Calc. of Variations – Optimization

I Basics of calc. of vars:

a) Elsgolc QA 315 E573

b) Hildebrand, "Methods of Appl. Math." QA 37 H52 1965 Ch. 2

c) Halfman, "Dynamics" Vol. 2

d) Lawden (Mc.)

(II) Leitmann (ed.) "Optimization Techniques" 1962

"Space Mathematics", Vol. 2 Hestenes American Mathematical ~~Association~~ Society

applications:

Astronautica Acta Vol. 12, No. 2, pp. 169–179 (Mar./Apr. 1966)

~~Vol.~~ AIAA Jour. ~~1~~, ~~3 115~~ (Tech Note) 1229–1231 ~~Jan. 1964~~ May 1963 (see also Faulkner)

Breakwell and refs.

Schrumer M.S. Thesis 1965

L. Ting, "Optimum Orbital Transfer by Impulses" J. Amer. Rocket Soc. 30 (1960), pp. 1013–1018.

Particular Applications & Problems:

a) various illustrative exps. in Edsgle
b) Faulkner pp. 59–85
c) Edelbaum 19 – 23
d) Miele – "Green's Theorem"
e) Miele paper on Calc of Vars. in Appl Aero.
f) Leitmann – Bounded Control
g) ~~Lawden the. esp. Ch. 2 (whatever pages is [illegible])~~
h) Lawden on transfer

Calculus

of

Variations

Obtained from:

Elsgolc, L.E. Calculus of Variations

Hildebrand, F.B. Methods of Applied Mathematics

Variational Calculus. $v(y(x))$

$\delta y = y(x) - y_1(x)$ of $v(y(x))$; $\Delta v = v(y(x)+\delta y) - v(y(x))$

2 curves are of closeness order k if $|y^{(k)} - y_1^{(k)}| < \epsilon$

$v(y(x))$ is C along $y = y_0(x)$ of order k if for $\forall \epsilon$ $\exists$ $\delta > 0$

$\ni$ $|v(y(x)) - v(y_0(x))| < \epsilon$ whenever $|y^{(k)}(x) - y_0^{(k)}(x)| < \delta$.

linear fnl $L(y(x))$ exists if

$$L(cy(x)) = c L(y(x))$$

$$L(y_1(x) + y_2(x)) = L(y_1(x)) + L(y_2(x))$$

if $\Delta v = v(y(x)+\delta y) - v(y(x))$ is of the form $L(y(x), \delta y) + \beta(y(x), \delta y) \max|\delta y|$

the part which is linear in δy — is the variation of the fnl $= \delta v$.

if $\Delta v = v(y(x) + \alpha \delta y) - v(y(x))$ then $\delta v = \frac{\partial}{\partial \alpha} v(y(x) + \alpha \delta y)_{\alpha = 0}$

consider $\frac{\partial f(x+\alpha\Delta x)}{\partial(x+\alpha\Delta x)} \frac{\partial(x+\alpha\Delta x)}{\partial\alpha} = \frac{\partial f(x+\alpha\Delta x)}{\partial\alpha}$

$$\lim_{\alpha \to 0} f'(x+\alpha\Delta x)\Delta x = f'(x)\Delta x = df$$

def: $v(y)$ is M along $y = y_0(x)$ if $v(y) - v(y_0) \le 0$ in nbd of $y = y_0(x)$

$v(y)$ is m " " " $v(y) - v(y_0) \ge 0$ " " " "

if $v(y)$ is M or m on $y = y_0$ $\ni$ $y(x) - y_0(x) < \epsilon$ (0 order closeness) M or m is strong; if $v(y)$ is M or m on $y = y_0$ $\ni$ $y^{(k)}(x) - y_0^{(k)}(x) < \epsilon$ $k = 0, 1$ M or m is weak

if $v(y)$ has an extremum on $y = y_0$ then

$$\partial/\partial\alpha \; v(y_0 + \alpha\delta y)_{\alpha=0} = 0 \quad \& \quad \frac{\partial}{\partial\alpha} v(y(x,\alpha))_{\alpha=0} = 0$$

FIXED BOUNDARIES

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if $\underline{v(y(x)) = \int_{x_0}^{x_1} F(x, y(x), y'(x))\,dx}$ choose any $\eta(x) \in C^1$ $\ni$ $\eta(x_0) = \eta(x_1) = 0$

for $y \to y + \epsilon\eta$ then for a fixed y and η $v(y(x) + \epsilon\eta(x)) \to v(\epsilon)$

$\eta(x) = y^*(x) - y(x) = \delta y$ for $\epsilon = 1$

by previous proofs $\frac{\partial}{\partial \epsilon} v(\epsilon)\Big|_{\epsilon=0} = 0$ let $F(x, y+\epsilon\eta, y'+\epsilon\eta') = \tilde{F}$

as $\epsilon \to 0$ $\tilde{F} \to F$

$$\therefore \quad \frac{\partial}{\partial \epsilon} v = \int_{x_0}^{x_1} \left[\frac{\partial \tilde{F}}{\partial x} \cancelto{0}{\frac{\partial x}{\partial \epsilon}} + \frac{\partial \tilde{F}}{\partial (y+\epsilon\eta)} \frac{\partial (y+\epsilon\eta)}{\partial \epsilon} + \frac{\partial \tilde{F}}{\partial (y'+\epsilon\eta')} \frac{\partial (y'+\epsilon\eta')}{\partial \epsilon} \right] dx$$

$$\delta v = \frac{\partial}{\partial \epsilon} v \Big|_{\epsilon \to 0} = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' \right] dx = 0$$

integration by parts of $\int_{x_0}^{x_1} \frac{\partial F}{\partial y'} \eta' \, dx$ let $\frac{\partial F}{\partial y'} = u$ $d\varphi = \eta' dx$

$$\therefore \int_{x_0}^{x_1} \frac{\partial F}{\partial y'} \eta' \, dx = \frac{\partial F}{\partial y'} \eta \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] \eta \, dx \qquad \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] dx = du \qquad \varphi = \eta$$

note: since $\eta(x_0) = \eta(x_1) = 0$ the first term vanishes and δv becomes,

$$\int_{x_0}^{x_1} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] \right\} \eta \, dx = 0$$ since $\eta(x)$ need not be 0 $\forall x_0 \le x \le x_1$

then

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0} \qquad (1)$$

<u>Fundamental Lemma of Calculus of Variations</u>

if $\Phi(x) \in C$, $x_0 \le x \le x_1$ and if $\int_{x_0}^{x_1} \Phi(x)\eta(x)\,dx = 0$

where $\eta(x)$ is an arbitrary fn. subject to some conditions of general character only, then $\Phi(x) \equiv 0 \in I(x_0, x_1)$.

Some Conditions of $\eta(x)$: $\eta(x) \in C^k$ and $\eta(x_0) = \eta(x_1) = 0$ and $|\eta(x)| < \epsilon$ or $|\eta(x)| < \epsilon$ and $|\eta'(x)| < \epsilon$.

if $\Phi(x,y) \in D$, and if $\iint_D \Phi(x,y)\eta(x,y)\,dx\,dy = 0$

then $\Phi(x,y) \equiv 0$.

since y, y' were treated as independent variables but are really functions of x eq (1) can be rewritten as: (since $\frac{d}{dx} F_{y'} = \frac{\partial}{\partial x} F_{y'} + \frac{\partial}{\partial y} F_{y'} \frac{dy}{dx} + \frac{\partial}{\partial y'} F_{y'} \frac{dy'}{dx}$)

$$\boxed{F_{y} - F_{xy'} - F_{yy'} \frac{dy}{dx} - F_{y'y'} \frac{d^2y}{dx^2} = 0} \qquad (2)$$

Eq (2) is known as Euler's Equation. The curves satisfying eq (2) are known as extremals.

Example: find all curves which make $v(y) = \int_0^{\pi/2} ((y')^2 - y^2)\,dx$ an extremum subject to $y(0)=0$ $y(\pi/2)=1$

$$F = (y')^2 - y^2 \qquad \frac{\partial F}{\partial y} = -2y \qquad \frac{\partial F}{\partial y'} = 2y'$$

$$\therefore \quad \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = -2y - 2y'' = 0 \qquad \text{or} \quad y'' + y = 0$$

$$y = C_1 \cos x + C_2 \sin x \qquad \text{at } x=0 \;\; y=0 \implies C_1 = 0$$
$$\text{at } x = \pi/2 \;\; y = 1 \implies C_2 = 1$$

$$\therefore \quad y = \sin x$$

Example: find all curves which make $v(y) = \int_0^1 ((y')^2 + 12xy)\,dx$ an extremum subject to $y(0)=0$ $y(1)=1$

$$\therefore \quad \frac{\partial F}{\partial y} = 12x \qquad \frac{\partial F}{\partial y'} = 2y'$$

$$\therefore \quad \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 12x - 2y'' = 0 \qquad \therefore \quad 6x - y'' = 0$$

$$\therefore \quad y'' = 6x \implies y = x^3 + C_1 x + C_2$$

using $y(0)=0$ $C_2 \Rightarrow 0$ $y(1)=1$ $C_1 \Rightarrow 0$ $\therefore$ $y = x^3$

if $F \neq F(y')$ then $F_{,y} = 0$ since $\frac{\partial F}{\partial y'} \equiv 0$ (usually no solution)
if $F \neq F(x)$ then $F - y'\frac{\partial F}{\partial y'} = C$ since $\frac{\partial F}{\partial x} \equiv 0$
if $F \neq F(y)$ then $F_{,y'} = C$

an extremum which satisfies the appropriate end conditions is called a stationary fn.

if F is linear in y' or $v(y(x)) = \int_{x_0}^{x_1} \left[M(x,y) + N(x,y)\frac{dy}{dx}\right] dx$
then Euler's Equation reduces to

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$$ which usually has no solution (b.c. not satisfied)

however if $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \equiv 0$ then $M\,dx + N\,dy$ is a perfect differential

if $F = F(y')$ only then $F_{,y'y'} \frac{d^2y}{dx^2} = 0 \Leftrightarrow F_{,y'y'} = 0$ or $y'' = 0$
if $y'' = 0$ $y = C_1 x + C_2$

Consider $\underline{I = \int_{x_0}^{x_1} F(x, y_i, y_i')\,dx}$ $i = 1, n$

if $y_i(x_0)$ and $y_i(x_1)$ are given

we can evaluate the extremum by fixing $n-1$ variables and varying the one left $\ni$ $v(y_1, y_2, \ldots, y_n) = \tilde{v}(y_i)$

As before

$$\boxed{F_{,y_i} - \frac{d}{dx}F_{,y_i'} = 0} \qquad (3)$$

however n such equations exist

Consider

$$\underline{I = \int_{x_0}^{x_1} (x, y, y', y'', \ldots, y^{(n)})\,dx} \qquad \text{where } F \in C^{n+2}$$

and $y^{(k)}(x_0)$ and $y^{(k)}(x_1)$ $k = 0, 1, 2, \ldots, n-1$ are given

if $y = y(x)$ is $2n$ times differentiable and gives an extremum and a curve $y = y^*(x)$ also $2n$ times differentiable: $y(x,\alpha) = y(x) + \alpha(y^*(x) - y(x))$ or

$$= y(x) + \alpha\,\delta y \qquad 0 \le \alpha \le 1$$

for an extremum at $\alpha = 0$ the fnl. must satisfy

$$\frac{\partial}{\partial \alpha} v(y(x,\alpha))_{\alpha=0} = 0. \text{ Then}$$

$$\delta v = \int_{x_0}^{x_1} \left(F_{,y}\,\delta y + F_{,y'}\,\delta y' + \dots + F_{,y^{(n)}}\,\delta y^{(n)} \right) dx.$$

integration of all terms of form $F_{y^{(k)}}\,\delta y^{(k)}$ $\quad k = 1, \dots, n$ leads to

$$F_{,y^{(k)}}\,\delta y^{(k-1)}\Big|_{x_0}^{x_1} - \frac{d}{dx} F_{,y^{(k)}}\,\delta y^{(k-2)}\Big|_{x_0}^{x_1} + \dots + (-1)^k \int_{x_0}^{x_1} F_{,y^{(k)}}\,\delta y\,dx.$$

since $x = x_0$, $y = y_0$ at end pts the variations must vanish since pts are fixed.

$\therefore \quad \delta y = \delta y' = \delta y'' = \dots = \delta y^{(n-1)} = 0$ therefore

$$\delta v = \int_{x_0}^{x_1} \left[F_{,y} - \frac{d}{dx} F_{,y'} + \frac{d^2}{dx^2} F_{,y''} + \dots + (-1)^n \frac{d^n}{dx^n} F_{,y^{(n)}} \right] \delta y\,dx = 0$$

therefore if $\delta y \neq 0$

$$\boxed{F_{,y} - \frac{d}{dx} F_{,y'} + \frac{d^2}{dx^2} F_{,y''} + \dots + (-1)^n \frac{d^n}{dx^n} F_{,y^{(n)}} \equiv 0} \qquad (*)$$

This is the Euler - Poisson equation. (diff. equation of order $2n$)

Example: Find the extremal of the fnl.

$$v(y) = \int_0^{\pi/2} \left(y''^2 - y^2 + x^2 \right) dx \qquad \text{whose boundary}$$

conditions are $y(0) = 1$ $\quad y'(0) = 0$ $\quad y(\pi/2) = 0$ $\quad y'(\pi/2) = -1$

$$F_{,y} = -2y \qquad F_{,y'} = 0 \qquad F_{,y''} = 2y'' \quad \text{therefore}$$

The Euler - Poisson equation is

$$-2y + \frac{d^2}{dx^2} 2y'' = 0 \qquad \text{or} \qquad y^{(4)} - y = 0$$

this equation is solved by

$$y(x) = C_1 e^{-x} + C_2 e^{x} + C_3 \cos x + C_4 \sin x$$

$y(0) = C_1 + C_2 + C_3 = 1$ $\qquad$ $y(\pi/2) = C_1 e^{-\pi/2} + C_2 e^{\pi/2} + C_4 = 0$

$y'(0) = -C_1 + C_2 + C_4 = 0$ $\qquad$ $y'(\pi/2) = -C_1 e^{-\pi/2} + C_2 e^{\pi/2} - C_3 = -1$

$C_1 = C_2 = C_4 = 0 \qquad C_3 = 1$

Thus $\qquad y(x) = \cos x$

Example: Find the extremal of the fnl.

$$v(y(x)) = \int_{-l}^{l} \left(\tfrac{1}{2}\mu y''^2 + \rho y\right) dx \quad ; \quad \rho, \mu \text{ constant.}$$

$F_y = \rho \qquad F_{y'} = 0 \qquad F_{y''} = \mu y''$, (study of buckled axis of elastic cylindrical beam with both ends fixed.)

∴ The Euler - Poisson Equation reads

$$\rho + \mu y^{(4)} = 0, \qquad y = -\frac{\rho}{24\mu} x^4 + \frac{C_1}{6} x^3 + \frac{C_2}{2} x^2 + C_3 x + C_4$$

however $y(-l) = 0 \quad y'(-l) = 0 \quad y(l) = 0 \quad y'(l) = 0$

$$\therefore \quad y = -\frac{\rho}{24\mu}\left[x^4 - 2l^2x^2 + l^4\right]$$

if $\quad v(y(x), z(x)) = \displaystyle\int_{x_0}^{x_1} F(x, y^{(k)}(x), z^{(j)}(x))\, dx \qquad k = 1, \ldots, n; \; j = 1, \ldots, m$

The solution to this problem is:

$$\boxed{\begin{aligned} &F_y - \frac{d}{dx}F_{y'} + \ldots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0 \\ &F_z - \frac{d}{dx}F_{z'} + \ldots + (-1)^m \frac{d^m}{dx^m} F_{z^{(m)}} = 0 \end{aligned}} \qquad (5)$$

if $\quad v(y_1, y_2, y_3, \ldots, y_m) = \displaystyle\int_{x_0}^{x_1} F(x, y_i^{(k_i)})\, dx \qquad i = 1, \ldots, m; \; k = 1, \ldots, n$

the solution is found by

$$\boxed{F_{y_i} - \frac{d}{dx}F_{y_i'} + \ldots + (-1)^{n_i} \frac{d^{n_i}}{dx^{n_i}} F_{y_i^{(n_i)}} = 0 \quad \text{where } i = 1, 2, \ldots, m} \qquad (6)$$

if $z = z(x,y)$ then

$$v\,(z(x,y)) = \iint_D F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dx\,dy \qquad \text{if } \partial D = C$$

where $z(x,y) \in \partial D$ is given by C^* where all admissible surfaces pass through it allow $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$

Consider $z(x,y,\alpha) = z(x,y) + \alpha\,\delta z$ where $\delta z = z^*(x,y) - z(x,y)$ where extremum exists when $\alpha = 0$ therefore $\frac{\partial}{\partial \alpha} v(z(x,y,\alpha))_{\alpha=0} = 0$ then

$$\delta v = \left[\frac{\partial}{\partial \alpha} \iint_D F(x, y, z(x,y,\alpha), p(x,y,\alpha), q(x,y,\alpha))\,dx\,dy\right]_{\alpha=0}$$

$$= \iint_D (F_{,z}\,\delta z + F_{,p}\,\delta p + F_{,q}\,\delta q)\,dx\,dy;$$

$$\iint_D (F_{,p}\,\delta p + F_{,q}\,\delta q)\,dx\,dy = \iint_D \left[\frac{\partial}{\partial x}\{F_{,p}\,\delta z\} + \frac{\partial}{\partial y}\{F_{,q}\,\delta z\}\right] dx\,dy$$

$$- \iint_D \left[\frac{\partial}{\partial x}[F_{,p}] + \frac{\partial}{\partial y}[F_{,q}]\right] \delta z\,dx\,dy$$

$$\frac{\partial}{\partial x} F_{,p} = F_{,px} + F_{,pz}\frac{\partial z}{\partial x} + F_{,pp}\frac{\partial p}{\partial x} + F_{,pq}\frac{\partial q}{\partial x}$$

$$\frac{\partial}{\partial y} F_{,q} = F_{,qy} + F_{,qz}\frac{\partial z}{\partial y} + F_{,qp}\frac{\partial p}{\partial y} + F_{,qq}\frac{\partial q}{\partial y}$$

by Green's theorem. $\iint_D \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) dx\,dy = \int_C (N\,dy - M\,dx) = 0$, since $\delta z = 0$ ∂D

$\therefore$ the first part of RHS is 0.

thus the necessary condition is

$$\iint_D (F_{,z}\,\delta z + F_{,p}\,\delta p + F_{,q}\,\delta q)\,dx\,dy = 0 \qquad \text{for extremum } \underline{\text{or}}$$

$$\iint_D (F_{,z} - F_{,px} - F_{,qy})\,\delta z\,dx\,dy = 0 \qquad \delta z \in C \quad = 0 \ \partial D$$

$$\boxed{F_{,z} - F_{,p\,x} - F_{,q\,y} = 0} \qquad (7)$$

This is the Ostrogradski Equation (Euler-Lagrange Equation)

Problem:

$$v(z(x,y)) = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] dx\,dy$$ where $z = f(x,y)$ on ∂D where $f(x,y)$ is given a priori on ∂D.

The Ostrogradski Equation becomes $\nabla^2 z = 0$ (Laplace Equation)

if $v(z(x,y)) = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 2z f(x,y)\right] dx\,dy$ where $z = f(x,y)$ on ∂D etc.

The Ostrogradski Equation becomes $\nabla^2 z = f(x,y)$ (Poisson Equation)

If the fnl is $v(z(x_1, x_2, x_3, \ldots, x_n)) = \iint_D \cdots \int F(x_i, z, p_i)\, dx_i \quad i = 1, \ldots, n$

where $p_i = \partial z / \partial x_i$ for $\delta v = 0$

$$\boxed{F_{,z} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[F_{,p_i}\right] = 0} \qquad (8)$$

if $v(z(x,y)) = \iint_D F(x, y, z, z_{,x}, z_{,y}, z_{,xx}, z_{,yx}, z_{,yy})\, dx\,dy$

the equation for extremum is

$$\boxed{\begin{array}{l} F_{,z} - F_{,p\,x} - F_{,q\,y} + F_{,r\,xx} + F_{,s\,yx} + F_{,t\,xx} = 0 \\ \text{where } p = z_{,x}\,;\ q = z_{,y}\,;\ r = z_{,xx}\,;\ s = z_{,yx}\,;\ t = z_{,yy} \end{array}} \qquad (9)$$

if $v(z(x,y)) = \iint_D F(x, y, z, z_{,x}, z_{,y}, z_{,xx}, z_{,yx}, z_{,yy}, z_{,xxx}, z_{,yxx}, z_{,yyx}, z_{,yyy}, \ldots)\, dx$

the equation for extremum is

$$F_{,z} - \left(F_{,p\,x} + F_{,q\,y}\right) + \left(F_{,r\,xx} + F_{,s\,yx} + F_{,t\,xx}\right) - \left(\frac{\partial^3}{\partial x^3} F_{,z_{,xxx}} + \frac{\partial^3}{\partial^2 x\,\partial y} F_{,z_{xxy}} + \frac{\partial^3}{\partial^2 y\,\partial x} F_{,z_{,yyx}} + \frac{\partial^3}{\partial y^3} F_{,z_{,yyy}}\right) + \left(\frac{\partial^4}{\partial x^4} F_{,z_{,xxxx}} + \cdots\right) - \cdots = 0 \qquad (10)$$

Parametric Representation of Variational Problems.

finding solutions of form $x, y = f(t)$

$$v(y(x)) = \int_{x_0}^{x_1} F(x, y, y')\,dx \rightarrow v(x(t), y(t)) = \int_{t_0}^{t_1} F\left(x(t), y(t), \frac{\dot{y}}{\dot{x}}\right)\dot{x}\,dt$$

does not depend on t explicitly, homogeneous of first order wrt $\dot{x}, \dot{y}$.

cannot take form of $\Phi(t, x(t), y(t), \dot{x}, \dot{y})$; introduction of parametric representation

$x = x(\tau)$ $y = y(\tau)$ leads $v(x, y) \rightarrow \int_{\tau_0}^{\tau_1} F\left(x, y, \frac{\dot{y}_\tau}{\dot{x}_\tau}\right)\dot{x}_\tau\,d\tau$

then v depends only on path of integration.

However

$$v(x(t), y(t)) = \int_{t_0}^{t_1} (t, x, y, \dot{x}, \dot{y})\,dt \quad \text{doesn't depend on } t \text{ explicitly}$$

& if v is first order homogeneous wrt $\dot{x}, \dot{y}$, v is only dependent on $x = x(t)$, $y = y(t)$

if:

$$v(x(t), y(t)) = \int_{t_0}^{t_1} \Phi(x(t), y(t), \dot{x}(t), \dot{y}(t))\,dt \quad \text{where} \tag{11}$$

$$\Phi(x, y, k\dot{x}, k\dot{y}) = k\,\Phi(x, y, \dot{x}, \dot{y}) \tag{12}$$

set: $\tau = \varphi(t)$ where $\dot{\varphi} \neq 0$ $x = x(\tau)$ $y = y(\tau)$

then

$$\int_{t_0}^{t_1} \Phi(x, y, \dot{x}, \dot{y})\,dt = \int_{\tau_0}^{\tau_1} \Phi\left(x(\tau), y(\tau), \dot{x}_\tau \dot{\varphi}(t), \dot{y}_\tau \dot{\varphi}(t)\right) \frac{d\tau}{\varphi^{\cdot}(t)} \tag{13}$$

by use of eq (12), eq (13) can be written as:

$$\int_{t_0}^{t_1} \Phi(x, y, \dot{x}, \dot{y})\,dt = \int_{\tau_0}^{\tau_1} \dot{\varphi}(t)\,\Phi(x(\tau), y(\tau), \dot{x}_\tau, \dot{y}_\tau)\frac{d\tau}{\dot{\varphi}(t)}$$

$$= \int_{\tau_0}^{\tau_1} \Phi(x(\tau), y(\tau), \dot{x}_\tau, \dot{y}_\tau)\,d\tau \tag{14}$$

thus integrand remains unchanged by parametric transformation.

a solution to such a problem leads to 2 diff equations of type

$$\Phi_{,x} - \frac{d}{dt}\Phi_{,\dot{x}} = 0; \quad \Phi_{,y} - \frac{d}{dt}\Phi_{,\dot{y}} = 0$$

however the equations are no longer independent since $\exists$ other pairs of fns. giving other parametric representation of same curve and $\forall$ must satisfy the Euler Equations. Therefore if they were independent a contradiction to Gen. Theor. of Exist. & Unique of solution of syst. of diff. equat. would exist. Thus it is only necessary to solve one of them with an equation determining parameter.

Applications:

Principle of Ostrogradski - Hamilton. Motion of system of points, compatible with the constraints, gives rise to motion tracing curve giving extremum to integral $\int_{t_0}^{t_1}(T-U)\,dt$ where T = k.e. U = p.e.

Example: give a system of masses m_i whose coordinates are (x_i, y_i, z_i) and upon which a force F_i acts where $F = F(U)$ where

$$F_{i_x} = -\frac{\partial U}{\partial x_i}, \quad F_{i_y} = -\frac{\partial U}{\partial y_i}, \quad F_{i_z} = -\frac{\partial U}{\partial z_i}$$

Since $T = \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$ and potential energy of syst. is U then

$$\int_{t_0}^{t_1}(T-U)\,dt \rightarrow \left.\begin{aligned} -U_{,x_i} - \frac{d}{dt}T_{,\dot{x}_i} &= 0 \\ -U_{,y_i} - \frac{d}{dt}T_{,\dot{y}_i} &= 0 \\ -U_{,z_i} - \frac{d}{dt}T_{,\dot{z}_i} &= 0 \end{aligned}\right\} \left\{\begin{aligned} m_i\ddot{x}_i - F_{i_x} &= 0 \\ m_i\ddot{y}_i - F_{i_y} &= 0 \\ m_i\ddot{z}_i - F_{i_z} &= 0 \end{aligned}\right.$$

if the system is subject to independent constraints of the form

$\varphi_j(t, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n) = 0 \quad j = 1, 2, \ldots m \quad m < 3n$

m variables $= f(3n-m)$ indep. var. exclusive of t $\therefore$ $3n$ var. can be replaced by $3n-m$ new var. $q_1, q_2, \ldots, q_{3n-m}$

when T and U are fns of $(q_1, q_2, \ldots, q_{3n-m}, t)$

$$T = T(q_1, q_2, q_3, \ldots, q_{3n-m}, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{3n-m}, t)$$

$$U = U(q_1, q_2, \ldots, q_{3n-m}, t)$$

Euler équat. are

$$\frac{\partial(T-U)}{\partial q_i} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = 0 \qquad i = 1, 2, \ldots, 3n-m.$$

Lagrange's Eq. of motion.

Problems.

1. $$v(y(x)) = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{y}\,dx. \qquad F_y = \sqrt{1+y'^2}\,(-y^{-2}) \qquad F_{y'} = \frac{1}{2}(1+y'^2)^{-1/2}\,\frac{2y'}{y}$$

$$F - y' F_{y'} = \frac{\sqrt{1+y'^2}}{y} - \frac{y'^2}{y\sqrt{1+y'^2}} = C_1$$

by manipulation and setting $y' = \tan t$ leads to the result

$$y = C_1 \cos t \qquad \text{also since} \quad dx = \frac{dy}{y'} = \frac{-C_1 \sin t}{\tan t}\,dt$$

$$x = -\int C_1 \cos t\,dt \qquad \therefore \qquad x = -C_1 \sin t + C_2$$

Some points to be considered:

1. The derivative of the variation = variation of the derivative

$$\frac{d}{dx}\,\delta y = \frac{d}{dx}\left[y_1(x) - y(x)\right] = \frac{d}{dx}\,\epsilon\,\eta(x) = \epsilon\,\eta'(x)$$

$$\delta \frac{d}{dx} y(x) = y_1'(x) - y'(x) = y'(x) + \epsilon\eta'(x) - y'(x) = \epsilon\,\eta'(x)$$

2. The variation of the definite integral = definite integral of the variation

$$\delta \int_a^b F(x)\,dx \equiv \int_a^b \delta F(x)\,dx$$

3. For extremals to occur the first variation

$$\left.\frac{\delta I(\epsilon)}{\delta\epsilon}\right|_{\epsilon=0} = \int_a^b \left(F_{,y}\,\eta + \eta' F_{,y'}\right)dx = 0$$

For maximum or minimum the second variation

$$\delta I_2 = \frac{\epsilon^2}{2!}\int_a^b \left(\eta^2 F_{,yy} + 2\eta\eta' F_{,yy'} + \eta'^2 F_{,y'y'}\right)dx < 0 \quad (>0)$$

$$= \frac{\epsilon^2}{2!}\int_a^b F_{,y'y'}\left(\eta' - \eta\frac{u'}{u}\right)^2 dx \quad \text{where } \eta \text{ is replaced by } u.$$

from this it can be seen that the sign of I_2 depends solely on $F_{,y'y'}$.

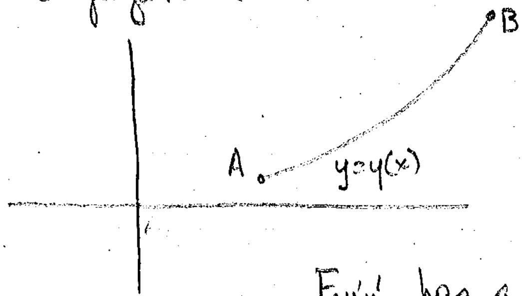
The Legendre Test . for small arcs & if $F_{,y'y'}$ has a constant sign throughout this arc then I is Max if $F_{,y'y'} < 0$, I is a min if $F_{,y'y'} > 0$

Jacobi's Test -

The Accessory Equation - $\left[F_{,yy} - \frac{d}{dx}F_{,yy'}\right]u - \frac{d}{dx}\left[F_{,y'y'}\frac{du}{dx}\right] = 0$

Conditions regarding the solution of this equation lead to the definition of conjugate points:

if $u(x)$ solves the equation, a is the abscissa of pt A, $u(a)=0$ then the roots of $u(x)=0$ are the abscissae of points on the curve $y=y(x)$ conjugate to A.



<u>**Theorem:**</u> if $y=y(x)$ is an extremal through A & B for $I=\int_a^b F(x,y,y')dx$, A' is the first pt conjugate of A along AB, B lies between A & A', then $F_{,y'y'}$ has a constant sign. Jacobi's test establishes criteria for max permissible length of AB.

Given $v(y(x))=\int_{x_0}^{x_1}\frac{\sqrt{1+y'^2}}{y}\,dx$ since $v(y) - F(y',y)$

$F_{,y}= -y^{-2}\sqrt{1+y'^2}$ $\qquad F_{,y'}=\frac{y'}{y\sqrt{1+y'^2}}$

$$F-y'F_{,y'}=\frac{\sqrt{1+y'^2}}{y}-\frac{y'^2}{y}\left(\frac{1}{\sqrt{1+y'^2}}\right)=c$$

$\frac{1}{\sqrt{1+y'^2}}\,\frac{1}{y}\left[\,1\,\right]=c$ $\qquad \frac{1}{y\sqrt{1+y'^2}}=c$

let $y\sqrt{1+y'^2}=c_1$ $\qquad$ let $y'=\tan t$ $\quad\therefore\quad y=\frac{c_1}{\sec t}=c_1\cos t$

Legendre Test $\qquad F=\frac{\sqrt{1+y'^2}}{y}$, $\quad F_{,y'}=\frac{y'}{y}(1+y'^2)^{-1/2}$

$$F_{,y'y'}=\frac{(1+y'^2)^{-1/2}}{y}+\frac{y'^2}{y}\left(-(1+y'^2)^{-3/2}\right)$$

$$=\frac{1}{y(1+y'^2)^{+3/2}}\left[(1+y'^2)-y'^2\right]=\frac{1}{y(1+y'^2)^{3/2}}>0$$

Variable Boundary Conditions

Unlike the fixed boundary problems the solutions will involve a broader number of curves. However since a curve $y_0(x) = y$ produces an extremal for a problem with variable boundaries. The same curve gives an extremal wrt a more restricted class of curves having the same boundary points as the curve $y = y_0(x)$. This allows use to use the fundamental necessary condition for extremums: $F_y - \frac{d}{dx} F_{y'} = 0$ however boundary conditions will not hold. Curve must be one of the integral curves $y = y(x, C_1, C_2)$ of Euler Eq.

1. Simplest form - one pt fixed. (x_0, y_0) $(x_1 + \delta x_1, y_1 + \delta y_1)$

$y = y(x)$ & $y = y(x) + \delta y$ as close to each other if $|\delta y|$ & $|\delta y'|$ are small & $|\delta x_1|$ & $|\delta y_1|$ are small.

Calculation of $\Delta v(y)$ of $y = y(x, C_1)$ when one pt moves from (x_1, y_1)

$$\Delta v = \int_{x_0}^{x_1 + \delta x_1} F(x, y + \delta y, y' + \delta y')\, dx - \int_{x_0}^{x_1} F(x, y, y')\, dx$$

$$= \int_{x_1}^{x_1 + \delta x_1} F(x, y + \delta y, y' + \delta y')\, dx + \int_{x_0}^{x_1} (F_y\, \delta y + F_{y'}\, \delta y')\, dx + R_1$$

use of mean value theorem on part I leads to

$$\int_{x_1}^{x_1 + \delta x_1} F(x, y + \delta y, y' + \delta y')\, dx = F\Big|_{x = x_1 + \theta_1 \delta x_1} \delta x_1 = F\Big|_{x = x_1} \delta x_1 + \epsilon_1 dx_1$$

when $0 < \theta_1 < 1$

part II can, by partial integration, be found as $F_{y'}\, \delta y \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} (F_y - \frac{d}{dx} F_{y'})\, \delta y\, dx$

using necessary condit and that $\delta y|_{x = x_0} = 0$ (fixed pt)

$$\int_{x_0}^{x_1} (F_y\, \delta y + F_{y'}\, \delta y')\, dx = [F_{y'}\, \delta y]_{x = x_1}$$

since $\delta y|_{x = x_1} \cong \delta y_1 - y'(x_1)\, \delta x_1$ and since $\epsilon_1 \to 0$ when $\left.\begin{matrix}\delta y_1 \\ \delta x_1\end{matrix}\right\} \to 0$

then $\Delta v = (F - y' F_{y'})_{x = x_1}\, \delta x_1 + F_{y'}\Big|_{x = x_1} \delta y_1 = 0$ for extremum

Thus the governing equation for a single variable boundary is

$$(F - y'F_{,y'})_{x=x_1}\,\delta x_1 + F_{,y'}\,\delta y_1\big|_{x=x_1} = 0 \qquad (1)$$

if δx_1 & δy_1 are independent then we obtain

$$(F - y'F_{,y'})\big|_{x=x_1} = F_{,y'}\big|_{x=x_1} = 0$$ however many times $y_1 = \phi(x_1)$

thus $\delta y_1 \cong \phi'(x_1)\,\delta x_1$ which leads to the transversality condition

$$\left[F - (y' - \phi')F_{,y'}\right]_{x=x_1} = 0 \qquad (2)$$

if the other boundary varied i.e. $y_0 = \Theta(x_0)$ a similar condition would lead to

$$\left[F - (y' - \Theta')F_{,y'}\right]_{x=x_0} = 0$$ since $\phi' = \frac{dy_1}{dx_1}$ $\Theta' = \frac{dy_0}{dx_0}$

equation (2) can be made to read

$$\left[(F - y'F_{,y'})\,dx + F_{,y'}\,dy\right]_0^1 = 0$$ **both variable boundaries** (3)

Therefore the requirements for extremum is

$$F_{,y} - \frac{d}{dx}F_{,y'} \equiv 0$$
$$\left(\left[F - y'F_{,y'}\right]dx + F_{,y'}\,dy\right]_0^1 = 0 \qquad (4)$$

The second variation investigation would have led to complex equations which would define what max and min I meant.

However in satisfying (4) we have established an extremal arc. Max & min may be established by considering arc as a fixed end pt problem. Legendre & Jacobi tests may be used; however one must note that these tests are applicable to one pt only. For other end pts, the tests must be performed again.

movable boundaries

$$I = \int_{x_0}^{x_1} F(x, y, z, y', z')\,dx$$ must satisfy $F_{,y} - \frac{d}{dx}F_{,y'} \equiv 0$

$F_{,z} - \frac{d}{dx}F_{,z'} \equiv 0$

Such an integral must have a transversality condition at pts 1 & 0

if $y_1 = \varphi(x_1)$ and $z_1 = \psi(x_1)$ then

$$F - (y' - \varphi')F_{,y'} - (z' - \psi')F_{,z'}\Big|_{x=x_1} = 0 \qquad (5)$$

usually (5) $y_1 = \varphi(x_1)$ and $z_1 = \psi(x_1)$ do not evaluate all the arbitrary constants

however if the boundary pt moves on a surface .$\mathcal{S}$. $z_1 = \varphi(x_1, y_1)$

then $\delta z_1 = \varphi_{,x_1}\,\delta x_1 + \varphi_{,y_1}\,\delta y_1$ and since

$$[F - y'F_{,y'} - z'F_{,z'}]_{x=x_1}\,\delta x_1 + F_{,y'}\Big|_{x=x_1}\delta y_1 + F_{,z'}\Big|_{x=x_1}\delta z_1 = 0$$

then we get

$$\left[F - y'F_{,y'} - z'F_{,z'} + F_{,z'}\,\varphi_{,x_1}\right]_{x=x_1}\delta x_1 + \left[F_{,y'} + F_{,z'}\varphi_{,y_1}\right]_{x=x_1}\delta y_1 = 0$$

again since δx_1 and δy_1 are independent, the conditions for extremum must be

$$\underline{\left[F - y'F_{,y'} - z'F_{,z'} + F_{,z'}\,\varphi_{,x_1}\right]_{x=x_1}} = \underline{\left[F_{,y'} + F_{,z'}\varphi_{,y_1}\right]_{x=x_1}} \equiv 0$$

if $z_0 = \psi(x_0, y_0)$ then

$$\underline{\left[F - y'F_{,y'} - z'F_{,z'} + F_{,z'}\psi_{,x_0}\right]_{x=x_0}} = \underline{\left[F_{,y'} + F_{,z'}\psi_{,y_0}\right]_{x=x_0}} \equiv 0$$

If $\underline{J = \int_{x_0}^{x_1} F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\,dx}$

then for one variable boundary pt $B(x_1, y_{11}, y_{21}, \ldots, y_{n1})$ the transversality condition is

$$\underline{\left(F - \sum_{i=1}^{n} y_i' F_{,y_i'}\right)\Big|_{x=x_1}\delta x_1 + \sum_{i=1}^{n} F_{,y_i'}\Big|_{x=x_1}\delta y_{i1} = 0}$$

Example: $v(y(x)) = \int_0^{\pi/4} (y^2 - y'^2)\,dx \quad y(0) = 0 \quad x$ can move along $x_1 = \frac{\pi}{4}$

Since $x_0 = 0 \quad y_0 = 0 \quad x_1 = \pi/4 \qquad \delta x_0 = \delta y_0 = \delta x_1 = 0$

the transversality condition reduces to $F_{,y'}\Big|_{x=x_1} = 0$

$\therefore \quad F_{,y'} = -2y'\Big|_{x=x_1} = 0$ which means $y = C$ at $x = x_1$ or that the curve

should intersect $x_1 = \pi/4$ at right angles.

Solving the Euler-poisson equation: $F - \frac{d}{dx} F_{,y'} = 0$

leads to $y = C_1 e^{-x} + C_2 e^{x}$ at $x=0$ $y=0$ $\therefore$ $C_1 + C_2 = 0$

also by the condition of $y'(x_1) = 0 \Rightarrow -C_1 e^{-\pi/4} + C_2 e^{\pi/4} = 0$

from the two conditions $C_2 = 0 \Rightarrow C_1 = 0$ or $y \equiv 0$.

Example: Making use of necessary condition of $\delta v = 0$ only, find the curve that makes

$$v(y(x)) = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{y} dx \quad y(0)=0$$ an extremum and (x_1, y_1) moving on $(x-9)^2 + y^2 = 9$

$(F - y'F_{,y'})\big|_{x=x_1} \delta x_1 + F_{,y'}\big|_{x=x_1} \delta y_1 = 0$ is the necessary condition for $\delta v = 0$

$$\left(\frac{\sqrt{1+y'^2}}{y} - y' \frac{y'}{y\sqrt{1+y'^2}}\right)_{x=x_1} \delta x_1 + \frac{y'}{y\sqrt{1+y'^2}}\Bigg)_{x=x_1} \delta y_1 = 0 \; ; \qquad \delta x_1 + y' \delta y_1 = 0$$

$$y' = -1 / \frac{\delta y_1}{\delta x_1}$$

$$2y\,\delta y\big|_{x=x_1} + 2(x-9)\,\delta x\big|_{x=x_1} = 0$$

$$\delta y_1 / \delta x_1 = -\frac{(x-9)}{y}\Bigg|_{\substack{x=x_1 \\ y=y_1}}$$

however the function $F(x,y,y') = \frac{\sqrt{1+y'^2}}{y}$ leads to $(x-C_1)^2 + y^2 = C_2^2$

using $y(0)=0 \Rightarrow C_1 = C_2$

$(x-C_1)^2 + y^2 = C_1^2$; solution of $(x-C_1)^2 + y^2 = C_1^2$ & $(x-9)^2 + y^2 = 9$

leads to $x_1(18-2C_1) = 72$; because of $y' = -1/y_1' = +\frac{y}{(x-9)}$

$y' = -\frac{(x-C_1)}{y}$ leads to $(x_1 - C_1)(x_1 - 9) + y_1^2 = 0$. by manipulation of the two resulting equations $x_1 = \frac{9C_1}{9-C_1}$. It is found $C_1 = 4$ thus the curve that produces the extremal

$$y = \pm\sqrt{16-(x-4)^2} = \pm\sqrt{8x - x^2}$$

Problems with movable boundaries for fnls of the form.

$$v(y(x)) = \int_{x_0}^{x_1} F(x, y, y', y'')\,dx$$

same argument – extremum existing for C will have extremum for C wrt a more restricted class of curves whose end points are in common with curve C and the same direction of tangents at these points. Thus the Euler-Poisson Equat. comes down to

$$F_{,y} - \frac{d}{dx} F_{,y'} + \frac{d^2}{dx^2} F_{,y''} = 0 \quad \text{whose solution is } y = y(x, c_{1 \ldots 4})$$

for solving the four constants relations involving the fundamental condition for extremum $\delta v = 0$ must be found.

if $y(x_0) = y_0$ $y'(x_0) = y_0'$ are known, using same technique as before we obtain

$$F_{,y''}\Big|_{x=x_1} \delta y_1' + \left(F - y'F_{,y'} - y''F_{,y''} + y'\frac{d}{dx}F_{,y''}\right)_{x=x_1} \delta x_1 + \left(F_{,y'} - \frac{d}{dx}(F_{,y''})\right)_{x=x_1} \delta y_1 = 0$$

if $y_1 = \varphi(x_1)$, $y_1' = \psi(x_1)$ leads to

$$\boxed{\left[F - y'F_{,y'} - y''F_{,y''} + y'\frac{d}{dx}F_{,y''} + \left(F_{y'} - \frac{d}{dx}F_{,y''}\right)\varphi' + F_{,y''}\psi'\right]_{x=x_1} \delta x_1 = 0}$$

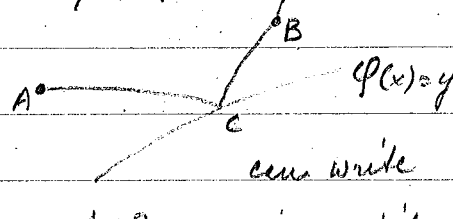
this with $y_1 = \varphi(x_1)$ & $y_1' = \psi(x_1)$ $\rightarrow$ x_1, y_1, y_1'.

if x_1, y_1, y_1' are related by one equation $\varphi(x_1, y_1, y_1') = 0$ then 2 of the variations are arbitrary and the third is dependent by $\varphi_{,x_1}\delta x_1 + \varphi_{,y_1}\delta y_1 + \varphi_{,y_1'}\delta y_1' = 0$ if $\delta y_1' = f(\delta x_1, \delta y_1)$ then

$$\delta y_1' = -(\varphi_{,x_1}\delta x_1 + \varphi_{,y_1}\delta y_1)/\varphi_{,y_1'} \quad \text{where } \varphi_{,y_1'} \neq 0$$

since $\delta x_1 \neq f(\delta y_1)$ then the coefficients of δx_1 & δy_1 must vanish leading to 2 equas. at $x = x_1$; along with $\varphi(x_1, y_1, y_1') = 0$ x_1, y_1, y_1' are evaluated. This is also true at $A(x_0, y_0)$ if A varies.

Extremals with cusps: finding a curve which makes the fnl. $v = \int_{x_0}^{x_2} F(x,y,y')\,dx$ an extremum & that passes through $A(x_0,y_0)$ and $B(x_2,y_2)$ after having been reflecting (refracting) off an arc given by $y = \varphi(x)$.



note that $y'_{x=x_1^-} \neq y'_{x=x_1^+}$ thus we can write v as $\int_{x_0}^{x_1} F\,dx + \int_{x_1}^{x_2} F\,dx$

since A & B are given while C is variable each of the segments is a curve with one variable boundary. For extremal each of the segments must be a solution to Euler's Equation, thus we obtain the condition of $\delta v = 0$

$$\delta v = \left[F + (\varphi' - y')F_{,y'}\right]_{x=x_1^-}\delta x_1 - \left[F + (\varphi' - y')F_{,y'}\right]_{x=x_1^+}\delta x_1 = 0$$

the second integral has a minus sign since the upper limit is the fixed pt. To make both integrals similar i.e. $\int_{\text{fixed pt}}^{\text{movable bdy}}$ by changing limits we must place the minus sign.

Since δx_1 is arbitrary then

$$\left[F + (\varphi' - y')F_{,y'}\right]_{x=x_1^-} = \left[F + (\varphi' - y')F_{,y'}\right]_{x=x_1^+}$$

If points A and B are situated so that $y = y(x)$ is not univalent one should investigate the problem in parametric representation.

In refraction consider a curve $y = \varphi(x)$ in which F is discontin as it crosses it & A & B lie on either side. let $F = F_1$ on one side $F = F_2$ on the other. suppose F_1 & F_2 have y'''s. the same reasoning as before leads $\delta v = 0$ to

$$\boxed{\left[F_1 + (\varphi' - y')F_{1,y'}\right]_{x=x_1^-} = \left[F_2 + (\varphi' - y')F_{2,y'}\right]_{x=x_1^+}}$$

this along with $y_1 = \varphi(x_1)$ establishes x_1 & y_1

(CORNERS)

cusps are used sometimes because $F \geq 0 \Rightarrow v \geq 0$. Smooth curves from A to B will always produce $v > 0$. The lower most bound $v = 0$ would only be attained by piecewise smooth curves.

Conditions for solution of cusp problem to give extremal for $v(y(x)) = \int_{x_0}^{x_2} F(x, y, y')\,dx$. Since an extremal with a cusp consists of smooth piecewise curves these curve should coincide with the integral curves of the Euler's Equation. By fixing all but one and varying this one piecewise curve we have a problem with fixed end pts. FOR 1 CUSP we obtain

$$v = \int_{x_0}^{x_2} F\,dx = \int_{x_0}^{x_1} F\,dx + \int_{x_1}^{x_2} F\,dx$$

$$0 = \delta v = (F - y' F_{y'})_{x = x_1^-}\,\delta x_1 + F_{y'}\big|_{x = x_1^-}\,\delta y_1 - (F - y' F_{y'})_{x = x_1^+}\,\delta x_1 - F_{y'}\big|_{x = x_1^+}\,\delta y_1$$

this leads to

$$\boxed{(F - y' F_{y'})\big|_{x = x_1^-} = (F - y' F_{y'})\big|_{x = x_1^+}, \quad F_{y'}\big|_{x = x_1^-} = F_{y'}\big|_{x = x_1^+}}$$

These conditions along with continuity condit of cusp determine coordinates of cusp.

Example: find solution for 1 corner for problem

$$v(y) = \int_0^4 (y' - 1)^2 (y' + 1)^2\,dx \qquad y(0) = 0 \quad y(4) = 2$$

since $F \geq 0$ we must use

$$(F - y' F_{y'})\big|_{x = x_1^-} = (F - y' F_{y'})\big|_{x = x_1^+}$$

$$\left[(y'-1)^2 (y'+1)^2 - 2y'(y'-1)^2 (y'+1) - 2y'(y'+1)^2 (y'-1)\right] = \ldots\ldots$$

$$(y'-1)^2 (y'+1)^2 - 2y'\left[(y'^2 - 1)(2y')\right] = \ldots\ldots$$

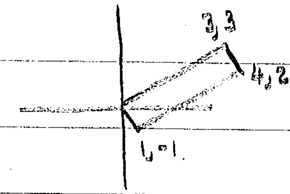
and $F_{y'}\big|_{x = x_1^-} = F_{y'}\big|_{x = x_1^+}$

lead to $y'_{x = x_1^-} = 1 \qquad y'_{x = x_1^+} = -1$

$y'_{x = x_1^-} = -1 \qquad y'_{x = x_1^+} = 1$

for the first set of eq we get

$$y = x \qquad 0 \le x \le 3$$
$$y = -x + 6 \qquad 3 \le x \le 4$$



for the second set of eq we get

$$y = -x \qquad 0 \le x \le 1$$
$$y = x - 2 \qquad 1 \le x \le 4$$

Examples: are there any solutions with cusps for the problem of extrema of fnl.

$$v(y(x)) = \int_0^{x_1} (y'^4 - 6y'^2)\,dx \quad y(0) = 0 \quad y(x_1) = y_1 ?$$

To find out one must look at

$$F_{y'}\big|_{x = x_0^-} = F_{y'}\big|_{x = x_0^+}$$
$$(4y'^3 - 12y')_{x = x_0^-} = (4y'^3 - 12y')_{x = x_0^+}$$
$$4y'(y' - \sqrt{3})(y' + \sqrt{3})\big|_{x = x_0^-} = 4y'(y' - \sqrt{3})(y' + \sqrt{3})\big|_{x = x_0^+}$$

it is seen that solutions of the form $y = \sqrt{3}x + C_1$, $y = -\sqrt{3}x + C_2$ do solve the problem.

Example: are there any solutions with cusps for problem of extrema of fnl.

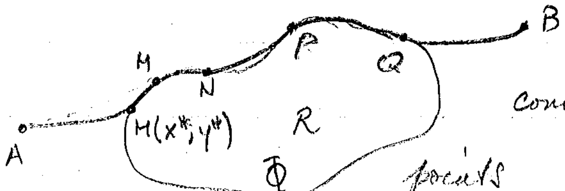
$$v(y(x)) = \int_{x_0}^{x_1} (y'^2 + 2xy - y^2)\,dx \quad y(x_0) = y_0 \quad y(x_1) = y_1$$

One must check

$$F_{y'}\big|_{x = x^{*-}} = F_{y'}\big|_{x = x^{*+}}$$
$$2y'\big|_{x = x_*^-} = 2y'\big|_{x = x_*^+};$$ this equat leads to an equation of $\dot{y} = C_1$ & $\dot{y} = C_2$ but because the cusp must satisfy both equations it turns out that $C_1 = C_2$ or that the curve is cont.

at the cusp. Therefore no cusp exists and ~~no solution does either~~.
However an extremum does exist only on a smooth curve.

One sided variations: for some problems problems of finding extrema restrictions on the curve may be they must not pass through a certain domain R and $\Phi(x,y)=0 \subset \partial R$. A curve for such a problem would pass entirely outside ∂R or lie partly on ∂R & partly outside ∂R. The first case would be treated as normal since R or ∂R would not influence extrema. For second case where **curve lies on ∂R one sided variations would occur (no variations in R is allowed)**. Consider the following



conditions at pt. M can be duplicated for other points

$$\int_{x_0}^{x_1} F\,dx = \int_{x_0}^{x^*} F\,dx + \int_{x^*}^{x_1} F\,dx \quad \text{since } x^* \text{ can move}$$

$\int_{x_0}^{x^*} F\,dx$ bounded by $\Phi(x,y)=0$ produces $\left[F+(\varphi'-y')F_{y'}\right]_{x=x^*}\delta x^*$
where $\Phi(x,y)=0 \Rightarrow y=\varphi(x)$. $\int_{x^*}^{x_1} F\,dx$ has a variable point (x^*,y^*) but at the neighborhood of this point $y=\varphi(x)$ doesn't vary. $\therefore$ δv becomes a fn of $\delta x^*, \delta y^*$

$$\therefore \int_{x^*+\delta x^*}^{x_1} F\,dx - \int_{x^*}^{x_1} F\,dx = -\int_{x^*}^{x^*+\delta x^*} F(x,\varphi(x),\varphi'(x))\,dx \quad \text{since } y=\varphi(x),\ y'=\varphi'(x)$$

by mean value theorem

$$-\int_{x^*}^{x^*+\delta x^*} F(x,\varphi(x),\varphi'(x))\,dx = \left[-F(x,\varphi(x),\varphi'(x))\Big|_{x=x^*} + \beta\right]\delta x^* \quad \text{where } \beta \to 0 \ \ \delta x^* \to 0$$

$$\therefore \ \delta v = \left\{\left[F+(\varphi'-y')F_{y'}\right]_{x=x^*} - F(x,y,\varphi')\Big|_{x=x^*}\right\}\delta x^* \quad \text{for } y(x^*)=\varphi(x^*)$$

which leads to $\left[F(x,y,y') - F(x,y,\varphi') - (y'-\varphi')F_{y'}(x,y,y')\right]_{x=x^*} = 0$

use of mean val theorem on preceding leads to

$$(y'-\varphi')\left[F_{y'}(x,y,q)-F_{y'}(x,y,y')\right]_{x=x^*}=0 \quad \varphi'(x^*)\leq q\leq y'(x^*)$$

use of MV theorem on preceding leads to

$$(y'-\varphi')(q-y')\,F_{y'y'}(x,y,q^*)_{x=x^*}=0 \quad \text{where} \quad q\leq q^*\leq y'(x^*)$$

if $F_{y'y'}(x,y,q^*)\neq 0$ then condit at M is $y'(x^*)=\varphi'(x^*)$ for $q=y'$ only when $y'(x^*)=\varphi'(x^*)$ since $\varphi'(x^*)\leq q\leq \varphi'(x^*)$.

Thus the curve must join tangent to the boundary if it must parallel the boundary ∂R.

Mixed problems: if $v=\int_{x_0}^{x_1}F(x,y,y')dx+\Phi(x_0,y_0,x_1,y_1)$ where $y_0=\varphi(x_0)$ & $y_1=\psi(x_1)$ or if $v=\iint_D F\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)dx\,dy+\int_C\Phi\,ds$ where C is ∂D & ds is arclength or

$$v=\iiint_W F\left(x,y,z,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)dx\,dy\,dz+\iint_S\Phi\,dS$$

where S is boundary surface of domain W, solution of problem are the same is solution of fundamental var. problems but boundaries are more complicated.

Consider first case: must solve Euler eq $F_y-\frac{d}{dx}F_{y'}=0$ since it only varies from $\int_{x_0}^{x_1}F\,dx$ by the constant term $\Phi(x_0,y_0,x_1,y_1)$.

$$\therefore\ \delta v=\left[(F-y'F_{y'})\delta x_1+F_{y'}\delta y_1\right]_{x=x_1}-\left[(F-y'F_{y'})\delta x_0+F_{y'}\delta y_0\right]_{x=x_0}+$$
$$\partial\Phi/\partial x_0\,\delta x_0+\partial\Phi/\partial y_0\,\delta y_0+\partial\Phi/\partial x_1\,\delta x_1+\partial\Phi/\partial y_1\,\delta y_1$$

if $y_0=\varphi(x_0)$ & $y_1=\psi(x_1)$ this can be written as:

$$\left[F+F_{y'}(\psi'-y')+\partial\Phi/\partial x_1+\partial\Phi/\partial y_1\,\psi'\right]_{x=x_1}\delta x_1-$$
$$\left[F+F_{y'}(\varphi'-y')-\partial\Phi/\partial x_0-\partial\Phi/\partial y_0\,\varphi'\right]_{x=x_0}\delta x_0=0$$

since both δx_1 & δx_0 are independent of each other the above reduces to

$$\left[F + F_{,y'}(\varphi' - y') + \Phi_{,x_1} + \Phi_{,y_1}\right]_{x=x_1} = 0$$
$$\left[F + F_{,y'}(\varphi' - y') + \Phi_{,x_0} - \Phi_{,y_0}\right]_{x=x_0} = 0$$

To calculate variation of problems with mult. integrals, we must calc. variation of mult. integral with variable boundary.

Example: Starting from point $A(x_0, y_0)$, where $y_0 \neq \varphi(x_0)$, reach pt $C(x_2, y_2)$, where $y_2 = \varphi(x_2)$, in shortest time. outside $y = \varphi(x)$ $v = c = v_1$; on $y = \varphi(x)$ $v = c = v_2 > v_1$.

$$v(y(x)) = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{v_1}\,dx + \int_{x_1}^{x_2} \frac{\sqrt{1+\varphi'^2}}{v_2}\,dx$$

the problem is to find the shortest distance from pt. 0 to 1 on curve

$$\left[F + (\varphi' - y')F_{,y'}\right]_{x=x_1^-} = \left[F + (\varphi' - y')F_{,y'}\right]_{x=x_1^+}$$

$$\left.\frac{1+\varphi' y'}{v_1\sqrt{1+y'^2}}\right|_{x=x_1^-} = \left.\frac{\sqrt{1+\varphi'^2}}{v_2}\right|_{x=x_1^+} \quad \text{hold.}$$

Example: Find the curves that make the fnl $v(y(x)) = \int_0^{10} y'^3\,dx$ $y(0) = 0$ $y(10) = 0$ an extremum subject to condition that no curve passes through interior of $(x-5)^2 + y^2 = 9$

Since $y'(x^*) = \varphi'(x^*)$ we obtain $\varphi' = \dfrac{-(x-5)}{\sqrt{9-(x-5)^2}}$

use of Euler's eq $\rightarrow 6y'' \cdot y' = 0$ or $y = \pm C$ or $y = \pm C_1 x + C_2$; $y = C$ turns out to be $y = 0$ but $y = 0$ runs through the domain $\therefore$ solution is $y = \pm C_1 x + C_2$

since $y(0) = 0$ $C_2 = 0$ or $y = \pm C_1 x$

solution of $y = \pm c_1 x$, $y = \pm\sqrt{9-(x-5)^2}$ and $\overset{y' = \varphi'}{c_1 = \dfrac{-(x-5)}{\sqrt{9-(x-5)^2}}}$

leads to $x_1^* = 16/5$ and $c_1 = \pm 3/4$

$\therefore$ curve (pt 1) runs from $0 \le x \le 16/5$ on $y = \pm 3/4 x$

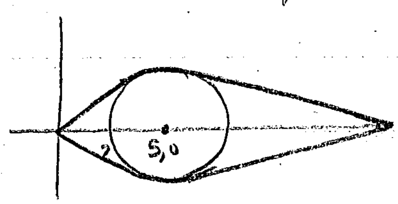
from the other side $y(10) = 0$ leads $y = \pm c_1 x + c_2$ to $c_2 = \mp 10 c_1$

$\therefore$ $y = \pm c_1 (x-10)$ & $y = \pm\sqrt{9-(x-5)^2}$ along with $\pm c_1 = \dfrac{-(x-5)}{\pm\sqrt{9-(x-5)^2}}$

pt(2)

leads to $x_2^* = 34/5$ and $c_1 = \mp 3/4$ $\therefore$ $y = \mp 3/4 (x-10)$ on $\frac{34}{5} \le x \le 10$

for part 3 curve runs parallel to circle for $\frac{16}{5} \le x \le 34/5$

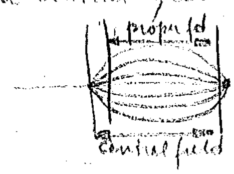


Sufficiency Conditions for an Extremum

Field of Extremals:

if for each pt in a domain D $\exists !$ a curve $y = y(x, C_1) \in y = y(x, C)$ then $y = y(x, C)$ is a proper field; the slope $p(x, y)$ to the curve $y = y(x, C)$ is the slope of the field. if for each pt $\exists$ more than 1 curve which satisfies it, the family of curves which these curves belong to do not constitute a field. If a family of curves $y = y(x, c)$ pass through one pt (x_0, y_0) and cover the domain D and never reintersect in D except at (x_0, y_0) this family of curves is known as a central field.

Example: the family of curves $y = C \sin x$ form a central field for $0 \le x < \pi$ but for a proper field for $0 < x < \pi$



If a proper or central field is generated by a family of extremals for some variational problem, then it is called a field of extremals.

An extremal curve $y = y(x)$ (simple fnl with fixed bc) is admissible in a field of extremals if $\exists$ a family of extremals $y = y(x, C)$ which is a field & for some $C = C_0$ turns into the extremal $y = y(x)$ not lying on the ∂D in which the family is a field. If the extremal passes through a point A & $\exists$ a pencil of extremals which forms a central field about A then the central field admits the extremal. The parameter for the family is the slope of the tangent to the curve at A.

Jacobi's Condition

Consider a family of extremals $y = y(x, C)$ which all pass through $A(x_0, y_0)$. The envelope of the family is found by use of equas. $F(x, y, C) = 0$ & $\partial y(x, C)/\partial C = 0$. Any 2 curves in this family will intersect at a pt near the envelope $\Phi(x, y) = 0$. If the arc of the extremal has no pt in common with the envelope those curves close to the extremal will form a central field since they do not intersect.

If the extremal does have a pt, other than (x_0, y_0), where it intersects the envelope then those curves close to the extremal will intersect near this point. Therefore they do not form a field. This point is known as the conjugate of $A(x_0, y_0)$

Thus Jacobi's condition is: to form a central field of extremals with center at A admitting an arc of the extremal curve, it is sufficient that the conjugate point not lie on the arc.

To find Jacobi's Equation: since $\Phi(x,y)=0$ is found by $y=y(x,c)$ & $\frac{\partial y(x,c)}{\partial c}=0$ also since $\partial y(x,c)/\partial c$ take along an arbitrary curve of the family turns into a fn of x only, use of these and the fact that $y=y(x,c)$ must satisfy Eulers Eq. allows us to differ. Euler's Eq with respect to C $\Rightarrow$,

$$\frac{\partial}{\partial c}\left[F_{,y} - \frac{d}{dx} F_{,y'}\right] = F_{,yy}\, u + F_{,yy'}\, u' - \frac{d}{dx}\left(F_{,yy'}\, u + F_{,y'y'}\, u'\right) = 0$$

or $$u\left(F_{,yy} - \frac{d}{dx} F_{,yy'}\right) - \frac{d}{dx}\left(F_{,y'y'}\, u'\right) = 0 \qquad \text{where } u = \frac{\partial y(x,c)}{\partial c}$$

if the solution to this equat is $u = \frac{\partial}{\partial c} y$ at A i.e. $x = x_0$ and also some other pt in the interval then the conjugate pt is found by $y=y(x,c_0)$ & $\frac{\partial}{\partial c} y = 0$ or $u=0$ & it lies on the extremal. If Jacobi's equat. vanishes at only 1 pt (center of pencil) then $\nexists$ no conjugate pt & the Jacobi condition is satisfied

The Weierstrass Condition & Equation:

1. Jacobi condition must hold. → extremal is admitted to central field; slope fn of field is $p(x,y)$. if C^* & C' are 2 extremal curves then ΔU in field is

$$\Delta U = \int_{C^*} F\,dx - \int_{C'} F\,dx$$

consider a fn ∋ $\int_{C^*}\{F(x,y,p) + (y'-p)\,F_p(x,y,p)\}\,dx$ becomes $\int_{C'} F\,dx$ when $p = y'$. It is the integral of an exact differential since it is, it is not a fn of path it then follows $\int_{C'} F\,dx = \int_{C^*} (F + (y'-p)F_{,p})\,dx$

we call the integrand of the implicit fn. the Weierstrass Fn. or

$$\underline{E(x,y,p,y') \equiv F(x,y,y') - F(x,y,p) - (y'-p)\,F_{,p}(x,y,p)}$$

or $\Delta U = \int_{C^*} E(x,y,p,y')\,dx$

WEIERSTRASS

Conditions for weak extremum.

1. C is an extremal curve satisfying boundary conditions
2. Jacobi condit.
3. E has a constant sign for $\forall (x,y)$ close to C & arbitrary values of y' close to p.

if $E \geq 0$ min if $E \leq 0$ max.

Conditions for strong extremum

1. Same as 1 above 2. Same as 2 above

3. E has constant sign for $\forall (x,y)$ close to curve C & arbitrary y' value

if $E \geq 0$ min if $E \leq 0$ max.

LEGENDRE

Use of Taylor's formula on $F(x,y,y')$ leads to $E(x,y,p,y') = \frac{(y'-p)^2}{2!} F_{,y'y'}(x,y,q)$

where $p \leq q \leq y'$ ∴ if $F_{,y'y'} > 0$ $E \geq 0$ min

$F_{,y'y'} < 0$ $E \leq 0$ max.

these are the Weak Legendre Condition if $F_{,y'y'} \neq 0$ along C because of continuity sign remains constant $\forall$ points close to C. Strong Legendre Cond. are $F_{y'y'} \leq (\geq)\, 0$ & arbitrary q

if $\quad v(y_i) = \int_{x_0}^{x_1} F(x, y_i, y_i')\,dx \quad \& \quad y_i(x_0) = y_{i0}\,,\ y_i(x_1) = y_{i1}$

$\quad i=1,n$

then $\quad E = F(x, y_i, y_i') - F(x, y_i, p_i) - \sum_{i=1}^{n} (y_i' - p_i) F_{,p_i}(x, y_k, p_k)$

$\quad k=1,\dots n$

Legendre Condition $\quad F_{,y'y'} \geq 0 \quad$ is replaced by

$$F_{,y_1'y_1'} \geq 0 \qquad \begin{vmatrix} F_{,y_1'y_1'} & F_{,y_1'y_2'} \\ F_{,y_2'y_1'} & F_{,y_2'y_2'} \end{vmatrix} \geq 0 \ \dots\dots \qquad \begin{vmatrix} F_{,y_1'y_1'} & F_{,y_1'y_2'} & \cdots & F_{,y_1'y_n'} \\ \vdots & & \ddots & \\ F_{,y_n'y_1'} & F_{,y_n'y_2'} & \cdots & F_{,y_n'y_n'} \end{vmatrix} \geq 0$$

for weak Extremum Sufficient Conditions depend on sign of second variation

$$\Delta v = \int_{x_0}^{x_1} [F(x, y+\delta y, y'+\delta y') - F(x,y,y')]\,dx = \int_{x_0}^{x_1} (F_{,y}\,\delta y + F_{,y'}\,\delta y')\,dx$$
$$+ \frac{1}{2}\int_{x_0}^{x_1} (F_{,yy}\,\delta y^2 + 2F_{,yy'}\,\delta y\,\delta y' + F_{,y'y'}\,\delta y'^2)\,dx + R\ (\text{order} > 2)$$

first variation $\equiv 0$ along C. $\therefore$ sign of Δv depends on $\delta^2 v$. Legendre & Jacobi guarantee $\delta^2 v$ sign to be const. if an integral $\int_{x_0}^{x_1} \frac{d}{dx}(\omega\,\delta y^2)\,dx$ is introduced (note: $\delta y|_{x_0} = \delta y|_{x_1} = 0$) to $\delta^2 v$ we get

$$\underline{\delta^2 v} = \int_{x_0}^{x_1} \left[(F_{,yy} + \omega')\,\delta y^2 + 2(F_{,yy'} + \omega)\,\delta y\,\delta y' + F_{,y'y'}\,\delta y'^2\right] dx$$

Choose $\omega(x)$ $\ni$ integrand becomes perfect square $\quad \omega(x)$ must satisfy

$$F_{,y'y'}(F_{,yy} + \omega') - (F_{,yy'} + \omega)^2 = 0$$ then $\delta^2 v$ becomes

$$\delta^2 v = \int_{x_0}^{x_1} F_{,y'y'} \left(\delta y' + \frac{F_{,yy'} + \omega}{F_{y'y'}}\,\delta y\right)^2 dx$$ then sign of $\delta^2 v$, Δv is fn of $F_{,y'y'}$

→ let $\omega = -F_{,yy'} - F_{,y'y'}\,\frac{u'}{u}$ where u is an unknown fn. we obtain from above Jacobi's equation (from setting integrand to perfect square)

if Jacobi's Condit holds $\exists$ for some x a $\omega(x)$ set above. Thus Jacobi's & Legendre conditions ensure ~~sign~~ stability of $\delta^2 v$ sign

Problems of Sufficient Conditions

$v(y(x)) = \int_0^1 (y'^2 + y^2 + 2ye^{2x})\,dx \quad y(0) = \frac{1}{3} \quad y(1) = \frac{1}{3}e^2$

Eulers equation becomes $y'' - y = e^{2x}$ or $y(x, C) = C_1 e^x + C_2 e^{-x} + \frac{1}{3}e^{2x}$

solving the boundary conditions leads to $y = \frac{1}{3}e^{2x}$ $\therefore$ the curve satisfying boundary condition. Jacobi's Equation becomes $u'' - u = 0$ or $u = C_1 \sinh x + C_2 \cosh x$

$\therefore$ since u vanishes at $x = x_0$ then $u(0) = 0$ iff $C_2 = 0$ $\therefore$ $u = C_1 \sinh x$

by finding roots of $u \rightarrow$ conjugate pts. however u vanishes only at $x = x_0 = 0$

$\therefore$ Jacobi Condition holds. note: $F_{y'y'} = 2 > 0$ for $\forall y'$ $\therefore$ a strong minimum exists on $y = \frac{1}{3}e^{2x}$

$v(y(x)) = \int_0^{\pi/4} (4y^2 - y'^2 + 8y)\,dx \quad y(0) = -1 \quad y(\pi/4) = 0$

Euler's Eq becomes $8y + 8 + 2y'' = 0$ then $y = C_1 \sin 2x + C_2 \cos 2x - 1$

$\therefore$ when $x = 0$ $C_2 = 0$ when $x = \pi/4$ $C_1 = 1$ C satisfies boundary condits.

$\therefore$ $y = \sin 2x - 1$. Solving Jacobi's Equat leads to $u'' + 4u = 0$ or

$u(x) = C_1 \sin 2x + C_2 \cos 2x$ since $x = x_0 = 0$ is a solution $C_2 = 0$ $\therefore$ $u = C_1 \sin 2x$

note that for $0 < x \le \pi/4$ u depends only on C and does not vanish in the interval $\therefore$ $u(x) = 0$ only at $x = 0$ $\therefore$ no conjugate pt exists $\rightarrow$ Jacobi Cond. satisfied. $F_{y'y'} = -2 < 0$ $\forall y'$ $\therefore$ a strong max exists on $y = \sin 2x - 1$.

$v(y(x)) = \int_0^a (y'^2 + 2yy' - 16y^2)\,dx \quad a > 0 \quad y(0) = 0 \quad y(a) = 0$

Euler's Eq leads to $y = C_1 \sin 4x$ for the second boundary cond. if $C_1 = 0$ $y = 0$

if $\sin 4a = 0$ $4a = k\pi$ $a = \frac{k\pi}{4}$. Solution of Jacobi's Equat leads to $u'' + 17u = 0$

or $u(x) = C_1 \cos\sqrt{17}x + C_2 \sin\sqrt{17}x$ since $u(x) = 0$ for $x = x_0 = 0$ $C_1 = 0$

$\therefore$ $u(x) = C_2 \sin\sqrt{17}x$ note for $u(x) \equiv 0$ $\sqrt{17}x = k\pi$; $x = \frac{k\pi}{\sqrt{17}}$

but from above the interval is $0 \le x \le \frac{k\pi}{4}$ but $\frac{k\pi}{4} > \frac{k\pi}{\sqrt{17}}$

therefore another pt will always lie in the interval : $\exists$ a conjugate pt.
thus no extremum can exist for this class of curves.

$$v(y(x)) = \int_0^{x_1} \frac{dx}{y'^2} \qquad y(0) = 0 \quad y(x_1) = y_1 \qquad x_1, y_1 > 0$$

Euler's Equat. $-2y''(-3y'^{-4}) = 0 \qquad y = C_1 x + C_2$

or $y = C_1 x$ since $y_1 = y(x_1)$ $\qquad y = \frac{y_1}{x_1} x$

Jacobi's Eq reads : $-\frac{d}{dx}(F_{y'y'}u') = -\frac{d}{dx}(6y'^{-4}u') = 0$

or $6y'^{-4}u' = C \qquad u' = \frac{C}{6} y'^4 = \frac{C}{6}\left(\frac{y_1}{x_1}\right)^4$

$u = \frac{C}{6}\left(\frac{y_1}{x_1}\right)^4 x + C_2$ since $u(0) = 0 \qquad C_2 = 0$

for $0 \le x \le x_1$ $\exists$ no other point which makes $u(x) \equiv 0$

$\therefore$ Jacobi condit is satisfied.

note for $F_{y'y'} = -2(-3y'^{-4}) = \frac{6}{(y')^4} > 0$ for $\forall y'$ <u>except</u> $y' \equiv 0$

There exists a weak minimum along $y = (y_1/x_1)\,x$

Variational Problems of Constrained Extrema.

<u>Constraints of form $\varphi(x, y_i) = 0$ where $i = 1, 2, \ldots, n$</u>

Consider $v(y_i) = \int_{x_0}^{x_1} F(x, y_i, y_i')\,dx$ with constraint $\varphi_j(x, y_i) = 0$

$i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, m \quad m < n.$

Possible method of solution solve $\varphi_j = 0$ with respect to m arguments $\ni$

$$y_{ii} = y_{ii}(y_{jj}) \quad \text{when } ii = 1, \ldots m \quad jj = m+1, \ldots, n$$

taking y_{ii} substitute into $v(y_i) \Rightarrow$ fnl $W(jj)$ depends only on $n-m$ independent arguments. <u>or</u> Lagrange Multipliers

<u>Form a new fnl $v^* = \int_{x_0}^{x_1} F^* dx$ where $F^* = F + \sum_{j=1}^{m} \lambda_j(x)\varphi_j$</u>

solve system of Euler Equations $F^*_{,y_i} - \frac{d}{dx} F^*_{,y_i'} = 0 \quad i = 1, \ldots, n$

with additional equations of constraints $\varphi_j = 0 \quad j = 1, 2, \ldots, m.$

This gives $m+n$ equat. to determine y_i's & λ_j's. & b.c. $y_i(x_0) = y_{i0}$ $y_i(x_1) = y_{i1}$ compatible with constraints are used to determine $2n$ constants.

if v^* = extremum $\lambda_j(x)$ & y_i must satisfy Euler then $\varphi_j = 0 \ \forall j$

$\therefore v^* = v.$

<u>Theorem</u>: a sequence of fns. y_i which makes $v = \int_{x_0}^{x_1} F(x, y_i, y_i')\,dx$ have an extremum with $\varphi_j(x, y_i) = 0 \quad j = 1, 2, \ldots, m \quad m < n$ satisfies for suitable chosen $\lambda_j(x)$ the Euler equat. for

$$v^* = \int_{x_0}^{x_1} F^* dx = \int_{x_0}^{x_1} \left(F + \sum_{j=1}^{m} \lambda_j(x)\varphi_j\right) dx$$

the fns. $\lambda_j(x)$ & $y_j(x)$ can be determined from.

$$F^*_{,y_i} - \frac{d}{dx} F^*_{,y_i'} = 0 \text{ and the constraints } \varphi_j = 0$$

The equats $\varphi_j = 0$ can be considered Euler Equations for fnl v^* if we consider not only y_i as arguments of v^* but also $\lambda_j(x)$. The eqs $\varphi_j = 0$ are assumed indep. ie $\exists$ a Jacobian of order $m \ni \frac{D(\varphi_j)}{D(y_i)} \neq 0$

Proof: since $\delta v = 0 = \int (F_{,y_i}\,\delta y_i + F_{,y_i'}\,\delta y_i')\,dx = 0$ or $\int_{x_0}^{x_1} \sum_{i=1}^{n} \left(F_{,y_i} - \frac{d}{dx} F_{,y_i'}\right) \delta y_i \, dx = 0$

since $\varphi_j(y_i) = 0$ δy_i is not arbitrary since $i > j$. Thus δy_i must satisfy $\delta \varphi_j = 0$ or $\sum_{i=1}^{n} \frac{\partial \varphi_j}{\partial y_i} \delta y_i = 0$ ∴ n-m are independent the remaining are determined from the above.

Take $\int_{x_0}^{x_1} \lambda_j \sum \frac{\partial \varphi_j}{\partial y_i} \delta y_i \, dx = 0$ take the following & the $\delta v = 0$ Eq

$$\int_{x_0}^{x_1} \sum_{i=1}^{n} \left(F_{,y_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial \varphi_j}{\partial y_i} - \frac{d}{dx} F_{,y_i'} \right) \delta y_i \, dx = 0$$ using F^* we get

$$\int_{x_0}^{x_1} \sum_{i=1}^{n} \left(F^*_{,y_i} - \frac{d}{dx} F^*_{,y_i'} \right) \delta y_i \, dx = 0$$ since δy_i is not arbitrary

we choose m multipliers $\lambda_j(x)$ $j = 1, \dots, m$ ∋ $F^*_{,y_i} - \frac{d}{dx} F^*_{,y_i'} = 0$ $i = 1, \dots, m$ or

$$F_{,y_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial \varphi_j}{\partial y_i} - \frac{d}{dx} F_{,y_i'} = 0 \qquad i = 1, \dots, m$$

This is a system of linear eq wrt λ_j where $\frac{D(\varphi_j)}{D(y_j)} \neq 0$

∴ ∃ a solution for $\lambda_j(x)$ $j = 1, 2, \dots, m$.

with the choice of $\lambda_j(x)$ then $\int_{x_0}^{x_1} \sum_{i=1}^{n} \left(F^*_{,y_i} - \frac{d}{dx} F^*_{,y_i'} \right) \delta y_i \, dx = 0$ becomes,

$$\int_{x_0}^{x_1} \sum_{i=m+1}^{n} \left(F^*_{,y_i} - \frac{d}{dx} F^*_{,y_i'} \right) \delta y_i \, dx = 0$$ since we have written the extremum in terms of the n-m independent terms δy_i becomes arbitrary and

$$F^*_{,y_i} - \frac{d}{dx} F^*_{,y_i'} = 0 \qquad j = m+1, m+2, \dots, n$$

also it is noted from before that if $\lambda_j(x)$ is chosen $F^*_{,y_i} - \frac{d}{dx} F^*_{,y_i'} = 0$ $j = 1, \dots, m$

Thus a fct. which has a constrained extremum must satisfy

$$F^*_{,y_i} - \frac{d}{dx} F^*_{,y_i'} = 0 \qquad i = 1, 2, \dots, n \quad \&$$

$$\varphi_j(x, y_i) = 0 \qquad j = 1, 2, \dots, m.$$

Constraints of form $\varphi(x, y_i, y_i') = 0$ acting on $v = \int_{x_0}^{x_1} F(x, y_i, y_i')\,dx$

assume $\dfrac{D(\varphi_j)}{D(y_j')} \neq 0$ (constraints independent) $\Rightarrow y_j' = \varphi_j(x, y_i, y_{jj}')$

varying the constants leads to $\displaystyle\sum_{i=1}^{n} \frac{\partial \varphi_j}{\partial y_i}\delta y_i + \sum_{i=1}^{n} \frac{\partial \varphi_j}{\partial y_i'}\delta y_i' = 0$

use same approach as before we finally get

$$\left[F^*_{,y_i} - \frac{d}{dx} F^*_{,y_i'} = 0 \quad i = 1, \dots, n \quad \& \quad \varphi_j = 0 \quad j = 1, \dots, m. \right]$$

Isoperimetric problems - finding geometrical fig producing maxima when constrained, or find extrema of

$$v = \int_{x_0}^{x_1} F(x, y_i, y_i')\,dx \quad \text{while} \quad \int_{x_0}^{x_1} F_j(x, y_i, y_n') = l_j \quad \text{when } m >, <, = n$$

Define a $z_j(x) = \displaystyle\int_{x_0}^{x} F_j\,dx \Rightarrow z_j(x_0) = 0$ & $z_j(x_1) = l_j$ for $\displaystyle\int_{x_0}^{x_1} F_j\,dx = l_j$

$\partial z_j/\partial x$ gives $z_j'(x) = F_j(x, y_i, y_i')$ applying Lagrange multipliers

define $v^* = \displaystyle\int_{x_0}^{x_1} \left(F + \sum_{j=1}^{m} \lambda_j (F_j - z_j')\right)dx = \int_{x_0}^{x_1} F^*\,dx$

Euler's Equations become.

$$F^*_{,y_i} - \frac{d}{dx} F^*_{,y_i'} = 0 \qquad i = 1, 2, \dots, n$$

$$F^*_{,y_j} - \frac{d}{dx} F^*_{,y_j'} = 0 \qquad j = 1, 2, \dots, m \qquad \text{also } \frac{d}{dx}\lambda_j(x) = 0$$

Rule: to obtain fundamental necessary condition for isoperimet. problem of finding extrema of fnl $v \int_{x_0}^{x_1} F\,dx$ with constraint $\int_{x_0}^{x_1} F_j\,dx = l_j$ set up a new fnl.

1. write $v^{**} = \displaystyle\int_{x_0}^{x_1} \left(F + \sum_{j=1}^{m} \lambda_j F_j\right)dx$
2. write Euler Equations, arbitrary constants $C_1, \dots, C_{2n}$ & λ_j are found from b c & isoperimetric condit.
3. multiply v^{**} by $\mu_0 \Rightarrow \mu_0 v^{**} = \displaystyle\int_{x_0}^{x_1} \sum_{j=0}^{m} \mu_j F_j\,dx$ where $F_0 = F$ and $\mu_j = \lambda_j \mu_0$ $j = 1, 2, \dots, m$.

now $\forall$ fns F_i are involved symmetrically $\therefore$ extremal curves for variational problems of find'g extrema of fnl $\int_{x_0}^{x_1} F_s dx$ with isoper. condit $\int_{x_0}^{x_1} F_j dx = l_j$ where $j = 0, 1, \ldots, s-1, s+1, m$ & the extremal curves for the original variational problem are the same. **Principle of Reciprocity**

Examples:

Find extremals of isoperimetric problem

$$v(y(x)) = \int_{x_0}^{x_1} y'^2 dx \quad \text{w.c.c.} \int_{x_0}^{x_1} y\,dx = a$$

$$F^* = y'^2 + \lambda y \qquad F^*_{,y} - \frac{d}{dx} F^*_{,y'} = \lambda - 2y'' = 0$$

$$y'' = \frac{\lambda}{2} \qquad y = \frac{\lambda}{2}\frac{x^2}{2} + C_1 x + C_2$$

where λ, C_1, C_2 are found from constraint condit & b.c.

Find differential equations of extremals for isoperimetric problem

$$\int_0^{x_1} \left(p(x)\, y'^2 + q(x)\, y^2\right) dx \quad \text{wcc} \int_0^{x_1} r(x)\, y^2 dx$$

where $y(0) = 0 \quad y(x_1) = 0$

$$F^* = p(x)\, y'^2 + q(x)\, y^2 + \lambda\, r(x)\, y^2$$

$$F^*_{,y} - \frac{d}{dx} F^*_{,y'} = 2y(\lambda r(x) + q(x)) - \frac{d}{dx}\left(2p(x)y'\right) = 0$$

this equation along with boundary condit & constraint would lead to solution.

Find the extremals for the isoperimetric problem

$$v(y(x)) = \int_0^1 (y'^2 + x^2)\,dx \quad \text{w.c.c.} \int_0^1 y^2 dx = 2 \quad y(0) = 0 \quad y(1) = 0$$

$F^*_{,y} - \frac{d}{dx} F^*_{,y'} = 0 \qquad 2\lambda y - 2y'' = 0 \qquad$ or $\quad y'' - \lambda y = 0$

such a solution leads to $y = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$

b.c. give
$$C_1 + C_2 = 0$$
$$C_1 e^{\sqrt{\lambda}} + C_2 e^{-\sqrt{\lambda}} = 0 \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ e^{\sqrt{\lambda}} & e^{-\sqrt{\lambda}} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

a trivial solution is $C_1, C_2 = 0$ for $\lambda > 0$. If $\lambda < 0$, for $\lambda = 0$ no solution

$y'' + \lambda_1 y = 0 \quad$ or $\quad y = C_1 \sin\sqrt{\lambda_1}\, x + C_2 \cos\sqrt{\lambda_1}\, x \quad$ with b.c

$\sqrt{\lambda_1} = n\pi \quad$ or $\quad y = C_1 \sin n\pi x$

therefore $\int_0^1 y^2 dx = \int_0^1 C_1^2 \sin^2 n\pi x \, dx = C_1^2 \int_0^1 \left(\frac{1 - \cos 2n\pi x}{2} \right) dx$

$$= C_1^2 \left[\frac{x}{2} - \frac{\sin 2n\pi x}{4n\pi} \right]_0^1 = C_1^2 \left[\frac{1}{2} \right] = 2 \qquad C_1^2 = 4 \qquad C_1 = \pm 2$$

$\therefore \quad y = \pm 2 \sin n\pi x.$

Find the extremal of an isoperimetric problem of $v(y,z) = \int_0^1 (y'^2 + z'^2 - 4xz' - 4z) dx$

with c.c. $\int_0^1 (y'^2 - xy' - z'^2) dx = 2 \qquad y(0) = 0 \quad y(1) = 1 \quad z(0) = 0, \; z(1) = 1$

$F^* = (y'^2 + z'^2 - 4xz' - 4z) + \lambda(y'^2 - xy' - z'^2)$

$F^*_{,y} - \frac{d}{dx} F^*_{,y'} = 0 \qquad -\frac{d}{dx}\left[2y' + \lambda(2y' - x)\right] = 0$

$F^*_{,z} - \frac{d}{dx} F^*_{,z'} = 0 \qquad (-4) - \frac{d}{dx}\left(2z' - 4x + \lambda(-2z')\right) = 0$

$2z'' - 4 + 4 + 2z''\lambda = 0 \qquad 2(\lambda + 1) z'' = 0$

$2y'' + 2\lambda y'' - \lambda = 0 \qquad 2y''(\lambda + 1) = \lambda \qquad y'' = \frac{\lambda}{2(\lambda+1)}$

$z = C_1 x + C_2 \rightarrow z = x$ with b.c.

$y = \frac{\lambda}{2(\lambda+1)} \frac{x^2}{2} + C_1 x + C_2 \quad \rightarrow \quad y = \frac{\lambda x^2 + (4 + 3\lambda) x}{4(\lambda + 1)}$

which leads to $\lambda = -10/11$ & $y = -\frac{5}{2}x^2 + \frac{7}{2}x$

Direct Methods of Solving Variational problems.

that $v(y(x))$ = fn of an infinite set of variables thus we can use $y(x) = \sum_{i=0}^{n} a_i x^i$ or fourier $y(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

or any series of form $y(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$ where $\varphi_n(x)$ are given fns of x.
one must determine the a_n's $\Rightarrow$ $v(y(x)) = \varphi(a_i)$ $i = 1, \dots, n)$

Euler method of finite differences, Ritz's method, Kantorovič method applies to fnls of several independent variables.

1. Euler's method of finite differences. main idea: $v(y(x))$ considered only along polygonal curves which consist of prescribed no. of line segments with fixed abscissae of vertices, $x_0 + \Delta x, x_0 + 2\Delta x, \dots, x_0 + (n-1)\Delta x$ when $\Delta x = \frac{x_1 - x_0}{n}$
thus $v(y(x)) \Rightarrow$ fn of $\varphi(y_1, y_2, \dots, y_{n-1})$ choose y_i's $\ni$ $\varphi(y_i)$ becomes extremum, i.e. we get y_i's by $\partial\varphi/\partial y_i = 0$ $i = 1, 2, \dots, n-1$ then let $n \to \infty$
thus $\int_{x_0}^{x_1} F(x, y, y')\, dx = \sum_{k=0}^{n-1} \int_{x_0 + k\Delta x}^{x_0 + (k+1)\Delta x} F\left(x, y, \frac{y_{k+1} - y_k}{\Delta x}\right) dx$

or $\sum_{i=1}^{n} F\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) \Delta x$ gives solution $F_y\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{F_{y'}(x_i, y_i, \Delta y_i/\Delta x)}{\Delta x} +$

$F_{y'}(x_{i-1}, y_{i-1}, \Delta y_{i-1}/\Delta x)/\Delta x = 0$ or $F_y\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{\Delta F_{y'}}{\Delta x} = 0$

2. Ritz's Method.

Consider $v(y(x))$ only along all possible linear combinations of $y_n = \sum_{i=1}^{n} \alpha_i W_i(x)$ where α_i are constant coeff. & y_n's should be admissible fns. thus in the problem the fnl $\Rightarrow \varphi(\alpha_1, \alpha_2, \dots, \alpha_n)$
if α_n's are chosen $\ni$ $\varphi(\alpha_n$'s$)$ is an extremum then α_n's are found

by $\partial\varphi/\partial\alpha_i = 0 \quad i = 1, 2, \ldots, n.$ then let $n \to \infty$.

y_n must satisfy b.c., continuity, smoothness etc.

note: if b.c. are linear & homogeneous choose $W_n(x) \ni$

$$W_i(x) = (x - x_0)(x - x_1)\varphi_i(x) \quad \text{or} \quad W_k(x) = \sin \frac{k\pi(x - x_0)}{x_1 - x_0}$$

if b.c. are non homogeneous. $y(x_0) = y_0 \quad y(x_1) = y_1$ where either is at least non zero; $\quad y_n = \sum_{i=1}^{n} \alpha_i W_i(x) + W_0(x)$

where $W_0(x_0) = y_0 \quad W_0(x_1) = y_1$ & all the remaining W_i satisfy homogeneous conditions, $\therefore \; W_0(x) = \dfrac{y_1 - y_0}{x_1 - x_0}(x - x_0) + y_0$

if $v(z(x_i))$ then W's must be functions of x_i's

3. Kantorovič Method: in Ritz method choose $W_j(x_i) \quad i = 1, \ldots, n \quad j = 1, \ldots, m$

& solution of form. $\quad z_m = \sum_{k=1}^{m} a_k W_k(x_i) \quad a_k$ are constants.

in Kantrovič choose again $W_j(x_i)$ but find solution of form $z_m = \sum_{k=1}^{m} a_k(x_i) W_k(x_i)$

where $a_k(x_i)$ are fns. of one of the independent variable.

in doing so $v(z) \Rightarrow \tilde{v}(\alpha_j(x_i))$

for instance find extrema of fnl. $v = \int_{x_0}^{x_1}\int_{\varphi_1(x)}^{\varphi_2(x)} F(x, y, z, z_{,x}, z_{,y})\,dx\,dy$,

where integration in D where $y = \varphi_1(x) \quad y = \varphi_2(x) \quad x = x_0 \quad x = x_1$

& $z(x,y) \subset \partial D$ is given. Choose $W_i(x,y) \ni \quad z_m = \sum_{k=1}^{m} a_k(x) W_k(x,y)$

$\therefore \; v(z_m(x,y)) = \int_{x_0}^{x_1} dx \int_{\varphi_1(x)}^{\varphi_2(x)} F\left(x, y, z_m(x,y), \frac{\partial z_m}{\partial y}, \frac{\partial z_m}{\partial x}\right) dy$

since integrand depends only on y, first integration is done $\ni$

$v(z_m(x,y)) = \int_{x_0}^{x_1} \varphi(x, u_j(x), u_j'(x))\,dx$; to make $v(z_m(x,y))$ an extremum $\quad \varphi_{,u_j} - \frac{d}{dx}\varphi_{,u_j'} = 0 \quad$ the constants must satisfy b.c.

Faulkner, Frank D   Direct Methods.

Adjoint system of diff eq used to derive formulas. Max principle defines optimum trajectory. Opt. traj = accel vector chosen to max. integral of $|\bar{a}|$ · vector defined by adjoint system.

3 problems - ① interception with minimum fuel consumption, ② rendezvous in min time. gravity = linear fn of coordinates. ③ program of thrust magnitude of rocket to move in opt manner along arbitrary path in space when thrust is tangent to path & time is not in the diff equa. or constraints. Acceleration is used rather than mass of rocket. Speed of gases is constant ⇒ $T \propto \frac{dM_p}{dt}$. ~~Initial conditions are given.~~

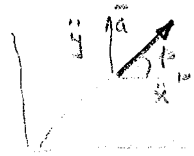
Problem 1 - when diff eq linear simple integral formulas found. two general principles of opt. for thrust programing above atmosphere. adjoint system generates vector → direct of thrust is ⇒ ① the accel has max proj on adjoint vector. ② the accel is max when |adjoint vector| > given value, accel vector is min when |adjoint| < given value. Use of successive approx determines trajectory. choose parameters, initial values for adjoint system. see if results satisfy constraints on terminal values of var. if no take differentials of terminal values in terms of diff of parameters → chosen to reduce errors in terminal values to 0. System of equations is inverted to get change in parameters & new trajectory is obtained repeated. Method of G A Bliss used to calculate diff in ballistics.

⇛ If diff eq non-linear adjoint system is adjoint to linear system of eq for the variation of original variables.

Basic Eq. $\ddot{\bar{r}} = \bar{g} + \bar{a}$ (1) where $\bar{a} = \dfrac{c'\,(\dot{q}t/m_0)}{1 - qt/m_0}\,\bar{e}$ (2) where q is rate of fuel mass, m_0 initial mass, $\bar{e}$ is unit vector in thrust direction, c' is speed of gasses

thus $$\int_0^t |\bar{a}|\,dt = c' \int \frac{(\dot{q}t/m_0)}{1 - qt/m_0}\,dt = -c' \log\left(1 - qt/m_0\right) \quad (3)$$

from (1) $$\ddot{x} = a\cos p \qquad (4)$$
$$\ddot{y} = a \sin p - g$$



if gravitational field of central body is approx. as a linear fn. of displacement then (4) becomes
$$\ddot{x} = -b^2 x + a\cos p \qquad \text{where } b^2 = g/r_e = B^2/2$$
$$\ddot{y} = B^2 y - g + a \sin p \quad (5)$$

Introduce 2 new variables λ_1 and λ_2, T is that value of t which solves ✓
$$\int_0^T \left[\lambda_1\left(\ddot{x} + b^2 x - a\cos p\right) + \lambda_2\left(\ddot{y} - B^2 y + g - a\sin p\right)\right] dt = 0 \quad (6)$$
for some x and y satisfying (5). Integrate (6) so that derivatives of x & y vanish
$$\int_0^T \left[x\left(\ddot{\lambda}_1 + b^2\lambda_1\right) + y\left(\ddot{\lambda}_2 - B^2\lambda_2\right) - a\left(\lambda_1\cos p + \lambda_2 \sin p\right) + g\lambda_2\right] dt$$
$$+ \left[\dot{x}\lambda_1 - x\dot{\lambda}_1 + \dot{y}\lambda_2 - y\dot{\lambda}_2\right]_0^T = 0 \quad (7)$$
choose λ_1 and λ_2 $\ni$ the coeff of x & y go to zero
$$\ddot{\lambda}_1 + b^2\lambda_1 = 0 \qquad \ddot{\lambda}_2 - B^2\lambda_2 = 0 \quad (8)$$
this is the adjoint system. Note that 4 multipliers may be used for $x, y, \dot{x}, \dot{y}$

thus if λ_1 and λ_2 solve the adjoint system (7) reduces to
$$\int_0^T \left[a\left(\lambda_1\cos p + \lambda_2 \sin p\right) - g\lambda_2\right] dt = \left[\dot{x}\lambda_1 - x\dot{\lambda}_1 + \dot{y}\lambda_2 - y\dot{\lambda}_2\right]_0^T \quad (9)$$
note that if $x(T)$ is needed set $\lambda_1(T) = \lambda_2(T) = \dot{\lambda}_2(T) = 0$ & $\dot{\lambda}_1(T) = -1$

Solving (8) $\lambda_1(t) = C_1 \cos bt + C_2 \sin bt$; $C_1 \cos bT + C_2 \sin bT = 0$
$-bC_1 \sin bT + bC_2 \cos bT = -1$

$$C_1 = \frac{\begin{vmatrix} 0 & \sin bT \\ -1 & b\cos bT \end{vmatrix}}{b} \qquad C_2 = \frac{\begin{vmatrix} \cos bT & 0 \\ -b\sin bT & -1 \end{vmatrix}}{b}$$

$$C_1 = \frac{1}{b}\sin bT \qquad C_2 = -\frac{1}{b}\cos bT \qquad \boxed{\frac{1}{b}\sin b(T - t) = \lambda_1(t)}$$

$\lambda_2(t) = C_3 e^{-Bt} + C_4 e^{Bt}$; $C_3 e^{-BT} + C_4 e^{BT} = 0$ $\quad C_3 = C_4 = 0 \quad \boxed{\lambda_2(t) = 0}$
$-BC_3 e^{-BT} + BC_4 e^{BT} = 0$

by substitution of 10 into 9 one gets

$$x = x_0 \cos bT + \frac{\dot{x}_0}{b} \sin bT + \frac{1}{b}\int_0^T a \cos p \sin b(T-t)\,dt$$

$$y = \left(y_0 - g/_{B^2}\right)\cosh BT + \left(\frac{\dot{y}_0}{B}\right)\sinh BT + g/_{B^2} + \frac{1}{B}\int_0^T a \sin p \sinh B(T-t)\,dt$$

if 5 is used.

To maximize (9) let $\bar{a} = a(i \cos p + j \sin p)$ & $\bar{W} = \lambda_1 i + \lambda_2 j$ thus (9) becomes

$$[\dot{x}\lambda_1 - x\dot{\lambda}_1 + \dot{y}\lambda_2 - y\dot{\lambda}_2]_0^T = \int_0^T (\bar{a}\cdot\bar{W} - g\lambda_2)\,dt$$

suppose λ_1 & λ_2 are found & solve adjoint. Consider how $\bar{a}$ must be found to max the integral. $0 \leq a \leq a_{max}$ where $a_{max} = a_{max}(t)$ dependent of rocket char. the amount of fuel is fixed $\ni$ $\int_0^T a\,dt \leq -c' \log\left(1 - \left(\frac{qt}{m_0}\right)_{max}\right)$ where $(qt/m_0)_{max}$ is largest burned fuel ratio. From the two principals set $\bar{a}$ to give max projection on $\bar{W}$ or $\tan p = \frac{\lambda_2}{\lambda_1}$ for $a \neq 0$ (p chosen to give max) note that the total quantity of fuel is limited $\ni$

$$\int_0^T a_{max} > -c' \log(1 - m_{max})$$

to determine the time to apply $\bar{a}$ note the second principle

$$a = \begin{cases} a_{max} & w > L \\ 0 & w < L \end{cases}$$

and L is a constant $\ni$ $\forall$ fuel is burned or

$$\int_0^T a\,dt = -c' \log\left(1 - (qt/m_0)_{max}\right)$$

Suppose $\exists$ C^* & a set of mults λ_1, λ_2 which satisfy the following

1. trajectory is admissible - prescribed end values & constraints on a satisfied
2. λ_1, λ_2 solve adjoint also $\dot{u}(T) < 0$ & if $y(T), \dot{y}(T), x(T)$ is not specified

the coeff $-\lambda_2'(T)$, $\lambda_1(T)$, $\lambda_2(T) \to 0$

3. for each t p maximizes $\bar{a} \cdot \bar{w}$.

4. $\exists$ L and $a = \begin{cases} a_{max} & \omega > L \\ 0 & \omega < L \end{cases}$ & $\omega = L$ only at some pts.

and $\int_0^T a\,dt = -c' \log \left(1 - \left(\frac{q t}{M_0}\right)_{max}\right)$

then if 1–4 are satisfied $x(T)$ is a max, is also unique.

→ Numerical Routine for a Simple Optimum Trajectory attaining a prescribed pt X, Y in a prescribed time T with min fuel consumption is taken up here. Since final velocity is not needed set $\lambda_1(T) = \lambda_2(T) = 0$. Consider only an initial period of thrust, $(0, t_1)$.

$\tan p = \lambda_2/\lambda_1 = C$ iff* g is const or $\tan p = \dfrac{(C/B)\sinh B(T-t)}{1/b \sin b(T-t)}$

where $C = (\tan p)_{T-}$ use of these with $\cos p = \lambda_1/\omega$ $\sin p = \lambda_2/\omega$

Thus $x = x_0 + \dot{x}_0 T + \cos p \int_0^{t_1} (T-t)\,a\,dt$

$y = y_0 - \dot{y}_0 T - gT^2/2 + \sin p \int_0^{t_1} (T-t)\,a\,dt$ if $p =$ const.

and

$$x = x_0 \cos bT + \frac{\dot{x}_0}{b} \sin bT + 1/b^2 \int_0^{t_1} \frac{a}{\omega} \sin^2 b(T-t)\,dt$$

$$y = (y_0 - g/B^2) \cos BT + \frac{\dot{y}_0}{B} \sinh BT + g/B^2 + C/B^2 \int_0^{t_1} \frac{a}{\omega} \sinh^2 B(T-t)\,dt$$

where $\omega^2 = (1/b)^2 \sin^2 b(T-t) + (C/B)^2 \sinh^2 B(T-t)$

If thrust of the rocket is constant its acceleration has a bound $a_{max}(t)$

$$a \leq \begin{cases} c' \dot{m}_{max}/(1-\dot{m}_{max}t) & \text{when } t < m_{max}/\dot{m}_{max} \\ c'\dot{m}_{max}/(1-m_{max}) & \text{when } t > m_{max}/\dot{m}_{max} \end{cases}$$

Consider a trajectory defined by a choice of t_1, c in the equations written prior, and another trajectory defined by $t_1+\delta t_1$, $c+\delta c$. Then $\delta x(T)$ & $\delta y(T)$ are given by

$$\delta x(T) = \left[\frac{a\sin^2 b(T-t)}{b^2\omega}\right]_{t_1}\delta t_1 - \frac{1}{c}\int_0^{t_1} a\omega\sin^2 p\cos^2 p\,dt\,\delta c$$

$$\delta y(T) = \left[\frac{ac\sinh^2 B(T-t)}{B^2\omega}\right]_{t_1}\delta t_1 + \frac{1}{c^2}\int_0^{t_1} a\omega\sin^2 p\cos^2 p\,dt\,\delta c$$

where $\tan p = bc\sinh B(T-t)/B\sin b(T-t)$

Routine - guess t_1 & c calculate the corresponding trajectory. Error in $x(T)$ & $y(T)$. Set $\delta x(T) = X - x(T)$

$\delta y(T) = Y - y(T)$ solve for δt_1, δc

to get new estimate of t_1 & c compute new trajectory repeat until $[X - x(T)]^2 + [Y - y(T)]^2 < \epsilon$.

Same type of equations would be generated for uniform field.

→ Interception with Min Fuel Consumpt. uniform grav field is only discussed. then $x = y = 0$

$$X_0 + \dot{X}_0 T - \cos p\int_0^{t_1}(T-t)a\,dt = 0$$

$$Y_0 + \dot{Y}_0 T - \sin p\int_0^{t_1}(T-t)a\,dt = 0$$

where X_0, Y_0 are relative position

if the trajectory is a fn of T, t_1, p the variation of the preceding eq. are

$$\left(\dot{X}_0 - \cos p \int_0^{t_1} a\,dt\right)\delta T - \cos p\,(T-t_1)\,a(t_1)\,\delta t_1 + \sin p \int_0^{t_1}(T-t)\,a\,dt\;\delta p = 0$$

$$\left(\dot{Y}_0 - \sin p \int_0^{t_1} a\,dt\right)\delta T - \sin p\,(T-t_1)\,a(t_1)\,\delta t_1 - \cos p \int_0^{t_1}(T-t)\,a\,dt\;\delta p = 0$$

eliminating δp obtain

$$\left[\left(\dot{X}_0 - \cos p \int_0^{t_1} a\,dt\right)\cos p + \left(\dot{Y}_0 - \sin p \int_0^{t_1} a\,dt\right)\sin p\right]\delta t = (T-t_1)\,a(t_1)\,\delta t_1$$

if t_1 is stationary hence $m(T)$ then

$$\left[\dot{X}_0 - \cos p \int_0^{t_1} a\,dt\right]\cos p + \left(\dot{Y}_0 - \sin p \int_0^{t_1} a\,dt\right)\sin p = 0$$

i.e. thrust direction $\perp$ to rel veloc. for $t > t_1$. Thus whenever final velocity is not needed $(\bar{V}_{rel} \cdot \dot{\bar{W}})_T = 0$

Comp routine: guess t_1, T, p. Errors in trajectory is

$$E_1 = X_0 + \dot{X}_0 T - \cos p \int_0^{t_1}(T-t)\,a\,dt$$

$$E_2 = Y_0 + \dot{Y}_0 T - \sin p \int_0^{t_1}(T-t)\,a\,dt$$

change $t_1 \to t_1 + \delta t_1$, $T \to T + \delta T$, $p \to p + \delta p$ then

$$\delta E_1 = \left(\dot{X}_0 - \cos p \int_0^{t_1} a\,dt\right)\delta T - (T-t_1)\,a(t_1)\cos p\,\delta t_1 + \sin p \int_0^{t_1}(T-t)\,a\,dt\,\delta p$$

$$\delta E_2 = \left(\dot{Y}_0 - \sin p \int_0^{t_1} a\,dt\right)\delta T - (T-t_1)\,a(t_1)\sin p\,\delta t_1 - \cos p \int_0^{t_1}(T-t)\,a\,dt\,\delta p$$

the new estimate of p is found from $(\bar{V}_{rel} \cdot \dot{\bar{W}}) = 0 \Rightarrow \delta p$,

set $\delta E_1 = -E_1$ & $\delta E_2 = -E_2$ & use δp found giving $\delta t_1, \delta T$ thus giving changes $t_1 \to t_1 + \delta t_1$, $T \to T + \delta T$, $p \to p + \delta p$ etc. check until $E^2 + F^2 < \epsilon$.

~~this~~ this will yield a stationary value of t_1 which is a min

Rendezvous in Min Time

transfer from one orbit in a uniform grav. field in min time.

Optimum thrust sched. is ∥ to plane defined by initial relative pos. & veloc.

choose axes ∋ init pos. of target rel. to rocket is X_0, Y_0 with $Y_0 > 0$

& the relative veloc. has components $\dot{X}_0, 0$ with $\dot{X}_0 > 0$. define errors.

$$E_1' = X_0 + \dot{X}_0 T - \int_0^T (T-t)\, a \cos p \, dt$$

$$E_2' = Y_0 - \int_0^T (T-t)\, a \sin p \, dt$$

$$E_3' = \dot{X}_0 - \int_0^T a \cos p \, dt$$

$$E_4' = -\int_0^T a \sin p \, dt$$

It is necessary & sufficient to determine a, p as fns of t ∋ at some $t = T$

$E_1' = E_2' = E_3' = E_4' = 0$ if $E_3' = E_4' = 0$ we can replace E_1', E_2' by new fns E_1'', E_2'' and write (mult E_3' by T, E_4' by T put into E_1', E_2')

$$E_1'' = X_0 + \int_0^T t a \cos p \, dt$$

$$E_2'' = Y_0 + \int_0^T t a \sin p \, dt$$

$$E_3' = \dot{X}_0 - \int_0^T a \cos p \, dt \qquad \text{all must go to } 0$$

$$E_4' = -\int_0^T a \sin p \, dt \qquad \text{for rendezvous.}$$

take $C_3 E_1'' + C_2 E_2'' - E_3' - C_1 E_3''$ we get

$$\dot{X}_0 - C_3 X_0 - C_2 Y_0 = \int_0^T a\left[(1 + C_3 t)\cos p + (C_1 + C_2 t)\sin p\right] dt$$

This is a form similar to (9) where grav term is 0 & $\lambda_1 = 1 + C_3 t$

it is noted that λ_1 & λ_2 satisfy the adjoint system $\begin{cases} \ddot{\lambda}_1 = 0 \\ \ddot{\lambda}_2 = 0 \end{cases}$ $\lambda_2 = C_1 + C_2 t$

Suppose we satisfy the max. principles. let $a = a_{max}$ $0 < t < T$

& let p be chosen to max the integrand $\tan p = \dfrac{\lambda_2}{\lambda_1} = \dfrac{C_1 + C_2 t}{C_3 t + 1}$

numerical routine find error formulas.

$$\delta E_1'' = (ta\cos p)_T\,\delta T - \int_0^T ta\sin p\,\delta p\,dt$$
$$\delta E_2'' = (ta\sin p)_T\,\delta T + \int_0^T ta\cos p\,\delta p\,dt$$
$$\delta E_3' = -(a\cos p)_T\,\delta T + \int_0^T a\sin p\,\delta p\,dt$$
$$\delta E_4' = -(a\sin p)_T\,\delta T - \int_0^T a\cos p\,\delta p\,dt$$

δp is found by variation of tan p $\quad \delta p = \left[(1+c_3t)(\delta c_1 + t\,\delta c_2) - (c_1+c_2t)(t\,\delta c_3)\right] \Big/ \left[(1+c_3t)^2 + (c_1+c_2t)^2\right]$

guess values for c_1, c_2, c_3, T calculate E_1'', E_2'', E_3', E_4' & $\delta E_1'', \delta E_2'', \delta E_3'$ and $\delta E_4'$. Then set $\delta E_1'' = -E_1'' \ldots \delta E_4' = -E_4'$ these four equat. for $\delta c_1, \delta c_2, \delta c_3, T$ giving new $c_1 - c_3, T$.

All integrals were set up as diff equations & Runga-Kutta method was used which allowed varying time steps along t_1, T.

For corners - defined as ~~of~~ a pt where $\bar{a}$ is discontinuous. 2 kinds of corners are throttling corner where $|\bar{a}|$ is discontinuous, and a steering corner where the direction of $\bar{a}$ is discontinuous. Steering corners occur when $\omega = 0$. Throttling corners occur when ω reaches L.

Optimum Thrust Programming along a Given Curve.

Reduction of diff equations to canonical form.

choose a path in space - Equation of motion along path is written

$$P\,dv + Q\,dM + R\,ds = 0$$ where P, Q, R are fns of v, M, s.

change in variable leads to canonical form

$$u - u_0 = \int_0^s K(u, v, s)\,ds$$ which should be minimized since integral represents loss in velocity.

to get best speed v^* for each M, s or u, s take $\frac{\partial K}{\partial v} = 0$

$\therefore$ if $v < v^*$ Thrust is max; $v > v^*$ coast.

$Mdx + Ndy$ let $M = \frac{\partial f}{\partial y}$ etc.

Consider $M dv + c\, dM + [Mg \sin\theta + D(s, v)]\, ds/v = 0$ y θ x

as for total diff eq treat one variable as constant & integrate, the resulting diff eq is

$$0 = M dv + c\, dM \rightarrow U(v, M) = -v - c \log M = u \quad \text{now}$$

eliminate M $\therefore$ $-\frac{u+v}{c} = \log M$ $\quad M = e^{-\frac{u+v}{c}}$

~~therefore $du = [Mg \sin\theta + D(s,v)]\, ds/v$~~

thus $$dv + c\frac{dM}{M} + \left[g \sin\theta + D/M\right] ds/v = 0$$

but $du = -dv - c\frac{dM}{M}$ $\therefore$ $$du = \left[g \sin\theta + D e^{\frac{u+v}{c}}\right] ds/v$$

$$\therefore \quad u - u_0 = \int_0^s \Big[\underbrace{g \frac{\sin\theta}{v}}_{\text{loss of velocity due to grav}} + \underbrace{\frac{D}{v} e^{\frac{u+v}{c}}}_{\text{due to drag}}\Big] ds$$

to minimize the integral maximize $-u = v + c \log M$

since the diff eq. holds no constraints on v minimize integral by $\frac{\partial K}{\partial v} = 0$ define $v = v^*$ which minimizes K $\therefore$ $K(u, v^*(u,s), s)$ is $< K(u, v, s)$ $\forall$ $v \neq v^*(u, s)$

note that $\frac{\partial K}{\partial u} \geq 0$ assume $\operatorname{sgn} \frac{\partial K}{\partial v} = \operatorname{sgn}(v - v^*)$

Consider the problem of max range along given path & amount of fuel & given init condit. Goddard problem with 3 subarcs.

Theorem: The Goddard problem carried out on a path C^* maximizes the range along the given curve & is unique.

Proof by contradiction $U^*, V^*, M^* \to C^*$ $U, V, M \to$ any other program.

assume $\exists S \ni U^* > U$ consider $U(S) - U^*(S) = \int_0^S K(U,V,s) - K(U^*,V^*,s)$

since $U = U^*$ initially then $K(U,V,s) - K(U^*,V,s) + K(U^*,V,s) - K(U^*,V^*,s)$

note that $K(U,V,s) \geq K(U^*,V,s)$ since $U^* \leq U$ $V \leq V^*$

also $K(U^*,V,s) \geq K(U^*,V^*,s)$ since $V \leq V^*$ thrust max & $V^*(s) < U^*(U^*$

$\therefore$ the integrand ≥ 0 $\forall s\ (0,S)$ $\therefore S$ cannot lie on $(0,s_1)$

for second segment $U(S) - U^*(S) = U(s_1) - U^*(s_1) + \int_{s_1}^{S} K(U,V,s) - K(U^*,V^*,s)$

from previous $U(s_1) \geq U^*(s_1)$ if $K(U,V,s) - K(U^*,V,s) + K(U^*,V,s) + K(U^*,V^*,s)$

$K(U,V,s) \geq K(U^*,V,s)$ since $\frac{\partial K}{\partial U} \geq 0$ since $U \geq U^*$ on (s_1, S) also

since $K(U^*,V,s) \geq K(U^*,V^*,s)$ $V^* = V$. $\therefore$ right hand side is ≥ 0 $\therefore \forall s \subset$ (s_1, s_2) S cannot be one of them.

let $U(S) - U^*(S) = U(s_2) - U^*(s_2) + \int_{s_2}^{S} \left(K(U,V,s) - K(U^*,V^*,s)\right) ds$

as before $U(s_2) \geq U^*(s_2)$ the integrand cannot be negative since $U \geq U^*$

$V \leq V^*$ also $V^* + c \log M^* = V^* + c \log M_f \geq V + c \log M$ since $u = -v - c$

since $M_f \leq M$ $\therefore K(U^*,V,s) \geq K(U^*,V^*,s)$ since $V < V^* < U^*(U^*,s)$

$\therefore S \subset (s_2, s_f)$ $\therefore U \geq U^*$ for $\forall s$ $\therefore V^* \geq V$ for last segment.

it is found that if V differs from V^* for Δs then $U^* < U$ for large s

$\therefore$ $V^* > V$ on 3rd interv. At pt where stall speed for compared curve is reached. $V^* > V$ $\therefore$ stall pt not reached on C^* yet.

singularity exists at $v = 0$ $\therefore$ t should be reintroduced.

$$\therefore \quad M \frac{dU}{dt} + c \frac{dM}{dt} + Mg \sin\theta + D(v,s) = 0$$

Enter parameters, initial values

subroutine for calculating $V_f(M, \Delta)$ if $V \le V_f$ stop

V^* is calculated, fuel consumption needed to maintain V^*. $\Delta m = M(V^* - V)/c + (Mg\sin\theta + D)\frac{\Delta t}{c}$

$m + M = M_0$ this gives Δm_1 for this time step.

then compare this with Δm_{max} which rocket can burn in that period. the smaller is $\Delta m_2 \rightarrow \Delta m$

Compare $M - M_f$ with Δm^2 the smaller becomes Δm. If all fuel used up $\Delta m = 0$ in the rest of the iterations when this occurs eliminate routine to find Δm

Then $\Delta V, V, \Delta \Delta, M, V_f$ is calculated & V & V_f compared. If $V > V_f$ & there is still fuel go back & determine Δm. the process repeated until convergence of $V \rightarrow V_f$

Optimization Techniques: Leitmann G.   Theory of Maxima & Minima - Edelbaum.

Optimization of Low Thrust trajectories & Propulsion Systems for 24 hr equatorial Satellite:

Problem - payload max of vehicle placed in 24 hrs orbit in a given time.

if launch site is not on equator ΔR & Δi must be made to put into equat plane

$$M_0 = M_{ENG} + M_p + M_L \quad (1) \qquad M_{ENG} \text{ is assumed } \propto HP \qquad HP = TV_{EXIT}/2 \quad (2)$$

$$M_{ENG} = \alpha T V_{EXIT}/2 \quad (3) \qquad dM_p/dt = T/V_{EXIT} \quad (4)$$

if T varies when $HP = C$

$$\frac{dM_p}{dt} = -\frac{dM}{dt} = \frac{\alpha T^2}{2M_{ENG}} = \frac{\alpha}{2M_{ENG}}\left(\frac{T}{M}\right)^2 M^2 \quad (5)$$

thus

$$-\frac{dM}{M^2} = \frac{\alpha}{2M_{ENG}}\int_0^{t_1}\left(\frac{T}{M}\right)^2 dt \quad (6) \qquad \text{when } t=0 \quad M=M_0 \quad \text{when } t=t_1 \quad M=M_1$$

$$\frac{M_0}{M_1} = 1 + \frac{\alpha M_0}{2M_{ENG}}\int_0^{t_1}\left(T/M\right)^2 dt \quad (7)$$

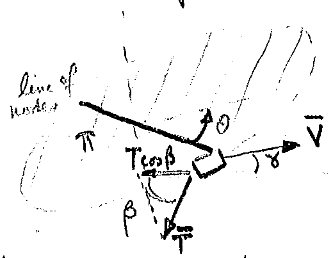
if propellant consumption is minimized by optimization of thrust vector (in magnitude & direction) as a fn of time.

Use of perturbation technique & sums over n. Assume changes in orbit (intermediate orbit) is quasi-circular. Large changes in orbit inclinations involve elliptic intermediate orbits.

$$\frac{dR}{R} = \frac{2}{V}\frac{T}{M}\cos\beta\cos\gamma\, dt$$

$$(8)$$

$$di = \frac{1}{V}\frac{T}{M}\sin\beta\cos\theta\, dt$$



if perturbation techniques used we say that orbit rad. will not change much during any revolution

then $dt = \frac{2R\,d\theta}{V}$ $\quad 0 \le \theta \le \pi \quad$ then

$$\frac{\Delta R}{R} = \frac{4R}{V^2}\int_0^{\pi}\frac{T}{M}\cos\beta\cos\gamma\, d\theta \quad (9) \quad \& \quad \Delta i = \frac{2R}{V^2}\int_0^{\pi} T/M \sin\beta\cos\theta\, d\theta \quad (10)$$

also

$$\frac{M_0}{M_1} = 1 + \frac{\alpha R M_0}{V M_{ENG}}\int_0^{\pi}\left(T/M\right)^2 d\theta \quad (11)$$

note that $\frac{M_0}{M_1}$ is constrained by $\frac{\Delta R}{R}$ & Δi

$$\therefore \quad \frac{M_0}{M} = 1 + \int_0^{\pi}\left[\frac{\alpha R M_0}{V M}\left(T/M\right)^2 + \lambda_1\frac{4R}{V^2}\left(T/M\right)\cos\beta\cos\gamma + \lambda_2\frac{2R}{V^2}\left(T/M\right)\sin\beta\cos\theta\right]d\theta \quad (12)$$

if M_0/M_1 are fns of β, γ & T/M then take $\frac{\partial(M_0/M_1)}{\partial\beta}, \frac{\partial(M_0/M_1)}{\partial\gamma}, \frac{\partial(M_0/M_1)}{\partial(T/M)} = 0$ (13)

this gives $-\lambda_1 \frac{4R}{V^2}\left(T/M\right)\sin\beta\cos\gamma + \lambda_2 \frac{2R}{V^2}\left(T/M\right)\cos\beta\cos\theta = 0$ (14)

$$-\lambda_1 \frac{4R}{V^2}\left(T/M\right)\cos\beta\sin\gamma = 0 \quad (15)$$

$$\frac{2M_0}{M_{ENG}}\frac{\alpha R}{V}\left(T/M\right) + \lambda_1 \frac{4R}{V^2}\cos\beta\cos\gamma + \lambda_2 \frac{2R}{V^2}\cos\theta\sin\beta = 0 \quad (16)$$

$$\Rightarrow \sin\gamma = 0 \; (17), \quad \tan\beta = \frac{\lambda_2}{2\lambda_1}\cos\theta \; (18), \quad T/M = \frac{2\lambda_1}{V}\frac{M_{ENG}}{\alpha M_0}\sqrt{1+\left(\frac{\lambda_2^2}{4\lambda_1^2}\right)\cos^2\theta} \; (19)$$

use of these conditions in to $\Delta R/R$, Δi, M_0/M_1 gives changes in parameters after 1 revolution. Taking the average T/M for 1 revolution & setting

$$\cos k = \frac{2\lambda_1}{\sqrt{4\lambda_1^2 + (\lambda_2^2/2)}} \qquad \sin k = \frac{\lambda_2/\sqrt{2}}{\sqrt{4\lambda_1^2+(\lambda_2^2/2)}} \quad (20) \quad \text{get}$$

$$\Delta R/R = \frac{4\pi R}{V^2}\left(\overline{T/M}\right)\cos k \; (21), \quad \Delta i = \frac{\sqrt{2}\pi R}{V^2}\left(\frac{\overline{T}}{M}\right)\sin k \; (22)$$

$$\frac{M_0}{M_1} = 1 + \pi \frac{\alpha M_0}{M_{ENG}}\frac{R}{V}\left(\overline{T/M}\right)^2 \; (23)$$ (note set $\overline{T/M}$ into 11 equate to that found by putting (14) into 11 → this gives 20).

Trajectory Optimization for Large Changes in the Orbital Elements.

take results of last section & sum for n. Assume (21-23) hold/revolution

thus $dR/R = \frac{2}{V}\left(\overline{T/M}\right)\cos k \, dt$ (24) $\quad di = \frac{\sqrt{2}}{2V}\left(\overline{T/M}\right)\sin k \, dt$ (25)

$$M_1 \frac{dM}{M^2} = -\frac{\alpha M_0}{2M_{ENG}}\left(\overline{T/M}\right)^2 dt \; (26)$$ for circular orbits $dV = -\frac{V\,dR}{R}$ (27)

then $dV = -\left(\overline{T/M}\right)\cos k \, dt$ (28) making a change of variable to dV

leads $$\frac{M_0}{M_1} = 1 - \int_{V_0}^{V_1} \left[\frac{M_0}{M_{Eng}} \frac{\alpha}{2} \frac{\bar{T}}{M} \sec k - \lambda_3 \frac{\sec k}{(\bar{T}/M)} + \lambda_4 \frac{\sqrt{2} \tan k}{2V} \right] dV$$

problem of max. final weight for specified time & change in inclination

Maximization of final weight for a given ΔV

take $$\frac{\partial (M_0/M_1)}{\partial (\bar{T}/M)} = 0 \quad \& \quad \frac{\partial (M_0/M_1)}{\partial K} = 0$$

define $\lambda_5 = -\lambda_3$

leads to $$\frac{M_0}{M_{Eng}} \frac{\alpha}{2} \sec k - \frac{\lambda_5 \sec k}{(\bar{T}/M)^2} = 0$$

$$\frac{M_0}{M_{Eng}} \frac{\alpha}{2} \left(\frac{\bar{T}}{M}\right)(\sec k \tan k) + \frac{\lambda_5}{(\bar{T}/M)} \sec k \tan k + \frac{\lambda_4 \sqrt{2} \sec^2 k}{2V} = 0$$

thus $$\frac{\bar{T}}{M} = \sqrt{\frac{2 M_{Eng}}{\alpha M_0} \lambda_5} \qquad \sin k = \frac{\sqrt{2}\, \lambda_4}{2V\alpha \left(M_0/M_{Eng}\right)\left(\bar{T}/M\right)} = \frac{V_0 \sin k_0}{V}$$

by using these results

$$M_0/M_1 = 1 + \frac{M_0}{M_{Eng}} \frac{\alpha t}{2} \left(\frac{\bar{T}}{M}\right)^2, \quad t = \frac{V_0 \cos k_0 \mp \sqrt{V^2 - V_0^2 \sin^2 k_0}}{\bar{T}/M} \quad \left\langle \frac{(\bar{T}/M)t}{V_0} \lessgtr \cos k_0 \right.$$

$$i = \begin{cases} \frac{\sqrt{2}}{2} \arcsin \frac{V_0 \sin k_0}{V} - \frac{\sqrt{2}}{2} k_0 & \frac{(\bar{T}/M)t}{V_0} \le \cos k_0 \\ 127.28^\circ - \frac{\sqrt{2}}{2} \arcsin \frac{V_0 \sin k_0}{V} - \frac{\sqrt{2}}{2} k_0 & \ge \cos k_0 \end{cases}$$

To decrease propellant required to accomodate transfer increase transfer time

Since $$\frac{M_p}{M_0} = \frac{M_0 - M_1}{M_0} = 1 - \left(1 + \frac{M_0}{M_{Eng}} \frac{\alpha}{2} \frac{[(\bar{T}/M)t]^2}{t}\right)$$

when transfer time increased power plant mass can decrease to maximize payload

$$\frac{M_L}{M_0} = 1 - \frac{M_{Eng}}{M_0} - \frac{M_p}{M_0} = \left(1 + \frac{M_0}{M_W} \frac{\alpha}{2} \frac{([\bar{T}/M]t)^2}{t}\right)^{-1} - \frac{M_W}{M_0}$$

take $$\frac{d(M_L/M_0)}{d(M_W/M_0)} = 0 \quad \text{this gives} \quad \frac{M_{Eng}}{M_{0\,opt}} = \sqrt{\frac{\alpha}{2}(\bar{T}/M)t} - \frac{\alpha}{2}(\bar{T}/M)^2 t$$

therefore $$\frac{M_L}{M_0} = \left[1 - \sqrt{\frac{\alpha}{2}\left(\bar{F}/M\right)^2 t}\, \right]^2$$

$$\frac{M_P}{M_0} = \sqrt{\frac{\alpha}{2}\left[\left(\bar{F}/M\right)^2 t\right]}$$

thus $$\sqrt{\frac{M_L}{M_0}} - \frac{M_L}{M_0} = \frac{M_W}{M_0}$$ and $$\frac{M_P}{M_0} = 1 - \sqrt{\frac{M_L}{M_0}}$$

mv | if $dm = 0$

$(m - dm)(v + dv)$ $\square$ $dm\,(v + dv)$

$$\frac{d}{dt}(mv) = \frac{d}{dt}\left[(m - dm)(v + dv) + dm\,(v + dv)\right]$$

$$\dot{m}v + \dot{v}m = (\dot{m} - \ddot{m})(v + dv) + (m - dm)(\dot{v} + \ddot{v}) + \ddot{m}(v + dm)$$

$$\dot{m}(v + dv) + m(\dot{v} + \ddot{v}) \qquad d\,\dot{m}(\dot{v} + \ddot{v})$$

$$\dot{m}v + Am = \dot{m}v + \mathring{m}A + mA.$$

Schnur

Method of Lagrange variable end points

if $I = \int F(x, y, y', z', z)\,dx$ subject to $\varphi(x, y, y', z, z') = 0$

for $K = F + \lambda\varphi$ where $\lambda = \lambda(x)$

therefore $\left\{ \left[K - y' F_{,y'} - z' K_{,z'}\right] dx + K_{,y'}\,dy + K_{,z'}\,dz \right\}_0^1 = 0$

an alternative form $\frac{d}{dx}\left[K - y' K_{,y'} - z' K_{,z'}\right] = K_{,x}$

Technique of Mayer:

$I = h(x_a, y_{1a}, \ldots, y_{na}, x_b, y_{1b}, \ldots, y_{nb})$ subject to $\varphi_i(x, y_j, y_j') = 0$ $i = 1, \ldots, n$

and boundary conditions $\psi_K = 0$

formulate $F = \sum_{i=1}^{m} \lambda_i \varphi_i$ solve the Euler-Lagrange $n + m$ eq. of form

$$F_{,y_j} - \frac{d}{dx} F_{,y_j'} = 0 \quad \& \quad \varphi_i = 0 \qquad \begin{matrix} j = 1, \ldots, n \\ i = 1, \ldots, m \end{matrix}$$

with the alternative $\frac{d}{dx}\left(F - \sum_{j=1}^{n} y_j' F_{,y_j'}\right) = F_{,x}$ subject to $\psi_K = 0$

additional boundary relations from transversality cond. are incorporated with relations among the differentials from $\psi_K = 0$. Transversality condit is

$$\left[\left(F - \sum_{j=1}^{n} y_j' F_{,y_j'}\right) dx + \sum_{j=1}^{n} F_{,y_j'}\,dy_j\right]_a^b + dh = 0$$

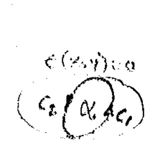
Extremization of linear integ. - Miele

if the integrand is linear in nature, i.e. $F(x, y, y') = \varphi(x, y) + \psi(x, y)\,y'$

if the class of arcs are contained within a region bounded by $\epsilon(x, y) = 0$

and the initial & final pts lie on ϵ i.e. $\epsilon(x_i, y_i) = \epsilon(x_f, y_f) = 0$.

if $\Delta I = \int_{c_1} (\varphi\,dx + \psi\,dy) - \int_{c_2} (\varphi\,dx + \psi\,dy) = \oint_{c_1 - c_2} (\varphi\,dx + \psi\,dy)$



use of green's thm leads to

$$\Delta I = \oint_{c_1 - c_2} (\varphi\,dx + \psi\,dy) = \iint_\alpha \left(\frac{\partial\psi}{\partial x} - \frac{\partial\varphi}{\partial y}\right) dx\,dy = \iint_\alpha \omega(x, y)\,dx\,dy$$

when $\omega = 0$ perfect differential

for linear isoperimetric problems if a constraint of form $\int_i^f [\varphi_1 + \psi_1\,y']\,dx = C$

$\Delta I = \oint_{c_1 - c_2} (\varphi\,dx + \psi\,dy)$, because of isoperimetricity $\oint_{c_1 - c_2} [\varphi_1\,dx + \psi_1\,dy] \equiv 0$

from $\varphi_* = \varphi + \lambda\varphi_1$ & $\psi_* = \psi + \lambda\psi_1$ $\therefore$ since $\Delta I = \oint_{C_1 - C_2} (\varphi_* dx + \psi_* dy)$

by green's theorem $\Delta I = \iint_\alpha \omega_*(x,y)\,dx\,dy$ where $\omega_*(x,y) = \frac{\partial \psi_*}{\partial x} - \frac{\partial \varphi_*}{\partial y}$

when $\omega_* = 0$ perfect differential

→ Burning Program for short Range, Non-Lifting Missile

A) Lift $= 0$ B) $D = KV^2$ C) $T = c\beta$ D) $\tan\alpha \sim \alpha \ll 1$

E) weight component on the tangent to the flight path is negligible wrt thrust or drag.

$\dot{x} - V = 0$
$\dot{y} - V\gamma = 0$
$T - D = m\dot{V}$
$\dot{m} = -\beta$
$\dot{\gamma} + g/V = 0$

using the mass as independent variable changes

$x_{,m} = -V/\beta$
$y_{,m} = -V\gamma/\beta$
$V_{,m} = \frac{KV^2}{m\beta} - c/m$
$\gamma_{,m} = g/V\beta$
$t_{,m} = -1/\beta$

but the equations are subject to $0 \leq \beta \leq \beta_{max}$

→ Extremizing the range

$$dX = \varphi\, dV + \psi\, dm \qquad \varphi = -\frac{m}{KV} \qquad \psi = -\frac{c}{KV}$$

if $X_i = 0$ $X_f = X$ then

$$X = \int_i^f \varphi\, dV + \psi\, dm$$

we can find $\omega(V,m) = \frac{\partial \psi}{\partial V} - \frac{\partial \varphi}{\partial m} = \frac{V + c}{KV^2} > 0$

for max range fly $\beta = 0$ one sub arc and $\beta = \beta_{max}$ on the other
min " " $\beta = \beta_{max}$ " " " $\beta = 0$ " " "

→ Extremizing time

$$dt = \varphi_1\, dV + \psi_1\, dm \qquad \varphi_1 = -\frac{m}{KV^2} \qquad \psi_1 = -\frac{c}{KV^2}$$

if $t_i = 0$ $t_f = t$

$t = \int_i^f \varphi_1\, dV + \psi_1\, dm$ we can find $\omega_1(V,m) = \frac{\partial \psi_1}{\partial V} - \frac{\partial \varphi_1}{\partial m} = \frac{V + 2c}{KV^3}$,

same type of set up $\beta = 0$ $\beta = \beta_{max}$ max range
$\beta = \beta_{max}$ $\beta = 0$ min range