Control Problems   G 63.2852

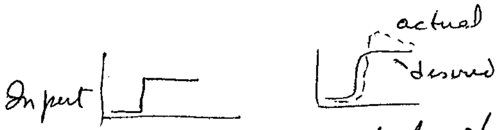
1. Trajectories

Offline Control - compute best results prior to need and program into data banks

On line Control - compute best results as one goes along

Regulator   Height is to be kept constant

Servo   Pointing a gun



Aircraft   meet some desirable response   system need not be stable

diff eq $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, \underset{\sim}{u}, t)$

$\underset{\sim}{x}$ is an n-dim vector representing system

$\underset{\sim}{u}$ is an m-dim vector which is used to control the 'state' x.

Control problem is to choose $u(t)$ so as to produce desired $\underset{\sim}{x}(t)$.

e.g. $\underset{\sim}{x}(t_f) = \underset{\sim}{x}^*$ given.

or minimize some loss fn.

$$J = \int_{t_0}^{t_f} L(\underset{\sim}{x}, \underset{\sim}{u}, t)\,dt + H(\underset{\sim}{x}(t_f))$$

Linear System

$$\dot{\underset{\sim}{x}} = A(t)\,\underset{\sim}{x}(t) + B(t)\,u(t)$$

A is a $n \times n$ matrix

B is a $n \times m$ matrix

Feed back Control $\underset{\sim}{u} = D\underset{\sim}{x} + \underset{\sim}{u}_0$

Estimation Theory

Suppose you can only measure

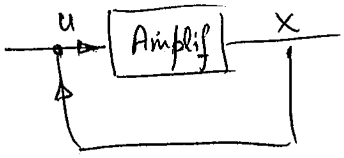
$$y = C\underbrace{\underset{\sim}{x} + \omega}_{\text{white noise, perturbation}}, \quad y \text{ is a } p \text{ vector} \qquad C \text{ is } p \times n \qquad p < n$$

Can you estimate $\underset{\sim}{x}$ and hence construct feedback control $\underset{\sim}{u} = D\underset{\sim}{x}$

Stability

Given $\dot{\underset{\sim}{x}} = \overset{\text{open loop}}{A}\underset{\sim}{x} + B\underset{\sim}{u}$ set $\underset{\sim}{u} = D\underset{\sim}{x}$

$$\dot{\underset{\sim}{x}} = \underbrace{(A + BD)}_{\text{closed loop matrix}}\underset{\sim}{x} = F\underset{\sim}{x}$$



Solution $\underset{\sim}{x}(t) = e^{F(t-t_0)}\,\underset{\sim}{x}(t_0)$

$e^{F(t-t_0)}\,\underset{\sim}{x}(t_0)$ suppose $F = V \Lambda V^{-1}$, $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

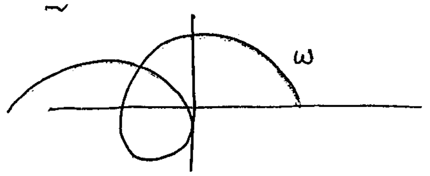
$$= V e^{\Lambda(t-t_0)} V^{-1}\,\underset{\sim}{x}(t_0)$$

$$\begin{bmatrix} e^{\lambda_1(t-t_0)} & & \\ & \ddots & \\ & & e^{\lambda_n(t-t_0)} \end{bmatrix}$$

if real part of λ_1 are > 0 we have possible blow up of system

Hurwitz, Routh, Nyquist devised tests to find out if system is stable

$\underset{\sim}{y} = c^T \underset{\sim}{x}$ $\quad \dot{\underset{\sim}{x}} = A\underset{\sim}{x} + b\underset{\sim}{u}$



Nyquist curves.

Lyapunov Stability Criterion

Suppose you can find $V(x) \geq 0$ $\quad \underline{x}^T Q \underline{x}$

such that $\frac{dV}{dt} < 0$

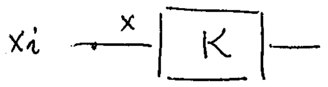
using $\dot{x} = Fx$

$$\frac{d}{dt} x^T Q x = x^T (F^T Q + QF) x$$

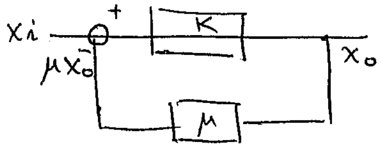
Parameter Variation

$\dot{\underline{x}} = A_1 \underline{x} + B_1 \underset{\sim}{u}$ see if solution is sensitive to variations in system.

negative feedback



$x_0 = kx_i$ open loop



$$x_0 = K(x_i - \mu x_0)$$

$$x_0 = \frac{K}{1+\mu K} x_i \quad \text{if } K\mu >> 1$$

$$\Longrightarrow x_0 = \frac{x_i}{\mu}$$

For a variation δK

$$\delta x_0 = \delta K\, x_i \qquad \frac{\delta x_0}{x_0} = \frac{\delta K}{K} \quad \text{open loop}$$

$$\text{closed loop} \quad \delta x_0 = \left[\frac{1}{1+\mu K} - \frac{\mu K}{(1+\mu K)^2}\right]\delta K\, x_i = \frac{1}{(1+\mu K)^2}\,\delta K\, x_i$$

$$\frac{\delta x_0}{x_0} = \frac{1}{K(1+\mu K)}\,\frac{\delta K}{K}$$

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + B\underset{\sim}{u}$$

If A is replaced by $A+\delta A$ then

$\delta\dot{x}_o = A\delta x + \delta A x$ if u is fixed

if $u = Dx$, $\delta u = D\delta x$

$\delta\dot{x}_c = (A+BD)\delta x + \delta A\, x$ for closed loop control.

Perkins & Cruz

For good results $\int(\delta x_c^T Z\, \delta x_c)dt \le \int(\delta x_o^T Z \delta x_o)dt$

Z is a matrix of Rank =

Suppose $\underset{\sim}{u} = D\underset{\sim}{x} + E\underset{\sim}{z}$ where $\dot{z} = Gz + H\underset{\sim}{x}$ Dynamic Control

may give better results

Typical Linear Optimal Control

Given

$$\underset{\sim}{\dot{x}} = A\underset{\sim}{x} + B\underset{\sim}{u}$$

Choose $\underset{\sim}{u}$ to minimize

penalty on U soft constraint

$$J = x^T(t_f)\, F\, x(t_f) + \int_{t_0}^{t_f}(x^T Q x + u^T R u)dt$$

Solution: is a linear feed back

$u = Dx$

where $D = -R^{-1}B^TP$ P is square

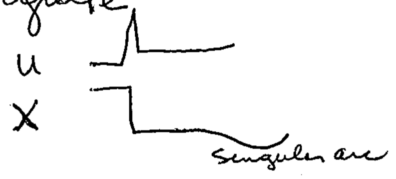
and $-\dot{P} = Q + A^TP + PA - PBR^{-1}B^TP \quad P(t_f) = F$

Matrix Riccati Equation. can blow up in finite time $\dot{x} = -x^2$

ex. $x = \frac{1}{(t-t_0)}$

if Q is positive deffinite we can integrate

if $R \to 0$ singular proble



Hard Constraint $(u_i - k)(k - u_i) =$

$-k \le u_i \le k$

Then if problem $x(t_f) = x^*$ in minimum time

Solution is 'bang bang' control

Consider rotating a disk from 0 to 90° in min time with $|T| \le T_{max}$

since $\ddot{\theta} = T$ apply max torque until $\theta = 45^\circ$

apply -max torque from $\theta = 45^\circ$ to 90°

switching fn determines switching time for the various conditions.

Differential Game

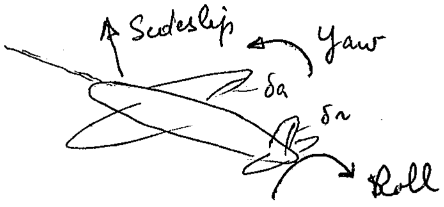
$$\dot{x} = Ax + B_1u_1 + B_2u_2$$

$$J = \int (x^TQx + u_1^TRu_1 + u_2^TRu_2)\,dt$$

u_1 tries to minimize J

u_2 " " maximize J

Solution if it exists should be a saddle point.



Lateral Control

ϕ = roll angle

p = roll rate

β = sideslip angle

r = yaw rate

$$\begin{pmatrix}\dot{\phi}\\ \dot{p}\\ \dot{\beta}\\ \dot{r}\end{pmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & L_p & L_\beta & L_r\\ g/V & 0 & Y_\beta & -1\\ 0 & N_p & N_\beta & N_r\end{bmatrix}\begin{bmatrix}\phi\\ p\\ \beta\\ r\end{bmatrix} + \begin{bmatrix}0 & 0\\ L_{\delta r} & L_{\delta a}\\ Y_{\delta r} & 0\\ N_{\delta r} & N_{\delta a}\end{bmatrix}\begin{bmatrix}\delta r\\ \delta a\end{bmatrix}$$

$$\begin{bmatrix}\dot{\delta r}\\ \dot{\delta a}\end{bmatrix} = \begin{bmatrix}-K_r & 0\\ 0 & -K_a\end{bmatrix}\begin{bmatrix}\delta r\\ \delta a\end{bmatrix} + \begin{bmatrix}u_r\\ u_a\end{bmatrix}$$

Desired Roll rate is $p^* = K\Theta_s$

$$\ddot{\Theta}_s + a\dot{\Theta}_s + b\Theta_s = 0$$

Performance Index

$$J = \int e^{-\alpha t}\left[Q_1(p-p^*)^2 + Q_2 r^2 + Q_3\beta^2\right]dt + \int e^{-\alpha t}(R_1 \underset{u_r^2}{\underline{\underline{\delta_r^2}}} + R_2 \underset{u_a^2}{\underline{\underline{\delta_a^2}}})dt$$

$u = Dx$, $D = -R^{-1}B^T P$ indep of $\underline{x}(t_0)$

Solution

$$u_R = D_{11}\phi + D_{12}p + D_{13}\beta + D_{14}r + D_{15}\delta_r + D_{16}\delta_a$$

$$u_a = \underbrace{D_{21}\phi}_{\text{Cannot have feed back from roll angle}} + D_{22}p + D_{23}\beta + D_{24}r + \underbrace{D_{25}\delta_r + D_{26}\delta_a}_{\text{Usually optimized before hand}}$$

use constraint optimization: Decide in advance which D_{ij} are allowed Try to optimize these.

Transition Matrix

Assume $\dfrac{d^n x}{dt^n} + a_{n-1}\dfrac{d^{n-1}x}{dt^{n-1}} + \dots + a_0 x_0 = b u_0$

Let
$x_1 = x$
$x_2 = \dot{x}$
$x_3 = \ddot{x}$
$\vdots$
$x_n = \dfrac{d^{n-1}x}{dt^{n-1}}$

$\rightarrow$
$\dot{x}_1 = x_2$
$\dot{x}_2 = x_3$
$\vdots$
$\dot{x}_n = -a_{n-1}x_{n-1} - a_{n-2}x_{n-2} \cdots - a_0 x_1$

$$A = \begin{bmatrix} & 1 & & \\ O & & \ddots & \\ & & & 1 \\ -a_{n-1} & \cdots & & -a_0 \end{bmatrix}$$

Companion Matrix Companion Form
Frobenius Form.

$x(t) = \phi(t,t_0)\,x(t_0)$ is ~~a~~ solution of $\dot{x} = A(t)\,x(t)$

then uniqueness requires

$\phi(t_0,t_0) = I$ $\qquad \phi(t_0,t_1)\,\phi(t_1,t_0) = \phi(t,t_0)$

ϕ is non singular

For all $x(t_0)$ we have

$$\frac{d}{dt}\phi(t,t_0)\,x(t_0) = A(t)\phi(t,t_0)\,x(t_0)$$

$$\Rightarrow \frac{d}{dt}\phi(t,t_0) = A(t)\phi(t,t_0) \text{ where } \phi(t_0,t_0) = I$$

Also $\frac{d}{dt_0}[\phi(t,t_0)]x(t_0) + \phi(t,t_0)\dot{x}(t_0) = 0$

$$\frac{d}{dt_0}\phi(t,t_0) = -\phi(t,t_0)A(t_0)$$

If A is a const.

$$\phi(t,t_0) = e^{A(t-t_0)} = I + (t-t_0)A + \frac{(t-t_0)^2}{2!}A^2 + \dots$$

$$\frac{d}{dt}\phi(t,t_0) = A\left[I + (t-t_0)A + \frac{(t-t_0)^2}{2!}A^2 + \dots\right] = A\phi(t,t_0)$$

$$x(t) = x(t_0) + (t-t_0)\dot{x}(t_0) + \frac{(t-t_0)^2}{2!}\ddot{x}(t_0) + \dots$$

$$\dot{x}(t_0) = Ax(t_0)$$

$$\ddot{x}(t_0) = A\dot{x}(t_0) = A^2x(t_0)$$

$$\vdots$$

$$x(t) = \cancel{x(t_0)}\left[I + A(t-t_0) + \dots\right]x(t_0) = e^{A(t-t_0)}x(t_0)$$

For $\dot{x} = Ax + Bu$

solution is

$$x(t) = \phi(t,t_0)x(t_0) + \int_{t_0}^{t}\phi(t,\tau)B(\tau)u(\tau)d\tau$$

Verify by diff

$$\dot{x}(t) = \underbrace{A(t)\phi x(t_0) + \int_{t_0}^{t}A\phi B u d\tau}_{} + \overset{I}{\phi(t,t)}B(t)u(t)$$

$$A\overset{\swarrow}{x} + Bu$$

<u>Controllability Simple Definition</u> A, B const.

$$\dot{x} = Ax + Bu$$

Assume A has distinct EV

Then $A = V\Lambda V^{-1}$ V is $\vec{EV}$ matrix Λ is diagonal

Set $x = Vz$ $z = V^{-1}x$

Then $\dot{x} = V\dot{z} = AVz + Bu = V\Lambda V^{-1}Vz + Bu = V\Lambda z + Bu$

$$\dot{z} = \Lambda z + V^{-1}Bu = \Lambda z + \beta u$$

we have decoupled the equations

$z_1 = \lambda_1 z_1 + \beta_{11} v_1 + \beta_{12} u_2$

$\dot{z}_2 = \lambda_2 z_2 + \beta_{21} u_1 + \beta_{22} u_2$ if $\beta_{21}, \beta_{22} = 0 \Rightarrow z_2$ is indep and system cannot be controllable

$\dot{z}_3 = \lambda_3 z_3 + \beta_{31} u_1 + \beta_{32} u_2$

Note: $\dot{x}_1 = \alpha x_1 + u$

$\dot{x}_2 = \alpha x_2 + u$ is not controllable since $(x_1 - x_2)' = \alpha(x_1 - x_2)$

In general for a constant ~~definition~~ matrix

System is controllable at t_0 if $x_1 = 0$ is reachable at some time $t_1 > t_0$ from $x(t_0)$ by a suitable choice of u.

Condition for a constant matrix:

$$x(t_1) = e^{A(t_1 - t_0)} x(t_0) + \int_{t_0}^{t_1} e^{A(t-t_0)} B u(t)\, dt$$

$$= e^{A(t_1 - t_0)} x(t_0) + \int_{t_0}^{t_1} \left\{ I + (t-t_0)A + (t-t_0)^2 A^2 + \dots \right\} B u(t)\, dt$$

By Cayley-Hamilton Theorem A^n and higher powers can be expressed in terms of $I, A, \dots, A^{n-1}$

$$x(t_1) = e^{A(t_1 - t_0)} x(t_0) = \sum_{k=1}^{n} A^{k-1} B v_k$$

Solvable only if

$$[B, AB, A^2B, \dots, A^{n-1}B] \text{ has rank } n.$$

Bryson Applied Optimal Control

Anderson & Moore Opt. Control (Linear)

Optimal Control Athans & Falb McGraw Hill

Applied Optimal Control Bryson & Ho Blaisdell

Linear Optimal Control Anderson & Moore Prentice Hall

Mathematical theory of optimal processes Pontryagin Boltyenskii Gemkrelidze, Mishenko Interscience

Selected Papers in control theory Bellman & Kalaba Dover

Matrix Theory Gantmacher Chelsea

Stability by Liapunov's Direct Method LaSalle Lefschetz AP

Papers by R. E. Kalman RIAS

Controllability Constant system

$$\dot{x} = Ax + Bu \qquad A, B \text{ constant matrices}$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Question is: can

$$z = \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \text{ be arbitrary}$$

$$e^{A(t-\tau)} = I + A(t-\tau) + A\frac{(t-\tau)^2}{2!} + \dots$$

Eliminate A^n and higher powers by Cayley-Hamilton

$$x_0A^n + a_1A^{n-1} + \dots + a_nI = 0$$

$$z = \Sigma A^{n-1}B\int_{t_0}^{t}\overset{\alpha_k(\tau)}{\cancel{(t-\tau)^{k-1}}}u(\tau)d\tau$$

Columns of

$$[B, AB, A^2B, \dots, A^{n-1}B] \text{ must span the space}$$

Time Varying linear system

$$x(t) = \phi(t,t_0)x(t_0) + \int_{t_0}^{t}\phi(t,\tau)B(\tau)u(\tau)d\tau$$

Define

$$w = \int_{t_0}^{t_1}\phi(t_1,\tau)B(\tau)B^T(\tau)\phi^T(t_1,\tau)d\tau$$

Then x is reachable if x is in range space of W

ie $x = Wy$ for some vector y

Sufficiency If $x = Wy$ then required control is

$$u(t) = B^T(t)\,\phi^T(t_1,t)\,y$$

Necessity

Suppose $x \neq 0$ and lies in nullspace of the

$$0 = x^T y x = x^T\left\{\int_{t_0}^{t_1}\phi(t_1,\tau)\,B(\tau)\,B^T(\tau)\,\phi^T(t_1,\tau)\,d\tau\right\}x$$

Can only be zero if $B^T\phi^T x = 0$ in interval $[t_0,t_1]$

If x is reachable for some $u(t)$

$$x = \int\phi(t_1,\tau)\,B(\tau)\,u(\tau)\,d\tau$$

$$x^Tx = \int\underbrace{x^T\phi\,B}_{=0^T}u\,d\tau = 0 \quad\Rightarrow\quad x = 0 \quad\rightarrow\leftarrow$$

System is controllable if Gramian matrix W is nonsingular
(positive definite)

If W is invertible and it is desired to reach x^* one solution is

$$u(t) = u^* = B^T\phi^T(t,\tau)\,W^{-1}x^*$$

Then of all controlls u which would bring $x(t)$ to x^*, u^* is that which minimizes $\int_{t_0} u^Tu\,dt$

We have

$$x^* = \int\phi\,Bu^*\,d\tau = \int\phi\,Bu\,d\tau$$

$$\therefore$$

$$\int(u-u^*)^Tu^*d\tau = \int u^TB^T\phi^TW^{-1}x^*d\tau - \int u^*B^T\phi^TW^{-1}x^*d\tau$$

$$= \int x^*W^{-1}x^* - x^*W^{-1}x^* = 0$$

$$\int u^{*T}u^*d\tau + \int(u-u^*)^T(u-u^*)\,d\tau = \int u^{*T}\cancel{u^*}d\tau + \int u^Tu\,d\tau$$
$$-2\int u^{*T}\cancel{u^*}d\tau + \int\cancel{u^{*T}}u^*d\tau$$

Observability

Given $\dot{x} = Ax + Bu$ $\quad y = Cx$, y of dim $p <$ dim of x

Given y from t_0 to t and u from t_0 to t can you determine $x(t_0)$?

If you minimize $J = \int (y^T y + u^T R u)\,dt$ you may cause the system in minimized state to be unstable

If you can determine it, system is observable

Constant case $\quad$ A has distinct eigenvalues

$A = V \Lambda V^{-1}$ $\quad \Lambda$ diag

Let $z = V^{-1} x$

$$\dot{x} = Ax + Bu$$

$$V\dot{z} = VAV^{-1}Vz + Bu \qquad y = CVz$$

$$\dot{z} = V^{-1}VAz + V^{-1}Bu = \Lambda z + \beta u \;, y = \Gamma z \qquad \Gamma = CV$$

$$\dot{z}_1 = \lambda_1 z_2 + \beta_1 u$$

$$z_2 = \lambda_2 z_2 + \beta_2 u$$

$$\vdots$$

if Γ has a zero i th col then y is independent of z_i

Assume $y(t) = Cx(t) = C\left\{ e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau \right\}$

if $u(t)$ is known.

$$\dot{y} = CAe^{A(t-t_0)} x(t_0)$$

$$\ddot{y} = CA^2 e^{A(t-t_0)} x(t_0)$$

$$\vdots$$

if $\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full rank then it is possible to get $x(t_0)$

rank condition for observability

or if for t_i $\quad i = 1, \dots, k$

$$y(t_i) = Ce^{A(t_i - t_0)} x(t_0)$$

For time varying case   Assume u is known

$$\dot{x} = Ax$$

$$y(t_1) = c(t_1)\,\phi(t_1, t_0)\,x(t_0)$$

$$\int_{t_0}^{t_1} \phi^T(\tau, t_0)\, c^T(\tau)\, y(\tau)\, d\tau = M x(t_0)$$

Look at Hoppensteadt's Notes

where $M = \int_{t_0}^{t_1} \phi^T(\tau, t_0)\, c^T(\tau)\, c(\tau)\, \phi(\tau, t_0)\, d\tau$

Nyquist : Single input-single output system

$$\dot{x} = Ax + bu$$

$$y = c^T x$$

$$x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_0 x = u \qquad u = b_0 + b_1 \dot{x} + \dots$$

$$y = c_0 x + c_1 \dot{x} + c_2 \ddot{x} + \dots$$

Solution $x = \sum \alpha_i e^{\lambda_i t}$ where λ_i are roots of poly

$$z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0$$

$$w = (z-\lambda_1)(z-\lambda_2) \cdots (z-\lambda_n) = 0$$

$$\mathrm{Arg}(w) = \arg(z-\lambda_1) + \dots + \arg(z-\lambda_n)$$

Stability of linear system

$$\dot{x} = Ax + Bu \qquad u = Dx$$

Then $\dot{x} = (A+BD)x = Fx$

$$x = e^{F(t-t_0)} x(t_0)$$ & when diag

$$= V e^{\Lambda(t-t_0)} V^{-1} x(t_0)$$

Stability

Suppose $\dot{x} = f(x)$, $f(0) = 0$

The solution $x = 0$ is a *stable* solution if for every $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni$ if $\|x(t_0)\| < \delta$ $\|x(t)\| < \epsilon$ $\forall$ $t \geq t_0$

System is asymptotically stable if

1. System is stable
2. $\exists$ a number $r(t_0)$ $\ni$ for every $\mu > 0$ $\exists$ a time T $\ni$ $\|x(t_0)\| < r(t_0)$ then $\|x(t)\| < \mu$ for $t > t_0 + T$

System is called completely stable if condition 2 holds for all r

Phase Plane Diagrams. for 2^{nd} order system

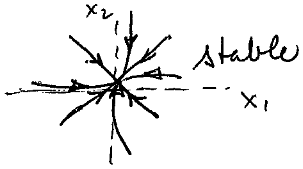
$$\ddot{x} + a\dot{x} + bx = 0$$

or

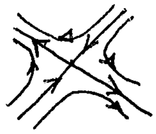
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Roots are $\lambda = -\frac{a}{2} \pm \sqrt{a^2/4 - b}$

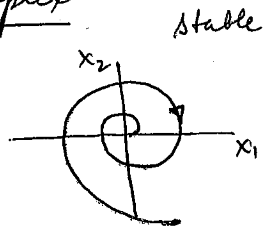
Real roots $\lambda_1 < 0$, $\lambda_2 < 0$

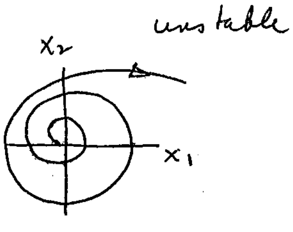


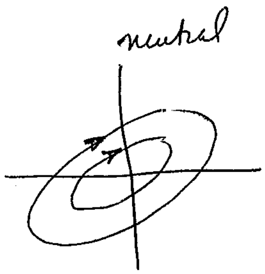
$\lambda_1 < 0$ $\lambda_2 > 0$



Complex





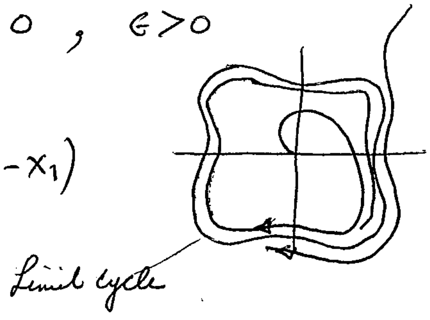


Van der Pol's Eq & Limit cycles

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0, \quad \epsilon > 0$$

$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = x_1 - \epsilon(x^3/3 - x_1)$$



Liapunov's Theorem

Let $\dot{x} = f(x)$, $f(0) = 0$

Suppose we can find an 'energy' fn $V(x)$ such that

1. $V(x)$ is continuous and has 1ST partial derivatives
2. $V(x) > 0$ when $x \neq 0$
3. $V(0) = 0$
4. $\frac{\partial \dot{V}}{\partial t}(x,t) = \nabla V \cdot \dot{x} \leq 0$ $\qquad \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \cdots$

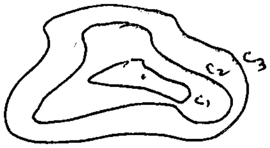
Then $x=0$ is a stable solution

Proof

$V(x) = c_1$

$V(x) = c_2$

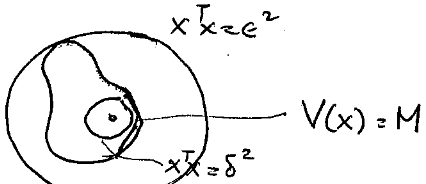
$\vdots$



Consider the hypersphere $x^T x = \epsilon^2$

Let minimum of $V(x)$ on this sphere be M

Then let minimum distance of $V(x) = M$ from the origin be δ



If $\|x(t_0)\| < \delta$ then $V(x(t_0)) < M$ and since $\dot{V} \leq 0$

$V(x(t)) < M$, so $\|x(t)\| < \epsilon$

Theorem 2 (Asymptotic Stability)

If $\dot{V} < 0$ then $x=0$ is an asymptotically stable solution

Proof Suppose $\|x(t)\| \geq m > 0$ Since $V(x)$ is positive definite

$V(x) \geq l > 0$, $x \neq 0$

But $\dot{V}(x) \leq -k < 0$ $\quad \therefore \; V(x(t)) = V(x(t_0)) - \int_{t_0}^{t} \dot{V}\, dt$

$\leq V(x(t_0)) - k(t - t_0)$

Theorem 3 (Kalman & Bertram)

If $\dot{V} \leq 0$ ~~then~~ and $\dot{V}(x) \neq 0$ $\forall\, t > t_1$

for any solution x of $\dot{x} = f(x)$ then $x=0$ is asymptotically stable

Linear Syste $\dot{x} = Ax$

Let $V = x^T Q x$ $Q = \text{const}$

$$\dot{V} = \dot{x}^T Q x + x^T Q \dot{x} = x^T (A^T Q + QA) x$$

Suppose $A^T Q + QA = -P$ where P is positive definite

then V is a liapunov fn. Turns out if true of one $P \Rightarrow \forall$ positive definite P V is a liapunov fn $\Rightarrow$ we can solve for Q

$$V[x(t_0)] - V[x(t)] = \int_{t_0}^{t} x^T P x \, d\tau > 0$$

for large t $V[x(t)] \to 0$ $\Rightarrow V[x(t_0)] > 0$

Given $\dot{x} = f(x)$

If $\exists$ energy fns $V(x) \geq 0$ such that $\dot{V} - \sum \frac{\partial V}{\partial x_i} f_i < 0$

Positive Limiting Set.: A point P is in a positive limiting set Γ^+ if $\exists$ a sequence of times $t_n \to \infty$ as $n \to \infty$ $\ni$. $x(t_n) \to P$.

An invariant set: A set M such that if $x(t_0)$ is in M, then $x(t)$ is in M for all t $t > t_0$ or $t < t_0$

Theorem on extent of Stability (La Salle)

Let Ω_ℓ be a region where $V(x) < \ell$ Suppose that Ω_ℓ is bounded and in Ω_ℓ $V(x) > 0$ for $x \neq 0$ $\dot{V} \leq 0$

Let R be the set where $\dot{V} = 0$

Let M be the largest invariant subset of R then $x(t) \to M$ as $t \to \infty$

Proof: Since $V > 0$ $\dot{V} \leq 0$ $\Rightarrow$ x remains in Ω_ℓ if $x(t_0) \in \Omega_\ell$

$\therefore$ $V(x)$ has a limit $l_0 < l$ as $t \to \infty$. Also the limiting set Γ^+ must be in Ω_ℓ. Then we have $V(x) = l_0$ on Γ^+ since V is cont. Thus $\dot{V} = 0$ on Γ^+ & Γ^+ is in R by the ~~theorem of extent of stability~~ def of invariant set $x(t) \to M$ as $t \to \infty$

Asymptotic Stability

If $\dot{V} < 0$ in Ω_ℓ when $x \neq 0$ $\dot{V}(0) = 0$ Then $x = 0$ is an asymp stable solution & $x(t) \to 0$ if $x(t_0)$ is in Ω_ℓ

Duffing equation

$$\ddot{x} + a\dot{x} + x - bx^3 = 0 \quad , \quad a > 0, \; b > 0$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -ax_2 - x_1 + bx_1^3$$

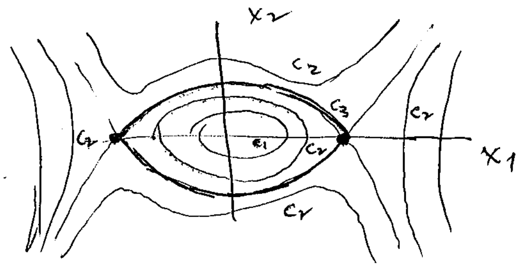
Think V = energy

$$KE = \frac{1}{2} x_2^2 \qquad PE = \int_0^{x} (x - bx^3)\,dx = \frac{x_1^2}{2} - \frac{bx_1^4}{4}$$

$$V = KE + PE = \frac{1}{2}\left\{x_2^2 + x_1^2 - \frac{bx_1^4}{2}\right\}$$

$$\dot{V} = x_2\dot{x}_2 + x_1\dot{x}_1 - bx_1^3\dot{x}_1 = -x_2\left\{ax_2 + \cancel{x_1} - b\cancel{x_1^3}\right\} + x_1\cancel{x_2} - b\cancel{x_1^3}x_2$$

$\dot{V} = -ax_2^2$



(0,0) invariant subset

c_3 provides bound for Ω_ℓ

to find size of largest closed region

$$\left.\frac{\partial V}{\partial x_1}\right|_{x_2=0} = 0 = x_1 - bx_1^3 \Rightarrow x_1 = 0 \text{ or } x_1 = \pm\sqrt{b}$$

$$c_3 = x_2^2/_2 + x_1^2/_2 - \frac{bx_1^4}{4} = \frac{1}{4b}$$

Ω_ℓ is the region where $V < \frac{1}{4b}$ and inside closed contour.

This region is an under estimate of actual region of stability

Theorem on Complete Stability 1

Let $V(x) > 0$

$\dot{V}(x) \leq 0$ Let R be region where $\dot{V} = 0$

M largest invariant subset in R

then all bdd solution $\rightarrow$ M as $t \rightarrow \infty$

Proof: All bdd solns lie inside some closed hypersphere S_r, $x^T x = r^2$

V since non neg must have a min inside S_r and $\dot{V} \leq 0$

so lim $V(x) = \ell_0$ since bdd from below

Since S_r closed & bdd solns $\exists$ a limiting set in S_r, Γ^+.

By same argument as before M contains Γ^+ & hence $x(t) \rightarrow M$

Theorem of Complete Stability 2

If $V(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $V(x)$ satisfies other assumptions then system is completely stable

Since $\dot{V}(x) \leq 0$, $V(x) \leq V(x(t_0))$

Routh & Hurwitz Stability Criterion

Given $$\sum_{i=0}^{n} a_i \frac{dx^{n-i}}{dt^{(n-i)}} = 0$$

All roots be in left half plane (system is stable) if all principal minors of following $n \times n$ matrix are positive

$$\begin{bmatrix} a_1 & a_2 & 0 & 0 & \\ a_3 & a_4 & a_1 & a_2 & \\ a_5 & a_6 & a_3 & a_4 & \cdots\cdots \\ \vdots & \vdots & & & \\ a_k & a_{k+1} & a_5 & a_6 & \\ 0 & 0 & \vdots & \vdots & \\ \vdots & \vdots & a_k & a_{k+1} & \\ \vdots & & 0 & 0 & \\ 0 & 0 & 0 & 0 & \end{bmatrix}$$

<u>Liapunov fn for linear system</u> $\dot{x} = Ax$

$$A = \begin{bmatrix} -\frac{a_1}{a_0} & -\frac{a_2}{a_0} & \cdots & -\frac{a_n}{a_0} \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

$x_n = x$

$\dot{x}_n = x_{n-1}$

$\dot{x}_{n-1} = x_{n-2}$

$\vdots$

$\dot{x}_1 = -\frac{a_1}{a_0} x_1 - \frac{a_2}{a_0} x_2 \cdots$

Let $V = x^T P x$, $\dot{V} = \dot{x}^T P x + x^T P \dot{x}$

$= -x^T Q x$ where $Q + A^T P + PA = 0$

Take

$$P = \begin{bmatrix} a_0 a_1 & 0 & a_0 a_3 & 0 \\ 0 & a_1 a_2 - a_0 a_3 & 0 & a_1 a_4 - a_0 a_5 \\ a_0 a_3 & 0 & a_0 a_5 - a_1 a_4 + a_2 a_3 & 0 \\ 0 & a_1 a_4 - a_0 a_5 & 0 & a_1 a_6 - a_2 a_5 + a_3 a_4 - a_0 a_7 \end{bmatrix}$$

$P_{ij} = 0$ if $i + j$ odd

$\sum_{k=0}^{i-1} (-1)^{k+i-1} a_k a_{i+j-1-k}$ $\quad i + j$ even

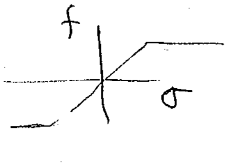
$$\dot{V} = -(a_1 x_1 + a_3 x_3 + a_5 x_5 \cdots\cdots)^2 \leq 0$$

Also $P > 0$ if principal minors are positive

Absolute Stability (Problem of Lurie)

$\dot{x} = Ax + b\xi$

$\dot{\xi} = f(\sigma)$ $\quad \sigma = c^T x + d\xi$



if $f(\sigma) = \sigma$ $\quad \dot{\xi} = c^T x + d\xi$

Variational Calculus

Let $\dot{x}_i = f_i(x,u,t)$ x n vector   u m vector $\dot{x}_i(0)$ given (1)

Find u to minimize

$$J = K(x(t_f)) + \int_{t_0}^{t_f} L(x,u,t)\,dt \qquad (2)$$

Since $\dot{x}_i - f_i = 0$ we get same minimum for $J' = K(x(t_f)) + \int\Big[L + \sum_i \lambda_i\{\dot{x}_i - f_i\}\Big]dt$

Integrate by parts

$$J' = K(\,) - \sum \lambda_i x_i\Big|_{t_f} + \sum \lambda_i x_i\Big|_{t_0} + \int \Big(H + \sum \dot{\lambda}_i x_i\Big)dt$$

$$H = \sum \lambda_i f_i + L$$

Consider effect of replacing $u(t)$ by $u + \delta u$

This causes a change δx in x. and hence $\delta x_i\big|_{t_0} = 0$

$$\delta J = \sum \frac{\partial K}{\partial x_i}\delta x_i\Big|_{t_f} - \sum \lambda_i \delta x_i\Big|_{t_f} + \int\Big\{\sum\Big(\frac{\partial H}{\partial x_i} + \dot{\lambda}_i\Big)\delta x_i$$
$$+ \sum \frac{\partial H_i}{\partial u_i}\delta u_i\Big\}dt$$

Choose λ_i to satisfy $\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} = -\Big\{\sum \lambda_k \frac{\partial f_k}{\partial x_i} + \frac{\partial L}{\partial x_i}\Big\}$

$$\Rightarrow \quad \delta J = \sum\Big(\frac{\partial K}{\partial x_i} - \lambda_i\Big)\delta x_i\Big|_{t_f} + \int \sum \frac{\partial H}{\partial u_i}\delta u_i\,dt$$

now set $\lambda_i(t_f) = \frac{\partial K}{\partial x_i}$

$$\Rightarrow \quad \delta J = \int \sum \frac{\partial H}{\partial u_i}\delta u_i\,dt$$

for max or min. since δu_i are arbitrary $\Rightarrow \delta J = 0 \Rightarrow \frac{\partial H}{\partial u_i} = 0$

provided there is no additional constraint on u_j.

Euler-Lagrange Equations.

$x_i(t_0) = x_{i0}$ Given $\dot{x}_i = f_i$ $\dot{\lambda}_i = -\sum_k \frac{\partial f_k}{\partial x_i}\lambda_k - \frac{\partial L}{\partial x_i}$

$\lambda_i(t_f) = \frac{\partial K}{\partial x_i}$

where on path $\frac{\partial H}{\partial u_j} = 0$, $H = \sum \lambda_i f_i + L$

$\dot{x}_i = \frac{\partial H}{\partial \lambda_i}$ $\quad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i}$ $\quad H = \sum \lambda_i f_i + L$

u is chosen so that $\frac{\partial H}{\partial u_j} = 0$.

Dynamic Programming

Let $\dot{x} = f_i(x,u,t)$ $\quad \dot{x}_i(0)$ give

Find u to minimize

$$J = K\left(x(t_f)\right) + \int_{t_0}^{t_f} L(x,u,t)\,dt$$

Let

$$V(x,t) = \min_u \left(K + \int_t^{tf} L\,dt\right)$$

Consider an interval from $t, t+\delta t$

~~$V(x,t+\delta t) = \min_u \left(K + \int_{t+\delta t}^{t_f} L\,dt\right)$~~ we presume $V(x+\delta x, t+\delta t)$ is known

$$V(x,t) = \min_u \left\{ L\,\delta t + V(x+\delta x, t+\delta t)\right\}$$

where $\delta x_i = f_i \delta t$; assuming V is differentiable $\Rightarrow \frac{\partial V}{\partial x_i}, \frac{\partial V}{\partial t}$ exist

$$V(x+\delta x, t+\delta t) = V(x,t) + \sum \frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial t}\delta t$$

$$= V(x,t) + \left\{\sum \frac{\partial V}{\partial x_i} f_i + \frac{\partial V}{\partial t}\right\}\delta t$$

$$\cancel{V(x,t)} = \min_u \left\{ L\,\delta t + \cancel{V(x,t)} + \left\{\sum \frac{\partial V}{\partial x_i} f_i + \frac{\partial V}{\partial t}\right\}\delta t \right\}$$

$$\frac{\partial V}{\partial t} = -\min_u \left\{ \sum \frac{\partial V}{\partial x_k} f_k + L \right\} \qquad (3)$$

Hamilton-Jacobi equations

if $\lambda_i = \frac{\partial V}{\partial x_i}$ we have Hamiltonian H in brackets.

To derive Euler-Lagrange equations

$$\dot{\lambda}_i = \sum_k \frac{\partial^2 V}{\partial x_i \partial x_k}\dot{x}_k + \frac{\partial^2 V}{\partial x_i \partial t}$$

diff (3) wrt x_i

$$\frac{\partial V}{\partial x_i \partial t} = -\min_u \left\{ \sum \frac{\partial^2 V}{\partial x_i \partial x_k} f_k + \sum \frac{\partial V}{\partial x_k}\frac{\partial f_k}{\partial x_i} + \frac{\partial L}{\partial x_i} + \sum \underset{0}{\cancel{\frac{\partial H}{\partial u_j}}} \frac{\partial u_j}{\partial x_i} \right\}$$

$\frac{\partial H}{\partial u_j} = 0$ since on optimal path.

$$\Rightarrow \quad \overset{\circ}{\lambda}_i = -\sum \frac{\partial f_k}{\partial x_i}\lambda_k - \frac{\partial L}{\partial x_i}$$

$\overset{\circ}{x}_i = f_i$

$$J = K(x(t_f)) + \int_{t_0}^{t_f} L(x,u,t)\,dt$$

under a variation δu

$$\delta \overset{\circ}{x}_i = \sum \frac{\partial f_i}{\partial x_k}\delta x_k + \sum \frac{\partial f_i}{\partial u_j}\delta u_j \qquad \delta x_i(t_0) = 0$$

Define costate variables λ_i satisfying adjoint differential equation.

$$\overset{\circ}{\lambda}_i = -\sum \frac{\partial f_k}{\partial x_i} - \frac{\partial L}{\partial x_i}$$

$$\frac{d}{dt}\Big(\sum_i \lambda_i \delta x_i\Big) = \cancel{\sum_i\sum_k \lambda_i \frac{\partial f_i}{\partial x_k}\delta x_k} + \sum_i\sum_j \lambda_i \frac{\partial f_i}{\partial u_j}\delta u_j + \sum_j \frac{\partial L}{\partial u_j}\delta u_j$$

$$\cancel{-\sum_k\sum_i \lambda_k \frac{\partial f_k}{\partial x_i}\delta x_i} \underbrace{-\sum_i \frac{\partial L}{\partial x_i}\delta x_i - \sum_j \frac{\partial L}{\partial u_j}\delta u_j}_{-\delta L}$$

$$\sum \lambda_i \delta x_i\Big|_{t_f} = -\int \delta L\,dt + \int \sum \frac{\partial H}{\partial u_j}\delta u_j\,dt\,, \quad H = \sum \lambda_i f_i + L$$

But

$$\delta J = \sum \frac{\partial K}{\partial x_i}\delta x_i(t_f) + \int \delta L\,dt$$ so if now make $\lambda_i(t_f) = \frac{\partial K}{\partial x_i}$

$$\therefore \quad \delta J = \int \sum \frac{\partial H}{\partial u_j}\delta u_j\,dt$$

$\sum \frac{\partial H}{\partial u_j}\delta u_j = \delta H$ whereas u is varied and λ_i & f_i are evaluated on original trajectory

$$\delta J = \int \delta H\,dt \qquad \boxed{H = \sum \lambda_i f_i + L}$$

Then choose u to minimize H pointwise. δu need not be continuous

to tighten argument set $\Delta u = \epsilon h(t)$

Results in Δx, ΔJ Define

$$\delta u = \lim_{\epsilon\to 0}\frac{\Delta u}{\epsilon} = h(t)$$

$$\delta x = \lim_{\epsilon\to 0}\frac{\Delta x}{\epsilon} \qquad \delta J = \lim_{\epsilon\to 0}\Delta J$$

$\dot{x}_i = f(x,u,t)$

$J = K\,(x(t_f), t_f) + \int_{t_0}^{t_f} L(x,u,t)\,dt$

One variable end pt

$g_m(x(t_f)) = 0 \quad , \; m = 1, \dots, q$

$J = \quad M + \int_{t_0}^{t_f} L\,dt \qquad M = K + \sum \nu_m g_m$

$$\delta \dot{x}_i = \sum \frac{\partial f_i}{\partial x_j}\delta x_j + \sum \frac{\partial f_i}{\partial u_k}\delta u_k$$

$$\delta J = \sum \frac{\partial M}{\partial x_i}\left[\partial x_i(t_f) + f_i(t_f)\delta t_f + \frac{\partial M}{\partial t}\delta t_f\right] + \int_{t_0}^{t_f} \delta L\,dt + L\,\delta t_f$$

$$\delta L = \sum \frac{\partial L}{\partial x_i}\delta x_i + \sum \frac{\partial L}{\partial u_k}\delta u_k$$

define $\qquad H = \sum \lambda_i f_i + L$

$$\dot{\lambda}_i = -\sum \lambda_j \frac{\partial f_j}{\partial x_i} - \frac{\partial L}{\partial x_i} = -\frac{\partial H}{\partial x_i}$$

$$\frac{d}{dt}(\lambda_i\,\delta x_i) = -\cancel{\sum\sum \lambda_j \frac{\partial f_j}{\partial x_i}\delta x_i} - \sum \frac{\partial L}{\partial x_i}\partial x_i + \cancel{\sum\sum \lambda_i \frac{\partial f_i}{\partial x_j}\delta x_j} + \sum \lambda_i \frac{\partial f_i}{\partial u_k}\delta u_k$$

$$= \sum \frac{\partial H}{\partial u_k}\delta u_k - \delta L$$

$$\int_{t_0}^{t_f} \delta L\,dt = \int \sum \frac{\partial H}{\partial u_k}\delta u_k\,dt - \sum \lambda_i\,\delta x_i\Big|_{t_f}$$

$$\delta J = \sum\left(\frac{\partial M}{\partial x_i} - \lambda_i\right)\delta x_i\Big|_{t_f} + \left(\sum \frac{\partial M}{\partial x_i} f_i + \frac{\partial M}{\partial t} + L\right)\delta t_f + \int \sum \frac{\partial H}{\partial u_k}\delta u_k\,dt$$

$$= \sum\left(\frac{\partial M}{\partial x_i} - \lambda_i\right)\delta x_i\Big|_{t_f} + \sum\left(\frac{\partial M}{\partial x_i} - \lambda_i\right) f_i\,\delta t_f + \left(H + \frac{\partial M}{\partial t}\right)\delta t_f + \int \sum \frac{\partial H}{\partial u_k}\delta u_k\,dt$$

Choose $\lambda_i = \frac{\partial M}{\partial x_i}$ to knock out first two terms.

Then we need for an extremal path $\frac{\partial H}{\partial u_k} = 0$

and $H + \frac{\partial M}{\partial t} = 0$ Transversality Conditions.

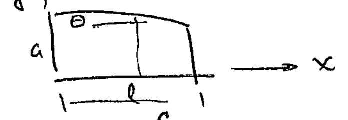
So we have for an extremal path

$\dot{x}_i = \frac{\partial H}{\partial x_i} \qquad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \qquad u$ minimizes $H \quad \Big\}$ along path

$g_m(x(t_f)) = 0 \quad q$ cond. $\quad ; \; x_i(t_0) = $ given, $\frac{\partial K}{\partial x_i} + \sum \nu_m \frac{\partial g_m}{\partial x_i} - \lambda_i = 0$ @ $t = t_f$

$H + \frac{\partial M}{\partial t} = 0$ at $t = t_f$

Minimum Drag nose for rocket



$$\text{Drag} = 2\pi q \int C_p(\theta)\, y\, dy$$

$$\frac{dy}{dx} = -\tan\theta \qquad \theta \text{ is control variable}$$

$$C_p = 2\sin^2\theta \quad \theta \geq 0$$
$$\quad\;\; 0 \quad\;\; \theta < 0$$

To find best $y(x)$, given $y(0) = a$

$$\frac{dy}{dx} = u$$

$$J = \frac{1}{2} y(l)^2 + \int_0^l y u \frac{u^2}{1+u^2}\, dx$$

Then

$$H = \frac{yu^3}{1+u^2} - \lambda u$$

$$\dot\lambda = -\frac{\partial H}{\partial y} = \frac{-u^3}{1+u^2} \qquad \lambda(l) = -y(l)$$

$$0 = \frac{\partial H}{\partial u} = \frac{yu^2(3+u^2)}{(1+u^2)^2} - \lambda$$

Eliminating λ we have

$$H = \frac{2yu^3}{(1+u^2)^2} = \text{const}$$

conditions at $x = l$

$$y(l)\left[1 - \frac{u^2(3+u^2)}{(1+u^2)^2}\right] = 0$$

Satisfied by $y(l) = 0$ or $u(l) = 1$ with $u(l) = 1$ we have

$$H = -\frac{y(l)}{2}$$

Thus $$\frac{y}{y(l)} = \frac{(1+u^2)^2}{4u^3} \qquad (A)$$

Also $$\frac{dx}{du} = \frac{dx}{dy}\frac{dy}{du} = -\frac{1}{u}\frac{dy}{du} = -\frac{y(l)}{u}\frac{d}{du}\frac{(1+u^2)^2}{4u^3}$$

Giving

$$\frac{l-x}{y(l)} = \frac{1}{4}\left(\frac{3}{4u^2} + \frac{1}{u^2} - \frac{7}{4} - \log\frac{1}{u}\right) \qquad (B)$$

If h & f are not explicit fns of time

$$\dot H = \sum \frac{\partial H}{\partial x_i}\dot x_i + \sum \frac{\partial H}{\partial u_j}\dot u_j + \dot\lambda_i f_i$$
$$= \sum \dot\lambda_i(-\dot x_i + f_i) = 0$$

$\Rightarrow H = \text{const}$

to get slope u_0 at $x=0$ subst $y=a, x=0$

$$\dot{x}_i = f(x_i) + \sum B_{ik} u_k$$

$$J = \int L(x)\,dt$$

$$H = \lambda^T(f+Bu) + L$$

$$\frac{\partial H}{\partial u_k} = \sum B_{ik}\lambda_i \neq \text{fn of } u \rightarrow \text{min by setting } u = \infty$$

We need constraint of form $a_i \le u \le b_i$

Bang Bang control u_i switches from one limit to another whenever

$\{B_{ik}\lambda_i\}$ changes sign

Minimum Time Case

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \qquad M = K + \sum \nu_m g_m$$

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \qquad H = \sum \lambda_i f_i + L$$

End cond

$$\frac{\partial M}{\partial x_i} - \lambda_i = 0$$

$$H + \frac{\partial M}{\partial t} = 0$$

Take $K=0$ $\qquad g_m(x) = x_m - x_{om} \qquad m = 1, \dots, q$

$$\frac{\partial g_m}{\partial x_i} = \delta_{mi}$$

We get $\lambda_m = \nu_m \qquad m = 1, \dots, q$

x_m = given at final time

$\lambda_i = 0 \qquad i = q+1, \dots, n$

For time optimal control $L = 1$

$$H = \sum \lambda_i f_i + 1$$

so we get

$$\dot{x}_i = f_i$$

$$\dot{\lambda}_i = -\sum \frac{\partial f_i}{\partial x_i}\lambda_j$$

Transversality $\frac{\partial M}{\partial t} = 0$ $\quad H = 0 \implies \sum \lambda_i f_i = -1$ @ $t = t_f$

x_i = given @ $t = t_f$ q conditions

$\lambda_i = 0$ @ t_f , $i = q+1, \dots, n$ conditions

$\dot{x} = Ax + bu$ $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Simplest time optimal problem

$\ddot{x} = u$

$\dot{x}_1 = x_2$

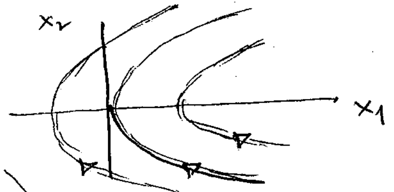
$\dot{x}_2 = u$ Minimize time to reach origin with $-1 \le u \le 1$

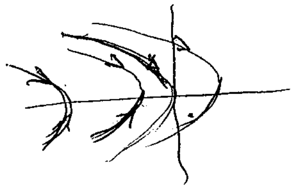
$H = \lambda_1 x_2 + \lambda_2 u + 1$

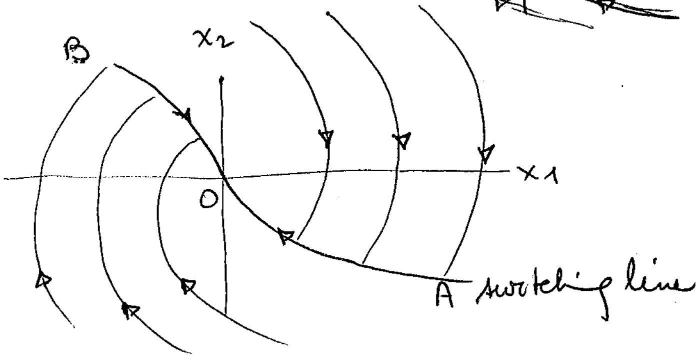
$\dot{\lambda}_1 = 0$ $\lambda_1 = \text{const} = c_1$

$\dot{\lambda}_2 = -\lambda_1$ $\lambda_2 = c_2 - c_1 t$

Optimal $u = \text{sign}\ \lambda_2$







$u = +1$ above AOB
and on BO
$u = -1$ below AOB
& on AO

Bang Bang Control exple 2 Harmonic Oscillator

$\ddot{x} + x = u$

Reach Origin in min time with $-1 \le u \le 1$

$\dot{x}_1 = x_2$ $\dot{x} = Ax + bu$ $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\dot{x}_2 = -x_1 + u$

Then $H = \lambda_1 x_2 - \lambda_2 x_1 + \lambda_2 u + 1$

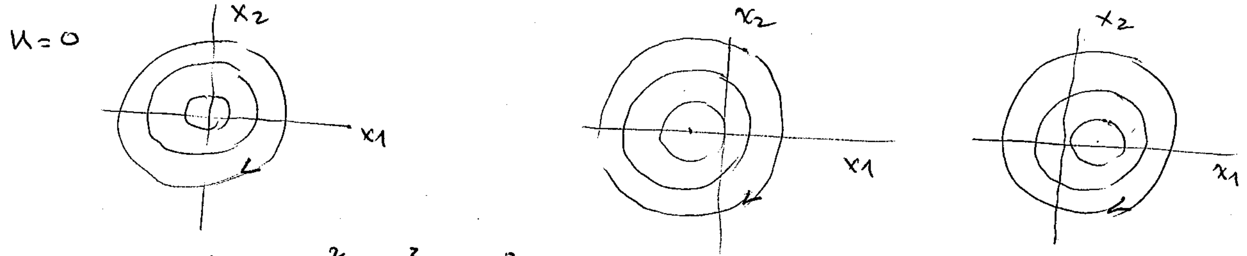
$\dot{\lambda}_1 = \lambda_2$

$\dot{\lambda}_2 = -\lambda_1$ $\lambda_2 = A \sin(t - t_0)$

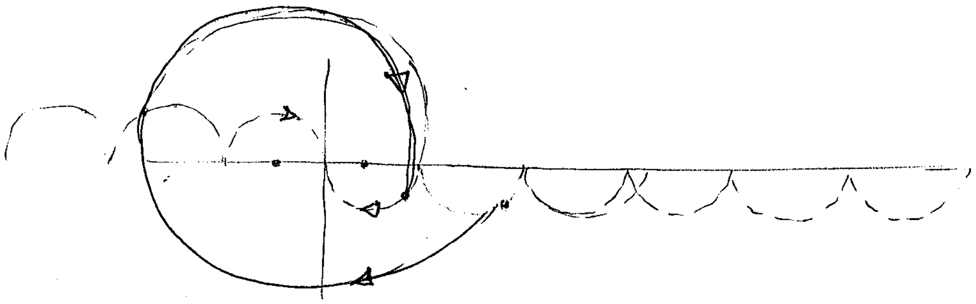
$\lambda_1 = -A \cos(t - t_0)$

Optimal control on Min H

$u = \text{sign}(\lambda_2)$



$$(x_1 \pm 1)^2 + x_2^2 = R^2 \quad \text{under} \quad u = \pm 1$$



Linear Optimal Control

$\dot{x} = Ax + Bu$

Choose m vector u to minimize

$$J = \frac{1}{2}\int_{t_0}^{t_f} (x^T Q x + u^T R u)\, dt$$

where $Q \geq 0$, $R > 0$ (positive definite)

$$H = \lambda^T (Ax + Bu) + \frac{1}{2}\left[x^T Q x + u^T R u\right]$$

$\frac{\partial H}{\partial u_k} = 0$ gives $Ru + B^T\lambda = 0$ $\qquad u = -R^{-1}B^T\lambda$

$$\dot{x} = Ax - BR^{-1}B^T\lambda \qquad \text{with} \qquad x(t_0) = x(0)$$
$$\dot{\lambda} = -A^T\lambda - Qx \qquad \lambda(t_f) = 0$$

$$\begin{pmatrix}\dot{x}\\ \dot{\lambda}\end{pmatrix} = M\begin{pmatrix}x\\ \lambda\end{pmatrix}, \quad M = \begin{bmatrix}A & -BR^{-1}B^T\\ -Q & -A^T\end{bmatrix}$$

Let $V = e^{M(t_f - t)} = \begin{bmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{bmatrix}$.

$$\begin{bmatrix}x(t_f)\\ \lambda(t_f)\end{bmatrix} = \begin{bmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{bmatrix}\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix}$$

But $\lambda(t_f) = 0 = V_{21}x + V_{22}\lambda \Rightarrow \lambda(t) = V_{22}^{-1}V_{21}x(t)$

if V_{22} is invertible $\qquad = P(t)\,x(t)$

so we set $\lambda = Px$ in Hamiltonian

$$\overset{\circ}{x} = Ax - BR^{-1}B^TPx$$

$$\overset{\circ}{\lambda} = \overset{\circ}{P}x + P\overset{\circ}{x} = -A^TPx - Qx$$

$$\overset{\circ}{P}x + PAx - PBR^{-1}B^TPx = -A^TPx - Qx$$

since must hold at initial time for arbitrary given x

$$\Rightarrow \overset{\circ}{P} + Q + A^TP + PA - PBR^{-1}B^TP = 0$$

$$P(t_f) = 0$$

Integrate matrix ricatti equation backwards from final time.

Solution : linear feedback control $u = Dx$ where

$$D = -R^{-1}B^TP$$

and

$$-\overset{\circ}{P} = Q + A^TP + PA - D^TRD$$

To establish that solution is a minimum we have if $\overset{\circ}{x} - Bu = Ax$ $P(t_f)=0$

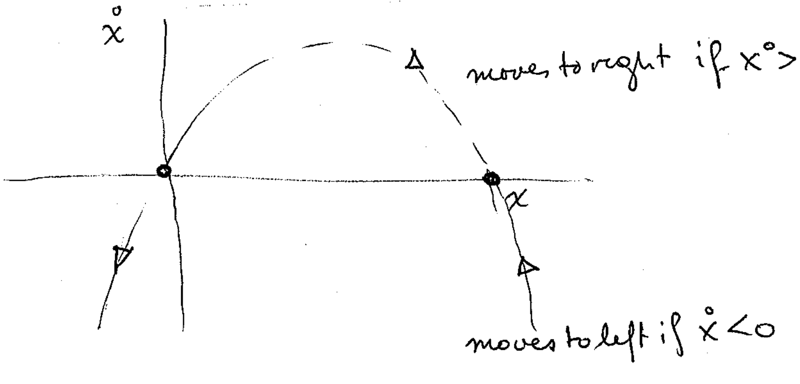
$$-x^T\overset{\circ}{P}x = x^TQx + \left[\overset{\circ}{x}^TPx - u^TB^TPx\right] + x^TP\overset{\circ}{x} - x^TPBu - x^TD^TRDx$$

$$-\frac{d}{dt}(x^TPx) = x^TQx + u^TRDx + x^TD^TRu - x^TD^TRDx + u^TRu - u^TRu$$

$$= x^TQx + u^TRu - (u-Dx)^TR(u-Dx)$$

$$\int(x^TQx + u^TRu)dt = x(0)^TP(0)x(0) + \int_{t_0}^{t_f}(u-Dx)^TR(u-Dx)\,dt$$

Ricatti Eq



Problem #1 Calculate optimal control for

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{\circ} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u$$

$$\ddot{y} + y = u$$
$$y = x_2$$
$$\overset{\circ}{y} = x_1$$

to minimize

$$J = \int_0^\infty (8x_2^2 + u^2)\,dt$$

write steady state with deriv → 0

② from Bryson – Airplane in Wind

Air speed V Wind speed U

$\dot{x} = V\cos\theta + U$, control is direction θ

$\dot{y} = V\sin\theta$

Find path to enclose maximum area in a time T.

$$\int y\,dx = \int_0^t y\,\dot{x}\,dt$$

8/7/73

$$\dot{x} = Ax + Bu$$

$$J = \tfrac{1}{2}\int_{t_0}^{t_f} (x^TQx + u^TRu)\,dt \qquad + \tfrac{1}{2}x^T(t_f)\,F\,x(t_f)$$

$$H = \lambda^T(Ax + Bu) + \tfrac{1}{2}x^TQx + \tfrac{1}{2}u^TRu$$

$$\frac{\partial H}{\partial u} = u = -R^{-1}B^T\lambda$$

$$\dot{x} = Ax - BR^{-1}B^T\lambda$$

$$\dot{\lambda} = -Qx - A^T\lambda \qquad , \lambda(t_f) = \partial F$$

$$\begin{bmatrix}\lambda(t_f)\\ \lambda(t_f)\end{bmatrix} = \Phi(t_f,t)\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix} \qquad \text{Let } \phi = \begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix}$$

$$\lambda(t_f) = \phi_{21}\,x(t) + \phi_{22}\,\lambda(t)$$

$$\lambda = \phi_{22}^{-1}\,\phi_{21}\,x$$

$\boxed{\lambda = Px}$ gives

$$-\dot{P} = Q + A^TP + PA - \overset{D^TRD}{PBR^{-1}B^TP} \qquad , \quad u = Dx$$

$$\text{Then } x^TQx + u^TRu = -x^T\dot{P}x - x^TA^TPx - x^TPAx - x^TD^TRDx + u^TRu,$$

$$= -\frac{d}{dt}(x^TPx) + (u-Dx)^TR(u-Dx) \qquad P(t_f)=1$$

$$\int (x^TQx + u^TRu)\,dt = x^T(t_0)\,P(t_0)\,x(t_0) - x(t_f)\,F\,x(t_f) + \int (u-Dx)^TR\,(u-Dx)\,dt$$

$$J = \tfrac{1}{2}x^T(t_0)\,P(t_0)\,x(t_0) + \int \ldots$$

$$J(x,t) = \frac{1}{2}\int_{t_0}^{t_f}(x^TQx + U^TRU) + x^TFx.$$

$$V(x,t) = J_{opt}(x,t)$$

$$-\frac{\partial V}{\partial t} = \min_{U}\left\{\tfrac{1}{2}(x^TQx + U^TRU) + \frac{\partial V^T}{\partial x}(Ax + BU)\right\}$$

Assuming you have a quadratic form, to check Hamiltonian equation

$$-\frac{1}{2}(x^T\dot{P}x) = \min_{U}\left\{\tfrac{1}{2}(x^TQx + U^TRU) + x^TP(Ax + BU)\right\}$$

differentiating w.r.t. U to get min.

$$\frac{\partial}{\partial U}\left\{\tfrac{1}{2}(x^TQx + U^TRU) + x^TP(Ax + BU)\right\}$$

$$= RU + B^TPx = 0, \quad U = -R^{-1}B^TPx$$

$$-\frac{1}{2}x^T\dot{P}x = \frac{1}{2}x^T(Q + A^TP + PA - PBR^{-1}\underset{(B^T)}{B^tP})x$$

~~x^TQx~~ ~~opt.~~ ~~cost~~ ~~coeff~~ **Summary:**

Given $\dot{x} = Ax + BU$

$$J = \frac{1}{2}x^T(t_f)F\,x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(x^TQx + U^TRU)dt$$

Solution

$$U = Dx$$

where $D = -R^{-1}B^TP$

$$-\dot{P} = Q + A^TP + PA - PBR^{-1}B^TP, \quad P(t_f) = F$$

and

~~$J_{opt}(x,t) = x^T(t)P(t)x(t)$~~

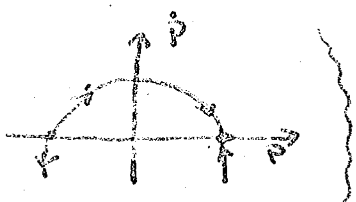
$$J_{opt}(x,t) = x^T(t)P(t)x(t)$$

Ref: R.E. Kalman B of Soc Math Mexicana 1960

Contributions to the theory of optimal control

— under inf. time ~~limit~~ limit, prove under what cond.'s sol. would exist.

EXISTENCE OF SOLUTION OF RICCATI EQUATION



$\dot{p} = a + bp - cp^2$

For fixed t_f P exists for all $t < t_f$ when $Q \geq 0$, $R > 0$, $F \geq 0$.

enough to [illegible] that you can integrate [illegible]

First $P \geq 0$ since $J_{opt} \geq 0 = x^T(t) P(t) x(t)$.

Suppose $P(t_1)$ is unbdd (finite escape time).

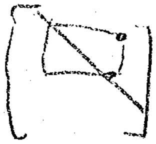
Then $P(t_1 + \epsilon)$ is unbdd as $t \to 0$

But with $u = 0$

$$J = \tfrac{1}{2}\int x^T Q x \, dt + x^T F x \quad ; \quad \dot{x} = Ax$$

so $J(x, t_1 + \epsilon) \geq x^T(t_1 + \epsilon)\, P(t_1 + \epsilon)\, x(t_1 + \epsilon)$

If P is unbdd, then at least one diag. element P_{ii} is unbdd because $P \geq 0$



Then setting $x = e_i$, $e_i = i^{th}$ unit vector we have

$$J(e_i, t) \geq P_{ii}$$

Infinite time interval case

Consider

$$\dot{x} = Ax + Bu$$

$$J = \int_{t_0}^{\infty} (x^T Q x + u^T R u)\, dt$$

Ex. $V = J_{opt}$ does not necessarily exist.

Suppose you too

$$\dot{x}_1 = x_1 \qquad J = \int (x_1^2 + u^2)\, dt \qquad x_1(t_0) = 1$$

$$\dot{x}_2 = x_2 + u \qquad\qquad x_2(0) = 0$$

$x_1 = e^t$

$$J = \int_{t_0}^{\infty} (e^{2t} + u^2)\, dt$$

↑ example of uncontrollable system

∴ To ensure a solution we assume ~~a~~ system is controllable for all t.

Let $P(t, t_f)$ be solution of

$-\dot{P} = Q + A^T P + PA - PBR^{-1}B^T$

We will show that $\bar{P}(t) = \lim_{t_f \to \infty} P(t, t_f)$ exists
and satisfy satisfies Riccati equation

Also $U_{opt} = -R^{-1}B^T P(t) x$

Since system is controllable $\exists$ a control U s.t. $x(t_1)=0$
$t_1 < t_f$, Then for $t > t_f$ solution is $x=0$, $U=0$.
This control gives finite cost $J(x,t)$ independent of t_f
for fixed t_1.

~~$J(x,t) \geq x^T P x$~~

$$J(x,t) \geq x^T(t) P(t, t_f) x(t)$$

But also since $Q \geq 0$, $R > 0$, $x^T(t) P(t, t_f) x(t)$ increases
as t_f increases

Thus $\lim_{t_f \to \infty} x(t) P(t, t_f) x(t)$ exists for all $x(t)$

Taking $x(t) = e_i$, $\lim_{t_f \to \infty} P_{ii}$ exists

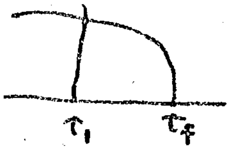
Also $2P_{ij} = (e_i + e_j)^T P (e_i + e_j) - P_{ii} - P_{jj}$

to show that it actually satisfies the Riccati equ'ns. you can do the following.

let $P(t, t_f, F)$ be solution for $P(t_f) = F$ ← final value

Then

$$P(t, t_f, 0) = P(t, t_1, P(t_1, t_f, 0))$$



Then in limit $t_f \to \infty$

$$\bar{P}(t) = P(t, t_1, \bar{P}(t_1))$$

---

now need to show that

$U_{opt} = -R^{-1}B^T \bar{P}(t) x$ is in fact ~~a~~ optimal

<u>let $U_{opt} = U^*$</u>

$$J(x,t,t_f,U^*) = x^T(t)\bar{P}(t)x(t) - x(t_f)\bar{P}(t_f)\alpha(t_f)$$

$$\leq x^T(t)\bar{P}(t)x(t)$$

Also,

$$J(x,t,t_f,U^*) \geq x(t)P(t,t_f)x(t)$$

Thus,

$$\lim_{t_f\to\infty} J(x,t,t_f,U^*) = x^T(t)\bar{P}(t)x(t)$$

Also optimal cost

$$V(x,t,\infty) \leq \lim_{t_f\to\infty} J(x,t,t_f,U^*) = x(t)\bar{P}(t)x(t)$$

Suppose This is strict inequality.

Then

$$\lim_{t_f\to\infty} V(x,t,t_f) = \lim_{t_f\to\infty} x^T(t)P(t,t_f)x(t)$$

$$< x(t)\bar{P}(t)x(t) \quad \Rightarrow \text{contradiction}$$

Time invariant case

Let A, B, Q, R are constant $\qquad \int_{t_0}^{\infty}(x^TQx + u^TRu)\,dt$

Then

$$\bar{P}(t) = \lim_{t_f\to\infty} P(t,t_f) \text{ exists}$$

Since cost is independent of t, $\bar{P}(t)$ is constant P_0 say.

Also

$$P(t,t_f) = \lim_{t_f\to\infty} P(0,t_f-t) = \lim_{t\to-\infty} P(0,t_f-t)$$

$$= \lim_{t\to-\infty} P(t,t_f)$$

Since $\bar{P}(t)$ is constant, we get $U = -R^{-1}B^TP_0x$ where

$$0 = Q + A^TP_0 + P_0A - P_0BR^{-1}B^TP_0$$

We have a constant feedback system

$$U = Dx, \quad D = -R^{-1}B^TP_0$$

$$\dot{x} = (A+BD)x \qquad \text{close loop system}$$

Question

(Can show that null cond. which ⇒ stability is satisfied)
ie Observability.

Let $Q = C^T C$ so $J = \int_0^\infty y^T y + u^T R u \, dt$, $y = Cx$

Then a sufficient condition for stability of closed loop systems is A, C are observable. That is

$$\int_{t_0}^{t_1} e^{A^T(t-t_0)} C^T C e^{A(t-t_0)} dt > 0 \quad \text{for } t_1 > t_0$$

which means that $C e^{A(t-t_0)} x(t_0) \neq$ identically zero for any $x(t_0) \neq 0$

We can write

$$(A+BD)^T P + P(A+BD) = A^T P - D^T R D + PA - D^T R D$$
$$= -(Q + D^T R D)$$

Thus

$$V(x) = x^T P x$$

has derivative

$$\dot V = -x^T (Q + D^T R D) x = -x^T Q x - u^T R u$$
$$\leq 0$$

⇒ V is a Liapunov func.
cost function

∴ Optimal system is in fact stable

Suff. cond.'s { Controllability guarantees solution for $t_f = \infty$
Observability guarantees the solution to be asymptotically stable.

$$x^T Q x + u^T R u = -\frac{d}{dt}(x^T P x) + (u - Dx)^T R (u - Dx)$$

$$\int_0^{t_f} (x^T Q x + u^T R u) dt = x(t) P x(t) = \underbrace{x(t_f)^T P x(t_f)}_{0 \text{ if system stable}} + \int \ldots\ldots$$

if not observable → This term may not

$\dot{x}_1 = x_1 + u_1$ $\qquad J = \int_0^\infty (x_1^2 + u_1^2 + u_2^2)\,dt$

$\dot{x}_2 = x_2 + u_2$

$A = I,\ B = I,\ R = I$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} - \begin{bmatrix} P_{11}^2 + P_{12}^2 & P_{11}P_{12} + P_{12}P_{22} \\ P_{11}P_{12} + P_{12}P_{22} & P_{12}^2 + P_{22}^2 \end{bmatrix} = 0$$

(Riccati equation)

$$2P_{12} - P_{12}(P_{11} + P_{22}) = 0\ ,\quad P_{12} = 0$$

$$(\text{or } P_{11} + P_{22} = 2)$$

$$P_{11}^2 - 2P_{11} - 1 = 0\ ,\quad P_{11} = 1 \pm \sqrt{2}$$

$$P_{22}^2 - 2P_{22} + 0 = 0\ ,\quad P_{22} = 0, 2$$

more than one solution since this is an example which is not observable.

~~Performance index with cross product term~~

Shifted stability

$$\dot{x} = Ax + Bu$$

$$J = \int_0^\infty e^{2\alpha t}(x^TQx + u^TRu)\,dt$$

A, B, Q, R constant

Consider $\hat{x} = e^{\alpha t}x\ ,\ \hat{u} = e^{\alpha t}u$

$$J = \int_0^\infty (\hat{x}^TQ\hat{x} + \hat{u}R\hat{u})\,dt$$

Also

$$\begin{aligned} \dot{\hat{x}} &= e^{\alpha t}\dot{x} + \alpha e^{\alpha t}x \\ &= (A + \alpha I)e^{\alpha t}x + e^{\alpha t}Bu \\ &= (A + \alpha I)\hat{x} + B\hat{u} \end{aligned}$$

Optimal control is $\hat{u} = -R^{-1}B^TP\hat{x}$ where

$$0 = Q + (A + \alpha I)^TP + P(A + \alpha I) - PBR^{-1}B^TP$$

Then $u = -R^{-1}B^TPx$

From $\wedge$ problem $A + \alpha I + BD$ has eigenvalues in left-half plane

$\therefore$ $A + BD$ has e.values to left of $z = -\alpha$

Problem:

Performance index with cross product term

$$\dot{x} = Ax + Bu$$

$$J = \int (x^T Q x + 2u^T K x + u^T R u)\,dt$$

Set

$$u = \hat{u} - R^{-1}Kx$$

$$2u^T K x + u^T R u = \cancel{2\hat{u}^T K x} - 2x^T K^T R^{-1} K x$$
$$+ \hat{u}^T R \hat{u} \cancel{- 2\hat{u}^T K x} + x^T K^T R^{-1} K x$$

Thus

$$J = \int \{ x^T (Q - K^T R^{-1} K) x + \hat{u}^T R \hat{u} \}\,dt$$

$$\dot{x} = (A - BR^{-1}K)x + B\hat{u}$$

Optimal solution $\hat{u} = -R^{-1}B^T P x$

$$u = Dx \;, \; D = -R^{-1}(B^T P + K)$$

$$-\dot{P} = Q - K^T R^{-1} K + (A - BR^{-1}K)^T P + P(A - BR^{-1}K)P$$
$$- PBR^{-1}B^T P$$
$$= Q + A^T P + PA - (B^T P + K) R^{-1} (B^T P + K)$$
$$= Q + A^T P + PA - D^T R D$$

Tracking problem - you want system to follow another

$\underline{\dot{z} = Fz}$ $\quad \dot{x} = Ax + Bu \;, \; y = Cx$ $\quad$ i.e ~~tech~~ you want one system to behave like another.

desired response

We want $\dot{y} = Fy$ as near as possible

So we ~~defined~~ define,

$$J = \int \{ (\dot{y} - Fy)^T Q (\dot{y} - Fy) + u^T R u \}\,dt$$

$$\dot{y} - Fy = C\dot{x} - FCx = C(Ax + Bu) - FCx$$
$$= (CA - FC)x + CBu$$

$$J = \int x^T (CA - FC)^T Q (CA - FC) x \, dt + 2\int u^T B^T C^T Q (CA - FC) x$$

$$+ \int u^T (R + B^T C^T C B) u \, dt.$$

Bridge to designing system called Model for Performance System.

<u>Ex.</u> 8/9/73

$\dot{x} = Ax + Bu$, $J = \int 1 \, dt$

$H = \lambda^T (Ax + Bu) + 1$

$\frac{\partial H}{\partial u} = B^T \lambda$ $\qquad a_i \leq u \leq b_i$

if $B^T \lambda \neq 0$ $\qquad$ u must hit constraint

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u \end{aligned}$$

$H = \lambda_1 x_2 + \lambda_2(-x_1 + u) + 1$

$\frac{\partial H}{\partial u} = \underline{\underline{\lambda_2}}$

$\underline{\lambda_2} = A \sin(t - \alpha)$

set $u = \pm 1$ to minimize H

$\dot{z} = Fz$

$\dot{x} = Ax + Bu$, $y = Cx$

We want y to behave like z

(A) <u>Model in performance index</u>

$$J = \int \{ (\dot{y} - Fy)^T Q (\dot{y} - Fy) + u^T R u \} dt$$

(B) <u>Model in the system</u>

(B) Model in the System.

$$\dot{x} = Ax + BU$$
$$\dot{z} = Fz$$

$$\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}A & 0\\ 0 & F\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + \begin{pmatrix}B\\ 0\end{pmatrix}U$$

$$J = \int\{(y-z)^T Q(y-z) + U^TRU\}\,dt$$

$$J = \int\{x^TC^TQCx - 2z^TQCx + z^TQz + U^TRU\}\,dt$$

$$= \int\left\{[x\ \ z]^T\begin{bmatrix}C^TQC & QC\\ C^TQ & Q\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + U^TRU\right\}$$

in Partition form the Riccati eq. that corresponds to this effect

$$-\begin{bmatrix}\dot{P}_{11} & \dot{P}_{12}\\ \dot{P}_{12} & \dot{P}_{22}\end{bmatrix} = \begin{bmatrix}C^TQC & C^TQ\\ QC & Q\end{bmatrix} + \begin{bmatrix}A^T & 0\\ 0 & F\end{bmatrix}\begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix}$$

$$+\begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix}\begin{bmatrix}A & 0\\ 0 & F\end{bmatrix} - \begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix}\begin{bmatrix}BR^{-1}B^T & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}P_{11} & P_{1}\\ P_{12} & P_{2}\end{bmatrix}$$

$-\dot{P}_{11} = C^TQC + A^TP_{11} + P_{11}A - P_{11}BR^{-1}B^TP_{11}$ ← Solve first

$-\dot{P}_{12} = CTQ + A^TP_{12} + P_{12}F - P_{11}BR^{-1}B^TP_{12}$ ← Linear in P_{12}

$-\dot{P}_{22} = Q + F^TP_{22} + P_{22}F - P_{12}^TBR^{-1}B^TP_{12}$ ← Linear in P_{22}

$$U = -R^{-1}[B^T0]\begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} = \underbrace{-R^{-1}B^TP_{11}}_{\text{Optimal feedback}}x \underbrace{- R^{-1}B^TP_{12}}_{\text{optimal forcing term}}z$$

— the same regardless of the system

feedbacks here are completely independent of the model.

Solving Riccati equation in steady-state form

neat solution proposed by Potter J. SIAM Appl. Math
Vol. 14 1966 pp 496-499

$$0 = Q + A^T P + PA - PSP \quad , \quad S = BR^{-1}B^T$$

Consider,

$$M = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix}$$

Let $\begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector

$$Au - Sv = \lambda u$$
$$-Qu - A^T v = \lambda v$$

Consider,

$$M^T \begin{bmatrix} -v \\ u \end{bmatrix} = \begin{pmatrix} A^T & -Q \\ -S & -A \end{pmatrix} \begin{pmatrix} -v \\ u \end{pmatrix} = \begin{pmatrix} -A^T v & -Qu \\ +Sv & -Au \end{pmatrix} = \begin{pmatrix} \lambda v \\ -\lambda u \end{pmatrix}$$

$$= -\lambda \begin{pmatrix} -v \\ u \end{pmatrix}$$

(can always pick n stable or unstable e. values)

Let $$V^{-1} M V = \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}$$

$$\begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}$$

$$AV_{11} - SV_{21} = V_{11}\Lambda$$
$$-QV_{11} - A^T V_{21} = V_{21}\Lambda$$

$$-Q - A^T V_{21} V_{11}^{-1} = V_{21} \Lambda V_{11}^{-1} = -V_{21} V_{11}^{-1} A - V_{21} V_{11}^{-1} S V_{21} V_{11}^{-1}$$

Thus $V_{21} V_{11}^{-1}$ satisfy steady-state Riccati eq.

Suppose you don't use (or have) all possible feedbacks

or suppose you can't measure all the feedbacks (second method shows how to approximate them)

Direct Calculation of Optimal Feedbacks.

Suppose

$$\dot{x} = Ax + Bu$$

$$J = \int (x^T Q x + u^T R u)\,dt$$

and we decide to use feedback controls

$$u = Dx$$

and try to optimize D. Then

$$\dot{x} = Fx \qquad , F = A + BD$$

$$J = \int_{t_0}^{t_f} x^T S x \, dt \quad , \; S = Q + D^T R D$$

$$x(t) = \phi(t, t_0)\, x(t_0) \quad \Big[\; x(t) = e^{F(t - t_0}\, x(t_0)$$

$$J = x^T(t_0)\, P(t_0)\, X(t_0)$$

where

$$P(t) = \int_t^{t_f} \Phi^T(\tau, t)\, S\, \Phi(\tau, t)\, d\tau$$

$$\dot{P} = -S - F^T P - PF$$

now calculate cost

$$\phi(t,\tau)\, x(\tau) = x(t) \quad , \dot{x} = Fx$$

$$\frac{d\Phi}{dt}(t,\tau)\, x(\tau) = F\, \Phi(t,\tau)\, x(\tau)$$

~~$\frac{d}{d\tau}\, e^{F(t-\tau)} = -F e^{F(t-\tau)}$~~

$$\frac{d}{dt}\, e^{F(\tau - t)} = -F e^{F(\tau - t)}$$

$$\frac{d}{dt}(t,\tau) = F\phi$$

$$\frac{d\Phi}{d\tau}(t,\tau)\, x(\tau) + \phi(t,\tau)\, F x(\tau) = 0$$

$$\frac{d}{d\tau}\, \Phi(t,\tau) - \Phi(t,\tau)\, F \, .$$

~~$\frac{d}{d\tau}\, \phi^T(\tau, t)$~~

~~$= -F^T \phi^T(\tau, t)$~~

~~$\phi(t,\tau)\, \phi(\tau, t) = I$~~

~~$\frac{d\,\Phi(t,\tau)}{d\tau} = F\, \Phi(t,\tau)$~~

~~$\frac{d}{d\tau}\left[\phi(t,\tau)\, \phi(\tau, t)\right]$~~

~~$= F \phi(t,\tau)\, \phi(\tau, t)$~~

~~$=$~~

not best way of doing this

To calculate cost

$$J = x(t_0)^T P(t_0)\, x(t_0)$$

where

$$-\dot{P} = F^T P + P F + S \quad , P(t_f) = 0 \qquad (A)$$

Now if we make a variation δD in D

$$\delta J = x^T(t_0)\ \delta P(t_0)\, x(t_0)$$

$$-\delta \dot{P} = F^T \delta P + \delta P F \quad + \delta D^T(B^T P + RD) + (B^T P + RD)^T \delta D$$

allowing for changes in F, S

$$\delta P(t_f) = 0$$

Suppose

$$D = -R^{-1} B^T P$$

Then $\delta P(t) = 0$ for all t.

Hence $\delta J = 0$ for all $x(t_0)$ (standard solution)

To calculate P you substitute $D = -R^{-1}B^T P$ in (A) giving

$$-\dot{P} = Q + A^T P + PA - D^T R D \qquad \text{Riccati eq.}$$

<u>Unconstrained solution</u>

In constrained case $\underline{B^T P + RD}$ cannot be made to vanish

δP cannot vanish

only δJ can vanish for particular $x(t_0)$

~~⇒ must take~~

<u>Condition for a minimum</u>

$$\frac{\partial J}{\partial D_{ij}} = 0 .$$

To evaluate $G_{ij} = \dfrac{\partial J}{\partial D_{ij}}$

introduce $X = x x^T \quad , \; X_{ij} = x_i x_j \quad n \times n$ matrix

path has something to do with it

Since $\dot{x} = Fx$,

$$\dot{X} = \dot{x}x^T + \dot{x}x^T = FX + XF^T, \quad X(t_0) = x(t_0)\,x(t_0)^T$$

also

$$x^T S x = \mathrm{tr}\,(S X) \qquad \text{where } \mathrm{tr}\, A = \sum A_{ii}$$

$$\sum_{ij} x_i S_{ij} x_i = \sum_i \Big(\sum_j S_{ij} x_j x_i\Big) = \sum_i \Big(\sum_j S_{ij} X_{ji}\Big)$$

Note that $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$ for any A, B

$$\sum_i \sum_j A_{ij} B_{ji} = \sum_j \sum_i B_{ij} A_{ji}$$

$$J = \int_0^\infty \mathrm{tr}(SX)\,dt$$

$$\delta J = \int \mathrm{tr}(S\,\delta X) + 2\,\mathrm{tr}(\delta D^T R D X)\}\,dt$$

where

$$\delta\dot{X} = F\,\delta X + \delta X F^T + B\,\delta D\,X + X\,\delta D^T B^T$$

unless F func. of D

$$\frac{d\,\mathrm{tr}(PSX)}{dt} = 2\,\mathrm{tr}(\delta D^T B^T P X) - \mathrm{tr}(S\,\delta X)$$

$$\int \mathrm{tr}\,(S\,\delta X)\,dt = 2\int \mathrm{tr}\,(\delta D^T B^T P X)\,dt$$

$$\delta J = 2\int \mathrm{tr}\,\delta D^T (B^T P + R D) X\,dt$$

where

$$G = 2\int_{t_0}^{t_f} (B^T P + R D) X\,dt \qquad \text{, treat } G \text{ as a matrix}$$

$$\delta J = \sum_{i,j} \frac{\partial J}{\partial D_{ij}}\,\delta D_{ij} = \mathrm{tr}(\delta D^T G)$$

In case A, B, are constant and $t_f = \infty$ Then P is constant.

$$G = 2\,(B^T P + R D)\,W$$

where $W = \int_0^\infty X\,dt$

$$\dot{X} = FX + XF^T \qquad \leftarrow \text{if integrate This from } 0 \text{ to } \infty \text{ \& } F \text{ stable}$$

$X(\infty) - X(0)$

Thus for the time invariant case, if F is stable then

$$G = 2(B^TP + RD)W$$

where

$$F^TP + PF + S = 0$$

$$FW + WF^T + X(0) = 0$$

to solve this equation use a desent method.

Example

Worst case design   to $\overset{get}{\wedge}$ rid of dependence on initial cond's

Let $$M = \max_{x_0} \frac{J}{x_0^T x_0} = \max_{x_0} \frac{x_0^T P(0) x_0}{x_0^T x_0}$$

since P is symmetric $P = V^T \Lambda V$, where $V^TV = I$

Set $z = Vz_0$ then

$$\frac{J}{x_0^T x_0} = \frac{z^TV^TPVz}{z^TV^TVz} = \frac{z^TVz}{z^Tz} = \frac{\sum \lambda_i z_i^2}{\sum z_i^2}$$

$$\lambda_{min} \sum z_i^2 \le \sum \lambda_i z_i^2 \le \lambda_{max} \sum z_i^2$$

Thus, $M = \lambda_{max}(P)$

To get $\frac{\partial M}{\partial P_{ij}}$ - note that if v is normalized ~~eigenvector~~ e. vector

then $\lambda = v^TPv$, $v^Tv = 1$

~~the~~ Under a change δP

$$\delta\lambda = \delta v^T P v + v^T \delta P v + v^T P \delta v$$

$$= v^T\delta P v + \lambda\,\cancel{\delta v^T v + v^T \delta v})$$

since $\delta v^T v + v^T \delta v = 0$

$$\underline{\delta M = v^T \delta P(0) v}$$

Average design

Let $M = \underset{x_0}{E}(J)$

But $J = x(t_0)\, P(0)\, x(t_0) = tr(PX)$

$M = tr(P\bar{X})$

where $\bar{X} = E(X(t_0))$

$\bar{X} = I$ if all x_0 are equally likely.

Example Harmonic oscillator

$$\ddot{y} + y = 0$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$y = x_2 \quad , \dot{y} = x_1$

$$J = \int_0^\infty (8y^2 + u^2)\, dt$$

$u = d_1 x_1 + d_2 x_2 \sim d_1 \dot{y} + d_2 y$

$$\begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} P_{11}^2 & P_{1} P_{12} \\ P_{11}P_{12} & P_{12}^2 \end{bmatrix} = 0$$

$2P_{12} - P_{11}^2 = 0$

$P_{22} - P_{11} - P_1 P_{12} = 0$

$-2P_{12} + 8 - P_{12}^2 = 0$

$P_{12} = 2, -4$

take $P_{12} = +2$

~~$P_{11} = 4$~~

$$P = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} \qquad [d_1 \; d_2] = -[1 \;\; 0] \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

$d_1 = -2, d_2 = -2$

Optimal system $\ddot{y} + 2\dot{y} + 3y = 0$ instead of $\ddot{y} + y = 0$

$\ddot{y} + 2\nu n + n^2 y = 0$, n spring constant , ν damping.

Then $\nu = \sqrt{3}$

Suppose we only allow $v = d_1 \dot{y} = d_1 x_1$

$F^T P + PF + S$ because

$$\begin{bmatrix} d_1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & \cdots \\ \cdots & \cdots \end{bmatrix} + \begin{bmatrix} P_{11} & \cdots \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} d_1 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} d_1^2 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$P = \begin{bmatrix} -\left(\frac{d_1}{2} + \frac{4}{d_1}\right) & 4 \\ 4 & -\left(\frac{9d_1}{2} + \frac{4}{d_1}\right) \end{bmatrix}$$

Initial condition
$x_1 = a_1$
$x_2 = a_2$

Gradient is $[g_1 \ g_2] = 2(B^T P + RD)W$

$$= 2\left\{[1 \ 0] \begin{bmatrix} P_{11} & \cdots \\ \cdots & \cdots \end{bmatrix} + [d_1 \ 0]\right\} \begin{bmatrix} W_{11} & W_{12} \\ W_{12} & W_{22} \end{bmatrix}$$

where

$$\begin{pmatrix} d_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_{11} & \\ & \end{pmatrix} + \begin{pmatrix} w_{11} & \\ & \end{pmatrix} \begin{pmatrix} d_1 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{pmatrix} = 0$$

check if squared or not

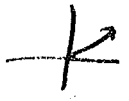
$$W = \begin{bmatrix} \frac{a_1^2 + a_2^2}{d_1^{\textcircled{2}}} & -a_2^2 \\ -a_2^2 & \frac{a_1^2 + a_2^2}{d_1^2} - a_2^2 d_1^2 + 2a_1 a_2 \end{bmatrix}$$

$$g_1 = 2(P_{11} + d_1)w_{11} + 2P_{12}W_{21}$$

$$\cancel{= \frac{4}{d_1^2}} \quad = \left(\frac{4}{d_1^2} - \frac{1}{2}\right)a_1^2 + \left(\frac{4}{d_1^2} - \frac{9}{2}\right)a_2^2$$

For minimum $g_1 = 0$

$$d_1 = -2\sqrt{\frac{2a_1^2 + 2a_2^2}{a_1^2 + 9a_2^2}}$$



If system has initial ~~value~~ velocity but not displacement
$a_2 = 0, \ a_1 = -2\sqrt{2}$

Damping of optimal system $= \sqrt{2}$

If system has initial displacement but not velocity

$$a_1 = 0 \;,\; d_1 = \frac{-J\sqrt{2}}{3}$$

What happens when you optimize the worst case

Worst case

$$\lambda^2 + \lambda\left(5d_1 + \frac{8}{d_1}\right) + \frac{9d_1^2}{4} + 20 + \frac{16}{d_1^2} = 0$$

$$\lambda_{max} = M = -\left(\frac{5}{2}d_1 + \frac{4}{d_1}\right) + 2\sqrt{d_1^2 + 4}$$

Corresponding e. vector is

$$v = \frac{1}{\sqrt{c^2+1}}\begin{bmatrix} c \\ 1 \end{bmatrix} \text{ where } c = \frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + 1}$$

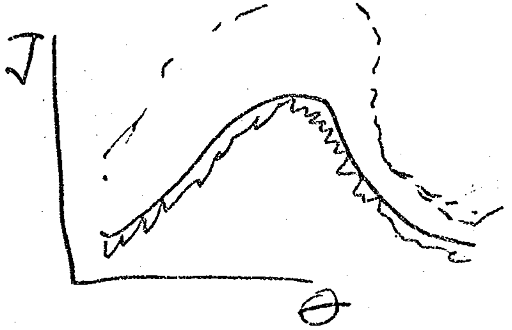
$$\frac{\partial M}{\partial d_1} = \frac{4}{d_1^2} - \frac{5}{2} - \frac{2}{\sqrt{4/d_1^2 + 1}}$$

Substituting

$$d_1 = \frac{-2}{\sqrt{\frac{4}{3} + \frac{7}{3}\cos\theta}}$$

$$\frac{\partial M}{\partial d_1} = 0 \quad \text{when } \cos 3\theta = \frac{89}{343} \;,\; d_1 = -1.077$$

Damping constant .539



θ = angle of x_0 vector with x_1 axis

Aircraft in cross wind. (H.W. problem)

$$\dot{x} = V\cos\theta + u$$
$$\dot{y} = V\sin\theta$$

Maximize enclosed area of path

$$J = \int_0^T y\,dx = \int_0^T y\dot{x}\,dt$$

$$H = (\lambda_1 + y)(V\cos\theta + u) + \lambda_2 V\sin\theta$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = 0 \quad , \lambda_1 = c_1$$

$$\dot{\lambda}_2 = -(x + c_2)$$

Then H is maximized when

$$0 = \frac{\partial H}{\partial \theta} = -(y + C_1)V\sin\theta - (x + C_2)V\cos\theta$$

$$\sin\theta = -\frac{x + C_2}{y + C_1}\cos\theta$$

Also Hamiltonian is constant, since t does not appear explicitly,

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial \lambda_1}\dot{\lambda}_1 + \frac{\partial H}{\partial \lambda_2}\dot{\lambda}_2 + \frac{\partial H}{\partial \theta}\dot{\theta}$$

$$C_3 = H = \left(y + C_1 + \frac{(x + C_2)^2}{y + C_1}\right)V\cos\theta + (y + C_1)u$$

where

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{1}{\sqrt{1 + \left(\frac{x + C_2}{y + C_1}\right)^2}}$$

$$C_3 = (y + C_1)\left\{\sqrt{1 + \left(\frac{x + C_2}{y + C_1}\right)^2} + \frac{u}{v}\right\}$$

$$1 + \left(\frac{x + C_2}{y + C_1}\right)^2 = \left(\frac{C_3}{y + C_1} - \frac{u}{v}\right)^2$$ equation of an ellipse

State estimation

Let $\dot{x} = Ax + Bu$ and suppose $y = Cx$ is measured, where y is a p-dimensional, $C = p \times n$ matrix.

$$y = Cx$$
$$\dot{y} = C\dot{x} = CAx + CBu$$
$$\ddot{y} = CA\dot{x} + CB\dot{u} = CA^2x + CABu + CB\dot{u}$$

From observability the system

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \cdots \\ \cdot \end{bmatrix} x = z$$

can be solved for x

Actually $y = Cx + v$, v = noise
and you differentiation magnifies noise

(Better approach) So instead suppose we have a model with output $z = Wx$
where W is invertible
Then we can use $x = W^{-1}z$ as state.

Let actual output of model be
$$z = Wx + e$$
where e is error. In order for $z \to Wx$ as $t \to \infty$ let e satisfy
$$\dot{e} = Fe$$, F a stable matrix ∋ $Re\,\lambda(F) < 0$

Then
$$\dot{z} = W\dot{x} + \dot{e}$$
$$= WAx + WBu + Fe$$, where $e = z - Wx$
~~$= WA - F$~~ $= Fz + (WA - FW)x + WBu$

This requires input x which is not available
But suppose for some matrix G
$$WA - FW = GC$$
Then
$$\dot{z} = Fz + Gy + WBu$$ Realizable

One alternative is to set $W = I$
~~$\dot{z} = Az + Bu +$~~ Then
$$\dot{x} = Ax + Bu , \; y = Cx$$
$$\dot{z} = (A - GC)z + Bu + Gy$$
$$= Az + Bu + G(y - Cz)$$
Then we need $F = A - GC$ stable. , $F^T = A^{+} - C^T G^T$ stable

Consider the problem of minimizing
$$J = \int (x_1^T Q x_1 + u_1^T R u_1)\,dt$$
where
$$\dot{x}_1 = A^T x_1 - C^T u_1$$
Take solution
$$u_1 = G^T x_1$$

~~Flow~~ Take solution

$$U_1 = -G^T x_1$$

Then by observability of original system, this system is controllable. So if Q positive definite yields a stable $A^T - G^T C^T$.

Since $z \to x$ if desired control was $v = Dx$ we now set

$$v = Dz$$

Thus control is actually

$$v = D(x + e)$$

where $\dot{e} = Fe$

Also $\dot{x} = Ax + Bv = (A + BD)x + BDe$

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A+BD & BD \\ 0 & F \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

Thus the eigenvalues of the total system are eigenvalues of A+BD + eigenvalues of F.

<u>Observers</u> of reduced dimension (Luenberger - IEEE Trans. Milita Electronics, April 1964

We have

$$\dot{x} = Ax + Bv, \quad y = Cx \text{ output}$$

Let output of model be

$$z = Wx + e,$$

where e is error, and

$$\begin{bmatrix} C \\ W \end{bmatrix} = M \quad \text{is invertible}$$

of order p

$$\begin{matrix} y = \\ z = \end{matrix} \begin{bmatrix} C \\ W \end{bmatrix} x$$

of order n-p

Then if e = 0

$$x = M^{-1} \begin{bmatrix} Cx \\ W \end{bmatrix} = M^{-1} \begin{bmatrix} y \\ z \end{bmatrix}$$

We want the error to decay

$$\dot{e} = Fe$$

Then, $\dot{z} = W\dot{x} + \dot{e}$

$$= W(Ax + Bv) + F(z - Wx)$$

$$= Fz + WBv + (WA - FW)x$$

We need

$$WA - FW = GC$$

for some G, and also $\begin{bmatrix} C \\ W \end{bmatrix}$ invertible

W is p-n x n matrix

Note that our estimate of x is

$$\hat{x} = M^{-1}\begin{bmatrix} y \\ z \end{bmatrix} = M^{-1}\begin{bmatrix} Cx \\ Wx+e \end{bmatrix} = M^{-1}\begin{bmatrix} C \\ W \end{bmatrix} x + M^{-1}\begin{bmatrix} 0 \\ e \end{bmatrix}$$

$$= x + Ke \quad , \qquad M^{-1} = [N \;\; K]$$

(partition of M^{-1})

Now control is

$$u = D\hat{x} = Dx + DKe$$

so $\dot{x} = Ax + Bu = (A+BD)x + BDKe$

and

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A+BD & BDK \\ & F \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix}$$

order of F $n-p$

Note that $e(t_0) = z(t_0) - Wx(t_0)$

If you pick F with very fast decay then W depends on F in such a way $e(t_0)$ becomes la

It was proved by Bongiorno & Youla that you cannot cause increase in cost due to e to $\to 0$ by choosing t except when F has dimension n.

Solution for observer in case of single output

<u>Example</u> A solution for single output system:

Suppose $y = x_n$, $C = [0\,0\,0 \quad 1]$ row vector

observer satisfies

$$\dot{z} = Fz + WBu + gy$$, where g is a vector, z is $n-1$ vect

The equation for W becomes

$(W_{11}$ $n-1 \times n-1)$ $$[W_{11} \,\vdots\, W_{12}]\begin{bmatrix} A_{11} & a_{12} \\ a_{21}^T & a_{22} \end{bmatrix} - F[W_{11} \;\; W_{12}] = g[0\,0\,0 \quad 1]$$

or

$$W_{11}A_{11} + W_{12}a_{21}^T - FW_{11} = 0$$

$$W_{11}a_{12} + W_{12}a_{22} - FW_{12} = g$$

If we set $W_{11} = I_{n-1}$, then

$$F^T = A_{11}^T + a_{21}W_{12}^T$$

This can be made stable if A_{11}^T is controllable through vector a_{21}

By observability system

$$\begin{bmatrix} \dot{\bar{x}} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} A_{11}^T & a_{21} \\ a_{12}^T & a_{nn} \end{bmatrix}\begin{bmatrix} \bar{x} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

is controllable. But

$$\dot{\bar{x}} = A_{11}^T \bar{x} + a_{21}x_n .$$

Observers of reduced dimension

If we had a desired control $v = Dx$ instead of trying to estimate x as $\hat{x}$ and setting $v = D\hat{x}$ we can try to estimate only the linear combination Dx

Suppose that

$$v = Dx + e$$

and let error e satisfy

$$e = Kv$$

where ~~$\dot{v} =$~~ v decays

$$\dot{v} = Fv$$

Now let model output z satisfy

$$z = Wx + v$$

Then

$$\begin{aligned} \dot{z} &= W\dot{x} + \dot{v} \\ &= WAx + WBu + F(z - Wx) \\ &= Fz + (WA - FW)x + WBu \end{aligned}$$

Also

$$\begin{aligned} v &= Dx + Kv \\ &= (D - KW)x + Kz \end{aligned}$$

So suppose we can find W, G, A, K such that

$$WA - FW = GC \qquad (A)$$

$$D - KW = HC \qquad (B)$$

Then we have dynamic control

$$u = Hy + Kz$$

$$\dot{z} = Hz + Gy + WBu$$

and only output $y = Cx$ is used.

We want to know smallest dimension r of z such th (A) and (B) can be simultaneously satisfied.

Consider single input system where $v = d^T x$ is a scalar. Then we need

$$WA - FW = GC \qquad (A)$$

$$d^T - k^T W = k^T C \qquad (B) \quad \text{just an eq. in row vectors}$$

Can this be satisfied with model of order 1? ①

Then z is a scalar

$$v = k^T y + kz$$

$$\dot{z} = fz + g^T y + w^T bu$$

just a 1st order eq.

where we need

$$w^T(A - fI) = g^T C \quad (A)$$

$$d^T - kw^T = k^T C \quad (B) \quad \text{with } f, k \text{ are scalars} \Big\} \text{ constraint equations}$$

(A) gives

$$w = (A^T - sI)^{-1} C^T g$$

Second gives

$$C^T h + k(A^T - sI)^{-1} C^T g = d$$

Writing $\begin{bmatrix} g \\ h \end{bmatrix}$ as composite vector of dimension $2p$ where p dimension of y, this is

$$\left[k C^T \,\vdots\, (A - sI)^T C^T \right] \begin{bmatrix} g \\ h \end{bmatrix} = (A - sI)^T d$$

We need rank $[C^T, A^T C^T] = n$

This may be satisfied if $p \geq \frac{n}{2}$

That is to say if no. of outputs is at least 1/2 order of system, an estimator of order 1 may be sufficient

$$v_1 = d_1^T x$$
$$u_2 = d_2^T x$$

Solve for $WA - FW = GC$

$d^T - k^T W = k^T C$ where F is now a matrix

Let $\det(\lambda I - F) = \lambda^r + f_1 \lambda^{r-1} + \dots + f_r$

Thus $F^r + f_1 F^{r-1} + \dots + f_r I = 0$

Let $C_0 = 0$

$$C_1 = GC + FC_0 = WA - FW$$
$$C_2 = GCA + FC_1 = WA^2 + F^2 W$$
$$C_3 = GCA^2 + FC_2 = WA^3 \underset{\pm}{-} F^3 W$$
$$\dots\dots$$

$\left\{ WA^2 - \cancel{FWA} + \cancel{FWA} - F^2 W \right.$

(may be all. + or −)

Explicitly

Multiplying by characteristic coefficients f_r and using Cayley-Hamilton Thm.

$$W(A^r + f_1 A^{r-1} + \dots + f_r I)$$
$$= C_r + f_1 C_{r-1} + \dots + f_{r-1} C_1$$

$$W(A^r + f_1 A^{r-1} \dots\dots)$$
$$= GCA^{r-1} + (F + f_1 I)GCA^{r-2} \dots + (F^{r-1} + f_1 F^{r-2} + \dots + f_{r-1} I)$$

Multiplying on left by k^T, W is eliminated giving

$$k^T C(A^r + f_1 A^{r-1} \dots)$$
$$+ \rho_1^T GCA^{r-1} + \rho_2^T GCA^{r-2} + \dots + \rho_r^T GC$$

where ρ_i is ith column of

$$S = \left[k \,\vdots\, (F + f_1 I)^T k \,\vdots\, \dots \,\middle|\, (F^{r-1} + f_1 F^{r-2} \dots f_r I)^T k \right]$$

This has to be solved for h^T and r rows of G.
given F, k
A sufficient condition for a solution to exist is $r \geq \alpha$.
Where α is observability index, That is least integer such that

$$\text{rank}\left[C^T \,|\, A^T C^T \,|\, \cdots\cdots A^{T\alpha-1} C^T\right] = n$$

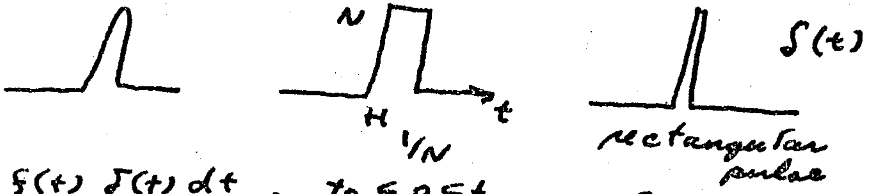
more details in: Sidney Int. J. of Control
vol 13 1971 p1041

$y = Cx + v$
v = noise
Minimize $E \ \Sigma (x_i - \hat{x}_i)^2$
solution is n^{th} order system.

8/16/73

Delta func.

limit of a spike



unit area

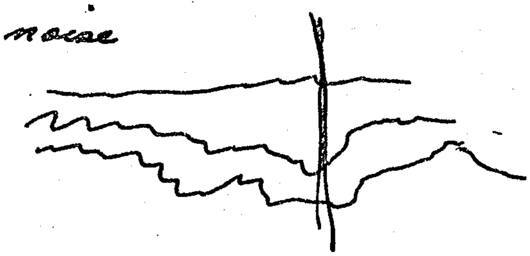
Def: $\int_{t_0}^{t_1} f(t)\,\delta(t)\,dt$, $t_0 < 0 < t_1$
$= f(0)$

(Definition of delta func)

$$\int_{t_0}^{t_2} f(t)\,\delta(t_1-\tau)\,dt = f(t_1) \quad , \quad t_0 < t_1 < t_2$$

Noise input

$\dot{x} = Ax + Bv$, v = noise
Stochastic process



$$E(f(x)) = \int f\, p(x)\,dx$$

$$\int p(x)\,dx = 1$$

$dx = \quad \sigma\, dw$

Covariant Matrix:
$E\{(x-\bar{x})(x-\bar{x})^T\}$

$$\begin{bmatrix} (x_1-\bar{x}_1)^2 & (x_1-\bar{x}_1)(x_2-\bar{x}_2) & \cdots \\ (x_2-\bar{x}_2)(x_2-\bar{x}_2) & \cdots & \\ \cdots & & \end{bmatrix}$$

as usual except max. when $t_1 = t_2$

Also we define

$X(t_1, t_2) = E(x(t_1)\,x^T(t_2))$

For white noise we take

$E(v) = 0$
$E(v(t_1)\,v^T(t_2)) = \boxed{Q(t_1)\,\delta(t_1 - t_2)}$

(white noise is some disturbance process)

$\dot{x} = Ax + Bv$,
v noise , $E(v) = 0$, $E(v(t_1)\,v^T(t_2))$
$= Q(t_1)\,\delta(t_1 - t_2)$

Stochastic diff eq.'s:
$dx = Ax\,dt + \sigma\,\underline{dw}$

Books on Stochastic Processes
Doob
Ito
Karlin

To get
$E(x) = \bar{x}$ take expectation
$\dot{\bar{x}} = A\bar{x} + 0$

for mean value just integrate original equation without the noise

We also want to know the variance

$$X(t_1, t_2) = E\{x(t_1) x^T(t_2)\}$$

Solution for x is

$$x(t) = \phi(t, t_0)\, x(t_0) + \int_{t_0}^{t} \phi(t, T)\, G(T)\, v(T)\, dT$$

$$X(t_1, t_2) = E\{\phi(t_1, t_0)\, x(t_0)\, x(t_0)^T \phi^T(t_2, t_0)\}$$

$$+ E\{\int_{t_0}^{t_1}\int_{t_0}^{t_2} \phi(t_1, T_1)\, G(T_1)\, v(T_1)\, v^T(T_2)\, G^T(T_2)\, \phi^T(t_2,$$

$$+ \text{ zero from cross terms} \qquad dT_1 d$$

Second term is

$$\int_{t_0}^{t_1}\int_{t_0}^{t_2} \phi(t_1, T_1)\, G(T_1)\, Q(T_1)\, \delta(T_1 - T_2)\, G^T(T_2)\ \phi^T(t_2, T_2)\, dT_1\, d$$

$$= \int_{t_0}^{t_2} \phi(t_1, T)\, G(T)\, Q(T)\, G^T(T)\, \phi^T(t_2, T)\, dT$$

Let

$$X(t) = X(t, t)$$

Then covariance of the terms with themselves

$$X(t) = \phi(t, t_0)\, X(t_0)\, \phi^T(t - t_0) + \int_{t_0}^{t} \phi(t, T)\, G(T)\, Q(T)\, G^T(T)\, \phi^T(t, T)\, dT$$

Since $\frac{d}{dt}\phi(t, T) = A\,\phi(t, T)$, $\phi(t, T) = I$

we get

$$\underline{\dot{X} = AX + XA^T + GQG^T}$$

$\longleftarrow$ (equation which is used to def. essentially ~~stochastic~~ stochistical process)

<u>Optimization in presence of noise</u>

Given

$$\dot{x} = Ax + Bu + Gv, \quad E(v(t_1) v(t_2)^T) = Q(t_1)\, \delta(t_1 - t_2)$$

$$J = E\int (x^T Q x + u^T R u)\, dt$$

try to find best feedback control $u = Dx$

Then

$$\dot{x} = Fx + Gv, \quad F = A + BD$$

$$J = E\int (x^T S x)\, dT, \quad S = Q + D^T R D$$

$$= E\int tr(S x x^T)\, dt = \int tr(SX)\, dt$$

where $X = E(x x^T)$

Also

$$\dot{X} = FX + XF^T + \underline{GQG^T}$$

Then under a variation δD we have

$$\delta J = \int \{ tr(S \delta X) + 2 tr(\delta D^T R D X) \} dt$$

where,

$$\delta \dot{X} = F \delta X + \delta X F^T + B \delta D X + X \delta D^T B^T$$

Variational equations are same as without noise

For more general random ~~distribution~~ disturbances we can take

$$\dot{x} = Ax + Bu + Cy$$

$$\dot{y} = Fy + Gv \,, \qquad v = \text{white noise}$$

y = random disturbance.

Optimum filters (for processing the measures)

Original theory - Norbert Wiener
for time invariant processes
using Fourier Transforms

Solutions for time varying case Kalman + Bucy
Trans. ASME Series D Journal of basic Engineering
Vol 83 1961 pp 95 - 108

$$\dot{x} = Ax + Bu \,,$$

measure output $y = Cx$

we can estimate x as $\hat{x}$ where

$$\dot{\hat{x}} = A\hat{x} + Bu + \boxed{K}(y - C\hat{x}) \qquad , K \text{ matrix}$$

such that $A - KC$ is stable.

if you want to measure x
W n-m x n matrix s.t.

$$\begin{matrix} Wx \\ Cx \end{matrix} \qquad \begin{bmatrix} W \\ C \end{bmatrix}^{-1}$$

(Read Bryson & Ho
discrete case

Now, some sort of justification

Optimum statistical ~~estimation~~ estimator (Kalman Bucy filter)

Let $\dot{x} = Ax + v$
with measurement
$y = Cx + w$
where v, w are white noise

$$E(v(t_1) v^T(t_2)) - Q(t_1) \delta(t_1 - t_2)$$

$$E(v) = 0$$

$$E(w(t_1) w(t_2)) = R(t_1) \delta(t_1 - t_2)$$

Try to find best linear estimate of $G^T x$ that is

$$\beta = \int_{t_0}^{t_1} s^T(\tau) y(\tau)\, d\tau$$

to minimize

$$E(b^T x - \beta)^2$$

(must assume all the statistics are guassian,

Let

$$\dot r = -A^T r + C^T s \quad , \quad r(t_1) = b$$

Then

$$\begin{aligned} d/dt\,(r^T x) &= \dot r^T x + r^T \dot x \\ &= -\cancel{r^T A x} + s^T C x + \cancel{r^T A x} + r^T v \\ &= s^T y - s^T w + r^T v \end{aligned}$$

Integrating

$$b^T x(t_1) - \int s^T(\tau) y(\tau)\, d\tau = r^T(t_0) - \int_{t_0}^{t_1} s^T(\tau) w(\tau)\, d\tau + \int_{t_0}^{t_1} r^T(\tau) v(\tau)\, d\tau$$

Since $x(t_0)$, v, w are independent

$$E(b^T x(t_1) - \beta)^2 = E(r^T(t_0) x(t_0))^2 + E\left(\int_{t_0}^{t_1} s^T w\, d\tau\right)^2 + E\left(\int_{t_0}^{t_1} r^T v\, d\tau\right.$$

Now

$$\begin{aligned} E\left(\int_{t_0}^{t_1} s^T w\, d\tau\right)^2 &= E\left\{\int_{t_0}^{t_1}\int_{t_0}^{t_1} s^T(\tau_1)\, w(\tau_1)\, w^T(\tau_2)\, s(\tau_2)\, d\tau_1\, d\tau_2\right\} \\ &= \int_{t_0}^{t_1}\int_{t_0}^{t_1} s^T(\tau_1)\, E(w(\tau_1)\, w^T(\tau_2))\, s(\tau_2)\, d\tau_1\, d\tau_2 \\ &= \iint_{t_0\,t_0}^{t_1\,t_1} s^T(\tau_1) R(\tau_1)\, \delta(\tau_1 - \tau_2)\, s(\tau_2)\, d\tau_1\, d\tau_2 \\ &= \int_{t_0}^{t_1} s^T(\tau) R(\tau)\, d\tau \end{aligned}$$

Also

$$E\left(\int r^T v\, d\tau\right)^2 = \int_{t_0}^{t_1} r^T Q r\, d\tau$$

$$E\left(r(t_0)\, x(t_0)\, x^T(t_0)\, r(t_0)\right) = r^T(t_0)\, P_0\, r(t_0)$$

where $P_0 = E(x(t_0)\, x^T(t_0))$

Thus

$$J = r^T(t_0) P_0\, r(t_0) + \int_{t_0}^{t_1} (r^T Q r + s^T R s)\, d\tau$$

where

$$\dot r = -A^T r + C^T s \quad , \quad r(t_1) = b$$

This is standard problem in backwards time (reversed)

Solution is of course the usual solution:

$$s(t) = R^{-1} C P\, r(t)$$

where $\dot P = PA^T + AP - PC^T R^{-1} C P + Q \; , \; P(t_0) = P_0$

Then

$$\dot{r} = -F^T r \quad , r(t_1) = b$$

where

$$F = A - PC^T R^{-1} C$$

then

$$r(t) = \phi^T(t_1, t) b \quad , \text{ where } \frac{d}{dt}\phi(t,\tau) = F(t)\phi(\tau,\tau)$$

Also

$$\rho = R^{-1}(t)\, c(t)\, P(t)\, \phi^T(t,t)\, b$$

taking $b = e_i$: we find [illegible] is The i^{th} entry of $\hat{x}$ wher

$$\hat{x}(t) = \int_{t_0}^{t} \phi(t,\tau) P(\tau) C^T(\tau) R^{-1}(\tau)\, y(\tau)\, d\tau$$

Then

$$\dot{\hat{x}} = PC^T R^{-1} y + F\hat{x} \quad , F = A - PC^T R^{-1} C$$

$$\dot{\hat{x}} = A\hat{x} + \underline{K}(C\hat{x} - y)$$

where $K = PC^T R^{-1}$
$\uparrow$ optimum gain

$$\dot{x} = Ax + Bu + v$$

$$\dot{r} = -A^T r + C^T \rho$$

$$E(b^T x - \int \rho^T y\, d\tau)^2 = r^T(t_0) P_0\, r(t_0) + \int_{t_0}^{T} (r^T Q r + \rho^T R \rho)\, dt$$

where if $\rho^* = R^{-1} C P r$
we have

$$r^T Q r + \rho^T R \rho = r^T \dot{P} r + r^T P A^T r + r^T A P r - \rho^{*T} R \rho^* + \rho^T R \rho$$

$$= \frac{d}{dt}(r^T R r) + (\rho - \rho^*)^T R (\rho - \rho^*)$$

<u>Optimal value of variance</u> is

$$E(b^T x - \int \rho^T y\, dt)^2 = b^T P(t_1) b$$

Since This is true for any b we deduce

$$P(t) = E\{(x - \hat{x})(x - \hat{x})^T\}$$

If $t_0 \to -\infty$, A, C are observable, A^T, C^T are controllable Then can establish

$$\lim_{t_0 \to -\infty} P \text{ exists}$$

So for time invariant case we obtain P as positive definite solution of

$$PA^T + AP - PC^T R^{-1} CP + Q = 0$$

(more problems from Bryson)

Problem 3:

Find the path of minimum time for a tunnel through the earth from A to B where particle falls freely under gravity with no friction.

Gravity varies linearly with distance from earth

Problem 4:

(Bryson p82)

Thrust direction programming

$\dot{u} = a \cos \beta$

$\dot{v} = a \sin \beta$ $\qquad \beta$ = control = thrust angle ↗

$\dot{x} = u$

$\dot{y} = v$

For time optimal problem show that

$$\tan \beta = \frac{-c_2 t + c_4}{-c_1 t + c_3}$$

Problem 5:

Find switching surface to minimize time to reach origin for

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

- Hamiltonian will be constant.
- with 2 real roots

Problem 6:

Given a pair of coupled oscillators

$$\dot{x} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

Let measured output vector be

$$y = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

(a) is the system controllable?
(b) is it observable from y
(c) If (b) is satisfied construct an observer to estimate state x.
(d) If desired control is $u = 4x_2 - 4x_3 - 4x_4$
construct an observer of order one to reproduce u

see last wks lecture notes

Optimal linear estimator is independent:

Given

$$\dot{x} = Ax + v \qquad E\{v(t_1)v^T(t_2)\} = Q(t_1)\delta(t_1-t_2)$$

$$y = Cx + w \qquad E\{w(t_1)w^T(t_2)\} = R(t_1)\delta(t_1-t_2)$$

Consider estimate

$$\hat{x} = \int_{t_0}^{t} S(t,\tau)y(\tau)\,d\tau$$

to minimize

$$E\{(x-\hat{x})^T Z (x-\hat{x})\} \quad , \quad Z \geq 0$$

Let

$$\dot{M} = -A^T M + C^T S \quad , \quad M(t_1) = I$$

$$\frac{d}{dt}(M^T x) = -\cancel{M^TA}x + S^TCx + \cancel{M^TA}x + M^Tv$$

$$= S^Ty - S^Tw + M^Tv$$

Then,

$$x - \hat{x} = M^T(t_0)x(t_0) - \int_{t_0}^{t_1} S^T w\,d\tau + \int M^T v\,d\tau$$

$$E\{(x-\hat{x})^T Z (x-\hat{x})\} = E(x^T M Z M^T x) + E\iint w^T S Z S^T w\,d\tau_1 d\tau_2$$

$$+ E\iint v(\tau_1)\,M Z M^T v(\tau_2)\,d\tau_1 d\tau_2$$

$$E\left\{\int_{t_0}^{t_1}\int_{t_0}^{t_1} w^T S Z S\; w\,d\tau_1 d\tau_2\right\} = E\,tr\left\{\iint S Z S^T w w^T d\tau_1 d\tau_2\right\}$$

(since $x^T Z x = tr(\underbrace{Z x x^T}_{\text{matrix}})$)

$$= tr\int_{t_0}^{t_1} Z S^T R S\,d\tau$$

from δ function property of ww^T

Thus we must choose S to minimize

$$J = tr\left\{Z\left[M^T(t_0)P_0M(t_0) + \int_{t_0}^{t_1}(M^TQM + S^TRS)\,d\tau\right]\right\}$$

where

$$\dot{M} = -A^T M + C^T S \quad , \quad M(t_0) = I \quad ; \quad P_0 = E(x(t_0)x^T(t_0))$$

Let

$$S^* = R^{-1}CPM$$

where

$$\dot{P} = Q + AP + PA^T - PC^TR^{-1}CP \quad , \quad P(t_0) = P_0$$

Then

$$M^TQM + S^TRS = M^T\dot{P}M - M^TAPM - M^TPA^TM + S^{*T}RS^* + S^TR$$

$$= \dot{M}^TPM + M^T\dot{P}M + M^TP\dot{M}$$

$$- S^TCPM - M^TPC^TS + S^{*T}RS^* + S^TRS$$

$$= \frac{d}{dt}(M^TPM) + (S-S^*)^TR(S-S^*)$$

$$M^T P_0 M + \int (M^T Q M + S^T R S)\, d\tau = P(t_1) + \int (S - S^*) R (S - S^+)\, dt$$

$$= P(t_1) + F$$

where $F \geq 0$

so $F = GG^T$ for some G

$$J = \operatorname{tr}(Z P(t_1)) + \operatorname{tr}(Z G G^T)$$

$$= \operatorname{tr}(Z P(t_1)) + \operatorname{tr}(G^T Z G)$$

(∴ Optimal estimator is ind. of ~~the~~ any particular waiting matu put into the estimate.

(If we were working with gaussian random vectors Then This solution would be the def. of the random process)

For Gaussian random vectors

$$p(x) = \frac{1}{(2\pi)^{n/2} |P|^{1/2}} \exp\left\{ -\tfrac{1}{2} (x - \bar{x})^T P^{-1} (x - \bar{x}) \right\}$$

$$E(x) = \bar{x}$$

$$E\{(x - \bar{x})(x - \bar{x})^T\} = P$$

If $y = Ax + b$, then y is also Gaussian with covariance

$$P_y = A P_x A^T$$

Separation Theorem:

Let

$$\dot{x} = Ax + Bu + v \qquad v = \text{white noise}$$

$$y = Cx + w \qquad w = \text{white noise}$$

Find u to minimize

$$J = E\left\{ x^T(t_f) F x(t_f) + \int_{t_0}^{t_f} (x^T Q x + u^T R u)\, dt \right\}$$

Optimal control is

$$u^* = D\hat{x}$$

Where

$$u_0 = Dx$$

is optimal deterministic control to minimize

$$J = x^T F x + \int (x^T Q x + u^T R u)\, dt$$

in absence of noise

case when plant is not what we thought it to be

$\dot{x} = Ax + Bu$ original system

Suppose A and B are replaced by $A + \Delta A$, $B + \Delta B$

Suppose $u = u(t)$ is a fixed time history

$$\dot{x} + \Delta\dot{x} = (A + \Delta A)(x + \Delta x) + (B + \Delta B)u$$

$$\Delta\dot{x} = (A + \Delta A)\Delta x + \Delta A x + \Delta B u$$

~~$\dot{x}$ +~~ suppose plant variations are small

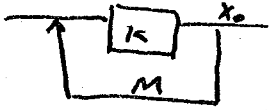
Let $\Delta A = \epsilon\,\delta A$, $\Delta B = \epsilon\,\delta B$,

and $\delta x = \lim_{\epsilon \to 0} \frac{\Delta x}{\epsilon}$

Open loop trajectory deviation satisfies

$$\delta\dot{x}_o = A\,\delta x_o + \delta A\,\underline{\underline{x}} + \delta B\,\underline{\underline{u}} \qquad \delta x_o(t_o) = 0$$

idea of feedback was to make things less sensitive



$x_o = K x_i$ open loop

$x_o = K(x_i - M x_o)$ closed loop

$x_o = \frac{K}{1 + MK} x_i$

If $MK >> 1$ $x_o \approx \frac{1}{M} x_i$

Consider also clod loop control

$u = Dx$

which is identical $u = u(t)$ when $\Delta A, \Delta B = 0$

now when plant changes so does the control — (before the control did not change)

Now

$$\dot{x} = (A + BD)x \qquad \text{on nomial path}$$

$$\Delta\dot{x} = (A + \Delta A + (B + \Delta B)D)\Delta x + \Delta A x + \Delta B D x$$

or for small variations

$$\delta\dot{x}_c = (A + BD)\,\delta x_c + \delta A x + \delta B D \underline{\underline{x}}$$

$$= (A + BD)\,\delta x_i + \delta A x + \delta B u$$

where u is control on nomial path.

Let $v = \delta x_o - \delta x_c$

then

$$\dot{v} = Av - BD\,\delta x_c \quad , v(t_o) = 0$$

Perkins & Cruz (IEEE Trans. Automatic Control Vol 9 (1964) pp 216-223) proposed as criterion for sensitivity reduction

$$\int_{t_o}^{t_1} \delta x_o^T Z\,\delta x_o\,dt \geq \int \delta x_c^T Z\,\delta x_c\,dt \qquad (A)$$

That is

$$\int_{t_o}^{t_1} (\delta x_c + v)^T Z (\delta x_c + v) \geq \int_{t_1}^{t_1} \delta x_c^T Z\,\delta x_c\,dt$$

where

$$\dot{v} = Av - BD\,\delta x_c \ , \quad v(t_0) = 0$$

In general δx_c is not necessarily arbitrary.
If you can solve

$$\delta A x = \delta \dot{x}_c - (A + BD)\,\delta x_c - \delta B v$$

for δA, and δA is arbitrary then δx_c is an arbitrary smooth function.

If $A = A(\mu)$, $\delta A = \dfrac{\partial A}{\partial \mu}\,\delta\mu$.

If we assume sufficiently general variations so that δx_c is arbitrary, we can state (A) as

$$\int_{t_0}^{t_1} (v+w)^T Z\,(v+w)\,dt \ \geq \int_{t_0}^{t_1} w^T Z w\,dt \qquad (B)$$

where v is solution of

$$\dot{v} = Av - BDw \ , \quad v(t_0) = 0 \qquad \text{for arbitrary } w.$$

alternative expression equivalent to above involving frequency domain.

For time invariant case we can take Laplace transforms

$$(sI - A)\,\bar{v}(s) = -BD\,\delta\bar{x}_i\,(s)$$

(bars indicate that these are the transforms)

$$\delta x_o = \delta x_c + v$$

$$\delta \bar{x}_o(s) = (I - (sI - A)^{-1}BD)\,\delta x_c(s)$$

Parseval's Theorem says

$$\int_0^\infty x^T Z x\,dt = \int_{-\infty}^{\infty} x^T(-j\omega)\,Z\,x(jW)\,d\omega$$

This criterion becomes

$$\int_{-\infty}^{\infty} \delta x_c^*\,\{(I - L)^* Z\,(I - L) - Z\}\,\delta x_c\,d\omega$$

where

$$L(j\omega) = (j\omega I - A)^{-1}BD = \text{transfer function}$$

This is true for arbitrary δx_c if

$$\underline{(I - L)^* Z\,(I - L) - Z \geq 0} \qquad \text{for all } \omega \qquad (B^*)$$

Sensitivity inequality in freq. domain

(equ. (B) can't be satisfied for arb. Z)

Z cannot be arbitrary non-negative

To prove this assume

(1) A, B is controllable from an output $w = Dx$

(2) D has linearly independent rows.

Then

$$\Gamma = \int_{t_0}^{t} \Phi^T(t,\tau) B D D^T B^T \Phi(t,\tau) > 0$$

Suppose

$$w = \xi + \overset{\text{scalar}}{\rho}\,\eta \qquad (\xi, \eta \text{ vectors})$$

where ρ is a scalar, ξ and η are vectors and η is any vector such that

$$D\eta = 0$$

We have

$$v(t) = \int_{t_0}^{t} \Phi(t,\tau) B D\, w(\tau)\, d\tau \qquad \text{since } v(t_0) = 0$$

~~Define~~ Also (B) requires

$$J = \int_{t_0}^{\tau} Z v^T Z w\, d\tau + \int_{t_0}^{t} v^T Z v\, d\tau \geq 0$$

Let

$$\alpha(t) = \xi^T Z \int_{t_0}^{t} \Phi(t,\tau) B D \xi\, d\tau$$

$$B(t) = \eta^T Z \int_{t_0}^{t} \Phi(t,\tau) B D \xi\, d\tau$$

$$\gamma(t) = \left\{ \int \Phi(t,\tau) B D \xi\, dt \right\}^T Z \left\{ \int \Phi(t,\tau) B D \xi\, d\tau \right\}$$

Then

$$J = 2\int_{t_0}^{t_1} \alpha(t)\, dt + 2\int_{t_0}^{t_1} \rho(t) \beta(t)\, dt + \int_{t_0}^{t_1} \gamma(t)\, dt$$

For fixed ξ, η let us choose

$$\rho(t) = -\epsilon\beta(t)$$

For large "enough" ϵ $\Longrightarrow$ $J < 0$ unless

$$\beta(t) \equiv 0$$

Now set $\xi = D^T B^T \Phi(t,\tau)\lambda$, for some vector λ

Then

$$\beta(t) = \eta^T(t) Z \Gamma(t) \lambda$$

Since Γ is non singular, λ is arbitrary $\beta \equiv 0$ requires

$$Z\eta = 0$$

Since η is any vector in null space of D, rows of Z

must be in subspace spanned by rows of D, or

$$Z = LD \quad \text{for some } L$$

But $Z^T = Z$ so we have $Z = D^T M D$ for some

That means rank $Z \leq$ rank D = no of control inputs

(Z must have rank one)

$\underline{Dx}$

We now have that inequality that can be satisfied must be of form

$$\int_{t_0}^{t_1} (v+w)^T D^T M D (v+w)\,dt \geq \int_{t_0}^{t_1} w^T D^T M D w$$

where $\dot{v} = Av - BDw$, $v(t_0) = 0$.

On setting:

$$-Dw = u,$$

what this says is that

$$\int_{t_0}^{t_1} (u - Dv)^T M (u - Dv)\,dt \geq \int_{t_0}^{t_1} u^T M u\,dt \qquad (C)$$

Where v is solution for arbitrary u of $\dot{v} = Av + Bu$, $v(t_0) = 0$

now proposed to prove (C) can be satisfied.

optimal control systems with non-neg. waiting matrix do satisfy this

Optimal ~~control~~ linear controls do satisfy (C)

Pf: Suppose

$$\dot{x} = Ax + Bu$$

and u is chosen to minimize

$$J(t_0) = x^T(t_f) F x(t_f) + \int_{t_0}^{t_f} (x^T Q x + u^T R u)\,dt$$

where $Q \geq 0$, $R > 0$, $F \geq 0$ Then

$$u = Dx$$

where $D = -R^{-1} B^T P$

and $\dot{P} = Q + A^T P + PA - D^T R D$, $P(t_f) = F$

Then

$$x^T Q x + u^T R u = -\frac{d}{dt}(x^T P x) + (u - Dx)^T R (u - Dx)$$

Integrating from t to t_f gives

$$J(t) = x^T(t) P(t) x(t) + \int_{t}^{t_f} (u - Dx)^T R (u - Dx)\,dt$$

Then $x^T(t) P(t) x(t) = \min J(t) \geq 0$

so $P \geq 0$

Integrate from t_0 to t and set $x(t_0) = 0$ then

$$\int_{t_0}^{t} (u - Dx)^T R (u - Dx)\, dt - \int_{t_0}^{t} U^T R U \, dt = x^T(t) P(t) x(t) + \int_{t_0}^{t} x^T$$

$$\geq 0$$

Sensitivity for dynamic control

Let

$$\dot{x} = Ax + Bu$$

and

$$u = Hx + Kz$$ most general u

where $\dot{z} = Fz + Gx + Eu$

Will this do better than a feedback control

For comparison with a feedback control $u_1 = Dx$ we need to ensure $u = u_1$ when A and B are not perturbed.

Let $z = Wx + v$

where we want

$$\dot{v} = Fv, \quad F \text{ stable}$$

Then $v(t) = 0$ if $v(t_0) = 0$

~~Let~~

$$\dot{v} = \dot{z} - W\dot{x} = F(v + Wx) + Gx + Eu - WAx - WBu$$

Now let

estimators
$$\boxed{\begin{aligned} G &= WA - FW \\ E &= WB \end{aligned}}$$

Then $\dot{v} = Fv$ as described.

Also if $\boxed{H = D - KW}$ Then $U = Dx + Kv$

Now equations on ~~nomind~~ nominal path are

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} A + BD & BK \\ & F \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \qquad \begin{aligned} x(t_0) &= x_0 \\ v(t_0) &= 0 \end{aligned}$$

Under a parameter change $\delta A, \delta B$ F, G, H, K, W are fix

$$\left.\begin{aligned} \delta\dot{x} &= (A + BD)\delta x + BK\delta v + \delta A x + \delta B u \\ \delta\dot{v} &= F\delta v - W(\delta A x + \delta B u) \end{aligned}\right\} \text{deviation equations}$$

We can suppose $\delta A x + \delta B u = Mx$ for some M

Then question is can we make δv such that

$BK\delta v$ cancels (or opposes) forcing term Mx in 1st equation

8/23/'73

variat end time
end up with transversality condition

$$\delta J = \sum \left(\frac{\partial M}{\partial x_i} \cancel{- \lambda_i} \right) \delta x_i \Big|_{t_f} + \sum \left(\frac{\partial M}{\partial x_i} \cancel{- \lambda_i} \right) f_i \, \delta t_f$$

$$+ \left(H + \frac{\partial M}{\partial t} \right) \delta t_f + \int \sum \frac{\partial H}{\partial u_k} \delta u_k \, dt$$

M based on contraints you are trying to solve
If $J = K(x(t_f)) + \int \cdots$, $g_m(x(t_f)) = 0$, $m = 1, \ldots, p$
$M = K + \sum \nu_m g_m$

H vanishes at t_f if M is not a function of time.
Also $H = \text{const.}$ if t does not enter explictly

Time optimal

$$J = \int 1 \, dt$$

$$H = \sum \lambda_i f_i + 1 = 0$$

called transversality cond.

$$\int (U - Dx)^T M (u - Dx) \, dt \geq \int u^T R u \, dt$$

where x is solution of

$$\dot{x} = Ax + Bu \quad , \quad x(t_0) = 0$$

for arbitrary u.
Note that $u = \tfrac{1}{2} Dx$ leads to $x \equiv 0$ since $x(t_0$

Sensitivity with dynamic controls

Given

$$\dot{x} = Ax + Bu$$

Let

$$u = Hx + Kz$$

where

$$\dot{z} = Fz + Gx + Bv$$

~~Including~~ Includes feedba control by setting $H = D$, $K = 0$

$U = Dx$
$U(t)_{opt}$ can be generated as $u = Dx$

To make $u(t) = Dx$ when plant is unperturbed we make $z \Rightarrow Wx$, i.e. $z = Wx + v$
where $\dot{v} = Fv$

Leads to constraints

$$G = WA - FW,$$
$$E = WB$$

Then

$$\dot v = \dot z - W\dot x = F(v + Wx) + Gx + Eu + WBu = Fv$$

Then

$$U = Dx + Kv = Dx \quad \text{if } v(t_0) = 0.$$

$$\begin{bmatrix} \dot x \\ \dot v \end{bmatrix} = \begin{bmatrix} A+BD & BK \\ 0 & F \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

Then

$$\delta \dot x = (A+BD)\delta x + \delta A x + \delta B(Dx + Kv) + BK\delta v$$
$$= (A+BD)\delta x + \underline{BK}\,\delta v + \boxed{\delta A x + \delta B u}$$
$$\delta \dot v = \delta \dot z - W\delta \dot x$$
$$= F(\delta v + W\delta x) + G\delta x + E\delta u$$
$$- W(A\delta x + B\delta u + \boxed{\delta A x + \delta B u})$$
$$\text{where} \quad WA - FW = 0, \quad WB = E$$

forcing both equa

Let $\delta A + \delta B D = M$, since $U = Dx$ on nominal path we have

$$\delta \dot x = (A+BD)\delta x + BK\delta v + Mx$$
$$\delta \dot v = F\delta v - WMx$$

Under a sinusoidal input

$$\delta v = -(j\omega I - F)^{-1} WMx$$

approximately equals $\approx F^{-1}WMx$

if F has very large e. values

Then forcing term for δx is

$$(BKF^{-1}W + I)Mx$$

Unfortunately rank $B = m < n$ in general, so complete cancellation is impossible. Suppose however,

$$M = BL$$

This would be case for a single input system

$$\begin{bmatrix} \dot x_1 \\ \dot x_2 \\ \dot x_3 \\ \dot x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} U$$

Phase variable form
Companion form

only variation possible are in a_1, a_2, a_3, a_4

Now we want,

$$B(\cancel{B}KF^{-1}W + I)BL = 0$$

or $KF^{-1}WB = -I_m$

This can be satisfied with a model of order equal to the number of controls, so that K is square. If we choose K non-singular, desired solution for ~~this~~ ~~W~~ E is $E = WB = -FK^{-1}$

(This is not a complete solution to the problem

Let $\delta A + \delta BD = M$, since $v = Dx$ on nomial path we have

$$\delta\dot{x} = (A+BD)\delta x + BK\delta v + Mx$$

$$\delta\dot{v} = F\delta v - WMx$$

~~$\delta(t_0) =$~~ $\delta\dot{v}(t_0) = 0$

Under a sinusoidal input

$$\delta v = -(j\omega I - F)^{-1} WMx \approx F^{-1}WM$$

More precisely

$$\delta v = -\int_{t_0}^{t} e^{F(t-\tau)} WMx\,d\tau$$

Since F is nonsingular we can integrate by parts to get

$$\delta v(t) = F^{-1}WMx(t) - F^{-1}e^{Ft}WMx(0) \cancel{+ \int e^{F(t-\tau)} F^{-1}WM(\theta +} + \int e^{F(t-\tau)} F^{-1}WM(\theta + BD)x\,dt$$

Repeating process

$$\delta v(t) = Z\,x(t) - e^{Ft}Z\,x(t_0)$$

where

$$Z = F^{-1}WM + F^{-2}WM(A+BD) + F^{-3}WM(A+BD)^2 \cdots$$

We are taking first term

Good approximation if $|\lambda_{min}(F)| >> |\lambda_{max}(A+BD)|$

More general case

In general, if F is fast enough so that $\delta v \approx F^{-1}W$ we have

$$\delta\dot{x} = (A+BD)x + (BKF^{-1}W + I)Mx$$

Forcing term $(I - BV)Mx$ where $V = KF^{-1}W$

We want $I - BV$ as small as possible. A reasonable approach would be to minimize

$$(Bv_i - e_i)^T(Bv_i - e_i) = v_i^T B^T B v_i - 2 v_i^T B^T e_i + e_i^T e_i$$

where e_i is the i^{th} column of I, v_i is the i^{th} column of V. Then

$$v_i = (B^TB)^{-1}B^Te_i$$

or $V = \cancel{(B^TB)^{-1}B^T}(B^TB)^{-1}B^T$

So we try to satisfy $KF^{-1}W = -(B^TB)^{-1}B^T$
This can be done with model of order equal to the dimension of u.
Then K is square, so if we take K nonsingular

Then, $W = -FK^{-1}(B^TB)^{-1}B^T$

$$E = WB = -FK^{-1}$$

Then model is

$$\dot{z} = Fz + Gx + Ev = Fz + Gx - FK^{-1}(Hx + Kz)$$
$$= (G - FK^{-1}H)x$$

$\Rightarrow$ solution you asked for is a pure integrator

<u>Nonlinear systems with parameter variations</u>

Let

$$\dot{x} = f(x, u, \mu, t)$$ where μ is a variable vector Parameter

Then for small variations $\delta\mu$

$$\delta\dot{x} = f_x\,\delta x + f_u\,\delta u + f_\mu\,\delta\mu$$

Open loop control $u = u(t)$ is fixed

$$\delta\dot{x}_o = f_x\,\delta x_o + f_\mu\,\delta\mu$$ where f_μ is evaluated on original path

With feedback control $u = k(x)$

$$\delta u = k_x\,\delta x$$

so what you now get is closed loop deviation satisfies

$$\delta\dot{x}_c = (f_x + f_u k_x)\,\delta x + f_\mu\,\delta\mu \qquad \delta x_c(t_0) = 0$$

Thus $\delta x_o = \delta x_c + v$ where

$$\dot{v} = f_x v - f_u k_x\,\delta x_c$$

Then you can satisfy criterion

$$\int \delta x_o^T Z\,\delta x_o\,dt \geq \int \delta x_c^T Z\,\delta x_c\,dt$$

for arbitrary δx_c only if Z has form $Z = k_x^T M k_x$ for some M.

~~$\dot{x} = A(\mu)x + B(\mu)u$~~
~~$\delta\dot{x} = A\delta x + A_\mu x + B_\mu u$~~
~~where $\delta x =$ [illegible]~~

another way to go about it

Direct approach to parameter variations

Let

$$\dot{x}_1 = A_1 x_1 + B_1 u_1$$
$$\dot{x}_2 = A_2 x_2 + B_2 u_2$$
$$\dot{x}_3 = A_3 x_3 + B_3 u_3$$
$$\cdots$$

be equations for plant in different conditions.

Let

$$J_1 = \int x_1^T Q_1 x_1 \, dt + \int u_1^T R_1 u_1 \, dt \text{ etc.}$$

Minimize

$$J = \Sigma J_i$$

by a control

$$u_1 = D x_1$$
$$u_2 = D x_2$$
$$u_3 = D x_3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & A_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_1 & & \\ & B_2 & \\ & & B_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

subject to $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} D & & 0 \\ 0 & D & \\ & & D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

You get $G = \Sigma G_i$

where G_i are gradient for J_i. where $G_{ij} = \dfrac{\partial J}{\partial D_{ij}}$

(can use method of steepest descent as a procedure)

~~Designing~~ Design for given eigenvalues (poles)

single input system.

$$\dot{x} = Ax + bu$$
$$U = d^T x = d_1 x_1 + d_2 x_2 \dots d_n x_n$$

we have n parameters d_i so we can hope to fix n eigenvalues.

Can be solved for exactly for a controllable single input system.

Consider matrix (square)

$$W = [b, Ab, A^2 b, \dots, A^{n-1} b]$$

W is non-singular by controllability.

Then if we make a transformation $x = Wy$, $y = W^{-1}x$

$$W\dot{y} = AWy + bu$$
$$\dot{y} = W^{-1}AWy + W^{-1}bu$$

$$w^{-1}AW = w^{-1}[Ab, A^2b, \dots, A^nb]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & a_1 \\ 1 & 0 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 & a_3 \\ 0 & 0 & 1 & 0 & a_4 \end{bmatrix}$$

$$w^{-1} = \begin{bmatrix} \overline{} \\ \\ \end{bmatrix} \quad [Ab, A\overline{b^2}, A^3b$$

$$W^{-1}b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

~~$\dot{y} = \begin{bmatrix} 0 & 0 & 0 & 0 & a_1 \\ 1 & 0 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 & a_3 \\ 0 & 0 & 1 & 0 & a_4 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} v$~~

for 4th order case

$$\dot{y} = \begin{bmatrix} 0 & 0 & 0 & a_1 \\ 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & a_3 \\ 0 & 0 & 1 & a_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$\dot{y}_1 = a_1y_4 + u$$

$$\dot{y}_2 = y_1 + a_2y_4$$

$$\dot{y}_3 = y_2 + a_3y_4$$

$$\dot{y}_4 = y_3 + a_4y_4$$

$$\ddot{y}_4 = a_4\dot{y}_4 + \dot{y}_3 = a_4\dot{y}_4 + a_3y_4 + y_2$$

$$\dddot{y}_4 = a_4\ddot{y}_4 + a_3\dot{y}_4 + \dot{y}_2 = a_4\ddot{y}_4 + a_3\dot{y}_4 + a_2y_4 + y_1$$

$$y_4^{(4)} = a_4\dddot{y}_4 + a_3\ddot{y}_4 + a_2\dot{y}_4 + a_1y_4 + u$$

$$(D^4 + a_4D^3 \cdots)y = u$$

$$u = k_1y_4 + k_2\dot{y}_4 + k_3\ddot{y}_4 + k_4\dddot{y}_4$$

Then

$$(D^4 + (a_4+k_4)D^3 + (a_3+k_3)D^2 \cdots$$

For a given set λ_i evaluate coefficients you need c_i say and ~~so~~ ~~ki~~ b_ia $k_i = b_i - a_i$

You can then express v in terms of $y + \ddot{y} = —$.

We can treat $y_4, \dot{y}_4$ etc as new state variables

$z_4 = y_4$

$z_3 = \dot{y}_4$, $y_3 + a_4z_4 = z_3$

$z_2 = \ddot{y}_4$, $y_2 + a_4z_3 + a_3z_4 = z_2$

$z_1 = \dddot{y}_4$, $y_1 + a_4z_2 + a_3z_3 + a_2z_4 = z_1$

or $y = Qz$

where

$$Q = \begin{bmatrix} 1 & -a_4 & -a_3 & -a_2 \\ & 1 & -a_4 & -a_3 \\ & & 1 & -a_4 \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} a_4 & a_3 & a_2 & a_1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

Transformation $x = WQz$ gives phase variable form.

Problem 7:

For coupled oscillators

$$\dot{x} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

Calculate feedbacks to give

$\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = -1$

Problem 8: (Bonus Problem) time invariant

Does solution to problem 7 minimize $\int_0^\infty (x^T Q x + U^2)\,dt$ for some $Q \geq 0$?

Design for given eigenvalues (poles)

Suppose $U = d^T y$ where $y = Cx$

and C is given $m \times n$ matrix

$$\dot{x} = (A + bd^T C)x$$

Now we have m parameters only.

We may wish to position m eigenvalues

Let ~~$(\lambda I - A)x$~~ $(\lambda I - A)^{-1} = F(\lambda)/g(\lambda)$

where $g(\lambda)$ is characteristic polynomial

$F(\lambda)$ is adj. matrix (Gantmacher)

$F(\lambda) = I\lambda^{n-1} + F_1 \lambda^{n-2} + F_{n-1}$

To get F_k use Leverrier algorithm

$$A_1 = A,\quad a_1 = tr(A_1),\quad F_1 = A_1 + a_1 I$$

$$A_1 = A F_{k-1},\quad a_k = \frac{-1}{k} tr(A_k),\quad F_k = A_k + a_k I$$

To verify this multiply by $\lambda I - A$ & equate powers of λ

a_k are characteristic coeff's

~~$(\lambda I - F(\lambda))$~~

$$(\lambda I - A)F(\lambda) = q(\lambda) I$$ and equate powers of λ:

New characteristic polynomial is

$$r(\lambda) = |\lambda I - A - bd^T C| = |I - bd^T C F(\lambda)/q(\lambda)|$$

$\cdot|\lambda I - A$

$$= |I - bd^T C F(\lambda)/q(\lambda)|\, q(\lambda)$$

But

$$|I_m - XY| = |I_m - YX|$$

for any conformable X, Y since

$$\begin{bmatrix} I & 0 \\ -Y & I \end{bmatrix}\begin{bmatrix} I & X \\ Y & I \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & I - YX \end{bmatrix}$$

and

$$\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}\begin{bmatrix} I & X \\ Y & I \end{bmatrix} = \begin{bmatrix} I - XY & 0 \\ Y & I \end{bmatrix}$$

(partitions)

Thus,

$$r(\lambda) = q(\lambda)\left(1 - d^T C F(\lambda) b / q(\lambda)\right)$$

$$= q(\lambda) - (CF(\lambda)b)^T d$$

For m λ_{D_i}'s get m equations

$$0 = r(\lambda_{D_i}) = q(\lambda_{D_i}) - (CF(\lambda_{D_i})b^T d$$

if you have 2 equal roots there's a simple way out.

$\overset{\circ}{x} = Ax + Bu$ $\quad y = Cx$ $\quad$ y is m vector.

& $\quad u = d^T y$

Let $\quad g(x) = |\lambda I - A|^{-1}, \quad \frac{F(\lambda)}{g(\lambda)} = (\lambda I - A)^{-1}$

and $r(\lambda) = |\lambda I - A - bd^T c|$

Then

$$r(\lambda) = g(\lambda) - (cF(\lambda)b)^T d$$

If you wand a root λ_{d_i}, $i = 1, 2, \dots, m$

Then $\quad 0 = r(\lambda_{D_i}) = g(\lambda_{D_i}) - (CF(\lambda_{D_i})b)^T d$

we can write this as

$$E W^T d = e \quad \text{where}$$

$$W = c[b, F_1 b, F_2 b, \dots, F_{n-1} b]$$ F is a poly in A up to powe k

$$E = \begin{bmatrix} \lambda_{D_1}^{n-1} & \cdots & \lambda_{D_1} & 1 \\ \vdots & & & \\ \lambda_{D_m}^{n-1} & \cdots & \lambda_{D_m} & 1 \end{bmatrix} \qquad e_i = g(\lambda_{D_i})$$

For 2 equal roots. $\quad \lambda_{D_{i+1}} = \lambda_{D_i}$ $\quad$ we need

$$r(\lambda_{D_i}) = 0$$

$$\frac{d}{d\lambda} r(\lambda) \Big|_{\lambda = \lambda_{D_i}} = 0$$

Then $(i+1)^{ST}$ row of E must be replaced $\left[(n-1)\lambda_{D_i}^{n-2} \cdots 2\lambda_{D_i}, 1, 0\right]$

and $(i+1)^{ST}$ element of e by $e_{i+1} = \frac{dg}{dx}\Big|_{\lambda = \lambda_{D_i}}$

For multiinput system pole placement is non unique but if A is controllable by B we can find a feedback $u = Cx$ such that $A + BC$ is controllable through input, b_1, say.

Set $\quad u = \hat{u} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

<u>To prove:</u> we have that we can find $n \times n$ matrix

$$B = [b_1, \dots,] \ \ni \ Q = [b_1, Ab_1, \dots, A^{k_1-1}b_1, \cancel{A}b_2, \dots, A^{k_2-1}b_2, \dots]$$

Define mxn matix S

$$S = \begin{bmatrix} 0 & 0 & \cdots & e_2 & \cdot & 0 & 0 & \cdots & e_3 & 0 & 0 & \cdots \end{bmatrix}$$

Col of I matrix

Take $C = S Q^{-1}$

Since $CQ = S$ Now consider controllability matrix

$$\hat{Q} = [\,b_1, \hat{A}_1 b_1, \ldots \hat{A} b_2, \ldots$$

where $\hat{A} = A + BC$

Then
$$\left.\begin{aligned} b_1 &= b_1 \\ \hat{A} b_1 &= (A+BC) b_1 = A b_1 \\ \hat{A}^{k_1-1} b_1 &= \ldots \quad A^{k_1-1} b_1 \\ \hat{A}^{k_1} b_1 &= b_2 \end{aligned}\right\} \Rightarrow \hat{Q} = Q$$

Method devised by Heyman IEEE transact Automatic Control Dec 68 p748

Singular Optim Problem

This can be if H is linear in u

$\frac{\partial H}{\partial u}$ is independent of u indicating a bang-bang control unless

$\frac{\partial H}{\partial u} = 0$ Consider $\dot{x} = Ax + Bu$ $\quad J = \frac{1}{2}\int_{t_0}^{t_f} x^T Q x \, dt$

Then $H = \lambda^T (Ax + Bu) + \frac{1}{2} x^T Q x$

To min H we have $\frac{\partial H}{\partial u} = B^T \lambda$

we taken $u = +\infty$ if $B^T\lambda > 0$
$\qquad\quad -\infty \quad B^T \lambda < 0$

Try $u = \rho\,\delta(t - t_0)$ $\quad x(t) = e^{A(t)} x(t_0) + \int_{t_0^-} e^{A(t-\tau)} B \rho\, \delta(T - t_0) dt$

$$x(t_0^+) = x(t_0^-) + B\rho$$

We may have $H_{,u} = 0$ or $B^T \lambda = 0$

Then $\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) = 0 \ldots \frac{d^n}{dt^n}\left(\frac{\partial H}{\partial u}\right) = 0$ etc.

Now $\dot{\lambda} = -A^T \lambda + Q x$ $\qquad \lambda(t_f) = 0$

Assuming A, B are const we have

$$0 = B^T\lambda$$

$$0 = B^T\mathring{\lambda} = -B^TA^T\lambda - B^TQx$$

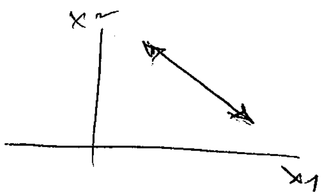
$$0 = -B^TA^T\mathring{\lambda} - B^TQ\mathring{x} = -\left[B^TA^T\{-A^T\lambda - Qx\} + B^TQBu\right\} + B^TQ$$

if B^TQB is non sengular or >0 we can solve it for u

$$u = \frac{B^TA^T\{A^T\lambda + Qx\} + B^TQAx}{B^TQB}$$

Example

$$\left.\begin{aligned}\mathring{x}_1 &= x_2 + u\\ \mathring{x}_2 &= -u\end{aligned}\right\} \text{ with } J = \tfrac{1}{2}\int_{t_0}^{t_f} x_1^2\,dt$$



Then $H = \lambda_1(x_2+u) + \lambda_2(-u) + \frac{1}{2}x_1^2$

$\mathring{\lambda}_1 =$ [illegible] $-x_1$

$\mathring{\lambda}_2 = -\lambda_1$

$\frac{\partial H}{\partial u} = \lambda_1 - \lambda_2$ If $\lambda_1 - \lambda_2 > 0$ $u = +\infty$

< 0 $= -\infty$

Singular arc is $\lambda_1 - \lambda_2 = 0$

$$0 = \mathring{\lambda}_1 - \mathring{\lambda}_2 = -x_1 + \lambda_1$$

$$\overset{\circ\circ}{\lambda}_1 - \overset{\circ\circ}{\lambda}_2 = -\mathring{x}_1 + \mathring{\lambda}_1 = -x_2 - u - x_1$$

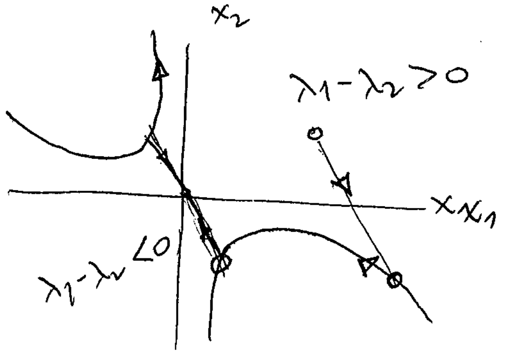
$$u = -(x_1 + x_2)$$

On a singular arc $\mathring{x}_1 = -x_1$ $x_1 = e^{-t}$

$\mathring{x}_2 = -x_2 + x_1$

$\mathring{\lambda}_1 = -x_1$ $\lambda_1 = e^{-t}$

$\mathring{\lambda}_2 = -\lambda_1$ $\left.\lambda_2 = e^{-t}\right\}\lambda_1 - \lambda_2 = 0$



If $u = \rho\,\delta(t_0 - t)$

$x_1(t_0^+) = x_1(t_0^-) + \rho$

$x_2(t_0^+) = x_2(t_0^-) - \rho$

For time opt. control $H = \lambda^T(Ax+Bu)+1$

$$\frac{\partial H}{\partial u} = +B^T\lambda$$

$$\& \quad \dot{\lambda} = -A^T\lambda$$

Relay Controls & chatter

Consider single input relay control

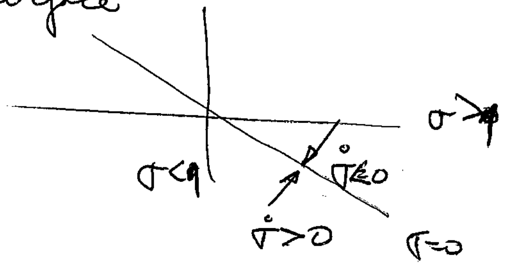
$$\dot{x} = Ax + Bu$$

where $u = \text{sign}(\sigma)$, $\text{sign}(\sigma) = 1$ if $\sigma > 0$

& $\sigma = d^T x$ $\qquad$ 0 if $\sigma = 0$

$\qquad -1$ if $\sigma < 0$

this might be an approx to an opt. bang bang control with a linear switching surface



If $\dot{\sigma} < 0$ when $\sigma > 0$ & $\dot{\sigma} > 0$ when $\sigma < 0$ we get stuck on the switching surface at one end pt.

We have $\dot{\sigma} = d^T\dot{x} = d^T Ax + d^T bu = d^T Ax + d^T b$ when $\sigma > 0$

$d^T Ax - d^T b$ when $\sigma < 0$

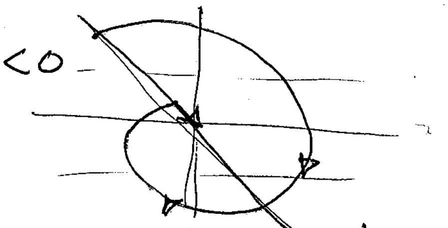
so this occurs if

$$d^T Ax + d^T b < 0 \quad \& \quad d^T Ax - d^T b > 0$$

or

$$d^T b < -d^T Ax < -d^T b$$

which can only occur if $d^T b < 0$



When this occurs control can no longer be in a bang bang mode

To continue the trajectory we require $\sigma = 0$ & hence $\dot{\sigma} = 0$

$$d^T\dot{x} = d^T Ax + d^T bu \Rightarrow u = -\frac{1}{d^T b} Ax \quad \text{a linear law}$$

also $\overset{\circ}{x} = \left(I - \frac{bd^T}{d^Tb}\right)Ax$ rank of $n-1$

Chattering: To realize this with relays suppose switch occurs with delay τ.

Let f^+ & f^- as $\overset{\circ}{x}$ with $u=+1$, $u=-1$

If we reach switching surface at t_1 & x_1 with $u=-1$

then we still use f^- for an extra period τ to reach

$$x_2 = x_1 + \tau f^- + O(\tau^2)$$

$\overset{\circ}{x} = f^+$ for $t > t_1 + \tau$ bringing system back to switching surface

at $t + \Delta t$ & x_3 where

$$x_3 = x_2 + (\Delta t - \tau)f^+ + O(\Delta t - \tau)^2$$
$$= x_1 + \Delta t f^+ + \tau(f^- - f^+) + O(\tau^2)$$

since $\sigma = 0$ when $x = x_1$ & $x = x_3$ we have $d^Tx_1 = 0$, $d^Tx_3 = 0$

Thus $\Delta t = -\tau\left[\frac{d^T(f^+ - f^-)}{d^Tf^+}\right]^{-1}$ Then $\frac{x_3 - x_1}{\Delta t} = f^+ - (f^- + f^+)\left[\frac{d^T(f^- - f^+)}{d^Tf^+}\right]^{-1}$

In limit as $\tau \to 0$ $\overset{\circ}{x} = f^+ - (f^- + f^+)\left[\frac{d^T(f^- - f^+)}{d^Tf^+}\right]^{-1}$

so that $d^T\overset{\circ}{x} = 0$

Example of Chattering control

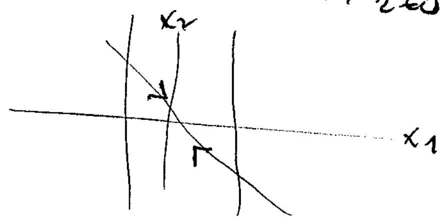
Let $\overset{\circ}{x} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}x + \begin{bmatrix}0\\ 1\end{bmatrix}u$ or $\overset{\circ\circ}{x}_1 + 3\overset{\circ}{x}_1 + 2x_1 = u$

Take control

$$u = \text{sign}(d^Tx), \quad d^T = [-3, -1]$$

Then $d^Tb = -1$ $\qquad d^TAx = 2x_1$

So there are points when $3x_1 + x_2 = 0$ $\quad -1 < -2x_1 < 0$



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Given

$$\dot{x} = Ax + Bv, \quad y = Cx, \quad y = m \text{ vector}$$

and $v = d^T y$

Let $g(\lambda) = |\lambda I - A|^{-1}$, $\dfrac{F(\lambda)}{g(\lambda)} = (\lambda I - A)^{-1}$

and $r(\lambda) = |\lambda I - A - bd^T c|$

Then

$$r(\lambda) = g(\lambda) - (cF(\lambda)b)^T d$$

If you want a root λ_{D_i}, $i = 1, 2, \ldots, m$

Then

$$0 = r(\lambda_{D_i}) = g(\lambda_{D_i}) - (cF(\lambda_{D_i})b)^T d$$

We can write this as

$$E W^T d = e$$

where

$$W = C[b, F_1 b, F_2 b, \ldots, F_{n-1} b]$$

F_k is polynomial in A up to power

$$E = \begin{bmatrix} \lambda_{D_1}^{m-1} & \lambda_{D_1}^{m-2} & \cdots\cdots & 1 \\ \lambda_{D_m}^{n-1} & \lambda_{D_m}^{n-2} & \cdots\cdots & 1 \end{bmatrix}$$

$$e_i = g(\lambda_{D_i})$$

For 2 equal roots $\lambda_{D_{i+1}} = \lambda_{D_i}$

We need

$$r(\lambda_{D_i}) = 0$$

$$\left.\frac{d}{d\lambda} r(\lambda)\right|_{\lambda = \lambda_{D_i}} = 0$$

Then $(i+1)^{th}$ row of E must be replaced by

$$[(n-1)\lambda_{D_i}^{n-2}, (n-2)\lambda_{D_i}^{n-3}, \ldots, 1, 0]$$

and $(i+1)^{th}$ element of e by

$$e_{i+1} = \left.\frac{dg}{d\lambda}\right|_{\lambda = \lambda_{D_i}}$$

For multi input system pole replacement is non-unique. But if A is controllable by B we can find a feedback $\hat{u} = Cx$ such that $A + BC$ is controllable through input, b_1, say

$$\text{set } u = \hat{u} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

ⓐ

To prove this we have that we can find $m \times n$ matrix $B = [b_1, b_2, \cdots]$

$$Q = [b_1, Ab_1, \cdots, A^{k_1-1}b_1, b_2, Ab_2, \cdots, A^{k_2-1}b_2, \cdots]$$

Define $m \times n$ matrix S

$$S = [0\,0\,e_2\,0\,0\,e_3\,0\,0 \quad]$$

Take

$$C = S Q^{-1}$$

Since $CQ = S$ we now Consider controllability matrix

$$\hat{Q} = [b_1, \hat{A}b_1, \hat{A}^2 b_1, \cdots]$$

where $\hat{A} = A + BC$

Then

$$b_1 = b_1$$
$$A b_1 = (A+BC)b_1 = Ab_1$$
$$\hat{A}^{k-1} b_1 = \quad A^{k-1} b_1 \quad \Longrightarrow \quad \hat{Q} = Q$$
$$\hat{A}^k b_1 = b_2$$

idea derived by Heyman IEEE Trans. Automatic control Dec 1968 p 748

Singular Arcs & Singular Optimization Problems

This can occur if H is linear in u

$\frac{\partial H}{\partial u}$ is independent of u indicating bang-bang control unless $\frac{\partial H}{\partial u} = 0$

Consider

$$\dot{x} = Ax + Bu \qquad J = \int_{t_0}^{t_f} x^T Q x \, dt$$

Then

$$H = \lambda^T(Ax + Bu) + \tfrac{1}{2} x^T Q x$$

to minimize H we have

$$\frac{\partial H}{\partial u} = B^T \lambda$$

so we take $u = +\infty$ if $B^T\lambda > 0$

$-\infty$ if $B^T\lambda < 0$

Try $u = \rho\,\delta(t - \tau_0)$

$$x(t) = e^{At}x(t_0) + \int_{t_0^-} e^{A(t-\tau)} B\rho\,\delta(\tau - \tau_0)\,d\tau$$

$$x(t_0^+) = x(t_0^-) + B\rho$$

We may have $\frac{\partial H}{\partial u} = 0$ or $\underline{B^T\lambda = 0}$

If this is true for an extended interval we have singular arc.

Then $\frac{d}{dt}\frac{\partial H}{\partial u} = 0 \quad \frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) = 0$ etc.

Now

$$\dot\lambda = -A^T\lambda - Qx \qquad \lambda(t_f) = 0$$

Assuming A, B are constant we have

$$0 = B^T\lambda$$
$$0 = B^T\dot\lambda = -B^TA^T\lambda - B^TQx$$
$$0 = B^TA^T\dot\lambda - B^TQ\dot x$$

or $0 = B^TA^T\lambda + B^TA^TQx - B^TQAx - B^TQBu$

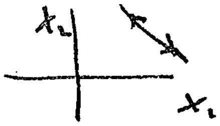
So if B^TQB non singular or >0 we can solve for u

$$u = (B^TQB)^{-1}B^T\{(A^TQ - QA)x + A^T\overset{2}{\lambda}\}$$

if not satisfied $\Rightarrow$ not enough information to determine u

<u>Example</u>: (Bryson) Singular arc

$$\dot x_1 = x_2 + u$$
$$\dot x_2 = -u$$



under an impulse all you can do is move it by 45°

with

$$J = \tfrac{1}{2}\int_{t_0}^{t_f} x_1^2\,dt$$

Then

$$H = \lambda_1(x_2 + u) - \lambda_2 u + \tfrac{1}{2}x_1^2$$
$$\dot\lambda_1 = -\frac{\partial H}{\partial x_1} = -x_1$$
$$\dot\lambda_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1$$

Also $\dfrac{\partial H}{\partial u} = \lambda_1 - \lambda_2$

If $\lambda_1 - \lambda_2 > 0 \qquad u = +\infty$

$\qquad\quad < 0 \qquad\;\; = -\infty$

Singular arc is

$$\lambda_1 - \lambda_2 = 0$$

differentiate that then

$$0 = \dot\lambda_1 - \dot\lambda_2 = -x_1 + \lambda_1$$

differentiating again

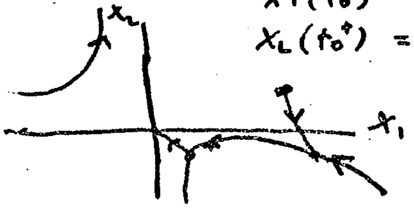
$$0 = -\dot x_1 + \dot\lambda_1 = -x_2 - u - x_1$$

so $u = -(x_1 + x_2)$

If $u = \rho\,\delta(t_0 - t)$

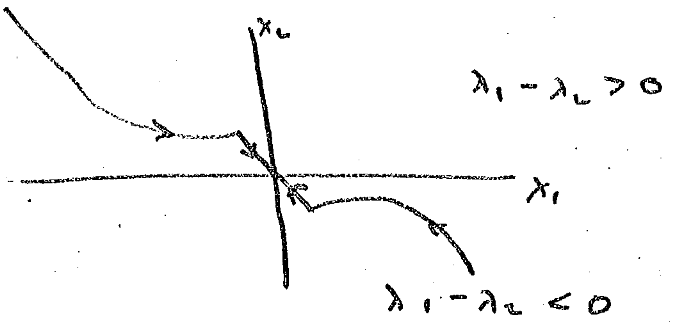
$x_1(t_0^+) = x_1(t_0^-) + \rho$

$x_2(t_0^+) = x_2(t_0^-) - \rho$



On singular arc

$$\dot x_1 = -x_1 \qquad x_1 = e^{-t}$$
$$\dot x_2 = x_2 + x_1$$
$$\left.\begin{array}{l}\dot\lambda_1 = -x_1 = -e^{-t},\ \lambda_1 = e^{-t}\\ \dot\lambda_2 = -\lambda_1 = -e^{-t},\ \lambda_2 = e^{-t}\end{array}\right\}\ \lambda_1 - \lambda_2 = 0$$



For time optimal control $H = \lambda^T(Ax + Bu) + 1$

$$\frac{\partial H}{\partial u} = B^T\lambda$$

and $\dot{\lambda} = -A^T\lambda$ (Clean switch in time op. prob's)

Relay controls and chattering

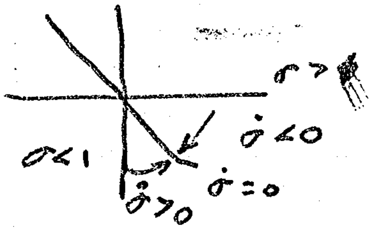
Consider single input relay control

$$\dot{x} = Ax + bu$$

where $u = \text{sign}(\sigma)$, $\text{sign}(\sigma) = 1$ if $\sigma > 0$
0 if $\sigma = 0$
-1 if $\sigma < 0$

and $\sigma = d^T x$

This might be an approximation to an optimal bang-bang control with a linear switching surface.



If $\dot{\sigma} < 0$ when $\sigma > 0$ and $\dot{\sigma} > 0$ when $\sigma < 0$ we get stuck on the switching surface at an endpoint. (leads to chattering)

(Condition for chattering)

We have

$$\dot{\sigma} = d^T\dot{x} = d^T A x + d^T b u = d^T A x + d^T b \quad \text{when } \sigma > 0$$
$$d^T A x - d^T b \quad \text{when } \sigma < 0$$

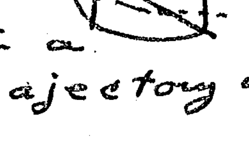
so this occurs if

$$d^T A x + d^T b < 0 \quad \text{and} \quad d^T A x - d^T b > 0$$

or $d^T b < -d^T A x < -d^T b$

which can only occur if $d^T b < 0$

so what you now have is a stripe



When this occurs control can no longer be in a bang-bang mode. ~~Instead~~ To continue trajectory we require $\sigma = 0$ and hence $\dot{\sigma} = 0$ so

$$d^T\dot{x} = d^T A x + d^T b u$$

and hence $u = -\frac{1}{d^T b} A x$ a linear law

Also then $\dot{x} = \left(I - \frac{b\, d^T}{d^T b}\right) A x$

Chattering (mode) To realize this with relays suppose switch occurs with time delay T.

Let f^+ and f^- as $\dot{x}$ with $U = +1$ and -1

If we reach switching surface at t_1 and x_1 with $U = -1$ Then we still use f^- for an extra period τ to reach

$$x_2 = x_1 + \tau f^- + O(\tau^2)$$

Then $\dot{x} = f^+$ for $t > t_1 + \tau$ bringing system back to switching surface at $t + \Delta t$ and x_3 where

$$x_3 = x_2 + (\Delta t - \tau) f^+ + O(\Delta t - \tau)^2$$
$$= x_1 + \Delta t\, f^+ + \tau (f^- - f^+) + O(\tau^2)$$

Since $\sigma = 0$ when $x = x_1$ and $x = x_3$ we have

$$d^T x_1 = 0 \quad , \quad d^T x_3 = 0$$

Thus

$$\Delta t = -\tau \left[\frac{d^T(f^- - f^+)}{d^T f^+}\right]^{-1}$$

then

$$\frac{x_3 - x_1}{\Delta t} = f^+ - (f^- - f^+)\left[\frac{d^T(f^- - f^+)}{d^T f^+}\right]^{-1}$$

In limit as $T \to 0$

$$\dot{x} = f^+ - (f^- - f^+)\left[\frac{d^T(f^- - f^+)}{d^T f^+}\right]^{-1}$$

so that

$$d^T \dot{x} = 0$$

Example of chattering ~~mode~~ control

Let $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$ or $\ddot{x}_1 + 3\dot{x}_1 + 2x_1 = U$

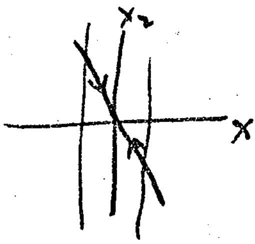
Take control

$$U = \text{sign}(d^T x), \quad d^T = [-3, 1]$$

then $d^T b = -1$, $d^T A x = [-3 \ -1]\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1$

so there are end points when

$3x_1 + x_2 = 0 \quad , \quad -1 < -2x_1 < 1$



On the chattering regime

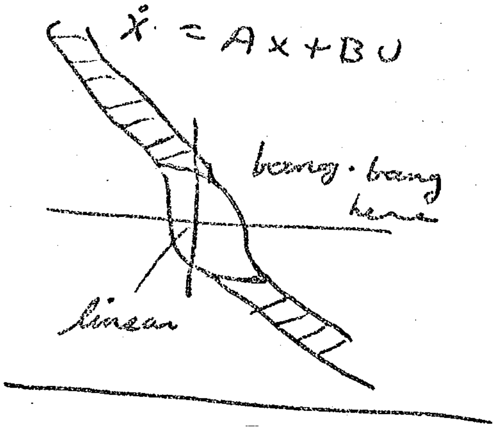
$$U = d^T A x = 2x_1$$
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} x$$

notice this stable situation

$$\dot{x} = \left[I - \frac{b d^T}{d^T b}\right] A$$

$\dot{x} = Ax + Bu$ $\qquad J = \int x^T Q x \, dt$ ⑥

$|u| \leq 1$



Transformation of singular problem

Given

$$\dot{x} = Ax + Bu \quad , \quad J = \int_{t_0}^{t_f} x^T Q x \, dt$$

(where A and B are constant but method works in time varying case)

Transform to new variables x_1, u_1 by setting

$$x = x_1 + Bu_1 \quad , \quad u = \dot{u}_1$$

Then

$$\dot{x}_1 = \dot{x} - B\dot{u}_1$$
$$= Ax + \cancel{Bu} - \cancel{B\dot{u}_1}$$
$$= Ax_1 + ABu_1$$

Also

$$J = \int_{t_0}^{t_f} \{x_1^T Q x_1 + 2u_1^T B^T Q x_1 + u_1^T B^T Q B u_1\} dt$$

Provided $B^T Q B > 0$ we have standard problem with cross product term. Set

$$\hat{u} = u_1 - (B^T Q B)^{-1} B^T Q x_1$$

Then

$$J = \int (x_1^T Q_1 x_1 + \hat{u}^T B^T Q B \hat{u}) \, dt$$

where

$$\dot{x}_1 = A_1 x_1 + AB\hat{u}$$

where $A_1 = A(I - B(B^T Q B)^{-1} B^T Q)$ $\quad , A_1 B = 0$

$Q_1 = Q(I - B(B^T Q B)^{-1} B^T Q$ $\quad , Q_1 B = 0$

to check Q_1 is non-neg note that

$$(I - QB(B^T Q B)^{-1} B^T) Q (I - B(B^T Q B)^{-1} B^T Q)$$

$1 - QB(B^TQB)^{-1}B^TQ$ ~~$= Q_1 - QB(B^TQB)^{-1}B^TQ$~~ $- QB(B^TQB)^{-1}B^TQ + QB(B^TQB^{-1})(B^TQB)(B^TQB)^{-1}B^TQ = Q_1$ where $Q \geq 0$ Form $G^T Q G$

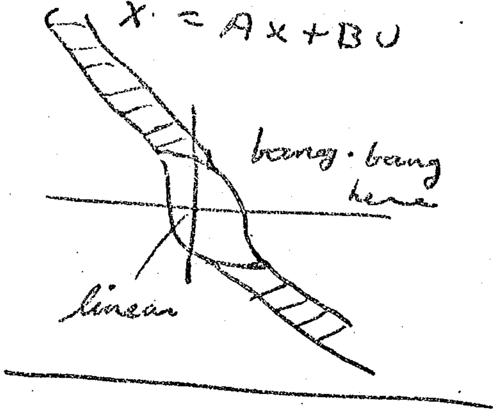
solution in standard fashion from Riccati equation as

$$u_1 = Dx_1$$

Where $D_1 = -(B^T Q B)^{-1} B^T (A^T P + Q)$

$\dot{x} = Ax + Bu$ $\qquad J = \int x^T Q x \, dt$

$|u| \leq 1$



Transformation of singular problem
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$$\dot{x} = Ax + Bu \quad , \quad J = \int_{t_0}^{t_f} x^T Q x \, dt$$

(where A and B are constant but method works in time varying case)

Transform to new variables x_1, u_1 by setting

$$x = x_1 + Bu_1 \quad , \quad u = \dot{u}_1$$

Then

$$\dot{x}_1 = \dot{x} - B\dot{u}_1$$
$$= Ax + \cancel{Bu} - B\cancel{\dot{u}_1}$$
$$= Ax_1 + ABu_1$$

Also

$$J = \int_{t_0}^{t_f} \{ x_1^T Q x_1 + 2u_1^T B^T Q x_1 + u_1^T B^T Q B u_1 \} dt$$

Provided $B^T Q B > 0$ we have standard problem with cross product term. Set

$$\hat{u} = u_1 - (B^T Q B)^{-1} BTQ x_1$$

Then

$$J = \int (x_1^T Q_1 x_1 + \hat{u}^T B^T Q B \hat{u}) \, dt$$

where

$$\dot{x}_1 = A_1 x_1 + AB\hat{u}$$

where $A_1 = A(I - B(B^T Q B)^{-1} B^T Q) \qquad , A, B = 0$

$Q_1 = Q(I - B(B^T Q B)^{-1} B^T Q \qquad , Q, B = 0$

to check Q_1 is non-neg note that

$$(I - QB(B^T Q B)^{-1} B^T) Q (I - B(B^T Q B)^{-1} B^T Q)$$

$) - QB(B^TQB)^{-1}B^TQ$ ~~$= Q - QB(B^TQB)^{-1}B^TQ$~~ $- QB(B^TQB)^{-1}B^TQ + QB\cancel{(B^TQB^{-1})}\cancel{(B^TQB)}(B^TQB)^{-1}B^TQ = Q_1$ where $Q \geq 0$

Form G^TQG

solution ~~in~~ standard fashion from Riccati equation as

$$u_1 = Dx_1$$

Where $D_1 = -(B^T Q B)^{-1} B^T (A^T P + Q)$

where
$$\overset{\circ}{P} = Q + A^TP + PA - D^TB^TQBD$$
$$= Q_1 + A_1^TP + PA_1 - PAB(B^TQB)^{-1}B^TA^TP$$
$$P(t_f)=0$$

Since $A_1B=0$, $Q_1B=0$ we have
$$-\frac{d}{dt}(PB) = \overset{\circ}{P}B = \cancel{QB} + A_1^TQB + P\cancel{A_1}B - PAB(B^TQB)^{-1}B^TA^TPB$$

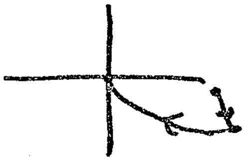
so since
$PB = 0$ at $t = T_f$, $PB=0$ for whole path

related to fact That a singular arc

then $\overline{DB = -\cancel{(B^TQB)^{-1}B^TA^TPB}} - (B^TQB)^{-1}B^TQB = -I$

Then on singular arc (solution)
$$\underline{Dx} = Dx_1 + DBu_1 = Dx_1 - u_1 = 0$$

$$B^T\lambda = 0$$



Under an initial impulse $u = \rho\delta(t_0-t)$ we get
$$x(t_0^+) = B\rho + x(t_0^-)$$
So we want
$$0 = Dx(t_0^+) = Dx(t_0^-) + DB\rho = Dx(t_0^-) - \rho$$
~~initial~~ so impulse is
$$u = Dx(t_0^-)\delta(t_0-t)$$

Note
$$u_1 = \int u\,dt$$
so $u_1(t_0^+) = Dx(t_0^-)$

Also
$$x(t_0^+) = x(t_0^-) + BDx(t_0^-) =$$
so $Dx(t_0^+) = 0$

then $x(t_0^+) = x(t_0^+) - Bu_1(t_0^+)$
$$= x(t_0^+) - BDx(t_0^-)$$
and $Dx_1(t_0^+) = Dx(t_0^-) = u_1(t_0^+)$

To verify That that transformation continues to be possible-
i.e. To verify $x = x_1 + Bu_1$ is continued properly
$$\frac{d}{dt}(x - x_1 - Bu_1) = Ax + \cancel{Bu} - Ax_1 - ABu_1 - B\dot{u}_1$$
$$= A(x - x_1 - Bu_1)$$

Finally $u = \dot{u}_1 = D\dot{x}_1 = D(Ax_1 + ABu_1) = DAx$

(solution for control)

Then we can verify that u forces $Dx = 0$ to be maintained

$$
\begin{aligned}
D\dot{x} &= D(Ax + Bu) \\
&= DAx + DBDAx \\
&= 0 \text{ since } DB = -I
\end{aligned}
$$

Complete solution if $B^T Q B > 0$

In the chattering regime $d^TAx = u = 2x_1$

$$\mathring{x} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} x.$$

$$\mathring{x} = \left[I - \frac{bd^T}{d^Tb} \right] A$$

Last week 8/30/73

$$\dot{x} = Ax + Bu$$

$$J = \int_{t_0}^{t_f} x^T Q x \, dt \qquad \text{Solution } B^T Q B > 0$$

Now consider case $B^TQB = 0$, $B^TA^TQAB > 0$

Since $B^TQB = 0$ $\quad B^TQ = 0$ & $Q \geq 0 \Rightarrow Q = C^TC$

So $B^TC^TCB = 0 \Rightarrow CB = 0$

1st transform

$$x = x_1 + Bu_1 \qquad u = \dot{u}_1$$

~~then~~ ~~[illegible]~~ gives $\dot{x}_1 = Ax_1 + ABu_1$

and $J = \int_{t_0}^{t_f} x_1^T Q x_1 \, dt$ since $B^TQ = 0$

2nd transfor

$$x_1 = x_2 + ABu_2 \, , \; u_1 = \dot{u}_2$$

then $\dot{x}_2 = \dot{x}_1 - AB\dot{u}_2 = Ax_1 + A\cancel{Bu_1} - A\cancel{B\dot{u}_2} = Ax_2 + A^2Bu_2$

$$J = \int_{t_0}^{t_f} \left(x_2^T Q x_2 + 2u_2^T B^TA^TQx_2 + u_2^T \underbrace{B^TA^TQAB}_{>0} u_2 \right) dt$$

Solution

$$u_2 = Dx_2 \qquad D = -(B^TA^TQAB)^{-1} B^TA^T(A^TP + Q)$$

& P satisfies $-\dot{P} = Q + A^TP + PA - D^TB^TA^TQABD$

$$= Q_2 + A_2^TP + PA_2 - PA^2B(B^TA^TQAB)B^TA^{T^2}P$$

$$P(t_f) = 0$$

Here $A_2 = A\left(I - AB(B^TA^TQAB)^{-1}B^TA^TQ\right)$

$$Q_2 = Q - QAB(B^TA^TQAB)^{-1}B^TA^TQ \geq 0$$

also

$$A_2AB = 0 \, , \quad A_2B = AB \text{ (since } QB = 0)$$

$Q_2AB = 0$, $Q_2B = 0$ more constraints

Then $-\frac{d}{dt}(PAB) = \overset{0}{\cancel{Q_2AB}} + A_2^TPAB + \overset{0}{\cancel{PA_2AB}} - PA^2B(B^TA^TQAB)^{-1} B^TA^{T^2}PAB$

& $PAB = 0$ at $t = t_f$ thus $PAB = 0$

Then $-\frac{d}{dt}(PB) = \cancel{Q_2B} + A_2^TPB + PA_2B - PA^2B(\;)^{-1}B^TA^{T^2}PB$

$$= A_2^TPB + \cancel{PAB} - PA^2B(\;)^{-1}B^TA^{T^2}PB$$

thus $PB = 0$

Then $DAB = -I$
$DB = 0$ (we can now push this back onto trajectory)

Thus $Dx_1 = Dx_2 + DABu_2 = Dx_2 - u_2 = 0$

and $u_1 = \overset{\circ}{u}_2 = D\overset{\circ}{x}_2 = DAx_2 + DA^2Bu_2 = DAx_1$

$$DAx = DAx_1 + DABu_1 = DAx_1 - u_1 = 0 \quad (1)$$

and

$$Dx = Dx_1 + DBu_1 = Dx_1 = 0 \quad (2)$$

thus $x(t_0^+)$ must satisfy two constraints. This requires a higher order impulse. Let $u = \rho_0\,\delta(t-t_0) + \rho_1\,\delta'(t-t_0)$

Thus $x = e^{At}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}B\{\rho_0\,\delta(\tau-t_0) + \rho_1\,\delta'(\tau-t_1)\}d\tau$

Integ By parts $= e^{At}x(t_0) + e^{At}B\rho_0 + B\rho_1\,\delta(t-t_0) + \int_{t_0}^{t} Ae^{A(t-\tau)}B\rho_1\,\delta(\tau-t_0)d\tau$

Thus $x(t_0^+) = x(t_0^-) + B\rho_0 + AB\rho_1$ after x has made a delta fn excursion. $B\rho_1\delta(t-t_0)$ which has zero cost since $QB=0$ (It would have infinite cost if QB were not zero)

Then we require $0 = Dx(t_0^+) = Dx(t_0^-) + DB\rho_0 + DAB\rho_1$
$= Dx(t_0^-) - \rho_1$

$0 = DAx(t_0^+) = DAx(t_0^-) + DAB\rho_0 + DA^2B\rho_1$
$= DAx(t_0^-) - \rho_0 + DA^2B\rho_1$

So required impulse is

$$u = \rho_0\,\delta(t-t_0) + \rho_1\,\delta'(t-t_0)$$

where $\rho_0 = DA[x(t_0^-) + AB\rho_1]$ $\quad \rho_1 = Dx(t_0^-)$

Singular perturbation approach

Consider $\dot{x} = Ax + Bu$

$$J = \tfrac{1}{2}\int_{t_0}^{t_f}(x^TQx + \epsilon^2u^TRu)\,dt$$

Consider as lim $\epsilon \to 0$

$$H = \lambda^T(Ax+Bu) + \tfrac{1}{2}x^TQx + \tfrac{1}{2}\epsilon^2 u^TRu$$

where $\overset{\circ}{\lambda} = -Qx - A^T\lambda \qquad \lambda(t_f) = 0$

$$0 = \frac{\partial H}{\partial u} = \epsilon^2 Ru + B^T\lambda$$

or $\quad u = -\frac{1}{\epsilon^2} R^{-1}B^T\lambda$

$$\overset{\circ}{x} = Ax - \frac{1}{\epsilon^2} BR^{-1}B^T\lambda \qquad x(t_0) = x_0$$

$$\overset{\circ}{\lambda} = -Qx - A^T\lambda \qquad \lambda(t_f) = 0.$$

let $\quad x = \sum x_j\epsilon^j \qquad \lambda = \sum \lambda_j \epsilon^j$

put into eq & equate like powers of j

$$BR^{-1}B^T\lambda_0 = 0 \qquad (1)$$

$$BR^{-1}B^T\lambda_1 = 0 \qquad (2)$$

$$\overset{\circ}{x}_0 = Ax_0 - BR^{-1}B^T\lambda_2 \;(3). \quad \text{let } R^{-1}B^T\lambda_2 = \text{control}$$

$$\overset{\circ}{\lambda}_0 = -Qx_0 - A^T\lambda_0 \qquad (4)$$

$$(1) \Rightarrow \lambda^T_0 BR^{-1}B^T\lambda_0 = 0 \Rightarrow B^T\lambda_0 = 0 \qquad (5)$$

Similarly $B^T\lambda_1 = 0 \qquad (6)$

Now (3,4,5) are just eq for singular arc when $\epsilon = 0$ if we take $u = R^{-1}B^T\lambda_2$

Different. (5) twice $\qquad 0 = B^T\overset{\circ}{\lambda}_0 = B^TQx_0 + B^TA^T\lambda_0$

$$B^T\overset{\circ\circ}{\lambda}_0 = -B^TQ\overset{\circ}{x}_0 + B^TA^T\overset{\circ}{\lambda}_0$$

$$= \cancel{-B^TQAx_0} + B^TQBR^{-1}B^T\lambda_2$$

$$\cancel{+B^TA^TQx_0} + B^TA^TA^T\lambda_0$$

so if $B^TQB > 0$

$$u = -(B^TQB)^{-1}B^T\{(QA + A^TQ)x_0 + A^{T^2}\lambda_0\}$$

and $\quad R^{-1}B^T\lambda_2 = u$ can be elim

To solves 3,4,5 it is convenient to use transformation method

Let $x_0 = \bar{x} + B\bar{u} \qquad u = \overset{\circ}{\bar{u}}$

then $\quad \overset{\circ}{\bar{x}} = A\bar{x} + AB\bar{u}$

$$\overset{\circ}{\lambda}_0 = -Q\bar{x} - QB\bar{u} - A^T\lambda_0$$

Also

$$0 = B^T\overset{\circ}{\lambda}_0 = -B^TQ\bar{x} - B^TQB\bar{u} - B^TA^T\lambda_0$$

$$\bar{u} = -(B^TQB)^{-1}B^T(Q\bar{x} + A^T\lambda_0)$$

Then $\overset{\circ}{\bar{x}} = \bar{A}\bar{x} - S\lambda_0$ $\quad \lambda_0(t_f) = 0$

$\overset{\circ}{\lambda}_0 = -\bar{Q}\bar{x} - \bar{A}^T\lambda_0$

where $\bar{\bar{A}} = A\left(I - B(B^TQB)^{-1}B^TQ\right)$ $\qquad \bar{A}B = 0$

$\bar{Q} = Q - Q\,B(B^TQB)^{-1}B^TQ \geq 0$ $\qquad \bar{Q}B = 0$

$S = AB(B^TQB)^{-1}B^TA^T$

This has a solution $\lambda_0 = P\bar{x}$

where $-\overset{\circ}{P} = \bar{Q} + \bar{A}^TP + P\bar{A} - PSP$, $P(t_f) = 0$

Riccati sol exists because $\bar{Q} \geq 0$.

Then $-\frac{d}{dt}(PB) = \cancel{\bar{Q}B} + A^{-T}PB + P\bar{A}B - PSPB$

so $\quad PB = 0$

$\lambda_0 = P\bar{x} = P(x_0 - B\bar{u}) = Px_0$

and $\quad B^T\lambda_0 = B^TPx_0 = 0$

But $x_0(t_0)$ is *not* arbitrary because at $t = t_0$

$0 = B^T\overset{\circ}{\lambda}_0 = -B^T(Q + A^TP)x_0$

singular perturbation situation must add a boundary layer correction

To bridge gap between $x_0(t_0)$ & $x(t_0)$ we need a boundary layer correction, so we seek a solution in more general form

$x = X(t,\epsilon) + m(\tau,\epsilon)$

$\lambda = \Lambda(t,\epsilon) + \frac{1}{\epsilon}f(\tau,\epsilon)$

where X, Λ are outer sol just obtained. $m, \frac{1}{\epsilon}f$ are boundary layer correction which $\to 0$ as stretched time $\tau \to \infty$ where

$\tau = \frac{t - t_0}{\epsilon}$

bl correction must satisfy original de since de are linear and we can add & subtract sol

$\frac{dm}{dt} = Am - \frac{1}{\epsilon}BR^{-1}B^Tf$ $\qquad \frac{dm}{d\tau} = \epsilon\frac{dm}{dt}$

$\epsilon\frac{df}{dt} = Qm - \epsilon A^Tf$ $\qquad \frac{df}{d\tau} = \epsilon\frac{df}{dt}$

$$\frac{dm}{d\tau} = \epsilon Am - BR^{-1}B^T f$$

$$\frac{df}{d\tau} = -Qm - \epsilon A^T f$$

Set $m = \Sigma m_j \epsilon^j \quad f = \Sigma f_j \epsilon^j$

Then
$$\frac{dm_0}{d\tau} = -BR^{-1}B^T f_0$$
$$\frac{df_0}{d\tau} = -Qm_0$$

$f_0, m_0 \to 0$ as $\tau \to \infty$

$$\underset{m_0 \to \text{dim } n}{\frac{d^2 m_0}{d\tau^2}} = +\underbrace{BR^{-1}B^T}_{B \to \text{rank } m} Q m_0 .$$ Roots are equal & opposite pairs

Now $\frac{dm}{d\tau}$ is in range space of B and $m_0(\infty) = 0$ so m_0 must always been in the range space of B, that is $m_0 = BM$

where
$$B\frac{d^2M}{d\tau^2} = BR^{-1}B^T QBM$$

mult By B^TQ on left $\to$ $B^TQB > 0$ inverse exists

$$\frac{d^2M}{d\tau^2} = (B^TQB)^{-1} B^TQBR^{-1}B^TQBM = R^{-1}B^TQBM$$

Let $R^{-1} = CC^T$ $C > 0$ since $R > 0$; set $M = Cy$

$$\frac{d^2y}{d\tau^2} = C^TB^TQBCy$$ $C^TB^TQBC > 0$ it has square root F

so $y(\tau) = e^{\pm F(\tau)} y(0)$

Also $\frac{d^2 f_0}{d\tau^2} = QBR^{-1}B^T f_0$ But since m_0 lies in range space of B

$$\Rightarrow \frac{df_0}{d\tau} = Qm_0$$

f_0 lies in range space of QB or $f_0 = QBF$

where $\frac{d^2F}{d\tau^2} = R^{-1}B^TQBF$ then $F = CZ$ where

$$Z(\tau) = e^{\pm F(\tau)} Z(0)$$

To match complete solution we note that

$$m_0(t_0) = BM_0(t_0)$$

so $X_0(t_0) = x_0(t_0) - BM_0(t_0)$ mult by B^TQ gives

$$B^TQBM_0(t_0) - B^TQ(x(t_0) - X(t_0)) = 0$$

$$M_0(t_0) = (B^TQB)^{-1}B^TQ(x(t_0) - X_0(t_0))$$

$$(I - B(B^TQB)^{-1}B^TQ)X_0(t_0) = (I - B(B^TQB)^{-1}B^TQ)x(t_0) \quad \text{n-m condition}$$

m more conditions on $X_0(t_0)$ from $B^T\dot{X}_0 = 0$ or

$$B^T(Q + A^TP)X_0(t_0) = 0$$

B.L. leads to δ fn behavior for u

Contribution to u from B.L. is $\frac{1}{\epsilon}BR^{-1}B^Tf$

where f behaves like

$$QBCe^{-F\tau}z(0) = QBCe^{-F(\frac{t-t_0}{\epsilon})}z(0)$$

so we have term of form

$$\frac{1}{\epsilon}e^{-\lambda/\epsilon\,(t-t_0)}$$

This is a δ fn as ε → 0 since for any smooth $y(t)$

$$\frac{1}{\epsilon}\int_{t_0}^{t} e^{-\lambda\tau}y(t)\,dt = \frac{1}{\lambda}\left(y(t_0) - e^{-\frac{\lambda}{\epsilon}(t-t_0)}y(t)\right) + \int_{t_0}^{t} e^{-\lambda/\epsilon\,(t-t_0)}y'(t)\,dt$$

$$\to \frac{y(t_0)}{\lambda} \quad \text{as } \epsilon \to 0$$

Nonlinear singular problem

Minimize $K(x(t_f))$ where $\dot{x} = f(x) + g(x)u$

Then $H = \lambda^T(f + gu)$ $\quad \dot{\lambda} = -\frac{\partial H}{\partial u} = -(f_x + g_x u)^T\lambda \quad \lambda(t_f) = K_x$

Then $0 = H_u = \lambda^T g$

$$0 = \frac{d}{dt}H_u = \lambda^T\dot{g} + \dot{\lambda}^T g = \lambda^T g_x\dot{x} + \dot{\lambda}^T g$$

$$= \lambda^T g_x(f + gu) - \lambda^T(f_x + g_x u)g$$

$0 = \lambda^T q(x)$ where $q = g_x f - f_x g$

$$0 = \frac{d}{dt} H_u = \lambda^T \dot{q} + \dot{\lambda}^T q$$

$$= \lambda^T q_x (f + gu) - \lambda^T (f_x + g_x u) q$$

$$= \lambda^T (q_x f - f_x q) - \lambda^T (q_x g - g_x q) u$$

If $\lambda^T (q_x g - g_x q) \neq 0$ —— (A) is ensured by general convexity condition

we can solve for

$$u = \frac{\lambda^T (q_x f - f_x q)}{\lambda^T (q_x g - g_x q)}$$

$$(-1)^k \frac{\partial}{\partial u}\left[\left(\frac{d}{dt}\right)^{2k} H_u \right] \geq 0$$

necessary cond $k = 1, 2, \ldots$

derived by Bryson p257

Kelley, Kopp, Moyer in Topics of Optimization edited by Leitmann AP 1966