T63.1050 Prof. Papanicolaou ELEM. NUMERICAL METHODS

Books of Interest

ALGORITHM 1. Conte-de Boor - Elementary Numerical Anal. Course Text. 1972

2. P. Henrici, Elements of Numerical Anal 1964

RIGOROUS * 3. Isaacson & Keller - Analysis of Numerical Methods 1966

Recitation Sessions: Tues 4-5 Wed 2-3 STAMPF {4th TECH}

LAST WK BEFORE EXAMS IS READING WEEK.

Numbers & Bases.

$(53781)_{10} = 5\cdot 10^4 + 3\cdot 10^3 + 7\cdot 10^2 + 8\cdot 10 + 1\cdot 10^0$

$(a_n a_{n-1} \dots a_0)_{10} = a_n \cdot 10^n + a_{n-1} 10^{n-1} + \dots + a_1 \cdot 10^1 + a_0 10^0 \quad 0 \le a_j \le 9$

Binary repres. of integers

$N = (a_n \dots a_0) = a_n \cdot 2^n + a_{n-1} 2^{n-1} + \dots + a_1 2^1 + a_0 2^0 \quad a_j = 0,1$

$(1001)_2 = (9)_{10}$

Algorithm of Polynomial Evaluation by nested operation

Let $p(x) = \sum_{i=0}^{n} a_i x^i$

Evaluate $p(x)$ at $x = \beta$

$p(x) = (\dots(((a_n x + a_{n-1})x + a_{n-2})x + a_{n-3})\dots)x + a_0$

To evaluate $p(\beta)$ evaluate $0 \le a_j \le 9$ $b_n \dots b_i \dots b_0$

$b_n = a_n$

$b_{n-1} = b_n \beta + a_{n-1}$

$b_{n-2} = b_{n-1} \beta + a_{n-2}$

$\vdots$

$b_0 = b_1 \beta + b_0$

$b_0 = p(\beta)$

$(11100101)_2 = ?$ decimal

Apply Algo $n = 7$

$b_7 = 1$ $b_5 = 3.2 + 1 = 7$ $b_3 = 14.2 = 28$

$b_6 = 1.2 + 1 = 3$ $b_4 = 7.2 + 0 = 14$ $b_2 = 56 + 1 = 57$

$b_1 = 57 \cdot 2 = 114$

$b_0 = 114 \cdot 2 + 1 = 229$ $\qquad (11100101)_2 = (229)_{10}$

Convert decimal to binary

$(229)_{10} = ?_2 \qquad 2 \cdot 10^2 + 2 \cdot 10 + 9 \qquad (10)_{10} = (1010)_2$

$(10)_2 (1010)_2^2 + (10)_2 (1010)_2 + (1001)_2$

$(1010)_2^2 = (1100100)_2$

$$\begin{array}{lrl} (10)_2 (1010)_2^2 = & 11001000 & 2 \\ & 10100 & 2 \\ & 1001 & 9 \\ \hline & 11100101 & \checkmark \end{array}$$

Arbitary Real Numbers

$X = (a_n a_{n-1} \ldots a_1 a_0 . b_1 b_2 b_3 \ldots) \qquad 0 \le a_j \le 9$

$X = a_n \cdot 10^n + a_{n-1} 10^{n-1} + a_{n-2} 10^{n-2} + \ldots + a_0 + b_1 10^{-1} + b_j \ldots b_n 10^{-n}$

$X_{INT} = (a_n a_{n-1} \cdots a_0)$

$X_{FRAC} = (b_1 b_2 \ldots) \qquad X = X_I + X_F$

Conversion of Decimal Fractions to Binary Fractions

$X_F = a_1/10 + a_2/10^2 + \ldots + a_n/10^n$

$b_1/2 + b_2/2^2 + \ldots + b_n/2^n$

$2X_F = b_1 + b_2/2 + b_3/2^2 + \ldots + b_n/2^{n-1}$

≥ 1

$(2X_F)_F = b_2/2 + b_3/2^2 + \ldots \qquad$ Integ part of $2X_F = b_1$

$= (2X_F - b_1)$

$b_2 =$ Integral part $= 2(2X_F - b_1)$

$.65321 \rightarrow ?$ binary

$2(.65321) = 1.3064 \qquad b_1 = 1$

$2(.3064) = .6128 \qquad b_2 = 0$

$2(.6128) = 1.2256 \qquad b_3 = 1$

$(.65321)_{10} = (.101\ldots)_2$

Floating Point Representation of Real Numbers

$x \longrightarrow fp(x)$ $\quad x$ = has always an exact decimal expansion

$fp(x) = \pm(\underbrace{.d_1 d_2 \ldots d_n}_{\text{mantissa}})_\beta \cdot \underbrace{\beta^e}_{\text{exponent}}$ $\quad \beta$ = base, $\quad 0 \le d_i \le \beta - 1$

$m \le e \le M \qquad m = -M$

n = no. of significant figures.

CDC 6600 $\quad n = 38 \quad \beta = 2 \quad 2^{\pm M} \sim 10^{\pm 400} \quad d_1 \ne 0$

$|x| > \beta^M$ overflow

$0 < |x| < \beta^{m-1}$

Given: $fl(x+y) \overset{?}{=} fl(x) + fl(y)$ no.

$x \to fp(x) = \begin{cases} \text{rounding} & d_n \text{ does not change } d_{n+1} < \beta/2; \quad d_n \to d_{n+1} \text{ if } d_{n+1} \ge \beta/2 \\ \text{chopping} \end{cases}$

$$\boxed{\text{Round off error} = \frac{|x - fl(x)|}{|x|} = \tfrac{1}{2}\beta^{1-n}}$$

$x = .666\ldots \qquad n = 3 \quad \beta = 10$

$fl(x) = +.667 \times 10^0 \qquad$ Round off $= \dfrac{.666\ldots - .66700}{.666} < .5 \cdot 10^{-2}$

Warning: loss of significant figures.

$x^* = fp(x) = .123456789 \qquad$ # sig figures = 9

$y^* = fl(y) = .123456781 \qquad$ # sig figures = 9

$z^* = x^* - y^* = .8 \cdot 10^{-8} \qquad$ # of sig figures = 1

Ex: $ax^2 + bx + c = 0 \qquad x_\pm = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$b^2 \gg 4ac$

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+c/a\right)$$
$$= a(x-x_+)(x-x_-)$$
$$= a\left[x^2-(x_++x_-)+x_+x_-\right.$$
$$x_+x_- = c/a$$
$$x_+ = \frac{c/a \quad 2a}{-b-\sqrt{b^2-4ac}} = \frac{-2c}{b+\sqrt{b^2-4ac}}$$

for stability

<u>ORDER SYMBOLS</u>

if $a_1 a_2 a_3 \dots$ is a seq of real numbers
then $a_n \to a$ if for any $\epsilon > 0$ $|a_n - a| < \epsilon$
provided $n > n_0(\epsilon)$

Speed of Convergence

If $b_1 b_2 \dots$ is another seq suppose $b_n \to 0$ as $n \to \infty$
Example: $b_n = 1/n$ or γ^n where $0<\gamma<1$ Comparison Sequence
we write
$$a_n = a + O(b_n)$$
if $\left|\frac{a_n - a}{b_n}\right| \le k$ some constant for $\forall$ large n.

<u>Example</u>:
let $a_n = \sum_{i=0}^{n} r^i$ $|r|<1$ $a_n \to \sum_{i=0}^{\infty} r^i = \frac{1}{1-r} = a$

$$a_n = \frac{1-r^{n+1}}{1-r} = a + \left(\frac{r}{1-r}\right)r^n = a + O(r^n) \quad |r|<1$$

if the above is true then $a_n \to a$ geometrically fast.

Definition

if $\dfrac{|a_n - a|}{b_n} < \epsilon$ where $\epsilon > 0$ for n sufficient large n

then $a_n = a + o(b_n)$ goes to a faster than b_n

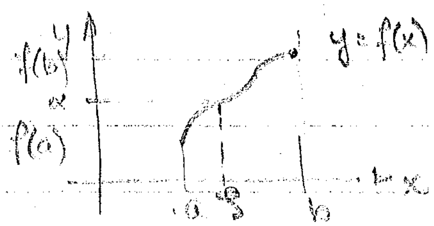
$a_n = a + O(b_n)$

$c_n = c + O(b_n)$

$a_n + c_n = a + c + O(b_n)$

$a_n c_n = ac + O(b_n)$ if $a^2 + c^2 \neq 0$ if a & c are 0 then $O(b_n^2)$

Intermediate Value Theorem



if $f(a) < \alpha < f(b)$ & f is continuous

then $\exists\ a \leq \xi \leq b \ \ni f(\xi) = \alpha$

Taylor's Theorem

let $f(x) \in C^{n+1}$ in $[a,b]$

then $f(x) = f(c) + \displaystyle\sum_{k=1}^{n} f^{(k)}(c) \frac{(x-c)^k}{k!} + R_{n+1}$

$$R_{n+1} = \frac{1}{n!}\int_c^x f^{(n+1)}(\alpha)\,(x-\alpha)^n\,d\alpha$$

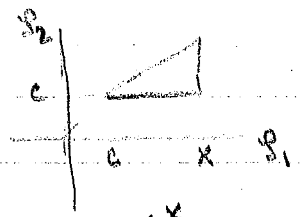
Note that $f(x) = f(c) + \int_c^x f'(\xi_1)\,d\xi_1$ Fund theorem of Cal

$f'(\xi_1) = f'(c) + \int_c^{\xi_1} f''(\xi_2)\,d\xi_2$

$$f(x) = f(c) + \int_c^x \left[f'(c) + \int_c^{\xi_1} f''(\xi_2)\,d\xi_2 \right] d\xi_1 =$$

$$f(c) + f'(c)[x-c] + \int_c^x \int_c^{\xi_1} f''(\xi_2)\,d\xi_2\,d\xi_1$$

$$\int_c^x \int_{\xi_2}^x f''(\xi_2)\, d\xi_1\, d\xi_2 = \int_c^x \int_c^{\xi_1} f''(\xi_2)\, d\xi_2\, d\xi_1$$



$$\int_c^x (x-\xi_2) f''(\xi_2)\, d\xi_2$$

HW Verify by integration

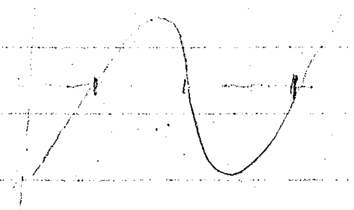
$$\int_c^x \int_c^{\xi_1} \int_c^{\xi_2} \cdots\cdots \int_c^{\xi_n} f^{(n+1)}(\xi_{n+1})\, d\xi_{n+1}\, d\xi_n \cdots d\xi_1$$

Prob 1-32, 33, 43, 41

1-76

locate by computer : roots of $f(x) = x^3 - 6x^2 + 11x - 6$ $[1, 2, 3]$

by bisection, ordinary regula falsi & newton



problem 2, 14, 11 pg 34

2 Oct. 1972

1. Review $fp(\cdot)$ floating point
2. Solving NON-LINEAR EQS.

x a real number

$$x = \pm (.d_1 d_2 \dots\dots) \beta^e \qquad 0 \le d_1 \le \beta - 1 \qquad d_1 \ne 0$$

$fp(x) = \pm (.\delta_1 \delta_2 \dots \delta_n) \beta^e$ Two methods of floating a number

1. Chopping $\delta_1 \delta_2 \dots \delta_n = d_1 d_2 \dots d_n$
2. Rounding
   (i) $d_{n+1} < \beta/2 \qquad \delta_1 \delta_2 \cdots \delta_{n-1} \delta_n = d_1 \cdots\cdots d_n$
   (ii) $d_{n+1} \ge \beta/2 \qquad \delta_1 \delta_2 \cdots \delta_{n-1} = d_1 \cdots d_{n-1};\ \delta_n = d_{n+1}$

Remark

Assume $x \ge 0$

(i) $x \ge x^* = fp(x)$ by chopping and $\dfrac{|x - x^*|}{|x|} \le \beta^{(1-n)}$

(ii) In rounding $x > x^*$ or $x \le x^*$

$$\frac{|x - x^*|}{|x|} \le \frac{1}{2} \beta^{(1-n)}$$

Verify (i) $|x - x^*| = (.\underbrace{000\cdots0}_{n} d_{n+1} d_{n+2} \dots) \beta^e = (.d_{n+1} d_{n+2} \dots) \beta^{e-n}$

$$= \frac{(.d_{n+1} d_{n+2} \dots\dots)}{(.d_1 d_2 \dots\dots)} (.d_1 d_2 \dots) \beta^e \beta^{-n}$$

largest no. can be

$$= \frac{(.d_{n+1} d_{n+2} \dots\dots)}{(.d_1 d_2 \dots\dots)} |x| \beta^{-n} \le \frac{(.\beta-1, \beta-1, \dots)}{(.1000 \dots\dots)} |x| \beta^{-n}$$

smallest no. can be

$$\le \beta \ |x| \beta^{-n} = |x| \beta^{1-n}$$

Order Symbols – $\alpha_1, \alpha_2, \dots\dots \qquad \alpha_n \to \alpha$

$\beta_1 \beta_2$ ... Converging seq. ...

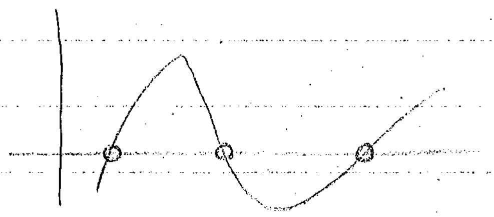
$\alpha_n = \alpha + O(\beta_n) \iff \left|\frac{\alpha_n - \alpha}{\beta_n}\right| \le K$ for sufficiently large n

$\alpha_n = \alpha + o(\beta_n) \iff \left|\frac{\alpha_n - \alpha}{\beta_n}\right| \le \epsilon$ for every choice of $\epsilon > 0$
provided $n = n(\epsilon)$ is sufficiently large.

$\alpha_n = \alpha + O(\beta_n) \iff \alpha_n \sim \alpha + K\beta_n$

Solution of non-linear Equations

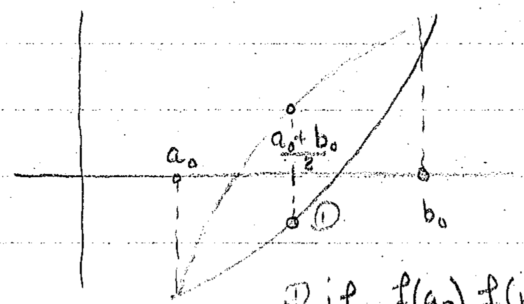
f(x) is a given function.



find the roots s ∋. $f(s) = 0$

$x \in R$ and $f \in R$

① Method of Bisection



Given $f(x)$ ∋. $f(a_0) < 0 < f(b_0)$

begin with $(a_0, b_0) \to$ construct $M = \frac{a_0+b_0}{2}$

check $f(a_0)\, f(m)$

① if $f(a_0)\, f(m) > 0$ then set $a_1 = m$, $b_1 = b_0$ repeat

② if $f(a_0)\, f(m) \le 0$ set $a_0 = a_1$ $b_1 = m$

Suppose a_0, b_0 are points ∋. the continuous $f(x)$ satisfies $f(a_0) \cdot f(b_0) \le 0$

then for $n = 0, 1, 2, \ldots$ Do

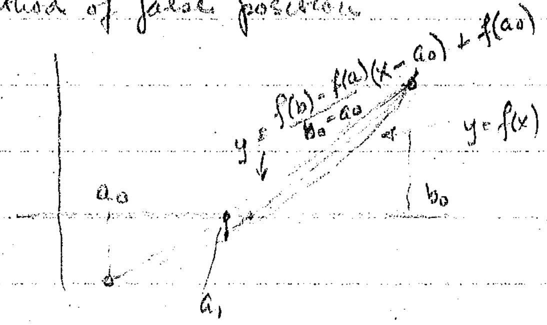
set $m = \frac{a_n + b_n}{2}$

if $f(a_0)\, f(m) > 0$ set $a_{n+1} = m$, $b_{n+1} = b_n$

≤ 0 set $a_{n+1} = a_n$, $b_{n+1} = m$

1. doesn't use properties of function

② Method of false position



$f(x)$ cont

$f(a_0)\,f(b_0) < 0$

$$-f(a_0) = \frac{f(b_0)-f(a_0)}{b_0-a_0}(a_1-a_0)$$

$$a_1 = \frac{f(b_0)\,a_0 - f(a_0)\,b_0}{f(b_0)-f(a_0)}$$ (numerically stable)

ALGORITHM:

If $f(a_0)\,f(b_0) < 0$ then for $n = 0, 1, 2 \ldots$ DO

Compute $m = \dfrac{f(b_n)\,a_n - f(a_n)\,b_n}{f(b_n) - f(a_n)}$

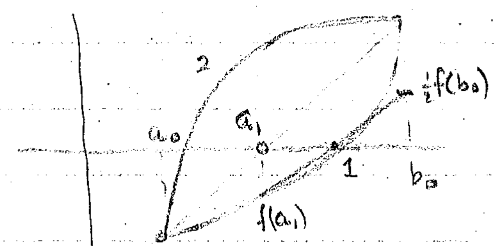
If $f(a_n)\,f(m) > 0$ set $a_{n+1} = m$, $b_{n+1} = b_n$

≤ 0 set $a_{n+1} = a_n$, $b_{n+1} = m$

then $[a_{n+1}, b_{n+1}]$ contains a root of $f(x)$

1. does not use the fact that one of the interval endpoints doesn't move

③ Method of Modified Regula Falsi



Statement of Algorithm:

Suppose $f(x)$ is cont.

and $f(a_0) f(b_0) < 0$, set $w_0 = a_0$ & $F = f(a_0)$ $G = f(b_0)$

For $n = 0, 1, 2, \ldots$ DO

Compute $w_{n+1} = \dfrac{G a_n - F b_n}{G - F}$

1 If $f(a_n) f(w_{n+1}) > 0$, set $a_{n+1} = w_{n+1}$, $b_{n+1} = b_n$, $F = f(w_{n+1})$

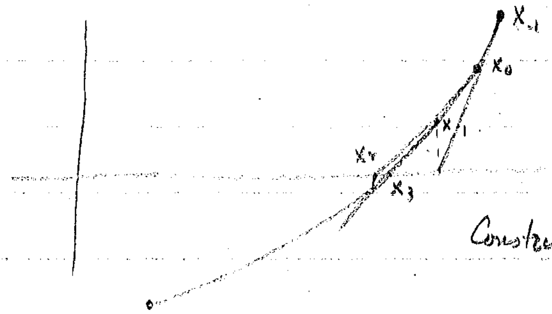
If <u>also</u> $f(w_n) f(w_{n+1}) > 0$, set $G = G/2$ repeat.

2 If $f(a_n) f(w_{n+1}) < 0$, set $a_{n+1} = a_n$, $b_{n+1} = w_{n+1}$ $G = f(w_{n+1})$

If <u>also</u> $f(w_n) f(w_{n+1}) > 0$ set $F = F/2$ repeat

Then $[a_{n+1}, b_{n+1}]$ contains a root of $f(x)$

(4) Secant Method



Given x_{-1}, x_0

$f(x)$ continuous

then for $n = 0, 1, 2, \ldots$ DO

Construct $x_{n+1} = \dfrac{f(x_n) x_{n-1} - f(x_{n-1}) x_n}{f(x_n) - f(x_{n-1})}$

$$= x_n - f(x_n) \frac{1}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

Then .... $x_n \to$ root

(5) Newton Method

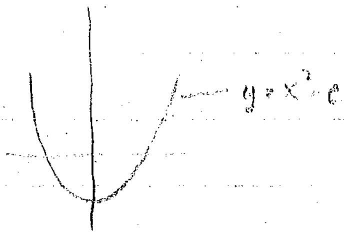
Assume f is diff & x_0 is given then for $n = 0, 1, \ldots$

DO $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

then $x_n \to$ root

Example: (Newton)

Given $c > 0$ find $\sqrt{c}$ let $f(x) = x^2 - c$



$y' = 2x \qquad y = x^2 - c$

By newton's method $x_{n+1} = x_n - \dfrac{(x_n^2 - c)}{2x_n}$

$= \frac{1}{2}\left(x_n + c/x_n\right)$

Try to find $\sqrt{2} = 1.414214$

let $x_0 = 1 \qquad x_1 = \frac{1}{2}(1 + 2/1) = 3/2$

$x_2 = 1/2\left(3/2 + \frac{2}{3/2}\right) = 17/12 = 1.41$

$x_3 = 1/2\left(17/12 + \frac{2}{17/12}\right)$

$\sqrt{3} = 1.732$

$x_0 = 1 \qquad x_3 = \frac{97}{\sqrt{6}} = 1.732 \to$ Root

Some theoretical Analysis

$f(x) = 0 \qquad x \in [a, b]$

For theor purpose better to study $x = g(x)$, $[a, b]$

For example $g(x) = x - h(x) f(x) \qquad h(x) > 0$ in $[a, b]$

How to find fixed pts of $g(x)$ i.e. $\xi \in [a, b]$ $\ni$. $g(\xi) = \xi$

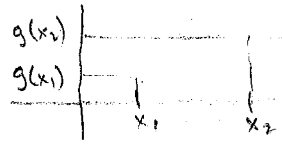
let $x_0 \in [a, b]$ & construct

$$x_{n+1} = g(x_n)$$

If $x_{n+1} \to$ convergence then $x_{n+1} \to \xi$, $g(\xi) = \xi$

Theorem:

⊛ Let $g(x)$ map $[a, b] \to [a, b]$ and assume $|g(x_1) - g(x_2)| \leq K |x_1 - x_2|$ $\forall x_1, x_2 \in [a, b]$ $\quad 0 < K < 1 \qquad g(x)$ is called a contraction



then $x_{n+1} = g(x_n) \quad x_0 =$ arbitrary $\in [a, b]$

converges to a necessarily unique fixed point

ξ of $g(x)$.

Remark if $|g'(x)| \le K < 1$ for $\forall x$ in $[a,b]$ then ⊛ is true.

<u>Proof</u>:

To show $[x_n]$ is a Cauchy seq i.e. given $\epsilon > 0$ $|x_m - x_n| < \epsilon$
$\forall$ suff large $m \geq n(\epsilon)$

From a theorem in calc then x_n converges to a point ξ in $[a,b]$

ξ is unique.

claim: $|x_m - x_n| \le \frac{K^n}{1-K} |x_1 - x_0| \quad \forall m$

$$|x_m - x_n| = |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \dots + (x_{n+1} - x_n)|$$
$$\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots |x_{n+1} - x_n|$$

$$|x_{j+1} - x_j| = |g(x_j) - g(x_{j-1})| \le K|x_j - x_{j-1}| = K|g(x_{j-1}) - g(x_{j-2})|$$
$$\le K^2 |x_{j-1} - x_{j-2}|$$
$$\vdots$$
$$\le K^j |x_1 - x_0|$$

geometric

$$|x_m - x_n| < (K^{m-1} + K^{m-2} + \dots + K^n)|x_1 - x_0| \le K^n(1 + K + K^2 + \dots)|x_1 - x_0|$$
$$\le \frac{K^n}{1-K} |x_1 - x_0|$$

Since $[x_n]$ is a cauchy seq theorem follows

Remark: let $\xi = \lim x_n$

let $e_{n+1} = x_{n+1} - \xi = g(x_n) - g(\xi) = g'(\alpha_n)(x_n - \xi) = g'(\alpha_n)e_n$
when $x_n < \alpha_n < \xi$ $\qquad \sim g'(\xi)e_n$

$\therefore$ $e_{n+1} \sim g'(\xi)e_n$ error shows that x_n converges linearly to ξ

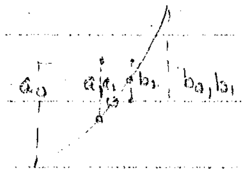
note that $g'(\xi) < 1$ by theorem.

a clever pick of g shows $e_{n+1} \sim (\ ) e_n^2 \qquad h = \frac{1}{f'}$

9 Oct 1972

Review (5 methods)

1. Bisection



$b_0 - a_0 = M$

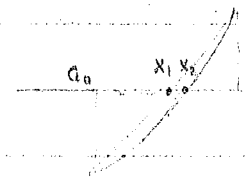
$|x_{i+1} - x_i| = \frac{M}{2^{i+2}} < \epsilon$

$2^{-i} < 4\epsilon/M$

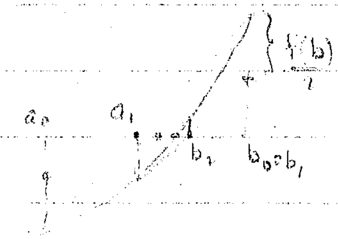
$-i \log 2 < 2 \log 2 + \log(\epsilon/M)$

$i > -2 - \frac{\log(\epsilon/M)}{\log 2}$

2. Regula Falsi

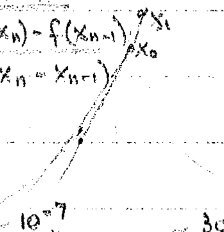


3. Modified Regula Falsi



4. Secant Method $x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n) - f(x_{n-1})}{(x_n - x_{n-1})}}$

5. Newton $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$




```
                                     10^-7         30
SUBROUTINE NEWTON(F, FDER, X0, XTOL, FTOL, NTOL, IFLAG)
 IFLAG=0                          F & FDER are fn subroutine
  DO 20 I = 1,NTOL
   FX0 = F(X0)
```

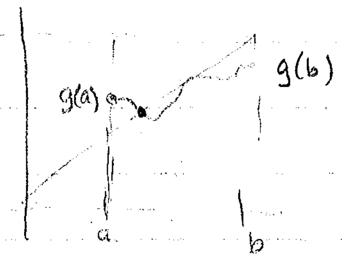
```
      IF (ABS(FXO).LE.FTOL)  RETURN               *  IFLAG=1
       DERIV= FDER(XO)                               RETURN
       IF (ABS(DERIV).LE.FTOL)  GO TO *
       XO = XO - FXO/DERIV
        IF (ABS(FXO/DERIV).LE.XTOL) RETURN
20      CONTINUE
```

Function Iteration

$G(\xi)=\xi$ is needed to be solved $\qquad x_{n+1}=g(x_n) \qquad x_0$ given

1) If $x\in[a,b]$ then $g(x)\in[a,b]$

2) a $\quad |g'(x)|\le K<1 \quad x\in[a,b]$

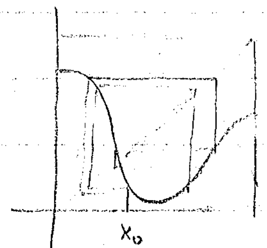


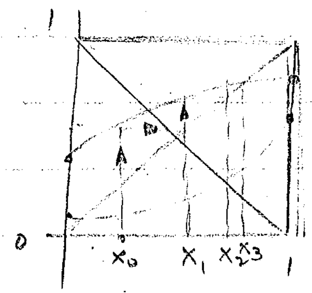
2b $\quad |g(x_1)-g(x_2)|\le K|x_1-x_2|$

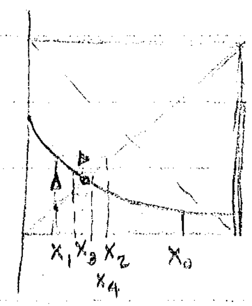
$2b \Rightarrow 2a$ because $g(x_1)-g(x_2)=g'(\eta)(x_1-x_2) \quad x_1\le\eta\le x_2$ by mean value theorem.

$$|g(x_1)-g(x_2)| = |g'(\eta)||x_1-x_2| \le K|x_1-x_2|$$

If 1 & 2 hold then $\{x_n\}$ converges for any $x_0\in[a,b]$ to the unique $\xi\in[a,b]$ .s. $\xi=g(\xi)$







Error in functional Iteration

$$e_{n+1} = \xi - x_{n+1} = g(\xi)-g(x_n) = g'(\eta_n)(\xi-x_n) = g'(\eta_n)e_n$$

$\eta_n\in[\xi,x_n] \quad \therefore$ since $x_n\to\xi,\quad \eta_n\to\xi \qquad e_{n+1}\sim g'(\xi)e_n$

x_n converges linearly to ξ whenever $e_{n+1} \sim g'(\xi)\, e_n$ holds

Suppose $g'(\xi) = 0$ then M.V.T.

$$e_{n+1} = \frac{1}{2} g''(\eta_{1_n})\, e_n^2 \quad \text{where} \quad \eta_{1_n} \in [x_n, \xi]$$

$$e_{n+1} \sim \frac{1}{2} g''(\xi)\, e_n^2 \quad \text{quadratic Error}$$

Verify that Newton's method is quadratic convergent

$$x = g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(\xi) = 1 - \frac{f'(\xi)}{f'(\xi)} + \frac{f(\xi)\, f''(\xi)}{f'^2(\xi)} = 0$$

Secant Method Error

$$x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

$\xi - x_{n+1} = e_{n+1} \sim \text{const}\ e_n e_{n-1}$ $(1+\sqrt{5})/2$

$= e_{n+1} \sim \text{const}\ e_n^{\alpha}$ $\quad 1 < \alpha < 2$

$$x_{n+1} = \frac{x_n\,(\cancel{f(x_n)} - f(x_{n-1})) - f(x_n)\,(\cancel{x_n} - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f(x_n)\, x_{n-1} - f(x_{n-1})\, x_n}{f(x_n) - f(x_{n-1})}$$

$$\xi - x_{n+1} = \xi - \frac{f(x_n)\, x_{n-1} - f(x_{n-1})\, x_n}{f(x_n) - f(x_{n-1})} = \frac{f(x_n)[\xi - x_{n-1}] - f(x_{n-1})[\xi - x_n]}{f(x_n) - f(x_{n-1})}$$

$$= -e_n e_{n-1} \frac{\left[\dfrac{f(x_n) - f(\xi)}{x_n - \xi} - \dfrac{f(x_{n-1}) - f(\xi)}{x_{n-1} - \xi} \right]}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

Assume $x_n \to \xi$ $\quad -e_n e_{n-1} \dfrac{\frac{1}{2} f''}{f'}$, $\text{const} = \dfrac{\frac{1}{2} f''(\xi)}{f'(\xi)}$

$e_{n+1} = C\, e_n e_{n-1}$

$a_{n+1} = (\ )a_n + (\ )a_{n+1}$ *Isaakson & Keller*

Aitken's Acceleration Method. (Delta^2)

$$\xi - x_{n+1} = g(\xi) - g(x_n) \qquad x_{n+1} = g(x_n) \quad x_1, x_2, \dots x_i$$

$$g'(\eta_n)[\xi - x_n]$$

$$\xi[1 - g'(\eta_n)] = x_{n+1} - g'(\eta_n)x_n$$

$$= (x_{n+1} - x_n) + x_n[1 - g'(\eta_n)] \qquad \eta_n \in (\xi, x_n)$$

$$\xi = x_n + \frac{x_{n+1} - x_n}{1 - g'(\eta_n)}$$

$$g'(\eta_n) \cong \frac{g(x_n) - g(x_{n-1})}{x_n - x_{n-1}} = \frac{x_{n+1} - x_n}{x_n - x_{n-1}} = k_n$$

Define $\hat{x}_n = x_n + \dfrac{x_{n+1} - x_n}{1 - k_n} = x_{n+1} + \dfrac{x_{n+1} - x_n}{\frac{1}{k_n} - 1}$

Theorem: If $x_n \to \xi$ then $\hat{x}_n = \xi + o(\xi - x_n)$

$$\Delta x_k = x_{k+1} - x_k$$

$$\Delta^2 x_k = \Delta(\Delta x_k) = \Delta(x_{k+1}) - \Delta x_k = x_{k+1} - 2x_{k+1} + x_k$$

$$\hat{x}_n = x_{n+1} - \frac{(\Delta x_n)^2}{\Delta^2 x_{n-1}}$$

Steffensen's Iteration Proc — to be programmed

$g(x) = x$

Given: x_0 and $g(x)$ $\qquad y_0 = x_0$

for $n = 0, 1, 2 \dots$ Calculate

1. $\xi_0 = y_n$
2. $\xi_1 = g(\xi_0)$, $\xi_2 = g(\xi_1)$
3. $d = \Delta\xi_1 = \xi_2 - \xi_1 \qquad \Delta\xi_0 = \xi_1 - \xi_0$

$$r^{-1} = \frac{\Delta\xi_0}{\Delta\xi_1} = \frac{\xi_1 - \xi_0}{d}$$

Set $y_{n+1} = S_2 + \frac{d}{r^{-1} - 1}$
repeat

program Prob. 2.4.3

$$g(x) = \frac{1}{2} + \frac{1}{5} \sin x$$

p. 43 2-2.3 part (a) 2-3.2, 5.3

Study $x = \tan x$; $x^n + \dots + k = 0$ plot roots as k varies

2.3-8 $f(x) = e^x - 1 - x - x^2/2$

$$= 1 + \sum \frac{x^n}{n!} - 1 - x - x^2/2 = \sum_{n=3}^{\infty} \frac{x^n}{n!}$$

$$e^x = 1 + x + x^2/2 + \frac{1}{2!} \int_0^x \underset{u}{(x-s)^2} \underset{dv}{e^s \, ds} \qquad \int u\,dv = uv - \int v\,du$$

$$+ \frac{1}{2!} \left[\cancel{(x-s)^2 e^s \Big|_0^x} + \cancel{2\int (x-s) e^s ds} \right]$$

$$+ \frac{1}{2!} \left[\cancel{-x^2 + 2(x-s) e^s \Big|_0^x} + \cancel{2\int_0^x e^s ds} \right]$$

$$\frac{2(e^x - 1)}{2}$$

$$e^x = 1 + x + \frac{x^2}{2} + e^{\xi} \frac{(x)^3}{3!} \qquad 0 < \xi < x$$

$$e^x - 1 - x - \frac{x^2}{2} = e^{\xi} \frac{(x)^3}{3!} \qquad \frac{e^{\xi}}{3!} \neq 0 \qquad \therefore \text{ if } f(x) = 0 \quad \underline{x = 0}$$

23 Oct 1972

1. Polynomial Evaluation
2. Newton's Method for Poly.
3. Fortran of (2)
4. Muller's Method
5. Problem
6. Non-linear systems of Equns (Newton's Method)

Given

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$p(x) = a_0 + x\,(a_1 + a_2 x + \dots + a_{n-1} x^{n-2} + a_n x^{n-1})$$

$$= a_0 + x\,(a_1 + x\,(a_2 + x\,(+ \dots + x(a_{n-1} + x\,(a_n))\dots)$$

To evaluate $p(x)$ at $x = z$, Set $b_n = a_n$

Do $k = n-1, n-2, \dots, 0$

$$b_k = a_k + z\,b_{k+1} \qquad (*)$$

Then $b_0 = p(z)$

Note that If we define $q(x) = b_1 + b_2 x + b_3 x^2 + \dots + b_n x^{n-1}$

then $p(x) = q(x)(x-z) + p(z) \qquad (p(z) = b_0)$

$q(x)$ is the quotient of p divided by $x-z$ $\quad p(z)$ is remainder

Proof:

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = (b_1 + b_2 x + \dots + b_n x^{n-1})(x-z) + b_0$$

$$= b_0 + b_1 x + b_2 x^2 + \dots + b_{n-1} x^{n-1} + b_n x^n$$

$$- b_1 z - b_2 z x - b_3 z x^2 - \dots - b_n z x^{n-1}$$

$$= (b_0 - b_1 z) + (b_1 - b_2 z) x + \dots + (b_{n-1} - b_n z) x^{n-1}$$

$$+ b_n x^n$$

because of $(*)$ Q.E.D.

Note that $p'(z) = q(z)$

because $p'(x) = q(x) + q'(x)(x-z)$; let $x = z$

$$p'(z) = q(z)$$

2. Newton's Method for polynomials

Let x_0 and coeff. $a_0, a_1, \dots, a_n$ be given

Then: Do for $m = 0, 1, 2, \dots$ iteration scheme

set $z = x_m$, $\quad b_n = a_n$, $\quad c_n = a_n$

Do for $k = n-1, n-2, \dots, 1$

$b_k = a_k + z\, b_{k+1}$

$c_k = b_k + z\, c_{k+1}$ defines $c_1 = p'(z)$

set $b_0 = a_0 + z b_1$, $x_{m+1} = z - b_0/c_1$

```
3.     DIMENSION  A(N)
C      READ ( Array A, M = # of iterations, X0 initial guess)
       X=X0
       DO 10  I=1,M
       B = A(N)
       C = A(N)
       DO 100  K= 1,N
       KK = N-K+1
       B = A(KK) + X * B
100    C = B + X * C
       B = A(1) + X*B
       ΔX = B/C
       IF (|DELTX| < 10^-7 .AND. |B| < 10^-7) RETURN
10     X = X - DELTX
```

Disadvantages of Newton's Method:

1. X_0 must be very good
2. Problems arise for high degree polynomials.
3. Behavior near a double root.

$p(x) = (x-\xi)^2 \varphi(x) \qquad \varphi(\xi) \neq 0$

Newtons method $f(x) = 0$ * $x_{n+1} = g(x_n)$

$g(x) = x - \frac{f(x)}{f'(x)}$ $\qquad e_n = x_n - x_{n-1}$

Notice $e_{n+1} = g(x_n) - g(x_{n-1}) \sim g'(\xi)e_n + \frac{1}{2} g''(\xi) e_n^2$

for newton's method.

Note that $g'(x) = 1 - 1 + \frac{f(x)\, f''(x)}{(f'(x))^2}\Big|_{x=\xi} = 0$

If $f'(\xi) = 0$ but $f''(\xi) \neq 0$ then

Newton's Method converges to ξ (for sufficient close x_0)

but $e_{n+1} \sim g'(\xi)\, e_n$ this is not zero now.

to see this we must study

$$\lim_{x\to\xi} \frac{f f''}{f'^2} = \lim_{x\to\xi} \frac{f}{f'^2} \lim_{x\to\xi} f'' = \frac{f'}{2f'f''} f''(\xi)$$

$$= \frac{1}{2}$$

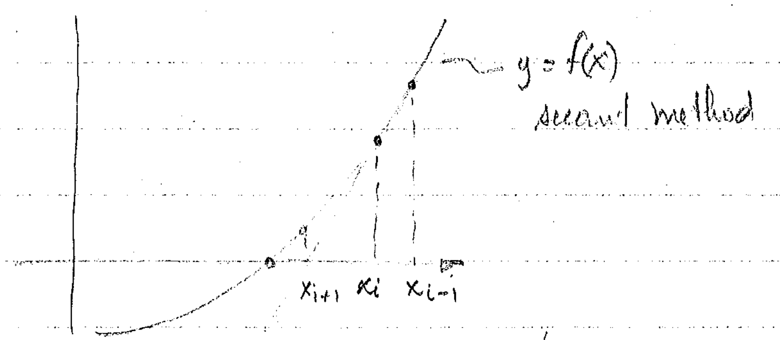
Then $e_{n+1} \sim \frac{1}{2} e_n$ when $f'(\xi) = 0$ & $g(x) = x - f/f'$

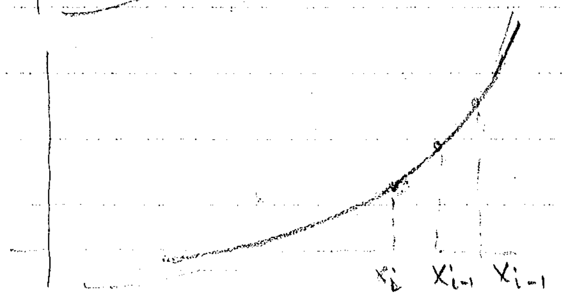
then make quadratic convergence by changing

$$\tilde{g}(x) = x - \frac{2f}{f'}$$

then $e_{n+1} \sim e_n^2$ i.e. $\tilde{g}'(\xi) = 0$

Muller's Method:





suppose x_i, x_{i-1}, x_{i-2} are known

$$f(x_i),\ f(x_{i-1}),\ f(x_{i-2}) = f_i\ f_{i-1}\ f_{i-2}$$

$$f[x_i, x_{i-1}] = \frac{f_i - f_{i-1}}{x_i - x_{i-1}}$$

$$f[x_i, x_{i-1}, x_{i-2}] = \frac{f[x_i, x_{i-1}] - f[x_{i-1}, x_{i-2}]}{x_i - x_{i-2}}$$

The parabola with eq $y = p(x)$ that passes through (x_i, f_i) $(x_{i-1}, f_{i-1}), (x_{i-2}, f_{i-2})$ is given by

$$p(x) = f_i + f[x_i, x_{i-1}](x - x_i) + f[x_i, x_{i-1}, x_{i-2}](x - x_i)(x - x_{i-1})$$

$p(x_i) = f_i$ $\quad$ $p(x_{i-1}) = f_i - f_i + f_{i-1} = f_{i-1}$

$$p(x_{i-2}) = f_i + f[x_i, x_{i-1}](x_{i-2} - x_i) + \frac{f[x_i, x_{i-1}] - f[x_{i-1}, x_{i-2}]}{x_i - x_{i-2}}$$

$$\times\ [x_{i-2} - x_i][x_{i-2} - x_{i-1}]$$

Write: $p(x) = a_2 x^2 + a_1 x + a_0$

$$a_2 = f[x_i, x_{i-1}, x_{i-2}]$$

$$a_1 = f[x_i, x_{i-1}] - (x_i + x_{i-1}) f[x_i, x_{i-1}, x_{i-2}]$$

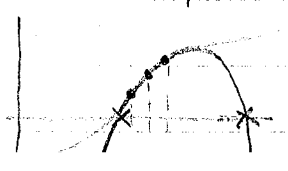
$$a_0 = -f[x_i, x_{i-1}] x_i + f[\ ,\ ,\ ](x_i x_{i-1})$$

Set x_{i+1} = root of $p(x) = 0$

$$x_{i+1} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} = \frac{2a_0}{-a_1 \mp \sqrt{a_1^2 - 2a_2 a_0}}$$

<u>Sign rule:</u> If $a_1 > 0$ take $-$ $\quad$ produces next iterates closer to root

If $a_1 < 0$ take $+$



Problem:

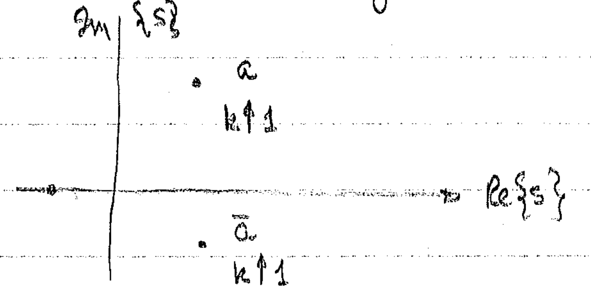
Let $0 \le k \le 1$ let $g(s) = (s-a)(s-\bar{a})(s-b)$

where $a = 2i+1$ $\bar{a} = 1-2i$ $b = -\frac{1}{2}$

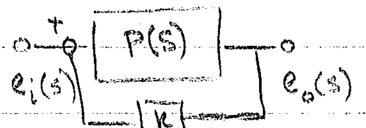
Let $G(s,k) = g(s) + k$ find the roots $G(s,k) = 0$

for $0 \le k \le 1$ $k = 0, .1, .1, .2, \ldots, 1$

what is the smallest k for which $\mathrm{Re}\{a(k)\} \le 0$?



Use Muller's Method.
& possibly deflation.



$s = \lambda + i\omega$

$E_i(t)$ = input signal

$E_o(t)$ = output signal $= \int_0^t P(t-\tau)\, E_i(\tau)\, d\tau$ ⊗

$e_{i,o}(s) = \int_0^\infty E_{i,o}(t)\, e^{-st}\, dt$ then ⊗ $e_o(s) = p(s)\, e_i(s)$.

$p(s) = \int_0^\infty P(t)\, e^{-st}\, dt$

Let $p(s) = \dfrac{1}{(s-a)(s-\bar{a})(s-b)}$

with feedback $\dfrac{e_o}{e_i} = \dfrac{p(s)}{1+kp(s)}$ find roots of $\dfrac{1}{p(s)} + k$

Algorithm of Muller's Method:

x_i, x_{i-1}, x_{i-2} are given

1 Compute f_i, f_{i-1}, f_{i-2}

2. Compute $h_i = x_i - x_{i-1}$ and $\lambda_i = \dfrac{h_i}{x_{i-1} - x_{i-2}}$

$$x_{i+1} = x_i + h_i \lambda_{i+1}$$

$$= x_i + \frac{\cancel{(x_i - x_{i-1})}(x_{i+1} - x_i)}{\cancel{x_i - x_{i-1}}} = x_{i+1}$$

3. $g_i = (1+2\lambda_i)(f_i - f_{i-1}) - \lambda_i^2 (f_{i-1} - f_{i-2}$

4. $\lambda_{i+1} =$ see book $-(\ \ ) \pm \sqrt{\ \ }$

Choose $\pm$ so that λ_{i+1} is min

5. $x_{i+1} = x_i + h_i \lambda_{i+1}$, $h_{i+1} = h_i \lambda_{i+1}$, $f_{i+1} = f(x_{i+1}$

6. Increase i by 1 and go to 3 or return

System of Nonlin Eq

$f(x,y)$, $g(x,y) = 0$ find roots

write $F(x,y) = x$

$G(x,y) = y$

write $\underset{\sim}{F}(\underset{\sim}{x}) = \underset{\sim}{x}$

$$\underset{\sim}{f}(\underset{\sim}{x}) = 0$$

$$\underset{\sim}{f}'(\underset{\sim}{x}) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

$$\underset{\sim}{x}_{n+1} = \underset{\sim}{F}(\underset{\sim}{x}_n)$$

$$\underset{\sim}{F}(\underset{\sim}{x}) = \underset{\sim}{x} - [\underset{\sim}{f}'(\underset{\sim}{x})]^{-1} \underset{\sim}{f}(\underset{\sim}{x})$$

10-30-72

1. Newton's Method for nonlinear systems.
2. Review of Linear Algebra.
3. Num. Soln for linear Systems
   (a) Back Substitution
   (b) Gaussian Elimin.

$x_{i+1} = x_i + \lambda_{i+1} h_i$ $\quad h_i = x_i - x_{i-1}$ $\quad$ problem: $\frac{1}{2} \pm 2i$ instead of $1 \pm 2i$

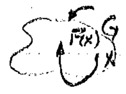
$$\underset{\sim}{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \underset{\sim}{F}(\underset{\sim}{x}) = \begin{pmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{pmatrix}$$

Problem find solns of $\quad f(x) = 0$

$$f_1(x_1, \dots, x_n) = 0$$
$$f_2(x_1, \dots, x_n) = 0$$
$$\vdots$$
$$f_n(x_1, \dots, x_n) = 0$$

<u>Theorem</u> Let $G \subset R^N$ (closed subset)

& suppose $F : G \longrightarrow G$.



for any $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in G$

$$\| \underset{\sim}{F}(\underset{\sim}{x}_1) - \underset{\sim}{F}(\underset{\sim}{x}_2) \| \leq K \| \underset{\sim}{x}_1 - \underset{\sim}{x}_2 \| \quad , \quad K < 1$$

then for any $\underset{\sim}{x}_0$ in G the sequence of vectors $\underset{\sim}{x}_n$ where $\underset{\sim}{x}_{n+1} = \underset{\sim}{F}(\underset{\sim}{x}_n)$ converges to a unique vector $\underset{\sim}{\xi}$. ie $\|\underset{\sim}{x}_n - \underset{\sim}{\xi}\| \to$ as $n \to \infty$

& $\underset{\sim}{\xi} = \underset{\sim}{F}(\underset{\sim}{\xi})$ <u>prove</u>:

<u>def</u>: norm $\| \ \|$ if $\underset{\sim}{x} \in R^N$ $\quad \|\underset{\sim}{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$

<u>Error</u> analysis of fixed point iteration

$$e^n \equiv \underset{\sim}{x}^n - \underset{\sim}{x}^{n-1} \qquad e^{n+1} \equiv \underset{\sim}{x}^{n+1} - \underset{\sim}{x}^n = \underset{\sim}{F}(\underset{\sim}{x}^n) - \underset{\sim}{F}(\underset{\sim}{x}^{n-1}) =$$
$$= \underset{\sim}{F}'(\underset{\sim}{\eta}^n)\left[\underset{\sim}{x}^n - \underset{\sim}{x}^{n-1}\right]$$

$$\underset{\sim}{F}'(\underset{\sim}{x}) = \left[\frac{\partial F_i(x)}{\partial x_j} \right]$$

$$e_i^{n+1} = \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j}(\underset{\sim}{\eta}^n)\, e_j^n$$

$\underset{\sim}{\eta}^n$ is on the line connecting $\underset{\sim}{x}^n$ & $\underset{\sim}{x}^{n-1}$

$\therefore \quad e^{n+1} \simeq \underset{\sim}{F}'(\underset{\sim}{\xi})\, e^n \qquad \underset{\sim}{F}'(\underset{\sim}{\xi})$ has "norm" less than 1

error is first order unless $\underset{\sim}{F}'(\underset{\sim}{\xi}) = \underset{\sim}{0}$

to have $\underset{\sim}{F}'(\underset{\sim}{\xi}) = 0$ choose $\underset{\sim}{X} = \underset{\sim}{F}(\underset{\sim}{x}) = \underset{\sim}{x} - [\underset{\sim}{f}'(\underset{\sim}{x})]^{-1} \underset{\sim}{f}(\underset{\sim}{x})$

Recall $\underset{\sim}{f}(\underset{\sim}{x}) = 0$ when $\underset{\sim}{X} = \underset{\sim}{\xi}$

Apply this to two equas. $\left.\begin{array}{l} f(x,y)=0 \\ g(x,y)=0 \end{array}\right\} \underset{\sim}{f}(\underset{\sim}{x}) = 0$

$\left.\begin{array}{l} F(x,y)=x \\ G(x,y)=y \end{array}\right\} \underset{\sim}{F}(\underset{\sim}{x}) = \underset{\sim}{x}$

$$\underset{\sim}{f}'(\underset{\sim}{x}) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \qquad [\underset{\sim}{f}'(\underset{\sim}{x})]^{-1} = \frac{\begin{bmatrix} g_y & -f_y \\ -g_x & f_x \end{bmatrix}}{f_x g_y - f_y g_x}$$

$$\begin{matrix} F(x,y) \\ G(x,y) \end{matrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{f_x g_y - g_x f_y} \begin{bmatrix} g_y & -f_y \\ -g_x & f_x \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$$

Newton's Method (x_0, y_0) $\quad x_{n+1} = F(x_n, y_n)$

$\quad y_{n+1} = G(x_n, y_n)$

Example $x^2 + xy^3 - 9 = 0$

$3x^2y - y^3 - 4$

Derivative Matrix:

$$A = \begin{bmatrix} 2x + y^3 & 3xy^2 \\ 6xy & 3x^2 - 3y^2 \end{bmatrix} \qquad \det A = [3(x^2 - y^2)(2x + y^3) - 18x^2y^3] = J(x,y)$$

$$F(x,y) = x - \frac{(x^2 + xy^3 - 9)(3x^2 - 3y^2) - (3x^2y - y^3 - 4)(3xy^2)}{J}$$

$$G(x,y) = y - \frac{\qquad\qquad}{J}$$

Error analysis for N.M. in R^N is identical to R^1. & Same general remarks hold.

Review of L.A.

$\underset{\sim}{x}, \underset{\sim}{y}, \ldots, \underset{\sim}{a}, \underset{\sim}{b}$. vectors $\quad Ax = b \quad \sum_{j=1}^{n} a_{ij} x_j = b_j$

$A, B, \ldots$ matrices

$A = (a_{ij}) \quad x = (x_i)$

1. Vectors form a vector space i.e. R^N is a vector space.

(i) $\underset{\sim}{x} + \underset{\sim}{y} \in R^N \quad \underset{\sim}{x} + \underset{\sim}{y} = \underset{\sim}{y} + \underset{\sim}{x}$

(ii) $\alpha \underset{\sim}{x} \in R^N \quad$ 0 is the additive identity $\underline{0} + \underline{x} = \underline{x}$

2. $n \times n$ matrices for a vector space

(i) $A + B \in V$

(ii) $\alpha A \in V \quad$ 0 matrix is $\ni$. $A + 0 = A$

3. Matrices have a mult. rule where

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

the mult. satisfies $A(B+C) = AB + AC$

$$A(BC) = (AB)C$$

non-commutative. $AB \neq BA$ in general $\quad I = \delta_{ij} \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$BI = B \quad IA = A$.

Linear Dependence

A set of vectors $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, x_k$ are linear indep. iff.

for $\alpha_1 \underset{\sim}{x}_1 + \alpha_2 \underset{\sim}{x}_2 + \ldots + \alpha_n \underset{\sim}{x}_n = 0 \ ; \quad \alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_n = 0$

Let A be $n \times n$ matrix & let $a_1, a_2, \ldots, a_n$ be its column vectors

Def: The largest number of L.I. a's is called the rank of A.

Thm: If rank $A = r$ then there are exactly r. l.i. row vectors of A. Hence rank (A) = column rank (A) = row rank (A)

Let $A=(a_{ij})$. Suppose, without loss of generality that the first r columns are L.I. then if $\sum_{i=1}^{r} c_i a_{ij} = 0$ then $c_1 = c_2 = \cdots = c_r = 0$.

$j = 1, 2, \ldots, n$

Let d_j be some constants & add $\sum_{j=1}^{n}\sum_{i=1}^{r} d_j c_i a_{ij} = 0$

$\sum_{i=1}^{r} c_i \left(\sum_{j=1}^{n} d_j a_{ij}\right) = 0$; To deduce that if $\sum d_j a_{ij} = 0$ then

$i = 1, 2, \ldots, n$ it cannot be that $d_1 = d_2 = \cdots = d_n = 0$

if it were zero.

Problem: prove the above.

We wish to solve the linear system $AX = b$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Given A & B does $\exists$ a soln?

① Alternative theorem: If $AX = 0 \Leftrightarrow X = 0$ then $AX = b$ has a unique soln. for every b & rank$(A) = n$. column vectors of A are L.I.

② Otherwise $\exists$ vectors $X^{(1)}, X^{(2)}, X^{(3)}, \ldots, X^{(p)}$ $\ni$ if $AX = 0$ then $X = \sum_{j=1}^{p} \alpha_j X^{(j)}$ $\quad n \geq p \geq 1$

In this case $AX = b$ is not uniquely solvable for every b.

$AX = b$ is solvable if rank(A) = rank$(A, b) < n$

where A, b is the set of column vectors of A & the column vector b.

Numerical Treatment

1) Back Substitution Algorithm

If A is upper triangular $\begin{pmatrix} \diagdown \\ 0 \end{pmatrix}$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$+ a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\ddots \quad a_{n-1\,n-1}x_{n-1} \; a_{n-1\,n}x_n = b_{n-1}$$
$$a_{nn}x_n = b_n$$

Assume $a_{KK} \neq 0$

$$x_n = \frac{b_n}{a_{nn}}, \quad x_{n-1} = \frac{b_{n-1} - a_{n-1\,n}x_n}{a_{n-1\,n-1}}, \dots$$

For $k = n, \dots, 1$ Do

$$x_k = \frac{b_k - \sum_{j=k+1}^{n} a_{kj}x_j}{a_{kk}}$$

where $\sum_{j=n+1}^{n}$ is interpreted as zero.

2) Gaussian Elimination

a) Assume $a_{11} \neq 0$

mult 2, 3, ... n by $\frac{a_{11}}{a_{j1}}$ $j = 2, 3, \dots, n$

& subtract from 1st. eq

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{22}^{(1)}x_2 + \dots + a_{2n}^{(1)}x_n = b_2^{(1)} \qquad a_{2j}^{(1)} = a_{1j} - \frac{a_{11}}{a_{21}}a_{2j}$$
$$a_{32}^{(1)}x_2 + \dots + a_{3n}^{(1)}x_n = b_3^{(1)} \qquad b_2^{(1)} = b_1 - \frac{a_{11}}{a_{21}}b_2$$
$$\vdots$$
$$a_{n2}^{(1)}x_2 + \dots + a_{nn}^{(1)}x_n = b_n^{(1)}$$

for $i = 2, 3, 4, \dots$

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{11}}{a_{i1}} a_{ij}$$

$$b_i^{(1)} = b_1 - \frac{a_{11}}{a_{i1}} b_i$$

Assume $a_{22}^{(1)} \neq 0$

Next G.E. with partial pivoting & retaining of multipliers

11/6/72

1. Row & Column Rank
2. Gauss Elimination
3. Pivoting
4. Fortran of Gauss
5. Triangular Factorization Substitution Algorithm
6. Fortran for Subst. Algo
7. Norms for vectors & matrices

If $A = (a_{ij})$ $a_1, a_2, \dots, a_n$ are its columns

then c-rank (A) = largest # of L.I. c-vectors

if A has c-rank $(A) = n$ then row rank$(A) = n$. Also.

Proof: Hypothesis If $\sum_{j=1}^{n} a_{ij} x_j = 0$ $i = 1, 2, \dots, n$ then $x_1 = x_2 = \dots = x_n = 0$

To show that row rank $= n$ i.e. If $\sum_{i=1}^{n} a_{ij} y_i = 0$ ⊗ $y_1 = y_2 = \dots = y_n = 0$

Suppose ⊗ holds but some $y_k \neq 0$ then $\sum_{i=1}^{n} y_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) = 0$

$\Rightarrow$ $a_{kj} = 0$ $j = 1, \dots, n$ $\Rightarrow$ the k^{th} entry of the column vectors $a_1, a_2, \dots, n$ is zero $\Rightarrow$ the vectors are L. Dependent.

Gauss Elimination $Ax=b$

$$A=\begin{pmatrix}1&2&3\\3&-1&1\\4&0&2\end{pmatrix}\qquad b=\begin{pmatrix}2\\3\\4\end{pmatrix}$$

No pivoting

$$W=\left(\begin{array}{ccc|c}1&2&3&2\\3&-1&1&3\\4&0&2&4\end{array}\right)\xrightarrow{*}\left(\begin{array}{c|ccc}1&2&3&2\\3&-7&-8&-3\\4&-8&-10&-4\end{array}\right)\xrightarrow{*}\left(\begin{array}{c|c|cc}1&2&3&2\\3&-7&-8&-3\\4&8/7&-6/7&-\frac{4}{7}\end{array}\right)$$

$$Ax=b \iff Ux=\tilde{b}$$

use back Subst. now to solve triang matrix

Pivoting uses the idea of non division by a small no.

Let $A=(a_{ij})$

$d_i = \max_{1\le j\le n} |a_{ij}|$ to eliminate x_1 choose that row p_1 for pivot for which $\dfrac{|a_{p_1 1}|}{d_{p_1}} \geq \dfrac{|a_{i1}|}{d_i}$ $i=p_2,\dots p_n$

compute $\dfrac{|a_{i1}|}{d_i}$ find row $\ni$ $\max_{1\le i\le n}\dfrac{|a_{i1}|}{d_i}$ & interchange rows

do this for all matrices.

<u>Outline of Algorithm</u> (without maximal pivoting)

1. Initialize the pivot vector $p=(1,2,\dots,n)$ and $W=A$ initially

2. DO $k=1,\dots,n-1$

search for pivot p_j $j\geq k$ $\ni$ $w_{p_j,k}\neq 0$

Exchange contents of row p_k & row p_j

DO $I=k+1,\dots,n$ $\quad m=\dfrac{w_{p_i k}}{w_{p_k,k}}$, $w_{p_i k}=\dfrac{w_{p_i k}}{w_{p_k k}}$

DO $J=k+1,\dots,n$

$w_{p_i j}=w_{p_i j}-m\,w_{p_k j}$

```
C Gauss Elimination   N-dim
      DIMENSION  A(N,N), W(N,N), IPVOT(N), D(N)
      IFLAG=1
C   INITIALIZE IPIVOT, D
      DO 10 I=1,N
      IPIVOT(I)=I
      ROWMAX=0.
      DO 9 J=1,N
      IF (ABS(W(I,J)).LE. ROWMAX) GO TO 9
      ROWMAX = ABS (W(I,J))
    9 CONTINUE
      IF (ROWMAX.LE. 1.0E-07)  GO TO 999
   10 D(I)=ROWMAX.
C   BEGIN ELIMINATION
      NM1 = N-1
      DO 20 K = 1,NM1
      J = K
      KP1 = K+1
      IP = IPVOT(K)
      COLMAX = ABS (W(IP,K))/ D(IP)
      DO 11 I= KP1,N
      IP = IPVOT (I)
      AW = ABS(W(IP,K))/D (IP)
      IF (AW.LE.COLMAX) GO TO 11
      COLMAX = AW
      J = I
   11 CONTINUE
C exchange of indices
      IPK = IPIVOT(J)
      IPIVOT(J) = IPIVOT(K)
      IPIVOT(K) = IPK
C LAST 2 STEPS IN ELIM.
      DO 20 I = KP1,N
      IP= IPIVOT(I)
      W(IP,K)= W(IP,K)/W(IPK,K)
      RATIO= - W(IP,K)
      DO 20 J= KP1,N
      W(IP,J) = W(IP,J)+RATIO*W(IPK,J)
   20 CONTINUE.
      RETURN
  999 IFLAG=2
      RETURN
```

Problem on Linear Equations

Let

$$H_n = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots & 1/n \\ 1/2 & & & & & \\ 1/3 & & & 1/(i+j) & & \\ \vdots & & & & & \\ 1/n & 1/(n+1) & & & & 1/(2n-1) \end{bmatrix}$$

For $n = 5$ DO Gaussian Elimination to triangular form.

Find inverse of matrix.

Solve also by hand

5) Forward & Backward Substitution

Backward Subst. clear i.e. how to solve when system is in triangular form.

$Ax = b$

$W = [A, b]$ if no pivoting

$$\begin{pmatrix} \diagdown & U \\ M & \diagdown \end{pmatrix} \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} \tilde{b} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \\ -1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \qquad W = \begin{pmatrix} 2 & 3 & 1 & 3 \\ 3 & 1 & 0 & 2 \\ -1 & -2 & 0 & 5 \end{pmatrix} \qquad b^{(0)} = b$$

$$\begin{pmatrix} 2 & 3 & 1 & 3 \\ 3/2 & \times & \times & 2 - 9/2 \\ -1/2 & \boxed{-1/2} & \times & 5 + 1/2 \cdot 3 \end{pmatrix} \qquad \begin{aligned} b_1^{(1)} &= b_1^{(0)} \\ b_2^{(1)} &= b_2^{(0)} - m_{21} b_1^{(0)} \\ b_3^{(1)} &= b_3^{(0)} - m_{31} b_1^{(0)} \end{aligned}$$

$$\begin{pmatrix} \\ \\ \\ \end{pmatrix} \qquad \begin{aligned} b_1^{(2)} &= b_1^{(1)} \\ b_2^{(2)} &= b_2^{(1)} \\ b_3^{(2)} &= b_3^{(1)} - m_{32} b_2^{(1)} \end{aligned}$$

$b_1^{(2)} = b_1^{(0)}$

$b_2^{(2)} = b_2^{(0)} - m_{21}\, b_1^{(0)=(2)}$

$b_3^{(2)} = b_3^{(0)} - m_{31}\, b_1^{(0)} - m_{32}\, b_2^{(1)}$

Given

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad Ax = b \quad \text{the } \tilde{b} \text{ in } Ux = \tilde{b}$$

$$\tilde{b}_k = b_k - \sum_{j=1}^{k-1} m_{kj}\tilde{b}_j \qquad k = 1, 2, \ldots, n$$

What if we had pivoting? The IPIVOT vector $\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$ is stored. Recall $m_{kj} = W_{kj}$ in routine $k > j$. In general then $\tilde{b}_k = b_{p_k} - \sum_{j=1}^{k-1} W_{p_k j}\, \tilde{b}_j$

Fortran program Assumes for forward & backward subst.

Backward Subst. $X_k = \dfrac{\tilde{b}_k - \sum_{j=k+1}^{n} W_{p_k j}\, x_j}{w_{p_k k}} \qquad k = n, n-1, \ldots, 1$

```
DIM W(N,N)  B(N)  X(N), IPIVOT(N), BTL(N)        N≥2
IP = IPIVOT(1)
BTL(1) = B(IP)
DO 11  K=2,N
KM1=K-1
SUM=0.0
IP= IPIVOT(K)
DO 12  J=1,KM1
12  SUM=SUM + W(IP,J) * BTL(J)
11  BTL(K) =B(IP) -SUM
X(N) =  BTL(N)/W(IPVOT(N),N)
DO 13   K=2,N
KK = N+1-K          KK = N-1, 1
KKP1 = KK+1         KKP1  N, 2
IP = IPIVOT(KK)
SUM=0.
DO 14  J= KKP1,N
14 SUM= SUM + W(IP,J)* X(J)
13. X(KK)= (BTL(KK)-SUM)/W(IP,K)
RETURN
```

Let V be a vector space. denote vector by x, y

$\|x\|$ is a positive number associated with each x called norm of x

with the following $\|x\| \geq 0$ $\|x\| = 0 \iff x = 0$

$\|\alpha x\| = |\alpha| \, \|x\|$

$\|x+y\| \leq \|x\| + \|y\|$

Example

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

$$\|x\|_{p,w} = \left(\sum_{i=1}^{n} |x_i|^p w_i \right)^{1/p} \qquad w_i \geq 0$$

Check these are norms

11-13-72

1. Tridiagonal Factoriz
2. Error - Residual Norms
3. Condition Number
4. Iterative improvement
5. Iterative Methods.

Recall that Gauss Elimination transforms a given nonsingular matrix A into a matrix W and a pivotal vector p.

Let $U = (u_{ij})$ be the matrix $u_{ij} = \begin{cases} w_{p_i j} & i \leq j \\ 0 & i > j \end{cases}$

U is upper triangular & the system

$$Ax = b \iff Ux = \tilde{b} \qquad \text{where}$$

$$\tilde{b}_k = b_{p_k} - \sum_{j=1}^{k-1} w_{p_k j}\tilde{b}_j \quad \| \quad b_{p_k} = \tilde{b}_k + \sum_{j=1}^{k-1} w_{p_k j}\tilde{b}_j$$

Let L be the matrix defined as follows

$$l_{ij} = \begin{cases} w_{p_i j} & i > j \\ 1 & i = j \\ 0 & i < j \end{cases}$$

Let $\check{b} = (b_{p_j})$ permuted b-vector

then $\check{b} = L\tilde{b}$

Now from $Ux = \tilde{b}$ $\quad LUx = L\tilde{b} = \check{b} = P^{-1}b$

where P is a permutation matrix.

$$P\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}(j) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}(p_j)$$

ie P permutes the rows of I according to the pivot strategy p.

Note that if P is any perm matrix then $P^2 = I$ i.e. $P = P^{-1}$

From $LUx = P^{-1}b \Rightarrow PLUx = b \Rightarrow \underbrace{P}_{\text{based on } P}\underbrace{L}_{\text{lower triang}}\underbrace{U}_{\text{upper triang}} = A$

since it is true for $\forall x$.

2) Problem $Ax = b$ By G.E. $\hat{x}$ is obtained

Error $x - \hat{x} = e$

Residual $r = Ae = A(x - \hat{x}) = b - A\hat{x}$

Since r is computable how can we use it to estimate e.

if r is small e may be tremendously large

Vector Matrix Norms $x, y, \ldots$ will be vectors. By $\|x\|$ we define a non negative no. for each x . ∋.

1. $\|x\| \geq 0$; $\|x\| = 0$ iff $x \equiv 0$
2. $\|ax\| = |a|\|x\|$
3. $\|x+y\| \leq \|x\| + \|y\|$

Examples of norms.

$$\|x\|_\infty = \max_{1\le j\le n} |x_j| \qquad \|x\|_{p,w} = \left(\sum_{j=1}^{n} |x_j|^p w_j\right)^{1/p} \quad w_j \geq 0$$

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|$$

$$\|x\|_2 = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$$

$$\boxed{\text{Homework: } \lim_{p\to\infty} \|x\|_p = \|x\|_\infty}$$

Verify 3) $\|x+y\|_\infty = \max_{1\le j\le n} |x_j+y_j| \le \max_{1\le j\le n}\{|x_j| + |y_j|\} \le \|x\|_\infty + \|y\|_\infty$

$$\|x+y\|_1 = \sum_{j=1}^{n} |x_j+y_j| \le \sum_{j=1}^{n}(|x_j| + |y_j|) = \|x\|_1 + \|y\|_1$$

$\|x+y\|_2$ note that if $x\cdot y = \sum_{i=1}^{n} x_i y_i = x^T y = (x_1, \dots, x_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$\|x\|_2 = \sqrt{x\cdot x}$ Now consider $\|x+y\|_2^2 = (x+y)\cdot(x+y)$

$$= x\cdot x + 2x\cdot y + y\cdot y$$

$\dfrac{x\cdot y}{\|x\|_2\|y\|_2} = \cos(x,y) \le 1 \quad \therefore \quad \|x\cdot y\| \le \|x\|_2\|y\|_2$ Schwartz ineq

$$\|x+y\|_2^2 \le \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 \le (\|x\|_2 + \|y\|_2)^2 \quad \text{QED}$$

Natural (induced) matrix norms: Given a vector norm and a matrix A then

$$\|A\| = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

$\| \ \|_\infty$ induces the matrix norm $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$

$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$

$\|A\|_2 = \sqrt{\rho(A^T A)}$

what properties does $\|A\|$ have?

1) $\|A\| \geq 0$; $\|A\| = 0 \quad A = 0$
2) $\|aA\| = |a|\,\|A\|$
3) $\|A+B\| \leq \|A\| + \|B\|$
4) $\|AB\| \leq \|A\|\,\|B\|$

4) from definition $\|Ax\| \leq \|A\|\,\|x\|$

$\|ABx\| \leq \|A\|\,\|Bx\| \leq \|A\|\,\|B\|\,\|x\|$

$\max_{x\neq 0} \dfrac{\|ABx\|}{\|x\|} \leq \|A\|\,\|B\| \quad = \|AB\|$

To show $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$ one must show that $\|Ax\|_\infty \leq \|A\|_\infty \|x\|_\infty$
for every x & that $\exists$ some x for which $=$ holds.

$\|Ax\|_\infty = \max_{1\le i\le n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|\,|x_j|$

$\leq \|x\|_\infty \, \|A\|_\infty$

Let i_0 be the index for which $\sum_{j=1}^{n} |a_{ij}|$ is max.

Let $x_j = \begin{cases} 1 & a_{i_0 j} \geq 0 \\ -1 & a_{i_0 j} < 0 \end{cases}$ Then $\|Ax\|_\infty = \max_{1\le i\le n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \geq \left| \sum a_{i}\right.$ x_j

but we have chosen x_j $\therefore$ $\sum_{j=1}^{n} |a_{i_0 j}| = \|A\|_\infty = \|A\|_\infty \|x\|_\infty$

$\therefore$ $\|Ax\|_\infty = \|A\|_\infty \|x\|_\infty$ for this vector.

Check: $\|A\|_1$ = column sum.

4) Iterative Improvement.

$Ax = b$, $\hat{x}^{(1)}$ computed solution (G.E.)

Compute $r^{(1)} = Ae = b - A\hat{x}^{(1)}$

Then solve $Ae = r^{(1)}$, call $\hat{e}^{(1)}$ the computed solution

Construct $\hat{x}^{(2)} = \hat{x}^{(1)} + \hat{e}^{(1)}$

continue   This will usually provide improvement.

⋮

3) Back to Condition number

$\|Ax\| \le \|A\| \, \|x\|$ $\dfrac{\|x\|}{\|A^{-1}\|} \le \|Ax\|$

$\|x\| = \|A^{-1}Ax\| \le \|A^{-1}\| \, \|Ax\|$

$\therefore \quad \dfrac{1}{\|A^{-1}\|} \le \dfrac{\|Ax\|}{\|x\|} \le \|A\|$

To estimate bounds for $\dfrac{\|e\|}{\|x\|} = \dfrac{\|x - \hat{x}\|}{\|x\|}$

$$\frac{\|r\|}{\|A\|} \le \|A^{-1}r\| \le \|A^{-1}\| \, \|r\|$$

$x = A^{-1}b \qquad \dfrac{\|b\|}{\|A\|} \le \|x\| \le \|A^{-1}\| \, \|b\|$

$$\frac{1}{\|b\|} \frac{\|r\|}{\|A^{-1}\| \, \|A\|} \le \frac{\|e\|}{\|x\|} \le \frac{\|A^{-1}\| \, \|r\|}{\|b\|} \cdot \|A\|$$

Let $\operatorname{cond}(A) = \|A\| \, \|A^{-1}\| \ge 1$

$$\boxed{\frac{1}{\operatorname{cond}(A)} \frac{\|r\|}{\|b\|} \le \frac{\|e\|}{\|x\|} \le \operatorname{cond}(A) \frac{\|r\|}{\|b\|}}$$ relative residual

$\frac{\|e\|}{\|x\|} \leq \text{cond}(A) \cdot n \cdot 2^{-t}$ t-digit binary G.E. only.

Clearly $\|A\|$ is computable ; $\|A^{-1}\|$ is not. Let B be any singular matrix & let x be ∋ $Bx = 0$

$$\|A-B\|\,\|x\| \geq \|(A-B)x\| = \|Ax\| \geq \frac{\|x\|}{\|A^{-1}\|}$$

$\therefore \quad \frac{1}{\|A^{-1}\|} \leq \|A-B\|$ find a B ∋ $\|A-B\|$ is very small

5) Iterative Methods

$Ax = b \Rightarrow (b-Ax) = 0$ Let C be nonsingular ie $\det C \neq 0$

$$\Rightarrow C^{-1}(b-Ax) = 0$$

$$x + C^{-1}(b-Ax) = x$$

$$(I - C^{-1}A)x + C^{-1}b = x \Rightarrow x = Bx + \bar{c}$$

Fixed pt. scheme where $B = I - C^{-1}A$ & $\bar{c} = C^{-1}b$.

Iterative Scheme

$$x^{(m+1)} = Bx^{(m)} + \bar{c}, \quad x^{(0)} = 0$$

Q: does $x^{(m)}$ converge? how to choose C to do the job "best."

A: $x^{(m)}$ converges $\{\|x^{(m)} - x\| \xrightarrow{m\to\infty} 0\}$ iff $\|B\| < 1$

Find C ∋. C^{-1} exists $\|I - C^{-1}A\| < 1$ C^{-1} must be easily computed

Decompose A into $\hat{L} + \hat{D} + \hat{U}$ lower, diag, upper

Jacoby Let $C = \hat{D}$ (assume $a_{ii} \neq 0$)

$$\therefore \quad x_i^{m+1} = \sum_{j\neq i} -\frac{a_{ij}}{a_{ii}} x_j^{(m)} + \frac{b_i}{a_{ii}}$$

Jacoby converges when $|a_{ii}| > \sum_{i \neq j} |a_{ij}|$ diagonal dominance

Proof: $\|B\|_\infty = \max_{1\le i\le n} \sum_{j\ne i} \left|\frac{a_{ij}}{a_{ii}}\right| < 1$

2) Gauss-Seidel Choose $\hat{L}+\hat{D}$

$$\|I-(\hat{L}+\hat{D})^{-1}A\| \le \|I-\hat{D}^{-1}A\| \quad \text{not true in General}$$

$$x^{m+1} = x^m + (\hat{L}+\hat{D})^{-1}(b-Ax^m)$$

$$(\hat{L}+\hat{D})\,x^{m+1} = (\hat{L}+\hat{D})\,x^m + b - Ax^m$$

$$\hat{D}\,x^{m+1} = -\hat{L}\,x^{m+1} - \hat{U}\,x^m + b$$

G.S. $$x^{m+1} = -\hat{D}^{-1}\{+\hat{L}\,x^{m+1} + \hat{U}\,x^m - b\}$$

$i=1,\dots,n$ $$x_i^{(m+1)} = -\frac{1}{a_{ii}}\left[\sum_{j=1}^{i-1} a_{ij}x_j^{m+1} + \sum_{j=i+1}^{n} a_{ij}x_j^m - b_i\right]$$

Let $A=A^T$ then A has real eigenvalues & n linearly indep eigenvectors $\Rightarrow$ the k^{th} eigenvector $\cdot$ j^{th} eigenvector $=\delta_{kj}$

Nov. 20, 1972

1. Linear Algebra (Review)
2. Matrix Norms
3. Convergence of It Schemes (Review)
4. Power method for λ-vector.

$$H_n = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & & & & \\ 1/3 & & & & \\ 1/4 & & & & \\ 1/5 & & & & 1/9 \end{pmatrix} \qquad H_n^{-1}$$

$$\text{cond}(A) = \|A\|_\infty\,\|A^{-1}\|_\infty = \|A\|_1\,\|A^{-1}\|_1 = 5\times 10^5$$

$$\det A = \sum_{(\sigma)} \operatorname{sign}(\sigma)\; a_{1 i_1} a_{2 i_2} \cdots a_{n i_n}$$

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$$ $n!$ terms in sum

$$\operatorname{sign}(\sigma) = \begin{cases} +1 & \sigma \text{ even} \\ -1 & \sigma \text{ odd} \end{cases}$$

Theorem : $\det(AB) = \det(A)\det(B)$

Theorem : A is invertible iff $\det(A) \neq 0$

Proof :

$$A = PLU$$

P is a permutation matrix

$$L = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ x & & 1 \end{pmatrix} \qquad U = \begin{pmatrix} \ddots & & \vdots \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$$

$$\begin{aligned} \det A &= \det P \det L \det U \\ &= \operatorname{sign}(P)\; 1 \cdot u_{11} u_{22} \cdots u_{nn} \end{aligned}$$

if $\exists$ a $u_{ii} = 0$ then no back sub is possible $\therefore$ A^{-1} is not invertible

Cramer's Rule : $Ax = b$

$$x_j = \frac{\det A^j}{\det A}$$ where A^j has j col replaced by b

Eigenvalues λ of a matrix is a no. real or complex $\ni$

$$\det(A - \lambda I) = 0$$

$\therefore$ $\exists\ x \neq 0, \ni$ $(A - \lambda I)x = 0$ or $Ax = \lambda x$

Every matrix has n - eigenvalues (with multiplicities)

one term is $\prod_{i=1}^{n} (a_{ii} - \lambda)$. Polynomial of degree n in λ

Def: $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$

$\rho(A)$ ... $= \max |\lambda_i(A)|$

Examples:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A \qquad \det(A - \lambda I) = (1-\lambda)^2 \qquad \lambda = 1$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} s \\ 0 \end{pmatrix} \quad s \text{ is any no.}$$

When does A have n-linear indep eigenvectors?

1. If eigenvalues are all distinct
2. if $A = A^T$ matrix has n linear indep eigenvectors:
3. If $A A^T = A^T A$

If A has n lin, indep eigenvectors $\exists$ a non singular matrix X s.t. $A = X \Lambda X^{-1}$
where Λ is diagonal

$A X = X \Lambda$ this says that the column of X are eigenvectors

$$\Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

<u>Theorem:</u> Let $x^{(0)}, x^{(1)}, \ldots$ be a sequences of column vectors generated by iteration scheme

$$x^{(m+1)} = B x^{(m)} + c$$

Then $x^{(m)} \longrightarrow x$ where $x = Bx + c$ <u>iff</u> $\rho(B) < 1$

Proof: Let

$$e^{(m)} = x^{(m)} - x$$

$$e^{(m+1)} = B e^{(m)} = B^2 e^{(m-1)} = B^{m+1} e^{(0)}$$

When B has n lin indep eigenvectors then $B^{(m+1)} e^{(0)} \to 0$

iff $\rho(B) < 1$ $\quad m \to \infty$

if B has n linear indep eigenvector

$$e^{(0)} = \sum_{j=1}^{n} \alpha_j \xi_j \qquad A\xi_j = \lambda_j \xi_j$$

$$B e^{(0)} = \sum_{j=1}^{n} \alpha_j B \xi_j = \sum_{j=1}^{n} \alpha_j \lambda_j \xi_j$$

$$B^{(m)} e^{(0)} = \sum_{j=1}^{n} \alpha_j \lambda_j^{(m)} \xi_j \longrightarrow 0 \quad \text{if } \rho(B) < 1$$

because $\lambda_i^{(m)} \to 0 \quad m \to \infty \qquad i = 1, \dots, n$

How is $\rho(A)$ related to norm A

where $\| \ \|$ is some natural norm. i.e. a matrix norm derived or induced from a vector norm by

$$\|A\| = \max_{\underline{x} \neq 0} \frac{\|Ax\|}{\|x\|}$$

Result: $\rho(A) \leq \|A\|$

Proof: Let $|\lambda| = \rho(A) \quad A\underline{u} = \lambda \underline{u}$

$$\|A\| = \max_{\underline{x} \neq 0} \frac{\|Ax\|}{\|x\|} \geq \frac{\|Au\|}{\|u\|} = \frac{|\lambda| \, \|u\|}{\|u\|} = \rho(A)$$

Result: Given any A & $\epsilon > 0$ $\exists$ a natural matrix norm $\ni$. $\rho(A) \leq \|A\| \leq \rho(A) + \epsilon$

Examples: $\begin{pmatrix} .9 & 1000 \\ 0 & .9 \end{pmatrix}$ $\qquad \|A\|_\infty = 1000.9$, $\quad \|A\|_1 = 1000.9$

$\rho(A) = .9$

If $A = A^T$ then $\rho(A) = \|A\|_2$

In general $\|A\|_2 = \sqrt{\rho(AA^T)}$

if $A = A^T$ $\quad \rho(AA^T) = \rho(A^2) = \rho^2(A)$

If $Ax = \lambda x$, $\quad A^2 x = A\,Ax = A\lambda x = \lambda^2 x$

Proof of $\|A\|_2 = \sqrt{\rho(AA^T)}$

Put $B = AA^T$ then $B^T = B$

Recall B has n-lin indep eigenvectors

$$Bu_s = \lambda_s u_s \qquad s = 1, \dots, n$$

Since B is sym $\quad u_s^T u_{s'} = \delta_{ss'} = \begin{cases} 1 & s = s' \\ 0 & s \neq s' \end{cases}$

$$\|A\|_2^2 = \max_{x \neq 0} \frac{\|Ax\|_2^2}{\|x\|_2} = \max_{\|x\|_2 = 1} \|Ax\|_2^2 = \max_{\|x\|_2 = 1} (Ax)^T \cdot (Ax)$$

$$= \max_{\|x\|_2 = 1} x^T(A^T)Ax = \max_{\|x\|_2 = 1} x^T B x = \rho(AA^T)$$

Let λ be the eigenvalue of $A^TA = B$ for which $|\lambda| = \rho(B)$ & u the corresp eigenvector, Then $\quad$ Arrange $\|u\|_2 = 1$

$$\max_{\|x\|_2 = 1} x^T(A^TA)x \geq u^T B u = u^T \lambda u = \lambda u^T u = \lambda = \rho(B)$$

To get $\leq$ use the representation $\quad x = \sum_{j=1}^{n} \alpha_j u_j \quad$ plug in & it works

Example:

$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \qquad \|A\|_\infty = 2 \qquad \|A\|_1 = 2$

$A^T = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \qquad \|A\|_2 = \sqrt{\dfrac{3+\sqrt{5}}{2}}$

* Check get eigenval of AA^T

$AA^T = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \qquad \begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 1 = 1 - 3\lambda + \lambda^2 = \frac{3+\sqrt{5}}{2}$ ok

* $\quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \|A\|_1 = 3 \qquad \|A\|_2 = \rho(A)$

$A^T = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \|A\|_\infty = 3 \qquad = 3$

③ Review iteration scheme   HW P.180   3.91–95   due 2 wks

$Ax=b \qquad Ax-b=0$

$$x = x - C^{-1}(Ax-b) \qquad C \text{ is invertible}$$

$$x = (I - C^{-1}A)x + C^{-1}b$$

Fixed Pt form $\quad x = Bx + c$

Iterates $\quad x^{(m+1)} = Bx^{(m)} + c \qquad x^{(0)}$ is given

$\rho(B)$ is what controls convergence of scheme.

a) Jacoby corresponds $\quad C = \hat{D}$

$$x^{m+1} = (I - \hat{D}^{-1}A)x^{(m)} + \hat{D}^{-1}b$$

$$x_i^{(m+1)} = \sum_{j\neq i} -\frac{a_{ij}}{a_{ii}} x_j^{(m)} + \frac{b_i}{a_{ii}}$$

Convergence: if $\|I - \hat{D}^{-1}A\|_{1,\infty} < 1 \qquad \rho(I - \hat{D}^{-1}A)$

$\|\ \|_1$ A is strictly column diag dominant

$\|\ \|_\infty$ A " " row " "

b) Gauss-Seidel $\quad C = \hat{L} + \hat{D}$

$$x_i^{(m+1)} = \frac{-1}{a_{ii}}\left[\sum_{j=1}^{i-1} a_{ij} x_j^{(m+1)} + \sum_{j=i+1}^{n} a_{ij} x_j^{(m)} - b_i\right]$$

Convergence: If $\quad k = \max_{1\le i\le n} \sum_{j\neq i} \left|\frac{a_{ij}}{a_{ii}}\right| < 1$

A is strictly rows diagonally dominant.

c) Iterative Correction method

$C = \tilde{A}^{-1}$ = Approx inverse from Gauss elim.

$$x^{(m+1)} = (I - \tilde{A}^{-1}A)\, x^{(m)} + \tilde{A}^{-1} b$$

$x^{(0)}$ arbitrary

Let $x^{(0)}$ be arbitrary define

$$r^{(0)} = b - A x^{(0)} \qquad \text{residual}$$

solve by GE

$$A\, e^{(0)} = r^{(0)}$$

$$x^{(1)} = x^{(0)} + e^{(0)}$$

$$r^{(1)} = b - A x^{(1)}$$

$$A e^{(1)} = r^{(1)}$$

$$x^{(2)} = x^{(1)} + e^{(1)}$$

$$e^{(0)} = \tilde{A}^{-1} r^{(0)}$$

$$e^{(1)} = \tilde{A}^{-1} r^{(1)}$$

in principle this is faster & more accurate than the others.

4) Power Method for computing $\rho(A)$

where $|\lambda_1| > |\lambda_2| \geq |\lambda_2| \geq \dots \geq |\lambda_r| \quad r \leq n$

$\rho(A) = |\lambda_1|$

Method: Let z be a vector to be described, compute the sequence of vectors

$z,\ Bz,\ B^2 z,\ B^3 z, \dots, B^m z$

Let $y_1, \dots, y_r$ be r lin indep eigenvectors corresp $\lambda_1, \lambda_2, \dots, \lambda_r$

Let $z = \sum_{i=1}^{r} y_i$

$$B^m z = \sum_{i=1}^{r} \lambda_i^m y_i$$

$$\left(\frac{1}{\lambda_1} B\right)^m z = y_1 + \sum_{i=2}^{r} \left(\frac{\lambda_i}{\lambda_1}\right)^m y_i \qquad \text{as } m \to \infty \text{ sum} \to 0$$

$$\left(\frac{1}{\lambda_1} B\right)^m z \sim y_1$$

$$z^{(m)} = \frac{B^m z}{i\text{th entry of } B^m z} = \frac{\left(\frac{1}{\lambda_1} B\right)^m z}{i\text{th entry of } \left(\frac{1}{\lambda_1} B\right)^m z} \sim \frac{y_1}{i\text{th entry of } y_1} = y_1^*$$

provided $\neq 0$

$$B z^{(m)} \sim B y_1^* = \lambda_1 y_1^* \qquad \lambda_1 \approx i\text{th entry } B z^{(m)} = \frac{i\text{th entry of } B^{m+1} z}{i\text{th entry of } B^m z}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -\lambda & & \\ 0 & -\lambda & \\ 0 & & -\lambda \end{bmatrix} \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = -(\lambda+1)\left[(1-\lambda)^2 - 4\right] = 0 \qquad \lambda = -1$$

$$\lambda^2 - 2\lambda + 3 \qquad (\lambda-3)(\lambda+1)$$

$$\lambda_1 = 3 \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \lambda_2 = -1 \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \lambda_3 = -1 \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

no. of iter	0	1	2	3	4	5
$B^m z$	$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 4 \\ -3 \end{pmatrix}$	$\begin{pmatrix} 13 \\ 14 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 41 \\ 40 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 121 \\ 122 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 365 \\ 365 \\ 3 \end{pmatrix}$

$$z = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

m	1	2	3	4	5	
λ_1	5	$13/5$	$41/13$	$\frac{121}{41}$	$\frac{365}{121}$	$i=1$
	2	$14/4$	$40/14$	$\frac{122}{40}$	$\frac{365}{122}$	$i=2$

Nov 27, 1972

Given $n+1$ pts : $x_0, x_1, \ldots, x_n$ & the values of $f(x)$ at these pts find a polynomial of degree $\leq n$ $\ni$.

$$p(x_i) = f(x_i) \qquad i = 0, \ldots, n$$

Solution (Lagrange)

Define

$$l_k(x) = \prod_{i=0}^{n} \frac{(x - x_i)}{(x_k - x_i)} = \text{poly of degree} \leq n$$

$$\ell_k(x_j) = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases} = \delta_{kj}$$

Solution $p_n(x) = \sum_{k=0}^{n} f(x_k)\, \ell_k(x)$

$$p_n(x_j) = \sum_{i=0}^{n} f(x_k)\, \ell_k(x_j) = f(x_j), \therefore p_n(x) \text{ is a solution}$$

& it is unique

assume $q_n(x)$ is another; take the difference $p_n(x) - q_n(x) = h_n(x)$ is a poly of degree n with $n+1$ zeroes is impossible $\therefore \equiv 0$

<u>Example:</u> 2 point $n=1$ x_0, x_1

$$\ell_0(x) = \frac{x-x_1}{x_0-x_1} \qquad \ell_1(x) = \frac{x-x_0}{x_1-x_0}$$

$$p_2(x) = f(x_0)\frac{x-x_1}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0}$$

$$p_2(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0)$$

$p_2(x)$ is the line through $(x_0, f(x_0))$ & $(x_1, f(x_1))$

$n=2$

$$p_3(x) = \frac{f(x_0)\,(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_1)\,(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$+ \frac{f(x_2)\,(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{1}{x_0-x_2}\left[f(x_0)\frac{(x-x_1)(x-x_2)}{x_0-x_1} + f(x_1)\frac{(x-x_0)(x-x_2)(x_0-x_2)}{(x_1-x_0)(x_1-x_2)} \right.$$

$$\left. + f(x_2)\,(x-x_0)(x-x_1) \right]$$

$$f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \left[\frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}\right](x - x_0)(x - x_1)$$

In general let $f[x_0] = f(x_0)$

& $f[x_0, x_1, \dots, x_n] = \dfrac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

We expect than

Newtons form

$$p_n(x) = \sum_{k=0}^{n} f(x_k)\, l_k(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots + f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

Where $l_k(x)$ & $f[\ \ ]$ have been defined

$$P_{n+1} = P_n + f[x_0, \dots, x_n, x_{n+1}](x - x_0)(x - x_1)\cdots(x - x_n)$$

Let $\psi(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$

then

$$l_k(x) = \frac{\psi(x)}{(x - x_k)\,\psi'(x_k)} = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)}$$

$$\therefore\ p_n(x) = \psi(x) \sum_{k=0}^{n} \frac{f(x_k)}{(x - x_k)\,\psi'(x_k)}$$

If $f(x) \equiv 1$

$$P_n(x) = \psi(x) \sum_{k=0}^{n} \frac{1}{(x - x_k)\,\psi'(x_k)} = 1 = \sum_{k=0}^{n} l_k(x)$$

Lemma

if $f(x) = p_n^*(x)$ a poly of degree $\le n$ and $p_n(x)$ = Lagrange Int poly with $f = p_n^*$

Then $p_n(x) \equiv p_n^*(x)$

Proof

$p_n(x) - p_n^*(x) = h(x)$ = poly of degree $\le n$ & vanishes at $n+1$ pts & therefore must be ident $\equiv 0$.

To get $1 = \sum l_k(x)$ apply lemma with $n=0$

Observe further $f[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} \frac{f(x_k)}{\prod_{j \ne k} (x_k - x_j)}$

$\therefore$ $f[\ \ ]$ is a symmetric function of $(x_0, x_1, \ldots, x_n)$

To show this fact use $\circledast$ $p_n = p_{n-1} + f[\ \ ](\ )(\ )\cdots$

& equate coeff of X^n of both sides.

$= f[x_0, \ldots, x_n] X^n$

$$l_k(x) = \frac{1}{\prod_{j \ne k} (x_k - x_j)} X^n + \ldots$$

$$\sum_{k=0}^{n} \frac{f(x_k)}{\prod_{j \ne k} (x_k - x_j)} X^n + \ldots$$

Nested Multiplication

$p(x) = a_0 + a_1 x + \ldots + a_n x^n$

set $b_n = a_n$

for $k = n-1, n-2, \ldots, 0$ Do

$b_k = a_k + (z \cdot) b_{k+1}$

Then $p(x) = b_0 + (x - z) \underbrace{(b_1 + b_2 x + \ldots + b_n x^{n-1})}_{q(x)}$

$p(z) = b_0$

Generalize to Newton type poly

$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)\cdots(x-x_{n-1})$

Find $p(z)$

Set $b_n = a_n$

For $k = n-1, n-2, \dots, 0$ DO

$b_k = a_k + (z - x_k)\, b_{k+1}$

Then: $p(x) = b_0 + (x-z)\left(b_1 + b_2(x-x_0) + b_3(x-x_0)(x-x_1) + b_n(x-x_0)\cdots(x-x_{n-2})\right)$

$\therefore \quad p(z) = b_0$

To show nested multiplication algorithm

$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)\cdots(x-x_{n-1})$

$= (b_0 - (z-x_0)b_1) + [b_1 - (z-x_1)b_2](x-x_0) + (b_2 - (z-x_2)b_3](x-x_0)(x-x_1)\cdot$

$b_0 + b_1(x-z + b_2(x-x_0)(x-z) + \dots$ QED

Note & recall the Newton form. $p(x) = \sum a_i (x-x_i)(x-x_{i-1})\cdots$

$\therefore \quad a_i = f[x_0, x_1, x_2, \dots x_i]$

$b_n = f[x_0, x_1, \dots, x_n]$

$f[x_0, x_1, \dots, x_k] = b_k - (z-x_k)\, b_{k+1}$ set $k = n-1$

$f[x_0, x_1, \dots, x_{n-1}] = b_{n-1} - (z - x_{n-1})\, f[x_0, x_1, \dots, x_n]$

$\therefore \quad f[x_0, x_1, \dots, x_n] = \dfrac{b_{n-1} - f[x_0, x_1, \dots, x_{n-1}]}{z - x_{n-1}}$

Suppose $f[x_0, \dots, x_n]$ was not given as a formula

let $z = x_n$

ok above

$f[x_0, x_1, \dots x_n] = \dfrac{b_{n-1} - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_{n-1}}$

$$p_n(x) = \sum_{k=0}^{n} f[x_0, x_1, \dots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

Probl : Given $x_0, \dots, x_n$ & $f(x_0) \cdots f(x_n)$

Find an efficient method for computing diff.

$$\begin{array}{lllll}
x_0 & f[x_0] & & & \\
 & & f[x_0,x_1] & & \\
x_1 & f[x_1] & & f[x_0,x_1,x_2] & \\
 & & f[x_1,x_2] & & \\
\vdots & \vdots & \vdots & \vdots & \cdots \quad f[x_0,\dots,x_n] \\
x_n & f[x_n] & f[x_{n-1},x_n] & f[x_{n-2},x_{n-1},x_n] &
\end{array}$$

write

$$p_n(x) = \sum_{k=0}^{n} f[x_k, x_{k+1}, \cdots x_n] \prod_{j=k+1}^{n} (x - x_j)$$

Algorithm for computation of coeff in Newton

Given $x_0, x_1, \dots, x_n$; $f(x_0) \cdots f(x_n)$

Initialize $d_i = f(x_i)$ $\quad i = 0, \dots, x_n$

For $k = 1, 2, \dots, n$ Do

for $i = 0, 1, \dots, n-k$ Do

$d_i = (d_{i+1} - d_i)/(x_{i+k} - x_i)$

Then $d_k = f[x_k, x_{k+1}, \dots x_n]$, $i = 0, 1, \dots, n$

for $k = 1$ $\quad i = 0, \dots, n-1$

$$d_0 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \cdots \quad d_{n-1} = \frac{f(x_n) - f(x_{n-1})}{x_n - x_0}$$

for $k = 2$ $\quad i = 0, \dots, n-2$

$$d_{n-2} = f[x_{n-2}, x_{n-1}, x_n]$$

Example

$\frac{1}{1+x^2}$ $\quad -5 \le x \le 5$ $\quad$ Use poly interp

$x_0, x_1, \dots, x_n$ $\quad x_k = -5 + \frac{10k}{n}$ $\quad$ n even

let E_n be $\max\limits_{-5\le x\le 5} \left| f(x) - P_n(x) \right|$

n	E_n (exp)
2	6.4 (-1)
4	4.3 (-1)
6	6.1 (-1)
8	1 (0)
10	|
12	
14	|
16	1.4 (1)

$p_2(x) = \frac{1}{26} + \frac{5}{26}(x+5) - \frac{1}{26}(x-5)x$

use low degree poly in smaller intervals.

Error: $x_0, x_1, \dots, x_n \in [a,b]$

$f(x_0), \dots, f(x_n)$

$f(x) - P_n(x) = e_n(x)$ $\quad$ To get a formula for $e_n(x)$

let $\bar{x}$ be another interp pt $(x_0, \dots x_n, \bar{x})$

$$P_{n+1}(x) = P_n(x) + f[x_0, \dots, x_n, \bar{x}](x-x_0)(x-x_1)\cdots(x-x_n)$$

set $x = \bar{x}$

$$f(\bar{x}) = P_n(\bar{x}) + f[x_0, \dots, x_n, \bar{x}](\bar{x}-x_0)(\bar{x}-x_1)\cdots(\bar{x}-x_n)$$

$$\therefore \quad e_n = f[x_0, x_1, \dots, x_n, x](x-x_0)(x-x_1)\cdots(x-x_n)$$

$$\exists \text{ a } \xi \ni e_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n) \quad \min x_i < \xi < \max x_i$$

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$$\begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}$$

3.95 program power method and different starting vectors

1. Review Interpol Poly.
2. Error radius of convergence & lemniscate of convergence
3. Forward & Backward differences
4. Spline Interpolation (Cubic)
5. Least Squares & Orthogonal polynomials.

1 $x_0, x_1, \ldots, x_n$

$f(x_0), f(x_1), \ldots, f(x_n)$

Construct a poly $\leq n$ $\ni$ $p(x_i) = f(x_i)$

Lagrange

$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{k} \frac{(x - x_i)}{(x_k - x_i)} \qquad l_k(x_j) = \delta_{kj}$$

$$P_n(x) = \sum_{k=0}^{n} f(x_k)\, l_k(x)$$

Newton's form

$$f[x_0] = f(x_0) \qquad f[x_0, \ldots, x_k] = \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}$$

$$P_n(x) = \sum_{k=0} f[x_0, \ldots, x_k](x - x_0) \cdots (x - x_{k-1})$$

$$P_{n+1}(x) = P_n(x) + f[x_0, x_1, \ldots, x_{n+1}](x - x_0) \cdots (x - x_n)$$

2) <u>Error:</u> $e_n(x) = f(x) - P_n(x) = f[x_0, \ldots, x_n, x](x - x_0)(x - x_1) \cdots (x - x_n)$

Construct $P_{n+1}(x) = P_n(x) + f[x_0, \dots, x_n, \bar{x}](x-x_0)\cdots(x-x_n)$

$P_{n+1}(\bar{x}) = f(\bar{x})$

$e_n(x) = f(x) - P_n(x) = f[x_0, \dots, x_n, x](x-x_0)\cdots(x-x_n)$

Theorem: $f[x_0, x_1, \dots, x_n] = \dfrac{f^{(n)}(\xi)}{n!}$ $\quad \min_i x_i \le \xi \le \max_i x_i$

For $n=1$ $\quad f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\xi)$ $\quad$ M.V.T.

For $n=2$ $\quad \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \dfrac{f^{(1)}(\xi_2) - f^{(1)}(\xi_1')}{x_2 - x_0} = f^{(2)}(\xi_2)$

$e_n(x)$ has $n+1$ zeroes in $[a,b]$

$e_n'(x)$ has n zeroes in $[a,b]$ by Rolle's Theorem

$e_n''(x)$ has $n-1$ zeroes i $[a,b]$

$\vdots$

$e_n^{(n)}(x)$ has 1 zero in $[a,b]$

$\therefore \quad e_n^{(n)}(\xi) = 0 \quad \xi \in [a,b]$

$e_n^{(n)}(\xi) = f^{(n)}(\xi) - \overbrace{n!\, f[x_0, x_1, \dots, x_n]}^{P_n^{(n)}(x)} = 0$ $\quad$ QED

Error: $\quad e_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$ $\quad \min(x_i, x) \le \xi \le \max(x_i, x)$

Recall Taylor's expansion error

Find x_i's so that $g(x; x_0, \dots, x_n)$ is max for all

$g(x) = \prod_{i=0}^{n} (x - x_i)$

$\min_{x \in [a,b]} |g(x; x_0, \dots, x_n) - g(x, x_0^*, x_1^*, \dots, x_n^*)| = \min.$ over all

3. Forward & Backward Differences.

Interpolation pts equally spaced

$x_k = x_0 + k.h \qquad k=0,1,\dots,n$

For a general $x = x_0 + sh$ $\quad$ s not an integer ; $f(x_0+sh) = f_s$

$\Delta^0 f_s = f_s \qquad \Delta^i f_s = \Delta^{i-1} f_{s+1} - \Delta^{i-1} f_s$

$$P_n(x) = \sum_{k=0}^{n} f[x_0, x_1, \dots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

$$f[x_k, x_{k+1}, \dots, x_{k+n}] = \Delta^n f_k \left(\frac{1}{n!\,h^n}\right)$$

for $n=0$

$$f[x_k] = f(x_k) = f_k = f(x_0+kh)$$

Assume true for n & induce for n+1

$$f[x_k, \dots, x_{k+n}, x_{k+n+1}] = \frac{f[x_{k+1}, x_{k+2}, \dots, x_{k+n}, x_{k+n+1}] - f[x_k, x_{k+1}, \dots, x_{k+n}]}{x_{k+n+1} - x_k}$$

$$= \frac{\frac{1}{n!\,h^n}\Delta^n f_{k+1} - \frac{1}{n!\,h^n}\Delta^n f_k}{(n+1)h}$$

$$= \frac{1}{(n+1)!\,h^{n+1}}\left[\Delta^n f_{k+1} - \Delta^n f_k\right] = \Delta^{n+1} f_k \frac{1}{(n+1)!\,h^{n+1}}$$

Plug into newton's form.

$$P_n(x_0+sh) = \sum_{i=0}^{n} f[x_k, x_{k+1}, \dots, x_{k+i}] \prod_{j=0}^{i-1} (x_s - x_{k+j})$$

$$= \sum_{i=0}^{n} \frac{1}{i!\,h^i} \Delta^i f_k \; h^i \prod_{j=0}^{i-1} (s-k-j) = \sum_{i=0}^{n} \Delta^i f_k \prod \frac{(s-k-j)}{j+1}$$

$$= \binom{s-k}{i} \; ; \; \binom{y}{i} = \frac{y(y-1)\cdots(y-i+1)}{1\cdots i}$$

$$\therefore\ p(x_0+sh) = \sum_{i=0}^{n} \binom{s-k}{i} \Delta^i f_k$$

Example

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	1				
		4			
1	5		22		
		26		42	
2	31		64		24
		90		66	
3	121		130		
		220			
4	341				

find $f(5)$ $\quad n=4 \quad s=5 \quad k=0$

$$\binom{5}{0}=1 \quad \binom{5}{1}=5 \quad \binom{5}{2}=10 \quad \binom{5}{3}=10 \quad \binom{5}{4}=5$$

$$\binom{5}{5}=1$$

$$f(5) = 1\cdot 1 + 4\cdot 5 + 10\cdot 22 + 10\cdot 42 + 5\cdot 24 = 1+20+220+420+120$$
$$=781$$

Backward Form of Newton Interpolation polyn.

$$p_n(x_0+sh) = \sum_{i=0}^{n} f[x_{n-i}, x_{n-i+1}, \ldots, x_k] \prod_{j=0}^{i-1} (x-x_{n-j})$$
$$= \sum_{i=0}^{n} \frac{1}{i!\,h^i} \Delta^i f_{k-i} \prod_{j=0}^{i-1} h(s-n+j)$$
$$= \sum_{i=0}^{n} \frac{1}{i!} \Delta^i f_{k-i} \prod_{j=0}^{i-1} (s-n+j)$$
$$= \sum_{i=0}^{n} \frac{(-1)^i}{i!} \Delta^i f_{k-i} \prod_{j=0}^{i-1} (n-s-j)$$
$$= \sum_{i=0}^{n} (-1)^i \Delta^i f_{k-i} \binom{n-s}{i}$$

Spline Interpolation. (Cubic)

Piecewise polyn interpolation

$x_1, x_2, \ldots, x_{n+1}$

$a = f(x_1), \ldots, f(x_{n+1}) = b$

Problem find $g_3(x)$ = cubic polyn interpolating the points whose $f(x)$ is defined at given pts.

Notation: in each interval $\{x_i, x_{i+1}\}$ $i = 1, \ldots, n$ construct

$$P_i(x) = c_{i,1} + c_{i,2}(x - x_i) + c_{i,3}(x - x_i)^2 + c_{i,4}(x - x_i)^3$$

To determine Coeff:

$P_i(x_i) = f(x_i)$ $\quad$ $P_i(x_{i+1}) = f(x_{i+1}) = P_{i+1}(x_i)$ $\quad \therefore g_3(x)$ cont.

Various possibilities to obtain the other two constants in P_i

I) Hermite interpolation (requires $f'(x_i)$)

require $P_i'(x_i) = f'(x_i)$ $\quad$ $P_i'(x_{i+1}) = f'(x_{i+1})$

$$P_i(x) = f(x_i) + \overset{f'(x_i)}{f[x_i, x_i]}(x - x_i) + f[x_i, x_i, x_{i+1}](x - x_i)^2 + f[x_i, x_i, x_{i+1}, x_{i+1}](x - x_i)^2 \underbrace{(x - x_{i+1})}_{(x - x_i) - \Delta x_i}$$

$c_{i,1} = f(x_i)$

$c_{i,2} = f'(x_i) = s_i$

$$c_{i,3} = f[x_i, x_i, x_{i+1}] - f[x_i, x_i, x_{i+1}, x_{i+1}]\Delta x_i = \frac{f[x_i, x_{i+1}] - f[x_i, x_i]}{x_{i+1} - x_i} - c_{i,4}\Delta x_i = \frac{f[x_i, x_{i+1}] - s_i}{\Delta x_i} - c_{i,4}\Delta x_i$$

$$c_{i,4} = f[x_i, x_i, x_{i+1}, x_{i+1}] = \frac{s_{i+1} + s_i - 2f[x_i, x_{i+1}]}{(\Delta x_i)^2}$$

$g_3(x)$ is now determined $\Rightarrow$ $g_3(x) \in C^1$ in $[a, b]$

2) Bessel interpol $\quad s_i = f[x_{i-1}, x_i]$ @ end pts extra data

3) Spline Interpolation : Fix $c_{i,1,2,3,4}$ $\ni$ $g_3(x)$ has 2 derivatives

Now $P_i(x)$ are determined globally

Require $\quad P''_{i-1}(x_i) = P''_i(x_i)$

$$2c_{3,i-1} + 6c_{4,i-1}\Delta x_{i-1} = 2c_{3,i} \qquad i = 2,\dots,N$$

put into formulas to get s_i

$$2\left(\frac{f[x_{i-1},x_i]-s_{i-1}}{\Delta x_{i-1}}\right) + 4c_{i-1,4}\,\Delta x_{i-1} = \frac{2(f[x_i,x_{i+1}]-s_i)}{\Delta x_i} - 2c_{i,4}\Delta x_i$$

$$(\Delta x_i)s_{i-1} + 2(\Delta x_{i-1}+\Delta x_i)s_i + (\Delta x_{i-1})s_{i+1} = 3\left(f[x_{i-1},x_i]\Delta x_i + f[x_i,x_{i+1}]\Delta x_{i-1}\right)$$

$i = 2,\dots,N$

$N-1$ eqs $\quad N+1$ s's $\quad$ 2 must be given for b.c.

Then is tridiag system of LE for s's.

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Least Squares

Given $x_i \quad i=1,N \quad f(x_i) \quad i=1,N \quad \in [a,b]$

Choose $\varphi_1(x), \varphi_2(x), \dots, \varphi_K(x)$ $\quad$ Try to fit data

by expressions of the form

$$\sum_{i=1}^{K} c_i\varphi_i(x) \qquad \text{choose } N >> K$$

Fit in least square norm

$$e = \sum^{N}\sum^{K}(c_i\varphi_i(x_j) - f_j)^2$$

To find c's diff & set deriv = 0

$$-2 \sum_{n=1}^{N} \left[f_n - \sum_{i=1}^{k} c_i \varphi_i(x_n) \right] \varphi_j(x_n) = 0$$

$$\sum_{i=1}^{k} c_i \left(\sum_{n=1}^{N} \varphi_i(x_n) \varphi_j(x_n) \right) = \sum_{n=1}^{N} f_n \phi_j(x_n)$$

Define

$$\langle f, g \rangle = \sum_{n=1}^{N} f_n g_n$$

Then $\displaystyle\sum_{n=1}^{N} \varphi_i(x_n)\varphi_j(x_n) = \langle \varphi_i, \varphi_j \rangle$ ∴ Eqs for c's take form

$$\sum_{i=1}^{k} \langle \varphi_j, \phi_i \rangle c_i = \langle f, \varphi_j \rangle \quad \text{normal eqs for } c\text{'s.}$$

Way to get c's is to find ϕ's so that they are orthogonal

$\ni \langle \phi_i, \phi_j \rangle = 0 \quad i \neq j$ Then the c's are given by

normalized $\langle \phi_i, \phi_i \rangle = 1 \quad i = j$ $c_j = \dfrac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}$

Problem now is to find the orthogonal fns $\phi_j(x)$.

<u>Orthogonal Polynomials</u> $[a,b]$, $f(x)$, $g(x)$ real fns on $[a,b]$

Inner Product (i) $\displaystyle\int_a^b f g w \, dx = \langle f, g \rangle \quad w(x) > 0$ (given) continuous

(ii) $\displaystyle\sum_{i=1}^{N} f g w = \langle f, g \rangle$ discreet

Examples

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx \qquad \langle f,g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}dx$$

$$\langle f,g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\,dx \qquad \langle f,g\rangle = \sum_{j=0}^{N} f(-1+jh)\,g(-1+jh)$$

$$h = \frac{2}{N}$$

f,g are orthogonal if $\langle f,g\rangle = 0$

Define $P_0(x), P_1(x), P_2(x), \ldots$ for a system of orthog polyn

if (i) $P_K(x)$ is of degree $K = \alpha_k X^k$ + poly deg $< k$ $\alpha_k \neq 0$

(ii) $\langle P_K, P_j\rangle = 0$ $j \neq k$

Example: if $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

then $P_0 = 1$ $P_1(x) = x$, $P_2(x) = 3x^2 - 1$ are orthog.

$\langle P_0, P_1\rangle = 0$

$\langle P_0, P_2\rangle = 0$

$\langle P_1, P_2\rangle = 0$

To construct P's need following general properties

1) If $p(x)$ is a poly of deg k then we can write

$$p(x) = \sum_{j=0}^{k} d_j \varphi_j(x) \qquad \text{where } d_j = \langle p, P_j\rangle / \langle P_j, P_j\rangle$$

Example: $x^2 = \frac{1}{3}P_2(x) + \frac{1}{3}P_0(x)$

2) If $p(x)$ has degree $< k$ then $\langle p, P_K\rangle = 0$

$$p = \sum_{i=1}^{k-1} c_i\varphi_i \quad \therefore \langle p, P_K\rangle = \sum c_i\, P_K\varphi_i = c_i\langle P_K, P_i\rangle = 0$$

since $i \neq K$

Aside $[a,b] = [0,1]$

$\varphi_j(x) = x^j$ $j = 0, 1, 2, \ldots$

$\langle f, g \rangle = \int_0^1 f g \, dx$

$H_n = \langle \phi_i, \phi_j \rangle = \int_0^1 x^{i+j} \, dx = \frac{1}{i+j+1} \quad i, j = 0, 1, \dots$

Hilbert Ill-Conditioned Matrix

(iii) $P_K(x)$ has exactly K real & distinct roots in $[a, b]$

Proof:

where $P_K(x) = \alpha_K (x - \xi_{1,K})(x - \xi_{2,K})(x - \xi_{3,K}) \cdots (x - \xi_{K,K})$

$a \le \xi_{1,K} < \xi_{2,K} < \dots < \xi_{K,K} \le b$

suppose $\xi_{1,K}, \xi_{2,K}, \dots, \xi_{r,K}$ are zeros in $[a,b]$ of $P_K(x)$ $r < K$

Consider $P_K(x)$ and the polyn $q = P_K(\xi)(x - \xi_{1,K})(x - \xi_{2,K}) \cdots (x - \xi_{r,K})$ have the same sign in $[a, b]$ where ξ is the largest of $(b, \xi_{r,K})$

degree $q(x) = r < K$ $\therefore$ by (ii) $\langle q, P_K \rangle = 0$

$\therefore$ This is impossible since q & $P_K(x)$ has the same sign in $[a, b]$ & hence $\langle q, P_K \rangle = \int_a^b q \, P_K(x) \, W(x) \, dx > 0$

unless $r = 0$

(iv) Any 3 consecutive orthogonal polynomial satisfy the relation

$$P_{n+1}(x) = A_n (x - B_n) P_n(x) - C_n P_{n-1}(x) \qquad n = 1, 2, \dots \quad P_{-1} \equiv 0$$

where

$$A_n = \frac{\alpha_{n+1}}{\alpha_n} \qquad B_n = A_n \frac{\langle P_n, x P_n \rangle}{\langle P_{n-1}, P_{n-1} \rangle} \; ; \; C_n = \frac{A_n}{A_{n-1}} \frac{\langle P_n, P_n \rangle}{\langle P_{n-1}, P_{n-1} \rangle}$$

Proof:

$$P_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} x P_n(x) = q_n(x) = \text{poly of degree} \le n$$

$$\therefore \quad q_n = \sum_{j=0}^{n} \gamma_j P_j(x) \qquad \text{by (ii)}$$

$$\langle q_n, P_k \rangle = 0 \qquad k \ne n-1, n$$

$$\langle P_{n+1} - \frac{\alpha_{n+1}}{\alpha_n} x P_n, P_k \rangle = \langle P_{n+1}, P_k \rangle - \frac{\alpha_{n+1}}{\alpha_n} \langle x P_n, P_k \rangle$$
$$\langle P_n, x P_k \rangle$$

if $k = n-1$ $\quad xP_k$ poly of degree k $\quad \therefore \langle P_n, xP_k \rangle \ne 0$ poly of $k+1$

when $k = n$ $\quad \langle q_n, P_n \rangle = \gamma_n \langle P_n, P_n \rangle = -\frac{\alpha_{n+1}}{\alpha_n} \langle P_n, x P_n \rangle$

when $k = n-1$

$$\langle q_n, P_{n-1} \rangle = \gamma_{n-1} \langle P_{n-1}, P_{n-1} \rangle = -\frac{\alpha_{n+1}}{\alpha_n} \langle P_n, x P_{n-1} \rangle$$

$$x P_{n-1} = \frac{\alpha_{n-1}}{\alpha_n} P_n + q_{n-1}$$

$$\therefore \quad \gamma_{n-1} \langle P_{n-1}, P_{n-1} \rangle = -\frac{\alpha_{n+1}}{\alpha_n} \frac{\alpha_{n-1}}{\alpha_n} \langle P_n, P_n \rangle$$

Set $A_n = \frac{\alpha_{n+1}}{\alpha_n}$; $\gamma_n = -B_n = A_n \frac{\langle P_n, x P_n \rangle}{\langle P_n, P_n \rangle}$

$$\gamma_{n-1} = -C_n = \frac{A_n}{A_{n-1}} \frac{\langle P_n, P_n \rangle}{\langle P_{n-1}, P_{n-1} \rangle}$$

Examples

1) $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$; $P_0(x) = 1 \quad P_k(+1) = 1$

then $P_0, P_1, P_2, \ldots$ are the Legendre polyn

$$P_{n+1} = \frac{(2n+1) x P_n - n P_{n-1}}{n+1} \qquad P_0 = 1 \quad P_{-1} \equiv 0$$

2) Chebyshev Poly

$$\langle f,g\rangle = \int_{-1}^{1} \frac{fg}{\sqrt{1-x^2}}\,dx \qquad T_0(x) = 1 \qquad T_{-1} \equiv 0$$

Then $T_{n+1} = 2x T_n - T_{n-1}$

$$T_n(\cos\theta) = \cos n\theta$$

Nested Mult. Algorithm for evaluating $p(x)$
where

$$p(x) = d_0 P_0(x) + d_1 P_1(x) + \dots + d_k P_k(x)$$

$(d_0, d_1, \dots, d_k$ given$)$

To find $p(\bar{x})$

$$p(\bar{x}) = d_0 P_0 + d_1 P_1 + \dots + (d_{k-2} + C_{k-2} d_k) P_{k-2} + [d_{k-1} + d_k A_{k-1}(x - B_{k-1})] P_{k-1}$$

set $\bar{d}_k = d_k$

$$\bar{d}_{k-1} = d_{k-1} + \bar{d}_k A_{k-1}(x - B_{k-1}$$

$$p(\bar{x}) = d_0 P_0 + \dots + (d_{k-2} + C_{k-2}\bar{d}_k) P_{k-2} + \bar{d}_{k-1} P_{k-1}$$

Repeat the above

Algorithm Nested Mult.:

Given $A_j, B_j, C_j \quad j = 0, 1, \dots \quad , d_0, d_1, \dots, d_k$

$\alpha_0 = P_0(1)$ and a pt $\bar{x}$ then

set $\bar{d}_k = d_k$

if $k = 0$ stop

set $\bar{d}_{k-1} = d_{k-1} + \bar{d}_k A_{k-1}(\bar{x} - B_{k-1})$

If $k = 1$ stop

for $j = k-2, \ldots, 0$ Do

$$\bar{d}_j = d_j + \bar{d}_{j+1} A_j (\bar{x} - B_j) - \bar{d}_{j+2} C_{j+1}$$

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Example: Back to Least Squares

Find polynomial $p^*(x)$ deg $\leq 3 \ni \int_{-1}^{1} [e^x - p^*(x)]^2 dx$ is min over all polyn of degree ≤ 3.

ie find $p^*(x) = c_0^* + c_1^* x + c_2^* x^2 + c_3^* x^3$

$$\sum_{j=1}^{k} \langle \phi_i, \phi_j \rangle c_j = \langle f, \phi_i \rangle \qquad f = e^x$$

$$\int_{-1}^{1} f g \, dx = \langle f, g \rangle$$

Orthogonal Polyn are Legendre Poly

$\phi_j(x)$ and $p^*(x) = d_0^* P_0 + d_1^* P_1 + d_2^* P_2 + d_3^* P_3$

where $d_j^* = \dfrac{\langle e^x, P_j \rangle}{\langle P_j, P_j \rangle}$

$P_0 = 1 \qquad P_1 = x \qquad P_2 = \dfrac{3x^2 - 1}{2} \qquad P_3 = \dfrac{5x^3}{2} (x^3 - \tfrac{3}{5} x)$

$$\langle P_k, P_k \rangle = \frac{2}{2k+1} = \int_{-1}^{1} P_k(x)^2 dx$$

$$\langle e^x, P_0 \rangle = \int_{-1}^{1} e^x dx = e - \frac{1}{e}$$

$$\langle e^x, P_1 \rangle = \int_{-1}^{1} e^x x \, dx = \frac{2}{e}$$

Generation & Use of discrete orthogonal polyn

$x_1, x_2, \ldots, x_m \in [a, b] \qquad \langle g, h \rangle = \sum^{n} g(x_n) h(x_n) w(x_n)$

Assume $A_n = 1 \quad \alpha_0 = 1$

Step 0

$P_0 = 1$

Compute $\langle P_0, P_0 \rangle = S_0 = \sum_{n=1}^{N} W(x_n)$

If $N \geq 1$ & $W(x) > 0$ $S_0 \neq 0$

using $P_{n+1} = A_n(x - B_n)P_n - C_n P_{n-1}$; $A_n = 1$, $B_n = \frac{\langle xP_n, P_n \rangle}{\langle P_n, P_n \rangle}$

Calculate $P_1(x) = (x - B_0) P_0 = x - B_0 = x - \frac{\langle xP_0, P_0 \rangle}{S_0}$ $\quad C_n = \frac{\langle P_n, P_n \rangle}{\langle P_{n-1}, P_{n-1} \rangle}$

$= x - \frac{1}{S_0} \sum_{n=1}^{\infty} x_n W(x_n)$

Once $P_0, P_1, \ldots, P_j$ calculated

Step j

Compute $S_j = \langle P_j, P_j \rangle = \sum_{n=1}^{N} P_j(x_n)^2 W(x_n)$ if > 0

$B_j = \frac{\langle xP_j, P_j \rangle}{S_j}$ ($\neq 0$ if there are more than j distinct points among $x_1, x_2, \ldots x_n$)

$= \frac{1}{S_j} \sum P_j(x_n)^2 x_n W(x_n)$

$C_N = \frac{S_j}{S_{j-1}}$

Continue.

Dec 18, 1972

Final Exam Mon Jan 22 11:30 - 1:30

Exam Closed Book - work up to Least Squares

Least Squares - orthogonal polyn.

Numerical Integration

Given (x_1, f_1) (x_2, f_2) --- (x_n, f_n)

and a sequence of fns $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ ....

Find $c_1, \ldots, c_k$

$$\ni \quad E(c_1, \ldots, c_k) = \sum_{n=1}^{N} \left[f_n - \sum_{i=1}^{k} c_i \phi_i(x_n)\right]^2 = \min$$

$\frac{\partial E}{\partial c_j} = 0$ we obtain normal equations

$$\sum_{i=1}^{k} \langle \phi_i, \phi_j \rangle c_i = \langle f, \phi_j \rangle$$ ← system of lin eqns

where $\langle \phi_j, \phi_i \rangle = \sum_{n=1}^{N} \phi_j(x_n) \phi_i(x_n)$ $\quad \langle f, \phi_j \rangle = \sum_{n=1}^{N} \phi_j(x_n) f_n$

Construct polynomials with property $\langle \phi_j, \phi_i \rangle = 0$ $i \neq j$

polynomial class satisfies 4 properties expounded especially 4th property of recurrence

More General inner product $\langle f, g \rangle = \int_a^b f g w \, dx$ $\quad w > 0$

$\langle f, g \rangle = \sum_{n=1}^{N} f_n g_n w(x_n)$

$\langle , \rangle$ inner product has properties

① $\langle f, f \rangle \geq 0$ $\quad = 0 \implies$ (cont case $f \equiv 0$)

② $\langle a_1 f + a_2 f_2, g \rangle = a_1 \langle f_1, g \rangle + a_2 \langle f_2, g \rangle$

$\langle f, b_1 g_1 + b_2 g_2 \rangle = b_1 \langle f, g_1 \rangle + b_2 \langle f_2, g_2 \rangle$

Fix a notion of $\langle , \rangle$ Then $P_0(x)$, $P_1(x)$, ... is a seq of orthogonal polynomial if (i) $P_j(x) = \alpha_j x^j$ + polyn of degree $< j$ $\quad \alpha_j \neq 0$

(ii) $\langle P_i, P_j \rangle = 0$ $\quad i \neq j$

Prop of orthog poly

① if $P(x)$ is poly of degree k then $p(x) = d_0 P_0(x) + d_1 P_1(x) + \ldots + d_k P_k(x)$

② if $p(x)$ is poly $< k$ then $\langle p, P_k \rangle = 0$

③ $x \in [a,b]$ $P_k(x)$ has exactly k distinct zeros in $[a,b]$

④ Recurrence Relation

$$P_{i+1}(x) = A_i (x - B_i) P_i(x) - C_i P_{i-1}(x) \qquad i = 0, 1, \ldots$$

$P_{-1}(x) \equiv 0$ $\quad A_i = \dfrac{\alpha_{i+1}}{\alpha_i}$ $\quad i = 0, 1, \ldots$ $\quad s_i = \langle P_i, P_i \rangle$

$B_i = \frac{1}{s_i} \langle x P_i, P_i \rangle$ $\quad C_j = \dfrac{A_j}{A_{j-1}} \dfrac{s_j}{s_{j-1}}$ $\quad j = 1, \ldots$

Usually Assume $P_0(x) \equiv 1$

Possible normalizations:

Examples Find a poly of degree ≤ 3 ∋. $\int_{-1}^{1} [e^x - p(x)] dx = \min$

Inner product

$\langle f, g \rangle = \int_{-1}^{1} f g \, dx$ $\quad P_0(x), P_1(x), P_2(x) \ldots$ socalled Legendre Poly

$p(x) = d_0 P_0 + d_1 P_1 + d_2 P_2 + d_3 P_3$ $\quad d_j = \dfrac{\langle e^x, P_j \rangle}{\langle P_j, P_j \rangle}$

Construction of O.P. when $\langle f, g \rangle = \sum f_n g_n w(x_n)$ $\quad w(x_n) > 0$

Assume $\alpha_i = 1$ and hence $A_i = 1$ $\quad P_{-1} = 0$ $\quad P_0 = 1$

$$P_{i+1} = (x - B_i) P_i - C_i P_{i-1}$$

$S_i = \langle P_i, P_i \rangle$, $B_i = \langle x P_i, P_i \rangle \frac{1}{S_i}$; $C_i = S_i / S_{i-1}$

To construct $P_j(x)$

① Set $P_0(x) = 1$, compute $S_0 = \langle P_0, P_0 \rangle = \sum_{n=1}^{N} w(x_n)$

$B_0 = \dfrac{\langle x P_0, P_0 \rangle}{S_0} = \frac{1}{S_0} \sum_{n=1}^{N} x_n w(x_n)$

② if $P_0, P_1 \ldots P_j$ are obtained

then compute:

$$S_j = \langle P_j, P_j \rangle = \sum_{n=1}^{N} P_j^2(x_n)\, W(x_n)$$

$$B_j = \frac{1}{S_j} \langle x P_j, P_j \rangle = \frac{1}{S_j} \sum_{n=1}^{N} x_n W(x_n)\, P_j^2(x_n)$$

Example:

$x_n = 10 + \frac{(n-1)}{5}$ $\quad n = 1, \dots, 6$

$f_n = f(x_n)$ $\quad f(x) = 10 - 2x + \frac{1}{10} x^2$

find by least squares a polyn of degree ≤ 2 fitting above data

$p(x) = c_0 + c_1 x + c_2 x^2 \ni \sum [f_n - p(x_n)]^2 = \min$

Set $P_0(x) = 1$ $\quad S_0 = \sum_{n=1}^{6} 1 = 6$ $\quad B_0 = \frac{1}{6} \sum_{n=1}^{6} 10 + \frac{(n-1)}{5} = 10.5$

$P_1(x) = (x - 10.5)$

$S_1 = \langle P_1, P_1 \rangle = \sum_{n=1}^{6} \left(\frac{n-1}{5} - .5 \right)^2 = .7$

$B_1 = \frac{1}{.7} \langle x P_1, P_1 \rangle = \frac{1}{.7} \sum_{n=1}^{6} x (x - 10.5)^2 = 10.5$

$P_2(x) = (x - 10.5)^2 - \frac{.7}{6}$

One gets $\quad d_j = \frac{\langle f, P_j \rangle}{\langle P_j, P_j \rangle}$

$$d_0 = \frac{1}{6} \sum_{n=1}^{6} \left[10 - 2\left(10 + \frac{n-1}{5}\right) + \frac{1}{10}\left(10 + \frac{n-1}{5}\right)^2 \right] = 0.0366$$

Fortran Implementation for algorithm above

DATA $\quad X(n) = x_n, \quad F(n) = f_n, \quad W(n) = w(x_n), \quad n = 1, \dots, NPOINT$

NTERMS-1 = degree of polyn

To construct $B(J)$, $C(J)$ $\quad J = 1, \dots, NTERMS-1$

$D(J)$, $J = 1, \dots, NTERMS$

$p(x) = D(1) + D(2) P_1(x) + D(3) P_2(x) + \dots + D(NTERMS)\, P_{NTERMS-1}(x)$

With orthog polyn eval routine the above gives complete answer to the above original problem.

$Error(n) = [f_n - f^*_n]$

```
      SUBROUTINE ORTPOL (NP,X,F,W,ER,PJM1,PJ)
      DIM X(100),F(100),W(100),ER(100),PJ(20),PJM1(20),S(20)
      COMMON /POLY/, NT, B(20),C(20),D(20)
      DO 9 J=1,NT
      B(J)=0.
      D(J)=0.
    9 S(J)=0.
      C(1)=0.
      DO 10 I=1,NP
      B(1)=B(1)+X(I)*W(I)
      D(1)=D(1)+F(I)*W(I)
   10 S(1)=S(1)+W(I)
      D(1)=D(1)/S(1)
      DO 11 I=1,NP
   11 ER(I)=F(I)-D(I)
      IF (NT.EQ.1) RETURN
      B(1)=B(1)/S(1)
      DO 12 I=1,NP
      PJM1(I)=1.
   12 PJ(I)=X(I)-B(1)
      J=1
   20 J=J+1
      DO 21 I=1,NP
      S(J)=S(J)+PJ(I)*PJ(I)*W(I)
      B(J)=B(J)+X(I)*PJ(I)*PJ(I)*W(I)
   21 D(J)=D(J)+ER(I)*PJ(I)*W(I)
```

$\sum f_n P_k(x_n) W(x_n)$

$= \sum [f_n - \underbrace{d_1 P_0 - d_2 P_1 - \dots}_{\text{degree} < k}] P_k W(x_n)$

```
      D(J)=D(J)/S(J)
      DO 22 I=NP
   22 ER(I)=ER(I)-D(J)*PJ(I)
      IF (J.EQ.NT) RETURN
      B(J)=B(J)/S(J)
      C(J)=S(J)/S(J-1)
      DO 27 I=1,NP
      P=PJ(I)
      PJ(I)=(X(I)-B(J))*PJ(I)-C(J)*PJM1(
   27 PJM1(I)=P
      GO TO 20
      END
```

$f_i = d_1 P_0 + d_2 P_1$

Homework. 11-2 4.11-3

Numerical Diff & Integration

an $f(x)$ $\in$ smooth in some interval

$$f'(a) = \frac{f(a+h) - f(a)}{h} + \frac{1}{2} f''(a) h$$

$$f'(a) = \frac{f(a+h) - f(a-h)}{2h} + \frac{1}{6} f'''(\eta) h^2$$

$$f''(a) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

$$I(f) = \int_a^b f(x)\,dx$$

numerically integrate two types of formulas : $\underbrace{\text{Ordinary}}_{\text{local}}$ & $\underbrace{\text{Composite}}_{\text{global}}$

$$\int_{-1}^{1} \frac{x^2 \, dx}{1 + x^2 + x^4 + \sin^5 x + 11} = \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} (\quad)\,dx \qquad -1 = x_1 < x_2 < \dots < x_{i+1} = +1$$

Now apply ordinary numerical integration in $\int_{x_i}^{x_{i+1}} (\quad)\,dx$

Ordinary Num. Integration in $[a,b]$

write $f(x) = \underbrace{P_k(x)}_{k^{th} \text{ degree interp poly based on } x_0, x_1, x_2 \dots x_k} + f[x_0, x_1, \dots, x_k, x]\, \underbrace{\psi_k(x)}_{(x-x_0)(x-x_1)\dots\dots(x-x_k)}$

$$I(f) = \int_a^b f(x)\,dx = I(P_k) + \int_a^b f[x_0, x_1, \dots, x_k, x]\,\psi_k(x)\,dx$$

Example $k=1$ $x_0 = a$ $x_1 = b$ $\quad f(x) = f(a) + \frac{f(b) - f(a)}{b-a}(x-a) + f[a,b,x](x-a)(x-b)$

$$I(f) = f(a)(b-a) + \frac{f(b) - f(a)}{b-a} \frac{(b-a)^2}{2}$$

trapezoidal rule $\quad (x-a)(x-b) < 0$

$$\text{error} = \frac{f''(\eta)}{2} \frac{(b-a)^3}{12}$$

for $h = 2$ $\quad x_0 = a \quad x_1 = \frac{a+b}{2} \quad x_2 = b$

$$P_2(x) = f(a) + f\left[a, \tfrac{a+b}{2}\right](x-a) + f\left[a, \tfrac{a+b}{2}, b\right](x-a)(x-b)$$

$$I(f) = \int_a^b f(x)\,dx = \left[f(a) + 4f\left(\tfrac{a+b}{2}\right) + f(b)\right]\frac{b-a}{6} - \frac{1}{90}(b-a)^5 f^{iv}(\eta)$$

Gaussian Quadrature

$$f(x) = f(c) + f'(c)(x-c) + \dots + f^{(n-1)}(c)\frac{(x-c)^{n-1}}{(n-1)!} + \int_c^x \frac{(x-S_{n-1})^{n-1}}{(n-1)!} f^{(n)}(S_{n-1})\,dS_{n-1}$$

$$f^{(n)}(c)\frac{(x-c)^n}{n!} + \frac{1}{n!}\int_c^x (x-S_n)^n f^{(n+1)}(S_n)\,dS$$

(A) but $\int_c^x \int_c^{S_1} f''(S_2)\,dS_2\,dS_1$

when $f''(S_2) = f''(c) + \int_c^{S_2} f'''(S_3)\,dS_3$ is placed in it gives

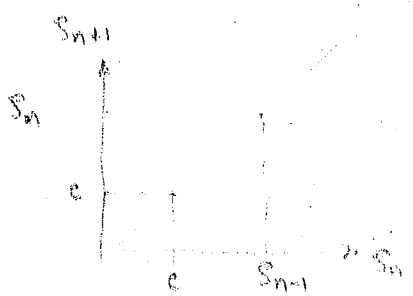
$$\int_c^x \int_c^{S_1} f''(c)\,dS_2\,dS_1 + \int_c^x \int_c^{S_1}\int_c^{S_2} f'''(S_3)\,dS_3\,dS_2\,dS_1$$

$\Rightarrow$ we can by induction show that

$$f(x) = f(c) + \int_c^x f'(c)\,dS_1 + \int_c^x\int_c^{S_1} f''(c)\,dS_2\,dS_1 + \dots \int_c^x\int_c^{S_1}\cdots\int_c^{S_{n-1}} f^{(n)}(c)\,dS_n\,dS_{n-1}\dots dS_1$$

$$+ \int_c^x\int_c^{S_1}\cdots\int_c^{S_n} f^{(n+1)}(S_{n+1})\,dS_{n+1}\dots dS_1$$

$$f(c) + f'(c)[x-c] + f''(c)\frac{(x-c)^2}{2} + \dots f^{(n)}\frac{(x-c)^n}{n!} + \int_c^x\int_c^{S_1}\cdots\int_c^{S_n} f^{(n+1)}(S_{n+1})\,dS_{n+1}$$



$$\int_c^x\int_c^{S_2}\cdots\int_c^{S_{n-1}}\int_c^{S_n} f^{(n+1)}(S_{n+1})\,dS_{n+1}\,dS_n\,dS_{n-1}\dots dS_1$$

$$= \int_c^x\int_c^{S_2}\cdots\int_c^{S_{n-1}}\int_{S_{n+1}}^{S_n} f^{(n+1)}(S_{n+1})$$

$$f(x) = f(c) + \int_c^x f'(S_1)\,dS_1$$

$$= f(c) + \int_c^x f'(c)\,dS_1 + \int_c^x\int_c^{S_1} f''(S_2)\,dS_2\,dS_1$$

$$+ \int_c^x\int_c^{S_1} f''(c)\,dS_2\,dS_1 + \int_c^x\int_c^{S_1}\int_c^{S_2} f'''(S_3)\,dS_3\,dS_2\,dS_1$$

assume $''' = n+1$

$'' = n$

$' = n-1$

by Taylors expansion $f(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2!} + f'''(c)\frac{(x-c)^3}{3!} + \ldots$

$$+ \frac{f^{(n)}(c)\,(x-c)^n}{n!} + \frac{1}{n!}\int_c^x f^{(n+1)}(S)\,(x-S)^n\,dS \qquad (1)$$

by the fundamental theorem of calculus

$$f(x) = f(c) + \int_c^x f'(S_1)\,dS_1 \qquad (a)$$

& $$f'(S_1) = f'(c) + \int_c^{S_2} f''(S_2)\,dS_2 \qquad (b)$$

substitution of (b) into (a) gives

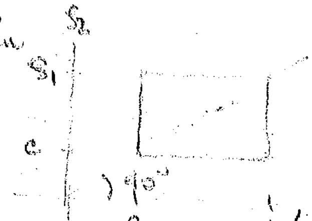
$$f(x) = f(c) + \int_c^x \left[f'(c) + \int_c^{S_1} f''(S_2)\,dS_2\right] dS_1 \qquad (c)$$

Since $f'(c)$ is a constant then (c) can be reduced to

$$f(x) = f(c) + \int_c^x f'(c)\,[dS_1] + \int_c^x \int_c^{S_1} f''(S_2)\,dS_2\,dS_1$$

$$= f(c) + f'(c)(x-c) + \int_c^x \int_c^{S_1} f''(S_2)\,dS_2\,dS_1$$

Consider the domain (A)



$\int_c^x \int_c^{S_1} \int_c^{S_2} f'''(S_3)\,dS_3\,dS_2\,dS_1$

$$\int_c^x\int_c^{S_1} f''(S_2)\,dS_2\,dS_1 = \int_c^x \int_{S_2}^x f''(S_2)\,dS_1\,dS_2 = \int_c^x (x-S_2)\,f''(S_2)\,dS_2$$

by applying integration by parts

$$\int_c^x (x-S_2)\,f''(S_2)\,dS_2 = -\frac{(x-S_2)^2}{2} f''(S_2)\Big|_c^x + \int_c^x \frac{(x-S_2)^2}{2} f'''(S_2)\,dS_2$$

$(x-S_2)\,dS_2 = dv \quad v = (x-S_2)^2/2$

$f'' = u \quad du = f'''\,dS_2$

$$+ \frac{(x-c)^2}{2} f''(c) + \int_c^x \frac{(x-S_2)^2}{2} f'''(S_2)\,dS_2$$

$$f(x) = f(c) + f'(c)(x-c) + \frac{(x-c)^2}{2} f''(c) + \int_c^x \frac{(x-S_2)^2}{2} f'''(S_2)\,dS_2$$

& $$f''(S_2) = f''(c) + \int_c^{S_2} f'''(S_3)\,dS_3 \quad \text{. by induction}$$

$$f^{(n)}(S_{n-1}) = f^{(n)}(c) + \int_c^{S_n} f^{(n+1)}(S_{n+1})\,dS_{n+1} \Rightarrow$$

choose

$x = + (.d_1 d_2 \dots d_n)_\beta \beta^e$

$fl(x) = + (.\delta_1 \delta_2 \dots \delta_n)_\beta \beta^e$

where $\delta_i = d_i$ $\quad d_{n+1} \neq \delta_{n+1}$ since $\delta_{n+1} = 0$

$x = -(.d_1 d_2 \dots d_n)_\beta \beta^e$

$fl(x) =$

1.32

$-x = \pm (.d_1 d_2 \dots d_n)_\beta \beta^e$ $\quad$ if only i digits are stored $\quad i < n$

$fl(-x) = \pm (.d_1 d_2 \dots d_i)_\beta \beta^e$ $\quad$ let $x_1 = -x$

$x_1 = \mp (.d_1 d_2 \dots d_n)_\beta \beta^e$ $\quad$ if only i digits are stored $\quad i < n$

$fl(x_1) = \mp (.d_1 d_2 \dots d_i)_\beta \beta^e$

$-fl(x_1) = \pm (.d_1 d_2 \dots d_i)_\beta \beta^e = fl(-x)$

let $\beta^r x = \pm (.d_1 d_2 \dots d_n)_\beta \beta^{e+r}$ $\quad$ if only i digits are stored $i < n$

$fl(\beta^r x) = \pm (.d_1 d_2 \dots d_i)_\beta \beta^{e+r}$

$fl(x) = \pm (.d_1 d_2 \dots d_i)_\beta \beta^e$ $\quad i < n$

$\therefore \beta^r fl(x) = \pm (.d_1 d_2 \dots d_i) \beta^e \cdot \beta^r = fl(\beta^r x)$

1.33

$fl(x)$ is given by chopping $\quad fl(x) = x(1+\delta)$

assume $x^* = \pm (.d_1 d_2 \dots d_n)_\beta \beta^e$ $\quad$ assume i digits stored

$|x - x^*|$ $\quad$ where x^* is number by chopping, x is "

$|x - x^*| = \mp (\underbrace{.00 \dots 0}_{n} d_{n+1} d_{n+2} \dots)_\beta \beta^e = \mp (.d_{n+1} d_{n+2} \dots)_\beta \beta^{e-n}$

$= \dfrac{\mp (.d_{n+1} d_{n+2} \dots)_\beta}{\pm (.d_1 d_2 \dots)_\beta} \pm (.d_1 d_2 \dots)_\beta \beta^e \beta^{-n}$

$= \dfrac{\mp (.d_{n+1} d_{n+2} \dots)_\beta}{\pm (.d_1 d_2 \dots)_\beta} \pm |x| \beta^{-n} \leq \dfrac{\pm (.\beta{-}1\ \beta{-}1 \dots)}{\pm (.1000 \dots)} |x| \beta^{-n}$

$\leq \beta |x| \beta^{-n} = |x| \beta^{1-n}$ $\qquad \dfrac{x - x^*}{x} \leq -\beta^{1-n}$

1.4-1 $x^2 + .4002 \cdot 10^0 x + .8 \cdot 10^{-4} = 0$

$$x_1 = \frac{-.4002 + \sqrt{(.4002)^2 - 4(.8\times10^{-4})}}{2}$$

$$\frac{-.4002 + \sqrt{.1598}}{2} =$$

$$x_1 = \frac{-.4002 + .3999}{2} = \frac{-.0003}{2} = -.0002$$

$$\frac{-.00016}{.4002 + \sqrt{.1598}} = \frac{-.0002}{.8001}$$

$$= -.0003$$

.4002 .4002 .16 .16008000 .16016004 .1602 .4001 $\sqrt{.1602}$ 16 02 .4001 .4001 .4002 4001 16004000 .16008001 .16016004 −.00032 .1598|4004 .39 .39 0351 1170 .3999 .3999 .00035991 .0035991.0 .0359910.0 .11997000 .11997001

$.4002 + .3999 = .8001$

1.43

$$f(x) = \frac{\cos x\,(x) - \cos x \sin x}{\sin(x)\quad \sin x} = x\cot x - \cos x$$

$$f(x) = \frac{x+\sqrt{x^2-\alpha}}{x+\sqrt{x^2-\alpha}}\left[x - \sqrt{x^2-\alpha}\right] = \frac{x^2 - x^2 + \alpha}{x+\sqrt{x^2-\alpha}} = \frac{\alpha}{x+\sqrt{x^2-\alpha}}$$

1-76 $$\cancel{\frac{h^{n+1}}{(n+1)!} f^{(n+1)}(c+\Theta h) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(c) + \epsilon}$$

$$\cancel{f^{(n+1)}(c+\Theta h) = f^{(n+1)}(c) + \Theta h f^{(n+2)}(c)}$$

$$R_{N+1}(x) = \frac{1}{n!}\int_c^x (x-s)^n f^{(n+1)}(s)\,ds$$

$$= \frac{f^{(n+1)}(c)\,(x-c)^{n+1}}{(n+1)!} + \frac{1}{(n+1)!}\int_c^x (x-s)^{n+1} f^{(n+2)}(s)\,ds$$

$$= \frac{f^{(n+1)}(c)\,h^{n+1}}{(n+1)!} + \frac{(x-s)^{n+1}}{(n+1)!}\int_c^x f^{(n+2)}(s)\,ds \qquad s\in[x,c]$$

$$f^{(n+1)}(x) - f^{(n+1)}(c)$$

$$\frac{(x-s)^{n+1}}{(n+1)!} f^{(n+1)}(x) - f^{(n+1)}(c) \le \frac{h^{n+1}}{(n+1)!}\left[f^{(n+1)}(x) - f^{(n+1)}(c)\right]$$

$$\lim_{h\to 0} h^{n+1}\left[\frac{f^{(n+1)}(x) - f^{(n+1)}(c)}{(n+1)!}\right] \to 0 \implies o(h^{n+1})$$

$x^2 - 2x - 2$

$x = 1 \quad f = -3$
$x = 2 \quad f = -2$
$x = 3 \quad f = 1$
$x = 0 \quad f = -2$
$x = -1 \quad f = 1$

$x_R \in [-1, 0]$
$x_R \in [2, 3]$

$2 \pm 2\sqrt{3} \,/\, 2 = 1 \pm \sqrt{3}$

$1 + 1.732$
$1 - 1.732$
$= -.732$

$x = -.5 \qquad f = +.25 + 1 - 2 = -.75$

$x = -.75 \qquad f = 9/16 + 3/2 - 2 = 1/16$

$x = -.625 \qquad f = \frac{25}{64} + \frac{80}{64} - \frac{128}{64} = -\frac{23}{64}$

$x = -.6875 \qquad < 0$

$x = -.71875 \qquad < 0$

$x = -.734375 \qquad > 0$

$x = -.7265625$

$x_n = \frac{F x_{n-1} - f(x_{n-1}) b}{F - f(x_{n-1})}$

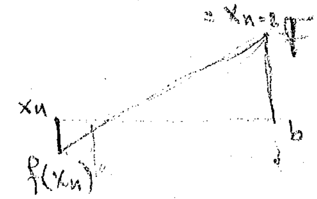
$.75$
$.6875$
1.4375

$x_n - b = F x_{n-1} - bF$

$\frac{x_n - b}{F} = x_{n-1} - b$

$-.75$
$.71875$
1.46875

$\frac{x_n - b}{F^2} = \frac{x_{n-1} - b}{F}$
$= x_{n-2} - b$



$x = \frac{1 \cdot 0 - (-2)(-1)}{1 + 2} = \frac{-2}{3} = -.67 \qquad f = -2/9$

$x = \frac{1(-2/3) - (-2/9)(-1)}{1 + 2/9} = \frac{-8/9}{11/9} = -8/11 \approx -.798 \qquad > 0$

$x_{n+1} = \frac{F x_n - f(x_n) b}{F - f(x_n)}$

$F x_{n+1} - f(x_n) x_{n+1} = F x_n - f(x_n) b$

$f(x_n) = \frac{F(\epsilon)}{F^{n+1} M}$

$F(x_{n+1} - x_n) = f(x_n)[x_{n+1} - b]$
$F^{n+1} M$

$|x_1 - x_0| = \frac{M}{4} \qquad 2^{-i} < \frac{4\epsilon}{M}$

$|x_2 - x_1| = M/8$

$\vdots$

$|x_{i+1} - x_i| = \frac{M}{2^{i+2}} < \epsilon \qquad -i \log 2 < \log(\epsilon/M) + \log 2$

$i > -2 - \frac{\log(\epsilon/M)}{\log 2}$

$\frac{.001}{1} = \log .001$
-3

$f(b_n) b_n - f(b_n) b_n$

$\xi - x_{n+1} = \xi - \frac{f(b_n) a_n - f(a_n) b_n}{f(b_n) - f(a_n)} = \frac{f(b_n)[\xi - a_n] + f(a_n)[b_n - \xi]}{f(b_n) - f(a_n)}$

$x_{n+1} = \frac{f(b_n)[a_n - b_n] + b_n[f(a_n) - f(b_n)]}{f(b_n) - f(a_n)}$

$x_{n+1} - b_n = \frac{f(b_n)[a_n - b_n]}{f(b_n) - f(a_n)}$

$e_{n+1} = \frac{f(b_n)}{\frac{f(b_n) - f(a_n)}{a_n - b_n}}$ constant $\rightarrow \frac{f(\xi)}{f'(\xi)} \qquad \xi \in (a_n, b_n)$

$i > \log\left[\frac{\epsilon}{M(F-1)}\right] \Big/ \log F \qquad$ where $F = f(b)$
$M = x_0 - b$

2.3-5 1. $x = x^3 - 1$ funct at $x=1$ $f=1$ $x=0$ $f=-1$ $f=-1$ $x=-1$

3 $g'(x) = 3x-1$ for $|x| \le 1/3$ $g'(x) \le 1$

at $-1/3$ $-\frac{1}{27} + \frac{1}{3} - 1 < 1$ $-\frac{27}{125} + \frac{3}{5} - 1$

$-1/8 + 1/2 - 1$

$$x = \sqrt[3]{x+1} = g(x)$$

$$\frac{1}{3\sqrt[3]{(x+1)^2}} = g'(x) < 1 \qquad 1 < x < 2$$

$x = \tan x = g(x)$ $x=0$

$\sec^2 x = g'(x)$

2 roots bet 0 & $\pi/2$
0 & $-\pi/2$

$g(x) = \tan^{-1} x$

$g'(x) \le 1$ between $\pi/2$ & 0
$-\pi/2$ & 0

~~$g(x) = \tan^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$~~

$e^{-x} = \cos x$ ~~$x = \cos^{-1}(e^{-x})$~~

$g(x) = x = -\ln(\cos x)$ .707

$x \ne 0$ in nbhd

$|x| < \pi/4$ $f=0$

$$\frac{-\sin x}{-\cos x} = |\tan x| \le 1 \qquad \pi/4 > |x|$$

at $x = .707$ $g(x) = -\ln(.707) = -[\ln(7.07) - \ln 10] = -.303$

$\ln 1$

$\ln a = \frac{\log_{10} a}{\log_{10} e} = \frac{\log_{10} e}{.434}$

$e^x + e^x e^{-x} - e^x e^{-x} + e^{-x}$

$e^x(1+e^{-x}) + e^{-x}(1+e^{x})$

$\ln\left[\frac{e^x + e^{-x}}{2}\right]$

$f\left[\frac{(e^x + e^{-x})}{2}\right] = x$

2.5-3 $x = \pi/4$

$e = \frac{27}{5}$

~~$g(x) = x = \ln(4\cos x)$~~

~~$g'(x) = \frac{-4\sin x}{4\cos x}$~~ $= -\tan x$

~~$|\cos(\eta)| \le 1$~~

$$g(x) = x = \cos^{-1}\left(\tfrac{1}{4}e^x\right)$$

$\frac{\cancel{20}}{4} = \frac{1}{5}$

$g(x) = x = \cosh^{-1}(2\cos x)$ $0 < x < \pi/2$

bounded $|0, +2|$

T63.1050

C. LEVY 097-42-1895

Prove: $\int_c^x \int_c^{\xi_1} \int_c^{\xi_2} \cdots \int_c^{\xi_n} f^{(n+1)}(\xi_{n+1})\, d\xi_{n+1}\, d\xi_n \cdots d\xi_1 = \frac{1}{n!} \int_c^x (x-\xi_2)^n f^{(n+1)}(\xi_2)\, d\xi$

by fundamental theorem of calc:

$$f(x) = f(c) + \int_c^x f'(\xi_1)\, d\xi_1 \qquad \text{and}$$

$$f(\xi_1) = f(c) + \int_c^{\xi_1} f''(\xi_2)\, d\xi_2$$

$$\vdots$$

$$f^{(n)}(\xi_n) = f^{(n)}(c) + \int_c^{\xi_n} f^{(n+1)}(\xi_{n+1})\, d\xi_{n+1}$$

this then leads to

$$f(x) = f(c) + \int_c^x f'(c)\, d\xi_1 + \int_c^x \int_c^{\xi_1} f''(c)\, d\xi_2\, d\xi_1 + \cdots + \int_c^x \int_c^{\xi_1} \cdots \int_c^{\xi_{n-1}} f^{(n)}(c)\, d\xi_n\, d\xi_{n-1} \cdots$$

$$+ \int_c^x \int_c^{\xi_1} \cdots \int_c^{\xi_n} f^{(n+1)}(\xi_{n+1})\, d\xi_{n+1}\, d\xi_n \cdots d\xi_1$$

since $f^{(i)}(c)$ = constant we can perform the first n integrals to produce

$$f(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2} + \cdots f^{(n)}(c)\frac{(x-c)^n}{n!} +$$

$$\int_c^x \int_c^{\xi_1} \cdots \int_c^{\xi_n} f^{(n+1)}(\xi_{n+1})\, d\xi_{n+1}\, d\xi_n \cdots d\xi_1 \qquad (a)$$

again by fundamental theorem of calculus

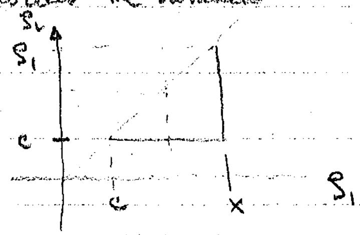
$$f(x) = f(c) + \int_c^x f'(\xi_1)\, d\xi_1$$

Evaluate the integral by parts

$$f(x) = f(c) + \int_c^x f'(\xi_1)\, d\xi_1 = f(c) + f'(c)\int_c^x d\xi_1 + \int_c^x \int_c^{\xi_1} f''(\xi_2)\, d\xi_2 \ldots$$

$$= f(c) + f'(c)(x-c) + \int_c^x \int_c^{\xi_1} f''(\xi_2)\, d\xi_2\, d\xi_1$$

Consider the domain



$$\int_c^x \int_c^{s_1} f''(s_2)\,ds_2\,ds_1 = \int_c^x \int_{s_2}^x f''(s_2)\,ds_1\,ds_2$$

$$\therefore \int_c^x \int_c^{s_1} f''(s_2)\,ds_2\,ds_1 = \int_c^x (x-s_2)\, f''(s_2)\,ds_2$$

by limit interchange; thus by integration by parts

$$f(x) = f(c) + f'(c)(x-c) - \frac{(x-s_2)^2}{2} f''(s_2)\Big|_c^x + \int_c^x \frac{(x-s_2)^2}{2} f'''(s_2)\,ds_2$$

carrying out n-2 such partial integrations leads to

$$f(x) = f(c) + \sum_{i=1}^{n} f^{(i)}(c)\frac{(x-c)^i}{i!} + \frac{1}{n!}\int_c^x (x-s_2)^n f^{(n+1)}(s_2)\,ds_2 \quad (b)$$

if (a) & (b) are proper representations of the same function f(x) then

$$\int_c^x \int_c^{s_1} \cdots \int_c^{s_n} f^{(n+1)}(s_{n+1})\,ds_{n+1}\,ds_n \cdots ds_1 = \frac{1}{n!}\int_c^x f^{(n+1)}(s_2)\,(x-s_2)^n\,ds_2$$

C. Levy 763.1050

1. Theorem: Let $G \subset R^N$ & let $F: G \to G$ for any $x_1, x_2 \in G$ and $\| F(x_1) - F(x_2) \| \le K \| x_1 - x_2 \|$ $K<1$

Then for any $x_0 \in G$ the sequence of vectors x_n where $x_{n+1} = F(x_n)$ converges to a unique vector ξ, i.e. $\| x_n - \xi \| \to 0$ as $n \to \infty$ & $\xi = F(\xi)$

Assume a sequence of vectors $x_1, x_2, \dots$ exist for $m \geq n$

$$\| x_m - x_n \| = \| (x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \dots + (x_{m+1} - x_n) \|$$
$$\le \| x_m - x_{m-1} \| + \| x_{m-1} - x_{m-2} \| + \dots + \| x_{n+1} - x_n \|$$

$$\| x_m - x_{m-1} \| = \| F(x_{m-1}) - F(x_{m-2}) \| \le K \| x_{m-1} - x_{m-2} \|$$
$$\le K \| F(x_{m-2}) - F(x_{m-3}) \| \le K^2 \| x_{m-3} - x_{m-4} \|$$
$$\dots \le K^{m-1} \| x_1 - x_0 \|$$

in the same manner

$$\| x_m - x_n \| \le K^n (1 + K + K^2 + \dots + K^{m-1-n}) \| x_1 - x_0 \| \le \frac{K^n}{1-K} \| x_1 - x_0 \|$$

if $K<1$ as $n \to 0$ $K^n \to 0$ $\therefore \| x_m - x_n \| \to 0$ $\xi = F(\xi)$

note $\| a + b \| = \max_i |a_i + b_i| \le \max_i (|a_i| + |b_i|) \le \max_i |a_i| + \max_i |b_i| = \|a\| + \|b\|$

2. Theorem: If rank$(A) = r$ then $\exists$ exactly r l.i. row vectors of A

$\Rightarrow$ rank(A) = column rank (A) = row rank(A)

Suppose that the first r columns are linear indep. then

$$\sum_{i=1}^{r} c_i a_{ij} = 0 \iff c_1 = c_2 = \dots = c_r = 0, \quad j = 1, 2, 3, \dots, n$$

Let us take n constants d_j .$\ni$. we form the sum

$$\sum_{j=1}^{n} \sum_{i=1}^{r} d_j c_i a_{ij} = 0$$

Let us switch the sum signs $\therefore \quad \sum_{i=1}^{k} c_i \sum_{j=1}^{n} d_j a_{ij} = 0$

but $\sum_{j=1}^{n} d_j a_{ij} \Rightarrow \sum_{j=1}^{r} d_j a_{ij} + \sum_{j=r+n}^{n} d_j a_{ij}$

since r a's are l.i. then we can express $n-r$ vectors in terms
the r remaining l.i. vectors
because we can rewrite this sum as a $A\underset{\sim}{d}$. where $A = (a)_{ij}$ & $\underset{\sim}{d} = d_j$
the solution to $A\underset{\sim}{d} = 0$ will have an ident matrix in the first r columns
since they are l.i. and the $n-r$ remaining a_{ij} can be expressed in terms
on the other r vectors. $\therefore$ if $\sum_{j=1}^{k} d_j a_{ij} = 0$ then $k = r$.

C. LEVY
T63.1050

1.32 Let $-x = \pm(.d_1d_2\ldots d_n)_\beta\,\beta^e$; if only i digits are stored $i<n$

$$fl(-x) = \pm(.d_1d_2\ldots d_i)_\beta\,\beta^e. \qquad \text{Let } x_1 = -x$$

$x_1 = \mp(.d_1d_2\ldots d_n)_\beta\,\beta^e$; if only i digits are stored $i<n$

$$fl(x_1) = \mp(.d_1d_2\ldots d_i)_\beta\,\beta^e$$

$$-fl(x_1) = \pm(.d_1d_2\ldots d_i)_\beta\,\beta^e = fl(-x)$$

Let $\beta^r x = \pm(.d_1d_2\ldots d_n)_\beta\,\beta^{e+r}$ if only i digits are stored $i<n$

$$fl(\beta^r x) = \pm(.d_1d_2\ldots d_i)_\beta\,\beta^{e+r}$$

$$fl(x) = \pm(.d_1d_2\ldots d_i)_\beta\,\beta^e \qquad i<n$$

$$\beta^r fl(x) = \pm(.d_1d_2\ldots d_i)_\beta\,\beta^{e+r} = fl(\beta^r x)$$

1.33

Let $fl(x)$ be given by chopping & $\delta = \delta(x) \ni fl(x) = x(1+\delta)$

Assuming $x^* = \pm(.d_1d_2\ldots d_n)_\beta\,\beta^e$ and i digits are stored

$$x - x^* = \mp(.\underbrace{000\ldots0}_{i}\,d_{i+1}d_{i+2}\ldots)_\beta\,\beta^e = \mp(.d_{i+1}d_{i+2}\ldots)_\beta\,\beta^{e-i}$$

$$= \mp\frac{(.d_{i+1}d_{i+2}\ldots)_\beta(.d_1d_2\ldots)_\beta}{(.d_1d_2\ldots)_\beta}\,\beta^e\beta^{-i}$$

$$= \mp\frac{(.d_{i+1}d_{i+2}\ldots)_\beta}{\pm(.d_1d_2\ldots)_\beta}\,x^*\,\beta^{-i}$$

$$x - x^* = fl(x^*) - x^* = \delta x^* \quad \therefore \quad \delta = \mp\frac{(.d_{i+1}d_{i+2}\ldots)_\beta}{\pm(.d_1d_2\ldots)_\beta}\,\beta^{-i}$$

Let |denominator| be the lowest number permitted in the base >0

Let |numerator| be the largest " " in the base $<\beta$

$$\delta \geq \mp\frac{(.(\beta-1)(\beta-1)\ldots)_\beta}{\pm(.100\ldots)_\beta}\,\beta^{-i} \quad \text{note that this number is} \leq 0$$

$$= -\beta^{1-i} \qquad \therefore \quad -\beta^{1-i} \leq \delta \leq 0$$

1.41 $x^2 + .4002\,x + .8 \cdot 10^{-4} = 0$

$$x_1 = \frac{-.4002 + \sqrt{.4002^2 - 4(.8\times10^{-4})}}{2} = \frac{-.4002 + \sqrt{.1598}}{2} = -\frac{.0003}{2} = -.0002$$

$$x_1 = \frac{-1.6\times10^{-4}}{.4002 + \sqrt{.1598}} = \frac{-.0002}{.8001} = -.0003$$

1.43 $$f(x) = \frac{x - \sin x}{\tan x} = \frac{x\cos x - \cos x \sin x}{\cos x} = \frac{x\cos x - .5\sin 2x}{\cos x}$$

$\cos x$ acts as a magnifier on x whereas $\sin 2x \to 0$ much faster & denominator finite at $x \to 0$

$$f(x) = x - \sqrt{x^2 - \alpha} = \frac{x^2 - x^2 + \alpha}{x + \sqrt{x^2 - \alpha}} = \frac{\alpha}{x + \sqrt{x^2 - \alpha}}$$

1.76 $$R_{N+1}(x) = \frac{1}{n!}\int_c^x (x-s)^n f^{(n+1)}(s)\, ds$$

$$= \frac{f^{(n+1)}(c)\,(x-c)^{n+1}}{(n+1)!} + \frac{1}{(n+1)!}\int_c^x (x-s)^{n+1} f^{(n+2)}(s)\, ds$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!} h^{n+1} + \frac{(x-\xi)^{n+1}}{(n+1)!}\int_c^x f^{(n+2)}(s)\, ds \qquad \xi \in [x, c]$$

$$\frac{(x-\xi)^{n+1}}{(n+1)!}\left[f^{(n+1)}(x) - f^{(n+1)}(c)\right] \le \frac{h^{n+1}}{(n+1)!}\left[f^{(n+1)}(x) - f^{(n+1)}(c)\right]$$

$$\lim_{h\to 0} h^{n+1}\left[\frac{f^{(n+1)}(x) - f^{(n+1)}(c)}{(n+1)!}\right] \to 0 \Rightarrow o(h^{n+1})$$

2.11 root $\in [2, 3]$

x^* has 2 significant figures in about 10 iterations for bisection, 5 iterations for regula falsi

for bisection $|x_{i+1} - x_i| = \frac{M}{2^{i+2}} < \epsilon \to \quad i > -2 - \frac{\log(\epsilon/M)}{\log 2}$

$\epsilon = 10^{-7} \quad M = 1$

Assume $b_n \not\!/ n$ remains const $\therefore$

$$x_{n+1}-x_n = [Fx_n - f(x_n)b]/(F-f(x_n)) - x_n$$

$$= \frac{Fx_n - f(x_n)b - Fx_n + x_n f(x_n)}{F-f(x_n)} = \frac{f(x_n)[x_n-b]}{F-f(x_n)}$$

$$\left|\frac{f(x_n)[x_n-b]}{F-f(x_n)}\right| < \epsilon \qquad [F-f(x_n)]\epsilon > f(x_n)[x_n-b]$$

$$x_n - b = (Fx_{n-1} - f(x_{n-1})b)/(F-f(x_{n-1})) - b = F[x_{n-1}-b]$$

$$\Rightarrow \quad \frac{x_n-b}{F^n} = x_0 - b = M \quad \text{where } x_0 \text{ is } a$$

$$[F-f(x_n)]\epsilon > f(x_n)F^n[x_0-b]$$

$$F\epsilon - f(x_n)\epsilon > F^n f(x_n) M \qquad \text{but } f(x_n) = \frac{F\epsilon}{x_{n+1}-b} = \frac{F\epsilon}{F^{n+1}M}$$

$$F\epsilon - \frac{\epsilon^2}{F^n M} > F^n \frac{F\epsilon}{F^{n+1}M} M > \epsilon$$

$$F - \epsilon/_{F^n M} > 1 \qquad F^{n+1}M - \epsilon > F^n M$$

$$F^n(M-FM) < -\epsilon \qquad F^n > \frac{\epsilon}{FM-M} \qquad n > \log\left[\epsilon/_{M(F-1)}\right]/\log F$$

2.14 $|x_0 - a| = \frac{(a-b)}{2}$ $\qquad |x_1-x_0| = \frac{x_0-a}{2} \quad \dots \quad |x_{i+1}-x_i| = \frac{|x_i - x_{i-1}|}{2}$

$$|x_2-x_1| = \frac{x_1-x_0}{2}$$

$$|x_{i+1}-x_i| = \frac{|x_{i-1}-x_{i-2}|}{4} = \dots = \frac{|x_{i-k}-x_{i-(k+1)}|}{2^{k+1}} \quad \text{if } k = i-1$$

$$\frac{|x_1-x_0|}{2^i} = |x_{i+1}-x_i| = \frac{x_0-a}{2^{i+1}} = \frac{(a-b)}{2^{i+2}} = \frac{M}{2^{i+2}}$$

2.23 $\quad e^x = 1+x+\frac{x^2}{2}+\frac{1}{2!}\int_0^x (x-s)^2 e^s ds = 1+x+\frac{x^2}{2}+\frac{e^\xi}{3!}(x)^3 \qquad 0 \le \xi \le x$

$$f(x) = e^x - 1 - x - x^2 = \frac{e^\xi}{3!}x^3 \qquad f(x) \to 0 \text{ when } x \to 0$$

2.3-2

for (a) root $\in [1,2]$ take $x = \sqrt[3]{(x+1)}$

since $g'(x) = \frac{1}{3}(x+1)^{-2/3} < 1$ in I

for (b) root $\in [\frac{\pi}{4}, \frac{\pi}{2}]$ take $x = \tan^{-1}(x)$

since $g'(x) = \frac{1}{1+x^2} < 1$ in I

for c root $\in [\frac{\pi}{4}, \frac{3\pi}{4}]$ tan $x = \text{arc}\cos(e^{-x})$

since $g'(x) = \frac{1}{\sqrt{1+e^{-2x}}} < 1$ in I

2.5-3 let $x=0$ $f(x) = 4\cos x - e^x$ $f'(x) = -4\sin x - e^x$

root lies between 0 & 1 $f'(x)$ varies between -1 & -6 in the interval

$f''(x) = -4\cos x - e^x$; f, f', f'' are continuous fns in the interval

$\exists$ $0 < \xi < 1$.∋. $f(\xi) = 0$ and $-6 < f'(\xi) < -1$

let $x = 1$ $f(x) = 2\cos x - \cosh x$ $f'(x) = -2\sin x - \sinh x$

root lies between ½ & 1 $f'(x)$ varies between -1.4 & -2.7

$f''(x) = -2\cos x - \cosh x$; f, f', f'' are continuous fns in this interval. $\exists$ $\frac{1}{2} < \xi < 1$.∋. $f(\xi) = 0$ & $-2.7 < f'(\xi) < -1.4$

let $x = \pi/4$ $f(x) = e^{-x^2} - \cos x$ $f'(x) = -2xe^{-x^2} + \sin x$

root lies between $\pi/4$ & $\frac{3\pi}{4}$ $f'(x)$ varies between -9 & $-.5$

$f''(x) = +4x^2e^{-x^2} - 2e^{-x^2} + \cos x$; f, f', f'' are cont. fns in this interval, $\exists$ $\frac{\pi}{4} < \xi < \frac{3\pi}{4}$.∋. $f(\xi) = 0$ & $-9 < f'(\xi) < -.5$

```
              PROGRAM MULLER(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
000003        EXTERNAL FN
000003        COMPLEX FM,XM,FN
000003        COMMON M
000003        DIMENSION A(5),FM(3),XM(3)
000003        READ(5,1) M,NDEG
000013    1   FORMAT(2I5)
         COMMENT  N IS THE ACTUAL DEGREE OF POLYNOMIAL, I.E. X**3 IS A THIRD DEGREE
         COMMENT  THEREFORE N=3
         COMMENT  M IS THE NUMBER OF ITERATIONS ALLOWED BEFORE EXITING ON NONCONVER
000013        N=NDEG+1
         COMMENT  N IS THE DEGREE OF THE POLYNOMIAL + 1 TO ACCOMODATE THE ARRAY OF
         COMMENT  COMPUTER SINCE A0 IS A(1)....
000015        READ(5,2) (A(I),I=1,N)
000030    2   FORMAT(8F10.0)
000030        DO 10 I=1,3
000032        FM(I)=CMPLX(0.000,0.000)
000040   10   XM(I)=CMPLX(0.000,0.000)
000051        DO 400 I=1,21
000052        AK=(1-I)*0.1
         C        GENERATION OF INITIAL DATA
         C        XM(I) I=1,3  ARE X(I-2),X(I-1),X(I)
         C        FM(I) I=1,3  ARE F(I-2),F(I-1),F(I)
000055        XM(1)=CMPLX(0.5,-1.7)
000060        XM(2)=CMPLX(0.5,-1.8)
000063        XM(3)=CMPLX(0.5,-2.1)
000065        FM(1)=FN(XM(1),AK,A,N)
000071        FM(2)=FN(XM(2),AK,A,N)
000075        FM(3)=FN(XM(3),AK,A,N)
000101        IF(I.EQ.1) WRITE(6,15)
000107   15   FORMAT(1H1,14X,≠REAL≠,12X,≠IMAG≠,18X,≠REAL≠,12X,≠IMAG≠,//)
000107        CALL MULL(XM,FM,FN,A,AK,N)
000113        IF(AK.NE.1.00000) WRITE(6,20)
000121   20   FORMAT(1X,//)
000121  400   CONTINUE
000123        CALL EXIT
000124        END
```

RUN VERSION 2.3 - CDC LEVEL 26 - NYU LEVEL 11

MULLER

PROGRAM LENGTH INCLUDING I/O BUFFERS
004303

FUNCTION ASSIGNMENTS

STATEMENT ASSIGNMENTS

1	-	000137	2	-	000141	15	-	000155	20	-	000165
400	-	000121									

BLOCK NAMES AND LENGTHS
  - 000001

VARIABLE ASSIGNMENTS

A	-	000226	AK	-	000236	FM	-	000212	I	-	000235
M	-	000000C01	N	-	000234	NDEG	-	000233	XM	-	000220

START OF CONSTANTS
000126

START OF TEMPORARIES
000167

START OF INDIRECTS
000207

UNUSED COMPILER SPACE
013000

```
              SUBROUTINE MULL(X,FX,G,A,AK,NDEG)
000011        COMPLEX X,FX,G,EPSI,HI,ALI,FI,SQ,ANU,ADE,XIP1,FXIP1
000011        COMMON M
000011        DIMENSION X(3),FX(3),A(NDEG)
000011        EPSI=CMPLX(1,0E-04,1,0E-04)
000013        J=1
000014        IFLAG≠0
000015     5  I=1
000016        HI=X(3)-X(2)
000023        ALI=HI/(X(2)-X(1))
000040    10  FI=(1.+2.*ALI)*(FX(3)-FX(2))-ALI*ALI*(FX(2)-FX(1))
000077        SQ=FI*FI-4.*FX(3)*(ALI+ALI*ALI)*(FX(3)-FX(2)-ALI*(FX(2)-FX(1)))
000146        ANU=-2.*FX(3)*(1.+ALI)
000163        ADE=FI+CSQRT(SQ)
000172        IF(CABS(FI-CSQRT(SQ)).GT.CABS(ADE)) ADE=FI-CSQRT(SQ)
000231        ALI=ANU/ADE
000241        XIP1=X(3)+HI*ALI
000251        HI=HI*ALI
000257        FXIP1≠G(XIP1,AK,A,NDEG)
000272        FX(1)≠FX(2)
000275        FX(2)≠FX(3)
000300        FX(3)≠FXIP1
000303        X(1)=X(2)
000306        X(2)=X(3)
000311        X(3)=XIP1
000314        IF(CABS(X(3)-X(2)).LE.CABS(EPSI)) GO TO 20
000336        I=I+1
000340        IF(I.GT. M) GO TO 22
000343        GO TO 10
000343    20  IF(CABS(FX(3)).LE.CABS(EPSI)) GO TO 30
000363    22  WRITE(6,25) X(3),FX(3),I,AK
000407    25  FORMAT(//,5X,≠ROOT CONVERGENCE ≠,E14.7,2X,E14.7,≠NOT FNL CONVERGE≠
             1,≠NCE ≠,E14.7,2X,E14.7,/,5X,≠ITER COUNT= ≠,I5,≠  FOR A K= ≠,F10.5)
000407        CALL EXIT
000410    30  IF(ABS(AIMAG(X(3))).LT.AIMAG(EPSI)) GO TO 50
000422        WRITE(6,35) X(3),FX(3),I,AK
000446    35  FORMAT(5X,≠X=≠,2(2X,E14.7),≠   FX=≠,2(2X,E14.7),≠  ITER CNT=≠,I3,
             1≠  K=≠,F10.5)
000446        AX=REAL(X(3))
000454        BX=-AIMAG(X(3))
000456        X(3)=CMPLX(AX,BX)
000464        FX(3)≠G(X(3),AK,A,NDEG)
000503        WRITE(6,40) X(3),FX(3)
000522    40  FORMAT(5X,≠X=≠,2(2X,E14.7),≠   FX=≠,2(2X,E14.7))
000522        IFLAG≠IFLAG+1
000524        J=J+2*(IFLAG)
000525        IF(J.EQ.NDEG) GO TO 100
          C      HERE WOULD BE PLACED SOME METHOD OF DETERMINING NEXT ROOT APPROX.
000533        X(1)=CMPLX(-0.3,0.000)
000540        X(2)=CMPLX(-0.4,0.000)
000546        X(3)=CMPLX(-0.45,0.000)
000553        FX(1)≠G(X(1),AK,A,NDEG)
000572        FX(2)≠G(X(2),AK,A,NDEG)
000611        FX(3)≠G(X(3),AK,A,NDEG)
000630        GO TO 5
000630    50  WRITE(6,35) X(3),FX(3),I,AK
000654        J=J+1
```

```
000656         IF(J,EQ,NDEG) GO TO 100
      C           HERE WOULD BE PLACED SOME METHOD OF DETERMINING NEXT ROOT APPROX,
      C            I,E,  DEFLATION
000663   100   IF(AK,NE,1,00000) RETURN
000666         N=J-1-2*IFLAG
000671         WRITE(6,110) IFLAG,N
000701   110   FORMAT(//,10X,≠THERE ARE ≠,I3,≠ PAIRS OF COMPLEX ROOTS AND ≠,I3,
              1≠ REAL ROOTS≠,//)
000701         RETURN
000702         END
```

```
RUN VERSION 2.3 = CDC LEVEL 26 - NYU LEVEL 11

MULL

SUBPROGRAM LENGTH
001110

FUNCTION ASSIGNMENTS

STATEMENT ASSIGNMENTS
5       =  000015    10     =  000040    20     =  000343    22     =  000363
25      =  000714    30     =  000410    35     =  000732    40     =  000743
50      =  000630    100    =  000663    110    =  000757

BLOCK NAMES AND LENGTHS
        =  000001

VARIABLE ASSIGNMENTS
ADE     =  001074    ALI    =  001064    ANU    =  001072    AX     =  001105
BX      =  001106    EPSI   =  001060-   FI     =  001066    FXIP1  =  001100
HI      =  001062    I      =  001104    IFLAG  =  001103    J      =  001102
M       =  000000C01 N      =  001107    SG     =  001070    XIP1   =  001076

START OF CONSTANTS
000704

START OF TEMPORARIES
000770

START OF INDIRECTS
001034

UNUSED COMPILER SPACE
010600
```

```
              COMPLEX FUNCTION FN(X,AK,A,N)
000007        COMPLEX X,B,CHTR,CAK
000007        DIMENSION A(N)
000007        B=CMPLX(A(N),0,000)
000012        NM1=N-1
000014        DO 10 K=1,NM1
000015        KK=NM1-K+1
000017        CHTR=CMPLX(A(KK),0,000)
000023    10  B=CHTR+X*B
000036        CAK=CMPLX(AK,0,000)
000040        FN=B+CAK
000045        RETURN
000047        END
```

```
RUN VERSION 2.3 - CDC LEVEL 26 - NYU LEVEL 11

FN

SUBPROGRAM LENGTH
000100

FUNCTION ASSIGNMENTS

STATEMENT ASSIGNMENTS

BLOCK NAMES AND LENGTHS

VARIABLE ASSIGNMENTS
B        -  000067    OAK      -  000073    CHTR     -  000071    FN       -  000065
K        -  000076    KK       -  000077    NM1      -  000075

START OF CONSTANTS
000051

START OF TEMPORARIES
000055

START OF INDIRECTS
000064

UNUSED COMPILER SPACE
013300
```

```
  $$        $$$$$$$   $$$$$$$$    $$$$$$     $$$$$$    $$     $$
  $$        $$$$$$$   $$$$$$$$    $$$$$$     $$$$$$    $$     $$
 $$$       $$     $$        $$   $$     $$  $$     $$  $$     $$
 $$$       $$     $$        $$   $$     $$  $$     $$  $$     $$
$$$$             $$        $$          $$$  $$     $$  $$     $$
$$$$             $$        $$          $$$  $$     $$  $$     $$
  $$           $$        $$              $$ $$     $$  $$$$$$$$$$
  $$           $$        $$              $$ $$     $$  $$$$$$$$$$
  $$         $$        $$        $$     $$  $$     $$         $$
  $$         $$        $$        $$     $$  $$     $$         $$
$$$$$$     $$$$$$$$$  $$          $$$$$$     $$$$$$           $$
$$$$$$     $$$$$$$$$  $$          $$$$$$     $$$$$$           $$
```

************* NYU/CIMS SCOPE 3.2/32 ***** 10/09/72 ********************

NEW VERSIONS OF THE RUN COMPILER AND LIBRARY ARE NOW AVAILABLE FOR TESTING. THEY MAY BE OBTAINED FROM THE PERMANENT FILES NAMED RUN AND RUNLIB.

AS OF MONDAY,9 OCT, SNOBOL 3.9 (PURDUE) WILL BE AVAILABLE FROM THE CIMLIB TAPE IN TWO FORMATS - SNOBOLP (ABSOLUTE OVERLAY) AND SNOBOLR (RELOCATABLE). THE NEW VERSION CORRECTS SEVERAL PROBLEMS FOUND IN THE OLDER VERSION, BUT DOES NOT INCORPORATE ANY NEW FEATURES.

THE FOLLOWING CDC MANUALS ARE PRESENTLY AVAILABLE IN RM 505 AT THE PRICES LISTED . CHECK THIS LIST FOR MANUAL UPDATES.

PRICE	REV	NAME OF MANUAL
$4.50	C	CDC SCOPE USERS GUIDE (VERY USEFUL FROM USER/S STANDPOINT)
$2.80	B	CDC UPDATE REFERENCE MANUAL
$2.70	N	CDC 6600 COMPUTER SYSTEM REFERENCE MANUAL
$9.00	L	CDC SCOPE 3.2 OPERATING SYSTEM REFERENCE MANUAL
$3.70	E	CDC FORTRAN REFERENCE MANUAL
$5.50	J	CDC FORTRAN EXTENDED REFERENCE MANUAL
$5.20	C	CDC 6000/7000 COMPASS 2.0 REFERENCE MANUAL
$2.00	C	CDC BASIC 2.0 REFERENCE MANUAL
$2.40		CDC SORT/MERGE REFERENCE MANUAL
$2.00	A	CDC INTERCOM I USER/S MANUAL
$2.00		NYU USER/S GUIDE TO THE CDC 6600
$3.00		NYU SNOBOL IV MANUAL
$3.00		NYU ARTSPEAK MANUAL
$1.00		NYU GUIDE TO COMPUTER ART FOR THE PRINTER - CAP

PLEASE REPORT ANY PROBLEMS OR DIFFICULTIES ENCOUNTERED IN USING THIS VERSION OF THE SYSTEM TO THE APPLICATIONS ANALYSTS IN RM 305 (460-7391)

****

```
          PROGRAM STEFF(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      C   PROGRAM TO FIND X SUCH THAT X=G(X)
)00003    WRITE(6,1)
)00007  1 FORMAT(1H1,4X,≠STEFFENSEN ITERATION TECHNIQUE≠,//)
)00007    I=1
)00010  5 READ(5,3) X0
)00016  3 FORMAT(F15,10)
)00016    WRITE(6,4) X0
)00024  4 FORMAT(//,5X,18HSTARTING POINT IS ,E14.7)
)00024    ITER=1
)00025    Y0=G(X0)
)00030 10 Y1=G(Y0)
)00032    Y2=G(Y1)
      C   OBTAIN DELTA1
)00034    DL=Y2-Y1
      C   CHECK TO SEE IF DELTA1 IS LESS THAN EPSILON...IF YES WE HAVE RESULTS
)00036    IF(ABS(DL),LE,1,0E-07) GO TO 100
      C   OBTAIN THE INVERSE OF THE DERIVATIVE
)00042    RINV=(Y1-Y0)/DL
      C   IF THE DERIVATIVE IS GREATER THAN 1 USE AITKEN≠S DELTA SQUARED
      C   PROCESS...IF NOT, START STEFFENSEN≠S ITERATION
)00044    IF(ABS(RINV),LE,1,0) GO TO 50
)00050    NITER=ITER
      C   OBTAIN THE NEW GUESS AT THE ROOT OF THE FUNCTION
)00051    Y0=Y2-DL/(RINV-1,)
      C   CHECK TO SEE IF THE ITERATION PROCEDURE CONVERGES...IF NOT STOP
)00055 40 IF(ITER,GE,50) GO TO 75
)00060    ITER=ITER+1
)00061    GO TO 10
)00062 50 Y0=Y2
)00064    GO TO 40
      C   PRINT OUT THE VALUES WHEN ITERATION COUNT WAS EXCEEDED...THEN STOP
)00064 75 GP=1,/RINV
)00066    WRITE(6,80) ITER,Y2,Y1,Y0,GP
)00104 80 FORMAT(5X,≠NO CONVERGENCE AFTER≠,I3,≠ ITERATIONS≠,/,5X,≠Y2=≠,E14,7
        1,2X,≠Y1=≠,E14,7,2X,≠Y0=≠,E14.7,2X,≠GPRIME=≠,E14.7)
)00104    CALL EXIT
      C   PRINT OUT THE RESULTS IF THE ITERATION PROCEDURE CONVERGED
)00105 100 GP=1,/RINV
      C   PRINT OUT THE NUMBER OF TIMES AITKENS WAS USED
)00107    WRITE(6,2) NITER
)00115  2 FORMAT(5X,≠GPRIME WAS .GE. 1 FOR ≠,I3,≠ ITERATIONS PRIOR TO CONFO≠
        1,≠RMING TO CONDITIONS FOR CONVERGENCE≠)
)00115    WRITE(6,101) Y2,Y1,Y0,ITER,GP
)00133 101 FORMAT(5X,≠Y2=≠,E14.7,2X,≠Y1=≠,E14.7,2X,≠Y0=≠,E14,7,/,5X,≠CONVERG≠
        1,≠ED AFTER≠,I3,≠ ITERATIONS WITH GPRIME EQUAL TO≠,E14.7)
)00133    I=I+1
)00135    IF(I-5) 5,5,102
)00137 102 CALL EXIT
)00140    END
```

RUN VERSION 2,3 - CDC LEVEL 26 - NYU LEVEL 11

STEFF

PROGRAM LENGTH INCLUDING I/O BUFFERS

004335

FUNCTION ASSIGNMENTS

STATEMENT ASSIGNMENTS

1	-	000153	2	-	000210	3	-	000162	4	-	000164
5	-	000010	10	-	000030	40	-	000055	50	-	000062
75	-	000064	80	-	000173	100	-	000105	101	-	000224
102	-	000137									

BLOCK NAMES AND LENGTHS

VARIABLE ASSIGNMENTS

DL	-	000265	GP	-	000270	I	-	000257	ITER	-	000261
NITER	-	000267	RINV	-	000266	XO	-	000260	YO	-	000262
Y1	-	000263	Y2	-	000264						

START OF CONSTANTS

000142

START OF TEMPORARIES

000241

START OF INDIRECTS

000257

UNUSED COMPILER SPACE

002700

```
              FUNCTION G(TSI)
000003        G=0,5+0,2*SIN(TSI)
000010        RETURN
000010        END
```

```
RUN VERSION 2.3 - CDC LEVEL 26 - NYU LEVEL 11

G

SUBPROGRAM LENGTH
000022

FUNCTION ASSIGNMENTS

STATEMENT ASSIGNMENTS

BLOCK NAMES AND LENGTHS

VARIABLE ASSIGNMENTS
G       -  000021

START OF CONSTANTS
000012

START OF TEMPORARIES
000015

START OF INDIRECTS
000021

UNUSED COMPILER SPACE
003400
```

```
CORE MAP  16.34.13.  NORMAL              CONTROL
        ---TIME---LOAD MODE --L1--L2-----TYPE---------------USEP---+---CALL------
        FWA LOADER  033771  FWA TABLES  032162
        -PROGRAM---=ADDRESS-                       --LABELED---COMMON--
        STEFF        000100
        G            004435
        SINCOS       004457
        SYSTEM       004556                                 SCOPE2       004556
        GETBA        005662
        INPUTC       005701
        KODER        006031
        KRAKER       007354
        OUTPTC       010474
        SIO$         010566
        ----UNSATISFIED EXTERNALS-----                              REFERENCES
```

Jan 8, 1972

Exam: Jan 22 11:30 – 1:30 T2–332

Review of Num. Methods

1. Floating pt representation of real numbers

Rounding & Chopping methods. Bounds on Relative Error

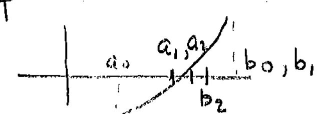
(Advantages of Rounding & Chopping).

Propagation of error & loss of significant figure in operation with floating pt. #'s.

Definition & familiarity with order symbols. $O(\,)$ & $o(\,)$

2. Soln of Non linear Eqs.

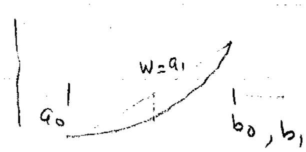
(1) Bisection



Given $[a_0, b_0]$ $f(a_0)f(b_0) < 0$

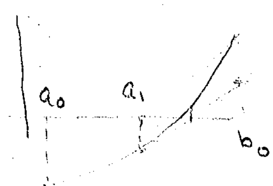
(2) Regula Falsi

$[a_0, b_0]$ $f(a_0)f(b_0) < 0$



(3) Modified Regula Falsi

$[a_0, b_0]$ $f(a_0)f(b_0) < 0$



Algorithm for mod regula falsi

Set $F = f(a_0)$, $G = f(b_0)$

$w_0 = a_0$

DO $n = 0, 1, \ldots$

Compute $w_{n+1} = \dfrac{F b_n - G a_n}{F - G}$

if $f(a_n)f(w_{n+1}) \leq 0$, $a_{n+1} = a_n$ $b_{n+1} = w_{n+1}$ $G = f(w_{n+1})$

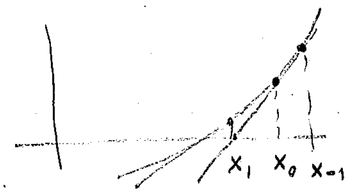
" " > 0, $a_{n+1} = w_{n+1}$ $b_{n+1} = b_n$ $F = f(w_{n+1})$

If also $f(w_n)f(w_{n+1}) > 0$ set $F = F/2$

If also $f(w_n)f(w_{n+1}) > 0$ set $G = G/2$

(4) Secant method

$[x_{-1}, x_0]$ $\quad f(x)$



$$x_{n+1} = \frac{f(x_n)\,x_{n-1} - f(x_{n-1})\,x_n}{f(x_n) - f(x_{n-1})}$$

do not know if root will be encountered since don't know root lies in interval

(5) Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

limit of (4) must know deriv.

Fixed Pt iteration

$f(x) = 0 \iff x = g(x)$ $\qquad g(x) = x - \frac{f}{f'}$ for newton's method.

General Result:

Given x_0, $x_{n+1} = g(x_n)$

$x_{n+1} \to \xi$ where $\xi = g(\xi)$; if (i) $g : [a,b] \to [a,b]$

ξ is unique $\qquad$ (ii) $|g'(x)| \leq K < 1$ $\quad x \in [a,b]$

Quest: describe fixed pt iteration using Newton's function for the evaluation of $\sqrt{2}$. Specify $[a,b]$ & check (ii)

Error in fixed pt iteration $\quad x_{n+1} = g(x_n)$

$e_n = \xi - x_n \qquad e_{n+1} \sim g'(\xi)\,e_n - \frac{1}{2} g''(\xi)\,e_n^2 + \ldots \qquad n \to \infty$

if $g'(\xi) \neq 0$ linear conv.

$g'(\xi) = 0$ quad.

On adjusting $g(x)$ so that $g'(\xi) = 0$ we obtain $g(x) = x - \frac{f(x)}{f'(x)}$

$\therefore$ newtons' method converges quad

Q: What happens if $f'(\xi)=0$ in newton's method

newton linear put $\frac{2f}{f'}$ newton $\to$ quad.

Acceleration of Convergence in fixed pt iteration $x_{n+1}=g(x_n)$ $\xi=g(\xi)$

Construct $x_0, x_1, \ldots, x_n \to \xi$

$e_n = \xi - x_n$ $x_n = \xi + O(e_n)$

construct $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_n$ $\ni$ $\hat{x}_n = \xi + o(e_n)$

Aitken Δ^2 method $\hat{x}_n = x_{n+1} - \frac{(\Delta x_n)^2}{\Delta^2 x_{n-1}}$ $\Delta x_n = x_{n+1} - x_n$

Be able to argue through this formula

$\xi - x_{n+1} \sim g'(\xi)(\xi - x_n)$

On using aitken in conj with $x_{n+1} = g(x_n)$ $\Rightarrow$ Steffensen It scheme

Set $y_0 = x_0$ Given

DO $n=0,1,\ldots$

$$\left.\begin{aligned} x_1 &= g(x_0) \\ x_2 &= g(x_1) \end{aligned}\right\} \quad \begin{aligned} &\Delta x_1 = x_2 - x_1 \\ &\Delta^2 x_0 \end{aligned}$$

Calculate $y_{n+1} = x_2 - \frac{(\Delta x_1)^2}{\Delta^2 x_0}$

<u>Polynomials</u>: (i) Nested Multiplication (Synthetic Div)

(ii) Newton's Method for real zeros.

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}$$

(iii) Muller's Method: Analogous to secant method except pass parabola through 3-pts.

(1st) Why is Mullers' Method good? (Isaakson & Keller)

Compare Newton & Muller's Method

3. Linear Eqs:

1. Elementary Linear Algebra

2. The Alternative Theorem. *

When is $Ax=b$ Solvable? if $Ax=0 \Leftrightarrow x=0$ then $Ax=b$ has ! solut. otherwise $\exists$ vectors $\ni$ if $Ax=0$ $x=\sum_{j=1}^{p}\alpha_j x^j$

3. Back Substitution for Triangular System.

4. Ordinary Gauss elimination.

5. Gauss Elimination with max. partial pivot

(describe: Work out an example)

Purpose of pivoting No pivoting $\Rightarrow$ inaccurate results.

2nd (*) 6. Describe $PLU=A$ factorization

Assuming you have G. Elim available.

7). Matrix Norms & Condition Numbers.

$e = x - \hat{x}$ ← Compute

$$r = Ae \quad \frac{1}{\|A\|\|A^{-1}\|}\frac{\|r\|}{\|b\|} \le \frac{\|e\|}{\|x\|} \le \|A\|\|A^{-1}\| \frac{\|r\|}{\|b\|}$$

$Cond = \|A\| \|A^{-1}\|$

8) Remarks on Condition Numbers.

3 (*) 9) Describe iterative methods for Lin System.

$A = \hat{L} + \hat{U} + \hat{D}$ Jacobi

G.S.

Iterative Correct

Solve $Ax - b = 0$

$x = x + C^{-1}(b - Ax) \Leftrightarrow Ax - b = 0$ C–no sing

$x = (I - C^{-1}A)x + C^{-1}b$

Jacobi $C = \hat{D}$ if $\hat{D} = (0)$ permute rows $\ni \hat{D} \neq (0)$

$x_{n+1} = Bx_n + c$ $x_n \xrightarrow{?} \xi$

G.S. $C = \hat{D} + \hat{L}$

(10) When does $x_{n+1} = Bx_n + c \Rightarrow$ convergence

Ans $\|x_n - \xi\| \to 0$ as $n \to \infty$ $\xi = B\xi + c \Leftrightarrow$ mod E.V. of B are less

ie $Bz = \lambda z$ $z \neq 0$ $|\lambda| = 1$

Sufficient Cond for Conv. Jacoby: if A diag dominant $a_{kk} > \sum_{j \neq k} a_{kj}$

$\|I - C^{-1}A\|$ has small norm

G.S.: same as Jacoby

11) Power method for eigenvalues $\lambda_1 > \lambda_2 > \dots > \lambda_n$

B $Bz, B z^2, \dots$

$$\lambda_1 \sim \frac{j\text{th entry } Bz^{m+1}}{j\text{th entry } Bz^m}$$

Why is this O.K.

Iterative Correction: up to us.

4. Interpolation & Least Squares.

1) what is interpolation problem: given $x_0, x_1, \dots, x_n$ & $f_0, f_1, \dots, f_n$

Find a polyn $p_n(x)$ degree $\leq n$ $\ni$ $p_n(x_k) = f(x_k)$ $k = 0, \dots, n$.

2) Lagrange form of $p_n(x) = \sum_{k=0}^{n} f(x_k) \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{(x-x_j)}{(x_k-x_j)}$

3) Newton's Form

$$P_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + \dots + f[x_0,\dots,x_n](x-x_0)\cdots(x-x_{n-1})$$

$$f[x_0,x_1,\dots,x_n] = \frac{f[x_1,\dots,x_n] - f[x_0,\dots,x_{n-1}]}{x_n - x_0}$$

4) Error of $P_n(x)$

5) Piecewise polyn interpolation   cubic interpolation

Q: describe piecewise cubic poly interpolation (i) Bessel
(ii) Hermite
(iii) Spline

& Comments on all three

$x_0 \quad x_1 \quad\quad k^{th}$ int $\quad\quad x_{n-1} \quad x_n$

$$P_k(x) = c_0^k + c_1^k x + c_2^k x^2 + c_3^k x^3 \qquad x_k \to x_{k+1}$$

if $P_k(x_k) = f(x_k)$ $\quad\quad P_k'(x_k) = P_{k-1}'(x_k)$

$P_k(x_{k+1}) = f(x_{k+1})$ $\quad\quad P_k'(x_{k+1}) = P_{k+1}'(x_{k+1})$

$P_k'(x_k) = f'(x_k)$

Hermite: fn & deriv are needed at $x_0, x_1, \dots, x_n$ $\quad$ use limit of interp pts

Bessel: f' not given, same as above but $f_k' \sim \frac{\Delta f}{\Delta x}$

Spline: Adjust $c_j^k \Rightarrow$ $P_k(x_k) = f(x_k)$

$P_k(x_{k+1}) = f(x_{k+1})$

other constants are determined by requiring that P.Wise poly has 2 continuous deriv.

5. Least Squares

(i) State problem

(ii) deriv normal eq.s.

(iii) Inner product & orthog pol def. Examples.

(iv) Major How does one use orthog poly for the numerical implementation of least square data fitting method: easy to program recursiveness

C. Levy
763.1050
Prof. Papanicolau

4/4

3.91 if λ is an eigenvalue of A $\exists$ an $\underset{\sim}{x}$ $\ni$ $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ $\quad \underset{\sim}{x} \neq \underset{\sim}{0}$

if a is a scalar then $aA\underset{\sim}{x} = a\lambda\underset{\sim}{x}$

if b is a scalar then $bI\underset{\sim}{x} = b\underset{\sim}{x}$

then $aA\underset{\sim}{x} + bI\underset{\sim}{x} = a\lambda\underset{\sim}{x} + b\underset{\sim}{x}$

Using distributive law of addition over multiplication

$$(aA+bI)\underset{\sim}{x} = aA\underset{\sim}{x} + bI\underset{\sim}{x}$$

$$(a\lambda+b)\underset{\sim}{x} = a\lambda\underset{\sim}{x} + b\underset{\sim}{x}$$

$\therefore \quad (aA+bI)\underset{\sim}{x} = (a\lambda+b)\underset{\sim}{x}$

denote $aA+bI$ by B and $a\lambda+b$ by μ

then $B\underset{\sim}{x} = \mu\underset{\sim}{x}$ where B is a matrix (square) & μ is a scalar; then μ is an eigenvalue of B since $\exists$ an $\underset{\sim}{x} \neq \underset{\sim}{0}$ $\ni$ $(B-\mu I)\underset{\sim}{x} = 0$

3.92 $\quad p(A) = c_0A^0 + c_1A^1 + c_2A^2 + \dots + c_kA^k$

where $A^0 = I$ and $A^j = A(A^{j-1})$

post multiply $p(A)$ by a non zero vector $\underset{\sim}{x}$ which satisfies $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$;

then $p(A)\underset{\sim}{x} = \sum_{i=0}^{k} c_i A^i \underset{\sim}{x} = c_0I\underset{\sim}{x} + \sum_{i=1}^{k} c_i A^{i-1} A\underset{\sim}{x} = c_0I\underset{\sim}{x} + \sum_{i=1}^{k} c_i A^{i-1}\lambda\underset{\sim}{x}$

$= c_0I\underset{\sim}{x} + c_1\lambda I\underset{\sim}{x} + \lambda\sum_{i=2}^{k} A^{i-2}A\underset{\sim}{x} = c_0I\underset{\sim}{x} + c_1\lambda I\underset{\sim}{x} + \sum_{i=2}^{k} A^{i-2}\lambda^2\underset{\sim}{x}$

$= \dots = c_0I\underset{\sim}{x} + c_1\lambda I\underset{\sim}{x} + c_2\lambda^2 I\underset{\sim}{x} + \dots + c_k\lambda^k I\underset{\sim}{x} = p(\lambda)I\underset{\sim}{x}$

where $p(\lambda) = \sum_{i=0}^{k} c_i\lambda^i$; then $p(A)\underset{\sim}{x} = p(\lambda)I\underset{\sim}{x} = p(\lambda)\underset{\sim}{x}$

let

B be a matrix $= p(A)$ and $\mu = p(\lambda)$ a scalar then let A be tridiag $\ni$. $a_{ii} = 0$ $\quad a_{i,i+1} = a_{i+1,i}$ therefore

$$\begin{pmatrix} 0 & r_1 & & \\ r_1 & \ddots & \ddots & \\ & \ddots & \ddots & r_{n-1} \\ & & r_{n-1} & 0 \end{pmatrix}$$

then $\exists$ a nonzero $\underset{\sim}{x}$ $\ni$. $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

let $\underset{\sim}{x}_i = \sin(i\pi/n+1)$ $\quad$ for an $n \times n$ matrix, argument of sin $0 < \frac{i\pi}{n+1} < \pi$

$\therefore$

$(A - \lambda I)\underset{\sim}{x} = 0$ $\quad$ iff $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} -\lambda & r_i & & \\ r_i & \ddots & \ddots & \\ & \ddots & \ddots & r_{n-1} \\ & & r_{n-1} & -\lambda \end{pmatrix}$$

write $A - \lambda I$ as LU where L is a lower triang matrix & U is upper

$$L = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & \beta_n & \alpha_n \end{pmatrix} \qquad U = \begin{pmatrix} 1 & \gamma_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \gamma_{n-1} \\ & & & 1 \end{pmatrix}$$

where $\alpha_i, \beta_i, \gamma_i$ are defined by

$\alpha_1 = -\lambda$ $\qquad \beta_i = r_{i-1}$ $\qquad \gamma_i = r_i / -\lambda$

and $\alpha_i = -\lambda - \beta_i \gamma_{i-1}$ $\Rightarrow$ $\alpha_i = \underline{-\lambda^2 + r_{i-1}^2}$

$\det U = 1$ $\quad \det L = \alpha_1 \alpha_2 \cdots \alpha_n = \dfrac{-\lambda(-\lambda^2 + r_1^2) \cdots (-\lambda^2 + r_{n-1}^2)}{\lambda^{n-1}}$

if λ is nonzero

$\det L = 0 = -\dfrac{(r_1^2 - \lambda^2) \cdots (r_{n-1}^2 - \lambda^2)}{\lambda^{n-2}} = 0 \Rightarrow \lambda = r_i$ $\qquad i = 1, \dots, n-1$

but $r_1 = r_2 = \cdots = r_{n-1} = 1$ $\quad$ then $\lambda = \pm 1$ $\quad$ of LU

take a look at ith equation of jth eigenvector

$\sin \underline{(i-1)j\pi} + \sin \underline{(i+1)j\pi} = \lambda^j \sin \underline{ij\pi}$

but $\sin\frac{(i+1)j\pi}{n+1} = \sin\frac{ij\pi}{n+1}\cos\frac{j\pi}{n+1} + \cos\frac{ij\pi}{n+1}\sin\frac{j\pi}{n+1}$

$\sin\frac{(i-1)j\pi}{n+1} = \sin\frac{ij\pi}{n+1}\cos\frac{j\pi}{n+1} - \cos\frac{ij\pi}{n+1}\sin\frac{j\pi}{n+1}$

add the two gives

$$\sin\frac{ij\pi}{n+1}\left(2\cos\frac{j\pi}{n+1}\right) = \lambda^j \sin\frac{ij\pi}{n+1}$$

$\therefore \quad \lambda^j = 2\cos\frac{j\pi}{n+1}$

$\therefore \quad A\underset{\sim}{x}^j = 2\cos\frac{j\pi}{n+1}\ \underset{\sim}{x}^j$

3.94 we can write $B = [e, d, e]$ (note: $[x, x, x]$ means tridiag form with elements e on super & subdiag & elements d on main diag)

$$B = dI + e[1,0,1]$$

by 3.91

$B\underset{\sim}{x} = \mu\underset{\sim}{x}$ has eigenvalues $\mu = e\lambda + d$

where $\lambda = 2\cos\frac{j\pi}{n+1}$ solving $A\underset{\sim}{x}^j = \lambda^j\underset{\sim}{x}$ $\quad A: [1,0,1]$

$\therefore \quad \mu = d + 2e\cos\frac{j\pi}{n+1}$

3.95 Using power method to estimate spectral radius and corresponding eigenvector, for 20x20 matrix $A: [-1, 4, -1]$ and compare with exact results from 3.94

$\lambda_{max} = 4 - 2\cos\left(\frac{20\pi}{21}\right) = 4 - 2\cos(171.5^\circ) = 4 - 2(-.989) = 5.978$

no. of iter.

$B^m z$	0	1	2	3	4	5	6	7
	1	3	10	37	150	654	3012	14475
	1	2	3	-2	-54	-396	-2427	-13858
	1		4	9	80	189	1138	7145
	1			8	15	14	-166	-1888
	1				16	33	86	447
	1					32	63	102
	1						64	129
	1							128

	0	1	2	3	4	5	6	7
								128
							64	129
						32	63	102
					16	33	86	447
				8	15	14	-166	-1888
			4	9	80	189	1138	7145
		2	3	-2	-54	-396	-2427	-13858
		3	10	37	150	654	3012	14475

m	1	2	3	4	5	6	7
λ_1	3	3.333	3.7	4.4	4.34	~4.56	~4.67
	2	1.5	-.67	27	~8	6.+	~5.6
	2	2	2.5	3.33	6.3	6.0	6.26
			2	1.875	1.+	-12	~11.3
				2	2+	~2.63	5.14
					2	~2	~1.67
						2	~2
							2

	1	2	3	4	5	6	7
							2
						2	~2
					2	~2	~1.67
				2	2+	~2.63	5.14
			2	1.875	1+	-12	~11.3
		2	2.5	3.33	6.3	6.0	6.26
	2	1.5	-.67	27	~8	6+	~5.6
	3	3.33	3.7	4.4	4.34	~4.56	~4.67

etcetera

The eigenvector is obtained by taking eigenvalue obtained & forming $\left(\frac{1}{\lambda_1}B\right)^m z \sim y_1$

after fifty iterations using an average $\lambda_{max} = \sum_{i=1}^{20} \lambda_i / 20$

$\lambda_{max} = 5.9402219$, closest to this average value

where entries used are 6th or 15th

$$B z^{(m)} = \begin{bmatrix} 1.0 \\ -10.63 \\ 13.70 \\ -14.76 \\ 14.02 \\ -11.99 \\ 9.29 \\ -6.43 \\ 3.72 \\ -1.21 \\ -1.21 \\ 3.72 \\ -6.43 \\ 9.29 \\ -11.99 \\ 14.02 \\ -14.76 \\ 13.70 \\ -10.63 \\ 5.82 \end{bmatrix}$$

since $B y_1^* = \lambda y_1^* \sim B z^{(m)}$

then $y_1^* = \frac{B}{\lambda} y_1^* = \begin{bmatrix} .168 \\ -1.79 \\ 2.31 \\ -2.49 \\ 2.36 \\ -2.02 \\ 1.56 \\ -1.08 \\ 6.26 \\ -2.04 \\ -2.04 \\ 6.26 \\ -1.08 \\ 1.56 \\ -2.02 \\ 2.36 \\ -2.49 \\ 2.31 \\ -1.79 \\ \cancel{0}.168 \end{bmatrix}$

C. Levy
T63.1050

1. $P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{3}{2}\left[x^2 - \frac{1}{3}\right]$

$S_0 = 2 \qquad S_1 = \frac{2}{3} \qquad S_2 = \frac{2}{5}$

$f(x) = \sin \pi x$

$$\langle f, P_0 \rangle = \int_{-1}^{1} \sin \pi x \, dx = -\frac{1}{\pi} \cos \pi x \Big|_{-1}^{1} = 0 \qquad \Rightarrow d_0 = 0$$

$$\langle f, P_1 \rangle = \int_{-1}^{1} x \sin \pi x \, dx = -\frac{x}{\pi} \cos \pi x \Big|_{-1}^{1} + \frac{1}{\pi^2} \sin \pi x \Big|_{-1}^{1} = \frac{2}{\pi} \qquad \Rightarrow d_1 = \frac{2/\pi}{2/3} \quad .95493$$

$$\langle f, P_2 \rangle = \int_{-1}^{1} \frac{3}{2}\left(x^2 - \frac{1}{3}\right) \sin \pi x \, dx = +\frac{1}{2\pi} \cos \pi x \Big|_{-1}^{1} + \frac{3}{2}\left[-\frac{x^2}{\pi} \cos \pi x + \frac{2x}{\pi^2} \sin \pi x + \frac{2}{\pi^3} \cos \pi x\right]_{-1}^{1} = 0$$

$$\Rightarrow d_2 = 0$$

$$f(x) = d_0 + d_1 P_1(x) + d_2 P_2(x) = 0 + .95493x + 0 \cdot \frac{3}{2}\left(x^2 - \frac{1}{3}\right) = .95493x$$