Extremization of Range for a Gwentini

we must extremize $X=\int_i^f(\varphi dV+\psi dm)$ with the constraint $t=\int_i^f(\varphi_1 dV+\psi_1 dm)=c$

$\therefore$ if $\varphi_* = \varphi+\lambda\varphi_1$ $\psi_* = \psi+\lambda\psi_1$ then $\omega_*(V,m,\lambda)= \frac{\partial\psi_*}{\partial V}-\frac{\partial\varphi_*}{\partial M}$

then $\omega_* = \frac{V^2+(\lambda+c)V+2\lambda c}{kV^3}$

$\omega_*\to 0$ along a constant velocity line if $V^2+(\lambda+c)V+2\lambda c\to 0$

for max range subarcs along $\omega_*=0$ & boundary subarcs $\beta=0$, $\beta=\beta_{max}$

their combinations described by boundary condits & isoperimetric constraint.

$X=A(\lambda)$ $t=B(\lambda)$

Calculus of Var in Applied Aerodynamics & Flight Mechanics

Bolza Problem - one independent variable

End conditions are separated.

Problem: Consider a class of fns. $y_k(x)$ $k=1,\dots,n$ satisfying $\varphi_j(x,y_k,\dot y_k)=0$

$j=1,\dots,p<n$

which involve $n-p$ degrees of freedom (difference bet no of dep variable & number of equations)

Assume these fns must be consistent with end-condits.

$\omega_r(x_i,y_{k_i})=0$ $r=1,\dots,q$

$\omega_r(x_f,y_{kf})=0$ $r=q+1,\dots,s\le 2n+2$

find that special set which minimizes the ful form

• - diff with respect to independent variable

$$\psi\equiv[G(x,y_k)]_i^f+\int_{x_i}^{x_f}H(x,y_k,\dot y_k)dx$$

Solution: can be treated by use of Lagrange mult $\lambda_j(x)$ $j=1,\dots,p$

where $F=H+\sum_{j=1}^{p}\lambda_j\varphi_j$ and the extremal must satisfy

$$\frac{d}{dx}\left(F,\dot y_k\right)-F,_{y_k}=0 \quad k=1,\dots,n.$$

Note: problems whose conditions are prescribed in part at init & final pts. must be dealt with trial & error techniques.

Corner Conditions are known as Erdmann-Weierstrass corner-condit.

Notice that the Euler-Lagrange eq are subjected to $2n+2$ b.c. of which s are supplied by the end condits while $2n+2-s$ are supplied by transversality condi.

The Weierstrass condit is useful for systems of strong var while the Legendre-Clebsch condit for weak variations

Weierstrass - ψ attains min if $\forall$ (x, y etc) satisfy $\Delta F-\sum_{k=1}^{n}\frac{\partial F}{\partial\dot y_k}\Delta\dot y_k\ge 0$

Legendre - Clebsch cond weak variation $\min$ if $\sum_{k=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial \dot{y}_k \partial \dot{y}_j} \delta\dot{y}_k \delta\dot{y}_j \geq 0$

consistent with $\sum_{k=1}^{n} \frac{\partial \varphi_j}{\partial \dot{y}_k} \delta\dot{y}_k = 0 \quad j = 1, \dots, p$

Mayer Problem $H \equiv 0 \qquad F = 0$

Lagrange Prob $G \equiv 0$ & transversality condition $dG = 0$ (dh in Schim paper)

Transformation of a Lagrange Problem into a mayer problem

if $\psi \equiv \int_{x_i}^{x_f} F(x, y, \dot{y})\,dx$ and $x, y\big|_i^f$ are given

rewrite $\dot{z} - F(x, y, \dot{y}) = 0 \implies \psi \equiv z_f - z_i$

if $\psi \equiv \int_{x_i}^{x_f} H(x, y, \dot{y}, \ddot{y})\,dx$ assume $\varphi \equiv \dot{y} - z = 0$

$\implies \psi \equiv \int_{x_i}^{x_f} H(x, y, z, \dot{z})\,dx$ & is lagrange type

Problems Involving Inequalities

$\psi = \int_{x_i}^{x_f} F(x, y, \dot{y})\,dx$ subject to $\dot{y} \geq K$

this can be reduced to writing a constraint $\varphi \equiv \dot{y} - K - z^2 = 0$

if it is subjected to $K_1 \leq \dot{y} \leq K_2$ write constraint

$\varphi \equiv (\dot{y} - K_1)(K_2 - \dot{y}) - z^2 = 0$

Optimum Trajectories for Rocket Flight in a Vacuum

Generalized Approach - (a) flat earth approximation $g = c$
(b) flight in Vacuum
(c) rocket engine exit velocity = const
(d) engine capable of delivering mass flows bet lower & upper limit

$$\varphi_1 \equiv \dot{x} - u = 0$$
$$\varphi_2 \equiv \dot{y} - v = 0$$
$$\varphi_3 \equiv \dot{u} - c\beta\cos\psi/m = 0$$
$$\varphi_4 \equiv \dot{v} + g - c\beta\sin\psi/m = 0$$
$$\varphi_5 \equiv \dot{m} + \beta = 0$$
$$\varphi_6 \equiv \beta(\beta_{max} - \beta) - \alpha^2 = 0$$

φ_6 comes from $0 \le \beta \le \beta_{max}$

• derivative wrt time

eight dependent var $(x, y, u, v, m, \beta, \alpha, \psi)$
t independent, 8 var 6 diff eq $\Rightarrow$ 2 degrees of freedom
requirements are to be imposed on $\beta(t)$ & $\psi(t)$

∴ in the class of fns $x(t), y(t), u(t), v(t), m(t), \beta(t), \alpha(t), \psi(t)$ & certain prescribed End conditions whose number < 12 find that particular set which minimizes the difference $\Delta G = G_f - G_i$ where $G = G(t, x, y, u, v, m)$

$$F = \lambda_1(\dot{x} - u) + \lambda_2(\dot{y} - v) + \lambda_3(\dot{u} - c\beta\cos\psi/m) + \lambda_4(\dot{v} + g - c\beta\sin\psi/m) + \lambda_5(\dot{m} + \beta)$$
$$+ \lambda_6(\beta[\beta_{max} - \beta] - \alpha^2)$$

Use of Euler-Lagrange Equations lead to

$$\dot{\lambda}_1 = 0 \;;\; \dot{\lambda}_2 = 0 \;;\; \dot{\lambda}_3 - \lambda_1 = 0 \;;\; \dot{\lambda}_4 - \lambda_2 = 0$$

$$\dot{\lambda}_5 - \left[\lambda_3 c\beta\cos\psi/m^2 + \lambda_4 c\beta\sin\psi/m^2\right] = 0 \;;\; -\left[-\lambda_3 c\cos\psi/m - \lambda_4 c\sin\psi/m + \lambda_5 + \right.$$
$$\left.\lambda_6\langle\beta(-1) + (\beta_{max} - \beta)\rangle\right] = 0 \;;\; + 2\lambda_6\alpha = 0 \;;\; + \lambda_4\frac{c\beta}{m}\cos\psi - \lambda_3\frac{c\beta}{m}\sin\psi = 0$$

thus $\lambda_1 = C_1$; $\lambda_2 = C_2$; $\lambda_3 = C_1 t + C_3$; $\lambda_4 = C_2 t + C_4$;

$$\dot{\lambda}_5 = \frac{c\beta}{m^2}\left[\lambda_3\cos\psi + \lambda_4\sin\psi\right] \;;\; (c/m)\left[\lambda_3\cos\psi + \lambda_4\sin\psi\right] - \lambda_5 = \lambda_6(\beta_{max} - 2\beta)$$

$$\lambda_4\cos\psi - \lambda_3\sin\psi = 0 \qquad \boxed{\tan\psi = \frac{C_2 t + C_4}{C_1 t + C_3}}$$

tangent of inclination of thrust wrt horizon is a bilinear fn of time.

Since F is not formally a function of time $-F + \sum \frac{\partial F}{\partial \dot{y}_k}\dot{y}_k = C \Rightarrow$

$$C = \lambda_1 u + \lambda_2 v - \lambda_4 g + \left\{(c/m)\left[\lambda_3\cos\psi + \lambda_4\sin\psi\right] - \lambda_5\right\}\beta - \lambda_6[\beta(\beta_{max} - \beta) - \alpha^2$$

(d) if fnl is $\ni \partial G/\partial x = 0$ & if final x coord is free transversality leads to $\lambda_{1f} = 0 \Rightarrow C_1 = 0$

$\tan\psi = \frac{C_2}{C_3} t + \frac{C_4}{C_3}$ tangent of inclination of thrust wrt horizon is linear fn of time.

(e) if $\partial G/\partial y = 0$ and if final altitude is free transversality leads to $\lambda_{2f} = 0 \Rightarrow C_2 = 0$ & that.

$\tan\psi = \frac{C_4}{C_1 t + C_2}$ or cotangent of inclination of thrust wrt horizon is a linear fn of time.

Sequence of Subarcs

if corner conditions exist $F_{\dot{y}_{K+}} = F_{\dot{y}_{K-}}$ $\lambda_1 \rightarrow \lambda_5$ are continuous

$$\left(-F + \sum_{K=1}^{n} F_{\dot{y}_K} \dot{y}_K\right)_- = \left(-F + \sum_{K=1}^{n} F_{\dot{y}_K} \dot{y}_K\right)_+ \quad C \text{ is const}$$

for each junction pt. Since u and v & m. are continuous fns of time a discontinuity can only exist in the mass flow iff $(K_\beta)_- = (K_\beta)_+ = 0$ while that in thrust direction can exist iff $(K_\psi)_- = (K_\psi)_+ = 0$
discontinuity by π exists since $K_\psi = K_{\psi+\pi}$
from the Legendre - Clebsch Condition & the Sixth Euler-Lagrange Eq

$$\Delta G \text{ is min if } \frac{2K_\beta}{2\beta - \beta_{max}} \left[(\delta\beta)^2 + (\delta\alpha)^2\right] + \frac{c\beta}{m} K_\psi (\delta\psi)^2 \geq 0$$

Everywhere

thus $K_\beta \gtrless 0$ if $\beta = \begin{cases} \beta_{max} \\ 0 \end{cases}$

thrust direction is an optimum iff $K_\psi \geq 0$

if they are employed in combination (a) no more than 3 subarcs may enter into the extremal arc & (b) discontinuity in thrust direction may occur only iff $\lambda_3 = \lambda_4 = 0$

Totality of Extremal Arcs

if equations of motion are integrated subject to $\beta = {}^{max}_{0}$

for β_{max} $\quad m e^{\left(\frac{V}{c} - \frac{mg}{c\beta_{max}}\right)} = \text{Const.} \rightarrow m e^{V/c}$ for $\beta_{max} \Rightarrow \infty$

$\beta = 0 \quad m = \text{const.}$

Max Altitude Increase

assume burning program is determined, Eq of motion integrated subject to $y_i, v_i, v_f = 0$

if final dimensionless altitude, unit thrust to weight ratio, propellant mass ratio

$$\eta_f = h_f g/c^2 \qquad \tau_i = c\beta_{max}/m_i g \qquad \zeta = (m_i - m_f)/m_i$$

peak altitude is found $\eta_f = \frac{1}{2}\log^2(1-\zeta) + \dfrac{\zeta + \log(1-\zeta)}{\tau_i}$

for constant accel - zero thrust $\beta \sim m \rightarrow \beta = 0 \qquad \eta_f = \dfrac{\tau_i - 1}{2\tau_i}\log^2(1-\zeta)$

when $\tau_i \rightarrow \infty \qquad \eta_f = \frac{1}{2}\log^2(1-\zeta)$

Maximum Range $\qquad G = -X$

$t_i = X_i = y_i = u_i = v_i = 0 \qquad m_i = \text{given} \qquad y_f = 0 \qquad m_f = \text{given}.$

use of euler-Lagrange, transversality, & switching fns

(a) extremal has one & only 1 corner pt $\Rightarrow$ 2 subarcs $<^{\text{max thrust}}_{\text{0 thrust.}}$

the thrust direction during powered flight is constant & $\perp$ to final velocity at end of coasting phase.

$$\xi_f = x_f g/c^2, \quad \tau_i = c\beta_{max}/m_i g \qquad \zeta = (m_i - m_f)/m_i$$

the thrust direction solves $\dfrac{2\sin^3\psi}{2\sin^2\psi - 1} + \dfrac{\tau_i \log^2(1-\zeta)}{\zeta + \log(1-\zeta)} = 0$

and the range is $\xi_f = \dfrac{\cos\psi}{2\sin^3\psi}\log^2(1-\zeta)$ and $\xi_f = A(\tau_i, \zeta)$

Constant accel - zero thrust leads to $\dfrac{\sin^3\psi}{2\sin^2\psi - 1} - \tau_i = 0$ or $\xi_f = B(\tau_i, \zeta)$

for $\tau_i \rightarrow \infty \qquad \xi_f = \log^2(1-\zeta)$

since F is not a fn of time explicitly first integral becomes.

$$(\lambda_1 \cos\gamma + \lambda_2 \sin\gamma) V - g\left(\lambda_3 \sin\gamma + \frac{\lambda_4}{V}\cos\gamma\right) + \beta K_\beta - \frac{K_L}{m} = C$$

if $K_\beta = c/m\left[\lambda_3 \cos\epsilon + \lambda_4/_V \sin\epsilon\right] - \lambda_5$

$K_\epsilon = \lambda_3 \cos\epsilon + (\lambda_4/V)\sin\epsilon$

$K_L = \lambda_3 D - (\lambda_4/V) L.$

the transversality condition becomes.

$$\left[dG - C dt + \lambda_1 dx + \lambda_2 dh + \lambda_3 dV + \lambda_4 d\gamma + \lambda_5 dm\right]_i^f = 0$$

$\lambda_1 = C_1$ from $\lambda_6 \alpha = 0$ if $\beta = 0$ $\beta = \beta_{max}$ when $\alpha = 0$

if $\lambda_6 = 0$ $K_\beta = 0$, $\dot{K}_\beta = 0$

thus the arc is composed of subarcs made up of β_{max} , β_{min} , β_{var}.

Thrust direction

for non trivial solution for λ_3, λ_4 then

$$\begin{vmatrix} \sin\epsilon \frac{c\beta}{m} & -\frac{\cos\epsilon \, c\beta}{mV} \\ \frac{1}{m}\partial D/\partial L & -\frac{1}{mV} \end{vmatrix} = 0$$

$$\sin\epsilon\left[-\frac{c\beta}{m^2 V}\right] + \cos\epsilon\left(\frac{\partial D}{\partial L}\right)\left(\frac{c\beta}{m^2 V}\right) = 0$$

$$\boxed{\epsilon = \arctan(\partial D/\partial L)}$$

if $D = f_1(V,h) + f_2(V,h) L^2$

$\epsilon = \arctan(2BL)$ if $2BL \ll 1$

$\epsilon = 2BL$

Sequence of Subarcs.

from first integral since $\lambda_1 \to \lambda_5$ & C are continuous at the junction's then mass flow may be discont iff $(K_\beta)_- = (K_\beta)_+ = 0$

in the thrust direction by π $(K_\epsilon)_- = (K_\epsilon)_+ = 0$

in the lift iff $(K_L)_- = (K_L)_+ = 0$ providing $(L/D)_- = (L/D)_+$

from thrust direction $(\partial D/\partial L)_- = (\partial D/\partial L)_+$

it is observed that the last 2 eq allow $L_- = L_+$ $\therefore$ the optimum lift program is a continuous fn of time

Variational Problems with Bounded Control Variables .. G. Leitmann

Discuss limited to trajectories traveling in force fields which give rise to accels depending on position & time only, prime interest: gravitational force field. Flight in atmosphere involve forces which are velocity dependent as well.

Control variables considered subject to inequality constraints are β, thrust acceleration, propuls power.

Consider $G = G(q_{ji}, q_{jf})$ where $q_j = x, y, u, v, m, t$

Assumptions

1. rocket is particle of variable mass
2. thrust magnitude $T = -c\dot{m}$
3. instantaneous change in thrust direction
4. acceleration of fns of position & time $\ddot{x} = f(x,y,t)$, $\ddot{y} = f(x,y,t)$

Planar flight & equations written relative to inertial space

$$\dot{u} = \ddot{x} + \frac{c\beta}{m}\cos\psi \qquad \dot{x} = u \qquad \dot{m} = -\beta$$

$$\dot{v} = \ddot{y} + \frac{c\beta}{m}\sin\psi \qquad \dot{y} = v$$

$x, y, u, v, m, \beta, c, \psi$

5 eq 3 degrees $c(t), \beta(t), \psi(t)$ to adjust

β limited systems most rockets $c = \text{const}$, $\beta_{min} \leq \beta \leq \beta_{max}$

$\therefore \quad (\beta - \beta_{min})(\beta_{max} - \beta) - \gamma^2 = 0$ this introduces 1 equation & 1 unknown more γ. $\therefore$

the equations of Euler are

$$\dot{\lambda}_1 + \lambda_3 = 0, \quad \dot{\lambda}_3 + \lambda_2 \ddot{y}_{,x} + \lambda_1 \ddot{x}_{,x} = 0, \quad \frac{c\beta}{m^2}(\lambda_1\cos\psi + \lambda_2\sin\psi) - \dot{\lambda}_5$$

$$\dot{\lambda}_2 + \lambda_4 = 0, \quad \dot{\lambda}_4 + \lambda_2 \ddot{y}_{,y} + \lambda_1 \ddot{x}_{,y} = 0, \quad -K_\beta + \lambda_7(\beta_{max} + \beta_{min} - 2\beta) = 0$$

where $K_\beta = \frac{c}{m}(\lambda_1\cos\psi + \lambda_2\sin\psi) - \dot{\lambda}_5$, $\quad -2\lambda_7\gamma = 0$, $\quad \frac{c\beta}{m}(\lambda_1\sin\psi - \lambda_2\cos\psi) = 0$

$$\therefore \quad \tan\psi = \frac{\lambda_2}{\lambda_1} \qquad \sin\psi = \pm\lambda_2/\sqrt{\lambda_2^2 + \lambda_1^2} \qquad \cos\psi = \pm\lambda_1/\sqrt{\lambda_2^2 + \lambda_1^2}$$

ambiguity of sign because $\psi = \psi(t)$, $\psi = \psi(t) + \pi$

note that if $\lambda_7 = 0$ $\beta_{min} < \beta < \beta_{max}$; if $\gamma = 0$ &/or $\lambda_7 = 0$ | $\beta = \beta_{min}$ or $\beta = \beta_{max}$

Since the control variables ψ and β do not appear in derivated form, they may possess a finite no. of discontinuities - implying that $\dot{u}, \dot{v}$ are discont. Which again imply the use of the Weierstrass - Erdmann Corner Cond.

$$F_{,\dot{q}_{j-}} = F_{,\dot{q}_{j+}} \qquad \lambda_{1\text{-}5_-} = \lambda_{1\text{-}5_+}$$

$$F - \sum \dot{q}_j F_{,\dot{q}_j} = C_- = C_+$$

Must go to Weierstrass E fn to find condi for G is min $\therefore E \geq 0$

this condition is equivalent to maxing $L \equiv \beta\left[\frac{c}{m}(\lambda_1\cos\psi + \lambda_2\sin\psi) - \lambda_5\right]$

in order for L to be max $\frac{\partial L}{\partial\psi} = 0$, $\frac{\partial^2 L}{\partial\psi^2} \leq 0$ which leads to the + sign for $\sin\psi$ & $\cos\psi$

radial field $\vec{F} = \nabla\phi$ where $\phi \sim \frac{1}{r}$ and the inequality (existence of flow rate) becomes $\cos^2\theta \le \frac{1}{3}$, where θ is the angle between the thrust & local vertical.

Note: for flight in inverse square rad. field $\dot{K} = -\frac{c}{m}(\lambda_3\lambda_1 + \lambda_2\lambda_4)(\lambda_1^2+\lambda_2^2)^{-1/2}$ and is discont at $\lambda_1 = \lambda_2 = 0$ when ψ experiences a discontinuous jump

<u>for propulsive power limited systems.</u>

$\alpha_{min} \le \alpha \le \alpha_{max}$ where $\alpha = \beta c^2 = 2P$. Thrust magnitude can be varied by adjusting β, c. Choose control var ψ, α, c 6 eq $u, v, x, y, m, c, \psi, \alpha, \gamma, \beta$

t indep

Get same Euler eq as before 1-6, 8 however

also get $\frac{\alpha}{mc^2}(\lambda_1 \cos\psi + \lambda_2 \sin\psi) - 2\lambda_m \frac{\alpha}{c^3} = 0$ F_c

$\frac{1}{mc}(\lambda_1 \cos\psi + \lambda_2 \sin\psi) - \frac{1}{c^2}\lambda_m + \lambda_8(2\alpha - \alpha_{min} - \alpha_{min}) = 0$ F_α

again if $\lambda_8 = 0$ $\alpha_{min} < \alpha < \alpha_{max}$ Same transversality & corner conditions

$\gamma = 0$ $\alpha = \alpha_{min}$ $\alpha = \alpha_{max}$.

a min G leads to $L \equiv \alpha\left[\frac{1}{mc}(\lambda_1\cos\psi + \lambda_2 \sin\psi) - \frac{1}{c^2}\lambda_m\right]$ max

wrt ψ, c, α

$\frac{\partial L}{\partial\psi}$ leads to $\tan\psi = \frac{\lambda_2}{\lambda_1}$ $\frac{\partial L}{\partial c}$ leads to

$\frac{\partial L}{\partial\alpha} = \frac{1}{mc}(\lambda_1^2+\lambda_2^2)^{1/2} - \frac{1}{c^2}\lambda_m \ge 0$ if $\frac{\partial L}{\partial \alpha} = 0$ $\lambda_1 = \lambda_2 = \ldots = 0$ in a non zero int

opt prog. $\alpha = \alpha_{max}$

Thrust Accel. Limited Systems. $a = \frac{c\beta}{m}$ problems of min G & bounded accel. power limited rocket with constant power

let $\mu = \frac{2P}{m}$ then 8 deps 6 eq

$\dot{u} = \ddot{x} + a\cos\psi$
$\dot{v} = \ddot{y} + a\sin\psi$
$\dot{x} = u$
$\dot{y} = v$
$\dot{\mu} = a^2$ $-(a - a_{min})(a - a_{max}) = \gamma^2$ if zero thrust accel is nonzero interval

then $G(x, y, u, v, \mu, \psi, \gamma, \alpha_{g}, \ldots)$ is same Euler 1-4 $\dot{\lambda}_\mu = 0$

$a(\lambda_u \sin\psi - \lambda_v \cos\psi) = 0$

$\lambda_u \cos\psi + \lambda_v \sin\psi + 2\lambda_\mu a + \lambda_\gamma(2a - a_{max} - a_{min}) = 0$

ψ, a are control var.

$\lambda_\gamma \gamma = 0$

Transversality
$dG + \left[\lambda_u du + \lambda_v dv + \lambda_x dx + \lambda_y dy + \lambda_\mu d\mu - C\,dt\right]_1^f = 0$
where $C = \lambda_u \dot{u} + \lambda_v \dot{v} + \lambda_x \dot{x} + \lambda_y \dot{y} + \lambda_\mu \dot{\mu}$

Since external force accel are known we can use bang bang $\kappa > 0$ $\beta = \beta_{max}$
$\kappa < 0$ $\beta = \beta_{min}$

allowing use of $K = \frac{c}{m}\left(\lambda_u^2 + \lambda_v^2\right)^{1/2} - \lambda_m$ it is clear that the initial portion of the trajectory must be powered. If integration is started at $t = t_f$, one must get information about latter part. From

$$K = \frac{c}{m}\left(\lambda_u^2 + \lambda_v^2\right)^{1/2} - \lambda_m \qquad K_f = -\lambda_{m_f} \text{ since } \lambda_{u_f} = \lambda_{v_f} = 0$$

however $\lambda_{m_f} = \frac{\partial m_i}{\partial m_f}$ depends on transversality conds

since $K_f = -\lambda_{m_f}$ & $K_f < 0$ for $\beta = \beta_{min}$ then $\frac{\partial m_i}{\partial m_f} > 0$

for final portion of trajectory to be unpowered. If integration starts at $t = 0$ the five parameter iteration & requirement to match $G_f = 0$ since t_f is not spec. will be reduced to a 3 parameter iteration without need to match $G_f = 0$

Eq. 1-4 of Euler (first set) can be integrated to yield

$$\lambda_{u_i} = \frac{\lambda_{xf}}{\epsilon} \sin \epsilon t_f$$

$$\lambda_{v_i} = \frac{\lambda_{yf}}{\epsilon\sqrt{2}} \sinh \epsilon\sqrt{2}\, t_f$$

$$\lambda_{x_i} = \lambda_{xf} \cos \epsilon t_f$$

$$\lambda_{y_i} = \lambda_{yf} \cosh \epsilon\sqrt{2}\, t_f, \text{ subst of}$$

$$\dot{u} = X + \frac{c\beta}{m}\cos\psi \qquad \dot{x} = u \qquad \dot{m} = -\beta \qquad \text{into } G_i = 0 \text{ leads to}$$

$$\dot{v} = Y + \frac{c\beta}{m}\sin\psi \qquad \dot{y} = v$$

$$\dot{m}_i = c\beta_{max}\left(\lambda_{u_i}^2 + \lambda_{v_i}^2\right)^{1/2}\left[-\beta_{max} + \lambda_{v_i}\left(g_0 - 2\epsilon^2 y_i\right) - \lambda_{x_i} u_i\right]^{-1}$$

which satisfies $G_f = 0$. In view of λ_u, λ_v when integrated from $t = t_f$ $\lambda_{u_f} = \lambda_{v_f} = 0$ are met. Assuming $t_f, \lambda_{xf}, \lambda_{yf}$ one has $(\lambda_u, \lambda_v, \lambda_x, \lambda_y, m)_i$ from above. Integration then from $t = 0$ to $t = t_f$ is made & values of x, y, m must be matched.

Appendix

Desired to determine state var. $y_j(x)$ and control var. $u_k(x)$ $\ni$

$$G = G(x_i, x_f, y_1(x_i) \ldots y_m(x_i), y_1(x_f), \ldots, y_m(x_f)) \text{ be min}$$

subject to contraints $\phi_j = y_j' - f_j(x, y_1 \ldots, y_m, u_1 \ldots u_n)$ $j = 1, 2, \ldots, m$ and at most $2m+1$ end conditions involving x & y_j, as well as inequality constraints on control var.

Propulsive Efficiency of Rockets G. Leitmann

Propulsive Efficiency - Point Fn.

$\eta =$ rate of increase of rocket kinetic energy / rate of increase of rocket & jet kinetic energy

$$= \frac{-\dot{m} c v}{-\dot{m} c v - \frac{1}{2}\dot{m}(c-v)^2} = \frac{2R}{1+R^2}$$ c = exhaust speed, v rocket speed, $R = v/c$

$\eta =$ rate of change of kinetic energy of rocket / rate of change of rocket & jet kinetic energy

$$= \frac{-\dot{m} c v + \frac{1}{2}\dot{m} v^2}{-\frac{1}{2}\dot{m} c^2} = R(2-R)$$ depends on instantaneous operating condit

$\eta = \eta_{max} = 1$ can occur 1 time in flight when c = const & v varies

or c varies & $\eta = \eta_{max}$ can be made to be satisfied

for $c = v$ one must look at the frame of reference

1. Constant Exhaust Speed - at which value of c will η_{mean} occur

$R_1 = v_1/c$ mean value

$$\eta_{mean} = \frac{1}{R_1}\int_0^{R_1} \frac{2R}{1+R^2}\,dR = \frac{\ln(1+R_1^2)}{R_1}$$

$$\eta_{mean} = \frac{1}{R_1}\int_0^{R_1} (2R - R^2)\,dR = R_1 - \frac{1}{3}R_1^2$$

thus setting $\frac{d\eta}{dR_1} = 0 \implies \eta_m$ is max when $R_1 = 1.98$ $\quad \frac{m_0}{m_1} = 7.17$

$R_1 = 1.50 \quad \frac{m_0}{m_1} = 4.48$

2. Variable Exhaust one can find exhaust speed program $c(v)$ & $m_0/m_1 \ni \eta$ is max. one must put $R = 1$ into

$m\,dv + c\,dm = 0 \longrightarrow \frac{m_0}{m_1} = \frac{v_1}{v_0} \Leftarrow$ this must be satisfied for prop. η to be max

however η is not a fn of coordinate system & $v_0 \neq 0$.

Use of equation of motion into the preceding integral leads to

$$\Delta T_j = -\frac{1}{2}\int_{m_0}^{m_1} \left(m\frac{dv}{dm} + v\right)^2 dm \quad \text{gives Euler Lagrange}$$

$$m\, v_{mm} + 2\, v_m = 0 \qquad \text{along with} \quad \frac{m_0}{m_1} = \frac{v_1}{v_0}$$

gives $dv = dc \Rightarrow c - c_0 = v - v_0$ where $c_0 = \dfrac{v_1 - v_0}{m_0/m_1 - 1}$

Direct Solution ΔT_j must be min $= 0$ in one of inferential coord system

when $\Delta T_j = 0$
$c = \tilde{v}$ speed in that coord system from $m\,dv = -c\,dm$

then $d(m\tilde{v}) = 0$ & $\dfrac{m_0}{m_1}\tilde{v}_0 = \tilde{v}_1$ & $\left(\dfrac{m_0}{m_1} - 1\right)\tilde{v}_0 = \tilde{v}_1 - \tilde{v}_0$

but $\tilde{v}_0 = c_0$ & for linear motion $\tilde{v} - \tilde{v}_0 = v - v_0 \Rightarrow c_0 = \dfrac{v_1 - v_0}{m_0/m_1 - 1}$

which leads to $c - c_0 = v - v_0$

In terms of prop eff $\eta = \dfrac{m_0/m_1 - 1}{m_0/m_1}$

comparison of $\eta = \dfrac{\ln^2(m_0/m_1)}{m_0/m_1 - 1}$ shows that for $c = \text{const}$ η is less than that for opt variable exhaust speed prog $\forall\ m_0/m_1$.

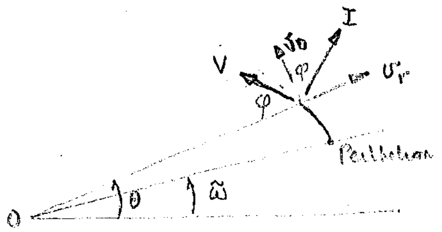
For same ΔT_1 ratio of $\dfrac{\Delta E_{const}}{\Delta E_{var}} = \dfrac{(m_0/m_1 - 1)^2}{(m_0/m_1)\ln^2(m_0/m_1)} > 1$

η possesses max < 1 w.r.t. m_0/m_1 for const exhaust whereas the var. exhaust η increases monotonically as $m_0/m_1 \uparrow$. Use of var. exhaust speed as opposed to const exhaust speed becomes more pronounced at higher mass ratios.

Impulsive transfer between elliptical Orbits — Lawden, D. F.

It was shown in a previous article by the same author that bang-bang conditions were attained if at certain junction points impulsive thrusts were applied & coasting existed between these pts.

11.2.



if γ/r^2 is attraction/unit mass at distance r from O.
Orbit of body is $l/r = 1 + e\cos(\theta - \tilde{\omega})$ (1)
l is semilatus rectum, e eccentricity, $\tilde{\omega}$ is longitude of perihelion.

let $\frac{1}{r} = s,\ \frac{1}{l} = p,\ \frac{e}{l} = q$ (2) then (1) becomes $s - p = q\cos(\theta + \tilde{\omega})$ (3)

$\boxed{p, q, \tilde{\omega} \text{ determine orbit}}$ the position of a body given by $(\frac{1}{s}, \theta)$. The velocity components v_θ, v_r are given by $r\dot{\theta}, \dot{r}$

$$\dot{r} = \frac{le\sin(\theta-\tilde{\omega})\frac{d\theta}{dt}}{(1+e\cos(\theta-\tilde{\omega}))^2} = \frac{le}{l^2/r^2}\sin(\theta-\tilde{\omega})\frac{d\theta}{dt} = q\sin(\theta-\tilde{\omega})\,r^2\frac{d\theta}{dt}$$

$$r^2\frac{d\theta}{dt} = l\sqrt{\gamma/l} = \sqrt{\frac{\gamma}{p}} \quad \therefore \quad v_r = q\sqrt{\gamma/p}\,\sin(\theta-\tilde{\omega}) \quad (4)$$

as for $r\dot{\theta}$

$$-\frac{l}{r^2}\frac{dr}{dt} = -e\sin(\theta-\tilde{\omega})\frac{d\theta}{dt}$$

$$\dot{r} = q\sin(\theta-\tilde{\omega})\,r\,v_\theta$$

$$q\sqrt{\frac{\gamma}{p}}\sin(\theta+\tilde{\omega}) = q\sin(\theta-\tilde{\omega})\,r\,v_\theta \qquad v_\theta = \sqrt{\frac{\gamma}{p}}\left[p + q\cos(\theta-\tilde{\omega})\right] \quad (5)$$

from the figure let V = veloc. component at P making an angle φ with v_r

$$\therefore \quad V = v_\theta\sin\varphi - v_r\cos\varphi = s\sqrt{\frac{\gamma}{p}}\sin\varphi - q\sqrt{\frac{\gamma}{p}}\sin(\theta-\tilde{\omega})\cos\varphi \quad (6)$$

then $$q\sin(\theta-\tilde{\omega}) = -V\sqrt{p/\gamma}\,\sec\varphi + s\tan\varphi$$

$$= \left[s - Zp^{1/2}\right]\tan\varphi \quad \text{where } Z = V/\gamma^{1/2}\csc\varphi \quad (7,8)$$

if an impulse is applied making an angle φ with v_θ, V remains unchanged. Also (s, θ) will be relatively unaltered. Thus $p, q, \tilde{\omega}$ of the new orbit will sat. (7) and thus (3). $p, q, \tilde{\omega}$ of any orbit into which body is transferred by impulse application making φ ∡ with v_θ will satisfy (3) → 9 (7) → (10). This leaves 1 degree of freedom since in 2 eq. & 3 parameters. Magnitude of I will be 3rd eq.

Obtaining formula for Magnitude of Impulse. let U be the component of velocity in direction of Impulse ∴ $v_\theta = V\sin\varphi + U\cos\varphi$ (16)

Optimal n-Impulse transfer between 2 terminal Orbits —

going from $(p_0, q_0, \tilde{\omega}_0)$ to $(p_n, q_n, \tilde{\omega}_n)$ using n impulsive thrusts.

going from $p_0, q_0, \tilde{\omega}_0$ to $(p_1, q_1, \tilde{\omega}_1)$ I is applied at $(1/s_1, \theta_1)$, going from $(p_1, q_1, \tilde{\omega}_1)$ to $(p_2, q_2, \tilde{\omega}_2)$ applying impulse at $(1/s_2, \theta_2)$ etc.

$$\Delta U_i = \gamma^{1/2} s_i \left(p_i^{-1/2} - p_{i-1}^{-1/2}\right) \sec \varphi_i \qquad (37) \quad \text{so that}$$

$$W = \gamma^{1/2} \sum_{i=1}^{n} s_i \left(p_i^{-1/2} - p_{i-1}^{-1/2}\right) \sec \varphi_i \qquad (38)$$

since $\Delta U_i = \Delta U_i\left(p_{i-1}, q_{i-1}, \tilde{\omega}_{i-1}, p_i, q_i, \tilde{\omega}_i\right) \Rightarrow W = W\left(p_{i-1}, q_{i-1}, \tilde{\omega}_{i-1}, \ldots, p_n, q_n, \tilde{\omega}_n\right)$

or $3n-3$ variables since $(p_0, q_0, \tilde{\omega}_0, p_n, q_n, \tilde{\omega}_n)$ are constants given.

$$W \text{ is stationary iff } \frac{\partial W}{\partial p_i} = \frac{\partial W}{\partial q_i} = \frac{\partial W}{\partial \tilde{\omega}_i} = 0 \qquad i = 1, \ldots, n-1 \qquad (39)$$

since p_i appears in both $\Delta U_i, \Delta U_{i+1}$ $\quad \frac{\partial W}{\partial p_i} = \gamma^{1/2}\left(\frac{\partial \Delta U_i}{\partial p_i} + \frac{\partial \Delta U_{i+1}}{\partial p_i}\right)$ (40)

using eq (11.41) $\quad \frac{\partial W}{\partial p_i} = 0$ if $\frac{\partial \Delta U_i}{\partial p_i} = -\frac{\partial \Delta U_{i+1}}{\partial p_i}$ or

$$\left[\frac{s_i}{2 p_i^{3/2}} \sec^2 \varphi_i - \frac{z_i}{2 p_i} \tan^2 \varphi_i - \frac{1}{p_i^{1/2}} - \frac{1}{z_i}\right] \cos \varphi_i =$$
$$\left[\frac{s_{i+1}}{2 p_i^{3/2}} \sec^2 \varphi_{i+1} - \frac{z_{i+1}}{2 p_i} \tan^2 \varphi_{i+1} - \frac{1}{p_i^{1/2}} - \frac{1}{z_{i+1}}\right] \cos \varphi_{i+1} \qquad (42)$$

again $\frac{\partial W}{\partial q_i} = 0$ if $\frac{\partial \Delta U_i}{\partial q_i} = -\frac{\partial \Delta U_{i+1}}{\partial q_i}$ & $\frac{\partial W}{\partial \tilde{\omega}_i} = 0$ if $\frac{\partial \Delta U_i}{\partial \tilde{\omega}_i} = -\frac{\partial \Delta U_{i+1}}{\partial \tilde{\omega}_i}$

$$(s_i - p_i)\left(1 + \frac{p_i^{1/2}}{z_i}\right) \cos \varphi_i + \left(s_i - p_i^{1/2} z_i\right) \tan \varphi_i \sin \varphi_i =$$
$$(s_{i+1} - p_i)\left(1 + \frac{p_i^{1/2}}{z_{i+1}}\right) \cos \varphi_{i+1} + \left(s_{i+1} - p_i^{1/2} z_{i+1}\right) \tan \varphi_{i+1} \sin \varphi_{i+1} \qquad (43)$$

$$\left(z_i - \frac{s_i}{z_i}\right) \sin \varphi_i = \left(z_{i+1} - \frac{s_{i+1}}{z_{i+1}}\right) \sin \varphi_{i+1} \qquad (44)$$

20-23 determine $(s_i, \theta_i, z_i, \varphi_i) \ni$

$$q_{i-1} \cos\left(\theta_i - \tilde{\omega}_{i-1}\right) = s_i - p_{i-1} \qquad 45$$
$$q_{i-1} \sin\left(\theta_i - \tilde{\omega}_{i-1}\right) = \left(s_i - z_i p_{i-1}^{1/2}\right) \tan \varphi_i \qquad 46$$
$$q_i \cos\left(\theta_i - \tilde{\omega}_i\right) = s_i - p_i \qquad 47$$
$$q_i \sin\left(\theta_i - \tilde{\omega}_i\right) = \left(s_i - z_i p_i^{+1/2}\right) \tan \varphi_i \qquad 48$$

$i = 1, 2, \ldots, n-1$

$$(s-p)\left(1+\frac{p^{1/2}}{Z}\right)\cos\varphi + (s-Zp^{1/2})\frac{\sin^2\varphi}{\cos\varphi} = 0 \qquad (65)$$

$$\left(1+\frac{\Delta+p}{Zp^{1/2}}\right)\cos\varphi = 0 \qquad \text{for } \cos\varphi = 0 \quad \varphi = \frac{\pi}{2} \quad \tan\varphi = \infty \qquad (66)$$

$$\therefore \left(1+\frac{\Delta+p}{Zp^{1/2}}\right) = 0 \ (67) \quad \text{letting } x = p/s = r/1 \quad \therefore \frac{Z}{P^{1/2}} = -\left(\frac{x+1}{x}\right) \qquad 68$$

la formulation de l'exemple se trouve dans l'équation (65)

Nous trouvons que

$$\tan^2\varphi = \frac{x-1}{(x+1)(x+2)} \quad \text{si } \tan\varphi = 0 \quad 64 \text{ devient } q\sin\theta = 0$$

et puisque $\sin\theta \neq 0 \Rightarrow q = 0$ et $e = 0$ mais nous avons dit que le program de transfère se trouve élliptique ($e > 0$). Alors $\tan\varphi \neq 0$ et x doit être ≥ 1.

$\tan^2\varphi$ est max si $x = 1+\sqrt{6}$ $\therefore$ φ est égue et $\varphi_{max} = 17°38'$

(63 & 64) deviennent

$$e\cos\theta = \frac{1}{x} - 1 \qquad (70) \qquad e = q/p$$

$$e\sin\theta = \frac{x+2}{x}\tan\varphi \qquad (71) \qquad \text{transfer ellipse}$$

$$\tan\theta_0 = \frac{x+2}{x-1}\tan\varphi. \qquad (72)$$

61 & 62 deviennent

$$E\cos(\theta+\alpha) = y^2/x - 1 \qquad (73)$$

$$E\sin(\theta+\alpha) = y\frac{(x+y+1)}{x}\tan\varphi \qquad (74)$$

$$\& \ E = Q/P \ \text{terminus ellipse} \quad y = \sqrt{P/p} \qquad (75)$$

$$\tan(\theta+\alpha) = \frac{y(x+y+1)}{y^2-x}\tan\varphi \qquad (76)$$

note given E and α 1169 – 1174 determine x, y, e, θ, φ.

11.19 given $\Delta U = \gamma^{1/2}\underline{X}$ & $P^{1/2}\underline{X} = \frac{2}{x}(1-y)(y\sec\varphi)$ 77

$\therefore$ if $E = E_0$ & $\alpha = \alpha_0$ then $y = y_0$, $x = x_0$, $e = e_0$, $\theta = \theta_0$, $\varphi = \varphi_0$ (78, 79)

if $E = E_0$ & $\alpha = \pi - \alpha_0$, $y = y_0$, $x = x_0$, $e = -e_0$, $\theta = \pi - \theta_0$, $\varphi = 2\pi - \varphi_0$ (80, 81)

if $E = E_0$ & $\alpha = \pi + \alpha_0$, $y = y_0$, $x = x_0$, $e = -e_0$, $\theta = \pi + \theta_0$, $\varphi = \varphi_0$ (82, 83)

Case 2 $\quad s_1 = p_1 - q_1 \qquad s_2 = p_2 + q_2$

$$p = \tfrac{1}{2}(p_1 + p_2 - q_1 + q_2) \qquad a = p_1 - q_1 \qquad b = p_2 - q_2$$

$$q = \tfrac{1}{2}(-p_1 + p_2 + q_2 + q_1) \qquad a' = p_1 + q_1 \qquad b' = p_2 + q_2$$

using these into 11.88 & 11.89 gives Z_1 & Z_2

since $\Delta U_{21} = \gamma^{1/2} X_{21} = \gamma^{1/2} s_1 (p^{-1/2} - p_1^{-1/2}) \sec\varphi_1$

$$= \gamma^{1/2} (p_1 - q_1)\left[\frac{1}{\underbrace{(p_1 + p_2 - q_1 + q_2)}_{a+b'}^{1/2}} - \frac{1}{p_1^{1/2}}\right]$$

for case 2 only

$$= \gamma^{1/2}\sqrt{2}\, a\left[\frac{1}{(a+b')^{1/2}} - \frac{1}{(a+a')^{1/2}}\right]$$

$$\Delta U_{22} = \gamma^{1/2} X_{22} = \gamma^{1/2} s_2 (p_1^{-1/2} - p_2^{-1/2}) \sec\varphi_2$$

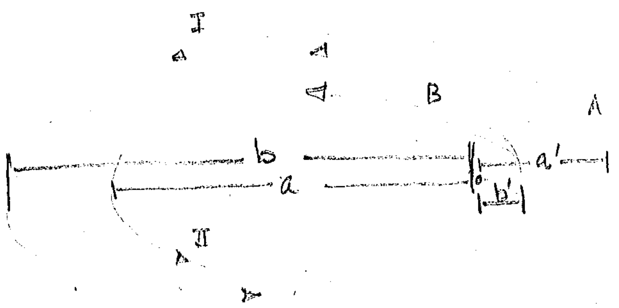
$$= \gamma^{1/2}(p_2 + q_2)\left[\frac{1}{p_1^{-1/2}} - \frac{1}{p_2^{-1/2}}\right]$$

$$= \gamma^{1/2}\sqrt{2}\, b'\left[\frac{1}{(a+b')^{1/2}} - \frac{1}{(b+b')^{1/2}}\right]$$

where , a, b, a', b' are reciprocal distances from center of attraction of the 4 apses on the terminals

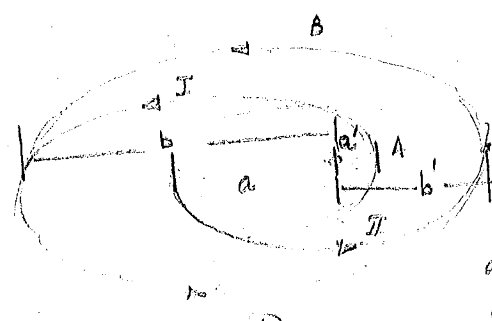
$$\therefore \quad \Delta U_{TOT} = \Delta U_1 + \Delta U_2 \quad \text{for} \quad \theta_1 = \pi \;,\; \theta_2 = 0$$

Suppose axes of terminals are in same sense & either orbits intersect or do not



$a > b$, $a' < b'$, $b' > b$, $a' > a$

(a)



(b)

$a' > b'$, $a' > a$, $a > b$, $b' > b$

For intersection (a) from case 1 and 2

$$\frac{1}{\sqrt{2}}\left(\underbrace{X_{11} + X_{12}}_{X_1} - \underbrace{X_{21} - X_{22}}_{X_2}\right) = (a'+b)^{1/2} - (a'+a)^{1/2} - (b'+b)^{1/2} + (b'+a)^{1/2}$$

$$= f(a', a, b) - f(b', a, b) \qquad 93$$

$$\text{where} \quad f(x, a, b) = (x+b)^{1/2} - (x+a)^{1/2} \qquad 94$$

from 93 if $a' < b'$ as in (a) then $X_1 < X_2$ for $\underline{a > b}$

Function $f(x,a,b)$

$$\frac{\partial f}{\partial x} = \tfrac{1}{2}(x+b)^{-1/2} - \tfrac{1}{2}(x+a)^{-1/2} \quad 99 \qquad \text{if } a>b \text{ and } x>0 \quad \frac{\partial f}{\partial x}>0$$

and $f \uparrow$ in x.

Function $g(x,a,b)$ since $g(a,a,b) = g(b,a,b) = 0$ use Rolle's theorem

$\therefore \exists$ a ξ, $a<\xi<b \ni g'(\xi)=0$ suppose $a>b>0$

$$g'(x) = \frac{x+3b}{2(x+b)^{3/2}} - \frac{x+3a}{2(x+a)^{3/2}} \quad 101 \qquad \text{if } h(x) = x^{1/2} g'(x)$$

$$h'(x) = x^{1/2} g''(x) + g'(x)\,\tfrac{1}{2}x^{-1/2} = \frac{3b(b-x)}{4x^{1/2}(x+b)^{5/2}} + \frac{3a(x-a)}{4x^{1/2}(x+a)^{5/2}} \quad 102$$

if $b \le x \le a$; therefore $h'(x)<0$ in interval $\therefore h(x)$ is decreasing

but $\exists\ \xi \ni g'(\xi)=0 \ \therefore h'(\xi)=0 \ \therefore h(\xi)=0$. $\therefore h(x)>0$

$b \le x < \xi$ and $h(x)<0$ when $\xi < x \le a$

it follows
$$\begin{array}{ll} g'(x)>0 & b \le x < \xi \\ g'(x)<0 & \xi < x \le a \end{array} \quad 103$$

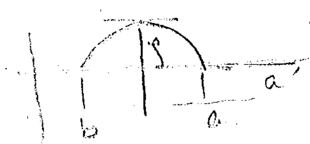


Fig 9

if x were fixed & a variable

$$k(a) = \frac{x+3a}{(x+a)^{3/2}} \quad 104 \qquad k'(a) = \frac{3(x-a)}{2(x+a)^{5/2}} \quad 105$$

$k'(a)>0 \quad x>a \implies k(a)\uparrow \quad x>a>b$ we have then

$$k(b)<k(a) \quad 106 \qquad \text{and } g'(x) = \tfrac{1}{2}\{k(b)-k(a)\}<0 \quad 107$$

for $x>a$. as $x\to\infty$ $g(x)\to 0$ this with 103 & 107

shows fig 9 as $g(x,a,b)$

Abstract

A survey has been made of existing literature dealing with the calculus of variations and its applications to trajectory, orbits & rendezvous problems?. The general theory of variational calculus is set forth and several sample problems dealing with the general theory are worked out. The results of the general theory is then put to use on put number problems dealing with trajectories, orbits rendezvous;

1) State goal of problem 1

2) " " " " 2 etc.

Introduction

In the last twenty years or so, the use of variational methods has become more pronounced, in fields dealing with the sciences and engineering. The versatility of / The calculus of variations is shown in its application ~~has been applied~~ to problems in dynamics, elasticity, optics electromagnetic theory and fluid dynamics. These methods so popular today were ~~unheard~~ uncommon ~~of~~ of in Optimal problems in ~~in the aeronautics and~~ astronautics field two decades ago (1). It is not unusual to find problems ~~when solving problems~~ in applied aerodynamics and applied mechanics being solved today by variational techniques.

See [illegible] sect

Because of the complexities of the problems encountered in actual practice, necessity often dictates the use of numerical techniques for solutions to be obtained. ~~However~~ it must be noted that the solution usually is independent of the data involved, thus allowing a problem to be analytically solved. ~~and then resolved using actual~~ This solution can be then used as a first approximation to the actual solution.

The calculus of variations is a branch of calculus whose function is similar to the theory of maxima and minima. It concerns itself with the variation of functional expressions and their behavior when minima or maxima of these expressions must be established. Whereas the theory of maxima and minima deals with distinct points, variational calculus deals with an infinite set of points which

However, problems can often be idealized and simplified in such a manner that analytical solutions may be available. Such solutions can then serve as first approximations and afford insight into related practical, though more complicated, problems.

I Basic theory by variational means.
(Euler-Lagrange, Corner conditions, Jump points, Weierstrass and PS, etc. Bolza, Mayer probs.
Exposition of the basic theory: Elsgolc, Miele, Breakwell

II ~~See Ting paper [J. Aero. Rocket Soc. 30 (1960) pp. 1013-1018~~

III
II Lawden book - Ch. 2. (perhaps some of the probs on p. 52-53).
III Fried paper Jet Propulsion 27, June 1957 pp. 641-643.
Derives $\tan \psi = a - bt$. a Miele

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$$(y'^2-1)^2 - 2y'\left[(y'^2-1)\langle y'-1+y'+1\rangle\right]$$

$$= 4y'^2(y'^2-1)$$

$$(y'^2-1)\left[y'^2-1-4y'^2\right]$$

$$(y'^2-1)\left[-1-3y'^2\right]$$

$$(y'^2-1)\left[3y'^2+1\right]$$

$(y'=\pm 1)$ soln

no $y'^2 = -\frac{1}{3}$

if $y'=0$
$y=c$

$$2(y'-1)(y'+1)^2 + 2(y'-1)^2(y'+1)$$

$$2(y'^2-1)\left[2y'\right]$$

$y'=\pm 1$ soln $y'=0$

$F - y'F_{y'}$

$y'^2 + 2xy - y^2 - 2y'y'$

$2xy - y^2 - y'^2$

$y' - y'^2\Big|_{x^*, y^*} = -$

$2xy - y^2 - y'^2$

$F_y = 2x - 2y$

$F_{y'} = 2y'$

$2xc - c^2$

$2x = \left| x = \frac{c}{2} \right|$

$2x - 2y - 2\ddot{y} = 0$

$\ddot{y} + y - x = 0$

$y = x$

Theory

General: The Bolza Problem with fixed end conditions Sec. 2

Consider the following problem:

~~Suppose~~ Consider $\exists$ a class of functions

$$y_k(x) \qquad k = 1, \dots, n$$

satisfying the constraints

$$\varphi_j(x, y_k, \dot{y}_k) = 0 \qquad j = 1, \dots, p < n$$

and also ~~are~~ consistent with the end conditions

$$\omega_r(x_i, y_{ki}) = 0 \qquad r = 1, \dots, q$$

$$\omega_r(x_f, y_{kf}) = 0 \qquad r = q+1, \dots, s \le 2n+2.$$

Find the such

That set of ~~points~~ functions y_k which minimize the functional

$$\mathcal{J} = A + I = G(x, y_k)\Big|_i^f + \int_{x_i}^{x_f} H(x, y_k, \dot{y}_k)\,dx \; * \qquad (1)$$

~~satisfies the Bolza Problem.~~

To take into account the constraints $\varphi_j(x, y_k, \dot{y}_k) = 0$ a set of variable Lagrange multipliers

$$\lambda_j(x) \qquad j = 1, \dots, p$$

can be used and a new integrand can be formed

$$F = H + \sum_{j=1}^{p} \lambda_j \varphi_j \qquad (2)$$

which must satisfy the conditions set above.[3]

Before the Bolza Problem is attempted, solution for $\mathcal{J}$ will be given for $G(x, y_k) \equiv 0$, the special case of Lagrange.

Euler – Lagrange Equations – single independent variable

Suppose $\exists$ a $y(x)$ which minimizes the integral $\int_{x_0}^{x_1} H(x, y(x), \dot{y}(x))\,dx$ and $y(x_0)$ and $y(x_1)$ are given.

---

* i indicates initial, f indicates final and the dot represents a derivative with respect to the

Choose any function $\eta(x) \in C^1$.∋. $\eta(x_0) = \eta(x_1) = 0$ for which a family of admissible functions of the form

$y^*(x,\epsilon) = y(x) + \epsilon\eta(x)$ can be formed and ϵ is a parameter ~~which is constant for any one function but which varies from function to function.~~

If one replaces $y(x)$ by $y(x)+\epsilon\eta(x)$ then it follows that the integral becomes a function of ϵ for fixed y & η: $I(\epsilon) = \int_{x_0}^{x_1} H(x, y+\epsilon\eta, \dot{y}+\epsilon\dot{\eta})\,dx = \int_{x_0}^{x_1} \tilde{H}(x, y^*(x,\epsilon), \dot{y}^*(x,\epsilon))dx$

Then $I(\epsilon)$ becomes a minimum when $\epsilon = 0$ and it follows that

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 0$$

$$\frac{dI(\epsilon)}{d\epsilon} = \int_{x_0}^{x_1}\left[\frac{\partial\tilde{H}}{\partial x}\frac{\partial x}{\partial\epsilon} + \frac{\partial\tilde{H}}{\partial(y+\epsilon\eta)}\frac{\partial(y+\epsilon\eta)}{\partial\epsilon} + \frac{\partial\tilde{H}}{\partial(\dot{y}+\epsilon\dot{\eta})}\frac{\partial(\dot{y}+\epsilon\dot{\eta})}{\partial\epsilon}\right]dx$$

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{x_0}^{x_1}\left[\frac{\partial H}{\partial y}\eta + \frac{\partial H}{\partial\dot{y}}\dot{\eta}\right]dx = 0$$

Integration by parts of $\int_{x_0}^{x_1}\frac{\partial H}{\partial\dot{y}}\dot{\eta}\,dx$ gives

$$\int_{x_0}^{x_1}\frac{\partial H}{\partial\dot{y}}\dot{\eta}\,dx = \left.\frac{\partial H}{\partial\dot{y}}\eta\right|_{x_0}^{x_1} - \int_{x_0}^{x_1}\frac{d}{dx}\left[\frac{\partial H}{\partial\dot{y}}\right]\eta\,dx$$

Because of the manner in which $\eta(x)$ was picked the first term vanishes. Therefore

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{x_0}^{x_1}\left[\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial\dot{y}}\right)\right]\eta\,dx = 0$$

Since $\eta(x)$ is arbitrary ~~need not be 0~~ $\forall x \in x_0 < x < x_1$ ~~then~~ it follows that

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial\dot{y}}\right) = 0 \qquad (3)$$

This equation is known as the Euler-Lagrange Equation.

In the case when the integral is of the form

$$I = \int_{x_0}^{x_1} H(x, y_1, y_2, \dots y_n, \dot{y}_1, \dot{y}_2, \dots, \dot{y}_n)\, dx$$

an evaluation of the integral for which a minimum is sought is done by fixing any $n-1$ variables and performing the preceding method on the remaining variable. However instead of obtaining one Euler equation, n such equations will exist, ~~on~~ namely

$$\frac{\partial H}{\partial y_k} - \frac{d}{dx}\left[\frac{\partial H}{\partial \dot{y}_k}\right] = 0 \qquad k = 1, \dots, n \qquad (4)$$

provided $y_k(x_0)$ and $y_k(x_1)$ are given.

It must be noted that the preceding work is also true for maximization of the integrals discussed. That solution which satisfies the Lagrange equation is known as an extremal; the extremal will cause the integral to be stationary, i.e., will minimize (or maximize) the integral

The Euler-Lagrange Equations - for two independent variables

Consider the following integral

$$I = \iint_D H\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dx\,dy$$ for which $\partial D = C$. Let $z(x,y)$ be an extremal of I.

All the permissible surfaces passing through the boundary of D are given by

$$z^*(x,y,\alpha) = z(x,y) + \alpha\, \eta(x,y)$$ ~~for $z(x,y)$ an extremal.~~

where $\eta(x,y) = 0$ on ∂D and α is a parameter ~~which remains constant for any function but varies from function to function~~. Substitute z^* for z, If the integral is rewritten as

$$I(\alpha) = \iint_D \tilde{H}\left(x, y, z^*(x,y,\alpha), p^*(x,y,\alpha), q^*(x,y,\alpha)\right) dx\,dy$$

for fixed $z(x,y)$ and $\eta(x,y)$.

The integral can be minimized if $\left.\frac{dI(\alpha)}{d\alpha}\right|_{\alpha=0} = 0$ as before

$$\left.\frac{dI(\alpha)}{d\alpha}\right|_{\alpha=0} = \left[\frac{\partial}{\partial\alpha}\iint_D \tilde{H}\left(x, y, z^*(x,y,\alpha), p^*(x,y,\alpha), q^*(x,y,\alpha)\right) dx\,dy\right]_{\alpha=0} = 0$$

$$= \iint_D \left(H_{,z}\,\eta + H_{,p}\,\eta_{,x} + H_{,q}\,\eta_{,y}\right) dx\,dy = 0$$

when $\alpha\eta, \alpha\frac{\partial\eta}{\partial x}, \alpha\frac{\partial\eta}{\partial y}$ are the variations of z, p, q. and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

~~Since~~ Note that

$$\iint_D \left(H_{,p}\,\eta_{,x} + H_{,q}\,\eta_{,y}\right) dx\,dy = \iint_D \left[\frac{\partial}{\partial x}\left\{H_{,p}\,\eta\right\} + \frac{\partial}{\partial y}\left\{H_{,q}\,\eta\right\}\right] dx\,dy$$

$$- \iint_D \left[\frac{\partial}{\partial x}\left\{H_{,p}\right\} + \frac{\partial}{\partial y}\left\{H_{,q}\right\}\right]\eta\; dx\,dy$$

The first double integral on the right side ~~preceding~~ ~~following~~ can be reduced by means of Green's Theorem.

Use of Green's Theorem shows that

$$\iint_D \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) dx\,dy = \int_c (N\,dy - M\,dx) = 0 \quad \text{since } \eta = 0 \text{ on } \partial D$$

Application of Green's theorem on the first double integral reduces the necessary condition for minimization to

$$\iint_D \left(H_{,z}\,\eta + H_{,p}\,\eta_{,x} + H_{,q}\,\eta_{,y}\right) dx\,dy = \iint_D \left(H_{,z} - \frac{\partial}{\partial x} H_{,p} - \frac{\partial}{\partial y} H_{,q}\right)\eta\; dx\,dy = 0$$

Since $\eta \neq 0$ except at the boundary, ~~then~~ it follows that

$$\frac{\partial H}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial q}\right) = 0 \qquad (5)$$

This defines the Euler-Lagrange equation for two independent variables. Elsgolc also makes reference to this equation as the Ostrogradski Equation.

Example 1:

Find all admissible functions, with fixed values at the end points ~~x=a and x=b~~, which make $I = \int_a^b \{(y'x)^2 + 12x^2y\}\,dx$

an extremum. Find the stationary function if the extremum is to satisfy
$y(0)=0 \quad y(1)=1$.

Solution:

By use of Euler's Equation $\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial \dot{y}}\right) = 0$, the following differential equation is obtained:

$$12x^2 - 2y'(2x) - 2x^2y'' = 0$$

If $y = Ax^m + C$, the differential equation reduces to

$$12x^2 - 4Amx^m - 2Am(m-1)x^m = 0$$

For a solution to exist $m = 2$ which gives rise to $A = 1$ and

$$y = x^2 + C \qquad \text{(extremum of the integral)}$$

Application of the end conditions leads to $C = 0$ and

$$y = x^2 \qquad \text{(stationary function)}$$

Example 2:

Derive the Ostrogradski Equation for the integral

$$I = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 2z\, f(x,y)\right] dx\,dy \quad \text{where } z = f(x,y) \text{ on } \partial D$$

and is given a priori on ∂D.[4]

Solution:

By use of the Euler-Lagrange equation

$$\frac{\partial H}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial q}\right) = 0,$$ the following differential equation is obtained

$$2f(x,y) - 2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = 0 \quad \text{or} \quad \nabla^2 z = f(x,y) \quad \text{Poisson's Equation}$$

since $z = g(x,y)$ on ∂D

Variable Boundary Conditions

Because no restrictions exist on the end conditions, a greater number of solutions to a variable end point problem exist. However, the solution which produces an extremal for this type of problem, will also be an extremal with respect to a more restricted set of curves having the same end conditions as the extremal. This fact then allows for the use of the fundamental necessary condition for extrema – the Euler - Lagrange equations derived earlier.+

The simplest variable boundary problem is the one-end-fixed problem. Consider an extremum to exist if the function $y^*(x)$ has its variable end point at (x_1, y_1). Consider a second function $y_N = y^* + \Delta y$ close to the function $y^*(x)$ formed by the movement of the variable end point from (x_1, y_1) to $(x_1 + \Delta x_1, y_1 + \Delta y_1)$. The change incurred by the integral $I = \int H(x, y, \dot{y})\,dx$ in its evaluation for $y = y^*(x)$ and $y = y_N(x)$ is

$$\Delta I = \int_{x_0}^{x_1+\Delta x_1} H(x, y_N, \dot{y}_N)\,dx - \int_{x_0}^{x_1} H(x, y^*, \dot{y}^*)\,dx$$

$$= \int_{x_1}^{x_1+\Delta x_1} H(x, y_N, \dot{y}_N)\,dx + \int_{x_0}^{x_1} \{ H(x, y_N, \dot{y}_N) - H(x, y^*, \dot{y}^*) \}\,dx$$

$$= \int_{x_1}^{x_1+\Delta x_1} H(x, y_N, \dot{y}_N)\,dx + \int_{x_0}^{x_1} (H_{,y}\,\Delta y + H_{,\dot{y}}\,\Delta \dot{y})\,dx + R$$

where Taylor's expansion is used on the second integral and the first two terms of the expansion yield the variation of H – a first-order approximation to the change of the integral from curve to curve. R is the higher order remainder term contributed by Taylor's Expansion.

+ Because no end points exist, and for the integral to be stationary it is necessary to find other conditions which must also hold in order to obtain the function ... the extremal

Use of the Mean-Value theorem on the first integral leads to

$$\int_{x_1}^{x_1+\Delta x_1} H(x, y_N, \dot{y}_N)\,dx = \Delta x_1 \left\{ H(\bar{x}, y, \dot{y}) \right\} = \left\{ H(x_1, y, \dot{y}) \right\} \Delta x_1 + \epsilon_1 \Delta x_1$$

where $\bar{x} = x_1 + \theta_1 \Delta x_1$ and $0 < \theta_1 < 1$

Integration of the second integral results in

$$\int_{x_0}^{x_1} \left\{ H_{,y}\,\Delta y + H_{,\dot{y}}\,\Delta\dot{y} \right\} dx = H_{,\dot{y}}\,\Delta y \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} \left(H_{,y} - \frac{d}{dx} H_{,\dot{y}} \right) \Delta y\, dx$$

Using the fundamental necessary condition (cf. (3)) and ~~that~~ $\Delta y \big|_{x=x_0} = 0$ (fixed point) the above reduces to

$$\int_{x_0}^{x_1} \left\{ H_{,y}\,\Delta y + H_{,\dot{y}}\,\Delta\dot{y} \right\} dx = \left[H_{,\dot{y}}\,\Delta y \right] \Big|_{x=x_1}$$

Since $\Delta y \big|_{x=x_1} \cong \Delta y_1 - \dot{y}(x_1)\,\Delta x_1$ and since $\epsilon_1, R \to 0$ when $\Delta y_1, \Delta x_1 \to 0$ the change incurred by the integral is

$$\Delta I = H_{,\dot{y}} \Big|_{x=x_1} \Delta y_1 + \left\{ H - \dot{y}\,H_{,\dot{y}} \right\} \Big|_{x=x_1} \Delta x_1 \qquad (6)$$

One requires ~~and~~ $\Delta I = 0$ for the extremal to exist. If Δy_1 and Δx_1 are independent of each other, ~~then~~ this implies

$$H_{,\dot{y}} \Big|_{x=x_1} = \left\{ H - \dot{y}\,H_{,\dot{y}} \right\} \Big|_{x=x_1} = 0 \qquad (7)$$

If $y_1 = \pi(x_1)$ then (6) would be subject to $\Delta y_1 \cong \dot{\pi}(x_1)\,\Delta x_1$; then

$$\Delta I = \left\{ H - (\dot{y} - \dot{\pi})\,H_{,\dot{y}} \right\} \Big|_{x=x_1} \Delta x_1 = 0 \quad \text{or}$$

$$\left[H - (\dot{y} - \dot{\pi})\,H_{,\dot{y}} \right] \Big|_{x=x_1} = 0 \qquad (8)$$

If a variable boundary also existed at the end point (x_0, y_0) a similar result would be obtained. The condition described in its varied form as shown by equations (6-8) is known as the transversality condition. In the preceding development infinitesimal increments in y and x of higher order were neglected. Therefore if both end points are variable, then for the integral to be stationary the following conditions must be satisfied:

$$H_{,y} - \frac{d}{dx}\left(H_{,\dot{y}} \right) = 0$$

$$H_{,\dot{y}}\Big|_{x=x_0} \Delta y_0 + \left\{ H - \dot{y} H_{,\dot{y}} \right\}\Big|_{x=x_0} \Delta x_0 = 0$$

$$H_{,\dot{y}}\Big|_{x=x_1} \Delta y_1 + \left\{ H - \dot{y} H_{,\dot{y}} \right\}\Big|_{x=x_1} \Delta x_1 = 0$$

If the integral is of the form

$$I = \int_A^B H(x, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n)\, dx$$

~~and whose~~ with variable boundary points ~~take~~ of the form

$$A(x_0, y_{10}, y_{20}, \ldots, y_{n0}) \text{ and } B(x_1, y_{11}, y_{21}, \ldots, y_{n1})$$

the conditions for which the integral becomes stationary are

$$\frac{\partial H}{\partial y_k} - \frac{d}{dx}\left[\frac{\partial H}{\partial \dot{y}_k}\right] = 0 \qquad k = 1, \ldots, n$$

$$\left\{ H - \sum_{k=1}^{n} \dot{y}_k \frac{\partial H}{\partial \dot{y}_k} \right\}\Bigg|_{x=x_0} \Delta x_0 + \sum_{k=1}^{n} \frac{\partial H}{\partial \dot{y}_k}\Bigg|_{x=x_0} \Delta y_{k_0} = 0 \qquad (9)$$

$$\left\{ H - \sum_{k=1}^{n} \dot{y}_k \frac{\partial H}{\partial \dot{y}_k} \right\}\Bigg|_{x=x_1} \Delta x_1 + \sum_{k=1}^{n} \frac{\partial H}{\partial \dot{y}_k}\Bigg|_{x=x_1} \Delta y_{k_1} = 0$$

Example 4:

Find the curve that makes

$I = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{y}\, dx$ an extremum knowing that $y(0) = 0$ and that (x_1, y_1) ~~moves~~ lies on the circle $(x-9)^2 + y^2 = 9$ (*)

Solution:

For $\Delta I = 0$

$$\Delta I = \left[H - \dot{y} \frac{\partial H}{\partial \dot{y}} \right]\Big|_{x=x_1} \Delta x_1 + \frac{\partial H}{\partial \dot{y}}\Big|_{x=x_1} \Delta y_1 = 0$$

Because (x_1, y_1) moves on a circle, then $y_1 = \pi(x_1)$ and $\Delta y_1 \cong \dot{\pi}(x_1)\Delta x_1$; in particular ~~then~~ $2y\big|_{x=x_1} \Delta y_1 + 2(x-9)\big|_{x=x_1} \Delta x_1 = 0$ or $\Delta y_1/\Delta x_1 = -\frac{(x-9)}{y}\Big|_{\substack{x=x_1\\y=y_1}}$

Use of the Euler-Lagrange equation leads to $(x-c_1)^2 + y^2 = c_2^2$.
Using the fixed end condition $y(0)=0$ results in $c_1 = c_2$
Therefore $(x-c_1)^2 + y^2 = c_1^2$. Use of the transversality condition gives

$$\left(\frac{\sqrt{1+\dot{y}^2}}{y} - \frac{\dot{y}^2}{y\sqrt{1+\dot{y}^2}} \right)_{x=x_1} \Delta x_1 + \frac{\dot{y}}{y\sqrt{1+\dot{y}^2}}\Bigg|_{x=x_1} \Delta y_1 = 0 \quad \text{or}$$

$$\Delta x_1 + \dot{y}\big|_{x=x_1} \Delta y_1 = 0 \text{ . Therefore}$$

$$\dot{y}\Big|_{\substack{x=x_1\\y=y_1}} = -\frac{\Delta x_1}{\Delta y_1}$$

Because of this result the tangent to the extremum must be perpendicular to the tangent to the curve $(x-9)^2 + y^2 = 9$ at (x_1, y_1). The solution of

$$(x-c_1)^2 + y^2 = c_1^2$$
$$(x-9)^2 + y^2 = 3^2 \quad \text{leads to} \quad x_1(18-2c_1) = 72$$

Since $\dot{y} = -1\Big/\frac{\Delta y_1}{\Delta x_1} = \frac{y}{(x-9)}\Big|_{\substack{x=x_1\\y=y_1}} = -\frac{(x-c_1)}{y}\Big|_{\substack{x=x_1\\y=y_1}}$ the following equation is found: $(x_1-c_1)(x_1-9) + y_1^2 = 0$. Since $(x_1-c_1)^2 + y_1^2 = c_1^2$, substitution for y_1^2 and solution for x_1 results in $x_1 = 9c_1/9-c_1$. Since two

Equations exist in x_1 and c_1 :

$$x_1(18-2c_1)=72$$

$$x_1 = 9c_1/9-c_1$$

solution leads to $c_1 = 4$; thus the extremal is found to be

$$y = \pm\sqrt{16-(x-4)^2} = \pm\sqrt{8x-x^2}$$

the end point (x_1, y_1) is found to be $(7.2, 2.4)$.

Extremals as a Consequence of Corners (Discontinuities)

Consider the fixed end point problem which does not obey the Euler-Lagrange equation, i.e. the integral cannot be minimized for any function. The solution to such a problem can be realized in the following manner: Let $i = 1, \ldots, n$

Suppose $\exists$ points $c_i(x_i, y_i)$ which connect the two end points by means of piecewise continuous curves. For these piecewise curves to minimize the integral they must satisfy the Euler-Lagrange equation individually. By fixing all but one of these points one obtains $n-2$ fixed-end-point-problems and two one-variable-end-point problems.

Consider now the simplest case, i.e. the existence of only one point $c_1(x_1, y_1)$ which produces two piecewise continuous curves. As in the case of the variable end point condition, the equations found are applicable in this case. Because each curve depends on the variable point if

$$I = \int_{x_0}^{x_2} H(x,y,y')dx = \int_{x_1}^{x_2} H(x,y,y')dx - \int_{x_1}^{x_0} H(x,y,y')dx \quad ,$$

then
$$\Delta I = \left(H - \dot{y}\frac{\partial H}{\partial \dot{y}}\right)\Big|_{x=x_1^-}\Delta x_1 + \frac{\partial H}{\partial \dot{y}}\Big|_{x=x_1^-}\Delta y_1 - \left[\left(H - \dot{y}\frac{\partial H}{\partial \dot{y}}\right)\Big|_{x=x_1^+}\Delta x_1 + \frac{\partial H}{\partial \dot{y}}\Big|_{x=x_1^+}\Delta y_1\right] = 0. \quad (9)$$

The negative sign is due to the fact that the variable end point for the second integral is made the lower limit.

If Δx_1 & Δy_1 are independent (9) gives the two corner conditions

$$\left(H - \dot{y}\frac{\partial H}{\partial \dot{y}}\right)\Big|_{x=x_1^-} = \left(H - \dot{y}\frac{\partial H}{\partial \dot{y}}\right)\Big|_{x=x_1^+} \tag{10}$$

$$\frac{\partial H}{\partial \dot{y}}\Big|_{x=x_1^-} = \frac{\partial H}{\partial \dot{y}}\Big|_{x=x_1^+} \tag{11}$$

It should be noted that the first test used to determine whether or not these equations should be used is whether if the integrand $H \geq 0$[4]. If so the integral $I \geq 0$.

Example 1:

Find the solution for 1 discontinuity for the problem

$$I = \int_0^4 (\dot{y}-1)^2(\dot{y}+1)^2\,dx \qquad \text{and} \quad y(0)=0 \qquad y(4)=2$$

Solution:

Because the integrand always is greater than or equal to zero, then equations (10,11) can be used. Equation (10) leads to

$$-(\dot{y}^2-1)(3\dot{y}^2+1)\Big|_{x=x_1^-} = -(\dot{y}^2-1)(3\dot{y}^2+1)\Big|_{x=x_1^+}$$

Equation (11) leads to

$$(4\dot{y})(\dot{y}^2-1)\Big|_{x=x_1^-} = (4\dot{y})(\dot{y}^2-1)\Big|_{x=x_1^+}$$

The only solution to both equations, ~~and~~ for a discontinuity to exist, is

$$\dot{y}_{x=x_1^-} = 1 \qquad \dot{y}_{x=x_1^+} = -1$$

$$\dot{y}_{x=x_1^-} = -1 \qquad \dot{y}_{x=x_1^+} = 1$$

Solution of the first set leads to

$$y = x^- + C_1 \qquad y = -x^+ + C_2$$

application of $y=0$ @ $x^-=0$ $\qquad$ $y=2$ @ $x^+=4$ $\qquad$ leads to

$$y = x^- \qquad y = -x^+ + 6$$

The domain for which these hold is found by equating the two and finding the point of intersection — in this case $x = 3$; therefore

$$y = x \qquad 0 \le x \le 3$$

$$y = -x + 6 \qquad 3 \le x \le 4$$

For the other set of solutions to the problem, the result is

$$y = -x \qquad 0 \le x \le 1$$

$$y = x - 2 \qquad 1 \le x \le 4$$

<u>Example 2:</u>

Are there any solutions with discontinuities for the problem

$$I = \int_{x_0}^{x_1} (\dot{y}^2 + 2xy - y^2)\,dx \qquad y(x_0) = y_0, \quad y(x_1) = x_1$$

<u>Solution:</u>

Consider a discontinuity at (x^*, y^*). From equation (10)

$$2xy - y^2 - \dot{y}^2 \Big|_{x=x^{*-}} = 2xy - y^2 - \dot{y}^2 \Big|_{x=x^{*+}}$$

Because x & y do not depend on whether one approaches (x^*, y^*) from the left (−) or from the right (+), the above reduces to

$$-\dot{y}^2 \Big|_{x=x^{*-}} = -\dot{y}^2 \Big|_{x=x^{*+}}$$

From equation (11)

$$2\dot{y} \Big|_{x=x^{*-}} = 2\dot{y} \Big|_{x=x^{*+}}$$

. The solution that satisfies both these equation is

$$\dot{y} \Big|_{x=x^{*-}} = -2 \qquad \dot{y} \Big|_{x=x^{*+}} = -2$$

This implies that no cusp exists; therefore no solution using discontinuities can be found.

Sufficiency Conditions for Extrema — Jacobi Condition

Before mathematical formulas are found regarding sufficiency conditions, it is necessary to define certain terms:

Consider a domain D; if for each point in the domain D $\exists(!)$ a curve $y = y(x, c_1)$ belonging to the family of curves $y = y(x, c)$ then the family of curves $y = y(x, c)$ defines a proper field. If a family of curves $y = y(x, c)$ passes through only one point $A(x_0, y_0)$ and also covers the domain D without intersecting (except at (x_0, y_0)) this family defines a central field. The parameter for the family is the slope at (x_0, y_0)

In order to obtain the condition attributed to Jacobi it is necessary to define an envelope. By the envelope of a family is meant a curve touched by all the members of the family.

Definition: a family of curves $f(x, y, c) = 0$ has an envelope

$$x = g(c), \quad y = h(c) \qquad \text{iff for each } c = c_0 \text{ the point}$$

$(g(c_0), h(c_0))$ of the curve defined above lies on the curve $f(x, y, c_0) = 0$ and both curves have the same tangent line there.

Theorem: If $f(x, y, c)$, $g(c)$, $h(c)$ are continuous and have continuous first derivatives, if $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \neq 0$, if $\left(\frac{dg}{dc}\right)^2 + \left(\frac{dh}{dc}\right)^2 \neq 0$, and if $f(g(c), h(c), c) \equiv \frac{\partial f}{\partial c}(g(c), h(c), c) \equiv 0$ then the family of curves $f(x, y, c) = 0$ has the curve $x = g(c)$ $y = h(c)$ as an envelope.

Consider initially a central field formed by a family of extremals which has the initial boundary point as the common point. The central field is defined away from the envelope. Because of the nature of the envelope any two curves of this family will intersect at a point near the envelope.

If the arc of the extremum (formed by the function satisfying the Euler-Lagrange equation) does not have a common point with the envelope, those curves close to the extremal will form a central field. If the extremal does intersect the envelope the curves close to the extremal will intersect at a point $A'(x_1, y_1)$. This point is known a the conjugate of the point $A(x_0, y_0)$. The Jacobi Condition states that an extremal to a problem exists if the extremal does not have a conjugate point lying on its arc.

To find whether or not a conjugate point exists it is necessary to find the envelope of the family of extremals. Since an extremal must satisfy the Euler-Lagrange Equation and since the envelope of the family is found by differentiating the governing equat. of the family with respect to the parameter c, then

$$\frac{\partial}{\partial c}\left[H_{,y} - \tfrac{d}{dx} H_{,\dot{y}}\right] = 0 \quad \text{or}$$

$$u\left(\frac{\partial^2 H}{\partial y^2} - \frac{d}{dx}\frac{\partial^2 H}{\partial y \partial \dot{y}}\right) - \frac{d}{dx}\left(\frac{\partial^2 H}{\partial \dot{y}^2}\dot{u}\right) = 0 \quad \text{where} \quad u = \frac{\partial y(x,c)}{\partial c} \qquad (12)$$

Therefore if a solution to this equation exists, i.e. $u = \frac{\partial y(x,c)}{\partial c}$ is found, and the solution to $u(x) = 0$ exists for $x = x_0$ (point A) as well as other points in the interval of definition, then conjugate points exist. If $u = \frac{\partial y(x,c)}{\partial c}$ is found and the only solution to $u(x) = 0$ is at $x = x_0$ then the Jacobi condition is satisfied. The existence of conjugate points indicate that the function satisfying the Euler-Lagrange Equation does not produce a minimal arc between the end points.

Example:

Determine whether or not the Jacobi condition holds for

$$I = \int_0^a (\dot{y}^2 + 2y\dot{y} - 16y^2)\,dx \qquad a > 0 \quad y(0) = 0 \quad y(a) = 0$$

Solution:

Use of the Euler-Lagrange Equation leads to $2(\ddot{y} + 16y) = 0$ or the solution $y = C_1 \sin 4x + C_2 \cos 4x$. Use of the boundary conditions lead to $C_2 = 0$ and $\frac{k\pi}{4} = a$. Therefore the interval of definition is

$$0 \leq x \leq \frac{k\pi}{4}$$

Solution to the Jacobi accessory equation

$$u\left(\frac{\partial^2 H}{\partial y^2} - \frac{d}{dx}\frac{\partial^2 H}{\partial y \partial \dot{y}}\right) - \frac{d}{dx}\left(\frac{\partial^2 H}{\partial \dot{y}^2}\dot{u}\right) = 0$$

leads to $\ddot{u} + 17u = 0$. The solution to this is

$$u(x) = D_1 \sin\sqrt{17}\,x + D_2 \cos\sqrt{17}\,x$$

The point $A(x_0, y_0)$ is the point $(0,0)$. Therefore $u(0) = 0$ leads to $D_2 = 0$. To find conjugate points set $u(x) = 0$. Therefore if $D_1 \neq 0$ then $\sqrt{17}\,x = k\pi$ or $x = \frac{k\pi}{\sqrt{17}}$ if conjugate points are to exist. Let us examine whether if conjugate points exist in the interval of definition for this problem. For any k it is found that $\frac{k\pi}{4} > \frac{k\pi}{\sqrt{17}}$; therefore there will always be k conjugate points in the interval of definition. Thus the function does not produce a minimizing arc in the interval $0 \leq x \leq \frac{k\pi}{4}$.

Sufficiency Conditions for Extremum - Weierstrass & Legendre Conditions

Consider the integral

$$I = \int_{x_0}^{x_1} H(x, y, \dot{y})\,dx$$

Also consider that the Jacobi Condition holds, which implies that the function satisfying the Euler-Lagrange Equation and also the boundary conditions is admitted into a central field formed by the extremals with center at $A(x_0, y_0)$. Now let us look at the change incurred by the integral when one moves along two different curves connecting (x_0,y_0) & (x_1,y_1). If the admitted curve is to minimize the integral between the two end points then the change incurred by the integral is

$$\Delta I = \int_{C^*} H(x, y, y')\,dx - \int_{C} H(x, y, p)\,dx \geq 0$$

where C is the minimizing curve, C^* is any neighboring curve and

$p = p(x,y)$ is the slope function of the field (therefore the minimizing curve).
The change incurred by the integral is due to the change incurred in the integrand.
Consider the following auxiliary integral (also known as Hilbert's Invariant Integral)

$$I = \int_{C^*} \left\{ H(x,y,p) + (y'-p)\frac{\partial H}{\partial p}(x,y,p) \right\} dx$$

and assume that when $y' = p$ then the integral reduces to

$$I = \int_C H(x,y,y')\,dx.$$

[illegible]

It is noted that the auxiliary integral is also the integral of an exact differential: Because exact differentials are not functions of path and because the auxiliary integral reduces to the above integral, ~~then~~

$$I = \int_C H(x,y,y')\,dx = \int_{C^*} \left[H(x,y,p) + (y'-p)\frac{\partial H}{\partial p}(x,y,p) \right] dx$$

for arbitrary arcs C^*.

Then $$\Delta I = \int_{C^*} \{H(x,y,y')\}\,dx - \int_{C^*} \left[H(x,y,p) + (y'-p)\frac{\partial H}{\partial p}(x,y,p) \right] dx$$

Therefore $$\Delta I = \int_{C^*} \left[H(x,y,y') - H(x,y,p) - (y'-p)\frac{\partial H}{\partial p}(x,y,p) \right] dx \qquad (13)$$

The integrand of ΔI is written as

$$E(x,y,p,y') = H(x,y,y') - H(x,y,p) - (y'-p)\frac{\partial H}{\partial p}(x,y,p) \qquad (14)$$

and is known as the excess function of Weierstrass.
Because the interval of definition is positive a minimum occurs when

$E(x,y,p,y') \geq 0$ and a maximum when
$E(x,y,p,y') \leq 0$. Note that these minimuma or maximuma are strong in that $p(x,y)$ can be arbitrarily large or small.

The Weierstrass Condition can also be obtained by expanding the integrand of the neighboring curve C^* about the minimized curve C, noting that the parameter of family of extremals in a central field is the slope to the curves at center. Therefore

$$H(x_0, y_0, y_0') = H(x_0, y_0, p(x_0, y_0)) + \frac{\partial H}{\partial p}(x_0, y_0, p(x_0, y_0))\,[y_0' - p(x_0, y_0)] + \epsilon(0^2)$$

However if the lower limit is made to move along the extremal curve to another point, and the change in the integrand is found, then it will be a function of the coordinate of the point; therefore it can be done for any point on the extremal curve, or

$$H(x,y,y') = H(x,y,p) + \frac{\partial H}{\partial p}(x,y,p)\{y'-p\} + \epsilon(0^2)$$

. By transposition of terms

$$H(x,y,y') - H(x,y,p) - (y'-p)\frac{\partial H}{\partial p}(x,y,p) = \epsilon(0^2) = E(x,y,p,y')$$

Returning to equation (13) and applying Taylor's Expansion with respect to $p(x,y)$ only, the integral reduces to

$$\Delta I = \int_{C^*} \frac{(y'-p)^2}{2!}\frac{\partial^2 H}{\partial \dot{y}^2}(x,y,q)\,dx \qquad \text{where } p \le q \le y'$$

$$\text{Therefore} \quad \Delta I \geq 0 \quad \text{for} \quad \frac{\partial^2 H}{\partial \dot{y}^2} \geq 0 \qquad (15)$$

$$\Delta I \leq 0 \quad \text{for} \quad \frac{\partial^2 H}{\partial \dot{y}^2} \leq 0$$

These conditions are known as the Legendre conditions for extremum.

In summary, the following must hold if:

A The curve is to have a weak extremum —

1. The curve must be an extremal satisfying the boundary conditions
2. The Jacobi condition must be true
3. The use of the Weierstrass Excess function shows that E has a constant sign $\forall$ (x,y) close to the curve and arbitrary values of y' close to $p(x,y)$; $E \geq 0$ for minimum ; $E \leq 0$ for maximum
4. The Legendre test shows that $\frac{\partial^2 H}{\partial \dot{y}^2} \neq 0$ and has constant sign $\forall$ (x,y) close to the curve ; $H_{,\dot{y}\dot{y}} > 0$ for minimum ; $H_{,\dot{y}\dot{y}} < 0$ for max

B. The curve is to have a strong extremum —

1. Conditions 1 & 2 of A must hold.
2. E has a constant sign $\forall (x,y)$ close to the curve and arbitrary value of $\dot{y}$'s
   $E \geq 0$ for minimum ; $E \leq 0$ for maximum

It must be stressed that the excess function and the Legendre test are necessary conditions for an extremum to occur. The Jacobi condition must hold for the conditions to become sufficient for extrema to exist.

In the case where the integral is of the form

$$I = \int H(x, y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n)\, dx$$

the Weierstrass condition becomes Consider the mass problem

$$\Delta H - \sum_{k=1}^{n} \frac{\partial H}{\partial \dot{y}_k} \Delta \dot{y}_k = E(x, y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n, p_1, \dots, p_n) \quad (16)$$

where $\Delta H = H(x, y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n) - H(x, y_1, \dots, y_n, p_1, \dots, p_n)$ and

$\Delta \dot{y}_k = (\dot{y}_k - p_k)$ for $k = 1, \dots, n$. involves

The Legendre Condition $Le(x, y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n, p_1, \dots, p_n)$

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 H}{\partial \dot{y}_k \partial \dot{y}_j} (\dot{y}_k - p_k)(\dot{y}_j - p_j) = Le(x, y_1, \dots y_n, \dot{y}_1 \dots \dot{y}_n, p_1, \dots, p_n) \quad (17)$$

Some Definitions Concerning the Calculus

1. The variation of a function $y(x)$ is defined as

$$\delta y = y(x) - y_1(x) \quad \text{where } y(x) \text{ and } y_1(x) \text{ are neighboring curves}$$

Unlike differential calculus, where a change dy is caused by a change dx, the variation of a function can be made without having a variation in its independent variable occurring.

2. Whereas in differential calculus $y = y(x)$ is read as "y is a function of the independent variable x," $v(y(x)) = \int_a^b H(x, y, y')\,dx$ is read as "$v(y(x))$ is a functional of $y(x)$."

3. Two curves are of closeness order k if

$$|y^{(k)} - y_1^{(k)}| < \epsilon \quad \text{where } \epsilon > 0$$

The variation of a functional $v(y(x))$ is continuous along a given curve $y = y_0(x)$ of order k if

$$\forall \epsilon > 0 \ \exists \ \delta > 0 \ \ni \ |v(y(x)) - v(y_0(x))| < \epsilon \text{ whenever}$$

$$|y^{(k)} - y_0^{(k)}| < \delta$$

4. Consider a linear functional $L(y(x))$; then

$$L(cy(x)) = c\,L(y(x))$$

$$L(y_1(x) + y_2(x)) = L(y_1(x)) + L(y_2(x))$$

If the change in variation due to δy

$$\Delta v = v(y(x) + \delta y) - v(y(x)) \quad \text{is of the form}$$

$$L(y(x), \delta y) + \beta(y(x), \delta y)\max|\delta y|,$$ that part which is linear in δy, namely $L(y(x), \delta y)$, is known as the variation of the functional and is denoted by δv.

5. A functional achieves a maximum along a curve $y=y_0(x)$ if for any neighboring curve $y(x)=y$ the difference in the ~~function~~ functional is less than or equal to zero. A functional achieves a minimum along a curve $y=y_0(x)$ if for any neighboring curve $y(x)=y$ (the difference in the ~~function~~ functional is greater than or equal to zero.

6. A maximum or a minimum occurring ~~by~~ $y=y_0(x)$ is considered strong if the order of closeness between any neighboring curve and $y=y_0(x)$ is of order zero. A maximum or a minimum occurring ~~on~~ for $y=y_0(x)$ is considered weak if the orders of closeness between any neighboring curve and $y=y_0(x)$ are of orders zero and one.

Parametric Representation

Consider the integral

$$I=\int_{x_0}^{x_1} H(x,y,y')\,dx \quad \text{where } x=x(t) \text{ and } y=y(t)$$

The above can be rewritten so that

$$I=\int_{t_0}^{t_1} H\left(x(t),y(t),\frac{dy/dt}{dx/dt}\right)\frac{dx}{dt}\,dt \qquad (18)$$

If the integrand is not a function of t explicitly and if the integrand is homogeneous of the first order, i.e.

$$H(x,y,k\dot{x},k\dot{y})=k\,H(x,y,\dot{x},\dot{y}) \quad \text{then the integral}$$

depends only on $x=x(t)$, $y=y(t)$, and no dependence on the parametric representation chosen exists. Let $\tau=\psi(t)$ so that $\dot{\psi}(t)\neq 0$

Then $x=x(\tau)$ and $y=y(\tau)$, and equation (18) is rewritten as

$$\int_{t_0}^{t_1} H\left(x,y,\frac{\dot{x}}{\dot{y}}\right)\dot{x}\,dt=\int_{t_0}^{t_1}\Phi(x,y,\dot{x},\dot{y})\,dt=\int_{\tau_0}^{\tau_1}\Phi\left[x(\tau),y(\tau),\frac{dx}{d\tau}\dot{\varphi},\frac{dy}{d\tau}\dot{\varphi}\right]\frac{d\tau}{\dot{\varphi}}$$

Because the integrand is homogeneous of first order

$$\int_{\tau_0}^{\tau_1} \Phi\left\{ x(\tau), y(\tau), \frac{dx}{d\tau}\dot{\varphi}, \frac{dy}{d\tau}\dot{\varphi} \right\} \frac{d\tau}{\dot{\varphi}} = \int_{\tau_0}^{\tau_1} \Phi\left(x(\tau), y(\tau), \frac{dx}{d\tau}, \frac{dy}{d\tau} \right) d\tau$$

(Euler-Lagrange eqs.)?

Thus the integrand is unchanged. The solution to this problem leads to

$$\Phi_{,x} - \frac{d}{dt}\Phi_{,\dot{x}} = 0$$
$$\Phi_{,y} - \frac{d}{dt}\Phi_{,\dot{y}} = 0$$

Note that these equations are no longer independent since other pairs of functions giving other parametric representation of the same curve exist. Since they must all satisfy the Euler-Lagrange equations, it is only necessary to solve one equation with the condition determining the parameter τ, to obtain the solution for the problem. , i.e. $x = x(\tau)$, $y = y(\tau)$, $t = \psi(t)$

Solution to the Mayer Problem

Consider the problem of minimizing

$$A = G(x_i, y_{k_i}, x_f, y_{k_f}) \qquad \text{where} \quad k = 1, \dots, n$$

subject to the constraints

$$\varphi_j(x, y_k, \dot{y}_k) = 0 \qquad j = 1, \dots, p < n$$

and consistent with the end conditions

$$\omega_r(x_i, y_{k_i}) = 0 \qquad r = 1, \dots, q$$
$$\omega_r(x_f, y_{k_f}) = 0 \qquad r = q+1, \dots, s \leq 2n+2$$

Consider the following integral

$$I = \int_{x_i}^{x_f} H(x, y_k, \dot{y}_k)\, dx \qquad k = 1, \dots, n$$

where $\lambda_k(x)$ are introduced so that

$$\dot{z}_k = f_k(x, y_k, \dot{y}_k)$$

and $\quad \sum_{k=1}^{n} f_k(x, y_k, \dot{y}_k) = H(x, y_k, \dot{y}_k)$

By making this transformation and by use of equation (2) one can transform $\int_{x_i}^{x_f} H(x, y_k, \dot{y}_k)\,dx$ to $\int_{x_i}^{x_f} F(x, y_k, \dot{y}_k)\,dx$

so that $I = \int_{x_i}^{x_f} \{\dot{z}_k + \lambda(x)_j \varphi_j(x, y_k, \dot{y}_k)\}\,dx$ (19). Since $\varphi_j(x, y_k, \dot{y}_k) = 0$ (19)

then $\sum_{k=1}^{n} z_k(x)\Big|_{x_i}^{x_f} = G(x_i, y_{k_i}, x_f, y_{k_f})$. Therefore one can use the necessary equations for minimization previously derived as well as the transformation

$$\dot{z}_k = f_k(x, y_k, \dot{y}_k) \tag{20}$$

In other words, the problem is that of finding those functions $y_k(x)$, $z_k(x)$ which satisfy equation (20) and minimize the functional given by equation (19).

The Equivalence of the Mayer, Lagrange, & Bolza Problem

The Bolza Problem can be solved by reducing it to a Mayer or Lagrange type problem.

1. In terms of the Mayer problem, functions $z_k(x)$ are introduced so that $\dot{z}_k - f_k(x, y_k, \dot{y}_k) = 0$, where $H(x, y_k, \dot{y}_k) = \sum_{k=1}^{n} f_k(x, y_k, \dot{y}_k)$ and $J = G(x, y_k)\Big|_i^f + z_k\Big|_i^f$ is to be minimized. (21)
2. In terms of the Lagrange problem, a function $y_{n+1}(x)$ is introduced so that $y'_{n+1} = 0$ and $y_{n+1}(x_i) - \dfrac{G(x, y_k)\Big|_i^f}{x_f - x_i} = 0$ (22)

and the integral to be minimized is

$$I = \int_{x_i}^{x_f} \left(H(x, y_k, \dot{y}_k) + Y_{n+1}(x) \right) dx \qquad (23)$$

Problems Involving Inequalities

Suppose that equation (1) were subjected to inequality constraints of the form

$$\dot{y}_i \geqslant \Gamma_i \qquad i = 1, \dots, p < n \qquad (24)$$

For such problems, since

$\dot{y}_i - \Gamma_i \geqslant 0$ define new variables $z_i(x)$ so that

$$\dot{y}_i - \Gamma_i = z_i^2 . \qquad (25)$$

In that manner a set of constraints

$\varphi_i = \dot{y}_i - \Gamma_i - z_i^2 = 0$ can be introduced and used in the form of equation (2).

Suppose that equation (1) were subjected to the constraints

$$\Gamma_{1_i} \leqslant \dot{y}_i \leqslant \Gamma_{2_i} \qquad i = 1, \dots, p < n \qquad (26)$$

From these two-side inequalities one obtains the following

$$\Gamma_{1_i} - \Gamma_{2_i} \leqslant \dot{y}_i - \Gamma_{2_i} \leqslant 0 \quad \text{or}$$

$$\Gamma_{2_i} - \Gamma_{1_i} \geqslant \Gamma_{2_i} - \dot{y}_i \geqslant 0 \qquad (a)$$

also $\Gamma_{2_i} - \Gamma_{1_i} \geqslant \dot{y}_i - \Gamma_{1_i} \geqslant 0 \qquad (b)$;

(a) & (b) can be rewritten

$$(\Gamma_{2_i} - \Gamma_{1_i}) - (\Gamma_{2_i} - \dot{y}_i) \geqslant 0 \qquad (c)$$

$$(\Gamma_{2_i} - \Gamma_{1_i}) - (\dot{y}_i - \Gamma_{1_i}) \geqslant 0 \qquad (d)$$

Multiplying (c) by $\dot{y}_i - \Gamma_{1_i}$ and (d) by $\Gamma_{2_i} - \dot{y}_i$ one obtains

$$(\dot{y}_i - \Gamma_{1_i})(\Gamma_{2_i} - \Gamma_{1_i}) - (\Gamma_{2_i} - \dot{y}_i)(\dot{y}_i - \Gamma_{1_i}) \geq 0 \qquad (e)$$
$$(\Gamma_{2_i} - \dot{y}_i)(\Gamma_{2_i} - \Gamma_{1_i}) - (\Gamma_{2_i} - \dot{y}_i)(\dot{y}_i - \Gamma_{1_i}) \geq 0 \qquad (f)$$

If (e) and (f) are rewritten as functions of new variables $\rho_i(x), \mu_i(x)$ then

$$(\dot{y}_i - \Gamma_{1_i})(\Gamma_{2_i} - \Gamma_{1_i}) - (\Gamma_{2_i} - \dot{y}_i)(\dot{y}_i - \Gamma_{1_i}) = \rho_i^2 \qquad (g)$$
$$(\Gamma_{2_i} - \dot{y}_i)(\Gamma_{2_i} - \Gamma_{1_i}) - (\Gamma_{2_i} - \dot{y}_i)(\dot{y}_i - \Gamma_{1_i}) = \mu_i^2 \qquad (h)$$

Adding (g) and (h) and defining new variables $z_i(x)$ in the following manner

$$z_i^2 = \left[(\Gamma_{2_i} - \Gamma_{1_i})^2 - (\rho_i^2 + \mu_i^2)\right]/2$$

the two-sided inequalities reduce to

$$(\Gamma_{2_i} - \dot{y}_i)(\dot{y}_i - \Gamma_{1_i}) = z_i^2 \qquad (27)$$

In this form a new set of constraints of the form

$\varphi_i = (\Gamma_{2_i} - \dot{y}_i)(\dot{y}_i - \Gamma_{1_i}) - z_i^2 = 0$ could be introduced and used in the form of equation (2).

In both the single inequality as well as the two-side inequality, the new functions $z_i(x)$ must be found along with the functions $y_i(x)$.

Hilbert's Invariant Integral

As stated before Hilbert's Invariant Integral is the integral of a perfect differential, in the case of extremal curves

Proof:

$$I = \int_{C^*} \left\{ H(x,y,p) + (y'-p)\frac{\partial H}{\partial p}(x,y,p) \right\} dx$$; this can be rewritten as

$$I = \int_{C^*} \left\{ \left[H(x,y,p) - p\frac{\partial H}{\partial p}(x,y,p)\right] dx + \frac{\partial H}{\partial p}(x,y,p)\, dy \right\}$$

The integral is of the form

$$I = \int_{C^*} (M\,dx + N\,dy) \quad \text{where} \quad M = H - p\frac{\partial H}{\partial p} \quad , \quad N = \frac{\partial H}{\partial p}$$

Let us look at $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$.

$$\frac{\partial M}{\partial y} = \frac{\partial H}{\partial y} - \frac{\partial p}{\partial y} H_{,p} - p\,\frac{\partial H_{,p}}{\partial y}$$

$\frac{\partial N}{\partial x} = \frac{\partial H_{,p}}{\partial x}$. Now let us take a look at $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$

$\frac{\partial H}{\partial y} - \frac{\partial p}{\partial y} H_{,p} - p\,\frac{\partial H_{,p}}{\partial y} - \frac{\partial H_{,p}}{\partial x}$. However

$$\frac{\partial H_{,p}}{\partial x} + p\,\frac{\partial H_{,p}}{\partial y} + \frac{\partial p}{\partial y} H_{,p} = \frac{d}{dx} H_{,p} \quad \therefore$$

$$\frac{\partial H}{\partial y} - \frac{d}{dx} H_{,p} = \frac{\partial H}{\partial y} - \left[\frac{\partial H_{,p}}{\partial x} + p\frac{\partial H_{,p}}{\partial y} + \frac{\partial p}{\partial y} H_{,p}\right]$$

Because the curves are extremals then

$$\frac{\partial H}{\partial y} - \frac{d}{dx} H_{,p} = 0 \quad \text{or} \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$$

But this is the condition which a differential must satisfy for it to be exact; therefore

$I = \int_{C^*} M\,dx + N\,dy = \int_{C} (M\,dx + N\,dy)$ or the integral depends only on the end points and not the path of integration.

Miele, etc.?

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