763.2130 Prof. Seymour Method of P.D.E.

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Text : P.D.E. of Math. Phys. - Tychonov & Samarski Vol I

linear Second Order P.D.E. - hyperbolic, parabolic, elliptic

$u_x = \frac{\partial u}{\partial x}$ $u_{xx} = \frac{\partial^2 u}{\partial x^2}$

2nd order - highest derivative is u_{xy} or u_{xx}, p.d.e. with 2 independent variables x & y. u = dependent variable u(x,y)

$$\Phi(u_{xx}, u_{yy}, u_{xy}, u_x, u_y, u, x, y) = 0$$

consider linear in highest derivative - quasi linear 2nd order P.D.E.

$a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} + F(x, y, u, u_x, u_y) = 0$

a_{ij} can depend only on u_x, u_y, u, x, y

quasi linear coeff do not depend on highest derivative

linear in highest deriv 2nd order P.D.E. - a_{ij} depend only on (x,y)

Strictly linear - $a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} + b_1 u_x + b_2 u_y + cy + f$

a_{ij}, b_i, c, f depend only on x & y

Simplest type of 2nd order P.D.E has constant coefficients

$\xi = \phi(x,y)$ $\eta = \psi(x,y)$ new coordinates used for transformation of our PDE into one of ξ & η. The chain rule can be used only if new variables are linearly independent thus $\frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$. ie $\begin{vmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{vmatrix} \neq 0$

$u_x = u_\xi \xi_x + u_\eta \eta_x$

$u_y = u_\xi \xi_y + u_\xi \eta_y$

$u_{xx} = u_{\xi\xi} \xi_x^2 + 2u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$

$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$

$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}$

$\bar{a}_{11}u_{\xi\xi} + 2\bar{a}_{12}u_{\xi\eta} + \bar{a}_{22}u_{\eta\eta} + \bar{F}(u_\xi, u_\eta, u, \xi, \eta) = 0$

note $\bar{F}$ can be non zero even if F were zero.

$\bar{a}_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2$

$\bar{a}_{22} = a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2$

$\bar{a}_{12} = a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \xi_y\eta_x) + a_{22}\xi_y\eta_y$

if for linear eq $F(u_x, u_y, u, x, y)$ ⇒ then $\bar{F}(u_\xi, u_\eta, u, x, y)$ is line

choose some fn so that some coeff go to zero

$a_{11}z_x^2 + 2a_{12}z_xz_y + a_{22}z_y^2 = 0 = \bar{a}_{11}$ (A)

1st order nonlinear p.d.e. nonlinear since z_xz_y is included

this is related to o.d.e.

$a_{11}y'^2 - 2a_{12}y' + a_{22} = 0$ (B)

look at $\phi(x,y) = C$ if slope of ϕ satisfies the diff eq

the $z = \phi(x,y)$ is a solution to the P.D.E.

Lemma

If $z = \phi(x,y)$ is a particular solution

of (A) then $\phi(x,y) = C$ is a general integral of the ode (B)

Also the converse

Proof:

if $z = \phi(x,y)$ satisfies (A) $z_x = \phi_x$ $z_y = \phi_y$

if $\phi_y \neq 0$ then A ⇒ $a_{11}\left(\frac{\phi_y}{\phi_y}\right)^2 - 2a_{12}\left(-\frac{\phi_x}{\phi_y}\right) + a_{22} = 0$

But slope of $\phi(x,y) = C$ $\phi_x + \phi_y\frac{dy}{dx} = 0$ ⇒ $\frac{dy}{dx} = -\frac{\phi_x}{\phi_y}$

(B) ⇒ $a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22} = a_{11}\left(-\frac{\phi_x}{\phi_y}\right)^2 - 2a_{12}\left(-\frac{\phi_x}{\phi_y}\right) + a_2$

This nonlinear O.D.E. is a characteristic equation which define
$\xi = \phi(x,y)$ $\eta = \psi(x,y)$, the characteristics of the P.D.E.

this eq has 3 sets of roots : real & distinct, real & eq, complex

thus $$\frac{dy}{dx} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}$$

3 types : Hyperbolic when $a_{12}^2 - a_{11}a_{22} > 0$
Elliptic when $a_{12}^2 - a_{11}a_{22} < 0$
parabolic " " $= 0$

Since a_{ij} can depend on x,y (at the pt you are at) classification is only pointwise. If a_{ij} are nonlinear dependence on derivatives & solution also influence classification

1. Hyperbolic 2 real distinct characteristics - wave phenomena

Let $\xi = \phi(x,y)$ $\eta = \psi(x,y)$
where ϕ & ψ are solutions of characteristic eq (of B)

lemma $\Rightarrow$ $\tilde{a}_{11}, \tilde{a}_{22} = 0$

$\Rightarrow$ A Canonical Form for a hyperbolic p.d.e.
is $u_{\xi\eta} + \Phi(u_\xi, u_\eta, u, \xi, \eta) = 0$

this reduction in form is at the expense that the coeff of the original eq must satisfy the characteristic Equation

2nd Canonical Form is useful since it usually occurs physically

let $\xi = \alpha + \beta$ $\eta = \alpha - \beta$ $\alpha = (\eta + \xi)/2$
$\beta = (\xi - \eta)/2$

$u_\xi = u_\alpha \alpha_\xi + u_\beta \beta_\xi = (u_\alpha + u_\beta)/2$
$u_\eta = u_\alpha \alpha_\eta + u_\beta \beta_\eta = (u_\alpha - u_\beta)/2$

so that $u_{\alpha\alpha} - u_{\beta\beta} + \bar{\Phi}_1 = 0$ where $\bar{\Phi}_1 = \bar{\Phi}_1(u_\alpha, u_\beta, u, \alpha, \beta)$

2. Parabolic   diffusion phenomena   equal roots

one distinct characteristics

$\xi = \phi(x,y)$ where $\phi = C$ satisfying (B)

$\eta = \psi(x,y)$ where ψ is arbitrary independent of ϕ

$a_{12} = \sqrt{a_{11}a_{22}}$

$$\bar{a}_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2 = \left(\sqrt{a_{11}}\,\xi_x + \sqrt{a_{22}}\,\xi_y\right)^2 = 0$$

$$\frac{dy}{dx} = +\frac{a_{12}}{a_{11}} = -\frac{\xi_x}{\xi_y} = -\frac{\phi_x}{\phi_y} = \sqrt{\frac{a_{22}}{a_{11}}}$$

$$\bar{a}_{12} = a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \xi_y\eta_x) + a_{22}\xi_y\eta_y$$

$$\bar{a}_{12} = \underbrace{\left(\sqrt{a_{11}}\,\xi_x + \sqrt{a_{22}}\,\xi_y\right)}_{\overset{\shortparallel}{0}}\left(\sqrt{a_{11}}\,\eta_x + \sqrt{a_{22}}\,\eta_y\right) = 0$$

Canonical form

$$u_{\eta\eta} + \bar{\Phi}(u_\xi, u_\eta, u, x, y) = 0$$

if $\bar{\Phi}$ is independent of u_ξ ⇒ ode   u as a fn of η with ξ as a parameter.

3. Elliptic $a_{12}^2 - a_{11}a_{22} < 0$ distinct complex roots   complex transfor,

$\phi(x,y) = C$ complex eqn. complex integral (B)   Static process [illegible] than 1-D

define $\phi^*(x,y) = C$ complex conjugate

choose $\eta = \phi$ $\xi = \phi^*$ transform again ⇒ $\alpha = \frac{\phi + \phi^*}{2}$

$\beta = \frac{\phi - \phi^*}{2i}$

α & β are real variables

$$\bar{a}_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2$$

$$= (a_{11}\alpha_x^2 + 2a_{12}\alpha_x\alpha_y + a_{22}\alpha_y^2) - (a_{11}\beta_x^2 + 2a_{12}\beta_x\beta_y + a_{22}\beta_y^2$$

$\bar{a}_{11} = 0$ this implies real & imaginary parts are zero

$\bar{\bar{a}}_{11} u_{\alpha\alpha} + 2\bar{\bar{a}}_{12} u_{\alpha\beta} + \bar{\bar{a}}_{22} u_{\beta\beta} + \ldots$

$\bar{\bar{a}}_{11} = \bar{\bar{a}}_{22} \quad \bar{\bar{a}}_{12} = 0 \implies$ Canonical Fn

$u_{\alpha\alpha} + u_{\beta\beta} + \Phi(u_\alpha, u_\beta, u, \alpha, \beta) = 0$

Hyperbolic space & time $\quad u_{xx} - u_{tt} + \phi = 0$

Parabolic $\quad u_{xx} + \Phi(u_t, u_x) = 0$

Example $\quad u_{xx} + y\, u_{yy} = 0$

$a_{12}^2 - a_{11}a_{22} = -y$

Hyperbolic for $-y > 0 \quad y < 0$ | E

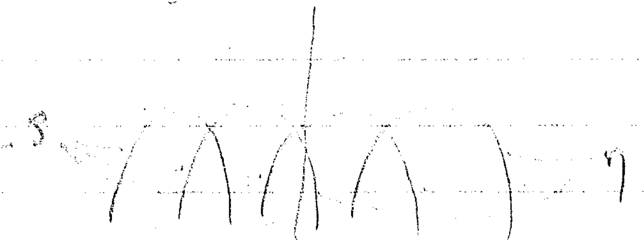
Parabolic for $-y = 0 \quad y = 0$ | P

Elliptic for $-y < 0 \quad y > 0$ | H

$\frac{dy}{dx} = \pm\sqrt{-y} \qquad c + x = \pm 2\sqrt{-y}$

$\xi = x + 2\sqrt{-y}$

$\eta = x - 2\sqrt{-y}$



ξ left hand branch $\quad \eta$ right hand branch (of parabola)

$\xi_x = 1 \quad \xi_y = -\frac{1}{\sqrt{-y}}$

$\eta_x = 1 \quad \eta_y = \frac{1}{\sqrt{-y}}$

$\bar{a}_{11} = 0 = \overset{1}{a_{11}} \xi_x^2 + 2\overset{0}{a_{12}} \xi_x \xi_y + \overset{y}{a_{22}} \xi_y^2$

$= 1 + y\left(-\frac{1}{\sqrt{-y}}\right)^2 = 0$

$\bar{a}_{12} = 2 \qquad \bar{a}_{22} = 0 \qquad \xi_{yy} = -\frac{1}{2(-y)^{3/2}} = -\eta_{yy}$

$4u_{\xi\eta} + y\left(-\frac{1}{2(-y)^{3/2}} u_\xi + \frac{1}{2(-y)^{3/2}} u_\eta\right) = 0$

CANONICAL $4u_{\xi\eta} + (u_\xi - u_\eta)\frac{1}{\ldots} = u_{\xi\eta} + (u_\xi - u_\eta)\frac{1}{\ldots} = 0$

do same as example in class P. 11 2a, c, f Read Ch 2.1 – 2.6

P11 4a,d P24 2

3 Oct. 1972

For parabolic case take $\eta = \eta(x,y)$ equal to either one or the other independent variables

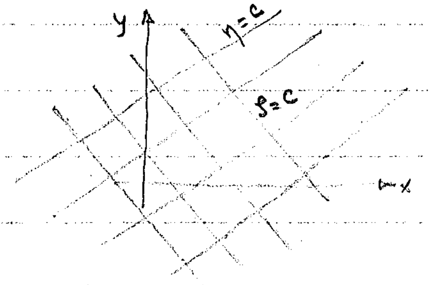
Simplifications of Linear Equations with constant Coeff

$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\left(\frac{dy}{dx}\right) + a_{22} = 0$ can be integrated directly

$$\frac{dy}{dx} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}$$

linear constant coeff characteristics $\quad y = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} x + C$ they are straight lines

$$\xi = y - (+)x \qquad \eta = y - (-)x$$



Canonical form

Hyperbolic: $\begin{cases} u_{\xi\eta} + b_1 u_\xi + b_2 u_\eta + cu + f = 0 \\ u_{\xi\xi} - u_{\eta\eta} + b_1 u_\xi + b_2 u_\eta + cu + f = 0 \end{cases}$ $\quad b_i, c, f =$ constant

Parabolic $\quad u_{\xi\xi} + b_1 u_\xi + b_2 u_\eta + cu + f = 0$

Elliptic $\quad u_{\xi\xi} + u_{\eta\eta} + b_1 u_\xi + b_2 u_\eta + cu + f = 0$

You can make another transformation to get rid of first order derivatives

Let $u = e^{\lambda\xi + \mu\eta} V$ where λ, μ are arbitrary constants. Choose them to eliminate first derivative terms.

$$u_\xi = e^{\lambda\xi+\mu\eta}(V_\xi + \lambda V)$$

$$u_\eta = e^{\lambda\xi+\mu\eta}(V_\eta + \mu V)$$

$$u_{\xi\xi} = e^{\lambda\xi+\mu\eta}(V_{\xi\xi} + 2\lambda V_\xi + \lambda^2 V)$$

$$u_{\eta\eta} = e^{\lambda\xi+\mu\eta}(V_{\eta\eta} + 2\mu V_\eta + \mu^2 V)$$

$$u_{\xi\eta} = e^{\lambda\xi+\mu\eta}(V_{\xi\eta} + \lambda V_\xi + \mu V_\eta + \lambda\mu V)$$

substituting into cannon forms

Elliptic

$$V_{\xi\xi} + V_{\eta\eta} + (b_1 + 2\lambda) V_\xi + (b_2 + 2\mu) V_\eta + (\lambda^2 + \mu^2 + b_1\lambda + b_2\mu + c) V + f_1 = 0$$

where $f_1 = f e^{-(\lambda\xi + \mu\eta)}$

Choose $\lambda = -b_1/2 \qquad \mu = -b_2/2$

$$V_{\xi\xi} + U_{\eta\eta} - \frac{1}{4}(b_1^2 + b_2^2 - 4c) V + f_1(\eta, \xi) = 0$$

Hyperbolic

$$V_{\xi\xi} - V_{\eta\eta} + \gamma V + f_1 = 0 \qquad \mu = b_2/2 \quad \lambda = -b_1/2$$

$$V_{\xi\eta} + \gamma V + f_1 = 0$$

Parabolic

$$V_{\xi\xi} + b_2 V_\eta + \gamma V + f_1 = 0 \qquad \lambda = -b_1/2 \qquad \mu \; .ɔ. \; \gamma = 0$$

Example : $u_{xx} - \frac{1}{a^2} u_{yy} = \alpha u_x + \beta u_y + \gamma u$

let $ay = z$

$$u_{xx} - u_{zz} = \alpha u_x + a\beta u_z + \gamma u$$

$u = V e^{\lambda x + \mu z}$

$$V_{xx} + 2\lambda V_x + \lambda^2 V - (V_{zz} + 2\mu V_z + \mu^2 V) = \alpha(V_x + \lambda V) + a\beta(V_z + \mu V) + \gamma V$$

$$V_{xx} - V_{zz} + (2\lambda - \alpha) V_z + (-2\mu - a\beta) V_z + (\lambda^2 - \mu^2 - \alpha\lambda - a\beta\mu - \gamma) V = 0$$

set $\lambda = \alpha/2 \qquad \mu = -a\beta/2$

$$V_{xx} - V_{zz} + \delta V = 0 \qquad \delta = \left(\frac{\alpha^2}{4} - \frac{a^2\beta^2}{4} - \alpha^2/2 + \frac{a^2\beta^2}{2} - \gamma\right) = \left(\frac{a^2\beta^2}{4} - \frac{\alpha^2}{4} - \gamma\right)$$

$u = V e^{(\alpha x - a\beta z)/2}$

2-D

Hyperbolic Equation : $u_{xx} + u_{yy} - \frac{1}{a^2} u_{tt} = 0$ linear wave Eq

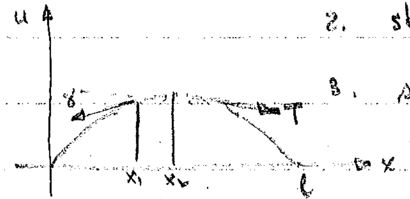
Parabolic Equation : $u_{xx} + u_{yy} - \nu u_t = 0$ Diffusion

Elliptic Eq : $u_{xx} + u_{yy} = 0$ Potential

H-P-E Equation : $u_{xx} + u_{yy} - \frac{1}{a^2} u_{tt} - \nu u_t = 0$ telegraph eq obtain 3 by steady state soln $u = u(x,y)$ only. Both 1 & 2 they describe disturbances $t > 0$, given initial conditions at $t = 0$ & boundary conditions ∀ time. If heat conduction coeff or viscosity is small $\nu u_t \to 0$ to get wave eq. If $a \to \infty$ disturbance travels with infinite speed $\Rightarrow$ $\frac{1}{a^2} u_{tt} \to 0$ to get diffusion eq.

Derivation of Wave Equation - Transverse Vibration of a string

Assumptions: 1. Small vibrations $\frac{\partial u}{\partial x} << 1$ u is vertical displacement.



2. string moves only in one plane $\Rightarrow u(x,t) = u$

3. string has no bending resistance $\Rightarrow$ T acts tangentially at each point.

Newton's Second Law

Elongation of (x_1, x_2) $S' = \int_{x_1}^{x_2} \sqrt{1+(u_x)^2}\, dx \overset{1}{\approx} x_2 - x_1$ = original length

$\Rightarrow T = \text{constant}$

Hooke's law Tension ∝ elongation $\sigma \sim \epsilon$

let T_x = horizontal component of tension $= T\cos\gamma \to T_x(x_1) = T_x(x_2)$

$T\cos\gamma = \frac{T}{\sqrt{1+(u_x)^2}} \sim T(x) \Rightarrow T(x_1) = T(x_2)$ since x_1 & x_2 are arbitrary

$T(x) = T_0$

Resolve vertically $T_u(x) = T(x)\sin\alpha = T_0 \frac{u_x}{\sqrt{1+(u_x)^2}} \sim T \frac{\partial u}{\partial x}$

Vertical Rate of change of momentum in a time $(t_2 - t_1)$ = impulsive force acting over $(t_2 - t_1)$ $\rho = \rho(x)$ force due to tension + external force

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho(\xi)\, u_t(\xi,t)\, d\xi = \int_{t_1}^{t_2} (\text{difference in vertical tension + external force})$$

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho(\xi)\, u_t(\xi,t)\, d\xi = \frac{\partial}{\partial x}\int_{t_1}^{t_2} T_0\, u(x,\tau)\, d\tau + \int_{x_1}^{x_2}\int_{t_1}^{t_2} \rho(\xi) f(\xi,\tau)\, d\xi\, d\tau$$

$f, u \in C^2(x,t)$ use mean value theorem of differential & integral calculus.

$$y(x_2) - y(x_1) = y_{x}(x_1 + \theta)(x_2 - x_1) \quad \text{where } 0 \le \theta \le x_2 - x_1$$

$$(t_2-t_1)\int_{x_1}^{x_2} \rho(\xi)\, u_{tt}(\xi, t^*)\, d\xi \qquad , t^{***}, t^{**}, t^* \in (t_1, t_2), x^{***}, x^{**}, x^* \in (t_1, t_2)$$

$$= (x_2-x_1)\int_{t_1}^{t_2} T_0\, u_{xx}(x^*, \tau)\, d\tau + (x_2-x_1)\int_{t_1}^{t_2} \rho(x^{**}) f(x^{**}, \tau)\, d\tau$$

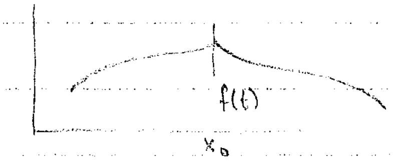
use MV Th of Integral Calc. leads to

$$\therefore\ \Delta t\, \Delta x \quad \rho(x^{***})\, u_{tt}(x^{***}, t^*) = \Delta x\, \Delta t\ T_0\, u_{xx}(x^*, t^{**})$$
$$+ \Delta t\, \Delta x\ \rho(x^{**}) f(x^{**}, t^{***})$$

let $t_1 \to t_2 \quad x_1 \to x_2 \qquad \rho(x) u_{tt} = T_0 u_{xx} + \rho(x) f(x,t)$

when $f(x,t) \equiv 0 \qquad u_{tt} = \frac{1}{a^2} u_{xx} \qquad a = \sqrt{T_0/\rho}$ speed of disturbance

Assume f is discontinuous at $x = x_0$



$x \in (0, x_{0_-}) \qquad u_{tt} - \overset{1}{a} u_{xx} = 0$

$x \in (x_{0_+}, l) \qquad u_{tt} - a^2 u_{xx} = 0$

Jump Condition $\quad u(x_{0_-}, t) = u(x_{0_+}, t)$ contin string

resolve forces $\quad T_0 \left[u_x(x_{0_+}, t) - u_x(x_{0_-}, t)\right] = -f(t)$

Longitudinal Vibrations of a string or rod.

Coordinate Systems

1) Lagrangian tells you where particle x is at time t $\quad x = x(\underset{\sim}{X}, t)$ cartesian

2) Eulerian tells you which particle x is at the pt X $\quad X = X(x,$

let displacement of particle x be $u(x,t)$ $\qquad$ Eulerian $X = x + u(x,t)$

$T(x_1) \longleftarrow \qquad \Delta x \qquad T(x_2)$

$x_1 \qquad\qquad x_2$

Tension $\propto$ relative elongation

$$X = x + u(x, t)$$
$$X = (x + \Delta x) + u(x + \Delta x, t)$$

relative extension of (x_1, x_2) $= \dfrac{\left[\cancel{x} + \cancel{\Delta x} + u(x+\Delta x, t) - \cancel{x} - u(x,t)\right] - \cancel{\Delta x}}{\Delta x}$

$\lim\limits_{\Delta x \to 0} = u_x(x,t) \qquad$ Hooke's law $\Rightarrow \quad T = k(x)\, u_x \quad k$ · Young's modulus

10 Oct 1972

Longitudinal Vibrations

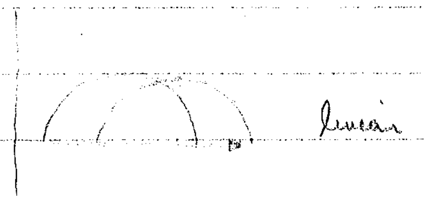
$[k(x)u_x(x)]_x = \rho(x)\, u_{tt} - f(x,t)$ $a = \sqrt{\frac{k(x)}{\rho(x)}}$

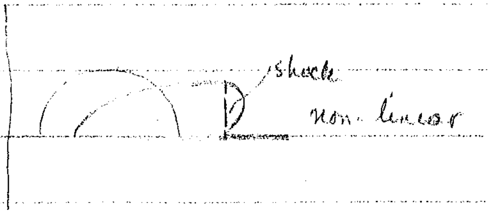
Hooke's Law $T = k(x)u_x$ - linear law - from linear Elasticity

Nonlinear Elasticity $T = k(x,u_x)u_x$

$[k(x,u_x)\,u_x]_x = \rho(x)\,u_{tt}$ $a = \sqrt{\frac{k(x,u_x)}{\rho(x)}}$ wave distortion

since $k = k(u_x)$ also





Electric Disturbances in Conductors (one-dimensional)

i = current strength

V = voltage

$i_{xx} = (CL)\,i_{tt} + (CR+GL)\,i_t + GR\,i$

$v_{xx} =$

C - capacitance R = resistance

L = inductance G = insulation loss   if $G, R \sim 0$ $i_{xx} \sim CL\,i_{tt}$

$a \sim \frac{1}{\sqrt{CL}}$ damping $\sim e^{-(CR+GL)t/2}$ if $i = e^{-(CR+GL)t/2}\, I$

Acoustics - pressure waves or sound

Consider a volume of ideal gas (no friction, no viscous resistance)

Let $\{x_1(t), x_2(t), x_3(t)\}$ be the path of a particle & let $\underset{\sim}{V}(\underset{\sim}{x}, t)$ be the velocity of the particle

Euler   convective

Acceleration of a particle $\frac{D\underset{\sim}{V}}{Dt} = \frac{\partial \underset{\sim}{V}}{\partial t} + \overbrace{\frac{\partial \underset{\sim}{V}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \underset{\sim}{V}}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial \underset{\sim}{V}}{\partial x_3}\frac{dx_3}{dt}}$

$$= \frac{\partial \underset{\sim}{V}}{\partial t} + V_1 \frac{\partial \underset{\sim}{V}}{\partial x_1} + V_2 \frac{\partial \underset{\sim}{V}}{\partial x_2} + V_3 \frac{\partial \underset{\sim}{V}}{\partial x_3}$$

$$= \underset{\sim}{V}_t + (\underset{\sim}{V}\cdot\nabla)\underset{\sim}{V} \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$$

Newton's second law on T $\underset{\sim}{n}$ is the outward normal vector.

$$\iiint_T \rho \frac{D\underset{\sim}{V}}{Dt}\, d\tau = -\iint_S p\underset{\sim}{n}\, ds + \iiint_T \rho \underset{\sim}{F}\, d\tau$$

Green's Theorem

$$\iint_S p\underset{\sim}{n}\, ds = \iiint_T \nabla p\, d\tau$$ & since T is arbitrary

$$\rho \frac{D\underset{\sim}{V}}{Dt} = -\nabla p + \rho\underset{\sim}{F}$$ divide by ρ & use definition of substantial deriv

$$\underset{\sim}{V}_t + (\underset{\sim}{V}\cdot\nabla)\underset{\sim}{V} = -\frac{1}{\rho}\nabla p + \underset{\sim}{F}$$ – 3 equat 5 unknowns

Conservation of mass

$$\frac{\partial}{\partial t}\iiint_\tau \rho\, d\tau = -\iint_S \rho\underset{\sim}{V}\,\underset{\sim}{n}\, ds = -\iiint_\tau \nabla\cdot(\rho\underset{\sim}{V})\, d\tau$$

$$\rho_t + (\underset{\sim}{V}\cdot\nabla) + \rho(\nabla\cdot\underset{\sim}{V}) + \underset{\sim}{V}\cdot\nabla\rho = 0 \qquad \frac{D\rho}{Dt} + \rho\nabla\cdot\underset{\sim}{V} = 0$$

linearize about $\rho_0, p_0, \underset{\sim}{V}=0$ Equation of state $p = f(\rho, S)$ S - entropy

for isentropic flow $p = f(\rho)$ for a polytropic gas $p/p_0 = (\rho/\rho_0)^\gamma$

γ = ratio of specific heats = c_p/c_v

$\rho = \rho_0(1+s)$ $|s| \ll 1$ s - condensation of gas $\frac{\rho - \rho_0}{\rho_0}$

5 eq for $\underset{\sim}{V}, \rho, p$

$$p = p_0 \left(1+s\right)^{\gamma} = p_0\left[1 + \gamma s + O(s^2)\right]$$

$$\underset{\sim}{V}_t + (\underset{\sim}{V}\cdot\nabla)\underset{\sim}{V} = -\frac{1}{\rho}\nabla p + \underset{\sim}{F} \qquad |s| << 1$$

$$\rho_t + \nabla(\rho \underset{\sim}{V}) = 0 \qquad |\underset{\sim}{V}| << a \quad \text{nondim with } a$$

linearize and neglect ∀ products $s\underset{\sim}{V},\ V^2,\ s^2$

$$\nabla p = p_0 \gamma \nabla s + \dots \qquad \frac{1}{\rho}\nabla p = \frac{\gamma p_0}{\rho_0}\left[\frac{\nabla s}{1+s} + \dots\right] = \frac{\gamma p_0}{\rho_0}\nabla s + O(s^2)$$

$$\therefore \quad \boxed{\underset{\sim}{V}_t = -\frac{\gamma p_0}{\rho_0}\nabla s + \underset{\sim}{F}} \quad (1)$$

by cont. $\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\rho_0 + \rho_0 s) = \rho_0 \frac{\partial s}{\partial t}$; $\nabla\cdot(\rho V) = \nabla\cdot(\rho_0 \underset{\sim}{V} + \rho_0 s \underset{\sim}{V}) \approx \nabla\cdot(\rho_0 \underset{\sim}{V}) = \rho_0 \nabla\cdot\underset{\sim}{V}$

since we neglect $s\underset{\sim}{V}$ product

$$\cancel{\rho_0} s_t + \cancel{\rho_0}(\nabla\cdot\underset{\sim}{V}) = 0 \qquad s,\ \underset{\sim}{V}$$

$$\boxed{s_t + \nabla\cdot\underset{\sim}{V} = 0.} \quad (2)$$

$$\nabla(1) - \frac{\partial}{\partial t}(2) = \frac{p_0\gamma}{\rho_0}\nabla^2 s - \nabla\cdot\underset{\sim}{f} - s_{tt} = 0 \qquad \nabla^2 = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)$$

if no body force $\quad a^2\nabla^2 s = s_{tt} \qquad a = \sqrt{\frac{\gamma p_0}{\rho_0}}$

$p, \rho, \underset{\sim}{V}$ satisfy the same eq

1-D $\quad a^2 s_{xx} = s_{tt} \quad$ speed depends on gas constant & equilib const

$a = \sqrt{\frac{dp}{d\rho}}$ for non linear solution $\quad a(s)$

Irrotational flow: $\quad \nabla\times\underset{\sim}{V} = 0 \qquad \underset{\sim}{V} = \nabla u \qquad u$ = scalar veloc. potential

$\nabla\times\nabla u = 0 \qquad a^2\nabla^2 u = u_{tt}$

Initial Conditions & Boundary Condition

$$u_{tt} - a^2 u_{xx} = 0$$

initial conditions $\quad u$ & $u_t,\ u_x$ @ $t = t_0 \quad \forall x \quad$ are specified

boundary cond. $\quad u$ & $u_t,\ u_x$ @ $x = \partial R \quad \forall t \quad$ are specified

if the right no. of IC & BC are prescribed the problem is well posed.
if < right no. problem is non-unique
if > right no. perhaps no solution.

types BC

1) rigid boundaries $u(0,t)=0$ $u(l,t)=0$

2) prescribed moving boundaries $\begin{cases} u(0,t) = \mu_1(t) \\ u(l,t) = \mu_2(t) \end{cases}$

3) Stress Conditions $T \propto u_x$ $\begin{cases} u_x(0,t) = \nu_1(t) \\ u_x(l,t) = \nu_2(t) \end{cases}$

$u_x(0,t)=0$ ⇒ stress free end.

$u_x(l,t) = -\alpha u(l,t)$ where α = impedance $\alpha \to 0$ stress free $\alpha \to \infty$ rigid end

$u_x(l,t) = -\alpha[u(l,t) - \theta(t)]$ elastic const + forcing term $\alpha = k\rho_1/\rho_2$

outside / inside material

Linear B.C.

Type 1) Prescribed displacement $u(0,t) = \mu(t)$

2) " force or stress $u_x(0,t) = \nu_1(t)$

3) elastic constraint $u_x(l,t) = -\alpha u(l,t)$

Initial Conditions

prescribed $u(x,t_0)$, $u_t(x,t_0)$ $0 \le x < l$

Linear Eq + B.C ⇒ superposition principle

i.e. if $u_1(x,t)$ & $u_2(x,t)$ are solutions of wave equation

then $u = u_1 + u_2$ is also a solution.

$$u_{tt} - a^2 u_{xx} = f(x,t) \quad \text{between} \quad 0 \le x \le l \quad t > 0$$

subject to $u(x,t_0) = \phi(x)$ $u_t(x,t_0)\ \psi(x)$

BC. $u(0,t) = \mu_1(t)$ $u(l,t) = \mu_1(t)$

Let $u(x,t) = \sum_{i=1}^{4} u_i$ & u_i satisfies $\frac{\partial^2 u_i}{\partial t^2} = a^2 \frac{\partial^2 u_i}{\partial x^2}$ $i = 1, 2, 3$

$$\frac{\partial^2 u_4}{\partial t^2} = a^2 \frac{\partial^2 u_4}{\partial x^2} + f(x,t)$$

$u_1(0,t) = 0$ $\quad u_2 = \mu_1(t)$ $\quad u_3 = 0$ $\quad u_4 = 0$

$\sum u_i(0,t) = \mu_1(t)$

$u_1(l,t) = 0$ $\quad u_2 = 0$ $\quad u_3 = \mu_2(t)$ $\quad u_4 = 0$

$\sum u_i(l,t) = \mu_2(t)$

$u_1(x,0) = \phi(x)$ $\quad u_2 = 0.$ $\quad u_3 = 0$ $\quad u_4 = 0$

$\sum u_i(x,0) = \phi(x)$

$u_{1_t}(x,0) = \psi(x)$ $\quad u_{2_t}(x,0) = 0$ $\quad u_{3,t} = 0$ $\quad u_{4,t} = 0$

$\sum u_{i,t}(x,0) = \psi(x)$

if boundaries are far enough away for $t < t_{travel}$ solve semi-infinite or infinite case.

thus split it up a) $0 < t < t_1$ $\quad t \geq t_1$ - first part of disturbance reaches bound.

p. 34 3,6,9

17 Oct, 1972

Typical Problem in Acoustics

s = condensation: $S_{tt} = c^2 \nabla^2 s$ ← small amplitudes only $\quad p, \rho, U$ $\quad \underset{\sim}{v} = \nabla U$ for $\nabla \times \underset{\sim}{v} = 0$

Explosion in the atmosphere ⟹ large amplitude

"Explosion" Theory needed. By geometry @ $t = t_0$ The disturbance is small amplitude & it occupies the region D_0. we now know that all disturbances travel with speed c.

Ask: at time $t_1 > t_0$ at the point $\underset{\sim}{y}$ is the atmosphere disturbed

$c(t_1-t_0)$ is distance disturbance in D_0 has moved

denote a sphere with $r = c(t_1-t_0)$. If D_0 intersects the sphere then $\underset{\sim}{y}$ will be disturbed at time t_1. i.e. if at t_0, D_0 is $c(t_1-t_0)$ away from $\underset{\sim}{y}$ then $\underset{\sim}{y}$ will feel the disturbance at t_1; if not, either D_0 is outside the sphere or inside the sphere but $\underset{\sim}{y} \notin D_0$

Given: $t = t_0$ disturbance occupies D_0

Explosion theory would give us everything about the disturbance in D_0

$$\text{I.C.:} \quad u(\underset{\sim}{x}, t_0) = \begin{cases} a(\underset{\sim}{x}) & \underset{\sim}{x} \in D_0 \\ 0 & \underset{\sim}{x} \notin D_0 \end{cases}$$

u is velocity potential

$$u_t(\underset{\sim}{x}, t_0) = \begin{cases} b(\underset{\sim}{x}) & \underset{\sim}{x} \in D_0 \\ 0 & \underset{\sim}{x} \notin D_0 \end{cases}$$

B.C. on the earth $E(\underset{\sim}{x}) = 0$ where $N(\underset{\sim}{x})$ is the normal to E

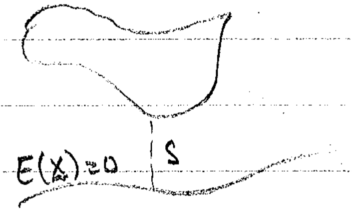
$\underset{\sim}{V} \cdot \underset{\sim}{n} = 0$ on $E(\underset{\sim}{x}) = 0$ no normal comp of velocity to earth

since $\underset{\sim}{V} = \nabla u$ $\quad n_1 \frac{\partial u}{\partial x_1} + n_2 \frac{\partial u}{\partial x_2} + n_3 \frac{\partial u}{\partial x_3} = 0$ B.C. on $E(\underset{\sim}{x})$

since dist travels at finite speed we can treat 2 separate problems

1 before earth contact 2. after earth contact

Problem 1 if $S = \min |D_0 - E|$, then time taken for disturbance to first reach $E = S/c = t_m$

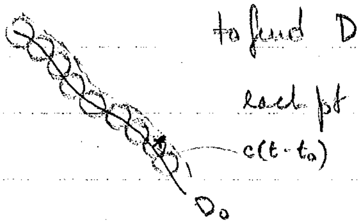


For $t < t_0 + t_m$ solve a pure initial value problems i.e. given I.C. - solve in infinite domain - Cauchy Initial Value Problem

Problem 2 - an initial B.V. Problem: Given u, u_t $\forall \underset{\sim}{x}$ @ $t = t_0 + t_m$

Given B.C. on $E(\underset{\sim}{x}) = 0$

Huygen's Principle - a surface governed by the wave equation moves normal to itself at speed c



to find D_0 at $t > t_0$ draw a circle about each pt with radius $c(t-t_0)$ & draw envelope

Type 1 : prescribed initial & boundary conditions

① $u(x,0) = \phi(x)$

$u_t(x,0) = \psi(x)$ $\quad 0<x<\ell$

$u(0,t) = \mu_1(t)$

$u(\ell,t) = \mu_2(t)$

is there a unique solution of

$$\rho u_{tt} - \frac{\partial}{\partial x}(k u_x) = F(x,t) \quad \text{subject to (1) ?} \qquad (2)$$

Assume there are two distinct solns $u_1(x,t)$, $u_2(x,t)$

Consider $u_1 - u_2 = v$ $\quad$ show $v \equiv 0$

since equas are linear if u_1, u_2 are solns the v must also be a soln

V satisfies ② with $F=0$ $\quad \left[\rho V_{tt} = \frac{\partial}{\partial x}(k V_x)\right.$

I.C. for V : $\quad V(x,0) = V_t(x,0) \equiv 0 \qquad u_1(x,0) - u_2(x,0) = V(x,0) = \phi(x) - \phi(x)$

B.C. for V : $\quad V(0,t) = V(\ell,t) \equiv 0$

Define a fn $E(t) = \frac{1}{2}\int_0^\ell \left[k(V_x)^2 + \rho(V_t)^2\right] dx$

differentiate wrt t $\quad E(t)_{,t} = \int_0^\ell \left[k V_x V_{xt} + \rho V_t V_{tt}\right] dx$

$$= \int_0^\ell \left[\rho V_t V_{tt}\right] dx + k V_x V_t \Big|_0^\ell - \int_0^\ell V_t \frac{\partial}{\partial x}(k V_x)\, dx$$

$$= \int_0^\ell (\rho V_t V_{tt}\, \underset{0}{\cancel{dx - V_t \frac{\partial}{\partial x}(k V_x)}})\, dx + \left[k V_x V_t\right]_0^\ell$$

since $V(0,t) = V(\ell,t) = 0$

$\therefore \quad E(t)_{,t} = 0 \qquad \therefore \quad E(t) = \text{constant} = E(0)$

But $E(0) \equiv 0$ by I.C. $V_t = 0$ $V_x = 0$

$\Rightarrow E(t) \equiv 0 \ \forall t$

Since k & $\rho > 0$ $V_x, V_t \equiv 0 \ \forall t, x$

$\therefore$ $V(x,t) \equiv 0 \ \forall (t,x)$ since $V = 0$ at $t = 0$

$\Rightarrow u_1 = u_2$ & soln unique.

Type $\underline{\text{III}}$ prescribed initial & b.c.

$$u(x,0) = \phi(x)$$

$$u_t(x,0) = \psi(x)$$

$$u_x(0,t) = h_1[u(0,t) - \theta_1(t)]$$

$$u_x(l,t) = -h_2[u(l,t) - \theta_2(t)]$$

$$\rho u_{tt} - \frac{\partial}{\partial x}(k u_x) = F(x,t)$$

Show uniqueness $V = u_1 - u_2$

I.C. $V(x,0) = V_t(x,0) = 0$

B.C. $V_x(0,t) = h_1 V(0,t)$

$V_x(l,t) = -h_2 V(l,t)$

$$\frac{dE}{dt} = \int_0^l [k V_x V_{xt} + \rho V_t V_{tt}] dx$$

$$[k V_x V_t]_0^l = [k V_x(l,t) V_t - k V_x(0,t) V_t]$$

$$= -k h_2 V V_t(l,t) - k h_1 V V_t(0,t)$$

$$= -\frac{k}{2}\frac{d}{dt}[h_2 V^2(l,t) + h_1 V^2(0,t)]$$

$$\frac{dE}{dt} = \int_0^l V_t \left(\rho V_{tt} - \frac{\partial}{\partial x}(k V_x)\right) dx - \frac{k}{2}\frac{d}{dt}[h_2 V^2(l,t) + h_1 V^2(0,t)]$$

Integrate wrt t

0 Equation

$$E(t) - E(0) = \int_0^t\int_0^l V_t\left(\rho V_{tt} - \frac{\partial}{\partial x}(k V_x)\right) dx\, dt - \frac{k}{2}[h_2\{V^2(l,t) - V^2(l,0)\}$$

$E(t) = -\frac{k}{2}\left\{ h_2 V^2(l,t) + h_1 V^2(0,t) \right\}$

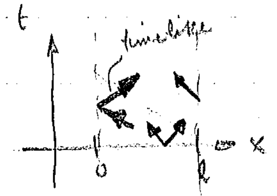
$h_1 \,\&\, h_2, k \geq 0 \implies E(t) \leq 0$

but by definition $E(t) \geq 0$ then $E(t) = 0$

again $V(x,t) \equiv 0$ & $u_1 = u_2$ soln is unique.

Type II similar :

Phase space



$y = x \pm ct$

Rule : you need as many boundaries condition at a pt as there are time
initial
like characteristics at that pt.

time like characteristics - one with increasing time leaving a boundary (of a domain in (x,t) space.

D'Alembert Solution

For the pure initial value problem for an infinite string

$u_{tt} - a^2 u_{xx} = 0$

with IC $u(x,0) = \phi(x)$, $u_1(x,0) = \psi(x)$

Characteristics : $dx/dt = \pm a$

choose $\xi = x - at$ $\eta = x + at$

$u_{\xi\eta} = 0$ ①

$u_\xi = \bar{f}_1(\xi)$ $\bar{f}_1$ is arbitrary fn of ξ

$u(\xi,\eta) = \int^{\xi} \bar{f}_1(s)\,ds + f_2(\eta)$

$u(\xi,\eta) = f_1(\xi) + f_2(\eta)$ - General Solution of ①

$u(x,t) = f_1(x-at) + f_2(x+at)$

wave propagating $\forall x > 0$ at speed a → wave propagating $\forall x < 0$ speed a

$u(x,t) = f(x-at)$

$\bar{u}(\eta,t) = f(\xi)$ indep of t

@ $t=0$ $u = \varphi(x)$ $f_1(x)+f_2(x) = \varphi(x)$

@ $t=0$ $u_t = \psi(x)$ $-af_1'+af_2' = \psi(x)$

$-f_1+f_2 = \frac{1}{a}\int_{x_0}^{x}\psi(s)ds$

$2f_2(x) = [\phi + \frac{1}{a}\int_{x_0}^{x}\psi(s)ds$

$2f_1(x) = [\varphi - \frac{1}{a}\int_{x_0}^{x}\psi(s)ds]$

$$u(x,t) = \frac{1}{2}\left|\varphi(x-at) - \frac{1}{2a}\int_{x_0}^{x-at}\psi ds\right| + \frac{1}{2}\left|\varphi(x+at) + \frac{1}{2a}\int_{x_0}^{x+at}\psi ds\right.$$

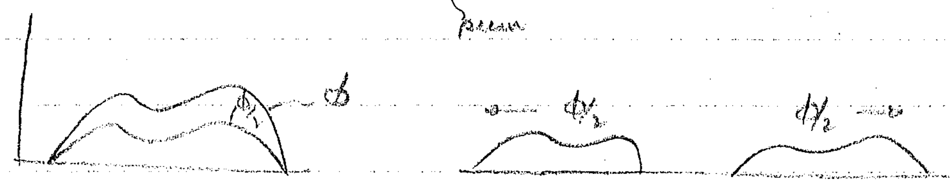
$$= \frac{1}{2}[\varphi(x-at)+\varphi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(s)ds$$

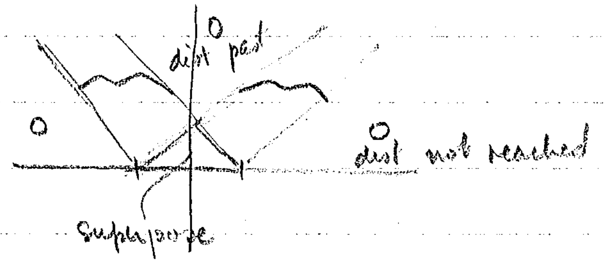
if $\Psi(y) = \frac{1}{2a}\int_{x_0}^{y}\psi(s)ds$

Solution is superposition of 2 solns

1) $\Psi \equiv 0$ $\phi(x) \neq 0$

$$u(x,t) = \frac{1}{2}[\phi(x-at) \underbrace{+}_{\text{sum}} \phi(x+at)]$$





0 - no disturbance

2) $\Psi \neq 0 \quad \varphi(x) = 0$

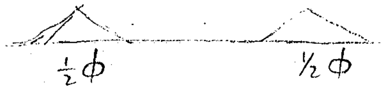
$$u(x,t) = \Psi(x+at) - \Psi(x-at)$$

difference

Given $\psi(x)$ workout $\Psi(x)$

P. 63 1a, 2, 5

24 Oct 1972



if $\phi \neq 0$ but $\psi = 0$ @ $t=0$

2) $u_t(x,0) = \psi(x) \qquad \phi = 0 \quad @ \ t=0$

$$u(x,t) = \frac{\phi(x+at) + \phi(x-at)}{2} + \frac{1}{2a}\left[\Psi(x+at) - \Psi(x-at)\right]$$

$$\Psi(y) = \frac{1}{2a}\int^{y} \psi(s)\,ds$$

Suppose you're given $\psi(x) = 0 \quad x < x_1$

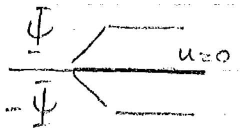
$\psi(x) = k \quad x_1 < x < x_2$

$= 0 \quad x > x_2$

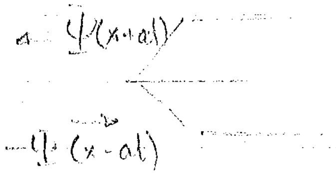
$$\Psi(y) = 0 \qquad x < x_1$$

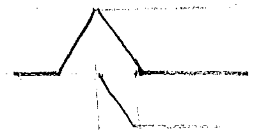
$$= \frac{k}{2a}(x - x_1) \qquad x_1 \leq x \leq x_2$$

$$= \frac{k}{2a}(x_2 - x_1)$$



$t > 0$





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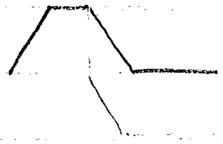
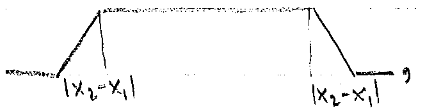
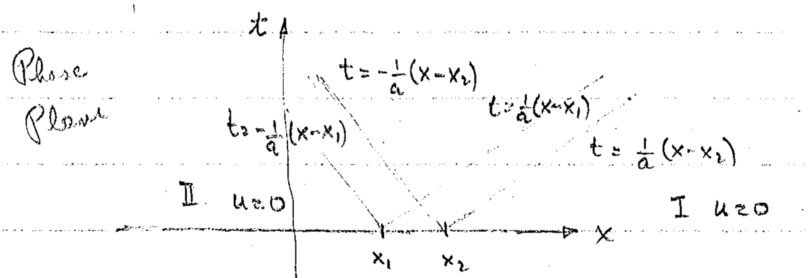


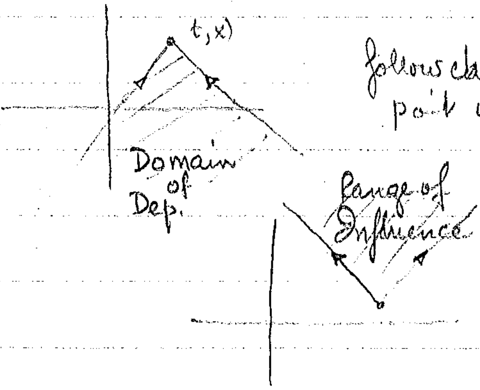


, as $|x_2 - x_1| \to 0$



Domain of Dependence

Range of Influence



follows characteristics from a point in phase plane

Range of influence

Continuous Dependence of Soln on Initial Data - if change data by a little, soln changes only by a little bit.

Stability of Solutions or continuous dependence.

For any interval $0<t<t_0$ and any $\epsilon>0$, $\exists$ a $\delta=\delta(\epsilon,t_0)>0$ $\ni$ two solutions $u_1(x,t)$ & $u_2(x,t)$ differ by an amount less than ϵ provided the initial cond differ by an amount less than δ

$$|u_1(x,t) - u_2(x,t)| < \epsilon$$

whenever $|\phi_1 - \phi_2| < \delta$ & $|\psi_1 - \psi_2| < \delta$

$$u_i(x,t) = \frac{1}{2}\left[\phi_i(x+at) + \phi_i(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \psi_i(s)\,ds$$

$$u_1 - u_2 = \frac{1}{2}\left[\phi_1(x+at) - \phi_2(x+at)\right] + \frac{1}{2}\left[\phi_1(x-at) - \phi_2(x-at)\right]$$

$$+ \frac{1}{2a}\int^{x+at} \left[\psi_1(s) - \psi_2(s)\right] ds$$

$$|u_1 - u_2| < \frac{1}{2}\delta + \frac{1}{2}\delta + \frac{1}{\cancel{2a}}\delta \cdot \cancel{2a}t \le \delta + \delta t_0$$
$$< \delta(1+t_0)$$

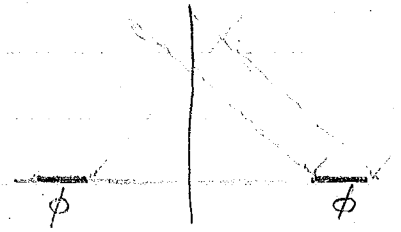
choose $\delta = \epsilon / _{1+t_0}$

if ϵ is fixed then when $t_0 \uparrow, \delta \downarrow$.

Bounded Interval fixed or free boundary.

$u=0$ b.c. $\quad \varphi(x) \Rightarrow -\varphi(-x)$

$\frac{\partial u}{\partial x} = 0$ bc $\quad \varphi(x) \Rightarrow \varphi(-x)$



Continuation evenly or oddly about $x=0$

1) If ψ & ϕ are odd wrt x_0 then $u(x_0,t) = 0$

2) If ψ & ϕ are even wrt x_0 then $u_x(x_0,t) = 0$

Choose $x_0 = 0$

1) $\quad \phi(x) = -\phi(-x)$
$\quad \psi(x) = -\psi(-x)$

$$u = \frac{1}{2}\left[\phi(x-at) + \phi(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(s)ds$$

$$u(0,t) = \frac{1}{2}\left[\phi(-at) + \phi(at)\right] + \frac{1}{2a}\int_{-at}^{at}\psi(s)ds$$
$$= 0$$

2) $\quad \phi(x) = \phi(-x) \qquad \phi'(x) = -\phi'(-x)$
$\quad \psi(x) = \psi(-x)$

$$u_x(x,t) = \frac{1}{2}\left[\phi'(x-at) + \phi'(x+at)\right] + \frac{1}{2a}\left[\psi(x+at) - \psi(x-at)\right]$$

$$u_x(0,t) = \frac{1}{2}\left[\phi'(-at) + \phi'(at)\right] + \frac{1}{2a}\left[\psi(at) - \psi(-at)\right]$$

Solve ①   I.C. $\left.\begin{matrix} u = \phi(x) \\ u_x = \psi(x) \end{matrix}\right\}$ $x > 0$ @ $t = 0$

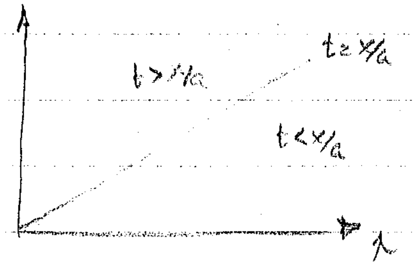
B.C. $u(0,t) = 0 \quad \forall t$

Define new I.C. on $-\infty < x < \infty$

$$\Phi(x) = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \end{cases}$$

$$\Psi(x) = \begin{cases} \psi(x) & x > 0 \\ -\psi(-x) & x < 0 \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[\Phi(x+at) + \Phi(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \Psi \, ds$$

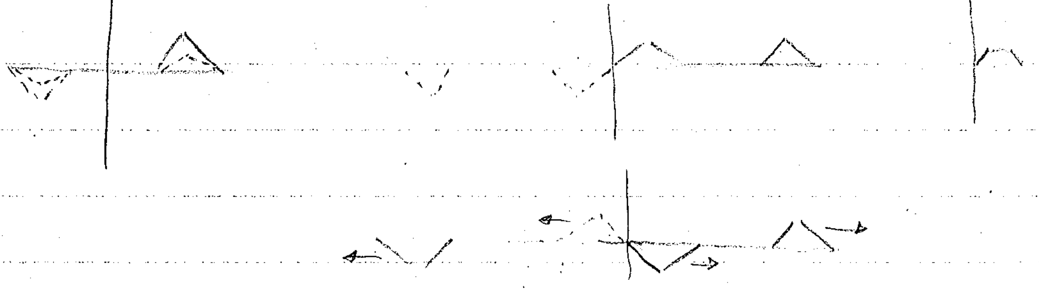


for $t < x/a$ Domain of dep. doesn't depend on b.c.

$t < x/a$ prob is original soln $u(x,t) = \frac{1}{2}\left[\phi(x+at) + \phi(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s) ds$

$t > \frac{x}{a}$ $u = \frac{1}{2}\left[\phi(x+at) - \phi(at-x)\right] + \frac{1}{2a}\int_0^{x+at} \psi(s) ds + \frac{1}{2a}\int_{at-x}^{0} \psi(y) dy$

Example: $\psi = 0$ $\phi = \wedge$



2) $\Phi(x) = \begin{cases} \phi(x) & x>0 \\ \phi(-x) & x<0 \end{cases}$

$\Psi(x) = \begin{cases} \psi(x) & x>0 \\ \psi(-x) & x<0 \end{cases}$

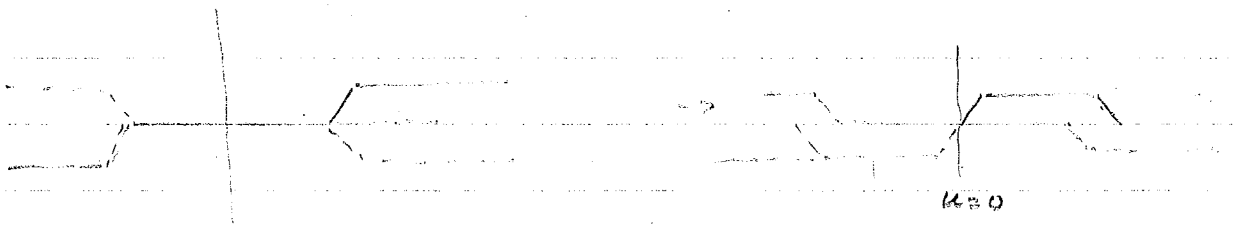
$$u(x,t) = \frac{1}{2}\left[\Phi(x+at) + \Phi(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(s)ds$$

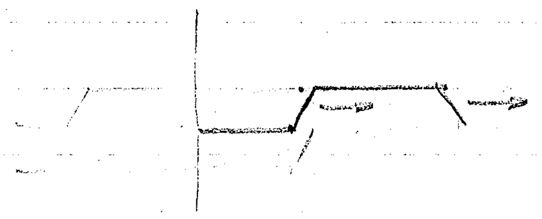
$t < x/a \quad u(x,t) = \frac{1}{2}\left[\phi(x+at) + \phi(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)ds$

$t > x/a \quad u(x,t) = \frac{1}{2}\left[\phi(x+at) + \phi(at-x)\right] + \frac{1}{2a}\int_{0}^{x+at} \psi(s)ds$

$+ \frac{1}{2a}\int_{0}^{at-x} \psi(y)dy$

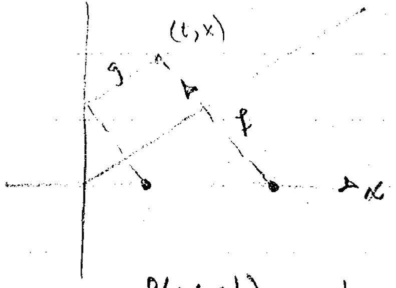
$\phi = 0 \quad \psi \neq 0$





final shape moving to right
fixed shape

2^{nd} method for $t > x/a$



follow charact back

Initially $\frac{1}{2}\left[\phi(x+at) + \phi(x-at)\right] + \frac{1}{2a}\left[\Psi(x+at) - \Psi(x-at\right.$

$\Psi = \int_0^y \psi(s)ds$

$u(x,t) = f(x+at) + g(x-at)$

$f(x+at) = \frac{1}{2}\phi(x+at) + \frac{1}{2a}\Psi(x+at)$

$g(x+at)$ at $x=0$, $u=0$ $u(0,t)=0 = g(-at) + f(at)$

from $u = g+f$ $f(at) = -g(-at)$

or $f(y) = -g(-y)$

or $g(y) = -f(-y)$

$$f(z) = \frac{1}{2}\phi(z) + \frac{1}{2a}\Psi(z)$$

$$g(x-at) = -f(at-x) = -\frac{1}{2}\phi(at-x) - \frac{1}{2a}\Psi(at-x)$$

$$u(x,t) = \frac{1}{2}\left[\phi(x+at) - \phi(at-x)\right] + \frac{1}{2a}\int_{at-x}^{x+at}\Psi(s)\,ds$$

P. 63 1b, 4, 6

10-31-72

$x = f(t)$ Eulerian coord of piston i 4

$u_{tt} = a^2 u_{xx}$ Lagrangian Coord. $X = x + u(x,t)$ $\frac{\partial X}{\partial x} = 1 + u_x$

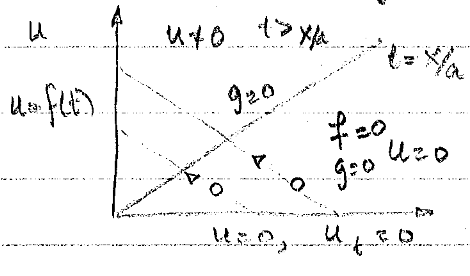
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0$

$$\left.\frac{\partial u}{\partial t}\right|_{X} = \left.\frac{\partial u}{\partial t}\right|_{x} + \frac{\partial X}{\partial t}\frac{\partial u}{\partial x} = \left.\frac{\partial u}{\partial t}\right|_{x} + v\frac{\partial}{\partial x}(u)$$

$$\left.u_{tt}\right|_{X} = \frac{\partial}{\partial t}\left[u_t + cu_x\right] + c\frac{\partial}{\partial x}\left[u_t + cu_x\right] = a^2 u_{xx}$$

$$u_{tt} + 2c\,u_{xt} + c^2 u_{xx} = a^2 u_{xx} \quad \text{or} \quad u_{tt} + 2c\,u_{xt} + (c^2 - a^2)u_{xx} = 0$$

Homogeneous & Non homogeneous b. c.



$x \geq 0$

$u = F(x-at) + G(\cancel{x+at})$

$f(t) = F(-at)$ @ $x=0$

$f(-z/a) = F(z)$

$$u = f\left(-\frac{1}{a}(x-at)\right) = f\left(t - x/a\right)$$

$u_{tt} = a^2 u_{xx}$ $x > 0$

$$\left.\begin{array}{l} t=0 \quad u=u_t=0 \\ \text{Soln:} \quad u(x,t)=0 \quad \text{for} \quad t<x/a \\ \qquad f(t-x/a) \quad " \quad t>x/a \end{array}\right\} \text{B.C.}$$

at $x=0$ $u=0$

$$\left.\begin{array}{l} t=0 \quad u=\phi \;, u_t=\psi \\ u(x,t)=\begin{cases} \dfrac{\phi+\phi}{2}+\int\psi & (t<x/a) \\ \dfrac{\phi-\phi}{2}+\displaystyle\int_{-(x-at)}^{x+at}\psi & (t>x/a) \end{cases} \end{array}\right\} \text{I.C.}$$

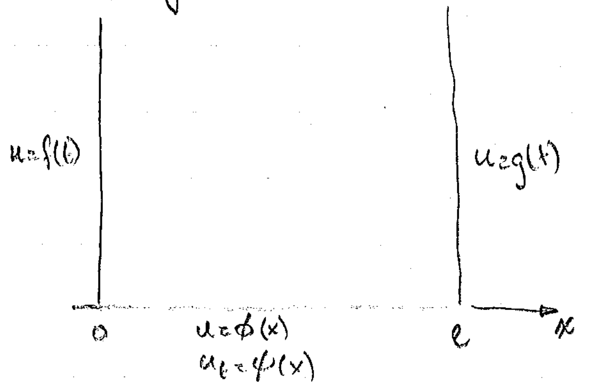
Superposition gives

$X=0 \quad u=f(t) \qquad t=0 \quad u=\phi \quad u_t=\psi$

$$u(x,t)=\begin{cases} \dfrac{\phi+\phi}{2}+\int\psi & t<x/a \\ f(t-x/a)+\dfrac{\phi+\phi}{2}+\displaystyle\int_{at-x}\psi & t>x/a \end{cases}$$

This is only done when problem is linear

Bounded Interval.



homogeneous b.c. & zero initial b.c.

$u=f(t-x/a)+g(\frac{x}{a}+t)$ @ $x=0$ $u=0$ $\therefore -f(t)=g(t)$

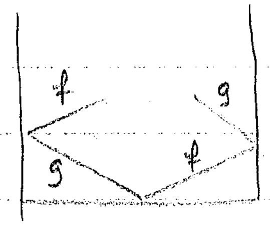
$\therefore g(y)=-f(y)$ ① @ $x=l$ $u=0$ $\therefore f(t-l/a)+g(t+l/a)=0$

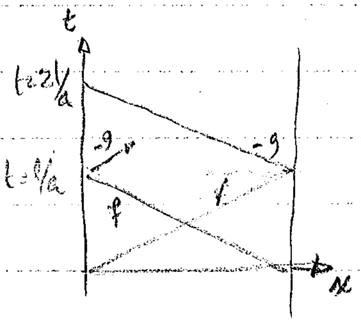
$\therefore -f(t-l/a)=g(t+l/a)=-f(t+l/a)$ ② $\Rightarrow$ f period of period $\frac{2l}{a}$

$u=f(t-x/a)-f(t+x/a)$ where $f(y)=f(y\pm\frac{2ln}{a})$ $n=1,2,\ldots$

f is unknown ϕ & ψ are known

find $f(\phi, \psi)$





find f as a fn of its argument from $t=0, 2l$

between $0 \le t \le l/a$ f is determined by initial conditions but

$$g(x/a+t) = \frac{1}{2}\phi(x+at) + \frac{1}{2a}\int_0^{x+at}\psi(s)\,ds$$ dalembert Soln for infinite Line wave propagating to left

but $g(y) = -f(y)$ at $x=0$ by ①

$$f(t) = -g(t) = -\tfrac{1}{2}\phi(at) - \frac{1}{2a}\int_0^{at}\psi(s)\,ds \qquad 0 < t \le l/a$$

now we need $f(t)$ for $l/a < t < \frac{2l}{a}$

Leaving $t=0$

$$f(t-x/a) = \phi/2\,(x-at) + \frac{1}{2a}\int_0^{x-at}\psi(s)\,ds$$

on $x=l$ $$f(t-l/a) = \frac{\phi}{2}(l-at) + \frac{1}{2a}\int_0^{l-at}\psi(s)\,ds \qquad 0<t<l/a$$

$$= -g(t+l/a)$$ by ② from B.C.

$$g(t+l/a) = -\frac{1}{2}\phi(l-at) - \frac{1}{2a}\int_0^{l-at}\psi(s)\,ds$$

$$= -f(t+l/a)$$ by ①

let $t+l/a = T$

$$f(T) = \frac{1}{2}\phi(-aT+2l) + \frac{1}{2a}\int_0^{2l-aT}\psi(s)\,ds \qquad \frac{l}{a} < T < 2l/a$$

$$f(t) = -\tfrac{1}{2}\phi(at) - \frac{1}{2a}\int_0^{at}\psi(s)\,ds \qquad 0<t<l/a$$

$$\tfrac{1}{2}\phi(2l-at) + \frac{1}{2a}\int_0^{2l-at}\psi(s)\,ds \qquad l/a<t<2l/a$$

$$\frac{2(n-1)l}{a} < t < \frac{2nl}{a} \qquad \text{for some } n$$

$$0 < t - \frac{2(n-1)l}{a} < \frac{2l}{a}$$

$$u = f(t - x/a) - f(t + x/a)$$

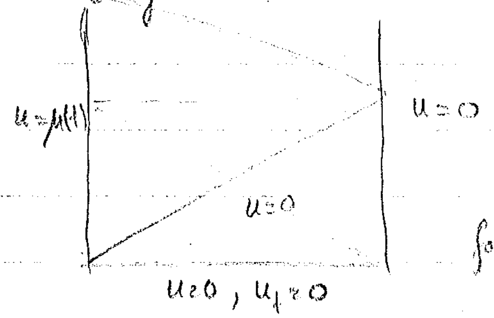
$$\pm 2nl/a$$

Given t, x choose m & n $\ni$. $t - x/a - \frac{2nl}{a}$ & $t + \frac{x}{a} - \frac{2ml}{a}$ lie in an interval $(0, 2l/a)$

Suppose $t_0 - \frac{x_0}{a} = 7.5$ $\quad \frac{2l}{a} = 1$ $\quad t_0 + \frac{x_0}{a} = 8.1$

$$f\left(t_0 - \frac{x_0}{a}\right) = f\left(t_0 - x/a_0 - 7\right) = f(.5)$$
$$f\left(t_0 + \frac{x_0}{a}\right) = f\left(t_0 + \frac{x_0}{a_0} - 8\right) = f(.1)$$
$$u(x_0, t_0) = f(.5) - f(.1)$$

Non Homogeneous B.C. & zero initial cond.



$$u = f(t - x/a) + g(t + x/a)$$

for $0 < t < 2l/a$ no reflected wave at $x = 0$ bound $\therefore$ $g(t) = 0$

B.C. on $x = l$ gives

$$f(t - l/a) = -g(t + l/a) \quad g(z) = -f(z - 2l/a) \quad ①$$

$$u = f(t - x/a) - f(t + x/a - 2l/a)$$

f is unknown & we need $f(\mu)$

B.C. on $x=0$ $\quad u = \mu(t) = f(t) - f(t - 2l/a)$ $\quad$ difference equation to determine f.

Let $\frac{2l(n-1)}{a} < t \leq \frac{2ln}{a}$ $\quad$ since $\mu(t)$ is being added at each period ∴

$$f(t) - f\left(t - \frac{2l}{a}\right) = \mu(t)$$

$$f\left(t - \frac{2l}{a}\right) - f\left(t - \frac{4l}{a}\right) = \mu(t - 2l/a)$$

$$\vdots$$

$$f\left(t - \frac{2l(n-1)}{a}\right) - 0 = \mu\left(t - \frac{2l(n-1)}{a}\right) \quad \text{by ① } \quad g(t) = 0$$

add all

$$f(t) - 0 = \sum_{k=0}^{n-1} \mu\left(t - \frac{2lk}{a}\right) ; \quad u(x,t) = \sum_{k=0}^{n-1} \mu\left(t - \frac{x}{a} - \frac{2lk}{a}\right) - \sum_{k=0}^{n-1} \mu\left(t + x/a - \frac{2l(k+}{a}\right.$$

for $\frac{2(n-1)l}{a} < t - x/a < \frac{2ln}{a}$

If μ is periodic Assume $\mu = H(\omega t)$ $\quad \frac{1}{\omega}$ = period $\quad$ freq = ω

Assume f has period $\quad$ ″ $\quad$ ″

Consider $\frac{1}{\omega} = \frac{2l}{a}(1+\epsilon)$

$$f(t) - f\left(t - \frac{2l}{a} + \frac{n}{\omega}\right) = H(\omega t)$$

$$-f\left(t - \frac{2l}{a} + \frac{1}{\omega}\right) \quad = H(\omega t) \quad \text{since}$$

$$-f\left(t - \frac{1}{\omega}(1+\epsilon) + \frac{1}{\omega}\right)$$

$$-f\left(t - \frac{\epsilon}{\omega}\right)$$

$$f(t) - f(t - \epsilon/\omega) = H(\omega t)$$

$$f(t) - f(t) + \frac{\epsilon}{\omega} f'(t) \cong H(\omega t)$$

$$f(t) = \frac{\omega}{\epsilon} \int^{t} H(\omega t)\, dt$$

1. HW. solve $f(t) - f(t - \frac{2l}{a}) = A \sin(2\pi\omega t)$

solve for $f(t)$

use periodicity & let $\frac{2l}{a} = \frac{1}{\omega}(1+\epsilon)$

2. Form the diff equation for the same problem

when $u = f\left[t - \frac{x}{a}\{1 + M f(t - x/a)\}\right] - f\left[t + (\frac{x}{a} - \frac{2l}{a})\{1 + M f(t + \frac{x}{a})\}\right]$

use this in b.c. $u = A \sin 2\pi\omega t$ on $x = 0$

Let $\omega = \frac{a}{2l}$ & show that if $|f| \ll 1$ that f is not unbounded but $\propto A^{1/2}$ (use periodicity M is a constant.

P65 #8,9,10 read section 2.2(6)

11-14-72

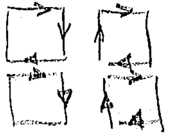
Integral Equn for linear wave equation

$$\int_{x_1}^{x_2} [u_t(t_2) - u_t(t_1)]\, dx = a^2 \int_{x_1}^{t_2} [u_x(x_2) - u_x(x_1)]\, dt - \iint_{x_1 t_1}^{x_2 t_2} f(x,t)\, dt\, dx$$

Prove $\int_c \left(\frac{\partial u}{\partial t} dx + a^2 \frac{\partial u}{\partial x} dt\right) + \iint_G f\, dx\, dt = 0$ ①

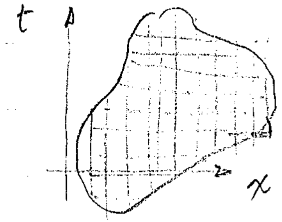
Where C is any contour in (x,t) space with interior G $C = \partial G$

divide G into rectangles



interior contribution = 0

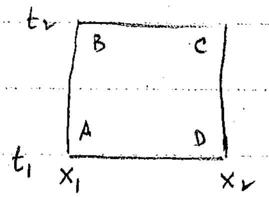
only contribution will be on approx of C. take limit as meshspace → 0



$\lim_{\square \to 0} \bar{C} \to C$

Show $\int_C + \iint_G = \int_x - a^2 \int_t + \iint_{xt}$

① $\Rightarrow$



$$\int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} + \iint_{\Box} = 0$$

AB: $dx=0$

$$\int_{AB} = a^2 \int_{t_1}^{t_2} \frac{\partial u}{\partial x}(x_1)\,dt$$

BC: $dt=0$

$$\int_{BC} = \int_{x_1}^{x_2} \frac{\partial u}{\partial t}(t_2)\,dx$$

CD

$$\int_{CD} = -a^2 \int_{t_1}^{t_2} \frac{\partial u}{\partial x}(x_2)\,dt$$

$$\int_{DA} = -\int_{x_1}^{x_2} \frac{\partial u}{\partial t}(t_1)\,dx$$

$$-a^2 \int_{t_1}^{t_2} [u_x(x_2) - u_x(x_1)]\,dt + \int_{x_1}^{x_2} \left[\frac{\partial u}{\partial t}(t_2) - \frac{\partial u}{\partial t}(t_1)\right] dx + \iint_{\Box} f\,dx\,dt$$

$\Rightarrow$ original equns of motion

ie equn (1) equiv to equat of motion around any rectangle

Take path $\int$ around all rectangles of an arbitrary grid covering G

Then all interior lines covered twice (in opposite directions) giving zero contrib

Result is $\int_{\bar{C}(n)} (u_t dx + u_x dt) + \iint_{\bar{G}(n)} f = 0$ where $\bar{C}(n)$ is rectangular path approx C with n grid lines.

as $n \to \infty$ $\quad \bar{C}(n) \to C \quad \bar{G}(n) \to G$

(C is an arbitrary path) Hence result.

Initial Value Problem

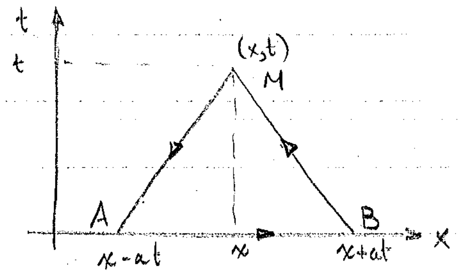
Consider $u(x,t)$ is piecewise smooth on $(-\infty, \infty)$ & satisfies

$$\int (u_t dx + a^2 u_x dt) + \iint f = 0$$

subject to the init conds $u(x,0) = \varphi(x)$ $u_t(x,0) = \psi(x)$

φ is piecewise smooth & ψ is piecewise cont.

C lies in $t \geq 0$



MA $\frac{dx}{dt} = a$

$$u_t dx + a^2 u_x dt = a\left[u_t dt + u_x dx\right] = a\,du$$

BM $\frac{dx}{dt} = -a$ $u_t dx + a^2 u_x dt = -a\left[u_t dt + u_x dx\right] = -a\,du$

$$\int_M^A (a\,du) = a\left[u(A) - u(M)\right] \;;\; \int_B^M (-a\,du) = -a\left[u(M) - u(B)\right]$$

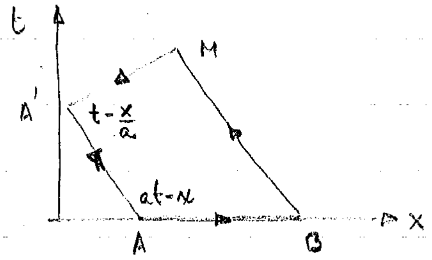
$$\int_M^A + \int_A^B + \int_B^M + \iint_G f\,dx\,dt = a\left[U(A) - U(M)\right] - a\left[U(M) - U(B)\right] + \int_{x-at}^{x+at} \psi(s)\,ds + \iint$$

$$\int_A^B u_t\,dx = \int_{x-at}^{x+at} \psi(s)\,ds \quad \text{hence } u_t(x,0) = \psi(s)$$

$$u(M) = \frac{1}{2}\left\{u(A) + u(B) + \frac{1}{a}\left[\int_{x-at}^{x+at} \psi(s)\,ds + \iint_G f\right]\right\}$$

$$u(x,t) = \frac{1}{2}\left[\phi(x-at) + \phi(x+at) + \frac{1}{a}\left[\int_{x-at}^{x+at} \psi(s)\,ds + \iint_G f\right]\right]$$

D'Alembert Soln



reflection

$$\int_M^{A'} + \int_{A'}^{A} + \int_A^B + \int_B^M + \iint_G f = 0$$

$$\int_M^{A'} = a\left[U(A') - U(M)\right] \quad , \quad \int_{A'}^{A} = -a\left[u(A) - u(A')\right]$$

$$\int_A^B = \int_{at-x}^{x+at} \psi(s)\,ds \quad ; \quad \int_B^M = -a\left[u(M)-u(B)\right]$$

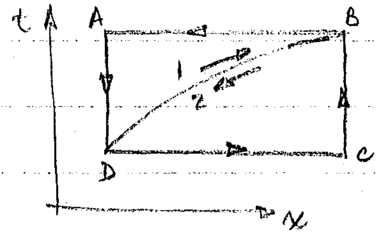
$$u(M) = u(A') + \frac{1}{2}\left[u(B)-u(A)\right] + \frac{1}{2a}\int_{at-x}^{x+at}\psi\,ds + \iint_G f$$

if $u(0,t)=0$ $\quad u(A')=0$

if $u(0,t)=\mu(t-x/a)$ $\quad u(A')=\mu(t-x/a)$

$$u(x,t) = \mu(t-x/a) + \frac{1}{2}\left[\phi(x+at)-\phi(at-x)\right] + \frac{1}{2a}\int_{at-x}^{x+at}\psi\,ds + \iint_G f$$

Propagation of Discontinuity $\quad$ (no external force)



$x=x(t)$ along which u_x or u_t is discont

$u_t(1) \neq u_t(2)$

ABD $\quad \int_{BA} + \int_{AD} + \int_{DB_1} = 0$

BDC

$$\int_{DC} + \int_{CB} + \int_{BD_2} = 0$$

ABCD $\quad \int_{BA} + \int_{AD} + \int_{DC} + \int_{CB} = 0$

$$ABD + BDC = ABCD + \int_{DB_1} + \int_{BD_2} = 0 \quad \Rightarrow \int_{DB_1} + \int_{BD_2} = 0$$

$$\int_{DB}\left(u_t\,dx + a^2 u_x\,dt\right)_1 - \int_{DB}\left(u_t\,dx + a^2 u_x\,dt\right)_2 = 0$$

$$\int_{DB}\left\{\left[\frac{\partial u}{\partial t}\right]dx + a^2\left[\frac{\partial u}{\partial x}\right]dt\right\} = 0 \qquad [g] = g(1) - g(2)$$

DB is arbitrary: $\frac{dx}{dt}\left[\frac{\partial u}{\partial t}\right] + a^2\left[\frac{\partial u}{\partial x}\right] = 0 \qquad x = x(t)$

But u is cont. on $x = x(t)$

$\therefore \quad \frac{d}{dt}\left[u(x(t), t)\right] = u_x\Big|_{1,2} x' + u_t\Big|_{1,2}$ true on either side of $x = x(t)$

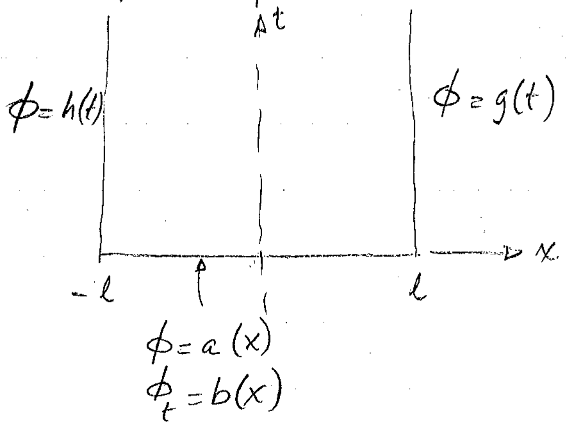
$$u_x(1)\, x' + u_t(1) = u_x(2)\, x' + u_t(2)$$

$\therefore \quad [u_x]\, x' + [u_t] = 0$

Non triv sol if $\begin{vmatrix} x' & a^2 \\ 1 & x' \end{vmatrix} = 0 \Rightarrow x' = \pm a$

discont propagate along characteristics <u>soln can only be discont across characteristics.</u>

Separation of Variables



$$\phi_{tt} - a^2\phi_{xx} = 0$$

$$\phi = \phi_1 + \phi_2 + \phi_3$$

ϕ_1 : $a(L) - a = b = g = 0 \quad h \neq 0$

ϕ_2 : $a = b = h = 0 \quad g \neq 0$

ϕ_3 : $g = h = 0 \quad a \neq 0, b \neq 0$

Consider pure initial value problem

Define $\Phi(x,t) = \Psi(x,t) - \left(\frac{d}{L}x + e\right)$

$$d = \tfrac{1}{2}\left[a(L) - a(-L)\right] \qquad e = \tfrac{1}{2}\left[a(L) + a(-L)\right]$$

Ψ satisfies $\quad \Psi_{tt} - a^2\Psi_{xx} = 0$

$$\left.\begin{array}{l}\psi(t,L)=0\\ \psi(t,-L)=0\\ \psi(x,0)=a(x)+\frac{d}{L}x+e=\bar{a}(x)\\ \frac{\partial\psi}{\partial t}(x,0)=b(x)\end{array}\right\}\quad -L<x<L$$

Prove : only necessary to solve for $\bar{a}\neq 0$ $b=0$.

<u>Proof:</u> Write

$$-L\leq x\leq L\left\{\begin{array}{l}\psi=\psi_1+\psi_2\quad,\quad \psi_1=\psi_2\equiv 0 \text{ on } L,-L\\ \psi_1(x,0)=\bar{a}\quad \psi_{1_t}(x,0)=0\ ;\ \psi_2(x,0)=0\ ,\ \psi_{2,t}(x,0)=b(x)\end{array}\right.$$

Define $\bar{\psi}_1=\frac{\partial\psi_2}{\partial t}$: ① $\bar{\psi}_1$ satisfies wave Equation, ② $\bar{\psi}_1(-L,t)=\bar{\psi}_1(L,t)=0$ (since $\psi_2\equiv 0$ on bdy), ③ $\bar{\psi}_1(x,0)=b(x)$

④ $\frac{\partial\bar{\psi}_1}{\partial t}(x,0)\equiv 0$ $\left[\psi_2(x,0)\equiv 0\Rightarrow\frac{\partial^2\psi_2}{\partial t^2}(x,0),\frac{\partial^2\psi_2}{\partial x^2}(x,0)=0\right]$

These are precisely what ψ_1 must satisfy if $b(x)$ is replaced by $\bar{a}(x)$ i.e. ψ_1 & $\bar{\psi}_1=\frac{\partial\psi_2}{\partial t}$ satisfy same equa & b.c. & I.C. with $\bar{a}(x)$ replaced by $b(x)$ $\Rightarrow$ only need solve for ψ_1

Set of conditions ① → ④ called canonical form for Pure Initial Value Problem

Look for solns which are separable Assume

$$\psi(x,t)=y(x)z(t)\qquad\qquad \lambda:\text{arbitrary}$$

$$\psi_{tt}-a^2\psi_{xx}=yz''-a^2zy''=0$$

$$f(t)\rightarrow\frac{1}{a^2}\frac{z''}{z}=\frac{y''}{y}\leftarrow f(x)\qquad=\text{constant}=-\lambda$$

$$y(-L)=y(L)=0\ ,\ z(0)\neq 0\quad z'(0)=0$$

$z'' + \lambda a^2 z = 0$

$y'' + \lambda y = 0$

if $\lambda < 0$ let $\lambda = -k^2$

$\therefore$ $y = Ae^{kx} + Be^{-kx}$ A & B are arbitrary

$$\left.\begin{array}{l} 0 = Ae^{-kL} + Be^{kL} \\ 0 = Ae^{kL} + Be^{-kL} \end{array}\right\} \text{for non triv } A = B = 0$$

if $\lambda > 0$ $\lambda = k^2$

$y = A \sin kx + B \cos kx$

$$\left.\begin{array}{l} 0 = A \sin kL + B\cos kL \\ 0 = -A \sin kL + B\cos kL \end{array}\right\} \begin{array}{l} \text{for non triv } \sin 2kL = 0 \\ 2kL = n\pi \quad k = \pm\frac{n\pi}{2L} = k_n \end{array}$$

$\lambda_n = \frac{n^2\pi^2}{4L^2}$ — characteristic values or eigenvalues for homogeneous probl.

when n is odd $0 = A \sin \frac{n\pi}{2} + B\cos\frac{n\pi}{2} \Rightarrow A = 0$

$\therefore$ $y = B_n \cos k_n x$ even fn. of x

when n is even $B = 0$

$\therefore$ $y = A_n \sin k_n x$ odd fn of x.

$Z(t) = C \sin kat + D \cos kat$

$\psi = \sin\left(\frac{n\pi x}{2L}\right)\cos\left(\frac{n\pi t a}{2L}\right)$ satisfies $y(-L) = y(L) = 0$, $z(0) = 1$, $z'(0) = 0$

normal mode solution. $\psi = 0$ @ $x = \pm L$

Get Solutions

H.W. 1) $\frac{\partial\psi}{\partial x}(-L,t) = \frac{\partial\psi}{\partial x}(L,t) = 0$

2) $\psi(-L,t) = \frac{\partial\psi}{\partial x}(L,t) = 0$

3) $\psi(L,t) = \frac{\partial\psi}{\partial x}(-L,t) = 0$

} Find Normal Mode Soln

11 P. 66

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homogeneous problem, zero boundary cond

$$\psi = y(x)z(t)$$

$$y(-L) = y(L) = 0 \qquad z(0) = 1 \quad z'(0) = 0$$

$$\psi_t(x,0) = 0 \qquad \psi(x,0) = f(x)$$

$$y = A_n \cos\left(\frac{n\pi x}{2L}\right) \quad n \text{ odd}$$

$$y = B_n \sin\left(\frac{n\pi x}{2L}\right) \quad n \text{ even}$$

$$z = C_n \cos\left(\frac{n\pi ct}{2L}\right) + D_n \sin\left(\frac{n\pi ct}{2L}\right) \implies D_n = 0$$

$$\psi(x,t) = A_n \cos\frac{n\pi x}{2L} \cos\frac{n\pi ct}{2L} \qquad n \text{ odd}$$

$$B_n \sin\frac{n\pi x}{2L} \cos\frac{n\pi ct}{2L} \qquad n \text{ even.}$$

Change coord.

$$0 \le x \le l \qquad x \longrightarrow x + \frac{l}{2}$$

$$\sin\left(\frac{n\pi x}{2L}\right) = \sin\left(\frac{n\pi x}{l} + \frac{n\pi}{2}\right)$$

$$= \sin\frac{n\pi x}{l}\cos\left(\frac{n\pi}{2}\right)$$

$$= (-1)^k \sin\frac{n\pi x}{l}$$

$$\cos\left[\frac{n\pi}{l}\left(x + \frac{l}{2}\right)\right] = \cos\left(\frac{n\pi x}{l} + \frac{n\pi}{2}\right)$$

$$= -\sin\frac{n\pi x}{l} \sin\frac{n\pi}{2}$$

$$= (-1)^k \sin\frac{n\pi x}{l}$$

$$\therefore \quad \psi(x,t) = C_n \sin\frac{n\pi x}{} \cos\left(\frac{n\pi ct}{}\right) \quad \text{normal mode soln} \qquad n = 1, 2, 3 \ldots$$

$0 \le x \le l$

$\psi(0,t) = \psi(l,t) \equiv 0$ & $\frac{\partial \psi}{\partial t}(x,0) = 0$

$\Rightarrow \lambda = k_n^2 = \left(\frac{n\pi}{l}\right)^2$ Eigenvalues of o.d.e. <u>integer n.</u>

Frequency: $z = \cos(\omega_n t)$ $\omega_n = \frac{n\pi c}{l} = n\omega$ $\omega = \frac{\pi c}{l}$ - fundamental freq.

ω_n - freq of n^{th} mode

period: $T_n = 2\pi/\omega_n = \frac{2\pi l}{n\pi c} = \frac{2l}{nc} = \frac{T}{n}$ $T = \frac{2l}{c}$ = fundamental period.

$z_n(t)$ has period $T_n \rightarrow T = nT_n$

periodic with T also.

$\psi = C_n \sin\left(\frac{n\pi x}{l}\right) z(t)$ $-1 \le z(t) \le 1$

<u>Superposition principle</u>

if $\psi_1, \psi_2, \ldots, \psi_n$ are solns of $\psi_{tt} = c^2 \psi_{xx}$

then $\psi(x,t) = \sum_{k=1}^{n} \psi_k(x,t)\, a_k$ is also a soln.

Result of linearity of Problem.

Uniform Convergence allows diff: if both the series & diff series are uniform convergent

if $\epsilon > 0$ $\exists$ $N(\epsilon)$ $\left|f(x) - \sum_{1}^{n} f_m(x)\right| < \epsilon$ for $n > N$

N is indep of x

point wise convergence $N(\epsilon, x)$

<u>Question A</u> How well can any fn $\psi = f(x)$ defined for $0 \le x \le l$ with $f(0) = f(l) = 0$ be approx by expressions like $A_n \sin\left(\frac{n\pi x}{l}\right)$?

Fourier's Theorem

If $f(x)$ is any periodic function of x with period $2P$

$$\ni \quad F(x) = F(x+2P)$$

and $\ni$

$$\int_{-P}^{P} |f(x)|\,dx \text{ exists}$$

and if

$$a_n = \frac{1}{P}\int_{-P}^{P} F(x)\,\cos\frac{m\pi x}{P}\,dx$$

$$b_n = \frac{1}{P}\int_{-P}^{P} F(x)\sin\frac{m\pi x}{P}\,dx$$

then the series

$$\bar{F}(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{P} + \sum_{m=1}^{\infty} b_n \sin\left(\frac{m\pi x}{P}\right) \quad *$$

converges at each interior pt x of any interval in which $F(x)$ is bounded to the value

$$\bar{F}(x) = \frac{1}{2}\{F(X)_+ + F(X)_-\}.$$

At values of x where $F(x)$ is continuous

$$\bar{F}(x) = F(x)$$

* 1) is the series called the Fourier series representation of $\bar{F}(x)$

2) The coeff a_n & b_n are called the Fourier coefficients

3) if $F(x)$ is odd about $x=0$, $F(x) = -F(-x)$

Then since $F(x)\cos\left(\frac{n\pi x}{P}\right)$ is odd & $F(x)\sin\left(\frac{n\pi x}{2}\right)$ is even

$a_n = 0 \;\forall n$

$$b_n = \frac{2}{P}\int_0^P F(x)\sin\frac{n\pi x}{P}\,dx$$

then $\bar{F}(x) = \sum_{1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$ Fourier Sine Series

4) if $F(x)$ is even about $x=0$ yields $b_n = 0 \ \forall_n$

Then $a_n = \frac{2}{p}\int_0^p F(x)\cos\frac{n\pi x}{p}\,dx$

& $\bar{F}(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos\frac{n\pi x}{p}$ Fourier Cosine Series

$$\bar{F}(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{p}}$$

where $2P\,C_n = \int_{-p}^{p} F(x)\, e^{\frac{i n\pi x}{P}}\,dx = \frac{1}{2}(a_n - i b_n) \quad n>0$

$\frac{1}{2}(a_n + i b_n) \quad n<0$

$\frac{1}{2}a_0 \quad n=0$

Check on the coeff.

$$\bar{F} = \frac{a_0}{2} + \sum a_n$$

$$\int_{-p}^{p} \bar{F}\cos\frac{n\pi x}{p}\,dx = \int_{-p}^{p}\frac{a_0}{2}\cos\frac{n\pi x}{p}\,dx + \sum a_m \overset{\begin{smallmatrix}0 & m\neq n\\ P & m=n\end{smallmatrix}}{\int_{-p}^{p}\cos\frac{m\pi x}{p}\cos\frac{n\pi x}{p}\,dx}$$

$$+ \sum b_m \int_{-p}^{p}\cos\frac{m\pi x}{p}\sin\frac{n\pi x}{p}\,dx$$

$$\int_{-p}^{p}\bar{F}\sin\frac{n\pi x}{p}\,dx = \overset{0}{\int_{-p}^{p}\frac{a_0}{2}\sin\frac{n\pi x}{p}\,dx} + \sum a_m \overset{0}{\int_{-p}^{p}\sin\frac{n\pi x}{p}\cos\frac{n\pi x}{p}\,dx}$$

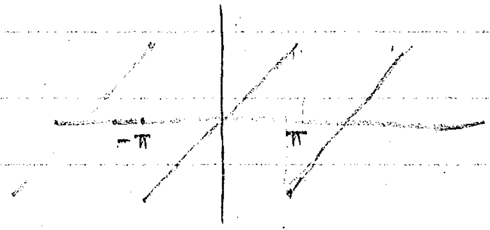
$$+ \sum b_m \overset{\begin{smallmatrix}0 & m\neq n\\ P & m=n\end{smallmatrix}}{\int_{-p}^{p}\sin\frac{n\pi x}{p}\sin\frac{m\pi x}{p}\,dx}$$

UC allows switch of sum & integral sign

1. $f(x) = x$ for $-\pi < x < \pi$

$F(x) = f(x) \qquad -\pi < x < \pi$

and $F(x) = F(x + 2n\pi) \qquad p = \pi$ & $F(x)$ is odd about 0



$a_n = 0 \ \forall n$

$$b_n = \frac{2}{\pi}\int_0^\pi x \sin\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{2}{\pi}\int_0^\pi x \sin(nx)\, dx$$

$$\frac{2}{\pi}\left[\frac{-x\cos nx}{n}\Big|_0^\pi + \frac{\sin nx}{n^2}\Big|_0^\pi\right]$$

$$\frac{2}{\pi}\left[\frac{-\pi}{n}(-1)^n + 0\right] = \frac{2(-1)^{n+1}}{n}$$

$$x = F(x) = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n}\sin nx = 2\left[\sin x - \sin\frac{2x}{2} + \sin\frac{3x}{3} - \dots\right]$$

let $x = \frac{\pi}{2}$ $= 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \frac{(-1)^{n+1}}{2n-1} + \dots\right]$

Gregory's Series

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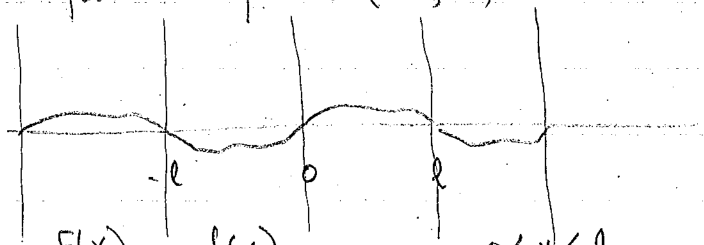
$\phi_{tt} - c^2\phi_{xx} = 0$ (i) $0<x<l$

$\left.\begin{array}{l}\phi(x,0) = f(x)\\ \phi_t(x,0) = 0\end{array}\right\} t \geq 0 \; , 0<x<l$

$\phi(l,t) = \phi(0,t) = 0$ $t>0$

$f(0) = f(l) = 0$

Extend to pure initial value problem by continuing initial condition periodically over $(-\infty, \infty)$



Define $F(x) = f(x)$ $0 \leq x \leq l$

$-f(-x)$ $-l \leq x \leq 0$

$F(x) = F(x+2l)$

$F(x)$ is odd about $x=0$ since zero displ

Φ satisfies (1) on $(-\infty, \infty)$

subject to

$\left.\begin{array}{l}\Phi(x,0) = F(x)\\ \Phi_t(x,0) = 0\end{array}\right\} t \geq 0$

Since F is odd wrt $x=0$ & periodic it can be represented as a Fourier Sine Series

$$F(x) = \sum_1^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

can write down solution of homogeneous equation 1 as a superposition

$$\Phi(x,t) = \sum_1^\infty B_n \sin\frac{n\pi x}{l} \cos\omega_n t$$

since

$$\Phi(x,0) = \sum b_n \sin\left(\frac{n\pi x}{l}\right) \qquad \omega_n = \frac{n\pi c}{l}$$

$$B_n = b_n$$

Solution is

$$\left.\begin{aligned} \phi(x,t) &= \sum_1^\infty b_n \sin\left(\frac{n\pi x}{l}\right) \cos\omega_n t \\ \text{where} \quad b_n &= \frac{2}{l}\int_0^l f(x) \sin\left(\frac{2\pi x}{l}\right) dx \end{aligned}\right] \quad 0 \le x \le l$$

$$\phi(x,t) = \sum \frac{b_n}{2}\left\{\sin\left(\frac{n\pi}{l}[x+ct]\right) + \sin\left(\frac{n\pi}{l}[x-ct]\right)\right\}$$

$$= \frac{1}{2}\left\{F(x+ct) + F(x-ct)\right\}$$ – travelling wave representati

where $F(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l}$ $\{ = f(x)$ for $0 \le x \le l$

B.C. Check $\phi(0,t) = \phi(l,t) = 0$

$$\phi(x,t) = \frac{1}{2}\left\{F(x+ct) + F(x-ct)\right\}$$

$$\phi(0,t) = \frac{1}{2}\left\{F(ct) + F(-ct)\right\} = 0$$ sin F is odd about zero

$$\phi(l,t) = \frac{1}{2}\left\{F(l+ct) + F(l-ct)\right\}$$

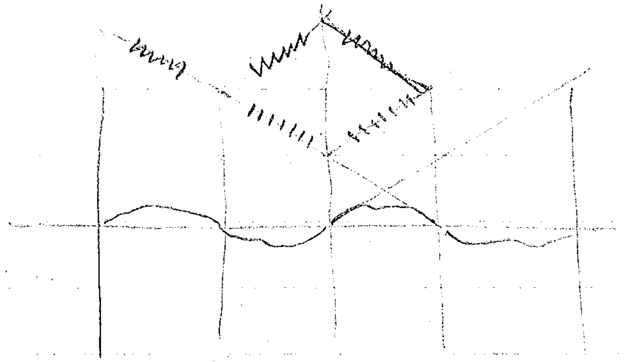
$$= \frac{1}{2}\left\{-F(ct-l) + F(ct-l+2l)\right\}$$

$$= \frac{1}{2}\left\{-F(ct-l) + F(ct-l)\right\} = 0$$ since F is periodi

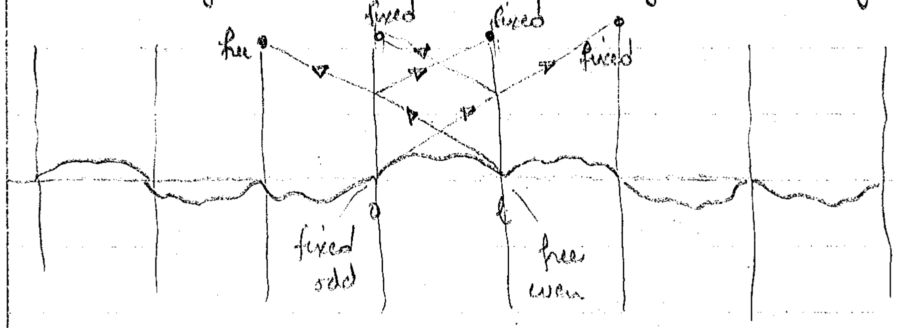
in $2l$.

I.C.

$$\phi(x,0) = \frac{1}{2}\{F(x) + F(x)\} = F(x)$$



following characteristics of travelling wave using odd continuation



period is 4l

do as homework fixed-free as previous example & check solution.

Soln for full problem

$\phi_{tt} - c^2\phi_{xx} = 0$

$\phi(x,0) = f(x)$

$\phi_t(x,0) = g(x)$

$\phi(0,t) = \phi(l,t) = 0$

$\phi = \phi_1 + \phi_2$

where ϕ_1 satisfies $g \equiv 0$ ϕ_2 satisfies $f \equiv 0$

When we found canonical problem we showed soln for $\frac{\partial \phi_2}{\partial t}(x,t)$ is the same as soln for ϕ_1 with f replaced by g.

$$\phi_1 = \sum b_n \sin\frac{n\pi x}{l} \cos\omega_n t$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$\frac{\partial \phi_2}{\partial t} = \sum a_n \sin\left(\frac{n\pi x}{l}\right) \cos \omega_n t$$

$$a_n = \frac{2}{l}\int_0^l g(x) \sin \frac{n\pi x}{l}\, dx$$

Assume UC

$$\phi_2 = \sum \frac{a_n}{\omega_n} \sin\left(\frac{n\pi x}{l}\right) \sin \omega_n t + K, \; \phi_2(x,0) \equiv 0 \Rightarrow K = 0$$

$$\therefore \; \phi(x,t) = \phi_1 + \phi_2 = \sum_{n=1}^{\infty} \left\{ b_n \cos \omega_n t + \frac{a_n}{\omega_n} \sin \omega_n t \right\} \sin \frac{n\pi x}{l}$$

$$= \sum_1^{\infty} \phi_n(x,t)$$

$$\phi_n(x,t) = \left[b_n \cos \omega_n t + \frac{a_n}{\omega_n} \sin \omega_n t \right] \sin \frac{n\pi x}{l}$$

$$= \alpha_n \cos\{\omega_n(t+\delta_n)\} \sin \frac{n\pi x}{l}$$

$$\alpha_n = \sqrt{b_n^2 + \frac{a_n^2}{\omega_n^2}} \; ; \quad \omega_n \delta_n = -\arctan\left(\frac{b_n \omega_n}{a_n}\right)$$

amplitude — phase change

$$E_n = \frac{1}{2}\int_0^l (\overset{KE}{\rho u_t^2} + \overset{P.E.}{T u_x^2})\, dx \qquad u_n = \text{displ of } n\text{th mode}$$

$$= \frac{\alpha_n^2}{2}\int_0^l \left\{ \rho \omega_n^2 \sin^2[\omega_n(t+\delta_n)] \sin^2\left(\frac{n\pi x}{l}\right) + T\frac{\omega_n^2}{c^2} \cos^2[\omega_n(t+\delta_n)] \cos^2\left(\frac{n\pi x}{l}\right) \right\}$$

$$\int_0^l \sin^2 = \int_0^l \cos^2 = \frac{l}{2} \qquad c^2 = T/\rho$$

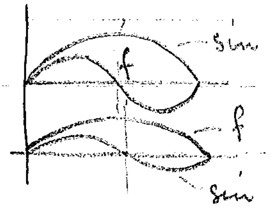
$$= \frac{\alpha_n^2 \omega_n^2}{2} \frac{l}{2} \rho = M\left(\frac{\alpha_n \omega_n}{2}\right)^2 \qquad \text{where } M = \rho l = \text{mass of the string}$$

for series to be uniform convergence conditions on a_n & b_n must exist to bound them

$$b_n = \frac{2}{\ell} \int_0^\ell f(x) \sin\left(\frac{n\pi x}{\ell}\right)$$

n odd f odd about $\ell/2$ $b_n = 0$

n even f even about $\ell/2$ $b_n = 0$



tone depends on frequency ω_n amplitude of tone depends on ω_n & α_n

Inhomogeneous problem – forcing fn $f(x,t)$

$$u_{tt} = a^2 u_{xx} + f(x,t)$$

$u(x,0) = \phi(x)$ $u(0,t) = u(\ell,t) = 0$ } odd contin

$u_t(x,0) = \psi(x)$

$$\therefore \quad u(x,t) = \sum u_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

By definition continue f, ϕ, ψ oddly about 0 & ℓ

$$\phi(x) = \sum \phi_n \sin\frac{n\pi x}{\ell} \qquad \phi_n = \frac{2}{\ell}\int_0^\ell \phi(x) \sin\frac{n\pi x}{\ell}\,dx$$

$$\psi(x) = \sum \psi_n \sin\frac{n\pi x}{\ell} \qquad \psi_n = \frac{2}{\ell}\int_0^\ell \psi(x) \sin\frac{n\pi x}{\ell}\,dx$$

$$f(x,t) = \sum f_n(t) \sin\left(\frac{n\pi x}{\ell}\right) \qquad f_n(t) = \frac{2}{\ell}\int_0^\ell f(x,t) \sin\frac{n\pi x}{\ell}\,dx$$

$$\sum_1^\infty \sin\frac{n\pi x}{\ell}\left\{-\omega_n^2 u_n(t) - u_n''(t) + f_n(t)\right\} = 0$$

$\therefore \quad u_n'' + \omega_n^2 u_n = f_n$ *

$u_n = u_n^{(1)} + u_n^{(2)}$ where $u_n^{(1)}$ satisfies inhomogeneous eq zero I.C.

$u_n^{(2)}$ satisfies homogeneous eq $(f_n \equiv 0)$

nonzero I.C.

ie $\left.\begin{array}{l} u_n^{(2)}(0) = \Phi_n \\ u_{n,t}^{(2)}(0) = \Psi_n \end{array}\right\}$ Know $u_n^{(2)}$ (previous soln)

$$u_n^{(1)} = \frac{1}{\omega_n}\int_0^t \sin[\omega_n(t-\tau)] f_n(\tau)\, d\tau$$

Indep solns of homogeneous eqns ($f_n \equiv 0$) are $\cos\omega_n t$, $\sin\omega_n t$

Look for a soln of form $u_n^{(1)} = A(t)\sin\omega_n t + B(t)\cos\omega_n t$

$$u_n^{\prime(1)} = A'\sin + B'\cos + \omega_n(A\cos - B\sin)$$

① $A'\sin + B'\cos \equiv 0$

$$u_n^{\prime\prime(1)} = \omega_n[A'\cos - B'\sin] \underbrace{- \omega_n^2[A\sin + B\cos]}_{-\omega_n^2 u_n^{(1)}}$$

② $\omega_n(A'\cos - B'\sin) = f_n$

$$A' = \frac{f_n(t)}{\omega_n}\cos\omega_n t \qquad B' = -\frac{f_n(t)}{\omega_n}\sin\omega_n t$$

$$A = \frac{1}{\omega_n}\int_0^t f_n(\tau)\cos\omega_n\tau\, d\tau \qquad B = \frac{-1}{\omega_n}\int_0^t f_n(\tau)\sin\omega_n\tau\, d\tau$$

$$u_n^{(1)} = \frac{1}{\omega_n}\int_0^t f_n(\tau)\sin\omega_n(t-\tau)\, d\tau$$

$$u(x,t) = \sum_1^\infty \frac{1}{\omega_n}\int_0^t \sin\omega_n(t-\tau)\sin\frac{n\pi x}{l} f_n(\tau)\, d\tau \; ; \quad f_n(\tau) = \frac{2}{l}\int_0^l f(\xi,\tau)\sin\frac{n\pi\xi}{l}d$$

$$+ \sum_1^\infty \left(\Phi_n\cos\omega_n t + \frac{\Psi_n}{\omega_n}\sin\omega_n t\right)\sin\frac{n\pi x}{l}$$

$$u_n^{(1)} = \int_0^t\int_0^l \Big\{\underbrace{\frac{2}{l}\sum_1^\infty \frac{1}{\omega_n}\sin(t-\tau)\sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}}_{G(x,\xi,t-\tau)}\Big\} f(\xi,\tau)\, d\xi\, d\tau$$

$$= \int^t\int^l G(x,\xi,t-\tau)\, f(\xi,\tau)\, d\xi\, d\tau$$

$u_n^{(1)}$ is soln of vibrating string with fixed ends under action of external force $f(x,t)$ & zero I.C. zero b.c.

$$\sum\left(\Phi_n \cos\omega_n t + \frac{\Psi_n}{\omega_n}\sin\omega_n t\right)\sin\frac{n\pi x}{l} \qquad \begin{matrix}\text{homogeneous} & \text{bc}\\ \text{non hom} & \text{I.C.}\end{matrix}$$

Special case – pt load ie impulse I @ $x = x_0$, $t = t_0$

$$f(x_0,t_0) = \frac{I}{\rho} \qquad \xi \to x_0,\ \tau \to t_0 \quad \text{in solution.}$$

$$u_n^{(1)} = G(x,x_0,t-t_0)\,f(x_0,t_0) = G(x,x_0,t-t_0)\frac{I}{\rho}$$

More complicated b.c.

$u(0,t) = \mu_1(t)$

$u(l,t) = \mu_2(t)$

$u = U(x,t) + V(x,t)$

choose $U \ni.$ V satisfies zero b.c.

$V_{tt} = a^2 V_{xx} + \bar{f}(x,t) \qquad \bar{f} = f - [U_{tt} - a^2 U_{xx}]$

$V(x,0) = \bar{\Phi}(x) = \phi(x) - U(x,0)\ ;\ 0$

$V_t(x,0) = \bar{\Psi}(x) = \psi(x) - U_t(x,0)\ ;$

$V(0,t) = \mu_1(t) - U(0,t) = 0$

$V(l,t) = \mu_2(t) - U(l,t) = 0$

need to choose $U(x,t)$ $\ni.$ $V(0,t) = V(l,t) = 0$

ie st $U(0,t) = \mu_1(t)$

$U(l,t) = \mu_2(t)$

choose $U(x,t) = \mu_1(t) + \frac{x}{l}\left[\mu_2(t) - \mu_1(t)\right] \qquad U_{xx} = 0$

we now get $u_{tt} - a^2 u_{xx} = f(t)$

$u(0,t) = \mu_1(t) \qquad u(x,0) = \phi(x)$

Dec. 5, 1972

Final Exam – Tuesday Jan 23 3–6 P.M.

$u = A \sin \omega t$

$u = u_I + u_B$ I.C. are replaced by periodicity

$u_{tt} = a^2 u_{xx} - \alpha u_t \qquad \alpha \geq 0$

B.C. $u(0,t) = 0$

$u(l,t) = A \cos \omega t$ or $B \sin \omega t$

No. I.C. but periodicity i.e. assuming u has period $\frac{2\pi}{\omega}$

Use complex representation

$$u(l,t) = A e^{i\omega t}$$

$$u(x,t) = u_1(x,t) + i u_2(x,t)$$

Let $u(x,t) = X(x)\, e^{i\omega t}$ periodicity

X satisfies $X'' + k^2 X = 0$

$X(0) = 0 \qquad X(l) = A \qquad k^2$ is complex $= \frac{\omega^2}{a^2} - i \frac{\alpha \omega}{a^2}$

$$X = \frac{A}{\sin kl} \sin kx = X_1 + i X_2$$

$u_1 = (X_1 \cos \omega t - X_2 \sin \omega t) \qquad u_2 = (X_1 \sin \omega t + X_2 \cos \omega t$

$k^2 = (\bar{a} + i b)^2 \qquad \bar{a}^2 - b^2 = \omega^2/a^2 \qquad 2\bar{a} b = -\frac{\alpha \omega}{a^2}$

$$k = \frac{\omega}{a}\left[\left(\frac{1 + \sqrt{1 + \alpha^2/\omega^2}}{2} \right)^{1/2} + i \left(\frac{\sqrt{1 + \frac{\alpha^2}{\omega^2}} - 1}{2} \right)^{1/2} \right]$$

$$= z_1 + i z_2$$

as $\alpha \to 0 \qquad z_1 \longrightarrow \omega/a \qquad z_2 \longrightarrow 0$

$\underline{\sin kx} = \underline{\sin[(z_1 + i z_2)x]} = \underline{\sin z_1 x \cosh z_2 x + i \sinh z_2 x \cos z_1 x}$

use conjugate to clear denominator

$= \frac{1}{A}(X_1 + iX_2)$ let $\omega \to \omega_n = \frac{n\pi a}{l}$

$$X_1 = \frac{A}{\sin\left(\frac{\omega l}{a}\right) + \alpha^2} + O(\alpha \ldots)$$

if $\omega \to \omega_n$ $X_1 \to \frac{A}{\alpha^2}$ must have $|X_1| << 1$

$|A| << 1$

$\boxed{A^{1/2} \ll \alpha}$ $|\alpha| << 1$

if we want to simplify the equations

$$\boxed{A^{1/2} << \alpha << 1}$$

for arbitrary $u = f(t)$ where f is periodic in $\frac{2\pi}{\omega}$

$f(t) = \sum A_n \sin(\omega_n t)$ in fourier components

$+ \sum B_n \cos(\omega_n t) + \frac{A_0}{2}$

Soln for $\alpha = 0$

$$u(x,t) = \frac{A_0 x}{2l} + \sum \left(A_n \sin \omega_n t + B_n \cos \omega_n t\right) \frac{\sin\left(\frac{\omega_n x}{a}\right)}{\sin\left(\frac{\omega_n l}{a}\right)}$$

+ any soln of homogeneous problem.

for $\alpha \neq 0$ $\frac{\omega_n x}{a}$ would be replaced by $\frac{kx}{a}$ $\frac{kl}{a}$ complex
and homog problem would be mult by e^{-At}

General Method for Separation of Variables

Inhomogeneous material with damping

$$(k(x)u_x)_x - q(x)u = \rho(x)u_{tt}$$

let $L[u] = \left\{\frac{\partial}{\partial x}\left(k\frac{\partial}{\partial x}\right) - q\right\}u$

Separation of Variables $u = XT$

$$\left.\begin{array}{l} T'' + \lambda T = 0 \\ L[X] + \lambda\rho(x)X = 0 \\ X(0) = X(l) = 0 \end{array}\right\} *$$

need basic properties of *

They are:

1) there are a countably many infinite no. of eigenvalues

$\lambda_1 < \lambda_2 < \lambda_3 \ldots\ldots < \lambda_n < \lambda_{n+1} \ldots$

which each correspond to a non trivial soln of the Boundary Value problem given by the eigenfuns $X_1, X_2, X_3, \ldots$

$$\begin{cases} L[X_n] + \lambda_n\rho X_n = 0 \\ X_n(0) = X_n(l) = 0 \end{cases}$$

infinite set but they can be ordered & counted like integers

2) For $q \geq 0$ all the λ_n are positive. damped system

3) Eigenfunctions X_n are orthonormal on interval $[0, l]$ wrt the density fn

$$\int_0^l X_i(x)X_j(x)\rho(x)dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

4) An arbitrary fn $F(x)$ which is twice differentiable & satisfies $F(0) = F(l) = 0$ can be developed in a uniformly convergent series wrt X_n

$$F(X) = \sum_1^\infty F_n X_n(x) \qquad F_n = \int_0^l F(x)X_n(x)\rho(x)dx$$

prove 283

Next a form of Greens Theorem

$$L[u] = (ku_x)_x - qu$$

Consider $uL[v] - vL[u] = u(kv')' - v(ku')' = [k(uv' - vu')]'$

$$\Rightarrow \int_a^b (uL[v] - vL[u])\,dx = k(uv' - vu')\Big|_a^b \quad *$$

Let $u = X_m$ $\quad v = X_n$ $\qquad a, b = 0, l$ in $*$

$X_m(0) = X_n(0) = X_m(l) = X_n(l) = 0 \Rightarrow$ right hand side $= 0$

$$\int_0^l (X_m L[X_n] - X_n L[X_m])\,dx = 0$$

But $L[X] = -\lambda\rho(x)X$

$$\int_0^l \left(X_m[-\lambda_n \rho X_n] - X_n[-\lambda_m \rho X_m]\right) dx = 0$$

$(\lambda_m - \lambda_n)\int_0^l X_m X_n \rho\,dx = 0 \Rightarrow$ for $m \neq n$ $\int_0^l X_n X_m \rho\,dx = 0$

$\int_0^l \rho X_n^2\,dx = 1$ since if $\tilde{X}_n$ is an eigenfunction so is $A_n \tilde{X}_n$

choose $A_n = \dfrac{1}{\sqrt{\int_0^l \tilde{X}_n^2 \rho\,dx}}$ $\qquad$ let $A_n \tilde{X}_n = X_n$

$$\int_0^l X_n^2 \rho\,dx = A_n^2 \int_0^l \tilde{X}_n^2 \rho\,dx = 1$$

for $q \geq 0$ $\quad \lambda_n > 0$

$$L[X_n] = -\lambda_n \rho X_n$$

$$\int_0^l X_n L[X_n]\,dx = \int_0^l -\lambda_n \rho X_n^2\,dx = -\lambda_n$$

$$\lambda_n = \int_0^l q\, X_n^2\, dx \; \underbrace{- \cancel{k X_n' X_n \Big|_0^l}}_{0 \text{ by b.c. on } X_n} + \int_0^l k (X_n')^2 dx$$

if $q > 0$, since $k > 0$ then $\lambda_n > 0$

$T = A_n \cos\sqrt{\lambda_n} t + B_n \sin\sqrt{\lambda_n} t$

Full Soln

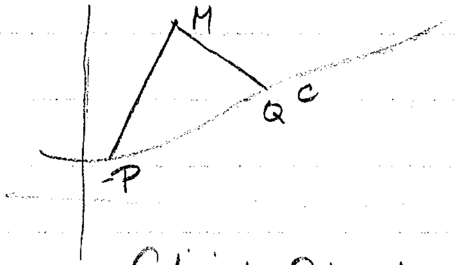
$u(x,t) = \sum (A_n \cos\sqrt{\lambda_n} t + B_n \sin\sqrt{\lambda_n} t) X_n$

Special Case $\rho' = k' = q = 0 \Rightarrow \sqrt{\lambda_n} = \frac{n\pi}{l}$ $X_n = \sin\left(\frac{n\pi x}{l}\right)$

I.C. $u(x,0) = \varphi(x) = \sum A_n X_n(x)$

$u_t(x,0)$ $\psi(x) = \sum B_n \sqrt{\lambda_n} X_n(x)$

Dalembert Soln for General linear hyperbolic equation.



$u_{xx} - u_{yy} = a u_x + b u_y + c u$

$L[u] = u_{xx} - u_{yy} + a(x,y) u_x + b(x,y) u_y + c(x,y)$

$L[u] = 0$

Adjoint Operator : operators L & M are adjoint

if $v L[u] - u M[v] = \frac{\partial H}{\partial x} + \frac{\partial K}{\partial y}$

if $L[u] = M[u]$ they are called self adjoint.

$L[u] = u_{xx} - u_{yy}$ is self adjoint

Use

$v u_{xx} = (v u_x)_x - (v_x u)_x + u v_{xx}$

$v u_{yy} = (v u_y)_y - (v_y u)_y + u v_{yy}$

$$vau_x = (vau)_x - u(av)_x$$
$$vbu_y = (vbu)_y - u(bv)_y$$
$$vcu = ucv$$

$$M[v] = v_{xx} - v_{yy} - (av)_x - (bv)_y + cv$$
$$H = (vu)_x - (2v_x - av)u \qquad K = (uv)_y - (2u_y - bu)v$$

$$vL[u] - uM[v] = \frac{\partial H}{\partial x} + \frac{\partial K}{\partial y}$$

Using Green's Theorem

$$\iint_G \{vL[u] - uM[v]\}\, d\xi\, d\eta = \iint_G \left(\frac{\partial H}{\partial \xi} + \frac{\partial K}{\partial \eta}\right) d\xi\, d\eta = \int_C H d\eta - K d\xi$$

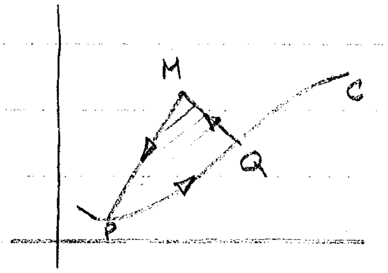
C is piecewise smooth. Clive Chester: Techniques in P.D.E. p223

there is needed to solve

$$L[u] = -f[x,y] \quad \text{subject to I.C.} \quad \left.u\right|_c = \varphi(x), \quad \left.u_n\right|_c = \psi(x)$$

} on curve $c: y = g(x)$, normal deriv to c.

Restriction: $|g'(x)| < 1$ to keep solution singlevalue



Apply Green's Theorem to MPQ

$$\iint_{MPQ} (vL - uM)\, d\xi\, d\eta = \int_Q^M + \int_M^P + \int_P^Q (Hd\eta - Kd\xi)$$

on QM: $d\xi = -d\eta = -\frac{ds}{\sqrt{2}}$ s = distance along character

MP: $d\xi = d\eta = -\frac{ds}{\sqrt{2}}$

using def of H & K

$$\int_Q^M (Hd\eta - Kd\xi) = -(uv)_M + (uv)_Q + \int_Q^M \left[2\frac{\partial v}{\partial s} - \frac{(b+a)}{\sqrt{2}}v\right] u\, ds$$

$$\int_M^P (H\,d\eta - K\,d\xi) = -(uv)_M + (uv)_P + \int_P^M \left[2\frac{\partial v}{\partial s} - \frac{(b-a)}{\sqrt{2}}\, v\right] u\,ds$$

choose $v \ni 2v_s - \frac{(b-a)}{\sqrt{2}}\, v = 0$

also $M[v] = 0$ in MPQ

also $v_M = 1$

$$\therefore \quad u_M = \frac{(uv)_P + (uv)_Q}{2} + \frac{1}{2}\int_P^Q \left[v \cdot \overset{H d\eta - K d\xi}{\text{first deriv terms}} \right]$$

$$+ \frac{1}{2}\iint_{MPQ} v(M, M')\, f(M')\, d\sigma_{M'}$$

$M = (x, y)$

$M' = (\xi, \eta) \qquad d\sigma_{M'} = d\xi\, d\eta$

for dalambert soln $a = b = c = 0 \implies \frac{dv}{ds} = 0$ on both bdrys.

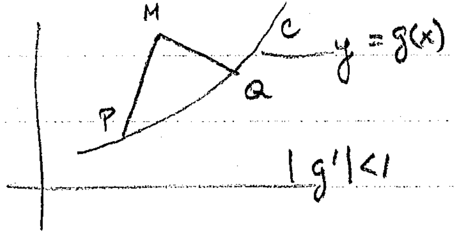
$v \equiv 1$

$$u(M) = \frac{u(P) + u(Q)}{2} + \frac{1}{2}\int_P^Q \underline{\qquad} + \frac{1}{2}\iint_{MPQ} f(\xi, \eta)\, d\xi\, d\eta$$

H.W. : Read 2.54: P 113-116 P100-18 & P116-3.

Dec 12, 1972

$$\iint_G (v L[u] - u M[v])\, d\xi\, d\eta = \int_C H\, d\eta - K\, d\xi$$



$$(uV)_m = \frac{(uV)_P + (uV)_Q}{2} + \int_P^M \left(\frac{\partial V}{\partial s} - \frac{(b-a)}{2\sqrt{2}} V \right) u\,ds$$

$$+ \int_Q^M \left(\frac{\partial V}{\partial s} - \frac{(a+b)}{2\sqrt{2}} V \right) u\,ds + \frac{1}{2}\int_P^Q (H\,d\eta - K\,d\xi - \frac{1}{2}\iint_{MPQ} (vL[u] - uM[v])\,d\xi\,d\eta$$

Want u to satisfy $L[u] = -f(x,y)$, V- arbitrary choose for convenience

Let V be $\ni$

$$\left.\begin{aligned} &\frac{\partial V}{\partial s} = \left(\frac{b-a}{2\sqrt{2}} V\right) \text{ on } MP \\ &\frac{\partial V}{\partial s} = -\left(\frac{b+a}{2\sqrt{2}}\right) V \text{ on } MQ \\ &V(M) = 1 \\ &M[V] = 0 \quad \text{in } G = MPQ \end{aligned}\right\}$$

V is called the Riemann Fn

$$u(M) = \frac{(uV)_P + (uV)_Q}{2} + \frac{1}{2}\int_P^Q \Big[V(u_\xi\,d\eta + u_\eta\,d\xi) - u(V_\xi\,d\eta + V_\eta\,d\xi) + uV(a\,d\eta - b\,d\xi) \Big]$$

$$+ \frac{1}{2}\iint_G V(M,M')f(M')\,d\sigma_{m'} \qquad d\sigma_{m'} = d\xi\,d\eta$$

Check that we know u_x & u_y on C

$$u\big|_c = \varphi(x)$$

$$u_{sn}\big|_c = \psi(x) \qquad y = g(x)$$

$$u_x = u_s \cos(s,x) + u_n \cos(x,n) = u_x X_s \cos(s,x) + \psi(x)\left[\frac{-g'}{\sqrt{1+g'^2}}\right]$$

$$= \frac{\phi'(x) - \psi g'\sqrt{1+g'^2}}{1+g'^2} \checkmark$$

Similarly $u_y = \dfrac{\phi' g' + \psi\sqrt{1+g'^2}}{1+g'^2}$ ✓

$\Big\}$ on $y = g(x)$

$a = b = 0$

PQ : $y = \text{const}$

$V = 1$ on G & ∂G

$$d\eta = 0$$

$$u(M) = \underbrace{u(P) + u(Q)}_{2} + \frac{1}{2}\int_P^Q u_\eta \, d\xi + \frac{1}{2}\iint_G f(\xi,\eta)\, d\xi \, d\eta$$

$$u_\eta = \left.\frac{\partial u}{\partial n}\right|_c = \psi(x)$$

$$u(M) = \underbrace{\phi(P) + \phi(Q)}_{2} + \frac{1}{2}\int_P^Q \psi \, d\xi + \frac{1}{2}\iint_G f(\xi,\eta)\, d\xi \, d\eta$$

Parabolic Diff Eq.

$$u_{xx} - u_y = 0$$

Diffusion & Heat Conduction $y \equiv t$

Linear Heat Conduction

A rod of length l which is insulated at its lateral surface & the

compared with l (so that disturbances are 1-D)

Fourier's law of Heat Conduction

$$dQ = -k(x)\,\frac{\partial u}{\partial x}\,S\,dt$$

Q = heat flow S = cross sectional area

$k(x)$ = Thermal conductivity

Generalization of linear temperature gradient

if temp $u = u_1$ @ $x = 0$ $u = u_2$ @ $x = l$

then $u = u_1 + \frac{x}{l}(u_2 - u_1)$

$Q = -k \frac{(u_2 - u_1)}{l} S = -k \frac{\partial u}{\partial x} S$ per unit time

Consider a portion $\Delta x = x_2 - x_1$ for time $\Delta t = t_2 - t_1$
& look at energy balance i.e. look at heat in vs heat out

$$\int_{t_1}^{t_2} \left[k \frac{\partial u}{\partial x}(x,\tau)\Big|_{x_2} - k \frac{\partial u}{\partial x}(x,\tau)\Big|_{x_1} \right] d\tau + \iint_{t_1\, x_1}^{t_2\, x_2} F(\xi,\tau)\, d\xi\, d\tau$$

ext. heat source
chem react
E-M effects

$$= \int_{x_1}^{x_2} c\rho \left[u(\xi,t)\Big|_{t_2} - u(\xi,t)\Big|_{t_1} \right] d\xi$$

c: specific heat $c(x)$
Use MV Theorem & $\Delta x, \Delta t \to 0$

$$\iint \left[\left(k \frac{\partial u}{\partial x}\right)_x + F(x,t) - c(x)\rho\, u_t \right] dt\, dx = 0$$

for k = const $u_t = a^2 u_{xx} + f(x,t)$ $\qquad f(x,t) = \frac{F}{c\rho} \qquad a^2 = \frac{k}{c\rho}$

note: analysis on

$$u_t = a^2 u_{xx} + f(x,t) \qquad (1)$$

$u_t = a^2 u_{xx} + f(x,t) + b u_x + c u$ can also be put into form (1)
by $v = u e^{\lambda x + \mu t}$

Some applications are discussed in 3.1.2 – Diffusion
Look at (1) where $u(x,t)$ is temp along bar

B.C.

1) Prescribed temperature $u(0,t) = \mu_1(t)$ $u(l,t) = \mu_2(t)$

2) Prescribed Heat Flow $Q(l,t) = \nu(t) \cong -k\, u_x(l,t)$

3) newtons law of cooling (heat exchange) $u_x(l,t) = -\lambda[u(l,t) - \theta(t)]$

Cooling at $x = l$ + prescribed heat source of temp $\Theta(t)$

Need only one condition on each boundary. Also for parab. need only one initial conditions.

Cauchy I.V. Problem

Given $u(x,t_0) = \varphi(x)$ for $-\infty < x < \infty$

Find $u(x,t)$ for $t > t_0$

1. Soln on infinite line is soln in a region where influence of boundaries has not been felt yet.

2. Semi infinite line - only one bdy has been felt.

$u(x,t_0) = \varphi(x)$, $0 \leq x < \infty$ & $u(0,t) = \mu(t)$ $t > 0$

3. no initial conditions (large time behavior replaces it)

Special case periodic boundary data & look for solns with same period.

- replace init conditions by periodicity cond.

First B.V. Proble.

1) $u(x,t)$ defined & cont for $0 \leq x \leq l$ $t_0 \leq t \leq T$

2) satisfies $u_t = a^2 u_{xx} + f(x,t)$ in $[0,l]$ $[t_0,T]$

3) $u(x,t_0) = \varphi(x)$

$u(0,t) = \mu_1(t)$ and $u(l,t) = \mu_2(t)$

where $\mu(t_0) = \varphi(0)$ $\varphi(l) = \mu(t_0)$

Show: Soln exists, unique & stability Δ(init cond) $\Rightarrow \Delta$(soln)

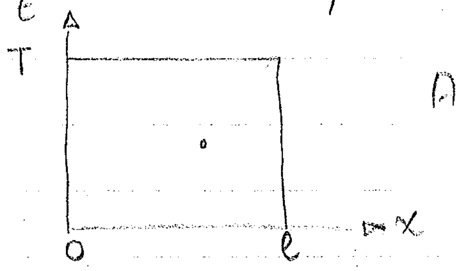
Principle of the Maximum

The fn $u(x,t)$ satisfying $u_t = a^2 u_{xx}$ in $0<t<T$, $0<x<l$ assumes its max or min at the initial moment $t=0$ or at the boundary points $x=0$ & $x=l$.

ie no temp in rod higher than max initial or max. boundary temp.

Proof for max. (for min replace u by $-u$)

Let $M =$ max of u for $t=0$ $x=0, x=l$



Assume u attains its max at some interior (x_0,t_0) $0<x_0<l$
$0<t_0 \leq T$

Let $u(x_0,t_0) = M+\epsilon$ $\epsilon>0$

Then at $(x_0,t_0) \rightarrow \frac{\partial u}{\partial x}(x_0,t_0) = 0$ $\frac{\partial^2 u}{\partial x^2}(x_0,t_0) < 0$

Also a max in $t \rightarrow \frac{\partial u}{\partial t}(x_0,t_0) \geq 0$

Consistent for $\frac{\partial^2 u}{\partial x^2} = 0$ & $\frac{\partial u}{\partial t} = 0$

Define $V(x,t) = u(x,t) + k\,(t_0 - t)$ $k = \text{const}$

need not satisfy HCE

Again $V(x_0,t_0) = u(x_0,t_0) = M+\epsilon$

Since $t_0 < T$ $k(t_0 - t) \leq kT$

choose $k \ni k.T < \epsilon/2$

Thus on $x = 0, l$ $t = 0$ $V(x,t) < M + \epsilon/2$

For $0 < t < t_0$ $k(t_0 - t) > 0$

$\therefore$ Max $V(x,t) \geq$ max $u(x,t)$

Since V is cont it assumes its max $\Rightarrow \exists$

$(x_1, t_1) \ni$ $V(x_1,t_1) \geq V(x_0,t_0) = M+\epsilon$

Some $t_1 > 0$ $0 < x_1 < l$ must be in interior

since V on bdy $\leq M + \epsilon/2$

$\Rightarrow v_{xx} \leq 0$ & $V_t \geq 0$ at (x_1, t_1)

$\Rightarrow$ at (x_1,t_1) $V_{xx} \leq 0 \Rightarrow u_{xx}(x_1,t_1) \leq 0$

$V_t \geq 0 \Rightarrow u_t(x_1,t_1) - k \geq 0$

$\Rightarrow u_t(x_1,t_1) \geq k > 0$

$\Rightarrow$ $u_{xx} \leq 0$ $u_t > 0$ @ (x_1,t_1)

Contradiction $\Rightarrow$ since they don't satisfy HCE

$\therefore$ max doesn't occur at interior pt.

<u>Uniqueness For first B.V. Problem</u>

Let u_1, u_2 satisfy $u_t = a^2 u_{xx}$

$0 < t < T$ $0 < x < l$

and both satisfy $u(x,0) = \phi(x)$

$u(0,t) = \mu_1(t)$

$u(l,t) = \mu_2(t)$

Define $u_2 - u_1 = V(x,t)$

$\therefore$ $V_t = a^2 V_{xx}$ $V(x,0) = 0$ $V(0,t) = 0 = V(l,t)$

max = min = 0 on bdy

Principle of Max $\Rightarrow V \equiv 0$ $\therefore u_1 \equiv u_2$

More Consequences (unproven)

1) If 2 solns u_1 & u_2 satisfy $u_1(x,0) \leq u_2(x,0)$
$u_1(0,t) \leq u_2(0,t)$ $\quad u_1(l,t) \leq u_2(l,t)$ then
$u_1(x,t) \leq u_2(x,t)$ $\quad$ and satisfy HCE

2) If u, $\bar{u}$ & $\underline{u}$ satisfy HCE & $\underline{u} \leq u \leq \bar{u}$ on $t=0, x=0, x=l$
then $\underline{u} \leq u \leq \bar{u}$ for $\forall$ $x \in [0,l]$ $\quad t \in [0,T]$

3) If u_1 & u_2 satisfy HCE & $|u_1 - u_2| \leq \epsilon$ for $t=0, x=0, x=l$
then $|u_1 - u_2| \leq \epsilon$ all $x \in [0,l]$
$t \in [0,T]$

3) $\Rightarrow$ continuous dependence on initial & bdy data.

Uniqueness on infinite line

extra restriction $u(x,t)$ bounded
$|u| < M$

Let u_1 & u_2 satisfy H.C.E with $u(x,0) = \phi(x)$
then $V = u_1 - u_2 \rightarrow$

$$V_t = a^2 V_{xx} \qquad V(x,0) = 0 \quad \& \quad |V| < 2M$$

Consider region $|x| \leq l$ $\quad$ (L arbitrary param)

Consider $\bar{V}(x,t) = 4M/L^2 \left(\frac{x^2}{2} + a^2 t\right)$
which satisfies $\bar{V}_t = a^2 \bar{V}_{xx}$

$$|\bar{V}(x,0)| \geq |V(x,0)| = 0$$

$$\bar{V}(\pm L, t) \geq 2M \geq V(\pm L, t)$$

applying principle of max to $[-L, L]$ any $t > 0$

$$\Rightarrow -\frac{4M^2}{L^2}\left(\frac{x^2}{2} + a^2 t\right) \leq V(x,t) \leq \frac{4M}{L^2}\left(\frac{x^2}{2} + a^2 t\right)$$

Consider any (x,t) fixed and let $L \rightarrow \infty$ $\qquad 0_- \leq V(x,t) \leq 0_+$

$$\Rightarrow \quad V(x,t) \equiv 0 \quad \rightarrow \quad u_1 \equiv u_2$$

Separation of Variables Soln.

$$u_t = a^2 u_{xx} \qquad \left.\begin{array}{l} u(x,0) = \varphi \\ u(0,t) = u(l,t) = 0 \end{array}\right\} \begin{array}{l} 0<x<l \\ t>0 \end{array}$$

Look for soln $u = XT$

$XT' = a^2 X'' T$ looks for soln without initial data

$$\frac{T'}{a^2 T} = \frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0 \qquad T' + \lambda_n a^2 T = 0 \qquad X(0) = X(l) = 0$$

$\Rightarrow$ $X_n = \sin\frac{n\pi x}{l}$ for $\lambda_n = \left(\frac{n\pi}{l}\right)^2$

$$T_n = c_n e^{-a^2 \lambda_n t}$$

$$u_n(t) = c_n e^{-a^2\lambda_n t} \sin\frac{n\pi x}{l}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{a^2 n^2 \pi^2}{l^2} t} \sin\frac{n\pi x}{l}$$

To set $u(x,0) = \varphi(x)$ $\quad \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = \varphi(x)$

$$c_n = \frac{2}{l}\int_0^l \varphi(x) \sin\frac{n\pi x}{l}\, dx$$

Hw. P. 187 #4,5,6

Dec 19, 1972

$u(x,t) = \sum c_n e^{-\lambda_n a^2 t} \sin \frac{n\pi x}{l}$ $\qquad c_n = \frac{2}{l}\int_0^l \phi(\xi) \sin\left(\frac{n\pi\xi}{l}\right) d\xi$

$\lambda_n = \left(\frac{n\pi}{l}\right)^2$

Must show uniform convergence of $\sum \frac{\partial u_n}{\partial t}$, $\sum \frac{\partial^2 u_n}{\partial x^2}$

$\left|\frac{\partial u_n}{\partial t}\right|$ (when $u = \sum u_n$) $= \left|-c_n \left(\frac{n\pi a}{l}\right)^2 e^{-\lambda_n a^2 t} \sin \frac{n\pi x}{l}\right|$

$< |c_n| \left(\frac{\pi}{l}\right)^2 a^2 n^2 e^{-\lambda_n a^2 t}$

$u(x,0) = \phi(x)$ bounded $\quad |\phi| < M$

Defn of $c_n \Rightarrow |c_n| < 2M$

$\left|\frac{\partial u_n}{\partial t}\right| < 2M \left(\frac{\pi}{l}\right)^2 a^2 n^2 e^{-\lambda_n a^2 t}$ $\qquad$ same on $\left|\frac{\partial^2 u_n}{\partial x^2}\right|$

$\left|\frac{\partial^{k+l} u_n}{\partial x^l \partial t^k}\right| < 2M \left(\frac{\pi}{l}\right)^{2k+l} n^{2k+l} a^{2k} e^{-\left(\frac{n\pi}{l}\right)^2 a^2 t}$

$\alpha_n = N n^q e^{-\beta n^2 t}$

use ratio test $\lim_{n\to\infty} \left|\frac{\alpha_{n+1}}{\alpha_n}\right| = 0 \Rightarrow$ uniformly convergent $\forall t$

since it is U.C. interchange $\sum$ & $\int$

$u(x,t) = \int_0^l G(x,\xi,t)\, \phi(\xi)\, d\xi$

$G(x,\xi,t) = \frac{2}{l}\sum e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin \frac{n\pi x}{l} \sin \frac{n\pi}{l}\xi = \lim_{\Delta\xi\to 0} \sum \left|G(x,\xi,t)\phi(\xi)\Delta\xi\right|$

G is Green's Function for homogeneous h.c. eqs with zero b.c & given init cond

Green's Function is also called Source Fn.

represents unit source at $x=\xi$ at $t=0$ & u is soln for $\forall x \in [0,l]$ $t>0$

For wave eqs

$$f(x,t) + a^2 u_{xx} = u_{tt} \qquad 0 \le x \le l$$

$$u = \int_0^t \int_0^l G(x,\xi,t-\tau)\, f(\xi,\tau)\, d\xi\, d\tau$$

$$G = \frac{2}{l} \sum \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi \xi}{l}\right) \cos \omega_n (t-\tau)$$

represents point force at $t=\tau$ & $x=\xi$ $\qquad$ u represents effect of force at $t>\tau$ & $0 \le x \le l$

Consider an amount of heat Q released between $\xi+\epsilon$, $\xi-\epsilon$ at $t=0$

ie $\quad c\rho \int_{\xi-\epsilon}^{\xi+\epsilon} \phi_\epsilon(s)\, ds = Q$

$$u(x,0) = \begin{cases} \phi_\epsilon & \xi-\epsilon \le x \le \xi+\epsilon \\ 0 & \forall \text{ other } x \end{cases}$$

$$u(x,t) = \int_{\xi-\epsilon}^{\xi+\epsilon} G(x,s,t)\, \phi_\epsilon(s)\, ds$$

using MVT & cont of G $\quad = G(x,s^*,t) \int_{\xi-\epsilon}^{\xi+\epsilon} \phi_\epsilon(s)\, ds = \frac{Q}{c\rho} G(x,s^*,t)$

where $s^* \in (\xi-\epsilon, \xi+\epsilon)$

choosing a unit temp distrib at ξ $\quad \int_{\xi-\epsilon}^{\xi+\epsilon} \phi_\epsilon(s)\, ds = 1$

ie $Q = \rho c$ & let $\epsilon \to 0$

$$\Rightarrow \quad u(x,t) = G(x,\xi,t)$$

Principle of max $\Rightarrow$ $G(x,\xi,t) \geq 0$

$$\begin{cases} \phi \text{ piecewise differentiable} \\ \phi(0) = \phi(l) = 0 \end{cases}$$

restrictiveness of ϕ is removed $\ni$ ϕ need only be piecewise continuous

Proof: in many stages

1st Stage show $u = \int_0^l G\phi \, ds$ is good for arbitrary continuous ϕ with $\phi(0) = \phi(l) = 0$ (no longer ϕ piecewise diff)

Let us consider a sequence $\phi_n(x)$ continuous with piecewise cont deriv $\ni$ $\phi_n \to \phi$ uniformly then $\exists$ $u_n(x,t) = \int_0^l G\phi_n \, ds$

Then for $\epsilon > 0$ $\exists$ $n(\epsilon)$ $\ni$ $|\phi_{n_1}(x) - \phi_{n_2}(x)| < \epsilon$ for $n_1, n_2 > n(\epsilon)$ Cauchy convergence criterion

Then by principle of max $\exists$ u_{n_1} & u_{n_2} which satisfy

$$|u_{n_1}(x,t) - u_{n_2}(x,t)| < \epsilon \quad \text{for} \quad n_1, n_2 > n(\epsilon)$$

$\Rightarrow$ $\exists$ $u(x,t)$ $\ni$ $u_n(x,t) \xrightarrow{\text{uniformly}} u(x,t)$

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \lim_{n\to\infty} \int_0^l G\phi_n \, ds$$

By U.C. $\int \lim_{n\to\infty} = \lim_{n\to\infty} \int$

then $u(x,t) = \int_0^l G\phi \, ds$

Result: good for arbitrary piecewise continuous ϕ

Inhomogeneous Heat Conduction Eq.

$$u_t = a^2 u_{xx} + f(x,t)$$

$$u(x,0) = 0 \quad u(0,t) = u(l,t) = 0$$

Let $u(x,t) = \sum u_n(t) \sin \frac{n\pi x}{l}$

$f(x,t) = \sum f_n(t) \sin \frac{n\pi x}{l}$

put back into eq

$$\sum \sin \frac{n\pi x}{l} \left\{ \left(\frac{\pi n}{l} a\right)^2 u_n + u_n' - f_n \right\} = 0$$

$$\Rightarrow \quad u_n' + \left(\frac{a\pi n}{l}\right)^2 u_n = f_n(t)$$

$$u_n(t) = \int^t e^{-\left(\frac{an\pi}{l}\right)^2 (t-\tau)} f_n(\tau) d\tau$$

$$u(x,t) = \sum_1^\infty \left\{ \int_0^t e^{-\left(\frac{n\pi a}{l}\right)^2 (t-\tau)} f_n(\tau)\, d\tau \right\} \sin \frac{n\pi x}{l} \qquad f_n = \frac{2}{l}\int_0^l f(\xi,\tau) \sin \frac{n\pi \xi}{l}\, d\xi$$

by U.C. $\sum \int = \int \sum$

$$u(x,t) = \int_0^t \int_0^l G(x,\xi,t-\tau)\, f(\xi,\tau)\, d\tau$$

where $G(x,\xi,t-\tau) = \frac{2}{l} \sum e^{-\left(\frac{n\pi a}{l}\right)^2 (t-\tau)} \sin \frac{n\pi x}{l} \sin \frac{n\pi \xi}{l}$

ie same Green Fn as for homogeneous eq.

To understand the meaning of the green's Function

Consider unit source at pt (ξ_0, τ_0)

Then Total Heat Produced

$$\dot{Q} = \rho c \int_t \int_0^l f(x,t)\, dx\, dt$$

$$\text{Define } f = \begin{cases} 0 & x \notin (\xi_0, \xi_0+\Delta\xi) \\ & t \notin (\tau_0, \tau_0+\Delta\tau) \\ f(\xi,\tau) & \text{otherwise} \end{cases}$$

∋ such that $\int_{\tau_0}^{\tau_0+\Delta\tau} \int_{\xi_0}^{\xi_0+\Delta\xi} f(\xi,\tau)\, d\xi\, d\tau = 1$

$$u(x,t) = \int_0^t \int_0^l G(x,\xi,t-\tau) f(\xi,\tau)\, d\xi\, d\tau$$

$$= \int_{\tau_0}^{\tau_0+\Delta\tau} \int_{\xi_0}^{\xi_0+\Delta\xi} G(x,\xi,t-\tau) f(\xi,\tau)\, d\xi\, d\tau \text{ ; use MVT.}$$

$$\Rightarrow u(x,t) = \underbrace{\int_{\tau_0}^{\tau_0+\Delta\tau} \int_{\xi_0}^{\xi_0+\Delta\xi} f(\xi,\tau)\, d\xi\, d\tau}_{1} \left[G(x,\xi^*,t-\tau^*) \right]$$

$\xi^* \in (\xi_0, \xi_0+\Delta\xi)$

$\tau^* \in (\tau_0, \tau_0+\Delta\tau)$

$$u(x,t) = G(x,\xi^*,t-\tau^*)$$

Let $\Delta\xi, \Delta\tau \to 0$ ∋

$$u(x,t) = G(x,\xi_0,t-\tau_0)$$

Green's Fn represents temp distrib that results from unit source

Again can reverse process - add up weighted external source fn $f(x,t)$ $\quad u = \sum_t \sum_x G f \, \Delta x \Delta t$

as $\Delta x, \Delta t \to 0$ $\quad u \to \int_0^t \int_0^l G f \, d\xi \, d\tau$

Inhomogeneous B.C.

$u(x,t) = U(x,t) + v(x,t)$ $\qquad$ let V satisfy $V_t = a^2 V_{xx} + \bar{f}$

$\qquad\qquad\qquad\qquad\qquad\qquad\;\; U \quad " \quad U_t = a^2 U_{xx} + f$

$V(x,0) = \bar{\phi}(x)$
$V(0,t) = 0$
$V(l,t) = 0$
$U(x,0) = \phi(t)$
$U(0,t) = \mu_1(t)$
$U(l,t) = \mu_2(t)$

then
$$\bar{f} = f - \left[U_t - a^2 U_{xx}\right]$$
$$\bar{\phi} = \phi - U(x,0)$$

$$\left.\begin{array}{l} V(0,t) = \mu_1(t) - U(0,t) \\ V(l,t) = \mu_2(t) - U(l,t) \end{array}\right\} \quad U(x,t) = \mu_1(t) + \frac{x}{l}\left[\mu_2(t) - \mu_1(t)\right]$$

known chosen fn

$V_1 = \int_0^l G \bar{\Phi} \, ds$ $\qquad V_2 = \int_0^t \int_0^l G \bar{f} \, d\xi \, d\tau$ $\qquad V = V_1 + V_2$

hom eq, inhom init cond $\qquad$ homo b.c. inhomo eq

Green's fn for Infinite line $(-\infty, \infty)$

Transform $(0,l) \to (-l/_2, l/_2)$ then let $l \to \infty$

$$G_l(x,\xi,t) = \frac{2}{l} \sum e^{-\left(\frac{an\pi}{l}\right)^2 t} \sin \frac{n\pi x}{l} \sin \frac{n\pi \xi}{l}$$

$x = x' + l/_2 \qquad \xi = \xi' + l/_2$

let $x' = x - l/_2 \qquad \xi' = \xi - l/_2$

$$G_l(x,\xi,t) = \frac{2}{l} \sum_{k=1}^{\infty} e^{-\left(\frac{2k\pi a}{l}\right)^2 t} \sin \frac{2k\pi x'}{l} \sin \frac{2k\pi \xi'}{l} + \frac{2}{l} \sum_{k=1}^{\infty} e^{-\left(\frac{[2k-1]\pi a}{l}\right)^2 t} \cos \frac{(2k-1)}{l}\pi x' \cos \left(\frac{2k-1}{l}\right)\pi \xi'$$

Over even sm.

write $\frac{2}{l} \sum e^{-\lambda_k^2 a^2 t} \sin(\lambda_k x') \sin(\lambda_k \xi') = \frac{1}{\pi} \sum f_1(\lambda_k) \, d\lambda$

$\lambda_k = \frac{2k\pi}{l} \qquad f_1(\lambda) = e^{-\lambda_n^2 a^2 t} \sin(\lambda x') \sin(\lambda \xi')$

$d\lambda = \frac{4\pi}{l}$

Consider lim as $l \to \infty$

i.e. as $d\lambda \to 0$

Then $\frac{1}{\pi}\sum f_1(\lambda_n)\,d\lambda \longrightarrow \frac{1}{\pi}\int_0^\infty f_1(\lambda)\,d\lambda$

Over the odd sum

Define $G(x,\xi,t) = \lim_{l\to\infty} G_l(x,\xi,t) = \frac{1}{\pi}\int_0^\infty e^{-\lambda^2 a^2 t}\left\{\sin\lambda x'\sin\lambda\xi' + \cos\lambda x'\cos\lambda\xi'\right\}d\lambda$

$f_2(\lambda) = e^{-\lambda^2 a^2 t}\cos\lambda x'\cos\lambda\xi'$

$$\underline{G(x,\xi,t)} = \frac{1}{\pi}\int_0^\infty e^{-\lambda^2 a^2 t}\cos\lambda(x-\xi)\,d\lambda$$

Green's fn for infinite line.

to evaluate G

Consider $I(\beta) = \int_0^\infty e^{-\lambda^2\alpha}\cos\lambda\beta\,d\lambda \quad (\alpha>0)$

$$I'(\beta) = -\int_0^\infty \lambda e^{-\lambda^2\alpha}\sin(\lambda\beta)\,d\lambda$$

$$\overset{\text{by parts}}{=} \frac{e^{-\lambda^2\alpha}}{2\alpha}\sin\lambda\beta\Big|_0^\infty - \beta/_{2\alpha}\int_0^\infty e^{-\lambda^2\alpha}\cos(\lambda\beta)\,d\lambda$$

$$= -\beta/_{2\alpha}\, I$$

$\frac{dI}{d\beta} + \beta/_{2\alpha}\, I = 0 \qquad I = Ce^{-\beta^2/4\alpha}$

$C = I(0) = \int_0^\infty e^{-\lambda^2\alpha}\,d\lambda = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$ since $\int_0^\infty e^{-z^2}dz = \frac{\sqrt{\pi}}{2}$

$$I(\beta) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\, e^{-\beta^2/4\alpha}$$

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi a^2 t}}\, e^{-\frac{(x-\xi)^2}{4a^2t}}$$

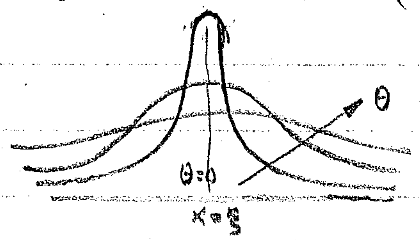
Fundamental Sol of Heat Eq on Infinite Line
Verify G is a sol of heat eq by differ.

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{\theta}} e^{-\frac{(x-\xi)^2}{4\theta}} \qquad \theta = a^2 t$$

Find shape of $G(x,\xi,0)$

at $x=\xi$ $\quad G = \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{\theta}} \to \infty$ at $t \to 0$

$\theta \to 0$, $x \neq \xi$ $\quad G \to 0$ (L'Hopital's Rule) or $e^{\alpha} \to 0$ faster than $\sqrt{\alpha}$



$\theta \neq 0 \quad G \neq 0$ any x

Amount of Heat present on $(-\infty, \infty)$ = Q

$$Q = c\rho \int_{-\infty}^{\infty} G(x,\xi,t-t_0)\, dx$$

$$= \frac{c\rho}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4a^2(t-t_0)}}}{2\sqrt{a^2(t-t_0)}}\, dx \qquad \text{let} \quad \alpha = \frac{(x-\xi)}{2a\sqrt{t-t_0}}$$

$$= \frac{c\rho}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha = c\rho \left[2 \frac{\sqrt{\pi}}{2}\right] = c\rho$$

Amount of Heat is Conserved int(no heat sources or sinks present)

General Soln to Cauchy Init Val Probl

Given $u(x,0) = \varphi(x)$

$$u(x,t) = \int_{-\infty}^{\infty} G(x,\xi,t)\,\varphi(\xi)\,d\xi$$

Special case $u(x,0) = T_1 \quad x < 0$

$T_2 \quad x > 0$

$$u(x,t) = \frac{T_1}{\sqrt{\pi}} \int_{-\infty}^{\frac{-x}{2\sqrt{a^2t}}} e^{-\alpha^2} d\alpha + \frac{T_2}{\sqrt{\pi}} \int_{\frac{-x}{2\sqrt{a^2t}}}^{\infty} e^{-\alpha^2} d\alpha$$

$$= \frac{T_1 + T_2}{2} + \frac{T_1 - T_2}{2}\, \Phi\left(\frac{x}{2\sqrt{a^2t}}\right)$$

where $\Phi(z) = erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\alpha^2} d\alpha \qquad erf(\infty) = 1$

Probls P. 213

Jan 9, 1972

Boundary Valued Problems on semi infinite line

$u_t = a^2 u_{xx} \quad x > 0 \quad t > 0$

I.C. $u(x,0) = \phi(x) \quad x > 0$

B.C. 1. $u(0,t) = \mu_1(t)$

2. $u_x(0,t) = \mu_2(t)$

3. $u_x = \lambda\,[u - \Theta(t)]$

$u = u_1 + u_2$ $\quad$ $u_1 : \{ \underset{u}{\phi}(x) \;, u(0,t) = 0 \text{ or } u_x(0,t) = 0 \}$

$u_2 : \{ (x,0) = 0 \text{ prescribed bc} \}$

u_1 : extend ϕ evenly or oddly about $x = 0$

Extension of Results for Poisson Integral (Soln to pure IV problem)

A) if $u(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \psi(\xi) d\xi$ (Poisson Int)

and if $\psi(x) = -\psi(-x)$ ie odd then $u(0,t) = 0 \;\; \forall t > 0$

B) if $\psi(x) = +\psi(-x)$ ie even then $u_x(0,t) = 0 \;\; \forall t > 0$

Proof split integral into 2 parts $\int_{-\infty}^{0} + \int_{0}^{\infty}$ & using property of $\psi(x)$

then $U(x,t)$ be defined on the whole line & satisfying

$$U_t = a^2 U_{xx}$$

$$U(0,t) = 0$$

$$U(x,0) = \Psi(x) = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \end{cases}$$

Then $U(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \Psi(\xi) d\xi)$

$$U = -\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{0} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \phi(-\xi) d\xi + \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \phi(\xi) d\xi$$

let $\eta = -\xi$

$$U = \frac{-1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x+\eta)^2}{4a^2 t}} \phi(\eta) d\eta + \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \phi(\xi) d\xi$$

$$= \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{a^2 t}} \phi(\xi) \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} - e^{-\frac{(x+\xi)^2}{4a^2 t}} \right\} d\xi$$

if for $\frac{\partial u}{\partial x}$

$$U(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{a^2t}} \left\{ e^{-\frac{(x-\xi)^2}{4a^2t}} + e^{-\frac{(x+\xi)^2}{4a^2t}} \right\} \phi(\xi)\, d\xi$$

$$\boxed{\begin{aligned} &\text{in integ } - \ \equiv u(0,t) = 0 \\ &\qquad\quad + \ \equiv u_x(0,t) = 0 \end{aligned}}$$

if $\phi = \text{const} = T = u(x,0)$

$u(0,t) = 0$

$$u = \frac{T}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{a^2t}} \left\{ e^{-\frac{(x-\xi)^2}{4a^2t}} - e^{-\frac{(x+\xi)^2}{4a^2t}} \right\} d\xi$$

let $\alpha = \frac{-(x-\xi)}{2\sqrt{a^2t}}$ $\qquad \beta = \frac{+(x+\xi)}{2\sqrt{a^2t}}$

$d\alpha = \frac{d\xi}{2\sqrt{a^2t}}$ $\qquad d\beta = \frac{d\xi}{2\sqrt{a^2t}}$

$$u = \frac{T}{\sqrt{\pi}} \left[\int_{-\frac{x}{2\sqrt{a^2t}}}^{\infty} e^{-\alpha^2} d\alpha - \int_{\frac{x}{2\sqrt{a^2t}}}^{\infty} e^{-\beta^2} d\beta \right] = \frac{T}{\sqrt{\pi}} \int_{-\frac{x}{2\sqrt{a^2t}}}^{\frac{x}{2\sqrt{a^2t}}} e^{-\alpha^2} d\alpha$$

$$= \frac{2T}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{a^2t}}} e^{-\alpha^2} d\alpha = T \operatorname{erf}\left(\frac{x}{2\sqrt{a^2t}}\right) = T\, \overline{\Phi}\left(\frac{x}{2\sqrt{a^2t}}\right)$$

$$\operatorname{erf}(z) = \Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\alpha^2} d\alpha$$

u_2 : $u_2(x,t_0)=0$ $\quad$ $u_2(0,t)=\mu(t)$

Find u_2 in stages

Case 1: $\mu(t)=\mu_0=\text{const}$ $\quad$ replace $t \to t-t_0$

Let $\bar{v}(x,t_0)=\mu_0$ $\quad$ $\bar{v}(0,t)=0$ $\quad$ $t>t_0$

then $\bar{v}(x,t)=\mu_0\,\Phi\left(\frac{x}{2\sqrt{a^2(t-t_0)}}\right)$ $\quad$ $t>t_0$

Let $u_2=\mu_0-\bar{v}$

$u_2(x,t_0)=\mu_0-\mu_0=0$

$u_2(0,t)=\mu_0-\bar{v}(0,t)=\mu_0$

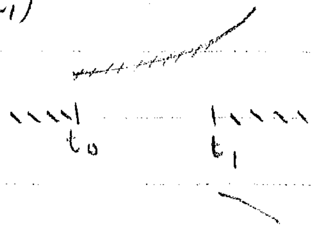
Thus $u_2(x,t)=\mu_0\left\{1-\Phi\left(\frac{x}{2\sqrt{a^2(t-t_0)}}\right)\right\}=\mu_0\,U(x,t)$

$$U(x,t)=\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{a^2t}}}^{\infty}e^{-\alpha^2}d\alpha \qquad \text{for } t>0$$

Extend the definition of U by choosing $U(x,t)\equiv 0$ $\quad$ $t<0$

Look for v $\ni$ $v(0,t)=\begin{cases}\mu_0 & t_0<t<t_1\\ 0 & \forall t\notin(t_0,t_1)\end{cases}$

$v(x,t)=\mu_0\{U(x,t-t_0)-U(x,t-t_1)\}$



Step fn $\mu(t)=$ $\quad$ μ_0 $\quad$ $t_0<t<t_1$

μ_1 $\quad$ $t_1<t<t_2$

$\vdots$

μ_{n-1} $\quad$ $t_{n-1}<t<t_n$

Soln:

$$u(x,t) = \sum_{i=0}^{n-2} \mu_i \left[U(x, t-t_i) - U(x, t-t_{i+1}) \right] + \mu_{n-1} U(x, t-t_{n-1})$$

to keep soln open ended Σ is to $n-2$ & not to $n-1$

Use M.V.T.: $u(x,t) = \sum_{i=0}^{n-2} \mu_i \frac{\partial U}{\partial t}(x, t-\tau_i)\Big|_{\tau_i} \Delta\tau + \mu_{n-1} U(x, t-t_{n-1})$

$t_i < \tau_i < t_{i+1}$ $\Delta\tau = t_{i+1} - t_i$

$$\boxed{\begin{array}{ll} \text{Let } \Delta\tau \to 0 & \text{General BV} \\ \quad u(x,t) = \int_0^t \mu(\tau) \frac{\partial U}{\partial t}(x, t-\tau)\, d\tau & \begin{array}{l} 0 \text{ I.C.} \\ u(0,t) = \mu(t) \end{array} \end{array}}$$

Since $t - t_{n-1} \to 0$ $U(x, t-t_{n-1}) = 0$ $U(x,0) = 0$; $U(0,t) = 1$ $t>0$

$U(0,t) = 0$ $t<0$

$$U = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{a^2 t}}}^{\infty} e^{-\alpha^2} d\alpha$$

$$U_t = \frac{1}{2\sqrt{\pi}} \frac{a^2 x}{(a^2 t)^{3/2}} e^{-x^2/4a^2 t} = -2a^2 \frac{\partial G}{\partial x}(x, 0, t)$$

$$= -2a^2 G_\xi (x, 0, t)$$

where G is Green's Fn for ∞ line ie fundamental Sol of HCE

$$G(x, \xi, t) = \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}}$$

$$u_2(x,t) = 2a^2 \int_0^t G_\xi (x, 0, t-\tau)\, \mu(\tau)\, d\tau$$

$u = u_1 + u_2$

$$= \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{a^2 t}} \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} \pm e^{-\frac{(x+\xi)^2}{4a^2 t}} \right\} \phi(\xi) d\xi + 2a^2 \int_0^t \frac{\partial G}{\partial \xi}(x, 0, t-\tau) \mu(\tau) d\tau$$

Duhamel's Principle

if a linear p.d.e exists with a b.c.

$$u(0,t) = \mu(t) \qquad t > 0$$

subject to $u(x,0) = 0$ & homogeneous b.c. (when they exist at $x = l$)

Then the soln of the problem can be expressed in the form

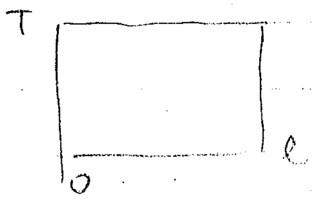
$$u(x,t) = \int_0^t \bar{U}_t(x, t-\tau) \mu(\tau) d\tau$$

where $\bar{U}(x,t)$ is the solution of the corresponding b.v. problem for $\bar{U}(0,t) = 1$

1) $\bar{U}$ satisfies eqn
2) Eq be linear
3) $\bar{U}(x,0) = 0$
4) $\bar{U}(0,t) = 1 \qquad t > 0$

   $= 0 \qquad t < 0$

If (other B.C.s) $\bar{U}$ must satisfy homogeneous b.c. at $x = l$ if exist



$V(x,t) = U(x,t) + k(t_0 - t)$

u satisfies HCE

$V_{xx} \leq 0 \quad \Rightarrow u_{xx} \leq 0$

$V_t \geq 0 \quad \Rightarrow u_t \geq k \neq 0$

$\Rightarrow$ contradiction

$$\xi+\eta = 4/3\, x^{3/2} \qquad 2x^{3/2} = \tfrac{3}{2}(\xi+\eta)$$

$$\eta-\xi = 4\sqrt{-y} \qquad 2\sqrt{-y} = \cancel{4}\ \frac{\eta-\xi}{2}$$

$$4u_{\xi\eta} + (u_\xi+u_\eta)\frac{2}{3(\xi+\eta)} + (u_\xi-u_\eta)\frac{2}{(\eta-\xi)} = 0$$

Final Canonical form $$2u_{\xi\eta} + \frac{(u_\xi+u_\eta)}{3(\eta+\xi)} + \frac{(u_\xi-u_\eta)}{(\eta-\xi)} = 0$$

$$4u_{\xi\eta} + y\left(-\frac{1}{2(-y)^{3/2}}\right)u_\xi + y\frac{1}{2(-y)^{3/2}}u_\eta = 0$$

$$4u_{\xi\eta} + (u_\eta-u_\xi)\left[\frac{1}{2\sqrt{-y}}\right] = 0 \qquad \xi-\eta = 4\sqrt{-y}$$

$$\frac{\xi-\eta}{2}$$

$$1u_{\xi\eta} + (u_\eta-u_\xi)\frac{1}{2(\xi-\eta)} = 0$$

$$C \doteq \quad e^{2x}u_{xx} + 2e^{(x+y)}u_{xy} + e^{2y}u_{yy} = 0$$

$$a_{11} = e^{2x} \quad a_{12} = 2e^xe^y \quad a_{22} = e^{2y}$$

$$(2e^xe^y)^2 - e^{2x}e^{2y} = (4-1)(e^{2x}e^{2y}) = 3e^{2x}e^{2y} > 0$$

for all x & y

$$\frac{dy}{dx} = \frac{2e^xe^y \pm \sqrt{4e^{2x}e^{2y} - e^{2x}e^{2y}}}{e^{2x}} = \frac{2e^xe^y \pm e^xe^y\sqrt{3}}{e^{2x}} = (2\pm\sqrt{3})\frac{e^y}{e^x}$$

$$\frac{dy}{e^y} = (2\pm\sqrt{3})\frac{dx}{e^x} \implies -e^{-y} = -(2\pm\sqrt{3})e^{-x} + c$$

$$\xi = -e^{-y} + (2+\sqrt{3})e^{-x} \qquad \xi_x = -(2+\sqrt{3})e^{-x} \quad \xi_y = e^{-y}$$

$$\eta = -e^{-y} + (2-\sqrt{3})e^{-x} \qquad \xi_{xx} = (2+\sqrt{3})e^{-x} \quad \xi_{yy} = -e^{-y}$$

$$\xi_{xy} = 0 \qquad \eta_x = -(2-\sqrt{3})e^{-x} \quad \eta_y = e^{-y}$$

$$\xi_{yx} = 0$$

$$\eta_{xy} = 0 \qquad \eta_{xx} = (2-\sqrt{3})e^{-x} \quad \eta_{yy} = -e^{-y}$$

$$\bar{a}_{11} = \overset{e^{2x}}{a_{11}}\xi_x^2 + 2\overset{4e^xe^y}{a_{12}}\xi_x\xi_y + \overset{e^{2y}}{a_{22}}\xi_y^2 = (2+\sqrt{3})^2 - 4(2+\sqrt{3}) + 1 = \cancel{4}+\cancel{3}+\cancel{4\sqrt{3}}-\cancel{8}-\cancel{4\sqrt{3}}+\cancel{1} =$$

$$\bar{a}_{12} = a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \xi_y\eta_x) + a_{22}\xi_y\eta_y = e^{2x}\cdot 1\, e^{-2x} + 2e^xe^y\left[-(2+\sqrt{3})-(2-\sqrt{3})\right]e^{-x}e^{-y} + e^{2y}e^{-2y} = 2+2(-4) =$$

$$(2+\sqrt{3})(2-\sqrt{3})$$

$z_1 = \alpha_1^{-1}(-\lambda\Delta^2 u_1)$

$z_i = \alpha_i^{-1}\left(-\lambda\Delta^2 u_i - B_i z_{i-1}\right)$

2a, c, f

2a $u_{xx} + xy\, u_{yy} = 0$ $\qquad a_{12} = 0 \quad a_{11} = 1 \quad a_{22} = xy$

$a_{12}^2 - a_{11}a_{22} > 0 \qquad -xy > 0$ either $\begin{matrix} x>0 & y<0 \\ x<0 & y>0 \end{matrix}$

$\frac{dy}{dx} = \pm\sqrt{-xy} \qquad \frac{dy}{\sqrt{-y}} = \pm\, dx\sqrt{x}$

$\pm 2\sqrt{-y} = \cancel{\pm}\ \frac{2}{3}x^{3/2} + C$

$\xi = \frac{2}{3}x^{3/2} - 2\sqrt{-y} \qquad \xi_x = \sqrt{x} \qquad \xi_y = +\frac{1}{\sqrt{-y}}$

$\eta = \left(+\frac{2}{3}x^{3/2} + 2\sqrt{-y}\right) \qquad \eta_x = \sqrt{x} \qquad \eta_y = -\frac{1}{\sqrt{-y}}$

$\bar{a}_{11} = \overset{1}{\cancel{a_{11}}}\xi_x^2 + 2\overset{0}{\cancel{a_{12}}}\xi_x\xi_y + \overset{xy}{\cancel{a_{22}}}\xi_y^2 = x + xy\frac{1}{-y} = 0$

$\bar{a}_{12} = \overset{1}{\cancel{a_{11}}}\xi_x\eta_x + \overset{0}{\cancel{a_{12}}}(\xi_x\eta_y + \eta_x\xi_y) + \overset{xy}{\cancel{a_{22}}}\xi_y\eta_y = \sqrt{x}(\sqrt{x}) + xy\left(-\frac{1}{-y}\right)$
$= x + xy\left(\frac{1}{y}\right) = 2x$

$\bar{a}_{22} = \overset{1}{\cancel{a_{11}}}\eta_x^2 + 2\overset{0}{\cancel{a_{12}}}\eta_x\eta_y + \overset{xy}{\cancel{a_{22}}}\eta_y^2 = x + xy\left(\frac{1}{-y}\right) = 0$

$\xi_{xx} = \frac{1}{2\sqrt{x}} = \eta_{xx}, \xi_{yy} = +\frac{1}{2}(-y)^{-3/2} = -\eta$

Canonical $u_{xx} + xy\, u_{yy} = 0$

$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx}$

$= \cancel{u_{\xi\xi}x} + 2u_{\xi\eta}x + \cancel{u_{\eta\eta}x} + u_\xi\frac{1}{2\sqrt{x}} + u_\eta\frac{1}{2\sqrt{x}}$

$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}$

$= u_{\xi\xi}\left(\frac{1}{-y}\right) + 2u_{\xi\eta}\frac{1}{y} + u_{\eta\eta}\left(-\frac{1}{y}\right) + u_\xi\frac{1}{2(-y)^{3/2}} + u_\eta\left(-\frac{1}{2(-y)^{3/2}}\right)$

$xy\, u_{yy} = -\cancel{u_{\xi\xi}x} + 2u_{\xi\eta}x - \cancel{u_{\eta\eta}x} + u_\xi\frac{x}{2\sqrt{-y}} + u_\eta\frac{x}{2\sqrt{-y}}$

$4u_{\xi\eta}x + (u_\xi + u_\eta)\frac{x}{2\sqrt{x}^3} + (u_\xi - u_\eta)\frac{x}{2\sqrt{-y}} = 0$

$4u_{\xi\eta}x + u_\xi\left(\frac{x}{} + \frac{x}{}\right) + u_\eta\left(\frac{x}{} - \frac{x}{}\right) = 0$

~~$\bar{a}_{22} = a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2 = e^{2x}(2-\sqrt{3})^2e^{-2x} + 4e^xe^y(2-\sqrt{3})e^{-x}e^{-y} + e^{2y}e^{-2y} =$~~

~~$\cancel{4}-4\sqrt{3}+\cancel{3}-\cancel{8}+4\sqrt{3}+\cancel{1}=0$~~

~~$e^{2x}u_{xx} = u_{\xi\xi}(2+\sqrt{3})^2 + 2u_{\xi\eta}(2+\sqrt{3})(2-\sqrt{3}) + u_{\eta\eta}(2-\sqrt{3})^2 + u_\xi(2+\sqrt{3}) + u_\eta(2-\sqrt{3})$~~

~~$2e^{(x+y)}u_{xy} = -2u_{\xi\xi}(+2+\sqrt{3}) + 2u_{\xi\eta}[-(2+\sqrt{3})-(2-\sqrt{3})] - 2u_{\eta\eta}(2-\sqrt{3}) + 2u_\xi\cdot 0 + 2u_\eta\cdot 0$~~

~~$e^{2y}u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} + u_\xi - u_\eta$~~

~~$u_{\xi\xi}[4+3+4\sqrt{3}-4-2\sqrt{3}+1] + 2u_{\xi\eta}[1-+1] + u_{\eta\eta}[4-4\sqrt{3}+\cancel{3}-\cancel{4}+2\sqrt{3}+\cancel{1}] + u_\xi(2+\sqrt{3}-1) + u_\eta(2-\sqrt{3}-1) = 0$~~

~~$(4+2\sqrt{3})u_{\xi\xi} + u_{\xi\eta}[-12] + u_{\eta\eta}[4-2\sqrt{3}] + u_\xi[1+\sqrt{3}] + u_\eta[1-\sqrt{3}] = 0$~~

c $\quad e^{2x}u_{xx} + 2e^xe^yu_{xy} + e^{2y}u_{yy} = 0$

$(e^xe^y)^2 - e^{2y}e^{2x} = 0$ eq is parabolic

$dy/dx = \frac{e^xe^y}{e^{2x}} = \frac{e^y}{e^x}$ $\qquad c = -e^{-y} + e^{-x}$

since eq is parabolic let $\eta = \eta(x,y) = x-y$

$\eta_x =$ [illegible] $\quad \eta_{xx} = 0 \quad \xi_x = -e^{-x} \quad \xi_y = +e^{-y}$

$\eta_y =$ [illegible] $\quad \eta_{yy} = 0 \quad \xi_{xx} = e^{-x} \quad \xi_{yy} = -e^{-y}$

$\eta_{xy} = 0 \qquad \xi_{xy} = 0$

$\bar{a}_{11} = e^{2x}e^{-2x} + 2e^xe^y(-e^{-x}e^{-y}) + e^{2y}e^{-2y} = 0$

$\bar{a}_{12} = e^{2x}\cdot\overset{-e^{-x}}{} + e^xe^y\overset{(-e^{-x}+e^{-y})}{()} + e^{2y}\cdot\overset{e^{-y}}{} = -e^x + e^y + e^x - e^y = 0$

$\bar{a}_{22} = e^{2x} + 2e^xe^y + e^{2y} = (e^x+e^y)^2$

$u_{xx} = u_{\xi\xi}e^{-2x} - 2u_{\xi\eta}e^{-x} + u_{\eta\eta} + u_\xi e^{-x}$

$u_{xy} = -u_{\xi\xi}e^{-x}e^{-y} + u_{\xi\eta}(+e^{-x}+e^{-y}) - u_{\eta\eta} + \cancel{u_\xi}$

$u_{yy} = u_{\xi\xi}e^{-2y} - 2u_{\xi\eta}e^{-y} + u_{\eta\eta} - u_\xi e^{-y}$

~~$2(e^{-y}-e^{-y})$~~

$\eta + y = x$

$\xi = +e^{-\eta}e^{-y} - e^{-y}$

$\xi = e^{-y}(e^{-\eta}-1)$

$u_{\xi\xi}(\underbrace{1-\cancel{2}+1}_{0}) + u_{\xi\eta}(\underbrace{-2e^x+2e^y+\cancel{2e^x}-2e^y}_{0}) + u_{\eta\eta}(e^{2x}-2e^xe^y+e^{2y})$

$+ u_\xi(e^x - e^{+y}) = 0$

$u_{\eta\eta}\cdot(e^x-e^y)^2 + u_\xi(e^x-e^y) = 0$

~~$u_{\eta\eta}\left(\frac{e^y}{\xi e^y+1}+e^y\right)^2$~~

~~$\eta - y = x$~~

~~$e^{-y}+\xi = e^{-x}$~~

~~$\frac{1}{\xi+e^{-y}} = e^x$~~

~~$\xi = e^{-(\eta-y)}e^{-y}$~~

~~$\xi = e^{-\eta}e^{y}e^{-y}$~~

$u_{\eta\eta}(e^x-e^y)^2 + u_\xi(e^x-e^y) = 0$ if $e^x - e^y \neq 0$

$u_{\eta\eta}(e^x-e^y) + u_\xi = 0$

$e^x = e^\eta e^y$

$e^x - e^y = e^y(e^\eta - 1)$

$\xi = e^{-y}(e^{-\eta}-1)$

$$u_{\eta\eta} + \frac{u_\xi(\xi e^\eta)}{-(e^\eta-1)^2} = 0$$

$\cancel{1+\xi e^\eta = e^{-y}}$ $\qquad \dfrac{\xi}{e^{-\eta}-1} = e^{-y}$

$\cancel{\dfrac{1}{1+\xi e^\eta} = e^y}$ $\qquad \dfrac{e^{-\eta}-1}{\xi} = e^y$

$\cancel{\dfrac{e^\eta}{1+\xi e^\eta}}$

$$\left(\frac{e^{-\eta}-1}{\xi}\right)(e^\eta-1) = e^x - e^y$$

$$\frac{\left(\frac{1}{e^\eta} - e^\eta\right)(e^\eta-1)}{\xi e^\eta}$$

$$-\frac{(e^\eta-1)^2}{\xi e^\eta}$$

$(x-y)u_{xx} + (x-y)(y-1)u_{xy} = 0$ Assume $x-y \neq 0$

$u_{xx} + (y-1)u_{xy} = 0$

$a_{11} = 1 \quad a_{12} = \frac{1}{2}(y-1) \quad a_{22} = 0$

Equation is hyperbolic since $a_{12}^2 - a_{11}a_{22} = a_{12}^2 > 0$

$$\frac{dy}{dx} = \frac{a_{12}+a_{12}}{a_{11}}, \quad \frac{dy}{dx} = \frac{a_{12}-a_{12}}{a_{11}} = 0$$

$$\frac{dy}{dx} = y-1$$

$x = \ln(y-1) + \ln C \qquad e^x = C(y-1) \qquad C = e^x/y-1$

$\xi = e^x/y-1$

$\eta = 0$

$$\bar{a}_{11} = 1\cdot\frac{e^{2x}}{(y-1)^2} + \frac{2}{2}(y-1)\frac{e^{2x}}{(y-1)^3} \neq 0$$

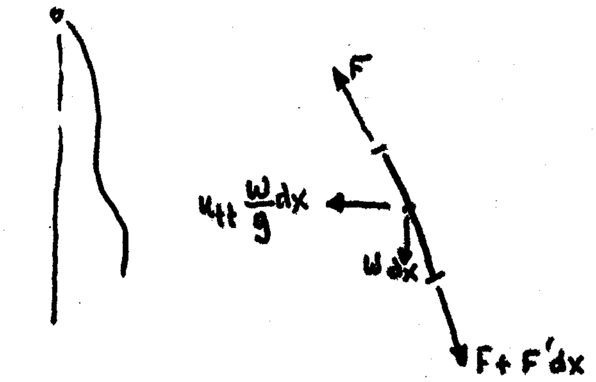
$\bar{a}_{12} = \cancel{1\cdot 0 + 0 + 0 = 0}\ e^y/2$

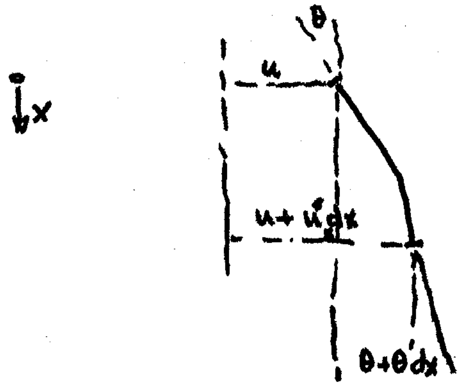
$\bar{a}_{22} = 0$

$\xi_x = \dfrac{e^x}{y-1} \qquad \xi_y = \dfrac{-e^x}{(y-1)^2}$

$\xi_{xy} = \dfrac{-e^x}{(y-1)^2} \qquad \xi_{xx} = \dfrac{e^x}{y-1} \qquad \xi_{yy} = \dfrac{2e^x}{(y-1)^3}$

$u_{xx} = u_{\xi\xi}\, e^{2x} + u_\xi\, e^x$





$$\sin\theta \sim \theta = \cancel{ds}\ \frac{u'dx}{ds} \sim u'$$

$$ds = dx\sqrt{(1+u_x^2)} \sim dx$$

$$\sum F_x = 0 \;\to\; (F+F'dx)\cos(\theta+\theta'dx) - F\cos\theta$$

$$+ w\,dx = 0$$

$$\cos(\theta+\theta'dx) = \cos\theta\cos(\theta'dx) - \sin\theta\sin(\theta'dx)$$

$$\cong \cos\theta - \cancel{\theta' dx \sin\theta}$$

$$\cancel{(F+F'dx)}(\cos\theta \cancel{- \sin\theta[\theta'dx]}) = \cancel{F\cos\theta + F'\cos\theta\,dx}$$

$$\cancel{- dF\sin\theta\ \theta'dx}$$

$$F\cos\theta + F'\cos\theta' dx - F\cos\theta + w\,dx = 0$$

$$F'\cos\theta + w = 0 \qquad \cos\theta \approx 1 \to F' = -w$$

$$F = -wx + C$$

$$C = WL$$

$$= W(L-x)$$

$$\sum F_u = (F+F'dx)\sin(\theta+\theta'dx) - F\sin\theta - u_{tt}\frac{w}{g}dx = 0$$

$$(F+F'dx)(\theta+\theta'dx) - F\theta - u_{tt}\frac{w}{g}dx = 0$$

$$F\theta'dx + F'\theta dx - F'\theta' dx^2 - u_{tt}\frac{w}{g}dx = 0$$

$$(F\theta)' - u_{tt}\frac{w}{g} = 0$$

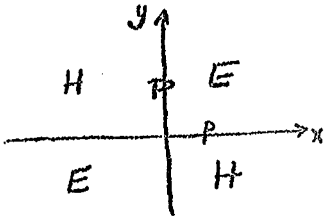
$$W[(l-x)\theta]' - u_{tt}\frac{w}{g} = 0$$

$$g[(l-x)u']' = u_{tt}$$

Methods of P.D.E. I

P.11, 2 a, c, f.

a) $u_{xx} + xy\,u_{yy} = 0$; $a_{12}^2 - a_{11}a_{12} = -xy \Rightarrow$ hyperbolic $\begin{cases} x>0 \text{ \& } y<0 \\ x<0 \text{ \& } y>0 \end{cases}$



Characteristics (for $xy<0$)

$$\frac{dy}{dx} = \pm\sqrt{-xy}$$

$\Rightarrow$ for $x>0, y<0$ (other case similar)

$$\xi = \tfrac{2}{3}x^{3/2} - 2\sqrt{-y}\ , \quad \eta = \tfrac{2}{3}x^{3/2} + 2\sqrt{-y} \quad \left(\Rightarrow \xi+\eta = \tfrac{4}{3}x^{3/2},\ \eta-\xi = 4\sqrt{-y}\right)$$

$$\bar{a}_{11} = \bar{a}_{22} = 0\ , \quad \bar{a}_{12} = (\sqrt{x})^2 + xy\left(\frac{-1}{-y}\right) = 2x$$

$u_{xx} =$ 2nd deriv terms $+ u_\xi \frac{1}{2\sqrt{x}} + u_\eta \frac{1}{2\sqrt{x}}$; $u_{yy} = \cdots + u_\xi \frac{1}{2(-y)^{3/2}} - u_\eta \frac{1}{2(-y)^{3/2}}$

$$\Rightarrow u_{\xi\eta} + (u_\xi + u_\eta)\frac{1}{8x^{3/2}} - \frac{1}{8(-y)^{1/2}}(u_\xi - u_\eta) = 0 \Rightarrow \underline{u_{\xi\eta} + \frac{1}{3(\xi^2-\eta^2)}\left[u_\xi(2\xi+\eta) - u_\eta(2\eta+\xi)\right] = 0}$$

c) $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$; $a_{12}^2 - a_{11}a_{22} = 0 \Rightarrow$ parabolic

$\frac{dy}{dx} = e^{y-x} \Rightarrow \underline{\xi = e^{-x} - e^{-y}}$, η arbitrary — choose $\underline{\eta = y}$ say

Then $\bar{a}_{11} = 0$, $\bar{a}_{12} = 0$ $\bar{a}_{22} = a_{22} = e^{2y}$; $u_{xx} = \cdots + u_\xi e^{-x}$

$u_{yy} = \cdots + u_\xi(-e^{-y})$, $u_{xy} =$ 2nd order terms only.

$\Rightarrow$ $u_{\eta\eta} + u_\xi(e^{x} - e^{y})e^{-2y} = 0$ But $e^{-x} = \xi + e^{-\eta}$, $e^{-y} = e^{-\eta}$

hence we obtain $\underline{u_{\eta\eta} - \frac{\xi}{1+\xi e^{\eta}}u_\xi = 0}$

f) $(x-y)\left[u_{xx} + (y-1)u_{xy}\right] = 0$ When $x=y$, u is indeterminate

$x \neq y$ $a_{12}^2 - a_{11}a_{12} = \frac{(y-1)^2}{4} \geq 0$ $y=1$, parabolic ; $y \neq 1$ hyperbolic.

$y \neq 1$. $\frac{dy}{dx} = (y-1)$ or 0. $\Rightarrow$ characteristics $y = $ const. & $\ln(y-1) - x = $ const $\Rightarrow \xi = y, \eta = x - \ln$

then $\bar{a}_{11} = \bar{a}_{22} = 0$, $\bar{a}_{12} = -\frac{1}{2}$, u_{xx} & u_{yy} contain no first derivatives in

& thus the canonical form is $\underline{u_{\xi\eta} = 0}$. Check : general soln. of this is

$u = f(y) + g(x - \ln(y-1))$ (f & g arbitrary) Then $u_{xx} + (y-1)u_{xy} = g''(x-\ln(y-1)) + (y-1)g''(x-\ln(y-1))\cdot\left(\frac{-1}{y-1}\right)$
$= 0.$

(0)

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C. LEVY 097-42-1895

2a. $u_{xx} + xy\,u_{yy} = 0$ $a_{11} = 1$ $a_{12} = 0$ $a_{22} = xy$

$a_{12}^2 - a_{11}a_{22} = -xy$. For hyperbolic type $-xy > 0$ $x>0\ \ y<0$ / $y>0\ \ x<0$

$$\frac{dy}{dx} = \pm\sqrt{-xy} \quad \text{or} \quad \xi = \tfrac{2}{3}x^{3/2} - 2\sqrt{-y}$$

$$\eta = \tfrac{2}{3}x^{3/2} + 2\sqrt{-y}$$

$\xi_x = \sqrt{x}$, $\xi_y = 1/\sqrt{-y}$, $\eta_x = \sqrt{x}$, $\eta_y = -1/\sqrt{-y}$, $\xi_{xx} = \eta_{xx} = 1/2\sqrt{x}$,

$\xi_{yy} = -\eta_{yy} = 1/2\sqrt{(-y)}^3$

$$\bar{a}_{11} = x + xy\left(\frac{1}{-y}\right) = 0$$

$$\bar{a}_{12} = (\sqrt{x})^2 + xy\left(-\frac{1}{-y}\right) = 2x$$

$$\bar{a}_{22} = x + xy\left(\frac{1}{-y}\right) = 0$$

$$u_{xx} = u_{\xi\xi}\,x + 2u_{\xi\eta}\,x + u_{\eta\eta}\,x + u_\xi \frac{1}{2\sqrt{x}} + u_\eta \frac{1}{2\sqrt{x}}$$

$$u_{yy} = u_{\xi\xi}\left(\frac{1}{-y}\right) + 2u_{\xi\eta}\,\frac{1}{y} + u_{\eta\eta}\left(\frac{-1}{y}\right) + u_\xi \frac{1}{2(-y)^{3/2}} + u_\eta\left(-\frac{1}{2(-y)^{3/2}}\right)$$

$$u_{xx} + xy\,u_{yy} = 4x\,u_{\xi\eta} + (u_\xi + u_\eta)\frac{x}{2\sqrt{x^3}} + (u_\xi - u_\eta)\frac{x}{2\sqrt{-y}} = 0$$

$$2x^{3/2} = \tfrac{3}{2}(\xi + \eta) \qquad 2\sqrt{-y} = \frac{\eta - \xi}{2}$$

$$u_{xx} + xy\,u_{yy} = 4u_{\xi\eta} + (u_\xi + u_\eta)\frac{2}{3(\xi+\eta)} + (u_\xi - u_\eta)\frac{2}{(\eta - \xi)} = 0$$

⑤

or $$u_{\xi\eta} + \frac{(u_\xi + u_\eta)}{6(\xi+\eta)} + \frac{(u_\xi - u_\eta)}{2(\eta - \xi)} = 0$$

2c. $e^{2x}u_{xx} + 2e^x e^y u_{xy} + e^{2y}u_{yy} = 0$ $a_{11} = e^{2x}$ $a_{12} = e^{x+y}$ $a_{22} = e^{2y}$

$a_{12}^2 - a_{11}a_{12} = 0$ Thus equation is parabolic

$dy/dx = e^y/e^x \qquad C = e^{-x} - e^{-y} = \xi$

since it is parabolic let $\eta = \eta(x,y) = x - y$

$\eta_x = -\eta_y = 1, \; \eta_{xx} = \eta_{yy} = 0, \; \xi_x = -\xi_{xx} = -e^{-x}, \; \xi_y = -\xi_{yy} = e^{-y}$

$\bar{a}_{11} = 1 - 2 + 1 = 0$

$\bar{a}_{12} = -e^x + e^y + e^x - e^y = 0$

$\bar{a}_{22} = (e^x - e^y)^2$

$u_{xx} = u_{\xi\xi} e^{-2x} - 2u_{\xi\eta} e^{-x} + u_{\eta\eta} + u_\xi e^{-x}$

$u_{xy} = -u_{\xi\xi} e^{-x} e^{-y} + u_{\xi\eta}(e^{-x} + e^{-y}) - u_{\eta\eta}$

$u_{yy} = u_{\xi\xi} e^{-2y} - 2u_{\xi\eta} e^{-y} + u_{\eta\eta} - u_\xi e^{-y}$

The transformed equation reduces to

$$u_{\eta\eta}(e^x - e^y)^2 + u_\xi(e^x - e^y) = 0$$

if $(e^x - e^y) \neq 0$ then

$$u_{\eta\eta}(e^x - e^y) + u_\xi = 0 \quad ; \quad x = \eta + y \Rightarrow e^x = e^\eta e^y$$

$$e^x - e^y = e^y(e^\eta - 1)$$

$$\xi = e^{-y}(e^{-\eta} - 1) \quad \text{or} \quad e^y = \frac{e^{-\eta} - 1}{\xi}$$

this substitution yields

$$u_{\eta\eta} - \frac{\xi e^\eta}{(e^\eta - 1)^2} u_\xi = 0$$

2f. $(x-y)u_{xx} + (xy - y^2 - x + y)u_{xy} = 0$ assume $(x-y) \neq 0$ then

$$u_{xx} + (y-1)u_{xy} = 0 \qquad a_{11} = 1 \quad a_{12} = \tfrac{1}{2}(y-1) \quad a_{22} = 0$$

This then is a hyperbolic eq since $a_{12}^2 - a_{11}a_{22} = \frac{1}{4}(y-1)^2$

$\frac{dy}{dx} = \frac{2a_{12}}{a_{11}} = y - 1 \qquad \frac{dy}{dx} = 0$

$\boxed{e^x}/y-1 = \xi \qquad y = \eta$

$\xi_x = \xi_{xx} = e^x/_{y-1}$ $\xi_y = -e^x/(y-1)^2$ $\xi_{yy} = 2e^x/(y-1)^3$ $\xi_{xy} = -e^x/_{(y-1)^2}$

$\eta_x = \eta_{xx} = 0$ $\eta_y = 1$ $\eta_{yy} = 0$ $\eta_{xy} = 0$

$\bar{a}_{11} = e^{2x}/_{(y-1)^2} - e^{2x}/_{(y-1)^2} = 0$

$\bar{a}_{12} = e^x/_2$

$\bar{a}_{22} = 0$

5

$u_{xx} = u_{\xi\xi}\ e^{2x}/_{(y-1)^2} + u_\xi\ e^x/_{(y-1)}$

$u_{xy} = u_{\xi\xi}\left(-\frac{e^{2x}}{(y-1)^3}\right) + u_{\xi\eta}\left(e^x/_{(y-1)}\right) + u_\xi\left(-e^x/_{(y-1)^2}\right)$

$u_{xx} + (y-1)\,u_{xy} = u_{\xi\xi}\cdot 0 + u_{\xi\eta}\,e^x + u_\xi\cdot 0 = 0$

or $u_{\xi\eta} = 0$

2a. if $\sqrt{-xy} = 0$ then $y = \xi$, take $\eta = \eta(x,y) = x$.

$u_{xx} = u_{\eta\eta}$

$u_{yy} = u_{\xi\xi}$

then $u_{xx} + xy\,u_{yy} = u_{\eta\eta} = 0$ since either x is zero or y is zero $\Rightarrow \xi = 0$

or $\eta = 0$ $\Rightarrow \xi\eta\,u_{\xi\xi} = 0$

ⓐ $u_{xx} + u_{yy} + \alpha u_x + \beta u_y + \gamma u = 0$ $\qquad$ $v = u e^{\lambda x + \mu y}$

~~$u_x = e^{\lambda x + \mu y} (v_x + \lambda v)$~~ $\qquad$ $u = v e^{-(\lambda x + \mu y)}$

~~$u_y =$~~

$$u_x = v_x e^{-(\lambda x + \mu y)} - v\lambda e^{-(\lambda x + \mu y)} = e^{-(\lambda x + \mu y)} [v_x - \lambda v]$$

$$u_y = e^{-(\lambda x + \mu y)} [v_y - \mu v]$$

$$u_{xx} = [v_{xx} - \lambda v_x] e^{(\;)} - \ast [v_x - \lambda v] e^{-(\;)} = [v_{xx} - 2\lambda v_x + \lambda^2 v] e^{-(\;)}$$

$$u_{xy} = [v_{xy} - \lambda v_y] e^{(\;)} - \mu [v_x - \lambda v] e^{-(\;)} = [v_{xy} - \lambda v_y - \mu v_x + \lambda \mu v] e^{-(}$$

$$u_{yy} = [v_{yy} - 2\mu v_y + \mu^2 v] e^{-(\;)}$$

$$[v_{xx} - 2\lambda v_x + \lambda^2 v] + [v_{yy} - 2\mu v_y + \mu^2 v] + \alpha [v_x - \lambda v] + \beta [v_y - \mu v] + \gamma v = 0$$

$$v_{xx} + v_{yy} + v_x [\alpha - 2\lambda] + v_y [\beta - 2\mu] + v [\lambda^2 + \mu^2 - \lambda\alpha - \beta\mu + \gamma] = 0$$

chose $\lambda = \alpha/2 \quad \mu = \beta/2$ $\qquad$ $[\alpha^2/4 + \beta^2/4 - \alpha^2/2 - \beta^2/2 + \gamma]$

$$v_{xx} + v_{yy} \ast - \tfrac{1}{4} [\alpha^2 + \beta^2 - 4\gamma] v = 0$$

$$v = u e^{1/2(\alpha x + \beta y)}$$

d $u_{xy} \ast \alpha u_x - \beta u_y = 0$

$$[v_{xy} - \lambda v_y - \mu v_x + \lambda \mu v] - \alpha [v_x - \lambda v] - \beta [v_y - \mu v] = 0$$

$$v_{xy} \ast v_x [\mu + \alpha] - v_y [\lambda + \beta] + v [\lambda \mu + \alpha \lambda + \beta \mu] = 0$$

let $-\alpha = \mu \quad -\beta = \lambda$ $\qquad$ $\alpha\beta \ast \alpha^2 - \beta^2$

$v_{xy} \ast [\alpha^2 - \alpha\beta + \beta^2] v = 0$ $\qquad$ $v_{xy} - [(\alpha - \beta)^2 \ast \alpha\beta] v = 0$

$$v = u e^{-(\beta x + \alpha y)}$$

Methods of P.D.E. I ②

P.12 4, a, d , P34. 2.

4a). $u_{xx} + u_{yy} + \alpha u_x + \beta u_y + \gamma u = 0$. Let $u = v e^{\lambda x + \mu y}$

Then $v_{xx} + v_{yy} + v_x(2\lambda + \alpha) + v_y(2\mu + \beta) + v(\mu^2 + \lambda^2 + \alpha\lambda + \mu\beta + \gamma) = 0$

Choose $\lambda = -\alpha/2$, $\mu = -\beta/2$, then

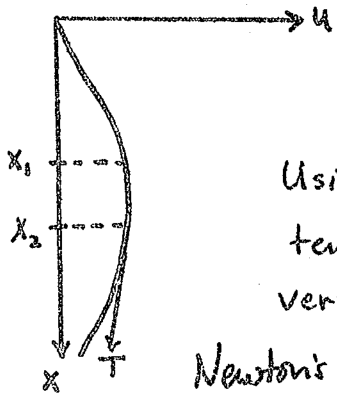
$\underline{v_{xx} + v_{yy} + \gamma_1 v = 0}$ where $\gamma_1 = \gamma - \frac{\alpha^2 + \beta^2}{4}$.

d) $u_{xy} = \alpha u_x + \beta u_y$ Let $u = v e^{\lambda x + \mu y}$

Then $v_{xy} + v_x(\mu - \alpha) + v_y(\lambda - \beta) + v(\lambda\mu - \alpha\lambda - \beta\mu) = 0$

Choose $\lambda = \beta$, $\mu = \alpha$, then $\underline{v_{xy} = \gamma v}$ where $\gamma = \alpha\beta$.

34, 2.



In the undisturbed (vertical) state the tension is given by: $T_0(x_1) = T_0(x_2) + \rho g(x_2 - x_1) \Rightarrow \frac{dT_0}{dx} = -\rho g$

$\Rightarrow T_0(x) = \rho g(l - x)$.

Using same arguments as on p.13 there is no additional tension or extension for small vibrations. Thus resolving vertically $T_x(x_1) = T_x(x_2) + \rho g \Delta x$, horizontally $T_u(x) = T_0(x) u_{x}$

Newton's 2nd Law for (x_1, x_2) yields

$$\int_{x_1}^{x_2} [u_t(\xi, t_2) - u_t(\xi, t_1)] \rho \, d\xi = \int_{t_1}^{t_2} [T_0(x_2) u_x(x_2, \tau) - T_0(x_1) u_x(x_1, \tau)] \, d\tau$$

Mean Value Theorem $\Rightarrow$

$$u_{tt}(\xi^*, t^*) \rho \Delta t \Delta x = \frac{\partial}{\partial x}\left(T_0(\xi^{**}) u_x(\xi^{**}, t^{**})\right) \Delta t \Delta x \quad \xi^*, \xi^{**} \in (x_1, x_2)$$
$$t^*, t^{**} \in (t_1, t_2)$$

$x_2 \to x_1$, $t_2 \to t_1$

$$\Rightarrow \rho u_{tt} = \frac{\partial}{\partial x}(T_0(x) u_x) = \rho g \frac{\partial}{\partial x}([l - x] u_x)$$

$$\Rightarrow \underline{u_{tt} = a^2 \frac{\partial}{\partial x}([l - x] u_x)} \; ; \; a^2 = g$$

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⑧

$u_{xx} + u_{yy} + \alpha u_x + \beta u_y + \gamma u = 0 \qquad (1)$

$$u = v\, e^{-(\lambda x + \mu y)}$$

$$u_x = e^{-(\lambda x + \mu y)} [v_x - \lambda v]$$

$$u_y = e^{-(\lambda x + \mu y)} [v_y - \mu v]$$

$$u_{xx} = [v_{xx} - 2\lambda v_x + \lambda^2 v] e^{-(\lambda x + \mu y)}$$

$$u_{yy} = [v_{yy} - 2\mu v_y + \mu^2 v] e^{-(\lambda x + \mu y)}$$

$$u_{yx} = [v_{xy} - \lambda v_y - \mu v_x + \lambda \mu v] e^{-(\lambda x + \mu y)}$$

we can the write (1) as

$$v_{xx} + v_{yy} + v_x[\alpha - 2\lambda] + v_y[\beta - 2\mu] + v[\lambda^2 + \mu^2 - \lambda\alpha - \mu\beta + \gamma] = 0$$

choosing $\alpha/2 = \lambda$ $\beta/2 = \mu$ equation (1) is transformed to

② $v_{xx} + v_{yy} - \frac{1}{4}[\alpha^2 + \beta^2 - 4\gamma] v = 0$ ✓ where $v = u\, e^{-\frac{1}{2}(\alpha x + \beta y)}$

$u_{xy} - \alpha u_x - \beta u_y = 0$ (2) can be written as

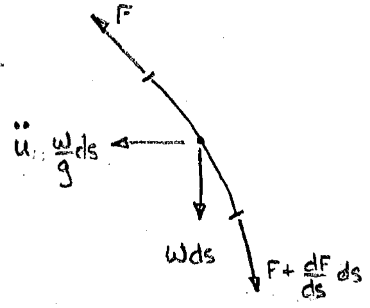
$$v_{xy} - v_x[\mu + \alpha] - v_y[\lambda + \beta] + v[\lambda\mu + \alpha\lambda + \beta\mu] = 0$$

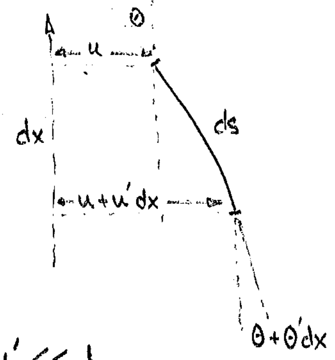
if $\mu = -\alpha$ & $\beta = -\lambda$ then (2) is transformed to

① $v_{xy} - [(\alpha - \beta)^2 + \alpha\beta] v = 0$ where $v = u\, e^{-(\beta x + \alpha y)}$

take any section

W is weight / unit length





$ds^2 = dx^2 + u'^2 dx^2 = dx^2(1 + u'^2)$ for small vibrations $u' \ll 1$

$\cong dx^2$ we shall denote $(\ )' = \frac{d(\ )}{dx}$ & $(\ddot{\ }) = \frac{d^2(\ )}{dt^2}$

$\sum F_x = 0$ (since no longitudinal oscillations occur)

$$\left(F + \frac{dF}{dx}dx\right)\cos(\theta + \theta' dx) - F\cos\theta + W dx = 0$$

since small vibrations $\theta \gg \theta' dx$ ∴

$$\left(F + \frac{dF}{dx}dx\right)\cos\theta - F\cos\theta + W dx = \left(\frac{dF}{dx}dx\right)\cos\theta + W dx = 0$$

since $\theta \ll 1$ then $\cos\theta \sim 1$ or $\frac{dF}{dx} + W = 0$

$F = -Wx + \text{constant}$

at this point forces acting on the fastener is

$F = WL$

$\therefore$ at $x=0$ $F = WL$ or

$F = W(L-x)$

$\sum F_y = 0$ (By D'alembert Principle)

$$\left(F + \frac{dF}{dx}dx\right)\sin(\theta + \theta' dx) - F\sin\theta - \ddot{u}\,\frac{w}{g}\,dx = 0$$

For small vibrations $\sin(\theta + \theta' dx) \sim \theta + \theta' dx$ or
$\sin\theta \sim \theta$

$$\left(F + \frac{dF}{dx}dx\right)(\theta + \theta' dx) - F\theta - \ddot{u}\,\frac{w}{g}\,dx = 0$$

$$F\theta' dx + F'\theta\,dx + F'\theta' dx^2 - \ddot{u}\,\frac{w}{g}\,dx = 0$$

because $F'\theta' dx^2$ is second order small neglect it with respect to dx terms

$$\therefore \quad \left[(F\theta)' - \ddot{u}\,\frac{w}{g}\right]dx = 0$$

from diagram $\sin\theta = \frac{u' dx}{ds} \sim u'$ $\quad \sin\theta \sim \theta \sim u'$ $\therefore$

$$\left[(F\theta)' - \ddot{u}\,w/g\right]dx = 0 \implies \left\{\left[W(L-x)\,u'\right]' - \ddot{u}\frac{W}{g}\right\}dx = 0$$

since W & dx are not zero then

$$g\left[(L-x)u'\right]' = \ddot{u} \quad \text{or}$$

⑤ $$\frac{\partial^2 u}{\partial t^2} = a^2\left[(l-x)\frac{\partial u}{\partial x}\right]_{,x} \quad \text{where } a^2 = g \quad ✓$$

35

3. Same as Q 2. except there is an external force (the centrifugal force) acting because of the rotation. For a constant angular rotation with velocity ω the inward acting force per unit mass is $\mp \omega^2 u$. Thus $f(x,t) = \rho\omega^2 u$ and the final equation is $u_{tt} = g\frac{\partial}{\partial x}([\ell - x]u_x) + \omega^2 u$

---

5. Hooke's Law $\Longrightarrow$ shear force or torque $T(x) = GJ\theta_x$ where G is modulus of torsion and J is polar moment of inertia. Conservation of angular momentum $\Longrightarrow$ $I \times$ (change in angular velocity) $=$ impulsive torque where I = moment of inertia per unit length. Consider length of rod (x_1, x_2) then

$$I\int_{x_1}^{x_2}[\theta_t(\xi,t_2) - \theta_t(\xi,t_1)]\,d\xi = \int_{t_1}^{t_2}[T(x_2,\tau) - T(x_1,\tau)]\,d\tau$$

M.V.Th. + $\Delta x, \Delta t \to 0$ yields

$$\underline{I\theta_{tt} = GJ\theta_{xx}}.$$

Let I_1, I_2 be moments of inertia of discs at $x=0$ and $x=\ell$ respectively.

Conserv. of ang. momentum for discs $\Longrightarrow$ $\left.\begin{aligned} I_1\theta_{tt} &= GJ\theta_x \quad \text{at } x=0 \\ I_2\theta_{tt} &= -GJ\theta_x \quad \text{at } x=\ell \end{aligned}\right\}$.

---

7. In text it is shown that s, ρ, U satisfy $z_{tt} = a^2\nabla^2 z$, (1). Since for small vibrations $\frac{p}{p_0} = 1+\gamma s$, $s_{tt} = \frac{1}{\gamma p_0}p_{tt}$ etc. & p also satisfies (1). Taking ∇ (eqn (1)) $\Longrightarrow$ $\underset{\sim}{v} = \nabla U$ also satisfies (1). If displacement is given by $\underset{\sim}{X}$, we have

$$\underline{\underset{\sim}{X}_{tt}} = \underset{\sim}{v}_t = -a^2\nabla s = -a^2\nabla\left(\int s_t\,dt\right) = -a^2\nabla\left(-\int \nabla\underset{\sim}{v}\,dt\right)$$
$$= a^2\nabla(\nabla\underset{\sim}{X}) = \underline{a^2\nabla^2\underset{\sim}{X}}.$$

B.C I : prescribed displacement or velocity of boundary

II : prescribed pressure on a boundary

III : forced boundary between to elastic media – free surface between two fluids.

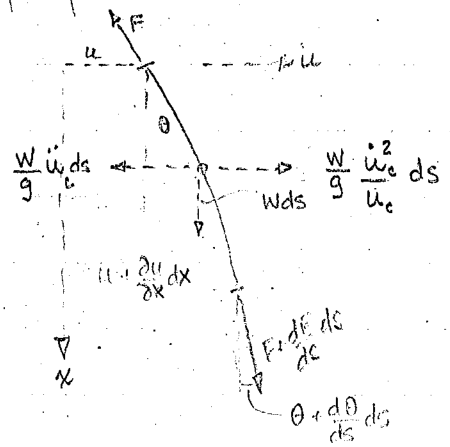
⑭

12 Oct. 1972

Cesar Levy 097-42-1875

T63.2130

④ Problem 3: a heavy cable of length l is fastened at its upper end (x=0) of a vertical axis; it rotates about it with a uniform angular velocity ω. Find the diff equns. for small vibrations about the vertical state of equilib.



where $\frac{W}{g}\ddot{u}\,ds$ is the dalambert force
$\frac{W}{g}\frac{\dot{u}_c}{u_c}ds$ is the centripetal force of the segment and subscript c denotes the segments centroid

$$\Sigma F_x \downarrow_+ = \left[F+\frac{dF}{ds}ds\right]\cos\left[\theta+\theta_s\,ds\right]-F\cos\theta+Wds=0$$ since no acceleration occurs in x directions

since $\cos\theta,\ \cos\left(\theta+\frac{d\theta}{ds}ds\right)\sim 1$ then

$$\frac{dF}{ds}ds+W\,ds=0 \quad\Longrightarrow\quad F=-Ws+C$$

at $s=0,(x=0)$ $\quad \overset{\uparrow F_0}{\underset{\downarrow WL}{////}} \quad\Longrightarrow\quad F_0=C=WL \quad \therefore\ F=W(L-S)$ ✓

$$\Sigma F_u \xrightarrow{+} = \left[F+\frac{dF}{ds}ds\right]\sin\left[\theta+\frac{d\theta}{ds}ds\right]-F\sin\theta+\frac{W}{g}\frac{\dot{u}_c^2}{u_c}ds-\frac{W}{g}\ddot{u}_c\,ds=0$$

since $\sin\theta\sim\theta$, $\sin(\theta+\frac{d\theta}{ds}ds)\ \theta+\frac{d\theta}{ds}ds$ then

$$=[F\theta]_s\,ds+F_s\theta_s\,ds^2+\frac{W}{g}ds\left[\frac{\dot{u}_c^2}{u_c}-\ddot{u}_c\right]=0$$

neglect second order terms

$$[F\theta]_s\,ds+\frac{W}{g}ds\left[\frac{\dot{u}_c^2}{u_c}-\ddot{u}_c\right]=0$$

since $\theta\ll 1$ $\quad u_c,\dot{u}_c,\ddot{u}_c\sim u$ and $ds\cong dx\left(\sqrt{1+u_x^2}\right)\cong dx$

therefore $[F\theta]_{,x} + \frac{W}{g}\left[\frac{\dot{u}^2}{u} - \ddot{u}\right] = 0$

but $\dot{u} = \omega u$ ∴ $\frac{\dot{u}^2}{u} = \omega^2 u$

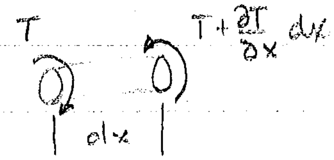
and since $\sin\Theta \sim \Theta = \frac{\partial u}{\partial x}dx/ds \sim \frac{\partial u}{\partial x}$ then

$$\left[W(L-x)\frac{\partial u}{\partial x}\right]_{,x} + \frac{W}{g}\left[\omega^2 u - u_{tt}\right] = 0 \quad \text{or}$$

$$a^2\left[(L-x)\frac{\partial u}{\partial x}\right]_{,x} + \omega^2 u = u_{tt} \quad ✓ \quad \text{where } a^2 = g$$

Problem 6: find the diff equas & auxiliary conditions for the torsional vibrations of a rod, at both ends of which discs are fastened

⑤



Looking at the equil. state the unit angle of twist is found as $\frac{\Delta\Theta}{\Delta l} = \frac{T}{GJ}$ where l is the total length of the bar T is the torque applied to both ends & G - is the shear modulus while J is the polar moment of inertia of the section. Since $\Theta = \Theta(x,t)$ then $\frac{\partial\Theta}{\partial x} = \frac{T}{GJ}$ ✓

Going to the torsional rod $T + \frac{\partial T}{\partial x}dx - T = \frac{\partial T}{\partial x}dx$

$\frac{\partial T}{\partial x}dx = \frac{\partial}{\partial x}\left[GJ\frac{\partial\Theta}{\partial x}\right]dx = I\ddot{\Theta}$ ✓ If G, J, I (moment of inertia) are not fns of x then $a^2\,\Theta_{xx} = \Theta_{tt}$ ✓ where $a = \sqrt{\frac{GJ}{I}}$

✓2

if +↻ torque then at the left end (+↻ positive convention)

$$\frac{\partial\theta}{\partial x} = \frac{T_1}{GJ} = \frac{I_1\ddot{\theta}}{GJ} \Rightarrow \theta_{tt}(0,t) = \frac{GJ}{I_1}\,\theta_x(0,t) \checkmark$$

where I_1 is the moment of inertia of the left disc

at the right end

$$-\frac{\partial\theta}{\partial x} = \frac{T_2}{GJ} = \frac{I_2\ddot{\theta}}{GJ} \Rightarrow \theta_{tt}(l,t) = -\frac{GJ}{I_2}\,\theta_x(l,t) \checkmark$$

where I_2 is the moment of inertia of the right disc

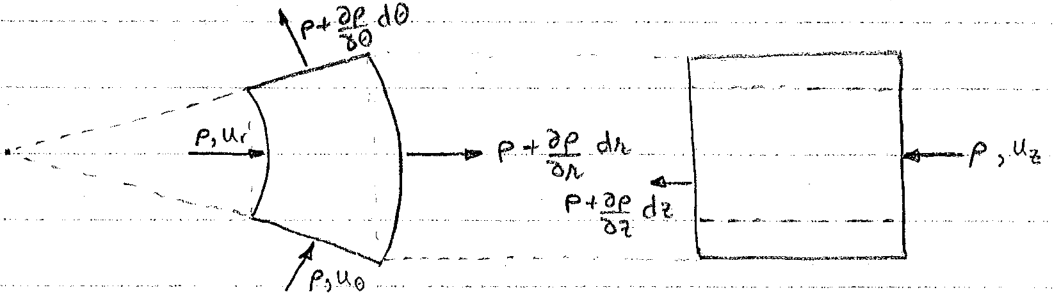
denote $\alpha_i^2 = \frac{GJ}{I_i}$ ✓

Problem 9: Consider small vibrations of an ideal gas in a cylinder tube. Derive fundamental diff equa. of hydrodynamics & assuming that the process is adiabatic find equations for ρ, p, velocity potential φ, veloc v, displacement u. In addition construct an example which realizes the B.C. of types 1,2,3 for these diff equations.

⑤

Conservation of Mass : Change of mass rate = mass influx – mass outflow

In cylindrical coordinates r, θ, z an elemental volume is $r\,dr\,d\theta\,dz$



rate of change of $\frac{\text{mass}}{\text{unit volume}}$ in the volume $= \frac{\partial}{\partial t}\left[\rho\, r\, dr\, d\theta\, dz\right]$

mass influx – mass outflow $= \rho u_r r d\theta dz - \left(u_r + \frac{\partial u_r}{\partial r} dr\right)\left(\rho + \frac{\partial \rho}{\partial r} dr\right)[r + dr] d\theta dz$

$+ \rho u_z r d\theta dr - \left(u_z + \frac{\partial u_z}{\partial z} dz\right)\left(\rho + \frac{\partial \rho}{\partial z} dz\right) r dr d\theta + \rho u_\theta \left[\sin\left(\frac{d\theta}{2}\right)\right]_2^2 dr dz$

$- \left(\rho + \frac{\partial \rho}{\partial \theta} d\theta\right)\left(u_\theta + \frac{\partial u_\theta}{\partial \theta} d\theta\right)_2 \left[\sin\left(\frac{d\theta}{2}\right)\right]^2 dr dz + \rho u_\theta \left[\cos\left(\frac{d\theta}{2}\right)\right]^2 dr dz - \left(\rho + \frac{\partial \rho}{\partial \theta} d\theta\right) \cdot$

$\left(u_\theta + \frac{\partial u_\theta}{\partial \theta} d\theta\right)\left[\cos\left(\frac{d\theta}{2}\right)\right] dr dz \quad \cong -u_r \frac{\partial \rho}{\partial r} r dr d\theta dz + \rho \frac{\partial u_r}{\partial r} r dr d\theta dz - \frac{\partial u_r}{\partial r}\frac{\partial \rho}{\partial r} r dr^2 d\theta dz$

$- \left(u_r + \frac{\partial u_r}{\partial r} dr\right)\left(\rho + \frac{\partial \rho}{\partial r} dr\right) dr d\theta dz - u_z \frac{\partial \rho}{\partial z} r dr d\theta dz - \rho \frac{\partial u_z}{\partial z} r dr d\theta dz$

$- \frac{\partial u_z}{\partial z}\frac{\partial \rho}{\partial z} r dr d\theta dz^2 - \rho \frac{\partial u_\theta}{\partial \theta} \frac{d\theta^3}{2} dr dz - u_\theta \frac{\partial \rho}{\partial \theta} \frac{d\theta^3}{2} dr d\theta dz - \frac{\partial \rho}{\partial \theta}\frac{\partial u_\theta}{\partial \theta} \frac{d\theta^4}{2} dr dz$

$- \rho \frac{\partial u_\theta}{\partial \theta} d\theta dr dz - u_\theta \frac{\partial \rho}{\partial \theta} d\theta dr dz - \frac{\partial \rho}{\partial \theta}\frac{\partial u_\theta}{\partial \theta} \frac{d\theta^2}{2} dr dz$

if both are equated & the equation is divided by $r dr d\theta dz$ & the elemental volume $\longrightarrow 0$ one obtains

$$\frac{\partial}{\partial t}\rho = -u_r \frac{\partial \rho}{\partial r} - \rho \frac{\partial u_r}{\partial r} - \frac{1}{r} u_r \rho - u_z \frac{\partial \rho}{\partial z} - \rho \frac{\partial u_z}{\partial z} - \frac{1}{r}\rho \frac{\partial u_\theta}{\partial \theta} - \frac{1}{r} u_\theta \frac{\partial \rho}{\partial \theta}$$

or

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r \rho u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta) + \frac{\partial}{\partial z}(\rho u_z) = 0 \quad ✓$$

This is the conservation of mass equation in cylindrical coordinates,

Conservation of Momentum : time rate of change of momentum in the volume = momentum influx – momentum outflow + external forces acting on the volume + body forces

In the r-direction & assuming the flow is frictionless

$$\frac{\partial}{\partial t}(\rho u_r dV) = \int_{-d\theta/2}^{d\theta/2} \cos^2\theta(\rho u_r)u_r\, r d\theta dz - \int_{-d\theta/2}^{d\theta/2}\left[(\rho u_r) + \frac{\partial}{\partial r}(\rho u_r)\, dr\right]\left(u_r + \frac{\partial u_r}{\partial r}dr\right)(r+dr)\cos^2\theta\, d\theta dz$$

$$+ \int_{-d\theta/2}^{d\theta/2} p\cos\theta\, r d\theta dz - \int_{-\frac{d\theta}{2}}^{\frac{d\theta}{2}}\left(p + \frac{\partial p}{\partial r}dr\right)\cos\theta\,\underset{[r+dr]}{\lambda}\, d\theta dz + (\rho u_\theta)u_\theta \sin^2\left(\frac{d\theta}{2}\right) dr dz$$

$$-\left(\rho u_\theta + \frac{\partial}{\partial\theta}(\rho u_\theta)\, d\theta\right)\left[u_\theta + \frac{\partial u_\theta}{\partial\theta}d\theta\right]\sin^2\left(\frac{d\theta}{2}\right) dr dz + R\, r dr d\theta dz + 2p\sin\left(\frac{d\theta}{2}\right) drd$$

$$= -\rho u_r\frac{\partial u_r}{\partial r} dr d\theta dz \cdot r + \frac{\partial}{\partial r}(\rho u_r)\left(u_r + \frac{\partial u_r}{\partial r}dr\right) r dr d\theta dz - \left[(\rho u_r) + \frac{\partial}{\partial r}(\rho u_r)d\right.$$

$$\left[u_r + \frac{\partial u_r}{\partial r}dr\right] dr d\theta dz + p r d\theta dz - \left(p + \frac{\partial p}{\partial r}dr\right)(r+dr)d\theta dz$$

$$-\rho u_\theta \frac{\partial u_\theta}{\partial\theta}\, d\theta \sin^2\left(\frac{d\theta}{2}\right) dr dz - \frac{\partial}{\partial\theta}(\rho u_\theta)\left[u_\theta + \frac{\partial u_\theta}{\partial\theta}d\theta\right] d\theta \sin^2\left(\frac{d\theta}{2}\right) dr dz$$

dividing by $r dr d\theta dz$ and allowing the volume to shrink to zero leads to

$$\frac{\partial}{\partial t}(\rho u_r) = -\frac{\partial u_r}{\partial r}\rho u_r - u_r\frac{\partial}{\partial r}(\rho u_r) - \frac{1}{r}(\rho u_r)u_r - p/r - \frac{\partial p}{\partial r} + R + p/r$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}[r\rho u_r u_r] - \frac{\partial}{\partial r}[p] + R \quad ✓$$

In the z-direction

$$\frac{\partial}{\partial t}(\rho u_z\, r dr d\theta dz) = \rho u_z u_z r dr d\theta - \left[(\rho u_z) + \frac{\partial}{\partial z}(\rho u_z)dz\right]\left(u_z + \frac{\partial u_z}{\partial z}dz\right) r dr d\theta$$

$$+ p r dr d\theta - \left(p + \frac{\partial p}{\partial z}dz\right) r dr d\theta + Z\, r dr d\theta dz$$

dividing by $r dr d\theta dz$ and allowing the volume to shrink to zero leads to

$$\frac{\partial}{\partial t}(\rho u_z) = -(\rho u_z)\frac{\partial u_z}{\partial z} - u_z\frac{\partial}{\partial z}(\rho u_z) - \frac{\partial p}{\partial z} + Z$$

$$= -\frac{\partial}{\partial z}[\rho u_z\ u_z] - \frac{\partial p}{\partial z} + Z \quad ✓$$

In the θ-direction

$$\frac{\partial}{\partial t}(\rho u_\theta\, r dr d\theta dz) = (\rho u_\theta)u_\theta\cos^2\left(\frac{d\theta}{2}\right) dr dz - \left[(\rho u_\theta) + \frac{\partial}{\partial\theta}(\rho u_\theta)d\theta\right]\left[u_\theta + \frac{\partial u_\theta}{\partial\theta}d\theta\right]\cos^2\left(\frac{d\theta}{2}\right) drd$$

$$+ p\cos\left(\frac{d\theta}{2}\right) dr dz - \left(p + \frac{\partial p}{\partial\theta}d\theta\right)\cos\left(\frac{d\theta}{2}\right) dr dz + \Theta\, r dr d\theta dz$$

dividing by $r dr d\theta dz$ & letting the volume shrink to zero leads to

$$\frac{\partial}{\partial t}(\rho u_\theta) = -\left(\frac{\rho u_\theta}{r}\right)\frac{\partial u_\theta}{\partial\theta} - \left(\frac{u_\theta}{r}\right)\frac{\partial}{\partial\theta}(\rho u_\theta) - \frac{1}{r}\frac{\partial}{\partial\theta}p + \Theta$$

$$= -\frac{1}{r}\left[\frac{\partial}{\partial\theta}[\rho u_\theta\ u_\theta] + \frac{\partial p}{\partial\theta}\right] + \Theta \quad ✓$$

therefore if the gradient is defined by

$$\nabla f = \frac{\partial f}{\partial r}\underset{\sim}{i}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\underset{\sim}{i}_\theta + \frac{\partial f}{\partial z}\underset{\sim}{k} \quad ✓$$

and the divergence of a vector

$$\nabla \cdot \bar{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial}{\partial \theta}F_\theta + \frac{\partial F_z}{\partial z} \quad ✓$$

where $\bar{F} = F_r \underset{\sim}{i_r} + F_\theta \underset{\sim}{i_\theta} + F_z \underset{\sim}{i_z}$

then conservation of momentum is

$$\frac{D}{Dt}(\rho \underset{\sim}{V}) + (\rho \underset{\sim}{V})\nabla \cdot \underset{\sim}{V} = (R\underset{\sim}{i_r} + \Theta \underset{\sim}{i_\theta} + Z \underset{\sim}{i_z}) - \nabla p \quad ✓$$

the conservation of mass is

$$\frac{D}{Dt}\rho + \rho \nabla \cdot \underset{\sim}{V} = 0 \quad ✓$$

since it is adiabatic and small vibrations assume

$p = p_\infty + p'$ $\quad \underset{\sim}{V} = \underset{\sim}{V}_\infty + \underset{\sim}{V}'$ $\quad \rho = \rho_\infty + \rho'$ where the ' quantities as the perturbation value due to some disturbance & the infinity conditions are those values when no disturbance exists.

then the equation of continuity becomes

$$\frac{\partial}{\partial t}(\rho_\infty + \rho') + (\rho_\infty + \rho')\nabla \cdot (\underset{\sim}{V}_\infty + \underset{\sim}{V}') + (\underset{\sim}{V}_\infty + \underset{\sim}{V}') \cdot \nabla(\rho_\infty + \rho') = 0 \quad ✓$$

Let $\underset{\sim}{V}_\infty = 0$

$$\left(\frac{\partial}{\partial t}\rho_\infty + \rho_\infty \nabla \cdot \underset{\sim}{V}_\infty + \underset{\sim}{V}_\infty \cdot \nabla \rho_\infty\right) + \frac{\partial \rho'}{\partial t} + \rho_\infty \nabla \cdot \underset{\sim}{V}' + \cancel{\rho' \nabla \cdot \underset{\sim}{V}_\infty} + \rho' \nabla \cdot \underset{\sim}{V}'$$
$$+ \cancel{\underset{\sim}{V}_\infty \cdot \nabla \rho'} + \underset{\sim}{V}' \cdot \nabla \rho_\infty + \underset{\sim}{V}' \cdot \nabla \rho' = 0$$

$$\left(\frac{\partial}{\partial t}\rho_\infty + \rho_\infty \nabla \cdot \underset{\sim}{V}_\infty + \underset{\sim}{V}_\infty \cdot \nabla \rho_\infty\right) + \left(\frac{\partial \rho'}{\partial t} + \rho' \nabla \cdot \underset{\sim}{V}' + \underset{\sim}{V}' \cdot \nabla \rho'\right)$$
$$+ \left[\nabla \cdot (\rho_\infty \underset{\sim}{V}') + \nabla \cdot (\rho' \underset{\sim}{V}_\infty)\right] = 0$$

First term is zero since that is the conservation of mass for no disturbance, assuming $\rho' \ll \rho_\infty$ then , $\rho' \nabla \cdot \underset{\sim}{V}'$, & $\underset{\sim}{V}' \cdot \nabla \rho'$ can be neglected. assuming $\underset{\sim}{V}' \ll a_\infty$ then

The equation can be linearized to

$$\frac{\partial \rho'}{\partial t} + \rho_\infty \nabla \cdot \underset{\sim}{V}' = 0 \quad ✓$$

now $a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho} = \frac{\gamma(p_\infty + p')}{\rho_\infty + \rho'} = \gamma \frac{p_\infty}{\rho_\infty}\left(1 + p'/p_\infty\right)\left(1 + \rho'/\rho_\infty\right)^{-1}$

but for small ρ' $(1+\rho'/\rho_\infty)^{-1} = (1-\rho'/\rho_\infty + \cdots\cdots)$

$$\therefore \quad a^2 = a_\infty^2\,(1+p'/p_\infty)(1-\rho'/\rho_\infty) + \text{Higher order terms}$$
$$= a_\infty^2 + a_\infty^2\,(p'/p_\infty - \rho'/\rho_\infty) + \text{'' \quad ' terms.}$$
$$= a_\infty^2 + a'^2$$

For momentum conservation & neglecting body forces,

$$\frac{\partial}{\partial t}\left[(\rho_\infty+\rho')\underset{\sim}{V}'\right] + \left[\underset{\sim}{V}'\circ\nabla[(\rho_\infty+\rho')\underset{\sim}{V}'\right] + \left[(\rho_\infty+\rho')\underset{\sim}{V}'\right]\nabla\circ\underset{\sim}{V}' = -\nabla(p_\infty+p')$$
$$\frac{\partial}{\partial t}(\rho_\infty\underset{\sim}{V}') + \underset{\sim}{V}'\circ\nabla(\rho_\infty\underset{\sim}{V}') + (\rho_\infty\underset{\sim}{V}')\nabla\circ\underset{\sim}{V}' = -\nabla p_\infty - \nabla p'$$

or $\quad \dfrac{\partial\underset{\sim}{V}'}{\partial t} + \dfrac{1}{\rho_\infty}\cdot\nabla p' = 0$ $\quad$ but $p = k\rho^\gamma$ then

$$\nabla p = a^2\nabla\rho = \nabla\rho\,\frac{\partial p}{\partial\rho} \qquad \& \quad \nabla p' = (a_\infty^2 + a'^2)\nabla\rho' \cong a_\infty^2\nabla\rho'$$

therefore

$$\frac{\partial\underset{\sim}{V}'}{\partial t} + \frac{a_\infty^2}{\rho_\infty}\nabla\rho' = 0 \quad ✓$$

Also $\quad p_\infty + p' = K(\rho_\infty+\rho')^\gamma = K\rho_\infty^\gamma\,(1+\rho'/\rho_\infty)^\gamma$

$$= K\rho_\infty^\gamma\,(1+\gamma\rho'/\rho_\infty + \text{H.O.T.})$$
$$p' = K\rho_\infty^\gamma\,(\gamma\rho'/\rho_\infty) = \frac{p_\infty\gamma}{\rho_\infty}\,\rho' = a_\infty^2\rho'$$

Now take the mass conservation equation & differentiate wrt t

$$\therefore \quad \frac{1}{\rho_\infty}\frac{\partial^2\rho'}{\partial t^2} + \frac{\partial}{\partial t}(\nabla\cdot\underset{\sim}{V}') = 0$$

Take the divergence of the momentum conservation equati

$$\nabla\cdot\left(\frac{\partial\underset{\sim}{V}'}{\partial t}\right) + \frac{a_\infty^2}{\rho_\infty}\nabla\cdot(\nabla\rho') = 0$$

<u>because of continuity</u> $\nabla\cdot\dfrac{\partial}{\partial t}\underset{\sim}{V}' = \dfrac{\partial}{\partial t}\nabla\cdot\underset{\sim}{V}'$ ✓ good!

$\therefore$

$$\frac{1}{\rho_\infty}\rho'_{,tt} = \frac{a_\infty^2}{\rho_\infty}\nabla^2\rho' \quad \text{or} \quad \rho'_{,tt} = a_\infty^2\nabla^2\rho' \ ✓$$

now $p' = a_\infty^2\rho'$ $\quad \therefore$

$$\frac{1}{a_\infty^2}p'_{,tt} = \frac{a_\infty^2}{a_\infty^2}\nabla^2 p' \quad \text{or} \quad p'_{,tt} = a_\infty^2\nabla^2 p' \ ✓$$

if one takes the div of mass conservation & differentiates the momentum conservation wrt t then

$$\nabla\cdot\frac{\partial\rho'}{\partial t} + \rho_\infty \nabla^2\circ\underset{\sim}{V}' = 0$$

and $\frac{\partial^2}{\partial t^2}\underset{\sim}{V}' + \frac{a_\infty^2}{\rho_\infty}\frac{\partial}{\partial t}\nabla\rho' = 0$ then

since $\nabla\cdot\frac{\partial}{\partial t}\rho' = \frac{\partial}{\partial t}\nabla\rho'$ then

$$\partial^2/\partial t^2\,\underset{\sim}{V}' = a_\infty^2\nabla^2\underset{\sim}{V}' \checkmark$$

if $\underset{\sim}{V}' = \nabla\varphi$ then $\partial^2/\partial t^2\,\nabla\varphi = a_\infty^2\nabla^2\cdot(\nabla\varphi)$

but $\partial^2/\partial t^2\,\nabla\varphi = \nabla\frac{\partial^2\varphi}{\partial t^2}$ & $a_\infty^2\nabla^2\cdot\nabla\varphi = \nabla(a_\infty^2\nabla^2\varphi)$

or $\nabla\left(\frac{\partial^2\varphi}{\partial t^2} - a_\infty^2\nabla^2\varphi\right) = 0$ ✓

if $\underset{\sim}{V}' = \frac{d\underset{\sim}{r}'}{dt}$ then $\int\underset{\sim}{V}'dt = \int d\underset{\sim}{r}'$

then $\int\partial^2/\partial t^2\,\underset{\sim}{V}'dt = \int a_\infty^2\nabla^2\underset{\sim}{V}'dt$, since $\nabla \neq$ fn of time

$$\frac{\partial^2\underset{\sim}{r}'}{\partial t^2} = \frac{\partial\underset{\sim}{V}'}{\partial t} = a_\infty^2\nabla^2\int\underset{\sim}{V}'dt = a_\infty^2\nabla^2\circ r'$$

$$\frac{\partial^2\underset{\sim}{r}'}{\partial t^2} = a_\infty^2\nabla^2\underset{\sim}{r}' \checkmark$$

Good!

a fluid column moves with prescribed end velocities ✓ and is having its ends bent so that end moments exist and the curvature is constrained

B.C. of 1st, 2nd, 3rd types?

P63 1a, 2, 5

I.V. $\begin{cases} u(x,t) = f(x-at) \\ u(x,0) = f(x) \end{cases}$

$$u_t(x,0) = \frac{\partial f(\alpha)}{\partial \alpha}\,\frac{d\alpha}{dt} = f'(x-at)[-a]\Big|_{t=0} = -af'(x)$$

$$u(x,t) = f_1(x+at) + f_2(x-at)$$

$$u(x,0) = f_1(x) + f_2(x) = f(x)$$

$$u_t(x,0) = af_1'(x) - af_2'(x) = -af'(x)$$

$$f_1 - f_2 = -\int_{x_0}^{x} f'(\alpha)\,d\alpha + C$$

$$f_1 + f_2 = f(x)$$

$$f_1 = \frac{f(x)}{2} \text{ [illegible] } \int_{x_0}^{x} \frac{f'(\alpha)\,d\alpha}{2} + \frac{C}{2}$$

$$f_2 = \frac{f(x)}{2} + \int_{x_0}^{x} \frac{f'(\alpha)}{2}\,d\alpha - \frac{C}{2}$$

$$f_1(x+at) = \frac{f(x+at)}{2} - \left[\frac{f(x+at) - f(x_0)}{2}\right] + c/2$$

$$f_2(x-at) = \frac{f(x-at)}{2} + \left[\frac{f(x \text{ [illegible] } at) - f(x_0)}{2}\right] - c/2$$

$$u(x,t) = \frac{f(x\cancel{+at}) + f(x-at)}{2} + \frac{f(x-at) - f(\cancel{x+}at)}{2}$$

Ask about 5

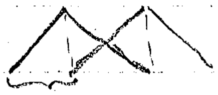
a. wave $u(x,t) = f(x-at)$

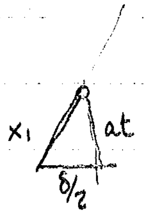
@ $u(x,0) = f(x) = \varphi(x)$

$u_t(x,0) = -af'(x) = \psi(x)$

$u_{tt} - a^2 u_{xx} = 0$

$u(x,t) = \frac{1}{2} f(x-at) + \frac{1}{2a}\int_{x_0}^{x-at} -af'(x)\,dx$

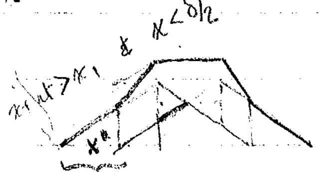




$\delta/2 = x - at = x_1$

$x_1^2 = (at)^2 + (\delta/2)^2$

$\sqrt{\frac{1}{a^2}\left[x_1^2 - \left(\frac{x_2 - x_1}{2}\right)^2\right]} = t$

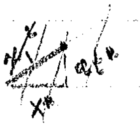


$\frac{\delta}{2} - at = x_1$

$t = \frac{x_1 - \delta/2}{-a}$ x^*

$t = \frac{-x_1 + \delta/2}{a}$

x^* $\delta/2$



$\frac{\delta - x_1}{2a}$

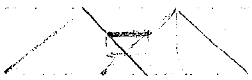
$x + at > x_1$

$x_1 + at > x > x_1 - at$

$x < x$

$x - at < x_1$

$x < x_1 + at$



$\frac{\delta}{2} - at < x_1$

$-\delta/2 + at > -x_1$

$at > \delta/2 - x_1$

$t > \frac{\delta/2 - x_1}{a}$

$x - x_1 \leq \delta/2$

$x < \delta/2 + x_1$

1 1a $\varphi(x-at) = 0 \quad \forall\ x \leq x_1 \quad x \geq x_2$

$$\varphi(x-at) = \frac{h}{\delta - 2x_1}(x - at - x_1) \qquad x_1 \leq \overset{x-at}{(x)} \leq \frac{x_2 - x_1}{2}$$

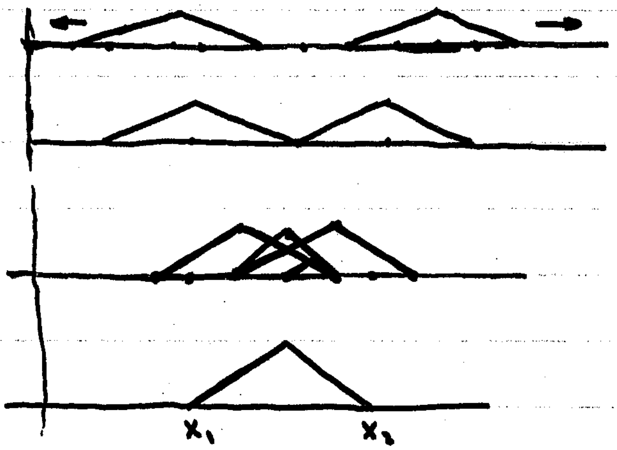
$$\varphi(x-at) = \frac{h}{\delta + 2x_1}(x_2 + at - x) \qquad \frac{x_2-x_1}{2} \leq x \leq x_2$$

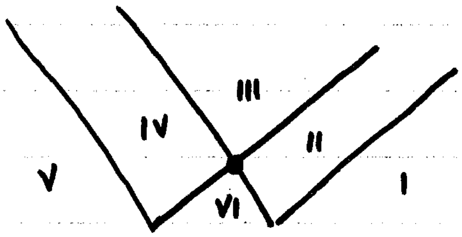
$\varphi(x+at) = 0 \quad \forall\ x \leq x_1 \quad x \geq x_2$

$$\varphi(x+at) = \frac{h}{\delta - 2x_1}(x + at - x_1) \qquad x_1 \leq x \leq \frac{x_2 - x_1}{2}$$

$$\varphi(x+at) = \frac{h}{\delta + 2x_1}(x_2 - x - at) \qquad \frac{x_2-x_1}{2} \leq x \leq x_2$$

$\delta = x_2 - x_1$





$u(x,t) = 0$ in I & V $\forall x$ & t

for $t < \frac{x_2 - x_1}{a}$

for $t > \frac{x_2 - x_1}{a}$ $\quad \varphi(x-at)$ holds in Region 2

$\varphi(x+at)$ " " " 4

$u = 0$ in Region 3

$\varphi(x-at) = 0 \qquad x - at \leq x_1$

$\varphi(x+at) = 0 \qquad x + at \leq x_1$

$$\int_c^x \frac{h}{2c^2}(2c\alpha - \alpha^2)\,d\alpha = \frac{h}{2c^2}\int_c^x \left(c\alpha^2 - \alpha^3/3\right) = \frac{h}{2c^2}\left[cx^2 - \frac{x^3}{3}\right.$$

$$\frac{h}{2c^2}\left[cx^2 - x^3/3 - 2c^3/3\right] \qquad \left. 3c\ 4c^2 - 8c^3 - 2c^3 - c^3 + c^3/3\right]$$

$\frac{1}{2a}(x - at - x_1)\psi_0 \qquad -\frac{1}{2a}a\psi_0$

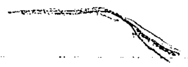
$\frac{1}{2a}(x + at - x_1)\psi_0 \qquad \frac{1}{2a}a\psi_0$

for a travelling wave propagating to the right

$\psi(x) = 0$ for $x \geqslant x_2$

$= \frac{1}{2a}(x_2 - x)\psi_0$ for $x_2 \geqslant x \geqslant x_1$

$= \frac{1}{2a}(x_1 - x_2)\psi_0$ for $x \leq x_1$



① $\psi(x-at) = 0$ $\quad x^{-at} \leq x_1$

② $\psi(x+at) = 0$ $\quad x^{+at} \leq x_1$ $\quad \frac{1}{2a}(x_2 - x + at)$

③ $\psi(x-at) = \frac{1}{2a}(x - at - x_1)\psi_0$ $\quad x_1 \leq (x-at) \leq x_2$

④ $\psi(x+at) = \frac{1}{2a}(x + at - x_1)\psi_0$ $\quad x_1 \leq (x+at) \leq x_2$

⑤ $\psi(x-at) = \frac{1}{2a}(x_2 - x_1)\psi_0$ $\quad x - at \geqslant x_2$

⑥ $\psi(x+at) = \frac{1}{2a}(x_2 - x_1)\psi_0$ $\quad x + at \geqslant x_2$

$x+at = x_1$
$x-at = x_1$
$t = \frac{x_2 - x_1}{2a}$

$u(x,t) = 0$ in reg. V $\quad$ ~~$\forall x, t$~~ $x+at \leq x_1$ $\quad x - at \leq x_1$, $x + at \leq x_1$ $\quad x \leq x_1$

$u(x,t) = 0$ in reg. I $\quad x - at \geqslant x_2$

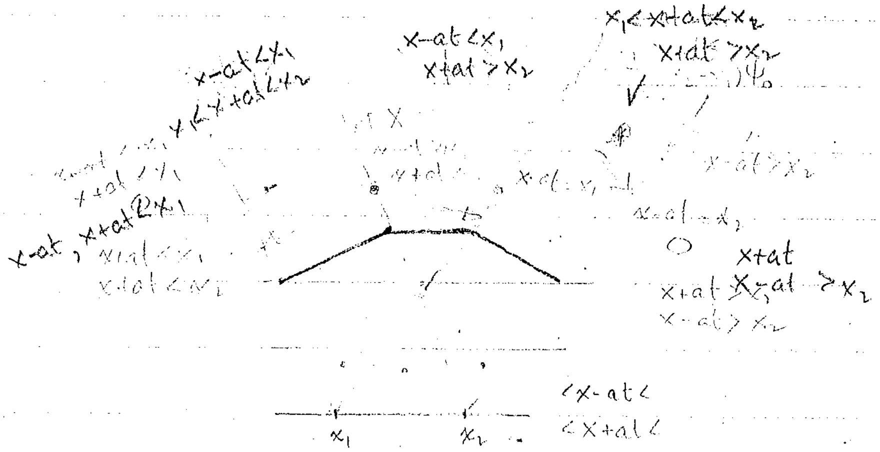
for $t \geqslant (x_2 - x_1)/2a$ $\quad$ region 4 has $u(x,t) =$ ④

region 3 has $u(x,t) =$ ⑤ or ⑥

region 2 has $u(x,t) =$ ③

in the region VI, IV, II $\quad x - at \geqslant x_1$, $x + at \leq x_2$ $\quad t \leq (x_2 - x_1)/2a$

$\psi = t\psi_0$



in 1 $u(x,t) = 0$

3 $u(x,t) = \frac{1}{2a}(x_2 - x_1)\psi_0$

2 $u(x,t) = \frac{1}{2a}(x_2 - x_1)\psi_0 - \frac{1}{2a}(x - at - x_1)\psi_0$

4 $\frac{1}{2a}(x + at - x_1)\psi_0$

5 $u(x,t) = 0$

6 $u(x,t) = \frac{1}{2a}\psi_0 (2at) = \psi_0 t$

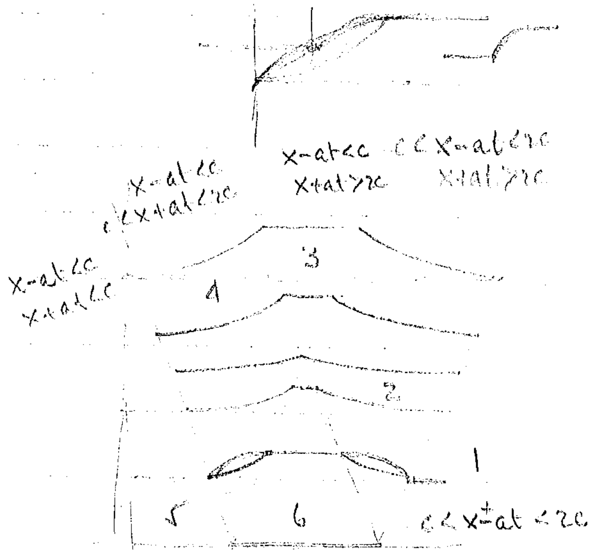
since $\varphi(x) = 0$ then

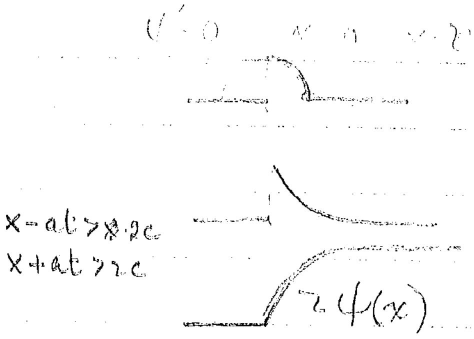
$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} \psi(\alpha)\,d\alpha = \Psi(x+at) - \Psi(x-at)$$

$$\Psi(x) = 0 \qquad x < c$$

$$\psi(x) = \frac{h}{2a6c^2}\left[3cx^2 - x^3 - 2c^3\right] \qquad c < x < 2c$$

$$\psi(x) = \frac{h}{2a6c^2}\left[2c^3\right] = \frac{hc}{6a} \qquad x > 2c$$





$x - at < c$
$x + at < 2c$
$2at = 4c$
$t = c/2a$

$$\psi(x^{\mp}-at) = 0 \qquad x - at < c$$

$$\psi(x+at) = 0 \qquad x + at < c$$

$$\psi(x-at) = \frac{h}{12ac^2}\left[3c(x-at)^2 - (x-at)^3 - 2c^3\right] \qquad c < x-at < 2c$$

$$\psi(x+at) = \frac{h}{12ac^2}\left[3c(x+at)^2 - (x+at)^3 - 2c^3\right] \qquad c < x+at < 2c$$

$$\psi(x-at) = \frac{h}{12ac^2}\left[2c^3\right] \qquad x - at > 2c$$

$$\psi(x+at) = \frac{h}{12ac^2}\left[2c^3\right] \qquad x + at > 2c$$

$u(x,t)$ in region 1 $= 0$

$$u(x,t) \text{ " " } 2 = \frac{h}{12ac^2}\left\{\left[2c^3\right] + 2c^3 - 3c(x-at)^2 + (x-at)^3\right\}$$

$$4c^3 - 3c(x-at)^2 + (x-at)^3$$

$$u(x,t) \text{ " " } 3 = \frac{h}{12ac^2}\left[2c^3\right]$$

$u(x,t)$ in region 4 $= \frac{h}{12ac^2}\left[3c(x+at)^2 - (x+at)^3 - 2c^3\right]$

$u(x,t)$ in reg 5 $= 0$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$
$(a+b)^2 - ab$

$u(x,t)$ in region 6 $= \frac{\cancel{3}\,h}{12ac^2}\left[3c\{(x+at)^2 - (x-at)^2\} - \{(x+at)^3 - (x-at)^3\}\right]$

$a \quad b \quad a+b$

$3c\{2at \cdot 2x\} - \{2at\}\{4x^2 - x^2 + a^2t^2\}$

Methods of P.D.E I

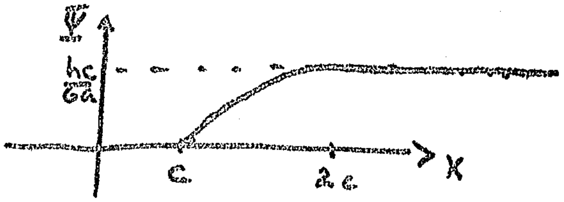
P63 1a, 2, 5.

1 a) Parts (1) and (2) done in class and the text

(3) Define $\Psi(x) = \frac{1}{2a}\int_d^x \psi(s)\,ds$ where $d<c$.

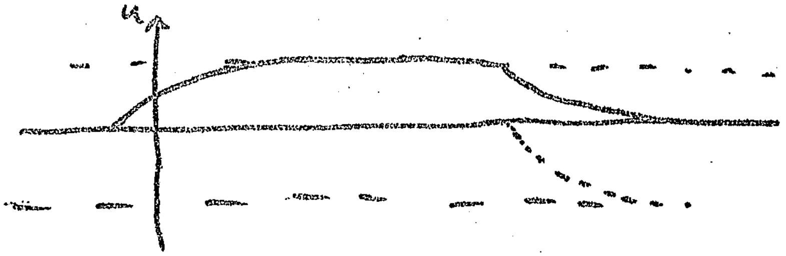
Then
$$\Psi(x) = 0 \quad , x < c$$
$$= \frac{h}{12ac^2}(x-c)(2c^2+2xc-x^2) \quad , c<x<2c$$
$$= \frac{hc}{6a} \quad , x>2c$$

Graph of $\Psi(x)$:
(Note $\Psi'(x) = 0$ at $x = 2c$)



Soln is then $u = \Psi(x+at) - \Psi(x-at)$

Eg.

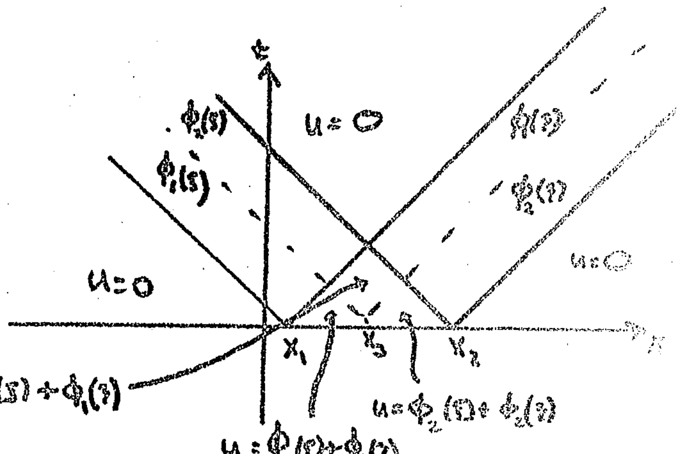


2. $\phi(x) = \begin{cases} 0 & [x<x_1] \\ \frac{2h}{(x_2-x_1)}(x-x_1) & [x_1<x<x_3 = \frac{x_2-x_1}{2}] \\ \frac{2h}{(x_2-x_1}(x_2-x) & [x_3<x<x_2] \\ 0 & [x>x_2] \end{cases}$

$2\phi_1(x)$

$2\phi_2(x)$

Define
$\xi = x+at$
$\eta = x-at$



5. $u(x,0) = f(x) \quad , u_t(x,0) = -af'(x)$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[f(x+at)+f(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} -af'(s)\,ds$$
$$= \frac{1}{2}\left[f(x+at)+f(x-at)\right] - \frac{1}{2}\left[f(x+at)-f(x-at)\right]$$
$$= \underline{f(x-at)}$$
i.e. progressing wave continues to move to the right inducing no component to the left.

~~⑧~~ 11

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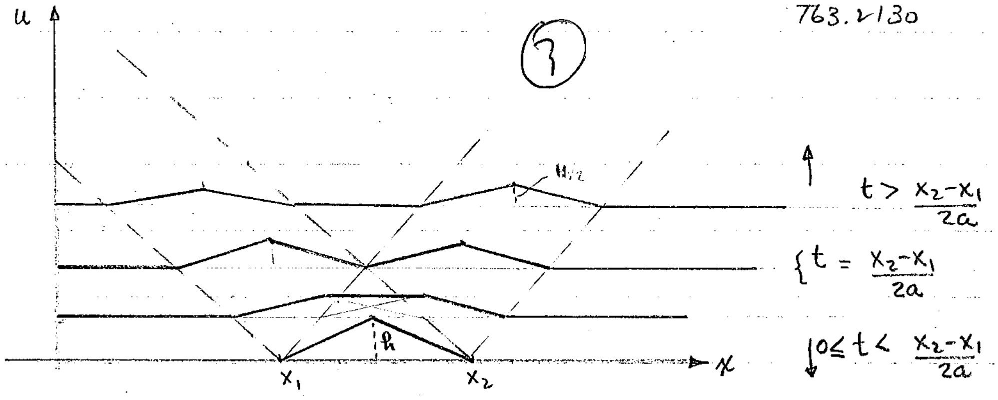
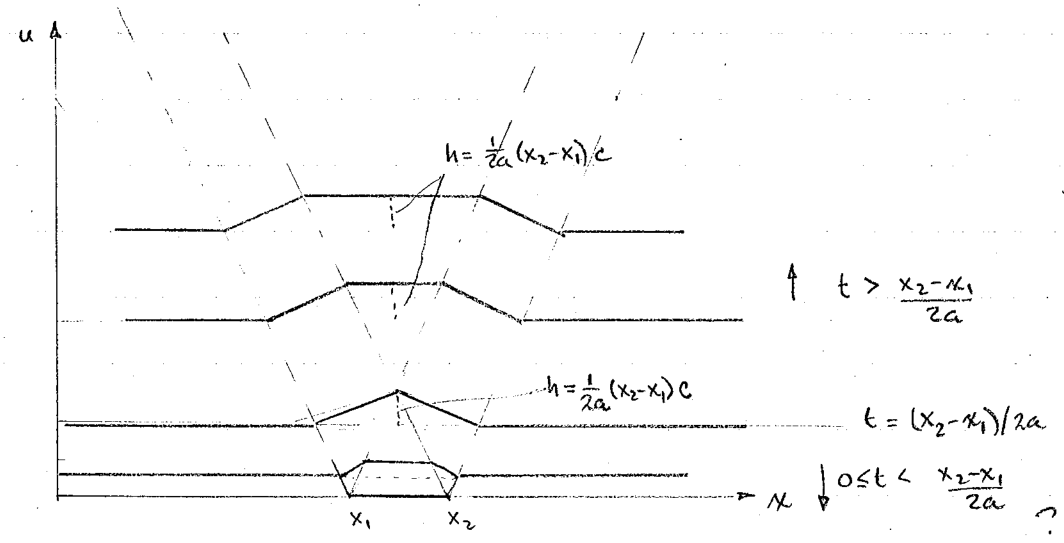
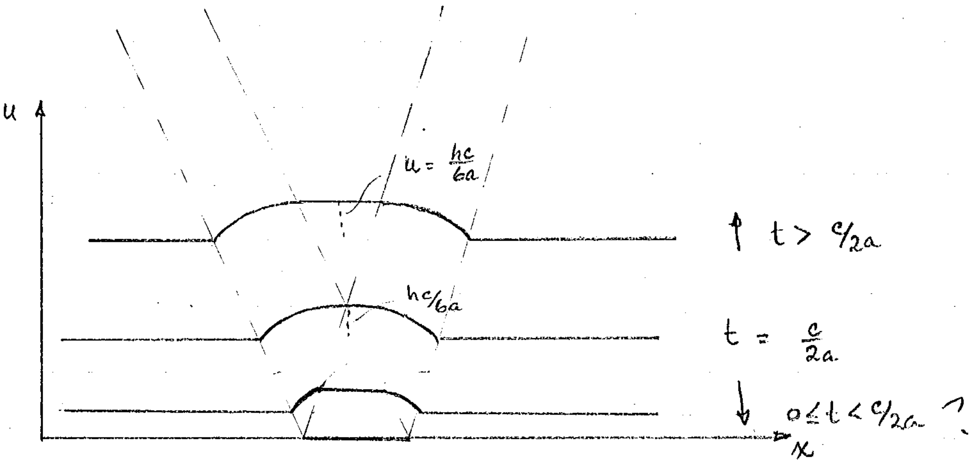
1.

⑦







Problem #2

$\psi = 0 \qquad \varphi(x) = 0 \qquad x \leq x_1 \qquad$ at $t=0$

$\varphi(x) = \frac{2h}{\delta}(x - x_1) \qquad x_1 \leq x \leq \delta/2 \qquad$ at $t=0$

$\varphi(x) = \frac{2h}{\delta}(x_2 - x) \qquad \delta/2 \leq x \leq x_2 \qquad$ at $t=0$

$\delta = x_2 - x_1$

$\varphi(x) = 0 \qquad x \geq x_2 \qquad$ at $t=0$ ✓

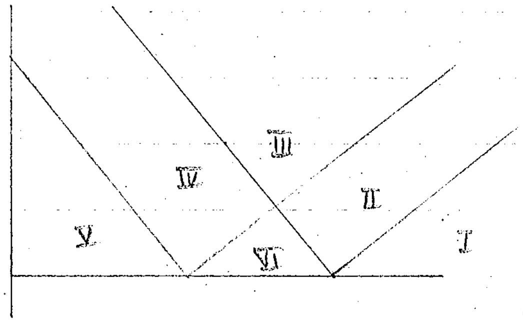
for $t > 0$

$\varphi(x-at) = \varphi(x+at) = 0 \qquad x \pm at \leq x_1$

$\varphi(x \pm at) = \frac{h}{\delta}(x \pm at - x_1) \qquad x_1 \leq x \pm at \leq \delta/2$

$\varphi(x \pm at) = \frac{h}{\delta}(x_2 - (x \pm at)) \qquad \delta/2 \leq x \pm at \leq x_2$

$\varphi(x-at) = \varphi(x+at) = 0 \qquad x \pm at \geq x_2$



In region I $\qquad x - at \geq x_2 \;,\; x + at \geq x_2$

$u(x,t) = 0$

In region II $\qquad x + at \geq x_2 \;,\; x_1 \leq x - at \leq x_2$

for right half of wave $\quad u(x,t) = \frac{h}{\delta}(x_2 - (x - at))$

for left half of wave $\quad u(x,t) = \frac{h}{\delta}(x - at - x_1)$

In region III $\qquad x + at \geq x_2 \;,\; x - at \leq x_1$

$u(x,t) = 0$

In region IV $\qquad x - at \leq x_1 \quad x_1 \leq x + at \leq x_2$

for right half of wave $\quad u(x,t) = \frac{h}{\delta}(x_2 - (x + at))$

for left half $u(x,t) = \frac{h}{\delta} \; (x+at-x_1)$

In region 5 $x-at \leq x_1$ & $x+at \leq x_1$

$$u(x,t)=0$$

In region 6 $x_1 \leq x \pm at \leq x_2$

depending upon the time the waves will have a combination of

$$\varphi(x \pm at) = \frac{h}{\delta}(x \pm at - x_1) \qquad x_1 \leq x \leq \delta/_2$$

$$\varphi(x \pm at) = h/_\delta(x_2 - (x \pm at)) \qquad \delta/_2 \leq x \leq x_2$$

✓

③

Consider

$u(x,t) = f(x-at)$ @ $t=0$

$u(x,0) = f(x)$ ✓

$u_t(x,0) = \frac{\partial f(x-at)}{\partial(x-at)} \frac{d(x-at)}{dt}\Big|_{t=0} = -a f'(x)$ ✓

using the wave equation & method of characteristics one finds

$$u(x,t) = f_1(x+at) + f_2(x-at)$$

using the above i.c.

$u(x,0) = f_1(x) + f_2(x) = f(x)$ ①

$u_t(x,0) = a[f_1'(x) - f_2'(x)] = -af'(x)$ ② ✓

differentiate ① wrt x

$f_1'(x) + f_2'(x) = f'(x)$ ③

$f_1'(x) - f_2'(x) = -f'(x)$ ②

Add ②&③ one obtains

$$f_1'(x) = 0 \implies f_1(x) = c_1$$

Subtract ②&③ one obtains

$$f_2'(x) = f'(x) \implies f_2(x) = f(x) + c_2$$

$\therefore \quad u(x,0) = f_1(x) + f_2(x) = f(x) = c_1 + f(x) + c_2$

let $c_1 = -c_2 = c \quad \therefore$

$$f_1(x) = c \quad ; \quad f_2(x) = f(x) - c$$

since the arguments are to satisfy arbitrary values then

$$f_1(x+at) = c \qquad f_2(x-at) = f(x-at) - c$$

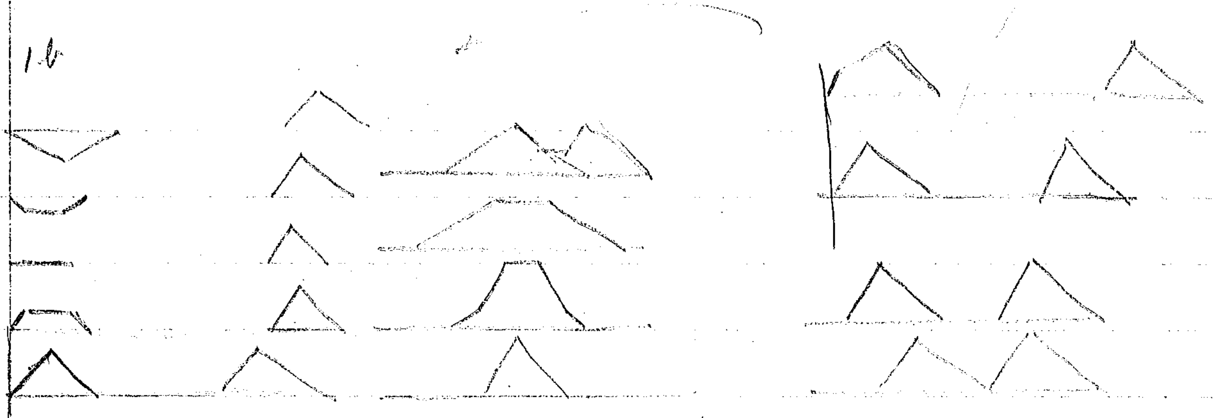
hence, $u(x,t) = f(x-at)$

Where as problem 1 is a non-motional but initially displaced wave propagation, This problem has an initial velocity equal to the derivative of the displacement

Also the non-motional initial displacement propagation produces two waves in opposite directions, motional initial propagation produces a single wave moving to the right & a constant wave front moving to the left. ?

1b

4



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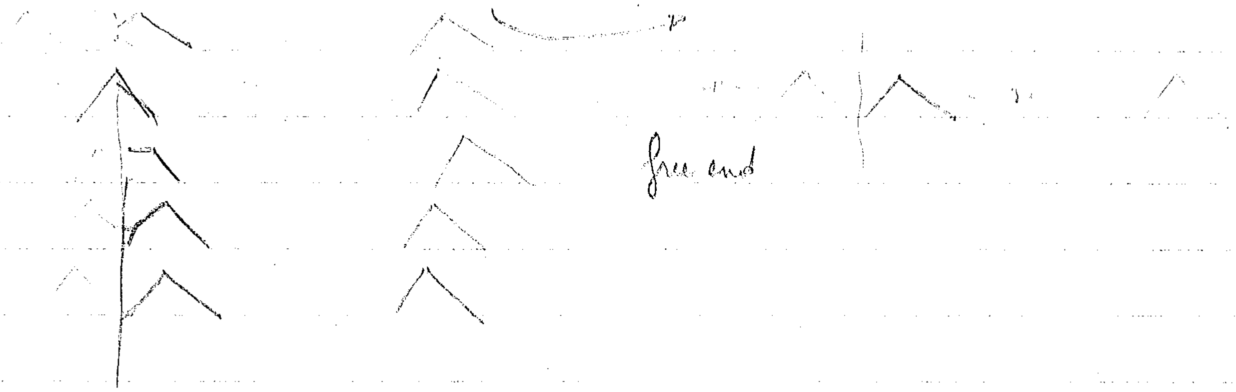
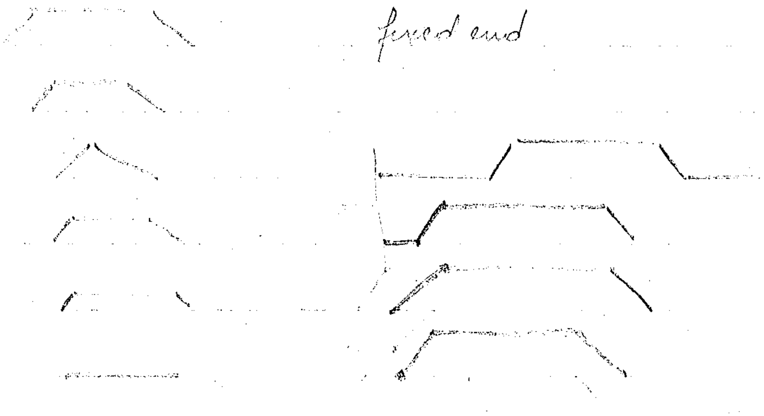
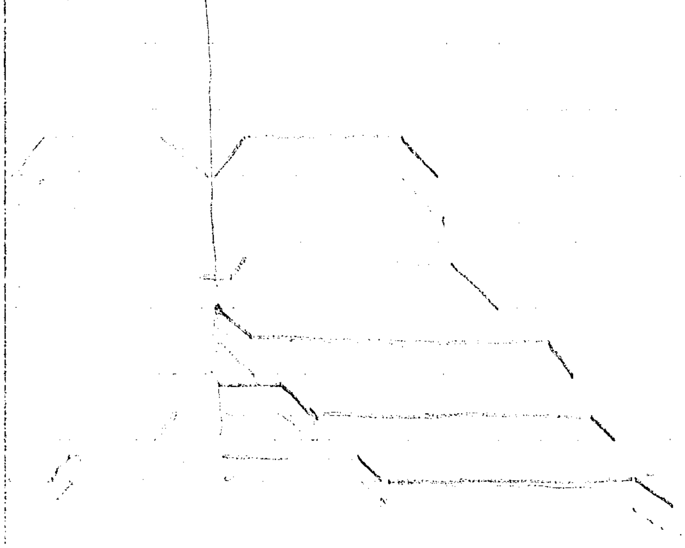


6



7. free end



Consider the linearized ~~momentum~~ acoustic equations

$$\frac{\partial \rho'}{\partial t} + \rho_\infty \nabla \cdot \underset{\sim}{V}' = 0 \qquad \rho'_{tt} = a_\infty^2 \nabla^2 \rho'$$

$$\frac{\partial \underset{\sim}{V}'}{\partial t} + \frac{a_\infty^2}{\rho_\infty} \nabla \rho' = 0 \qquad p'_{tt} = a_\infty^2 \nabla^2 p'$$

$$\underset{\sim}{V}'_{tt} = a_\infty^2 \nabla^2 \underset{\sim}{V}' \qquad \underset{\sim}{V}' = \nabla \varphi$$

$$p' = a_\infty^2 \rho' \qquad \varphi_{tt} = a_\infty^2 \nabla^2 \varphi$$

In 1-D $\qquad r'_{tt} = a_\infty^2 \nabla^2 \underset{\sim}{r}'$

$\rho' = f(x-at) + g(x+at)$ density of gas

$u' = \frac{a_\infty}{\rho_\infty}\left[f(x-a_\infty t) - g(x+a_\infty t)\right]$ velocity of particle

@ $t=0$ & [illegible] $u'=0$ & $r'=0$

$$0 = \frac{a_\infty}{\rho_\infty}\left[f(x) - g(x)\right] \Rightarrow f(x) = g(x) \quad \text{if } g=0 \ \forall x \,\&\, t$$

then $f(x) = 0$

$$r' = \int_0^t \frac{a_\infty}{\rho_\infty}\left[f(x-a_\infty t) - g(x+a_\infty t)\right] dt$$

at $t=0$ $\qquad r' = \int_0^t \frac{a_\infty}{\rho_\infty}\left[f(x) - g(x)\right] d\tau$

$$r' = \frac{a_\infty}{\rho_\infty} t \left[f(x) - g(x)\right]$$

$$r' = \frac{a_\infty}{\rho_\infty} t \left[f(x-at) - g(x+at)\right] \quad @\ x = x_0$$

$x = f(t) \qquad v = f'(t) = c$

if $v < a$

$f(0) = 0 \qquad f'(0) = 0$

if

from linearized equations

$$\rho'_{tt} = a_\infty^2 \nabla^2 \rho' \qquad \& \text{ 1-D} \Rightarrow \quad \rho'_{tt} = a_\infty^2 \rho'_{xx}$$

or $\rho' = f(x-at) + g(x+at)$

since $\bar{q}'_t + \frac{a_\infty^2}{\rho_\infty} \nabla \rho' = 0$ is lin mom we can write

$$u'_t + \frac{a_\infty^2}{\rho_\infty} \rho'_x = 0 \qquad \rho'_x = f'(x-at) + g'(x+at)$$

$$u'_t + \frac{a_\infty^2}{\rho_\infty} \left[f'(x-at) + g'(x+at) \right] = 0$$

$$u' = +\frac{a_\infty}{\rho_\infty} \left[f^*(x-at) \pm g^*(x+at) \right]$$

$$u'_t = \frac{a_\infty}{\rho_\infty} \left[af' - a g' \right] \rightarrow \qquad u' = \frac{a_\infty}{\rho_\infty} \left[f(x-at) - g(x+a_\infty t) \right]$$

$$r'_{x,tt} = a_\infty^2 r'_{x,xx}$$

$$\int \underset{\sim}{v}' dt = \underset{\sim}{r}' \qquad \int \frac{a_\infty}{\rho_\infty} \left[f(x-at) - g(x+at) \right] dt = r'_x$$

$$\frac{}{\rho_\infty} \left[-F(x-at) - G(x+at) \right] = r'$$

$$-\frac{1}{\rho_\infty} \left[F(x-at) + G(x+at) \right] = r'$$

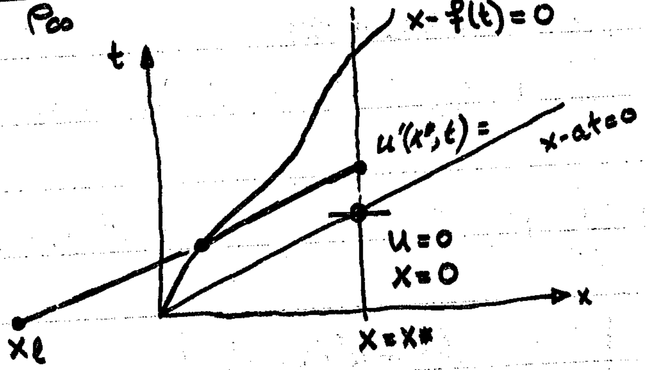
where $\frac{-1}{a_\infty} F(x-at) = \int f(x-at)\, dt \qquad \frac{1}{a_\infty} G(x+at) = \int f(x+at)\, dt$

when wave move to right $G(x+at)$ & $g(x+at)$ $=0$

$$\rho' = f(x-at)$$

$$u' = \frac{a_\infty}{\rho_\infty} f(x-at)$$

$$-\frac{F(x-at)}{\rho_\infty} = r'$$



$t = \frac{x^*}{a}$

$x - at = x_\ell$ $x_i - x_\ell = at_i$ $t = (x_i - x_\iota)/a$

$x_i - f(t_i) = 0$

u'_{at} $t - t^* = f(t_i)$

$$u' = \frac{a_\infty}{\rho_\infty} f(x^* + at^* - at)$$ t

$f(0) = 0$

$f'(0) = 0$

$$u' = \frac{a_\infty}{\rho_\infty} f(x^* - at)$$

$t = t^*$ $u' = 0$

$t < t^*$ $u' = 0$

$t > t^*$ $u' = f'(t$

$\frac{\Delta t}{\Delta x} = \frac{1}{a}$ $\frac{1}{a_\infty} = \frac{(t - t_i)}{x^* - x_i}$

$t - t^* > 0$ $v = f(t_i$

$x_i = f(t_i)$ $x^* - x_i = a_\infty t - a_\infty t_i$

$x^* - f(t_i) = a_\infty t - a_\infty t_i$

$$u' = \frac{a_\infty}{\rho_\infty} f\left[x^* - a_\infty\left(t - \frac{x^*}{a_\infty}\right)\right]$$

LEMENT.

$$\frac{1}{2}\varphi(x-a_1t) + \frac{a_2-a_1}{2(a_1+a_2)}\varphi(x-a_1t) = \frac{a_2}{a_1+a_2}\varphi(x-a_1t)$$

a_1+a_2

$$\frac{1}{2}\varphi' + g' = f_2'$$

$$\frac{1}{2}\varphi' - g' = \frac{a_1}{a_2} f_2'$$

$$\varphi' = f_2'\left[1+\frac{a_1}{a_2}\right]$$

$$\frac{a_2+a_1}{a_2} \qquad f_2' = \frac{a_2}{a_2+a_1}\varphi'$$

$$2g' = f_2' - \frac{a_1}{a_2} f_2'$$

$$g' = \frac{a_2-a_1}{2a_2} f_2' = \frac{a_2-a_1}{2a_2}\cdot\frac{a_2}{a_2+a_1}$$

$$\frac{a_2-a_1}{2(a_2+a_1)}$$

$$T_c - R_c \overset{?}{=} \frac{1}{2}$$

$$\cancel{\frac{a_2-a_1}{2(a_2+a_1)}} + \frac{2a_1}{2(a_2+a_1)} + \frac{a_1-a_2}{2(a_1+a_2)} \qquad \frac{2(a_2)}{2(a_1+a_2)} + \frac{a_1-a_2}{2(a_1+a_2)}$$

$$\frac{1}{2}$$

$$u(x,t) = \begin{cases} f(t-x/c_1) & -\infty < x < -c_1t \\ f(t-x/c_1) + \underset{\text{reflected}}{g(t+x/c_1)} & -c_1t < x < 0 \\ f_2(t-x/c_2) \text{ transmitted} & 0 < x < c_2t \\ 0 & c_2t < x < \infty \end{cases}$$

$u^{(1)}(0,t) = u^{(2)}(0,t)$ displ at interface

$u_x^{(1)}(0,t) = u_x^{(1)}(0,t)$ Stress at interface

$$f + g = f_2 \qquad 2g = (1 - c_1/c_2) f_2$$

$$-\frac{1}{c_1}f' + \frac{1}{c_1}g' = -\frac{1}{c_2}f_2' \qquad g = \frac{c_2-c_1}{2c_2}\cancel{\varphi} f_2 = \frac{c_2-c_1}{c_1+c_2} f$$

$$f' + g' = f_2'$$

$$2f' = \left(1+\frac{c_1}{c_2}\right) f_2 \qquad T_c = \frac{c_2+c_2}{c_1+c_2}$$

$$f_2 = \frac{2c_2}{c_1+c_2} f \qquad R_c = \frac{c_2-c_1}{c_1+c_2}$$

$$T_c - R_c = 1$$

MULLER ROUTINE TO SOLVE EQUATION FOR COMPLEX ROOTS

```
@MAD,IS  S1,R1
RALPH  51,12 10/30-20:49-S1(0)
<<<<< ERROR 002513 .... ILLEGAL INPUT PARAMETERS.
                        COMPILATION TERMINATED -- NO RELOCATABLE
```

$u(x,t) = \phi(t - x/c)$ for $x < ct$

$\cancel{\phi}\ 0$ $x > ct$

$u(x,0) = \phi(-x/c)$ $x < 0$

$u_t(x,0) = \phi'(-x/c)$

$u(x,0) = u_t(x,0) = 0$ $x > 0$

$u_{xx} - c_1^{-2} u_{tt} = 0$ $x < 0$

$u_{xx} - c_2^{-2} u_{tt} = 0$ $x > 0$

$u(x,t) = \phi_1(x - c_1 t) + \psi_1(x + c_1 t)$ $x < 0$

$u(x,t) = \phi_2(x - c_2 t) + \psi_2(x + c_2 t)$ $x > 0$

@ interface $u(x,0) = \phi(-x/c_1)$ $-\infty < x < 0$

$u_t(x,0)$ $\phi'(-x/c_1)$

$u(x,0) = u_t(x,0) = 0$ $0 < x < \infty$

⑫

Cesar Levy
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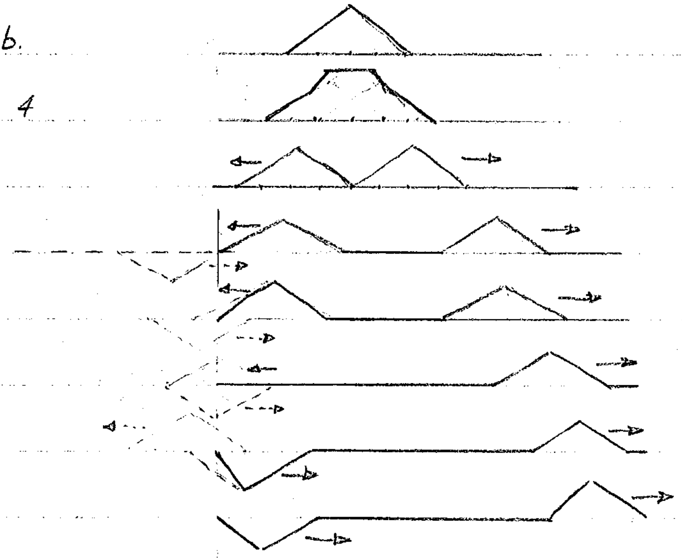
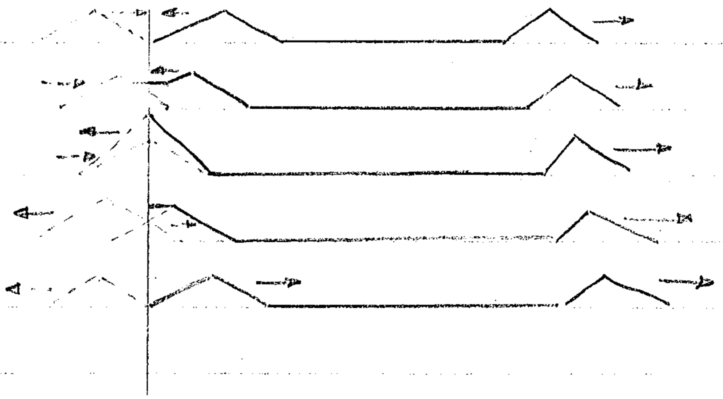
4



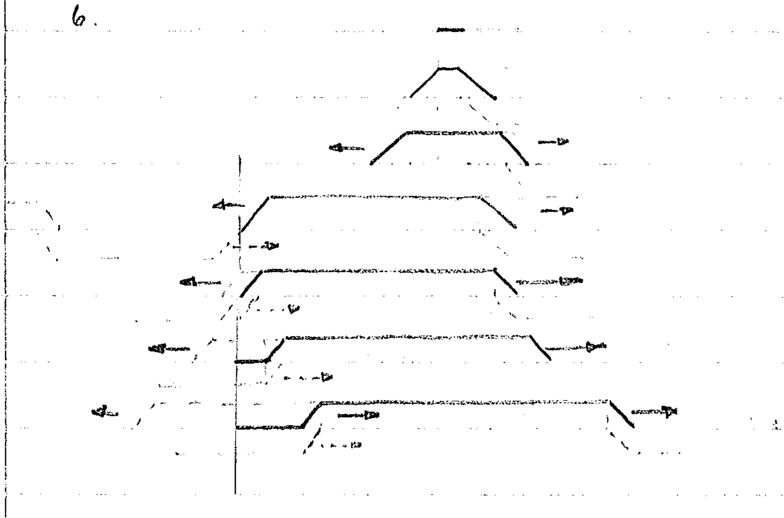
✓ ②

5. same first 3 pictures of (4)



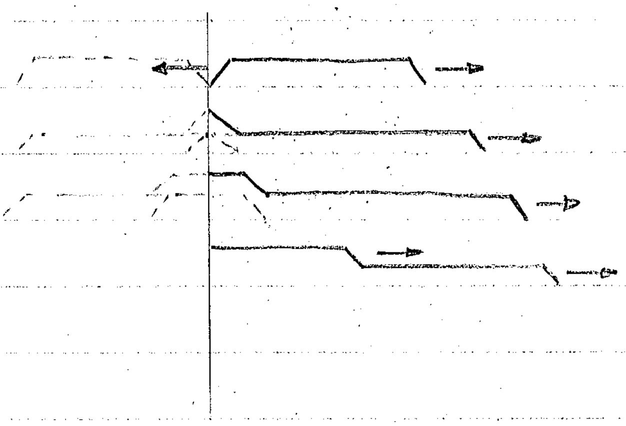
✓ ②

6.



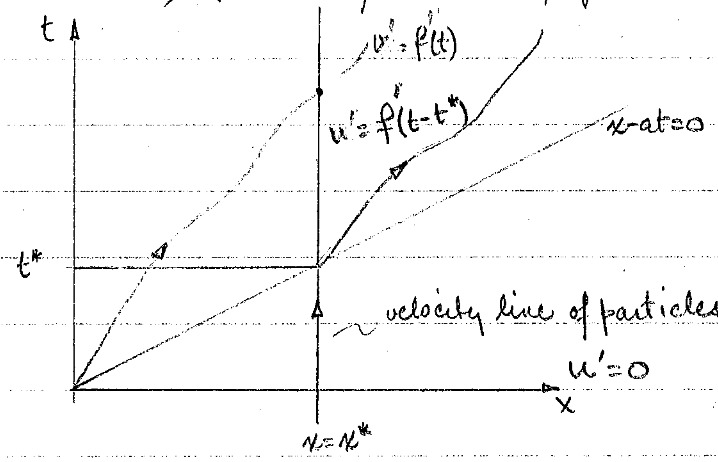
✓ ②

7. Same as the first two of (6)



✓ ②

4. A gas filled tube has a piston which moves according to some relation $x = f(t)$ & whose velocity is $v' = f'(t) < a_\infty$. Given that x_{part} & v_{part} are zero at $t=0$, find displacement of gas at plane $x = x^*$



Assume that at $t=0^+$ & $x=0$ the piston has propagated a wave forward ∴ the characteristic drawn from the origin is $x-at=0$ because $v' << a_\infty$ we will assume linearized acoustic theory

$$\therefore \rho'_{tt} = a_\infty^2 \nabla^2 \rho' \Rightarrow \rho' = f(x-a_\infty t) + g(x+a_\infty t)$$

also since $\frac{\partial \underset{\sim}{V}'}{\partial t} + \frac{a_\infty^2}{\rho_\infty}\nabla \rho' = 0$ & 1-D $\Rightarrow$ $v' = \frac{a_\infty}{\rho_\infty}\left[f(x-a_\infty t) - g(x+a_\infty t)\right]$

at $x = x^*$ & $t < t^* = x^*/a_\infty$ the particles are at rest and have no velocity. at $t > t^*$ the velocity of the particle is the velocity of the piston at the intersection of the $x - a_\infty t$ characteristic & $v' = f'(t)$

$\therefore$ the particle distance $= 0$ $\forall\ t < t^*$ (with respect to x^*)

& $\int_{x^*/a_\infty}^{t} v'(\tau - x^*/a_\infty)\,d\tau \qquad t > x^*/a_\infty$

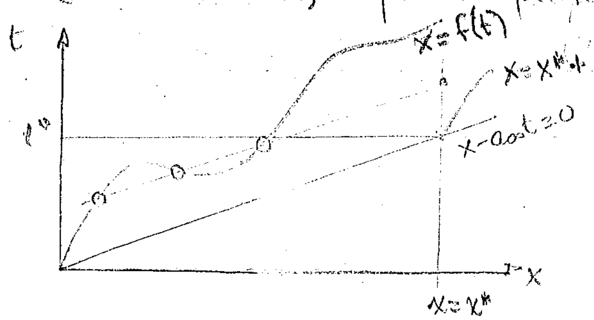
when $v' =$ constant c, then $D = v'\int_{x^*/a_\infty}^{t} (\tau - x^*/a_\infty)\,d\tau$

$$= v'\left[\frac{\tau^2}{2} - \frac{x^*}{a_\infty}\tau\right]_{x^*/a_\infty}^{t}$$

$$D = c\left[\frac{t^2}{2} - \frac{x^* t}{a_\infty} + \left(\frac{x^*}{a_\infty}\right)^2\right] \qquad t \geq x^*/a_\infty$$

total distance is $x^* + c\left[t^2/2 - \frac{x^*}{a_\infty}t + \left(\frac{x^*}{a_\infty}\right)^2\right]$ wrt origin

b. if a time t $v' > a_\infty$: Instead of a sound wave a shock wave is set up. One must then treat the full equations & not the linearized versions of continuity & momentum. If one takes a look at the x,t phase plane for $v' > a_\infty$



this implies that the particles for $t > t^*$, there will exist multivalued solution to the displacements. Look at Eulerian solution for $X = \overset{\text{piston disp}}{x} + \overset{\text{relative particle disp}}{u}$ where X are fixed coord. The wave eq becomes $u'_{tt} + 2cu'_{Xt} + (c^2 - a^2)u'_{XX} = 0$ giving $u' = f[X - (c-a)t] + g[X - (c+a)t]$. for $c > a$ we will get 2 waves travelling to right and at some pt. they will intersect forming shock wave.

6. A wave travelling in medium 1 has a speed of propagation

$a_1 = \sqrt{k_1/\rho_1}$, in medium 2 $a_2 = \sqrt{k_2/\rho_2}$

③ it also has the solution that for $u_{xx} - a_i^{-2} u_{tt} = 0$ $i = 1,2$

$u(x,t) = f_1(x-a_1t) + g_1(x+a_1t)$ or $\bar{f}_1(-x/a_1+t) + \bar{g}_1(x/a_1+t)$

$u(x,t) = f_2(x-a_2t) + g_2(x+a_2t)$ or $\bar{f}_2(-x/a_2+t) + \bar{g}_2(x/a_2+t)$

For $x+a_1t<0$ $x<0$

$u(x,t) = f_1(x-a_1t)$ since it is the only wave travelling in the region

For $x+a_1t>0$ & $x<0$

$u(x,t) = f_1(x-a_1t) + g_1(x+a_1t)$ because of reflection

For $x-a_2t<0$ & $x>0$

$u(x,t) = f_2(x-a_2t)$ because wave is partly transmitted

For $x-a_2t>0$ & $x>0$

$u(x,t) = 0$ wave has not disturbed any particles there.

The conditions to be solved are that

$u(0_-,t) = u(0_+,t)$ continuity in displacement

$u_x(0_-,t) = u_x(0_+,t)$ continuity in force or stress note $k_1 \neq k_2$ here.

∴ @ $x=0$

$f_1(-a_1t) + g_1(a_1t) = f_2(-a_2t)$; $\bar{f}_1(t) + \bar{g}_1(t) = \bar{f}_2(t)$ ①

$-\frac{1}{a_1}\bar{f}_1'(t) + \frac{1}{a_1}\bar{g}_1'(t) = -\frac{1}{a_2}\bar{f}_2'(t)$ ②

differentiate ① wrt t

$\bar{f}_1'(t) + \bar{g}_1'(t) = \bar{f}_2'(t)$

$\bar{f}_1'(t) - \bar{g}_1'(t) = \frac{a_1}{a_2}\bar{f}_2'(t)$

or $\bar{f}_1'(t) = \frac{1}{2}\left(1+\frac{a_1}{a_2}\right)\bar{f}_2'(t)$ or $\bar{f}_2'(t) = \frac{2a_2}{a_2+a_1}\bar{f}_1'(t)$

$\bar{g}_1'(t) = \frac{1}{2}\left(1-\frac{a_1}{a_2}\right)\bar{f}_2'(t) = \frac{a_2-a_1}{2a_2} \cdot \frac{2a_2}{a_2+a_1}\bar{f}_1'(t) = \frac{a_2-a_1}{a_2+a_1}\bar{f}_1'(t)$

Since a_1 & a_2 are not fns of time then

$$\bar{f}_2(t) = \frac{2a_2}{a_2+a_1}\bar{f}_1(t)$$

$$\bar{g}_1(t) = \frac{a_2-a_1}{a_2+a_1}\bar{f}_1(t) \checkmark$$

?

∴ since $\bar{f}_2(t)$ is transmitted wave $\frac{2a_2}{a_2+a_1} = T_c$ = transmission coeff

& $\frac{a_2-a_1}{a_2+a_1} = R_c$ reflection coeff of reflected wave $\bar{g}_1(t)$

note: $T_c - R_c = 1$ ✓ MARK 1.

at $a_1 = a_2$ $R_c \equiv 0$; this is plausible for this means that the elastic rods are made of <u>same materials</u> ? & has same characteristics ∴ there is no need for a wave to reflect at $x = 0$.

b. Given that

$$u(x,0) = 0 \qquad x < x_1$$

① $$= \varphi(x) \qquad x_1 < x < x_2 < 0$$

$$= 0 \qquad x > x_2$$

and $\varphi(x)$ is equal to zero

for $$u(x,t) = \frac{\varphi(x-a_1t) + \varphi(x+a_1t)}{2} \qquad x<0$$

$$= \bar{\varphi}(t - x/a_1) + \bar{\varphi}(t + x/a_1)$$

As $t\uparrow > 0$ $\varphi(x+a_1t)$ keeps moving to the left whereas $\varphi(x-a_1t)$ is reflected and transmitted at $x=0$

∴ let us only take a look at $\varphi(x-a_1t)$

$$u(x,t) = \tfrac{1}{2}\bar{\varphi}(t - x/a_1) \qquad x+a_1t < 0 \quad x<0$$

$$u(x,t) = \tfrac{1}{2}\bar{\varphi}(t - x/a_1) + g_1(t + x/a_1) \qquad x+a_1t > 0 \quad x<0$$

$$u(x,t) = f_2(t - x/a_2) \qquad x - a_2t < 0 \quad x>0$$

$$u(x,t) = 0 \qquad x - a_2t > 0 \quad x>0$$

✓

at $x=0$ $\quad u(0_-,t) = u(0_+,t)$

$$u_x(0_-,t) = u_x(0_+,t)$$ note $k_1 \neq k_2$

$$\therefore \quad \tfrac{1}{2}\bar{\varphi}(t) + \bar{q}_1(t) = \bar{f}_2(t)$$

$$-\frac{1}{2a_1}\bar{\varphi}'(t) + \frac{1}{a_1}\bar{g}_1'(t) = -\frac{1}{a_2}\bar{f}_2'(t)$$

or

$$\bar{\varphi}'(t) = \left(1 + \frac{a_1}{a_2}\right)\bar{f}_2'(t) \qquad \bar{\varphi}(t)\left[\frac{a_2}{a_1+a_2}\right] = \bar{f}_2(t)$$

$$2\bar{g}_1'(t) = \left(1 - \frac{a_1}{a_2}\right)\bar{f}_2'(t) \qquad \bar{\varphi}(t)\left[\frac{a_2-a_1}{2(a_1+a_2)}\right] = \bar{g}_1(t)$$

$$\therefore \quad T_c = \frac{a_2}{a_1+a_2} \qquad R_e = \frac{a_2-a_1}{2(a_1+a_2)}$$ note: $T_c - R_e = 1/2$

$\therefore$ for $\quad t \leq x_1/a_1 \quad x \leq 0$

$$u(x,t) = \frac{\bar{\varphi}(t - x/a_1) + \bar{\varphi}(t + x/a_1)}{2}$$

for $\quad t \geq x_2/a_1$

$$u(x,t) = \frac{a_2}{a_1+a_2}\bar{\varphi}(t - x/a_1) \qquad x \geq 0$$

$$u(x,t) = \tfrac{1}{2}\bar{\varphi}(t + x/a_1) + \frac{a_2-a_1}{2(a_1+a_2)}\bar{\varphi}(t - x/a_1) \qquad x \leq 0$$ ✓

Prof Seymour: I forgot which problems were assigned for 9 Nov 1972
I have enclosed problems 8, 9, 10 P65 and will hand in
The other problem on 16 Nov 1972. I hope this does not
inconvenience the grader.

⑨ C. LEVY 097421896

763.2130

8. for the wave equation $u(x,t) = f(x-at)$

⑤

$$M u_{tt}(0,t) = S\gamma p_0 \, u_x(0,t)$$

$$\therefore \quad M a^2 f''(-at) = S\gamma p_0 f'(-at)$$

$$\therefore \quad f''(-at)/f'(-at) = \frac{S\gamma p_0}{Ma^2}$$

$\therefore$ if we write $-at$ as ξ and $f(\xi) = A + Be^{r\xi}$

$$\therefore \quad f''(\xi) = Br^2 e^{r\xi} \quad \& \quad f'(\xi) = Bre^{r\xi}$$

or $$f''(\xi)/f'(\xi) = r = \frac{S\gamma p_0}{Ma^2}$$

$$\therefore \quad f(\xi) = A + Be^{\frac{S\gamma p_0}{Ma^2}\xi} \quad ✓$$

now $$f(x-at) = A + Be^{(x-at)\frac{S\gamma p_0}{Ma^2}} \quad ✓$$

but at time $t=0$ & $x=0$ $v = v_0$ ✓

$$\therefore \quad -a f'(x-at) = -aB\frac{S\gamma p_0}{Ma^2} e^{(x-at)\frac{S\gamma p_0}{Ma^2}}$$

$$\therefore \quad -a f'(0) = v_0 = -aB\frac{S\gamma p_0}{Ma^2} \quad ; \quad B = -\frac{v_0 Ma}{S\gamma p_0}$$

$$\therefore \quad f(x-at) = A - \frac{v_0 Ma}{S\gamma p_0} e^{\rho} \quad \text{where } \rho = (x-at)\frac{S\gamma p_0}{Ma^2}$$

but at $t=0$ & $x=0$ $u=0$ $\therefore$ ✓

$$f(0) = A - \frac{v_0 Ma}{S\gamma p_0} = 0 \qquad A = \frac{v_0 Ma}{S\gamma p_0} \quad \therefore$$

$$u(x,t) = f(x-at) = \frac{v_0 Ma}{S\gamma p_0}\left(1 - e^{(x-at)\frac{S\gamma p_0}{Ma^2}}\right) \qquad x-at<0 \quad ✓$$

since wave hasn't propagated past particles $\forall$ $x-at>0$ ✓

$u(x,t) = 0$

9. from the wave equation $u(x,t) = f(x-at) - g(x+at)$

④ but the conditions that it must solve is

$$\frac{M}{2} u^1_{tt}(0,t) = \frac{M}{2} u^2_{tt}(0,t) = T u^1_x(0,t) = -T u^2_x(0,t)$$

where 1 & 2 superscripts represent wave propagating to right & to left respect.

∴ $u_1(x,t) = f(x-at)$ or

$$u_{1_{tt}} = a^2 f''(x-at) \qquad u_{1_x} = f'(x-at)$$

at $x=0$ letting $\xi = -at$

$$\frac{M}{2} a^2 f''(\xi) = T f'(\xi) \quad \text{or} \quad f'/f'' = \frac{Ma^2}{2T}$$

∴ $f(\xi) = A + Be^{r\xi}$ where $r = 2T/Ma^2$

at $t=0$ & $x=0$ $v = v_0$

∴ $f(x-at) = A + Be^{(x-at)2T/Ma^2}$ & $-af'(0) = v_0 = -a\cdot\frac{2T}{Ma^2}B$

∴ $B = \frac{-Mav_0}{2T}$

∴ $f(x-at) = A - \frac{Mav_0}{2T} e^{(x-at)2T/Ma^2}$

at $x=0$ & $t=0$ the mass is at equilib. Let us assume that

$u(0,0) \equiv 0$ ∴ $f(x-at) = \frac{Mav_0}{2T}\left(1 - e^{(x-at)2T/Ma^2}\right)$

for $\forall\, x > at$ $u_1(x,t) = 0$ since no wave has propagated into that region yet.

The same development for $u_2(x,t) = g(x+at)$ leads to

$$\frac{M}{2} a^2 g''(\eta) = T g'(\eta) \quad \text{where } \eta = at \quad \text{or} \quad g'/g'' = \frac{Ma^2}{2T}$$

using the condition that $u_{1_t}(0,0) = u_{2_t}(0,0) = v_0$ leads to the fact that $f'(0) = -g'(0)$ and $g(\eta) = C + De^{-r\eta}$

∴ $r = 2T/Ma^2$ and $ag'(0) = -a\frac{2T}{Ma^2}D = v_0$ $\quad D = -\frac{Mav_0}{2T}$

∴ $g(x+at) = C - \frac{Mav_0}{2T} e^{-(x+at)(2T/Ma^2)}$

+

and since $g(0) = 0$ $\quad C = \frac{Mav_0}{2T}$

∴ $g(x+at) = \frac{Mav_0}{2T}\left(1 - e^{-(x+at)(2T/Ma^2)}\right)$

$u_2(x,t) = 0 \quad \forall \; x < -at$ since the propagated wave is not affecting that region yet.

$$\therefore \quad u_1(x,t) = \frac{Mav_0}{2T}\left[1 - e^{\frac{2T}{Ma^2}(x-at)}\right] \qquad x-at<0$$

$$= 0 \qquad x-at>0$$

$$u_2(x,t) = \frac{Mav_0}{2T}\left[1 - e^{-\frac{2T}{Ma^2}(x+at)}\right] \qquad x+at>0$$

$$= 0 \qquad x+at<0$$

+, if you don't use $-Tu_{2,x}(0,t)$.

✓

$$g\left[\left(\xi - \frac{2l}{a}\right) - \frac{lM}{a} f(\xi)\right] = f\left[\left(\xi - \frac{2l}{a}\right) - \frac{lM}{a} f\left(\xi - \frac{2l}{a}\right)\right]$$

$$\therefore \quad u = f\left[t - x/a \left\{1 + M f(t - x/a)\right\}\right]$$

$$u = f\left[t - \frac{2l}{a}\left\{1 + M f\left(t - \frac{2l}{a}\right)\right\}\right] \quad @ \quad \underline{\underline{x = 2l}}$$

$$- g[t]$$

$$g(\xi) = f\left[\xi - \frac{2l}{a}\left\{1 + M f\left(\xi - \frac{2l}{a}\right)\right\}\right]$$

$$g\left[t + \left(x/a - \frac{2l}{a}\right)\left\{1 + M f(t + x/a)\right\}\right]$$

$$= f\left[t + \left(x/a - \frac{2l}{a}\right)\left\{1 + M f(t + x/a)\right\} - \frac{2l}{a}\left\{1 + M f(t + \cdots\right.\right.$$

$f(t) = \mu(t)$ @ $x = 0$

$$u = f\left[t - \frac{x}{a}\left\{1 + M f(t - x/a)\right\}\right] - f\left[t + \frac{x}{a}\left\{1 + M f(t + x/a)\right\} - \frac{2l}{a}\left\{1 + M f\left(t + \frac{x}{a}\right)\right\}\right]$$

at $x = l$

$$u = f\left[t - \frac{l}{a}\left\{1 + M f\left(t - \frac{l}{a}\right)\right\}\right] - f\left[t + \frac{l}{a}\left\{1 + M f\left(t + \frac{l}{a}\right)\right\} - \frac{2l}{a}\left\{1 + M f\left(t + \frac{l}{a}\right)\right\}\right]$$

$$f\left[\left(t + \frac{l}{a}\right) + \frac{lM}{a} f\left(t + \frac{l}{a}\right) - \frac{2l}{a}\right.$$

$$f\underbrace{\left[t - \frac{l}{a} - \frac{lM}{a} f\left(t - \frac{l}{a}\right)\right]}_{\xi - \frac{lM}{a} f\left(\eta - \frac{2l}{a}\right)} = g\underbrace{\left(t - \frac{l}{a} - \frac{lM}{a} f\left(t + \frac{l}{a}\right)\right\}}_{\xi - \frac{lM}{a} f(\eta)}$$

$$\xi = \eta - \frac{2l}{a}$$

tota fn of $t - \frac{l}{a}$

$$u = f\left[t - \frac{x}{a}\left\{1 + M f\left(t - \frac{x}{a}\right)\right\}\right] - f\left[t + \left(\frac{x}{a} - \frac{2l}{a}\right)\left\{1 + M f\left(t + \frac{x}{a}\right)\right\}\right]$$

@ $x = 0$

$$u = f[t] - f\left[t - \frac{2l}{a}\{1 + M f(t)\}\right] = A \sin 2\pi\omega t$$

$$f\left[t - \frac{2l}{a}\right] - f\left[t - \frac{2l}{a} - \frac{2l}{a}\left\{1 + M f\left(t - \frac{2l}{a}\right)\right\}\right] = A \sin 2\pi\omega\left(t - \frac{2l}{a}\right)$$

$$f\left[t - \frac{4l}{a}\right] - f\left[t - \frac{4l}{a} - \frac{2l}{a}\left\{1 + M f\left(t - \frac{4l}{a}\right)\right\}\right] = A \sin 2\pi\omega\left(t - \frac{4l}{a}\right)$$

$$f(t) - f\left(t - \frac{2l}{a}\left\{1 + M f\left(t + \frac{1}{\omega}\right)\right\} + \frac{1}{\omega}\right) = H(\omega t)$$

$$f\left\{t - \frac{2lM}{a} f\left(t + \frac{1}{\omega}\right)\right\}$$

$$f(t) - \frac{2lM}{a} \underbrace{f\left(t + \frac{1}{\omega}\right)}_{f(t)} f'(t) \cong H(\omega t)$$

$$f(t) + \frac{1}{\omega} f'(t)$$

$$+ \frac{2lM}{a} f f' + \underbrace{M f'^2}_{\text{small wrt } f} \cong H(\omega t)$$

$$f[t] - \left\{f\left[t + \frac{2l}{a}\right] - \frac{2l}{a} M f\, f'\left[t - \frac{2l}{a}\right]\right\} \overset{= A \sin 2\pi\omega t}{f f'} \cong \frac{M}{\omega} H$$

$f\left[t - \frac{2l}{a}\right] = f(t)$

$f'\left[t - \frac{2l}{a}\right] = f'[t]$ ∴

$$f^2 \cong \int \frac{M}{\omega} H \, dt$$

$$A \int \frac{M}{\omega} \frac{H}{A} dt$$

$$f - f + \frac{2l}{a} M f f' = A \sin 2\pi\omega t$$

$$f \cong A^{1/2} \left[\int \frac{M}{\omega} \frac{H}{A} dt\right]^{1/2}$$

$$ff' = \frac{A}{} \sin 2\pi\omega t$$

$$f(t) - f\left(t - \frac{2l}{a} + \frac{n}{\omega}\right) = A \sin(2\pi\omega t)$$
$$f(t) - f\left(t - \frac{2l}{a} + \frac{1}{\omega}\right) = A \sin(2\pi\omega t)$$
$$f(t) - f\left(t - \frac{1}{\omega}(1+\epsilon) + \frac{1}{\omega}\right) = A \sin(2\pi\omega t)$$
$$f(t) - f\left(t - \frac{\epsilon}{\omega}\right) = A \sin(2\pi\omega t)$$

$f(t - \epsilon/\omega) = f(t) - \epsilon/\omega\, f'(t)$ by taylor expansion

$$f(t) - f(t) + \epsilon/\omega\, f'(t) = A \sin(2\pi\omega t)$$
$$f'(t) = A\frac{\omega}{\epsilon} \sin(2\pi\omega t)$$

$$f(t) = A\frac{\omega}{\epsilon}\int^t \sin(2\pi\omega t)\,dt = \frac{A\omega}{\epsilon}\left(\frac{-\cos 2\pi\omega t}{2\pi\omega}\right)\Bigg|_a^t$$
$$= \frac{-A}{2\pi\epsilon}\cos 2\pi\omega\tau\Big|_a^t$$

$$\sin 2\pi\omega t \cos 2\pi\epsilon k - \cos 2\pi\omega t \sin 2\pi\epsilon k$$
$$\sin(2\pi\omega t - 2\pi\epsilon k)$$
$$f(t) = A\sum_{k=0}^{n-1} \sin(2\pi\omega t - 2\pi\epsilon k)$$
$$\sin[2\pi(\omega t - \epsilon k)]$$

$$u = f\left[t - \frac{x}{a}\left(1 + M f(t - x/a)\right)\right] - f\left[t + \left(\frac{x}{a} - \frac{2l}{a}\right)\left\{1 + M f\left(t + \frac{x}{a}\right)\right\}\right]$$

Zero init cond. at $x = 0$ $u = f[t] - \underline{f\left[t - \frac{2l}{a}(1 + Mf(t))\right]}$

at $x = l$ 0 on bc @ $x = 0$

$$0 = f\left[t - l/a\left\{1 + Mf(t - l/a)\right\}\right] - f\left[t + \frac{l}{a}\left\{1 + Mf\left(t + \frac{l}{a}\right)\right\}\right]$$
$$f\left[t - l/a\left\{1 + Mf(t - l/a)\right\}\right] = f\left[t - l/a\left\{1 + Mf(t + l/a\right\}\right]$$
$$f\left[\xi + \frac{lM}{a} f(\xi)\right] = f\left[\xi - \frac{lM}{a} f(\xi + 2l/a)\right]$$

$$u=\mu(t) = f(t) - f(t-2l/a) = A \sin(2\pi\omega t)$$

$$f(t-2l/a) - f(t-4l/a) = A \sin\left(2\pi\omega t - 4\frac{\pi\omega l}{a}\right)$$

$$f(t-4l/a) - f(t-8l/a) = A \sin\left(2\pi\omega t - 8\frac{\pi\omega l}{a}\right)$$

$$\vdots$$

$$f\left(t-\frac{2l(n-1)}{a}\right) - 0 = A \sin\left(2\pi\omega t - \frac{4\pi\omega l}{a}(n-1)\right)$$

$$f(t) - 0 = A \sum_{k=0}^{n-1} \sin\left[2\pi\omega t - \frac{4\pi\omega l\,k}{a}\right]$$

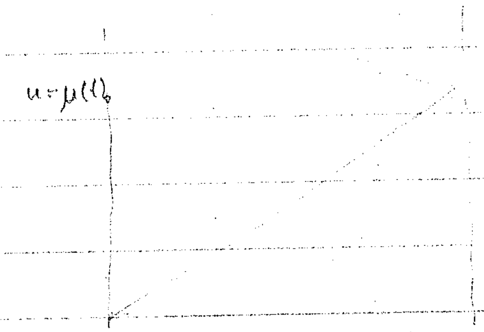
$$f(t) = A \sum_{k=0}^{n-1} \sin\left[2\pi\omega t - \frac{4\pi\omega l}{a}k\right] \qquad \text{if} \quad \frac{2l}{a} = \frac{1}{\omega}(1+\epsilon)$$

$$f(t) = A \sum_{k=0}^{n-1} \sin\left[2\pi\omega t - 2\pi(1+\epsilon)k\right]$$

$$u = f\left[t - x/a\left\{1 + M f(t - x/a)\right\}\right]$$

$$- f\left[t + \left(x/a - \frac{2l}{a}\right)\left\{1 + M f(t + x/a)\right\}\right]$$

Hold



at $x=0$

$$f(t) = A \sum \sin\left[2\pi\omega t - 2\pi k - 2\pi\epsilon k\right]$$

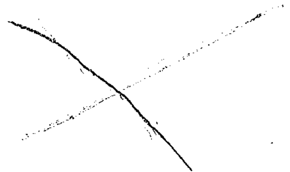
$$\sin\left[2\pi\omega t - 2\pi(1+\epsilon)k\right] = \sin 2\pi\omega t \cos 2\pi(1+\epsilon)k$$
$$- \cos 2\pi\omega t \sin 2\pi(1+\epsilon)k$$

$$\cos(2\pi k + 2\pi\epsilon k) = \cos 2\pi k \cos 2\pi\epsilon k - \sin 2\pi k \sin 2\pi\epsilon k$$

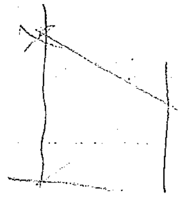
$$\cos 2\pi k = 1 \quad \sin 2\pi k = 0 \quad , \cos(2\pi k + 2\pi\epsilon k) = \cos 2\pi\epsilon k$$

$$\sin(2\pi k + 2\pi\epsilon k) = \sin 2\pi k \cos 2\pi\epsilon k + \sin 2\pi\epsilon k \cos 2\pi k$$

$$\sin(2\pi k + 2\pi\epsilon k) = \sin(2\pi\epsilon k)$$



$$g\left[t-\frac{2l}{a}(1+Mf(t)\right]$$



at $x=0$

$u=f[t]$

$\frac{2l}{a}$

~~$g(l,t)$~~ $u(l,t)=f\left[t-\frac{l}{a}\left\{1+Mf(t-l/a)\right\}\right]$

$$-g\left[t-\frac{l}{a}\left\{1+Mf(t+l/a)\right\}\right]=0$$

$$f\left[t-\frac{l}{a}\left\{1+Mf(t-l/a)\right\}\right]=g\left[t-\frac{l}{a}\left\{1+Mf(t+l/a)\right\}\right]$$

let $z=t-\frac{l}{a}\left\{1+Mf(t+l/a)\right\}$ $\qquad z=\gamma(t)$

$$t-l/a\left\{1+Mf(t-l/a)\right\}=z+\frac{Ml}{a}\left[f(t+l/a)-f(t-l/a)\right]$$

$$g(z)=f\left[z+\frac{Ml}{a}\left\{f\left(t+\frac{l}{a}\right)-f(t-l/a)\right\}\right]$$

$$u=f(t)-f\left[t-\frac{2l}{a}\left\{1+Mf(t)\right\}\right]$$

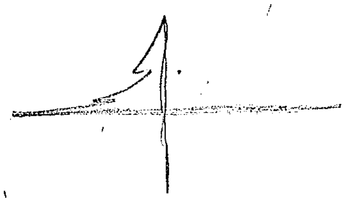
$$f\left(t-\frac{2l}{a}\right)-f\left[t-\frac{2l}{a}-\frac{2l}{a}\left\{1+Mf\left(t-\frac{2l}{a}\right)\right\}\right]=$$

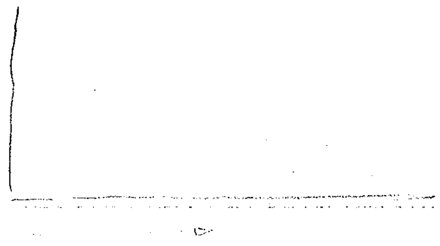
$$f(t)-f\left(t\cancel{+}\frac{2l}{a}\left\{1+Mf(t)\right\}+\frac{1}{\omega}\right)$$

$$-f\left(t\cancel{+}\frac{M}{\omega}f(t)\right)\qquad\qquad f^2=\frac{\omega}{M}\int H$$

$$f(t)+\frac{M}{\omega}f\,f'(t)\simeq H(\omega t)\qquad f=\sqrt{\frac{\omega}{M}\int H}=$$

$$\frac{M}{\omega}ff'\simeq H(\omega t)$$





wave travelling

$$u = f\Big[\underbrace{\left(t - \frac{x}{a}\right)}_{\text{wave to right}} - \underbrace{\frac{Mx}{a} f(t - x/a)}_{\text{wave to left.}}\Big]$$

$$+ \quad f\Big[\underbrace{(t + x/a) + \frac{Mx}{a} f(t + x/a)}_{\text{out}} - \frac{2l}{a}\left\{1 + M f(t + x/a)\right\}\Big]$$

no wave travelling to the left

$$u = f\left[(t - x/a)\right] - f\left[-\frac{2l}{a} - \frac{2l}{a} M f(t + x/a)\right]$$

Q. a) The <u>linear</u> difference equation $f(t) - f(t - \frac{2l}{a}) = \mu(t)$ arises in the study of forced vibrations of a string or gas filled tube of length l when one end is fixed while the other has a periodic displacement $\mu(t)$. Find the <u>explicit</u> solution of this equation when $\mu(t) = A\sin(2\pi\omega t)$ and show this is unbounded when $\omega \to \frac{an}{2l}$, $n = 1, 2, \dots$

b) When <u>nonlinear</u> theory is used u can be represented (after using the boundary condition at the fixed end, $x = l$,) by

$$u(x,t) = f\left(t - \frac{x}{a}\left[1 + Mf\left(t - \frac{x}{a}\right)\right]\right) - f\left(t + \left[\frac{x}{a} - \frac{2l}{a}\right]\left\{1 + Mf\left(t + \frac{x}{a} - \frac{2l}{a}\right)\right\}\right).$$

Show that if $u(0,t) = A\sin(2\pi\omega_0 t)$, where $\omega_0 = \frac{an}{2l}$, $n = 1, 2, \dots$, then the amplitude is now $\propto A^{1/2}$. Assume f has period $1/\omega_0$.

A. a) Let $\frac{a}{2l} = \omega_0$ and $\frac{1}{\omega} - \frac{1}{\omega_0} = \varepsilon$, i.e. $\omega\varepsilon = 1 - \frac{\omega}{\omega_0}$.

Then, since f has period $1/\omega$ (like $\mu(t)$), $f(t) - f(t+\varepsilon) = A\sin(2\pi\omega t)$.

Look for soln. of the form $f(t) = B\cos(2\pi\omega t + \delta)$, then

$$f(t) - f(t+\varepsilon) = -2B\sin(2\pi\omega t + \delta + \pi\omega\varepsilon)\sin(-\pi\omega\varepsilon)\{= A\sin(2\pi\omega t)\}$$

Thus, equating coeffs. $B = -\frac{A}{2\sin(-\pi\omega\varepsilon)}$ and $\delta = -\pi\omega\varepsilon$

Simplifying yields $$\underline{f(t) = \frac{A\cos\left(2\pi\omega t + \pi\frac{\omega}{\omega_0}\right)}{2\sin\left(\frac{\pi\omega}{\omega_0}\right)}}$$

This is unbounded as $\sin\left(\pi\frac{\omega}{\omega_0}\right) \to 0$ i.e. as $\underline{\omega \to n\omega_0}$.

b) Using the b.c. on $x = 0$ we obtain $f(t) - f\left(t - \frac{2l}{a}\left[1 + Mf\left(t - \frac{2l}{a}\right)\right]\right) = A\sin 2\pi\omega_0 t$

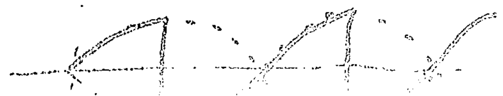
Since f has period $\frac{1}{\omega_0} = \frac{2l}{an}$ this becomes $f(t) - f\left(t - \frac{2l}{a}Mf(t)\right) = A\sin 2\pi\omega_0 t$

Since $|f| \ll 1$, $f(t) - f\left(t - \frac{2lM}{a}f(t)\right) = f(t) - \left[f(t) - 2\frac{Ml}{a}f(t)f'(t) + O(f^3)\right] = A\sin 2\pi\omega_0 t$

Thus $\frac{2lM}{a}f(t)f'(t) = A\sin 2\pi\omega_0 t$, $f^2(t) = \frac{-a}{2\pi l M\omega_0}A\overset{\cos}{\sin}(2\pi\omega_0 t) + K = \frac{-a}{2\pi l M\omega_0}A\cos\left(2\pi\omega_0 t - \frac{\pi}{2}\right) + K$

Choose $K = \frac{-aA}{lM} \implies f^2(t) = \frac{aA}{lM}\left(\cos\left(2\pi\omega_0 t - \pi/2\right) - 1\right) = \frac{aA}{lM}\cos^2\left(\pi\omega_0 t - \pi/4\right)$

$\Rightarrow$ $\underline{f(t) = \pm\sqrt{\frac{aA}{lM}}\cos\left(\pi\omega_0 t - \pi/4\right)}$ i.e. $\underline{f \sim A^{1/2}}$



Continued

P.65 No. 10.

$$\begin{rcases} ① \quad i_x + CV_t + GV = 0 \\ ② \quad V_x + L i_t + Ri = 0 \end{rcases} \quad \begin{array}{l} v \text{ and } i \text{ satisfy} \\ y_{xx} = CLy_{tt} + (CR+GL)y_t + GRy \end{array}$$

Let $y = e^{\lambda t} q(x,t)$. Then choosing $\lambda = -\frac{G}{C}$ and using the given relation $GL = CR$ yields : $\underline{a^2 q_{xx} = q_{tt}}$ where $a^2 = \frac{1}{CL}$

Thus $v(x,t) = e^{-\frac{R}{L}t}\{v_1(x-at)+v_2(x+at)\}$, $i(x,t) = e^{-\frac{R}{L}t}\{i_1(x-at)+i_2(x+at)\}$

Let $v(x,0) = f(x)$, $i(x,0) = \sqrt{\frac{C}{L}}\,F(x)$.

Thus $v_1(x)+v_2(x) = f(x)$, $i_1(x)+i_2(x) = \sqrt{\frac{C}{L}}\,F(x)$ ③

① $\Rightarrow$ (at $t=0$) $i_1'(x)+i_2'(x) - aC(v_1'(x)-V_2'(x)) + \cancel{C\left(-\frac{G}{C}\right)(v_1(x)+V_2(x)) + G(v_1+v_2)} = 0$

i.e $i_1'+i_2' = aC(V_1'-V_2') = \sqrt{\frac{C}{L}}\,F'(x)$

(by ③)

Similarly $V_1'+V_2' = aL(i_1'-i_2') = f'(x)$

Thus we have four equns. for i_1', i_2', v_1' & V_2', yielding (using relations on G, C, a, L, R)

$i_1' = \sqrt{\frac{C}{L}}\frac{1}{2}(f'+F')$, $i_2' = \frac{1}{2}\sqrt{\frac{C}{L}}(F'-f')$, $v_1' = \frac{1}{2}(f'+F')$, $V_2' = \frac{1}{2}(f'-F')$

i.e $i_1 = \frac{1}{2}\sqrt{\frac{C}{L}}(f+F)+K_1$, $i_2 = \frac{1}{2}\sqrt{\frac{C}{L}}(F-f)-K_1$, $V_1 = \frac{1}{2}(f+F)+K_2$, $V_2 = \frac{1}{2}(f-F)-K_2$

where K_1 & K_2 are constants. Thus in equns for v and i

$$v = e^{-\frac{R}{L}t}(\phi(x-at)+\psi(x+at))\,;\ i = e^{-\frac{R}{L}t}\sqrt{\frac{C}{L}}(\phi(x-at)-\psi(x+at))$$

where $\phi = \frac{f+F}{2}$ and $\psi = \frac{f-F}{2}$

and constants K_1 & K_2 cancel in final soln.

rec'd 11/15/72

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⑫.

Given $f(t) - f(t - \frac{2l}{a}) = A \sin 2\pi\omega t$ @ $x=0$

since $A \sin 2\pi\omega t$ is being added at each period then

$$f(t) - f(t - 2l/a) = A \sin 2\pi\omega t$$
$$f(t-\frac{2l}{a}) - f(t-\frac{4l}{a}) = A \sin 2\pi\omega (t - \frac{2l}{a})$$
$$f(t-\frac{4l}{a}) - f(t-\frac{6l}{a}) = A \sin 2\pi\omega (t - \frac{4l}{a})$$
$$\vdots$$
$$f(t - \frac{2l}{a}[n-1]) - 0 = A \sin 2\pi\omega (t - \frac{2l}{a}[n-1])$$

Letting $\frac{2l(n-1)}{a} \le t \le \frac{2l}{a} n$; also since $g(t) = 0$ @ $x=0$

$$f(t - \frac{2ln}{a}) = 0$$

Adding all these differences:

$$f(t) = \sum_{k=0}^{n-1} A \sin 2\pi\omega (t - \frac{2lk}{a})$$

if $\frac{2l}{a} = (1+\epsilon)\frac{1}{\omega}$ then $f(t) = \sum_{k=0}^{n-1} A \sin 2\pi[\omega t - (1+\epsilon)k]$

but $\sin[2\pi\xi \pm 2\pi k] = \sin 2\pi\xi$ by periodicity since integer k

∴

$f(t) = \sum_{k=0}^{n-1} A \sin 2\pi(\omega t - \epsilon k)$ Accordingly, f(t) is bded by n|A|, instead of and unbded! ①

Given $u = f[t - \frac{x}{a}\{1 + Mf(t - \frac{x}{a})\}] - f[t + (\frac{x}{a} - \frac{2l}{a})\{1 + Mf(t + \frac{x}{a})\}]$

at $x = l$ $u = f[t - \frac{l}{a}\{1 + Mf(t - \frac{l}{a})\}] - f[t - \frac{l}{a}\{1 + Mf(t + \frac{l}{a})\}]$

by periodicity $f(\xi) = f(\xi + \frac{2l}{a})$

∴ $f(t - l/a) = f(t + l/a)$ ∴ $u(l,t) = 0$

at $x = 0$ $u = f[t] - f[t - \frac{2l}{a}\{1 + M\underline{f(t)}\}] = A \sin 2\pi\omega t$ ✓

$$f[t] - f\left[t - \frac{1}{\omega} - \frac{M}{\omega} f(t)\right] = A \sin 2\pi\omega t$$

Expand $f\left[t - \frac{1}{\omega} - \frac{M}{\omega} f(t)\right]$ about $f\left[t - \frac{1}{\omega}\right]$

$$\therefore \quad f\left[t - \frac{1}{\omega} - \frac{M}{\omega} f(t)\right] = f\left[t - \frac{1}{\omega}\right] - \frac{M}{\omega} f(t) f'\left(t - \frac{1}{\omega}\right) + \dots \text{ H.O.T.}$$

$$\therefore \quad A \sin 2\pi\omega t = f[t] - f[t - 1/\omega] + \frac{M}{\omega} f(t) f'(t - 1/\omega) + \dots \text{ HOT}$$

but $f\left[t - \frac{1}{\omega}\right] = f[t]$ by periodicity

$$\therefore \quad A \sin 2\pi\omega t = \frac{M}{\omega} f\left(t - \frac{1}{\omega}\right) f'\left(t - \frac{1}{\omega}\right) + \dots \text{ H.O.T. in } \left(t - \frac{1}{\omega}\right)$$

since $|f(t)| << 1$ then higher order terms can be neglected

$$\therefore \quad A \sin 2\pi\omega t \approx \frac{M}{\omega} f\left(t - \frac{1}{\omega}\right) f'\left(t - \frac{1}{\omega}\right)$$

but $f(\xi) f'(\xi) = \frac{1}{2}\left[\overline{f(\xi)}^2\right]'$ prime denotes differentiation wrt argument

$$\therefore \quad \int^t \frac{2\omega}{M} A \sin 2\pi\omega \tau \, d\tau \approx \overline{f\left[t - \frac{1}{\omega}\right]}^2$$

$$\text{or} \quad f\left(t - \frac{1}{\omega}\right) \approx \left[\frac{2\omega}{M} A \int^t \sin 2\pi\omega\tau \, d\tau\right]^{1/2} = A^{1/2} k$$

where $k = \frac{2\omega}{M} T \qquad T = \int^t \sin 2\pi\omega\tau \, d\tau$; because T is a cosine fn

T is bounded ; since ω, M, T are bounded, k is bounded.

$\therefore \; f(t - 1/\omega) \propto A^{1/2}$ & by periodicity $f(t) \propto A^{1/2}$

$$f \text{ as a fn of its argument} = \left[\frac{2\omega}{M} A \int^{\text{fn of its argument}} \sin 2\pi\omega\tau \, d\tau\right]^{1/2}$$ ✓ ④

10. Given $GL = RC$

⑦ the diff equations for electrical vibrations is

$$i_x = -Cv_t + Gv$$

$$v_{xx} = CL\,v_{tt} + (CR + GL)v_t + GRv$$

Since $GL - RC = 0$ the eqns become

$$v_{xx} = CL\,v_{tt} + GRv + 2GL\,v_t$$

with conditions $v(x,0) = \varphi(x)$ $\quad i(x,0) = \psi(x)$

The equations are of the form

$$a_{11}u_{xx} + 2a_{12}u_{xt} + a_{22}u_{tt} + b_1u_x + b_2u_t + cu = 0$$

$a_{11} = 1 \quad a_{12} = 0 \quad a_{22} = -CL \quad b_1 = 0 \quad b_2 = -2GL \quad c = -GR$

$$a = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{22}} = \frac{0 \pm \sqrt{CL}}{CL} = \pm\frac{1}{\sqrt{CL}}$$

$$\frac{dx}{dt} = \pm\frac{1}{\sqrt{CL}} \quad \therefore \quad x = \frac{1}{\sqrt{CL}}t + c_1 \qquad x = \frac{-1}{\sqrt{CL}}t + c_2$$

$$\xi = x - \frac{1}{\sqrt{CL}}t \qquad \eta = x + \frac{1}{\sqrt{CL}}t$$

now $\xi_x = 1$, $\xi_t = \frac{-1}{\sqrt{CL}}$, $\eta_x = 1$, $\eta_t = \frac{1}{\sqrt{CL}}$

$$v_t = \frac{1}{\sqrt{CL}}\left[v_\eta - v_\xi\right]$$

$$v_{tt} = \left[v_{\xi\xi} - 2v_{\xi\eta} + v_{\eta\eta}\right]\frac{1}{CL}$$

$$v_{xx} = v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}$$

Putting into diff equation then

$$4v_{\xi\eta} - GRv + \frac{2GL}{\sqrt{CL}}\left(v_\xi - v_\eta\right) = 0$$

rewrite $v = e^{\lambda\xi + \mu\eta} V$ so that the diff eqns become

$$4V_{\xi\eta} + V_\eta\left[4\lambda - \frac{2GL}{\sqrt{CL}}\right] + V_\xi\left[4\mu + \frac{2GL}{\sqrt{CL}}\right] + V\left[4\lambda\mu - RG + \frac{\lambda 2GL}{\sqrt{CL}} - \frac{\mu 2GL}{\sqrt{CL}}\right] = 0$$

Let $\lambda = \frac{GL}{2\sqrt{CL}}$ $\mu = \frac{-GL}{2\sqrt{CL}}$

$$4\lambda\mu - RG + \frac{\lambda 2GL}{\sqrt{CL}} - \mu\frac{2GL}{\sqrt{CL}} \equiv 0$$

$\therefore$ $V_{\xi\eta} = 0$ and $V = f(\xi) + g(\eta)$

or

$$v = e^{\lambda\xi + \mu\eta}\left[f(\xi) + g(\eta)\right]$$

$$\lambda\xi + \mu\eta = -G/C\, t = -\frac{Rt}{L}$$

$$\therefore \quad v(x,t) = e^{-\frac{Rt}{L}}\left[f\left(x - \frac{t}{\sqrt{LC}}\right) + g\left(x + \frac{t}{\sqrt{LC}}\right)\right] \checkmark$$

$$v_t(x,t) = \frac{e^{-Rt/L}}{\sqrt{LC}}\left[-f'\left(x - t/\sqrt{LC}\right) + g'\left(x + \frac{t}{\sqrt{CL}}\right)\right] - \frac{R}{L}e^{-\frac{Rt}{L}}\left[f\left(x - \frac{t}{\sqrt{LC}}\right) + g\left(x + \frac{t}{\sqrt{LC}}\right)\right]$$

$$i_x = -Cv_t - Gv = -e^{-\frac{Rt}{L}}\Big[\left(G - \frac{CR}{L}\right)\{(f\left(x - \frac{t}{\sqrt{LC}}\right) + g\left(x + \frac{t}{\sqrt{LC}}\right)\} - \frac{C}{\sqrt{LC}}\{f'\left(x - \frac{t}{\sqrt{LC}}\right) - g'(x + \frac{t}{\sqrt{LC}}$$

note that $G = \frac{CR}{L}$ $\therefore$

$$i_x = \sqrt{\frac{C}{L}}\, e^{-\frac{Rt}{L}}\left\{f'\left(x - \frac{t}{\sqrt{LC}}\right) - g'\left(x + \frac{t}{\sqrt{LC}}\right)\right\}$$

$$\therefore \quad i = \sqrt{\frac{C}{L}}\, e^{-\frac{Rt}{L}}\left\{f\left(x - \frac{t}{\sqrt{LC}}\right) - g\left(x + \frac{t}{\sqrt{LC}}\right)\right\} \checkmark$$

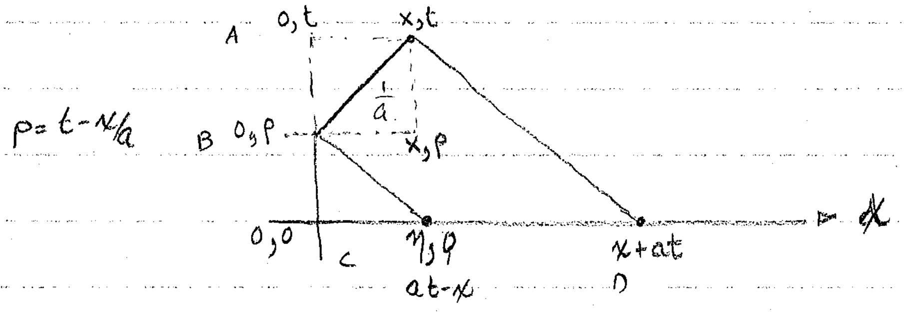
Consider $v(x,0) = f(x) + g(x)$ $\quad i(x,0) = \sqrt{\frac{C}{L}}\left[f(x) - g(x)\right]$

$\varphi(x) = f(x) + g(x)$ $\quad \psi(x) = \sqrt{\frac{C}{L}}\left[f(x) - g(x)\right]$

$f(x) = \dfrac{\sqrt{C/L}\,\varphi(x) + \psi(x)}{2\sqrt{C/L}}$ $\quad g(x) = \dfrac{\sqrt{C/L}\,\varphi(x) - \psi(x)}{2\sqrt{C/L}}$ ✓

If you used the IC prescribed in the text, you would obtain the exact answer.

① $\int_c (u_t dx + a^2 u_x dt) + \iint_G f dx dt = 0$



$\frac{t-\rho}{x} = \frac{1}{a}$ $\quad \frac{x}{a} = t - \rho \quad \rho = t - x/a$

$\frac{0-\rho}{\eta - 0} = -\frac{1}{a} \quad + \frac{\eta}{a} = +\rho \quad \eta = a\rho = at - x$

$u(x,0) = \varphi(x) \quad u_t(x,0) = \Psi(x) \quad u_x(0,t) = h[u(0,t) - \theta(t)]$

$k \cancel{p} u_x = T_0$

$\int_A^B + \int_B^C + \int_C^D + \int_D^A + \iint_G f = 0$

$dx = a\,dt \quad \therefore \quad a\int_A^B du = \{u[B] - u[A]\}a$

$\int_B^C - a\,du = -a\{u[C] - U[B]\}$

$\int_C^D \frac{\partial u}{\partial t}\Big|_{t=0} dx \quad + \int_D^A - a\,du = -a[U(A) - U(D)]$

$-U[A] + U[B] - \frac{U[C]}{2} \overset{+U(D)}{+} \frac{1}{2a}\int_C^D u_t\Big|_{t=0} dx + \frac{\iint_{ABCD} f dx dt}{2a} = 0$

$U[A] = U[B] - \frac{U[C]}{2} \overset{+U(D)/2}{+} \frac{1}{2a}\int_{at-x}^{x+at} u_t\Big|_{t=0} dx + \frac{1}{2a}\iint f dx dt$

$U(x,t) = U(0, t - \frac{x}{a}) + \frac{U(x+at, 0) - U(\overset{at-x}{\cancel{x-at}}, 0)}{2} + \frac{1}{2a}\int_{at-x}^{x+at} u_t\Big|_{t=0} dx$
$+ \frac{1}{2a}\iint f dx dt$

$$u[0, t - x/a] = \frac{1}{h} u_x[0, t - x/a] + \theta[t - x/a]$$

denote $u_x(0, t - x/a) = \nu(t - x/a)$

$\therefore$

$$u(x,t) = \left[\frac{1}{h} \nu(t - x/a) + \theta(t - x/a)\right] + \frac{\varphi(x+at) - \varphi(at-x)}{2}$$

$$+ \frac{1}{2a}\int_{at-x}^{x+at} \psi(s)\,ds + \frac{1}{2a}\int_0^t d\tau \int_{|x-a(t-\tau)|}^{x+a(t-\tau)} f(\xi, \tau)\,d\xi$$

P66 11. By the method in the text

$$u(x,t) = \mu\left(t - \frac{x}{a}\right) + \frac{1}{2}\left[\phi(x+at) - \phi(at-x)\right] + \frac{1}{2a}\int_{at-x}^{at+x} \psi(s)\,ds \quad , t > \frac{x}{a} \quad (1)$$

Must find $\mu(t)$ using b.c. : $u_x(0,t) = h\,(u(0,t) - \theta(t))$, $t > 0$ (2)

Putting (1) in b.c.(2) yields

$$\mu'(t) + ah\,\mu(t) = \psi(at) + a\,\phi'(at) + ah\,\theta(t) \quad (3)$$

Soln. of (3) is $\mu(t) = e^{-aht}\int_0^t e^{ahs}\left[\psi(as) + a\,\phi'(as) + ah\,\theta(s)\right]ds + C e^{-aht}$

Using in. cond. (continuity of $u(x,t)$ at $(0,0)$) , $C = \phi(0)$. Then integrating by parts yields $\mu(t) = \phi(at) + \int_0^t e^{ah(s-t)}\left[\psi(as) + ah\{\theta(s) - \phi(as)\}\right]ds$

so that

$$u(x,t) = \frac{1}{2}\left[\phi(x+at) + \phi(at-x)\right] + \frac{1}{2a}\int_{at-x}^{at+x}\psi(s)\,ds + \int_0^t e^{ah(s-t)}\left[\psi(as) + ah\{\theta(s) - \phi(as)\}\right]ds$$

P2. a) $\psi_x(-L,t) = \psi_x(L,t) = 0$

b) $\psi(-L,t) = \psi_x(L,t) = 0$

c) $\psi_x(-L,t) = \psi(L,t) = 0$

[Check this carefully yourself].

$\psi = z(t)\,y(x)$, $y'' + k_n^2 y = 0$

$z'' + (k_n a)^2 z = 0$

all cases $z(0) = 1$, $z'(0) = 0 \Rightarrow \underline{z = \cos(k_n a t)}$

Use b.c. to find y which is always of the form $y = A_n \sin k_n x + B_n \cos k_n x$.

For a) b.c. $\Rightarrow$ for non-trivial soln. $k_n = \frac{n\pi}{2L}$, $n = 1, 2, \ldots$. n odd $B = 0$, n even $A = 0$

Transforming to interval $(0,\ell)$ yields $\underline{\psi = C_n \cos\left(\frac{n\pi x}{\ell}\right)\cos(\omega_n t)}$, $\omega_n = k_n a = \frac{n\pi a}{\ell}$.

b) b.c. on $y \Rightarrow$ for non-trivial soln. $k_n = \left(\frac{2n+1}{4L}\right)\pi$, $n = 0, 1, 2$. Then $B_n = A_n \tan(k_n L)$

and $y(x) = C_n \sin k_n(x+L)$. Using transf. to $(0,\ell)$, $y(x) = C_n \sin(k_n x)$

and $\underline{\psi = C_n \sin(k_n x)\cos(\omega_n t)}$ $n = 0, 1, 2\ldots$

c) b.c. on $y \Rightarrow$ for non-trivial soln. $k_n = \left(\frac{2n+1}{4L}\right)\pi$ $n = 0, 1, 2\ldots$

Then $B_n = -A_n \tan(k_n L) \Rightarrow y = C_n \sin(k_n(x - L))$. Using transf. to $(0,\ell)$

$y(x) = C_n \sin\left(k_n x + [2n+1]\frac{\pi}{2}\right)$ $n = 0, 1, 2\ldots$. Thus $y = \pm C_n \cos k_n x$

and $\psi = \pm C_n \cos(k_n x)\cos(\omega_n t)$

(11)

Cesar Levy T63.2130

1. $\Psi(x,t) = y(x)\, z(t)$

leads to $\frac{1}{a^2}\frac{z''}{z} = \frac{y''}{y} = \lambda = k^2$

or $y(x) = A \sin kx + B \cos kx$

$z(t) = C \sin kat + D \cos kat$

a) $\Psi_{,x}(-L,t) = \Psi_{,x}(L,t) = 0 \quad , \quad \Psi(x,0) = 1 \,, \quad \Psi_{,t}(x,0) = 0$

or $y'(-L) = y'(L) = 0$ and $z(0) = 1 \,,\ z'(0) = 0$

this leads to

$$\begin{vmatrix} k \cos kL & k \sin kL \\ k \cos kL & -k \sin kL \end{vmatrix} = -\left[2k \sin kL \cos kL\right] = -\sin 2kL = 0 \Rightarrow k_n = \frac{n\pi}{2L} \checkmark$$

$z(0) = D = 1 \quad , \quad z'(0) = kaC = 0 \Rightarrow C = 0$

$$\boxed{z(t) = \cos \frac{n\pi a}{2L} t} \checkmark$$

$$\Psi(x,t) = \sum_{n=1}^{\infty} \left\{A_n \sin \frac{m\pi}{2L} x + B_n \cos \frac{n\pi x}{2L}\right\} \cos \frac{n\pi a}{2L} t \checkmark$$

if $n = 2m$ & $m = 0, 1, 2, \ldots$ then

$$\Psi(x,t) = \sum_{m=0}^{\infty} \left\{B_m \cos \frac{m\pi x}{L} \cos \frac{m\pi a t}{L}\right\}$$

if $n = 2m+1$ & $m = 0, 1, 2, \ldots$

$$\Psi(x,t) = \sum_{m=0}^{\infty} \left\{A_n \sin \frac{(2m+1)\pi x}{2L} \cos \frac{(2m+1)\pi a t}{2L}\right\} \checkmark$$

(3)

b) $\Psi(-L,t) = \Psi_{,x}(L,t) = 0 \quad , \quad \Psi(x,0) = 1 \,, \quad \Psi_{,t}(x,0) = 0$

or $y(-L) = y'(L) = 0$ and $z(0) = 1 \,,\ z'(0) = 0$

this leads to

$$\begin{vmatrix} -\sin kL & \cos kL \\ k \cos kL & -k \sin kL \end{vmatrix} = k\left[\sin^2 kL - \cos^2 kL\right] = -k \cos 2kL = 0 \Rightarrow k_m = \frac{(2m+1)\pi}{4L}$$

$\checkmark$

$$\Psi(x,t) = \sum_{m=0,1,2,\ldots} \left\{ A_m \sin\frac{(2m+1)\pi x}{4L} + B_m \cos\frac{(2m+1)\pi x}{4L} \right\} \cos\frac{(2m+1)\pi a t}{4L}$$ ✓

since $\Psi(x,t) = 0$ @ $x = -L$ & $\Psi_{,x}(x,t) = 0$ @ $x = L$ yields $B_m = (-1)^m A_m$

$$\therefore \quad \Psi(x,t) = \sum A_m \cos\frac{(2m+1)\pi a t}{4L} \left\{ \sin\frac{(2m+1)\pi x}{4L} + (-1)^m \cos\frac{(2m+1)\pi x}{4L} \right\}$$ ✓ ③

c) $\Psi_{,x}(-L,t) = \Psi(L,t) = 0$, $\Psi(x,0) = 1$, $\Psi_{,t}(x,0) = 0$

or $y'(-L) = y(L) = 0$ and $z(0) = 1$, $z'(0) = 0$

this leads to

$$\begin{vmatrix} k\cos kL & +k\sin kL \\ \sin kL & \cos kL \end{vmatrix} = k\left[\cos^2 kL - \sin^2 kL\right] = +k\cos 2kL = 0 \Rightarrow k_m = \frac{(2m+1)\pi}{4L}$$ ✓

$$\Psi(x,t) = \sum_{m=0,1,2,\ldots} \left\{ A_m \sin\frac{(2m+1)\pi x}{4L} + B_m \cos\left(\frac{(2m+1)\pi x}{4L}\right) \right\} \cos\frac{(2m+1)\pi a t}{4L}$$ ✓

since $\Psi(x,t) = 0$ @ $x = L$ and $\Psi_{,x}(x,t) = 0$ @ $x = -L$ yields $B_m = (-1)^{m+1} A_m$

$$\therefore \quad \Psi(x,t) = \sum A_m \cos\frac{(2m+1)\pi a t}{4L} \left\{ \sin\frac{(2m+1)\pi x}{4L} + (-1)^{m+1} \frac{\cos(2m+1)\pi x}{4L} \right\}$$ ✓ ③

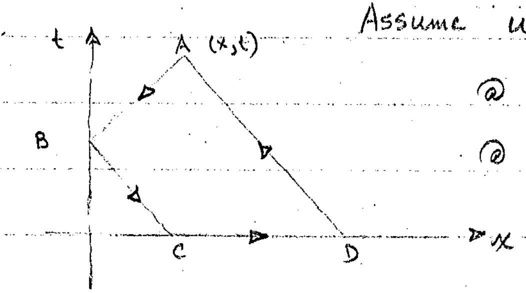
11. Find soln of integral wave eqn. ②

$$\int_{\bar{C}} (u_t\,dx + a^2 u_x\,dt) + \iint_{G} f\,dx\,dt = 0$$

(homogeneous string $f = \frac{F}{\rho}$) $\ni.$ $u(x,0) = \varphi(x)$ $u_t(x,0) = \psi(x)$

and $u(0,t) = \frac{1}{h}[u_x(0,t)] + \theta(t) = \frac{1}{h}\nu(t) + \theta(t)$

Assume $u(x,t)$ is of the form below



@ $B(0,\rho)$ $\frac{1}{a} = \frac{t-\rho}{x} \Rightarrow \rho = t - x/a$

@ $C(\eta,0)$ $-\frac{1}{a} = \frac{-\rho}{\eta} \Rightarrow \eta = at - x$

$$\int_{\bar{C}} = \int_A^B + \int_B^C + \int_C^D + \int_D^A$$

$\int_A^B (u_t\,dx + a^2 u_x\,dt)$ using $dx = a\,dt \Rightarrow a\int_A^B dU = a[U(\overset{\times}{B}) - U(\overset{\times}{A})]$

$\int_B^C (u_t\,dx + a^2 u_x\,dt)$ using $dx + a\,dt = 0 \Rightarrow -a\int_B^C du = -a[U(\overset{\times}{C}) - U(\overset{\times}{B})]$

$\int_C^D (u_t\,dx + a^2 u_x\,dt)$ using $dt = 0 \Rightarrow \int_C u_t\,dx = \int_{at-x}^{x+at} \psi(\xi)\,d\xi$

$\int_D^A (u_t\,dx + a^2 u_x\,dt)$ using $dx + a\,dt = 0 \Rightarrow -a\int_D^A du = -a[U(\overset{\times}{A}) - U(\overset{\times}{D})]$

$\therefore$ the integral equation implies

$$-2aU(A) + 2aU(B) + aU(D) - aU(C) + \int_{at-x}^{x+at} \psi(\xi)\,d\xi + \iint_{ABCD} f\,dx\,dt = 0$$

or $u(x,t) = u(0, t - x/a) + \frac{u(x+at,0) - u(at-x,0)}{2} + \frac{1}{2a}\int_{at-x}^{x+at} \psi(\xi)\,d\xi$

$$\frac{1}{2a}\int_0^t d\tau \int_{|x-a(t-\tau)|}^{x+a(t-\tau)} f(\xi,\tau)\,d\xi$$

✓

using initial condition & b.c.

$$u(x,t) = \frac{1}{h}\nu(t) + \theta(t) + \frac{\varphi(x+at) - \varphi(at-x)}{2} + \frac{1}{2a}\int_{at-x}^{x+at}\psi(\xi)d\xi + \frac{1}{2a}\int_0^t d\tau\int_{|x-a(t-\tau)|}^{x+a(t-\tau)} f(\xi,\tau)d\xi$$

where $h = \frac{\alpha}{T_0}$ T_0 is constant tension α characterizes rigidity of constraint

For $t < \frac{x}{a}$, you would have (2-2.24)

$t > \frac{x}{a}$, you have (2-2.25) ;

using prescribed BC and ICs, you can determine $\mu(t)$.

where $\mu(t) = Ce^{-aht} + e^{-aht}\int e^{aht}[\psi(at) + a\phi' + ah\theta(t)]\,dt$.

ES 2130 Problems # 9 Due 11/30/72

1. Find the Fourier expansions for the functions which are periodic with period 2π and which in $-\pi < x < \pi$ are defined by

a) e^{ax} b) $(x^2-\pi^2)^2$ c) $\sin ax\,(1+\cos x)$

d) $f(x) = \begin{cases} 1 & a \le x \le b \\ 0 & -\pi < x < a,\ b < x < \pi \end{cases}$ $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos$

---

2. The polynomials $B_n(t)$ (Bernoulli polynomials) are defined by

a) $B_1(t) = t - \frac{1}{2}$

b) $B_n'(t) = n B_{n-1}(t)$ $B' = 2(t - 1/2)$ $t^2 - t + 1/6$

$B = t^2 - t + c$

c) $\int_0^1 B_n(t)\,dt = 0$ $t^3/3 - t^2/2 + ct \Big|_0^1 =$

$1/3 - 1/2 + c = 0$ $c = 1$

Find $B_2(t)$, $B_3(t)$ and $B_4(t)$ [Note: $B_n(0)$ are rationals called Bernoulli nos.]

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3. Show that

$$B_1(t) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\pi t}{n}, \quad B_2(t) = \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos 2n\pi t}{n^2}$$

$B_2(t) = 2(t - 1/2) = 2(\frac{t^2}{2} - \frac{t}{2}$

take $= t^2 - t + 1/6$

$B_2(0) = 1/6$

---

4. Prove that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

$u = \sin ax \qquad dv = \cos x \, dx$

$du = a\cos ax \qquad v = \sin x$

$$\int_{-\pi}^{\pi} \sin ax + \sin ax \cos x \; dx$$

$$-\frac{\cos ax}{a}\Big|_{-\pi}^{\pi} = \frac{1}{a}\left[-\cos a\pi + \cos a\pi\right]$$

$\sin x \sin ax - a\int \cos ax \sin x$

$\cos ax \qquad \sin x$

$-a\sin ax \qquad -\cos x$

$$a\cos x \cos ax\Big|_{-\pi}^{\pi} - a\left[-a\int \sin ax \cos x \, dx\right]$$

$a\cos\pi\cos a\pi - a\cos\pi\cos a\pi$

$\int = a^2 \int$

$\cos(a+m)x = \cos a \cos m - \sin a \sin m$

$\cos(a-m)x = \cos a \cos m + \sin a \sin m$

$$\frac{1}{2(a^2-m^2)}\left\{(a+m)\sin(a-m)x - (a-m)\sin(a+m)x\right\}$$

$$(\overset{\checkmark}{a}+\overset{\checkmark}{m})\left[\sin a \overset{\checkmark}{\cos} m - \cos \overset{\checkmark}{a} \sin m\right] + (\overset{\checkmark}{-a}+\overset{\checkmark}{m})\left[\sin a \overset{\checkmark}{\cos} m + \cos \overset{\checkmark}{a} \sin m\right]$$

$= -a\cos a \sin m + m\sin a \cos m =$

$\cos(2m+1)x = \cos 2m \cos - \sin 2x \sin$

$\cos(2m-1)x \quad \cos 2m \cos + \sin 2x \sin$

$$\left(\frac{1-\cos 2mx}{2}\right)\cos x$$

$$\frac{1}{2}\cos x - \frac{1}{4}\left[\cos(2m+1)x + \cos(2m-1)x\right]$$

$\frac{1}{a}e^{ax}\sin mx - \frac{m}{a}\int e^{ax}\cos mx$

$-\frac{m}{a}\left[\frac{1}{a}e^{ax}\cos mx + \frac{m}{a}\int e^{ax}\sin\right]$

$\int e^{ax}\cos mx\,dx$ $\qquad \int u\,dv = uv - \int v\,du$

let $e^{ax}\,dx = dv \qquad u = \cos mx$

$\frac{1}{a}e^{ax} = v \qquad du = -m\sin mx\,dx$

$$\frac{1}{a}e^{ax}\cos mx + \frac{m}{a}\int e^{ax}\sin mx\,dx$$

$\int e^{ax}\sin mx\,dx$

let $e^{ax}\,dx = dv \qquad u = \sin mx$

$\frac{1}{a}e^{ax} = v \qquad du = m\cos mx\,dx$

$$\frac{1}{a}e^{ax}\sin mx - \frac{m}{a}\int e^{ax}\cos mx\,dx$$

$$\frac{a}{a^2}e^{ax}\cos mx + \frac{m}{a^2}e^{ax}\sin mx - \frac{m^2}{a^2}\int e^{ax}\cos mx\,dx = \int e^{ax}\cos mx\,dx$$

$$\frac{me^{ax}\sin mx + ae^{ax}\cos mx}{a^2} = \frac{a^2+m^2}{a^2}\int$$

$$\left[\frac{me^{ax}\overset{0}{\cancel{\sin}}\,mx + ae^{ax}\cos mx}{a^2+m^2}\right]_{-\pi}^{\pi} = \int$$

ok $\quad \frac{a}{a^2+m^2}\left[e^{a\pi}(-1)^m - e^{-a\pi}(-1)^m\right] = \frac{2a(-1)^m}{a^2+m^2}\left[\sinh a\pi\right]$

$$\int e^{ax}\sin mx\,dx = \frac{1}{a}e^{ax}\sin mx - \frac{m}{a}\int e^{ax}\cos mx\,dx$$

$$\int e^{ax}\cos mx\,dx = \frac{1}{a}e^{ax}\cos mx + \frac{m}{a}\int e^{ax}\sin mx\,dx$$

$$\frac{a}{a^2}e^{ax}\sin mx - \frac{m}{a^2}e^{ax}\cos mx - \frac{m^2}{a^2}\int$$

$$\left[\frac{ae^{ax}\overset{0}{\cancel{\sin mx}} - me^{ax}\cos mx}{a^2+m^2}\right] \qquad -\frac{m}{a^2+m^2}\left[e^{a\pi}(-1)^m - e^{-a\pi}(-1)^m\right]$$

$$\frac{-2m\,(-1)^m}{a^2+m^2}\left[\sinh a\pi\right] = \frac{2m\,(-1)^{m+1}}{a^2+m^2}\sinh a\pi$$

$$\frac{a_0}{2} = \frac{\sinh a\pi}{a\pi} \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\,dx = \frac{2}{a\pi}\sinh(a\pi)$$

$$a_m = \frac{2}{a\pi}\,\frac{a^2}{a^2+m^2}\,(-1)^m \sinh a\pi$$

$$b_m = \frac{2}{a\pi}\,\frac{am\,(-1)^{m+1}}{a^2+m^2}\sinh a\pi$$

$$e^{ax} = \frac{\sinh a\pi}{a\pi}\left[1 + 2\sum_{m=1}^{\infty}\frac{a^2(-1)^m}{a^2+m^2}\cos mx + 2\sum_{m=1}^{\infty}\frac{am(-1)^{m+1}}{a^2+m^2}\sin mx\right]$$

$$= \frac{\sinh a\pi}{a\pi}\left[1 + 2a\sum_{m=1}^{\infty}\left\{\frac{(-1)^m}{a^2+m^2}\left[a\cos mx - m\sin mx\right]\right\}\right]$$

$\cdot\cdot\ (x^2-\pi^2)^2$ is an even fn since $f(x) = f(-x)$

$\therefore\ b_m$ is zero $\forall m$

$\frac{3}{15} - \frac{10}{15} + \frac{15}{15}$

then $a_0 = \frac{2}{\pi}\int_0^\pi (x^2-\pi^2)^2 dx = \frac{2}{\pi}\int_0^\pi (x^4-2x^2\pi^2+\pi^4)dx = \frac{2}{\pi}\left[\frac{\pi^5}{5}-\frac{2}{3}\pi^5+\pi^5\right]$

$= \frac{2}{\pi}\left[\frac{8}{15}\pi^5\right] = \frac{16}{15}\pi^4$

$a_m = \frac{2}{\pi}\int_0^\pi (x^4-2x^2\pi^2+\pi^4)\cos mx\,dx$

$\int_0^\pi \pi^4\cos mx\,dx = \frac{\pi^4}{m}\sin mx\Big|_0^\pi = 0$

$x^2 \qquad \cos mx\,dx$

$2x\,dx \qquad \frac{\sin mx}{m}$

$\int_0^\pi 2x^2\pi^2\cos mx\,dx = 2\pi^2\left[\cancelto{0}{\frac{x^2}{m}\sin mx} - \frac{2}{m}\int x\sin mx\,dx\right] = -\frac{4\pi^2}{m}\int x\sin mx\,dx$

$x \qquad \sin mx\,dx$

$1 \qquad -\frac{\cos mx}{m}$

$\int x\sin mx\,dx = \qquad -\frac{x}{m}\cos mx + \cancelto{0}{\int\frac{\cos mx}{m}dx}$

$-\frac{4\pi^2}{m}\int x\sin mx\,dx = +\frac{4x\pi^2}{m^2}\cos mx = \frac{4\pi^3}{m^2}(-1)^m$

$a_m = \frac{2}{\pi}\int x^4\cos mx\,dx - \frac{2}{\pi}\left[\frac{4\pi^3}{m^2}(-1)^m\right] = \frac{2}{\pi}\int x^4\cos mx\,dx + \frac{8\pi^2}{m^2}(-1)^{m+1}$

$x^4 \qquad \cos mx$

$4x^3 \qquad \frac{\sin mx}{m}$

$\cancelto{0}{x^4\frac{\sin mx}{m}} - \frac{4}{m}\int x^3\sin mx\,dx$

$-\frac{4}{m}\int x^3\sin mx\,dx \qquad \begin{matrix} x^3 & \sin mx \\ 3x^2 & -\frac{\cos mx}{m}\end{matrix} \qquad = +\frac{4x^3}{m^2}\cos mx - \frac{12}{m^2}\int x^2\cos mx\,dx$

$\frac{8}{\pi}\frac{x^3}{m^2}\cos mx - \frac{24}{\pi m^2}\int x^2\cos mx\,dx$

$\cancel{\frac{8\pi^2}{m^2}(-1)^m} - \frac{24}{\pi m^2}\int x^2\cos mx\,dx - \cancel{\frac{8\pi^2}{m^2}(-1)^m} = -\frac{24}{\pi m^2}\cdot\frac{2\pi}{m^2}(-1)^m$

$= \frac{48}{m^4}(-1)^{m+1}$

$$\therefore \quad (x^2-\pi^2)^2 = \frac{8\pi^4}{15} + \sum_{m=1}^{\infty} \frac{48}{m^4}(-1)^{m+1} \cos mx$$

$$\frac{1}{2}(a^2-m^2)\Big[(a+m)\sin(a-m) - (a-m)\sin(a+m)$$

$$(a+m)\left[\sin a\cos m - \sin m\cos a\right] - (a-m)\left[\sin a\cos m + \cos a\sin m\right]$$

$$\sin a\cos m\left[(a+m)-(a-m)\right] - \cos a\sin m\left[(a+m)+(a-m)\right]$$

$$2m\sin a\cos m - 2a\cos a\sin m$$

$\sin ax\,(1+\cos x)$ is an odd fn $\therefore\ a_m = 0 \quad \forall m$

$$\therefore\quad b_m = \frac{2}{\pi}\int_0^{\pi}\left(\sin ax + \sin ax\cos x\right)\sin mx\,dx$$

Assume a is positive

$$\int_0^{\pi}\sin ax\,\sin mx\,dx = \frac{1}{2}\int_0^{\pi}\left[\cos(a-m)x - \cos(a+m)x\right]dx \qquad \text{for } a \neq m$$

$$= \int_0^{\pi}\left[\frac{1-\cos 2mx}{2}\right]dx \qquad \text{for } a = m$$

for $a \neq m$

$$\frac{1}{2}\left[\frac{\sin(a-m)x}{a-m} - \frac{\sin(a+m)x}{a+m}\right] = \cancel{2(a^2-m^2)}\ \frac{m}{\cancel{2}(a^2-m^2)}\left[\sin ax\cos mx\right]_0^{\pi}$$

$$= \frac{m(-1)^m}{\cancel{2}(a^2-m^2)}\sin a\pi$$

for $a = m$ $\qquad \frac{x}{2}\Big|_0^{\pi} = \frac{\pi}{2}$

$$\int_0^{\pi}\sin ax\cos x\,\sin mx\,dx = \frac{1}{2}\int_0^{\pi}\left\{\cos(a-m)x - \cos(a+m)x\right\}\cos x\,dx \qquad \text{for } a \neq m$$

$$= \frac{1}{2}\int_0^{\pi}\cos x\left(1 - \cancel{\sin}\,2mx\right)dx \qquad \text{for } a = m$$

Since $a = m \Rightarrow a$ is an integer

$$= \frac{1}{2}\int_0^{\pi}\left\{\cos x - \frac{1}{2}\left[\cos(2m+1)x + \cos(2m-1)x\right]\right\}dx$$

problems would arise if $2m-1 = 0 \Rightarrow m = \frac{1}{2} \neq$ integer therefore this can never occur $\therefore\ \int_0^{\pi} \equiv 0$

if $a \neq m$

$$= \frac{1}{4}\int_0^{\pi}\left[\cos(a-m+1)x + \cos(a-m-1)x - \cos(a+m-1)x - \cos(a+m+1)x\right]dx$$

$$= \frac{1}{4}\int_0^{\pi}\frac{\sin(a-m+1)x}{(a-m+1)} + \frac{\sin(a-m-1)x}{a-m-1} - \frac{\sin(a+m-1)x}{a+m-1} - \frac{\sin(a+m+1)x}{a+m+1}$$

$= \cancel{0}$ if none of cases below

problem arise if $a-m+1 = 0$

or $a-m-1 = 0$

or $a+m-1 = 0 \Rightarrow$ if $a>0$ $m = 0$ but series begins from $m = 1$

$\therefore \Rightarrow a = 0$ which is trivial case of $f(x) = 0$

$a = m-1$

$$\int_0^\pi \sin(m-1)x \sin mx \cos x \, dx = \frac{1}{2}\int_0^\pi [\cos x - \cos(2m-1)x] \cos x \, dx$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \cos(2m-1)x\cos x = \frac{1}{2}\cos(2mx) + \frac{1}{2}\cos(2m-2)x$$

trouble if $m=1$ but $m=1 \Rightarrow a=0$ trivial case

$$\frac{1}{2}\int_0^\pi \left(\frac{1+\cos 2x}{2} - [\cos 2mx + \cos(2m-2)x]\right) dx = \begin{matrix} \frac{\pi}{4} & \text{if } m \neq 1 & a=m-1 \\ 0 & \text{if } m=1 & a=0 \end{matrix}$$

$a = m+1$

$$\int_0^\pi \sin(m+1)x \cos x \sin mx \, dx = \frac{1}{2}\int_0^\pi [\cos x - \cos(2m+1)x] \cos x \, dx$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \cos(2m+1)x\cos x = \frac{1}{2}\cos 2mx + \frac{1}{2}\cos(2m+2)x$$

$$\frac{1}{2}\int_0^\pi \left(\frac{1+\cos 2x}{2} - [\cos(2m+2)x + \cos 2mx]\right) dx = \frac{\pi}{4}$$

$$\frac{1}{4}\left[\frac{\overset{\overset{(-1)^{m+1}}{\sin a\cos(m+1)\pi}}{\sin a \cos(m+1)x} - \cos a \cancel{\sin(m+1)x}}{a-m+1} + \sin a \cos m+1 - \cos a \overset{\to 0}{\cancel{\sin m+1}} - [\sin a \cos m-1 + \cos a \cancel{\sin m-1}\right.$$

$$\frac{\sin a\pi \, (-1)^{m+1}}{a-(m+1)} + \frac{\sin a\pi \, (-1)^{m+1}}{a-(m+1)} - \frac{\sin a\pi}{a+(m-1)}(-1)^{m-1} - \frac{\sin a\pi}{a+(m+1)}(-1)^{m+1}$$

$$(-1)^{m+1}\sin a\pi \left[\frac{\overset{2(m-1)}{a+(m-1) - a + (m-1)}}{a^2-(m-1)^2} + \frac{\overset{2(m+1)}{a+(m+1) - a + (m+1)}}{a^2-(m+1)^2}\right] \quad a^2(m-1)$$

$$2(-1)^{m+1}\sin a\pi\left[\frac{\{a^2-(m+1)^2\}(m+1) + \{a^2-(m-1)^2\}(m+1)}{[a^2-(m-1)^2][a^2-(m+1)^2]}\right]$$

$$(a^2-(m+1)^2)(m-1)$$
$$+ (a^2-(m-1)^2)(m+1)$$
$$-(m-1)(m+1)\underbrace{[m+1+m-1]}_{2m}$$

$$2ma^2 + [-(m+1)(m-1)]$$

$$-(m+1)(m-1)\underbrace{[m+1+m+1]}_{2m}$$

$$\sin a\pi \, (-1)^{m+1} \frac{4m[a^2-m^2+1]}{[a^2-(m-1)^2][a^2-(m+1)^2]}$$

⑦

$$\text{"}b_m\text{"} = \frac{2}{\pi}\left[\frac{m(-1)^m}{a^2-m^2}\sin a\pi\right] \quad \text{if} \quad \begin{array}{l} a \neq m-1 \\ a \neq m+1 \\ a \neq 1-m \end{array}$$

$$\frac{2}{\pi}\left[\frac{\pi}{2}\right] = 1 \qquad a = m$$

$$= \frac{2}{\pi}\left[\frac{m(-1)^m}{1-2m}\cdot 0 + \frac{\pi}{4}\right] = \frac{1}{2} \quad \text{if} \quad a = m-1 \qquad m \neq 1$$

$$= \frac{2}{\pi}\left[\frac{(-1)}{-1}\cdot 0 + 0\right] = 0 \quad \text{if} \quad a = m-1 \qquad m = 1$$

$$= \frac{2}{\pi}\left[\frac{m(-1)^m}{2m+1}\cdot 0 + \frac{\pi}{4}\right] = \frac{1}{2} \;\text{if} \quad a = m+1 \qquad \cancel{m \neq 1}$$

~~if a = m~~

$$\sin ax\,(1+\cos x) = \cancel{\sum_{m=1}^{\infty} \sin m\pi}$$

if a is not an integer then

$$b_m = \frac{2}{\pi}\left[\frac{\overset{\checkmark}{m}\,\overset{\checkmark}{(-1)^m}}{(a^2-m^2)}\,\overset{\checkmark}{\sin a\pi} - \frac{\overset{\checkmark}{(-1)^m}\,\overset{\checkmark}{\sin a\pi}\;\overset{\checkmark}{m}\,(a^2-m^2+1)}{(a^2-(m-1)^2)\,(a^2-(m+1)^2)}\right]$$

$$\sin ax\,(1+\cos x) = \cancel{\frac{2\sin a\pi}{\pi}\left[\sum_{m=1}^{\infty}\frac{m(-1)^m}{(a^2-m^2)}\sin mx\right]}$$

$$= \frac{2}{\pi}\sin a\pi\left\{\sum_{m=1}^{\infty} m(-1)^m \sin mx\left[\frac{1}{(a^2-m^2)} - \frac{(a^2-m^2+}{(a^2-(m-1)^2)\,(a^2-(m+1)}\right.\right.$$

$$= \frac{2}{\pi}\sin a\pi\left\{\sum_{m=1}^{\infty} m(-1)^m\left[\frac{1}{a^2-m^2} - \frac{a^2-m^2+1}{(a^2-(m-1)^2)(a^2-(m+1)^2)}\right]\sin mx\right\}$$

$B(t) = t - 1/2$

$B(2\pi t - \pi)$, $B(\pi)$ ~~when t=1~~ $B(-\pi)$ where $t=0$

$B[2\pi t - \pi] = 2\pi t - \pi - 1/2$

$$2\int_0^1 (t - 1/2)\,dt$$

$$2\int_0^1 (t - 1/2)\,dt = 2\left[t^2/2 - \frac{t}{2}\right] = (t^2 - t)\Big|_0^1 = 0 \checkmark$$

$a_m = 0 \ \forall m$

$$2\int_0^1 (t - 1/2)\sin 2m\pi t\,dt$$

$$2\int_0^1 t \sin 2m\pi t\,dt - \int_0^1 \sin 2m\pi t\,dt$$

$$+ \frac{\cos 2m\pi t}{2m\pi}\Big|_0^1 = \frac{1}{2m\pi} - \frac{1}{2m\pi} = 0$$

t $\qquad \sin 2m\pi t$

dt $\qquad -\frac{\cos 2m\pi t}{2m\pi}$

$$-\frac{t\cos 2m\pi t}{2m\pi}\Big|_0^1 + \underbrace{\int \frac{\cos 2m\pi t}{2m\pi}}_{\cancel{\frac{\sin 2m\pi t}{4m^2\pi^2}\Big|_0^1} \to 0}$$

$$-\frac{1}{2m\pi}$$

$$\therefore \quad B_1(t) = -\sum_{m=1}^{\infty} \frac{1}{m\pi} \sin 2m\pi t$$

$B_2(t) = (t^2 - t + 1/6)$ is even about $t = 1/2$ $\therefore$ $b_m = 0$

$$\therefore \quad a_0 = 2\int_0^1 (t^2 - t + 1/6)\,dt = 2\left[t^3/3 - t^2/2 + t/6\right]_0^1 = 2\left[1/3 - 1/2 + 1/6\right] = 0$$

$B_1(t) = t - 1/2$

$B_2'(t) = 2(t - 1/2) = 2t - 1$

$B_2(t) = t^2 - t + c \qquad \int B_2(t) = \frac{t^3}{3} - \frac{t^2}{2} + ct\Big|_0^1 = -\frac{1}{6} + C = 0 \quad C = 1/6$

$B_3'(t) = 3\left(t^2 - t + 1/6\right) = 3t^2 - 3t + 1/2$

$$\boxed{B_2(t) = t^2 - t + 1/6}$$

$B_3(t) = t^3 - \frac{3t^2}{2} + \frac{t}{2} + C$

$\int B_3(t) = \frac{t^4}{4} - \frac{t^3}{2} + \frac{t^2}{4} + ct\Big|_0^1 = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} + C = 0 \qquad C = 0$

$$\boxed{B_3(t) = t^3 - \frac{3t^2}{2} + \frac{t}{2}}$$

$B_4'(t) = 4t^3 - 6t^2 + 2t$

$B_4(t) = t^4 - 2t^3 + t^2 + C$

$\int B_4 = \frac{t^5}{5} - \frac{t^4}{2} + \frac{t^3}{3} + Ct\Big|_0^1 = 1/5 - \frac{1}{2} + \frac{1}{3} + C$

$\frac{6 - 15 + 10}{30} + C = \frac{1}{30} + C = 0$

$C = -1/30$

$$\boxed{B_4(t) = t^4 - 2t^3 + t^2 - 1/30}$$

~~$B_1(t)$~~

~~$\frac{1}{\pi}\int_{-\pi}^{\pi}(t - 1/2)\,dt = \frac{1}{2\pi}\left[t^2 - t\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\pi^2 - \pi - \pi^2 - \pi\right] = -1$~~

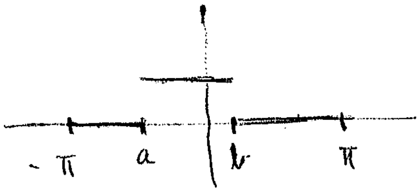
~~$\frac{1}{\pi}\int_{-\pi}^{\pi}(t - 1/2)\cos mt\,dt - \frac{1}{\pi}\int t\cos mt - \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin mt\,dt^{\nearrow 0}$~~

~~$t \qquad \cos$~~

~~$\frac{1}{\pi}$~~

~~originally $a_m = \frac{1}{P}\int_{-P}^{P} f\cos\frac{m\pi t}{P}\,dt$ ok~~

~~$\int_0^1 \to$ periodic $2P = 1 \quad P = 1/2 \qquad \frac{1}{1/2}\int_0^1 f\cos 2m\pi t\,dt$~~



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{a} + \int_{a}^{b} + \int_{b}^{\pi}\right] = \frac{1}{\pi}\int_{a}^{b} dx = \frac{b-a}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{1}{\pi}\int_{a}^{b} \cos mx\,dx = \frac{1}{\pi}\left[\frac{\sin mb}{m} - \frac{\sin ma}{m}\right]$$

$$\frac{1}{m\pi}\left[\sin mb - \sin ma\right]$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \frac{1}{\pi}\int_{a}^{b} \sin mx\,dx = \frac{-1}{m\pi}\left[\cos mb - \cos ma\right]$$

$$f(x) = \frac{b-a}{2\pi} + \sum \frac{1}{\pi m}\Big[\cos mx \sin mb - \cos mx \sin ma + \cos ma \sin mx - \cos mb \sin mx\Big]$$

$$f(x) = \frac{b-a}{2\pi} + \sum_{m=1}^{\infty} \frac{1}{\pi m}\left[\sin m(b-x) - \sin m(a-x)\right]$$

$$a_m = 2\int_0^1 (t^2 - t + 1/6)\cos 2m\pi t\, dt$$

u t^2 $\cos 2m\pi t$ dv

du $2t$ $\frac{\sin 2m\pi t}{2m\pi}$ v

t $\sin 2m\pi t$

dt $-\frac{\cos 2m\pi t}{2m\pi}$

$$\int t^2 \cos 2m\pi t\,dt = \frac{t^2}{2m\pi}\overset{0}{\cancel{\sin 2m\pi t}} - \frac{1}{m\pi}\int t \sin 2m\pi t\, dt = \frac{+1}{2m^2\pi^2}$$

t $\cos 2m\pi t$

dt $\frac{\sin 2m\pi t}{2m\pi}$

$$\int t\cos 2m\pi t\,dt = \overset{0}{\cancel{t\,\frac{\sin 2m\pi t}{2m\pi}}} - \int \frac{\sin 2m\pi t}{2m\pi} = \frac{\cos 2m\pi t}{4m^2\pi^2}\Big|_0^1 = 0$$

$$\int \frac{1}{6}\cos 2m\pi t\,dt\Big|_0^1 = 0$$

$$a_m = \frac{1}{m^2\pi^2}$$

$$B_2(t) = \frac{1}{\pi^2}\sum_{m=1}^{\infty} \frac{\cos 2m\pi t}{m^2}$$

@ $t=0$ $\quad B_2(0) = \frac{1}{6} = \frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}$

$$\therefore \quad \frac{\pi^2}{6} = \sum_{m=1}^{\infty}\frac{1}{m^2}$$

⑭ C. Levy 763.2/30

1. 1. $f(x) = e^{ax}$

$$\int e^{ax}\cos mx\,dx = \frac{1}{a}e^{ax}\cos mx + \frac{m}{a}\int e^{ax}\sin mx\,dx$$

$$\int e^{ax}\sin mx\,dx = \frac{1}{a}e^{ax}\sin mx - \frac{m}{a}\int e^{ax}\cos mx\,dx$$

$$\therefore \int_{-\pi}^{\pi} e^{ax}\cos mx\,dx = \frac{1}{a^2+m^2}\Big[m\,e^{ax}\sin mx + a\,e^{ax}\cos mx\Big]_{-\pi}^{\pi} = \frac{2a(-1)^m}{a^2+m^2}\sinh a\pi \quad ✓$$

$$\int_{-\pi}^{\pi} e^{ax}\sin mx\,dx = \frac{1}{a^2+m^2}\Big[a e^{ax}\sin mx - m\,e^{ax}\cos mx\Big]_{-\pi}^{\pi} = \frac{2m(-1)^{m+1}}{a^2+m^2}\sinh a\pi \quad ✓$$

$$\int_{-\pi}^{\pi} e^{ax}\,dx = \frac{2}{a}\sinh a\pi \quad ✓$$

$$\therefore\ e^{ax} = \frac{a_0}{2} + \sum a_m\cos mx + \sum b_m\sin mx$$

$$= \frac{\sinh a\pi}{a\pi} + \sum \frac{2}{a\pi}\,\frac{a^2}{a^2+m^2}(-1)^m\sinh a\pi\cos mx$$

$$+ \sum \frac{2}{a\pi}\,\frac{am}{a^2+m^2}(-1)^{m+1}\sinh a\pi\sin mx$$

$$\text{or } e^{ax} = \frac{\sinh a\pi}{a\pi}\left[1 + 2a\sum_{m=1}^{\infty}\left\{\frac{(-1)^m}{a^2+m^2}\big[a\cos mx - m\sin mx\big]\right\}\right] \quad ✓ \quad ①$$

2. $f(x) = (x^2 - \pi^2)^2$

this is an even fn in x $\therefore\ b_m = 0\ \forall m$ ✓

then

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(x^2-\pi^2)^2\,dx = \frac{16}{15}\pi^4 \quad ✓$$

$$a_m = \frac{2}{\pi}\int_0^{\pi}(x^4 - 2x^2\pi^2 + \pi^4)\cos mx\,dx$$

$$\int_0^{\pi} \pi^4 \cos mx\,dx = 0$$

$$\int_0^{\pi} 2x^2\pi^2 \cos mx\,dx = 2\pi^2\left\{ \frac{x^2}{m}\cancelto{0}{\sin mx}\Big|_0^{\pi} - \frac{2}{m}\left[-\frac{x}{m}\cos mx\Big|_0^{\pi} + \int_0^{\pi}\cancelto{0}{\frac{\cos mx}{m}dx}\right]\right\}$$

$$= \frac{4\pi^3}{m^2}(-1)^m$$

$$\int_0^{\pi} x^4 \cos mx\,dx = \cancelto{0}{\frac{x^4 \sin mx}{m}} - \frac{4}{m}\left[-\frac{x^3\cos mx}{m}\Big|_0^{\pi} + \frac{3}{m}\int_0^{\pi} x^2 \cos mx\,dx\right]$$

$$\therefore\ a_m = \frac{8\pi^2}{m^2}(-1) - \frac{24}{\pi m^2}\int x^2 \cos mx\,dx - \frac{8\pi^2}{m^2}(-1)^m = \frac{48}{m^4}(-1)^{m+1}$$ ✓

$$\therefore\ (x^2-\pi^2)^2 = \frac{a_0}{2} + \sum a_m \cos mx$$

$$= \frac{8\pi^4}{15} - \sum_{m=1}^{\infty}\frac{48}{m^4}(-1)^m \cos mx$$ ✓ ②

3. $f(x) = \sin ax\,(1+\cos x)$ $a \neq$ integer

$f(x)$ is an odd fn $\therefore$ $a_m = 0\ \forall m$ ✓

$$\therefore\ b_m = \frac{2}{\pi}\int_0^{\pi}(\sin ax + \sin ax \cos x)\sin mx\,dx$$

$$\int_0^{\pi}\sin ax \sin mx\,dx = \frac{1}{2}\int_0^{\pi}\left[\cos(a-m)x - \cos(a+m)x\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin(a-m)x}{a-m} - \frac{\sin(a+m)x}{a+m}\right]_0^{\pi}$$

$$= \frac{1}{a^2-m^2}\left[m\sin ax\cos mx - a\cos ax\,\cancelto{0}{\sin mx}\right]_0^{\pi}$$

$$= \frac{m}{a^2-m^2}\left[(-1)^m \sin a\pi\right]$$

$$\int_0^\pi \sin ax \sin mx \cos x\,dx = \frac{1}{2}\int_0^\pi \left[\cos(a-m)x - \cos(a+m)x\right]\cos x\,dx$$

$$= \frac{1}{4}\int_0^\pi \left[\cos(a-m+1)x + \cos(a-m-1)x - \cos(a+m-1)x - \cos(a+m+1)x\right]dx$$

since a is not an integer $a-m+1 \neq 0$, $a-m-1 \neq 0$, $a+m-1 \neq 0$, $a+m+1 \neq 0$

$$= \frac{1}{4}\left[\frac{\sin(a-m+1)x}{a-m+1} + \frac{\sin(a-m-1)x}{a-m-1} - \frac{\sin(a+m-1)x}{a+m-1} - \frac{\sin(a+m+1)x}{a+m+1}\right]_0^\pi$$

for lower limit the integral is zero ; for upper limit

$$\frac{1}{4}\left[\frac{\sin(a-m+1)\pi}{a-m+1} + \frac{\sin(a-m-1)\pi}{a-m-1} - \frac{\sin(a+m-1)\pi}{a+m-1} - \frac{\sin(a+m+1)\pi}{a+m+1}\right]$$

$$\sin(a-m+1)\pi = \sin a\pi\cos(m-1)\pi - \cos a\pi\,\overset{=0}{\sin(m-1)\pi} = (-1)^{m-1}\sin a\pi$$

$$\sin(a-m-1)\pi = \sin a\pi\cos(m+1)\pi - \cos a\pi\,\overset{=0}{\sin(m+1)\pi} = (-1)^{m+1}\sin a\pi$$

$$\sin(a+m-1)\pi = \sin a\pi\cos(m-1)\pi + \cos a\pi\,\overset{=0}{\sin(m-1)\pi} = (-1)^{m-1}\sin a\pi$$

$$\sin(a+m+1)\pi = \sin a\pi\cos(m+1)\pi + \cos a\pi\,\overset{=0}{\sin(m+1)\pi} = (-1)^{m+1}\sin a\pi$$

we can then write

$$\frac{1}{4}(-1)^{m+1}\sin a\pi\left[\frac{1}{a-m+1} + \frac{1}{a-m-1} - \frac{1}{a+m-1} - \frac{1}{a+m+1}\right]$$

$$= \frac{1}{4}(-1)^{m+1}\sin a\pi\left[\frac{2(m-1)}{a^2-(m-1)^2} + \frac{2(m+1)}{a^2-(m+1)^2}\right]$$

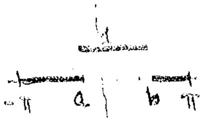
$$= \frac{1}{2}(-1)^{m+1}\sin a\pi\left[\frac{a^2(2m) - (m^2-1)2m}{[a^2-(m-1)^2][a^2-(m+1)^2]}\right] = -m(-1)^m\sin a\pi\frac{\left[a^2-m^2+1\right]}{[a^2-(m-1)^2][a^2-(m+1)^2]}$$

$$\therefore f(x) = \sum b_m \sin mx$$

$$= \frac{2}{\pi}\sin a\pi\left\{\sum_{m=1}^{\infty}(-1)^m m\left[\frac{1}{a^2-m^2} - \frac{\overset{2a}{\boxed{a^2-m^2+1}}}{(a^2-(m-1)^2)(a^2-(m+1)^2)}\right]\sin mx\right\}$$

$$= \frac{\sin a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^n\left\{\frac{2n}{a^2-n^2} - \frac{n}{(a-1)^2-n^2} - \frac{n}{(a+1)^2-n^2}\right]\sin nx. \quad ①$$

1. $$f(x) = \begin{cases} 1 & a \leq x \leq b \\ 0 & -\pi < x < a \;,\; b < x < \pi \end{cases}$$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{a} f + \int_{a}^{b} f + \int_{b}^{+\pi} f\right] = \frac{1}{\pi}\int_{a}^{b} dx = \frac{b-a}{\pi}$$ ✓

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{1}{\pi}\int_{a}^{b}\cos mx\,dx = \frac{1}{\pi m}\left[\sin mb - \sin ma\right]$$ ✓

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \frac{1}{\pi}\int_{a}^{b}\sin mx\,dx = \frac{-1}{\pi m}\left[\cos mb - \cos ma\right]$$ ✓

$$\therefore\; f(x) = \frac{a_0}{2} + \sum a_m \cos mx + \sum b_m \sin mx$$

$$= \frac{b-a}{2\pi} + \sum_{m=1}^{\infty} \frac{1}{\pi m}\left[\left(\sin mb - \sin ma\right)\cos mx - \left(\cos mb - \cos ma\right)\sin mx\right]$$

$$= \frac{b-a}{2\pi} + \sum_{m=1}^{\infty}\frac{1}{m\pi}\left[\sin m(b-x) - \sin m(a-x)\right]$$ ✓ ①

2. $B_1(t) = t - \frac{1}{2}$

$B_2'(t) = 2(t - \frac{1}{2}) = 2t - 1$

$B_2(t) = t^2 - t + c$

$\int_0^1 B_2(t)\,dt = t^3/3 - t^2/2 + ct \Big|_0^1 = -\frac{1}{6} + c = 0$

$\therefore \quad B_2(t) = t^2 - t + \frac{1}{6}$ ✓ ①

$B_2(t) = t^2 - t + \frac{1}{6}$

$B_3'(t) = 3(t^2 - t + \frac{1}{6}) = 3t^2 - 3t + \frac{1}{2}$

$B_3(t) = t^3 - 3t^2/2 + t/2 + c$

$\int_0^1 B_3(t)\,dt = t^4/4 - t^3/2 + t^2/4 + ct \Big|_0^1 = 0 + c = 0$

$\therefore \quad B_3(t) = t^3 - 3t^2/2 + t/2$ ✓ ①

$B_3(t) = t^3 - 3t^2/2 + t/2$

$B_4'(t) = 4t^3 - 6t^2 + 2t$

$B_4(t) = t^4 - 2t^3 + t^2 + c$

$\int_0^1 B_4(t)\,dt = t^5/5 - t^4/2 + t^3/3 + ct \Big|_0^1 = \frac{1}{30} + c = 0$

$\therefore \quad B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}$ ✓ ①

3. Originally we had $a_m = \frac{1}{p}\int_{-p}^{p} f \cos \frac{m\pi t}{p}\,dt$

and $b_m = \frac{1}{p}\int_{-p}^{p} f \sin \frac{m\pi t}{p}\,dt$

The Bernoulli numbers are periodic of period $2P = 1$

$\therefore \quad P = \frac{1}{2} \qquad \therefore$ about $t = \frac{1}{2}$

$$a_m = 2\int_0^1 f \cos 2m\pi t \, dt$$

$$b_m = 2\int_0^1 f \sin 2m\pi t \, dt$$

a) $B_1(t) = t - \frac{1}{2}$

but about $t = \frac{1}{2}$ $B_1(t)$ is odd therefore $a_m = 0 \; \forall m$ ✓

$$b_m = 2\int_0^1 \left(t - \tfrac{1}{2}\right) \sin 2m\pi t \, dt$$

$$= 2\int_0^1 t \sin 2m\pi t \, dt - \int_0^1 \cancelto{0}{\sin 2m\pi t \, dt}$$

$$= 2\left[-\frac{t \cos 2m\pi t}{2m\pi} \Big|_0^1 + \cancelto{0}{\int_0^1 \frac{\cos 2m\pi t \, dt}{2m\pi}} \right]$$

$$= -\frac{1}{m\pi}$$ ✓

$$\therefore \quad B_1(t) = -\sum_{m=1}^{\infty} \frac{1}{m\pi} \sin 2m\pi t$$ ✓ ②

b) $B_2(t) = t^2 - t + \frac{1}{6}$

but about $t = \frac{1}{2}$ $B_2(t)$ is even therefore $b_m = 0 \; \forall m$ ✓

$$a_0 = 2\int_0^1 \left(t^2 - t + \tfrac{1}{6}\right) dt = 2\left[\frac{t^3}{3} - \frac{t^2}{2} + \frac{1}{6} t \right]_0^1 = 0$$ ✓

$$a_m = 2\int_0^1 \left(t^2 - t + \tfrac{1}{6}\right) \cos 2m\pi t \, dt$$

$$\int t^2 \cos 2m\pi t\,dt = \frac{t^2}{2m\pi}\,\cancel{\sin 2m\pi t}^{\,0}\Big|_0^1 - \frac{1}{m\pi}\int_0^1 t \sin 2m\pi t\,dt$$

$$-\frac{1}{m\pi}\int_0^1 t \sin 2m\pi t\,dt = \frac{-1}{m\pi}\left[-\frac{t\cos 2m\pi t}{2m\pi}\Big|_0^1 + \int_0^1 \cancel{\frac{\cos 2m\pi t}{2m\pi}\,dt}^{\,0}\right]$$

$$= \frac{1}{2m^2\pi^2}$$

$$\int_0^1 t\cos 2m\pi t\,dt = \cancel{\frac{t\sin 2m\pi t}{2m\pi}}^{\,0}\Big|_0^1 - \int_0^1 \cancel{\frac{\sin 2m\pi t}{2m\pi}}^{\,0}\,dt$$

$$\int_0^1 \frac{1}{6}\cos 2m\pi t\,dt = 0$$

$\therefore \quad a_m = \frac{1}{m^2\pi^2}$ ✓

and $\quad B_2(t) = \frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos 2m\pi t}{m^2}$ ✓ ②

9. $B_2(0) = t^2 - t + 1/6\Big|_0 = 1/6 = \frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}$

$\therefore \quad \frac{\pi^2}{6} = \sum_{m=1}^{\infty}\frac{1}{m^2}$ ✓ ②

T63.2130

⑨

1 a) Define $a_n = \frac{1}{p}\int_{-p}^{p} F(x)\cos\left(\frac{n\pi x}{p}\right)dx$, $b_n = \frac{1}{p}\int_{-p}^{p} F(x)\sin\left(\frac{n\pi x}{p}\right)dx$

$F(x) = e^{ax}$, $p = \pi \implies a_n = \frac{2a(-1)^n}{\pi(n^2+a^2)}\sinh(a\pi)$, $b_n = \frac{-2n(-1)^n}{\pi(n^2+a^2)}\sinh(a\pi)$

b) $F(x) = (x^2-\pi^2)$, $p = \pi \implies a_n = -\frac{48}{n^4}(-1)^n, n \neq 0$, $a_0 = \frac{16}{15}\pi^4$, $b_n = 0$.

c) $F(x) = \sin ax(1+\cos x)$, $p = \pi \implies a_n = 0$, $b_n = \frac{\sin(a\pi)}{\pi}(-1)^n n\left\{\frac{2}{a^2-n^2} - \frac{1}{(a-1)^2-n^2} - \frac{1}{(a+1)^2-n^2}\right\}$

d) $F(x) = \begin{cases} 1 & a \le x \le b \\ 0 & \text{other } x \end{cases}$ $p = \pi$, $a_n = \frac{1}{n\pi}\left[\sin nb - \sin na\right]$ $b_n = \frac{-1}{n\pi}\left[\cos nb - \cos na\right]$

$a_0 = \frac{b-a}{\pi}$

combining $\implies$ $f(x) = \frac{b-a}{2\pi} + \frac{2}{\pi}\sum^{1}\frac{1}{n}\cos\left\{n\left[\frac{b+a}{2} - x\right]\right\}\sin\left\{n\left(\frac{b-a}{2}\right)\right\}$

2. a) $B_1(t) = t - \frac{1}{2}$ (odd about $t = \frac{1}{2}$)
b) $B_n'(t) = nB_{n-1}(t)$
c) $\int_0^1 B_n(t)\,dt = 0$

General scheme : knowing B_{n-1} use b) and c) to determine B_n.

Thus $B_2(t) = t^2 - t + \frac{1}{6}$, $B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{t}{2}$, $B_4 = t^4 - 2t^3 + t^2 - \frac{1}{30}$

B_1 odd about $t = \frac{1}{2} \implies a_n = 0$ $\left\{\text{taking } a_n = \left(\frac{1}{\frac{1}{2}}\right)\int_0^1 B_1(t)\cos\left(\frac{n\pi t}{\frac{1}{2}}\right)dt\right\}$

Similarly $b_n = -\frac{1}{n\pi}$ and $B_1(t) = -\frac{1}{\pi}\sum_1^{\infty}\frac{\sin(2n\pi t)}{n}$

$B_2 = \left(t - \frac{1}{2}\right)^2 + \frac{1}{12}$: even about $t = \frac{1}{2} \implies b_n = 0$

$a_n = \frac{1}{n^2\pi^2} \implies B_2(t) = \frac{1}{\pi^2}\sum_1^{\infty}\frac{\cos(2n\pi t)}{n^2}$

4. Setting $t = 0$ in B_2 :

$$\frac{1}{6} = \frac{1}{\pi^2}\sum_1^{\infty}\frac{1}{n^2}$$

$$\implies \frac{\pi^2}{6} = \sum_1^{\infty}\frac{1}{n^2}$$

---

since it is odd about zero
and periodic in $4l$

$$\begin{aligned} F'(x) &= f'(x) & \overset{\text{define}}{F(x)} &= f(x) & 0 &\le x \le l \\ &= -f'(2l-x) & &= f(2l-x) & l &\le x \le 2l \\ &= f'(-x) & &= -f(-x) & -l &\le x \le 0 \\ &= -f'(x+2l) & &= -f(x+2l) & -2l &\le x \le -l \end{aligned}$$

$$\begin{aligned} &f'(x) & F(x) &= F(x+4l) & f(x) \quad & 0\le x\le 2l \\ &f'(-x) & F(x) &\text{ is odd about } x=0 & -f(-x) \quad & -2l\le x\le 0 \end{aligned}$$

Φ satisfy (1) $(-\infty,\infty)$

subject to $\left.\begin{aligned} \Phi(x,0) &= F(x) \\ \Phi_t(x,0) &= 0 \end{aligned}\right\} t=0$

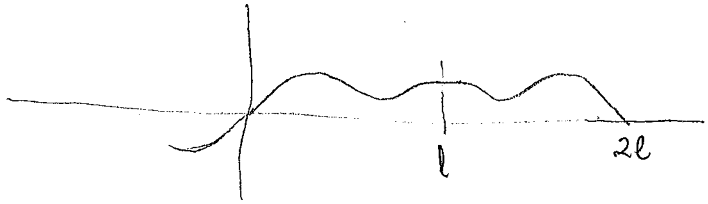
$$F(x) = \sum b_n \sin \frac{n\pi x}{2l}$$

where $b_n = \dfrac{2}{2l}\displaystyle\int_0^{2l} f(x) \sin \frac{n\pi x}{2l}$

$$\Phi(x,t) = \sum b_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi a t}{2l}$$

$$\Phi(x,0) = \sum b_n \sin \frac{n\pi x}{2l} = F(x)$$

$$\Phi_t(x,0) = -\frac{n\pi a}{2l} \sum b_n \sin \frac{n\pi x}{2l} \overset{0}{\cancel{\sin \frac{n\pi a t}{2l}}} = 0 \checkmark$$



$$F(x) = f(x) \qquad 0 \leq x \leq 2\ell$$
$$F(x) = -f(-x) \quad -2\ell \leq x \leq 0$$

$\Phi(x,t)$ must satisfy

$$f(0)=0 \quad f'(\ell)=0$$

$$F(x) = \sum b_n \sin \frac{n\pi x}{2\ell}$$

n must be odd

$$F(x) = \sum b_m \sin \frac{(2m-1)\pi x}{2\ell}$$

$$F(0) = \sum b_m \sin \frac{(2m-1)\pi \cdot 0}{2\ell} = 0$$

ok

$$F'(\ell) = \sum b_m \frac{(2m-1)\pi}{2\ell} \cos \frac{(2m-1)\pi}{2}$$

if $m =$ ~~0~~ $1, 2, \ldots$

$\cos\left(\frac{\text{odd}\,\pi}{2}\right) = 0$ ok

$$\omega_m = \frac{(2m-1)\pi a}{2\ell}$$

$$u(x,t) = \sum b_m \sin \frac{(2m-1)\pi x}{2\ell} \cos \omega_m t$$

ok

$$u_t(x,0) = -\omega_m \sum b_m \sin(\ ) \sin \omega_m t \Big|_0 = 0$$

$$u(x,t) = \sum_{m=1}^{\infty} b_m \sin \frac{(2m-1)\pi x}{2\ell} \cos \omega_m t$$

$$b_m = \frac{1}{\ell} \int_0^{2\ell} F(x) \sin \frac{(2m-1)\pi x}{2\ell}\, dx$$

$$= \frac{1}{\ell}\left[\int_0^{\ell} f(x) \sin \frac{(2m-1)\pi x}{2\ell}\, dx + \int_{\ell}^{2\ell} f(2\ell - x) \sin \frac{(2m-1)\pi x}{2\ell}\, dx\right]$$

$\sin \boxed{\frac{(2m-1)\pi}{2l}} x \cos \omega_m t$

$\sin r_m x \cos \omega_m t = \frac{1}{2}\left[\sin(r_m x + \omega_m t) + \sin(r_m x - \omega_m t)\right]$

$sc + \cancel{c}s \quad + sc - \cancel{c}s$

$$u(x,t) = \sum_{m=1}^{\infty} \frac{1}{2}\left[b_m \sin(x) + b_m \sin(x)\right]$$

$$= \frac{1}{2}\left\{F(x+at) + F(x-at)\right\}$$

$$u_t(x,0) = \frac{a}{2}\left\{F'(x+at) - F'(x-at)\right\}\Big|_{t=0} = 0$$

$$u(x,0) = F(x)$$

$$u(0,t) = \frac{1}{2}\left\{F(at) + F(-at)\right\}$$

$$- F(at)\} = 0$$

$$u'_x(l,t) = \frac{1}{2}\left\{F'(l+at) + F'(\underbrace{l-at})\right\}$$

$l - at = 2l - (at+l) \qquad F'(at-l)$

$-F(-x) = F(x) \qquad F'(at+l-2l)$

$F'(-x) = F'(x)$

$$\frac{1}{2}\left\{F'(l+at) + F'(l-at)\right\}$$

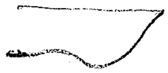
~~$\frac{1}{2}\left\{F'(4l+[at-3l]) + F'([4l-(at+3l)])\right\}$~~

because

$$\frac{1}{2}\left\{F'(l+at)\right.$$

$F(x) = f(x) \qquad 0 \le x \le l$

$ f(2l-x) \qquad l \le x \le 2l$



$-f(-x) \qquad -l \leq x \leq 0$

~~$-f(-x+2l)$~~ $\qquad -2l \leq x \leq -l$

$-f(2l+x)$

$$F'(x) = f'(x)$$

$$-f'(2l-x)$$

$$f'(x)$$

$$-f'(2l+x)$$

$$\boxed{\begin{aligned} F'(l-at) &= -f'(l+at) \\ F'(l+at) &= f'(l+at) \end{aligned}}$$

$$f(x) = \begin{cases} hx/c & 0 \le x \le c \\ h\left(1 - \frac{x-c}{l-c}\right) & c \le x \le l \end{cases}$$

odd cont since $u(0,t) = u(l,t) = 0$

$f = b_n \sin$

$$b_n = \frac{2}{l} \int_0^c + \int_c^l$$

$$b_n = \frac{2hl^2}{n^2\pi^2(cl - c^2)} \sin \frac{n\pi c}{l}$$

$$u = f(x) \cos \frac{n\pi a t}{l}$$

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$$$$$$$$$$
$$$$$$$$$$
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T63.2130 Problems #10 Due 12/5/72

#1.

Solve $u_{tt} - a^2 u_{xx} = 0$ $\quad 0<x<l \quad , t>0$

subject to
$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= 0 \end{aligned}\right\} \quad t=0 \quad , 0<x<l$$

and boundary conditions
$$u(0,t) = u_x(l,t) = 0 \quad , f(0) = f'(l) = 0,$$

by reducing to a pure initial value problem over $-\infty < x < \infty$, by replacing the b.c. by suitable periodicity conditions. Use the normal mode solutions. Interpret the solution as the sum of two travelling waves. Check your solution satisfies the correct boundary and initial data.

Sum of 2 travelling wave

98—

T&S. p.100.

Nos. 1 and 14.

⑮

Cesar Levy 763.2130

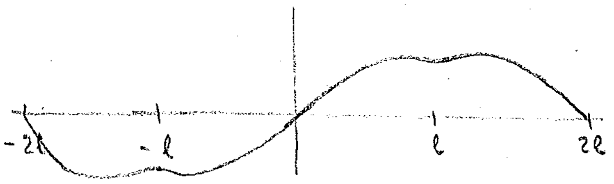
Given $u_{tt} - a^2 u_{xx} = 0$ $\quad 0<x<l \quad t>0$

Subject to $\left.\begin{array}{l} u(x,0) = f(x) \\ u_t(x,0) = 0 \end{array}\right\} \quad t=0 \quad 0<x<l$ ⑤

and b.c. $u(0,t) = u_x(l,t) = 0$

from $u(x,0) = f(x)$ and $u(0,t) = 0 \Rightarrow f(0) = 0$

from $u_x(x,0) = f'(x)$ and $u_x(l,t) = 0 \Rightarrow f'(l) = 0$



because $f(0)=0$ $\quad f(x)$ is continued oddly about $x=0$

because $f'(l)=0$ $\quad f(x)$ is continued evenly about $x=l$

$\therefore \; f(0) = f(\pm 2l) = \dots \; f(\pm 2nl) = 0 \qquad n=0,1,\dots$

$\therefore \qquad f(x) = f(x \pm 4l)$ or f is periodic in $4l$ ✓

$$\begin{aligned} F(x) &= f(x) & 0 \le x \le l \\ &= -f(-x) & -l \le x \le 0 \\ &= f(2l-x) & l \le x \le 2l \\ &= -f(x+2l) & -2l \le x \le -l \end{aligned}$$ ✓

$$\begin{aligned} F'(x) &= f'(x) & 0 \le x \le l \\ &= f'(-x) & -l \le x \le 0 \\ &= -f'(2l-x) & l \le x \le 2l \\ &= -f'(x+2l) & -2l \le x \le -l \end{aligned}$$ ✓

since F is periodic in $4l$ and F is odd about $x=0$

choose $F(x) = \sum b_n \sin \frac{n\pi x}{2l}$ ✓

$F(0) = f(0) = 0$; $\quad F'(l) = \sum \frac{n\pi}{2l} b_n \cos \frac{n\pi l}{2l}$

$F'(l) = 0$ iff $\frac{n\pi}{2}$ is an odd multiple of $\frac{\pi}{2}$ $\therefore$ let $m = 1, \dots \quad n = 2m-1$

$\therefore \quad F(x) = \sum_{m=1}^{\infty} b_m \sin \frac{(2m-1)\pi x}{2l}$ ✓

where $b_m = \frac{1}{l} \int_0^{2l} F(x) \sin \frac{(2m-1)}{2l} \pi x \, dx = \frac{1}{l} \int_0^{l} f(x) \sin \frac{(2m-1)\pi x}{2l} dx$

$+ \frac{1}{l} \int_l^{2l} f(2l-x) \sin \frac{(2m-1)}{2l} \pi x \, dx$ ✓

$$\therefore \quad u(x,t) = \sum_{1}^{\infty} b_m \sin \frac{(2m-1)\pi x}{2l} \cos \omega_m t \quad ✓ \qquad \omega_m = \frac{(2m-1)\pi a}{2l}$$

$$b_m = \frac{1}{l}\int_0^l f(x) \sin\frac{(2m-1)\pi x}{2l}\, dx + \frac{1}{l}\int_l^{2l} f(2l-x) \sin \frac{(2m-1)\pi x}{2l}\, dx \quad ✓$$

since $\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$ then

$$u(x,t) = \sum_{1}^{\infty} \frac{b_m}{2}\left[\sin\left\{\frac{(2m-1)\pi}{2l}(x+at)\right\} + \sin\left\{\frac{(2m-1)\pi}{2l}(x-at)\right\}\right]$$

$$= \frac{1}{2}\left[F(x+at) + F(x-at)\right] \quad ✓$$

$$u(x,0) = \frac{1}{2}\left[2F(x)\right] = F(x)$$

$$u_t(x,0) = \frac{a}{2}\left[F'(x) - F'(x)\right] = 0$$

$$u(0,t) = \frac{1}{2}\left[F(at) + F(-at)\right] \quad ✓$$

if $\quad 0 \le at \le l$

$$u(0,t) = \frac{1}{2}\left[f(at) - f(at)\right] = 0 \quad ✓$$

if $\quad l \le at \le 2l$

$$u(0,t) = \frac{1}{2}\left[f(2l-at) - f(2l-at)\right] = 0$$

$$u_x(l,t) = \frac{1}{2}\left[F'(l+at) + F'(l-at)\right] \quad ✓$$

if $\quad 0 \le at \le l$

$$u_x(l,t) = \frac{1}{2}\left[-f'(l-at) + f'(l-at)\right] = 0 \quad ✓$$

if $\quad l \le at \le 2l$

$$u_x(l,t) = \frac{1}{2}\left[-f'(3l+at) + f'(at-l)\right]$$

$$= \frac{1}{2}\left[-f'(at-l+4l) + f'(at-l)\right] = 0 \quad \text{since period} = 4l \quad ✓$$

Cesar Levy 763.2130

Given $u(0,t) = u(l,t) = 0$

$u(x,0) = f(x) \Rightarrow u(c,0) = h$ and $u_t(x,0) = 0$

⑤ for the string $u_{tt} = a^2 u_{xx}$

$\therefore$ by separation of variables $u(x,t) = X(x)\,T(t)$

$$X(x) = A\sin\lambda x + B\cos\lambda x$$
$$T(t) = C\sin\lambda a t + D\cos\lambda a t$$

$u(0,t) = u(l,t) = 0 \Rightarrow X(0) = X(l) = 0 \Rightarrow B = 0\,;\ \lambda l = n\pi \ \therefore\ \lambda = \frac{n\pi}{l}$

$u(c,0) = h$

$u_t(x,0) = 0 \Rightarrow T'(0) = 0 \quad \therefore C = 0$

$$\therefore\ u(x,0) = \begin{cases} hx/c & 0 \le x \le c \\ h\left(1 - \frac{x-c}{l-c}\right) & c \le x \le l \end{cases} \quad ✓$$

since the ends are fixed then continue the results oddly

$$\Rightarrow \quad F(x) = f(x) \qquad 0 \le x \le l$$
$$= -f(-x) \qquad -l \le x \le 0 \quad ✓$$

since $F(x)$ is odd about $x = 0$ and periodic since $F(x) = F(x+2l)$ then $a_n = 0 \ \therefore$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}h\left[\int_0^c \frac{x}{c}\sin\frac{n\pi x}{l}\,dx + \int_c^l \left(1 - \frac{x-c}{l-c}\right)\sin\frac{n\pi x}{l}\,dx\right.$

$$b_n = \frac{2h}{l}\left[\frac{l^2}{cn^2\pi^2} - \frac{l^2}{(c-l)n^2\pi^2}\right]\sin\frac{n\pi c}{l} = \frac{2hl^2}{n^2\pi^2(cl-c^2)}\sin\frac{n\pi c}{l} \quad ✓$$

$$\therefore\ u(x,t) = \frac{2hl^2}{\pi^2(cl-c^2)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi c}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l} \quad ✓$$

to check that $u(x,t)\Big|_{\substack{x=c\\t=0}} = h$

let $c = l/2$: $u(c,0) = \frac{2h}{\pi^2\,1/4}\sum\frac{1}{n^2}\sin^2\frac{n\pi}{2} = \frac{8h}{\pi^2}\sum\frac{1}{n^2}\sin^2\frac{n\pi}{2}$

$$\sum\frac{1}{n^2}\sin^2\frac{n\pi}{2} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{3\pi^2}{24} = \frac{\pi}{8}$$

$\therefore \quad u(c,0) = \frac{8h}{\pi^2} \cdot \frac{\pi^2}{8} = h$ which checks

$\therefore$

$$1 = \frac{2\ell^2}{\pi^2(c\ell - c^2)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2 \frac{n\pi c}{\ell}$$

$$1 = \frac{2\ell^2}{\pi^2(c\ell - c^2)} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[1 - \cos^2 \frac{n\pi c}{\ell} \right]$$

$$1 = \frac{\ell^2}{3(c\ell - c^2)} - \frac{2\ell^2}{\pi^2(c\ell - c^2)} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos^2 \frac{n\pi c}{\ell}$$

for $c = \ell/2$

$$1 = \frac{4}{3} - \frac{8}{\pi^2} \sum \frac{1}{n^2} \cos^2 \frac{n\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos^2 \frac{n\pi}{2} = \sum_{k=1}^{\infty} \left(\frac{1}{2k} \right)^2 = \frac{\pi^2}{24}$$

Cesar Levy 763.2130

⑤

$$\left.\begin{array}{l} u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \\ u(0,t) = u(l,t) = 0 \end{array}\right\} \text{satisfy} \quad \begin{array}{l} u_{tt} = a^2 u_{xx} + f(x,t) \\ f(x,t) = \Phi(x) \sin \omega t \end{array}$$

Homogeneous Soln

$$u_h = \sum \left(A_n \cos \omega_n t + B_n \sin \omega_n t \right)\left(D_n \cos \lambda x + E_n \sin \lambda x \right)$$

From $u(0,t) = u(l,t) = 0 \qquad D_n = 0 \quad \& \quad \lambda = \frac{n\pi}{l} \qquad \omega_n = \frac{n\pi a}{l}$

$$\therefore \quad u(x,t) = \sum_1^\infty \left(a_n \cos \omega_n t + b_n \sin \omega_n t \right) \sin \frac{n\pi x}{l} \quad ✓$$

$$u(x,0) = \sum_1^\infty a_n \sin \frac{n\pi x}{l} \qquad a_n = \frac{2}{l}\int_0^l \varphi(x) \sin \frac{n\pi x}{l}\, dx$$

$$u_t(x,0) = \sum_1^\infty b_n \omega_n \sin \frac{n\pi x}{l} \qquad b_n = \frac{1}{\omega_n}\, \frac{2}{l}\int_0^l \psi(x) \sin \frac{n\pi x}{l}\, dx$$

let $a_n = \varphi_n \qquad b_n = \frac{\psi_n}{\omega_n}$

$$u_h(x,t) = \sum_{n=1}^\infty \left(\varphi_n \cos \omega_n t + \frac{\psi_n}{\omega_n} \sin \omega_n t \right) \sin \frac{n\pi x}{l} \quad ✓$$

Non-Homogeneous Soln

Let $f(x,t) = \sum f_n(t) \sin \frac{n\pi x}{l} \qquad f_n(t) = \frac{2}{l}\int_0^l f(x,t) \sin \frac{n\pi x}{l}\, dx$

$$\Phi(x) = \sum_1^\infty \Phi_n \sin \frac{n\pi x}{l} \qquad \Phi_n = \frac{2}{l}\int_0^l \Phi(x) \sin \frac{n\pi x}{l}\, dx$$

$$u_p(x,t) = \sum_1^\infty u_n(t) \sin \frac{n\pi x}{l} \qquad u_n(t) = \frac{2}{l}\int_0^l u(x,t) \sin \frac{n\pi x}{l}\, dx$$

$$u_{tt} = \sum_1^\infty \ddot{u}_n(t) \sin \frac{n\pi x}{l} \qquad a^2 u_{xx} = -\sum_1^\infty u_n(t)\, \omega_n^2 \sin \frac{n\pi x}{l}$$

$$\therefore \quad u_{tt} - a^2 u_{xx} - f(x,t) = \sum_1^\infty \sin \frac{n\pi x}{l} \left[\ddot{u}_n(t) + \omega_n^2 u_n(t) + f_n(t) \right] = 0$$

iff $\ddot{u}_n(t) + \omega_n^2 u_n(t) = f_n(t)$ ✓

Using method of variation of parameters

$u_n = A(t) \sin \omega_n t + B(t) \cos \omega_n t$

$$u_n(t) = \frac{1}{\omega_n}\int_0^t f_n(\tau) \sin \omega_n (t-\tau)\, d\tau$$

$$u_n(t) = \frac{1}{\omega_n}\int_0^t \sin \omega_n (t-\tau) \left\{ \frac{2}{l}\int_0^l \Phi(x) \sin \omega \tau \sin \frac{n\pi x}{l}\, dx \right\} d\tau$$

$$= \frac{1}{\omega_n}\, \Phi_n \int_0^t \sin \omega_n (t-\tau) \sin \omega \tau \, d\tau$$

using $\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

$$u_n(t) = \frac{\Phi_n}{2\omega_n}\int_0^t \left\{\cos\left[\omega_n t - \omega_n \tau - \omega\tau\right] - \cos\left[\omega_n t - \omega_n\tau + \omega\tau\right]\right\} d\tau$$

$$= \frac{\Phi_n}{2\omega_n}\left[\frac{\sin\left[\omega_n t - \omega_n\tau - \omega\tau\right]}{-(\omega_n+\omega)} + \frac{\sin\left[\omega_n t - \omega_n\tau + \omega\tau\right]}{(\omega_n - \omega)}\right]_0^t$$

provided $\omega_n \neq \omega$ ✓

$$= \frac{\Phi_n}{2\omega_n}\left[\frac{\sin\left[\omega_n(t-2\tau)\right]}{-2\omega_n} - \tau\cos\omega_n t\right]_0^t$$

if $\omega_n = \omega$

Note that the second term oscillates between t & $-t$. As $t \to \infty$ this term will become unbounded.

For $\omega_n \neq \omega$

$$u_n(t) = \frac{\Phi_n}{\omega_n}\left\{\frac{\omega_n \sin\omega t - \omega\sin\omega_n t}{\omega_n^2 - \omega^2}\right\}$$ ✓

For $\omega_n = \omega$

$$u_n(t) = \frac{\Phi_n}{2\omega_n}\left[\frac{\sin\omega_n t}{\omega_n} - t\cos\omega_n t\right]$$ ✓

$$\therefore\ u_p(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty}\frac{\Phi_n}{\omega_n}\left\{\frac{\omega_n\sin\omega t - \omega\sin\omega_n t}{\omega_n^2 - \omega^2}\right\}\sin\frac{n\pi x}{l}$$

$(\omega_n \neq \omega)$

$$u_p(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty}\frac{\Phi_n}{2\omega_n}\left\{\frac{\sin\omega_n t}{\omega_n} - t\cos\omega_n t\right\}\sin\frac{n\pi x}{l}$$ ✓

$(\omega = \omega_n)$

Total Soln

$u = u_h + u_p$

$(\omega_n \neq \omega)$

$$u(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\left\{\varphi_n\cos\omega_n t + \frac{\psi_n}{\omega_n}\sin\omega_n t + \frac{\Phi_n}{\omega_n}\left[\frac{\omega_n\sin\omega t - \omega\sin\omega_n t}{\omega_n^2 - \omega^2}\right]\right\}$$

$(\omega_n = \omega)$

$$u(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\left\{\varphi_n\cos\omega_n t + \frac{\psi_n}{\omega_n}\sin\omega_n t + \frac{\Phi_n}{2\omega_n}\left[\frac{\sin\omega_n t}{\omega_n} - t\cos\omega_n t\right]\right\}$$

if $\left|\frac{\Phi_n^0 t}{2\omega_n}\right| << 1$ then u remains bounded for finite time

$\Phi_n^0 = \max\limits_{1 \leq n < \infty}|\Phi_n|$

✓

T63.2130

#1 Extend $f(x)$ to $F(x)$ oddly w.r.t. $x=0$ and evenly w.r.t $x=\ell$, such that

$$F(x) = \begin{cases} f(x) & 0 \le x \le \ell \\ -f(-x) & -\ell \le x \le 0 \\ f(2\ell - x) & \ell \le x \le 2\ell \\ -f(x+2\ell) & -2\ell \le x \le -\ell \end{cases}$$

$F(x) = F(x+4\ell)$ $F(x) = -F(x+2\ell$

$F(x) = -F(-x)$

$F(x - m\ell) = (-1)^{m+1} F(m\ell - x)$ $*$.

Thus $F(x) = \sum_1^\infty b_n \sin\left(\frac{n\pi x}{2\ell}\right)$, $b_n = \frac{1}{\cancel{2}\ell}\int_0^{2\ell} F(x) \sin\left(\frac{n\pi x}{2\ell}\right) dx$

$$b_n = \frac{1}{\cancel{2}\ell}\left[\int_0^\ell + \int_\ell^{2\ell}\right] = \frac{1}{\cancel{2}\ell}\left[\int_0^\ell F(x) \sin\frac{n\pi x}{2\ell} dx + \int_0^\ell F(s+\ell) \sin\frac{n\pi}{2\ell}(s+\ell) ds\right]$$

$$= \frac{1}{\cancel{2}\ell}\left[\int_0^\ell F(x) \sin\frac{n\pi x}{2\ell} dx + \int_0^\ell F(\ell - s) \sin\frac{n\pi}{2\ell}(s+\ell) ds\right]$$

$$= \frac{1}{\cancel{2}\ell}\left[\int_0^\ell F(x)\left\{\sin\frac{n\pi x}{2\ell} + \sin\left(n\pi - \frac{n\pi x}{2\ell}\right)\right\} dx\right] \quad (\text{put } \ell - s = x)$$

$$b_n \begin{cases} = \frac{2}{\ell}\int_0^\ell F(x) \sin\left(\frac{n\pi x}{2\ell}\right) dx & n \text{ odd} \\ = 0 & n\text{-even} \end{cases}$$

i.e. $b_{2m+1} = \frac{2}{\ell}\int_0^\ell f(x) \sin(m+\frac{1}{2})\frac{\pi x}{\ell} dx$

$b_{2m} = 0$

Then u can be written $u(x,t) = \frac{1}{2}\left[F(x+at) + F(x-at)\right]$ (same $F(x)$)

Obviously $u(x,0) = F(x)$ and $u_t(x,0) = 0$. $u(0,t) = \frac{1}{2}\left[F(at) + F(-at)\right] = 0$ (by odd prop.)

$u_x(\ell,t) = \frac{1}{2}\left[F'(\ell+at) + F'(\ell-at)\right] = 0$ (by diffn. of $*$)

P18 no1.

$$u(x,0) = \left\{\begin{array}{ll} \frac{hx}{c} & , 0 \le x \le c \\ h\left(\frac{\ell - x}{\ell - c}\right) & , c \le x \le \ell \end{array}\right\} = \phi(x), \quad u_t(x,0) = 0, \quad u(0,t) = u(\ell,t) = 0.$$

Continue ϕ oddly about $x=0$ & $x=\ell$: $F(x) = \begin{cases} \phi(x) & 0 \le x \le \ell \\ -\phi(-x) & -\ell \le x \le 0 \end{cases}$

and $F(x+2\ell) = F(x)$.

Then solve. pure in.val problem. Because F odd about zero it can be represented as F.S.S. $F(x) = \sum_1^\infty b_n \sin\left(\frac{n\pi x}{\ell}\right)$, $b_n = \frac{2}{\ell}\int_0^\ell \phi(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$.

Using defn. of $\phi(x)$ & int. by parts, $b_n = \frac{2h}{c(\ell - c)} \sin\left(\frac{n\pi c}{\ell}\right)\left(\frac{\ell}{n\pi}\right)^2$; $u(x,t) = \sum_1^\infty b_n \sin\left(\frac{n\pi x}{\ell}\right) \times \cos\left(\frac{n\pi a t}{\ell}\right)$

P100, 14

$$u_{tt} = a^2 u_{xx} + \Phi(x) \sin\omega t$$

$$\left.\begin{array}{l} u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u(0,t) = u(\ell,t) = 0 \end{array}\right\} \Rightarrow \quad u = u^{(H)} + u^{(I)}$$

where $u^{(H)}$ is soln. to homogeneous problem ($\Phi \equiv 0$) & $u^{(I)}$ is soln. to inhom. problem

<u>cont.</u>

From notes $u^{(H)}(x,t) = \sum_1^\infty \left(\phi_n \cos\omega_n t + \frac{\psi_n}{\omega_n} \sin\omega_n t \right) \sin\frac{n\pi x}{\ell}$.

where $\phi_n = \frac{2}{\ell}\int_0^\ell \phi(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$ $\qquad \psi_n = \frac{2}{\ell}\int_0^\ell \psi(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$.

Let $\Phi(x) = \sum_1^\infty \Phi_n \sin\left(\frac{n\pi x}{\ell}\right)$ where $\Phi_n = \frac{2}{\ell}\int_0^\ell \Phi(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$

(extend all f.s. oddly about $x=0$ & $x=\ell$).

Then let $u^{(I)}(x,t) = \sum_1^\infty u_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$.

$$u_{tt} - c^2 u_{xx} - \Phi(x)\sin\omega t = \sum_1^\infty \sin\left(\frac{n\pi x}{\ell}\right) \left[u_n'' + \omega_n^2 u_n - \Phi_n \overset{\sin\omega t}{\wedge} \right] = 0$$

$$\Rightarrow \quad u_n(t) = \frac{1}{\omega_n}\int_0^t \Phi_n \sin\omega\tau \sin\omega_n(t-\tau) d\tau.$$

$$= \frac{\Phi_n}{2\omega_n}\int_0^t \left[\cos(\omega_n[t-\tau]-\omega\tau) - \cos(\omega_n(t-\tau)+\omega\tau) \right] d\tau$$

$$\text{Thus } u_n(t) \begin{cases} = \frac{\Phi_n}{\omega_n}\left\{ \frac{\omega_n \sin\omega t - \omega \sin\omega_n t}{\omega_n^2 - \omega^2} \right\} & \text{for } \omega \neq \omega_n \\ = \frac{\Phi_n}{2\omega_n}\left[\frac{\sin\omega_n t}{\omega_n} - t\cos\omega_n t \right] & \text{for } \omega = \omega_n \end{cases}$$

and $u(x,t) = \sum_1^\infty \sin\left(\frac{n\pi x}{\ell}\right) \left\{ \phi_n \cos\omega_n t + \frac{\psi_n}{\omega_n}\sin\omega_n t + u_n(t) \right\}$

Note: for $\omega = \omega_n$ $|u_n| \to \infty$ as $t \to \infty$

Soln good only $|u_n| \ll 1$ i.e. for $t \ll \left| \frac{2\omega_n}{\Phi_n} \right|$

$a^2u''+A\sinh x=0$

let $u=-\sinh x$

$u'=-\cosh x$

$u''=-\sinh x$

∴ $\boxed{u_p=-\frac{A}{a^2}\sinh x}$

u_h must sat $u''=0$

$u=c_1x+c_2$

∴ $\bar{u}=c_2+c_1x-\frac{A}{a^2}\sinh x$

let $\bar{u}(0)=B$

$\bar{u}(\ell)=C$

$\bar{u}=B+c_1x-\frac{A}{a^2}\sinh x$

$\bar{u}(\ell)=B+c_1\ell-\frac{A}{a^2}\sinh \ell$

$x\frac{(C-B)}{\ell}+\frac{xA}{\ell a^2}\sinh \ell=c_1x$

∴ $\bar{u}=B+(C-B)\frac{x}{\ell}+\frac{xA}{\ell a^2}\sinh \ell-\frac{A}{a^2}\sinh x$

$a^2\bar{u}''+f_0(x)=0$ let $f_0(x)=0$

∴ $a^2\bar{u}''=0 \Rightarrow \bar{u}=c_1+c_2x$

$\bar{u}''(x)=-\frac{f_0(x)}{a^2}$

$\frac{d}{dx}[\bar{u}'(x)]=-\frac{f_0(x)}{a^2}$

$\int_0^\xi d[\bar{u}'(x)]=-\frac{1}{a^2}\int_0^\xi f_0(x)\,dx$

$\bar{u}'(\xi)-\bar{u}'(0)=-\frac{1}{a^2}\int_0^\xi f_0(x)\,dx$

$$\bar{u}'(\xi) - \bar{u}'(0) = -\frac{1}{a^2}\int_0^\xi f_0(x)dx$$

$$\bar{u}'(\xi) = \frac{d}{d\xi}[u(\eta)]$$

$$\int_0^\eta d[u(\xi)] - \int_0^\eta \bar{u}'(0)\,d\xi = -\frac{1}{a^2}\int_0^\eta\int_0^\xi f_0(x)dx\,d\xi$$

$$u(\eta) - u(0) = +\int_0^\eta \bar{u}'(0)d\xi - \frac{1}{a^2}\int_0^\eta\int_0^\xi f_0(x)dx\,d\xi$$

$$u(\eta) = u(0) + \bar{u}'(0)[\eta] - \frac{1}{a^2}\int_0^\eta\int_0^\xi f_0(x)dx\,d\xi$$

$$$

let $u(0) = u_1$

$u(l) = u_2$

$$u(0) = u(0)$$

$$u(\eta) = u_1 + \bar{u}'(0)[\eta] - \frac{1}{a^2}\int_0^\eta\int_0^\xi f_0(x)dx\,d\xi$$

$$u(l) = u_1 + \bar{u}'(0)\,l - \frac{1}{a^2}\int_0^l\int_0^\xi f_0(x)dx\,d\xi$$

Solve for $\bar{u}'(0)\eta$ $\quad \eta\frac{(u_2-u_1)}{l} + \frac{\eta}{la^2}\int_0^l\int_0^\xi f_0(x)dx\,d\xi = \bar{u}'(0)\eta$ ✓

$$\therefore ✓\; u(\eta) = u_1 + (u_2-u_1)\frac{\eta}{l} + \frac{\eta}{la^2}\int_0^l\int_0^\xi f_0(x)dx\,d\xi - \frac{1}{a^2}\int_0^\eta\int_0^\xi f_0$$

$$\therefore\; \bar{u}(x) = u_1 + (u_2-u_1)\frac{x}{l} + \frac{x}{la^2}\int_0^l\int_0^\xi f_0(\eta)d\eta\,d\xi - \frac{1}{a^2}\int_0^x\int_0^\xi f_0(\eta)d\eta$$

$f_0(x) = A\sinh x$ $\qquad u(0) = B$

$u(l) = C$

$$\bar{u}(x) = B + (C-B)\frac{x}{l} + \frac{x}{la^2}A\int_0^l\int_0^\xi \sinh\eta\,d\eta\,d\xi - \frac{A}{a^2}\int_0^x\int_0^\xi \sinh$$

$$\cosh\eta\Big|_0^\xi$$

$$\bar{u}(x) = B + (C-B)\frac{x}{l} + \frac{xA}{la^2}\int_0^l (\cosh\xi - 1)\,d\xi - \frac{A}{a^2}\int_0^x (\cosh\xi - 1)\,d\xi$$

$$\sinh\xi - \xi\Big|_0^l \qquad \sinh\xi - \xi\Big|_0^x$$

$$\bar{u} = B + (C-B)\frac{x}{l} + \frac{xA}{la^2}(\sinh l - l) - \frac{A}{a^2}(\sinh x - x)$$

now solve

$$v_{tt} = a^2 v_{xx} \quad \text{under} \quad \begin{array}{l} v(0,t)=0 \\ v(l,t)=0 \\ v(x,0) = -\bar{u}(x) \\ v_t(x,0) = 0 \end{array}$$

$$v = XT$$

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda^2 \qquad \begin{array}{l} X = x_1\cos\lambda x + x_2\sin\lambda x \\ T = t_1\cos\lambda at + t_2\sin\lambda at \end{array}$$

$$\text{B.C.} \Rightarrow \lambda = \frac{n\pi}{l},\ x_1 = 0$$

$$X = x_2 \sin\frac{n\pi x}{l}$$

$$v = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi at}{l}\sin\frac{n\pi x}{l}$$

$$\text{I.C.} \quad T(0) = t_1$$

$$T'(0) = 0 \Rightarrow t_2 = 0$$

$$T = t_1\cos\lambda at$$

$$\ni \quad v(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\text{where} \quad b_n = -\frac{2}{l}\int_0^l \bar{u}(x)\sin\frac{n\pi x}{l}\,dx$$

$$\int_0^l \bar{u}(x)\sin\frac{n\pi x}{l}\,dx = B\int_0^l \sin\frac{n\pi x}{l}\,dx + \left(\frac{C-B}{l}\right)\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

$$+ \frac{A}{la^2}(\sinh l - l)\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

$$- \frac{A}{a^2}\int_0^l (\sinh x - x)\sin\frac{n\pi x}{l}\,dx$$

$$\int_0^l \sin\frac{n\pi x}{l}\,dx = -\frac{l}{n\pi}\cos\frac{n\pi x}{l}\Big|_0^l = -\frac{l}{n\pi}\left((-1)^n - 1\right)$$

$$\int_0^l x\sin\frac{n\pi x}{l}\,dx = -\frac{xl}{n\pi}\cos\frac{n\pi x}{l}\Big|_0^l + \frac{l^2}{n^2\pi^2}\cancel{\sin\frac{n\pi x}{l}}\Big|_0^l$$

x s

dx $-\frac{l}{n\pi}\cos$

$$= -\frac{l^2}{n\pi}(-1)^n$$

$$\int_0^l \sinh x\,\sin\frac{n\pi x}{l}\,dx = \cosh x\,\sin\frac{n\pi x}{l}\Big|_0^l - \frac{n\pi}{l}\int_0^l \cosh\cos\frac{n\pi x}{l}\,dx$$

dv u

sh $\sin$

v du

$\cosh$ $\frac{n\pi}{l}\cos\frac{n\pi x}{l}$

$$\int_0^l \cosh x\,\cos\frac{n\pi x}{l}\,dx = \sinh x\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\int_0^l \sinh x\,\sin\frac{n\pi x}{l}\,dx$$

dV u

$\cosh$ $\cos$

V du

$\sinh$ $-\frac{n\pi}{l}\sin\frac{n\pi x}{l}$

$$1+\frac{n^2\pi^2}{l^2}\int = \frac{\cosh x\cancel{\sin}\frac{n\pi x}{l}\Big|_0^l - \frac{n\pi}{l}\sinh x\cos\frac{n\pi x}{l}}{1+\frac{n^2\pi^2}{l^2}}\Big|_0^l \cancel{-\frac{n^2\pi^2}{l^2}\int} = \frac{-\frac{n\pi l}{l}\sinh l\,(-1)^n}{l^2+n^2\pi^2}$$

$$\int_0^l \sinh x\,\sin\frac{n\pi x}{l}\,dx = -\frac{l}{n\pi}\sinh x\cos\frac{n\pi x}{l} + \frac{1}{n\pi}\int_0^l \cosh x\cos\frac{n\pi x}{l}\,dx$$

$\sinh x = u$ $dv = \sin$

$\cosh = du$ $v = -\frac{l}{n\pi}\cos$

$\cosh x = u$ $\cos x = dv$

$\sinh x = du$ $\frac{1}{n\pi}\sin\frac{n\pi x}{l}$

$$\frac{-\frac{l}{n\pi}\overset{\sinh l\quad(-1)^n}{\sinh x\cos\frac{n\pi x}{l}}\Big|_0^l + \frac{l^2}{(n\pi)^2}\cancel{\cosh x\sin}\frac{n\pi x}{l}\Big|_0^l}{1+l^2/n^2\pi^2} \cancel{-\frac{l^2}{n^2\pi^2}\int}$$

$$\frac{A}{a^2}\,\mathrm{si}\,(-1)^n\left\{-\frac{l}{n\pi}+\cancel{\frac{n\pi l}{l^2+n^2\pi^2}}\right\}$$

$$\cancel{\frac{-l n^3\pi^2 + l^3 + n^2\pi^2 l}{(l^2+n^2\pi^2)\,n\pi}}$$

$$b_n = -\frac{2}{l}\left\{\frac{l}{n\pi}\left[B-C(-1)^n\right]-\frac{l^3A}{a^2}\,\mathrm{si} \qquad\right\}$$

$$\frac{2}{n\pi}\left[C(-1)^n-B\right]+\frac{2A}{a^2}\,\frac{l^2\sinh l\,(-1)^n}{n\pi\,(l^2+n^2\pi^2)}$$

Solution

$u = \bar{u}+v$

$$= B+(C-B)\frac{x}{l}+\frac{xA}{la^2}(\sinh l-l)-\frac{A}{a^2}(\sinh x-x)$$

$$+\sum_{n=1}^{\infty} b_n\cos\frac{n\pi a t}{l}\,\sin\frac{n\pi x}{l}$$

$$b_n = -\frac{2}{l}\int_0^l \bar{u}(x)\sin\frac{n\pi x}{l}\,dx =$$

$$\frac{2}{n\pi}\left[C(-1)^n-B\right]+\frac{2}{n\pi}\,\frac{Al^2}{a^2}\,\frac{\sinh l\,(-1)^n}{l^2+n^2\pi^2}$$

$$i_{xx} - CL\, i_{tt} - i_t(RB+GL) - GRi = 0$$

let $y = \frac{1}{\sqrt{CL}}\, t$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial t}\sqrt{CL} \qquad f_t = \frac{1}{\sqrt{CL}} f_y$$

$$\frac{\partial}{\partial y} f_y = \frac{\partial}{\partial t}\left(f_t \sqrt{CL}\right)\sqrt{CL} \qquad f_{tt} = \frac{1}{CL} f_{yy}$$

$$i_{xx} - i_{yy} - i_y\left(\frac{RC+GL}{\sqrt{CL}}\right) - GRi = 0$$

$$y = \frac{t}{\sqrt{CL}}$$

$$i(x,0) = \varphi(x) \qquad i\big|_{y=0} = \varphi(x)$$

$$i_t\sqrt{CL} = i_y(x,0) = -\frac{\sqrt{CL}}{L}\,\psi'(x) - \frac{R}{L}\sqrt{CL}\,\varphi(x) = \psi_0(x) \qquad i_y\big|_{y=0} = \psi_0(x)$$

$$a = 0 \qquad b = -\frac{(RC+GL)}{\sqrt{CL}} \qquad c = -GR$$

define $c_1 = \frac{1}{4}(4c + b^2)$ $\quad \frac{1}{4}\left(\frac{(RC+GL)^2}{CL} - 4GR\right)$

~~$+\frac{(RC)^2 \quad 2GL(RC+GL)}{CL}$~~

~~$c_1 = \frac{RC+GL}{\sqrt{CL}}$~~

$$\varphi_1(x) = \varphi(x)\, e^{a/2\, x} = \varphi(x) \quad \text{since } a = 0$$

$$\psi_1(x) = \left(\psi_0(x) - \frac{b}{2}\varphi(x) e^{(a/2)x} = \psi_0(x) - \frac{b}{2}\varphi(x)\right.$$

and let $\lambda = 0 \qquad \mu = -\frac{b}{2}$

$\therefore$

$$i(x,y) = \frac{\varphi(x-y)\, e^{-((a-b)/2)y} + \varphi(x+y)\, e^{((a+b)/2)y}}{2}$$

$$-\frac{1}{2} e^{(b/2)y} \int_{x-y}^{x+y} \left\{ \frac{b}{2} J_0\left(\sqrt{c_1 \langle (x-\xi)^2 - y^2 \rangle}\right.\right.$$

$$\left. + \sqrt{c_1}\, y\, \frac{J_1\left(\sqrt{c_1 \langle (x-\xi)^2 - y^2 \rangle}\right.}{\sqrt{(x-\xi)^2 - y^2}} \right\} e^{-\frac{a}{2}(x-\xi)}\, \varphi(\xi)\, d\xi$$

$$$$$$

$$+\frac{1}{2}e^{\frac{by}{2}}\int_{x-y}^{x+y}J_0\left(\sqrt{c_1\langle(x-\xi)^2-y^2\rangle}\right)e^{-\frac{a}{2}(x-\xi)}\psi_0(\xi)\,d\xi$$

reduces to

$$i(x,y)=\underline{\varphi(x-y)e^{\frac{by}{2}}+\varphi(x+y)}e^{\frac{by}{2}}$$

$$-\frac{1}{2}e^{\frac{by}{2}}\int_{x-y}^{x+y}\left\{\frac{b}{2}J_0\left(\sqrt{c_1\langle(x-\xi)^2-y^2\rangle}\right)\right.$$

$$\left.+\sqrt{c_1}\,y\,\frac{J_1\left(\sqrt{c_1\langle(x-\xi)^2-y^2\rangle}\right)}{\sqrt{(x-\xi)^2-y^2}}\right\}\varphi(\xi)\,d\xi$$

$$+\frac{1}{2}e^{\frac{by}{2}}\int_{x-y}^{x+y}J_0\left(\sqrt{c_1\langle(x-\xi)^2-y^2\rangle}\right)\psi_0(\xi)\,d\xi$$

where

$$b=-\frac{(RC+GL)}{\sqrt{CL}}\qquad c=-GR$$

$$c_1=\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2\qquad y=\frac{t}{\sqrt{CL}}$$

$$\psi_0(x)=-\frac{C}{\sqrt{CL}}\psi'(x)-\frac{CR}{\sqrt{CL}}\varphi(x)$$

~~$i(x,y)=\left\{\frac{\varphi(x-y)+\varphi(x+y)}{2}-\frac{1}{2}\left[\int_{x-y}^{x+y}\frac{b}{2}\right.\right.$~~

$$i(x,t)=\frac{\varphi\left(x-\frac{t}{\sqrt{CL}}\right)+\varphi\left(x+\frac{t}{\sqrt{CL}}\right)}{2}e^{-\frac{(RC+GL)t}{2CL}}$$

$$-\frac{1}{2}e^{-\frac{(RC+GL)}{2CL}t}\int_{x-\frac{t}{\sqrt{CL}}}^{x+t/\sqrt{CL}}\left\{-\frac{(RC+GL)}{2\sqrt{CL}}J_0\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2\left\langle(x-\xi)^2-\frac{t^2}{CL}\right\rangle}\right)\right.$$

$$\left.+\frac{RC-GL}{2CL}t\,\frac{J_1\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2\left\langle(x-\xi)^2-\frac{t^2}{CL}\right\rangle}\right)}{\sqrt{(x-\xi)^2-\frac{t^2}{CL}}}\right\}\varphi(\xi)\,d\xi$$

$$-\frac{1}{2}e^{-\frac{(RC+GL)}{2CL}t}\int_{x-\frac{t}{\sqrt{CL}}}^{x+\frac{t}{\sqrt{CL}}}J_0\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2\left\langle(x-\xi)^2-\frac{t^2}{CL}\right\rangle}\right)\left\{\frac{C}{\sqrt{CL}}\psi'(\xi)\right.$$

$$\left.+\frac{CR}{\sqrt{CL}}\varphi(\xi)\right\}d\xi$$

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$$$$$$$$$$$
$$$$$$$$$$$
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$$i(x,t) = e^{-\frac{(RC+GL)}{2CL}t} \frac{\varphi\left(x-\frac{t}{\sqrt{CL}}\right)+\varphi\left(x+\frac{t}{\sqrt{CL}}\right)}{2}$$

$$-\frac{e^{-\frac{(RC+GL)}{2CL}t}}{2} \int_{x-\frac{t}{\sqrt{CL}}}^{x+\frac{t}{\sqrt{CL}}} \left\{ \frac{(RC-GL)}{2\sqrt{CL}} J_0\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2 \left\langle (x-\xi)^2-\frac{t^2}{CL}\right\rangle}\right)\right.$$

$$\left. + \frac{RC-GL}{2\sqrt{CL}} t \frac{J_1\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2\left\langle (x-\xi)^2-\frac{t^2}{CL}\right\rangle}\right)}{\sqrt{(x-\xi)^2-\frac{t^2}{CL}}} \right\} \varphi(\xi)d\xi$$

$$-\frac{1}{2}e^{-\frac{(RC+GL)}{2CL}t} \int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}} \sqrt{\frac{C}{L}} \left\{ J_0\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2\left\langle (x-\xi)^2-\frac{t^2}{CL}\right\rangle}\right)\right\} \psi'(\xi)d\xi$$

$$J_0(x) = 1 - \frac{(x/2)^2}{1!} + \frac{(x/2)^4}{(2!)^2} - \dots$$

$$J_1(x) = \frac{x}{2}\left[1 - \frac{(x/2)^2}{1 \cdot 2} + \frac{(x/2)^4}{1\cdot 2\cdot 2\cdot 3} - \dots\right]$$

$$J_0(\sqrt{x}) = 1 - \frac{(x/4)}{1!} + \frac{(x/4)^2}{(2!)^2} - \dots$$

$$J_1(\sqrt{x}) = \frac{\sqrt{x}}{2}\left[1 - \frac{(x/4)}{1\cdot 2} + \frac{(x/4)^2}{1\cdot 2\cdot 2\cdot 3} - \dots\right]$$

for $G \to 0, R \to 0$

$$i(x,t) = \frac{\varphi\left(x-\frac{t}{\sqrt{CL}}\right) + \varphi\left(x+t/\sqrt{CL}\right)}{2}$$

$$-\frac{1}{2}\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}} \sqrt{\frac{C}{L}}\, \psi'(\xi)d\xi \qquad \frac{1}{\sqrt{CL}}$$

$$-\frac{(RC+GL)}{2\sqrt{CL}} + \frac{2CR}{2\sqrt{CL}}$$

$$\frac{CR-GL}{2\sqrt{CL}}$$

$$-\frac{1}{2}\sqrt{\frac{C}{L}}\left\{\psi\left(x+t/\sqrt{CL}\right) - \psi\left(x-t/\sqrt{CL}\right)\right\}$$

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$$$$$$$$$$$
$$$$$$$$$$$
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using perturbation

let $u(x,t,\epsilon) = \sum u_i \epsilon^i$

by use of diff eq

$$\sum_{i=2}^{\infty}\left[(u_i)_{xx} - (u_i)_{yy} - (u_{i-1})_y \frac{(c+L)}{\sqrt{cL}} - u_{i-2}\right]\epsilon^i = 0$$

$$(u_0)_{xx} - (u_0)_{yy} = 0$$

$$(u_1)_{xx} - (u_1)_{yy} - (u_0)_y \frac{(c+L)}{\sqrt{cL}} = 0$$

let $i(x,0,\epsilon) = \sum \varphi_i(x)\epsilon^i \quad = \varphi(x)$

$$i_y(x,0,\epsilon) = \sum \psi_i(x)\epsilon^i \quad = -\frac{\sqrt{cL}}{L}\psi'(x) - \frac{\epsilon\sqrt{cL}}{L}\varphi(x)$$

$$\varphi_0 = \varphi(x) \qquad \varphi_1 = 0 \qquad \varphi_2 = 0 \quad \cdots\cdots\cdots$$

$$\psi_0 = -\frac{\sqrt{cL}}{L}\psi'(x) \qquad \psi_1 = -\frac{\sqrt{cL}}{L}\varphi \qquad \psi_2 = 0 \quad \cdots\cdots$$

$$u_0 = f_1(x+y) + f_2(x-y) \qquad u_0\Big|_{y=0} = \varphi(x)$$

$$(u_0)_y\Big|_{y=0} = -\frac{\sqrt{cL}}{L}\psi'(x)$$

$$u_0(x,y) = \frac{\varphi(x+y)+\varphi(x-y)}{2} - \sqrt{\frac{c}{L}}\left\{\frac{\psi(x+y)-\psi(x-y)}{2}\right\} \qquad f_1 = \frac{\varphi - \sqrt{\frac{c}{L}}\,\psi}{2}$$

$$f_2 = \frac{\varphi + \sqrt{c/L}\,\psi}{2}$$

$$(u_0)_y = \frac{\varphi'(x+y) - \varphi'(x-y)}{2} - \sqrt{\frac{c}{L}}\left\{\frac{\psi'(x+y)+\psi'(x-y)}{2}\right\}$$

$$(u_1)_{xx} - (u_1)_{yy} = \frac{\varphi'(x+y) - \varphi'(x-y)}{2\sqrt{cL}}(c+L) - \frac{(c+L)}{L}\left\{\frac{\psi'(x+y) + \psi'(x-y)}{2}\right\}$$

~~$u_1(x,t) = \frac{c+L}{2\sqrt{cL}}\int_{x-y}^{x+y}\varphi(\xi)\,d\xi - \frac{c+L}{2L}\int_{x-y}^{x+y}\psi(\xi)\,d\xi$~~ ‖

$$u_{1x} = \frac{C+L}{2\sqrt{CL}}\left[\varphi(x+y)-\varphi(x-y)\right] - \frac{C+L}{2L}\left[\psi(x+y)-\psi(x-y)\right]$$

$$u_{1xx} = \frac{C+L}{2\sqrt{CL}}\left[\varphi'(x+y)-\varphi'(x-y)\right] - \frac{C+L}{2L}\left[\psi'(x+y)-\psi'(x-y)\right]$$

$$u_{1y} = \frac{C+L}{2\sqrt{CL}}\left[\varphi(x+y)+\varphi(x-y)\right] - \frac{C+L}{2L}\left[\psi(x+y)+\psi(x-y)\right]$$

$$u_{1yy} = \frac{C+L}{2\sqrt{CL}}\left[\varphi'(x+y)-\varphi'(x-y)\right] - \frac{C+L}{2L}\left[\psi'(x+y)-\psi'(x-y)\right]$$

$$\left| \; u_{1xx} - u_{1yy} = 0 \; \right|$$

$$i(x,y) = + \frac{1}{2}\sqrt{\frac{C}{L}}\int_{x-y}^{x+y}\varphi(\xi)\,d\xi + \frac{C+L}{2}\int_0^y\int_{x-(y-\eta)}^{x+(y-\eta)} \frac{\varphi'(\xi+\eta)-\varphi'(\xi-\eta)}{2\sqrt{CL}}\,d\xi\,d\eta$$

$$- \frac{C+L}{2}\int_0^y\int_{x-(y-\eta)}^{x+(y-\eta)} \frac{\psi'(\xi+\eta)+\psi'(\xi-\eta)}{2L}\,d\xi\,d\eta$$

$$i(x,y) = -\frac{1}{2}\sqrt{\frac{C}{L}}\int_{x-y}^{x+y}\varphi(\xi)\,d\xi + \frac{C+L}{2}\int_0^y \left.\frac{\varphi(\xi+\eta)-\varphi(\xi-\eta)}{2\sqrt{CL}}\right|_{x-(y-\eta)}^{x+(y-\eta)} d\eta$$

$d\eta \to 0 \to t/\sqrt{CL}$

$dt = \sqrt{CL}\,d\eta \quad 0 \to t$

$$+ \frac{C+L}{4\sqrt{CL}}\int_0^y \left[\varphi(x+y) - \varphi(x-y+2\eta) - \varphi(x+y-2\eta) + \varphi(x-y)\right],$$

$$= -\frac{1}{2}\sqrt{\frac{C}{L}}\int_{x-y}^{x+y}\varphi(\xi)\,d\xi + \frac{C+L}{4\sqrt{CL}}\int_0^y\left[\varphi(x+y)+\varphi(x-y) - \left\{\varphi(x-y+2\eta)+\varphi(x+y-2\eta)\right\}\right]d\eta$$

$$- \frac{C+L}{4L}\int_0^y\left[\psi(x+y)-\psi(x-y) - \left\{\psi(x-y+2\eta)-\psi(x+y-2\eta)\right\}\right]d\eta$$

$$\bar{i} = i_0 + \epsilon i_1$$

$$i_0 = \frac{\varphi(x+y)+\varphi(x-y)}{2} - \sqrt{\frac{C}{L}}\left\{\frac{\psi(x+y)-\psi(x-y)}{2}\right\}$$

$$i_0 = \frac{\varphi(x + t/\sqrt{CL}) + \varphi(x - t/\sqrt{CL})}{2} - \sqrt{\frac{C}{L}}\left\{\frac{\psi(x + t/\sqrt{CL}) - \psi(x - t/\sqrt{CL})}{2}\right\}$$

$$i_1 = -\frac{1}{2}\sqrt{\frac{C}{L}}\int_{x - t/\sqrt{CL}}^{x + t/\sqrt{CL}} \varphi(\xi)d\xi + \frac{C+L}{4\sqrt{CL}}\int_0^t \Big[\varphi(x + t/\sqrt{CL}) + \varphi(x - t/\sqrt{CL}) - \Big\{\varphi\Big(x - \frac{[t-2\tau]}{\sqrt{CL}}\Big) + \varphi\Big(x + \frac{[t-2\tau}{\sqrt{CL}}$$

$$- \frac{C+L}{4L}\int_0^t \Big[\psi\Big(x + \frac{t}{\sqrt{CL}}\Big) - \psi\Big(x - \frac{t}{\sqrt{CL}}\Big) - \Big\{\psi\Big(x - \frac{[t-2\tau]}{\sqrt{CL}}\Big) - \psi\Big(x + \frac{[t-2\tau]}{\sqrt{CL}}\Big)\Big\}\Big] d\tau$$

---

let R & G be ϵ $\therefore$ $e^{-\frac{(RC+GL)}{2CL}t} = e^{-\epsilon\frac{(C+L)}{2CL}t} = e^{-\epsilon\delta}$

$$e^{-\epsilon\delta} \cong 1 - \epsilon\frac{\delta}{1!} + \epsilon^2\frac{\delta^2}{2!}$$

$$J_0\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2 \langle (x-\xi)^2 - y^2 \rangle}\right) = J_0\left(\sqrt{\frac{\epsilon^2}{4}\frac{(C-L)^2}{CL}\langle \qquad \rangle}\right)$$

$$\cong 1 - \frac{\frac{\epsilon^2}{4}\frac{(C-L)^2}{CL}\langle (x-\xi)^2 - y^2 \rangle}{4 \cdot 1!}$$

$$\frac{J_1}{x} \cong \frac{\frac{\epsilon(C-L)}{2\sqrt{CL}}}{2}\left[1 - \frac{\frac{\epsilon^2}{4}\frac{(C-L)^2}{CL}\langle x-\xi)^2 - y^2 \rangle}{1 \cdot 2 \cdot 4}\right]$$

$$\frac{\varphi\left(x - \frac{t}{\sqrt{CL}}\right) + \varphi\left(x + \frac{t}{\sqrt{CL}}\right)}{2} - \epsilon\frac{C+L}{2CL}t\;\frac{\varphi(x - t/\sqrt{CL}) + \varphi(x + t/\sqrt{CL})}{2}$$

$$-\frac{1}{2}\left(1 - \epsilon\frac{C+L}{2CL}t + \epsilon^2\frac{(C+L)^2 t^2}{4C^2L^2 \cdot 2!}\right)\left(\epsilon\frac{(C-L)}{2\sqrt{CL}}\right)\int_{x - t/\sqrt{CL}}^{x + t/\sqrt{CL}}\left\{1 - \frac{\frac{\epsilon^2}{4}\frac{(C-L)^2}{CL}\langle (x-\xi)^2 - t^2/CL \rangle}{4 \cdot 1!}\right\}\varphi(\xi)d\xi$$

$$-\frac{1}{2}\left(1 - \epsilon\frac{C+L}{2CL}t + \epsilon^2\frac{(C+L)^2 t^2}{4C^2L^2 \cdot 2!}\right)\left(\epsilon\frac{(C-L)t}{2CL}\right)\int_{x - t/\sqrt{CL}}^{x + t/\sqrt{CL}}\left\{\frac{\frac{\epsilon(C-L)}{2\sqrt{CL}}}{2}\left[1 - \frac{\frac{\epsilon^2}{4}\frac{(C-L)^2}{CL}\langle (x-\xi)^2 - t^2/CL \rangle}{1 \cdot 2 \cdot 4}\right]\right\}\varphi(\xi)$$

$$-\frac{1}{2}\left(\overset{\checkmark}{1}-\epsilon\,\overset{\checkmark}{\frac{C+L}{2CL}}t+\epsilon^2\frac{(C+L)^2t^2}{4C^2L^2\cdot 2!}\right)\sqrt{\frac{C}{L}}\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}}\left\{\overset{\checkmark}{1}-\frac{\frac{\epsilon^2}{4}\frac{(C-L)^2}{CL}\left\langle(x-\xi)^2-t^2/CL\right\rangle}{4\cdot 1!}\right\}\psi'(\xi)\,d\xi$$

Collect all zeroth order terms.

$$i_0(x,t)=\frac{\varphi(x-t/\sqrt{CL}+\varphi(x+t/\sqrt{CL})}{2}-\frac{1}{2}\sqrt{\frac{C}{L}}\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}}\psi'(\xi)\,d\xi$$

$$=\frac{\varphi(x-t/\sqrt{CL})+\varphi(x+t/\sqrt{CL})}{2}-\frac{1}{2}\sqrt{\frac{C}{L}}\left\{\psi(x+t/\sqrt{CL})-\psi'(x-t/\sqrt{CL})\right\}$$

Collect all 1st order terms

$$i_1(x,t)=-\frac{C+L}{2CL}t\left\{\frac{\varphi(x-t/\sqrt{CL})+\varphi(x+t/\sqrt{CL})}{2}\right\}-\frac{(C-L)}{4\sqrt{CL}}\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}}\varphi(\xi)\,d\xi$$

$$+\frac{(C+L)t}{4CL}\sqrt{\frac{C}{L}}\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}}\psi'(\xi)\,d\xi$$

$$=-\frac{C+L}{2CL}t\left\{\frac{\varphi(x-t/\sqrt{CL})+\varphi(x+t/\sqrt{CL})}{2}\right\}-\frac{(C-L)}{4\sqrt{CL}}\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}}\varphi(\xi)\,d\xi$$

$$+\frac{C+L}{4CL}t\sqrt{\frac{C}{L}}\left\{\psi(x+t/\sqrt{CL})-\psi(x-t/\sqrt{CL})\right\}$$

16

Cesar Levy
763.2130

Given: $u_{tt} = a^2 u_{xx} + A\sinh x$

$u(x,0) = 0 \qquad u_t(x,0) = 0$

$u(0,t) = B \qquad u(l,t) = C$

where A, B, C are constant

this is the same as solving

$$u(x,t) = \bar{u}(x) + v(x,t) \quad ✓$$

where

$$a^2 \bar{u}''(x) + A\sinh x = 0$$

$$\bar{u}(0) = B \qquad \bar{u}(\ell) = C \quad ✓$$

and

$$v_{tt} = a^2 v_{xx}$$

$$v(0,t) = 0$$

$$v(\ell,t) = 0$$

$$v(x,0) = -\bar{u}(x)$$

$$v_t(x,0) = 0 \quad ✓$$

$$\bar{u}(x) = B + (C-B)\frac{x}{\ell} + \frac{x}{\ell a^2}\int_0^\ell\int_0^\xi A\sinh\eta\, d\eta\, d\xi - \frac{1}{a^2}\int_0^x\int_0^\xi A\sinh\eta\, d\eta\, d\xi$$

$$\bar{u}(x) = B + (C-B)\frac{x}{\ell} + \frac{xA}{\ell a^2}(\sinh\ell - \ell) - \frac{A}{a^2}(\sinh x - x) \quad ✓$$

By separation of variables

$$v = XT \implies v_{tt} - a^2 v_{xx} = T''X - a^2 X''T = 0$$

and

$$X = a\sin\lambda x + b\cos\lambda x$$

$$T = c\sin a\lambda t + d\cos a\lambda t$$

with bc $\quad X = a_n \sin\lambda_n x \qquad \lambda_n = \frac{n\pi}{\ell}$ ✓

and $v(x,t) = \sum_{n=1}^{\infty}\left(A_n \sin a\lambda_n t + B_n \cos a\lambda_n t\right)\sin\lambda_n x$ ✓

where $-\bar{u}(x) = \sum_{n=1}^{\infty} B_n \sin\lambda_n x \qquad B_n = -\frac{2}{\ell}\int_0^\ell \bar{u}(x)\sin\lambda_n x\, dx$

and $0 = \sum_{n=1}^{\infty} A_n a\lambda_n \sin\lambda_n x \implies A_n = 0 \quad \forall n$ ✓

$$\int_0^l \bar{u}(x)\sin\frac{n\pi x}{l}\,dx = B\int_0^l \sin\lambda_n x\,dx + \frac{(C-B)}{l}\int_0^l x\sin\lambda_n x\,dx$$

$$+\frac{A}{la^2}(\sinh l - l)\int_0^l x\sin\lambda_n x\,dx - \frac{A}{a^2}\int_0^l(\sinh x - x)\sin\lambda_n x\,dx$$

$$\int_0^l \sin\lambda_n x\,dx = -\frac{l}{n\pi}\cos\frac{n\pi x}{l}\Big|_0^l = -\frac{l}{n\pi}\left[(-1)^n - 1\right]$$

$$\int_0^l x\sin\lambda_n x\,dx = -\frac{xl}{n\pi}\cos\frac{n\pi x}{l}\Big|_0^l + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\Big|_0^l = -\frac{l^2}{n\pi}(-1)^n$$

$$\int_0^l \sinh x\sin\frac{n\pi x}{l}\,dx = \cosh x\sin\frac{n\pi x}{l}\Big|_0^l - \frac{n\pi}{l}\sinh x\cos\frac{n\pi x}{l}\Big|_0^l - \frac{n^2\pi^2}{l^2}\int_0^l \sinh x\sin\lambda_n x\,dx$$

$$\therefore \int_0^l \sinh x\sin\frac{n\pi x}{l}\,dx = \frac{-\frac{n\pi}{l}\sinh l\,(-1)^n}{(l^2+n^2\pi^2)/l^2} = \frac{-n\pi l\sinh l\,(-1)^n}{l^2+n^2\pi^2}$$

putting all these into equation for $-\frac{2}{l}\int_0^l \bar{u}(x)\sin\frac{n\pi x}{l}\,dx$

gives
$$B_n = \frac{2}{n\pi}\left[C(-1)^n - B\right] + \frac{2A}{a^2}\,\frac{l^2\sinh l\,(-1)^n}{n\pi\,(l^2+n^2\pi^2)} \quad \checkmark$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n\cos\frac{n\pi a t}{l}\sin\frac{n\pi x}{l} + B + (C-B)\frac{x}{l} + \frac{xA}{la^2}(\sinh l - l) - \frac{A}{a^2}(\sinh x - x)$$

$$B_n = -\frac{2}{l}\int_0^l \bar{u}(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{n\pi}\left[C(-1)^n - B\right] + \frac{2}{n\pi}\,\frac{Al^2}{a^2}\,\frac{\sinh l\,(-1)^n}{l^2+n^2\pi^2}$$

✓

1. taking both eqs & solving for i

$$i_{xx} - CL\, i_{tt} - i_t\,(RC+GL) - GR\,i = 0 \quad \checkmark \quad (1)$$

let

$$y = \frac{t}{\sqrt{CL}} \checkmark \qquad \therefore \qquad i_{yy} = CL\; i_{yy} \qquad i_y = \sqrt{CL}\; i_t$$

then (1) can be reduced to

$$i_{xx} - i_{yy} - i_y\ \frac{(RC+GL)}{\sqrt{CL}} - GR\,i = 0 \qquad (2)$$

conditions are

$$i(x,0) = \varphi(x)$$

$$\sqrt{CL}\; i_t(x,0) = i_y(x,0) = -\frac{\sqrt{CL}}{L}\,\psi'(x) - \frac{R\sqrt{CL}}{L}\,\varphi(x) = \psi_0(x) \checkmark$$

we can reduce (2) to a form

$$I_{xx} - I_{yy} - c_1 I = 0$$

by letting

$$I = i\,e^{\lambda x + \mu y} \qquad \lambda = 0 \quad \mu = -\frac{(RC+GL)}{\sqrt{CL}} \checkmark$$

$$c_1 = \frac{1}{4}\left\{-4GR + \frac{(RC+GL)^2}{CL}\right\} = \left(\frac{RC-GL}{2\sqrt{CL}}\right)^2 \checkmark$$

by using (2.5-35)

$$i(x,y) = \left[\frac{\varphi(x-y)+\varphi(x+y)}{2}\right] e^{\frac{\mu y}{2}} - \frac{1}{2}e^{\frac{\mu y}{2}}\int_{x-y}^{x+y}\left\{\frac{\mu}{2}\,J_0\left(\sqrt{c_1\langle(x-\xi)^2-y^2\rangle}\right)\right.$$

$$\left. + \sqrt{c_1}\,y\ \frac{J_1\left(\sqrt{c_1\langle(x-\xi)^2-y^2\rangle}\right)}{\sqrt{(x-\xi)^2-y^2}}\right\}\varphi(\xi)\,d\xi$$

$$+ \frac{1}{2}e^{\frac{\mu y}{2}}\int_{x-y}^{x+y} J_0\left(\sqrt{c_1\langle(x-\xi)^2-y^2\rangle}\right)\psi_0(\xi)\,d\xi$$

making all substitution results in

$$i(x,t) = \left[\frac{\varphi(x-t/\sqrt{CL}) + \varphi(x+t/\sqrt{CL})}{2}\right] e^{-\frac{(RC+GL)t}{2CL}} - \frac{1}{2}e^{-\frac{(RC+GL)t}{2CL}}\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}}\left\{\frac{(RC-GL)}{2\sqrt{CL}}\cdot\right.$$

$$\left. J_0\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2\langle(x-\xi)^2-t^2/CL\rangle}\right) + \frac{RC-GL}{2CL}\,t\ \frac{J_1\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2\langle(x-\xi)^2-t^2/CL\rangle}\right)}{\sqrt{(x-\xi)^2-t^2/CL}}\right\}\cdot$$

④

$$\varphi(\xi)\,d\xi - \frac{1}{2}e^{-\frac{(RC+GL)t}{2CL}}\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}}\sqrt{\frac{C}{L}}\ J_0\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2\langle(x-\xi)^2-t^2/CL\rangle}\right)\Bigg\}\,\psi'(\xi)\,d\xi \checkmark$$

Similarly for $v(x,t)$

$$v(x,t) = \left[\frac{\psi(x - t/\sqrt{CL}) + \psi(x + t/\sqrt{CL})}{2}\right] e^{-\left(\frac{RC+GL}{2CL}\right)t}$$

$$- \frac{1}{2} e^{-\left(\frac{RB+GL}{2CL}\right)t} \int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}} \left\{ \frac{(RC-GL)}{2\sqrt{CL}} \; J_0\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2 \left\langle (x-\xi)^2 - t^2/CL \right\rangle}\right)\right.$$

$$\left. + \frac{RC-GL}{2CL} t \; \frac{J_1\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2 \left\langle (x-\xi)^2 - t^2/CL \right\rangle}\right)}{\sqrt{(x-\xi)^2 - t^2/CL}} \right\} \psi(\xi)\, d\xi$$

$$- \frac{1}{2} e^{-\left(\frac{RC+GL}{2CL}\right)t} \int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}} \sqrt{\frac{L}{C}} \; J_0\left(\sqrt{\left(\frac{RC-GL}{2\sqrt{CL}}\right)^2 \left\langle (x-\xi)^2 - t^2/CL \right\rangle}\right) \varphi'(\xi)\, d\xi$$

④

for $G = R = 0$ $\quad e^{\frac{\mu y}{2}} = 1 \quad J_0(\text{argument}) = 1$ ✓

$$\frac{J_1(\text{argument})}{\eta} = \frac{1}{2}\sqrt{c_1} = 0 \; ✓ \qquad \eta = \sqrt{(x-\xi)^2 - t^2/CL} \qquad \text{argument} = \sqrt{c_1}\,\eta$$

$\therefore$

$$i(x,t) = \frac{\varphi(x - t/\sqrt{CL}) + \varphi(x + t/\sqrt{CL})}{2} - \frac{1}{2} \int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}} \sqrt{\frac{C}{L}}\, \psi'(\xi)\, d\xi$$

$$= \frac{\varphi(x - t/\sqrt{CL}) + \varphi(x + t/\sqrt{CL})}{2} - \sqrt{\frac{C}{L}} \left\{ \frac{\psi(x + t/\sqrt{CL}) - \psi(x - t/\sqrt{CL})}{2} \right\}$$

this is the solution to $i_{xx} - i_{yy} = 0$ $\qquad y = t/\sqrt{CL}$

with $i(x,0) = \varphi(x)$

$u_y(x,0) = -\frac{\sqrt{CL}}{L}\psi'(x)$

for $G = R = \epsilon$

$$e^x \cong 1 - x$$

$$J_0(x) \cong 1 - \frac{(x/2)^2}{1!}$$

$$J_1(x) \cong \frac{x}{2}\left[1 - \frac{(x/2)^2}{1 \cdot 2}\right] \; ✓$$

$$\therefore \quad i(x,t) \cong \frac{\varphi(x-t/\sqrt{CL})+\varphi(x+t/\sqrt{CL})}{2} - \epsilon\,\frac{C+L}{2CL}\,t\left[\frac{\varphi(x-t/\sqrt{CL})+\varphi(x+t/\sqrt{CL})}{2}\right]$$

$$-\frac{1}{2}\left(1-\epsilon\,\frac{C+L}{2CL}\,t\right)\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}} \epsilon\left(\frac{C-L}{2\sqrt{CL}}\right)\left\{1-\frac{\frac{\epsilon^2}{4}\frac{(C-L)^2}{CL}\langle (x-\xi)^2-\frac{t^2}{CL}\rangle}{4\cdot 1!}\right.$$

$$\left.+\,\epsilon\,\frac{(C-L)}{2CL}\,t \qquad \frac{\epsilon\,(C-L)}{4\sqrt{CL}}\left[1-\frac{\frac{\epsilon^2}{4}\frac{(C-L)^2}{CL}\langle (x-\xi)^2-t^2/CL\rangle}{1\cdot 2\cdot 4}\right]\right\}\varphi(\xi)\,d\xi$$

$$-\frac{1}{2}\left(1-\epsilon\,\frac{C+L}{2CL}\,t\right)\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}} \sqrt{C/L}\left\{1-\frac{\frac{\epsilon^2}{4}\frac{(C-L)^2}{CL}\langle (x-\xi)^2-t^2/CL\rangle}{4\cdot 1!}\right\}\psi'(\xi)\,d\xi$$

Collect terms of zeroth power of epsilon

$$\frac{\varphi(x-t/\sqrt{CL})+\varphi(x+t/\sqrt{CL})}{2} - \frac{1}{2}\sqrt{\frac{C}{L}}\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}} \psi'(\xi)\,d\xi \quad = i_0(x,t)$$

Collect terms of 1st power of epsilon

$$-\frac{C+L}{2CL}\,t\left[\frac{\varphi(x-t/\sqrt{CL})+\varphi(x+t/\sqrt{CL})}{2}\right] - \frac{C-L}{4\sqrt{CL}}\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}} \varphi(\xi)\,d\xi$$

$$+\ \frac{C+L}{2CL}\,t\sqrt{\frac{C}{L}}\left\{\frac{1}{2}\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}}\psi'(\xi)\,d\xi \quad = i_1(x,t)\right.$$

collect terms in 2nd power of epsilon

$$\frac{(C+L)^2}{8C^2L^2}\,t^2\left[\frac{\varphi(x-t/\sqrt{CL})+\varphi(x+t/\sqrt{CL})}{2}\right] + \int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}}\left\{\frac{C+L}{4CL}\,t\left(\frac{C-L}{2\sqrt{CL}}\right) - \frac{C-L}{4CL}\,t\left(\frac{C-L}{4\sqrt{CL}}\right)\right\}\varphi(\xi)\,d\xi$$

$$+\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}} \sqrt{\frac{C}{L}}\left\{\frac{\frac{(C-L)^2}{CL}\langle (x-\xi)^2-t^2/CL\rangle}{8\cdot 4\cdot 1} - \frac{(C+L)^2}{8C^2L^2}\,t^2\right\}\psi'(\xi)\,d\xi \quad = i_2(x,t)$$

$$i(x,t) \cong i_0(x,t) + \epsilon\, i_1(x,t) + \epsilon^2\, i_2(x,t)$$

By collecting more & more terms we can write

$i(x,t) = \sum_{j=0}^{\infty} i_j(x,t)\,\epsilon^j$ which is a power series in epsilon in a similar manner one can solve for $v(x,t)$ for $G=R=\epsilon$ and show that for $\epsilon \to 0$

$$v_{xx} - v_{yy} = 0 \qquad y = t/\sqrt{CL}$$

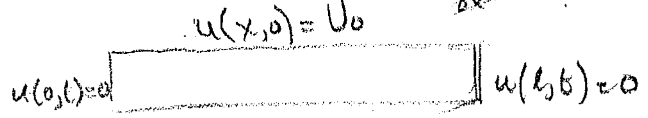
with initial condition $v(x,0) = \phi(x)$
and $v_y(x,0) = -\sqrt{\frac{L}{C}}\,\phi'(x)$

returns the D'alembert Formula

$$v(x,t) = \frac{\phi(x+t/\sqrt{CL}) + \phi(x-t/\sqrt{CL})}{2} - \frac{1}{2}\sqrt{\frac{L}{C}}\int_{x-t/\sqrt{CL}}^{x+t/\sqrt{CL}} \phi'(\xi)\,d\xi$$

✓ ①

~~$\frac{\partial u}{\partial x}(x,t)$~~



$$u_t - a^2 u_{xx} = 0$$

$$\frac{X''}{X} = \frac{T'}{a^2 T} = -\lambda^2$$

$$X'' + \lambda^2 X = 0$$

~~$X = a\sin\lambda x + b\cos\lambda x$~~

$\sin\frac{n\pi x}{l}$

$$X = a\sin\lambda x + b\cos\lambda x$$

$X(0)=0 \quad b=0$

$X(l)=0 \quad \lambda = \frac{n\pi}{L}$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\lambda_n x \, e^{-\lambda^2 a^2 t}$$

$$u(x,0) = \sum c_n \sin\lambda_n x$$

$$U_0 = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{L}x$$

$$c_n = \frac{2}{L}\int_0^L U_0 \sin\frac{n\pi x}{L}\,dx$$

since uniformly heated $\quad U_0 = \text{constant}$

$$c_n = \frac{2U_0 L}{L n\pi}\cos\frac{n\pi x}{L}\Big|_0^L = -\frac{2U_0}{n\pi}\left[(-1)^n - 1\right]$$

n even $\quad c_n = 0$

n odd $\quad c_n = \frac{4U_0}{n\pi} \qquad$ let $n = 2k-1$

$k = 1, 2, \ldots$

$$u(x,t) = \sum_{k=1}^{\infty} \frac{4U_0}{(2k-1)\pi}\sin\frac{(2k-1)\pi x}{L}\, e^{-\frac{(2k-1)^2 a^2\pi^2 t}{L^2}}$$

F

$$\bar{u}(x) = u_1 + \frac{x}{l}(u_2-u_1)$$

$$v_t = a^2 v_{xx}$$

$$v(x,0) = u_0 - u_1 - \frac{x}{l}(u_2-u_1)$$

$$v(0,t) = 0 \quad v(l,t) = 0$$

$$v \quad \sum c_n \sin \frac{n\pi}{l} x \; e^{-\frac{n^2\pi^2}{l^2}a^2 t}$$

$$v(x,0) = \sum c_n \sin \frac{n\pi x}{l} = (u_0 - u_1) - \frac{x}{l}(u_2-u_1)$$

$$c_n = \frac{2}{l}\int_0^l (u_0-u_1)\sin\frac{n\pi x}{l}\,dx \; - \frac{2}{l^2}(u_2-u_1)\int_0^l x\sin\frac{n\pi x}{l}dx$$

$$- \frac{2(u_0-u_1)}{l}\,\frac{l}{n\pi}\cos\frac{n\pi x}{l}\Big|_0^l$$

$$- \frac{2(u_0-u_1)}{n\pi}\Big[(-1)^n - 1\Big] - \frac{2(u_2-u_1)}{l^2}\Big[-\frac{xl}{n\pi}\cos\frac{n\pi x}{l}\Big|_0^l + \cancel{\frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi x}{l}\right)_0^l}$$

$$x \sin\frac{n\pi x}{l}\,dx$$

$$dx \; -\frac{l}{n\pi}\cos\frac{n\pi x}{l}\,dx \qquad \Big[-\frac{l^2}{n\pi}(-1)^n\Big]$$

$$-\frac{2(u_0-u_1)}{n\pi}\Big[(-1)^n - 1\Big] + \frac{2(u_2-u_1)}{n\pi}(-1)^n$$

$$v(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\Big\{(u_2-u_1)(-1)^n + (u_1-u_0)\big[(-1)^n - 1\big]\Big\}\sin\frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2}{l^2}a^2 t}$$

$$u_2(-1)^n - u_1(-1)^n + u_1(-1)^n - u_0(-1)^n - u_1 + u_0$$

$$(u_2-u_0)(-1)^n + u_0 - u_1$$

$$u(x,t) = \bar{u}(x) + v(x,t)$$

$$-\frac{2(u_0)}{n\pi} \qquad -\frac{2}{n\pi}\Big[u_0(-1)^n - u_0 - u_1(-1)^n + u_1 + u_1(-1)^n - u_2(-1)^n$$

$$-\frac{2}{n\pi}\Big[(u_1-u_0) + (u_0-u_2)(-1)^n\Big]$$

4. Give a physical setup of the following boundary cond in problems of the theory of heat conduction & diffusion

$$u_t = a^2 u_{xx} + f(x,t)$$

a) $u(0,t) = 0$ | wall is kept at a constant temp of zero

b) $u_x(0,t) = 0$ | Insulated wall condition

c) $u_x(0,t) - h\,u(0,t) = 0$

$u_x(l,t) + h\,u(l,t) = 0$

$u_x(l,t) = -h u(l,t)$

5. Solve problem of cooling of uniformly heated homogeneous rod at whose ends the temp is $= 0$ under the assumption that no heat loss occurs on lateral surface

Soln: $u(x,0) = U_0$

$$u(x,t) = \frac{4U_0}{\pi} \sum \frac{e^{-(a^2(2k-1)^2\pi^2/l^2)t}}{2k-1} \sin \frac{(2k-1)\pi}{l} x$$

6. Let initial temp of rod be give $u(x,0) = u_0$ const of $0 < x < l$

the temp at ends held const $u(0,t) = u_1$

$u(l,t) = u_2$ for $0 < t < \infty$

Find the temp of rod when no heat exchange occurs on lateral temp

Determine Stationary temp.

i.e. temp at which we can find a soln $\bar{u} \neq f(t)$

$$u(x,t) = \bar{u}(x) + v(x,t)$$

$\bar{q}\cdot\bar{n} = 0$ no oscillating body, porous surface

$\bar{q}\times\bar{n} = 0$ on surface no slip

$T = T_w$ = const isothermal wall cond

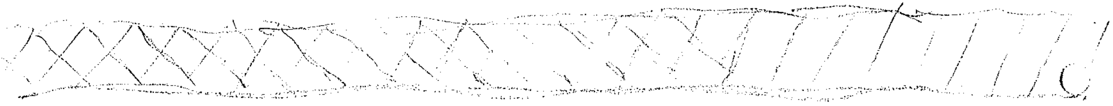
$\tilde{q}_{wall} = 0$ $\left.\frac{\partial T}{\partial n}\right|_{wall} = 0$ adiabat wall

1. ρ_A or w_A mass density or fraction
2. mass flux $\bar{n}_a$ or $\bar{N}_A$
3. $\bar{n}_A = k\,\rho_A$ catalytic wall cond
4. liquid or solid having gaseous interface a convect b.c. applied

$$\left.\bar{n}_A\right|_{\substack{\text{normal}\\\text{to surface}}} = \rho_A q_{A_{\text{normal}}} = -\rho D_{AB}\frac{\partial w_A}{\partial n} + \rho_A v_{\text{intersurf}}$$

$$= k_c(\rho_A - \rho_{A\infty})$$

5. If injection $v_{wall} \neq 0$



Prob 18. $u_{tt} = a^2 u_{xx} + A \sinh x$ $\quad u(0,t) = B, \; u(l,t) = C$; $u(x,0) = u_t(x,0) = 0$

Let $u(x,t) = \bar{u}(x) + v(x,t)$

then $v_{tt} = a^2 v_{xx}$, $\quad a^2 \bar{u}_{xx} + A \sinh x = 0 \quad \bar{u}(0) = B, \; \bar{u}(l) = C$

$v(0,t) = v(l,t) = 0, \quad v(x,0) = -\bar{u}(x)$

$v_t(x,0) = 0$

$$\Longrightarrow \bar{u}(x) = B + \frac{1}{l}\left(\frac{A}{a}\sinh l + C - B\right)x - \frac{A}{a^2}\sinh x$$

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi a t}{l}\right), \quad A_n = -\frac{2}{l}\int_0^l \bar{u}(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\Longrightarrow A_n = \frac{2}{n\pi}\left\{\frac{A(-1)^n \sinh l}{a^2\left(1+\frac{n^2\pi^2}{l^2}\right)} + (-1)^n C - B\right\}$$

Prob 3, $i_x + C v_t + G v = 0 = v_x + L i_t + R i$, $\quad i(x,0) = \phi(x)$

combine to: $i_{xx} = CL\, i_{tt} + (GL + RC)\, i_t + RG\, i$. $\quad v(x,0) = \psi(x)$

Let $y = \frac{t}{\sqrt{CL}}$; $\quad i_{xx} - i_{yy} - \frac{(CR+GL)}{\sqrt{CL}}\, i_y - GR\, i = 0$, $\quad$ Let $a = \frac{1}{\sqrt{CL}}$

Standard Soln., from text, gives

$$i(x,t) = \frac{1}{2} e^{-\frac{a^2}{2}(CR+GL)t}\left[\left\{\phi(x-at) + \phi(x+at)\right\} - a\int_{x-at}^{x+at}\left\{\frac{(CR+GL)}{2} J_0\left(\sqrt{c_1}\sqrt{(x-\xi)^2 - y^2}\right)\right.\right.$$

$$\left. + \sqrt{c_1}\, t\, \frac{J_1\left(\sqrt{c_1}\sqrt{(x-\xi)^2-y^2}\right)}{\sqrt{(x-\xi)^2-y^2}}\right\}\phi(\xi)\,d\xi + \int_{x-at}^{x+at} J_0\left(\sqrt{c_1}\sqrt{(x-\xi)^2-y^2}\right)\psi_0(\xi)\,d\xi$$

where $c_1 = \underset{\text{from st. soln}}{\frac{1}{4}\left(4c^2 - a^2 - b^2\right)} = -\left[GR + \frac{a^2}{4}(CR+GL)^2\right]$, $\psi_0 = -\frac{1}{L}(\psi$

As $G \to 0$, $R \to 0$, $\quad c_1 \to 0$, As $x \to 0$, $J_1(x) \to 0$, $J_0(x) \to$

$$\Longrightarrow i(x,t) \longrightarrow \frac{1}{2}\left[\phi(x-at) + \phi(x+at)\right] + \frac{1}{2}\int_{x-at}^{x+at}\psi_0(\xi)\,d\xi$$

---

similarly for $v(x,t)$

(10) Levy A

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Heat Eq

4. 1. $u(0,t)=0$ the left end boundary is held against a temperature reservoir of 0° absolute

2. $u_x(0,t)=0$ no heat flux across the left end boundary

3. $u_x(0,t) - h\,u(0,t) = 0$ the amount of heat entering the left end boundary is proportional to the temperature at the left end

4. $u_x(l,t) + h\,u(l,t) = 0$ the amount of heat leaving the right hand boundary is proportional to the temperature at the right end.

In both these cases the temperature of the surrounding medium is zero absolute.

Diffusion Eq.

1. $u_A(0,t)=0$ the mass density or fraction at the cross-section $x=0$ of species A is zero

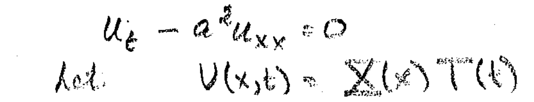
2. $u_{A_x}(0,t)=0$ the mass flux of species A across the cross-section $x=0$ is zero.

3. $u_{Ax}(0,t) - h\,u_A(0,t)=0$ catalytic wall condition: the mass flux of species A across $x=0$ is proportional to the mass density of the species at $x=0$.

4. $u_{A_x}(l,t) + h\,u_A(l,t)=0$ catalytic boundary condition:

in both the concentration of the surrounding species B is << species A.
∴ in (3) species A is moving from a low concentration med. to a higher concentration medium whereas in (4) the opposite is true.

5.



$$u_t - a^2 u_{xx} = 0$$

let $U(x,t) = X(x)T(t)$

then $XT' - a^2X''T = 0 \Rightarrow X = a\sin\lambda x + b\cos\lambda x$

where λ is a constant

From boundary cond $X(0) = X(l) = 0 \Rightarrow b=a$ $\lambda = \frac{n\pi}{l}$

$T(t) = c_n e^{-\lambda^2 a^2 t}$ then

$$U(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2 a^2 t}{l^2}}$$

but $U(x,0) = U_0 = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}$

where $c_n = \frac{2}{l}\int_0^l U_0 \sin\frac{n\pi x}{l}\,dx$; since rod was uniformly heated prior to cooling $u_0 = \text{const}$

$$\therefore c_n = \frac{2}{l}U_0\int_0^l \sin\frac{n\pi x}{l}\,dx = -\frac{2U_0}{n\pi}\cos\frac{n\pi x}{l}\Big|_0^l$$

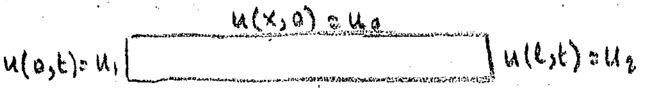
$$= -\frac{2U_0}{n\pi}\left[(-1)^n - 1\right] = \begin{cases}\frac{4U_0}{n\pi} & n \text{ odd}\\ 0 & n \text{ even}\end{cases}$$

Let $n = 2k-1$ $\quad k = 1, 2, \ldots$ then

$$U(x,t) = \sum_{k=1}^{\infty} \frac{4U_0}{(2k-1)\pi} \sin\frac{(2k-1)\pi x}{l}\, e^{-\left[\frac{(2k-1)\pi a}{l}\right]^2 t}$$

since no heat loss in y direct this is soln since problem is 1-D

6.



Let us assume that as $t \to \infty$ $\exists$ a $\bar{u}(x)$

$\Rightarrow \bar{u}(0) = u_1 \quad \bar{u}(l) = u_2$

since $\bar{u}(x)$ must satisfy heat eqs & $\bar{u} \neq f(t)$

then $\bar{u}''(x) = 0 \Rightarrow \bar{u}(x) = A + Bx$

then

$\bar{u}(0) = u_1 = A$; $\bar{u}(l) = u_2 = u_1 + Bl$ or $B = (u_2 - u_1)/l$

then

$\bar{u}(x) = u_1 + \frac{x}{l}(u_2 - u_1)$ Stationary temperature

Now let us assume $u(x,t) = \bar{u}(x) + v(x,t)$; (1)

using the heat eqs. on (1) leads to

$v_t - a^2(\bar{u}_{xx} + v_{xx}) = 0$ But since $\bar{u}_{xx} = 0$ from our soln then

$$v_t - a^2 v_{xx} = 0$$

also since
$u(0,t) = \bar{u}(0) + v(0,t) = u_1 \Rightarrow v(0,t) = 0$
$u(l,t) = \bar{u}(l) + v(l,t) = u_2 \Rightarrow v(l,t) = 0$
$u(x,0) = \bar{u}(x) + v(x,0) = u_0 \Rightarrow v(x,0) = u_0 - \bar{u}(x)$

Soln to $v_t - a^2 v_{xx} = 0$ $\quad v(0,t) = v(l,t) = 0$
is same as problem 5.

$$v(x,t) = \sum c_n \sin\frac{n\pi x}{l}\, e^{-\left(\frac{an\pi}{l}\right)^2 t}$$

$$v(x,0) = \sum c_n \sin\frac{n\pi x}{l} = u_0 - u_1 - \frac{x}{l}(u_2-u_1)$$

$$c_n = \frac{2}{l}\int_0^l \left[u_0-u_1-\frac{x}{l}(u_2-u_1)\right] \sin\frac{n\pi x}{l}\,dx$$

$$c_n = \frac{2}{l}\left[u_0-u_1\right]\left\{-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\Big|_0^l\right\} - \frac{2}{l}\frac{(u_2-u_1)}{l}\int_0^l x \sin\frac{n\pi x}{l}\,dx$$

$$= -\frac{2(u_0-u_1)}{n\pi}\left[(-1)^n - 1\right] - \frac{2(u_2-u_1)}{l^2}\left[-\frac{xl}{n\pi}\cos\frac{n\pi x}{l}\Big|_0^l + \frac{l^2}{n^2\pi^2}\underset{0}{\cancel{\sin\frac{n\pi x}{l}}}\Big|_0^l\right]$$

$$= -\frac{2(u_0-u_1)}{n\pi}\left[(-1)^n-1\right] + \frac{2(u_2-u_1)}{n\pi}\left[(-1)^n\right]$$

$$\therefore v(x,t) = \frac{2}{\pi}(u_0-u_1)\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}\, e^{-\left(\frac{an\pi}{l}\right)^2 t} + \frac{2}{\pi}(u_2-u_0)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}\, e^{-\left(\frac{an\pi}{l}\right)^2 t}$$

$$V(x,t) = u_1 + \frac{x}{l}(u_2-u_1) + \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n}\sin\frac{n\pi x}{l}\, e^{-\left(\frac{an\pi}{l}\right)^2 t}\left[(u_0-u_1) + (u_2-u_0)(-1)^n\right]\right\}$$

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		H.E	D.E
P127 4.	1. $u(0,t)=0$	Heat sink keeps $x=0$ at zero temp.	Concentration or mass density is zero
	2. $u_x(0,t)=0$	No heat flux across $x=0$	No mass flux across $x=0$
	3. $u_x \pm hu = 0$	heat exchange across $x=0$	Exchange of mass flow

---

5. $u_t = a^2 u_{xx}$, $u(0,t) = 0 = u(\ell,t)$, $u(x,0) = U_0 = \text{const.}$

B.C. $\Rightarrow$ $u(x,t) = \sum_1^\infty C_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-a^2\lambda_n t}$; $U_0 = \sum_1^\infty C_n \sin\left(\frac{n\pi x}{\ell}\right)$

$C_n = \frac{2}{\ell}\int_0^\ell U_0 \sin\left(\frac{n\pi x}{\ell}\right)dx \Longrightarrow C_n = \begin{cases} \frac{4U_0}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$$\Longrightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \frac{4U_0}{(2n-1)\pi} e^{-a^2\lambda_n t} \sin\left(\frac{n\pi x}{\ell}\right)$$

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6. $u_t = a^2 u_{xx}$, $u(x,0) = u_0$, $u(0,t) = u_1$, $u(\ell,t) = u_2$

let $u = \bar{u}(x) + v$, $\bar{u}''(x) = 0$, $\bar{u}(0) = u_1$, $\bar{u}(\ell) = u_2 \Rightarrow \bar{u} = u_1 + \frac{x}{\ell}(u_2-u$

$v_t = a^2 v_{xx}$, $v(0,t) = v(\ell,t) = 0$, $v(x,0) = u_0 - \bar{u}(x) = \phi(x)$

$v(x,t) = \sum_1^\infty C_n e^{-a^2\lambda_n t} \sin\left(\frac{n\pi x}{\ell}\right)$, $C_n = \frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$

$$C_n = \frac{2}{\pi n}\left[(u_0-u_1) + (-1)^n(u_2-u_0)\right]$$

$$u(x,t) = u_1 + \frac{x}{\ell}(u_2-u_1) + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[u_0-u_1+(-1)^n(u_2-u_0)]}{n} e^{-a^2\lambda_n t} \sin\left(\frac{n\pi x}{\ell}\right)$$

[illegible] Methods of P.D.E. of Applied Math. FINAL EXAMINATION JAN. '73

Answer all questions.

1. Determine the region in which

$$u_{xx} + (1+e^{-y})u_{xy} + e^{-y}u_{yy} + (1-e^{y})^{-1}(u_x+u_y) = 0$$

is hyperbolic. In this region transform it to canonical form and hence write down its general solution.

[If $\xi = \xi(x,y)$, $\eta = \eta(x,y)$, you may use the relations :

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx}$$

and

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}.$$]

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2. a) Derive the D'Alembert solution to the wave equation,

$$u_{tt} = a^2u_{xx} \quad , t>0 \; , -\infty < x < \infty \, ,$$

subject to $u(x,0) = \phi(x)$, $u_t(x,0) = \Psi(x)$.

b) If, for a semi-infinite string $(x \geq 0)$ with transverse displacement $u(x,t)$, $u_t(x,0) \equiv 0$ while $u(x,0) = \begin{cases} g(x) & x_1 \leq x \leq x_2 \\ 0 & \text{elsewhere} \end{cases}$ and the end $x=0$ is held rigidly for all time, indicate on a phase plane diagram the regions of nonzero displacement and give the values of the displacement in those regions.

---

3. Solve $u_{tt} = a^2u_{xx} + f$, $0<x<l$, $t>0$

subject to the initial conditions :

$$u(x,0) = \frac{f}{2a^2}(lx - x^2) \quad , \quad u_t(x,0) = \frac{f}{a}(l-x)$$

and boundary conditions :

$$u(0,t) = u_1 \quad , \quad u(l,t) = u_2 \quad ,$$

where f, u_1 and u_2 are constants.

4. State the principle of the maximum for a function $u(x,t)$, defined and continuous in the closed region $0 \leq t \leq T$, $0 \leq x \leq \ell$, which satisfies the heat conduction equation,

$$u_t = a^2 u_{xx},$$

in the interior of this region.

Use the principle to show that the function $u(x,t)$ which satisfies

$$u_t = a^2 u_{xx} + f(x,t) \quad , 0<x<\ell, \; 0<t<T$$

together with $u(x,0) = \phi(x)$, $u(0,t) = \mu_1(t)$, $u(\ell,t) = \mu_2(t)$ is unique.

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5. <u>Derive</u> the Green's Function, $G(x,\xi,t)$, for the homogeneous heat conduction equation, $u_t = a^2 u_{xx}$, subject to the homogeneous boundary conditions $u(0,t) = u(\ell,t) = 0$ and the initial condition $u(x,0) = \phi(x)$. Interpret G in terms of the temperature distribution in a rod of length ℓ.

Using $G(x,\xi,t-\tau)$ find the solution to the equation

$$u_t = a^2 u_{xx} + f(x,t) \quad , \; 0<x<\ell, \; 0<t,$$

subject to $u(x,0) = u(0,t) = u(\ell,t) \equiv 0$, where

$$f(x,t) = \begin{cases} F_0 \text{ (constant)}, & 0 \leq t \leq T \\ 0 & t > T. \end{cases}$$