CHAPTER I

THE SINGLE FIRST ORDER EQUATION

1. The linear and quasi-linear equations.

The first order equations, in general, present interesting geometric interpretations. It will be convenient then to restrict the discussion to the case of two independent variables, but it will be made clear that the theory can be extended immediately to any number of variables. We consider then equations of the form

$$(1) \qquad F(x,y,u,u_x,u_y) = F(x,y,u,p,q) = 0$$

where we have used the notation $u_x = p$, $u_y = q$. A solution $z = u(x,y)$, when interpreted as a surface in three dimensional space, will be called an integral surface of the differential equation.

We begin with the general linear equation

$$(2) \qquad a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y).$$

We notice that the left hand side of this equation represents the derivative of $u(x,y)$ in the direction $(a(x,y),b(x,y))$. Thus when we consider the curves in the x,y-plane whose tangents at each point have those directions, i.e. the one parameter family of curves defined by the ordinary differential equations

$$(3) \qquad \frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}, \quad \text{or} \quad \frac{dx}{dt} = a(x,y), \quad \frac{dy}{dt} = b(x,y),$$

they will have the property that along them $u(x,y)$ will satisfy the ordinary differential equation

$$(4) \qquad \frac{du}{dx} = \frac{c(x,y)u + d(x,y)}{a(x,y)}, \quad \text{or} \quad \frac{du}{dt} = c(x,y)u + d(x,y).$$

The one parameter family of curves defined by equations (3) are called the <u>characteristic curves</u> of the differential equation.

Suppose now $u(x,y)$ is assigned an "initial" value at a point (x_0, y_0) in the x,y-plane. From the existence and uniqueness of the initial value problem for ordinary differential equations, equations (3) will define a unique characteristic curve, say

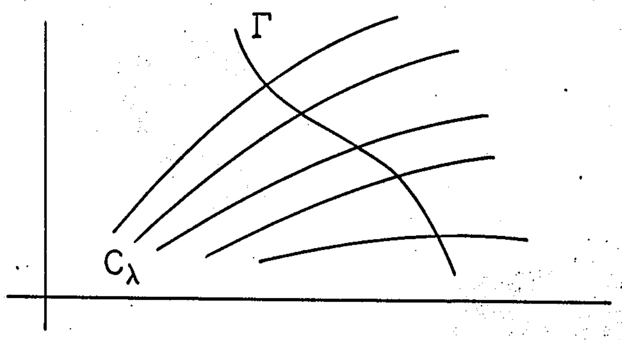
$$x = x(x_0,y_0,t), \quad y = y(x_0,y_0,t) \tag{5}$$

along which

$$u = u(x_0,y_0,t) \tag{6}$$

will be uniquely determined by equation (4). That is, if u is given at a point, it is determined along a whole characteristic curve through the point.

This suggests that if we were to assign initial values for u along some curve, say Γ of the figure below,



intersecting the characteristics C_λ, we may determine a unique solution $u(x,y)$ in the whole region covered by C_λ by means of (5) and (6).

The curve Γ, which we may call the initial curve, may not be chosen quite arbitrarily. For clearly it must not at any point coincide with a characteristic, since there u is determined as a solution of an ordinary differential equation.

A precise formulation of this initial problem, called the Cauchy initial

value problem, will be given for the more general quasi-linear equations to follow.

As an example we consider the P.D.E.

$$xu_x + yu_y = \alpha u$$

with initial conditions $u = \phi(x)$ for $y = 1$. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{y}{x} .$$

having solutions

$$y = cx.$$

Along such a curve u satisfies the equation

$$\frac{du}{dx} = \frac{\alpha u}{x} ,$$

whose solution is

$$u = kx^{\alpha}.$$

As k may differ from characteristic to characteristic, i.e. depend on c, we have the general solution

$$u = k(c)x^{\alpha} = k(\frac{y}{x})x^{\alpha}$$

where k is an "arbitrary" function. If we apply the initial condition for $y = 1$ we obtain

$$\phi(x) = k(\frac{1}{x})x^{\alpha}$$

or

$$k(s) = \phi(\frac{1}{s})s^{\alpha},$$

and hence the required solution

$$u = \phi(\frac{x}{y})y^{\alpha}.$$

The general quasi-linear equation may be written

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u). \tag{7}$$

A solution $u(x,y)$ defines an integral surface $z = u(x,y)$ in the x,y,z-space. The direction numbers of the normal to the surface are $(u_x, u_y, -1)$, so that equation (7) can be interpreted as the condition that the integral surface at each point has the property that the vector (a,b,c) is tangent to the surface.

Thus the P.D.E. defines a direction field (a,b,c), called the characteristic directions, having the property that a surface $z = u(x,y)$ is an integral surface if and only if at each point the tangent plane contains the characteristic direction.

It is suggestive then that we consider the integral curves of this field, i.e. the family of space curves whose tangent coincides with the characteristic direction. They are called the characteristic curves and are given by the equations

$$\frac{dx}{a(x,y,z)} = \frac{dy}{b(x,y,z)} = \frac{dz}{c(x,y,z)} . \tag{8}$$

Calling the common value of these ratios dt, we can write (8) also in the form

$$\text{(9)} \qquad \frac{dx}{dt} = a(x,y,z), \quad \frac{dy}{dt} = b(x,y,z), \quad \frac{dz}{dt} = c(x,y,z).$$

[This notion differs from the one used in the linear case. The projection of the present curves on the x,y-plane will be the curves previously called characteristic.] Through each point (x_0, y_0, z_0) there passes one characteristic curve

$$x = x(x_0, y_0, z_0, t), \quad y = y(x_0, y_0, z_0, t), \quad z = u(x_0, y_0, z_0, t).$$

One important property of the characteristic curves is immediately evident from the geometric interpretation of equation (7). Namely, every surface generated by a one parameter family of characteristics is an integral surface. Moreover, the converse is also true. For suppose $z = u(x,y)$ is a given integral surface Σ. Consider the solution of

$$\frac{dx}{dt} = a(x,y,u(x,y)), \qquad \frac{dy}{dt} = b(x,y,u(x,y))$$

with $x = x_0$, $y = y_0$ for $t = 0$. Then for the corresponding curve

$$x = x(t), \quad y = y(t), \quad z = u(x(t), y(t))$$

also

$$\frac{dz}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) = c(x,y,z)$$

from (7). Hence the curve satisfies condition (9) for characteristic curves, and also lies on Σ by definition. Thus Σ contains with each point also the characteristic curve through the point. Therefore Σ consists of integral curves. Furthermore, if two integral surfaces intersect in a point then they intersect along the whole characteristic through this point; and the curve of intersection of two integral surfaces must be characteristic.

At this point the solution to the Cauchy initial value problem, i.e. of finding a solution $u(x,y)$ satisfying prescribed initial values along a curve in the x,y-plane, becomes evident. For we may take as the solution the integral surface consisting of the family of characteristics passing through each initial point in space.

We will again have to exclude initial curves which are characteristic at any point, i.e. satisfy (8). We shall even have to exclude initial curves satisfying the one equation $\frac{dx}{a} = \frac{dy}{b}$, as otherwise u would have unbounded derivatives. The precise formulation and proof of the existence theorem follows:

Theorem: Consider the first order quasi-linear partial differential equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u), \tag{10}$$

where a,b,c have continuous partial derivatives with respect to x,y,u. Suppose that along the initial curve $x = x_0(s)$, $y = y_0(s)$ the initial values $u = u_0(s)$ are prescribed, x_0, y_0, u_0 being continuously differentiable functions for $0 \leq s \leq 1$. Furthermore, let

$$\frac{dy_0}{ds} a(x_0(s),y_0(s),u_0(s)) - \frac{dx_0}{ds} b(x_0(s),y_0(s),u_0(s)) \neq 0. \tag{11}$$

Then there exists one and only one solution $u(x,y)$ defined in some neighborhood of the initial curve, which satisfies the P.D.E. and the initial conditions

$$u(x_0(s),y_0(s)) = u_0(s). \tag{12}$$

Proof: We consider the ordinary differential equations

$$\frac{dx}{dt} = a(x,y,u)$$

(13)
$$\frac{dy}{dt} = b(x,y,u)$$

$$\frac{du}{dt} = c(x,y,u).$$

From the existence and uniqueness theorem for ordinary differential equations we may solve for a unique family of characteristics

(14)
$$\begin{aligned} x &= x(x_0,y_0,u_0,t) = x(s,t) \\ y &= y(x_0,y_0,u_0,t) = y(s,t) \\ u &= u(x_0,y_0,u_0,t) = u(s,t), \end{aligned}$$

whose derivatives with respect to the parameters s, t are continuous and such that they satisfy the initial conditions

$$\begin{aligned} x(s,0) &= x_0(s) \\ y(s,0) &= y_0(s) \\ u(s,0) &= u_0(s). \end{aligned}$$

We note that the Jacobian

$$\frac{\partial(x,y)}{\partial(s,t)}_{t=0} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix}_{t=0} = \left(\frac{dx_0}{ds}\,b - \frac{dy_0}{ds}\,a\right) \neq 0$$

by condition (11). Thus in (14) we may solve for s, t in terms of x, y in the neighborhood of the initial curve $t = 0$, obtaining from (14) a candidate for the solution

$$\phi(x,y) = u(s(x,y),\ t(x,y)).$$

$\phi(x,y)$ clearly satisfies the initial conditions; for

$$\phi(x,y)_{t=0} = u(s,0) = u_0(s).$$

Moreover it satisfies the differential equations. For

$$\begin{aligned} a\phi_x + b\phi_y &= a(u_s s_x + u_t t_x) + b(u_s s_y + u_t t_y) \\ &= u_s(as_x + bs_y) + u_t(at_x + bt_y) \\ &= u_s(s_x x_t + s_y y_t) + u_t(t_x x_t + t_y t_t) \\ &= u_s \cdot 0 + u_t \cdot 1 \\ &= c, \end{aligned}$$

since from the equations

$$\begin{aligned} s &= s(x,y) \\ t &= t(x,y), \end{aligned}$$

we have

$$\begin{aligned} s_t &= 0 = s_x x_t + s_y y_t \\ t_t &= 1 = t_x x_t + t_y y_t. \end{aligned}$$

Moreover, $\phi(x,y)$ is unique. For suppose $\overline{\Phi}(x,y)$ is any other solution satisfying the initial conditions and x', y' an arbitrary point in the neighborhood of the initial curve. We consider the characteristic curve

$$x = x(s',t), \quad y = y(s',t), \quad u = u(s',t)$$

where $s' = s(x',y')$. At $t = 0$ this curve passes through both surfaces since here it passes through the initial curve at the point

$$x(s',0) = x_0(s'), \quad y(s',0) = y_0(s'), \quad u(s',0) = u_0(s')$$

But if a characteristic curve has one point in common with an integral surface it lies entirely on the surface. Thus the characteristic curve lies on both surfaces, and in particular for t' we have

$$\overline{\Phi}(x',y') = \overline{\Phi}(x'(s',t'),y'(s',t')) = u(s',t') = \phi(x',y').$$

As an example consider the P.D.E.

$$uu_x + u_y = 1$$

with initial conditions $x = s$, $y = s$, $u = \frac{1}{2}s$ for $0 \leq s \leq 1$. We note that condition (11) is satisfied; for

$$\frac{dy}{ds}a - \frac{dx}{ds}b = \frac{1}{2}s - 1 \neq 0 \quad \text{for} \quad 0 \leq s \leq 1.$$

Solving the ordinary differential equations

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = 1, \quad \frac{du}{dt} = 1$$

with the initial conditions

$$x(s,0) = s, \quad y(s,0) = s, \quad u(s,0) = \frac{1}{2}s,$$

we find the family of characteristics

$$x = \frac{1}{2}t^2 + \frac{1}{2}st + s$$

$$y = t + s$$

$$u = t + \frac{s}{2}.$$

When we solve for s and t in terms of x and y, we obtain

$$s = \frac{x - \frac{y^2}{2}}{1 - \frac{y}{2}}, \quad t = \frac{y - x}{1 - \frac{y}{2}},$$

and finally the solution

$$u = \frac{2(y-x) + (x - \frac{y^2}{2})}{2 - y}.$$

2. The general first order equation for a function of two variables.

The general first order partial differential equation for a function of two variables $z(x,y)$ and its derivatives $z_x = p$, $z_y = q$ can be written

$$F(x,y,z,p,q) = 0. \tag{1}$$

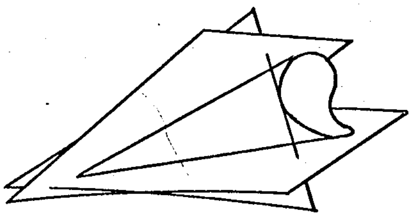
It will be assumed that F has continuous second derivatives with respect to its variables x, y, z, p, q.

Surprisingly enough the problem of solving even the most general first order equation reduces to that of solving a system of ordinary differential equations. The geometry, though, is not nearly as simple as for the quasi-linear equations, where we were concerned principally with integral curves. In the general case we will require, as we will see, more complicated geometric objects, called "strips".

Suppose now at some point in space (x_0, y_0, z_0) we consider a possible integral surface $z = z(x,y)$ and the direction numbers $(p, q, -1)$ of its tangent plane. The equation states that there is a relation

$$F(x_0, y_0, z_0, p, q) = 0$$

between the direction numbers p and q. That is, the differential equation will restrict its solutions to those surfaces having tangent planes belonging to a one parameter family.



In general this one parameter family of planes will envelopea cone (see figure) called the Monge cone. Thus the differential equation (1) describes a field of cones having the property that a surface will be an integral surface if and only if it is tangent to a cone at each point. We note that in the quasi-linear case the cone degenerates into a straight line.

Let us consider for a moment that we have a one parameter family of integral surfaces

$$z = f(x,y,c), \tag{2}$$

where we assume f has continuous second derivatives with respect to its variables x, y, c. As we may suspect from the geometric interpretation of the differential equation, the envelope, if it exists, will again be a solution.

We may find the envelope of a family of surfaces by considering the points of intersection of "neighboring surfaces"

$$z = f(x,y,c)$$

and

$$z = f(x,y,c + \Delta c).$$

Subtracting and dividing by Δc,

$$0 = \frac{f(x,y,c) - f(x,y,c + \Delta c)}{\Delta c},$$

and passing to the limit $\Delta c \to 0$, we have the envelope defined by the two equations

$$(3) \qquad \begin{aligned} z &= f(x,y,c) \\ 0 &= f_c(x,y,c). \end{aligned}$$

If we may solve for c in the second equation and eliminate in the first, we have the envelope expressed as

$$z = g(x,y) = f(x,y,c(x,y)).$$

The envelope will satisfy the differential equation. For

$$(4) \qquad \begin{aligned} g_x &= f_x + f_c c_x = f_x \\ g_y &= f_y + f_c c_y = f_y, \end{aligned}$$

since $f_c \equiv 0$. That is, the envelope will have the same derivatives as a member of the family, and the differential equation is just a relation between these derivatives to be satisfied.

Finding one more solution when we already have a whole family of them is not much gain. However, suppose we know a two parameter family of integral surfaces, say

$$(5) \qquad z = f(x,y,a,b).$$

Then we can find solutions depending on an arbitrary function. For if we let

$$b = \phi(a)$$

where ϕ is differentiable, we obtain the family

$$z = f(x,y,a,\phi(a)),$$

and its envelope

$$z = f(x,y,a,\phi(a))$$
$$0 = f_a + f_b\phi'(a)$$

will be an integral surface depending on the function ϕ.

This suggests that if we were given a two parameter family of integral surfaces, we may be able to select a one parameter family of these surfaces whose envelope contains a given curve in space, i.e. find a solution to the initial value problem. We note that it is quite reasonable to expect the existence of a two parameter family of solutions. For suppose we are given to begin with an arbitrary family of surfaces, say

$$z = f(x,y,a,b). \tag{6}$$

If we may solve for the parameters a and b in the two derivatives

$$p = z_x = f_x(x,y,a,b)$$
$$q = z_y = f_y(x,y,a,b),$$

we can eliminate in equation (6) and obtain the partial differential equation

(7) $$z - f(x,y,a(x,y,p,q),\ b(x,y,p,q)) = 0$$

having the given family as solutions. We will call a two parameter family of solutions a complete solution of the differential equation.

Suppose now we have a complete solution $z = f(x,y,a,b)$ and wish to find an envelope containing the initial curve, say

$$x = x(s), \quad y = y(s), \quad z = z(s).$$

We consider the two equations

(8) $$G(s,a,b) \equiv z(s) - f(x(s),y(s),a,b) = 0$$

and

(9) $$G_s(s,a,b) = z'(s) - f_x x'(s) - f_y y'(s) = 0,$$

obtaining a relation between a and b, say in terms of the parameter s: $a = a(s)$, $b = b(s)$. The envelope

(10) $$\begin{aligned} z &= f(x,y,a(s),b(s)) \\ 0 &= f_a a'(s) + f_b b'(s) \end{aligned}$$

will contain the initial curve. For both equations are satisfied identically in s by $x(s)$, $y(s)$, $z(s)$; the first as a direct consequence of equation (8), and the second from the derivative of the first

$$z'(s) = f_x x'(s) + f_y y'(s) + f_a a'(s) + f_b b'(s),$$

or

$$0 = f_a a'(s) + f_b b'(s),$$

where we used equation (9).

For example, we consider the two parameter family of planes which are of unit distance from the origin, i.e. the planes touching the unit sphere. They are given by the equation

$$z = \frac{-a}{\sqrt{1-(a^2+b^2)}}\,x - \frac{b}{\sqrt{1-(a^2+b^2)}}\,y + \frac{1}{\sqrt{1-(a^2+b^2)}},$$

and can be shown to be a complete solution of the P.D.E.

$$(z - px - qy)^2 - (1 + p^2 + q^2) = 0.$$

If we wish to find an integral surface containing the initial curve, say the circle of radius $\frac{1}{2}$ about the z-axis,

$$z = 1, \quad x = \frac{1}{2}\cos\theta, \quad y = \frac{1}{2}\sin\theta, \quad 0 \le \theta \le 2\pi$$

we would obtain from the given family of planes the equations

$$G(\theta,a,b) = \sqrt{1-(a^2+b^2)} + \frac{a}{2}\cos\theta + \frac{b}{2}\sin\theta - 1 = 0$$

$$G_\theta(\theta,a,b) = a\sin\theta - b\cos\theta = 0,$$

which lead to the relations

$$a = \frac{4}{5}\cos\theta, \quad b = \frac{4}{5}\sin\theta, \quad \text{or} \quad a^2 + b^2 = \frac{16}{25}.$$

The required integral surface is then the envelope of the family

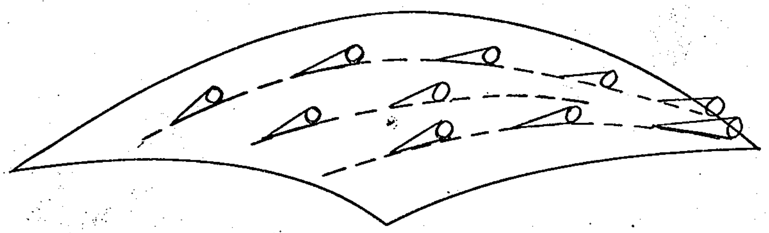
$$z = -\frac{4}{3} x \cos \theta - \frac{4}{3} y \sin \theta + \frac{5}{3},$$

which can be computed to be the cone

$$z = -\frac{4}{3}\sqrt{x^2 + y^2} + \frac{5}{3}.$$

All this is very well if we already have a two parameter family of integral surfaces. We continue therefore with a more systematic attack with the object in mind to describe a system of ordinary differential equations the solutions of which will lead to a solution of the given partial differential equation.

Suppose we are given an integral surface $z = z(x,y)$ having continuous second derivatives with respect to x and y. At each point the surface will be tangent to a Monge cone. See figure below. The lines of contact between the tangent



planes of the surface and the cones define a field of directions on the surface called the characteristic directions, and the integral curves of this field define a family of characteristic curves.

In order to describe the characteristic curves, we obtain first an analytic expression for the Monge cone at some fixed point (x_0, y_0, z_0). It is the envelope of the one parameter family of planes

$$z - z_0 = p(x-x_0) + q(y-y_0)$$

where p and q satisfy

(11) $$F(x_0,y_0,z_0,p,q) = 0, \quad \text{or} \quad q = q(x_0,y_0,z_0,p),$$

and therefore can be given by the equations

(12) $$\begin{aligned} z - z_0 &= p(x-x_0) + q(x_0,y_0,z_0,p)(y-y_0) \\ 0 &= (x-x_0) + (y-y_0)\frac{dq}{dp} . \end{aligned}$$

From equations (11) we obtain

(13) $$\frac{dF}{dp} = F_p + F_q \frac{dq}{dp} = 0$$

so that $\frac{dq}{dp}$ may be eliminated from (12) and the equations describing the Monge cone written

(14) $$\begin{aligned} F(x_0,y_0,z_0,p,q) &= 0 \\ z - z_0 &= p(x-x_0) + q(y-y_0) \\ \frac{x-x_0}{F_p} &= \frac{y-y_0}{F_q} . \end{aligned}$$

We note that given p and q the last two equations define a generating line of the cone, i.e. the line of contact between the tangent plane and the cone.

Thus on our given integral surface, where at each point $p_0 = p(x_0,y_0)$ and $q_0 = q(x_0,y_0)$ are known, the tangent plane

$$z - z_0 = p_0(x-x_0) + q_0(y-y_0)$$

together with the equation

$$\frac{x-x_0}{F_p} = \frac{y-y_0}{F_q}$$

determine the line of contact with the Monge cone

$$\frac{x-x_0}{F_p} = \frac{y-y_0}{F_q} = \frac{z-z_0}{pF_p + qF_q},$$

or characteristic direction

$$(F_p, F_q, pF_p + qF_q).$$

It follows then that the characteristic curves are determined by the system of ordinary differential equations

(14) $$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q}$$

or

(15) $$\frac{dx}{dt} = F_p, \quad \frac{dy}{dt} = F_q, \quad \frac{dz}{dt} = pF_p + qF_q.$$

If the integral surface is as yet unknown, it is clear that the three equations (14) or (15) will not be enough to determine the characteristic curves comprising the surface. For one thing the equations contain two too many unknown functions, namely p and q. However more information concerning the behavior of p and q along a characteristic curve can be obtained. For along such a curve on the given integral surface we have

(16) $$\begin{aligned} \frac{dp}{dt} &= p_x \frac{dx}{dt} + p_y \frac{dy}{dt} = p_x F_p + p_y F_q, \\ \frac{dq}{dt} &= q_x \frac{dx}{dt} + q_y \frac{dy}{dt} = q_x F_p + q_y F_q. \end{aligned}$$

Returning to the differential equation (1) and differentiating first with respect to x and then with respect to y, we have

$$F_x + F_z p + F_p p_x + F_q q_x = 0$$

$$F_y + F_z q + F_p p_y + F_q q_y = 0,$$

so that equations (16) may be written

$$\text{(17)} \qquad \begin{aligned} \frac{dp}{dt} &= -F_x - F_z p \\ \frac{dq}{dt} &= -F_y - F_z q \end{aligned}$$

where we have used $p_y = q_x$.

We have then associated with the given integral surface $z = z(x,y)$ a family of characteristic curves on the surface such that the coordinates of the curve $x(t)$, $y(t)$, $z(t)$, and along the curve, the numbers $p(t)$, $q(t)$ are related by the system of five ordinary differential equations (15) and (17). These five ordinary differential equations are called the <u>characteristic differential equations</u> related to the given P.D.E. (1).

Suppose now the integral surface is as yet to be determined. We are led by the previous discussion to consider the partial differential equation (1) together with the system of characteristic equations (15) and (17), as a system of six equations

$$F(x,y,z,p,q) = 0$$

$$\text{(18)} \qquad \begin{aligned} \frac{dx}{dt} &= F_p \\ \frac{dy}{dt} &= F_q \\ \frac{dz}{dt} &= pF_p + qF_q \\ \frac{dp}{dt} &= -F_x - F_z p \\ \frac{dq}{dt} &= -F_y - F_z q \end{aligned}$$

for the five unknown functions $x(t)$, $y(t)$, $z(t)$, $p(t)$, $q(t)$. This system is overdetermining; however the finite equation $F(x,y,z,p,q) = 0$ is not much of a restriction. For along a solution of the last five equations,

$$\frac{dF}{dt} = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} + F_p \frac{dp}{dt} + F_q \frac{dq}{dt}$$

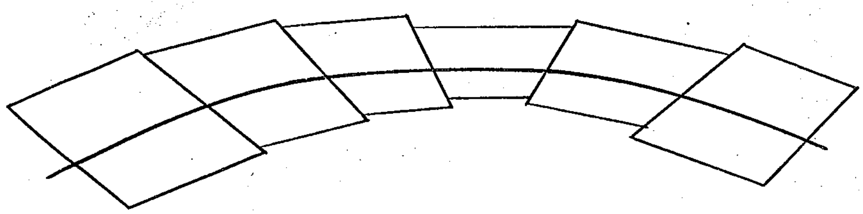
$$= F_x F_p + F_y F_q + pF_z F_p + qF_z F_q - F_p F_x - F_p F_z p - F_q F_y - F_q F_z q$$

$$= 0.$$

Showing that $F = \text{const.}$ is an integral of the ordinary differential equations.

It is clear then that if $F = 0$ is satisfied at an initial "point", say x_0, y_0, z_0, p_0, q_0 for $t = 0$, the five characteristic equations will determine a unique solution $x(t)$, $y(t)$, $z(t)$, $p(t)$, $q(t)$ passing through this point and along which $F = 0$ will be satisfied for all t.

A solution to (18) can be interpreted as a strip. That is, a space curve $x = x(t)$, $y = y(t)$, $z = z(t)$ and along it a family of tangent planes defined by the direction numbers $(p, q, -1)$. See figure below.



For fixed t_0 the five numbers x_0, y_0, z_0, p_0, q_0 will be said to define an element of the strip, i.e. a point on the curve and the corresponding tangent plane.

Note not any set of five functions can be interpreted as a strip. Namely we require that the planes be tangent to the curve, which is the condition

(19) $$\frac{dz(t)}{dt} = p(t)\,\frac{dx(t)}{dt} + q(t)\,\frac{dy(t)}{dt},$$

called the strip condition. In our case the strip condition is guaranteed by the first three characteristic equations.

We will call the strips which are solutions to (18) characteristic strips and their corresponding curves characteristic curves.

We will show that if a characteristic strip has one element x_0, y_0, z_0, p_0, q_0 in common with an integral surface $z = u(x,y)$, it lies completely on the surface.

For, given a solution u, consider the two ordinary differential equations

(20) $$\frac{dx}{dt} = F_p(x,y,u(x,y),\ u_x(x,y)\ u_y(x,y))$$
$$\frac{dy}{dt} = F_q(x,y,u(x,y),\ u_x(x,y)\ u_y(x,y))$$

for $x(t)$, $y(t)$ with initial conditions $x(0) = x_0$, $y(0) = y_0$. They will uniquely determine a curve $x = x(t)$, $y = y(t)$ along which the corresponding curve on the integral surface

(21) $$x = x(t),\ y = y(t),\ z = u(x(t),y(t))$$

satisfies

(22) $$\frac{du}{dt} = u_x\frac{dx}{dt} + u_y\frac{dy}{dt} = u_xF_p + u_yF_q,$$

(23) $$\frac{du_x}{dt} = u_{xx}F_p + u_{xy}F_q,$$

and

(24) $$\frac{du_y}{dt} = u_{yx} F_p + u_{yy} F_q,$$

where $u(0) = u(x_0,y_0) = z_0$, $u_x(0) = u_x(x_0,y_0) = p_0$, and $u_y(0) = u_y(x_0,y_0) = q_0$.

By assumption

(25) $$F(x,y,u(x,y),u_x(x,y),u_y(x,y)) = 0,$$

and thus

$$F_x + F_u u_x + F_{ux} u_{xx} + F_{uy} u_{yx} = 0$$

$$F_y + F_u u_y + F_{ux} u_{xy} + F_{uy} u_{yy} = 0,$$

so that equations (22) and 23) can be written

(26) $$\frac{du_x}{dt} = -F_x - F_z u_x,$$

and

(27) $$\frac{du_y}{dt} = -F_y - F_z u_y,$$

where we used $u_{xy} = u_{yx}$.

Examine now the five functions $x = x(t)$, $y = y(t)$, $z = u(x(t),y(t))$, $p = u_x(x(t),y(t))$, $q = u_y(x(t),y(t))$. They determine a characteristic strip. For they satisfy the five characteristic equations, (20), (21), (22), (26), (27), and the finite equation (25). Moreover, they determine the unique characteristic strip with the initial element x_0, y_0, z_0, p_0, q_0. But this strip lies on the surface by definition, and thus the theorem is proved.

It is clear now from previous considerations how we may proceed to solve the Cauchy initial value problem with the help of these characteristic strips. For consider some arbitrary initial curve

$$x = x_0(s), \quad y = y_0(s), \quad z = z_0(s).$$

If along this curve we can assign functions $p_0(s)$ and $q_0(s)$ such that together with the initial curve $x_0(s)$, $y_0(s)$, $t_0(s)$ we will have defined a family of appropriate initial elements, i.e. satisfying the equation

$$F(x_0(s), y_0(s), z_0(s), p_0(s), q_0(s)) = 0, \tag{28}$$

and being tangent to the initial curve, i.e. satisfying as well the strip condition

$$\frac{dz_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds}, \tag{29}$$

we may expect to construct the integral surface by means of the characteristic strips issuing from the initial elements.

The initial elements $x_0(s)$, $y_0(s)$, $z_0(s)$, $p_0(s)$, $q_0(s)$ satisfying equations (28) and (29) will be said to define an <u>initial strip</u> through the initial curve.

Note that there can be more than one integral surface passing through the initial curve. For there can be more than one pair of functions p_0 and q_0 satisfying the two equations (28) and (29). However, once $p_0(s)$ and $q_0(s)$ are chosen, that is once an <u>initial strip</u> is determined we can expect the solution to be unique. In the quasi-linear case we note that both equations (28) and (29) are linear in p and q and thus only one solution in general occurs.

Again we will require that the initial strip be non-characteristic. In fact, again we will require a more stringent condition

$$\frac{dx_0}{F_p} \neq \frac{dy_0}{F_q}.$$

The precise formulation of the initial value theorem and proof follows:

Theorem. Consider the partial differential equation

$$F(x,y,z,p,q) = 0, \tag{30}$$

where F has continuous second derivatives with respect to its variables x,y,z,p,q. Suppose that along the initial curve $x = x_0(s)$, $y = y_0(s)$, $0 \leq s \leq 1$, the initial values $z = z_0(s)$ are assigned, x_0, y_0, z_0 having continuous second derivatives. Suppose further that continuously differentiable functions $p_0(s)$, $q_0(s)$ have been determined satisfying the two equations

$$\begin{aligned} &F(x_0(s),y_0(s),z_0(s),p_0(s),q_0(s)) = 0 \\ &\frac{dz_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds} . \end{aligned} \tag{31}$$

Finally, suppose that the five functions x_0, y_0, z_0, p_0, q_0 satisfy

$$\frac{dx_0}{ds} F_q(x_0,y_0,z_0,p_0,q_0) - \frac{dy_0}{ds} F_p(x_0,y_0,z_0,p_0,q_0) \neq 0. \tag{32}$$

Then in some neighborhood of the initial curve there will exist one and only one solution $z = u(x,y)$ of (30) containing the initial strip, i.e. such that

$$z(x_0(s),y_0(s)) = z_0(s), \quad z_x(x_0(s),y_0(s)) = p_0(s),$$

$$z_y(x_0(s),y_0(s)) = q_0(s).$$

Proof. We consider the system of characteristic equations

$$\begin{aligned} &\frac{dx}{dt} = F_p, \\ &\frac{dy}{dt} = F_q, \qquad \frac{dp}{dt} = -F_x - pF_z, \\ &\frac{dz}{dt} = pF_p + qF_q, \qquad \frac{dq}{dt} = -F_y - qF_z, \end{aligned} \tag{33}$$

with the family of initial conditions $x = x_0(s)$, $y = y_0(s)$, $z = z_0(s)$, $p = p_0(s)$, $q = q_0(s)$ for $t = 0$. From the existence and uniqueness theorem of the initial value problem for ordinary differential equations we can obtain a family of solutions depending on the initial parameter s

$$x = X(s,t), \quad y = Y(s,t), \quad z = Z(s,t), \quad p = P(s,t), \quad q = Q(s,t), \tag{34}$$

where X, Y, Z, P, Q have continuous derivatives with respect to s and t and such that they satisfy the initial conditions

$$X(s,0) = x_0(s), \quad Y(s,0) = y_0(s), \quad Z(s,0) = z_0(s), \quad P(s,0) = p_0(s), \quad Q(s,0) = q_0(s). \tag{35}$$

Having determined then the characteristic strips issuing from the initial elements we would like to show that the characteristic curves of these strips

$$x = X(s,t), \quad y = Y(s,t), \quad z = Z(s,t)$$

indeed form a surface. Namely we would like to solve for s and t in terms of x and y in the first two equations then, substituting in the last, obtain the surface $z = Z(s(x,y),t(x,y)) = z(x,y)$ as a function of the two variables x and y. This can be done for some neighborhood $N(\xi,\eta)$ about each point (ξ,η) on the initial curve since along the initial curve the Jacobian

$$\left.\frac{\partial(X,Y)}{\partial(s,t)}\right|_{t=0} = \begin{vmatrix} X_s & X_t \\ Y_s & Y_t \end{vmatrix}_{t=0} = \frac{dx_0}{ds} F_q - \frac{dy_0}{ds} F_p \neq 0, \tag{36}$$

by condition (32).

We have then defined in $N(\xi,\eta)$ the functions

$$s = s(x,y), \quad x \equiv X(s(x,y),t(x,y)),$$

$$t = t(x,y), \quad y \equiv Y(s(x,y),t(x,y)),$$

(37) $$z = Z(s(x,y),t(x,y)) = z(x,y),$$

$$p = P(s(x,y),t(x,y)) = p(x,y), \quad q = Q(s(x,y),t(x,y)) = q(x,y).$$

We wish to show that $z(x,y)$ as a solution to the P.D.E. (30), i.e.

$$F(x,y,z(x,y),z_x(x,y),z_y(x,y)) = 0.$$

Since we know from previous considerations of characteristic strips that

$$F(x,y,z,p,q) = 0,$$

all that remains to be proved is that

$$p = z_x \quad \text{and} \quad q = z_y.$$

We consider the expression

(38) $$U(s,t) \equiv Z_s - PX_s - QY_s.$$

For $t = 0$

$$U(s,0) = \frac{dz_0}{ds} - p_0\frac{dx_0}{ds} - q_0\frac{dy_0}{ds} = 0,$$

by the strip condition for the initial elements (31). We would like to show that $U = 0$ for all t, which expresses the fact that the characteristic strips fit smoothly together. To do this we consider the derivative of U with respect to t

$$\frac{\partial U}{\partial t} = Z_{st} - P_t X_s - Q_t Y_s - P X_{st} - Q Y_{st}$$

$$= \frac{\partial}{\partial s}(Z_t - P X_t - Q Y_t) + P_s X_t + Q_s Y_t - Q_t Y_s - P_t X_s$$

$$= 0 + F_p P_s + F_q Q_s + (F_x + F_z P) X_s + (F_y + F_z Q) Y_s,$$

where we made use of the characteristic equations (33). We have further, by adding and subtracting $F_z Z_s$ and then rearranging terms

$$\frac{\partial U}{\partial t} = F_x X_s + F_y Y_s + F_z Z_s + F_p P_s + F_q Q_s - F_z(Z_s - P X_s - Q Y_s)$$

$$= F_s - F_z U$$

$$= -F_z U,$$

since $F_s \equiv 0$ in s and t.

That is for fixed s the function U satisfies the ordinary differential equation

$$\frac{dU}{dt} = -F_z U,$$

having the solution

$$U = U(0) e^{-\int_0^t F_z dt}.$$

Since $U = 0$ for $t = 0$ it follows that $U \equiv 0$ for all t, i.e.

$$Z_s = P X_s + Q Y_s.$$

We observe now the four equations

$$(39) \qquad \begin{aligned} Z_s &= P X_s + Q Y_s \\ Z_t &= P X_t + Q Y_t \end{aligned}$$

$$Z_s = z_x X_s + z_y Y_s$$
$$Z_t = z_x X_t + z_y Y_t.$$

The first follows from the previous discussion, the second is just the third characteristic equation, and the last two are obtained by differentiating the identities of (37).

The four quantities P, Q, z_x, z_y can be considered as two solutions for two linear equations in two unknowns. However, since near (ξ,η) the determinant

$$\begin{vmatrix} X_s & Y_s \\ X_t & Y_t \end{vmatrix} \neq 0$$

by virtue of equation (36) the two solutions must be identical, i.e.

$$P(s,t) = z_x(x(s,t),y(s,t))$$
$$Q(s,t) = z_y(x(s,t),y(s,t)),$$

or

$$p(x,y) = z_x(x,y)$$
$$q(x,y) = z_y(x,y), \tag{40}$$

as was to be shown.

The solution $z = z(x,y)$ contains the initial strip. For

$$z(x_0,y_0) = z(x(s,0),y(s,0)) = Z(s,0) = z_0(s)$$
$$z_x(x_0,y_0) = p(x_0,y_0) = p(x(s,0),y(s,0)) = P(s,0) = p_0(s)$$
$$z_y(x_0,y_0) = q(x_0,y_0) = q(x(s,0),y(s,0)) = Q(s,0) = q_0(s),$$

by virtue of equations (35), (37), and (40). To show that $z = z(x,y)$ so determined is unique we suppose there exists some other solution $z = z'(x,y)$ defined in $N(\xi,\eta)$ and containing the initial strip. We choose an arbitrary point (x', y') in $N(\xi,\eta)$ and solve for the s' and t' associated with it from the equations

$$
\begin{aligned}
s &= s(x,y) \\
t &= t(x,y).
\end{aligned}
$$

We consider now the initial element $x_0(s'), y_0(s'), z_0(s'), p_0(s'), q_0(s')$. By assumption this element lies on both integral surfaces. Thus the uniquely determined characteristic strip issuing from this element

$$
x = X(s',t), \quad y = Y(s',t), \quad z = Z(s',t), \quad p = P(s',t),
$$
$$
q = Q(s',t),
$$

must also be contained in both surfaces. That is

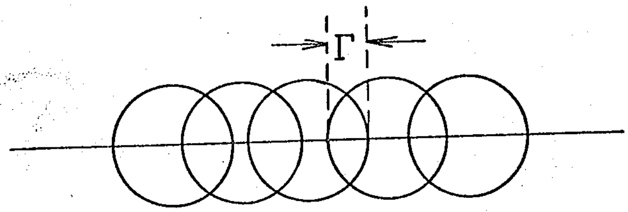
$$
z'(X(s',t),Y(s',t)) = Z(s',t) = z(X(s',t),Y(s',t)),
$$

and in particular for t' we have

$$
\begin{aligned}
z'(x',y') = z'(X(s',t'),Y(s',t')) &= Z(s',t') \\
&= z(X(s',t'),Y(s',t')) = z(x',y').
\end{aligned}
$$

Note that so far we have only constructed a unique solution for a neighborhood $N(\xi,\eta)$ about a point (ξ,η) on the initial curve. We would like to have the solution extended to include the complete curve. This can be done with the help of the uniqueness proof and a proper covering of the initial curve with such neighborhoods $N(\xi,\eta)$.

We suppose then that a region about the initial curve is mapped homeomorphically onto say the u,v-plane such that the initial curve maps into a portion Γ of the line $u = 0$. The system of neighborhoods $N(\xi,\eta)$ will map into a system of neighborhoods $N(u,v)$ which by properly restricting the size of the $N(\xi,\eta)$ we may assume to be circular. We consider now a finite covering S of Γ by the $N(u,v)$. This can be done since the initial curve and thus its image Γ is compact. The intersections of this covering will have a minimum distance, say r. See figure below.



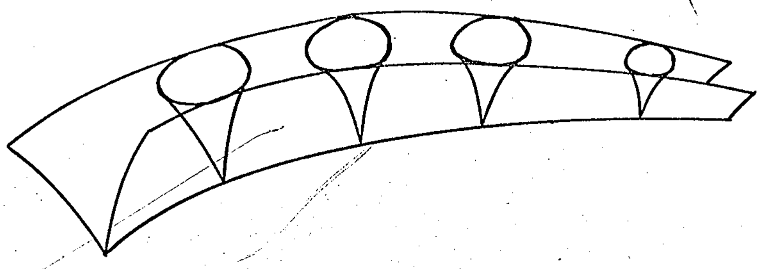
Suppose now we cover Γ with a second set T of the $N(u,v)$ having a maximum diameter of $r/2$. This covering T will then have the property that intersections of any of its neighborhoods will lie entirely within at least one of the neighborhoods of the covering S.

Consider the solutions $z_T(x,y)$ which have been constructed for the neighborhoods of the covering T. It is clear that they will define a solution of the Cauchy problem for the whole curve. For all that has to be shown is that the z_T agree along the intersections of their respective neighborhoods. But this is clear by the uniqueness proof for the neighborhoods of the first covering S.

We have seen before that we can solve the Cauchy problem with the help of a complete integral. There is yet another type of solution which is also convenient for this purpose. Namely, at each point (x_0, y_0, z_0) in space there is a one parameter family of elements $x_0, y_0, z_0, p_0(s), q_0(s)$ through each of which we can pass characteristic strips. This one parameter family of strips will in general form a certain conical surface with a singularity at the point (x_0, y_0, z_0).

It can be proved in a manner similar to what has been done in the existence theorem that this surface, called a conoid, will be an integral surface. Note that the first approximation to the conoid at the singular point is given by the Monge cone.

The figure below indicates how a conoid may be used to solve the initial problem.



Namely, it is plausible that the envelope of the family of conoids whose singular points lie on the given initial curve will be a solution containing this curve. This envelope may consist of several sheets corresponding to the various integral surfaces which may pass through the given curve.

A complete integral $z = f(x,y,a,b)$, besides being useful to solve the Cauchy problem, can also be used to obtain all characteristic strips and conoids. For suppose the parameter b is given as a function of a. We consider the envelope

$$b = b(a)$$

$$z = f(x,y,a,b)$$

$$0 = f_a + f_b b'(a)$$

for a fixed a the above expression represents a curve of contact between the envelope and family. The curve of contact is bound to be characteristic. The associated functions p and q are given by

$$p = f_x$$

$$q = f_y$$

We need not take b as a function of a. We can consider instead the equations

$$(41)\qquad \begin{aligned} z &= f(x,y,a,b) \\ 0 &= f_a + f_b c \\ p &= f_x \\ q &= f_y. \end{aligned}$$

For any a, b, c we will obtain a characteristic strip. For a function b can be found for which $b \equiv b(a)$ and $c \equiv b'(a)$. The conoids are obtained by taking the family of strips which pass through a given point.

3. <u>The general first order equation for a function of n independent variables.</u>

The general first order P.D.E. for a function of n variables $z = u(x_1,\ldots,x_n)$ with first partial derivatives $p_i = u_{x_i}$, $i = 1,\ldots,n$, can be written

$$(1)\qquad F(x_1,\ldots,x_n,z,p_1,\ldots,p_n) = 0.$$

A solution $z = u(x_1,\ldots,x_n)$ will appear as a certain n dimensional hypersurface imbedded in the $n+1$ space $(z,x_1,\ldots,x_n)$.

The theory of characteristics can be extended to equations (1) in a manner simply analogous to what has been done for the case $n = 2$. Namely, we consider the P.D.E. (1) together with the system of <u>characteristic equations</u>

$$(2)\qquad \begin{aligned} \frac{dx_i}{dt} &= F_{p_i} && i = 1,\ldots,n \\ \frac{dz}{dt} &= \sum_{i=1}^{n} p_i F_{p_i} \\ \frac{dp_i}{dt} &= -F_{x_i} - p_i F_z && i = 1,\ldots,n. \end{aligned}$$

2-13-73

6:30-8:30 on Tuesday at lounge; next week on Thursday ask sec about room

Elliptic 2nd order PDE

$$c^2(u_{xx}+u_{yy}+u_{zz}) = u_{tt} + \nu u_t \qquad \text{3-D telegraphic equation}$$

$\nu \to 0$ wave equation

$c \to \infty$ parabolic

$\frac{\partial}{\partial t} = 0$ static deformations

$$\Delta u = 0 \qquad \text{Laplace's Equation}$$

$$\Delta u = -f(x,y,z,t) \qquad \text{Poisson's Equation}$$

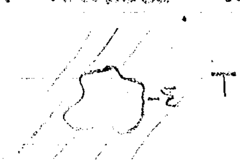
B.C.:

1. Specify $u|_\Sigma = f_1$: Dirichlet Region T and surface Σ
2. $\frac{\partial u}{\partial n}\Big|_\Sigma = f_2$: Neumann $\frac{\partial}{\partial n}$ = deriv in direct of normal to Σ
3. $\frac{\partial u}{\partial n} + h(u - f_3) = 0$: mixed

applied to Exterior and interior regions

Interior Σ is bounded and closed and contains T

Exterior Σ is bounded and closed T is outside Σ & unbounded



Applications: static fields

u : temp field, stress field, E-M potential, velocity potential in incompressible, inviscid, irrotational flow.

Cartesian $u_{xx} + u_{yy} + u_{zz} = 0$ x, y, z

Spherical $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$ r, θ, ϕ

Cylindrical $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0$ ρ, φ, z

3 dim $\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi} = 0 \Rightarrow \frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0 \Rightarrow u = \frac{c_1}{r} + c_2$

Fund to 3 dim Laplace Equ is $u(r) = \frac{1}{r}$ $(r \neq 0)$

$$u(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

Cylindrical $\frac{\partial}{\partial\phi}, \frac{\partial}{\partial z} \equiv 0$ (2 dim)

$\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) = 0$ $u = c_1 \ln\rho + c_2$

Fund to 2 dim $u = \ln\left(\frac{1}{\rho}\right)$ $\rho \neq 0$

$w = f(z)$ where $w = u + iv$ $z = x + iy$

Define $\frac{dw}{dz} = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right]$

as $\Delta z \to 0$ in any direction

If lim exists then w is called analytic at z

$$\frac{dw}{dx} = u_x + iv_x = \frac{dw}{dz}\frac{dz}{dx} = \frac{dw}{dz}$$

$$\frac{dw}{dy} = u_y + iv_y = \frac{dw}{dz}\frac{dz}{dy} = i\frac{dw}{dz}$$

$$\frac{dw}{dz} = u_x + i v_x = -i u_y + v_y$$

then $u_x = v_y \quad v_x = -u_y$ Cauchy-Riemann Equations

$\Rightarrow u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$ i.e. Real & Imaginary parts of an analytic function satisfy Laplace's Equation - called conjugate harmonic functions.

In general a fn satisfying Laplace's Equation is called a harmonic function

Properties of Harmonic Fns.

$\left.\begin{array}{l}\text{① Principles of the maximum} \\ \text{② Uniqueness} \\ \text{③ Boundary value problems}\end{array}\right\}$ Green's Formula.

$$\iiint_T \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) d\tau = \iint_\Sigma (P\cos\alpha + Q\cos\beta + R\cos\gamma)\, d\sigma$$

$d\tau = dx\,dy\,dz \qquad \underline{n}\cdot\underline{i} = \cos\alpha \qquad \underline{n}\cdot\underline{k} = \cos\gamma$

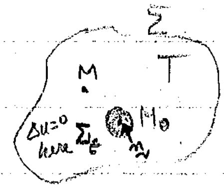
n - exterior normal $\qquad \underline{n}\cdot\underline{j} = \cos\beta$

Let $P = u\dfrac{\partial v}{\partial x} \quad Q = u\dfrac{\partial v}{\partial y} \quad R = u\dfrac{\partial v}{\partial z}$

$$\iiint_T (u\Delta v)\, d\tau = \iint_\Sigma u\frac{\partial v}{\partial n}\, d\sigma - \iiint_T (\nabla u)\cdot(\nabla v)\, d\tau$$

Let $P = v\dfrac{\partial u}{\partial x} \quad Q = v\dfrac{\partial u}{\partial y} \quad R = v\dfrac{\partial u}{\partial z}$
and subtract

$$\iiint_T (u\Delta v - v\Delta u)\, d\tau = \iint_\Sigma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma \qquad (1)$$



test fn $U_0(M) = \frac{1}{r}$ — satisfies Laplace's Equation
where r = distance between M and $M_0 \in T$
Let $u(M)$ satisfy $\Delta u = 0$
& $u \in C'$ in T & Σ
In (1) Let $v = \frac{1}{r} = U_0(M)$ - singular at M_0

Consider a sphere K_ϵ center M_0, radius ϵ. Use (1) in region $T-K_\epsilon$

$$\iiint_{T-K_\epsilon} (u\Delta v - v\Delta u)\,d\tau = 0$$

$$0 = \iint_\Sigma \left(u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right)d\sigma + \iint_{\Sigma_\epsilon} u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)d\sigma - \iint_{\Sigma_\epsilon}\frac{1}{r}\frac{\partial u}{\partial n}\,d\sigma$$

$\frac{\partial}{\partial n}$ on sphere is $-\frac{\partial}{\partial r}$ $\qquad \frac{\partial}{\partial n}\left(\frac{1}{r}\right) = -\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = \frac{1}{r^2} = \frac{1}{\epsilon^2}$ on Σ_ϵ

$$\iint_{\Sigma_\epsilon} u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)d\sigma = \frac{1}{\epsilon^2}\iint_{\Sigma_\epsilon} u\,d\sigma = \frac{1}{\epsilon^2}4\pi\epsilon^2 u^* \quad (2)$$

where u^* is an average value of u over Σ_ϵ ; $u^* = \frac{1}{4\pi\epsilon^2}\iint_{\Sigma_\epsilon} u\,d\sigma$

$$\iint_{\Sigma_\epsilon}\frac{1}{r}\frac{\partial u}{\partial n}\,d\sigma = \frac{1}{\epsilon}4\pi\epsilon^2\left(\frac{\partial u}{\partial n}\right)^{*} \ (3) ; \quad \left(\frac{\partial u}{\partial n}\right)^* = \frac{1}{4\pi\epsilon^2}\iint_{\Sigma_\epsilon}\frac{\partial u}{\partial n}\,d\sigma$$

Let $\epsilon \to 0 \Rightarrow \Sigma_\epsilon \to M_0$ $\qquad$ (2) $\to 4\pi u|_{M_0}$ $\epsilon \to 0$

$$\iint_\Sigma \left(u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right)d\sigma + 4\pi u|_{M_0} \overset{(3)}{=} 0 \to 0$$

$$u(M_0) = \frac{1}{4\pi}\iint_\Sigma\left[\frac{1}{r}\frac{\partial u(P)}{\partial n} - u(P)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\sigma$$

distance between M_0, P $\qquad P \in \Sigma$

i.e. the value of harmonic fn $u(M_0)$ at an arbitrary interior pt M_0 is determined in terms of its values & its normal derivs on the boundary of the region [D'Alembert ?]

Consequences

1. If v is harmonic in T then $\iint_\Sigma \frac{\partial v}{\partial n}\, d\sigma = 0$

<u>Proof:</u> Put $u = 1$ in Green's formula

$$\Delta v = 0 \qquad \Delta u = 0 \qquad \frac{\partial u}{\partial n} = 0$$

2. Prescribe $\frac{\partial u}{\partial n}$ on Σ (Neumann) $= f$ then $\iint_\Sigma f\, d\sigma = 0$ for a soln to exist

3. Spherical Region Σ_a, center M_0

$$u(M_0) = \frac{1}{4\pi a^2} \iint_{\Sigma_a} u\, d\sigma$$

Proof: $$4\pi u(M_0) = -\iint_{\Sigma_a} \Big[u \frac{\partial}{\partial n}(1/r) - \underbrace{\frac{1}{r}\frac{\partial u}{\partial n}}_{\frac{1}{a}\iint_{\Sigma_a} \frac{\partial u}{\partial n} d\sigma \,=\, \frac{1}{a}\cdot 0 = 0} \Big] d\sigma$$

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\Big|_{\Sigma_a} = \frac{-1}{a^2}$$

$$u(M_0) = \frac{1}{4\pi a^2} \iint_{\Sigma_a} u\, d\sigma \qquad \text{spherical mean}$$

4. <u>Principle of the Max</u>

If a fn $u(M)$ is continuous and defined on a closed region $T + \Sigma$ and satisfies $\Delta u = 0$ in the interior of T, then it assumes its max and min on the boundary Σ

<u>Proof</u>: by contradiction

Let u attain its max at M_0 interior to T

Then $u(M) \le u(M_0) \quad \forall M \in T$

Enclose M_0 with a sphere, center M_0, rad ρ

lying entirely in T. Then $u(M_0) = \frac{1}{4\pi\rho^2} \iint_{\Sigma_\rho} u(M)\, d\sigma$

$$\le \frac{1}{4\pi\rho^2} \iint_{\Sigma_\rho} u(M_0)\, d\sigma = u(M_0)$$

Let $u(M) < u(M_0)$ some $M \in \Sigma_\rho \Rightarrow$ contradiction

otherwise

$$u(M_0) < u(M_0) \quad \therefore \quad u(M) = u(M_0) \quad \forall M \in \Sigma_\rho$$

this argument holds for all radii $\le \rho$ of initial sphere $\Rightarrow u(M) = u(M_0)$ in sphere

Take any M_1 in surface Σ_ρ. $u(M_1) = u(M_0) = \max$ of u

Consider sphere center M_1, radius ρ_1, entirely in T

Same argument $\Rightarrow u(M) = u(M_1) \quad \forall M$ in Σ_{ρ_1}

Type of Heine-Borel Covering Thm. of T

By continuation $u(M) = u(M_0)$ in T

$\Rightarrow u(M) =$ constant in T.

Consequences of Principle of Maximum

1. If u & U are continuous in $T+\Sigma$ & harmonic in T and if $u \le U$ on Σ then $u \le U$ inside T

   Proof: $v = u - U$ $\quad v \le 0$ on $\Sigma \implies v \le 0$ in T

2. Similarly if $|u| \le U$ on Σ then $|u| \le U$ in T

Uniqueness of Solution

For First B.V. Problem: Dirichlet

Consider region T bounded by Σ. Dirichlet Problem for $\Delta u = 0$ is: determine u $\ni$.

1. $\Delta u = 0$ interior of T
2. u is defined and continuous on $T + \Sigma$
3. u assumes a prescribed value on Σ

Theorem Soln of Dirichlet Problem is unique

Proof Let u_1 and u_2 satisfy (1–3)

Define $v = u_1 - u_2$; Then v satisfies 1) and 2) and $v\big|_{\Sigma} \equiv 0$

v satisfies condition for principle of maximum $\implies v \equiv 0$ in T

i.e. $u_1 = u_2$ + Soln is unique

Stability of Soln – continuous dependence on boundary data

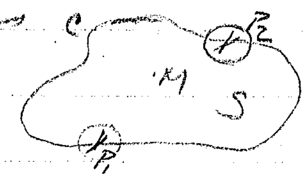
Small changes in boundary data induces small changes in Solution

Let u_1 and u_2 satisfy (1 − 3) ∋. on bdy $|u_1-u_2|<\epsilon$
Then by consequence 2 $|u_1-u_2|<\epsilon$ in $\bar{T}$

First B.V. problem (2-D)
for discontinuous boundary data

Let a piecewise continuous fn $f(P)$ be defined on a curve C which bounds the region S. We seek $u(M)$ such that

1) $u(M)$ is harmonic in S
2) $u(M)$ continuously approaches $f(P)$ as $M\to P$ wherever f is continuous



3) $u(M)$ is bounded in $S+C$ [only needed near to P_i]

Thm: The soln of Dirichlet problem with piecewise continuous boundary data is unique

Proof: Let u_1 & u_2 be 2 soln $V=u_1-u_2$

V satisfies
1) V is harmonic
2) $V(M)\to 0$ as $M\to P$ except at P_i
3) V is bounded in $S+C$ $(|V|<A)$

Construct a fn $U(M)$ $M\in S$ $U(M)=\epsilon\sum_{i=1}^{n}\ln\frac{d}{r_i(M)}$

where D is $\max|M_1-M_2|$ $M_1, M_2\in S+C$

$r_i(M)=|M-P_i|$ $\Rightarrow$ $D>r_i(M)\Rightarrow U(M)>0$

Enclose each P_i with a circle K_i : center P_i, rad δ
sufficiently small $\Rightarrow$ $\epsilon \ln (D/\delta) > A$

$U(M) > A$ on K_i $\Rightarrow$ on K_i $U(M) > |V|$

In $S - K_i$ $U(M)$ and $|V|$ satisfy $\Delta u = 0$ and are continuous

$\Rightarrow$ $u(M) > |V|$ $\forall M$ in $S - K_i$

Then fix M and let $\epsilon \to 0$ $\Rightarrow u(M) \to 0 \Rightarrow |V| \to 0$

But δ arbitrarily small $\Rightarrow$ let $\delta \to 0$ when $K_i \to P_i$

$\Rightarrow$ $V = u_1 - u_2 \equiv 0$ everywhere in $S + C$ except at P_i

Proved uniqueness except at P_i

Prepare Lecture

Separation of Variables § 4.3,4 Pg 270

2-27-73

<u>Extension</u>

All pts inside a region u can be made continuous if not already or $u = \infty$ at the pt no jump discontinuities allowable

ie If a sol for $u(M)$ is harmonic in the interior of S with the exception of P, then $u(M)$ at P can be defined $\ni$ $u(M)$ is harmonic for all of S.

harmonic fn can only have unbounded singularity in the interior of a region

1. $\lim_{M \to P} u(M) = \infty$ or $\lim_{M \to P} u(M)$ exists = finite

Define $u(P) = \lim_{M \to P} u(M) \Rightarrow u$ harmonic in all of S

Same thm holds for 3 dimensions. $U = \frac{1}{r}$

if $u(M)$ harmonic in nbd of P & ∋ $|u| < \frac{\varepsilon(r)}{r}$
& $\varepsilon(r) \to 0$ as $r \to 0$ Then u is bdd there &
can choose $u(P)$ ∋ u is harmonic at P.

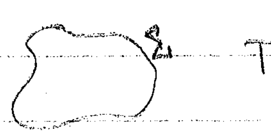
<u>Regularity of Harmonic Fns at ∞.</u>
Conditions for regularity at ∞ are $|u| < \frac{A}{r}$ $\left|\frac{\partial u}{\partial x}\right|, \left|\frac{\partial u}{\partial y}\right|, \left|\frac{\partial u}{\partial z}\right| <$
$\Rightarrow$ u regular at ∞

<u>Thm</u>
if a fn u is harmonic <u>outside</u> of a closed surface Σ & as
$r \to \infty$ $u \to 0$ uniformly, then u is regular at ∞.

<u>Proof</u> Kelvin Transformation: if u harmonic then v is harmonic
where $v(r', \theta, \varphi) = r\,u(r, \theta, \varphi)$ $r' = \frac{1}{r}$
$u \to 0$ $r \to \infty \Rightarrow \exists$ a fn $\epsilon^*(r)$ ∋ $|u| < \epsilon^*(r)$ & $\epsilon^*(r) \to 0$
as $r \to \infty$. V is harmonic & satisfies
$$|v| < \epsilon^*\left(\frac{1}{r'}\right)\frac{1}{r'} = \frac{\varepsilon(r')}{r'} \quad \text{where } \epsilon(r') \to 0 \text{ as } r' \to 0$$
$\Rightarrow$ limit exists as $r' \to 0$ $\Rightarrow |v| < A$ as $r' \to 0$
$\Rightarrow |ru| < A$ as $r \to \infty$ $\Rightarrow |u| < \frac{A}{r}$ as $r \to \infty$

By writing $u = \frac{v}{r}$ & differentiating wrt x, y, z
derivatives of v are bdd as $r \to \infty$ (v harmonic)
$\left|\frac{\partial u}{\partial x_i}\right| < \frac{A}{r^2}$ as $r \to \infty$ $\Rightarrow$ u regular at ∞

Exterior ~~F~~ int BVP Dirichlet



T unbdd

Σ bdd closed curve (surface) in T

Find $u(x,y,z)$ $\ni$.

1. $\Delta u = 0$ in T
2. $u \in C^2$ in T and ∂T
3. $u|_\Sigma = f(x,y,z)$
4. $u(M) \xrightarrow{\text{unif}} 0$ as $M \to \infty$

(3-dim case) ($u(M)$ must be bdd for 2-dim case)

Condition 4 is necessary for uniqueness

choose Σ = sphere radius R

$f = f_0$ = constant on R

$u_1 = f_0 \quad u_2 = f_0 \frac{R}{r}$ — non uniqueness if $|u| \not\to 0$ as $r \to \infty$

<u>Theorem</u> The exterior FBVP for the 3-dim Laplace equation has a unique soln.

<u>Proof</u> Let u_1, u_2 be solns to 1)–4)

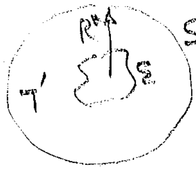
define $u = u_1 - u_2 \Rightarrow u$ satisfies 1) – 4) except $u|_\Sigma = 0$

Condition 4) tells us any $\epsilon > 0$ $\exists R^* \ni. |u| < \epsilon$ for

$r = |M| \quad r > R^*$

Let S_Σ be any sphere $\ni.$ for r on surface of sphere

then $r > R^* \Rightarrow$ On S_Σ $|u| < \varepsilon$ Let T' be the region between Σ and S_Σ. $|u| < \epsilon$ on S_Σ and $|u| \equiv$ on Σ



$\Rightarrow |u| < \epsilon$ on bdy of T'

$\Rightarrow$ principle of max $\Rightarrow |u| < \epsilon \ \forall M \in T'$

But ϵ arbitrary small $\Rightarrow u \equiv 0$

Result also true $\forall\, r > R^*$ $\Rightarrow$ $u \equiv 0$ in T

2-D case 1-3 same 4) $u(M)$ bdd as $M \to \infty$

i.e. $\exists\ N \ni |u(M)| \le N \ \forall M$

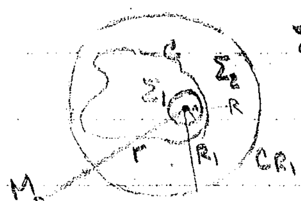
Theorem: solution to 2-dim exterior fbvp is unique

Proof: then let u_1 & u_2 satisfy 1-4

Define $u = u_1 - u_2$

$\Rightarrow u$ satisfies 1,2 except $u|_C = 0$ and $|u| \le N_1 + N_2$

triangle inequality



Σ

$\Delta u = 0$ in Σ

i.e. $M_0 \in \Sigma_1$ let circle radius R, center M_0 lie in Σ_1 then if $r = r_{MM_0}$, where $M \in \Sigma$

$r/R > 1$ and $\ln\left(r/R\right) > 0$ & harmonic Σ

Consider $R_1 \ni C_{R_1}$ about M_0 is entirely in Σ

Construct $U_{R_1}(M) = N \dfrac{\ln(r/R)}{\ln(R_1/R)}$

$\underset{\text{const}}{\ln\left(R_1/R\right)} > 0$, $U_{R_1}(M)$ harmonic $= N$ on C_{R_1}

$u = 0$ on C $\Rightarrow$ $|u| \le N$ on bdy of Σ_2

where $C < \Sigma_2 < C_{R_1}$

$\Rightarrow |u| \le |U_{R_1}(M)|$ on bdy of Σ_2

principal of maximum $\Rightarrow$ $|u| \le |U_{R_1}(M)|$ in Σ_2

keep M fixed and let $R_1 \to \infty$ $\Rightarrow$ $|U_{R_1}| \to 0$

$|u| \leq |U_{R_1}| \Rightarrow |u| = 0$ & $u_1 \equiv u_2$

Proves uniqueness of exterior f.bvp for 2-D.

Neumann Problem $\qquad$ second BVP

Case where $\left.\frac{\partial u}{\partial n}\right|_\Sigma = f(M)$

Theorem

Solution to SBVP is uniquely determined to within an arbitrary constant. $\qquad$ Interior problem

Proofs

Assume u_1, u_2 satisfying

1) $\Delta u = 0$ $\qquad$ 3. $u \in C^2$ in $T \cup \partial T$

2) $\left.\frac{\partial u}{\partial n}\right|_\Sigma = f$

$\Rightarrow u$ satisfies 1)–3) and $\left.\frac{\partial u}{\partial n}\right|_\Sigma = 0$

Green's Formula

$$\iiint_T u \Delta v \, d\tau = \iint_\Sigma u \frac{\partial v}{\partial n} \, d\sigma - \iiint_T (\nabla u \cdot \nabla v) \, d\tau$$

In this formula let $u = v \Rightarrow \Delta v = 0 \quad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$

$$GF \Rightarrow \iiint_T \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] d\tau = 0$$

by condition 3 $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0$ in $T \Rightarrow u \equiv$ const

$u_1 \equiv u_2 +$ Const.

Exterior 2nd BVP

Same result but must show that G. Formula is true for u & V regular at ∞. Then proof same.

MY LECTURE

HW. P.330 1,2 done

3 hr class 2 wks.

§ 4.3 Solution of the B. V. Problems for simple Regions by Separation of variables

Limit our discussion to those problems which can be solved by trig functions alone.

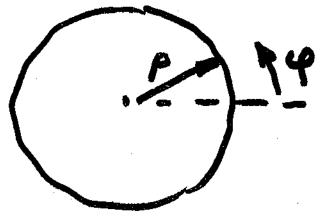
I. The first BVP for the circle

Q: Can we find a fn u which satisfies Laplaces equation of the interior of a circle along with the condition of a prescribed displacement f on the boundary

Find a fn u .∋. $\Delta u = 0 \quad u$ in R

$u = f \quad u$ on ∂R

A. Assume $f \in C^1$ and use ρ, φ as coord.



$$\Delta u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} = 0 \qquad (1)$$

Let $u(\rho, \varphi) = R(\varphi)\,\Phi(\varphi)$ (2)

placing (2) into (1) and assuming $\rho, \Phi, R \neq 0$

$$\frac{\rho}{R}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial R}{\partial \rho}\right) = -\Phi''/\Phi = \lambda^2 \qquad (3)$$

then (3) gives

$$\Phi'' + \lambda^2 \Phi = 0 \qquad (4a,b)$$

and

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - \lambda^2 R = 0$$

let $\rho\frac{d}{d\rho} = \frac{d}{dq}$ $\qquad R = e^{-c_1 q} + e^{+c_2 q}$

$dq = \frac{d\rho}{\rho} \qquad q = \ln\rho \qquad R = e^{-c_1 \ln\rho}, e^{c_2 \ln\rho}$

$R = \rho^{-c_1}, \; R = \rho^{c_2}$ ✓

from (4a) $\Phi_\lambda(\varphi) = A\cos\lambda\varphi + B\sin\lambda\varphi$

With the stipulation that φ is periodic in 2π
$\Rightarrow \lambda =$ integer call it n

$\therefore \quad \Phi_n(\varphi) = A_n \cos n\varphi + B_n \sin n\varphi \qquad (5a)$

to solve (4b) let $R(\rho) = \rho^k$

put into 4b one obtains $(k^2-\lambda^2)\rho^k = 0$

then $k^2 = \lambda^2 = n^2 \quad \therefore \quad k = \pm n \quad n > 0$

then $R(\rho) = C\rho^n + D\rho^{-n} \qquad (5b)$

1. Interior problem $D \equiv 0$ if solution is to be harmonic ie. no singularities in region
2. Exterior problem $C \equiv 0$ if solution is to remain bounded in region

Then for a circle of radius a and let

$\underline{A}_n = CA_n \qquad \underline{B}_n = CB_n$

$\overline{A}_n = DA_n \qquad \overline{B}_n = DB_n$ the solutions are

$u_n(\rho,\varphi) = (\underline{A}_n \cos n\varphi + \underline{B}_n \sin n\varphi)\rho^n \quad \rho \le a$

$u_n(\rho,\varphi) = (\overline{A}_n \cos n\varphi + \overline{B}_n \sin n\varphi)\rho^{-n} \quad \rho \ge a$

by superposition principle of linear operators

(6)
$$u(\rho,\varphi) = \sum_{n=0}^{\infty} (\underline{A}_n \cos n\varphi + \underline{B}_n \sin n\varphi)\rho^n \quad \text{interior}$$
$$u(\rho,\varphi) = \sum_{n=0}^{\infty} (\overline{A}_n \cos n\varphi + \overline{B}_n \sin n\varphi)\rho^{-n} \quad \text{exterior}$$

since f is prescribed on ∂R the $f = f(\varphi)$ only

and @ $\rho = a$ for ~~interior~~ exterior problem

$$u(a,\varphi) = \sum_{n=0}^{\infty} (\bar{A}_n \cos n\varphi + \bar{B}_n \sin n\varphi)\, a^n = f(\varphi) \quad (7)$$

we can use fourier series ∴

$$f(\varphi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos n\varphi + \beta_n \sin n\varphi) \quad (8)$$

where $\alpha_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\varphi)\,d\varphi$ $\quad \alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\varphi)\cos n\varphi\,d\varphi$

$$\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\varphi)\sin n\varphi\,d\varphi$$

comparision of (7) and (8)

$$\bar{A}_0 = \frac{\alpha_0}{2} \qquad \bar{A}_n = \frac{\alpha_n}{a^n} \qquad \bar{B}_n = \frac{\beta_n}{a^n} \qquad (9a)$$

for ~~exterior~~ interior problem

$$\underline{A}_0 = \frac{\alpha_0}{2} \qquad \underline{A}_n = \alpha_n a^n \qquad \underline{B}_n = \beta_n a^n \quad (9b)$$

therefore place (9a) and (9b) in (6a, 6b)

$$\text{inter} \quad u(\rho,\varphi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\frac{\rho}{a}\right)^n \left[\alpha_n \cos n\varphi + \beta_n \sin n\varphi\right] \quad (10)$$

$$\text{exter} \quad u(\rho,\varphi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a}{\rho}\right)^n \left[\alpha_n \cos n\varphi + \beta_n \sin n\varphi\right]$$

B To prove solution to problems are given by (10) must show the superposition principle applies by showing convergence of series, term wise differentiability and continuity of fns on the circumference.

for interior problem $t = \rho/a \le 1$ $\quad \rho \le a$

Exterior problem $t = a/\rho \le 1$ $\quad \rho \ge a$

write (10) as

$$u(\rho,\varphi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} t^n (\alpha_n \cos n\varphi + \beta_n \sin n\varphi)$$

suppose $\exists$ an [redacted] M $\ni$ $|\alpha_n|, |\beta_n| < M \quad \forall n$

then the fourier coefficients are bounded ~~above~~.

To prove differentiability of each term set

$$u_n = t^n (\alpha_n \cos(n\varphi) + \beta_n \sin(n\varphi)) \qquad t<1$$

and calculate $\dfrac{\partial^k u_n}{\partial \varphi^k} = n^k t^n \left\{ \alpha_n \cos(n\varphi + \frac{k\pi}{2}) + \beta_n \sin(n\varphi + \frac{k\pi}{2}) \right\}$

$$\leq n^k t^n \left[|\alpha_n| + |\beta_n| \right] \leq 2M n^k t^n$$

choose some $\rho_0 < a \qquad t_0 = \rho_0/a < 1$ interior problem

$a^2/\rho_0 = \rho_1 > a \qquad t_0 = a/\rho_1 < 1$ exterior problem

Then $|t| \leq t_0 < 1$ and arbitrary k

$$\sum_{n=1}^{\infty} t^n n^k (|\alpha_n| + |\beta_n|) \leq \sum_{n=1}^{\infty} 2M t_0^n n^k \implies \text{U.C. of series on } k$$

$\implies$ (10) can be ~~arbitrarily~~ diff as often as we please wrt φ.

thus the superposition principle is applicable and (10) define fns which satisfy

$$\Delta u = 0 \quad \text{for } t<1$$

B. to show continuity of solutions in closed region $t \leq 1$

note: $f(\varphi) = \sum_{n=1}^{\infty} (\alpha_n \cos n\varphi + \beta_n \sin n\varphi) + \frac{\alpha_0}{2}$

but $f(\varphi) - \frac{\alpha_0}{2} \leq \sum_{n=1}^{\infty} (|\alpha_n| + |\beta_n|)$ converges

but $|t^n \alpha_n \cos n\varphi| \leq |\alpha_n|$
$|t^n \beta_n \sin n\varphi| \leq |\beta_n|$

so that (10) converges uniformly for $t \leq 1$

note: $\sum_{n=1}^{\infty} (|\alpha_n| + |\beta_n|)$ shows that the solution of exterior prob is bdd at infinity.

II. Poisson's Integral

Placing (9) into (10) and because of U.C. we can interchange limit and integral, for interior problem

$$u(\rho,\varphi)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\left\{\frac{1}{2}+\sum_{n=1}^{\infty}(\rho/a)^n\left[\cos n\psi\cos n\varphi+\sin n\psi\sin n\varphi\right]\right\}d\psi$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\left\{\frac{1}{2}+\sum_{n=1}^{\infty}(\rho/a)^n\left[\cos n(\psi-\varphi)\right]\right\}d\psi \qquad (11)$$

let $t=\rho/a<1$

but $t^n\cos n(\varphi-\psi)=\frac{1}{2}\left[t^n e^{in(\varphi-\psi)}+t^n e^{-in(\varphi-\psi)}\right]$

Can be obtained from De Moivre's Rule $e^{i\theta}=\cos\theta+i\sin\theta$
$e^{-i\theta}=\cos\theta-i\sin\theta$

note that $\sum_{n=1}^{\infty}t^n e^{in(\varphi-\psi)} = te^{i(\varphi-\psi)}\underbrace{\sum_{n=0}^{\infty}\left[te^{i(\varphi-\psi)}\right]^n}_{\text{power series}} = te^{i(\varphi-\psi)}\frac{1}{1-te^{i(\varphi-\psi)}}$

then $$u(\rho,\varphi)=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(\psi)}{2}\left\{1+\frac{te^{i(\varphi-\psi)}}{1-te^{i(\varphi-\psi)}}+\frac{te^{-i(\varphi-\psi)}}{1-te^{-i(\varphi-\psi)}}\right\}d\psi$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(\psi)}{2}\left[\frac{1-t^2}{1-2t\cos(\varphi-\psi)+t^2}\right]d\psi$$

and $$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\psi)\,\frac{a^2-\rho^2}{a^2-2a\rho\cos(\varphi-\psi)+\rho^2}\,d\psi \qquad (12)$$

(12) Is the solution to first bvp for interior of circle and is known as poisson integral

$$K(\rho,\varphi,a,\psi)=\frac{a^2-\rho^2}{a^2-2a\rho\cos(\varphi-\psi)+\rho^2}$$ is the Poisson Kernel

which is >0 for $\rho<a$

For $\rho = a$ Poisson integral degenerates to

$$u(a,\varphi) = f(\varphi) \qquad (12b)$$

$\therefore$ for interior problem

$$u(\rho,\varphi) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\psi)\,\dfrac{a^2-\rho^2}{\rho^2 - 2a\rho\cos(\varphi-\psi)+a^2}\,d\psi & \rho<a \\ f(\varphi) & \rho = a \end{cases} \qquad (12)$$

for exterior problem

$$u(\rho,\varphi) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\psi)\,\dfrac{\rho^2-a^2}{\rho^2 - 2a\rho\cos(\varphi-\psi)+a^2}\,d\psi & \rho > a \\ f(\varphi) & \rho = a \end{cases} \qquad (13)$$

from the above note that $f(\varphi)$ need only be continuous

A. Proof of the continuity of $u(\rho,\varphi)$ as $\rho \to a$

Choose any sequence of continuous & diff fns

$f_1(\varphi), \ldots, f_k(\varphi), \ldots$ which are U.C. toward $f(\varphi)$

~~the sequence of bdy fns correspond to a sequence of harmonic fns defined by (13)~~

by U.C. it follows that for $\epsilon > 0$ $\exists$ $k_0(\epsilon)$ $\ni$

$$|f_k(\varphi) - f_{k+l}(\varphi)| < \epsilon \quad \text{whenever } k > k_0(\epsilon) \quad l > 0$$

by the principle of the maximum

$$|u_k(\rho,\varphi) - u_{k+l}(\rho,\varphi)| < \epsilon \quad \text{for } \rho \le \rho_0 \quad k > k_0(\epsilon)$$

then the sequence $\{u_k\}$ converges uniformly. The limit fn $u(\rho,\varphi)$ is continuous in the closed region since all the fns $u_k(\rho,\varphi)$ are continuous in the closed region

Since $u(\rho,\varphi) = \lim\limits_{k\to\infty} u_k(\rho,\varphi) = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} K(\rho,\varphi,a,\psi)\, f(\psi)\,d\psi \\ f(\varphi) \end{cases}$ then

since $f_k(\varphi)$ converges uniformly to $f(\varphi)$

B. Discontinuous b.v.

We will show that (12) and (13) represent the solution of b.v.p. for every piecewise continuous fn $f(\varphi)$

Problem: Show that $\exists\, \delta>0 \;\forall\, \epsilon>0 \;\ni\; |u(\rho,\varphi) - f(\varphi_0)| < \epsilon$

$$\text{if } |\rho-a| < \delta(\epsilon) \;\&\; |\varphi-\varphi_0| < \delta(\epsilon)$$

By continuity of $f(\varphi)$ a $\delta_0(\epsilon)$ can be found $\ni$

$$|f(\varphi) - f(\varphi_0)| < \epsilon/2$$

for $\quad |\varphi-\varphi_0| < \delta_0(\epsilon)$

Consider 2 cont. diff fns $\bar{f}(\varphi)$ and $\underline{f}(\varphi)$ where

$$\bar{f}(\varphi) = f(\varphi_0) + \epsilon/2 \qquad |\varphi-\varphi_0| < \delta_0(\epsilon)$$

$$\bar{f}(\varphi) \geq f(\varphi) \qquad |\varphi-\varphi_0| > \delta_0(\epsilon)$$

and

$$\underline{f}(\varphi) = f(\varphi_0) - \epsilon/2 \qquad |\varphi-\varphi_0| < \delta_0(\epsilon)$$

$$\underline{f}(\varphi) \leq f(\varphi) \qquad |\varphi-\varphi_0| > \delta_0(\epsilon)$$

which are otherwise arbitrary and which satisfy (10a)

define $\bar{u}(\rho,\varphi)$ $\underline{u}(\rho,\varphi)$ which satisfy (10a)

such that $\bar{u}(\rho,\varphi) \to \bar{f}(\varphi)$ as $\rho\to a$ $\underline{u}(\rho,\varphi) \to \underline{f}(\varphi)$

Since Poisson's Kernel >0

$$\underline{u}(\rho,\varphi) \leq u(\rho,\varphi) \leq \bar{u}(\rho,\varphi)$$

since

$$\underline{f} \leq f \leq \bar{f}$$

from continuity of $\bar{u}(\rho,\varphi)$ & $\underline{u}(\rho,\varphi)$ on bdy for $\varphi=\varphi_0$

$\exists\ \delta_1(\epsilon)\ \ni.$

$$|\bar{u}(\rho,\varphi)-\bar{f}(\varphi_0)|\le \epsilon/2 \qquad (1')$$

for $|\rho-a|<\delta_1(\epsilon) \qquad |\varphi-\varphi_0|<\delta_1(\epsilon)$

and
$$|\underline{u}(\rho,\varphi)-\underline{f}(\varphi_0)|\le \epsilon/2 \qquad (2')$$

for $|\rho-a|<\delta_1(\epsilon) \qquad |\varphi-\varphi_0|<\delta_1(\epsilon)$

from (1') $\qquad \bar{u}(\rho,\varphi)\le \bar{f}(\varphi_0)+\epsilon/2 = f(\varphi_0)+\epsilon$

(2') $\qquad \underline{u}(\rho,\varphi)\ge \underline{f}(\varphi_0)-\epsilon/2 = f(\varphi_0)-\epsilon$

for $\quad |\rho-a|<\delta(\epsilon)\ ,\ |\varphi-\varphi_0|<\delta(\epsilon) \qquad \delta=\min(\delta_0,\delta_1)$

Now since
$$\underline{u}\le u\le \bar{u} \qquad \text{then}$$

$$f(\varphi_0)-\epsilon\le \underline{u}\le u\le \bar{u}\le f(\varphi_0)+\epsilon$$

or $\qquad |u(\rho,\varphi)-f(\varphi_0)|<\epsilon \quad$ for $\ |\rho-a|<\delta\ ,\ |\varphi-\varphi_0|<\delta$

and thus $u(\rho,\varphi)$ is cont at (a,φ_0)

c. *To show boundedness of solution*

since $K(\rho,\varphi,a,\psi)>0$ then if $|f(\varphi)|<M$ then

$$u(\rho,\varphi)<\frac{M}{2\pi}\int_{-\pi}^{\pi}\frac{a^2-\rho^2}{a^2+\rho^2-2a\rho\cos(\varphi-\psi)}\,d\psi$$

as $\rho\to a \qquad \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\frac{a^2-\rho^2}{a^2+\rho^2-2a\rho\cos(\varphi-\psi)}\,d\psi \to 1$

thus $u(\rho,\varphi)<M \qquad$ and therefore is bdd.

Consider now a problem which doesn't give trig fns solution, i.e. Axisym temp distrib in solid sphere

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi \frac{\partial}{\partial\varphi} u\right) = 0$$

where $u(a,\varphi) = f(\varphi)$

Let $u(r,\varphi) = R(r)\,\Phi(\varphi)$ this gives

$$\Phi \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{R}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi \frac{\partial\Phi}{\partial\varphi}\right) = 0$$

or $$\frac{1}{R}(r^2 R')' = -\frac{1}{\Phi \sin\varphi}(\sin\varphi\,\Phi')' = k^2$$

then $$(r^2R')' - k^2 R = 0$$

$0 \le r \le a$
$0 \le \theta \le 2\pi$
$0 \le \varphi \le \pi$

$$\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi \frac{d\Phi}{d\varphi}\right) + k^2 \Phi = 0$$

which is a legendre equation if $k^2 = n(n+1)$

then $$\Phi = A_n P_n(\cos\varphi) + B_n Q_n(\cos\varphi)$$

$$R = C_n r^n + D_n r^{-n-1}$$

for finiteness of solution at z axis $\varphi = 0, \pi$ $B_n = 0$ n integer
$D_n = 0$ for finiteness of $R(r)$ at $r=0$

then $$u(r,\varphi) = \sum_{n=0}^{\infty} c_n \left(\frac{r}{a}\right)^n P_n(\cos\varphi) \quad \text{where } A_n C_n = \frac{c_n}{a^n}$$

and $$f(\varphi) = \sum c_n P_n(\cos\varphi)$$

and since the legendre fns are an orthonormal set of fns like the sin, cos series in fourier expansion we can define c_n by

$$\int_\pi^0 -f(\varphi) P_m(\cos\varphi)\sin\varphi\, d\varphi = c_n \int_\pi^0 -P_n(\cos\varphi) P_m(\cos\varphi) \sin\varphi\, d\varphi$$

by the fact that $\int_{\pi}^{0} -P_n(\cos\varphi) P_m(\cos\varphi) \sin\varphi \, d\varphi = \begin{cases} \frac{2}{2n+1} & n=m \\ 0 & n \neq m \end{cases}$

then $c_n = \frac{2n+1}{2} \int_0^{\pi} f(\varphi) P_n(\cos\varphi) \sin\varphi \, d\varphi$

§ 4-4 Green's Function (Source Fn)

Discuss basic properties of G. Fn. of $\Delta u = 0$ as well as construction of G. Fn. for a class of simple regions by method of images in electrostatics.

1. Let $u(M)$ be C^1 on $\Sigma = \partial T$
   $u(M)$ be C^2 in T

then

$$u(M_0) = \frac{1}{4\pi} \iint_\Sigma \left[\frac{1}{r_{PM_0}} \frac{\partial u}{\partial n} - u(P) \frac{\partial}{\partial n}\left(\frac{1}{r_{PM_0}}\right)\right] d\sigma_P - \frac{1}{4\pi} \iiint_T \frac{\Delta u}{r_{MM_0}}\, d\tau \qquad (1)$$

if u is harmonic $\Delta u = 0$ if u satisfies $\Delta u = f$

Let $v(M)$ be a harmonic fn which nowhere is singular $T \cup \partial T$ then by greens second formula

$$\iiint_T [u \Delta v - v \Delta u]\, d\tau = \iint_\Sigma \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) d\sigma$$

then yields

$$0 = \iint_\Sigma \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right) d\sigma - \iiint_T v\, \Delta u\, d\tau \qquad (2)$$

ADD 1 & 2 then

$$u(M_0) = \iint_\Sigma \left[G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}\right] d\sigma - \iiint_T \Delta u \cdot G\, d\tau \qquad (3)$$

where $G =$ [redacted] $G(M, M_0) = \dfrac{1}{4\pi r_{MM_0}} + v \qquad (4)$

$M_0 = M_0(x, y, z) \qquad M = M(\xi, \eta, \zeta)$

In interior of T G is harmonic except at $M = M_0$

Choose $v(M) \ni G|_\Sigma = 0$ ie $v|_\Sigma = -\dfrac{1}{4\pi r}$

****

knowledge of G allows the BVP to be solved for any bc $u|_\Sigma = f$ while G itself is determined only for BVP having $v|_\Sigma = -1/4\pi r$

G is G.F. of first BVP for Laplacian on u, $\Delta u = 0$

G is also defined as an instantaneous pt source of the F.B.V.P. of $\Delta u = 0$; from (3)

$$u(M_0) = -\int_\Sigma\int u \frac{\partial G}{\partial n}\, d\sigma = -\int_\Sigma\int f \frac{\partial G}{\partial n}\, d\sigma \qquad f = u|_\Sigma \qquad (5)$$

A.M. Liapunov showed that under general assumptions on u (5) is a representation of the solution of FBVP for a wide class of boundaries.

* * * *

Properties of Green's Fn. where G exists in T and $\frac{\partial G}{\partial n}$ is cont. and Greens formula applicable on Σ

1a. G.F. > 0 in interior of T; Proof: $G = 0$ on $\partial T = \Sigma$ and $G > 0$ on surface of small sphere about pole. G must be positive according to max principle in T.

1b. $dG/dn|_\Sigma \leq 0$ since $G > 0$ in T $G = 0$ on $\partial T = \Sigma$

2. G.F. symmetric with respect to arguments
$G(M, M_0) = G(M_0, M) \Rightarrow$ source at M has same effect at M_0 as " " M_0 having effect at M

for 2-D case GF is defined by
1. $\Delta G = 0$ in S except at $M = M_0$
2. at $M = M_0$ $G(M, M_0)$ has singularity of form $\frac{1}{2\pi} \ln \frac{1}{r_{MM_0}}$
3. $G|_C = 0$ C is curve bdy of S.

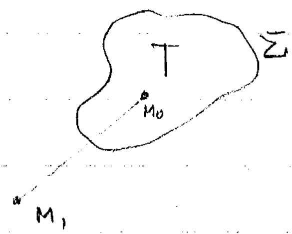
$$G(M, M_0) = \frac{1}{2\pi} \ln \frac{1}{r_{MM_0}} + v(M, M_0)$$

where v satisfie $\Delta v = 0$ in S $v|_C = -\frac{1}{2\pi} \ln \frac{1}{r_{MM_0}}$

and solution of FBVP for $\Delta u = 0$ is given by

$$u(M_0) = -\int_C f \frac{\partial G}{\partial n}\, ds, \quad f = u|_C$$

Calculation of Green's fn: Method of Images

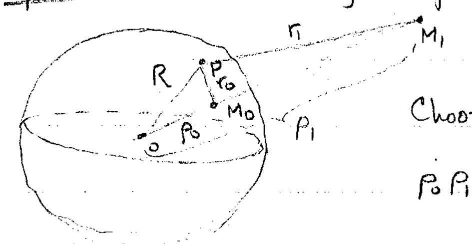


$\Delta G = 0$ have singularity $\frac{1}{r}$

and $G|_{\Sigma} = 0$

pick an image pt ∋ influence of singularity of M_0, M_1 on $\Sigma = 0$

Sphere G.F. for sphere of radius R M_0 inside sphere P on ∂T



Choose M_1 ∋. $|OM_0||OM_1| = R^2$

$\rho_0 \rho_1 = R^2$

$\Delta OPM_0 \sim \Delta OM_1P$

1) common ∡ P_0M_0 = ∡ P_0M_1

2) $OM_0/OP = \dfrac{OP}{OM_1}$

$$\frac{r_0}{r_1} = \frac{\rho_0}{R} = \frac{R}{\rho_1} \qquad r_0 = \frac{\rho_0 r_1}{R} \qquad \frac{1}{r_0} = \frac{R}{\rho_0 r_1}$$

$$G(P, M_0) = \frac{1}{4\pi r_{PM_0}} + v = \frac{1}{4\pi}\left(\frac{1}{r_0} - \underbrace{\frac{R}{\rho_0 r_1}}_{v}\right)$$

satisfies Laplace, $G|_{\Sigma} = 0$ and singularity of $\frac{1}{r}$

$$\frac{\partial G}{\partial n} = \frac{1}{4\pi}\left[\frac{\partial}{\partial n}\left(\frac{1}{r_0}\right) - \frac{R}{\rho_0}\frac{\partial}{\partial n}\left(\frac{1}{r_1}\right)\right]$$ where n is exterior normal

$$\frac{\partial}{\partial n}\left(\frac{1}{r_0}\right) = \frac{\partial}{\partial r_0}\left(\frac{1}{r_0}\right)\frac{\partial r_0}{\partial n} = -\frac{1}{r_0^2}\cos(r_0, \underline{n})$$

$$\frac{\partial}{\partial n}\left(\frac{1}{r_1}\right) = \frac{\partial}{\partial r_1}\frac{\partial r_1}{\partial n} = -\frac{1}{r_1^2}\cos(r_1, \underline{n})$$

Cosine law

$$\cos(r_0, \underline{n}) = \frac{R^2 + r_0^2 - \rho_0^2}{2Rr_0}, \quad \cos(r_1, n) = \frac{R^2 + r_1^2 - \rho_1^2}{2Rr_1}$$

$$\text{on } \Sigma, \quad r_1 = \frac{Rr_0}{\rho_0} \qquad \cos(r_1, n) = \frac{\rho_0^2 + r_0^2 - R^2}{2\rho_0 r_0}$$

$$\left.\frac{\partial G}{\partial n}\right|_{\Sigma} = \frac{1}{4\pi}\left[-\frac{1}{r_0^2}\frac{R^2 + r_0^2 - \rho_0^2}{2Rr_0} + \frac{\rho_0^2}{R^2 r_0^2}\frac{R}{\rho_0}\frac{\rho_0^2 + r_0^2 - R^2}{2\rho_0 r_0}\right]$$

$$= -\frac{1}{4\pi}\frac{R^2 - \rho_0^2}{Rr_0^3}$$

F.B.V.P.

$$u(M_0) = \frac{1}{4\pi R}\iint_{\Sigma} f(P)\left(\frac{R^2 - \rho_0^2}{r_0^3}\right)d\sigma_P$$

In spherical polars $(\rho_0, \theta_0, \varphi_0)$

$$u(\rho_0, \theta_0, \varphi_0) = \frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta, \varphi)\frac{R^2 - \rho_0^2}{(R^2 - 2R\rho_0\cos\gamma + \rho_0^2)^{3/2}}\sin\theta \, d\theta \, d\varphi$$

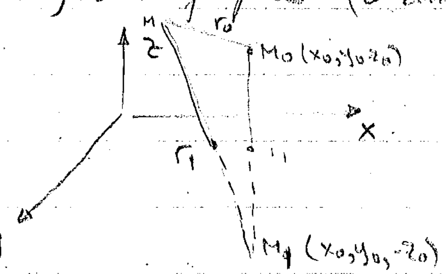
where $\cos\gamma = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0)$

poisson integral for sphere

In 2-D for circle $\log\left(\frac{1}{r}\right)$ as fund sol.

$$u(\rho_0,\theta_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-\rho_0^2}{R^2+\rho_0^2-2R\rho_0\cos(\theta-\theta_0)} f(\theta)\,d\theta$$

G.F. for half space (3 dim)



$$\sim (z-z_0)^2 \quad (z+z_0)^2$$

$$G = \frac{1}{4\pi}\left(\frac{1}{r_0} - \frac{1}{r_1}\right)$$

as $M_0 \to \pi\ (z=0)$ $\quad G \to 0$

$r_0 = \overline{MM_0}$
$r_1 = \overline{M_1M_1}$

$$\left.\frac{\partial G}{\partial n}\right|_{z=0} = -\left.\frac{\partial G}{\partial z}\right|_{z=0} = -\frac{1}{4\pi}\left[\frac{-(z-z_0)}{r_0^3} + \frac{(z+z_0)}{r_1^3}\right]_{z=0}$$

$$\left.\frac{\partial G}{\partial n}\right|_{z=0} = \frac{-z_0}{2\pi r_0^3}$$

$$u(M_0) = \frac{1}{2\pi}\iint_{z=0} \frac{z_0}{r_{M_0P}^3} f(P)\,d\sigma_P$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{z_0 f(x,y)\,dx\,dy}{\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{3/2}}$$

where $u = f(x,y)$ on $z=0$

$\Delta u = -f \qquad u = u_1 + u_2$

$\Delta u_1 = 0 \qquad \Delta u_2 = -f$ partic sol.

3 D $\quad u_2 = \dfrac{1}{4\pi}\iiint_T \dfrac{f}{r}\,d\tau$

2-D $\quad u_2 = \dfrac{1}{2\pi}\iint_\Sigma f \ln r\,d\sigma$

HW P331 3,6,7,8

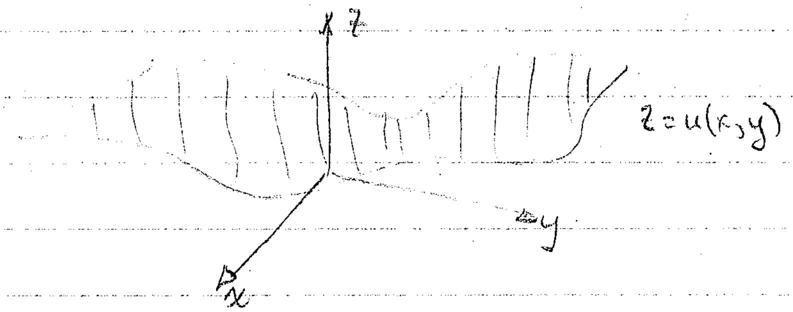
Single First Order PDE

2 indep variables (can be generalized to n indep variables)

$$F(x,y,u,u_x,u_y) = F(x,y,u,p,q) = 0 \quad (1)$$

where $p = u_x$ $q = u_y$

Any solution is called an integral surface



1. Linear Case

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y) \quad (2)$$

Consider LHS as directional deriv

$$\frac{du}{dt} = u_x x_t + u_y y_t = c(x,y)\,u + d \quad 3^*$$

choose $x_t = a$ $\quad y_t = b$ $\quad \dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)} \quad (3)$

distance in some sense along char curve

(3) gives curves in x,y plane : characteristic curves of diffeq:

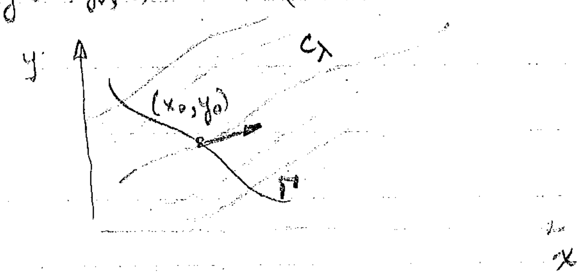
$$\frac{du}{dx} = \frac{cu+d}{a} \quad \text{along} \quad \frac{dy}{dx} = \frac{b}{a}$$

i.e. one first order pde reduced to 2 ode

We will assume all existence, uniqueness theorems of ODE.

Given $u = u_0$ at (x_0, y_0) (3) $\Rightarrow \exists$ char curve $x = x(x_0, y_0, t)$

$y = y(x_0, y_0, t)$ $(3^*) \Rightarrow u = u(x_0, y_0, t)$



$\Rightarrow$ given $u = u(x_0, y_0)$ we know u along whole char curve thru (x_0, y_0)

Γ = initial curve

C_λ = char curves intersecting Γ

C_λ exist & are indep of Γ however Γ cannot be chosen indep of C_λ

Γ cannot be parallel to char curves anywhere since along C_λ u satisfies a definite eqn & cannot be prescribed arbitrarily.

Example:

$$x u_x + y u_y = \alpha u$$

prescribe $u = \phi(x)$ along initial line $y = 1$

Characteristics: $\dfrac{dy}{dx} = \dfrac{y}{x}$ $\ln y = \ln x + C$ $y = cx$

$\left.\dfrac{du}{dx}\right|_{char} = \dfrac{\alpha u}{x}$ $u = kx^\alpha$ k is constant along a charact

$k = k(c)$ since $u = \phi(x)$ along $y = 1$

$k = k(y/x)$ $u = k(y/x)\, x^\alpha$ — unknown $k(s)$

on $y = 1$ $u = \phi(x) \Rightarrow \phi(x) = k(1/x)\, x^\alpha$ get k as fn of arg

$\therefore k(s) = \phi(1/s)\, s^\alpha$

$$u = \phi(x/y)\left(y/x\right)^{\alpha} x^{\alpha} = \phi(x/y)\, y^{\alpha} \quad \text{solution}$$

satisfy $x u_x + y u_y = \alpha x$ and $y=1$ $u = \phi(x)$

let $y = x$ $u = \phi(x)$

$$u = k(y/x)\, x^{\alpha}$$

$$\phi(x) = k(1)\, x^{\alpha} \quad \text{find } k \text{ as fn of its argument.}$$

ie ϕ must be a soln of equn. does not determine $k(s)$

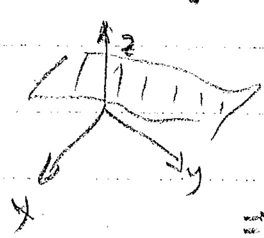
General Quasi linear Case

linear in highest derivative

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$$

$$(a,b,c)\cdot\begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix} = 0$$

if $z = u(x,y)$ is an integral surface



$(u_x, u_y, -1)$ is vector normal to surface

$\Rightarrow (a,b,c)$ must be tanget to plane

Directional field (a,b,c) are called the char directions

The properties of char dir are that $z = u(x,y)$ is an integral surface iff at each point the tangent plane contains the char direct.

$$\frac{dx}{a(x,y,z)} = \frac{dy}{b(x,y,z)} = \frac{dz}{c(x,y,z)}$$

In terms of param t along charac

$$\frac{dx}{dt} = a \quad \frac{dy}{dt} = b \quad \frac{dz}{dt} = c$$

define an integral surface from a one param family of char curves

Consider soln of

$$\frac{dx}{dt} = a(x,y,u(x,y)) \, , \quad \frac{dy}{dt} = b(x,y,u(x,y))$$

where $u(x,y)$ is a soln of diff eq

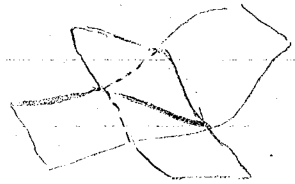
with $x = x_0$, $y = y_0$ when $t = 0$

Then the char curves are $x = x(t)$ $y = y(t)$ $z = u(x(t), y(t))$

$$\frac{dz}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt} = au_x + bu_y = c \quad \text{by diff eq}$$

$\Rightarrow$ curve is characteristic since it satisfies all three eqs.

If two integral surfaces intersect they must intersect along a whole char line



Cauchy IVP $u/_c = f$ & u cannot be specified along characteristic

3-20-73

$au_x + bu_y = c$ $\quad a, b, c$ fns of (x, y, u)

Quasi linear - linear in highest derivatives

Special form of $F(x, y, u, u_x, u_y) = 0$ General first order PDE in 2 indep var linear case $\quad \frac{dy}{dx} = -\frac{b}{a}(x,y)$

$\frac{du}{dx}$

linear case uncouples; for quasi linear case eqs are coupled

Can't prescribe initial conditions along a curve which is anywhere tangent to a characteristic

Theorem Cauchy IVP for the quasilinear case

Consider a first order p.d.e.

$$a(x,y,u)\,u_x + b(x,y,u)\,u_x = c(x,y,u)$$

where a, b, c have cont partial deriv wrt x, y, u

Suppose that along the initial curve $x = x_0(s)$, $y = y_0(s)$ the initial values $u = u_0(s)$ are prescribed, x_0, y_0, u_0 being continuously differentiable fns. for $0 \le s \le 1$. Let $\frac{dy_0}{ds}\, a(x_0,y_0,u_0) - \frac{dx_0}{ds}\, b(x_0,y_0,u_0) \neq 0$

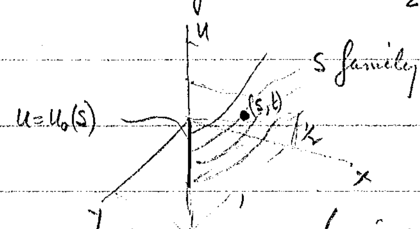
Then there exists one & only one solution $u(x,y)$ defined in some nbd of the initial curve which satisfies the P.D.E. + Init Cond.

$$u(x_0(s), y_0(s)) = u_0(s)$$

Proof: in notes give

$$u u_x + u_y = 1$$

I.C. $\quad x = s \quad y = s \quad u = \frac{1}{2}s \qquad 0 \le s \le 1$



look at characteristic crossing this curve. It will be a one param family, $f(s)$

Characteristic equns

$$\frac{dx}{dt} = u \qquad \frac{dy}{dt} = 1 \qquad \frac{du}{dt} = 1$$

subject to $x(s,0) = s \quad y(s,0) = s \quad u(s,0) = s/2$

$$u(s,t) = t + s/2$$

$$y(s,t) = t + s \qquad\qquad s = \frac{x - y^2/2}{1 - y/2} \qquad t = \frac{y-x}{1-y/2}$$

$$x(s,t) = t^2/2 + \frac{st}{2} + s$$

$$u = \frac{2(y-x) + (x - y^2/2)}{2-y}$$

$u = x/2,\ y/2$ when $x=y$

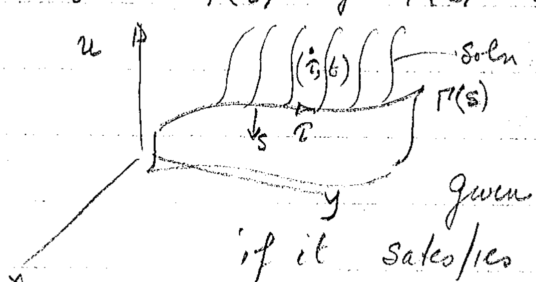
$$u = \frac{x - x^2/2}{2-x} = \frac{x(2-x)}{2(2-x)} = \frac{x}{2} \quad \checkmark$$

check that $\frac{dy}{ds}a - \frac{dx}{ds}b \neq 0 \qquad -(1\cdot a - 1b) = -(a-b) = (u-1) = (s/2 - 1) \neq 0$

on initial line $s \neq 2$

In general Initial curve will be given in one param

$$\Gamma: \quad x = \varphi(\tau) \quad y = \psi(\tau) \quad u = \omega(\tau)$$



Soln $\quad x = x(\tau,t)$
$\qquad y = y(\tau,t)$
$\qquad u = u(\tau,t)$

given in terms of two parameters
if it satisfies diff eq it is called an integral surface.

Integral surface generated by a one param family of characteristics

<u>Methods to find general soln</u> both theoretical

1) Take an initial curve Γ & vary it with respect to a new param
2) Take a 2 param family of characteristics & from them choose any one param family

each of these will be an integral surface.

2) is usually better; Theoretically:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

Find two independent integrals i.e. $\frac{dx}{a} = \frac{dy}{b}$, $\frac{dy}{b} = \frac{du}{c}$

$$\Phi(x,y,u) = \alpha$$
$$\Psi(x,y,u) = \beta \qquad \alpha, \beta \text{ arbitrary constants of integration}$$

each represents one param family of surfaces. Intersection of the two will give characteristics. Together they give a two param family of curves; their curves of intersection happens to be their characteristics.

Let one be a fn of the other symmetrically, let

$F(\alpha,\beta) = 0$ where F is an arbitrary fn.

This yields a one parameter family of curves each of which lies on the surface $F(\Phi(x,y,u), \Psi(x,y,u)) = 0$

This does satisfy the diffeq & doesn't depend on initial conditions & is called the general soln.

Note: general solution doesn't contain all soln of pde. Usually contains most.

Since F is arbitrary general soln is not unique.

Example

$$u(x u_x - y u_y) = y^2 - x^2$$

$$\boxed{\frac{dx}{ux} = \frac{dy}{-uy}} = \frac{du}{y^2 - x^2} \qquad \text{char eq}$$

$\ln xy = \log \alpha \qquad \boxed{xy = \alpha}$

$\frac{dx}{ux} = \frac{du}{y^2 - x^2} \quad ; \quad \frac{y^2}{x}dx - x\,dx = u\,du$

$\frac{dx}{x} = -\frac{dy}{y} \quad \therefore \quad -y\,dy - x\,dx = u\,du$

$-y^2/2 - x^2/2 = u^2/2 + C$

$\boxed{y^2 + x^2 + u^2 = \beta}$

$\phi(x,y,u) = xy = \alpha$

$\Psi(x,y,u) = y^2 + x^2 + u^2 = \beta$

$F(xy,\, x^2 + y^2 + u^2) = 0$ General Soln

for an arbitrary fn F. Write $\beta = G(\alpha)$

$\therefore \quad y^2 + x^2 + u^2 = G(xy)$

$u = \sqrt{G(xy) - (x^2 + y^2)} \Leftarrow$ arbitrary fn G General Soln.

or $\frac{dx + dy}{u(x-y)} = \frac{du}{y^2 - x^2}$ since $d(xy) = 0$

$(dx + dy)(y + x) = -u\,du$

$\frac{(x+y)^2}{2} = -u^2/2 + C^* \qquad \boxed{(x+y)^2 + u^2 = \beta^*} \qquad \beta^* = G^*(\alpha)$

$u = \sqrt{G^*(xy) - (x+y)^2} \qquad G^*(\alpha) = G(\alpha) + 2\alpha$

The inverse problem : complete integral

O.D.E. $\quad \phi(x,y,c) = 0 \qquad c =$ arbitrary param

Quest: what ODE does this family of curves satisfy?

the o.d.e must be independent of c

usually eliminated by differentiation of ϕ

Question: for 1st order PDE what do you do

Given an arbitrary fn $F(\alpha,\beta)=0$ & two fixed (i.e. definite) fns $\Phi(x,y,u)$ & $\Psi(x,y,u)$ so that $F(\Phi,\Psi)=0$ what is the p.d.e. for which this is the general solution.

Must eliminate F; but the resulting p.d.e. will depend on the choice of Φ & Ψ.

Differentiate F wrt x,y; treat u as a fn of x,y

$$F(\alpha,\beta)=0 \qquad \alpha=\Phi \quad \beta=\Psi$$

$$F_\alpha\left(\Phi_x+\Phi_u u_x\right)+F_\beta\left(\Psi_x+\Psi_u u_x\right)=0$$

$$F_\alpha\left(\Phi_y+\Phi_u u_y\right)+F_\beta\left(\Psi_y+\Psi_u u_y\right)=0$$

assuming $F_\alpha^2+F_\beta^2\neq 0$

then
$$\begin{vmatrix} \Phi_x+\Phi_u u_x & \Psi_x+\Psi_u u_x \\ \Phi_y+\Phi_u u_y & \Psi_y+\Psi_u u_y \end{vmatrix}=0$$

$$(\Phi_x+\underline{\Phi_u u_x})(\Psi_y+\underline{\Psi_u u_y})-(\Phi_y+\underline{\Phi_u u_y})(\Psi_x+\underline{\Psi_u u_x})=$$

$$\underbrace{\Phi_x\Psi_y-\Psi_x\Phi_y}_{c(x,y,u)}=u_x\underbrace{(\Psi_u\Phi_y-\Phi_u\Psi_y)}_{a(x,y,u)}+u_y\underbrace{(\Phi_u\Psi_x-\Phi_x\Psi_u)}_{b(x,y,u)}$$

Independent of F

<u>Method 2</u> from a fn depending on 2 parameters

$$u=f(x,y,\alpha,\beta)$$

define implicitly by $F(x,y,u,\alpha,\beta)=0$

now F is not arbitrary α, β arbitrary

Eliminate α, β to find P.D.E. Differentiate F wrt x, y

$$\left.\begin{aligned} &F_x + F_u u_x = 0 \\ &F_y + F_u u_y = 0 \\ &F = 0 \end{aligned}\right\}$$ 3 eq we can "in theory" eliminate α & β to yield a first order p.d.e.

this method will not necessarily yield a quasi linear Eq

2nd method usually is more useful

Definition of a complete integral

A fn of two parameters which is an integral of a first order pde in 2 indep variable is called a complete integral of the p.d.e

The complete integral & the general solution are connected in that one can be obtained from the other

Thm The general soln of the P.D.E. can, in theory, be obtained from a complete integral of the P.D.E.

Proof Let $F(x,y,u,\alpha,\beta) = 0$ be a complete integral

Geometrically this is a two param family of surfaces - must represent this in terms of an arbitrary fn - Do this by letting one parameter depend on the other in an arbitrary way - $\beta = G(\alpha)$

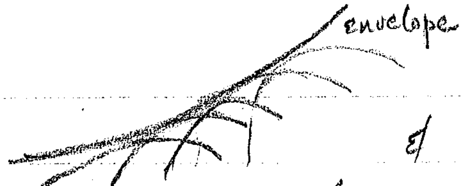
Then u defined implicitly

$$F(x,y,u,\alpha,G(\alpha)) = 0 \quad (1)$$

These are one parameter family of surfaces depending on α.

We must find sol of pde which satisfies (1) but which is indep of α

do this by constructing the envelope of all solns of (1) as α varies



Find envelope by diff wrt α

& eliminating α

i.e. $\left.\begin{array}{l} F_\alpha + G'(\alpha) F_\beta = 0 \\ F = 0 \end{array}\right\} \Rightarrow$ envelope

"in theory" $\alpha = \alpha(x,y,u)$ from $F_\alpha + G'(\alpha) F_\beta = 0$

then $F[x, y, u, \alpha(x,y,u), G(\alpha(x,y,u))] = 0$

is the general soln to same p.d.e. as was complete integral

now depends on the arbitrary fn $\underline{\underline{G}}$ instead of (α, β)

<u>Note</u>: this holds for all first order P.D.E.

in practice take particular G, at $F_\alpha + G'(\alpha) F_\beta = 0$, stage

& get F in terms of G.

<u>Ex</u> Clairaut's Eqn.

$$u = x u_x + y u_y + u_x u_y$$

Complete integral

$$u = \alpha x + \beta y + \alpha\beta$$

Choose $\beta = 1/\alpha$ $\therefore$ $u = \alpha x + y/\alpha + 1$ (1)

$\frac{\partial}{\partial \alpha}$ (1) $0 = x - y/\alpha^2$ $\alpha = \sqrt{y/x}$

$$u = \sqrt{y/x}\, x + y/\sqrt{y/x} + 1 = 2\sqrt{yx} + 1$$ <u>new soln</u>

<u>H/W</u>

Solve linear 1st order pde subject to I.C

1. $(x+2) u_x + 2y\, u_y = 2u$ IC : $u(-1, y) = \sqrt{y}$
2. $x^2 u_x - y^2 u_y = 0$ IC : $u(1, y) = F(y)$

} draw sketch of char & init line

3. Find general solns of

a) $x u_x (u - 2y^2) = (u - u_y y)(u - y^2 - 2x^3)$

b) $u_x + u^2 u_y = 1$

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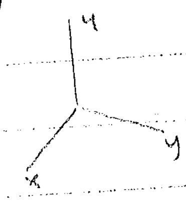
$a(x,y,u)\, u_x + b(x,y,u)\, u_y = c(x,y,u)$

General Nonlinear First Order P.D.E.

$F(x,y,u,p,q) = 0 \qquad p = u_x \quad q = u_y$



for quasi linear case given x_0, y_0, u_0

then a, b, c are determined uniquely

– know that normals to integral surfaces

through (x_0, y_0, u_0) are all perpendicular to $(a_0, b_0, c_0$

$(a,b,c)\begin{pmatrix} u_x \\ u_y \end{pmatrix} = 0 \qquad (a,b,c)$ tangent

one parameter family of normals all perpendicular to (a_0, b_0, c_0)

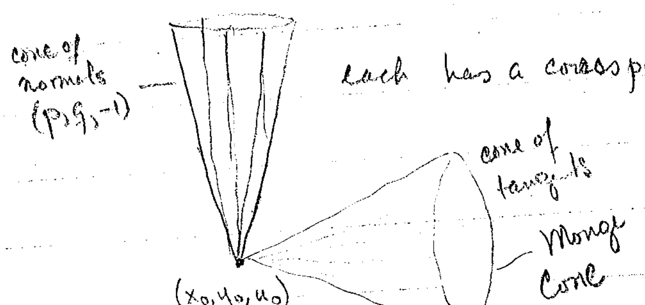
yield one parameter family of integral surfaces thru (a_0, b_0, c_0)

General Nonlinear

Given $(x_0, y_0, u_0) \Rightarrow F(x_0, y_0, u_0, p, q) = 0 \quad (1)$

yields a one parameter family of normals satisfying (1)

all go through pt (x_0, y_0, u_0)



each has a corresponding tangent

Monge Cone: envelope of the tangent planes thru (x_0, y_0, u_0)

in quasi linear case cone collapses onto line (a_0, b_0, c_0) ...

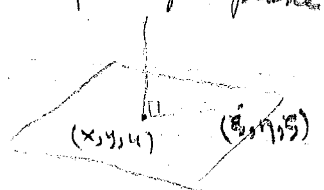
By definition generators of Monge cone are characteristics

To find characteristics - we find the eqs of the generators of Monge cone.

Let $\Theta(x,y,u,\alpha) = 0$ be one param family of tangent planes

$\Theta_\alpha(x,y,u,\alpha) = 0$ $\longrightarrow$ eliminate α to give envelope

Equation of tangent plane



$$p(\xi - x) + q(\eta - y) = \zeta - u \quad (2)$$

where p, q are connected thru $F(x,y,u,p,q)=0$

Assume we can solve for $q = q(p)$ from $F=0$.

insert $q = q(p)$ into (2) yielding a one param family of tangent planes depending on p

Differentiate wrt p

$$(\xi - x) + q'(p)(\eta - y) = 0 \quad (3)$$

Now equation of monge cone is obtained by eliminating p from (2,3) - usually this cannot be done explicitly,

need generators of cone.

Eliminate $q'(p)$ by differentiating F wrt p

$F=0 \quad \Rightarrow \quad F_p + q' F_q = 0$

$$q' = -F_p / F_q$$

$$\frac{(\xi - x)}{F_p} = \frac{(\eta - y)}{F_q}$$

Original eqn of tangent plane $\quad p\dfrac{(\xi - x)}{(\eta - y)} + q = \dfrac{(\zeta - u)}{(\eta - y)}$

$$F_p/F_q = \frac{(\xi - x)}{(\eta - y)}$$

$$p\frac{F_p}{F_q} + q = \frac{\zeta - u}{\eta - y}$$

$$\frac{\eta - y}{F_q} = \frac{\zeta - u}{pF_p + qF_q} = \frac{\xi - x}{F_p}$$

Eqs of generators of Monge cone

"In the small" $\frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = \frac{dx}{F_p}$

<u>Characteristic equns</u>

$$\frac{dx}{dt} = F_p \qquad \frac{dy}{dt} = F_q \qquad \frac{du}{dt} = pF_p + qF_q$$

rhs depends on x, y, u, p, q

From $F = 0$ these give one parameter family of characteristic direction from (x, y, u)

Quasi-linear

$$F(x, y, u, p, q) = ap + bq - c \quad (= 0)$$

$$\left.\begin{aligned} F_p &= a(x,y,u) \\ F_q &= b(x,y,u) \\ pF_p + qF_q &= ap + bq = c(x,y,u) \end{aligned}\right\} \begin{array}{l} \text{indep of } p \,\&\, q \\ \Longrightarrow \text{unique char} \\ \text{direction } (a,b,c) \end{array}$$

$F = 0$ is a redundant equn.

For general case – need two more equns.

look for equns for dp/dt & dq/dt

$$p = p(x,y) \qquad q = q(x,y)$$

$$dp/dt = p_x x_t + p_y y_t = p_x F_p + p_y F_q$$

$$dq/dt = q_x x_t + q_y y_t = q_x F_p + q_y F_q$$

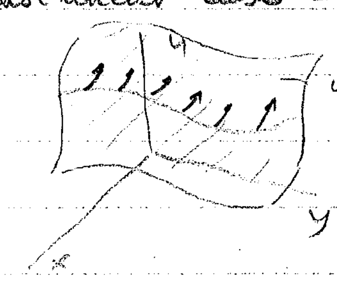
Differentiate $F(x,y,u,u_x,u_y)$ wrt x,y

$$F_x + F_p p_x + F_q q_x + F_u p = 0 \qquad F_x + \frac{dp}{dt} + F_u p = 0$$
$$F_y + F_p p_y + F_q q_y + F_u q = 0 \qquad F_y + \frac{dq}{dt} + F_u q = 0$$

$$\therefore \left.\begin{array}{|l} dp/dt = -F_x - F_u p \\ \\ dq/dt = -F_y - F_u q \\ \\ \dfrac{dx}{dt} = F_p \quad \dfrac{dy}{dt} = F_q \quad \dfrac{du}{dt} = pF_p + qF_q \end{array}\right\} \begin{array}{l}\text{Char eq} \\ \text{for} \\ F(x,y,u,p,q)=0\end{array}$$

Cauchy problem — IVP

Quasilinear case — initial curve nowhere characteristic



$C: x(\tau), y(\tau), u(\tau)$

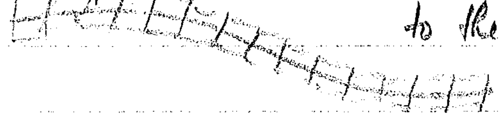
then each pt on C determines a unique character direction

→ yields unique integral surface.

General Nonlinear

fun. fns $x(\tau), y(\tau), u(\tau), p(\tau), q(\tau)$ needed to give a well posed problem ie specify a curve together with a direction at each pt on the curve - this is called an initial strip

by choosing p & q @ each pt we are picking a particular generator



to the monge cone @ each pt.

Need extra condition for smooth data: strip condition

$$u' = px' + qy' \qquad ' \equiv \frac{d}{d\tau}$$

this ensures we have a "smooth" initial strip
i.e. that we are picking generators of the monge cones in a "continuous" manner.

A strip is a set of five fns (x, y, u, p, q) satisfying the strip condition.
A soln to diff eq can be interpreted as a strip since it is defined in terms of x, y, u, p, q & automatically satisfies strip condition

$$p x_t + q y_t = p F_p + q F_q = u_t$$

Strips which are also solutions to the original pde are called characteristic strips. (x, y, u, p, q)
- their corresponding curves are called characteristic curves (x, y, u)

For fixed $\tau = \tau_0$, $(x_0, y_0, u_0, p_0, q_0)$ defines an element of the strip i.e. an element is a pt + a tangent plane.

If a characteristic strip has one element $(x_0, y_0, u_0, p_0, q_0)$ in common with an integral surface $u = u(x, y)$ then it lies completely on the surface.

Given the solution $u = u(x, y)$ & one element
show that this defines a unique characteristic strip lying in u.

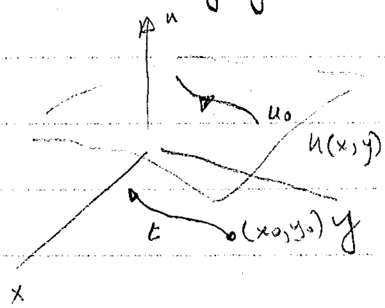
Consider 2 ode

$$x_t = F_p(x, y, u(x,y), u_x(x,y), u_y(x,y)) \qquad u(x,y) \text{ is known}$$
$$y_t = F_q(x, y, u(x,y), u_x(x,y), u_y(x,y))$$

for $x(t), y(t)$ subject to initial conditions

$$x(0) = x_0, \; y(0) = y_0$$

These define $x = x(t), y = y(t)$ in x-y plane.



x, y will determine a unique curve in the integral surface

$x = x(t), \; y = y(t)$

$u = u(x(t), y(t))$

Now determine equns satisfied by this curve

$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt} = u_x F_p + u_y F_q$$

$$\frac{du_x}{dt} = u_{xx} F_p + u_{xy} F_q$$

$$\frac{du_y}{dt} = u_{xy} F_p + u_{yy} F_q$$

where $u(0) = u_0 = u(x_0, y_0)$, $u_x(0) = u_x(x_0, y_0) = p_0$

$u_y(0) = u_y(x_0, y_0) = q_0$

use $F(x, y, u(x,y), u_x(x,y), u_y(x,y)) = 0$

$$\frac{dF}{dx} = \frac{dF}{dy} = 0$$

$$\frac{dF}{dx} = F_x + F_u u_x + F_p u_{xx} + F_q u_{xy} = 0$$

$$\frac{dF}{dy} = F_y + F_u u_y + F_p u_{xy} + F_q u_{yy} = 0$$

use fact $u_{xy} = u_{yx}$

$$\left.\begin{array}{l} \dfrac{du_x}{dt} = -F_x - F_u u_x \\ \\ \dfrac{du_y}{dt} = -F_y - F_u u_y \end{array}\right\} \text{strip } \underline{\text{Equns}}$$

i.e. $u(x,y)$ together with init cond (initial element) determines a unique characteristic strip including the element x_0, y_0, u_0, p_0, q_0
- this strip by defn of u lies entirely on the integral surface
- hence if char strip has one element in common with $u(x,y)$ it lies entirely on u.

Formulation of Cauchy problem (IVP)
details of proof in printed handout notes:
Prescribe initial strip by five fns

$$\left.\begin{array}{ll} x = \phi(\tau) & \\ y = \psi(\tau) & p = \rho(\tau) \\ u = \omega(\tau) & q = \sigma(\tau) \end{array}\right\} \text{subject to strip conditions } \omega' = \rho\phi' + \sigma\psi', \text{ each}$$

then: find a soln of $F(x, y, u, p, q) = 0$ which contains initial strip.

<u>Method of Solution</u> :- write down five eq.

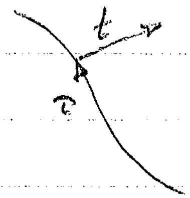
$$\left.\begin{array}{l} \dfrac{dx}{dt} = \\ \vdots \\ \dfrac{dq}{dt} = \end{array}\right\}$$

& solve them subject to initial conditions

$x(0,\tau) = \phi(\tau)$
$y(0,\tau) = \psi(\tau)$ etc
$u(0,\tau) = \omega(\tau)$

These yield:

$$\left.\begin{aligned} x &= f(t,\tau) \\ y &= g(t,\tau) \\ u &= h(t,\tau) \\ &\text{etc.} \end{aligned}\right\} (4)$$



$x, y, u, p, q = (x, y, u, p, q)(t,\tau)$

All that is needed that the initial strip is nowhere characteristic is that Jacobian of transformation is nonzero

$$\left.\begin{aligned} x_t &= F_p \\ y_t &= F_q \end{aligned}\right\} \quad \phi' F_q - \psi' F_p \neq 0$$

Soln is given in terms of 2 parameters (t,τ)

Theoretically: solve first two of (4) for t & τ in terms of (x,y) then substitute them into 3^{rd} of (4) to give $u(x,y)$

- will also get $p(x,y)$ & $q(x,y)$

Check $u_x = p$ & $u_y = q$ @ any pt.

$\equiv$ showing that strip condition holds at any point:

$$u_\tau(t,\tau) = p(t,\tau)\, x_\tau(t,\tau) + q(t,\tau)\, y_\tau(t,\tau)$$

Initially this is true by definition of initial strip - i.e. holds for $t=0$. Must show $U(t,\tau) = u_\tau - p x_\tau - q y_\tau \equiv 0$

We know that $U(0,\tau) \equiv 0$

Show that $U_t(t,\tau) = 0$

$$U_t = u_{\tau t} - p_t x_\tau - p x_{\tau t} - q_t y_\tau - q y_{\tau t}$$

$$= (pF_p + qF_q)_\tau + (F_x + F_u p)x_\tau - p(F_p)_\tau - q(F_q)_\tau$$

$$+ (F_y + qF_u) y_\tau \quad \text{by char equations.}$$

$$= p_\tau F_p + q_\tau F_q + (F_x + F_u p)x_\tau + (F_y + qF_u) y_\tau$$

since $F=0$

$$0 = F_x x_\tau + F_y y_\tau + F_u u_\tau + F_p p_\tau + F_q q_\tau$$

$$F_u u_\tau = -(\qquad\qquad)$$

$$\therefore \quad U_t = F_u(p x_\tau + q y_\tau - u_\tau)$$

$$U_t = -F_u \, U \quad \text{linear ode for } U$$

$\Rightarrow U_t \equiv 0$ since $U(0,\tau) = 0$

$\Rightarrow U \equiv 0$

Must show soln satisfies $F \equiv 0$

$$F_t = F_x x_t + F_y y_t + F_u u_t + F_p p_t + F_q q_t$$

$$= F_x F_p + F_y F_q + F_u(pF_p + qF_q) - F_p(F_x + pF_u) - F_q(F_y + qF_u)$$

$$= 0$$

$F_t = 0$ along a char strip

But $F=0$ at initial element

$\Rightarrow F \equiv 0$ on strip

Charpits Method for finding complete integral

1) Write down five characteristic eq
find one integral from these
$\Phi(x, y, u, p, q) = \alpha$ const of integration

2) Solve Φ & F for p, q in terms of x, y, u, α
$\Rightarrow P(x, y, u, \alpha) = p \quad Q(x, y, u, \alpha) = q$
then form $du = P\,dx + Q\,dy$ (note that P, Q contain u)
$\rightarrow$ this can be integrated = in form of perfect differential

$\Rightarrow u = u(x, y, \alpha, \beta)$

Example: $p^2 + qy - u = 0$

$\frac{dx}{dt} = F_p = 2p \quad \frac{dy}{dt} = F_q = y \quad \frac{du}{dt} = 2p^2 + qy = u$

$$\frac{dx}{2p} = \frac{dy}{y} = \frac{du}{u} = \frac{dp}{p} = \frac{dq}{0} \qquad F_y = q,\ qF_u = -q$$

$Q = q = \alpha \quad p = \sqrt{u - \alpha y} = P$

$$du = \sqrt{u - \alpha y}\,dx + \alpha\,dy$$

$$\frac{(du - \alpha\,dy)}{\sqrt{u - \alpha y}} = dx$$

$$2\sqrt{u - \alpha y} = x + \beta$$
$$2\sqrt{u - \alpha y} - x = \beta$$

$u = \alpha y + \frac{1}{2}(x+\beta)^2$

HW find complete integral of

1. $p^2 + q^2 - px - qy + \frac{1}{2}xy = 0$
2. $\sqrt{p} + \sqrt{q} = 2x$
3. $u - x + \log(pq) = 0$

April 10, 1973

if the deriv occur in complicated way and coeff occur in simpler way. if we interchange the roles we can simplify the problem

Legendre Transformation $(u_x, u_y) \to (x, y)$

General Idea change from point coords to line or tangent plane coords

if (x, y, u) are a pt on $u = u(x,y)$ then the tangent plane is $\alpha\xi - \beta\eta - \zeta = \gamma$

where (ξ, η, ζ) are running coords (corres to (x, y, u))

$$\alpha = u_x \quad \beta = u_y \quad \gamma = \alpha x + \beta y - u$$

The transformation $(x,y) \to (\alpha, \beta)$ is 1:1 provided $\frac{\partial(\alpha,\beta)}{\partial(x,y)} \neq 0$

$= u_{xx}u_{yy} - u_{xy}^2 \neq 0$

If this is true: given (x, y, u) then (α, β) are determined
$\Rightarrow \gamma$ is determined uniquely $\Rightarrow$ transformation $(x, y, u) \to (\alpha, \beta, \gamma)$ is 1:1.

we need inverse transformation

$$\gamma_\alpha = \alpha x_\alpha + x + \beta y_\alpha - u_\alpha$$
$$\gamma_\beta = \alpha x_\beta + y + \beta y_\beta - u_\beta$$

But $u_\alpha = u_x x_\alpha + u_y y_\alpha = \alpha x_\alpha + \beta y_\alpha$

$u_\beta = u_x x_\beta + u_y y_\beta = \alpha x_\beta + \beta y_\beta$

$$\left.\begin{array}{ll} \Rightarrow \gamma_\alpha = x & x = \gamma_\alpha \\ \gamma_\beta = y & y = \gamma_\beta \\ \gamma = x\alpha + y\beta - u \; ; & u = \alpha\gamma_\alpha + \beta\gamma_\beta - \gamma \end{array}\right\} \text{inverse Legendre transformation}$$

Example $x u_x^2 + y u_y^2 = 1$

Nonlinear in u_x & u_y

$x = \gamma_\alpha \qquad u_x = \alpha$

$y = \gamma_\beta \qquad u_y = \beta$

$\gamma = \ldots$

Linear in γ_α & γ_β $\alpha^2\gamma_\alpha + \beta^2\gamma_\beta = 1.$ $\qquad \frac{d\beta}{d\alpha} = \frac{\beta^2}{\alpha^2} \qquad \frac{d\gamma}{dt} = \frac{1}{\alpha^2}$

Can be extended to higher orders

$\alpha_j = u_{x_j}$

$\gamma = \sum \alpha_j x_j - u \qquad j = 1, \ldots, n$

There are n independent variables

Systems of First Order P.D.E.

Linear

$$\sum_{k=1}^{n} \left[a_{jk}(x,y)\, u_x^{(k)} + b_{jk}(x,y)\, u_y^{(k)} \right] = \underbrace{\sum_{k=1}^{n} c_{jk}(x,y)\, u^{(k)}}_{c_j(x,y,\underset{\sim}{u})} \qquad j = 1, \ldots, n$$

an equation where

r.h.s. can be nonlinear is called semilinear

dependent variables are $(u^{(1)}, \ldots, u^{(n)})$

rewrite as $A(x,y)\,U_x + B(x,y)\,U_y = C(x,y,U)$

where $A = (a_{jk})$ $B = (b_{jk})$ $C = (C_j)$

and $U^t = [u^{(1)}, u^{(2)}, \ldots, u^{(n)}]$

Classify → simplify by writing in terms of directed derivatives by taking linear combinations of the n equations so that all derivatives are in same direction.

notation $D_{jk} = a_{jk}\frac{\partial}{\partial x} + b_{jk}\frac{\partial}{\partial y}$

⇒ n^2 directional derivatives since $j\ \&\ k$ go from $1,\ldots,n$

$$\sum_1^n D_{jk}\, u^{(k)} = C_j$$

each $u^{(k)}$ differentiated in a different direction

take linear combinations. i.e. mult jth equation by λ_j and add

$$\sum_{j=1}^n \sum_{k=1}^n \left\{ \lambda_j a_{jk} u_x^{(k)} + \lambda_j b_{jk} u_y^{(k)} \right\} = \sum_{j=1}^n \lambda_j C_j$$

Now direction nos of derivatives are $(\sum \lambda_j a_{jk}, \sum \lambda_j b_{jk})$

We want these to be same direction for all k

$$\Rightarrow \sum \lambda_j a_{jk} = \mu \sum \lambda_j b_{jk} \qquad k=1,\ldots,n$$

$$\sum_{j=1}^n \lambda_j (a_{jk} - \mu b_{jk}) = 0 \qquad k=1,\ldots,n$$

for non triv sol must have $\det |a_{jk} - \mu b_{jk}| = 0$

ie that μ is ev for $A - \mu B$ & that $\Lambda = (\lambda_1, \ldots, \lambda_n)$ is the corresponding $\vec{EV}$.

$\det \|A(x,y) - \mu B(x,y)\| = 0$ is nth order eq for $\mu(x,y)$

these are local conditions

Suppose there are r distinct real roots $\mu_1, \ldots, \mu_r$ $r \le n$ $\left\{ \begin{array}{l} r=0 \\ r=n \end{array} \right.$ elliptic / physical for each root there is a corresponding left EV Λ_k $k=1,\dots,r$ $(\lambda_{1k},\lambda_{2k},\dots)$

Using each component will yield r distinct equns in which all u's differentiated in the same direction

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j b_{jk}(\mu u_x^{(k)} + u_y^{(k)}) = \sum \lambda_j c_j$$

$$\sum_j \lambda_j^{(l)} b_{jk} = \beta_k^{(l)}(x,y)$$

$$\sum \underline{\beta_k}(\mu u_x^{(k)} + u_y^{(k)}) = \sum \lambda_j^{(l)} c_j^{(l)}$$

$$\sum \beta_k^{(l)} D u^{(k)} \qquad D_l = \mu_l \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \quad l=1,\dots,r$$

$$\sum \beta_k^{(l)} D u^{(k)} = D\sum \beta_k^{(l)} u^{(k)} - \sum (D\beta_k^{(l)}) u^{(k)}$$

$$\hat{u}^{(l)} = \sum \beta_k^{(l)} u^{(k)} \qquad \hat{c} = \sum \lambda_k^{(l)} c_k^{(l)} + \sum (D\beta_k^{(l)}) u^{(k)}$$
$$= \sum [\lambda_k c_k + (D\beta_k^{(l)}) u^{(k)}]$$
$$= \hat{c}(x,y,\underset{\sim}{u})$$

$$D_l \hat{u}^{(l)} = \hat{c}^{(l)} \qquad l=1,2,\dots,r$$

r^{th} order system.

For $r<n$ must add $n-r$ of original equations to determine $u^{(k)}$. This mixed problem difficult will not be considered

$r=0$ no real root — elliptic

$r=n$ all real roots hyperbolic

$r=n$ with all distinct totally hyperbolic

Example Cauchy Riemann Eq $u_x = v_y$ $\underset{\sim}{u} = \begin{pmatrix} u \\ v \end{pmatrix}$

$u_y = -v_x$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

EV of $A - \mu$

$$\det \begin{vmatrix} 1 & -\mu \\ \mu & 1 \end{vmatrix} = \mu^2 + 1 = 0$$ no real roots elliptic

C-R satisfy $\Delta u = \Delta v = 0$ ✓

$u_x = v_y$

$u_y = v_x$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underset{\sim}{u}_x + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\underset{\sim}{u}_y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & -\mu \\ \mu & -1 \end{pmatrix} = \mu^2 - 1 = 0 \quad \mu = \pm 1$$ two distinct real roots totally hyperbolic

$$\overline{\nabla}\overline{\nabla} u = \overline{\nabla}\overline{\nabla} v = 0$$

char direction $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, $\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ $\frac{dy}{dx} = \pm 1$

$D_\ell = \mu_\ell \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ Directional deriv in x, y plane have slope $1/\mu_\ell$ — these are called char direction. Curves ∥ to char dir are given by $dy/dx = \frac{1}{\mu_\ell(x,y)}$

these define char curves.

Totally hyperbolic case $\quad j=1,2,\ldots,n$

$$\left.\begin{aligned} u_x - v_y &= 0 \\ u_y - v_x &= 0 \end{aligned}\right\} \quad \frac{dy}{dx} = \pm 1$$

$C^{\pm}$ – label two char curves

$y = x + \alpha \qquad \alpha = y - x$

$y = -x + \beta \qquad \beta = y + x$

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \qquad \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

$$C^{+} \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(u - v) = 0 \qquad D_1 \hat{u}^{(1)} = 0 \qquad (u - v) = 2f(\beta)$$

$$C^{-} \quad \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(u + v) = 0 \qquad D_2 \hat{u}^{(2)} = 0 \qquad u + v = 2g(\alpha)$$

$$u = f(\beta) + g(\alpha) \qquad v = g(\alpha) - f(\beta)$$

Quasi linear case – all the same except everything depends on u as well

$$A(x,y,u)\,u_x + B(x,y,u)\,u_y = C(x,y,u)$$

Characteristics

$$\frac{dy}{dx} = \frac{1}{\mu_\ell(x,y,u)}, \quad \lambda_j(x,y,u)$$

i.e. Characteristics depend on the soln.

Full nonlinear case → unnecessary since it reduces to quasilinear

we can always reduce a single nonlinear p.d.e. into a system of 3 quasilinear. We can reduce n nonlinear system to a system of $3n$ quasilinear.

$F(x,y,u,p,q) = 0$ reduced to a system of 3 in p, q, u

F_x, F_y are obtained

$$F_x + F_p p_x + F_u p + F_q q_x = 0$$

$$F_y + F_p p_y + F_u q + F_q q_y = 0$$

$F_p p + F_q q = F_p u_x + F_q u_y$

Char direct.

$\frac{dx}{dt} = F_p \quad \frac{dy}{dt} = F_q$ using these into the two give

$$\frac{dp}{dt} = -(F_x + F_u p) \qquad \frac{dq}{dt} = -(F_y + F_u q) \qquad \frac{du}{dt} = pF_p + qF_q$$

p, q, u are the $\hat{u}$

$$\begin{pmatrix} p \\ q \\ u \end{pmatrix}' = \begin{pmatrix} -F_u & 0 & 0 \\ 0 & -F_u & 0 \\ F_p & F_q & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ u \end{pmatrix} + \begin{pmatrix} -F_x \\ -F_y \\ 0 \end{pmatrix}$$

<u>The hodograph transformation</u>

particularly useful for one dim unsteady & two dim steady compressible gas flow

Euler's equns steady 2d, irrot, isentropic - two quasilin

Hodograph transf converts them into linear equations

(x,y) don't occur in coefficients

(u,v) are components of velocity in (x,y) directions

$$u_y - v_x = 0 \qquad \text{Continuity}$$

$$[c^2(q) - u^2]u_x - uv(u_y + v_x) + [c^2(q) - v^2]v_y = 0$$

$q = \sqrt{u^2+v^2}$ speed at a pt. $c^2(q)$ c^2 as a fn of q

Coeff are not fns of x, y; depend on u, v only

$(x,y) \to u, v$

$$J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \neq 0 \qquad u_x = Jy_v \quad u_y = -Jx_v \qquad v_x = -Jy_u \quad v_y = Jx_u$$

put into 2 eq

$$x_v - y_u = 0 \quad \& \quad (c^2-u)y_v + uv(x_v + y_u) + (c^2 - v)x_u = 0$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 0 \\ c^2-v & uv & uv & c^2-u \end{pmatrix} \begin{pmatrix} x_u \\ x_v \\ y_u \\ y_v \end{pmatrix}$$

One dim Gas Eq

Eulerian Co-ord

Cont $\rho_t + (\rho u)_x = 0$ } fluid Eq ρ = density

Moment $u_t + u u_x + \frac{1}{\rho} p_x = 0$ p = pressure

Entropy is const $S_t + u S_x = 0$ Energy Eq u = velocity in x direct

Eq of State defines particular material S = entropy

$$p = f(\rho, S)$$

$\frac{\partial f}{\partial \rho} > 0$ & has dimensions of velocity squared

$$c^2 = \frac{\partial f}{\partial \rho}(\rho, S)$$

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} u \\ \rho \\ S \end{pmatrix}_x + \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} u \\ \rho \\ S \end{pmatrix}_t = 0$$

<u>HW</u> a) Find characteristics

b) Classify System (where possible)

c) Put in canonical form $D_\ell \hat{U}^\ell = \hat{C}^\ell$

1. $u_x + 2u_t + v_x + 3v_t - u + v = 0$

$3u_x + u_t - 2v_x - v_t - 2v = 0$

2. $\rho u_x + u\rho_x + \rho_t = -2\rho\frac{u}{x}$

$\rho u u_x + \rho u_t + c^2 \rho_x = 0$

3) $(x+y)\,u_x + v_y = 0$
$(x-y)\,v_x + u_y = 0$

4) $u_x + u_y + u - v_x = 0$
$2u_y + u^2 - 1 - v_y = 0$

p 9
25

One Dimensional Compressible Flow

Eulerian equns [field equns]

$$\rho_t + (\rho u)_x = 0 \qquad \text{continuity}$$

$$u_t + u u_x + \frac{1}{\rho} p_x = 0 \qquad \text{Newton 2nd Law}$$

$$S_t + u S_x = 0 \qquad \text{Entropy does not change at a particle}$$

ρ = density (lineal)
u = velocity in x direction
p = pressure
S = entropy

Equ of State

$$p = f(\rho, S) \qquad \text{describes particular ga[s]}$$

Physically $f_\rho > 0$ & has dimension of velocity^2

put $c^2 = f_\rho(\rho, S)$

$$\Rightarrow \quad p_t = c^2 \rho_t + f_S S_t$$

$$u p_x = u c^2 \rho_x + u f_S S_x$$

adding $p_t + u p_x = c^2 \underbrace{(\rho_t + u\rho_x)}_{= -c^2 \rho u_x} + f_S \underbrace{(S_t + u S_x)}_{\shortparallel}$

$$\Rightarrow \quad p_t + u p_x + c^2 \overset{=\rho(p,S)}{\rho} u_x = 0$$

Using $\quad u_t + u u_x + \frac{1}{\rho} p_x = 0$

and $\quad S_t + u S_x = 0$

we have, looking on $\rho = \rho(p,S)$, a system of 3 quasilinear p.d.e in p, u, S

Characteristics

$$\begin{pmatrix} u & c^2\rho & 0 \\ 1/\rho & u & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} p_x \\ u_x \\ S_x \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_t \\ u_t \\ S_t \end{pmatrix} =$$

$$\Rightarrow \begin{vmatrix} u-\mu & c^2\rho & 0 \\ 1/\rho & u-\mu & 0 \\ 0 & 0 & u-\mu \end{vmatrix} = 0$$

$$\Rightarrow (u-\mu)\left((u-\mu)^2 - c^2\right) = 0$$

$\Rightarrow \underline{\mu = u \ , u \pm c}$ three char directions

C_0 : $\dfrac{dx}{dt} = u$ (particle trajectories)

$C_\pm^\pm$: $\dfrac{dx}{dt} = u \pm c$

To find diffl. eqns along characteristics must find eigenvectors. Let $(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)})$ be eigenvector for $\mu = u$.

$$(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}) \begin{pmatrix} 0 & c^2\rho & 0 \\ 1/\rho & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$\Rightarrow \lambda_2^{(0)} = \lambda_1^{(0)} = 0$, $\lambda_3^{(0)}$ arb. $\Rightarrow (0,0,1)$ is eigenvector

$\Rightarrow$ $S_t + u S_x = 0$ given eqn.

i.e. along particle trajectories no change in entropy – no dissipative mechanism present.

For $C^\pm$: $(\lambda_1^\pm, \lambda_2^\pm, \lambda_3^\pm) \begin{pmatrix} \mp c & c^2\rho & 0 \\ 1/\rho & \mp c & 0 \\ 0 & 0 & \mp c \end{pmatrix} = 0$

$$\Rightarrow \left.\begin{array}{l} c\lambda_1^\pm \mp \lambda_2^\pm = 0 \\ c^2\rho\lambda_1^\pm \mp c\lambda_2^\pm = 0 \\ \lambda_3^\pm = 0 \end{array}\right\} \Rightarrow c\rho\,\lambda_1^\pm = \pm\lambda_2^\pm$$

$\Rightarrow$ $\underline{(1, \pm c\rho, 0)}$ eigenvectors

$$\Rightarrow \quad p_t + u p_x + c^2 \rho u_x \pm c\rho\left(u_t + u u_x + \frac{1}{\rho} p_x\right) = 0$$

$$\Longrightarrow \quad p_t + (u \pm c) p_x \pm c\rho\left[u_t + (u \pm c) u_x\right] = 0$$

Both p & u diff. in direction with direction nos.

$(1, u \pm c)$. If $s_\pm$ parameter in $C^\pm$ directions

$$\frac{\partial p}{\partial s_\pm} \pm c\rho \frac{\partial u}{\partial s_\pm} = 0 \quad \text{on} \quad C^\pm$$

Since c depends on S these last two equns. coupled

to entropy equn. $\rightarrow$ simplifies if S constant

$\Longrightarrow$ <u>isentropic flow</u> $\quad f_{,S} \equiv 0$

Then $\quad p = f(\rho) \quad , \quad c^2 = f'(\rho)$

Now more convenient to eliminate p from equns

$$p_x = c^2 \rho_x$$

$$\Rightarrow \quad \left.\begin{aligned} u_t + u u_x + \frac{c^2}{\rho} \rho_x &= 0 \\ \rho_t + \rho u_x + u \rho_x &= 0 \end{aligned}\right] \quad \text{two equns for } (\rho, u)$$

$$\begin{pmatrix} u & \rho \\ c^2/\rho & u \end{pmatrix} \begin{pmatrix} \rho_x \\ u_x \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho_t \\ u_t \end{pmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} u-\mu & \rho \\ c^2/\rho & u-\mu \end{vmatrix} = 0$$

$$(u-\mu)^2 - c^2 = 0$$

$$\mu = u \pm c$$

Eigenvectors: $(\lambda_1^\pm, \lambda_2^\pm)$ are soln of

$$(\lambda_1^\pm, \lambda_2^\pm) \begin{pmatrix} \mp c & \rho \\ c^2/\rho & \mp c \end{pmatrix} = 0$$

ie. $c\lambda_1^\pm = \pm \frac{c^2}{\rho} \lambda_2^\pm$

$$\pm \frac{c}{\rho} (\rho_t + \rho u_x + u\rho_x) + \left(u_t + uu_x + \frac{c^2}{\rho}\rho_x\right) = 0$$

$$\Rightarrow \pm \frac{c}{\rho} \left[\rho_t + (u \pm c)\rho_x\right] + \left[u_t + (u\pm c) u_x\right] = 0$$

$$\pm \frac{c}{\rho} d\rho + du = 0 \quad \text{on} \quad C^\pm$$

Most conveniently written in terms of Riemann Invariants

Since $c^2 = f'(\rho)$, write $c = c(\rho)$

& $$L(\rho) = \int_{\rho_0}^{\rho} \frac{c(\rho)}{\rho}\, d\rho$$

Then $$d\left[L(\rho) \pm u\right] = 0 \quad \text{on } C^{\pm}$$

or $$L(\rho) \pm u = \text{const on } C^{\pm}$$

Let $$L(\rho) + u = 2r \quad \text{on } C^{+}$$

$$L(\rho) - u = 2s \quad \text{on } C^{-}$$

r & s called <u>Riemann Invariants</u>

r is a constant on a C^+, s is a const. on a C^-
but not the <u>same</u> constant on ~~eq~~ each curve.

$\Rightarrow$ r varies on a C^- + s varies on a C^+

$\Rightarrow$ take s as parameter on C^+, r as parameter on C^-

$$\left.\begin{aligned} \frac{dx}{ds} &= (u+c)\frac{dt}{ds} \quad \text{on } C^+ \\ \frac{dx}{dr} &= (u-c)\frac{dt}{dr} \quad \text{on } C^- \end{aligned}\right\} \quad \text{Regarding } \begin{aligned} x &= x(r,s) \\ t &= t(r,s) \end{aligned}$$

these become two p.d.e's for x, t.

$$x_s = (u+c)\, t_s$$
$$x_r = (u-c)\, t_r$$

Solve them for $x, t\ (r,s)$ – invert to give r, s as fns of x, t. Then solve
(u, c known by below)

$L \pm u = 2\frac{r}{s}$ to give ρ, u

: $\left.\begin{array}{l} L(\rho) = r+s \\ u = r-s \end{array}\right]$ gives u & ρ as fns of r, s

Thus original problem reduced to solving linear system for x, t in terms of r, s. Can eliminate x from them yielding:

$$(u_r + c_r)\, t_s + (u+c)\, t_{rs} = (u-c)\, t_{rs} + (u_s - c_s)\, t_r$$

But $u_r = 1$, $u_s = -1$, $L'(\rho)\,\rho_r = L'(\rho)\,\rho_s = 1$

$$\Rightarrow \quad \rho_r = \rho_s = \rho/c$$

and $\quad c_r = \frac{dc}{d\rho}\rho_r = \frac{\rho}{c}\frac{dc}{d\rho}$, $c_s = \frac{\rho}{c}\frac{dc}{d\rho}$

$$\Rightarrow \quad \left(1+\frac{\rho}{c}c_\rho\right)t_s + 2c\,t_{rs} = -\left(1+\frac{\rho}{c}c_\rho\right)t_r$$

$$\Rightarrow$$

$$t_{rs} + [\phi(r+s)]\,(t_r+t_s) = 0$$

where $$\phi(r+s) = \frac{1+\frac{\rho}{c}\frac{dc}{d\rho}}{2c}$$

is just a fn. of $r+s$ since it is just a fn. of ρ

<u>Special case</u> : polytropic gas - large no. of actual gases

$$p = A\rho^\gamma \qquad \gamma = \text{ratio of spec. heats} = 1.4 \text{ air.}$$

$$\Rightarrow \quad p_\rho = \gamma A \rho^{\gamma-1} \qquad ($$

$$\Rightarrow \quad c(\rho) = \sqrt{A\gamma}\,\rho^{\frac{\gamma-1}{2}} \qquad (\rho_0 \doteq 0)$$

$$L(\rho) = \frac{2\sqrt{A\gamma}}{\gamma-1}\,\rho^{\frac{\gamma-1}{2}} = \frac{2c}{\gamma-1}$$

To find $\phi(r+s)$, $$\frac{dc}{d\rho} = \frac{c}{\rho}\cdot\left(\frac{\gamma-1}{2}\right)$$

$$\Rightarrow \frac{\rho}{c}\frac{dc}{d\rho} = \frac{\gamma-1}{2}, \qquad \phi(r+s) = \frac{\gamma+1}{2(2c)}$$

$$= \frac{\gamma+1}{2(\gamma-1)L}$$

But $\quad L = r+s$

$$\Rightarrow \quad \phi(r+s) = \frac{\gamma+1}{2(\gamma-1)(r+s)}$$

Let $\quad K = \frac{1}{2}\frac{\gamma+1}{\gamma-1}$

$$\underline{t_{rs} + \frac{K}{r+s}(t_r + t_s) = 0}$$

Can eliminate first deriv. terms with subst.

$t_r = \psi' \omega + \psi \omega_r \quad t_{rs} = \psi'' \omega + \psi' \omega_s + \psi' \omega_r + \psi \omega_{rs}$

$t_s = \psi' \omega + \psi \omega_s$

$$t = \psi(r+s)\,\omega(r,s)$$

where ω is new dep. var. + ψ chosen s.t. $\omega_r + \omega_s$ terms vanish

$$\Rightarrow \psi\, \omega_{rs} + \left(\psi' + \frac{K\psi}{r+s}\right)(\omega_r + \omega_s) + \left(\psi'' + \frac{2K\psi'}{r+s}\right)\omega =$$

Let $\quad \psi = (r+s)^{-K}$

so that $\quad t(r,s) = \dfrac{\omega(r,s)}{(r+s)^K}$

$$\underline{\omega_{rs} + \frac{K(1-K)}{(r+s)^2}\,\omega = 0}$$

lin. telegraph eqn. with variable coeff.

soln. straightforward. — not trivial

<u>Simple Waves</u> : all previous analysis presupposes

can change variables $(x,t) \to (r,s)$

- can't do this if r & s are not independent

i.e. if $\exists$ $F(r,s)$ s.t $F(r,s)=0$
but $F_r^2 + F_s^2 \neq 0$

$\Rightarrow$ $F_r\,dr + F_s\,ds = 0$

Consider characteristic C^+ — r is a constant
$\Longrightarrow dr = 0$
& $F_s\,ds = 0$ on C^+
$F_r\,dr = 0$ on C^-

If $F_s \neq 0$ s is also constant on C^+
If $F_r \neq 0$ r is also constant on C^-

$\Rightarrow$ if $F_r \neq 0$ & $F_s \neq 0$ r & s must both be

constants in whole region covered by characteristics.

$\Rightarrow$ u & ρ both constant (& p also) $\Longrightarrow$ have <u>constant state</u>
(not at rest $u \neq 0$ rec.)

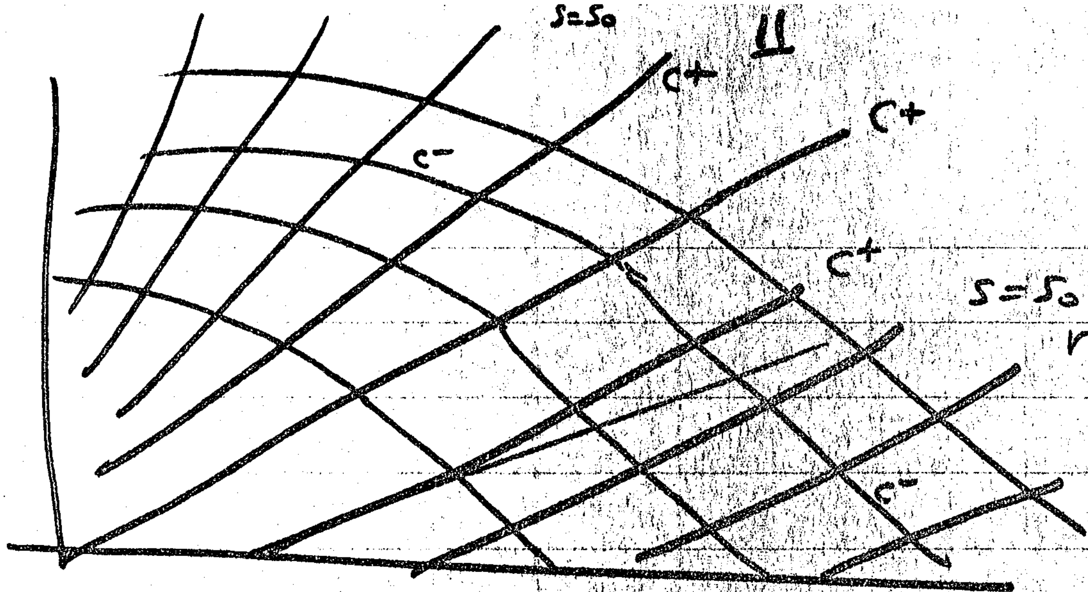
Now consider cases for <u>one</u> of F_r, F_s nonzero.

1) $F_r = 0$, $F_s \neq 0$ 2) $F_s = 0$, $F_r \neq 0$

1) $F_r = 0, F_s \neq 0 \Rightarrow$ both r & s constant along C^+
(since $F_s\, ds = 0$ on C^+ & r already const. there)

$\Rightarrow u$ & ρ constant on C^+'s $\Rightarrow$ slopes of C^+ characteristics $\left(\frac{dx}{dt} = u + c\right)$ are constant i.e. straight lines. However they are not parallel since r varies with C^- i.e r is a different constant on each C^+. Hence u & ρ are different constants on each C^+ & each C^+ has different slope.

A flow in which one family of the characteristics consists of straight lines is called a <u>simple wave</u>. In fact the flow in a region adjacent to a constant state is always a <u>simple</u> wave.



Consider constant state $r=r_0$, $s=s_0$, and an adjacent region where u & ρ vary. Curve separating two regions must be a characteristic — let it be a C^+.

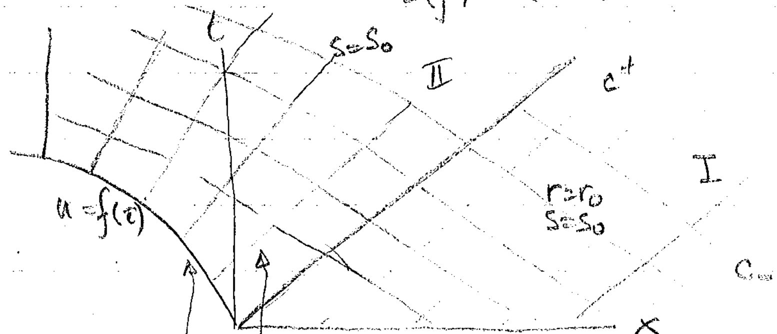
Then the curves entering region II from I are C^- characteristics. Along these s is a constant & it must be same const. on each C^-, $s=s_0$. Now consider any C^+ char. in II. On it r is a constant by defn. & also s is a constant $\Rightarrow$ u & ρ const on any C^+ in II

$\Rightarrow$ characteristics straight lines in II

$\Rightarrow$ simple wave

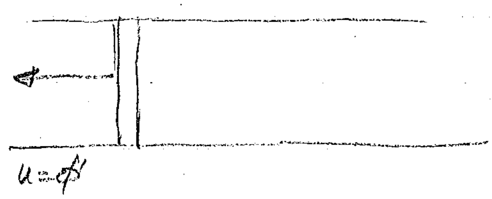
4-24-73

Riemann Invariants $L(\rho)+u=2r$ on C^+, $L(\rho)-u=2s$ on C^- $\Big\}\ \frac{dx}{dt}=u\pm c(\rho)$



Simple wave region - straight line char

Piston Problem $x=\phi(t)$ piston path $u=\phi'$



1) Rarefaction wave piston accelerates in $-x$ direction

$$\phi(0)=0 \qquad \phi'(0)=0 \qquad \phi''(t)<0$$

at piston $u=\phi'(t)$

Region I : $\frac{dx}{dt}=\pm c_0=\pm c(\rho_0)$

II $\frac{dx}{dt}=u+c(\rho)$ on C^+ : const on C^+

i.e. char C^+ are straight lines

To find slope - know $u=\phi'(t)$ - must find ρ

$L(\rho)=u+L(\rho_0)\rightarrow$ gives ρ as a fn of τ

$\Rightarrow$ have c as a fn of τ

$\Rightarrow$ have $u+c=\frac{dx}{dt}$ as a fn of τ, param along piston path to show non convergence of char

find $m(\tau) = u + c$ slope of C^+

what property of ϕ do we want for char to be diverg'g

i.e. $m' < 0$.

$$m'(\tau) = u'(\tau) + c_\tau = \phi''(\tau) + c_\tau$$

$$L(\rho) = L(\rho_0) + u = L(\rho_0) + \phi'(\tau)$$

$$L'(\rho)\rho_\tau = \phi''(\tau) \qquad L = \int \frac{c(\rho)}{\rho} d\rho$$

$$\frac{c}{\rho}\rho_\tau = \phi'(\tau)$$

$$c^2(\rho) = f'(\rho) \Rightarrow 2c\, c_\tau = f''(\rho)\rho_\tau \qquad p = f(\rho)$$

$$c_\tau = \frac{f''\rho_\tau}{2c} = \frac{f''}{2c}\frac{\phi''\rho}{c}$$

$$m'(\tau) = \left(1 + \frac{\rho f''}{2c^2}\right)\phi''(\tau)$$

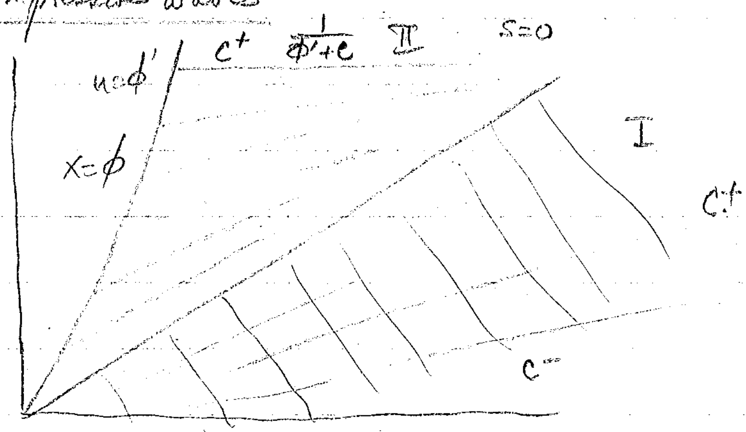
physically $f'' > 0 \Rightarrow m'(\tau) = \text{sign}(\phi''(\tau)) \Rightarrow$ characteristics fan out for ϕ'' negative. Rarefaction wave.

$$u = L(\rho) - L(\rho_0) = \int_{\rho_0}^{\rho} \frac{c(r)}{r} dr$$

Since $\rho \geq 0$ & $c > 0$ if $u \downarrow \rightarrow \rho \downarrow$

for veloc at which piston leave gas behind ie $\rho = 0$

if $u_\infty = \int_{\rho_0}^{0} \frac{c(r)}{r} dr = -\int_0^{\rho_0} \frac{c(r)}{r} dr$ Escape veloc, Compression waves



For convenience $r=r_0$ $s=0$

$$\begin{cases} L(\rho)+u=2r & C^+ \\ L(\rho)-u=2s=0 & C^- \end{cases}$$

$u=L(\rho)=r=$ const on C^+

@ pt of intersect $u_t \to \infty$: Characteristics intersect each carries different value of r or u $\Rightarrow u$ is multivalued at a $(t,x) \Rightarrow$ for this to happen $u_t \to \infty$ for some (t,x)

$*$ $u=\phi'(\tau)$

$\frac{dt}{dx}=\frac{1}{u+c}=$ const on C^+

Coupled equations

$t=\frac{x}{u+c}+\beta$ when $x=\phi$ $t=\tau$

implicit eq for τ as a fn of x,t

$*$ $t=\tau+\frac{(x-\phi(\tau))}{(u+c)[\tau]}$ eq of characteristic

$u_t=\frac{u_\tau}{t_\tau}=\phi''(\tau)/t_{,\tau}$

bdd

if $u_t \to \infty$ then $t_{,\tau} \to 0$

$$t_\tau = 1 + \frac{[x-\phi(\tau)]\, m(\tau)}{(u+c)^2} - \frac{\phi'(\tau)}{u+c} \to 0$$

$$\frac{c(u+c) - (x-\phi)m'}{(u+c)^2} = 0$$

$$m' = \left(1 + \frac{\rho f''}{2c^2}\right)\phi''$$

for char to intersect @ $x = x_s$ $\quad c(u+c) - (x_s - \phi)\left[1 + \frac{\rho f''}{2c^2}\right]\phi'' = 0$

cond: $\underbrace{c(u+c) > 0 \quad x_s > \phi}$

$\Downarrow$

$\underline{\phi'' > 0}$ for intersection to occur

i.e. char intersect only after piston accelerates in positive direction
$\Rightarrow$ if $\phi'' > 0$ characteristics converge & will always intersect @ some x_s.

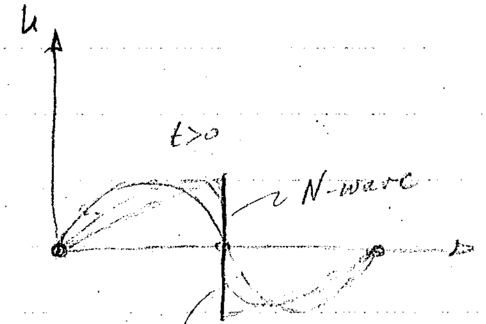
$$\left.\begin{aligned} u(\rho) &= u = \phi'(\tau) \\ t &= \tau + \frac{x - \phi(\tau)}{(u+c)} \end{aligned}\right\}$$

$u=0$ $\rho=\rho_0$ is a characteristic $\quad t = \tau_0 + \dfrac{x - \phi(\tau_0)}{c(\rho_0)}$

i.e. those characteristics carry ρ $u=0$
have the linear speed $c(\rho_0)$, those carrying $u>0$ have speed $u+c > c_0$
Those carry $u<0$ have speed $u+c < c_0$

Suppose $u = X' = -\epsilon 2\pi \sin 2\pi t$

$X = \epsilon \cos 2\pi t$



$t - \left(\frac{x-b}{c_0}\right)$

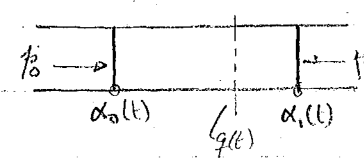
Shock condition needed.

how fast shock forms depends on $\epsilon \phi''(\tau)$ not on $|u|$; that which determines how fast shock forms is $|u'|$

N-wave asymptotic shape that a wave will take for large t.

magnitude of N-wave depends on distance between zeroes.

Shock relations



① Conservation of mass

$$\frac{d}{dt}\int_{\alpha_0}^{\alpha_1} \rho\, dx = 0$$

② Conservation of Mom

$$\frac{d}{dt}\int_{\alpha_0}^{\alpha_1} \rho u\, dx = p(\alpha_0, t) - p(\alpha_1, t)$$

③ Conservation of Energy

$$\frac{d}{dt}\int_{\alpha_0}^{\alpha_1} \rho\left(\frac{u^2}{2} + e\right) dx = p(\alpha_0)u(\alpha_0) - p(\alpha_1)u(\alpha_1)$$

e internal energy

④ non decreasing in entropy $\Delta S \geq 0$

determines sign of discont.

$$\frac{d}{dt}\int_{\alpha_0}^{\alpha_1} \rho S\, dx \geq 0$$

$\frac{dQ}{dt}$, $Q = \int_{\alpha_0}^{\alpha_1} \Psi(x)\,dx$

Let $Q = \int_{\alpha_0}^{q} + \int_{q}^{\alpha_1}$

Def $\Psi_0 = \lim_{x\to q_-} \Psi \qquad \Psi_1 = \lim_{x \to q_+} \Psi$

Define: speed of discont $\frac{dq}{dt} = U(t)$; $u_j = \frac{d\alpha_j}{dt}$ $j = 0,1$

$$\frac{dQ}{dt} = \Psi_0 U - \Psi(\alpha_0)u_0 + \int_{\alpha_0}^{q} \Psi_{,t}\,dx + \Psi(\alpha_1)u_1 - \Psi_1 U + \int_{q}^{\alpha_1} \Psi_{,t}\,dx$$

$$= (\Psi_0 - \Psi_1)U + \underset{\downarrow \Psi_1}{\Psi(\alpha_1)u_1} - \underset{\downarrow \Psi_0}{\Psi(\alpha_0)u_0} + \underset{\to 0}{\int_{\alpha_0}^{\alpha_1} \Psi_t\,dt}$$

Now let $\alpha_0 \to q^-$ $\alpha_1 \to q^+$ always $\alpha_0 < q < \alpha_1$

$$\frac{dQ}{dt} = \Psi_0(U - u_0) - \Psi_1(U - u_1)$$
$$= v_1\Psi_1 - v_0\Psi_0$$

Let $v_j = u_j - U$ relative flow speed on either side of discont

$Q = \rho$ ① $\rho_0 v_0 = \rho_1 v_1 = m$ mass flux across discont

ρu ② $\rho_1 u_1 v_1 - \rho_0 u_0 v_0 = p_0 - p_1$ or $m u_0 + p_0 = m u_1 + p_1$

or $\rho_0 v_0^2 + p_0 = \rho_1 v_1^2 + p_1 = p$ total mom flux

$\rho(u^2/2 + e)$ ③ $v_1\rho_1(e_1 + \frac{1}{2}u_1^2) - \rho_0(e_0 + \frac{1}{2}u_0^2)v_0 = p_0 u_0 - p_1 u_1$

or $m(e_1 + \frac{1}{2}u_1^2) - m(e_0 + \frac{1}{2}u_0^2) = p_0 u_0 - p_1 u_1$

define $\tau = \frac{1}{\rho}$ specific vol $\quad m(e_0 + \frac{1}{2}v_0^2 + p_0\tau_0) = m(e_1 + \frac{1}{2}v_1^2 + p_1\tau_1)$

ρS ④ $\rho_1 v_1 S_1 \geq \rho_0 v_0 S_0$ or $m S_1 \geq m S_0$

2 cases

(1) $m=0$ - no mass flow across discon contact discontinuity

(2) $m\neq 0$ - Gas flow across discont called a Shock

Case 1

since $m=0$ & $\rho_s' \neq 0 \Rightarrow V_1 = V_0 \Rightarrow \boxed{u_0 = u_1}$ velocity & press are cont.

$\Rightarrow \boxed{p_0 = p_1}$

Case 2 $m\neq 0$ define Enthalpy $j = e + p\tau$

(3) $\rightarrow \quad j_0 + \frac{1}{2}V_0^2 = j_1 + \frac{1}{2}V_1^2$

Equation of state $e = e(p,\tau)$

3 equations for seven quantities $u_0, u_1, \rho_0, \rho_1, p_0, p_1, U$

Must be given four of them to determine other 3.

e.g. $u_0, \rho_0, p_0 + \begin{cases} u_1 \\ \rho_1 \\ p_1 \end{cases}$ or U all fns of (x,t)

Given $e = e(p,\tau)$ 3) is written as $(\tau_0 - \tau_1)\dfrac{(p_0+p_1)}{2} = e(\tau_1, p_1) - e(\tau_0, p_0)$

Define $H(\tau,p) = e(\tau,p) - e(\tau_0,p_0) + (\tau - \tau_0)\left(\dfrac{p+p_0}{2}\right)$

Hugoniot Function

State on other side of shock given by $H(\tau,p) = 0$

Curve $H(\tau,p)=0$ in (τ,p) space is called the Hugoniot curve.

For polytropic gas : $e = \dfrac{p\tau}{\gamma - 1}$ define $\mu = \dfrac{\gamma-1}{\gamma+1}$

$$2\mu^2 H(\tau,p) = (\tau - \mu^2\tau_0)p - (\tau_0 - \mu^2\tau)p_0$$

$H=0$

$$\frac{p_1}{p_0} = \frac{\tau_0 - \mu^2\tau_1}{\tau_1 - \mu^2\tau_0} = \frac{\rho_1 - \mu^2\rho_0}{\rho_0 - \mu^2\rho_1}$$

$$\frac{\tau_0}{\tau_1} = \frac{\rho_1}{\rho_0} = \frac{p_1 + \mu^2 p_0}{p_0 + \mu^2 p_1}$$

Since $\gamma p = \rho c^2$ $\quad j = e + p\tau = \frac{\gamma}{\gamma-1} p/\rho = \left(\frac{1-\mu^2}{2\mu^2}\right) c^2$

$$\mu^2 V_0^2 + (1-\mu^2) c_0^2 = \mu^2 V_1^2 + (1-\mu^2) c_1^2 = c_*^2$$

After much algebra can then show

$$V_0 V_1 = c_*^2 \qquad \text{critical speed.}$$

$$\Longrightarrow \quad (1-\mu^2)(U-u_0)^2 - (u_1-u_0) \times (U-u_0) = (1-\mu^2) c_0^2$$

gives U in terms of $u_1, u_0, c_0 = c_0(\rho_0)$

HW Consider Ahead of shock $u = 0$ $\rho = \rho_0$ $p = p_0$

Piston starts instantaneously from rest with $u_p > 0$ const.

Find U & rest of unknowns behind shock & draw (t, x) diagram explaining the sol. look at case $u_p = u_p(t)$

H.W. look at non lin wave eq

$$w_{tt} = c^2(w_x)\, w_{xx}$$

Change variables to put it in a system of 2 first order pde

Find Characteristics & Riemann Invariants: suggest

if $c = c(w_x, \epsilon x)$

5-1-73

$$\underset{\sim}{u}_{,t} + \underset{\sim}{A}(\underset{\sim}{u}, x)\underset{\sim}{u}_{,x} = \underset{\sim}{B}(\underset{\sim}{u}, x)$$

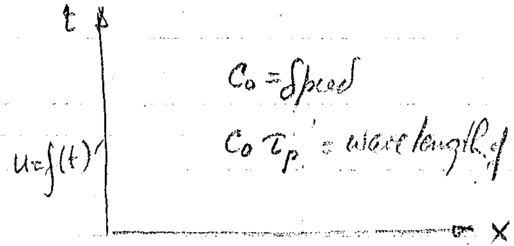
x due to stratification $\quad \underset{\sim}{B}$ relaxation or geometric effects

high frequency : if L_d = length scale defined by the dissipative mechanisms :
1. relaxation or rate dependence of medium
2. stratification (reflection due to change in media prop.)
3. geometric dissipation (length scale : radius of curvature)

L_d
1. distance of prop. $\Rightarrow$ wavelength is reduced by ½
2. $|A|/|A'|$
3. radius of curvature

Always talk about results wrt what occurs at bdy

$x, t \geq 0$



c_0 = speed

$c_0 \tau_p$ = wavelength of b.c. $= L_w$

① pulse τ_p $f \equiv 0$ except $0 \le t \le \tau_p$

② time periodic bc : period $= \tau_p$

$f(y+1) = f(y)$

$u(0,t) = f(t/\tau_p)$ Small error since data depends only on small t interv

$f(y+1) = f(y) \;\to\; \int_0^1 f = 0$ to give small error.

Let $\dfrac{L_d}{L_w} = \omega$ $\qquad$ high freq $\omega \gg 1 \xrightarrow{\text{AKA}}$ geometrical acoustics or optics

high freq $\Longrightarrow$ Small stratification or small relaxation or geometric effects. (almost plane wave)

Let $\bar{t} = t/L_d c_0^{-1}$ $\qquad L_d c_0^{-1}$ = dissipation time

$$u(0,t) = f(\bar{t}\, L_d c_0^{-1}/\tau_p) = f\left(\bar{t}\,\frac{L_d}{L_w}\right) = f(\bar{t}\omega)$$

when $\begin{cases} \underset{\sim}{A}(\underset{\sim}{u}, x) = \underset{\sim}{A}(x) & \text{linear geometrical acoustics Developed by Keller} \\ \underset{\sim}{B}(\underset{\sim}{u}, x) = & \text{linear in } \underset{\sim}{u} \end{cases}$

We will look at.

1) Acceleration wave: discontinuity in du/dt = accel. $\tau_p \to 0$ $\omega \to \infty$

a) exact   b) introduces naturally the length scales.

2) linear Geometrical acoustics

a) expansion scheme for $\omega >> 1$

3. N.L. geom. acoustics.

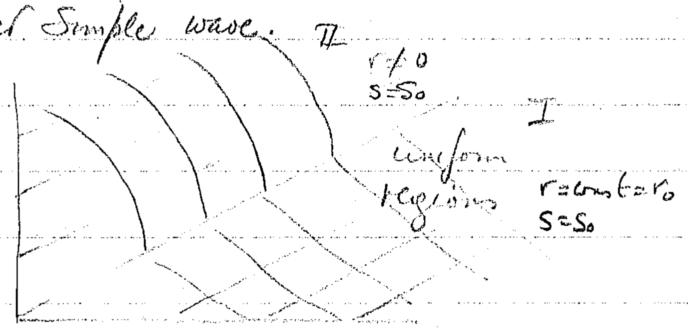
a) ad hoc "rules" - Whitham '56 & '53

keep linear eq but adjust characteristic as nonlinear char.

b) expansion schemes for $\omega >> 1$ & $|\underset{\sim}{u}| << 1$

c) Shock propagation

4) Modulated Simple wave.



Modulated $s = s_0(x)$ slow varying

r varies very fast as compared to s.

finite amplitude case.

I Accel Wave

$$u_t - c^2(v, x)\, v_x = \psi(u, x) \qquad (*)$$
$$v_t - u_x \qquad = 0$$

$t \geq 0$ $x \geq 0$

Acceleration front: a propagating surface $\alpha = \phi(x,t) = \text{const}$ where the particle "acceleration" u_t changes discontinuously @ x as the wave front passes while u remains continuous

Change variables $(x,t) \to (x,\alpha)$

Define $G(x,\alpha) = g(x,t)$

$G_{,x} = g_{,x} + T_{,x}\, g_{,t}$ where $t = T(x,\alpha)$ is the arrival time of $\alpha =$ const at position x.

$u_t + T_x\, c^2 v_t = c^2 V_x + \psi$ cont in x

$v_t + T_x u_t = U_x$ denote $[f] = f_+ - f_-$

Characteristic Cond $\begin{pmatrix} 1 & c_0^2 T_x \\ T_x & 1 \end{pmatrix} \begin{pmatrix} [u_t] \\ [v_t] \end{pmatrix} = \underline{0} \quad \sim ([\underset{\sim}{u}_t])$

$c = c(v,x)$

$c_0 = c(0,x)$

(take jumps & evaluate everything ahead of front.)

Assume ahead of front $u = v = 0$

$c_0(x)$ is measure of stratification ahead of wave.

$T_{,x}\, c_0(x) = \pm 1 \implies T_{,x} = 1/c_0(x)$ for waves to right

$\begin{pmatrix} 1 & c_0(x) \\ 1/c_0 & 1 \end{pmatrix} \begin{pmatrix} [u_t] \\ [v_t] \end{pmatrix} = \underline{0} \implies [u_t] = -c_0(x)[v_t] = \sigma(x)$ strength of the accel front

Differentiate original eq (*) w.r.t. t & then transform to (x,α) coord.

$u_{tt} - c^2(v,x)\, v_{xt} = 2cc_v v_x v_t + \psi_u u_t$

$v_{tt} - u_{xt} = 0$

let $^\circ$ be diff wrt t $\quad u_{xt} = U^\circ_x - T_{,x} u_{tt}$

$v_{tt} + T_{,x} u_{tt} = U^\circ_x$

$u_{tt} + c^2 T_{,x} v_{tt} = c^2 V^\circ_x + 2cc_v v_x v_t + \psi_u u_t$

now take jumps & eval everything at front.

$\begin{pmatrix} T_{,x} & 1 \\ 1 & c^2 T_{,x} \end{pmatrix} \begin{pmatrix} [u_{tt}] \\ [v_{tt}] \end{pmatrix} = \begin{pmatrix} [U^\circ_x] \\ c^2[V^\circ_x] + 2cc_v[v_t v_x] + \psi_u[u_t] \end{pmatrix}$

use $T_{,x} = 1/c_0$

$$c_0^2 [V_x^\circ] + 2c_0 c_v [V_t V_x] + \psi_u [u_t] - c_0 [U_x^\circ] = 0$$

$$[U_x^\circ] = \underbrace{[u_t]_x}_{=\sigma_x} - T_x \underbrace{[u_t]_t}_{0}$$

$$[u_x^\circ] = \sigma_x(x) \qquad [V_x^\circ] = \left(\frac{\sigma(x)}{-c_0}\right)_{,x}$$

$$[u_t] = \sigma \qquad [V_t V_x] = -1/c_0 [V_t^2]$$

$$[g^2] = g_+^2 - g_-^2 = (g_+ - g_-)(-g_+ + g_- + 2g_+)$$
$$= -[g]^2 + [g]\,2g_+ \quad \text{if } g_+ = 0 \text{ then } [g^2] = -[g]$$

$$[V_t V_x] = 1/c_0 [V_t^2] = \sigma^2/c_0^3$$

Transport Eq. how jump varies @ α = const

$$-c_0^2 \left(\frac{\sigma}{c_0}\right)_{,x} + 2\frac{c_v}{c_0^2}\sigma^2 + \psi_u \sigma - c_0 \sigma_x = 0$$

stratific   dissip $c_v = c_v(0,x) \quad \psi_u = \psi_u(0,x)$

$$-2c_0 \sigma_x + \sigma c_0' + 2\frac{c_v(x)}{c_0^2}\sigma^2 + \psi_u \sigma = 0$$

$$\sigma_x + \Omega(x)\sigma - A(x)\sigma^2 = 0$$

relax. strat

$$\Omega(x) = \frac{c_0' + \psi_u}{-2c_0} = L_r^{-1} + L_s^{-1} \qquad L_r = -\frac{2c_0}{\psi_u} \qquad L_s = -\frac{2c_0}{c_0'}$$

$$A(x) = -\frac{c_v}{c_0^3} \quad \text{only non lin contrib}$$

For 3d case
$$L_c^{\text{curvature}} = \frac{1 - 2\Omega_0(a,b)X + K_0(a,b)X^2}{\Omega_0(a,b) - K_0(a,b)X}$$
X corresponds to distance along charac.

where $\underset{\sim}{X} = \underset{\sim}{X}(a,b)$ is initial surface & Ω_0, K_0 are mean & Gaussian curvatures.

$\Omega_0 = K_0 \equiv 0$ corresponds to a plane surface.

$\sigma_x + \Omega(x)\sigma - \Lambda(x)\sigma^2 = 0$ Riccati Eq. in $\frac{1}{\sigma}$

rewrite as $-\left(\frac{1}{\sigma}\right)_{,x} + \Omega(x)\left(\frac{1}{\sigma}\right) - \Lambda(x) = 0$ subject to $\sigma = \sigma(0)$ @ $x=0$

$$\sigma(x) = \frac{\exp\left(-\int_0^x \Omega(r)\,dr\right)\sigma(0)}{\left[1 - \sigma(0)\int_0^x \Lambda(r)\exp\left(-\int_0^r \Omega(s)\,ds\right)dr\right]}$$

linear sol $\Lambda \equiv 0$ $\sigma_L(x) = \sigma(0)\exp\left(-\int_0^x \Omega(r)\,dr\right)$

HW. 1. For $0 \le x \le y$ under what conditions is $\sigma_L(x)$ a good approx to $\sigma(x)$

2. $A_t + (Au)_x = 0$ Go through accel front and
$A(u_t + uu_x) + p_x = -\nu u$ propagating into $A = A_0(x)$, $u=0$
$p = K(A, x)$

Flow in a flexible tube u = velocity A - crossectional area.
p = pressure. Should get Riccati Eq.

References TY Thomas "Plastic Flow & Fracture" 1961
Varley & Cumberbatch (65) Journal of Inst of Math & its Applic

May 8, 1973

No final HW's will be final.

$u_t + A(u, x)\,u_x = B(u, x)$

$[u_{,t}] = r(x)\,\sigma(x)$

Geometrical acoustics $\equiv$ high frequency — assoc with BC is large wrt natural freq of material $\Rightarrow$ wavelength (x) $\ll$ dissipation length (x) $= \Omega^{-1}$

High freq is local condition we will use it as a global condition

1) Pulse <time which we prescribe bdy data> $0 \le t \le \tau_p$

2) time periodic bdy data. τ_p = period

non dim eq   non dim X wrt wavelength or dissipative length.

dissipative length determined by eq & not bc will enter only in b.c.

wavelength " " bd " not eq " " " in eq.

Example $u(0,\bar{t}) = u_0 f(\bar{t}/\tau_p)$ normalize let $t = \bar{t}/\tau_d$

$$u(0,\tau_d t) = u_0 f\left(\frac{\tau_d}{\tau_p} t\right) \qquad \text{let } \omega = \tau_d/\tau_p \gg 1 \Rightarrow \tau_d \gg \tau_p$$

$$u(0, \omega t \tau_p) = u_0 f(\omega t) \qquad \tau_d \sim \frac{1}{|B|} \quad \tau_d \sim \frac{1}{|A_{,x}|}$$

Linear Geometrical Acoustics   linearize eq $\overset{B}{A(u,x)} = \overset{B}{A(0,x)} + \overset{B_u}{A_u(0,x)}u + \dots$

$$\underset{\sim}{u}_t + \underset{\sim}{A}_0(x)\underset{\sim}{u}_x = \underset{\sim}{B}_0(x)\underset{\sim}{u} \qquad A_0(x) = \underset{\sim}{A}(0,x)$$

non dim wrt dissipative length $B_0(x) = \underset{\sim}{B}_u(0,x)$

$\underset{\sim}{B}(0,x) = 0$ in eq.

Char condition $\det|c(x)\underset{\sim}{I} - \underset{\sim}{A}| = 0$

$$\frac{dt}{dx} = \frac{1}{c_r(x)} \quad \text{on } C_r \quad r = 1,\dots,n \quad \text{where } c_r \text{ is an EV of } A_0(x)$$

Consider only $c = c_0(x)$ $\alpha = t - \int_0^x \frac{ds}{c_0(s)}$

Choose α ∋. $\alpha = t$ on $x = 0$.

α = slow characteristic variable.

boundary condition is written in terms of a fast variable $\beta = \omega\alpha = \omega\left(t - \int_0^x \frac{ds}{c_0(s)}\right)$

on $x = 0$ $\beta = \omega t \Rightarrow u = f(\beta)$

choose β, x as new variables $(x,t) \to (x,\beta)$ $\beta = \beta(x,t)$

Change to new coords & use $\omega \gg 1$

$\underset{\sim}{u}(x,t) = \underset{\sim}{U}(x,\beta)$

$\underset{\sim}{u}_t = \underset{\sim}{U}_\beta \omega \qquad \underset{\sim}{U}_x - \frac{\omega}{c_0}\underset{\sim}{U}_\beta = \underset{\sim}{u}_x$

$\omega(\underset{\sim}{U}_\beta + A_0 \underset{\sim}{U}_x - \frac{\omega}{c_0} A_0 \underset{\sim}{U}_\beta = \underset{\sim}{B}_0 u$

$(c_0 I - A_0)\underset{\sim}{U}_\beta = \frac{c_0}{\omega}[B_0 \underset{\sim}{U} - \underset{\sim}{A}_0 \underset{\sim}{U}_x]$

Geometrical Acoustics Expansion: progressing wave valid until freq dispersion is not negligible. ie $x \ll \omega$

$\underset{\sim}{u}(x,t) = \underset{\sim}{U}(x,\beta,\omega) = \sum_{n=0}^{\infty} \omega^{-n} h_n(\beta) \underset{\sim}{U}^{(n)}(x)$

h_n are phase fns.; $\underset{\sim}{U}^{(n)}$ amplitude fns.; β is the phase variable

$\beta = \phi(x,t) \qquad u_1(0,\beta) = f(\beta) \Rightarrow h_0(\beta) = f(\beta) \quad U_1^{(0)} = 1$ on $x=0$

1st comp of U

$h'_{n+1}(\beta) = h_n(\beta) \qquad f$ is prescribed signal fn.

1) pulse : $f \equiv 0$ except for $0<\beta<1$

2) time periodic : $f(\beta) = f(\beta+1)$ measure $f \ni \int_0^1 f(s)ds = 0$ by choice of reference state.

1) pulse $\quad h_n(\beta) = \frac{1}{(n-1)!}\int_0^\beta s^{n-1} f(\beta - s)\, ds \qquad (h_n(0) = 0)$

error will be small due to $\beta \ll$

2) periodic $\quad (\int_0^1 h_n(s)ds = 0)$ error will have zero mean

$h_n(\beta) = -\frac{1}{n!}\int_0^1 B_n(s) f(\beta - s) ds$

$B_n(s) = n^{th}$ degree Bernoulli polyn $\quad B_n$ satisfies ① $B'_n = n B_{n-1}$

② $B_n(1) = B_n(0) \quad (n \geq 2) \qquad B_0(s) \equiv 1, \quad B_1(1) = -B_1(0) = 1/2$

define an iteration scheme.

$(c_0 I - A_0)\underset{\sim}{U}^{(n+1)} = c_0 [B_0 \underset{\sim}{U}^{(n)} - A_0 \underset{\sim}{U}^{(n)}_x]$ with $\underset{\sim}{U}^{(-1)}(x) \equiv 0$

Zeroth Order

$(c_0 I - A_0)\underset{\sim}{U}^{(0)}(x) = 0 \qquad \underset{\sim}{U}^{(0)}(x) = \underset{\sim}{r}(x)\, \sigma_L(x) \qquad r(x)$ is right EV of A_0

corresponding to $c = c_0(x)$ where $r_1(x) \equiv 1$ $\Rightarrow$ choose $r \ni$, $\sigma_L(x)$ measures

1st component $\qquad$ variation in U.

Compatability Condition $\Rightarrow \underset{\sim}{l}(x)\left[\underset{\sim}{B_0}\underset{\sim}{U}^n - \underset{\sim}{A_0}\underset{\sim}{U}_x^n\right] = 0$

where $\underset{\sim}{l}$ is any left EV of A_0. Known: transport Equation

$\underset{\sim}{U}^0 = \underset{\sim}{r}(x)\,\sigma_L(x)$

$$l\left(B_0 \underset{\sim}{r}\, \sigma_L - A_0\left\{\underset{\sim}{r}'\sigma_L + \underset{\sim}{r}\,\sigma_L'\right\}\right) = 0$$

$l(c_0 I - A) r = 0 \quad \Rightarrow \quad c_0 l r = l A r$

where σ_L solves

$$\sigma_L' + \Omega_L \sigma_L = 0 \qquad \Omega = L_r^{-1} + L_s^{-1} \qquad L_r = -\frac{c_0 \underset{\sim}{l}\underset{\sim}{r}}{\underset{\sim}{l}\underset{\sim}{B_0}\underset{\sim}{r}} \quad L_s = \frac{l \cdot r}{\underset{\sim}{l}\cdot\underset{\sim}{r}'}$$

$$\underset{\sim}{U}(x,\beta) = f(\beta)\,\underset{\sim}{r}(x)\exp\left(-\int_0^x \Omega_L(s)\,ds\right) \qquad \sigma_L = f(\beta)\exp\left(-\int_0^x \Omega(s)\,ds\right)$$

for general n

$$\left(I - A_0/c_0\right)\underset{\sim}{U}^{(n+1)} = B_0 U^{(n)} - A_0 U_x^n \quad \text{with } U^{(-1)}_{(x)} = 0$$

$(m-1)$ indep eq

m^{th} eq is $\quad l A_0 U_x^{(n+1)} - l B_0 U^{(n+1)} = 0$

form m eq for m unknowns $\underset{\sim}{U}^{(n+1)}$ in terms of $\underset{\sim}{U}^{(n)}$

Example

$$u_t - c^2 v_x = \psi \qquad c(v) \quad \text{with } c_x = 0 \qquad c^2(v,x) = c(0,0) + c_v(0,0)v + c_x(0,0)$$

$$v_t - u_x = 0$$

linearized $\quad \bar{u}_{\bar{t}} - c^2 \bar{v}_{\bar{x}} = \psi_{\bar{u}}\,\bar{u} \qquad$ all coeff are const $\quad c, \psi_u = \text{const}$

also let $() = \bar{()}$ $\quad \bar{v}_{\bar{t}} - \bar{u}_{\bar{x}} = 0$

Normalize wrt relax length $\quad \frac{-2c}{\psi_u} = L_r \qquad L_s = \infty$

non dim(x,t) wrt $(L_r \,\&\, c) \quad \Rightarrow \quad c \equiv 1 \quad \psi_u = -2$

define new variable $\quad u = \bar{u}/u_0 \qquad v = \bar{v}/u_0 c^{-1} \qquad t = \bar{t}/c^{-1}L_r$

$$x = \bar{x}/L_r$$

$$u_t \frac{u_0}{c^{-1}L_r} - c^2 v_x \frac{u_0}{c L_r} = \psi_u u_0 u$$

$$v_t - u_x = 0$$

normalized

or $v_t - u_x = 0$

$$u_t - v_x = \frac{\psi_u L_r}{c} u \quad -2$$

$$A_0 \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} \quad B_0 u = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \underset{\sim}{r} = \underset{\sim}{l}^t = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\sigma_L' + \Omega \sigma_L = 0 \qquad \Omega = \frac{1}{L_r} = \frac{-l \cdot r}{\underset{\sim}{l} \cdot B_0 \underset{\sim}{r}} = \frac{-2}{-2} = 1$$

$$\sigma_L' + \sigma_L = 0 \qquad \boxed{\sigma_L = e^{-x}}$$

$$\underset{\sim}{u} = \begin{pmatrix} U \\ V \end{pmatrix} = f(\beta) e^{-x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

or

$$u_t - v_x = -2u \qquad (x,t) \to (\beta, x) \qquad \beta = \omega(t - x)$$

$$v_t - u_x = 0$$

$$U_\beta + V_\beta = \omega^{-1}(V_x - 2U)$$

$$V_\beta + U_\beta = \omega^{-1}(U_x)$$

using the eqs for nth order

$$U^{(n+1)} + V^{(n+1)} = (V_x^{(n)} - 2U^{(n)}) \quad (1)$$

$$V^{(n+1)} + U^{(n+1)} = U_x^{(n)} \quad (2)$$

Compatibility $\quad V_x^{(n)} - 2U^{(n)} = U_x^{(n)}$ use $n = n+1$ (3)

use 2 & 3

$$U^{(0)} = -V^{(0)} \qquad U_x^{(0)} = -V_x^{(0)} \quad \therefore \quad -2U^{(0)} = U_x^{(0)} - V_x^{(0)} = 2U_x^{(0)} \text{ from (3)}$$

for 1 with $U^{(-1)} = 0 \quad V^{(-1)} = 0 \Rightarrow V_x^{(-1)} = 0$

Zeroth order Sol
$$V^{(0)} = -U^{(0)} \qquad V^{(0)} = -e^{-x}$$
$$U_x^{(0)} + U^{(0)} = 0 \Rightarrow U^{(0)} = e^{-x}$$

diff wrt x
$$V^{(1)} + U^{(1)} = U_x^{(0)} = -e^{-x}$$
$$V_x^{(1)} - 2U^{(1)} = U_x^{(1)}$$
$$V_x^{(1)} + U_x^{(1)} = e^{-x}$$
subtract
$$2U_x^{(1)} + 2U^{(1)} = e^{-x}$$

$$U^{(1)} = \frac{xe^{-x}}{2} \quad \text{with } U^{(1)}(0) = 0$$
2 throws out homog term.

$$V^{(1)} = -U^{(1)} - e^{-x} = -e^{-x}\left(\frac{x}{2} + 1\right)$$

$$\underset{\sim}{u} = \begin{pmatrix} U \\ V \end{pmatrix} = f(\beta)\, \underset{\sim}{r}\, e^{-x} + \omega^{-1} h_1(\beta) \left\{ \frac{x}{2} \underset{\sim}{r} + r^{(1)} \right\} e^{-x}$$
dispersive term

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \underset{\sim}{r}^{(1)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

n th term will have a term $\propto (x\omega^{-1})^n$

when $y = x\omega^{-1}$

$$e^x U = \sum_0^\infty g_n(\beta)\, y^n$$

error of expansion for exp up to ω^{-n} is $\omega^{-(n+1)}$
range of validity $(n+1)$ term $\sim$ nth term. Because of growth with x.

Since linear separable.

HW Do linear geom acoustics for
$$\left.\begin{aligned} u_t - c^2(x)\, V_x &= 0 \\ V_t - u_x &= 0 \end{aligned}\right]$$
Calculate first two terms.
$L_r = \infty \qquad L_s = \frac{l r}{l r'}$

Subject to $u(0,t) = f(\omega t)$ where $\beta = \omega(t - \int^x ds)$

5-15-73

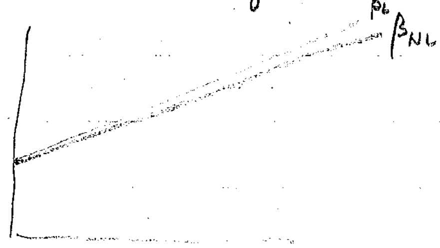
if rth comp of U is excited right EV $\underset{\sim}{r}(x)$ must have a 1 in the $r^{\underline{th}}$ row we have done work for 1 component going in 1 direction only!

for linear case sol is variable separable now characteristics depend on sol for nonlinear case.

Whitham 53 or 56 Com of Pure & Applied Math J Fluid Mech.

Whitham's Rule $u = f(\beta_L) E(x)$ linear sol $\beta_L = t - \int_0^x \frac{ds}{c}$ linear char.

near zero $\beta_{NL} \doteq \beta_L$ keep variation of u on a characteristic to be same but change variation of characteristic.



$$t_x = \frac{1}{C(u,x)} \quad \text{Expand for small } u = \frac{1}{c(0,x)} - \frac{c_u(0,x)\,u}{c^2(0,x)} + \cdots$$

Whitham's Rule

$$\boxed{\begin{array}{l} \text{sol for } u = f(\beta) E(x) \quad \text{with } |u| \ll 1 \\ \beta = t - \displaystyle\int_0^x \frac{ds}{c(0,s)} - f(\beta) \int_0^x \frac{c_u(0,s) E(s) ds}{c^2(0,s)} \end{array}}$$

for shock formation $t_\beta = 0$

$$t_\beta = 1 + f'(\beta) \int_0^x \frac{c_u}{c^2} E\, ds = 0$$

if $\exists\ x_s \ni t_\beta = 0$ shock will form.

another dependent variable is time of arrival of wave @ a pt.

for linear acoustics time of arrival is known since char is known
use same technique as linear

$$u(x,t) = U(x,\beta) \qquad \beta = \beta_{NL}$$

$$t_x\big|_\beta = \frac{1}{c(u,x)}$$

for high freq $\quad |U_x|_\beta \ll |u_x|_t \sim |u_t|_x$

motivation is same as linear (iteration procedure) and an extra eq on arrival time.

$$\underset{\sim}{u}_t + \underset{\sim}{A}(\underset{\sim}{u},x)\underset{\sim}{u}_x = \underset{\sim}{B}(\underset{\sim}{u},x)$$

Non dim wrt $|\Omega_l|^{-1}$ $\left(\underset{\sim}{I} - T_{,x}\underset{\sim}{A}(\underset{\sim}{u},x)\right)\underset{\sim}{u}_t = \underset{\sim}{B} - \underset{\sim}{A}\,\underset{\sim}{U}_x$ & $t = T(x,\beta)$ arrival time at x of wavelet β = const.

with fact that $\underset{\sim}{U}_x = \underset{\sim}{u}_x + T_x \underset{\sim}{u}_t \qquad T_x = T_x\big|_\beta$

Subject to $u_1 = f(\omega t) \qquad \omega = \frac{c^{-1}L}{\tau_p} \gg 1$

because of choose on $x=0$ $\quad \beta = \omega t \therefore u_1 = f(\beta) \quad$ β is measure of fast variation, x " " of slow stratification

$\Longrightarrow T_\beta = O\left(\frac{1}{\omega}\right) \quad$ since $\beta = \omega t \quad 1 = \omega t_\beta = \omega T_\beta \Rightarrow T_\beta \sim O(\frac{1}{\omega})$

$$\underset{\sim}{u} = \underset{\sim}{U}(x,\beta) \qquad U_1(0,\beta) = f(\beta)$$

& assume $U(x,0)=0$

$$t = T(x,\beta) \qquad T(0,\beta) = \beta/\omega \qquad f(0)=0$$

Convert $\left(I - T_x A(u,x)\right)u_t = B - A U_x$ into β coord

$$u_t = U_\beta\, \beta_t = U_\beta\, \omega = U_\beta \frac{1}{T_\beta}$$

$$\therefore \quad (I - T_x A(u,x))\, U_\beta = (B - A\, U_x)\, T_\beta$$

T_x is $\frac{1}{EV \text{ of } A} = \frac{1}{c(u,x)}$

$$\therefore \quad \left(I - \frac{A}{c}\right) U_\beta = (B - A U_x)\, T_\beta$$

or $\quad (Ic - A) = 0$

Small amplitude nonlinear solution ie $|u| \ll 1$

$\Rightarrow$ expand c, A, B in powers of u

$$\underset{\sim}{u} = \underset{\sim}{U} = \underbrace{\omega^{-1}}_{\text{where } U \text{ is small}} \sum_{n=0}^{\infty} \omega^{-n}\, \underset{\sim}{U}^{(n)}(x,\beta) \qquad |U| \ll 1$$

$$T = \int_0^x \frac{ds}{c(0,s)} = \omega^{-1} \sum_{n=0}^{\infty} \omega^{-n}\, T_n(x,\beta)$$

Compatibility $\quad \underset{\sim}{l}(\underset{\sim}{u},x)\left[B(\underset{\sim}{u},x) - \underset{\sim}{A}(\underset{\sim}{u},x)\underset{\sim}{U}_{,x}\right] = 0$ transport eq for arbitrary amplitude

put Expansions into Governing eq

order $\frac{1}{\omega}$ eq: $\quad \left[I - \frac{A(0,x)}{c(0,x)}\right] \underset{\sim}{U}^{(0)}_{,\beta} = 0 \quad \Rightarrow \quad \underset{\sim}{U}^{0}_{\beta}(x,\beta) = \underset{\sim}{r}^{(0)}(x)\, g'(\beta)$

$\underset{\sim}{r}^{(0)}(x)$ is right EV for $A(0,x)$ & $c(0,x)$

$$\therefore \quad U^0(x,\beta) = \underset{\sim}{r}^{(0)}(x)\, g(\beta) + \underset{\sim}{V}(x)$$

use $\underset{\sim}{r}^{(0)}$ for variation of U_1 $\qquad$ scalar

$$= \bar{\underset{\sim}{r}}^{(0)}(x)\, \sigma(x)\, f(\beta) + \underset{\sim}{V}(x)$$

where $\bar{r}_1^{(0)}(x) \equiv 1 \quad \Rightarrow \quad \sigma$ measures variation of u_1 with x

& $\sigma(0) = 1$ to satisfy to BC

to satisfy initial cond $\quad \underset{\sim}{V}(x) \equiv 0$

ie $\quad \sigma(x) = r_1^{(0)}(x)$

$$T_{,x} = \frac{1}{c(0,x)} - \frac{c_u u}{c^2(0,x)} + O(u^2)$$

$$\cancel{\frac{1}{c(0,x)}} + \omega^{-1} T_{0,x} + O\left(\frac{1}{\omega^2}\right) = \cancel{\frac{1}{c(0,x)}} - \frac{c_u(0,x)}{c^2(0,x)} r^{(0)}(x)\sigma(x) f(\beta)\, \omega^{-1}$$

$$T_0(\beta, x) = -f(\beta)\int_0^x \frac{c_u(0,s)\, r^{(0)}(s)\, \sigma(s)}{c^2(0,s)}\, ds + \text{const}$$

for bc & expansion of $T \Rightarrow T_0(0,\beta) = \beta \quad T_n(0,\beta) = 0 \;\; n>0$

$\therefore$ const $= \beta$

to find σ get to transport from compatability
put in expansion for U $\quad B(0,x) = 0$ since you propagate into equil state

zeroth order,

$$l^{(0)}(x)\left[B_u(0,x) U^{(0)} - A(0,x)\, U^{(0)}_x \right] = 0$$

linear transport eq $\quad$ slut process

$$l B_u\, r\sigma - lA(r\sigma' + r'\sigma) = 0$$

$\Omega = \dfrac{lB_u r - lAr'}{-lAr}$ $\quad$ relax process $\quad$ we get $\sigma' + \Omega\sigma = 0$ $\quad \Omega = L_r' + L_s'$

$$\sigma = \sigma(0)\exp\left(-\int_0^x \Omega(s)\,ds\right) \qquad \sigma(0) = 1 \text{ previously}$$

Zeroth order sol

$$U^{(0)}(x,\beta) = r^{(0)}(x)\, f(\beta)\exp\left(-\int_0^x \Omega(s)\,ds\right)$$

where $T = \displaystyle\int_0^x \frac{ds}{c(0,s)} = \frac{\beta}{\omega} - \frac{f(\beta)}{\omega}\int_0^x \frac{c_u(0,s)}{c^2(0,s)}\, r^{(0)}(s)\exp\left(-\int_0^s \Omega(r)\,dr\right)ds$

this is Whithams rule. Note: Soln is implicit

Ex: $B = -2u \quad \sigma = e^{-x} \quad \underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for linear case

$$\underset{\sim}{U}^{(0)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} f(\beta)\, e^{-x} \quad \text{with } c = 1$$

now let $\quad c = 1 + kv \quad k = c_v$

$c(0,x) = 1 \qquad c = c(v)$ only $\therefore c_v = \text{const}$

$\therefore$ secularity $\Rightarrow$

$$T - x = \frac{\beta}{\omega} - \frac{f(\beta)}{\omega} \int_0^x \frac{c_v \cdot \underset{\sim}{r}^{(0)}}{1} e^{-s}\, ds$$

$$c_{\underset{\sim}{u}} \cdot r = \begin{pmatrix} 0 \\ c_v \end{pmatrix}^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -c_v$$

$$T - x = \beta/\omega + \frac{f(\beta)}{\omega} \int_0^x c_v\, e^{-s}\, ds$$

$$= \beta/\omega + \frac{f(\beta)}{\omega} c_v \left[1 - e^{-x}\right]$$

$$= \omega^{-1} \left\{ \beta + f(\beta)\, c_v \left(1 - e^{-x}\right) \right\}$$

Shocks form when $T_\beta = 0$

$$\therefore \quad T_\beta = \omega^{-1} \left\{ 1 + f'\, c_v \left(1 - e^{-x}\right) \right\} = 0$$

or when $\displaystyle f' = \frac{-1}{c_v(1 - e^{-x})}$ $\qquad$ for small x $\quad 1 - e^{-x} \sim x$

Simple wave $B \equiv 0 \quad T_\beta = \omega^{-1}\left(1 + f' c_v\, x\right) \Rightarrow f' c_v < 0$ for some β then a shock forms.

if in nl case $-f' c_v < 1$ no shocks form

if f is velocity & if $-f'(0)$ accel on boundary) $< \frac{1}{c_v} = A_c$

is a critical accel.

must put an accel $> A_c$ to have shock form.

HW. do second order sol for this problem.

$$\left\{ \begin{array}{l} U^{(0)} = \omega^{-1}\binom{1}{-1} f(\beta)\, e^{-x} + \left\{ \int_0^{\beta} f(s)\, ds\, X \right\} \\ \qquad + \omega^{-2} c_v\, f^2(\beta)\, X(x) \end{array} \right\}$$

System $\left. \begin{array}{l} u_t - c^2(v)\, v_x = -2u - Bu^2 \\ v_t - u_x = 0 \end{array} \right\}$

$$c(v) = 1 + c_v V + c_{vv} V^2 + \ldots$$

Subject to $u(0,t) = f(\omega t) = f(\beta)$

$T(0,\beta) = \beta/\omega \quad \omega \gg 1$

$\underset{\sim}{U} = \omega^{-1} \sum \underset{\sim}{y}^{(n)}(x,\beta)\, \omega^{-n}$ $\qquad U = \binom{u}{v}$

get U^0, U'

$T - x = \omega^{-1} \sum \omega^{-n} T_n$ $\qquad T^0, T'$

with $\underset{\sim}{u}(x,0) = 0 \quad f(0) = 0$

Tidal Bore
Appl
Atmospheric pressure pulse

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1. from probl. $f(\varphi) = A\cos\varphi$

$$\alpha_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} A\cos\varphi \, d\varphi = \frac{A}{\pi} \sin\varphi \Big|_{-\pi}^{\pi} = 0$$

$$\alpha_1 = \frac{A}{\pi}\int_{-\pi}^{\pi} \cos^2\varphi \, d\varphi = \frac{A}{2\pi}\int_{-\pi}^{\pi} (1 + \cos 2\varphi) \, d\varphi$$

$$\alpha_1 = \frac{A}{2\pi}\left[\varphi + \frac{\sin 2\varphi}{2}\right]_{-\pi}^{\pi} = \frac{A}{2\pi}\left[\pi + 0 + \pi + 0\right] = A$$

$\therefore$ solution is $u(\rho,\varphi) = \sum_{n=1}^{\infty}$

$$\alpha_n = \frac{A}{\pi}\int_{-\pi}^{\pi} \cos\varphi \cos n\varphi \, d\varphi \qquad \frac{A}{2\pi}\int_{-\pi}^{\pi} \{\cos(n+1)\varphi + \cos(n-1)\varphi\} \, d\varphi$$

since f is even $\beta_n = 0$ $\qquad \frac{A}{2\pi}\left[\frac{\sin(n+1)\varphi}{n+1} + \frac{\sin(n-1)\varphi}{n-1}\right]_{-\pi}^{\pi} = 0$

$A_n \; n>1 = 0$

$$\therefore \quad u(\rho,\varphi) = \frac{\rho}{a} A\cos\varphi$$

$f(\varphi) = A + B\sin\varphi$ $\qquad$ this is an odd function $\therefore \alpha_n = 0$

$$\alpha_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (A + B\sin\varphi) \, d\varphi = \frac{1}{\pi}\left[A\varphi - B\cos\varphi\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[A\pi + B + A\pi - B\right]$$

$$= 2A$$

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} A\cos n\varphi \, d\varphi + \frac{B}{2\pi}\int_{-\pi}^{\pi} \{\sin(1+n)\varphi + \sin(1-n)\varphi\} \, d\varphi$$

$$= 0 \qquad -\left[\frac{\cos(1+n)\varphi}{1+n} + \frac{\cos(1+n)\varphi}{1-n}\right]_{-\pi}^{\pi} = 0$$

if n even $\frac{-1}{1+n} + \frac{-1}{1+n} - \frac{(-1)}{1+n} - \frac{(-1)}{1-n}$

odd $\frac{1}{1+n} + \frac{1}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}$

$\beta_1 = B \qquad \beta_n \; n>1 = 0$

$$\therefore \quad u(\rho,\varphi) = A + \frac{\rho}{a} B\sin\varphi$$

(2) $\Delta u = u_{xx} + u_{yy} = 0$ $\quad X''Y + Y''X = 0$

$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2 \quad \lambda > 0$

$X'' + \lambda^2 X = 0$

$Y'' - \lambda^2 Y = 0$

$X = A_\lambda \sin \lambda x + B_\lambda \cos \lambda x$

$Y = C_\lambda e^{\lambda y} + D_\lambda e^{-\lambda y} = C_\lambda \sinh \lambda y + D_\lambda \cosh \lambda y$

$X(x)\,Y(y) = A_\lambda C_\lambda \sin \lambda x\, e^{\lambda y} + B_\lambda C_\lambda e^{\lambda y} \cos \lambda x + D_\lambda A_\lambda \sin \lambda x\, e^{-\lambda y}$

$+ B_\lambda D_\lambda e^{-\lambda y} \cos \lambda x$

$f_1(y) = B_\lambda C_\lambda e^{\lambda y} + B_\lambda D_\lambda e^{-\lambda y}$

$f_1(y) =$

$u(0,y) = f_1(y) \quad u(x,0) = f_2(x) \quad u(a,y) = 0 \quad u(x,b) = 0$

$f_1(y) = C_\lambda e^{\lambda y} + D_\lambda e^{-\lambda y} = C_\lambda \sinh \lambda y + D_\lambda \cosh \lambda y$

$f_2(x) = A_\lambda \sin \lambda x + B_\lambda \cos \lambda x$

$0 = A_\lambda \sin \lambda a + B_\lambda \cos \lambda a$

$0 = C_\lambda e^{\lambda b} + D_\lambda e^{-\lambda b} \Leftarrow C_\lambda \sinh \lambda b + D_\lambda \cosh \lambda b$

$$\begin{bmatrix} \sinh \lambda y & \cosh \lambda y \\ \sinh \lambda b & \cosh \lambda b \end{bmatrix} \begin{pmatrix} C_\lambda \\ D_\lambda \end{pmatrix} = \begin{pmatrix} f_1(y) \\ 0 \end{pmatrix}$$

$$C_\lambda = \frac{f_1(y) \cosh \lambda b}{\sinh \lambda y \cosh \lambda b - \cosh \lambda y \sinh \lambda b}$$

$$C_\lambda = \frac{f_1(y) \cosh \lambda b}{\sinh \lambda (y-b)} \qquad D_\lambda = \frac{-f_1(y) \sinh \lambda b}{\sinh \lambda (y-b)}$$

$$\therefore\ Y = \frac{f_1(y)\left[\cosh \lambda b \sinh \lambda y - \sinh \lambda b \cosh \lambda y\right]}{\left[\sinh \lambda y \cosh \lambda b - \cosh \lambda y \sinh \lambda b\right]} = \frac{\sinh \lambda (y-b)}{\sinh \lambda (y-b)}$$

where b_n are defined as $\quad b_n = \frac{2}{b}\int_0^b f_1(y) \sin\frac{m\pi y}{b}\, dy$

$$U(x,y) = u(x,y) + v(x,y) = \sum_{n=1}^{\infty}\left\{a_n \frac{\sinh\frac{n\pi}{a}(b-y)}{\sinh\frac{n\pi}{a}b}\sin\frac{n\pi x}{a} + b_n\frac{\sinh\frac{m\pi}{b}(a-x)}{\sinh\frac{m\pi a}{b}}\sin\frac{m\pi y}{b}\right\}$$

where $\quad a_n = \frac{2}{a}\int_0^a f_2(x)\sin\frac{n\pi x}{a}\,dx \qquad b_n = \frac{2}{b}\int_0^b f_1(y)\sin\frac{m\pi y}{b}\,dy$

$$U(0,y) = u(0,y) + v(0,y) = \sum_{n=1}^{\infty} b_n \sin\frac{m\pi y}{b} = f_1(y)$$

$$U(a,y) = u(a,y) + v(a,y) = 0$$

$$U(x,0) = u(x,0) + v(x,0) = \sum a_n \sin\frac{n\pi x}{a} = f_2(x)$$

$$U(x,b) = u(x,b) + v(x,b) = 0$$

$$\Delta U = 0 \qquad \Delta(u+v) = \Delta u + \Delta v = 0$$

to show convergence of serie. if $|f_2(x)|$ and $|f_1(y)|$ are bounded by M

then $\quad |a_n| \le \frac{2}{a} M\int_0^a \left|\sin\frac{n\pi x}{a}\right| dx \le 2M \qquad |b_m| \le \frac{2}{b}M\int |\quad| \le 2M$

$$U(x,y) \le 2M\sum\left\{\sinh \quad \sin\frac{n\pi x}{a} + \underline{\quad}\sin\right\} \le 2M\sum\left\{\frac{\sinh}{\sinh\frac{n\pi}{a}b} + \frac{\sinh}{\sinh}\right\}$$

$$u_n = a_n \boxed{\frac{\sinh \frac{n\pi}{a}(b-y)}{\sinh \frac{n\pi b}{a}}} \sin \frac{n\pi x}{a}$$

at $x=0$ $u=0$

at $x=a$ $u=0$

at $y=0$ $\sin \frac{n\pi x}{a}$

at $y=b$ $u=0$

$$u = \sum u_n$$

$$u(x,0) = \sum u_n(x,0) = f_2(x) = \sum a_n \sin \frac{n\pi x}{a}$$

$$f_2(x) \sin \frac{m\pi x}{a} = \sum a_n \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a}$$

$$\int_0^a f_2(x) \sin \frac{n\pi x}{a} = a_n \int_0^a \sin^2 \frac{n\pi x}{a}\, dx$$

$$\frac{a_n}{2} \qquad 1 - \cos \frac{2n\pi x}{a}$$

$$a_n = \frac{2}{a}\int_0^a \cancel{\sin 2n\pi x}\, f_2(x) \sin \frac{n\pi x}{a}\, dx \qquad a \qquad \frac{\sin \frac{2n\pi x}{a}}{a/2n\pi}\Bigg|_0^a$$

$u(x,y)$ which satisfies $\Delta u = 0$ in R &

$x=0$	$u=0$
$x=a$	$u=0$
$y=0$	$u=f_1(x)$
$y=b$	$u=0$

now find a fn which satisfies

$x=0$	$u=f_2(y)$
$x=a$	$u=0$
$y=0$	$u=0$
$y=b$	$u=0$

$$v(x,y) = \sum u_n(x,y) = \sum b_n \frac{\sinh \frac{m\pi}{b}(a-x)}{\sinh \frac{m\pi}{b} a} \sin \frac{m\pi y}{b}$$

thus $U(x,y) = \sum_{n=1}^{\infty} \left\{ a_n \frac{\sinh \frac{n\pi}{a}(b-y)}{\sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a} + b_m \frac{\sinh \frac{m\pi}{b}(a-x)}{\sinh \frac{m\pi a}{b}} \sin \frac{m\pi y}{b} \right\}$

where $a_n = \frac{2}{a}\int_0^a f_2(\xi) \sin \frac{n\pi\xi}{a}\, d\xi \qquad b_m = \frac{2}{b}\int_0^b f_1(\xi) \sin \frac{m\pi \xi}{b}\, d\xi$

to show convergence of $U(x,y)$ if $|f_2(x)|, |f_1(y)|$ are bdd by M

then $|a_n| \le \frac{2}{a} M \int_0^a \left|\sin \frac{n\pi\xi}{a}\right| d\xi \le 2M \qquad |b_m| \le \frac{2}{b} M \int_0^b \left|\sin \frac{n\pi\xi}{b}\right| d\xi \le 2M$

then $|U(x,y)| \le 2M \sum_{n=1}^{\infty} \left\{ \left| \frac{\sinh \frac{n\pi}{a}(b-y)}{\sinh \frac{n\pi b}{a}} \right| + \left| \frac{\sinh \frac{m\pi}{b}(a-x)}{\sinh \frac{m\pi a}{b}} \right| \right\}$

but $\sinh \frac{\pi b}{a} < \sinh \frac{n\pi b}{a} \Rightarrow \frac{1}{\sinh \frac{\pi b}{a}} > \frac{1}{\sinh \frac{n\pi b}{a}}$ now take the smaller of $\left(\frac{1}{\sinh \frac{n\pi b}{a}}, \frac{1}{\sinh \frac{\pi a}{b}}\right)$

$|U(x,y)| \le \frac{2M}{r} \sum_{n=1}^{\infty} \left\{ \left|\sinh \frac{n\pi}{a}(b-y)\right| + \left|\sinh \frac{m\pi}{b}(a-x)\right| \right\}$

but sinh is an exponential fn and the exponential fn is known to converge everywhere

$\frac{\partial u_n}{\partial x} = a_n \frac{\sinh}{\sin} \left(\frac{n\pi}{a} \cos \frac{n\pi x}{a}\right) + b_m \frac{m\pi}{b} \frac{\cosh \frac{m\pi}{b}(a-x)}{\sinh} \sin \frac{m\pi y}{b}$

$\frac{\partial^2 u_n}{\partial x^2} = a_n \frac{\sinh}{\sin} \left(\frac{n^2\pi^2}{a^2} \sin \frac{n\pi x}{a} + b_m \frac{m^2\pi^2}{b^2} \cancel{\frac{\sinh \frac{m\pi}{b}(a-x)}{\sinh}} \sin \frac{m\pi y}{b}\right.$

$\frac{\partial u_n}{\partial y} = -a_n \frac{n\pi}{a} \frac{\cosh}{\sin} \sin \frac{n\pi x}{a} + b_m \frac{m\pi}{b} \frac{\sinh}{\sinh} \cos \frac{m\pi y}{b}$

$\frac{\partial^2 u_n}{\partial y^2} = a_n \frac{n^2\pi^2}{a^2} \frac{\sinh}{\sinh} \sin \frac{n\pi x}{a} - b_m \left(\frac{m^2\pi^2}{b^2}\right) \frac{\sinh}{\sinh} \sin \frac{m\pi y}{b}$

$$\text{let } \cos\frac{\pi}{2a}x\,dx = dv \qquad u = \sin\frac{n\pi x}{a}$$

$$\int_a^b \cos\frac{\pi}{2a}x \sin\frac{n\pi x}{a}\,dy \qquad \frac{2a}{\pi}\sin\frac{\pi}{2a}x = v \qquad du = \frac{n\pi}{a}\cos\frac{n\pi x}{a}$$

$$\frac{2a}{\pi}\sin\frac{n\pi x}{a}\sin\frac{\pi x}{2a} - \int \frac{2a}{\cancel{\pi}}\cdot\frac{n\cancel{\pi}}{\cancel{a}}\sin\frac{\pi}{2a}x\cos\frac{n\pi x}{a}\,dx$$

$$\frac{2a}{\pi}\sin\frac{n\pi x}{a}\cancel{\sin\frac{\pi x}{2a}}^{\,0}\Big|_0^a - 2n\int_0^a \sin\frac{\pi}{2a}x \cos\frac{n\pi x}{a}\,dx$$

$$\sin\frac{\pi}{2a}x\,dx = dv \qquad u = \cos\frac{n\pi x}{a}$$

$$-\frac{2a}{\pi}\cos\frac{\pi}{2a}x = v \qquad du = -\frac{n\pi}{a}\sin\frac{n\pi x}{a}$$

$$-2n\left[-\frac{2a}{\pi}\cos\frac{\pi x}{2a}\cos\frac{n\pi x}{a}\Big|_0^a - 2n\int_0^a \cos\frac{\pi}{2a}x\sin\frac{n\pi x}{a}\,dx\right]$$

$$\int_0^a = \frac{4an}{\pi}\cos\frac{\pi x}{2a}\cos\frac{n\pi x}{a}\Big|_0^a + 4n^2\int_0^a \cos\frac{\pi}{2a}x\sin\frac{n\pi x}{a}\,dx$$

$$= \frac{1}{1-4n^2}\cdot\frac{4an}{\pi}\cos\frac{\pi x}{2a}\cos\frac{n\pi y}{a}\Big|_0^a = \frac{4an}{\pi(1-4n^2)}\left[\cancel{\cos\frac{\pi}{2}\cos n\pi} - \cos 0\cdot\cos 0\right]$$

$$\frac{4an}{\pi(4n^2-1)}$$

$$b\int y\sin\frac{m\pi y}{b}\,dy - \int y^2\sin\frac{m\pi y}{b}\,dy$$

$$dv = \sin\frac{m\pi y}{b}\,dy \qquad u = y \qquad u = y^2$$

$$v = -\frac{b}{m\pi}\cos\frac{m\pi y}{b} \qquad du = dv \qquad du = 2y\,dy$$

$$b\left[-\frac{yb}{m\pi}\cos\frac{m\pi y}{b}\Big|_0^b + \frac{b}{m\pi}\frac{b}{m\pi}\sin\frac{m\pi y}{b}\Big|_0^b\right] - \left[-\frac{y^2 b}{m\pi}\cos\frac{m\pi y}{b}\Big|_0^b + \frac{2b}{m\pi}\int_0^b\right]$$

$$\frac{2b}{m\pi}\int_0^b y\cos\frac{m\pi y}{b}\,dy$$

$$\text{let } dv = \cos\frac{m\pi y}{b}\,dy \qquad u = y$$

$$v = \frac{b}{m\pi}\sin\frac{m\pi y}{b} \qquad du = dy$$

$$\int_0^b y\cos\frac{m\pi y}{b}\,dy = \frac{by}{m\pi}\sin\frac{m\pi y}{b}\Big|_0^b + \frac{b}{m\pi}\frac{b}{m\pi}\cos\frac{m\pi y}{b}\Big|_0^b$$

$$b\left[-\frac{yb}{m\pi}\cos\frac{m\pi y}{b}\Big|_0^b + \frac{b^2}{m^2\pi^2}\cancel{\sin\frac{m\pi y}{b}}^{\,0}\Big|_0^b\right] + \frac{y^2 b}{m\pi}\cos\frac{m\pi y}{b}\Big|_0^b - \frac{2b^2}{m^2\pi^2}y\cancel{\sin\frac{m\pi y}{b}}^{\,0}\Big|_0^b - \frac{2b^3}{m^3\pi^3}\cos\frac{m\pi y}{b}\Big|_0^b$$

$$-\frac{b^3}{m\pi}\left\{(-1)^m\right\} + \frac{b^3}{m\pi}\left\{(-1)^m\right\} - \frac{2b^3}{m^3\pi^3}\left[(-1)^m - 1\right]$$

$\frac{13}{13}$

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3-6-73

Problem # 1 Find the fn u $\ni$ $\Delta u = 0$ in a circle of radius a and on the circumference C assumes the value

(a) $u|_c = A\cos\varphi$

(b) $u|_c = A + B\sin\varphi$

(a) since $\Delta u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = 0$

we obtain $u(\rho,\varphi)$ by assuming $u(\rho,\varphi) = R(\rho)\Phi(\varphi)$ to which the solutions are

$$R(\rho) = C\rho^n$$
$$\Phi_n(\varphi) = A_n\cos n\varphi + B_n\sin n\varphi$$

from this

$$u(\rho,\varphi) = \sum_{n=0}^{\infty}\rho^n(A_n\cos n\varphi + B_n\sin n\varphi)$$

since $u(a,\varphi) = A\cos\varphi$ then $A\cos\varphi = \sum_{n=0}^{\infty} a^n(A_n\cos n\varphi + B_n\sin n\varphi)$

note that we can define A_n, B_n by fourier analysis $\ni$.

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} A\cos\psi\,d\psi \qquad A_n = \frac{1}{a^n\pi}\int_{-\pi}^{\pi} A\cos\psi\cos n\psi\,d\psi \quad n\geq 1$$

$$B_n = \frac{1}{a^n\pi}\int_{-\pi}^{\pi} A\cos\psi\sin n\psi\,d\psi \quad n\geq 0$$

but $A\cos\psi$ is an even fn $\therefore$ $B_n \equiv 0$ $\quad A_0 = 0$ & $A_n = 0\ \forall\, n\geq 2$

the only term left is $A_1 = \frac{A}{a\pi}\int_{-\pi}^{\pi}\cos^2\psi\,d\psi = \frac{A}{2a\pi}\left[\psi + \frac{\sin 2\psi}{2}\right]_{-\pi}^{\pi} = \frac{A}{a}$

$\therefore$ $u(\rho,\varphi) = A\frac{\rho}{a}\cos\varphi$ ✓

(b) since $u(a,\varphi) = A + B\sin\varphi$ then $A + B\sin\varphi = \sum_{n=0}^{\infty} a^n(A_n\cos n\varphi + B_n\sin n\varphi)$

define as before $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(A + B\sin\psi)\,d\psi$, $A_n = \frac{1}{a^n\pi}\int_{-\pi}^{\pi}(A + B\sin\psi)\cos n\psi\,d\psi$ $n\geq 1$

$$B_n = \frac{1}{a^n\pi}\int_{-\pi}^{\pi}(A + B\sin\psi)\sin n\psi\,d\psi \quad n\geq 0$$

but $A + B\sin\psi$ is an odd fn (A is even, $B\sin\varphi$ is odd ; ~~even·odd=odd~~)

$\therefore$ $A_n = 0\ \forall\, n\geq 1$ $\qquad B_n = 0\ \forall\, n\geq 2$

the only terms left are $A_0 = \frac{1}{2\pi}\left[A\psi - B\cos\psi\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[A\cdot 2\pi + B - B\right] = A$

& $B_1 = \frac{1}{a\pi}\left[-A\cos\psi + \frac{B}{2}\left(\psi - \frac{\sin 2\psi}{2}\right)\right]_{-\pi}^{\pi} = \frac{1}{a\pi}\left[+A - A + \frac{B}{2}(2\pi - 0)\right] = \frac{B}{a}$

$\therefore \quad u(\rho,\varphi) = A + B\frac{\rho}{a}\sin\varphi$

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Problem #2 Find a fn $U(x,y) \ni. \Delta U = 0$ in a rectangle $0 \le x \le a, 0 \le y \le b$ and on ∂R

$$U(0,y) = f_1(y) \quad U(a,y) = 0 \quad U(x,0) = f_2(x) \quad U(x,b) = 0$$

we will solve the problem in two parts

1) find a fn $u \ni \Delta u = 0$ in R $\quad u(0,y) = u(a,y) = 0 \quad u(x,0) = f_2(x)$, $u(x,b) = 0$

2) find a fn $v \ni. \Delta v = 0$ in R $\quad v(0,y) = f_1(y), v(a,y) = 0 \quad u(x,0) = 0$, $u(x,b) = 0$

By separation of variables in part 1 $X(0) = X(a) = 0$ where $X = A_1 \sin\lambda x + B_1 \cos\lambda x$ this leads to a normal mode solution $X(x) = \sin \frac{n\pi x}{a}$

also since $Y(0) = \text{constant}$ & $Y(b) = 0$ one obtains for $Y(y) = C_1 \sinh\lambda y + D_1 \cosh\lambda y$ that the form of

$$Y(y) = a_n \frac{\sinh \frac{n\pi}{a}(b-y)}{\sinh \frac{n\pi}{a} b}$$

$$\therefore \quad u_n(x,y) = a_n \frac{\sinh \frac{n\pi}{a}(b-y)}{\sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a}$$

now let $u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$ then $u(x,0) = \sum_{n=0}^{\infty} u_n(x,0) = f_2(x) = \sum_{n=0}^{\infty} a_n \sin \frac{n\pi x}{a}$

using fourier series one now shows that $a_n = \frac{2}{a}\int_0^a f_2(\xi) \sin \frac{n\pi\xi}{a}\, d\xi$

in the same manner one shows that for $v(x,y)$, the form of $Y(y)$ is

$$Y(y) = \sin \frac{m\pi y}{b}$$

and that the form of $X(x)$ is

$$X(x) = b_m \frac{\sinh \frac{m\pi}{b}(a-x)}{\sinh \frac{m\pi}{b} a}$$

$$\therefore \quad v_m(x,y) = b_m \frac{\sinh \frac{m\pi}{b}(a-x)}{\sinh \frac{m\pi a}{b}} \sin \frac{m\pi y}{b}$$

let $v(x,y) = \sum_{n=0}^{\infty} v_n(x,y)$ then $v(0,y) = \sum_{n=0}^{\infty} v(0,y) = \sum_{n=0}^{\infty} b_m \sin \frac{m\pi y}{b}$

using fourier series one shows that $b_m = \frac{2}{b}\int_0^b f_1(\xi) \sin \frac{m\pi\xi}{b}\, d\xi$

then $U(x,y) = \sum_{n=1}^{\infty} \left\{ a_n \frac{\sinh \frac{n\pi}{a}(b-y)}{\sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a} + b_m \frac{\sinh \frac{m\pi}{b}(a-x)}{\sinh \frac{m\pi a}{b}} \sin \frac{m\pi y}{b} \right\}$

where $a_n = \frac{2}{a}\int_0^a f_2(\xi) \sin \frac{n\pi\xi}{a} d\xi \qquad b_m = \frac{2}{b}\int_0^b f_1(\zeta) \sin \frac{m\pi\zeta}{b} d\zeta$

note that $f_2(\xi)$ and $f_1(\zeta)$ need not be differentiable but only piecewise continuous. If piecewise continuous then the integral can be separated into several integrals over 2 adjacent pts having a differentiable discont ie suppose that $\frac{d}{d\xi} f_2(\xi)$ is discont at pts $\xi_1, \xi_2, \dots, \xi_n = a$

then $a_n = \frac{2}{a}\int_0^a f_2(\xi) \sin \frac{n\pi\xi}{a} d\xi = \frac{2}{a}\left[\int_0^{\xi_1} + \int_{\xi_1}^{\xi_2} + \dots + \int_{\xi_{n-1}}^{\xi_n}\right] f_2(\xi) \sin \frac{n\pi\xi}{a} d\xi$

where $f_2(\xi)$ is piecewise continuous in each of the intervals $[0,\xi_1] \dots [\xi_{n-1},\xi_n]$ and this can be done also for $f_1(\zeta)$.

if $f_1(y) = Ay(b-y)$ and $f_2(x) = B \cos \frac{\pi}{2a} x$

then $a_n = \frac{2B}{a}\int_0^a \cos\frac{\pi}{2a}x \sin\frac{n\pi x}{a} dx = \frac{2B}{a\pi}\left(\frac{4na}{4n^2-1}\right) = \frac{8Bn}{\pi(4n^2-1)}$

since $\int_0^a \cos\frac{\pi}{2a}x \sin\frac{n\pi x}{a} dx = \frac{1}{1-4n^2}\left[\cancel{\frac{2a}{\pi}\sin\frac{n\pi x}{a}\sin\frac{\pi x}{2a}}\Big|_0^a + \frac{4an}{\pi}\cos\frac{\pi x}{2a}\cos\frac{n\pi x}{a}\Big|_0^a\right]$

and $b_m = \frac{2}{b}A\int_0^b y(b-y)\sin\frac{m\pi y}{b} dy = \frac{2A}{b}\left[\frac{2b^3}{m^3\pi^3}\left(1-(-1)^m\right)\right] = \frac{4Ab^2}{m^3\pi^3}\left[1+(-1)^{m+1}\right]$

since $\int_0^b (by-y^2)\sin\frac{m\pi y}{b} dy = b\left[\frac{-yb}{m\pi}\cos\frac{m\pi y}{b}\Big|_0^b + \frac{b^2}{m^2\pi^2}\cancel{\sin\frac{m\pi y}{b}}\Big|_0^b\right] + \frac{y^2 b}{m\pi}\cos\frac{m\pi y}{b}\Big|_0^b - \frac{2b^2 y}{m^2\pi^2}\cancel{\sin\frac{m\pi y}{b}}\Big|_0^b - \frac{2b^3}{m^3\pi^3}\cos\frac{m\pi y}{b}$

$\therefore$ if $f_1(y) = Ay(b-y)$ & $f_2(x) = B\cos\frac{\pi x}{2a}$ then

$U(x,y) = \sum_{n=1}^{\infty}\left\{a_n \frac{\sinh\frac{n\pi}{a}(b-y)}{\sinh\frac{n\pi b}{a}}\sin\frac{n\pi x}{a} + b_m \frac{\sinh\frac{m\pi}{b}(a-x)}{\sinh\frac{m\pi a}{b}}\sin\frac{m\pi y}{b}\right\}$

$a_n = \frac{8Bn}{\pi(4n^2-1)} \qquad b_m = \frac{4Ab^2}{m^3\pi^3}\left[1+(-1)^{m+1}\right]$

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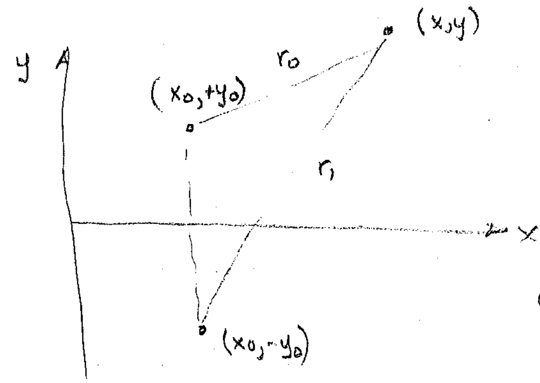
since $\quad u(M_0) = \int_\Sigma \left(\overset{0}{\cancel{G \frac{\partial u}{\partial n}}} - u\frac{\partial G}{\partial n} \right) ds - \cancel{\iint_T \overset{0}{\Delta u} \cdot G\, d\sigma}$



$$r_0 = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$r_1 = \sqrt{(x-x_0)^2 + (y+y_0)^2}$$

$$G(M, M_0) = \frac{1}{2\pi} \ln \frac{1}{r_0} - \frac{1}{2\pi} \ln \frac{1}{r_1}$$

$$\left.\frac{\partial G}{\partial n}\right|_{y=0} = -\left.\frac{\partial G}{\partial y}\right|_{y=0} = \frac{1}{2\pi}\left(\frac{r_0}{1}\right)\frac{\partial}{\partial y}\left(\frac{1}{r_0}\right) - \frac{1}{2\pi}\left(\frac{r_1}{1}\right)\frac{\partial}{\partial y}\left(\frac{1}{r_1}\right)$$

$$\frac{\cancel{r_0}}{2\pi}\left(-\frac{1}{r_0^2}\right)\frac{1}{2\cancel{r_0}} \cdot \cancel{2}(y-y_0) - \frac{\cancel{r_1}}{2\pi}\left(-\frac{1}{r_1^2}\right)\frac{1}{\cancel{2r_1}} \cdot \cancel{2}(y+y_0)$$

$$\frac{\partial G}{\partial y} = -\frac{1}{2\pi r_0^2}(y-y_0) + \frac{1}{2\pi r_1^2}(y+y_0)$$

$$\therefore \; -\left.\frac{\partial G}{\partial y}\right|_{y=0} = \frac{1}{2\pi r_0^2}\left[-y_0\right] - \frac{1}{2\pi r_1^2}\left[y_0\right] = \frac{-y_0}{\pi r_0^2}$$

where $\quad r_0 = \sqrt{(x-x_0)^2 + y_0^2}$

$$\therefore \quad u(M_0) = -\int_c u \frac{\partial G}{\partial n} ds = \frac{1}{\pi}\int_c \frac{y_0}{(x-x_0)^2 + y_0^2} \cdot f(x)\, dx$$

$$u(M_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y_0}{(x-x_0)^2 + y_0^2} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{0} \frac{y_0}{(x-x_0)^2 + y_0^2} \cdot 0\, dx + \frac{1}{\pi}\int_0^{+\infty} \frac{y_0 u_0}{(x-x_0)^2 + y_0^2} dx$$

$$\arctan\left(\frac{x-x_0}{y_0}\right) = \frac{1}{1 + \left(\frac{x-x_0}{y_0}\right)^2} \quad \frac{dx}{y_0}$$

$$\frac{1}{\frac{y_0^2 + (x-x_0)^2}{y_0^2}} \quad \frac{dx\, y_0}{y_0^2}$$

$$\frac{y_0}{y_0^2 + (x-x_0)^2} \quad \text{ok}$$

$$\begin{array}{l} u(M_0) \\ u(x_0, y_0) \end{array} = \frac{u_0}{\pi} \arctan\left(\frac{x-x_0}{y_0}\right)\Bigg|_0^\infty = \frac{u_0}{\pi}\Big[\arctan$$

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Cesar Levy T63.2131

#3 Solve $\Delta u = 1$ for a circle of radius a with b.c. $u|_{\rho=a} = 0$

let u_1 be $\ni$. $\Delta u_1 = 1$ a solution is $u_1 = \rho^2/4$

but $u_1|_{\rho=a} = a^2/4$

let u_2 be $\ni$. $\Delta u_2 = 0$ with $u_2|_{\rho=a} = -a^2/4$

$$u_{2_n} = R(\rho)\,\Phi(\varphi) = (C\rho^n + D\rho^{-n})(A_n \cos n\varphi + B_n \sin n\varphi)$$

since $\Delta u_2 = 0$ in interior of circle $D = 0$

$$\therefore \quad u_2 = \sum_{n=0}^{\infty} (\underline{A}_n \cos n\varphi + \underline{B}_n \sin n\varphi)\rho^n$$

$$u_2|_{\rho=a} = -a^2/4 = \sum_{n=0}^{\infty} (\underline{A}_n \cos n\varphi + \underline{B}_n \sin n\varphi) a^n$$

by use of fourier series $\frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos n\varphi + \beta_n \sin n\varphi)$ where $\underline{A}_0 = \frac{\alpha_0}{2}$, $\underline{A}_n = \alpha_n a^n$

$\underline{B}_n = \beta_n a^n$

one obtains $\frac{\alpha_0}{2} = -\frac{a^2}{4} \cdot \frac{1}{\pi} \cdot \frac{2\pi}{2} = -\frac{a^2}{4}$

$$\alpha_n = \frac{1}{\pi}\left(-\frac{a^2}{4}\right)\int_{-\pi}^{\pi} \cos n\varphi\, d\varphi = 0$$

$$\beta_n = \frac{1}{\pi}\left(-\frac{a^2}{4}\right)\int_{-\pi}^{\pi} \sin n\varphi\, d\varphi = 0$$

$$\therefore \quad u_2 = -a^2/4$$

hour $u = u_1 + u_2 = \frac{\rho^2 - a^2}{4}$

u satisfies $\Delta u = 1$ with $u|_{\rho=a} = 0$

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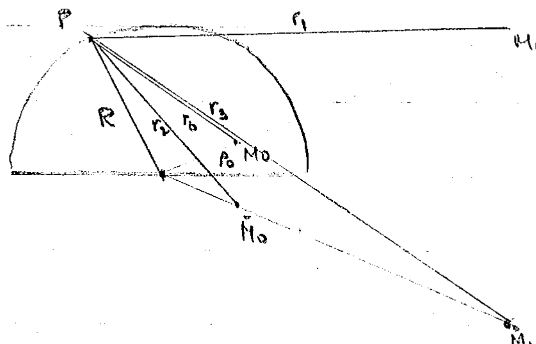
#6 Find the green's fn for a half circle, ring, a layer $0 \le z \le l$

1. half circle: place two points of singularity one at (ρ_0, θ_0) & one at $(\rho_0, -\theta_0)$



let $P\bar{M}_0 = r_2$ $PM_0 = r_0$

$P\bar{M}_1 = r_3$ $PM_1 = r_1$

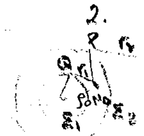
5

$$\therefore \quad G(P, M_0, \bar{M}_0) = \frac{1}{2\pi}\left[\left(\ln\frac{1}{r_0} - \ln\frac{R}{\rho_0}\frac{1}{r_1}\right) - \left(\ln\frac{1}{r_2} - \ln\frac{R}{\rho_0}\frac{1}{r_3}\right)\right]$$

notice that when P lies on $(r, 0)$ or (r, π) $r_0 = r_2$, $r_1 = r_3$

thus $G\big|_c = 0$ & $\Delta G = 0$ in the interior of the half-circle

2. ring $G(P, M_0) = \frac{1}{2\pi}\ln\frac{1}{r_{PM_0}} + v$ where $\Delta v = 0$ & $v\big|_\Sigma = -\frac{1}{2\pi}\ln\frac{1}{r_{PM_0}}$



Solution to v is $v(\rho, \theta) = a_0 + b_0\ln\rho + \sum_{n=1}^{\infty}\left\{(a_n\rho^n + b_n\rho^{-n})\cos n\theta + (a_n\rho^n + b_n\rho^{-n})\sin n\theta\right\}$

on Σ_1 $v\big|_{\Sigma_1} = -\frac{1}{2\pi}\ln\frac{a}{\rho_0}\frac{1}{r_1}$ on Σ_2 $v\big|_{\Sigma_2} = -\frac{1}{2\pi}\ln\frac{R}{\rho_0}\frac{1}{r_2}$

$$\therefore v\big|_{\Sigma_1} = -\frac{1}{2\pi}\ln\frac{a}{\rho_0}\frac{1}{r_1} = a_0 + b_0\ln a + \sum_{n=1}^{\infty}\left\{\left[a_n(a^n) + b_n(a^{-n})\right]\cos n\theta + \left[a_n(a^n) + b_n(a^{-n})\right]\sin n\theta\right\}$$

$$v\big|_{\Sigma_2} = -\frac{1}{2\pi}\ln\frac{R}{\rho_0}\frac{1}{r_2} = a_0 + b_0\ln R + \sum_{n=1}^{\infty}\left\{\left[a_n(R^n) + b_n(R^{-n})\right]\cos n\theta + \left[a_n(R^n) + b_n(R^{-n})\right]\sin n\theta\right\}$$

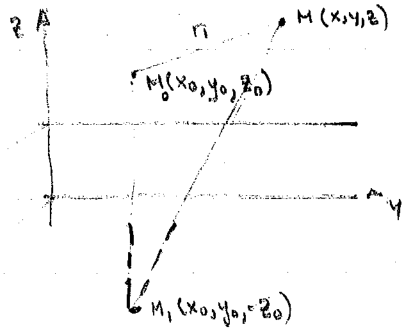
& a_0, b_0, a_n, b_n can be obtained

$$\therefore \quad G(P, M_0) = \frac{1}{2\pi}\ln\frac{1}{r_{PM_0}} + a_0 + b_0\ln\rho + \sum_{n=1}^{\infty}\left\{(a_n\rho^n + b_n\rho^{-n})\cos n\theta + (a_n\rho^n + b_n\rho^{-n})\sin\right.$$

where a_0, b_0, a_n, b_n are from the above two eqs.

5

3. a layer $0 \le z \le l$: the green's fn is of the form $\frac{1}{4\pi r}$. We must place an infinite number of pts of singularity along a line $x = x_0$, $y = y_0$



$\frac{1}{4\pi r_1} - \frac{1}{4\pi r_0}$ will be zero at $z=0$, but at $z=l$

G will have a residual of $\frac{1}{4\pi \bar{r}_1} - \frac{1}{4\pi \bar{r}_0}$ $\quad \bar{r}_1 = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-l)^2}$

$\bar{r}_0 = \sqrt{\quad " \qquad " \quad +(z+l)^2}$

now add $\frac{1}{4\pi \bar{r}_2} - \frac{1}{4\pi \bar{r}_3}$ $\ni$. $\bar{r}_2 = \bar{r}_0$ & $\bar{r}_3 = \bar{r}_1$ at $z=l$

but now these pts will contribute to lower bdy so one must add two new pts to counteract their effect, etc.

$$\therefore \quad G(M, M_0, M_1, \ldots) = \frac{1}{4\pi} \sum_{i=0}^{\infty} \left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0 \mp 2il)^2}} - \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0-2il)^2}} \right]$$

where $2il$ is the necessary placement of an image pt so that its correspond'g singularity is zero either at $z=l$ or $z=0$

3

Question: if one solves $\Delta G = 0$ in the slab with $G = 0$ at $z = 0, 1$ will one obtain G of form: $\sin n\pi z \; e^{-k|y|} \sin\sqrt{k^2-n^2\pi^2}\, x$ $\quad k^2 > n^2\pi^2$

& $\quad \sin n\pi z \; e^{-k|y| - \sqrt{n^2\pi^2 - k^2}|x|}$ for $k^2 < n^2\pi^2$

should be no preference for x or y

7 Determine a harmonic fn in a ring $a \le \rho \le b$ and satisfying

$$u|_{\rho=a} = f_1(\theta) \qquad u|_{\rho=b} = f_2(\theta)$$

Solution: $\nabla^2 u = u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\theta\theta} = 0$ with $u = R(\rho)\Theta(\theta)$

gives

$$\rho^2 R'' + \rho R' - k^2 R = 0$$

$$\Theta'' + k^2 \Theta = 0$$

or

$$\left.\begin{aligned} R &= A_k \rho^k + B_k \rho^{-k} \\ \Theta &= C_k \cos k\theta + D_k \sin k\theta \end{aligned}\right\} k \neq 0$$

$$R = A_0 + B_0 \log \rho$$

$$\Theta = C_0 + D_0 \theta$$

since u is single valued $c_0 + d_0 = 0$ where $c_0 = A_0 D_0$, $d_0 = D_0 B_0$

and therefore $k =$ integer n

$$\therefore \quad u = (a_0 + b_0 \ln \rho) + \sum_{n=1}^{\infty} \left[(a_n \rho^n + b_n \rho^{-n}) \cos n\theta + (c_n \rho^n + d_n \rho^{-n}) \sin n\theta\right]$$

at bdy

$$f_1(\theta) = (a_0 + b_0 \ln a) + \sum_{n=1}^{\infty} \left[(a_n a^n + b_n a^{-n}) \cos n\theta + (c_n a^n + d_n a^{-n}) \sin n\theta\right]$$

$$f_2(\theta) = (a_0 + b_0 \ln b) + \sum_{n=1}^{\infty} \left[(a_n b^n + b_n b^{-n}) \cos n\theta + (c_n b^n + d_n b^{-n}) \sin n\theta\right]$$

by fourier series $a_0 = \dfrac{\frac{1}{2\pi}\int_0^{2\pi} \left[f_1(\theta) \ln b - f_2(\theta) \ln a\right] d\theta}{\ln b - \ln a}$ $\qquad b_0 = \dfrac{\frac{1}{2\pi}\int_0^{2\pi} \left[f_2(\theta) - f_1(\theta)\right] d\theta}{\ln b - \ln a}$

$$a_n = \frac{\frac{1}{\pi}\int_0^{2\pi} \left[f_1(\theta)\, b^{-n} \cos n\theta - f_2(\theta)\, a^{-n} \cos n\theta\right] d\theta}{\left(\frac{a}{b}\right)^n - \left(\frac{b}{a}\right)^n}$$

2

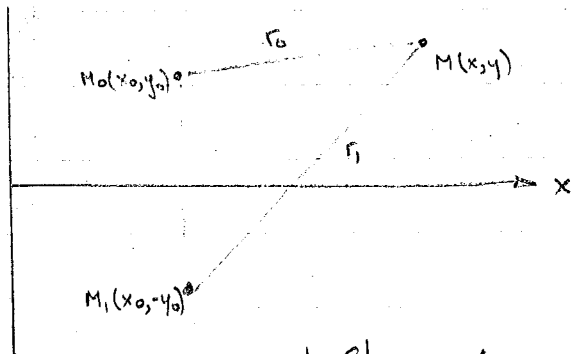
$$b_n = \frac{1}{\pi} \int_0^{2\pi} \frac{\left[f_2(\theta)\, a^n \cos n\theta - f_1(\theta)\, b^n \cos n\theta \right]}{\left(\frac{a}{b}\right)^n - \left(\frac{b}{a}\right)^n}\, d\theta$$

$$c_n = \frac{1}{\pi} \int_0^{2\pi} \frac{\left[f_1(\theta)\, b^{-n} \sin n\theta - f_2(\theta)\, a^{-n} \sin n\theta \right]}{\left(a/b\right)^n - \left(b/a\right)^n}\, d\theta$$

$$d_n = \frac{1}{\pi} \int_0^{2\pi} \frac{\left[f_2(\theta)\, a^n - f_1(\theta)\, b^n \right] \sin n\theta\, d\theta}{\left(a/b\right)^n - \left(b/a\right)^n}$$

\# 8 Determine solution to $\Delta u = 0$ in half plane $y \geq 0$ with b.c.

$$u(x,0) = \begin{cases} 0 & x<0 \\ u_0 & x>0 \end{cases}$$



$$G(M,M_0) = \frac{1}{2\pi}\ln\frac{1}{r_0} - \frac{1}{2\pi}\ln\frac{1}{r_1}$$

since $u(M_0) = \int_\Sigma \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right)ds - \iint_T \Delta u \cdot G\,d\tau$

and $G\big|_\Sigma = 0$, $\Delta u = 0$ in T then

$$u(M_0) = -\int_\Sigma u\frac{\partial G}{\partial n}\,ds \qquad u\big|_\Sigma = f(s)$$

now $r_0 = \sqrt{(x-x_0)^2+(y-y_0)^2}$ $\quad r_1 = \sqrt{(x-x_0)^2+(y+y_0)^2}$

$$\left.\frac{\partial G}{\partial n}\right|_{y=0} = -\left.\frac{\partial G}{\partial y}\right|_{y=0} = -\frac{y_0}{\pi r_0'^2} \qquad r_0' = \sqrt{(x-x_0)^2+y_0^2}$$

$$\therefore \quad u(x_0,y_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y_0}{(x-x_0)^2+y_0^2}f(x)\,dx = \frac{1}{\pi}\int_0^\infty \frac{y_0 u_0\,dx}{(x-x_0)^2+y_0^2}$$

$$= \frac{u_0}{\pi}\arctan\left(\frac{x-x_0}{y_0}\right)\Bigg|_0^\infty = \frac{u_0}{2} + \frac{u_0}{\pi}\tan^{-1}\left(\frac{x_0}{y_0}\right)$$

$$\therefore \quad u(x,y) = \frac{u_0}{2} + \frac{u_0}{\pi}\tan^{-1}\left(\frac{x}{y}\right)$$

5

P330

1. Find u harmonic in circle of radius a and on the circle assumes the value

a) $u|_c = A\cos\phi$

b) $u|_c = A + B\sin\phi$

Seperation of Variables $\Rightarrow$ $u(\rho,\phi) = \sum_{n=0}^{\infty} \rho^n (A_n \cos n\phi + B_n \sin n\phi)$ *

where, if $u|_c = f(\phi)$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,ds \quad , A_n = \frac{1}{a^n \pi}\int_{-\pi}^{\pi} f(s)\cos ns\,ds \quad n \geq 1$$

$$B_n = \frac{1}{a^n \pi}\int_{-\pi}^{\pi} f(s)\sin ns\,ds \quad n \geq 0$$

Then a) $\Rightarrow$ $u = A\frac{\rho}{a}\cos\phi$

b) $\Rightarrow$ $u = A + B\frac{\rho}{a}\sin\phi$

(these solns can be obtained by finding A_n, B_n through the integrals or simply by equating coefficients in *)

2. Solve $\Delta u = 0$ in rectangle $0 \le x \le a$, $0 \le y \le b$ with

$$u|_{x=0} = f_1(y), \quad u|_{y=0} = f_2(x), \quad u|_{x=a} = 0, \quad u|_{y=b} = 0$$

Let $u = u_1 + u_2$ where $\Delta u_1 = 0$, $\Delta u_2 = 0$ in rectangle

and $u_1 = 0$ on boundary except $u_1|_{x=0} = f_1(y)$,
$u_2 = 0$ on boundary except $u_2|_{y=0} = f_2(x)$.

Seperation of variables $\Longrightarrow$ $u_2 = \sum_{n=0}^{\infty} u_n^{(2)}(x,y)$

where
$$u_n^{(2)} = \frac{a_n \sinh \frac{\pi n}{a}(b-y) \sin\left(\frac{n\pi x}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}$$

so that
$$u_n^{(2)}(x,0) = a_n \sin\left(\frac{n\pi x}{a}\right)$$

$$\Longrightarrow \quad a_n = \frac{2}{a}\int_0^a f_2(\xi)\sin\left(\frac{n\pi\xi}{a}\right)d\xi$$

Similarly,
$$u_n^{(1)} = \frac{b_m \sinh \frac{m\pi}{b}(a-x) \sin\left(\frac{m\pi}{b}y\right)}{\sinh\left(\frac{m\pi a}{b}\right)}$$

$$u_n^{(1)}(0,y) = b_m \sin\left(\frac{m\pi}{b}\right)y$$

$$\Longrightarrow \quad b_m = \frac{2}{b}\int_0^b f_1(\xi)\sin\left(\frac{m\pi\xi}{b}\right)d\xi$$

When $f_1 = Ay(b-y)$, $\quad b_m = \frac{4Ab^2}{m^3\pi^3}\left[1 + (-1)^{m+1}\right]$

$f_2 = B\cos \frac{\pi x}{\ }$, $\quad a_n = \frac{4B}{\pi}\left[\frac{2n}{4n^2-1}\right]$

3. $\Delta u = 1$ for circle radius a, s.t. $u\big|_{\rho=a} = 0$.

Let $u = u_1 + u_2$ s.t. $\Delta u_1 = 1$, $\Delta u_2 = 0$

$u_2\big|_a = -u_1\big|_a$

Let $u_1 = \rho^2/4$, $\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u_1}{\partial \rho}\right) = 1$

on $\rho = a$, $u_1\big|_{\rho=a} = a^2/4 \Rightarrow u_2\big|_a = -a^2/4$

Seperation of variables $\Rightarrow$

$$u_2 = \sum_{n=0}^{\infty} (A_n \cos n\phi + B_n \sin n\phi)\rho^n$$

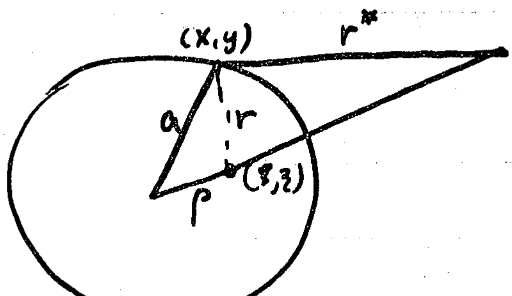
Equating coefficients $\Rightarrow$ $B_n = 0$ all n, $A_n = 0$ $n \geq 1$

$A_0 = -a^2/4$

$$\Rightarrow \quad \underline{u = \frac{\rho^2 - a^2}{4}}$$

6 a) G.F. for ½ circle, radius a

G.F. for disk, $G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln r - \frac{1}{2\pi}\ln\left(\frac{\rho}{a} r^*\right)$



$$= \frac{1}{2\pi}\ln\left(\frac{a r}{\rho r^*}\right)$$

$$r^2 = (x-\xi)^2 + (y-\eta)^2$$

$$r^{*2} = \left(x\frac{a^2}{\rho^2} - \xi\right)^2 + \left(y\frac{a^2}{\rho^2} - \eta\right)^2$$

In polar co-ords $(x,y) \to (\rho,\theta)$

$(\xi,\eta) \to (v,\phi)$

$$r^2 = v^2 - 2\rho v\cos(\phi-\theta) + \rho^2$$
$$r^{*2} = v^2 - 2a^2\frac{v}{\rho}\cos(\phi-\theta) + \frac{a^4}{\rho^2}$$

$\Rightarrow$

$$G(v,\theta;\rho,\phi) = \frac{1}{4\pi}\ln\left\{\frac{a^2(v^2-2v\rho\cos(\theta-\phi)+\rho^2)}{a^4-2a^2v\rho\cos(\theta-\phi)+v^2\rho^2}\right\}$$

Thus $G=0$ when $v=a$.

$G(v,\theta;\rho,-\phi)$ is G.F for a disk of radius a with pole at $(\rho,-\phi)$ i.e. image of (ρ,ϕ) in $\phi=0$.

Thus $G^*(v,\theta;\rho,\phi) = G(v,\theta;\rho,\phi) - G(v,\theta;\rho,-\phi)$

satisfies $G^*=0$ when $v=a$ and $G^*=0$ when $\phi=0$

i.e it is G.F. for semi-circle.

Hence $$G^*(v,\theta;\rho,\phi) = \frac{1}{4\pi}\ln\left\{\frac{(v^2-2v\rho\cos(\theta-\phi)+\rho^2)}{(a^4-2a^2v\rho\cos(\theta-\phi)+v^2\rho^2)}\cdot\frac{(a^4-2a^2v\rho\cos(\theta+\phi)+v^2\rho^2)}{(v^2-2v\rho\cos(\theta+\phi)+\rho^2)}\right\}$$

---

ring $a \le \rho \le b$ $(= D)$

$(x,y) \to (\rho,\theta)$
$(\xi,\eta) \to (\nu,\phi)$

6 b) $G(x,y;\xi,\eta) = \frac{1}{2\pi} \ln \frac{1}{r} + V$

where $\Delta V = 0$ in D, $V|_{\partial D} = -\frac{1}{2\pi} \ln \frac{1}{r}$

Sepn. of variables $\Rightarrow$

$$V = \alpha_0 + \beta_0 \ln \rho + \sum_{n=1}^{\infty} \left[(\alpha_n \rho^n + \beta_n \rho^{-n}) \cos n\theta \right] + \sum_{n=1}^{\infty} \left[(\gamma_n \rho^n + \delta_n \rho^{-n}) \sin n\theta \right] \quad (1)$$

$$r^2 = \nu^2 - 2\rho\nu \cos(\phi - \theta) + \rho^2$$

$\Rightarrow$ on $\rho = a$ $\quad V = -\frac{1}{2\pi} \ln \frac{1}{r_a}$

$$r_a = (\nu^2 - 2a\nu\cos(\phi-\theta) + a^2) = \bar{f}(\theta)$$
$$r_b = (\nu^2 - 2b\nu\cos(\phi-\theta) + b^2) = \bar{g}(\theta)$$

let $-\frac{1}{2\pi} \ln \frac{1}{\bar{f}(\theta)} = f(\theta)$, $-\frac{1}{2\pi} \ln \frac{1}{\bar{g}(\theta)} = g(\theta)$

B.C. are now: $V|_{\rho=a} = f(\theta)$, $V|_{\rho=b} = g(\theta)$ $\quad (2)$

substituting (1) into (2) & using expansion of f & g

yields =

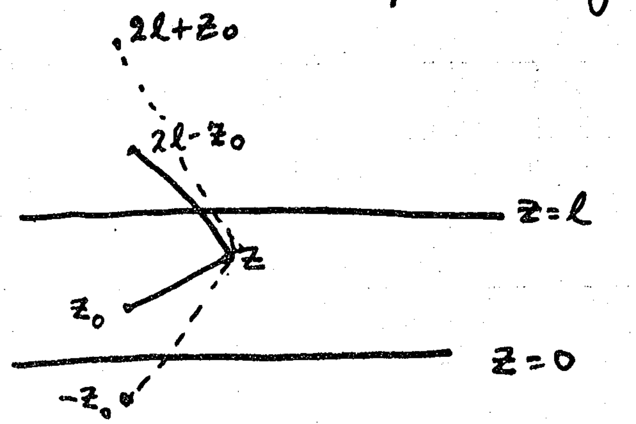
$$\left.\begin{aligned} \alpha_0 + \beta_0 \ln b &= a_0 \\ \alpha_n b^n + \beta_n b^{-n} &= a_n \\ \gamma_n b^n + \delta^n b^{-n} &= b_n \end{aligned}\right\} \qquad \begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta \, d\theta \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta \, d\theta \end{aligned}$$

$$\left.\begin{aligned} \alpha_0 + \beta_0 \ln a &= c_0 \\ \alpha_n a^n + \beta_n a^{-n} &= c_n \\ \gamma_n a^n + \delta^n a^{-n} &= d_n \end{aligned}\right\} \qquad \begin{aligned} c_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta \\ d_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta \end{aligned}$$

These determine v and hence G.

This also solves <u>Q. 7</u>.

6 c) G.F. for a layer $0 \le z \le l$.

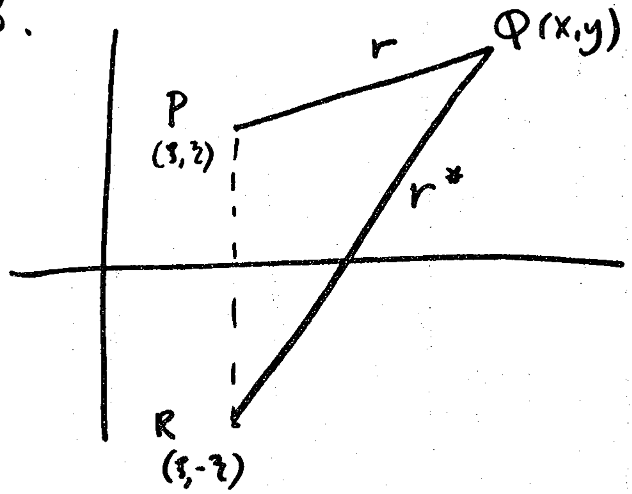


z_0 reflected in $z=0$ & $z=l$.
<u>both</u> images reflected in <u>both</u> $z=0$ & $z=l$
yields a series of images at $z = \pm z_0 \pm 2nl$

Green's Fn is

$$G = \frac{1}{4\pi}\sum_{n=0}^{\infty}\left\{\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0\pm 2nl)}} - \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0\pm 2nl)^2}}\right\}$$

8.



$$G(Q,P) = \frac{1}{2\pi} \ln\left(\frac{r}{r^*}\right)$$

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$$

$$r^* = \sqrt{(x-\xi)^2 + (y+\eta)^2}$$

$$y = 0 \quad , \quad G = 0$$

$$\left.\frac{\partial G}{\partial n}\right|_{y=0} = -\left.\frac{\partial G}{\partial y}\right|_{y=0} = -\frac{\eta}{\pi r_0^2}$$

$$r_0 = \sqrt{(x-\xi)^2 + \eta^2}$$

$$u(\xi,\eta) = -\int_{y=0} f \frac{\partial G}{\partial n}\, ds \qquad \text{where } u = f \text{ on } y = 0.$$

$$s = x$$

$$\Rightarrow \quad u = \frac{1}{\pi}\int_0^\infty \frac{\eta\, u_0\, dx}{(x-\xi)^2+\eta^2} = \frac{u_0}{\pi} \arctan\left(\frac{x-\xi}{\eta}\right)\Bigg|_{x=0}^{x=\infty}$$

$$= \underline{\frac{u_0}{2} + \frac{u_0}{\pi}\arctan\left(\frac{\xi}{\eta}\right)} \qquad \eta \geq 0 \; , -\infty < \xi < \infty .$$

① $(x+2)u_x + 2y\,u_y = 2u$ IC: $u(-1,y) = \sqrt{y}$

$$\frac{dy}{dx} = \frac{2y}{x+2} \qquad y = c_1 (x+2)^2 \qquad \text{Characteristic}$$

$$\frac{du}{dx} = \frac{2u}{x+2} \qquad u = k_1 (x+2)^2 \qquad \text{where } k_1 = k_1(c_1)$$

on $u(-1,y) = \sqrt{y}$

$$\sqrt{y} = k_1(c_1)\cdot 1 = k_1(y) \qquad \therefore \qquad k_1(s) = s^{1/2}$$

$$\therefore \qquad u = \frac{\sqrt{y}}{(x+2)}(x+2)^2 = \sqrt{y}\,(x+2)$$

② $x^2 u_x - y^2 u_y = 0$ IC: $u(1,y) = F(y)$

$$\frac{dy}{dx} = \frac{-y^2}{x^2} \qquad \therefore \qquad \frac{1}{y} + \frac{1}{x} = c \qquad \frac{xy}{x+y} = c$$

$$\frac{du}{dx} = \frac{0}{x^2} = 0 \qquad \therefore \qquad u = k(c)$$

$$@\ x=1 \qquad F(y) = k\left(\frac{1+y}{y}\right) \qquad \therefore \qquad k(s) = F\left(\frac{1}{s-1}\right)$$

$$\therefore \qquad u = F\left(\frac{xy}{x+y-xy}\right)$$

③ b) $u_x + u^2 u_y = 1$

$$\frac{dx}{1} = \frac{dy}{u^2} = \frac{du}{1} \qquad x - u = c_1 \qquad 3y - u^3 = c_2$$

$$\therefore \quad F(c_1, c_2) = 0 \quad \text{or} \quad F(x-u,\ 3y-u^3) = 0$$

a) $x u_x (u - 2y^2) = (u - u_y\, y)(u - y^2 - 2x^3)$

$$\frac{dx}{x(u-2y^2)} = \frac{dy}{y(u-y^2-2x^3)} = \frac{du}{u(u-y^2-2x^3)}$$

$$\ln y = \ln u + \ln c_1 \qquad y = u c_1$$

$$\frac{dx}{x}\left[u - c_1^2 u^2 - 2x^3\right] = \frac{du}{u}\left(u - 2c_1^2 u^2\right) = du\left(1 - 2c_1^2 u\right) \qquad \text{since } u = f(y) \text{ only}$$
$$\text{\& } y \text{ is indep of } x$$

$$(u - y^2)\ln x - \frac{2}{3}x^3 = (u - y^2) + c_2$$

$$\therefore \quad (u-y^2)[\ln x - 1] - \tfrac{2}{3}x^3 = c_2 \qquad \therefore \qquad F\left(y/u,\ (u-y^2)[\ln x - 1] - \tfrac{2}{3}x^3\right) = 0$$

C

$$\frac{dx}{x(u-2y^2)} = \frac{dy}{y(u-y^2-2x^3)} = \frac{du}{u(u-y^2-2x^3)} \qquad d(\quad)=C$$

$k_1(xu-2y^2x) + k_2(yu-y^3-2x^3y) = k_3(u^2-uy^2-2x^3u)$

$y(xu-2y^2x) \qquad (xyu-y^2x-2x^4y)$

$x^2(xu-2y^2x) \qquad c_1y=u$

$2x^2(xu-2y^2x) \to \underline{2x^3u} - \underline{4y^2x^3}$

$-2y(yu-y^3-2x^3y) \to -2y^2u+2y^4+\underline{4x^3y^2}$

$(u^2-uy^2-2x^3u) \longrightarrow u^2-uy^2-\underline{2x^3u}$

$u(u-y^2) \qquad 2y^2(y^2-u) \qquad (y^2-u)$

$-u\,(\cancel{u-y^2})\frac{1}{c_1}2y^2(y \qquad c_1y$

$2u^2c_1^2 \qquad 2y^2$

$xy(c_1-2y) \qquad y(u-y^2-2x^3) \qquad u(u-y^2-2x^3) \qquad \frac{u}{y} \quad uy$

$y^2(c_1-y)-2yx^3$

$$\frac{dx}{x(u-2y^2)} = \frac{dy}{y(u-y^2-2x^3)} = \frac{du}{u(u-y^2-2x^3)}$$

$-2\frac{u^3}{c_1^2}+\frac{2u^4}{c_1^4}$

$2x^2(xu-2y^2x) \qquad \cancel{2x^3u-4y^2x^3}$

$2y(yu-y^3-2x^3y) \qquad \cancel{yu^2-y^3u-2x^3yu} \quad y^2u-y^4-2x^3y^2 \quad -2y^2u+2y^4+4x^3y^2$

$(u^2-uy^2-2x^3u) \qquad u^2-uy^2-2x^3u$

$$\frac{(2x^2dx-2ydy+du)}{\cancel{u^2-uy^2-2xu}}=0 \qquad 2x^3$$

$2x^4u-2x^4y^2$

$2y^4-3uy^2+u^2$

$2y$

$\left(\frac{2x^3}{3}-y^2+u\right)=C$

$x^2u(xu-2y^2x) \to x^3u^2-2y^2x^3u$

$-2y \quad (yu-y^3-2x^3y)$

$$\frac{-2y\,dy + xu\,du}{(u-y^2)} = \frac{dx}{1} \qquad \frac{u\,dx}{ux(u-2y^2)} = \frac{x\,du}{xu(u-y^2-2x^3)}$$

$$\frac{u\,dx - x\,du}{-ux\,2y^2 + xu\,y^2}$$

$$\int \frac{2x^3}{u-y^2}dx + \ln(u-y^2) = \ln x + C \qquad -3xuy^2 + xy\;2x^3$$

$$-xu\;\frac{u\,dx - x\,du}{y^2-2x^3}$$

$$C = \ln\left(\frac{u-y^2}{x}\right) + \int \frac{2x^3\,dx}{(u-y^2)} \qquad \frac{u\,dx - x\,du}{-xu(y^2-2x^3)} \qquad u\,dx + x\,du$$

$$\frac{u\,dx + x\,du}{2u^2x - 3xuy^2 + xu(-2x^3)} \qquad y\,\frac{u\,dx + x\,du}{2x^2u}$$

$$yxu(2u - 3y^2 - 2x^3)$$

$$\frac{\cancel{x}\;u\,dy + y\,du}{2yu^2 - 2uy^3 - 4x^3uy} \qquad \frac{xu\,dy + xy\,du}{xuy(2u - 2y^2 - 4x^3)}$$

$$\frac{xu\,dy + xy\,du}{xuy(2u-2y^2-4x^3)} + \frac{yu\,dx - yx\,du}{yxu(2u-3y^2-2x^3)}$$

$$\frac{xu\,dy - yu\,dx + 2xy\,du}{xuy(y^2-2x^3)}$$

let $\frac{dx}{x(u-y^2)} = dv \qquad u = 2x^4$

$du = 8x^3$

$$y(\quad)\,dx = x(u-2y^2)\,dy$$

$$\frac{y(u-y^2-2x^3)\,dx}{x(u-y^2)} = (xu - 2y^2x)\,dy = \frac{\cancel{x(u-y^2)dy} - xy^2dy}{y(u-y^2-2x^3)\quad xy(u-y^2-2x^3)(u-y^2)}$$

$$\frac{dx}{x(u-y^2)} = \frac{dy}{y(\;)}$$

$$u_x\, x(u-2y^2) = (u - u_y y)(u - y^2 - 2x^3)$$

$$\frac{dx}{x(u-2y^2)} = \frac{dy}{y(u-y^2-2x^3)} = \frac{du}{u(u-y^2-2x^3)}$$

$$\frac{y\,d\cancel{x}}{xy(u-2y^2)} + \frac{x\,dy}{xy(u-y^2-2x^3)}$$

$$\frac{2}{3} + \frac{4}{6} \cdot \frac{6}{9} \cdot \frac{2}{3}$$

$$\frac{4}{6} + \frac{4}{6} = \frac{8}{6} = \frac{4}{3}$$

$$\frac{x\,dx}{ux^2} + \frac{y\,dy}{-uy^2}$$

$$\frac{y\,dx + x\,dy}{\cancel{xyu} - 2xy^3 + \cancel{xyu} + xy^3 + 2x^4 y} = \frac{du}{u(u-y^2-2x^3)}$$

$$\frac{y\,dx - x\,dy}{-xy^3 + 2x^4 y} = \frac{du}{u(u-y^2-2x^3)} \qquad \frac{y\,dx - x\,dy}{-xy(y^2-2x^3)} = \frac{xy\,du}{xyu(u-y^2-2x)}$$

$$-xy(y^2 + 2x^2)$$

$$\frac{u\,dx}{xu(u-2y^2)} \;\; \frac{+2x\,du}{-xu(u-y^2-2x^3)} = \frac{dy}{y(u-y^2-2x^3)}$$

$$\frac{u\,dx - 2x\,du}{xu^2 - \cancel{2y^2xu} - 2xu^2 + \cancel{2xuy^2} + 4x^4u} \qquad \frac{u\,dx - x\,dy}{xu^2 - 2xy^2u - xu^2 + xuy^2 + 2x^4u}$$

$$\frac{u\,dx - 2x\,du}{-xu^2 + 4x^4u} =$$

$$-\frac{u\,dx - 2x\,du}{(u+4x^3)} = \frac{u\,dx}{u-2y^2}$$

$$\frac{u\,dx - 2x\,du}{-xu(u-4x^3)} = \frac{u\,dx}{ux(u-2y^2)}$$

$$(u-2y^2)u\,dx - 2x($$

Van Dyke – Perturbation Probl in Fluid Dyn

A. Erdelyi –

$$\frac{dx}{x(u-2y^2)} = \frac{dy}{y(u-y^2-2x^3)} = \frac{du}{u(u-y^2-2x^3)}$$

$$x\,d(u-y^2) = \cancel{x\,d}(u-y^2)$$

$$dx\,(u-y^2-2x^3) = \cancel{dx}\ \ dy\ \ x(u-2y^2)$$
$$dx\,(u-y^2-2x^3) = \ \ du\ \ x(u-2y^2)$$

$$d[(u-y^2-2x^3)x] = (u-y^2-2x^3)dx + x\,d(u-y^2-2x^3)$$
$$dy\,[x(u-2y^2)]$$
$$x\,du - 2xy\,dy - 6x^3dx$$
$$\cancel{xu}\,dy - 2xy^2dy$$

$$-6x^3dx - 2xy(1+y)dy$$
$$(u-y^2-2x^3)dx = x\,d(u-y^2) + (u-y^2)dx$$
$$x\,du - 2yx\,dy - 6x^3dx =$$

$$xu(u-y^2-2x^3) - 2y^2x(u-y^2-2x^3) - 6x^4(u-2y^2)$$
$$xu^2 - xuy^2 - 2x^4u - 2y^2xu + 2y^4x + 4y^2x^4 - 6x^4u + 6x^4y^2$$
$$xu^2 - 3xuy^2 + 2y^4x \quad 10x^4y^2 - 8x^4u$$

$$u = y\,g\left[-\tfrac{x^4}{2} + x(u-y^2)\right]$$
$$u_x = y\,g'[\quad]\{-2x^3 + (u-y^2) + u_x x\}$$
$$u_y = g + y\,g'[\quad]\{x(u_y - 2y)\}$$
$$u_x[1 - xy\,g'] = y\,g'[-2x^3 + (u-y^2)]$$

$$\cancel{[u_y - u/y]} \qquad u_y - xyg'u_y = g + yg'(-2xy)$$

$$\cancel{u_y[1 - xyg'] = yg'[2x^3 + (u-y^2)]} + g$$

$$\cancel{[yu_y - u] =} \qquad \cancel{u_x[\quad] = yg'}$$

$$[-u + yu_y][1 - xyg'] = yg + yg'(-2xy) - u[1 - xyg']$$

$$u_x[1 - xyg'] = yg'\,[-2x^3 + (u - y^2)]$$

$$(u - y^2 - 2x^3)[+u - yu_y](1 - xyg') = xyg'(2y^2 - u)(u - y^2 - 2x^3)$$

$$x(u - 2y^2)u_x(1 - xyg') = xyg'[(u - 2y^2)(-2x^3 + (u - y^2))$$

$$(u - 2y^2)(u - y^2 - 2x^3)$$

$$u = yg\left[-\frac{x^4}{2} + x(u - y^2)\right]$$

$$u_x = yg'[\ ]\{-2x^3 + (u - y^2) + x(u_x)\}$$

$$u_x[1 - yxg'] = yg'\{u - y^2 - 2x^3\}$$

$$u_y = g + yg'\{x(u_y - 2y)\}$$

$$(u - yu_y)[1 - xyg'] = -yg - xyg'(-2y^2) + u(1 - xyg')$$

$$= -u + u - xyg'(u - 2y^2)$$

$$(u - y^2 - 2x^3)(u - yu_y)[\quad] = -xyg'(u - 2y^2)(u - y^2 - 2x^3)$$

$$xu_x(u - 2y^2)(1 - yg'x) = xyg'[u - y^2 - 2x^3][u - 2y^2]$$

$$\frac{y\,dx - x\,dy}{-xy\,(y^2 - 2x^3)}$$

$$\frac{2x^2dx}{2x^3(u-2y^2)} + \frac{du}{u^2 - uy^2 - 2x^3u}$$

$$2x^2dx + du - 2y\,dy$$

$$\cancel{2x^3u} - \cancel{4x^3y^2} + u^2 - uy^2 - \cancel{2x^3u} - 2y^2u + 2y^4 + \cancel{4x^3y^2}$$

$$u^2 - 3y^2u + 2y^4$$

$$\frac{2x^2dx + du - 2y\,dy}{(u-2y^2)(u-y^2)} = \frac{dx}{x\,(u-2y^2)}$$

$$2x^3dx + x(du - 2y\,dy) = (u-y^2)\,dx$$

$$2x^3dx + d\left[(u-y^2)x\right]$$

$$2x^4dx + x(du - 2y\,dy) - (u-y^2)\,dx = 0$$

$$2x^2dx + du - 2y\,dy$$

$$2x^2(xu - 2y^2x) + (u^2 - uy^2 - 2x^3u) - 2y(yu - y^3 - 2x^3y)$$

$$\cancel{2x^3u} - \cancel{4x^3y^2} + u^2 - uy^2 - 2x^3u - 2y^2u + 2y^4 + \cancel{4x^3y^2}$$

$$d(u-y^2) \qquad u^2 - 3y^2u + 2y^4$$

$$\frac{2x^2dx + du - 2y\,dy}{(u-2y^2)(u-y^2)} = \frac{dx}{x\,(u-2y^2)}$$

$$2x^3dx + x\,d(u-y^2) = (u-y^2)\,d(x)$$

$$2x^3dx + d\left[x(u-y^2)\right] = 0$$

$$\frac{x^4}{2} + x(u-y^2) = C \qquad x(u-y^2-2x^3) = C$$

$$(u - y^2 - 2x^3)\,dx = x\,d(u-y^2)$$

18

March 23, 1973

763.2131

C. LEVY

1. $(x+2)u_x + 2y\, u_y = 2u$ $\qquad$ IC: $u(-1,y) = \sqrt{y}$

$$\frac{dy}{dx} = \frac{2y}{x+2} \quad \Rightarrow \quad y = c_1 (x+2)^2 \qquad \text{Characteristic}$$

$$\frac{du}{dx} = \frac{2u}{x+2} \quad \Rightarrow \quad u = k_1 (x+2)^2 \quad \text{where } k_1 = k_1(c_1)$$

Since $u(-1,y) = \sqrt{y}$: $\sqrt{y} = k_1(c_1) \cdot 1 = k_1(y)$ $\therefore$ $k_1(s) = s^{1/2}$

$$u = \frac{\sqrt{y}}{x+2}(x+2)^2 = \sqrt{y}\,(x+2)$$

5

2. $x^2 u_x - y^2 u_y = 0$ $\qquad$ IC: $u(1,y) = F(y)$

$$\frac{dy}{dx} = \frac{-y^2}{x^2} \quad \therefore \quad \frac{1}{y} + \frac{1}{x} = c \qquad \text{Characteristic}$$

$$\frac{du}{dx} = \frac{0}{x^2} = 0 \quad \therefore \quad u = k(c)$$

Since $u(1,y) = F(y)$: $F(y) = k\left(\frac{1+y}{y}\right)$ $\therefore$ $k(s) = F\left(\frac{1}{s-1}\right)$

$$u = F\left(\frac{xy}{x+y-xy}\right)$$

5

3. b $u_x + u^2 u_y = 1$

$$\frac{dx}{1} = \frac{dy}{u^2} = \frac{du}{1} \quad \Rightarrow \quad x - u = c_1$$

$$3y - u^3 = c_2$$

5

$\therefore$ $F(c_1, c_2) = 0$ or $F(x-u, 3y-u^3) = 0$

3a $\quad u_x\, x(u-2y^2) = (u-u_y y)(u-y^2-2x^3)$

$$\frac{dx}{x(u-2y^2)} = \frac{dy}{y(u-y^2-2x^3)} = \frac{du}{u(u-y^2-2x^3)}$$

$$\Rightarrow \frac{dy}{y} = \frac{du}{u} \quad \text{or} \quad y = C_1 u$$

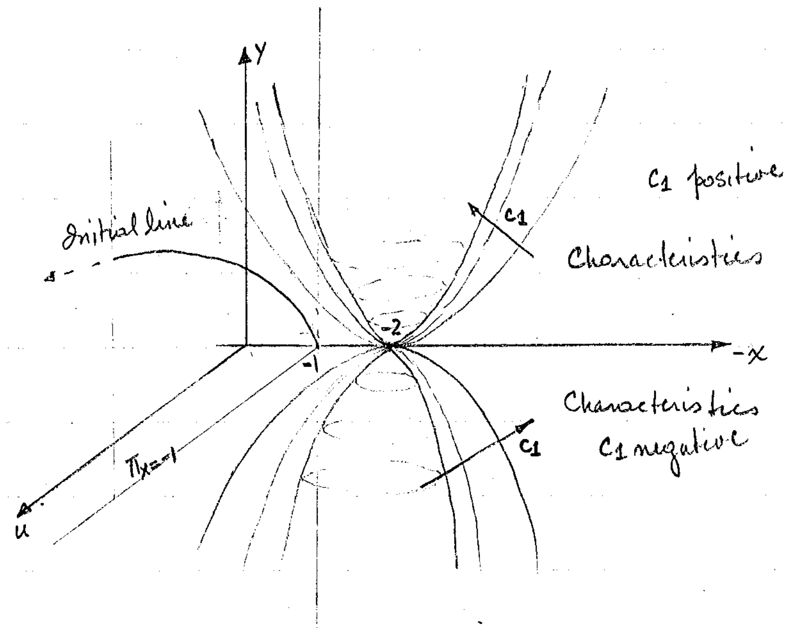
$$\frac{2x^2dx + du - 2y\,dy}{(u-2y^2)(u-y^2)} = \frac{dx}{x(u-2y^2)}$$

$$\therefore\ d\left(\frac{2x^3}{3} + u - y^2\right) = \left(\frac{u-y^2}{x}\right)dx$$

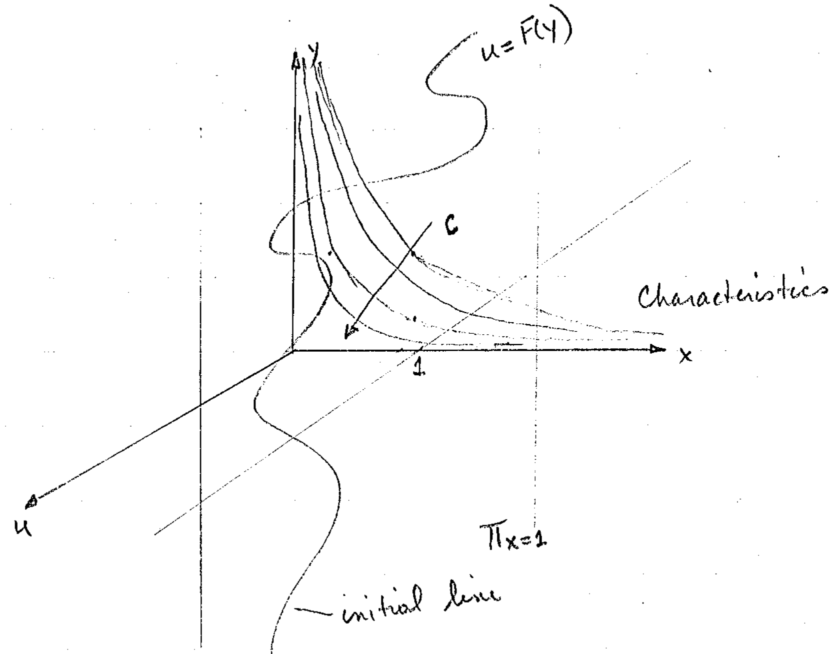
$$\text{or} \quad \frac{2x^3}{3} + (u-y^2) - \int \frac{u-y^2}{x}dx = C_2$$

$$\therefore \quad F(C_1, C_2) = 0 \quad \text{or} \quad F\left(y/u,\ \frac{2x^3}{3} + (u-y^2) - \int \frac{u-y^2}{x}dx\right) = 0$$

3



y
Initial line
-2
-x
-1
$\Pi_{x=-1}$
u
c_1
c_1 positive
Characteristics
Characteristics
c_1 negative
c_1



$u = F(y)$
y
c
Characteristics
x
1
u
$\Pi_{x=1}$
initial line

#2

1. $(x+2)u_x + 2yu_y = 2u$, $u(-1,y) = \sqrt{y}$

$$\frac{dy}{dx} = \frac{2y}{x+2} , \qquad \frac{du}{dx} = \frac{2u}{x+2}$$

$$\frac{1}{2}\ln y = \ln(x+2) + \frac{1}{2}\ln k \qquad \frac{1}{2}\ln u = \ln(x+2) + \ln c$$

$$y = k(x+2)^2 \qquad u = c(x+2)^2$$

where $c = c(k)$

$= c\left(\frac{y}{(x+2)^2}\right)$

When $x = -1$ $u = \sqrt{y}$

$$\Rightarrow \quad \sqrt{y} = c\left(\frac{y}{1^2}\right)1^2 \Rightarrow c(\alpha) = \sqrt{\alpha}$$

$$\Rightarrow \text{ soln. is } \quad u = \left(\frac{y}{(x+2)^2}\right)^{1/2}(x+2)^2$$

$$\underline{u = \sqrt{y}\,(x+2)}$$

2. $x^2u_x - y^2u_y = 0$, $u(1,y) = F(y)$

$$\frac{dy}{dx} = -\frac{y^2}{x^2} \qquad \frac{du}{dx} = 0 \Rightarrow u = \theta(k)$$

$$\frac{1}{y} = -\frac{1}{x} + 1 + \frac{1}{k} \qquad \text{B.C.} \Rightarrow F(y) = \theta(y)$$

$$\Rightarrow \quad \underline{u = F\left(\frac{xy}{x+y-xy}\right)}$$

3 a) $$x u_x (u-2y^2) = (u-u_y y)(u-y^2-2x^3)$$

b) $$u_x + u^2 u_y = 1$$

b) $$dx = \frac{dy}{u^2} = du$$

$$\Rightarrow \quad u = x + \alpha \qquad \frac{u^3}{3} = y + \beta$$

$$\underline{F\left(u-x, \frac{u^3}{3} - y\right) = 0}$$

a) $$\frac{dx}{x(u-2y^2)} = \underbrace{\frac{dy}{y(u-y^2-2x^3)} = \frac{du}{u(u-y^2-2x^3)}}$$

$$\Longrightarrow \frac{dy}{y} = \frac{du}{u} \quad \Rightarrow \underline{y = \alpha u}$$

$\alpha = \text{const.}$

$$\frac{dx}{x(u-2y^2)} = \frac{dy(u-2y^2)}{y(u-y^2-2x^3)(u-2y^2)} = \frac{du - 2y\,dy}{\cancel{y}(u-y^2-2x^3)(u-2y^2)}$$

$\frac{z}{x}dx - \frac{2x^3}{x}dx = dz \qquad -2x\,dx = \frac{x^2 dz - \cancel{x}z\,dx}{x^2}$ ok

$$u - y^2 = z \Rightarrow \quad \frac{dx}{x} = \frac{dz}{z - 2x^3} \quad \Rightarrow -2x\,dx = d\left(\frac{z}{x}\right)$$

$d(z/x)$

$$\Rightarrow \quad \underline{-x^2 = \frac{z}{x} - \beta} \quad \Rightarrow \quad \beta = \frac{z + x^3}{x}$$

$$F\left(y/u, \underline{\cancel{z}u - y^2 + x^3}\right) = 0$$

(15)

Cesar Levy T63.2131

Find the complete integral of

1. $\sqrt{p}+\sqrt{q}-2x=0$
2. $u-x+\log(pq)=0$
3. $p^2+q^2-px-qy+\frac{1}{2}xy=0$

1. $F_p=\frac{1}{2\sqrt{p}}$ $F_q=\frac{1}{2\sqrt{q}}$ $F_u=0$ $F_x=-2$ $F_y=0$

since $\frac{dq}{dt}=\mathcal{F}(F_u,F_y)=0$ $q=\alpha$

from differential eq $\sqrt{p}=2x-\sqrt{\alpha}$ $p=(2x-\sqrt{\alpha})^2$

$$\therefore\ du=(2x-\sqrt{\alpha})^2dx+\alpha dy$$
$$u=\frac{1}{6}(2x-\sqrt{\alpha})^3+\alpha y+\beta$$

$$\boxed{u=\frac{1}{6}(2x-\sqrt{\alpha})^3+\alpha y+\beta}$$

5

2. $F_p=\frac{1}{p}$ $F_q=\frac{1}{q}$ $F_u=1$ $F_x=-1$ $F_y=0$

$$\frac{dx}{1/p}=\frac{dy}{1/q}=\frac{du}{2}=\frac{dp}{1-p}=\frac{dq}{-q}$$

this leads to $u=-2\ln q+\alpha$ or $\frac{u-\alpha}{-2}=\ln q$; $q=e^{\frac{\alpha-u}{2}}$

from differential equation $\log p=-\log q+x-u=\frac{u-\alpha}{2}+x-u=\frac{2x-\alpha-u}{2}$

or $p = e^{x - \left(\frac{\alpha+u}{2}\right)}$

$$\therefore \quad du = e^{x-\left(\frac{\alpha+u}{2}\right)} dx + e^{\left(\frac{\alpha-u}{2}\right)} dy$$

$$= e^{\frac{\alpha-u}{2}} \left\{ e^{x-\alpha} dx + dy \right\}$$

or $\quad e^{\frac{u-\alpha}{2}} du = e^{x-\alpha} dx + dy$

$$2 e^{\frac{u-\alpha}{2}} = e^{x-\alpha} + y + \beta$$

$$e^{u/2} = \frac{1}{2} e^{\frac{2x-\alpha}{2}} + \frac{1}{2}(y+\beta) e^{\alpha/2}$$

or $\quad u = 2 \ln \left[\frac{1}{2} e^{\frac{2x-\alpha}{2}} + \frac{1}{2}(y+\beta) e^{\alpha/2} \right]$

$$\boxed{u = 2 \ln \left[\frac{1}{2} e^{\alpha/2} \left\{ e^{x-\alpha} + (y+\beta) \right\} \right]} \quad 5$$

3. $F_p = 2p - x \quad F_q = 2q - y \quad F_u = 0 \quad F_x = -p + y/2 \quad F_y = -q + x/2$

$$\frac{dx}{2p-x} = \frac{dy}{2q-y} = \frac{du}{p^2+q^2-\frac{1}{2}xy} = \frac{2dp}{2p-y} = \frac{2dq}{2q-x}$$

one obtains $\quad p+q = \left(\frac{x+y+\alpha}{2}\right)$
using diff eq & substituting for p
gives

$$q = \frac{2y+\alpha}{4} \pm \frac{\sqrt{2(x-y)^2 - \alpha^2}}{4}$$

$$p = \frac{2x+\alpha}{4} \mp \frac{\sqrt{2(x-y)^2 - \alpha^2}}{4}$$

$$\therefore \quad du = \frac{2y+\alpha}{4} dy + \frac{2x+\alpha}{4} dx \mp \frac{1}{4}\sqrt{2R^2 - \alpha^2}\, dR \quad \text{where } R = (x-y)$$

$$\int \frac{1}{\sqrt{2}} \sqrt{2R^2 - \alpha^2}\, \sqrt{2} dR = \left\{ \frac{\sqrt{2} R}{2} \sqrt{2R^2 - \alpha^2} - \frac{\alpha^2}{2} \ln\left(\sqrt{2}R + \sqrt{2R^2 - \alpha^2}\right) \right\} \frac{1}{\sqrt{2}}$$

$$\therefore u = \beta + \frac{y^2+x^2+\alpha(y+x)}{4} \mp \left\{ \frac{(x-y)}{8}\sqrt{2(x-y)^2-\alpha^2} - \frac{\alpha^2}{8\sqrt{2}} \ln\left(\sqrt{2}(x-y)+\sqrt{2(x-y)^2-\alpha^2}\right)\right\}$$

where β & α are integration constants

$$\boxed{u = \frac{y^2+x^2+\alpha(x+y)}{4} \mp \left\{ \frac{(x-y)}{8}\sqrt{2(x-y)^2-\alpha^2} - \frac{\alpha^2}{8\sqrt{2}} \ln\left(\sqrt{2}(x-y)+\sqrt{2(x-y)^2-\alpha^2}\right)\right\} + \beta}$$

5

$\sqrt{p} + \sqrt{q} - 2x = 0$

$F_p = \frac{1}{2\sqrt{p}}$ $\quad F_q = \frac{1}{2\sqrt{q}}$ $\quad F_u = 0$ $\quad F_x = -2$ $\quad F_y = 0$

$\therefore \quad \frac{dx}{dt} = \frac{1}{2\sqrt{p}} \quad \frac{dy}{dt} = \frac{1}{2\sqrt{q}} \quad \frac{du}{dt} = \frac{\sqrt{p}}{2} + \frac{\sqrt{q}}{2}$

$\frac{dp}{dt} = -(-2) \quad \frac{dq}{dt} = 0$

$q = \alpha = Q$ $\quad \frac{dx}{\frac{1}{2\sqrt{p}}} = \frac{dy}{\frac{1}{2\sqrt{q}}} = \frac{du}{\frac{2}{\sqrt{p}+\sqrt{q}}} = \frac{dp}{2} = \frac{dq}{0}$

$\frac{dp}{2} = \frac{dx}{\frac{1}{2\sqrt{p}}}$ $\quad (2x - \sqrt{\alpha})^2 = p = P$

$du = (2x - \sqrt{\alpha})^2 dx + \alpha\, dy$

$\boxed{u = \frac{1}{6}(2x - \sqrt{\alpha})^3 + \alpha y + \beta}$

$\frac{2pdy}{4pq - 2yp} = \frac{2ydp}{2py - y^2}$

$\frac{2pdy + 2ydp}{4pq - y^2}$

$p^2 + q^2 - px - qy + \frac{1}{2}xy = 0$

$F_p = 2p - x \quad F_q = 2q - y \quad F_u = 0 \quad F_x = -p + \frac{1}{2}y$

$F_y = -q + \frac{1}{2}x$

$\frac{dx}{dt} = 2p - x \quad \frac{dy}{dt} = 2q - y \quad \frac{du}{dt} = 2p^2 - px + 2q^2 - qy$

$\frac{dp}{dt} = -(-p + \frac{1}{2}y) \quad \frac{dq}{dt} = -(-q + \frac{1}{2}x)$

$\frac{dx + dy}{2p - x + 2q - y} = \frac{2dp + 2dq}{2p - y + 2q - x}$ $\quad \frac{dx}{2p-x} = \frac{dy}{2q-y} = \frac{du}{(\ )} = \frac{2dp}{2p-y} = \frac{2}{?}$

$dx + dy \quad x + y = 2p + 2q + \alpha$ $\quad \frac{pdy}{2pq - yp} = 2q\,dp$

$\frac{2pdx}{4p^2-2px}$ $\frac{2qdy}{4q^2-2qy}$ $\frac{2xdp}{2px-yy}$ $\frac{\cancel{2}2ydq}{2qy-xy}$ $2p^2-px+2q^2-qy$ $\frac{2pdx+2xdp}{4p^2-xy} = \frac{2qdy+2ydq}{4q^2-yx}$

$2p^2+2q^2-px-qy = p^2+q^2-\frac{1}{2}xy$

$u - x + \log(pq) = 0$

$\frac{2pdx+2xdp-2qdy-2ydq}{4p^2-4q^2} = \frac{dx-2}{2(p-}$

$4(p-q)(p+q)$

$2(p+q)$

$F_p = \frac{q}{pq} = \frac{1}{p}$ $F_q = \frac{p}{pq} = \frac{1}{q}$ $F_u = 1$

$F_x = -1$ $F_y = 0$ $dp = -(F_x + F_u p)$ $dq = -(F_y + F_u q)$

$\frac{2d(px)-2d(py)}{2(p+q)} = dx-2d$

$\frac{dx}{1/p} = \frac{dy}{1/q} = \frac{du}{2} = \frac{dp}{1+p} = \frac{dq}{-q}$ $e^{\frac{\alpha-u}{2}} = q$

$\frac{u}{2} = -\ln q + \alpha$ $\cancel{u=\alpha}$ $\frac{\alpha-u}{2} = \ln q$

$-2\ln q + \alpha - x + \log p + \log q = 0$ $u - x + \log p + \frac{\alpha-u}{2} = 0$

$\log p + \alpha - x - \log q = 0$ $e^{\frac{u+\alpha-2x}{2}} = p = P$

$\frac{du-2dp}{2p}$

$q\,dy = \frac{dq}{q}$ $dy = \frac{dq}{q^2}$ $\log p + \frac{u-2x+\alpha}{2} = 0$

$2pdx + 2xdp - 2qdy - 2ydq = 2pdx + 2qdx - 4pdq - 4qdq$ $y = \frac{1}{q} + c$ $p = e^{\frac{2x-u-\alpha}{2}}$

$2xdp - 2qdx + 4pdq$ $q = e^{\frac{\alpha-u}{2}}$

$y - c = \frac{1}{q}$ $q = \frac{1}{y-c}$

$u - x + \log p + \log\frac{1}{y-c} = 0$

$u - x + \log p/_{y-c} = 0$ $p/_{y-c} = e^{(u-x)}$

$p = (y-\alpha)e^{(u-x)}$ $q = \frac{1}{y-\alpha}$ $du = e^{\frac{2x-u-\alpha}{2}}dx + e^{\frac{\alpha-u}{2}}dy$

$du = (y-\alpha)e^{(u-x)}dx + \frac{1}{y-\alpha}dy$ $du\ e^{\frac{\alpha-u}{2}}\left[e^{x+\alpha}dx + dy\right]$

$du - \frac{1}{y-\alpha}dy = (y-\alpha)e^{(u-x)}dx$ $e^{\frac{u-\alpha}{2}}du = e^{x-\alpha}dx + dy$

$du = e^{\frac{(u+\alpha-2x)}{2}}dx + e^{\left(\frac{\alpha-u}{2}\right)}dy$ $2e^{\frac{u-\alpha}{2}} = e^{x-\alpha} + y + \beta$

$2e^{u/2} = \frac{e^{x-\alpha/2}}{2} + \frac{(y+\beta)}{2}e^{\alpha/2}$

$e^{\left(\frac{\alpha-u}{2}\right)}\left[e^{(u-x)}dx + dy\right]$ $\frac{u}{2} = \log\left(\frac{e^{x-\alpha/2}}{2} + (\ )\right)$

$e^{\frac{(u-\alpha)}{2}}du = e^{(u-x)}dx + dy$ $u = 2\log(e$

$$\frac{dx}{2p-x} = \frac{dy}{2q-y} = \frac{du}{2p^2-px+2q^2-qy} = \frac{2dp}{2p-y} = \frac{2dq}{2q-x}$$

$$\frac{2dp-dy}{2p-y+y-2q} \quad \frac{2dp-dy}{2(p-q)} = \frac{-2dp+dx}{-2q+x+2p-x} \qquad \frac{2pdp+2qdq-du}{2p^2-py+2q^2-qx-2p^2+px-2q^2+qy}$$

$$\frac{-dx+2dq}{-2p+x+2p-y} \quad \frac{-dx+2dq}{x-y} \qquad \frac{2pdp+2qdq-du}{(p-q)(x-y)} = \frac{-dx+2dq}{x-y}$$

$2pdp+2qdq-du=(p-q)dx+2(p-q)dq \quad u = f(x)+g(y) \qquad f \; g$

$2qdq$

$2pdp$

$2dq-dy = dx-2dp \qquad (f')^2+(g')^2-xf'-yg'+\frac{1}{2}xy=0$

$\underline{2(p+q)-(y+x)=\alpha} \qquad f'(f'-x)+g'(g'-y)=-\frac{1}{2}xy$

$p=p(q)$

$2p^2+2q^2-2px-2qy+xy=0 \quad (f')^2g^2+g'^2f^2-xf'g-yg'f^2+\frac{1}{2}xy=0$

$2p^2+2pq-py-px-p\alpha=0$

$\cancel{2q^2-2}$

$f'=x \qquad g'=y$

$\cancel{x^2\frac{y^2}{2}+y^2\frac{x^2}{2}-x^2y-y^2x+\frac{1}{2}xy}$

$py+p\alpha-2pq+2q^2-px-2qy+xy=0$

$2q(q-p-y)-p(x-y-\alpha)+xy$

$\cancel{f=\frac{x^2}{2}+c} \qquad \cancel{g=\frac{y^2}{2}+k}$

$\cancel{x^2\left(\frac{y^2}{2}+k\right)^2+y^2\left(\frac{x^2}{2}+c\right)^2}$

$f'=1 \qquad g'=1$

$f=x+c \qquad g=y+k$

$(y+k)^2+(x+c)^2-y(x+c)+\frac{1}{2}xy=0$

$-x(y+k)$

$\frac{x+y-\alpha}{2} - p = q$

$\left(\frac{x+y-\alpha}{2}\right)^2 = (p+q)^2$

$(\quad)^2 = p^2+q^2+2pq$

$px+qy+2pq-\frac{1}{2}xy$

$p(x+2q)+qy+\frac{1}{2}xy$

$$\frac{1-\left(\frac{x+y-\alpha}{2}\right)^2+qy+\frac{1}{2}xy}{x+2q} = p$$

$$\frac{x+y-\alpha}{2} + \frac{qy+\frac{1}{2}xy-\left(\frac{x+y-\alpha}{2}\right)^2}{x+2q} = q$$

$a=\frac{x+y-\alpha}{2}$ $\quad a(x+2q)+q(x-y)+\frac{1}{2}xy-a^2=2q^2$ $\quad a^2=\frac{x^2+2xy-2x\alpha+y^2-2y\alpha+\alpha^2}{4}$

$2a=x+y-\alpha$ $\quad ax+q(x-y+2a)+\frac{1}{2}xy-a^2=2q^2$

$2q^2-ax-q(2x-\alpha)-\frac{1}{2}xy+a^2=0$

$\frac{1}{2}xy+ax=\frac{x^2+xy-\alpha x}{2}+\frac{1}{2}xy = \frac{x^2+2xy-\alpha x}{2}=\frac{x(x+2y-\alpha)}{2}=b$

$2q^2-b+a^2-q(2x-\alpha)=0$

$-b+a^2 = \frac{-2x^2-4xy+2x\alpha}{4} + \frac{x^2+2xy-2x\alpha+y^2-2y\alpha+\alpha^2}{4}$

$\frac{-x^2-2xy+y^2-2y\alpha+\alpha^2}{4}$ $\quad -\frac{1}{4}(x^2+2xy-y^2+2y\alpha-\alpha^2)$

$(x+y\quad)(x+y$

$(y-\alpha+x)(y-\alpha-x)$ $\quad -(y-\alpha+x)(y-\alpha+x)$

$(2x-\alpha)\pm\sqrt{(2x-\alpha)^2+2[x^2+2xy-y^2+2y\alpha-\alpha^2]}$

$4x^2-4x\alpha+\alpha^2+2x^2+4xy-2y^2+4y\alpha-2\alpha^2$

$6x^2-4x\alpha+4xy+4y\alpha-2y^2-\alpha^2$

$\cancel{(3x-\alpha-y)(2x+\alpha+y)}$

#3. Find complete integrals.

1. $$p^2+q^2-px-qy+\tfrac{1}{2}xy=0 \qquad (1)$$

$F_p = 2p-x$, $F_q = 2q-y$, $F_u = 0$, $F_x = -p+y/2$, $F_y = -q+x/2$

$$\Rightarrow \quad \frac{dx}{2p-x} = \frac{dy}{2q-y} = \frac{2\,dp}{2p-y} = \frac{2\,dq}{2q-x} = \frac{du}{p^2+q^2-\frac{1}{2}xy} \leftarrow \text{using (1)}$$

Since first four denominators linear in p, q, x, y, take linear combination of the four. Coefficients can be chosen to yield:

$$2(dp+dq)-(dx+dy)=0$$

i.e $$p+q=\tfrac{1}{2}(x+y+\alpha) \qquad (\alpha = \text{constant.}) \qquad (2)$$

(1) & (2) imply

$$p = \frac{1}{4}\left(2x+\alpha \mp \sqrt{2(x-y)^2-\alpha^2}\right) = P(x,y)$$

$$q = \frac{1}{4}\left(2y+\alpha \pm \sqrt{2(x-y)^2-\alpha^2}\right) = Q(x,y)$$

Then $$du = P\,dx + Q\,dy$$

$$\Rightarrow \quad du = \frac{1}{4}\left\{(2y+\alpha)\,dy + (2x+\alpha)\,dx \mp \sqrt{2(x-y)^2-\alpha^2}\;d(x-y)\right\}$$

2.

$$\Longrightarrow$$

$$u = \beta + \frac{1}{4}\left\{x^2+y^2+\alpha(x+y) \mp \frac{(x-y)}{2}\sqrt{2(x-y)^2-\alpha^2} \pm \frac{\alpha^2}{2\sqrt{2}} \ln\left(\sqrt{2}(x-y) + \sqrt{2(x-y)^2-\alpha^2}\right)\right\}$$

α & β are the constants of integration.

2. $$\sqrt{p} + \sqrt{q} - 2x = 0$$

$$F_p = \frac{1}{2\sqrt{p}}, \quad F_q = \frac{1}{2\sqrt{q}}, \quad F_u = 0, \quad F_x = -2, \quad F_y = 0$$

$$2\sqrt{p}\,dx = 2\sqrt{q}\,dy = \frac{2\,du}{\sqrt{p}+\sqrt{q}} = \frac{dp}{2} = \frac{dq}{0}$$

Last $\Rightarrow$ $\quad q = \alpha = \text{const} = Q(x,y)$

Eqn $\Rightarrow$ $\quad \sqrt{p} = 2x - \sqrt{\alpha} \Rightarrow p = (2x-\sqrt{\alpha})^2 = P(x,y)$

$$\Rightarrow du = P\,dx + Q\,dy = (2x-\sqrt{\alpha})^2 dx + \alpha\,dy$$

$$\Rightarrow u = \frac{1}{6}(2x-\sqrt{\alpha})^3 + \alpha y + \beta$$

3.

$$u - x + \log p + \log q = 0 \qquad (I)$$

$$F_p = 1/p, \quad F_q = 1/q, \quad F_u = 1, \quad F_x = -1, \quad F_y = 0$$

$$p\,dx = q\,dy = \frac{du}{2} = \frac{dp}{1-p} = \frac{dq}{-q}$$

Many choices for first integral – use du & dq since soln. contains ~~&~~ $\ln(q)$ & no p – $\ln q$ can be eliminated from (I)

$$\ln q = -\tfrac{1}{2}(u-\alpha)$$

$$\text{(I)} \Rightarrow \qquad q = e^{\frac{1}{2}(\alpha-u)} \quad \& \quad p = e^{x-\frac{1}{2}(u+\alpha)}$$

$$du = e^{x-\frac{1}{2}(u+\alpha)}\,dx + e^{\frac{1}{2}(\alpha-u)}\,dy$$

$$2e^{u/2} = e^{x-\alpha/2} + ye^{\alpha/2} + 2\beta$$

$$\underline{u = 2\ln\left\{\tfrac{1}{2}\left(e^{x-\alpha/2} + ye^{\alpha/2}\right) + \beta\right\}}$$

$1\frac{8}{20}$

Coheny T63.2131

Notation: $\begin{pmatrix} u \\ v \end{pmatrix}_{,x} = \underset{\sim}{u}_{,x} \qquad \begin{pmatrix} u \\ v \end{pmatrix}_{,t} = \underset{\sim}{u}_{,t}$

1. $$\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \underset{\sim}{u}_{,x} + \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \underset{\sim}{u}_{,t} = \begin{pmatrix} u-v \\ 2v \end{pmatrix}$$

$$\det|A-\mu B| = -5(\mu^2 - 3\mu + 1) = 0 \qquad \mu^{(1)} = 2.618 \qquad \mu^{(2)} = .382 \qquad \mu^{(1)} \neq \mu^{(2)}$$

Since eigenvalues are real system is totally hyperbolic ✓

for $\mu^{(1)}$ the left $\vec{EV} = (1.6175 \ \ -1)$ for $\mu^{(2)}$ the left $\vec{EV} = (1 \ \ 1.6175)$

$\beta_1^{(1)}, \beta_2^{(1)}$ for $\mu^{(1)}$ are $(2.235 \ \ 5.8525)$

$\beta_1^{(2)}, \beta_2^{(2)}$ for $\mu^{(2)}$ are $(3.6175 \ \ 1.3825)$

$\hat{u}^{(1)} = \beta_1^{(1)} u + \beta_2^{(1)} v = 2.235u + 5.8525v \qquad D_1 = 2.618\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$ 5

$\hat{c}^{(1)} = \lambda_1^{(1)} c_1 + \lambda_2^{(1)} c_2 = 1.6175u - 3.6175v$

$\hat{u}^{(2)} = \beta_1^{(2)} u + \beta_2^{(2)} v = 3.6175u + 1.3825v \qquad D_2 = .382\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$

$\hat{c}^{(2)} = \lambda_1^{(2)} c_1 + \lambda_2^{(2)} c_2 = u + 2.235v$

Characteristics: $\frac{dt}{dx} = \frac{1}{2.618} \Rightarrow t - \frac{x}{2.618} = \alpha \qquad \frac{dt}{dx} = \frac{1}{.382} \Rightarrow t - \frac{x}{.382} = \beta$

2. $$\begin{pmatrix} \rho & u \\ \rho u & c^2 \end{pmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ \rho & 0 \end{pmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}_t = \begin{pmatrix} -\frac{2\rho u}{x} \\ 0 \end{pmatrix}$$

$$\det|A-\mu B| = \rho[c^2 - (u-\mu)^2] = 0 \qquad \mu^{(1)} = u-c \qquad \mu^{(2)} = u+c \qquad \mu^{(1)} \neq \mu^{(2)}$$

Since eigenvalues are real system is totally hyperbolic ✓

$\ell = (\pm c, 1)$?

?

for $\mu^{(1)}$ the left $\vec{EV} = (c \; -\rho)$ for $\mu^{(2)}$ the left $\vec{EV} = (c \; \rho)$

$\beta_1^{(1)}, \beta_2^{(1)}$ for $\mu^{(1)}$ are $(-\rho^2 \; c)$

$\beta_1^{(2)}, \beta_2^{(2)}$ for $\mu^{(2)}$ are $(\rho^2 \; c)$

3

$\hat{u}^{(1)} = \beta_1^{(1)} u + \beta_2^{(1)} \rho = \rho(c - \rho u)$ $\quad D_1 = (u-c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$

$\hat{c}^{(1)} = \lambda_1^{(1)} c_1 + \lambda_2^{(1)} c_2 + (D_1\beta_1^{(1)})u + (D_1\beta_2^{(1)})\rho = -\frac{2\rho u c}{x} + \left\{\left[(u-c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right](-\rho^2)\right\}u$ if c is a const.

$\hat{u}^{(2)} = \beta_1^{(2)} u + \beta_2^{(2)} \rho = \rho(c + \rho u)$ $\quad D_2 = (u+c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$

$\hat{c}^{(2)} = \lambda_1^{(2)} c_1 + \lambda_2^{(2)} c_2 + (D_2\beta_1^{(2)})u + (D_2\beta_2^{(2)})\rho = -\frac{2\rho u c}{x} + \left\{\left[(u+c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]\rho^2\right\}u$ if c is a const

if c is not const add $\left\{\left[(u \mp c)\frac{\partial}{\partial t} + \frac{\partial}{\partial t}\right]c\right\}\rho$ to $\hat{c}^{(1)}$ & $\hat{c}^{(2)}$ respectively

Characteristic $\frac{dt}{dx} = \frac{1}{u+c}$; $\frac{dt}{dx} = \frac{1}{u-c}$

3. $\begin{pmatrix} x+y & 0 \\ 0 & x-y \end{pmatrix} \underset{\sim}{u}_{,x} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\sim}{u}_{,y} = \underline{0}$

$\det|A - \mu B| = (x+y)(x-y) - \mu^2 = 0 \qquad \mu^{(1),(2)} = \pm\sqrt{x^2-y^2}$ totally hyperbolic for $x^2 > y^2$

elliptic if $x^2 < y^2$

for $\mu^{(1)}$ the left $\vec{EV} = (x-y \;\; \sqrt{x^2-y^2})$ for $\mu^{(2)}$ the left $\vec{EV} = (x-y \;\; -\sqrt{x^2-y^2})$

$\beta_1^{(1)}, \beta_2^{(1)}$ for $\mu^{(1)}$ are $(\sqrt{x^2-y^2} \;\; x-y)$

$\beta_1^{(2)}, \beta_2^{(2)}$ for $\mu^{(2)}$ are $(-\sqrt{x^2-y^2} \;\; x-y)$

Let $\sqrt{} = \sqrt{x^2-y^2}$

5

$\hat{u}^{(1)} = \beta_1^{(1)} u + \beta_2^{(1)} v = \sqrt{x^2-y^2}\, u + (x-y) v$ $\quad D_1 = \sqrt{x^2-y^2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

$\hat{c}^{(1)} = (D_1\beta_1^{(1)})u + (D_1\beta_2^{(1)})v = \left\{\left[\sqrt{}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]\sqrt{}\right\}u + \left\{\left[\sqrt{}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right](x-y)\right\}v$

$\hat{u}^{(2)} = \beta_1^{(2)} u + \beta_2^{(2)} v = -\sqrt{x^2-y^2}\, u + (x-y) v$ $\quad D_2 = -\sqrt{x^2-y^2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

$\hat{c}^{(2)} = (D_2\beta_1^{(2)})u + (D_2\beta_2^{(2)})v = \left\{\left[-\sqrt{}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right](-\sqrt{})\right\}u + \left\{\left[-\sqrt{}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right](x-y)\right\}v$

the characteristics are $\frac{dy}{dx} = \pm \frac{1}{\sqrt{}}$

4. $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \underset{\sim}{u}_x + \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \underset{\sim}{u}_y = \begin{pmatrix} -u \\ 1-u^2 \end{pmatrix}$

det $|A - \mu B| = (1-\mu)\mu - 2\mu = 0$ $\quad \mu^{(1),(2)} = 0, -1$ Since EV are real system is totally hyperbolic

for $\mu^{(1)}$ the left $\bar{EV} = (0, 1)$ $\quad$ for $\mu^{(2)}$ the left $\bar{EV} = (1 \ -1)$

for $\mu^{(1)}$ $\beta_1^{(1)}, \beta_2^{(1)}$ are $(2 \ -1)$

$\mu^{(2)}$ $\beta_1^{(2)}, \beta_2^{(2)}$ are $(-1 \ 1)$

$\hat{u}^{(1)} = 2u - v \qquad D_1 = \frac{\partial}{\partial y} \qquad \hat{c}^{(1)} = 1 - u^2$ since $D_1 \beta_i^{(1)} = 0$

$\hat{u}^{(2)} = -u + v \qquad D_2 = -\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \qquad \hat{c}^{(2)} = (u^2 - u - 1)$

5

$$c u_x - c v_t + u_t - c^2 v_x = 0$$

$$u_t + c u_x - c(v_t + c v_x) = 0$$

$$-c u_x + c v_t + u_t - c^2 v_x = 0$$

$$u_t - c u_x + c(v_t - c v_x) = 0$$

$$\frac{d}{ds} = \frac{\partial}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial}{\partial t}\frac{\partial t}{\partial s}$$

$$\frac{\partial t}{\partial s} = 1 \quad \frac{\partial x}{\partial s} = m \quad \frac{dt}{dx} =$$

$x = ms + f(t)$

$$d\cancel{s} = \frac{\partial x}{\partial s}\frac{\partial t}{\cancel{\partial x}} + \partial x \qquad \frac{\partial t}{\partial x} = \frac{1}{m}$$

$t = \frac{x}{m} + a$

$\cancel{t = mx} + a$

$$\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_{,x} + \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_{,t} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u-v \\ 2v \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$|A - \mu B| = \begin{vmatrix} 1-2\mu & 1-3\mu \\ 3-\mu & \mu-2 \end{vmatrix} = \mu - 2\mu^2 - 2 + 4\mu - 3 + 9\mu + \mu - 3\mu^2$$

$$= -5(\mu^2 - 3\mu + 1) = 0 \qquad \mu = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\mu = \frac{3 \pm \sqrt{5}}{2} = \frac{3 \pm 2.236}{2} = 2.618,\ .382$$

$\mu_1 = 2.618$ $\qquad \begin{pmatrix} -4.236 & -6.854 \\ .382 & .618 \end{pmatrix}$ $\qquad .382\overline{\smash{)}.618}$ $= 1.6175$, 382

$\overline{EV}_1 = (1.6175 \quad -1)$

$\mu_2 = .382$ $\qquad \begin{pmatrix} .236 & -.146 \\ 2.618 & -1.618 \end{pmatrix}$ $\qquad .146\overline{\smash{)}.236}$ $= 1.6175$, $.146$

$\overline{EV}_2 = (1 \quad 1.6175)$

$$(1.6175 \quad -1)\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} = (2.235 \quad 5.8585) = \beta_1^{(1)}, \beta_2^{(1)} \text{ for } \mu_1$$

$$(1 \quad 1.6175)\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} = (3.6175 \quad 1.3825) = \beta_1^{(2)}, \beta_2^{(2)} \text{ for } \mu_2$$

$$\hat{u}^{(1)} = \beta_1^{(1)}u + \beta_2^{(1)}v = 2.235u + 5.8525v \qquad D_1 = 2.618\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

$$\hat{c} = \lambda_1^{(1)}c_1 + \lambda_2^{(1)}c_2 + (D\beta_1)u + (D\beta_2)v = 1.6175(u-v) - 2v + \left[\left(2.618\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)2.235\right]u + \left[\left(2.618\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)2.2\ldots\right.$$

since β is not a fn. of x or t

$$\therefore \quad \hat{c}_1 = 1.6175(u-v) - 2v \qquad D_1 = 2.618\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \qquad \hat{u}^{(1)} = 2.235u + 5.8525v$$

$$= 1.6175u - 3.6175v$$

$$\hat{u}^{(2)} = \beta_1^{(2)} u + \beta_2^{(2)} v = 3.6175u + 1.3825 v$$

$$D_2 = .382 \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

$$\hat{c}_2 = \lambda_1^{(2)} c_1 + \lambda_2^{(2)} c_2 = (u-v) + 3.235 v = u + 2.235 v$$

$$\begin{pmatrix} \rho & u \\ \rho u & c^2 \end{pmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ \rho & 0 \end{pmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}_t = \begin{pmatrix} -2\rho/x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}$$

$$\begin{vmatrix} \rho & u-\mu \\ \rho u - \rho\mu & c^2 \end{vmatrix} = 0 \qquad \rho c^2 - (u-\mu)^2 \rho = 0 \qquad (u-\mu)^2 = c^2$$

$\mu = u - c \quad \mu = u + c$

totally hyperbolic

$\mu = u - c$

$$\begin{pmatrix} \rho & c \\ \rho c & c^2 \end{pmatrix} \Rightarrow \begin{pmatrix} \rho & c \\ 0 & 0 \end{pmatrix}$$

$(c \quad -\rho)$ should be $(c \pm 1)$ dual case

$\mu = u + c$

$$\begin{pmatrix} \rho & -c \\ -\rho c & c^2 \end{pmatrix}$$

$(c \quad \rho)$

$$(c \quad -\rho) \begin{pmatrix} 0 & 1 \\ \rho & 0 \end{pmatrix} = (-\rho^2 \quad c) = \beta_1^{(1)} \quad \beta_2^{(1)} \text{ for } u-c$$

$$(c \quad \rho) \begin{pmatrix} 0 & 1 \\ \rho & 0 \end{pmatrix} = (\rho^2 \quad c) = \beta_1^{(2)} \quad \beta_2^{(2)} \text{ for } u+c$$

$$\hat{u}_1^{(1)} = \beta_1^{(1)} u + \beta_2^{(1)} \rho = -\rho^2 u + c\rho \qquad D_1 = (u-c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

$$\hat{c}_1^{(1)} = \lambda_1^{(1)} c_1 + \lambda_2^{(1)} c_2 + (D\beta_1) u + (D\beta_2)\rho = c\left(-\frac{2\rho u}{x}\right) + \left\{\left[(u-c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right](-\rho^2)\right\} u + \left\{\left[(u-c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right] c\right\} \rho$$

$$\hat{u}^{(2)} = \beta_1^{(2)} u + \beta_2^{(2)} \rho = \rho^2 u + c\rho \qquad D_2 = (u+c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

$$\hat{c}^{(2)} = c\left(-\frac{2\rho u}{x}\right) + \left\{\left[(u+c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]\rho^2\right\} u + \left\{\left[(u+c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right] c\right\} \rho$$

$$\begin{pmatrix} x+y & 0 \\ 0 & x-y \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_{,x} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_{,y} = \underline{0}$$

$$|A-\mu B| = \begin{pmatrix} x+y & -\mu \\ -\mu & x-y \end{pmatrix} \qquad (x+y)(x-y)-\mu^2=0 \qquad \mu = \pm\sqrt{x^2-y^2}$$

totally hypob if $x^2 > y^2$
Elliptic if $y^2 > x^2$

$$\mu_1 = +\sqrt{x^2-y^2} \quad (x-y \ \ \sqrt{\ \ }) \begin{pmatrix} x+y & -\sqrt{\ \ } \\ -\sqrt{\ \ } & x-y \end{pmatrix} = (0 \ \ 0)$$

$$\overline{EV}_1 = \left(x-y \quad \sqrt{x^2-y^2}\right)$$

$$\mu_2 = -\sqrt{x^2-y^2} \qquad \begin{pmatrix} x+y & \sqrt{\ \ } \\ \sqrt{\ \ } & x-y \end{pmatrix} \qquad \overline{EV}_2 = \left(x-y \quad -\sqrt{x^2-y^2}\right)$$

$$\mu_1 \quad (x-y \ \ \sqrt{\ \ })\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\sqrt{\ \ } \quad x-y) \qquad \beta_1^{(1)}, \beta_2^{(1)}$$

$$\mu_2 \quad (x-y \ \ -\sqrt{\ \ })\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (-\sqrt{\ \ } \quad x-y) \qquad \beta_1^{(2)} \quad \beta_2^{(2)}$$

$$\hat{u}^{(1)} = \beta_1^{(1)} u + \beta_2^{(1)} v = \sqrt{x^2-y^2}\, u + (x-y) v \qquad D_1 = \sqrt{x^2-y^2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

$$\hat{u}^{(2)} = \beta_1^{(2)} u + \beta_2^{(2)} v = -\sqrt{x^2-y^2}\, u + (x-y) v \qquad D_2 = -\sqrt{x^2-y^2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

$$\hat{c}^{(1)} = \cancel{\lambda_1^{(1)} c_1} + \cancel{\lambda_2^{(1)} c_2} + (D^{(1)}\beta_1)u + (D\beta_2)v = \left\{\left[\sqrt{x^2-y^2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]\sqrt{x^2-y^2}\right\}u + \left\{\left[\sqrt{\ \ }\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right](x-y)\right\}v$$

$$\frac{1}{2\sqrt{\ }}2x \qquad \frac{1}{2\sqrt{\ }}\cdot -2y$$

$$\hat{c}^{(2)} = (D^{(2)}\beta_1)u + (D^{(2)}\beta_2)u = \left\{\left[-\sqrt{\ \ }\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right] - \sqrt{\ \ }\right\}u^{x} \ \frac{-y}{\sqrt{\ }}u + \left\{\left[-\sqrt{\ \ }\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right](x-y)\right\}v$$

$$1 \qquad -1$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_{x} + \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_{,y} = \begin{pmatrix} -u \\ 1-u^2 \end{pmatrix}$$

$$|A-\mu B| = \begin{pmatrix} 1-\mu & -1 \\ -2\mu & +\mu \end{pmatrix} = (1-\mu)\mu - 2\mu = 0 \qquad -\mu^2-\mu=0 \qquad \mu=0 \text{ or } \mu=-1$$

$$\mu = 0 \quad \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \qquad (0 \quad 1) = \overline{EV}_1$$

$$\mu = -1 \quad \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \qquad (1 \quad -1) = \overline{EV}_2$$

$$\mu = 0 \quad (0 \quad 1) \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \qquad (2 \quad -1) \qquad \beta_1^{(1)} \; \beta_2^{(1)}$$

$$\mu = -1 \quad (1 \quad -1) \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \qquad (-1 \quad 1) \qquad \beta_1^{(2)} \; \beta_2^{(2)}$$

$$\hat{u}^{(1)} = \beta_1^{(1)} u + \beta_2^{(1)} v = 2u - v \qquad D_1 = \frac{\partial}{\partial y}$$

$$\hat{u}^{(2)} = \beta_1^{(2)} u + \beta_2^{(2)} v = -u + v \qquad D_2 = -\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

$$\hat{c}^{(1)} = \left(D_1 \beta_1^{(1)}\right) u + \left(D_1 \beta_1^{(1)} v\right) + \lambda_1^{(1)} c_1 + \lambda_2^{(1)} c_2 = (1 - u^2)$$

$$\hat{c}^{(2)} = \text{---} \qquad \text{---} \qquad + \lambda_1^{(2)} c_1 + \lambda_2^{(2)} c_2 = -u + \overset{-1+u^2}{\cancel{1 - u^2}} = (-1 - u + u^2)$$

$$-u - 1 + u^2$$

Cesar Levy T63.2131

H.W. Look at $W_{tt} = c^2(W_x)\, W_{xx}$

Change variables to put it into a system of 2 first order p.d.e.
Find characteristics & Riemann Invariants. Suggest method if $c = c(W_x, \epsilon x)$

Let $W_t = u \quad W_x = v \quad$ then $W_{tx} = W_{xt}$ or $u_x - v_t = 0 \quad (1)$

$W_{tt} - c^2(W_x)\, W_{xx} = 0 \quad \Longrightarrow \quad u_t - c^2(v)\, v_x = 0$

then $$\begin{pmatrix} 1 & 0 \\ 0 & -c^2(v) \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

to find EV
$$\det|A - \mu B| = 0 \qquad \det\begin{vmatrix} 1 & \mu \\ -\mu & -c^2(v) \end{vmatrix} = 0 \quad \text{or} \quad \mu = \pm c(v)$$

the $\vec{EV}$ are

for $\mu_1 = +c(v)$ on C_+ $\quad (c(v) \quad 1)\begin{pmatrix} 1 & c(v) \\ -c(v) & -c^2(v) \end{pmatrix} = (0 \quad 0) \qquad (c(v) \quad 1) \qquad \beta_1^{(1)} = 1,\ \beta_2^{(1)} = -c(v)$

for $\mu_2 = -c(v)$ on C_- $\quad (-c(v) \quad +1)\begin{pmatrix} 1 & -c(v) \\ c(v) & -c^2(v) \end{pmatrix} = (0 \quad 0) \qquad (-c(v) \quad +1) \qquad \beta_1^{(2)} = +1,\ \beta_2^{(2)} = +c($

$\therefore \quad D_1 = c(v)\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial t} \qquad \hat{u}^{(1)} = u - cv \qquad$ since $c \neq c(x,t)$ then $\hat{c}^{(1)} = 0$

$D_2 = -c(v)\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial t} \qquad \hat{u}^{(2)} = +(u + cv) \qquad$ since $c \neq c(x,t)$ then $\hat{c}^{(2)} = 0$

or: $\quad D_1 u - c\, D_1 v = 0 \quad$ on C_+
$\qquad D_2 u + c\, D_2 v = 0 \quad C_-$

since $c = c(v)$ let $L(v) = \int_{v_0}^{v} c(z)\,dz$ where $v_0 = $ const $\therefore DL(v) = c(v)Dv$

$\therefore$ on $C_\pm$ $\left.\begin{array}{l} D_1[u - L(v)] = 0 \\ D_2[u + L(v)] = 0 \end{array}\right\}$ on C^+ let the const be r
on C^- let the const be s

$\therefore \quad u - L(v) = r \quad$ on C^+
$\qquad u + L(v) = s \quad$ on C^-

In the case where $c = c(W_x, \epsilon x)$ we do the same as before i.e. $W_t = u \quad W_x = v$

then $\quad u_x - v_t = 0$

& $\quad u_t - c^2(v, \xi)\, v_x = 0 \qquad$ where $\xi = \epsilon x$

again we write in matrix notation $A\underset{\sim}{U}_x + B\underset{\sim}{U}_t = 0$
and solve. Everything remains the same
i.e.

$$D_1 = c(v,\xi)\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \qquad \hat{u}^{(1)} = u - cv \qquad \text{but} \quad \hat{c}^{(1)} = D_1[-c(v,\xi)]\,v$$

$$D_2 = -c(v,\xi)\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \qquad \hat{u}^{(2)} = u + cv \qquad \hat{c}^{(2)} = D_2[+c(v,\xi)]\,v$$

we can also write this as

$$D_1 u - c D_1 v = 0$$

$$D_2 u + c D_2 v = 0$$

now since $c = c(V,\xi)$ write the Taylor expansion about $\xi = 0$

$\therefore$ $c(V,\xi) = c(V,0) + \frac{\partial c}{\partial \xi}(V,0)\xi + O(\xi^2)$ terms; since ξ is slowly varying

neglect the terms of order ξ^2 or higher. Note that $\frac{d\xi}{dx} = \epsilon$ a small number.

now $c(V,0)$ & $\frac{\partial c}{\partial \xi}(V,0)$ are fns of V only

$\therefore$ let $L_0(V) = \int_{V_0}^{V} c(z,0)\,dz$ & $L_1(V) = \int_{V_0}^{V} \frac{\partial c}{\partial \xi}(z,0)\,dz$

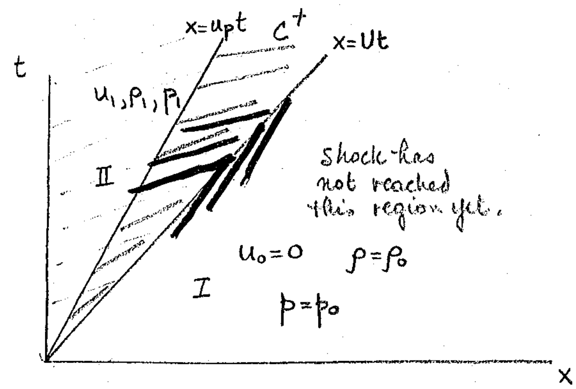
then $D\{L_0 + \xi L_1\} \simeq c(V,\xi)\,DV$ for slowly varying ξ

$\therefore$ $u + [L_0 + \xi L_1](V) = s$ on C^-

$u - [L_0 + \xi L_1](V) = r$ on C^+

C. LEVY 763.2131

Given: Ahead of shock $u=0$ $\rho=\rho_0$ $p=p_0$. The piston starts instantaneously from rest. $u_p>0$ $u_p=\text{const}$. Find U and the rest of unknowns behind shock and draw (t,x) diagram explaining solution.



in region 2 the velocity $u_1=u_p<U$. This is due to the fact that the shock has passed this region and the velocity there is subsonic. By subsonic theory the data at a pt depends on the data at the piston i.e. $u_1=u_p$.

from $\quad (1-\mu^2)(U-u_0)^2-(u_1-u_0)(U-u_0)=(1-\mu^2)c_0^2$

we find $\quad U=u_p\left\{\frac{1}{2(1-\mu^2)}+\sqrt{\left[\frac{1}{2(1-\mu^2)}\right]^2+\left(\frac{c_0}{u_p}\right)^2}\right\}$

Using $\rho_0(u_0-U)=\rho_1(u_1-U)$

we find $\quad \rho_1=\frac{\rho_0}{1-u_p/U}$ $\quad$ where $\quad U/u_p=\frac{1}{2(1-\mu^2)}\left\{1+\sqrt{1+\left\{\frac{2c_0(1-\mu^2)}{u_p}\right\}^2}\right\}$

using $m u_0+p_0=m u_1+p_1$

we find $\quad p_1=p_0-m u_1=p_0+\rho_0 u_p U=p_0+\frac{\rho_0 u_p^2}{2(1-\mu^2)}\left\{1+\sqrt{1+\left(\frac{2c_0(1-\mu^2)}{u_p}\right)^2}\right\}$

note. I, II are simplewave regions since u,ρ,p are constant.

1. When $u_p=u_p(t)$ we had shown previously that the characteristics will intersect only when the piston accelerates in positive direction i.e. when $u_p'(t)>0$

ⓐ If $u_p'>0\ \forall t$ then we would have compression waves and shock analysis would go through in same manner, i.e. where $c_0, u_p, U, u_0, u_1, p_1, p_0, \rho_1, \rho_0$ were written as constants they now become time dependent.

ⓑ If $u_p'<0\ \forall t$ then we have rarefaction waves and no shock will form.

③ If $u_p=u_p(t)$ is such that $u_p'(t)>0\ \forall\ t<T_1$ and $u_p'(t)<0\ \forall t>T_1$

then we can use the results of (a) to find out where shock occurred and all pertinent data.

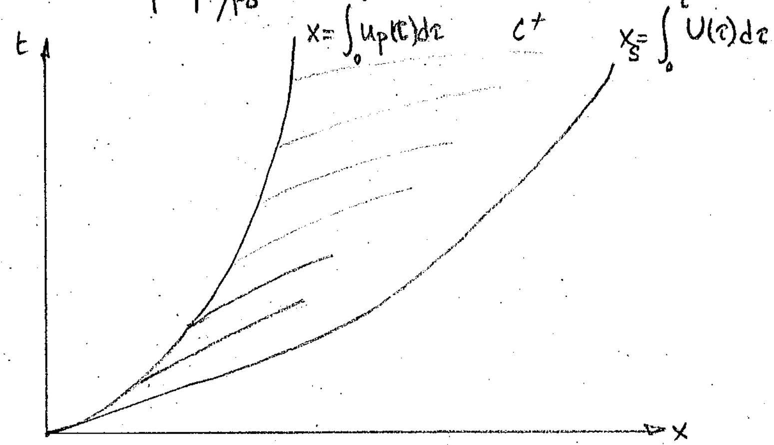
If $u_p'(t) < 0 \quad \forall\, t < T_1$ and $u_p'(t) > 0 \quad \forall\, t > T_1$ then @ $t = T_1^+$

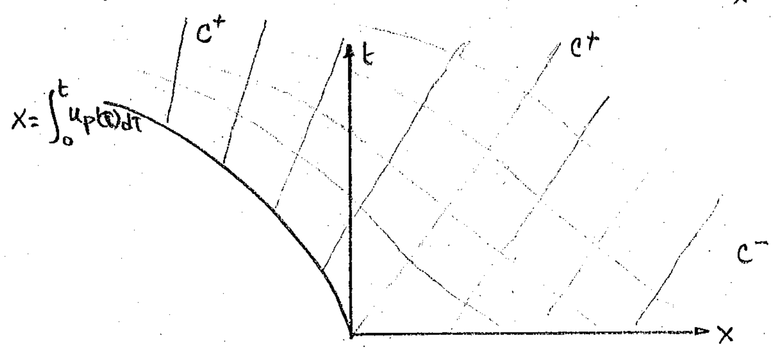
where $u_p'(t) = 0^+$ is when first characteristic of converging nature will meet last characteristic of diverging nature, i.e. a rarefaction wave will intersect a compression wave.

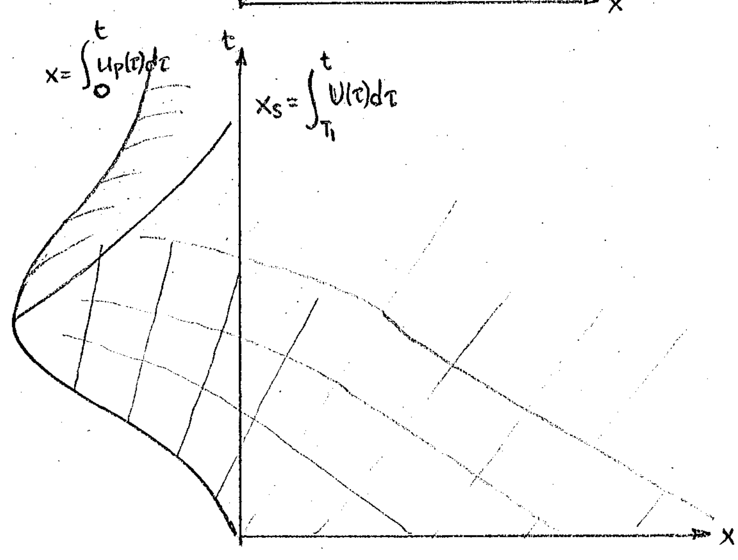
Behind the shock all the values will be a fn of time and position (can be param wrt t)

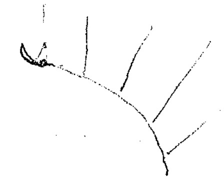
Before " " " " " " " " " " " " "

The shock strength can be defined by the pressure change due to shock formation namely p_1/p_0





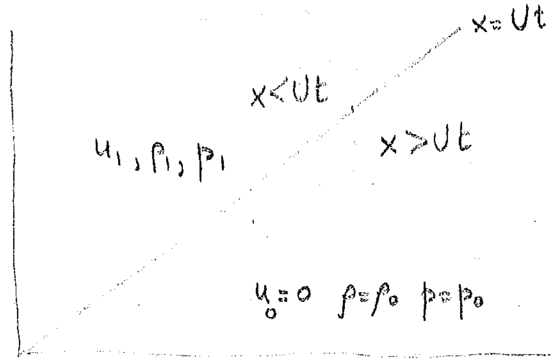




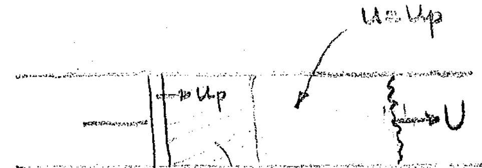
Ahead of the shock $u=0$ $\rho=\rho_0$ $p=p_0$

Piston starts instantaneously from rest $u_p>0$ $u_p=\text{const}$

Find U & rest of unknowns behind shock & draw (t,x) diag explaining sol. look at case $u_p=u_p(t)$



make assumption that $u_1=u_p$



if $u>u_p$ but $u<U$ then $\exists$ a time t_∞ such that $\rho=0 \Rightarrow$ rarefaction waves this is impossible. if $u<u_p$ that implies that piston will catch up the slower moving gas.

$$(U-u_0)^2 = c_0^2 + \frac{(u_1-u_0)^2}{1-\mu^2}$$

$$U = u_0 \pm \sqrt{c_0^2 + \frac{(u_1-u_0)^2}{1-\mu^2}} \qquad u_0=0 \quad \therefore \quad U = \sqrt{c_0^2 + \frac{u_p^2}{1-\mu^2}}$$

$$\mu^2 U^2 + (1-\mu^2)c_0^2 = \cancel{\mu^2(u_1-U)^2 +}$$

$$\mu^2 U^2 + (1-\mu^2)c_0^2 = -U(u_1-U) = +U^2 - Uu_1$$

$$U^2(\mu^2-1) + (1-\mu^2)c_0^2 + Uu_1 = 0$$

$$U^2 - \frac{u_1}{1-\mu^2}U \quad (1-\mu^2)c_0^2 =$$

$$U = \frac{u_p}{2(1-\mu^2)} + \sqrt{\frac{u_1^2}{4(1-\mu^2)^2} + c_0^2}$$

$$\mu^2(u_0-U)^2 + (1-\mu^2)c_0^2 = \mu^2(u_1-U)^2 + (1-\mu^2)c_1^2 = (u_0-U)(u_1-U)$$

$$(1-\mu^2)(U-u_0)^2 - (1-\mu^2)c_0^2 - (U-u_0)(u_1-U) = \cancel{\otimes}\ (U-u_0)^2$$

$$(1-\mu^2)(U-u_0)^2 = (1-\mu^2)c_0^2 + (U-u_0)\underbrace{\left[(U-u_0)+(u_1-U)\right]}_{(u_1-u_0)(U-u_0)}$$

$$\rho_0(\cancel{u_0}-U) = \rho_1(u_1-U)$$

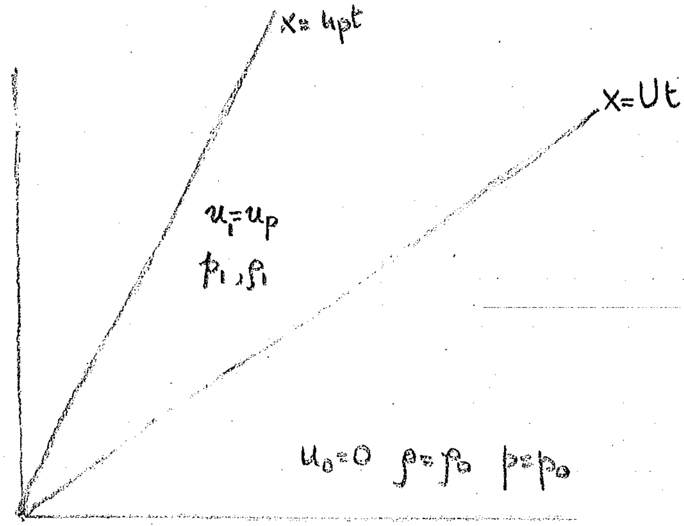
$$\frac{-\rho_0 U}{u_p-U} = \rho_1 \qquad \rho_1 = \frac{\rho_0 U}{U-u_p} = \frac{\rho_0}{1-u_p/U}$$ where

$$U/u_p = \frac{1}{2(1-\mu^2)} + \sqrt{\left(\frac{c_0}{u_p}\right)^2 + \frac{1}{4}\left(\frac{1}{1-\mu^2}\right)^2}$$

$$mu_0 - mu_1 + p_0 = p_1 \qquad m(u_0-u_1) + p_0 = p_1$$

$$m = \rho_0 v_0 = \rho_0(u_0 - U) = -\rho_0 U$$

$$\therefore \quad p_1 = p_0 - mu_1 = p_0 + \rho_0 u_p U = p_0 + \rho_0 u_p\left[\frac{u_p}{2(1-\mu^2)} + u_p\sqrt{\left(\frac{c_0}{u_p}\right)^2 + \frac{1}{4}\left(\frac{1}{1-\mu^2}\right)^2}\right]$$



Since $\quad (1-\mu^2)(U-u_0)^2 - (u_1-u_0)(U-u_0) = (1-\mu^2)c_0^2$

then by $u_0 = 0 \quad u_1 = u_p$

$$U = \frac{u_p}{2(1-\mu^2)} + \sqrt{\left[\frac{u_p}{2(1-\mu^2)}\right]^2 + c_0^2}$$

Since $\rho_0(u_0 - U) = \rho_1(u_p - U)$ then

$$\rho_1 = \frac{\rho_0}{1 - u_p/U} \qquad \text{where} \qquad U/u_p = \frac{1}{2(1-\mu^2)} + \sqrt{\left[\frac{1}{2(1-\mu^2)}\right]^2 + \left(\frac{c_0}{u_p}\right)^2}$$

also $\quad m u_0 - m u_1 + p_0 = p_1 \qquad$ then

$$p_1 = p_0 + \rho_0 u_p U = p_0 + \rho_0 \left\{ \frac{u_p^2}{2(1-\mu^2)} + u_p^2 \sqrt{\left[\frac{1}{2(1-\mu^2)}\right]^2 + \left(\frac{c_0}{u_p}\right)^2} \right.$$

$$\begin{pmatrix} 1/c_0 & 1 \\ 1 & c_0 \end{pmatrix}\begin{pmatrix} [\underset{\sim}{u}_{tt}] \end{pmatrix} = \begin{bmatrix} [U_x^\circ] \\ c_0^2[V_x^\circ] + 2c_0c_y[V_tV_x] + \psi_u[u_t] \end{bmatrix}$$

$$\begin{pmatrix} 1 & c_0 \\ & c_0 \end{pmatrix}\Big(\qquad c_0[U_x^\circ]$$

$$\boxed{\begin{aligned} P_x &= K_{yA}A_x + K_x \\ P_t &= K_yA\,A_t \end{aligned}}$$

$$\frac{1}{\sigma} = \frac{1}{\sigma(0)} e^{\int_0^x \Omega(s)ds}$$

$$\sigma = \sigma_0 e^{-\int_0^x \Omega(s)ds}$$

$$0 = c_0[U_x^\circ] - c_0^2[V_x^\circ] + 2c_0c_y[V_tV_x] - \psi_u[u_t]$$

$$\frac{dK}{dA}A_x$$

$$[u_t] = -c_0(x)[V_t] = \sigma_L(x)$$

$$y' - \Omega y = A(x)$$

$$P_x = \frac{P_t}{A_t}A_x + K_x$$

$$[u_t]_x \qquad [u_{xt}] = [U_x^\circ] - [T_{gx}u_{tt}]$$

$$[U_x^\circ] = V_{tt} + T_x u_{tt} = [u_{xt}] + [T_x u_{tt}]$$

$$\frac{1}{\sigma} = \frac{1 - \int \sigma_L A}{\sigma_L}$$

$$[u_t]_x = \sigma(x) \qquad [U^\circ]_x = [u_t]_x + [T_x u_{tt}]$$

$$\frac{1}{\sigma_L}\int A\sigma_L$$

$$u_{tt} - c^2V_{xt} - c_V^2 V_t V_x \overset{\sigma_x}{=} \psi_u u_t$$

$$y = -$$

$$[u_t]_x = [u_{xt}] -$$

$$y' = -A\frac{\sigma_L}{\sigma_L'}$$

$$V_{xt} = u_{xv}$$

$$[U_x^\circ] = [u_{xt}] - [T_{gx}u_{tt}]$$

$$[V_x^\circ] = [v_{xt}] - [T_{gx}V_{tt}]$$

$$y = -\sigma_L A\sigma_L + \frac{\sigma_L'\int A\sigma_L}{\sigma_L^2}$$

$$u_{tt} - c^2u_{xx} - c_V^2u_x$$

$$y' = -A + \frac{\sigma_L'\int A\sigma_L}{\sigma_L^2}$$

$$\cancel{u_{xt}}\ \cancel{- c^2V_{xx} - c_V^2 V_x^2} - c_x^2 V_x$$

$$\sigma_L' - \sigma_L\Omega_L = 0$$

$$[v_t]_x = \left(\frac{\sigma}{-c_0}\right)_x$$

$$\sigma_L$$

$$\sigma_L' = \sigma_L\Omega$$

$$y - \frac{\sigma_L'\int A\sigma_L}{\sigma_L^2}V_x$$

$$-\Omega y = \frac{\sigma_L\Omega\int A\sigma_L}{\sigma_L^2}$$

$$\frac{d}{dx}\int_0^x f(z)\,dz = f(x)$$

the sol to Riccati Eq gave

$$\sigma(x) = \frac{\exp\left(-\int_0^x \Omega(r)\,dr\right)\sigma(0)}{1-\sigma(0)\int_0^x A(r)\exp\left(-\int_0^r \Omega(s)\,ds\right)dr} \qquad e^u = \frac{\text{num}}{\text{den}}$$

Linear Sol $A \equiv 0 \Rightarrow \sigma_L(x) = \sigma(0)\exp\left(-\int_0^x \Omega(r)\,dr\right)$

$$\left.\frac{d\sigma(x)}{dx}\right|_{x=0} = \left[1-\cancel{\sigma(0)\int_0^x A(r)\exp\left(-\int_0^r \Omega(s)\,ds\right)dr}\right]\exp\left(-\int_0^x \Omega(r)\,dr\right)\sigma(0) \quad (-\Omega(x))$$

$$\frac{(\text{denom})(\text{num})(-\Omega(x)) + \text{num}\left(\sigma(0)\,A(x)\exp\left(-\int_0^x \Omega(s)\,ds\right)\right)}{1-\sigma(0)\int_0^x A(r)\exp\left(-\int_0^r \Omega(s)\,ds\right)dr}$$

$$\left|\sigma(0)\int_0^x A(r)\exp\left(-\int_0^r \Omega(s)\,ds\right)dr\right| \ll \epsilon$$

$$|\sigma(0)|\left|\int_0^x A(r)\exp\left(-\int_0^r \Omega(s)\,ds\right)dr\right|$$

$$1 \gg |\sigma(0)|\int_0^x \left|A(r)\exp\left(-\int_0^r \Omega(s)\,ds\right)\right|dr \geq |\sigma(0)|\int_0^x |A(r)$$

$$A(x)\exp\left(-\int_0^x \Omega(s)\,ds\right) = 0 \qquad \text{Assume } \sigma_L \text{ solves}$$

$$\sigma(x) = \frac{\sigma_L(x)}{1-\int_0^x A(r)\,\sigma_L(r)\,dr}$$

$$\left|\frac{\sigma(x)}{\sigma_L(x)}-1\right| = \left|\frac{1}{1-\int_0^x A(r)\,\sigma_L(r)\,dr}-1\right| < \epsilon$$

$$\left|\frac{\int_0^x A(r)\,\sigma_L(r)\,dr}{1-\int_0^x A(r)\,\sigma_L(r)\,dr}\right| < \epsilon$$

$\sigma_x + \Omega(x)\,\sigma = 0$

non hom

$\sigma_x + \Omega(x)\,\sigma = A(x)\sigma^2$

Suppose $\sigma_p(x)$ solves this & suppose that σ_p is slowly varying i.e. $\sigma_p \approx c$

$\Omega(x)\,\sigma_p = A\,\sigma_p^2$

$\sigma_p = \frac{\Omega(x)}{A(x)}$

(15)

Cesar Levy 763.2/31

1. Given: $\sigma(x) = \dfrac{\sigma_L(x)}{1 - \int_0^x A(r)\sigma_L(r)\,dr}$ and $\sigma_L(x) = \sigma(0)\exp\left(-\int_0^x \Omega(s)\,ds\right)$

For $0 \le x \le y$ under what conditions is $\sigma_L(x)$ a good approx to $\sigma(x)$

we want the conditions $\ni$. $\left|\dfrac{\sigma(x)}{\sigma_L(x)} - 1\right| < \epsilon$ for $\epsilon > 0$ but small.

$$\left|\frac{\sigma(x)}{\sigma_L(x)} - 1\right| = \left|\frac{1}{1 - \int_0^x A(r)\sigma_L(r)dr} - 1\right| = \left|\frac{\int_0^x A(r)\sigma_L(r)dr}{1 - \int_0^x A(r)\sigma_L(r)dr}\right| < \epsilon$$

$$\left|\int_0^x A(r)\sigma_L(r)dr\right| < \epsilon\left|1 - \int_0^x A(r)\sigma_L(r)dr\right| \le \epsilon + \epsilon\left|\int_0^x A(r)\sigma_L(r)dr\right|$$

or

$$\left|\int_0^x A(r)\sigma_L(r)dr\right| \le \frac{\epsilon}{1-\epsilon} \qquad \forall\, x \in [0,y]$$

In other words, if we were given $\sigma_L(x)$, then $\forall\, A(x) \ni$. the integral condition is satisfied then σ/σ_L approximates 1, to a tolerance of ϵ.

2.
$$A_t + (Au)_x = 0$$
$$A(u_t + uu_x) + p_x = -\nu u$$
$$p = K(A,x)$$

propagating into $A = A_0(x)$ $u = 0$

$$\left.\begin{aligned} p_t &= K_A A_t + K_x x_t \\ p_x &= K_A A_x + K_x \end{aligned}\right\} \quad p_t + up_x = -K_A A u_x + K_x(u + x_t)$$

Since x,t are independent variables $x_t = 0$

$\therefore$
$$p_t + up_x + K_A A u_x = uK_x \qquad (1)$$
$$u_t + uu_x + p_x/A = -\nu u/A$$

Define $G(x,\alpha) = g(x,t)$

where $G_{,x} = g_{,x} + g_{,t}T_{,x}$ and $t = T(x,\alpha)$ is the arrival time

then we can rewrite (1) as

$$p_t(1 - uT_x) - K_A A T_x u_t = uK_x - K_A A U_x - uP_x$$
$$p_t\left(-\frac{T_x}{A}\right) + (1 - uT_x)u_t = -\frac{\nu u}{A} - \frac{P_x}{A} - uU_x$$

1.

taking jumps and noting the right hand side of eqs. to be continuous

$$\begin{pmatrix} 1-uT_x & -K_A A T_x \\ -\frac{1}{A} T_x & 1-uT_x \end{pmatrix} \begin{pmatrix} [p_t] \\ [u_t] \end{pmatrix} = \underset{\sim}{0} \qquad (2)$$

and $(1-uT_x)^2 + K_A T_x^2 = 0$ or $(1-uT_x) = \pm\sqrt{K_A}\, T_x$

and $T_x = \dfrac{1}{u \pm \sqrt{K_A}}$; take $T_x = \dfrac{1}{u+\sqrt{K_A}}$; ahead of front $T_x = \dfrac{1}{\sqrt{K_A}}$

from (2) one obtains that $[u_t] = \dfrac{1}{A_0\sqrt{K_A}}[p_t]$ strength of acceleration front.

now differentiating (1) wrt t and taking jumps gives

$$\begin{pmatrix} 1-uT_x & -K_A A T_x \\ -T_x/A & 1-uT_x \end{pmatrix} \begin{pmatrix} [p_{tt}] \\ [u_{tt}] \end{pmatrix} = \begin{pmatrix} -[u_t p_x] - [u \overset{\circ}{P_x}] - [(K_{AA}A + K_A)A_t u_x] - [K_A A \overset{\circ}{U_x}] + [K_x u_t] + [u K_{xt}] \\ -[u_t u_x] - [u \overset{\circ}{U_x}] - [\overset{\circ}{P_x}/A] + [p_x A_t/A^2] - [\frac{\nu u_t}{A}] + [\frac{\nu u A_t}{A^2}] \end{pmatrix} \qquad (3)$$

multiply 1st eq by $\frac{1}{A\sqrt{K_A}}$ and evaluate @ head of front $(u=0,\ A=A_0(x))$ making use of

$\frac{A_t}{A}\left(p_x/A + \frac{\nu u}{A}\right) = \frac{(Au)_x}{A}(u_t + u u_x)$ and $u_t(K_x - p_x) = -(u_t K_A A_x)$. Then (3) leads to

$$0 = \frac{1}{A_0\sqrt{K_A}}\left\{[u_t K_A A_0'] - [K_A A_0 \overset{\circ}{U_x}] - [(K_{AA}A_0^2 + K_A A_0)\, u_x^2]\right\} + [\overset{\circ}{P_x}/A_0] - \left[\frac{\nu u_t}{A_0}\right]$$

but $[\overset{\circ}{U_x}] = [u_t]_x - T_x\overset{\nearrow 0}{[u_t]_t}$ and $[u_x^2] = \overset{\nearrow 0}{[U_x^2]} - 2T_x\overset{\nearrow 0}{[U_x u_t]} + T_x^2[u_t^2]$

$= -T_x^2[u_t]^2 + 2T_x^2[u_t]\, u_t + \overset{\nearrow 0}{} = -T_x^2[u_t]^2 = -\frac{1}{K_A}[u$

also $[\overset{\circ}{P_x}] = [p_t]_x - T_x\overset{\nearrow 0}{[p_t]_t} = (A_0\sqrt{K_A}[u_t])_x$

thus $0 = K_A^2 A_0'[u_t] - K_A^2 A_0[u_t]_x + (K_{AA}A_0^2 + K_A A_0)[u_t]^2 - A_0 K_A\sqrt{K_A}\left\{\frac{1}{A_0}\left((A_0\sqrt{K_A}[u_t])_x + \nu[u\right.\right.$

and one obtains the Riccati Eq

$$-2K_A^2 A_0[u_t]_x - \left(\frac{A_0 K_A}{2}(K_A)' + K_A\sqrt{K_A}\,\nu\right)[u_t] + (K_{AA}A_0^2 + K_A A_0)[u_t]^2 = 0$$

or $$[u_t]_x + \left(\frac{(K_A)'}{4K_A} + \frac{\nu\sqrt{K_A}}{2K_A A_0}\right)[u_t] - \left(\frac{K_{AA}A_0}{2K_A^2} + \frac{1}{2K_A}\right)[u_t]^2 = 0$$

Let $\Omega(x) = Lr^{-1} + Ls^{-1}$ where $Lr = \dfrac{4K_A}{(K_A)'}$ $Ls = \dfrac{2K_A A_0}{\nu\sqrt{K_A}}$

$$A(x) = \left(\frac{K_{AA} A_0 + K_A}{2K_A^2}\right)$$

$$[u_t]^x = \frac{[u_t]^0 \exp\left(-\int_0^x \Omega(s)ds\right)}{1 - [u_t]^0 \left\{\int_0^x A(r) \exp\left(-\int_0^r \Omega(s)ds\right)dr\right\}}$$

where $[u_t]^x$ is the strength at position x, and $[u_t]^0$ is the strength @ $x=0$.

$$A_t + (Au)_x = 0 \quad \longrightarrow \quad A_t + Au_x + A_x u = 0 \implies A_t + uA_x = -Au_x$$

$$A(u_t + uu_x) + p_x = -\nu u$$

$$p = K(A,x)$$

$$p_t = K_A A_t + K_x x_t^{\;\to 0}$$

$$p_x = K_A A_x + K_x \quad \longrightarrow \quad up_x + p_t = K_A[A_t + uA_x] + uK_x$$

$$p_t + up_x + K_A A u_x = uK_x$$

$$u_t + uu_x + p_x/A = -\frac{\nu u}{A}$$

now make change of variables to (x,α) i.e. for any $G(x,\alpha) = g(x,t)$

then $G_{,x} = g_x + g_t t_x \qquad t_x = \frac{\partial}{\partial x}(T(x,\alpha)) = T_x$

Change all x deriv of g to x deriv of G

$\therefore \quad g_x = G_x - g_t T_x$

$$p_t + uP_x - up_t T_x + K_A A U_x - K_A A T_x u_t = uK_x$$

$$u_t + uU_x - uu_t T_x + \frac{1}{A}(P_x - p_t T_x) = -\frac{\nu u}{A}$$

or $$p_t(1 - uT_x) + u_t(-K_A A T_x) = u(K_x - P_x) - K_A A U_x$$

$$p_t\left(-\frac{T_x}{A}\right) + u_t(1 - uT_x) = -u\left(\frac{\nu}{A} + U_x\right) - \frac{P_x}{A}$$

now take jumps

Characteristic Condition $$\begin{pmatrix} 1 - uT_x & -K_A A T_x \\ -T_x/A & 1 - uT_x \end{pmatrix} \begin{pmatrix} [p_t] \\ [u_t] \end{pmatrix} = \underset{\sim}{0}$$ since all are cont wrt x

this gives rise to $T_x = \dfrac{1}{u \pm \sqrt{K_A}}$ as the EV of the matrix; take $+\sqrt{\ }$ for wave to right

Now differentiate original eq wrt t and transform to (x,α) coord.

$$1 - uT_x = \frac{\sqrt{K_A}}{u + \sqrt{K_A}} \qquad \frac{\sqrt{K_A}}{u + \sqrt{K_A}}[p_t] - \frac{K_A A}{u + \sqrt{K_A}}[u_t] = 0$$

or $[u_t] = \dfrac{1}{A\sqrt{K_A}}[p_t]$ strength of the acceleration front.

$$p_{tt} + u_t p_x + up_{xt} + K_{At} A u_x + K_A A_t u_x + K_A A u_{xt} = u_t K_x + uK_{xt}$$

$$u_{tt} + u_t u_x + uu_{xt} + p_{xt}/A - \frac{p_x}{A^2}A_t = -\frac{\nu u_t}{A} + \frac{\nu u}{A^2}A_t$$

$$p_{tt} - uT_x p_{tt} - K_A A T_x u_{tt} = -u_t p_x - uP_x^{\circ} - K_{At} A u_x - K_A A_t u_x - K_A A U_x^{\circ} + u_t K_x + uK_x$$

$$u_{tt} - uT_x u_{tt} - \frac{T_x}{A}p_{tt} = -u_t u_x - uU_x^{\circ} - \frac{P_x^{\circ}}{A} + \frac{p_x A_t}{A^2} - \frac{\nu u_t}{A} + \frac{\nu u A_t}{A^2}$$

$$
\begin{array}{l} 1 \\ \\ 2 \end{array}
\begin{pmatrix} 1-uT_x & -K_A A T_x \\ & \\ -T_x/A & 1-uT_x \end{pmatrix}
\begin{pmatrix} [p_{tt}] \\ \\ [u_{tt}] \end{pmatrix}
=
\begin{pmatrix} -[u_t p_x] - u[P_x^\circ] - (K_{AA}A + K_A)[A_t u_x] - K_A A[U_x^\circ] + K_x[u \\ \quad + u[K_{xt}] \\ -[u_t u_x] - u[U_x^\circ] - \frac{1}{A}[P_x^\circ] + \frac{1}{A^2}[p_x A_t] - \frac{\nu}{A}[u_t] + \frac{\nu u}{A^2}[A_t \end{pmatrix}
$$

mult 1 by $\dfrac{1}{A\sqrt{K_A}}$ $\therefore$

$$
0 = \frac{1}{A\sqrt{K_A}}\Big\{ -[u_t p_x] - [u P_x^\circ] - [(K_{AA}A + K_A)A_t u_x] - [K_A A U_x^\circ] + [K_x u_t] + [uA_t K
$$
$$
-[u_t u_x] - [u U_x^\circ] - [P_x^\circ/A] + [1/A^2\, p_x A_t] - \left[\frac{\nu}{A} u_t\right] + \left[\frac{\nu u A_t}{A^2}\right]
$$

$$
[u_t p_x] = [u_t(P_x - T_x p_t)] = [u_t P_x] - T_x[u_t p_t]
$$

note also $\dfrac{A_t}{A}\left(p_x/A + \dfrac{\nu u}{A}\right) = -\dfrac{A_t}{A}(u_t + u u_x)$ & $A_t = -(Au)_x$

$$
p_x = -\nu u - A(u_t + u u_x)
$$

$$
\therefore \left[\frac{A_t}{A}\left(p_x/A + \frac{\nu u}{A}\right)\right] = \left[\frac{(Au)_x}{A}(u_t + u u_x)\right] = \left[\frac{(Au)_x u_t}{A}\right] = [u_x u_t] + \left[\frac{u A_x u_t}{A}\right.
$$

$$
\therefore\ -[u_t u_x] + \left[\frac{1}{A^2} p_x A_t\right] + \left[\frac{\nu u A_t}{A^2}\right] = \left[\frac{u A_x u_t}{A}\right]
$$

$$
A_t u_x = -(Au)_x u_x = -(A u_x^2 + u A_x u_x)
$$

$$
[u_t(K_x - p_x)] = -[u_t K_A A_x]
$$

$$
0 = \frac{1}{A\sqrt{K_A}}\Big\{[u_t K_A A_x] - [u P_x^\circ] - [K_A A U_x^\circ] + [u(Au)_x K_{Ax}] - [(K_{AA}A + K_A)(Au)_x u_x]\Big\}
$$
$$
+ \left[\frac{u A_x u_t}{A}\right] - [u U_x^\circ] - [P_x^\circ/A] - \left[\frac{\nu}{A} u_t\right]
$$

$$
\frac{(K_A)_x}{K_A}
$$

Evaluated in front of front:

$$0 = \frac{1}{A_0\sqrt{K_A}}\left\{ K_A A_0' [u_t] - K_A A_0 [u_t]_x - \left[(K_{AA}A^2 + K_A A)\, u_x^{2^*}\right]\right\}$$

$$- \frac{1}{A_0}\left(A_0\sqrt{K_A}\,[u_t]\right)_x - \nu/A_0 [u_t]$$

$$[u_x^2]^* = -T_x^2 [u_t]^2 = -\frac{1}{K_A}[u_t]^2 \quad \text{since } [u_x^2] = -[u_x]^2 + 2[u_x]\,\underset{\nearrow = 0 \text{ since } u=0 \text{ @ } f \text{ of wave}}{u_{x_+}}$$

$$\left\{ K_A A_0' [u_t] - K_A A_0 [u_t]_x + (K_{AA}A_0^2 + K_A A_0)\frac{1}{K_A}[u_t]^2 \right\}$$

$$-\sqrt{K_A}\,\underbrace{\left(A_0\sqrt{K_A}\;[u_t]\right)_x} - \sqrt{K_A}\,\nu\,[u_t] = 0$$

$$\cancel{(K_A A_0 - \sqrt{K_A}\nu)[u_t]} \qquad A_0'\sqrt{K_A}\,[u_t] + A_0(\sqrt{K_A})_x\,[u_t] + A_0\sqrt{K_A}\,[u_t]_x$$

$$\frac{1}{2\sqrt{K_A}}\left(K_{AA}A_0' + K_{Ax}\right) \qquad A_0 K_A\sqrt{K_A}\,[u_t]\,\frac{1}{2\sqrt{K_A}}(K_A)_x$$

$$\cancel{K_A^2 A_0' [u_t]} - K_A^2 A_0 [u_t]_x + (K_{AA}A_0^2 + K_A A_0)[u_t]^2 - A_0' K_A^2 [u_t]$$

$$- \frac{A_0 K_A}{2}\left(K_{AA}A_0' + K_{Ax}\right)[u_t] - A_0 K_A^2 [u_t]_x - \sqrt{K_A}\,K_A\nu\,[u_t] = 0$$

$$-2K_A^2 A_0 [u_t]_x - \left[\frac{A_0 K_A}{2}\overset{(K_A)_x}{\left(K_{AA}A_0' + K_{Ax}\right)} + K_A^{3/2}\nu\right][u_t] + (K_{AA}A_0^2 + K_A A_0)[u_t]^2 = 0$$

$$\frac{[u_t]_x}{[u_t]^2} + \left[\frac{A_0 K_A\left(K_{AA}A_0' + K_{Ax}\right) + 2K_A^{3/2}\nu}{4K_A^2 A_0}\right]\frac{1}{[u_t]} - \frac{(K_{AA}A_0^2 + K_A A_0)}{2K_A^2 A_0} = 0$$

$$\Omega(x) = \left(\frac{K_{AA}A_0' + K_{Ax}}{4K_A}\right) + \frac{\nu}{2K_A^{1/2}A_0} = \frac{(K_A)_x}{4K_A} + \frac{\nu}{2K_A^{1/2}A_0}$$

$$L_r = \frac{4K_A}{(K_A)_x} \qquad L_s = \frac{2K_A^{1/2}A_0}{\nu}$$

$$A(x) = -\left(\frac{K_{AA}A_0^2 + K_A A_0}{2K_A^2 A_0}\right)$$

Solution is $$[u_t] = \frac{[u_t]_0 \exp\left(-\int_0^x \Omega(s)\,ds\right)}{1 - [u_t]_0 \int^x A(r)\exp\left(-\int^r \Omega(s)\,ds\right)dr}$$

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C. LEVY 763.2021

$$\bar{u}_{\bar{t}} - c^2(\bar{x})\,\bar{V}_{\bar{x}} = 0$$

$$\bar{V}_{\bar{t}} - \bar{u}_{\bar{x}} = 0 \qquad \text{with} \quad u\left(0, \tfrac{\bar{t}c}{L_s}\right) = f\left(\tfrac{\bar{t}}{\tau_p}\right) \qquad (1)$$

$$\omega = \frac{L_s}{\tau_p c} \gg \quad \text{or} \quad \frac{L_s}{c} \gg \tau_p$$

i.e. $\min\left(\frac{L_s}{c}\right) \gg \tau_p \quad x \in [0, y]$; suppose this occurs @ $x = x^*$

then let $L_s(x^*) = L^0$ & $c(x^*) = c_0$

then $\min L_s \le L^0$ and $\max c \geq c_0$

$\therefore \quad L^0/c_0 \geq \frac{\min L_s}{\max c}$; take as condition on ω to be

$$\frac{\min L_s}{\max c} = \frac{\underline{L}}{\bar{c}} \gg \tau_p .$$

Normalize wrt $\underline{L}$ and non dim $(\bar{x}, \bar{t})$ wrt $(\underline{L}, \bar{c})$

let $u = \bar{u}/u_0$, $V = \bar{V}\bar{c}/u_0$, $t = \bar{t}\bar{c}/\underline{L}$, $x = \bar{x}/\underline{L}$

then (1) becomes

$$u_t - \frac{c^2(x\underline{L})}{\bar{c}^2} V_x = 0$$

$$V_t - u_x = 0 \qquad \text{with } u(0,t) = f(\omega t)$$

or

$$\begin{pmatrix} u \\ V \end{pmatrix}_t + \begin{pmatrix} 0 & -\gamma^2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ V \end{pmatrix}_x = 0 \qquad \text{where } \gamma(x) = \frac{c(x\underline{L})}{\bar{c}}$$

Characteristic Condition leads to λ's: $\pm\gamma$ pick $+\gamma$

the right EV becomes $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1/\gamma \end{pmatrix}$

the left EV becomes $\underset{\sim}{l} = (1 \quad -\gamma)$

the characteristic $\frac{dt}{dx} = \frac{1}{\gamma(x)}$ and $\alpha = t - \int_0^x \frac{ds}{\gamma(s)}$ $\therefore$ $\beta = \omega\alpha$

the solution to this problem is $U^{(0)}(x) = \underset{\sim}{r}\,\sigma_L$ where

σ_L solves $\quad \sigma_L' + \Omega\,\sigma_L = 0 \quad$ & $\quad \Omega = \dfrac{\underset{\sim}{l}\cdot r}{\underset{\sim}{l}\cdot r'} = \dfrac{-\,dc/dx}{2c}$

$$\therefore \quad \sigma_L(x) = \sigma(0)\exp\left(-\int_0^x -\frac{dc}{2c}\right) = f(\beta)\exp\left[\tfrac{1}{2}\ln c(x_b) - \tfrac{1}{2}\ln c(0)\right] = f(\beta)\left\{\frac{c(x_b)}{c(0)}\right\}^{1/2}$$

Let $\delta = c(x_b)/c(0)$

$$\therefore \quad \underset{\sim}{U}^{(0)} = f(\beta)\,\delta^{1/2}\begin{pmatrix}1\\ -1/\gamma\end{pmatrix} = \begin{pmatrix}U^0\\ V^0\end{pmatrix}$$

rewriting the equations we find

$$\left.\begin{aligned} U_\beta + \gamma V_\beta &= \gamma^2/\omega\, V_x\\ U_\beta + \gamma V_\beta &= \gamma/\omega\, U_x\\ \gamma/\omega\, U_x &= \gamma^2/\omega\, V_x \end{aligned}\right\} = \left\{\begin{aligned} U^{(n+1)} + \gamma V^{(n+1)} &= \frac{\gamma^2}{\omega} V_x^{(n)}\\ U^{(n+1)} + \gamma V^{(n+1)} &= \frac{\gamma}{\omega} U_x^{(n)}\\ U_x^{n} &= \gamma V_x^{(n)} \end{aligned}\right.$$

one finds that

$$\begin{aligned} U^{(1)} + \gamma V^{(1)} &= \frac{\gamma^2}{\omega} V_x^{(0)}\\ U^{(1)} + \gamma V^{(1)} &= \gamma/\omega\, U_x^{(0)} = \frac{\gamma}{\omega}\frac{d}{dx}\left(f(\beta)\,\delta^{1/2}\right) = \frac{\gamma}{\omega} f(\beta)\,\delta^{1/2}\,\frac{\gamma_x}{2\gamma}\\ U_x^{(1)} &= \gamma V_x^{(1)} \end{aligned}$$

then

$$\left.\begin{aligned} U^{(1)} + \gamma V^{(1)} &= \frac{\gamma_x}{2\omega} f(\beta)\,\delta^{1/2}\\ \& \quad U_x^{(1)} &= \gamma V_x^{(1)} \end{aligned}\right\} \Rightarrow U^{(1)} - \frac{2\gamma}{\gamma_x} U_x^{(1)} = Z(x) - \frac{\gamma}{\gamma_x}\frac{d}{dx} Z(x)$$

where $Z(x) = \dfrac{\gamma_x}{2\omega} f(\beta)\,\delta^{1/2}$

$U^{(1)} = Z(x) + \text{const}$ where $U^{(1)}(0) = 0 \quad \therefore$ const $= -Z(0)$

$$\therefore \quad U^{(1)} = Z(x) - Z(0)$$

and $\quad V^{(1)} = \dfrac{1}{\omega} f(\beta)\,\delta^{1/2}\,\dfrac{\gamma_x}{2\gamma} - \dfrac{U^{(1)}}{\gamma} = \dfrac{1}{\gamma}\left\{Z(x) - Z(x) + Z(0)\right\} = \dfrac{Z(0)}{\gamma}$

$$\therefore \quad \underset{\sim}{U}^{(1)} = Z(x)\begin{pmatrix}1\\0\end{pmatrix} - Z(0)\begin{pmatrix}1\\ -1/\gamma\end{pmatrix} \qquad \& \quad Z(x) = \frac{\gamma_x}{2\omega} f(\beta)\,\delta^{1/2}$$

$$\therefore \quad \underset{\sim}{u} = \underset{\sim}{U}^{(0)} + \frac{h_1(\beta)}{\omega}\,U^{(1)}$$

C. LEVY 763.2021

$\bar{u}_{\bar{t}} - c^2(\bar{x})\,\bar{v}_{\bar{x}} = 0$ (1)

$\bar{v}_{\bar{t}} - \bar{u}_{\bar{x}} = 0$ with $u(0,\bar{t}) = f(\omega\bar{t})$

$$\omega = \frac{L_s}{\tau_p C} \gg 1 \quad \text{or} \quad \frac{L_s}{C} \gg \tau_p$$

i.e. $\min\left(\frac{L_s}{c}\right) \gg \tau_p$ $x \in [0,y]$; suppose this occurs @ $x = x^*$

then let $L_s(x^*) = L^0$ & $c(x^*) = c_0$

then $\min L_s \leqslant L^0$ and $\max c \geqslant c_0$

$\therefore$ $L^0/c_0 \geqslant \dfrac{\min L_s}{\max c}$; take as condition on ω to be

$$\frac{\min L_s}{\max c} = \frac{\underline{L}_s}{\bar{c}} \gg \tau_p .$$

normalize wrt $\underline{L}_s$ and non dim(x,t) wrt $(\underline{L}_s$ & $\bar{c})$

let $u = \bar{u}/u_0$ $v = \bar{v}\bar{c}/u_0$ $t = \bar{t}\,\bar{c}/\underline{L}_s$ $x = \bar{x}/\underline{L}_s$

then (1) becomes

$$u_t - \frac{c^2(x\underline{L}_s)}{\bar{c}^2} v_x = 0$$

$$v_t - u_x = 0 \quad \text{with } u(0, \tfrac{\underline{L}_s}{\bar{c}}t) = f\left(\frac{\underline{L}_s^2\, t}{\bar{c}^2 \tau_p}\right)$$

or $$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & -c^2/\bar{c}^2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x = 0 \quad \text{where } c = c(x\underline{L}_s)$$

Characteristic condition leads to 2 EV : $\pm\, c/\bar{c}$ pick $c/\bar{c}$

then

$$\alpha = t - \int_0^x \frac{\bar{c}\, d\xi}{c(\xi \underline{L}_s)} \quad \text{\&} \quad \beta = \omega\left(t - \int_0^x \frac{\bar{c}\, d\xi}{c(\xi\underline{L}_s)}\right)$$

the right EV becomes $\underline{r} = \begin{pmatrix} 1 \\ -\bar{c}/c \end{pmatrix}$

the left EV becomes $\underline{l} = \begin{pmatrix} 1 & -c/\bar{c} \end{pmatrix}$

$$\Omega = \frac{\underline{l}\cdot\underline{r}'}{\underline{l}\cdot\underline{r}} = \frac{\begin{pmatrix} 1 & -c/\bar{c} \end{pmatrix}\begin{pmatrix} 0 \\ \frac{\bar{c}c'\underline{L}_s}{c^2} \end{pmatrix}}{\begin{pmatrix} 1 & -c/\bar{c} \end{pmatrix}\begin{pmatrix} 1 \\ -\bar{c}/c \end{pmatrix}} = -\frac{c'}{2c}\underline{L}_s = \frac{\underline{L}_s}{L_s}$$

where $L_s = -\dfrac{2c}{c'}$

and $v_L'(x) + \Omega_L(x)\, v_L(x) = 0$

$$U^{(0)} - \frac{2\gamma}{\gamma_x} U_x^{(0)} = 0 \qquad \therefore \qquad U_x^{(0)} = U^{(0)} \frac{\gamma_x}{2\gamma} = f(\beta)\delta^{1/2} \frac{\gamma_x}{2\gamma}$$

$$U^{(\cancel{n+1})} + \gamma V^{\cancel{n+1}} = \frac{\cancel{\gamma}}{\omega} f(\beta)\delta^{1/2} \frac{\gamma_x}{2\cancel{\gamma}} = \frac{\gamma_x}{2\omega} f(\beta)\delta^{1/2}$$

$$U^{(1)} + \gamma V^{(1)} = \frac{\gamma_x}{2\omega} f(\beta)\delta^{1/2}$$

$$\& \quad U_x^{(1)} = \gamma V_x^{(1)}$$

$$U_x^{(1)} + \gamma_x V^{(1)} + \gamma V_x^{(1)} = \frac{d}{dx}\left\{\frac{\gamma_x}{2\omega} f(\beta)\delta^{1/2}\right\}$$

$$2U_x^{(1)} + \gamma_x V^{(1)} = \frac{d}{dx}\left\{\frac{\gamma_x}{2\omega} f(\beta)\delta^{1/2}\right\}$$

$$\gamma V^{(1)} = -\frac{\gamma 2 U_x^{(1)}}{\gamma_x} + \frac{\gamma}{\gamma_x}\frac{d}{dx}\left\{\frac{\gamma_x}{2\omega} f(\beta)\delta^{1/2}\right\}$$

$$U^{(1)} - \frac{2\gamma}{\gamma_x} U_x^{(1)} + \underbrace{\frac{\gamma}{\gamma_x}\frac{d}{dx}\left\{\frac{\gamma_x}{2\omega} f(\beta)\delta^{1/2}\right\}}_{\frac{\gamma}{\gamma_x}\frac{d}{dx}Z(x)} = \underbrace{\frac{\gamma_x}{2\omega} f(\beta)\delta^{1/2}}_{Z(x)}$$

with $U^{(1)}(0) = 0$

$$U^{(1)} - \frac{2\gamma}{\gamma_x} U_x^{(1)} = Z(x) - \frac{\gamma}{\gamma_x}\frac{d}{dx}Z(\cancel{x})$$

let $U^{(1)} = Z(x) + C \qquad U^{(1)} = Z(x) - Z(0)$

$$\cancel{U^{(1)}} + \cancel{\gamma} V^{(1)} = \cancel{\gamma}/\omega\, f(\beta)\delta^{1/2}\frac{\gamma_x}{2\gamma^{2}} - \frac{U^{(1)}}{\gamma}$$

$$V^{(1)} = \frac{Z(x)}{\gamma} - \frac{U^{(1)}}{\gamma} = \frac{1}{\gamma}\left(Z - Z + Z_0\right)$$

$$\therefore \quad \underset{\sim}{U}^{(1)} = \begin{pmatrix} Z(x) - Z(0) \\ \frac{Z(0)}{\gamma} \end{pmatrix} = Z(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1/\gamma \end{pmatrix} Z(0)$$

$$\therefore \quad \underset{\sim}{U}^{0} = \underset{\sim}{r}(x)\sigma_1(x) = f(\beta)\begin{pmatrix}1\\ -\bar{c}/c\end{pmatrix}\exp\left(-\int_0^x \frac{L_s}{L(\xi L_s)}\,d\xi\right)$$

$$\underset{\sim}{U}^{0} = f(\beta)\begin{pmatrix}1\\ -\bar{c}/c\end{pmatrix}\exp\left(+\int_0^x \frac{dc}{2c}\right) = f(\beta)\begin{pmatrix}1\\ -1/\gamma\end{pmatrix}\delta^{1/2}$$

$$= f(\beta)\begin{pmatrix}1\\ -1/\gamma\end{pmatrix}\delta^{1/2} \qquad \text{where } \gamma = c/\bar{c} \ \ \& \ \ \delta = c/c(0)$$

$$\frac{1}{2}\delta^{-1/2}\cdot\frac{c' L_s}{c(0)}$$

$$(c_0 I - A_0)\underset{\sim}{U}^{(n+1)} = c_0\left(B_0 U^{(n)} - A_0 \underset{\sim}{U}^{n}_x\right)$$

$$\left[\begin{pmatrix}\gamma & 0\\ 0 & \gamma\end{pmatrix} - \begin{pmatrix}0 & -\gamma^2\\ -1 & 0\end{pmatrix}\right] = \begin{pmatrix}\gamma & \gamma^2\\ 1 & \gamma\end{pmatrix}\underset{\sim}{U}^{(2)} = -\gamma\begin{pmatrix}0 & -\gamma^2\\ -1 & 0\end{pmatrix}\underset{\sim}{U}^{1}_x$$

$$\& \quad \ell A_0 U_x^{n+1} = 0$$

$$\underset{\sim}{U}^{(0)}_x = \frac{d}{dx}f(\beta)\left[\delta^{1/2}\begin{pmatrix}1\\ -1/\gamma\end{pmatrix}\right] + f(\beta)\left[\frac{d}{dx}\delta^{1/2}\right]\begin{pmatrix}1\\ -1/\gamma\end{pmatrix} + f(\beta)\delta^{1/2}\frac{d}{dx}\begin{pmatrix}1\\ -1/\gamma\end{pmatrix}$$

$$= \cancel{\frac{d}{d\beta}} f'(\beta)\left(-\frac{\omega}{\gamma}\right)\delta^{1/2}\begin{pmatrix}1\\ -1/\gamma\end{pmatrix} + \frac{f(\beta)}{2}\frac{\delta^{-1/2}}{c(0)}c' \underline{L}\begin{pmatrix}1\\ -1/\gamma\end{pmatrix} + f(\beta)\delta^{1/2}\begin{pmatrix}0\\ \frac{d\gamma/dx}{\gamma^2}\end{pmatrix}$$

$$U_x^{(0)} = U^{(0)}\frac{\gamma_x}{2\gamma}$$

$$(1 \quad -\gamma)\begin{pmatrix}0 & -\gamma^2\\ -1 & 0\end{pmatrix} = \cancel{\begin{pmatrix}\gamma\\ -\gamma^3\end{pmatrix}} \ (\gamma \quad -\gamma^2)$$

$$u_x^{(0)}$$

$$U^{(0)} = f(\beta)\delta^{1/2}$$

$$U_x^{(0)} = \delta^{1/2}f'(\beta)\beta_x + f(\beta)\tfrac{1}{2}\delta^{-1/2}\delta_x$$

$$U^{(0)}$$

$$U = U(0)\left(\frac{c}{c(0)}\right)^{1/2} = f(\beta)\begin{pmatrix}1\\ -1/\gamma\end{pmatrix}\delta^{1/2}$$

$$\omega \quad (c_0 I - A_0)\underset{\sim}{U}_\beta = -\frac{c_0}{\omega}\underset{\sim}{A}_0\underset{\sim}{U}_x$$

where $c_0 = \gamma(x)$

$$\frac{U}{U_x} = \frac{2\gamma}{\gamma_x} \qquad U/U_0 = \left[\gamma(x)/\gamma(0)\right]^{1/2}$$

$$\begin{pmatrix}\gamma & +\gamma^2\\ 1 & \gamma\end{pmatrix}\begin{pmatrix}U_\beta\\ V_\beta\end{pmatrix} = -\frac{\gamma}{\omega}\begin{pmatrix}0 & -\gamma^2\\ -1 & 0\end{pmatrix}\begin{pmatrix}U_x\\ V_x\end{pmatrix}$$

$$\ln U = +\tfrac{1}{2}\ln\gamma$$

$$U = \frac{2\gamma}{\gamma_x}U_x \qquad U = \gamma^{1/2}$$

$$\cancel{\gamma}(U_\beta + \gamma V_\beta) = +\cancel{\gamma}/\omega \ \gamma^2 V_x$$

$$U_\beta + \gamma V_\beta = \gamma/\omega\, U_x$$

$$V = -\frac{2U_x}{\gamma_x} \qquad \text{with Comp} \qquad \gamma/\omega\, U_x = \gamma^2/\omega\, V_x$$

$$U_x + \gamma_x V + \gamma V_x = 0 \qquad U^{n+1} + \gamma V^{n+1} = \gamma^2/\omega\, V_x^n \qquad \& \quad U_x^n = \gamma V_x^n$$

$$2U_x + \gamma_x V = 0 \qquad U^{n+1} + \gamma V^{n+1} = \gamma/\omega\, U_x^n$$

C. Levy T63.2131

Given:
$$\left.\begin{aligned} u_t - c^2(v)\,v_x &= -2u - Bu^2 \\ v_t - u_x &= 0 \end{aligned}\right\} (1)$$

And suppose we have non-dim equations with respect to $|\Omega^{-1}|$; we can now rewrite the eqns for $\underset{\sim}{u}(x,t) = \underset{\sim}{U}(x,\beta)$ with respect to arrival time $t = T(x,\beta)$

$\therefore$ (1) becomes

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - T_x\begin{pmatrix}0 & -c^2\\ -1 & 0\end{pmatrix}\right]\underset{\sim}{U}_\beta = \left[\begin{pmatrix}-2U-BU^2\\ 0\end{pmatrix} - \begin{pmatrix}0 & -c^2\\ -1 & 0\end{pmatrix}\underset{\sim}{U}_x\right]T_\beta \qquad (2)$$

let $U(x,\beta) = \frac{1}{\omega}\sum \omega^{-n}\,U^{(n)}(x,\beta)$

let $T - \int_0^x \frac{ds}{c(0,s)} = \frac{1}{\omega}\sum \omega^{-n}\,T^{(n)}(x,\beta)$ where $T_x = \frac{1}{c(u,x)} = \frac{1}{EV \text{ of } \underset{\sim}{A}}$

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \left(\frac{1}{c(0)} + \frac{1}{\omega}\sum\omega^{-n}T_x^{(n)}\right)\begin{pmatrix}0 & -[c(0)^2+2c_v c(0)V + c_v^2V^2+\ldots]\\ -1 & 0\end{pmatrix}\right]\sum\omega^{-n}\underset{\sim}{U}^{(n)}_{,\beta}(x,\beta)$$
$$= \left[\begin{pmatrix}-2\frac{1}{\omega}\sum\omega^{-n}U^{(n)} - B\{\frac{1}{\omega}\sum\omega^{-n}U^{(n)}\}^2\\ 0\end{pmatrix} - \begin{pmatrix}0 & -[c(0)^2+2c_v c(0)V+c_v^2V^2+\ldots]\\ -1 & 0\end{pmatrix}\frac{1}{\omega}\sum\omega^{-n}\underset{\sim}{U}_x^{(n)}\right] \quad (3)$$

but $c(V) = c(0) + c_v(0)V + \ldots$, $c(0) = 1$ $c_v(0) = \text{const}$

for $O\left(\frac{1}{\omega}\right)$ equation let $n=0$; thus we obtain

$$\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\underset{\sim}{U}^{(0)}_\beta = \left[\begin{pmatrix}-2\frac{1}{\omega}U^{(0)}\\ 0\end{pmatrix} - \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\frac{1}{\omega}\underset{\sim}{U}^{(0)}_x\right] \qquad (4)$$

and $T_x^{(0)} = -\frac{c_v(0)}{c(0)}V = -c_v(0)V = \begin{pmatrix}0 & -c_v\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$

we found $\underset{\sim}{U}^{(0)} = \begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}f(\beta)e^{-x}$ and thus $T_x^{(0)} = -\begin{pmatrix}0 & -c_v\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}f(\beta)e^{-x}$ $(*)$

for $O\left(\frac{1}{\omega^2}\right)$ equation

$$\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\frac{1}{\omega}\underset{\sim}{U}^{(1)}_\beta - \begin{bmatrix}0 & -2c_vV^{(0)}\\ 0 & 0\end{bmatrix}\frac{1}{\omega}\underset{\sim}{U}^{(0)}_\beta + \frac{1}{\omega}T_x^{(0)}\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}\underset{\sim}{U}^{(0)}_\beta = \left[\begin{pmatrix}-2\frac{1}{\omega^2}U^{(1)} - B\frac{1}{\omega^2}U^{2(0)}\\ 0\end{pmatrix} - \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\frac{1}{\omega^2}\underset{\sim}{U}^{(1)}_x - \frac{1}{\omega^2}\begin{pmatrix}0 & -2c_vV^{(0)}\\ 0 & 0\end{pmatrix}\underset{\sim}{U}_x\right.$$

(5)

$$\frac{1}{\omega}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\underset{\sim}{U}^{(1)}_\beta - \frac{1}{\omega}\begin{bmatrix}0 & +2c_V f(\beta)e^{-x}\\ 0 & 0\end{bmatrix}\begin{pmatrix}1\\-1\end{pmatrix}f'(\beta)e^{-x} + \frac{1}{\omega}\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}(-c_V f(\beta)e^{-x})\begin{pmatrix}1\\-1\end{pmatrix}f'(\beta)e^{-x} = \text{left hand side of eq(5)}$$

$$= \frac{1}{\omega}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\underset{\sim}{U}^{(1)}_\beta - \frac{1}{\omega}c_V f(\beta)f'(\beta)e^{-2x}\begin{pmatrix}-1\\-1\end{pmatrix} = 0 \qquad \begin{pmatrix}6a\\6b\end{pmatrix}$$

the left hand side of (5) gives

$$l(x)\begin{bmatrix}-2U^{(1)} + V_x^{(1)} + (2c_V - B)f^2(\beta)e^{-2x}\\ U_x^{(1)}\end{bmatrix} = 0 \qquad (7)$$

after multiplication by a left EV of A.

Using $\qquad U^{(1)}_\beta + V^{(1)}_\beta = -c_V f(\beta)f'(\beta)e^{-2x}$

one obtains $\quad U^{(1)} + V^{(1)} = -c_V e^{-2x} f^2(\beta) + g(x) \qquad (8)$

but $\underset{\sim}{U}(x,0) = 0 \implies g(x) = 0$ since $f(0) = 0$

Using (8) into (7) gives

$$(\,1 \quad -1\,)\begin{bmatrix}2V^{(1)} + 2c_V f^2(\beta)e^{-2x} + V_x^{(1)} + (2c_V - B)f^2(\beta)e^{-2x}\\ +2c_V f^2(\beta)e^{-2x} - V_x^{(1)}\end{bmatrix} = 0$$

and the transport equation is of the form,

$$V_x^{(1)} + \frac{2\cdot 1}{1-(-1)}V^{(1)} + f^2(\beta)e^{-2x}\left\{\frac{2c_V + 2c_V - B - 2c_V}{2}\right\} = 0$$

or $\qquad V_x^{(1)} + V^{(1)} = -\left[\frac{2c_V - B}{2}\right]f^2(\beta)e^{-2x} \qquad (9)$

thus (9) yields $\qquad V^{(1)} = f^2(\beta)\left[\frac{2c_V - B}{2}\right]e^{-2x} \qquad (10)$

put (10) into (8) giving $\quad U^{(1)} = [1 - e^{-2x}]f^2(\beta)\left[\frac{4c_V - B}{2}\right]$ with $U^{(1)}(0,t) = 0 \qquad (11)$

$$\therefore \quad \underset{\sim}{U}^{(1)} = \begin{pmatrix}-1\\1\end{pmatrix}V^{(1)} + \begin{pmatrix}-1\\0\end{pmatrix}c_V e^{-2x}f^2(\beta) + f^2(\beta)\frac{4c_V - B}{2} \quad (12)$$

$$T_X = \frac{1}{c(0)} = \frac{1}{\omega}\sum \omega^{-n}T_x^{n}(x,\beta) = -\frac{c_V(0)}{c^2(0)}V + \left\{\frac{2c_V^2(0)}{c^3(0)} - \frac{c_{VV}(0)}{c^2(0)}\right\}V^2$$

$$T_x^{(0)} = -\frac{c_V(0)}{c^2(0)}V^{(0)} \qquad T_x^{(1)} = -c_V V^{(1)} + \{2c_V^2(0) - c_{VV}(0)\}V^{2(0)}$$

$$T^{(1)} = \int_0^X -c_V V^{(1)}dx + \int_0^X\{2c_V^2(0) - c_{VV}(0)\}V^{2(0)}dx$$

where $V^{(1)}$ is given by (10) and $V^{(0)}$ is given by (*)