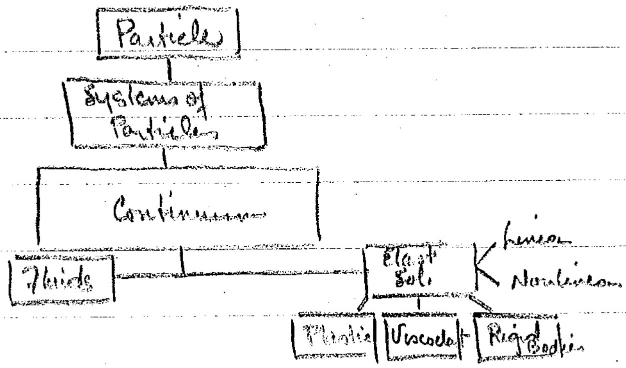
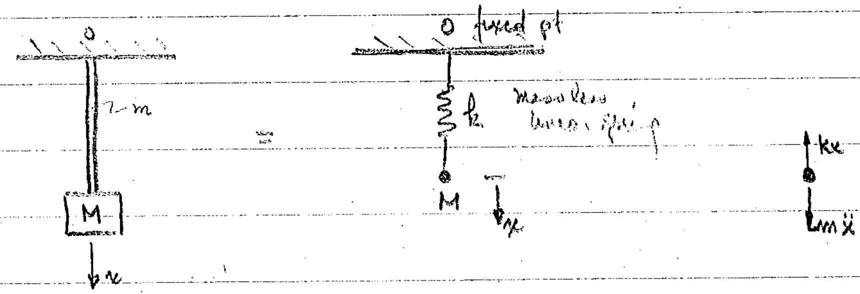
AE 212   Drs. Nardo & Goldberg

Mathematical Model

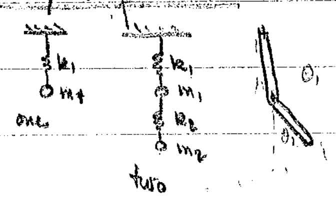


Example



$kx + m\ddot{x} = 0$

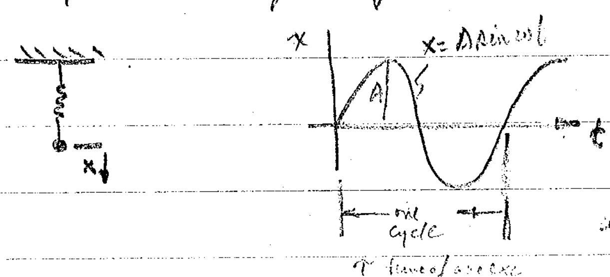
Degrees of Freedom



Free or Forced Vibration System.

Linear or Non Linear System.

Damped or Undamped System



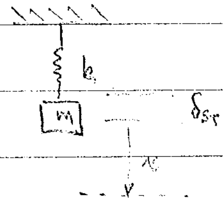
$(x(0)) = 0 \quad \dot{x}(0) = v_0$

$\omega = \sqrt{k/m}$ circular freq

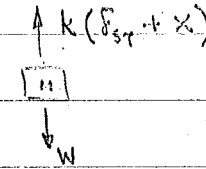
$\omega = 2\pi f$

$\boxed{Tf = 1}$

AE 212 Free Vibrations



$W = k\,\delta_{st}$



$$\frac{W}{g}\ddot{x} = W - k(\delta_{st} + x)$$

$$m\ddot{x} = W - k\delta_{st} - kx = -kx \qquad m\ddot{x} + kx = 0$$

Concerned with small motions about equilib. position

$$\ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{K/M}$$

let $x = Ce^{st}$ $\qquad Cs^2e^{st} + \omega^2 C e^{st} = 0$

$$(s^2 + \omega^2)Ce^{st} = 0$$

$s^2 + \omega^2 \qquad s = \pm\omega i \qquad x = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$

coeff can be complex $\quad C_1 = A_1 + iB_1 \qquad C_2 = A_2 + iB_2$

$$x = (A_1 + iB_1)e^{i\omega t} + (A_2 + iB_2)e^{-i\omega t}$$

$$x = \left[(A_1 + A_2)\cos\omega t - (B_1 - B_2)\sin\omega t\right] + i\left[(A_1 - A_2)\sin\omega t + (B_1 + B_2)\cos\omega t\right]$$

since x is real the the complex argument must be zero since trig fns are real they must be indep

$$(A_1 - A_2)\sin\omega t + (B_1 + B_2)\cos\omega t = 0 \qquad \therefore\ A_1 = A_2 \quad B_1 = -B_2$$

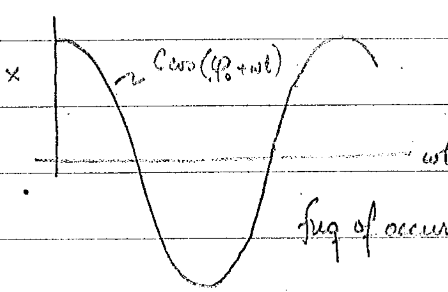
$$x = \left[A\cos\omega t + B\sin\omega t\right] \quad \text{if } t = 0 \quad x = x_0,\ \dot{x} = v_0$$

$$x_0 = A \qquad v_0 = B\omega \qquad B = \frac{v_0}{\omega} \qquad x = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

can be defined as $x = C\cos(\varphi_0 + \omega t)$

$$x_0 = C\cos\varphi_0 \qquad \frac{v_0}{\omega} = -C\sin\varphi_0 \qquad \left[x_0^2 + \left(\frac{v_0}{\omega}\right)^2\right]^{1/2} = C$$

$$\tan\varphi_0 = \frac{-v_0}{\omega x_0}$$



max height is amplitude

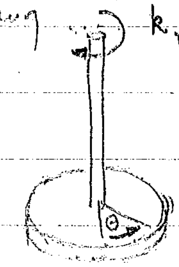
$$\tau = \frac{2\pi}{\omega}$$

freq of occurrence $= \frac{1}{\tau} = \frac{\omega}{2\pi}$

$\dot{x} = v = -C\omega \sin(\varphi_0 + \omega t)$

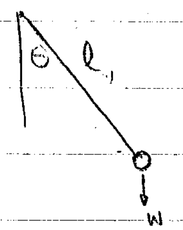
$\ddot{x} = a = -C\omega^2 \cos(\varphi_0 + \omega t)$

Torsional Spring $\quad k_T\theta = T$ $\qquad \delta = \frac{PL}{AE} \qquad \theta = \frac{TL}{JG}$



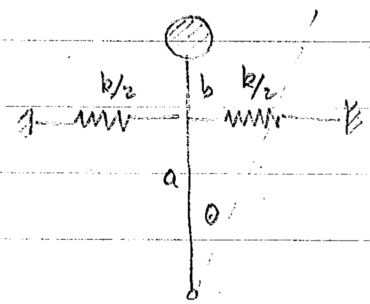
$$I\ddot{\theta} = -T = -K_T\theta$$

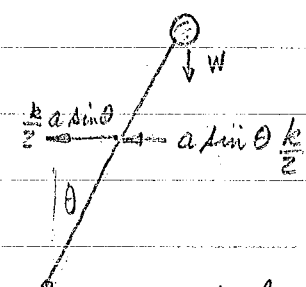
$$I\ddot{\theta} + K_T\theta = 0 \qquad \omega^2 = \frac{K_T}{I}$$



$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$





moment of inertia about a pt is $I_0 + ml^2$

if $I_0 \sim 0$ then $ml^2\ddot{\theta} = Wl\sin\theta - (ka\sin\theta)a$

$$ml^2\ddot{\theta} = (Wl - ka^2)\theta \qquad \text{if } \sin\theta \sim \theta$$

$$\ddot{\theta} + \left(\frac{ka^2 - Wl}{ml^2}\right)\theta$$

$\theta = C\cos(\varphi + \omega t)$ where $\omega = \left(\frac{ka^2 - Wl}{ml^2}\right)^{1/2}$ $\quad ka^2 > Wl$

if $ka^2 = Wl \qquad ml^2\ddot{\theta} = 0$

$$\theta = C_1 t + C_2$$

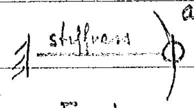
if $ka^2 < Wl$

$$ml^2\ddot{\theta} - \frac{(Wl - ka^2)}{ml^2}\theta = 0$$

$$\theta = C_1 \cosh Pt + C_2 \sinh(Pt)$$

where $P^2 = \dfrac{Wl - ka^2}{ml^2}$

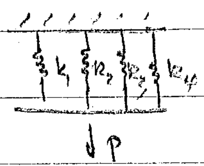
assumption: moment of mass in linear direct.



$F = kx$

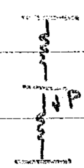
$$\delta = \frac{PL^3}{3EI} \qquad \frac{F}{x} = k = \frac{P}{\delta} = \frac{3EI}{l^3}$$

H.W. 2, 8, 11, 23 look at last part P31-33 on Energy methods.



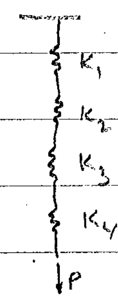
$$P = \Delta K_2 + \Delta K_3 + \Delta K_4 + \Delta K_1 + \dots$$

$$= \Delta(K') \qquad K' = \sum_{i=1} K_i$$



in parallel

Δk_1 Δk_2 are same.



$$P = K_1\Delta_1 = K_2\Delta_2 = K_3\Delta_3 = \dots$$

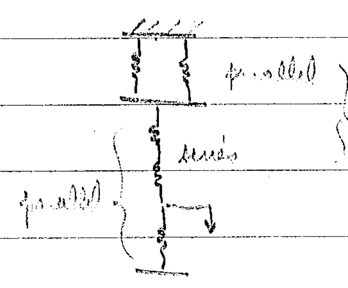
$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$$

since $P = k'\Delta$

$$\frac{P}{K'} = \frac{P}{K_1} + \frac{P}{K_2} + \frac{P}{K_3} + \frac{P}{K_4}$$

$$\frac{1}{K'} = \sum_i \frac{1}{K_i}$$

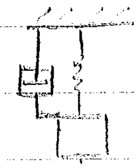
Development of Equivalent Spring

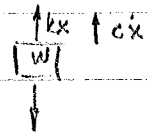


9-28-71

Energy of System $T+V=C$ conservative system.

$$\frac{d}{dt}(T+V)=0$$





$$m\ddot{x}=-kx-c\dot{x}$$

$$m\ddot{x}+kx+c\dot{x}=0$$

$$\ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=0$$

Assume $x=e^{st}$ $\quad \left(s^2+\frac{c}{m}s+\frac{k}{m}\right)e^{st}=0$

$$s^2+\frac{c}{m}s+\frac{k}{m}=0 \qquad s=-\frac{c}{2m}\pm\sqrt{\left(\frac{c}{2m}\right)^2-\left(\frac{k}{m}\right)}$$

$x=Ae^{-s_1t}+Be^{-s_1t}$ $\quad$ when $\frac{c}{2m}=\sqrt{\frac{k}{m}}$ then $s_1=s_2$

if $s_1=s_2$ $\quad (A+Bt)e^{-st}$ $\quad s=\frac{c}{2m}$ $\quad$ threshold solution, crit damp

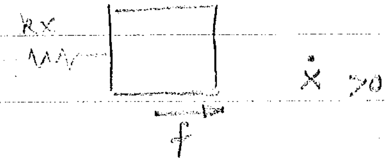
look at $\frac{c}{m}>2\sqrt{\frac{k}{m}}$ exponential $\quad \frac{c}{m}<2\sqrt{\frac{k}{m}}$ $\quad s_1=\alpha+i\beta$, $s_2=\alpha-i\beta$

$$x=Ae^{-(\alpha+i\beta)t}+Be^{-(\alpha-i\beta)t}$$

$e^{-\alpha t}\left(Ae^{-i\beta t}+Be^{i\beta t}\right)$ oscillating motion.

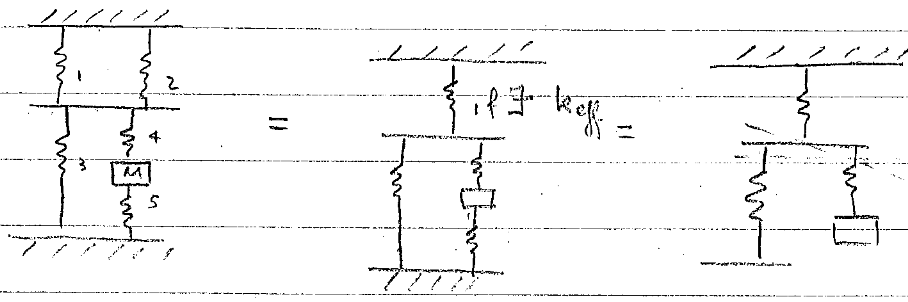
damping changes cycle time & decreases amplitude.

Coulomb Damping

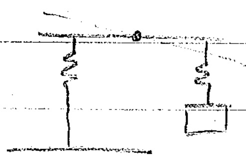


$m\ddot{x}+kx=F_d$ $\qquad\qquad$ $m\ddot{x}+kx=-F_d$

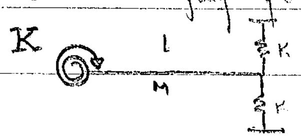
9-30-71



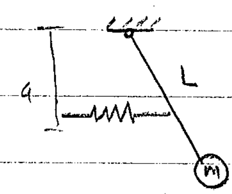
Consider



Problem 2-32, 33 uniform stiff rod to move vertically using linear & torsional spring Calculate freq of vertical Oscill



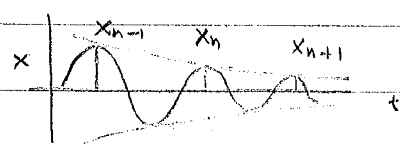
$$\omega_n = \sqrt{\frac{3K+6kL^2}{ML^2}}$$



$$\omega_n = \sqrt{\frac{mgL+ka^2}{mL^2}}$$

3.5, 3.6, 3.12, 3.16

Damped Motion



$\frac{X_n}{X_{n-1}}$ rate of decay

$$\left(\frac{X_n}{X_{n-1}}\right)\left(\frac{X_{n+1}}{X_n}\right) = \frac{X_{n+1}}{X_{n-1}}$$

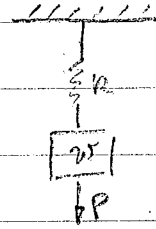
$$\delta = \ln \frac{X_{n-1}}{X_n}$$

$$X_{n-1} = \sqrt{1-\zeta^2}\ X\ e^{-\zeta\omega t_{n-1}}$$

$$X_n = \sqrt{1-\zeta^2}\ X\ e^{-\zeta\omega(t_{n-1}+\tau)}$$

$$\frac{X_{n-1}}{X_n} = e^{\zeta\omega\tau}$$

$$\delta = \zeta\omega\tau = \ln \frac{X_{n-1}}{X_n}$$



$m\ddot{x} + kx = P$ if $P = P(t)$ motion

$P = P_0 \sin \omega_f t$

$\ddot{x} + \omega^2 x = \frac{P_0}{m} \sin \omega_f t$

$x = A \sin \omega_f t$ $\left[-A\omega_f^2 + A\omega^2 - P_0/m \right] \sin \omega_f t = 0$

particular soln.

$A = \frac{P_0}{m(\omega^2 - \omega_f^2)}$ $x = \frac{P_0}{m(\omega^2 - \omega_f^2)} \sin \omega_f t$ when $\omega \neq \omega_f$

if $\frac{\omega_f}{\omega} = r$ then $x = \frac{P_0}{m(1-r^2)\omega^2} \sin \omega_f t$ ω depends on system
ω_f frequency of applied load

if $r < 1$ situation is well ordered $A < \infty$

if $r > 1$ $x = \frac{P_0}{m\omega^2(r^2-1)} (-\sin \omega_f t)$ $A < \infty$

if $r = 1$ $\omega = \omega_f$ resonant frequency. $A \to \infty$

$x = C_1 \sin \omega t + C_2 \cos \omega t + \frac{P_0/m}{\omega^2 - \omega_f^2} \sin \omega_f t$

$x = x_0$ $\dot{x} = \dot{x}_0$ $t = t_0$ $C_2 = x_0$

$C_1 = \frac{\dot{x}_0 - P_0/m\,(\omega^2 - \omega_f^2)^{-1}\omega_f}{\omega}$

$$x = \left[\frac{\dot{x}_0}{\omega} - \frac{P_0/m}{\omega^2 - \omega_f^2} \frac{\omega_f}{\omega} \right] \sin \omega t + x_0 \cos \omega t + \frac{P_0/m}{\omega^2 - \omega_f^2} \sin \omega_f t$$

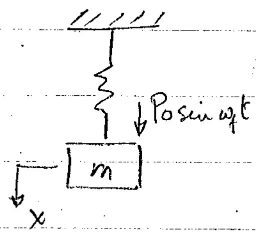
$\lim_{\omega \to \omega_f} x =$ using $\omega - \omega_f = 2\Delta$

$(\omega - \omega_f)^2 = 4\Delta^2$ $\omega^2 - \omega_f^2 = 4\Delta^2 + 2\omega_f(2\Delta)$

$= 4\Delta(\Delta + \omega_f)$

10 - 5 - 71

Due One week from Thursday 4.2, 5, 12



$$m\ddot{x} + kx = P_0 \sin \omega_f t$$

$$x_p + x_c = x$$

$$\ddot{x} + \omega^2 x = \frac{P_0}{m} \sin \omega_f t$$

$\omega^2 = \frac{k}{m}$

$$x_p = A \sin \omega_f t$$

$\omega^2 m = k$

$$\left[A\left(-\omega_f^2 + \omega^2\right) - P_0/m\right] \sin \omega_f t = 0$$

$P_0 = k\delta_{st}$

$$A = \frac{P_0/m}{\omega^2 - \omega_f^2}$$

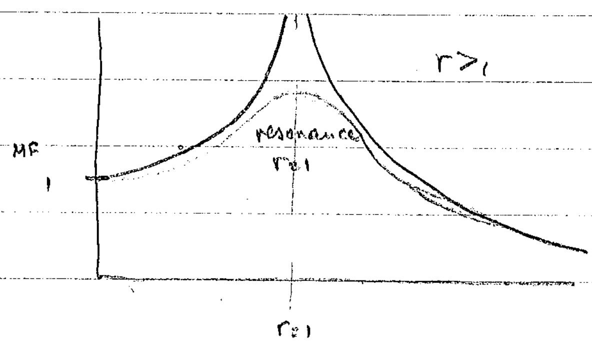
$\frac{P_0}{k} = \delta_{st}$ fictitious

$$x_p = \frac{P_0/m}{\omega^2\left(1 - \frac{\omega_f^2}{\omega^2}\right)} \sin \omega_f t$$

$$x_p = \frac{P_0}{m\omega^2 (1 - r^2)} \sin \omega_f t \qquad r = \frac{\omega_f}{\omega}$$

$$x_p = \frac{\delta_{st} \sin \omega_f t}{(1 - r^2)}$$

Mag Factor $= \frac{\delta_{st}}{(1-r^2)\delta_{st}} = \frac{1}{1-r^2}$ $\qquad \frac{\text{amplitude}}{\delta_{st}} = \text{M.F.}$



$r > 1 \qquad x_p = \frac{\delta_{st}}{r^2 - 1} - \sin \omega_f t$

$$x = M \cos \omega t + N \sin \omega t + \frac{P_0/k}{(1-r^2)} \sin \omega_f t$$

$x = x_0 \quad \dot{x} = \dot{x}_0 \quad t = 0$

$M = x_0 \qquad N = \frac{\dot{x}_0}{\omega} - \frac{(P_0/k)}{(1-r^2)}\frac{\omega_f}{\omega}$

$$x = x_0\cos\omega t + \frac{1}{\omega}\left(\dot{x}_0 - \frac{(P_0/K)\omega_f}{1-r^2}\right)\sin\omega t + \frac{P_0/K}{1-r^2}\sin\omega_f t$$

$$x = x_0\cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t + \frac{P_0/K}{1-r^2}\sin\omega_f t - \frac{(P_0/K)\omega_f}{1-r^2}\sin\omega t$$

$\omega - \omega_f = 2\Delta$

$\omega + \omega_f \approx 2\omega_f \approx 2\omega$ $\qquad \omega^2 - \omega_f^2 = \overbrace{(\omega-\omega_f)}^{2\Delta}\overbrace{(\omega+\omega_f)}^{2\omega_f} \approx 4\Delta\omega_f$

$\sin\omega_f t\cos 2\Delta t + \cos\omega_f t\sin 2\Delta t$

$$\frac{P_0/K\ \omega}{\omega^2-\omega_f^2}\left(\omega\sin\omega_f t - \omega_f\sin\omega t\right) = \frac{(P_0/K)\omega}{4\Delta\omega_f}\left(\omega\sin\omega_f t - \omega_f\sin(\omega_f t + 2\Delta t)\right)$$

if Δt is small $\quad \frac{(P_0/K)\omega}{4\Delta\omega_f}\left(\omega\sin\omega_f t - \omega_f\sin\omega_f t - \omega_f\cdot 2\Delta t\cos\omega_f t\right)$

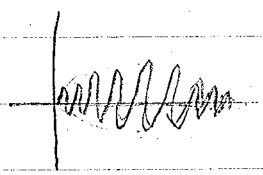
$\qquad 2\Delta\left(\sin\omega_f t - \omega_f t\cos\omega_f t\right)$

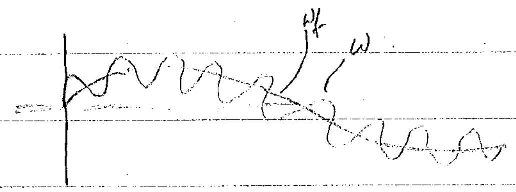
as $\omega \to \omega_f$ $\quad \frac{(P_0/K)\omega}{4\omega_f}\left[-2\omega_f t\cos\omega_f t\right]$ $\quad \therefore$

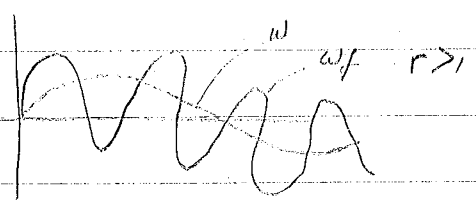
$$x = x_0\cos\omega_f t + \frac{\dot{x}_0}{\omega_f}\sin\omega_f t - \frac{(P_0/K)}{2}\omega_f t\cos\omega_f t$$ x = linear comb. of t & has

Beats

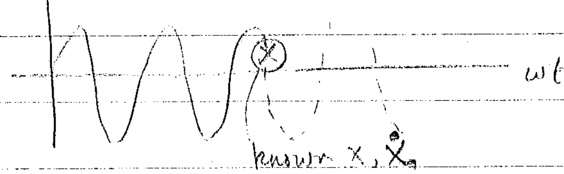
$$x = \left(-\frac{P_0/m}{2\omega_f\Delta}\cos\omega_f t\right)\sin\Delta t$$ If $\omega \sim \omega_f$ but $\omega \neq \omega_f$





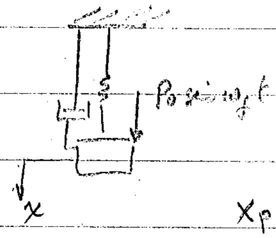


Suppose body under force freq has freq removed



$$\therefore\ x = x_1 \cos \omega_1 t + \frac{\dot{x}_1}{\omega_1} \sin \omega_1 t \qquad \omega = \omega_1 \qquad t \geq t_1$$

Damped Forced Vibrations



$$m\ddot{x} + c\dot{x} + kx = P_0 \sin \omega_f t$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \omega^2 x = (P_0/m) \sin \omega_f t$$

$$x_p = A \cos \omega_f t + B \sin \omega_f t$$

$$\dot{x}_p = -A\omega_f \sin \omega_f t + B\omega_f \cos \omega_f t$$

$$\ddot{x}_p = -A\omega_f^2 \cos \omega_f t - B\omega_f^2 \sin \omega_f t$$

$$-A\omega_f^2 + B\omega_f \frac{c}{m} + \omega^2 A = 0$$

$$-B\omega_f^2 - A\omega_f \frac{c}{m} + B\omega^2 = P_0/m$$

$$A(\omega^2 - \omega_f^2) + B\omega_f \frac{c}{m} = 0$$

$$B(\omega^2 - \omega_f^2) - A\omega_f \frac{c}{m} = P_0/m$$

$$A = \frac{\begin{vmatrix} 0 & \omega_f c/m \\ P_0/m & \omega^2 - \omega_f^2 \end{vmatrix}}{\Delta} \qquad B = \frac{\begin{vmatrix} \omega^2 - \omega_f^2 & 0 \\ -\omega_f \frac{c}{m} & P_0/m \end{vmatrix}}{\Delta}$$

$$\Delta = \begin{vmatrix} \omega^2 - \omega_f^2 & \omega_f \frac{c}{m} \\ -\omega_f \frac{c}{m} & \omega^2 - \omega_f^2 \end{vmatrix} = (\omega^2 - \omega_f^2)^2 + \omega_f^2 \frac{c^2}{m^2}$$

$$\frac{-P_0 \omega_f c}{(\omega^2 - \omega_f^2)^2 m^2 + \omega_f^2 c^2} = A \qquad B = \frac{(\omega^2 - \omega_f^2) P_0 m}{(\omega^2 - \omega_f^2)^2 m^2 + \omega_f^2 c^2}$$

$$x_p = D \sin(\omega_f t - \psi) = D \sin \omega_f t \cos \psi - D \cos \omega_f t \sin \psi$$

$$B = D \cos \psi \qquad A = -D \sin \psi$$

$$\sqrt{B^2 + A^2} = D \qquad \tan \psi = -\frac{A}{B}$$

$$D = \frac{P_0}{(\omega^2 - \omega_f^2)^2 m^2 + \omega_f^2 c^2} \left[(\omega^2 - \omega_f^2)^2 m^2 + \omega_f^2 c^2\right]^{1/2} = \frac{P_0/m}{(\omega^2 - \omega_f^2)} \frac{1}{\sqrt{1 + \frac{\omega_f^2 c^2/m^2}{(\omega^2 - \omega_f^2)^2}}}$$

$$D = \frac{P_0/M}{(\omega^2 - \omega_f^2)} \frac{1}{\sqrt{E}} \quad \text{where } E = \omega_f^2 c^2/m^2 \div (\omega^2 - \omega_f^2)^2$$

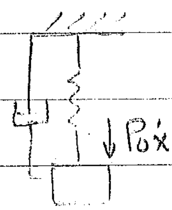
$$\tan\psi = \frac{\omega_f\, c/m}{\omega^2 - \omega_f^2} = \frac{\omega_f c}{(K - m\omega_f^2)}$$

$$\zeta = c/c_c \qquad r = \frac{\omega_f}{\omega} \qquad \omega^2 = \frac{K}{m} \qquad \frac{c_c}{2m} = \sqrt{\frac{K}{m}} = \omega$$

$$\tan\psi = \zeta\,\frac{2m\omega\omega_f}{(K - m\omega_f^2)} = \zeta \cdot \frac{2\omega\omega_f}{(\omega^2 - \omega_f^2)} = \frac{2\zeta r}{(1 - r^2)}$$

$$MF = \frac{D}{\delta_{st}} = \frac{P_0/m}{\omega^2\,\delta_{st}\,\sqrt{(1-r^2)^2 + 4\zeta^2 r^2}} = \frac{1}{\sqrt{(1-r^2)^2 + 4\zeta^2 r^2}}$$

negative damping flutter —



$$m\ddot{x} + c\dot{x} + kx = P_0\dot{x}$$

$$m\ddot{x} + (c - P_0)\dot{x} + kx = 0$$

if $P_0 > c$

$P_0 < c$ damped motion, free motion

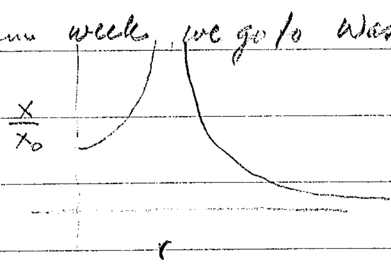
$P_0 = c$ no damping

$$X = e^{st} \qquad ms^2 + (c - P_0)s + k = 0 \qquad s^2 - \left(\frac{P_0 - c}{m}\right)s + \frac{k}{m} = 0$$

$$s = \frac{P_0 - c}{2m} \pm \sqrt{\left(\frac{P_0 - c}{2m}\right)^2 - \omega^2}$$

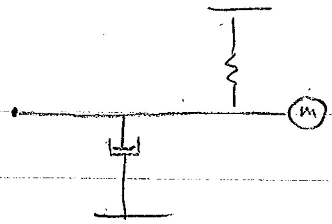
10-12-71

Exam week, we go to Wash D.C.



$$\omega = \sqrt{\frac{K}{m}} \rightarrow (X/X_0)$$

$$\omega_f > \omega$$



$\frac{a}{l} c\dot{x}_2 = c\frac{a}{l}\dot{x}\frac{a}{l}$ $\quad k x_1 \frac{b}{l} = k\frac{b}{l}x \cdot \frac{b}{l}$

using moments

$$\underbrace{-\frac{c}{2m}\left(\frac{a}{l}\right)^2}_{\text{damped motion}} \pm j\underbrace{\sqrt{\frac{k}{m}\left(\frac{b}{l}\right)^2 - \left(\frac{c}{2m}\left(\frac{a}{l}\right)^2\right)^2}}_{\text{frequency}}$$

if forcing fn leads x $\quad r>1$

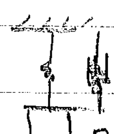
if " " is in phase with x $\quad r<1$

$$\left.\begin{array}{l} P = P_0 \sin \omega_f t \\ X = \bar{X} \sin \omega_f t \end{array}\right\} r<1$$

$$x = \bar{X}(-\sin \omega_f t) \; \} \; r>1$$

forced part only. in phase / lags.

$s^2 + as + b = 0$



$P = P_0\dot{x}$

$m\ddot{x} + (c - P_0)\dot{x} + kx = 0$ $\quad$ if $P > c$ divergent

$P = 0$ no damping

$P_0 < c$ damping.

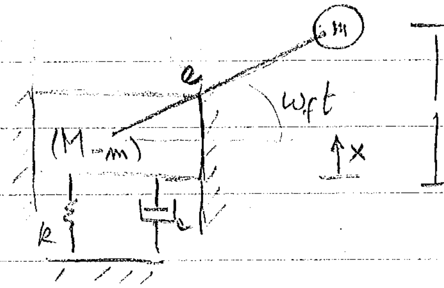
if $a, b > 0$ $\quad \alpha + \beta = a$ $\quad (s - \alpha)(s + \beta) = 0$

$\alpha\beta = b$ $\quad \therefore$ α & b must have negative sign

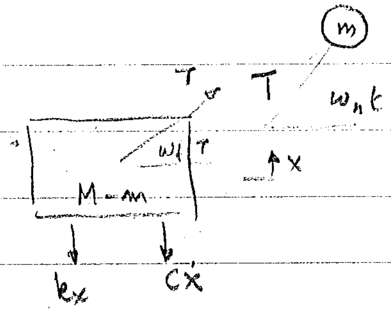
$(s - (\alpha + i\beta))(s - (\alpha - i\beta)) = s^2 - 2s\alpha + (\alpha^2 + \beta^2)$

$\therefore$ α is >0

If equat is $s^2 + as + b = 0$ & a & $b > 0$ motion is convergent



$$y = x + e\sin\omega_f t$$



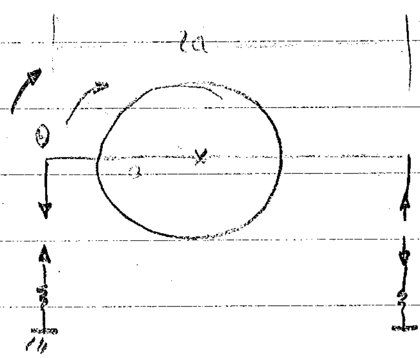
$$m\ddot{y} = -T\sin\omega_f t$$

$$(M-m)\ddot{x} = T\sin\omega_f t - kx - c\dot{x}$$

$$(M-m)\ddot{x} = -m\frac{d^2}{dt^2}(x + e\sin\omega_f t) - kx - c\dot{x}$$

$$M\ddot{x} + c\dot{x} + kx = me\omega_f^2\sin\omega_f t$$

Example:



$$I\ddot{\theta} = -2Fa$$

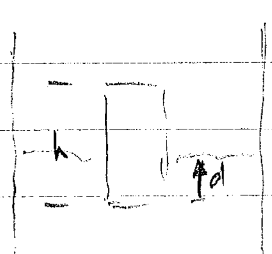
$$F = ka\theta \qquad F = k\Delta = ka\sin\theta \doteq ka\theta$$

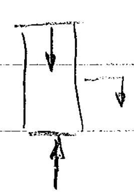
$$I\ddot{\theta} + 2Fa = 0$$

$$I\ddot{\theta} + 2a^2k\theta = 0$$

$$\omega = \sqrt{\frac{2a^2k}{I}} \text{ rad/sec}$$

Example:





$$W = Ahs(64.4)$$

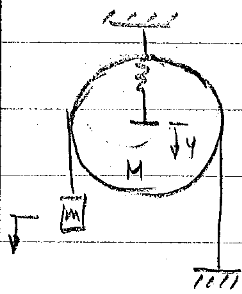
$$F = Ad(64.4)$$

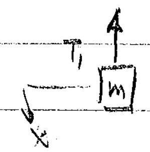
$$F = A(d+x)\,64.4$$

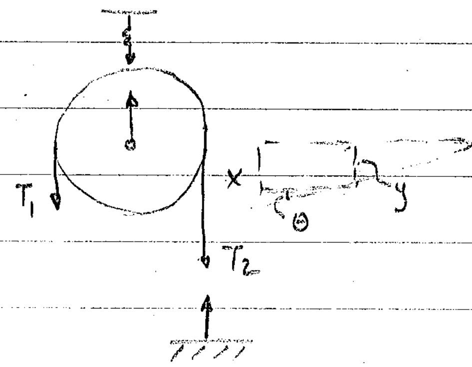
$$M\ddot{x} = W - A(d+x)\,64.4$$

$$\frac{Ahs(64.4)}{g}\ddot{x} = -A(64.4)x$$

$$\ddot{x} + \frac{g}{hs}x = 0 \qquad \omega = \sqrt{g/hs}$$







$$I\ddot{\theta} = r(T_1 - T_2)$$

$$M\ddot{y} = -k(y_0+y) + Mg + T_1 + T_2$$

$$m\ddot{x} = mg - T_1$$

$$y = \frac{x}{2} \qquad \frac{x}{2} = r\theta$$

$$T_1 = (mg - m\ddot{x}) \qquad T_2 = M\ddot{y} - Mg + k(y-y_0) - (mg - m\ddot{x})$$

$$I\ddot{\theta} = rmg - rm\ddot{x} - r\left[M\ddot{y} - Mg + k(y-y_0) - (mg - m\ddot{x})\right]$$

$$I\ddot{\theta} + rM\ddot{y} + 2rm\ddot{x} = rmg + rMg - rk(y-y_0) + rmg$$

in static equip $Mg + 2mg = k\delta$

$$I\ddot{\theta} + rM\ddot{y} + 2mr\ddot{x} + kry = 0$$

$$\frac{1}{2}Mr^2\left(\frac{\ddot{x}}{2r}\right) + rM\frac{\ddot{x}}{2} + 2rm\ddot{x} + kr\frac{x}{2} = 0$$

$$\frac{Mr\ddot{x}}{4} + \frac{rM\ddot{x}}{2} + 2rm\ddot{x} + kr\frac{x}{2} = 0$$

$$\frac{3}{2}M\ddot{x} + 4m\ddot{x} + kx = 0$$

freq $\left(\frac{2k}{8m+3M}\right)^{1/2} = \omega$

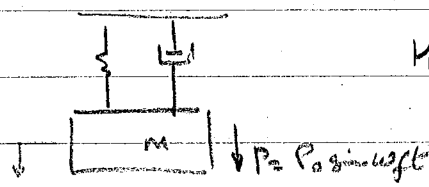
use energy methods

$V+T = C$

$\frac{1}{2} m\dot{x}^2 + \frac{1}{2} M\dot{y}^2 + \frac{1}{2} I\dot{\Theta}^2 = KE$

$\frac{1}{2} Ky^2 = PE$

Force of Transmission



$M\ddot{x} = F = -kx - c\dot{x}$

Support feels $kx + c\dot{x} = F$

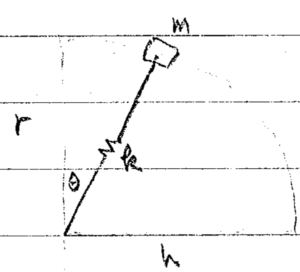
above becomes $M\ddot{x} = F = -kx - c\dot{x} + P_0 \sin \omega_f t$

Support still feels $F = kx + c\dot{x}$

$x = X \sin(\omega_f t + \psi)$ $\qquad X = \dfrac{X_0}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

$\psi = \left(\dfrac{2r\zeta}{1-r^2}\right)$

10-14-71



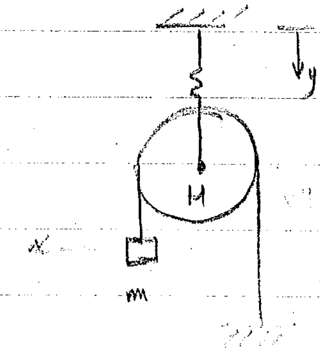
$KE = \frac{1}{2} mv^2$

$V = mg\, r\cos\theta + \frac{1}{2} K(l-r)^2$

$T_1 + V_1 = T_2 + V_2$

$0 + mgr + \frac{1}{2} K(l-r)^2 = \frac{1}{2} mv^2 + \frac{1}{2} K(l-h)^2$

where l is datum.



$$KE = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} M\dot{y}^2 + \frac{1}{2} I\dot{\theta}^2$$

$$V = -mgx - Mgy + \frac{1}{2}k(y+y_0)^2$$

since $\frac{d}{dt}(T+V) = 0$

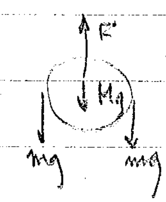
$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + \frac{I}{2}\dot{\theta}^2\right) - mg\dot{x} - Mg\dot{y} + \frac{1}{2}k(y+y_0)\dot{y} = 0$$

since $\frac{x}{2} = y$ & $r\theta = \frac{x}{2}$

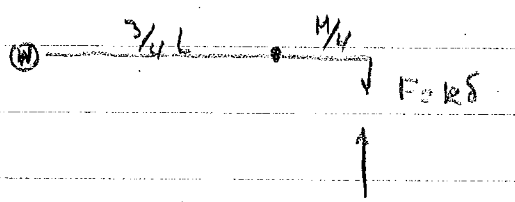
$$m\ddot{x}\dot{x} + M\dot{y}\ddot{y} + I\dot{\theta}\ddot{\theta} - mg\dot{x} - Mg\dot{y} + k(y+y_0)\dot{y} = 0$$

$$\dot{x}\left[\left(m\ddot{x} + \frac{M}{4}\ddot{x} + \frac{M}{8}\ddot{x}\right) + \left\{-mg + \frac{Mg}{2} + \frac{k}{2}(x/2 + y_0)\right\}\right] = 0$$

$$\frac{1}{2}\left[\underbrace{-2mg - Mg + ky_0}_{0}\right]$$ since



$Mg + 2mg = ky_0$

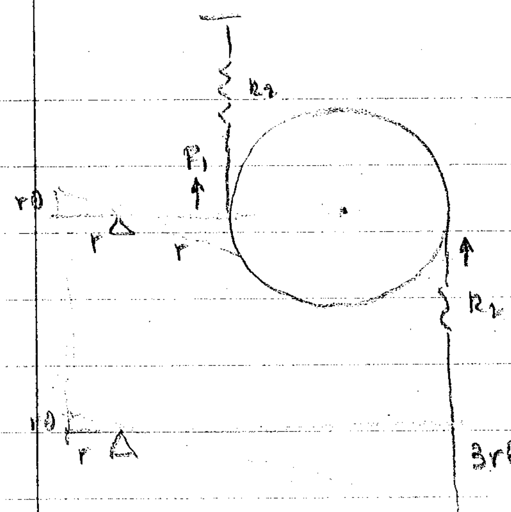


$$I\ddot{\theta} = \frac{W}{4}(3L) - \frac{L}{4}F$$

$$I\ddot{\theta} = \underbrace{\frac{3WL}{4} - \frac{L}{4}k\left(y_0\right.}_{\text{balance}} + \frac{L}{4}\theta\Big)$$

$$I\ddot{\theta} + \frac{KL^2}{4}\theta = 0$$

$$\frac{9}{16}L^2 m$$



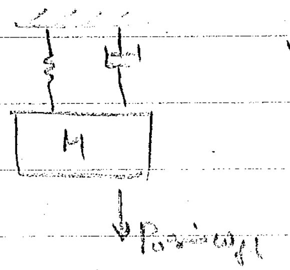
$I\ddot{\theta} = -F_1 r - F_2 r \qquad I = \frac{1}{2}Mr^2$

$F_1 = r\theta k_1$

$F_2 = 4r\theta k_2$

$\frac{1}{2}Mr^2\ddot{\theta} + r^2\theta k_1 + 4r^2\theta k_2 = 0$

Check



$F_{STR} = kx + c\dot{x}$

$m\ddot{x} + c\dot{x} + kx = P_0 \sin\omega_f t$

$x_p = X = \dfrac{X_0}{\sqrt{(1-r^2)^2 + 2\zeta r^2}} \sin(\omega_f t - \psi)$

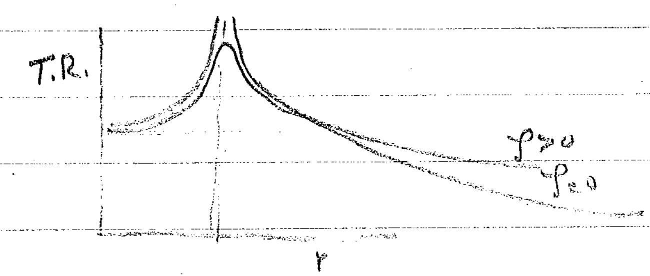
$\tan\psi = \dfrac{2r\zeta}{(1-r^2)}$

$F = kx + c\dot{x}$

$= \dfrac{X_0}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \left\{ k\sin(\omega_f t - \psi) + c\omega_f \cos(\omega_f t + \psi) \right\}$

$F = \dfrac{X_0}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sqrt{k^2 + (c\omega_f)^2} \sin(\omega_f t - \psi - \beta)$

$\dfrac{F_{Trans}}{P_0} = \dfrac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

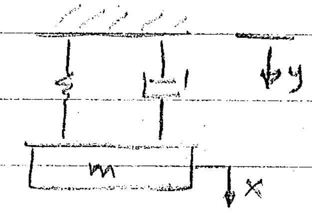


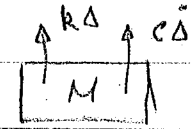
as $r \to \infty$ $T.R. \to 0$

$\therefore F_T \to 0 \Rightarrow$

$m\ddot{x} \to P_0 \sin\omega_f t$

oscillating supported





$\Delta = x - y$

$\dot\Delta = \dot x - \dot y$

$m\ddot x = -k(x-y) - c(\dot x - \dot y)$ if $y = y_0 \sin\omega_f t$

$m\ddot x + kx + c\dot x = ky + c\dot y$

let $z = x - y$ $\quad m(\ddot z + \ddot y) = -kz - c\dot z$

$$m\ddot z + c\dot z + kz = -m\ddot y = +\underbrace{m y_0 \omega_f^2}_{P_0} \sin\omega_f t \qquad z \to x$$

use material we had before z - relative motion

$\frac{X}{y_0}$ is now found as $\frac{F_T}{P_0}$ was before one is MF, TR

Fourier Series Solution

$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + a_k \cos kx + b_0 + b_1 \sin x$
$\quad + b_2 \sin 2x + \ldots + b_j \sin jx$

One form of orthogonal fns obtained from Sturm-Liouville Equation.

$$\int_0^{2\pi} \cos kx \cos nx \, dx = \begin{cases} \pi & n = k \\ 0 & n \neq k \end{cases}$$

$$\int_0^{2\pi} f(x)dx = \int_0^{2\pi} a_0 \, dx + a_1 \int_0^{2\pi} \cos x dx \ldots + b_1 \int_0^{2\pi} \sin x \, dx + \ldots$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

$$\int_0^{2\pi} f(x)\cos mx\,dx = a_0\int_0^{2\pi}\cos mx\,dx + a_k\int_0^{2\pi}\cos kx\cos mx\,dx$$

$$+ b_j\int_0^{2\pi}\sin jx\cos mx\,dx$$

10-19-71

$$P(z) = a_0 + \sum a_n \cos nz + \sum b_n \sin nz$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} P(z)\,dz \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} P\cos nz\,dz$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} P\sin nz\,dz$$

$m\ddot{x} + c\dot{x} + kx = P(t) = \sum_{n=1}^{\infty} P_n(t)$ suppose $= \sum b_n \sin \omega_f t$

$m\ddot{x} + c\dot{x} + kx = b_1 \sin \omega_f t$ $\qquad x_{p_1}$

$= b_2 \sin \omega_f t$ $\qquad x_{p_2}$

$$x = \sum_{i=1}^{\infty} x_{p_i}$$

$$m\ddot{x} + c\dot{x} + kx = P(t) = a_0 + \sum a_n \cos \omega_f t + \sum b_n \sin \omega_f t$$

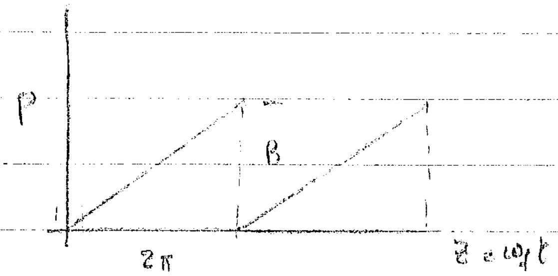
$$X_b = \frac{a_0}{K} \qquad X_b = \frac{a_n/K}{\sqrt{(1-r_n^2)^2+(2\zeta r_n)^2}}\cos(n\omega_f t - \psi_n)$$

$$r_n = n\left(\frac{\omega_f}{\omega}\right) \qquad X_b = \frac{b_n/K}{\sqrt{(1-r_n^2)^2+(2\zeta r_n)^2}}\sin(n\omega_f t - \psi_n)$$

$$m\ddot{x} + c\dot{x} + kx = a_n \cos n\omega_f t$$

$X_b = A\cos n\omega_f t + B\sin n\omega_f t$ is assumed,

$$X_b = \frac{a_0}{K} + \sum_{n=1}^{\infty}\frac{(a_n/K)\cos(n\omega_f t - \psi_n)}{\sqrt{(1-r_n^2)^2+(2\zeta r_n)^2}} + \sum_{n=1}^{\infty}\frac{(b_n/K)\sin(n\omega_f t - \psi_n)}{\sqrt{(1-r_n^2)^2+(2\zeta r_n)^2}}$$



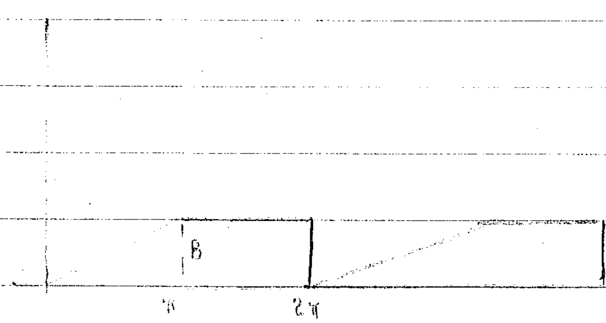
$$P = \frac{B z}{2\pi} \qquad 0 \le z \le 2\pi$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} P(z)\,dz = \frac{1}{2\pi}\int_0^{2\pi} \frac{Bz}{2\pi}\,dz = \frac{B}{4\pi^2}\,2\pi^2 = \frac{B}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{Bz}{2\pi}\cos nz\,dz = \frac{B}{2\pi^2}\int_0^{2\pi} z\cos nz\,dz = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{Bz}{2\pi}\sin nz\,dz = \frac{B}{2\pi^2}\int_0^{2\pi} z\sin nz\,dz = \frac{-B}{\pi n}$$

$$\therefore\ P(z) = \frac{B}{2} + \sum \frac{-B}{\pi n}\sin nz$$

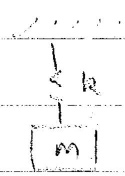


$$P = \frac{B}{\pi} z \qquad 0 \le z \le \pi$$

$$= B \qquad \pi \le z \le 2\pi$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} P\,dz = \left\{\int_0^{\pi} P\,dz + \int_{\pi}^{2\pi} P\,dz\right\}\frac{1}{2\pi}$$

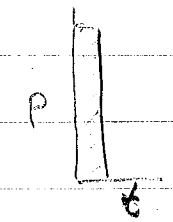
$$= \frac{1}{2\pi}\left\{\left[\frac{B}{2\pi} z^2\right]_0^{\pi} + Bz\Big|_{\pi}^{2\pi}\right\} =$$



$$m\ddot{x} + kx = 0$$

$$x = A\sin\omega t + B\cos\omega t$$

$$t=0 \quad x=0$$



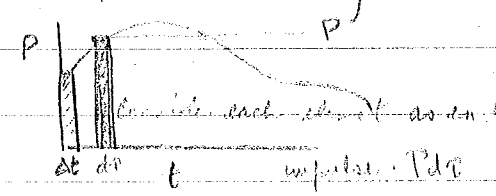
Impulsive Force $\lim_{t \to 0} \int F(t)\,dt = 1$

Impulsive force: $m\dot{x} = 1$ $\quad \dot{x} = 1/m$ at $t=0$

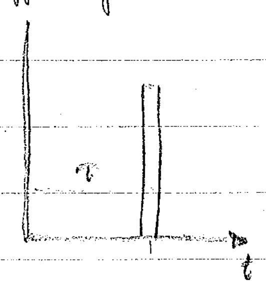
$$A = \frac{1}{m\omega} \qquad x = \frac{1}{m\omega}\sin\omega t \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\therefore\ x = \frac{1}{\sqrt{km}}\sin\omega t = \left.\frac{1}{\sqrt{km}}\sin\omega(t-t_1)\right\} \begin{matrix} 0 & t<t_1 \\ & \end{matrix}$$ for unit impulse

If forcing fn.



for each element effect must be zero before its time but takes effect for all time after.



$$dx = \frac{(P\,d\tau)}{\sqrt{km}}\sin(t-\tau)$$

$$\int_0^x dx = \int_0^t \frac{P\,d\tau}{\sqrt{km}}\sin(t-\tau)$$

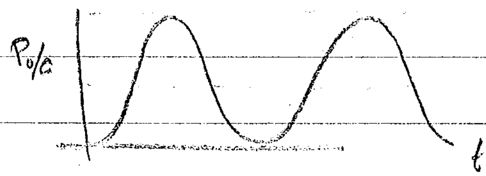
$$x = \frac{1}{\sqrt{km}}\int_0^t P\sin(t-\tau)\,d\tau$$ Duhamel's equation

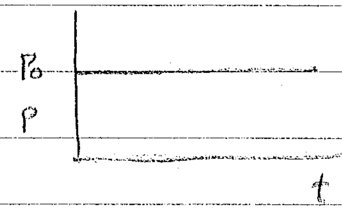
gives Particular solution only

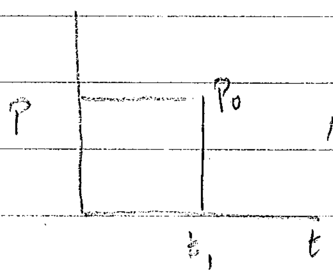
Impulse $= \int F\,dt$ $\quad$ Impulse force $= \lim_{t \to 0}\int F\,dt$

If $P = $ const $\quad x = \frac{P_0}{\sqrt{km}}\int_0^t \sin\omega(t-\tau)\,d\tau$

$$= \frac{P_0}{\sqrt{km}}\left[\frac{\cos\omega(t-\tau)}{\omega}\right]_0^t = \frac{P_0}{\sqrt{km}}\left[\frac{1}{\omega} - \frac{1}{\omega}\cos\omega t\right]$$







H.W. 5, 2, 14 Complete this problem V-4

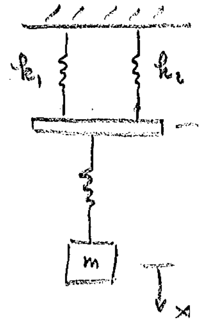
$$x = \frac{1}{\sqrt{km}}\left\{P_0\int_0^{t_1}\sin\omega(t-\tau)\,d\tau\right\}$$

$$= \frac{P_0}{\sqrt{km}}\left[\frac{\cos\omega(t-\tau)}{\omega}\right]_0^{t_1} = \frac{P_0}{\omega\sqrt{km}}\left[\cos\omega(t-t_1)-\cos\omega t\right]$$

3 quest.

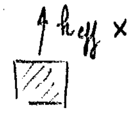
1. get effective spring const
2. Free vibration of spring mass system
3. Forced Vibration W or W/o damping.

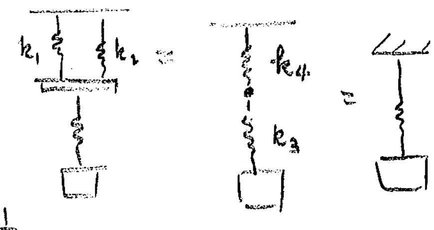
Senior Class Mtg
Wed Nov. 3 4th Hr. Rm 176
Atlantic Gulf & Pacific Co.



— mass horiz

$k_1 = 10$
$k_2 = 40$
$k_3 = 50$





$k_4 = k_1 + k_2$

$$\frac{1}{k_5} = \frac{1}{k_4} + \frac{1}{k_3}$$

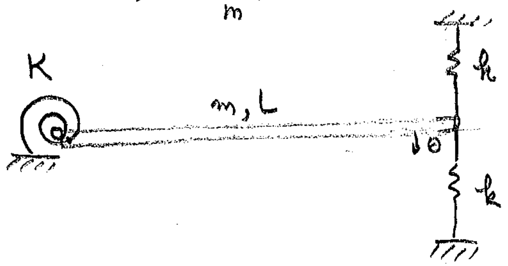
$$= \frac{1}{k_1 + k_2} + \frac{1}{k_3}$$

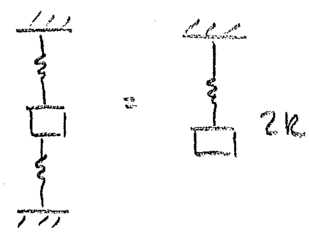
$$\frac{1}{k_5} = \frac{k_3 + k_1 + k_2}{k_3(k_1 + k_2)} = \frac{100}{2500} = .04$$

$$\boxed{k_5 = 25 = k_{eff}}$$

$$m\ddot{x} + k_{eff}x = 0$$

$$\ddot{x} + \frac{25}{m}x = 0$$

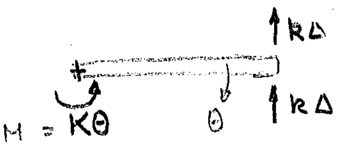




$k\Delta = P_1$

$k\Delta = P_2$

$P_{TOT} = P_1 + P_2 = 2k\Delta$
$= k'\Delta \Rightarrow k' = 2k$

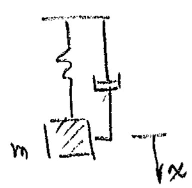


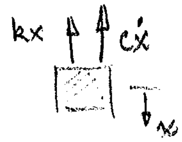
$\Delta = l\theta$

$$I_o\ddot{\theta} = -2k\Delta l - K\theta$$

$$\frac{mL^2}{3}\ddot{\theta} + K\theta + 2kL^2\theta = 0$$

$$\ddot{\theta} + \frac{3(K + 2kL^2)}{mL^2}\theta = 0$$





$$-kx - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$x = C_1 e^{\left(-\frac{c}{2m} + \sqrt{\quad}\right)t} + C_2 e^{\left(-\frac{c}{2m} - \sqrt{\quad}\right)t}$$

$$= e^{-\frac{c}{2m}t}\left[C_1 e^{\sqrt{\quad}\,t} + C_2 e^{-\sqrt{\quad}\,t}\right]$$

underdamped $\frac{c}{2m} < \frac{k}{m}$ $x = e^{-\frac{c}{2m}t}\{C_1 e^{i\sqrt{}\,t} + C_2 e^{-i\sqrt{}\,t}\}$

$c_c = 2\sqrt{mk} = 2m\omega$ $\frac{c}{2m} = \zeta\omega$

$\therefore$ $x = e^{-\zeta\omega t}\{A\cos\underbrace{\sqrt{1-\zeta^2}\,\omega}_{\omega d}t + B\sin\sqrt{1-\zeta^2}\,\omega t\}$

if $c = 10\ \frac{lb-sec}{ft}$

$k = 600\ \frac{lb}{ft}$ $W = 32.2\ \#$ $x(0) = 0.1\ ft$
$\dot{x}(0) = 0$

$\omega = \sqrt{k/m} = \sqrt{\frac{600}{1}} = 24.3\ rad/sec$

$c_c = 2m\omega = 2(1)(24.3) = 48.6\ \frac{lb-sec}{ft}$

$\zeta = \frac{10}{48.6} = .205$ $\zeta\omega = .205(24.3) = 4.4$

$\sqrt{1-\zeta^2} = \sqrt{1-.04} = .98$

$x = e^{-4.4t}[A\cos(.98)(24.3)t + B\sin 23.8t]$

$x = .1 = A = .1$

$\dot{x} = e^{-4.4t}[-A(23.8)\sin 23.8t + B(23.8)\cos 23.8t]$

$+ \{A\cos 23.8t + B\sin 23.8t\} - 4.4e^{-4.4t}$

$= 23.8B - 4.4A$ $B = \frac{.44}{23.8} = .01845$

$$1 = A_1' \sin\varphi + A_2' \sin\varphi_1$$
$$1 = A_1' \sin\varphi - A_2' \sin\varphi_1$$

$$2 = 2A_1' \sin\varphi$$
$$1 = A_1' \sin\varphi$$
$$0 = A_2' \sin\varphi_1$$

$$\dot{x}_1 = \omega_1 A_1' \cos(\omega_1 t + \varphi) + A_2' \omega_2 \cos(\omega_2 t + \varphi_1)$$
$$\dot{x}_2 = \omega_2 A_1' \cos(\omega_1 t + \varphi) - \omega_2 A_2' \cos(\omega_2 t + \varphi_1)$$

$A_1' \cos\varphi = 0$ $\quad$ $A_2' \cos\varphi_1 = 0$ $\quad$ take $A_2' = 0$ $\quad \varphi = \pi/2$

$$x_1 = \sin(\omega_1 t + \pi/2)$$
$$x_2 = \sin(\omega_1 t + \pi/2)$$

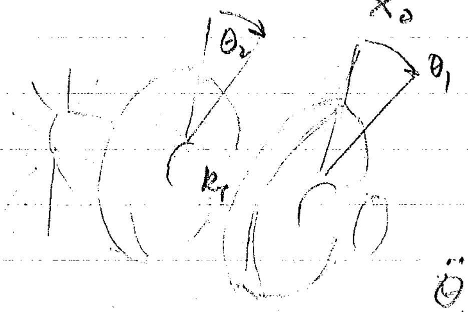
Problem $\quad x_1 = 1 \quad \dot{x}_1 = 0 \quad t = 0$

HW. $\quad x_2 = 0 \quad \dot{x}_2 = 0$

If @ $t = 0$ $\quad x_1 = x_0 \quad \dot{x}_1 = \dot{x}_0$ $\qquad x_0 = A_1 \sin\varphi \quad \dot{x}_0 = A_1 \omega \cos\varphi$

$$A_1 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega}\right)^2}$$

$$\tan\varphi = \frac{x_0 \omega}{\dot{x}_0}$$



$$I_1 \ddot{\theta}_1 = -k_T(\theta_1 - \theta_2)$$
$$I_2 \ddot{\theta}_2 = k_T(\theta_1 - \theta_2)$$

$$\ddot{\theta}_1 + \frac{k_T}{I_1}\theta_1 - \frac{k_T}{I_1}\theta_2 = 0$$

$$\ddot{\theta}_2 + \frac{k_T}{I_2}\theta_2 - \frac{k_T}{I_2}\theta_1 = 0$$

Assume $\quad \theta_1 = A_1 \sin(\omega t + \varphi)$

$\qquad \theta_2 = A_2 \sin(\omega t + \varphi)$

$$\left(-\omega^2 + \frac{k_T}{I_1}\right)A_1 - \frac{k_T}{I_1}A_2 = 0$$

$$\left(-\frac{k_T}{I_2}\right)A_1 + \left(-\omega^2 + \frac{k_T}{I_2}\right)A_2 = 0$$

$$\left(-\omega^2 + \frac{k_T}{I_1}\right)\left(-\omega^2 + \frac{k_T}{I_2}\right) - \frac{k_T^2}{I_1 I_2} = 0$$

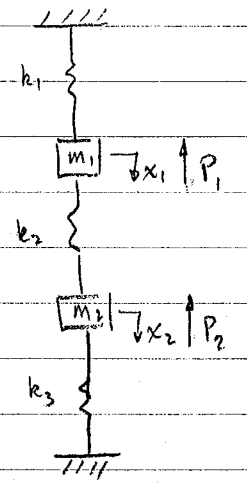
$$\omega^4 - k_T\left(\frac{1}{I_1} + \frac{1}{I_2}\right)\omega^2 + \frac{k_T^2}{I_1 I_2} - \frac{k_T^2}{I_1 I_2} = 0$$

$$\omega = 0, 0 \qquad \omega^2 = k_T\left(\frac{1}{I_1} + \frac{1}{I_2}\right) \qquad \omega$$

$$\omega = \sqrt{\frac{k_T(I_1 + I_2)}{I_1 I_2}} \quad ; \qquad \text{when } \omega = 0 \quad A_1 = A_2$$

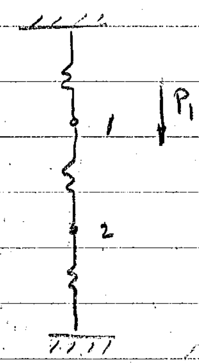
$$\frac{A_1}{I_1} = -\frac{A_2}{I_2} \quad \text{for above } \omega = \sqrt{\frac{k_T(I_1+I_2)}{I_1 I_2}}$$

11-9-71



$$m_1\ddot{x}_1 = -P_1$$
$$m_2\ddot{x}_2 = -P_2$$

forces prop. to deflect.



deflect at pt 1 due to P_1

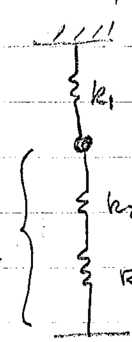
deflect at pt 1 due to P_2

deflection at x_1

$$x_1 = P_1\alpha_{11} + P_2\alpha_{12}$$
$$x_2 = P_1\alpha_{21} + P_2\alpha_{22}$$

$$\left.\begin{aligned} x_1 &= -\alpha_{11} m_1 \ddot{x}_1 - \alpha_{12} m_2 \ddot{x}_2 \\ x_2 &= -\alpha_{21} m_1 \ddot{x}_1 + \alpha_{22} m_2 \ddot{x}_2 \end{aligned}\right\} \text{dynamically coupled}$$

Apply unit force at pt 1



unit force

$$1\left\{k_1 + \frac{k_2 k_3}{k_2 + k_3}\right\} = \frac{1}{\alpha_{11}}$$

if $P_2 = 0$ we can use superpos

$$k_e = \frac{k_1 k_3}{k_1 + k_2}$$

$$\alpha_{11} = \frac{k_2 + k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1}$$

$$\alpha_{22} = \frac{k_1+k_2}{k_1k_2+k_2k_3+k_3k_1}$$

α's are influence coefficients

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\underbrace{k_2 (\alpha_{11} - \overbrace{\alpha_{21}}^{\text{deformation at 2 due to load at } P_1})}_{\text{deformation due to load } P_1} = k_3(\alpha_{21})$$

$$\frac{k_2\alpha_{11}}{k_2+k_3} = \alpha_{21}$$

$$k_2(\alpha_{22} - \alpha_{12}) = k_1\alpha_{12} \qquad \alpha_{12} = \frac{k_2\alpha_{22}}{k_2+k_1}$$

$\alpha_{21} = \alpha_{12}$ (maxwell - Beatty Reciprocity Relationship)

$$\frac{k_2\alpha_{22}}{k_2+k_1} = \frac{k_2}{k_2+k_1}\;\frac{k_1+k_2}{k_1k_2+k_2k_3+k_3k_1} = \frac{k_2}{\text{denom.}}$$

$\therefore$ it is proved.

$$\frac{k_2\alpha_{11}}{k_2+k_3} = \frac{k_2}{k_2+k_3}\;\frac{k_2+k_3}{k_1k_2+k_2k_3+k_3k_1} = \frac{k_2}{\text{denom}}$$

11-11-71

$$m_1\left(\frac{k_2+k_3}{k_1k_2+k_2k_3+k_1k_3}\right)\ddot{x}_1 + m_2\left(\frac{k_2}{k_1k_2+k_2k_3+k_1k_3}\right)\ddot{x}_2 + x_1 = 0$$

$$m_1\left(\frac{k_2}{k_1k_2+k_2k_3+k_1k_3}\right)\ddot{x}_1 + m_2\left(\frac{k_1+k_3}{(k_1k_2)+k_2k_3+k_1k_3}\right)\ddot{x}_2 + x_2 = 0$$

From previous results

$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2x_1 = 0$$

$$x_1 = \frac{1}{k_2}\left[m_2\ddot{x}_2 + (k_2+k_3)x_2\right] \quad \text{then} \quad \frac{m_1}{k_2}\left[m_2\ddddot{x}_2 + (k_2+k_3)\ddot{x}_2\right] + \left(\frac{k_1+k_2}{k_2}\right)\left[m_2\ddot{x}_2 + (k_2+k_3)x_2\right]$$

$$+ x_2k_2 = 0$$

$$A_1x_2^{(4)} + A_2x_2'' + A_3x_2 = 0$$

get 4 constants from the solution for $x_1 = f(\ddot{x}_2, x_2)$ & bc on x_2

usual method the equas are treated simultaneously.

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{\frac{3k}{m}} \qquad \text{if } m_1 = m_2 \quad k_1 = k_2 = k_3$$

let $m = m_1 = m_2$ $\quad k = k_1 = k_2 = k_3$

$$m\frac{2k}{3k^2}\ddot{x}_1 + m\frac{k}{3k^2}\ddot{x}_2 + x_1 = 0$$

$$m\frac{k}{3k^2}\ddot{x}_1 + m\frac{2k}{3k^2}\ddot{x}_2 + x_2 = 0$$

$$x_1 = A_1 \sin(\omega t + \varphi)$$

$$x_2 = A_2 \sin(\omega t + \varphi)$$

$$-\frac{2m}{3k}A_1\omega^2 - \frac{m}{3k}A_2\omega^2 + A_1 = 0$$

$$-\frac{m}{3k}A_1\omega^2 - \frac{2m}{3k}A_2\omega^2 + A_2 = 0$$

$$A_1\left(1 - \frac{2m}{3k}\omega^2\right) - A_2\frac{m\omega^2}{3k} = 0$$

$$A_2\left(1 - \frac{2m\omega^2}{3k}\right) - A_1\frac{m\omega^2}{3k} = 0 \qquad \text{let } \frac{k}{m\omega^2} = \lambda$$

$$A_1\left(1 - \frac{2}{3\lambda}\right) - A_2\frac{1}{3\lambda} = 0$$

$$-A_1\frac{1}{3\lambda} + A_2\left(1 - \frac{2}{3\lambda}\right) = 0$$

$$\left(1 - \frac{2}{3\lambda}\right)^2 - \left(\frac{1}{3\lambda}\right)^2 = 0 \qquad \left(\lambda - \frac{2}{3}\right)^2 - \frac{1}{9} = 0$$

$$\left(\lambda - \frac{2}{3}\right)^2 = \left(\frac{1}{3}\right)^2$$

$$\left(\lambda - \frac{2}{3}\right) = \pm\frac{1}{3}$$

$$\frac{k}{m\omega^2} = 1 \qquad \omega = \sqrt{\frac{k}{m}} \checkmark \longleftarrow \lambda = 1$$

$$\frac{k}{m\omega^2} = \frac{1}{3} \qquad \omega = \sqrt{\frac{3k}{m}} \checkmark \longleftarrow \lambda = 1/3$$

if $\lambda = 1 \longrightarrow A_1(1/3) - A_2\left(\frac{1}{3}\right) = 0 \qquad A_1 = A_2$ ✓

$\lambda = \frac{1}{3} \longrightarrow A_1(-1) - A_2(1) = 0 \qquad A_1 = -A_2$ ✓

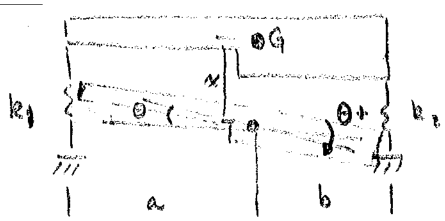
$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \qquad \text{if} \quad m = m_1 = m_2$$
$$m\ddot{x}_2 + 2kx_2 - kx_1 = 0 \qquad k = k_1 = k_2 = k_3$$

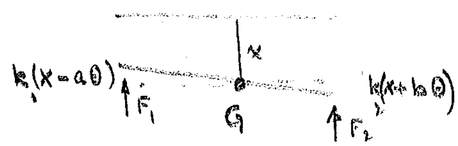
$$m(\ddot{x}_1 + \ddot{x}_2) + 2k(x_1 + x_2) - k(x_1 + x_2) = 0$$
$$m(\ddot{x}_1 - \ddot{x}_2) + 2k(x_1 - x_2) + k(x_1 - x_2) = 0$$

$$m(\ddot{x}_1 + \ddot{x}_2) + k(x_1 + x_2) = 0 \qquad \text{let} \quad x_1 + x_2 = q_1$$
$$m(\ddot{x}_1 - \ddot{x}_2) + 3k(x_1 - x_2) = 0 \qquad x_1 - x_2 = q_2$$

$$\therefore \quad m\ddot{q}_1 + kq_1 = 0 \qquad q_1 = B_1 \sin(\omega_1 t + \varphi_1) \qquad \omega_1 = \sqrt{\frac{k}{m}}$$
$$m\ddot{q}_2 + 3k\,q_2 = 0 \qquad q_2 = B_2 \sin(\omega_2 t + \varphi_2) \qquad \omega_2 = \sqrt{\frac{3k}{m}}$$

these are principal modes (independent modes shapes)





$$m\ddot{x} = -k_1[x - a\theta] + [x + b\theta](-k_2)$$
$$I\ddot{\theta} = +k_1(x - a\theta)a - k_2(x + b\theta)b$$
$$m\ddot{x} + (k_1 + k_2)x - \theta(ak_1 - bk_2) = 0$$
$$I\ddot{\theta} + (k_1a^2 + k_2b^2)\theta - (k_1a - bk_2)x = 0$$

if $\quad \beta = \dfrac{k_1 + k_2}{m} \qquad \gamma = \dfrac{k_2 b - k_1 a}{m} \qquad \eta = \dfrac{k_1 a^2 + k_2 b^2}{I}$

then
$$\ddot{x} + \beta x + \gamma\theta = 0$$
$$\ddot{\theta} + \eta\theta + \frac{m\gamma}{I}x = 0$$

Assume $\quad x = A \sin(\omega t + \varphi)$
$\qquad\qquad \theta = B \sin(\omega t + \varphi)$

$$-A\omega^2 + A\beta + \gamma B = 0 \qquad A(\beta - \omega^2) + \gamma B = 0$$
$$-B\omega^2 + \eta B + \frac{m\gamma}{I}A = 0 \qquad B(\eta - \omega^2) + \frac{m\gamma}{I}A = 0$$
$$(\beta - \omega^2)(\eta - \omega^2) - \frac{m\gamma^2}{I} = 0 \quad \rightarrow \quad \omega^4 - (\eta + \beta)\omega^2 + \beta\eta - \frac{m\gamma^2}{I} = 0$$
$$\omega^2 = \frac{(\eta + \beta) \pm \sqrt{(\eta + \beta)^2 - 4\left(\beta\eta - \frac{m\gamma^2}{I}\right)}}{2} = \left(\frac{\eta + \beta}{2}\right) \pm \sqrt{\left(\frac{\eta + \beta}{2}\right)^2 - \left(\beta\eta - \frac{m\gamma^2}{I}\right)}$$

if $\gamma = 0 \qquad \omega^2 = \dfrac{\eta + \beta}{2} \pm \left(\dfrac{\eta - \beta}{2}\right) \quad$ or $\quad \omega^2 = \eta$ or β

$$\omega_1 = \sqrt{\left(\frac{\eta+\beta}{2}\right) + \frac{1}{2}\sqrt{(\beta+\eta)^2 - 4\left(\beta\eta - \frac{\gamma^2 m}{I}\right)}}$$

$$\omega_2 = \sqrt{\left(\frac{\beta+\eta}{2}\right) - \frac{1}{2}\sqrt{(\beta+\eta)^2 - 4\left(\beta\eta - \frac{\gamma^2 m}{I}\right)}}$$

KE of system $\quad \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I\dot{\theta}^2$

PE of system $\quad \frac{1}{2} k_1(\delta_1 + \delta_{10})^2 + \frac{1}{2} k_2(\delta_2 + \delta_{20})^2 - mgx$

$\delta_1 = x_1 - a\theta \qquad k_1\delta_{10} + k_2\delta_{20} = mg$

$\delta_2 = x_1 + b\theta$

PE + KE = C $\quad \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} k_1(\delta_1+\delta_{10})^2 + \frac{1}{2} k_2(\delta_2+\delta_{20})^2 - mgx = \text{const}$

$$\frac{d}{dt}(PE + KE) = m\ddot{x}\dot{x} + I\ddot{\theta}\dot{\theta} + k_1(\delta_1+\delta_{10})\frac{d}{dt}(\delta_1+\delta_{10}) + k_2(\delta_2+\delta_{20})\frac{d}{dt}(\delta_2+\delta_{20})$$

$$- mg\dot{x} = 0$$

$$\frac{d}{dt}(\delta_1+\delta_{10}) = \frac{d}{dt}(x_1 - a\theta + \delta_{10}) = \dot{x} - a\dot{\theta}$$

$$\frac{d}{dt}(\delta_2+\delta_{20}) = \frac{d}{dt}(x_1 + b\theta + \delta_{20}) = \dot{x} + b\dot{\theta}$$

$$m\ddot{x}\dot{x} + I\ddot{\theta}\dot{\theta} + k_1(x_1 - a\theta + \delta_{10})(\dot{x} - a\dot{\theta}) + k_2(x_1 + b\theta + \delta_{20})(\dot{x} + b\dot{\theta}) - mg\dot{x} = 0$$

$$\dot{x}\left[m\ddot{x} + k_1(x - a\theta + \cancel{\delta_{10}}) + k_2(x + b\theta + \cancel{\delta_{20}}) - \cancel{mg}\right]$$

$$+ \dot{\theta}\left[I\ddot{\theta} - ak_1(x - a\theta + \cancel{\delta_{10}}) + bk_2(x + b\theta + \cancel{\delta_{20}})\right] = 0$$

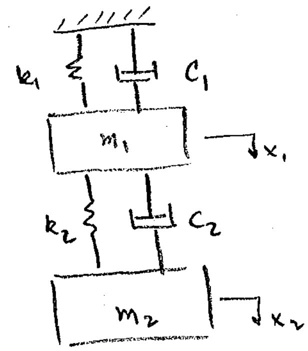
& sum of moments = 0 $\qquad$ for it to be 0 $\quad$ coeff of $\dot{x}$ & $\dot{\theta}$ = 0

$$m\ddot{x} + (k_1+k_2)x - (k_1a - k_2b)\theta = 0$$

$$I\ddot{\theta} + (a^2k_1 + b^2k_2)\theta - (ak_1 - bk_2)x = 0$$

HW Problem: 7.11 neglect plotting. Use energy & free-body

11-16-71



$$m_1\ddot{x}_1 = -k_1x_1 - c_1\dot{x}_1 - k_2(x_1-x_2) - c_2(\dot{x}_1-\dot{x}_2) \quad ①$$

$$m_2\ddot{x}_2 = -k_2(x_2-x_1) - c_2(\dot{x}_2-\dot{x}_1) \quad ②$$

$$m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 + (k_1+k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2-\dot{x}_1) + k_2(x_2-x_1) = 0$$

Consider $x_1 = B_1e^{st}$ $\quad x_2 = B_2e^{st}$

$$\{m_1s^2 + (c_1+c_2)s + (k_1+k_2)\}B_1 - (c_2s+k_2)B_2 = 0$$

$$\{m_2s^2 + c_2s + k_2\}B_2 - \{c_2s+k_2\}B_1 = 0$$

$$[m_2s^2 + c_2s + k_2][m_1s^2 + (c_1+c_2)s + (k_1+k_2)] - [c_2s+k_2]^2 = 0$$

$$m_1m_2s^4 + [m_2(c_1+c_2) + c_2m_1]s^3 + [c_2(c_1+\cancel{c_2}) + k_2m_1 + m_2(k_1+k_2)]s^2$$
$$+ [c_2(k_1+\cancel{k_2}) + k_2(c_1+\cancel{c_2})]s + k_2(k_1+\cancel{k_2}) - c_2^2s^2 + 2c_2k_2s - k_2^2 = 0$$

$$m_1m_2s^4 + [m_2(c_1+c_2) + c_2m_1]s^3 + [c_1c_2 + k_2(m_1+m_2) + k_1m_2]s^2$$
$$+ [c_2k_1 + c_1k_2]s + k_2k_1 = 0$$

Soln: $\quad s_1 = -p_1 + iq_1 \qquad s_3 = -p_2 + iq_2 \qquad x_2 = B_3e^{(-p_2+iq_2)t} + B_4e^{(-p_1+iq_1)t}$

$\quad s_2 = -p_1 - iq_1 \qquad s_4 = -p_2 - iq_2$

$$(s^2+2sp_1+p_1^2)(s^2+2p_2s+p_2^2) + q_2^2(s^2+2sp_1+p_1^2) + q_1^2(s^2+2p_2s+p_2^2) + q_1^2q_2^2 = 0$$

$m_1m_2 = 1 \qquad s^4$

$(2p_2+2p_1) = m_2(c_1+c_2) + c_2m_1 \qquad s^3$

$p_2^2 + p_1^2 + 4p_2p_1 + q_2^2 + q_1^2 = [c_1c_2 + k_2(m_1+m_2) + k_1m_2] \qquad s^2$

$2p_1^2p_2 + 2p_1p_2^2 + 2p_1q_2^2 + 2p_2q_1^2 = c_2k_1 + c_1k_2 \qquad s$

$p_1^2p_2^2 + q_2^2p_1^2 + p_2^2q_1^2 + q_1^2q_2^2 = k_2k_1$

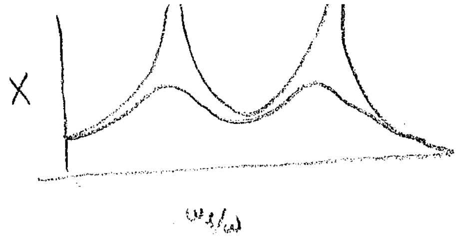
if a forcing function is added on mass 1 = $P_0\sin\omega_f t$ equation ① adds a $P_0\sin\omega_f t$ on right side

particular solution $\quad x_1 = A_1\sin\omega_f t + A_2\cos\omega_f t \qquad$ placing into eq(1)

$\quad x_2 = A_3\sin\omega_f t + A_4\cos\omega_f t$

$$-m_1\omega_f^2A_1 - (c_1+c_2)A_2\omega_f + c_2A_4\omega_f - k_2A_3 + (k_1+k_2)A_1 = 0$$

$$-m_1\omega_f^2A_2 + (c_1+c_2)A_1\omega_f - c_2A_3\omega_f - k_2A_4 + (k_1+k_2)A_2 = 0$$



2 degree system

if no damping $\quad m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = P_0 \sin\omega_f t$

$$m_2\ddot{x}_2 + k_2x_2 - k_1x_1 = 0$$

partic solution is assumed

gives

$$-m_1\omega_f^2 A_1 - k_2A_3 + (k_1+k_2)A_1 = P_0$$

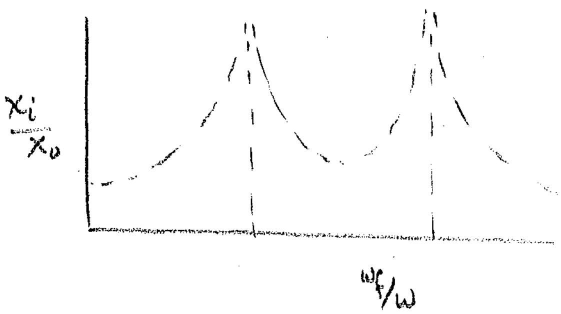
$$-m_1\omega_f^2 A_2 - k_2A_4 + (k_1+k_2)A_2 = 0$$

$$-m_2\omega_f^2 A_3 + k_2A_3 - k_1A_1 = 0$$

$$-m_2\omega_f^2 A_4 + k_2A_4 - k_1A_2 = 0$$

$$\frac{X_1}{X_0} = \frac{1-r_2^2}{\left(1+\frac{k_2}{k_1}-r_1^2\right)(1-r_2^2)-k_2/k_1} \qquad r_2 = \frac{\omega_f}{\omega_2} \quad r_1 = \omega_f/\omega_1$$

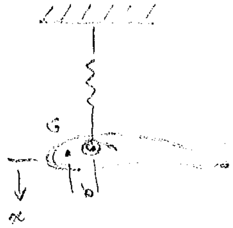
$$\frac{X_2}{X_0} = \frac{1}{(1+k_2/k_1-r_1^2)(1-r_2^2)-k_2/k_1}$$



if $(1+k_2/k_1 - r_1^2)(1-r_2^2) - k_2/k_1 = 0$ we have resonance

Problem 7.23. (Model for flutter type bodies) 7.17 (influence coeff.)

11-23-71



$$KE = \tfrac{1}{2} m v_G^2 + \tfrac{1}{2} I \dot\theta^2$$
$$= \tfrac{1}{2} m \{ (v_b + b\dot\theta\cos\theta)^2 + (b\dot\theta\sin\theta)^2 \} + I\dot\theta^2$$

$$\bar v_G = \bar v_B + (b\dot\theta)\bar e_r$$
$$\bar v_G = b\dot\theta \sin\theta\, \bar i + (v_B + b\dot\theta\cos\theta)\bar j$$

$$PE = \tfrac{1}{2} k x^2 + \tfrac{1}{2} k_r \theta^2 \quad \text{at equil}$$

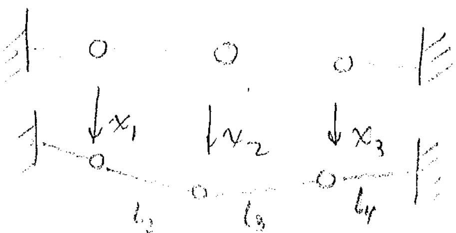
$$\tfrac{1}{2} m \{ (\dot x + b\dot\theta\cos\theta)^2 + (b\dot\theta\sin\theta)^2 \} + \tfrac{1}{2} I\dot\theta^2 + \tfrac{1}{2} k x^2 + \tfrac{1}{2} k_r \theta^2 = \text{const.}$$

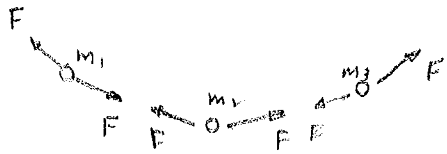
$$\frac{d}{dt}\{\ c\ \} = 0$$

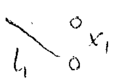
$$m\ddot x + mb\ddot\theta + kx = 0 \qquad \theta +\circlearrowright$$

$$mb\cos\theta(\ddot x + b\ddot\theta\cos\theta) + mb^2\sin^2\theta\,\ddot\theta + I\ddot\theta + k_r\theta = 0$$

$$\underbrace{I\ddot\theta + mb^2\ddot\theta}_{I_B\ddot\theta} + mb\ddot x + k_r\theta = 0$$







$$m_1\ddot x_1 = -F\frac{x_1}{l_1} + F\frac{(x_2 - x_1)}{l_2}$$

$$m_2\ddot x_2 = -F\left(\frac{x_2 - x_1}{l_2}\right) - F\left(\frac{x_2 - x_3}{l_3}\right)$$

$$m_3\ddot x_3 = F\frac{(x_2 - x_3)}{l_3} - F\frac{x_3}{l_4}$$

$$m_1\ddot x_1 + F\left(\tfrac{1}{l_1} + \tfrac{1}{l_2}\right)x_1 - F\frac{x_2}{l_2} = 0$$

$$m_2\ddot x_2 + F\left(\tfrac{1}{l_2} + \tfrac{1}{l_3}\right)x_2 - F\frac{x_1}{l_2} - F\frac{x_3}{l_3} = 0$$

$$m_3\ddot x_3 + F\left(\tfrac{1}{l_3} + \tfrac{1}{l_4}\right)x_3 - F\frac{x_2}{l_3} = 0$$

let $l_i = l$

$m_1 = m \quad m_2 = 3m \quad m_3 = 2m$

$$m\ddot x_1 + \frac{2F}{l}x_1 - \frac{Fx_2}{l} = 0$$

$$3m\ddot x_2 + \frac{2F}{l}x_2 - \frac{Fx_1}{l} - \frac{Fx_3}{l} = 0$$

$$2m\ddot x_3 + \frac{2F}{l}x_3 - \frac{Fx_2}{l} = 0$$

$$x_1 = A_1\sin(\omega t + \varphi)$$
$$x_2 = A_2\sin(\omega t + \varphi)$$
$$x_3 = A_3\sin(\omega t + \varphi)$$

$$A_1\left(-m\omega^2 + \frac{2F}{l}\right) - \frac{F}{l}A_2 = 0$$

$$-A_1 F/l + A_2(-3m\omega^2 + 2F/l) - \frac{F}{l}A_3 = 0$$

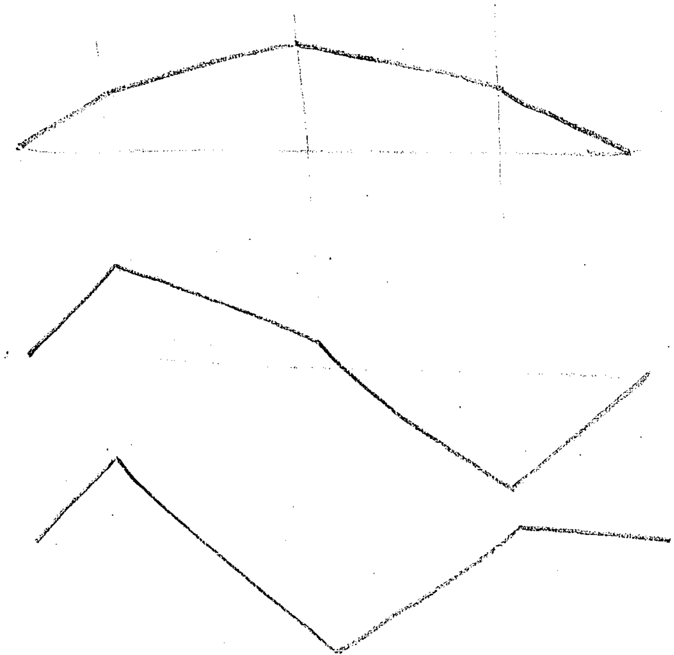
$$-F/l\,A_2 + A_3(-2m\omega^2 + 2F/l) = 0$$

$$\begin{vmatrix} -m\omega^2 + 2F/l & -F/l & 0 \\ -F/l & -3m\omega^2 + 2F/l & -F/l \\ 0 & -F/l & (-2m\omega^2 + 2F/l) \end{vmatrix} = 0$$

$$\rho = \frac{lm}{F}\omega^2 \qquad \begin{cases} \rho = .25282 \\ \rho = 1.1810 \\ \rho = 2.2329 \end{cases}$$

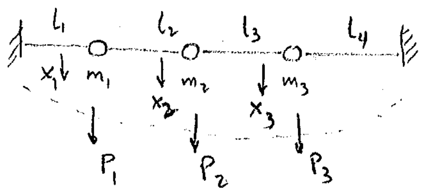
$$6\rho^3 - 22\rho^2 + 21\rho - 4 = 0$$

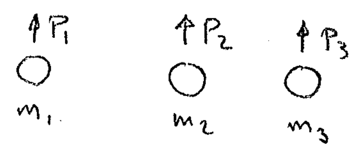
Mode shapes.



Near Coupled - mass pt is influenced by nearby masses.
Far Coupled - " " " " by far masses spread throughout system.

11-30-71





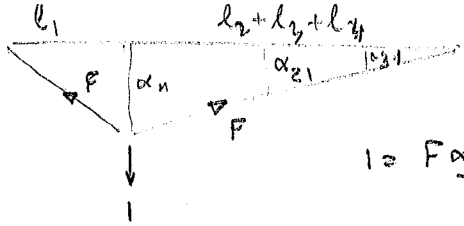
$$x_1 = P_1\alpha_{11} + P_2\alpha_{12} + P_3\alpha_{13}$$

$$x_2 = P_1\alpha_{21} + P_2\alpha_{22} + P_3\alpha_{23} \qquad \alpha_{ij} = \alpha_{ji}$$

$$x_3 = P_1\alpha_{31} + P_2\alpha_{32} + P_3\alpha_{33}$$

$m_1\ddot{x}_1 = -P_1$
$m_2\ddot{x}_2 = -P_2$
$m_3\ddot{x}_3 = -P_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} = -\begin{bmatrix} m_1\ddot{x}_1 \\ m_2\ddot{x}_2 \\ m_3\ddot{x}_3 \end{bmatrix} \begin{bmatrix} \bar{\alpha} \end{bmatrix}$$



$$1 = F\frac{\alpha_{11}}{l_1} + F\left(\frac{\alpha_{11}}{l_2+l_3+l_4}\right) = F\left(\frac{l_2+l_3+l_4+l_1}{l_1(l_2+l_3+l_4)}\right)\alpha_{11}$$

$$\alpha_{11} = \frac{l_1(l_2+l_3+l_4)}{F\left[l_2+l_3+l_4+l_1\right]}$$

$$\frac{\alpha_{31}}{\alpha_{11}} = \frac{l_4}{l_2+l_3+l_4} \qquad \frac{\alpha_{21}}{\alpha_{11}} = \frac{l_3+l_4}{l_2+l_3+l_4}$$

Assume $x_1 = A_1 \sin\omega t$

$x_2 = A_2 \sin\omega t$

$x_3 = A_3 \sin\omega t$

$$-\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} + \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}\begin{bmatrix} m_1\omega^2 A_1 \\ m_2\omega^2 A_2 \\ m_3\omega^2 A_3 \end{bmatrix} = 0$$

$$-\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}^{-1}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} + \begin{bmatrix} m_1\omega^2 A_1 \\ m_2\omega^2 A_2 \\ m_3\omega^2 A_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}^T = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{13} & \alpha_{33} \end{bmatrix}$$

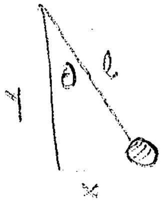
$$\text{adj}\,[\ ]^T = \begin{bmatrix} \begin{vmatrix} \alpha_{22} & \alpha_{32} \\ \alpha_{23} & \alpha_{33} \end{vmatrix} & -\begin{vmatrix} \alpha_{12} & \alpha_{32} \\ \alpha_{13} & \alpha_{33} \end{vmatrix} & \begin{vmatrix} \alpha_{12} & \alpha_{22} \\ \alpha_{13} & \alpha_{23} \end{vmatrix} \\ & & \end{bmatrix} \qquad [A]^{-1} = \frac{\text{adj}\,[A]}{|A|}$$

$$\text{let } [K] = [\alpha]^{-1} \qquad k_{ij} = k_{ji}$$

$$-[K][A] + \begin{bmatrix} m_1\omega^2 A_1 \\ m_2\omega^2 A_2 \\ m_3\omega^2 A_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} K_{11}-m_1\omega^2 & K_{12} & K_{13} \\ K_{21} & K_{22}-m_2\omega^2 & K_{23} \\ K_{31} & K_{32} & K_{33}-m_3\omega^2 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0$$

An independent coordinate system   Generalized Coordinate system



$$x^2+y^2=l^2$$

Lagrange coordinates.

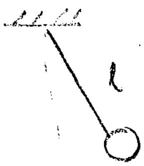
$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = 0$$

$V = V(q_k)$ T = K.E.

$T = T(q_k, \dot{q}_k, t)$ V = PE

$$L = T - V \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

found from $I = \int F(q_k, \dot{q}_k, t)\, dt$ $\delta W = Q\, \delta q_k$



$\frac{1}{2} m (r\dot{\theta})^2 = KE$

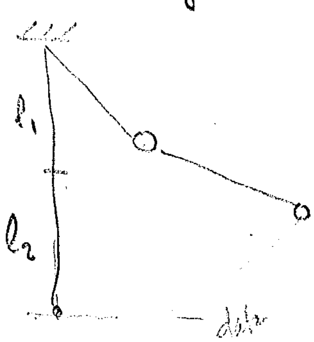
$mg(l - l\cos\theta) = PE$

$$L = \frac{1}{2} m l \dot{\theta}^2 - mgl(1-\cos\theta)$$

$\dot{q}_1 = \dot{\theta}$ $\frac{\partial}{\partial t}\left(m r^2 \dot{\theta}\right) + mgl\sin\theta = 0$

$$\boxed{ml^2\ddot{\theta} + mgl\sin\theta}$$

$$V = m_1 g l_1 (1-\cos\theta_1) + m_2 g\left[l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)\right] + m_1 g l_2$$



$$m_1 g l_2 + m_1 g l_1 \frac{\theta_1^2}{2} + m_2 g\left(\frac{\theta_1^2 l_1}{2} + \frac{\theta_2^2 l_2}{2}\right)$$

$$KE = \frac{1}{2} m_1 (l_1\dot{\theta}_1)^2 + \frac{1}{2} m_2 (l_1\dot{\theta}_1 + l_2\dot{\theta}_2)^2$$

relative

$$\frac{\partial L}{\partial \theta_1} = m_1 g l_1 \theta_1 + m_2 g l_1 \theta_1$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 g l_2 \theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta} + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

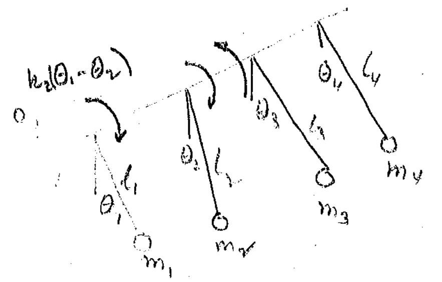
$$(m_1 l_1^2 + m_2 l_1^2)\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 + m_1 g l_1 \theta_1 + m_2 g l_1 \theta_1 = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 + m_2 g l_2 \theta_2 = 0$$

Prob 8.4, 8.15, 8.17

12-7-71

derive lagranges equations of motion. (see Fowles)



$\theta_1 > \theta_2 > \theta_3$

$$m_1 l_1^2 \ddot{\theta}_1 = -k_2(\theta_1-\theta_2) - m_1 g l_1 \sin\theta_1$$

$$m_1 l_1^2 \ddot{\theta}_1 + k_2(\theta_1-\theta_2) + m_1 g l_1 \theta_1 = 0$$

$$m_1 l_1^2 \ddot{\theta}_1 + (W_1 l_1 + k_2)\theta_1 - k_2\theta_2 = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 = k_2(\theta_1-\theta_2) - k_3(\theta_2-\theta_3) - m_2 g l_2 \theta_2$$

$$m_3 l_3^2 \ddot{\theta}_3 = k_3(\theta_2-\theta_3) - k_4(\theta_3-\theta_4) - m_3 g l_3 \theta_3$$

$$m_4 l_4^2 \ddot{\theta}_4 = +k_4(\theta_3-\theta_4) - m_4 g l_4 \theta_4$$

$$\begin{bmatrix} m_1 l_1^2 \ddot{\theta}_1 \\ m_2 l_2^2 \ddot{\theta}_2 \\ m_3 l_3^2 \ddot{\theta}_3 \\ m_4 l_4^2 \ddot{\theta}_4 \end{bmatrix} + \begin{bmatrix} k_2+m_1 g l_1 & -k_2 & 0 & 0 \\ -k_2 & (k_2+k_3+m_2 g l_2) & -k_3 & 0 \\ 0 & -k_3 & (k_3+k_4+m_3 g l_3 & -k_4 \\ 0 & 0 & -k_4 & (k_4+m_4 g l_4) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = 0$$

$\theta_1 = A_1 \sin(\omega t + \varphi)$
$\theta_2 = A_2 \sin(\omega t + \varphi)$ substituting
$\vdots$

$$\begin{bmatrix} k_2+m g l_1 - m_1 l_1^2 \omega^2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3+m_2 g l_2 - m_2 l_2^2 \omega^2 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4+m_3 g l_3 - m_3 l_3^2 \omega^2 & -k_4 \\ 0 & 0 & -k_4 & k_4+m_4 g l_4 - m_4 l_4^2 \omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0$$

if $\ddot{x}_1 + \left(\frac{k_1+k_2}{m_1}\right) x_1 - \frac{k_2}{m_1} x_2 = 0$

$\ddot{x}_2 - \frac{k_2}{m_2} x_1 + \left(\frac{k_2+k_3}{m_2}\right) x_2 = 0$

using $x_1 = A_1 \sin(\omega t + \varphi)$
$x_2 = A_2 \sin(\omega t + \varphi)$

$$\begin{bmatrix} \frac{k_1+k_2}{m_1} - \omega^2 & -k_2/m_1 \\ -k_2/m_2 & \frac{k_2+k_3}{m_2} - \omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

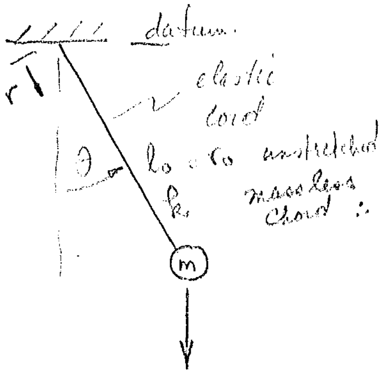
$$\ddot{x}_i = x_i(q_1, q_2, q_3, q_4 \cdots)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j$$ ~ general force terms.

$$\delta W = F_x \delta x_i + F_y \delta y_i + F_z \delta z_i$$

$$\delta x_i = \frac{\partial x_i}{\partial q_j}\delta q_j \qquad \delta y_i = \frac{\partial y_i}{\partial q_j}\delta q_j \qquad \delta z_i = \frac{\partial z_i}{\partial q_j}\delta q_j$$

$$\delta W = \delta q_j \underbrace{\left[F_x \frac{\partial x_i}{\partial q_j} + F_y \frac{\partial y_i}{\partial q_j} + F_z \frac{\partial z_i}{\partial q_j} \right]}_{Q_j} = Q_j \delta q_j$$



$$KE = \frac{1}{2} m (\vec{v}\cdot\vec{v}) = \frac{1}{2} m (\dot{r} e_r + r\dot{\theta} e_\theta)(\dot{r} e_r + r\dot{\theta} e_\theta)$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = -mg(r\cos\theta) + \frac{1}{2}k(r-r_0)^2$$

$$L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + mg r\cos\theta - \frac{1}{2}k(r-r_0)^2$$

$q_1 = r$

$q_2 = \theta$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

Conservative System

$$\frac{\partial L}{\partial r} = m r \dot{\theta}^2 + mg\cos\theta - k(r-r_0)$$

$$\frac{\partial L}{\partial \theta} = -mg r \sin\theta$$

$$m\ddot{r} - m r\dot{\theta}^2 - mg\cos\theta + k(r-r_0) = 0$$

$$m r^2 \ddot{\theta} + mg r \sin\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i$$

Rayleigh's Method.

Given a system in harmonic motion

mean values of KE = mean values of PE
max values of KE = max values of PE } conservative systems.

$T = \frac{1}{2} m \dot{x}^2$

$V = \frac{1}{2} k x^2$

let $x = A \sin(\omega t + \varphi)$ $\qquad T = \frac{A^2}{2} m \omega^2 \cos^2(\omega t + \varphi)$

$V = \frac{A^2}{2} k \sin^2(\omega t + \varphi)$

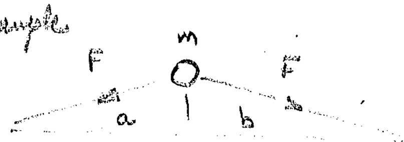
$$T_{mean} = \frac{1}{\tau}\int_0^{\tau} T\,dt = \frac{1}{\tau}\ \frac{A^2}{2} m\omega^2 \int_0^{\tau} \cos^2(\omega t + \varphi)\,dt = \frac{1}{\tau}\ \frac{A^2}{4} m\omega^2 \cdot \tau = \frac{A^2}{4} m\omega^2$$

period = τ

$$V_{mean} = \frac{kA^2}{2\tau}\int_0^{\tau} \sin^2(\omega t + \varphi)\,dt = \frac{kA^2}{4\tau}\cdot\tau = \frac{A^2}{4} m\omega^2 \qquad \boxed{k = \omega^2 m}$$

if $T_{max} = \omega^2 T^* = V_{max}$ then $\omega^2 = \frac{V_{max}}{T^*}$

Example



$T = \frac{1}{2} m \dot{x}^2$

$x = A \sin(\omega t + \varphi)$

$T_{max} = \frac{A^2}{2} m\omega^2$

$$V = -\int (F\cos\alpha + F\cos\beta)(-dx)$$

$$= F\int (x/a)\,dx + \left(\frac{x}{l-a}\right)dx = \frac{F}{a(l-a)}\int x(l-a+a)\,dx$$

$$= \frac{Fl}{a(l-a)}\int x\,dx = \frac{Fl\,x^2}{2a(l-a)} = \frac{Fl\,A^2}{2a(l-a)} \sin^2(\omega t + \varphi)$$

$V_{max} = \frac{FlA^2}{2a(l-a)}$

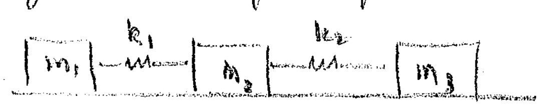
$\underset{T_{max}}{T} = \frac{1}{2} m A^2 \omega^2 \cos^2(\omega t + \varphi)$

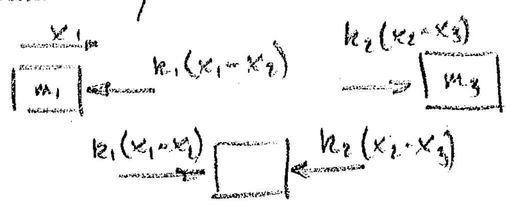
$$\omega = \sqrt{\frac{V_{max}}{T^*}} = \sqrt{\frac{FlA^2}{2a(l-a)}\cdot\frac{2}{A^2 m}} = \sqrt{\frac{Fl}{ma(l-a)}}$$

used to get fundamental frequency. will give freq > freq of system (actual)

Holzer Method

gets amplitude ratio, mathematically solved





$m_1 \ddot{x}_1 + k_1 x_1 - k_1 x_2 = 0$

$m_2 \ddot{x}_2 + (k_1 + k_2) x_2 - k_1 x_1 - k_2 x_3 = 0$

$m_3 \ddot{x}_3 + k_2 x_3 - k_2 x_2 = 0$

let $x_i = A_i \sin(\omega t + \varphi)$

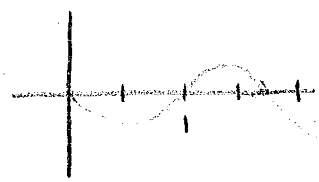
let $m_1 = m_2 = m_3 = 1$
$k_1 = k_2 = 1$

$$(1-\omega^2)A_1 - A_2 = 0 \qquad \text{divide by } A_1 \qquad (1-\omega^2) - A_2' = 0$$
$$-A_1 + (2-\omega^2)A_2 - A_3 = 0 \qquad\qquad -1 + (2-\omega^2)A_2' - A_3' = 0$$
$$-A_2 + (1-\omega^2)A_3 = 0 \qquad\qquad -A_2' + (1-\omega^2)A_3' = 0$$

$$\boxed{\begin{aligned} A_2' &= (1-\omega^2) \\ (2-\omega^2)A_2' - A_3' &= 1 \\ -A_2' + (1-\omega^2)A_3' &= 0 \end{aligned}} \longleftarrow \text{control eq}$$

guess ω to get A_2'
get A_3' then put into control eq & plot remainder of control Eq with ω

$\omega = .5$
$A_2 = .75$ the error $= +.514$
$A_3 = .3125$



$\omega = .75$
$A_2 = .4375$ error =
$A_3 = -.37$

$\omega = 1.0$
$\omega = \sqrt{3}$

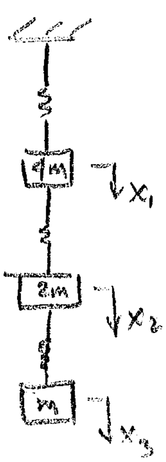
$\omega = \sqrt{3}$
$A_2 = -2$ error $= 0$
$A_3 = +1$

$\omega = 0$
$A_2 = 1$
$A_3 = 1$
error $= 0$

Rayleigh – Holzer Method

Holzer Method gives amplitude ratio $\rightarrow$ used in Rayleigh technique to get ω.

Influence coeff



$$-x_1 = \alpha_{11} 4m\ddot{x}_1 + \alpha_{12} 2m\ddot{x}_2 + \alpha_{13} m\ddot{x}_3$$
$$-x_2 = \alpha_{21} 4m\ddot{x}_1 + \alpha_{22} 2m\ddot{x}_2 + \alpha_{23} m\ddot{x}_3$$
$$-x_3 = \alpha_{31} 4m\ddot{x}_1 + \alpha_{32} 2m\ddot{x}_2 + \alpha_{33} m\ddot{x}_3$$

$\alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{31} = 1/3K$
$\alpha_{22} = \alpha_{32} = \alpha_{23} = 4/3K$
$\alpha_{33} = 7/3K$

$x_1 = A_1 \sin(\omega t + \varphi) \qquad x_3 = C_1 \sin(\omega t + \varphi)$
$x_2 = B_1 \sin(\omega t + \varphi)$

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \omega^2 m \begin{bmatrix} 4\alpha_{11} & 2\alpha_{12} & \alpha_{13} \\ 4\alpha_{21} & 2\alpha_{22} & \alpha_{23} \\ 4\alpha_{31} & 2\alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix}$$

let $A_1 = 1$
$B_1 = 2$
$C_1 = 4$

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 12 \\ 36 \\ 48 \end{bmatrix} = \frac{12\omega^2 m}{3K} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

let $A_1 = 1$
$B_1 = 3$
$C_1 = 4$

$$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 14 \\ 44 \\ 56 \end{bmatrix} = \frac{14\omega^2 m}{3K} \begin{bmatrix} 1 \\ 3.2 \\ 4.0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3.2 \\ 4 \end{bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 14.4 \\ 45.6 \\ 57.6 \end{bmatrix} = \frac{14.4\omega^2 m}{3K} \begin{bmatrix} 1 \\ 3.18 \\ 4.0 \end{bmatrix}$$

Gives only one frequency mode   to get diff mode shapes look at orthogonality

$$14.4 \frac{\omega^2 m}{3K} = 1$$

$$\omega_i^2 [\alpha][M]\{q^{(i)}\} = [K]\{q^{(i)}\}$$

abc

$$\omega_j^2 [\alpha][M]\{q^{(j)}\} = [K]\{q^{(j)}\}$$

stiffness matrix   identity for above

$$\omega_i^2 [M][\alpha][M]\{q^{(i)}\} = [M][K]\{q^{(i)}\}$$

symmetrizing

$$\omega_j^2 [M][\alpha][M]\{q^{(j)}\} = [M][K]\{q^{(j)}\}$$

$$\omega_i^2 \{q^{(j)}\}^T [M][\alpha][M]\{q^{(i)}\} = \{q^{(j)}\}^T [M][K]\{q^{(i)}\}$$

$$\omega_j^2 \{q^{(i)}\}^T [M][\alpha][M]\{q^{(j)}\} = \{q^{(i)}\}^T [M][K]\{q^{(j)}\}$$

trans of product = product of transposes reversed

$$\omega_i^2 \{q^{(i)}\}^T [M][\alpha][M]\{q^{(j)}\} = \{q^{(i)}\}^T [M][K]\{q^{(j)}\}$$

since $[M]^T = [M]$ $\quad [\alpha]^T = [\alpha]$ $\quad \{[M][K]\}^T = \{K\}^T\{M\}^T = (K)(M)$

$$(\omega_i^2 - \omega_j^2)\{q^{(i)}\}^T [M][\alpha][M]\{q^{(j)}\} = 0$$ from the above & previous

if $\omega_i \neq \omega_j$ then

$$\{q^{(i)}\}^T [M][\alpha][M]\{q^{(j)}\} = 0$$

from first mode $\omega_1^2, A_1, B_1, C_1$

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \frac{\omega^2 M}{3K} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix}$$

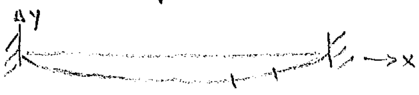
$$\begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 0 & -\frac{a_2 B_1}{a_1 A_1} & -\frac{a_3 C_1}{a_1 A_1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix}$$

if $a_1 A_1 A_2 + a_2 B_1 B_2 + a_3 C_1 C_2 = 0$ from orthogonality

$$\therefore \quad A_2 = \frac{-a_2 B_1 B_2 - a_3 C_1 C_2}{a_1 A_1} \qquad B_2 = B_2 \quad C_2 = C_2$$

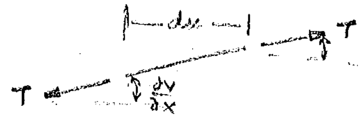
12-14-11

Continuous Systems



$y = f(x,t)$

Consider an element



$T \quad \frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2} dx$

$\frac{\partial y}{\partial x}$

in y direct $\quad T\left(\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2} dx\right) - T\frac{\partial y}{\partial x} = \gamma \frac{\partial^2 y}{\partial t^2} dx$

mass/length unit

$$T\frac{\partial^2 y}{\partial x^2} = \gamma \frac{\partial^2 y}{\partial t^2} \qquad \text{or}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad c^2 = T/\gamma$$

Separation of variable

$$Y = X(x)\,\Theta(t)$$

$$\Theta \ddot{X} - \frac{1}{c^2} X \ddot{\Theta} = 0 \qquad \text{or} \qquad \frac{\ddot{X}}{X}c^2 = \frac{\ddot{\Theta}}{\Theta} = \text{constant } \mu$$

$\ddot{\Theta} - \mu\Theta = 0 \qquad \mu \neq 0$

$\ddot{X} - \frac{\mu}{c^2} X = 0 \qquad \mu < 0 \quad$ let $\mu = -\omega^2$ then

$$\ddot{\Theta} + \omega^2\Theta = 0 \qquad \text{or} \qquad \Theta = C_1 \sin\omega t + C_2\cos\omega t$$

$$\ddot{X} + \left(\frac{\omega}{c}\right)^2 X = 0 \qquad X = A\sin \omega/c\, x + B\cos \omega/c\, x$$

$$Y = \left(A\sin \omega/c\, x + B\cos \omega/c\, x\right)\left(C_1\sin\omega t + C_2\cos\omega t\right)$$

at $x = 0 \quad y = 0 \quad \rightarrow \quad B = 0$

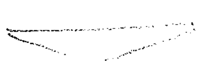
$x = L \quad y = 0 \qquad A\sin \omega/c\, L = 0 \qquad \frac{\omega L}{c} = n\pi \qquad \omega = \frac{n\pi c}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\gamma}}$

$$y = \left(A\sin\frac{\omega x}{c}\right)\left(C_1 \sin\omega t + C_2\cos\omega t\right)$$

when $t = 0 \quad \dot{y} = 0 \rightarrow C_1 = 0 \quad \therefore \quad y = AC_2 \sin\frac{\omega}{c}x\cos\omega t = y_0 \sin\frac{\omega x}{c}\cos\omega t$

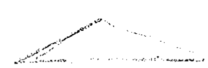
gives rise to standing waves (permanent node points) $\qquad y = \frac{y_0}{2}\left[\sin\omega\left(\frac{x}{c} + t\right) + \sin\left(\frac{x}{c} - t\right)\right]$

if mode shape



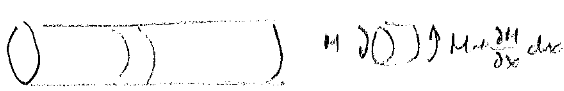
$$y = \sum a_n \sin\frac{n\pi x}{L}$$

$$y = \left(\frac{2y_0}{L}\right)x \qquad 0 \le x \le L/2$$



$$y = \left(\frac{2y_0}{L}\right)(L - x) \qquad \frac{L}{2} \le x \le L$$

$$y = \sum A_n \sin\frac{n\pi x}{L}\cos\omega t$$



$\Theta = \frac{ML}{GJ} \qquad \frac{d\theta}{dx} = \frac{M}{GJ}$

$$M + \frac{\partial M}{\partial x}dx - M = I\frac{\partial^2\theta}{\partial t^2}$$

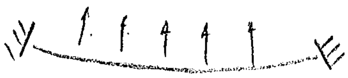
$$\frac{\partial M}{dx}dx = I\frac{\partial^2\theta}{\partial t^2}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial M}{\partial x}\frac{1}{GJ}$$

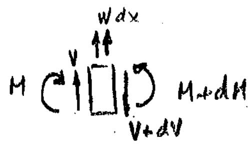
$$GJ\, dx\, \frac{\partial^2\theta}{\partial x^2} = \gamma\, dx\, J\frac{\partial^2\theta}{\partial t^2}$$

or $G\dfrac{\partial^2\theta}{\partial x^2} = \gamma\dfrac{\partial^2\theta}{\partial t^2}$ $\qquad \theta_{,xx} = \dfrac{\gamma}{G}\,\theta_{,tt}$

for free boundary $M=0$ $\qquad M = GJ\dfrac{\partial\theta}{\partial x}$

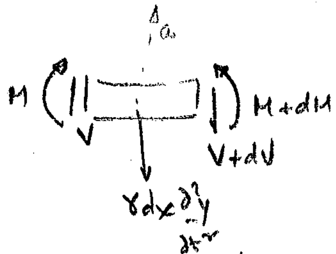


$M = EI\dfrac{d^2y}{dx^2}$



$V = \dfrac{dM}{dx}$ $\qquad$ static

$W = \dfrac{dV}{dx}$ $\qquad W = EI\dfrac{d^4y}{dx^4}$



inertia d'alembert

$$EI\frac{\partial^4 y}{\partial x^4} = -\gamma\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^4 y}{\partial x^4} = -a^4\frac{\partial^2 y}{\partial t^2} \qquad \therefore \frac{\partial^4 y}{\partial x^4} + a^4\frac{\partial^2 y}{\partial t^2} = 0$$

$a^4 = \left(\dfrac{\gamma}{EI}\right)$

$Y = X(x)\,T(t)$

$X^{(4)}(T) + a^4 X T'' = 0$

$$\frac{X^{(4)}}{a^4 X} = -\frac{T''}{T} = +\mu = +\omega^2$$

$X^{(4)} - a^4\mu X = 0$

$T'' + \mu T = 0$

let $\underbrace{\left(\dfrac{d^2}{dx^2} + a^2\omega\right)}_{L_1}\underbrace{\left(\dfrac{d^2X}{dx^2} - a^2\omega X\right)}_{L_2} = 0$ $\qquad \dfrac{d^2}{dx^2}L_2 + a^2\omega L_2 = 0 \Rightarrow$

$L_2 = A\sin(a^2\omega)^{1/2}x + B\cos(a^2\omega)^{1/2}x$

$L_1 = A_1\sinh(a^2\omega)^{1/2}x + B_1\cosh(a^2\omega)^{1/2}x$

$\therefore\ \overline{X} = A\sin(a^2\omega)^{1/2}x + B\cos(a^2\omega)^{1/2}x + A_1\sinh x(a^2\omega)^{1/2}x + B_1\cosh(a^2\omega)^{1/2}x$

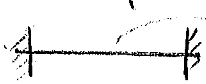
$T = C_1\sin\omega t + C_2\cos\omega t$

S.S. $\quad X=0$ @ $x=0,L$ $\quad$ will give $\quad B_1=0 \quad B=0$

$\dfrac{d^2X}{dx^2}=0$ @ $x=0,L$ $\qquad A_1=0,\ A=0$ or $(a^2\omega)^{1/2}L = n\pi$

since $\sinh(k) = 0$ at $k=0$ $\qquad \boxed{\omega = \dfrac{n^2\pi^2}{L^2a^2}}$

$A\sin(a^2\omega)^{1/2}x\left[C_1\sin\omega t + C_2\cos\omega t\right]$ $\quad$ if $\dot{y}=0$ $\quad C_1\to 0$

HW  fixed beam first freq no continuous systems.

Potential is − work done

mass = mgh

spring = $\frac{1}{2} k \Delta^2$

Rayleigh guess mode shape   max potential = max kinetic   get freq

Holzer   iterate on guess at freq

Holzer

$$
\left.\begin{aligned}
(-m_1\omega^2 + k_1)A_1 - k_1 A_2 &= 0 \\
-k_1 A_1 + (-m_2\omega^2 + k_1 + k_2)A_2 - k_2 A_3 &= 0 \\
-k_2 A_2 + (-m_3\omega^2 + k_2)A_3 &= 0
\end{aligned}\right\}
$$

gives ratio of mode shapes
assume $A_1 = 1$ (normalize

Find $A_2 = A_2(A_1)$

$A_3 = A_3(A_2, A_1)$ put into 3rd eq   get remainder ; pick $\omega \to$ get A_2, A_3 put into 3

& find remainder   keep doing it until ω gives 3rd eq = 0

CESAR LEVY

AE 212

2. Since $W = K\Delta$ $K = W/\Delta$

$$\omega = \sqrt{\frac{K}{M}} = \left(\frac{W/\Delta}{W/g}\right)^{1/2} = \left(g/\Delta\right)^{1/2} = 47.7 \text{ rad/sec}$$

$$f = \frac{\omega}{2\pi} = \frac{47.7}{6.283} = 7.59 \text{ cps}$$

8. Break up into 2 series springs $\therefore$ $W = K_1\Delta_1 = K_1\Delta_2 \Rightarrow \frac{\Delta}{2} = \ldots$
& $K_1 = K_2 = 2K$ $\therefore$ put in parallel $\therefore$ $K' = 4K$
if $\Delta = \delta$ of old system & $\Delta' = \delta$ of new system $\Delta = 12\Delta'$
since $f = \frac{1}{3}$ cps $f' = \sqrt{12} f = \sqrt{12}(20 cpm) = 69.25$ cp...

note: $2\pi f = \omega = \sqrt{\frac{K}{M}} = \sqrt{g/\Delta}$

$$2\pi f' = \omega' = \sqrt{\frac{K'}{M}} = \sqrt{g/\Delta'} \quad \text{but} \quad \frac{K}{M} = \frac{g}{12\Delta'}$$

11. The equation of motion of the physical pendulum

$$I\ddot{\theta} + mgl\sin\theta = 0 \quad \text{can be reduced}$$

to $I\ddot{\theta} + mgl\theta = 0$ if $\sin\theta \sim \theta$

therefore

$$\omega = \sqrt{\frac{mgl}{I}} = \sqrt{gl/K^2}$$

if one were to solve for k^2 using $f = 1.2\,cps$ then

$$\frac{gl}{4\pi^2 f^2} = k^2 = \frac{386 \cdot 3}{4\pi^2(1.44)}\ in^2 = 20.35\ in^2$$

but $k^2 = k_{cm}^2 + l^2 \quad \therefore\ k_{cm}^2 = 11.35\ in^2$

now for $f' = .9\,cps \qquad k'^2 = \dfrac{gl'}{4\pi^2 f'^2}$

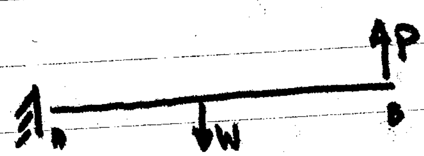
which leads to $l'^2 - 12.05\,l' + 11.35 = 0$

since $k'^2 = k_{cm}^2 + l'^2$

the solution gives $l' = GB = 1.025\,in$

$\therefore\quad OB = 3 - l' = 1.975\ in$

23



$\Sigma M_A = PL - Wa = 0 \qquad P = \dfrac{Wa}{L}$

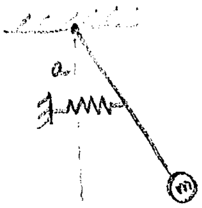
but $P = k_1\Delta_1 \quad \therefore\ \Delta_1 = \dfrac{Wa}{k_1 l}$ note that due to $P\ \exists$ a Δ_p at pt $A = Wa^2/k_1 l^2$; due to $W\ \exists\ \Delta_W = W/k_2$;

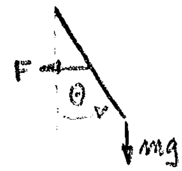
$$\therefore\ \Delta_{stat} = \Delta_p + \Delta_W = \frac{W}{l^2}\left(\frac{a^2k_2 + l^2k_1}{k_1k_2}\right)$$

define a $\dfrac{1}{k'} = \dfrac{1}{l^2}\left(\dfrac{a^2k_2 + l^2k_1}{k_1k_2}\right) \quad \therefore\ m\ddot{y} = -k'y$

or $\quad m\ddot{y} + l^2\left(\dfrac{k_1k_2}{a^2k_1 + l^2k_2}\right)y = 0 \quad \& \quad \dfrac{\omega}{2\pi} = f = \dfrac{l}{2\pi}\sqrt{\dfrac{k_1k_2}{(a^2k_1 + l^2k_2)m}}$

9-30-71
Cesar Levy





$$I_0\ddot{\theta} = -mgl\sin\theta - Fa$$

$F = k\delta \qquad \dfrac{\delta}{a} = \dfrac{l\sin\theta}{l\cos\theta} \qquad \delta = a\tan\theta$

$I_0\ddot{\theta} = -mgl\sin\theta - ka^2\tan\theta$ if $\theta < 15^\circ$ $\sin\theta \sim \tan \sim \theta$

$I_0\ddot{\theta} = -mgl\theta - ka^2\theta$ $I_0 = ml^2$

$ml^2\ddot{\theta} + (mgl + ka^2)\theta = 0$

$$\omega_n = \sqrt{\left(\frac{mgl + ka^2}{ml^2}\right)}$$

since Δ is the same then $P = k\delta + k\delta = 2k\delta$

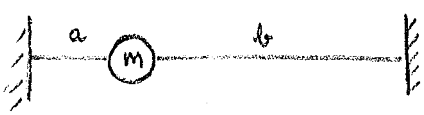
$I_0\ddot{\theta} = -K\theta - 2kl(l\sin\theta)$ for small θ $\sin\theta \sim \theta$

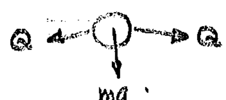
$= -K\theta - 2kl^2\theta$ moment of inertia about the spring is $ml^2/3$

$\therefore \dfrac{ml^2}{3}\ddot{\theta} = -(K + 2kl^2)\theta$

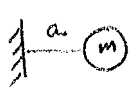
$$\therefore \omega_n = \sqrt{\frac{3K + 6kl^2}{ml^2}}$$

2.32





since deflection is the same we can find an equivalent spring system for



$I_1\ddot{\theta} = -Q\sin\theta\, a$ $I_1 = ma^2$ $\therefore$ since $\dfrac{k_1}{m} = \dfrac{+Qa}{ma^2}$

if θ is small

$k_1 = \dfrac{Q}{a}$ similarly for

$I_2\ddot{\delta} = -Q\sin\delta\, b$ where $I_2 = mb^2$

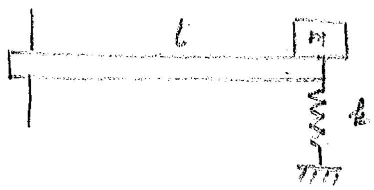
$\therefore k_2 = \dfrac{Q}{b}$ $\therefore$ if δ is the same $k' = k_1 + k_2$

$= \dfrac{Q(a+b)}{ab}$

therefore $\ddot{y} + \frac{k'}{m} y = 0$ $\qquad \therefore \quad \omega = \sqrt{\frac{k'}{m}} = \sqrt{\frac{Q}{m}\left(\frac{a+b}{ab}\right)}$

also $\quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{mab}{Q(a+b)}}$

2.33



since beam bends we can find an equivalent spring constant. If a load were applied at the end the deflection obtained would be

$$\Delta = Pl^3/3EI$$

$P/\Delta = k_1 = \frac{3EI}{l^3}$ since Δ is experienced by m and the spring the system is in parallel $\therefore \quad k' = k_1 + k = \frac{3EI + kl^3}{l^3}$

$$\therefore \quad \ddot{y} + \frac{3EI + kl^3}{ml^3} y = 0$$

$$\omega = \sqrt{\frac{3EI + kl^3}{ml^3}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{3EI + kl^3}{ml^3}}$$

3.5, 6, 12, 16

5 $\quad m\ddot{x} + kx = 0 \qquad \omega = \sqrt{k/m} \qquad f_1 = \frac{1}{2\pi}\sqrt{k/m}$

$m\ddot{x} + c\dot{x} + kx = 0$ since a frequency exists, the roots of system are

$$\frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\frac{c}{2m} \pm j\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

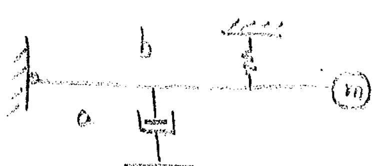
$$\therefore \quad e^{-\frac{c}{2m}t}\left(A\cos\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t + B\sin\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t\right) \text{ or}$$

$$C e^{-\frac{c}{2m}t}\cos\left\{\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t + \varphi_0\right\}$$ ω of this system is

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \implies f_2 = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

$$\therefore \quad \boxed{c = 4\pi m\sqrt{f_1^2 - f_2^2}}$$

6



$$J\ddot{\theta} = -kb\,b\sin\theta - cv\,a\sin\theta$$

$$v = a\dot{\theta} \qquad \text{small } \theta \quad \sin\theta \approx \theta$$

$$\therefore \quad J\ddot{\theta} + kb^2\theta + ca^2\dot{\theta} = 0 \qquad J = ml^2$$

let $\quad J = \bar{M} \qquad kb^2 = \bar{K} \qquad ca^2 = \bar{C}$

now
$$C_c = 2\sqrt{\bar{M}\bar{K}} = 2lb\sqrt{mk}$$

$$\omega = \frac{C_c}{\bar{M}} = \frac{b}{ml}\sqrt{mk}$$

$$\zeta = \frac{\bar{C}}{C_c} = \frac{ca^2}{2lb\sqrt{mk}}$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega = \frac{1}{2l^2 m}\sqrt{4l^2b^2mk - c^2a^4}$$

$$\therefore \boxed{f_d = \frac{1}{4\pi l^2 m}\sqrt{4l^2b^2mk - c^2a^4}}$$

12 $\quad k = 45$ lb/in $\qquad mg = 19.3$ lb $\qquad$ for critically damped system

$$\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = 0$$

$$\therefore \quad c = \sqrt{\frac{4mk}{12g}} = \sqrt{\frac{4\cdot 19.3\cdot 45}{32.2\cdot 12}} = 3 \ \frac{\text{lb}\cdot\text{sec}}{\text{in}}$$

$$.05\ddot{y} + 3\dot{y} + 45y = 0$$

$$y = (4 + C_2 t)e^{-30t} \qquad C_2 = 120 \text{ iff } \dot{y}(0) = 0$$

$$y = (4 + 120t)e^{-30t}$$

at $t = .01 \quad y = 5.2\,e^{-.3} = 5.2(.74) = 3.85$ in

at $t = .1 \quad y = 16\,e^{-3} = 16(.0495) = .792$ in

at $t = 1 \quad y = 124\,e^{-30} \approx 0$

16. $\quad \zeta = .1 = \frac{c}{C_c} \qquad m = .05 \ \frac{\text{lb-sec}^2}{\text{in}} \qquad k = 45$ lb/in

$$C_c = 2\sqrt{mk} = 2(1.5)\ \frac{\text{lb-sec}}{\text{in}} \implies c = 3 \ \frac{\text{lb-sec}}{\text{in}}$$

$$\omega = \frac{C_c}{2m} = \frac{.3}{.1} = 3 \text{ rad/sec}$$

$$f_d = \frac{1}{2\pi}\sqrt{1-\zeta^2}\,\omega = \frac{1}{2\pi}(.995)(30) = 4.75 \text{ cps}$$

$x = \bar{X} e^{-3t} \sin(\omega_d t + \varphi)$ (This solves diff eq. $.05\ddot{x} + 3\dot{x} + 45x = 0$) . . .

$\omega_d = 2\pi f_d = 29.9$

from $t=0$ $x = 4$ in

$t=0$ $\dot{x} = 0$ $\bar{X} = 4.02$ $\varphi = 84°15'$

$\therefore$ $x = 4.02 e^{-3t} \sin(29.9t + 84°15')$

amplitude after ½ & 1 cycle is found by setting $\dot{x} = 0$

this gives the equation $\omega_d t = n\pi$ for ½ cycle $n=1$

1 cycle $n=2$

$\therefore$ $t_{1/2} = .105$

$t_1 = .211$

$\therefore$ $x_{1/2} = 4.02 e^{-3(.158)} \sin(264°15') = 4.02(.729)(-.995) = -2.92$ in

$x_1 = 4.02 e^{-3(.211)} \sin(444°15') = 4.02(.532)(+.995) = 2.13$ in

Cesar Levy
10-7-71

4-2 $\sqrt{k/m} = \omega \qquad \sqrt{\frac{25(386)}{2.44}} = 62.75 \text{ rad/sec}$

$\omega_f = \frac{2\pi}{T} = \frac{2\pi}{1/3} = 18.85 \text{ rad/sec}$

$\frac{\omega_f}{\omega} = \frac{18.85}{62.75} = .3$

$MF = \frac{1}{1-\left(\frac{\omega_f}{\omega}\right)^2} = \frac{1}{1-.09} = 1.1$

$X_0 = P_0/k = \frac{100}{25} = 4$

$MF = \frac{X}{X_0} = \frac{X}{4} = 1.1 \qquad X = 4.4 \text{ in}$ forced amplitude

4-5 $\sqrt{\frac{k}{m}} = \omega \qquad \sqrt{\frac{20(386)}{19.3}} = 20 \text{ rad/sec}$

$X_0 = \frac{P_0}{K} = \frac{42}{20} = 2.1 \text{ in}$

$X = 3 \text{ in} \qquad \therefore \qquad MF = \frac{X}{X_0} = \frac{10}{7} = \frac{1}{1-r^2}$

$\therefore \quad r^2 = .3 \qquad r = .5475$

$\omega_f = \omega r = 20(.5475) = 10.95 \text{ rad/sec}$

$2\pi f_f = \omega_f \implies f_f = \frac{\omega_f}{2\pi} = \frac{10.95}{6.28} = 1.74 \text{ cps.}$

4-12 $m\ddot{x} + kx = P_0 \sin \omega_f t$

solution to x_c is $x_c = A_1 \sin \omega t + A_2 \cos \omega t$ where $\omega = \sqrt{k/m}$

solution to x_p is $x_p = \frac{P_0/m}{\omega^2 - \omega_f^2} \sin \omega_f t$

$\therefore$

$$x = \frac{P_0/m}{\omega^2 - \omega_f^2} \sin \omega_f t + A_1 \sin \omega t + A_2 \cos \omega t$$

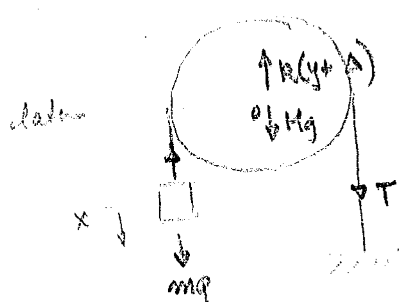
since @ $t=0$ $x=x_0$ $\qquad A_2 = x_0$

since @ $t=0$ $\dot{x}=\dot{x}_0$ $\qquad A_1\omega + \omega_f \dfrac{P_0/m}{\omega^2-\omega_f^2} = \dot{x}_0$

$$A_1 = \frac{\dot{x}_0}{\omega} - \frac{\omega_f}{\omega}\,\frac{P_0/m}{\omega^2-\omega_f^2}$$

$\therefore$

$$x = \frac{P_0/m}{\omega^2-\omega_f^2}\left[\sin\omega_f t - \frac{\omega_f}{\omega}\sin\omega t\right] + \left[\frac{\dot{x}_0}{\omega}\sin\omega t + x_0\cos\omega t\right]$$



$$KE = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M\dot{y}^2 + \tfrac{1}{2}I\dot{\theta}^2$$

$$KE = \int_y^0 [Mg - k(y+\Delta)]\,dy + \int_x^0 mg\,dx$$

From equil $T = mg$ $(\Sigma M_0 = 0)$

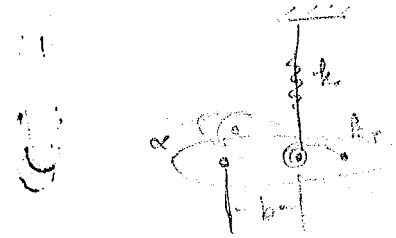
From statics $mg + Mg + T = k\Delta$ $\qquad$ also $x = 2y$

$$\therefore \quad PE = \int_y^0 [Mg - k(y) - Mg - 2mg]\,dy + \int_y^0 2mg\,dy$$

$$\therefore \quad PE = \int_y^0 -ky\,dy = \frac{ky^2}{2}$$

$\therefore$ $T+V = C$ $\qquad \dfrac{d}{dt}\left(\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M\dot{y}^2 + \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}ky^2\right) = 0$

$\therefore$ $4m\ddot{x} + \tfrac{3}{2}M\ddot{x} + kx = 0$ $\qquad$ QED



$m\ddot{y} = -ky - k_T\alpha$ $\qquad\qquad$ $m(\ddot{y} - b\ddot{\alpha}) = -ky$

$I_B\ddot{\alpha} =$

$KE = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I_B\dot{\theta}^2$

$PE = mg(h - b\alpha) + \frac{kx^2}{2} + k_T\frac{\theta^2}{2}$

$b\alpha = l$

$m(\ddot{y} - b\ddot{\alpha}) = -ky$ $\qquad\qquad$ $\ddot{\alpha}$

$m\ddot{y} + ky - mb\ddot{\alpha} = 0$

$I_B\ddot{\alpha} = -k_T\alpha$

$I_B(\ddot{\alpha}) = -k_T\alpha - mgb(b\alpha)$ $\quad \ddot{y} - b\ddot{\alpha}$

$I_B(\ddot{\alpha}) = -k_T\alpha - m(\ddot{y} - b\ddot{\alpha})b$

$m(\ddot{y} - b\ddot{\alpha}) = -ky$

Assume $y = A_1\sin(\omega_1 t + \varphi)$

$\theta = A_2\sin(\omega_1 t + \varphi)$

$I_B(-A_2\omega_1^2) = -k_T A_2 + mb(A_1\omega_1^2 - b\omega_1^2 A_2)$

$m(A_1\omega_1^2 - b\omega_1^2 A_2) = kA_1$

$A_1[mb\omega_1^2] + A_2[I_B\omega_1^2 - k_T - mb^2\omega_1^2] = 0$

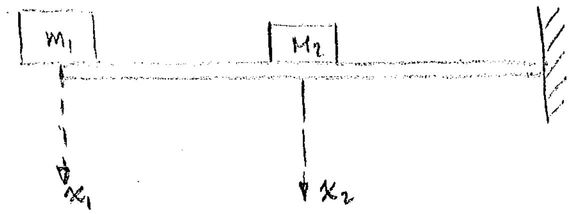
$A_1[m\omega_1^2 - k] + A_2[-mb\omega_1^2] = 0$

$(mb\omega_1^2)^2 + (m\omega_1^2 - k)(I_B\omega_1^2 - k_T - mb^2\omega_1^2) = 0$

$\omega_1^4(m^2b^2 + I_Bm) - \omega_1^2(kI_B + kw - kmb^2) + kk_T = 0$

$\omega_1^2 = \frac{(kI_B + kw - kmb^2)}{2} \pm \sqrt{\left(\frac{kI_B + kw - kmb^2}{2}\right)^2 - kk_T(m^2b^2}$

7, 23



$$x_1 = P_1\alpha_{11} + P_2\alpha_{12} \qquad P_1 = -m_1\ddot{x}_1$$

$$x_2 = P_1\alpha_{21} + P_2\alpha_{22} \qquad P_2 = -m_2\ddot{x}_2$$

$$x_1 = -m_1\ddot{x}_1\alpha_{11} + m_2\ddot{x}_2\alpha_{12}$$

$$x_2 = -m_1\ddot{x}_1\alpha_{21} - m_2\ddot{x}_2\alpha_{22}$$

$$k\cancel{y}x_1 \qquad k\alpha_{11} = 1$$

$$P_1\alpha_{11} = x_1$$

$$\alpha_{11} = \frac{1}{3}\frac{Wl^3}{EI}$$

$1\ \alpha_{11}$ $\qquad$ 1

$$\cancel{W} = k\delta = \frac{K}{3}\frac{Wl^3}{EI}$$

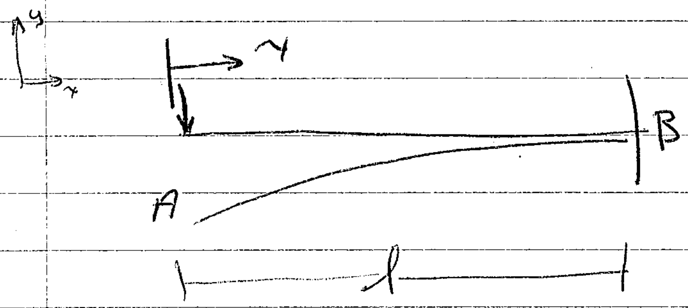
$$\alpha_{11} = \frac{1}{K} = \frac{1}{3}\,l^3/EI \qquad \frac{1}{K} = \frac{1}{3}\,l^3/EI$$

$$\text{if } a = 1/2 \qquad \text{Max } y = \frac{1}{3}\,W/EI\left(3\frac{l^3}{4} - \frac{l^3}{8}\right) = \frac{1}{3}\,W/EI\left(5\frac{l^3}{8}\right)$$

$$\delta = \frac{1}{3}\,\cancel{W}\frac{5l^3}{8}$$

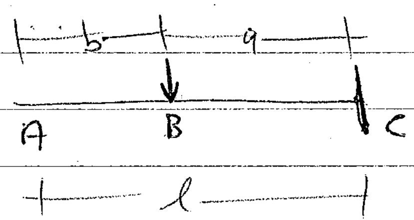
$$Y = \frac{1}{3}\frac{W}{EI}\frac{l^3}{8} \qquad \frac{1}{K} = \frac{l^3}{24EI} \qquad K = \frac{1}{\alpha_{22}}$$

$$Y = -\frac{1}{6}\frac{W}{EI}\left[-\frac{l^3}{8} + \frac{3l^3}{4}\right] = \frac{5}{48}\,l^3/EI$$



$$y = -\frac{1}{6}\frac{W}{EI}\left(x^3 - 3l^2x + 2l^3\right)$$

$$\text{Max } y = -\frac{1}{3}\frac{Wl^3}{EI}$$



A to B $\quad y = -\frac{1}{6}\frac{W}{EI}\left(-a^3 + 3a^2l - 3a^2x\right)$

B to C $\quad y = -\frac{1}{6}\frac{W}{EI}\left[(x-b)^3 - 3a^2(x-b) + 2a^3\right]$

$$\text{MAX } y = -\frac{1}{6}\,W/EI\left(3a^2l - a^3\right)$$

$$A_1 - \frac{m_1}{3} \rho A_1 \omega^2 - \frac{5}{16} m_2 \rho A_2 \omega^2 = 0$$

$$A_2 - \frac{m_2}{8} \rho A_2 \omega^2 - \frac{5}{16} m_1 \rho A_1 \omega^2 = 0$$

$$A_1 \left[1 - \frac{m_1}{3} \rho \omega^2 \right] + A_2 \left[- \frac{5 m_2}{16} \rho \omega^2 \right] = 0$$

$$A_1 \left[-\frac{5}{16} m_1 \rho \omega^2 \right] + A_2 \left[1 - \frac{m_2}{8} \rho \omega^2 \right] = 0$$

$$\left[1 - \frac{m_1 \rho \omega^2}{3} \right] \left[1 - \frac{m_2}{8} \rho \omega^2 \right] - \left[\frac{5}{16} m_1 \rho \omega^2 \right] \left[\frac{5}{16} m_2 \rho \omega^2 \right] = 0$$

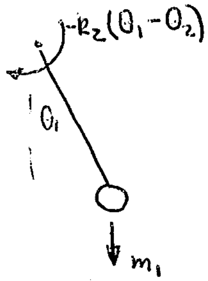
$$1 - \frac{m_2}{8} \rho \omega^2 - \frac{m_1}{3} \rho \omega^2 + \frac{m_1 m_2}{24} \rho^2 \omega^4 - \frac{25}{256} m_1 m_2 \rho^2 \omega^4 = 0$$

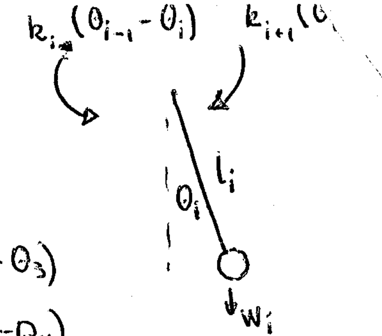
(10.67) $\left(\frac{24}{256}\right)$ $$\omega^4 \left[\frac{m_1 m_2}{24} \rho^2 - \frac{25}{256} m_1 m_2 \rho^2 \right] - \left[\frac{m_1 \rho}{3} + \frac{m_2 \rho}{8} \right] \omega^2 + 1 = 0$$

$$\omega^4 \left[- \frac{14.33}{256} m_1 m_2 \rho^2 \right] - \left[\omega^2 \right] \left(\frac{8 m_1 + 3 m_2}{24} \right) \rho + 1$$

$$\omega^4 \left(\frac{14.33}{256} m_1 m_2 \rho^2 \right) + \left(\frac{8 m_1 + 3 m_2}{24} \right) \rho \omega^2 - 1 = 0$$

8.4





$$m_1 l_1^2 \ddot{\theta}_1 = -W_1 l_1 \sin\theta_1 - k_2(\theta_1-\theta_2)$$

$$m_2 l_2^2 \ddot{\theta}_2 = -W_2 l_2 \sin\theta_2 + k_2(\theta_1-\theta_2) - k_3(\theta_2-\theta_3)$$

$$m_3 l_3^2 \ddot{\theta}_3 = -W_3 l_3 \sin\theta_3 + k_3(\theta_2-\theta_3) - k_4(\theta_3-\theta_4)$$

$$m_4 l_4^2 \ddot{\theta}_4 = -W_4 l_4 \sin\theta_4 + k_4(\theta_3-\theta_4)$$

$$m_1 l_1^2 \ddot{\theta}_1 + (W_1 l_1 + k_2)\theta_1 - k_2\theta_2 = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + (W_2 l_2 + k_2 + k_3)\theta_2 - k_2\theta_1 - k_3\theta_3 = 0$$

$$m_3 l_3^2 \ddot{\theta}_3 + (W_3 l_3 + k_3 + k_4)\theta_3 - k_3\theta_2 - k_4\theta_4 = 0$$

$$m_4 l_4^2 \ddot{\theta}_4 + (W_4 l_4 + k_4)\theta_4 - k_4\theta_3 = 0$$

8.15

$$T = \tfrac{1}{2} I_1 \dot{\theta}_1^2 + \tfrac{1}{2} I_2 \dot{\theta}_2^2 + \tfrac{1}{2} I_3 \dot{\theta}_3^2 + \tfrac{1}{2} I_4 \dot{\theta}_4^2$$

$$V = \tfrac{1}{2} k_2(\theta_1-\theta_2)^2 + \tfrac{1}{2} k_3(\theta_2-\theta_3)^2 + \tfrac{1}{2} k_4(\theta_3-\theta_4)^2 + \cancel{}$$

$$L = T - V$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = I_1\dot{\theta}_1 \qquad \frac{\partial L}{\partial \dot{\theta}_2} = I_2\dot{\theta}_2 \qquad \frac{\partial L}{\partial \dot{\theta}_3} = I_3\dot{\theta}_3 \qquad \frac{\partial L}{\partial \dot{\theta}_4} = I_4\dot{\theta}_4$$

$-k_2(\theta_1-\theta_2)$ $\qquad +k_2(\theta_1-\theta_2)-k_3(\theta_2-\theta_3)$ $\qquad +k_3(\theta_2-\theta_3)-k_4(\theta_3-\theta_4)$ $\qquad +k_4(\theta_3-\theta_4)$

$$\frac{\partial L}{\partial \theta_1} = \cancel{k_2\theta_1} \qquad \frac{\partial L}{\partial \theta_2} = \cancel{k_2(-\theta_2)+k_3\theta_2} \qquad \frac{\partial L}{\partial \theta_3} = \cancel{\theta_3(k_4-k_3)} \qquad \frac{\partial L}{\partial \theta_4} = \cancel{k_4(-\theta_4)}$$

$+k_2(\theta_1-\theta_2)$ $\qquad +k_3(\theta_2-\theta_3)+k_4(\theta_3-\theta_4)$

$$I_1\ddot{\theta}_1 - \cancel{k_2\theta_1} = 0 \qquad I_3\ddot{\theta}_3 - \cancel{\theta_3(k_4-k_3)} = 0$$

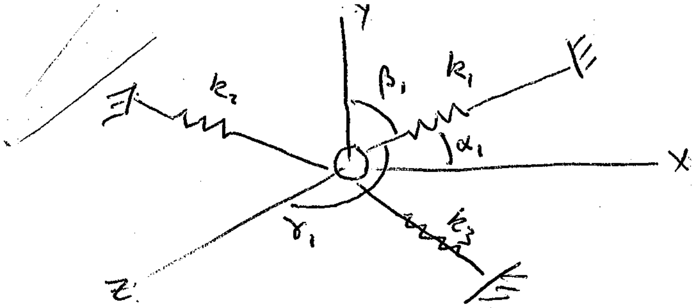
$+k_2(\theta_1-\theta_2)+k_3(\theta_2-\theta_3)$ $\qquad +k_4(\theta_3-\theta_4)$

$$I_2\ddot{\theta}_2 - \cancel{\theta_2(k_3-k_2)} = 0 \qquad I_4\ddot{\theta}_4 + \cancel{k_4\theta_4} = 0$$

8.17

$\cancel{T = \tfrac{1}{2}m(}$ $\qquad \cancel{T = \tfrac{1}{2}m(\dot{x}+\dot{y}+\dot{z})^2}$

$\cancel{V}$ $\qquad \cancel{V} =$



direction angles

$$V = \tfrac{1}{2}k_1(x\cos\alpha_1 + y\cos\beta_1 + z\cos\gamma_1)^2$$
$$+\tfrac{1}{2}k_2(x\cos\alpha_2 + y\cos\beta_2 + z\cos\gamma_2)^2$$
$$+\tfrac{1}{2}k_3(x\cos\alpha_3 + y\cos\beta_3 + z\cos\gamma_3)^2$$

$$\cancel{T = \tfrac{1}{2}k_1(\dot{x}\cos\alpha_1 + \dot{y}\cos\beta_1 + \dot{z}\cos\gamma_1)^2}$$

$T - V$ $\qquad T = \frac{1}{2}m(\dot{x} + \dot{y} + \dot{z})^2$ $\quad \cancel{+\tfrac{1}{2}k_2(\dot{x}\cos\alpha_2 + \dot{y}\cos\beta_2 + \dot{z}\cos\gamma_2)^2}$

$$\cancel{+\tfrac{1}{2}k_3(\dot{x}\cos\alpha_3 + \dot{y}\cos\beta_3 + \dot{z}\cos\gamma_3)^2}$$

$$\frac{\partial L}{\partial x} = -\Big[k_1(x\cos\alpha_1 + y\cos\beta_1 + z\cos\gamma_1)\cos\alpha_1 + k_2(x\cos\alpha_2 + y\cos\beta_2 + z\cos\gamma_2)\cos\alpha_2$$
$$+ k_3(x\cos\alpha_3 + y\cos\beta_3 + z\cos\gamma_3)\cos\alpha_3\Big]$$

$$\frac{\partial L}{\partial y} = -\Big[k_1(\qquad\qquad)\cos\beta_1 + k_2(\qquad\qquad)\cos\beta_2 + k_3(\qquad\qquad)\cos\beta_3\Big]$$

$$\frac{\partial L}{\partial z} = -\Big[k_1(\qquad\qquad)\cos\gamma_1 + k_2(\qquad\qquad)\cos\gamma_2 + k_3(\qquad\qquad)\cos\gamma_3\Big]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = m\ddot{y} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = m\ddot{z}$$

$$m\ddot{x} + x\big[k_{11}\big] + y\big[k_{12}\big] + z\big[k_{13}\big] = 0$$

$$k_1\cos\alpha_1\cos\alpha_1 + k_2\cos\alpha_2\cos\alpha_2 + k_3\cos\alpha_3\cos\alpha_3 = k_{11}$$
$$k_1\cos\alpha_1\cos\beta_1 + k_2\cos\alpha_2\cos\beta_2 + k_3\cos\alpha_3\cos\beta_3 = k_{12}$$
$$k_1\cos\alpha_1\cos\gamma_1 + k_2\cos\alpha_2\cos\gamma_2 + k_3\cos\alpha_3\cos\gamma_3 = k_{13}$$

$$m\ddot{y} + x\big[k_{21}\big] + y\big[k_{22}\big] + z\big[k_{23}\big] = 0$$

x y z 0 0 0
0 0 0

$$\cancel{\begin{bmatrix} k_1\cos\alpha_1 & k_2\cos\alpha_2 & k_3\cos\alpha_3 \end{bmatrix}} \begin{bmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \cos\gamma_1 & \cos\gamma_2 & \cos\gamma_3 \end{bmatrix} \begin{bmatrix} k_1\cos \\ k_2\cos \\ k_3\cos \end{bmatrix}$$

$\bar{\lambda}$ $\qquad x\cos$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$x\Big[\cos\alpha_1\cos\beta_1 + \cos\alpha_2\cos\beta_2 + \cos\alpha_3\cos\beta_3$$

Cesar Levy

AE212

$x_1 = A_1' \sin(\omega_1 t + \varphi) + A_2' \sin(\omega_2 t + \varphi_1)$ @ $t=0$ $x_1 = 1$ $\dot{x}_1 = 0$

$x_2 = A_1' \sin(\omega_1 t + \varphi) - A_2' \sin(\omega_2 t + \varphi_1)$ $x_2 = 0$ $\dot{x}_2 = 0$

$x_1 = A_1' \sin\varphi + A_2' \sin\varphi_1 = 1$

$x_2 = A_1' \sin\varphi - A_2' \sin\varphi_1 = 0 \quad \therefore \quad 2A_1' \sin\varphi = 1$

$A_1' \sin\varphi = 1/2 \quad ; \quad A_2' \sin\varphi_1 = 1/2$

$\dot{x}_1 = A_1'\omega_1 \cos(\omega_1 t + \varphi) + A_2'\omega_2 \cos(\omega_2 t + \varphi_1)$

$\dot{x}_2 = A_1'\omega_1 \cos(\omega_1 t + \varphi) - A_2'\omega_2 \cos(\omega_2 t + \varphi_1)$

using BC

$A_1'\omega_1 \cos\varphi + A_2'\omega_2 \cos\varphi_1 = 0$

$A_1'\omega_1 \cos\varphi - A_2'\omega_2 \cos\varphi_1 = 0 \quad \therefore \quad 2A_1'\omega_1 \cos\varphi = 0$

$A_1' \cos\varphi = 0 \quad A_2' \cos\varphi_1 = 0$

$\therefore$ either $A_1' = 0$ or $\cos\varphi = 0$; by deduction $\varphi = \pi/2$ $A_1' = \frac{1}{2}$

$\varphi_1 = \pi/2$ $A_2' = 1/2$

$\therefore \; x_1 = \frac{1}{2}\left[\sin(\omega_1 t + \pi/2) + \sin(\omega_2 t + \pi/2)\right] = \frac{1}{2}\left[\cos\omega_1 t + \cos\omega_2 t\right]$

$x_2 = \frac{1}{2}\left[\sin(\omega_1 t + \pi/2) - \sin(\omega_2 t + \pi/2)\right] = \frac{1}{2}\left[\cos\omega_1 t - \cos\omega_2 t\right]$

Levy
AE 212

1. Since the kinetic energy of a system is usually a function of the position and the particles velocity whereas the potential energy is only a function of the position.

Suppose $\exists$ an integral

$$I = \int_{t_0}^{t_1} F(y_i(t), y_i'(t), t)\,dt \qquad \text{choose any } \eta_i(t) \in C^1 \ \ni \ \eta_i(t_0) = \eta_i(t_1) = 0$$

let $y_i(t)$ become $y_i(t) + \epsilon\eta_i(t)$

$y_i'(t)$ become $y_i'(t) + \epsilon\eta_i'(t)$ so that I is a function of ϵ

Using the fact that if a functional $M(y)$ is to have an extremum at $y = y_0$ then by letting $y = y_0 + \alpha\delta y$, the extremum will occur when $\frac{\partial}{\partial\alpha} M(y_0 + \alpha\delta y)_{\alpha=0} = 0$

since now $M(y) = M(\alpha)$ & $\frac{\partial}{\partial\alpha} M(\alpha)_{\alpha=0} = M'(\alpha)_{\alpha=0} = M'(y_0) = 0$

Therefore $$I = \int_{t_0}^{t_1} F(y_i + \epsilon\eta_i, y_i' + \epsilon\eta_i', t)\,dt = \int_{t_0}^{t_1} \tilde{F}\,dt \qquad \text{note } \tilde{F} \underset{\epsilon \to 0}{\to} F$$

since $I = I(\epsilon)$ take $\left.\frac{\partial I}{\partial \epsilon}\right|_{\epsilon\to 0}$ then

$$\delta I = \left.\frac{\partial I}{\partial\epsilon}\right|_{\epsilon\to 0} = \int_{t_0}^{t_1} \left[\frac{\partial\tilde{F}}{\partial t}\frac{\partial t}{\partial\epsilon} + \frac{\partial\tilde{F}}{\partial(y_i+\epsilon\eta_i)}\frac{\partial(y_i+\epsilon\eta_i)}{\partial\epsilon} + \frac{\partial\tilde{F}}{\partial(y_i'+\epsilon\eta_i')}\frac{\partial(y_i'+\epsilon\eta_i')}{\partial\epsilon}\right]dt_{\epsilon\to 0}$$

$$= \int_{t_0}^{t_1}\left[\frac{\partial F}{\partial y_i}\eta_i + \frac{\partial F}{\partial y_i'}\eta_i'\right]dt = 0$$

integration by parts of $\int_{t_0}^{t_1} \frac{\partial F}{\partial y_i}\eta_i'\,dt = \left.\frac{\partial F}{\partial y_i'}\eta_i\right|_{t_0}^{t_1} - \int_{t_0}^{t_1}\frac{d}{dt}\left[\frac{\partial F}{\partial y_i'}\right]\eta_i\,dt$

note that $\eta_i(t_0) = \eta_i(t_1) = 0$ as assumed $\therefore$

$\int_{t_0}^{t_1} \frac{\partial F}{\partial y_i}\eta_i'\,dt = -\int_{t_0}^{t_1}\frac{d}{dt}\left[\frac{\partial F}{\partial y_i'}\right]\eta_i\,dt$ putting this into $\delta I = 0$

$$\int_{t_0}^{t_1}\left[\frac{\partial F}{\partial y_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial y_i'}\right)\right]\eta_i\,dt = 0 \qquad \text{since } \eta_i(t) \neq 0 \ \forall t \ \ t_0 \le t \le t_1 \text{ then}$$

$$\left[\frac{\partial F}{\partial y_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial y_i'}\right)\right] = 0$$

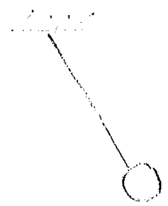
Now since the energy of the system must be minimized

$$I = \int_{t_0}^{t_1}(T-V)\,dt \qquad \text{if } T(y_i, y_i'; t) \ \& \ V(y_i, t) \text{ then}$$

$F = T - V = L$ Lagrange fnl.

$\therefore \quad I = \int_{t_0}^{t_1} L\,dt$ will produce $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) - \frac{\partial L}{\partial y_1} = 0$ for min energy

2.



$$T = \tfrac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_2$$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial T}{\partial r} = mr\dot{\theta}^2 \qquad \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial T}{\partial \theta} = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 = Q_r$$

$$mr^2\ddot{\theta} = Q_\theta$$

$$x = r\sin\theta \qquad\qquad y = r\cos\theta$$

$$dx = r\cos\theta\, d\theta + \sin\theta\, dr \qquad\qquad dy = -r\sin\theta\, d\theta + \cos\theta\, dr$$

$$Q_r = F_x \frac{\partial x}{\partial r} + F_y \frac{\partial y}{\partial r} \qquad\qquad Q_\theta = F_x \cdot \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta}$$

from equilib $\quad F_x = -k(r-r_0)\sin\theta$

$$F_y = mg - k(r-r_0)\cos\theta$$

$$Q_r = -k(r-r_0)\sin\theta \cdot \sin\theta + (mg - k[r-r_0]\cos\theta)\cos\theta$$

$$= -k(r-r_0)\cdot 1 + mg\cos\theta$$

$$Q_\theta = -k(r-r_0)\sin\theta\,(+r\cos\theta) + (mg - k(r-r_0)\cos\theta)(-r\sin\theta) = -mgr\sin\theta$$

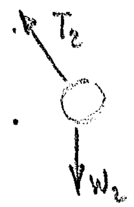
$$\therefore \quad Q_r = mg\cos\theta - k(r-r_0)$$

$$Q_\theta = -mg\, r\sin\theta \qquad \text{or}$$

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + k(r-r_0) = 0$$

$$mr^2\ddot{\theta} + mgr\sin\theta = 0$$

Cesar Levy
11-16-71



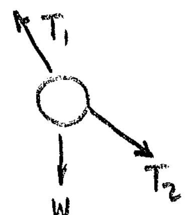
$$m_2 \ddot{a}\, l_2 = -w_2 l_2 \sin\theta_2$$

$$\ddot{a} = l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2$$

$$\therefore \quad m_2 l_2 (l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2) = -m_2 g l_2 \sin\theta_2$$

$$T_2 \cos\theta_2 = m_2 g$$

$$T_2 \sin\theta_2 = -m_2 (l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1)$$



$$m_1 l_1 (l_1 \ddot{\theta}_1) = -m_1 g l_1 \sin\theta_1 - T_2 \cos\theta_2\, l_1 \sin\theta_1 + T_2 \sin\theta_2\, l_1 \cos\theta_1$$

$$\therefore \quad m_1 l_1^2 \ddot{\theta}_1 = -m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1 - m_2 l_2 l_1 \ddot{\theta}_2 \cos\theta_1 - m_2 l_1^2 \ddot{\theta}_1 \cos\theta_1$$

if $m_1 = m_2 = m$; $l_1 = l_2 = l$ & θ_1 & $\theta_2 < 15°$

$$m l^2 (\ddot{\theta}_1 + \ddot{\theta}_2) = -mgl\theta_2$$

$$m l^2 \ddot{\theta}_1 + m l^2 \ddot{\theta}_2 = -2mgl_1\theta_1 - ml_1^2\ddot{\theta}_1$$

$$\therefore \quad \ddot{\theta}_1 + \ddot{\theta}_2 + g/l\, \theta_2 = 0$$

$$\ddot{\theta}_1 + \ddot{\theta}_2 + 2g/l\, \theta_1 = 0$$

if $\theta_1 = A_1 \sin(\omega t + \varphi)$
$\theta_2 = A_2 \sin(\omega t + \varphi)$

this gives $-\omega^4 + 2(g/l - \omega^2)^2 = 0$

$$\omega = \left[g/l \left(2 \pm \sqrt{2}\right)\right]^{1/2}$$

$$\omega_1 = 1.8475 \sqrt{g/l}$$

$$\omega_2 = .766 \sqrt{g/l}$$

$$KE = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2\right)^2$$

$$PE = -m_1 g l_1 \cos\theta_1 - m_2 g \left(l_1 \cos\theta_1 + l_2 \cos\theta_2\right)$$

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore \left[m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 + m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1 \right] \dot{\theta}_1$$

$$+ \left[m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 l_1 \ddot{\theta}_1 + m_2 g l_2 \sin\theta_2 \right] \dot{\theta}_2 = 0$$

$$\therefore \quad m_2 l_2 \left[l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2 \right] = -m_2 g l_2 \theta_2 \qquad \text{for } \theta_2 < 15^\circ$$

$$m_2 l_1 \left[l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2 \right] + m_1 l_1^2 \ddot{\theta}_1 = -m_1 g l_1 \theta_1 - m_2 g l_1 \theta_1 \quad \text{for } \theta_1 < 15^\circ$$

5.12

Cesar Levy

$P = B \qquad 0 \le z \le \pi$

$P = -B \qquad \pi \le z \le 2\pi$

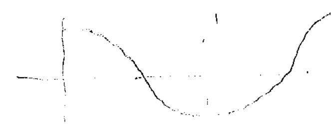
$$P(z) = a_0 + a_n \cos nz + b_n \sin nz$$

$$a_0 = \frac{1}{2\pi}\left\{\int_0^{\pi} B\,dz + \int_{\pi}^{2\pi} -B\,dz\right\} = \frac{B}{2\pi}\left[\pi - (2\pi - \pi)\right] = 0$$

$$a_n = \frac{1}{\pi}\left\{\int_0^{\pi} B\cos nz\,dz + \int_{\pi}^{2\pi} -B\cos nz\,dz\right\} = \frac{1}{\pi}\left\{\left[B\frac{\sin nz}{n}\right]_0^{\pi} - \left[B\frac{\sin nz}{n}\right]_{\pi}^{+2\pi}\right\} = 0$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{\pi} B\sin nz\,dz + \int_{\pi}^{2\pi} -B\sin nz\,dz\right\} = \frac{1}{\pi}\left\{\left[-B\frac{\cos nz}{n}\right]_0^{\pi} + B\left[\frac{\cos nz}{n}\right]_{\pi}^{2\pi}\right\}$$

$$\frac{B}{n\pi}\left\{\left[-\cos n\pi + 1\right] + \left[\overset{1}{\cos 2n\pi} - \cos n\pi\right]\right\}$$



when n is even $\cos n\pi = 1$ when n is odd $\cos n\pi = -1$

$$\sum_{k=1}^{\infty} \frac{2B}{(2k-1)\pi}\left[1 - \cos(2k-1)\pi\right] = P$$

5.14 if subject to $m\ddot{x} + c\dot{x} + kx = P$ $\qquad P = \begin{cases} B & 0 \le \omega_f t \le \pi \\ -B & \pi \le \omega_f t \le 2\pi \end{cases}$

$$x_b = \frac{b_n/K}{\sqrt{(1-r_n^2)^2 + (2r_n\zeta)^2}} \sin(n\omega_f t - \psi_n)$$

and $\tan\psi_n = \dfrac{2r_n\zeta}{1-r_n^2}$ $\qquad r_n = rn = n\left(\dfrac{\omega_f}{\omega}\right)$

since all odd terms exist only then $r_n = r_k = (2k-1)r$

$$\tan\psi_k = \frac{2(2k-1)r\zeta}{1-(2k-1)r^2} \quad \& \quad x_b = \sum_{k=1}^{\infty} \frac{b_k/K \,\sin([2k-1]\omega_f t - \psi_k)}{\sqrt{(1-[2k-1]r^2)^2 + (2[2k-1]r\zeta)^2}}$$

$$\therefore\ r_1 = r \qquad \tan\psi_1 = \frac{2r\zeta}{1-r^2} \qquad x_{b_1} = \frac{b_1/K}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\sin(\omega_f t - \psi_1) \qquad b_1 = \frac{4B}{\pi}$$

$$r_2 = 3r \qquad \tan\psi_2 = \frac{6r\zeta}{1-9r^2} \qquad x_{b_2} = \frac{b_2/K}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}}\sin(3\omega_f t - \psi_2) \qquad b_2 = \frac{4B}{\pi}$$

$$\therefore\ x_b = \frac{4B}{\pi K}\sum_{k=1}^{\infty}\left[\frac{\sin((2k-1)\omega_f t - \psi_k)}{\sqrt{(1-[2k-1]r^2)^2 + (2[2k-1]r\zeta)^2}}\right]$$

5-4 Example

$$P = P_0 \qquad 0 \le \tau \le t_1$$

$$P = 0 \qquad \tau \ge t_1$$

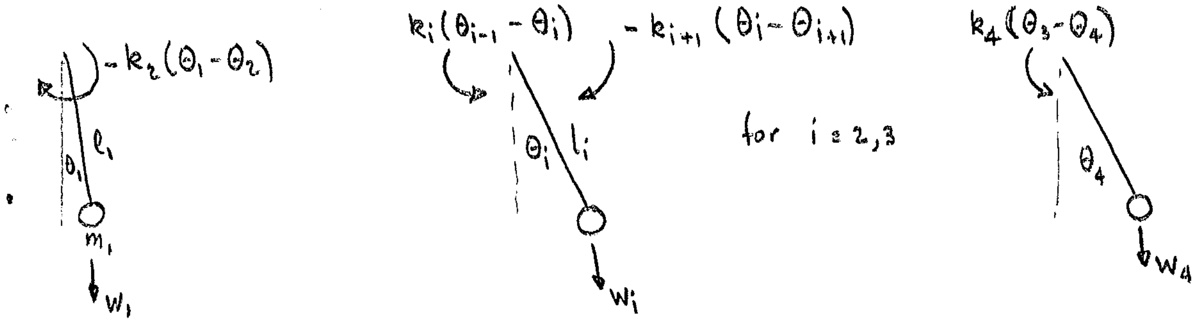
$$x = \frac{1}{\sqrt{mk}} \int_0^{t_1} P_0 \sin(\omega t - \omega\tau)\,d\tau + \frac{1}{\sqrt{mk}} \int_{t_1}^{t} [0] \sin(\omega t - \omega\tau)\,d\tau$$

$$t \ge t_1$$

$$\therefore \quad x = \frac{1}{\sqrt{mk}} \frac{P_0}{\omega} \left[+ \cos\omega(t - \tau) \right] \Big|_0^{t_1} = \frac{P_0}{\omega\sqrt{mk}} \left[\cos\omega(t - t_1) - \cos\omega t \right]$$

Cesar Levy

8.4



$$m_1 l_1^2 \ddot{\theta}_1 = -W_1 l_1 \sin\theta_1 - k_2(\theta_1-\theta_2)$$
$$m_2 l_2^2 \ddot{\theta}_2 = -W_2 l_2 \sin\theta_2 + k_2(\theta_1-\theta_2) - k_3(\theta_2-\theta_3)$$
$$m_3 l_3^2 \ddot{\theta}_3 = -W_3 l_3 \sin\theta_3 + k_3(\theta_2-\theta_3) - k_4(\theta_3-\theta_4)$$
$$m_4 l_4^2 \ddot{\theta}_4 = -W_4 l_4 \sin\theta_4 + k_4(\theta_3-\theta_4)$$

if $\theta_1, \theta_2; \theta_3, \theta_4 \ll 15^\circ$ $\sin\theta_i \sim \theta_i$ $i = 1 \rightarrow 4$

therefore

$$m_1 l_1^2 \ddot{\theta}_1 + (W_1 l_1 + k_2)\theta_1 - k_2\theta_2 = 0$$
$$m_2 l_2^2 \ddot{\theta}_2 + (W_2 l_2 + k_2 + k_3)\theta_2 - k_2\theta_1 - k_3\theta_3 = 0$$
$$m_3 l_3^2 \ddot{\theta}_3 + (W_3 l_3 + k_3 + k_4)\theta_3 - k_3\theta_2 - k_4\theta_4 = 0$$
$$m_4 l_4^2 \ddot{\theta}_4 + (W_4 l_4 + k_4)\theta_4 - k_4\theta_3 = 0$$

8.15

$$T = \tfrac{1}{2} I_1 \dot{\theta}_1^2 + \tfrac{1}{2} I_2 \dot{\theta}_2^2 + \tfrac{1}{2} I_3 \dot{\theta}_3^2 + \tfrac{1}{2} I_4 \dot{\theta}_4^2$$
$$V = \tfrac{1}{2} k_2(\theta_1-\theta_2)^2 + \tfrac{1}{2} k_3(\theta_2-\theta_3)^2 + \tfrac{1}{2} k_4(\theta_3-\theta_4)^2$$

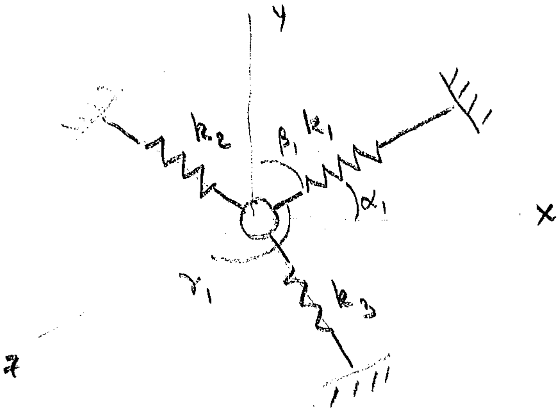
$$L = T - V$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = I_1\dot{\theta}_1, \quad \frac{\partial L}{\partial \dot{\theta}_2} = I_2\dot{\theta}_2, \quad \frac{\partial L}{\partial \dot{\theta}_3} = I_3\dot{\theta}_3, \quad \frac{\partial L}{\partial \dot{\theta}_4} = I_4\dot{\theta}_4, \quad \frac{\partial L}{\partial \theta_4} = k_4(\theta_3-\theta_4)$$

$$\frac{\partial L}{\partial \theta_1} = -k_2(\theta_1-\theta_2), \quad \frac{\partial L}{\partial \theta_2} = k_2(\theta_1-\theta_2) - k_3(\theta_2-\theta_3), \quad \frac{\partial L}{\partial \theta_3} = k_3(\theta_2-\theta_3) - k_4(\theta_3-\theta$$

Since $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0$

$$I_1\ddot{\theta}_1 + k_2(\theta_1-\theta_2) = 0, \quad I_3\ddot{\theta}_3 + k_4(\theta_3-\theta_4) - k_3(\theta_2-\theta_3) = 0$$
$$I_2\ddot{\theta}_2 + k_3(\theta_2-\theta_3) - k_2(\theta_1-\theta_2) = 0, \quad I_4\ddot{\theta}_4 - k_4(\theta_3-\theta_4) = 0$$



$$T = m(\dot{x}+\dot{y}+\dot{z})^2$$

$$V = \sum_{i=1}^{3} \tfrac{1}{2} k_i (x\cos\alpha_i + y\cos\beta_i + z\cos\gamma_i)^2$$

$$\bar{x}_i = x\cos\alpha_i\,\bar{i} + y\cos\beta_i\,\bar{j} + z\cos\gamma_i\,\bar{k}$$

$$x_i\bar{\lambda} = x\cos\alpha_i\,\bar{i} + y\cos\beta_i\,\bar{j} + z\cos\gamma_1\,\bar{k}$$

$$L = T - V$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = m\ddot{y} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = m\ddot{z}$$

$$\frac{\partial L}{\partial x} = -\left[\rho_1\cos\alpha_1 + \rho_2\cos\alpha_2 + \rho_3\cos\alpha_3\right] \qquad \frac{\partial L}{\partial y} = -\left[\rho_1\cos\beta_1 + \rho_2\cos\beta_2 + \rho_3\cos\beta_3\right]$$

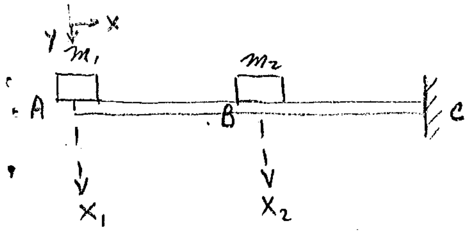
$$\frac{\partial L}{\partial z} = -\left[\rho_1\cos\gamma_1 + \rho_2\cos\gamma_2 + \rho_3\cos\gamma_3\right]$$

where $\qquad \rho_i = x\cos\alpha_i + y\cos\beta_i + z\cos\gamma_i$

now the equations are $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$ or

$$m\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} + \begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}\cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3\\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3\\ \cos\gamma_1 & \cos\gamma_2 & \cos\alpha_3\end{bmatrix}\begin{bmatrix}k_1\cos\alpha_1 & k_1\cos\beta_1 & k_1\cos\gamma_1\\ k_2\cos\alpha_2 & k_2\cos\beta_2 & k_2\cos\gamma_2\\ k_3\cos\alpha_3 & k_3\cos\beta_3 & k_3\cos\gamma_3\end{bmatrix} = 0$$

7.17 Cesar Levy



if m_2 were 0 and $m_1 = W/g$ then the cantelever beam with a W_2 load at then end A would have a displacement equation

$$y = \frac{1}{6}\frac{W_1}{EI}\left[x^3 - 3l^2x + 2l^3\right]$$

the displacement y due to a W load would be (at A)

$$y = \frac{1}{6}\frac{W_1}{EI}\left[2l^3\right] = \frac{W_1L^3}{3EI} = \delta_1$$

since $X_1 = P_1\alpha_{11} + P_2\alpha_{12}$, $P_2 = 0$ & $P_1 = W_1$ then

$$x_1 = k_1x_1\alpha_{11} = \frac{W_1}{\delta_1}\alpha_{11} \quad \therefore \quad \alpha_{11} = \frac{1}{k_1}$$

$$\alpha_{11} = l^3/3EI$$

if m_2 were W_2/g and $m_1 = 0$ then the cantelever beam with a W_2 load at B would have a displacement eq

$$y = \frac{1}{6}\frac{W_2}{EI}\left[-\frac{l^3}{8} + \frac{3}{4}l^3 - \frac{3l^2}{4}x\right] \qquad 0 \le x \le l/2$$

at $x = \frac{l}{2}$ $\quad y = \delta_2 = \frac{W_2\, l^3}{24EI}$ $\quad$ however $P_2 = k_2x_2$

$$k_2 = W_2/\delta_2$$

$$\therefore\ X_2 = P_2\alpha_{22} = k_2X_2\alpha_{22} \qquad \alpha_{22} = \frac{1}{k_2}$$

or

$$\alpha_{22} = \frac{l^3}{24EI}$$

if m_1 where w_1/g and $m_2 = 0$ and displacement at $x = 0$ were to be looked at then

$$Y = \frac{1}{6}\frac{W_1}{EI}\left(\frac{5l^3}{8}\right) = \delta_{21}$$

$X_2 = P_1 \, \alpha_{21}$ $\qquad X_1 = \alpha_{21} P_1 = k_{21} X_1 \alpha_{21}$ $\qquad$ then $\quad \alpha_{21} = \dfrac{1}{k_{21}}$

$k_{21} = \dfrac{W_1}{\delta_{21}}$ $\qquad$ therefore $\qquad \alpha_{21} = \alpha_{12} = \dfrac{5}{48}\dfrac{l^3}{EI}$

now

$$X_1 = -m_1 \ddot{X}_1 \left(\frac{l^3}{3EI}\right) - m_2 \ddot{X}_2 \left(\frac{5}{48}\frac{l^3}{EI}\right)$$

$$X_2 = -m_1 \ddot{X}_1 \left(\frac{5}{48}\frac{l^3}{EI}\right) - m_2 \ddot{X}_2 \left(\frac{l^3}{24EI}\right)$$

Assume $\quad X_1 = A_1 \sin(\omega t + \varphi)$

$\qquad\qquad X_2 = A_2 \sin(\omega t + \varphi)$ $\qquad$ let $\rho = \dfrac{l^3}{EI}$

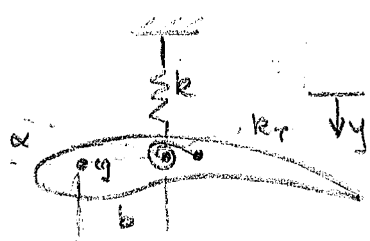
$$A_1 = -\frac{m_1}{3}\rho\left[-A_1\omega^2\right] - \frac{5m_2}{16}\rho\left[-A_2\omega^2\right]$$

$$A_2 = -\frac{5m_1}{16}\rho\left[-A_1\omega^2\right] - \frac{m_2}{8}\rho\left[-A_2\omega^2\right]$$

this will give us the frequency equation

$$\omega^4\left[\frac{14.33}{256} m_1 m_2 \rho^2\right] + \left[\frac{8m_1 + 3m_2}{24}\right]\rho\omega^2 - 1 = 0$$

7-23



$\Sigma F = 0$

$m(\ddot{y} - b\ddot{\alpha}) = -ky$

$\Sigma M = 0$

$I_B \ddot{\alpha} = -k_r \alpha + m(\ddot{y} - \underline{b\ddot{\alpha}})b$

not necessary
use only relative accel
and Dalembert force
since I not at CG

Assume $\quad y = A_1 \sin(\omega t + \varphi)$

$\qquad\qquad \alpha = A_2 \sin(\omega t + \varphi)$ $\qquad$ this gives

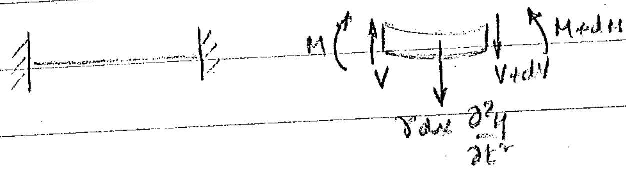
$$A_1(mb\omega^2) + A_2(I_B\omega^2 - k_r - mb^2\omega^2) = 0$$

$$A_1(m\omega^2 - k) + A_2(-mb\omega^2) = 0$$

or

$$\omega_1 = \sqrt{\left(\frac{kI_B + k\omega - kmb^2}{2}\right) + \sqrt{\left(\frac{kI_B + k\omega - kmb^2}{2}\right)^2 - kk_r\left(m^2b^2 + I_B m\right)}}$$

12-14-71



$\therefore \quad EI\, y_{xxxx} = -\gamma\, y_{tt}$ or $\frac{\partial^4 y}{\partial x^4} + a^4 \frac{\partial^2 y}{\partial t^2} = 0$

where $a^4 = \gamma/EI$

assume $\quad Y = X(x)\, T(t)$ this leads to

$$X^{(IV)}\, T + a^4 X\, T'' = 0$$

$$\therefore \quad \frac{X^{(IV)}}{a^4 X} = -\frac{T''}{T} = \omega^2 \quad \text{or}$$

$$X^{(IV)} = a^4\omega^2 X \quad \longrightarrow \quad X^{(IV)} - a^4\omega^2 X = 0$$

$$T'' = -\omega^2 T \quad \longrightarrow \quad T'' + \omega^2 T = 0$$

from first eq $\quad X = A\sin(a^2\omega)^{1/2}x + B\cos(a^2\omega)^{1/2}x + A'\sinh(a^2\omega)^{1/2}x$
$\qquad + B'\cosh(a^2\omega)^{1/2}x$

from second eq $\quad T = C\cos\omega t + D\sin\omega t$

for fixed ends $\quad x = 0, L \quad y = 0$
$\qquad x = 0, L \quad \frac{\partial Y}{\partial X} = 0$

$$\therefore \quad 0 = T[B + B']$$

$$0 = T\left[A\sin(a^2\omega)^{1/2}L + B\cos(a^2\omega)^{1/2}L + A'\sinh(a^2\omega)^{1/2}L + B'\cosh(a^2\omega)^{1/2}L\right]$$

$$0 = T(a^2\omega)^{1/2}[A + A']$$

$$0 = T(a^2\omega)^{1/2}\left[A\cos(a^2\omega)^{1/2}L - B\sin(a^2\omega)^{1/2}L + A'\cosh(a^2\omega)^{1/2}L + B'\sinh(a^2\omega)^{1/2}\right]$$

this leads to $\quad -2 + 2\cosh kL \cos kL = 0 \quad$ for any solution to exist

$\cosh kL \cos kL = 1 \quad$ if one writes $\cos Z = \cos(x+iy)$

& $\cosh(x+iy) = \cosh Z$ then

$\frac{1}{2}(\cos Z + \cosh Z) = \cosh kL \cos kL = 1 \quad$ when $Z = kL(1+i)$

or $\cosh kL = \sinh kL$

trivial freq occurs when $kL = 0$ $\omega = 0$ trivial

first freq occurs bet: $0 < kL < \pi/2$

$$\therefore \quad X = A\left[\cos kx - \cosh kx\right] + B\left[\sin kx - \sinh kx\right]$$

where k is defined from $\cosh kL \cos kL = 1$

$$X = A \sum_{n=0}^{\infty} \left[(-1)^n - 1\right] \frac{z^{2n}}{(2n)!} + B \sum_{n=0}^{\infty} \left[(-1)^n - 1\right] \frac{z^{2n+1}}{(2n+1)!}$$

$$y = TX$$

$$= \left\{C \cos \omega t + D \sin \omega t\right\} \left[\sum_{n=0}^{\infty} \left[(-1)^n - 1\right] \left\{A \frac{z^{2n}}{(2n)!} + B \frac{z^{2n+1}}{(2n+1)!}\right\}\right]$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

if $m\ddot{x} + c\dot{x} + k = P$ then

particular $x_p = \frac{a_0}{K} + \frac{a_n/K}{\sqrt{(1-r_n)^2 + (2\zeta r_n)^2}} \cos(n\omega_f t - \psi_n)$

when $r_n = n\,\omega_f/\omega = rn$ & $\tan\psi_n = \frac{2\zeta r_n}{1-r_n^2}$

$$+ \frac{b_n/K}{\sqrt{(1-r_n^2)^2 + (2\zeta r_n)^2}} \sin(n\omega_f t - \psi_n)$$

homogeneous $x_a = e^{-\zeta\omega t}(A\cos\omega_d t + B\sin\omega_d t)$ where $\omega_d = \omega\sqrt{1-\zeta^2}$

where $\zeta = \frac{c}{2m\omega}$

Duhamel's Integral

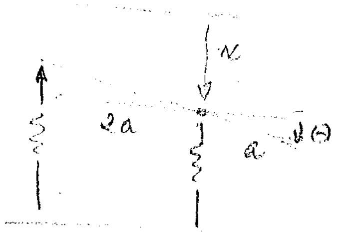
$dx = \frac{P}{m\omega}\sin\omega(t-\tau)\,d\tau$ obtain from impulse on undamped system.

$dx = \frac{P}{m\omega_d} e^{-\zeta\omega(t-\tau)} \sin\omega_d(t-\tau)\,d\tau$ particular solution only influence coefficient $x_1 = P_1\alpha_{11} + P_2\alpha_{12}$

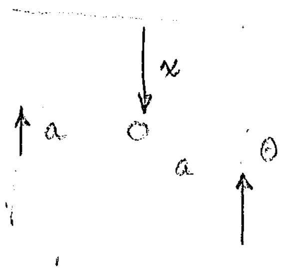
$x_2 = P_1\alpha_{21} + P_2\alpha_{22}$

Rayleigh Method get V_{max} & set $= T_{max}$ use influence coeff to get displacements ~~normalize~~ set $x_i = A_i \sin(\omega t + \varphi)$ normalize wrt A_1 & put into eq. for V_{max} & T_{max} $\rightarrow \omega$ freq > system freq

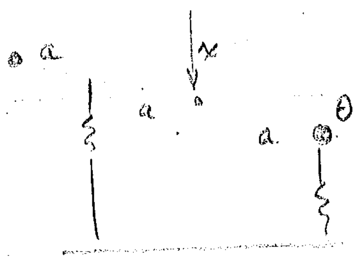
Holzer Method use normal equations of motion normalize wrt A_1 pick and ω to get relation for $A_2', A_3', \dots A_i'$ put into error equation & see what ω will make error $\rightarrow 0$.



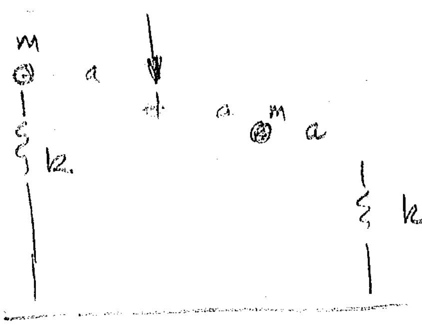
$$3m\ddot{x} = -k(x - 2a\theta) - kx$$
$$3mr^2\ddot{\theta} = -k(x - 2a\theta)\,2a$$



$$3m\ddot{x} = -k(x - a\theta) - 2k(x + a\theta)$$
$$3mr^2\ddot{\theta} = +k(x - a\theta)a - 2k(x + a\theta)a$$

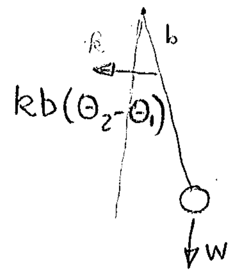


$$3m\ddot{x} = -k(x - a\theta) - 2k(x + a\theta)$$
$$3mr^2\ddot{\theta} = k(x - a\theta)a - 2k(x + a\theta)a$$



$$2m\ddot{x} = -k(x - a\theta) - k(x + 2a\theta)$$
$$2mr^2\ddot{\theta} = k(x - a\theta)a - 2k(x + 2a\theta)2a$$

$$\dot{x}_{m_1} = \dot{x}_{cm} - a\dot{\theta} \qquad \dot{x}_{m_2} = \dot{x}_{cm} + a\dot{\theta}$$



$x_2 = b\theta_2$

$\cancel{m_2 l}$ $I_2\ddot{\theta}_2 = -m_2 l_2 \theta_2 - kb^2(\theta_2-\theta_1)$

$I_1\ddot{\theta}_1 = -m_1 l_1 \theta_1 + kb^2(\theta_2-\theta_1)$

$I_2\ddot{\theta}_2 + \theta_2(m_2 l_2 + kb^2) - kb^2\theta_1 = 0$

$I_1\ddot{\theta}_1 + \theta_1(m_1 l_1 + kb^2) - kb^2\theta_2 = 0$

assume $\theta_2 = A_2 \sin(\omega t + \varphi)$

$\theta_1 = A_1 \sin(\omega t + \varphi)$

$-I_2\omega^2 A_2 + A_2(m_2 l_2 + kb^2) - kb^2 A_1 = 0$

$-I_1\omega^2 A_1 + A_1(m_1 l_1 + kb^2) - kb^2 A_2 = 0$

$$\begin{vmatrix} (m_1 l_1 + kb^2 - I_1\omega^2) & -kb^2 \\ -kb^2 & (m_2 l_2 + kb^2 - I_2\omega^2) \end{vmatrix} = 0$$

$m_1\ddot{x}_1 = -k(x_1 - x_2)$

$m_2\ddot{x}_2 = k(x_1 - x_2)$

if $x_1 > x_2$

$m_1\ddot{x}_1 \cancel{+} m_2\ddot{x}_2 = -2k(x_1 - x_2)$

if $m_1 = m_2$ $m_1(\ddot{y}) = -2ky$

let $-A_1\omega^2 + a_1 A_1 + a_2 A_2 = 0$

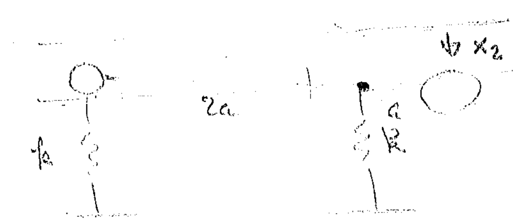
$-A_2\omega^2 + b_1 A_1 + b_2 A_2 = 0$

$\frac{a_1}{a_2} = \frac{b_1}{b_2}$

$(-\omega^2 + a_1)(b_2 - \omega^2) - b_1 a_2 = 0$

$\cancel{a_1 b_2} - \omega^2(a_1 + b_2) + \omega^4 - \cancel{b_1 a_2} = 0$

$\omega^2 = 0$ $\omega = 0$ deg.



$m(a + x) = 2m(2a - x)$

$\cancel{ma} + xm = \cancel{4am} - 2xm$

$3xm = 3am$

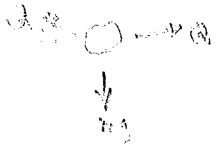
$-3mx + 2m \cdot 3a$ $(2a - x)m = 2m(a + x)$ only if $x = 0$

$x = 2a$ $\cancel{2am} - xm = \cancel{2am} + 2xm$

$-3mx + 2m \cdot 3a = 0$

$-3m \cdot x + ma + 2ma$ $-2mx + ma + 3ma = 0$ $x = 2a$

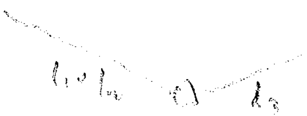
$x = 2a$



$$+1 = Q\left[\frac{x}{l_2+l_3} + \frac{x}{l_1}\right]$$

$$W = Q\left[\frac{\delta}{l_2+l_3} + \frac{\delta}{l_1}\right] \qquad 1 = Qx\left[\frac{l_1+l_2+l_3}{(l_2+l_3)l_1}\right]$$

$$\frac{1}{k} = \frac{l_1(l_2+l_3)}{Q l_0} = \alpha_{11}$$



$$1 = Qx\left[\frac{1}{l_3} + \frac{1}{l_1+l_2}\right] = Qx\left[\frac{l_0}{l_3(l_1+l_2)}\right]$$

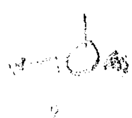
$$kx = Qx\left[\frac{l_0}{l_3(l_1+l_2)}\right] \qquad \frac{1}{k} = \frac{l_3(l_1+l_2)}{Q l_0} = \alpha_{22}$$

$$\frac{x_a}{x} = \frac{l_3}{l_2+l_3} \qquad x_a = \frac{k}{} \qquad x_a\frac{(l_2+l_3)}{l_3} = x$$

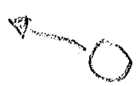
$$k = Q\,\frac{l_2+l_3}{l_3}\left[\frac{l_0}{(l_2+l_3)l_1}\right] = \frac{Q\,l_0}{l_3 l_1}$$

$$\therefore \quad \alpha_{12} = \alpha_{21} = \frac{l_3 l_1}{Q l_0}$$

assume $x_1 > x_2$



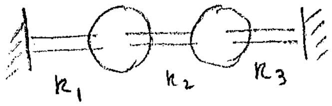
$$m\ddot{x}_1 = -Q\,\frac{x_1}{l_1} - Q\,\frac{(x_1-x_2)}{l_2}$$



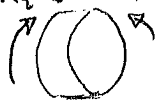
$$m\ddot{x}_2 = +Q\,\frac{(-x_2+x_1)}{l_2} - Q\,\frac{x_2}{l_3}$$

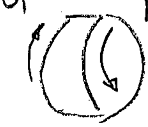
$$m\ddot{x}_1 = -Q\left[\frac{x_1}{l_1} + \frac{x_1-x_2}{l_2}\right] = -Q\left[\frac{x_1(l_1+l_2) - x_2 l_1}{l_1 l_2}\right]$$

$$m\ddot{x}_2 = -Q\left[\frac{x_2}{l_3} - \frac{x_1-x_2}{l_2}\right]$$



Assume $\Theta_2 > \Theta_1$





$\cancel{I_2\ddot{\theta}_2 = -k_2(\theta_2-\theta_1}$

$$I_2\ddot{\theta}_2 = -k_3\theta_2 + k_2(\theta_2-\theta_1)$$

$$I_1\ddot{\theta}_1 = -k_2(\theta_2-\theta_1) + k_1\theta_1$$

$$\frac{T}{JG} = \frac{k_1\theta_1}{JG}$$

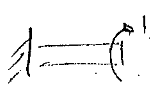
$k_1\theta_1$

$$\cancel{\frac{k\theta_1}{JG}} \quad 1 = \cancel{\frac{k_1\theta_1}{JG}} (+ \cancel{k_2} \cancel{\frac{(k_2+k_3)\theta_1}{JG}}$$

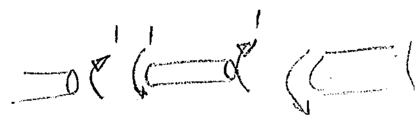
$$\cancel{k = k_1 + k_2+k_3}$$

$\cancel{\alpha_{11} =}$

$$1 = \frac{k_1\theta_1}{\cancel{JG}} + k_2(\theta_2-\theta_1) + k_3(\theta_2)$$

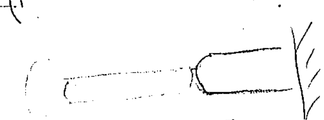


$$\alpha = \frac{T}{JG}$$



$k\,\Theta_1$

$$\frac{1}{G_1J_1} = \frac{1}{G_2J_2} \qquad \frac{T}{GJ_1} = \alpha_1$$



$k =$

$$\theta = \frac{Tl}{JG} \qquad \cancel{\frac{k_t}{\theta}}$$

$$\alpha = \frac{T}{JG} \qquad \frac{T}{GJ_{2}} = \alpha_2$$

$$k_T\theta = T$$

$$\frac{Tl_1}{G_1J_1} = \frac{l_2(\cancel{\theta_1 - \theta_2})}{\cancel{G_2}J_2} + \frac{l_3\cancel{\theta_2}}{\cancel{G_3}J_3}$$

$$k_T = \frac{T}{\theta} = \frac{GJ}{l}$$

$$\frac{Tl_1}{G_1J_1} = \frac{Tl_3}{G_3J_3} - \frac{Tl_2}{G_2J_2}$$

l

$$k_1 = k_3 - k_2$$