THE CALCULUS OF VARIATIONS AND ITS APPLICATION TO TRAJECTORY PROBLEMS

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AE 399

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Abstract

The Theory of the Calculus of Variations is very useful in solving problems of the Bolza Type. In view of this fact, the general theory is set forth and several problems dealing with the pertinent sections of the theory are worked out.

Because of the setup of trajectory problems, they lend themselves well to the application of the theory. The trajectory problems dealt with in this paper are:

A. Non - constrained problems
   1. No drag case
   2. Drag as a linear ~~variation~~ function of velocity ($\bar{D} = k\bar{q}$)

B. Constrained problem
   No drag case

The problems dealt with assume a flat earth approximation, constant acceleration of gravity independent of altitude, and when dealing with the drag case, the coefficient of drag and the density are constant and independent of velocity and altitude. The objective of all three problems was to maximize the range.

An IBM-360 Digital Computer aided in producing numerical output and the following graphs were drawn:

1. The effect of drag on plots of altitude versus range for various mass propellant flow rates;
2. Plots of altitude versus range for constant mass propellant flow rate but various K (drag proportionality constant)
3. Time History of velocity & Lagrange multipliers for the no drag and drag ($\bar{D} = k\bar{q}$) cases. ($k = 1$).

From solution of these problems and the numerical output it is apparent that the Calculus of variations is useful in solving simple problems. However the difficulty of the variational calculus lies in solving for the Lagrange multiplier for this reason as the complexities of the problems increase so does the engineer's reliance on numerical methods to solve these problems.

Introduction

In the last twenty years or so, the use of variational methods has become more pronounced in fields dealing with the sciences and engineering. The versatility of the variational methods is shown in its application to problems in dynamics, elasticity, optics and fluid dynamics. These methods so popular today were uncommon in optimal problems in the astronautics field two decades ago. Through the work of Kelley's steepest descent method, Bellman's dynamic programming, and contributions to the art by people such as Leitmann, Breakwell, Miele and Lawden, the use of variational techniques in the aerospace fields has become a commonplace occurrence.

Because of the complexities of the problems encountered in actual practice, necessity often dictates the use of numerical techniques for solutions to be obtained. However, problems can often be idealized and simplified in such a manner that analytical solutions may be available. Such solutions can then serve as first approximations, and afford insight into related practical, though more complicated, problems.[1]

The most general variational method is the calculus of variations; it is a branch of calculus whose function is similar to the theory of maxima and minima. It concerns itself with the variation of functional expressions and their behavior as maxima or minima conditions are imposed. Whereas the theory of maxima and minima deals with distinct points, variational calculus deals with an infinite set of points which identify a curve in two-dimensional problems, a surface in three-dimensional problems or more complex surfaces for higher dimensional problems.[2]

The most general problem to be solved by variational methods is the Bolza problem. The problem involves the minimization or maximizat of a functional of the form $\mathcal{J} = A + I$, where A depends on the end condition of the state and control functions while I is an integral depending on the path of integration of the state and control functions. Moreover the functions must obey the constraints placed on its control and state functions and must also be consistent with its end conditions.

In the following sections a general theory of the calculus of variations will be set down, leading to the method of solving the Bolza Problem. Special cases of the Bolza problem will be dealt with and finally the (general) calculus will be used to solve several problems pertinent to trajectories.

1 Lawden - Optimal Trajectories for Space Navigation

2. Bliss - Lectures on the Calculus of Variations, obtained from Miele

Theory

General: The Bolza Problem with fixed end conditions

Consider a class of functions

$$y_k(x) \qquad k=1,\ldots,n$$

satisfying the constraints

$$\varphi_j(x, y_k, \dot{y}_k) = 0 \qquad j=1,\ldots,p<n$$

and also consistent with the end conditions

$$e_r(x_i, y_{ki}) = 0 \qquad r=1,\ldots,q$$
$$e_r(x_f, y_{kf}) = 0 \qquad r=q+1,\ldots,s \le 2n+2$$

Find the set of such functions $y_k(x)$ which minimize the functional

$$\mathcal{J} = A + I = G(x, y_k)\Big|_i^f + \int_{x_i}^{x_f} H(x, y_k, \dot{y}_k)\,dx \; * \qquad (1)$$

To take into account the constraints $\varphi_j(x, y_k, \dot{y}_k) = 0$, a set of variable Lagrange multipliers

$$\lambda_j(x) \qquad j=1,\ldots,p$$

can be used and a new integrand, known as the augmented integrand, can be formed

$$F = H + \sum_{j=1}^{p} \lambda_j \varphi_j \qquad (2)$$

which must satisfy the above set of conditions.[3]

Some Definitions Concerning the Calculus of Variations

1. The variation of a function $y(x)$ is defined as
$$\delta y = y(x) - y_1(x)$$
where $y(x)$ and $y_1(x)$ are neighboring curves. Unlike differential calculus, where a change in dy is caused by a change in dx, the variation of a function can be made without a variation in its independent variable occurring.

2. Whereas in differential calculus $y = y(x)$ is read as "y is a function of the independent variable x",
$$v[y(x)] = \int_a^b H(x, y, y')\,dx$$
is read as "$v[y(x)]$ is a functional of $y(x)$".

* i indicates initial, f indicates final, the dot indicates the derivative with respect to the independent variable.

3. Two curves are of closeness order k if

$$|y^{(k)} - y_1^{(k)}| < \epsilon \quad \text{where } \epsilon > 0$$

The variation of a functional $v[y(x)]$ is continuous along a given curve $y = y_0(x)$ of order k if

$$\forall\, \epsilon>0 \;\exists\, \delta>0 \;\ni\; |v[y(x)] - v[y_0(x)]| < \epsilon \text{ whenever } |y^{(k)} - y_0^{(k)}| < \delta$$

4. Consider a linear functional $L[y(x)]$; then

$$L[cy(x)] = c\,L[y(x)]$$
$$L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)]$$

If the change in the variation due to δy

$$\Delta v = v[y(x)+\delta y] - v[y(x)] \text{ is of the form}$$

$$L[y(x),\delta y] + \beta[y(x),\delta y]\max|\delta y|,$$ that part which is linear in δy, namely

$$L[y(x),\delta y],$$

is known as the variation of the functional and is denoted by δv.

5. A functional achieves a maximum along a curve $y = y_0(x)$ if for any neighboring curve $y = y(x)$ the difference in the functional is less than or equal to zero. A functional achieves a minimum along a curves $y = y_0(x)$ if for any neighboring curve $y = y(x)$ the difference in the functional is greater than or equal to zero.

6. A maximum or a minimum occurring for $y = y_0(x)$ is considered strong if the order of closeness between any neighboring curve and $y = y_0(x)$ is of order zero. A maximum or a minimum occurring for $y = y_0(x)$ is considered weak if the orders of closeness between any neighboring curve and $y = y_0(x)$ are of orders zero and one.

Euler - Lagrange Equations - one independent variable

Before the Bolza Problem is attempted, a solution for ζ will be given for $G(x, y_k) \equiv 0$, the special case of Lagrange.

Consider the following problem:

Suppose $\exists$ a $y(x)$ which minimizes the integral

$$\int_{x_0}^{x_1} H[x, y(x), \dot{y}(x)]\,dx \quad \text{and } y_0 = y(x_0),\ y_1 = y(x_1) \text{ are given.}$$

Let us choose any function

$$\eta(x) \in C^1$$

$\ni$ $\eta(x_0) = \eta(x_1) = 0$ for which a family of admissible functions of the form

$$y^*(x,\epsilon) = y(x) + \epsilon\eta(x)$$

can be formed and ϵ is a parameter. If one replaces $y(x)$ by $y(x) + \epsilon\eta(x)$ then it follows that for any given $y(x)$ and $\eta(x)$ the integral becomes a function of ϵ:

$$I(\epsilon) = \int_{x_0}^{x_1} H[x, y+\epsilon\eta, \dot{y}+\epsilon\dot{\eta}]\,dx = \int_{x_0}^{x_1} H[x, y^*(x,\epsilon), \dot{y}^*(x,\epsilon)]\,dx$$

Then $I(\epsilon)$ becomes a minimum when $\epsilon = 0$; it follows that

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 0.$$

$$\frac{d}{d\epsilon} I(\epsilon) = \int_{x_0}^{x_1} \left[\frac{\partial \tilde{H}}{\partial x}\frac{\partial x}{\partial \epsilon} + \frac{\partial \tilde{H}}{\partial (y+\epsilon\eta)}\frac{\partial (y+\epsilon\eta)}{\partial \epsilon} + \frac{\partial \tilde{H}}{\partial(\dot{y}+\epsilon\dot{\eta})}\frac{\partial(\dot{y}+\epsilon\dot{\eta})}{\partial\epsilon} \right] dx$$

$$\left.\frac{d}{d\epsilon} I(\epsilon)\right|_{\epsilon=0} = \int_{x_0}^{x_1} \left[\frac{\partial H}{\partial y}\eta + \frac{\partial H}{\partial \dot{y}}\dot{\eta} \right] dx$$

Integration by parts of $\int_{x_0}^{x_1} \frac{\partial H}{\partial \dot{y}}\dot{\eta}\,dx$ gives

$$\int_{x_0}^{x_1} \frac{\partial H}{\partial \dot{y}}\dot{\eta}\,dx = \left.\frac{\partial H}{\partial \dot{y}}\eta\right|_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx}\left[\frac{\partial H}{\partial \dot{y}}\right]\eta\,dx$$

Because of the manner in which $\eta(x)$ was picked the first term vanishes. Therefore

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{x_0}^{x_1} \left[\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial \dot{y}}\right)\right]\eta\,dx = 0$$

Since $\eta(x)$ is arbitrary $\forall x \in x_0 < x < x_1$ it follows that

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial \dot{y}}\right) = 0 \qquad (3)$$

This is the Euler-Lagrange Equation.

In the case when the integral is of the form

$$I = \int_{x_0}^{x_1} H[x, y_1, y_2, \ldots, y_n, \dot{y}_1, \dot{y}_2, \ldots, \dot{y}_n]\, dx$$

an evaluation of the integral for which a minimum is sought is done by fixing $n-1$ variables and performing the preceding method on the remaining variable. However instead of obtaining one Euler-Lagrange Equation, n such equations will exist, namely

$$\frac{\partial H}{\partial y_k} - \frac{d}{dx}\left[\frac{\partial H}{\partial \dot{y}_k}\right] = 0 \qquad k = 1, \ldots, n \qquad (4)$$

provided $y_k(x_0)$ and $y_k(x_1)$ are given.

It must be noted that the preceding work is also true for maximization of the integrals discussed. That solution which satisfies the Euler-Lagrange Equation is known as an extremal; the extremal will cause the integral to be stationary, i.e. will minimize (or maximize) the integral.

<u>Euler-Lagrange Equations - two independent variables.</u>

Consider the following integral

$$I = \iint_D H\left[x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right] dx\,dy \quad \text{for which } \partial D = C.$$

Let $z(x,y)$ be an extremal of I. All the permissible surfaces passing through the boundary of D are given by

$$z^*(x, y, \alpha) = z(x,y) + \alpha\eta(x,y)$$

where $\eta(x,y) = 0$ on ∂D and α is a parameter. Substituting $z^*(x,y,\alpha)$ for $z(x,y)$, the integral is rewritten as

$$I(\alpha) = \iint_D H\left[x, y, z^*(x,y,\alpha), p^*(x,y,\alpha), q^*(x,y,\alpha)\right] dx\,dy$$

for fixed $z(x,y)$ and $\eta(x,y)$. The integral can be minimized if

$$\left.\frac{d\,I(\alpha)}{d\alpha}\right|_{\alpha=0} = 0 \qquad \text{as before.}$$

$$\left.\frac{d\,I(\alpha)}{d\alpha}\right|_{\alpha=0} = \frac{\partial}{\partial\alpha} \iint_D H\left[x, y, z^*, p^*, q^*\right] dx\,dy \Big|_{\alpha=0} = 0$$

$$= \iint_D \left[\frac{\partial H}{\partial z}\eta + \frac{\partial H}{\partial p}\eta_{,x} + \frac{\partial H}{\partial q}\eta_{,y}\right] dx\,dy = 0$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $\eta_{,x} = \frac{\partial \eta}{\partial x}$, $\eta_{,y} = \frac{\partial \eta}{\partial y}$.

Note that

$$\iint_D \left[H_{,p}\,\eta_{,x} + H_{,q}\,\eta_{,y}\right] dx\,dy = \iint_D \left[\frac{\partial}{\partial x}\{H_{,p}\,\eta\} + \frac{\partial}{\partial y}\{H_{,q}\,\eta\}\right] dx\,dy$$
$$- \iint_D \left[\frac{\partial}{\partial x}H_{,p} + \frac{\partial}{\partial y}H_{,q}\right]\eta\,dx\,dy$$

The first double integral on the right side can be reduced by means of Green's Theorem:

$$\iint_D \left[\partial N/\partial x + \partial M/\partial y\right] dx\,dy = \int_C (N\,dy - M\,dx) .$$ Therefore

$$\iint_D \left[\partial/\partial x\{H_{,p}\,\eta\} + \partial/\partial y\{H_{,q}\,\eta\}\right] dx\,dy = \int_C \left[H_{,p}\,dy - H_{,q}\,dx\right]\eta = 0$$

since $\eta(x,y) = 0$ on ∂D. Thus

$$\left.\frac{d\,I(\alpha)}{d\alpha}\right|_{\alpha=0} = \iint_D \left[\frac{\partial H}{\partial z} - \frac{\partial}{\partial x}H_{,p} - \frac{\partial}{\partial y}H_{,q}\right]\eta\,dx\,dy = 0$$

Since $\eta(x,y)$ was arbitrary, it follows that

$$\frac{\partial H}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial q}\right) = 0 \qquad (5)$$

This defines the Euler-Lagrange Equation for two independent variables. Elsgolc makes reference to this equation as the Ostrogradski Equation.[4]

Example 1:

Find all admissible functions with fixed endpoints which makes

$$I = \int_a^b \{ (\dot{y}x)^2 + 12x^2 y \} dx \quad \text{stationary}$$

. Find that function which makes the integral stationary and satisfies $y(0)=0$ and $y(1)=1$

Solution:

Use of the Euler-Lagrange Equation (3) leads to the differential equation:

$$12x^2 - 2y'(2x) - 2x^2 y'' = 0$$

If a solution of the form $y = Ax^m + C$ is placed in the differential equation, the equation reduces to

$$12x^2 - 4Amx^m - 2Am(m-1)x^m = 0$$

For a solution to exist $m = 2$ which gives rise to $A=1$; then

$y = x^2 + C$ (Extremum of the integral)

Application of the end condition leads to $C=0$ and

$y = x^2$ (stationary function)

Example 2:

Derive the Ostrogradski Equation for the integral

$$I = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 2z f(x,y) \right] dx dy$$

where $z = f(x,y)$ on ∂D and is given a priori on the boundary.[4]

Solution:

By use of the Euler-Lagrange Equation (5), the following differential equation is obtained:

$$2f(x,y) - 2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = 0 \quad \text{or} \quad \nabla^2 z = f(x,y)$$

This is Poisson's Equation, encountered in the study of fluid mechanics.

Variable Boundary Conditions

Because no restrictions exist on the end conditions, a greater number of solutions to a variable end point problem exists. However, the solution which produces an extremal for this type of problem, will also be an extremal with respect to a more restricted set of curves having the same end condition as the extremal. This fact then allows for the use of the fundamental necessary condition for extrema - the Euler-Lagrange equations derived earlier. Because no end points exist, and for the integral to be stationary, it is necessary to find other conditions which must also hold in order to obtain the function which produces the extremal.

The simplest variable boundary problem is the one-end-fixed problem. Consider an extremum to exist if the function $y^*(x)$ has its variable end point at (x_1, y_1). Consider a second function $y_N = y^* + \delta y$ close to y^* formed by the movement of the variable end point from (x_1, y_1) to $(x_1+\Delta x_1, y_1+\Delta y_1)$.

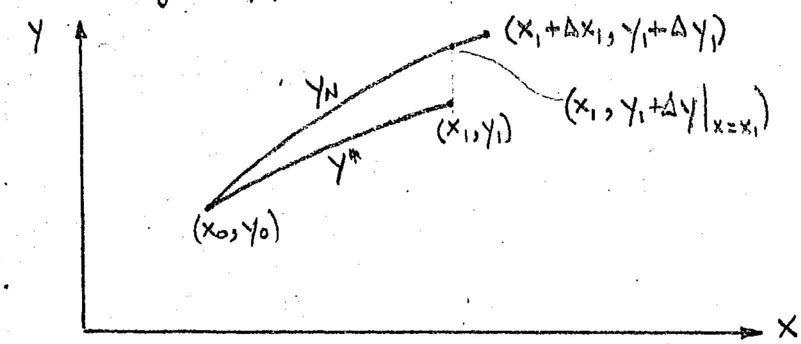


FIGURE 1

Let us examine the change in

$$I = \int H(x, y, \dot{y})\,dx \qquad \text{in its evaluation of } y = y^*(x)$$

and $y = y_N(x)$.

$$\Delta I = \int_{x_0}^{x_1+\Delta x_1} H(x, y_N, \dot{y}_N)\,dx - \int_{x_0}^{x_1} H(x, y^*, \dot{y}^*)\,dx$$

$$= \int_{x_1}^{x_1+\Delta x_1} H(x, y_N, \dot{y}_N)\,dx + \int_{x_0}^{x_1} \left\{H(x, y_N, \dot{y}_N) - H(x, y^*, \dot{y}^*)\right\} dx$$

$$= \int_{x_1}^{x_1+\Delta x_1} H(x, y_N, \dot{y}_N)\,dx + \int_{x_0}^{x_1} \left\{ H,_{y^*}\delta y + H,_{\dot{y}^*}\delta \dot{y} \right\} dx + R$$

where Taylor's expansion is used on the second integral and the first two terms of the expansion yield the variation in H – a first order approximation to the change in the integral from curve to curve. R is the higher order remainder term contributed by the expansion.
Use of the Mean Value theorem on the first integral yields

$$\int_{x_1}^{x_1+\Delta x_1} H(x, y_N, \dot{y}_N)\,dx = \Delta x_1 \left\{ H(\bar{x}, y_N, \dot{y}_N) \right\} = \left\{ H(x_1, y_N, \dot{y}_N) \right\} \Delta x_1 + \epsilon_1 \Delta x_1$$

where $\bar{x} = x_1 + \theta_1 \Delta x_1$ and $0 < \theta_1 < 1$.
Integration of the second integral results in

$$\int_{x_0}^{x_1} \left\{ H,y^* \,\delta y + H,\dot{y}^* \,\delta\dot{y} \right\} dx = H,\dot{y}^* \,\delta y \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} \left(H,y^* - \frac{d}{dx} H,\dot{y}^* \right) \delta y\, dx$$

Using the fundamental necessary condition [eq.(3)] and $\delta y \Big|_{x=x_0} = 0$ the above reduces to

$$\int_{x_0}^{x_1} \left\{ H,y^* \,\delta y + H,\dot{y}^* \,\delta\dot{y} \right\} dx = \left[H,\dot{y}^* \,\delta y \right]_{x=x_1}$$

From figure 1 it is noted that since $\delta y \Big|_{x=x_1} \cong \Delta y_1 - \dot{y}_N(x_1)\Delta x_1$

and because $\epsilon_1, R \to 0$ as $\Delta y_1, \Delta x_1 \to 0$ the change incurred by the integral is

$$\Delta I = H,\dot{y}^* \Big|_{x=x_1} \Delta y_1 + \left\{ H_N - \dot{y}_N H,\dot{y}^* \right\} \Big|_{x=x_1} \Delta x_1 \qquad (6)$$

However one requires $\Delta I = 0$ for the extremal to exist. If Δy_1 and Δx_1 are independent of each other, this implies

$$H,\dot{y}^* \Big|_{x=x_1} = \left\{ H_* - \dot{y}^* H,\dot{y}^* \right\} \Big|_{x=x_1} = 0 \qquad (7)$$

If $y_1 = \pi(x_1)$ then (6) would be subject to $\Delta y_1 \cong \dot{\pi}(x_1)\Delta x_1$; then

$$\Delta I = \left\{ H_N - (\dot{y}_N - \dot{\pi})\, H,\dot{y}^* \right\} \Big|_{x=x_1} \Delta x_1 = 0 \text{ or}$$

$$\left[H_* - (\dot{y}_* - \dot{\pi})\, H,\dot{y}^* \right]_{x=x_1} = 0 \qquad (8)$$

If a variable boundary also existed at the end point (x_0, y_0) a similar result would be obtained. The condition described in its varied form as shown by equations (6-8) is known as the transversality condition. In the preceding development infinitesimal increments in x and in y of higher order were neglected. Therefore if both end points are variable, then for the integral to be stationary the following conditions must be satisfied:

$$H_{,y} - \frac{d}{dx}(H_{,\dot{y}}) = 0$$

$$H_{,\dot{y}}\Big|_{x=x_0} \Delta y_0 + \{H - \dot{y} H_{,\dot{y}}\}\Big|_{x=x_0} \Delta x_0 = 0$$
$$H_{,\dot{y}}\Big|_{x=x_1} \Delta y_1 + \{H - \dot{y} H_{,\dot{y}}\}\Big|_{x=x_1} \Delta y_1 = 0$$

If the integral is of the form

$$I = \int_A^B H(x, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n)\, dx$$

with variable boundary points of the form
$A(x_0, y_{10}, y_{20}, \ldots, y_{N0})$ and $B(x_1, y_{11}, y_{21}, \ldots, y_{n1})$
the condition for which the integral becomes stationary are

$$\frac{\partial H}{\partial y_k} - \frac{d}{dx}\left[\frac{\partial H}{\partial \dot{y}_k}\right] = 0 \qquad k = 1, \ldots, n$$

$$\left\{H - \sum_{k=1}^{n} \dot{y} \frac{\partial H}{\partial \dot{y}_k}\right\}\Bigg|_{x=x_0} \Delta x_0 + \sum_{k=1}^{n} \frac{\partial H}{\partial \dot{y}_k}\Bigg|_{x=x_0} \Delta y_{k_0} = 0 \qquad (9)$$

$$\left\{H - \sum_{k=1}^{n} \dot{y} \frac{\partial H}{\partial \dot{y}_k}\right\}\Bigg|_{x=x_1} \Delta x_1 + \sum_{k=1}^{n} \frac{\partial H}{\partial \dot{y}_k}\Bigg|_{x=x_1} \Delta y_{k_1} = 0$$

Example: [4]

Find that curve which makes

$$I = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{y}\, dx$$ an extremum knowing that $y(0) = 0$

and (x_1, y_1) lies on the circle $(x-9)^2 + y^2 = 9$

Solution:

Because (x_1, y_1) lies on a circle, $y_1 = \pi(x_1)$ and $\Delta y_1 \cong \dot{\pi}(x_1)\Delta x_1$; in particular

$$2y\Big|_{x=x_1} \Delta y_1 + 2(x-9)\Big|_{x=x_1} \Delta x_1 = 0 \quad \text{or}$$

$$\Delta y_1/\Delta x_1 = -\frac{(x-9)}{y}\Bigg|_{x=x_1} \qquad (a)$$

Use of the Euler-Lagrange equation leads to

$$(x-c_1)^2+y^2=c_2^2 \qquad (b)$$

Using the fixed end condition $y(0)=0$ results in

$$c_1=c_2 \qquad (c)$$

Use of the transversality condition (equation 6) yields

$$\left[\frac{\sqrt{1+\dot{y}^2}}{y}-\frac{\dot{y}^2}{y\sqrt{1+\dot{y}^2}}\right]_{x=x_1}\Delta x_1+\frac{\dot{y}}{y\sqrt{1+\dot{y}^2}}\bigg|_{x=x_1}\Delta y_1=0 \qquad (d)$$

this reduces to

$$\Delta x_1+\dot{y}\big|_{x=x_1}\Delta y_1=0 \qquad \text{provided } y\sqrt{1+\dot{y}^2}\Big|_{x=x_1}\neq 0.$$

Then equation (d) gives

$$\dot{y}\big|_{x=x_1}=-\Delta x_1/\Delta y_1 \qquad (e)$$

Because of this result the tangent to the extremum must be perpendicular to the tangent to the curve $(x-9)^2+y^2=3^2$ at (x_1,y_1). The solution of equation (b), with condition (c) imposed, and the equation of the circle leads to

$$x_1(18-2c_1)=72 \qquad (f)$$

Since

$$\dot{y}=-1\Big/\frac{\Delta y_1}{\Delta x_1}=\frac{y}{(x-9)}\bigg|_{x=x_1}=-\left(\frac{x-c_1}{y}\right)\bigg|_{x=x_1}$$ the following is observed:

$$(x_1-c_1)(x_1-9)+y_1^2=0 \qquad (g)$$

Substitution of equation (b) with $(x,y)=(x_1,y_1)$ imposed, and solution for x_1 results in

$$x_1=9c_1/9-c_1 \qquad (h)$$

Substitution of (h) for x_1 in (f) leads to $c_1=4$ and the extremal is found to be

$$y=\pm\sqrt{16-(x-4)^2}=\pm\sqrt{8x-x^2} \qquad (i)$$

The end point (x_1,y_1) is found to be $(7.2,2.4)$.

Extremals as a consequence of Corners (Discontinuities)

The possibility will be considered here of extremizing solutions which may have discontinuities at one or more points of the interval. Such points will be called "corners".

Suppose $\exists$ points $C_i(x_i, y_i)$ $i = 1, \ldots, n$ which connect the two end points by means of piecewise continuous curves. For these piecewise curves to extremize the integral they must satisfy the Euler-Lagrange equation individually. By fixing all but one of the points one obtains $n-2$ fixed-end-point problems and two one-variable-end-point problems.

Consider now the simplest case, i.e. the existence of only one point $C_1(x_1, y_1)$ which produces two continuous curves. As in the case of the variable end point condition, the equations are applicable in this case.

Then

$$I = \int_{x_0}^{x_2} H(x,y,y')\,dx = \int_{x_1}^{x_2} H(x,y,y')\,dx - \int_{x_1}^{x_0} H(x,y,y')\,dx,$$

$$\Delta I = \left(H - \dot{y}\frac{\partial H}{\partial \dot{y}}\right)\Big|_{x=x_1^-}\Delta x_1 + \frac{\partial H}{\partial \dot{y}}\Big|_{x=x_1^-}\Delta y_1 - \left[\left(H - \dot{y}\frac{\partial H}{\partial \dot{y}}\right)\Big|_{x=x_1^+}\Delta x_1 + \frac{\partial H}{\partial \dot{y}}\Big|_{x=x_1^+}\Delta y_1\right] = 0 \quad (1$$

The negative sign is due to the fact that the variable end point for the second integral is made the lower limit. If Δx_1 & Δy_1 are independent (10) gives the two corner conditions

$$\left(H - \dot{y}\frac{\partial H}{\partial \dot{y}}\right)_{x=x_1^-} = \left(H - \dot{y}\frac{\partial H}{\partial \dot{y}}\right)_{x=x_1^+} \qquad (11a)$$

$$\frac{\partial H}{\partial \dot{y}}\Big|_{x=x_1^-} = \frac{\partial H}{\partial \dot{y}}\Big|_{x=x_1^+} \qquad (11b)$$

It should be noted that the first test used to determine whether or not these equatio should be used is whether the integrand $H \geq 0$.[4] If so the integral $I \geq 0$.

Example 1:

Find the solution for 1 discontinuity for the problem

$$I = \int_0^4 (\dot{y}-1)^2(\dot{y}+1)^2\,dx \quad \text{and} \quad y(0)=0 \quad y(4)=2$$

Solution:

Because the integrand is greater than or equal to zero, equations (11a, can be used. Equation (11a) leads to

$$-(\dot{y}^2-1)(3\dot{y}^2+1)\Big|_{x=x_1^-} = -(\dot{y}^2-1)(3\dot{y}^2+1)\Big|_{x=x_1^+} \quad (a)$$

Equation (11b) yields

$$(4\dot{y})(\dot{y}^2-1)\Big|_{x=x_1^-} = (4\dot{y})(\dot{y}^2-1)\Big|_{x=x_1^+} \quad (b)$$

The only solution to both equations is
$$(\dot{y}^2-1)\Big|_{x=x_1^-} = (\dot{y}^2-1)\Big|_{x=x_1^+} \qquad (c)$$
For a discontinuity to exist
$$\dot{y}_{x=x_1^-} = 1 \qquad \dot{y}_{x=x_1^+} = -1 \qquad (d)$$
$$\dot{y}_{x=x_1^-} = -1 \qquad \dot{y}_{x=x_1^+} = 1 \qquad (e)$$
Solution of the first set yields
$$y = \underline{x} + C_1 \qquad y = -\bar{x} + C_2 \qquad (f)$$
Application of: $y = 0$ @ $\underline{x} = 0$ $\quad$ $y = 2$ @ $\bar{x} = 4$ $\quad$ leads to
$$y = \underline{x} \qquad y = -\bar{x} + 6 \qquad (g)$$
The domain for which these hold is found by equating the two and finding the point of intersection (the corner) – in this case $\underline{x} = \bar{x} = 3$; therefore
$$y = x \qquad 0 \le x \le 3 \qquad (h)$$
$$y = -x+6 \qquad 3 \le x \le 4$$
For the other set of solutions to the problem, the result is
$$y = -x \qquad 0 \le x \le 1 \qquad (i)$$
$$y = x-2 \qquad 1 \le x \le 4$$
Note: $\underline{x}$ was used to denote x to the left of the discontinuity and $\bar{x}$ was used to denote x to the right of the discontinuity.

Example 2:

Are there any solutions with discontinuities for the problem
$$I = \int_{x_0}^{x_1} (\dot{y}^2 + 2xy - y^2)\,dx \qquad y(x_0) = y_0, \; y(x_1) = y_1$$
Solution:

Consider a discontinuity at (x^*, y^*). From equation (11a)
$$(2xy - y^2 - \dot{y}^2)\Big|_{x=x^*-} = (2xy - y^2 - \dot{y}^2)\Big|_{x=x^*+} \qquad (a)$$

Because x & y are not dependent on the direction one approaches the discontinuity equation (a) reduces to
$$-\dot{y}^2\big|_{x=x^*-} = -\dot{y}^2\big|_{x=x^*+} \qquad (b)$$
From equation (11b)
$$2\dot{y}\big|_{x=x^*-} = 2\dot{y}\big|_{x=x^*+} \qquad (c)$$
The solution that satisfies both equations is
$$\dot{y}\big|_{x=x^*-} = \dot{y}\big|_{x=x^*+} = -2$$
This implies that no corner exists; therefore no solution using discontinuities can be found.

Sufficiency Conditions for Extrema - Jacobi Condition

Before mathematical formulas are found regarding sufficiency conditions, it is necessary to define certain terms:

Consider a domain D; if for each point in the domain D $\exists(!)$ a curve $y = y(x, C_1)$ belonging to a family of curves $y = y(x, C)$ then the family of curves $y = y(x, C)$ defines a proper field. If a family of curves $y = y(x, C)$ passes through only one point $A(x_0, y_0)$ and also covers the domain D without intersecting [except at (x_0, y_0)] this family defines a central field. The parameter for the family is the slope at (x_0, y_0).

In order to obtain the condition attributed to Jacobi it is necessary to define an envelope. By the envelope of a family, one means a curve touched by all the members of the family.

Definition:[5]

A family of curves $f(x, y, C) = 0$ has an envelope

$x = g(c)$, $y = h(c)$ iff for each $C = C_0$ the point $[g(C_0), h(C_0)]$ of the curve defined above lies on the curve $f(x, y, C_0) = 0$ and both curves have the same tangent line there.

Theorem:[5]

If $f(x, y, c)$, $g(c)$, $h(c)$ are continuous and have continuous first derivatives, if

$$1)\ \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \neq 0 \quad , 2)\ \left(\frac{dg}{dc}\right)^2 + \left(\frac{dh}{dc}\right)^2 \neq 0 \quad \text{and}$$

$$3)\ f[g(c), h(c), c] \equiv \frac{\partial f}{\partial c}[g(c), h(c), c] = 0$$

then the family of curves $f(x, y, C) = 0$ has the curve $x = g(c)$, $y = h(c)$ as an envelope.

Consider initially a central field formed by a family of extremals which has the initial boundary point as the common. The central field is defined away from the envelope. Because of the nature of the envelope, any two curves of this family will intersect at a point near the envelope. If the arc of the extremum (formed by the function satisfying the Euler-Lagrange equation) does not have a common point with the envelope, those curves close to the extremal will form a central field. If the extremal does intersect the envelope the curves close to the extremal will intersect at a point $A'(x_1, y_1)$. This point is known as the conjugate of the point

$A(x_0, y_0)$. The Jacobi Condition states that an extremal to a problem exists if the extremal does not have a conjugate point lying on its arc.

To find whether or not a conjugate point exists it is necessary to find the envelope of the family of extremals. Since an extremal must satisfy the Euler-Lagrange equation and since the envelope of the family is found by differentiating the governing equation of the family with respect to the parameter C,

$$\frac{\partial}{\partial C}\left[H_{,y} - \frac{d}{dx} H_{,\dot{y}}\right] = 0 \quad \text{or}$$

$$u\left[H_{,yy} - \frac{d}{dx} H_{,y\dot{y}}\right] - \frac{d}{dx}\left[H_{,\dot{y}\dot{y}}\,\dot{u}\right] = 0 \quad \text{where } u = \frac{\partial y(x,C)}{\partial C} \qquad (12)$$

Therefore if a solution to this equation exists, i.e. $u = \frac{\partial y(x,C)}{\partial C}$ is found, and the solution to $u(x)=0$ exists for $x = x_0$ (point A) as well as other points in the interval of definition, then conjugate points exist. If $u = \partial y(x,C)/\partial C$ is found and the only solution to $u(x)=0$ is at $x = x_0$ then the Jacobi condition is satisfied. The existence of conjugate points indicate that the function satisfying the Euler-Lagrange Equation does not produce a minimal arc between the end points.

Example:

Determine whether or not the Jacobi Condition holds for

$$I = \int_0^a (\dot{y}^2 + 2y\dot{y} - 16y^2)\,dx \quad a>0 \quad y(0)=0 \quad y(a)=0$$

Solution:

Use of the Euler-Lagrange Equation leads to

$$2(\ddot{y} + 16y) = 0 \qquad (a)$$

to which the solution is

$$y = C_1 \sin 4x + C_2 \cos 4x \qquad (b)$$

Use of the boundary conditions leads to

$$C_2 = 0 \quad \text{and} \quad \frac{k\pi}{4} = a$$

Therefore the interval of definition is

$$0 \le x \le \frac{k\pi}{4}$$

Solution to the Jacobi accessory equation (equation 12) leads to

$$\ddot{u} + 17u = 0 \qquad (c)$$

to which the solution is

$$u(x) = D_1 \sin \sqrt{17}x + D_2 \cos \sqrt{17}x \tag{d}$$

the point $A(x_0, y_0)$ is the point $(0,0)$. Therefore $u(0)=0$ yields $D_2 = 0$. To find conjugate points set $u(x)=0$. Therefore if $D_1 \neq 0$ then

$$\sqrt{17}\,x = k\pi . \tag{e}$$

Thus if conjugate points are to exist

$$x = \frac{k\pi}{\sqrt{17}} \tag{f}$$

Let us examine whether conjugate points exist in the interval of definition for this problem. For any k it is found that

$\frac{k\pi}{4} > \frac{k\pi}{\sqrt{17}}$; therefore there will always be k conjugate points in the interval of definition. Thus the function does not produce a minimizing arc in the interval

$$0 \leq x \leq \frac{k\pi}{4}$$

Hilbert's Invariant Integral

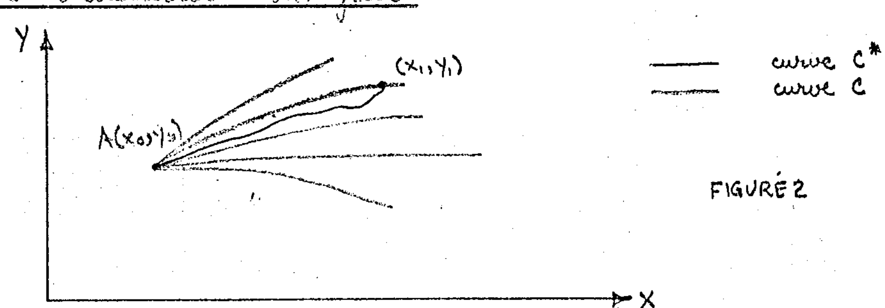


FIGURE 2

Consider two curves C and C^* with the following conditions:
Curve C is a curve satisfying the Euler-Lagrange equation, the Jacobi Condition and the boundary conditions. It is admitted into a central field with center at $A(x_0,y_0)$ which has a slope function $p(x,y)$. C^* is any other admissible curve between points (x_0,y_0) and (x_1,y_1) (FIGURE 2).
Let us look at the following integral

$$I^* = \int_{C^*} \left[H(x,y,p) + (\dot{y} - p)\frac{\partial H}{\partial p}(x,y,p)\right] dx \tag{13}$$ this can be rewritten as

$$I^* = \int_{C^*} \left\{ \left[H(x,y,p) - p\frac{\partial H}{\partial p}(x,y,p)\right] dx + \frac{\partial H}{\partial p}(x,y,p)\, dy \right\} \tag{a}$$

The rewritten integral is of the form

$$I^* = \int_{C^*} (M dx + N dy) \quad \text{where } M = H - p\frac{\partial H}{\partial p} \;, \quad N = \frac{\partial H}{\partial p}$$

Now let us look at $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$.

$$\frac{\partial M}{\partial y} = \frac{\partial H}{\partial y} - \frac{\partial p}{\partial y} H_{,p} - p\frac{\partial H_{,p}}{\partial y} \qquad (b)$$

$$\frac{\partial N}{\partial x} = \frac{\partial H_{,p}}{\partial x} \qquad (c)$$

Note that

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = (b)-(c) = \frac{\partial H}{\partial y} - \frac{\partial p}{\partial y} H_{,p} - p\frac{\partial H_{,p}}{\partial y} - \frac{\partial H_{,p}}{\partial x} \qquad (d)$$

Note also that

$$\frac{d}{dx} H_{,p} = \frac{\partial H_{,p}}{\partial x} + p\frac{\partial H_{,p}}{\partial y} + \frac{\partial p}{\partial y} H_{,p} \qquad (e)$$

Substituting (e) into (d)

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial p}\right) \qquad (f)$$

Because the curves are extremal they must satisfy the Euler-Lagrange Equation. Therefore

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial p}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 \qquad (g)$$

But (g) is the condition which a differential must satisfy for it to be exact, therefore

$$I^* = \int_{C^*} Mdx + Ndy = \int_C (Mdx + Ndy) .$$

Thus the integral depends only on the end points and not the path of integration. Equation (13) is known as Hilbert's Invariant Integral for extremal curves.

Sufficiency conditions for Extrema - Weierstrass and Legendre Conditions.

Consider the integral

$$I = \int_{x_0}^{x_1} H(x, y, \dot{y})\, dx .$$

Again consider that the Jacobi Condition holds and the function satisfying Equation (3) and the boundary conditions is admitted into a central field formed by extremals with center at $A(x_0, y_0)$. Let us take a look at the change incurred by the integral when one moves along two different curves connecting (x_0, y_0) and (x_1, y_1) (FIGURE 2). If the admitted curve is to minimize the integral between the two end points then the change incurred by the integral is

$$\Delta I = \int_{C^*} H(x,y,\dot{y})\,dx - \int_C H(x,y,p)\,dx \quad \geq 0$$

where C is the minimizing curve, C^* is any neighboring curve and $p = p(x,y)$ is the slope function of the field (therefore the minimizing curve).

Consider Hilbert's Invariant Integral

$$I^* = \int_{C^*} \left\{ H(x,y,p) + (\dot{y}-p)\frac{\partial H}{\partial p}(x,y,p) \right\} dx$$

and assume that when $\dot{y} = p$ the integral reduces to

$$I = \int_C H(x,y,\dot{y})\,dx .$$

It was noted that Hilbert's Integral was the integral of an exact differential and therefore not a function of path. Because I^* reduces to I,

$$I = \int_C H(x,y,\dot{y})\,dx = \int_{C^*} \left[H(x,y,p) + (\dot{y}-p)\frac{\partial H}{\partial p}(x,y,p)\right] dx$$

for arbitrary arcs C^*.

Then
$$\Delta I = \int_{C^*} H(x,y,\dot{y})\,dx - \int_{C^*} \left[H(x,y,p) + (\dot{y}-p)\frac{\partial H}{\partial p}(x,y,p)\right] dx$$

Therefore
$$\Delta I = \int_{C^*} \left[H(x,y,\dot{y}) - H(x,y,p) - (\dot{y}-p)\frac{\partial H}{\partial p}(x,y,p)\right] dx . \tag{14}$$

The integrand of ΔI is referred to as the Excess Function of Weierstrass and is written as

$$E(x,y,p,\dot{y}) = H(x,y,\dot{y}) - H(x,y,p) - (\dot{y}-p)\frac{\partial H}{\partial p}(x,y,p) \tag{15}$$

Because the interval of definition is positive, i.e. $x_0 \leq x \leq x_1$, a minimum occurs when

$E(x,y,p,\dot{y}) \geq 0$ and a maximum occurs when
$E(x,y,p,\dot{y}) \leq 0$.

Note that these minima or maxima are strong in that $p(x,y)$ can be arbitrarily large or small.

The Weierstrass condition can also be obtained by expanding the integrand of the neighboring curve C^* about the minimized curve C, noting that the parameter of the family of extremals in a central field is the slope to the curves at $A(x_0,y_0)$. Therefore

$$H(x_0,y_0,\dot{y}_0) = H(x_0,y_0,p(x_0,y_0)) + \frac{\partial H}{\partial p}(x_0,y_0,p(x_0,y_0))\left[\dot{y}_0 - p(x_0,y_0)\right] + \epsilon(O^{2}$$

However if $A(x_0, y_0)$ is made to move along the extremal curve to another point, and the change in the integrand is found, then it is seen that the integran will be a function of the coordinate of the point; this can be done for any point on the extremal curve, or

$$H(x,y,\dot{y}) = H(x,y,p) + \frac{\partial H}{\partial p}(x,y,p)\{\dot{y}-p\} + \epsilon(0^2)$$. By transposition of terms

$$H(x,y,\dot{y}) - H(x,y,p) - (\dot{y}-p)\frac{\partial H}{\partial p}(x,y,p) = \epsilon(0^2) = E(x,y,p,\dot{y})$$

Returning to equation (14) and applying Taylor's Expansion with respect to $p(x,y)$ only, the integral reduces to

$$\Delta I = \int_{C^*} \frac{(\dot{y}-p)^2}{2!} \frac{\partial^2 H}{\partial \dot{y}^2}(x,y,q)\,dx \qquad \text{where} \quad p \leq q \leq \dot{y}$$

Therefore

$$\Delta I \geqslant 0 \quad \text{for} \quad \frac{\partial^2 H}{\partial \dot{y}^2} \geqslant 0$$

$$\Delta I \leqslant 0 \quad \text{for} \quad \frac{\partial^2 H}{\partial \dot{y}^2} \leqslant 0 \qquad (16)$$

These conditions are known as the Legendre - Clebsch conditions for extremum. In summary, the following must hold if:

A. The curve is to have a weak extremum —
1. The curve must be an extremal satisfying the boundary conditions,
2. The Jacobi condition must hold.
3. If $\forall (x,y)$ close to the curve and arbitrary $\dot{y}$ close to $p(x,y)$ $E(x,y,p,\dot{y})$ has a constant sign:
   $E \geqslant 0$ for minimum
   $E \leqslant 0$ for maximum.
4. If $\forall (x,y)$ close to the curve and $\frac{\partial^2 H}{\partial \dot{y}^2} \neq 0$ and has constant sign:
   $\frac{\partial^2 H}{\partial \dot{y}^2} > 0$ for minimum
   $\frac{\partial^2 H}{\partial \dot{y}^2} < 0$ for maximum:

B. The curve is to have a strong extremum —
1. Conditions (1) and (2) of A must hold.
2. If $\forall (x,y)$ close to the curve and arbitrary $\dot{y}$ $E(x,y,p,\dot{y})$ has a constant sign:
   $E \geqslant 0$ for minimum

$E \leq 0$ for maximum.

It must be stressed that the excess function and the Legendre test are necessary condition for an extremum to occur. The Jacobi condition must hold for the conditions to become sufficient for extrema to exist.

In the case where the integral is of the form

$$I = \int H(x, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n)\, dx$$

the Weierstrass Excess function takes the form

$$\Delta H - \sum_{k=1}^{n} \frac{\partial H}{\partial \dot{y}_k} \Delta \dot{y}_k = E(x, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n, p_1, \ldots, p_n) \qquad (17)$$

where $\Delta H = H(x, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n) - H(x, y_1, \ldots, y_n, p_1, \ldots, p_n)$

and $\Delta \dot{y}_k = (\dot{y}_k - p_k)$ for $k = 1, \ldots, n$.

In the case of the Legendre type condition, the Legendre-Clebsch condition for extremum becomes

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 H}{\partial \dot{y}_k \partial \dot{y}_j} \delta \dot{y}_k \delta \dot{y}_j \qquad (18)$$

where $\delta \dot{y}_k = (\dot{y}_k - p_k)$ and $\delta \dot{y}_j = (\dot{y}_j - p_j)$

<u>Parametric Representation</u>

Consider the Integral

$$I = \int_{x_0}^{x_1} H(x, y, \dot{y})\, dx \qquad \text{where } x = x(t),\ y = y(t)$$

The above can be rewritten so that

$$I = \int_{t_0}^{t_1} H\left[x(t), y(t), \frac{dy/dt}{dx/dt}\right] \frac{dx}{dt}\, dt \qquad (19)$$

If the integrand is not a function of t explicitly and if the integrand is homogeneous of the first order, i.e.

$H(x, y, c\dot{x}, c\dot{y}) = c\, H(x, y, \dot{x}, \dot{y})$ then the integral depends only on $x = x(t)$ and $y = y(t)$ and no dependence on the parametric representation chosen exists. Let $\tau = \psi(t)$ so that $\dot{\psi}(t) \neq 0$. Then $x = x(\tau)$, $y = y(\tau)$ and (19) can be rewritten as

$$\int_{t_0}^{t_1} H(x,y,\frac{\dot{x}}{\dot{y}})\dot{x}\,dt = \int_{t_0}^{t_1}\Phi(x,y,\dot{x},\dot{y})\,dt = \int_{\tau_0}^{\tau_1}\Phi\left[x(\tau),y(\tau),\frac{dx}{d\tau}\dot{\psi},\frac{dy}{d\tau}\dot{\psi}\right]\frac{d\tau}{\dot{\psi}}$$

Because the integrand is homogeneous of first order

$$\int_{\tau_0}^{\tau_1}\Phi\left[x(\tau),y(\tau),\frac{dx}{d\tau}\dot{\psi},\frac{dy}{d\tau}\dot{\psi}\right]\frac{d\tau}{\dot{\psi}} = \int_{\tau_0}^{\tau_1}\Phi\left[x(\tau),y(\tau),\frac{dx}{d\tau},\frac{dy}{d\tau}\right]d\tau$$

Thus the integrand is unchanged. The solution to this problem leads to

$$\Phi_{,x} - \frac{d}{dt}\Phi_{,\dot{x}} = 0,\quad \Phi_{,y} - \frac{d}{dt}\Phi_{,\dot{y}} = 0$$

Note that these equations are no longer independent since other pairs of functions giving other parametric representation of the same curve exist. Since they must all satisfy the Euler-Lagrange equations, it is only necessary to solve on equation with the condition determining the parametric representation i.e $x = x(\tau)$, $y = y(\tau)$, $\tau = \psi(t)$, to obtain the solution of the problem.

Solution to the Mayer Problem

Consider the problem of minimizing

$$A = g(x_i, y_{k_i}, x_f, y_{k_f}) \qquad \text{where } k = 1,\dots,n$$

subject to the constraints

$$\varphi_j(x, y_k, \dot{y}_k) = 0 \qquad j = 1,\dots,p<n$$

and consistent with the end conditions

$$w_r(x_i, y_{k_i}) = 0 \qquad r = 1,\dots,q$$

$$w_r(x_f, y_{k_f}) = 0 \qquad r = q+1,\dots,s \leq 2n+2.$$

Consider the following integral

$$I = \int_{x_i}^{x_f} H(x, y_k, \dot{y}_k)\,dx \qquad k = 1,\dots,n$$

where $z_k(x)$ are introduced so that

$$\dot{z}_k = f_k(x, y_k, \dot{y}_k) \quad \text{and} \quad \sum_{k=1}^{n} f_k(x, y_k, \dot{y}_k) = H(x, y_k, \dot{y}_k)$$

By making this transformation and by the use of equation (2) we can transform

$$\int_{x_i}^{x_f} H(x, y_k, \dot{y}_k)\,dx \quad \text{to} \quad \int_{x_i}^{x_f} F(x, y_k, \dot{y}_k)\,dx$$

so that

$$I = \int_{x_i}^{x_f}\left\{\sum_{k=1}^{n}\left[\dot{z}_k(x) + \sum_{j=1}^{p}\lambda_j\varphi_j(x, y_k, \dot{y}_k)\right]\right\}dx \qquad (20)$$

Since $\varphi_j(x, y_k, \dot{y}_k) = 0$ then

$$\sum_{k=1}^{n} z_k(x)\Big|_{x_i}^{x_f} = g(x_i, y_{k_i}, x_f, y_{k_f}) = G(x, y_k)\Big|_{x_i}^{x_f}.$$

Therefore one can use the necessary equations for extremization previously derived as well as the transformation

$$\dot{z}_k(x) = f_k(x, y_k, \dot{y}_k) \qquad (21)$$

In other words, the problem is that of finding those functions $y_k(x)$, $z_k(x)$ which satisfy equation (21) and minimize the functional given by equation (20).

The Equivalence of the Mayer, Lagrange & Bolza Problem

The Bolza Problem can be solved by reducing it to a Mayer or Lagrange type problem.

1. In terms of the Mayer Problem, functions $z_k(x)$ are introduced so that $\dot{z}_k = f_k(x, y_k, \dot{y}_k) = 0$ where $H(x, y_k, \dot{y}_k) = \sum_{k=1} f_k(x, y_k, \dot{y}_k)$

and

$$\varphi = \left[G(x, y_k) + z_k(x)\right]_i^f \text{ is to be extremized.} \qquad (22)$$

2. In terms of the Lagrange problems a set of functions $y_{n+1_k}(x)$ are introduced so that

$$\dot{y}_{n+1_k} = 0 \quad \text{and} \quad y_{n+1_k}(x_i) - \frac{G(x, y_k)\big|_i^f}{x_f - x_i} = 0 \qquad (23)$$

and the integral to be minimized is

$$I = \int_{x_i}^{x_f} \left\{ H(x, y_k, \dot{y}_k) + \sum_{k=1}^{n} y_{n+1_k}(x) \right\} dx \qquad (24)$$

Problems Involving Inequalities

Suppose that equation (1) were subjected to inequality constraints of the form

$$y_i \geq \Gamma_i \qquad i = r, \dots, p < n \qquad (25)$$

For such problems, since $y_i - \Gamma_i \geq 0$ define new variables $z_i(x)$ so that

$$y_i - \Gamma_i = z_i^2 \qquad (26)$$

In that manner a set of constraints

$\varphi_i = y_i - \Gamma_i - z_i^2 = 0$ can be introduced and used in the form of equation (2).

Suppose that equation (1) were subjected to contraints of the form

$$\Gamma_{1_i} \le y_i \le \Gamma_{2_i} \qquad i = 1, \dots, p < n \tag{27}$$

From eq. (27) one obtains

$$\Gamma_{2_i} - \Gamma_{1_i} \geq \Gamma_{2_i} - y_i \geq 0 \qquad (a)$$

also

$$\Gamma_{2_i} - \Gamma_{1_i} \geq y_i - \Gamma_{1_i} \geq 0 \qquad (b);$$

(a) & (b) can be rewritten

$$(\Gamma_{2_i} - \Gamma_{1_i}) - (\Gamma_{2_i} - y_i) \geq 0 \qquad (c)$$

$$(\Gamma_{2_i} - \Gamma_{1_i}) - (y_i - \Gamma_{1_i}) \geq 0 \qquad (d)$$

Multiplying (c) by $y_i - \Gamma_{1_i}$ and (d) by $\Gamma_{2_i} - y_i$ one obtains

$$(y_i - \Gamma_{1_i})(\Gamma_{2_i} - \Gamma_{1_i}) - (\Gamma_{2_i} - y_i)(y_i - \Gamma_{1_i}) \geq 0 \qquad (e)$$

$$(\Gamma_{2_i} - y_i)(\Gamma_{2_i} - \Gamma_{1_i}) - (\Gamma_{2_i} - y_i)(y_i - \Gamma_{1_i}) \geq 0 \qquad (f)$$

If (e) and (f) are rewritten as functions of new variables $\rho_i(x)$ and $\mu_i(x)$ then

$$(y_i - \Gamma_{1_i})(\Gamma_{2_i} - \Gamma_{1_i}) - (\Gamma_{2_i} - y_i)(y_i - \Gamma_{1_i}) = \rho_i^2 \qquad (g)$$

$$(\Gamma_{2_i} - y_i)(\Gamma_{2_i} - \Gamma_{1_i}) - (\Gamma_{2_i} - y_i)(y_i - \Gamma_{1_i}) = \mu_i^2 \qquad (h)$$

Adding (g) and (h) and defining new variables $z_i(x)$ in the following manner

$$z_i^2(x) = \left[(\Gamma_{2_i} - \Gamma_{1_i})^2 - (\rho_i^2 + \mu_i^2)\right]/2$$

the constraint equations (26) reduce to

$$(\Gamma_{2_i} - y_i)(y_i - \Gamma_{1_i}) = z_i^2 \tag{28}$$

In this form a new set of constraints of the form

$$\varphi_i = (\Gamma_{2_i} - y_i)(y_i - \Gamma_{1_i}) - z_i^2 = 0$$

could be introduced and used in the form of equation (2).

In both forms of the constraints presented, the new functions $z_i(x)$ must be found along with the functions $y_i(x)$.

Equations and Necessary Conditions for the Mayer Problem

Consider the following

$$\frac{d}{dx} H(x, y_k, \dot y_k) = \frac{\partial}{\partial x} H + \sum_{k=1}^{n} \frac{\partial H}{\partial y_k} \dot y_k + \sum_{k=1}^{n} \frac{\partial H}{\partial \dot y_k} \ddot y_k \qquad (a)$$

$$\frac{d}{dx} \sum_{k=1}^{n} \left(\frac{\partial H}{\partial \dot y_k}\right) \dot y_k = \sum_{k=1}^{n} \left[\frac{d}{dx}\left(\frac{\partial H}{\partial \dot y_k}\right) \dot y_k + \frac{\partial H}{\partial \dot y_k} \ddot y_k \right] \qquad (b)$$

Subtracting equation (a) from equation (b) yields

$$\frac{d}{dx}\left[-H + \sum_{k=1}^{n} \left(\frac{\partial H}{\partial \dot y_k}\right) \dot y_k\right] = -\frac{\partial H}{\partial x} + \sum_{k=1}^{n} \dot y_k \left[\frac{d}{dx}\left(\frac{\partial H}{\partial \dot y_k}\right) - \frac{\partial H}{\partial y_k}\right] \qquad (c)$$

Noting that the second term of the right hand side of equation (C) are the Euler Lagrange Equations

$$\frac{\partial H}{\partial x} + \frac{d}{dx}\left[-H + \sum_{k=1}^{n}\left(\frac{\partial H}{\partial \dot{y}_k}\right)\dot{y}_k\right] = 0 \qquad (29)$$

Equation (29) is obtained from equation (C) as a result of the Euler-Lagrange equations (4). If the integrand $H(x, y_k, \dot{y}_k)$ is not an explicit function of the independent variable, i.e. $H(y_k, \dot{y}_k)$ only, then equation (29) can be written as

$$-H + \sum_{k=1}^{n}\left(\frac{\partial H}{\partial \dot{y}_k}\right)\dot{y}_k = C \qquad (30)$$

where C is a constant due to the integration of (29).

For the problems which we will be dealing with

$$F = H + \lambda_j \varphi_j \qquad (2)$$

Because the problems will be of the Mayer type

$$F = \lambda_j \varphi_j \qquad (31)$$

Substitution of equation (31) into (30) and remembering that F is not an explicit function of the independent variable , as well as $\varphi_j(x, y_k, \dot{y}_k) = 0$, equation (30) reduces to

$$\sum_{j=1}^{p} \lambda_j(x) \sum_{k=1}^{n} \frac{\partial \varphi_j(x, y_k, \dot{y}_k)}{\partial \dot{y}_k}\dot{y}_k = C \qquad (32)$$

The constraints can be written as

$$\dot{y}_k = f_j(x, y_k) \qquad j = 1, 2, \ldots, p \qquad (33)$$

for minimization the following must hold

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}_k}\right) = 0 \qquad (4)$$

The transversality condition will take the form

$$dG\Big|_i^f + \left[\sum_{j=1}^{p}\lambda_j(x)\sum_{k=1}^{n}\frac{\partial \varphi_j}{\partial \dot{y}_k}\,dy_k - \left(\sum_{j=1}^{p}\lambda_j(x)\sum_{k=1}^{n}\frac{\partial \varphi_j}{\partial \dot{y}_k}\dot{y}_k\right)dx\right]_i^f = 0 \qquad (34)$$

The Excess function given by equation (17) will take the form

$$F(x, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n) - F(x, y_1, \ldots, y_n, p_1, \ldots, p_n) - \sum_{k=1}^{n}(\dot{y}_k - p_k)\frac{\partial F}{\partial \dot{y}_k} \geqslant 0$$

Since $F = \lambda_j \varphi_j$ then

$$\sum_{j=1}^{p}\left[\lambda_j \varphi_j(x, y_k, \dot{y}_k) - \lambda_j \varphi_j(x, y_k, p_k) - \sum_{k=1}^{n}(\dot{y}_k - p_k)\lambda_j \frac{\partial \varphi_j}{\partial \dot{y}_k}\right] \geqslant 0$$

Because $\varphi_j(x, y_k, \dot{y}_k) = \dot{y}_k - f_j(x, y_k) = 0$, the above can be rewritten as

$$\sum_{j=1}^{p} \lambda_j p_k \geq \sum_{j=1}^{p} \lambda_j \dot{y}_k , \tag{35}$$

knowing that

$$\dot{y}_k = f_j(x, y_k)$$
$$p_k = \theta_j(x, y_k)$$ are admissible functions which cause

$$\varphi_j(x, y_k, \dot{y}_k) = \varphi_j(x, y_k, p_k) = 0 , \quad \text{and} \quad \frac{\partial \varphi_j}{\partial \dot{y}_k} = 1 \text{ by equation (33).}$$

Define

$$L = \sum_{j=1}^{p} \lambda_j p_k = \sum_{j=1}^{p} \lambda_j \theta_j(x, y_k) \; ; \tag{36a}$$

by virtue of the Euler-Lagrange equations, equation (36a) must satisfy

$$\frac{\partial L}{\partial y_k} = 0 , \quad \frac{\partial^2 L}{\partial y_k^2} \leq 0 \quad \text{for minimization,} \tag{36b}$$

where the y_k's referred to here are those functions mentioned in eq (25,27), and must also satisfy the imposed constraints. This form of the excess function is equivalent to Pontryagin's maximum principle.

In the case of the Legendre-Clebsch Condition (Equation 18)

Look at

$$\varphi_j(x, y_k, \dot{y}_k) = 0 \tag{a}$$

Using Taylor's Expansion

$$\varphi_j(x, y_k, \dot{y}_k) = \varphi_j(x, y_k, p_k) + \sum_{k=1}^{n} \frac{\partial \varphi_j}{\partial p_k}(\dot{y}_k - p_k) = 0 \tag{b}$$

since

$$\varphi_j(x, y_k, \dot{y}_k) = \varphi_j(x, y_k, p_k) = 0$$

equation (b) reduces to

$$\sum_{k=1}^{n} \frac{\partial \varphi_j}{\partial p_k}(\dot{y}_k - p_k) = 0 \qquad j = 1, \ldots, p \tag{37}$$

Therefore for a system of weak variations, equation (18) must be satisfied along with equation (37).

In the following pages three problems dealing with trajectories have been worked out. All three deal with maximizing the range. The first problem is analyzed without drag & without constraints; the second problem approximates drag linearly, but does not include constraints. The last problem deals with no drag but constraints are placed on the mass flow rate.

Consider the following problem:

A rocket is launched at time $t=0$ from the point $x=0$, $y=0$ with initial velocities $u=0$ and $v=0$. It is assumed that the propellent mass flow is prescribed, but its thrust direction remains to be determ The only restriction on the thrust direction is that it lie in a vertical plane through the launch point. The description on the mass propellent flow is such that between time $t=0$ and time $t=T$ a mass propellent flow shall exist. At time $t=T$ the propellent flow will be discontinued (the engine shut down) and the rocket will then move under the influence of gravity along a ballistic trajectory. Drag will be neglected in this analysis, with the propellant mass flow given. With the above stated conditions the problem shall be to find the maximum horizontal range.

The following shall be taken as the arrangement of the coordinate axes:

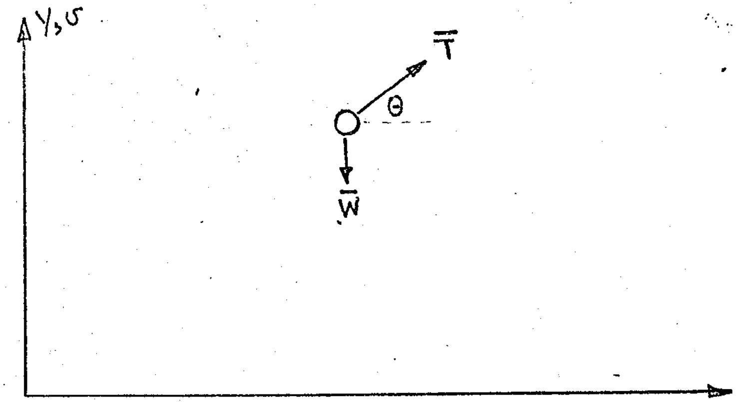


FIG 3

u and v shall be the horizontal and vertical velocities respectively, $\bar{T}$ shall be the thrust vector and θ the angle between the thrust vector and the x coordinate axis; the weight will be subject to a uniform gravitational field.

Let the acceleration due to thrust be given by

$$a = |\bar{a}| = \left|\frac{\bar{T}}{m}\right|$$

Thus the equations of motion of the rocket are:

$$\left.\begin{aligned} \dot{u} &= a\cos\theta \\ \dot{v} &= a\sin\theta - g \\ \dot{x} &= u \\ \dot{y} &= v \end{aligned}\right\} \quad (\text{I } a,b,c,d)$$

Assume that at $t = T$ the rocket has achieved a downrange distance x_1, an altitude y_1, a horizontal velocity u_1 and a vertical velocity v_1. At engine shut down the rocket follows a ballistic trajectory so that the equations of motion are

$$\left.\begin{aligned} \dot{u} &= 0 \\ \dot{v} &= -g \\ \dot{x} &= q_1 \cos\theta_1 \\ \dot{y} &= q_1 \sin\theta_1 - g t^B \end{aligned}\right\} \quad (\text{II } a,b,c,d)$$

where now t^B is the time measured from engine shut down, q_1 is the magnitude of the velocity vector at engine shut down and θ_1 is the angle between q_1 and the x coordinate axis. Superscript B will denote those quantities relating to the ballistic trajectory arc only. Integrating the equations (II c,d) leads to

$$\left.\begin{aligned} X^B &= q_1 \cos\theta_1\, t^B \\ Y^B &= q_1 \sin\theta_1\, t^B - \tfrac{1}{2} g {t^B}^2 + y_1 \end{aligned}\right\} \quad (\text{III } a,b)$$

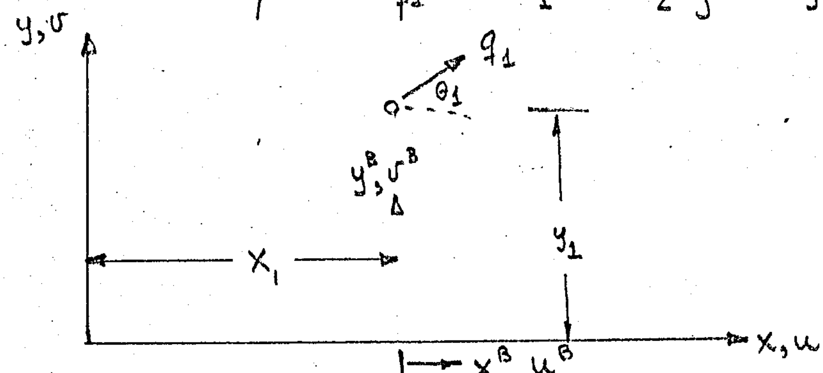


FIG 4

at time t^B_*, $y^B = 0$ (impact). Solving for t^B_* leads to

$$t^B_* = \frac{q_1 \sin\theta_1 + \sqrt{q_1^2 \sin\theta_1^2 + 2 g y_1}}{g} = \frac{v_1 + \sqrt{v_1^2 + 2 g y_1}}{g} \qquad (\text{IV})$$

Substitution of t^B_* into x^B (IIIa) leads to

$$X^B = \frac{u_1}{g}\left[v_1 + \sqrt{v_1^2 + 2 g y_1} \right] \qquad (\text{V})$$

Therefore the total distance to be maximized

$$x = x_1 + X^B = x_1 + \frac{u_1}{g}\left[v_1 + \sqrt{v_1^2 + 2 g y_1} \right] \qquad (\text{VI})$$

Defining Lagrange multipliers $\lambda u(t), \lambda v(t), \lambda x(t), \lambda y(t)$ the augmented function is formed:

$$F = \lambda_u[\dot{u} - a\cos\theta] + \lambda_v[\dot{v} - a\sin\theta + g] + \lambda_x[\dot{x} - u] + \lambda_y[\dot{y} - v] \qquad \text{(VII)}$$

Using the Euler Lagrange Equation one obtains

$$\left.\begin{aligned} \dot{\lambda}_x &= 0 \\ \dot{\lambda}_y &= 0 \\ \lambda_x + \dot{\lambda}_u &= 0 \\ \lambda_y + \dot{\lambda}_v &= 0 \\ a(\lambda_u \sin\theta - \lambda_v\cos\theta) &= 0 \end{aligned}\right\} \text{VIII (a-e)}$$

Integrating eq 8(a-d) :

$$\left.\begin{aligned} \lambda_x &= C_1 \\ \lambda_y &= C_2 \\ \lambda_u &= C_{11} - C_1 t \\ \lambda_v &= C_{22} - C_2 t \end{aligned}\right\} \text{IX (a-d)}$$

From eq 8e & 9c,d

$$\tan\theta = \frac{\lambda_v}{\lambda_u} = \frac{C_{22} - C_2 t}{C_{11} - C_1 t} \qquad \text{IXe.}$$

This is the bilinear equation attributed to Lawden.

Since the function to be maximized is $G = x = x(x_1, y_1, u_1, v_1)$. If G is taken in this manner, inequality of eq (36b) changes. Differentiation of equation (VI) and application to the transversality leads to

$$dG - \bar{C}\,dt + \lambda_u\,du + \lambda_v\,dv + \lambda_x\,dx + \lambda_y\,dy \Big|_i^f = 0 \qquad \text{(X)}$$

where $\bar{C} = \lambda_u\dot{u} + \lambda_v\dot{v} + \lambda_x\dot{x} + \lambda_y\dot{y}$ (eq 30). This is a result obtained from the first integral since the augmented function is not an explicit function of time. Since no end conditions are prescribed at time $t = T$ the following data is obtained from equation (X):

$$\lambda_{u_f} = -\frac{(v_1 + r)}{g} \qquad \lambda_{v_f} = -\frac{u_1}{r}\lambda_{u_f} \qquad \lambda_{y_f} = -u_1/r \qquad \lambda_{x_f} = -1$$

(XI (a-d))

Application of the corner conditions shows that $C_- = C_+$, and that the multipliers are continuous (11a,b); $\therefore \lambda_{x_f} = \lambda_x = -1$ and $\lambda_{y_f} = \lambda_y = -u_1/r$

Note: $r = \sqrt{v_1^2 + 2gy_1}$

Substitution of $t = T$ into IX c & d and equating the result to eq (XI a, b) leads to

$$C_{11} = -\left[T + \left(\frac{v_1 + r}{g}\right)\right]$$

$$C_{22} = -\frac{u_1}{r}\ C_{11}$$

Thus

$$\left.\begin{aligned} \lambda u &= (t-T) - \left(\frac{v_1+r}{g}\right) \\ \lambda v &= u_1/_r\ \lambda u \\ \lambda x &= -1 \\ \lambda y &= -u_1/_r \end{aligned}\right\} \text{XII } a-d$$

Using the Weierstrass Condition on Θ shows that Θ must lie in the first quadrant for maximum range (Equation 36b with inequality reversed

$$\frac{\partial L}{\partial \Theta} = 0 \text{ shows } \tan\Theta = \frac{\lambda v}{\lambda u} = u_1/r \quad ;\text{ note } \sin\Theta = \frac{\lambda v}{-\sqrt{\lambda u^2 + \lambda v^2}}$$

$$\frac{\partial^2 L}{\partial \Theta^2} = +\left(1 + u_1^2/_{r^2}\right)\cos\Theta \geq 0 \quad \text{shows } 0 \leq \Theta \leq \pi/_2 \text{ for } u_1 > 0$$

Θ is not discontinuous because it is a solution to the equation

$\tan\Theta =$ constant, a result attributed to Fried & Richardson.

Assuming the thrust acceleration $a = \frac{c\beta}{m}$ where c is the exhaust velocity of the gases and β is the mass propellant flow rate and that the mass flow rate is a constant, integrating the equations of motion lead to:

$$\left.\begin{aligned} u &= c\cos\Theta \ln\left(m_0/m_0 - \beta t\right) \\ v &= c\sin\Theta \ln\left(m_0/m_0 - \beta t\right) - gt \\ x &= c\cos\Theta\left[t\left\{\ln\left(m_0/_{m_0-\beta t}\right) + 1\right\} - \frac{m_0}{\beta}\ln\left(\frac{m_0}{m_0 - \beta t}\right)\right] \\ y &= c\sin\Theta\left[t\left\{\ln\left(\frac{m_0}{m_0-\beta t}\right) + 1\right\} - \frac{m_0}{\beta}\ln\left\{\frac{m_0}{m_0-\beta t}\right\}\right] - gt^2/_2 \end{aligned}\right\} \text{XIII } a-d$$

@ $t = T$

$$v_1 = c \ln B \sin\theta - gT$$
$$u_1 = c \ln B \cos\theta$$
$$y_1 = c \sin\theta \left[T(\ln B + 1) - \frac{m_0}{\beta} \ln B \right] - gT^2/2$$
$$x_1 = c \cos\theta \left[T(\ln B + 1) - \frac{m_0}{\beta} \ln B \right]$$

XIV (a-d)

where $B = m_0/m_0 - \beta T$ and m_0 is $\frac{W_0}{g}$ @ $t = 0$.

Defining
$$A = c \ln B$$
$$D = 2gcT$$
$$E = -2g m_0/\beta$$

The solution to

$$\sin^3\theta + \left(\frac{2A^2}{D+EA}\right)\sin^2\theta - \frac{A^2}{D+EA} = 0$$ gives the angle

lying in the first quadrant and thus defines $u(t)$, $v(t)$, $x(t)$, $y(t)$ completely. In the ballistic trajectory the following equations hold:

$$u = u_1$$
$$v = v_1 - g(t-T)$$
$$x = x_1 + u_1(t-T)$$
$$y = y_1 + v_1(t-T) - \tfrac{1}{2}g(t-T)^2$$

$T \leq t \leq$ flight time

XV (a-d)

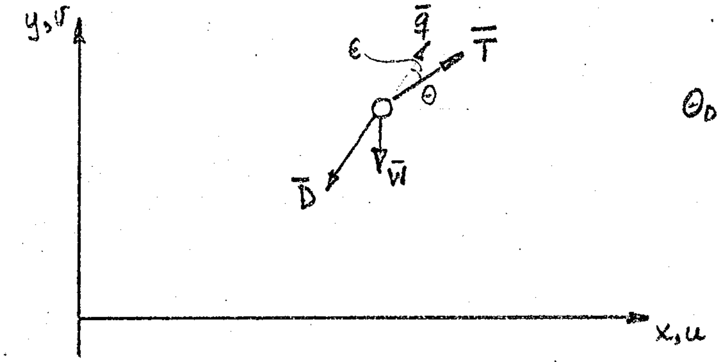
where flight time $= T + \left(\frac{v_1 + r}{g}\right)$.

The results of this analysis are shown on figures 1, 2, 3. They have been plotted for $c = 9000$ ft/sec, $W_0 = 60{,}000$ lb, $\beta = 400, 500, 600 \frac{\text{lbs}}{\text{sec}}$, $T = \ell * \frac{W_0}{\beta}$ where $\ell = \frac{\text{initial weight} - \text{structural weight}}{\text{initial weight}} = .88$

The results of other runs made for this (no drag) case and the following ($\bar{D} = k\bar{q}$) case are given in tables 1 & 2.

see App. A for a copy of computer program used.

This problem is identically the same as the previous one except that drag is included in this analysis:



$\theta_D = \theta + \epsilon$; $\bar{q} = u\bar{i} + v\bar{j}$

FIG 5

Drag will be taken to be $\bar{D} = k\bar{q}$ (a linear function of the velocity)

The equations of motion are:

$$\dot{u} = a\cos\theta - \frac{kq}{m}\cos\theta_D$$

$$\dot{v} = a\sin\theta - \frac{kq}{m}\sin\theta_D - g \qquad \text{I (a-d)}$$

$$\dot{x} = u$$

$$\dot{y} = v$$

again at shutdown $(t = T)$

$$\dot{u} = -\frac{kq}{m_1}\cos\theta_D$$

$$\dot{v} = -\frac{kq}{m_1}\sin\theta_D - g \qquad \text{II (a-d)}$$

$$\dot{x} = u$$

$$\dot{y} = v \qquad \text{where } m_1 = m \text{ @ } t = T$$

Since $u = q\cos\theta_D$, $v = q\sin\theta_D$
integration of II (a-d) leads to

$$u^B = u_1 e^{-\frac{kt^B}{m_1}}$$

$$v^B = -\frac{m_1 g}{k} + \left(v_1 + \frac{m_1 g}{k}\right)e^{-\frac{kt^B}{m_1}} \qquad \text{III (a-d)}$$

$$x^B = \frac{u_1 m_1}{k}\left[1 - e^{-\frac{kt^B}{m_1}}\right]$$

$$y^B = y_1 - \frac{m_1 g}{k}t^B + \frac{m_1}{k}\left(v_1 + \frac{m_1 g}{k}\right)\left[1 - e^{-\frac{kt^B}{m_1}}\right]$$

see figure 4 for definition of u_1, v_1, y_1.

Defining t_*^B when $y^B = 0$, the maximum range is given by

$$x = x_1 + \frac{u_1}{k} m_1 \left[1 - e^{-\frac{k t_*^B}{m_1}} \right] \quad \text{(total range from } t=0\text{)} \qquad \text{IV}$$

Forming the augmented matrix & defining Lagrange multipliers as before:

$$F = \lambda_u \left(\dot{u} - a\cos\theta + \frac{ku}{m} \right) + \lambda_v \left(\dot{v} - a\sin\theta + \frac{kv}{m} + g \right) + \lambda_x(\dot{x} - u) + \lambda_y(\dot{y} - $$

one obtains the Euler-Lagrange equations:

$$\begin{aligned}
&\dot{\lambda}_x = 0 \\
&\dot{\lambda}_y = 0 \\
&\lambda_x - \frac{k}{m}\lambda_u + \dot{\lambda}_u = 0 \\
&\lambda_y - \frac{k}{m}\lambda_v + \dot{\lambda}_v = 0 \\
&a\,(\lambda_u \sin\theta - \lambda_v \cos\theta) = 0
\end{aligned} \qquad \text{V (a-e)}$$

Integration of V (a-d) leads to

$$\begin{aligned}
\lambda_x &= c_1 \\
\lambda_y &= c_2 \\
\lambda_u &= c_1 \left[\frac{m}{k+\beta} \right] + c_{11}\, m^{-k/\beta} \\
\lambda_v &= c_2 \left[\frac{m}{k+\beta} \right] + c_{22}\, m^{-k/\beta}
\end{aligned} \qquad \text{VI (a-d)}$$

Equation Ve leads to

$$\tan\theta = \frac{\lambda_v}{\lambda_u} \qquad (\text{VI e})$$

Using the transversality condition

$$dG - \bar{C}dt + \lambda_x dx + \lambda_y dy + \lambda_v dv + \lambda_u du \Big|_i^f = 0 \qquad (\text{VII})$$

where $G = X = X(x_1, u_1, t_*^B(y_1, v_1))$ and $\bar{C}$ is defined by the first integra

Since the end conditions at $t = T$ are not prescribed the transversality condition leads to

$$\lambda_{x_f} = -1 \qquad \lambda_{u_f} = -\frac{m_1}{k}\left[1 - e^{-\frac{k t_*^B}{m_1}} \right]$$

$$\lambda_{y_f} = \frac{u_1 e^{-\frac{k t_*^B}{m_1}}}{\frac{m_1 g}{k}\left[e^{-\frac{k t_*^B}{m_1}} - 1\right] + v_1 e^{-\frac{k t_*^B}{m_1}}} \qquad \lambda_{v_f} = \frac{\frac{m_1}{k} u_1 e^{-\frac{k t_*^B}{m_1}}\left[1 - e^{-\frac{k t_*^B}{m_1}}\right]}{\frac{m_1 g}{k}\left[e^{-\frac{k t_*^B}{m_1}} - 1\right] + v_1 e^{-\frac{k t_*^B}{m_1}}}$$

VIII (a-d)

by the corner conditions [eq 11a, b] the Lagrange multipliers are continuous as well as the constant of the first integral, $\bar{C}$.
By setting $t = T$ C_{11} & C_{22} are found; Because of the continuity of the multipliers $\lambda_x = \lambda_{x_f} = C_1$, $\lambda_y = \lambda_{y_f} = C_2$. These results lead to

$$\lambda_u = \frac{-1}{(K+\beta)K}\left[\left(m_1\beta - (K+\beta)m_1 e^{-\frac{k t_*^B}{m_1}}\right)\left(\frac{m_1}{m}\right)^{K/\beta} - mK\right]$$

$$\lambda_v = \frac{u_1 e^{-\frac{k t_*^B}{m_1}}}{\frac{m_1 g}{K}\left(1 - e^{-\frac{k t_*^B}{m_1}}\right) - v_1 e^{-\frac{k t_*^B}{m_1}}} \cdot \lambda_u$$

IX (a-d)

$$\lambda_x = -1$$

$$\lambda_y = \frac{-u_1 e^{-\frac{k t_*^B}{m_1}}}{\frac{m_1 g}{K}\left(1 - e^{-\frac{k t_*^B}{m_1}}\right) - v_1 e^{-\frac{k t_*^B}{m_1}}}$$

Integration of the equations of motion I(a-d) leads to

$0 \leq t \leq T$

$$u = \frac{\beta c}{K} \cos\theta \left[1 - \left(\frac{m}{m_0}\right)^{K/\beta}\right]$$

$$v = \frac{c\beta}{K} \sin\theta \left[1 - \left(\frac{m}{m_0}\right)^{K/\beta}\right] - \frac{g}{K-\beta}\left[m - m_0\left(\frac{m}{m_0}\right)^{K/\beta}\right]$$

X (a-d)

$$x = \frac{m_0 c \cos\theta}{K}\left[\frac{K}{K+\beta} + \left(\frac{m}{m_0}\right)^{1+K/\beta}\left(\frac{\beta}{K+\beta}\right) - \frac{m}{m_0}\right]$$

$$y = \frac{m_0 c}{K} \sin\theta \left[\left(\frac{m}{m_0}\right)^{1+K/\beta}\frac{\beta}{K+\beta} + \frac{K}{K+\beta} - \frac{m}{m_0}\right] + \frac{g m_0^2}{\beta(K-\beta)}\left[\frac{1}{2}\left(\frac{m}{m_0}\right)^2 - \frac{\beta}{K+\beta}\left(\frac{m}{m_0}\right)^{1+K/\beta} - \frac{K-\beta}{2(\beta+K)}\right]$$

It should be noted that the thrust direction angle is found constant, as in the previous analysis.
Simultaneous solution of the thrust direction equation (VIa) and equation (III) where y^B is set equal to zero, yields solutions for t_*^B and θ. Knowledge of these parameters defines $u(t)$, $v(t)$, $x(t)$ and $y(t)$ completely.

Thus in the ballistic trajectory arc the governing equations become

$$u = u_1 e^{-\frac{k(t-T)}{m_1}}$$
$$v = -\frac{m_1 g}{K} + \left(v_1 + \frac{m_1 g}{K}\right) e^{-\frac{k(t-T)}{m_1}} \qquad T \le t \le T + t_*^B \qquad \text{XI(a-d)}$$
$$x = x_1 + \frac{u_1 m_1}{K}\left(1 - e^{-\frac{k(t-T)}{m_1}}\right)$$
$$y = y_1 - \frac{m_1 g}{K}(t-T) + \frac{m_1}{K}\left(v_1 + \frac{m_1 g}{K}\right)\left(1 - e^{-\frac{k(t-T)}{m_1}}\right)$$

The results of this analysis are shown on figures 1,2,4. They have been plotted for $c = 9000$ ft/sec, $w_0 = 60{,}000$ lb $\beta = 400, 500, 600$ lbs/sec and $T = \ell * \frac{w_0}{\beta}$ where $\ell = \frac{m_0 - m_1}{m_0}$ as defined previously.

In order to verify the analysis performed one should take the pertinent equations and see if in the limit as K tends to zero, the equations tend to those found for the no drag case. This has been done, and all the equations have been verified, but will not be shown here.

see App. B for computer program used.

Bill Ingebel
Cary Kunze

Consider now the problem discussed previously — that of no drag. Previously no constraints had been placed on any of the variables. Let us now contemplate the same problem but now impose a constraint on the propellant mass flow rate,

$$0 \leq \beta \leq \beta_{max}$$

Therefore the governing equations become

$$\dot{x} = u$$
$$\dot{y} = v \qquad\qquad \mathrm{I}(a\text{-}f)$$
$$\dot{u} = \frac{c\beta}{m}\cos\theta$$
$$\dot{v} = \frac{c\beta}{m}\sin\theta - g$$
$$\dot{m} = -\beta \qquad \text{(change in mass of rocket is negative propellant mass flow rate)}$$
$$0 \leq \beta \leq \beta_{max} \quad \text{or} \quad \beta(\beta_{max} - \beta) - \gamma^2 = 0 \qquad \text{(obtained by use of equation 28)}$$

From the above there are six equations and eight unknowns $x, y, u, v, \beta, m, \theta, \gamma$. Since two degrees of freedom exist, $\beta(t)$ and $\theta(t)$ will then be maximized with respect to the problem at hand, i.e. maximum range.

Setting up the augmented function:

$$F = \lambda_x(\dot{x} - u) + \lambda_y(\dot{y} - v) + \lambda_u\left(\dot{u} - \frac{c\beta}{m}\cos\theta\right) + \lambda_v\left(\dot{v} - \frac{c\beta}{m}\sin\theta + g\right) + \lambda_m(\dot{m} + \beta) + \lambda_\gamma\left(\beta[\beta_{max} - \beta] - \gamma^2\right)$$

From the Euler-Lagrange Equations:

$$-\dot{\lambda}_x = 0$$
$$-\dot{\lambda}_y = 0$$
$$-(\lambda_x + \dot{\lambda}_u) = 0 \qquad\qquad \mathrm{II}(a\text{-}h)$$
$$-(\lambda_y + \dot{\lambda}_v) = 0$$
$$-\frac{c}{m}(\lambda_u\cos\theta + \lambda_v\sin\theta) + \lambda_m + (\beta_{max} - 2\beta)\lambda_\gamma = 0$$
$$-2\gamma\lambda_\gamma = 0$$
$$\frac{c\beta}{m}\left[\lambda_u\sin\theta - \lambda_v\cos\theta\right] = 0$$
$$\frac{c\beta}{m^2}\left[\lambda_u\cos\theta + \lambda_v\sin\theta\right] - \dot{\lambda}_m = 0$$

This leads to

$$\lambda_x = c_1$$
$$\lambda_y = c_2 \qquad\qquad \mathrm{III}(a\text{-}f)$$
$$\lambda_u = c_{11} - c_1 t$$
$$\lambda_v = c_{22} - c_2 t$$
$$\tan\theta = (c_{22} - c_2 t)/(c_{11} - c_1 t)$$
$$\text{either } \gamma = 0 \text{ or } \lambda_\gamma = 0$$

Application of 36a with respect to θ:

$$\frac{\partial L}{\partial \theta} = -\frac{c\beta}{m}\left(\lambda u \sin\theta - \lambda v \cos\theta\right) = 0 \quad \text{as per } (\mathrm{II}\,g).$$

$$\frac{\partial^2 L}{\partial \theta^2} = -\frac{c\beta}{m}\left(\lambda u \cos\theta + \lambda v \sin\theta\right) \geq 0 \quad \text{for maximization};$$

but since $\tan\theta = \frac{\lambda v}{\lambda u}$, $\sin\theta = \frac{\lambda v}{\pm\sqrt{\lambda u^2 + \lambda v^2}}$ and $\cos\theta = \frac{\lambda u}{\pm\sqrt{\lambda u^2+\lambda v^2}}$,

the negative square root must therefore be used.

Attempt to use this technique on the variable β is impossible since $\frac{\partial^2 L}{\partial \beta^2} = 0$. Application of the theory of minima & maxima of a function of two variables show that $L_{,\theta\theta} L_{,\beta\beta} - L_{,\beta\theta}^2 = 0$. This implies that further investigation is necessary to establish extrema. Therefore if maximum exists for β, investigation shows that it will occur at one of the end points of the interval $0 \leq \beta \leq \beta_{max}$.

By use of the transversality condition (eq 31) one obtains

$$-\bar{C}\, dt \Big|_{t_0}^{t_f} + \lambda_x dx + \lambda_y dy + \lambda v\, dv + \lambda u\, du + \lambda m\, dm + d\psi_f = 0 \qquad \mathrm{IV}$$

$$\bar{C}_f = 0 \qquad \lambda v_f = 0 \qquad dy_f = 0 \qquad dy_i = 0$$
$$\lambda x_f = -1 \qquad \lambda u_f = 0 \qquad dx_i = 0 \qquad dm_f = dm_i = 0$$

The above are obtained by applying the conditions that $u = v = x = y = 0$, $m = m_0$ @ $t = 0$ and $y = 0$, $m = m_1$ @ $t = T_{TOT}$, where T_{TOT} is total flight time from launch to impact.

Because the augmented function is not explicit in time

$$\lambda_x \dot{x} + \lambda_y \dot{y} + \lambda v \dot{v} + \lambda u \dot{u} + \lambda m \dot{m} = \bar{C} \qquad \mathrm{V}$$

Our interest lies at $t = T_{TOT}$. At this point $\bar{C} = \bar{C}_f = 0$, $\lambda_x = \lambda_{x_f} = -1$, $\lambda_y = \lambda_{y_f}$, $\lambda v = \lambda v_f = 0$, $\lambda u = \lambda u_f = 0$, $\dot{m} = -\beta$. Therefore V reduces to

$$-\dot{x}_f + \lambda y_f \dot{y}_f - \lambda m_f \beta = 0 \qquad \mathrm{VI}$$

Application of the corner conditions, if a discontinuity were to exist, shows that $\lambda x, \lambda y, \lambda u, \lambda v, \lambda m$ are continuous at the point of discontinuity, as well as the constant of the first integral, $\bar{C}$.

Using this result on equations III (a-c) yields

$$\lambda_x = -1$$
$$\lambda_y = C_2$$
$$\lambda_u = -(T_{TOT} - t) \qquad \text{VII (a-e)}$$
$$\lambda_v = C_2(T_{TOT} - t)$$
$$\tan\theta = -C_2$$

Most important of these results is equation VII e which states that the thrust vector is maintained at a constant angle. Returning the equation VI one can see that

$$C_2 = \frac{\dot{x}_f + \lambda_{m_f}\beta}{\dot{y}_f} \qquad \text{VIII}$$

a result due to Miele.

Let us investigate the mass flow rates possible. Define a switching function

$$K_\beta = \frac{c}{m}\left[\lambda_u \cos\theta + \lambda_v \sin\theta\right] - \lambda_m \qquad \text{IX}$$

differentiation of K_β with respect to time and use of the Euler-Lagrange equa II (c,d,g,h) yields

$$\dot{K}_\beta = -c/m\,(\lambda_x \cos\theta + \lambda_y \sin\theta) \qquad \text{X}$$

If λ_γ were 0, from eq II e one would obtain $K_\beta = 0$ and thereby leading to $\dot{K}_\beta = 0$. Such a result would lead to

$\dot{K}_\beta = \lambda_x^2 + \lambda_y^2 = 0$. This is not possible since λ_x and λ_y are real constants not equal to zero (actually C_2 is unknown; even if it were zero, λ_x is still -1; therefore $\dot{K}_\beta$ could not be zero). Therefore $\lambda_\gamma \neq 0$ and $\gamma = 0$. Therefore $\beta = \beta_{max}$ or $\beta = 0$. This result confirms the fact that L defined by eq (36 a) does not have a relative maximum with respect to β and therefore one must look at the endpoints for the right value of β.

Returning to the first integral for a moment, it can be rewritten as

$\bar{C} = \lambda_x \dot{x} + \lambda_y \dot{y} + K_\beta \beta$. If a discontinuity exists it must exist in β from the results obtained. This implies that $K_\beta = 0$ at the discontinuity since $\bar{C}, \lambda_x, \dot{x}, \lambda_y, \dot{y}$ are all continuous there.

By the description of the problem, i.e. a rocket which is to maximize its range, it is necessary that the initial phase be powered to get the rocket off the launch pad

Let us assume that the total trajectory is powered; therefore from equation (II h) one obtains

$$\lambda_m = c\left[\frac{T_{tot}-t}{m} + \int_0^{T_{TOT}} \frac{dt}{m}\right]\sqrt{1+C_2^2}$$

and since $\beta = \beta_{max}$ the $m = m_0 - \beta_{max} t$ from eq (I e), then

$$\lambda_m = C\sqrt{1+C_2^2}\left[\frac{T_{TOT}-t}{m} + \frac{1}{\beta_{max}} \ln \frac{m_0}{m}\right]$$

@ $t = T_{TOT}$ $\quad \lambda_m = \lambda_{m_f} = C\sqrt{1+C_2^2}\left[\frac{1}{\beta_{max}} \ln \frac{m_0}{m_1}\right]$. By rewriting eq IV

$$\dot{y}_f = \frac{\dot{x}_f + c\sqrt{1+C_2^2} \ln m_0/m_1}{C_2} = \frac{c}{\sqrt{1+C_2^2}} \ln \frac{m_0}{m_1}\left[\frac{2+C_2^2}{C_2}\right]$$

If $C_2 > 0$, $\dot{y}_f > 0$; this is physically impossible if impact is to occur. If $C_2 < 0$, $\dot{y}_f < 0$, $y_f < 0$; this is not possible for it violates the condition on y @ $t = T_{TOT}$, namely $y_f = 0$. Therefore $\beta \neq \beta_{max}$ @ impact point. Thus the trajectory is made up of two subarcs : the first is a powered arc flown at $\beta = \beta_{max}$; the second is a coasting arc flown at $\beta = 0$. Thus equation reduces to

$$C_2 = \dot{x}_f / \dot{y}_f \qquad \text{a result due to Miele.}$$

Note that $\dot{x}_f > 0$, $\dot{y}_f < 0$ for $C_2 < 0$.

Let t_1 be the time at which the mass propellant flow rate discontinuity exists. Then

$$m = m_0 - \beta_{max} t \qquad 0 \le t \le t_1$$
$$m = m_1 \qquad t_1 \le t \le T_{TOT}$$

Integration of eq (II h) along with the condition of K_β

$$\lambda_m = -C\sqrt{1+C_2^2}\left[\frac{T_{tot}-t}{m} + \frac{1}{\beta_{max}} \ln \frac{m}{m_1}\right] \qquad 0 \le t \le t_1$$

$$\lambda_m = -C\sqrt{1+C_2^2}\left[\frac{T_{tot}-t}{m_1}\right] \qquad t_1 \le t \le T_{tot}$$

Using equation (II e) and the above yields

$$\lambda_\gamma = -\frac{C\sqrt{1+C_2^2}}{\beta_{max}^2} \ln \frac{m}{m_1} \qquad 0 \le t \le t_1$$

$$\lambda_\gamma = c\,\frac{\sqrt{1+C_2^2}}{\beta_{max}}\left[\frac{t_1-t}{m_1}\right] \qquad t_1 \le t \le T_{tot}$$

In summary

$$m = m_0 - \beta_{max} t$$
$$\lambda_m = -c\sqrt{1+c_2^2}\left[\frac{T_{tot}-t}{m} + \frac{1}{\beta_{max}} \ln \frac{m}{m_1}\right]$$

$$\beta = \beta_{max}$$
$$\lambda_\gamma = -\frac{c\sqrt{1+c_2^2}}{\beta_{max}^2} \ln \frac{m}{m_1}$$

XI (a-h)

$$v = c \sin\theta \ln \frac{m_0}{m} - gt$$

$0 \le t \le t_1$

$$u = c \cos\theta \ln \frac{m_0}{m}$$

$$y = \frac{c \sin\theta}{\beta_{max}}\left\{m \ln \frac{m}{m_0} + \beta_{max} t\right\} - gt^2/2$$
$$x = \frac{c\cos\theta}{\beta_{max}}\left\{m \ln \frac{m}{m_0} + \beta_{max} t\right\}$$

$$m = m_1$$
$$\lambda_m = -\frac{c\sqrt{1+c_2^2}}{m_1}\left[T_{tot} - t_1\right]$$

$$\beta = 0$$
$$\lambda_\gamma = \frac{c\sqrt{1+c_2^2}}{\beta_{max}}\left[\frac{t_1 - t}{m}\right]$$

XII (a-h)

$$v = c \sin\theta \ln \frac{m_0}{m_1} - gt$$

$t_1 \le t \le T_{tot}$

$$u = c \cos\theta \ln \frac{m_0}{m_1}$$

$$y = \frac{c \sin\theta}{\beta_{max}}\left\{\bar{M} \ln \frac{m_1}{m_0} + \beta_{max} t_1\right\} - gt^2/2$$
$$x = \frac{c\cos\theta}{\beta_{max}}\left\{\bar{M} \ln \frac{m_1}{m_0} + \beta_{max} t_1\right\}.$$

where $\bar{M} = m_0 - \beta_{max} t$

To obtain c_2:

$c_2 = \dot{x}_f / \dot{y}_f$. Solution of

$$\sin^3\theta + \frac{2A^2}{D+EA}\sin^2\theta - \frac{A^2}{D+EA} = 0$$

where $B = m_0/m_1$
$A = c \ln B$
$D = 2gc\,t_1$
$E = -2g\,m_0/\beta_{max}$

leads to three roots for θ.

The root leading to a value of θ in the first quadrant produces the maximum range condition. From

$$\tan\theta = -C_2$$

one obtains the value of C_2.

C_2 must be negative if y is to decrease in the ballistic subarc, and $\dot{y}$ be less than zero at impact.

Note that all the results obtained depend on satisfying the multipliers and the conditions implied on the mass propellant flow switching function. There exists another switching function which depends on θ and is known as the thrust direction switching function. It is defined by

$$K_\theta = \lambda_u \cos\theta + \lambda_v \sin\theta$$

The switching function goes to zero when a discontinuity of $180°$ exists in the thrust direction. This however occurs only when λ_u & λ_v pass through zero at the same time. In this problem this only occurs at impact and is not discussed during the solution of the problem.

Discussion & Conclusion

From the results given in table 1, it is noticed that the thrust direction angle decreases with increasing mass propellant flow rate (Figure 5) for const initial weight and exhaust gas velocity. It is also noticed that as the thrust direction angle decreases the range increases. Note runs 1 and 4: The angles are approximately the same, the ranges are approximately the same yet run 1 was performed at one half the initial weight and one half the propellant flow rate. Therefore, if a mission were designed for a 500 mile range, the data from run 1 would be used. (Note M_1 RUN 1 IS ½ M_1 RUN 4).

Note the effect of increasing the exhaust velocity of the gases (Runs 4 & 8. The angle has decreased by approximately two degrees, the horizontal velocity at the end of powered phase as well as the total flight time has almost doubled while the maximum altitude as well as the range has almost quadrupled.

Note that for increasing mass propellant flow rate, the angles of the thrust vector decreases. If the mass propellant flow rate keeps increasing the angle will decrease to a minimum angle of forty-five degrees. This occurs because the flow rate will be so high that the rocket will in effect have a powered flight phase of extremely short duration. This is nothing more than the definition of an impulse. This case will then approach the limiting problem of a particle given an initial velocity at time $t=0$. The maximum range is found to occur when the launching angle is forty-five degrees as predicted.

From the results given in table 2, it is noticed that the thrust direction angle decreases with increasing mass propellant flow rate for constant initial weight, proportionality constant and exhaust gas velocity. When the angle decreases the range increases to a maximum & then decreases (fig 6). Note that as K becomes large, the overall effect tends to decrease the thrust vector angle as well as the range. As K goes to zero the results tend to approach the no drag results thus providing a good check on the analytical work.

The plotting of the results of tables 1 & 2 was affected in graphs 1-6.

Graph 1. - range versus altitude for varying beta indicates that when the drag proportionality constant goes from $K=0$ to a non-zero value, namely 1, the trajectory shrinks such that the range decreases by a factor of three and a maximum altitude by a factor of about four

Note that in the initial powered phase, the trajectory is not parabolic in either the drag or no drag case. It should also be noted that at the end of the powered phase the acceleration is discontinuous as shown in Graphs 3&4.
Graph 2 - range versus altitude for varying K shows that as K increases the effect is to decrease the maximum altitude as well as the range.
Graph 3 - the time history of a rocket with no drag shows the variation of $u, v, \lambda u, \lambda v$ as a function of time. Note the linearity of the Lagrange multipliers as predicted in the first problem dealt with. The intersection of the multipliers at impact time was predicted in the third problem dealt with.
The velocity diagrams show the non linear behavior expected during the powered phase. The discontinuity in their derivatives occur at engine shutdown due to the loss of the thrust acceleration.
Graph 4 - the time history of a rocket with drag shows the variation of $u, v, \lambda u, \lambda v$ as a function of time. Note the non linearity of the Lagrange multipliers in the ballistic phase and their linearity in the powered phase. This is due to the fact that the thrust vector angle is constant in the powered phase and that λu and λv must satisfy the condition that at impact λu and λv are zero and be continuous at transition between powered & unpowered phase.
The velocity diagrams are non-linear in both subarcs because of the drag term. Unlike the no drag case, u in the ballistic arc is a decreasing exponential function of time.

The overall effect of these results were to confirm the effect of drag on a trajectory. These results have also confirmed what one expects in limiting cases, i.e. high flow rates for short durations and its relation to the initial velocity projectile and as the drag proportionality constant tends to zero the equations reduce to the no drag equations.
However one should also discuss the calculus with respect to the problems it can solve. Because of the great difficulty in obtaining the Lagrange multipliers (due to coupling, non linear terms, etc.) only simple problems can be solved in this manner. More complex problems must be solved by numerical methods, this however tends to decrease the usefulness of the calculus in obtaining the analytical solutions to the family of these complex problems.

POLYTECHNIC INSTITUTE OF BROOKLYN

TITLE The Calculus of Variations & Its Application to Trajectory Problems

EXPERIMENT NO. ____ DATE ____ COURSE NO. AE 399

OBSERVERS Cesar Levy

TABLE 1

NO DRAG Run #	W_o lbs	BETA lbs/sec	C ft/sec	Powered Flight Time sec	Total Flight Time sec	Θ degrees	U * AT END OF POWER ft/sec	V^+ AT END OF POWER ft/sec	|V| AT IMPACT ft/sec	MAX ALT. * Mi.	MAX RANGE Mi.	COEFF. $1 - M_1/...$	W_1 lbs		
1.	30,000	300	5000	83	434.5	49.937	5910.2	5181.1	5904.5	102.3	491.937	0.88	3600		
2	"	400	"	66	443.4	48.160	7095.6	5670.9	6353.0	116.6	436.780	"	"		
3	60,000	500	"	105.6	426.6	50.695	6123.9	4671.6	5504.5	89.0	453.061	"	7200		
4	"	600	"	88	433.6	49.491	6595.1	4172.4	5590.7	102.1	489.745	"	"		
5	"	300	9000	176	173.0	50.142	12245.2	8570.9	10223.3	307.3	1521.118	"	"		
6	"	400	"	132.	791.2	49.586	12636.9	9821.1	11148.1	365.4	1683.733	"	"		
7	"	500	"	105.6	801.5	47.757	12645.0	10502.6	11664.9	400.2	1779.021	"	"		
8a	"	600	"	88.	808.2	47.240	12912.1	11130.1	11995.1	423.1	1841.747	"	"		
... MAX RANGE															
CALCULATION															
8b	"	"	"	"	527.516	50.00	11546.8	6655.3	7432.7	162.4	1469.683	"	"		

VALUES FROM IBM 360 COMPUTER PROGRAM

* ALSO SAME AT IMPACT. $\ast\ast$ THIS VALUE IS OBTAINED FOR THE TIME VALUE PRIOR TO THE CHANGE IN SIGN OF THE VELOCITY.

+ THIS VALUE IS OBTAINED FROM THE [illegible]

12M-2-67

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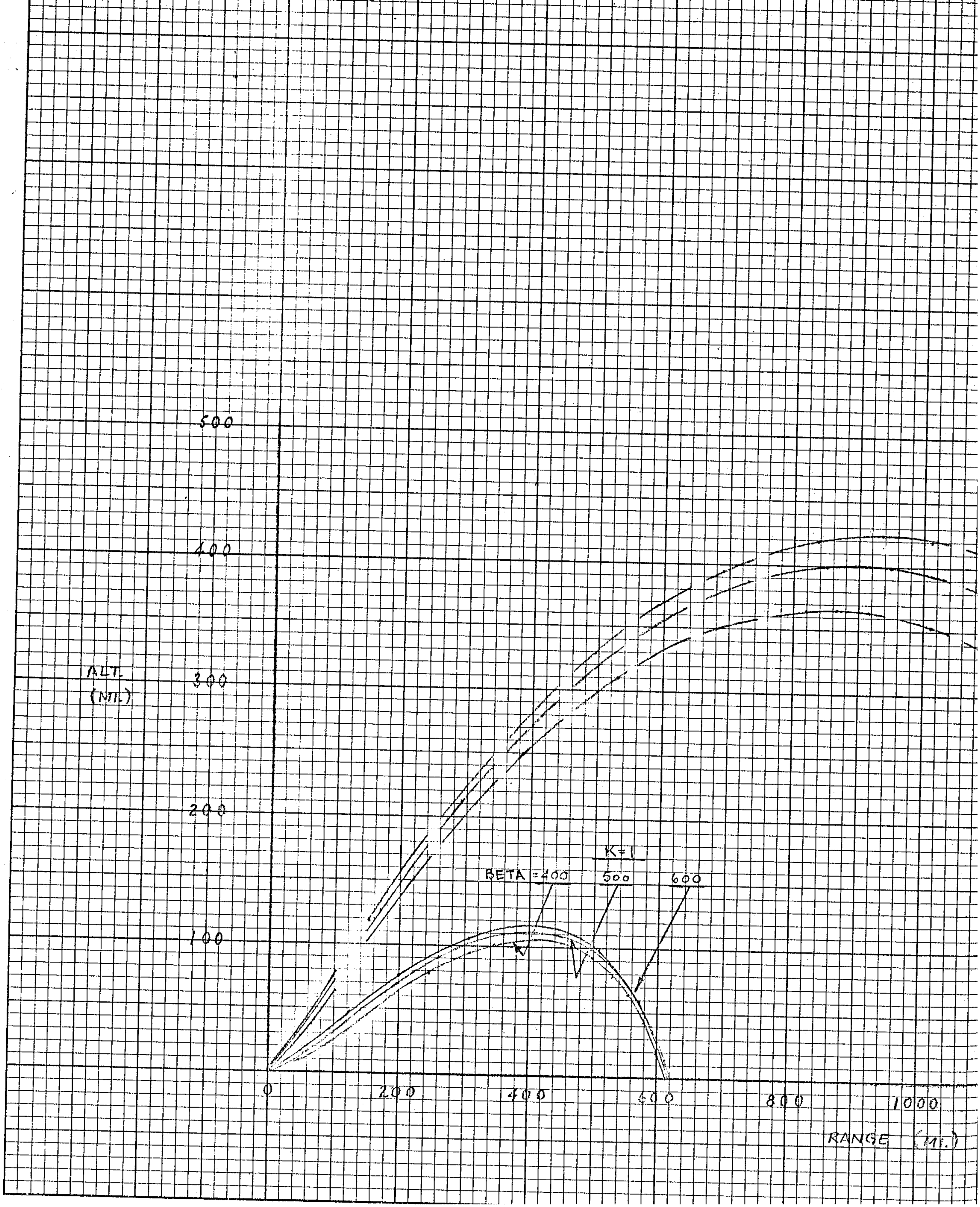
TABLE 2

Drag $\bar{D} = K\bar{q}$ Run #	K lbs-sec/ft	W_o lbs	BETA lbs/sec	C ft/sec	POWERED FLT TIME sec	TOTAL FLT TIME sec	Θ DEGREES	U^* AT END OF PWR'D FLT ft/sec	U AT IMPACT ft/sec	V^* AT END OF PWR'D FLT ft/sec	$\lvert V \rvert$ AT IMPACT ft/sec	MAX. ALT$^+$ MI.	MAX. RANGE MI.	COEFF $1-M_1/M_o$	W_1 lbs
1	0.5	60,000	600	9000	88	571.6	37.047	14734.2	5021.0	8344.9	6657.8	192.0	912,531	0.88	7200
2	1.0	"	400	"	132	483.6	34.107	14014.6	3015.1	5573.4	4452.0	105.1	614.029	"	"
3	"	"	500	"	105.6	473.2	32.533	14745.0	2905.9	6218.4	4555.6	112.5	617.855	"	"
4a	"	"	600	"	88	465.9	31.571	15230.3	2835.4	6690.1	4614.1	116.8	618.850	"	"
5	1.5	"	500	"	105.6	411.4	29.064	14643.1	1938.4	5113.4	3487.8	77.3	477.052	"	"
6	"	"	600	"	88	402.7	28.008	15281.7	1876.2	5565.8	3527.4	80.2	474.257	"	"
√ MAX. RANGE															
CALCULATION															
4b	1.0	"	600	"	88	611.08	45.0	12640.3	1229.4	9971.3	5530.0	211.3	561.206	"	

+ THIS VALUE WAS OBTAINED FROM THE FIRST INCREMENT IN THE BALLISTIC TRAJECTORY; ACTUALLY THE VELOCITY IS HIGHER THAN THAT QUOTED.

FIG 1: RANGE VS. ALTITU



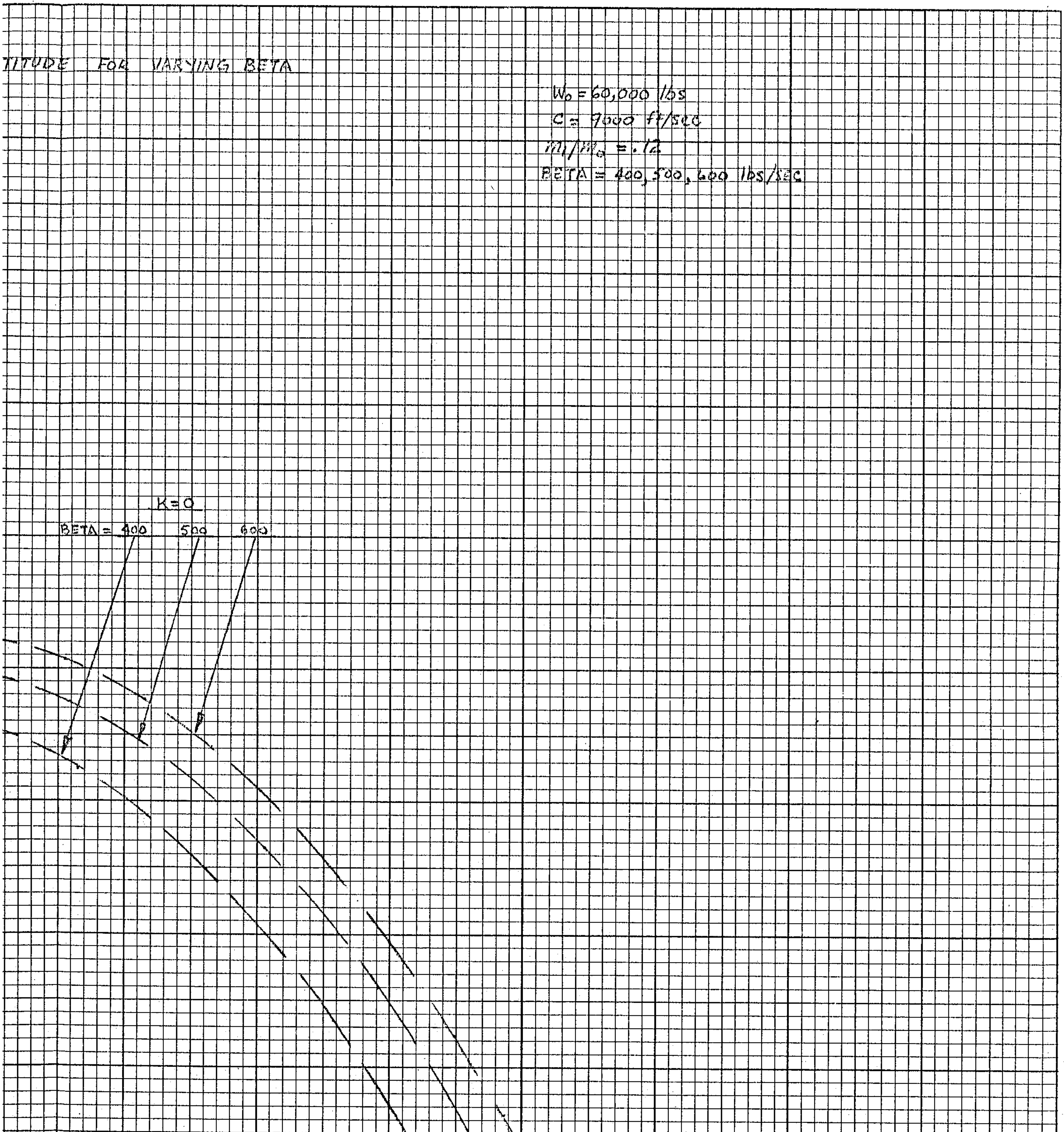
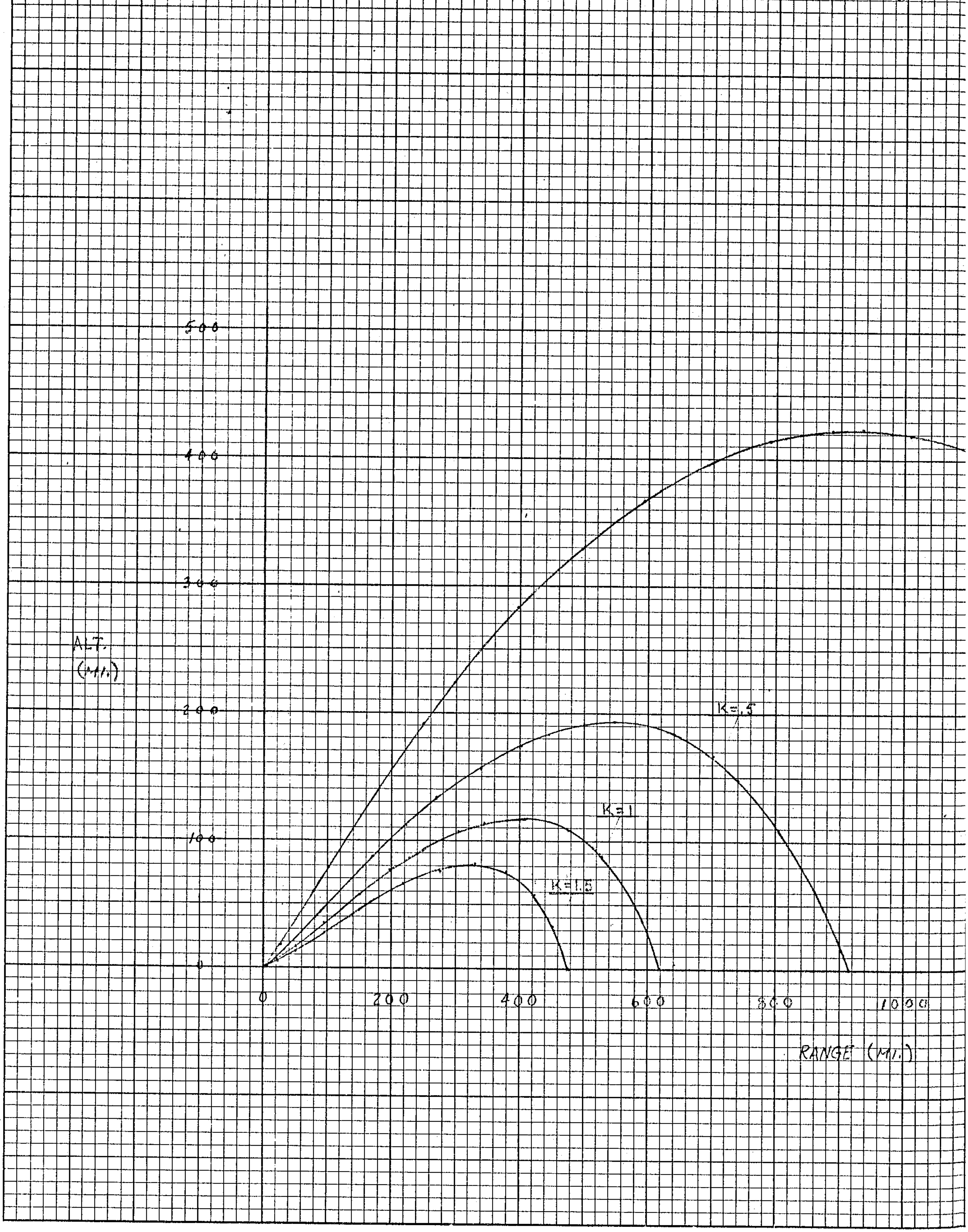
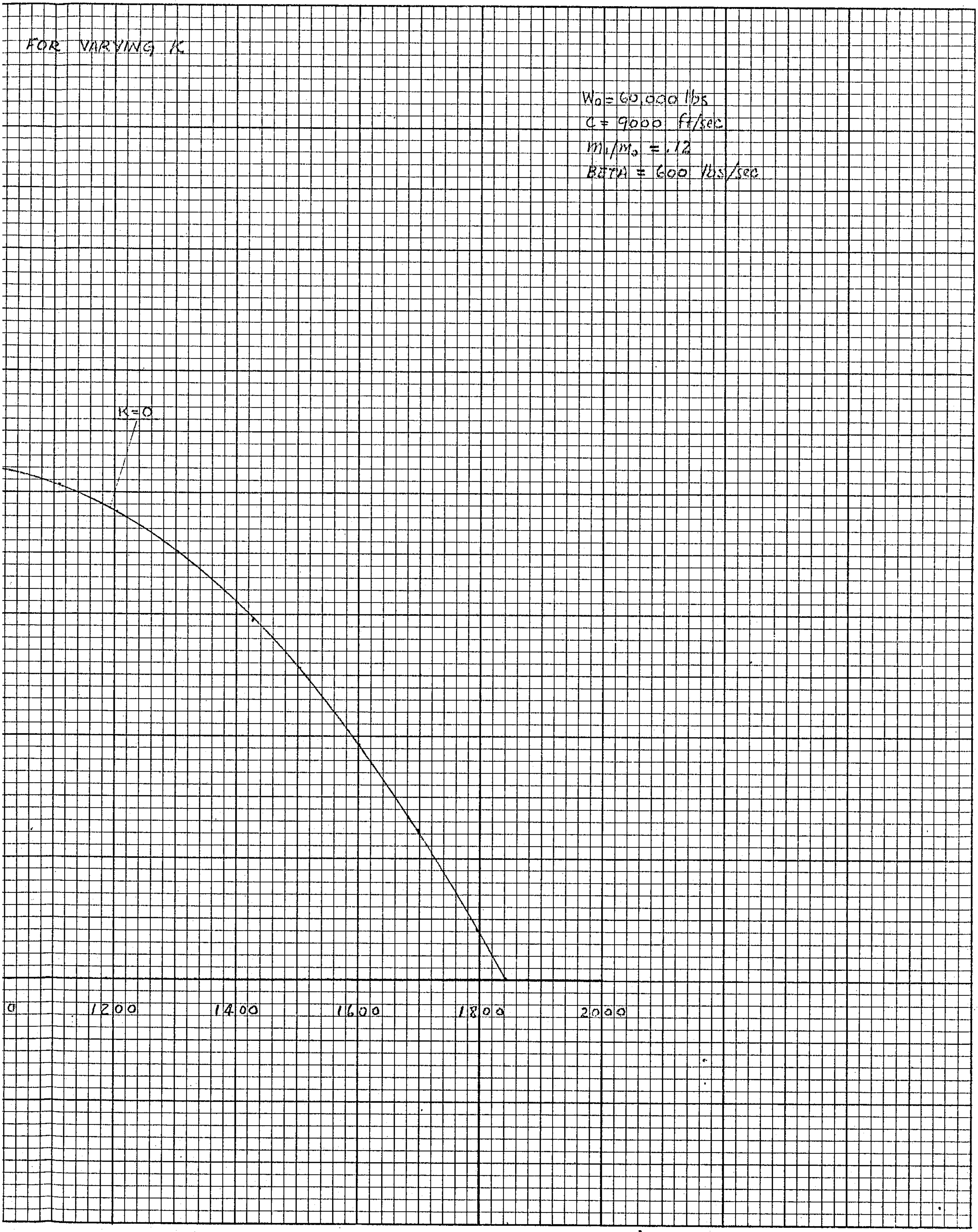
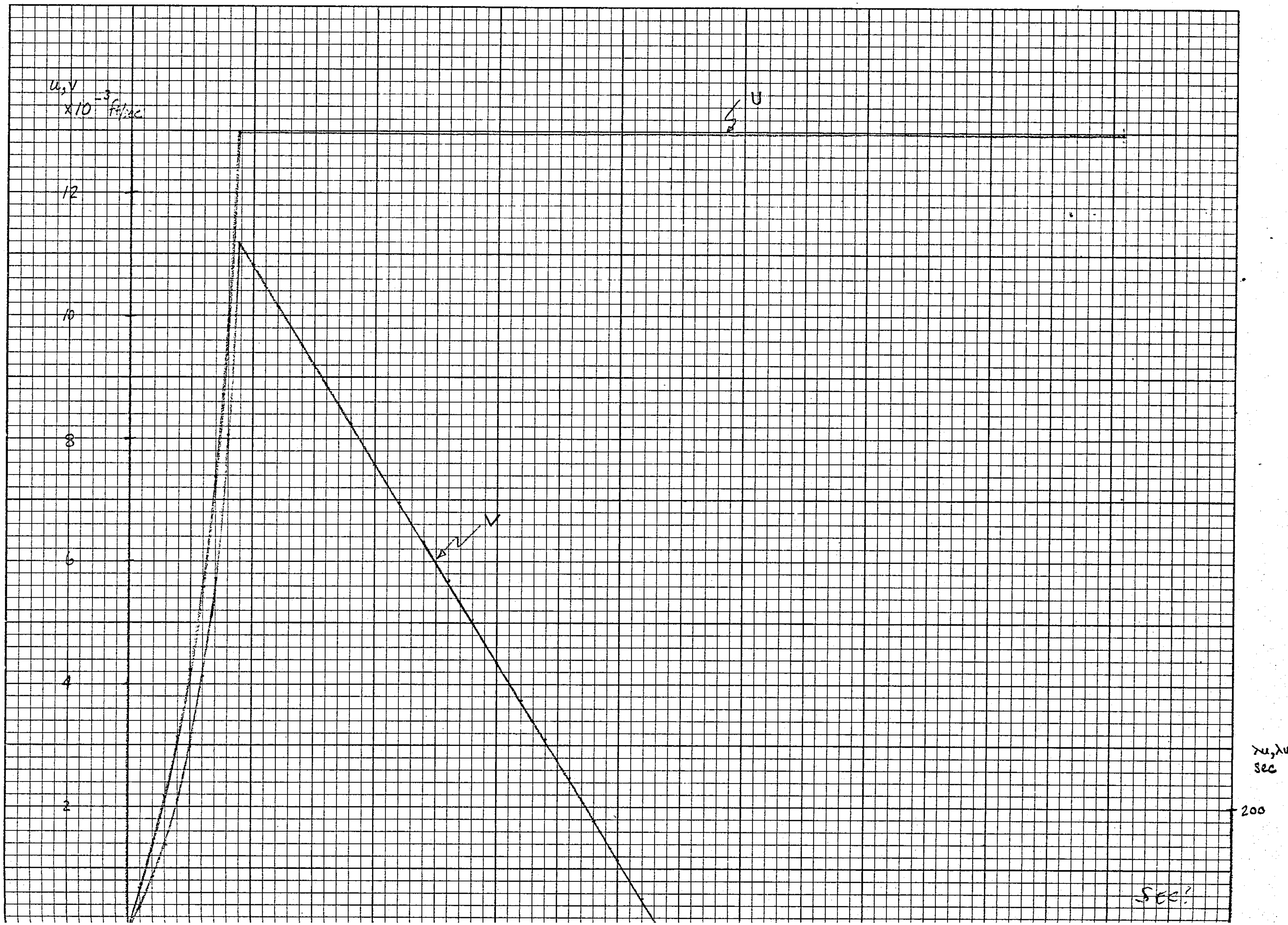

TITUDE FOR VARYING BETA
W_0 = 60,000 lbs
C = 9000 ft/sec
m_1/m_0 = .12
BETA = 400, 500, 600 lbs/SEC
K=0
BETA = 400
500
600
0
1200
1400
1600
1800
2000
MI.)

FIG 2: RANGE VS. ALTITUDE FO




FOR VARYING K
$W_0 = 60,000$ lbs
$C = 9000$ ft/sec
$m_1/m_0 = .12$
BETA = 600 lbs/sec
K=0
0
1200
1400
1600
1800
2000


u, v
×10⁻³ ft/sec
12
10
8
6
4
2
U
V
λu, λv
sec
200
Sec?

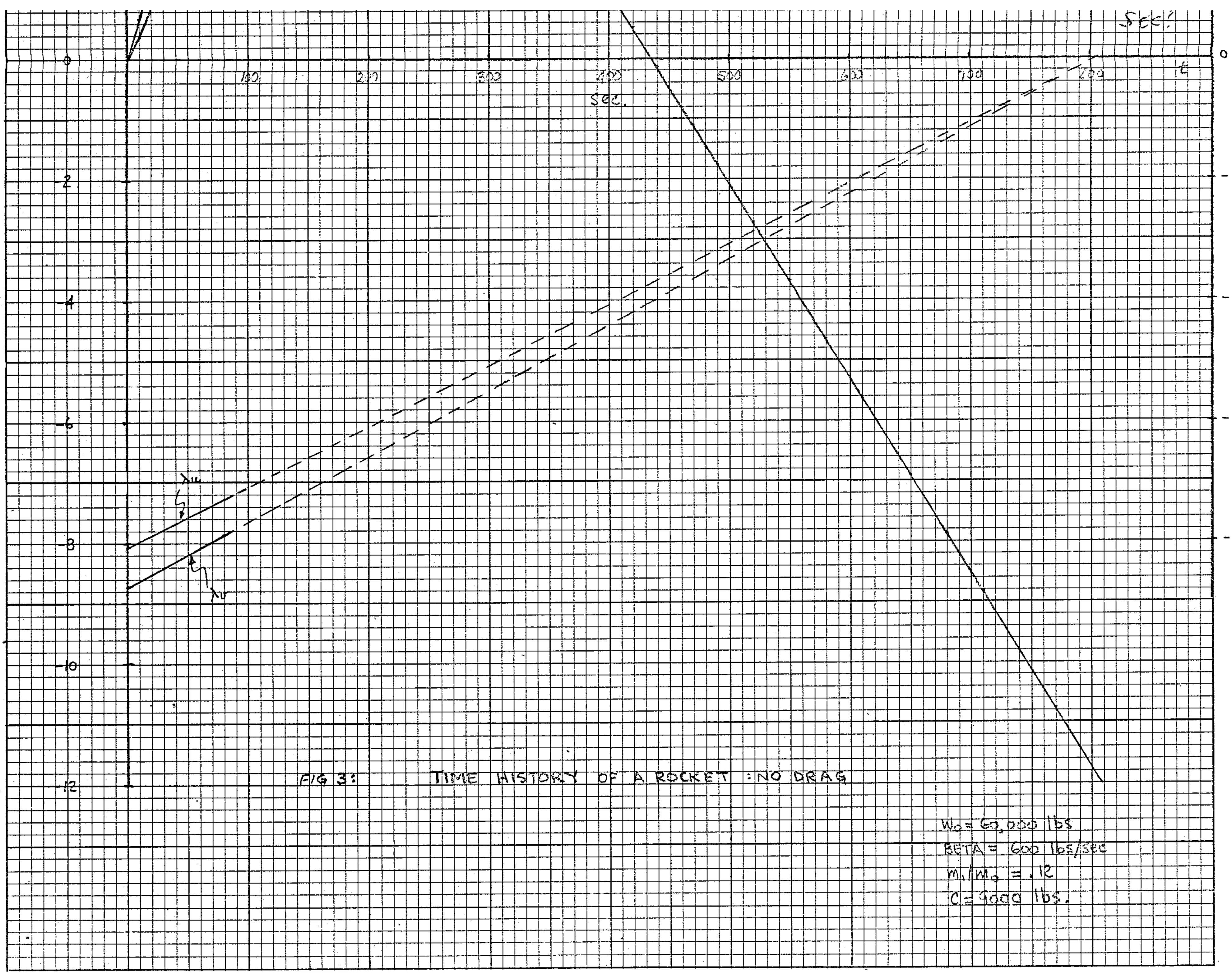
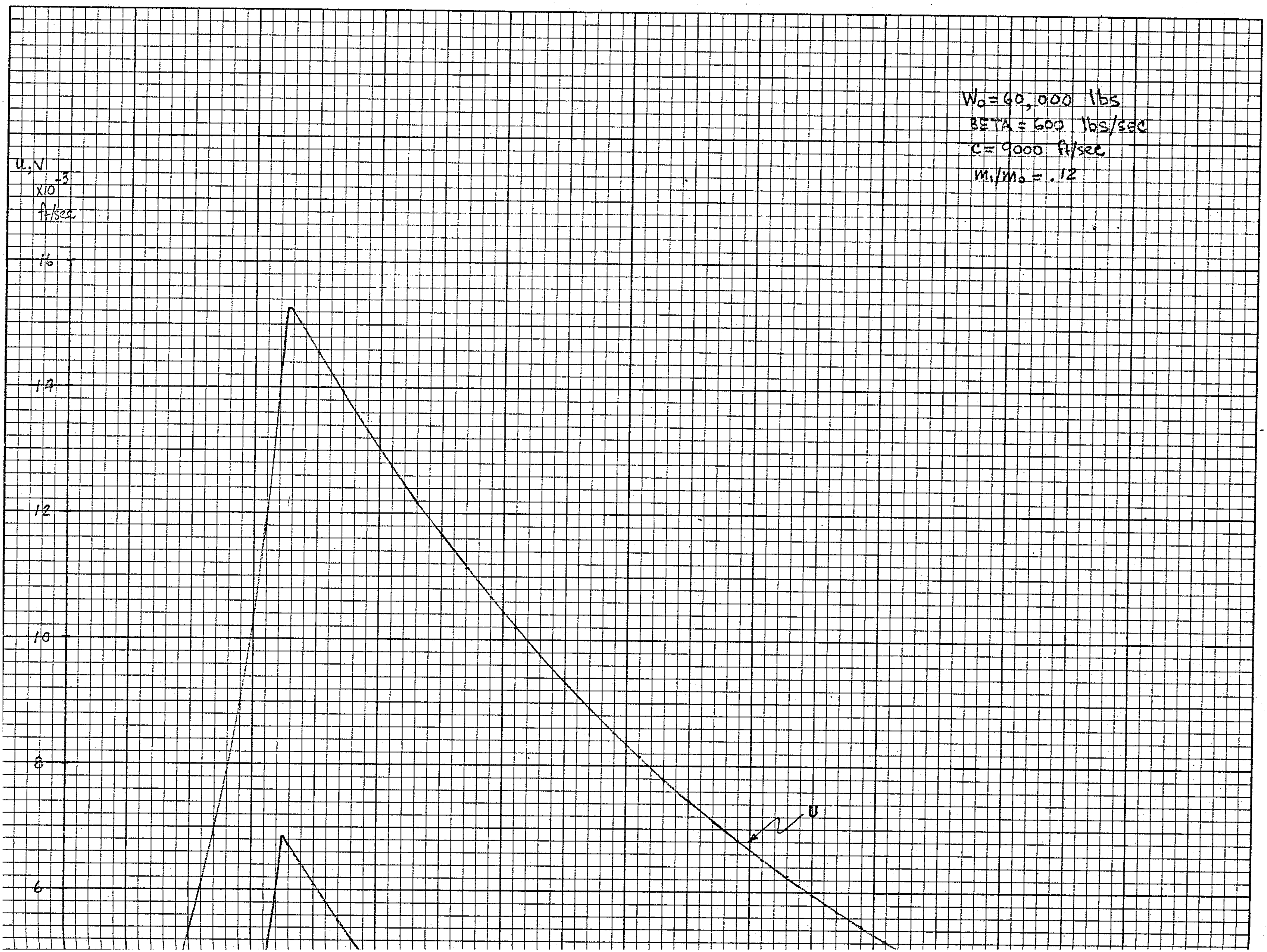


FIG 3: TIME HISTORY OF A ROCKET : NO DRAG


$W_o = 60,000$ lbs
BETA = 600 lbs/sec
C = 9000 ft/sec
$m_1/m_o = .12$
u, v $\times 10^{-3}$ ft/sec
16
14
12
10
8
6
u

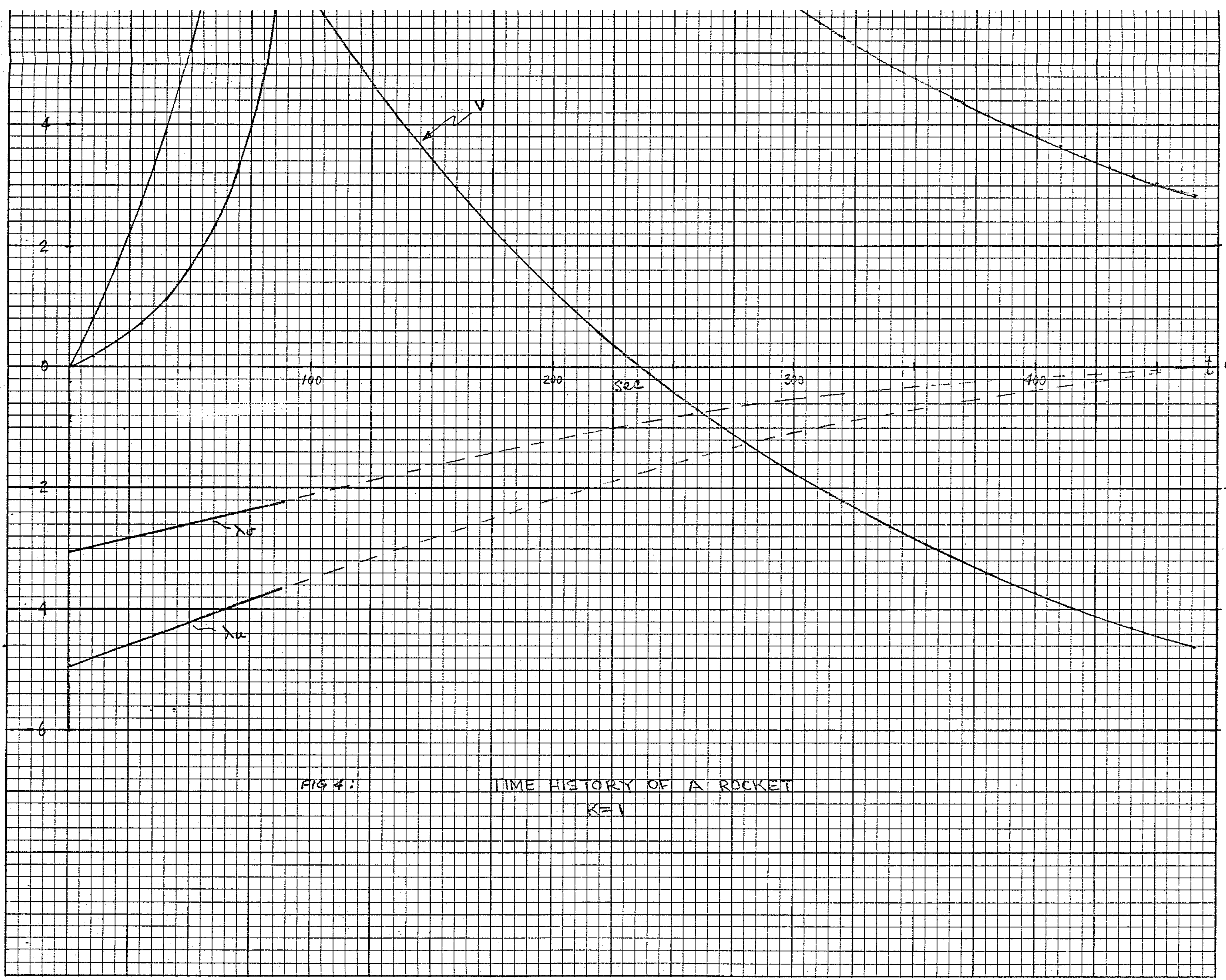
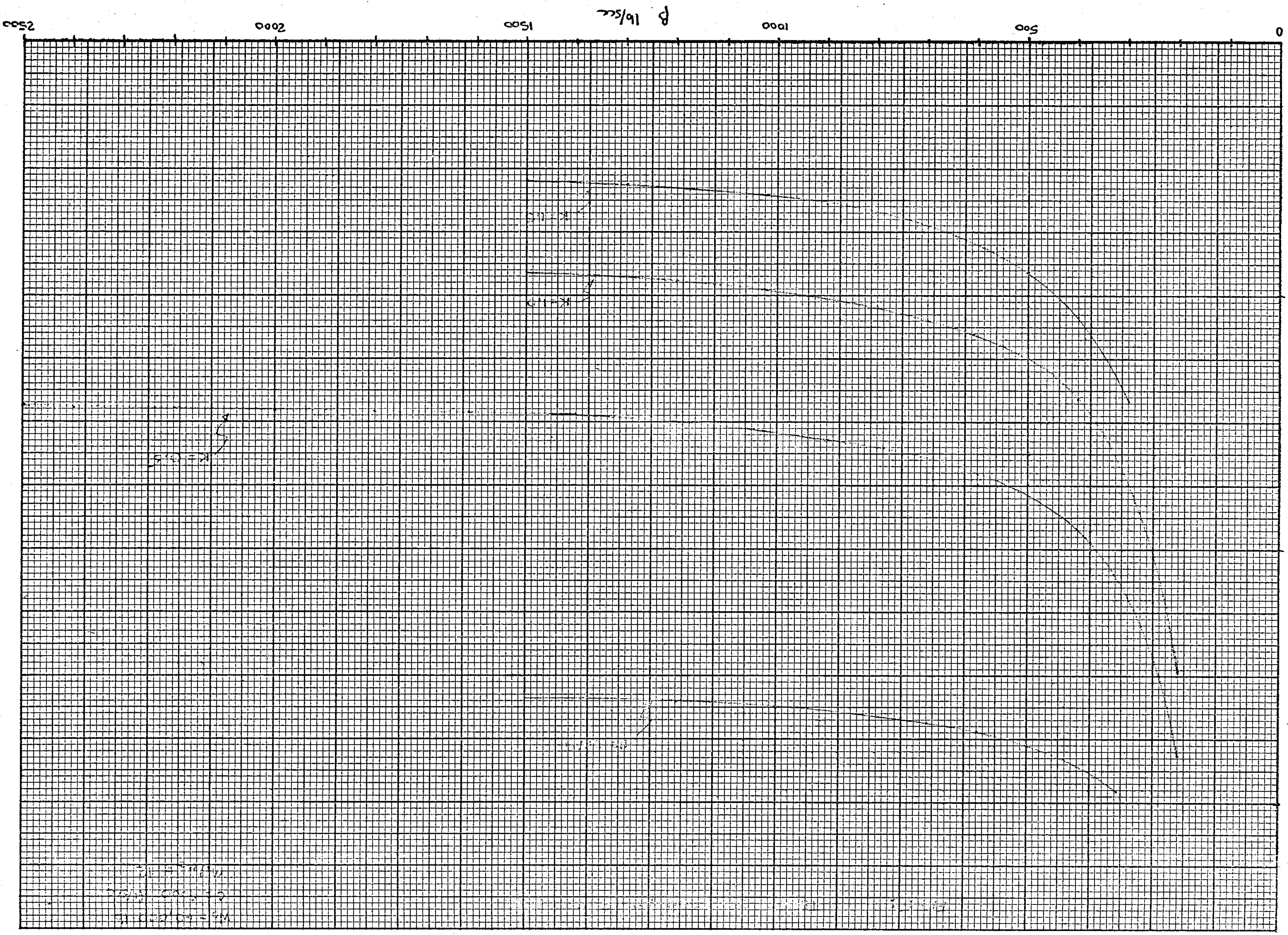


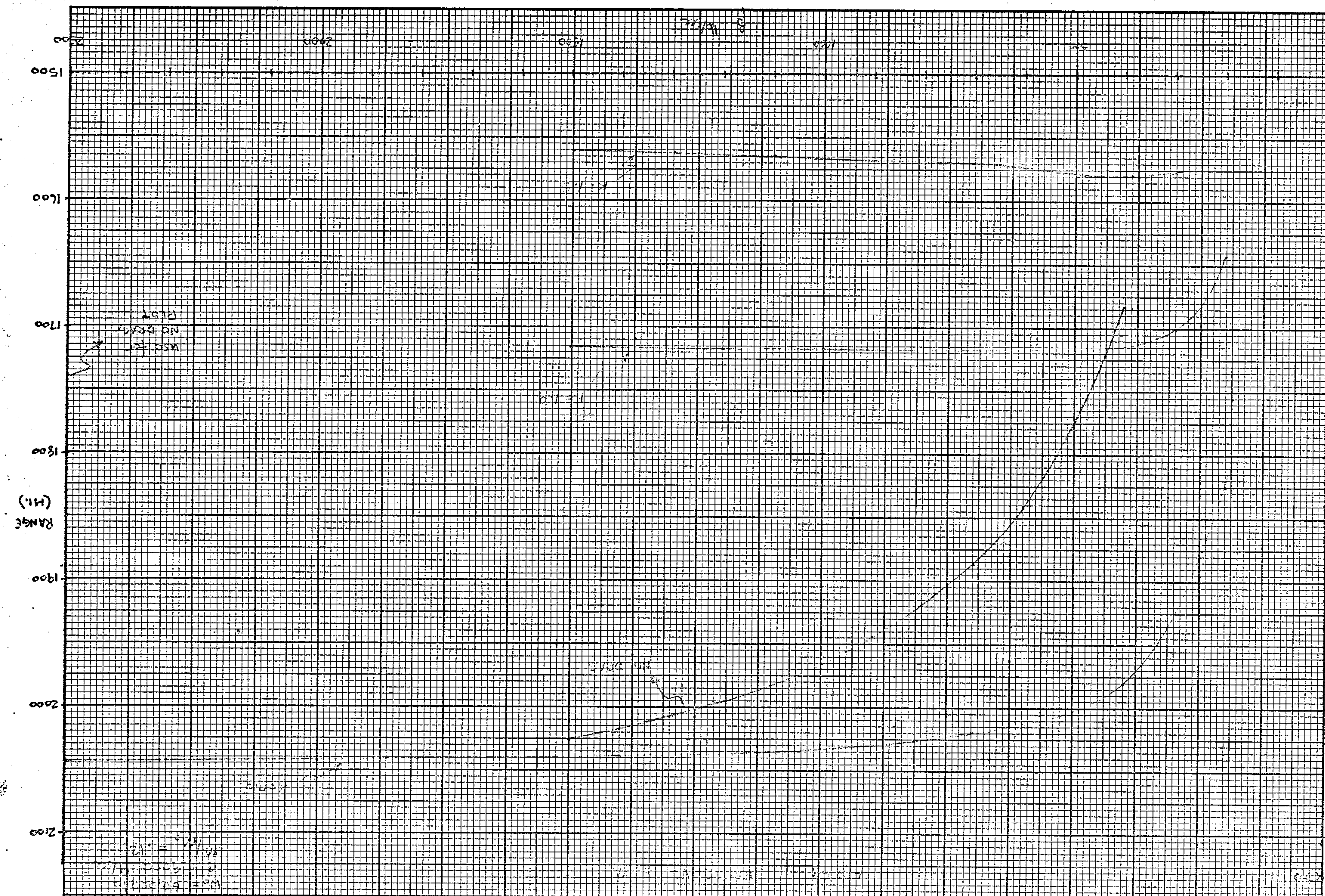
FIG 4: TIME HISTORY OF A ROCKET

K=1

K+E 10 X 10 TO ½ INCH 46 1323
7 X 10 INCHES MADE IN U.S.A.
KEUFFEL & ESSER CO.



K+E 10 X 10 TO 1/2 INCH 46 1323
7 X 10 INCHES MADE IN U.S.A.
KEUFFEL & ESSER CO.



Bibliography

1. LAWDEN, D.F.: OPTIMAL TRAJECTORIES FOR SPACE NAVIGATION; BUTTERWORTH & CO. LTD.; LONDON, 1963.
2. BLISS, G.A.: LECTURES ON THE CALCULUS OF VARIATIONS; THE UNIVERSITY OF CHICAGO PRESS; CHICAGO, 1946.
3. MIELE, A: THE CALCULUS OF VARIATIONS IN APPLIED AERODYNAMICS AND FLIGHT MECHANICS; FROM OPTIMIZATION TECHNIQUES - LEITMANN, G. EDITOR; ACADEMIC PRESS, NEW YORK 1962
4. ELSGOLC, L.E.: CALCULUS OF VARIATIONS; ADDISON - WESLEY PUBLISHING CO, INC; READING, MASS, 1962.
5. WIDDER, D.V.: ADVANCED CALCULUS; PRENTICE-HALL, INC.; ENGLEWOOD CLIFFS, N.J., 1961.
6. LEITMANN, G: VARIATIONAL PROBLEMS WITH BOUNDED CONTROL VARIABLES; FROM OPTIMIZATION TECHNIQUES - LEITMANN, G. EDITOR; ACADEMIC PRESS; NEW YORK 196

General References

a. HILDEBRAND, F.B.: METHODS OF APPLIED MATHEMATICS; 2nd EDITION, PRENTICE-HALL; ENGLEWOOD CLIFFS, NEW JERSEY, 1965.
b. SCHNUR, A.A.: MASTER'S THESIS IN AERONAUTICS & ASTRONAUTICS; POLYTECHNIC INSTITUTE OF BROOKLYN; 1955.
c. FAULKNER, F.D.: DIRECT METHODS; FROM OPTIMIZATION TECHNIQUES - LEITMANN, G. EDITOR; ACADEMIC PRESS; NEW YORK 1962.
d. EDELBAUM, T.N.: THEORY OF MAXIMA & MINIMA; FROM OPTIMIZATION TECHNIQUES - LEITMANN, G. EDITOR; ACADEMIC PRESS; NEW YORK 1962.
e. MIELE, ANGELO: EXTREMIZATION OF LINEAR INTEGRALS BY GREEN'S THEOREM; FROM OPTIMIZATION TECHNIQUES - LEITMANN, G. EDITOR; ACADEMIC PRESS; NEW YORK, 1962.
f. LAWDEN, D.F.: IMPULSIVE TRANSFER BETWEEN ELLIPTICAL ORBITS; FROM OPTIMIZATION TECHNIQUES - LEITMANN, G. EDITOR; ACADEMIC PRESS; NEW YORK, 1962.
g. LEITMANN, G.: PROPULSIVE EFFICIENCY OF ROCKETS; FROM OPTIMIZATION TECHNIQUES - LEITMANN, G. EDITOR; ACADEMIC PRESS, NEW YORK, 1962.
h. SPURLOCK, O.F. & TEREN, F.: OPTIMUM LAUNCH TRAJECTORIES FOR THE ATS-E MISSION; JSR VOLUME 8 NUMBER 12 - DEC 1971, Pgs 1202-1208.
i. SPEYER, J.L. & WOMBLE, M.E.: APPROXIMATE OPTIMAL ATMOSPHERIC ENTRY TRAJECTORIES; JSR VOLUME 8 NUMBER 11 - NOV. 1971, Pgs 1120-1126.

APPENDIX A

G - acceleration of gravity

EPSI - value for testing accuracy of results

WBETA - propellant flow rate lbs/sec

BETA - " " " slugs/sec

WZERO - initial weight lbs

MZERO - " mass slugs

COEFF - 1.0 - final weight/initial weight

CAPT - powered flight time

THETA - angle between thrust vector & horizon

Program uses newton-Raphson method on statement 22

ULAM1 = λu_f

VLAM1 = λv_f

N - no. of printouts per set of data

U(I), V(I) - velocities for time = 10(I-1)

X(I), Y(I) - distances for time = 10(I-1)

MA(I) - mass of rocket for time = 10(I-1)

ULAM(I), VLAM(I) - λu, λv for time = 10(I-1)

T - time

XLAM - λx

YLAM - λy

I - index

OMEGA - THETA IN RADIANS

```
$JOB 0900P,NAME=LEVY
      DIMENSION U(200),V(300),X(300),Y(300),MA(200),ULAM(200),VLAM(200)
      REAL MA, MZERO
  100 READ(5,1) C,WZERO,WBETA,COEFF
    1 FORMAT(4F20.0)
      G=32.200
      EPSI=.00002
      BETA=WBETA/G
      MZERO=WZERO/G
      IF(MZERO.LE.0.000) STOP
      CAPT=COEFF*MZERO/BETA
      MONE=MZERO-BETA*CAPT
      B=MZERO/MONE
      A=C*ALOG(B)
      D=2.*G*C*CAPT
      E=-2.*G*MZERO/BETA
      THETA=40.00
      OMEGA=THETA*(4.*ATAN(1.00))/180.
    5 F1=(SIN(OMEGA))**3+((2.*A**2)/(D+E*A))*(SIN(OMEGA))**2-((A**2)
     1/(E*A+D))
      OMEGA=OMEGA+0.08727
      F2=(SIN(OMEGA))**3+((2.*A**2)/(D+E*A))*(SIN(OMEGA))**2-((A**2)
     1/(E*A+D))
      IF(F1*F2) 10,10,5
   10 FTHETA=(SIN(OMEGA))**3+((2.*A**2)/(D+E*A))*(SIN(OMEGA))**2-((A**2)
     1/(E*A+D))
      DF=(1.5*SIN(OMEGA)+(2.*A**2)/(D+E*A))*SIN(2.*OMEGA)
      IF(ABS(DF).LE.EPSI) STOP
      OMA=OMEGA-FTHETA/DF
      IF(ABS(OMA-OMEGA)-EPSI) 20,20,15
   15 OMEGA=OMA
      GO TO 10
   20 THETA=OMA*180./(4.*ATAN(1.00))
      T=0.00
      I=1
      AA=C*ALOG(B)
      BB=C*CAPT*(ALOG(B)+1.00)
      CC=(C*MZERO/BETA)*ALOG(B)
      U1=AA*COS(OMA)
      V1=AA*SIN(OMA)-G*CAPT
      X1=(BB-CC)*COS(OMA)
      Y1=(BB-CC)*SIN(OMA)-G*CAPT**2/2.
      R=SQRT(V1**2+2.*G*Y1)
      ULAM1=-(V1+R)/G
      VLAM1=U1/R*ULAM1
      FLTT=CAPT+(V1+R)/G
      N=IFIX(FLTT/10.)+2
   30 MA(I)=MZERO-BETA*T
      B=MZERO/MA(I)
      AA=C*ALOG(B)
      BB=C*T*(ALOG(B)+1.00)
      CC=(C*MZERO/BETA)*ALOG(B)
      U(I)=AA*COS(OMA)
      V(I)=AA*SIN(OMA)-G*T
      X(I)=(BB-CC)*COS(OMA)
      Y(I)=(BB-CC)*SIN(OMA)-G*T**2/2.
      ULAM(I)=(T-CAPT)-(V1+R)/G
      VLAM(I)=(U1/R)*ULAM(I)
   35 T=T+10.
      IF(T.GE.CAPT)GO TO 40
```

```
      GO TO 30
   40 M=I
      T=T-10.
   50 T=T+10.
      I=I+1
60    V(I)=V1-G*(T-CAPT)
      X(I)=X1+U1*(T-CAPT)
      Y(I)=Y1+V1*(T-CAPT)-0.5*G*(T-CAPT)**2
      IF(I.LE.(N-2)) GO TO 50
      IF(I.EQ.N) GO TO 70
      T=FLTT
      I=I+1
      GO TO 60
70    XLAM=-1.000
      YLAM=U1/R
      WRITE(6,75) WZERO,WBETA,C,CAPT,FLTT,XLAM,YLAM,THETA
75    FORMAT(/,4(5X,E15.7),/,4(5X,E15.7),//)
      M=M+1
      DO 30 I=1,M
      X(I)=X(I)/5280.
      Y(I)=Y(I)/5280.
COMMENT  NOTE THAT MA PRINTED OUT IS ACTUALLY THE WEIGHT OF THE PARTICLI
      IF(I.EQ.M) ULAM(I)=ULAM1
      IF(I.EQ.M) VLAM(I)=VLAM1
      IF(I.EQ.M) MA(I)=MONE
      IF(I.EQ.M) U(I)=U1
      MA(I)=MA(I)*G
COMMENT  NOTE THAT THE DISTANCES ARE IN MILES...
COMMENT  NOTE M VALUE FOR LAMBDAS U AND V AND THE MASS ARE VALUES AT CA
80    WRITE(6,85) I,ULAM(I),VLAM(I),U(I),V(I),X(I),Y(I),MA(I)
85    FORMAT(5X,I5,7(2X,E14.7))
      M=M+1
      DO 90 I=M,N
      X(I)=X(I)/5280.
      Y(I)=Y(I)/5280.
90    WRITE(6,95) I,V(I),X(I),Y(I)
95    FORMAT(5X,I5,48X,3(2X,E14.7))
      GO TO 100
      END
```

APPENDIX B

K - drag proportionality const.
C - speed of exhaust gases
G - acceleration of gravity
EPSI - accuracy check value
WZERO - initial weight in lbs
MZERO - " mass in slugs
WBETA - mass propellant flow rate in lbs
BETA - " " " " in slugs
COEFF - 1. - weight at burnout/weight initial
CAPT - powered flight time
Program uses Newton-Raphson method on statement 35
BALT - time in ballistic trajectory
N - number of printouts obtained (maximum)
T - time
I - time increment count.
THETA - angle of thrust vector with horizontal
OMA - THETA in radians
U(I), V(I) - velocities at time = 10(I-1)
X(I), Y(I) - distances at time = 10(I-1)
ULAM(I), VLAM(I) - λu, λv at time = 10(I-1)
MA(I) - mass at time = 10(I-1)
XLAM - λx
YLAM - λy

j. MESCHLER, P.A.: ROCKET GUIDANCE WITH A STEERING ANGLE-RATE CONSTRAINT; JSR VOLUME 8 NUMBER 11 NOV 1971, PPGS 1167-1168.

k. POWERS, W.F. & MCDANELL, J.P.: SWITCHING CONDITIONS & A SYNTHESIS TECHNIQUE FOR THE SINGULAR SATURN GUIDANCE PROBLEM; JSR VOLUME 8 NUMBER 10 OCT 1971, PPGS 1027-1032.

l. TING, L.: OPTIMUM ORBITAL TRANSFER BY IMPULSES; ARS JOURNAL VOLUME 30 NUMBER 11 NOV. 1960, PPGS 1013-1018

m. EGGLESTON, J.M.: OPTIMUM TIME TO RENDEVOUS; ARS JOURNAL VOLUME 30 NUMBE NOV. 1960, PPGS 1089-1091

n. MACKAY, J.S.: APPROXIMATE SOLUTION FOR ROCKET FLIGHT WITH LINEAR-TANGENT THRUST ATTITUDE CONTROL; ARS JOURNAL VOLUME 30 NUMBER 11 NOV 1960, PPGS 1091-1098.

```
$JOB 0900P,NAME=LEVY
      DIMENSION U(300),V(300),X(300),Y(300),MA(200),ULAM(200),VLAM(200)
      REAL MA,K,MZERO,MONE
  100 READ(5,1) K,C,WZERO,WBETA,COEFF
    1 FORMAT(5F16.0)
      J=1
      G=32.200
      EPSI=.00001
      MZERO=WZERO/G
      IF(MZERO.LE.0.000) STOP
      BETA=WBETA/G
      CAPT=COEFF*MZERO/BETA
      MONE=MZERO-BETA*CAPT
      A=(BETA*C/K)*(1.00-(MONE/MZERO)**(K/BETA))
      B=G/(BETA-K)*(MONE-MZERO*(MONE/MZERO)**(K/BETA))
      D=C*MZERO/K*(K/(K+BETA)+BETA/(K+BETA)*(MONE/MZERO)**(K/BETA+1.00)-
     1MONE/MZERO)
      E=G*MZERO**2/(BETA*(K-BETA))*(0.5*(MONE/MZERO)**2-BETA/(BETA+K)*
     1(MONE/MZERO)**(1.0+K/BETA)-(K-BETA)/(2.*(BETA+K)))
      T=300.00
    5 ETA=1.00-EXP(-K*T/MONE)
      DELTA=(MONE*G*T/K-MONE/K*(B+MONE*G/K)*ETA-E)/(D+MONE*A*ETA/K)
      OMA=ARSIN(DELTA)
      FT1=TAN(OMA)-A*COS(OMA)*(1.-ETA)/(MONE*G*ETA/K-(A*SIN(OMA)+B)*(1.
     1ETA))
      T=T+25.
      ETA=1.00-EXP(-K*T/MONE)
      DELTA=(MONE*G*T/K-MONE/K*(B+MONE*G/K)*ETA-E)/(D+MONE*A*ETA/K)
      OMA=ARSIN(DELTA)
      FT2=TAN(OMA)-A*COS(OMA)*(1.-ETA)/(MONE*G*ETA/K-(A*SIN(OMA)+B)*(1.
     1ETA))
      IF(FT1*FT2) 10,10,5
   10 ETA=1.00-EXP(-K*T/MONE)
      J=J+1
      IF(J.GE.100) WRITE(6,12) J
   12 FORMAT(5X,I5,2X,'NOT CONVERGING RAPIDLY')
      IF(J.GE.100) STOP
      DELTA=(MONE*G*T/K-MONE/K*(B+MONE*G/K)*ETA-E)/(D+MONE*A*ETA/K)
      OMA=ARSIN(DELTA)
      FTO=TAN(OMA)-A*COS(OMA)*(1.-ETA)/(MONE*G*ETA/K-(A*SIN(OMA)+B)*(1.
     1ETA))
      DTHDT=((D+MONE*A*ETA/K)*(MONE*G*ETA/K-B*(1.-ETA))-(MONE*G*T/K-MON
     1/K*(B+MONE*G/K)*ETA-E)*A*(1.-ETA))/((D+MONE*A*ETA/K)**2*COS(OMA))
      GAMMAA=(MONE*G*ETA/K-(A*SIN(OMA)+B)*(1.-ETA))*(SIN(OMA)*DTHDT+K/
     1MONE*COS(OMA))*A*(1.-ETA)
      GAMMAB=(A*COS(OMA)*(1.-ETA))*(G+K/MONE*(A*SIN(OMA)+B)-A*COS(OMA)*
     1DTHDT)*(1.-ETA)
      SECOMA=1.00/COS(OMA)
      DFTO=(SEC OMA )**2*DTHDT+(GAMMAA+GAMMAB)/(MONE*G*ETA/K-(A*SIN(OMA
     1+B)*(1.-ETA))**2
      IF(ABS(DFTO).LE.EPSI) STOP
      TNEW=T-FTO/DFTO
      IF(ABS(TNEW-T)-EPSI) 20,20,15
   15 T=TNEW
      GO TO 10
   20 THETA=OMA*180./(4.*ATAN(1.000))
      BALT=T
      FLTT=CAPT+BALT
      N=IFIX(FLTT/10.)+2
      T=0.00
```

```
      U1=A*COS(OMA)
      V1=A*SIN(OMA)+B
      X1=D*COS(OMA)
      Y1=D*SIN(OMA)+E
      ULAM1=-MONE/K*ETA
      VLAM1=ULAM1*U1*(1.-ETA)/(MONE*G/K*ETA-V1*(1.-ETA))
   30 MA(I)=MZERO-BETA*T
      A=(BETA*C/K)*(1.00-(MA(I)/MZERO)**(K/BETA))
      B=G/(BETA-K)*(MA(I)-MZERO*(MA(I)/MZERO)**(K/BETA))
      D=C*MZERO/K*(K/(K+BETA)+BETA/(K+BETA)*(MA(I)/MZERO)**(K/BETA+1.00
     1-MA(I)/MZERO)
      E=G*MZERO**2/(BETA*(K-BETA))*(0.5*(MA(I)/MZERO)**2-BETA/(BETA+K)*
     1(MA(I)/MZERO)**(1.0+K/BETA)-(K-BETA)/(2.*(BETA+K)))
      U(I)=A*COS(OMA)
      V(I)=A*SIN(OMA)+B
      X(I)=D*COS(OMA)
      Y(I)=D*SIN(OMA)+E
      ULAM(I)=-1./((K+BETA)*K)*((MONE*BETA-(K+BETA)*MONE*(1.-ETA))*(MON
     1/MA(I))**(K/BETA)+MA(I)*K)
      VLAM(I)=ULAM(I)*K*U1*(1.-ETA)/(MONE*G*ETA-K*V1*(1.-ETA))
   35 T=T+10.
      IF(T.GE.CAPT)  GO TO 40
      I=I+1
      GO TO 30
   40 M=I
      T=T-10.
   50   T=T+10.
      I=I+1
   60 U(I)=U1*EXP(-K*(T-CAPT)/MONE)
      V(I)=-MONE*G/K+(V1+MONE*G/K)*EXP(-K*(T-CAPT)/MONE)
      X(I)=X1+U1*MONE/K*(1.-EXP(-K*(T-CAPT)/MONE))
      Y(I)=Y1-MONE*G*(T-CAPT)/K+MONE/K*(V1+MONE*G/K)*(1.-EXP(-K*(T-CAPT
     1/MONE))
      IF(I.LE.(N-2)) GO TO 50
      IF(I.EQ.N) GO TO 70
      T=FLTT
      I=I+1
      GO TO 60
   70 XLAM=-1.000
      YLAM=U1*EXP(-K*BALT/MONE)/(-MONE*G/K*ETA+V1*(1.-ETA))
      WRITE(6,75) K,WZERO,WBETA,C,CAPT,FLTT,XLAM,YLAM,THETA
   75 FORMAT(4(5X,E15.7),/,4(5X,E15.7),/,5X,E15.7,//)
      M=M+1
      DO 80 I=1,M
C     COMMENT  NOTE THE DISTANCES ARE IN MILES....
      X(I)=X(I)/5280.
      Y(I)=Y(I)/5280.
      IF(I.EQ.M) ULAM(I)=ULAM1
      IF(I.EQ.M) VLAM(I)=VLAM1
      IF(I.EQ.M) MA(I)=MONE
COMMENT  NOTE M VALUES FOR LAMBDAS U AND V AND THE MASS ARE AT CAPT
C     COMMENT  NOTE THAT THE MASS IS ACTUALLY THE WEIGHT OF THE ROCKET
      MA(I)=MA(I)*G
   80 WRITE(6,85) I,ULAM(I),VLAM(I),U(I),V(I),X(I),Y(I),MA(I)
   85 FORMAT(5X,I5,7(2X,E14.7))
      M=M+1
      DO 90 I=M,N
      X(I)=X(I)/5280.
      Y(I)=Y(I)/5280.
   90 WRITE(6,95)I,U(I),V(I),X(I),Y(I)
   95 FORMAT(5X,I5,32X,4(2X,E14.7))
      GO TO 100
      END
```