# EXAMINATION BOOK

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Subject AE 331

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Exam Seat No. ______ Section ______

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AE331 Mid-Term Exam.

1. (a). Continuum : P. 7-10 , Prandtl and Tietjens, "Hydro- & Aeromechanics".

(b). Pressure at a point : p. 2 , Kuethe and Schetzer.

(c). Compressibility : P. 225-229 , Prandtl and Tietjens.

(d). Vorticity : p. 32-33 , Kuethe and Schetzer.

(e). Lagrangian and Eulerian : p. 69 , Prandtl and Tietjens.

(f). stagnation value : p. 146 , Kuethe and Schetzer.

(g). state of equation : p. 129-132 , Kuethe and Schetzer

2. (a). $\nabla^2 F = 0$ if $F \equiv \varphi$ , $F_{xx} = 4a - 6cx$ , $F_{yy} = -2 + 2x$

$\therefore \quad a = \frac{1}{2}$ , $c = \frac{1}{3}$

(b). $\frac{dy}{dx} = \frac{v}{u}$ streamline , or $\frac{dy}{dx} = \frac{xy^2}{x^2y} = \frac{y}{x}$

or $\frac{dy}{y} = \frac{dx}{x}$ , $y = cx$ , $c = 2$ , $y = 2x$ ←

(c). steady flow , $\frac{dx}{dt} = u = x^2 y = x^2(2x) = 2x^3$

or $2t = c_1 - \frac{1}{2x^2}$ , $t = 0, x = 1$ , $c_1 = \frac{1}{2}$ , $2t = \frac{1}{2}(1 - \frac{1}{x^2})$

$\frac{dy}{dt} = v = xy^2 = \frac{1}{2}y^3$ , $c_2 - y^{-2} = t$ , $t = 0$ , $y = 2$ , $c_2 = \frac{1}{4}$

$\therefore \quad t = \frac{1}{4} - \frac{1}{y^2}$ and $4t = 1 - \frac{1}{x^2}$ , when $t = 0.1$ , $x = \sqrt{\frac{5}{3}}$ , $y = \sqrt{\frac{20}{3}}$

(d). $\frac{Du}{Dt} = \cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = x^2y(2xy) + y^2x^3 = 3x^3y^2 = 12$

$\frac{Dv}{Dt} = \cancel{\frac{\partial v}{\partial t}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = x^2y(y^2) + y^2x(2xy) = 3x^2y^3 = 24$

(e). steady : $(\rho u)_x + (\rho v)_y = 0$ , $\rho = xy$

$\frac{\partial}{\partial x}[(xy)(x^2y)] + \frac{\partial}{\partial y}[(xy)(y^2x)] = 3x^2y^2 + 3x^2y^2 \neq 0$

... this density distribution is not possible.

3. $\frac{\partial \rho'}{\partial t} + \rho_\infty \frac{\partial u'}{\partial x} = 0$ , $\frac{\partial u'}{\partial t} = \frac{1}{\rho_\infty}\frac{\partial p'}{\partial x} = \frac{a_\infty^2}{\rho_\infty}\frac{\partial \rho'}{\partial x}$

by cross-differentiation, we get

$$\frac{\partial^2 u'}{\partial t^2} - a_\infty^2 \frac{\partial^2 u'}{\partial x^2} = 0 \ , \quad \frac{\partial^2 \rho'}{\partial t^2} - a_\infty^2 \frac{\partial^2 \rho'}{\partial x^2} = 0 \ , \quad p' = \rho' a_\infty^2$$

These are wave equations, and disturbance travels at speed $a_\infty$. Then

$\rho' = f(x - a_\infty t)$

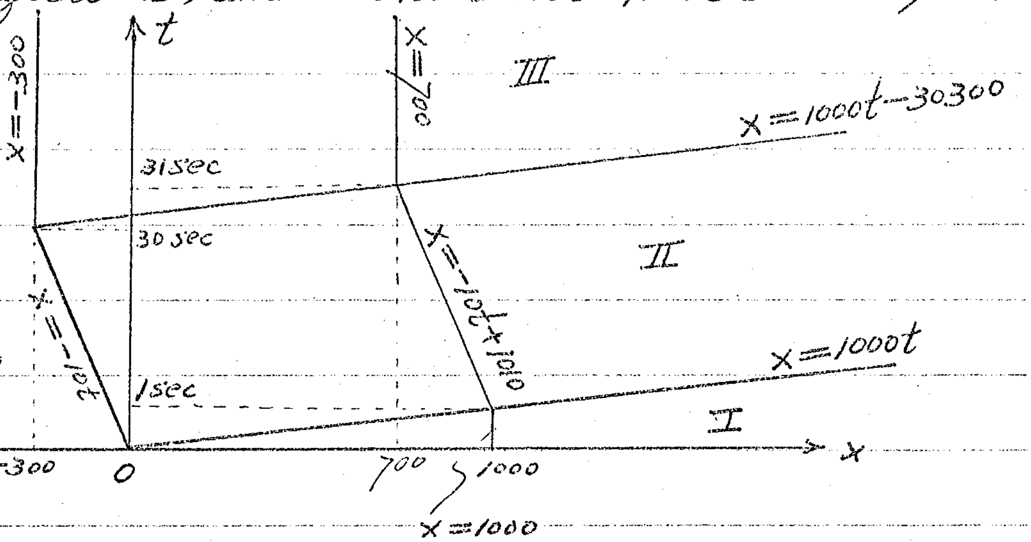
$u' = \frac{a_\infty}{\rho_\infty} f(x - a_\infty t)$

$= u_p = \begin{cases} -10 & 0 < t < 30 \\ 0 & t > 30 \end{cases}$ For PARTICLE

$\therefore$ region I, flow is undisturbed

II, $\rho' < 0$, $p' < 0$, $1 < t < 31$

III, undisturbed, $t > 31$




4. (a) $p_0 = 200$ psia, $T_0 = 1000°R$ , $\gamma = 1.4$

Sonic condition at the throat, $\frac{A^*}{A_3} = \frac{A_T}{A_4} = 0.2$

from Table 3, p. 426, $\frac{p_0}{p_4} = 0.021$ , $p_4 = 4.2$ psia

Thus $p_a \leq 4.2$ psia. $M_{EXIT} \approx 3.17$ ←

(Since $A_3 = A_4$, under quasi-one dimensional flow approximation, the flow properties are constant between 3 and 4). $\left(A_3/A^* = A_4/A^*\right)$ $M_3 = M_4$

(b). $\frac{A_T}{A_2} = 0.333$ , from Table, $M_2 = 2.64$, $\frac{T_2}{T_0} = 0.4177$, $T_2 = 417.7$ °R

$\frac{A_T}{A_1} = 0.333$, for subsonic flow, $M_1 = 0.2$, $\frac{p_1}{p_0} = 0.9725$

$\therefore p_1 = 194.5$, $\frac{T_1}{T_0} = .9921$, $T_1 = 992.1$ °R, $M_1 = 0.2$

$$\frac{u_2}{u_1} = \frac{M_2 a_2}{M_1 a_1} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} = 8.55$$

(c). The mass flow will decrease ($\dot{m}$ is Max when $M_T = 1$)

(d). $\ln \frac{A^*}{A} = \ln M - \frac{(\gamma+1)}{2(\gamma-1)}\left[\ln\left(\frac{2}{\gamma+1}\right) + \ln\left(1 + \left(\frac{\gamma-1}{2}\right)M^2\right)\right]$

$$\frac{d \ln A^*/A}{dM} = \frac{1}{M} - \frac{(\gamma+1)}{2\cancel{(\gamma-1)}}\left[\frac{\cancel{(\gamma-1)}M}{\left(1 + \frac{\gamma-1}{2}M^2\right)}\right] = 0 \ , \text{ or } \ 1 + \frac{\gamma-1}{2}M^2 = \frac{(\gamma+1)}{2}M^2$$

$1 = \left(\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\right)M^2$ , or $M^2 = 1$ , $M = 1$. ←

1. (a) <u>Continuum</u> - we look at a conglomerate of molecules and not one
4 molecule. We then define the properties of the particle as the average property of all the molecules in the fluid particle. The restriction is that the particle volume must be small and there must be a large no. of molecules in the volume defined. For continuum to hold $M/R_N << 1$. , properties are continuous in space

(b) the pressure at a point is defined as the lim. of the force / unit area as the $\Delta$ unit area is shrunk to zero & $\Delta F$
3

$$p = \lim_{\substack{\Delta A \to 0 \\ \Delta F \to 0}} \frac{\Delta F}{\Delta A}$$

force determined by molecular transfer of momentum

(c) <u>Compressibility</u> - the ability for molecules in a unit
4 volume to be compressed so that for the same volume a larger no. of molecules can be put into in, due to action of pressure force.

(d) Vorticity - the rotation of the diagonal of a
5 square from its initial position at $t=0$ to its new position at time equals $t_1$ or $\frac{\Delta \theta}{\Delta t} = \omega$

(e) <u>Eulerian</u> - we take a look a the properties of a
3 fluid particle as it moves through a pt in space.

<u>Lagrange</u> - start with a specific ~~point~~ particle in space, We investigate the properties of ~~the~~ particle as it moves through all the pts ~~at the specific time~~.

5 (f) <u>stagnation value</u> - those properties of a flow when the flow is brought to rest isentropically

(g) $p/p_r = (\rho/\rho_r)^{\gamma} \left[ e^{\frac{s - s_r}{c_v}} \right]$

4

to use the perfect gas equation

1. Intermolecular interaction must be small
2. particle volume must be > dimension of molecule
   particle dimension " " > mean free path of molecule
3. no chemical reactions or excessively high temperatures.

what is general state eqn

2. (a) $\nabla\cdot\bar{q} = 0$ for incompressible flow.

$\bar{q} = \nabla\varphi$ if $\nabla\times\bar{q} = 0$ if $\varphi$ exist

10 $\frac{\partial\varphi}{\partial x} = u \quad \frac{\partial\varphi}{\partial y} = v$

$\frac{\partial F}{\partial x} = 4ax - 2y + y^2 - 3cx^2 \qquad \frac{\partial F}{\partial y} = -2y - 2x + 2xy$

$\frac{\partial^2 F}{\partial x\partial y} = \frac{\partial^2 F}{\partial y\partial x} \;;\; -2+2y = \frac{\partial^2 F}{\partial x\partial y} \;;\; \frac{\partial^2 F}{\partial y\partial x} = -2+2y$

$$\nabla\times\bar{q} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & 0 \end{vmatrix}$$

$k\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = 0 \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \;;\; \frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial^2\varphi}{\partial y\partial x}$

$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$ if $\nabla\cdot\bar{q} = 0$

$[4a - 6cx] = \frac{\partial^2\varphi}{\partial x^2} \;;\; [-2 + 2x] = \frac{\partial^2\varphi}{\partial y^2}$

$\therefore \quad 4a - 2 + 2x - 6cx = 0$

$4a - 2 = 0 \qquad \boxed{a = 1/2}$

$2x - 6cx = 0 \qquad \boxed{c = 1/3}$

(b) For streamline

$\frac{dy}{dx} = \frac{v}{u} = \frac{y^2x}{x^2y} = \frac{y}{x}$

$\frac{dy}{y} = \frac{dx}{x} \rightarrow \boxed{\ln y = \ln x + \ln c}$

at $x=1, y=2 \;\therefore\; \ln 2 = \ln 1 + \ln c \;\therefore\; \ln c = \ln 2$

1 Streamline $\therefore \quad y = 2x$

for steady flow pathline = streamline good!

$y = 2x \quad \therefore \quad u = x^2 \cdot 2x = 2x^3 \; ; \; v = 4x^2 \cdot x = 4x^3 = \frac{y^3}{2}$

c)

10 $u = \frac{dx}{dt} \rightarrow u\,dt = dx \qquad 2x^3 dt = dx$

$$t = \int \frac{dx}{2x^3} + C \qquad t = -\frac{x^{-2}}{4} + C$$

at $t = 0 \quad x = 1 \quad \therefore \quad \frac{1}{4x^2} = C \qquad C = \frac{1}{4}$

$$\boxed{t = \frac{1}{4}\left[1 - \frac{1}{x^2}\right]}$$

at $t = .1 \qquad .1 = \frac{1}{4}\left[1 + \frac{1}{x^2}\right]; \; +.6 = \frac{1}{x^2}$

$$x^2 = \frac{1}{.6} = 1.66 \qquad x = \pm 1.29$$

$v = \frac{dy}{dt} \longrightarrow v\,dt = dy \qquad v = \frac{y^3}{2} \qquad \int dt = \int \frac{2\,dy}{y^3} + C$

$$t = -y^{-2} + C$$

at $t = 0 \quad y = 2 \quad \therefore \quad -\frac{1}{4} + C = 0 \qquad C = \frac{1}{4}$

$$\boxed{t = \frac{1}{4} - \frac{1}{y^2}} \qquad y_{t=.1} = \pm 2.58$$

at $t = .1 \qquad (x, y) = \pm(1.29, 2.58)$

(d) $\frac{Du}{Dt} = \cancel{\frac{\partial u}{\partial t}}^{0} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \cancel{\frac{\partial u}{\partial z}\frac{dz}{dt}}^{0}$

1°

$a_x = 2xy(x^2y) + x^2(y^2x)$

$a_x\big|_{1,2} = \left(2x^3y^2 + x^3y^2\right)_{1,2} = 3\left(x^3y^2\right)_{1,2} = 12$

$\frac{Dv}{Dt} = \cancel{\frac{\partial v}{\partial t}}^{0} + \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} + \cancel{\frac{\partial v}{\partial z}\frac{dz}{dt}}^{0}$

$a_y = y^2(x^2y) + 2yx(y^2x)$

$a_y\big|_{1,2} = \left(y^3x^2 + 2y^3x^2\right)_{1,2} = 3\left(y^3x^2\right)_{1,2} = 24$

(e) steady flow $\partial/\partial t = 0$ $\therefore$

1°

$\cancel{\partial/\partial t\,\rho} + \bar{q}\cdot\nabla\rho + \rho\nabla\cdot\bar{q} = 0$

$u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} = 0$

$x^2y(y) + y^2x^2 + 2xy^2 + 2x^2y^2 \neq 0$

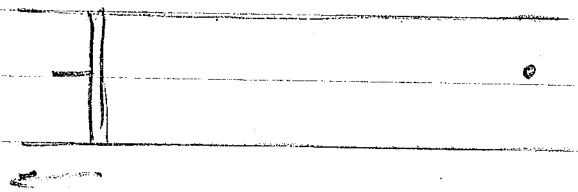
$x^2y^2 + y^2x^2 + 2x^2y^2 + 2x^2y^2 \neq 0$

if $\rho(x,y) = -xy$ may be

but $\rho(x,y) \neq xy$ for 2(b).

3.

6/20



$$\rho' = f(x - a_\infty t) + g(x + a_\infty t) \quad \begin{cases} \cancel{u_p = -10 \quad x<0} \\ 0 \quad x \geq 0 \end{cases}$$

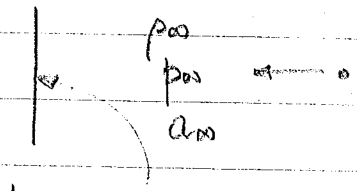
$$u' = \frac{a_\infty^2}{\rho_\infty}\left[f(x - a_\infty t) - g(x + a_\infty t)\right] < \begin{matrix} u_p = -10 & x<0 \\ 0 & x \geq 0 \end{matrix}$$

$$\rho' = + u_p \frac{a_\infty^2}{\rho_\infty} \qquad \nabla p' = a_\infty^2 \nabla \rho'$$

$$\frac{\partial u'}{\partial x} = -\frac{1}{\rho_\infty}\frac{\partial \rho'}{\partial t} = + \frac{a_\infty}{\rho_\infty} f'(x - a_\infty t) - \frac{a_\infty}{\rho_\infty} g'(x + a_\infty t) = + \frac{a_\infty}{\rho_\infty}\left[f' - g'\right]$$

$$\cancel{- \frac{\partial u'}{\partial t} = \frac{a_\infty^2}{\rho_\infty}\left[f'(x - a_\infty t) + g'(x + a_\infty t)\right]}$$

$t = 0$




$$\frac{x}{u_p} = t \quad 30\,\mu s$$

$$p = p_\infty - p'$$
$$\rho_\infty = \rho_\infty - \rho'$$
$$u = u_\infty - u'$$

Continuum - investigation of a conglomeration of molecules. Conglomeration has high no. of molecules in a small volume. Properties of conglomerate is the average properties of molecules

Fluid Particle - a conglomeration of molecules at a given point in space.

$\rho = M/V$ ; $p = \frac{F}{A}$ - independent of direction.

Mean Free Path - distance a specific molecule travels before intermolecular collision

$$\bar{\lambda} = \frac{\sum \lambda_i}{n}$$

Transport Properties - transfer of mass, momentum, energy

For continuum to hold $\frac{M}{R_N} \ll 1$ $\quad M = u/a \quad R_N = \frac{\rho u x}{\mu}$

Vorticity - the amount the diagonal of a square rotates in time $\Delta t$.

$$\nabla \times \bar{q} = \Omega$$

if irrotational $\nabla \times \bar{q} = 0$ and $\bar{q} = \nabla \varphi$ where $\varphi$ is a potential of $\bar{q}$

if $u$ is continuous & has continuous derivatives $\int_V \nabla u \, dV = \int_S u \bar{n} \, ds$

Divergence Theorem - $\int_V \nabla \cdot \bar{q} \, dv = \int_S \bar{q} \cdot \bar{n} \, ds$

Stokes' Theorem - $\int_S \bar{n} \cdot \nabla \times \bar{q} \, ds = \oint_c \bar{q} \cdot d\bar{l} = \Gamma$

Fluid Motion

Lagrange - looking at a specific ~~point~~ particle we investigate the properties of fluid particles through ~~the~~ all point.

$$u(x_0, y_0, z_0, t) = \left. \frac{\partial x}{\partial t} \right|_{x_0, y_0, z_0}$$

Eulerian - look at the particles as they move past the point at a specific time

$$u(x, y, z, t) = dx/dt$$

$$\frac{Da}{Dt} = \frac{\partial a}{\partial t} + \frac{\partial a}{\partial x} u + \frac{\partial a}{\partial y} v + \frac{\partial a}{\partial z} w = \frac{\partial a}{\partial t} + \nabla a \cdot \bar{q}$$

Pathline - locus of all points of the path of a particle (long time picture)

Streamline - locus of tangents to the instantaneous velocity vector.

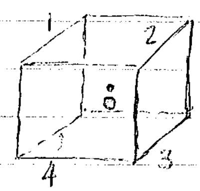
~~[illegible]~~ Streakline:- locus of all particles that pass through a particular point. (long time picture)

Steady flow - velocity at a given point in space $\neq f(t)$

Path-, Stream-, Streaklines are equal

Streamline $\quad dy/dx = v/u$

$$\frac{\partial}{\partial t}\int_V \rho \, dV = -\int_S \rho \bar{q}\cdot\bar{n}\, dS \qquad \text{no body forces or heat addition}$$




$dV = dx\,dy\,dz \qquad \bar{q}\cdot\bar{n}$ in x direct. on 1,4 $\left(u_0 - \frac{\partial u}{\partial x}dx\right)$

2,3 $\left(u_0 + \frac{\partial u}{\partial x}dx\right)$

in limit x, y, z direct. $\quad \frac{\partial \rho}{\partial t} + \nabla\cdot\rho\bar{q} = 0$

or $\quad \frac{\partial \rho}{\partial t} + \bar{q}\cdot\nabla\rho + \rho\nabla\cdot\bar{q} = 0 \qquad \frac{\partial \rho}{\partial t} + \bar{q}\cdot\nabla\rho = \frac{D\rho}{Dt}$

$\therefore \quad \frac{D\rho}{Dt} + \rho\nabla\cdot\bar{q} = 0$

Note - fixed volumes : $\partial/\partial t \int \ldots$
fixed mass : $D/Dt \int \ldots$

Equations

$$\int_V \frac{\partial A}{\partial t}\, dV + \int_S A\, \bar{q}\cdot\bar{n}\, dS = \int_S B\, dS + \int_V C\, dV \quad ; \text{ Inviscid}$$

1. Mass Conservation $A = \rho$ ; $B = C = 0$
2. Momentum $A = \rho\bar{q}$ ; $B = -p\bar{n}$ , $C = \rho\bar{F}$
3. Energy $A = \rho(e + q^2/2)$ ; $B = -p\, \bar{n}\cdot\bar{q}$ , $C = \rho\bar{F}\cdot\bar{q} + \dot{Q}$

for steady flow $\int_V \frac{\partial A}{\partial t}\, dV = 0$

no body forces and heat addition $\int_V C\, dV = 0$

$\frac{D\rho}{Dt} + \rho\nabla\cdot\bar{q} = 0$ Continuity ; if $\frac{D\rho}{Dt} = 0$ $\nabla\cdot\bar{q} = 0$ incompress.

Lagrange - Continuity
fixed Mass - $\int_V \rho\, dV = \int_{V_0} \rho_0\, dV_0$

$dV = J\, dV_0$ $\therefore$ $\int_{V_0} (\rho J - \rho_0)\, dV_0 = 0$ iff* $\rho J - \rho_0 = 0$

$\therefore$ $\boxed{\rho_0/\rho = J}$

Eulerian - fixed mass

$$\frac{D}{Dt}\int_V \rho\, dV = \frac{D}{Dt}\int_{V_0} \rho J\, dV_0 = 0 \; ; \quad \int_{V_0} \frac{D\rho J}{Dt}\, dV_0 = 0$$

$$\int_{V_0}\left(J\frac{D\rho}{Dt} + \rho\frac{DJ}{Dt}\right) dV_0 = \int_{V_0} J\left(\frac{D\rho}{Dt} + \rho\, \mathrm{div}\, \bar{q}\right) dV_0 = 0$$

$$\boxed{\left(\frac{D\rho}{Dt} + \rho\nabla\cdot\bar{q}\right) = 0}$$

Momentum. fixed mass

$$\int_V \rho\frac{D\bar{q}}{Dt}\, dV = -\int_V \nabla p\, dV + \int_V \rho\bar{F}\, dV$$

fixed volume ; $\frac{D(\rho\bar{q}\, dV)}{Dt} = -\nabla p\, dV + \rho\bar{F}\, dV$

Equation of State

$p = \rho R T \qquad p = p(\rho, T)$

1. Intermolecular forces small.
2. molecule dim small wrt particle volume
   " " " " mean free path
3. no chem reactions; T not very large

Local Thermodynamic Equil

Each particle is in thermodynamic equil with itself even if it touches another particle

Specific Heat - heat required to raise temp of unit mass by 1°

Enthalpy $h = e + pv$; $\quad C_p = \left(\frac{\partial h}{\partial T}\right)_p \quad C_v = \left(\frac{\partial e}{\partial T}\right)_v \qquad C_p/C_v = \gamma$

from $T\,ds = de + p\,dv$ & $p = \rho R T$ ; $C_p$ & $C_v$ are const for calorically perfect gases.

$$p/p_r = (\rho/\rho_r)^\gamma \left[ e^{\frac{s - s_r}{C_v}} \right]$$

for frictionless-adiabatic flow $\dot{Q} = 0 \quad T\frac{Ds}{Dt} = 0 \quad s = C.$ along streamline

homentropic flow $p = K\rho^\gamma$ $\quad$ if $\rho = f(p)$ barotropic

Energy relationships - inviscid, fixed mass

rate of work on system by pressure forces, body forces,

$$\frac{D}{Dt}\int_V \left[\rho(e + q^2/2)\right] dV = \int_V -\nabla\cdot \bar{q}p\,dV + \int_V \rho\bar{F}\cdot\bar{q}\,dV + \int_V \dot{Q}\,dV$$ rate at which heat is being added to system

$$\rho\frac{D}{Dt}(e + q^2/2) + \underbrace{(e + q^2/2)\frac{D\rho}{Dt} + \rho(e + q^2/2)\nabla\cdot\bar{q}}_{0} + \nabla\cdot p\bar{q} = \rho\bar{F}\cdot\bar{q} + \dot{Q}$$

$$\boxed{\rho\frac{D}{Dt}(e + q^2/2) + \nabla\cdot\bar{q}p = \rho\bar{F}\cdot\bar{q} + \dot{Q}}$$

Compressible Bernoulli

Steady flow $\partial/\partial t = 0$ $\quad$ Conservative force $\bar{F} = \nabla U$ ; $\quad \rho = \rho(p)$

$\therefore \quad \rho\frac{D\bar{q}}{Dt} + \nabla p = \rho\nabla U \;\rightarrow\; q\frac{\partial q}{\partial \ell} + \frac{1}{\rho}\frac{\partial p}{\partial \ell} = \frac{\partial U}{\partial \ell}$ component along streamline

Constant density flow $\rightarrow$ $q^2/2 + p/\rho - U = C$

$q\frac{\partial q}{\partial \ell} + \frac{\partial}{\partial \ell}\int_{p_0}^{p}\frac{dp}{\rho(p)} = \partial U/\partial \ell \;\rightarrow\; q^2/2 + \int\frac{dp}{\rho} - U = C$ along streamline

if $p = K\rho^\gamma \;\rightarrow\; \frac{\gamma}{\gamma - 1}p/\rho + q^2/2 - U = C \qquad \frac{\gamma}{\gamma-1}p/\rho = C_pT = h$

$h + q^2/2 = H_0 \quad$ if $U = 0$

if $\bar{F} = Q = \partial/\partial t = 0 \qquad$ Energy: $\rho \frac{D H_0}{Dt} = \frac{\partial p}{\partial t} \rightarrow \rho \frac{D H_0}{Dt} = 0$

if $H_0 \equiv$ constant flow is isentropic

1. $a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ use $p = f(s)\rho^\gamma \rightarrow a = \sqrt{\gamma R T}$

Incompressible flow $\partial p \neq 0 \quad \rho = C$ ; $\partial \rho = 0 \quad \therefore \left(\frac{\partial p}{\partial \rho}\right)_s = \infty$ ; $a = \infty$

Stagnation

Conditions at a pt where flow is brought to rest isentropically.

$$T_0/T = 1 + \frac{\gamma R M^2}{2 C_p} = 1 + \frac{\gamma - 1}{2} M^2 \qquad \frac{C_p}{R} = \frac{\gamma}{\gamma - 1}$$

$$p/p_0 = (\rho/\rho_0)^\gamma \quad \text{iff } f(s) = f(s_0) \rightarrow \text{stagnation cond.}$$

$$p/p_0 = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/\gamma - 1}$$

if inviscid steady flow $p_0 = C$ along streamline
if steady flow, $\dot{Q} = 0 \quad T_0 = C$ along streamline.

Governing Equations - Unsteady flow $\qquad \bar{F} = 0$, Inviscid.

(1) $(\partial/\partial t)\rho + \rho \nabla \cdot \bar{q} + (\bar{q} \cdot \nabla)\rho = 0$

(2) $\frac{\partial \bar{q}}{\partial t} + \bar{q} \cdot \nabla \bar{q} + \frac{1}{\rho} \nabla p = 0 \qquad$ flow homentropic, isentropic. $\bar{q} = \bar{q}'$, perturbation small $\rho' \ll \rho_\infty$, $p' \ll p_\infty$

Use taylor's expansion, neglect square terms ; $\bar{q}' \ll a_\infty$
$a^2 = \frac{\gamma p}{\rho}$ where $p = p_\infty + p'$ $\quad \rho = \rho' + \rho_\infty$

(A) from (1) $\qquad \frac{\partial \rho'}{\partial t} + \rho_\infty \operatorname{div} \bar{q}' = 0 \qquad$ linearized Continuity $\quad \rho' \nabla \cdot \bar{q} = \bar{q} \cdot \nabla \rho' = 0$ ; $\frac{\partial \rho_\infty}{\partial t} = 0$

from (2) $\qquad \frac{\partial \bar{q}'}{\partial t} + \frac{a_\infty^2}{\rho_\infty} \nabla \rho' = 0 \qquad$ Linearized Momentum $\quad \rho' \ll \rho_\infty$

$p' = a_\infty^2 \rho'$ from homentropic

from (A) $\quad \frac{\partial^2 \rho'}{\partial t^2} = a_\infty^2 \nabla^2 \rho' \quad$ wave equat. ; $\quad \frac{\partial^2 \bar{q}'}{\partial t^2} = -\frac{a_\infty^2}{\rho_\infty} \nabla \rho'$

for sphere $\rho' = \frac{1}{r}\left[f(r-a_\infty t) + g(r+a_\infty t)\right]$

cylinder $\rho' = (a_\infty^2 t^2 - r^2)^{-1/2}$

1-D $f(x-a_\infty t) + g(x+a_\infty t) = \rho'$

<u>Steady Flow & Linearize Equations</u> $F=0,\ s=c$

neglect square terms $\rho' << \rho_\infty$

$\partial/\partial t = 0$ ; $u_\infty = C$ let $\bar q = \bar q'$ $p = p_\infty + p'$ $\rho = \rho_\infty + \rho'$

Continuity $\bar q \cdot \nabla\rho + \frac{\partial \rho}{\partial t} + \rho \nabla\cdot \bar q = 0$ $\frac{\partial \rho}{\partial t} = 0$; $\bar q = (U_\infty + u')i + v'j$

$u' \,\&\, v' << u_\infty$, $\rho_\infty >> \rho'$

$\therefore\ u_\infty \cdot \frac{\partial \rho'}{\partial x} + \rho_\infty\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$ since $\nabla\rho_\infty = 0$ $\nabla\cdot\bar q = \nabla\cdot(U_\infty i + u'i + v'j) = \nabla\cdot(u'i + v'j)$

Momentum $(\bar q\cdot\nabla)\bar q + \frac{1}{\rho}\nabla p = 0$

$(\bar q \cdot \nabla \bar q') + \frac{1}{\rho_\infty}\nabla p' = 0 \longrightarrow U_\infty \frac{\partial \bar q'}{\partial x} + \frac{1}{\rho_\infty}\nabla p' = 0$ $\nabla p_\infty = 0$

from x-moment & continuity $M^2 \rho_\infty \frac{\partial u'}{\partial x} - \rho_\infty\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$

$(1-M_\infty^2)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$ if irrot $\nabla\times\bar q' = 0$ $u' = \frac{\partial\varphi}{\partial x}$ $v' = \frac{\partial \varphi}{\partial y}$ $\therefore$

$(1-M_\infty^2)\frac{\partial^2\varphi}{\partial x^2} + \left(\frac{\partial^2\varphi}{\partial y^2}\right) = 0$ let $\beta_\infty^2 = (1-M_\infty^2) = -m_\infty^2$

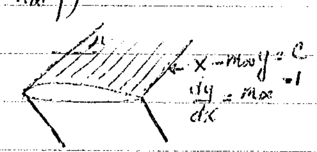
if $M=0$ Laplace

$M=1$ $\partial^2\varphi/\partial y^2 = 0$

$1>M>0$ $\varphi_{xx} = -\beta^2\varphi_{yy}$

$M>1$, $\varphi_{xx} = m^2\varphi_{yy}$ $\varphi = f(x-m_\infty y) + g(x+m_\infty y)$

for supersonic flow $\varphi = f(x-m_\infty y) + g(x+m_\infty y)$




Mach wave weakest form of shock wave. No disturbance to front or rear of mach lines

Quasi - 1-D Flow

2-D flows, use boundary conditions, inviscid, steady flows.

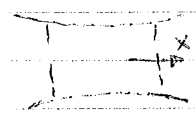
homentropic, frictionless adiabatic.

Assume.
1. flow across the channel doesn't change too much.
2. Long slender channels, edge effects neglected, [Entrance / exit flow don't dominate]
3. Assume area variation small wrt x-direct. $\frac{A'(x)}{A(x)} \ll 1$
4. Viscous Effects small.

Mass conservation

$$\int_S \rho \bar{q} \cdot \bar{n} \, dS = 0 \qquad \bar{q} \cdot \bar{n} = 0 \text{ on walls} \qquad \bar{q} \text{ at wall} = 0$$

$$\rho_{av} = \frac{1}{A(x)} \int_0^{A(x)} \rho \, dy \quad \text{where } \rho = \rho_{av} + \hat{\rho} \quad \hat{\rho} \ll \rho_{av}$$



$$\rho A u = \text{const.}$$

from continuity $\frac{\partial}{\partial x} \rho u + \frac{\partial}{\partial y} \rho v = 0$ if $\rho v|_0 = 0$

$$\int_0^A \frac{\partial \rho u}{\partial x} dy + \int_0^A \frac{\partial \rho v}{\partial y} dy = 0 \quad \text{By Leibnitz's rule} \quad \frac{\partial}{\partial x} \int_0^A \rho u \, dy = \int_0^A \frac{\partial \rho u \, dy}{\partial x} + A'(x) \rho u \Big|_{y = A(x)}$$

$$\therefore \frac{\partial}{\partial x} \int_0^A \rho u \, dy - A'(x) \rho u \Big|_{y = A(x)} + \rho v \Big|_0^{A(x)} = 0 \quad \text{but} \quad \frac{dy}{dx} = \frac{v}{u} = A'(x) \text{ at } y = A(x) \therefore$$

$$\partial/\partial x \int_0^A \rho u \, dy = 0 \implies \frac{\partial}{\partial x}\Big[\rho u A\Big] = 0 \quad \text{or} \quad \rho u A = \text{constant}$$

Momentum: Steady flow, inviscid $\bar{F} = 0$

$$\int_S \rho \bar{q}\, \bar{q} \cdot \bar{n} \, dS + \int_S p \bar{n} \, dS = 0$$

$dS = \sqrt{1 + y'^2}\, dx \qquad y = \frac{A}{2} \therefore y' = \frac{A'}{4}$

since $\frac{A'}{4} \ll 1 \therefore dS \sim dx$

$$\rho \bar{q} = \rho u \qquad \bar{q} \cdot \bar{n} = q_n$$

$\bar{n}_x = -\frac{A'(x)}{2}$ on wall

$$-\rho u^2 A_0 + \rho u^2 A - p A_0 + p A - 2 \int_{x_0}^{x} p \frac{A'}{2} dt = 0 \qquad \frac{d}{dx}\Big[ \quad \Big] = 0$$

$$\rho u A (u') + p' A + \cancel{p A'} - \cancel{p A'} = 0 \implies \rho u u' + p' = 0$$

Shockless isentropic flow (homentropic flow)

from momentum & def of $a^2 = \frac{\partial p}{\partial \rho}$ and $\rho u A = \text{const.}$ we get

$$(M^2 - 1)\frac{u'}{u} = \frac{A'}{A} = -\frac{(M^2-1)}{M^2} \frac{p'}{p}$$

| | | | | | |
|---|---|---|---|---|---|
| | $A'<0$ | $M<1$ | $u'>0$ | $p'<0$ | $\rho'<0$ |
| Conv. | | $M>1$ | $u'<0$ | $p'>0$ | $\rho'>0$ |
| | $A'>0$ | $M<1$ | $u'<0$ | $p'>0$ | $\rho'>0$ |
| diverg | | $M>1$ | $u'>0$ | $p'<0$ | $\rho'<0$ |

if $M=1$ $A'=0$ & $A=A^*$

Mass flow $\rho u A = \rho^* u^* A^*$ $A^* \le A_T$ ; $a^2 = \gamma R T$ $u^* = a^*$

$$\frac{A}{A^*} = \frac{\rho^* u^*}{\rho u} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{a^*}{a}\frac{a}{u} = \frac{1}{M}\frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\sqrt{\frac{T^*}{T_0}}\sqrt{\frac{T_0}{T}}$$

$M^* = 1$ $\therefore \frac{T_0}{T^*} = \frac{\gamma+1}{2}$ $\rho_0/\rho = \left(\frac{T_0}{T}\right)^{1/\gamma-1}$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

max mass flow is at $\frac{A^*}{A}$ when $M=1$ Chocked flows

1. Assume $A_T = A^*$ given $A_{exit}$ $\frac{A_{exit}}{A^*} \to M_{exit} \to \frac{p_{exit}}{p}$

if $\frac{p_{exit}}{p} > \frac{p_{exit}}{p_0}$ then start $\frac{p_e}{p_0} \to M_e \to \frac{A_e}{A^*}$ & we obtain $A^*$

$A^* = A(x)$ x can be determined

# Fluid Dynamics

Dr. M. H. Bloom taught first class for Prof G. Moretti
Dr. G. Moretti was replaced by Dr. S. Rubin
Rm 524

9-14-70

Course will deal in acoustics (Compressibility Effect) Superposition of waves as vehicule moves thru gases will produce shock waves.

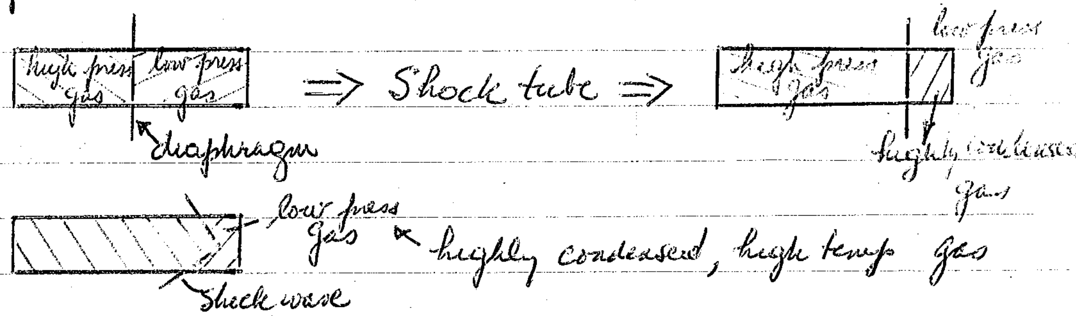



Fig. 1

9-16-70

Continuum - in order for mathematics to apply it is not necessary to investigate a molecule, but we can investigate a conglomerate of molecules and define properties from this conglomeration. We assume that this conglomerate has a small volume which contains a large no. of molecules from which we get the average properties of the molecules.

Fluid Particles - a conglomeration of molecules at a given point in space.

1. Density $\rho \rightarrow \frac{\Delta m}{\Delta V}$ (mass contained within fluid particle) (volume of fluid particle)

$$\boxed{\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}}$$

$\rho = \rho(x, y, z)$ continuous funct.

2. Pressure

$$\boxed{p = \lim_{\Delta A \to 0} \frac{\Delta F_{\perp \Delta A}}{\Delta A}}$$

fluid pressure acts normal to a surface

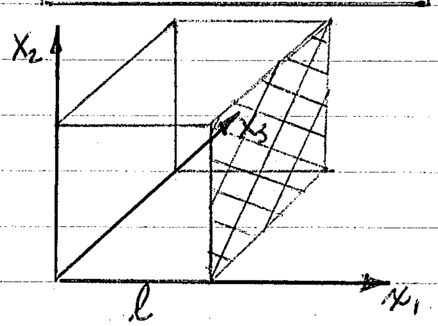



Fig 2

$\bar{c}_1$ = average molecular velocity in the $x_1$ - direction.

$n$ = number density of molecules

$m$ = mass of molecule

$2m\bar{c}_1$ = change in momentum/coll.

average time between collision $= \frac{2l}{\bar{c}_1}$

no consideration of intermolecular collisions; consider the molecule to hit wall rebound to other wall & back.

rate of change of momentum / molecule $= \frac{2m\bar{c}_1}{2l/\bar{c}_1} = \frac{m\bar{c}_1^2}{l}$

total rate of change of momentum $= \frac{nm\bar{c}_1^2}{l}$ in one direct.

in 3 directions we get total in $x_1, x_2, x_3$ direct

total in 1 direct with respect to any of the three axes

$$= \frac{\frac{1}{3} \sum_{i=1}^{k} n_i m_i \left(c_{i_1}^2 + c_{i_2}^2 + c_{i_3}^2\right)}{l^3}$$

if ~~...~~ $= n = n_i / l^3$

$$P = \frac{2}{3}\,(K.E.)_{\text{molecules}}$$

$$n_i\, m_i = \rho_i$$

$$c_{i_1}^2 + c_{i_2}^2 + c_{i_3}^2 = \bar{c}^2$$

$$P = \rho R T$$

$$\ell^3 = V$$

$$\frac{1}{2} m_i \bar{c}^2 = K.E.$$

from the above we can see that $T \sim \bar{c}^2$

## Mean Free Path $\bar{\lambda}$

$\lambda$ is the distance of a specific molecule travels before an intermolecular collision occurs.

$\bar{\lambda}$ is the average distance that all molecules travel, before a collision occurs.

$\Longrightarrow$ 4R  R - ~~radius of molecule~~ of the box ; for continuum to hold $\bar{\lambda}/R \ll 1$

## Transport Properties

transfer of mass, momentum, energy of a molecules.

property $a$ at fixed $x$  $a = a(x,y)$

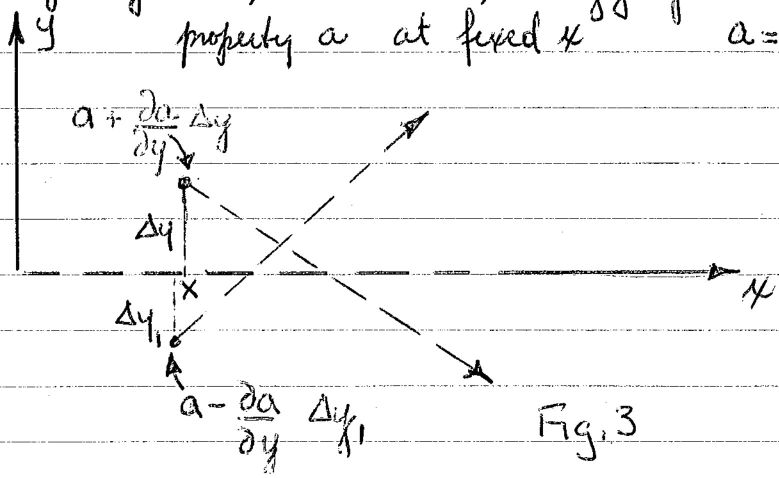



Fig. 3

if $a = mu$  where $u = V_x$  x-momentum

$\frac{\partial u}{\partial y}$ can be related to a shearing strain on a fluid particle.

$\tau$ can be related to $\frac{\partial u}{\partial y}$ by a proportionality coeff.
this coeff $\mu$ is related by

$$\tau = \mu \frac{\partial u}{\partial y}$$

$\tau = \frac{\Delta F_{\parallel \Delta A}}{\Delta A}$

$\mu$ is the coefficient of viscosity dependent on motion of particles.

total transfer of tangential momentum / unit area
Calculation.

$n\bar{c}$ = number flux of molecules / unit time.
$m \frac{\partial u}{\partial y} \propto$ transfer of tangential momentum across boundary $\Delta y, \Delta y_1 \propto \bar{\lambda}$ therefore we can say

$$\tau = n\bar{c}\bar{\lambda} m \frac{\partial u}{\partial y} \propto \frac{\text{transfer of tangential momentum / unit time}}{\text{unit area}}$$

$\mu \propto \rho \bar{\lambda} \bar{c}$ if $nm = \rho$ $\quad \rho\bar{\lambda} \approx$ constant
as $\rho$ increases $\bar{\lambda}$ decreases
as $\rho$ decreases $\bar{\lambda}$ increases

Gas only $\therefore$ $\mu \propto \bar{c}$ but $\bar{c}^2 \propto T$ therefore $\mu \propto \sqrt{T}$

$q = K \frac{\partial T}{\partial y}$ heat transfer thermal conductivity - K

$T = D \frac{\partial x_i}{\partial y}$ mass transfer diffusion coeff - D

since $\mu \propto \rho \bar{\lambda} \bar{c}$ & $\bar{\lambda}/R \ll 1$ then

$\mu/\rho\bar{c}R \ll 1$ then $\dfrac{\bar{u}/\bar{c}}{\rho\bar{u}R/\mu}$ $\quad \bar{c} \propto a$ speed of sound

$\bar{u}/\bar{c}$ can be equated to $M$ mach no.
$\rho\bar{u}R/\mu$ can be equated to $R_N$ reynolds' no.

therefore $\dfrac{\bar{u}/\bar{c}}{\rho\bar{u}R/\mu} = \dfrac{M}{R_N} = K_N$ ; if Knudsen no $\ll 1$ we can use the continuum theory.

Compressibility $- \rho \dfrac{\partial p}{\partial \rho} = K$ Compressibility Factor

flow is compressible when $K$ is finite.
" " incompressible when $K \longrightarrow \infty$.

9-21-70

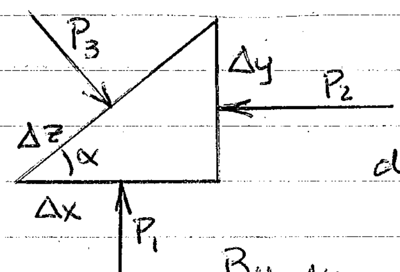


Fig. 4a

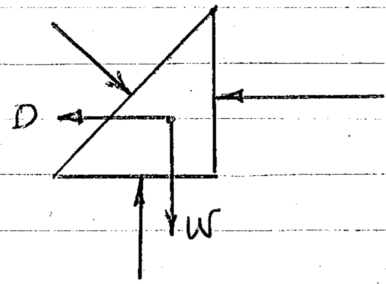
Pressure is a scalar
tangential momentum is unchanged due to elastic collisions.

By summing forces in x-direction we get that $P_1, P_2, P_3$ are equal $\sum P_x = 0$.

$P_3 \Delta z \sin\alpha = P_2 \Delta y$ since $\Delta z \sin\alpha = \Delta y$
$P_2 = P_3$ and by same procedure $P_1 = P_2 = P_3$.

We have neglected the mass of fluid & its movement in the above (Body Forces).

$W$ & $D$ are $\propto$ volume




by adding D & W we see $P_3 \neq P_2$
however in the limit we get
$$P_3 = P_2 + \frac{D\Delta x}{2}$$
and the limit at $\Delta x \to 0$
shows $P_3 = P_2$

Fig. 4b

<u>Vorticity</u>

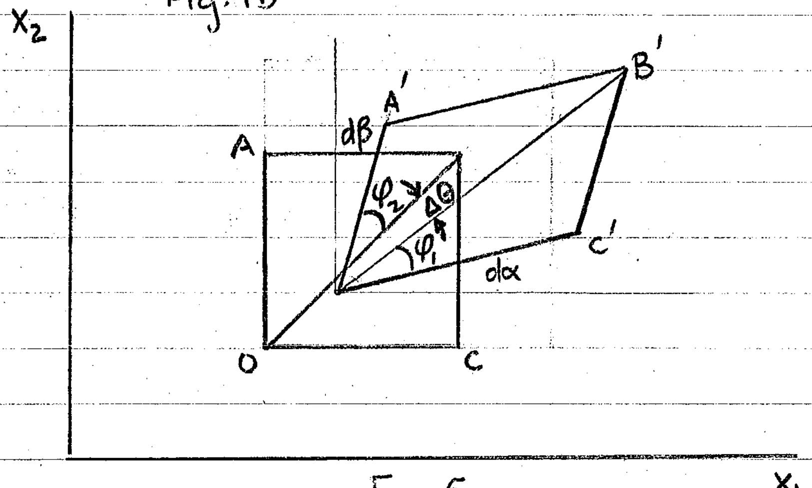



Fig. 5

Body forces will not affect the results at the limit
thus <u>pressure is independent of direction</u> (press. is scalar)

Velocity (Fig 5)

$O: \quad (u_1, u_2)$

$$A: \quad \left(u_1 + \frac{\partial u_1}{\partial x_2} dx_2,\ u_2 + \frac{\partial u_2}{\partial x_2} dx_2\right)_0$$

$$B: \quad \left(u_1 + \frac{\partial u_1}{\partial x_1} dx_1,\ u_2 + \frac{\partial u_2}{\partial x_1} dx_1\right)_0$$

shearing strain: deviation of the right angle;
strain $\propto$ stress.

<u>Vorticity</u> is defined as the amount the diagonal
rotates after time $\Delta t$. rotation of OB. $\frac{\Delta\theta}{\Delta t}$

$$d\alpha = \frac{\left(u_2 + \frac{\partial u_2}{\partial x_1} dx_1\right) dt - u_2 dt}{\left(u_2 + \frac{\partial u_1}{\partial x_1} dx_1\right) dt - u_1 dt + dx_1}$$

$$= \frac{\partial u_2}{\partial x_1} dt \Big/ \left(1 + \frac{\partial u_1}{\partial x_1} dt\right) \approx \frac{\partial u_2}{\partial x_1} dt \text{ ; since } \frac{\partial u_1}{\partial x_1} dt \ll 1$$

$d\beta = \dfrac{\partial u_1}{\partial x_2} dt$ by same thing $d\beta = \dfrac{\left(u_1 + \frac{\partial u_1}{\partial x_2} dx_2\right) dt - u_1 dt}{\left(u_2 + \frac{\partial u_2}{\partial x_2} dx_2\right) dt - u_2 dt + dx_2}$

$$\Delta\theta = \frac{d\beta - d\alpha}{2} = \frac{dt}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right)$$

$$\lim_{\substack{\Delta\theta \to 0 \\ \Delta t \to 0}} 2\omega = \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \qquad 2\omega_z = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = -\Omega_z$$

vorticity

$$\Omega = \text{curl } \bar{q}$$

$$= \nabla \times \bar{q}$$

$$\text{curl } \bar{q} = \begin{bmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{bmatrix} = i\,(w_y - v_z) - j\,(w_x - u_z) + k\,(v_x - u_y)$$

vorticity $= -2$ angular velocity of element.
$\Omega = 0$ for irrotational flow.

Irrotational $\nabla \times \bar{q} = 0$

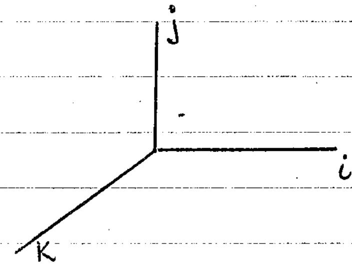
if $u = \dfrac{\partial \varphi}{\partial x}$, $v = \dfrac{\partial \varphi}{\partial y}$ $\varphi$ is a potential

$$\bar{q} = \nabla \varphi$$

9-23-70

## Vector Analysis

$$\bar{A} = A_x i + A_y j + A_z k$$




$$|\bar{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$|\bar{A}| = \sqrt{\bar{A}\cdot\bar{A}}$$

## Multiplication of Vectors

1. Dot Product

$$\bar{A}\cdot\bar{A} = A^2$$

$$\bar{V}\cdot\bar{u} = \bar{u}\cdot\bar{V} = uV\cos\theta_{uV} = u_x V_x + u_y V_y + u_z V_z$$

$$\bar{u}\cdot\bar{V} = 0 \text{ if } \bar{u}\perp\bar{V}$$

2. Cross Product

$$\bar{w} = \bar{u}\times\bar{V} = uV\sin\theta_{uV}\ \bar{\epsilon} \qquad \bar{\epsilon}\perp\bar{u},\ \bar{\epsilon}\perp\bar{V}$$

$$\bar{u}\times\bar{V} = -\bar{V}\times\bar{u} = \begin{vmatrix} i & j & k \\ u_x & u_y & u_z \\ V_x & V_y & V_z \end{vmatrix}$$

$$= i\left(u_y \cancel{V}_z - u_z V_y\right) - j\left(u_x V_z - u_z V_x\right) + k\left(u_x V_y - V_x u_y\right)$$

3. Triple Product

(a) $\bar{S}\cdot(\bar{u}\times\bar{V}) = \det\begin{pmatrix} S_x & S_y & S_z \\ u_x & u_y & u_z \\ V_x & V_y & V_z \end{pmatrix}$

(b) $\bar{S}\times(\bar{p}\times\bar{V}) = (\bar{S}\cdot\bar{V})\bar{p} - (\bar{S}\cdot\bar{p})\bar{V}$

4. Vector Derivatives

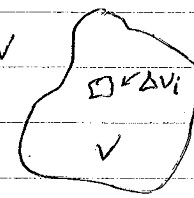
(i) $\text{Grad} = \nabla u = \frac{\partial u}{\partial x} i + \frac{\partial u}{\partial y} j + \frac{\partial u}{\partial z} k$

$$\nabla u = \text{grad } u = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_S u \bar{n} \, ds \qquad \bar{n} \cdot \bar{n} = 1$$

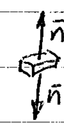
$\bar{n}$ + outfrom surface $\bar{n} \perp$ surface

$u$ continuous & has cont. deriv:

$$\lim_{\Delta V \to 0} \sum_i \nabla u \cdot \Delta V_i = \sum_i \int_{S_i} u \bar{n} \, ds \quad ; \quad \sum_i \Delta V_i = V$$




$$\boxed{\int_V \nabla u \, dV = \int_S u \bar{n} \, ds}$$




$\therefore \int_{top-bot} = 0$
Except outer surface

(ii) Divergence

$$\text{div } \bar{q} = \nabla \cdot \bar{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$$

$$\text{div } \bar{q} = \nabla \cdot \bar{q} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_S \bar{q} \cdot \bar{n} \, ds$$

$$\int_V \nabla \cdot \bar{q} \, dV = \int_S \bar{q} \cdot \bar{n} \, ds$$

Green's Theorem
Divergence Theorem.

(iii) Curl

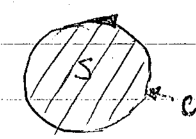
$$\text{Curl } \bar{q} = \nabla \times \bar{q} = \det \begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ q_x & q_y & q_z \end{pmatrix}$$

$$\nabla \times \bar{q} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_S \bar{n} \times \bar{q} \, ds$$

## Stokes' Theorem

$$\int_S \underset{\bar{n} \cdot (\nabla \times \bar{q}) ds}{\bar{n} \cdot \text{curl } \bar{q} \, ds} = \oint_C \bar{q} \cdot d\bar{l} = \Gamma \quad \text{circulation}$$

l represents path around surface
S surface




<u>Description of Fluid Motion</u>

<u>Lagrangian</u>: $(x_0, y_0, z_0)$ — start out here; <u>look at the properties of particles as particles moves through a</u> ~~specific point~~ space

Velocity of fluid particle is the average velocity of all molecules making up the particles.

$$x = x(x_0, y_0, z_0, t)$$
$$y = y(x_0, y_0, z_0, t)$$
$$z = z(x_0, y_0, z_0, t)$$

what is velocity

$$u(x_0, y_0, z_0, t) = \lim_{\Delta x, \Delta t \to 0} \Delta x/\Delta t = \partial x/\partial t \Big|_{x_0, y_0, z_0}$$

$$v(x_0, y_0, z_0, t) = \partial y/\partial t \Big|_{x_0, y_0, z_0}$$

$$w(x_0, y_0, z_0, t) = \partial z/\partial t \Big|_{x_0, y_0, z_0}$$

<u>Eulerian</u>: <u>look at what happens at a point in space at a specific time when a particle moves past the point.</u>

$$u = u(x, y, z, t) = dx/dt$$
$$v = v(x, y, z, t) = dy/dt$$
$$w = w(x, y, z, t) = dz/dt$$

What is the rate of change of a property of a particle as its moves through the point. The change of the property, a, at a specific point is given by:

$$\frac{da}{dt} = \frac{\partial a}{\partial x}\frac{dx}{dt} + \frac{\partial a}{\partial y}\frac{dy}{dt} + \frac{\partial a}{\partial z}\frac{dz}{dt} + \frac{\partial a}{\partial t}$$

the Eulerian Substantial Derivative $\frac{Da}{Dt}$ when

$\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$, $\frac{dz}{dt} = w$. is $\frac{Da}{Dt} = \frac{\partial a}{\partial x}u + \frac{\partial a}{\partial y}v + \frac{\partial a}{\partial z}w + \frac{\partial a}{\partial t}$

this is the directional derivative relating motion of a fluid particle.

$\frac{\partial a}{\partial t}$ – local derivative

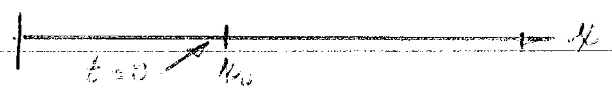
$\frac{\partial a}{\partial x}u + \frac{\partial a}{\partial y}v + \frac{\partial a}{\partial z}w$ – convective derivative

$\frac{Da(x,y,z,t)}{Dt}$ – Euler; $\frac{\partial a(x_0,y_0,z_0,t)}{\partial t}$ – Lagrangian

at a specific point the rate of change of property at point many never change however we want to find the rate of change of a property at the point of the molecule passing through the point.

Example:

$t=0$




$u = kx_0$

Lagrangian approach:

$$kx_0 = \left.\frac{\partial x}{\partial t}\right|_{x_0}$$

$$kx_0 t + C = x \qquad C = x_0$$

$$x = x_0(1+kt)$$

$\frac{\partial u}{\partial t} = 0$ then $u$ is a constant

Eulerian Approach:

$$x = x_0(1+kt) \qquad u = \frac{kx}{1+kt}$$

$$\frac{Du}{Dt} = \underset{\displaystyle \frac{\partial u}{\partial t}}{\underset{\shortparallel}{\frac{-k^2x}{(1+kt)^2}}} + \underset{\displaystyle u}{\underset{\shortparallel}{\frac{kx}{(1+kt)}}} \cdot \underset{\displaystyle \frac{\partial u}{\partial x}}{\underset{\shortparallel}{\frac{k}{(1+kt)}}} = 0$$

then $u$ = constant.

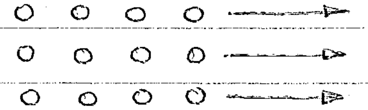
9-28-70

Pathline - locus of all points of the path of a particle describes motion of single particle (long time picture)

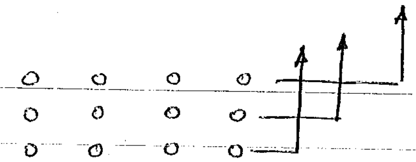
Streamline - locus of tangents to the instantaneous velocity vector. (instantaneous picture)

Streakline - locus of all particles that pass through a particular point (long time picture)

Steady flow - at a given point in space the velocity is not a function of time. Pathline, Streamline, Streakline are equal.

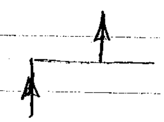


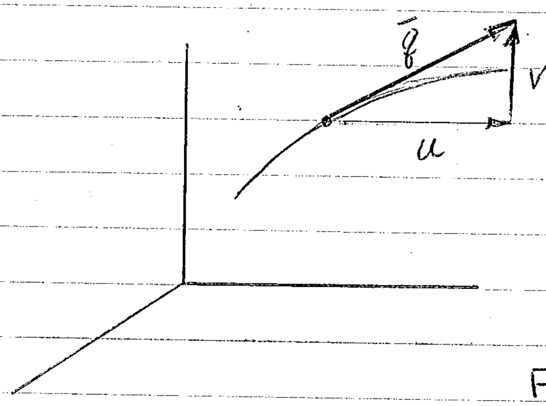
pathline, streamline, streakline are equal.



If a left flank

Pathline is  Streamline 

Streak line 




$$\frac{dy}{dx} = \frac{v}{u}$$

mathematical defenction of Streamline.

Fig 6

Example:

$$u = \frac{2Ay}{x^2+y^2} \qquad V = \frac{-2Ax}{x^2+y^2}$$

$$\frac{dy}{dx} = -\frac{x}{y} \qquad x\,dx + y\,dy = 0$$

$$x^2 + y^2 = C$$ Vortex flow (Fig. 7)

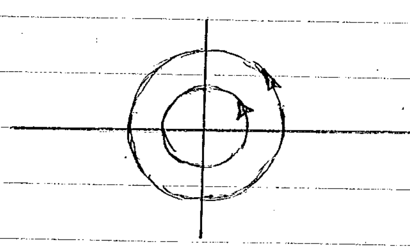


Fig. 7

Example:

$$u = \frac{Ax}{x^2+y^2} \qquad v = \frac{Ay}{x^2+y^2}$$

$$\frac{dy}{dx} = y/x$$

$\ln y = \ln x + \ln c \qquad cx = y$

Source & Sink flow.
(Fig. 8)

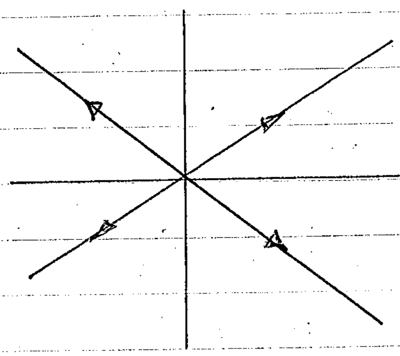


Fig. 8

<u>Boundary Condition at a solid surface</u>

$$q_n \equiv 0 \qquad \frac{dy}{dx} = \frac{V}{u} \quad \text{at the surface}$$

<u>Two-Dimensional</u>

$$y = Y(x) \qquad Y(x) - y = F(x,y) = 0$$

$$\frac{DF}{Dt} = 0 \qquad \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{DF}{Dt}$$

$$F \neq F(t) \qquad \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + V\frac{\partial F}{\partial y}$$

if steady flow $\frac{\partial F}{\partial t} = 0 \qquad \frac{\partial F}{\partial y} = -1 \qquad \frac{\partial F}{\partial x} = Y'(x)$

$$u.Y'(x) - V = 0 \qquad Y'(x) = V/u$$

F is constant on the streamline

Equation of mass conservation or Equation of Continuity

$$\frac{\partial}{\partial t}\int_V \rho\, dV = -\int_S \rho\, \bar{q}\cdot\bar{n}\, ds + \int_V \dot{m}\, dV$$

where $\bar{q}\cdot\bar{n} = q_n$ ; $\bar{n}$ is a unit vector directed outward from surface

mass conservation equation

9-30-70

Review -

Euler - $\dfrac{Da}{Dt} = \dfrac{\partial a}{\partial t} + \bar{q}\cdot\nabla a = \dfrac{dA}{dt}(x,y,z,t)$

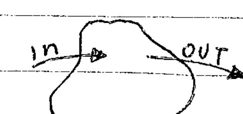
where $\dfrac{dx}{dt} = u,\quad \dfrac{dy}{dt} = v,\quad \dfrac{dz}{dt} = w,$

Lagrange - if $A = A(x_0, y_0, z_0, t)$

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial A}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial A}{\partial z}\frac{\partial z}{\partial t}$$

where $x = x(x_0, y_0, z_0, t)$, etc.

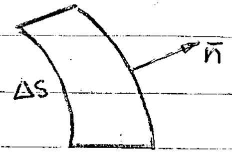
Mass Conservation Equation
fixed volume




$$\frac{\partial}{\partial t}\int_V \rho\, dV = -\int_S \rho\, \bar{q}\cdot\bar{n}\, dS \quad \text{(if no mass source)}$$

$$\rho(\bar{q}\cdot\bar{n})\, ds = \frac{\text{mass}}{\text{vol}}\ \frac{\text{ft}}{\text{sec}}\ \text{ft}^2 = \frac{\text{mass}}{\text{sec}}$$

Fig. 9




if $\bar{q}$ & $\bar{n}$ are in the same direct: $+\int_S \rho\bar{q}\cdot\bar{n}\, dS < 0$

$\Delta t\,(q_n) = (\bar{q}\cdot\bar{n})\Delta t$

$\rho \cdot$ volume $= \Delta S \rho \bar{q} \cdot \bar{n} \Delta t =$ total mass crossing the region.

$$\lim_{\substack{\Delta t \to 0 \\ \Delta S \to 0}} \frac{\rho \bar{q} \cdot \bar{n} \Delta t \Delta S}{\Delta t} \doteq \text{mass / unit time.}$$

if mass comes in $\bar{q} \cdot \bar{n} < 0$ but $\frac{\partial}{\partial t} \int \rho \, dV$ is increasing $\therefore$ we need the negative sign.

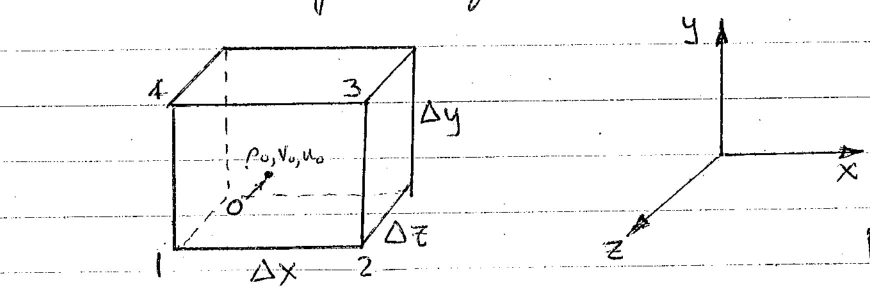


Fig. 10

By Taylor series expansion the properties on 1-4

Change on x-axis only since y above, y below $\to y_0$ z above, z below $\to z_0$ are equal & opposite

On wall

1-4 $$\rho = \rho_0 - \left(\frac{\partial \rho}{\partial x}\right)_{x_0} \frac{\Delta x}{2}$$

2-3 $$\rho = \rho_0 + \left(\frac{\partial \rho}{\partial x}\right)_{x_0} \frac{\Delta x}{2}$$

<u>Proof:</u> Mass flow: 1-4

$$\frac{\partial}{\partial t}(\rho_0 \Delta x \Delta y \Delta z) = \left[\rho_0 - \left(\frac{\partial \rho}{\partial x}\right)_{x_0} \frac{\Delta x}{2}\right]\left[u_0 - \left(\frac{\partial u}{\partial x}\right)_{x_0} \frac{\Delta x}{2}\right] \Delta y \Delta z$$

Mass flow: 2-3

$$- \left[\rho_0 + \left(\frac{\partial \rho}{\partial x}\right)_{x_0} \frac{\Delta x}{2}\right]\left[u_0 + \left(\frac{\partial u}{\partial x}\right)_{x_0} \frac{\Delta x}{2}\right] \Delta y \Delta z$$

$$= \left(-u_0 \frac{\partial \rho_0}{\partial x} - \rho_0 \frac{\partial u_0}{\partial x}\right) \Delta x \Delta y \Delta z = \left(-\frac{\partial \rho u}{\partial x}\right) \Delta x \Delta y \Delta z$$

then if we take the same relations in y & z directions

$$\boxed{\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w = 0}$$ true at the limit $\Delta x, \Delta y, \Delta z \to 0$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \bar{q} = 0} \qquad \bar{q} = ui + vj + wk$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \bar{q} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \underbrace{\bar{q} \cdot \nabla \rho}_{u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}} + \rho\,(\underbrace{\operatorname{div} \bar{q}}_{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}) = 0}$$

$$\boxed{\frac{D\rho}{Dt} + \rho\left(\nabla \cdot \bar{q}\right) = 0}$$

Incompressible flow – $\frac{D\rho}{Dt} = 0 \quad \therefore \quad \nabla \cdot \bar{q} = 0$

$$\frac{\partial}{\partial t}\int_V \rho\, dV + \int_S \rho \bar{q} \cdot \bar{n}\, dS = 0$$

$$\int_V \frac{\partial \rho}{\partial t}\, dV + \int_V \nabla \cdot \rho \bar{q}\, dV = 0$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \bar{q}\right) dV = 0$$

this is only possible if $\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \bar{q} = 0}$ as shown above.

<u>Lagrangian</u>

<u>fixed mass</u> – $\int_V \rho\, dV = \int_{V_0} \rho_0\, dV_0$

$$\boxed{dV = J\, dV_0} \qquad J = \frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)} = \begin{vmatrix} \partial x/\partial x_0 & \partial y/\partial x_0 & \partial z/\partial x_0 \\ \partial x/\partial y_0 & \partial y/\partial y_0 & \partial z/\partial y_0 \\ \partial x/\partial z_0 & \partial y/\partial z_0 & \partial z/\partial z_0 \end{vmatrix}$$

$$\int_V (\rho J - \rho_0)\, dV_0 = 0$$

$$\boxed{\rho/\rho_0 = \frac{1}{J}}$$ continuity equation of Lagrange.

Eulerian

fixed mass -

$$\frac{D}{Dt}\int_V \rho\, dV = 0 \qquad \frac{D}{Dt}\int_{V_0} \rho J\, dV_0 = 0$$

$$\int_{V_0}\left(J\frac{D\rho}{Dt} + \rho\frac{DJ}{Dt}\right) dV_0 = 0$$

$$\boxed{\text{Prove: } \frac{DJ}{Dt} = J \operatorname{div} \bar{q} \qquad \text{let } J = \begin{vmatrix} \partial x/\partial x_0 & \partial y/\partial x_0 \\ \partial x/\partial y_0 & \partial y/\partial y_0 \end{vmatrix}}$$

$$\int_{V_0} J\left(\frac{D\rho}{Dt} + \rho \operatorname{div} \bar{q}\right) dV_0 = 0$$

$$\int_V \left(\frac{D\rho}{Dt} + \rho \operatorname{div} \bar{q}\right) dV = 0$$

Momentum Equation or Equation of Motion

viscosity is not considered

Eulerian view - fixed mass

$$\frac{D}{Dt}\int_V \rho \bar{q}\, dV = -\int_S \overset{\text{pressure}}{p}\bar{n}\, dS + \int_V \rho \overset{\text{body force, force/unit mass}}{\bar{F}}\, dV + \int_S \cancelto{0}{\text{friction force}}\, dS$$

$$\int_{V_0} \frac{D}{Dt}(\rho\bar{q}J)\, dV_0 = \int_{V_0}\left(J\frac{D}{Dt}\rho\bar{q} + \rho\bar{q}\frac{DJ}{Dt}\right) dV_0$$

must be retained if work with boundary layer, shock wave, areas of high gradients, etc

$$= \int_{V_0} J\left[\frac{D}{Dt}\rho\bar{q} + \rho\bar{q}\operatorname{div}\bar{q}\right] dV_0 = \int_V \left(\frac{D}{Dt}\rho\bar{q} + \rho\bar{q}\operatorname{div}\bar{q}\right) dV$$

$$= \int_V \left( \frac{\partial}{\partial t}\rho\bar{q} + \underbrace{\bar{q}\cdot\nabla(\rho\bar{q}) + \rho\bar{q}\,\mathrm{div}\,\bar{q}}_{\nabla\cdot(\rho\bar{q}\bar{q})} \right) dV$$

$$\boxed{\int_V \frac{\partial}{\partial t}\rho\bar{q}\,dV + \int_S \rho\bar{q}\bar{q}\cdot\bar{n}\,dS = -\int_S p\bar{n}\,dS + \int_V \rho\bar{F}\,dV}$$

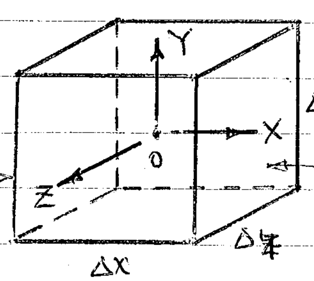
from $\displaystyle\int_V \left(\frac{D}{Dt}\rho\bar{q} + \rho\bar{q}\,\mathrm{div}\,\bar{q}\right)dV = \int_V \Big[\rho\frac{D\bar{q}}{Dt} + \underbrace{\bar{q}\frac{D\rho}{Dt} + \rho\bar{q}\,\mathrm{div}\,\bar{q}}_{\substack{0 \\ \text{by continuity}}}\Big]d$

$$\int_V \rho\frac{D\bar{q}}{Dt}\,dV = -\int_V \nabla p\,dV + \int_V \rho\bar{F}\,dV$$

Eulerian fluid particle $\boxed{\rho\dfrac{D\bar{q}}{Dt} = -\nabla p + \rho\bar{F}}$ Momentum Equation

$\boxed{\dfrac{D\rho}{Dt} + \rho\nabla\cdot\bar{q} = 0}$ Continuity

Assume

$p_0 - \left(\dfrac{\partial p}{\partial x}\right)_{x_0}\dfrac{\Delta x}{2}$  $p_0 + \left(\dfrac{\partial p}{\partial x}\right)_{x_0}\dfrac{\Delta x}{2}$



$\dfrac{D}{Dt}\left(\rho\bar{q}\,\Delta x\Delta y\Delta z\right) =$ Change of Momentum $\bar{q} = ui + vj + wk$

for x direction only $\dfrac{D}{Dt}\left(\rho u\,\Delta x\Delta y\Delta z\right) = -\left(\dfrac{\partial p}{\partial x}\right)_{x_0}\Delta x\Delta y\Delta z$

$+ \rho F_x \Delta x\Delta y\Delta z$

$$\rho' \frac{Du}{Dt} = -\left(\frac{\partial p}{\partial x}\right)_{x_0} + \rho F_x$$

$$\therefore \; \rho \frac{D\bar{q}}{Dt} = -\nabla p + \rho \bar{F} \quad \text{in 3-D.}$$

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Inviscid

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \bar{q} = 0 \qquad \int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho \bar{q} \cdot \bar{n} \, ds = 0$$

$$\rho \frac{D\bar{q}}{Dt} + \nabla p = \rho \bar{F} \qquad \int_V \frac{\partial \rho \bar{q}}{\partial t} dV + \int_S \rho \bar{q} (\bar{q} \cdot \bar{n}) ds = -\int_S p \cdot \bar{n} \, ds + \int_V \rho \bar{F} dV$$

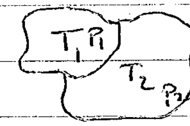
From continuity, momentum, state, energy.
we get $1 + 3 + 1 + 1 = 6$ equations

and we have $u, v, w, \rho, p, T$ unknowns.

<u>Equation of State</u>
Ideal (Perfect) gas: $p = \rho R T$, $R = \frac{\bar{R}}{M}$ ← universal gas const / molecular wt

$p = p(\rho, T)$; 2 independent variables, 1 dependent variable

<u>Local Thermodynamic Equilibrium</u>
1st particle has a set of temp & press
2nd particle has a diff set of temp & press
each is in local thermodynamic equilibrium even if they are touching



For the equation of state to hold, the following must be true:

1. Intermolecular forces are very small (negligible) $\Rightarrow$ [rarefied gas]
2. Molecular dimensions are small compared with particle volume
2b. If particle is flowing, dimensions small compared with mean free path.
3. No chemical reactions which alter state of gas. Temperatures are not excessively large.

<u>Van der Walls Equation (Corrected Equation of State)</u>

$(p + a/V^2)(V - b) = RT$; where $V = 1/\rho$

<u>$1^{ST}$ Law of Thermodynamics (Conservation of Energy)</u>

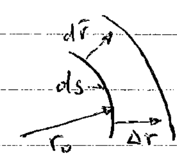
e: Internal energy / unit mass:

$e = e(T, p)$ $\qquad p = p(\rho, e)$

State variables do not depend on path.

$\Delta e = Q + W = Q' + W_p$ $\qquad$ where $Q' = \underbrace{W_{\rho}}_{\text{irreversible work}} + Q$

$$dW_p = -\int_{r_0}^{r_0+\Delta r} p\bar{n}\,ds \cdot d\bar{r}$$




$$W_p = -\int_s \int_{r_0}^{r_0+\Delta r} p\,ds(\bar{n} \cdot d\bar{r}) = -\int_V p\,dV$$

$$de = \underbrace{\delta Q'}_{\text{path function}} - p\,dV$$

$$\frac{De}{Dt} = \frac{\delta Q'}{\delta t} - p\frac{DV}{Dt} = \frac{\delta Q'}{\delta t} + p\frac{1}{\rho^2}\frac{D\rho}{Dt} \quad \text{where } V = \frac{1}{\rho}$$

<u>Specific Heat</u> - heat required to raise temperature of unit mass by $1^\circ$

$$c_p = \left(\frac{\Delta Q'}{\Delta T}\right)_p$$

$$c_v = \left(\frac{\Delta Q'}{\Delta T}\right)_v = \left(\frac{\Delta e}{\Delta T}\right)_v = \left(\frac{\partial e}{\partial T}\right)_v$$

<u>Specific enthalpy</u> : $h$

$$h = e + pV = e + p/\rho$$

$$de = dh - pdV - Vdp$$

$$dh = \delta Q' + Vdp \qquad \left(\frac{\partial h}{\partial T}\right)_p = C_p$$

<u>Second Law of Thermodynamics</u>

Stability condition: limits those processes satisfying 1ST LAW

<u>specific entropy</u> : $s$

$$Tds \geq \delta Q' \geq 0 \quad \text{Stability} \qquad \text{irreversible}$$

in an infinitesimal reversible process - $Tds = \delta Q'$ rev.

$$de = Tds - pdV \qquad \frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{Dv}{Dt}$$

$$dh = Tds + vdp$$

$$T\frac{Ds}{Dt} = \frac{De}{Dt} + p\frac{Dv}{Dt} \geq 0$$

<u>Special Relations For an Ideal (Perfect) Gas:</u> $V = 1/\rho$

$$Tds = de + pdV = dh - \frac{1}{\rho}dp$$

$$p = \rho RT \quad \text{or} \quad pV = RT$$

Assume $s = s(T,V) \quad e = e(T,V)$

$$T\left(\frac{\partial s}{\partial T}dT + \frac{\partial s}{\partial v}dv\right) = \frac{\partial e}{\partial T}dT + \frac{\partial e}{\partial v}dv + pdV$$

$$\therefore \left(\frac{\partial s}{\partial T}\right)_v = \frac{1}{T}\left(\frac{\partial e}{\partial T}\right)_v \quad ; \quad \left(\frac{\partial s}{\partial v}\right)_T = \frac{1}{T}\left(\frac{\partial e}{\partial v}\right)_T + p/T$$

$$\frac{\partial}{\partial v}\left[\frac{1}{T}\left(\frac{\partial e}{\partial T}\right)_v\right] = \frac{1}{T}\frac{\partial^2 e}{\partial T \partial v}$$

$$\frac{\partial}{\partial T}\left[\frac{1}{T}\left(\frac{\partial e}{\partial v}\right)_T + \frac{R}{v}\right] = -\frac{1}{T^2}\left(\frac{\partial e}{\partial v}\right)_T + \frac{1}{T}\frac{\partial^2 e}{\partial v \partial T}$$

$$\left(\frac{\partial e}{\partial v}\right)_T = 0 \qquad e = e(T) \qquad h = e + pV = e + RT$$

$$\frac{de}{dT} = C_v ; \quad \frac{dh}{dT} = C_p$$

if calorically perfect gases: $C_v, C_p$ = constants

$$\therefore \quad e = C_v T + C \quad ; \quad h = C_p T + C$$

$\frac{C_p}{C_v} = \gamma$ $\qquad \gamma = 1 + \frac{2}{N}$ $\qquad N$ = no. of degrees of freedom of the molecules.

monatomic: $\gamma = 1.67$

diatomic: $\gamma = 1.4$

$$T ds = de + p dV$$
$$= C_v dT - \frac{P}{\rho^2} d\rho$$

if $p = \rho R T$ $\qquad \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$

$$ds = C_v \frac{dp}{p} - (R + C_v)\frac{d\rho}{\rho}$$

$$= C_v \frac{dp}{p} - C_p \frac{d\rho}{\rho}$$

$$\frac{ds}{C_v} = \frac{dp}{p} - \gamma \frac{d\rho}{\rho} \quad ; \quad \frac{S - S_r}{C_v} = \ln(P/P_r) - \ln(\rho/\rho_r)^\gamma$$

$$\boxed{P/P_r = (\rho/\rho_r)^\gamma \left[e^{\frac{S-S_r}{C_v}}\right]}$$

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$R = C_p - C_v \qquad pv = RT$

$$e = h - pv$$

$$de = dh - d(pv) = C_v dT$$

$$= C_p dT - R dT = C_v dT \iff C_v = C_p - R \quad \text{if } dT \neq 0$$

$$T\frac{Ds}{Dt} = \frac{De}{Dt} + p\frac{D\,1/\rho}{Dt} = Q'$$

for frictionless-adiabatic flow $Q' = 0$

$\therefore\ T\frac{Ds}{Dt} = 0 \iff s$ is a constant along a streamline
or an isentropic flow.

if $s \equiv$ constant we have a homentropic flow.

from previous work $p/p_r = \left(\exp\frac{s-s_r}{c_v}\right)\left(\rho/\rho_r\right)^{\gamma}$

for diatomic molecules $\gamma = 1.4$ if a homentropic flow we can use $p = k\rho^{\gamma}$ (polyentropic relation) if the ~~pressure~~ density depends only on the pressure we have a barotropic relationship

$$p = k\rho^{\gamma}$$

Energy Relationships

1) Particle - fixed mass - Inviscid (no viscosity)

$$\frac{D}{Dt}\int_V \rho\,(\underbrace{e}_{\text{internal}} + \underbrace{q^2/2}_{\text{kinetic}})\,dV = -\int p\bar{n}\cdot\bar{q}\,dS + \int_V \rho\bar{F}\cdot\bar{q}\,dV + \int_V \underbrace{\dot{Q}}_{\substack{\text{rate at which heat is}\\ \text{being added to the}\\ \text{system}}}\,dV$$

Tot. Energy   rate of work done on system by, press forces, body forces

$$\Delta W = \bar{F}\cdot\Delta\bar{r} \quad \text{or} \quad \frac{dW}{dt} = \frac{1}{dt}\int_{r_0}^{r+dr}\bar{F}\cdot d\bar{r} \cong \bar{F}\cdot\frac{d\bar{r}}{dt} = \bar{F}\cdot\bar{q}$$

$\bar{F} = -p\bar{n}\,dS$   - sign denotes $\bar{F}$ positive inwards.

$$\frac{D}{Dt}\int_{V_0}\rho\left(e+\frac{q^2}{2}\right)J\,dV_0 = \int_{V_0}\rho\left(e+\frac{q^2}{2}\right)\frac{DJ}{Dt}\,dV_0$$

$$+\int_{V_0} J\,\frac{D}{Dt}\left[\rho\left(e+q^2/2\right)\right]dV_0$$

$$= \int_V \rho\left(e + q^2/2\right) \operatorname{div} \bar{q}\, dV + \int \frac{D}{Dt}\left[\rho\left(e + q^2/2\right)\right] dV$$

$$= -\int \nabla \cdot (p\,\bar{q})\, dV + \int \rho \bar{F} \cdot \bar{q}\, dV + \int_V \dot{Q}\, dV$$

if put under 1 integral sign then we get $\int (\mu)\, dV = 0$ but $dV \neq 0$

$\therefore\ \mu \equiv 0$ or

$$\rho\left(e + q^2/2\right) \operatorname{div} \bar{q} + \frac{D}{Dt}\left[\rho\left(e + q^2/2\right)\right] + \operatorname{div}(p\bar{q}) - \rho \bar{F} \cdot \bar{q} - \dot{Q} = 0$$

where $q^2 = \bar{q} \cdot \bar{q}$

expanding the above, we get:

$$\rho\left(e + q^2/2\right) \operatorname{div} \bar{q} + \rho \frac{D}{Dt}\left(e + q^2/2\right) + \left(e + q^2/2\right) \frac{D\rho}{Dt} + \operatorname{div} p\bar{q} = \rho \bar{F} \cdot \bar{q} + \dot{Q}$$

but from continuity $\frac{D\rho}{Dt} + \rho \operatorname{div} \bar{q} = 0$ $\therefore$ we get

$$\boxed{\rho \frac{D}{Dt}\left(e + q^2/2\right) + \operatorname{div}(p\bar{q}) = \rho \bar{F} \cdot \bar{q} + \dot{Q}}$$

also if $h = e + p/\rho$ then

$$\boxed{\frac{D}{Dt}\left(h + q^2/2\right) = \frac{1}{\rho} \frac{\partial p}{\partial t} + \rho \bar{F} \cdot \bar{q} + \frac{\dot{Q}}{\rho}}$$

where $h + q^2/2 = H_0$ the stagnation enthalpy.

for the particle - fixed volume (no local derivative)

$$\frac{\partial}{\partial t} \int_V \rho\left(e + q^2/2\right) dV = -\int_S p\,\bar{n} \cdot \bar{q}\, dS + \int \rho \bar{F} \cdot \bar{q}\, dV + \int \dot{Q}\, dV$$
$$+ \int_S \rho\left(e + q^2/2\right) \bar{q} \cdot \bar{n}\, dS$$

All Integral Equations

$$\int_V \frac{\partial A}{\partial t}\, dV + \int_S A\, \bar{q}\cdot\bar{n}\, dS = \int_S B\, dS + \int_V C\, dV$$

1) Mass conservation $A = \rho$ ; $B = C = 0$

2) Momentum " $A = \rho\bar{q}$ , $B = -p\bar{n}$ , $C = \rho\bar{F}$

3) energy " $A = \rho(e + q^2/2)$ , $B = -p\bar{n}\cdot\bar{q}$ , $C = \rho\bar{F}\cdot\bar{q} + \dot{Q}$

for steady flow $\int_V \frac{\partial A}{\partial t}\, dV = 0$

no body forces $\longrightarrow$ $\int_V C\, dV = 0$

heat and addition $\longrightarrow$

Particle equations

$$\frac{D\rho}{Dt} + \rho\, \mathrm{div}\, \bar{q} = 0 \qquad \text{Continuity}$$

$$\rho \frac{D\bar{q}}{Dt} = -\nabla p + \rho\bar{F} \qquad \text{Momentum}$$

$$\frac{D}{Dt}\left(h + q^2/2\right) = \frac{1}{\rho}\frac{\partial p}{\partial t} + \bar{F}\cdot\bar{q} + \frac{\dot{Q}}{\rho} \qquad \text{Energy}$$

$$p = \rho R T;\ p/p_r = \left(\exp\frac{S - S_r}{c_v}\right)\left(\rho/\rho_r\right)^{\gamma} \qquad \text{State}$$

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Unknowns: $p, \rho, T, u, v, w, S, h, e, \Omega, \cancel{\mu}$

Equations

1. Continuity or Mass Conservation
3. Momentum or Momentum Integral

1. State
1. Energy.

1. Combining of $1^{st}$ & $2^{nd}$ laws of Thermodynamics
1. Calorically perfect gas. $h = \int C_p dT = C_p T$
1. $e = C_v T$

$$R = C_p - C_v \,, \quad \gamma = \frac{C_p}{C_v} = 1 + 2/N$$

$N = 3$ - monatomic
$N = 5$ - diatomic

1. viscosity
1. Vorticity $\nabla \times \bar{q} = \bar{\Omega} = 2\omega$

Irrotational $\bar{q} = \nabla \phi \qquad \bar{\Omega} = 0$

Frictionless adiabatic flow.

$Ds/Dt = 0$ isentropic flow $s =$ constant $\therefore p = K\rho^{\gamma}$

this is homentropic if $p \propto \rho^{\gamma}$

& $s =$ constant along a stream line

Compressible Bernoulli Equation

$$\rho \frac{D\bar{q}}{Dt} + \nabla p = \rho \bar{F}$$

Steady flow $\partial/\partial t = 0$
Conservative force $\bar{F} = \nabla U$

$$\boxed{\rho\,(\bar{q}\cdot\nabla\bar{q}) + \nabla p = \rho \nabla U}$$

Component along the stream line

$$q\frac{\partial q}{\partial l} + \frac{1}{\rho}\frac{\partial p}{\partial l} = \frac{\partial U}{\partial l}$$

Constant density flow

$$p/\rho + q^2/2 - U = \text{constant along streamline}$$

if $\bar{F} = 0 \quad U = 0$

Compressible flow

Barotropic fluid $\rho = \rho(p)$

$$q\frac{\partial q}{\partial l} + \frac{\partial}{\partial l}\int_{p_0}^{p} \frac{dp}{\rho(p)} = \frac{\partial U}{\partial l}$$

$$\int \frac{dp}{\rho} + q^2/2 - U = \text{constant along streamline}$$

if Homentropic flow $p = K\rho^{\gamma}$ we get.

$$\boxed{\frac{\gamma}{\gamma-1}\frac{p}{\rho} + q^2/2 - U = \text{constant.}}$$ if $U = 0$ then

$$\frac{C_p/C_v}{C_p/C_v - 1} = \frac{C_p}{R} \quad ; \quad \frac{\gamma}{\gamma-1} p/\rho = \frac{\gamma}{\gamma-1} RT = C_p T = h$$

$$H_0 = h + q^2/2 = \text{constant}$$

if $\bar{F} = 0$, $Q = 0$ $\frac{\partial}{\partial t} = 0$

Energy equation : $\rho \frac{DH_0}{Dt} = \frac{\partial p}{\partial t}$ ;

$$\rho(q \cdot \nabla)H_0 = 0 \quad \therefore \quad \rho \bar{q} \frac{\partial H_0}{\partial l} = 0$$

if $H_0 \equiv$ constant we have an isoenergetic flow.

1) Speed of Sound

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \qquad p = p(\rho, s)$$

Perfect gas: $p = f(s)\rho^{\gamma}$ ; $a = \sqrt{\gamma f(s)\rho^{\gamma-1}} = \sqrt{\frac{\gamma f(s)\rho^{\gamma}}{\rho}}$

or $a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT}$

Incompressible flow if $\partial p$ occurs $\rho$ = constant; $\partial \rho = 0$
or $\left(\frac{\partial p}{\partial \rho}\right)_s = \infty$ or $a = \infty$

2) Mach number

$$M = \frac{u}{a} \; ; \; M^2 = \frac{u^2}{a^2} = \frac{\rho u^2}{\gamma p} = \frac{u^2}{\gamma RT}$$

$\frac{\rho u^2}{\gamma p}$ = ratio of $\frac{\text{dynamic press}}{\text{static pressure}}$ ; ratio of $\frac{\text{Kinetic energy}}{\text{thermal energy}} = \frac{u^2}{\gamma RT}$

if $a = \infty$ (incompressible) $M = 0$

in inviscid problems M must be considered.

Stagnation Condition.

Conditions at a point if you bring the flow to rest isentropically.

$$H_0 = h + q^2/2$$

$$= C_pT + q^2/2 = C_pT_0$$

or $$\frac{T_0}{T} = 1 + \frac{q^2}{2C_pT} = \frac{1 + \gamma R M^2}{2C_p} = 1 + \frac{\gamma-1}{2} M^2$$

$$\boxed{\frac{T_0}{T} = 1 + \frac{\gamma-1}{2} M^2}$$

$p = f(s)\rho^\gamma$ but at the same pt. $f(s) = f(s_0)$ by stagnation

$p_0 = f(s_0)\rho_0^\gamma$ definition, therefore

$$\boxed{p/p_0 = (\rho/\rho_0)^\gamma}$$ good even if entropy $\neq$ constant [$p = k\rho^\gamma$ good only at $s = c$.]

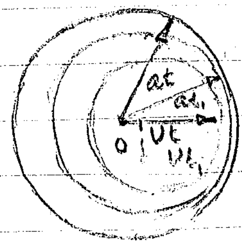
from $p/p_0 = \rho/\rho_0 \; T/T_0 \iff (p/p_0)^\gamma = (\rho/\rho_0)^\gamma (T/T_0)^\gamma = (p/p_0)(T/T_0)^\gamma$

or

$$\boxed{p_0/p = \left(\frac{T_0}{T}\right)^{\gamma/\gamma-1} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\gamma/\gamma-1}}$$

if inviscid steady flow $p_0 = C$ along streamline

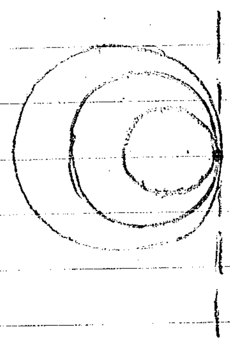
if steady flow, no heat addition $T_0 = C$ along stream line




$a > U \quad M < 1$ subsonic

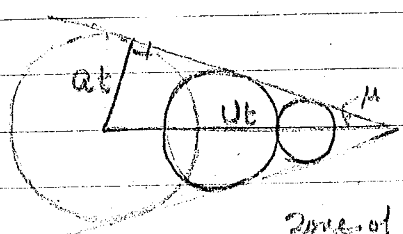
disturbance is felt everywhere.

$a = U \quad M = 1$ transonic



to the left everything is disturbed
to the right nothing is disturbed.

$a < U \quad M > 1$ supersonic




$$\sin\mu = \frac{at}{Ut} = \frac{a}{U} = \frac{1}{M}$$

Zone of Silence

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Governing Equations: $\bar{F} = 0$, Inviscid

$$\frac{\partial\rho}{\partial t} + \rho \operatorname{div}\bar{q} + (\bar{q}\cdot\nabla)\rho = 0$$

$$\frac{\partial\bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q} + \frac{1}{\rho}\nabla p = 0$$

__Unsteady flows__

homentropic : $S =$ constant - energy

$p = k\rho^{\gamma}$ - state

$\bar{q} = \bar{q}'$ (small components), $\rho = \rho_\infty + \rho'$ perturbation value due to disturbance, $\rho' \ll \rho_\infty$

$p = p_\infty + p'$, $p' \ll p_\infty$

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho}$$

$$a^2 = \frac{\gamma(p_\infty + p')}{\rho_\infty + \rho'} = \frac{\gamma p_\infty}{\rho_\infty}\left(\frac{1 + p'/p_\infty}{1 + \rho'/\rho_\infty}\right) \; ; \; (1 + \rho'/\rho_\infty)^{-1} = 1 - \rho'/\rho_\infty + \ldots$$

$$= a_\infty^2 \,(1 + p'/p_\infty + \ldots)(1 - \rho'/\rho_\infty + \ldots)$$

$$= a_\infty^2 \,(1 + p'/p_\infty - \rho'/\rho_\infty + \text{square terms} + \ldots)$$

$$= a_\infty^2 + a_\infty^2 \,(p'/p_\infty - \rho'/\rho_\infty) + \ldots$$

$\therefore \; a^2 = a_\infty^2 + a'^2$

neglect all square terms ; $\bar{q}' \ll a_\infty$

Linear Theory

$$\frac{\partial \rho}{\partial t} + \rho \operatorname{div} \bar{q} + \bar{q} \cdot \nabla \rho = 0$$

$$\frac{\partial \rho'}{\partial t} + (\rho_\infty + \rho')\operatorname{div} \bar{q}' + \bar{q}' \cdot \nabla \rho' = 0$$

second order small

$$\boxed{\frac{\partial \rho'}{\partial t} + \rho_\infty \operatorname{div} \bar{q}' = 0}$$ Linearized Continuity

$$\frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot \nabla)\bar{q} + \frac{1}{\rho}\nabla p = 0$$

second order small

$$\frac{\partial \bar{q}'}{\partial t} + (\bar{q}' \cdot \nabla)\bar{q}' + \frac{1}{\rho_\infty + \rho'}\nabla p' = 0$$

$$\frac{1}{\rho_\infty}\left(\frac{1}{1 + \rho'/\rho_\infty}\right)\nabla p' = \frac{1}{\rho_\infty}\left(1 - \frac{\rho'}{\rho_\infty} + \ldots\right)\nabla p'$$

$$\frac{1}{\rho_\infty}\nabla p' + h(\theta t)$$

note that $\nabla p_\infty = 0$

$$\frac{\partial \bar{q}'}{\partial t} + \frac{1}{\rho_\infty}\nabla p' = 0 \qquad \text{since } p = k\rho^\gamma$$

$$\nabla p = a^2 \nabla \rho = \nabla \rho \left(\frac{\partial p}{\partial \rho}\right)$$

$$\nabla p' = (a_\infty^2 + a'^2)\nabla \rho'$$

$$\therefore \; \boxed{\frac{\partial \bar{q}'}{\partial t} + \frac{a_\infty^2}{\rho_\infty}\nabla \rho' = 0}$$ Linearized Momentum

$$p_\infty + p' = K(\rho_\infty + \rho')^\gamma$$
$$= K\rho_\infty^\gamma (1 + \rho'/\rho_\infty)^\gamma$$

$$= K\rho_\infty^\gamma \left(1 + \frac{\gamma\rho'}{\rho_\infty} + \text{square terms}\right)$$

$$p' = K\rho_\infty^\gamma \left(\frac{\gamma\rho'}{\rho_\infty}\right) \quad ; \quad a_\infty^2 = \frac{\gamma p_\infty}{\rho_\infty}$$

$$\boxed{p' = a_\infty^2 \rho'}$$

$$\frac{1}{\rho_\infty}\frac{\partial^2\rho}{\partial t^2} + \frac{\partial}{\partial t}(\operatorname{div}\bar{q}') = 0$$

$$\operatorname{div}\left(\frac{\partial \bar{q}'}{\partial t}\right) + \frac{a_\infty^2}{\rho_\infty}\operatorname{div}(\nabla\rho') = 0$$

because of continuity $\operatorname{div}\frac{\partial}{\partial t}\bar{q}' = \frac{\partial}{\partial t}\operatorname{div}\bar{q}'$

$$\therefore \quad \frac{\partial^2\rho'}{\partial t^2} = a_\infty^2 \operatorname{div}(\nabla\rho')$$

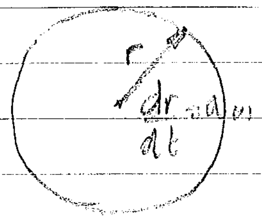
since $\nabla\rho' = \left[\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right]\rho'$

$\therefore \quad \nabla\cdot\nabla\rho' = \nabla^2\rho'$ Laplacian of $\rho'$

$\therefore \quad \boxed{\frac{\partial^2\rho'}{\partial t^2} = a_\infty^2 \nabla^2\rho'}$ wave equation
governing equation of perturbation density.

a) Spherical symmetry $\rho' = \frac{1}{r}\left[f(r - a_\infty t) + g(r + a_\infty t)\right]$

explosion at $r = 0$ ; if $r - a_\infty t$ = constant




b) Cylindrical Symmetry

$$\rho' = \left(a_\infty^2 t^2 - r^2\right)^{-1/2}$$

c) One-dimensional flow:

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$$

$$\rho' = f(x - a_\infty t) + g(x + a_\infty t)$$

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$$\frac{\partial \rho'}{\partial t} + \rho_\infty \nabla \cdot \bar{q}' = 0 \quad ; \quad p' = a_\infty^2 \rho'$$

$$\frac{\partial \bar{q}'}{\partial t} = -\frac{1}{\rho_\infty} \nabla p' = -\frac{a_\infty^2}{\rho_\infty} \nabla \rho' \quad ; \qquad \frac{\partial^2 \bar{q}'}{\partial t^2} = a_\infty^2 \nabla^2 \bar{q}'$$

$$\frac{\partial^2 \rho'}{\partial t^2} = a_\infty^2 \nabla^2 \rho' \quad \text{wave equation} \qquad \nabla^2 \rho' = \operatorname{div}(\operatorname{grad} \rho') = \nabla \cdot (\nabla \rho')$$

1-D wave equation

$$\frac{\partial^2 \rho'}{\partial t^2} = a_\infty^2 \frac{\partial^2 \rho'}{\partial x^2} \quad ; \quad \rho' = f(x - a_\infty t) + g(x + a_\infty t)$$

If we let $\xi = x - a_\infty t$ ; $\eta = x + a_\infty t$ (Characteristic directions

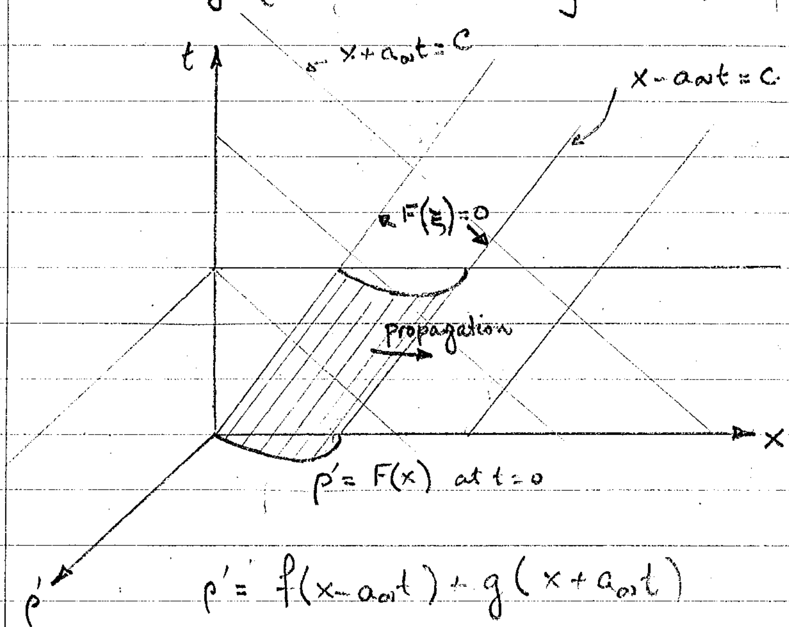
$$\frac{\partial}{\partial t} = -a_\infty \frac{\partial}{\partial \xi} + a_\infty \frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial t^2} = a_\infty^2 \frac{\partial^2}{\partial \xi^2} - 2a_\infty^2 \frac{\partial^2}{\partial \xi \partial \eta} + a_\infty^2 \frac{\partial^2}{\partial \eta^2}$$

$$a_\infty^2 \frac{\partial^2}{\partial x^2} = a_\infty^2 \frac{\partial^2}{\partial \xi^2} + 2a_\infty^2 \frac{\partial}{\partial \xi \partial \eta} + a_\infty^2 \frac{\partial^2}{\partial \eta^2}$$

from this we get $\dfrac{\partial^2 \rho'}{\partial \xi \partial \eta} = 0 \rightarrow \dfrac{\partial \rho'}{\partial \eta} = \bar{g}(\eta)$;

$\rho' = g(\eta) + f(\xi) \Longrightarrow g(x + a_\infty t) + f(x - a_\infty t)$



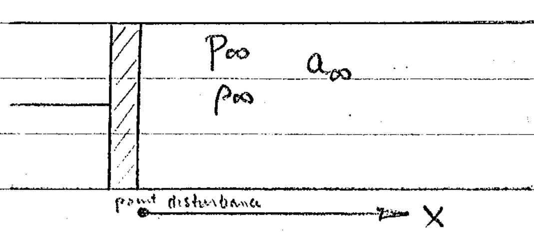

$\rho' = f(x - a_\infty t) + g(x + a_\infty t)$

at $t = 0$ when $g = 0$ $f(x) = F(x)$

$\therefore \rho' = F(x - a_\infty t)$ satisfies initial condition & diff equat ∴ it is a solution

if viscosity is not considered then propagation goes on forever but propagation dissipates when viscosity is taken into effect

## Linearized Acoustic Theory (Unsteady flow)




$\rho' = f(x - a_\infty t) + g(x + a_\infty t)$

$p' = a_\infty^2 \rho'$

to get $u$ we use $\dfrac{\partial \bar{q}'}{\partial t} = -\dfrac{a_\infty^2}{\rho_\infty} \nabla \rho'$

1-D gives us $\dfrac{\partial u'}{\partial t} = -\dfrac{a_\infty^2}{\rho_\infty}\dfrac{\partial \rho'}{\partial x}$

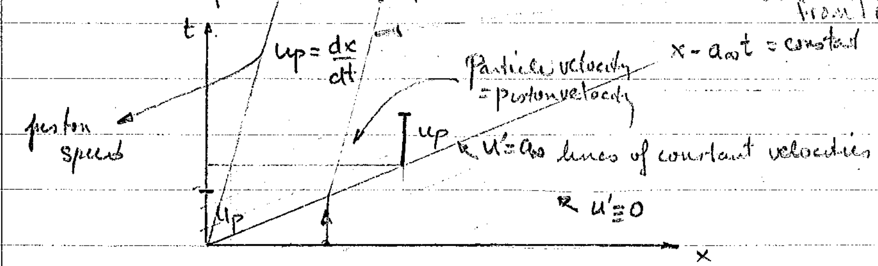
$$\therefore\quad u' = \frac{a_\infty}{\rho_\infty}\left[f(x-a_\infty t) - g(x+a_\infty t)\right]$$

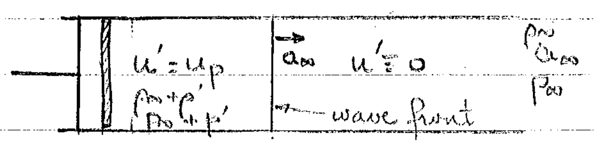
$$\frac{\partial u'}{\partial t} = -\frac{a_\infty^2}{\rho_\infty}\left[\frac{df(x-a_\infty t)}{d(x-a_\infty t)} + \frac{dg(x+a_\infty t)}{d(x+a_\infty t)}\right] = -\frac{a_\infty^2}{\rho_\infty}\left[\frac{df(\xi)}{d\xi} + \frac{dg(\eta)}{d\eta}\right]$$

at $t=0$ $\quad u' = u_p =$ constant $\qquad u_p \ll a_\infty$

$$u' = \frac{a_\infty}{\rho_\infty}\left[f(x-a_\infty t)\right] = u_p \quad \text{at } t=0 \text{ \& } x=0$$
$$= 0 \qquad x>0$$

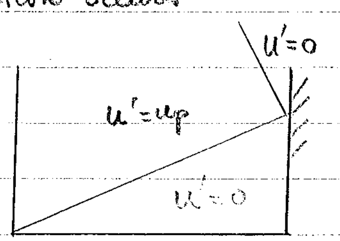
$$u' = \frac{a_\infty}{\rho_\infty} f(x) = \frac{a_\infty}{\rho_\infty} f(0) = u_p$$




if closed off $\quad$ use $\quad u' = \dfrac{a_\infty}{\rho_\infty}\left(f(x-a_\infty t) - g(x+a_\infty t)\right) = 0$

wave reflection occurs

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When wave hits the wall

$$\rho' = f(x - a_\infty t) + g(x + a_\infty t)$$

$$u' = \frac{a_\infty}{\rho_\infty}\left[f - g\right]$$

$$p' = a_\infty^2 \rho'$$

before hitting the wall $g = 0$

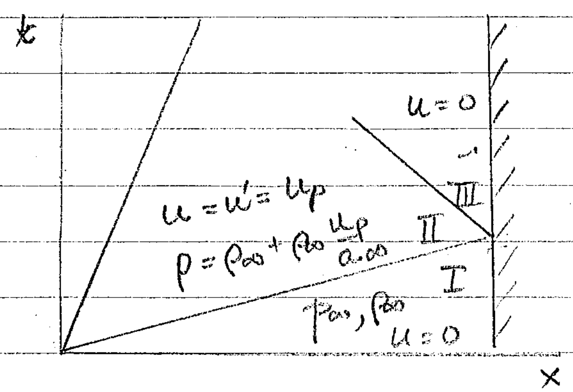
$x < a_\infty t \qquad u' = u_p = \frac{a_\infty}{\rho_\infty} f \qquad t = 0,\ x = 0$

$x > a_\infty t \qquad = 0 \qquad t = 0,\ x > 0$

$\rho' = \frac{\rho_\infty u_p}{a_\infty} \qquad \frac{u_p}{a_\infty} \ll 1 \quad \therefore \ \rho'/\rho_\infty \ll 1 \qquad x < a_\infty t$

$\rho' = 0 \qquad x > a_\infty t$

at wall $u = 0$ boundary condition. $f = g \qquad g \neq 0$




$\rho'_{III} = 2\rho'_{II}$

$f_{II} = \frac{\rho_\infty u_p}{a_\infty}$

$g_{III} = f_{II} = f_{III}$

Going back to Governing Equation

Assume a steady flow & Linearize Equations, $\bar{F} = 0$

$s \equiv$ constant

$\bar{q} = U_\infty i + \bar{q}' \qquad \bar{q}' = u'i + v'j$

$p = p_\infty + p' \qquad \rho = \rho_\infty + \rho'$

Since steady flow $\partial/\partial t = 0$ ; $U_\infty$ = constant

Continuity

$$\frac{\partial \rho}{\partial t} + \bar{q}\cdot\nabla\rho + \rho\nabla\cdot\bar{q} = 0$$ neglect square terms

$$U_\infty \frac{\partial \rho'}{\partial x} + \rho_\infty\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

Momentum

$$(\bar{q}\cdot\nabla)\bar{q} + \frac{1}{\rho}\nabla p = 0$$

$$U_\infty \frac{\partial \bar{q}'}{\partial x} + \frac{1}{\rho_\infty}\nabla p' = 0 \left\{ \begin{array}{l} U_\infty \frac{\partial u'}{\partial x} + \frac{1}{\rho_\infty}\frac{\partial p'}{\partial x} = 0 \\ U_\infty \frac{\partial v'}{\partial x} + \frac{1}{\rho_\infty}\frac{\partial p'}{\partial y} = 0 \end{array}\right.$$

$$p = k\rho^\gamma \quad a^2 = \frac{\gamma p}{\rho} \implies p' = a_\infty^2 \rho'$$

from the above momentum equation we get:

$$U_\infty \frac{\partial u'}{\partial x} + \frac{a_\infty^2}{\rho_\infty}\frac{\partial \rho'}{\partial x} = 0$$ X-momentum eq.

$$U_\infty \frac{\partial v'}{\partial x} + \frac{a_\infty^2}{\rho_\infty}\frac{\partial \rho'}{\partial y} = 0$$ Y-momentum eq.

from the X-momentum & continuity equations we get:

$$\frac{U_\infty^2}{a_\infty^2}\rho_\infty \frac{\partial u'}{\partial x} - \rho_\infty\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

$$(1 - M_\infty^2)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$ ; Assume flow is irrotational $\nabla\times\bar{q} = 0$

$$\bar{q} = \nabla\varphi$$

$$u' = \frac{\partial\varphi}{\partial x} \quad v' = \frac{\partial\varphi}{\partial y}$$

$$\therefore (1 - M_\infty^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \boxed{\beta_\infty^2 \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0}$$

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Test 2 wks from today

Assume Steady flow ; $s \equiv$ constant

$\bar{q} = U_\infty i + \bar{q}'$ where $\bar{q}' = u'i + v'j$ $\quad u', v' << U_\infty$

$\rho = \rho_\infty + \rho'$

$p = p_\infty + p'$ where $p' = a_\infty^2 \rho'$

and $a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ ; $a_\infty^2 = \frac{\gamma p_\infty}{\rho_\infty}$ & $p = K\rho^\gamma$

Neglecting products of primed terms we get:

$$(1 - M_\infty^2)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

Flow is taken irrotational .∋. $u' = \frac{\partial \varphi}{\partial x}$ ; $v' = \frac{\partial \varphi}{\partial y}$

$$\therefore \quad \beta_\infty^2 \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{where } \beta_\infty^2 \equiv 1 - M_\infty^2 = -m_\infty^2$$

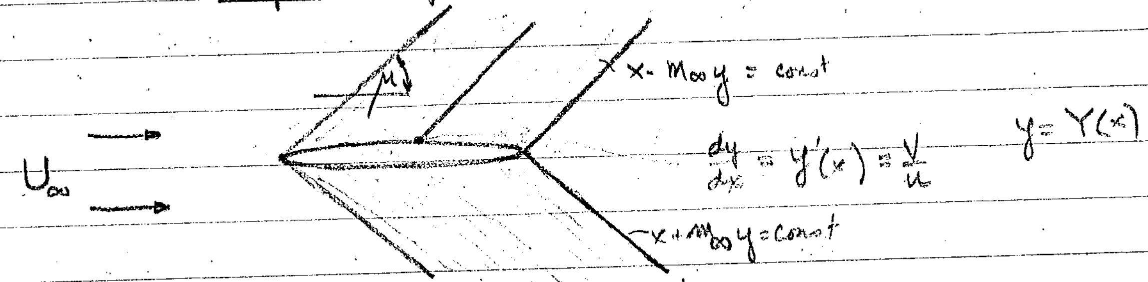
$M_\infty > 1$ — if $1 - M_\infty^2 < 0$ then $\frac{\partial^2 \varphi}{\partial y^2} = m_\infty^2 \frac{\partial^2 \varphi}{\partial x^2}$ wave equation

$M_\infty = 0$ — if $1 - M_\infty^2 = 1$ then we get Laplace equation

$M_\infty = 1$ — if $1 - M_\infty^2 = 0$ then we get $\frac{\partial^2 \varphi}{\partial y^2} = 0$ { Breaks down in transonic flow

$1 > M_\infty > 0$ — if $1 - M_\infty^2 > 0$ then $\beta_\infty^2 \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$

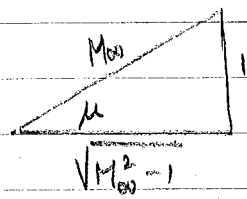
## Supersonic flow.




$$\varphi = f(x - m_\infty y) + g(x + m_\infty y)$$

top line $x - m_\infty y = $ const.

$$dy/dx = \frac{1}{m_\infty} = \frac{1}{\sqrt{M_\infty^2 - 1}} = \tan\mu$$

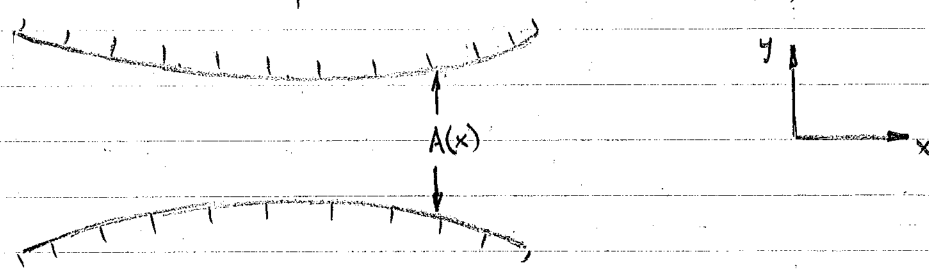



$\sin\mu = \dfrac{1}{M_\infty}$

no disturbance to rear or forward of trailing or leading mach lines. (Different from subsonic where all is disturbed)
Mach wave is weakest form of shock wave.

<u>Quasi - 1-D Flows.</u>

2-D flows, use boundary conditions, inviscid, steady flows




homentropic flow $S \equiv C$ $\quad T_0 = \text{const}$ $\quad$ frictionless adiabatic flow

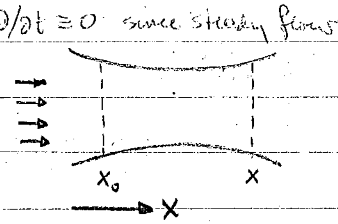
$p_0 = \text{const}$

flow across the channel doesn't change too much

1. long-slender channels (entrance & exits flow donot dominate)
   Flow changes along x-direct not very large. Edge effects neglected
2. Assume area variation should be small wrt x-direct
   $\dfrac{A'(x)}{A(x)} \ll 1$
3. Viscous effects are small.

1. <u>Mass Conservation</u> $\quad \partial/\partial t \equiv 0$ since steady flows

$$\int_S \rho \bar{q}\cdot\bar{n}\, dS = 0$$




$\bar{q}\cdot\bar{n} = 0$ on the walls $\qquad \bar{q}$ at wall $= 0$

$$\rho_{av}(x) = \frac{1}{A(x)} \int_0^{A(x)} \rho \, dy$$

$$\rho = \rho_{av}(x) + \hat{\rho}(x,y) \text{ where } \hat{\rho} \ll \rho_{av}$$

$$-\rho_{av}(x_0)\; u_{av}(x_0)\, A(x_0) + \rho_{av}(x)\, u_{av}(x)\, A(x) = 0$$

since normal is opposite to flow    since normal is in same direct. as flow

$$\therefore \boxed{\rho A u = \text{constant}}$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \qquad \text{Continuity}$$

$$\int_0^{A(x)} \frac{\partial \rho u}{\partial x}\, dy + \int_0^{A(x)} \frac{\partial \rho v}{\partial y}\, dy = 0$$

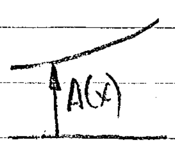
Leibnitz's rule:

$$\frac{\partial}{\partial x}\int_0^{A(x)} \rho u \, dy = \int_0^{A(x)} \frac{\partial}{\partial x} \rho u \, dy + A'(x)\, \rho u \Big|_{y = A(x)}$$

by Leibnitz's rule we get

$$\frac{\partial}{\partial x}\int_0^{A(x)} \rho u \, dy - A'(x)\, \rho u \Big|_{y=A(x)} + \rho v \Big|_0^{A(x)} = 0$$

$$\quad " \quad - A'(x)\, \rho u \Big|_{y=A(x)} + \rho v \Big|_{y=A(x)} = 0$$




$\rho v|_0 = 0$

but $\frac{dy}{dx} = \frac{v}{u} = A'(x)$ at $y = A(x)$ $\therefore$ we come to the following

since $A'u = v$ $\quad \frac{\partial}{\partial x}\int_0^{A(x)} \rho u \, dy = 0 \;;\; \frac{\partial}{\partial x}\left[A \rho_{av} u_{av}\right] = 0$

and $\rho A u = \text{constant}$

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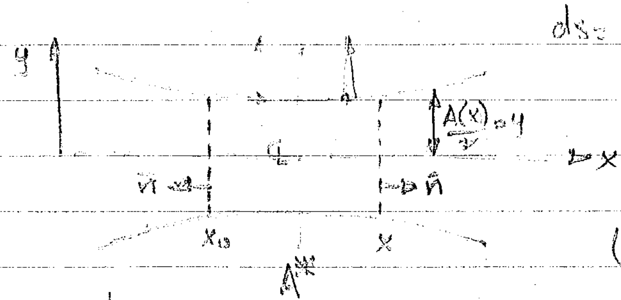
2. <u>Momentum Integral Equation – steady flow, inviscid, $\bar{F}=0$</u>

$$\int_S \rho \bar{q}\, \bar{q}\cdot\bar{n}\,ds + \int_S p\,\bar{n}\,ds = 0$$

$ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} = dx\sqrt{1+\frac{A'^2}{4}}$



since $\frac{A'}{4} \ll 1 \therefore ds \approx dx$

$(\bar{n})_x = -\frac{A'(x)}{2}$

x-component

$$-\rho(x_0)\,u^2(x_0)\,A(x) + \rho(x)\,u^2(x)A(x) - p(x_0)A(x_0) + p(x)A(x)$$

$$-2\int_{x_0}^{x} p\,\frac{A'(t)}{2}\,dt = 0$$

differentiate with respect to x:

$$(\rho u A)\,u'(x) + p'A + pA' - pA' = 0$$

Momen – $\boxed{p' + \rho u\, u' = 0}$ $\qquad p = K\rho^{\gamma}$

Cons – $\boxed{\frac{d}{dx}(\rho u A) = 0}$ $\qquad a^2 = \frac{\gamma p}{\rho}$

<u>Shockless isentropic flow</u>

From moment

$$\rho u\, u' + p' = \rho u\, u' + a^2\rho' = 0 \qquad \frac{dp}{dx} = \frac{dp}{d\rho}\frac{d\rho}{dx} = a^2\rho'$$

$\ln(\rho u A) = $ const.

$$\therefore \quad \rho'/\rho + \frac{u'}{u} + \frac{A'}{A} = 0$$

$$\frac{u u'}{a^2} + \rho'/\rho = 0 \qquad M^2\frac{u'}{u} + \rho'/\rho = 0$$

Eliminating the density

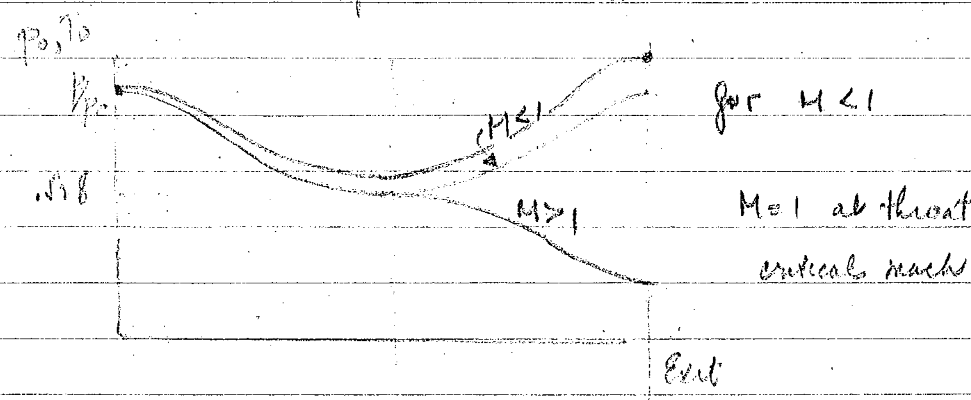
$$(M^2-1)\frac{u'}{u} = \frac{A'}{A} = -\frac{(M^2-1)}{M^2}\rho'/\rho = -\frac{(M^2-1)}{M^2a^2}\frac{p'}{\rho}$$

$$= -\frac{(M^2-1)}{\gamma M^2} p'/p$$

| | | | | | |
|---|---|---|---|---|---|
| Converging | $A'<0$ | $M<1$ | $u'>0$ | $\rho'<0$ | $p'<0$ |
| | | $M>1$ | $u'<0$ | $\rho'>0$ | $p'>0$ |
| Diverging | $A'>0$ | $M<1$ | $u'<0$ | $\rho'>0$ | $p'>0$ |
| | | $M>1$ | $u'>0$ | $\rho'<0$ | $p'<0$ |

if $M=1$ then it must occur when $A'=0$ or at $A=A^*$

pressure in channel




$A=A_T$

$M_T<1$ if $p/p_0$ at exit for $M_T<1$ is $>$ $p/p_0$ at exit for $M^*=1$

<u>Mass Flow</u>

$$\rho u A = \rho^* u^* A^* \qquad A^* \le A_T \; ; \; a^2 = \gamma RT$$

$$\frac{A}{A^*} = \frac{\rho^* u^*}{\rho u} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{a^*}{a}\frac{a}{u}$$

$$= \frac{1}{M}\frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\sqrt{\frac{T^*}{T_0}}\sqrt{\frac{T_0}{T}}$$

but $\frac{T_0}{T} = 1+\frac{\gamma-1}{2}M^2$ ; $\rho_0/\rho = \left(\frac{T_0}{T}\right)^{1/\gamma-1}$ ; $\frac{T_0}{T^*} = \frac{\gamma+1}{2}$ ;

$$\frac{\rho_0}{\rho^*} = \left(\frac{\gamma+1}{2}\right)^{1/\gamma-1}$$

$$\therefore \quad \frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

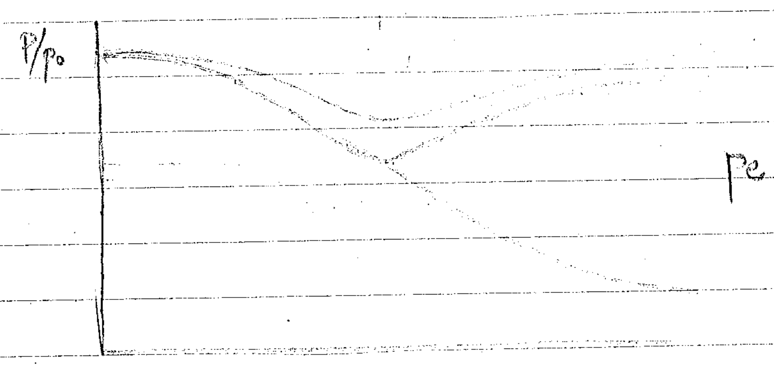
$$\dot m = \rho u A = \rho^* u^* A^* = \rho^* u^* \frac{A^*}{A} A$$

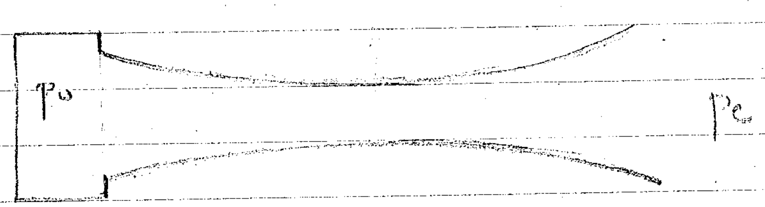
$$\frac{\dot m}{A} = \frac{A^*}{A}(M)\,\rho^* a^* = \frac{A^*}{A}(M)\sqrt{\gamma R T^*}\,\frac{p^*}{RT^*}$$

$$= \frac{p_0}{\sqrt{T_0}}\frac{A^*}{A}(M)\sqrt{\frac{\gamma}{R}}\;\frac{p^*}{p_0}\sqrt{\frac{T_0}{T^*}}$$

$$\boxed{\frac{\dot m}{A} = f(\gamma)\,\frac{p_0}{\sqrt{T_0}}\,\frac{A^*}{A}(M)} \qquad \frac{A^*}{A}(M)_{max} \text{ at } M=1$$

max mass will pass at Area where $M=1$ — choked flow




1. Assume $A_t = A^*$ $\qquad \frac{A_{exit}}{A^*} \rightarrow M_{exit}$ (2 values $M<1$, $M>1$)
2. Given $A_{exit}$ →

3. $\frac{p_{exit}}{p_0} \leftarrow M_{exit}$

If $\frac{p_{exit}}{p_0} >$ given $p/p_0$ then

calculate $M_e \rightarrow \frac{A_e}{A^*}$ & we obtain $A^*$ since $A^* = A(x)$

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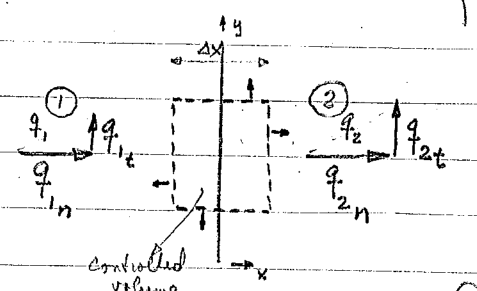
Exam: Know assumptions and conditions for equations

Exam is long

<u>Investigation of other flows</u> (Discontinuity in flows, s≠c)

Q: Can a discontinuity exist in a steady inviscid flow?

fixed discontinuity assumed.




① $\int_S \rho \bar{q}\cdot\bar{n}\, ds = 0$

② $\int_S \rho \bar{q}\, \bar{q}\cdot\bar{n}\, ds + \int p\bar{n}\, ds = 0$

③ $\int_S \rho\left(e + \frac{q^2}{2}\right)(\bar{q}\cdot\bar{n})\, ds + \int_S p\, \bar{q}\cdot\bar{n}\, ds = 0$

Steady flow, $\bar{F} = 0$

Viscosity is unimportant outside of the discontinuity

In ①, ②, ③ the contributions in $(x_0, y_0)$ and $(x_0, -y_0)$ are opposite or $(x_0, -y_0) \equiv -(x_0, y_0)$

① Continuity Eq. $\quad -\rho_1 q_{1n} \Delta y + \rho_2 q_{2n} \Delta y = 0$

$$\boxed{\rho_1 q_{1n} = \rho_2 q_{2n}}$$

② Momentum eq: x comp

$$-\rho_1 q_{1n}\, q_{1n} \Delta y + p_1(-1)\Delta y + \rho_2 q_{2n}\, q_{2n} \Delta y + p_2 \Delta y = 0$$

$$\boxed{p_1 + \rho_1 q_{1_n}^2 = p_2 + \rho_2 q_{2_n}^2}$$

(3) Energy Eq

$$-\rho_1\left(e_1 + \frac{q_1^2}{2}\right)(q_{1_n}\Delta y) - p_1 q_{1n}\Delta y + \rho_2\left(e_2 + \frac{q_2^2}{2}\right)(q_{2_n}\Delta y) + p_2 q_{2_n}\Delta y =$$

$$\left(e_1 + \frac{p_1}{\rho_1} + \frac{q_1^2}{2}\right)\rho_1 q_{1n} = \left(e_2 + \frac{p_2}{\rho_2} + \frac{q_2^2}{2}\right)\rho_2 q_{2n}$$

if $\rho_1 q_{1n} = \rho_2 q_{2n}$ then $\boxed{H_{o_1} = H_{o_2}}$

(2) Mom: y-comp

$$-\rho_1 q_{1_t} q_{1_n} + \rho_2 q_{2_t} q_{2_n} = 0$$

$$\boxed{\rho_1 q_{1_t} q_{1_n} = \rho_2 q_{2_t} q_{2_n}}$$

if $q_n \neq 0$ $\quad q_n \rho = C$ across discontinuity

$p + \rho q_n^2 = C$ across discontinuity

$q_t = C$ " "

$H_o = C$ " "

<u>Normal Shocks:</u> $q_t = 0$ let $q_n = u$

1. $\rho_2 q_{2_n} = \rho_1 q_{1n}$ — 3a. Calorically perfect gas $h = C_pT$

2. $p_2 + \rho_2 q_{2_n}^2 = p_1 + \rho_1 q_{1_n}^2$ — 4. $P = \rho RT$

3. $h_2 + \frac{q_{2n}^2}{2} = h_1 + \frac{q_{1n}^2}{2}$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}$$

$$M_1 = \frac{u_1}{a_1} \; ; \quad a_1^2 = \frac{\gamma p_1}{\rho_1}$$

$$p_2/p_1 = 1 + \frac{2\gamma}{\gamma+1}(M_1^2-1)$$

$$T_2/T_1 = 1 + \frac{2(\gamma-1)}{(\gamma+1)^2}\left[\frac{\gamma M_1^2+1}{M_1^2}(M_1^2-1)\right]$$

$$M_2^2 = \frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}}$$

$$\frac{S_2-S_1}{R} = \ln\frac{P_{o_1}}{P_{o_2}} = \ln\left\{\left[1+\frac{2\gamma}{\gamma+1}(M_1^2-1)\right]^{1/\gamma-1}\left[\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}\right]^{-\gamma/\gamma-1}\right\}$$

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Normal Shock

$q_n = u \neq 0$

$q_t = 0$

$\rho u = \text{const}$

$p + \rho u^2 = \text{const}$

$h + \frac{u^2}{2} = H_0 = \text{const.} = C_p T_0$

$h = C_p T$

$p = \rho R T$

mach behind the shock with respect to mach in front of shock

$$M_2^2 = \frac{1+\frac{(\gamma-1)}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}}$$

$$\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} \; ; \quad \frac{T_2}{T_1} = 1 + \frac{2(\gamma-1)}{(\gamma+1)^2}\left[\frac{\gamma M_1^2+1}{M_1^2}(M_1^2-1)\right]$$

1st & 2nd law of thermo

$T_o\, ds_o = dh_o - v_o\, dp_o$

no heat & work $\Rightarrow dh_o = 0$ since $T_o$ is unchange

$\Rightarrow T_o\, ds_o = -v_o\, dp_o = -\frac{RT_o}{p_o}\, dp_o \Rightarrow \frac{ds_o}{R} = $

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \quad ; \quad \frac{S_2 - S_1}{R} = \ln \frac{p_{o_1}}{p_{o_2}}$$

this is obtained from

$$p/p_r = \exp\left(\frac{s-s_r}{c_v}\right)\left(\rho/\rho_r\right)^{\gamma}$$

$$p/p_o = \left(\rho/\rho_o\right)^{\gamma} = \left(T/T_o\right)^{\gamma/\gamma-1}$$

$$\frac{S_2 - S_1}{R} = \ln\left\{\left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right]^{1/\gamma-1}\left[\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2}\right]^{-\gamma/\gamma-1}\right\}$$

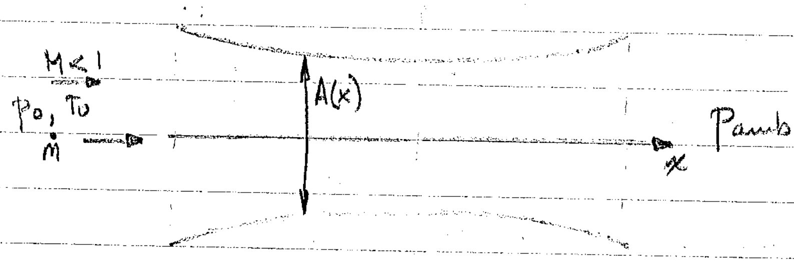
$S_2 \geq S_1$ only if $M_1 \geq 1$

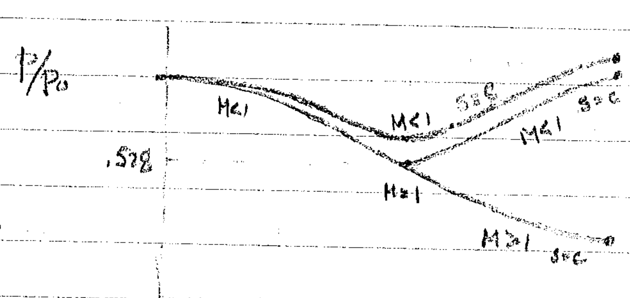
$M_1 = 1 + \epsilon \quad \epsilon \ll 1$ weak discontinuity or shock

$M_1 \gg 1$, $\frac{\rho_2}{\rho_1} \to \frac{\gamma+1}{\gamma-1}$ strong wave (shock)

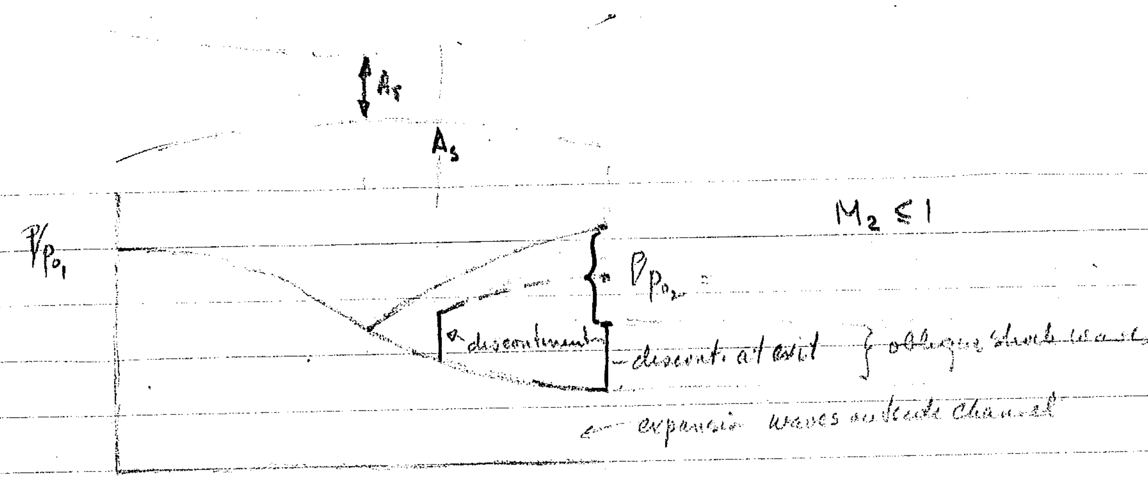
$$\left.\begin{array}{ll} \rho_2/\rho_1 \geq 1 & S_2 - S_1 \geq 0 \\ u_2/u_1 \leq 1 & p_{o2}/p_{o1} \leq 1 \\ T_2/T_1 \geq 1 & \\ p_2/p_1 \geq 1 & \end{array}\right\}$$

if $M_1 \geq 1$ this is a compression discontinuity




$\gamma = 1.4$




$A^*$ changes across shock.

$$\frac{A}{A^*} = f(M)$$ . A is the same $f(M)$ changes $\therefore A^*$ change

$A_2^*$, $p_2$, $\rho_2$, $T_2$, $M_2$ can be obtained.
when $M=1$

if $p/p_0$ is a value betw $\frac{P_{sub}\ isentrop}{P_0}$, $\frac{P}{P_0}$ shock at exit
the discontinuity is in the channel

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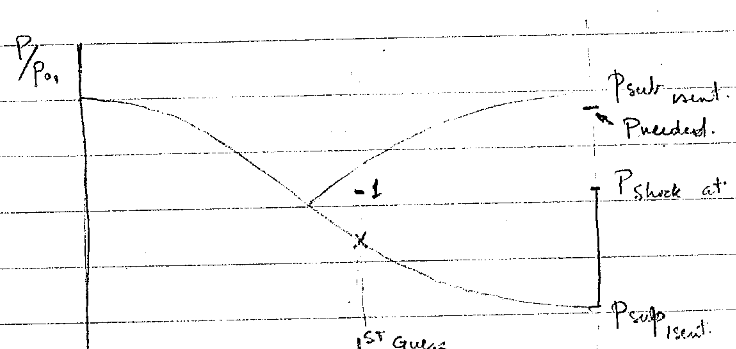


find all values at first guess on Psup isent.
use shock jump cond. to get press after shock.
knowing this and $A^*$ at 1 use equat to get new
M and new $P/P_0$ & check against $P/P_0$. If it is greater
new guess s/b closer to throat; if lower move new guess
farther from throat.

## Heat Additions in a Constant - Area Channel, Rayleigh

Governing Equations

Conserv. of Mass: $\rho u =$ constant

Momentum: $p' + \rho u u' = 0$ ; $\rightarrow p + \rho u^2 =$ constant.

Energy: $T_0 \neq$ constant.

Steady flow, $F = 0$, inviscid

$$\rho u \frac{\partial H_0}{\partial x} + \rho v \frac{\partial H_0}{\partial y} = \dot{Q}$$

$$\int \rho \left(u + \frac{q^2}{2}\right) q_n \, dA + \int p \, \underset{\sim}{q} \cdot \underset{\sim}{n} \, dA = \int \dot{Q} \, dA$$

or $\left(u + p/\rho + \frac{q^2}{2}\right) \rho \, \underset{\sim}{q} \cdot \underset{\sim}{n} \, dA$

State: - $p = \rho R T$ or $p = f(s) \rho^{\gamma}$

Quasi - 1D flow:

$$\rho u H_0' = \dot{Q}(x) \qquad \text{heat added } T_0' > 0$$

$$\rho u c_p T_0' = \dot{Q}(x)$$

$$T_0 = T\left(1 + \frac{\gamma - 1}{2} M^2\right)$$

$\rho a M =$ constant. $\qquad a^2 = \gamma R T$

$$p + \rho u^2 = \rho\left(RT + u^2\right) = \rho\left(\frac{a^2}{\gamma} + u^2\right) = \frac{\rho a^2}{\gamma}\left(1 + \gamma M^2\right) = \text{const.}$$

$$\boxed{\frac{T_0'}{T_0} = \frac{T'}{T} + \frac{(\gamma - 1) M M'}{1 + \frac{\gamma - 1}{2} M^2}}$$

; hence $\rho a M =$ const $\rightarrow \frac{a}{M \gamma}\left(1 + \gamma M^2\right) =$ const.

$$\frac{T^{1/2}}{M}\left(1 + \gamma M^2\right) = \text{const.}$$

$$\boxed{\frac{T'}{2T} - \frac{M'}{M} + \frac{2 \gamma M M'}{1 + \gamma M^2} = 0}$$

this leads to:

$$\frac{T_0'}{T_0}\left(1 + \gamma M^2\right)\left(1 + \frac{\gamma - 1}{2} M^2\right) = 2\left(1 - M^2\right) \frac{M'}{M}$$

if $M < 1$ $\quad M' > 0$ when heat is added $\quad T_0' > 0$

if $M > 1$ $\quad M' < 0$ when " " " $\quad T_0' > 0$

tend to drive $M \rightarrow 1$ (you are increasing mass flow toward maximum) Note: mass max occurs when $M = 1$ chocked flow

$A = A^*$

Compensation for this occurs outside channel.

with respect to $T$: when $T_0' > 0$

$T' > 0 \qquad M > 1$

$T' > 0 \qquad M < \frac{1}{\sqrt{\gamma}}$

$T' < 0 \qquad 1 > M > \frac{1}{\sqrt{\gamma}}$

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$A^* = A_T$ $\qquad$ $A$ at discont. $\qquad \frac{A}{A^*} \to M_1,\ p/p_0$

use jump cond. $M_2 = f(M_1)$; $p_2/p_1$; look up $p_{0_1}/p_{0_2}$

using $M_2$ look for $A/A^*$ $\qquad$ $A$ is known

$A^*$, $p_{0_2}$ is known, $T_{0_2} = T_{0_1}$

Equations with Heat Addition, A = cond.

mass Cons: $\rho u = m = \text{constant}$

moment: $p + \rho u^2 = p + \frac{m^2}{\rho} = \text{constant}$

state: $p = \rho R T$

$T_0 \neq C$

energy: $\rho u H_0' = \dot{Q}(x)$

$$m\, C_p \left( T_0(x) - T_0(x_0) \right) = \int_{x_0}^{x} \dot{Q}(x)\, dx$$

$$dp - \frac{m^2}{\rho^2}\, d\rho = 0 \ ; \quad \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

Eliminate $d\rho \longrightarrow \left(\rho - \frac{m^2}{p}\right) dp + \frac{m^2\, dT}{T} = 0$

$$p + \frac{m^2}{\rho} = p_A + \frac{m^2}{\rho_A} = C \ ; \quad \rho = \frac{m^2}{p_A + \frac{m^2}{\rho_A} - p}$$

$$\left( \frac{m^2}{p_A + \frac{m^2}{\rho_A} - p} - \frac{m^2}{p} \right) dp + \frac{m^2\, dT}{T} = 0$$

$$\left(\frac{1}{p_A + \frac{m^2}{\rho_A} - p} - \frac{1}{p}\right) dp + \frac{dT}{T} = 0$$

$$-\ln\left(p_A + \frac{m^2}{\rho_A} - p\right)\Bigg|_{p_A}^{p} - \ln p\Bigg|_{p_A}^{p} + \ln T\Bigg|_{T_A}^{T} = 0$$

$$-\ln\left(p_A + \frac{m^2}{\rho_A} - p\right) + \ln\left(\frac{m^2}{\rho_A}\right) - \ln p + \ln p_A + \ln T - \ln T_A = 0$$

$$-\ln\left(\frac{p_A + m^2/\rho_A - p}{m^2/\rho_A}\right) - \ln\left(p/p_A\right) + \ln\left(T/T_A\right) = 0$$

$$T/T_A = \left(\frac{p_A + m^2/\rho_A - p}{m^2/\rho_A}\right)\left(p/p_A\right)$$

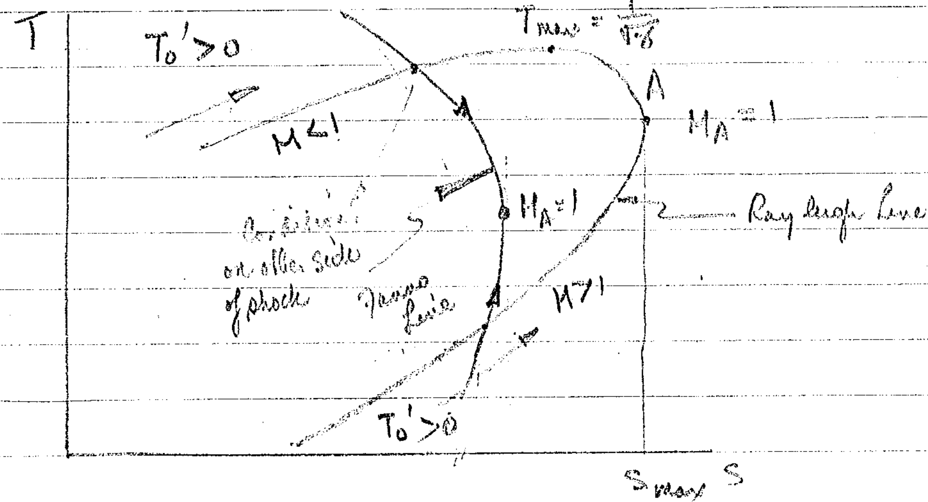
$$= \left(\frac{1 + m^2/\rho_A p_A - p/p_A}{m^2/\rho_A p_A}\right)\left(p/p_A\right)$$

$$T/T_A = \left(\frac{1 + \gamma M_A^2 - p/p_A}{\gamma M_A^2}\right)\left(p/p_A\right)$$

$$\boxed{\begin{gathered}\left(p/p_A\right)^2 - p/p_A\underbrace{\left(1 + \gamma M_A^2\right)}_{2B} + \underbrace{\gamma M_A^2 \frac{T}{T_A}}_{C} = 0 \\[1em] p/p_A = B - \sqrt{B^2 - C}\end{gathered}}$$

$$\frac{S - S_A}{c_p} = \ln\left[\left(\frac{T}{T_A}\right)\left(p/p_A\right)^{-\left(\frac{\gamma-1}{\gamma}\right)}\right] \quad \text{Equation of state.}$$

→ use the above & state equations giving Rayleigh Equat choose A state as point of Max. entropy




H.W. prove $M_A = 1$ @ $S_A$

prove $T_{max} = \frac{1}{\sqrt{\gamma}}$

<u>Flow with friction A = const. no heat addition</u>

$\rho u = m = \text{const}$

$p = \rho R T$

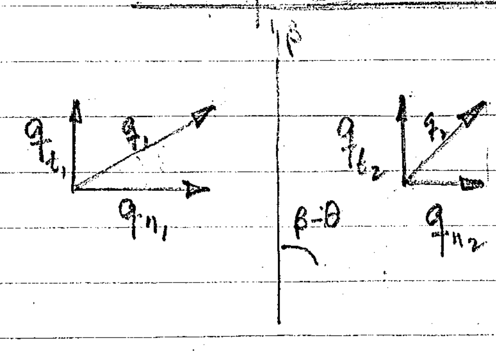
$$T_0 = T\left(1 + \frac{\gamma-1}{2} M^2\right) = \text{Const.}$$

If A (ref state) is chosen $\ni$ $M_A = 1$

$$\frac{S - S_A}{C_p} = \ln\left[\left(\frac{2}{\gamma-1}\right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{T}{T_A}\right)^{1/\gamma} \left(\frac{\gamma+1}{2} - \frac{T}{T_A}\right)^{\frac{\gamma-1}{2\gamma}}\right]$$

Fanno Equation

## <u>Oblique Shock Waves</u>




$q_{n_2} < q_{n_1}$ $q_{t_1} = q_{t_2}$

$\rho_2 > \rho_1$

$\theta$ represents change in flow direction

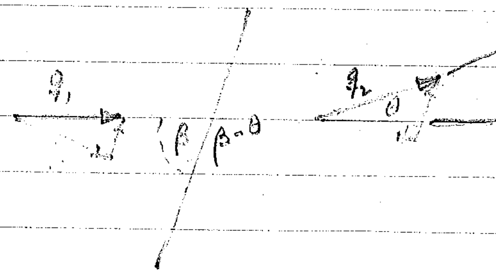
Compressive Shock.

1. $[\rho q_n] = 0$ $\quad [\ ] = (\ )_2 - (\ )_1$

2. $[p + \rho q_n^2] = 0$

3. $[H_0] = 0$ $\quad$ 4. $p = \rho R T$ $\quad$ 5. $[q_t] = 0$

$$H_0 = h + q^2/2 = c_p T + q^2/2 \; ; \quad q^2 = q_t^2 + q_n^2 \quad \therefore [q^2] = [q_n^2]$$



let $q_t = V$ $\quad q_n = u$

all conditions for normal shock can be used for oblique shock wave, iff Mach no. in front of shock is interpreted as $M_{normal}$ & not $M_{actual}$

as $\theta \uparrow$ shock becomes weaker $\Delta$ of $S$, $p \downarrow$ after shock

$M_n = \frac{q_n}{a} \geq 1$ $\quad M_1 \sin\beta = M_{n_1}$ $\quad M_1 = \frac{q_1}{a_1}$

$\boxed{M_1 \sin\beta = 1 \quad \text{weak shock limit}}$ $\quad \sin\beta = \frac{1}{M_1} = \sin\mu$

$a_2 > a_1$ since $T_2 > T_1$

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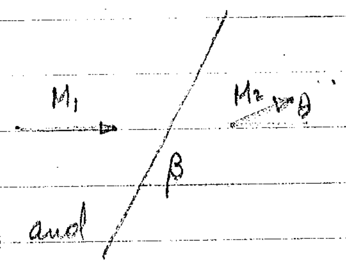
$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_{1n}^2}{(\gamma-1) M_{1n}^2 + 2} \qquad M_{n_1} = \frac{q_{1n}}{a_1} = M_1 \sin\beta$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_1^2 \sin^2\beta}{(\gamma-1) M_1^2 \sin^2\beta + 2} = \frac{u_1}{u_2}$$

$$\frac{p_2}{p_1} = \frac{1+2\gamma(M_1^2 \sin^2\beta - 1)}{\gamma+1}$$

$$M_{n_2} = M_2 \sin(\beta - \theta)$$




$\beta$ is measured wrt the incoming $M_1$ and the shock wave.

$$\tan\beta = \frac{u_1}{V_1} \qquad \tan(\beta-\theta) = \frac{u_2}{V_2} \qquad V_2 = V_1, \quad g_t = g_n$$

$$\frac{\tan\beta}{\tan(\beta-\theta)} = \frac{u_1}{u_2}$$ which we have.

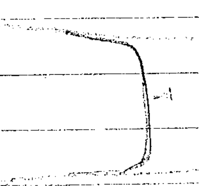
For every value of $\theta$ there are 2 solutions of $\beta$

1. for the physical problems (External-flow problems) weak shock values
2. for internal flow use strong shock values [channel problems] normal

when turning angle is greater than max turning angle than the shock wave is detached.

strong shocks give subsonic $M_2$ in oblique shock waves

weak shocks usually gives supersonic mach no. behind oblique shock waves except of within a range of max. turning angle where flow becomes subsonic



≈ normal shock

actual shock formation in a channel

$$\tan\theta = 2\cot\beta \, \frac{M_1^2 \sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}$$

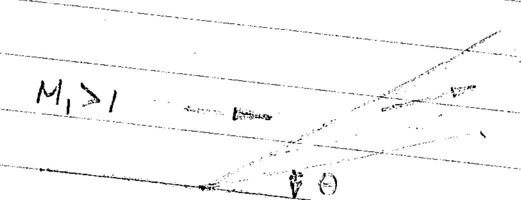
if $M_1 \sin\beta \sim 1$ $\quad \tan\theta << 1 \quad \therefore \tan\theta \sim \theta$

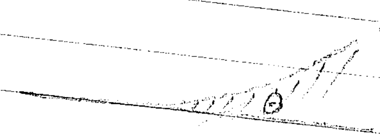
for weak shocks (very weak) $\quad M_1^2 \sin^2\beta \sim 1 + A(M_1, \beta \approx \mu)\,\theta$

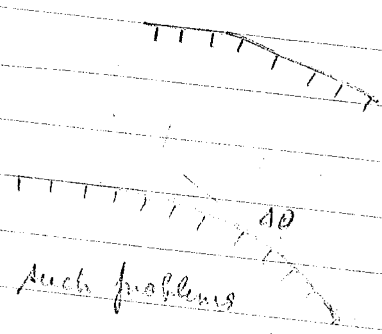
$$p_2/p_1 = 1 + B\theta$$

$$\frac{\Delta p}{p_1} = \frac{p_2 - p_1}{p_1} \sim B\theta$$

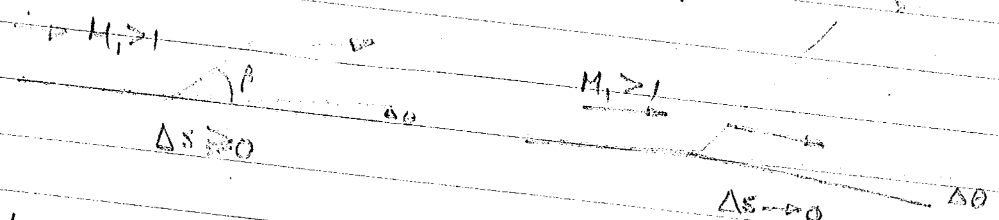
$$S_2 - S_1 = \Delta S \sim C\theta^3$$




Compressive waves do not satisfy such problems




$\frac{\Delta p}{p} \sim \theta$

divide the radius into $n$ line segments & find $\frac{\Delta p}{p}$ for each line seg then

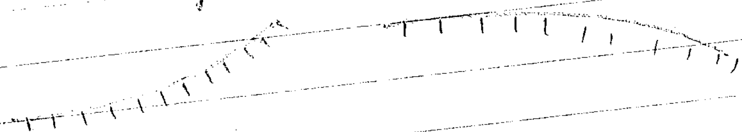
$n \frac{\Delta p}{p} \sim n\Delta\theta \qquad n \to \infty \quad \Delta\theta \to 0 \quad n\Delta\theta \to d\theta$

$$\left( \frac{dp}{p} \sim d\theta \right.$$

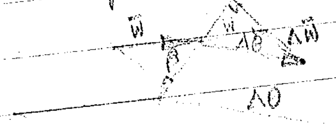
$n\Delta s \sim n(\Delta\theta)^3$ this is an isentrop

$dp \sim d\theta(\Delta\theta)^2 \to 0$ in limit this allows me to use oblique shock eq

flow.

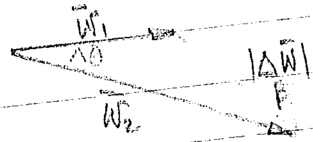


Expansion Case $\beta = \mu + A\,\Delta\theta$ as $\Delta\theta \to$



$\Delta W = -\Delta u_n$

law of sines



$|\Delta\vec{W}| = \Delta u_n$

$$\frac{W_1}{\sin\left(\frac{\pi}{2} - (\beta + \Delta\theta)\right)} = \frac{-\Delta u_n}{\sin\Delta\theta}$$

$\Delta\theta < 0 \quad \Delta u_n > 0$

expanding terms

$$W_1 \Delta\theta = -\Delta u_n \cos(\beta + \Delta\theta) + C\theta^3$$

$$W_1 \Delta\theta = -\Delta u_n \cos\beta + h(\Delta\theta^2, \Delta\theta^3, \Delta\theta\,\Delta u_n) \text{ terms}$$

$$W_1 \Delta\theta = -\Delta u_n \underbrace{\ }_{\text{taylor expansion terms}}$$

$u_{t_1} = u_{t_2}$

$u_{n_2} = u_{n_1} + \Delta u_n$

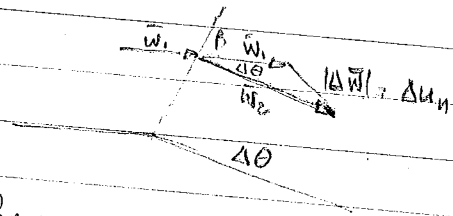
$$\Delta W = W_2 - W_1 = \sqrt{(u_{n_2}^2 + u_{t_2}^2)} - \sqrt{(u_{n_1}^2 + u_{t_1}^2)}$$

$$\Delta W = \Delta u_n \sin\beta + K\,\Delta u_n^2 + \dots \sim \Delta u_n \sin\beta$$

$$W_1 \Delta\theta = -\Delta W \cot\beta + \text{square terms}$$

$$\int W_1\, n\,\Delta\theta = -\int n\,\Delta W \cot\beta + n \text{ square terms}$$

$$\int W_1\, d\theta = -\int \cot\beta\, dw$$

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from
$$\frac{w_1}{\sin(\pi/2-(\beta+\Delta\theta))} = \frac{-\Delta u_n}{\sin\Delta\theta}$$
we obtain:
$$w_1\,\Delta\theta = -\Delta u_n \cos\beta + (\Delta\theta)^2 \text{ terms.}$$

$$\Delta w = w_2 - w_1 = \sqrt{(u_{n_2}^2 + u_{t_2}^2)} - \sqrt{(u_{n_1}^2 + u_{t_1}^2)}$$
$$\sqrt{u_{n_1}^2 + u_{t_1}^2}\left[\sqrt{1 + \frac{2u_{n_1}\Delta u_n}{u_{n_1}^2 + u_{t_1}^2}} - 1\right]$$
expand by binomial expansi.

$\therefore$
$$\Delta w = \Delta u_n \sin\beta + (\Delta u_n^2) \text{ terms.}$$
$$\frac{\Delta w}{\Delta\theta} = -w_1\tan\beta + (\Delta\theta, \Delta u_n) \text{ terms.}$$

$$\therefore \frac{dw}{d\theta} = -w\tan\beta \rightarrow \int \cot\beta\,\frac{dw}{w} = -\theta + C$$

$$\sum n\frac{\Delta w}{w}\cot\beta = -\sum n\Delta\theta + \sum n(\Delta\theta^2, \Delta u_n^2)$$ this leads to

the above is called $\nu = f(M)$ $\therefore$ $\nu + \theta = \text{constant}$

all that happens before with other process is reversed 12/16/70

$\cot\beta = \cot\mu$ in the limit & $\cot\mu = \sqrt{M^2-1}$ $M = \frac{w}{a}$

as $\Delta\theta \rightarrow 0$ $n\Delta\theta \rightarrow$ finite as $n \rightarrow \infty$ the shock angle is the mach angle in the limit

$$a^2 = \gamma R T \rightarrow 2\frac{da}{a} = \frac{dT}{T} \quad (2)$$

$$\frac{dw}{w} = \frac{dM}{M} + \frac{da}{a} \quad (1)$$

$$\frac{T}{T_0} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{-1}$$ steady isentropic flow – no heat added.

$$\frac{dT}{T} = -\frac{(\gamma-1)\,M\,dM}{1+\left(\frac{\gamma-1}{2}\right)M^2} \quad (3)$$

from (1), (2), (3)

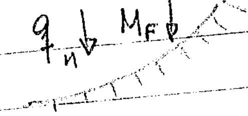
$$\frac{dw}{w} = \frac{dM}{M}\left(\frac{1}{1+\frac{\gamma-1}{2}M^2}\right) = \frac{dM^2}{2M^2}\left(\frac{1}{1+\left(\frac{\gamma-1}{2}\right)M^2}\right)$$

$$\int \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM^2}{2M^2} = -\theta + \text{const.}$$

let $M^2 = S$

$$\underbrace{\int \frac{\sqrt{S-1}}{1+\frac{\gamma-1}{2}S}\,\frac{dS}{2S}}_{\nu(M)} = -\theta + C$$

at $M=1$ $\nu(M) = 0 \leftarrow$ defined

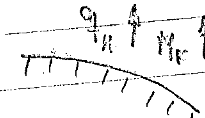


$q_n\downarrow$ $M_F\downarrow$ $\theta\uparrow$ $\nu\downarrow$ $M\downarrow$ Compression

$\nu + \theta = \text{const.}$ invariant

$q_n\uparrow$ $M_F\uparrow$ $\theta\downarrow$ $\nu\uparrow$ $M\uparrow$ expansion. $|\Delta\nu| = |\Delta\theta|$

$\Delta\nu = -\Delta\theta$



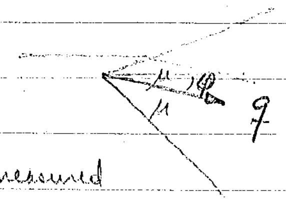
since $\Delta S = 0$ $p_0$, $T_0$ are constant.

given $M$ get $\nu$ knowing flow direct $\rightarrow \theta$

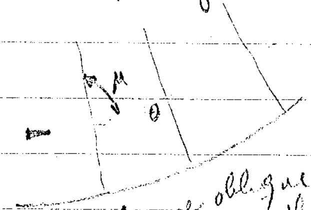
at new point find $\theta$, this gives $\nu \rightarrow M$

from $M$ find $p/p_0$, $T/T_0 \rightarrow p, T$, from $T \rightarrow a$, knowing $M, a \rightarrow w$

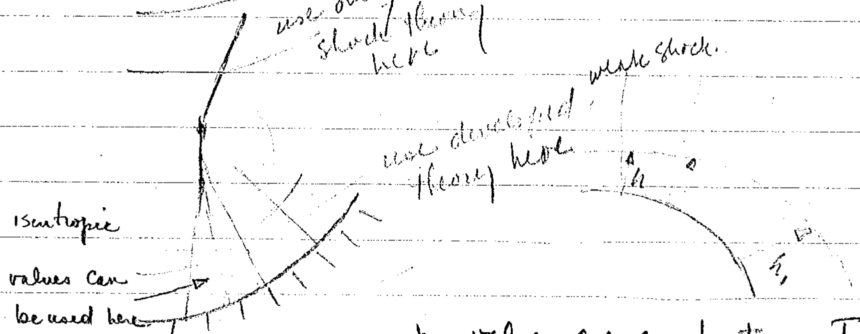
Mach lines $\varphi \pm \mu = \frac{dy}{dx}$



$\varphi$ is the local direction of flow measured from the horizontal. the above gives mach lines measured from horizontal



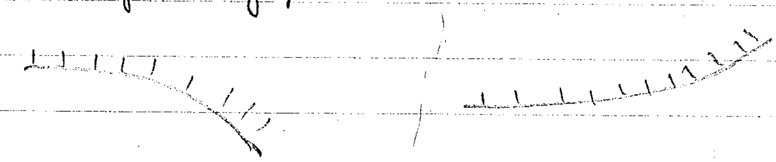
|  | $\mu + \Theta$ | |
|---|---|---|
| Compress | ↑ | ↑ |
| Expansion | ↓ | ↓ |

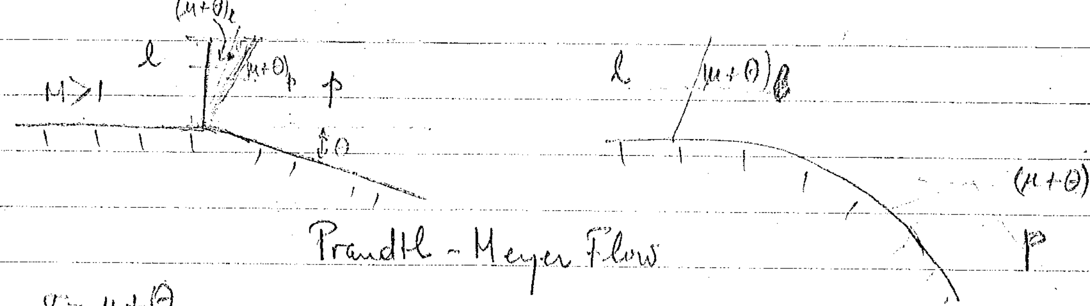



$h < h_1$
Mach lines diverge

$p_0$ values are constant, $T_0$ never changes.

For the following flow $\nu - \Theta = \text{const.}$






Prandtl - Meyer Flow

$\nu = \mu + \Theta$
$\nu + \Theta = \text{Const.}$

$\nu + \theta = C$ along a line having a slope $\frac{dy}{dx} = \tan(\theta - \mu)$

$\nu - \theta = C$ " " " " " " " $= \tan(\theta + \mu)$

Simple wave Regions Prandtl Meyer Region — one of the above 2 are straight

uniform Region —

non-uniform Regions — both lines are cu

12/21/70

Contact Discontinuity

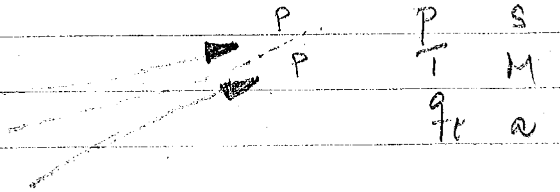
Slip Stream

Vortex Discontinuity

Jump conditions are satisfied.

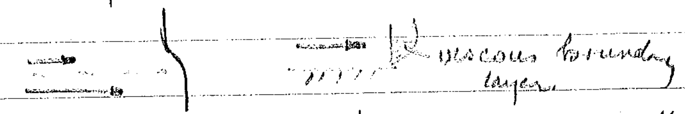
1. $[\rho q_n] = 0$
2. $[p + \rho q_n^2] = 0$ $\qquad q^2 = q_n^2 + q_t^2$
3. $[H + q^2/2] = 0 = [H_0]$
4. $[\rho q_n q_t] = 0$ $\qquad [\rho q_n \neq 0$ shock wave case$]$

Consider $q_n = 0$



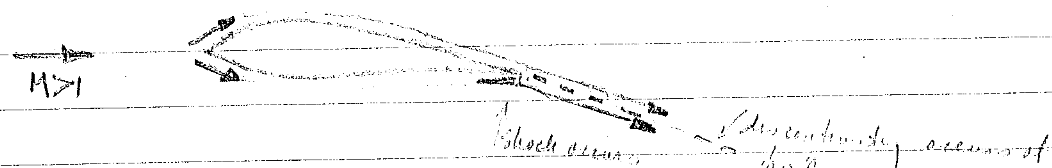

2. $[p] = 0$ $\qquad$ p is constant across discontinuity

$q = q_t$ $\qquad$ flow direct is the same across discontinuity if $V_{above} \neq V_{below}$ a free shear layer is formed across contact discontinuity $\Rightarrow$ the $V$ through the layer adjusts velocity so that $V$ above disc. in shear layer = $V_{above}$ & $V_1$ below disc. in shear layer = $V_{below}$.




We will not consider this case. this shear layer produces vorticity

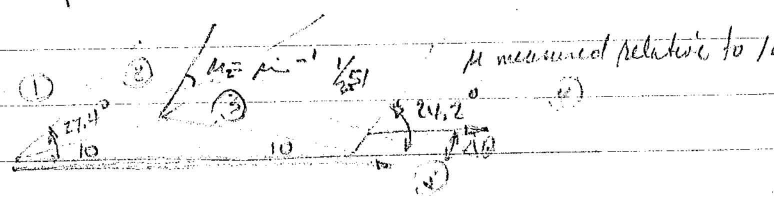
Consider flow over supersonic wing




flow direct & $p$ must be the same for particles moving over upper surface & lower surface.

Occurs only in supersonic cases.

Example




$M_1 = 3$

$T_{o_1}$ $\qquad$ $M_{1_n} = 3 \sin 27.4^\circ = 1.38$

$p_{o_1}$ $\qquad$ $p_2/p_1 = 2.055$ $\qquad$ $p_{o_2}/p_{o_1} = .9630$

$p_1/p_{o_1} = .02722$ $\qquad$ $T_2/T_1 = 1.248$

$M_{2_n} = .7483$ $\qquad$ $M_2 = \dfrac{M_{2n}}{\sin 17.4^\circ} = 2.51$

look up mach angle for $M_2 = 2.51$

$\nu_{2.51} = 39.357$

$\nu + \theta = C$ $\qquad$ $\nu(M_3) = 59.357$

$M_3 = 3.55$ $\qquad$ $p_3/p_1 = \dfrac{p_3}{p_{o_3}} \cdot \dfrac{p_{o2}}{p_{o1}} \dfrac{p_{o1}}{p_1}$, $p_{o3} = p_{o2}$ isentropic flow

$\dfrac{T_3}{T_1} = \dfrac{T_3/T_{o3}}{T_1/T_{o1}} = \dfrac{.2841}{.3571} = .796$, $\quad .01204 \cdot .9630 \cdot \dfrac{1}{.02722} = .426$

$\mu_3 = \sin^{-1}\frac{1}{3.55}$

What happens at trailing edge.
Assume flow comes out horizontally need shock wave to turn

$M_{3_n} = 3.55 \sin 24.2^\circ = 1.46$

$p_4/p_3 = 2.32 \qquad M_{4_n} = .7157 \qquad \frac{p_{o4}}{p_{o3}} = .9420$

$M_4 = \frac{M_{4_n}}{\sin 14.2^\circ} = 2.92$

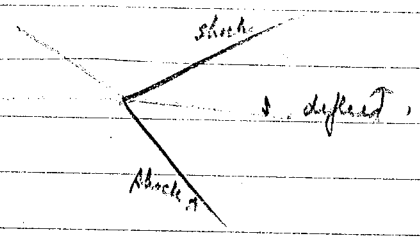
$p_4/p_1 = \frac{p_4}{p_3} \cdot \frac{p_3}{p_1} = 1.06$

$\frac{T_4}{T_1} = \frac{T_4/T_0}{T_1/T_0} = \frac{.3696}{.3571} = 1.04$

$p_1 = p_5$

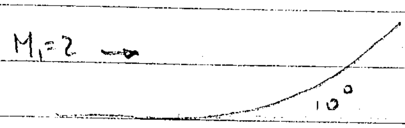
but $p_4 \neq p_1$ ∴

to increase press below & press above decreased
slip stream must be deflected at some angle.




12-22-70

Problem

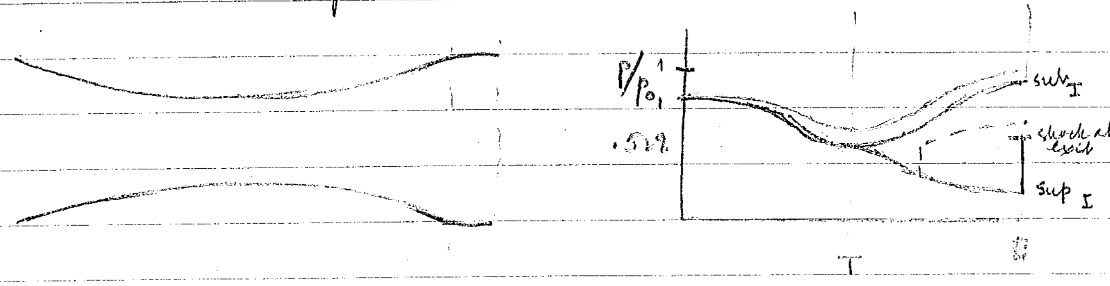



$\nu(2) = 26.38 \qquad \theta = 10^\circ \qquad \nu(M_2) = 16.38 \qquad M_2 = 1.65$

$\frac{p_2/p_{o_2}}{p_1/p_{o_1}} = \frac{p_2}{p_1}$

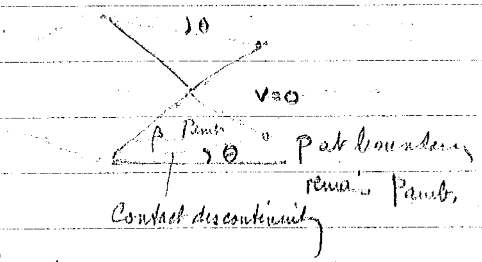
$\mu$'s are measured from local flow direction

Channel problems

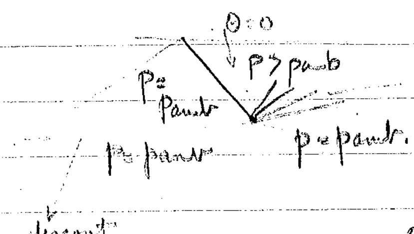



1. below shock at exit & greater than isentropic supersonic, low altitude
 if $P_e < P_{amb}$. Overexpanded exhaust.
 flow exits at a pres equal to supersonic isentropic

you are given $P_a/P_e > 1$ $P_a$ = $P_{ambient}$
 you know $M_e$



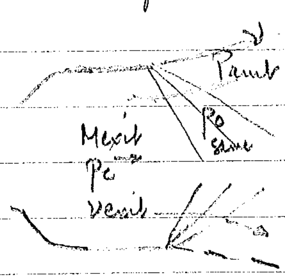

$$P_a/P_e = 1 + \frac{2\gamma}{\gamma+1}\left(M_{e_n}^2 - 1\right)$$




expansion fan to reduce pressure

knowing $P_{amb}/P_0 \rightarrow M \rightarrow \nu$
knowing $\nu$ before expansion $\rightarrow$ c
$\therefore$ $\theta$ can be calculated at the flow after expansion & new mach no.

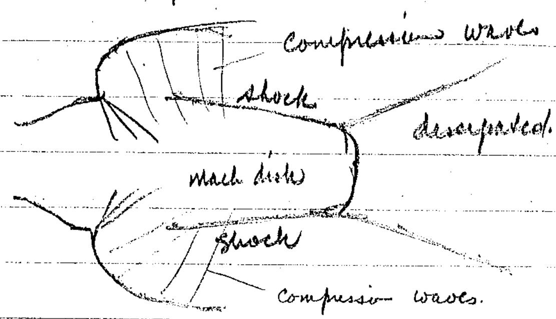
2. $\frac{P_a}{P_e} < 1$ below isentropic supersonic underexpanded exhaust
 high altitude case



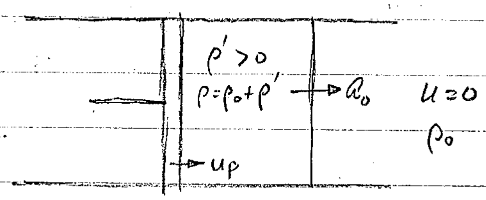

when reflecting, expansion reflects as compression at constant pressure lines.

LowestMexit is 1 when exit$_{\wedge}^{ing}$ supersonic isentropic $\nu(\infty) \sim 130.5°$

in a very high altitude

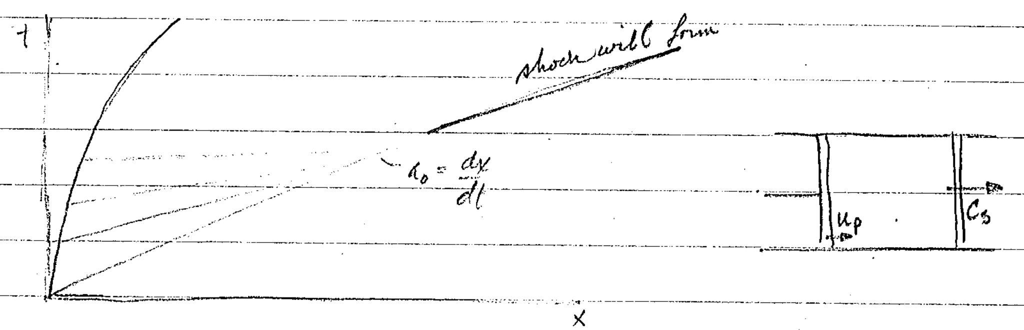


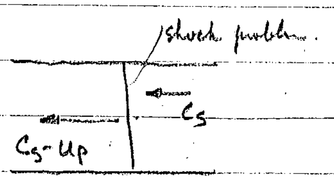

## Acoustic Waves




$a^2 = \frac{\gamma p}{\rho} = \gamma RT$

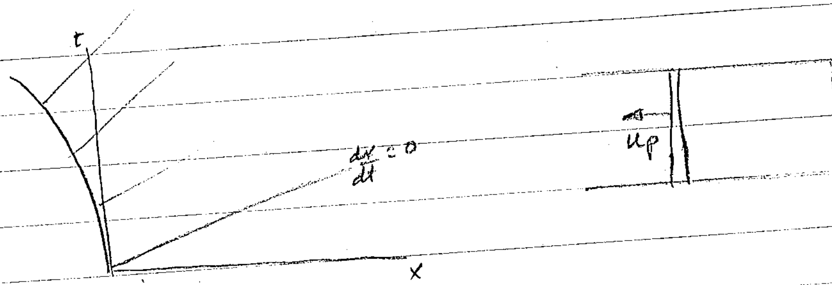
since piston is accelerating, at some new time $a$ will intersect with $a_0$ where $a_n > a_{n-1} > \ldots a_2 > a_1 > a_0$




$\frac{C_s}{a_0} > 1$ $\quad \frac{C_s}{a_n} > 1$

fixed coordinate sys




if impulsed quickly shock wave forms immediately
" " slowly " " degenerates to sound waves

1/4/71

Unsteady 1-D Flow : Treat full equation ; no linearization
Homentropic ( $s \equiv const$ ) , no body forces , viscous forces etc

$\partial/\partial y = \partial/\partial z = 0$
$v = w = 0$

Continuity : $\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$ ②

x - Mom : $\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0$

Energy & State: $p = K\rho^\gamma$ since Homentropic

$$a^2 = \frac{\gamma p}{\rho} = \frac{dp}{d\rho}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial \rho}{\partial x}\left(\frac{\partial p}{\partial \rho}\right) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial x} = 0 \quad ①$$

Mult. cont by $\frac{a}{\rho}$ & add ① :
Mult cont by $\frac{a}{\rho}$ & subt ① :

$$③ \quad \frac{a}{\rho}\frac{\partial \rho}{\partial t} + \frac{au}{\rho}\frac{\partial \rho}{\partial x} + a\frac{\partial u}{\partial x} \pm \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial x}\right] = 0$$

$$\left(\frac{a}{\rho}\frac{\partial \rho}{\partial t} \pm \frac{\partial u}{\partial t}\right) + (u \pm a)\left[\frac{a}{\rho}\frac{\partial \rho}{\partial x} \pm \frac{\partial u}{\partial x}\right] = 0$$

$\frac{a}{\rho}\frac{\partial \rho}{\partial t}$ ; $\quad a^2 = \frac{\gamma p}{\rho} = \gamma K \rho^{\gamma-1}$

$$\gamma^{1/2}K^{1/2}\rho^{\frac{\gamma-3}{2}}\frac{\partial \rho}{\partial t} = \frac{2\gamma^{1/2}K^{1/2}}{\gamma-1}\frac{\partial}{\partial t}\rho^{\frac{\gamma-1}{2}} = \frac{2}{\gamma-1}\frac{\partial a}{\partial t} \qquad \text{since} \quad a = \gamma^{1/2}K^{1/2}\rho^{\frac{\gamma-1}{2}}$$

$$\therefore ③ \quad \frac{\partial}{\partial t}\left[\frac{2a}{\gamma-1} \pm u\right] + (u \pm a)\frac{\partial}{\partial x}\left[\frac{2a}{\gamma-1} \pm u\right] = 0$$

let $\frac{2a}{\gamma-1} + u = P$ , $\frac{2a}{\gamma-1} - u = Q$ Riemann Invariants.

$$\frac{\partial P}{\partial t} + (u+a)\frac{\partial P}{\partial x} = 0 \qquad P = f(x-(u+a)t)$$

$$\frac{\partial Q}{\partial t} + (u-a)\frac{\partial Q}{\partial x} = 0 \qquad Q = f(x-(u-a)t)$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial x}\frac{dx}{dt} = 0 \quad \text{when} \quad \frac{dx}{dt} = u+a \quad \text{for } P = \text{const}$$

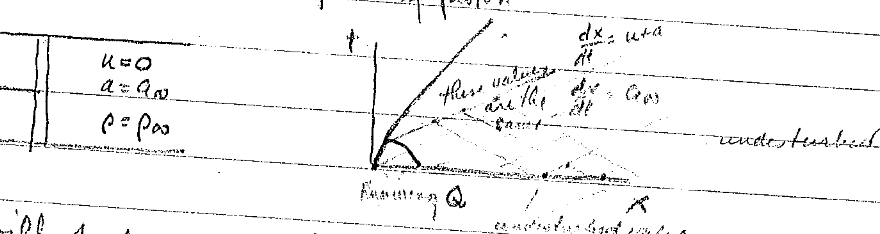
$$\frac{dx}{dt} = u-a \quad \text{for } Q = \text{const.}$$

almost the substantial derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\frac{dx}{dt} \quad \text{is} \quad \frac{D}{Dt} \quad \text{if} \quad \frac{dx}{dt} = u$$

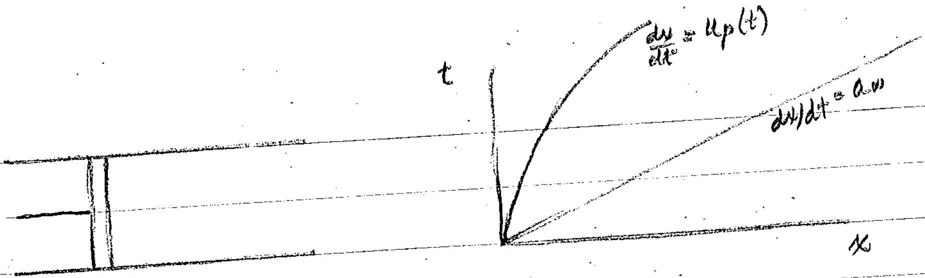
You will always have a wave in unsteady flow.

flow in front of piston




the values will produce a straight line emanating from piston
Knowing Q in undisturbed flow & u at piston this gives a at piston
Knowing a & u at piston gives P.

1/6/71




$P = \frac{2a}{\gamma-1} + u =$ constant on a line having slope $\frac{dx}{dt} = u+a$

$Q = \frac{2a}{\gamma-1} - u =$ constant on a line having slope $\frac{dx}{dt} = u-a$

In acoustics $u = u'$ $a = a_\infty + a'$ $\frac{u'}{a_\infty}, \frac{a'}{a_\infty} \ll 1$, $\frac{u_p}{a_\infty} \ll 1$

Therefore $u+a = u' + a_\infty + a' \sim a_\infty$ ; $u-a = u' - a_\infty - a' \sim -a_\infty$

reduces to linearized theory.

In lin theory all lines drawn back to piston where $dx/dt = a_\infty$ by approx.

But non-lin theory : $u$ & $a$ are not constant

$$\bar{P} = P - \frac{2a_\infty}{\gamma-1} = \frac{2a'}{\gamma-1} + u' \qquad u' = \frac{1}{2}(\bar{P} - \bar{Q})$$

$$\bar{Q} = Q - \frac{2a_\infty}{\gamma-1} = \frac{2a'}{\gamma-1} - u' \qquad a' = \frac{\gamma-1}{4}(\bar{P} + \bar{Q})$$

in linear theory: if $u' = f(x - a_\infty t) + g(x + a_\infty t)$

$$f = \tfrac{1}{2}\bar{P}, \quad g = \frac{-\bar{Q}}{2} \rightarrow u'$$

& $a' = \frac{\gamma-1}{2}(f - g)$ $\qquad$ $\bar{P}$ is constant along $\frac{dx}{dt} = a_\infty$

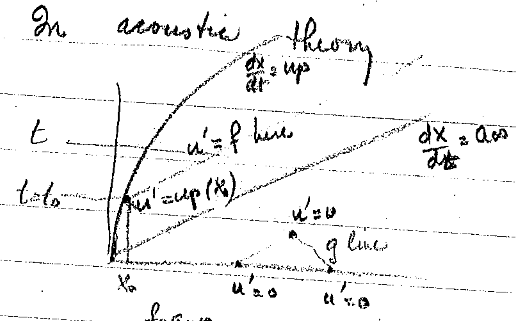
$\bar{Q}$ " " " $dx/dt = -a_\infty$

In non-lin theory

$u = \frac{P-Q}{2}$ $\qquad$ $P$ is const along $\frac{dx}{dt} = u + a$

$a = \frac{\gamma-1}{4}(P+Q)$

In acoustic theory




$g$ is const along line $g = 0$ at $t = 0$

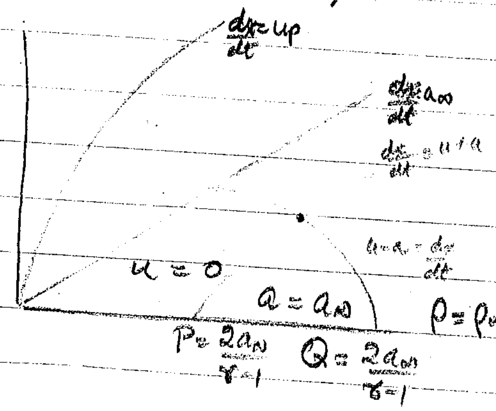
$\therefore$ $g = 0$ everywhere

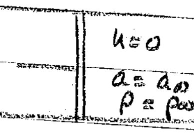
$$u' = u_p\left(x - a_\infty(t - t_0)\right)$$

given $u_p(x_0)$

$\frac{dx}{dt} = a_\infty$ at $t = t_0$ $u = u_p(x_0)$

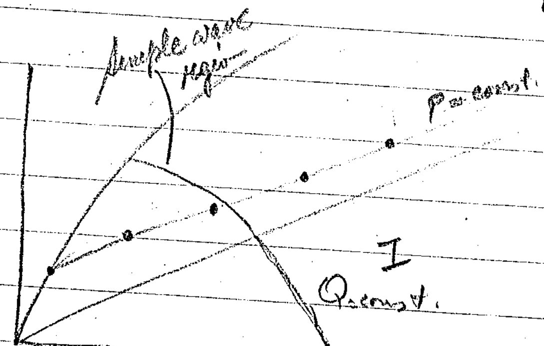
$\therefore$ $u' = u_p\left(x - a_\infty[t - t_0]\right)$




along $u + a$

$$\left.\begin{aligned} \frac{2a}{\gamma - 1} + u &= \frac{2a_\infty}{\gamma - 1} \\ \frac{2a}{\gamma - 1} - u &= \frac{2a_\infty}{\gamma - 1} \end{aligned}\right\} \quad u = 0 \quad a = a_\infty$$




① $\frac{2a}{\gamma - 1} - u = \frac{2a_\infty}{\gamma - 1}$ $Q$ in I

true everywhere

between $1^{st}$ wave & piston

Given piston velocity $u_p(t)$ = give ②

③ $a_p(t)$

$$\frac{2a}{\gamma-1} + u = \frac{2a_p}{\gamma-1} + u_p.$$

$$" \quad = 2u_p + \frac{2a_\infty}{\gamma-1}$$ true along any line where $\frac{dx}{dt} = u+a$

solve 1 & 4 for $a$ & $u$

what happens to $u + a$ : Region II

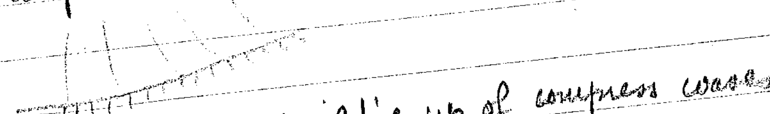
$u + a$

$a - \frac{\gamma-1}{2}u = a_\infty$

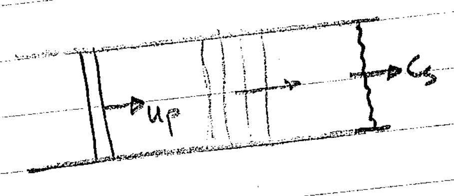
$u + a - \frac{\gamma-1}{2}u - u = a_\infty$

$u + a = a_\infty + \left(\frac{\gamma+1}{2}\right)u$

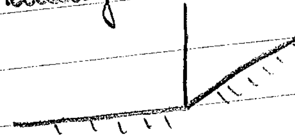
Subtract 4 from 1 ⇒ $u = u_p$  $a = a_p$
along a constant X;
wave speed is increasing since $u$ is increasing, shock formation will occur since wave catches up with preceding wave, of this type/
this is analogous to compression waves of this type



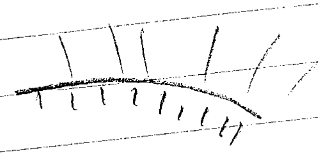
building up of compress waves produce a shock wave.

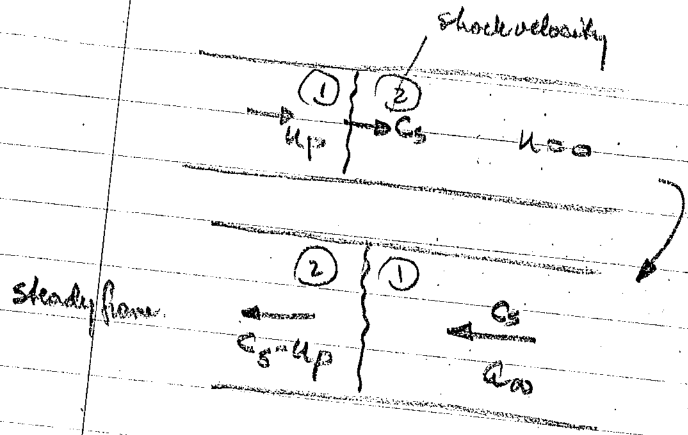



if we move piston impulsively we immediately get a shock forming; this is analogous to this type of flow.



If it moves backwards we have an expansion type
no shock forms
formation






1-D shock problem

$u_1 = C_s \quad a_1 = a_\infty \quad \rho_1 = \rho_\infty$

$u_2 = C_s - u_p$

$M_1 = \dfrac{C_s}{a_\infty} > 1 \implies C_s > a_\infty$

$$u_2/u_1 = f(M_1) = f(C_s/a_\infty)$$

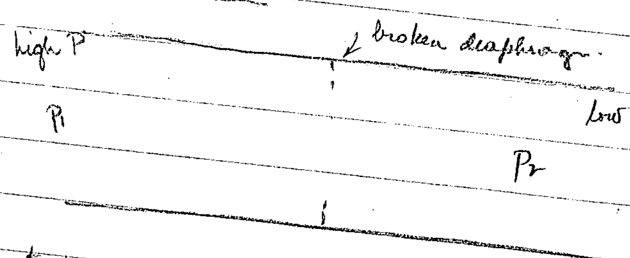
$$\frac{C_s - u_p}{C_s} = f(C_s/a_\infty)$$

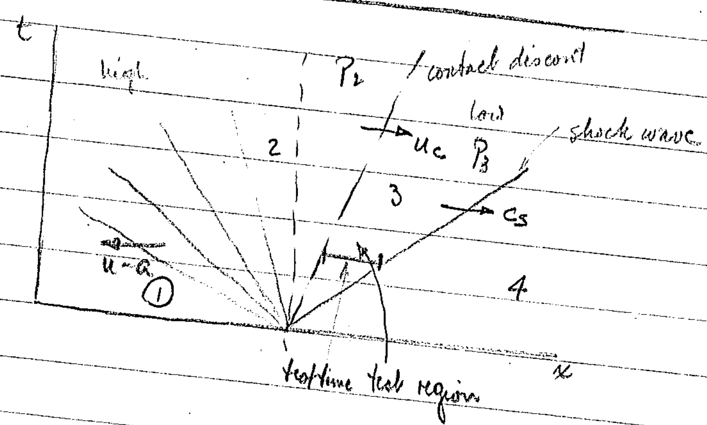
$$= \frac{(\gamma-1)M_1^2 + 2}{(\gamma+1)M_1^2}$$

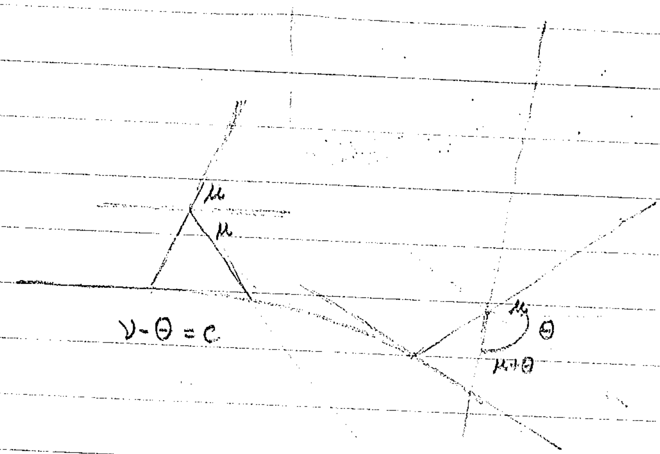
$$C_s = a_\infty \left( \frac{\gamma-1}{2\gamma} + \frac{\gamma+1}{2\gamma}\, p_2/p_1 \right)^{1/2}$$

$$u_p = \frac{a_\infty}{\gamma}\left(p_2/p_1 - 1\right)\left[\frac{2\gamma/\gamma+1}{p_2/p_1 + \frac{\gamma-1}{\gamma+1}}\right]^{1/2}$$

## Shock Tube

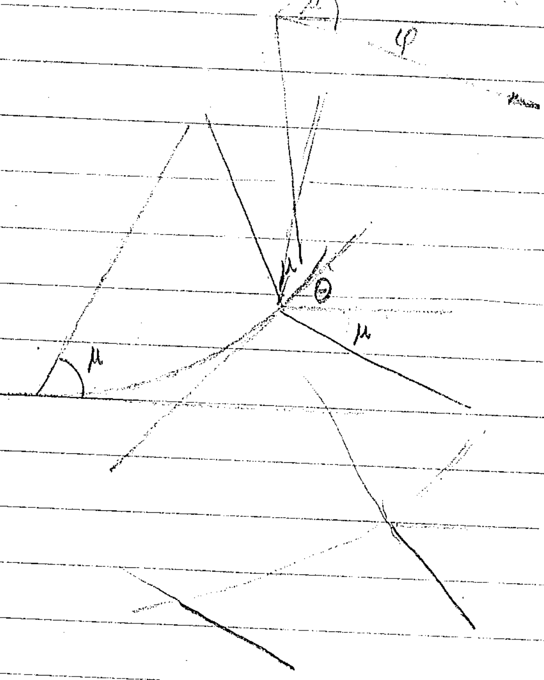


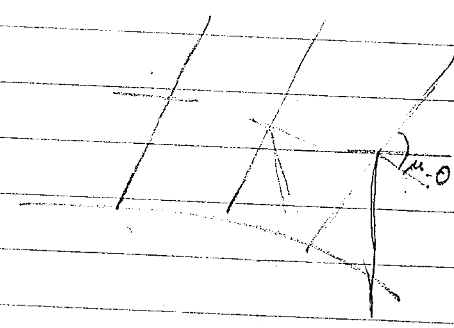





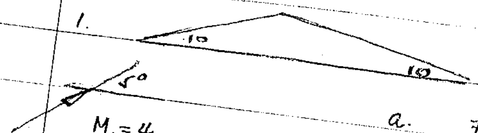

$\mu + \theta$ because they don't interfect

$\mu - \varphi$ above
$\mu + \varphi$ below } horiz



1.



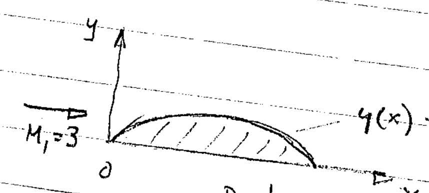

$M_1 = 4$

$T_0$, $p_0$, given

a. find $p/p_1$ at all points on the body.

b. Find flow deflection behind trailing edge.

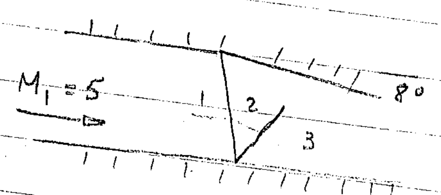
2.




$y(x) = 1/5\,(x - x^2)$

Determine pressure pressure distribution on body & deflect. behind t.c.

3.




determine $M_2$, $M_3$, $\frac{p_2}{p_1}$, $\frac{p_3}{p_1}$

Sunday Jan. 31 ~ 3:30 PM Rm 418 Inst.

7:00 PM Dinner Junior's Burgundy Room.

$40 -- Fee (Will be initiated in spring)

Write an essay on a non-technical topic due Jan 31

HW # 5

1. For weak shock waves ($\theta \ll 1$) prove that

a) $\dfrac{p_2 - p_1}{p_1} = \dfrac{\gamma M_1^2}{\sqrt{M_1^2 - 1}}\,\theta + \theta^2$ terms

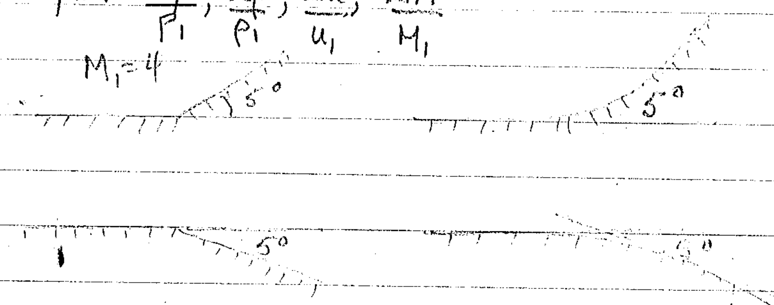
b) $s_2 - s_1 = A\theta^3 + \theta^4$ terms

c) If $\beta = \mu_1 + \epsilon$ (shock angle) then $\epsilon = \left( \dfrac{\gamma+1}{4} \dfrac{M_1^2}{M_1^2 - 1} \right) \theta$ $\quad \mu_1 = \sin^{-1}\dfrac{1}{M_1}$

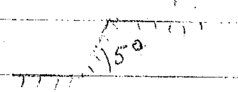
d) $\mu_2 = \mu_1 + \delta\theta$ and $\mu_2, \mu_1$ bisect $\beta$ $\quad A \,\&\, B$ are functions of $M_1$

2. find $\dfrac{\Delta p}{p_1}, \dfrac{\Delta \rho}{\rho_1}, \dfrac{\Delta u}{u_1}, \dfrac{\Delta M}{M_1}$

$M_1 = 4$




3. $M_1 = 6$




no intersection

AE 331 Homework # 1 Due: Sept. 28, 1970

1. Discuss in a few paragraphs:

   (i) How the coefficient of viscosity might be measured.
   (ii) The temperature dependence of the viscosity coefficient for a liquid as opposed to a gas.
   (iii) At an altitude of 250,000 ft., if a vehicle having a typical dimension L=50ft, were travelling at 1000 fps, would continuum theory be applicable?

2. Consider a flow in which the velocity components in the x and y directions are respectively,

$$u = Ax, \qquad v = -Ay$$

   (i) If at t=0, $x=x_o$ and $y=y_o$, determine the Lagrangian formulas for $x=x(x_o, y_o, t)$ and $y=y(x_o, y_o, t)$. Determine the rate of change of u for a particle passing through a point $(x_1, y_1)$ at $t=t_1$ by using both the Eulerian and Lagrangian approaches.

   (ii) Determine the shape of the streamlines associated with the above flow. Draw a sketch of this flow field.

3. Consider a flow in which the velocity components in the radial and tangential directions $(r, \theta)$ are respectively

$$u_r = U \cos\theta \left( \frac{R^2}{r^2} - 1 \right) \qquad , r \leq R$$

$$u_\theta = U \sin\theta \left( \frac{R^2}{r^2} + 1 \right)$$

Show that the streamline equation is given as

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{u_r}{u_\theta}$$

and prove that the circle r=R is in fact a streamline. Draw a qualitative sketch of the flow field.

4. Particles are dropping from rest off the edge of a table continuously. Express the velocity distribution by the Eulerian representation.

22/25

AE-331 Quiz #1 Fall 1970

Cesar Levy

Consider a flow in which the velocity components in the x and y directions are respectively,

$$u=Ax, \qquad v=-Ay$$

(i) If at $t=0$, $x=x_o$ and $y=y_o$, determine the Lagrangian formulas for $x=x(x_o,y_o,t)$ and $y=y(x_o,y_o,t)$. Determine the rate of change of u for a particle passing through a point $(x_1,y_1)$ at $t=t_1$ by using both the Eulerian and Lagrangian approaches.

(ii) Determine the shape of the streamlines associated with the above flow. Draw a sketch of this flow field, for $y>o$.

Lagrangian

$$u = \frac{\partial x}{\partial t}\Big|_{x_o}$$

$$v = \frac{\partial y}{\partial t}\Big|_{y_o}$$

for Eulerian

$$u = \frac{dx}{dt}$$

$$x = x(t)$$

① $Ax = \frac{dx}{dt}$ Eulerian

$$At = \ln x + \ln c$$

$$0 = \ln x_o + \ln c$$

$$\ln c = -\ln x_o$$

$$At = \ln \frac{x}{x_o}$$

$$\boxed{x_o e^{At} = x}$$ ✓

② $-Ay = \frac{dy}{dt}$

$$-At = \ln y + \ln c$$

$$0 = \ln y_o + \ln c$$

$$\ln c = -\ln y_o$$

$$\boxed{y_o e^{-At} = y}$$ ✓

(Over)

Eulerian

$$\frac{Du}{Dt} = \frac{du}{dt} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \; ;$$

$$= 0 + A^2 x \; ; \text{ at } x = x_\phi \text{ then } \frac{Du}{Dt} = A^2 x_\phi$$

Lagrange

$$\left.\frac{du}{dt}\right)_{x_0, y_0} = \cancel{\frac{\partial u}{\partial x_0}\frac{\partial x_0}{\partial t}} = A\left(x_0 A e^{At}\right) = A^2 x_0 e^{At} = A^2 x \; ; \text{ at } x = x_1$$

$$+ \frac{\partial u}{\partial y_0}\frac{\partial y_0}{\partial t} \quad \text{then } \frac{\partial u}{\partial t} = A^2 x_1$$

---

2.

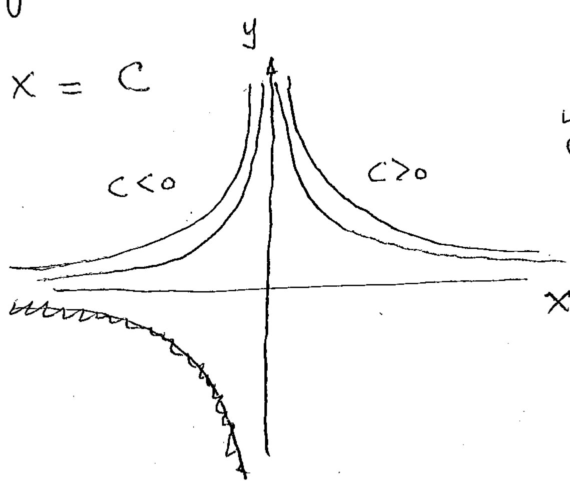
$$\frac{dy}{dx} = \frac{v}{u} = \frac{-Ay}{Ax} = -y/x$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\ln y = -\ln x + \ln c$$

$$yx = c$$

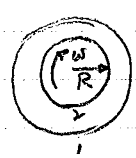
$y > 0$



Cesar Levy  AE 331

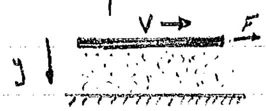
1.

(i) The coefficient of viscosity may be measured experimentally. Viscosity in a fluid can be observed because of its tendency to adhere to a surface; e.g.



if cylinder 2 & cylinder 1 are coaxial and a fluid is placed between 2 & 1, as 2 starts rotating there will be a time where the intervening fluid will start causing cylinder 1 to rotate even though cylinder 2 is not directly connected to cylinder 1.

Consider a plate a small distance y above a fixed boundary. The intervening space between the plate & the boundary is filled with a fluid. If the plate is



moving with a velocity V the particles next to the plate will have the same velocity. However a shear due to the movement is experienced such that $\tau = \frac{F}{A}$ where F is the force in the plane of the plate & A is the area of the plate. As we move further & further from the plate the particles act as the plate would act on the fluid producing shears between particles above & particles below producing a retardation of flow such that as y increased V decreased until at the boundary V=0. It has been found experimentally again that the shear $\tau = \frac{F}{A}$ is proportional to $\frac{V}{y}$; the constant of proportionality is the coefficient of viscosity, $\mu$.

(ii) The dependence of viscosity on temperature in a gas:

As one heats up a gas the fluid particles will have imparted to them increased kinetic energy thus allowing them to move faster. In a time t, because of this increased energy, it is expected that if one moves along the probability of meeting another particle will increase with respect to the initial probability (before the increase in K.E.); this means that the mean free path will decrease. All this means is that there will be an increase in the resistance to free motion or a increase in viscosity, which implies that as temperature increase, viscosity increase in a gas. ✓

The dependence of viscosity on temperature in a liquid:

as one heat up a liquid the fluid particles will have imparted to the an increase in K.E. thus allowing them to move faster, However unlike that of a gas they are bounded by an upper surface. The particles at the upper surface in fact may have imparted K.E. that will cause them to be freed. The other particles in a liquid will tend to move toward the surface to be freed also. Thus in the overall picture the mean free path will increase and the resistance to flow will decrease; this implies that in a liquid viscosity decrease with increasing temperature.

(iii) at 250,000 ft  a = 1048 ft/sec, $\rho = 1.5 \times 10^{-7}$ slugs/ft$^3$, $\mu = \frac{1}{3} \times 10^{-6}$ slug/ft$^3$

to find whether continuum holds $K_n << 1$
therefore

$$\frac{\bar{u}/\bar{c}}{\rho \bar{u} R/\mu} = \frac{M}{R_N} \qquad M = \frac{1000}{1080} = .96 \;, \qquad R_N = 22500$$

$$\frac{M}{R_n} = \frac{.96}{.225 \times 10^5} = 4.26 \times 10^{-5} << 1 \qquad \therefore \text{ Continuum holds.}$$

2. (i) $u = Ax = \frac{\partial x}{\partial t}$

$$u\,dt = dx = Ax\,dt \quad \therefore \quad \int A\,dt = \int \frac{dx}{x}$$

$$At = \ln x + \ln C$$

at $t=0$ $x = x_0$ $\therefore$ $0 = \ln x_0 + \ln C$ $\quad \ln C = -\ln x_0$

$$\therefore \quad At = \ln \frac{x}{x_0} \quad \text{or} \quad \frac{x}{x_0} = e^{At} \Rightarrow \boxed{x = x_0 e^{At}}$$

$$V = -Ay = \frac{\partial y}{\partial t}$$

$$V\,dt = -Ay\,dt = dy \quad \therefore \quad -\int A\,dt = \int \frac{dy}{y}$$

$$-At = \ln y + \ln C_1$$

at $t=0$ $y = y_0$ $\therefore$ $0 = \ln y_0 + \ln C_1$ $\quad \ln C_1 = -\ln y_0$

$$\therefore \quad -At = \ln y/y_0 \quad \text{or} \quad y/y_0 = e^{-At} \Rightarrow \boxed{y = y_0 e^{-At}}$$

Eulerian

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial (Ax)}{\partial t} + Ax\frac{\partial (Ax)}{\partial x} = A^2 x$$

Lagrangian

$$u = Ax_0 e^{At} \qquad x = x_0 e^{At} \qquad u = Ax$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} = \frac{\partial (Ax)}{\partial x} \cdot \frac{\partial (x_0 e^{At})}{\partial t} = A \cdot x_0 A e^{At} = A^2 x_0 e^{At} = A^2 x$$

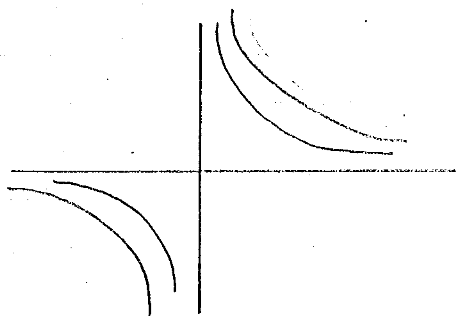
(ii)

$$u = Ax, \quad V = -Ay$$

$$\text{since} \quad \frac{dy}{dx} = \frac{V}{u} = -\frac{y}{x}$$

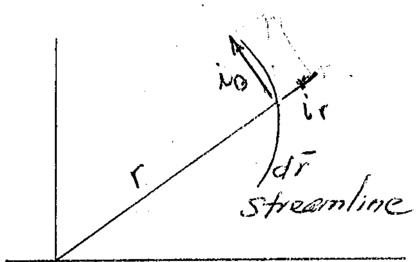
$dy/y = -dx/x \qquad \ln y = -\ln x + \ln c$

$\therefore \quad yx = c$



if $c \geq 0$

3.




if $\bar{r} = r i_r$

$$\frac{d\bar{r}}{dt} = \frac{dr}{dt} i_r + r \frac{di_r}{dt}$$

$$= \dot{r} i_r + r \frac{di_r}{d\theta} \cdot \frac{d\theta}{dt}$$

$$\frac{di_r}{d\theta} = i_\theta$$

$$= \dot{r} i_r + r\dot{\theta} \frac{di_r}{d\theta}$$

$$= \dot{r} i_r + r\dot{\theta} i_\theta$$

$\dot{r} = u_r$

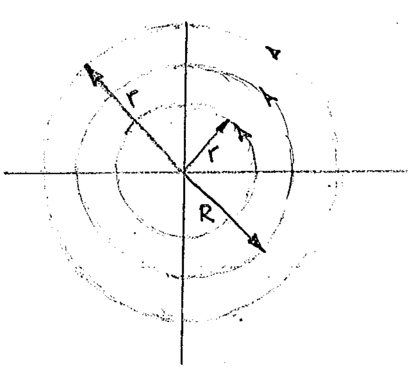
$r\dot{\theta} = u_\theta$

$$d\bar{r} \times \bar{q} = 0 \qquad \begin{vmatrix} dr & r d\theta \\ u_r & u_\theta \end{vmatrix}$$

$$\therefore \quad \frac{u_r}{u_\theta} = \frac{\dot{r}}{r\dot{\theta}} = \frac{1}{r} \frac{dr/dt}{d\theta/dt} = \frac{1}{r} \frac{dr}{d\theta}$$

for a circle $\dot{r} = 0 \qquad \therefore \quad u_r = U \cos\theta \left(\frac{R^2}{r^2} - 1\right) = 0$

but $U \neq 0$, $\cos\theta \neq 0 \quad \therefore \quad R^2/r^2 - 1 = 0 \quad \therefore \quad r = R$



4. $y = -\frac{1}{2} g_0 t^2 + V_0 t + y_0$

however if we start at rest $V_0 = 0$ and if x axis is at the top of the table $y_0 = 0$

$\therefore y = -\frac{1}{2} g_0 t^2$

$\frac{dy}{dt} = V_y = -g_0 t$

$\boxed{\therefore V_y = -g_0 t}$ ✓

or

$$\frac{Du}{Dt} = g = \cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x}$$

or $u = \sqrt{2gx}$ $t=0,\ x=0$ only

AE-331

Homework #2

Fall 1970

1. Derive the particle momentum equation

$$\rho \frac{D\underline{q}}{Dt} = - \text{grad } p + \underline{F}\rho$$

by considering the forces acting on a moving rectangular volume element, of dimensions $\Delta x, \Delta y, \Delta z$, having fixed mass. Consider the limit as the dimensions of the volume element vanish.

2. The velocity components in a two-dimensional x,y coordinate system are, respectively,

$$u = xy \;, \quad v = - y^2/2$$

a) Determine the components of acceleration, in the x and y directions, of a particle passing through the point x=1, y=2.

b) Determine the streamline equation, and the path line equation for a particle at $x=x_o$ when t=0.

3. For the above problem as well as problems (2) and (3) in homework #1, use both the Eulerian and Lagrangian methods to show that all of these are incompressible flows.

4. Prove that if $J = \frac{\partial(x,y)}{\partial(x_o,y_o)}$ for a two-dimensional flow that

(a) $\frac{DJ}{Dt} = J \text{ div } \underline{q}$

(Hint: Prove first that $\frac{Dx}{Dt} = \frac{dx}{dt} = u$ in the Eulerian description of motion.)

and (b) $\int_V \frac{D}{Dt} \, dv = \int_S \underline{q} \cdot \underline{n} \, dS$

Cesar Levy

very good 30/30

AE-331

Quiz #2

Fall 1970

The velocity components in a two-dimensional x,y coordinate system are, respectively,

$$u = xy \quad , \quad v = -y^2/2$$

a) Determine the components of acceleration, in the x and y directions, of a particle passing through the point x=1, y=2.

b) Determine the streamline equation, and the path line equation for a particle at $x=x_o$ when t=0.

c) Use both the Eulerian and Lagrangian methods to show that this is an incompressible flow.

a. $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v$

$= 0 + y^2x - \frac{xy^2}{2} = \frac{xy^2}{2}$ at $x_1 = 1 \quad y_1 = 2$ $\boxed{a_x = \frac{x_1 y_1^2}{2} = 2}$

$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v = 0 + 0 - y(-y^2/2) = y^3/2$

at $x_1 = 1, y_1 = 2$ $\boxed{a_y = \frac{y_1^3}{2} = 4}$

b. $\frac{dy}{dx} = \frac{v}{u} = \frac{-y^2/2}{yx} = \frac{-y}{2x}$

$\frac{dy}{-y} = \frac{dx}{2x} \Rightarrow -\ln y = \frac{1}{2}\ln x + \ln c$

$c_1 = yx^{1/2} \Rightarrow \boxed{y^2x = c}$

$\frac{dy}{dt} = v = -y^2/2 \qquad \frac{dy}{-y^2} = \frac{dt}{2} \Rightarrow \qquad y^{-1} = \frac{t}{2} + c$

at $y = y_0$, $t = 0$ $\quad \frac{1}{y_0} = c \quad \therefore \quad \frac{1}{y} = \frac{t}{2} + \frac{1}{y_0}$

$$\frac{1}{y} = \frac{2 + ty_0}{2y_0} \quad ; \quad y = \frac{2y_0}{2 + ty_0}$$

$$\frac{dx}{dt} = u = \frac{x(2y_0)}{2 + ty_0} \quad ; \quad \frac{dx}{x} = \frac{dt\,(2y_0)}{2 + ty_0}$$

$$\ln x = 2\ln(2 + ty_0) + \ln c$$

at $x = x_0$, $t = 0$ $\quad \therefore \quad \ln x_0 = 2\ln 2 + \ln c$

$$x_0 = 4c$$

$$\ln x = 2\ln(2 + ty_0) + \ln\frac{x_0}{4} \Rightarrow \boxed{x = (2 + ty_0)^2\frac{x_0}{4}}$$

$$x = \left(\frac{4y_0^2}{y^2}\right)\cdot\frac{x_0}{4} \Rightarrow \boxed{xy^2 = x_0 y_0^2}$$

notes same as streamline since flow is steady.

(c) $\frac{D\rho}{Dt} + \rho(\nabla\cdot\bar{q}) = 0$

$$\nabla\cdot\bar{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = y - y = 0 \quad \therefore \quad \nabla\cdot\bar{q} = 0$$

$$\therefore \quad \frac{D\rho}{Dt} = 0 \underset{\text{not nec. true}}{\Longleftrightarrow} \underbrace{\rho = c}_{\text{const density flow}} \quad \therefore \text{ flow is incompressible}$$

Lagrange $\quad \rho/\rho_0 = \frac{1}{J}$ flow may be stratified $\quad J = \begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial y}{\partial x_0} \\ \frac{\partial x}{\partial y_0} & \frac{\partial y}{\partial y_0} \end{vmatrix}$

$$\frac{\partial x}{\partial x_0} = \frac{1}{4}(2 + ty_0)^2 \qquad \frac{\partial y}{\partial x_0} = 0$$

$$\frac{\partial x}{\partial y_0} = 0 \qquad \frac{\partial y}{\partial y_0} = -2y_0(2 + ty_0)^{-2}t + (2 + ty_0)^{-1}(2)$$

$$\frac{-2y_0 t}{(2 + ty_0)^{-2}} + \frac{2(2 + ty_0)}{(2 + ty_0)^2} = \frac{4}{(2 + ty_0)^2}$$

$$\therefore \quad J = \frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} - \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0} = \frac{(2 + ty_0)^2}{4} \cdot \frac{4}{(2 + ty_0)^2} = 1$$

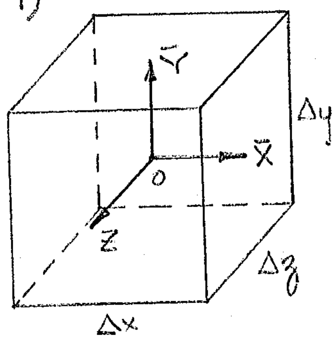
$$\therefore \quad \rho/\rho_0 = 1/J = 1 \quad \therefore \quad \rho = \rho_0 \quad \text{flow is incompressible}$$

good

Cesar Luy

AE 331

1)




at O : $\rho_0, u_0, v_0, w_0, p_0$

the rate of change in momentum $= \frac{D}{Dt}(\rho \bar{q}\,\Delta x \Delta y \Delta z)$

By Taylor's expansion the change of a property A in the x-direction from the initial $A_0$ is

$$A_0 \pm \left(\frac{\partial A}{\partial x}\right)_0 \frac{\Delta x}{2} \quad \text{depending on the direction}$$

also the rate of change of momentum due to the body force, $\bar{X}$ is $\rho F_x \,\Delta x \Delta y \Delta z$

$\therefore$ In the x-direction

$$\frac{D}{Dt}(\rho u \,\Delta x \Delta y \Delta z) = \left[p_0 + \left(\frac{\partial p}{\partial x}\right)_0 \frac{\Delta x}{2}\right]\Delta y \Delta z - \left[p_0 - \left(\frac{\partial p}{\partial x}\right)_0 \frac{\Delta x}{2}\right]\Delta y \Delta z + \rho F_x \,\Delta x \Delta y \Delta z$$

or

$$\frac{D}{Dt}(\rho u \,\Delta x \Delta y \Delta z) = \left(\frac{\partial p}{\partial x}\right)_0 \Delta x \Delta y \Delta z + \rho F_x \,\Delta x \Delta y \Delta z$$

but pressure is defined positive inwards; to agree with the sign convention we must write a $-$ sign in front of the $\frac{\partial p}{\partial x}$

therefore

$$\frac{D(\rho u \,\Delta x \Delta y \Delta z)}{Dt} = -\left(\frac{\partial p}{\partial x}\right)_0 \Delta x \Delta y \Delta z + \rho F_x \,\Delta x \Delta y \Delta z$$

in the same manner we can develop the equations in the y, z directions

$$\therefore \quad \frac{D}{Dt}(\rho \bar{q}\,\Delta x \Delta y \Delta z) = \left[-\left(\frac{\partial p}{\partial x}\right)_0 \underset{\sim}{i} - \left(\frac{\partial p}{\partial y}\right)_0 \underset{\sim}{j} - \left(\frac{\partial p}{\partial z}\right)_0 \underset{\sim}{k}\right]\Delta x \Delta y \Delta z + \rho \bar{F}\,\Delta x \Delta y \Delta z$$

but $\rho \,\Delta x \Delta y \Delta z$ is fixed & doesn't depend on time $\therefore$

$$(\rho \,\Delta x \Delta y \Delta z)\frac{D}{Dt}\bar{q} = \left[-\nabla p + \rho \bar{F}\right]\Delta x \Delta y \Delta z$$

$$\rho \frac{D}{Dt}\bar{q} = -\nabla p + \rho \bar{F}$$

2) $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$a_x = \frac{\partial u}{\ }\Big|$

$$a_x = \left.\frac{\partial u}{\partial t}\right|_{1,2} = y^2 x - x y^2/2 = \left.\frac{y^2 x}{2}\right|_{1,2} = 2$$

$$a_y = \frac{\partial v}{\partial t} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t}$$

$$a_y = \left.\frac{\partial v}{\partial t}\right|_{1,2} = -y\left(-y^2/2\right) = y^3/2 = 4$$

b) $$\frac{dy}{dx} = \frac{v}{u} = \frac{-y^2/2}{xy} = \frac{-y}{2x}$$

$$\frac{dy}{y} + \frac{dx}{2x} = 0 \qquad \ln y + \frac{1}{2}\ln x = \ln C_1$$

$$y x^{1/2} = C_1 \; ; \; \boxed{y^2 x = C}$$ streamline

① $v = -y^2/2$ $$\frac{dy}{-y^2} = \frac{1}{2}dt \longrightarrow 2y^{-1} = t + C_1$$

①b $y = \dfrac{2}{t + C_1}$ at $t = 0, y = y_0 \therefore C_1 = 2/y_0$

or ② $$y = \frac{2y_0}{y_0 t + 2}$$

① $u = xy$ $$\frac{dx}{dt} = \frac{2x}{t + C_1} \longrightarrow \frac{dx}{x} = \frac{2\,dt}{t + C_1}$$

$$\ln x = 2\left[\ln (t + C_1)\right] + \ln C_2$$

or ② $$x = (t + C_1)^2 C_2 = \left(\frac{y_0 t + 2}{y_0}\right)^2 C_2 = \frac{4}{y^2} C_2$$

at $t = 0$ $x = x_0$ $\therefore$ $C_2 = \dfrac{x_0 y_0^2}{4}$

and $x = \dfrac{x_0 y_0^2}{y^2} \iff \boxed{x y^2 = x_0 y_0^2}$ pathline

3. <u>Eulerian</u>

i $u = xy \quad v = -y^2/2$

$$\nabla \cdot \bar{q} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j\right) \cdot (ui + vj) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = y + (-y) = 0$$

$\therefore \quad \frac{D\rho}{Dt} = 0 \quad \therefore$ $\rho$ is a constant – flow is incompressible

ii $u = Ax \quad v = -Ay$

$$\nabla \cdot \bar{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = A + (-A) = 0$$

$\therefore \quad \frac{D\rho}{Dt} = 0 \quad \therefore$ $\rho$ is a constant – flow is incompressible

iii $u_r = U\cos\theta\left(\frac{R^2}{r^2} - 1\right) \quad u_\theta = U\sin\theta\left(\frac{R^2}{r^2} + 1\right)$

$$\nabla \cdot \bar{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{R}\left[\frac{\partial}{\partial r}\left(U\cos\theta\left[\frac{R^2}{r} - r\right]\right) + \frac{\partial}{\partial \theta}\left(U\sin\theta\left[\frac{R^2}{r^2} + 1\right]\right)\right]$$

$$= \frac{1}{r}\left[-U\cos\theta\left(\frac{R^2}{r^2} + 1\right) + U\cos\theta\left(\frac{R^2}{r^2} + 1\right)\right] = 0$$

$\therefore \quad \frac{D\rho}{Dt} = 0 \quad \therefore$ $\rho$ is a constant – flow is incompressible

<u>Lagrange</u>

$u = xy \quad v = -y^2/2 \qquad x = \frac{x_0}{4}(ty_0 + 2)^2 \quad y = \frac{2y_0}{ty_0 + 2}$

$$J = \begin{vmatrix} \partial x/\partial x_0 & \partial y/\partial x_0 \\ \partial x/\partial y_0 & \partial y/\partial y_0 \end{vmatrix}$$

$$\begin{vmatrix} \frac{1}{4}(ty_0+2)^2 & 0 \\ \frac{x_0}{2}(ty_0+2)t & \frac{4}{(ty_0+2)^2} \end{vmatrix} = 1$$

$\rho/\rho_0 = \frac{1}{J} = 1 \qquad \rho = \rho_0 \qquad$ flow is incompressible ii $u = Ax \qquad v = -Ay \qquad x = x_0 e^{At} \qquad y = y_0 e^{-At}$

$$J = \begin{vmatrix} \partial x/\partial x_0 & \partial y/\partial x_0 \\ \partial x/\partial y_0 & \partial y/\partial y_0 \end{vmatrix} = \begin{vmatrix} e^{At} & 0 \\ 0 & e^{-At} \end{vmatrix} = e^0 = 1$$

$\rho/\rho_0 = \frac{1}{J} = 1 \qquad \rho = \rho_0$ flow is incompressible

4a. Prove $\frac{DJ}{Dt} = J(\nabla \cdot \bar{q})$

$J(\nabla\cdot\bar{q}) = J\frac{\partial u}{\partial x} + J\frac{\partial v}{\partial y}$ by use of $\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x_0}\frac{\partial x_0}{\partial x} + \frac{\partial u}{\partial y_0}\frac{\partial y_0}{\partial x}\right)$

and $\frac{\partial v}{\partial y} = \left(\frac{\partial v}{\partial x_0}\frac{\partial x_0}{\partial y} + \frac{\partial v}{\partial y_0}\frac{\partial y_0}{\partial y}\right)$, multiplying and expanding and using

the relations $\cancel{\frac{\partial}{\partial x_0}\left(\frac{\partial x}{\partial x_0}\right) = \frac{\partial}{\partial x_0}\left(\frac{\partial x}{\partial y}\frac{\partial y}{\partial x_0}\right), \frac{\partial x}{\partial y_0} = \frac{\partial x}{\partial y}\frac{\partial y}{\partial y_0}}$ This is not Chain rule!

and $\cancel{\frac{\partial}{\partial y_0}\left(\frac{\partial y}{\partial y_0}\right) = \frac{\partial}{\partial y_0}\left(\frac{\partial y}{\partial x}\frac{\partial x}{\partial y_0}\right), \frac{\partial y}{\partial x_0} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial x_0}}$

we come down to the following:

$$J(\nabla\cdot\bar{q}) = \left[\frac{\partial^2 y}{\partial y_0^2}\frac{\partial y_0}{\partial x}\frac{\partial x}{\partial x_0} - \frac{\partial^2 x}{\partial y_0^2}\frac{\partial y_0}{\partial x}\frac{\partial y}{\partial x_0}\right]u + \left[\frac{\partial^2 x}{\partial x_0^2}\frac{\partial x_0}{\partial y}\frac{\partial y}{\partial y_0} - \frac{\partial^2 y}{\partial x_0^2}\frac{\partial x_0}{\partial y}\frac{\partial x}{\partial y_0}\right]v$$

by factoring we get

$$= \left[\frac{\partial}{\partial y_0}\frac{\partial y_0}{\partial x}\left[\frac{\partial y}{\partial y_0}\frac{\partial x}{\partial x_0}\right] - \frac{\partial}{\partial y_0}\frac{\partial y_0}{\partial x}\left[\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\right]\right]u$$

$$+ \left[\frac{\partial}{\partial x_0}\frac{\partial x_0}{\partial y}\left[\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0}\right] - \frac{\partial}{\partial x_0}\frac{\partial x_0}{\partial y}\left[\frac{\partial y}{\partial x_0}\frac{\partial x}{\partial y_0}\right]\right]v \qquad ?$$

by the fact that $\frac{\partial^2 y_0}{\partial y_0 \partial x} = \frac{\partial^2 y_0}{\partial x \partial y_0} = \frac{\partial}{\partial x}$, $\frac{\partial^2 x_0}{\partial x_0 \partial y} = \frac{\partial^2 x_0}{\partial y \partial x_0} = \frac{\partial}{\partial y}$

we get

$$J(\bar{\nabla}\cdot\bar{q}) = \frac{\partial}{\partial x}Ju + \frac{\partial}{\partial y}Jv$$

from the Eulerian view we look at a point & watch a particle move through it $\therefore$ $\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t} = 0$ since the point does not change position

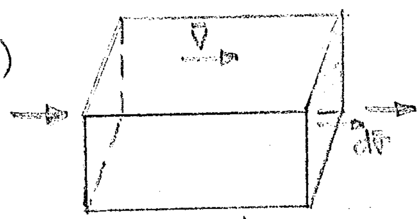
and since $x, y$ are independent of each other: $\frac{Dx}{Dt} = \frac{dx}{dt} = u$, etc.

by same token, $\frac{DJ}{Dt} = \frac{\partial}{\partial x} J \frac{dx}{dt} + \frac{\partial}{\partial y} J \frac{dy}{dt} = \frac{\partial J}{\partial x} u + \frac{\partial J}{\partial y} v$

$\therefore \quad \frac{DJ}{Dt} = J(\nabla \cdot \bar{q})$

let $\bar{Q} = \rho \bar{q}$ where $\rho = \frac{dm}{dV}$

4.b)




In x-direction:
flow entering at left $= -\bar{Q}_1(x,y,z)\Delta y \Delta z$
flow leaving at right $= \bar{Q}_1(x+\Delta x, y, z)\Delta y \Delta z$

$\Delta F_x$ = the net flow outward in x-direct $= \left[\bar{Q}_1(x+\Delta x, y, z) - \bar{Q}_1(x,y,z)\right]\Delta y \Delta z$

By taylor's expansion we get
$\Delta F_x = \frac{\partial Q_1}{\partial x} \Delta y \Delta x \Delta z$ ; by same manner we get $\Delta F_y, \Delta F_z$

$$\Delta F = \sum \Delta F_i = \left[\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial Q_3}{\partial z}\right]\Delta x \Delta y \Delta z$$
$$= \nabla \cdot \bar{Q}\, \Delta x \Delta y \Delta z = \nabla \cdot \bar{Q}\, \Delta V$$

however as $\Delta F, \Delta x, \Delta y, \Delta z \to 0$

$$F = \int_V \nabla \cdot \bar{Q}\, dV$$

let $d\bar{\sigma}$ = area $\perp$ to flow rate $\bar{Q}$ & $d\bar{\sigma} = \bar{n}\, dS$

$\bar{Q} \cdot d\bar{\sigma}$ = flow thru the area / unit time $\therefore$

$$F = \int_S \bar{Q} \cdot d\bar{\sigma} = \int_S \bar{Q} \cdot \bar{n}\, dS \quad ✓ \qquad \text{good.}$$

$$\therefore \quad \int_V \nabla \cdot \bar{Q}\, dv = \int_S \bar{Q} \cdot \bar{n}\, dS \quad \text{or} \quad \int_V \nabla \cdot (\rho \bar{q})\, dV = \int_S \rho \bar{q} \cdot \bar{n}\, dS$$

if mass source, the left hand side must include the term

$$\int_V \frac{\partial \rho}{\partial t}\, dV \quad \Rightarrow \quad \int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{q})\right] dV = \int_S \rho\bar{q}\cdot\bar{n}\, dS$$

$$\nabla\cdot\rho\bar{q} = \rho\nabla\cdot\bar{q} + \bar{q}\cdot\nabla\rho$$

in an incompressible flow $\nabla\cdot\bar{q} = 0$ $\therefore$ $\rho$ = constant in volume

then $\int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{q})\right] dV \rightarrow \int_V \left[\frac{\partial \rho}{\partial t} + \bar{q}\cdot\nabla\rho\right] dV \rightarrow \int_V \frac{D\rho}{Dt}\, dV$

$$\int_V \frac{D\rho}{Dt}\, dV \underset{X}{\Updownarrow} = \int_S \rho\bar{q}\cdot\bar{n}\, dS$$

since $\rho \neq f(x,y,z)$ then

we get $\rho\int_V \frac{D}{Dt}\, dV = \rho\int_S \bar{q}\cdot\bar{n}\, dS$ or

$\int\left(\frac{D\rho}{Dt}\right) dV = \int dV \left(\frac{D\rho}{Dt}\right)$

The operator $\frac{D}{Dt}$ does not apply on the "$dV$"!

$$\int_V \frac{D}{Dt}\, dV = \int_S \bar{q}\cdot\bar{n}\, dS$$

By the use of the Jacobian $dV = J\, dV_0$

$$\int_V \frac{D}{Dt}\, dV = \int_{V_0} \frac{DJ}{Dt}\, dV_0 = \int_{V_0} J\, \nabla\cdot\bar{q}\, dV_0 = \int_V \nabla\cdot\bar{q}\, dV$$

by the divergence theorem $\int_V \nabla\cdot\bar{q}\, dV = \int_S \bar{q}\cdot\bar{n}\, dS$

thus $\int_V \frac{D}{Dt}\, dV = \int_S \bar{q}\cdot\bar{n}\, dS$ ✓

AE-331

Homework #3

Fall 1970

1. From the basic differential relations of equilibrium thermodynamics, prove the Maxwell relationships.

$$\left(\frac{\partial s}{\partial v}\right)_T = \left(\frac{\partial p}{\partial T}\right)_v, \left(\frac{\partial s}{\partial v}\right)_p = \left(\frac{\partial p}{\partial T}\right)_s$$

$$\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p, \left(\frac{\partial s}{\partial p}\right)_v = -\left(\frac{\partial v}{\partial T}\right)_s$$

also show in general that

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma\left(\frac{\partial p}{\partial \rho}\right)_T;$$ verify this for a perfect gas

2. Prove that for a calorically perfect gas

$$e = A\exp(s/c_v)\,\rho^{\gamma-1}; \quad A = \text{constant}$$

3. Find the velocity potential, if one exists, of the velocity field whose components are

(a) $u=2xy, \ v=x^2+1$

(b) $u=3xy, \ v=x^2+1$

(c) $u=x^2-y^2+x, \ v= -(2xy+y)$

4. An airfoil is travelling through sea level air at 100 mph. What is the pressure at the stagnation point? What is the static pressure at a point on the airfoil where the velocity of the air relative to the airfoil is 200 fps. (Inviscid, Adiabatic, Homentropic Flow).

5. Consider a one-dimensional unsteady homentropic flow. A piston starts at $t=0$ to move with a constant velocity $U_p$ in a closed tube of length L. The initial state of the gas in the tube was undisturbed with known properties. (Linearized theory applies) How long will it take for the wave generated by the motion of the piston to reflect off the end wall and hit the piston? What is pressure at the piston after the wave reflects off the piston?

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$u = xy \qquad v = -y^2/2$
$u = Ax \qquad v = -Ay$

for incompress $\nabla \cdot \bar{q} = 0$
for irrot $\nabla \times \bar{q} = 0$
if irrot $\bar{q} = \nabla \varphi$

(a) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = y - y = 0$

(b) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = A - A = 0$

~~incompressibility proof~~

$$\nabla \times \bar{q} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & 0 \end{vmatrix} = i\left(-\frac{\partial v}{\partial z}\right) + j\left(\frac{\partial u}{\partial z} - 0\right) + k\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} = 0$

$$\therefore \quad \nabla \times \bar{q} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\underline{k}$$

(a) $0 - x \neq 0$ $\therefore$ (a) is rotat. & cannot have a $\varphi$

(b) $0 - 0 = 0$ $\therefore$ (b) is irrot & $\varphi$ can exist

$\frac{\partial \varphi}{\partial x} = u \quad \therefore \quad \varphi = \int u\,dx + g(y)$

$\varphi = \frac{Ax^2}{2} + g(y)$

$\frac{\partial \varphi}{\partial y} = 0 + g'(y) = -Ay \quad \therefore \quad g(y) = -\frac{Ay^2}{2} + C$

$$\therefore \quad \varphi = \frac{Ax^2}{2} - \frac{Ay^2}{2} + C$$

over please

Dr. Rubin,

Please accept the quiz paper. I came in late due to a completely mixed up morning. Someone I haven't seen for 14 yrs came in unexpectedly & and other complicating problems brought about a hectic morning leading to my lateness. I did the exam and gave it to Mrs Brownstein to give to you.

Thank you

C. Levy

Cesar Levy

ME 331

good

1. if $z = f(x,y)$ then $dz = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$ ①

if $dz$ is an exact differential (i.e.- a point function) we can then use the fact that

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad ②$$

let $M = \left(\frac{\partial f}{\partial x}\right)_y$ & $N = \left(\frac{\partial f}{\partial y}\right)_x$ ③ $\therefore$ $\left(\frac{\partial M}{\partial y}\right)_x = \left(\frac{\partial N}{\partial x}\right)_y$ ④

by Euler's Theorem.

Knowing that $s, T, V, p, \rho$ are point functions we can use this method to solve for the maxwell relationships.

a) $dA = -p\,dV - s\,dT$ where A is the Helmholtz function

$\therefore$ $-p = \left(\frac{\partial A}{\partial V}\right)_T$ $-s = \left(\frac{\partial A}{\partial T}\right)_V$ by equation ①

$\therefore$ by equation ④ $-\left(\frac{\partial p}{\partial T}\right)_V = -\left(\frac{\partial s}{\partial V}\right)_T$ ✓ or $\left(\frac{\partial p}{\partial T}\right)_V = \left(\frac{\partial s}{\partial V}\right)_T$

b) $dh = v\,dp + T\,ds$

$\therefore$ from ① $v = \left(\frac{\partial h}{\partial p}\right)_s$ & $T = \left(\frac{\partial h}{\partial s}\right)_p$

by equation ④ $\left(\frac{\partial v}{\partial s}\right)_p = \left(\frac{\partial T}{\partial p}\right)_s$ since $\left(\frac{\partial v}{\partial s}\right)_p = \frac{1}{\left(\frac{\partial s}{\partial v}\right)_p}$

we can invert our results $\Rightarrow$:

$$\left(\frac{\partial s}{\partial v}\right)_p = \left(\frac{\partial p}{\partial T}\right)_s \checkmark$$

c) $dG = v\,dp - s\,dT$ where G is the Gibbs Function

$\therefore$ $v = \left(\frac{\partial G}{\partial p}\right)_T$ & $-s = \left(\frac{\partial G}{\partial T}\right)_p$ from ①

by equation ④ $\left(\frac{\partial v}{\partial T}\right)_p = -\left(\frac{\partial s}{\partial p}\right)_T$ or $-\left(\frac{\partial v}{\partial T}\right)_p = \left(\frac{\partial s}{\partial p}\right)_T$ ✓

d) $du = T\,ds - p\,dV$

$\therefore$ $T = \left(\frac{\partial u}{\partial s}\right)_V$ & $-p = \left(\frac{\partial u}{\partial V}\right)_s$ from ① by equation ④ $\left(\frac{\partial T}{\partial v}\right)_s = -\left(\frac{\partial p}{\partial s}\right)_v$ or since $-\frac{1}{\left(\frac{\partial v}{\partial T}\right)_s} = -\left(\frac{\partial T}{\partial v}\right)_s$

we obtain $-\left(\frac{\partial v}{\partial T}\right)_s = \left(\frac{\partial s}{\partial p}\right)_v$ ✓

c) from $p = \rho RT$ we can differentiate $p$ with respect to $T$ & $\rho$

$\therefore \quad dp = R\,(\rho\, dT + d\rho\, T)$

by equation ① $\quad R\rho = \left(\frac{\partial p}{\partial T}\right)_\rho$ & $RT = \left(\frac{\partial p}{\partial \rho}\right)_T$

from $a^2 = \gamma RT$ we know $a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ $\therefore$

by equating $a^2 \quad \gamma\left(\frac{\partial p}{\partial \rho}\right)_T$ we get

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma\left(\frac{\partial p}{\partial \rho}\right)_T$$

Note: this is for ideal gas; see last page for general solution.

2. $T\,ds = de + p\,dV \qquad$ if $\quad V = 1/\rho$

$= C_v\,dT - p/\rho^2\, d\rho$

$ds = C_v \frac{dT}{T} - p/\rho^2 T\, d\rho \quad$ since $p/\rho = RT$ we get $ds = C_v \frac{dT}{T} - R\frac{d\rho}{\rho}$

if $p = \rho RT \iff \ln p = \ln\rho + \ln R + \ln T \Rightarrow \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$

$\therefore \quad ds = C_v\left(\frac{dp}{p} - \frac{d\rho}{\rho}\right) - R\frac{d\rho}{\rho} = C_v\frac{dp}{p} - (C_v + R)\frac{d\rho}{\rho}$

for ideal gas $C_p - C_v = R \iff C_p = C_v + R$

$\therefore \quad ds = C_v\frac{dp}{p} - C_p\frac{d\rho}{\rho} \;; \quad \frac{s}{C_v} = \ln p - \gamma \ln\rho + \ln C$

if gas is calorically perfect

$\boxed{\exp^{s/C_v} = C\,p/\rho^\gamma}$ ⓐ

from (α) we get $\rho^{\gamma-1} \exp^{s/c_v} = C\, p/\rho = CRT$ (β)

for a calorically perfect gas $\boxed{e = C_v T}$ $\therefore$ $CRT = \frac{1}{A} C_v T = \frac{1}{A} e$

$\therefore$ from (β) we get $\rho^{\gamma-1} \exp^{s/c_v} = \frac{1}{A} e$ or

$$\boxed{A \exp^{s/c_v} \rho^{\gamma-1} = e}$$ ✓

3. a. $u = 2xy$, $v = x^2+1$ if a potential exists $\nabla \times \bar{q} \equiv 0 \Rightarrow \bar{q} = \nabla\varphi$

$\therefore$ $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Rightarrow 2x = 2x$ $\therefore$ $\nabla \times \bar{q} \equiv 0$

since $\bar{q} = \nabla\varphi \rightarrow u = \frac{\partial\varphi}{\partial x}$ ; $v = \frac{\partial\varphi}{\partial y}$

$\varphi = \int (u\, dx) = x^2 y + f(y)$ to find $f(y)$ find $\frac{\partial\varphi}{\partial y}$

$\therefore$ $\frac{\partial\varphi}{\partial y} = x^2 + f'(y) \equiv x^2+1$ $\therefore$ $f'(y) = 1$ or $f(y) = y + c$

→ then $\varphi = x^2 y + y + c$

b. $u = 3xy$ $v = x^2+1$ $\nabla \times \bar{q} = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)k = 3x - 2x \neq 0$

→ $\therefore$ $\varphi$ doesn't exist. ✓

c. $u = x^2 - y^2 + x$ $v = -(2xy + y)$

$\nabla \times \bar{q} = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)k = -2y - (-2y) \equiv 0$

$\varphi = \int (u\, dx) = \frac{x^3}{3} - y^2 x + \frac{x^2}{2} + f(y)$

$\therefore$ $\frac{\partial\varphi}{\partial y} = -2yx + f'(y)$ $\therefore$ $f'(y) = -y$ or $f(y) = -y^2/2 + c$

→ $\therefore$ $\varphi = \frac{x^3}{3} - y^2 x + \frac{x^2}{2} - \frac{y^2}{2} + c$ ✓

4. Isentropic flow : $p_1 = f(s_1)\rho_1^{\gamma}$ , $p_2 = f(s_2)\rho_2^{\gamma}$

but adiabatic : $s_1 = s_2$

$$\therefore \quad p_1/p_2 = (\rho_1/\rho_2)^{\gamma}$$

Derived previously: $T_0/T = 1 + \frac{\gamma-1}{2} M^2$ and $p = \rho R T$

using fact that at $p = p_t$ $V = 0$ $\therefore$

$$p_t = p\left[1 + \frac{\gamma-1}{\gamma} M^2\right]^{\gamma/\gamma-1}$$

$$p = 14.7\,\text{psi} \qquad \gamma = 1.4 \qquad M = \frac{146.7}{1117} = .1313$$

$$p_t = 14.7\left[1 + .2(.1313)^2\right]^{3.5}$$

$$= 14.7\left[1.00345\right]^{3.5} = 14.91\,\text{psi}$$

$$p_t = p_{stat}\left[1 + \frac{\gamma-1}{2} M_{stat}^2\right]^{\gamma/\gamma-1} \qquad p_t = 14.91\,\text{psi} \qquad M_{stat} = \frac{200}{1117} = .179$$

$$14.91 = p_{stat}\left[1 + .2(.179)^2\right]^{3.5} = p_{stat}(1.024)$$

$$p_{stat} = \frac{14.91}{1.024} = 14.66\,\text{psi}$$

5. Since $U_p$ = piston velocity and $a_\infty$ = wave velocity

the time $t_1$ $\ni$ the wave reaches the wall is $t_1 = \frac{L}{a_\infty}$. The distance the piston would have travelled in $t_1$ is $\frac{U_p L}{a_\infty}$.

Upon reflection a time $t_2$ occurs $\ni$ the wave hits the piston. $t_2$ is obtained from $U_p t_2 + a_\infty t_2 = L - \frac{U_p L}{a_\infty}$ (initial length $t=0$ − unit length $t = t_1$ of tube)

$$\therefore \quad t_2 = \frac{a_\infty L - U_p L}{a_\infty (U_p + a_\infty)} = \frac{L}{a_\infty}\frac{(a_\infty - U_p)}{(a_\infty + U_p)}$$

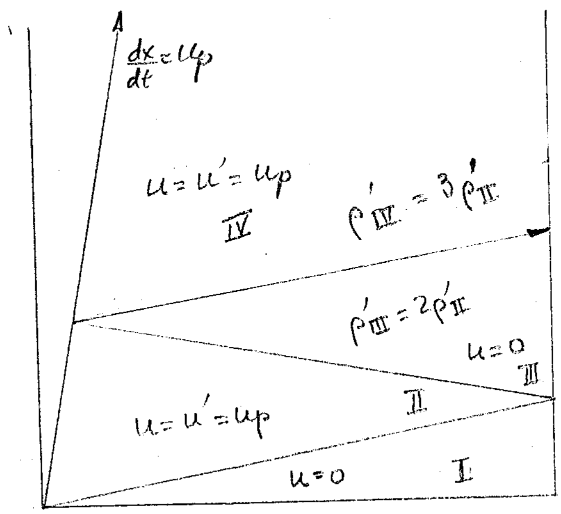
the total time is $t_1 + t_2 = \frac{L}{a_\infty}\frac{(a_\infty + U_p)}{(a_\infty + U_p)} + \frac{L(a_\infty - U_p)}{a_\infty(a_\infty + U_p)} = \frac{2L}{(a_\infty + U_p)}$

$$\text{total time} = \frac{2L}{a_\infty + U_p}$$

upon hitting the wall $u = 0$ , $u' = 0$

$\rho'_{III} = f(x - at) + g(x + a_\infty t)$

$u' = \frac{a_\infty}{\rho_\infty}[f - g]$ ; if $u' = 0$ then $f_{II} = g_{III}$ and




$f_{II} = \frac{\rho_\infty u_p}{a_\infty}$

$\rho'_{III} = f_{III} + g_{III}$

and $g_{III} = f_{II} = f_{III}$

$\therefore \rho'_{III} = 2\rho'_{II}$

$$\begin{cases} u_p & x, t = 0 \\ 0 & x > 0, t = 0 \end{cases}$$

$\rho'_{IV} = f_{IV} + g_{IV}$ $\qquad u'_{IV} = \frac{a_\infty}{\rho_\infty}[f_{IV} - g_{IV}]$

$f_{IV} - g_{IV} = \frac{\rho_\infty u_p}{a_\infty}$ ; $\rho'_{IV} = 2g_{IV} + \frac{\rho_\infty u_p}{a_\infty}$

but as before $g_{IV} = f_{III} = f_{II} = \frac{\rho_\infty u_p}{a_\infty}$

$\therefore \rho'_{IV} = 3\frac{\rho_\infty u_p}{a_\infty}$

now $p_{IV} = p_\infty + p'_{IV} = p_\infty + a_\infty^2 \rho'_{IV} = p_\infty + 3a_\infty \rho_\infty u_p$

1\. c General Solution: start with $s = s(v,T)$

$$ds = \left(\frac{\partial s}{\partial T}\right)_v dT + \left(\frac{\partial s}{\partial v}\right)_T dv$$

but $\left(\partial s/\partial T\right)_v = \frac{1}{T}\left(\frac{\partial u}{\partial T}\right)_v = \frac{C_v}{T}$ from $T\left(\frac{\partial s}{\partial T}\right)_v = \left(\frac{\partial u}{\partial T}\right)_v + p\left(\frac{\partial v}{\partial T}\right)_v^{\nearrow 0}$

also we have proven $\left(\frac{\partial s}{\partial v}\right)_T = \left(\frac{\partial p}{\partial T}\right)_v$ if $ds = 0$, solve for $C_v$:

$$-\frac{C_v}{T}dT = \left(\frac{\partial p}{\partial T}\right)_v dv \quad \Rightarrow \quad C_v = -T\left(\frac{\partial p}{\partial T}\right)_v\left(\frac{\partial v}{\partial T}\right)_s \checkmark$$

if $s = s(p,T)$

$$ds = \left(\frac{\partial s}{\partial p}\right)_T dp + \left(\frac{\partial s}{\partial T}\right)_p dT$$

but we have proven $\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p$

from $T\left(\frac{\partial s}{\partial T}\right)_p = \left(\frac{\partial h}{\partial T}\right)_p - v\left(\frac{\partial p}{\partial T}\right)_p^{\nearrow 0}$ we get $\left(\frac{\partial s}{\partial T}\right)_p = \frac{1}{T}C_p$

$\therefore$ let $ds = 0$

$$ds = 0 : -\left(\frac{\partial v}{\partial T}\right)_p dp + \frac{1}{T}C_p\, dT \quad \text{or}$$

$$C_p = T\left(\frac{\partial v}{\partial T}\right)_p\left(\frac{\partial p}{\partial T}\right)_s \checkmark$$

since $$\gamma = \frac{C_p}{C_v} = \frac{T\left(\frac{\partial v}{\partial T}\right)_p\left(\frac{\partial p}{\partial T}\right)_s}{-T\left(\frac{\partial p}{\partial T}\right)_v\left(\frac{\partial v}{\partial T}\right)_s} = -\frac{\left(\partial v/\partial T\right)_p\left(\partial p/\partial v\right)_s}{\left(\partial p/\partial T\right)_v}$$

$$\gamma = \frac{\left(\partial s/\partial p\right)_T\left(\partial p/\partial v\right)_s}{\left(\partial s/\partial v\right)_T} = \frac{\left(\partial p/\partial v\right)_s}{\left(\partial p/\partial v\right)_T}$$

<u>BUT</u> the definition of the adiabatic bulk modulus is

$$-v\left(\partial p/\partial v\right)_s = \rho\left(\partial p/\partial \rho\right)_s$$

the definition of the Isothermal bulk modulus is

$$-v\left(\partial p/\partial v\right)_T = \rho\left(\partial p/\partial \rho\right)_T$$

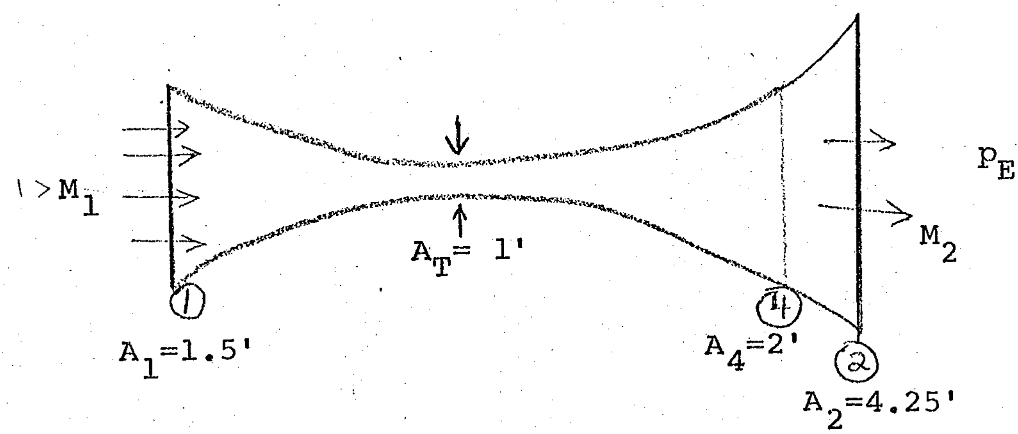
As you can see. $\gamma = \frac{\text{adiab Bulk Mod}}{\text{a thermal Bulk Mod}} = \frac{\left(\partial p/\partial \rho\right)_s}{\left(\partial p/\partial \rho\right)_T} \checkmark$

or $\quad \gamma \left(\partial p/\partial \rho\right)_T = \left(\partial p/\partial \rho\right)_s \quad$ In general

AE 331

HOMEWORK # 4

1. Consider a "quasi"-one - dimensional flow in a converging-diverging channel with a subsonic entrance (Steady with zero viscosity)




Given: At the entrance the stagnation pressure is $p_o=150$ psi

(i) Determine $p_2$ for a subsonic isentropic exit with sonic conditions (M=1) at the throat.

(ii) Determine $p_2$ for a supersonic isentropic exit.

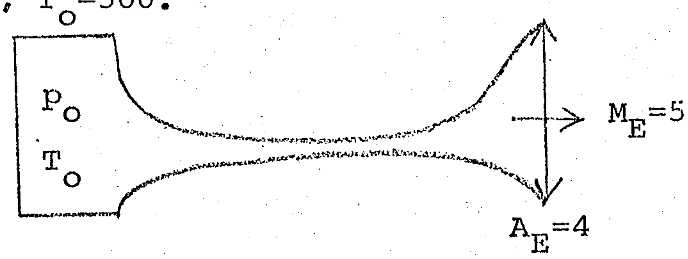
(iii) Determine $p_2$ (behind the shock) for a normal shock that is standing at the exit.

(iv) If $p_E$= 90 psi, what will the ultimate flow look like?

(v) Determine the final $p_2$ $p_{exit}$ for a shock located at $A_4=2'$

(vi) Determine $M_{THROAT}$ if $p_E=149$ psi.

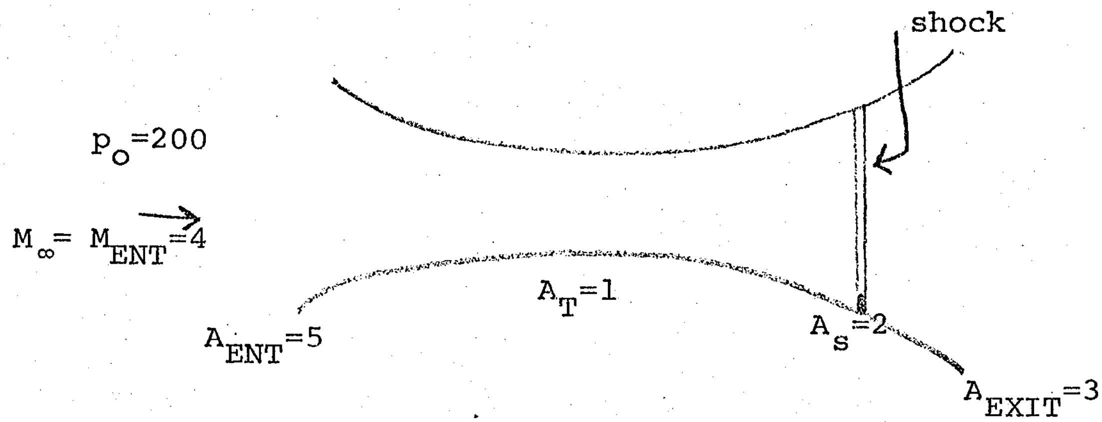
2. Given a channel, with exit area $A_E=4$ and exit Mach number $M_E=5$, $p_o=200$, $T_o=500$.




1. Determine the exit pressure.
2. Determine the exit Mach number if $T_o$ is raised to 700.
3. Determine the exit Mach number if $A_t^o$ is reduced by ½.

($p_o$ and $T_o$ are the original values.)

3. Consider a jet engine




1. Determine the Mach No. upstream of the shock.
2. Determine the Mach exit pressure.

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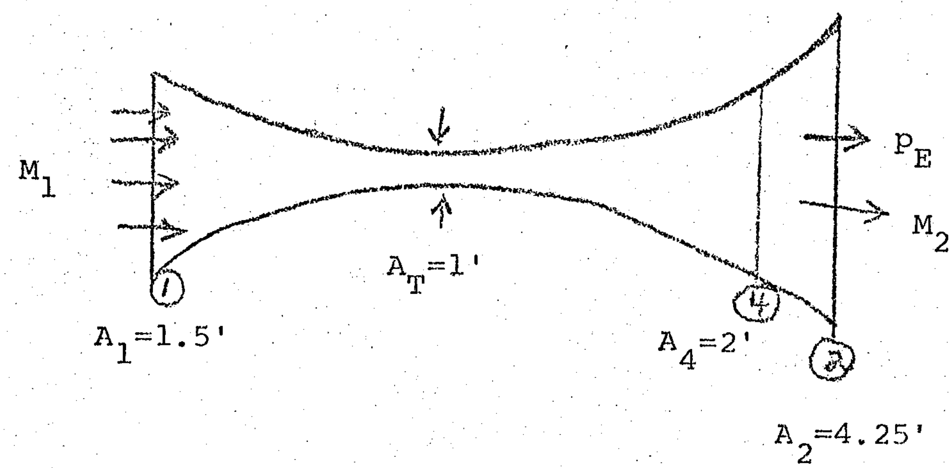
QUIZ #4

1. Consider a "quasi"-one-dimensional flow in a converging-diverging channel with a subsonic entrance (steady with zero viscosity).



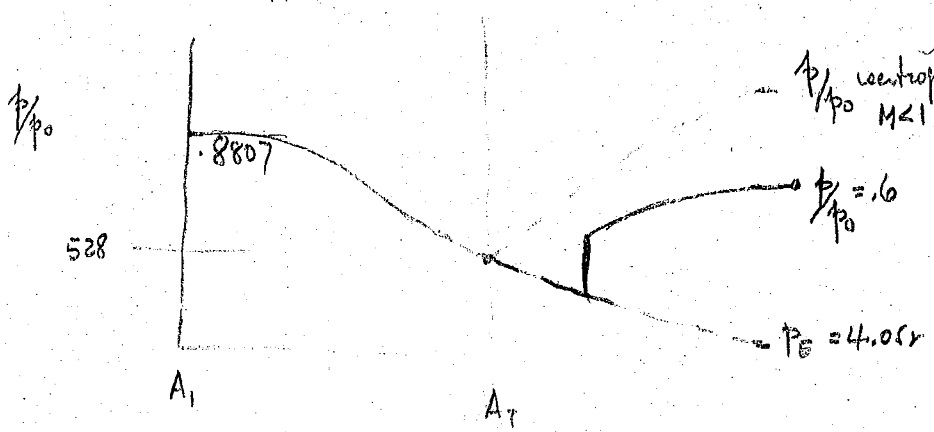

Given: At the entrance the stagnation pressure is $p_o = 150$ psi

(i) If $p_E = 90$ psi, what will the ultimate flow look like?

(ii) Determine the final $p_2$ for a shock located at $A_4 = 2'$.

(i) For the homework

$\frac{A_1}{A^*} = 1.5$ $\quad p/p_0 = .8807$ $\quad p_1 = 88.07 + 44.035 = 132.105$ psia




$\frac{A_2}{A_*} = 4.25$

$p_2/p_0 = .9868$

$p_2 = 148.02$ $\quad 98.68 + 49.34 = 148.02$

$\frac{p_2}{p_0}$ $.02702 =$ $\quad 2.702 + 1.350 = 4.052$

flow has a shock wave between throat & exit.

$P/P_0{}_{(M<1)} = .9868$

$P/P_0{}_{(M>1)} = .02702$

$p_E/p_0 = .6$

How do you know that there aren't shock waves outside the channel?

you must determine $P_2$ with shock at exit!

$\therefore \quad p_{E_{M>1}} < p_E < p_{E_{M<1}}$

2 $\quad \frac{A_4}{A_*} = 2 \qquad \frac{p_4}{p_0} = .09352 \qquad p_4 = p_1 = \begin{array}{r} 9.352 \\ 4.676 \\ \hline 14.028 \end{array}$

$M_1 = 2.20$

$p_2/p_1 = 5.480 \qquad p_2 = 5.480\,(14.028) = 77 \text{ psia}$

$\frac{p_{o_2}}{p_{o_1}} = .6281 \qquad p_{o_2} = .6881\,(150) = \begin{array}{r} 62.81 \\ 31.40 \\ \hline 94.21 \end{array}$

$\frac{p_2}{p_{o_2}} = \frac{77}{94.21} = .818 \qquad \frac{A}{A_*} = 1.2648$

$\frac{38}{59} = .64$
$\quad 36$

$A_* = \frac{2}{1.2648} = 1.58$

$\frac{A_2}{A_*} = \frac{4.25}{1.58} = 2.69$

$\frac{p_2}{p_{o_2}} = .9668 \qquad p_2 \approx 91 \text{ psia}$

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(i) $A^* = 1$ $A = 4.25$ $A^*/A = .2355$

| M | $A^*/A$ | $p/p_0$ |
|---|---|---|
| .13 | .2224 | .9883 |
| $M_x$ | .2355 | $p_x/p_0$ |
| .14 | .2391 | .9864 |

$M_x = .13785$

$p_x/p_0 = .98681$

$p_x = p_0(.98681) = 148.021$ psia ✓

(ii)

| M | $A^*/A$ | $p/p_0$ |
|---|---|---|
| 3.00 | .2362 | .02722 |
| $M_1$ | .2355 | $p_1/p_0$ |
| 3.01 | .2339 | .02682 |

$M_1 = 3.00304$

$p_1/p_0 = .0270984 \sim .0271$

$p_1 = p_0(.02741) = 4.06$ psia ✓

(iii) $p_2 = p_1 K$ K from tables → 10.33

$p_2 = (4.06)(10.33) = 42$ psia ✓

(iv) take $A = 2.03$ $A_1^*/A_1 = .4933$ $M_1 = 2.2125$

$p_1/p_0 = .09171$

$p_1 = (150)(.09171) = 13.757$ psia

$p_{o_2}/p_{o_1} = .6225$

$p_{o_2} = (150)(.6225) = 93.38$ psia

$p_2 = p_1(5.544) = 76.2$ psia $\quad p_2/p_{o_2} = \frac{76.2}{93.38} = .816$

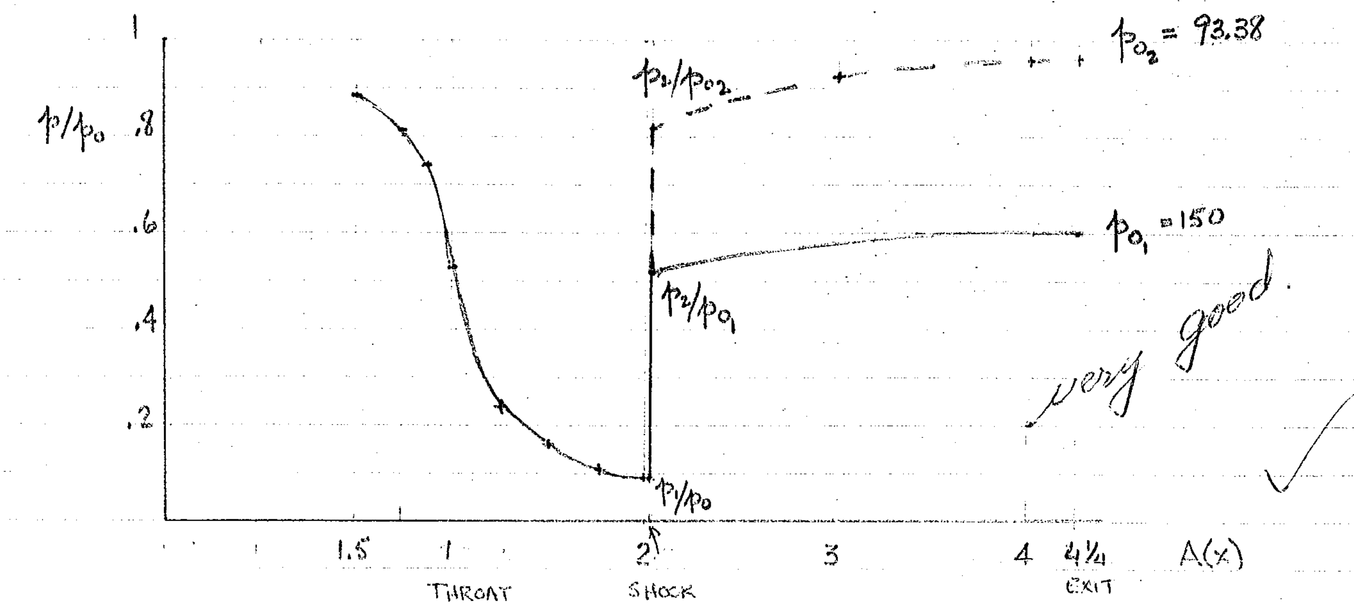
$A_2^*/A_1 = .7939$ $\quad A_2^* = (2.03)(.7939) = 1.61$

$A_2^*/A_E = \frac{1.61}{4.25} = .379$ $\quad p_E/p_{o_2} = .965$ $\quad p_E = .965(93.38) = 90$ psia

| A | $A^*/A$ | $p/p_0$ |
|---|---|---|
| 1.5 | .667 | .881 |
| 1.25 | .800 | .8122 |
| 1.125 | .890 | .7444 |
| 1.00 | 1.00 | .528 |

| A | $A^*/A$ | $p/p_0$ |
|---|---|---|
| 1.25 | .8 | .2353 |
| 1.50 | .667 | .1602 |
| 1.75 | .572 | .1197 |
| 2.00 | .5 | .0939 |

| A | $A^*/A$ | $p/p_0$ | $p/p_{0_2}$ |
|---|---|---|---|
| 2.03 | .4933 | .09171 | |
| 2.03 | .794 | .507 | .816 |
| 3.00 | .537 | .9248 | |
| 4.00 | .403 | .9602 | |
| 4.25 | .379 | .6 | .965 |




(v) $A_1^*/A_1 = .5$ $\qquad M_1 = 2.1975$ $\qquad p/p_0 = .0939$

$p_1 = 150(.0939) = 14.08$ psia

$p_2 = (14.08)(5.467) = 77$ psia $\qquad p_{0_2} = 150(.629) = 94.35$

$p_2/p_{0_2} = \frac{77}{94.35} = .817 \rightarrow M_s = .545 \qquad A_2^*/A_1 = .792$

$A_2^* = 2(.792) = 1.584$

$A_2^*/A_2 = \frac{1.584}{4.25} = .373 \qquad M_2 = .2223$

$p_E/p_{0_2} = .966 \qquad p_E = (94.35)(.966) = 91$ psia

(vi) $p_E/p_0 = \frac{149}{150} = .993 \qquad A^*/A_E = .1718 \qquad M_E = .1$

$A^* = 4.25(.1718) = .73$

$A^*/A_t = .73/1 = .73 \qquad M_t = .48491$

2.

(i) $p_E/p_0 = K = 1.89\times10^{-3}$

$p_E = 200(1.89\times10^{-3}) = .378$ psia

(ii) since $A^*/A_E = f(M)$ only if both $A^*$ & $A_E$ remain the same and $T_0$ increases then $M_E$ must remain the same as from the equation above. This means $T_{E_2}/T_{0_2} = T_{E_1}/T_{0_1}$ this means that if $p_0$ remains constant from the isentropic equations $p_E$ must remain constant from the mass flow equation:

before $T_0$ increased at a particular $A_1$:

$\rho_{1_b} u_{1_b} A_{1_b} = C_1$ since $A_{1_b} = A_{1_a}$ (b - before, a - after)

$\rho_{1_b} u_{1_b} A_{1_a} = C_1$ since $u_{1_a} > u_{1_b}$ (since $u_{1_b} = f(T_{1_b})$, $u_{1_a} = f(T_{1_a})$ and $T_{1_b} < T_{1_a}$ $\therefore$ $u_{1_a} > u_{1_b}$)

$\therefore$ $\rho_{1_b} u_{1_a} A_{1_a} > C_1$

now if $p_0 = \text{const}$ & $p_E$ remains constant $p_E/p_0 = (\rho_E/\rho_0)^\gamma$

$\therefore$ if $p_0 = \rho_0 R T_0$ if $p_0 = C$, $T_0$ increases, $\rho_0$ must decrease if $\rho_0$ decreases $\rho_E$ must decrease, at $A_1$ $\rho_{1_a} < \rho_{1_b}$

we can transform all the above saying $C_1 = \rho_{1_a} u_{1_a} A_1 \dfrac{\sqrt{T_{0a}}}{\sqrt{T_{0b}}}$

and since $T_{0b} < T_{0a}$ $\rho_{1_a} u_{1_a} A_1 < \rho_{1_b} u_{1_b} A_1$

or mass flow decreases

(iii) $A^*/A_E = .04$ $A^* = .16$ let $A^* = .08$

$A^*/A_E = .08/4 = .02$ $M_E = 5.943$

3. (i) $M_{ENT} = 4$ $A^*/A = .09329$ $A^* = .46645$

$A^*/A_S = .23323$ $M_1 = 3.06$ $M_2 = .472$

(iii) $p/p_0 = .02489 \qquad p_1 = 4.978$

$p_2/p_1 = 10.76 \qquad p_2 = 53.56$

$p_{o_2}/p_{o_1} = .312 \qquad p_{o_2} = 62.4$

$p_2/p_{o_2} = \frac{53.56}{62.4} = .8583 \qquad A^*/A = .7157$

$A^* = 1.4314 \qquad A^*/A_E = \frac{1.4314}{3.0} = .4771 \qquad M_E = .29$

$p_E/p_{o_2} = .9433 \qquad p_E = 58.86$ psia $\approx 61.0$

Cesar Levy
AE 331

1. $T\,ds = dh - v\,dp$

$$= C_p dT - C_p\left(\frac{\gamma-1}{\gamma}\right)T\frac{dp}{p} \longrightarrow \frac{ds}{C_p} = \frac{dT}{T} - \frac{\gamma-1}{\gamma}\frac{dp}{p}$$

from $p/p_A$:

$$\frac{dp}{p_A} = \frac{\gamma M_A^2\, dT}{2T_A}\frac{1}{\left[\left(\frac{1+\gamma M_A^2}{2}\right)^2 - \gamma M_A^2 T/T_A\right]^{1/2}}$$

$$\therefore \quad \frac{ds}{C_p} = \frac{dT}{T} - \frac{\gamma-1}{\gamma}\cdot\frac{\gamma M_A^2\, dT}{2T_A}\frac{p_A}{p}\frac{1}{\left[\left(\frac{1+\gamma M_A^2}{2}\right)^2 - \gamma M_A^2 T/T_A\right]^{1/2}}$$

$$\frac{1}{C_p}\frac{ds}{dT} = \frac{ds}{dT} = 0 = \frac{1}{T} - \frac{(\gamma-1)}{2T_A}M_A^2\frac{p_A}{p}\frac{1}{\left[\left(\frac{1+\gamma M_A^2}{2}\right)^2 - \gamma M_A^2 T/T_A\right]^{1/2}}$$

at $T = T_A$, $p = p_A$:

$$\frac{ds}{dT_A} = 0 = \frac{1}{T_A} - \frac{(\gamma-1)}{2T_A}M_A^2\frac{2}{(\gamma M_A^2 - 1)}$$

$$\text{or} \quad 0 = 1 - \frac{\gamma-1}{2}M_A^2\frac{2}{\gamma M_A^2-1} = \gamma M_A^2 - 1 - (\gamma-1)M_A^2 = -1 + M_A^2$$

$$M_A = \sqrt{1} = 1$$

2. $$C_p\frac{dT}{ds} = \left\{\frac{1}{T} - \frac{(\gamma-1)}{2T_M}M_M^2\frac{p_M}{p}\frac{1}{\left[\left(\frac{1+\gamma M_M^2}{2}\right)^2 - \frac{\gamma M_M^2 T}{T_M}\right]^{1/2}}\right\}^{-1} = 0$$

at $T = T_M$, $p = p_M$

$$\frac{dT_M}{ds} = \left\{\frac{1}{T_M} - \frac{(\gamma-1)M_M^2}{2T_M}\frac{2}{\gamma M_M^2 - 1}\right\}^{-1} = 0$$

$$\frac{dT_M}{ds} = T_M\left\{\frac{(\gamma M_M^2 - 1) - (\gamma-1)M_M^2}{\gamma M_M^2 - 1}\right\}^{-1} = \left\{\frac{M_M^2 - 1}{\gamma M_M^2 - 1}\right\}^{-1} = 0$$

$$\therefore \quad \gamma M_M^2 - 1 = 0 \qquad M_M = \sqrt{1/\gamma} \quad @ \quad T = T_M \quad \text{where } T_M = T_{max}$$

3.

$$\rho u = C \quad \longrightarrow \quad \rho\, du + u\, d\rho = 0 \qquad \text{or} \qquad -\frac{du}{u} = \frac{d\rho}{\rho}$$

$$T_0 = T\left(1 + \frac{\gamma-1}{2} M^2\right) = T_A\left(\frac{\gamma+1}{2}\right)$$

$$p = \rho R T \quad \longrightarrow \quad \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

from before:

$$\int_{s_A}^{s} \frac{ds}{c_p} = \int_{T_A}^{T} \frac{dT}{T} - \frac{\gamma-1}{\gamma} \int_{p_A}^{p} \frac{dp}{p} = \frac{1}{\gamma} \int_{T_A}^{T} \frac{dT}{T} + \frac{\gamma-1}{\gamma} \int_{u_A}^{u} \frac{du}{u}$$

$$\frac{s - s_A}{c_p} = \ln\left(\frac{T}{T_A}\right)^{1/\gamma} + \ln\left(u/u_A\right)^{\frac{\gamma-1}{\gamma}} = \ln\left[\left(T/T_A\right)^{1/\gamma} \left(u/u_A\right)^{\frac{\gamma-1}{\gamma}}\right]$$

$$\frac{T_0}{T} = 1 + \frac{(\gamma-1)u^2}{2\gamma R T} \quad ; \quad \frac{T_0}{T_A} = 1 + \frac{(\gamma-1)u_A^2}{2\gamma R T_A}$$

$$\frac{2\gamma R T}{(\gamma-1)}\left[\frac{T_0}{T} - 1\right] = u^2 \quad ; \quad \frac{2\gamma R T_A}{(\gamma-1)}\left[\frac{T_0}{T_A} - 1\right] = u_A^2$$

$$\frac{u}{u_A} = \sqrt{\frac{\frac{2\gamma R T}{\gamma-1}\left[\frac{T_0}{T} - 1\right]}{\frac{2\gamma R T_A}{\gamma-1}\left[\frac{T_0}{T_A} - 1\right]}} = \sqrt{\frac{T_0 - T}{T_0 - T_A}} = \left[\frac{T_A\left(\frac{\gamma+1}{2}\right) - T}{T_A\left(\frac{\gamma+1}{2}\right) - T_A}\right]^{1/2}$$

$$\frac{u}{u_A} = \left[\frac{\frac{\gamma+1}{2} - T/T_A}{\left(\frac{\gamma-1}{2}\right)}\right]^{1/2} \quad \therefore \quad \ln\left(u/u_A\right)^{\frac{\gamma-1}{\gamma}} = \ln\left[\left(\frac{2}{\gamma-1}\right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{\gamma+1}{2} - T/T_A\right)^{\frac{\gamma-1}{2\gamma}}\right]$$

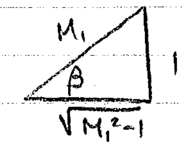
$$\therefore \quad \frac{s - s_A}{c_p} = \ln\left[\left(T/T_A\right)^{1/\gamma} \left(2/\gamma-1\right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{\gamma+1}{2} - T/T_A\right)^{\frac{\gamma-1}{2\gamma}}\right]$$

Cesar Luoy AE331

1.

a. $p_2/p_1 = 1 + \frac{2\gamma}{\gamma+1}(M_{1n}^2 - 1)$ ; $\tan\theta = 2\cot\beta \dfrac{M_{1n}^2 - 1}{M_1^2(\gamma + \cos 2\beta) + 2}$

$$M_{1n}^2 - 1 = \tan\theta \frac{\left(M_1^2[\gamma + \cos 2\beta] + 2\right)}{2\cot\beta}$$




$$p_2/p_1 = 1 + \frac{2\gamma}{\gamma+1}\left[\frac{\tan\theta\left(M_1^2[\gamma+\cos 2\beta]+2\right)}{2\cot\beta}\right]$$

$$\frac{\Delta p}{p_1} = \frac{2\gamma}{\gamma+1}\left[\frac{\theta\left(M_1^2\left[\gamma + \frac{M_1^2-2}{M_1^2}\right]+2\right)}{2\sqrt{M_1^2-1}}\right] \qquad \text{if } \theta \ll 1 \text{ ; } \tan\theta = \theta + B\theta^2 + \dots$$

$$= \frac{\gamma\theta}{\gamma+1}\left[\frac{\gamma M_1^2}{\sqrt{M_1^2-1}} + \frac{M_1^2-2}{\sqrt{M_1^2-1}} + \frac{2}{\sqrt{M_1^2-1}}\right] + \theta^2 \text{ terms}$$

$$= \frac{\gamma\theta}{\gamma+1}\left[\frac{(\gamma+1)M_1^2}{\sqrt{M_1^2-1}}\right] + \theta^2 \text{ terms} = \frac{\gamma M_1^2}{\sqrt{M_1^2-1}}\theta + \theta^2 \text{ terms}$$

1

b. $$\Delta s = \frac{R}{\gamma-1}\left\{\ln\left[\frac{2\gamma(M_1^2\sin^2\beta) - (\gamma-1)}{\gamma+1}\right] - \gamma\ln\left[\frac{(\gamma+1)M_1^2\sin^2\beta}{(\gamma-1)M_1^2\sin^2\beta + 2}\right]\right\}$$

for very weak shock $M_1^2\sin^2\beta = 1 + A\theta$ 

Using $\ln(1+\Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} + \dots$

$\frac{1}{1+\Delta} = 1 - \Delta + \Delta^2 + \dots$

$\Delta \ll 1$

$$\Delta s = R\ln\left\{\left[1 + \frac{2\gamma}{\gamma+1}(M_{1n}^2 - 1)\right]^{1/\gamma-1}\left[\frac{(\gamma+1)M_{1n}^2}{(\gamma-1)M_{1n}^2+2}\right]^{-\gamma/\gamma-1}\right\}$$

$$\Delta s = R\ln\left\{\left[1 + \frac{2\gamma}{\gamma+1}(2A\theta + A^2\theta^2)\right]^{1/\gamma-1}\left[\frac{(\gamma+1)(A\theta+1)^2}{(\gamma+1)(1+A\theta)^2+2}\right]^{-\gamma/\gamma-1}\right\}$$

Expansion by use of Binomial Theorem. neglect $\theta^3$ & $\theta^4$ terms.
Simplifying the equation leads to:

$$\Delta S = R\ln\left\{\frac{\alpha\beta}{\delta\epsilon}\right\}^{1/\gamma-1} \qquad \text{where}$$

$$\alpha\beta = \left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + 2A\theta\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + \gamma(2\gamma-1)A^2\theta^2\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + \frac{2\gamma}{\gamma-1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + 2\gamma^2A\theta\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma$$
$$\frac{(2\gamma-1)(\gamma-1)A^2\theta^2}{(\gamma+1)^\gamma} + \frac{2\gamma}{\gamma-1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + 4\gamma A\theta\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + 4\gamma(2\gamma-3)A^2\theta^2 + \frac{(4\gamma)A\theta}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma$$

$$+\frac{8\gamma^2 A^2\theta^2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + \frac{8\gamma^2 A\theta}{(\gamma+1)(\gamma-1)}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + \frac{8\gamma^3 A^2\theta^2}{(\gamma+1)}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + \frac{2\gamma A\theta}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma$$

$$+\frac{4\gamma^2}{\gamma+1}A^2\theta^2\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + \frac{4\gamma^2 A^2\theta^2}{(\gamma+1)(\gamma-1)}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + \frac{4\gamma^2 A^2\theta^2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma - 2\gamma A\theta\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma$$

$$-4\gamma^2 A^2\theta^2\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma - \frac{4\gamma^2 A\theta}{\gamma-1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma - 4\gamma^3 A^2\theta^2\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma - \frac{4\gamma^2}{\gamma+1}A\theta\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma$$

$$+8\gamma^2 A^2\theta^2\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma - \frac{8\gamma^2 A^2\theta^2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma - \frac{16\gamma^2 A^2\theta^2}{(\gamma-1)(\gamma+1)}\left(\frac{\gamma-1}{\gamma+1}\right)^2$$

$$-\frac{16\gamma^3}{\gamma^2-1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma A\theta - \frac{36\gamma^3 A^2\theta^2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma - \frac{4\gamma^2 A^2\theta^2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + \gamma(2\gamma+1)A^2\theta^2\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma$$

$$+\frac{2\gamma^2(2\gamma+1)A^2\theta^2}{\gamma-1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + \frac{2\gamma^2(2\gamma+1)A^2\theta^2}{\gamma-1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma + \frac{8\gamma^3 A^2\theta^2}{(\gamma+1)(\gamma-1)}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma$$

$$\delta\epsilon = (\gamma-1)^\gamma + 2\gamma(\gamma-1)^\gamma A\theta + (2\gamma-1)(\gamma-1)^\gamma A^2\theta^2 + \frac{2\gamma}{\gamma-1}(\gamma-1)^\gamma + 2\gamma A\theta(\gamma-1)^\gamma$$
$$+ 2\gamma(2\gamma-1)A^2\theta^2(\gamma-1)^\gamma + \frac{2\gamma}{\gamma-1}(\gamma-1)^\gamma + 4\gamma A\theta(\gamma-1)^\gamma + 4\gamma(2\gamma-3)A^2\theta^2(\gamma-1)^\gamma$$

Collecting terms and cancelling, one gets:

$$\Delta s = R\ln\{1\}^{1/\gamma-1} = R\ln 1 = 0$$

*Are you sure!*

$\Delta s = 0$ ; $\therefore$ $\Delta s$ must included the $\theta^3$ & $\theta^4$ terms.

& $\Delta s = A\theta^3 + \theta^4$ terms

$$A = \frac{\gamma(\gamma+1)}{12}\frac{M_1^6}{(M_1^2-1)^{3/2}}$$

c.

$$\sin\beta = \sin(\mu+\epsilon)$$
$$= \sin\mu\cos\epsilon + \sin\epsilon\cos\mu \qquad \text{for small } \epsilon \quad \sin\epsilon \sim \epsilon,\ \cos\epsilon \sim 1$$

$$\therefore \sin\beta = \sin\mu + \epsilon\cos\mu$$

$$\sin\beta = \frac{1}{M_1} + \epsilon\,\frac{\sqrt{M_1^2-1}}{M_1}$$


$$M_1\sin\beta = 1 + \epsilon\sqrt{M_1^2-1}$$

$$M_{1n}^2 = \left(1+\epsilon\sqrt{M_1^2-1}\right)^2 = \left(1 + 2\epsilon\sqrt{M_1^2-1} + \epsilon^2(M_1^2-1)\right) \qquad \text{if } \epsilon \ll \mu \quad \epsilon^2 \ll \epsilon$$

take it as 0

$$\therefore \quad \epsilon = \frac{M_{1n}^2 - 1}{2\sqrt{M_1^2-1}}$$

$$\text{but} \quad \tan\theta\ \frac{\left(M_1^2[\gamma+\cos 2\beta]+2\right)}{2\cot\beta} = M_{1n}^2 - 1 \qquad \text{if } \theta \ll 1$$

$$\therefore \frac{(\gamma+1)\theta}{2}\,\frac{M_1^2}{\sqrt{M_1^2-1}} + \theta^2 \text{ terms} = M_{1n}^2 - 1$$

$$\epsilon = \frac{\gamma+1}{4}\,\frac{M_1^2}{M_1^2-1}\,\theta + \theta^2 \text{ terms}$$

d.

$$\frac{u_2}{u_1} = \frac{(\gamma-1)M_1^2\sin^2\beta + 2}{(\gamma+1)M_1^2\sin^2\beta} \qquad M = u/a$$

?

$$\frac{M_2 u_2}{M_1 u_1} = \frac{(\gamma-1)(1+A\theta)+2}{(\gamma+1)(1+A\theta)} = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)(1+A\theta)}$$

$$= 1 - 2\,\frac{(1+A\theta) - 2}{(\gamma+1)(1+A\theta)} = 1 - \frac{2A\theta}{(\gamma+1)(1+A\theta)}$$

*algebra error!*

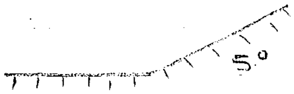
$$\therefore M_2 = M_1 - M_1\,\frac{(\cancel{1}+2A\theta)}{(\gamma+1)(1+A\theta)} = M_1 + (\quad)\theta$$

X *This is not true!*

$$\mu_2 = \sin^{-1}\frac{1}{M_2}; \quad \mu_1 = \sin^{-1}\frac{1}{M_1}$$

$\therefore \quad \mu_2 = \mu_1 + B\Theta$

2.




$\beta = 18^\circ$

$M_{1n} = 4 \sin\beta = 1.236 \qquad M_{2n} = .8183$

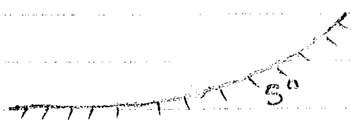
$M_2 = \dfrac{.8183}{\sin(\beta - \delta)} = \dfrac{.8183}{.225} = 3.64$

$\dfrac{\Delta p}{p_1} = p_2/p_1 - 1 = 1.627 - 1 = .627$

$\dfrac{\Delta \rho}{\rho_1} = \rho_2/\rho_1 - 1 = 1.411 - 1 = .411$

$\dfrac{\Delta M}{M_1} = \dfrac{3.64 - 4.00}{4} = -.09 \quad ; \quad \rho_2/\rho_1 = \dfrac{u_1}{u_2} = 1.411$

$\dfrac{\Delta u}{u_1} = \dfrac{u_2}{u_1} - 1 = .708 - 1 = -.292$ ✓




$\delta = 0 \qquad \nu(4) = 65.785 = c$

$\therefore \quad \nu(M_2) = 60.785 \qquad M_2 = 3.64+$

$p_2/p_{0_2} = .01076 \qquad p_{0_2} = p_{0_1} \qquad p_1/p_{0_1} = .006586$

$\rho_2/\rho_{0_2} = .03929 \qquad \rho_{0_2} = \rho_{0_1} \qquad \rho_1/\rho_{0_1} = .02766$

$p_2/p_1 = \dfrac{.01076}{.006586} = 1.64 \qquad p_2/p_1 - 1 = .64$

$\rho_2/\rho_1 = \dfrac{.03929}{.02766} = 1.42 \qquad \rho_2/\rho_1 - 1 = .42$

$\dfrac{u_2}{u_1} = \dfrac{1}{1.42} = .705 \qquad \dfrac{u_2}{u_1} - 1 = -.295 \qquad \dfrac{\Delta M}{M_1} = -.09$ ✓


$\nu(4) = 65.785 = c \; ; \quad \nu - \theta = c$

$\nu(M_2) = 70.785 \qquad M_2 = 4.41$

$\dfrac{M_2}{M_1} - 1 = \dfrac{.41}{4} = .1+$

$p_2/p_{0_2} = .003868 \qquad \rho_2/\rho_{0_2} = .01892$

$p_1/p_{0_1} = .006586 \qquad \rho_1/\rho_{0_1} = .02766$

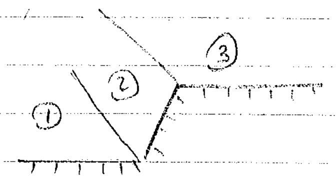
$p_2/p_1 = \frac{.003868}{.006586} = .587$ $\qquad \frac{\Delta p}{p_1} = -.413$

$\rho_2/\rho_1 = \frac{.01892}{.02766} = .684$ $\qquad \frac{\Delta \rho}{\rho_1} = -.316$

$u_2/u_1 = 1.46$ $\qquad \frac{\Delta u}{u_1} = .46$ ✓

d. Since flow is isentropic, same values appear.

3.




$\beta = 13.2°$ $\qquad \theta = 5°$

$M_{1n} = M_1 \sin\beta = 1.35$ $\qquad M_{2n} = .7618$

$M_2 = \frac{.7618}{.139} = 5.48$ $\qquad p_2/p_1 = 1.96$ $\qquad \rho_2/\rho_1 = 1.603$

$\theta = -5°$

$\nu(5.48) = 81.084$ $\qquad \nu(M_3) = 86.084$ $\qquad M_3 = 6.17$

$\frac{\Delta M}{M_1} = \frac{M_3 - M_1}{M_1} = \frac{.17}{6} = .0283$

$p_3/p_{o_3} = .0005331$ $\qquad \rho_3/\rho_{o_3} = .004592$

$p_{o_3} = p_{o_2}$, $\rho_{o_3} = \rho_{o_2}$ $\qquad p_{o_2}/p_{o_1} = .9697$

$\frac{p_3}{p_1} \doteq 0.98$

$p_1/p_{o_1} = .0006334$ $\qquad p_3/p_1 = p_3/p_{o_3} \cdot \frac{p_{o_1}}{p_1} \cdot \frac{p_{o_3}}{p_{o_1}} =$

$= (5.331 \times 10^{-4})(.9697) \cdot \frac{1}{(6.334 \times 10^{-4})}$

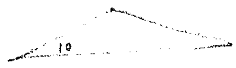
too low

$p_3/p_1 = .815$ $\qquad \frac{\Delta p}{p_1} = -.185$

$\rho_1/\rho_{o_1} = .005194$ $\qquad \rho_3/\rho_{o_3} \cdot \frac{\rho_{o_3}}{\rho_{2}} \cdot \frac{\rho_2}{\rho_1} = \frac{(.004592)(1.603)}{(.007697)} = .958$

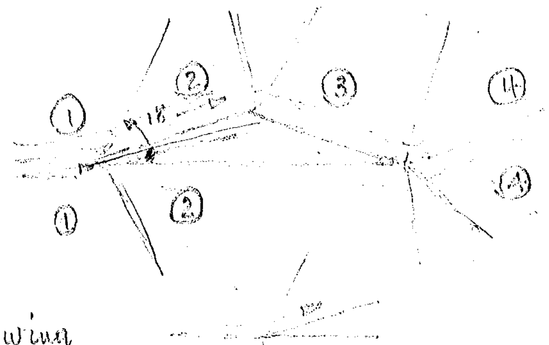
$\frac{\Delta \rho}{\rho_1} = -.042$ $\qquad \frac{u_3}{u_1} = 1.045$ $\qquad \frac{\Delta u}{u_1} = .045$ ✓

(1a)



$M_1 = 4$
$T_{0_1}$
$p_{0_1}$ ) given

find $p/p_1$ at all pts on body




above wing

$M_1 = 4$ flow must be turned $5°$ $\therefore$ $\beta = 18°$

$M_{1n} = 4 \sin 18° = 1.24$ @ $M_{1n} = 1.24$ $M_{2n} = .8183$

$p_2/p_1 = 1.627$ $T_2/T_1 = 1.153$ $p_{0_2}/p_{0_1} = .9884$

@ $M_1 = 4$ $p_1/p_{0_1} = .006586$ $M_2 = \dfrac{.8183}{\cancel{\sin 8°}\ \sin 13°} = \cancel{5.57}\ 3.64$ $\cancel{p_2/p_{0_2} = .9961 \times 10^{-3}}$

$\nu(\cancel{5.57}) = \cancel{81.798}\ 60.7$ $\nu(M_3) = \cancel{101.798}\ 80.7$ $M_3 = \cancel{9.81}\ 5.43$

$p_3/p_{0_3} = \cancel{.0000 2677}\ .001161$ $p_3/p_{0_3} \cdot \dfrac{p_{0_3}}{p_{0_1}} \cdot \dfrac{p_{0_1}}{p_1} = \dfrac{(\cancel{2.677 \times 10^{-4}}\ .001161)(.9884)}{.006586 \times \cancel{10^{-4}}} = \cancel{.402 \times 10^{-2}}\ .1745$

below the wing the flow is turned $5°$

$M_1 = 4$ $M_{1n} = 4 \sin 18° = 1.24$ $p_2/p_1 = 1.627$

(1) Assume horizontal flow $M_{3n} = M_3 \sin \cancel{14.7}\ 18.7° = \cancel{9.81}\ 5.43\ (\cancel{.300}\ .32) = \cancel{2.49}\ 1.74$

$p_4/p_3 = \cancel{7.067}\ 3.366$ $p_4/p_3 \cdot p_3/p_1 = \cancel{7.067}\ 3.366\ (\cancel{.00402}\ .1745) = \cancel{.028}\ .587$

$M_{4n} = .5140$ $M_4 = \dfrac{.5140}{\sin 10°} = \dfrac{.5140}{.174} = 2.96$

(2) Assume deflection from horizontal of $10°$ $\therefore$ $M_{3n} = M_3 \sin \beta$ $\beta = 26°$

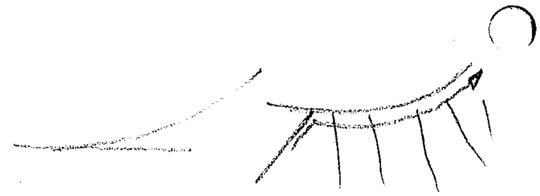
$M_{3n} = 4.3$ $p_4/p_3 \cdot p_3/p_1 = 21.41(.00402) = .085$

$\nu(5.57) = 81.798$ $\nu + \theta = c$ $\nu(M_4) = 71.798$ $M_4 = 4.496$

$\dfrac{p_4/p_{0_4}}{p_2/p_{0_2}} = \dfrac{.05945}{.0009961} =$

(2) if $\delta = 12°$ & $M_3 = 4.35$ $\beta = 23°$

$M_{3n} = 4.35 \sin 23°$



$M_1 = 3$

$p/p_1 = 1$

$y'(0) = \frac{1}{5}(1 - 2x)\Big|_0 = \frac{1}{5}$

slope $\frac{dy}{dx}\Big|_0 = \frac{1}{5}$ $\therefore$ $\tan\Theta = \frac{1}{5}$ $\Theta = 11°20'$

$\beta = 28°40'$

$M_2 = \frac{M_{2n}}{\sin 17°20'} = \frac{.7235}{.298} = 2.43$

$M_{1n} = 3 \sin 28°40' = 1.44$ $M_{2n} = .7235$

$p_2/p_1 = 2.253$

$\nu + \Theta = C$ $\Delta\Theta = -11°20' = -11.333$

$\Delta\nu = +11.333$

$\nu(M_2) = 37.469$ $\nu(M_3) = 48.802$

11.333

$a_1 M_1 = V_1$

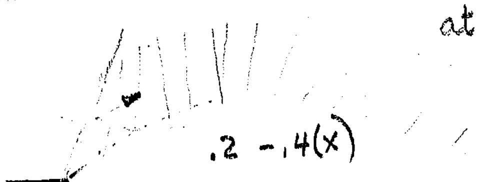
@ $M_1 = 3$ $p_1 = ?$

$(M_3) = 2.952$

$p_2/p_1$

at $0 \le x \le 1$

at $x = 0$ after shock $p_2/p_1 = 2.253$



$.2 - .4(x)$

$M_2 = 2.43$ $\nu(M_2) = 37.469$

$\Theta(M_2) = 11.333$

$\frac{dy}{dx} = \left(\frac{1}{5} - \frac{2}{5}x\right)$

$C(M_2) = 48.802$

at $x = .1$ $\tan\Theta = .16$ $\Theta = 9.125$

$\nu = 48.802 - 9.125 = 39.677$

$M_3 \approx 2.523$

$\mu = \sin^{-1}\frac{1}{2.523}$

$p_3/p_2 = \frac{p_3/p_{03}}{p_2/p_{02}} = \frac{.5647 - 1}{.6527 - 1} = .865$

$p_3/p_2 \cdot p_2/p_1 = 2.1$

at $x = .2$ $\tan\Theta = .12$ $\Theta = 6.835$ $\nu = 41.967$

$M_4 = 2.625$ $p_4/p_2 \cdot p_2/p_1 = \frac{.4822 - 1}{.6527 - 1}\ 2.253 = 1.665$

889
2310

38
9784
4822
167
831

137
10486
223

(1-b)

$M_2 = 3.64$ $\nu = 60.74$ $\theta = 0$ $p_2/p_{o_2} = .1076-1$

$M_4 = 4.02$ $\nu = \cancel{66.00}$ $65.95$ $\theta = -5\frac{1}{4}°$ $p_4/p_{o_4} = .6413-2$

$$\frac{p_4}{p_2}\cdot\frac{p_2}{p_1} = \frac{.6413-2}{.1076-1}(1.627) = .97$$

---

$M_3 = 5.43$ $\delta = 15.25°$ $\beta = 24°$

$.407$

$M_{3n} = 5.43(.407) = 2.215$ $p_4/p_3 = \cancel{5.531}$ $5.557$

$$p_4/p_1 = (\cancel{5.531}\ 5.557)(.1745) = \cancel{.965}\ .97$$

---

flow is $5\frac{1}{4}°$ from horizontal $5\frac{1}{4}°$

$5.557$

6456

---

~~Assume 5° from horiz~~

~~$M_2 = 3.64$ 60.7 $\theta = 0$ $p_2/p_{o_2} = .1076-1$~~

~~$M_4 = 3.993$ 65.7 $\theta = -5$ $p_4/p_{o_4} = .6645-2$~~

~~$p_4/p_1 = 1.005$~~

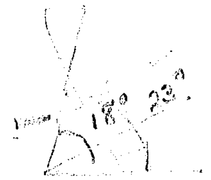
~~$\beta = 23.6°$ .404~~

~~$M_{3n} = 2.19$ $p_4/p_3 = 5.429$~~

$\nu + \theta = C$

$M_{3_n} = M_3 \sin(\beta - \theta)$

1




flow must be turned 5° $\therefore$ $\delta = 5$ $\quad \theta = 18°$ wrt 5°

$\therefore$ $\theta = 23°$ wrt horiz

$M_1 = 4 \qquad M_{1n} = 4 \sin 23° = 4(.3907) = 1.5628$

@ $M_{1n} = 1.563 \qquad p_2/p_1 = 2.684 \qquad M_{2n} = .6800 \qquad p_{o_2}/p_{o_1} = .9086$

@ $M_1 = 4 \qquad p_1/p_{o_1} = .006586 \qquad M_2 = \dfrac{.6800}{\sin 13°} = 3.02 \qquad p_2/p_{o_2} = .2642 \times 10^{-1}$

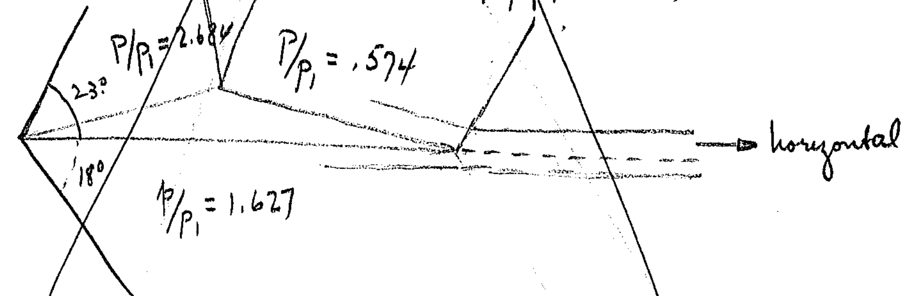
$\nu(3.02) = 50.142 \qquad \Delta\theta = -20° \qquad \Delta\nu = 20° \quad \therefore \quad \nu(M_3) = 70.142$

$M_3 \approx 4.352 \qquad p_3/p_{o_3} = .4165 \times 10^{-2}$

$$p_3/p_1 = p_3/p_{o_3} \cdot \overset{1}{\frac{p_{o_3}}{p_{o_2}}} \cdot \frac{p_{o_2}}{p_{o_1}} \cdot \frac{p_{o_1}}{p_1} = \frac{(.4165 \times 10^{-2})(.9086)}{.6586 \times 10^{-2}} = .574$$

below : deflection = 5° $\qquad \beta = 18°$ wrt horizontal.

$M_1 = 4 \qquad M_{1n} = 4 \sin 18° = 1.24 \qquad p_2/p_1 = 1.627$




if deflect. is 10° $\qquad M_{3_n} = 4.352 \sin 21.0 = 1.605 \qquad p_4/p_3 = \cancel{4.5}\ 2.838$

$p_4/p_1 = \cancel{2.58}$ 1.627

4.35
.46
1.240
.2610
2.001 ~ 2.0

Flow comes out horizontal.

(2) at $x = .3$ $\tan\theta = .08$ $4°35' = \theta = 4.583$

$\nu = 48.802 - 4.583 = 44.219$ $M_5 \approx 2.728$

$\frac{166}{214} \approx .8$ $P_5/P_{o_5} = .4115-1$ $P_5/p_1 = 1.357$

at $x = .4$ $\tan\theta = .04$ $\theta = 2.333$

$\nu = 46.469$ $M_6 = 2.835$ $P_6/P_{o_6} = .3494-1$

$P_6/p_1 = 1.206$

at $x = .5$ $\tan\theta = 0$ $\theta = 0$ $\nu = 48.802$ $M_7 = 2.951$ $\frac{19}{200}$

$P_7/P_{o_7} = .2930-1$ $p_7/p_1 = 1.011$

at $x = .6$ $\tan\theta = -.04$ $\theta = -2.333$ $\nu = 46.469$ $P/p_1 = 1.206$

at $x = .7$ $\tan\theta = -.08$ $\theta = -4.583$ $\nu = 44.219$ $P/p_1 = 1.357$

at $x = .8$ $\tan\theta = -.12$ $\theta = -6.835$ $\nu = 41.967$ $P/p_1 = 1.665$

at $x = .9$ $\tan\theta = -.16$ $\theta = -9.125$ $\nu = 39.677$ $P/p_1 = 2.1$

at $x = 1$ $\tan\theta = -.2$ $\theta = -11.333$ $\nu = 37.469$ $P/p_1 = 2.25$

$M_{x=1} = 2.43$ turn back to horiz flow $\theta = 11.333$

$\beta = 34°$ $M_n = 2.43 \sin 34°$ $.559(2.43) = 1.36$

$P_{aft}/P_{bef} = 1.991$ $\therefore$ we must $\frac{P_{bef}}{P_1} = 2.253$

$.6527-1$ 2

$.6527-1$

$\frac{33}{168}$

at $x = .5$ $\tan\theta = 0$ $\theta = 0$ $\nu = 48.802$ $M_7 = 2.951$

$\frac{45}{161} \approx \frac{1}{4}$ $P_7/P_{o_7} = .2930-1$ $p_7/p_1 = 1.011$

$\nu - \theta = C$

at $x = .6$ $\tan\theta = -.04$ $\theta = -2.333$ $\nu = \begin{array}{r} 48.802 \\ 2.333 \\ \hline 51.135 \end{array}$ $M_8 = 3.072$

$P_8/P_{o_8} = .2443-1$ $P_8/p_1 = .844$

at $x = .7$ $\tan\theta = -.08$ $\theta = -4.583$ $\nu = \begin{array}{r} 48.802 \\ 4.583 \\ \hline 53.381 \end{array}$ $M_9 = 3.195$

$P_9/P_{o_9} = .2038-1$ $P_9/p_1 = .704$

at $x = .8$ $\tan\theta = -.12$ $\theta = -6.835$ $V = 48.802$ $M_{10} = 3.323$

$\frac{13}{170}$ $6.835$

$55.637$

$P_{10}/P_{B_{10}} = .1690-1$ $P_{10}/P_1 = .583$

at $x = .9$ $\tan\theta = -.16$ $\theta = -9.125$ $V = 48.802$ $M_{11} = 3.462$

$9.125$

$57.927$

$\frac{29}{167}$ $P_{11}/P_{B_{11}} = .1383-1$ $P_{11}/P_1 = .477$

$V = 48.802$ $M_{12} = 3.603$

at $x = 1.0$ $\tan\theta = -.2$ $\theta = -11.333$ $11.333$

$60.135$

$\frac{44}{188}$ $P_{12}/P_{B_{12}} = .1134-1$ $P_{12}/P_1 = .392$

if horiz

let $\theta = 11\tfrac{1}{3}^\circ$ $\beta = 25.\cancel{4}^\circ$

$M_{12_n} = M_{12}\sin 25.\tfrac{1}{4}^\circ = 3.603(.426) = 1.53$ $P_A/P_B = 2.\cancel{582}$ $\sqrt{64}$

$P_A/P_B \cdot P_{12}/P_1 = P_A/P_1 = .392(\cancel{2.582}\ 2.564) = \cancel{1.01}$

flow comes out at an angle to horiz

let $\theta = 11^\circ$ $\beta = 24.8^\circ$

$1/6^\circ$ $M_{12_n} = M_{12}\sin 24.8^\circ = (3.603)(.42) = 1.471$ $P_A/P_B = 2.358$

$P_A/P_B \cdot P_{12}/P_1 = 2.358(.392) = .925$

below $\delta = 1/3^\circ$ $\beta = 19.8^\circ$

$M_{in} = 3$ $M_n = M_{in}\sin 19.8^\circ = 3(.339) = 1.02$ $P_A/P_B = 1.047$

$1/4^\circ$ if $P_A/P_B = 2.55 \cdot (.392) = \frac{21}{35}$ $\frac{3}{5}$ $1.526$

$\sin\beta = \frac{1.526}{3.603} = .424$ $\beta = 25^\circ 05$ $25.1$

$1.2$

$\frac{1.023}{1.0218}$ deflection is about $.04^\circ$ off horizontal. $\frac{14}{27}$

if $\theta = 11\tfrac{1}{4}$ $\beta = 25.1^\circ$ $\frac{271}{4}$ $\cancel{4242}$

$M_{B_n} = M_B\sin 25.1^\circ = 1.526$ $11$ $P_A/P_B = 2.55$ $P_A/P_1 = 1.00$

$M_{in} = 3$ $\theta = 1/4^\circ$ $\beta = 19.7^\circ$ $3(.337) = \cancel{1.001} = M_{in_n}$ $P_4/P_3 = 1.023$

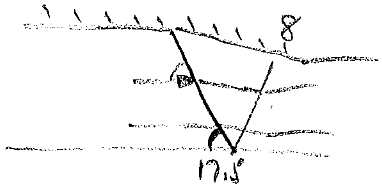
$2.(3)$ $.3365$ $1.0095$

.426 $M_n = 1.532$ $p_4/p_3 = 2.571$ 1007 $19°30' = M_{1n} = 1$

$M_{1n} = \frac{1.003}{3} \sin\beta$

(3)

$\beta = 17.5$




$$M_2 = \frac{M_{2n}}{\sin(9.5)} = \frac{.6999}{.165} = 4.24$$

$M_{1n} = 5\ (.3007) = 1.5035 = 1.50/3$

$35/3 = 12$

$p_2/p_1 = 2.470$ $M_{2n} = .6999$ 2.470

35

$M_2 = 4.24$ $\delta = 8°$ $\beta \approx 19.65$ $\frac{13}{20}$ .63 3

$M_{2n} = 4.24 \sin(\cancel{.3867})$ 19.65 $= 1.424$

$p_3/p_2 = 2.199$ $M_{3n} = .7338$ $M_3 = \frac{.7338}{\sin 11.65} = \frac{.7338}{\cancel{.219}}$ .202

$p_3/p_1 = \frac{p_3}{p_2} \cdot \frac{p_2}{p_1} = 2.199\ (2.470) = 5.43$ $\cancel{3.35} = M_3$ 3.63

If correct: $p_2/p_1 = 2.470$ $p_3/p_1 = 5.43$

$M_2 = 4.24$ $M_3 = 3.35$