - e^{-\frac{kt_1}{m_1}}\right]$$

$$-\frac{m}{K+\beta} + \frac{C_{11}}{m^{K/\beta}} = \left\{\frac{m_1}{K+\beta} - \frac{m_1}{K}\left[1 - e^{-\frac{kt_1}{m_1}}\right]\right\}\left(\frac{m_1}{m}\right)^{K/\beta} - \frac{m}{K+\beta}$$

$$\lambda_u = \left\{\frac{K m_1}{(K+\beta)K} - \frac{(K+\beta)\, m_1}{(K+\beta)K}\left[1 - e^{-\frac{kt_1}{m_1}}\right]\right\}\left(\frac{m_1}{m}\right)^{K/\beta} - \frac{mK}{(K+\beta)(K)}$$

$$\lambda_{v_f} = \frac{k u_1 e^{-\frac{kt_1}{m_1}}}{\left[kv_1 + m_1 g\right] e^{-\frac{kt_1}{m_1}} - m_1 g}\left(\frac{m_1}{K+\beta}\right) + C_{22}\, m_1^{-K/\beta} = \frac{m_1 u_1 e^{-\frac{kt_1}{m_1}}\left[1 - e^{-\frac{kt_1}{m_1}}\right]}{\left[kv_1 + m_1 g\right] e^{-\frac{kt_1}{m_1}} - m_1 g}$$

$$\frac{(K+\beta)\, m_1 u_1 e^{-\frac{kt_1}{m_1}}\left[1 - e^{-\frac{kt_1}{m_1}}\right] - k m_1 u_1 e^{-\frac{kt_1}{m_1}}}{\left\{\left[kv_1 + m_1 g\right] e^{-\frac{kt_1}{m_1}} - m_1 g\right\} K+\beta}\left(\frac{M_1}{m}\right)^{K/\beta} = \frac{C_{22}}{m^{K/\beta}}$$

$K\beta m_1 u_1 e^{-\frac{kt_1}{m_1}} + (K+\beta) k m_1 u_1 e^{-\frac{2kt_1}{m_1}}$ $\qquad k^2 m_1 u_1 e^{-\frac{kt_1}{m_1}}\left[1 - e^{-\frac{kt_1}{m_1}}\right]$

$$\frac{-K^2 m_1 u_1 e^{-\frac{2kt_1}{m_1}} + K\beta m_1 u_1 e^{-\frac{kt_1}{m_1}}\left[1 - e^{-\frac{kt_1}{m_1}}\right]}{\left\{\left[kv_1 + m_1 g\right] e^{-\frac{kt_1}{m_1}} - m_1 g\right\}(K+\beta)K}\left(\frac{m_1}{m}\right)^{K/\beta} + \frac{k^2 m u_1 e^{-\frac{kt_1}{m_1}}}{\left\{\left[kv_1 + m_1 g\right] e^{-\frac{kt_1}{m_1}} - m_1 g\right\}(K+\beta)}$$

$$\lambda_u = \cancel{\frac{Km\left\{\left[kv_1 + m_1 g\right] e^{-\frac{kt_1}{m_1}} - m_1 g\right\} - m_1(K+\beta)\left[1 - e^{-\frac{kt_1}{m_1}}\right]}{\left\{\left[kv_1 + m_1 g\right] e^{-\frac{kt_1}{m_1}} - m_1 g\right\}(K+\beta)K}} \qquad -\frac{K u_1 e^{-\frac{kt_1}{m_1}}}{\left[\quad\right]}\left[-\beta m_1 + (K+\beta) m_1 e^{-\frac{kt_1}{m_1}} + Km\right]$$

$$\lambda_u = \frac{-\beta m_1 + (K+\beta)\, m_1 e^{-\frac{kt_1}{m_1}}}{(K+\beta)K}\left(\frac{m_1}{m}\right)^{K/\beta} - \frac{mK}{(K+\beta)K}$$

$$u(x) = e^{K/\beta \ln m}\left(+\frac{\beta c}{K}\cos\theta \quad m^{-K/\beta} + \mathcal{S}\right)$$

$$\frac{e^{-K/\beta \ln m_1}}{m_1^{-K/\beta}}$$

$$0 = e^{K/\beta \ln m_1}\left(\frac{\beta c}{K}\cos\theta \quad m_1^{-K/\beta} + \mathcal{S}\right)$$

$$\mathcal{S} = -\frac{\beta c}{K}\cos\theta \quad m_1^{-K/\beta}$$

$$u(x) = e^{K/\beta \ln m}\left(\frac{\beta c}{K}\cos\theta\left[m^{-K/\beta} - m_1^{-K/\beta}\right]\right)$$

$$u = e^{K/\beta \ln m}\left\{\frac{\beta c}{K}\cos\theta\left[m^{-K/\beta} - m_1^{-K/\beta}\right]\right\}$$

$$u = \underbrace{m_1^{-K/\beta}}_{e^{-K/\beta \ln m}}\ \underbrace{e^{K/\beta \ln m}}_{e^{K/\beta \ln m}}\left\{\frac{\beta c}{K}\cos\theta\left[\left(\frac{m}{m_1}\right)^{-K/\beta} - 1\right]\right\}$$

$$\boxed{u = e^{K/\beta \ln(m/m_1)}\left\{\frac{\beta c}{K}\cos\theta\left[\left(\frac{m}{m_1}\right)^{-K/\beta} - 1\right]\right\}}$$

m_0 - initial mass

$$\dot{v} + \frac{kv}{m} = a\sin\theta - g$$

$$\mu(x) = e^{+k\int\frac{dt}{m}} = e^{-K/\beta \ln m}$$

$$v = e^{K/\beta \ln m}\left[\int(a\sin\theta - g)e^{-K/\beta \ln(m_1 - \beta t)}\,dt + C\right]$$

$$u = \left(\frac{m}{m_1}\right)^{K/\beta}\left\{\frac{\beta c}{K}\cos\theta\left[\left(\frac{m}{m_1}\right)^{-K/\beta} - 1\right]\right\}$$

$$\boxed{u = \frac{\beta c}{K}\cos\theta\left[1 - \left(\frac{m}{m_0}\right)^{K/\beta}\right]}$$

$dm = \beta dt$

$$v = e^{K/\beta \ln m}\left[c\beta\sin\theta\int\frac{e^{-K/\beta \ln m}}{m}\,dt - g\int e^{-K/\beta \ln(m_1 - \beta t)}\,dt + C\right]$$

$$\left[-c\sin\theta\int \cancel{\beta}\, m^{-(K/\beta+1)}\,dm + g/_{\beta}\int m^{-(K/\beta)}\,dm\right.$$

$$m^{K/\beta} \quad -c\sin\theta\left[\frac{m^{-K/\beta}}{-K/\beta}\right] + g/_{\beta}\ \frac{m^{-(K/\beta-1)}}{-(K/\beta-1)}$$

$$v = \frac{c\beta}{K}\sin\theta\Big[$$

$$v = e^{K/\beta \ln m}\left[\frac{c\beta}{K}\sin\theta\left(m^{-K/\beta}\right) - \frac{g}{K-\beta}(m)^{-(K/\beta-1)} + C\right]$$

$$0 = e^{K/\beta \ln m_1}\left[\frac{c\beta}{K}\sin\theta\left(m_1^{-K/\beta}\right) - \frac{g}{K-\beta}(m_1)^{-(K/\beta-1)} + C\right]$$

$m_1^{K/\beta}$

$$C = \frac{c\beta}{K}\sin\theta\left(m_1^{-K/\beta}\right) + g/_{K-\beta}\, m_1^{-(\ )}$$

$$v = e^{K/\beta \ln m}\left[\frac{c\beta}{K}\sin\theta\left[m^{-K/\beta} - m_1^{-K/\beta}\right] - g/_{K-\beta}\left(m(m)^{-K/\beta} - m_1(m_1)^{-K/\beta}\right)\right]$$

$$v = e^{(\ )} m_1^{-K/\beta}\left(\frac{c\beta}{K}\sin\theta\left[\left(\frac{m}{m_1}\right)^{-K/\beta} - 1\right] - g/_{K-\beta}\left[m\left(\frac{m}{m_1}\right)^{-K/\beta} - m_1\right]\right.$$

$$\left(\frac{m}{m_1}\right)^{+K/\beta}$$

$$\left[\frac{c\beta}{K}\sin\theta\left(1 - \left(\frac{m}{m_1}\right)^{K/\beta}\right) - g/_{K-\beta}\left[m - m_1\left(\frac{m}{m_1}\right)^{K/\beta}\right]\right]$$

m_0 - initial mass

$$\boxed{v = \left\{\frac{c\beta}{K}\sin\theta\left[1 - \left(\frac{m}{m_0}\right)^{K/\beta}\right] - g/_{K-\beta}\left[m - m_0\left(\frac{m}{m_0}\right)^{K/\beta}\right]\right\}}$$

$$x = \int u\,dt = -\frac{1}{\beta}\int u\,dm \qquad \frac{dm}{(m_0)} = -\beta\frac{dt}{m_0}$$

$$= -\frac{1}{\beta}\int\left[\frac{\beta c}{K}\cos\theta\left(1 - \frac{m^{K/\beta}}{m_1^{K/\beta}}\right)\right]dm$$

$$x = -\frac{c\cos\theta}{K}\int\left[dm - \frac{m^{K/\beta}\,dm}{m_1^{K/\beta}}\right] \qquad \frac{m^{K/\beta+1}}{K/\beta+1}$$

$$x = -\frac{c\cos\theta}{K}\left[m - \frac{\beta}{(K+\beta)m_1^{K/\beta}}m^{(K/\beta+1)} + C\right]$$

$$0 = m_1 - \frac{\frac{1}{K/\beta+1}\, m_1^{(K/\beta)}\, m_1}{m_1^{K/\beta}} + C$$

$$0 = m_1\left[1 - \frac{\beta}{K+\beta}\right] + C$$

$$C = -m_1\left[\frac{K}{K+\beta}\right]$$

$$m_1\left[\frac{K}{K+\beta}\right] + C = 0$$

$$x = -\frac{c\cos\theta\, m_1}{K}\left[\frac{m}{m_1} - \left(\frac{m}{m_1}\right)^{K/\beta}\frac{\beta}{K+\beta}\frac{m}{m_1} - \left[\frac{K}{K+\beta}\right]\right]$$

$$-\frac{c\cos\theta\, m_1}{K}\left[\frac{m}{m_1} - \left(\frac{m}{m_1}\right)^{K/\beta+1}\frac{\beta}{K+\beta} - \frac{K}{K+\beta}\right]$$

$$\boxed{x = \frac{c\cos\theta\, m_0}{K}\left[\frac{K}{K+\beta} + \left(\frac{m}{m_0}\right)^{\frac{K+\beta}{\beta}}\frac{\beta}{(K+\beta)} - \frac{m}{m_0}\right]}$$

$$y = \int v\,dt = -\frac{1}{\beta}\int v\,dm$$

$$= -\frac{c}{K}\sin\theta\int\left[1-\left(\frac{m}{m_1}\right)^{K/\beta}\right]dm + \frac{g}{\beta(K-\beta)}\int\left[m - m\left(\frac{m}{m_1}\right)^{K/\beta}\right]dm$$

$$y = -\frac{c}{K}\sin\theta\left[m - \frac{\beta\, m^{\frac{K+\beta}{\beta}}}{(K+\beta)m_1^{K/\beta}}\right] + \frac{g}{\beta(K-\beta)}\left[\frac{m^2}{2} - \frac{m\beta\, m^{\frac{K+\beta}{\beta}}}{(K+\beta)m_1^{K/\beta}}\right] + C$$

$$0 = -\frac{c}{K}\sin\theta\left[m_1 - \frac{m_1\beta}{K+\beta}\right] + \frac{g}{\beta(K-\beta)}\left[\frac{m_1^2}{2} - \frac{m_1\beta}{K+\beta}m_1\right] + C$$

$$C = +\frac{c}{K}\sin\theta\left[m_1 - \frac{m_1\beta}{K+\beta}\right] - \frac{g}{\beta(K-\beta)}\left[\frac{m_1^2}{2} - \frac{\beta}{K+\beta}m_1^2\right]$$

$$Y = -\frac{c}{K}\sin\theta\left[m - \frac{\beta}{K+\beta}\left(\frac{m}{m_1}\right)^{K/\beta}m - m_1 + \frac{m_1\beta}{K+\beta}\right] + \frac{g}{\beta(K-\beta)}\left[\frac{m^2}{2} - \frac{\beta m_1}{K+\beta}\left(\frac{m}{m_1}\right)^{K/\beta}m - \frac{m_1^2}{2} + \frac{\beta}{K+\beta}m_1^2\right]$$

$$Y = \frac{c}{K}\sin\theta\left[\left(\frac{m}{m_1}\right)^{K/\beta}\frac{m\beta}{K+\beta} - \frac{m_1\beta}{K+\beta} + m_1 - m\right] + \frac{g}{\beta(K-\beta)}\left[\frac{m^2}{2} - \frac{\beta m_1}{K+\beta}\left(\frac{m}{m_1}\right)^{K/\beta}m - \frac{m_1^2}{2} + \frac{\beta}{K+\beta}m_1^2\right]$$

$$Y = \frac{c}{K}m_1\sin\theta\left[\left(\frac{m}{m_1}\right)^{\frac{K+\beta}{\beta}}\left(\frac{\beta}{K+\beta}\right) - \frac{\beta}{K+\beta} + 1 - \frac{m}{m_1}\right]$$

$$\frac{c}{K}m_1\sin\theta\left[\left(\frac{m}{m_1}\right)^{\frac{K+\beta}{\beta}}\left(\frac{\beta}{K+\beta}\right) + \frac{K}{K+\beta} - \frac{m}{m_1}\right] + \frac{g m_1^2}{\beta(K+\beta)}\left[\frac{1}{2}\left(\frac{m}{m_1}\right)^2 - \left(\frac{\beta}{K+\beta}\right)\left(\frac{m}{m_1}\right)^{\frac{K+\beta}{\beta}} - \frac{1}{2}\right.$$

$$\boxed{Y = \frac{c}{K}m_0\sin\theta\left[\left(\frac{m}{m_0}\right)^{\frac{K+\beta}{\beta}}\frac{\beta}{K+\beta} + \frac{K}{K+\beta} - \frac{m}{m_0}\right] + \frac{g\, m_0^2}{\beta(K-\beta)}\left[\frac{1}{2}\left(\frac{m}{m_0}\right)^2 - \frac{\beta}{K+\beta}\left(\frac{m}{m_0}\right)^{\frac{K+\beta}{\beta}} - \frac{K-\beta}{2(\beta+K)}\right]}$$

$$-\frac{1}{2} + \frac{2\beta}{2(K+\beta)} \qquad \frac{2K+2\beta-2\beta-K+\beta}{2(\beta+K)} \qquad \frac{2(K+\beta)}{2(K+\beta)} - \frac{2\beta}{2(K+\beta)} - \frac{(K-\beta)}{2(\beta+K)}$$

$$-K-\beta+2\beta \qquad -K+\beta \qquad -(K-\beta) \qquad K+\beta \qquad \frac{1}{2} - \frac{\beta}{K+\beta} - \frac{K-\beta}{2(\beta+K)}$$

$$v_1^2 = \left(\frac{c\beta}{K}\right)^2 \sin^2\theta \left[1 - \left(\frac{m_1}{m_0}\right)^{K/\beta}\right]^2 + \left[\frac{g^2 m_0^2}{(K-\beta)^2}\left[\left(\frac{m_1}{m_0}\right) - \left(\frac{m_1}{m_0}\right)^{K/\beta}\right]^2\right]$$

$$- \frac{2c\beta g \sin\theta m_0}{K(K-\beta)}\left[1 - \left(\frac{m_1}{m_0}\right)^{K/\beta}\right]\left[\left(\frac{m_1}{m_0}\right) - \left(\frac{m_1}{m_0}\right)^{K/\beta}\right]$$

$$2gy_1 = \frac{2gc}{K}\sin\theta m_0\left[\frac{-K}{\quad}\qquad\right] + \frac{2g^2 m_0^2}{\beta(K-\beta)}\left[\qquad\right]$$

$$\frac{2gcK\sin\theta m_0}{K(K-\beta)}\left[\quad\right] - \frac{2gc\beta\sin\theta m_0}{K(K-\beta)}\left[\qquad\right]$$

$$v_1^2 = \left(\frac{c\beta}{K}\right)^2 \sin^2\theta - 2\left(\frac{c\beta}{K}\sin\theta\right)\left(\frac{m_1}{m_0}\right)^{K/\beta} + \left(\frac{c\beta}{K}\sin\theta\ \frac{m_1}{m_0}\right)^2$$

$$+ \frac{g^2 m_1^2}{(K-\beta)^2} - \frac{2g^2 m_0^2}{(K-\beta)^2}\left(\frac{m_1}{m_0}\right)^{\frac{K+\beta}{\beta}} + \frac{g^2 m_0^2}{(K-\beta)^2}\left(\frac{m_1}{m_0}\right)^{2K/\beta}$$

$$2gy_1 = \frac{2gc\sin\theta m_0}{K+\beta} + \frac{2gc\sin\theta m_0\beta}{K(K+\beta)}\left(\frac{m_1}{m_0}\right)^{\frac{K+\beta}{\beta}} - \frac{2gc}{K}\sin\theta m_1$$

$$+ \frac{g^2 m_1^2}{\beta(K-\beta)} - \frac{2g^2 m_0^2}{(K-\beta)(K+\beta)}\left(\frac{m_1}{m_0}\right)^{\frac{K+\beta}{\beta}} - \frac{g^2 m_0^2}{\beta(K+\beta)}$$

$$V_1 = A\sin\theta + B$$

$$Y_1 = D\sin\theta + E$$

$$r = \sqrt{V_1^2 + 2gY_1}$$

$$r = \sqrt{(A^2\sin^2\theta + 2BA\sin\theta + B^2 + 2gD\sin\theta + 2gE)}$$

$$u_1 = A\cos\theta \qquad \tan\theta = \frac{A\cos\theta\ e^{-\frac{k(A\sin\theta+B) + k\sqrt{A'\sin^2\theta + B'\sin\theta}}{m_1 g}}}{\sqrt{A'\sin^2\theta + B'\sin\theta + C'}}$$

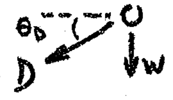
$$A' = A^2$$

$$B' = 2BA + 2gD$$

$$C' = 2gE$$

$$\sin^2\theta\left(A'\sin^2\theta + B'\sin\theta + C'\right) = A\cos^3\theta\ e^{(D'\sin\theta + E' - \sqrt{F'\sin^2\theta + G'\sin\theta + H'})}$$

where $D' = \frac{-Ak}{m_1 g}$ $\quad E' = \frac{-Bk}{m_1 g}$ $\quad F' = \frac{A'k^2}{m_1^2 g^2}$ $\quad G' = \frac{B'k^2}{m_1^2 g^2}$ $\quad H' = \frac{C'k^2}{m_1^2 g^2}$



$$\dot{u} = -\frac{D}{m_T}\cos\theta_D = -\frac{ku}{m_T}$$

$$\dot{v} = \frac{-D}{m_T}\sin\theta_D = -\frac{kv}{m_T} - g$$

this leads to

$$u = u_1 e^{-\frac{kt}{m_T}} = u_1 e^{-n_1 t} \qquad \text{where} \quad n_1 = k/m_T$$

$$v = -\frac{m_T g}{k} + \left(v_1 + \frac{m_T g}{k}\right) e^{-n_1 t}$$

$$x = \frac{u_1}{n_1}\left[1 - e^{-n_1 t}\right]$$

$$y = y_1 - \frac{m_1 g}{k} t + \frac{m_1}{k}\left(v_1 + \frac{m_1 g}{k}\right)\left[1 - e^{-kt/m_1}\right]$$

Assume drag is small as in the case of launching at high altitudes

$\therefore$ t_1 is the same as t_1 for the ballistic trajectory for drag-free flight

$$t_1 = \frac{v_1}{g} + \frac{r}{g} = \frac{v_1 + r}{g} \qquad \text{where} \quad r = \sqrt{v_1^2 + 2gy_1}$$

$$\therefore \quad x_{BALLISTIC} = \frac{u_1}{n_1}\left[1 - e^{-n_1\left[\frac{v_1+r}{g}\right]}\right]$$

$$F = \lambda_u\left(\dot{u} - a\cos\theta + \frac{k}{m}u\right) + \lambda_v\left(\dot{v} - a\sin\theta + \frac{kv}{m} + g\right) + \lambda_x(\dot{x} - u)$$
$$+ \lambda_y(\dot{y} - v)$$

E-L in u $\quad n\lambda_u - \lambda_x - \dot{\lambda}_u = 0$ $\qquad$ E-L in x $\quad -\dot{\lambda}_x = 0$

E-L in v $\quad n\lambda_v - \lambda_y - \dot{\lambda}_v = 0$ $\qquad$ E-L in y $\quad -\dot{\lambda}_y = 0$

E-L in θ $\quad a(\lambda_u \sin\theta - \lambda_v \cos\theta) = 0$

$$\therefore \quad C_1 = \lambda_x \qquad C_2 = \lambda_y \qquad \tan\theta = \frac{\lambda_v}{\lambda_u} \qquad \lambda_u = \cancel{\frac{C_1}{n} + C_{11} e^{nt}} + C_1\left[\frac{m}{K+\beta}\right] + \frac{C_{11}}{m}$$

$$\lambda_v = \cancel{\frac{C_2}{n} + C_{22} e^{nt}} + C_2\left[\frac{m}{K+\beta}\right] + \frac{C_{v}}{K}$$

$$dR = dx_1 + du_1\left[\frac{1}{n_1} - \frac{e}{n_1}^{-n_1\left[\frac{v_1+r}{g}\right]}\right] - \frac{u_1}{n_1} e^{-n_1\left[\frac{v_1+r}{g}\right]}\left\{-\frac{n_1}{g} - \frac{n_1 v_1}{gr}\right\} dv_1$$

$$- \frac{u_1}{n_1} e^{-n_1\left[\frac{v_1+r}{g}\right]}\left\{-\frac{n_1 g}{\ }\right\} dy_1$$

$$dR = dx_1 + du_1\left[\frac{1 - e^{-n_1\left(\frac{v_1+r}{g}\right)}}{n_1}\right] + \left[\frac{u_1}{g} + \frac{u_1 v_1}{gr}\right]e^{-n_1\left(\frac{v_1+r}{g}\right)} dv_1$$

$$+ \frac{u_1}{r}\, e^{-n_1\left(\frac{v_1+r}{g}\right)} dy_1$$

Transversality

$$-C dt + du\left(\lambda u + \frac{1-e^{-n_1\left(\frac{v_1+r}{g}\right)}}{n_1}\right) + dv\left(\lambda v + \left\{\frac{u_1}{g} + \frac{u_1 v_1}{gr}\right\} e^{-n_1\left(\frac{v_1+r}{g}\right)}\right)$$

$$dx\,(\lambda x + 1) + dy\left(\lambda y + \frac{u_1}{r}\, e^{-n_1\left(\frac{v_1+r}{g}\right)}\right) = 0$$

$$\lambda u_f = -\left[\frac{1}{n_1} - \frac{e^{-n_1\left(\frac{v_1+r}{g}\right)}}{n_1}\right] \qquad \lambda v_f = -\left[\frac{u_1(v_1+r)}{gr}\, e^{-n_1\left(\frac{v_1+r}{g}\right)}\right]$$

$$\lambda x_f = -1 \qquad \lambda y_f = -\frac{u_1}{r}\, e^{-n_1\left(\frac{v_1+r}{g}\right)}$$

Since $\lambda x = C_1 = -1 \qquad \lambda y = C_2 = -\frac{u_1}{r}\, e^{-n_1\left(\frac{v_1+r}{g}\right)}$

$$\therefore \quad \lambda u = -\frac{1}{n} + C_{11} e^{nt}; \text{ at } t=T \quad \lambda u_f = -\frac{1}{n_1} + C_{11} e^{n_1 T} = -\frac{1}{n_1} + \frac{e^{-n_1\left(\frac{v_1+r}{g}\right)}}{n_1}$$

$$C_{11} = \frac{1}{n_1}\, e^{-n_1\left(\frac{v_1+r+gT}{g}\right)} \qquad \lambda u = \frac{-1}{n} + \frac{1}{n_1}\, e^{nt - n_1\left(\frac{v_1+r+gT}{g}\right)}$$

$$\therefore \quad \lambda v = -\frac{u_1}{nr}\, e^{-n_1\left(\frac{v_1+r}{g}\right)} + C_{22} e^{nt}; \text{ at } t=T \quad \lambda v_f = -\frac{u_1}{n_1 r}\, e^{-n_1\left(\frac{v_1+r}{g}\right)} + C_{22} e^{n_1 T}$$

also $\lambda v_f = -\left[\frac{u_1(v_1+r)}{gr}\, e^{-n_1\left(\frac{v_1+r}{g}\right)}\right] \quad \therefore \quad \lambda v = \frac{u_1 e^{-n_1\left(\frac{v_1+r}{g}\right)}}{gr}\left[\frac{g - n_1(v_1+r)}{n_1}\, e^{nt - n_1 T} - \frac{g}{n}\right]$

$$\tan\Theta = \frac{\frac{u_1 e^{-\frac{n_1(v_1+r)}{g}}}{gr}\left[\frac{g-n_1(v_1+r)}{n_1 g} e^{nt-n_1T} - \frac{g}{n}\right]}{-\frac{1}{n} + \frac{e^{nt - n_1\left(\frac{v_1+r+gT}{g}\right)}}{n_1}}$$

note that if $m_T g >> k v_1$

$$\lambda_{x_f} = -1 \qquad \lambda_{u_f} = -\frac{1}{n_1}\left[1 - e^{-n_1\left(\frac{v_1+r}{g}\right)}\right]$$

$$\lambda_{v_f} = -\frac{u_1(r+v_1)}{gr} e^{-n_1\left(\frac{v_1+r}{g}\right)} \qquad \lambda_{y_f} = -\frac{u_1}{r} e^{-n_1\left(\frac{v_1+r}{g}\right)}$$

$$\lambda_x = -1 \qquad \lambda_u = \frac{-m}{K+\beta} + C_{11} m^{-K/\beta}$$

$$\lambda_y = -\frac{u_1}{r} e^{-n_1\left(\frac{v_1+r}{g}\right)} \qquad \lambda_v = \frac{-u_1}{r(K+\beta)} e^{-n_1\left(\frac{v_1+r}{g}\right)} m + C_{22} m^{-K/\beta}$$

$$\lambda_{v_f} = \frac{-u_1}{r(K+\beta)} e^{-n_1\left(\frac{v_1+r}{g}\right)} m_1 + C_{22} m_1^{-K/\beta} = -\frac{u_1(r+v_1)}{gr} e^{-n_1\left(\frac{v_1+r}{g}\right)}$$

$$\frac{u_1}{r} e^{-n_1\left(\frac{v_1+r}{g}\right)}\left[\frac{m_1}{K+\beta} - \frac{(r+v_1)}{g}\right] m_1^{K/\beta} = C_{22}$$

$$\lambda_v = \frac{-u_1}{r(K+\beta)} e^{-n_1\left(\frac{v_1+r}{g}\right)} m + \left[\frac{u_1}{r} e^{-n_1\left(\frac{v_1+r}{g}\right)}\left\{\frac{m_1}{K+\beta} - \frac{(r+v_1)}{g}\right\}\left(\frac{m_1}{m}\right)^{K/\beta}\right]$$

$$\boxed{\lambda_v = \frac{u_1}{r} e^{-n_1\left(\frac{v_1+r}{g}\right)}\left[\frac{-m}{K+\beta} + \left\{\frac{m_1}{K+\beta} - \frac{(r+v_1)}{g}\right\}\left(\frac{m_1}{m}\right)^{K/\beta}\right]}$$

$$\lambda_u = -\frac{m}{K+\beta} + C_{11} m^{-K/\beta} \qquad \lambda_{u_f} = \frac{-m_1}{K+\beta} + C_{11} m_1^{-K/\beta} = -\frac{m_1}{K}\left[1 - e^{-n_1\left(\frac{v_1+r}{g}\right)}\right]$$

$$m_1 g - (K+\beta)(r+v_1)$$
$$(K+\beta)g$$

$$\frac{-m_1 \cancel{K}}{K(\cancel{K}+\beta)} + C_{11} m_1^{-K/\beta} = -\frac{m_1(\cancel{K}+\beta)}{K(K+\beta)} + \frac{m_1}{K} e$$

$$y(x) = \frac{1}{\mu(x)}\left[\int^x \mu g \, ds + c\right] \qquad \dot{u} + \frac{ku}{m} = a\cos\theta$$

$$\mu(x) = e^{\int^x p\,dt} \qquad p = \frac{k}{m}$$

$$K\int^x \frac{dt}{m} = -\frac{K}{\beta}\ln[m_1 - \beta t]$$

$$\mu = e^{K\int^x \frac{dt}{m}} = e^{-K/\beta \ln[m_1-\beta t]}$$

$$\cancel{y(x)}\, u = \frac{1}{e^{-K/\beta \ln(m_1-\beta t)}}\left[\int e^{-K/\beta\ln[m_1-\beta t]} a\cos\theta + C\right.$$

$$u = e^{K/\beta \ln(m_1-\beta t)}\left[c\beta\cos\theta \int \frac{e^{-K/\beta \ln[m_1-\beta t]}}{m} dt + C\right]$$

$$\overset{u}{e^{-K/\beta \ln(m_1-\beta t)}} \qquad \overset{dv}{m^{-1}\,dt} \qquad m^{-K/\beta - 1} \qquad m^{-(K/\beta+1)} \qquad m^{-K/\beta}$$

$$\frac{K}{\beta}\frac{-\beta}{m_1-\beta t}\overset{du}{e^{-K/\beta \ln(\ )}} \qquad \overset{v}{-\frac{1}{\beta}\ln(m_1-\beta t)}$$

$$e^{-K/\beta \ln(m_1-\beta t)}\left[-\frac{1}{\beta}\ln(m_1-\beta t)\right] + \frac{K}{\beta}\int e^{-K/\beta \ln(m_1-\beta t)}$$

$$-\frac{1}{\beta}\int \frac{e^{-K/\beta \ln m}}{m}\ln m \, dm \qquad dm = -\beta\,dt$$

$$-\frac{c\beta\cos\theta}{\beta}\int \frac{e^{-K/\beta \ln m}}{m}dm = -c\cos\theta\int \frac{e^{-K/\beta\ln m}}{m}dm$$

$$u(x) = e^{+K/\beta \ln m}\left[-c\cos\theta\int \frac{e^{-K/\beta \ln m}}{m}dm + C\right]$$

$$e^{-K/\beta \ln m} = \frac{m^{-K/\beta}}{m^{-1}}$$

$$e^{-K/\beta \ln m} \qquad e^{\ln m} = K \qquad m^{-K/\beta}$$

$$\ln e^{\ln m} = \ln m = \ln K \qquad m = K$$

$$\ln m^{K/\beta} = K/\beta \ln m$$

$$\int m^{-K/\beta - 1} dm$$

$$m^{-(K/\beta+1)}\,dm. \qquad \int m^{-(K/\beta+1)}dm = \frac{m^{-(K/\beta+1)+1}}{-K/\beta}$$

$$-\beta/K \; m^{-K/\beta}$$

$$\tan\theta = \frac{u_1 e^{-\frac{kt_1}{m_1}}}{\frac{m_1 g}{K}\left(1 - e^{-\frac{kt_1}{m_1}}\right) - v_1 e^{-\frac{kt_1}{m_1}}} = \text{const}$$

$$V_1 = A\sin\theta + B \qquad y_1 = D\sin\theta + E \qquad u_1 = A\cos\theta$$

from eq $\quad \frac{m_1 g t_1}{K} - \frac{m_1}{K}\left[V_1 + \frac{m_1 g}{K}\right]e = y_1$

$$\frac{m_1 g t_1}{K} - \frac{m_1}{K}\left[A\sin\theta + B + \frac{m_1 g}{K}\right]e = D\sin\theta + E$$

Ans $\quad -\frac{m_1 A}{K} e = D \qquad -e = \frac{KD}{m_1 A} = e^{-\frac{kt_1}{m_1}} - 1$

$$\ln\left[1 + \frac{KD}{m_1 A}\right] = -\frac{k}{m_1} t_1 \qquad t_1 = -\frac{m_1}{K}\ln\left[1 + \frac{KD}{m_1 A}\right]$$

Check $\quad e = -\frac{KD}{m_1 A}$

$$-\frac{m_1^2}{K^2} g \ln\left[1 + \frac{KD}{m_1 A}\right] + \frac{m_1 B}{\cancel{K}} \frac{\cancel{K} D}{\cancel{m_1} A} + \frac{m_1^2}{K^{\cancel{2}}} g \frac{\cancel{K}}{\cancel{m_1}} \frac{D}{A} = E$$

$$-\frac{m_1^2}{K^2} g \ln\left[1 + \frac{KD}{m_1 A}\right] + \frac{BD}{A} + \frac{m_1}{K} g \frac{D}{A} = E$$

$$\frac{c\beta}{K}\left[1 - \left(\frac{m_1}{m_0}\right)^{K/\beta}\right] = A \qquad -\frac{g}{K-\beta}\left[m_1 - m_0\left(\frac{m_1}{m_0}\right)^{K/\beta}\right] = B$$

$$\frac{c}{K} m_0\left[\left(\frac{m_1}{m_0}\right)^{\frac{K+\beta}{\beta}} \frac{\beta}{K+\beta} + \frac{K}{K+\beta} - \frac{m_1}{m_0}\right] = D \quad E = \frac{g m_0^2}{\beta(K-\beta)}\left[\frac{1}{2}\left(\frac{m_1}{m_0}\right)^2 - \frac{\beta}{K+\beta}\left(\frac{m_1}{m_0}\right)^{\frac{K+\beta}{\beta}} - \frac{K-\beta}{2(\beta+K)}\right.$$

$$\frac{K}{m_1} D = \frac{c}{K+\beta}\left[\left(\frac{m_1}{m_0}\right)^{K/\beta} \beta + \frac{m_0}{m_1} K - (K\right]$$

$$\frac{c}{K+\beta}\left[\right.$$

$$\epsilon = 1 - e^{\Theta}$$
$$1-\epsilon = e^{\Theta}$$

$$\tan\Theta = \frac{A\cos\Theta\, e^{-\frac{kt_1}{m_1}}}{\frac{m_1 g}{K}\left(1-e^{-\frac{kt_1}{m_1}}\right) - (A\sin\Theta + B)\, e^{-\frac{kt_1}{m_1}}}$$

$$\frac{m_1 g t_1}{K} - \frac{m_1}{K}\Big[\underbrace{A\sin\Theta + B + \frac{m_1 g}{K}}_{V_1}\Big]\left(1-e^{-\frac{kt_1}{m_1}}\right) = \underbrace{D\sin\Theta + E}_{Y_1}$$

$$\tan\Theta\left[\frac{m_1 g}{K}\left(1-e^{-\frac{kt_1}{m_1}}\right) - (A\sin\Theta + B)\, e^{-\frac{kt_1}{m_1}}\right] = A\cos\Theta\, e^{-\frac{kt_1}{m_1}}$$

$$+\frac{\left[D\sin\Theta + E - \frac{m_1 g t_1}{K}\right]\frac{K}{m_1}}{\left(1-e^{-kt_1/m_1}\right)} + \frac{m_1 g}{K} = -(A\sin\Theta + B)$$

$$\tan\Theta\left[\frac{m_1 g}{K}\left(1-e^{-\frac{kt_1}{m_1}}\right) + \frac{m_1 g}{K} e^{-\frac{kt_1}{m_1}} + \frac{\left[D\sin\Theta + E - \frac{m_1 g t_1}{K}\right]\frac{K}{m_1} e^{-\frac{kt_1}{m_1}}}{\left(1-e^{-\frac{kt_1}{m_1}}\right)}\right] = \frac{A\cos^2\Theta}{\cos\Theta} e^{-\frac{kt_1}{m_1}}$$

$$\frac{m_1 g}{K}\tan\Theta - \frac{m_1 g}{K}\tan\Theta\, e^{-\frac{kt_1}{m_1}} - A\frac{\sin^2\Theta}{\cos\Theta} e^{-\frac{kt_1}{m_1}} - B\tan\Theta\, e^{-\frac{kt_1}{m_1}} = A\frac{\cos^2\Theta}{\cos\Theta} e^{-\frac{kt_1}{m_1}}$$

$$\frac{m_1 g}{K}\sin\Theta - \frac{m_1 g}{K}\sin\Theta\, e^{-\frac{kt_1}{m_1}} - B\sin\Theta\, e^{-\frac{kt_1}{m_1}} = A \quad e^{-\frac{kt_1}{m_1}}$$

$$\sin\Theta\ \frac{m_1 g}{K} = \left[A\sec\Theta + \frac{m_1 g}{K}\sin\Theta + B\sin\Theta\right] e^{-\frac{kt_1}{m_1}}$$

$$\frac{m_1 g t_1}{K} - \frac{m_1}{K}\left[A\sin\Theta + B + \frac{m_1 g}{K}\right] + \frac{m_1}{K}\left[A\sin\Theta + B + \frac{m_1 g}{K}\right] e^{-\frac{kt_1}{m_1}} = D\sin\Theta + E$$

$$\theta = \theta_0 - \frac{f(\theta_0)}{f'(\theta_0)}$$

$$\Theta = g(t_1)$$

$$f'(\Theta) = \frac{df}{dt}\frac{dt}{d\Theta}$$

$$\Theta = \Theta_0 - \frac{f(\Theta_0)}{f'(\Theta_0)}$$

$$\frac{m_1 g t_1}{K} - \frac{m_1}{K}\left[B + \frac{m_1 g}{K}\right]\epsilon$$

$$= D\sin\Theta + E + \frac{m_1 A}{K}\sin\Theta\,\epsilon$$

$$f'(g(t_1)) = \frac{df}{d\Theta}\frac{d\Theta}{dt_1}$$

$$\frac{\frac{m_1 g t_1}{K} - \frac{m_1}{K}\left[B + \frac{m_1 g}{K}\right]\epsilon - E}{\left(D + \frac{m_1 A}{K}\epsilon\right)} = \sin\Theta$$

$$\left(D+\frac{m_1A}{K}\epsilon\right)\left[\frac{m_1g}{K}-\left[B+\frac{m_1g}{K}\right]e^{-\frac{kt_1}{m_1}}\right]-\left[\frac{m_1gt_1}{K}-\frac{m_1}{K}\left[B+\frac{m_1g}{K}\right]\epsilon-E\right]Ae^{\frac{kt_1}{m_1}}$$

$$f(\theta(t_1)) = \sin\theta = \frac{\frac{m_1gt_1}{K}-\frac{m_1}{K}\left[B+\frac{m_1g}{K}\right]\epsilon - E}{D+\frac{m_1A}{K}\epsilon}$$

$$1-\epsilon = e^{-\frac{kt_1}{m_1}} \qquad \frac{d}{dt_1}(1-\epsilon) = -\frac{k}{m_1}e \qquad -\frac{k}{m_1}(1-\epsilon)$$

$$\epsilon = 1-e^{-\frac{kt_1}{m_1}}$$

$$\frac{d\epsilon}{dt} = \frac{k}{m_1}e^{-\frac{kt_1}{m_1}}$$

$$f(\theta(t_1)) = \tan\theta = \frac{A\cos\theta\,(1-\epsilon)}{\frac{m_1g}{K}\epsilon-(A\sin\theta+B)(1-\epsilon)} = \frac{A\cos\theta\,(1-\epsilon)}{[A\csc\theta+B]-[A\sin\theta+B]}$$

$$\frac{\cos\theta}{\frac{1}{\sin\theta}-\frac{\sin^2\theta}{\sin\theta}} = \frac{\cos\theta}{\frac{\cos^2\theta}{\sin\theta}}$$

$$\frac{\sin\theta}{\cos\theta}$$

$$\frac{m_1g}{K} = \left[A\csc\theta+\frac{m_1g}{K}+B\right](1-\epsilon)$$

$$\frac{m_1g}{K}-\frac{m_1g}{K}(1-\epsilon) = [A\csc\theta+B](1-\epsilon)$$

$$\frac{m_1g}{K}[1-1+\epsilon] = \frac{m_1g}{K}\epsilon = [A\csc\theta+B](1-\epsilon)$$

$$\theta(t_1) = \theta_0(t_1)-\frac{f(\theta_0(t_1))}{f'(\theta_0(t_1))} = \theta_0(t_1)-\frac{f(\theta_0(t_1))}{\frac{df}{d\theta_0}\frac{d\theta_0}{dt_1}}$$

$$\longrightarrow \quad \frac{d\theta}{dt_1} = \left\{\frac{\left(D+\frac{m_1A}{K}\epsilon\right)\left[\frac{m_1g}{K}\epsilon-B(1-\epsilon)\right]-\left[\frac{m_1gt_1}{K}-\frac{m_1}{K}\left[B+\frac{m_1g}{K}\right]\epsilon-E\right]A(1-\epsilon)}{\left(D+\frac{m_1A}{K}\epsilon\right)^2\cos\theta}\right\}$$

$$f'(\theta(t))\quad \sec^2\theta\,\frac{d\theta}{dt_1} = \left\{\frac{\left[\frac{m_1g}{K}\epsilon-(A\sin\theta+B)(1-\epsilon)\right]\left[-A(1-\epsilon)\sin\theta\frac{d\theta}{dt}-\frac{k}{m_1}A\cos\theta(1-\epsilon)\right]-[A\cos\theta(1-\epsilon)]\left[g(1-\epsilon)+\frac{k}{m_1}(A\sin\theta+B)(1-\epsilon)-(1-\epsilon)A\cos\theta\frac{d\theta}{dt}\right]}{\left[\frac{m_1g}{K}\epsilon-(A\sin\theta+B)(1-\epsilon)\right]^2}\right\}$$

$$f' = \frac{1}{\sec}$$

$$t_{new} = t_0-\frac{f(t_0)}{f'(t_0)}$$

$$t_{New} = t_0 - \frac{f(t_0)}{f'(t_0)}$$

$\epsilon = 1 - e^{-\frac{kt_1}{m_1}}$

$$f(t_0) = \tan\theta - \frac{A\cos\theta\,(1-\epsilon)}{\frac{m_1 g}{K}\epsilon - (A\sin\theta + B)(1-\epsilon)}$$ steering equation

input $t \longrightarrow$ use $\text{DELTA} = \dfrac{\frac{m_1 g t_1}{K} - \frac{m_1}{K}\left[B + \frac{m_1 g}{K}\right]\epsilon - E}{D + \frac{m_1 A}{K}\epsilon}$ to get θ $\left\{\frac{y\text{ eq}}{}\right\}$

get $\dfrac{d\theta}{dt_1} = \left\{\dfrac{\left(D + \frac{m_1 A}{K}\epsilon\right)\left[\frac{m_1 g}{K}\epsilon - B(1-\epsilon)\right] - \left[\frac{m_1 g t_1}{K} - \frac{m_1}{K}\left[B + \frac{m_1 g}{K}\right]\epsilon - E\right]A(1-\epsilon)}{\left(D + \frac{m_1 A}{K}\epsilon\right)^2\cos\theta}\right\}$

input into from y eq

$$f'(t_0) = \sec^2\theta\,\frac{d\theta}{dt} + \left(\frac{\text{GAMMAA} + \text{GAMMAB}}{\left[\frac{m_1 g}{K}\epsilon - (A\sin\theta + B)(1-\epsilon)\right]^2}\right)$$

$$\text{GAMMAA} = \left[\frac{m_1 g}{K}\epsilon - (A\sin\theta + B)(1-\epsilon)\right]\left[\sin\theta\,\frac{d\theta}{dt} + \frac{k_1}{m_1}\cos\theta\right]A(1-\epsilon)$$

$$\text{GAMMAB} = \left[A\cos\theta(1-\epsilon)\right]\left[g + \frac{k}{m_1}(A\sin\theta + B) - A\cos\theta\,\frac{d\theta}{dt}\right](1-\epsilon)$$

$$\theta = \text{ASIN}(\text{DELTA})$$

Space Program" Forum).

The number of pieces of paper processed by Leonard Rosenberg's office-services team loses meaning as it adds digits. A suggestion of the magnitude: Over 10,000,000 sheets of meeting papers alone were preprinted!

Managing the West Coast Headquarters in support of all national activities, and having responsibility for the Los Angeles Section and the Pacific Aerospace Library (PAL), was Joseph P. Ryan, Assistant Executive Secretary, Western Operations, assisted by Nan MacCandless, PAL Director, and by Jeff Moreau.

Fiscal 1967 confirmed the gradually changing service posture of the Pacific Aerospace Library. Once largely supported by half a dozen aerospace corporations, PAL's importance to these large aerospace companies diminished as their own technical-information centers strengthened through centralization of library facilities, mechanized operations, and the availability of services and documents from the federal government.

In 1967 the acting PAL Advisory Committee was augmented by six members from the six industries most critically involved in PAL's history. An operating and management plan was drafted to direct PAL's operations, cushion it through a period of transition, and assure its viable role in future information activities.

The Advisory Committee's study showed a continuing need for PAL by dozens of small aerospace subcontractors in the area, as well as confirming its limited need by the larger companies. A coupon service started late in 1966 generated more than 35 new clients for PAL's services in 1967.

Our other far-flung post, the three-man London office, was managed by Sanford M. Harris largely in support of IAA.

Keeping track of the ebb and flow of finances to cover these activities—almost $4,000,000 in FY 1967—were AIAA Controller Joseph J. Maitan and Assistant Controller Vincent Gallo and seven others in the Accounting Department.

The cost of operating the AIAA staff is considerable—payroll alone amounting to more than $1,620,000, or more than $40 per member for every one of our 33,000 members and 7000 student members. For the skeptical member who wonders if it's all worth it, and is nonplussed by the fact that the payroll/member figure is twice his dues payment, it seems relevant to bring up the fact that the $20 dues actually represents only about one-fifth of the $100/member that AIAA put into services to the profession in FY 1967. The other $80 comes largely from staff-administered services to aerospace firms, to NASA, DOD, and other government agencies, and to individuals, universities and other organizations in almost every country of the world.

James J. Harford
Executive Secretary

the space region in which the planet, rather than the Sun can be regarded as the center body and in which the concept of the hyperbolic flight path holds with good approx.

$$r_{act} = R_{pl}\left(\frac{K_{pl}}{K_{\odot}}\right)^{2/5}$$

Heliocentric departure velocity
transfer to outer planet

$$V^*_p = \sqrt{\frac{2n}{n+1}} \qquad n = R_A/R_p \qquad V_p = \left[\frac{K}{R_p}\left(\frac{2n}{n+1}\right)\right]^{1/2}$$

transfer to inner planet.

$$V^*_A = \sqrt{\frac{2}{n+1}} = \sqrt{\frac{K_{\odot}}{R_A}} \quad n = R_A/R_p \qquad K = GM$$

$$V_a = \left[\frac{K}{R_a}\left(\frac{2}{n+1}\right)\right]^{1/2}$$

Departure

$$V_p - 1 + \Delta V_1 = v_\infty$$

where

$$\Delta V_1 = |V_p - V_c|$$

Arrival

$$V_A - 1 - \Delta V_1 = v_\infty$$

where

$$\Delta V_1 = |V_A - V_c|$$

So that

Planetocentric escape velocity — V_{launch}

$$V_{launch} = \sqrt{\Delta V_1^2 + v_{esc}^2}$$

Or from orbit

$$\Delta v =$$

Arrival $\Delta V_2 = v_{\infty,2} = V_p - \sqrt{K_{\odot}/a_{pl}}$ inner planet

$\Delta V_2 = v_{\infty,2} = \sqrt{K_{\odot}/a_{pl}} - V_a$ outer planet

Capture energy requirement.

Conversion Factors

feet × 0.30480 = meters
kilometers × 3280.8 = feet
× 0.62137 = miles
meters × 39.37 = inches
× 3.2808 = feet
miles × 1.6093 = kilometers
pounds × 453.59 = grams
slugs × 32.174 = pounds
newtons × 0.225 = pounds

$V = \omega r$ - angular speed is ω
Acc. is independent of mass
$V_c \downarrow$ $H \uparrow$ K.E. $\uparrow$
W to E E $\downarrow$ Prograde Orbit
E to W E $\uparrow$ Retrograde Orbit.
Inertia ~ mass & d^2 from rev. AXIS
Orbital vel. ≈ both body masses
Time ZONES = 24 1 TZ = 15° LONG.
Prime Vertical is ⊥ to meridian circle
Ecliptical plane - earth rev. about sun plane [with axis tilted]
celestial " - earth equator projected.

feet per second × 0.68182 = miles per hour
× 1.0973 = kilometers per hour
× 0.30480 = meters per second
kilometers per hour × 0.91134 = feet per second
× 0.62137 = miles per hour
× 0.2777 = meters per second
meters per second × 3.2808 = feet per second
× 2.2369 = miles per hour
× 3.600 = kilometers per hour
miles per hour × 0.44704 = meters per second
× 1.6093 = kilometers per hour

g ~ I_0(rotational), ellipticity & distribution of mass

THE DOPPLER EFFECT

Suppose that a source of waves is moving directly (radially) away from a fixed observer at a velocity V_R, and let t be the time between the emission of two successive wave crests. During this time, the source moves farther away by a distance tV_R. If c is the velocity at which the waves travel (velocity of light) the second wave takes a time tV_R/c longer than the first wave to reach the observer. Hence, the time between successive waves reaching the observer is increased by a factor of V_R/c. The time between successive wave crests is inversely related to the frequency of the waves; hence, the frequency of the waves reaching the fixed observer from a receding source will be decreased by a factor of V_R/c of the emitted frequency. In the case of a source moving toward the observer, the frequency will be increased by the same factor.

If f_0 is the frequency of the waves as they leave the source, the frequency change noted by the observer, f_d (the Doppler frequency shift) is equal to

$$\frac{f_0 V_R}{c}$$

Therefore:

$$V_R = \frac{f_d}{f_0} c$$

$$I_t = F t_b \qquad \overset{\text{thrust.}}{F_i} = \dot{m}_i v_i = \frac{\dot{\omega}_i}{g} v_i \qquad I_{sp_i} = \frac{F_i t_{b_i}}{\omega_i}$$

$$v_i = I_{sp_i} g \qquad F_i = \dot{\omega}_i I_{sp_i}$$

$$v_f - v_o = \sum_i v_i \ln \frac{M_{o_i}}{M_{b_i}} \qquad M_{o_i} = M_{b_i} + \dot{\omega}_i t_{b_i}$$

@ $t=0$ $x=y=u=v=0$; $m=m_0$ ①

@ $y=0$; $m=m_1$

$T=-c\dot{m}=c\beta$

$\dot{x}=u$

$\dot{y}=v$

$\dot{u}=\frac{c\beta}{m}\cos\theta$

$\dot{v}=\frac{c\beta}{m}\sin\theta-g$

$\dot{m}=-\beta$

$0\le\beta\le\beta_{max}$

$$F=\lambda_x(\dot{x}-u)+\lambda_y(\dot{y}-v)+\lambda_u\left(\dot{u}-\frac{c\beta}{m}\cos\theta\right)+\lambda_v\left(\dot{v}-\frac{c\beta}{m}\sin\theta+g\right)+\lambda_m(\dot{m}+\beta)+\lambda_\gamma\left(\beta[\beta_{max}-\beta]-\gamma^2\right)=0$$

$\beta(\beta_{max}-\beta)-\gamma^2=0$

$\cancel{\lambda_x,\lambda_y,\lambda_u,\lambda_v,\lambda_m,\lambda_\gamma}$

$x,y,u,v,\beta,m,\theta,\gamma$

6 eq 8 unk

$$\frac{\partial F}{\partial x}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)=-\dot{\lambda}_x=0 \qquad \lambda_x=C_1$$

$$\frac{\partial F}{\partial y}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)=-\dot{\lambda}_y=0 \qquad \lambda_y=C_2$$

$$\frac{\partial F}{\partial u}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right)=-\lambda_x-\dot{\lambda}_u=0 \qquad C_1+\dot{\lambda}_u=0 \qquad C_1t+\lambda_u=C_{11}$$

$$\frac{\partial F}{\partial v}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}}\right)=-\lambda_y-\dot{\lambda}_v=0 \qquad C_2+\dot{\lambda}_v=0 \qquad C_2t+\lambda_v=C_{22}$$

$$\frac{\partial F}{\partial \beta}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\beta}}\right)=-\frac{c}{m}\cos\theta\cdot\lambda_u-\frac{c}{m}\sin\theta\,\lambda_v+[\beta_{max}-2\beta]\lambda_\gamma+\lambda_m=0$$

$$\frac{\partial F}{\partial m}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{m}}\right)=\lambda_u\frac{c\beta}{m^2}\cos\theta+\lambda_v\frac{c\beta}{m^2}\sin\theta-\dot{\lambda}_m=0$$

$$\frac{\partial F}{\partial \theta}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\theta}}\right)=\lambda_u\frac{c\beta}{m}\sin\theta-\lambda_v\frac{c\beta}{m}\cos\theta=0$$

$$\frac{\lambda_v}{\lambda_u}=\tan\theta \qquad \frac{\lambda_v}{\lambda_u}=\frac{C_{22}-C_2t}{C_{11}-C_1t}$$

$$\frac{\partial F}{\partial \gamma}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\gamma}}\right)=-2\gamma\lambda_\gamma=0 \qquad \gamma=0 \quad \lambda_\gamma=0$$

$$L=\lambda_x u+\lambda_y v+\lambda_u\left[\frac{c\beta}{m}\cos\theta\right]+\lambda_v\left[\frac{c\beta}{m}\sin\theta-g\right]-\lambda_m\beta+\ldots$$

$$\frac{\partial L}{\partial\theta}=-\lambda_u\frac{c\beta}{m}\sin\theta+\lambda_v\frac{c\beta}{m}\cos\theta=0 \qquad \tan\theta=\frac{\lambda_v}{\lambda_u}$$

$\lambda_u\sin\theta=\lambda_v\cos\theta$

$\frac{\lambda_u\sin\theta}{\lambda_v}=\cos\theta$

$$\frac{\partial^2 L}{\partial\theta^2}\ngtr 0 \qquad -\lambda_u\frac{c\beta}{m}\cos\theta-\lambda_v\frac{c\beta}{m}\sin\theta\geqslant 0$$

$-\frac{\lambda_u}{}$

$+\frac{c\beta}{m}(\lambda_u^2+\lambda_v^2)$

only if — region of regret

$$-C\,dt\Big|_{t_0}^{t_f} + \lambda_x\,dx + \lambda_y\,dy + \lambda_v\,dv + \lambda_u\,du + \lambda_m\,dm - d\psi_f$$

$C_f = 0$ $\quad \lambda_{x_f} + 1 = 0$ $\quad \lambda_x = -1$ $\quad dy_f = 0 \quad dy_i = 0$

$\lambda_{v_f} = 0 \quad \lambda_{u_f} = 0$ $\quad dm_f = 0 \quad dm_i = 0$

$\tan\theta = -(u/v)_f$ $\quad \lambda_x \dot{x} + \lambda_y \dot{y} + \lambda_v \dot{v} + \lambda_u \dot{u} + \lambda_m \dot{m} = 0$

how $\quad \lambda_{u_f} = 0 = C_{11} - C_1 T = C_{11} + T \qquad C_{11} = -T$

$\therefore \quad \lambda_u = -T - t = -(T-t)$

$\cos\theta = \dfrac{-1}{\sqrt{1+C_2^2}}$

$\lambda_{v_f} = C_{22} - C_2 T = 0 \qquad C_{22} = C_2 T$

$\sin\theta = \dfrac{C_2}{\sqrt{}}$

$\lambda_v = C_2(T-t) \qquad \lambda_v/\lambda_u = \tan\theta = -C_2\,,\quad \sin\theta = -C_2\cos\theta$

$\dot{\lambda}_v = -C_2$

if $\gamma = 0 \qquad \beta = \beta_{max}$ or $\beta = 0$

$-\lambda_u \frac{c\beta}{m}\cos\theta + C_2 \lambda_u \frac{c\beta}{m}\sin\theta \leq 0$

$-\lambda_u \frac{c\beta}{m}\cos\theta - C_2^2 \lambda_u \frac{c\beta}{m}\cos\theta < 0$

$\dfrac{\partial L}{\partial \beta} = \lambda_u \dfrac{c}{m}\cos\theta + \lambda_v \dfrac{c}{m}\sin\theta - \lambda_m = K \quad 0$

$\frac{c\beta}{m}(T-t)\left[\cos\theta + C_2^2\cos\theta\right] \leq 0$

$\dfrac{c}{m}\left[\sqrt{\lambda_u^2+\lambda_v^2}\right] = \lambda_m \qquad \dfrac{c}{m}(T-t)\sqrt{1+C_2^2} = \lambda_m$

$L = \beta K$

~~only if~~ λ_γ

$dL = \underset{0}{\underset{\parallel}{\dfrac{\partial L}{\partial \theta}}}\,d\theta + \underset{0}{\underset{\parallel}{\dfrac{\partial L}{\partial \beta}}}\,d\beta \qquad \dfrac{\partial^2 L}{\partial^2\theta} < 0$

$\dfrac{\partial^2 L}{\partial \beta^2} = 0$

$\cancel{-\lambda_u \frac{c}{m}\sin\theta + \lambda_v \frac{c}{m}\cos\theta}$

$\lambda_v = -C_2 \lambda_u$

$\cancel{-\lambda_u \frac{c}{m}\left[\sin\theta + C_2\cos\theta\right]}$

$\dfrac{\lambda_v}{\lambda_u} = \dfrac{\sin\theta}{\cos\theta} = -C_2$

$\cancel{+(T-t)\frac{c}{m}\left[\,0\,\right] = 0}$

$\dfrac{c\beta}{m^2}\left[\dfrac{\lambda_u \cos^2\theta + \lambda_u \sin^2\theta}{\cos\theta}\right] - \dot{\lambda}_m = 0$

$\dfrac{c\beta}{m^2}(T-t)\sqrt{1+C_2^2} - \dot{\lambda}_m = 0$

$f_{xx} f_{yy} > f_{xy}^2$ no max exists

$\lambda_m = \displaystyle\int_0^T \frac{c\beta}{m^2}(T-t)\sqrt{1+C_2^2}\,dt$

$\sqrt{1+C_2^2}\;c \displaystyle\int \frac{-\dot{m}}{m^2}(T-t)\,dt$

$-\left[\lambda_u \dfrac{c}{m}\cos\theta + \lambda_v \dfrac{c}{m}\sin\theta\right] = -\dot{\lambda}_m\left[\dfrac{m}{\beta}\right]$

$-\dot{\lambda}_m \dfrac{m}{\beta} + \left[\beta_{max} - 2\beta\right]\lambda_\gamma + \lambda_m = 0$

$$\frac{c}{\sqrt{1+C_2^2}} \ln \frac{m_0}{m_1}\left[\frac{2+C_2^2}{C^2}\right]$$

$c\cos\theta \ln \frac{m_0}{m_1}$

$\dot{x}_f =$

$u_f = c\cos\theta \ln\frac{m_0}{m_1}$

$-c\cos\theta\int_0^T -\frac{\beta\, dt}{m}$ $\quad \frac{m_1}{m_0}$ $\quad \frac{dm}{m}$

$c\cos\theta \frac{m_0}{m_1}$

Euler Lag

$\dot{x} = u$ $\qquad \dot{\lambda}_x = 0$

$\dot{y} = v$ $\qquad \dot{\lambda}_y = 0$

$\dot{u} = \frac{c\beta}{m}\cos\theta$ $\qquad \lambda_x + \dot{\lambda}_u = 0$

$\dot{v} = \frac{c\beta}{m}\sin\theta - g$ $\qquad \lambda_y + \dot{\lambda}_v = 0$

$\dot{m} = -\beta$ $\qquad K_\beta = \frac{c}{m}[\lambda_u\cos\theta + \lambda_v\sin\theta] - \lambda_m;\ K_\beta - \lambda_\gamma[\beta_{max} - 2\beta] = 0$

$\beta(\beta_{max} - \beta) - \gamma^2 = 0$ $\qquad \frac{c\beta}{m^2}[\lambda_u\cos\theta + \lambda_v\sin\theta] - \dot{\lambda}_m = 0$

$$\frac{c\beta}{m}[\lambda_u\sin\theta - \lambda_v\cos\theta] = 0$$

$$-2\gamma\lambda_\gamma = 0$$

this leads to $\lambda_x = C_1 \quad \lambda_y = C_2 \quad \lambda_u = C_{11} - C_1 t \quad \lambda_v = C_{22} - C_2 t$

$\gamma = 0$ or $\lambda_\gamma = 0 \qquad \tan\theta = \frac{\lambda_v}{\lambda_u} = \frac{C_{22} - C_2 t}{C_{11} - C_1 t} = \frac{C_2(T-t)}{+1(T-t)}$

Also @ $t=0$ $\quad x = y = u = v = 0 \quad m = m_0$

@ End $\quad y = 0 \quad m = m_1$

First integral $\quad C = \lambda_x\dot{x} + \lambda_y\dot{y} + \lambda_v\dot{v} + \lambda_u\dot{u} + \lambda_m\dot{m}$

trans. $\quad -C\,dt\Big|_{t_0}^{t_f} + \lambda_x dx + \lambda_y dy + \lambda_u du + \lambda_v dv + \lambda_m dm + dx_f = 0$

$dy_f = dy_i = 0$

or $\quad \cancel{C_f = 0 = C} \quad \lambda_x = +1 \quad \lambda_{u_f} = 0 \quad \lambda_{v_f} = 0 \qquad dm_f = dm_i = 0$

ok

Weierstrass $\lambda_u\frac{c\beta}{m}\cos\theta + \lambda_v\left(\frac{c\beta}{m}\sin\theta - g\right) + \lambda_x u + \lambda_y v - \lambda_m\beta \le$

$\lambda_u\frac{c\beta^*}{m}\cos\theta^* + \lambda_v\left(\frac{c\beta^*}{m}\sin\theta^* - g\right) + \lambda_x u + \lambda_y v - \lambda_m\beta^*$

$$\lambda_u\frac{c\beta}{m}\cos\theta + \lambda_v\frac{c\beta}{m}\sin\theta - \lambda_m\beta \le \lambda_u\frac{c\beta^*}{m}\cos\theta^* + \lambda_v\left(\frac{c\beta^*}{m}\sin\theta^*\right) - \lambda_m\beta^*$$

Over a null thrust arc $\beta = 0$ $\qquad K = \lambda_m - \frac{c}{m}\left[\lambda_u\cos\theta + \lambda_v\sin\theta\right] \qquad K \le 0$

$K\beta \ge K^*\beta^*$ only if $\quad \lambda_m \le \frac{c}{m}\lambda_i l_i^* = \frac{c}{m}[\lambda_u^2 + \lambda_v^2]^{1/2}$

Over an intermed $0 \le \beta^* \le \beta_{max}$ only poss if $\quad \lambda_m = \frac{c}{m}[\lambda_u^2 + \lambda_v^2]^{1/2}$

Over a max $\beta = \beta_{max}$ only if $\quad \lambda_m \ge \frac{c}{m}[\lambda_u^2 + \lambda_v^2]^{1/2} \qquad K \ge 0$

Since θ's are already defined $\qquad K \ge 0$

at final   first integral $\lambda_x = -1$ $\lambda_{\dot y_f} = 0$ $\lambda_y = 0$

$$0 = \lambda_y \dot y - \lambda_m \beta$$ if intermed then

$$\lambda_m = \frac{c}{m}\left[\lambda_u^2 + \lambda_v^2\right]^{1/2} = 0 \quad \therefore \quad \lambda_y = 0 \quad \text{or} \quad \dot y = 0$$

but $\dot y \neq 0$ $\lambda_y = 0$ ∴ $C_2 = 0$ ∴ no contraints would exist on y

this is a contradiction to $y = 0$ @ end of flight from transversality

if max thrust $\lambda_m \geq 0$ ~~$\dot y = \frac{\lambda_m \beta}{\lambda_y}$ Assuming~~ $\lambda_y > 0$

~~$\dot y \geq 0$ $y \geq 0$ but this is not possible since~~

~~this is not possible~~ since integration for $\dot\lambda_m$ shows $\lambda_m = c\left[\frac{(T-t)}{m} + \int_0^T \frac{dt}{m}\right]\sqrt{1+C_2^2}$

and $\int_0^T \frac{dt}{m}$ $\dot y = \frac{\lambda_m \beta}{\lambda_y}$ if $\lambda_y > 0$ $\tan\theta < 0$ θ in 2nd q

$\dot y > 0$ $y > 0$ this is not possible.

∴ final arc must be null thrust arc. ~~Since rocket velocity is~~

initially velocity must be positive

$$0 = -\dot x_f + \lambda_y \dot y_f - \lambda_m \beta \qquad \therefore \quad \lambda_y \dot y = \dot x \qquad C_2 = (\dot x_f/\dot y_f)$$

$\tan\theta = \frac{\lambda_v}{\lambda_u} = -\frac{\dot x}{\dot y}$ $\dot x/\dot y < 0$

90°   ~~if $\dot x = \dot y = 0$ this is impossible~~

Max thrust arc $0 = +\dot x_1 - \lambda_y \dot y_1 = -\lambda_m \beta_{max}$ $\dot x - \lambda_y \dot y < 0$

if $\lambda_\gamma = 0$ $K_\beta = 0$ $\dot K_\beta = 0$ but $\dot K_\beta = -\frac{c}{m}\left(\lambda_x \cos\theta + \lambda_y \sin\theta\right)$

for $\dot K_\beta = 0$ then $\lambda_x^2 + \lambda_y^2 = 0$ this is not possible ∴ $\lambda_\gamma \neq 0$

∴ $\gamma = 0$ $\beta = \beta_{max}, \beta$

from first integral $-\dot x_f + \lambda_y \dot y_f - \lambda_m \beta = 0$ if $\beta = \beta_{max}$ $\lambda_m \geq \frac{c}{m}\left[\lambda_u^2 + \lambda_v^2\right]$

$$\boxed{\lambda_y \dot y_f - \dot x_f} = +\lambda_m \beta_{max} \leq 0 \qquad \lambda_y = \frac{\dot x_f - \lambda_m \beta_{max}}{\dot y_f}$$

$$\dot y_f = \frac{\dot x_f - \lambda_m \beta_{max}}{\lambda_y}$$

Assume $\lambda_y > 0$ $dm = -\beta dt$

$$u = \int \frac{c\beta}{m}\cos\theta \, dt = \frac{c}{\sqrt{1+C_2^2}}\int_0^T \frac{dm}{m} = \frac{c}{\sqrt{1+C_2^2}}\left[\ln m \Big|_0^T\right] = \frac{c}{\sqrt{1+C_2^2}} \ln \frac{m_1}{m_0} \qquad \dot x_f < 0$$

$$\beta_{max} \lambda m_f = \frac{c}{\ } \ln \frac{m_0}{m_1} \sqrt{1+c_2^2} \qquad \dot{y}_f = \frac{\frac{c}{\sqrt{1+c_2^2}} \ln \frac{m_1}{m_0} + c \ln \frac{m_1}{m_0} \sqrt{1+c_2^2}}{c_2}$$

$$= \frac{c}{\sqrt{1+c_2^2}} \ln \frac{m_1}{m_0} \left[\frac{1 + 1 + c_2^2}{c_2} \right]$$

if $c_2 > 0$ $\quad \dot{y}_f < 0$ $\qquad \dot{y}_f = \frac{c}{\sqrt{1+c_2^2}} \ln \frac{m_1}{m_0} \left[\frac{2+c_2^2}{c_2} \right]$

K? ! physically impossible to max. range $[\,W \gg T\,]$

if $c_2 < 0$ $\quad \dot{y}_f > 0$ again physically impossible for $\dot{y} > 0$ for impact to occur.

$\therefore$ last phase reached by coasting.

At junction pt m is discont. $\therefore$ $\quad K = 0 \qquad c_- = c_+ \qquad \lambda_+ = \lambda$

$\therefore$ integration of eq for λ_m leads to

$m = m_0 - \beta_{max} t \qquad 0 \le t \le t_1 \qquad t_1$ is burnout

$m = m_1 \qquad t_1 \le t \le T$

$$\lambda m = -c\sqrt{1+c_2^2} \left[\frac{T-t}{m} + \frac{1}{\beta_{max}} \ln \frac{m}{m_1} \right] \qquad 0 \le t \le t_1$$

$$\lambda m = -\frac{c\sqrt{1+c_2^2}}{m_1} \left[T - t_1 \right] \qquad t_1 \le t \le T$$

$\lambda x = -1$ everywhere $\qquad t_1 = \frac{m_0 - m_1}{\beta_{max}}$

$\lambda y = c_2$

$\lambda u = -(T-t) \qquad 0 \le t \le T$

$\lambda v = c_2 (T-t)$

$\gamma = 0$

$\tan\theta = -c_2$

$\beta = \beta_{max} \qquad 0 \le t \le t_1$

$\beta = 0 \qquad t_1 \le t \le T$

$$\lambda \gamma = -\frac{c\sqrt{1+c_2^2}}{\beta_{max}^2} \ln \frac{m}{m_1} \qquad 0 \le t \le t_1$$

$$\lambda \gamma = c\sqrt{1+c_2^2} \left[\frac{t_1 - t}{m_1} \right] \qquad t_1 \le t \le T$$

$$\dot{v} = \frac{c\beta}{m}\sin\theta - g$$

$$v = c\sin\theta \ln\frac{m_0}{m} - gt \qquad 0 \le t \le t_1$$

$$v = c\sin\theta \ln\frac{m_0}{m_1} - gt \qquad t_1 \le t \le T$$

$$\dot{u} = \frac{c\beta}{m}\cos\theta$$

$$u = c\cos\theta \ln\frac{m_0}{m} \qquad 0 \le t \le t_1$$

$$u = c\cos\theta \ln\frac{m_0}{m_1} \qquad t_1 \le t \le T$$

$$\dot{y} = v$$

$$y = \frac{c\sin\theta}{\beta_{max}}\left\{ m \ln\frac{m}{m_0} + \beta t \right\} - gt^2/2 \qquad 0 \le t \le t_1$$

$$= \frac{c\sin\theta}{\beta_{max}}\left\{ \bar{M} \ln\frac{m_1}{m_0} + \beta_{max} t_1 \right\} - gt^2/2 \qquad t_1 \le t \le T$$

$$\bar{M} = m_0 - \beta_{max} t$$

$$\dot{x} = u$$

$$x = \frac{c\cos\theta}{\beta_{max}}\left\{ m \ln\frac{m}{m_0} + \beta t \right\} \qquad 0 \le t \le t_1$$

$$= \frac{c\cos\theta}{\beta_{max}}\left\{ \bar{M} \ln\frac{m_1}{m_0} + \beta t_1 \right\} \qquad t_1 \le t \le T$$

if $\quad \text{BRAVO} = C*\sin\theta*\ln m_1/m_0 \qquad \text{CHARLY} = -\frac{C*\sin\theta}{\beta_{max}}\left[M_0*\ln m_1/m_0 + \beta_{max}*t_1 \right]$

$\quad \text{ALPHA} = G/2$

then
$$T = \frac{-\text{BRAVO} + \sqrt{(\text{BRAVO})^2 - 4*\text{ALPHA}*\text{CHARLY}}}{2*\text{ALPHA}}$$

from 1st Integral
$$C_2 = \frac{\dot{x}_f}{\dot{y}_f} = \frac{c\cos\theta \ln m_0/m_1}{c\sin\theta \ln m_0/m_1 - gT} = \frac{c\cos\theta \ln m_0/m_1}{-\sqrt{(\text{BRAVO})^2 - 4*\text{ALPHA}*\text{CHARLY}}}$$

$$\cos\theta = \frac{C_2}{-\sqrt{1+C_2^2}} \qquad \sin\theta = \frac{1}{\sqrt{1+C_2^2}}$$

$$\sin\theta = \frac{C_2}{-\sqrt{1+C_2^2}} \qquad \sin\theta^* = \frac{C_2}{\sqrt{1+C_2^2}}$$

$$-gT = +c\sin\theta^* \ln m_1/m_0 - \sqrt{c^2\sin^2\theta^* \ln^2\frac{m_1}{m_0} + \frac{2gc\sin\theta^*}{\beta_{max}}\left(m_0\ln\frac{m_1}{m_0} + \beta_{max}t_1\right)}$$

$$C_2 \dot{y}_f - 1\,\dot{x}_f = 0 \qquad C_2 = \frac{\dot{x}_f}{\dot{y}_f} = \frac{c\cos\theta^* \ln\frac{m_0}{m_1}}{c\sin\theta^* \ln\frac{m_0}{m_1} - gT}$$

$$\tan\psi = C_2 = -\frac{\dot{x}_f}{\dot{y}_f} \qquad -C_2 =$$

$$y = \frac{c\sin\theta}{\beta_{max}}\left[m_0\ln\frac{m_1}{m_0} - \beta_{max}t\ln\frac{m_1}{m_0} + \beta_{max}t_1\right] - gt^2/2$$

$$\frac{c\sin\theta}{}\left[\frac{m_0}{\beta_{max}}\ln\frac{m_1}{m_0} - t\ln\frac{m_1}{m_0} + t_1\right] - gt^2/2$$

$$gt^2/2 + c\sin\theta\ln\frac{m_1}{m_0}\,t - c\sin\theta\left[\frac{m_0}{\beta_{max}}\ln\frac{m_1}{m_0} + t_1\right] = 0$$

$$T = \frac{c\sin\theta\ln m_0/m_1 + \sqrt{c^2\left(\ln\frac{m_0}{m_1}\right)^2\sin^2\theta + \frac{2gc\sin\theta}{\beta_{max}}\left[m_0\ln\frac{m_1}{m_0} + \beta_{max}t_1\right]}}{g}$$

$$C_2 = \frac{\dot{x}_f}{\dot{y}_f} = \frac{c\cos\theta\ln m_0/m_1}{c\sin\theta\ln m_0/m_1 - c\sin\theta\ln m_0/m_1 - \sqrt{c^2\left(\ln\frac{m_0}{m_1}\right)^2\sin^2\theta + \frac{2gc\sin\theta}{\beta_{max}}\left[m_0\ln\frac{m_1}{m_0} + \beta_{max}t\right.}}$$

$$\tan^2\theta = \frac{c^2\cos^2\theta\,(\ln m_0/m_1)^2}{c^2\left(\ln\frac{m_0}{m_1}\right)^2\sin^2\theta + \frac{2gc\sin\theta}{\beta_{max}}\left[m_0\ln\frac{m_1}{m_0} + \beta_{max}t_1\right]}$$

$$\sin^4\theta\left\{c\ln B\right\}^2 + 2gc\sin^3\theta\left\{\frac{-m_0}{\beta_{max}}\ln\frac{m_0}{m_1} + t_1\right\} = \left\{c\ln B\right\}^2\cos^4\theta$$

Call $B = m_0/m_1$ $\qquad D = 2gct_1$

$A = c\ln B$ $\qquad \therefore \quad A^2\sin^4\theta + \sin^3\theta\left\{AE + D\right\} = A^2(1 - 2\sin^2\theta + \sin^4\theta)$

$E = -\frac{2gm_0}{\beta_{max}}$ $\qquad \sin^3\theta + \frac{2A^2}{D+EA}\sin^2\theta - \frac{A^2}{D+EA} = 0$ ✓✓✓

```
      DIMENSION  U(100), V(150), X(150), Y(150), MA(100), ULAM(100), VLAM(100)
                                                    REAL MA,MZERO
100   READ (5,1)    , C, MZERO, CAPT, BETA, G, EPSI
  1   FORMAT
      B =  MZERO /(MZERO - BETA*CAPT)
      D =  2*G*C*CAPT
      E =  2*G*MZERO/BETA  *(-1.00)  <-- changes
      THETA = 30
      OMEGA = THETA* (4* ATAN(1.00))/ 180.
10    FTHETA = (SIN(OMEGA))**3 + ((2*A**2)/(D+E*A))*(SIN(OMEGA))**2 - ((A**2)/(D+EA))
      DF     = (1.5* SIN(OMEGA) + (2*A**2)/(D+E*A))* SIN(2*OMEGA)
      OMA    = OMEGA - FTHETA/DF
      IF ( ABS(DF) .LE. EPSI) STOP
      IF ( ABS(OMA - OMEGA) - EPSI) 20,20,15
15    OMEGA = OMA
      GO TO 10
                                                    (A)
20    THETA = OMA * 180./(4* ATAN(1.00))   -->   T=0
30                                               I=1
      AA = C* ALOG( MZERO/(MZERO -BETA*T))
      BB = C*T*(ALOG (MZERO / (MZERO - BETA*T)) + 1)
      CC =(C*MZERO/ BETA) * ALOG(MZERO/(MZERO - BETA*T))
      U(I) = AA* COS(OMA)            MA(I) = MZERO - BETA*T
      V(I) = AA * SIN (OMA) - G*T
      Y(I) = (BB-CC)* SIN(OMA) - G* T**2/2.   <-- change
      X(I) = (BB-CC) *COS(OMA)
      ULAM (I) = (T-CAPT) - (V1+R)/g
      VLAM (I) = (V1/R) * ULAM(I)
      IF ( T.EQ.CAPT) GO TO  40
        T=T+10
```

(A)

```
AA  =  C*ALOG(MZERO/(MZERO-BETA*CAPT))
BB  =  C*CAPT*(ALOG(MZERO/(MZERO-BETA*CAPT))+1)
CC  =  (C*MZERO/BETA)*ALOG(MZERO/(MZERO-BETA*CAPT))
U1  =  AA*COS(OMA)
V1  =  AA*SIN(OMA)-G*CAPT
Y1  =  (BB-CC)*SIN(OMA)-G*CAPT**2/2.      changes
X1  =  (BB-CC)*COS(OMA)
R   =  SQRT(V1**2+2*G*Y1)
FLTT = CAPT+(V1+R)/G
N   =  IFIX(FLTT/10+1)

      GO TO 30
40    M=I
50    T=T+10
      I=I+1
60
      V(I) = V1-G*(T-CAPT)
      X(I) = X1+U1*(T-CAPT)
      Y(I) = Y1+V1*(T-CAPT)-0.5*G*(T-CAPT)**2
      IF (I.LE.(N-2)) GO TO 50
      IF (I.EQ.N) GO TO 70
         T=FLTT
         I=I+1                         XLAM = -1,   YLAM = U1/R
      GO TO 60
70
      WRITE (6,75) , MZERO, BETA, C, CAPT, FLTT, XLAM, YLAM, THETA
75    FORMAT ( 4(5X,E15.7)/4(5X,E15.7))
      DO 80 I=1,M
80    WRITE (6,85) I, ULAM(I), VLAM(I), U(I), V(I), X(I), Y(I), MA(I)
85    FORMAT (5X,I5, 7(2X,E14.7))
      M=M+1
      DO 90 I=M,N
90    WRITE (6,95) I, V(I), X(I), Y(I)
95    FORMAT (5X,I5,57X,3(5X,E14.7))

      GOTO 100
```

B

DIMENSION U(100), V(100), X(150), Y(150), MA(100), ULAM(100), VLAM(100)

REAL MA, K, MZERO

100 READ (5,1) K, C, WZERO, CAPT, WBETA, G, EPSI

1 FORMAT

.88*MZERO/BETA = CAPT

MZERO = WZERO/G

BETA = WBETA/G

MONE = MZERO - BETA*CAPT

A = (BETA*C/K)*(1.0 - ((~~MZERO-BETA*CAPT~~ MONE)/MZERO)**(K/BETA))

B = G/(BETA-K) * ((~~MZERO-BETA*CAPT~~ MONE) - MZERO*((~~MZERO-BETA*CAPT~~ MONE)/MZERO)**(K/BETA))

D = (C*MZERO/K)*(K/(K+BETA) + β/(K+β) * ((~~MZERO-BETA*CAPT~~ MONE)/MZERO)**(K/BETA)
- (~~MZERO-BETA*CAPT~~ MONE)/MZERO)

E = G*MZERO**2/(BETA*(K-BETA))*(0.5*(MONE/MZERO)**2 - BETA/(K+BETA)*(MONE/MZERO)
**(1+K/BETA) - (K-BETA)/(2.*(BETA+K)))

T = 10.00

10 ETA = 1.0 - EXP(-K*T/MONE)

DELTA = (MONE*G*T - MONE/K*(B+MONE*G/K)*ETA - E)/(D + MONE*A*ETA/K)

OMA = ASIN(DELTA)

FTO = TAN(OMA) - A*COS(OMA)*(1-ETA)/(MONE*G*ETA/K - (A*SIN(OMA)
+B)(1-ETA))

DTHDT = ((D+MONE*A*ETA/K)*(MONE*G*ETA/K - B*(1-ETA)) - (MONE*G*T/K
- MONE/K*(B+MONE*G/K)ETA - E)*A*(1-ETA))/((D+MONE*A*ETA/K)
**2*COS(OMA))

GAMMAA = (MONE*G*ETA/K - (A*SIN(OMA)+B)*(1-ETA))*(SIN(~~THETA~~ OMA)*DTHDT
+ K/MONE*COS(~~THETA~~ OMA))*A*(1-ETA)

GAMMAB = (A*COS(~~THETA~~ OMA)*(1-E))*(G + K/MONE*(A*SIN(~~THETA~~ OMA)+B) -
A*COS(~~THETA~~ OMA)*DTHDT)*(1-E)

DFTO = SEC(~~THETA~~ OMA)**2*DTHDT + (GAMMAA+GAMMAB)/(MONE*G*ETA/K -

(A* sin(OMA) + B)*(1 - ETA))**2

TNEW = T - FTO/DFTO

IF (ABS(DFTO).LE.EPSI) STOP

IF (ABS(TNEW - T) - EPSI) 20,20,15

15 T = TNEW

GO TO 10

20 THETA = OMA * 180./(4.* ATAN(1.00))

~~U1 = A*COS(OMA)~~

~~V1 = A*Sin(OMA) + B~~

~~X1 = D* COS(OMA)~~

~~Y1 =~~

FLTT = CAPT + T

N = IFIX(FLTT/10. + 1)

T = 0

I = 1

V1 = A* Cos(OMA)

V1 = A* Sin(OMA) + B

30 MONE = MZERO - BETA*T

recopy A, B, D, E

U(I) = A* cos(OMA)

V(I) = A* sin(OMA) + B

X(I) = D* cos(OMA)

Y(I) = D* Sin(OMA) + E

VLAM(I) =

MA(I) = MZERO - BETA*T

VLAM(I) =

IF (T.EQ.CAPT) GO TO 40

T = T+10.

I = I+1

GO TO 30

$$X_u = \left\{ \frac{\cancel{Km_1} - \cancel{m_1K} - m_1\beta + (K+\beta)m_1 e^{-\frac{kt_1}{m_1}}}{(K+\beta)K} \right\} \left(\frac{m_1}{m}\right)^{K/\beta} - \frac{mK}{K(K+\beta)}$$

$$- \frac{1}{(K+\beta)K} \left\{ \left[m_1\beta - (K+\beta)m_1 e^{-\frac{kt_1}{m_1}} \right] \left(\frac{m_1}{m}\right)^{K/\beta} + mK \right\}$$

$R = m \ddagger \sqrt{c}$

$\frac{\ldots}{\ldots} = \frac{c}{m v}$

$\cancel{\frac{du}{dt} = -\frac{c}{m} \dot{m} \cos\theta - \frac{ku}{m}}$

Suppose that at burnout (i.e. at $t = T$), $x = x_1$, $y = y_1$, $u = u_1$, $v = v_1$. For present purposes may measure t from burnout. We consider this phase. Then $m = m_1$ (const.)

$$(kv + m_1 g)/k = \left(kv_1 + \frac{m_1}{k} g\right) e^{-\frac{kt}{m_1}}$$

when $x_1 + \frac{m_1 u_1}{k}$

$$\frac{dx}{dt} = u_1 e^{-\frac{kt}{m_1}}$$

$$\boxed{x = x_1 + u_1 \frac{m_1}{k}\left[1 - e^{-\frac{kt}{m_1}}\right]}$$

agrees with S.&G.

t measured from t

$$\left(1 - e^{-\frac{kt_1}{m_1}}\right)\frac{m_1}{k}\left(v_1 + \frac{m_1 g}{k}\right) = \frac{m_1}{k} g t_1 - y_1$$

$$1 - e^{-\frac{kt_1}{m_1}} = \frac{\left(\frac{m_1 g}{k} t_1 - y_1\right)\frac{k}{m_1}}{v_1 + \frac{m_1 g}{k}}$$

There is no [illegible] that the approx. 1.

If $\frac{k}{m_1}$ is very large:

[illegible] $= g t_1 - y_1 \frac{k_1}{m_1}$

Suppose we assume $\frac{k}{m_1}$ very large & $\frac{m_1 g t_1}{k} \ll y_1$.

It may be best to suppose very small drag, so that can expand.
Perhaps even assume that t_1 is unaffected by drag.

Looking at S.9 G. p. 146, eq. 6.227, this implies

$$\cancel{t}g \gg \cancel{t}\frac{k}{m_1}v_1 .$$ or $m_1 g \gg k v_1$

i.e. drag $\ll$ wt.

Means esp. if v_1 not very large.

Still, can agree that this is O.K. for suff. small k.
After all, we are looking for effect of drag. So it appears all that leads
first to this. Then t_1 is as in orig. $\cancel{p}$ 1 d/d problem (zero drag)

$$x_R = x_1 + \frac{u_1 m_1}{k}\left\{1 - \left(1 - \frac{k t_1}{m_1} + \ldots\right)\right\} = \text{as before.}$$

So to a first approx., jump-off point is as before.

$$\frac{1}{n} = \cancel{k}\,\frac{m}{k}$$

$$m_1 g \gg k v_1$$

$$e^{\ln m/k}$$

$$\frac{m_1 g - k(v_1 + r)}{m_1} e^{nt - n_1 T} - \frac{mg}{k\cancel{}}$$

$$m_1 g \qquad g e^{nt - n_1 T}$$

(11)

From Bell (Aero. Quart. Aug. 1965)
a Lawden (?)

Rocket missile launched with zero initial vel. at $t=0$; burnout occurs at a known instant $t=T$. Assume const. acceleration, f, caused by the motor thrust is const. Find thrust dir. to maximize total range of rocket.

Sol: $\varphi_1 = \dot{u} - f\cos\theta = 0$; $\varphi_2 = \dot{v} - f\sin\theta + g = 0$

$\varphi_3 = \dot{x} - u = 0$, $\varphi_4 = \dot{y} - v = 0$. (where $f = -c\dfrac{dm}{m\,dt}$)

End conds: $x_0 = y_0 = u_0 = v_0 = t_0 = 0$; $t_1 = T$.

To maximize $h = x_1 + \dfrac{u_1}{g}\left[v_1 + \sqrt{v_1^2 + 2gy_1}\right]$.

Write $F = \lambda_u(\dot{u} - f\cos\theta) + \lambda_v(\dot{v} - f\sin\theta + g) + \lambda_x(\dot{x} - u) + \lambda_y(\dot{y} - v)$. Variables are u, v, x, y, θ.

Euler-Lag. eqs for $F \Rightarrow$:

$$\dot{\lambda}_u = -\lambda_x;\quad \dot{\lambda}_v = -\lambda_y;\quad \dot{\lambda}_x = 0;\quad \dot{\lambda}_y = 0;$$

$$\lambda_u f\sin\theta - \lambda_v f\cos\theta = 0.$$

Transversality cond $\Rightarrow$ (since $dt_0 = dt_1 = 0$),

$$\left[\lambda_u du + \lambda_v dv + \lambda_x dx + \lambda_y dy\right]_0^1 + dx_1 + \frac{du_1}{g}\left[v_1 + r\right] + \frac{u_1}{g}\left[dv_1 + \frac{1}{r}\left(v_1 dv_1 + g\,dy_1\right)\right] = 0.$$

This $\Rightarrow$: $(\lambda_u)_1 = -\dfrac{(v_1 + r)}{g}$; $(\lambda_v)_1 = -\dfrac{u_1(v_1 + r)}{gr}$ (Cont'd)

$(\lambda_x)_1 = -1$, $(\lambda_y)_1 = -\frac{u_1}{r}$, where $r \equiv \sqrt{v_1^2 + 2gy_1}$

Sol. δ) Euler-Lag. eqs: $\lambda_u = at+b$, $\lambda_v = a't+b'$,

$$\lambda_x = -a, \quad \lambda_y = -a', \quad \tan\theta = \frac{a't+b'}{at+b}.$$

From the transversality cond: $a=1$, $a' = \frac{u_1}{r}$, $b = -k$,

$b' = -\frac{u_1 k}{r}$, where $k \equiv T + \frac{(v_1 + r)}{g}$. Whence

find $\tan\theta = \frac{u_1}{r}$. $\therefore \theta =$ const. throughout.

Bell does second variation to imply this indeed gives max. range.
Lawden shows that Weierstrass cond → θ acute & pos. (∴ θ must be ~~const~~ continuous).
Values at end are not specified in advance. To obtain explicit
solution: $u_1 = fT\cos\theta$; $v_1 = (f\sin\theta - g)T$

$v = (f\sin\theta - g)t = \dot{y}$. $\therefore y_1 = \frac{(f\sin\theta - g)T^2}{2}$

$$r = T\sqrt{(f\sin\theta - g)^2 + g(f\sin\theta - g)}$$

$$\therefore \tan\theta = \frac{u_1}{T\sqrt{\ }}\ ; \quad \tan^2\theta = \frac{f\cos^2\theta}{(f\sin\theta - g)\sin\theta}$$

Set [illegible]

must require $\tan\theta = \frac{u_1}{r}$, & thereby find θ.

(cont'd)

get $f(1-\sin^2\theta)^2 = \sin^3\theta\,(f\sin\theta - g)$, or (3)

$f(1-2\sin^2\theta+\sin^4\theta) = f\sin^4\theta - g\sin^3\theta$, or

$f(1-2\sin^2\theta) + g\sin^3\theta = 0$, or

$\sin^3\theta - 2\left(\frac{f}{g}\right)\sin^2\theta + \left(\frac{f}{g}\right) = 0.$

If $\left(\frac{f}{g}\right) \gg 1$, then approximately, $\sin^2\theta = \frac{1}{2}$, $\theta \approx 45°$.

$x = \frac{1}{2} f t^2 \cos\theta$, $y = \frac{1}{2}(f\sin\theta - g)t^2$.

$\therefore$ the opt. trajectory is a str. line at angle φ to horizontal, where $\tan\varphi = \tan\theta - \frac{g}{f}\sec\theta$.

~~If~~ ~~and~~ ~~z~~ = out. the Max. range will be $fT^2\left(\frac{f}{g}\cot\theta - \frac{1}{2}\cos\theta\right)$ (using $r = \frac{u_1}{\tan\theta}$).

I Include in Intro: Basically, an elementary survey.
At least 3 types of procedures for analysis of optimization problems: a) Calc. of vars. (b) Adjoint Eqs. (c) Pontryagin Max. Princ. ~~You~~ will emphasize (a) & (b).
Might give some idea of what has been done by others: e.g. Lawden, Bell, Leitmann, Miele, Faulkner, Bliss, Hoffman, Thomson, Pontryagin.

II Calc. of Vars. Problems of Bolza, Mayer & Lagrange, with emphasis on Mayer Prob.
Euler - Lag. eqs, transversality condition, Corner condition, Legendre - Clebsch, Weierstrass conds. Try to give clear idea of what these do (e.g. nec., suff., etc?), & how they can be derived.

III A Problem with ~~an~~ unbounded controls.
(a) "Bell's" problem. ~~same~~ without drag.
(b) Problem (a) with drag. Assume D ~ v^2 f ellips. - Might then still be able to use Cartesian coords.
Note that if in this problem (as we are assuming) $f = \text{const.}$, then

$$\frac{dm}{m} = -\frac{f}{c}\,dt, \text{ whence } m = m_0 e^{-\frac{f}{c}t}.$$

(May see Synge & Griffith for $D \sim v$, pp. 141-142).
~~(c) If feasible, as~~ in (a), ~~assume a different~~

IV A Problem with bounded controls.

(a) Miele one-dim. problem p. 128

(b) Range problem: Miele p. 141

V Method of adjoint Eqs.

Showing specifically for ~~Illustrate with~~ the max. range problem, i.e. max. $x(T)$ p. 41 of Faulkner. (Is this same as Miele range problem?). Would be desirable, if same, to show that get same answers for the optimal control.