T10.2713 Prof. Kerr Theory of Elastic Stability

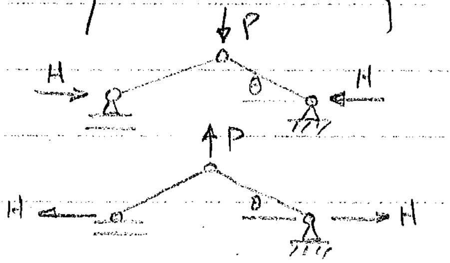
H. Ziegler

Timoshenko & Gere

Static Stability 1-Degree of freedom

Equilibrium - Equil Eqs

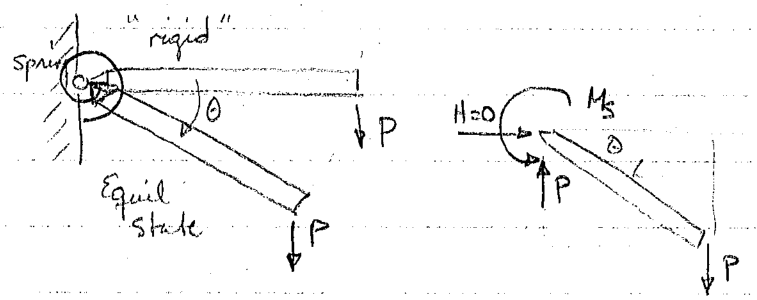
Stability - Stability Criterion



Equil, Unstab.

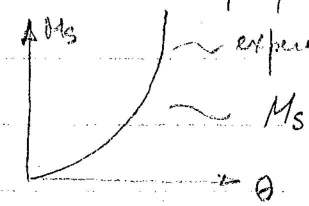
Equil, Stab

Example



$\Sigma M_1 = 0 \qquad PL\cos\theta - M_s = 0$ Equil equation

assume M_s satisfies



~ experimental data

~ $M_s = \frac{s}{n} \sinh(n\theta)$ non linear spring response

assume $M_s = k\theta$ linear spring

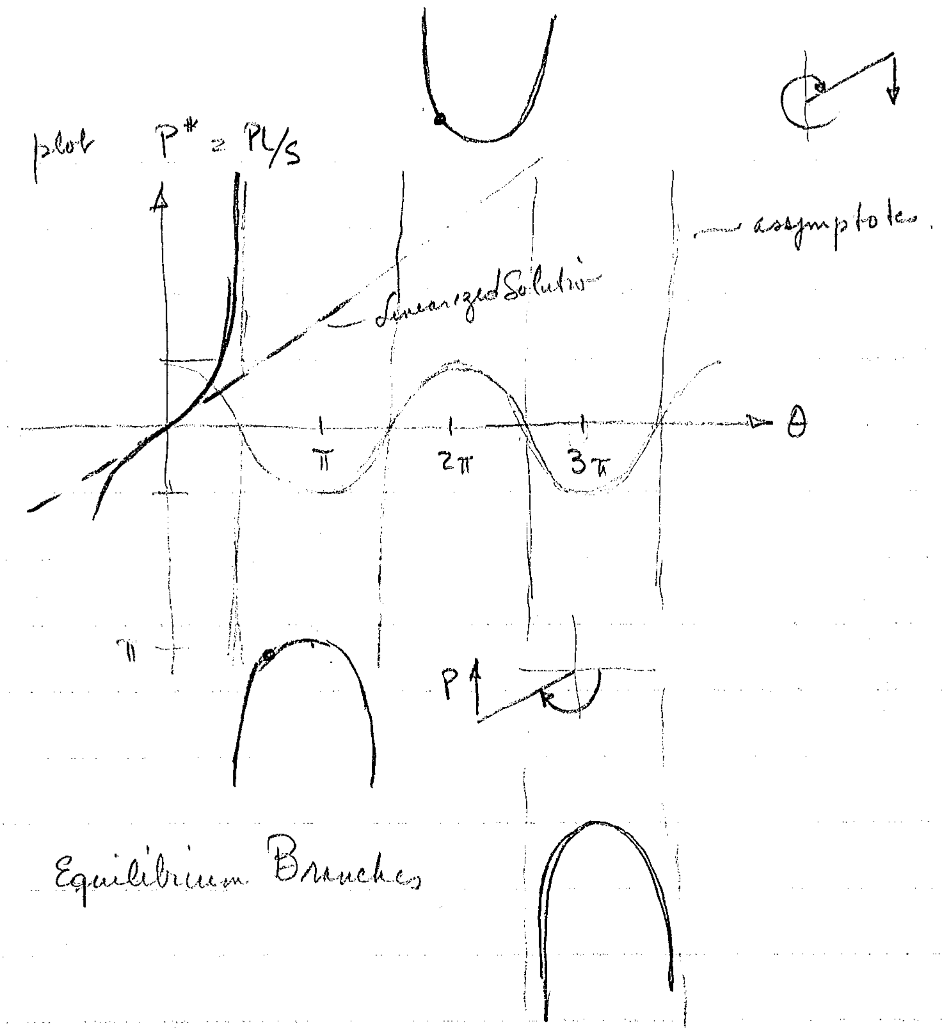
Non linearity can enter by geometric consideration or material propert

(1) $PL\cos\theta - s\theta = 0$ Eq equation of problem.

$P = \frac{s}{L}\frac{\theta}{\cos\theta}$ θ is deformation parameter

highly non-linear eq

plot $P^* = Pl/s$



Equilibrium Branches

<u>Linearization</u>

1) For small θ $\quad \cos\theta \sim 1 \quad \therefore \quad P^* = \theta$

2) Assume Taylor Expansion about $x=0$

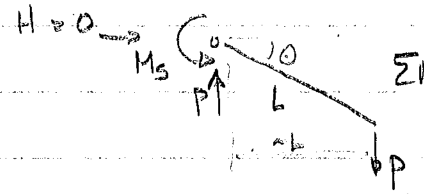
$$P(\theta) = P(0) + \left.\frac{dP}{d\theta}\right|_{\theta=0} \theta + \left(\frac{d^2P}{d\theta^2}\right)\Big|_{\theta=0} \frac{\theta^2}{2!} +$$

$P(0)=0$

$$\frac{dP}{d\theta}(\theta=0) = \frac{s}{l}\,\frac{\cos\theta + \theta\sin\theta}{\cos^2\theta} = \frac{s}{l} \qquad P(\theta) = 0 + \frac{s}{l}\theta$$

$P = \frac{s}{L}\theta$ is a linearized equil eq analogous to $EIW^{(IV)} = q$

3rd approach deriv result from free body diagram (used in Structural Mech) derivation of Equil Equation from free body diag assuming a priori that deformations are small



$\Sigma M \sim PL - M_s = PL - s\theta = 0$

$P = \frac{s}{L}\theta$

Energy Considerations Primitive Approach

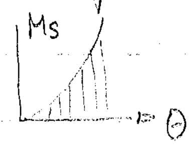
$PL\cos\theta - s\theta = 0$ Equil Eq

Observation Define Π (total potential energy)

then $\frac{d\Pi}{d\theta} = 0 \Rightarrow$ Equilib Equation

$\Pi = U - W$ U - strain energy stored during deformation
W - work potential of outside forces.

$W = +PL\sin\theta$



$\int_0^\theta M_s d\theta' = U_s(\theta)$

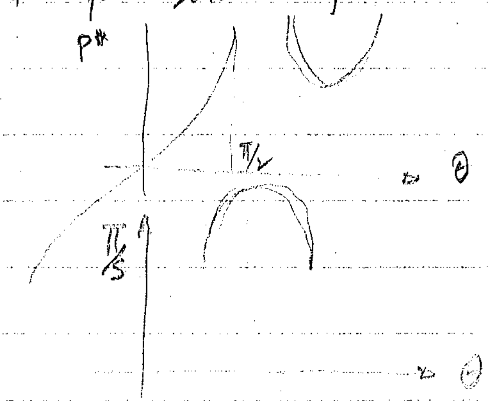
$M_s(\theta') = s\theta'$

$U_s(\theta) = s\frac{\theta^2}{2}$

$\Pi = s\theta^2/2 - PL\sin\theta$ since beam is rigid $U_{beam} = 0$

For $\frac{d\Pi}{d\theta} = s\theta - PL\cos\theta = 0$ Equil Equation

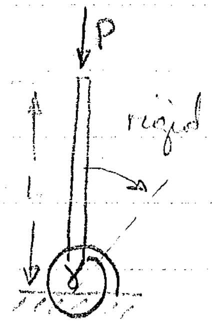
HW #1 Establish the relationship between energy level curves & equil branches for second example $|\theta| \leq \pi$



plot Π^* versus θ for different P^* 0.5, 1.0, 1.5, 2.0

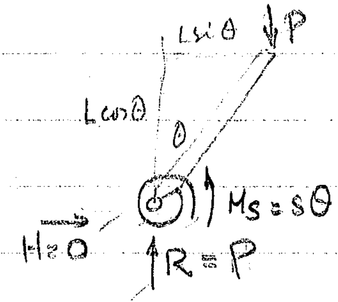
Equil equations may be obtained in two ways
1) Free body diagram
2) From condition that $\frac{d\Pi}{d\theta} = 0$

Example 2



$\theta = 0$ is an equil position $\forall N$

Determination of equil state $\theta \neq 0$



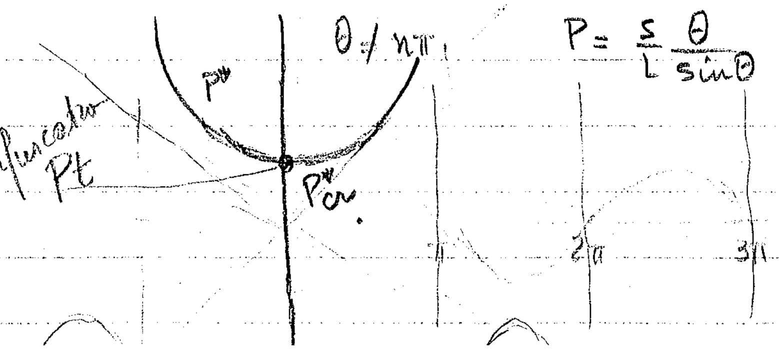
Assume existence of such equil states & consider FB Diag with $\theta \neq 0$

$-PL\sin\theta + s\theta = 0$

$s\theta - PL\sin\theta = 0$ Eq

$\theta \neq n\pi$

$P = \frac{s}{L}\frac{\theta}{\sin\theta}$

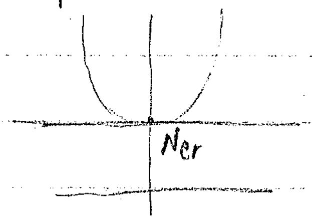


Equil Branches at P_{cr}^* are bifurcating

back to equil $S\theta - PL\sin\theta \sim S\theta - PL\theta = 0 \quad (S-PL)\theta = 0$

Equil eq is satisfied for any P when $\theta \equiv 0$ trivial case

if $\theta \neq 0 \Rightarrow P_{cr} = S/L$ Euler Buckling load



$$\Pi = \underbrace{\frac{S\theta^2}{2}}_{\text{Energy stored}} - PL(1-\cos\theta)$$

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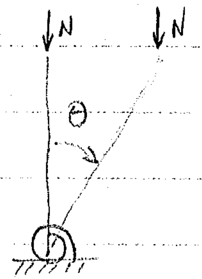
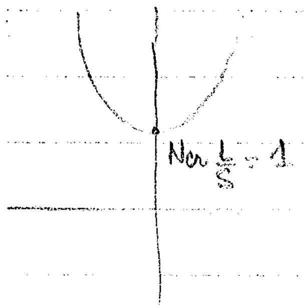


$NL\sin\theta - S\theta = 0$

$$\frac{NL}{S} = \frac{\theta}{\sin\theta}$$



Stability Criteria

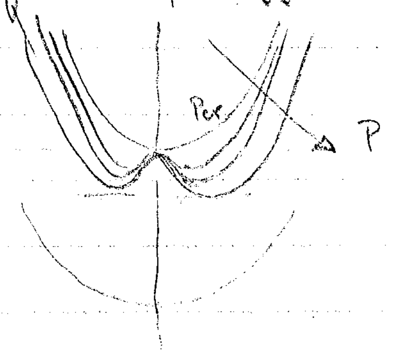
1) Dynamic Stability Analysis - Given a disturbance of the system for a given load study the dynamic response. If after initial disturbance the system oscillates about the equilib position - system is stable. If system move to another position and oscillates about it then original position of equil is unstable and the new position is a position of stable equil

2) Static Stability Analysis - based on the Lagrange stability criterion

"An equilib configuration of a conservative mechanical system is stable if the corresponding total potential energy Π has a proper minimum in this configuration with respect to all kinematically admissible states (which need not be states of equilibrium)"

Use of static stability criterion

(I) By means of energy level curves.



$P < P_{cr}$ stable structure for $\theta = 0$

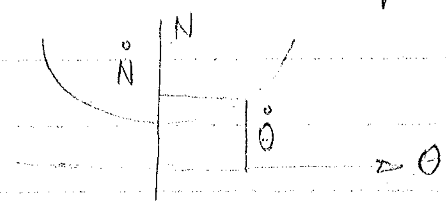
$P > P_{cr}$ " " for $\theta > 0$

$P > P_{cr}$ unstable " for $\theta = 0$

(II) By analytic formulation

Notation:

$\mathring{()}$ variables of a state of equil under investigation



perturbation from equil state by $\alpha\tilde{\theta}$ $\tilde{\theta}$ prescribed α is small param

then $\Pi(\mathring{\theta}+\alpha\tilde{\theta}, \mathring{N}) - \Pi(\mathring{\theta}, \mathring{N}) > 0$ stable

< 0 unstable

More suitable form for analysis noting that

$$\Pi(\overset{\circ}{\theta}+\alpha\tilde{\theta},\overset{\circ}{N}) = \Pi(\alpha)$$

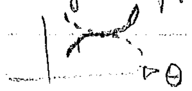
$$\Pi(\overset{\circ}{\theta}+\alpha\tilde{\theta},\overset{\circ}{N}) = \Pi(\overset{\circ}{\theta},\overset{\circ}{N}) + \left.\frac{\partial\Pi}{\partial\alpha}\right|_{\alpha=0}\alpha + \left.\frac{\partial^2\Pi}{\partial\alpha^2}\right|_{\alpha=0}\frac{\alpha^2}{2!} + \dots$$

Rewritten

$$\Pi(\overset{\circ}{\theta}+\alpha\tilde{\theta},\overset{\circ}{N}) - \Pi(\overset{\circ}{\theta},\overset{\circ}{N}) = \left.\frac{\partial^2\Pi}{\partial\alpha^2}\right|_{\alpha=0}\frac{\alpha^2}{2!} + \left.\frac{\partial^3\Pi}{\partial\alpha^3}\right|_{\alpha=0}\frac{\alpha^3}{3!} + \left.\frac{\partial^4\Pi}{\partial\alpha^4}\right|_{\alpha=0}\frac{\alpha^4}{4!} + \dots$$

since $\frac{\alpha^2}{2!} > 0$ then $\frac{\partial^2\Pi}{\partial\alpha^2} > 0$ for stability $\Big\|$ $\sum_{n=3}^{\infty}\left.\frac{\partial^n\Pi}{\partial\alpha^n}\right|_{\alpha=0}\frac{\alpha^n}{3!} < \frac{\partial^2\Pi}{\partial\alpha^2}\frac{\alpha^2}{2!}$

If $\frac{\partial^2\Pi}{\partial\alpha^2} = 0$ consider next order term; if $\left.\frac{\partial^3\Pi}{\partial\alpha^3}\right|_{\alpha=0} \neq 0$ unstable inflection pt



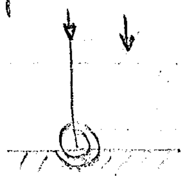
If $\left.\frac{\partial^3\Pi}{\partial\alpha^3}\right|_{\alpha=0} = 0$ then $\left.\frac{\partial^4\Pi}{\partial\alpha^4}\right|_{\alpha=0} > 0$ for stability

if $\frac{\partial^k\Pi}{\partial\alpha^k} = 0 \quad k = 1, \dots, i$

if $i+1$ is an even power then $\frac{\partial^{i+1}\Pi}{\partial\alpha^{i+1}} > 0$ for stability

if $i+1$ is an odd power then $\frac{\partial^{i+1}\Pi}{\partial\alpha^{i+1}} \neq 0$ system is unstable

Application of Analytic Procedure to Problems



$$\Pi = \tfrac{1}{2}S\theta^2 - NL(1-\cos\theta)$$

at equil $\overset{\circ}{N}L\sin\overset{\circ}{\theta} - S\overset{\circ}{\theta} = 0$

at equil $\Pi^0 = \tfrac{1}{2}S\overset{\circ}{\theta}^2 - \overset{\circ}{N}L(1-\cos\overset{\circ}{\theta})$

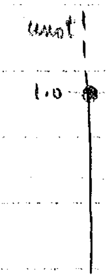
for perturbed state $\Pi(\overset{\circ}{\theta}+\alpha\tilde{\theta},\overset{\circ}{N}) = \tfrac{1}{2}S[\overset{\circ}{\theta}+\alpha\tilde{\theta},\overset{\circ}{N}] - \overset{\circ}{N}L[1-\cos(\overset{\circ}{\theta}+\alpha\tilde{\theta})]$

$$\frac{\partial \Pi}{\partial \alpha} = s(\mathring{\theta}+\alpha\tilde{\theta})\tilde{\theta} - \mathring{N}L \sin(\mathring{\theta}+\alpha\tilde{\theta})\tilde{\theta} \quad @\alpha=0 \quad \frac{\partial \Pi}{\partial \alpha}=0$$

$$\frac{\partial^2 \Pi}{\partial \alpha^2} = s\tilde{\theta}^2 - \mathring{N}L \cos(\mathring{\theta}+\alpha\tilde{\theta})\tilde{\theta}^2 \quad \text{at } \alpha=0 \quad s\tilde{\theta}^2 - \mathring{N}L\cos\mathring{\theta}\,\tilde{\theta}^2$$

if $s - \mathring{N}L\cos\mathring{\theta} > 0$ then $\mathring{\theta} > \arccos \frac{s}{\mathring{N}L}$

if $\mathring{\theta}=0$ $\quad \left.\frac{\partial^2 \Pi}{\partial \alpha^2}\right|_{\alpha=0} \begin{cases} >0 \text{ if } s-\mathring{N}L>0 & \text{stable} \quad N^* = \mathring{N}L/s < 1 \\ <0 \text{ if } s-\mathring{N}L<0 & \text{unstable} \quad N^* = \frac{\mathring{N}L}{s} > 1 \end{cases}$



$@\ \frac{\partial^2 \Pi}{\partial \alpha^2} = 0 \quad N^* = 1$ go to higher order

$$\left.\frac{\partial^3 \Pi}{\partial \alpha^3}\right|_{\alpha=0} = +\mathring{N}L\sin(\mathring{\theta}+\alpha\tilde{\theta})\tilde{\theta}^3\Big|_{\alpha=0} = (\mathring{N}L\sin\mathring{\theta})\tilde{\theta}^3 = 0 \text{ for } \mathring{\theta}=0$$

go to higher order

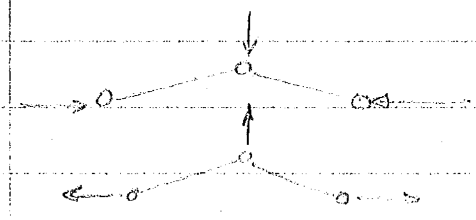
$$\left.\frac{\partial^4 \Pi}{\partial \alpha^4}\right|_{\alpha=0} = \mathring{N}L\cos(\mathring{\theta}+\alpha\tilde{\theta})\tilde{\theta}^4\Big|_{\alpha=0} = \mathring{N}L\tilde{\theta}^4\cos\mathring{\theta} > 0 \text{ for } \mathring{\theta}=0$$

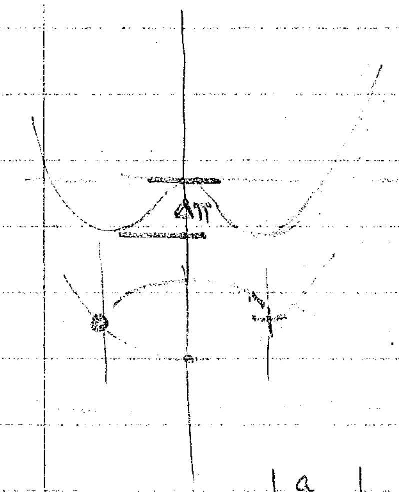
thus $N^*=1$ is a stable pt

HW #2 determine stability of equil state which corresponds to $N = \frac{3}{2}\frac{S}{L}$ and corresponding θ

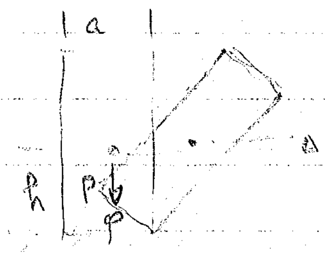
Determine stability of equil branch for $0 < \theta < \pi$

Using analytic criterion





to move from • to + one must add energy
to system this amount of energy is energy barrier
$\Delta\Pi$



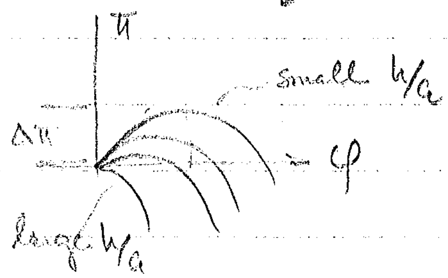
$$\Delta = h\cos\varphi + \frac{a}{2}\sin\varphi - h$$

$$\Pi = \underset{\substack{\downarrow \\ 0}}{\underline{U}} - W = -(-P\Delta) = P\Delta$$

$$\Pi = Ph\cos\varphi + \frac{Pa}{2}\sin\varphi - Ph$$

$$\frac{d\Pi}{d\varphi} = -Ph\sin\varphi + \frac{Pa}{2}\cos\varphi = 0$$

$$0 \le \varphi \le \pi/2$$



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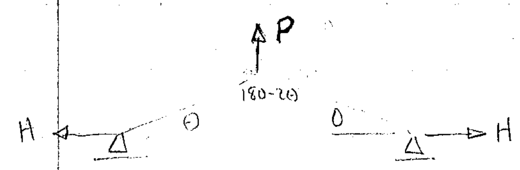
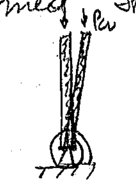


Adjacent "Equilibrium" Argument

Determine the smallest load P for which in addition to the undeformed state $\exists$ an adjacent equilibrium state for same load



Equil Eq

$$NL\sin\theta - S\theta = 0$$

Assume $N = \mathring{N}$ $\quad \theta = \mathring{\theta} + \tilde{\theta}$

both at equil $N = \mathring{N}$ $\quad \theta = \mathring{\theta}$

$$\mathring{N}L\sin\mathring{\theta} - S\mathring{\theta} = 0$$

$$\mathring{N}L\sin(\mathring{\theta} + \tilde{\theta}) - S(\mathring{\theta} + \tilde{\theta}) = 0$$

$$\mathring{N}L(\sin\mathring{\theta}\underbrace{\cos\tilde{\theta}}_{1} + \underbrace{\sin\tilde{\theta}}_{\tilde{\theta}}\cos\mathring{\theta}) - S\mathring{\theta} - S\tilde{\theta} = 0$$

$$\underline{\mathring{N}L\sin\mathring{\theta}} + \tilde{\theta}\mathring{N}L\cos\mathring{\theta} - \underline{S\mathring{\theta}} - S\tilde{\theta} = 0$$

$$\tilde{\theta}\left[\mathring{N}L\cos\mathring{\theta} - S\right] = 0$$ linearized EVP for buckling

$\cos\mathring{\theta} = \dfrac{S}{\mathring{N}L}$ $\quad \tilde{\theta} \neq 0$

eq is satisfied for $\tilde{\theta} = 0$ trivial case

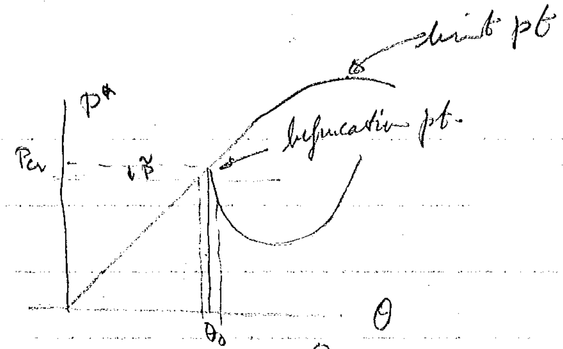
If $\tilde{\theta} \neq 0$ $\quad \mathring{N}L\cos\mathring{\theta} - S = 0$

Equivalent to determinant $= 0$ for EVP involving Diff Eq.

for undeformed state of eq $\mathring{\theta} = 0$ by def.

$$\therefore \quad \mathring{N}L - S = 0 \qquad \mathring{N} = \frac{S}{L}$$

Given



$$\left\{\begin{array}{ll} N = \overset{\circ}{N} = N_{cr} & \theta = \overset{\circ}{\theta} \\ N = \overset{\circ}{N} + \tilde{N} & \theta = \overset{\circ}{\theta} + \tilde{\theta} \end{array}\right.$$

$f(\overset{\circ}{\theta}, \overset{\circ}{N}) = 0$ Eq Equation

$f(\overset{\circ}{\theta} + \tilde{\theta}, \overset{\circ}{N} + \tilde{N}) = 0$ Equil Eq

$\rightarrow (\quad)\tilde{\theta} = (\quad)\tilde{P}$

Now look for nonuniqueness of $\tilde{\theta}$.

In the previous problem criterion for det of $\overset{\circ}{N}$ the existence of nonzero $\tilde{\theta}$, i.e. det of coeff $= 0$.

$a_{11}\tilde{\theta}_1 + a_{12}\tilde{\theta}_2 = 0$
$a_{21}\tilde{\theta}_1 + a_{22}\tilde{\theta}_2 = 0$

in non uniqueness
Fredholm alternative condition

$$\left\{\begin{array}{l} \text{det } \cancel{\text{of}} \text{ denom} = 0 \\ \text{det of num} = 0 \end{array}\right.$$

$a_{11}\tilde{\theta}_1 + a_{12}\tilde{\theta}_2 = b_1\tilde{P}$

$a_{21}\tilde{\theta}_1 + a_{22}\tilde{\theta}_2 = b_2\tilde{P}$

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$ for non uniq

$\tilde{P}\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = 0$ for consistency

$\tilde{P}\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = 0$

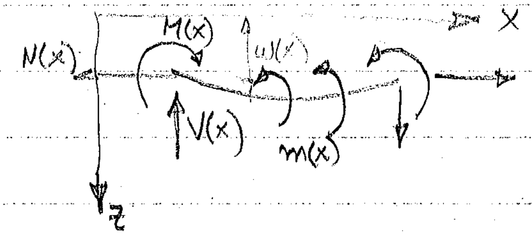
Note that both criterion, also correspond to completely different physical argument lead to the same buckling load.

J of Applied Mech. - 1969 Prof Kerr & discussion of Adjacent Equil method.

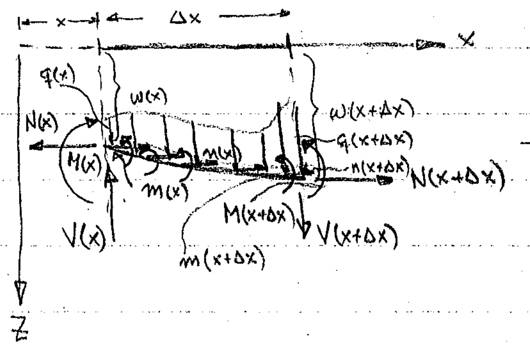
Derivation of Diff Equat for beam in $w(x)$ $\quad w(x) = \overset{\circ}{w}(x) + \tilde{w}(x)$

$\overset{\circ}{w}(x) = 0$ for perfectly straight beams.

Assume $M(x) = -EI \dfrac{d^2 w}{dx^2}$



Derivation of Equil Equations



$= \dfrac{dw}{dx}\Delta x + O(\Delta x^2)$

in modern track analysis horizontal buckling of tracks will have moments located at a joint of tie & track. if ties are close together then we can assume moment is continuous

taylor expansion

$\sum F_z = 0 \qquad \cancel{V(x)} - \underset{\cancel{V(x)} + \frac{dV}{dx}\Delta x + \dots}{V(x+\Delta x)} - q(x)\Delta x \underbrace{- \frac{1}{2}\frac{dq}{dx}\Delta x \Delta x}_{O(\Delta x^2)} = 0$

$$-\frac{dV}{dx} - q(x) + O(\Delta x) = 0 \qquad \text{if } \Delta x \neq 0$$

$$\lim_{\Delta x \to 0} \qquad \boxed{\frac{dV}{dx} = -q(x)} \quad (1)$$

$$\sum F_x = 0 \qquad N(x) - \left\{N(x) + \frac{dN}{dx}\Delta x + O(\Delta x^2)\right\} - \left\{m(x)\Delta x + O(\Delta x^2)\right\} = 0$$

$\boxed{\dfrac{dN}{dx} = -m(x)}$ .. if $\Delta x \neq 0$ the take lim

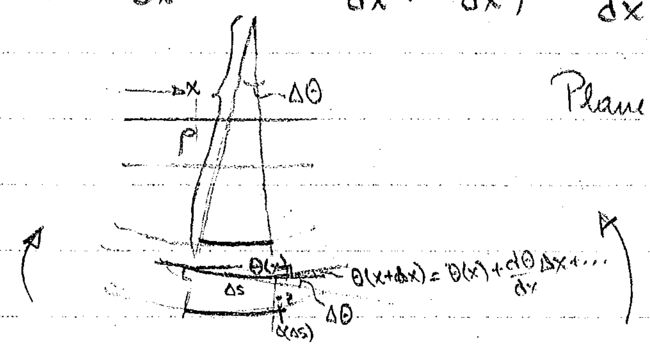
$$\sum M\Big|_{x+\Delta x} = 0 \qquad \cancel{M(x)} - \left\{\cancel{M(x)} + \frac{dM}{dx}\Delta x + O(\Delta x^2)\right\} + V(x)\Delta x$$

$$- N(x)\left\{\frac{dw}{dx}\Delta x + O(\Delta x^2)\right\} - q(x)\frac{\Delta x^2}{2} + O(\Delta x^3)$$

$$\sim + m(x)\Delta x\left\{\frac{dw}{dx}\frac{\Delta x}{2} + O(\Delta x^2)\right\} - \left\{m(x)\Delta x + O(\Delta x^2)\right\} = 0$$

$$\boxed{-\frac{dM}{dx} + V(x) - N(x)\frac{dw}{dx} - m(x) = 0} \quad (3)$$

$$\frac{d}{dx}(3) = -\frac{d^2M}{dx^2} - q(x) - \frac{d}{dx}\left(N\frac{dw}{dx}\right) - \frac{dm(x)}{dx} = 0 \quad (3')$$

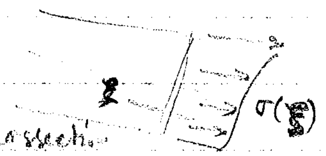


Plane Section Hypothesis

$$\frac{\rho}{\Delta s} = \frac{z}{\Delta(\Delta s)} \longrightarrow \frac{\Delta(\Delta s)}{\Delta s} = \frac{z}{\rho}$$

$$\epsilon = \frac{\sigma}{E} \qquad \sigma = \frac{E}{\rho}z$$

$M(x)$ is obtained



$$M(x) = \iint_A \sigma(\xi)\,\xi\,dA = \frac{E}{\rho}\iint_A \xi^2\,dA = \frac{EI}{\rho}$$

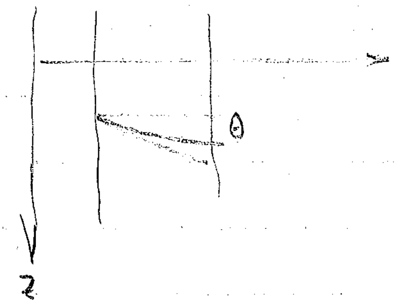
Assume E = const along cross section

determination of $\frac{1}{\rho}$ curvature

$$\rho\Delta\theta = \Delta s \qquad \frac{1}{\rho} = \lim_{\Delta x\to 0}\frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}$$

For assumed sign convention of moments $\Delta\theta < 0$. Since $\rho > 0$ $\frac{1}{\rho} > 0$ and since $\Delta s > 0$ it follows that $\frac{1}{\rho} = -\frac{d\theta}{ds}$

θ in our coordinate syste



$$\tan\theta = \frac{dw}{dx}$$

$$\theta = \arctan\frac{dw}{dx}$$

$$\frac{1}{\rho} = -\frac{d\theta}{ds} = -\frac{d\theta}{dx}\,\frac{dx}{ds} = -\frac{1}{1+\left(\frac{dw}{dx}\right)^2}\cdot\frac{d^2w}{dx^2}\,\frac{\cancel{dx}}{\sqrt{1+\left(\frac{dw}{dx}\right)^2}\,\cancel{dx}}$$

$$= \frac{-\frac{d^2w}{dx^2}}{\sqrt{\left(1+\left[\frac{dw}{dx}\right]^2\right)^3}}$$

$$\boxed{\text{if } \frac{dw}{dx} \ll 1 \quad \text{then} \quad \frac{1}{\rho} \approx -\frac{d^2w}{dx^2}}$$

Go to (3') then since $M = -EI\,w''(x)$

$$\boxed{\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - \frac{d}{dx}\left(N(x)\frac{dw}{dx}\right) - \frac{dm}{dx} = q(x)}$$

4^{th} O.D.E.

Note that when $m(x)$ = constant then $dm/dx = 0$

from (2)

$$N(x) = -\int n(x)\,dx + C$$

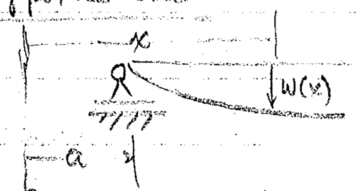
or $$N(x) = -\int_a^x n(\xi)\,d\xi + N(a)$$

Bdy conditions: 4 bc are needed

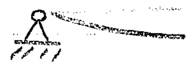
note from (3) $\rightarrow$ $V(x) = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) + N(x)\frac{dw}{dx} + m$

Simply supported end

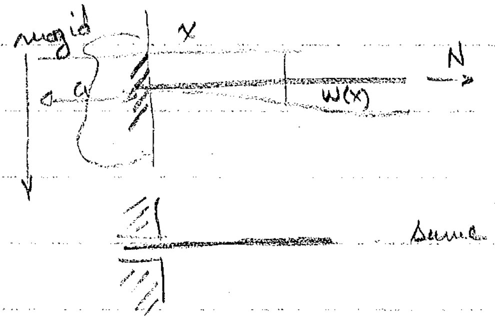
$w(a)=0$ (1)



$M(a)=0 \Rightarrow w''(a)=0$

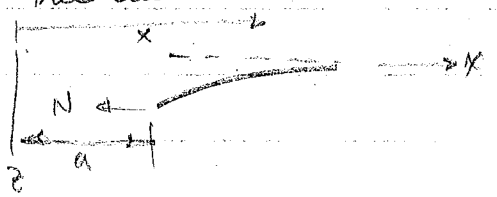


Fixed End



$w(a)=0$
$w'(a)=0$

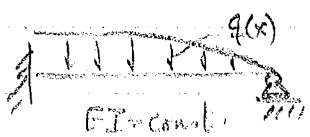
Free end



$M=0 \qquad w''(a)=0$

$V=0 \qquad -\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]+N(x)\frac{dw}{dx}+m(x)\Big|_a=0$

Example



Find $w(x)$

diff eq $\quad EI\, w^{IV} = \underbrace{q(x)}_{}$ given fn.

B.C. $\quad w(0)=0 \quad w'(0)=0 \quad w(l)=0 \quad w''(l)=0$

$$w(x) = \underbrace{w_0}_{\text{particular}} + \underbrace{A_1\varphi_1(x) + A_2\varphi_2(x) + A_3\varphi_3(x) + A_4\varphi_4(x)}_{w_{\text{homogeneous}}}$$

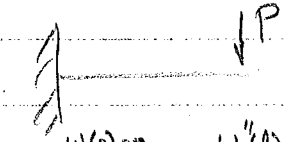
Determination of $A_1 \to A_4$. Subst $w(x)$ into 4 B.C.

4 linear non homogeneous algebraic eqns for $A_1 \to A_4$.

$$a_{11}A_1 + a_{12}A_2 + a_{13}A_3 + a_{14}A_4 = b_1$$
$$a_{21}A_1 + \dots \qquad = b_2$$
$$a_{31}A_1 + \dots \qquad = b_3$$
$$a_{41}A_1 + \dots \qquad + a_{44}A_4 = b_4$$

Use cramer's rule to solve $A_i \implies w(x)$

if det of matrix $\neq 0$ solution unique.

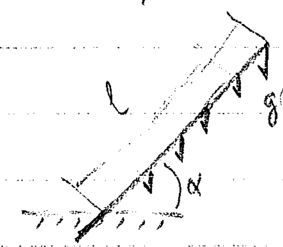


$w(0) = 0 \qquad w''(l) = 0$

$w'(0) = 0 \qquad V(l) = P = -w'''(l)$

stress anal problems are non homogeneous B.V.P.

with a unique expression for $w(x)$.

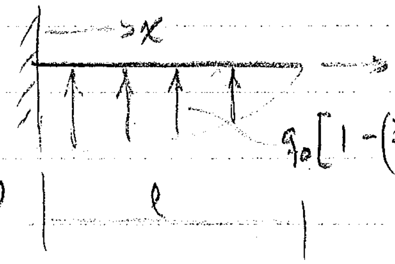


q = const/unit length

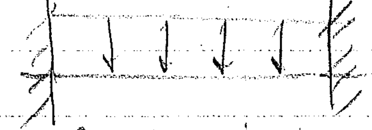
EI const.

Formulate problem for $w(x)$ D.E. & B.C.'s

Formulate problem find $w(x)$ at $x = l$



$q_0\left[1 - \left(\frac{x}{l}\right)^2\right]$

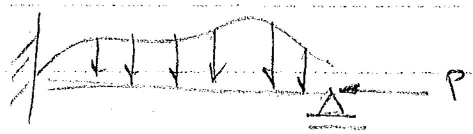


Find reactions

$w(x) = \underbrace{\mathring{w}(x)}_{\text{from fund state of equil}} + \underbrace{\tilde{w}(x)}_{\text{perturbation}}$

$$\frac{d^2\omega}{dx^2}\left(EI\frac{d^2\omega}{dx^2}\right) - \frac{d}{dx}\left(N\frac{d\omega}{dx}\right) - \frac{dm}{dx} = q(x) \qquad \mathrm{I}$$

3-8-73



Stress anal proble for $P \neq P_{cr}$

$$(EI\,w'')'' + P\,w'' = q$$

$$\omega = \omega_0 + \sum_{n=1}^{\infty} A_n \varphi_n(x)$$

4 be will give system of lin algebraic Eq

$$\sum a_{ij} x_j = b_i$$

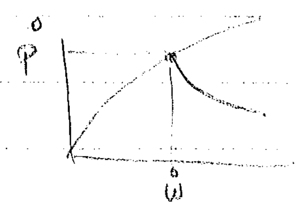
According to cramers rule $A_i = \dfrac{|\;\;|}{\Delta = |\;\;|}$

if $\Delta = 0$ & num $= 0$ is EV proble.

(I) m = const or zero $\frac{dm}{dx} = 0$

Adjacent Equil method

(I) $\rightarrow$ $Lw = q$ L lin op



$w = \mathring{w} + \alpha\tilde{w}$

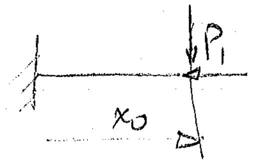
$Lw = L\mathring{w} + \alpha L\tilde{w} = q$

at bifurcation pts $L\mathring{w} = q$ $\alpha \neq 0$ $L\tilde{w} = 0$

BC.

$M_n w\big|_{x_0} = \gamma$ $\quad$ M_n lin op. $\quad$ $\gamma = \text{const}$

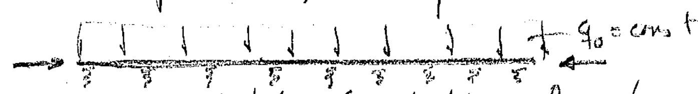
$[-EI(\ )''' + P(\ )']\,w(l) = \underbrace{P}_{\gamma}$



$M_n(\overset{\circ}{w} + \alpha\tilde{w}) = \gamma$

at bef $M_n\overset{\circ}{w} = \gamma$ $\quad \therefore \quad M_n\tilde{w} = 0$ homog condition

Example to show importance of retaining $\overset{\circ}{w}(x)$ term.



Winkler foundation $\overset{\circ}{w} = kw$

perturbation eq. for buckling $\boxed{L\tilde{w} = 0}$

For beam problems $L\tilde{w} = 0$ $\quad \frac{d^2}{dx^2}\left(EI\frac{d^2\tilde{w}}{dx^2}\right) - \frac{d}{dx}\left(N\frac{d\tilde{w}}{dx}\right) = 0$

Simply supported $\tilde{w}\big|_{x_0} = 0$

$\tilde{w}_{,xx}\big|_{x_0} = 0$

note in this EV problem the fn is not the deflection but the perturbation (as small as you please $\neq 0$) from the zero state

$$\frac{d^2}{dx^2}\left(EI\frac{d^2\tilde{w}}{dx^2}\right) - \frac{d}{dx}\left(N\frac{d\tilde{w}}{dx}\right) = 0 \quad \& \ 4 \text{ hom BC in } \tilde{w}$$

$$\tilde{w} = \sum_{n=1}^{4} A_n\varphi_n(x) \qquad \varphi_n \text{ are lin indep}$$

4 hom b.c.

$$\sum_{j=1}^{4} a_{ij}X_j = 0$$

Stability Criterion (according to the adjacent equil method)

Critical Pt on an equil branch is characterized by the existence of an adjacent state of equilib for $\overset{\circ}{P}$. That means that $\overset{\circ}{P}$ & $\overset{\circ}{w}$, $\overset{\circ}{P}$ & $\overset{\circ}{w}+\tilde{w}$ corresponding to equil states.

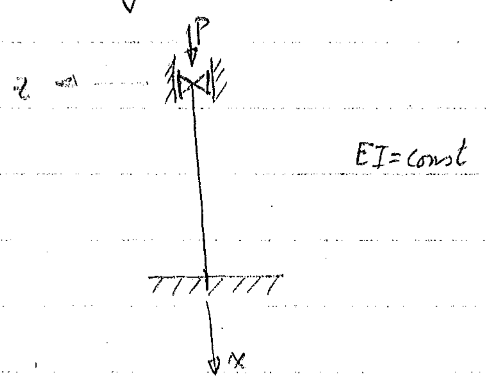
one solution $\underset{\sim}{X} = (x_1, x_2, x_3, x_4) = 0$ trivial sol

above criterion is equiv to existence of nonzero $\tilde{w}$.

Condition $\det|a_{ij}| = 0$

This approach only determines bifurcation pt only & nothing Else

EXAMPLES



Find $\overset{\circ}{P} = P_{cr}$ $\qquad \overset{\circ}{w}(x) \equiv 0$

$$EI\,\tilde{w}^{IV} + N\,\tilde{w}^{II} = 0 \quad 0 \le x \le l \quad N = P = \text{const}$$

$\tilde{w}(0) = 0 \qquad \tilde{w}(l) = 0$

$\tilde{w}_{xx}(0) = 0 \qquad \tilde{w}'(l) = 0$

ODE with CC $\qquad$ assume Ae^{nx}

$$n^2(n^2 + \lambda^2) = 0 \qquad n = 0,\ n = 0,\ \pm i\lambda$$

$$\tilde{w}(x) = A_1 + A_2 x + A_3 \cos\lambda x + A_4 \sin\lambda x$$

BC

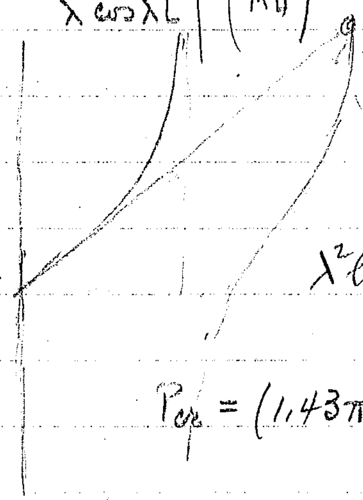
$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -\lambda^2 & 0 \\ 1 & L & \cos\lambda L & \sin\lambda L \\ 0 & 1 & -\lambda\sin\lambda L & \lambda\cos\lambda L \end{vmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = 0$$

$A_3 = 0 \quad A_1 = 0$

$$\begin{vmatrix} L & \sin\lambda L \\ 1 & \lambda\cos\lambda L \end{vmatrix} \begin{pmatrix} A_2 \\ A_4 \end{pmatrix} = 0$$

$\lambda L\cos\lambda L - \sin\lambda L = 0$

$\tan\lambda L = \lambda L$



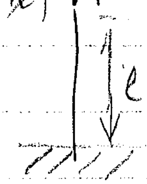
Nontrivial Solution

$$\lambda^2 \ell^2 = \frac{P\ell^2}{EI} = (1.43\pi)^2$$

$$P_{cr} = (1.43\pi)\frac{EI}{\ell^2}$$

HW #4 Get euler buckling load for

x P ℓ



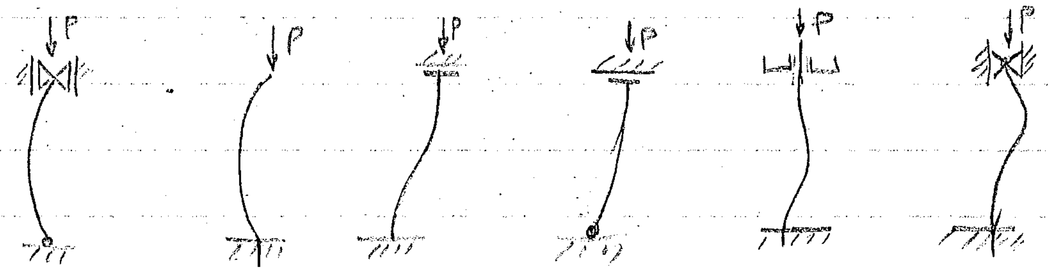
$EI = \text{const}$.

$w(0) = 0 \qquad w''(\ell) = 0$

$w'(0) = 0 \qquad w'''(\ell) = 0$

Tabulation of results

$$P_{cr} = \beta^2 \frac{EI}{\ell^2}$$



$\beta = \pi$ $\quad \beta = \pi/2$ $\quad \beta = \pi$ $\quad \beta = \pi/2$ $\quad \beta = 2\pi$ $\quad 1.43\pi$

Effect of Constraints of P_{cr}

1. The stronger the constraint the larger P critical
2. If constraint released P_{cr} will drop.

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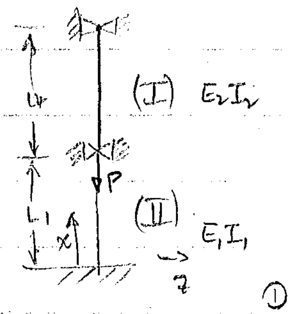


$$E_1 I_1 \tilde{w}_I^{IV} + P \tilde{w}_I'' = 0 \qquad 0 \le x \le L_1$$

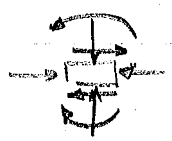
$$E_2 I_2 \tilde{w}_2^{IV} = 0 \qquad L_1 \le x \le L_1 + L_2$$

① $\tilde{w}(0) = 0$ ② $\tilde{w}'(0) = 0$

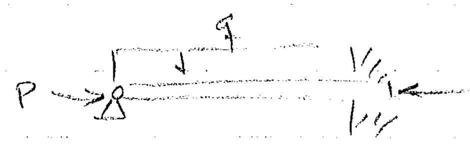
③ $\tilde{w}(l_1 + l_2) = 0$ ④ $\tilde{w}''(l_1 + l_2) = 0$

⑤, ⑥ $\tilde{w}_1(l_1) = \tilde{w}_2(l_1) = 0$

from plane sections remain same ⑦ $\tilde{w}_1'(l_1) = \tilde{w}_2'(l_1)$



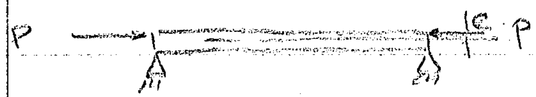
$$M_I(l_1) - M_{II}(l_1) = 0$$



$$EI w^{IV} + P w'' = q$$

$w(0) = 0 \qquad w(l) = 0$

$w''(0) \qquad w'(l) = 0$



$EIw^{IV} + Pw''' = 0$

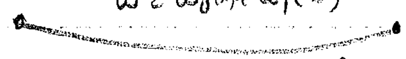
$w(0) = 0 \qquad w(\ell) = 0$

$-EIw^{IV}(0) = Pe \qquad -EIw''(0) = Pe$

all cases have imperfection & are stress analysis problems.

not EVP

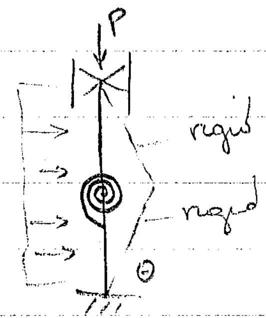
$w = w_0(x) + w_1(x)$

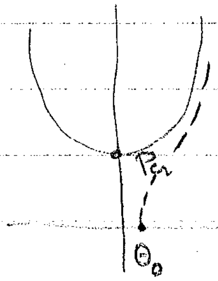


$EIw_1^{IV} + Pw_1'' = f(x)$

B.C. same as before.

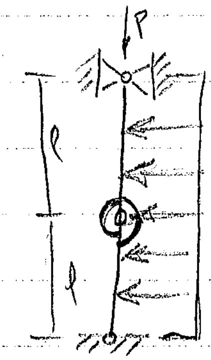
In ideal case





θ_0 when $P = 0$

HW.



for given model

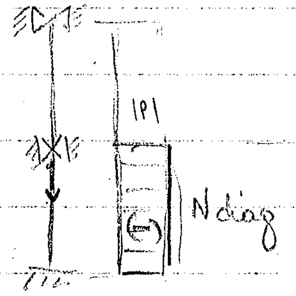
① Obtain Equil Eq using equil & energy

② Plot Equil branches $-\pi \le \theta \le \pi$

nondim of $q^* = \frac{q\ell^2}{6S}$

③ linearize equil eq wrt θ & determine P_{cr} from cond that def $\to \infty$ & plot branches in dashed line

④ determine if equil config on hand for $q^* = .1$ is stable at $P = P_{cr}$

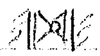


for $P < P_{cr}$ $\quad M \equiv 0$

$V \equiv 0$

close determination of P_{upper} & P_{lower}

1) If $E_{II} I_{II} \to 0$ $\quad P_{lower}$



2) If $E_{II} I_{II} \to \infty$ $\quad P_{upper}$

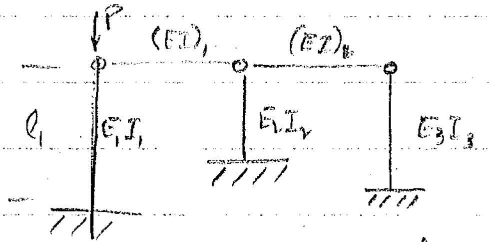
$1.43\pi < \beta < 2\pi$ $\qquad P/E_1 I_1 = \lambda^2$

Solution : $W_1(x) = A_1 + A_2 x + A_3 \cos \lambda x + A_4 \sin(\lambda x)$

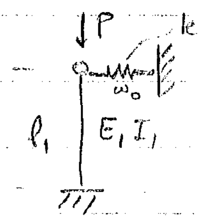
$W_2(x) = A_5 + A_6 x + A_7 x^2 + A_8 x^3$

Subst b.c. $\Rightarrow$ system of algebraic eq $\Rightarrow$ det $= 0$

$P_{cr} = \beta_{cr}^2 \dfrac{E_I I_I}{L_1^2}$

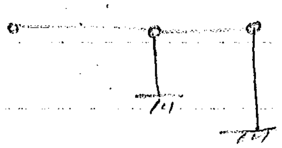


$\sim$



$k = X$ for $w_0 = 1$ $\qquad \bar{X} = k w_0$

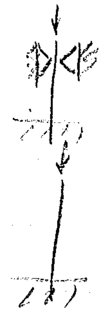
Find value of $\bar{X}$ for which $w_0 = 1$



If design is fixed & one has to determine if $P = 1$ TON is a permissible load, the deter of P_{lower} is sufficient if $P = 1$ ton $< P_l$

if $P > P_L$ the check if $P \gtrless P_u$

$P > P_l$ $\begin{cases} \text{if } P > P_u & \text{buckling} \\ \text{if } P < P_u & \text{the proceed with exact analysis} \end{cases}$

for upper bound  $1.43\,\pi$

for lower bound $\pi/2$

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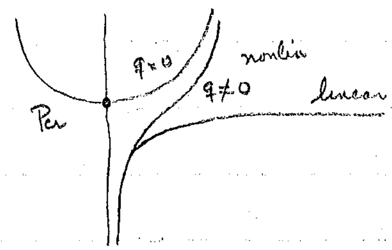
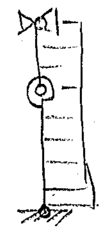
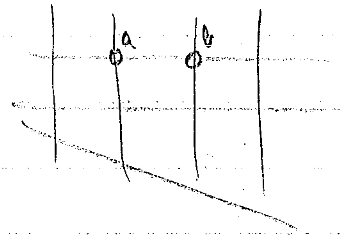
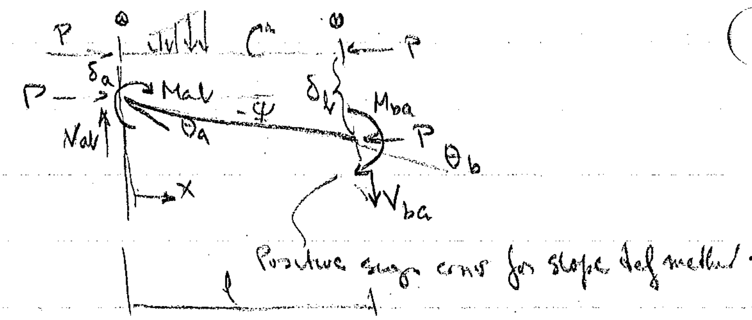




For a structure of more than one beam use of slope deflection method





DE $\quad w^{IV} + \lambda^2 w'' = q(x) \qquad EI = \text{const}$ & $N = \text{const}$

BC $\quad w(0) = \delta_a \qquad w(l) = \delta_b$

$\qquad w'(a) = \theta_a \qquad w'(l) = \theta_b$

$w(x) = w_0(x) + A_1 + A_2 x + A_3 \cos\lambda x + A_4 \sin\lambda x$ subs into BC. 4 nonhom alg eq

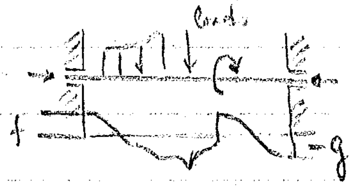
$M_{ab} = -IE \left.\dfrac{d^2w}{dx^2}\right|_{x=0} = S_1 k\theta_a + S_2 K\theta_b + (S_1+S_2)K\dfrac{\delta_a-\delta_b}{l} + f(w_0(0), w_0'(0), w_0''(\ldots)$

convention is satisfied $\quad M_{ba} = EI \left.\dfrac{d^2w}{dx^2}\right|_{x=l} = S_2 K\theta_a + S_1 K\theta_b + (S_1+S_2)K\dfrac{\delta_a-\delta_b}{l} + g(w_0(0), w_0'(0), w_0(l), w_0'(l), w_0''(\ldots)$; $w_0(l), w_0'(l)$

$$\boxed{S_1 = \frac{\dfrac{1-\beta\cot\beta}{2\tan(\beta/2)}}{\beta} - 1} \qquad \boxed{S_2 = \frac{\beta\csc\beta - 1}{\dfrac{2\tan(\beta/2)}{\beta} - 1}} \qquad \beta = \lambda l = \sqrt{\frac{Pl^2}{EI}}\ ;\ K = \frac{EI}{l}$$

Identification of f & g note that when $\theta_a = \theta_b = \delta_a - \delta_b = 0$

then $M_{ab} = f$ $\quad M_{ba} = g$



Let $f = M^{\circ}_{ab}$

$g = M^{\circ}_{ba}$

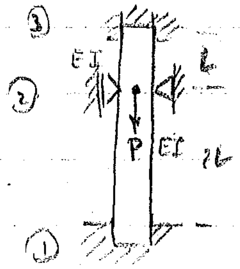
Special case $P = 0$ $\quad \beta = 0$ $\quad \lim_{\beta \to 0} S_1 = 4$ $\quad \lim_{\beta \to 0} S_2 = 2$

$$\boxed{\begin{aligned} M_{ab} &= 4K\theta_a + 2K\theta_b + 6K\frac{\delta_a - \delta_b}{l} + M^{\circ}_{ab} \\ M_{ba} &= 2K\theta_a + 4K\theta_b + 6K\frac{\delta_a - \delta_b}{l} + M^{\circ}_{ba} \end{aligned}} \quad (I)$$

Solution of stability problems

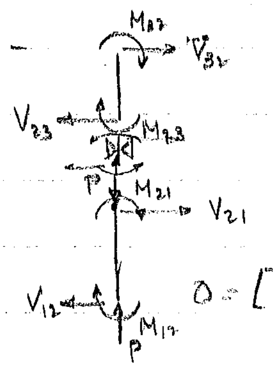
$q = 0 \implies M^{\circ}_{ab} = 0 \quad M^{\circ}_{ba} = 0$

Example



1) DE approach 2 DE's 8 BC's

2) S-D approach



Only 1 unknown θ_2

one eqn needed

$$\overset{M_{ba}}{M_{21}} + \overset{M_{ab}}{M_{23}} = 0$$

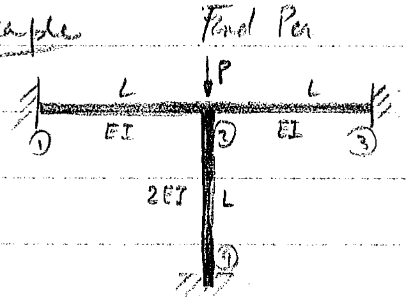
$$0 = [S_2K]_{21}\,\theta_1^{\,0} + \underbrace{[S_1K]_{21}\,\theta_2}_{M_{21}} + 0 + \underbrace{[S_1K]_{23}\,\theta_2}_{M_{23}} + 0 + 0$$

$$\therefore \quad \{[S_1K]_{21} + [S_1K]_{23}\}\underset{\neq 0}{\theta_2} = 0 \implies \text{stability } [S_1K]_{21} + [S_1K]_{23} = 0$$

$$(S_1)_{21}\frac{EI}{2L} + (4)\frac{EI}{L} = 0 \qquad (S_1)_{21} = -8 \quad \text{from tables } \beta_{21} = +5.7$$

since $\beta = \sqrt{\frac{Pl^2}{EI}}$ where $l = 2L$, $\quad (5.7)^2 = \frac{4L^2}{EI}P_{cr}$

Example Find P_{cr}

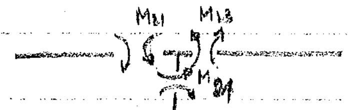


For $P < P_{cr}$

$P_{24} = P \qquad M \equiv 0 \qquad V \equiv 0$

$P_{12} = P_{23} = 0$

For $P = P_{cr}$ 1 unknown θ_2 1 eq is needed



$M_{21} + M_{23} + M_{24} = 0$

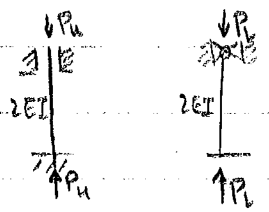
DE approach 3 diff eq
12 b.c.

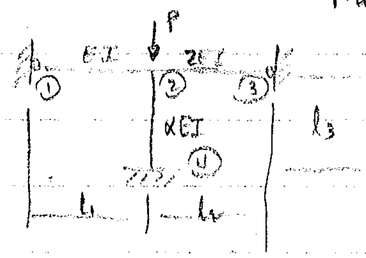
$$\underbrace{[S_1K]_{21}\,\theta_2}_{M_{21}} + \underbrace{[S_1K]_{23}\,\theta_2}_{M_{23}} + \underbrace{[S_1K]_{24}\,\theta_2}_{M_{24}} = 0$$

$$\left(4\frac{EI}{L} + 4\frac{EI}{L} + \frac{2EI}{L}\beta_{24}\right)\theta_2 = 0 \qquad \theta_2 \neq 0$$

$-4 = S_1 \qquad \beta_{24} = 5.3 \qquad P_{cr} = (5.3)^2\dfrac{2EI}{L^2}$

bound for P_{cr}



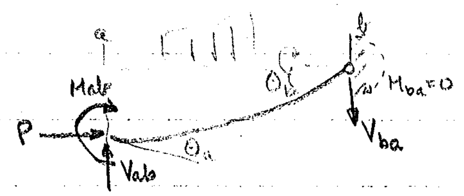


$P = P_{cr}$ 3 unknowns $\theta_1, \theta_2, \theta_3$

3 eqs are needed

$M_{12} = 0 \qquad M_{21} + M_{23} + M_{24} = 0 \qquad M_{32} = 0$

Modification of M_{ij} equation to take into consideration hinges on opposite sides without intro new variables.



$$M_{ba} = 0 = [S_2K]\theta_a + [S_1K]\theta_b + (S_1+S_2)K\frac{\delta_a-\delta_b}{l} + M^o_{ba}$$

$$\theta_b = -\frac{[S_2K]\theta_a + (S_1+S_2)K\frac{\delta_a-\delta_b}{l} + M^o_{ba}}{S_1K}$$

$$M_{ab} = [S_1K]\theta_a - \frac{S_2}{S_1}\left[(S_2K)\theta_a + (S_1+S_2)K\frac{\delta_a-\delta_b}{l} + M^o_{ba}\right] + (S_1+S_2)K\frac{\delta_a-\delta_b}{l} + M^o_{ab}$$

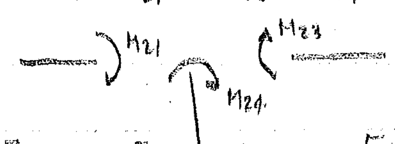
$$M_{ab} = \underbrace{\left(S_1 - \frac{S_2^2}{S_1}\right)}_{S_3} K\theta_a + \underbrace{(S_1+S_2)\left(1 - \frac{S_2}{S_1}\right)}_{S_3} K \frac{\delta_a - \delta_b}{l} - \frac{S_2}{S_1} K M_{ba}^0 + M_{ab}^0$$

$$\left.\boxed{\begin{aligned} M_{ab} &= S_3 K \theta_a + S_3 K \frac{\delta_a - \delta_b}{l} - \frac{S_2}{S_1} K M_{ba}^0 + M_{ab}^0 \\ M_{ba} &= 0 \qquad M_{ab} \end{aligned}}\right\} II$$

$$\boxed{S_3 = \frac{\beta^2}{1 - \beta \cot\beta}} \qquad \lim_{\beta \to 0} S_3 = 3$$

If use is made of eqs (II) then only 1 unknown θ_2 the one equ needed is

$$M_{21} + M_{23} + M_{24} = 0$$



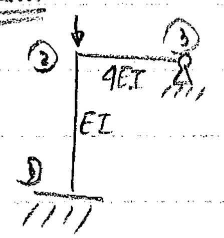
$$\underbrace{[S_3 K]_{21}}_{M_{21}} \theta_2 + \underbrace{[S_3 K]_{23}}_{M_{23}} \theta_2 + \underbrace{[S_1 K]_{24}}_{M_{24}} \theta_2 = 0 \qquad \text{Assume } l_i = l$$

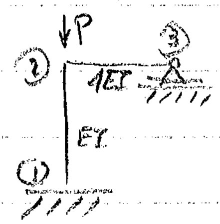
$$3\frac{EI}{l}\theta_2 + 3 \cdot \frac{2EI}{l}\theta_2 + S_{1_{24}} \frac{4EI}{l}\theta_2 = 0 \qquad \text{let } \alpha = 4$$

$$S_1 = \frac{-9}{4} = -2.25 \qquad \beta_{24} \cong 5.1$$

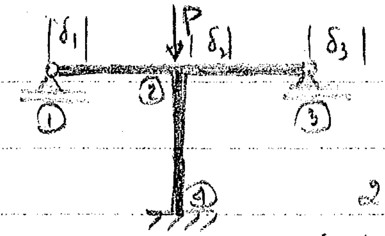
$$P_{cr} = (5.1)^2 \cdot \frac{4EI}{l^2} \qquad \underset{\text{rigid}}{2\pi} < \beta < \underset{\text{string}}{1.43\pi}$$

problem HW #6





Find $P_{critical}$ for both case compare effect of support change.



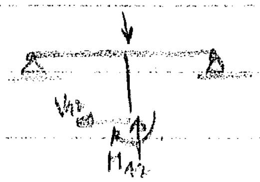
$\delta_1 = \delta_2 = \delta_3 = \delta$

2 unknowns If 2 eqs are used $\theta_2 \neq \delta$ using II

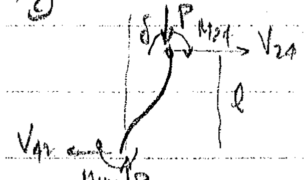
(If (I) is used 4 unknowns ie $\theta_1, \theta_2, \theta_3, \delta$)

① $M_{21} + M_{23} + M_{24} = 0$

to obtain second eq



$V_{42} = 0$ ②



$\Sigma M_2 = 0 \quad V_{42}L + M_{42} + M_{24} + P\delta = 0$

$V_{42} = -\frac{M_{42} - M_{24} + P\delta}{L}$

Express V_{42} in terms of M_{ij}

out of 2 follows $M_{42} + M_{24} + P\delta = 0$ (2')

$a_{11}\theta_a + a_{12}\delta = 0$

$a_{21}\theta_a + a_{22}\delta = 0$

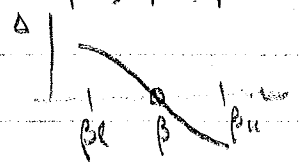
$\Rightarrow \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \Rightarrow \beta_{24}$

$f([S_i]_{24}) = 0$ ① estimate upper lower bound

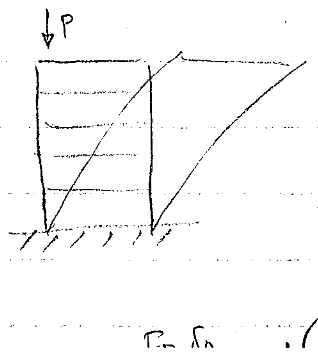
choose 3 or 4 pts in interval of $\beta_l < \beta < \beta_u$

determine δ for $\beta_1, \beta_2, \beta_3, \beta_4$

plot



April 12, 1973



unknown: 8 θ's 4 δ's 12 unknowns

8 eq $\Rightarrow$ from $\Sigma M_{i\,joints} = 0 \quad i = 1-8$

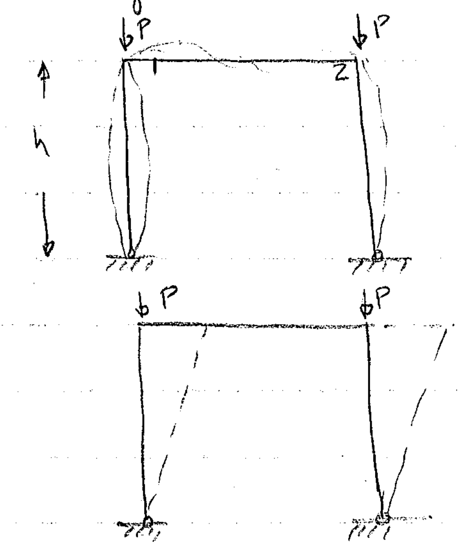
$V_{01} - V_{10} = 0 \quad$ 4 cuts – 4 eq

Final results:

$$\begin{bmatrix} a_{11} & \cdots & a_{1,12} \\ \vdots & & \vdots \\ a_{12,1} & \cdots & a_{12,12} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_8 \\ \delta_1 \\ \vdots \\ \delta_4 \end{bmatrix} = 0$$

Stress Analysis Problem: you can for instance calculate for P how much it contracts & from this you can calculate moment diagram throughout

Assignment



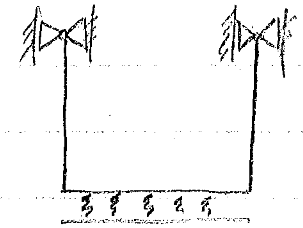
$\theta_1 = -\theta_2$ Symmetric case 1 unknown

$P_{cr} = 12.9 \frac{EI}{h^2}$ unstable

Asymmetric case 2 - unknown

θ, δ $P_{cr} = 1.8 \frac{EI}{h^2}$

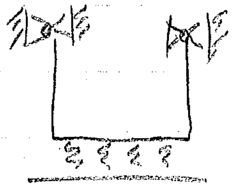
Note:



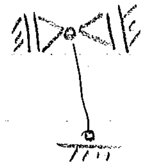
You can not use SD Method as derived since the winkler foundation introduces additional stiffness

S.D.M. would have to be derived from $W^{IV} + \lambda^2 W'' + \frac{K}{EI} W = 0$

To solve this either D.E. approach or get $P_{L\,min}$ & check to see if $P_L < P = 10$ tons If it is the structure is safe

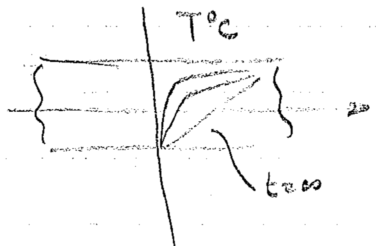


$P_L \Rightarrow$

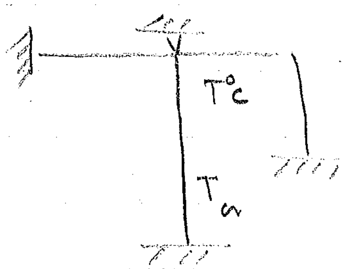


$$P_L = \pi^2 \frac{EI}{L^2}$$

Because of



corresponds to heat transfer through the bar, as a fn of time
therefore if you increase T 2x when cooking (frying) meat
you cannot make the cook faster since it is function
of time you may burn the outer layer



what is T at which the structure
will buckle out

Buckling due to thermal forces

$$\Delta l = \frac{\alpha T^\circ l}{E(1)} = \frac{\sigma l}{E}$$

$$T_{cr} = \frac{P_{cr}}{\alpha A E}$$

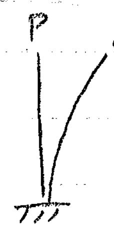
Determine P_{cr} & then use

Note: For P_{mech}

when P_{cr}, after large deformation
$P_{thermal}$ after P_{cr} is reached - relief of
axial force lead to very small deformation

Rd P48-51 Influence of shear on buckling we have so far neglected shear effects on buckling which gives you higher P_{cr}

Pd P 107 111-115



P nonconservative you cannot use Euler Anal for P buckling

P here is much higher (10x) than for Euler case

$I = Ar^2$ (rad of gyration) I section

l/r	20	50	100	150
P_{cr}'/P_{cr}	.995		.997	.998

slender bar ($l/r = 100$)

l/r slenderness ratio

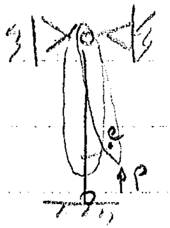
$\nu = .3$

P_{cr} = effect of shear forces is neglected

P_{cr}' = " ——— considered

Conclusion: Effect of neglecting deformation due to shear is negligible

Example



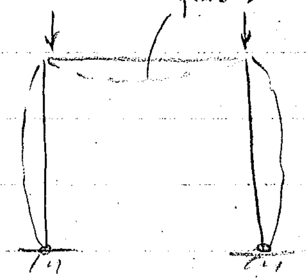
bar is in tension however due to Pe, there is moment which will buckle beam

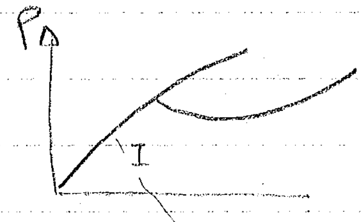
Formulation $EI\tilde{W}^{IV} - P\tilde{W}'' = 0$

$$EVP \begin{cases} \tilde{W}(0) = 0 & \tilde{W}(l) \\ \tilde{W}''(0) = 0 & M(l) = Pa\tilde{W}'(l) = 0 \end{cases}$$

fund state

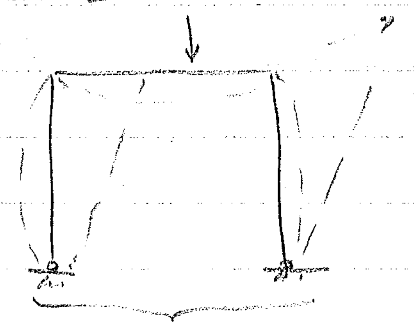
if nonlinear state problem



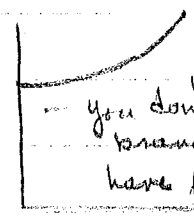


You have to use large deformation theory for fundamental branch I

or

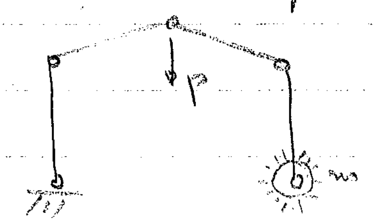


trivial state

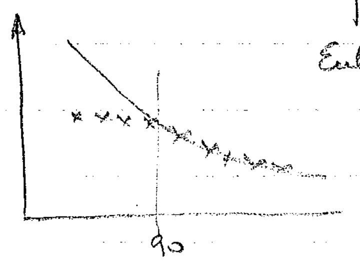


you don't have this branch here since you have first

you can solve this problem by considering model study



Inelastic buckling of columns

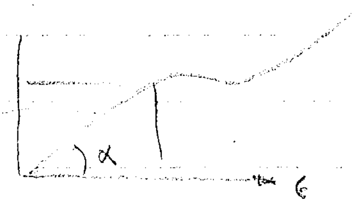


Euler hyperbola
xxx experimental results

big discrepancy between P_{cr} & test results for small l/r (inelastic range)

For l/r > 90 Euler theory good

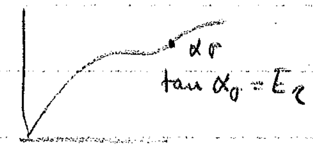
in this range $E = \sigma/\epsilon$ $\sigma = \sigma_{prop}$



$\tan\alpha = E$ $\qquad E = \left.\frac{d\sigma}{d\epsilon}\right|_{\epsilon=0}$

tangent modulus theory (Engesser 1889)

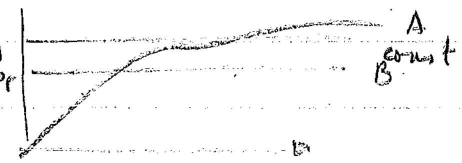
$E \xrightarrow[\text{replace}]{} E_t = \left.\frac{d\sigma}{d\epsilon}\right|_{\sigma}$



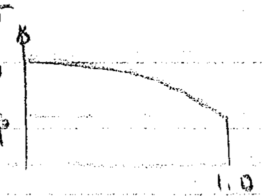
$E_t = \tau E$

$P_{cr} = \beta^2 \frac{E_t I}{l^2}$

Procedure

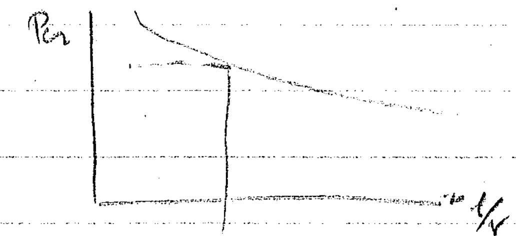


Construct

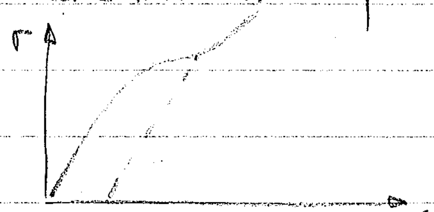


$\tau = \frac{E_t}{E}$

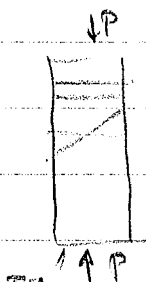
$P_{cr} = \beta^2 \frac{EI}{l^2} \tau$



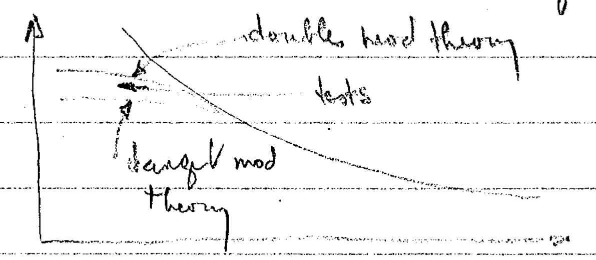
the double modulus theory



Loading elastic
unloading plastic



In section there are 2 regions governed by 2 EIs



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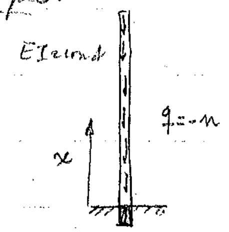
Example:



$$EI\,w^{IV} + (N w')' = 0 \qquad N(x) = -(l-x)q$$

$$EI\,w^{IV} + q[(l-x)\,w']' = 0$$

$$EI\,w^{IV} + q\,(l-x)w'' - q\,w' = 0$$

$w(0) = 0 \qquad w''(l) = 0$

$w'(0) = 0 \qquad V = -(EI\,w'')' + Nw' \quad @\ l \Rightarrow w''' = 0$

$$[EI\,w''' + q\,(l-x)w']' = 0$$

$$EI\,w''' + q(l-x)\,w' = C_1$$

$@\ x = l \qquad EI\,w'''(l) = C_1 = 0$

$[EI\,w''' + q(l-x)w'] = 0 \quad (a) \qquad$ Assume $\frac{dw}{dx} = u(x) \qquad l - x = \xi_1$

then (a) $\quad EI\,u'' + q\,\xi_1\,u = 0 \qquad ' = \frac{d}{d\xi_1} \qquad x = 0 \quad \xi_1 = l$

$x = l \quad \xi_1 = 0$

Set $K = \sqrt[3]{\frac{q}{EI}} \qquad K\xi_1 = \xi \qquad 0 \le \xi_1 \le l$

$$\frac{d^2u}{d\xi^2} + \xi\,u = 0 \qquad \text{Airy differential}$$

$$u(\xi) = A_1\sqrt{\xi}\;J_{1/3}\left(\tfrac{2}{3}\xi^{3/2}\right) + A_2\sqrt{\xi}\,J_{-1/3}\left(\tfrac{2}{3}\xi^{3/2}\right)$$

BC $\quad$ 2 here & 1 to get w from $\int u$

(2) $\longrightarrow \quad w_{,x}(0) = 0 \qquad u(\xi^*) = 0 \qquad \xi^* = Kl \qquad u(Kl) = 0$

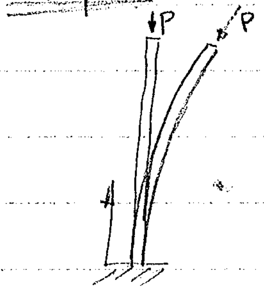
(3) $\longrightarrow \quad w_{,xx}(l) = 0 \qquad -u_\xi(\xi^{**}) = 0 \quad \xi^{**} = 0 \qquad u_\xi(0) = 0 \quad A_1 = 0$

for $u(Kl) = 0 \qquad J_{-1/3}\left(\tfrac{2}{3}\sqrt{\frac{q\,l^3}{EI}}\right) = 0 \qquad$ a non trivial sol will exist when bessel fn $= 0$

smallest root from table $\quad \tfrac{2}{3}\sqrt{\frac{q\,l^3}{EI}} = 1.866$

$(q\,l)_{cr} = 7.84\,\frac{EI}{l^2} \qquad l_{cr} = \sqrt[3]{\frac{7.84\,EI}{q}}$

Example



Straight state is definitely a State of Equil.

Determination of P_{cr}

$$\tilde{W}^{IV} + \lambda^2 \tilde{W}'' = 0$$

BC $\quad W(0) = 0 \qquad \tilde{W}''(l) = 0$

$\quad W'(0) = 0$

$$P_H = P\sin\theta \sim P W'(l)$$

$$V_l = -P W_x(l) = -EI W'''(l) - P\tilde{W}'$$

$$\tilde{W}'''(l) = 0$$

$$W(x) = A_1\cos(\lambda x) + A_2\sin(\lambda x) + A_3 x + A_4 \rightarrow 4 \text{ bc}$$

Next form

$$\Delta = \begin{vmatrix} \cos\lambda l & \sin\lambda l \\ \sin\lambda l & -\cos\lambda l \end{vmatrix} \neq 0$$

Adj equil states exist when $\Delta = 0$.

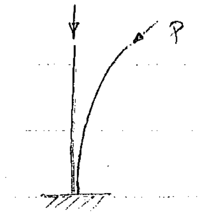
$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & \lambda & 1 & 0 \\ -\lambda^2\cos\lambda l & -\lambda^2\sin\lambda l & 0 & 0 \\ +\lambda^3\sin\lambda l & -\lambda^3\cos\lambda l & 0 & 0 \end{vmatrix} \qquad -1\begin{vmatrix} 0 & \lambda & 1 \\ -\lambda^2 c & -\lambda^2 s & 0 \\ \lambda^3 s & -\lambda^3 c & 0 \end{vmatrix}$$

$$-1\begin{vmatrix} -\lambda^2 c & -\lambda^2 s \\ \lambda^3 s & -\lambda^3 c \end{vmatrix} = -1\left\{\lambda^5 c^2 + \lambda^5 s^2\right\} = -\lambda^5 \neq 0$$

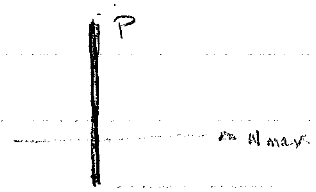
Rotational System is nonconservative & depends upon path of displacements allowed. For this problem energy criterion cannot be used to analyze stability of undeformed branch because W depends upon path of deformation & hence π isn't conservative. Use of dynamic criterion is necessary.

4-19-73

Beck's Problem



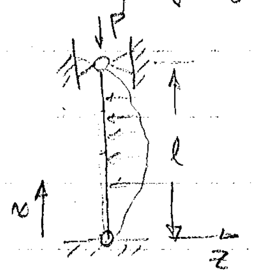
$\Delta = \left| \quad \right| = -1 \neq 0$



no adjacent equilib. state

influence coefficients $\delta_{ij} \neq \delta_{ji}$

Stability by Dynamic Analysis. (Static Stability Problems 1st)



Dynamic Diff Eq for beams

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial w}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(N \frac{\partial w}{\partial x}\right) = q(x,t)$$

For dynamic proble : $q(x,t) \longrightarrow -m \frac{\partial^2 w}{\partial t^2} + q(x,t)$

we will assume $q(x,t) = 0$

EI & N = const $\quad \underbrace{EI\, w_{,xxxx}}_{\text{bending}} + \underbrace{P\, w_{,xx}}_{\text{axial force}} + \underbrace{m\, w_{,tt}}_{\text{lateral inertia}} = 0 \quad (I) \qquad w = w(x,t)$

B.C. $\quad w(0,t) = 0 \qquad w(l,t) = 0 \qquad$ IC $\quad w(x,0) = f(x)$

$\quad w_{,xx}(0,t) = 0 \qquad w_{,xx}(l,t) = 0 \qquad w_{,t}(x,0) = g(x)$

for $t \leq 0$ $\quad q(x) \neq 0$

$t > 0$ $\quad q(x) = 0$ Steady response of beam

for $t > 0$ we will have (I) b.c. (1) → 4

$w(x,0) =$ deflection curve due to $q(x)$ for static problem

$w_{,t}(x,0) = 0$

to get $w(x,0)$ $\qquad EIw^{IV} + Pw'' = q(x) \qquad$ B.C (1) to (4)

Assume $W(x,t) = X(x)\,T(t)$ put into I

$$\lambda = P/EI$$

$$X^{IV}T + \lambda^2 X''T + \frac{m}{EI} X T'' = 0$$

$$\frac{X^{IV} + \lambda^2 X''}{\frac{m}{EI} X} = -\frac{T''}{T} = +\omega^2 \quad \begin{cases} T'' + \omega^2 T = 0 \\ X^{IV} + \lambda^2 X'' - \alpha^4 X = 0 \\ \underline{ODE} \end{cases}$$

$$\alpha^4 = \frac{m\omega^2}{EI} \geq 0$$

$T = A_1 \cos\omega t + A_2 \sin\omega t$

from (bc) 1 $\quad X(0) = 0 \qquad X(l) = 0$

(bc) 2 $\quad X''(0) = 0 \qquad X''(l) = 0$

EVP for $X(x) \rightarrow X_n(x)$ EV & ω_n EV

$$W(x,t) = \sum_{n=1}^{\infty} X_n(x)\left[A_{n1}\sin\omega_n t + A_{n2}\cos\omega_n t\right]$$

IC $\quad W(x,0) = \sum_{n=1}^{\infty} X_n(x) A_{n2} = f(x)$

$\quad W_{,t}(x,0) = \sum_{n=1}^{\infty} X_n(x)\,\omega_n A_{n1} = 0 \qquad A_{n1} = 0$

$$W(x,t) = \sum_{n=1}^{\infty} X_n(x) A_{n2} \cos\omega_n t$$

$$\int_0^l f(x) X_m(x)\,dx = \int_0^l \sum X_m(x) X_n(x) A_{n2}\,dx$$

Use orthog cond to get A_{n2}

if ω is complex we get cosh & sinh which go to ∞ as $t \to \infty$

let $\overline{X} = e^{sx}$ $\quad s^4 + \lambda^2 s^2 - \alpha^2 = 0$

$$(s^2 + b)(s^2 + a) = 0 \qquad a+b = \lambda^2, \quad ab = -\alpha^2$$

$$b^2 - \alpha^2 - \lambda^2 b = 0 \implies \frac{\lambda^2 \pm \sqrt{\lambda^4 + 4\alpha^2}}{2} = b_\pm \qquad a_\pm = \lambda^2 - b_\pm$$

Note that step 2 which determines if ω is Re or complex is sufficient for stability analysis. Since when $\omega \to i\omega$ then $\cos \omega t \to \cosh \omega t$. Therefore step 3 (solution of problem) is really not necessary if one is not interested in exact response after q is removed.

Simplified procedure: assume $w(x,t) = \overline{X}(x) e^{i\omega t} \to$ DE

get 2 ODE for X & T $\to \overline{X}_n, \omega_n$

Study properties of ω

If $\omega = a - ib$ $\quad w(x,t) = \overline{X}(x)\, e^{i(a-ib)t} = \overline{X}(x) e^{(ia+b)t}$

$$= \overline{X}(x)\, e^{iat}\, e^{bt}$$

assume if $b > 0$: $e^{bt} \to \infty$ for $t \to \infty$

---

or $\quad s = \pm\sqrt{\dfrac{-\lambda^2 \pm \sqrt{\lambda^4 + 4\alpha^4}}{2}} = \pm\sqrt{-\dfrac{\lambda^2}{2} \pm \sqrt{\dfrac{\lambda^4}{4} + \alpha^4}}$

Hence $\quad s_{1,2} = \pm p \quad$ and $\quad s_{3,4} = \pm iK$

$$p = \sqrt{-\frac{\lambda^2}{2} + \sqrt{\frac{\lambda^4}{4} + \alpha^4}} \qquad K = \sqrt{\frac{\lambda^2}{2} + \sqrt{\frac{\lambda^4}{4} + \alpha^4}}$$

$$\overline{X}(x) = A_1 \cosh px + A_2 \sinh px + A_3 \cos Kx + A_4 \sin Kx$$

$$\overline{X}''(x) = A_1 p^2 \cosh px + A_2 p^2 \sinh px - A_3 K^2 \cos Kx - A_4 K^2 \sin Kx$$

by using bc $\quad A_1 + A_3 = 0 \qquad A_3(p^2 + K^2) = 0 \qquad A_3 = 0 \quad A_1 = 0$

$\qquad A_1 p^2 - A_3 K^2 = 0$

from 3 & 4 $\qquad A_2 \sinh pl + A_4 \sin Kl = 0$

$\qquad\qquad\qquad A_2 p^2 \sinh pl - A_4 K^2 \sin Kl = 0$

For existence of non triv sol $\quad \begin{vmatrix} \sinh pl & \sin(Kl) \\ p^2 \sinh(pl) & -K^2 \sin Kl \end{vmatrix} = 0$

$(p^2 + K^2) \sinh pl \sin kl = 0 \quad$ if $p \neq 0 \quad kl = n\pi$

$$K_n = \frac{n\pi}{l}$$

$$\sqrt{\frac{\lambda^2}{2} + \sqrt{\frac{\lambda^4}{4} + \alpha_n^4}} = \frac{n\pi}{l} \qquad \frac{\lambda^2}{2} + \sqrt{\frac{\lambda^4}{4} + \alpha_n^4} = \left(\frac{n\pi}{l}\right)^2$$

$$\frac{m\omega_n^2}{EI} = \left[\left(\frac{n\pi}{l}\right)^2 - \frac{\lambda^2}{2}\right]^2 - \frac{\lambda^4}{4}$$

$$\omega_n^2 = \frac{EI}{m} \frac{n^4 \pi^4}{l^4} \left[1 - \frac{P}{\left(\frac{n^2 \pi^2 EI}{l^2}\right)}\right] \qquad P_{cr} = \frac{\pi^2 EI}{l^2}$$

$$\omega_1 = \pm \sqrt{\frac{EI}{m} \frac{\pi^4}{l^2} \left(1 - P/P_{cr}\right)}$$

$P < P_{cr} \quad \omega_1$ real

$P > P_{cr} \quad \omega_1$ im

$w(x,t) = \sum X_n \cos(\omega t) \quad$ stable

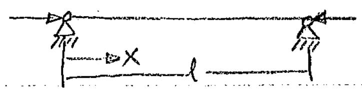
$w(x,t) \sim e^{\omega_1 t} \quad$ unstable

note that when $P = P_{cr} \quad \omega_1 = 0 \quad$ this is a stability criterion

4-26-73

1st Thursday of Exam week - 2hrs, of Exam week

$EI\, W^{IV} + P W'' + m \ddot{W} = 0 \quad (I)$



4 BC

2 IC $\quad W(x,0) = f(x)$

$\qquad w_{,t}(x,0) = g(x) \qquad$ our case $\dot{W}(x,0) = 0$

We assumed apriori $w(x,t) = X(x)e^{i\omega t}$

$T'' + \omega^2 T = 0$ $\quad$ ω^2 chosen positive

ω_n obtained from EVP in x $\quad$ $\overline{EV} = X_n(x)$

reduced to fourier analysis to satisfy I.C.

if ω_n is real system is stable

if ω_n is imaginary system is unstable

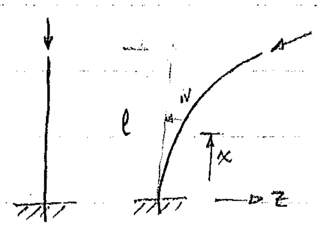
$$\omega_n^2 = \frac{n^4\pi^4}{l^4}\frac{EI}{m}\left[1 - \frac{P}{P_{cr}}\right] \qquad P_{cr} = \frac{n^2\pi^2 EI}{l^2} \qquad n = 1, 2$$

$$X_n(x) = A_2[\cdots]$$

Example

DE (I)

Same as before



$W(0,t) = 0$ $\quad$ $W''(l,t) = 0$

$W'(0,t) = 0$ $\quad$ $W'''(l,t) = 0$

$W(x,t) = X(x)\,T(t) \rightarrow$ 2 ODE $\quad LX = 0 \quad T'' + \omega^2 T = 0$

$$X(x) = A_1\cosh(\beta x) + A_2\sinh(\beta x) + A_3\cos(\gamma x) + A_4\sin(\gamma x)$$

$X(0) = 0$; $X''(l) = 0$

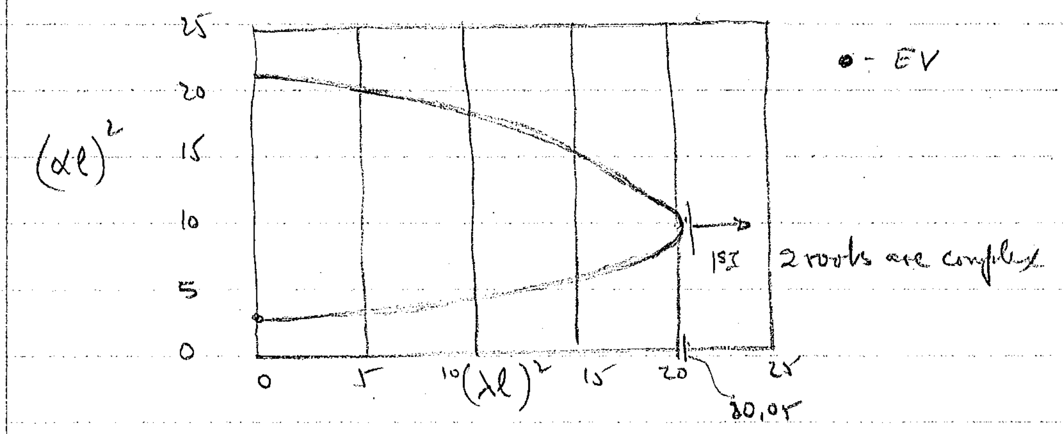
$X'(0) = 0$; $X'''(l) = 0$

$\Delta = 0$ gives $2\alpha^4 + \lambda^4 + 2\alpha^4\cosh(\beta l)\cos\gamma l + \lambda^2\alpha^2\sinh\beta l\sin\gamma l = 0$

frequency equation

for $P = 0 \Rightarrow \lambda^2 = 0$ $\quad 2\alpha^4 + 2\alpha^4\cosh(\beta l)\cos(\gamma l) = 0 \quad [1 + \cosh(\beta l)\cos\gamma l] = 0$

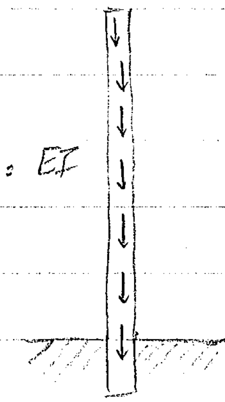
freq eq for cantilever beam



for $p >$ $20{,}05 \frac{EI}{l^2}$ system is unstable.

$P < 20{,}05 \frac{EI}{l^2}$ column is stable. let q = Weight/unit length.

const : EI



$N(x) = -q(l-x)$

Formulate stability problem
from dynamic point of view
(using dynamic stability criterion)

① PDE & BC ② Separate variables & state EVP in X

$EI\,W^{IV} + q[(l-x)W']' + m\ddot{W} = 0$ $W(0)=0$ $W''(l)=0$

$EI\,X^{IV}T + q[(l-x)X']'T + mT'' = 0$ $W'(0)=0$ $W'''(l)=0$

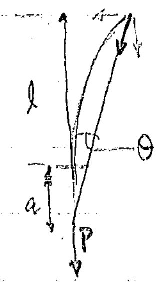
$\frac{EI}{m}X^{IV} + \frac{q}{m}[(l-x)X']' = -\frac{T''}{T} = \omega^2$

$T'' + \omega^2 T = 0$

$EI\,X^{IV} + q[(l-x)X']' = m\omega^2$ let $X_p = \frac{m\omega^2 x}{q}$

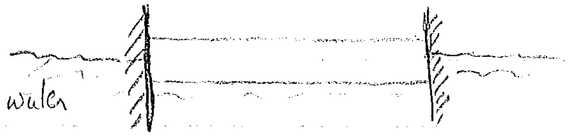
for homog problem same as before $X_p' = \frac{m\omega^2}{q}$

X from homog problem $+ X_p$



$$\frac{w(l,t)}{l+a} = \tan\theta$$

Stability of continuously supported beams



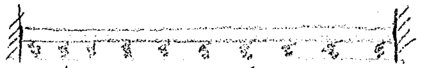
T° is temp increase (uniform)

$\Delta l = \alpha T l$ $\qquad \alpha$ = coeff of thermal expansion

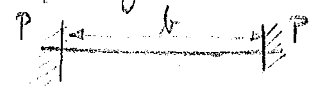
thermal strain is $\Delta l / l = \alpha T = \epsilon_T \quad \frac{\sigma}{E} = \frac{P}{AE} = \epsilon_m$

$\epsilon_T = \epsilon_m$; $\boxed{P = \alpha T A E}$

Largest force beam can exert on shore is P_{cr}.



In order to determine largest load the ice beam may exert on wall use strongest constraints possible at end pts.



DE: $\quad EI\tilde{w}^{iv} + P\tilde{w}'' + \overset{\gamma b}{k}\tilde{w} = 0$ $\qquad \gamma$ specific weight of water; γb buoyancy term

$\tilde{w}(0) = 0 \quad \tilde{w}(l) = 0$ $\qquad EIs^4 + Ps^2 + \gamma b = 0$ where $w = Ce^{sx}$

$\tilde{w}'(0) = 0 \quad \tilde{w}'(l) = 0$

matching cond $\begin{cases} w_1 = w_2 = 0 & w_2'' = w_1'' \\ w_1' = w_2' & \text{dependent on } \underline{a} \end{cases}$ $\quad w_1''' = w_2'''$

a obtained by shear cond.

buckling of highway

Regularity cond $\lim_{x\to\infty}(w_2, w_2', w_2'') = 0$

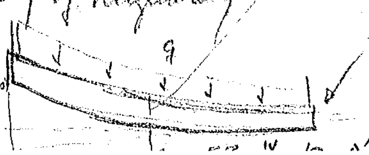
$r = q\mu$ — friction coeff

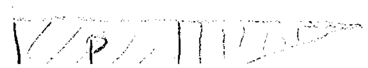
$w_1''(0) = 0$

$0 = -EI\, w_1'''(0) - Pw_1'(0)$

$P = P(a)$

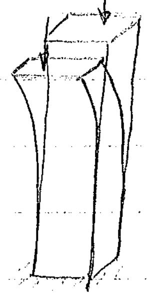
resistance

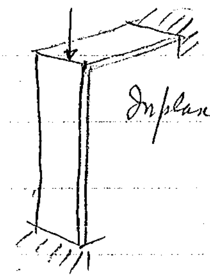




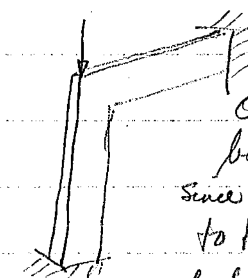
May 3, 1973.

Large deformation of beams.

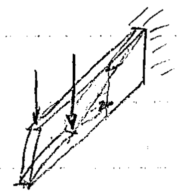




Inplane buckling



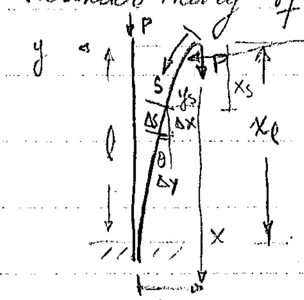
Out of plane buckling
Since buckling due to torsion is not looked at by our anal



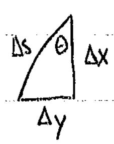
buckling out of plane
Lateral Stability

Torsional Buckling   Den Hartog: Advanced Strength of Materials P274-301

Elastica Theory - Large deformation of Beams - bending only
Nonlinear theory of axially compressed beams.



Inextensible theory - length of beam same



No longer assume $w_x \ll 1$
Plane Section hypothesis & validity of hooke's law are kept.

$$\frac{M}{EI} = \frac{1}{\rho} = -\frac{d\theta}{ds} \qquad M = Py$$

$$\frac{Py}{EI} = -\frac{d\theta}{ds} \qquad \text{Assume } EI = \text{const}$$

$$\frac{d}{ds}\left[\frac{Py}{EI} = -\frac{d\theta}{ds}\right] \Longrightarrow EI\,\theta_{,ss} + P\,y_{,s} = 0 \qquad y_{,s} = \sin\theta$$

$$\boxed{\theta_{,ss} + \lambda^2 \sin\theta = 0} \quad \text{NLDE of beam} \qquad (1) \qquad \lambda^2 = P/_{EI}$$

Solution Integrate (1) wrt $d\theta$

$$\int \frac{d^2\theta}{ds^2}\, d\theta = -\lambda^2 \int \sin\theta\, d\theta$$

$$\frac{1}{2}\int \frac{d}{ds}\left(\frac{d\theta}{ds}\right)^2 ds = \frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 = +\lambda^2\cos\theta + C \qquad (2)$$

Determin of C $\quad \left.\frac{d\theta}{ds}\right|_{\theta=\alpha} = 0$ since $\frac{d\theta}{ds} \sim M = Py$; when $\theta = \alpha$ $y = 0$

$\therefore \quad 0 = \lambda^2\cos\alpha + C \qquad \therefore \quad (\theta_{,s})^2 = 2\lambda^2(\cos\theta - \cos\alpha)$

$\frac{d\theta}{ds} = \pm\lambda\sqrt{2(\cos\theta - \cos\alpha)}$; for mode shown $\frac{d\theta}{ds} < 0$

$0 < \theta < \alpha \qquad \cos\theta > \cos\alpha$ since $\theta < \alpha$

$$\therefore \quad ds = \frac{-d\theta}{\lambda\sqrt{2(\cos\theta - \cos\alpha)}}$$

total length of bar $\quad \ell = \int_0^\ell ds = \int_\alpha^0 \frac{-d\theta}{\lambda\sqrt{2(\cos\theta - \cos\alpha)}}$

transformation of $\int$ into an elliptic integral

$$\cos\theta = 1 - 2\sin^2\theta/2$$

$$\cos\theta - \cos\alpha = 2\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)$$

$$\therefore \quad \ell = \int_0^\alpha \frac{d\theta}{2\lambda\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\theta/2}}$$

let $p = \sin\frac{\alpha}{2}$ and introduce a new variable $\sin\theta/2 = p\sin\phi$

$\sin\theta/2 = \sin\alpha/2 \sin\phi \quad (3)$

d③ $\quad \frac{1}{2}\cos\frac{\theta}{2}\, d\theta = p\cos\phi\, d\phi$

$$d\theta = 2p\frac{\cos\phi\, d\phi}{\cos\theta/2} = \frac{2p\cos\phi\, d\phi}{\sqrt{1-p^2\sin^2\phi}}$$

$$\sqrt{\sin^2\alpha/2 - \sin^2\theta/2} = \sin\frac{\alpha}{2}\sqrt{1-\sin^2\phi} = \sin\alpha/2\,\cos\phi = p\cos\phi$$

transformation of limits of integ
when $\quad 0 \le \theta \le \alpha \qquad \phi = 0 \quad \phi = \pi/2$

$$l = \int_0^{\pi/2} \frac{d\phi}{\lambda\sqrt{1-p^2\sin^2\phi}} \qquad \text{Elliptic Integral of first kind}$$

define $K(p) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} \qquad \therefore\ l = \frac{K(p)}{\lambda}$ (I)

Deflection $y_l \qquad \frac{dy}{ds} = \sin\theta \qquad dy = \sin\theta\, ds = -\frac{\sin\theta\, d\theta}{\lambda\sqrt{2(\cos\theta - \cos\alpha)}}$

$$y_l = \int_0^\alpha \frac{\sin\theta\, d\theta}{\lambda\sqrt{2(\cos\theta-\cos\alpha)}} = \frac{1}{2\lambda}\int_0^\alpha \frac{\sin\theta\, d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\theta/2}}$$

$$\sin\theta = 2\sin\theta/2\,\cos\theta/2 = 2p\sin\phi\sqrt{1-p^2\sin^2\phi}$$

$$y_l = \frac{1}{\lambda}\int_0^{\pi/2} 2p\sin\phi\, d\phi = -\frac{2}{\lambda}p\cos\phi\Big|_0^{\pi/2} = \frac{2}{\lambda}p$$

$y_l = \frac{2}{\lambda}\sin\alpha/2 \qquad$ (II)

Procedure for determination of end deflec

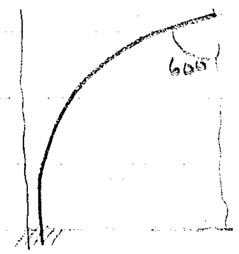
① Select a value of $\alpha \Rightarrow p$

Knowing p get $K(p)$ from tables

② using (I) get $\lambda = \sqrt{P/EI} \Rightarrow P$

③ using p and λ get y_e from (II).

Example choose $\alpha = 60^\circ$



$p = \sin 30^\circ = 1/2$

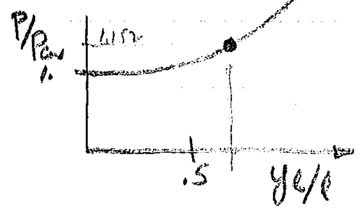
from (I) it follows that $\lambda l = K(1/2)$

from table $K(1/2) = 1.686$

$$\lambda^2 = \left(\frac{1.686}{l}\right)^2 = \frac{P}{EI}$$

$$\therefore \quad P = \frac{1.686}{l^2} EI = 2.842 \frac{EI}{l^2}$$

$$y_e = \frac{2p}{\lambda} = 2\cdot(1/2)\frac{1}{\lambda} = \frac{l}{1.686} = .593 l.$$



Determination of Pcrit.

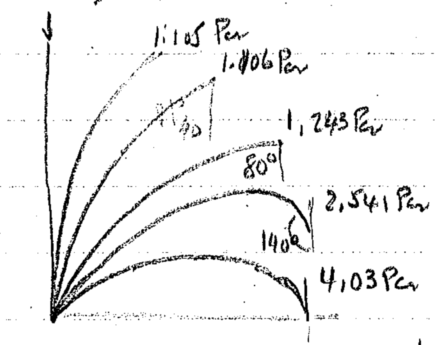
$$l = \frac{1}{\lambda}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} \cong \frac{\pi}{2\lambda} \text{ for small } \theta = \frac{\pi}{2}\sqrt{\frac{EI}{P_{cr}}}$$

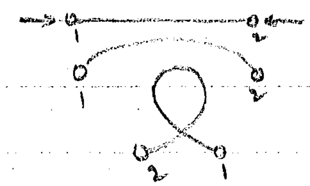
$$P_{cr} = \frac{\pi^2}{4}\frac{EI}{l^2} \simeq 2.5\frac{EI}{l^2}$$

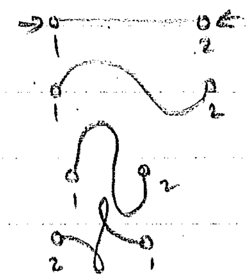
$$x_e = \int_0^{\alpha} \frac{\cos\theta\, d\theta}{\lambda\sqrt{2(\cos\theta - \cos\alpha)}} = \frac{2}{\lambda}\int_0^{\pi/2} \sqrt{1-p^2\sin^2\phi}\, d\phi - \frac{1}{\lambda}\frac{d\phi}{\sqrt{1-p^2\sin^2\phi}}$$

$E(p)$ complete elliptic integral second kind

$$X_\ell = \frac{2}{\lambda}E(p) - \ell$$

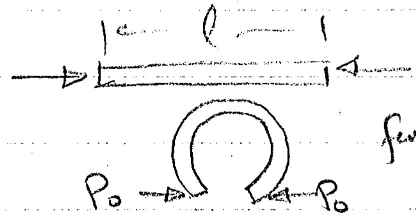






Mathematical Theory of Elasticity by Love. Dover Publication

HW #8



find value of p_0 so that both ends of beam will touch each other

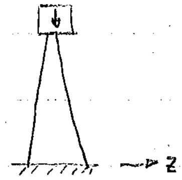
5/10/73

Final will be given the 24 May

Approximate Method: Stodolla - Vianello — Hildebrand

Solution of EVP's by successive approximation.

Description of Method



$(EI\,w'')'' + (Pw')' = 0$ DE

$w(0)=0 \quad w'(0)=0$

$w''(\ell)=0 \quad (EIw'')' + (Pw')\Big|_\ell = 0$

rewrite $EI(x) = (EI)_0\,\psi(x) \qquad P(x) = P_0\,\varphi(x)$

$(\psi w'')'' + \lambda^2(\varphi w')' = 0 \qquad$ let $\dfrac{P_0}{(EI)_0} = \lambda^2 \qquad \psi, \varphi$ are known fns.

bc $\quad (\psi w'')' + \lambda^2(\varphi w')\Big|_\ell = 0$

① Shift all terms containing λ^2 to right hand side and assign $n+1$ to LHS & n to RHS

DE $\quad (\psi w''_{n+1})'' = -\lambda^2(\varphi w'_n)'$

bc. $\quad w_{n+1}(0)=0 \quad w'_{n+1}(0)=0 \quad w''_{n+1}(\ell)=0 \quad (\psi w''_{n+1})'_\ell = -\lambda^2(\varphi w'_n)_\ell$

$\Rightarrow \quad w_{n+1}(x) = \lambda^2_{n+1} f_n(x) \quad \lambda$ is fn of x; for actual EVP $\dfrac{w_{n+1}}{f_n} = $ const

② Assume a $w_0(x)$ as a given fn. $n=0$ and solve resulting BVP.

$\to \quad w_{n+1}(x) = \lambda^2_{(n+1)} \underbrace{f_n(x)}_{\text{known}} \qquad$ usually $\dfrac{w_{n+1}}{f_n} \neq$ const

③ estimate of $\lambda^2_{(n+1)}$ One possibility $\int_0^\ell [w_{n+1} - w_n]\,dx = 0$

$= \int_0^\ell \left(\lambda^2_{n+1} f_n(x) - w_n(x)\right)dx \qquad \lambda^2_{n+1} = \dfrac{\int_0^\ell w_n(x)\,dx}{\int_0^\ell f_n(x)\,dx}$

lim as $n \to \infty$ $(\lambda^2_n) = \lambda^2$ lowest ev and $\vec{EV}$ converges.

Example: $\quad w'' + \lambda^2 w = 0 \qquad w(0) = w'(\ell) = 0 \quad m^{th}$ EV, $\vec{EV}$: $\lambda_m = \dfrac{m\pi}{\ell} \quad w_m = \sin\dfrac{m\pi x}{\ell}$

Stodolla-Vianello Method assume $W_1 = x(\ell - x)$

$$W''_{n+1} = -\lambda_n^2 W_n$$

$$W_{n+1}(0) = 0 \qquad W_{n+1}(\ell) = 0$$

$$W_2'' = -\lambda_1^2\, x(\ell - x)$$

$$W_2' = -\lambda_1^2 \left(\frac{x^2\ell}{2} - \frac{x^3}{3} + C_1\right)$$

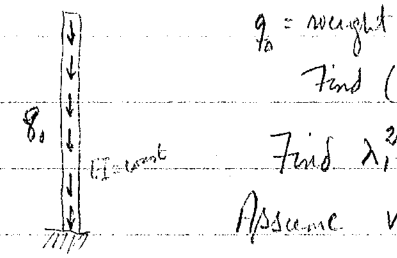
$$W_2 = -\lambda_1^2 \left(\frac{x^3\ell}{6} - \frac{x^4}{12} + C_1 x + \cancel{C_2}\right) \qquad W_2 = -\lambda_{(1)}^2\left(\frac{x^3\ell}{6} - \frac{x^4}{12} - \frac{\ell^3 x}{12}\right) = \frac{\lambda^2}{12}\left(x^4 - 2x^3\ell + \ell^3 x\right)$$

$$\lambda_{(1)}^2 = \frac{\int_0^\ell x(\ell - x)\,dx}{\frac{1}{12}\int_0^\ell (x^4 - 2x^3\ell + \ell^3 x)\,dx} = \frac{\ell^3/6}{\ell^5/60} = \frac{10}{\ell^2} \qquad \lambda_{(1)} = \frac{\sqrt{10}}{\ell} \qquad \pi^2 \sim 9.87.$$

For second iterative step $W_2 = (x^4 - 2x^3\ell + \ell^3 x) \longrightarrow W_3 = \frac{-\lambda_2^2}{30}\left[x^6 - 3\ell x^5 + 5\ell^3 x^3 - 3\ell^5 x\right]$

$$\lambda_2^2 = \frac{9.882}{\ell^2}$$

HW#9



q_0 = weight of beam/unit length.

Find $(\ell q_0)_{cr}$ for column using Stodolla-Vianello

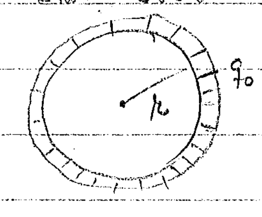
Find λ_1^2 & λ_2^2 compare them & with exact results.

Assume $W_1'(x) = \alpha x$

Ref. Advanced Strength of Materials Den Hartog.

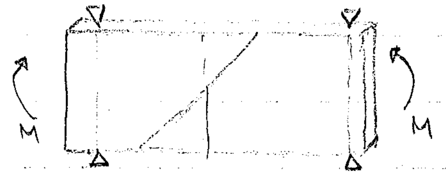
Note: look at largest pressure of the buckling Coeff.

1. hydrostatic press - normal to deformed surface $q_{cr} = 3\frac{EI}{R^2}$
2. centrally directed load $q_{cr} = 4.5\, EI/R^2$
3. Constant Directional load $q_{cr} = 4\frac{EI}{R^2}$



shallow arches - effect of extensibility of axis has to be included.

Lateral Buckling of Beams.



$$M_{cr} = \frac{\pi\sqrt{CEI}}{l}$$

$$\varphi'' + \lambda\varphi = 0$$

$$\lambda = \frac{M^2}{CEI_y}$$

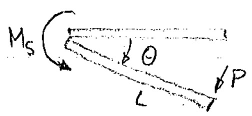
torsional stiffness

flexible bending stiffness

Torsion $\left(\frac{M_t}{M_{cr}}\right)^2 + \frac{P}{P_{cr}} = 1$

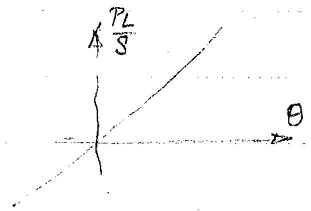
May 17, 1973

Midterm



$M_s - PL = 0$

$S\theta - PL = 0$ $\quad \frac{PL}{S} = \theta$



$\Pi = U - W$ $\quad W = P\sum l d\theta$

$= \frac{1}{2}S\theta^2 - PL\theta$

$\frac{d\Pi}{d\theta} = S\theta - PL = 0$ $\quad \frac{d^2\Pi}{d\theta^2} = S > 0$ stable

$w_1 = w_2$

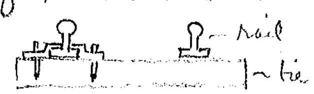
$w_1' = w_2'$

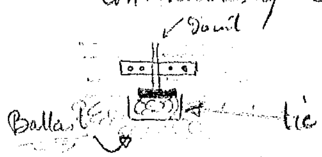
$w_1'' = w_2''$

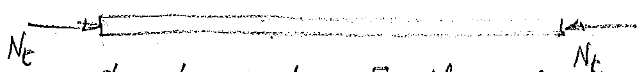
$V_1 = V_2 \Rightarrow -EIw_1''' - Pw_1' = +EIw_2'''$

Buckling of the railroad tracks

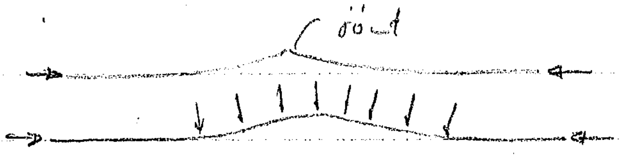
continuously welded rail (CWR)







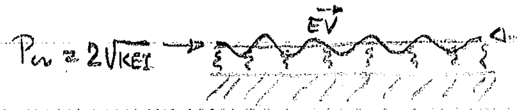
due to constrained thermal expansion

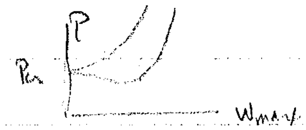


Vertical Buckling

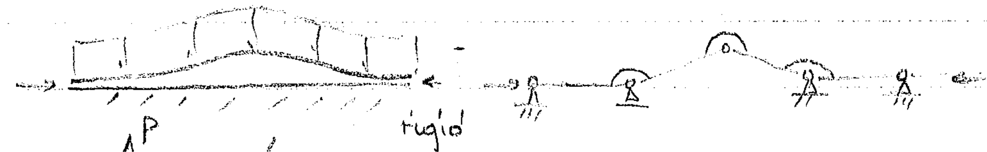
side view

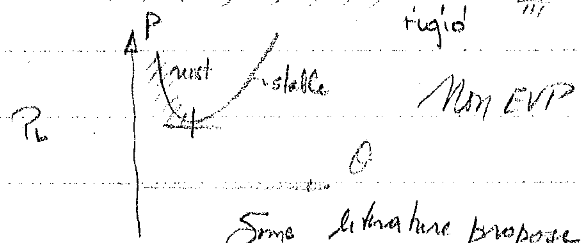
Assumptions: 1. Beam is continuously supported on Winkler Base (before & after buckling)





2. Rigid base.

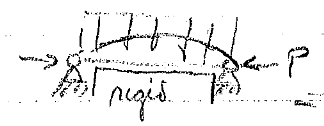


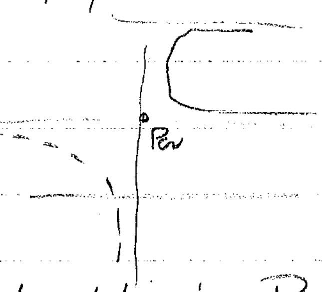


Non EVP

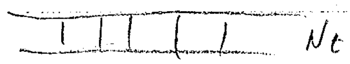
Some literature propose

$EIw^{IV} + Pw'' = -q$

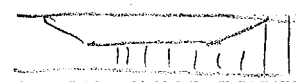




Linearized Analysis is not suitable to determine P_L.



before liftoff



after liftoff

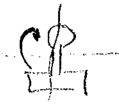
$N(x) \leftarrow \square \rightarrow N(x+\Delta x)$, $u(x)$, $r(x)$, $r(x+\Delta x)$

$\Sigma F_H = 0 \longrightarrow N_x(x) = r(x)$ & $N(x) = \int^x r(\xi)d\xi + B$

Assumption on $r(x)$

(1) $r(x) = \text{const} = r_0$ 2) $r(x) = h\,u(x)$

(2) $N = r_0 x + B$ (1) $\rightarrow$ $\frac{dN}{dx} - h\,u(x) = 0$



I_x

$$N = A\sigma = A\epsilon E = AE\frac{du}{dx}$$

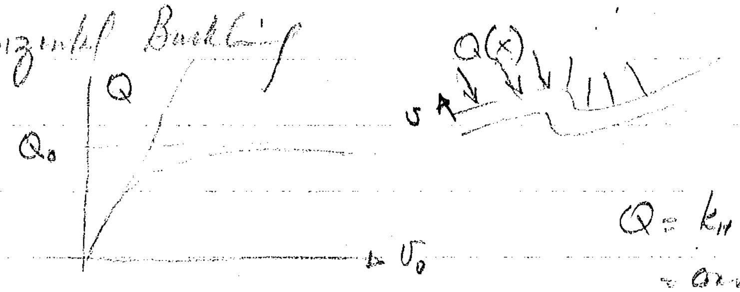
if $EA = \text{const}$ then $\frac{dN}{dx} = AE\,u''$

$\therefore$ (1) $\rightarrow$ $AE u'' - h u = 0$ or $u = \cosh\sqrt{\frac{h}{AE}}\,x + \sinh\sqrt{\frac{h}{AE}}\,x$

$\nu = \sqrt{\frac{h}{AE}}$ $u = B_1 e^{-\nu x}$ because of regularity

$$N = \int^x h B_1 e^{-\nu\xi} d\xi = \frac{hB_1}{-\nu} e^{-\nu\xi}\Big|_c^{\infty}$$

Horizontal Buckling

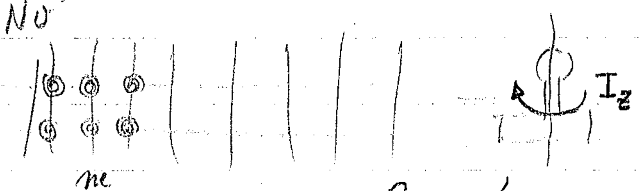


$Q = k_H v$

$= \text{order } \theta_0$

$$EI v'''' + (N v')' + Q = 0$$

EI is ~~...~~ not known put another term $(-\tau v')$ to include moment in Nv'



$m = \tau\theta = \tau v'$

$m' = \tau v''$ $\tau = \text{const.}$

A MODEL STUDY FOR VERTICAL TRACK BUCKLING

DEPARTMENT OF TRANSPORTATION

FEDERAL RAILROAD ADMINISTRATION

OFFICE OF RESEARCH, DEVELOPMENT AND DEMONSTRATIONS

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16. Abstract

The paper contains a study of two models which represent the mechanism of vertical buckling of a track when subjected to a mechanical or to a thermal compression force, respectively. The postbuckling equilibrium curves and their stability are discussed and a stability criterion is defined. The effect of various track model parameters upon the buckling load or buckling temperature, are shown. The nonlinear equilibrium equations were then linearized. It was found that the buckling loads, or temperatures, obtained from a linearized analysis have no relevance to the actual values obtained from a nonlinear analysis; the difference in results being substantial for buckling temperatures.

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A MODEL STUDY FOR VERTICAL TRACK BUCKLING*)

by

Arnold D. Kerr[1)]

SUMMARY

The paper contains a study of two models which represent the mechanism of vertical buckling of a track when subjected to a mechanical or to a thermal compression force, respectively. The postbuckling equilibrium curves and their stability are discussed and a stability criterion is defined. The effect of various track model parameters, upon the buckling load or buckling temperature, are shown. The nonlinear equilibrium equations were then linearized. It was found that the buckling loads, or temperatures, obtained from a linearized analysis have no relevance to the actual values obtained from a nonlinear analysis; the difference in results being substantial for buckling temperatures.

INTRODUCTION

The recent practice of eliminating expansion joints by welding the rail ends to each other increases the possibility of track buckling at high temperatures. A number of recent derailments, which are attributed to buckled tracks, are reported in Ref. [1].

Stability analyses of the welded track were conducted in the past by many investigators [2]. These analyses may be grouped into two main categories: when track buckles vertically and when track buckles in the horizontal plane. Although actual track buckling may proceed in a more complicated manner, the choice of these two special modes of deformation was apparently made in order to reduce the problem to two dimensions and thus to simplify the resulting analyses.

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The buckling process as described and analyzed in the literature, is shown in Fig. 1. There it is assumed that the straight rails rest on a "rigid" base and are subjected, due to a uniform temperature increase and constrained axial expansions, to an axial compression force N_t. For large values of N_t, the track buckles out vertically, as shown. In the lift-off region of length ℓ, part of the thermal expansions are released, which results in a reduction of the axial force, to $\tilde{N}_t$. In the adjoining regions, each of length a, due to ballast resistance to axial displacements of the track, the constrained thermal expansions vary; so does the axial force $\tilde{N}_t < N < N_t$.

According to the above observations, vertical buckling is a local phenomenon. That is, except for the length $(\ell + 2a)$ the track is not affected by it.

When analyzing a railroad track for vertical buckling, the following questions are of interest:

1) The magnitude of the buckling force and the corresponding temperature increase which causes it.

2) The effect of geometrical imperfections of the rails upon the buckling force.

3) The effect upon the buckling force of an increase of the track weight. For example, by increasing the weight of the ties.

A review of the papers on vertical buckling by K.N. Mishchenko [3], H. Lederle [4], M.A. Martinet [5], F. Corini [6], E. Engel [7], R. Sauvage [8], and others, reveals that these authors used the linear theory of beams for the determination of the buckling loads. However, because of the lift-up at the onset of buckling, these problems do not exhibit a bifurcation from the undeformed state, and therefore, it is not a priori certain that a linear theory is suitable for the analysis of vertical buckling.

The solution of the governing nonlinear differential equation is rather involved. In order to demonstrate the vertical buckling phenomenon and its relation to nonlinear and linearized analyses, as well as to obtain preliminary information about the questions listed above, in the following we analyze two models which exhibit the essential features of the anticipated buckling mechanism but are amenable to a simple exact analysis.

A MODEL STUDY FOR A TEST TRACK COMPRESSED BY JACKS

In this section, we study the simple model shown in Fig. 2. It represents the vertical buckling mechanism of a short track in a test stand, where the axial forces are induced by means of jacks; for example, as described in Ref. [9].

The model consists of four "rigid" bars constrained at the interconnecting joints by spiral springs. These springs represent the flexural rigidity of the track in the vertical direction. It is assumed that only joint 3 can lift-off from the base. The other joints can only slide horizontally. The model is of one degree of freedom and hence its equilibrium states are described by only one algebraic equation with θ as unknown.

From Fig. 2(a) it follows that $\theta = 0$ is an equilibrium state for any P. To determine if equilibrium states do exist for $\theta > 0$, we consider the free body diagram of bar $\overline{34}$, shown in Fig. 3, and set up the moment equilibrium about point 4.

The resulting equation is

$$2s\theta + s\theta + qL\,\frac{L\cos\theta}{2} - PL\sin\theta = 0$$

or rewritten

$$P^* = \frac{\theta + q^*\cos\theta}{\sin\theta} \qquad (1)$$

where

$$P^* = \frac{PL}{3s} \quad ; \quad q^* = \frac{qL^2}{6s} \qquad (2)$$

The corresponding equilibrium branches for $0 < \theta < \frac{\pi}{2}$ and $q^* = 0.04, 0.06, 0.08$ are shown in Fig. 4, as solid lines, and are denoted by II. Also shown, as a dashed line, is the equilibrium branch for $0 < \theta < \frac{\pi}{2}$ and $q = 0$, and is denoted by II_o. The $\theta = 0$ axis is an equilibrium branch for any $q \geqq 0$ and is denoted by I.

Note the different character of the equilibrium branches for $q > 0$ as compared to the one with $q = 0$. In particular that branch II_o intersects branch I at $P^* = 1.0$, whereas all branches II approach branch I asymptomatically at infinity.

Whereas branch II_o represents the usual post-buckling response of beams, branches II exhibit the characteristic response associated with the lift-off problem for $q > 0$.

In order to determine the <u>stability</u> of the equilibrium states on branches I and II, we utilize the Lagrange energy criterion. According to this criterion, an equilibrium configuration of a conservative mechanical system is stable if the corresponding total potential energy Π has a proper minimum with respect to all kinematically admissible displacements.

For the model shown in Fig. 2, the total potential energy Π for $\theta > 0$ is

$$\Pi = 2\left(\tfrac{1}{2} s\theta^2\right) + \tfrac{1}{2} s(2\theta)^2 - 2\left[PL(1-\cos\theta)\right] + 2\left[qL\left(\tfrac{1}{2} L\sin\theta\right)\right]$$

or rewritten

$$\frac{\Pi}{6s} = \tfrac{1}{2}\theta^2 - P^*(1-\cos\theta) + q^*\sin\theta \qquad (3)$$

Equ. (3) was numerically evaluated for $q^* = 0.06$ and the results are presented as energy level curves in Fig. 5(a). The corresponding equilibrium branch, based on equ. (1), is presented in Fig. 5(b).

First, it should be noted that according to the principle of stationary total potential energy,

$$\frac{\partial \Pi}{\partial \theta} = 0 \qquad (4)$$

yields the equilibrium equation (1). Hence, points on the energy level curves with a horizontal tangent, correspond to equilibrium configurations. This correspondence may be easily verified by correlating the graphs in Fig. 5(a) and (b).

According to the Lagrange stability criterion minima on the energy level curves correspond to stable equilibrium configurations, while maxima and horizontal inflection points correspond to unstable equilibrium configurations. Hence, according to the energy level curves shown in Fig. 5, the undeformed equilibrium states (with $\theta = 0$) are stable, the equilibrium positions on branch $\overline{AL}$ are unstable, and the equilibrium positions on branch $\overline{LB}$ are stable (L is defined as the lowest point on a branch II).

Consider branch II for a fixed q^*, say $q^* = 0.06$, shown in Fig. 4 and Fig. 5(b). It may be seen that when $P < P_L$, only one equilibrium configuration is possible, namely the stable undeformed one. However, when $P > P_L$, say $P^* = 1.2$, there exist two stable equilibrium configurations; one on branch I and the other on part $\overline{LB}$ of branch II. For the structure to snap-out from branch I to branch II, the energy barrier $\Delta\Pi_{1,2}$ has to be overcome. The necessary energy could be supplied by outside disturbances such as vibrations or impact loads (caused by a moving train) or by an excessive lift up (during maintenance of a track).

Therefore, for design purposes, it appears reasonable to consider P_L a "safe buckling load" and to stipulate

$$P < P_L \qquad (5)$$

as the stability criterion [10].

It should also be noted that with increasing P the energy barrier $\Delta\Pi$ is decreasing. Hence, for increasing P, smaller disturbances are needed to overcome $\Delta\Pi$ and snap the system into the deformed state with $\theta > 0$. Therefore, although the undeformed equilibrium state is theoretically stable for any P, from a practical point of view, the system becomes less stable with increasing P. For more details on energy barriers, the reader is referred to Ref. [10].

An actual railroad track contains a number of imperfections; although small they may have a profound effect upon the buckling load. To demonstrate this point, we assume that the model in Fig. 2 has a geometric imperfection θ_o, as shown in Fig. 6(a). It is assumed that in this state, the system is stressless. It is anticipated that for large compression forces P, equilibrium states exist for which the beams $\overline{234}$ do not touch the base, as shown in Fig. 6(b).

Considering the equilibrium of the free body diagram of beam $\overline{34}$, or using $\Pi(\theta,\theta_o)$ in conjunction with condition (4), the following equilibrium equation results:

$$P^* = \frac{(\theta-\theta_o) + q^* \cos\theta}{\sin\theta} \tag{6}$$

As anticipated, for $\theta_o = 0$ above equation reduces to equation (1).

The corresponding equilibrium branches are shown in Fig. 7. It may be seen that for $\theta_o = 1^o$ and P increasing from zero, the structure does not deform until P^* reaches the value P_u^*. At P_u^* it snaps out into a strongly deformed equilibrium state on branch II, as indicated in Fig. 7.

The load P_u is denoted in the stability literature as the "upper buckling load". Note, however, that under the influence of outside disturbances, such as vibrations or impulse loads, the system may snap-out at smaller loads $P_L < P < P_u$. Hence, also in this case, stability criterion (5) is valid.

It is of interest to note that P_u diminishes with increasing θ_o and that for $\theta_o > 3.43^o$ the loads P_u and P_L coalesce and no snap-out take place (Fig, 8). It may be shown, using the corresponding energy level curves, that in such cases the entire equilibrium branch is stable. However, a small increase of the axial force beyond $P_u = P_L$ leads to very large deformations (see branch for $\theta_o = 5^o$ in Fig. 7). Therefore, for reasonably small imperfections, criterion (5) is generally valid.

The response of an actual railroad track is governed not by an algebraic equation, such as (1) or (6), but by nonlinear differential equations. Because it is very difficult to solve these equations, the authors of References [3] to [8] analyzed buckling in the vertical plane using a linear differential equation*. It will be shown in the following that the linearization of the governing equations may lead to erroneous results.

In order to study the effect of linearization, equilibrium equation (6) was linearized in θ. It becomes

$$P^{*\ell} = \frac{(\theta-\theta_o) + q^*}{\theta} \tag{7}$$

It should be noted that for $q = 0$ and $\theta_o = 0$, equ. (7) yields

$$P^{*\ell} = P^*_{cr} = 1 \tag{8}$$

namely, the Euler load.

A comparison of the actual with the linearized equilibrium branches is shown in Fig. 9. It may be seen that for the considered $\theta_o > 0$ and $q > 0$, the linearized equilibrium branches are hyperbolas with $P^{*\ell}=1$ as an asymptote.

---

*) It should be noted that the assumption that Π is a quadratic functional for a given problem, is equivalent to the assumption that the problem is governed by linear equations.

By means of the energy level curves, it may be shown that they are unstable. It should also be noted that for the linearized equilibrium branches shown, $P_L^* = 1$ for any θ^o and $q > 0$, whereas for the actual branches the corresponding P_L^* values are higher and depend upon θ_o and q^*. This difference will be much more pronounced in the model for thermal buckling discussed in the following section, which in turn, suggests that the use of linear equations for the analysis of vertical buckling with lift-off may not be admissible.

A MODEL STUDY FOR VERTICAL TRACK BUCKLING DUE TO A UNIFORM TEMPERATURE INCREASE

The rail region to be affected by buckling is represented by four bars constrained at the interconnecting joints by linear spiral springs as shown in Fig. 10(a). It is assumed that the bars deform axially according to Hooke's law and the law of linear thermal expansion, but are "rigid" in bending. The flexural rigidity of the track is represented by the spiral springs at the joints. In order to study the effect of the energy released from the adjoining regions upon the buckling load, each of the two outside bars, $\overline{12}$ and $\overline{45}$, is assumed to be of length nL, where $0 < n < \infty$. To simulate the effect of the track beyond the region under consideration, it is assumed that joints 1 and 5 can not move horizontally.

We consider a model which in its weightless state exhibits the geometrical imperfection θ_o, or h_o, shown in Fig. 10(a). Due to the own weight of structure, the model deforms, as shown in Fig. 10(b). In order to simplify the analysis, it is assumed that tamping of the track model restores h to h_o as shown in Fig. 10(c). In the following it will be shown that in this state, also $\theta = \theta_o$ and hence the model is stressless. We then analyze the model when it is subjected to a small uniform temperature increase which causes only a horizontal motion of joints 2 and 4, as shown in

Fig. 10(d). This is followed by an analysis of the case when a further increase of the temperature causes also a lift-off of joint 3, as shown in Fig. 10(e).

To prove that the undeformed state, shown in Fig. 10(c), is in equilibrium when subjected to q, we form the corresponding Π. Since in this case the work potential of q is equal zero, it follows that

$$\Pi = 3s\,(\theta-\theta_o)^2 + 2\left[\tfrac{1}{2}\,k\,(\Delta L_1)^2 + \tfrac{1}{2}k_n(\Delta L_2)^2\right] \tag{9}$$

where

$$\begin{aligned} \Delta L_1 &= L - L' = L\left(1 - \frac{\sin\theta_o}{\sin\theta}\right) \\ \Delta L_2 &= nL - (nL + u_o) = -u_o \end{aligned} \tag{10}$$

Noting that

$$u_o = L\cos\theta_o - L'\cos\theta \quad ; \qquad k_n = k/n \text{ [1)]} \tag{11}$$

we obtain

$$\Pi(\theta) = 3s\,(\theta-\theta_o)^2 + kL^2\left(1 - \frac{\sin\theta_o}{\sin\theta}\right)^2 + \frac{kL^2}{n}\left(-\cos\theta_o + \sin\theta_o\frac{\cos\theta}{\sin\theta}\right)^2 \tag{12}$$

It may be easily verified, that $\partial\Pi/\partial\theta = 0$ is satisfied for $\theta = \theta_o$. Hence, for the equilibrium state shown in Fig. 10(c) the model is stressless.

Next we analyze the equilibrium state shown in Fig. 10(d). The corresponding Π is as given in (9), with

$$\begin{aligned} \Delta L_1 &= L(1+\alpha T) - L' = L\left(1 + \alpha T - \frac{\sin\theta_o}{\sin\theta}\right) \\ \Delta L_2 &= nL\,(1+\alpha T) - (nL + u) = L\left(n\alpha T - \cos\theta_o + \sin\theta_o\frac{\cos\theta}{\sin\theta}\right) \end{aligned} \tag{13}$$

where T is the temperature change. Hence

$$\Pi(\theta) = 3s\,(\theta-\theta_o)^2 + kL^2\left(1 + \alpha T - \frac{\sin\theta_o}{\sin\theta}\right)^2 + \frac{kL^2}{n}\left(n\alpha T - \cos\theta_o + \sin\theta_o\frac{\cos\theta}{\sin\theta}\right)^2 \tag{14}$$

---

1) For a beam of length L the axial force is $N = k\,\Delta L = EA\,\Delta L/L$, where A is cross sectional area and E is Young's modulus. For a beam of length (nL), $N = k_n \Delta L = EA\,\Delta L/(nL)$. Hence $k_n = k/n$.

and the corresponding equilibrium equation is

$$(\theta-\theta_o)\ \sin^2\theta + k^*\left(1+\alpha T - \frac{\sin\theta_o}{\sin\theta}\right)\sin\theta_o\cos\theta$$

$$-\frac{1}{n}\ k^*\left(n\alpha T - \cos\theta_o + \sin\theta_o\ \frac{\cos\theta}{\sin\theta}\right)\ \sin\theta_o = 0 \tag{15}$$

where

$$k^* = \frac{kL^2}{3s}$$

Solving above equation for $\alpha T k^*$ we obtain

$$\alpha T k^* = \frac{\frac{k^*}{n}\left(\cos\theta_o - \sin\theta_o\ \frac{\cos\theta}{\sin\theta}\right) + k^*\left(1 - \frac{\sin\theta_o}{\sin\theta}\right)\cos\theta + \frac{(\theta-\theta_o)\ \sin^2\theta}{\sin\theta_o}}{(1-\cos\theta)} \tag{16}$$

Next, we analyze the structure after lift-off takes place as shown in Fig. 10(e). Since q is the weight of the track, it follows that $qL = q'L'$. Setting

$$u = \varepsilon(nL) \tag{17}$$

and noting that for the present case

$$\begin{aligned} \Delta L_1 &= L\left(1+\alpha T - \frac{\cos\theta_o - \varepsilon n}{\cos\theta}\right) \\ \Delta L_2 &= nL(\alpha T - \varepsilon) \end{aligned} \tag{18}$$

we obtain

$$\begin{aligned} \Pi(\theta,\varepsilon) &= 3s(\theta-\theta_o)^2 + kL^2\left(1+\alpha T - \frac{\cos\theta_o - \varepsilon n}{\cos\theta}\right)^2 + knL^2\ (\alpha T - \varepsilon)^2 + \\ &\quad + qL^2\left[(\cos\theta_o - \varepsilon n)\ \frac{\sin\theta}{\cos\theta} - \sin\theta_o\right] \end{aligned} \tag{19}$$

The corresponding two equilibrium equations are derived from the two conditions

$$\frac{\partial\Pi}{\partial\theta} = 0 \quad ; \quad \frac{\partial\Pi}{\partial\varepsilon} = 0 \tag{20}$$

The first condition yields

$$(\theta-\theta_o)\ \cos\theta - k^*\left(1+\alpha T - \varepsilon^*\right)\ \varepsilon^*\sin\theta + q^*\varepsilon^* = 0 \tag{21}$$

where

$$\varepsilon^* = \frac{\cos\theta_o - \varepsilon n}{\cos\theta} \tag{22}$$

The second condition yields an equilibrium equation which can be solved explicitly for ε^*. It is

$$\varepsilon^* = \frac{n(1+\alpha T) + (\cos\theta_o - n\alpha T)\cos\theta - \frac{q^* n}{k^*}\sin\theta}{(n+\cos^2\theta)} \tag{23}$$

Thus, equations (21) and (23) are the two equilibrium equations. The axial force in the lift-off region is

$$\tilde{N} = k\Delta L_1 = kL(1+\alpha T - \varepsilon^*) \tag{24}$$

where ε^* is given in (22). Or rewritten

$$\tilde{N}^* = \frac{\tilde{N}L}{3s} = \frac{\left[\alpha T k^*(n+\cos\theta) + k^*(\cos\theta - \cos\theta_o)\right]\cos\theta + q^* n\sin\theta}{n+\cos^2\theta} \tag{25}$$

To study the response of the model under consideration, the above equilibrium equations were evaluated, at first, for the case when track model is perfectly straight, i.e. when $\theta_o = 0$. Equilibrium equation (15) is satisfied by $\theta = \theta_o = 0$ for any $q > 0$, αT, k^*, and n. The numerical evaluation of equ.'s (21) and (23) for $\theta > 0$ is shown in Fig. 11 (a).

The general characteristic of this equilibrium branch is very similar to the branches II shown in Fig. 7. Also this branch does not bifurcate from the undeformed branch, but approaches it asymptotically as $\theta \to 0$.

By means of energy level curves it may be shown that the undeformed branch ($\theta = 0$) is stable for any αT, that the branch to the left of point

L is unstable, and that the branch to the right of point L is stable. Hence, when $T < T_L/(\alpha k^*)$, the only possible state of equilibrium is the one with $\theta = 0$, which is stable. But when $T > T_L/(\alpha k^*)$ there exist three equilibrium states; two stable ones and an unstable one. Thus, any $T < T_L$ may be considered a "safe" temperature increase.

The axial force in the lift-off region obtained from equ. (25) for $\theta_o = 0$ is shown in Fig. 11(b). Note, that according to (25), to $\theta = \theta_o = 0$ there corresponds $\tilde{N}^* = \alpha T k^* = T^*$. From Fig. 11(b) it may be seen that, for example, when $T = 8/(\alpha k^*)$ the axial force drops, due to the snap-out, to about a quarter of its value.

To study the effect of geometrical imperfections, equ. (15) or (16) and (21) and (23) were evaluated for $q^* = 0.06$ and $\theta_o = 0°$, $\frac{1}{2}°$, $1°$, and $2°$ as well as for $q = 0$ and $\theta_o = 0°$. The obtained equilibrium branches are shown, as solid lines, in Fig. 12.

Although the general nature of these branches is similar to those shown in Fig. 7, an important difference should be noted. Namely, the deformed equilibrium states, which correspond to the points L, exhibit deformations of only a few degrees, whereas the corresponding equilibrium states in Fig. 7 exhibit much larger deformations (over 30°).

Thus, after snap-out takes place the deformations are of a different order of magnitude when track model is subjected to a thermal compression force N, which reduces after track buckles out, or to a mechanical force P, as shown in Fig. 2.

This finding also implies that a lift-up of joint 3 by a θ of only 1.5° is sufficient to cause a snap-out of the track model when subjected to thermal forces, whereas when model is subjected to a mechanical force P, much higher lift-ups are permissible. The question of "allowable lift-up" is of importance in connection with the maintenance of a track at high temperature.

Also shown in Fig. 12, as dashed lines, are the equilibrium branches which correspond to the linearized equ.'s (21) and (23) (with respect to θ) which become

$$(\theta-\theta_o) - k^* (1+\alpha T - \varepsilon^*) \varepsilon^* \theta + q^* \varepsilon^* = 0 \qquad (21')$$

and

$$\varepsilon^* = 1 - \frac{n}{1+n} \frac{q^*}{k^*} \theta$$

Using the corresponding energy level curves, it may be shown that the presented linearized branches are unstable.

It should be noted that the T_L^* - value for a track model with a small imperfection, is much higher than $T^* = 1$, which is the T_L^* - value for <u>all</u> linearized branches, and is also the Euler load when $q = 0$ and $\theta_o = 0$. For example, for $\theta_o = 0.5°$, the exact $T_L^* = 4.15$, whereas the linearized analysis yields $T_L^{*\ell} = 1.0$.

In formulating the differential equation for the response of an actual track, various assumptions are made in the literature regarding ballast resistance to axial displacements of the track. Some authors assume that the friction force at each point of the adjoining regions is constant, whereas others assume that the ballast resistance at each point is proportional to the axial displacements at this point. The different assumptions vary the elastic energy which is released by the adjoining regions during buckling which in turn affects the safe temperature increase, T_L.

To study this effect, equations (16), (21) and (23) were evaluated for different values of n. The results are shown in Fig. 13. From these graphs, it follows that the T_L^* - values are not very sensitive to small variations of n, which correspond to the different assumptions for ballast resistance.

Also shown in Fig. 13 is the equilibrium branch for the linearized equations. It may be seen that for the linearized equilibrium equation, $T_L^* = 1$ for any $n \geqq 0$. Thus, according to the linearized equilibrium equations, the variation of n has no effect upon T_L, which differs from the actual response.

When designing a railroad track it is of interest to know the effect of heavier ties upon T_L. To study this effect on the track model, equations (16), (21) and (23) were evaluated for different values of q^* and the results are shown in Fig. 14. From the obtained graph it follows that doubling of the weight of the track model (without changing its flexural rigidity), raises significantly the value of T_L.

Also shown in Fig. 14 are the equilibrium branches for the linearized equations and different q^*-values. It may be seen that irrespective of the value q^*, all these branches approach asymptotically $T_L^* = 1$. Thus, according to the linearized equilibrium equations also a change of q^* has no effect upon T_L, which differs significantly from the actual response.

CONCLUSIONS

The analyses of the two models, and the presented graphs, show that

(1) the response of the structure when $q \neq 0$ is very different from the response when $q = 0$.

(2) when $q \neq 0$, which is the case for the vertical buckling analysis of the structure under consideration, the linearized equilibrium equations are not suitable for predicting a "safe" buckling load, or a "safe" temperature increase.

(3) the Euler buckling load, which results from a linearized analysis [for example, eq. (8)], has no relevance to the actual problem under consideration.

These findings suggest the possibility that the results of vertical buckling analyses for an actual track, by means of differential equations, will exhibit a similar pattern. Namely:

(a) that the actual postbuckling response of the track will be of the same type as the <u>nonlinear</u> branches II, with a "safe" buckling load P_L or "safe" temperature increase T_L, respectively, and

(b) that the corresponding equilibrium branches of a <u>linearized</u> analysis will, with increasing deformations, approach the value of the corresponding Euler load. This second point may be easily verified using results of Reference [3-8].

It appears, in view of the above, that all published results for vertical buckling, which are based on linear formulations, should be used with caution.

note: Buckling load for thermal case $(N_L^*)_{mechanical} << (N_L^*)_{thermal}$

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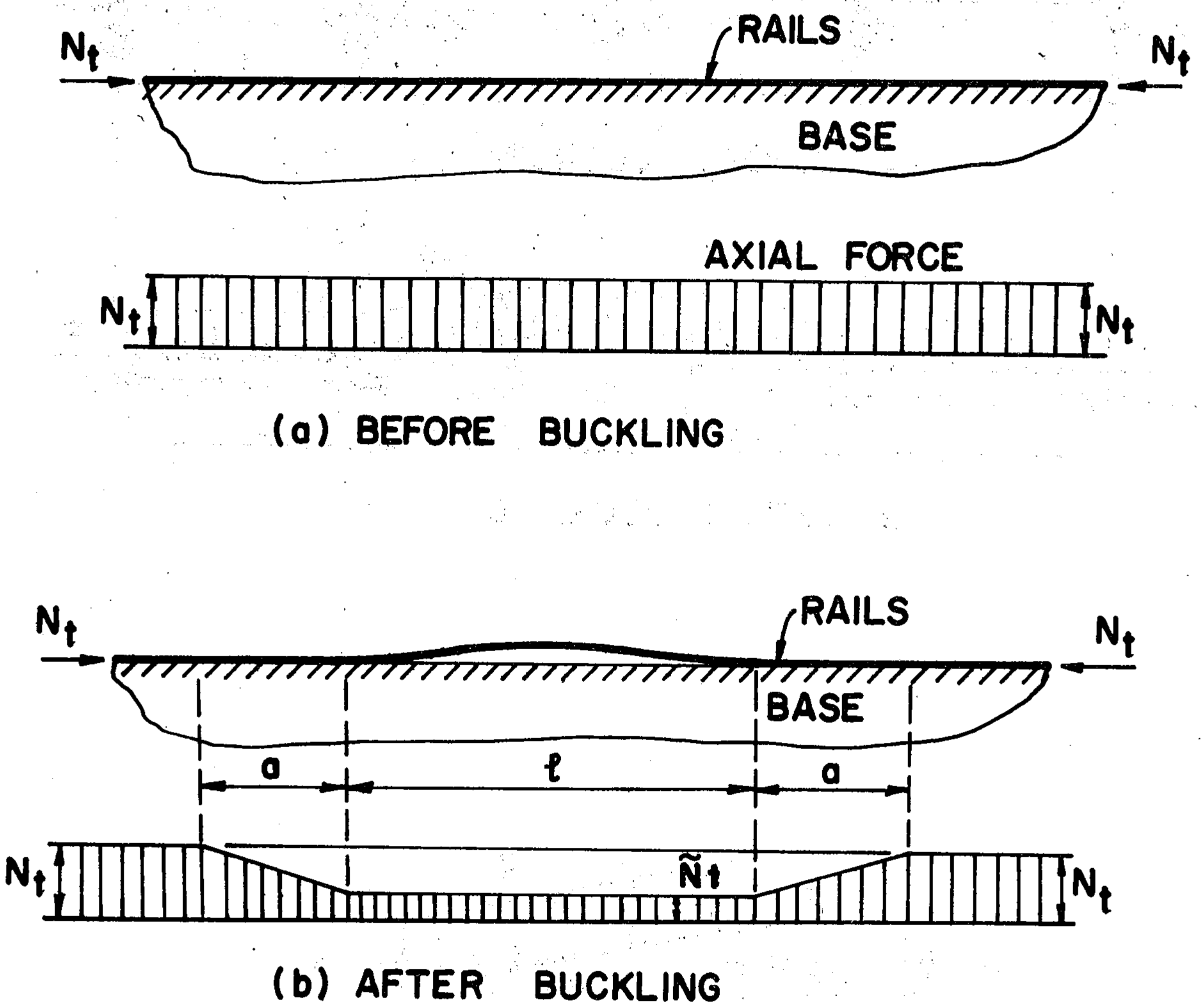


N t
RAILS
N t
BASE
AXIAL FORCE
N t
N t
(a) BEFORE BUCKLING
N t
RAILS
N t
BASE
a
ℓ
a
N t
Ñt
N t
(b) AFTER BUCKLING

Fig. 1

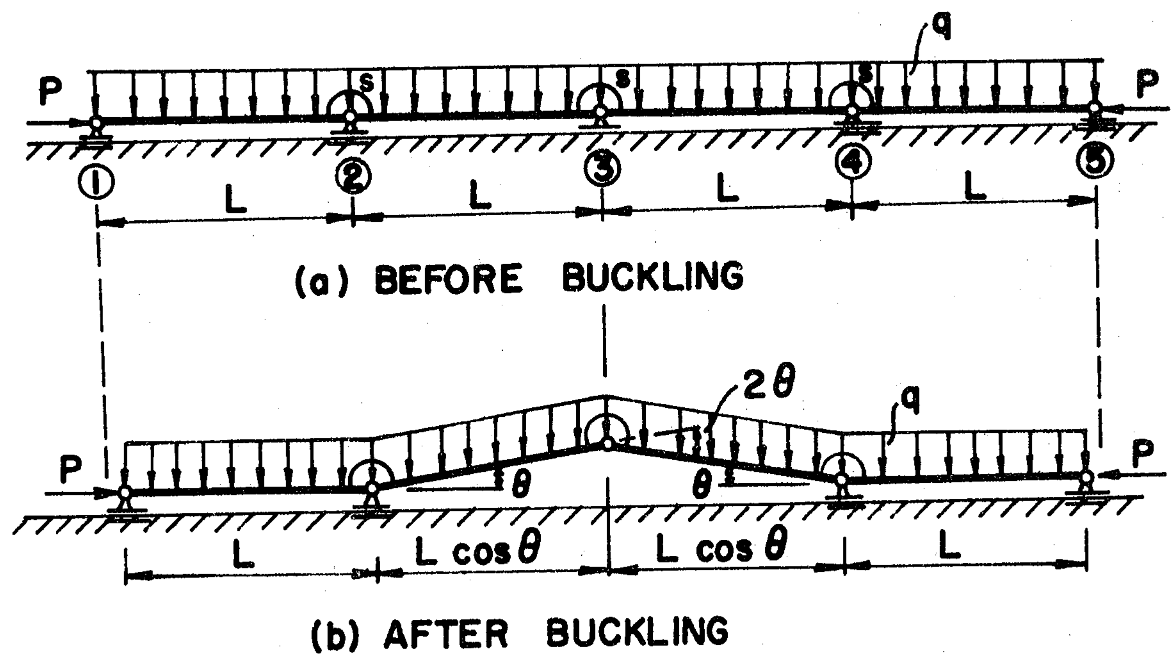

q
P
s
s
s
P
1
2
3
4
5
L
L
L
L
(a) BEFORE BUCKLING
2θ
q
P
θ
θ
P
L
L cos θ
L cos θ
L
(b) AFTER BUCKLING

Fig. 2

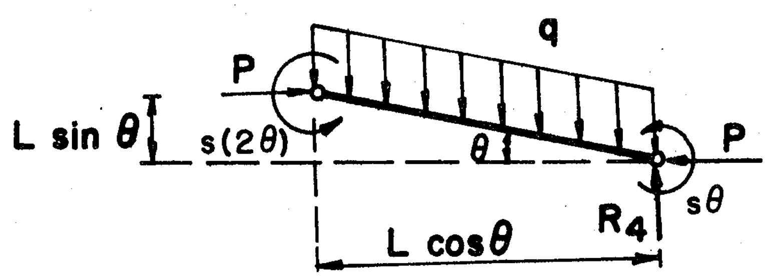

q
P
L sin θ
s(2θ)
θ
P
R4
sθ
L cosθ

Fig. 3

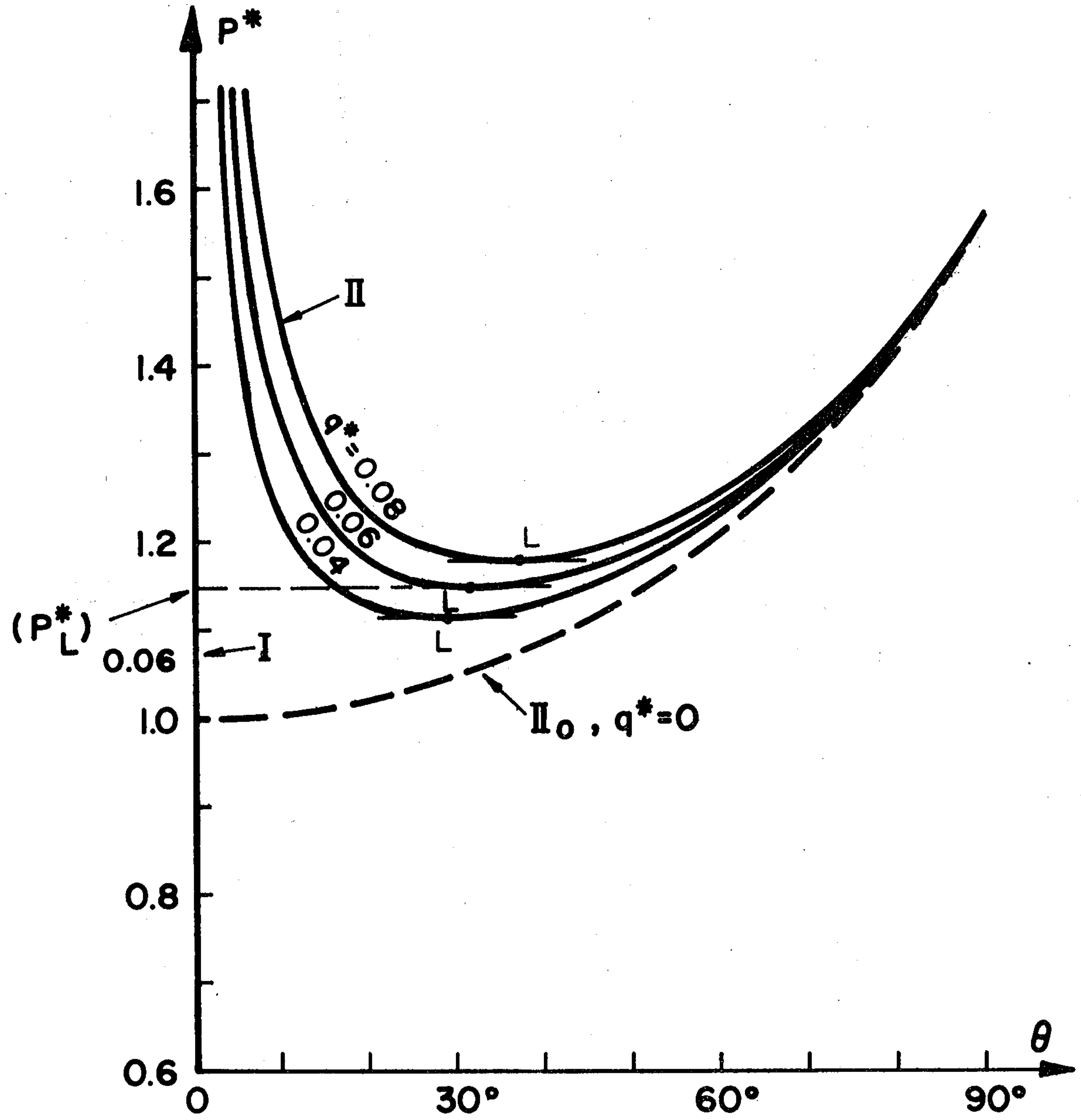
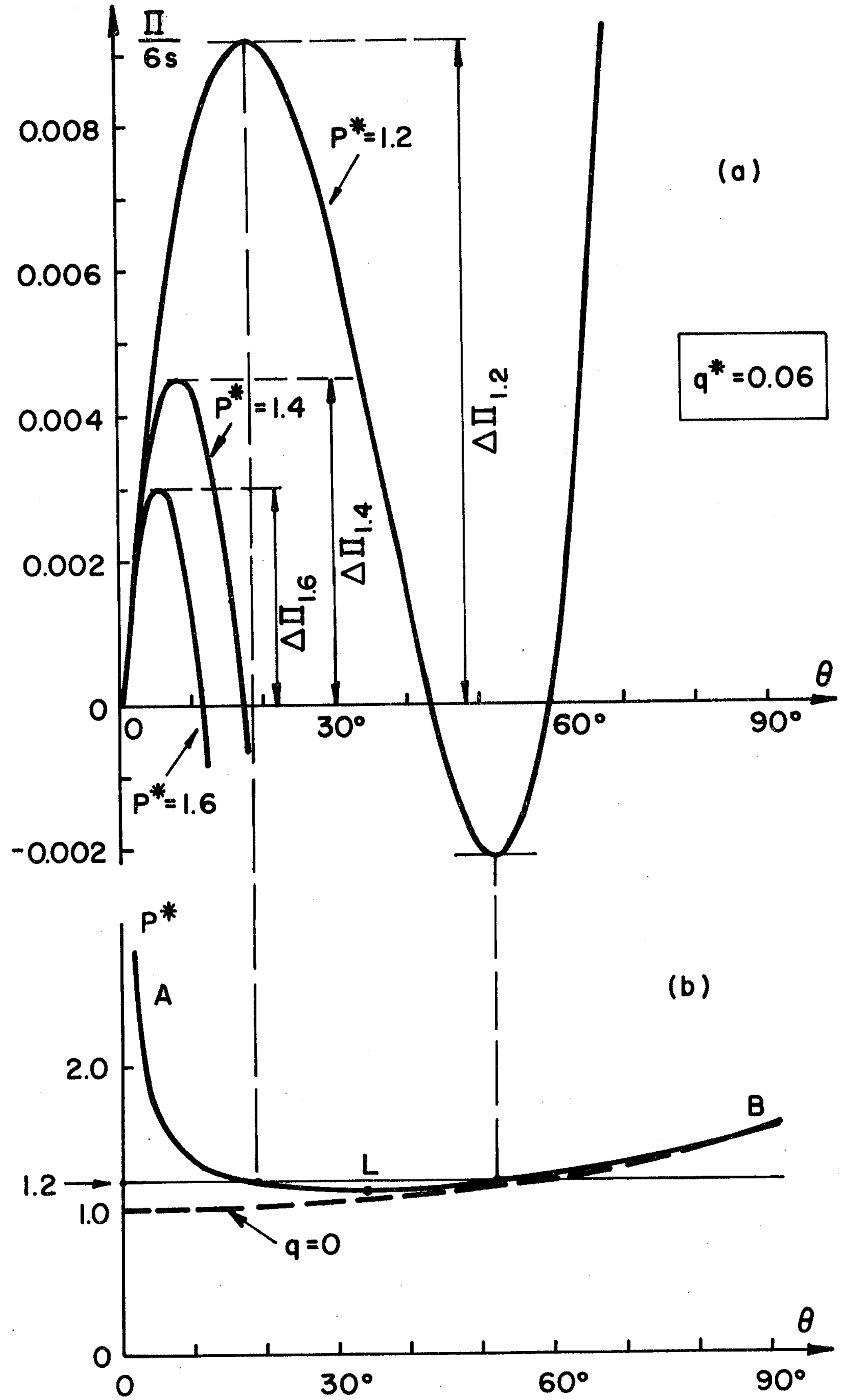
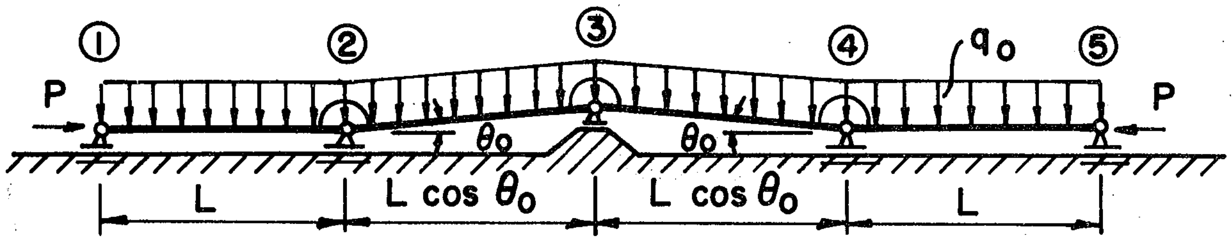


Fig. 4

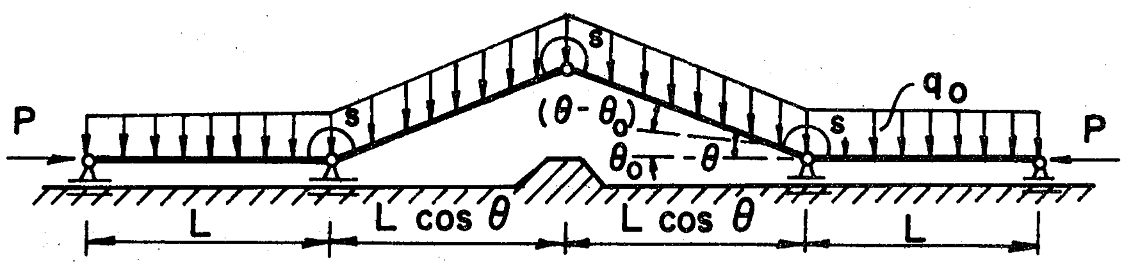


Π/6s
0.008
0.006
0.004
0.002
0
-0.002
P*=1.2
P*=1.4
P*=1.6
(a)
q*=0.06
ΔΠ1.2
ΔΠ1.4
ΔΠ1.6
θ
0
30°
60°
90°
P*
A
(b)
2.0
1.2
1.0
L
B
q=0
θ
0
30°
60°
90°

Fig. 5



(a) BEFORE LIFT-OFF



(b) AFTER LIFT-OFF

Fig. 6

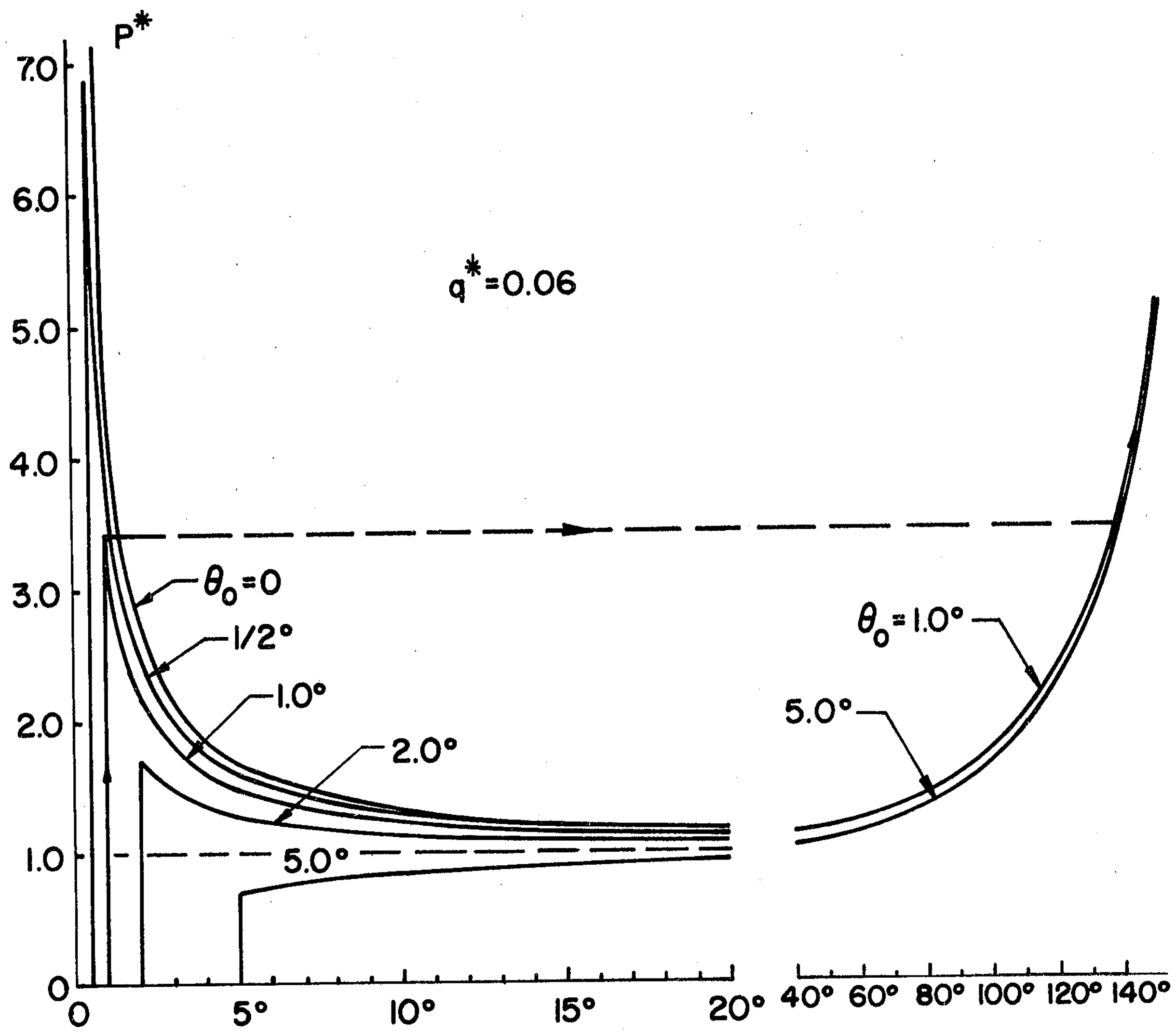

P*
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
q*=0.06
θ0=0
1/2°
1.0°
2.0°
5.0°
θ0=1.0°
5.0°
0
5°
10°
15°
20°
40° 60° 80° 100° 120° 140°

Fig. 7

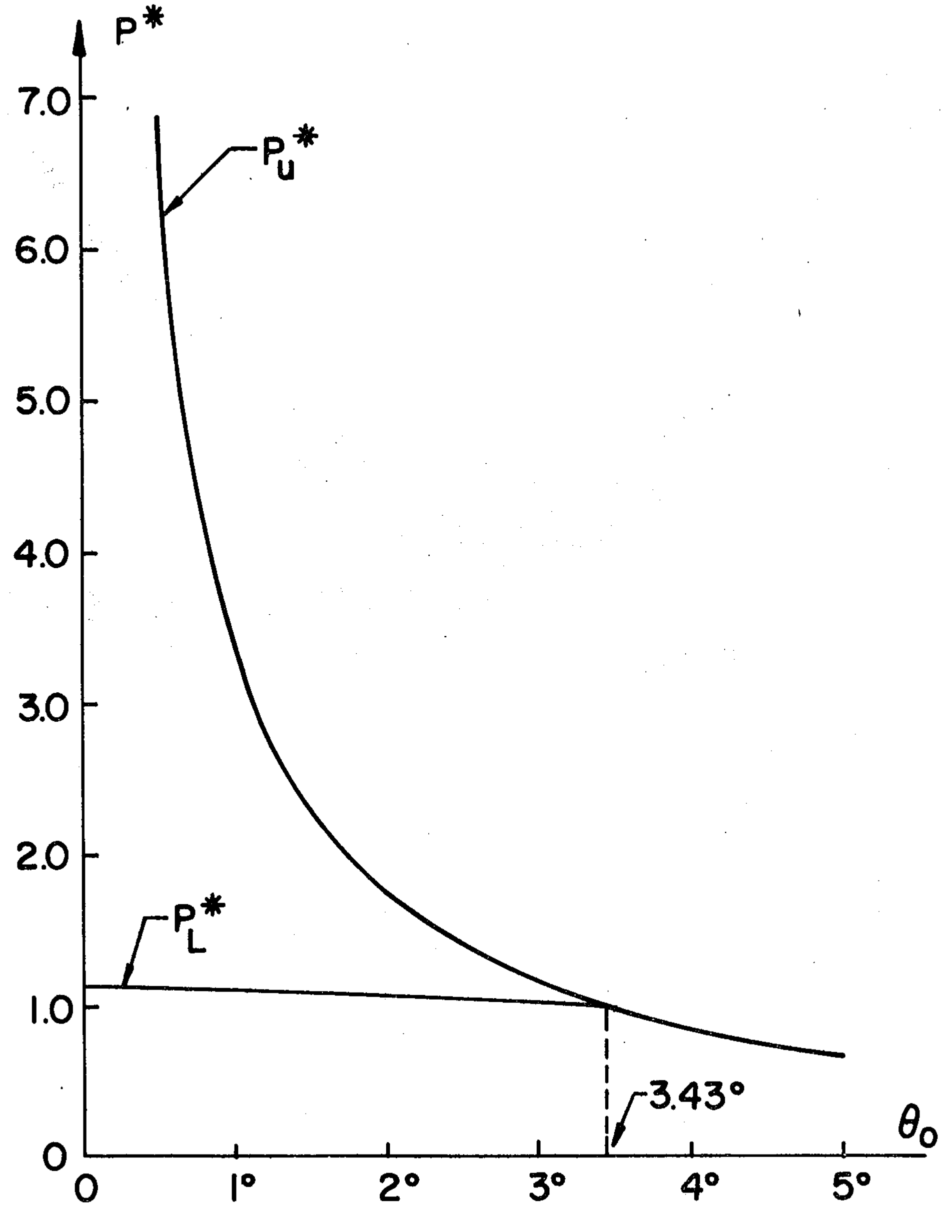


Fig. 8

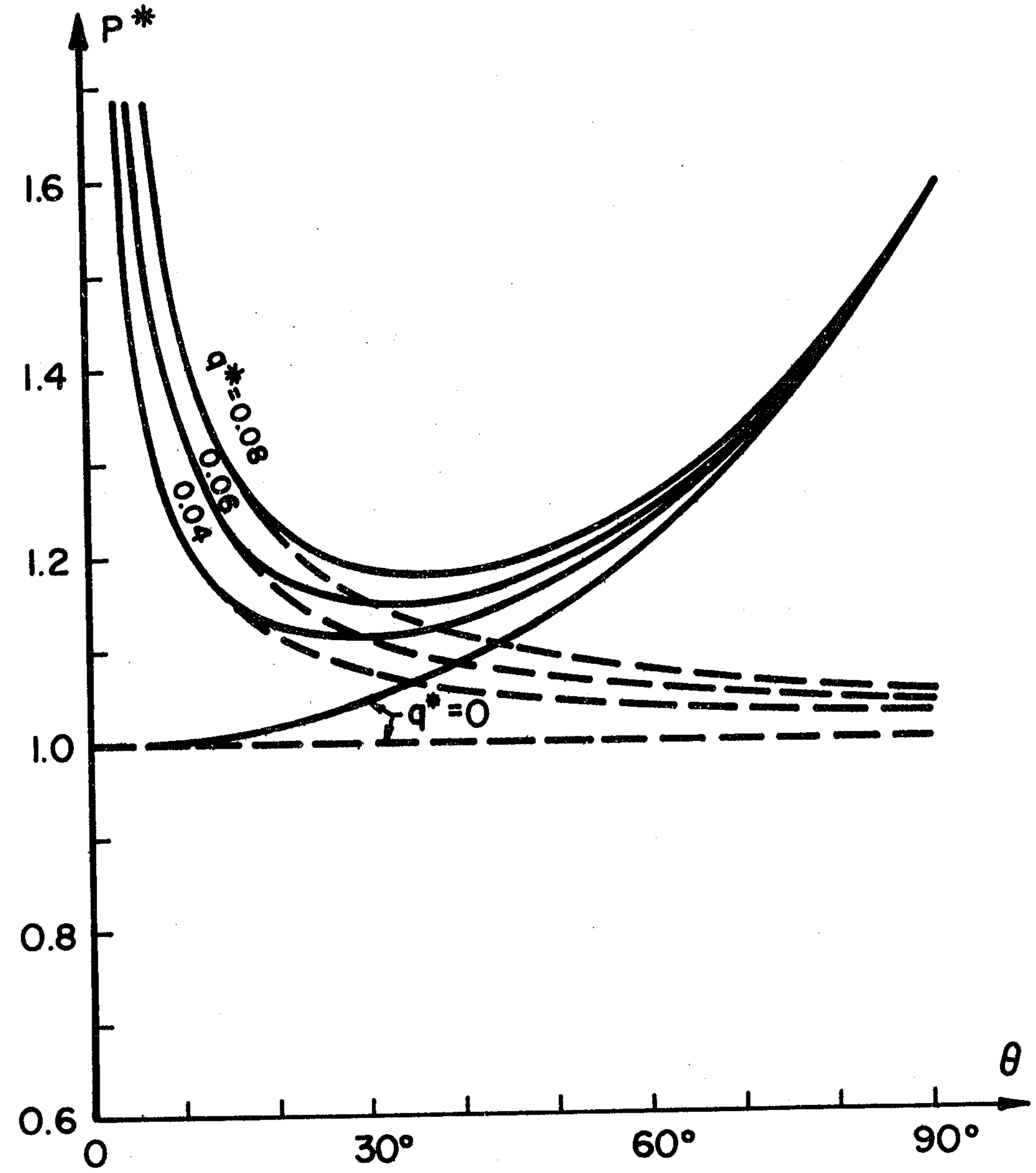
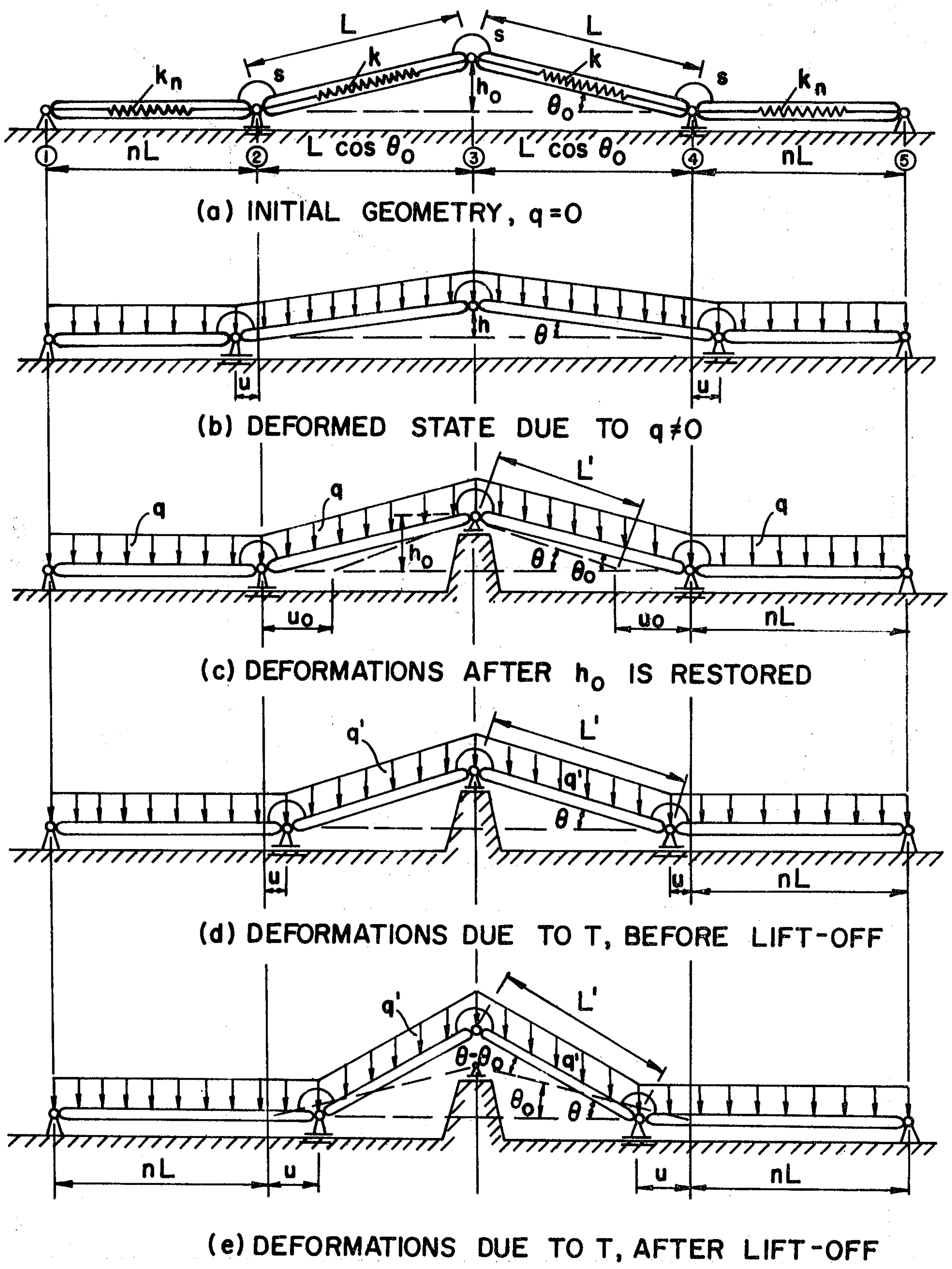


Fig. 9

(———— nonlinear, - - - - - linearized)



L
L
s
s
s
k
k
k_n
k_n
h_o
θ_o
1
2
3
4
5
nL
L cos θ_o
L cos θ_o
nL
(a) INITIAL GEOMETRY, q = 0
h
θ
u
u
(b) DEFORMED STATE DUE TO q≠0
L'
q
q
q
h_o
θ
θ_o
u_o
u_o
nL
(c) DEFORMATIONS AFTER h_o IS RESTORED
L'
q'
q'
θ
u
u
nL
(d) DEFORMATIONS DUE TO T, BEFORE LIFT-OFF
L'
q'
q'
$\theta-\theta_o$
θ_o
θ
nL
u
u
nL
(e) DEFORMATIONS DUE TO T, AFTER LIFT-OFF

Fig. 10

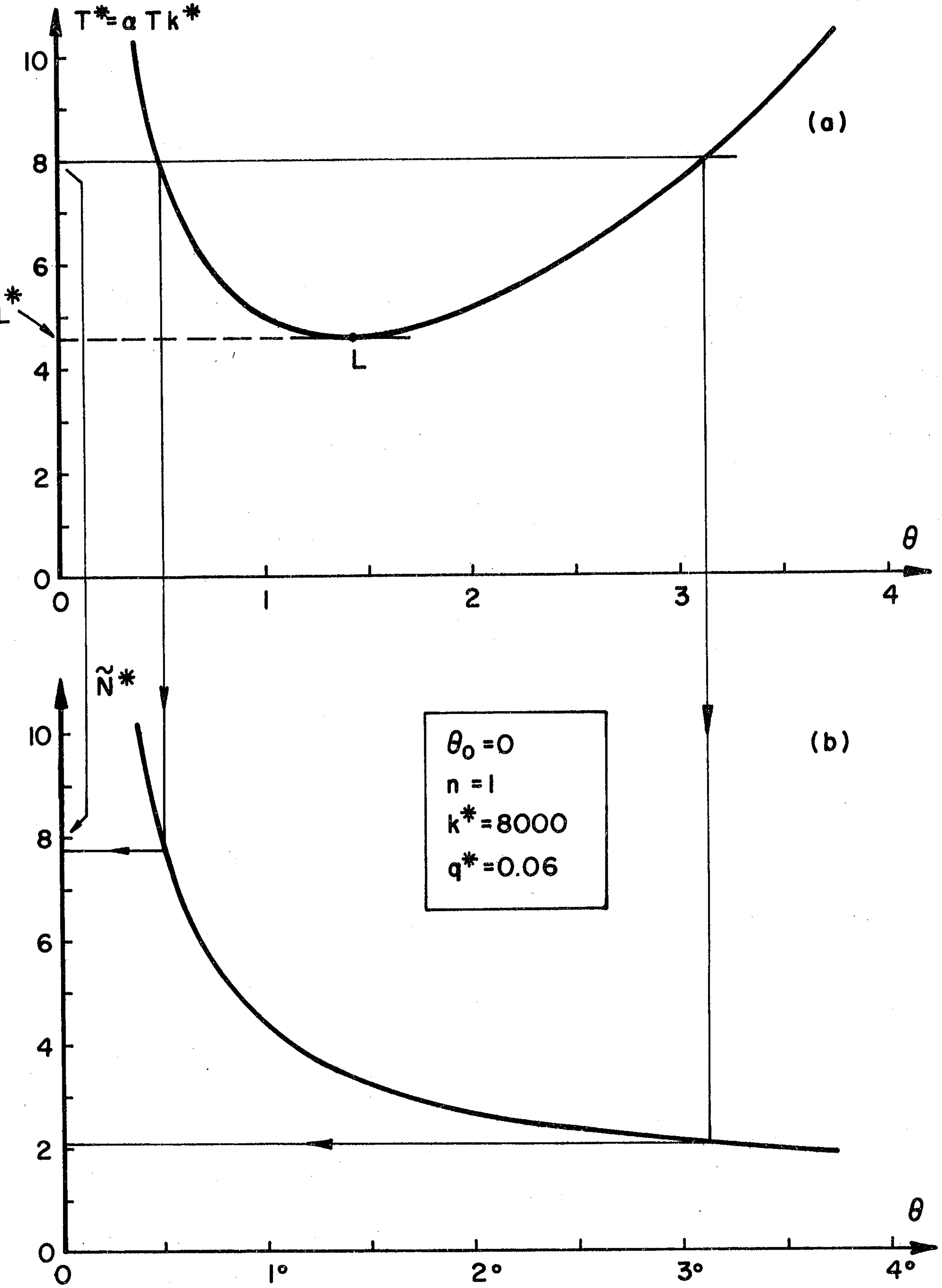


Fig. 11

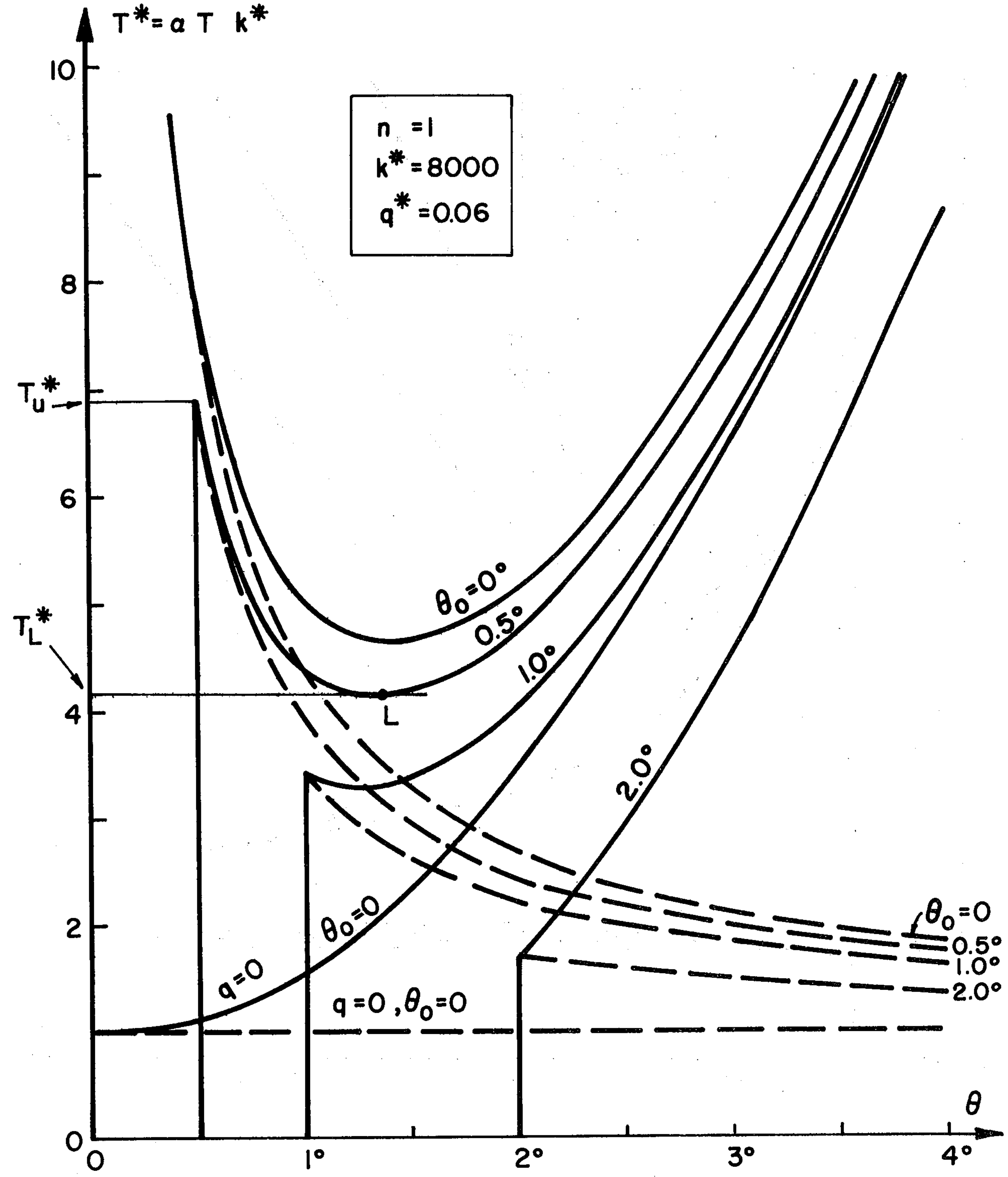


Fig. 12

(——— nonlinear, – – – – – linearized)

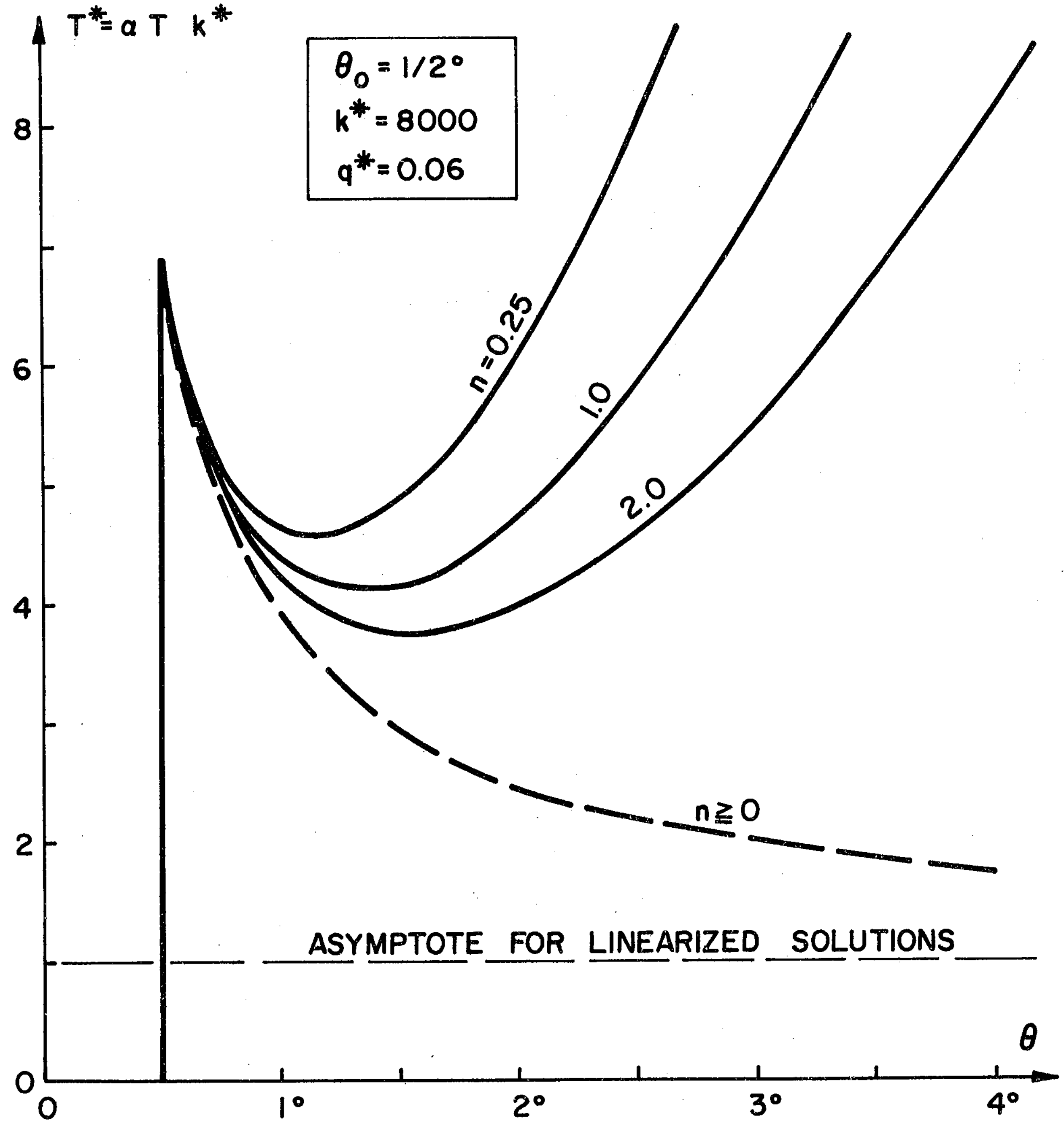


Fig. 13

(——— nonlinear, – – – linearized)

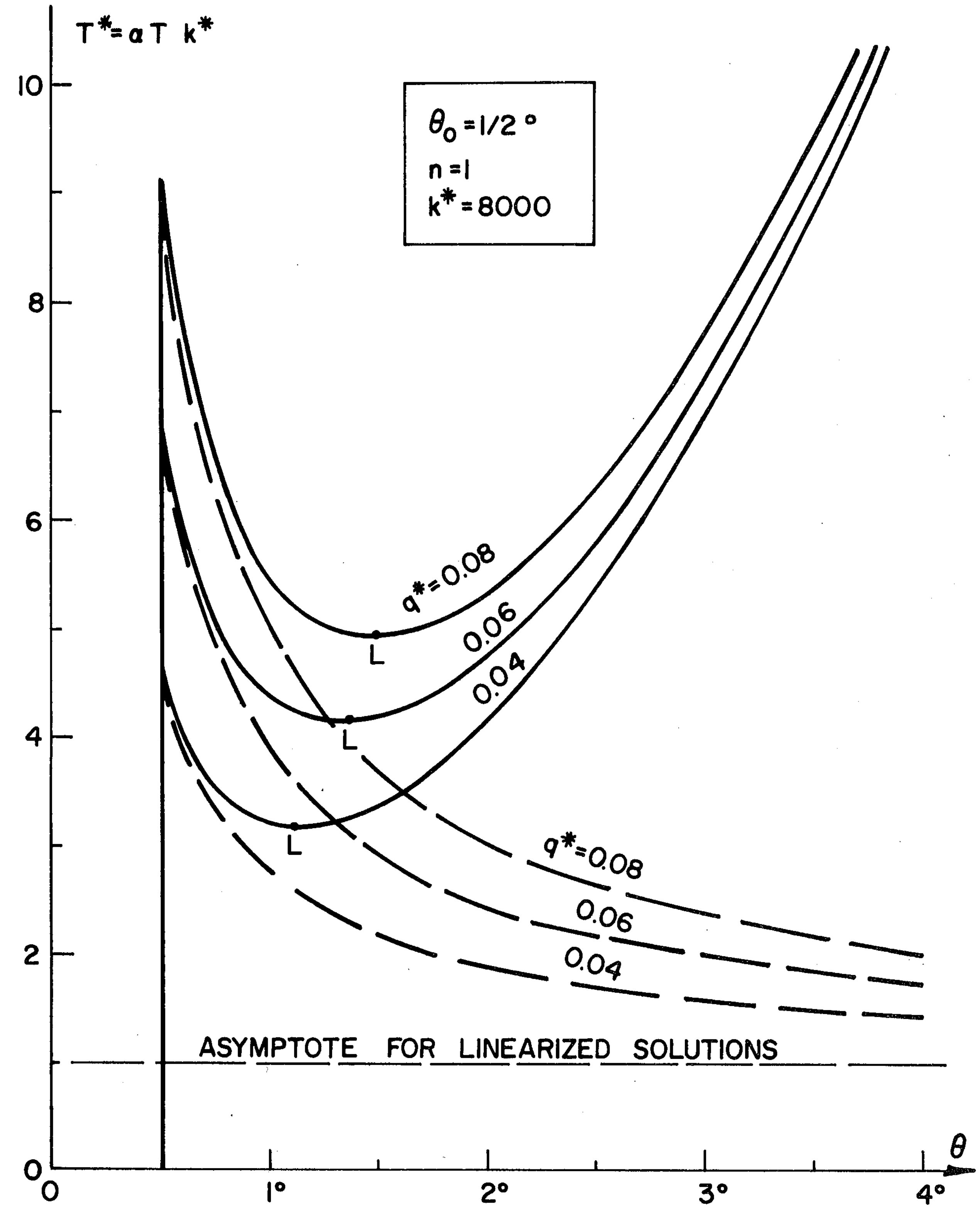


Fig. 14

(——— nonlinear, – – – – linearized)

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Point Set Topology

1. Point Set

<u>Equality</u> of Point Sets $E_1 = E_2 \iff E_1 \subset E_2$ and $E_2 \subset E_1$

<u>Complement</u> of a set; Let $E \equiv$ set in a given "universe"

$CE \equiv$ all pts not in E

<u>null set</u> ϕ is empty (no points)

<u>Product</u> (Intersection) of sets $E_1 \cap E_2$ all pts belonging to E_1 and E_2

If more than two sets $\cap E_i$ $i = 1, 2, \ldots$ all pts belong to all the sets

<u>Union of Sets</u>: $E_1 \cup E_2$ if $x \in (E_1 \cup E_2) \rightarrow x \in E_1$ or $x \in E_2$

$\cup E_i$ $i = 1, 2, \ldots$ $x \in E_k$ for one k

<u>Difference of two sets</u>: $E_1 - E_2$ if $x \in E_1$ but $x \notin E_2$

$$E_1 \cup E_2 = E_1 \cap E_2 + E_1 - E_2$$

<u>Disjoint Sets</u>: two sets E_1 and E_2 are disjoint if $E_1 \cap E_2 = \phi$

Generalize $E_1, E_2, \ldots$ are disjoint if $\cap E_i = \phi$

<u>Properties</u>

$E_1 \cup E_2 = E_2 \cup E_1$ (Commutative law for union)

$E_1 \cap E_2 = E_2 \cap E_1$ (" " " intersect)

$(E_1 \cup E_2) \cup E_3 = E_1 \cup (E_2 \cup E_3)$ (Assoc. law for union)

$(E_1 \cap E_2) \cap E_3 = E_1 \cap (E_2 \cap E_3)$ (" " " intersect)

$E_1 \cap (E_2 \cup E_3) = (E_1 \cap E_2) \cup (E_1 \cap E_3)$ Distrib

Proof: $x \in \{E_1 \cap (E_2 \cup E_3)\} = \begin{matrix} x \in E_1 \\ x \in E_2 \cup E_3 \end{matrix} = \begin{matrix} x \in E_1 \;\&\; x \in E_2 \\ \text{or} \\ x \in E_1 \;\&\; x \in E_3 \end{matrix} \quad \begin{matrix} x \in E_1 \cap E_2 \\ \text{or} \\ x \in E_1 \cap E_3 \end{matrix}$

if $x \in E_1 \cap E_2$ or $x \in E_1 \cap E_3$ then $x \in (E_1 \cap E_2) \cup (E_1 \cap E_3)$

Let $x \in \{(E_1 \cap E_2) \cup (E_1 \cap E_3)\} \Rightarrow \left\{ \begin{matrix} x \in E_1 \cap E_2 \\ x \in E_1 \cap E_3 \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} x \in E_1 \;\&\; x \in E_2 \\ x \in E_1 \;\&\; x \in E_3 \end{matrix} \right\}$

Show $C(CE) = E$

let $x \in C(CE) \longrightarrow x \notin CE \Rightarrow x \in E$

$x \in E \longrightarrow x \notin CE \longrightarrow x \in C(CE)$

Proper Subset of a set: E_1 is a proper subset of E_2 if

1. $E_1 \subset E_2$
2. $E_2 \not\subset E_1$

note; If $E_1 \cap E_2 = \phi$, then $E_1 - E_2 = E_1$

Problems

a) $E_1 - E_2 = E_1 \cap (CE_2)$

b) $E_1 \cap E_2 = C[CE_1 + CE_2]$

$$C(E_1 \cup E_2) = CE_1 \cap CE_2$$

$$C(E_1 \cap E_2) = CE_1 \cup CE_2$$

$$C(E_1 - E_2) = CE_1 \cup E_2$$

ϵ - nbd of a pt p of a set: $\epsilon > 0$

$$N_\epsilon(p) = \{x;\ |x-p| < \epsilon\}$$

Interior pts - $p \in S$ is an interior pt if $\exists\ N_\epsilon(p) \subset S$

Boundary pt of a set for every $\epsilon > 0$ $N_\epsilon(p) \cap S \neq \phi$
$N_\epsilon(p) \cap CS \neq \phi$

Deleted nbd of a pt p of a set S. $x \in S \ni 0 < |x-p| < \epsilon$

Limit Pt (Cluster Pt, Accumulation Pt) of a set:
if every deleted neighborhood of p contains points of set

p need not belong to S

Isolated pt of a set

deleted $N_\epsilon(p) \subset CS$

$\exists \{N_\epsilon(p) - p\} \cdot S \neq \phi$

<u>Derived Set</u> S' <u>of a set S</u>: set of all limit pts of S

$S = \{x; \quad 2 < x < 5\} \qquad S' = \{x; 2 \leq x \leq 5\}$

<u>Closure $\bar{S}$ of a set</u>:

$\bar{S} = S + S'$

<u>Closed Set</u> S is closed if $S' \subset S$

<u>Open Set</u> if every pt is an interior pt of S

<u>Note</u>; $S = \{2, 3, 4\} \quad S' = \phi$

Accept: null set is a member of every set.

$\therefore S = \{2, 3, 4\}$ is closed

Thms:

(a) if E is open then CE is closed

Let E be open show CE is closed, show CE contains all its limit pts

Let x be a limit pt of CE Show $x \in CE$

Suppose $x \notin CE \rightarrow x \in E \rightarrow$ x is an interior pt of $E \rightarrow \exists N_\epsilon(x) \subset E$

$\rightarrow N_\epsilon(x) \not\subset CE \rightarrow$ x is not a limit pt of CE

(b) if E is closed then CE is open

(c) The sum or union of any no of open sets is open

(d) The intersection of a finite no. of open sets is open.

(e) The product of an infinite no of closed sets is closed

(f) " union of a finite no of closed sets " "

(g) S' is closed

(h) Closure of set $\bar{S} = S + S'$ is closed

(i) every interior pt of a set is a limit pt.

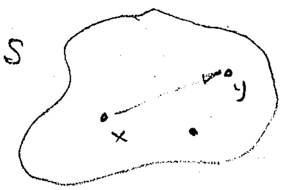
universe & null set both open & closed.

2-21-73

$d: S \times S \to R$ $\qquad (x,y) \xrightarrow{d} d(x,y)$

1. $d(x,y) \geq 0 \qquad d(x,y) = 0 \iff x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z)$

S



$|x-y| < \rho$ nbd.

$d(x,y) < \rho \qquad (\rho > 0)$

Courant - McShane (Vol II)

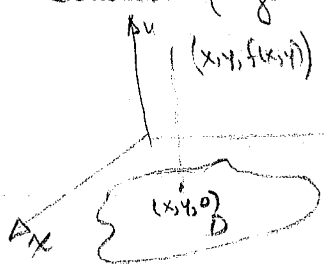
Chapter I Diff & Integral Calculus

Vectors in 2 & 3 space

Chapter II

functions of 2 or more variables

domain (region in a plane) $\qquad f: R \times R \to R,\ (x,y) \xrightarrow{f} f(x,y)$ image of x,y



$f: D \subset R \times R$

$ax + by + cu = d \qquad$ plane

$\log(1 - x^2 - y^2) \qquad \underline{x^2 + y^2 < 1}$

domain of def since log(arg) is defined for argument > 0

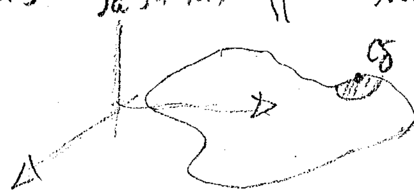
P.41 Examples of regions:

simply connected, multiply-connected regions.

Continuity $\quad$ single variable $\quad \forall \epsilon > 0\ \exists\ \delta = \delta(\epsilon, a) > 0 \ni |f(x) - f(a)| < \epsilon$ for $|x-a| <$

multiple var $\quad \forall \epsilon > 0\ \exists\ \delta = \delta(\epsilon, x_0, y_0) > 0 \ni |f(x,y) - f(x_0,y_0)| < \epsilon$ for $d[(x_0,y_0),(x,$ $< \delta$

in a closed interval U.C. exists $\Rightarrow \int_a^b f(x)\,dx$ $\quad \|$ at bdy $C_\delta(x_0,y_0) \cap R$



a fn may be continuous for each variable separately but not as a fn of all variables

i.e. $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} ; & x^2+y^2 > 0 \\ 0 & x=y=0 \end{cases}$

let $y=kx$ & $x \to 0$ $\qquad f(x,y) = \frac{2k}{1+k^2}$ at $x=y=0$

fn is discontinuous at origin as fn of 2 var. whereas $f(x,y)$ is cont for $x=$const or $y=$const.

$\left.\begin{array}{l} f(x,y) \longrightarrow l \quad \lim_{(x,y)\to(x_0,y_0)} f(x,y) = l \quad \text{double lim} \\ (x,y) \to (x_0,y_0) \\ \forall \epsilon > 0 \quad \exists\ \delta(\epsilon, x_0, y_0) > 0 \quad \ni \text{ for } 0 < (x-x_0)^2 + (y-y_0)^2 < \delta^2 \\ \text{then } |f(x,y) - l| < \epsilon. \quad \text{deleted nbd} \end{array}\right\}$ definition of limit

Iterated lim

$\lim_{x\to\xi} \lim_{y\to\eta} f(x,y) = l$

Iterated lim $\neq$ double lim in general

but double lim $\Rightarrow$ Iterated lim

o, O notation