AE 334 CLASS SCHEDULE

	Tues	Thurs
J	25	27
F	1	--
	8	10
	15	--
	22	24
	29	--
M	7	9
	14	--
	21	23
	X (Fri)	X
A	X	X
	11	--
	18	20
	25	--
M	2	4
	9	~~4~~ 11 Final

References:

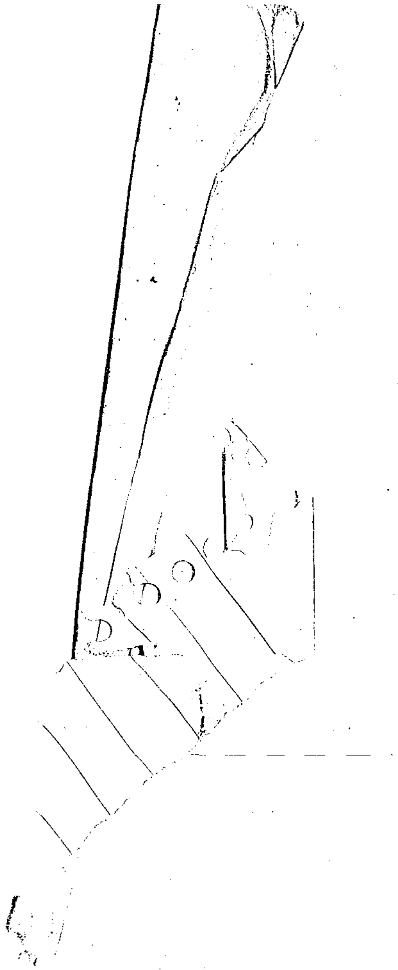
Eckert and Drake: Heat and Mass Transfer McGraw-Hill

Schlichting: Boundary Layer Theory McGraw-Hill

Carslaw and Jaeger: Conduction of Heat in Solids Oxford

Southwell: Relxation Methos in Engineering Science Oxford

Bird, Stewart and Lightfoot: Transport Phenomena Wiley



E 334 A Prof. Rubin 1-25-72

Heat & Mass Transfer w & w/o main flows.

Chapter 15 → on

transport properties of fluids - gradient in the transport properties exist due to molecular movement; another method is turbulent flow : transport by convection

We will only discuss only molecular transport properties. In solid mass, momentum, energy is transported by means of free electrons. In liquid intermolecular forces will be methods of transport. In gas molecular collisions produces transport.

Transport in Gases.

Each molecule velocity $\bar{V}$

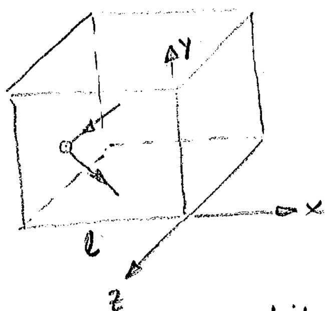
$\bar{V} = \bar{q} + \bar{q}'$ — $\bar{q}$ is average velocity for a given fluid particle (aggregate whose properties are the average of all those properties of molecules making up properties); $\bar{q}'$ is deviation of velocity of each molecule

define $\bar{c}^2 = \overline{u'^2 + v'^2 + w'^2}$ where $\bar{q}' = u'\hat{i} + v'\hat{j} + w'\hat{k}$ — $\bar{c}$ is root mean square & represents the random molecular velocity of molecules.

$$p = \frac{1}{3}\sum_i n_i m_i \bar{c}_i^2$$

n_i no of molecules

m_i mass of each molecule



c_x = random velocity in x direction

n = no of molecules in the volume.

m_i = molecule mass.

$2m_i c_x$ = change of momentum when molecule hits wall in x-direction assume elastic collision

ℓ^2 = area of any wall

$\frac{2\ell}{c_x}$ = average time between collisions in the y-z plane.

rate of change of momentum/unit area $= 2m_i c_x \Big/ \frac{2\ell}{c_x} \cdot \ell^2$

average rate of change of mom/unit area $= \frac{1}{3}\sum_{m_i}\left[m_i(c_x^2 + c_y^2 + c_z^2)\right)/\ell^3$

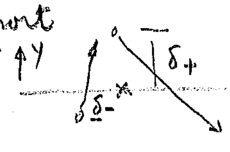
if m_i, etc are assumed the same then

$$\frac{1}{3}\frac{n m_i \bar{c}^2}{\ell^3} = \frac{1}{3}\rho\bar{c}^2 = p = \rho RT \quad \text{perfect gases}$$

$3RT = \bar{c}^2$ for perfect gas $a^2 = \gamma RT$ $\bar{c}^2 = \frac{3}{\gamma}a^2 \approx (1.4\sqrt{a})^2$

$\underbrace{}_{5/8 \sim 1.1}$

Transport



assume same no. of molecules moving up as down.

Assume macroscopic property is only a fn of y $\bar{c} = f(y)$

$\bar{\lambda}$ mean free path : average value of all free paths
free paths being distance travelled by molecules between collisions

Net transport of $A = \sum_i m_i (A_{\delta_-} - A_{\delta_+})$

define Z no. of molecules / unit area, unit time crossing the control surface

assume a continuum $\quad A_{\delta_\mp} = A_0 \mp \left(\frac{dA}{dy}\right)_0 \delta_\mp + \ldots$

then net transp of $A = -\sum_i^{Z} m_i \left(\frac{dA}{dy} \delta_- + \frac{dA}{dy} \delta_+ \right)$

assume δ_- & $\delta_+ \equiv \bar{\lambda}$ & $\sum_i^{Z} m_i = Zm$ for $\forall$ quantities

Net transport of A / unit area, unit time $= -2 Z \bar{\lambda} m \left.\frac{dA}{dy}\right|_0$

define N = no of molecules / unit vol (tot)

no of molecules / unit vol in $y+$ direct $= N/6$

assume $c_y = \bar{c}$ = random molecular veloc in y direct. s/B $\frac{1}{3}\bar{c}$

$\frac{N}{6} \frac{\bar{c}\, \Delta t}{\Delta t}$ = no of moles / unit area, unit time $= Z = \frac{N}{6} \bar{c}$

$$\therefore \quad = -\frac{N}{3} m \bar{\lambda} \bar{c} \frac{dA}{dy} = -\underbrace{\frac{1}{3}}_{\text{in actual results } = \frac{1}{2}} \rho \bar{\lambda} \bar{c} \frac{dA}{dy}$$

1. <u>Transport of momentum</u> $\quad A$ = velocity u mean velocity $\quad A$ = prop / unit mass

$-\frac{1}{3} \rho \bar{\lambda} \bar{c} \frac{du}{dy}$ $\quad$ define rate of shear $-\tau = -\frac{1}{3} \rho \bar{\lambda} \bar{c} \frac{du}{dy}$

for Newtonian Fluid $-\tau = -\mu \frac{du}{dy}$

$\rho \bar{\lambda} = \frac{m}{\sqrt{2} \pi d^2}$ depends of molecule properties $\quad \mu \sim \rho \bar{\lambda} \bar{c} \sim \bar{c} \sim T^{1/2}$

viscosity only a fn of temperature

2. <u>Transport of Energy</u> $\quad A = \frac{1}{2} \bar{c}^2$

internal energy = kinetic energy $= C_V T = \frac{3}{2} RT$
$C_V = \frac{3}{2} R$
$C_p - C_V = R \quad C_p = \frac{5}{2} R \quad C_p / C_V = \gamma = \frac{5}{3}$
monatomic gas.

$= -\frac{Nm}{3} \bar{\lambda} \bar{c}\, C_V \frac{dT}{dy}$

$\tilde{q} = -\frac{1}{3} \rho \bar{\lambda} \bar{c}\, C_V \frac{dT}{dy} = -k \frac{dT}{dy}$ rate of transfer of energy / unit area, unit time

Fourier's Law of Heat Conduction $\quad$ Conduction - molecular interaction
free electrons

$k = \frac{1}{3}\rho\bar{\lambda}\bar{c}\, C_v$ $\qquad k \sim \bar{c} \sim T^{1/2}$ $\qquad$ thermal conductivity

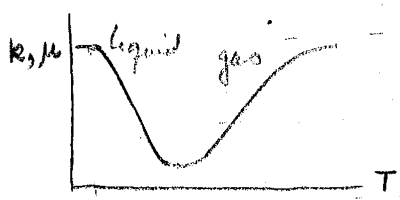
$k \sim \mu C_v \qquad \frac{\mu C_v}{k} = \text{const} \qquad \frac{\mu C_p}{k} = \text{const}\cdot\gamma = P_r \cong .73$ for perfect gas

3. Transport of mass $\qquad A = \omega_A = \frac{\rho_A}{\rho}$ mass fraction of species A

$$j_{A_y} = -\tfrac{1}{3}\rho\bar{\lambda}\bar{c}\,\frac{d\omega_A}{dy}$$

$$j_{A_y} = -\rho D_{AB}\frac{d\omega_A}{dy}$$ Fick's Law $\qquad D_{AB}$ is the diffusion coeff of A diffusing through B

$$D_{AB} \sim \bar{\lambda}\bar{c} \sim \frac{T^{1/2}}{\rho} \sim \frac{T^{3/2}}{P}$$



Convection - another method of transport

$\tilde{q} = \dot{q}/A = h\,\Delta T \qquad \Delta T$ diff between surface & fluid $\quad$ Newtonian Rate Equation

h is convection coefficient

Natural Convection - occurs when induced by means of heating gradients.

Forced Convection - occurs when mean motion exists, if forced by any means.

<u>Radiation</u>

depends on temp difference & actual level of temperature

depends on temp difference & actual level of temperature. Radiation depends on propagation of waves. Molecular transp depends only on temp diff. Radiation depends on propagation of waves. Takes form of light, X rays, Γ rays created by change in internal structures of atom. Thermal rad. in visible ultraviolet to infrared range discussed by freq & wavelength.

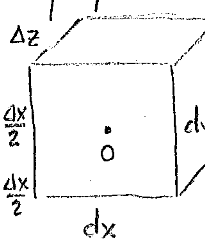
$\lambda: .01 - 100\ \mu$

1-27-72

HW #1 $\quad$ Chapt 15 $\quad$ 6, 8, 9, 15 $\quad$ Chapt 16 $\quad$ 1, 3, 11 $\quad$ Due 1 WK from 2-8-72

Continuity Equation (Mass Conservation)

define all properties at 0 $\qquad$ Continuum Exists



$\rho_- = \rho_0 - \frac{\partial\rho}{\partial x}\frac{\Delta x}{2} \qquad \rho_+ = \rho_0 + \frac{\partial\rho}{\partial x}\frac{\Delta x}{2}$

$u_- = u_0 - \frac{\partial u}{\partial x}\frac{\Delta x}{2} \qquad u_+ = u_0 + \frac{\partial u}{\partial x}\frac{\Delta x}{2}$

neglect variation in Δy direct since identical pts exist above & below 0 in y direct

<u>Assume Constant Volume</u>

Rate of Change of mass in $\Delta x\Delta y\Delta z$ = mass in - mass out

$$\underbrace{\frac{\partial}{\partial t}\left(\rho\Delta x\Delta y\Delta z\right)}_{\text{Rate of Change in volume}} = \underbrace{\left(\rho - \frac{\partial\rho}{\partial x}\frac{\Delta x}{2}\right)\left(u - \frac{\partial u}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z - \left(\rho + \frac{\partial\rho}{\partial x}\frac{\Delta x}{2}\right)\left(u + \frac{\partial u}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z}_{\text{due to flux through } \Delta y\Delta z}$$

$$\frac{\partial}{\partial t}(\rho\,\Delta x \Delta y \Delta z) = -u\frac{\partial \rho}{\partial x}\Delta x\Delta y\Delta z - \rho\frac{\partial u}{\partial x}\Delta x\Delta y\Delta z + O(\Delta^4)$$

since fixed volume

$$\frac{\partial \rho}{\partial t} = -u\frac{\partial \rho}{\partial x} - \rho\frac{\partial u}{\partial x} + O(\Delta)$$

$$\lim_{\Delta\to 0}\frac{\partial \rho}{\partial t} = -u\frac{\partial \rho}{\partial x} - \rho\frac{\partial u}{\partial x} + y \,\&\, z \text{ cont} = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w)$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{q}) = 0 \qquad \text{or} \qquad \underbrace{\frac{\partial \rho}{\partial t} + \bar{q}\cdot\nabla\rho}_{} + \rho\,\nabla\cdot\bar{q} = 0$$

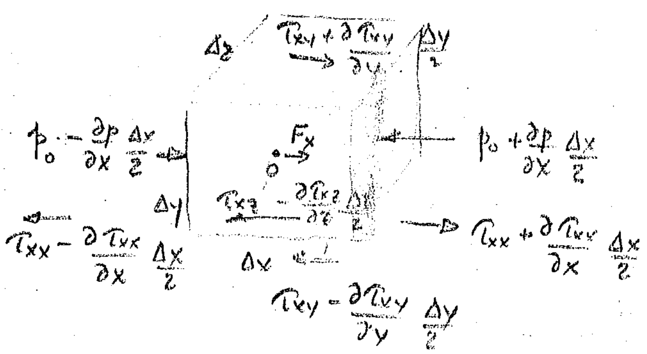
$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot\bar{q} = 0$$

$$\boxed{\frac{D\rho}{Dt} + \rho\,\nabla\cdot\bar{q} = 0} \qquad \frac{D\rho}{Dt} = 0 \text{ incompressible} \Rightarrow \nabla\cdot\bar{q} = 0$$

along a streamline

Momentum Equation

Newton's law $F \sim \bar{a}$



F_x = body force / unit Volume
p acts normal to face

τ_{ij} – viscous stress/unit area acting in i direction on a plane whose outward normal is in the j direction
positive stress in direct of positive axis if outward normal is in positive direction

Assume volume keeps its shape & keeps same mass inside. No mass cross boundary.

Rate of Change of Momentum = $\sum$ external forces $(p, \bar{F}, \tau_{ij})$

$$\frac{D}{Dt}(\rho u\,\Delta x\Delta y\Delta z) = \left(p - \frac{\partial p}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z - \left(p + \frac{\partial p}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z - \left(\tau_{xx} - \frac{\partial \tau_{xx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z$$
$$+ \left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z - \left(\tau_{xy} - \frac{\partial \tau_{xy}}{\partial y}\frac{\Delta y}{2}\right)\Delta x\Delta z + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}\frac{\Delta y}{2}\right)\Delta x\Delta z$$
$$- \left(\tau_{xz} - \frac{\partial \tau_{xz}}{\partial z}\frac{\Delta z}{2}\right)\Delta x\Delta y + \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z}\frac{\Delta z}{2}\right)\Delta x\Delta y + F_x\,\Delta x\Delta y\Delta z$$

since const mass

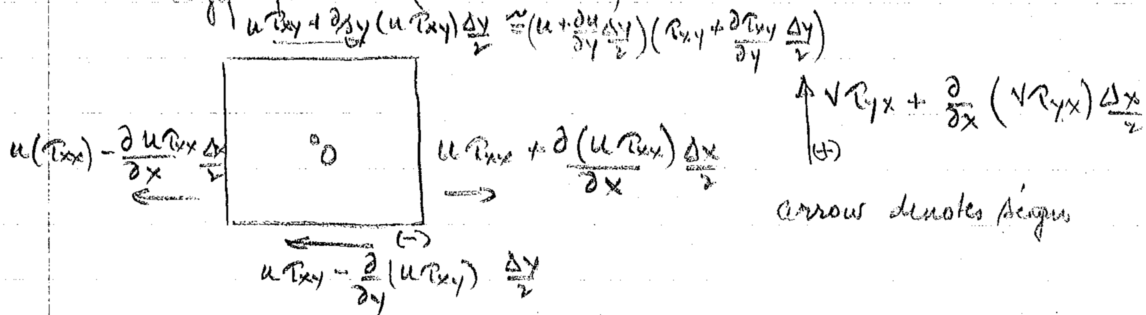
$$\rho\frac{D u}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x$$

$$\rho\frac{D v}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y$$

$$\rho \frac{D\bar{q}}{Dt} = -\nabla p + \frac{\partial \bar{\tau}_x}{\partial x} + \frac{\partial \bar{\tau}_y}{\partial y} + \frac{\partial \bar{\tau}_z}{\partial z} + \bar{F}$$

where $\quad \bar{\tau}_x = \tau_{xx}\,\bar{i} + \tau_{yx}\,\bar{j} + \tau_{zx}\,\bar{k}$

Energy Equation (Fixed Mass)



Rate of change of total energy (internal + kinetic) = the rate at which work is being done on volume by $(p, \tau_{ij}, \bar{F})$ + rate at which heat is being added by conduction & other forms of heat generation.

$$\lim_{\Delta t, \Delta r \to \Delta t} \frac{1}{\Delta t}\int_{r_0}^{r_0+\Delta \bar{r}} \bar{F}\cdot d\bar{r} = \text{rate of work which is being done at } r_0$$

$$= \lim_{\Delta t, \Delta r \to \Delta t} \frac{1}{\Delta t}\, \bar{F}_{AV}\cdot \Delta \bar{r} = \bar{F}\cdot\bar{q} = |\bar{F}||\bar{q}|\cos(\bar{F},\bar{q})$$

$$\frac{D}{Dt}\left\{\rho\,\Delta x\Delta y\Delta z\,(e + q^2/2)\right\} = \left(up - \frac{\partial}{\partial x}(up)\frac{\Delta x}{2}\right)\Delta y\Delta z - \left(up + \frac{\partial up}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z$$

$$+ \text{ y \& z pressure work terms } - \left(u\tau_{xx} - \frac{\partial u\tau_{xx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z + \left(u\tau_{xx} + \frac{\partial u\tau_{xx}}{\partial x}\frac{\Delta x}{2}\right)$$

$$-\left(v\tau_{yx} - \frac{\partial v\tau_{yx}}{\partial y}\frac{\Delta x}{2}\right)\Delta y\Delta z + \left(v\tau_{yx} + \frac{\partial v\tau_{yx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z - \left(w\tau_{zx} - \frac{\partial w\tau_{zx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z$$

$$+\left(w\tau_{zx} + \frac{\partial w\tau_{zx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z + \text{work due to stresses on other 4 planes} + \left(F_x u + F_y v + w F_z\right)\Delta V$$

$$+\left\{\dot{Q}\,(\text{heat addition, other than conduction}) + \dot{q}\,(\text{conduction})\right\}\Delta x\Delta y\Delta z$$

$$\rho\frac{D}{Dt}(e + q^2/2) = -\frac{\partial up}{\partial x} - \frac{\partial vp}{\partial y} - \frac{\partial wp}{\partial z} + \frac{\partial u\tau_{xx}}{\partial x} + \frac{\partial v\tau_{yx}}{\partial x} + \frac{\partial w\tau_{zx}}{\partial x}$$

$$+ \frac{\partial u\tau_{xy}}{\partial y} + \frac{\partial v\tau_{yy}}{\partial y} + \frac{\partial w\tau_{zy}}{\partial y} + \frac{\partial u\tau_{xz}}{\partial z} + \frac{\partial v\tau_{yz}}{\partial z} + \frac{\partial w\tau_{zz}}{\partial z}$$

$$+ F_x u + F_y v + F_z w + \dot{Q} + \dot{q}$$

$$\rho \frac{D}{Dt}\left(e+q^2/2\right) = -\left[p\nabla\cdot\bar{q} + \bar{q}\cdot\nabla p\right] + \frac{\partial}{\partial x}(\bar{q}\cdot\bar{\tau}_x) + \frac{\partial}{\partial y}(\bar{q}\cdot\bar{\tau}_y) + \frac{\partial}{\partial z}(\bar{q}\cdot\bar{\tau}_z)$$
$$+ \bar{q}\cdot\bar{F} + \dot{Q} + \dot{q}$$

Stagnation enthalpy $H_0 = \overbrace{e + p/\rho}^{h} + q^2/2$

$$-p\nabla\cdot\bar{q} - \bar{q}\cdot\nabla p = p/\rho \frac{D\rho}{Dt} - \frac{Dp}{Dt} + \frac{\partial p}{\partial t}$$ from cont

$$= -\rho\frac{D\,p/\rho}{Dt} + \frac{\partial p}{\partial t}$$

$$\rho\frac{DH_0}{Dt} = \frac{\partial p}{\partial t} + \bar{q}\cdot\bar{F} + \dot{Q} + \dot{q} + \Lambda$$

where $\Lambda = \frac{\partial}{\partial x}(\bar{q}\cdot\bar{\tau}_x) + \frac{\partial}{\partial y}(\bar{q}\cdot\bar{\tau}_y) + \frac{\partial}{\partial z}(\bar{q}\cdot\bar{\tau}_z)$

H_0 is const on streamline when steady, no body forces, no viscous work, no heat addition

$$\rho\frac{D}{Dt}\left(e+q^2/2\right) = \rho\frac{De}{Dt} + \rho\bar{q}\cdot\frac{D\bar{q}}{Dt} = -\left[p\nabla\cdot\bar{q} + \bar{q}\cdot\nabla p\right] + \bar{q}\cdot\left[\frac{\partial\bar{\tau}_x}{\partial x} + \frac{\partial\bar{\tau}_y}{\partial y} + \frac{\partial\bar{\tau}_z}{\partial z}\right]$$

viscous work term

$$+ \underbrace{\bar{\tau}_x\cdot\frac{\partial\bar{q}}{\partial x} + \bar{\tau}_y\cdot\frac{\partial\bar{q}}{\partial y} + \bar{\tau}_z\cdot\frac{\partial\bar{q}}{\partial z}}_{\text{viscous heat dissipation}} + \bar{q}\cdot\bar{F} + \dot{Q} + \dot{q}$$

$$\rho\frac{De}{Dt} = -p\nabla\cdot\bar{q} + \bar{\Phi} + \dot{Q} + \dot{q} \quad\leftarrow\quad -p\nabla\cdot\bar{q} = -\frac{p}{\rho}\frac{D\rho}{Dt} = -\rho\frac{D\,1/\rho}{Dt}$$

$$\bar{\Phi} = \bar{\tau}_x\cdot\frac{\partial\bar{q}}{\partial x} + \bar{\tau}_y\cdot\frac{\partial\bar{q}}{\partial y} + \bar{\tau}_z\cdot\frac{\partial\bar{q}}{\partial z}$$ viscous dissipation term

$$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} + \bar{\Phi} + \dot{Q} + \dot{q}$$

2-1-72

$$\rho\frac{De}{Dt} = -p\nabla\cdot\bar{q} + \bar{\Phi} + \dot{Q} + \dot{q}$$

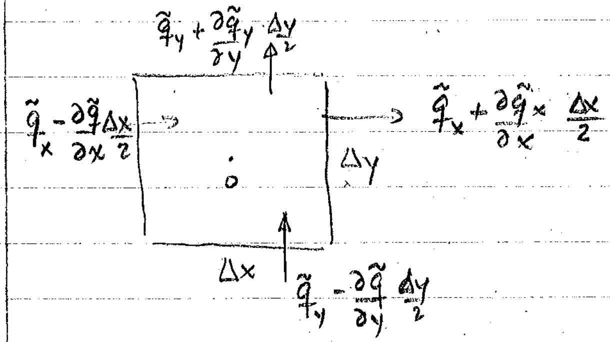
$$\rho\frac{DH_0}{Dt} = \frac{\partial p}{\partial t} + \bar{q}\cdot\bar{F} + \dot{Q} + \dot{q} + \Lambda$$

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \rho p \frac{D\,1/\rho}{Dt} - p\,\nabla\cdot\bar{q} + \bar{\Phi} + \dot{Q} + \dot{q}$$

$$\underbrace{-\frac{p}{\rho}\frac{D\rho}{Dt} - p\,\nabla\cdot\bar{q}}_{=\,0}$$

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \bar{\Phi} + \dot{Q} + \dot{q}$$

$$\rho \frac{De}{Dt} = \rho C_v \frac{DT}{Dt} \;, \quad \rho \frac{Dh}{Dt} = \rho C_p \frac{DT}{Dt} \quad \text{for perfect gas.}$$



$$\bar{\tilde{q}} = -k\nabla T \quad \text{Fourier's Law of Heat Conduction}$$

$$\tilde{q}_x = -k\frac{\partial T}{\partial x}$$

$$-\frac{\partial \tilde{q}_x}{\partial x}\Delta V - \frac{\partial \tilde{q}_y}{\partial y}\Delta V - \frac{\partial \tilde{q}_z}{\partial z}\Delta V = \dot{q}\,\Delta V$$

$$-\nabla\cdot\bar{\tilde{q}} = \dot{q} = +\nabla\cdot(k\nabla T) = +k\,\nabla^2 T \quad \text{if } k \neq k(x,y,z)$$

Continuity

$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot\bar{q} = 0$$

Momentum

$$\rho \frac{D\bar{q}}{Dt} = -\nabla p + \bar{F} + \frac{\partial \bar{\tau}_x}{\partial x} + \frac{\partial \bar{\tau}_y}{\partial y} + \frac{\partial \bar{\tau}_z}{\partial z}$$

Energy

$$\rho C_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \bar{\Phi} + \dot{Q} + \nabla\cdot(k\nabla T)$$

$$\rho C_v \frac{DT}{Dt} = -p\,\nabla\cdot\bar{q} + \bar{\Phi} + \dot{Q} + \nabla\cdot(k\nabla T)$$

$$\rho \frac{DH_0}{Dt} = \frac{\partial p}{\partial t} + \bar{q}\cdot\bar{F} + \dot{Q} + \Lambda + \nabla\cdot(k\nabla T)$$

where $\Lambda = \frac{\partial}{\partial x}(\bar{q}\cdot\bar{\tau}_x) + \frac{\partial}{\partial y}(\bar{q}\cdot\bar{\tau}_y) + \frac{\partial}{\partial z}(\bar{q}\cdot\bar{\tau}_z)$

$$\Phi = \bar{\tau}_x \cdot \frac{\partial \bar{q}}{\partial x} + \bar{\tau}_y \cdot \frac{\partial \bar{q}}{\partial y} + \bar{\tau}_z \cdot \frac{\partial \bar{q}}{\partial z}$$

$$\bar{\tau}_x = \tau_{xx}\,\bar{i} + \tau_{yx}\,\bar{j} + \tau_{zx}\,\bar{k}$$

$$\Phi = 2\mu\left[u_{,x}^2 + v_{,y}^2 + w_{,z}^2\right] + \mu\left[(u_{,y}+v_{,x})^2 + (v_{,z}+w_{,y})^2 + (w_{,x}+u_{,z})^2\right]$$

$$\tau_{xx} = -\frac{2}{3}\mu\,\nabla\cdot\bar{q} + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yy} = -\tfrac{2}{3}\mu\,\nabla\cdot\bar{q} + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

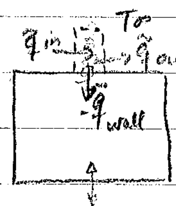
$$\tau_{zz} = -\tfrac{2}{3}\mu\,\nabla\cdot\bar{q} + 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

since $\mu > 0$ $\therefore$ Φ is positive definite $\quad \Phi \geq 0$

Boundary Conditions

$\bar{q}$ ⟨
1) $\bar{q}\cdot\bar{n} = 0$ no oscillating body, porous surface $\bar{q} \rightarrow \downarrow \bar{n}$ no flow thru body
2) $\bar{q}\times\bar{n} = 0$ on surface no slip
3) $T = T_W$ = const (isothermal wall condition)
4) $\tilde{q}_{wall} = 0 \quad \left.\frac{\partial T}{\partial n}\right|_{wall} = 0$ insulated (adiabatic wall)



$\tilde{q}_y = -k\frac{\partial T}{\partial y}$ heat cond to wall on upper surface

at low speeds $\quad \left. k\frac{\partial T}{\partial y}\right|_W = h(T_\infty - T)_W$

at lower surface $\quad \left. -k\frac{\partial T}{\partial y}\right|_W = h(T_\infty - T)_W$

<u>Steady State Heat Conduction</u>

no convection $\quad -\nabla p + \bar{F} = 0 \qquad$ small velocities or $\bar{q} = 0$

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \cancel{\Phi} + \dot{Q} + \nabla\cdot(k\nabla T)$$

steady $\dot{q} = 0$ $\quad$ steady $q = 0$ $\quad$ $q = 0$

$$\nabla \cdot (k \nabla T) + \dot{Q} = 0 \qquad \text{Governing Equation}$$

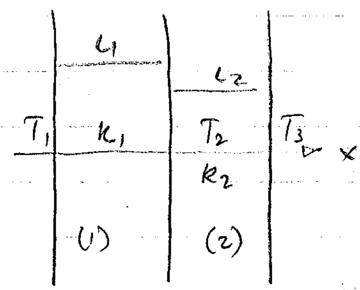
if $k = \text{const}$

$$k \nabla^2 T + \dot{Q} = 0 \qquad \text{Poisson Equation}$$

if $\dot{Q} = 0$

$$\nabla^2 T = 0$$

1-D Conduction $\quad k = \text{const} \quad \dot{Q} = 0$



q_x

$$T_{xx} = 0$$
$$T_1 = Ax + B \qquad T = T_1 \quad x = 0$$
$$T_2 = Cx + D \qquad T = T_2 \quad x = L_1$$
$$T = T_2 \quad x = L_1$$
$$T = T_3 \quad x = L_1 + L_2$$

$$T = T_1 - \left(\frac{T_1 - T_2}{L_1}\right) x$$

$$T = T_2 - \left(\frac{T_2 - T_3}{L_2}\right)(x - L_1) \qquad \tilde{q} = \dot{q}/A = -k\frac{dT}{dx}$$

$$\dot{q}_1 = k_1 A \left[\frac{T_1 - T_2}{L_1}\right]$$

$$\dot{q}_1 = \frac{k_1 A}{L_1}(T_1 - T_2) \qquad (1)$$

$$\dot{q}_2 = \frac{k_2 A}{L_2}(T_2 - T_3) \qquad (2)$$

$\dot{q}_1 = \dot{q}_2$ at interface but each is constant $\therefore \dot{q}_1 = \dot{q}_2 = \dot{q}$.

$$\dot{q}\left(\frac{L_1}{k_1 A} + \frac{L_2}{k_2 A}\right) = T_1 - T_3$$

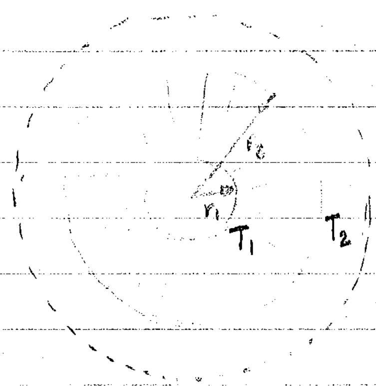
$$\dot{q} = \frac{T_1 - T_3}{\dfrac{L_1}{k_1 A} + \dfrac{L_2}{k_2 A}}$$

we assume a continuum $\qquad L/kA$ = thermal resistance

$\frac{kA}{L}$ = thermal conductance

$\nabla^2 T = 0$ — thermal field — $\dot{q} = \dfrac{\Delta T}{R_{TH}}$

$\nabla^2 E = 0$ — Electric field — $j = \dfrac{\Delta E}{R}$



$$\nabla^2 T = \frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

$$r\frac{dT}{dr} = \bar{A}$$

$$T = \bar{A}\ln r + B$$

$$T = T_1 + (T_2 - T_1)\frac{\ln r/r_1}{\ln \frac{r_2}{r_1}}$$

$$\dot{q}/A = -k\frac{dT}{dr} = \frac{k(T_1 - T_2)}{r \ln r_2/r_1}$$

$$A = 2\pi r L$$

$$\dot{q} = \frac{2\pi L k}{\ln r_2/r_1}(T_1 - T_2) \qquad R_{TH} = \frac{\ln r_2/r_1}{2\pi L k}$$

$$\dot{q} = h(2\pi r_2 L)(T_\infty - T_2) \qquad R_{TH_{conv}} = \frac{1}{2\pi r L h}$$

Example:

$$\frac{d}{dx}\left(\{1 + \beta_0 T\}\frac{dT}{dx}\right) = 0$$

$$\{1 + \beta_0 T\}\,dT = \bar{A}\,dx$$

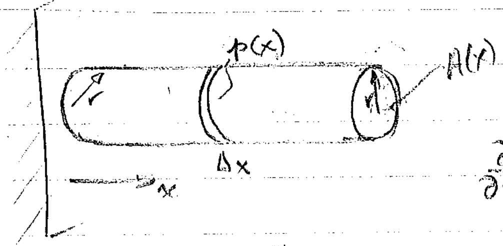
$$T + \beta_0\frac{T^2}{2} = \bar{A}x + B \qquad \begin{array}{ll} x = 0 & t = T_1 \\ x = l_1 & t = T_2 \end{array}$$

2-8-72

Extended Surfaces - Quasi 1-D Flow Conduction

$\dot{Q} = 0 \qquad k = \text{const}$

if almost circular



$$\nabla \cdot (k \nabla T) = 0$$

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) = 0$$

Cylindrical Coord, $\frac{\partial T}{\partial \theta} \sim 0$ roughly uniform in θ direction

$$\int_0^{r_0(x,\theta)} \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) 2\pi r dr + \int_0^{r_0(x,\theta)} \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) 2\pi r dr$$

$$\frac{\partial}{\partial x}\int_0^{r_0(x,\theta)} k\frac{\partial T}{\partial x} 2\pi r dr - \frac{\partial r_0}{\partial x}\left(2\pi r k \frac{\partial T}{\partial x}\right)_{r_0} + rk(2\pi)\frac{\partial T}{\partial r}\Big|_0^{r_0} = 0$$ Leibnitz's rule

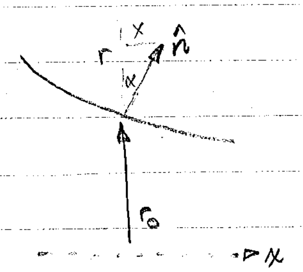
$$\frac{\partial}{\partial x}\int_0^{r_0} 2\pi k \frac{\partial T}{\partial x} r dr - \frac{\partial r_0}{\partial x}\left(2\pi r k \frac{\partial T}{\partial x}\right)_{r=r_0} + rk(2\pi)\frac{\partial T}{\partial r}\Big|_0^{r_0} = 0$$

$$\bar{V} = \text{average value} = \frac{1}{A(x)}\int_0^{r_0} 2\pi r V dr$$

$$\frac{\partial}{\partial x}\left(A(x)\, \overline{k\frac{\partial T}{\partial x}}\right) + 2\pi r_0 \left(k\frac{\partial T}{\partial r} - \frac{\partial r_0}{\partial x} k \frac{\partial T}{\partial x}\right)_{r=r_0} = 0$$

since average value $T \neq T(r,\theta)$

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + 2\pi r_0\left(k\frac{\partial T}{\partial r} - \frac{\partial r_0}{\partial x} k \frac{\partial T}{\partial x}\right)_{r=r_0} = 0$$



$$\tan\alpha = -\frac{\partial r_0}{\partial x} \qquad \sin\alpha = \frac{\partial x}{\partial n} \qquad \cos\alpha = \frac{\partial r}{\partial n}$$

$$\frac{\partial T}{\partial n} = \frac{\partial T}{\partial r}\cos\alpha + \frac{\partial T}{\partial x}\sin\alpha = \cos\alpha\left(\frac{\partial T}{\partial r} + \tan\alpha\frac{\partial T}{\partial x}\right)$$

$$= \cos\alpha\left(\frac{\partial T}{\partial r} - \frac{\partial r_0}{\partial x}\frac{\partial T}{\partial x}\right)$$

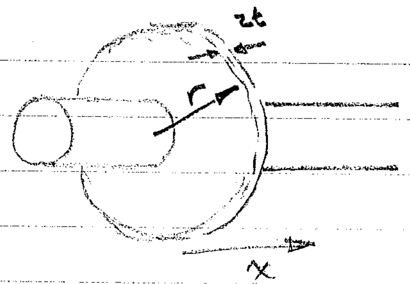
$\therefore$

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + \frac{2\pi r_0 k}{\cos\alpha}\frac{\partial T}{\partial n}\Big|_{r=r_0} = 0$$ if adiabatic $\frac{\partial T}{\partial n} = 0$

if balance of cond with convect $\qquad -k\frac{\partial T}{\partial n} = h(T - T_\infty)$

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) - \frac{2\pi r_0 h}{\cos\alpha}(T - T_\infty) = 0 \qquad 2\pi r_0 = P(x)$$

Changes in area cannot be large $\therefore \cos\alpha \sim 1 = \frac{1}{\sqrt{1 + \left(\frac{\partial r_0}{\partial x}\right)^2}} \qquad \frac{\partial r_0}{\partial x} \ll 1$



17.33 Derive

Assume small changes in x

Conduction to convect balance

Problem Assume $A(x) = A = const$

$P(x) = P = const$

$k = const$

$$\frac{d^2T}{dx^2} - \frac{hP}{Ak}(T - T_\infty) = 0$$

let $\theta = T - T_\infty$ $\qquad \frac{d^2\theta}{dx^2} - \frac{hP}{Ak}\theta = 0 \qquad m^2 = \frac{hP}{Ak}$

$\theta = \bar{A}e^{mx} + Be^{-mx}$ at $x = 0$ $\quad T = T_0 \quad \theta = T_0 - T_\infty = \theta_0$

$\theta_0 = \bar{A} + B \qquad \theta = \bar{A}e^{mx} + (\theta_0 - \bar{A})e^{-mx}$

at $x = L$ $\left\{ T = T_L \right. \qquad \theta = \theta_L$

or $\left\{ \frac{dT}{dx} = \frac{d\theta}{dx} = 0 \right.$ insulated

or $\left\{ -k\frac{dT}{dx} = h(T - T_\infty) = -k\frac{d\theta}{dx} = h\theta \right.$

$$\int_0^L \frac{d}{dx}\left(Ak\frac{dT}{dx}\right)dx = \int_0^L h\theta P(x)\,dx = \int_0^L \underbrace{h\theta\,dS(x)}_{q_{conv}}$$ = total heat transfer from the fin

$Ak\frac{dT}{dx}\Big|_0^L$ heat balance

define η_f = efficiency of fin = $\frac{\text{actual } \dot{q}}{\max \dot{q}}$; $\quad \dot{q}_{total} = \eta_f \dot{q}_{fin} + \dot{q}_{\text{rest of system}}$

if A_0 = exposed area of primary surface (rest of system) $\qquad \dot{q}_{rest} = hA_0(T_0 - T_\infty)$

$$\dot{q}_{fin} = \int_0^L h\theta P\,dx = hP\int_0^L \theta\,dx = hP(T_0 - T_\infty)L = hA_{fin}(T_0 - T_\infty)$$

$$\dot{q}_{tot} = h(T_0 - T_\infty)(A_0 + A_{fin}\eta_f)$$

2-D Steady State Conduction $\qquad \dot{Q} = 0, \quad k = const$

$$\nabla^2 T = 0$$

$$T_{xx} + T_{yy} = 0$$

$$T_{xx} + (rT_r)_r = 0$$

1) Separation of Variables
2) Fourier Transforms
3) Electrical Analogies X
4) Numerical Relaxation
5) Green's Functions X
6) Graphical Methods X

X will not be discussed

$T_{xx} + T_{yy} = 0$ assume $T(x,y) = X(x)\,Y(y)$

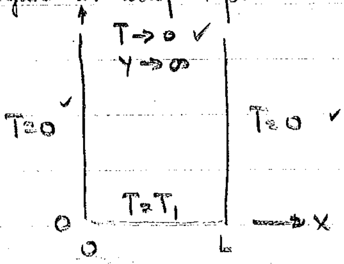
$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \qquad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

$$X'' + \lambda^2 X = 0 \qquad Y'' - \lambda^2 Y = 0$$

$$X = A\sin\lambda x + B\cos\lambda x \qquad Y = C\,e^{-\lambda y} + D e^{\lambda y}$$

$$T(x,y) = \sum_{\lambda} \left(A_\lambda \sin\lambda x + B_\lambda \cos\lambda x\right)\left(C_\lambda e^{-\lambda y} + D_\lambda e^{\lambda y}\right)$$

funct or its normal derivative must be specified on boundary. Only 1 cond must be specified at any 1 pt.



since funct is finite at ∞ $D_\lambda = 0$ $\lambda > 0$

if $B_\lambda = 0$ $\lambda = \frac{n\pi}{L}$

$$T = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\, e^{-\frac{n\pi}{L} y}$$

$$T_1 = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

bounded variation
→ any function having any number of finite discont.
{at discont the sol is the average of the two.}
can be written as a series.

$$A_n = \frac{2}{L}\int_0^L T_1 \sin\frac{n\pi x}{L}\,dx$$

if $T_1 = \text{const}$

$$A_n = \frac{2}{L} T_1 \left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right)\Big|_0^L = \frac{2T_1}{L}\left(\frac{L}{n\pi}\right)\left[1 - \cos n\pi\right]$$

$$A_n = \frac{2T_1}{n\pi}(2) \text{ when } n \text{ is odd} \longrightarrow A_n = \frac{4T_1}{n\pi}$$

0 when n is even,

$$T_1 = \sum_{k=1}^{\infty} \frac{4T_1}{(2k-1)\pi} \sin\frac{(2k-1)\pi x}{L}$$

$$T(x,y) = \sum_{k=1}^{\infty} \frac{4T_1}{(2k-1)\pi} \sin \frac{(2k-1)\pi x}{L} \, e^{-\frac{(2k-1)\pi y}{L}}$$

2-10-72

Fourier Transforms. whenever b.c. are at $\pm\infty$
Laplace Transforms " b.c. are at $0, \infty$

Fourier Transform
$\bar{T}(x,\lambda)$ or $T(x,y)$

defined $$\bar{T}(x,\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T(x,y)\, e^{-i\lambda y}\, dy$$ even fn. odd fn. $\cos\lambda y - i \sin\lambda y$

if $T(x,y)$ is even in y $\quad T(x,y) = T(x,-y)$

define $$\bar{T}(x,\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T(x,y) \cos\lambda y\, dy = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} T(x,y) \cos\lambda y\, dy$$

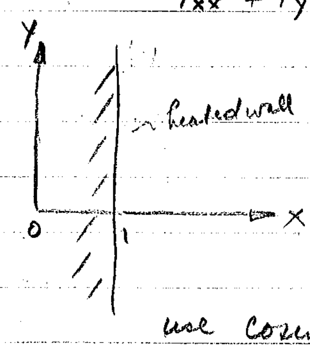
if $T(x,y)$ is odd in y $\quad T(x,y) = -T(x,-y)$

define $$\bar{T}(x,\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T(x,y) \sin\lambda y\, dy$$

Inversion formulas

$$T(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{T}(x,\lambda)\, e^{i\lambda y}\, d\lambda$$

Laplace Equation $\dot{q} = 0$ $\quad k = $ const
$$T_{xx} + T_{yy} = 0$$



$x = 1 \quad T = \frac{1}{y^2+1}$

localheat'g $\quad T, T_y \to 0 \quad$ as $y \to \pm\infty$

$T \to 0 \quad$ as $x \to \infty$

separation of variables NG since too many bc are at ∞

T is an even fn of Y, symmetric

use cosine type fourier

$$\sqrt{\frac{2}{\pi}}\int_0^\infty T_{xx}\cos\lambda y\,dy + \sqrt{\frac{2}{\pi}}\int_0^\infty T_{yy}\cos\lambda y\,dy = 0$$

because limits are const & do not depend on x ∴

$$\frac{\partial^2}{\partial x^2}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty T\cos\lambda y\,dy\right) + \sqrt{\frac{2}{\pi}}\left\{ T_y\cos\lambda y\Big|_0^\infty + \lambda\int_0^\infty T_y\,\sin\lambda y\,dy\right\} = 0$$

$$\frac{\partial^2}{\partial x^2}\bar{T}(x,\lambda) + \sqrt{\frac{2}{\pi}}\left\{ T_y\cos\lambda y\Big|_0^\infty + \lambda T\sin\lambda y\Big|_0^\infty - \lambda^2\int_0^\infty T\cos\lambda y\,dy\right\} = 0$$

$$\left\{\frac{\partial^2}{\partial x^2}\bar{T} - \lambda^2\bar{T}\right\} + 0 + 0$$

$T_y = 0$ at 0 because it is an even fn. $T_y, T \to 0$ because of b.c.

$\bar{T} = Ae^{-\lambda x} + Be^{\lambda x}$ since $T \to 0$ as $x \to \infty$ $\bar{T} \to 0$ ∴ $B = 0$

A & B may be fns of λ

$$\bar{T}(1,\lambda) = \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^\infty \frac{1}{y^2+1}\cos\lambda y\,dy = \sqrt{\frac{\pi}{2}}\,e^{-\lambda} \qquad \lambda \geq 0$$

$$\bar{T} = Ae^{-\lambda x} \qquad \bar{T}(1,\lambda) = Ae^{-\lambda} = \sqrt{\frac{\pi}{2}}\,e^{-\lambda}$$

$$\therefore \quad A = \sqrt{\frac{\pi}{2}}$$

$$\bar{T}(x,\lambda) = \sqrt{\frac{\pi}{2}}\,e^{-\lambda x}$$

$$T(x,y) = \int_0^\infty e^{-\lambda x}\cos\lambda y\,d\lambda = \frac{x}{x^2+y^2}$$

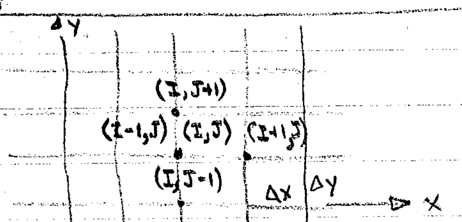
Must transform bc before inversion

Numerical Methods

1) Relaxation } steady state — boundary value problems
2) Successive Approximation }
3) Marching Techniques — unsteady & convect. initial value problem

finite-difference notation $T_{xx} + T_{yy} + \frac{\dot{Q}}{k} = 0$

2-D



must get same number of eq as unknown pts

$\dot{Q}/k$ at I, J $\left(\frac{\dot{Q}}{K}\right)_{I,J}$ $\dot{Q}$, K are known everywhere

$$T_{I+1,J} = T_{I,J} + \left(\frac{\partial T}{\partial x}\right)_{I,J} \Delta x + \left(\frac{\partial^2 T}{\partial x^2}\right)_{I,J} \frac{\Delta x^2}{2} + A\Delta x^3 + B\Delta x^4 + H.O.T.$$

$$T_{I-1,J} = T_{I,J} - \left(\frac{\partial T}{\partial x}\right)_{I,J} \Delta x + \left(\frac{\partial^2 T}{\partial x^2}\right)_{I,J} \frac{\Delta x^2}{2} - A\Delta x^3 + B\Delta x^4 + H.O.T.$$

$$T_{I+1,J} - T_{I-1,J} = 2\Delta x \left(\frac{\partial T}{\partial x}\right)_{I,J} + 2A\,\Delta x^3$$

$$\left(\frac{\partial T}{\partial x}\right)_{I,J} = \frac{T_{I+1,J} - T_{I-1,J}}{2\Delta x} + \underbrace{O(\Delta x^2)}_{\text{error}}$$

$$T_{I+1,J} + T_{I-1,J} = \left(\frac{\partial^2 T}{\partial x^2}\right)_{I,J} \Delta x^2 + 2B\,\Delta x^4 + 2T_{I,J}$$

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{I,J} = \frac{T_{I+1,J} + T_{I-1,J} - 2T_{I,J}}{\Delta x^2} + \underbrace{O(\Delta x^2)}_{\text{Error}}$$

Central Finite Difference

$$\left(\frac{\partial^2 T}{\partial y^2}\right)_{I,J} = \frac{T_{I,J+1} + T_{I,J-1} - 2T_{I,J}}{\Delta y^2} + \underbrace{O(\Delta y^2)}_{\text{error}}$$

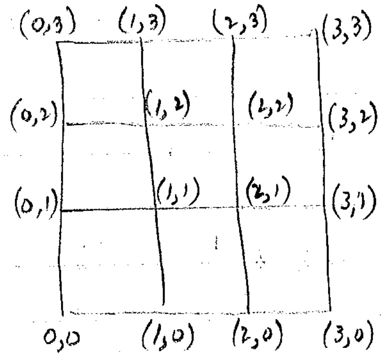
$$\frac{T_{I+1,J} + T_{I-1,J} + 2T_{I,J}}{\Delta x^2} + \frac{T_{I,J+1} + T_{I,J-1} - 2T_{I,J}}{\Delta y^2} + \left(\frac{\dot{Q}}{k}\right)_{I,J} = 0$$

Can generate for all interior points.

for 11 x 11 79 interior pts 42 b.p.

let $\dot{Q} = 0$ $\Delta x = \Delta y$

$$T_{I+1,J} + T_{I-1,J} + T_{I,J+1} + T_{I,J-1} - 4T_{I,J} = 0$$



$T = 1$ on boundary

Relaxation

guess $= 0 = T$ Residual at I,J $R_{I,J} = \frac{1}{4}(\quad) - T_{I,J}$

$$R_{1,1} = \frac{1}{4}(1+0+1+0) - 0 = \frac{1}{2}$$

$\frac{1}{4}(0+0+1+1) - 0 = R_{2,1} = \frac{1}{2}$ $R_{1,2} = \frac{1}{2}$ $R_{2,2} = \frac{1}{2}$

take largest value of residual & set it equal to 0

Step 1 let $R_{(2,1)} = 0$ if all points around are unchanged $\rightarrow \underline{T_{2,1} = 1/2}$

$$R_{1,1} = \frac{1}{4}(1+1+1/2) - 0 = 5/8$$
$$R_{2,2} = \frac{1}{4}(1+1+1/2) - 0 = 5/8$$
$$R_{1,2} = 1/2$$

0 5/8
0 1/2

Step 2 let $R_{2,2} = 0 \longrightarrow \underline{T_{2,2} = 5/8}$

$R_{1,1} = 5/8$ $R_{2,1} = 21/32 = \frac{1}{4}(1+1+5/8+0) - 0$

$R_{1,2} = 5/32 = \frac{1}{4}(1+1+5/8+0) - \frac{16}{32}$

Step 3 let $T_{2,1} = \frac{21}{32}$ $R_{2,1} = 0$

$\underline{R_{1,1} = 101/128}$

$R_{2,2} = \frac{21}{128}$ $R_{1,2} = 5/32$

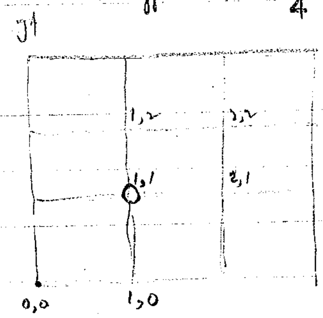
Step 4 let $T_{1,1} = \frac{101}{128}$ $\underline{R_{1,2} = 181/512}$ $R_{2,2} = \frac{21}{128}$ $R_{21} = \frac{101}{512}$

$$R_{1,2} = \frac{1}{4}\left(1+1+5/8+\frac{101}{128}\right) - 0$$

2-15-72

Successive Replacement

Finite Difference $\frac{1}{4}\left(T_{I+1,J} + T_{I-1,J} + T_{I,J+1} + T_{I,J-1}\right) - T_{I,J} = 0$



Guess an interior initial solution

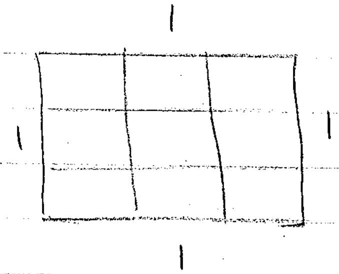
$$T^N_{I,J} = \frac{1}{4}\left(T_{I+1,J} + T_{I-1,J} + T_{I,J+1} + T_{I,J-1}\right)^0$$

Gauss-Seidel –

$$T^N_{I,J} = \frac{1}{4}\left(T^0_{I+1,J} + T^N_{I-1,J} + T^0_{I,J+1} + T^N_{I,J-1}\right)$$

$$\left|T^N_{I,J} - T^0_{I,J}\right| < \epsilon$$

2 ways
- immediately replace $T_{i,j}$ with that obtained & use it in terms of the next $T_{i,j}$ to its right. after total grid is solved return to (1,1) start at (1,1)
- save $T_{i,j}$; after total grid is solved return to (1,1)



ass $T=0$ — immediate replacement

$T_{11} = \frac{1}{4}(1+0+0+1) = \frac{1}{2}$

$T_{21} = \frac{1}{4}(\frac{1}{2}+1+1+0) = 5/8$

$T_{12} = \frac{1}{4}(1+1+\frac{1}{2}+0) = 5/8$

$T_{22} = \frac{1}{4}(5/8+5/8+1+1) = \frac{26}{32}$

$T_{11} = \frac{26}{32}$ $\qquad T_{21} = (1+1+5/8+\frac{26}{32})\frac{1}{4} = \frac{116}{128}$ $\qquad T_{12} = \frac{116}{128}$

$T_{22} = \frac{1}{4}(1+1+\frac{232}{128}) = \frac{488}{512}$

<u>Unsteady Conduction</u>

$\dot{Q} = 0 \qquad \rho C_p \frac{\partial T}{\partial t} = \nabla\cdot(k\nabla T)$

If $k = \text{const}$

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \qquad \alpha = \frac{k}{\rho C_p} = \text{thermal diffusivity}$$

(A) Newtonian Heating - thermal conductivity is so large & heat transfer is finite $\therefore \frac{\partial T}{\partial x_i} = 0 \quad \therefore T$ is uniform $\quad \dot{q} = k\frac{\partial T}{\partial x_i}$

variation in body is neglected we talk about T of body $\quad k \gg 1$

$$\rho C_p \int_V \frac{\partial T}{\partial t} dV = \int_V \nabla\cdot(k\nabla T)dV = \int_S k\frac{\partial T}{\partial n}\, dS$$

divergence theorem.

$$\rho C_p \frac{\partial T}{\partial t} V = kS\frac{\partial T}{\partial n} \qquad \frac{\partial T}{\partial t} = \frac{kS}{\rho C_p V}\frac{\partial T}{\partial n}$$

only b.c. heat balance $\quad k\frac{\partial T}{\partial n} = h(T_\infty - T)$

$$\frac{\partial T}{\partial t} = \frac{hS}{\rho C_p V}(T_\infty - T)$$

if $T = T_0$ at $t = 0 \qquad \frac{T - T_\infty}{T_0 - T_\infty} = e^{-\frac{hSt}{\rho C_p V}}$

$V/S = \frac{h}{2}$ for an infinite plate of thickness h

$= r/2$ " " " cylinder of radius r

$= h/6$ " " cube with side h

$= r/3$ " " sphere " radius r

(B) $h \gg 1$ $\quad T_s - T_\infty \to 0$ $\quad \dot{q}$ is finite at surface $\quad k = $ constant

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \qquad \text{1-D Example} \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}$$

$T = T_0(z)$ at $t = 0$ $\quad 0 \le z \le L$

$T = T_s$ at $z = 0$ $\quad t > 0$

$T = T_s$ at $z = L$ $\quad t > 0$

define $Y = \dfrac{T - T_s}{T_0 - T_s}$ where $T_0 = T_0(z = 0)$

$$\frac{\partial Y}{\partial t} = \alpha \frac{\partial^2 Y}{\partial z^2} \qquad Y = Y_0(z) \text{ at } t = 0 \quad 0 \le z \le L$$

$$Y = 0 \text{ on } z = 0, L \quad t > 0$$

let $Y = \overline{T}(t) Z(z)$

$$\frac{\overline{T}'}{\overline{T}} = \alpha \frac{Z''}{Z} = -\lambda^2 \qquad \overline{T} = e^{-\lambda^2 t}$$

$$Z'' + \frac{\lambda^2}{\alpha} Z = 0$$

$$Z = C_\lambda \cos \frac{\lambda}{\sqrt{\alpha}} z + D_\lambda \sin \frac{\lambda}{\sqrt{\alpha}} z$$

$$Y = \sum \left[\underbrace{C_\lambda \cos \frac{\lambda}{\sqrt{\alpha}} z}_{=0} + D_\lambda \sin \frac{\lambda}{\sqrt{\alpha}} z \right] e^{-\lambda^2 t}$$

$$\frac{\lambda}{\sqrt{\alpha}} L = n\pi \qquad Y = \sum D_n \sin \frac{n\pi z}{L} e^{-\frac{\alpha n^2 \pi^2}{L^2} t}$$

$$D_n = \frac{2}{L} \int_0^L Y_0(z) \sin \frac{n\pi z}{L} dz \qquad \text{let } Y_0(z) = \text{const}$$

$$D_n = \frac{2}{L} Y_0 \left(-\frac{L}{n\pi} \cos \frac{n\pi z}{L} \right) \Big|_0^L = \frac{2Y_0}{n\pi} [2, 0] \text{ odd, even}$$

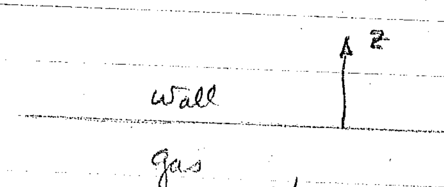
$$\to \frac{4Y_0}{(2k-1)\pi} \qquad k = 1 \ldots$$

$$\therefore \frac{T - T_s}{T_0 - T_s} = Y = \frac{4Y_0}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \left[\sin \frac{(2k-1)\pi}{L} z \; e^{-\frac{(2k-1)^2 \alpha \pi^2}{L^2} t} \right]$$

$$\dot{q} = -kAT_z = \frac{4kA}{L}(T_s - T_0) \sum_{n=0}^{\infty} \cos \frac{(2n+1)\pi z}{L} \, e^{\,''}$$

(C) 1-D Combination of both A & B

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}$$ Rayleigh Heating Problem.



$T = T_0$ at $t=0$ $\forall\, z>0$. $h=\infty,\ T_0=0$ $T_\infty \to U$ $\alpha = \nu$

$T \to T_0$ as $z\to\infty$ $\forall\, t>0$.

at surface $-kT_z = h(T_\infty - T)$ at $z=0$ $t>0$

Laplace Transform useful in initial value problem.

$$\tilde{T}(p,z) = \int_0^\infty T(t,z)\, e^{-pt}\,dt$$

$$\int_0^\infty \frac{\partial T}{\partial t} e^{-pt}dt = \alpha \int_0^\infty \frac{\partial^2 T}{\partial z^2} e^{-pt}dt$$

$$= \alpha \frac{\partial^2}{\partial z^2}\int_0^\infty T e^{-pt}dt = \alpha \frac{\partial^2 \tilde{T}}{\partial z^2}$$

$$\int_0^\infty \frac{\partial T}{\partial t} e^{-pt}dt = \alpha \frac{\partial^2 \tilde{T}}{\partial z^2}$$

$$T e^{-pt}\Big|_0^\infty + p\int_0^\infty T e^{-pt}dt = \alpha \frac{\partial^2 \tilde{T}}{\partial z^2}$$

$$T e^{-pt}\Big|_0^\infty + p\tilde{T} = \alpha \frac{\partial^2 \tilde{T}}{\partial z^2}$$

$$-T_0 + \tilde{T}p = \alpha \frac{\partial^2 \tilde{T}}{\partial z^2} \qquad \tilde{T}_{zz} - \frac{p}{\alpha}\tilde{T} = -T_0/\alpha$$

$$\tilde{T} = Ae^{-\sqrt{p/\alpha}\,z} + Be^{\sqrt{p/\alpha}\,z} + T_0/p$$

$T \to T_0$ as $z\to\infty$ $\forall t$ $\tilde{T} = \int_0^\infty T e^{-pt}dt$ as $z\to\infty$ $\sim \int_0^\infty T_0 e^{-pt}dt$ T_0/p

$\boxed{B=0}$; at $z=0$ $-k\frac{\partial T}{\partial z} = h(T_\infty - T)$

$$-k\int_0^\infty \frac{\partial T}{\partial z} e^{-pt}dt = h\int_0^\infty (T_\infty - T)e^{-pt}dt$$

$-k\frac{\partial \tilde{T}}{\partial z}$ since not fn of z

2-22-72

No class 2-24 class on 3-2-72

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} \qquad \begin{array}{l} T = T_0 \text{ at } t=0 \quad \forall z \\ T \to T_0 \text{ as } z \to \infty \quad t>0 \\ -k\frac{\partial T}{\partial z} = h(T_\infty - T) \text{ at } z=0 \quad t>0 \end{array}$$

$$\tilde{T}(p,z) = \int_0^\infty T(t,z)\, e^{-pt}\, dt \qquad \tilde{T}_{zz} - \frac{p}{\alpha}\tilde{T} = -\frac{T_0}{\alpha}$$

$$\tilde{T} = A e^{-\sqrt{p/\alpha}\, z} + B e^{\sqrt{p/\alpha}\, z} + T_0/p \qquad B=0$$

$$\tilde{T} = A(p)\, e^{-\sqrt{p/\alpha}\, z} + T_0/p$$

$$-k\frac{\partial \tilde{T}}{\partial z} = h T_\infty \int_0^\infty e^{-pt}\, dt - h\tilde{T} \Big)_{z=0} = h\frac{T_\infty}{p} - h\tilde{T} \qquad \text{at } z=0$$

$$-k A(p)\left(-\sqrt{p/\alpha}\, e^{-\sqrt{p/\alpha}\, z}\right) = h\left(\frac{T_\infty}{p} - A(p)\, e^{-\sqrt{p/\alpha}\, z} - T_0/p\right)$$

at $z=0$ $$+kA(p)\sqrt{p/\alpha} = h\left(\frac{T_\infty - T_0}{p} - A(p)\right)$$

$$A\left[h + k\sqrt{p/\alpha}\right] = h\,\frac{T_\infty - T_0}{p}$$

$$A = \frac{h}{p}\,\frac{T_\infty - T_0}{h + k\sqrt{p/\alpha}} = \frac{\sqrt{\alpha}\, h}{k}\,\frac{T_\infty - T_0}{p\left(\sqrt{p} + \frac{\sqrt{\alpha}\, h}{k}\right)}$$

$$\tilde{T}(p,z) = \frac{b}{p}(T_\infty - T_0)\,\frac{1}{\sqrt{p}+b}\, e^{-\sqrt{p/\alpha}\, z} + T_0/p \qquad b = \frac{\sqrt{\alpha}\, h}{k}$$

L.T.	Inverse
$1/p$	1
$\frac{1}{p}\frac{1}{\sqrt{p}+b} e^{-\sqrt{p/\alpha}\, z}$	$\frac{1}{b}\left[\operatorname{erfc}\left(\frac{z}{2\sqrt{\alpha t}}\right) - e^{b(bt + z/\sqrt{\alpha})}\operatorname{erfc}\left(b\sqrt{t} + \frac{z}{2\sqrt{\alpha t}}\right)\right]$

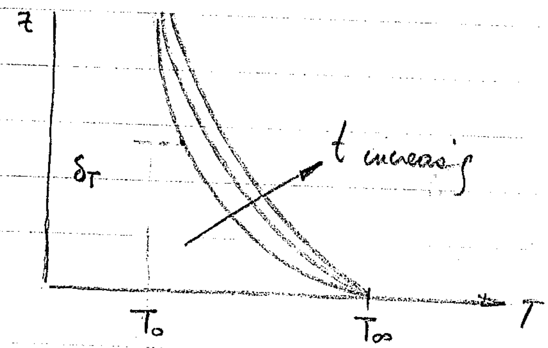
$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-x'^2}\, dx' \qquad \operatorname{erfc}(0) = 1 \qquad \operatorname{erf} = 1 - \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x'^2}\, dx'$$

$$T(t,z) = T_0 + (T_\infty - T_0)\left\{ \operatorname{erfc}\left[\frac{z}{2\sqrt{\alpha t}}\right] - \exp\left(\frac{\alpha h^2 t}{k^2} + \frac{zh}{k}\right) \operatorname{erfc}\left(\frac{\sqrt{\alpha t}\, h}{k} + \frac{z}{2\sqrt{\alpha t}}\right)\right\}$$

if $h \to \infty$ at $z = 0$ $T = T_\infty$ $t > 0$ (TJ) $\alpha = \nu$

at $z \to \infty$ $T \to T_0$ (0)

erfc( ) $\to 0$ faster than exp( ) $\to 0$



$$T = T_0 + (T_\infty - T_0)\operatorname{erfc}\frac{z}{2\sqrt{\alpha t}}$$

Rayleigh

if $h \to 0$ insulation $T = T_0$

if $\operatorname{erfc}\frac{z}{2\sqrt{\alpha t}} = \frac{T - T_0}{T_\infty - T_0} = \frac{.01}{\frac{T_\infty}{T_0} - 1}$

$\delta_T \sim z$ $\therefore$ $\delta_T \sim 4\sqrt{\alpha t}$ δ_T thermal layer thickness

define $\eta = \frac{z}{2\sqrt{\alpha t}}$ $T = T_0 + (T_\infty - T_0)\operatorname{erfc}\eta$ similarity solution

no length can be defined (remember Rayleigh Problem $\frac{dp}{dx} = 0$ since no length could be defined)

note for general problem k/h = length & is defined.

<u>Integral Method Approach</u>

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}$$

$$\int_0^{\delta_T} \frac{\partial T}{\partial t} dz = \alpha \int_0^{\delta_T} \frac{\partial^2 T}{\partial z^2} dz$$

$$\frac{\partial}{\partial t}\int_0^{\delta_T} T\,dz - \frac{\partial \delta_T}{\partial t}\Big[T\Big]_{\delta_T} = \alpha \left.\frac{\partial T}{\partial z}\right|_{\delta_T} - \alpha \left.\frac{\partial T}{\partial z}\right|_0$$

$$\frac{d}{dt}\int_0^{\delta_T} T\,dz - T_0 \frac{d\delta_T}{dt} = -\alpha \left.\frac{\partial T}{\partial z}\right|_{z=0}$$

when $z = \delta_T$ $T = T_0$, $\frac{\partial T}{\partial z} = 0$

define $\theta = T/T_0$

$$\frac{d}{dt}\int_0^{\delta_T} \theta\, dz - \frac{d\delta_T}{dt} = -\alpha \left.\frac{\partial \theta}{\partial z}\right|_{z=0}$$

define $\eta = z/\delta_T$

$$\frac{d}{dt}\delta_T\int_0^1 \theta\, d\eta - \frac{d\delta_T}{dt} = -\frac{\alpha}{\delta_T}\left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0}$$

at $z=\delta_T$ or $\eta=1$ $T=T_0$ $\frac{\partial^2\theta}{\partial\eta^2}=0$, $\theta=1$

at $z=0$ or $\eta=0$ $-k\frac{\partial T}{\partial z} = h(T_\infty - T)$

(1) $-\frac{k}{\delta_T}\frac{\partial\theta}{\partial\eta} = h\left(\frac{T_\infty}{T_0}-\theta\right)$

(2) $T=T_\infty$ or $\theta=\sigma$ (Rayleigh $h=\infty$)

(3) $\frac{\partial\theta}{\partial\eta}=0$ (insulated wall)

Rayleigh Assume $\theta = f(\eta) = A + B\eta + C\eta^2$

at $\eta=0$	$\theta=\sigma$	$A=\sigma$	
$\eta=1$	$\theta=1$	$1=\sigma+B+C$	$B=2(1-\sigma)$
$\eta=1$	$\frac{\partial\theta}{\partial\eta}=0$	$0=B+2C$	$C=\sigma-1$

$$\theta = \sigma + 2(1-\sigma)\eta + (\sigma-1)\eta^2$$

$$\int_0^1 \theta\, d\eta = \sigma + (1-\sigma) + \frac{(\sigma-1)}{3} = \frac{2+\sigma}{3}$$

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=0} = 2(1-\sigma) + 2(\sigma-1)\eta^{\to 0}$$

$$\frac{d\delta_T}{dt}\left[\frac{2+\sigma}{3}\right] - \frac{d\delta_T}{dt} = -\frac{\alpha}{\delta_T}(2-2\sigma)$$

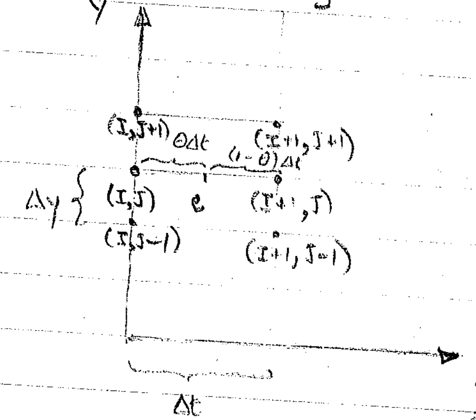
$$\left(\frac{\sigma-1}{3}\right)\frac{d\delta_T}{dt} = +\frac{2\alpha}{\delta_T}(\sigma-1) \longrightarrow \delta_T\, d\delta_T = 6\alpha\, dt$$

$$\delta_T^2 = 12\alpha t \quad \text{at } t=0 \;\; \delta_T=0$$

$$\delta_T = 3.464\sqrt{\alpha t}$$

Numerical Methods for 1-D Unsteady Conduction Problems.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2}$$



$$T_{I+1,J} = T_c + \frac{\partial T}{\partial t}\Big|_c (1-\theta)\Delta t + \frac{\partial^2 T}{\partial t^2}\Big|_c \frac{(1-\theta)^2}{2}\Delta t^2$$

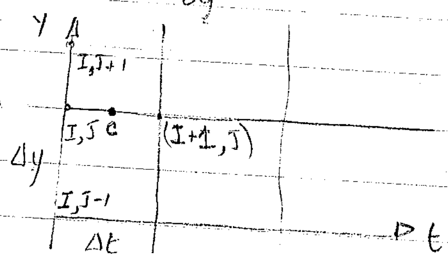
$$T_{I,J} = T_c - \frac{\partial T}{\partial t}\Big|_c \theta\Delta t + \frac{\partial^2 T}{\partial t^2}\Big|_c \frac{\theta^2\Delta t^2}{2} + O(\Delta t^3)$$

$$T_{I+1,J} - T_{I,J} = \frac{\partial T}{\partial t}\Delta t + \frac{\partial^2 T}{\partial t^2}\frac{\Delta t^2}{2}(1-2\theta) + O(\Delta t^3)$$

$$\frac{\partial T}{\partial t}\Big|_c = \frac{(T_{I+1,J} - T_{I,J})}{\Delta t} + \frac{\partial^2 T}{\partial t^2}\frac{\Delta t}{2}(2\theta - 1) + O(\Delta t^2)$$

to get better accuracy $\theta = 1/2$ ie error is $O(\Delta t^2)$

$\partial T/\partial t = \alpha \frac{\partial^2 T}{\partial y^2}$ 2-29-72



$$\left(\frac{\partial T}{\partial t}\right)_c = \left(\frac{T_{I+1,J} - T_{I,J}}{\Delta t}\right) - \frac{\partial^2 T}{\partial t^2}\frac{\Delta t}{2}(1-2\theta) + O(\Delta t^2)$$

$$T_{I,J+1} = T_c + \frac{\partial T}{\partial y}\Big|_c \Delta y - \frac{\partial T}{\partial t}\Big|_c \theta\Delta t + \frac{\partial^2 T}{\partial y^2}\frac{\Delta y^2}{2} + \frac{\partial^2 T}{\partial t^2}\frac{\theta^2\Delta t^2}{2}$$
$$- 2\frac{\partial^2 T}{\partial t\,\partial y}\frac{\theta\Delta y\Delta t}{2} + O(\Delta y^3) - O(\theta^3\Delta t^3)$$
$$- O(\theta\Delta y^2\Delta t) + O(\Delta y\theta^2\Delta t^2) + O(\Delta^4)$$

$$T_{I,J-1} = T_c - \frac{\partial T}{\partial y}\Big|_c \Delta y - \frac{\partial T}{\partial t}\Big|\theta\Delta t + \frac{\partial^2 T}{\partial y^2}\frac{\Delta y^2}{2}$$
$$+ \frac{\partial^2 T}{\partial t^2}\frac{\theta^2\Delta t^2}{2} + 2\frac{\partial^2 T}{\partial t\,\partial y}\frac{\theta\Delta y\Delta t}{2}$$
$$- O(\Delta y^3) - O(\theta^3\Delta t^3)$$
$$- O(\theta\Delta y^2\Delta t) - O(\Delta y\theta^2\Delta t^2) + O(\Delta^4)$$

$$T_{I,J} = T_c - \frac{\partial T}{\partial t}\theta\Delta t + \frac{\partial^2 T}{\partial t^2}\frac{\theta^2\Delta t^2}{2} - O(\theta^3\Delta t^3) + O(\Delta^4)$$

(1) $$(T_{I,J+1} + T_{I,J-1}) - 2T_{I,J} = \frac{\partial^2 T}{\partial y^2}\Big|_c \Delta y^2 - O(2\theta\Delta t\Delta y^2) + O(\Delta^4)$$

You can get rid of second term by expanding at $(I+1,J)$ $(I+1,J+1)$, $(I+1,J-1)$

make change for $-\theta\Delta t \longrightarrow (1-\theta)\Delta t$

(2) $$\therefore (T_{I+1,J+1} + T_{I+1,J-1}) - 2T_{I+1,J} = \frac{\partial^2 T}{\partial y^2}\Big|_c \Delta y^2 + O(2(1-\theta)\Delta t\Delta y^2) + O(\Delta^4)$$

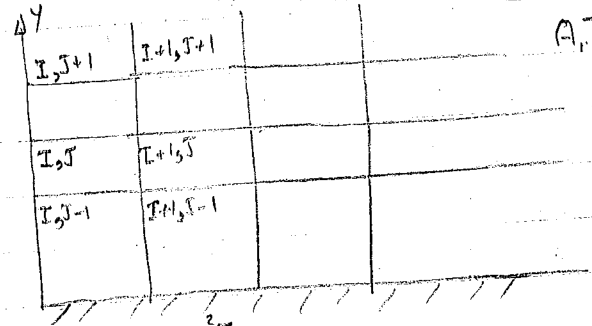
mult (1) by $(1-\theta)$ (2) by (θ) & add

$$(1-\theta)\left[T_{I,J+1} + T_{I,J-1} - 2T_{I,J}\right] + \theta\left[T_{I+1,J+1} + T_{I+1,J-1} - 2T_{I+1,J}\right]$$

$$= \frac{\partial^2 T}{\partial y^2}\bigg|_c \Delta y^2 + O(\Delta^4)$$

$$\therefore \quad \frac{(1-\theta)\left[T_{I,J+1} + T_{I,J-1} - 2T_{I,J}\right] + \theta\left[T_{I+1,J+1} + T_{I+1,J-1} - 2T_{I+1,J}\right]}{\Delta y^2} = \frac{\partial^2 T}{\partial y^2} + O(\Delta^2)$$

for $\theta = \frac{1}{2}$ error is Δ^2



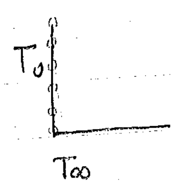
$$A_1T_1 + B_1T_2 + C_1T_3 = D_1$$
$$B_2T_2 + C_2T_3 + E_2T_4 = D_2$$

$\theta = 0 \qquad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2}$

$$\frac{T_{I+1,J} - T_{I,J}}{\Delta t} = \alpha\left[\frac{T_{I,J+1} + T_{I,J-1} - 2T_{I,J}}{\Delta y^2}\right]$$

$$T_{I+1,J} = \frac{\alpha \Delta T}{\Delta y^2}\left[T_{I,J+1} + T_{I,J-1}\right] + \left(1 - \frac{2\alpha\Delta t}{\Delta y^2}\right)T_{I,J}$$

Explicit Approach – calc pt by pt indep on other unknowns.



if $\theta \neq 0$ method is implicit. $\theta = \frac{1}{2}$ Crank-Nicholson tech.
$\theta = 1$ fully implicit

Stability of Calc.

$\theta = 0$	Expl.	Error $(\Delta t, \Delta y^2)$	Stability $\Delta t \leq \frac{\Delta y^2}{2\alpha}$
$\theta = \frac{1}{2}$	Imp	Error $(\Delta t^2, \Delta y^2)$	Unconditionally Stable.
$\theta = 1$	Fully Imp	Error $(\Delta t, \Delta y^2)$	Uncond. Stable.

allowing coupling to fullest extent.

Convection:

If molecular cond in fluid will excite velocity of fluid then we have Convection. If motion of fluid is created solely by heating of fluid

- Natural

Forced Convect veloc larger than those of Natural Conv.

Natural Conv - effect of buoyancy or hydrostatic effect on the fluid

Forced Convect - effects of inertia forces are great neglect buoyancy

Forced Convect

(1) Inertia forces are created but small enough ∴ low mach flow. $\rho \sim$ const, $\mu \sim$ const.

(2) Inertia forces are created in high mach flow $\rho \neq$ const, $\mu \neq$ const.

Schlichting - Thermal b.l.

Thermal boundary layer

$$\frac{D\rho}{Dt} + \rho \nabla\cdot\bar{q} = 0$$

$$\frac{\partial}{\partial x_i}\bar{\tau}_i = \sigma_{ji,i}e_j \quad \text{or} \quad \nabla\cdot\sigma = \sigma_{ji,i}e_j$$

$$\rho\frac{\partial \bar{q}}{\partial t} + \rho(\bar{q}\cdot\nabla)\bar{q} = -\nabla p + \bar{F} + \frac{\partial \bar{\tau}_x}{\partial x} + \frac{\partial \bar{\tau}_y}{\partial y} + \frac{\partial \bar{\tau}_z}{\partial z} = \rho\frac{D\bar{q}}{Dt}$$

$c_pT = h - u + p/\rho$

$$\rho C_p \frac{\partial T}{\partial t} + \rho C_p(\bar{q}\cdot\nabla)T = \frac{Dp}{Dt} + \Phi + \dot{Q} + \nabla\cdot(k\nabla T) = \rho C_p\frac{DT}{Dt}$$

$$p = \rho R T$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{3}{2}\mu\nabla\cdot\bar{q} \qquad \tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y} - \frac{3}{2}\mu\nabla\cdot\bar{q}$$

$$\Phi = \left\{2(u_x^2 + v_y^2) + (v_x + u_y)^2 - \frac{2}{3}(\nabla\cdot\bar{q})^2\right\}\mu$$

1. Steady flow
2. $\dot{Q} = 0$
3. Neglect buoyancy
4. $\frac{u_\infty}{\sqrt{\gamma R T_\infty}} = M_\infty \ll 1$ $\quad \rho, \mu, k \sim$ constant
5. Flow 2-D

Continuity

$$u_x + v_y = 0$$

Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -p_x + \frac{1}{Re}(u_{xx} + u_{yy})$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -p_y + \frac{1}{Re}(v_{xx} + v_{yy})$$

u & v are non dimens with u_∞

x & y " " " " L

p is " " " ρu_∞^2 ← obtained from $p' = -\rho u_e u_e'$

T " " " " T_∞

Energy

$$\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\frac{U_\infty^2}{Re\,c_p T_\infty}\left(2(u_x^2+v_y^2)+(v_x+u_y)^2\right)+\frac{1}{Pr\,Re}\left(T_{xx}+T_{yy}\right)$$

$$+\frac{U_\infty^2}{c_p T_\infty}\left(u p_x+v p_y\right)$$

$p=\rho RT$

$Re=\frac{\rho U_\infty L}{\mu}\gg 1$ $Pr=\frac{\mu c_p}{K}$ $\frac{U_\infty^2}{c_p T_\infty}=(\gamma-1)M_\infty^2$ for a perfect gas $=E$ Eckert No.

$\frac{\partial}{\partial x}=O(1)$ $\frac{\partial}{\partial y}=O\left(\frac{1}{\delta}\right)$ $\delta=O\left(\frac{1}{\sqrt{R_e}}\right)$

$u=O(1)$ $v=O(\delta)$ $u_{xx}\ll u_{yy}$

$v_{xx}\ll v_{yy}$ $p_y\sim O(\delta)$

$\Delta p\sim O(\delta^2)$ $p_x\sim O(1)$ $\therefore$ $p_x\gg p_y \rightarrow p=p(x)$ only.

from euler eq $p'(x)=-\rho u_e u_e'$ inviscid solution at wall.

in energy $\underset{O(\delta)}{v_x}\ll\underset{O(1/\delta)}{u_y}$ $u_y - O(1/\delta)$ $\therefore$ neglect $\overset{O(1)}{u_x^2}, v_y^2$

$T_{xx}\ll T_{yy}$, $\underset{O(1)}{u p_x}\gg v p_y\sim O(\delta^2)$

$u_x+v_y=0$ low speed flow

$u u_x+v u_y=\rho u_e u_e'+\frac{1}{Re}(u_{yy})$

second mom is 0.

$$uT_x+vT_y=\frac{U_\infty^2}{c_p T_\infty Re}(u_y^2)+\frac{1}{Pr\,Re}(T_{yy})+\frac{U_\infty^2}{c_p T_\infty}(u p_x)$$

if $M\ll 1$ $uT_x+vT_y=\frac{1}{Pr\,Re}T_{yy}+\cancel{\frac{U_\infty^2}{c_p T_\infty Re}u_y^2}+\cancel{\frac{U_\infty^2}{c_p T_\infty}(u p_x)}$

if $Pr=1$ $T=Au+B$ for flat plate

3-2-72

$$\left.\begin{aligned}&u_x+v_y=0\\&u u_x+v u_y=-p'(x)+\frac{1}{Re}u_{yy}\end{aligned}\right\}$$ low speed flow $\therefore$ ρ = const; 2 eq 2 unkn u & v no coupling for ρ

$$uT_x+vT_y=\frac{U_\infty^2}{Re\,c_p T_\infty}u_y^2+\frac{1}{Pr\,Re}T_{yy}+\frac{U_\infty^2}{c_p T_\infty}(u p_x)$$

$\frac{U^2}{c_p T_\infty}=\frac{M^2\gamma R T_\infty}{c_p T_\infty}=M^2\frac{\gamma R}{\gamma R}(\gamma-1)$ $(\gamma-1)M$

$p'(x)=-\rho u_e(x)u_e'(x)$

u & v are known

$$\frac{V_\infty^2}{c_p T_\infty} = (\gamma-1) M_\infty^2 = E \qquad \therefore \quad u T_x + v T_y = \frac{1}{P_r Re} T_{yy}$$

B.C. $T \to T_e(x)$ at b.l. edge $\quad T_e(x)$ inviscid case $\quad$ neglected

$T \to T_w$ at wall, $-k\,\partial T/\partial n = 0$ (insul), radiation $(-k\,\partial T/\partial n = h_w^{(x)}(T_w - T_\infty))$

$$u_{bl}(x,y)_{\text{at } y=\delta} = u_{INV}(x,y)_{\text{at } y=\delta} = u_e(x) + \frac{\partial u_{INV}}{\partial y}(x,0)\,\delta + \ldots\ldots$$

neglected since they are small

$$\begin{matrix} 1 & \delta & \frac{1}{\delta_T} & & \delta^2 & \frac{1}{\delta_T^2} \end{matrix}$$

$$u T_x + v T_y = \frac{1}{P_r Re} T_{yy}$$

δ_T = thermal b.l. thickness

δ = velocity b.l. "

$u = O(1) \qquad v = O(\delta)$

$T_x = O(1) \qquad \delta_T = O(\delta) \quad$ if $P_r = 1 \quad \frac{\partial T}{\partial y} = O\left(\frac{1}{\delta_T}\right)$

$$P_r = \mu c_p/k = \frac{\mu}{k/c_p} = O\left(\frac{\delta}{\delta_T}\right)$$

$\delta_T << \delta$

$P_r = O(1)$ or $P_r >> 1 \qquad k/c_p << \mu \quad$ viscosity >> thermal conductivity $\quad \delta > \delta_T$

at wall $u(x,0) = 0 \to u(x,y) = u_y(x,0)\,y + \ldots\ldots + u(x,0)$

$\Rightarrow u(x+\Delta x, 0) = 0 \Rightarrow \frac{\Delta u}{\Delta x} = 0$ at wall

$v(x,0) = 0 \quad v_y(x,0) = 0 \quad$ By cont.

$v(x,y) = v_{yy}(x,0)\,\frac{y^2}{2} + \ldots\ldots \qquad v_y \sim O(1)$ by cont. $\quad \therefore v_{yy} \sim O(\frac{1}{\delta})$

Use thermal equations.

$\delta \qquad \delta_T$

pick $y = \delta_T \,|\,$ order u = order of $u_y(x,0)\,y = \frac{\delta_T}{\delta}$

order of $v(x,y)$ = order of $v_{yy}(x,0)\,\frac{y^2}{2} = \frac{\delta_T^2}{\delta}$

T_x order = 1, $\quad T_y = \frac{1}{\delta_T}$

order of eq. is $\quad \frac{\delta_T}{\delta_1} = \frac{1}{1 \cdot P_r}\left(\frac{\delta^2}{\delta_T^2}\right) \qquad$ order $T_{yy} \approx \frac{1}{\delta_T^2}$

$$\delta_T = O\left(P_r^{-1/3} Re^{-1/3} \delta^{1/3}\right)$$

$$\delta = \frac{1}{\sqrt{Re}} \qquad \delta_T = O\left(P_r^{-1/3} Re^{-1/2}\right) \qquad \delta_T = O\left(P_r^{-1/3} \delta\right)$$

$$\boxed{\frac{\delta_T}{\delta} \sim P_r^{-1/3} \qquad P_r \sim O(1) \text{ or } >> 1}$$

$\delta_T >> \delta$

$$\boxed{P_r \to 0 \qquad \frac{\delta_T}{\delta} \sim P_r^{-1/2} \frac{Re^{-1/2}}{\delta} \sim P_r^{-1/2}}$$

$\delta_T \qquad u \sim O(1) \qquad v \sim (\delta) \qquad \delta$

$P_r \geq 1$

$$\begin{matrix} \frac{\delta_T}{\delta} & 1 & \frac{\delta_T^2}{\delta} & \frac{1}{\delta_T} & & ? & \delta^2 & 1/\delta_T^2 \end{matrix}$$

$$u T_x + v T_y = \frac{1}{P_r Re} T_{yy}$$

$P_r \sim O(\delta_T^3/\delta^3) \qquad O(P_r^{-1/3}\delta) = \delta_T$

$P_r \to 0$ $\quad$ plate $y=0$ $\quad$ $u\,\cdot_x + v\,\cdot_y = P_r R_N\,\cdot_{yy}$ $\quad$ same order of magnitude

$(1) = \frac{1}{P_r}\frac{\delta^2}{\delta_T^2}$ $\quad$ $\frac{1}{P_r} = \frac{\delta_T^2}{\delta^2}$ $\quad$ $\delta_T = P_r^{-1/2}$

Heat Transfer

$$\dot{q}/A = -k\frac{\partial T}{\partial y} = h\,(T_w - T_e) \qquad T_e \text{ - temp of external source}$$

nondim wrt $\left(\frac{T_w - T_e}{L}\right)$ $\therefore$ $\frac{\partial(1)}{\partial(\delta_T)}$

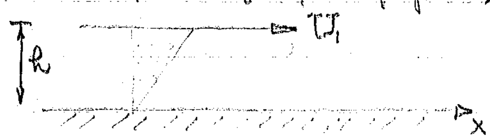
$$\frac{\dot{q}L}{A(T_w - T_e)k} = \frac{-\partial T/\partial y}{(T_w - T_e)/L} = \frac{hL}{k} = Nu \qquad \text{Nusselt No} \qquad Nu \sim \frac{1}{\delta_T}$$

$Nu \sim P_r^{1/3} R_e^{1/2}$ $\qquad$ $P_r = O(1)$ or $P_r >> 1$

$Nu \sim P_r^{1/2} R_e^{1/2}$ $\qquad$ $P_r << 1$

$$\text{Stanton No} = S_T = \frac{Nu}{P_r R_e} = \frac{\dot{q}}{A\rho u C_p (T_w - T_e)} = C_H$$

1\. Couette Flow $\quad$ fully develop flow $\partial/\partial x = 0$ $\qquad$ $u_x = 0$



from cont $\Rightarrow V_y = 0$ $\qquad V = 0$ $\qquad$ since $\frac{\partial V}{\partial y} = 0$, $\frac{\partial V}{\partial x} = 0$ FD flow $\Rightarrow V = \text{const}$ but $V|_{y=0} = 0$

$p'(x) = 0$ $\qquad u = \frac{U_1 y}{h}$

$$\rho(\underbrace{u u_x}_{FD} + \underbrace{v u_y}_{V=0}) = -\underbrace{\frac{\partial p}{\partial x}}_{FD} + \nu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{FD} + \frac{\partial^2 u}{\partial y^2}\right) \Rightarrow u = c + dy$$

$u|_{y=0} = 0$ no slip $\quad u|_{y=h} = U_1$

Nondim form $\qquad y \to y/h \qquad u \to \frac{u}{U_1}$

$u = y$

Nondim energy $\quad u T_x + v T_y = \frac{V_\infty^2}{C_p T_\infty R_e} u_y^2 + \frac{1}{P_r R_e} T_{yy} + \frac{V_\infty^2}{C_p T_\infty} u\, p'(x)$

fully developed $T_x = 0$ $\qquad$ non mechanism by which T has gradient in x direct since plate are infinitely long

$R_e = \frac{\rho U_1 h}{\mu}$ $\qquad P_r = 1$

$$T_{yy} = -P_r E u_y^2 = -P_r E \qquad T = A + By - \frac{P_r E y^2}{2}$$

b.c. $\quad y = 0 \quad T = T_0 \qquad T_0 \to T/T_1$ $\qquad$ T_1 - temp at upper plate

$y = 1 \quad T = 1 \qquad A = T_0 \qquad 1 = T_0 + B - \frac{P_r E}{2}$

$$B = 1 - T_0 + \frac{P_r E}{2}$$

$$\therefore \quad T = T_0(1 - y) + y + \frac{P_r E}{2}\left[y - y^2\right]$$

nondim

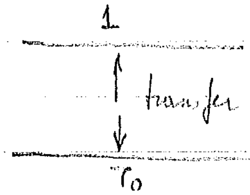
$$\dot{q}/A = -k\frac{\partial T}{\partial y} = -k\frac{T_1}{h}\frac{\partial T}{\partial y} = -k\frac{T_1}{h}\left[-T_0 + 1 + \frac{P_r E}{2}(1 - 2y)\right]$$

$$\dot{q}/A\Big|_{y=1} = -\frac{kT_1}{h}\left[-T_0 + 1 - \frac{P_r E}{2}\right] \qquad T_0 < 1$$

$$\dot{q}/A\Big|_{y=1} > 0 \quad \text{if} \quad P_r E > 2(1 - T_0)$$

$$< 0 \quad \text{"} \quad \text{"} \quad < 2(1 - T_0)$$

if $P_r E = 0$ $\qquad \dot{q}/A$ is constant.



$T_0 < 1$

if $\left|\frac{P_r E}{2}\right| \gg |1 - T_0|$ the temp dist is max at center & radiates in both direct

if $P_r E \leq 2(1 - T_0)$ heat flows from upper to lower

if at $y=0$ $T_y = 0$ $\qquad y_1 = 1 \quad T = 1$

$B = 0 \qquad T = \left(1 + \frac{P_r E}{2}\right) - \frac{P_r E}{2} y^2$ heat is always flowing away from hot wall

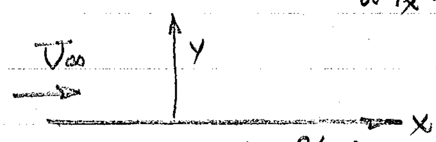
<Poiseuille Flow> See Problem P. 277 Schlichting

Flat Plate

$\dot{p}(x) = 0 \qquad u_x + v_y = 0$

$u u_x + v u_y = \nu u_{yy}$

$u T_x + v T_y = \frac{\nu}{c_p} u_y^2 + \alpha T_{yy} \qquad \alpha = \frac{k}{\rho c_p} = \frac{\nu}{P_r}$



$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \qquad \psi = \sqrt{\nu x U_\infty}\, f(\eta)$

$u = \psi_y = U_\infty f'(\eta)$

$v = -\psi_x = \frac{U_\infty}{2}\sqrt{\frac{\nu}{U_\infty x}}\left\{\eta f'(\eta) - f(\eta)\right\}$

$2f'''(\eta) + f(\eta) f''(\eta) = 0 \qquad v = u = 0 \quad f, f' = 0$ at wall

$u = U_\infty \quad \lim_{\eta\to\infty} f'(\eta) \to 1$

$\frac{\partial}{\partial x} = \frac{\partial}{\partial \eta}\left(-\frac{\eta}{2x}\right) \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}\left(\sqrt{\frac{U_\infty}{\nu x}}\right)$

$\frac{\partial^2}{\partial y^2} = \frac{U_\infty}{\nu x}\frac{\partial^2}{\partial \eta^2}$

$$-\frac{\eta U_\infty}{2x} f'(\eta) T' + U_\infty\sqrt{\frac{U_\infty}{\nu x}}\Big[\frac{1}{2}\sqrt{\frac{\nu}{U_\infty x}}\underbrace{\left\{\eta f' - f\right\}}_{\eta f' = 0}\Big] T' = \frac{\nu}{c_p}\frac{U_\infty}{\nu \cancel{x}} U_\infty^2 f''^2 + \alpha \frac{U_\infty}{\nu \cancel{x}} T''$$

$\frac{\alpha}{\nu} T'' + \frac{1}{2} f T' + \frac{U_\infty^2}{c_p} f''^2 = 0 \quad , \quad T'' + \frac{P_r}{2} f T' = -\frac{P_r U_\infty^2}{c_p} f''^2$

non dim T with T_∞ as $\eta \to \infty$ $\qquad \boxed{T'' + \frac{P_r}{2} f T' = -\frac{P_r E}{2} f''^2}$

as $\eta \to \infty \quad T(\eta) \to 1 \qquad$ f is the known blasius fn.

$\eta = 0 \quad T(0) = T_W$ (T_W/T_∞) or $\partial T/\partial \eta = 0$

3-7-72

Midterm Exam – Up to Natural Convection

Flat Plate Thermal b.l.

$p'(x)=0 \qquad u_x+v_y=0 \qquad uu_x+vu_y=\nu u_{yy}$

$u T_x + v T_y = \frac{\nu}{c_p} u_y^2 + \alpha T_{yy} \qquad \alpha = \frac{\nu}{Pr}$

$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \qquad \psi = \sqrt{\nu x U_\infty}\, f(\eta) \qquad u = \psi_y = U_\infty f'(\eta)$

$v = \frac{U_\infty}{2}\sqrt{\frac{\nu}{U_\infty x}}\{\eta f' - f) \qquad 2f''' + ff'' = 0 \qquad \overset{v}{f(0)} = \overset{u}{f'(0)} = 0 \quad \lim_{\eta\to\infty} f'(\eta) = 1$

$\theta'' + \frac{Pr}{2} f\,\theta' = -\frac{E}{2} Pr\, f''^2 \qquad \theta = T/T_\infty \qquad f = -2\frac{f'''}{f''}$

B.C. $\theta(0) = \theta_W$

$\theta'(0) = 0$ insulated case

$\theta(\eta) \to 1$ as $\eta\to\infty$

when no viscous dissipation closed form solution exists $\qquad E=0$

$\frac{\theta''}{\theta'} = -\frac{Pr}{2} f = Pr\frac{f'''}{f''}$

$\ln\theta' = Pr \ln f'' + \ln A$

$\theta' = A(f'')^{Pr}$

$\theta = A\int_0^\eta [f''(\xi)]^{Pr} d\xi + B \qquad \theta(0) = \theta_W$

$B = \theta_W$

$1 = A\int_0^\infty [f''(\xi)]^{Pr} d\xi + \theta_W$

$A = \frac{1-\theta_W}{\int_0^\infty [f''(\xi)]^{Pr} d\xi} \qquad \theta = \frac{(1-\theta_W)\int_0^\eta (f'')^{Pr} d\xi}{\int_0^\infty (f'')^{Pr} d\xi} + \theta_W$

f'' is Blasius Solution

if $Pr=1 \quad \delta_T = \delta \qquad \theta = \frac{(1-\theta_W)\, f'(\eta)\big|_0^\eta}{f'\big|_0^\infty} + \theta_W$

$f'(0)=0 \quad f'(\infty)=1$

$\theta = (1-\theta_W) f' + \theta_W = (1-\theta_W) u/U_\infty + \theta_W$

$\Theta = Cu + D$

from mom $\quad uu_x + vu_y = \nu u_{yy}$

$Pr = 1 \quad E = 0 \quad u\theta_x + v\theta_y = \alpha\theta_{yy} = \nu\theta_{yy}$

let $\theta = cu + d$

at $\theta = \theta_W \quad u = 0 \quad d = \theta_W$

$\theta = 1 \quad u = u_\infty \quad \dfrac{1-\theta_W}{U_\infty} = C \qquad \theta = (1-\theta_W)\underbrace{\dfrac{u}{U_\infty}}_{f'} + \theta_W$

Heat Transfer at wall $Pr = 1$

$$\dot{q}/A = -T_\infty k \frac{\partial\theta}{\partial y} = -T_\infty k \frac{\partial\theta}{\partial\eta}\left.\frac{\partial\eta}{\partial y}\right|_x = -T_\infty k\sqrt{\frac{U_\infty}{\nu x}}(1-\theta_w)f''(0)$$

$$\dot{q}/A = -.332\, T_\infty k\sqrt{\frac{U_\infty}{\nu x}}(1-\theta_w)$$

$$Nu_x = \frac{\dot{q}x}{Ak(T_W - T_\infty)} = \frac{\dot{q}x}{AT_\infty k(\theta_W - 1)} = .332\sqrt{\frac{U_\infty x}{\nu}} = .332\sqrt{Re_x}$$

$\dfrac{C_F}{Nu_x}$ = constant

$Nu_x = .564\, Pr^{1/2} Re_x^{1/2} \qquad P \to 0$

$Nu_x = .332\, Pr^{1/3} Re_x^{1/2} \qquad .6 < P < 10$

$Nu_x = .339\, Pr^{1/3} Re_x^{1/2} \qquad P >> 1$

grad. due to conduction $\quad \nabla\cdot\dot{q} = \nabla\cdot(k\nabla T)$

$$uT_x + vT_y = \frac{\nu}{c_p}\underbrace{u_y^2}_{\text{visc. dissip}} + \alpha T_{yy}$$

$u_x + v_y = 0 \quad \rightarrow$ moment integral equation [Karman Pohlhausen]

$$\frac{u}{U_e} = f(y/\delta) = f(\eta)$$

continuity $\times T = 0$

Integral Method $\qquad u_xT + v_yT = 0 +$ thermal eq

$$(uT)_x + (vT)_y = \frac{\nu}{c_p}u_y^2 + \alpha T_{yy}$$

$$\int_0^{\delta_T}(uT)_x dy + \int_0^{\delta_T}(vT)_y dy = \int_0^{\delta_T}\frac{\nu}{c_p}u_y^2 dy + \int_0^{\delta_T}\alpha T_{yy} dy$$

Aside $\quad \dfrac{d\theta}{dx} + \dfrac{\delta^*}{u_e}\dfrac{du_e}{dx}\left(\dfrac{\theta}{\delta^*} + 2\right) = \dfrac{c_f}{2} \qquad \theta = \displaystyle\int_0^\delta \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right)dy$

$$\delta^* = \int_0^\delta\left(1 - \frac{u}{u_e}\right)dy$$

$v=0$ at wall. 0 at upper li[illegible]

$$\int_0^{\delta_T}(uT)_x\,dy + (vT)\Big|_0^{\delta_T} = \int_0^{\delta_T}\frac{\nu}{c_p}u_y^2\,dy + \alpha T_y\Big|_0^{\delta_T}$$

$$\frac{\partial}{\partial x}\int_0^{\delta_T}(UT)\,dy - \frac{d}{dx}\delta_T\,(UT|_{\delta_T}) + (VT)_{\delta_T} = \int_0^{\delta_T}\frac{\nu}{c_p}u_y^2\,dy - \alpha T_y\Big|_0$$

$$T_{\delta_T}V_{\delta_T} = T_{\delta_T}\int_0^{\delta_T}V_y\,dy = -T_{\delta_T}\int_0^{\delta_T}u_x\,dy = -\frac{\partial}{\partial x}\int_0^{\delta_T}T_{\delta_T}u\,dy + u\Big|_{\delta_T}\frac{d\delta_T}{dx}T_{\delta_T}$$

$$\frac{d}{dx}\int_0^{\delta_T}(uT)\,dy - T_{\delta_T}\frac{d}{dx}\int_0^{\delta_T}u\,dy = \frac{\nu}{c_p}\int_0^{\delta_T}u_y^2\,dy - \alpha T_y\Big|_0$$

$$\frac{d}{dx}\int_0^{\delta_T}(uT_{\delta_T} - uT)\,dy = \alpha T_y\Big|_{y=0} - \frac{\nu}{c_p}\int_0^{\delta_T}u_y^2\,dy$$

Assume T_{δ_T} = const u_{δ_T} = const (flat plate soln) $\frac{\partial p}{\partial x}=0$

Assume δ_T & δ are of same order $\quad \frac{d}{dx}\int_0^{\delta_T}\frac{u}{u_{\delta_T}}\left(1-\frac{T}{T_{\delta_T}}\right)dy = \frac{\alpha}{u_{\delta_T}}\left(T/T_{\delta_T}\right)_y\Big| - \frac{\nu}{c_p}\frac{u_{\delta_T}}{T_{\delta_T}}\int_0^{\delta_T}\left(\frac{u}{u_{\delta_T}}\right)_y^2 dy$

$$\frac{d\delta_T}{dx}\int_0^1\frac{u}{u_{\delta_T}}\left(1-\frac{T}{T_{\delta_T}}\right)d(y/\delta_T) = \frac{\alpha}{u_{\delta_T}\delta_T}\left(\frac{T}{T_{\delta_T}}\right)_{(y/\delta_T)}\Bigg|_{(y/\delta_T)=0} - \frac{\nu}{c_pT_{\delta_T}}\frac{u_{\delta_T}\delta_T}{\delta^2}\int_0^1\left(\frac{u}{u_{\delta_T}}\right)^2_{(y/\delta)}d(y/\delta_T)$$

$$\frac{d}{dx}\delta_T\int_0^1 U(1-\theta)\,d\eta_T = \frac{\alpha}{U_{\delta_T}\delta_T}\theta_\eta\Big|_{\eta_T=0} - \frac{\nu u_1\delta_T}{c_pT_{\delta_T}\delta^2}\int_0^1 U_\eta^2\,d\eta_T$$

$$\frac{u}{u_{\delta_T}} = U \qquad \frac{y}{\delta_T} = \eta_T$$

$$\frac{T}{T_{\delta_T}} = \theta \qquad \frac{y}{\delta} = \eta$$

if $\delta = \delta_T$ one more unknown is needed $\therefore$ when solving θ leave one const unknown

Low speed, forced ~~BL~~ convect : Thermal B.L. can determ. velocity profile without knowing about temp.

<u>Natural Convection:</u> Heat body, density ↓, particles rise buoyancy effect. The Mechanism for creating the motion is the change in the gravity (weight) force. Small Velocity, Density variations aren't large

Neglect density var. as it effects continuity; no neglecting var. as it affects lift
Boussinesq App: $\Delta\rho$ is negligible except as it effects the force.

Boussinesq Approx

$u_{,x} + v_{,y} = 0$ $\qquad \rho$ neglected

$u u_{,x} + v u_{,y} = -\frac{p_{,x}}{\rho} + \frac{F_x}{\rho} + \nu u_{,yy}$

$F_y - p_{,y} = O(\delta)$

$u T_{,x} + v T_{,y} = \frac{\nu}{c_p} u_{,y}^2 + \frac{1}{\rho c_p} u p'(x) + \alpha T_{,yy}$

$\bar{F} = \rho \bar{g}$ $\qquad F_x = \rho g_x$

$\qquad F_y = \rho g_y$

no motion

$\nabla p = \rho_\infty \bar{g}$ $\qquad$ hydrostatic eq

$p_{,x} = \rho_\infty g_x$ $\qquad p_{,y} = \rho_\infty g_y$

$p_{,x} = p_{,x}^* + \rho_\infty g_x$ $\qquad$ with motion induced $p_{,x}^*$

$p_{,y} = p_{,y}^* + \rho_\infty g_y$

3-9-72

Natural Convection

$u_{,x} + v_{,y} = 0$

$u u_{,x} + v u_{,y} = -\frac{p_{,x}}{\rho} + \frac{F_x}{\rho} + \nu u_{,yy}$

$F_y - p_{,y} = O(\delta)$

$u T_{,x} + v T_{,y} = \frac{\nu}{c_p} u_{,y}^2 + \frac{1}{\rho c_p} u p'(x) + \alpha T_{,yy}$

$F_x = \rho g_x$

$F_y = \rho g_y$ — due to body force

$p_{,x} = p_{,x}^* + \rho_\infty g_x$ — hydrostatic

induced by motion

$p_{,y} = p_{,y}^* + \rho_\infty g_y$ $\qquad$ $u, v = 0$ @ wall $\quad u = 0$ @ bl

if $p_{,x}$ $p_{,y}$, ρg_x, ρg_y are neglected & $u, v = 0$ Equations reduce to $T_{yy} = 0$

leads to $u = v = 0$ everywhere

simple 1-D flow

when one considers $p_{,x}, p_{,y}$ & F_x, F_y then

$$u u_{,x} + v u_{,y} = -\frac{p^*_{,x}}{\rho} - \frac{\rho_\infty}{\rho} g_x + g_x + \nu u_{,yy}$$

$$\rho g_y - p^*_{,y} - \rho_\infty g_y = O(\delta)$$

if $Re = \frac{\rho_\infty U L}{\mu_\infty} \gg 1$ then b.l. equations that can be applied,

let $\bar{u}_m$ = mean velocity u $\quad Re = \frac{\rho_\infty \bar{u}_m L}{\mu_\infty} \gg 1$

$F_x - p_y$

$$p^*_{,y} = (\rho - \rho_\infty) g_y + O(\delta)$$

if $\rho - \rho_\infty \sim O(1)$

$p_{,y} \sim O(1)$ $\quad \Delta P \sim O(\delta)$ from wall to b.l.

$p^*_{,x} \to 0$ induced by motion, at b.l. $p^* = 0$ it is assumed that $p^*_{,x}$ any level $\sim p^*_{,x}$ at b.l.

$$\therefore \quad u_{,x} + v_{,y} = 0$$

$$\rho u u_{,x} + \rho v u_{,y} = \underbrace{(\rho - \rho_\infty) g_x}_{\text{buoyancy force}} + \mu u_{,yy}$$

$$u T_{,x} + v T_{,y} = \frac{\nu}{c_p} u^2_{,yy} + \alpha T_{,yy} \quad , \quad u p'(x) = u p^*_{,x} + u \rho_\infty g_x$$

cancelled in original derivation of temperature diff eq see page 3 d 4

B.C. $\quad u(x,0) = 0$

$v(x,0) = 0$

$u(x,y) \to 0 \quad y \to \delta$

$T(x,0) = T_w$

$T_{,y}(x,0) = 0$

$T(x,y) \to T_\infty \quad y \to \delta$

if $(\rho - \rho_\infty) g_x$ is small use Boussinesq

if " is large must include other terms with ρ.

if $\rho - \rho_\infty$ is small & $T - T_\infty$ is small $\quad$ note $(\rho - \rho_\infty) g_x$ causes motion

$$\rho(T,p) = \rho_\infty + \left(\frac{\partial \rho}{\partial T}\right)_\infty (T - T_\infty) + \left(\frac{\partial \rho}{\partial p}\right)_\infty \underbrace{(p - p_\infty)}_{\text{small}} + \ldots$$

use

$$\rho - \rho_\infty = \left(\frac{\partial \rho}{\partial T}\right)_{\substack{p=c \\ \infty}} (T - T_\infty) = -\beta_\infty \rho (T - T_\infty)$$

where β is coeff of thermal expansion. $\quad$ for perfect gas $\beta = \frac{1}{T}$

therefore $\quad u_{,x} + v_{,y} = 0$

$G/R_N^2 \quad \frac{u_0}{L}$

$$\rho u u_{,x} + \rho v u_{,y} = -\rho \beta_\infty (T - T_\infty) g_x + \ldots$$

$$u T_{,x} + v T_{,y} = \alpha T_{,yy} + \frac{\nu}{c_p} u_{,y}^2$$

$\beta_\infty = \frac{1}{T_\infty}$

$\rho \approx \rho_\infty$

$\rho - \rho_\infty << 1$

neglect $\frac{\nu u_{,y}^2}{c_p} \sim O\left(\frac{u_m^2}{c_p T_\infty}\right) \sim \frac{\Delta T}{T_\infty} << 1$

looking at momentum $u^2 \sim \Delta T$ if inertia becomes very large include $u_{,y}^2$ term and all density variation terms.

$$\frac{G}{Re^2} = \frac{\text{Grashof}}{\text{Reynolds}} \quad \frac{\text{buoyancy}/\text{viscous}}{\text{inertia}/\text{viscous}}$$

$$G = \frac{\beta \Delta T g L^3 \rho^2}{\mu_\infty^2} \qquad R_N^2 = \frac{\rho_\infty^2 U_\infty^2 L^2}{\mu_\infty^2}$$

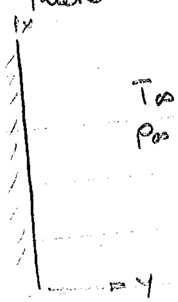
$$\frac{G}{R_N^2} = \frac{\beta \Delta T g}{U_\infty^2 / L} = \frac{\text{Buoyancy force}}{\text{inertia effect}}$$

forced conv. $<< 1$ neglect buoyancy

natural conv $\gtrsim 1$ retain

Richardson number — $\frac{\text{body force}}{\text{inertia effect}}$

Vertical Plate that is heated



$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \qquad \beta_\infty = \frac{1}{T_\infty}$$

$$u_{,x} + v_{,y} = 0$$

$$u u_{,x} + v u_{,y} = \frac{T_w - T_\infty}{T_\infty} g\, \theta + \nu u_{,yy}$$

$g_x = -g$ ⟵ note:

$$u \theta_{,x} + v \theta_{,y} = \alpha \theta_{,yy}, \quad \frac{\nu}{P_r} = \alpha$$

@ $y = 0$ $\theta = 1$ $u, v = 0$

$y >> 1$ $\theta \to 0$ $u \to 0$

$$\psi = 4\nu C x^{3/4} f(\eta) \qquad \eta = C \frac{y}{x^{1/4}}$$

$u = \psi_{,y}$ $v = -\psi_{,x}$

$$C = \left(\frac{g(T_w - T_\infty)}{4\nu^2 T_\infty}\right)^{1/4}$$

$$u = 4\nu x^{1/2} C^2 f'(\eta)$$

$$v = \nu C x^{-1/4} (\eta f' - 3f)$$

$$\eta = \frac{y}{x}\left(\frac{G_x}{4}\right)^{1/4}$$

$$G_x = \frac{g x^3}{\nu^2}\left(\frac{T_w - T_\infty}{T_\infty}\right)$$

Assume $\eta = C y / x^n$ $\psi = d x^m f(\eta)$

$u = \psi_{,y} = C d\, x^{m-n} f'(\eta)$

$v = -\psi_{,x} = -d x^{m-1} [m f(\eta) - \eta n f'(\eta)]$

$u_{,x} = cd\, x^{m-n-1}\left((m-n) g' - n\eta g''\right)$
$u_{,y} = c^2 d\, x^{m-2n}\; g''$
$u_{,yy} = c^3 d\, x^{m-3n}\; g'''$

$\theta = \theta(\eta)$

$\theta_{,x} = -n\eta x^{-1}\theta'$
$\theta_{,y} = c x^{-n}\theta'$
$\theta_{,yy} = c^2 x^{-2n}\theta''$

place in Governing Eq

Mom $\quad c^2 d^2 x^{2m-2n-1}\left((m-n) g'^2\right) - c^2 d^2 x^{2m-2n-1}\left(m g g''\right) =$

$\frac{T_w - T_\infty}{T_\infty} g\theta + \nu c^3 d\, x^{m-3n}\, g'''$

Energy $\quad -cd\, x^{m-n-1}\, m g\theta' = \alpha c^2 x^{-2n}\theta''$

cannot depend on x $\quad \therefore \quad 2m-2n-1 = m-3n = 0$

$m-n-1 = -2n \qquad m-1 = -n$

$m = 3/4 \quad n = 1/4$

$\therefore$

$c^2 d^2 (m-n) g'^2 - c^2 d^2 m g g'' = \frac{T_w - T_\infty}{T_\infty} g\theta + \nu c^3 d\, g'''$

$-cd\, m g\theta' = \alpha c^2 \theta''$

let $\frac{d}{\nu c} = 4 \quad \rightarrow \quad g''' + 3 g g'' - 2 g'^2 + \frac{T_w - T_\infty}{4 T_\infty}\,\frac{g}{\nu^2 c^4}\,\theta = 0$

$\theta'' + \frac{3\nu}{\alpha} g\theta' = 0$

let $\frac{T_w - T_\infty}{4 T_\infty}\,\frac{g}{\nu^2 c^4} = 1$

$c = \left(\frac{g(T_w - T_\infty)}{4\nu^2 T_\infty}\right)^{1/4}$

no characteristic length exists since plate is infinite

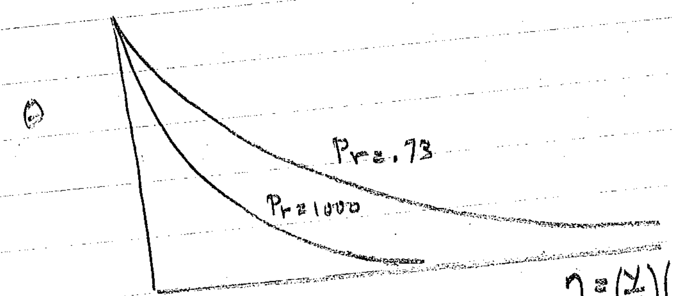
$g''' + 3 g g'' + 2 g'^2 + \theta = 0$

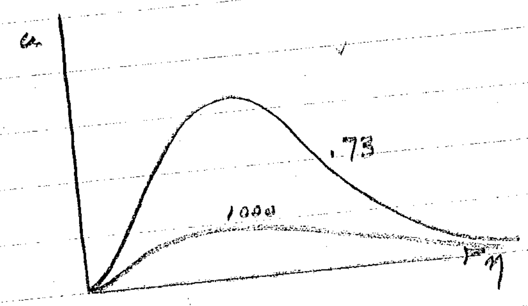
$\theta'' + 3 P_r\, g\theta' = 0$

$g(0) = g'(0) = 0 \qquad \theta(0) = 1$

$\theta(\eta) \underset{\eta\to\infty}{\longrightarrow} 0 \qquad g'(\eta) \underset{\eta\to\infty}{\longrightarrow} 0$

Solved numerically in Schlichting P301





$$\eta = \left(\frac{y}{x}\right)\left(\frac{G_x}{4}\right)^{1/4}$$

$$\frac{q}{A} = \hat{q} = -k T_y = -kC x^{-1/4}\, \theta'(0)\,(T_w - T_\infty)$$

$$Nu/\tfrac{\text{length}}{\text{area}} = \frac{\hat{q}}{(T_w - T_\infty)k} = -C x^{-1/4}\,\theta'(0)$$

3-14-72

Natural Convection

$$\zeta''' + 3\zeta\zeta'' - 2\zeta'^2 + \theta = 0 \qquad \zeta(0) = \zeta'(0) = 0 \quad \theta(0) = 1$$

$$\theta'' + 3P_r \zeta\theta' = 0 \qquad \underset{\eta\to\infty}{\theta(\eta)} \to 0 \quad \zeta'(\eta) \to 0$$

$$\eta = \left(\frac{y}{x}\right)\left(\frac{G_x}{4}\right)^{1/4};\; G_x = \frac{g x^3}{\nu^2}\,\frac{T_w - T_\infty}{T_\infty}$$

$$u = 4\nu x^{1/2} C \zeta' = 2(G_x)^{1/2}\frac{\nu}{x}\zeta'$$

Assume $u_\delta = .01\, u_{max} \rightarrow \eta = \eta_\delta$ for given $P_r \rightarrow \frac{y}{x} = A(P_r)\, G_x^{-1/4}$

as $G_x \uparrow$ $y \downarrow$

Heat Transfer @ wall $-kT_y = \hat{q} = -k(T_w - T_\infty)\theta_y = -kCx^{-1/4}\,\theta'(0)(T_w - T_\infty)$

$$Nu/\tfrac{\text{length}}{\text{area}} = \frac{\hat{q}}{(T_w - T_\infty)k} = -Cx^{-1/4}\theta'(0)$$

$$Nu_{mean} = \int_0^L Nu\, dx = -C\,\theta'(0)\int_0^L x^{-1/4}dx = -C\,\theta'(0)\cdot\frac{4}{3}L^{3/4} = -\frac{4}{3}\frac{\theta'(0)}{\sqrt{2}}\left(\frac{gL^3(T_w -}{\nu^2 T_\infty}\right.$$

$$Nu_{mean} = -\frac{2\sqrt{2}}{3}\,\theta'(0)\, G_L^{1/4} \qquad P_r = .73 \quad \theta'(0) = -.508$$

$$Nu = .478\; G_L^{1/4}$$

↑ EXAM

Compressible Boundary Layer Theory

$V \gg$ bouyancy effect neglected, $\rho \neq$ constant

equations are coupled, body forces terms neglected

2. $\mu = \mu(T)$, $k = k(T)$

$\mu \sim \frac{1}{2}\rho\bar{c}\bar{\lambda}$ $\quad \rho\bar{\lambda} \sim \text{const}$

$\bar{c} \sim T^{1/2}$ $\quad \therefore \quad \mu \sim T^{1/2}$

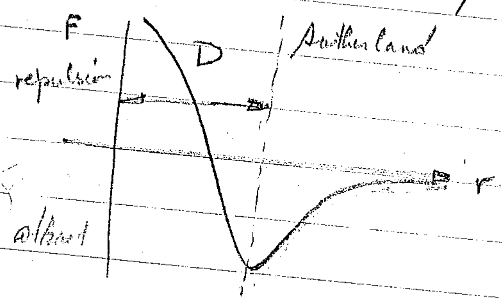
1) $\dfrac{\mu}{\mu_{ref}} = c\left(\dfrac{T}{T_{ref}}\right)$ $\quad$ c = Chapman - Rubesin Constant

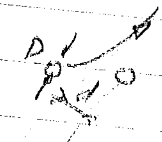
2) $\dfrac{\mu}{\mu_{ref}} = \left(\dfrac{T}{T_{ref}}\right)^{\omega}$ $\quad \mu = aT+b$

$\frac{1}{2} < \omega < 1$

3) $\dfrac{\mu}{\mu_{ref}} = \left(\dfrac{T}{T_r}\right)^{3/2}\left(\dfrac{T_r+S}{T+S}\right)$ $\quad S = 110°K$ or $198°R$ $\quad$ numerical techniques

Sutherland Viscosity Law.





collision (not actual however this indicates large repulsion force.)

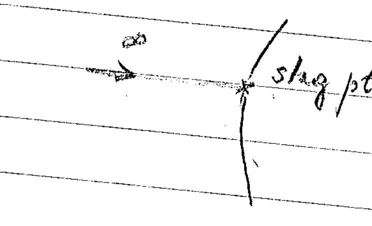
4) as $M\uparrow$ μ & ρ must be included

as $M\uparrow$ $T\uparrow$ ... gas will ioniz, radiate, dissociate

will exclude all real gas effects.

Flow is inviscid, irrot, isentropic

$C_pT_{o_\infty} = H_{o_\infty} = \text{const} = C_pT_\infty + \dfrac{V_\infty^2}{2}$



$H_{o_\infty} = C_pT_W$

$\dfrac{-T_\infty + T_{o_\infty}}{T_\infty} = \dfrac{V_\infty^2}{2C_pT_\infty} = \dfrac{\gamma-1}{2}M_\infty^2$ $\quad$ ($T_{o_\infty} = T_W$)

as $M_\infty\uparrow$ $T_{o_\infty}\uparrow$

heat due to friction leads to $T\uparrow$ in b.l.

compressibility effect.

Governing Equations $\quad \bar{F} = \dot{Q} = 0 \quad$ 2-D Steady

$\rho(\bar{q}\cdot\nabla)\bar{q} = -\nabla p + \dfrac{\partial}{\partial x}\bar{\tau}_x + \dfrac{\partial}{\partial y}\bar{\tau}_y$ $\quad$ assume $Pr = \text{const}$ $O(1)$

$C_p = \text{const}$

$\nabla \cdot \rho \bar{q} = 0$

$c_p(\bar{q}\cdot\nabla)T = \bar{q}\cdot\nabla p + \Phi + \nabla\cdot(k\nabla T)$

or $\rho\bar{q}\cdot\nabla H_0 = \Delta + \nabla\cdot(k\nabla T)$

$p = \rho R T$

$\bar{\tau}_x = \left(-\frac{2}{3}\mu\,\nabla\cdot\bar{q} + 2\mu\frac{\partial u}{\partial x},\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right)$

$\bar{\tau}_y = \left(\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),\ -\frac{2}{3}\mu\,\nabla\cdot\bar{q} + 2\mu\frac{\partial v}{\partial y}\right)$

Cont $(\rho u)_{,x} + (\rho v)_{,y} = 0$

x Mom: $\rho u u_{,x} + \rho v u_{,y} = -p_{,x} + \left[\frac{4}{3}\mu u_{,x} - \frac{2}{3}\mu v_{,y}\right]_{,x} + \left[\mu(u_{,y} + v_{,x})\right]_{,y}$

Non-Dim v, u wrt U_∞ μ wrt μ_∞ ρ with ρ_∞

T with $T_\infty M_\infty^2$ p wrt $\rho_\infty U_\infty^2$ x, y with L

$\therefore$ $(\rho u)_{,x} + (\rho v)_{,y} = 0$

x mom: $\rho u u_{,x} + \rho v u_{,y} = -p_{,x} + \frac{1}{R_{N_\infty}}\overbrace{\left[\frac{4}{3}\mu u_{,x} - \frac{2}{3}\mu v_{,y}\right]_{,x}}^{O(1)} + \frac{1}{R_{N_\infty}}\overbrace{\left[\mu(u_{,y} + v_{,x})\right]_{,y}}^{O(\frac{1}{\delta})}$

y Mom: $\rho u v_{,x} + \rho v v_{,y} = -p_{,y} + \frac{1}{R_{N_\infty}}\left[\mu(\underbrace{v_{,x}}_{\text{neglect}} + u_{,y})\right]_{,x} + \frac{1}{R_{N_\infty}}\left[\frac{4}{3}\mu v_{,y} - \frac{2}{3}u_{,x}\right]_{,y}$

Energy $\rho u T_{,x} + \rho v T_{,y} = (\gamma - 1)\left(\overbrace{u p_{,x}}^{O(1)} + \overbrace{v p_{,y}}^{O(\delta)}\right) + \frac{(\gamma-1)}{R_{N_\infty}}\Phi$

$+ \frac{1}{R_{N_\infty}}\frac{1}{P_r}\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial T}{\partial y}\right)\right]$

$\Phi = 2\mu\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right) + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$

mult $\frac{\mu_\infty U_\infty^2}{L^2}$ on Φ to non dim $\qquad \dfrac{\frac{\mu_\infty U_\infty^2}{L^2}}{\frac{c_p \rho_\infty U_\infty T_\infty M_\infty^2}{L}} = \frac{(\gamma - 1)}{R_N}$

Assume $R_{N_\infty} \gg 1$ $\qquad$ Assume $u = O(1)$ $\quad \rho = O(1)$

$\frac{\partial}{\partial x} = O(1)$

You would neglect R_N terms however no slip not satisfied (you get inviscid so must retain $\frac{1}{R_{N_\infty}}$ since $\frac{\partial}{\partial y}$ are large $\frac{\partial}{\partial y} = O(\frac{1}{\delta})$

from continuity $v \sim O(\delta)$

Got x mom $\qquad \mu \sim O(1)$ all order 1 except R_N neglect $,x$ te

neglect $v_{,x}$ term wrt $u_{,y}$ $\rightarrow$ $R_{N_\infty} \rightarrow O(\delta^{\frac{1}{2}})$ $\qquad$ not first... conds go to y mom $p_{,y} \sim o(\delta)$ $\Delta P = o(\delta^2)$

go to x mom $p \to p(x)$ only $\therefore p'(x) \sim o(1)$

Energy eq $\mu\left(\frac{\partial u}{\partial y}\right)^2$ is only held in Φ neglect $v p_{,y}$ & $\frac{\partial}{\partial x}\left(\mu \frac{\partial T}{\partial x}\right)$

$$\therefore \quad (\rho u)_{,x} + (\rho v)_{,y} = 0$$

$$\rho u u_{,x} + \rho v u_{,y} = -p'(x) + \frac{1}{R_{N\infty}}\left[\mu u_{,y}\right]_{,y}$$

nondim

$$\rho u T_{,x} + \rho v T_{,y} = (\gamma-1)\, u p_{,x} + \frac{\gamma-1}{R_{N\infty}}\mu\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{R_{N\infty}}\frac{1}{P_r}\frac{\partial}{\partial y}\left(\mu\frac{\partial T}{\partial y}\right)$$

dim

$$(\rho u)_{,x} + (\rho v)_{,y} = 0$$

$$\rho u u_{,x} + \rho v u_{,y} = -p'(x) + (\mu u_{,y})_{,y}$$

$$\rho u c_p T_{,x} + \rho v c_p T_{,y} = u p'(x) + \mu u_y^2 + \frac{c_p}{P_r}(\mu T_{,y})_{,y}$$

$$p = \rho R T$$

since $u u_x = \left(\frac{u^2}{2}\right)_{,x}$

x mom (u) + energy

$$\rho u H_{o,x} + \rho v H_{o,y} = u(\mu u_{,y})_{,y} + \mu u_{,y}^2 + \frac{1}{P_r}(\mu h_{,y})_{,y}$$

$$\frac{1}{P_r}(\mu H_{o,y})_{,y} + \left(1 - \frac{1}{P_r}\right)\left(\mu\left(\frac{u^2}{2}\right)_{,y}\right)_{,y}$$

B.C. $u = v = 0$ at $y = 0$ $T = T_w$ or $\frac{\partial T}{\partial y} = 0$

as $y \to \delta$ $u \to u_e$ $T \to T_e$, $\rho \to \rho_e$, $p = p_e(x)$

$p'(x) = p_e'(x)$ if flow outside is irrotational

On outside euler $\rho u u_{,x} + \rho v u_{,y} = -p_{,x}$ @ wall $\rho_e u_e \frac{\partial u_e}{\partial x} = -p_e'(x)$

Exam - Closed (1 HR), Open (3/4 HR) will not emphasize handbook problems.

3-21-72

2-D Steady $\dot{Q} = 0$ $\bar{F} = 0$

$$(\rho u)_x + (\rho v)_y = 0$$

$$\rho u u_x + \rho v u_y = -p'(x) + (\mu u_y)_y$$

$$\rho u\, C_p T_x + \rho v\, C_p T_y = u\, p'(x) + \mu u_y^2 + \frac{C_p}{Pr}(\mu T_y)_y$$

$$\rho u\, H_{o_x} + \rho v\, H_{o_y} = \frac{1}{Pr}(\mu H_{o_y})_y + \left(1 - \frac{1}{Pr}\right)(\mu u u_y)_y \qquad H_o = h + \frac{u^2}{2}$$

$p'(x) = -\rho_e(x)\, u_e(x)\, u_e'(x)$, $\quad u_e(x) = u_{INV}(x, 0)$

$H_{o_e} = h_e + \frac{u_e^2}{2} = \text{constant}$ $\qquad h_e' + u_e u_e' = 0 = C_p T_e' + u_e u_e'$

@ inviscid flow $\qquad p'(x) = +\rho_e C_p T_e'$

$p = \rho R T$

for $Pr = 1$: $\quad H_o$ eq simplifies

$\qquad H_o = \text{const}$ regardless of $p'(x)$

1) Busemann Integral $\quad$ look at H_o eq.

$h + \frac{u^2}{2} = \text{const} = C_p T + \frac{u^2}{2} = \text{const} = C_p T_e + \frac{u_e^2}{2}$

differentiate wrt y $\quad C_p T_y + u u_y = 0 \quad$ @ $y = 0 \quad T_y = 0$ insulated wall only.

$$\frac{T}{T_e} = 1 + \frac{\gamma - 1}{2} M_e^2 \left(1 - \left(\frac{u}{u_e}\right)^2\right)$$

$\frac{T}{T_e} = 1 + \frac{\gamma-1}{2} M_e^2$ at wall $\quad$ at high $M_e \quad T_e \uparrow$

2) Crocco Integral $\quad p'(x)$ is very small in x mom eq can be neglected $\quad Pr = 1$

$H_o = \text{const}$ looks at u mom $\quad H_o = Bu \;\therefore\; H_o = A + Bu$

@ edge of b.l. $\quad u = u_e \quad T = T_e$

@ wall $\quad T = T_w \quad u = 0 \qquad H_o = C_p T + \frac{u^2}{2} = C_p T_w = A$

$C_p T_e + \frac{u_e^2}{2} = C_p T_w + B u_e$

$$B = \frac{1}{u_e}\left[C_p(T_e - T_w) + \frac{u_e^2}{2}\right]$$

$$H_o = C_p T_w + \frac{u}{u_e}\left[C_p(T_e - T_w) + \frac{u_e^2}{2}\right]$$

$$T/T_e = \frac{T_w}{T_e} + \left[1 - \frac{T_w}{T_e} + \frac{u_e^2}{2 C_p T_e}\right]\frac{u}{u_e} - \frac{u^2}{2 C_p T_e}$$

$$\frac{u_e^2}{2 C_p T_e} = \frac{u_e^2 (\gamma R)}{M_e^2\, 2 C_p} = \frac{\gamma - 1}{2} M_e^2 \qquad \frac{u^2}{2 c_p T_e} = \left(\frac{u}{u_e}\right)^2 \frac{\gamma-1}{2} M_e^2$$

$$\frac{\rho_e}{\rho} = T/T_e = T_w/T_e + \left[1 - T_w/T_e + \frac{\gamma-1}{2} M_e^2\right]\frac{u}{u_e} - \left(\frac{u}{u_e}\right)^2 \frac{\gamma-1}{2} M_e^2$$

$1 = \frac{p}{p_e} = \frac{\rho R T}{\rho_e R T_e}$ $\quad$ but $p = p_e \;\therefore\; T/T_e = \rho_e/\rho$

$$\hat{q}\Big|_{y=0} = -kT_y \quad \text{for blunt body fluid.} = -\frac{kT_e}{u_e}\left[1-\frac{T_w}{T_e}+\frac{\gamma-1}{2}M_e^2\right]u_y$$

when $\frac{\mu c_p}{k} = 1$
$$= -c_p\frac{T_e}{u_e}\left[1-\frac{T_w}{T_e}+\frac{\gamma-1}{2}M_e^2\right]\mu u_y\Big|_{y=0}$$
$$= -c_p\frac{T_e}{u_e}\left[\qquad\right]\tau_w$$

heat transferred from fluid to body
$$\hat{q} = c_p\frac{T_e}{u_e}\left[1-\frac{T_w}{T_e}+\frac{\gamma-1}{2}M_e^2\right]\tau_w$$

T_r = recovery temperature at wall that would exist at wall if $\hat{q}_w = 0$

for Crocco-Busemann integral $\frac{T_r}{T_e} = 1+\frac{\gamma-1}{2}M_e^2 \quad (T_y = 0)$

$$\hat{q} = c_p\frac{T_e}{u_e}\left[\frac{T_r-T_w}{T_e}\right]\tau_w = \frac{c_p}{u_e}\tau_w\left[T_r-T_w\right]$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2} \quad \Rightarrow \quad \hat{q} = \frac{c_p\rho_e u_e}{2}\left[T_r-T_w\right]c_f$$

$$\frac{\hat{q}}{\rho_e u_e c_p\left[T_r-T_w\right]} = \frac{c_f}{2} = S_t = C_H \quad \text{Stanton number.}$$

$$\frac{2C_H}{c_f} = 1 \quad \text{Reynolds' Analogy}$$
$$= P_r^{-2/3} \qquad P_r \sim 1$$

Solutions for u, v $\quad P_r = 1$

1) $p'(x) = 0 \qquad \frac{\rho\mu}{\rho_e\mu_e} = \text{const} \qquad \left(\frac{\mu}{\mu_e} = \frac{T}{T_e}\right) \qquad u_e = u_\infty, \ T_e = T_\infty$

$$(\rho u)_x + (\rho v)_y = 0$$
$$\rho u u_x + \rho v u_y = (\mu u_y)_y$$

Howarth Transformation.

$$\xi = x$$
$$Y = \int_0^y \frac{\rho}{\rho_\infty}dy \qquad Y_y = \frac{\partial Y}{\partial y}\Big|_x = \frac{\rho}{\rho_\infty}(x,y); \ Y_x = \frac{\partial Y}{\partial x}\Big|_y = \int_0^y \frac{\partial}{\partial x}(\rho/\rho_\infty)\,dy$$

define $\frac{\rho}{\rho_\infty}u = \psi_y \qquad \frac{\rho}{\rho_\infty}v = -\psi_x$ satisfy continuity identically

$\psi(x,y) = \bar{\psi}(\xi, Y)$

$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial Y} Y_x \quad , \quad \frac{\partial}{\partial y} = \frac{\rho}{\rho_\infty}\frac{\partial}{\partial Y} + \frac{\partial}{\partial \xi} \cdot 0 = \frac{\rho}{\rho_\infty}\frac{\partial}{\partial Y}$

$\frac{\rho}{\rho_\infty} u = \frac{\rho}{\rho_\infty} \bar{\psi}_Y \qquad u = \bar{\psi}_Y(\xi, Y)$

$\frac{\rho}{\rho_\infty} v = -\bar{\psi}_\xi - \bar{\psi}_Y Y_x$

$\rho \bar{\psi}_Y \left[\bar{\psi}_{Y\xi} + \bar{\psi}_{YY} Y_x\right] - \rho_\infty \left[\bar{\psi}_\xi + \bar{\psi}_Y Y_x\right] \frac{\rho}{\rho_\infty} \bar{\psi}_{YY} = \mu_\infty \left(\frac{\mu}{\mu_\infty} \frac{\rho}{\rho_\infty} \bar{\psi}_{YY} \right)_Y$

$$\bar{\psi}_Y \bar{\psi}_{Y\xi} - \bar{\psi}_\xi \bar{\psi}_{YY} = \frac{\mu_\infty}{\rho_\infty} \bar{\psi}_{YYY}$$

$u = \bar{\psi}_Y = 0 \quad @ \ y = 0 \qquad \bar{\psi}_\xi = 0 \qquad \therefore \ \bar{\psi} = \text{const} \quad$ take it as zero.

$v = 0 \quad @ \ y = 0$

as $Y \to \delta \qquad \psi_Y \to U_\infty$

looks like incompressible flow b.l. eq in transformed plane

$\bar{\psi}(\xi, Y) = (2 \nu_\infty U_\infty \xi)^{1/2} f(\eta)$

$$\eta = Y \left(\frac{U_\infty}{2\nu_\infty \xi} \right)^{1/2}$$

$u = \bar{\psi}_Y = (2\nu_\infty U_\infty \xi)^{1/2} f'(\eta) \left(\frac{U_\infty}{2\nu_\infty \xi} \right)^{1/2} = U_\infty f'(\eta)$

$\bar{\psi}_{YY} = U_\infty f''(\eta) \left(\frac{U_\infty}{2\nu_\infty \xi} \right)^{1/2} \qquad \bar{\psi}_{YYY} = U_\infty f'''(\eta) \frac{U_\infty}{2\nu_\infty \xi}$

$\bar{\psi}_\xi = \frac{1}{2} (2\nu_\infty U_\infty)^{1/2} \xi^{-1/2} f(\eta) + (2\nu_\infty U_\infty \xi)^{1/2} f'(\eta) \left(-\frac{\eta}{2\xi} \right)$

$\bar{\psi}_{Y\xi} = U_\infty f''(\eta) \left[\frac{-\eta}{2\xi} \right]$

put into diff eq

$- U_\infty f'' \left(\frac{U_\infty}{2\nu_\infty \xi} \right)^{1/2} \left[\frac{1}{2} (2\nu_\infty U_\infty)^{\frac{1}{2}} \xi^{-1/2} f(\eta) \right] = \nu_\infty U_\infty f'''(\eta) \left(\frac{U_\infty}{2\nu_\infty \xi} \right)$

$f''' + f'' f = 0 \qquad u = U_\infty f'(\eta) \qquad f'(0) = 0$

$\eta \to \infty \quad f'(\eta) \to 1$

$v = 0 \quad f(0) = 0$

Blasius Problem.

Crocco Integral $\quad \dfrac{T}{T_\infty} = \dfrac{T_W}{T_\infty} + \left(1 - \dfrac{T_W}{T_\infty} + \dfrac{\gamma-1}{2} M_\infty^2\right) f'(\eta)$

$$- \frac{\gamma-1}{2} M_\infty^2 \, f'^2(\eta)$$

Shear stress $-\ \tau_W = (\mu u_y)_{y=0} = \rho \dfrac{\mu}{\rho_\infty} \bar{\Psi}_{YY}\Big|_{y=0} = \mu_\infty \dfrac{\rho\mu}{\rho_\infty \mu_\infty} U_\infty \left(\dfrac{U_\infty}{2\nu_\infty \xi}\right)^{1/2} f''(0)$

$$= U_\infty \left(\frac{\mu U_\infty \rho_\infty}{2\xi}\right)^{1/2} f''(0)$$

$$C_f = \frac{\tau_W}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{U_\infty \left(\frac{\mu_\infty U_\infty \rho_\infty}{2\xi}\right)^{1/2} f''(0)}{\frac{1}{2}\rho_\infty U_\infty^2} = \left(\frac{\mu_\infty}{2\rho_\infty U_\infty \xi}\right)^{1/2} f''(0)$$

$$= \sqrt{2}\, f''(0)\, Re_\xi^{-1/2} = .664\, Re_\xi^{-1/2}$$

3-11-72

$$(\rho u)_x + (\rho v)_y = 0$$

$$\rho u u_x + \rho v u_y = (\mu u_y)_y$$

$$H_0 = C_p T + u^2/2 = Au + B$$

$$p = \rho R T \qquad \text{assume } \rho\mu = \text{const.} \rightarrow \mu \sim T$$

Let $\xi = x \qquad Y = \int_0^y \rho/\rho_\infty \, dy$

$$\frac{\partial Y}{\partial y}\Big|_x = \rho/\rho_\infty \,, \quad \frac{\partial Y}{\partial x}\Big|_y = \int_0^y \frac{\partial}{\partial x}(\rho/\rho_\infty)\, dy$$

$$\psi(x,y) = \bar{\Psi}(\xi, Y) \qquad \frac{\rho u}{\rho_\infty} = \psi_{,y} \qquad \frac{\rho v}{\rho_\infty} = -\psi_{,x}$$

$$\bar{\Psi}_Y \bar{\Psi}_{Y\xi} - \bar{\Psi}_\xi \bar{\Psi}_{YY} = \nu_\infty \bar{\Psi}_{YYY}$$

$u_x \quad u \quad + \quad v \quad u_y \qquad\qquad u_{yy}$

where $u_{inc} = \bar{\Psi}_Y$

$v_{inc} = -\bar{\Psi}_x$

define $\eta = Y \left(\dfrac{U_\infty}{2\nu_\infty \xi}\right)^{1/2}$

$$\bar{\Psi} = (2\nu_\infty U_\infty \xi)^{1/2} f(\eta)$$

$u = 0$ @ $y=0 \quad f'(0) = 0 \qquad\qquad f f'' + f''' = 0$

$v = 0$ @ $y=0 \quad f(0) = 0$

$u \rightarrow U_\infty$ @ $y = \delta \quad \lim_{\eta\to\infty} f'(\eta) \rightarrow 1$

$$\frac{\rho_\infty}{\rho} = T/T_\infty = T_w/T_\infty + \underbrace{\left(1 - T_w/T_\infty + \frac{\gamma-1}{2}M_\infty^2\right)}_{I_1} f'(\eta) - \underbrace{\frac{\gamma-1}{2}M_\infty^2}_{I_2} f'^2(\eta)$$

Schlichting 324-336

$$R_N = \frac{\rho_\infty U_\infty x}{\mu_\infty}$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho_\infty u_\infty^2} = \frac{\mu\, u_y|_{y=0}}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{0.664}{R_{N_x}^{1/2}}$$

1) B.L. thickness

$$Y = \int_0^y \rho/\rho_\infty\, dy \qquad y = \int_0^Y \frac{\rho_\infty}{\rho}\, dY \qquad dY = \rho/\rho_\infty\, dy : x \text{ fixed}$$

$$y = \int_0^\eta \frac{\rho_\infty}{\rho}\left(\frac{2\nu_\infty \xi}{u_\infty}\right)^{1/2} d\eta$$

$$y/x = \frac{\sqrt{2}}{\sqrt{R_{N_x}}}\int_0^\eta \rho_\infty/\rho\, d\eta = \frac{\sqrt{2}}{\sqrt{R_{N_x}}}\int_0^\eta \left(T_w/T_\infty + I_1 f'(\eta) - I_2 f'^2\right) d\eta$$

$$ff'\Big|_0^\eta - \int_0^\eta ff''\, d\eta \quad \text{use governing eq} \quad + f''\Big|_0^\eta$$

$$= \frac{\sqrt{2}}{\sqrt{R_{N_x}}}\left[\frac{T_w}{T_\infty}\eta + I_1 f\Big|_0^\eta - I_2\int_0^\eta f'^2\, d\eta\right]$$

$$y/x = \frac{\sqrt{2}}{\sqrt{R_{N_x}}}\left\{\frac{T_w}{T_\infty}\eta + I_1 f(\eta) - I_2 f(\eta)f'(\eta) - I_2\left[f''(\eta) - \overset{.4696}{f''(0)}\right]\right\}$$

define B.L. thickness $y = \delta$ when $u/u_\infty = f'(\eta) \sim .99$ at $\eta = \eta^* \approx 3$

at b.l. edge $u/u_\infty \sim 1$, $u_y \sim f''(\eta^*) \sim 0$ $\quad f(\eta) \sim \eta^* - \beta$ $\quad \eta \gg 1$ $\qquad \beta = 1.21678$

$$\delta/x = \left\{\frac{T_w}{T_\infty}\eta^* + I_1(\eta^* - \beta) - I_2(\eta^* - \beta) + I_2(.4696)\right\}\frac{\sqrt{2}}{\sqrt{R_{N_x}}}$$

$$\delta/x = \frac{\sqrt{2}}{\sqrt{R_{N_x}}}\left\{\frac{T_w}{T_\infty}\eta^* + \left(1 - \frac{T_w}{T_\infty}\right)(\eta^* - \beta) + .4696\, I_2\right\}$$

$$\delta/x = \frac{\sqrt{2}}{\sqrt{R_{N_x}}}\left\{\eta^* - \beta\left(1 - T_w/T_\infty\right) + \frac{\gamma-1}{2}M_\infty^2 f''(0)\right\}$$

$$T_0/T_\infty = 1 + \frac{\gamma-1}{2}M_\infty^2$$

$$\delta/x = \frac{\sqrt{2}}{\sqrt{R_{N_x}}}\left\{\eta^* - \beta\left(1 - \frac{T_w}{T_{\infty}}\right) + \frac{\gamma-1}{2}M_\infty^2\left(f''(0) + \frac{T_w \beta}{T_{\infty}}\right)\right\}$$

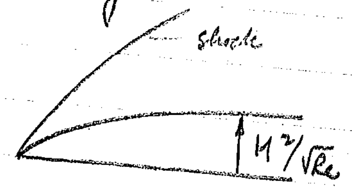
low speed flow $\delta/x \sim \frac{1}{\sqrt{R_{N_x}}}$ $\qquad$ heating increases b.l.
cooling decreases b.l.

high speed flow $\delta/x \sim \frac{M_\infty^2}{\sqrt{R_{N_x}}}$

$$C_H = \frac{C_f}{2} = \frac{\hat{q}}{\rho_e u_e c_p (T_r - T_w)} = \frac{.332}{\sqrt{R_{N_x}}}$$

$$T_r / T_\infty = 1 + \frac{\gamma - 1}{2} M_\infty^2$$

you can get interaction between shock wave & b.l. — b.l. becomes so thick a shock is formed by b.l.



Radiation – does not require any sort of media to transmit information. If a media is in between 2 radiating bodies impedance will occur. heat transfer depends on ΔT & absolute temperatures.

radiation 1. wave motion E-M waves
2. particle motion photons.

wave velocity is always the local value of speed of light.
local speed of light $\leq$ speed of light in vacuum.
different forms of rad are distinguished by λ & ν. $\lambda\nu = c$

$\mu = 10^{-6}$ m $= 3.94 \times 10^{-5}$ in.

If $\lambda < 10^{-1}\mu$ X-rays, Γ-rays, cosmic rays $\nu \gg$
$100\mu > \lambda > 10^{-1}\mu$ thermal radiation (UV, Visible, Infrared)
$\lambda > 100\mu$ radio waves $\nu \ll$

Energy $= E = h\nu$ h Planck's const $= 6.6256 \times 10^{-27}$ erg/sec.
Thermal radiation has very little penetration power.
radio waves because of long wavelengths will have more penetration than thermal rad.

$\frac{h\nu}{c}$ = momentum of the particles

$\frac{h\nu}{c}\bar{n}$ = vector momentum
velocity of the wave $= c\bar{n}$

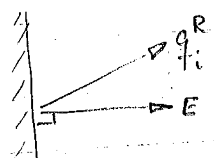
radiation – sudden transition of atomic & molec... bound-bound free electrons collide with one another. free-free

if electrons frees itself from atoms. bound-free

excitation of electrons will cause radiation to occur

1. Radiant Heat Flux Vector $\bar{q}^R$ or q_i^R amount of radiation / unit area that is transferred across a fictitious plane in space. total emissive power / unit area = net rate at which energy is transferred across a real or imaginary surface

2. E - total emissive power / unit area of thermal radiant energy from a surface and directed normal to that surface.



3. E_ν or E_λ = monochromatic radiant power / unit area

$$dE = E_\lambda \, d\lambda = E_\nu \, d\nu$$

$$E = \int_0^\infty E_\lambda \, d\lambda = \int_0^\infty E_\nu \, d\nu = -\int_\infty^0 E_\lambda \frac{c}{\nu^2} \, d\nu \quad \therefore \quad E_\lambda \frac{c}{\nu^2} = E_\nu$$

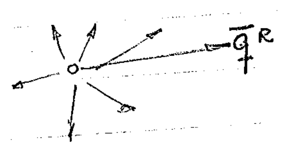
$E_{b,\lambda}$ = monochromatic radiant energy / unit area from a black body

black body - absorbs all radiation energy incident upon it.

4-18-72

if flow is isotropic $\bar{q}^R = 0$ heat is transferred equally in all directions

$E = 0$ at a pt; however at a surface radiation is experienced on the side of the outward normal $\therefore$ there exist an $E \,\&\, \bar{q}^R$ in direction of outward normal

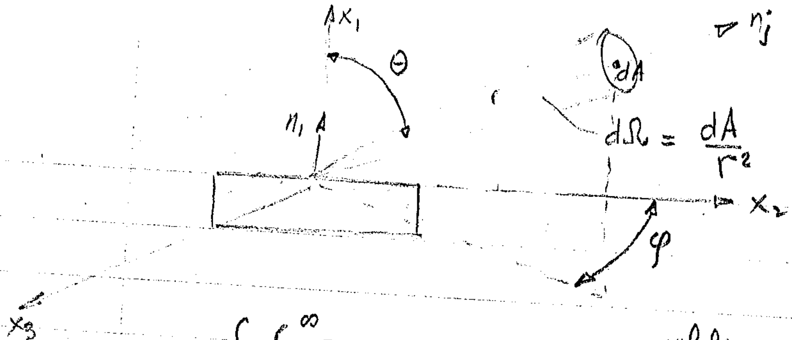


4. emissivity $\epsilon = E / E_b$

monochromatic $\epsilon_\lambda = E_\lambda / E_{\lambda,b}$

5. n^R - no. of photons / unit volume at a point in space = photon density

6. f_Ω^R - photon distribution function = $f^R(x_i, t, \bar{n}, \nu)$ = fraction of photons having a frequency between ν and $\nu + d\nu$ and included in a solid angle $d\Omega$ with unit area dA and normal $\bar{n}$ or n_j



$$d\Omega = \frac{dA}{r^2}$$

$$dA = r^2 \sin\theta \, d\theta \, d\varphi$$

$$d\Omega = \sin\theta \, d\theta \, d\varphi$$

$$\int_\Omega \int_0^\infty f_\Omega^R \, d\nu \, d\Omega = 1$$

whole space $\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{cases}$ wall $\begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi/2 \end{matrix}$

$$n^R = \int_\Omega \int_\nu n_\Omega^R f_\Omega^R \, d\nu \, d\Omega$$ photon density at a pt. $n_\Omega^R f_\Omega^R$ those values assoc with $d\Omega$

I_ν - monochromatic intensity of radiation

$$I_\nu \equiv c h \nu n_\Omega^R f_\Omega^R$$

$$n^R = \frac{1}{c} \int_\Omega \int_\nu \frac{I_\nu}{h\nu} \, d\Omega \, d\nu$$

5) energy density of photon

$$u_\nu^R = \int_\Omega \underbrace{h\nu}_{\text{energy of 1 photon}} \overbrace{n_\Omega^R f_\Omega^R}^{\text{elemental number density at a freq } \nu} \, d\Omega$$

$$u^R = \int_0^\infty u_\nu^R \, d\nu = \frac{1}{c} \int_\Omega \int_\nu I_\nu \, d\Omega \, d\nu$$

6) photon momentum density (Vector)

$$M_\nu^R = \int_\Omega \frac{h\nu}{c} \overbrace{n_\Omega^R f_\Omega^R}^{\text{number density}} \underbrace{\bar{n}}_{\text{vector mom.}} \, d\Omega$$

$\bar{n}$ - unit normal vector in direction of motion of photon.

$$M^R = \int_0^\infty M_\nu^R \, d\nu = \frac{1}{c^2} \int_\Omega \int_\nu I_\nu \, \bar{n} \, d\Omega \, d\nu$$

Fluxes

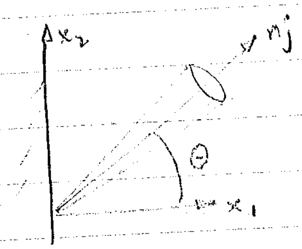
$$\underbrace{c n_j}_{\text{velocity of photon}} \ \overbrace{n_\Omega^R f_\Omega^R}^{\text{no. density}} \, d\Omega \, d\nu = \text{flux of photons/unit area}$$

$$\frac{n_j}{h\nu} I_\nu \, d\Omega \, d\nu$$

$$q_j^R = \int_\Omega \int_\nu \underbrace{h\nu}_{\text{energy}} \overbrace{\frac{\bar{n}_j}{h\nu}}^{\text{flux}} I_\nu \, d\Omega \, d\nu = \int_\Omega \int_\nu \bar{n}_j I_\nu \, d\Omega \, d\nu$$

Complete space

$$q_j^R = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} n_j I_\nu \, d\nu \, \sin\theta \, d\theta \, d\varphi$$



$n_j = \bar{n}$

$$q_1^R = E = \int_\Omega \int_0^\infty \cos\theta \, I_\nu \, d\nu \, d\Omega = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{\infty} \cos\theta \, I_\nu \, d\nu \, \sin\theta \, d\theta \, d\varphi$$

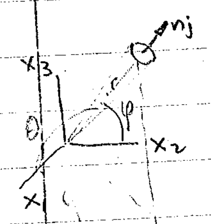
$$E = \bar{n}_j \cdot n_1$$

Momentum flux / unit area

$$\int_\Omega \int_0^\infty \frac{h\nu}{c} n_i \frac{n_j}{h\nu} I_\nu \, d\Omega \, d\nu = P_{ij}^R$$ negligible because $P_{ij}^R \sim \frac{1}{c}$

Isotropic medium $I_\nu(\theta, \varphi, \nu) = I_\nu(\nu)$

$$q_j^R = I \int_0^{2\pi} \int_0^{\pi} n_j \sin\theta \, d\theta \, d\varphi \qquad I = \int_0^\infty I_\nu \, d\nu$$



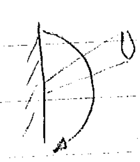
$$q_j^R = 0 = I_\nu \int_\Omega n_j \, d\Omega$$

$n_1 = \cos\theta$

$n_2 = \sin\theta \cos\varphi$

$n_3 = \sin\theta \sin\varphi$

$$E = I \int_0^{2\pi} \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \, d\varphi$$



$$E = \frac{2\pi I}{2} \int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$E = -\frac{\pi I}{2} \cos 2\theta \Big|_0^{\pi/2} = \pi I$$

$$u^R = \frac{I}{c} \int_\Omega d\Omega = \frac{4\pi I}{c}$$

$$P_{ij}^R = \frac{I}{c} \int_0^{2\pi} \int_0^{\pi} n_i n_j \sin\theta \, d\theta \, d\varphi \qquad \begin{matrix} 0 & i \neq j \\ \frac{4\pi I}{3c} & i = j \end{matrix}$$

4-20-72

ρ_λ - fractions of incident radiation reflected off surface
α_λ - " " " absorbed by "
τ_λ - " " " transmitted thru "

$$\rho_\lambda + \alpha_\lambda + \cancel{\tau_\lambda} = 1$$

very small most bodies are opaque

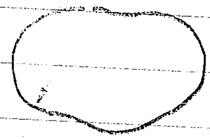
specular reflection $\measuredangle i = \measuredangle r$

diffuse refection reflected at all different angles.

for most solid bodies rad is absorbed close to body in a layer ~ (1μ) thick.

Black Body

$\rho_\lambda = 0 \quad \alpha_\lambda = 1$

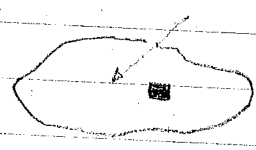


black body not hole. {In a gas $\rho_\lambda \sim 0$ If most of rad is transmitted & very small is absorbed - thin gas; otherwise it is a thick gas}

Assume Thermodynamic equil Radiation Incident = total Radiat from body

Kirchhoff's Law $\epsilon_\lambda = \alpha_\lambda$

$\epsilon_\lambda = \dfrac{E_\lambda}{E_{b,\lambda}}$



Assume hollow body with hole & a small black body inserted in hollow.

1) if emitted energy by the body $= E_b$
incident " on " " $= E_b$ Equil

Take out black body places new body but not black body

2) incident radiation $= E_b$
emitted radiation $= \epsilon_\lambda E_b$
Absorbed radiation $= \alpha_\lambda E_b$ Opaque body $\alpha_\lambda + \rho_\lambda = 1$
reflected radiation $= \rho_\lambda E_b$

3) Equil emitted = incident $\epsilon_\lambda E_b + \rho_\lambda E_b = E_b$

$\epsilon_\lambda + \rho_\lambda = 1 \quad \therefore \quad \epsilon_\lambda = \alpha_\lambda$

Good in quasi-equil whenever $\Delta T < 2000°F$

Consider emission from a Black Body

$$I_{b,\nu} = B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT}-1}$$ k - Boltzmann Constant

$$B_\lambda(T) = \frac{c}{\lambda^2} B_\nu = \frac{2hc^2\lambda^{-5}}{e^{hc/\lambda kT}-1}$$

Black Body radiation is Isotropic

$$E_{\lambda,b} = \pi I_{\lambda,b} = \pi B_\lambda(T)$$

upto 3000°R max rad lies in .1μ — 10μ.

$(\lambda T)_{B_\lambda(T)_{max}}$ when 5215.6 μ°R Wien's Displacement law

when $T = 2600$°R $\lambda \sim 2\mu$

Stefan - Boltzmann Law

$$E_b = \int_0^\infty E_{\lambda,b}\,d\lambda = 2hc^2\pi \int_0^\infty \frac{\lambda^{-5}}{e^{\frac{hc}{\lambda kT}}-1}\,d\lambda$$

let $S = \frac{hc}{\lambda kT}$ $\quad dS = -\frac{1}{\lambda^2}\frac{hc}{kT}\,d\lambda$

$$E_b = -2hc^2\pi \int_\infty^0 \left(\frac{SkT}{hc}\right)^3 \frac{kT/hc}{e^s-1}\,ds$$

$$= \frac{2k^4T^4\pi}{h^3c^2}\int_0^\infty \frac{s^3\,ds}{e^s-1} = \frac{2k^4T^4\pi}{h^3c^2}\left(\frac{\pi^4}{15}\right)$$

$$E_b = \pi B(T) = \sigma T^4$$

$$\sigma = \frac{2}{15}\frac{\pi^5 k^4}{h^3c^2} = .1714\times 10^{-8}\ \frac{BTU}{hr\,ft^2\,°R}$$ Stefan - Boltzmann Const.

Non black body

$$dE = E_\lambda\,d\lambda = \epsilon_\lambda E_{\lambda,b}\,d\lambda$$

$$= \epsilon_\lambda \pi B_\lambda\,d\lambda$$

$$= \pi\int_0^\infty \epsilon_\lambda B_\lambda\,d\lambda = \pi\bar{\epsilon}\,B$$

average emissivity

Ref. Pg. 415-418

Gray Surface - if $\alpha_\lambda = \alpha$ and if not a fn of λ, ν.
Quasi Equil $\alpha_\lambda = \epsilon_\lambda \leftrightarrow \alpha = \epsilon$ & $\epsilon_\lambda = \epsilon$

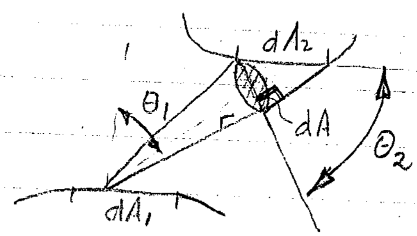
G_λ - irradiation = rate at which radiation is incident on a surface / unit area
Energy absorbed = $\int_0^\infty \alpha_\lambda G_\lambda \, d\lambda$
for Gray surface $\alpha \int_0^\infty G_\lambda \, d\lambda = \alpha G$

Radiant Heat Transfer Between Black Bodies
No Medium :

$$dq_{1\to 2} = \overset{= B_1(T)}{I_{b_1}} \cos\theta_1 \, d\Omega_{1-2} \, dA_1$$

θ between normal & solid angle

$$dE_{1\to 2} = dq_{1-2} / dA_1$$



$$d\Omega_{12} = \frac{dA}{r^2}$$

$$B_1(T) = \frac{E_{b_1}}{\pi} = I_{b_1}$$

$$dA = dA_2 \cos\theta_2 \qquad dq_{1\to 2} = E_{b_1} \, dA_1 \left[\frac{\cos\theta_1 \cos\theta_2 \, dA_2}{\pi r^2} \right]$$

$$dq_{2\to 1} = E_{b_2} \, dA_2 \left[\frac{\cos\theta_2 \cos\theta_1 \, dA_1}{\pi r^2} \right]$$

$$dq_{NET} = dq_{1\to 2} - dq_{2\to 1} = \frac{(E_{b_1} - E_{b_2})}{\pi r^2} \left\{ \cos\theta_1 \cos\theta_2 \, dA_1 dA_2 \right\}$$

$$q_{NET} = (E_{b_1} - E_{b_2}) \iint_{A_1 A_2} \frac{\cos\theta_1 \cos\theta_2 \, dA_1 \, dA_2}{\pi r^2}$$

$$q_{NET} = \sigma (T_1^4 - T_2^4) \left[\iint \quad \right]$$

$$= \sigma (T_1^4 - T_2^4) A_1 F_{12} = \sigma (T_1^4 - T_2^4) A_2 F_{21}$$

F_{12} - fraction of radiation emitted by body 1 that reaches body 2. $F_{12} \le 1.00$ it is known as a view factor.

$$F_{12} = \frac{1}{A_1}\iint \frac{\cos\theta_1 \cos\theta_2 \, dA_1 dA_2}{\pi r^2}$$

if k bodies $\sum_{N=1}^{k} F_{1N} = 1$ & $F_{12} A_1 = F_{21} A_2$

From two gray surfaces: $\alpha_\lambda = \alpha$ $\epsilon_\lambda = \epsilon$ $\alpha = \epsilon$ $\rho = \rho_\lambda$

J - radiosity = rate at which radiation leaves a surface / unit area

G - irradiation

$\therefore$ $J = \rho_\lambda G + \epsilon E_b$

emiss. absorp.

q_{NET} at surface 1 $= A(J - G) = \frac{A\epsilon}{\rho}\{E_b - J\}$

Two surfaces:

$$\left(dq_{NET}\right)_{1-2} = \left(J_1 - J_2\right)\frac{\cos\theta}{\pi} \, dA \, d\Omega_{12}$$

$$= \left(J_1 - J_2\right)\frac{\cos\theta_1 \cos\theta_2 \, dA_1 dA_2}{\pi r^2}$$

$$\left(q_{NET}\right)_{1-2} = \left(J_1 - J_2\right) A_2 F_{21} = \left(J_1 - J_2\right) A_1 F_{12}$$

if two surfaces this is q_{NET} between the two bodies

$$J_1 = -\frac{\rho_1}{A_1\epsilon_1}\left(q_{NET}\right)_1 + E_{b_1} \qquad J_2 = -\frac{\rho_2}{A_2\epsilon_2}\left(q_{NET}\right)_2 + E_{b_2}$$

$$J_1 - J_2 = \frac{\left(q_{NET}\right)_{12}}{A_1 F_{12}} = -\rho_1/A_1\epsilon_1 \left(q_{NET}\right)_1 + \frac{\rho_2}{A_2\epsilon_2}\left(q_{NET}\right)_2 + E_{b_1} - E_{b_2}$$

$$\left(q_{NET}\right)_1 = \left(q_{NET}\right)_{12} = -\left(q_{NET}\right)_2$$

$$\therefore \quad \left(q_{NET}\right)_{12} = \frac{E_{b_1} - E_{b_2}}{\rho_1/A_1\epsilon_1 + \rho_2/A_2\epsilon_2 + \frac{1}{A_1 F_{12}}} \sim \frac{\Delta E}{\Sigma R}$$

see Example 3 Pg. 428

Gases - $\alpha, \tau \neq 0$ $\rho \sim 0$. Radiative Energy is not emitted continuous as in a solid. Emission depends on ease of excitation of molecules & constitution of gases. Emission is not at one certain freq but a small band due to the uncertainty principle & doppler effect.

Radiative Transfer in Gases. 4-25-72

1. bound - bound (electron changes level) $E < E_{dissoc.}$ $E = h\nu$
2. bound - free (ionized) $E = E_{dissoc}$
3. free - free high frequencies

Conservation of Radiant Photons:

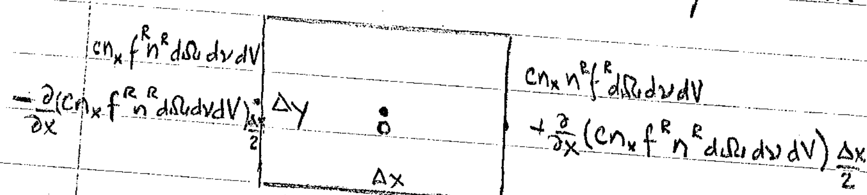
number of photons in a frequency range $(\nu, \nu+d\nu)$ in a volume element dV and in a solid angle $d\Omega$

$$n^R f^R \, d\Omega \, d\nu \, dV$$

cn_j = vector velocity of photon.

$$cn_j \, n^R f^R d\Omega \, d\nu \, dV = \text{differential flux of photons}$$

time rate of change of number of photons in volume dV = net rate at which photons are convected through the surface of dV + rate at which photons are created or destroyed in dV by emission or absorption.



fixed volume.

$$\frac{\partial}{\partial t}\int_V (n^R f^R d\Omega\, d\nu\, dV) = -\frac{\partial}{\partial x}(cn_x n^R f^R d\Omega\, d\nu)\Delta x \Delta y \Delta z - \frac{\partial}{\partial y}(cn_y n^R f^R d\Omega\, d\nu)\Delta y \Delta x \Delta z$$

$$-\frac{\partial}{\partial z}(cn_z n^R f^R d\Omega\, d\nu)\Delta z \Delta x \Delta y + \int_V (n^R f^R)_t \, d\Omega\, d\nu\, dV$$

created or destroyed.

shrink to 0 $\ni$ $n^R f^R \to n^R f^R$ at 0

In limit

$$\frac{\partial}{\partial t} n^R f^R \, d\Omega \, d\nu \, \Delta V + c\left[\frac{\partial}{\partial x}(n_x n^R f^R) + \frac{\partial}{\partial y}(n_y n^R f^R) + \frac{\partial}{\partial z}(n_z n^R f^R)\right] d\Omega\, d\nu\, dV$$

$$= (n^R f^R)_t \, d\Omega \, d\nu \, \Delta V$$

then $$\frac{\partial}{\partial t}(n^R f^R) + c\sum_{j=1}^{3}\frac{\partial}{\partial x_j}(n_j n^R f^R) = (n^R f^R)_t$$

$I_\nu = ch\nu \, n^R f^R$; $$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}(n_j I_\nu) = h\nu (n^R f^R)_t$$

$h\nu(n^R f^R)_{,t}$ rate at which energy is being created or destroyed
from quantum mechanics

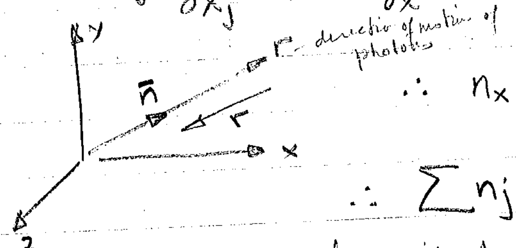
$$h\nu(n^R f^R)_{,t} = \rho(j_\nu - K_\nu I_\nu) = \alpha'_\nu(B_\nu(T) - I_\nu)$$

j_ν - mass emission coefficient
ρ - gas density
K_ν - mass absorption coefficient
α'_ν - volumetric absorption coefficient

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \sum_{j=1}^{3} n_j \frac{\partial I_\nu}{\partial x_j} = \alpha'_\nu(\underset{\text{emiss}}{B_\nu} - \underset{\text{absorption}}{I_\nu})$$

can be neglected since dependent on $\frac{1}{c}$
Assume r represents direction in which radiation is propagating

$$\sum n_j \frac{\partial I_\nu}{\partial x_j} = n_x \frac{\partial I_\nu}{\partial x} + n_y \frac{\partial I_\nu}{\partial y} + n_z \frac{\partial I_\nu}{\partial z} = -\frac{\partial I_\nu}{\partial r}$$



$$\therefore \quad n_x = \frac{\partial x}{\partial r} \qquad n_y = \frac{\partial y}{\partial r} \qquad n_z = \frac{\partial z}{\partial r}$$

$$\therefore \sum n_j \frac{\partial I_\nu}{\partial x_j} = -\frac{\partial I_\nu}{\partial r}$$

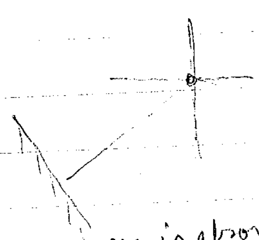
r is usually defined opposite to motion of photons that's why $-$ sign

$$-\frac{\partial I_\nu}{\partial r} = \alpha'_\nu(\underset{\text{emission}}{B_\nu} - \underset{\text{absorption}}{I_\nu}) \qquad \frac{\partial I_\nu}{\partial r} = \alpha'_\nu(I_\nu - B_\nu), \quad \text{look at } r=0$$

$$I_\nu = I_\nu(r_s)\, e^{-\tau_\nu(r_s)} + \int_0^{r_s} \alpha'_\nu B_\nu e^{-\tau_\nu(r)} dr$$

$\tau_\nu = \int_0^r \alpha'_\nu \, dr$ optical depth $= \alpha'_\nu r$
if α'_ν is independent of r

r_s location of an initial condition.



if r_s is very large τ_ν is very large $\quad I_\nu(r_s)\, e^{-\tau_\nu(r_s)}$ is negligible
if r_s is small but α'_ν is very large $\quad$ " $\quad$ " $\quad$ "

gas is absorbing energy (this is shown by $e^{-\tau_\nu}$)

if α'_ν is very large $\gg 1$ $\quad \tau_\nu(r_s) \gg 1$ if $r_s \sim O(1)$ $\quad$ Thick gas

$$I_\nu = I_\nu(r_s)\, e^{-\tau_\nu(r_s)} + \int_0^{\tau_\nu(r_s)} B_\nu(\tau_\nu)\, e^{-\tau_\nu(r)}\, d\tau_\nu$$

thick gas $I_\nu(r_s)\, e^{-\tau_\nu(r_s)}$ is negligible (opaque gas)

$\tau_\nu \gg 1$

for thick gas

$$I_\nu = \int_0^{\infty} B_\nu(\tau_\nu)\, e^{-\tau_\nu}\, d\tau_\nu \sim B_\nu(0) \int_0^{\infty} e^{-\tau_\nu}\, d\tau_\nu = B_\nu(0)$$

Intensity of radiation $\simeq$ emission at pt where you are at $r=0$

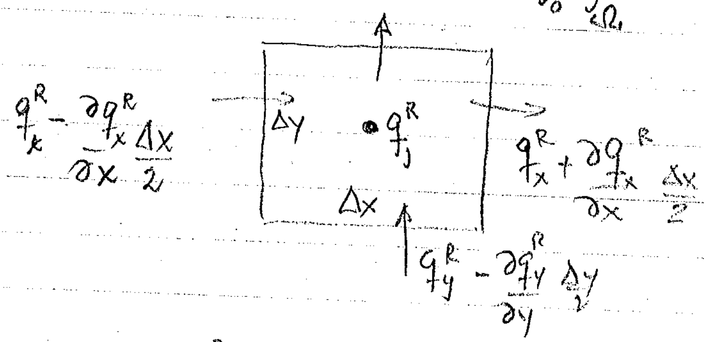
Thin Gas $\tau_\nu \ll 1$ $\quad \alpha'_\nu \ll 1$ (transparent gas)

$$I_\nu(r_s)\, e^{-\tau_\nu(r_s)} \sim I_\nu(r_s) + \int_0^{\tau_\nu} B_\nu\, d\tau_\nu$$

$$\frac{\partial I_\nu}{\partial r} = \alpha'_\nu(B_\nu - \cancel{I_\nu}) \quad \text{if no absorption} \quad I_\nu = \underbrace{I_\nu(r_s)}_{\text{initial condx}} + \int \alpha'_\nu B_\nu\, dr$$

Emission Dominated Flow (radiation)

Radiant Energy Flux $\quad q_j^R = \int_0^\infty \int_\Omega I_\nu n_j\, d\Omega\, d\nu \quad$ $h\nu\, c n_j\, n^R f^R\, d\Omega\, d\nu$ $\quad I_\nu = c h \nu n^2 f^R$



by rad. $\quad \dot{Q}_{/\text{unit vol}} = \left(\frac{\partial q_x^R}{\partial x} + \frac{\partial}{\partial y} q_y^R + \frac{\partial}{\partial z} q_z^R\right) = \nabla \cdot q_j^R$

$$\dot{Q}_{Rad} = \int_0^\infty \int_\Omega \underbrace{\sum n_j \frac{\partial I_\nu}{\partial x_j}}\, d\Omega\, d\nu$$

$$= \int_0^\infty \int_\Omega \frac{\partial I_\nu}{\partial r}\, d\Omega\, d\nu = -\int_0^\infty \alpha'_\nu \int_\Omega (I_\nu - B_\nu)\, d\Omega\, d\nu$$

independent of Ω

$E = \sigma T^4$?
$B = E/\pi$

$$= -\int_0^\infty \alpha_\nu' \left(\int_\Omega I_\nu \, d\Omega - 4\pi B_\nu \right) d\nu = -\alpha \left[\int_\Omega I \, d\Omega - 4\pi \sigma T^4 \right]$$

if $\alpha_\nu' = \text{const} = \alpha$ Gray gas approx

$$I = I(r_s) e^{-\tau(r_s)} + \alpha \int_0^{r_s} B(r) e^{-\tau(r)} dr \quad \text{graygas.} \qquad \tau_\nu = \alpha r_s$$

This results in an integral differential eq that says radiation at a pt depends on radiation everywhere

5-2-72

Thursday May 11 - 1 PM exam Closed Book Study as if closed book.
Tuesday May 9 - 2 PM

$$\dot{Q}_R = \sum_{j=1}^{3} \frac{\partial q_j^R}{\partial x_j} = -\int_0^\infty \alpha_\nu' \left(\int_\Omega (I_\nu - B_\nu) \, d\Omega \right) d\nu$$

$$I_\nu = I_\nu(r_s) e^{-\tau_\nu(r_s)} + \int_0^{\tau_\nu(r_s)} B_\nu(\tau_\nu) e^{-\tau_\nu} d\tau_\nu$$

Gray gas approx $\alpha_\nu' = \alpha$

$$\dot{Q}_R = \alpha \left\{ \int_\Omega I \, d\Omega - 4\pi \sigma T^4 \right\}$$

$$I = \int_0^\infty I_\nu \, d\nu = I(r_s) e^{-\tau(r_s)} + \alpha \int_0^{r_s} B(r) e^{-\tau(r)} dr$$

Milne-Eddington Approx. → differential expression
Exponential Approx → radiation intensity replaced by exponential forms.

Thick & Thin Gas Approx.

Thin Gas no absorption I_ν absorption term drops out $\tau_\nu \ll 1$

$$\dot{Q}_R = + \int_0^\infty \alpha_\nu' \, d\nu \left[4\pi \sigma T^4 \right] = 4\pi \int_0^\infty \alpha_\nu' B_\nu \, d\nu = 4\pi \alpha_p \sigma T^4$$

$$\alpha_p = \int_0^\infty \alpha_\nu' B_\nu \, d\nu \Big/ \int_0^\infty B_\nu \, d\nu = \text{Planck Mean Absorption Coefficient}$$

For Gray gas $\alpha_p = \alpha$

Thick Gas no emission $\tau_\nu \gg 1$ since $\tau_\nu = \int_0^{r_s} \alpha_\nu' \, dr$

$$I_\nu = \int_0^{\tau_\nu} B_\nu(\tau_\nu) e^{-\tau_\nu} d\nu$$

$\alpha_R = \frac{4\sigma T}{\pi \int \frac{1}{\alpha_\nu} \frac{\partial B_\nu}{\partial T} d\nu}$

if B_ν is continuous — C^1 expand $B_\nu(\tau_\nu)$ about $B_\nu(0)$

$$\therefore \quad B_\nu = B_\nu(0) + \frac{\partial B_\nu}{\partial \tau_\nu} \tau_\nu(0) + \ldots\ldots$$

$$= B_\nu(0) + \frac{\partial B_\nu}{\partial T} \frac{\partial T}{\partial \tau_\nu} \tau_\nu + \ldots$$

$$= B_\nu(0) + \frac{\partial B_\nu}{\partial T} \frac{\partial T}{\partial \lambda} \frac{\partial \lambda}{\partial \tau_\nu} \tau_\nu + \ldots$$

$$= B_\nu(0) \left[1 + \frac{\tau_\nu}{B_\nu(0)\alpha_\nu'} \left(\frac{\partial T}{\partial \lambda} \frac{\partial B_\nu}{\partial T} \right)_{\tau_{\nu 0}} \right] + \ldots$$

$$I_\nu = B_\nu(0) \left[\int_0^\infty e^{-\tau_\nu} d\tau_\nu + \frac{1}{\alpha_\nu' B_\nu(0)} \left(\frac{\partial T}{\partial \lambda} \frac{\partial B_\nu}{\partial T} \right)_{\tau_{\nu 0}} \int_0^\infty \tau_\nu e^{-\tau_\nu} d\nu \right]$$

$$I_\nu = B_\nu(0) \left[1 + \frac{1}{\alpha_\nu' B_\nu(0)} \left(\frac{\partial T}{\partial \lambda} \frac{\partial B_\nu}{\partial T} \right)_{\tau_\nu = 0} \right]$$

if $\alpha_\nu' \uparrow \infty \quad B_\nu(0) = I_\nu$

$$q_j^R = \int_\nu \int_\Omega I_\nu \, n_j \, d\Omega \, d\nu = \int_0^\infty \left[B_\nu(0) \int_\Omega n_j \, d\Omega + \frac{1}{\alpha_\nu'} \left(\frac{\partial B_\nu}{\partial T} \right) \underbrace{\int_\Omega n_j \frac{\partial T}{\partial \lambda} d\Omega}_{-\frac{4\pi}{3} \frac{\partial T}{\partial x_j}} \right] d\nu$$

$\int_\Omega n_j \, d\Omega = 0$ since isotropic field

$$q_j^R = -\frac{4\pi}{3} \frac{\partial T}{\partial x_j} \int_0^\infty \alpha_\nu' \frac{\partial B_\nu}{\partial T} d\nu = -\frac{16}{3} \frac{\sigma \pi T^3}{\alpha_R} \frac{\partial T}{\partial x_j}$$

$$\alpha_R = \frac{4\sigma T^3}{\int_0^\infty \frac{1}{\alpha_\nu'} \frac{\partial B_\nu}{\partial T} d\nu}$$ = Rosseland mean absorption coefficient.

$$\dot{Q}_R = \sum_{j=1}^{3} \frac{\partial q_j^R}{\partial x_j} = \sum_{j=1}^{3} -\frac{16}{3} \frac{\sigma \pi}{\alpha_R} \left[T^3 \frac{\partial^2 T}{\partial x_j^2} + 3T^2 \left(\frac{\partial T}{\partial x_j} \right)^2 \right]$$

Mass transfer

dependent on gradient of concentration of particular species in flow field
Binary species diffusion   diffusion of one thru another.

$q = -k\,\nabla T$ $\qquad k \sim \rho \bar{\lambda} \bar{c}\, c_p$ $\qquad \frac{\mu c_p}{k} = Pr \sim \text{const} \sim 3/4$ for air $\leftarrow$ momentum / Energy

$\mu \sim \rho \bar{\lambda} \bar{c}$

$\tau_{xy} = \mu\, u_{,y}$

$j_{A_y} = -\rho D_{AB}\, \omega_{A,y}$ $\qquad \omega_A = \rho_A/\rho$ species mass fraction. $\qquad$ Fick's law $\qquad D_{AB} \sim \bar{\lambda} \bar{c}$

$$\frac{\mu/\rho}{D_{AB}} = \frac{\nu}{D_{AB}} \approx \text{const.} = S_c \sim 1 \text{ for air} \qquad \frac{\text{momentum}}{\text{mass}}$$

$$\bar{c} \sim T^{1/2} \qquad D_{AB} \sim \frac{T^{3/2}}{p} \text{ for perfect gas}$$

$$\rho \bar{\lambda} \sim \text{const}$$

Gases only.

Definitions

ρ_A - mass density of species A.

$\omega_A = \rho_A/\rho$ - mass fraction of A.

$c_A = \rho_A/M_A$ - molar concentration of A. moles/unit vol.

Perfect Gas

$p_A V = n_A RT$ or $p_A = \rho_A RT$ $\qquad n_A$ - number of moles.

$y_A = c_A/c$ - mole fraction of A. $\qquad c$ - is for total gas.

$$c_A = \frac{n_A}{V} = \frac{p_A}{RT} \qquad y_A = p_A/p$$

$$\sum_A p_A = p \qquad \text{Dalton's law}$$

$$\sum_A \rho_A = \rho \qquad \sum_A \omega_A = 1$$

Mass Average (Macroscopic) Velocity of a fluid particle

$$\bar{q} = \frac{\sum_{i=1}^{n} \rho_i \bar{q}_i}{\rho}$$

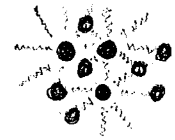
n - no. of species

ρ_i - mass density of species i

$\bar{q}_i$ - mean velocity of species i

$$\bar{q}_i = \underset{\text{molecule velocity}}{\bar{q}} + \bar{v}_i \quad \text{/ relative mass velocity of species}$$

$$\bar{V} = \frac{\sum_{i=1}^{n} c_i \bar{q}_i}{c} = \text{mean molar velocity} \qquad \bar{q}_i = \bar{V} + \bar{V}_i \quad \text{/ relative molar veloc of species}$$

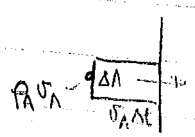


Fick's Law

$$\bar{j}_A = -\rho D_{AB} \nabla w_A \quad \text{net flux of mass.}$$

$$\bar{\bar{J}}_A = -c\, D_{AB} \nabla y_A \quad \text{net flux of moles} \qquad \text{P453-57}$$

$$\bar{j}_A = \rho_A \bar{v}_A \quad \text{mass flux/unit area, time} = \rho_A(\bar{q}_A - \bar{q}) = -\rho D_{AB} \nabla w_A$$



net flux $= \rho_A v_A \Delta A$

Δm - mass flux

Flux/unit time area. $= \dfrac{\rho_A v_A \Delta t \Delta A}{\Delta t \Delta A} = \rho_A v_A$

$$\bar{\bar{J}}_A = c_A \bar{V}_A = c_A(\bar{q}_A - \bar{V}) = -c D_{AB} \nabla y_A$$

$$c_A \bar{q}_A = -c D_{AB} \nabla y_A + c_A \bar{V}$$

$$\rho_A \bar{q}_A = -\rho D_{AB} \nabla w_A + \rho_A \bar{q}$$

define $c_A \bar{q}_A = \bar{N}_A \qquad \rho_A \bar{q}_A = \bar{n}_A$ total flux/unit area.

Binary Sys

$$\bar{q} = \frac{\rho_A \bar{q}_A + \rho_B \bar{q}_B}{\rho} \qquad \bar{V} = \frac{c_A \bar{q}_A + c_B \bar{q}_B}{c}$$

$$= \frac{\bar{n}_a + \bar{n}_B}{\rho} \qquad = \frac{\bar{N}_A + \bar{N}_B}{c}$$

$$\therefore \quad \bar{n}_a = -\rho D_{AB} \nabla w_A + w_A(\bar{n}_A + \bar{n}_B)$$

relative effect, random p conduction — bulk - effect due to fact species are transmitted Bulk or convective.

$$\bar{N}_A = -c_A D_{AB} \nabla y_A + y_A(\bar{N}_A + \bar{N}_B)$$

concentration gradient — bulk motion

Convection →

5-4-72

Emphasize material from compressible bl →

Differential Equations of Mass Transfer.

If multiple species you must take into account

1. Rate of production of a species in the mass ...

equation for that species
2. Energy produced by a reaction or any other mechanism that creates a species.
3. Boundary conditions due to mass transfer. mass transfer through a solid surface (porous solid body) chemical surface effects

1. Overall momentum eq with velocity $\bar{q}$, pressure p, density ρ etc. is obtained via Newton's Law of is unchanged by mass transfer.

$$p_A = \rho_A RT \qquad \sum p_A = p = \sum \rho_A RT = \rho RT$$

each species acts as perfect gas

2. Overall Continuity equation:

fixed volume $\frac{\partial}{\partial t}\int_V \rho\, dV = -\int_S \rho \bar{q}\cdot\bar{n}\, dS + \int_V \mathring{r}\, dV$ source sink in volume.

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{q})\right) dV = 0 \qquad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{q}) = 0$$

n densities unknown $\rightarrow$ $n-1$ unknown since n^{th} eq is eq of state

Derive Species Mass conservation

$$\frac{\partial}{\partial t}\int_V \rho_A\, dV = -\int_S \rho_A \bar{q}_A\cdot\bar{n}\, dS + \int_V \mathring{r}_A\, dV$$

$$\int_V \left[\frac{\partial}{\partial t}\rho_A + \nabla\cdot(\rho_A\bar{q}_A) - \mathring{r}_A\right] dV = 0$$

$$\boxed{\frac{\partial \rho_A}{\partial t} + \nabla\cdot\rho_A\bar{q}_A = \mathring{r}_A} \rightarrow \frac{\partial \rho_A}{\partial t} + \nabla\cdot\bar{n}_A = \mathring{r}_A$$

$$\sum_A \frac{\partial \rho_A}{\partial t} + \mathrm{div}\sum_A \rho_A\bar{q}_A = \sum_A \mathring{r}_A \qquad \text{since } \bar{q} = \frac{\sum \rho_A \bar{q}_A}{\rho}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\bar{q}) = 0$$ since no net production of overall gas (no source or sink)

if $\mathring{r}_A = 0$ frozen flow

add cont of A to cont overall $\qquad$ cont A $- \rho_A/\rho$ cont

$$\underbrace{\frac{\partial \rho_A}{\partial t} - \frac{\rho_A}{\rho}\frac{\partial \rho}{\partial t}} + \underbrace{\mathrm{div}\,\rho_A\bar{q}_A} - \rho_A/\rho\ \mathrm{div}\,\rho\bar{q} = \mathring{r}_A$$

$$\ldots\ \frac{\partial \rho_A/\rho}{\ldots} + \mathrm{div}(\rho_A(\bar{q}_A - \bar{q})) + \mathrm{div}\,\rho_A\bar{q} - \rho_A/\rho\ \mathrm{div}\,\rho\bar{q} = \mathring{r}_A$$

$\omega_a = \rho_A/\rho$

$$\rho \frac{\partial \omega_A}{\partial t} + \text{div}\,\bar{j}_A + \cancel{\rho_A \nabla\cdot\bar{q}} + \bar{q}\cdot\nabla\rho_A - \cancel{\rho_A \nabla\cdot\bar{q}} - \frac{\rho_A}{\rho}\bar{q}\cdot\nabla\rho = \dot{r}_A$$

$$\rho \frac{\partial \omega_A}{\partial t} + \text{div}\,\bar{j}_A + \rho\bar{q}\cdot\nabla \rho_A/\rho = \dot{r}_A$$

$$\rho \frac{D\omega_A}{Dt} + \nabla\cdot\bar{j}_A = \dot{r}_A$$

Fick's Law

$$\bar{j}_A = -\rho D_{AB} \nabla\omega_A$$

$$\rho \frac{D\omega_A}{Dt} = \nabla\cdot(\rho D_{AB} \nabla\omega_A) + \dot{r}_A$$

low speed case where $\rho \sim$ const $D \sim$ const

$$\rho \frac{D\omega_A}{Dt} = \rho D_{AB} \nabla^2\omega_A + \dot{r}_A$$

$$\frac{D\omega_A}{Dt} = D_{AB} \nabla^2\omega_A + \dot{r}_A/\rho$$

no product frozen flow $\frac{D\omega_A}{Dt} = D_{AB} \nabla^2\omega_A \rightarrow \frac{DT}{Dt} = \alpha \nabla^2 T$ $\quad \frac{k}{\rho c_p}$

$\bar{q} = 0$

if no convect pure cond. no mean flow $\bar{q} = 0$

$$\frac{\partial\omega_A}{\partial t} = D_{AB} \nabla^2\omega_A$$

if steady $\quad \frac{\partial\omega_A}{\partial t} = 0$

$$\bar{q}\cdot\nabla\omega_A = D_{AB} \nabla^2\omega_A$$

if steady with no conv. $= \frac{D\omega_A}{Dt} = 0$

$$\nabla^2\omega_A = 0$$

2-D Steady flow

$$u \frac{\partial\omega_A}{\partial x} + v \frac{\partial\omega_A}{\partial y} = D_{AB}\left(\frac{\partial^2\omega_A}{\partial x^2} + \frac{\partial^2\omega_A}{\partial y^2}\right)$$

non dim $\quad u, v, \to u_\infty \quad x, y \to L$

$$\bar{U} \frac{\partial\omega_A}{\partial\bar{x}} + \bar{V} \frac{\partial\omega_A}{\partial\bar{y}} = \frac{D_{AB}}{U_\infty L}\left(\frac{\partial^2\omega_A}{\partial\bar{x}^2} + \frac{\partial^2\omega_A}{\partial\bar{y}^2}\right)$$

$$\frac{D_{AB}}{U_{\infty} L} = \frac{D_{AB}}{\nu R_N}$$

$$\bar{U}\frac{\partial \omega_A}{\partial \bar{x}} + \bar{V}\frac{\partial \omega_A}{\partial \bar{y}} = \frac{1}{Sc\,Re}\left(\frac{\partial^2 \omega_A}{\partial \bar{x}^2} + \frac{\partial^2 \omega_A}{\partial \bar{y}^2}\right)$$

for $Re \gg$ neglect right hand side species is convected. if gradients are not large

for bl type drop $\frac{\partial^2 \omega_A}{\partial \bar{x}^2}$

talk about concentration b.l.

Boundary Conditions

1. specify ρ_A or ω_A
2. " mass flux by $\bar{n}_A$ or $\bar{N}_A$
3. $\bar{n}_A = -k_1(\rho_A)$ (catalytic wall condition)
4. a liquid or solid that has a gaseous interface a convect b.c. can be applied

$$(\bar{n}_A)_{\text{normal to surface}} = \rho_A (q_A)_{\text{normal}} = -\rho D_{AB}\frac{\partial \omega_A}{\partial n} + \rho_A v^{\text{veloc}}_{\text{interface}} = k_c(\rho_A - \rho_{A_\infty})$$

where k_c is a convection coeff.

5. if injection $v_{wall} \neq 0$

need equil eq to define production terms.

Low speed

$$\frac{D\omega_A}{Dt} = D_{AB}\nabla^2 \omega_A$$

Steady flow - no convection

$$\nabla^2 \omega_A = 0$$

1-D

$$\frac{d^2 \omega_A}{dz^2} = 0$$

2-D

$$\frac{\partial^2 \omega_A}{\partial y^2} + \frac{\partial^2 \omega_A}{\partial z^2} = 0$$

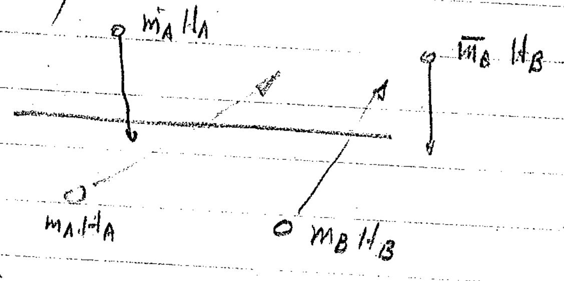
1. Separation of Variables
2. Fourier transforms.
3. Numerical Relaxation
4. Graphical methods
5. Electrical analogies
6. Green's Function

481-505

506-516

uses velocity from cont & moment & then go into low speed eqs

energy can be transferred without temp gradient
energy ~ velocity of molecules if $v_1 \gg v_2$ energy is carried across boundary



— represents movement of molecules downward.

$$\hat{q}_{mass} \sim \underbrace{\bar{q}_A(m_A - \bar{m}_A)H_A}_{energy} + \underbrace{\bar{q}_B(m_B - \bar{m}_B)H_B}_{energy} \quad \text{per unit area}$$

$$\bar{\bar{q}} = -k\nabla T + \hat{q}_{mass}$$

$$\hat{q}_{mass} = \rho H_A \bar{q}_A(\omega_A - \bar{\omega}_A) + \rho H_B \bar{q}_B(\omega_B - \bar{\omega}_B)$$

$$(\omega_A)_\lambda = \omega_A)_0 - \frac{\partial \omega_A}{\partial z}\bar{\lambda}$$

$$\bar{\omega}_A)_\lambda = \omega_A)_0 + \frac{\partial \omega_A}{\partial z}\bar{\lambda}$$

$$\hat{q}_{mass} = -2\rho H_A \bar{q}_A \frac{\partial \omega_A}{\partial z}\bar{\lambda} - 2\rho H_B \bar{q}_B \frac{\partial \omega_B}{\partial z}\bar{\lambda}$$

5-9-72

Unsteady flow $\quad \bar{q} = 0$ no convection $\quad \dot{r}_A = 0 \quad \rho, D_{AB} \sim$ const.

$$\frac{\partial \rho_A}{\partial t} = D_{AB}\nabla^2 \rho_A \quad \rightarrow \quad \omega_A, c_A, y_A$$

1. Separation of Variables
2. Laplace Transforms. initial value
3. numerical Hardship
4. Graphical methods.

Löschmidt Diffusion Cell.

liq A	Pg. 522. Finite cell Laplace trans
liq B	remove t=0 Infinite cell Fourier transf.

Mass Transfer By Convection:

Steady flow, 2-D, no reaction, low speed $\to \rho, D_{AB} \sim$ const

non dim $v, u \to u_\infty$ $x, y \to L$ $\rho_A \to \rho_{A\infty}$

$$\bar{q} \cdot \nabla \rho_A = D_{AB} \nabla^2 \rho_A$$

nondim

$$u \frac{\partial \rho_A}{\partial x} + v \frac{\partial \rho_A}{\partial y} = \frac{1}{S_c Re} \left(\frac{\partial^2 \rho_A}{\partial x^2} + \frac{\partial^2 \rho_A}{\partial y^2} \right)$$ same as $\frac{1}{P_r R_e}$ $\rho_A \to T$

concentration b.l.

$$\frac{\partial}{\partial x} << \frac{\partial}{\partial y}$$ neglect $\frac{\partial^2}{\partial x^2}$ for low speed

$$u \frac{\partial \rho_A}{\partial x} + v \frac{\partial \rho_A}{\partial y} = \frac{1}{S_c Re} \left(\frac{\partial^2 \rho_A}{\partial y^2} \right)$$ — same as thermal b.l.

if $S_c = 1$ Flat plate prob $p'(x) = 0$ mom. $u u_{,x} + v u_{,y} = \frac{1}{Re} \frac{\partial^2 u}{\partial y^2}$

then $\rho_A = Au + B$; since $u/u_\infty = f'(\eta) \Rightarrow$ $\boxed{\rho_A/\rho_{A\infty} = \left(1 - \frac{\rho_{Aw}}{\rho_{A\infty}}\right) f'(\eta) + \frac{\rho_{Aw}}{\rho_{A\infty}}}$

if $S_c \geq 1$ $\frac{\delta_c}{\delta} \sim S_c^{-1/3}$ $\frac{\delta_c}{\delta} \sim S_c^{-1/2}$ $S_c \to 0$

differences come in when chemical reaction comes into play

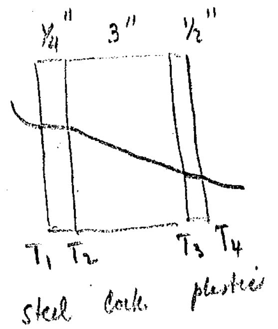
Ch 15 6, 8, 9, 15

16 1, 3, 11

6. $q_x/A = 900$ BTU/hr ft² $= -k\dfrac{\Delta T}{\Delta X} = -\ .1$ BTU/hr ft °F $\dfrac{(1600-500)}{\Delta X}\ \dfrac{°F}{ft}$

$$-900 = -.1\ \frac{(1100)}{\Delta X} = \frac{-110}{\Delta X}$$

$$\Delta X = \frac{-11}{-90} = .1224 \text{ ft}$$

8. 

$R_{\text{Stainless}} = \dfrac{1/48}{10\,A} = \dfrac{1}{480A}$

$R_{\text{cork}} = \dfrac{1/4}{.025\,A} = \dfrac{1}{.1\,A} = \dfrac{10}{A}$

$R_{\text{plast}} = \dfrac{\frac{1}{24}}{1.5\,A} = \dfrac{1}{36A}$

$$T_1 - T_4 = q_x\left(\frac{1}{480A} + \frac{10}{A} + \frac{1}{36A}\right) = \frac{q_x}{A}\left(.208\times10^{-2} + 10 + 2.78\times10^{-2}\right)$$

.00208 .0278 .0278

$$170 = \frac{q_x}{A}(10.02988)$$

$$\frac{170}{10.02988} = 16.95 \text{ BTU/hr ft}^2$$

$$T_1 - T_2 = \frac{16.95}{480} = .0353$$

$$T_2 = 249.9647\ °F$$

$$T_1 - T_3 = \frac{q_x}{A}(10.00208) = (16.95)(10.002) =$$

$$T_1 - T_3 = 169.5$$

$$T_3 = 80.5\ °F$$

10.02988
.2
.025

plastic $T_3 = 70.5°F$

9. $q/A = h\Delta T \qquad \Delta T = 180$

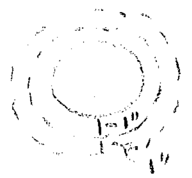
$$\cancel{\Delta T = q/A\left(\Sigma\frac{1}{h}\right) = q/A\left(\frac{1}{40} + \frac{1}{5}\right)} = q/A\,(.025 + .2) = q/A\,(.225)$$

$$\cancel{\frac{180}{.225} = q/A} \quad \Delta T = q_x\left(\frac{1}{40A} + \frac{10.02988}{A} + \frac{1}{5A}\right) = \frac{q_x}{A}(10.25488)$$

$$\frac{180}{10.25488} = \frac{q_x}{A} = 17.55 \text{ BTU/hr ft}^2 \qquad T_3 = 80 + \frac{17.55}{36} = 80.49°$$

controlling is corkwood

15 H_2O @ 40°F flows thru 1½ in pipe
85% Magn.



$$\frac{\ln r_i}{\ln r_i - \ln r_o}$$

1. if steady state $\frac{\partial T}{\partial t} = 0$ $\quad T = T(r)$ only
if radial transfer $\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = 0$

$$\alpha\left(\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right) = 0$$

$$c_1 \ln r = T + c_2$$

$$\alpha \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = \alpha\left(\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right) = 0$$

$$T + c_2 = c_1 \ln r$$

$$r\frac{dT}{dr} = c \qquad \cancel{-Mr\frac{dT}{dr} = q_r} \quad \text{let } \cancel{C = -q_r/M}$$

$$-k\frac{dT}{dr} = \frac{q}{A}$$

$$\frac{1}{c}\,dT = \frac{dr}{r}$$

$$\frac{-kA}{q} = \frac{r}{c}$$

where $\cancel{M = 2\pi L k}$

$$q_r = -\frac{kAC}{r}$$

$$\frac{1}{c}(T_i - T_o) = \ln(r_i/r_o)$$

$$T_i - T_o = C \ln(r_i/r_o) \qquad \cancel{\boxed{2\pi Lk(T_i - T_o)\ln(r_i/r_o) = q_r}}$$

$$\frac{q_r \cdot r}{-kA} = c$$

3. for steady state $\frac{\partial T}{\partial t} = 0$ if $T = T(\theta)$ then

$$T_i - T_o = \frac{q_r r}{-kA}\ln\frac{r_i}{r_o}$$

$$0 = \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = \frac{d^2 T}{d\theta^2} \qquad c_1\theta + c_2 = T$$

$$\boxed{q_r = \frac{(T_i - T_o)kA}{-r\ln r_i/r_o}}$$

at $T = T_0$ $\quad \theta = 0$ $\qquad T_0 = C_2$
$T = T_\pi$ $\quad \theta = \pi$ $\qquad c_1\pi + T_0 = T_\pi$ $\qquad c_1 = \frac{T_\pi - T_0}{\pi}$

$$T = \frac{T_\pi - T_0}{\pi}\theta + T_0$$

$$\frac{dT}{d\theta} = C_1$$

$$q_0 = -(r_o - r_i)\, k\, \frac{dT}{d\theta}$$

$$\frac{T_\pi - T_0}{\pi} = -(r_o - r_i)\, k/q_0 \qquad q_0/(r_o - r_i)k \qquad -(r_o - r_i)\, k/q_0$$

$$k\Delta T S = q_0$$
$$S = \frac{r_b - r_0}{\pi}$$

4. $T\,dS = dh - dP/\rho$

$$\rho T \frac{ds}{dt} = \rho\frac{dh}{dt} - \frac{dP}{dt} \qquad \frac{dS}{dt} = \frac{dh}{dt} - P\frac{dP}{dt}$$

$P = \rho R T \qquad P/R$

$$\rho\frac{dh}{dt} - \frac{dP}{dt} = \rho\bar{q}\cdot\nabla h + \rho T\frac{dS}{dt} = \bar{q}\cdot\nabla P + \bar{\Phi} + \dot{Q} + \dot{q}$$

$$\rho\bar{q}\cdot\nabla(h - \tfrac{P}{\rho})$$

$$\rho\bar{q}\cdot\nabla e + \rho T\frac{dS}{dt} = \bar{\Phi} + \dot{Q} + \dot{q}$$

$$\rho\frac{De}{Dt} - \rho\frac{\partial e}{\partial t} + \rho T\frac{dS}{dt} = \Phi + \dot{Q} + \dot{q}$$

$$-p\nabla\cdot\bar{q} + \cancel{\bar{\Phi} + \dot{Q} + \dot{q}} - \rho\frac{\partial e}{\partial t} + \rho T\frac{dS}{dt} = \cancel{\bar{\Phi} + \dot{Q} + \dot{q}}$$

$$\rho T\frac{ds}{dt} = \rho\frac{\partial e}{\partial t} + p\nabla\cdot\bar{q}$$

$$T\frac{Ds}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = \frac{1}{\rho}(\bar{\Phi} + \dot{Q} + \dot{q})$$

$s = s(t, x, y, z)$

$$\frac{dS}{dt} = \frac{\partial s}{\partial t}\frac{dt}{dt} + \frac{\partial s}{\partial x}\frac{dx}{dt} + \frac{\partial s}{\partial y}\frac{dy}{dt} + \frac{\partial S}{\partial z}\frac{dz}{dt}$$

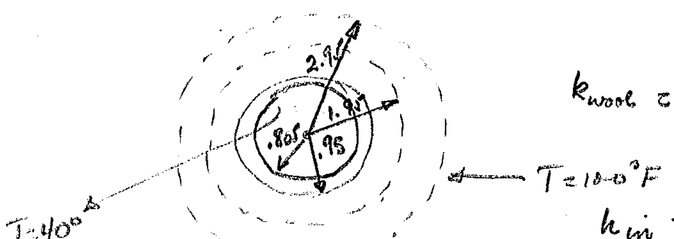
$$\frac{Ds}{Dt} = \frac{ds}{dt} \quad \text{if } \dot{x}, \dot{y}, \dot{z} \to u, v, w$$

---

40 steel pipe

OD = 1.9

ID = 1.61



$k_{wool} = .022$

$k_{mag} = .037$

$k_{steel} = 26$

$h_{in} = 100$

$h_{out} = 5$

$R_{conv\ inside}$ $R_{cond\ steel}$ $R_{cond\ wool}$ $R_{cond\ mag}$

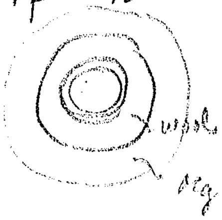
$R_{conv\ outside}$

$$\frac{1}{h_{in}} = .01$$

$$\frac{1}{h_{out}} = .2$$

$$R_{steel} = \frac{\ln(.95/.805)}{2\pi(26)(1)} = \frac{\ln(.95/.805)}{52\pi} = \frac{.167}{52\pi} = 1.024 \times 1$$

if R



$$R_{wool} = \frac{\ln(1.95/.95)}{2\pi(.022)(1)} = \frac{\ln(2.055)}{.044\pi} = \frac{.721}{.044\pi} = \cancel{5.18}\ 5.21$$

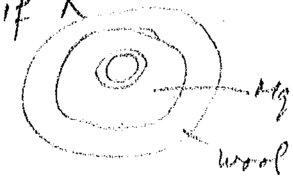
$$R_{Mg} = \frac{\ln(2.95/1.95)}{2\pi(.037)(1)} = \frac{\ln(1.51)}{.074\pi} = \frac{.413}{.074\pi} = 1.775$$

$$q = \frac{40-100}{\sum R} = \frac{-60}{7.196}$$

$.21 + 5.21 + 1.775 = 5.42$

$$\begin{array}{r} 5.42 \\ 1.775 \\ \hline 7.195 \\ .001 \end{array}$$

if R



$$R_{mag} = \frac{\ln(2.055)}{.074\pi} = \frac{.721}{.074\pi} = 3.1$$

$$R_{wool} = \frac{\ln(1.51)}{.044\pi} = \frac{.413}{.044\pi} = 2.99$$

$$\begin{array}{r} .211 \\ 6.09 \\ \hline 6.301 \end{array}$$

$$q = \frac{-60}{7.196} = -8.34 \text{ BTU/hr ft}$$

wool

2.95   3   6   5.9

$$\begin{array}{r} 6.0900 \\ .1295 \\ .0237 \\ \hline 6.2432 \end{array}$$

$$R_{in} = \frac{1}{100\pi\left(\frac{1.61}{12}\right)} = \frac{12}{161\pi} = .0237$$

$$R_{out} = \frac{12}{5\pi(5.9)} = \frac{12}{29.5\pi} = .1295$$

$$\begin{array}{r} 5.21 \\ 1.775 \\ \hline 6.985 \\ .1295 \\ .0237 \\ \hline 7.1382 \end{array}$$

$q = -8.41$ BTU/hr ft

$$q/\pi d = \frac{-8.41(12)}{\pi(5.9)} = -5.45 \text{ BTU/hr-ft}^2$$

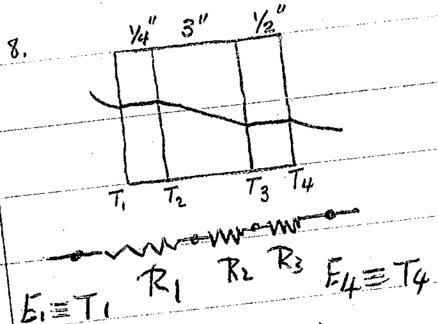
$$q = \frac{-8.41\ (12)}{\pi(5.9)}$$

Cesar Levy
AE 334

6. $q_x/A = -900 \ BTU/hr\,ft^2 = -k\frac{\Delta T}{\Delta x} = -.1 \ BTU/hr\,ft\,°F \ \frac{(1600-500)}{\Delta x}$

$\Delta x = -.1\left(\frac{1100}{900}\right) = .1224 \ ft$

8.



$R_{stainless} = \frac{L}{kA} = \frac{1/48}{10A} = \frac{1}{480A}$

$R_{corkwood} = \frac{L}{kA} = \frac{1/4}{.025A} = \frac{10}{A}$

$R_{plastic} = \frac{L}{kA} = \frac{1/24}{1.5A} = \frac{1}{36A}$

$T_1 - T_4 = 170°F$

$T_1 - T_4 = q_x/A\,(10.02988)$

$q_x/A = \frac{170}{10.02988} = 16.95 \ BTU/hr\,ft^2$

$T_1 - T_2 = \frac{q_x}{A}\left(\frac{1}{480}\right) = \frac{16.95}{480} = .0353$ $\quad T_1 = 250° \quad T_2 = 249.9647°F$

$T_1 - T_3 = q_x/A\,(10.002) = (16.95)(10.002) = 169.5$ $\quad T_1 = 250° \quad T_3 = 80.5°F$

9. $\Delta T = q/A\left(\frac{1}{h_{in}} + \frac{1}{h_{out}} + \sum R\right) = q/A\left(\frac{1}{40} + \frac{1}{5} + 10.02988\right)$

$\Delta T = 180°F \qquad q/A = \frac{180}{10.25488} = 17.55 \ BTU/hr\,ft^2$

~~$T_3 = 80° + \frac{q_x}{A}R = 80° + \frac{17.55}{36} = 80.49°$~~ ?

$T_3 - 70 = 17.55(0.2 + 0.0278)$ $\qquad T_3 = 74°F$

Note: Problem 15 is last problem done

1. if steady state $\frac{\partial T}{\partial t} = 0$, if radial transfer $\frac{\partial^2 T}{\partial \theta^2} = \frac{\partial^2 T}{\partial z^2} = 0$

$T = T(r)$ only $\therefore$

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0 \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

or $$r\frac{dT}{dr} = C_1 \Rightarrow T = C_1 \ln r - C_2$$

$$T_i = C_1 \ln r_i - C_2$$

$$T_o = C_1 \ln r_o - C_2$$

$$T_i - T_o = C_1 \ln r_i/r_o \qquad C_1 = \frac{(T_i - T_o)}{\ln r_i/r_o}$$

$$T_i = \frac{(T_i - T_o)\ln(r_i)}{\ln(r_i/r_o)} - C_2$$

$$C_2 = \frac{(T_i - T_o)\ln r_i}{\ln(r_i/r_o)} - T_i$$

$$T = \frac{(T_i - T_o)}{\ln(r_i/r_o)}\ln r - \frac{(T_i - T_o)\ln r_i}{\ln(r_i/r_o)} + T_i$$

$$= \frac{(T_i - T_o)}{\ln(r_i/r_o)}\ln(r/r_i) + T_i$$

$$q_r/A = -k\frac{dT}{dr} = -k\frac{C_1}{r} = -\frac{k}{r}\frac{(T_i - T_o)}{\ln(r_i/r_o)}$$

$$q_r = -\frac{kA}{r}\frac{(T_i - T_o)}{\ln(r_i/r_o)}$$

3. For steady state $\frac{\partial T}{\partial t} = 0$ and $T = T(\theta)$ then

$$\frac{\partial^2 T}{\partial \theta^2} = \frac{d^2 T}{d\theta^2} = 0$$

$T = C_1\theta + C_2$

$T_0 = C_2$

$T_\pi = C_1 \pi + T_0$ $\qquad$ $C_1 = \dfrac{T_\pi - T_0}{\pi}$

$\dfrac{q_\theta}{A} = -\dfrac{1}{r} k \dfrac{dT}{d\theta}$

$T = \dfrac{T_\pi - T_0}{\pi} \theta + T_0$

$dA = 1 \cdot dr$

shape factor $= \dfrac{1}{\pi} \ln \dfrac{r_o}{r_i}$

$q_\theta / A = -k \dfrac{\partial T}{\partial \theta} = -k \left(\dfrac{T_\pi - T_0}{\pi} \right)$

$q_\theta = -\dfrac{kA}{\pi} (T_\pi - T_0)$

$A = (r_o - r_i) l$ $\qquad$ l = length of the semi pipe

$S = \dfrac{A}{\pi} = \dfrac{(r_o - r_i) l}{\pi}$

11. $T dS = dh - dP/\rho$ $\qquad$ since $s = s(x,y,z,t)$

$h = h(x,y,z,t)$

$P = P(x,y,z,t)$

then $\dfrac{dS}{dt} = \dfrac{Ds}{Dt}$, $\dfrac{dh}{dt} = \dfrac{Dh}{Dt}$, $\dfrac{dP}{dt} = \dfrac{DP}{Dt}$

$T \dfrac{ds}{dt} = \dfrac{dh}{dt} - \dfrac{1}{\rho} \dfrac{dP}{dt} = T \dfrac{Ds}{Dt} = \dfrac{Dh}{Dt} - \dfrac{1}{\rho} \dfrac{DP}{Dt}$

$\rho \dfrac{Dh}{Dt} = \dfrac{DP}{Dt} + \Phi + \dot{Q} + \dot{q}$; $\rho T \dfrac{DS}{Dt} = \Phi + \dot{Q} + \dot{q}$

$\Phi \geq 0$, $\therefore S \uparrow$ or

$\therefore \dfrac{Ds}{Dt} = \dfrac{1}{\rho T} \left(\Phi + \dot{Q} + \dot{q} \right)$

15 Chapt 15

$$R_{conv\ inside} = 1/h_i A_i = \frac{1}{100\pi\left(\frac{1.61}{12}\right)} = .0237$$

$$R_{conv\ outside} = 1/h_o A_o = \frac{1}{5\pi\left(\frac{5.9}{12}\right)} = .1295$$

$$R_{steel} = \frac{\ln(.95/.805)}{2\pi(26)(1)} = 1.024 \times 10^{-3}$$

if wool first

$$R_{wool} = \frac{\ln(1.95/.95)}{2\pi(.022)(1)} = 5.21$$

$$R_{Mg} = \frac{\ln(2.95/1.95)}{2\pi(.037)(1)} = 1.775$$

$$\sum R = 7.1382$$

if Mg first

$$R_{Mg} = \frac{\ln(1.95/.95)}{2\pi(.037)(1)} = 3.1$$

$$R_{wool} = \frac{\ln(2.95/1.95)}{2\pi(.022)(1)} = 2.99$$

$$\sum R = 6.2432$$

since $q \downarrow$ as $R \uparrow$ then wool is placed first

$$q = \frac{\Delta T}{\sum R} = \frac{-60}{7.1382} = -8.41 \ \text{BTU/hr ft}$$

across outside surface area

$$q/A = \frac{q}{2\pi r_o(1)} = \frac{-8.41(12)}{\pi(5.9)} = -5.45 \ \frac{\text{BTU}}{\text{hr ft}^2}$$

Groups of 5

Homework #2

AE 334

Steady State Conduction

1. Derive Eqn. 17-33.

2. 17.3   17.15

3. Solve completely the problem done in class by separation
of variables, only with b.c. that $\dot{q}=0$ at x=0 and x=L. $\left.\frac{\partial T}{\partial x}=0\right|_{x=0,\,x=L}$
$T \to 0$ as $y \to \infty$ and ~~T=r at y=0~~ $\dot{q} = x(L-x)$ at $y=0$

4. Show by the separation of variables technique in cylindrical
coordinates, with no θ dependence, that if $\dot{Q}=0$, k=constant
$T \neq T(\theta)$ $T=R(r)X(x)$ and that $R(r)$ results in Bessel's equation.
Derive the general solution for T. Discuss the orthogonality
of the Bessel functions that result and their behavior
for $r \to 0$ and $r \to \infty$.

5. Consider LaPlace's equation $\nabla^2 T=0$ with boundary conditions.
T=x on y=0 and 1 and T=0 on x=0, T=1 on x=1. (The exact
solution is T=x).

   1. By the relaxation method, starting with T=0 at all
   interior points go ten (10) steps in the calculation of T.
   (choose 4 interior points as shown in class).

   2. If the computer is available, solve the same problem
   with 4 interior points equally spaced by the Gauss-Seidel
   method of successive approximation (work in groups of [redacted]
   [redacted] 5).

$R'' + \frac{1}{r}R' = 0$

$\frac{1}{r}\frac{d}{dr}(rR') = 0$

$rR' = C$

$R' = \frac{C}{r}$

R ... C_2

$\frac{\pi}{4 \cdot 144}(6.25 - 1) = \frac{\pi}{576}(5.25) = .0286$

$\frac{\pi D_o t}{144} = \pi \frac{(2.5)(.094)}{144} = .0051$

$\cancel{.0286}$	(.0572	.0287
$\cancel{.0005}$	.0005	.0287
$\cancel{.0291}$	.0579	.0574
		.005
		.0624
		514

$\frac{7}{140} = \frac{.01}{.20}$ $\frac{.01}{20}$ .0005

$\frac{.001}{2}$ .0005

$S_f = .0291$ 55 .43 ft .57 .53 .1075 spacing

$A_o = \pi D_o(1) = \pi(\frac{1}{12}) = .262 - 25.5 \frac{\pi D_o t}{144} = .262 - .052 = .210 \text{ ft}^2$ .1125 (25.5) 55

$A_f = 1.594 \text{ ft}^2$ $\frac{\pi 1(.094)(25.5)}{144}$

$r_b = 1.25$ $r_o = .5$ $t = .094$

$(\frac{.844}{12})^{3/2} = \cancel{.844(.919)} = (.0703)^{3/2} = (.0703)(.265) = .0186$

$\frac{r_b + t}{r_o} = \frac{1.344}{.5} = 2.69$ $\sqrt{\frac{80\ 60(32)(144)}{(26)(3)(.75)^3}} = \sqrt{\frac{80 \cdot 32 \cdot 144}{78}}$ 4725

3300 68.7

$\eta_f = .56$

$A_o + \eta_f A_f = .210 + .56(1.594) = \cancel{1.0455}$ 11400 1200 10200

.909
.21
1.119

.896
.1495
1.0455

$q = 10680$

.893
.2057
1.0987

.924
.21
1.134

$\Delta q = 10680 - 2680 = \frac{8000}{1.0455} =$ $\frac{11200}{1.594} =$ 11400 - 2680

1.179
.924
.25

11400

11280 - 2680

$\frac{8600}{1.119}$

9460
1138
11700

$\frac{872}{1.119}$

.896
.21
1.106
1.8

11400

$$\int_0^{2t} \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx + \int_0^{2t}\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)dx = 0$$

$$k\frac{\partial T}{\partial x}\Big|_0^{2t} + \frac{1}{r}\frac{\partial}{\partial r}\int_0^{2t}\left(rk\frac{\partial T}{\partial r}\right)dx - \cancel{\frac{1}{r}\frac{\partial 2t}{\partial r}}^{\,0} = 0$$

$$k\frac{\partial T}{\partial x}\Big|_0^{2t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\overline{k\frac{\partial T}{\partial r}}\right)\cdot 2t = 0 \quad \text{where} \quad \overline{k\frac{\partial T}{\partial r}} = \frac{1}{2t}\int_0^{2t} k\frac{\partial T}{\partial r}\,dx$$

$$\cancel{k\frac{\partial T}{\partial x}\Big|_{2t} = -h\left(T-T_\infty\right)}$$

T_∞ T

x_1 x_2

$$\cancel{-k\frac{T-T_\infty}{x_2-x_1} = h(T-T_\infty)}$$

T_1 T_2 $T_1 > T_2$

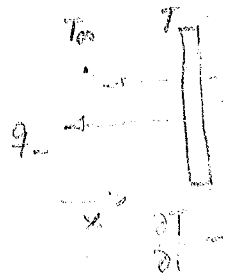
q_+

$\Delta T =$

$\Delta x =$

$$k\frac{-}{\Delta} = q_-$$

$$\cancel{+k\frac{\partial T}{\partial x} = h(T-T_\infty)}$$



$$\cancel{-h(T_\infty - T)}$$

$$-k\frac{+}{+}$$

$$q = h\,(T-T_\infty)$$

$$-h\,(T-T_\infty) = q_- = -k\frac{\partial T}{\partial x}$$

$$-2h(T-T_\infty) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\overline{k\frac{\partial T}{\partial r}}\right)\cdot 2t = 0$$

$$\overline{k\frac{\partial T}{\partial r}} = \frac{1}{A}\int_0^{2t} k\frac{\partial T}{\partial r}\,dx$$

$$-\frac{h}{kt}\left(T-T_\infty\right) + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = 0$$

$$\int_0^{2t} rk\frac{\partial T}{\partial r}\,dx$$

$$\overline{k\frac{\partial T}{\partial r}} = \frac{1}{2\pi r t}\int_0^{2t}(2\pi r)\,k\frac{\partial T}{\partial r}\,dx$$

$$\boxed{2rt\;\overline{k\frac{\partial T}{\partial r}}} = \int_0^{2\pi} kr\frac{\partial T}{\partial r}\,dx$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) = 0$$

$$\int_0^{2t}\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx + \int_0^{2t}\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)dx = 0$$

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}k\frac{\partial T}{\partial r}\right)$$

$$\cancel{rk\frac{\partial T}{\partial r} + r^2} \qquad -\frac{1}{r}$$

$$k\frac{\partial T}{\partial x}\Big|_0^{2t} + \frac{d}{dr}\int_0^{2t} rk\frac{\partial T}{\partial r}\left(\frac{1}{r}\right)dx - \cancel{\frac{\partial(2t)}{\partial x}\left[\frac{rk}{r}\frac{\partial T}{\partial r}\right]}^{\,0} = 0 \qquad r\left(-\frac{1}{r^2}\frac{k\partial T}{\partial r}\right)$$

$$k\frac{\partial^2 T}{\partial r^2}$$

$$k\frac{\partial T}{\partial x}\Big|_0^{2t} + r\frac{d}{dr}\int_0^{2t}\frac{k}{r}\frac{\partial T}{\partial r}\,dx = 0$$

$$\int_0^{2t} k\frac{\partial T}{\partial r}\,dx = 2t$$

$$k\frac{\partial T}{\partial x}\Big|_{2t} - k\frac{\partial T}{\partial x}\Big|_0$$

$$\bar{V} = \frac{1}{2t}\int_0^{2t} V\,dx = 2t\ \overline{k\frac{\partial T}{\partial r}}$$

$$\cancel{-h(T-T_\infty)}$$

$$k\frac{\partial T}{\partial x}\Big|_0^{2t} + \frac{d}{dr}\left(2t\ \overline{k\frac{\partial T}{\partial r}}\right) = 0 \qquad 2t\,k\frac{d}{dr}\left(\frac{\partial T}{\partial r}\right)$$

$$\begin{array}{r} .05870 \\ .00237 \\ .00001 \\ .00003 \\ \hline .06111 \end{array}$$

$$\cancel{\frac{d}{dr}\frac{k\partial T}{\partial r}} = \cancel{\frac{\partial}{\partial r}\frac{k\partial T}{\partial r}} + \frac{k\partial T}{\partial r}\frac{\partial}{\partial x} \qquad r\frac{d}{dr}\left(\frac{1}{r}\frac{dT}{dr}\right)$$

$$\int\frac{1}{r}\left[k\frac{\partial T}{\partial r}\right]dx + \frac{\partial}{\partial r}k\frac{\partial T}{\partial r}$$

$$\begin{array}{r} .00537 \\ .00003 \\ .00001 \\ \hline .00541 \end{array}$$

$$f(r,x)$$

$$f = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)$$

$$\frac{d}{dr}f(r,x) = \frac{\partial}{\partial r}f + \frac{\partial f}{\partial x}\frac{dx}{dr}$$

$$\frac{\partial}{\partial r}\int\left(T\ \ k\frac{dx}{r}\right.$$

$$\frac{df}{dr} = \frac{\partial}{\partial r}$$

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}k\frac{\partial T}{\partial r}\right)dx + \frac{2}{r}k\frac{\partial T}{\partial r}dx$$

$$\frac{-2h}{2kt}(T-T_\infty) + \qquad \frac{T - T_\infty}{\Delta x}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\int_0^{2t} rk\frac{\partial T}{\partial r}\,dx$$

$$T_\infty \quad T$$

$$\frac{T-T_\infty}{} \qquad \frac{T-T_\infty}{\Delta x}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\cdot 2t\ \overline{k\frac{\partial T}{\partial r}}\right]$$

$$\bar{V} = \frac{1}{2t}\int_0^{2t} V\,dx$$

$$T = T_\infty$$

$$\frac{\partial T}{\partial x}\Big|_{2t} - k\frac{\partial T}{\partial x}\Big|_0$$

$$k\frac{\partial T}{\partial x}\Big|_0^{2t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\cdot 2t\ \overline{k\frac{\partial T}{\partial r}}\right]$$

$$-k\frac{\partial T}{\partial n} = h(T-T_\infty)$$

$$-h(T-T_\infty) - h(T-T_\infty)$$

$\pi D L - \frac{12(1)}{12}(3/32) = A_0$

1'

$\frac{\pi}{12} = .262 - .094 = .168$

$A_f = \frac{12 \cdot 1}{12}(2 \times \frac{3}{4}) + .094 = 1.594$ $\Delta T = 170°$

$L + t = \frac{3}{4} + \frac{3}{32} = \frac{27}{32} = \left(\frac{2.25}{32}\right)^{3/2} = (.0703)^{3/2} = (.265)^3 = .0186$

$h = 60$ $k = 26$ $tL = \frac{9}{128 \cdot 144}$

$A_0 = \pi D_0 = .262$

$\frac{60(144)(128)}{26(9)} = \frac{42500}{9}$ 4725 68.7

$q_{00} = .262($

Δq $12\,800$ $2670\ q_{00}$

$(68.7)(.0186) = 1.28$ $\eta_f = .68$

$q_{Tot} = 60\left[.168 + 1.594(.68)\right] 170$ 8025 10130

$= 60\left[\begin{matrix}1.085\\ .168\end{matrix}\ \ 1.253\right] 170 = 12{,}800\ \frac{BTU}{hr}$ ~~8530~~

$\Delta q = 10130$

$12\pi D t = 12\pi\frac{(1)}{12}\ \frac{(3/32)}{12} = .262(.094) = .025$ $.067$

$A_f = \cancel{1.594}\ \cancel{.025}\ \cancel{1.569}$ $A_f = .133 = \frac{2\pi r_0 3}{32(144)} + \frac{2\pi}{144}(r_0^2 - .25)$

$.81$ $3(.133) - \frac{4}{60}$ $.133 - \frac{4}{60} = \frac{6\pi}{32(144)} r_0 + \frac{64\pi}{32(144)}(r_0^2 - .25)$

~~1.545~~ $1.31 - .5$ 1.639

$r_0 = \cancel{1.97}$ in 1.045 $.5$ $.9$ $32(144)\frac{4}{60} = 6\pi r_0 + 64\pi r_0^2 - \frac{64\pi}{4}$ 640000

1.545

~~1.97~~ $- .5 + .094 = 1.47$ $\frac{1.14}{12}$ 308 ~~1.201~~ 50 358

$.094$

$\frac{1.564}{12}$ $.075$ ~~816~~ $= 18.84 r_0 + 201 r_0^2$ ~~-816~~ ~~509~~ $160{,}000$

$.095$

~~(.1304)~~ $= .361$ $.308$ $.274(.075) =$ $-18.84 \pm \sqrt{(18.84)^2 + (804)(509)}$ 40000 $296{,}0$

$(.361)(.1304) =$ ~~.047~~ ~~.0293~~ $.0206$ ~~.7~~ $1.36 =$ 402 640 $\frac{621}{402}$ ~~410~~ 40

$\sqrt{\frac{60(32)\ 12 \cdot 12}{26 \cdot 3\ (.81)}} = \sqrt{3370} = \cancel{49.1}$ 58 $.66$ 365 656000

$= \cancel{-18.84 + 870}$ $\cancel{\frac{791}{402}}$ $\frac{\sqrt{44}}{-19}$ $\frac{\sqrt{25}}{402}$

2.3 $\frac{1.97 + .094}{.5} = \frac{2.064}{.5} =$

$\eta = .39$

$q = 60(.237 + \cancel{.29}(1.094))\,170 = 10200(.86) = \cancel{8760}$ $\quad 10570$

$\cancel{\frac{.624}{860}}$ $\quad .237 \quad .4$

$$\begin{array}{r} .797 \\ .237 \\ \hline 1.034 \end{array}$$

$\cancel{5810}$ $\quad 6630$

$$\begin{array}{r} 10570 \\ 2670 \\ \hline 7900 \\ \hline 1. \end{array}$$

$$.133 = \frac{2\pi r_0 l}{144} + \frac{2\pi(r_0^2 - .25)}{144}$$

$$\frac{.133(144)}{(2\pi)} = \frac{r_0\,3}{32} + r_0^2 - .25$$

$$3.05 = r_0(.094) + r_0^2 - .25$$

$$0 = r_0(.094) + r_0^2 - 3.3$$

$$\frac{-.094 \pm \sqrt{(.094)^2 + 13.2}}{2} =$$

$A_0C_0XY + A_0D_0X$

$B_0C_0Y + B_0D_0$

$\underbrace{B_0}_{0}\underbrace{C_0}_{0}Y + \underbrace{A_0D_0}_{0}$

$= \underbrace{A_0C_0}_{0}Y$

$\frac{\partial T}{\partial x} = 0 \quad \text{at } x=0$

$\frac{3.64 - .094}{2} = \frac{3.55}{2}$

1.77 in

$1.77 - .5 + .094$

$$\begin{array}{r} 6970 \\ 2670 \\ \hline 5300 \end{array}$$

$$\frac{\begin{array}{r} 1.27 \\ .094 \\ \hline 1.364 \end{array}}{12} = .114$$

$(.336)(.114)$

$\cancel{\frac{50(59)\;60 \cdot 32 \cdot 12 \cdot 12}{26.3(1.27)\;4}}$ $\quad 1.48$

$\sqrt{2790} \quad \sqrt{2.8}$

$X'' - \lambda^2 X = 0$

$Y'' + \lambda^2 Y = 0$

$$\begin{array}{r} 1.77 \\ .094 \\ \hline 1.864 \end{array} \times 2 = 3.73$$

$A_\lambda \sinh\lambda x + B_\lambda \cosh\lambda x = X$

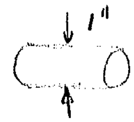
$C_\lambda \sin\lambda y + D_\lambda \cos\lambda y = Y$ $\quad .5$

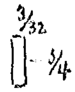
$$\begin{array}{r} .447 \\ .237 \\ \hline .684 \end{array}$$

$\frac{\partial X}{\partial x} = A_\lambda(\lambda)\cosh\lambda x + B_\lambda \sinh\lambda x$

$T_{out} = 250°F$

$T_\infty = 80°F$

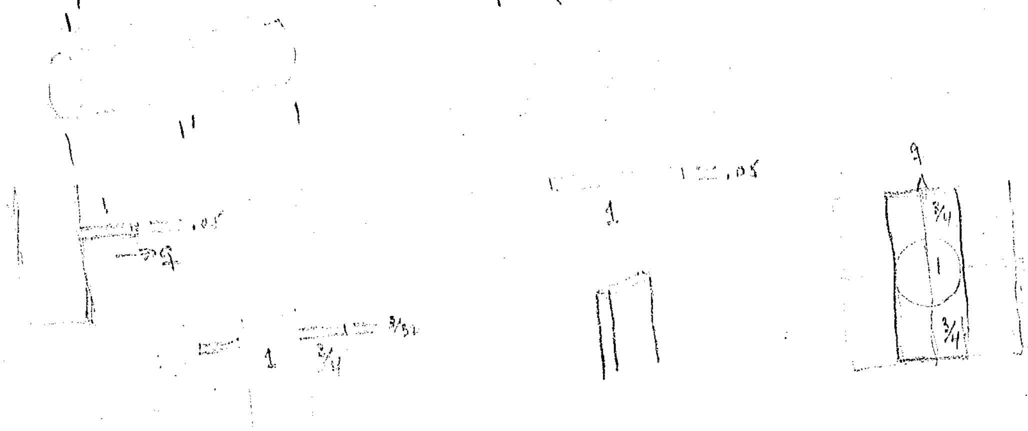


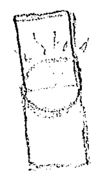


12 in.

$$A_f = \left(\frac{3}{32}\right)\left(\frac{3}{4}\right) = \frac{9}{128}$$

A_o





$$A_o = \cancel{2\pi}\ \pi D L - 12(\pi D t)$$

$$A_f = \frac{1}{12} \cdot 3$$

$$2(12)\left[\frac{1}{12}\left(\frac{3}{32}\right)\left(\frac{1}{12}\right)(\ \ ) + \frac{1.5}{12}\left(\frac{3}{32}\right)\left(\frac{1}{12}\right)\right]$$

$$A_o = \pi\left(\frac{1}{12}\right)(1) - 12(\pi)\left(\frac{1}{12}\right)\left(\frac{3}{32}\right)\left(\frac{1}{12}\right)$$

$$\frac{\pi}{12} - \frac{\pi}{128} = .262 - .0245 = .2596$$

$$A_f = \frac{6}{32 \cdot 12} + \frac{9}{32 \cdot 12} = \frac{15}{32 \cdot 12} = \frac{5}{32 \cdot 4} = \frac{5}{128} = .0391 + A_N = \begin{array}{r} .0391 \\ .295 \\ \hline .3341 \end{array}$$

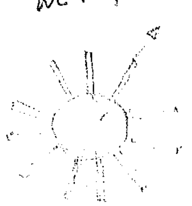
$h = 6 \quad k = 26 \quad L = 1.5 \quad t = 3/32$

$$A_N = 12\pi D t = 12\pi \frac{1}{12} \cdot \frac{3}{32} = \frac{3\pi}{32} = .295$$

$A_f = .0391$ $\quad .0391 = 12\pi D_L t \quad \dfrac{.0391(32)\ ft^2}{\frac{12\pi 3}{12}} = D_L\ ft = .1325\ \frac{ft \cdot 12}{ft}$

$$r_L = \frac{1.59}{2}\ in = .795\ in$$

$$k = \frac{BTU}{hr\,°F\,ft} \qquad h = \frac{BTU}{hr\,°F\,ft^2} \qquad \frac{h}{k}\ \frac{\frac{BTU}{hr\,°F\,ft^2}}{\frac{BTU\ ft}{hr\,°F\,ft^2}} \qquad \frac{1}{ft} \qquad \sqrt{\frac{h\,(L+t)^3}{k\,t\,L}} = \frac{1}{ft}\ \frac{ft^2}{ft\,ft}$$



$\nabla^2 T = 0 \qquad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad$ let $T(x,y) = X(x)\, Y(y)$

$$X''Y + Y''X = 0$$

$$\frac{Y''}{Y} = -\frac{X''}{X} = +\lambda^2 \qquad Y'' - \lambda^2 Y = 0$$

$$X'' + \lambda^2 X = 0$$

T_∞

$$X = A_\lambda \cos\lambda x + B_\lambda \sin\lambda x$$

$$Y = C_\lambda e^{-\lambda y} + D_\lambda e^{\lambda y}$$

bc are $\dot q = 0$ at $x = 0, L$

$$\dot q = \left| kA \frac{\partial T}{\partial x} \right|$$

$$\dot q = 0 \qquad \frac{\partial T}{\partial x} = 0$$

$$\frac{\partial T}{\partial x} = Y \frac{dX}{dx} = Y \left\{ -A_\lambda \lambda \sin\lambda x + B_\lambda \lambda \cos\lambda x \right\} = 0$$

if $\lambda \neq 0$ at $x = 0$ $\quad B_\lambda = 0$

if $\lambda \neq 0$ at $x = L$ $\quad \sin\lambda L = 0 \qquad \lambda L = n\pi \qquad \lambda = \frac{n\pi}{L}$

$$\therefore \quad X = A_\lambda \cos\frac{n\pi x}{L} \qquad n = 0 \to \infty$$

$T_\infty \cos\lambda L + B_\lambda \sin\lambda L = T_\infty$

if

$B_\lambda \sin\lambda L = 2T_\infty\,(1 - \cos\lambda L)$

$B_\lambda \sin\lambda L = 2T_\infty \sin^2\left(\frac{\lambda L}{2}\right)$

at $T = 0$ when $y \to \infty$

$D_\lambda = 0$

$$\therefore \quad Y = C_\lambda e^{-\frac{n\pi}{L} y}$$

$$\underline{A_0 = T(x,y) = 1}$$

$$\therefore \quad T(x,y) = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi y}{L}} \cos\frac{n\pi x}{L}$$

$$A_n = \frac{2}{L} L \cdot A_n = 2$$

when $y \to 0 \quad T = 1$

$$1 = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \qquad A_n = \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L}\, dx = \frac{2}{L}\frac{L}{n\pi} \sin\frac{n\pi x}{L}\Big|_0^L = 0$$

$$A_n \int c \;\; c \; dx = A_n \frac{L}{2} \qquad A_n$$

$$A_0 = \frac{2}{2} = 1 \qquad A_1 \to 0$$

$A_\lambda \cos \lambda y + B_\lambda \sin \lambda y$

$C_\lambda e^{-\lambda x} + D_\lambda e^{\lambda x}$ $\qquad \frac{\partial T}{\partial x} = 0$

$-\lambda\{C_\lambda e^{-\lambda x} + D_\lambda e^{\lambda x}\}$

$\{C_\lambda \cosh \lambda x + D_\lambda \sinh \lambda x\}$

$\lambda \{C_\lambda \sinh \lambda x + D_\lambda \cosh \lambda x\} = 0$

$D_\lambda = 0$

$C_\lambda = 0 \quad \lambda = 0$

$\lambda \{C_\lambda \sinh \lambda L\} = 0$

$A_\lambda C_\lambda$ $\qquad T(x,y) = 1$ const

$R(r)(X(x)) = T(r,x)$ $\qquad \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x^2}$

$R''X + \frac{1}{r}R'X + RX'' = 0$

$\frac{\left(R'' + \frac{1}{r}R'\right)}{R} = -\frac{X''}{X} = +\lambda^2$

$R'' + \frac{1}{r}R' - \lambda^2 R = 0$

$X'' + \lambda^2 X = 0$ $\qquad r^2R'' + rR' - r\cancel{\lambda}^2 R = 0$

Assume $R = \sum_{n=0}^{\infty} a_n r^{n+2} = a_n$

$a_n(n+2)(n+1)r^n + a_n(n+2)r^n - \lambda^2 a_n r^{n+2} = 0$

$a_n(n+2)(n+1)r^n + a_n(n+2)r^n - \lambda^2 a_{n-2} r^n = 0$

$a_n(n+2)(n+2)r^n$

$$R'' + \frac{1}{r}R' - \lambda^2 R = 0$$

$$R = \sum_{n=0}^{\infty} a_n r^{(\rho+n)}$$

$$R' = \sum_{n=0}^{\infty} a_n(\rho+n) r^{\rho+n-1}$$

$$R'' = \sum_{n=0}^{\infty} a_n(\rho+n)(\rho+n-1) r^{\rho+n-2}$$

$$\sum_{n=0}^{\infty}\left(a_n(\rho+n)(\rho+n-1) r^{\rho+n-2} + a_n(\rho+n) r^{\rho+n-2} + \lambda^2 a_n r^{\rho+n}\right) = 0$$

$$\sum_{n=0} a_n(\rho+n)(\rho+n-1) r^{\rho+n-2} + a_n(\rho+n) r^{\rho+n-2} + \sum_{n=2}\lambda^2 a_{n-2} r^{\rho+n-2} = 0$$

$$\left[a_0(\rho)(\rho-1) + a_0(\rho)\right] r^{\rho-2} + \left[a_1(\rho+1)(\rho) + a_1(\rho+1)\right] r^{\rho-1}$$

$$+ \sum_{n=2}\left\{a_n(\rho+n)(\rho+n-1) + a_n(\rho+n) + \lambda^2 a_{n-2}\right\} r^{\rho+n-2} = 0$$

$$a_0(\rho^2) = 0 \qquad \rho = 0$$

$$J_0(x)$$

$$\rho^2 + 2\rho + 1 = 0 \qquad \rho = -1 \quad \rho = -1$$

$$a_n(n-1)(n-2) + a_n(n-1) + \lambda^2 a_{n-2} = 0$$

$$a_{n-2} = a_n(n-1)(n-1)$$

$$a_n = \frac{\lambda^2 a_{n-2}}{(n-1)(n-1)}$$

$$\sum \lambda^2 \cancel{x}^{k-1} a_1$$

$$a_k = \frac{a_0}{(k-1)^2(k-3)^2\cdots 1^2}$$

$$a_{k+1} = \frac{\lambda^2 a_1}{k^2(k-2)^2\cdots 2^2}$$

$$a_4 = \frac{a_2}{3^2} = \frac{a_0}{3^2 1^2}$$

$$a_5 = \frac{a_3}{4^2} = \frac{a_1}{4^2 2^2}$$

$$a_2 = \frac{a_0}{\cancel{1}} = \frac{a_0}{1^2}$$

$$a_3 = \frac{a_1}{2(0)} = \frac{a_1}{2^2}$$

$\nabla^2 T = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(}$ let $T_{,rr} + \frac{1}{r} T_{,r} + T_{,xx} = 0$

$T_{,rr} = R''X$ $\qquad R''X + \frac{1}{r}R'X + RX'' = 0$

$T_{,r} = R'X$ $\qquad \left(R'' + \frac{1}{r}R'\right)X + RX'' = 0$

$T_{,xx} = RX''$

$$\frac{X''}{X} = -\frac{R'' + \frac{1}{r}R'}{R} = -\lambda^2$$

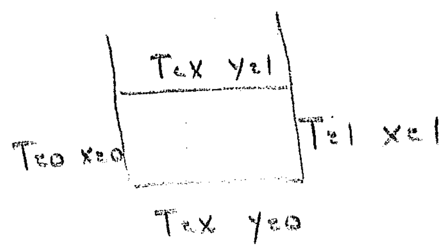
$X'' = -\lambda^2 X$

$X'' + \lambda^2 X = 0 \qquad \underline{X} = A_\lambda \cos\lambda x + B_\lambda \sin\lambda x$

$R'' + \frac{1}{r}R' = \lambda^2 R$ $\qquad$ let $R = e$

$R'' + \frac{1}{r}R' - \lambda^2 R = 0$ $\qquad$ check

$\nabla^2 T = 0 \qquad \frac{X''}{X} = \frac{Y''}{Y} = -\lambda^2$



$(A_\lambda \cos\lambda x + B_\lambda \sin\lambda x)(C_\lambda e^{-\lambda y} + D_\lambda e^{+\lambda y}) = T(x,y)$

$(C_\lambda \cosh\lambda y + D_\lambda \sinh\lambda y)$

$(A_\lambda \cosh\lambda x + B_\lambda \sinh\lambda x)$
$(C_\lambda \sin\lambda y + D_\lambda \cos\lambda y)$

$A_\lambda = 0 \qquad T=0 \quad x=0$

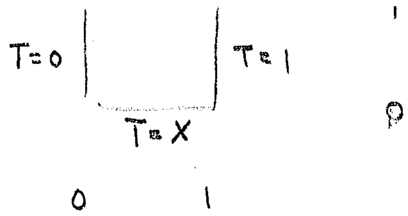
$(B_\lambda \sin\lambda)(C_\lambda \cosh\lambda y + D_\lambda \sinh\lambda y) = 1 \qquad T=1 \quad x=1$

$(B_\lambda \sin\lambda x)(C_\lambda) = x$

$\boxed{B_\lambda C_\lambda (\sin\lambda x) = x}$

$(B_\lambda \sin\lambda x)(C_\lambda \cosh\lambda + D_\lambda \sinh\lambda) = x$

$x\cosh\lambda + D_\lambda B_\lambda \sinh\lambda \sin\lambda x = x$


T=0
T=1
T=X
0
1
ρ

Let $(Ax+B)(Cy+D) = T$

$$\frac{X''}{X} = -\frac{Y''}{Y} = 0$$

when $y \to \infty$ $T \to 0$

$X'' = 0$

$A = 0$ $\quad B(\cancel{C}y + D)$

$X' = A$

$X = Ax + B$

$BD = T$

$T =$ constant

$D = 0$

$q_x = k\frac{\partial T}{\partial x} = 0$ $\quad \left.\frac{\partial T}{\partial x}\right|_{x=0} = \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0$

$A_\lambda \lambda \cos\lambda x - B_\lambda \lambda \sin\lambda x =$

$A_\lambda \lambda = 0$ $\quad A_\lambda = 0$ $\quad \lambda > 0$

$B_\lambda \lambda \sin\lambda L = 0$ $\quad \lambda L = n\pi$ $\quad \lambda = \frac{n\pi}{L}$

$$\left(B_\lambda \cos\frac{n\pi x}{L}\right) C_\lambda e^{-\frac{n\pi}{L}y}$$

$T(x,y) = A_n \cos\frac{n\pi x}{L} e^{-\frac{n\pi}{L}y}$ at $y=0$

$$1 = T(x,0) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$$

~~T(x,~~

$$A_n = \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L}\,dx = \frac{L\,2}{n\pi L}\sin\frac{n\pi x}{L}\Big|_0^L = \frac{2}{n\pi}(0-0) = 0$$

$A_n \equiv 0$ $\quad T$

$\lambda = 0$

$(Ax+B)(Cy+D) = T(x,y)$ $\quad C_0 = 0$ for finite

$(AD)x + BD = T(x,y)$

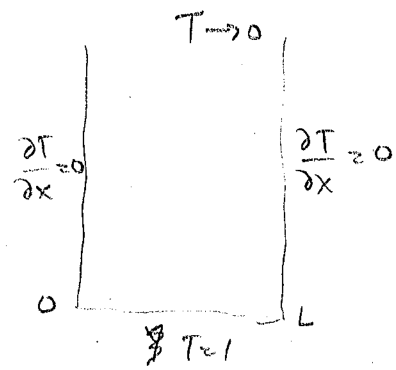
$AD = \left.\frac{\partial T}{\partial x}\right|_{x=0,L} = 0$ $\quad BD = T(x,y)$

$BD = 1$ $\quad T(x,y) = 1$

$T_{xx} + T_{yy} = 0$ on cond $\frac{\partial T}{\partial x} = 0$ at $x = L, 0$

$T \to 0$ at $y \to \infty$

$T = 1$ at $y = 0$



$$\frac{X''}{X} = -\frac{Y''}{Y} = 0$$

$$YX = (Ax+B)(Cy+D)$$

$A = 0 \quad BD = 1$

$$X = A \sin \lambda x + B \cos \lambda x$$

$$Y = Ce^{-\lambda y} + De^{\lambda y} \qquad D = 0 \quad \text{by } 3^{rd} \text{ BY}$$

$$Y = Ce^{-\lambda y}$$

$$T(x,y) = \sum_\lambda (A_\lambda \sin \lambda x + B_\lambda \cos \lambda x)(C_\lambda e^{-\lambda y})$$

$$\frac{\partial T}{\partial x} = A_\lambda \lambda \cos \lambda x - B_\lambda \lambda \sin \lambda x = 0$$

$A_\lambda = 0 \quad \cancel{X \neq 0} \to A_\lambda = 0 \qquad \cancel{B_\lambda \cos \lambda L = 0}$

$\therefore \cancel{\cos \lambda L} \quad \cancel{\lambda L = \frac{2n-1}{2}\pi} \qquad \lambda > 0$

$\cancel{\lambda = \frac{2n-1}{2}\frac{\pi}{L}} \qquad B_\lambda \lambda \sin \lambda L = 0$

$\cancel{B_\lambda \cos\left(\frac{2n-1}{2}\right)\frac{\pi x}{L} = X} \qquad \lambda L = n\pi \quad \lambda = \frac{n\pi}{L}$

~~if~~ $B_\lambda \lambda \sin \lambda L = 0 \qquad \lambda \neq 0$

$B_\lambda = 0 \quad \sin \lambda L = 0 \quad \frac{n\pi}{L} = \lambda$

$B_\lambda \cos \frac{n\pi x}{L} = X$ if $A_\lambda = 0 \quad B_\lambda \neq 0 \quad \lambda \neq 0$

if $\cancel{\lambda = 0} \qquad \cancel{X = B \quad Y = C}$

$$\sum \left(B_n \cos \frac{n\pi}{L} x\right) C_n e^{-\frac{n\pi}{L} y}$$

$$T = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi y}{L}} \cos \frac{n\pi}{L} x \qquad A_0 = 2 \qquad \cancel{1 = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{L} x} \qquad A_0 = 2$$

T

$$A_0 = \frac{2}{L}\int_0^L \cos \frac{n\pi x}{L}\, dx = \frac{2}{L}\left(\frac{L}{n\pi}\right) \sin \frac{n\pi x}{L}\Big|_0^L = 0$$

$T = 1e$

$$T_1 = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \qquad A_n = \frac{2}{L}\int_0^L T_1 \cos \frac{n\pi x}{L}\, dx = \frac{2T_1}{L}\left(\frac{L}{n\pi} \sin \frac{n\pi x}{L}\right.$$

$A_n \to 0$

$$\int_0^1 r\, J_0(\lambda_i r)\, J_0(\lambda_j r)\, dr = 0$$

$$J_{0_i} = \frac{(-1)^K (\lambda_i r)^{2K}}{(K!)^2\, 2^{2K}} \qquad J_{0_j} = \frac{(-1)^K (\lambda_j r)^{2K}}{(K!)^2\, 2^{2K}}$$

$$r\, J_{0_i} J_{0_j} = \frac{(\lambda_i \lambda_j)^{2K}\, r^{4K+1}}{(K!)^4\, 2^{4K}}$$

$$\int_0^1 \frac{(\lambda_i \lambda_j)^{2K}\, r^{4K+1}}{(K!)^4\, 2^{4K}}\, dr = \int_0^1 \mu\, r^{4K+1}\, dr = \mu \frac{r^{4K+2}}{4K+2}\Big|_0^1 = \frac{\mu}{4K+2}$$

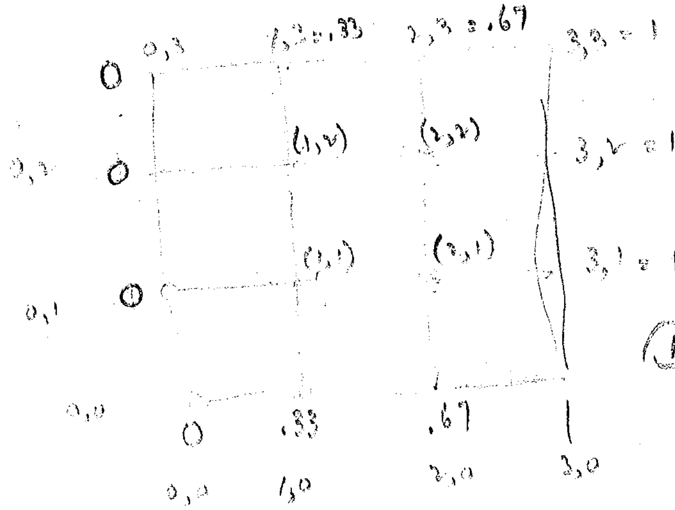
$$\lim_{K\to\infty} \frac{(\lambda_i \lambda_j)^{2K}}{(K!)^4\, 2^{4K}\,(4K+2)} \qquad \frac{a_{n+1}}{a_n} = \frac{\dfrac{(\lambda_i \lambda_j)(\lambda_i \lambda_j)^{2K}}{(K+1)^4 (K!)^4\, 2^4\, 2^{4K}\,(4K+6)}}{\dfrac{(\lambda_i \lambda_j)^{2K}}{(K!)^4\, 2^{4K}\,(4K+2)}}$$

$$4(K+1)+2$$

$$\left(\frac{c}{4}\right)^{2K} \frac{1}{(K!)^4\, 4K+2}\; 4K+6$$

$$\frac{(\lambda_i \lambda_j)\; 4K+2}{(K+1)^4\, 2^4\; 4K+6} \to 0$$

$$K! \overset{?}{>} 2^K$$



$-4T_{I,J} + T_{I+1,J} + T_{I-1,J} + T_{I,J+1} + T_{I,J-1}$

Assume $T = 0$

(1)

$R_{11} = \frac{1}{4}(0 + .33 + 0 + 0) - T_{11} = .08+$

$R_{2,1} = \frac{1}{4}(1 + 0 + 0 + .67) - 0 = \underline{.42-}$

$R_{1,2} = \frac{1}{4}(0 + .33 + 0 + 0) - 0 = .08+$

$R_{22} = \frac{1}{4}(1 + .67 + 0 + 0) - 0 = .42-$

let $R_{21} = 0 \quad \therefore \quad T_{21} = .42-$

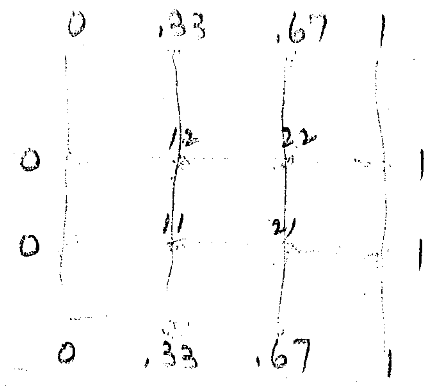
(2)

$R_{1,1} = \frac{1}{4}(.42^- + .33 + 0 + 0) - 0 = .19-$

$R_{2,2} = \frac{1}{4}(1 + .42 + 0 + .67) - 0 = \underline{\underline{.52}}$

$R_{1,2} = \frac{1}{4}(.33) = .08+$

$T_{22} = .52$ $\quad T_{21} = .42$



$R_{22} = 0$

$R_{11} = \frac{1}{4}(.33+.42) = .19-$

.55

(3) $R_{21} = \frac{1}{4}(.67+0+1+.52) - .42 = .13$

$R_{12} = \frac{1}{4}(.52+.33) = .21+$

$T_{12} = .21- \quad T_{21} = .42 \quad T_{22} = .52$

$R_{12} = 0$

(4) $R_{11} = \frac{1}{4}(.33+0+.42+.21) = .24$

.58

$R_{22} = \frac{1}{4}(.21+.42+1+.67) - .52 = +.06$

$R_{21} = \frac{1}{4}(0+.52+1+.67) - .42 = .13$

(5) $T_{11} = .24 \quad T_{12} = .21 \quad T_{22} = .52 \quad T_{21} = .42$

$R_{11} = 0$

.61

$R_{21} = \frac{1}{4}(.67+1+.52+.24) - .42 = .19$

.57

$R_{22} = \frac{1}{4}(1+.67+.21+.42) - .52 = .05$

.27

$R_{12} = \frac{1}{4}(0+.24+.33+.52) - .21 = .06$

(6) $T_{21} = .61 \quad T_{12} = .21 \quad T_{11} = .24 \quad T_{22} = .52$

$R_{21} = 0$

.62

$R_{22} = \frac{1}{4}(.21+.61+1+.67) - .52 = .1$

.29

$R_{11} = \frac{1}{4}(0+.33+.21+.61) - .24 = .05$

.27

$R_{12} = \frac{1}{4}(0+.24+.52+.33) - .21 = .06$

(7) $T_{22} = .62, \; T_{21} = .61, \; T_{12} = .21, \; T_{11} = .24$

$R_{22} = 0$

.64

$R_{21} = \frac{1}{4}(.24+.67+1+.62) - .61 = .03$

.29

$R_{11} = \frac{1}{4}(0+.33+.61+.21) - .24 = .05$

119 .30

$R_{12} = \frac{1}{4}(0+.24+.62+.33) - .21 = .09$

$R_{12} = 0$

.645

$R_{22} = \frac{1}{4}(.3+.61+1+.67) - .62 =$

.63

(8) $R_{21} = \frac{1}{4}(.24+.67+1+.62) - .61 =$

.31

$R_{11} = \frac{1}{4}(0+.33+.61+.3) - .24 =$

$T_{12} = .3 \quad T_{11} = .24 \quad T_{22} = .62 \quad T_{21} =$

$R_{11} = 0$

.32

(9) $R_{12} = \frac{1}{4}(0+.32+.62+.33) - .3 =$

.67

$R_{21} = \frac{1}{4}(.32+.67+1+.62) - .61 =$

.64

$R_{22} = \frac{1}{4}(.30+.61+1+.67) - .62 =$

$T_{12} = .3 \quad T_{11} = .32 \quad T_{22} = .62 \quad T_{21} = .61$

Comp no. 1006P

$R_{21} = 0$

.31

$R_{11} = \frac{1}{4}(.33+0+.61+.3) - .32 = 0$

.34

(10) $R_{12} = \frac{1}{4}(0+.32+.67+.33) - .3 = .04$

.645

$R_{22} = \frac{1}{4}(.3+.61+1+.67) - .67 = -$

$T_{21} = \cancel{.0}.67 \quad T_{12} = .3 \quad T_{11} = .32 \quad T_{22} = .62$

0	.33	.67	1
0	.30	.67	1
0	.32	.61	1
0	.33	.67	1

$T_{21} = .67 \quad T_{12} = .34 \quad T_{11} = .32 \quad T_{22} = .62$

Cesar Levy AE334

1. $$\nabla\cdot(k\nabla T) = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) = 0$$

$$\int_0^{2t}\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx + \int_0^{2t}\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)dx = 0 \quad \checkmark$$

Operation by means of Leibnitz Rule

$$\frac{1}{r}\frac{\partial}{\partial r}\int_0^{2t} rk\frac{\partial T}{\partial r}dx = \int_0^{2t}\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)dx + \frac{1}{r}\left\{\frac{\partial}{\partial r}(2t)\right\}\left(rk\frac{\partial T}{\partial r}\right)_{x=2t} - 0$$

but $2t = \text{const}$ $\therefore$ $$\frac{1}{r}\frac{\partial}{\partial r}\int_0^{2t} rk\frac{\partial T}{\partial r}dx = \int_0^{2t}\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)dx$$

then

$$k\frac{\partial T}{\partial x}\Big|_0^{2t} + \frac{1}{r}\frac{\partial}{\partial r}\int_0^{2t} rk\frac{\partial T}{\partial r}dx = 0$$

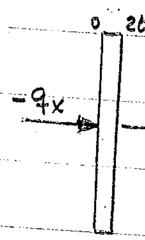
define an average value as $$\bar{V} = \frac{1}{A(r)}\int_0^{2t} 2\pi r\, V\, dx$$

Thus

$$\overline{k\frac{\partial T}{\partial r}} = \frac{1}{4\pi r t}\int_0^{2t} 2\pi r\, k\frac{\partial T}{\partial r}dx$$

but $$\frac{1}{r}\frac{\partial}{\partial r}\int_0^{2t} rk\frac{\partial T}{\partial r}dx = \frac{1}{r}\frac{\partial}{\partial r}\left(2rt\,\overline{k\frac{\partial T}{\partial r}}\right) \quad \therefore$$

$$k\frac{\partial T}{\partial x}\Big|_0^{2t} + \frac{1}{r}\frac{\partial}{\partial r}\left\{2rt\,\overline{k\frac{\partial T}{\partial r}}\right\} = 0$$



$$-q_x = -k\frac{\partial T}{\partial x} = -h(T-T_\infty)$$

$$q_x = -k\frac{\partial T}{\partial x} = h(T-T_\infty) \quad \checkmark$$

$$\therefore\ \underbrace{-h(T-T_\infty)}_{\text{due to conduction from } x=2t} - \underbrace{h(T-T_\infty)}_{\text{due to conduct from } x=0} + \frac{2tk}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0$$

or $$\frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} - \frac{h}{kt}(T-T_\infty) = 0$$

2.(a) a 2 inch sched No 40 has an OD = 2.375 in ID = 2.067 in

@ $t = 280$ $p = 49.2$ $\frac{10.8}{17.81} = \frac{x}{20}$

@ $t = 300$ $p = 67.01$

$x = 12.12$ $\therefore$ $t = 292.12\ °F$ on the inner surface

$L = 60$ ft, 85% mag has a $k = .037$ thickness = $1\frac{1}{2}$ in

$h_{in} = 1000$. $h_{out} = 5$ BTU/hr ft^2 °F

$$R_{cond} = \frac{\ln(2.375/2.067)}{2\pi(26)(60)} = \frac{.14}{(120\pi)(26)} = 1.43\times10^{-5}$$

$$R_{inst} = \frac{\ln(5.375/2.375)}{2\pi(.037)(60)} = \frac{.817}{120\pi(.037)} = .0587$$

$$R_{conv_{out}} = \frac{12}{(5)(\pi)(5.375)(60)} = \frac{1}{25\pi(5.375)} = 2.37\times10^{-3}$$

$$R_{conv_{in}} = \frac{12}{1000(\pi)(2.067)(60)} = 3.09\times10^{-5}$$

$R_{TOT} = .06111$ (BTU/hr °F)$^{-1}$ with insulation

for an outside temperature of 70°F

$$\dot{q}_{ins} = \frac{292.12 - 70}{.06111} = 3640 \text{ BTU/hr.}$$

without insulation: $R_{conv_{in}} = 3.09\times10^{-5}$, $R_{cond} = 1.43\times10^{-5}$

$$R_{conv\,out} = \frac{12}{5(\pi)(2.375)(60)} = 5.37\times10^{-3}$$

$R_{TOT} = .00541$ (BTU/hr °F)$^{-1}$ without insulation

$$\dot{q}_{UN} = \frac{292.12 - 70}{.00541} = 41100 \text{ BTU/hr.}$$

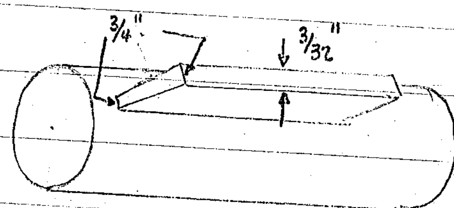
$\Delta\dot{q} = 37460$ BTU/hr

for 60 ft cost is \$31.80

also it takes 2.66 hrs to save \$.17 on the steam $\therefore$

$$2.66 \text{ hrs } \frac{(31.80)}{.17} = 497 \text{ fins}$$

2(b)



$r_o = .5$ in $\quad t = 3/32''$ $\quad L = .75$ in

take typical section 1' long

12 longitudinal fins

$$A_o = \pi D_o(1') - 12(1')(t) = \frac{\pi(1)}{12} - 12\left(\frac{3}{32}\right)\left(\frac{1}{12}\right) = .262 - .094 = .168 \text{ ft}^2$$

$$A_f = 12(1' \times L)(2) + 12(1')(t) = 24\left(\frac{.75}{12}\right) + 12\left(\frac{3}{32}\right)\left(\frac{1}{12}\right) = 1.5 + .094 = 1.594 \text{ ft}^2$$

$\Delta T = 250° - 80° = 170°$ $\quad h = 60$ BTU/hr ft^2 °F $\quad k = 26$ BTU/hr ft °F

This formula differs from that in the Textbook!

$(L + t)^{3/2}\sqrt{h/Kt L} = (.0186)(68.7) = 1.275$ $\quad \eta_f = .69$

$$q_{TOT} = h\left[A_o + \eta_f A_f\right]\Delta T = 60\left[.168 + 1.594(.69)\right]170 = 12820 \text{ BTU/hr.}$$

$$\Delta q = 12820 - \pi D_o(1) h \Delta T = 12820 - 2680 = 10140 \text{ BTU/hr}$$

area of 1 circular fin $t = 3/32''$ $r_i = 1.25''$ $r_o = 1''$ is .0624 ft $\therefore$ 25.5 fins are needed in 1 ft long section

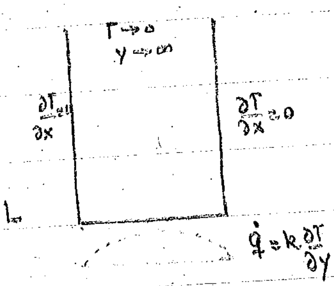
$A_o = \pi D_o(1') - 25.5\,\pi D_o t = .210 \text{ ft}^2$

$A_f = 1.594 \text{ ft}^2$ $\qquad (r_i - r_o + t)^{3/2}\sqrt{h/kt(r_i - r_o)} = 68.7$ $\quad \frac{r_i + t}{r_o} = 2.69$

$\eta_f = .57$ $\qquad A_o + \eta_f A_f = 1.119 \text{ ft}^2$ $\qquad q_{TOT} = 11400$ BTU/hr

$\Delta q = 11400 - 2680 = 8720$ BTU/hr.

3 $\nabla^2 T = 0$ leads to $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$ ✓



$X'' + \lambda^2 X = 0 \quad \longrightarrow \quad A_\lambda \cos\lambda x + B_\lambda \sin\lambda x = \mathfrak{X}$

$\frac{\partial T}{\partial x} = -A_\lambda \lambda \sin\lambda x + B_\lambda \lambda \cos\lambda x = 0 \quad @\ x = 0, L$

if $\lambda \neq 0 \quad B_\lambda = 0$ at $x = 0$

$\lambda L = n\pi$ at $x = L \quad \lambda = \frac{n\pi}{L}$

$Y'' - \lambda^2 Y = 0 \quad \longrightarrow \quad C_\lambda e^{-\lambda y} + D_\lambda e^{\lambda y} = \mathfrak{Y}$

since $y \to \infty \quad T \to 0 \quad D_\lambda = 0$

$\therefore \quad \mathfrak{Y} = C_\lambda e^{-\lambda y}$ or

$\mathfrak{Y} = C_\lambda e^{-\frac{n\pi y}{L}}$; for $\dot{q} = -kA\frac{\partial T}{\partial y} = x(L-x)$ at $y = 0$

$T(x,y) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-\frac{n\pi y}{L}}$; $-Ak\frac{\partial T}{\partial y} = -Ak\sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\left[-\frac{n\pi}{L} e^{-\frac{n\pi y}{L}}\right]$

$x(L-x) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\left(\frac{kn\pi}{L}\right)$

$A_n = \frac{2}{Akn\pi}\int_0^L x(L-x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{AK\varrho\pi}\left[\frac{-L^3}{2\varrho^2\pi^2}\right] = \frac{-L^3}{2K\varrho^3\pi^3 A}$

where $2\varrho = n$

$$T(x,y) = \sum_{\varrho=1}^{\infty} \frac{-L^3}{2KA\varrho^3\pi^3}\cos\frac{2\varrho\pi x}{L}\, e^{-\frac{2\varrho\pi y}{L}}$$ ✓

note: $\frac{+2}{KAn\pi}\int_0^L x(L-x)\cos\frac{n\pi x}{L}\,dx = \frac{+2}{KAn\pi}\left[L\int_0^L x\cos\frac{n\pi x}{L}\,dx - \int_0^L x^2\cos\frac{n\pi x}{L}\,dx\right]$

$L\int_0^L x\cos\frac{n\pi x}{L}\,dx = \frac{L^2}{n\pi}x\sin\frac{n\pi x}{L}\Big|_0^L - \frac{L^2}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\,dx = \frac{L^3}{n^2\pi^2}\left[\cos\frac{n\pi x}{L}\right]_0^L = \frac{L^3}{n^2\pi^2}\left[\cos n\pi - 1\right]$

$-\int_0^L x^2\cos\frac{n\pi x}{L}\,dx = -\frac{Lx^2}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L + \frac{2L}{n\pi}\int_0^L x\sin\frac{n\pi x}{L}\,dx = +\frac{2L}{n\pi}\left[\frac{-Lx}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\,dx\right] = -\frac{2L^3}{n^2\pi^2}\cos n\pi$ ✓

4. $\nabla\cdot(k\nabla T)=0$ leads to $k\frac{\partial^2 T}{\partial r^2}+\frac{k}{r}\frac{\partial T}{\partial r}+k\frac{\partial^2 T}{\partial x^2}=0$, $\frac{\partial T}{\partial \theta}=0$

let $T(r,x)=R(r)X(x)$ then $R''X+\frac{1}{r}R'X+X''R=0$ or

$$\frac{R''+\frac{1}{r}R'}{R}=-\frac{X''}{X}=-\lambda^2 \qquad \lambda>0$$

therefore $X''-\lambda^2X=0$ or $X_\lambda=A_\lambda\cosh\lambda x+B_\lambda\sinh\lambda x$

and $R''+\frac{1}{r}R'+\lambda^2R=0$ ✓ Let $Q(r)=\frac{1}{r}$ $P(r)=1$ $S(r)=+\lambda^2$

then $(r-r_0)\frac{Q(r)}{P(r)}=(r-r_0)\frac{1}{r}$ and $(r-r_0)^2\frac{S(r)}{P(r)}=(r-r_0)^2(+\lambda^2)$

if $r_0=0$ $\lim_{r\to 0}(r-r_0)\frac{1}{r}=1$ $\lim_{r\to 0}(r-r_0)(+\lambda^2)=0$ ✓

thus the equation is analytic about $r=0$ and $r=0$ is a regular singular point. Assume

$$R=r^m\sum_{n=0}^{\infty}a_nr^n\ ;\quad R'=\sum_{n=0}^{\infty}(m+n)a_nr^{m+n-1}$$

$R''=\sum_{n=0}^{\infty}(m+n)(m+n-1)a_nr^{m+n-2}$ placing these back into the differential equation then

$$\sum_{n=0}^{\infty}\{(m+n)(m+n-1)a_nr^{m+n-2}+(m+n)a_nr^{m+n-2}+\lambda^2a_nr^{m+n}\}=0$$

$$(m)(m-1)a_0r^{m-2}+ma_0r^{m-2}+(m+1)(m)a_1r^{m-1}+(m+1)a_1r^{m-1}$$
$$+\sum_{n=2}^{\infty}\{(m+n)(m+n)a_n+\lambda^2a_{n-2}\}r^{m+n-2}=0$$

Since $r^{m-2}\neq 0$ $a_0(m^2)=0$; also $r^{m-1}\neq 0$ $a_1(m+1)^2=0$

if $m=0$ $\therefore$ $a_1=0$ thus

$$a_n=\frac{-\lambda^2}{(m+n)(m+n)}a_{n-2}$$

or $a_2=\frac{-\lambda^2a_0}{2\cdot 2}$, $a_3=\frac{-\lambda^2}{3\cdot 3}a_1=0$, $a_4=\frac{-\lambda^2a_2}{4\cdot 4}=\frac{+\lambda^4a_0}{4\cdot 4\cdot 2\cdot 2}$

let $2K = n$ then $\qquad K = 1, \dots, \infty$

$$a_{2K} = \frac{(-1)^K \lambda^{2K} a_0}{K!\,K!\; 2^{2K}}$$

$$\therefore \quad R_1 = \sum_{k=0}^{\infty} \frac{(-1)^K \lambda^{2K} a_0}{(K!)^2\; 2^{2K}}\, r^{2K} \qquad \text{about } r=0$$

this is of the form $a_0 J_0(\lambda r)$ where J_0 is bessel's function of the first kind of order zero.

Because the differential equation is second order, there are two solutions

$$R = R_1 + R_2 \qquad R_2 \text{ is obtained as follows}$$

$$R_2 = R_1(\lambda r) \ln \lambda r + \sum_{n=1}^{\infty} \frac{d a_n}{dm}(0)\, r^n$$

$$a_{2K} = \frac{(-1)^K \lambda^{2K} a_0}{(2K+m)^2 (2K-2+m)^2 \cdots (2+m)^2} \qquad K = 1, \dots,$$

$$\frac{d a_n}{dm}(0) = \frac{d a_{2K}}{dm}(0) \quad \therefore \quad \ln \frac{a_{2K}}{(-1)^K \lambda^{2K} a_0} = -2\left[\ln(2K+m) + \ln(2K-2+m) + \dots + \ln(2+m)\right]$$

then
$$\frac{1}{a_{2K}(0)}\, a'_{2K}(0) = -2\left[\frac{1}{2K+m} + \frac{1}{2K-2+m} + \dots + \frac{1}{2+m}\right]_{m=0}$$

$$\therefore \quad a'_{2K}(0) = -2\left[\frac{1}{2K} + \frac{1}{2K-2} + \dots + \frac{1}{2}\right] a_{2K}(0)$$

$$a'_{2K}(0) = -\left[\frac{1}{K} + \frac{1}{K-1} + \dots + \frac{1}{1}\right] a_{2K}(0)$$

denotes $\sum_{a=0}^{K-1} \frac{1}{K-a} = H_K \quad \therefore$

$$a'_{2K}(0) = H_K\, a_{2K}(0) = -H_K \frac{(-1)^K \lambda^{2K} a_0}{(K!)^2\; 2^{2K}}$$

$$\therefore \quad R_2 = a_0 J_0(\lambda r) \ln \lambda r + \sum_{k=1}^{\infty} H_K \frac{(-1)^{K+1} \lambda^{2K} a_0\, r^{2K}}{(K!)^2\; 2^{2K}} \quad ✓$$

define $Y_0(\lambda r) = \frac{2}{\pi}\left[R_2 + (\gamma - \ln 2) a_0 J_0(\lambda r)\right]$ where $\gamma = \lim_{n\to\infty} (H_n - \ln n) \cong .577$

then
$$Y_0(\lambda r) = \frac{2}{\pi}\left[\left(\gamma + \ln \frac{\lambda r}{2}\right) a_0 J_0(\lambda r) + a_0 \sum_{k=1}^{\infty} \frac{(-1)^{K+1} H_K (\lambda r)^{2K}}{(K!)^2\; 2^{2K}}\right]$$

Note: take $a_0 = 1$ for convenience

then Bessels' Equation has a solution $R(\lambda r) = C_\lambda J_0(\lambda r) + D_\lambda Y_0(\lambda r)$

where Y_0 is bessels' function of the second kind of order zero.

note that if $\lambda = 0$ the solution degenerates to

$$R_0(r) = C_0 \ln r + D_0 \qquad \mathbb{X}_0(x) = A_0 x + B_0$$

note that if $\lambda < 0$ $Y_0(\lambda r)$ is not defined $\therefore$

$$T(x,r) = R_0 \mathbb{X}_0 + \mathbb{X}_\lambda R_\lambda$$

$$= (C_0 \ln r + D_0)(A_0 x + B_0) + (A_\lambda \cosh \lambda x + B_\lambda \sinh \lambda x)(C_\lambda J_0(\lambda r) + D_\lambda Y_0(\lambda r))$$

For orthogonality $\int_0^1 r\, J_0(\lambda_i r) J_0(\lambda_j r)\, dr = 0$ if $J_0(\lambda_{i,j} r) = \begin{matrix} 1 & r=0 \\ 0 & r=1 \end{matrix}$

and $\lambda_i \neq \lambda_j$

The series for $J_n(\lambda r) = \sum_{K=0}^{\infty} \dfrac{(-1)^K (\lambda r)^{n+2K}}{2^{n+2K} K!\, \Gamma(n+K+1)}$

as $r \to 0$ $J_n(\lambda r) \to \phi$ $\qquad 0! = 1, \quad \Gamma(1) = 1.$

as $r \to \infty$ the asymptotic form of $J_n(\lambda r)$ is

$$J_n(\lambda r) \approx \sqrt{\frac{2}{\pi \lambda r}} \left\{ \left[1 - \frac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8\lambda r)^2} + \frac{(4n^2-1^2)\ldots(4n^2-7^2)}{4!\,(8\lambda r)^4} - \ldots \right] \right.$$

$$\cos\left(\lambda r - \frac{\pi}{4} - \frac{n\pi}{2}\right) - \left[\frac{4n^2-1^2}{1!\,8\lambda r} - \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{3!\,(8\lambda r)^3} + \ldots \right].$$

$$\left. \sin\left(\lambda r - \frac{\pi}{4} - \frac{n\pi}{2}\right) \right\}$$

For $r \to \infty$ $Y_n(\lambda r)$ is like $J_n(\lambda r)$ with sine and cosine reversed and a plus sign before the second group of terms.

For $r \to 0$ $\quad Y_n(\lambda r) = \frac{2}{\pi}\left(\gamma + \ln \frac{\lambda r}{2}\right) J_n(\lambda r) - \frac{1}{\pi} \sum_{K=0}^{n-1} \frac{(n-K-1)!}{K!} \left(\frac{2}{\lambda r}\right)^{n-2K}$

$$- \frac{1}{\pi} \sum_{K=0}^{\infty} \frac{(-1)^K \left(\frac{\lambda r}{2}\right)^{n+2K}}{K!\,(n+K)!} \left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{K} + 1 + \frac{1}{2} + \ldots + \frac{1}{n+K} \right]$$

first term is indeterminate, second goes to ∞ and third goes to 0 as $r \to 0$.

5 (a) Assume $T \approx 0$ everywhere

Step 1 $R_{11} \approx .08$ $R_{2,1} \approx .42$ $R_{1,2} \approx .08$ $R_{22} \approx .42$

Step 2 If $T_{21} \approx .42$ $R_{21} \approx 0$ $R_{11} \approx .19$ $R_{22} \approx .52$ $R_{12} \approx .08$

Step 3 If $T_{21} \approx .42$ $T_{22} \approx .52$ $R_{22} \approx 0$ $R_{11} \approx .19$ $R_{21} \approx .13$ $R_{12} \approx .21$

Step 4 If $T_{21} \approx .42$ $T_{22} \approx .52$ $T_{12} \approx .21$ $R_{12} \approx 0$ $R_{11} \approx .24$ $R_{22} \approx .06$ $R_{21} \approx .13$

Step 5 If $T_{21} \approx .42$ $T_{22} \approx .52$ $T_{12} \approx .21$ $T_{11} \approx .24$ $R_{11} \approx 0$ $R_{21} \approx .19$ $R_{22} \approx .05$ $R_{12} \approx .06$

Step 6 If $T_{21} \approx .61$ " " " $R_{21} \approx 0$ $R_{22} \approx .1$ $R_{11} \approx .05$ $R_{12} \approx .06$

Step 7 If " $T_{22} \approx .62$ " " $R_{22} \approx 0$ $R_{21} \approx .03$ $R_{11} \approx .05$ $R_{12} \approx .09$

Step 8 If " " $T_{12} \approx .30$ " $R_{12} \approx 0$ $R_{22} \approx .02$ $R_{21} \approx .02$ $R_{11} \approx .08$

Step 9 If " " " $T_{11} \approx .32$ $R_{11} \approx 0$ $R_{12} \approx .02$ $R_{21} \approx .06$ $R_{22} \approx .02$

Step 10 If $T_{21} \approx .67$ " " " $R_{21} \approx 0$ $R_{11} \approx 0$ $R_{12} \approx .04$ $R_{22} \approx -$

Final Values

0	.33	.67	1
0	.34	.62	1
0	.32	.67	1
0	.33	.67	1

<u>AE 334 Homework #3 Unsteady Conduction</u>

1. Consider the heating of a semi-infinite wall with the Newtonian convection boundary condition applied at the surface. Let $\alpha=0.1$, $k/h=1$, $T_\infty/T_o=\sigma=5$.

   1) Determine T/T_o by the integral method.

   2) *Determine T/T_o by numerical <u>explicit</u> marching with $\Delta t=0.01$, $\Delta z=0.1$. $0\leq t\leq 1$.

   3) Compare these results with the exact transform solution at t=1 and three z locations.

*Note: For boundary condition at z=0

$$kT_z = h(T-T_\infty)$$

or

$$k\theta_z = h(\theta-\sigma) \qquad \theta=T/T_o \qquad \sigma=5$$

Use for $(\theta_z)_{z=0} = \dfrac{4\theta_1-\theta_2-3\theta_o}{2\Delta z}$, where 1 and 2 are the grid points Δz and $2\Delta z$ above z=0.

2. a) For $h=\infty$, solve the above problem by similarity methods with $\eta=z/4\sqrt{\alpha t}$ and $T/T_o=f(\eta)$.

   b) For $h=\infty$, from the ~~integral~~ transform solution found in class, determine δ_T (value of z when $T=1.01T_o$) for $\sigma=2,5,100$.

$$\int_0^{\delta_T} \frac{\partial T}{\partial t}\, dz = \alpha \int_0^{\delta_T} \frac{\partial^2 T}{\partial z^2}\, dz$$

$z = \delta_T \quad T = T_0$

$k \left.\frac{\partial T}{\partial z}\right|_{z=0} = h(T - T_\infty)$

$$\frac{\partial}{\partial t}\int_0^{\delta_T} T\,dz - \frac{\partial \delta_T}{\partial t} T\Big|_{\delta_T} = \alpha \left.\frac{\partial T}{\partial z}\right|_{\delta_T} - \alpha \left.\frac{\partial T}{\partial z}\right|_0$$

$$\frac{\partial}{\partial t}\int_0^{\delta_T} T\,dz - T_0 \frac{\partial \delta_T}{\partial t} = -\frac{\alpha h}{k}\left(T - T_\infty\right)$$

$$T_0 \frac{\partial}{\partial t}\left(\delta_T \int_0^1 \theta\, d\eta\right) - T_0 \frac{\partial \delta_T}{\partial t} = -\frac{\alpha h}{k} T_0 (\theta - \sigma) = -\frac{\alpha}{\delta_T} T_0 \left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0}$$

$$\frac{\partial}{\partial t} \delta_T \int_0^1 \theta\, d\eta - \frac{\partial \delta_T}{\partial t} = \frac{\alpha h}{k}(\sigma - \theta)$$

Assume $\theta = A + B\eta + C\eta^2$ ① at $\eta = 1 \quad \theta = 1$

$\frac{\partial \theta}{\partial \eta} = B + 2C\eta$ ② $\eta = 1 \quad \frac{\partial \theta}{\partial \eta} = 0$

at ② $B = -2C$ $\eta = 0 \quad \frac{k}{\delta_T}\frac{\partial \theta}{\partial \eta} = h(\theta - \sigma)$

at ① $1 = A - 2C + C = A - C$

at ③ $\frac{k}{\delta_T}\left[-2C\right] = h\left[A - \sigma\right]$ $A = \frac{k}{h\delta_T}\left[-2C\right] + \sigma = \frac{k(-2C) + \sigma h \delta_T}{h\delta_T}$

$$1 = \frac{k}{h\delta_T}\left[-2C\right] + \sigma - C = \frac{-2kC - h\delta_T C}{h\delta_T} + \sigma$$

$C = \frac{(1-\sigma) h\delta_T}{2k + h\delta_T}$ $B = \frac{2(1-\sigma)h\delta_T}{2k + h\delta_T}$ $A = \sigma + \frac{2k(1-\sigma)}{2k + h\delta_T}$

$$T/T_0 = \theta = \left(\frac{2k + h\delta_T\sigma}{2k + h\delta_T}\right) + \frac{2(1-\sigma)h\delta_T}{2k + h\delta_T}\eta - \frac{(1-\sigma)h\delta_T}{2k + h\delta_T}\eta^2$$

$$\int_0^1 \theta\, d\eta = \int_0^1 \left[\frac{2k + h\delta_T \sigma}{2k + h\delta_T}\right] d\eta + \int_0^1 \left[\frac{2(1-\sigma)h\delta_T}{2k + h\delta_T}\right]\eta\, d\eta - \int_0^1 \left[\frac{(1-\sigma)h\delta_T}{2k + h\delta_T}\right]\eta^2 d\eta$$

$$\frac{3}{3}\left[\frac{2k+h\delta_T\sigma}{2k+h\delta_T}\right]+\frac{3}{3}\left[\frac{(1-\sigma)h\delta_T}{2k+h\delta_T}\right]-\frac{1}{3}\left[\frac{(1-\sigma)h\delta_T}{2k+h\delta_T}\right]$$

$$\frac{3(2k+h\delta_T\sigma)}{6k+3h\delta_T}+\frac{2(1-\sigma)h\delta_T}{6k+3h\delta_T}=\frac{6k+3h\delta_T\sigma+2h\delta_T-2\sigma h\delta_T}{6k+3h\delta_T}$$

$$=\frac{6k+2h\delta_T+h\delta_T\sigma}{6k+3h\delta_T}=1+\frac{(\sigma-1)h\delta_T}{6k+3h\delta_T}$$

$$\frac{\partial}{\partial t}\left[\delta_T+\frac{(\sigma-1)h\delta_T^2}{6k+3h\delta_T}\right]-\frac{\partial}{\partial t}\delta_T=-\frac{\alpha}{\delta_T}\left[\frac{6(1-\sigma)h\delta_T}{6k+3h\delta_T}\right]$$

$$\frac{2(\sigma-1)h\,\delta_T\frac{\partial}{\partial t}\delta_T}{6k+3h\delta_T}-\frac{(\sigma-1)h\delta_T^2}{(6k+3h\delta_T)^2}\left[3h\frac{\partial}{\partial t}\delta_T\right]=-\frac{\alpha}{\cancel{\delta_T}}\left[\frac{6(1-\sigma)h\cancel{\delta_T}}{6k+3h\delta_T}\right]$$

$$2(6k+3h\delta_T)(\sigma-1)h\delta_T\frac{\partial}{\partial t}\delta_T-(\sigma-1)3h^2\delta_T^2\frac{\partial}{\partial t}\delta_T=6\alpha(\sigma-1)h(6k+3h\delta_T)$$

$$2(6k+3h\delta_T)h\delta_T\frac{\partial}{\partial t}\delta_T-3h^2\delta_T^2\frac{\partial}{\partial t}\delta_T=6\alpha(6k+3h\delta_T)h$$

$$\left(\frac{\partial}{\partial t}\delta_T\right)\left[12kh\delta_T+6h^2\delta_T^2-3h^2\delta_T^2\right]=6\alpha(6k+3h\delta_T)h$$

$$(12kh\delta_T+3h^2\delta_T^2)\frac{\partial}{\partial t}\delta_T=6\alpha(6k+3h\delta_T)h$$

$$\frac{\cancel{3}\cancel{h}\delta_T(4k+h\delta_T)\,d\delta_T}{\cancel{3}(2k+h\delta_T)\cancel{h}}=6\alpha\,dt$$

$$\frac{2k\delta_T}{(\quad)}+\frac{(2k+\cancel{h\delta_T})\delta_T\,d\delta_T}{(\quad)}$$

$$\frac{\delta_T(2k)\,d\delta_T}{2k+h\delta_T}+\delta_T\,d\delta_T=6\alpha\,dt$$

$$\int\frac{2\delta_T k\,d\delta_T}{2k+h\delta_T}+\frac{\delta_T^2}{2}=6\alpha t$$

$$2k\int\frac{\delta_T\,d\delta_T}{2k+h\delta_T}=2k\left\{\frac{1}{h^2}\left[2k+h\delta_T-2k\ln(2k+h\delta_T)\right]\right\}+c$$

$$2k\left[\frac{1}{h^2}\left[2k+h\delta_T-2k\ln(2k+h\delta_T)\right]\right]+c+\frac{\delta_T^2}{2}=6\alpha t$$

$$4k^2/h^2\,[1-\ln 2k]+c=0 \qquad c=4k^2/h^2\,[\ln 2k-1]$$

$$2k\left\{\frac{1}{h^2}\left[2k+h\delta_T-2k\left[\ln(2k+h\delta_T)-\ln 2k\quad+1\right]\right]\right\}+\frac{\delta_T^2}{2}=6\alpha t$$

$$2k\left\{\frac{1}{h^2}\left[\cancel{2k}+h\delta_T-2k\left(\ln\left(\frac{2k+h\delta_T}{2k}\right)\cancel{+1}\right)\right]\right\}+\frac{\delta_T^2}{2}=6\alpha t$$

$$2k\left\{\frac{\delta_T}{h}-\frac{2k}{h^2}\ln\left[\frac{2k+h\delta_T}{2k}\right]\right\}+\frac{\delta_T^2}{2}=6\alpha t$$

$$\checkmark \qquad \checkmark \qquad \checkmark \qquad \checkmark$$

$$\frac{2k}{h}\{\delta_T\}-\left(\frac{2k}{h}\right)^2\ln\left[1+\frac{h}{2k}\delta_T\right]+\frac{\delta_T^2}{2}=6\alpha t$$

$$\frac{\delta_T}{(h/2k)}-\frac{\ln\left[1+\frac{\delta_T}{(2k/h)}\right]}{(h/2k)^2}+\frac{\delta_T^2}{2}=6\alpha t$$

$$2\delta_T-4\ln\left[1+\frac{1}{2}\delta_T\right]+\frac{\delta_T^2}{2}=.6 \qquad \text{for } \alpha=.1 \quad t=1$$

assume $\delta_T=.5$ $\quad 1-4\ln[1.25]+.125=\epsilon$

$1-4(.223)+.125=1.125-.892=.233$

$\delta_T=.6$ $\quad 1.2-4\ln[1.3]+.18=\epsilon$

$1.38-4(.262)=1.38-1.048=.332$

$\delta_T=.7$ $\quad 1.4-4\ln[1.35]+.245=1.645-1.2=.445$

$\delta_T=.8$ $\quad 1.6-4\ln[1.4]+.32=1.92-4(.336)=1.92-1.344=.576$

$\delta_T=.9$ $\quad 1.8-4\ln[1.45]+.405=2.205-4(.372)=2.205-1.488=.717$

$\delta_T=.85$ $\quad 1.7-4\ln[1.425]+.3613=2.0613-4(.354)=2.0613-1.416=.645$

$\delta_T=.82$ $\quad 1.64-4\ln[1.41]+.336=1.976-4(.344)=1.976-1.376=.6$

$\boxed{\delta_T=.82}$

$$\theta=\frac{\left(1+\frac{\delta_T\sigma}{(2k/h)}\right)}{\left(1+\frac{\delta_T}{(2k/h)}\right)}+\frac{(1-\sigma)\left(\frac{h}{k}\right)\delta_T}{\left(1+\frac{\delta_T}{(2k/h)}\right)}\eta-\frac{(1-\sigma)\left(\frac{h}{2k}\right)\delta_T}{\left(1+\frac{\delta_T}{(2k/h)}\right)}\eta^2$$

$$\left(1+\frac{\delta_T}{(2K/h)}\right) = 1+.41 = 1.41 \qquad \sigma = 5$$

$$\theta = \frac{(1+2.05)}{1.41} + \frac{(-4)(.82)}{1.41}\eta - \frac{(-4)(.41)}{1.41}\eta^2$$

$$\theta = \frac{3.05}{1.41} - \frac{3.28}{1.41}\eta + \frac{1.64}{1.41}\eta^2$$

$\theta = 1$ at $\eta = 1$

$\theta = 2.163$ at $\eta = 0$

$\theta = 1.291$ at $\eta = .5$

$$2.163 - 2.326\eta + 1.163\eta^2$$

$$2.163 - 1.163 + .291$$

$$\theta = 1 + (\sigma - 1)\left\{ erfc\left[\frac{\delta_T \eta}{2\sqrt{\alpha t}}\right] - exp\left(\frac{\alpha h^2 t}{k^2} + \frac{\delta_T h}{K}\eta\right) erfc\left(\frac{\sqrt{\alpha t}h}{K} + \frac{\delta_T \eta}{2\sqrt{\alpha t}}\right)\right\}$$

$$\theta = 1 + (\sigma - 1)\left\{ erfc\left[\frac{.82\eta}{.632}\right] - exp\left(.1 + .82\eta\right) erfc\left(.316 + \frac{.82}{.632}\eta\right)\right\}$$

$$= 1 + 4\left\{ erfc(1.3\eta) - exp(.1 + .82\eta)\, erfc(.316 + 1.3\eta)\right\}$$

$$erfc(K) = 1 - erf(K)$$

$$erfc(0) = 1 - erf(0) = 1$$

when $\eta = 0$

$$\theta = 1 + 4\left\{ 1 - exp(.1)\left[1 - erf(.316)\right]\right\}$$

$$erf(.316) = .3446$$

$$= 1 + 4\left\{1 - exp(.1)[.6554]\right\} = 1 + 4\left\{1 - 1.105(.6554)\right\} = 1 + 4\left\{1 - .725\right\}$$

$$= 1 + 4\{.275\} = 2.1$$

$$\theta = 1 + 4\left\{ erfc(.65) - exp(.1 + .41)\, erfc(.966)\right\}$$

when $\eta = .5$

$$1 + 4\left\{.358 - 1.665(.172)\right\} = 1 + 4\left\{.359 - .287\right\} = 1 + 4(.072) = 1.288$$

$$\theta = 1 + 4\left\{ erfc(1.3) - exp(.92)\, erfc(1.616)\right\}$$

when $\eta = 1$

$$= 1 + 4\left\{.066 - 2.504(.0225)\right\} = 1 + 4\left\{.066 - .0564\right\} = 1 + 4\left\{.0096\right\} = 1 + .038 = 1.0$$

let $2\xi = \rho$

$2d\xi = d\rho$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} \qquad T_0 \frac{\partial f(\eta)}{\partial t} = \alpha T_0 \frac{\partial^2 f(\eta)}{\partial z^2}$$

$$\eta = \frac{z}{4\sqrt{\alpha t}} \qquad \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial T}{\partial \eta} - \frac{z}{4}\left((\alpha t)^{-3/2}\alpha\right) = \frac{\partial T}{\partial \eta}\left(\frac{z}{4}\right)\left(-\frac{1}{2}\frac{\alpha}{\alpha t\sqrt{\alpha t}}\right)$$

$$-\frac{\partial T}{\partial \eta}\left(\frac{z}{8t\sqrt{\alpha t}}\right)$$

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial z} = \frac{\partial T}{\partial \eta}\frac{1}{4\sqrt{\alpha t}}$$

$$\alpha \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) = \alpha \frac{\partial}{\partial \eta}\left(\frac{\partial T}{\partial z}\right)\frac{\partial \eta}{\partial z}\frac{1}{16\sqrt{\alpha t}} = \frac{\partial^2 T}{\partial \eta^2}\frac{\alpha}{16\alpha t} \qquad \alpha\left(\frac{\partial^2 T}{\partial \eta^2}\frac{1}{4\sqrt{\alpha t}}\right)\frac{1}{4\sqrt{\alpha t}}$$

$$\frac{\alpha}{16}\frac{\partial^2 T}{\partial \eta^2}\frac{1}{\alpha t} = \frac{\partial^2 T}{\partial \eta^2}\frac{1}{16t}$$

$$-\frac{\partial T}{\partial \eta}\frac{z}{8\cancel{t}\sqrt{\alpha t}} = \frac{\partial^2 T}{\partial \eta^2}\frac{1}{16\cancel{t}}$$

$$-\frac{\partial T}{\partial \eta}\frac{z}{2\sqrt{\alpha t}} = \frac{\partial^2 T}{\partial \eta^2} \qquad T = T_0 f \qquad -\frac{\partial T}{\partial \eta}\frac{\eta}{\cancel{2}} = \frac{\partial^2 T}{\partial \eta^2}\frac{1}{16}8$$

$$-T_0 \frac{\partial f}{\partial \eta}\eta = \frac{T_0 f''}{8}$$

$$-T_0 f'[\eta] = \frac{1}{2}T_0 f''$$

$$\cancel{T_0} f'' + 8\cancel{T_0} f'\eta = 0$$

$$f'' + 2\eta f' = 0 \qquad f'' + 8f'\eta = 0$$

let $f' = Ae^{-\eta^2}$

$$f = A\int_0^\eta e^{-\eta^2}d\eta + B \qquad \cancel{f = 1 + (\sigma - 1)\{erfc(\eta)\}}$$

$$\cancel{\eta = \frac{z}{4\sqrt{\alpha t}} \qquad \frac{\partial T}{\partial \eta}\frac{1}{4\sqrt{\alpha t}} = \frac{\partial T}{\partial z} \qquad \frac{\partial T}{\partial \eta}\frac{-z}{8t\sqrt{\alpha t}} = \frac{\partial T}{\partial t}}$$

$$\cancel{\frac{\partial^2 T}{\partial \eta^2}\frac{1}{16t} = -\frac{\partial T}{\partial \eta}\left[\frac{z}{8t\sqrt{\alpha t}}\right]}$$

$$\cancel{\frac{\partial^2 T}{\partial \eta^2}\frac{1}{8} = \frac{\partial T}{\partial \eta}[\eta]}$$

when $\eta = \infty$ $f = 1$

$$f_z = \frac{h}{k}(\theta - \sigma)$$

$$\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z} = \frac{f'}{2\sqrt{\alpha t}} = \frac{h}{k}(f - \sigma)\Big|_{\eta = 0}$$

$$f'(0) = (2\sqrt{\alpha t})\frac{h}{k}(f(0) - \sigma)$$

$$A = 2\sqrt{\alpha t}\,\frac{h}{k}(B - \sigma) \qquad A = 2\sqrt{\alpha t}\,\frac{h}{k}(B - \sigma)$$

$$f = 2\sqrt{\alpha t}\,\frac{h}{k}\,(B-\sigma) \qquad \text{if } h \to \infty \qquad z=0 \quad T = T_\infty \qquad T_\infty/T_0$$

$$z \to \infty \quad T = T_0 \qquad 1$$

$$\frac{T}{T_0} = B = \sigma$$

$$(1-\sigma) = A\int_0^\eta e^{-\eta^2}\,d\eta = A\frac{\sqrt{\pi}}{2}\,erf(\eta)$$

$$\frac{2}{\sqrt{\pi}}\frac{(1-\sigma)}{erf(\eta)} = A \qquad f = \frac{2}{\sqrt{\pi}}\frac{(1-\sigma)}{erf(\eta)}\int_0^\eta e^{-\eta^2}\,d\eta + \sigma$$

$$f = A\int_0^\eta e^{-\eta^2}\,d\eta + B \qquad \eta = z = 0 \quad f = \sigma$$

$$\eta = \infty \quad f = 1$$

$$f = A\int_0^\eta e^{-\eta^2}\,d\eta + \sigma$$

$$(1-\sigma) = A\int_0^\eta e^{-\eta^2}\,d\eta = \cancel{\frac{\sqrt{\pi}A}{2}}\;\underbrace{\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}\,d\eta}_{1} = \frac{\sqrt{\pi}}{2}A\,erf(\eta)$$

$$A = 1-\sigma$$

$$f = (1-\sigma)\,erf(\eta) + \sigma$$

$$= erf(\eta) + \sigma\,erfc(\eta)$$

$$\frac{z}{2\sqrt{\alpha t}} = \eta \qquad z = 2\eta\sqrt{\alpha t}$$

$$T = T_0 + (T_\infty - T_0)\,erfc(\eta)$$

$$\frac{T}{T_0} = 1 + (\sigma - 1)\,erfc(\eta) \qquad \text{for } \sigma = 2 \qquad 1 - erf(\eta) = erf(c)$$

$$1.01 = 1 + erfc(\eta) = 2 - erf(\eta) \qquad erf\,\eta = .99 \qquad \eta = 1.825$$

$$\delta_T = 13.65\sqrt{\alpha t} \qquad \eta = \frac{z}{2\sqrt{\alpha t}}$$

$$1.01 = 1 + 4\,erfc(\eta) = 1 + 4 - 4\,erf(\eta) = 5 - 4\,erf(\eta) \qquad \sigma = 5$$

$$\frac{3.99}{4} = erf(\eta) \qquad .9975 = erf(\eta) \qquad \eta = 2.157$$

$$\delta_T = 4.314\sqrt{\alpha t}$$

$$1.01 = 1 + 99\,erfc(\eta) = 1 + 99 - 99\,erf(\eta) = 100 - 99\,erf(\eta) \qquad \sigma = 100$$

$$98.99 \qquad erf(\eta) = \frac{98.99}{99} \qquad \frac{.01}{99} = 10^{-2}(1.01)\times 10^{-2} = 1.01\times 10^{-4} = .000101 \qquad .9999$$

$\eta = 2.8 \qquad \delta_r = 5.6\sqrt{\alpha b}$

$\dot{u} = a\cos\theta - \frac{ku}{m} \qquad u = e^{-k/m} \qquad \dot{u} = \frac{k}{m} e^{-k/m}\left(+\frac{\beta}{m^2}\right)$

$\dot{u} + \frac{ku}{m} = 0 \qquad (m_1 - \beta t)\dot{u} + ku = 0$

$u = \sum a_n t^{n+r}$

$m_1 \sum^{\infty} (n+r) a_n t^{n+r-1} - \beta (n+r) a_n t^{n+r} + k a_n t^{n+r} = 0$

$m_1 \sum_{n=0} (r+n+1) a_{n+1} t^{r+n} - \beta \sum_{n=1} (n+r) a_n t^{n+r} + k \sum_{n=1} a_n t^{n+r}$

$m_1 (r+1) a_1 + \sum_{n=1}^{\infty} \left((r+n+1) a_{n+1} m_1 - \beta a_n + k a_n\right) t^{n+r} = 0$

$m_1 (r+1) a_1 = 0 \qquad a_1 = 0 \quad \text{or} \quad r = -1$

$m_1 (r+n+1) a_{n+1} - \beta (n+r) a_n + k a_n = 0 \qquad + a_n (k - \beta(n+r))$

$r = -1 \qquad m_1 n a_{n+1} + a_n (k - \beta(n-1)) = 0$

$a_{n+1} = \frac{a_n (\beta[n-1] - k)}{m_1 n}$

$a_2 = \frac{a_1 (-k)}{m_1} \qquad a_3 = \frac{a_2 (\beta - k)}{2 m_1} \qquad a_4 = \frac{a_3 (2\beta - k)}{3 m_1} \qquad a_{n+1} = \frac{-a_1 (k)(\beta - k)(2\beta - k) \cdots ((n-1)\beta - k)}{(n!) m_1^n}$

$a_3 = \frac{-a_1 k (\beta - k)}{2 m_1^2} \qquad = \frac{-a_1 k (\beta - k)(2\beta - k)}{3!\, m_1^3}$

$m\dot{u} + ku = a\cos\theta\, (t^0)$

$m_1 (r+1) a_1 t^r + m_1 (r+2) a_2 t^{r+1} - \beta (r+1) a_1 t^{r+1} + k a_1 t^{r+1} + \sum_{n=2}$

$\text{if } r = 0 \qquad a_1 m_1 = c\beta\cos\theta \qquad m_1 2 a_2 - \beta a_1 + k a_1 = 0$

$a_2 = \frac{a_1 (\beta - k)}{2 m_1} = \frac{c\beta\cos\theta (\beta - k)}{2 m_1^2}$

$$\boxed{a_{n+1} = a_n \frac{(\beta n - \beta - k)}{m_1 n} \quad \text{where } a_1 = \frac{c\beta\cos\theta}{m_1}}$$

$r = 0 \qquad 3 m_1 a_3 - 2\beta a_2 + k a_2 = 0$

$a_3 = \frac{a_2 (2\beta - k)}{3 m_1} = \frac{c\beta\cos\theta (\beta - k)(2\beta - k)}{3!\, m_1^3}$

$$\boxed{a_{n+1} = a_n \frac{(\beta n - \beta - k)}{m_1 n} \quad \text{where } a_1 = a_1}$$

$r = -1$

$a_n = \frac{(c\beta\cos\theta)(\beta - k)(2\beta - k) \cdots ((n-1)\beta - k)}{n!\, m_1^n}$

1

Cesar Leung AE 334

1a.

$$\int_0^{\delta_T} \frac{\partial T}{\partial t} dz = \alpha \int_0^{\delta_T} \frac{\partial^2 T}{\partial z^2} dz$$

$$\frac{\partial}{\partial t}\int_0^{\delta_T} T dz - T\Big|_{\delta_T} \frac{\partial \delta_T}{\partial t} = \alpha \frac{\partial T}{\partial z}\Big|_{\delta_T} - \alpha \frac{\partial T}{\partial z}\Big|_0 \checkmark$$

from b.c. $k\frac{\partial T}{\partial z}\Big|_{z=0} = h(T-T_\infty)$ and $\frac{\partial T}{\partial z}\Big|_{\delta_T} = 0$, $T\Big|_{\delta_T} = T_0$, therefore

$$\frac{\partial}{\partial t}\int_0^{\delta_T} T dz - T_0 \frac{\partial \delta_T}{\partial t} = -\frac{\alpha h}{k}(T-T_\infty) \quad ⓐ \checkmark$$

making substitutions that $T = \theta T_0$, $T_\infty = T_0 \sigma$, $z = \delta_T \eta$ then

$$T_0 \frac{\partial}{\partial t}\left(\delta_T \int_0^1 \theta d\eta\right) - T_0 \frac{\partial \delta_T}{\partial t} = \frac{\alpha h}{k}(\sigma - \theta) = -\frac{\alpha}{\delta_T} T_0 \frac{\partial \theta}{\partial \eta}\Big|_{\eta=0} \quad ⓑ$$

Assume $\theta = A + B\eta + C\eta^2$ ✓ B.C. 1. @ $\eta = 1$ $\theta = 1$

$\frac{\partial \theta}{\partial \eta} = B + 2C\eta$ 2. @ $\eta = 1$ $\frac{\partial \theta}{\partial \eta} = 0$

3. @ $\eta = 0$ $\frac{k}{\delta_T}\frac{\partial \theta}{\partial \eta} = h(\theta - \sigma)$ ✓

using 2: $B = -2C$

using 1: $1 = A - 2C + C = A - C$ $C = A - 1$

using 3: $\frac{k}{\delta_T}[-2C] = h[A - \sigma]$ $A = \frac{-2kC + \sigma h \delta_T}{h\delta_T}$

from these relationships

$$C = -\frac{(1-\sigma)h\delta_T}{2k + h\delta_T} \qquad B = \frac{2(1-\sigma)h\delta_T}{2k + h\delta_T} \qquad A = \sigma + \frac{2k(1-\sigma)}{2k + h\delta_T}$$

$$T/T_0 = \theta = \left(\frac{2k + h\delta_T\sigma}{2k + h\delta_T}\right) + \frac{2(1-\sigma)h\delta_T}{2k + h\delta_T}\eta - \frac{(1-\sigma)h\delta_T}{2k + h\delta_T}\eta^2 \quad ⓒ$$

1c. To find δ_T

$$\int_0^1 \theta d\eta = 1 + \frac{(\sigma - 1)h\delta_T}{6k + 3h\delta_T}, \quad \frac{\partial \theta}{\partial \eta}\Big|_{\eta=0} = \frac{6(1-\sigma)h\delta_T}{6k + 3h\delta_T}$$

Substitution in (b)

$$\frac{\partial}{\partial t}\left[\delta_T + \frac{(\sigma-1)h\delta_T^2}{6k+3h\delta_T}\right] - \frac{\partial}{\partial t}\delta_T = -\frac{\alpha}{\delta_T}\left[\frac{6(1-\sigma)h\delta_T}{6k+3h\delta_T}\right] \quad (d)$$

Working with (d) the following differential form is obtained

$$\frac{2k}{2k+h\delta_T}\,\delta_T\,d\delta_T + \delta_T\,d\delta_T = 6\alpha\,dt \quad (e)$$

Integrating & setting $t=0$ $\delta_T=0$ to get the constant leads to

$$\frac{\delta_T}{(h/2k)} - \frac{\ln\left[1+\frac{\delta_T}{(2k/h)}\right]}{(h/2k)^2} + \frac{\delta_T^2}{2} = 6\alpha t \quad (f)$$

for $\alpha = .1$ $t=1$ $k/h = 1$

$$2\delta_T - 4\ln\left[1+\tfrac{1}{2}\delta_T\right] + \frac{\delta_T^2}{2} - .6 = \epsilon \text{ (should be zero)}$$

Assume $\delta_T = .7$ $\quad\epsilon = -.155$

$\delta_T = .8$ $\quad\epsilon = -.024$

$\delta_T = .9$ $\quad\epsilon = .117$

$\delta_T = .85$ $\quad\epsilon = .045$

$\delta_T = .82$ $\quad\epsilon = 0$

$\delta_T = .82$

Subst. this value into (c)

$$\theta = \frac{\left(1+\frac{\delta_T\sigma}{(2k/h)}\right)}{\left(1+\delta_T/(2k/h)\right)} + \frac{(1-\sigma)\left(\frac{h}{k}\right)\delta_T}{\left(1+\frac{\delta_T}{(2k/h)}\right)}\eta - \frac{(1-\sigma)\left(\frac{h}{2k}\right)\delta_T}{\left(1+\frac{\delta_T}{(2k/h)}\right)}\eta^2$$

$$\theta = 2.163 - 2.326\eta + 1.163\eta^2 \quad (g)$$

@ $\eta = 0$ $\quad\theta = 2.163$

@ $\eta = .5$ $\quad\theta = 1.291$

@ $\eta = 1$ $\quad\theta = 1.000$

The exact transform solution is

$$\theta = 1 + (\sigma - 1)\left\{ \operatorname{erfc}\left[\frac{\delta_T \eta}{2\sqrt{\alpha t}}\right] - \exp\left(\frac{\alpha h^2 t}{k^2} + \frac{\delta_T h}{k}\eta\right)\operatorname{erfc}\left(\frac{\sqrt{\alpha t}\, h}{k} + \frac{\delta_T \eta}{2\sqrt{\alpha t}}\right)\right\} \quad (h)$$

placing the values obtained

$$\theta = 1 + 4\left\{\operatorname{erfc}(1.3\eta) - \exp(.1 + .82\eta)\operatorname{erfc}(.316 + 1.3\eta)\right\} \quad (i)$$

when $\eta = 0$ $\quad \operatorname{erfc}(0) = 1$, $\operatorname{erfc}(.316) = .6554$, $\exp(.1) = 1.105$

$\therefore \quad \theta = 2.1$

when $\eta = .5$ $\quad \operatorname{erfc}(.65) = .358$, $\operatorname{erfc}(.966) = .172$, $\exp(.51) = 1.665$

$\therefore \quad \theta = 1.288$

when $\eta = 1$ $\quad \operatorname{erfc}(1.3) = .066$, $\operatorname{erfc}(1.616) = .0225$, $\exp(.92) = 2.504$

$\therefore \quad \theta = 1.038$

	Integral	Exact	Numerical
$\eta = 0$	2.163	2.1	2.112
$\eta = .5$	1.291	1.288	1.296
$\eta = 1.$	1.0	1.038	1.044

2a.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} \qquad \eta = \frac{z}{4\sqrt{\alpha t}} \qquad (a)$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial T}{\partial \eta}\left[\frac{-z\alpha}{8\alpha t\sqrt{\alpha t}}\right] = -\frac{\partial T}{\partial \eta}\frac{\eta}{2t}$$

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial z} = \frac{\partial T}{\partial \eta}\frac{1}{4\sqrt{\alpha t}}, \quad \frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial \eta \partial z}\frac{\partial \eta}{\partial z} = \frac{\partial^2 T}{\partial \eta^2}\frac{1}{4\sqrt{\alpha t}}\left(\frac{1}{4\sqrt{\alpha t}}\right)$$

$\therefore$ (a) $\quad -\frac{\partial T}{\partial \eta}\frac{\eta}{2t} = \frac{\partial^2 T}{\partial \eta^2}\frac{1}{16t}$ $\quad$ if $T = T_0 f(\eta)$ then

$$-f'\eta = f''/8$$

$$f'' + 8\eta f' = 0 \qquad \text{let } f' = Ae^{-4\eta^2}$$

then

$$f = A\int_0^{\eta} e^{-4\xi^2}\,d\xi + B \qquad \text{let } 2\xi = \rho \quad 2d\xi = d\rho$$

$$\therefore \quad f = A\int_0^{\rho'} e^{-\rho^2}\,d(\rho/2) + B = \frac{A}{2}\int_0^{\rho'} e^{-\rho^2}\,d\rho + B$$

at $\eta = \xi = \rho/2 = 0 \quad f = \sigma$

at $\eta = \xi = \rho/2 = \infty \quad f = 1 \qquad \therefore \quad f = (1-\sigma)\,\mathrm{erf}(2\eta) + \sigma$

$$f = \mathrm{erf}(2\eta) + \sigma\,\mathrm{erfc}(2\eta)$$

2b.

$$T = T_0 + (T_\infty - T_0)\,\mathrm{erfc}(\eta)$$

$$T/T_0 = 1 + (\sigma - 1)\,\mathrm{erfc}(\eta) = 1 + (\sigma-1)(1-\mathrm{erf}(\eta))$$

$$= \mathrm{erf}(\eta) + \sigma(1-\mathrm{erf}(\eta))$$

at $T/T_0 = 1.01 \quad \sigma = 2 \quad \mathrm{erf}(\eta) = .99 \quad \eta = 1.825$

$\therefore \quad \delta_T = 2\eta\sqrt{\alpha t} = 3.65\sqrt{\alpha t}$

at $T/T_0 = 1.01 \quad \sigma = 5 \quad \mathrm{erf}(\eta) = .9975 \quad \eta = 2.157$

$\therefore \quad \delta_T = 2\eta\sqrt{\alpha t} = 4.314\sqrt{\alpha t}$

at $T/T_0 = 1.01 \quad \sigma = 100 \quad \mathrm{erf}(\eta) = .9999 \quad \eta = 2.8$

$\therefore \quad \delta_T = 2\eta\sqrt{\alpha t} = 5.6\sqrt{\alpha t}$

A > 115

A

EXAMINATION BOOK

Name Cesar Levy

Subject AE 334

Instructor Prof. Ruben

Exam Seat No. Section A

Date 3-23-72 Grade

C. P. LESH PAPER CO., INDIANAPOLIS, IND.

1. 92

2. 37/60

129/165

a) 1. Conductivity – the means by which information (such as temperature, heat, energy) is passed in a body i.e. In a solid conduction is established by means of free electrons. In a liquid by molecular attraction. In a gas by means of inter molecular collision $q = kT_n$

b) In Convection the molecules must exhibit a mean velocity while in conduction the velocity of the elements involved is much smaller & relies on intermolecular forces to pass on information ~~very small~~.
10 In convection there are two methods by which this is achieved: 1) Natural convection where the buoyancy effect causes a change in density of the molecules and thus a velocity is induced by this effect. 2) Forced convection where an external

c) source will give the molecules a mean velocity. Three b.c.
10 that can occur is 1) $T = T_s$ T_s is surface temperature

2) $\frac{\partial T}{\partial n} = 0$ insulated wall n is the outward normal to surface

3) $k\frac{\partial T}{\partial n} = h(T - T_\infty)$ where convection & conduction are balanced h is the convective coeff & k is the conductive coefficient.

d) i The rate of change of internal energy & kinetic energy in a volume equals
10 the rate at which work is being done on the volume by the external forces (i.e. pressure, shears, body) + the rate at which heat is added to the volume by conduction + the rate at which heat is added by ~~other~~ means other than conduction.

ii The rate of change of heat moving ~~out~~ (in) of the walls of a body due to conduction is equal to the change in energy in the body ~~(produced by means other than conduction)~~

e) In steady state conduction method of successive approximation is used whereby the values of the boundary are taken into account
10 this method uses the fact that after a new approximation is found for a temperature at a point it is immediately used in the next

calculation where it has direct effect ie

$$\begin{array}{ccc} & \overset{\circ}{I,J+1} & \\ \overset{\circ}{I-1,J} & \overset{\circ}{I,J} & \overset{\circ}{I+1,J} \\ & \overset{\circ}{I,J-1} & \end{array} \qquad \overset{N}{T}_{I,J} = \frac{1}{4}\left(\overset{O}{T}_{I,J+1} + \overset{N}{T}_{I,J-1} + \overset{O}{T}_{I+1,J} + \overset{N}{T}_{I-1,J}\right)$$

where superscript N indicates new value
" O " old "

The boundary values must be satisfied ✓

8 In unsteady flow a marching technique is used whereby a grid is set up & a dummy pt is picked in the Δt direction & taylor's ~~Expansion is used~~ initial value problem as in the method of successive replacements. The differential equation is satisfied & is found to be a function of the variable θ. If $\theta = 0$ the method is fully explicit but the error is of order $(\Delta t, \Delta y^2)$. If $\theta = 1/2$ method is implicit-explicit error is $(\Delta t^2, \Delta y^2)$ - best. If $\theta = 1$ fully implicit error is $(\Delta t, \Delta y^2)$. Both partial & fully implicit methods are stable unconditionally. Explicit method is conditionally stable depending on Δt, Δy^2. These are initial value ~~for~~ type problems.

$$Y \uparrow \quad \overset{\circ}{I,J} \xrightarrow{\theta\Delta t} \overset{\circ}{C} \xrightarrow{(1-\theta)\Delta t} \overset{\circ}{I+1,J} \quad \rightarrow t \qquad \leftarrow \Delta t \rightarrow$$

$\Delta t \cdot \theta$ is distance bet. c & pt I,J

f -4 i. If b.c. are such that they are given at $\pm\infty$ fourier transforms are used (used in steady state - fourier)

see after part 9

If b.c. are such that they are given at 0 & ∞ Laplace is used. They apply in these manners because

10 of the way the transform is defined (i e limits of integration on the transforms) (used in unsteady state - Laplace transforms)

ii They are useful because they take the partial diff equation

√ & transform them to ordinary diff equations which are easier to handle

iii All b.c. are satisfied by taking these conditions

4 & transforming them before applying them to the transformed ordinary diff equation

some are involved in transform

iv – inversion of the transform solution tend to become messy
5 & complicated as shown by the problem involving $h \to \infty$
$T_s - T_\infty \to 0$ & $k \to \infty$ $\frac{\partial T}{\partial n} \to 0$ for finite $\dot{q}$ discussed in
class.

9 i if $G/Re^2 << 1$ frictional heating is important
0 $\boxed{E = O(1)}$ G is the Grashof Number
10 Re " " Reynolds Number

G/Re^2 shows the effect of buoyancy to effect of viscosity

5 ii If $G/Re^2 \geq O(1)$ the buoyancy effect becomes very
important

4 iii If the thermal boundary layer thickness is much or the same size or
larger than velocity b.l. thickness (the prandtl number smaller $\to 0$)
whenever T has a gradient as in or $Pr = O(1)$

iv If the velocity b.l. thickness is much larger than
the thermal bl thickness ($Pr = \frac{\mu c_p}{k} >> 1$)
2 in a b.l when $Re >> 1$ or in general when $Re << 1$.

f i In unsteady state usually t is given at $t = 0$
and the spacial coordinates usually have b.c. at $\pm\infty$
for unsteady state cond. ∴ use Laplace.
In steady state b.c. on spacial coord are usually
given at $\pm\infty$ ∴ use fourier

EXAMINATION BOOK

Name Cesar Levy

Subject AE 334

Instructor Prof Rubin

Exam Seat No. ______ Section A

Date 3-23-72 Grade ______

C. P. LESH PAPER CO., INDIANAPOLIS, IND.

37

OPEN BOOK PROBLEM

Upper and Lower channels are Couette flow solutions with $u_{yy}=0$ and $u=0$ at $y=\pm H/2$, $u=U_1$ $y=\frac{H}{2}+L$

$u=U_0$, $y=-\frac{H}{2}-L$.

$$\therefore \quad u_T = U_1\,\frac{y-H/2}{L} \qquad u_B = -\frac{U_0}{L}\left[y+H/2\right]$$

Non-dimensionalizing u with U_1, T with T_1, y with L, $\frac{H}{L}=h$

$\beta = T_0/T_1$, $\alpha = U_0/U_1$

$$u_T = y-h/2 \ , \quad u_B = -(y+h/2)\,\alpha$$

2) Governing Equations For T $\qquad E_1 = \frac{U_1^2}{c_pT_1} \quad E_0 = \frac{U_0^2}{c_pT_0} \quad P_r = \mu c_p/k$
(See Class Notes)

Fluid Top: $\qquad T_{yy} = -E_1P_r \quad \therefore \quad T_{F_T} = A + By - E_1P_r\,y^2/2$

Fluid Below: $\qquad T_{yy} = -E_0P_r\beta \quad \therefore \quad T_{F_B} = A' + B'y - E_0P_r\beta\,y^2/2$

Solid $\qquad T_{yy} = 0 \quad \therefore \quad T_S = Cy + D$

3) Boundary Conditions

At $y = 1+h/2$, $\quad T_{F_T}=1$, $\quad$ At $y=-1-h/2 \quad T_{F_B}=\beta$

At $y = h/2$, $\quad T_{F_T} = T_S$, $\quad k(T_{F_T})_y = k_s(T_S)_y$

At $y=-h/2$, $\quad T_{F_B}=T_S$, $\quad k(T_{F_B})_y = k_s(T_S)_y$

$$1 = A + B(1+h/2) - E_1 P_r (1+h/2)^2/2 \,^*, \quad \beta = A' - B'(1+h/2) - E_0 P_r \beta (1+h)$$

$$-\frac{Ch}{2} + D = A + Bh/2 - E_1 P_r h^2/8 \,^*, \quad -Ch/2 + D = A' - B'h/2 - E_0 P_r \beta h^2$$

Therefore $\frac{Ch}{2} + D = A + Bh/2 - E_1 P_r h^2/8$

$$k_s C = k(B - E_1 P_r h/2) \,^*, \quad k_s C = k(B' + E_0 P_r \beta h/2)$$

$$A = 1 - B(1+h/2) + E_1 P_r (1+h/2)^2/2$$

so

$$A' = \beta + B(1+h/2) + E_0 P_r \beta (1+h/2)^2/2 - (1+h/2)\, h/2\, P_r (E_0 \beta + E_1)$$

$$B' = B - \frac{P_r h}{2}(E_0 \beta + E_1), \quad C = k/k_s (B - E_1 P_r h/2)$$

$$D = -B\left(1 + \frac{k}{k_s}\frac{h}{2}\right) + 1 + \frac{h^2}{4} E_1 P_r \left(\frac{k}{k_s} - \frac{1}{2}\right) + E_1 P_r (1+h/2)^2/2$$

and

$$B = \frac{(1-\beta) + \frac{P_r E_1 h}{2}\left(h\frac{k}{k_s} + \left(1 + \frac{E_0}{E_1}\beta\right) + (1+h)\left(1 - \frac{E_0 \beta}{E_1}\right)\right)}{h k/k_s + 2}$$

and at the solid surface

$$q_B = q_T = -\frac{k_s}{L} T_1 C = -\frac{k}{L} T_1 (B - E_1 P_r h/2)$$

(1) No frictional heating $\quad E_0 = E_1 = 0$

$$B = \frac{1-\beta}{2 + h k/k_s}, \quad A = 1 - \frac{(1-\beta)(1+h/2)}{(2 + h k/k_s)}, \quad C = \frac{(1-\beta) k/k_s}{2 + h k/k_s}$$

$$A' = \beta + \frac{(1-\beta)(1+h/2)}{2 + h k/k_s}, \quad B' = B, \quad D = 1 - 2(1-\beta) =$$

$$q = -\frac{k}{L} T_1 (1-\beta)\Big/(2 + h k/k_s), \text{ if } \beta < 1 \quad q < 0$$

heat flows from hotter of two plates toward colder

$$= 0 \quad \text{if } \beta = 1.$$

$$T_{h/2} = C\frac{h}{2} + D = \frac{(1-\beta) k h/2k_s}{2 + h k/k_s} + 2\beta - 1 = 1 \quad \text{if } \beta = 1$$

(2) $U_0 = -U_1$, $T_0 = T_0$ + $U_0 = U_1$, $T_0 = T_1$ are same case as U only enters in E as square quantity $\beta = 1$, $E_0 = E_1$

$$-B' = B = \frac{P_r E h}{2}, \quad C = 0 \quad \therefore q = 0, \quad A = A' = 1 + \frac{EP_r}{2}\left(1 - \frac{h^2}{4}\right)$$

$$D = T_{SOLID} = 1 + \frac{EP_r}{2}$$

3) With no plate below b.c. at $y = -h/2$ is

$$\tilde{h}(T - T_0) = -k_s (T_y)_s \qquad \tilde{h} = \text{convection coefficient}$$

or non-dimensionalized $\tilde{h}(T - \beta) = -\frac{k_s}{L}(T_y)_s$

or

$$\tilde{h}\left(C\left(-\frac{h}{2}\right) + D - \beta\right) = -\frac{k_s}{L} C$$

$$\therefore \quad D - \beta = C\left(\frac{h}{2} - \frac{k_s}{L\tilde{h}}\right)$$

$$\boxed{C = \frac{D - \beta}{\frac{h}{2} - k_s/L\tilde{h}}}$$

Conditions at $y = h/2$ and $y = h/2 + 1$ are the same as before (The starred conditions on page 2)

$$\therefore \quad A = 1 - B(1 + h/2) + E_1 P_r (1 + h/2)^2/2$$

$$C = \frac{k_s}{k}(B - E_1 P_r h/2) \qquad D = \beta + \left(\frac{h}{2} - \frac{k_s}{L\tilde{h}}\right)\left(\frac{k_s}{k}\right)\left(B - E_1 P_r h/2\right)$$

and

$$B = \frac{(1 - \beta) + \frac{E_1 P_r}{2}\left((1 + h) - \frac{k_s^2}{L\tilde{h}k}\right)}{\left[1 + \frac{k_s h}{k} - \frac{k_s^2}{L\tilde{h}k}\right]}$$

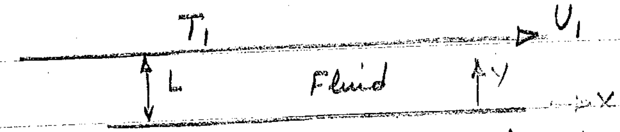
1. The equations:

18 $\rho u u_{,x} + \rho v u_{,y} = -p_{,x} + F_{,x} + \mu(u_{,xx} + u_{,yy})$, $\quad u_{,x} + v_{,y} = 0$

$\rho u v_{,x} + \rho v v_{,y} = -p_{,y} + F_{,y} + \mu(v_{,xx} + v_{,yy})$ ✓

$$\rho C_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \Phi + \dot{Q} + \nabla\cdot k(\nabla T)$$

where $\Phi = 2\mu[u_{,x}^2 + v_{,y}^2] + \mu[(u_{,y} + v_{,x})^2]$ ✓



this problem is of the couette flow type ✓

1. ~~fully developed~~ one dim flow is ~~assumed~~ given $u_{,x} = 0$ $\quad u_{,xx} = 0$ & F_x, F_y are neg.
2. Continuity sets $V_{,y} = 0$ but because @ $y=0$ $V=0$ $\quad V=0$ everywhere
3. the continuity eq lead to $p_{,x} = \mu u_{,yy}$ ✓ $\quad p = p(x)$ only

from this a solution for u is obtained ✓

4. the energy eq symplifies to $\rho C_p(uT_{,x} + vT_{,y}) = u\,p'(x) + \mu u_{,y}^2 + kT_{yy}$

since plates are infinite no variation i temp in x direct can be existing this leads to

5. $u\,p'(x) + \mu u_{,y}^2 + kT_{yy} = 0$ $\quad$ u is known, $p'(x)$ is known

one-dim. $u_{,y}$ is known $\therefore$ T can be found.

$p'(x) \equiv 0$

6. Solution to u must satisfy $u = U_1$ at $y = L$

$u = 0$ at $y = 0$ ✓

solution to T must satisfy $T = T_1$ at $y = L$ ✓

$T = T_{s_1}$ at $y = 0$ ✓

In the solid bc are at $y = H/2$ $\quad T = T_{s_1}$ $\forall x$ ✓

$y = -H/2$ $\quad T = T_{s_0}$ $\forall x$ ✓

at $y = \pm H/2$ $\quad kT_{yy}\Big|_0 + \mu u_{,y}^2\Big|_0 = 0$ ✓

~~T must satisfy k∇T~~

Using $T_{xx} + T_{yy} = 0$ & fourier transform

obtain a solution for temperature $\quad T = Ay + B$

For the lower fluid flow set up coord syste $\downarrow y$

and apply same type of equat as upper flow however use $U_o = u$ at $y = L$

$T_o = T$ at $y = L$

The wall temperature of the solid are found by the amount of heat conducted as the surfaces so that no discont exists i.e. $k \frac{\partial T}{\partial y}\Big|_{\substack{\text{fluid} \\ y=0}} = k_s \frac{\partial T}{\partial y}\Big|_{\substack{\text{wall} \\ y=\pm\frac{H}{2}}}$

2. ~~For heat transfer assume~~ $p'(x) \equiv 0$

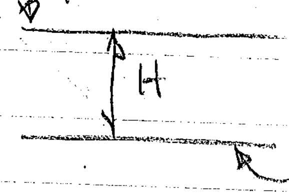
10 $\therefore$ the energy eq reduces to $T_{yy} = -P_r E u_y^2$

the solution to which can be found

now $\dot{q}/A = -k_s \frac{\partial T}{\partial y} = -\frac{T_1}{h}\left[-T_{s_1} + 1 + \frac{P_r E}{2}(1-2y)\right]$ at upper $y = 0$

$= +\frac{T_o}{h}\left[-T_{s_o} + 1 + \frac{P_r E}{2}(1-2y)\right]$ at lower $y = 0$

nondimensionalize wrt T_1 & T_o respectively



at surfaces $y = 0$

3. if ~~$\mu \to 0$~~ $\mu \neq 0$ $E \to 0$ $T_{yy} = 0$ $\frac{\partial T}{\partial y} = \text{const}$ $k\frac{\partial T}{\partial y} = \text{const}$ at both surfaces depending on the coff found from the energy balance

$k_s \frac{\partial T}{\partial y}\Big|_{y=\pm H/2} = k\frac{\partial T}{\partial y}\Big|_{y=0}$

b

c The temperature of the solid will be a constant so that

7 $\frac{\partial T}{\partial y} = 0$ at the walls.

1. solid $T = T_0$ at $y = -H/2$

2 for the fluid $u = y$ nondimens. will refer to $u \Rightarrow \frac{u}{U_1}$ $y \Rightarrow \frac{y}{L}$

T nond. will resp to T_0

$$T = T_{s_1}(1-y) + y + \frac{PrE}{2}[y - y^2]$$

$T = T_{s_1}$ at wall

$$\dot{q}/A = -\frac{kT_1}{h}\left[-T_{s_1} + 1 + \frac{PrE}{2}\right] \text{ at wall} \quad \text{as before}$$

$T_{xx} + T_{yy} = 0$ in body use $T = T_0$ at lower face

$T_x = 0$ at $y = -H/2$ $\forall x$

$k T_{yy} + \mu u_{,y}^2 = 0$ at upper face

also $k \frac{\partial T}{\partial y}\Big|_{y=0} = k_s \frac{\partial T}{\partial y}\Big|_{y=+H/2}$

AE-334

Homework #4

Connection

1. Determine the temperature field in fully developed incompressible Poiseuille flow where the wall temperatures are prescribed as T_1 and T_2. Discuss the heat transfer to the surface.

2. Determine the temperature distribution on a flat plate in a low speed flow by the integral method. $\mu c_p/k$ $\delta < \delta_T$

   (a) Consider $P_r=3/4$ and evaluate δ_T; compare with δ. Discuss the effects of frictional heating and compare with the results given in Schlichting from the similarity solution.

   (b) For $P_r=1$ and neglecting frictional heating show that if the same order profiles are chosen for u and T that

$$\frac{T}{T_\infty}=(1-\overline{T}_w)\frac{u}{u_\infty}+\overline{T}_w \quad \text{and} \quad \delta=\delta_T \qquad \overline{T}_w=\frac{T_w}{T_\infty}$$

3. By non-dimensionalizing the boundary layer equations show that the buoyancy term (in terms of $T-T_\infty$) can be neglected when $G/R^2 \ll 1$. What is a physical description of the Grashof number. (Ratio of what effects?) buoyancy to viscous.

4. Governing Equations: Show that for a constant density flow (liquid or solid) $c_p = c_v$ and the energy equations in terms of h and e become identical.

   (Hint: Equation of state is $\rho =$ constant; $p=\rho RT$ no longer applies. Using one of the Maxwell relationships and the 1st and 2nd laws of thermodynamics, prove that $(\partial h/\partial p)_T=1/\rho$ and therefore,

$$dh=C_p d_T+\frac{1}{\rho}dp = de+\frac{1}{\rho}dp = C_v dT+\frac{1}{\rho}dp$$

   and $C_p=C_v$).

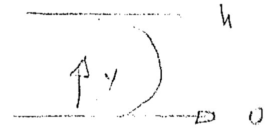
$$-\frac{Dp}{Dt}+\frac{\partial p}{\partial t}+\bar{q}\cdot\nabla p$$

$$\rho\frac{De}{Dt} = -p\nabla\cdot\bar{q}+\Phi+\dot{Q}+\dot{q}$$

$$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt}+\Phi+\dot{Q}+\dot{q}$$

$$-p\nabla\cdot\bar{q}-\bar{q}\cdot\nabla p = -\frac{Dp}{Dt} + p/\rho\frac{D\rho}{Dt} + \frac{\partial p}{\partial t}$$

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t}+p\nabla\cdot\bar{q}+\bar{q}\cdot\nabla p \quad 0$$



$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ since fully developed $\frac{\partial u}{\partial x} = 0$ $\therefore$ $\frac{\partial v}{\partial y} = 0$ $v = c$ @ wall $v = 0$

$\therefore$ $v = 0$ everywhere

mom x $\quad uu_x + vu_y = -\frac{p_{,x}}{\rho} + \nu\,(u_{,xx} + u_{,yy}) \Longrightarrow \quad -p_{,x} + \mu\,(u_{,yy}) = 0$

mom y shows $p_{,y} = 0$ $\quad p \neq f(y)$ $\quad \therefore \quad u_{,yy} = \frac{1}{\mu}\frac{dp}{dx}$

$u_{,y} = \frac{1}{\mu}\frac{dp}{dx}(y + A) \qquad u = \frac{1}{\mu}\frac{dp}{dx}\left(\frac{y^2}{2} + Ay + B\right)$

at $y = 0$ $\;u = 0$ $\quad$ at $y = h$ $\;u = 0$ $\qquad B = 0$

$u = \frac{1}{\mu}\frac{dp}{dx}\left(\frac{h^2}{2} + Ah\right) = 0 \qquad A = -\frac{h}{2} \quad \therefore \quad u = \frac{1}{\mu}\frac{dp}{dx}\left((y-h)\frac{y}{2}\right)$

$u = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - yh\right) \qquad v = 0$

@ $y = 0$ $\;T = T_1$ $\qquad$ @ $y = h$ $\;T = T_2$

$\cancel{u_{med} = \frac{1}{2\mu}\frac{dp}{dx}\left(\frac{h^2}{4} - \frac{h^2}{2}\right) = \frac{1}{2\mu}\frac{dp}{dx}\left(-\frac{h^2}{4}\right)}$

$\rho C_p\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right] = \frac{Dp}{Dt} + \Phi + \dot{Q} + \nabla\cdot(k\nabla T)$

since $v = 0$ & fully developed

$\Phi = \mu\, u_{,y}^2 \qquad \dot{Q} = 0 \qquad \frac{\partial T}{\partial t} = 0 \qquad k\left[\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2}\right] = \nabla\cdot k\nabla T$

since steady $\frac{\partial T}{\partial t} = 0$ $\qquad$ since $T = T_1 \;\forall x$, $T_2 \;\forall x$ $\quad$ then $\frac{\partial T}{\partial x}, \frac{\partial^2 T}{\partial x^2} = 0$

$\rho C_p(0) = \frac{Dp}{Dt} + \mu u_{,y}^2 + k\,T_{,yy}$

$0 = u(p_{,x}) + \mu\, u_{,y}^2 + k\,T_{,yy}$

$0 = 0 + \mu\, u_{,y}^2 + kT_{,yy} \qquad T_{,yy} = -\frac{\mu}{k}\, u_{,y}^2$

$u_{,y} = \frac{1}{\mu}\frac{dp}{dx}\left(y - \frac{h}{2}\right) \qquad u_{,y}^2 = \frac{1}{\mu^2}\left(\frac{dp}{dx}\right)^2\left(y^2 - yh + \frac{h^2}{4}\right)$

$T_{yy} = -\frac{1}{\mu k}\left(\frac{dp}{dx}\right)^2\left[y^2 - yh + \frac{h^2}{4}\right]$

$T_y = -\frac{1}{\mu k}\left(\frac{dp}{dx}\right)^2\left[\frac{y^3}{3} - \frac{y^2 h}{2} + \frac{h^2}{4}y + C\right]$

$$\dot{q}/A = -k\frac{\partial T}{\partial y} = -k\alpha\left[\frac{y^3}{3} - \frac{y^2 h}{2} + \frac{h^2 y}{4} + \left(\frac{T_2 - T_1}{\alpha h}\right)\right]$$

$$\text{at } y=0 \quad \dot{q}/A = -k\alpha\left(\frac{T_2-T_1}{\alpha h}\right) = \frac{k}{h}(T_2 - T_1)$$

$$T = \overbrace{-\frac{1}{\mu K}\left(\frac{dp}{dx}\right)^2}^{\alpha}\left[\frac{y^4}{12} - \frac{y^3 h}{6} + \frac{h^2}{8}y^2 + Cy + D\right]$$

$$\text{at } y=h \quad k\frac{\partial T}{\partial y} = -k\alpha\left[\frac{\alpha h^4}{12\,\alpha h} + \left(\frac{T_2-T_1}{\alpha h}\right)\right]$$

$$\text{at } y=0 \quad T = T_1 \quad T_1 = \cancel{-\frac{1}{\mu K}\left(\frac{dp}{dx}\right)^2}_{\alpha}[D] \quad D = \frac{T_1}{\alpha}$$

$$\frac{\dot{q}}{A} = -\frac{k}{h}\left[\frac{\alpha h^4}{12} + (T_2 - T_1)\right] @ h$$

$$T_2 = \overbrace{-\frac{1}{\mu K}\left(\frac{dp}{dx}\right)^2}^{\alpha}\left[\frac{h^4}{12} - \frac{h^4}{6} + \frac{h^4}{8} + Ch + \frac{T_1}{\alpha}\right]$$

$$\frac{h^3}{3} - \frac{h^3}{2} + \frac{h^3}{4} \qquad \frac{h^3}{3} - \frac{h^3}{4} \qquad \alpha\frac{h^3}{12}$$

$$y\left(\frac{T_2 - T_1}{\alpha h}\right) - \overbrace{\left[\frac{2h^4}{24h} - \frac{4h^4}{24h} + \frac{3h^4}{24h}\right]}^{\frac{1}{24}h^4/h} y = Cy$$

$$T = \alpha\left[\frac{y^4}{12} - \frac{y^3 h}{6} + \frac{h^2 y^2}{8} - \frac{h^3}{24}y + \left(\frac{T_2 - T_1}{\alpha h}\right)y + \frac{T_1}{\alpha}\right]$$

2. a

$$u_x + v_y = 0$$

$$u u_x + v u_y = -p'(x) + \frac{1}{R_N}u_{yy}$$

$$u T_x + v T_y = \frac{1}{P_r R_N}T_{yy} + \frac{E}{R_N}u_y^2$$

for flat plate $u=0$, $v=0$ @ $y=0$

$u = u_e$ @ $y = \delta$

$T = T_W$ @ $y=0$

$T = T_e$ @ $y = \delta_T$

$\frac{\partial u}{\partial y} = 0$ @ $y = \delta$

$p'(x) = 0$

$$\int_0^\delta \frac{\partial u}{\partial x}dy + \int_0^\delta \frac{\partial v}{\partial y}dy = 0 \qquad \int_0^\delta \frac{\partial u}{\partial x}dy = -v\Big|_\delta$$

put into eq for mom
$$\int_0^\delta u\frac{\partial u}{\partial x}dy - \int_0^\delta\left(\int_0^\delta \frac{\partial u}{\partial x}dy\right)\frac{\partial u}{\partial y}dy = -\int_0^\delta \frac{\partial p}{\partial x}dy + \mu\int_0^\delta \frac{\partial^2 u}{\partial y^2}dy$$

leads to
$$\frac{d\theta}{dx} + \theta\left[2 + \frac{\delta^*}{\theta}\right]\frac{1}{u_e}\frac{du_e}{dx} = \frac{c_f}{2}$$
where
$$\theta = \delta\int_0^1 \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right)d\left(\frac{y}{\delta}\right)$$
$$\delta^* = \delta\int_0^1 \left(1 - u/u_e\right)d\left(\frac{y}{\delta}\right)$$

Assume $\frac{u}{u_e} = A + B\eta + C\eta^2$ $\quad A = 0,\ u=0 \quad p'(x)=0 \rightarrow p u_e u_e' \rightarrow u_e' = 0$

$$1 = B + C \qquad -C = 1 \qquad C = -1$$
$$\cancel{0 = B + 2C} \qquad B = 2$$

$$\frac{u}{u_e} = 2\eta - \eta^2$$

$$\frac{1}{u_e}u_{y}\big|_1 = 0 = \frac{B}{\delta} + \frac{2C}{\delta}\eta \qquad \cancel{B\delta + 2C\delta = 0} \qquad -2C = B$$

$$\theta = \delta\int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\,d\eta$$

$$= \delta\int_0^1 \left[2\eta - \eta^2 - 4\eta^2 + 2\eta^3 + 2\eta^3 - \eta^4\right]d\eta = \eta^2 - \frac{5}{3}\eta^3 + \eta^4 - \frac{\eta^5}{5}\Big|_0^1 = 1 - \frac{5}{3} + 1 - \frac{1}{5}$$

$$2\eta - 5\eta^2 + 4\eta^3 - \eta^4$$

$$\frac{15}{15} - \frac{3}{15} = \frac{2}{15}$$

$$\theta = \frac{2}{15}\delta \qquad \frac{d\theta}{dx} = \frac{2}{15}\frac{d\delta}{dx} = \frac{\tau_w}{\rho u_e^2} = \frac{2\mu u_e}{\delta \rho u_e^2} = \frac{2\nu}{\delta u_e} = \frac{d\delta}{dx} =$$

$$\frac{1}{u_e}\frac{du}{dy} = \frac{2}{\delta} - \frac{2}{\delta}\eta = \frac{\nu}{\delta} \qquad \text{at wall } \tau = \mu\frac{du}{dy} = \mu\frac{2u_e}{\delta}$$

$$\delta = \sqrt{\frac{30\nu}{u_e}}\sqrt{x}$$

$$\delta\frac{2}{15}d\delta = \frac{2\nu}{u_e}dx \qquad \frac{\delta^2}{30} = \frac{\nu x}{u_e} \qquad \delta = 5.48x\sqrt{\frac{\nu}{u_e x}} = \frac{5.48x}{\sqrt{R_{N_x}}}$$

$$\frac{15\nu}{\delta u_e}$$

$$\cancel{\frac{\partial u}{\partial x} = u_e\left(\frac{2\eta}{\partial x} - \frac{\eta^2}{\partial x}\right)} \qquad \int_0^\delta \frac{\partial u}{\partial x}dy = \frac{\partial}{\partial x}\int_0^\delta u\,dy - u_e\frac{d\delta}{dx}$$

$$\frac{\partial\delta}{\partial x}u_e\int_0^1(2\eta-\eta^2)d\eta - u_e\frac{d\delta}{dx}$$

$$v = \frac{1}{3}u_e\frac{d\delta}{dx} = \frac{1}{3}u_e\frac{15\nu}{\delta u_e} = \frac{5\nu}{\delta} = \frac{5\nu\sqrt{R_{N_x}}}{5.\_\_x} \qquad \eta^2 - \frac{\eta^3}{3} \qquad \frac{2}{3}u_e \qquad -\frac{1}{3}u_e\frac{d\delta}{dx} = -v$$

$$\frac{v}{u_e} = \frac{5}{5.48\sqrt{R_{N_x}}} = \frac{.913}{\sqrt{R_{N_x}}}$$

$$uT_x + vT_y = \frac{1}{P_r R_N}T_{yy} + \frac{E}{R_N}u_y^2 \qquad u = u_e\left[2\eta - \eta^2\right]$$

for low speed flow $\quad uT_x + vT_y = \frac{\nu}{P_r}T_{yy} + \frac{\nu}{c_p}u_y^2$

$$\int_0^{\delta_T}(uT)_x\,dy + \int_0^{\delta_T}(vT)_y\,dy = \frac{\nu}{P_r}\int_0^{\delta_T}T_{yy}\,dy - \frac{\nu}{c_p}\int_0^{\delta_T}u_y^2\,dy$$

$$\frac{d}{dx}\int_0^{\delta_T}(uT_{\delta_T} - uT)dy = \alpha T_y\Big|_0 - \frac{\nu}{c_p}\int_0^{\delta_T}u_y^2\,dy$$

$$\frac{u}{u_{\delta_T}} = \frac{u}{u_e} \qquad \frac{d}{dx}\delta_T\int_0^1\frac{u}{u_e}(1-\theta)d\eta_T = \frac{\alpha}{u_e\delta_T}\theta_{\eta_T}\Big|_{\eta_T=0} - \frac{\nu u_e\delta_T}{c_p T_{\delta_T}\delta^2}\int_0^1\left(\frac{u}{u_e}\right)_\eta^2 d\eta_T$$

$$\frac{u}{u_e} = 2\eta - \eta^2 \qquad \left(\frac{u}{u_e}\right)_{\eta} = 2 - 2\eta$$

Assume $\quad \theta = A + B\eta_T + C\eta_T^2 \quad$ at $\eta_T = 0 \quad \theta = \frac{T_W}{T_\infty} \qquad \eta_T = 1 \quad T = 1$

$d\theta/d\eta_T = 0$ at $\eta_T = 1$

$$\theta = \theta_W + 2(1-\theta_W)\eta_\delta - (1-\theta_W)\eta_\delta^2$$
$$T = T_W + 2(T_\infty - T_W)\frac{y}{\delta_T} - (T_\infty - T_W)\frac{y^2}{\delta_T^2}$$

$\theta = \theta_W = A$ $\qquad 1 = \theta_W + B + C$

$$\frac{d\theta}{d\eta_T} = B + 2C\eta_T = \qquad \eta_T = 1 \qquad 2C = -B$$

$$1 - \theta_W = -C \qquad C = \theta_W - 1$$
$$B = 2(1-\theta_W)$$

$$1 - \theta = 1 - \theta_W - 2(1-\theta_W)\eta_T + (1-\theta_W)\eta_T^2 \qquad \frac{d\theta}{d\eta_T} = \quad + 2(1-\theta_W) - 2(1-\theta_W)\eta_T$$

$$(1-\theta)d\eta_T = (1-\theta_W)\left[1 - 2\eta_T + \eta_T^2\right]d\eta_T$$

$$\frac{u}{u_e}(1-\theta)d\eta_T = (2\eta - \eta^2)(1-\theta_W)\left[1 - 2\eta_T + \eta_T^2\right]d\eta_T$$

$$\frac{\delta_T}{\delta} \approx 1.155 \qquad \frac{\eta_T}{\eta} \approx 1.155$$

$$= 2\frac{\alpha}{u_e\delta_T}(1-\theta_W) - \frac{\nu u_e \delta_T}{c_p T_\infty \delta^2}\int_0^1 (2 - 2\eta)\,d\eta_T$$

8×10^{-3} $\quad -1.7778\times10^{-3}$ $\quad 7\times10^{-3}$ $\quad 1.777\times10^{-3}$ $\quad 1\times10^{-3}$

2.2716×10^{-3} $\quad -1.5556\times10^{-3}$ $\quad -3.95\times10^{-4}$ $\quad 2.22\times10^{-4}$

7.850×10^{-3} $\quad 2\times10^{-3}$ $\quad 8.75\times10^{-4}$

2.2716×10^{-3} $\quad 2.22\times10^{-4}$

1.875×10^{-3}

$$\left(2\frac{y\sqrt{R_{Nx}}}{5.48x} - \frac{y^2 u_e}{30\nu x}\right)$$

$$\frac{\partial T}{\partial y} = \frac{2(T_\infty - T_W)}{\delta_T}$$

$$\frac{\partial T}{\partial y} = \frac{2(T_\infty - T_w)}{\delta_T} - \frac{2(T_\infty - T_w)}{\delta_T} y/\delta_T$$

$$T = T_w + 2(T_\infty - T_w)\frac{y}{\delta_T} - (T_\infty - T_w) y^2/\delta_T^2$$

$$u_{,y} = u_e\left[\frac{2}{\delta} - \frac{2}{\delta}\frac{y}{\delta}\right] = \frac{2u_e}{\delta}[1 - y/\delta]$$

$$u = u_e\left[2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right] \qquad T_{\delta_T} = T_\infty \qquad T_\infty - T = T_\infty - T_w - 2(T_\infty - T_w)\frac{y}{\delta_T} + (T_\infty - T_w)\frac{y^2}{\delta_T^2}$$

$$\partial u/\partial y = u_e\left[\frac{2}{\delta} - \frac{2y}{\delta^2}\right] = \frac{2u_e}{\delta}[1 - y/\delta]$$

$$\frac{d}{dx}\left[\int_0^{\delta_T} u_e\left[\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right][T_\infty - T_w]\left\langle 1 - 2\frac{y}{\delta_T} + \frac{y^2}{\delta_T^2}\right\rangle dy = 2\alpha\frac{[T_\infty - T_w]}{\delta_T} - \frac{\nu}{c_p}\int_0^{\delta_T}\frac{4u_e^2}{\delta^2}[1 - y/\delta]^2 dy\right.$$

$$\frac{d}{dx}\left[u_e(T_\infty - T_w)\int_0^{\delta_T}\left(\frac{2y}{\delta} - \frac{4y^2}{\delta\delta_T} + \frac{2y^3}{\delta\delta_T^2} - \frac{y^2}{\delta^2} + \frac{2y^3}{\delta^2\delta_T} - \frac{y^4}{\delta^2\delta_T^2}\right)dy\right.$$

$$\delta = A\sqrt{x} = \sqrt{\frac{30\nu x}{u_e}}$$

$$\delta^2 = A^2 x \qquad A = \sqrt{\frac{30\nu}{u_e}}$$

$$\frac{d}{dx}\left[u_e(T_\infty - T_w)\left\{\frac{y^2}{\delta} - \frac{4y^3}{3\delta\delta_T} + \frac{y^4}{2\delta\delta_T^2} - \frac{y^3}{3\delta^2} + \frac{y^4}{2\delta^2\delta_T} - \frac{y^5}{5\delta^2\delta_T^2}\right\}_0^{\delta_T}\right.$$

$$\delta' = \frac{A}{2\sqrt{x}}$$

$$\left\{\frac{\delta_T^2}{\delta} - \frac{4\delta_T^2}{3\delta} + \frac{\delta_T^2}{2\delta} - \frac{\delta_T^3}{3\delta^2} + \frac{\delta_T^3}{2\delta^2} - \frac{\delta_T^3}{5\delta^2}\right\} = \frac{\delta_T^2}{6\delta} - \frac{\delta_T^3}{30\delta^2} \qquad \frac{5}{30}\delta - \frac{\delta}{30} = \frac{4\delta}{30}$$

$$\frac{5}{6} \qquad \frac{3}{5}$$

$$\frac{u_e(T_\infty - T_w)}{30}\frac{d}{dx}\left[5\frac{\delta_T^2}{\delta} - \frac{\delta_T^3}{\delta^2}\right] = 2\alpha\frac{[T_\infty - T_w]}{\delta_T} + \frac{4\nu u_e^2}{3c_p\delta}\left[1 - y/\delta\right]^3\Big|_0^{\delta_T} \qquad E\,\Delta T$$

$$\frac{T_w - T_\infty}{T_\infty - T_w}$$

$$\frac{u_e(T_\infty - T_w)}{30}\left\{5\left[\frac{\delta\cdot 2\delta_T\delta_T' - \delta_T^2\delta'}{\delta^2}\right] - \left[\frac{\delta^2\cdot 3\delta_T^2\delta_T' - \delta_T^3\cdot 2\delta\delta'}{\delta^4}\right]\right\} = 2\alpha\frac{[T_\infty - T_w]}{\delta_T} + \frac{4\nu u_e^2}{3c_p\delta}\left[\left\{1 - \frac{\delta_T}{\delta}\right\}^3 - 1\right]$$

$$1 - \frac{\delta_T}{\delta} + \frac{\delta_T^2}{\delta^2} - \frac{\delta_T^3}{\delta^3} - 1 \qquad \frac{4\nu}{3\delta}E(\Delta T)$$

$$+\cancel{\frac{8\nu(T_\infty - T_w)}{3\delta}}$$

$$\frac{u_e}{30}(T_\infty - T_w)\left[\frac{10\delta_T\delta_T'}{\delta} - \frac{5\delta_T^2\delta'}{\delta^2} - \frac{3\delta_T^2\delta_T'}{\delta^2} + \frac{2\delta_T^3\delta'}{\delta^3}\right] = \frac{\alpha\nu}{P_r}\frac{[T_\infty - T_w]}{\delta_T} - \frac{4\nu u_e^2}{3c_p\delta}\left[3\left(\frac{\delta_T}{\delta}\right) - 3\left(\frac{\delta_T}{\delta}\right)^2 + \left(\frac{\delta_T}{\delta}\right)^3\right]$$

$$\varphi\left[\frac{10\delta_T\delta_T'}{A\sqrt{x}} - \frac{5\delta_T^2}{2A\,x\sqrt{x}} - \frac{3\delta_T^2\delta_T'}{A^2x} + \frac{2\delta_T^{3}}{2A^2x^2}\right] = \frac{\Gamma}{\delta_T} - \Gamma'\left[\frac{3\delta_T}{A^2x} - \frac{3\delta_T^2}{A^3x\sqrt{x}} + \frac{\delta_T^3}{A^4x^2}\right]$$

$$\text{let } \delta_T = F\sqrt{x} \qquad \delta_T' = \frac{F}{2\sqrt{x}} \qquad \delta_T^2 = F^2x \qquad -\frac{U_e^2}{2c_p}(T_\infty - T_w)$$

$$\varphi\left[\frac{10F^2}{2A\sqrt{x}} - \frac{5F^2}{2A\sqrt{x}} - \frac{3F^3}{2A^2\sqrt{x}} + \frac{2F^3}{2A^2\sqrt{x}}\right] = \frac{\Gamma}{F\sqrt{x}} - \Gamma'\left[\frac{3F}{A^2\sqrt{x}} - \frac{3F^2}{A^3\sqrt{x}} + \frac{F^3}{A^4\sqrt{x}}\right]$$

$$\longrightarrow \varphi\left[\frac{5}{2A}F^3 - \frac{F^4}{2A^2}\right] = \Gamma - \Gamma'\left[\frac{3F^2}{A^2} - \frac{3F^3}{A^3} + \frac{F^4}{A^4}\right] \qquad F = F(A, \Gamma', \Gamma, \varphi)$$

$$\frac{u_e}{30} \qquad \frac{2\nu}{P_r} \qquad +\frac{8\nu}{3}$$

$\frac{1}{2} \quad -\frac{5}{2} \quad 0 \quad 0 \quad 2$ | $\Delta T = T_\infty$ $\quad \frac{1}{2}L^4 - \frac{5}{2}L^3 + 2$ $\quad$ 1539 −837

$\quad \frac{1}{2} \quad -2 \quad -2 \quad -2$

$\frac{1}{2} \quad -2 \quad -2 \quad -2$ ‖ $\quad \frac{1}{2}L^3 - 2L^2 - 2L - 2 = 0$

$$\frac{u_e}{30}\left(\frac{5}{2A}F^3 - \frac{F^4}{2A^2}\right) = \frac{2\nu}{P_r} - \quad + \frac{4\nu}{3}E\left(\frac{3F^2}{A^2} - \frac{3F^3}{A^3} + \frac{F^4}{A^4}\right) \quad 8\nu/3$$

$$\frac{u_e}{30\nu} = \frac{1}{A^2} \qquad \left(\frac{5}{2A^3}F^3 - \frac{F^4}{2A^4}\right) = \frac{2}{P_r} + \frac{4E}{3}\left(\frac{3F^2}{A^2} - \frac{3F^3}{A^3} + \frac{F^4}{A^4}\right) \quad 8/3$$

$$\frac{5}{2}[L^3] - \frac{1}{2}[L^4] = \frac{8}{3} + \frac{4E}{3}[3L^2 - 3L^3 + L^4] \quad 8/3$$

$L = 1.1 \qquad \frac{5}{2}[1.33] - \frac{1}{2}[1.464] = 2.67 \quad 5.33$

$\qquad 6.650 \quad -.732 \quad 1.464 = 5.186$

$L = 1.2 \qquad \frac{5}{2}[1.73] \quad -2.075$

$\qquad 8.650 \quad -2.075 \quad = 6.575$

$L = 1.11$ no friction

$$\boxed{\delta_T = 1.11\,\delta}$$

$\qquad 5[1.365] - \quad 1.515 \qquad 2.5 + 2.66 = 5.167$

$\qquad 6.825 \quad -1.515 \quad = \quad 5.310 \qquad .5 + 2.66 = 3.16$

Solution produces $\quad F = F(A; E)$ $\quad$ when $\quad A = \sqrt{\frac{30\nu}{u_e}}$

$2.31 - 3.38 = 3.93$

$20 - 5$

$1.33 \quad 1.47 \quad 6.27 \quad 2.25 - 3.375 + 5.06 \quad 1.21$

$3.31 - .74 \qquad 3.6 + 2.67 = 6.27 \qquad \frac{8}{3}(3.93)$

$$\frac{5}{2}[L^3] - \frac{1}{2}[L^4] = \frac{8}{3} \quad \frac{8}{3}[L^2 - L^3 + L^4]$$

$2.58 \qquad 2.5 + 2.66 \quad .74 \qquad .5 + 2.66 \qquad 1.21 - 1.33 + 1.47 \quad 1.35 \qquad 10.48 \quad -2.68 \quad 13.15$

$$\frac{5}{2}L^3 + \frac{8}{3}L^3 = \frac{15}{6} + \frac{16}{6} = \frac{31}{6}L^3 \qquad \frac{3}{6} + \frac{16}{6} = \frac{19}{6} \qquad 32 + 2\frac{2}{3}$$

$$\frac{31}{6}L^3 - \frac{19}{6}L^4 = \frac{8}{3} + \frac{8}{3}L^2 \qquad 2.16 + 2.66 \quad 2\tfrac{2}{3}(3.375)$$

$1.7 \qquad .75 \quad .656 \quad .81 \qquad 4.82$

$3.77 \quad -2.085 \qquad -16 \qquad L = 0$

$$-16L^2 + 31L^3 - 19L^4 - 16 = 0 \qquad -20 \quad L = 1$$

$$-16(2.25) + 31(3.375) - 19(5.06) - 16 = \qquad -42. \qquad L = 1.5$$

$9.3 - 52$

$-36 + 104.5 - 96.2 - 16 = \qquad 0 = 8/3 \qquad L = 0$

$1.0 \quad 4.82 \qquad L = .9$

$-64 + 248 - \quad 0 = 8/3 \quad L = 0 \qquad 2 = 11/6 \qquad L = 1$

$L = 1.1$

$\frac{31}{6}[1.15] \qquad 2 = \frac{16}{3} \; 5.33 \quad L = 1 \qquad = 38\frac{2}{3} \qquad L = 2$

$\frac{10}{3} \quad L = .5$

$\frac{31}{48} \qquad = 13.15 \qquad L = 1.5$

$\frac{8}{3} + \frac{2}{3} =$

$$T = T_W + 2(T_\infty - T_W)\, y/\delta_T$$

$$u = u_e\left[\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right] \qquad T_\infty - T = (T_\infty - T_W) - 2(T_\infty - T_W)\, y/\delta + (T_\infty - T_W)\left(\frac{y}{\delta}\right)^2$$

$$\frac{d}{dx}\left[u_e (T_\infty - T_W)\frac{4}{30}\delta\right] = 2\alpha\frac{[T_\infty - T_W]}{\delta}$$

$$\frac{4u_e}{30}\frac{d\delta}{dx} = \frac{2\alpha}{\delta}, \qquad \frac{2\alpha(30)}{4u_e}dx = \delta\, d\delta \qquad \delta^2 = \frac{2\alpha(30)x}{u_e} \quad \frac{\nu}{Pr}$$

$$\delta = 5.48\sqrt{\frac{\nu x}{u_e}}$$

for flat plate soln. $\quad \theta'' + \frac{Pr}{2} f\theta' = -\frac{E}{2}Pr f''^2$

$$f = -2\frac{f'''}{f''} \rightarrow$$

$$\frac{\theta''}{\theta'} = +Pr\frac{f'''}{f''}$$

at $\eta = 0 \quad \theta = \theta_W$

$$\theta' = A f'' \qquad \theta = \frac{(1-\theta_W)\int_0^\eta f''\,d\eta}{f'|_0^\infty} + \theta_W$$

$$\theta = (1-\theta_W)\, f'|_0^\eta + \theta_W \qquad f'_0 = 0 \quad f'^\eta = u/U_\infty$$

$$\theta = (1-\theta_W)\, u/U_\infty + \theta_W$$

Use eq from before

$$\frac{u_\infty}{30}\left[\frac{5}{2A}F^3 - \frac{F^4}{2A^2}\right] = 2\nu \qquad \frac{60\nu}{u_\infty}\cdot 2A = 5F^3 - \frac{F^4}{A} \quad 4A^3$$

if $F = A$

$$\frac{60\nu}{u_e}2\sqrt{\frac{30\nu}{u_e}} = 4\cdot\frac{30\nu}{u_e}\sqrt{\frac{30\nu}{u_e}} \quad ✓$$

$$\therefore \quad F = A \rightarrow \delta = \delta_T$$

$$\frac{u}{u_e} = 2\eta - \eta^2 \qquad T/T_\infty = \bar{T}_W + (1-\bar{T}_W)\left[2\eta - \eta^2\right] \rightarrow \frac{u}{u_e}$$

③ Thermal b.l.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -p_{,x} + \frac{1}{R_N}\left[u_{,xx} + u_{,yy}\right] + \frac{G}{R_N^2}\rho\theta\cos\alpha \quad T - T_\infty$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -p_{,y} + \frac{1}{R_N}\left[v_{,xx} + v_{,yy}\right] + \frac{G}{R_N^2}\rho\theta\sin\alpha$$

In bl order of eq is (1). If $G/R_N^2 \ll 1$ we can neglect grashof term

$$\oint dS = \oint \frac{dQ}{T}$$

$$\oint dQ = \oint dW$$

$$\oint dQ = \oint dU + \cancel{\oint p dv} + \oint v dp + dK + dP + dW$$

$$dh(p,T) = \left(\frac{\partial h}{\partial p}\right)_T dp + \frac{\partial h}{\partial T} dT$$

$$C_p\, dT$$

$$\oint dQ = \oint du + \oint v dp + \cancel{dK + dP} \oint dW$$

$$dh = -\oint v dp$$

$$dh = du + d(pv) = \underbrace{du}_{C_v dT} + v dp$$

$$C_v dT + v dp$$

$$dh = du + d(pv) = du + v dp$$

$$du = C_v dT \qquad \left(\frac{\partial h}{\partial p}\right)_T = v \quad ✓$$

$$\left(\frac{\partial h}{\partial p}\right)_T = \frac{1}{\rho}$$

$$dh = \left(\frac{\partial h}{\partial T}\right)_p dT + \left(\frac{\partial h}{\partial p}\right)_T dp = \quad v dp + C_p dT = C_v dT + v dp \quad C_p = C_v$$

$2.5L^3 - .5L^4 = 8/3 + \frac{8}{3}\left[3L^2 - 3L^3 + L^4\right]$

$= 8/3 + 8L^2 - 8L^3 + \frac{8}{3}L^4$

$10.5L^3 - 3.17L^4 = 8/3 + 8L^2$

$L = 0 \qquad 0 \neq 8/3$

$L = 1 \qquad 7.33 \neq 10.67$

$L = 2 \qquad 84 - 50.67 = 33.33 \neq 34.67$

$L = 3 \qquad 283.5 - 256.77 = 26.73 = 74.67$

$L = 3/2 \qquad 35.45 - 16.03 = 19.42 = 20.67$

$L = 1.7 \qquad 51.5862 - 26.4483 = 25.1379 = 25.79$

$L = 1.8 \qquad 61.2 - 33.2 = 28 = 25.92$ .59

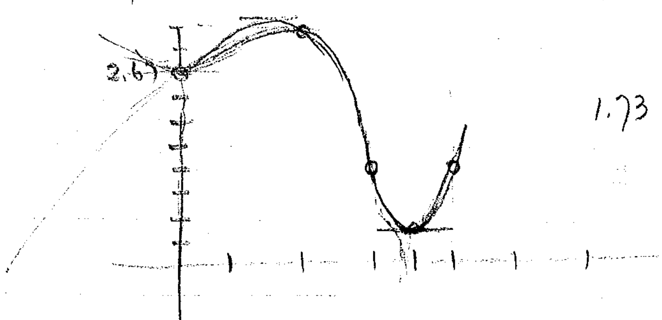
$\begin{array}{r} 25.92 \\ 2.67 \\ \hline 28.59 \end{array}$

$\begin{array}{r} 49.13 \\ 2.4562 \\ \hline 51.5862 \end{array}$

$L = 1.75 \qquad 56.3 - 29.7 = 26.6 =$ $\begin{array}{r} 24.5 \\ 2.67 \\ \hline 27.17 \end{array}$

$L = 1.71 \qquad 52.5 - 27.08 = 25.42 = 26.05$

$L = 1.73 \qquad 54.35 - 28.35 = 26 = 26.58$



1.73

$\begin{array}{r} 23.38 \\ 2.67 \\ \hline 26.05 \end{array}$

$\begin{array}{r} 23.91 \\ 2.67 \\ \hline 26.58 \end{array}$

$2.67 \times .5 = 3.17 \qquad 48 \qquad \frac{16}{6} = 2.67$

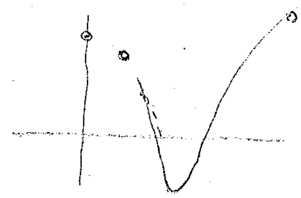
$\begin{array}{r} 3.17 \\ 16 \\ \hline 1902 \\ 3170 \\ \hline 50.72 \end{array}$

$\begin{array}{r} 3.17 \\ 81 \\ \hline 3.17 \\ 25360 \\ \hline 270 \\ 13.5 \end{array}$ 256.7

$\frac{283.5}{8}$

10.5 27

$\frac{256.77}{16}$ .0475



$\frac{1}{6}$ 1

$3.17\left(\frac{81}{16}\right) = 5\,1/16$

15.1875 167

.8333 $5/6$

$\frac{19}{6} \cdot \frac{81}{16} = \frac{}{96}$

16.03

$96)\overline{1539}$

960

579

576

810 729 1539

$L^2 = 2.89$

$L^3 = 4.913$

$L^4 = 8.3521$

$\begin{array}{r} 2.89 \\ 2.023 \\ 4.913 \\ 1.7 \\ \hline 4.913 \\ 3.4391 \\ \hline 8.3521 \end{array}$

$\begin{array}{r} 8.3521 \\ 3 \\ \hline 25.0563 \\ 1.392 \\ \hline 26.4483 \end{array}$ 8.352

$\begin{array}{r} 2.9 \\ 8 \\ \hline 23.2 \\ 23.12 \\ 2.67 \\ \hline 25.79 \end{array}$ 7 .01 8 .08

$\frac{1.22}{2} = .61$

$\frac{2.49 \pm 1.07}{2} = \frac{3.56}{3}$

1.73

$31.5L^2 - 12.67L^3 = + 16L$

$L = 0 \qquad 31.5L - 12.67L^2 = 16$

$6.2 - 5.06 = 1.14 \qquad 12.67L^2 - 31.5L + 16 = 0$

$L^2 - 2.44L + 1.263 = 0$

$\frac{2.49 \pm \sqrt{(2.49)^2 - 4(1.263)}}{2}$

$dh = T\,ds + v\,dp \qquad T = \left(\frac{\partial h}{\partial s}\right)_p \qquad \left(\frac{\partial h}{\partial p}\right)_s = v$

$dh = du + d(p/\rho) \qquad v = \frac{1}{\rho}$

$c_v = \left(\frac{\partial u}{\partial T}\right)_v \qquad c_p = \left(\frac{\partial h}{\partial T}\right)_p$

$\frac{dh}{dT}$,

$h = u + pv$

$dh = du + p\,dv + v\,dp$

$dA = -p\,dv - s\,dT \qquad -dA - s\,dT = p\,dv$

$dh = du - dA - s\,dT + v\,dp$

$h = h(p, T)$

$dh = \left(\frac{\partial h}{\partial p}\right) dp + \left(\frac{\partial h}{\partial T}\right) dT$

$W = Q$

$= \left(\frac{\partial h}{\partial p}\right)_T dp + c_p\,dT$

$dS = \frac{dQ}{T}$

$h = u + p/\rho \qquad \left(\frac{\partial h}{\partial s}\right)_p = T$

$du = dQ - p\,dv$

$dh = du + d(p/\rho)$

$\left(\frac{\partial p}{\partial s}\right)_T \left(\frac{\partial s}{\partial h}\right)_T \left(\frac{\partial h}{\partial p}\right)_T = 1$

$s = s(p, T) = \frac{\partial s}{\partial p} dp + \frac{\partial s}{\partial T} dT$

$\frac{v}{T}$

$dh = T\,ds + R\,dT$

$\left(\frac{\partial h}{\partial s}\right)_T = T \quad \left(\frac{\partial h}{\partial T}\right)_s = R$

$dh = du + d(RT) = R\,dT$

$\left(\frac{\partial h}{\partial u}\right)_T \qquad \left(\frac{\partial h}{\partial T}\right)_u = R$

Cesar Levy AE334

1. Governing Equations

$$u_{,x} + v_{,y} = 0$$

$$u u_{,x} + v u_{,y} = -\frac{p_{,x}}{\rho} + \nu (u_{,xx} + u_{,yy})$$

$$v v_{,y} + u v_{,x} = -\frac{p_{,y}}{\rho} + \nu (v_{,xx} + v_{,yy})$$

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \Phi + \dot{Q} + \nabla \cdot (K \nabla T) \checkmark$$

a. Since flow is fully developed from the continuity equation $v = 0 \ \forall (x,y)$ this shows $p = p(x)$ only. By the first momentum equation

$$u_{,yy} = \frac{1}{\mu} p'(x) \quad \text{with b.c.} \quad u = 0 \ @ \ y = 0, h$$

thus $u = \frac{1}{2\mu} p'(x) [y(y-h)] \checkmark$

b. Since $v = 0$ & fully developed, steady state & since walls are infinitely long

$$\Phi = \mu u_{,y}^2 \qquad \dot{Q} = 0 \qquad T_{,t} = 0 \qquad k T_{,yy} = \nabla \cdot (K \nabla T)$$

By the energy equation

$$T_{,yy} = -\frac{\mu}{K} u_{,y}^2 \quad \text{with b.c.} \quad T = T_1 \ @ \ y = 0 \quad T = T_2 \ @ \ y = h$$

thus if $\alpha = -\frac{1}{\mu K} \overline{p'(x)}^2$ then

$$T = \alpha \left[\frac{y^4}{12} - \frac{y^3 h}{6} + \frac{h^2 y^2}{8} - \frac{h^3 y}{24} + \left(\frac{T_2 - T_1}{\alpha h} \right) y + \frac{T_1}{\alpha} \right]$$

c. Since $\dot{q}/A = -k \frac{\partial T}{\partial y}$ then

$$\dot{q}/A = -k\alpha \left[y^3/3 - \frac{y^2 h}{2} + \frac{h^2 y}{4} + \left(\frac{T_2 - T_1}{\alpha h} \right) \right]$$

at $y = 0$

$$\dot{q}/A = -\frac{k}{h} (T_2 - T_1)$$

at $y = h$

$$\dot{q}/A = -\frac{k}{h} \left[\frac{\alpha h^4}{12} + (T_2 - T_1) \right] \checkmark$$

over

2.(a) Governing equations

$$u_{,x} + v_{,y} = 0$$

$$u u_{,x} + v u_{,y} = -p'(x) + \frac{1}{R_N} u_{,yy}$$

$$u T_{,x} + v T_{,y} = \frac{1}{P_r R_N} T_{,yy} + \frac{E}{R_N} u_{,y}^2$$

a. From integral equation $\quad v\Big|_\delta = -\int_0^\delta \frac{\partial u}{\partial x}\,dy$

into momentum

$$\int_0^\delta u\frac{\partial u}{\partial x}\,dy - \int_0^\delta\left(\int_0^\delta \frac{\partial u}{\partial x}\,dy\right)\frac{\partial u}{\partial y}\,dy = -\int_0^\delta \frac{\partial p}{\partial x}\,dy + \mu\int_0^\delta \frac{\partial^2 u}{\partial y^2}\,dy$$

leads to $\quad \frac{d\theta}{dx} + \theta\left[2 + \frac{\delta^*}{\theta}\right]\frac{1}{u_e}\frac{du_e}{dx} = \frac{c_f}{2}$ $\quad$ where $\quad \theta = \delta\int_0^1 \frac{u}{u_e}\left(1-\frac{u}{u_e}\right)d(y/\delta)$

$$\delta^* = \delta\int_0^1 (1 - u/u_e)\, d(y/\delta)$$

for flat plate $p'(x) = 0$ $\quad$ assume

$$\frac{u}{u_e} = A + B\eta + C\eta^2 \quad \text{with} \quad u_{,y}\Big|_e = 0 \quad u = 0\Big|_{y=0} \quad u = u_e\Big|_{y=\delta}$$

$\therefore \quad \frac{u}{u_e} = 2\eta - \eta^2 \longrightarrow \theta = \frac{2}{15}\delta \longrightarrow \delta\,d\delta = \frac{15\nu}{u_e}dx \longrightarrow \delta = \frac{5.48x}{\sqrt{R_{N_x}}}$

this leads to $\quad v = \frac{.913u_e}{\sqrt{R_{N_x}}}$

b. $$\int_0^{\delta_T}(uT)_x\,dy + \int_0^{\delta_T}(vT)_y\,dy = \frac{\nu}{P_r}\int_0^{\delta_T} T_{,yy}\,dy - \frac{\nu}{c_p}\int_0^{\delta_T} u_{,y}^2\,dy$$

this leads to $$\frac{d}{dx}\int_0^{\delta_T} u\,(T_{\delta_T} - T)\,dy = \alpha\, T_{,y}\Big|_0 - \frac{\nu}{c_p}\int_0^{\delta_T} u_{,y}^2\,dy$$

Assume $\quad \frac{T}{T_\infty} = A + B\eta_T + C\eta_T^2 \quad @\ \eta_T = 0 \quad T/T_\infty = \overline{T_W} \qquad \eta_T = 1 \quad \frac{T_{,y}}{T_\infty} = 0$

$$\eta_T = 1 \quad T/T_\infty = 1$$

this leads to

$$T = T_W + 2(T_\infty - T_W)\, y/\delta_T - (T_\infty - T_W)\, y^2/\delta_T^2$$

now $\quad u_{,y} = \frac{2u_e}{\delta}\left[1 - y/\delta\right] \quad ; \quad u = u_e\left[\, y/\delta\,(2 - y/\delta)\right]$

$$T_\infty - T = (T_\infty - T_W)\left[1 - 2\, y/\delta_T + y^2/\delta_T^2\right]$$

Placing these results into the integral energy equation results in

$$\varphi\left[\frac{10\,\delta_T^2\,\delta_T'}{\delta} - \frac{5\delta_T^2\,\delta'}{\delta^2} - \frac{3\,\delta_T^2\delta_T'}{\delta^2} + \frac{2\delta_T^3\,\delta'}{\delta^3}\right] = \frac{\Gamma}{\delta_T} - \frac{\Gamma'}{\delta}\left[3\,\frac{\delta_T}{\delta} - 3\,\frac{\delta_T^2}{\delta^2} + \frac{\delta_T^3}{\delta^3}\right]$$

where $\varphi = \frac{u_e(T_\infty - T_W)}{30} \qquad \Gamma = \frac{2\nu}{P_r}\left[T_\infty - T_W\right] \qquad \Gamma' = \frac{4\nu u_e^2}{3c_p}$

if $\delta = A\sqrt{x}$ & $\delta_T = F\sqrt{x}$ this equation reduces to

$$\varphi\left[\frac{5}{2A}F^3 - \frac{1}{2A^2}F^4\right] = \Gamma - \Gamma'\left[\frac{3}{A^2}F^2 - \frac{3}{A^3}F^3 + \frac{1}{A^4}F^4\right]$$

or

$$\left[T_\infty - T_W\right]\left\{\frac{5}{2}L^3 - \frac{1}{2}L^4\right\} = \frac{2\left[T_\infty - T_W\right]}{P_r} - \frac{4}{3}\frac{u_e^2}{c_p}\left[3L^2 - 3L^3 + L^4\right] \quad \text{where } L = F/A$$

Assuming that $T_W - T_\infty = \frac{u_e^2}{2c_p}$ this reduces to

$$5/2\, L^3 - \tfrac{1}{2}L^4 = \frac{2}{P_r} + \frac{8}{3}\left[3L^2 - 3L^3 + L^4\right]$$

for a prandtl number of 3/4 the solution to this equation are all complex. However taking $\frac{d}{dL}$ (eq) leads to $L=0$ $L=1.73$ $L=.61$ at $L=1.73$ is the lowest minimum possible. Since nature tends to a minimum energy state this would then be a valid assumption

$\therefore \quad \delta_T = 1.73\,\delta$

too high

c. If no frictional heating

over

$\frac{5}{2}L^3 - \frac{1}{2}L^4 = 8/3 \quad \rightarrow \quad L = 1.11$

or $\delta_T = 1.11\,\delta$

thus frictional heating tends to increase the thermal b.l.

2(b) Governing Equation

$$\frac{5}{2}L^3 - \frac{1}{2}L^4 = \frac{2}{P_r} + \text{frictional term}$$

for this problem $P_r = 1 \quad E = 0$

$$\frac{5}{2}L^3 - \frac{1}{2}L^4 = 2$$ this has a solution $L = 1$ or $\delta_T = \delta$

∴ going to

$$T = T_W + 2(T_\infty - T_W)\,y/\delta_T - (T_\infty - T_W)\,y^2/\delta_T^2$$

and letting $y/\delta_T = y/\delta$ then

$$\bar{T} = \bar{T}_W + 2(1-\bar{T}_W)\eta - (1-\bar{T}_W)\eta^2 = \bar{T}_W + (1-\bar{T}_W)[2\eta - \eta^2]$$

but $u/u_e = 2\eta - \eta^2$ ∴

$$\bar{T} = \bar{T}_W + (1-\bar{T}_W)\,u/u_e$$

3. Governing Equations

$$u_{,x} + v_{,y} = 0$$

$$u u_{,x} + v u_{,y} = -p_{,x}/\rho + \frac{F_x}{\rho} + \nu u_{,yy}$$

if $p_{,x} = p^*_{,x} + \rho_\infty g_x$ & $F_x = \rho g_x$ then

$$\rho u u_{,x} + \rho v u_{,y} = (\rho - \rho_\infty) g_x + \mu u_{,yy}$$ since $p^*\big|_\delta = 0$ & it is

assume $p^*_{,x}$ any level $\sim p^*_{,x}$ at δ

if $\rho = \rho(T, p) = \rho_\infty \frac{\partial\rho}{\partial T}(T - T_\infty) + \frac{\partial\rho}{\partial p}(p - p_\infty)$ → negligible then

$$\rho - \rho_\infty = \frac{\partial\rho}{\partial T}\Big|_\infty (T - T_\infty) = -\beta_\infty\, T - T_\infty$$

∴

$$\rho u\, u_{,x} + \rho v\, u_{,y} = -\rho\beta_\infty (T - T_\infty) g_x + \mu u_{,yy}$$

non dim V, U wrt U_∞ y, x wrt L then

$$\left[\frac{U_\infty^2}{L}\rho\right]\bar{U}\,\bar{U}_{,\bar{x}} + \left[\frac{U_\infty^2}{L}\rho\right]\bar{V}\,\bar{U}_{,\bar{y}} = -\rho\,\beta_\infty(T-T_\infty)g_x + \left[\mu\frac{U_\infty}{L^2}\right]\bar{U}_{,\bar{y}\bar{y}}$$

$$\bar{U}\,\bar{U}_{,\bar{x}} + \bar{V}\,\bar{U}_{,\bar{y}} = -\frac{\rho\,\beta_\infty(T-T_\infty)g_x}{\rho\,U_\infty^2/L} + \left[\frac{\mu}{\rho U_\infty L}\right]\bar{U}_{,\bar{y}\bar{y}}$$

$$\therefore \quad \bar{U}\,\bar{U}_{,\bar{x}} + \bar{V}\,\bar{U}_{,\bar{y}} = -\,G/R_N^2 + \frac{1}{R_N}\bar{U}_{,\bar{y}\bar{y}}$$

a same development can be made of y momentum equation

for b.l. $\bar{U} \sim O(1)$ $\bar{U}_{,\bar{x}} \sim O(1)$ $\bar{V} \sim O(\delta)$ $\bar{U}_{,\bar{y}} \sim O(\frac{1}{\delta})$

$\bar{U}_{,\bar{y}\bar{y}} \sim O\left(\frac{1}{\delta^2}\right)$ $R_N \sim O\left(\frac{1}{\delta^2}\right)$ $\therefore$

if G/R_N^2 is not of order 1 but of order δ; $\delta^2 \ldots$ it can be neglected

$$G = \frac{\beta\Delta T g L^3\rho^2}{\mu_\infty^2} = \frac{\text{buoyancy}}{\text{viscous}} \text{ effects}$$

4. $h(p, T)$ $\therefore$ $dh = \frac{\partial h}{\partial p}dp + \frac{\partial h}{\partial T}dT$

but $dh = du + d(pv) = du + vdp + pdv$ since ρ = const $v = \frac{1}{\rho}$ = const

$\therefore$ $dh = du + vdp = C_v\,dT + \frac{1}{\rho}dp$ $\therefore$ $\left(\frac{\partial h}{\partial p}\right)_T = \frac{1}{\rho}$

$\therefore$ $dh = \frac{\partial h}{\partial p}dp + \frac{\partial h}{\partial T}dT = \frac{1}{\rho}dp + C_p\,dT = C_v\,dT + \frac{1}{\rho}dp$

$\therefore$ $C_v = C_p$ QED

Prove that the energy equations in terms of h & e are identical!

$(1+S)\,\frac{C_0}{C_1}\,u_e\,\frac{du_e}{d\xi}\,\left(\frac{C_{0e}}{C_1}\right)^3$

$(1+S)\,u_e\,\frac{du_e}{d\xi}\left(\frac{\rho_0}{\rho_1}\right)^2$

AE-334

Homework #5

High-Speed Boundary Layers

Consider the high-speed flow over a surface (y=0) for which the external velocity field is prescribed by $u_e = U_o x^m$. Assume $\rho\mu$=constant, $P_r = 1$, no slip or injection at the surface where $T = T_w$.

Stewartson Transformation

$$\text{Let } \xi = \int_0^x \frac{p}{p_{o_\infty}}\left(\frac{T_e}{T_{o_\infty}}\right)^{\frac{1}{2}} dx, \quad Y = \left(\frac{T_e}{T_{o_\infty}}\right)^{\frac{1}{2}} \int_0^y \frac{\rho}{\rho_{o_\infty}}\, dy$$

$$\eta = Y\left(\frac{(m+1)\,u_e}{2\nu_{o_\infty}\xi}\right)^{1/2}, \quad \overline{\psi}(\xi, Y) = \left(\frac{2\nu_{o_\infty} u_e \xi}{m+1}\right)^{\frac{1}{2}} f_\beta(\eta)$$

$S(\eta) = \frac{H_o}{H_{o_e}} - 1$; subscript o denotes stagnation conditions.

$H_o =$

1) Show that the governing equations for f_β and S are

$$f_\beta''' + f_\beta'' f_\beta = \beta(-1 + f_\beta'^2 - S); \quad \beta = \frac{2m}{m+1}$$

$$S'' + fS' = 0$$

$$f_\beta(0) = f_\beta'(0) = 0 \qquad S(0) = \frac{T_w}{T_{o_e}} - 1$$

$$f_\beta'(\eta) \to 1 \quad \text{as } \eta \to \infty; \qquad S(\eta) \to 0 \quad \text{as } \eta \to \infty$$

2) Discuss what happens in the zero pressure gradient flat plate limit.

3) Obtain the skin friction and heat transfer at the wall.

FALKNER - SKAN SOLUTIONS

$U = C x^{m}$ — velocity at outer edge of boundary layer as given by potential flow theory.

Let $u = U f'(\eta) = C x^{m} f'(\eta)$ where $\eta = A y x^{\alpha}$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial \psi}{\partial \eta} A x^{\alpha}$$

$$\frac{\partial \psi}{\partial \eta} = \frac{1}{A x^{\alpha}} C x^{m} f' \qquad \boxed{\psi = \frac{C}{A} x^{m-\alpha} f}$$

$$\frac{\partial u}{\partial x} = C m x^{m-1} f' + C x^{m} f'' A y \alpha x^{\alpha-1} = m U \frac{1}{x} f' + U f'' \frac{\alpha}{x} = \frac{U}{x}\left[m f' + f'' \alpha \eta\right]$$

$$\frac{\partial u}{\partial y} = U f'' A x^{\alpha} \qquad \frac{\partial^2 u}{\partial y^2} = U f''' A^2 x^{2\alpha}$$

$$v = -\frac{\partial \psi}{\partial x} = -\left[\frac{C}{A} x^{m-\alpha-1}(m-\alpha) f + \frac{C}{A} x^{m-\alpha} f' A y \alpha x^{\alpha-1}\right]$$

$$= -\frac{C}{A} x^{m-\alpha}\left[\frac{m-\alpha}{x} f + A y x^{\alpha}\frac{\alpha}{x} f'\right] = -\frac{C}{A} x^{m-\alpha-1}\left[(m-\alpha) f + \alpha \eta f'\right]$$

$$= -\frac{U(m-\alpha)}{A x^{\alpha+1}}\left[f + \eta f' \frac{\alpha}{m-\alpha}\right] = -\frac{\tilde{u}_e (m-1)}{\sqrt{\frac{c}{\nu}\frac{m+1}{2}}} \qquad \frac{\frac{m-1}{2}}{\frac{m+1}{2}} = \frac{m-1}{m+1}$$

$$\frac{dp}{dx} = -\rho U \frac{dU}{dx} = -\rho U m C x^{m-1} = -\frac{\rho U^2 m}{x}$$

$$u\frac{\partial u}{\partial x} = U f' \frac{U}{x}\left[m f' + f'' \alpha \eta\right] = \frac{U^2 f'}{x}\left[m f' + f'' \alpha \eta\right]$$

$$v\frac{\partial u}{\partial y} = -A f'' U x^{\alpha}\frac{U(m-\alpha)}{A x^{\alpha+1}}\left[f + \eta f' \frac{\alpha}{m-\alpha}\right] = -\frac{U^2 f''(m-\alpha)}{x}\left[f + \eta f' \frac{\alpha}{m-\alpha}\right]$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{U^2 f'}{x}\left[m f' + f'' \alpha \eta\right] - \frac{U^2 f''(m-\alpha)}{x}\left[f + \eta f' \frac{\alpha}{m-\alpha}\right] = \frac{U^2 m}{x} + \nu U f''' A^2 x^{2\alpha}$$

$$m(f')^2 + \cancel{f' f'' \alpha \eta} - f f''(m-\alpha) - \cancel{\eta f' f'' \alpha} = m + \frac{\nu f''' A^2 x^{2\alpha+1}}{U}$$

$$m(f')^2 - f f''(m-\alpha) = m + \frac{\nu}{C} f''' A^2 x^{2\alpha+1-m}$$

For the equation to be independent of x let $2\alpha + 1 - m = 0$ $\quad \alpha = \frac{m-1}{2}$

$$\frac{\nu}{C} f''' A^2 - m\left[(f')^2 - 1\right] + f f''(m-\alpha) = 0 \qquad m - \alpha = \frac{m+1}{2}$$

$$\frac{\nu}{C}\frac{A^2}{m} f''' - \left[(f')^2 - 1\right] + \frac{m+1}{2m} f f'' = 0, \text{ Let } \frac{\nu}{C}\frac{A^2}{m} = \frac{m+1}{2m}, \quad A = \sqrt{\frac{C}{\nu}\frac{m+1}{2}}$$

$$\therefore \boxed{f''' + f f'' - \frac{2m}{m+1}\left[(f')^2 - 1\right] = 0} \qquad \eta = \sqrt{\frac{C}{\nu}\frac{m+1}{2}}\, y x^{\frac{m-1}{2}} = \sqrt{\frac{C x^m}{\nu}\frac{m+1}{2x}}\, y = \sqrt{\frac{U}{\nu x}\frac{m+1}{2}}\, y$$

$$\rho\left[\int_0^\delta u\frac{\partial u}{\partial x}\,dy - \int_0^\delta\left[\frac{d}{dx}\int_0^y u\,dy - u_e\frac{d\delta}{dx}\right]\frac{\partial u}{\partial y}\,dy\right] = \rho\int_0^\delta u_e\frac{\partial u_e}{\partial x}\,dy - \tau_w$$

$$\int_0^\delta\left\{\frac{d}{dx}(u_e\delta - u_e\delta^*) - u_e\frac{d\delta}{dx}\right\}\frac{\partial u}{\partial y}\,dy$$

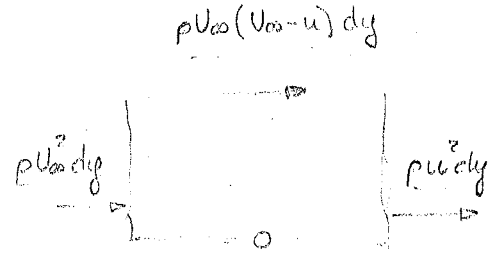
$$\delta\frac{du_e}{dx} + \cancel{u_e\frac{d\delta}{dx}} - \cancel{u_e\frac{d\delta}{dx}}$$

$$\int_0^\delta\left\{\delta\frac{du_e}{dx} - \frac{d}{dx}(u_e\delta^*)\right\}\frac{\partial u}{\partial y}\,dy$$

$$\cancel{\int_0^\delta u\frac{d}{dx}\int_0^\delta u\,dy} \qquad -\left[\delta\frac{du_e}{dx} - \frac{d}{dx}(u_e\delta^*)\right]u_e + \int_0^\delta\left(u\frac{\partial u}{\partial x}\,dy - u_e\frac{\partial u_e}{\partial x}\right) = -\frac{\tau_w}{\rho}$$

$$\frac{1}{2}\int_0^\delta\frac{\partial}{\partial x}(u^2)\,dy \qquad \delta\frac{du_e}{dx} - \frac{du_e}{dx}\delta^* + \frac{d\delta^*}{dx}u_e$$

$$\frac{1}{2}\left[\frac{d}{dx}\int_0^\delta u^2\,dy - u_e^2\frac{d\delta}{dx}\right] \qquad \left[\beta\frac{du_e}{dx} + \frac{d\delta^*}{dx}u_e\right] - \int_0^\delta(\qquad) = \frac{c_f}{2}u_e^2$$



$$\rho\left[U_\infty^2 - u^2 - U_\infty^2 + U_\infty u\right]dy$$

$$u\,(U_\infty - u)$$

$$\rho\int u\,(U_\infty - u)\,dy = \rho\,U_\infty^2\,\delta_2$$

$$\frac{d\theta}{dx} + \frac{\theta}{u_e}\left[2 + \frac{\delta^*}{\theta}\right]\frac{du_e}{dx} = \frac{c_f}{2} \qquad \int_0^y f'(1 - f')\,dy = \delta_2$$

denote $c_e^2 = (\gamma-1)\, c_p T_e$

$c_{\infty}^2 = (\gamma-1)\, c_p T_{\infty}$

$$\xi = \int_0^x \frac{P_e}{P_\infty}\frac{c_e}{c_\infty}\,dx \qquad dx = \frac{P_\infty}{P_e}\frac{c_\infty}{c_e}\,d\xi$$

$$Y = \frac{c_e}{c_\infty}\int_0^y \rho/\rho_\infty\, dy$$

$$\frac{\partial}{\partial x} = \frac{d}{dx} + \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y}$$

$$\bar{\Psi}(\xi, Y) = \Psi(x,y)$$

$$\frac{\partial}{\partial y} = \frac{d}{dy} + \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y}$$

since $\xi = \xi(x)$ only $\quad \dfrac{\partial \xi}{\partial y} = 0 \qquad \dfrac{\partial Y}{\partial y} = \dfrac{c_e}{c_\infty}\,\rho/\rho_\infty$

$$\rho/\rho_\infty\, u = \Psi_{,y} = \frac{c_e}{c_\infty}\,\rho/\rho_\infty\,\bar{\Psi}_{,Y} \longrightarrow u = \frac{c_e}{c_\infty}\bar{\Psi}_{,Y}$$

$$\rho/\rho_\infty\, v = -\bar{\Psi}_{,x} = -\left[\frac{P_e}{P_\infty}\frac{c_e}{c_\infty}\bar{\Psi}_{,\xi} + Y_{,x}\bar{\Psi}_{,Y}\right]$$

$$\frac{\partial u}{\partial x} = \frac{\bar{\Psi}_{,Y}}{c_\infty}\frac{dc_e}{dx} + \frac{P_e}{P_\infty}\frac{c_e}{c_\infty}u_{,\xi} + Y_{,x}u_{,Y}$$

$$= \frac{\bar{\Psi}_{,Y}}{c_\infty}\frac{dc_e}{dx} + \frac{P_e}{P_\infty}\left(\frac{c_e}{c_\infty}\right)^2\bar{\Psi}_{,Y\xi} + \frac{c_e}{c_\infty}Y_{,x}\bar{\Psi}_{,YY}$$

$$= \frac{\bar{\Psi}_{,Y}}{c_\infty}\frac{P_e c_e}{P_\infty c_\infty}\frac{dc_e}{d\xi} + \frac{P_e}{P_\infty}\left(\frac{c_e}{c_\infty}\right)^2\bar{\Psi}_{,Y\xi} + \frac{c_e}{c_\infty}Y_{,x}\bar{\Psi}_{,YY}$$

$$\frac{\partial u}{\partial y} = \frac{c_e}{c_\infty}\rho/\rho_\infty\, u_{,Y} = \left(\frac{c_e}{c_\infty}\right)^2\rho/\rho_\infty\,\bar{\Psi}_{,YY}$$

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = \rho\frac{c_e}{c_\infty}\bar{\Psi}_{,Y}\left[\frac{P_e}{P_\infty}\frac{c_e}{c_\infty^2}\frac{dc_e}{d\xi}\bar{\Psi}_{,Y} + \frac{P_e}{P_\infty}\left(\frac{c_e}{c_\infty}\right)^2\bar{\Psi}_{,Y\xi} + \cancel{\frac{c_e}{c_\infty}Y_{,x}\bar{\Psi}_{,YY}}\right]$$

$$-\rho_\infty\left[\frac{P_e}{P_\infty}\frac{c_e}{c_\infty}\bar{\Psi}_{,\xi} + \cancel{Y_{,x}\bar{\Psi}_{,Y}}\right]\left(\frac{c_e}{c_\infty}\right)^2\rho/\rho_\infty\,\bar{\Psi}_{,YY}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{P_e}{P_{o\infty}}\left(\frac{c_e}{c_{o\infty}}\right)^3\left[\frac{1}{c_e}\frac{dc_e}{d\xi}\bar{\Psi}_{,Y}^2+\bar{\Psi}_{,Y\xi}\bar{\Psi}_{,Y}-\bar{\Psi}_{,YY}\bar{\Psi}_{,\xi}\right]$$

$$=\frac{P_e}{P_{o\infty}}\left(\frac{c_e}{c_{o\infty}}\right)^3\left[\bar{\Psi}_{,Y\xi}\bar{\Psi}_{,Y}-\bar{\Psi}_{,YY}\bar{\Psi}_{,\xi}\right]+\frac{c_e}{c_{o\infty}^2}\frac{dc_e}{dx}\left(u^2\frac{c_{o\infty}^2}{c_e^2}\right)$$

$$=\frac{P_e}{P_{o\infty}}\left(\frac{c_e}{c_{o\infty}}\right)^3\left[\bar{\Psi}_{,Y\xi}\bar{\Psi}_{,Y}-\bar{\Psi}_{,YY}\bar{\Psi}_{,\xi}\right]+\frac{1}{c_e}u^2\frac{dc_e}{dx}$$

now $c_p\frac{dT_e}{dx}+u_e\frac{du_e}{dx}=0$ and $2c_e\frac{dc_e}{dx}=(\gamma-1)c_p\frac{dT_e}{dx}$

and $\frac{1}{c_e}\frac{dc_e}{dx}=-\left(\frac{\gamma-1}{2}\right)\frac{u_e}{c_e^2}\frac{du_e}{dx}$ $\therefore$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{P_e}{P_{o\infty}}\left(\frac{c_e}{c_{o\infty}}\right)^3\left[\bar{\Psi}_{,Y\xi}\bar{\Psi}_{,Y}-\bar{\Psi}_{,YY}\bar{\Psi}_{,\xi}\right]-\frac{\gamma-1}{2}u^2\frac{u_e}{c_e^2}\frac{du_e}{dx}$$

$$\mu\frac{\partial u}{\partial y}=\mu_{o\infty}\left(\frac{c_e}{c_{o\infty}}\right)^2\frac{P_e}{P_{o\infty}}\bar{\Psi}_{,YY}\qquad \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)=\nu_{o\infty}\frac{P_e}{P_{o\infty}}\left(\frac{c_e}{c_{o\infty}}\right)^3\bar{\Psi}_{,YYY}$$

$$\frac{\rho\mu}{\rho_{o\infty}\mu_{o\infty}}=\frac{\rho\mu}{\rho_e\mu_e}\frac{\rho_e\mu_e}{\rho_{o\infty}\mu_{o\infty}}=1\,\frac{P_e}{P_{o\infty}}$$

$$-\frac{1}{\rho}\frac{dp}{dx}=+\frac{T}{T_e}u_e\frac{du_e}{dx}\qquad \text{define}\quad S=\frac{c_pT+\frac{1}{2}u^2}{c_pT_{o\infty}}-1$$

$$1+S=\frac{T}{T_{o\infty}}+\frac{u^2}{2c_pT_{o\infty}}=\frac{T}{T_e}\left(\frac{c_e}{c_{o\infty}}\right)^2+\frac{(\gamma-1)u^2}{2(\gamma-1)c_pT_{o\infty}}=\frac{T}{T_e}\left(\frac{c_e}{c_{o\infty}}\right)^2+\frac{\gamma-1}{2c_{o\infty}^2}u^2$$

$$\left[(1+S)-\frac{\gamma-1}{2c_{o\infty}^2}u^2\right]\frac{c_{o\infty}^2}{c_e^2}=\frac{T}{T_e}=(1+S)\left(\frac{c_{o\infty}}{c_e}\right)^2-\frac{\gamma-1}{2c_e^2}u^2$$

$$\therefore\quad -\frac{1}{\rho}\frac{dp}{dx}=\left\{(1+S)\left(\frac{c_{o\infty}^2}{c_e^2}\right)-\frac{\gamma-1}{2}\frac{u^2}{c_e^2}\right\}u_e\frac{du_e}{dx}$$

$$\therefore\quad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\frac{1}{\rho}\left[\mu\frac{\partial u}{\partial y}\right]_{,y}\quad \text{becomes}$$

$$\frac{\rho_e}{\rho_{o\infty}}\left(\frac{c_e}{c_{o\infty}}\right)^3\left[\bar{\Psi}_{,Y\xi}\bar{\Psi}_{,Y}-\bar{\Psi}_{,YY}\bar{\Psi}_{,\xi}\right]=(1+S)\left[\frac{c_{o\infty}^2}{c_e^2}\right]u_e\frac{du_e}{dx}+\nu_{o\infty}\frac{\rho_e}{\rho_{o\infty}}\left(\frac{c_e}{c_{o\infty}}\right)^3\bar{\Psi}_{,YYY}$$

$$\frac{du_e}{dx}=\frac{\rho_e c_e}{\rho_{o\infty}c_{o\infty}}\frac{du_e}{d\xi}$$

$$\bar{\Psi}_{,Y}\bar{\Psi}_{,Y\xi}-\bar{\Psi}_{,\xi}\bar{\Psi}_{,YY}=(1+S)\left(\frac{c_{o\infty}}{c_e}\right)^2u_e\frac{du_e}{d\xi}+\nu_{o\infty}\bar{\Psi}_{,YYY}$$

let $u_e\frac{c_{o\infty}}{c_e}=\tilde{u}_e$

$$\therefore\quad\bar{\Psi}_{,Y}\bar{\Psi}_{,Y\xi}-\bar{\Psi}_{,\xi}\bar{\Psi}_{,YY}=(1+S)\tilde{u}_e\frac{d\tilde{u}_e}{d\xi}+\nu_{o\infty}\bar{\Psi}_{,YYY}$$

note that $\bar{\Psi}_{,Y}=u\frac{c_{o\infty}}{c_e}=\tilde{u}$ define $\tilde{v}=-\bar{\Psi}_{,\xi}$

$$\therefore\quad\tilde{u}\,\tilde{u}_{,\xi}+\tilde{v}\,\tilde{u}_{,Y}=(1+S)\tilde{u}_e\frac{d\tilde{u}_e}{d\xi}+\nu_{o\infty}\tilde{u}_{,YY}$$

if $\tilde{u}_e=K\xi^m$ where $m=\frac{\gamma-1}{3-5\gamma}$ and $u_e=U_0x^m$

& $u_e=\frac{c_e}{c_{o\infty}}\tilde{u}_e=\frac{c_e}{c_{o\infty}}K\xi^m=U_0x^m$

let us write $\tilde{u}=\tilde{u}_e f'(\eta)$ where $\eta=Y\left[\frac{(m+1)}{2}\frac{\tilde{u}_e}{\nu_{o\infty}\xi}\right]^{1/2}$

$$\tilde{u}_{,Y}=\tilde{u}_{,\eta}\eta_{,Y}=\tilde{u}_e f''\left[\frac{m+1}{2}\frac{\tilde{u}_e}{\nu_{o\infty}\xi}\right]^{1/2}=\tilde{u}_e f''\,\eta/Y$$

$$\tilde{u}_{,YY}=\tilde{u}_{,Y\eta}\eta_{,Y}=\tilde{u}_e f'''\left[\frac{m+1}{2}\frac{\tilde{u}_e}{\nu_{o\infty}\xi}\right]=\tilde{u}_e f'''\,\eta^2/Y^2$$

$$\tilde{u}_e\frac{d\tilde{u}_e}{d\xi}=\tilde{u}_eKm\xi^{m-1}=m\frac{\tilde{u}_e^2}{\xi}$$

$$\tilde{u}_{,\xi}=\tilde{u}_{,\eta}\eta_{,\xi}=\tilde{u}_e f''\left[\frac{Y}{2}\left[\frac{m+1}{2\nu_{o\infty}}\frac{\tilde{u}_e}{\xi}\right]^{-1/2}\left[\left(\frac{m-1}{\xi}\right)\left(\frac{m+1}{2\nu_{o\infty}}\frac{\tilde{u}_e}{\xi}\right)\right]\right]+m\frac{\tilde{u}_e}{\xi}f'$$

$$=m\frac{\tilde{u}_e}{\xi}f'+\frac{\tilde{u}_e}{\xi}f''\left(\frac{m-1}{2}\right)\eta$$

$$\tilde{v}=-\frac{\partial\bar{\Psi}}{\partial\xi}=-\left[\frac{1}{2}\left(\frac{2\nu_{o\infty}\tilde{u}_e\xi}{m+1}\right)^{-1/2}\left(2\nu_{o\infty}\tilde{u}_e\right)f+\left(\frac{2\nu_{o\infty}\tilde{u}_e\xi}{m+1}\right)^{1/2}f'\frac{m-1}{2\xi}\eta\right]$$

$$\tilde{v} = -\left[\frac{\nu_{0\infty} \tilde{u}_e (m+1)}{2\xi} \right]^{1/2} \left\{ f + \left(\frac{m-1}{m+1}\right) f' \eta \right\}$$

therefore $\tilde{v} \frac{\partial \tilde{u}}{\partial Y} + \tilde{u} \frac{\partial \tilde{u}}{\partial \xi} = (1+S)\, \tilde{u}_e \frac{d\tilde{u}_e}{d\xi} + \nu_{0\infty} \frac{\partial^2 \tilde{u}}{\partial Y^2}$ can be written as

$$\frac{m \tilde{u}_e^2}{\xi} f'^2 - \frac{\tilde{u}_e^2 (m+1)}{2\xi} f f'' = (1+S) \frac{m \tilde{u}_e^2}{\xi} + \frac{\tilde{u}_e^2 (m+1)}{2\xi} f'''$$

$$\frac{\tilde{u}_e^2 (m+1)}{2\xi} \left(f''' + f f''\right) = \frac{m \tilde{u}_e^2}{\xi} \left[f'^2 - 1 - S \right]$$

$$f''' + f f'' = \frac{2m}{m+1} \left[f'^2 - 1 - S \right] = \beta \left[f'^2 - 1 - S \right]$$

since $u = 0$ at the wall $\tilde{u} = 0 \rightarrow f'(0) = 0$ $\quad u = 0 @ y = 0 \rightarrow Y = 0 \rightarrow \eta = 0$

since $v = 0$ at the wall & $\bar{\Psi}_{,Y} = 0$ at the wall & because $\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = 0$ & since $\bar{\Psi}_{,YY} \neq 0 \rightarrow \bar{\Psi}_{,\xi} = 0 \rightarrow \tilde{v} = 0 \rightarrow f(0) = 0$. $\quad \tilde{u} = \tilde{u}_e \rightarrow f'(\eta) = 1 \quad y \rightarrow \infty \quad Y \rightarrow \infty \quad \eta \rightarrow \infty$

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) u + \rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = u \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu \left(\frac{\partial u}{\partial y}\right)^2$$

or

$$\rho u \frac{\partial}{\partial x} H_0 + \rho v \frac{\partial}{\partial y} H_0 = \frac{\partial}{\partial y} \left\{ \mu \frac{\partial}{\partial y} (H_0) \right\} \quad \text{for } Pr = 1$$

$$(S+1)\, C_p T_{0\infty} = C_p T + \tfrac{1}{2} u^2$$

$$\rho u \left[\frac{\partial}{\partial x} \left\{ (S+1) C_p T_{0\infty} \right\} \right] + \rho v \left[\frac{\partial}{\partial y} \left\{ (S+1) C_p T_{0\infty} \right\} \right] = \frac{\partial}{\partial y} \left\{ \mu \frac{\partial}{\partial y} \left\{ (S+1) C_p T_{0\infty} \right\} \right\}$$

$$\rho u\, C_p T_{0\infty} \frac{\partial S}{\partial x} + \rho v\, C_p T_{0\infty} \frac{\partial S}{\partial y} = \frac{\partial}{\partial y} \left\{ \mu\, C_p T_{0\infty} \frac{\partial S}{\partial y} \right\}$$

$$\rho u \frac{\partial S}{\partial x} + \rho v \frac{\partial S}{\partial y} = \frac{\partial}{\partial y} \left\{ \mu \frac{\partial S}{\partial y} \right\}$$

$$\frac{\partial S}{\partial x} = \frac{P_e}{P_{o\infty}} \frac{C_e}{C_{o\infty}} \frac{\partial S}{\partial \xi} + Y_{,x}\, S_{,Y}$$

$$\frac{\partial S}{\partial y} = \frac{C_e}{C_{o\infty}} \rho/\rho_{o\infty} \frac{\partial S}{\partial Y} \qquad \mu\frac{\partial S}{\partial y} = \frac{C_e}{C_{o\infty}} \frac{\mu\rho}{P_{o\infty}} \frac{\partial S}{\partial Y} = \mu_{o\infty} \frac{C_e}{C_{o\infty}} \frac{P_e}{P_{o\infty}} \frac{\partial S}{\partial Y}$$

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial S}{\partial y}\right) = \frac{\rho}{\rho_{o\infty}} \mu_{o\infty} \frac{C_e^2}{C_{o\infty}^2} \frac{P_e}{P_{o\infty}} \frac{\partial^2 S}{\partial Y^2}$$

$$\rho u \frac{\partial S}{\partial x} + \rho v \frac{\partial S}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial S}{\partial y}\right)$$

$$\left[\rho \frac{C_e}{C_{o\infty}} \tilde{u}\right]\left[\frac{P_e}{P_{o\infty}} \frac{C_e}{C_{o\infty}} \frac{\partial S}{\partial \xi} + \cancel{Y_{,x} S_{,Y}}\right] + \rho_{o\infty}\left[\frac{P_e}{P_{o\infty}} \frac{C_e}{C_{o\infty}} \tilde{v} - Y_{,x}\, \tilde{u}\right] \frac{C_e}{C_{o\infty}} \rho/\rho_{o\infty}\, S_{,Y}$$

$$= \rho\, \nu_{o\infty} \frac{C_e^2}{C_{o\infty}^2} \frac{P_e}{P_{o\infty}} S_{,YY}$$

or

$$\tilde{u} \frac{\partial S}{\partial \xi} + \tilde{v} \frac{\partial S}{\partial Y} = \nu_{o\infty} \frac{\partial^2 S}{\partial Y^2} \qquad \frac{\partial S}{\partial \xi} = \frac{\partial S}{\partial \eta} \eta_{,\xi} = S' \frac{Y}{2} \left[\frac{(m+1)\tilde{u}_e}{2\nu_{o\infty}\xi}\right]^{-1/2} \left[\frac{(m+1)(m-1)\tilde{u}_e}{2\nu_{o\infty}\xi^2}\right]$$

$$= S' \frac{m-1}{2\xi} \eta$$

$$\frac{\partial S}{\partial Y} = \frac{\partial S}{\partial \eta} \eta_{,Y} = S' \frac{\eta}{Y} \qquad \frac{\partial^2 S}{\partial Y^2} = \frac{\partial}{\partial \eta} S_{,Y} \frac{\partial \eta}{\partial Y} = S'' \eta^2/Y^2$$

$$\tilde{u} = \tilde{u}_e f' \qquad \tilde{v} = -\frac{\eta}{Y}\left[\nu_{o\infty}\right]\left\{f + \frac{m-1}{m+1} \eta f'\right\}$$

$$\tilde{u}_e f' S' \frac{m-1}{2\xi}\eta - \frac{\eta^2}{Y^2}\left[\nu_{o\infty}\right] f S' - \frac{\eta^2}{Y^2} \nu_{o\infty} \frac{m-1}{m+1} \eta f' S' = \nu_{o\infty} S'' \eta^2/Y^2$$

$$\eta^2/Y^2\, \nu_{o\infty} \left[S'' + f S'\right] = 0$$

when $u = 0$ $T = T_w$ @ wall $\therefore$ $S(\eta) = \dfrac{T_w}{T_{oe}} - 1$ since $y = 0$ $Y = 0 \to \eta = 0$

when $\eta \to 0$ $u \to u_e$ $T \to T_e$ $H_o \to H_{oe}$ $\therefore$ $S \to 0$

(2) the momentum equation reduces to

$$\bar{\Psi}_{,Y}\,\bar{\Psi}_{,Y\xi} - \bar{\Psi}_{,\xi}\,\bar{\Psi}_{,YY} = \nu_{0\infty}\,\bar{\Psi}_{,YYY}$$

which in turn reduces to

$$f''' + ff'' = \frac{2m}{m+1} f'^2$$

since $\frac{du_e}{dx} = 0$ T_e = constant. & $S = A + B\tilde{u}$ since $\tilde{u} = 0$ when $S = S_W$

and $S = 0$ when $\tilde{u} = \tilde{u}_e$ $S = S_W(1 - \tilde{u}/\tilde{u}_e)$

but $u/u_e = \tilde{u}/\tilde{u}_e$ $\therefore$ $S = S_W(1 - u/u_e)$ where $S_W = \frac{T_W}{T_{0e}} - 1$

and u/u_e can be found from $\frac{\sqrt{2C_p(T_0 - T)}}{\sqrt{2C_p(T_{0e} - T_e)}} = \sqrt{\left(\frac{T_0 - T}{T_{0e} - T_e}\right)}$ where T_0 is the stag temperature which ranges from $T_W \geqslant T_0 \geqslant T_{0e}$

$$\mu\frac{\partial u}{\partial y} = \mu_{0\infty}\left(\frac{C_e}{C_{0\infty}}\right)^2 \frac{P_e}{P_{0\infty}}\,\tilde{u}_e\, f''\eta/_Y$$

$\frac{1}{2}\rho_W u_e^2$ = pressure at wall

$$c_f = \frac{\mu_{0\infty}}{2\rho_W}\left(\frac{C_e}{C_{0\infty}}\right)^2 \frac{P_e}{P_{0\infty}}\,\frac{C_{0\infty}}{C_e}\,\frac{1}{u_e}\, f''\eta/_Y = \frac{\mu_{0\infty}}{2\rho_W}\,\frac{C_e}{C_{0\infty}}\,\frac{P_e}{P_{0\infty}}\left[\frac{(m+1)\,C_{0\infty}}{2\nu_{0\infty}\,\xi u_e C_e}\right]^{1/2} f''(0)$$

$$\dot{q}/_A = k\frac{\partial T}{\partial y} = k\,T_e\,\frac{\partial}{\partial y}\left[(1+S) - \frac{\gamma-1}{2C_{0\infty}^2}\,u^2\right]\frac{C_{0\infty}^2}{C_e^2} = k\,T_{0\infty}\left[\frac{\partial S}{\partial y} - \frac{\gamma-1}{2C_{0\infty}^2}\,\frac{\partial u^2}{\partial y}\right]$$

where $\frac{\partial S}{\partial y} = \frac{C_e}{C_{0\infty}}\,\frac{\rho_W}{\rho_{0\infty}}\left[\frac{(m+1)\,C_{0\infty}}{2\nu_{0\infty}\,\xi u_e C_e}\right]^{1/2} S'(0)$

$$\frac{\partial u^2}{\partial y} = \frac{C_e^2}{C_{0\infty}^2}\frac{\partial}{\partial y}\bar{\Psi}_{,Y}^{\,2} = \frac{C_e^2}{C_{0\infty}^2}\cdot\frac{C_e}{C_{0\infty}}\,\rho/\rho_{0\infty}\cdot 2\bar{\Psi}_{,Y}\,\bar{\Psi}_{,YY} = 2\left(\frac{C_e}{C_{0\infty}}\right)\frac{\rho_W}{\rho_{0\infty}}\,u_e^2\left[\frac{(m+1)\,C_{0\infty}}{2\nu_{0\infty}\,\xi u_e C_e}\right]^{1/2} f'(0)f''$$

or $$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial Y}\,\frac{\partial Y}{\partial y} = \frac{C_e}{C_{0\infty}}\,\rho/\rho_{0\infty}\,\frac{\partial T}{\partial \eta}\,\frac{\partial \eta}{\partial Y} = \frac{C_e}{C_{0\infty}}\,\frac{\rho}{\rho_{0\infty}}\left[\frac{(m+1)\,C_{0\infty}}{2\nu_{0\infty}\,\xi u_e C_e}\right]^{1/2} T'(0)$$

where $T'(0)$ is found from $S(\eta)$

Cesar Levy AE334

$\boxed{1.}$ denote $c_e^2 = (\gamma-1) c_p T_e$

$c_\infty^2 = (\gamma-1) c_p T_\infty$

$dx = \frac{P_\infty}{P_e} \frac{c_\infty}{c_e} \, d\xi$

$dY = \frac{c_e}{c_\infty} \rho/\rho_\infty \, dy$

$$\frac{\partial}{\partial x} = \frac{d}{dx} + \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y}$$

$$\frac{\partial}{\partial y} = \frac{d}{dy} + \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y}$$

since $\xi = \xi(x)$ only $\frac{\partial \xi}{\partial y} = 0$

define $\rho/\rho_\infty \, u = \bar{\Psi}_{,y} = \frac{c_e}{c_\infty} \rho/\rho_\infty \bar{\Psi}_{,Y} \implies u = \frac{c_e}{c_\infty}\bar{\Psi}_{,Y}$

$$\rho/\rho_\infty \, v = -\bar{\Psi}_{,x} = -\left[\frac{P_e}{P_\infty}\frac{c_e}{c_\infty}\bar{\Psi}_{,\xi} + Y_{,x}\bar{\Psi}_{,Y}\right]$$

$$\frac{\partial u}{\partial x} = \frac{\bar{\Psi}_{,Y}}{c_\infty}\frac{dc_e}{dx} + \frac{P_e}{P_\infty}\frac{c_e}{c_\infty} u_{,\xi} + Y_{,x} u_{,Y}$$

$$= \frac{\bar{\Psi}_{,Y}}{c_\infty}\frac{P_e c_e}{P_\infty c_\infty}\frac{dc_e}{d\xi} + \frac{P_e}{P_\infty}\left(\frac{c_e}{c_\infty}\right)^2 \bar{\Psi}_{,Y\xi} + \frac{c_e}{c_\infty} Y_{,x}\bar{\Psi}_{,YY}$$

$$\frac{\partial u}{\partial y} = \frac{c_e}{c_\infty}\rho/\rho_\infty u_{,Y} = \left(\frac{c_e}{c_\infty}\right)^2 \rho/\rho_\infty \bar{\Psi}_{,YY}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \rho \frac{c_e}{c_\infty}\bar{\Psi}_{,Y}\left[\frac{P_e}{P_\infty}\frac{c_e}{c_\infty^2}\frac{dc_e}{d\xi}\bar{\Psi}_{,Y} + \frac{P_e}{P_\infty}\left(\frac{c_e}{c_\infty}\right)^2\bar{\Psi}_{,Y\xi} + \cancel{\frac{c_e}{c_\infty} Y_{,x}\bar{\Psi}_{,YY}}\right]$$

$$- \rho_\infty\left[\frac{P_e}{P_\infty}\frac{c_e}{c_\infty}\bar{\Psi}_{,\xi} + \cancel{\bar{\Psi}_{,Y} Y_{,x}}\right]\left(\frac{c_e}{c_\infty}\right)^2 \rho/\rho_\infty \bar{\Psi}_{,YY}$$

$$= \frac{P_e}{P_\infty}\left(\frac{c_e}{c_\infty}\right)^3\left[\bar{\Psi}_{,Y\xi}\bar{\Psi}_{,Y} - \bar{\Psi}_{,YY}\bar{\Psi}_{,\xi} + \frac{1}{c_e}\frac{dc_e}{d\xi}\bar{\Psi}_{,Y}^2\right]$$

$$= \frac{P_e}{P_\infty}\left(\frac{c_e}{c_\infty}\right)^3\left[\bar{\Psi}_{,Y\xi}\bar{\Psi}_{,Y} - \bar{\Psi}_{,YY}\bar{\Psi}_{,\xi}\right] + \frac{1}{c_e}u^2\frac{dc_e}{dx}$$

now $c_p \frac{dT_e}{dx} + u_e \frac{du_e}{dx} = 0$ & $2c_e \frac{dc_e}{dx} = (\gamma-1) c_p \frac{dT_e}{dx}$ then

$$\frac{1}{c_e}\frac{dc_e}{dx} = -\frac{(\gamma-1)}{2}\frac{u_e}{c_e^2}\frac{du_e}{dx} \qquad \therefore$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{P_e}{P_{\infty}}\left(\frac{c_e}{c_{\infty}}\right)^3\left[\bar{\Psi}_{,Y\xi}\bar{\Psi}_{,Y} - \bar{\Psi}_{,YY}\bar{\Psi}_{,\xi}\right] - \frac{(\gamma-1)}{2}u^2\frac{u_e}{c_e^2}\frac{du_e}{dx}$$

$$\mu\frac{\partial u}{\partial y} = \mu_{\infty}\left(\frac{c_e}{c_{\infty}}\right)^2\frac{P_e}{P_{\infty}}\bar{\Psi}_{,YY} \qquad \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \nu_{\infty}\frac{P_e}{P_{\infty}}\left(\frac{c_e}{c_{\infty}}\right)^3\bar{\Psi}_{,YYY}$$

$$\frac{\rho\mu}{\rho_{\infty}\mu_{\infty}} = \frac{\rho\mu}{\rho_e\mu_e}\,\frac{\rho_e\mu_e}{\rho_{\infty}\mu_{\infty}} = 1\cdot P_e/P_{\infty}$$

$$-\frac{1}{\rho}\frac{dp}{dx} = \frac{T}{T_e}u_e\frac{du_e}{dx} \qquad \text{define} \qquad S = \frac{c_pT + \tfrac{1}{2}u^2}{c_pT_{\infty}} - 1$$

$$1+S = \frac{T}{T_{\infty}} + \frac{u^2}{2c_pT_{\infty}} = \frac{T}{T_e}\left(\frac{c_e}{c_{\infty}}\right)^2 + \frac{(\gamma-1)}{2c_{\infty}^2}u^2 \;;\quad \left[(1+S) - \frac{\gamma-1}{2c_{\infty}^2}u^2\right]\frac{c_{\infty}^2}{c_e^2} = \frac{T}{T_e}$$

$$\therefore\quad -\frac{1}{\rho}\frac{dp}{dx} = \left\{(1+S)\left(\frac{c_{\infty}}{c_e}\right)^2 - \frac{\gamma-1}{2}\frac{u^2}{c_e^2}\right\}u_e\frac{du_e}{dx}$$

thus the momentum equation becomes

$$\frac{P_e}{P_{\infty}}\left(\frac{c_e}{c_{\infty}}\right)^3\left[\bar{\Psi}_{,Y\xi}\bar{\Psi}_{,Y} - \bar{\Psi}_{,YY}\bar{\Psi}_{,\xi}\right] = (1+S)\left(\frac{c_{\infty}}{c_e}\right)^2 u_e\frac{du_e}{dx} + \nu_{\infty}\frac{P_e}{P_{\infty}}\left(\frac{c_e}{c_{\infty}}\right)^3\bar{\Psi}_{,YYY}$$

since $\frac{du_e}{dx} = \frac{P_e}{P_{\infty}}\left(\frac{c_e}{c_{\infty}}\right)\frac{du_e}{d\xi}$

$$\bar{\Psi}_{,Y}\bar{\Psi}_{,Y\xi} - \bar{\Psi}_{,\xi}\bar{\Psi}_{,YY} = (1+S)\left(\frac{c_{\infty}}{c_e}\right)^2 u_e\frac{du_e}{d\xi} + \nu_{\infty}\bar{\Psi}_{,YYY}$$

let $u_e\frac{c_{\infty}}{c_e} = \tilde{u}_e$

$$\therefore\quad \bar{\Psi}_{,Y}\bar{\Psi}_{,Y\xi} - \bar{\Psi}_{,\xi}\bar{\Psi}_{,YY} = (1+S)\,\tilde{u}_e\frac{d\tilde{u}_e}{d\xi} + \nu_{\infty}\bar{\Psi}_{,YYY}$$

note that $\bar{\Psi}_{,Y} = u \frac{C_{\infty}}{C_e} = \tilde{u}$ define $\tilde{v} = -\bar{\Psi}_{,\xi}$

$$\therefore \quad \tilde{u}\,\tilde{u}_{,\xi} + \tilde{v}\,\tilde{u}_{,Y} = (1+S)\,\tilde{u}\,\frac{d\tilde{u}_e}{d\xi} + \nu_{\infty}\,\tilde{u}_{,YY}$$

if $\tilde{u}_e = K\xi^m$ where $m = \frac{\gamma-1}{3-5\gamma}$ and $u_e = U_0 x^m$ we can write

$$u_e = \frac{C_e}{C_{\infty}}\tilde{u}_e = \frac{C_e}{C_{\infty}} K\xi^m = U_0 x^m$$

write $\tilde{u} = \tilde{u}_e f'(\eta)$ where $\eta = Y\left[\frac{(m+1)}{2}\frac{\tilde{u}_e}{\nu_{\infty}\xi}\right]^{1/2}$

$$\tilde{u}_{,Y} = \tilde{u}_{,\eta}\,\eta_{,Y} = \tilde{u}_e f''\,\eta/Y$$

$$\tilde{u}_{,YY} = \tilde{u}_{,Y\eta}\,\eta_{,Y} = \tilde{u}_e f'''\,\eta^2/Y^2$$

$$\tilde{u}_e\frac{d\tilde{u}_e}{d\xi} = m\frac{\tilde{u}_e^2}{\xi}$$

$$\tilde{u}_{,\xi} = \tilde{u}_{,\eta}\,\eta_{,\xi} = \frac{m\tilde{u}_e}{\xi}f' + \frac{\tilde{u}_e}{\xi}f''\left(\frac{m-1}{2}\right)\eta$$

$$\tilde{v} = -\frac{\partial\bar{\Psi}}{\partial\xi} = -\left[\frac{\nu_{\infty}\tilde{u}_e(m+1)}{2\xi}\right]^{1/2}\left\{f + \left(\frac{m-1}{m+1}\right)f'\,\eta\right\}$$

taking these & placing them in the momentum equation leads to

$$\frac{m\tilde{u}_e^2}{\xi}f'^2 - \frac{\tilde{u}_e^2(m+1)}{2\xi}ff'' = (1+S)\frac{m\tilde{u}_e^2}{\xi} + \frac{\tilde{u}_e^2(m+1)}{2\xi}f'''$$

or $$f''' + ff'' = \frac{2m}{m+1}\left[f'^2 - 1 - S\right] = \beta\left[f'^2 - 1 - S\right]$$

1. since $u=0$ @ wall $\tilde{u}=0 \Longrightarrow f'(\eta)=0$ $\quad u=0$ @ $y=0 \Longrightarrow Y=0 \Longrightarrow \eta=0$
2. since $v=0$ @ wall & $\bar{\Psi}_{,Y}=0$ @ wall & because $\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = 0$ & since $\bar{\Psi}_{,YY} \neq 0 \Longrightarrow \bar{\Psi}_{,\xi}=0 \Longrightarrow \tilde{v}=0 \Longrightarrow f(0)=0$
3. since $\tilde{u} \to \tilde{u}_e \Longrightarrow f'(\eta) \to 1$; $y \to \infty \Longrightarrow Y \to \infty \Longrightarrow \eta \to \infty$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)u+\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=u\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

or

$$\rho u\frac{\partial}{\partial x}H_0+\rho v\frac{\partial}{\partial y}H_0=\frac{\partial}{\partial y}\left(\mu\frac{\partial}{\partial y}H_0\right) \qquad \text{for } P_r=1$$

Since $\quad (S+1)c_pT_{0\infty}=c_pT+\frac{1}{2}u^2$

$$\rho u\left[\frac{\partial}{\partial x}\left\{(S+1)c_pT_{0\infty}\right\}\right]+\rho v\left[\frac{\partial}{\partial y}\left\{(S+1)c_pT_{0\infty}\right\}\right]=\frac{\partial}{\partial y}\left[\mu\frac{\partial}{\partial y}\left\{(S+1)c_pT_{0\infty}\right\}\right]$$

$$\rho u c_pT_{0\infty}\frac{\partial S}{\partial x}+\rho v c_pT_{0\infty}\frac{\partial S}{\partial y}=\frac{\partial}{\partial y}\left\{\mu c_pT_{0\infty}\frac{\partial S}{\partial y}\right\}$$

$$\rho u\frac{\partial S}{\partial x}+\rho v\frac{\partial S}{\partial y}=\frac{\partial}{\partial y}\left\{\mu\frac{\partial S}{\partial y}\right\} \qquad \text{now}$$

$$\frac{\partial S}{\partial x}=\frac{P_e}{P_{0\infty}}\frac{c_e}{c_{0\infty}}S_{,\xi}+Y_{,x}S_{,Y} \quad , \quad \frac{\partial S}{\partial y}=\frac{c_e}{c_{0\infty}}\rho/\rho_{0\infty}S_{,Y}$$

$$\mu\frac{\partial S}{\partial y}=\frac{c_e}{c_{0\infty}}\frac{\mu\rho}{\rho_{0\infty}}S_{,Y}=\mu_{0\infty}\frac{c_e}{c_{0\infty}}\frac{P_e}{P_{0\infty}}S_{,Y} \qquad \text{from a previously developed formula}$$

$$\frac{\partial}{\partial y}\left(\mu S_{,y}\right)=\rho/\rho_{0\infty}\,\mu_{0\infty}\frac{c_e^2}{c_{0\infty}^2}\frac{P_e}{P_{0\infty}}S_{,YY} \qquad \text{thus}$$

$$\left[\rho\frac{c_e}{c_{0\infty}}\tilde{u}\right]\left[\frac{P_e}{P_{0\infty}}\frac{c_e}{c_{0\infty}}S_{,\xi}+\cancel{Y_{,x}S_{,Y}}\right]+\rho_{0\infty}\left[\frac{P_e}{P_{0\infty}}\frac{c_e}{c_{0\infty}}\tilde{v}-\cancel{Y_{,x}\tilde{u}}\right]\left[\frac{c_e}{c_{0\infty}}\rho/\rho_{0\infty}S_{,Y}\right]$$

$$=\rho\nu_{0\infty}\frac{c_e^2}{c_{0\infty}^2}\frac{P_e}{P_{0\infty}}S_{,YY} \quad ; \quad \text{thus}$$

$$\tilde{u}\,S_{,\xi}+\tilde{v}\,S_{,Y}=\nu_{0\infty}S_{,YY}$$

now

$$S_{,\xi}=S_{,\eta}\,\eta_{,\xi}=S'\,\frac{m-1}{2\xi}\,\eta \quad , \quad S_{,Y}=S_{,\eta}\,\eta_{,Y}=S'\,\eta/Y$$

$$S_{,YY} = S_{,Y\eta}\ \eta_{,Y} = S''\ \eta^2/Y^2 \quad ; \quad \tilde{u} = \tilde{u}_e f' \qquad \tilde{v} = -\eta/_Y\ \nu_{o\infty}\left[f + \frac{m-1}{m+1}\eta f'\right]$$

$$\cancel{\tilde{u}_e f'S' \frac{m-1}{2\xi}\eta} - \eta^2/_{Y^2}\,\nu_{o\infty} f S' - \cancel{\eta^2/_{Y^2}\,\nu_{o\infty}\frac{m-1}{m+1}\eta f'S'} = \nu_{o\infty} S''\eta^2/Y^2$$

$$\eta^2/_{Y^2}\,\nu_{o\infty}\left[S'' + f S'\right] = 0$$

1. when $u = 0$ $\quad T = T_W$ @ wall $\therefore$ $S(\eta) = \frac{T_W}{T_{o\infty}} - 1$ $\quad$ where $\eta = 0$

2. when $\eta \to \infty$ $\quad u \to u_e$ $\quad T \to T_e$ $\quad H_o \to H_{o\infty}$ $\quad \therefore$ $S(\eta) = \frac{H_{oe}}{H_{oe}} - 1 = 0$

$\boxed{2.}$ The momentum equation reduces to

$$\bar{\Psi}_{,Y}\,\bar{\Psi}_{,Y\xi} - \bar{\Psi}_{,\xi}\,\bar{\Psi}_{,YY} = \nu_{o\infty}\,\bar{\Psi}_{,YYY}$$

which in turn reduces to

$$f''' + f f'' = \frac{2m}{m+1} f'^2$$

Since $\frac{du_e}{dx} = 0$ $\quad T_e = $ const $\quad$ & $S = A + B\tilde{u}$; since $\tilde{u} = 0$ when $S = \frac{T_W}{T_{o\infty}} - 1 = S_W$ and $S = 0$ when $\tilde{u} = \tilde{u}_e$ $\quad S = S_W\,(1 - \tilde{u}/\tilde{u}_e)$; but $u/u_e = \tilde{u}/\tilde{u}_e$

$\therefore$ $S = S_W\,(1 - u/u_e)$. u/u_e can be found from $\sqrt{\frac{(T_o - T)}{(T_{oe} - T_e)}}$ where T_o is the stagnation temperature which ranges from

$$T_{oe} \le T_o \le T_W$$

$\boxed{3.}$

a) $\mu\frac{\partial u}{\partial y} = \mu_{o\infty}\left(\frac{c_e}{c_{o\infty}}\right)^2\frac{P_e}{P_{o\infty}}\,\tilde{u}_e f''\,\eta/_Y$ $\qquad$ $\frac{1}{2}\rho(x,0)\,u_e^2$ = dynamic pressure

$$c_f = \frac{\mu_{o\infty}}{2\rho(x,0)}\left(\frac{c_e}{c_{o\infty}}\right)^2\frac{P_e}{P_{o\infty}}\,\frac{c_{o\infty}}{c_e}\,\frac{1}{u_e}\,f''\,\eta/_Y$$

b) $\dot{q}/A = k\frac{\partial T}{\partial y} = kT_{o\infty}\frac{\partial}{\partial y}\left[(1+S) - \frac{\gamma-1}{2c_{o\infty}^2}u^2\right]$ $\quad$ when $\frac{\partial S}{\partial y} = \frac{c_e}{c_{o\infty}}\frac{P(x,0)}{P_{o\infty}}\left[\frac{(m+1)\,c_{o\infty}}{2\nu_{o\infty}\,\xi u_e c_e}\right]^{1/2} S'(0)$

and $\dfrac{\partial u^2}{\partial y} = \dfrac{c_e^2}{c_{o\infty}^2} \dfrac{\partial}{\partial y} \bar{\Psi}_{,Y}^{\,2} = 2 \dfrac{c_e}{c_{o\infty}} \dfrac{\rho(x,0)}{\rho_{o\infty}} u_e^2 \left[\dfrac{(m+1)\, c_{o\infty}}{2\nu_{o\infty} \int u_e c_e}\right]^{1/2} f'(0)\, f''(0)$

or $\dfrac{\partial T}{\partial y} = \dfrac{\partial T}{\partial Y} \dfrac{\partial Y}{\partial y} = \dfrac{c_e}{c_{o\infty}} \dfrac{\rho(x,0)}{\rho_{o\infty}} \left[\dfrac{(m+1)\, c_{o\infty}}{2\nu_{o\infty} \int u_e c_e}\right]^{1/2} T'(0)$

Convective Heat Transfer

Natural – fluid motion due to density change; buoyancy effect $Nu = f(Pr, Gr)$

Forced – fluid motion " " external source $Nu = f(Re, Pr)$

$Pr = \frac{\mu c_p}{k}$ $\frac{\text{molecular diffusivity of mom.}}{\text{molecular diffusivity of heat}}$ $\frac{\text{viscous}}{\text{conduct}}$

$Nu = \frac{hL}{k}$ $\frac{\text{conductive thermal resistance}}{\text{convective thermal resistance}}$ $\frac{\text{convect}}{\text{conduct}}$

$Gr = \frac{\beta g \rho^2 L^3 \Delta T}{\mu^2}$ $\frac{\text{buoyancy forces}}{\text{viscous forces}}$

from molecular consideration $\bar{c} \sim T^{1/2}$

transport of momentum $\mu \sim \bar{c} \sim T^{1/2}$ $\tau = \mu \frac{du}{dy}$

Energy $k \sim \bar{c} \sim T^{1/2}$ $\bar{q} = -k \frac{\partial T}{\partial y}$ Fourier

Mass $D_{AB} \sim T^{1/2}/p$ $j = -\rho D_{AB} \frac{dw_A}{dy}$ Fick's $w_A = \rho_A/\rho$

Conduction – molecular interaction, free electrons.

Radiation depends on temp difference & actual level of temperature & due to internal molecule struct change.

Molecular transport depend on ΔT

Mass Conservation $\frac{\partial}{\partial t}[\rho \Delta V] = (\rho \cdots)(u \cdots)$ $\frac{D\rho}{Dt} + \rho \nabla \cdot \bar{q} = 0$

Navier Stokes rate of change of moment $= \sum$ external forces $\rho \frac{D\bar{q}}{Dt} = -\nabla p + \bar{F} + \nabla \cdot \bar{\tau}$

$\bar{\tau} = \bar{\tau}_x i + \bar{\tau}_y j + \bar{\tau}_z k$

Energy rate of change of energy = rate work is done on volume by external forces + rate of heat addition

$$\rho \frac{D(e + q^2/2)}{Dt} = -\nabla(up) + \frac{\partial}{\partial x}(\bar{q} \cdot \bar{\tau}_x) + \dots + \bar{q} \cdot \bar{F} + \dot{Q} + \dot{q}$$

$$\Phi = \bar{\tau}_x \cdot \frac{\partial \bar{q}}{\partial x} + \dots = \bar{\tau} \cdot \nabla \bar{q}$$ viscous dissipation

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \Phi + \dot{Q} + \nabla \cdot (k \nabla T)$$ $\Phi = 2\mu\left[u_x^2 + v_y^2 + w_z^2\right] + \mu\left[(u_y + v_x)^2 + \dots\right.$

$$\rho c_v \frac{DT}{Dt} = -p \nabla \cdot \bar{q} + \Phi + \dot{Q} + \nabla \cdot (k \nabla T)$$

2-D Steady State Conduction $\nabla^2 T = 0$

Fourier transf for bc at $\pm\infty$

Laplace " " bc at $(\pm)\infty$, 0

Relax

$$\begin{array}{ccc} & \circ & \\ \circ & \circ & \circ \\ & \circ & \end{array} \quad \sum_{j,i=1}^{4} T_{ij} - 4T_{i,i} = 0$$

Numerical Methods

1) Relaxation
2) Successive Approx } steady state boundary value
3) Marching tech – initial value, unsteady convect.

Unsteady Cond $\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T)$

Newtonian heating $k \to \infty$ $\dot{q}$ is finite $\frac{\partial T}{\partial x_i} \to 0$

Rayleigh $h \to \infty$ $\Delta T \to 0$

Marching $0 = \theta$ – explicit

$\frac{1}{2} = \theta$ – Implicit best error

$1 = \theta$ – fully implicit

Convection   molecular conduction will excite velocity of fluid

Velocity b.l.

$Re = \frac{1}{\sqrt{\delta}}$ $p'(x) \gg p_{,y} = o(\delta)$ $v(\delta)$ $u_{yy} \gg u_{,xx}$

Thermal bl
1. Steady flow
2. $\dot{Q} = 0$
3. neglect buoyancy
4. $M_\infty \ll 1$
5. Flow 2-D

$$\rho C_p \frac{DT}{Dt} = u p'(x) + k \frac{\partial^2 T}{\partial y^2} + \mu u_y^2$$

$$\frac{C_p - C_v}{C_p} = \frac{R}{C_p}$$

$$\frac{\gamma - 1}{\gamma} \qquad C_p = \frac{\gamma R}{\gamma - 1}$$

Natural Convect if $G/Re^2 \ll 1$ neglect buoyancy

$G/Re^2 \sim 1 \geqslant 1$ use boussinesq

1. neglect ρ in cont.
2. $F_y - P_{,y} = o(\delta)$ $p_{,x,y} = p^*_{,x,y} + \rho_\infty g_{x,y}$ induced by motion, hydrostatic   if $\rho - \rho_\infty \sim o(1)$ $P_{,y} \sim o(1)$ $\therefore \Delta P \sim o(\delta)$
3. assume $p^*_{,x} \to 0$

Equat   cont

$$\rho u u_{,x} + \rho v u_{,y} = -\rho \beta_\infty (T - T_\infty) g_x + \mu u_{,yy}$$

Convection - if molecular conduction induces velocity in the flow

Natural convection - due to heating buoyancy & hydrostatic effect large

Forced convection - inertia forces are large

1. If inertia forces are small low mach flow $\mu \sim$ const, $\rho \sim$ const
2. " " " " large high " " $\rho \not\sim$ const, $\mu \not\sim$ const

Thermal bl - Continuity $\frac{D\rho}{Dt} + \rho \operatorname{div} \bar{q} = 0$

Mom. $\rho \frac{D\bar{q}}{Dt} = -\nabla p + \bar{F} + \sum_{i=1}^{3} \frac{\partial \bar{\tau}_{x_i}}{\partial x_i}$

$\bar{\tau}_{x_i} = \sum_{j=1} \tau_{x_j x_i} \bar{x}_j$ (unit vector)

temp. $\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \Phi + \dot{Q} + \nabla \cdot (k \nabla T)$

where $\Phi = \left\{ 2\left(u_{,x}^2 + v_{,y}^2\right) + \left(v_{,x} + u_{,y}\right)^2 - \frac{2}{3}\left(\nabla \cdot \bar{q}\right)^2 \right\} \mu$

(1) Steady Flow (5) Flow 2-D

(2) $\dot{Q} = 0$

(3) Neglect Buoyancy

(4) $M_\infty << 1$

if non dim wrt $u_\infty, L, T_\infty, \rho u_\infty^2$

$$\bar{u}_{,\bar{x}} + \bar{v}_{,\bar{y}} = 0$$

$$\bar{u}\,\bar{u}_{,\bar{x}} + \bar{v}\,\bar{u}_{,\bar{y}} = -\bar{p}_{,\bar{x}} + \frac{1}{R_N}\left[\bar{u}_{,\bar{x}\bar{x}} + \bar{u}_{,\bar{y}\bar{y}}\right]$$

$$\bar{u}\,\bar{v}_{,\bar{x}} + \bar{v}\,\bar{v}_{,\bar{y}} = -\bar{p}_{,\bar{y}} + \frac{1}{R_{N_2}}\left[\bar{v}_{,\bar{x}\bar{x}} + \bar{v}_{,\bar{y}\bar{y}}\right]$$

$$\bar{u}\,\bar{T}_{,x} + \bar{v}\,\bar{T}_{,y} = \frac{E}{R_N}\left[2\left(\bar{u}_{,\bar{x}}^2 + \bar{v}_{,\bar{y}}^2\right) + \left(\bar{v}_{,\bar{x}} + \bar{u}_{,\bar{y}}\right)^2\right]$$

$$+ \frac{1}{P_r R_N}\left[\bar{T}_{,\bar{x}\bar{x}} + \bar{T}_{,\bar{y}\bar{y}}\right] + E\left[\bar{u}\,\bar{p}_{,\bar{x}} + \bar{v}\,\bar{p}_{,\bar{y}}\right]$$

where $E = \frac{U_\infty^2}{c_p T_\infty}$ $P_r = \frac{\mu c_p}{k}$ $\frac{\text{viscous}}{\text{conductive}}$ $R_N = \frac{\rho u_\infty L}{\mu}$ $\frac{\text{inertia}}{\text{viscous}}$

E is Eckert No Prandtl Reynolds

assume $x/L \sim O(1)$ $y/L \sim O(\delta)$ $u/u_\infty \sim O(1) \rightarrow v/u_\infty \sim O(\delta)$ || $T/T_\infty \sim O(1)$

from order of magnitude check $\bar{u}_{,\bar{y}\bar{y}} \gg \bar{u}_{,\bar{x}\bar{x}}$ $\bar{v}_{,\bar{y}\bar{y}} > \bar{v}_{,\bar{x}\bar{x}}$

@ edge $\bar{p}(\bar{x}) \sim O(1)$ $\therefore$ $R_N \sim O\left(\frac{1}{\sqrt{\delta}}\right)$ $\bar{p}_{,\bar{y}} \sim O(\delta) \Rightarrow \bar{p} = \bar{p}(\bar{x})$ only

from energy eq $\bar{u}_{,\bar{y}} \gg \bar{v}_{,\bar{x}}, \bar{u}_{,\bar{x}}, \bar{v}_{,\bar{y}}$ || $\bar{T}_{,\bar{y}\bar{y}} > \bar{T}_{,\bar{x}\bar{x}}$ || $\bar{u}\,\bar{p}_{,\bar{x}} \gg \bar{v}\,\bar{p}_{,\bar{y}}$

$\therefore$ Eqn reduce to

$$\bar{u}\,\bar{u}_{,\bar{x}} + \bar{v}\,\bar{u}_{,\bar{y}} = -\bar{p}_{,\bar{x}} + \frac{1}{R_N}\bar{u}_{,\bar{y}\bar{y}}$$

$$\bar{u}\,\bar{T}_{,\bar{x}} + \bar{v}\,\bar{T}_{,\bar{y}} = E(\bar{u}\,\bar{p}_{,\bar{x}}) + \frac{1}{P_r R_N}\bar{T}_{,\bar{y}\bar{y}} + \frac{E}{R_N}\bar{u}_{,\bar{y}}^{\,2}$$ ← viscous disp term.

$$\bar{u}_{,\bar{x}} + \bar{v}_{,\bar{y}} = 0$$

$$\bar{p}_{,\bar{x}} = \bar{u}\,\bar{u}_{,\bar{x}}$$

if low speed $E \rightarrow 0$ & eq reduce to

$$\bar{u}\,\bar{T}_{,\bar{x}} + \bar{v}\,\bar{T}_{,\bar{y}} = \frac{1}{P_r R_N}\bar{T}_{,\bar{y}\bar{y}}$$

use Taylor Expansion on u, v assuming $y = \delta_T$ if ① or $y = \delta$ if ②

① if $P_r \sim O(1)$ or greater viscosity $\gg$ conduct $\therefore \delta > \delta_T$ $\frac{\delta_T}{\delta} \sim P_r^{-1/3}$

② if $P_r \rightarrow 0$ " $\ll$ " $\therefore \delta < \delta_T$ " $\sim P_r^{-1/2}$

$O\left(\frac{1}{\delta_T}\right) \sim Nu = \frac{hL}{K}$ $\frac{\text{convective resistance}}{\text{conductive resistance}}$, $Gr = \frac{\beta g \rho^2 L^3 \Delta T}{\mu^2}$ $\frac{\text{buoyancy}}{\text{viscous}}$

Stanton $= \frac{Nu}{Pr\,Re} = \frac{\frac{conv}{con}}{\frac{vis}{con}\cdot\frac{int}{vis}} = \frac{\text{convective}}{\text{inertia term}}$, reynolds anal $C_H = \frac{C_f}{2}$

Integral Method

$$\frac{d\theta}{dx} + \frac{1}{u_e}\frac{du_e}{dy}\left(2\delta^* + \theta\right) = \frac{C_f}{2} \quad \text{where } \theta = \int_0^\delta \frac{u}{u_e}(1 - u/u_e)\,dy$$

$$\delta^* = \int_0^\delta (1 - u/u_e)\,dy$$

$$\frac{d}{dx}\int_0^{\delta_T}(u\,T_{\delta_T} - uT)\,dy = \frac{k}{\rho c_p}T_{,y}\Big|_{y=0} - \frac{\mu}{\rho c_p}\int_0^{\delta_T} u_{,y}^2\,dy$$

Natural Convection:

heat body so that density decreases & particles rise

Neglect density variation as it effects continuity but not if it affects lift

Boussinesq approx $\rho =$ const in continuity

$$u u_{,x} + v u_{,y} = -\frac{p_{,x}}{\rho} + \frac{F_x}{\rho} + \nu u_{,yy}$$

$$F_y - P_{,y} = O(\delta)$$

$$u T_{,x} + v T_{,y} = \frac{\nu}{c_p} u_{,y}^2 + \frac{1}{\rho c_p} u p'(x) + \alpha T_{,yy} \qquad \alpha = \frac{k}{\rho c_p}$$

$$F_x = \rho g_x \qquad F_y = \rho g_y \qquad \text{body force}$$

hydrostatic equation $\nabla p = \rho_\infty \bar{g}$

assume $p_{,x} = p^*_{,x} + \rho_\infty g_x$ \quad *quantity is induced by motion.

$$p_{,y} = p^*_{,y} + \rho_\infty g_y$$

if $R_N >> 1$ use b.l. eq $\qquad p^*_{,y} = (\rho - \rho_\infty) g_y + O(\delta)$

if $\rho - \rho_\infty \sim O(1) \qquad P_{,y} \sim O(1) \qquad \Delta P \sim O(\delta)$

$p^*_{,x} \Rightarrow 0$ at bl $p^* = 0$ assumed $p^*_{,x}$ not fn of y.

since $u p'(x) = u p^*_{,x} + u \rho_\infty g_x \}$ cancelled because it press work term.

$$\rho(T, p) = \rho_\infty + \left(\frac{\partial \rho}{\partial T}\right)(T - T_\infty) + \ldots \text{ HOT.}$$

$$\therefore \quad \rho - \rho_\infty = \frac{\partial \rho}{\partial T}\Big)_\infty (T - T_\infty) = -\beta_\infty \rho (T - T_\infty) \qquad \text{where } \beta = \frac{1}{T}$$

$$\therefore \quad u_{,x} + v_{,y} = 0$$

$$\sim \frac{U_\infty^2}{L} \frac{Gr}{R_N^2}$$

$$\rho u u_{,x} + \rho v u_{,y} = -\rho \beta_\infty (T - T_\infty) g_x + \mu u_{,yy}$$

$$u T_{,x} + v T_{,y} = \alpha T_{,yy} + \frac{\nu}{c_p} u_{,y}^2 \quad \text{neglect } O\left(\frac{\bar{u}_m^2}{c_p T_\infty}\right) \sim \frac{\Delta T}{T_\infty} << 1$$

$$R_i = \frac{\text{body force}}{\text{inertia effect}} \cong \frac{\text{body force}}{\text{viscous}} \cdot \frac{\text{visc}}{\text{inert}} = \frac{Gr}{Re}$$

Compressible b.l.

$V \gg$ buoyancy effect neglected, body force terms neglected

$\rho \neq$ const

1. $\rho = \rho(x,y,z,t)$ 2. $\mu = \mu(T)$, $k = k(T)$

2. neglect real gas effects

3. Flow is inviscid & irrotational.

Gov. Eq

1. $\bar{F} = \dot{Q} = 0$

2. 2-D

3. Assume Pr = const c_p = const.

Cont. $\nabla \cdot (\rho \bar{q}) = 0$

Mom. $\rho(\bar{q} \cdot \nabla)\bar{q} = -\nabla p + \frac{\partial}{\partial x}\bar{\tau}_x + \frac{\partial}{\partial y}\bar{\tau}_y$ $\bar{\tau}_x = \left[2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu \nabla \cdot \bar{q},\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]$

Energy $\rho c_p (\bar{q} \cdot \nabla) T = \bar{q} \cdot \nabla p + \Phi + \nabla \cdot (k \nabla T)$

non dim $v, u \to U_\infty$ $\mu \to \mu_\infty$ $\rho \to \rho_\infty$

$T \to T_\infty M_\infty^2$ $p \to \rho_\infty U_\infty^2$ $x, y \to L$

Assume $\bar{u} = O(1)$ $\bar{\rho} = O(1)$ $\frac{\partial}{\partial x} \sim O(1)$ $\frac{\partial}{\partial y} \sim O\left(\frac{1}{\delta}\right)$ $\bar{\mu} \sim O(1)$

cont $\Rightarrow v \sim O(\delta)$

Eq reduce to

$(\rho u)_{,x} + (\rho v)_{,y} = 0$

$\rho u\, u_{,x} + \rho v\, u_{,y} = -p'(x) + (\mu u_{,y})_{,y}$

$\rho u c_p T_{,x} + \rho v c_p T_{,y} = u p'(x) + \mu u_{,y}^2 + \frac{c_p}{Pr}(\mu T_{,y})_{,y}$

Busemann Integral H_0 = const

Crocco Integral $p'(x)$ is small & can be neglected in mom. eq. $H_0 = A + Bu$ $Pr = 1$ $C_H = \frac{C_f}{2}$

Howarth Transformation $\xi = x$ $Y = \int_0^y \rho/\rho_\infty\, dy$

low speed flow $\delta/x \sim \frac{1}{\sqrt{R_N}}$, high speed flow $\frac{\delta}{x} \sim \frac{M_\infty^2}{\sqrt{R_N}}$

recovery temperature - temperature at wall that would exist if $\dot{q}_{w/A} = 0$

Radiation - doesn't require any sort of media. heat transfer depends on ΔT

& absolute temperatures of each body

radiation characteristics are given by λ & ν $\quad$ $\lambda\nu = c$

Energy $E = h\nu$ $\quad$ Momentum $p = \frac{h\nu}{c}$

radiation sudden transition of atomic & molecular structure bound-bound

if electron frees itself $\quad$ bound free

if free electrons collide $\quad$ free-free

1. Radiant Heat Flux vector $\bar{q}^R$ - amount of radiation/unit area that is transferred across a fictitious plane in space.

2. E - total emissive power/unit area - net rate at which thermal radiant energy is transferred across a real or imaginary surface & directed normal to that surface.

3. E_ν or E_λ - monochromatic radiant power/unit area.

   $E_{b,\lambda}$ - monochr rad. power/unit area from a black body.

   black body - absorbs all radiation energy incident upon it.

Isotropic flow - heat transferred equally in all direction where $\bar{q}^R = E = 0$

4. emissivity $\epsilon = E/E_b$ $\quad$ total emissive power/total emissive power of an ideally radiating surface at same temp.

5. f_Ω^R - photon distribution function - fraction of photons between freq. ν & $\nu + d\nu$ and included in the solid angle $d\Omega$ with unit area dA and normal $\bar{n}$ or n_j

6. n^R - no. of photons/unit volume at a point - photon density.

7. I_ν - monochromatic intensity of radiation $= ch\nu\, n_\Omega^R f_\Omega^R$

8. $n^R = \int_\Omega \int_\nu n_\Omega^R f_\Omega^R \, d\nu \, d\Omega$ $\quad$ photon density at a pt.

9. energy photon density $u^R = \int_\Omega \int_\nu \underset{\text{energy}}{h\nu} \, n_\Omega^R f_\Omega^R \, d\Omega \, d\nu$

   mom. photon density M^R. $\quad$ $\underset{\text{mom.}}{\frac{h\nu}{c}\bar{n}}$

Fluxes

flux of photons / unit area $= \overset{\text{veloc}}{c n_j} \quad \overset{\text{no. density}}{n^R_\Omega f^R_\Omega\, d\Omega\, d\nu}$

$\bar{q}^R$ energy flux " " / unit area $= h\nu \; c n_j \; n^R_\Omega f^R_\Omega\, d\Omega\, d\nu$

p^R_{ij} momentum flux " " / unit area $= \frac{h\nu}{c} n_i \; c n_j \; n^R_\Omega f^R_\Omega\, d\Omega\, d\nu$

for an isotropic medium $I_\nu(\theta, \varphi, \nu) = I_\nu(\nu)$

$$\bar{q}^R = I \int_\Omega \bar{n}\, d\Omega \quad ; \quad E = \pi I$$

$$u^R = \frac{4\pi I}{c} \qquad p^R_{ij} \begin{cases} 0 & i \neq j \\ \frac{4\pi I}{3c} & i = j \end{cases}$$

$\rho_\lambda + \alpha_\lambda + \tau_\lambda = 1$ $\quad \tau_\lambda$ is very small since most bodies are opaque.

specular reflection $\measuredangle i = \measuredangle r$

diffuse " reflection is all direct.

Black body $\rho_\lambda = 0 \quad \alpha_\lambda = 1$

Kirchoffs Law $\quad \epsilon_\lambda = \alpha_\lambda \quad$ radiation incident on body = total rad from body
for thermo equil.

$$I_{b,\nu} = B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$

Stefan – Boltzmann Laws

$$E_b = \pi B = \sigma T^4$$

Gray surface – $\alpha_\lambda = \alpha \;\;\&\;\; \alpha \neq \alpha(\lambda, \nu) \quad \alpha = \epsilon$

G – irradiation is defined as the rate at which radiation is incident on the surface / unit area

J – radiosity rate at which radiation leaves a surface / unit area.

Radiant Transfer from 2 black bodies

$$q_{Net} = \sigma(T_1^4 - T_2^4) A_1 F_{12} = \sigma(T_1^4 - T_2^4) A_2 F_{21} \qquad F_{12}$$

F_{12} – fraction of radiation emitted by body 1 that reaches body 2.

$$F_{21} A_2 = F_{12} A_1 \qquad \sum_{N=1}^{k} F_{1N} = 1$$

$J = \rho G + \epsilon E$ $\qquad$ $q_{Net} = A(J - G)$

$q_{Net} = \frac{\Delta E}{\sum R}$ $\qquad$ $R = \rho_1 / A_1\epsilon_1 + \frac{\rho_2}{A_2\epsilon_2} + \dots + \frac{1}{A_1 F_{12}}$

Gases $\rho \sim 0$ $\alpha, \tau \neq 0$. Radiative Energy is intermittent, energy depends on ease of excitation & gas constituents. Emission is not at one certain freq but in a small band due to uncertainty principle & doppler effect.

time rate of change of number of photons in volume dV = net rate at which photons are convected through surfaces of volume + rate at which photons are created or destroyed in dV by emission or absorption.

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \sum \frac{\partial}{\partial x_j}(n_j I_\nu) = h\nu (n^R f^R)_t = \alpha_\nu' (B_\nu - I_\nu) = \rho(j_\nu - K_\nu I_\nu)$$

emission $\qquad$ absorption

α_ν' - volumetric absorption coefficient

j_ν - mass emission coeff $\quad$ K_ν - mass absorption coeff

since $\frac{1}{c}\frac{\partial I_\nu}{\partial t} \sim \frac{1}{c}$ eq. reduces $\sum \frac{\partial}{\partial x_j}(n_j I_\nu) = \alpha_\nu'(B_\nu - I_\nu) = -\frac{\partial I_\nu}{\partial r}$

r - direction of motion of photons,

τ_ν - optical depth $= \int_0^r \alpha_\nu' dr$ $\qquad$ $I_\nu = I_\nu(r_s) e^{-\tau_\nu(r_s)} + \int_0^{r_s} \alpha_\nu' B_\nu e^{-\tau_\nu(r)} dr$

r_s - location of initial condition.

Thick gas if $\alpha_\nu' \gg 1$ $\quad$ opaque gas $I_\nu e^{-\tau_\nu} \ll 1$ $\qquad$ $I_\nu = B_\nu(0)$

Thin gas $\quad \alpha_\nu' \ll 1$ $\quad$ transparent gas $I_\nu \sim I_\nu(r_s) + \int_0^{r_s} \alpha_\nu' B_\nu dr$

Energy Dominated Flow.

$$\dot{Q}^R = \nabla \cdot q^R = \nabla \cdot \iint I_\nu n_j \, d\Omega \, d\nu = -\int_0^\infty \int_\Omega \frac{\partial I_\nu}{\partial r} d\Omega \, d\nu = -\int_0^\infty \int_\Omega \alpha_\nu'(I_\nu - B_\nu) d\Omega$$

no absorption $\quad$ thin gas $\qquad \dot{Q}^R = 4\pi \alpha_P \sigma T^4$ $\qquad \alpha_P$ - plank mean absorption coeff

no emission $\quad$ thick gas $\qquad I_\nu = \int_0^{\tau_\nu} B_\nu e^{-\tau_\nu} d\nu$

$$\bar{q}^R = -\frac{16}{3}\frac{\sigma \pi T^3}{\alpha_R}\frac{\partial T}{\partial x_j} \qquad \alpha_R \text{ - rosseland mean absorption coeff,}$$

$$\dot{Q}^R = \sum_{j=1}^{3} \frac{\partial q_j^R}{\partial x_j}$$

Mass Transfer - dependent on gradient of concentration of particular species

$\frac{\nu}{D_{AB}} = S_c \quad \frac{mom}{mass}$ Fick $j_A = -\rho D_{AB} \nabla \omega_A$

Mass av. $\bar{q} = \frac{\sum \bar{n}_i}{\rho}$ $\quad \bar{n}_i = \rho_i \bar{q}_i$ $\quad \bar{q}_i = \bar{q} + \bar{v}_i$ mass veloc

Molar av. $\bar{V} = \frac{\sum \bar{N}_i}{\bar{c}}$ $\quad \bar{N}_i = \bar{c}_i \bar{q}_i$ $\quad \bar{q}_i = \bar{V} + \bar{V}_i$ molar veloc

Mass flux/unit area, time $\quad \bar{j}_A = \rho_A \bar{v}_A = \rho_A(\bar{q}_A - \bar{q}) = -\rho D_{AB} \nabla \omega_A$

$\bar{n}_A = -\rho D_{AB} \nabla \omega_A + \omega_A(\bar{n}_A + \bar{n}_B)$

Multiple species take into account

1. Rate of production of a species in mass conservation for that species
2. Energy produced by reaction or mechanism that creates species
3. BC due to mass transfer, chemical surface effects.

Mass Cons.

$$\frac{\partial \rho_A}{\partial t} + \nabla \cdot \rho_A \bar{q}_A = \dot{r}_A$$ rate of production in volume. $\quad \sum \dot{r}_A = 0$ if no source or sink in vol

$\dot{r}_A = 0$ frozen flow.

$$\frac{D\rho_A}{Dt} + \rho_A \nabla \cdot \bar{q}_A = \dot{r}_A$$

$$\rho \frac{D\omega_A}{Dt} + \nabla \cdot \bar{j}_A = \dot{r}_A \quad ; \quad \rho \frac{D\omega_A}{Dt} = \rho D_{AB} \nabla^2 \omega_A + \dot{r}_A$$ low speed $\rho \sim$ const $D \sim$ const

for frozen flow reduce to $\frac{D\omega_A}{Dt} = D_{AB} \nabla^2 \omega_A$

$$\bar{U}\frac{\partial \omega_A}{\partial \bar{x}} + \bar{V}\frac{\partial \omega_A}{\partial \bar{y}} = \frac{1}{S_c R_N}\left(\cancel{\frac{\partial^2 \omega_A}{\partial \bar{x}^2}} + \frac{\partial^2 \omega_A}{\partial \bar{y}^2}\right)$$ drop $\quad$ b.l.

if $S_c = 1$ flat plane problem $p'(x) = 0$ $\quad \rho_A = Au + B$

Boundary Condition

1. specify ρ_A or ω_A
2. " $\bar{n}_A$ or $\bar{N}_A$
3. $\bar{n}_A = k(\rho_A)$ catalytic wall condition
4. liq or solid with gaseous interface

$$\bar{n}_{A_\perp} = \rho_A \bar{q}_{A_\perp} = -\rho D_{AB} \frac{\partial \omega_A}{\partial n} + \rho_A v_{inter} = k_c(\rho_A - \rho_{A\infty})$$

5. injection $v_{wall} \neq 0$

energy can be transferred without ΔT

energy ~ velocity of particles or mass

if $Sc \geq 1$ $\frac{\delta_c}{\delta} \sim Sc^{-1/3}$ $\frac{\delta_c}{\delta} \sim Sc^{-1/2}$ $Sc \to 0$