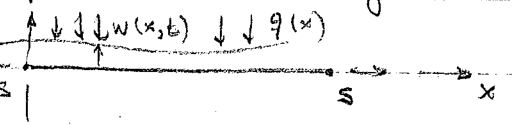
2-6-74

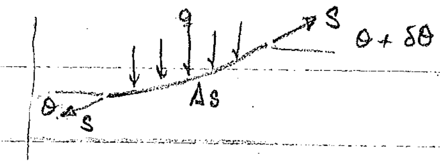
Wave Propagation – Stoker

Wave – unsteady & time dependent motion in a continuous media.

Perfect incompressible irrotational gravitating fluid.

We look at string 

Perfectly flexible $w(x,t)$ is small



$$\frac{\Delta s}{\Delta x} \sim \sqrt{1 + \left(\frac{\Delta w}{\Delta x}\right)^2} \sim 1$$

$$S \sin(\theta + \delta\theta) - S \sin\theta - q\,\Delta s = \rho\,\Delta s\,\frac{\partial^2 w}{\partial t^2} \qquad \Delta s \simeq \Delta x$$

$$S\ \ \theta + \delta\theta - S\theta - q\,\Delta x = \rho\,\Delta x\,\frac{\partial^2 w}{\partial t^2}$$

$$S\,\delta\theta - q\,\Delta x = \rho\, w_{tt} \qquad \theta \sim \tan\theta = \frac{\partial w}{\partial x}$$

$$\text{or} \qquad S w_{xx} - q = \rho\, w_{tt}$$

Look at $\rho = \text{const}$ $\quad q = 0 \quad$ with $S/\rho = c^2$

$$\therefore \quad c^2 w_{xx} - w_{tt} = 0$$

$$w = f(x - ct) + g(x + ct) \quad \text{is solution}$$

wave : $t \to t + \Delta t \qquad x \to c\Delta t + x \quad \to$ motion is translation of curve to right with speed c.

I.V.P. $\quad t = 0 \quad w(x,0) = F(x)$ $\quad$ Continuity is needed for $F(x)$

$\qquad\qquad w_t(x,0) = G(x)$ $\quad$ Continuity & piecewise cont deriv

d'Alembert Solution

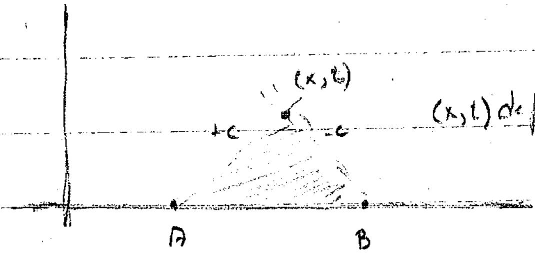
$$\left.\begin{aligned} F(x) &= f(x) + g(x) \\ G(x) &= -c(f' - g') \end{aligned}\right\} \quad f(x) - g(x) = -\frac{1}{c}\int^x G(s)\,ds$$

$$2f = F - \frac{1}{c}\int^x G(s)\,ds$$

$$2g = F + \frac{1}{c}\int^x G(s)\,ds$$

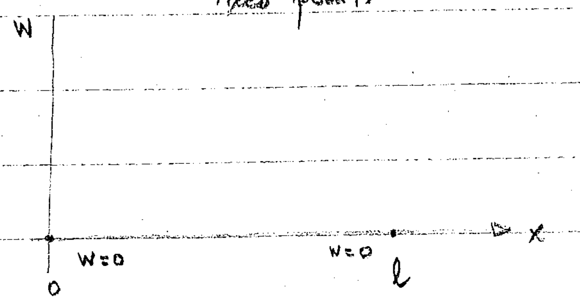
$$w(x,t) = \frac{1}{2}\left[F(x-ct) + G(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$$

<u>Problem</u>



(x,t) depends only on AB.

Fixed point.



look for sol of form $w(x,t) = X(x)T(t)$ Standing waves & separation of variables.

$$c^2 X''T - XT'' = 0$$

$$c^2\frac{X''}{X} = \frac{1}{T}T'' = -\nu^2 \Rightarrow T'' + \nu^2 T = 0 \quad \& \quad X'' + \frac{\nu^2}{c^2}X = 0$$

$$\boxed{\cos\nu t,\ \sin\nu t} \qquad \boxed{\sin\tfrac{\nu}{c}x,\ \cos\tfrac{\nu}{c}x}$$

$w(x,t)$ is assumed $\qquad T(0), T'(0)$ must be given.

These give rise to EV problems.

$$X'' + \frac{\nu^2}{c^2}X = 0 \qquad X(0) = X(l) = 0 \longrightarrow X^* = 0 \text{ trivial sol.}$$

$$X = A\cos\frac{\nu}{c}x + B\sin\frac{\nu}{c}x \qquad X(0) = 0 \Rightarrow X = B\sin\frac{\nu}{c}x$$

$$0 = A\sin\frac{\nu}{c}l \qquad \text{if } \frac{\nu}{c} = n\pi l$$

$$\therefore w(x,t) = B_n\sin\frac{n\pi x}{l} \qquad T = C_n\cos\frac{n\pi c}{l}t + D_n\sin\frac{n\pi c}{l}t$$

$$w\ \sum_n\left\{C_n\cos\frac{n\pi c}{l}t + D_n\sin\frac{n\pi c}{l}t\right\}\sin\frac{n\pi x}{l}$$

Impose I.C. $w(t=0) = F(x)$

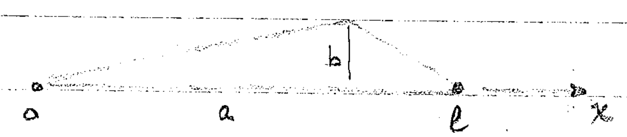
$w_t(t=0) = G(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx \qquad \begin{matrix} a_m \\ b_m \end{matrix} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \begin{pmatrix} \cos mx \\ \sin mx \end{pmatrix} dx$$

$f(x)$ is piecewise continuous & piecewise differentiable

Vibration of Harp String



$$f(x) = \begin{cases} \frac{b}{a} x & 0 \le x \le a \\ \frac{b}{(a-l)}(x-l) & a \le x \le l \end{cases}$$

$w_t = 0$ for $t = 0$ $\Rightarrow$ $G(s) = 0$ $\quad \therefore w(x,t) = \frac{1}{2}\{F(x+ct) + F(x-ct)\}$

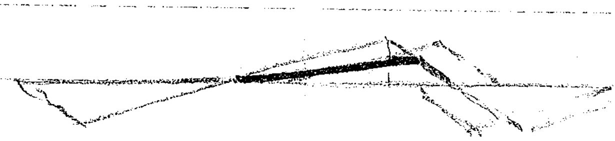
$w|_{t=0} = f(x)$

$$w(x,t) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi c}{l} t \, \sin \frac{n\pi x}{l}$$

$$w(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = f(x)$$

standing wave $\quad w = \dfrac{2bl^2}{\pi^2 a(l-a)} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi a}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l}$

progressive wave sol. $\quad = \dfrac{bl^2}{\pi^2 a(l-a)} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi a}{l} \left[\sin \frac{n\pi}{l}(x-ct) + \sin \frac{n\pi}{l}(x+ct) \right]$



$e^{imx} e^{i\omega t} = e^{i(mx \pm \omega t)}$

$$Re\left(e^{i[mx \pm \omega t]}\right) = \cos(mx \pm \omega t) = \cos m\left(x \pm \frac{\omega}{m} t\right)$$

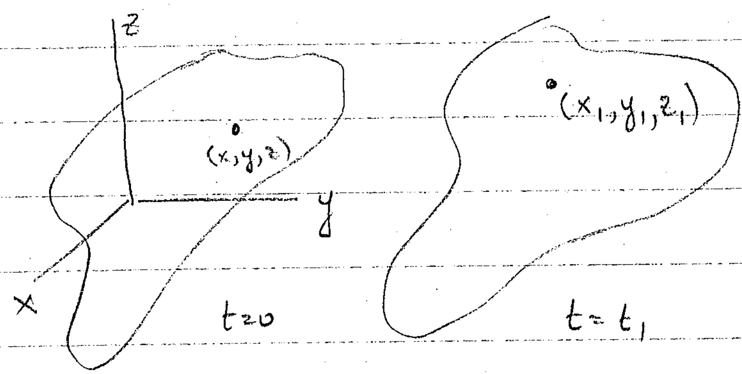
if t fixed wavelength $\lambda = \frac{2\pi}{m}$ prop speed $= \pm \omega/m$

m is wave number ω - circular freq freq $= \frac{\omega}{2\pi}$

$\nabla \cdot \underline{u} = 0$ incompressible $\iff$ $\rho =$ const for incompressible fluid

$\nabla \times \underline{u} = 0$ irrotational.

Continuum Mechanics



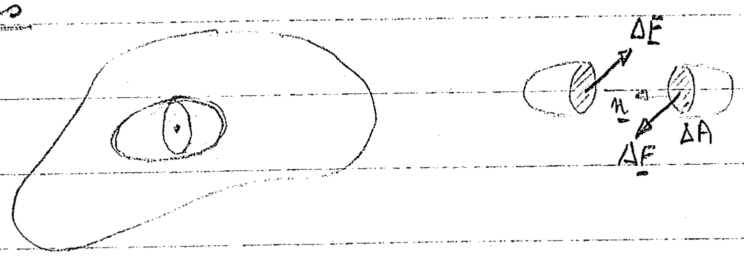
Assume topological mapping 1-1, continuous, differentiable to high order

Jacobian $\neq 0$

Assume curves remain curves in this mapping interior $\to$ interior

bdy $\to$ bdy.

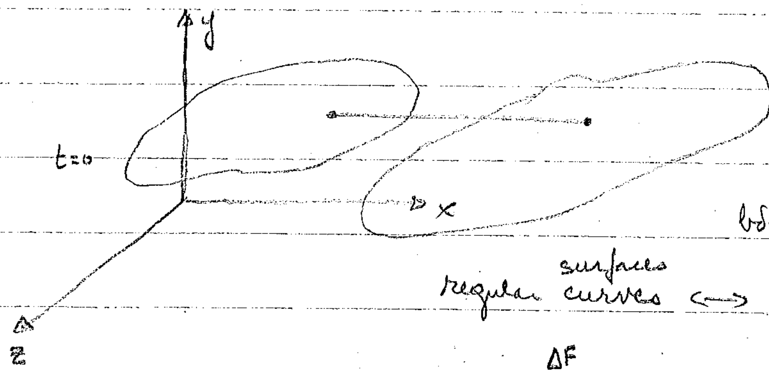
Stress



$$\lim_{\Delta A \to 0} \frac{\Delta \underline{F}}{\Delta A} = \underset{\sim}{\sigma}(x_1, y_1, z_1, t; \underline{n})$$ smooth, continuous

Perfect fluid: cannot support shear $\underset{\sim}{\sigma} = p\delta_{ij}$ $p(x_1, y_1, z_1, t)$

2-13-72

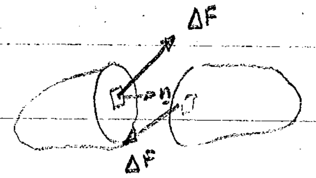


1-1 mapping inverse exists & differentiable

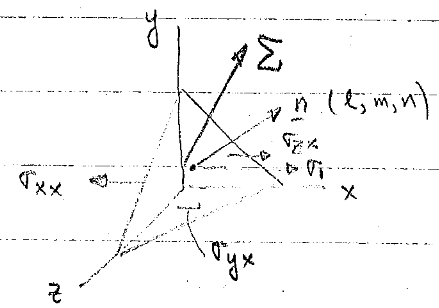
Jacobian $\delta/\beta > 0$

bdy pts $\Longleftrightarrow$ bdy pts   interior pts $\Longleftrightarrow$ interior pts

regular curves (surfaces) $\Longleftrightarrow$ regular curves (surfaces).

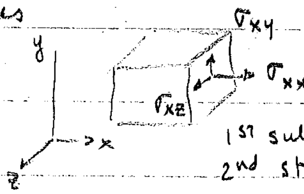


$$\sigma_{av} = \frac{\Delta F}{\Delta A} \quad ; \quad \sigma = (x, y, z, t, \underline{n})$$



$\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ direction cosines

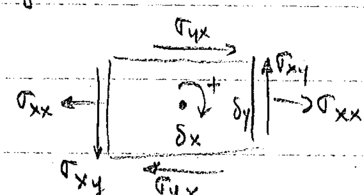
notation :



1st subscript acting face ⊥ to axis
2nd stress ∥ to this axis

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \sigma_{tensor}$$

Claim: symmetric tensor



$$\sigma_{yx}\,\delta x \delta z \frac{\delta y}{2} + \sigma_{yx}\,\delta x \delta z \frac{\delta y}{2} - \sigma_{xy}\,\delta z \delta y \frac{\delta x}{2} - \sigma_{xy}\,\delta z \delta y \frac{\delta x}{2} = I\alpha$$

$$\Rightarrow \quad I \text{ is } O(\delta^5) \quad \therefore \quad \sigma_{yx} = \sigma_{xy}$$

$$\sigma_1 \Delta A - l \Delta A\, \sigma_{xx} - m \Delta A\, \sigma_{yx} - n \Delta A\, \sigma_{zx} + \rho X \frac{\delta x \delta y \delta z}{6} = \rho V a(x)$$

divide by ΔA & let tetrahedron shrink to origin since

$$\sigma_1 = \sigma_{xx} l + \sigma_{yx} m + \sigma_{zx} n$$

$\vdots$

$$\Sigma = (\sigma)^T \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

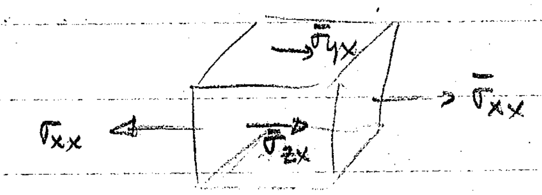
$\exists$ principal axis $\ni$ $\sigma = \begin{pmatrix} \sigma_{xx} & & 0 \\ & \sigma_{yy} & \\ 0 & & \sigma_{zz} \end{pmatrix}$

$\Rightarrow$ all shears are zero

in a perfect fluid no shear exists & all directions are principal axes hence σ is described by a single scalar called pressure

Equation of motion in $x \rightarrow$

$$\bar{F} = (X, Y, Z)$$



$\bar{\sigma}$ denotes $\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx$

$$(\bar{\sigma}_{xx} - \sigma_{xx})\,\delta z\,\delta y + (\bar{\sigma}_{xy} - \sigma_{xy})\,\delta z\,\delta x + (\bar{\sigma}_{xz} - \sigma_{xz})\,\delta y\,\delta x$$
$$+\rho X\,\delta x\,\delta y\,\delta z = \rho\,\delta x\,\delta y\,\delta z\; a(x)$$

to first order $\frac{\partial \sigma_{xx}}{\partial x}\,\delta x\,\delta y\,\delta z + \frac{\partial \sigma_{xy}}{\partial y}\,\delta x\,\delta y\,\delta z + \frac{\partial \sigma_{xz}}{\partial z}\,\delta x\,\delta y\,\delta z + \rho X\,\delta x\,\delta y\,\delta z$

$$= \rho\,\delta x\,\delta y\,\delta z\; a(x)$$

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,y} + \rho X = \rho\, a(x)$$

since we assume shears are zero & $\sigma_{xx} = -p$

$$-\frac{\partial p}{\partial x} + \rho X = \rho\, a(x)$$
$$-\frac{\partial p}{\partial y} + \rho Y = \rho\, a(y)$$
$$-\frac{\partial p}{\partial z} + \rho Z = \rho\, a(z)$$

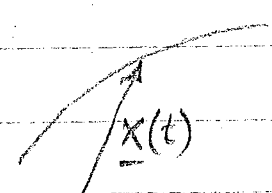
deformed position & differentiation is done with respect to deformed position } Lagrange form.

x, y, z are dependent variables since we really don't know the deformed position

$$-\frac{1}{\rho}\nabla p + \underline{F} = \underline{a}$$ In deformed position.

$F(x,y,z,t)$ what is $\frac{dF}{dt}$ along some curve

$x = x(t)$ $y = y(t)$ $z = z(t)$



$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}$$

$$\frac{d\underline{x}}{dt} = (\dot{x}, \dot{y}, \dot{z}) = (u, v, w) \qquad \frac{d}{dt}F = \left(\frac{\partial}{\partial t} + \underline{q}\cdot\nabla\right)F$$

if $\underline{x} = (x(t), y(t), z(t))$

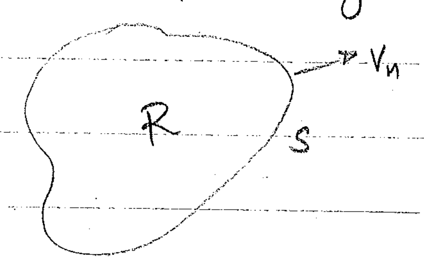
$\underline{q} = \frac{d\underline{x}}{dt} = (\dot{x}(t), \dot{y}(t), \dot{z}(t)) = (u, v, w)$ Euler fix & watch one pt in space

$\underline{a} = \frac{d\underline{q}}{dt}$

$$a_x = \frac{du}{dt} = u_t + uu_x + vu_y + wu_z$$

$$\left.\begin{aligned} u_{,t} + uu_{,x} + vu_{,y} + wu_{,z} &= X - \frac{1}{\rho}P_{,x} \\ v_{,t} + uv_{,x} + vv_{,y} + wv_{,z} &= Y - \frac{1}{\rho}P_{,y} \\ w_{,t} + uw_{,x} + vw_{,y} + ww_{,z} &= Z - \frac{1}{\rho}P_{,z} \end{aligned}\right\} \text{Conservation of Momentum}$$

Assume incompressible flow $\Rightarrow \Delta$ volume $= 0$; $\rho = $ const

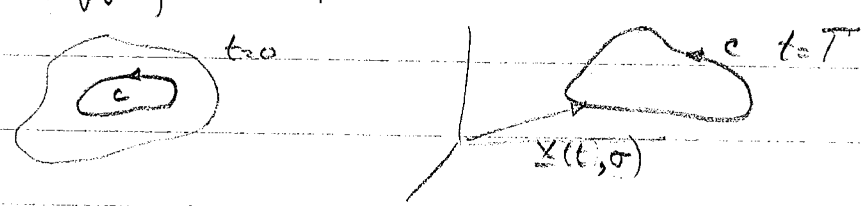


flow out of volume is $\iint v_n\,dS = \iiint \nabla\cdot\underline{V}\,d\tau$

if $V\cdot n = 0 \Rightarrow \nabla\cdot\underline{V} = 0 \Rightarrow u_x + v_y + w_z = 0$

$u_x + v_y + w_z = 0$ Conservation of mass

Another simplifying assumption



Circulation $\Gamma(t) = \oint_c u\,dx + v\,dy + w\,dz = \oint_c \underline{q}\cdot d\underline{x} = \oint \underline{q}\cdot x_\sigma\,d\sigma$

Helmholtz Law $\dot{\Gamma}(t) = 0$

For each time let s be arc length $\ni$, $0 < s \le l$

let $\sigma = s/l \qquad 0 \le \sigma \le 1$

then $\Gamma(t) = \int_0^1 \underline{q} \cdot \underline{x}_\sigma \, d\sigma$

$$\mathring{\Gamma}(t) = \int_0^1 \left(\mathring{\underline{q}} \cdot \underline{x}_\sigma + \underline{q} \cdot \mathring{\underline{x}}_\sigma \right) d\sigma$$

$\dot{\underline{x}}_\sigma = \dfrac{\partial^2 x}{\partial\sigma\partial t}$ by continuity $\dot{\underline{x}}_\sigma = \dfrac{\partial^2 \underline{x}}{\partial t \partial\sigma} = q_\sigma$

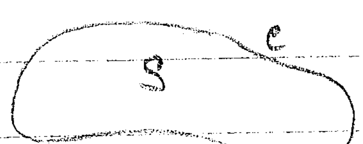
$$\mathring{\Gamma}(t) = \int_0^1 \mathring{\underline{q}} \cdot \underline{x}_\sigma + \underline{q} \cdot \mathring{\underline{q}}_\sigma \Big) d\sigma$$

$\mathring{q} = \underline{a}$ in Lagrangian $\quad \underline{a} = -\frac{1}{\rho}\nabla p + \bar{F} \qquad \bar{F} = (0, -g, 0)$ gravity

$$\mathring{\Gamma}(t) = \int_0^1 \left(-\frac{1}{\rho} \underline{x}_\sigma \cdot \nabla p - g \, \underline{x}_\sigma \cdot \nabla y + \underline{q} \cdot \underline{q}_\sigma \right) d\sigma$$

$$\int_0^1 \left[-\frac{1}{\rho} P_{,\sigma} - g \, y_{,\sigma} + \frac{1}{2} (\underline{q} \cdot \underline{q})_{,\sigma} \right] d\sigma = 0$$

since $\sigma = 0$, $\sigma = 1$ is same pt. $\Rightarrow \Gamma(t) = \text{const}$



$$\oint_C q_s \, ds = \Gamma = \iint_S (\nabla \times \underline{q}) \cdot n \, dA \quad \text{Stokes' Th}$$

if start from rest $\Gamma = 0$ if true for every closed curve

then $\nabla \times \underline{q} = 0$ irrotational $\Rightarrow q = \nabla \Phi \quad \Phi$ a scalar fn.

$$\Phi(x, y, z, t) = \int^{x,y,z} u \, dx + v \, dy + w \, dz$$

$$\Phi_x = u \quad \Phi_y = v \quad \Phi_z = w$$

$\Rightarrow$ continuity $\Rightarrow \Delta \Phi = 0$

$$\begin{cases} u_t + uu_x + vu_y + wu_z = -\frac{1}{\rho}P_x \\ v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho}P_y - g \\ \end{cases}$$

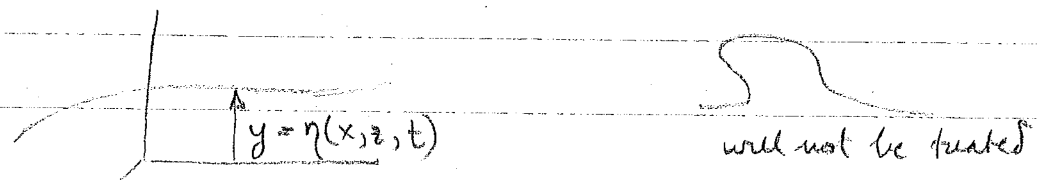
Can write the above as $\nabla\Phi_t + \frac{1}{2}\nabla(\nabla\Phi)^2 = -\nabla(P/\rho) - \nabla(gy)$

Bernoulli law: $\Phi_t + \frac{1}{2}(\Phi_x^2 + \Phi_y^2 + \Phi_z^2) = -p/\rho - gy + \frac{1}{\rho}C(t)$

$\Rightarrow$ p can now be found we can assume $C(t) = 0$

On fixed bdy $\underline{n}\cdot\nabla\Phi = 0$

Free surface



$$S:\ y - \eta(x,z,t) = 0$$

$$\frac{d}{dt}\left(\eta(x,z,t) - y\right) = \Phi_x\eta_x - \Phi_y + \Phi_z\eta_z + \eta_t = 0$$

Kinematic free surface condition.

If pressure prescribed on surface use Bernoulli law

$$\Phi_t + \frac{1}{2}(\nabla\Phi)^2 + g\eta = -p/\rho \quad \text{on } y = \eta(x,z,t)$$

Find a solution to $\Delta\Phi = 0$ with these two b.c. & two I.C.

I.C. initial position & velocity.

Assume $\Phi = \Phi_1\epsilon + \ldots$

$\eta = \epsilon\eta_1 + \ldots$

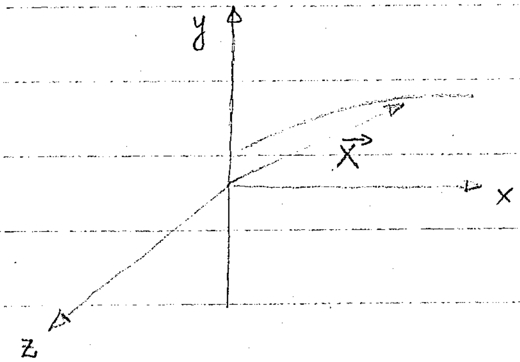
Let $\Phi = \Phi(x,y,z,t) = \Phi(x, \epsilon\eta_1(x,z,t) + \ldots, z, t)$ about some equil point.

$= \Phi(x,0,z,t) + \epsilon\Phi_y\big|_{\epsilon=0}$

for small first order case drop $\Phi_x\eta_x$, $\Phi_z\eta_z$

$\Rightarrow$ $\eta_t - \Phi_y = 0\ _{y=0}$ & $\Phi_t + g\eta = -p/\rho$ at $y = 0$

2-20-74



$\underline{q} = (u, v, w)$ velocity components

$p(x,y,z,t)$

$\underline{F} = (X, Y, Z) = (0, -g, 0)$

$\vec{X} = (x(t), y(t), z(t))$

$$\frac{d\vec{X}}{dt} = \left(\dot{x}(t), \dot{y}(t), \dot{z}(t)\right) = \underline{q}$$

$$\frac{dx}{dt} = u(x,y,z,t)$$

$$\frac{dy}{dt} = v(x,y,z,t)$$

Conserv of Mom

$$\frac{dz}{dt} = w(x,y,z,t)$$

$$\begin{cases} u_t + uu_x + vu_y + wu_z = \rho X - p_x \\ v_t + \qquad\qquad\qquad = \rho Y - g - p_y \\ w_t \qquad\qquad\qquad\quad = \rho Z - p_z \end{cases}$$

Conserv of mass $\quad \nabla \cdot \underline{q} = u_x + v_y + w_z = 0$

$$\Gamma(t) = \oint v_s \, ds \qquad \Phi(x,y,z,t) \qquad \underline{q} = \nabla\Phi = (\Phi_x, \Phi_y, \Phi_z)$$

$$\dot{\Gamma}(t) = 0 \qquad \nabla \times \underline{q} = 0 \quad \text{irrotational}$$

$$\Rightarrow \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0$$

Bernoulli Law $\quad \Phi_t + \frac{1}{2}\left(\Phi_x^2 + \Phi_y^2 + \Phi_z^2\right) + p/\rho + gy = 0$

Energy; $\quad E(t) = \rho \iiint_R \left[\frac{1}{2}\left(\Phi_x^2 + \Phi_y^2 + \Phi_z^2\right) + gy\right] dx\,dy\,dz$

$$= -\iiint_R (p + \rho\,\Phi_t)\, dx\,dy\,dz \qquad \text{no sources or sinks}$$

$$\frac{dE}{dt} = \frac{d}{dt}\iiint_R f(x,y,z,t)\,dx\,dy\,dz = \iiint_R f_t\,dx\,dy\,dz + \iint_S f\,\underline{q}\cdot\underline{n}\,dS$$

$$\frac{dE}{dt} = \rho\iiint_R (\Phi_x\Phi_{xt} + \Phi_y\Phi_{yt} + \Phi_z\Phi_{zt})\,dx\,dy\,dz - \iint_S (p+\rho\Phi_t)\,\underline{q}\cdot\underline{n}\,dS$$

$$\iiint_R \nabla f\cdot\nabla g\,dx\,dy\,dz = -\iiint_R f\,\nabla^2 g\,dx\,dy\,dz + \iint_S f\frac{\partial g}{\partial n}\,dS$$

$$\frac{dE}{dt} = \rho\iiint_R \nabla\Phi\cdot\nabla\Phi_t\,dx\,dy\,dz - \iint_S (p+\rho\Phi_t)\,\underline{q}\cdot\underline{n}\,dS$$

$$= \rho\iiint \Phi_t\frac{\partial\Phi}{\partial n}\,dS - \iiint_R \Phi_t\underbrace{\Delta\Phi}_{=0}\,dx\,dy\,dz - \iint_S (p+\rho\Phi_t)\,\underline{q}\cdot\underline{n}\,dS$$

$$\frac{dE}{dt} = \iint_S [\rho\Phi_t(\Phi_{,n} - V_n) - pV_n]\,dS \qquad V_n = \underline{q}\cdot\underline{n}$$

a) S a surface moving with fluid then $\dot{E} = -\iint pV_n\,dS$

b) " " " fixed in space & fluid moves along it $\dot{E} = 0$

c) " " " fixed in space but fluid moves through it. $\dot{E} = \rho\iint \Phi_t\Phi_n\,dS$

if $\Phi = \phi(x-ct, y, z)$ $\quad \xi = x-ct \quad \dot{E} = -\rho c\iint_S \Phi_\xi^2\,dS$

1. S fixed $\Phi_n = 0$

2. <u>free surface</u> <u>Kinematic condition</u> if $y = \eta(x,z,t)$ $\quad F = y - \eta(x,z,t) = 0$

particles in free surface stay in free surface.

$$\frac{dF}{dt} = \eta_t + u\eta_x - v + w\eta_z = 0$$

$$\eta_x\Phi_x - \Phi_y + \eta_z\Phi_z + \eta_t = 0 \quad \text{on } y = \eta$$

<u>dynamic free surface condition</u>: pressure prescribed on free surface given by Bernoulli's Law

$$\Phi_t + \tfrac{1}{2}(\Phi_x^2 + \Phi_y^2 + \Phi_z^2) + P/\rho + gy = 0 \quad \text{on } y = \eta$$

b.c. are not linear even though $\Delta\Phi=0$.

if $p=0$ $\eta=0$ satisfies the cond $\Rightarrow \Phi_y=0$ or $V_{vertical}=0$

$\Rightarrow \Phi=Ux \rightarrow$ Bernoulli' law $\Rightarrow \frac{1}{2}U^2$

we assume small perturbations, in some variable ε, about exact soln.

$\Phi(x,y,z,t,\varepsilon)$; $\eta(x,z,t,\varepsilon)$

$$\Phi = \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \dots \qquad \Phi_i = \Phi_i(x,y,z,t,0)$$

$$\eta = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \dots \qquad \eta_i = \eta_i(x,z,t,0)$$

$$p/\rho = p_0 + \varepsilon p_1 + \dots$$

$$\Delta\Phi = \varepsilon\Delta\Phi_1 + \dots \qquad \text{since } \Delta\Phi=0 \Rightarrow \Delta\Phi_i=0$$

$$\eta_x\Phi_x - \Phi_y + \eta_z\Phi_z + \eta_t = 0 \Rightarrow$$

$$(\varepsilon\eta_{1x}+\varepsilon^2\eta_{2x}+\dots)(\varepsilon\Phi_{1x}+\varepsilon^2\Phi_{2x}+\dots) - (\varepsilon\Phi_{1y}+\varepsilon\Phi_{2y}+\dots)$$
$$+(\varepsilon\eta_{1z}+\varepsilon^2\eta_{2z}+\dots)(\varepsilon\Phi_{1z}+\varepsilon^2\Phi_{2z}+\dots)+(\varepsilon\eta_{1t}+\dots)=0$$

Look at ε terms

$$-\Phi_{1y}+\eta_{1t}=0$$

If we assume $y=y_0$ plane & η measured from it

$\Rightarrow \eta(x,y_0,z,t) \Rightarrow$ from bernoulli'

$\frac{p_0}{\rho} + gy_0 + \eta(x,y_0,z,t)+\dots$ or $p_0 = -\rho g y_0$ hydrostatic pressu

$$\left.\begin{aligned} p_1/\rho + \Phi_{1t} + g\eta_1 &= 0 \\ -\Phi_{1y} + \eta_{1t} &= 0 \\ \Delta\Phi_1 &= 0 \end{aligned}\right\}$$

2-D $\quad \Delta\Phi = 0 \qquad 0 > y > -\infty \qquad |x| < \infty$

$p = 0$ on bdy

$$\left.\begin{aligned} \Phi_t + g\eta &= 0 \\ -\Phi_y + \eta_t &= 0 \end{aligned}\right\} y = 0$$

Bddness at ∞

$\Phi(x,y,t) = e^{i\sigma t}\varphi(x,y)$ $\qquad \Phi \equiv 0,\ \eta \equiv 0$ are soln.

diff 1st by t
$$\left.\begin{aligned} \Phi_{tt} + g\eta_t &= 0 \\ -g\Phi_y + g\eta_t &= 0 \end{aligned}\right\} \qquad \Phi_{tt} + g\Phi_y = 0$$

$(-\sigma^2\varphi + g\varphi_y)e^{i\sigma t} = 0 \qquad \Delta\Phi = \Delta\varphi = 0$

let $\frac{\sigma^2}{g} = m \qquad \varphi_y - m\varphi = 0 \quad$ for $y = 0$

φ & φ_y are bdd at infinity.

introduce $\psi(x,y) = \varphi_y - m\varphi$ since φ is harmonic $\Rightarrow \Delta\psi = 0$ with $\psi = 0$ at $y = 0$

ψ is bdd at ∞ use reflection principle since ψ is harmonic & bdd in whole plane $\psi = \text{const} = 0$ by bdy cond $\therefore \varphi_y - m\varphi = 0$ everywhere

$\varphi = c(x)e^{my} \qquad \Delta\varphi = 0 \Longrightarrow c_{xx} + m^2 c = 0$ or $c = e^{\pm ixm}$

$c(x) = a\cos mx + b\sin mx$

$$\Phi(x,y,t) = \begin{cases} \cos mx & \cos\sigma t \\ \sin mx & \sin\sigma t \end{cases}$$

Standing waves

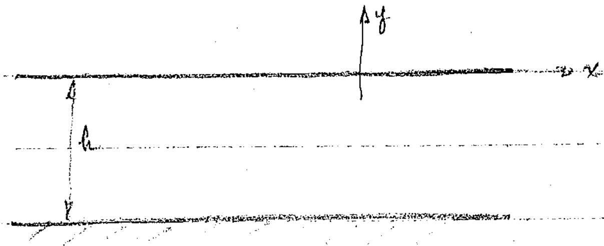
$$\Phi(x,y,t) = A\cos mx\cos\sigma t\, e^{my} \qquad \lambda = \frac{2\pi}{m}$$

Progressing waves. $\Phi(x,y,t) = \cos(mx \pm \sigma t)e^{my}$

$m(x \pm \frac{\sigma}{m}t)$ $\qquad$ phase speed ~~propagating~~ speed $= c = \frac{\sigma}{m} = \frac{\sigma\lambda}{2\pi}$

$\sigma^2 = gm = \frac{2\pi g}{\lambda}$; c: speed phase $\qquad T = \frac{2\pi}{\sigma} \qquad c = \sqrt{\frac{g\lambda}{2\pi}}$

dispersion



$\Delta\varphi = 0$

$\varphi_y - m\varphi = 0 \quad y = 0 \qquad \varphi = 0$ is a sol.

$\varphi_y = 0 \quad y = -h$

$$\varphi = \cosh m(y+h)\begin{cases}\cos mx\\ \sin mx\end{cases}$$

$$\varphi_y = m \sinh m(y+h)\{ \qquad \text{at } y=-h \;= 0$$

$$\varphi_y - m\varphi = m \sinh mh - m\cosh mh = 0$$

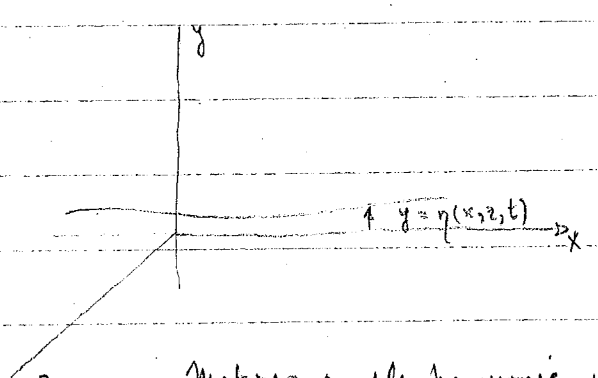
$$\Longrightarrow \tanh mh = 1 \quad \Longrightarrow$$

$$\sigma^2 = gm \tanh mh = g\frac{2\pi}{\lambda}\left[\frac{2\pi h}{\lambda} - \frac{1}{3}\left(\frac{2\pi h}{\lambda}\right)^3 + \cdots\right]$$

$$\text{let } \frac{h}{\lambda} \to 0 \qquad \sigma^2 \to \left(\frac{2\pi}{\lambda}\right)^2 gh = m^2 gh$$

$$c = \frac{\sigma}{m} \to \sqrt{gh}$$

2-27-74



Flow governed by $\Phi(x,y,z,t)$ $\quad (u,v,w) = (\Phi_x, \Phi_y, \Phi_z)$

flow perfect, irrotational $\quad \nabla^2\Phi = 0$

B.C. $\left.\begin{aligned} -\Phi_y + \eta_t &= 0 \\ \Phi_t + g\eta + p/\rho &= 0 \end{aligned}\right\} y = 0$

Motions simple harmonic in time studied

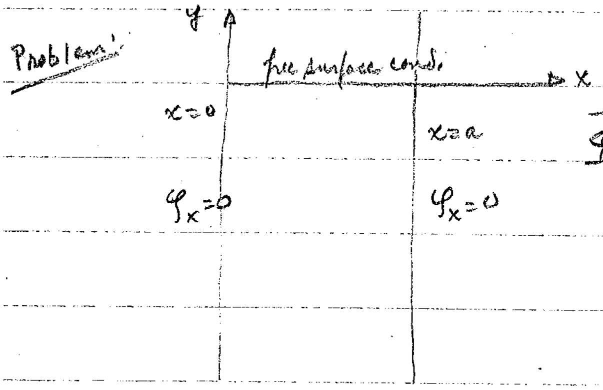
Consider $\Phi = e^{i\sigma t}\varphi(x,y)$ ansatz.

solutions $\varphi(x,y) = e^{my}\begin{cases}\cos mx\\ \sin mx\end{cases} \quad \to \quad \Phi = A e^{my}\cos(mx \pm \sigma t + \alpha)$

+ sign $\qquad$ − sign

$$\Phi = \Phi(x \pm ct, y)$$

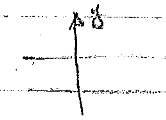
Problem:



$$\Phi = e^{i\sigma t}\varphi(x,y) \quad \text{point spectrum}$$

$$\nabla^2\varphi = 0 \; : \; \varphi = g(y)\cos\frac{n\pi x}{a} + \text{free body cond.}$$

Now consider 2-D case uniform but finite height h –



$$\varphi = \cosh(y+h)m \begin{cases} \cos mx \\ \sin mx \end{cases} \qquad \varphi_y = 0 \text{ at } -h$$

for free surface cond $\sigma^2 = gm \tanh mh \qquad m = \frac{2\pi}{\lambda}$

Soln $\varphi = \cosh m(y+h)\cos(mx - \sigma t + \alpha)$

$$c = \frac{\sigma}{m} \qquad \sigma^2 = \frac{2\pi g}{\lambda}\left[\frac{2\pi h}{\lambda} - \frac{1}{3}\left(\frac{2\pi h}{\lambda}\right)^3 \cdots\right.$$

$$\sigma^2 = \left(\frac{2\pi}{\lambda}\right)^2 gh = m^2 gh \text{ as } \frac{h}{\lambda} \to 0 \qquad c \to \sqrt{gh}$$

δx, δy

$$\frac{d}{dt}\delta x = \Phi_x = -mA\cos\sigma t\cosh(m+h)\sin mx$$

$$\frac{d\delta y}{dt} = \Phi_y = mA\cos\sigma t\tanh(m+h)\cos mx$$

$$\delta x = -\frac{mA}{\sigma}\sin\sigma t\cosh m(y+h)\sin mx$$

$$\delta y = \frac{mA}{\sigma}\sin\sigma t\sinh m(y+h)\cos mx$$

particle path are ellipses

$$\Phi = A\cosh m(y+h)\cos(mx \pm \sigma t + \alpha) \qquad \frac{\delta x^2}{a^2} + \frac{\delta y^2}{b^2} = 1$$

Assume $\Phi = e^{i\sigma t}\varphi(x,y,z) = e^{i\sigma t}\varphi(r,y)$ not dependent on θ

$$\nabla^2\varphi(r,y) = \frac{1}{r}\partial_r(r\partial_r\varphi) + \partial_y^2\varphi = 0 \qquad -\infty < y < 0 \quad 0 \le r < \infty$$

$\varphi_y - m\varphi = 0$ at $y = 0$

Look for $\varphi = e^{my} g(r)$

$\Rightarrow \quad \frac{1}{r}\partial_r(r\,\partial_r g) + m^2 g = 0$

Let $\rho = \log r \quad \Rightarrow \quad \nabla^2\varphi = e^{-2\rho}\dfrac{\partial^2\varphi}{\partial\rho^2} + \dfrac{\partial^2\varphi}{\partial y^2} = 0$

Let $\psi = \varphi_y - my \qquad \psi$ satisfies $\nabla^2\psi = 0$

$g = J_0(mr)\ ,\ Y_0(mr)$

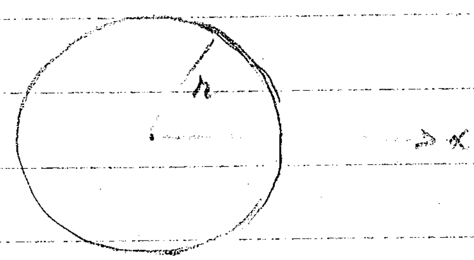
Entire fn. log singularity

Asymptotic large values of r $\quad J_0(mr) \simeq \sqrt{\dfrac{4}{\pi m r}}\cos\left(mr + \dfrac{\pi}{4}\right)$

$Y_0(mr) \simeq \sqrt{\dfrac{4}{\pi m r}}\sin\left(mr - \dfrac{\pi}{4}\right)$

take $H_0 = J_0 + iY_0$

$$\Phi(r, y, t) = Ae^{my}e^{-i\sigma t}\sqrt{\frac{4}{\pi m r}}\,e^{i(mr - \pi/4)}$$
$$= Ae^{my}\frac{2}{\sqrt{\pi m r}}\,e^{i(mr - \sigma t)}\,e^{-i\pi/4}$$

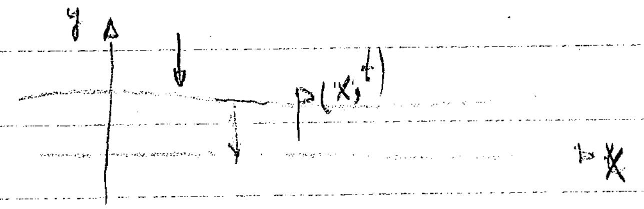


energy flux out of cylinder $\int$ for 1 period

$$F = \int_0^T \left\{ \int_0^{2\pi} \Phi_t\, \Phi_r\, r\, d\theta \right\} dt$$

$\Phi_t \Phi_r \cdot r$ are indep of r to first order approx

& hence $F \neq F(r) \neq 0 \qquad \Phi_t, \Phi_r$ have opp signs & hence $F < 0$



$\Phi(x, y, t) = 0$

$g\eta = -\Phi_t - P/\rho \qquad y = 0$

$$\eta_t = \Phi_y \qquad y=0$$

$$p(x,t) = \bar{p}(x)\sin\sigma t$$

$$\Phi = \varphi(x,y)\cos\sigma t + \gamma(x,y)\sin\sigma t$$

$$g\eta_t = -\Phi_{tt} - \frac{p_t}{\rho}$$

$$g\eta_t = \Phi_y g$$

$$g\Phi_y + \Phi_{tt} + p_t/\rho = 0$$

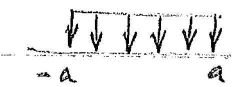
$$y=0 \begin{cases} \varphi_y - m\varphi = -\dfrac{\sigma}{\rho g}\bar{p}(x) \\ \gamma_y - m\gamma = 0 \end{cases}$$

γ satisfies previous cond. since harmonic

$$\therefore \quad \gamma(x,y) = Ae^{my}\begin{cases}\cos mx \\ \sin mx\end{cases}$$

we will assume $\bar{p}(x) = \text{const. } P \quad |x| < a$

$\qquad\qquad\qquad\qquad\quad 0 \qquad |x| > a$



$$\varphi_y - m\varphi = \begin{cases} c & |x| < a \\ 0 & |x| > a \end{cases} \qquad m = \sigma^2/g \qquad c = -\frac{P\sigma}{\rho g}$$

Boundary at ∞ $\quad \varphi, \varphi_x, \varphi_y$ bdd.

at $\pm a$ $\quad p$ is discont. we assume η is cont there $\Rightarrow \Phi_t$ must be discont; we assume $\varphi_x, \varphi_y \sim \dfrac{1}{\rho^{1-\varepsilon}} \quad \varepsilon > 0$

Let $x_1 = mx \quad y_1 = my \quad a_1 = ma \quad c_1 = \frac{c}{m}$

$$\Rightarrow \quad \nabla^2_{x_1,y_1}\varphi = 0 \qquad \varphi_{y_1} - \varphi = \begin{cases} c_1 & |x_1| \le a_1 \\ 0 & |x_1| > a_1 \end{cases}$$

drop subscript $\quad \nabla^2\varphi = 0 \quad \varphi_y - \varphi = \begin{cases} c & |x| \le a \\ 0 & |x| > a \end{cases}$

$\varphi(x,y)$ $\exists$ an anal $\psi(x,y)$ $\ni$. $\underline{\varphi + i\psi = f(z)}$

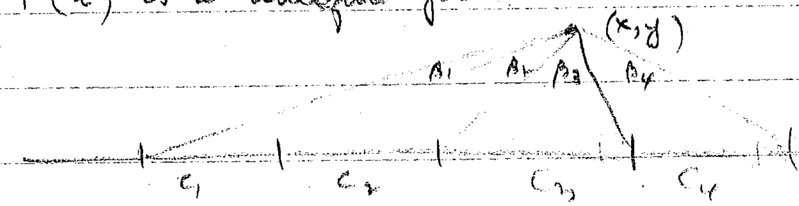
$\frac{df}{dz} = \varphi_x + i\psi_x = \varphi_x - i\varphi_y$ velocity: (φ_x, φ_y) $\varphi_x + i\varphi_y$ = complex velo.

$$(\varphi_y - \varphi) = \left(\frac{\partial}{\partial y} - 1\right)\varphi = Re\left(\frac{\partial}{\partial y} - 1\right)(\varphi + i\psi)$$
$$= Re\left(\frac{\partial}{\partial y} - 1\right) f(z) = Re\left(i\frac{df}{dz} - f\right)$$

$$\varphi_y - \varphi = \begin{cases} c \\ 0 \end{cases} \rightarrow Re\,(i f' - f) = \begin{cases} c \\ 0 \end{cases} \text{ on real axis}$$

Let $F(z) = if' - f$. for uniqueness let $G_1 - G_2 = G$

$Re\, G$ is real on real axis continue it by Schwartz Reflection at $\pm a$ we define the fn in a nbd about the pts $\Rightarrow$ since sing is less than pole it is a removable singularity $\Rightarrow$ that means G is harmonic & is bdd in entire plane hence $G = $ Const (real const) since zero on real axis $\Rightarrow G = 0$ $\therefore G_1 = G_2$ hence $F(z)$ is a unique fn.



$= \arctan \frac{y}{x-a}$ $\varphi = \frac{1}{\pi}\left(c_1\beta_1 + c_2\beta_2 + c_3\beta_3 + c_4\beta_4\right)$

$$F(z) = \frac{ic}{\pi}\log\frac{z-a}{z+a} = \frac{ic}{\pi}\left(\log r_1 + i\theta_1 - \log r_2 - i\theta_2\right)$$

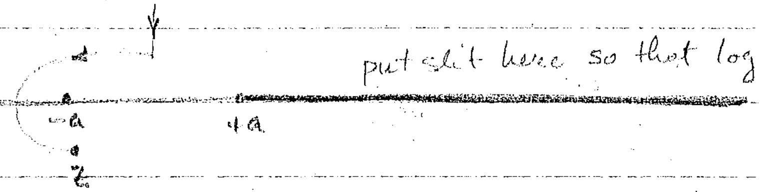
$$= -\frac{c}{\pi}(\theta_1 - \theta_2) + \text{Imag}.$$

$$i f' - f = \frac{ic}{\pi} \log \frac{z-a}{z+a}$$

$$i f' - f = 0 \qquad f = e^{-iz} = e^{y} e^{-ix} \quad \text{since } y < 0 \text{ O.K.}$$

sol to $|x| > a$

$$f(z) = \frac{c}{\pi} e^{-iz} \int_{z_0}^{z} e^{it} \log \frac{t-a}{t+a}\, dt + A e^{-iz}$$

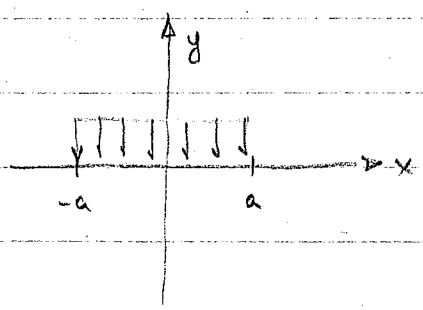


Integrate by parts

$$f(z) = \frac{ci}{\pi} \left[- \underbrace{\log \frac{z-a}{z+a}} + e^{-iz} \int_{i\infty}^{z} e^{it} \left(\frac{1}{t-a} - \frac{1}{t+a} \right) dt \right]$$

$$\log \frac{1 - a/z}{1 + a/z} = \log \left| 1 - \frac{2a}{z} + \ldots \right.$$

3-6-74



$$\Phi(x,y,t) = \varphi(x,y) e^{i\sigma t}$$

$$p(x,t) = \begin{cases} P \sin \sigma t & |x| \le a \\ 0 & |x| > a \end{cases} \qquad P = \text{const}$$

Introduce $f(z) = \varphi(x,y) + i\gamma(x,y) \qquad z = x + iy$

B.C. $\varphi_y - m\varphi = -\frac{\sigma}{\rho g} P, \; y = 0 \;\; |x| \le a \qquad \gamma_y - m\gamma = 0$

$\qquad\qquad = 0 \qquad\qquad |x| > a \qquad m = \sigma^2/g$

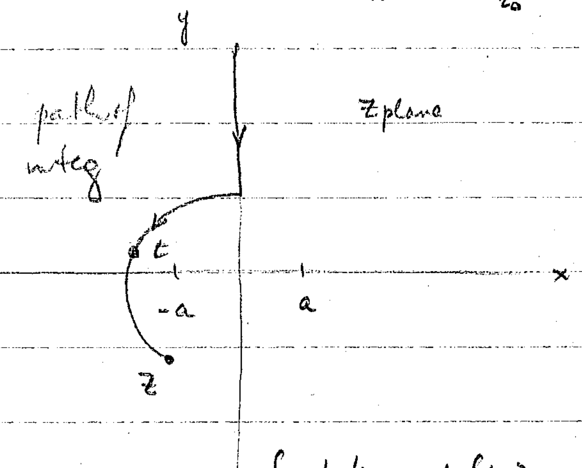
$$\text{Re}\left(i \frac{df}{dz} - f \right) = \begin{cases} c, & |x| \le a \\ 0 & |x| > a \end{cases}$$

$x_1 = mx, \; y_1 = my \quad a_1 = ma \quad c_1 = \frac{c}{m}$

Let $F(z) = i \frac{df}{dz} - f$

$$F(z) = i f'(z) - f = \frac{ic}{\pi}\log\frac{z-a}{z+a}$$

$$f(z) = Ce^{-iz} + \frac{c}{\pi}e^{-iz}\int_{z_0}^{z} e^{it}\log\frac{t-a}{t+a}\,dt$$



first term of $f(z)$ gives $e^{y}\{A\cos x + B\sin x$

Look at last integral

$$e^{-iz}\int_{z_0}^{z} e^{it}\log\frac{t-a}{t+a}\,dt = \frac{-cie^{-iz}}{\pi}\int_{z_0}^{z} e^{it}\left(\frac{1}{t-a} - \frac{1}{t+a}\right)dt + \frac{c}{\pi}\log\frac{(z-a)}{(z+a)}$$

last part $\to 0$ as $z \to \infty$

$$I(z) = e^{-iz}\int_{i\infty}^{z}\frac{e^{it}}{t\pm a}\,dt = \frac{-i}{z\pm a} - i\int_{i\infty}^{z}\frac{e^{i(t-z)}}{(t\pm a)^2}\,dt$$

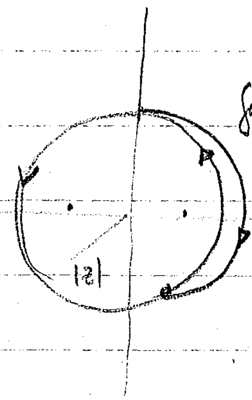
$$\frac{1}{t^2(1\pm \frac{a}{t})^2} = \frac{1}{t^2}\left[1 \mp \frac{2a}{t} + \dots\right]$$ $t-z$ is positive

Re part is negative since $y-y_t<$

$\therefore$ $e^{i(t-z)}$ is decreasing

$$\therefore\quad I(z) \le \frac{1}{|z\pm a|} + \int_{\pi|z|}^{\infty}\frac{|dt|}{|t|^2} \sim \frac{1}{|z|}$$

for $\operatorname{Re} z > 0$



for z in $\exists\, t$ s.t. $\operatorname{Re} z - t > 0$ we close off the circle
rt half plane and retrace it $\Rightarrow$ from closed circle we get
residue from $-a, a$ and from we get the
same thing as before.
hence we get from residue

$$2\pi i\, e^{-iz}\left(e^{ia} - e^{-ia}\right)$$
$$= e^{-iz} \sin a$$

z

$$f(z) \sim \begin{cases} \frac{1}{z} & \operatorname{Re} z < 0 \\ -4ci \sin a\, e^{-iz} + O\left(\frac{1}{z}\right) & \operatorname{Re} z > 0 \end{cases}$$

$$\varphi(x,y) \sim \operatorname{Re} f(z) \sim \begin{cases} 1/r, & x < 0 \\ -4c \sin a\, e^{y} \sin x + O\left(\frac{1}{r}\right) & x > 0 \end{cases}$$

$$\Phi(x,y,t) = \left(\varphi(x,y) + A e^{y} \sin x + B e^{y} \cos x\right) \cos \sigma t + O\left(\frac{1}{r}\right) \quad x < 0$$

we had assumed $\Phi(x,y,t) = \psi \sin \sigma t + \chi \cos \sigma t$

$$\Phi(x,y,t) = \left(C e^{y} \sin x + D e^{y} \cos x\right) \sin \sigma t$$

we want at $-\infty$:

$$\Phi \sim e^{y}\left[H \sin(x+\sigma t) + K \cos(x+\sigma t)\right] + O\left(\frac{1}{r}\right)$$

at $+\infty$:

$$\Phi = e^{y}\left[L \sin(x-\sigma t) + M \cos(x-\sigma t)\right] + \varphi \cos \sigma t$$

4 eq & 4 unknowns $\Rightarrow$ $L = -H \qquad L = H - 4c \sin a$

$M = K \qquad M = -K \Rightarrow 0 = M = K$

$L = -2c \sin a \qquad H = 2c \sin a.$

for right hand side alone because of symm about y axis

$$\Phi = \frac{2P\sigma}{\rho g m} \sin ma \, e^{my} \sin(mx - \sigma t) + O\left(\frac{1}{r}\right) \qquad x \geq 0$$

at ∞ $\quad m = \frac{2\pi}{\lambda}$; if $\sin ma = 0$ $\quad ma = n\pi$ then $2a$ = interval of the pressure application $\quad \therefore \quad 2a = n\lambda_\infty$

oscillatory pressure does no net work on the fluid since flux is $\int \Phi \Phi_t \, dx \, dt$

$$= \int O\left(\frac{1}{r}\right) O\left(\frac{1}{r}\right) dx \sim O\left(\frac{1}{r}\right) \to 0 \text{ as } r \to \infty$$

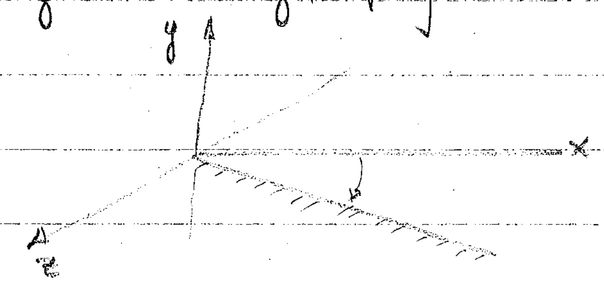
$\therefore$ no flux at ∞ for 1 period

let $2Pa = 1$ let $P\uparrow$ $a\downarrow$ $\Rightarrow$ we get delta fn in limit $\delta(x) = 1$

Exercise show that if $p(x,t) = \delta(x) \sin \sigma t$

$$f(z) = \frac{c}{\pi} e^{-iz} \int^{z} \frac{e^{it}}{t} \, dt$$

Progressing waves along sloping beach



Breaking of waves absorbs energy of incoming waves.

$P(x,t) = p(x) \sin \sigma t \qquad t > 0$

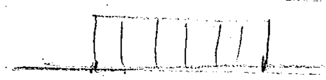
$\Phi(x,y,0) = 0$ $\qquad$ since Φ harmonic $\Rightarrow \eta$ is harmonic

$\Rightarrow p$ is harmonic

$$\left.\begin{aligned} -\Phi_y + \eta_t &= 0 \\ \Phi_t + g\eta &= -p/\rho \end{aligned}\right\} y = 0$$

since we assume ρ = const

$$g\Phi_y + \Phi_{tt} = \frac{-i\omega}{\rho_0} p(x) e^{i\omega t} \qquad t>0$$



$$\Delta \Phi = 0$$

$$\Phi = \Phi_t = 0 \text{ at } t=0$$

$\Phi_t + g\eta = 0$ at $y=0 \Rightarrow \eta = 0$ which is O.K.

Fourier Integral Th. $f(x)$: piecewise cont 1st deriv w/ finite discont.

$$-\infty < x < \infty \qquad f(x) = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^\infty f(\eta)\cos\alpha(\eta - x)\,d\eta$$

$$\& \quad \int_{-\infty}^\infty |f(x)|\,dx < \infty$$

$$f(x) = \frac{1}{\pi}\lim_{\xi\to\infty}\int_{-\infty}^\infty f(\eta)\,d\eta \int_0^\xi \cos s(\eta - x)\,ds$$

$$\int_{-\xi}^\xi \cos s(\eta - x)\,ds = 2\int_0^\xi$$

$$\int_{-\xi}^\xi \sin s(\eta - x)\,ds = 0$$

$$\int_0^\xi \cos s(\eta - x)\,ds = \frac{1}{2}\int_{-\xi}^\xi e^{is(x-\eta)}\,ds$$

$$\therefore \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{isx}\,ds \int_{-\infty}^\infty f(\eta) e^{-i\eta s}\,d\eta$$

$$\bar{f}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(\eta) e^{-i\eta s}\,d\eta$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \bar{f}(s) e^{isx}\,ds$$

$$\overline{\frac{d^n f}{dx^n}} = \frac{1}{\sqrt{2\pi}} \left. \frac{d^{n-1}}{dx^{n-1}} f \, e^{-is} \right|_{-\infty}^{\infty} + is \int_{-\infty}^{\infty} f^{(n-1)} e^{-isx} \, dx$$

we assume derivs are cont. we assume $f^{(n-1)} \to 0$ at $x \to \pm\infty$

then

$$\overline{\frac{d^n f}{dx^n}} = (-is)^n \bar{f}$$

$$\Phi_{xx} + \Phi_{yy} = 0$$

$$\left[(-is)^2 + \frac{d^2}{dy^2}\right] \bar{\Phi} = 0 \qquad \bar{\Phi} = a_{\pm}(s,t) e^{\pm sy} \quad \text{from bddness of } \bar{\Phi}$$

for $y \to -\infty \implies \bar{\Phi} = a_+(s,t) e^{+sy}$

3-13-74

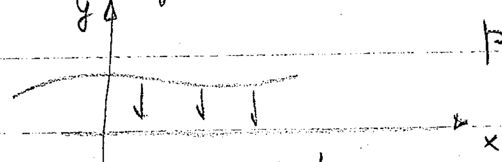
Fourier transform

$$\bar{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\eta) e^{-i\eta s} \, d\eta \qquad \int_{-\infty}^{\infty} |f(\eta)| \, d\eta < \infty$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(s) e^{+ixs} \, ds \qquad \text{Courant-Hilbert Vol 2}$$

$$\overline{\frac{d^n f}{dx^n}} = (is)^n \bar{f} \quad \text{write all deriv} \to 0 \text{ as } |x| \to \infty$$

Unsteady Motion



$p(x,t) = p(x) e^{i\omega t}$ at $y = 0$

veloc potential $\Phi(x,y,t)$: $\Delta \Phi = 0$ $y < 0$

with free surface cond. $g\Phi_y + \Phi_{tt} = -\frac{i\omega}{\rho} p(x) e^{i\omega t}$ $t > 0, y = 0$

apply fourier transf on $\Delta \Phi = 0 \implies -s^2 \bar{\Phi} + \bar{\Phi}_{yy} = 0$

$\bar{\Phi} = b(s,t) e^{-sy} + a(s,t) e^{sy}$ bddness cond $\Rightarrow b(s,t) = 0$

$\bar{\Phi} = a(s,t) e^{sy}$

put into b.c. $A_{tt} + gsA = -\frac{i\omega}{\rho} \bar{p}(s) e^{i\omega t}$ for $y = 0$

$$\bar{p} = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} c\, e^{-isx}\, ds = \frac{ci}{\sqrt{2\pi}\, s}\left[e^{-isa} - e^{isa}\right]$$

$$= \frac{ci}{\sqrt{2\pi}\, s}\left[-2i \sin sa\right] = \frac{2aci}{\sqrt{2\pi}}\left[\frac{-2i\sin sa}{2as}\right]$$

let $a \to 0$ & $2ac = 1$

$$\therefore \lim_{a\to 0} \bar{p} = \frac{1}{\sqrt{2\pi}}$$

$\therefore \quad A_{tt} + gsA = -\dfrac{i\omega}{\rho}\dfrac{1}{\sqrt{2\pi}} e^{i\omega t}$; we have also that fluid at rest at $t=0$ $\quad \Phi = \Phi_t = 0 \Rightarrow A, A_t = 0$ at $t=0$

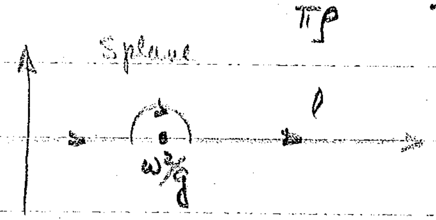
$A = \kappa(t)\cos\sqrt{gs}\,t + \lambda(t)\sin\sqrt{gs}\,t \qquad$ but $\Phi = 0$ at $t = 0 \Rightarrow$

$$A(s,t) = -\frac{1}{\sqrt{2\pi}}\frac{i\omega}{\rho}\int_0^t \frac{e^{i\omega(t-\tau)}}{\sqrt{gs}} \sin\sqrt{gs}\,\tau \, d\tau$$

$$\Phi(x,y,t) = -\frac{i\omega}{\rho\pi}\int_0^\infty e^{sy}\cos sx \int_0^t \frac{e^{i\omega(t-\tau)}}{\sqrt{gs}}\sin\sqrt{gs}\,\tau\, d\tau\, ds$$

let $\sin\sqrt{gs}\,\tau = \dfrac{e^{i\sqrt{gs}\,\tau} - e^{-i\sqrt{gs}\,\tau}}{2i}$

$$= -\frac{i\omega e^{i\omega t}}{\pi\rho}\int_0^\infty e^{sy}\cos sx \cdot \left\{ -\frac{1}{2\sqrt{gs}}\frac{e^{i(\sqrt{gs}-\omega)t}}{\sqrt{gs}-\omega} - \frac{1}{2\sqrt{gs}}\frac{e^{-i(\sqrt{gs}+\omega)t}}{\sqrt{gs}+\omega} + \frac{1}{gs-\omega^2}\right\} ds$$

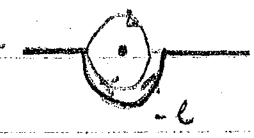


$\operatorname{Re}\, i(\sqrt{gs} \pm \omega) < 0$ (make branch cut along $-s$ axis) $\arg\sqrt{gs} < \frac{\pi}{2}$

we get no contrib from the exponentials at $t \to \infty$

$$\therefore \underset{t\ \text{large}}{\Phi(x,y,t)} \sim \frac{i\omega}{\pi\rho} e^{i\omega t}\int_\ell \frac{e^{sy}\cos sx}{gs-\omega^2}\, ds \qquad \text{Lamb}$$

want $x \to \infty$ let $\cos sx = \frac{e^{isx} + e^{-isx}}{2}$

& add & subtract an arc 

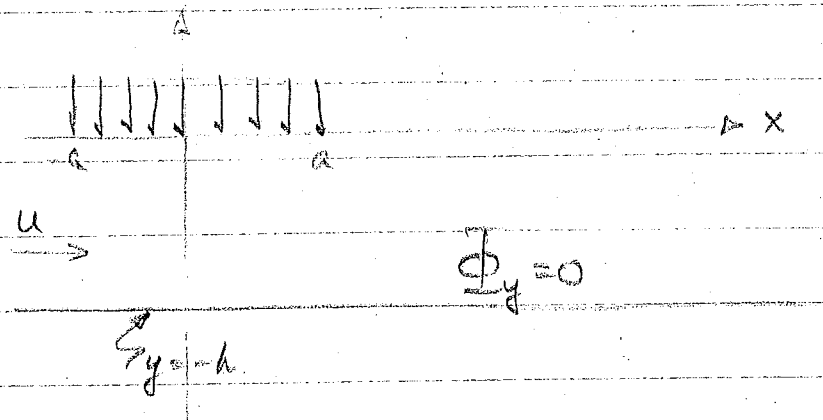
$$\Phi \sim \frac{\omega}{\rho} e^{i\omega t} \left[\int_{-\ell} \left(\frac{e^{sy} e^{isx}}{2(gs-\omega^2)} + \frac{e^{sy} e^{-isx}}{2(gs-\omega^2)} \right) ds \right]$$

$$\sim \frac{\omega}{\rho} e^{i\omega t} \left[\frac{1}{2\pi i} \oint \frac{e^{sy} e^{isx}}{gs-\omega^2} + \frac{1}{2\pi i} \oint \frac{e^{sy} e^{-isx}}{gs-\omega^2} \right] ds$$

$\sim$ $= 0$ since analytic in upper half plane; gives contrib $\frac{\omega i}{g\rho} e^{\frac{\omega^2}{g} y} \cdot e^{-i\left(\frac{\omega^2}{g} x - \omega t\right)}$

if $\omega^2/g = m$

$$\therefore \Phi \sim \frac{mi}{\sigma\rho} e^{my} e^{-i(mx - \sigma t)} \quad \text{let } \sigma = \omega$$



$$\Phi(x,y,t) = Ux + \varphi(x,y,t)$$

free surface $\quad \frac{p}{\rho} + g\eta + \varphi_t + U\varphi_x + \frac{1}{2}U^2 = 0 \quad$ absorb $U^2 \to p$

kinematic $\quad y=0 \quad \eta_t + U\eta_x - \varphi_y = 0$

found that the sol depends on U^2/gh

from before $U^2 = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda} \qquad \frac{U^2}{gh} = \frac{1}{2\pi} \frac{\lambda}{h} \tanh \frac{2\pi h}{\lambda}$

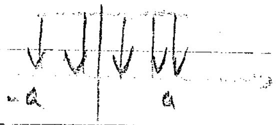
as $\lambda \to \infty \quad \therefore h/\lambda \to 0 \qquad U^2/gh \to 1$

$u^2/gh < 1$ need radiation condition

$u^2/gh > 1$ no radiation condition

for $u^2/gh = 1$ look at time dependent case by solving free surface for η & put into kinematic, use fourier transform now let $t \to \infty$ & then $x \to \infty$ $\Rightarrow$ $\Phi \sim t^{2/3}$ velocities behave like $t^{1/3}$

take infinite depth discuss motion of pressure impulse over some region



$I(x)$

$$\Phi(x,0,0) = -\frac{1}{\rho} I(x)$$

obtained from $\Phi_t + g\eta = -\frac{1}{\rho} p$; $y=0$

$$\int_0^\tau p\,dt = -\rho\,\Phi(x,0,z,\tau) - g\rho \int_0^\tau \eta\,dt$$

as $\tau \to 0$ η remains bdd

$$I = -\rho\,\Phi + 0 \quad \text{or} \quad \Phi(x,0,0) = -\frac{1}{\rho} I(x)$$

assume $\Phi_t(x,0,0) = 0$? we will go to diff problem

let $\Phi_t = -g\eta$ and let $\Phi = -g\int \eta\,dt = \delta$

$$\Phi = \frac{-\sqrt{g}}{\pi} \int_0^\infty e^{sy} \cos sx \sin \sqrt{gs}\,t \frac{ds}{\sqrt{s}}$$

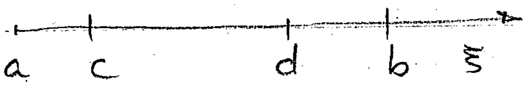
$$\eta = -\frac{1}{g} \Phi_t \Big|_{y=0} = \frac{1}{\pi} \lim_{y \to 0} \int_0^\infty e^{sy} \cos sx \cos \sqrt{gs}\,t\,ds$$

$$I(k) = \int_a^b \psi(\xi)\, e^{ik\varphi(\xi)}\, d\xi$$

want to know what happens when $k \gg 1$ assume ψ not oscillatory

ψ & φ real

Kelvin discusses:



$\varphi'(\xi) \neq 0$ in interval

disturb. dies out like $1/k$

if $\exists$ an α $\ni.$ $\varphi'(\alpha)=0$, $\varphi''(\alpha)=0$ disturbance goes like $\frac{1}{\sqrt{k}}$

principal of stationary phase.

Poincaré proved rigorously for ψ, φ analytic

integrate by parts if $\varphi'(\xi)\neq 0$ in interval

$$I(k)_c^d = \int_c^d \frac{\psi(\xi)}{ik\,\varphi'(\xi)} \frac{d}{d\xi}\left[e^{ik\varphi(\xi)}\right] d\xi$$

$$= \frac{\psi(\xi)}{ik\varphi'(\xi)}\, e^{ik\varphi(\xi)}\Big|_c^d - \frac{1}{ik}\int_c^d e^{ik\varphi}\frac{d}{d\xi}\left(\frac{\psi}{\varphi'}\right) d\xi$$

$$\xi = \alpha + x \qquad \varphi(\xi) = \varphi(\alpha) + \frac{1}{2}x^2\varphi''(\alpha^*) \quad \text{if } \varphi'(\alpha)=0$$

$$I = \psi(\alpha)\, e^{ik\varphi(\alpha)}\int_{\alpha-X}^{\alpha+X} e^{\frac{ik}{2}\varphi''(\alpha)x^2}\, dx$$

let $\tau^2 = \pm \frac{k}{2}\varphi''(\alpha)$

$$I = \psi(\alpha)\, e^{ik\varphi(\alpha)}\int_{\alpha-x}^{\alpha+x} e^{i(\pm x^2\tau^2)}\, dx$$

don't change integral by going $\int_{-\infty}^{\infty}$

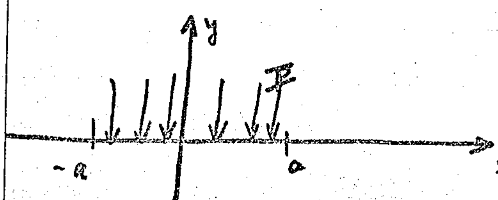
$$I = \psi(\alpha)\, e^{ik\varphi(\alpha)}\, \frac{\sqrt{\pi}}{\tau}\, e^{\pm i\pi/4} \simeq \frac{\sqrt{2\pi}\,\psi(\alpha)}{\sqrt{\pm k\varphi''(\alpha)}}\, e^{ik\varphi(\alpha) \pm i\pi/4}$$

$\pm$ $\ni.$ $\sqrt{|k\varphi''(\alpha)|}$ positive then take $\sum$ over all such pts where $\varphi'(x) = 0$

these are asymptotic expansions.

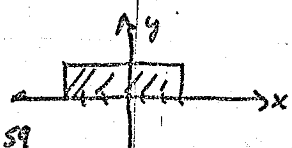
3/20/74

<u>Unsteady Motions</u>



$\swarrow$ impulse

$2aI$



$$\Phi(x,y,t) = -\frac{2Pa}{\pi\rho}\int_0^\infty \frac{\sin sa}{sa} e^{sy}\cos sx\cos(\sqrt{gs}\,t)\,dt$$

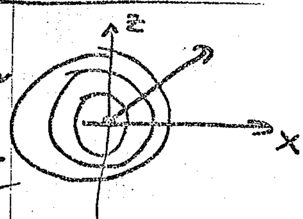
p 159

$$\eta(x,t) = -\frac{1}{g}\Phi_t\Big|_{y=0} = \frac{-2Pa}{\pi\rho\sqrt{g}}\lim_{y\to 0}\int_0^\infty \frac{\sin sa}{sa} e^{sy}\cos s(\sin\sqrt{gs}\,t)\sqrt{g}\,ds$$

$$\eta(x,t) = \frac{1}{\pi}\lim_{y\to 0}\int_0^\infty e^{sy}\cos sx\cos(\sqrt{gs}\,t)\,ds$$

(case of initial elevation)

See Sneddon Fourier Transform



$$\eta(r,t) = \frac{-1}{2\pi\sqrt{\rho g}}\lim_{y\to 0}\int_0^\infty e^{sy}J_0(sr)\sin(\sqrt{gs}\,t)s^{3/2}\,ds$$

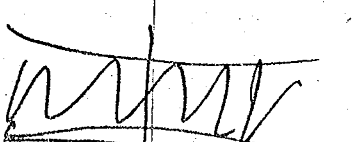
H. transf.

exact: $\eta(x,t) = \frac{1}{\pi x}\left\{\frac{gt^2}{2x} - \frac{1}{1\cdot 3\cdot 5}\left(\frac{gt^2}{2x}\right)^3 + \cdots\right\}$

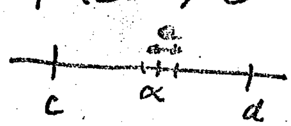
for $\frac{gt^2}{2x}$ small conv., entire fct.

$\therefore$ <u>as soon as disturbances starts, its felt throughout fluid</u>

(163) $I(k) = \int_a^b \psi(s)e^{ik\varphi(s)}\,ds$, k, φ real fct



for $\varphi'(s) \neq 0$ on $[c,d]$



Then $I(k) \sim \frac{1}{k}$

$$I(k) \simeq \frac{\sqrt{2\pi}\ \psi(\alpha)}{\sqrt{\pm\varphi''(\alpha)}}\ e^{i(k\varphi(\alpha)\pm\pi/4)} + O\left(\frac{1}{k}\right)$$

~~[illegible]~~

$\varphi'(\alpha) = \varphi''(\alpha) = 0$, $\varphi'''(\alpha) \neq 0$

$$\varphi(x) \simeq \varphi(\alpha) \pm 1/2\, \varphi''(\alpha)\, x^2$$

$$\varphi(x) = \varphi(\alpha) + \frac{1}{3!}\varphi'''(\alpha)\, x^3$$

$$I(k) \simeq \frac{1}{k^{1/3}} + O\left(\frac{1}{k^{2/3}}\right)$$

Asymptotic approx. for wels pot.

$$\Phi(x,y,t) = -\frac{\sqrt{g}}{\pi\rho} \int_0^\infty e^{sy} \cos sx \sin(\sqrt{gs}\,t) \frac{ds}{\sqrt{s}}$$

$$\int_0^b \psi(\xi)\ e^{ik\varphi(\xi)} d\xi$$

~~[illegible]~~ let $\sigma = \sqrt{gs}$, $2\sigma d\sigma = g\,ds$, $m = \frac{\sigma^2}{g}$

p161 $$\Phi(x,y,t) = -\frac{1}{\rho\pi}\left\{\int_0^\infty e^{\frac{\sigma^2}{g}y} \sin\left(\frac{\sigma^2 x}{g} + \sigma t\right) d\sigma \right.$$

$$\left. - \int_0^\infty e^{\frac{\sigma^2}{g}y} \sin\left(\frac{\sigma^2 x}{g} - \sigma t\right)\right\} d\sigma$$

let $k = \frac{gt^2}{4x}$, $\xi = \frac{2x}{gt}\sigma$

165

$$\Phi = \frac{-gt}{2\pi\rho x}\left\{\int_0^\infty e^{my} e^{ik(\xi^2+2\xi)} d\xi - \int_0^\infty e^{my} e^{ik(\xi^2-2\xi)} d\xi\right\}$$

$\varphi(s) = s^2 \pm 2s$

$\varphi'(s) = 2s \pm 2$

$s = 1$ (only pt. of stationary phase)

$\varphi''(s) = 2$

$\Phi \simeq \frac{1}{s}\sqrt{\frac{2}{\pi x}}\, e^{m\varphi}\, e^{i(\alpha k + \pi/4)}$

$\eta(x,t) \simeq \dfrac{g^{1/2}\, t}{\sqrt{\pi}\, x^{3/2}\, \rho} \cos\left(\dfrac{gt^2}{4x} - \pi/4\right) +$

$\eta(r,t) = -\dfrac{1}{2\pi\rho\sqrt{g}} \int_0^\infty e^{sy} J_0(sr)\,(\sin\sqrt{gs}\,t)\cdot s^{3/2}\, ds$

$J_0(sr) = \frac{2}{\pi}\int_0^{\pi/2} \cos(sr\cos\beta)\, d\beta$

$= \mathrm{Re}\ \frac{2}{\pi}\int_0^{\pi/2} e^{isr\cos\beta}\, d\beta$

$\frac{gt^2}{4r}$

p166

$\eta(r,t) \simeq \dfrac{-gt^3}{2^{5/2}\pi\rho r^4} \sin\dfrac{gt^2}{4r}$

---

phase

<u>propagation of phases</u>

$k = \dfrac{gt^2}{4x}$

$\dfrac{gt^2}{4x} = k = \text{const.}$

$x = \dfrac{gt^2}{4k}$ phase moving with an acceleration.

$\dfrac{dx}{dt} = \dfrac{gt}{2k}$

---

p169

$\varphi = \dfrac{gt^2}{4x}$

x_0, t_0

$\varphi = \dfrac{gt_0^2}{4x_0}\left(1 + 2\dfrac{\Delta t}{t_0} + \left(\dfrac{\Delta t}{t_0}\right)^2\right)$

$\Delta t / t_0$ small

$\Delta\varphi = \varphi - \varphi_0 \simeq \frac{2\Delta t}{t_0}\left(\frac{g t_0^2}{4x_0}\right)$

T period

$\Delta\varphi = 2\pi$, $T \simeq \frac{4\pi x_0}{g t_0}$

valid if

$\frac{T}{t_0}$ is small

$T/t_0 = \frac{4\pi x_0}{g t_0^2}$ has to be small in the region of validity

$\boxed{\lambda \simeq \frac{8\pi x_0^2}{g t_0^2} \text{ local wave length} \qquad \frac{\lambda_0}{x_0} \text{ is small}}$

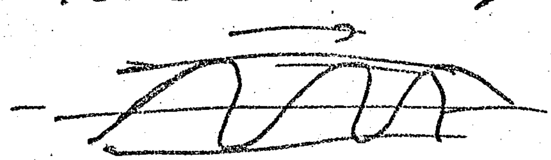
$g\frac{t^2}{4x} = k$

$\frac{dx}{dt} = \frac{gt}{2k} = \frac{2x}{t}$

$\simeq \sqrt{\frac{g\lambda}{2\pi}}$

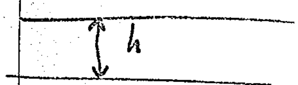
What's the propagation speed of λ?
how much must you move along to
observe the same wave length

$x \simeq \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}\, t$



group moves half as [illegible]

.170 group vel. is $\frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}$

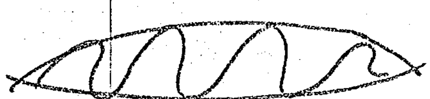


$$\boxed{c^2 = \frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}} \quad (1')$$

$$\Phi_1 = A\sin(mx - \sigma t)$$

$$\Phi_2 = A\sin[(m+\delta m)x - (\sigma+\delta\sigma)t]$$

$$\Phi = \Phi_1 + \Phi_2 = 2A\cos\tfrac{1}{2}(x\,\delta m - t\,\delta\sigma)\sin\left[\left(m+\frac{\delta m}{2}\right)x - \left(\sigma+\frac{\delta\sigma}{2}\right)t\right]$$



$$= B\sin(m'x - \sigma' t)$$

Sommerfeld $\int_{m_0-\epsilon}^{m_0+\epsilon} A(m)\, e^{i(mx-\sigma t)}\, dm$

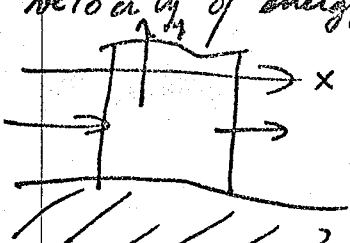
$$\sigma = \sigma(m)$$

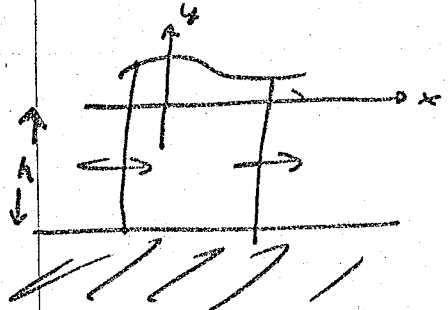
$$\Phi = C\, e^{i(m_0 x - \sigma_0 t)} \qquad \frac{\sigma-\sigma_0}{m-m_0}$$

modulated $C(x,t) = \int_{m_0-\epsilon}^{m_0+\epsilon} A(m_0)\, e^{i\{(m-m_0)x - (\sigma-\sigma_0)t\}}$

capillary waves (short wavelength, large curvature)

velocity of energy (flow of energy)





verage Flux $\rightarrow$

$$F_{av} = \frac{A^2 \rho\, \sigma m h}{4 f}\left(1 + \frac{\sinh 2mh}{2mh}\right)$$

$$\sigma^2 = g m \tanh mh$$

$$m = \frac{2\pi}{\lambda}$$

$$F_{av} = U \, \frac{A^2 \rho\, \sigma^2}{2g} \cosh^2 mh$$

$$U = \tfrac{1}{2}\, c \left(1 + \frac{2mh}{\sinh 2mh}\right)$$

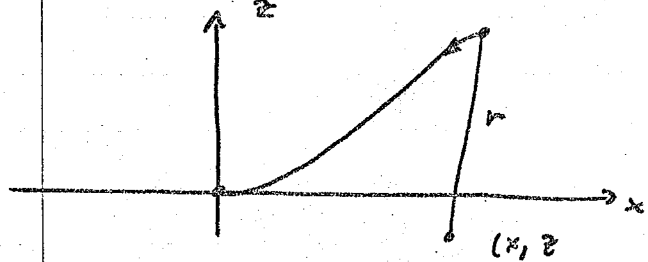
v. energy $\rightarrow$

$$\frac{E_{av}}{unit} = \frac{A^2 \rho\, \sigma^2}{2g} \cosh^2 mh \qquad , c \text{ given by } (1)$$

group velocity

p221

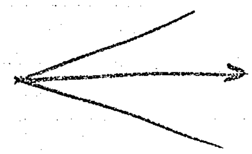
pt impulse
(idealization of a ship's course)



$x_1 = x_0(t)$ $\quad 0 \le t \le T$
$z_1 = z_0(t)$

$r^2 = (x-x_1)^2 + (z-z_1)^2$ $\Big|$ $\lim \sum I \, \Delta t \longrightarrow \int_0^T \eta(r, t) \, dt$

Simplest case, ship move along x-axis



let $\sigma^2 = gm$
$m = \frac{g\tau^2}{4\pi^2} \, \zeta^2$

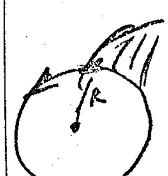
p222

$$\int_0^{\pi} \int_0^{\infty} \int_0^{\pi/2} \cdots \left\{ e^{i\kappa \frac{(2\zeta - \zeta^2 \cos\alpha)}{\ell}} + e^{i\kappa(+)} \right.$$

$\ell = \frac{r}{R}$
$\vec{c} = (\dot{x}_1, \dot{z}_1)$
$\kappa = \frac{gR}{4c^2}$

Wave pattern comes out very accurately

Bromberg



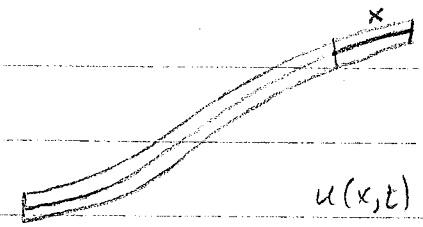
destroyer

R = ?

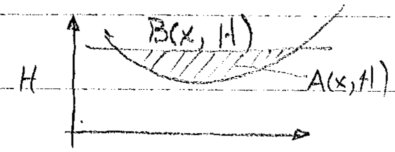
Speed = ?

4/24/74

Shock Waves



$u(x,t)$ – velocity



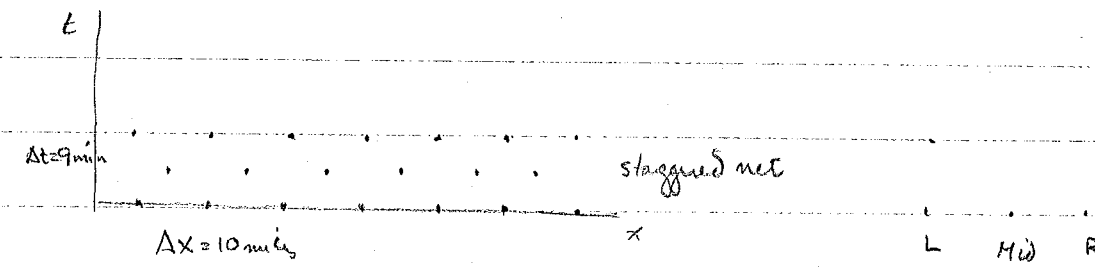
$$\underbrace{B}_{}H_t + (\underset{\text{cross sectional area of river}}{A}u)_x = \underset{\text{flowrate}}{}q(x,t) \quad \text{source term}$$

flowrate

Dynamical Condition $u_t + uu_x + \underbrace{gH_x}_{\text{force of gravity}} = -\underbrace{F(x,H)u|u|}_{\text{resistance term}}$

propagation speed $c = \sqrt{g\frac{A}{B}}$ characteristics $\frac{dx}{dt} = u \pm c$

Numerical Scheme



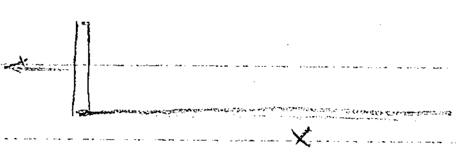
$U_{mid} = \frac{U_L + U_R}{2}$

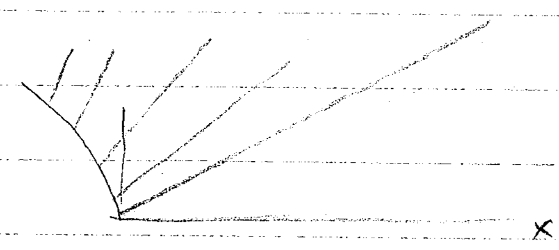
$U_t = \frac{U_P - U_M}{\Delta t}$

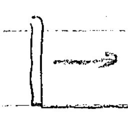
Coefficients are values taken at M

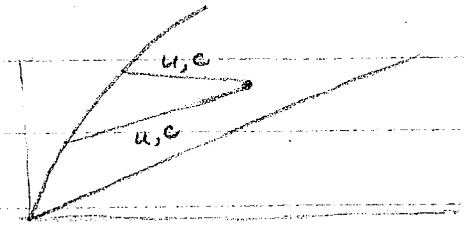
B, A, F contain unknown H.

Consider tank with movable end

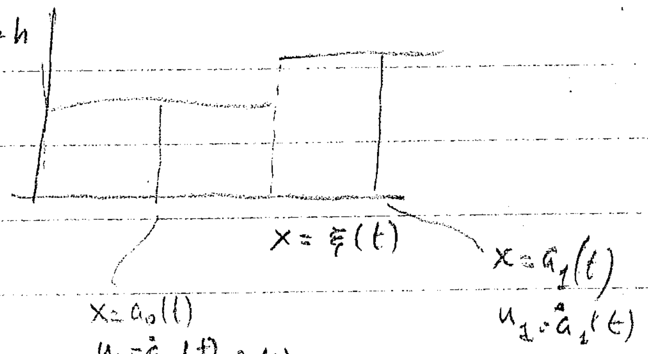








discontinuity in



Mass conservation $\frac{d}{dt}\int_{a_0(t)}^{a_1(t)} \rho(\eta+h)\,dx = 0$

Mom $\frac{d}{dt}\int_{a_0(t)}^{a_1(t)} \rho(\eta+h)u\,dx = \int_{-h}^{\eta_0} p_0\,dy - \int_{-h}^{\eta_1} p_1\,dy$

$$= \frac{1}{2} g\rho(\eta_0+h)^2 - \frac{1}{2} g\rho(\eta_1+h)^2$$

All integrals have form

$$I \int_{a_0(t)}^{a_1(t)} \psi(x,t)\,dx$$

$$\frac{dI}{dt} = \frac{d}{dt}\int_{a_0(t)}^{\xi(t)} \psi\,dx + \frac{d}{dt}\int_{\xi(t)}^{a_1(t)} \psi\,dx$$

use Leibnitz Rule $\frac{d}{dt}\int_{a_0}^{\xi} = \int \frac{\partial}{\partial t} + \psi(\xi_-,t)\,\dot{\xi}(t) - \psi(a_0,t)\,u_0$

$$\lim_{a_1\to a_0} \frac{dI}{dt} = \psi_1(u_1-\dot{\xi}) - \psi_0(u_0-\dot{\xi})$$
$$= \psi_1 v_1 - \psi_0 v_0$$

$$\rho(\eta_1+h)v_1 - \rho(\eta_0+h)v_0 = 0$$

$$\rho(\eta_1+h)u_1v_1 - \rho(\eta_0+h)u_0v_0 = \tfrac{1}{2}\rho g(\eta_0+h)^2 + \tfrac{1}{2}\rho g(\eta_1+h)$$

$$\bar\rho_1 v_1 - \bar\rho_0 v_0 = 0$$

$$\bar\rho_1 u_1 v_1 - \bar\rho\, u_0 v_0 = \bar p_0 - \bar p_1 \,;\quad \bar p = \frac{g}{2\rho}\bar\rho^{\,2}$$

define $\bar\rho_1 v_1 = \bar\rho_0 v_0 = m$

$$u_1 - u_0 = v_1 - v_0 \qquad \text{since} \quad v_1 - u_1 = -\dot\xi$$

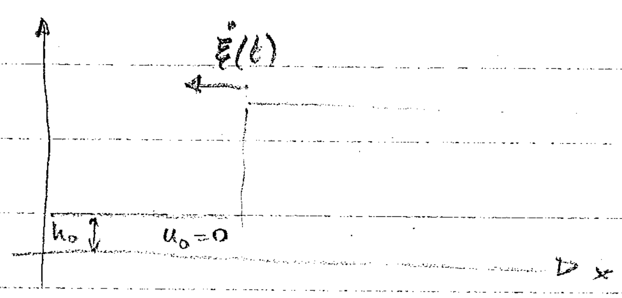
$$m(v_1 - v_0) = \bar p_0 - \bar p_1$$

$$\frac{dE}{dt} = \left\{ \int_{a_0}^{a_1}\left[\rho(\eta+h)\frac{u^2}{2} + \frac{g\rho}{2}(\eta+h)^2\right]dx + \int_{-h}^{\eta_1} p_1 u_1\, dy - \int_{-h}^{\eta_0} p_0 u_0\, dy \right\}$$

$$\frac{dE}{dt} = \tfrac{1}{2}\bar\rho_1 u_1^2 v_1 - \tfrac{1}{2}\bar\rho_0 u_0^2 v_0 + \bar p_1 v_1 - \bar p_0 v_0 + \bar p_1 u_1 - \bar p_0 u_0$$

$$\dot\xi\left[\bar\rho_1 u_1 v_1 - \bar\rho_0 u_0 v_0\right] - \dot\xi(p_0 - p_1) = 0$$

$$\frac{dE}{dt} = \frac{mg}{\rho}\,\frac{(\bar\rho_0 - \bar\rho_1)^3}{4\bar\rho_1\bar\rho_0} \qquad \frac{dE}{dt} < 0 \quad \text{energy is lost}$$



$m \neq 0 \qquad m > 0$

$v_1, v_0 > 0$

$$\frac{dE}{dt} < 0 \Rightarrow \bar\rho_0 < \bar\rho_1 \Rightarrow v_1 > v_0$$

$$v_1 v_0 = \frac{\bar p_0 - \bar p_1}{\bar\rho_0 - \bar\rho_1}$$

$u_0 = 0 \qquad v_0 = -\dot{\xi} \qquad v_1 = u_1 - \dot{\xi}$

$$-\dot{\xi}(u_1 - \dot{\xi}) = \left(\frac{g\bar{\rho}_0^{\,2}}{2\rho} - \frac{g\bar{\rho}_1^{\,2}}{2\rho}\right) \Big/ \bar{\rho}_0 - \bar{\rho}_1$$

$$= \frac{g}{2\rho}(\bar{\rho}_0 + \bar{\rho}_1)$$

$$\begin{cases} -\dot{\xi}(u_1 - \dot{\xi}) = \dfrac{g}{2\rho}(\bar{\rho}_0 + \bar{\rho}_1) \\ \bar{\rho}_1 (u_1 - \dot{\xi}) = -\bar{\rho}_0 \dot{\xi} \end{cases} \rightarrow \quad \dot{\xi}^2 = \frac{g\bar{\rho}_1}{2\rho}\left(1 + \frac{\bar{\rho}_1}{\bar{\rho}_0}\right)$$

or $\quad -\dot{\xi}(u_1 - \dot{\xi}) = \dfrac{g\bar{\rho}_1}{2\rho}\left(1 - \dfrac{u_1 - \dot{\xi}}{\dot{\xi}}\right)$

unknown $\dot{\xi}, u_1, \bar{\rho}_0, \bar{\rho}_1$

Case 1: $\bar{\rho}_1, \bar{\rho}_0$ given $\quad \Rightarrow \dot{\xi}^2 = f(\bar{\rho}_1, \bar{\rho}_0)$ gives $\dot{\xi}$

put into $-\dot{\xi}(u_1 - \dot{\xi}) = f(\bar{\rho}_0, \bar{\rho}_1) \Rightarrow u_1$

$$\dot{\xi} = -\sqrt{g\frac{h_1}{h_0}\left(\frac{h_1 + h_0}{2}\right)} \qquad h_1 = \eta_1 + h \qquad h_0 = \eta_0 + h$$

if $h_1 > h_0$

then $\quad |\dot{\xi}| = \sqrt{g\dfrac{h_1}{h_0}\left(\dfrac{h_1 + h_0}{2}\right)} > \sqrt{gh_0}$

$$v_1 = u_1 - \dot{\xi} \qquad u_1 = \dot{\xi}\left(1 - \frac{\bar{\rho}_0}{\bar{\rho}_1}\right) \quad \Rightarrow v_1 = -\frac{h_0}{h_1}\dot{\xi}$$

$$v_1 = \sqrt{g\frac{h_0}{h_1}\left(\frac{h_1 + h_0}{2}\right)} \qquad |v_1| < \sqrt{gh_1}$$

5/8/74

2 wks from today — Arthur Peters "Weiner-Hopf Technique"
(Integral equations)
tech. for linear prob.

Term paper (just summarizing, if you want can expand)
suggested topics

1. Lax, A., Decaying Shocks, Comm P.A. Math, Vol. 1, 1948
Section 7-1, 7-2

shock wave

rarefaction waves goes forward
what happens?

2. John, F. Motion of Floating Bodies I, II
CPAM, 1950
recommend this one

$u_{xx} + u_{yy} - c u_{tt} = 0$

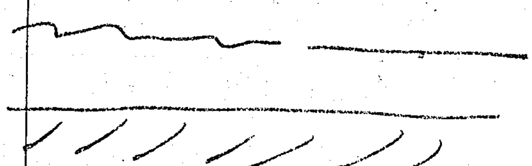
$u = e^{i\omega t} w(x, y)$

$w_{xx} + w_{yy} + c^2 \omega^2 w = 0$ reduced wave eq (elliptic)

how does this behave at ∞

3. Heins, A.E., Waves over channel of finite depth, Am. Journal Math., vol. 70, 1948   1-1, 1-2

deals with Weiner-Hopf Technique



wave comes in from ∞ into dock

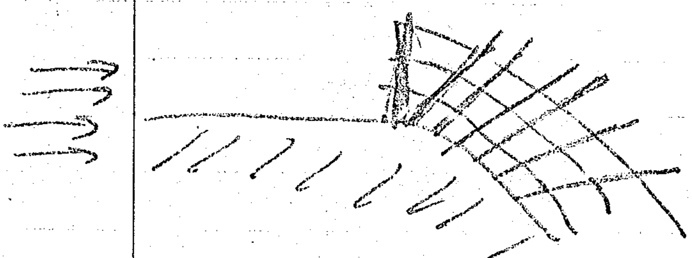
4. Two dimensional steady supercritical (supersonic) flow.

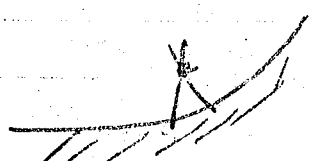
Anneli Lax CAM 1948 Vol 1 247-257
KO Friedrichs " 1948 vol 1 211-245

Supercritical - Steady flow
(in gas dynamics version very interesting)

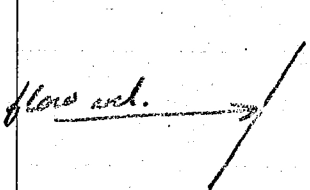
$$c = \sqrt{gh}$$

simple waves exist in same sense as in 1-dim hyperbolic problem
real characteristics
same Theorem holds
if ∃ a st. charac. which is constant
Theorem that is embedded in a whole field of st. charac.

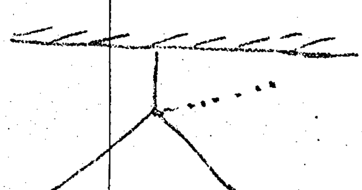




Shock waves develop here



Mach wave
need contact discontinuity

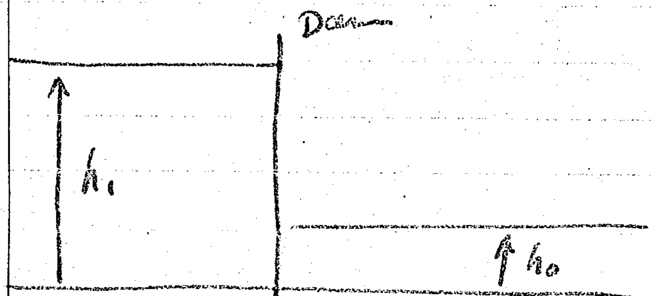


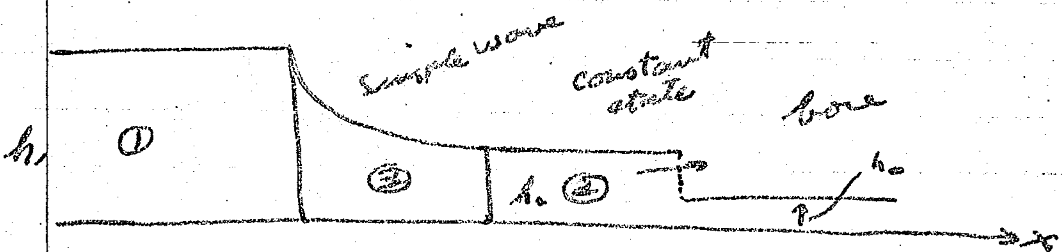
$$\frac{h_1}{h_0} = 10 \quad , \; h_1 - h_2 = 35(h_1 - h_0)$$

$$\bar{P} = \sim \left(\frac{h_1}{h_0}\right)^3$$

Breaking of a dam:

P333



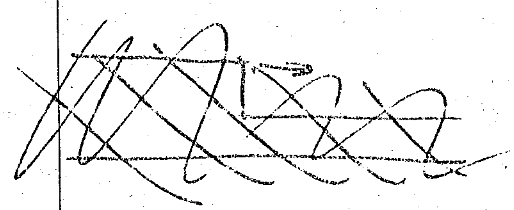


$$\frac{h_0}{h_1} = .176$$

$$\frac{h_2}{} \sim \frac{h_1}{3}$$

-

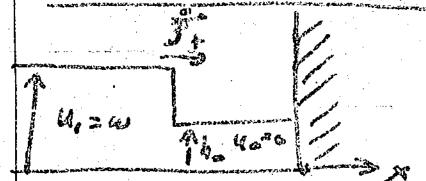
only steady state ^are when you have^ unif. depth + const velocity.



$$u = \frac{k}{c^2}$$

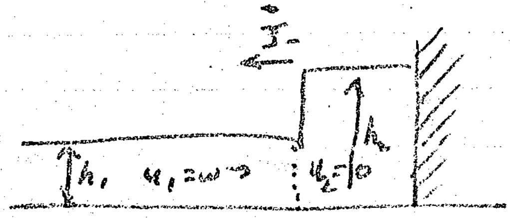
$$\frac{k^2}{c^4} + 2c^2 = k.$$

a)



what will happen when shock comes up to wall + get reflected?

b)



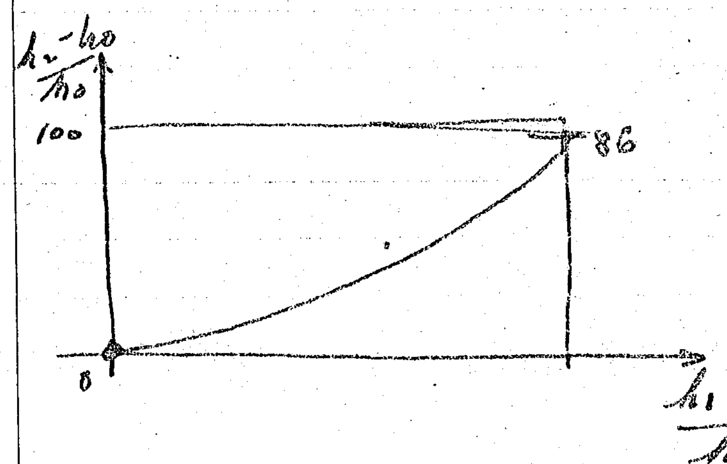
front + ~~back~~ back wave interchanged their roles

$$-\dot{\xi}(w - \dot{\xi}) = \frac{g h_0}{2}\left(1 - \frac{\dot{\xi}}{w - \dot{\xi}}\right)$$

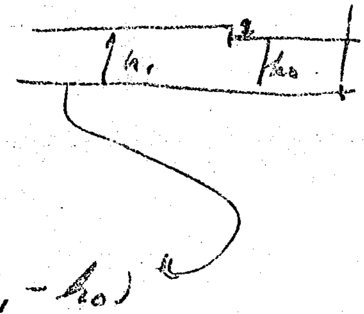
Case a) Choose $\dot{\xi}_+$, $h_1(w - \dot{\xi}_+) = h_0 \dot{\xi}_+$

Case b) Choose $\dot{\xi}_-$, $h_1(w - \dot{\xi}_-) = -h_0 \dot{\xi}_-$

p332



tremendous incr. in amplitude



Ex: $h_1 - h_0$ small, then $h_2 - h_0 \approx 2(h_1 - h_0)$

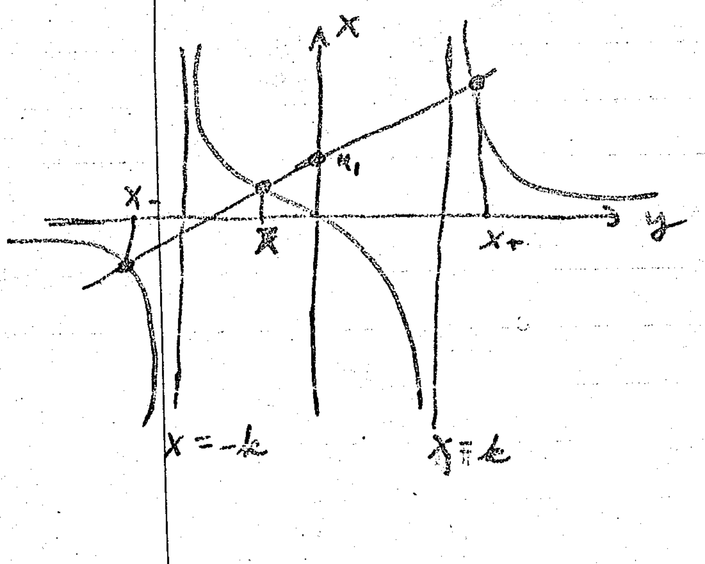
Eliminate $\bar{\rho}_0$ and

$$-\dot{\xi}(u_1-\dot{\xi}) = \frac{g\bar{\rho}_1}{2\rho}\left(1-\frac{u_1-\dot{\xi}}{\dot{\xi}}\right)$$

$$x = -\dot{\xi}\ ,\ y = u_1-\dot{\xi}\ ,\ k^2 = \frac{gh_1}{2} = \frac{g\bar{\rho}_1}{2\rho}$$

$$xy = -k^2\left(1+\frac{y}{x}\right)$$

$$y(x^2-k^2) = k^2x \Rightarrow \begin{cases} y = \dfrac{k^2x^2}{x^2-k^2} \\ y = u_1+x \end{cases}$$



$\bar{x}$ not physically significant since $\dot{\xi} = -x$

$\bar{\rho}_1(u_1+x) = \bar{\rho}_0 x$

$\bar{\rho}_1 y = \bar{\rho}_0 x$

true only when y and x have same sign

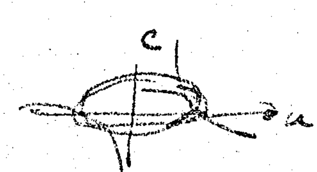
$\Rightarrow$ only 2 physically signif. roots.

$u_1 = 0 \Rightarrow$ no shock

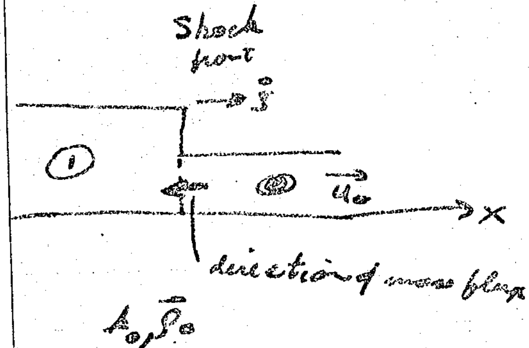
$\dot{\xi}_+, \dot{\xi}_-$ only roots useable

$$\begin{cases} u_t + uu_x + 2cc_x = 0 \\ 2c_t + 2uc_x + cu_x = 0 \end{cases}$$

are there any steady state solutions



$$\begin{cases} u\dfrac{du}{dx} + 2c\dfrac{dc}{dx} = 0 \\ 2u\dfrac{dc}{dx} + c\dfrac{du}{dx} = 0 \end{cases} \qquad \begin{cases} u^2+2c^2 = \text{const} \\ uc^2 = \text{const} \quad c^2 = \dfrac{\text{const}}{u} \end{cases}$$



$\dot{\xi}$ speed of propagation

$$\bar{\rho} = \rho(\eta + h)$$

$$\bar{p} = \frac{g}{2\rho}\bar{\rho}^2$$ adiabatic relation

relative flow velocity

$$V_1 = u_1 - \dot{\xi}$$
$$V_0 = u_0 - \dot{\xi}$$

Shock conditions

① $\bar{\rho}_1 V_1 = \bar{\rho}_0 V_0 = m$ — cons. of mass; represents mass flux through shock

② $\bar{\rho}_1 u_1 V_1 - \bar{\rho}_0 u_0 V_0 = \bar{p}_0 - \bar{p}_1$ — cons. of momentum

$$\frac{dE}{dt} = \frac{mg}{\rho}\frac{(\bar{\rho}_0 - \bar{\rho}_1)^3}{4\bar{\rho}_1\bar{\rho}_0} < 0$$ not zero due to turbulence

$$m(V_1 - V_0) = \bar{p}_0 - \bar{p}_1$$

$$V_1 V_0 = \frac{\bar{p}_0 - \bar{p}_1}{\bar{\rho}_0 - \bar{\rho}_1}$$

Let: $u_0 = 0$, $V_0 = -\dot{\xi}$, $V_1 = u_1 - \dot{\xi}$

$$-\dot{\xi}(u_1 - \dot{\xi}) = \frac{g}{2\rho}(\bar{\rho}_0 + \bar{\rho}_1)$$

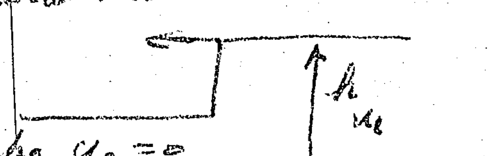
$$\bar{\rho}_1(u_1 - \dot{\xi}) = -\rho_0\dot{\xi} \quad \text{or} \quad u_1 = \dot{\xi} - \left(1 - \frac{\bar{\rho}_0}{\bar{\rho}_1}\right)$$

eliminate u_1 yields

$$\dot{\xi}^2 = \frac{2\bar{\rho}_1}{2\rho}\left(1 + \frac{\bar{\rho}_1}{\bar{\rho}_0}\right), \quad \dot{\xi} = -\sqrt{g\frac{h_1}{h_0}\left(\frac{h_1 + h_0}{2}\right)}$$

~~$h_0\ u_0 = 0$~~

velocity of shock always less velocity of wave into behind



$$\dot{\xi} < \sqrt{gh}$$

5/22/74

Prof A.S. Peters on Singular Integral Equations.

Wiener – Hopf

$$y\frac{\partial w(x,y)}{\partial x} + w(x,y) = \lambda\int_{-1}^{1} w(x,y')\,dy' \equiv \lambda\varphi(x)$$

subject to $w(0,y) = \begin{cases} I(y) & y>0 \\ 0 & y<0 \end{cases}$

Arose in radiative equilibrium problem

Equivalent system

$$\varphi(x) + g(x) = \lambda\int_0^\infty K(x-\xi)\varphi(\xi)\,d\xi$$

Example

Suppose we have potential $\Delta\varphi = 0 \quad y<0$

$\varphi_y(x,0) = 0 \quad x<0 \qquad \varphi_x - c\varphi_y = f(x) \quad x>0, y=0$

Assume poisson representation we can show

$$\varphi_x(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(\xi-x)\varphi_y(\xi,0)\,d\xi}{(\xi-x)^2+y^2}$$

use bdy cond & let $y\to 0$ with $x>0$

$$\varphi_x(x,y) = \frac{1}{\pi}\int_0^\infty$$

$$\varphi_x(x,0) = \frac{1}{\pi}\int_0^\infty\frac{\varphi_y(\xi,0)\,d\xi}{\xi-x}$$

using bdy cond $x>0 \quad y=0$

$$c\varphi_y(x,0) + f(x) = \frac{1}{\pi}\int_0^\infty\frac{\varphi_y(\xi,0)}{\xi-x}\,d\xi$$

This is a wiener-hopf equation singular in 2 ways infinite interval & infinite kernel

Example 2

dock at an infinite ocean

$\varphi_y(x,0)=0$

$$\varphi_y - q\varphi = f(x)$$
$$\Delta\varphi = 0$$

$\Rightarrow \quad x>0$

$$\varphi_x(x,0) = \frac{1}{\pi}\int_0^\infty \frac{f(\xi)+c\,\varphi(\xi)}{\xi - x}\,d\xi$$

Wiener Hopf comes up when solving some D.E. in a domain D which has as part of bdy the half line & data on the bdy.

Can circumvent W-H if we are willing to deal with "fourier eqns" we note that $K(t)$ must be defined $\forall$ values of its argument & $x>0$

Define $\varphi(x)=0 \qquad x<0$

then

$$\varphi(x) + g(x) = \lambda\int_{-\infty}^{\infty} K(x-\xi)\,\varphi(\xi)\,d\xi$$

this is convol kernel in fourier transform.

LHS unknown for $x<0$ $\therefore$

let for $x<0$

$$\psi(x) = \lambda\int_{-\infty}^{\infty} K(x-\xi)\,\varphi(\xi)\,d\xi$$

define RH fourier transform $\Phi_+(\xi) = \int_0^\infty e^{i\xi x}\varphi(x)\,dx$

" " " " $G_+(\xi) = \int_0^\infty e^{i\xi x} g(x)\,dx$

define $K(\xi) = \int_{-\infty}^{\infty} e^{i\xi t} K(t)\,dt$

define LH transform $\Psi_-(\xi) = \int_{-\infty}^{0} e^{i\xi x}\psi(x)\,dx$

$$\Phi_+(\zeta) + \Psi_-(\zeta) + G_+(\zeta) = K(\zeta)\,\Phi_+(\zeta)$$

What conditions are satisfied by K, Φ assuming $\exists$ a ζ $\ni$.
The untransformed eqns are equiv to transformed eqn. Assume this holds for $\operatorname{Im}\zeta = a$. $\therefore$ we have a fnl eqn to be solved

$$[1 - K(\zeta)]\,\Phi_+(\zeta) + \Psi_-(\zeta) = -G_+(\zeta)$$

$$\text{define} \quad F(z) = \begin{cases} \Phi_+(z) & \operatorname{Im} z > \operatorname{Im}\zeta \\ -\Psi_-(z) & \operatorname{Im} z < \operatorname{Im}\zeta \end{cases}$$

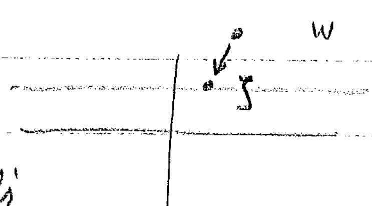
Analytic in upper half plane & lower half plane & the above eq must hold on the line $\operatorname{Im}\zeta = a$.

$$[1 - K(\zeta)]\,F^+(\zeta) - F^-(\zeta) = -G_+(\zeta) \quad \text{barrier eq}$$

find $F(z)$ which is sectionally holomorphic for which has desired properties.

$$f(z) = \int_{-\infty+ia}^{\infty+ia} \frac{h(w)\,dw}{w - z} \qquad \text{Representation for sectional holomorphic fn.}$$

$$f^+(z) = \pi i\, h(\zeta) + \int_L \frac{h(w)\,dw}{w - \zeta}$$



Plemelj

$$f^-(z) = -\pi i\, h(\zeta) + \int_L \quad ''$$

$$\boxed{f^+ - f^- = 2\pi i\, h(\zeta)}$$

the barrier eq looks like this to get it in same form
Let $F(z) = F_1(z)\,F_2(z)$ choose

$$[1-K(\zeta)]\, F_1^+ F_2^+ - F_1^- F_2^- = -G_+$$

choose $F_1 \ni$ $[1-K(\zeta)]\, F_1^+(\zeta) = F_1^-(\zeta)$

F_1 soln to homog eq then

$$F_2^+ - F_2^- = \frac{-G_+(\zeta)}{[1-K(\zeta)]F_1^+(\zeta)}$$

now use plemelj

$$F_2(z) = -\frac{1}{2\pi i}\int_L \frac{G_+(\zeta)\, d\zeta}{[1-K(\zeta)]F_1^+(\zeta)\,(\zeta - z)} + p(z) \quad \text{sectional}$$

where $p_1^+(\zeta) - p_1^-(\zeta) = 0$ p(z) can be picked so that the transform properties of F are satisfied

$$\log F_1^+ - \log F_1^- = -\log[1-K(\zeta)]$$

$$\ln F_1(z) = \frac{-1}{2\pi i}\int_L \frac{\log[1-K(\zeta)]\, d\zeta}{\zeta - z}$$

Example: $\varphi(x) = \lambda\int_0^\infty e^{-|x-\xi|}\varphi(\xi)\, d\xi$ are there solns.

$$K(\zeta) = \int_{-\infty}^{\infty} e^{i\zeta t} e^{-|t|}\, dt = \frac{2\lambda}{\zeta^2+1} \qquad -1<\zeta<1$$

$$\therefore \left[1 - \frac{2\lambda}{\zeta^2+1}\right]F^+(\zeta) - F^-(\zeta) = 0 \qquad -1<\zeta<1 \quad \text{exists in a strip}$$

$$[1-K(\zeta)] = \left[\frac{(\zeta-\zeta_1)(\zeta-\zeta_2)}{\zeta^2+1}\right] \qquad \zeta_1 = \sqrt{2\lambda} - 1 \quad \zeta_2 = -\sqrt{2\lambda} - 1$$

$$\log F^+ - \log F^- = \ln \frac{\zeta^2+1}{(\zeta-\zeta_1)(\zeta-\zeta_2)}$$

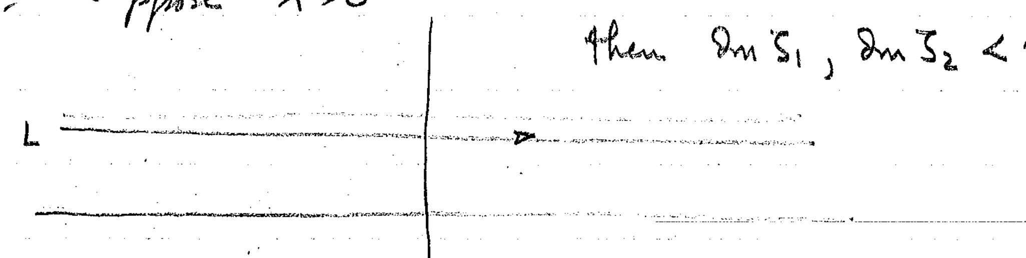
Take $\Im m\, \zeta = 1-\epsilon \qquad \epsilon>0$

$$\ln F(z) = \frac{1}{2\pi i}\int_{\substack{L\\ \Im m\zeta = 1-\epsilon}} \frac{1}{\zeta - z} \log \frac{(\zeta^2+1)}{(\zeta-\zeta_1)(\zeta-\zeta_2)}\, d\zeta + p(z)$$

with $p^+ - p^- = 0$

this integral doesn't exist by Cauchy Residue Th. But we will fix it. Suppose $\lambda > 0$

then $\Im m\, \zeta_1,\ \Im m\, \zeta_2 < 1$



we can rewrite (?!)

$$\ln F(z) = \frac{1}{2\pi i}\, z \int \frac{1}{\zeta}\, \frac{1}{\zeta - z}\, \ln \frac{\zeta^2+1}{(\zeta-\zeta_1)(\zeta-\zeta_2)}\, d\zeta + p(x)$$

as $\zeta \to z$ then $\ln F(\zeta) \to \frac{1}{2\pi i}\int \frac{1}{\zeta - z}\, L$

for $\Im m\, z > \Im m\, \zeta$

$$\Phi_+(z) = p(z)\, \frac{z+i}{(z-\zeta_1)(z-\zeta_2)} \qquad p^+ - p^- = 0$$

Φ_+ vanishes for large $z \Rightarrow p(z) = \text{const}$

since $\frac{z+i}{(z-\zeta_1)(z-\zeta_2)} \to 0$ as $z \to \infty$ & has no sing in upper half plane.

for $\lambda \le 0$ $\quad \zeta_1$ is in upper half plane

$$\Phi_+(z) = p(z)\, \frac{z+i}{z-\zeta_2}$$

$p(z)$ must be $\Rightarrow p^+ - p^- = 0$ & $p(z) \to 0$ as $z \to \infty$ $\quad p(z)$ cannot have zeros

$\Rightarrow p(z) = 0$ $\therefore$ we have trivial sol.

<u>Ex.</u>
$$e^{-x} = \int_0^\infty K_0(|x-\xi|)\,\varphi(\xi)\,d\xi$$ Bessel fn of second Kind

Sommerfeld diffraction problem

$$K(\zeta) = \int_{-\infty}^{\infty} e^{-i\zeta t} K_0|t|\,dt = \frac{\pi}{\sqrt{1+\zeta^2}}$$ analytic $-1 < \Im\zeta < 1$

$$\frac{\pi}{\sqrt{\zeta^2+1}} \quad F^+(\zeta) + F^-(\zeta) = G_+(\zeta) = \frac{1}{1-i\zeta}$$

$$\frac{\pi}{\sqrt{\zeta+i}} F^+(\zeta) + \sqrt{\zeta-i}\,F^-(\zeta) = \frac{\sqrt{\zeta-i}}{-i(\zeta+i)} \quad -1 < \Im\zeta < 1$$

Let
$$F(\zeta) = \begin{cases} \dfrac{\sqrt{z+i}}{\pi} F_2(z) & \Im z > \Im\zeta \\ \dfrac{1}{\sqrt{z-i}} F_2(z) & \Im z < \Im\zeta \end{cases}$$

$$F_2^+(\zeta) - F_2^-(\zeta) = \frac{i\sqrt{\zeta-i}}{\zeta+i}$$

$$F_2(z) = \frac{1}{2\pi}\int_L \frac{\sqrt{\zeta-i}}{(\zeta+i)(\zeta-z)}\,d\zeta + p(z)$$

For $\Im z > 0$ ζ

$$\Phi^{+} = \frac{\sqrt{z}}{\pi\sqrt{1-iz}} + p(z)\frac{\sqrt{z+i}}{\pi} \qquad \text{choose } p(z)=0$$

inversion gives solution

<u>At home</u> $\qquad g(x) = \int_0^\infty \frac{\varphi(\xi)}{x-\xi}\, d\xi$ $\qquad$ Cauchy Singular Integral Eqn.

$$K(\zeta) = \pi i \frac{|\zeta|}{\zeta} \qquad \Im \zeta = 0 \quad \text{must be}$$

$$\varphi(\lambda) = \frac{1}{\pi^2}\sqrt{\lambda}\int_0^\infty \frac{g(x)\,dx}{\sqrt{x}\,(\lambda-x)} + \frac{c_1}{\sqrt{\lambda}} \qquad \text{Most general}$$

A problem to show circumvention of W-H to go straight to barrier eqn.

$$\varphi(x,1) = 0$$

$$\Delta\varphi = 0 \qquad 0 < y < 1$$

$$\varphi_y(x,0) = 0 \qquad \varphi(x,0) = 1$$

$$\Phi(\zeta,y) = \int_{-\infty}^{\infty} e^{i\zeta x}\,\varphi(x,y)\,dx$$

$$\Phi_{yy}(\zeta,y) - \zeta^2\Phi = 0 \qquad \Phi(\zeta,1) = 0$$

$$\Phi(\zeta,y) = A(\zeta)\sinh \zeta(y-1)$$

$$\bar{\Phi}_y(\xi,y) = \xi\, A(\xi) \cosh \xi(y-1)$$

elim $A(\xi) \Rightarrow$

$$\bar{\Phi}_y = \xi \coth \xi(y-1)\, \bar{\Phi}(\xi,y)$$

let $y \to 0$ $\quad \bar{\Phi}_y(\xi,0_+) = -\xi \coth \xi \; \bar{\Phi}(\xi,0_+)$

$\bar{\Phi}_y = 0$ for $x<0$

$$\int_0^\infty e^{i\xi x} \varphi_y(x,0)\,dx = -\xi \coth \xi \left[\int_{-\infty}^0 e^{i\xi x}\varphi(x,0)\,dx + \int_0^\infty e^{i\xi x}\varphi(x,0)\,dx\right]$$

$$\bar{\Phi}_y^+ = -\xi \coth \xi \left\{ \bar{\Phi}_- + \frac{1}{i\xi} \right\}$$

or $\quad \dfrac{\tanh \xi}{\xi}\displaystyle\int_0^\infty e^{i\xi x}\varphi_y(x,0)\,dx + \int_{-\infty}^0 e^{i\xi x}\varphi(x,0) = \frac{+1}{i\xi}$

$$F(z) = \begin{cases} \int_0^\infty e^{izx}\varphi_y(x,0)\,dx & \Im z > \Im \xi = \epsilon \\ \int_{-\infty}^0 e^{izx}\varphi(x,0)\,dx \end{cases}$$

barrier eq $\quad \dfrac{\tanh \xi}{\xi} F^+ + F^- = \dfrac{1}{i\xi}$

Decaying Shocks by Anneli Lax

In a paper by K.O. Friedrichs "Formation & Decay of Shock Waves" an approximate method was developed to determine the behavior of decaying shock waves in a gas. Prof. Lax tests this method on a problem in surface waves in shallow water; this approximate method was then compared to a finite difference scheme, applicable to shocks of any strength, limited only by the mesh size of the scheme.

The Problem

A bore of constant speed U moves into still water of uniform depth h_0. Behind the bore the water moves with a velocity u_1 and has a uniform height h_1. This part in itself can be solved by using the shock jump conditions of gas dynamics. Now suppose that at time $t=0$ a wall is suddenly inserted into the moving water at position $x=0$. This creates a depression wave which will catch up to the bore, since its velocity is greater than the bore velocity, and will start to decay the bore. The problem is to find the position, velocity and strength as functions of time of the bore. Because the bores weaken a combination finite difference and approximate method is used, i.e. the finite difference is used until the bore is weakened and the continuation of the solution is given by the Friedrichs' approximation.

I. The Solution - First the bore jump conditions

Looking at the governing equations and non dimensionalizing

the velocities with respect $\sqrt{gh_0}$, the distance with respect to the distance between the wall and the bore x_0, and the time with respect to $x_0/\sqrt{gh_0}$ and defining $c^2 = h/h_0$, we obtain equations similar to the shock equations in gas dynamics, viz

$$\tilde{\rho}_0 V_0 = \tilde{\rho}_1 V_1 \quad : \quad \tilde{\rho} = \rho g h \qquad (1\,a,b)$$
$$p_0 + \tilde{\rho}_0 V_0^2 = p_1 + \tilde{\rho}_1 V_1^2 \quad : \quad p = (g/2)\rho h^2$$

thus defining the adiabatic equation of state not involving entropy:

$$p = \tilde{\rho}^2 \frac{1}{2\rho g} . \qquad (2)$$

For a barotropic gas it is noted that

$$p = K \rho^{\gamma} \qquad \gamma = 1.4 \quad ;$$

however in our case $\gamma = 2$. With the aid of (1) and (2) the shock velocity U and particle velocity u behind the shock can be found in terms of c behind the shock.

II. <u>The Solution</u> - Now add the depression wave

We note that a depression wave in fluids is like a rarefaction wave in gas dynamics. Hence $\exists$ a centered simple wave region having straight line characteristics eminating from the origin. The depression wave moves with sound speed c_1 into a medium whose velocity is u_1. The straight line characteristics C^+ are given by $\frac{dx}{dt} = u_1 + c_1$ and the simple wave solution is $\frac{x}{t} = u_1 + c_1$. Since $c_1 > U$ the depression wave catches up to the bore at $x = x^* = l$ & $t = t^* = \kappa$; κ and l can be found as follows:

Since we have nondimensionalized, at $t=0$ the bore is at $x = 1$. Hence in time $t = \kappa$ the bore is at $x = x^* = l = U\kappa + 1$. In time κ the depression wave has moved a distance l; therefore from $\frac{x}{t} = u_1 + c_1$, $\frac{l}{\kappa} = u_1 + c_1$.

These two equations can be solved for l & K. From these coordinates a new coordinate system $\bar{x}, \bar{t}$ is defined so that $\bar{x} = x - l$, $\bar{t} = t - K$. (It is noted that the reflection of the depression wave from the bore is ignored since this effect enters in terms of higher than second order.) Since the only common intersection point in a centered simple wave region of the C^+ characteristics is the origin then for $(\bar{x}, \bar{t}) = (\xi, 0)$

$$\frac{x}{t} = u + c \implies \frac{\xi + l}{K} = u(\xi) + c(\xi) \qquad (3)$$

It is assumed that the Riemann invariant s is constant across the bore which is consistent with the approximate theory, since changes in s appear in third order or higher terms; thus for a depression wave, on the C^- characteristic

$$\frac{u}{2} - \frac{c}{\gamma - 1} = s \qquad (4)$$

and since $\gamma = 2$ we have

$$u - 2c = 2s. \qquad (4')$$

Across the bore $h = h_0$, $u = 0 \implies c = 1$ $u = 0$ $\therefore$

$$u - 2c = -2. \qquad (5)$$

Behind the bore $u - 2c = u_1 - 2c_1$ (6)

Using (3) and (6) and (5)

$$u(\xi) = \frac{2}{3}\left[\frac{\xi + l}{K} - 1\right]$$
$$c(\xi) = \frac{1}{3}\left[\frac{\xi + l}{K} + 2\right]. \qquad (7a,b)$$

The bore curve $U = f[\bar{x}(\xi), \bar{t}(\xi)]$ can be found by obtaining a differential equation for $\bar{t}(\xi)$ and solving it. It is known that behind the shock

$$\frac{x}{t} = \frac{\bar{x} - \xi}{\bar{t}} = u(\xi) + c(\xi) ; \qquad (8)$$

thus by differentiating (8) with respect to ξ the following is obtained

$$\bar{t}_\xi \left[U - c(\xi) - u(\xi)\right] = \bar{t}\left[u'(\xi) + c'(\xi)\right] + 1 \qquad (9)$$

using $\bar{x}_{\bar{t}} = U$.

The homogeneous solution is

$$\ln \bar{t}_h(\xi) = \int_0^\xi \frac{u'+c'}{U-(c+u)}\, d\check{\xi} = I(\xi)$$

and the particular solution is

$$\bar{t}_p(\xi) = e^{I(\xi)} \int_0^\xi \frac{e^{-I(\check{\xi})}}{U-(u+c)}\, d\check{\xi} \qquad (10)$$

From (8)

$$\bar{x}(\xi) = \xi + (u+c)\,\bar{t}(\xi) \qquad (11)$$

It is found that only part of the depression wave interacts with the bore; this part of the wave has height greater than h_0, i.e. that region in the x, t space whose characteristics satisfy $x/t = u+c \geq 1$. Hence if we define $\sigma(\xi) = u+c-1$ a simple expression for $\bar{t}(\xi)$ can be obtained when u & U are written as functions of $\sigma(\xi)$. In doing so it is seen that the forward moving part decays the bore for an infinitely long time, i.e. $(\xi+l)/k = 1$.

If the wall moves with a given velocity $u_1 > w > u_1 - 2c_1$, the depression wave is narrowed, since the last characteristic has particle velocity $u = w$, and a constant state appears to the left of the variable zone. If $w \leq 0$ the depression wave is still wide enough to include that part which decays the shock. If $0 < w < u_1$, then $\exists$ a finite time period in which the interaction causes the bore to decay from one constant state to another. From the results shown the height of the

bore decreases rapidly at first and then approaches its asymptotic value slowly.

Solution by Finite Differences

The governing equations are one dimensional hyperbolic equations

$$u_t + uu_x + 2cc_x = 0 \qquad (12a,b)$$
$$2c_t + cu_x + 2uc_x = 0$$

which give the following:

$$\left.\begin{aligned} x_\alpha &= (u+c)\,t_\alpha \\ u+2c &= \text{const} \end{aligned}\right\} \text{ along } C^+ \text{ characteristics} \qquad (13a,b)$$

$$\left.\begin{aligned} x_\beta &= (u-c)\,t_\beta \\ u-2c &= \text{const} \end{aligned}\right\} \text{ along } C^- \text{ characteristics} \qquad (14a,b)$$

We replace 13a, 14a by

$$\Delta_\alpha x = (\tilde{u}+\tilde{c})\,\Delta_\alpha t \quad \text{along } \beta = \text{const } (C^+) \qquad (13a')$$
$$\Delta_\beta x = (\bar{u}-\bar{c})\,\Delta_\beta t \quad \text{along } \alpha = \text{const } (C^-) \qquad (14a')$$

where $\Delta_\alpha(\ ) = (\ )_{ik} - (\ )_{i-1,k}$ $\qquad (\tilde{\ }) = \left[(\ )_{ik} + (\ )_{i-1,k}\right]/2$

$\Delta_\beta(\ ) = (\ )_{ik} - (\ )_{i,k-1}$ $\qquad (\bar{\ }) = \left[(\ )_{ik} + (\ )_{i,k-1}\right]/2$

as noted in Courant & Friedrichs. From (13a') and (14a') we can solve for t_{ik} and x_{ik}. Upon using (14b) in the equations for t_{ik} and x_{ik}, u_{ik} and c_{ik} can be obtained.

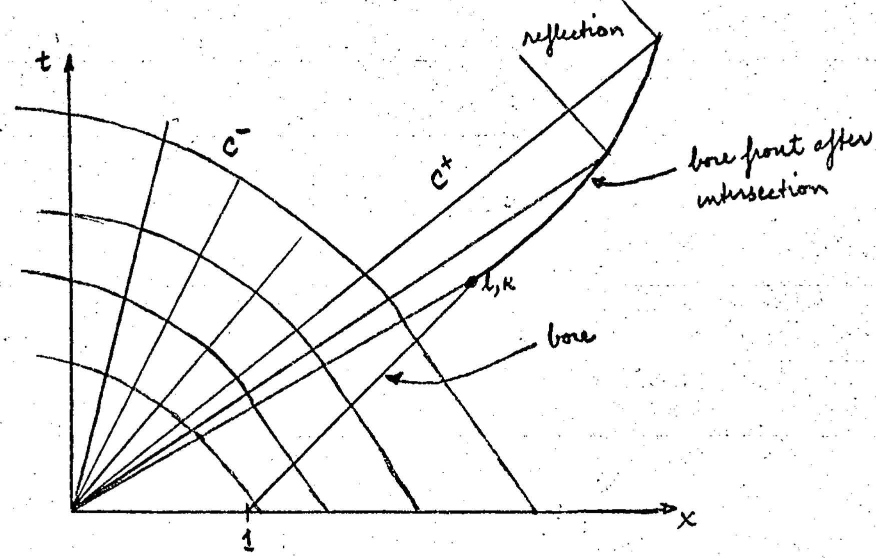
With these results and the jump conditions the velocity of the bore V_{ii} can be found from $V_{ii} = \sqrt{(c_{ii}^2+1)/2}\cdot c_{ii}$

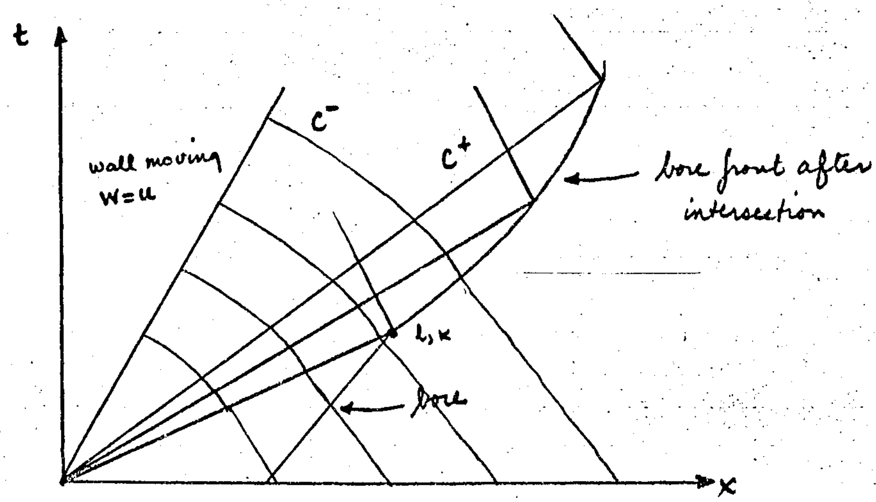
Results & Conclusions

Prof. Lax shows results in x,t space for bore strength $h_1/h_0 = 1.8, 3.0$; we note that the agreement is very good, even for such a high value of h_1/h_0.

between the finite difference and approximate method. For $x < 7$ both methods give coincident results on both graphs. However as x, t become large the bore curves $x = x(t)$ obtained no longer coincide, probably due the large mesh size of the Finite Difference scheme in that region. We note that Friedrichs' Method should become more accurate since the bore weakens when x, t become large.

Prof. Lax concludes that even higher than second order terms in the expansion of quantities across the bore differ little from what they would be in the case of a continuous wave and it seems reasonable to apply the approximate method in cases where the pressure ratios are high.





1

write up tomorrow

Tchaikovsky 1812 Overture
Sibelius #1 & Finlandia
Rimsky Korsakov - Russian Easter Overture
Saint Saens Symphony #3.

DVORAK #6, Carnival Overture
MENDELSSOHN - Hebrides
KING Stephen - Beethoven
Jupiter - Planets - Holst
Beethoven #6, 7, 3

Decaying Shocks by Annell Lax

In Paper by K.O. Friedrichs Formation & Decay of Shock Waves an approx method was developed to determine the behavior of decaying shock wave in a gas. To test this method on problem in surface waves in shallow water this approx method was used as well as a finite difference scheme applicable to shocks of any strength, which is only limited by the mesh size.

The physical problem: A shock wave (or bore) moves with constant speed V into still water of uniform depth h_0. The height of the water behind the shock is h_1, and its velocity is u_1. @ $t=0$ a vertical wall is suddenly inserted into the moving water @ $x=0$ so that a rarefaction (or depression) wave is created which will catch up with the shock, since such disturbances are propagated with a speed greater than that of the shock; thus shock will decay

Since strong shocks weaken due to the above then the following treatment of them is justified: a combination of Finite Difference & Friedrichs approximation; F.D. until the shock is weakened and then continuation of solution by K.O.F. Approx.

We non dimensionalize the velocities with respect to $(gh_0)^{1/2}$ the distance with respect to x_0 the distance between wall & discont & the time wrt $\frac{x_0}{(gh_0)^{1/2}}$.

Problem is to find the strength of the shock as well as the veloc of shock & its position as fns of time.

The approx method applicable to weak shocks uses the same shock equations as those for a gas

$$\tilde{\rho}_0 v_0 = \tilde{\rho}_1 v_1$$

$$p_0 + \tilde{\rho}_0 v_0^2 = p_1 + \tilde{\rho}_1 v_1^2$$

where the veloc are relative to shock & $\tilde{\rho} = \rho g h$ ρ = density of wat

& $p = (g/2)\rho h^2$ then $p = \frac{1}{2\rho g}\tilde{\rho}^2$.

$g/_2\, \rho_0 h_0^2 + \rho_0 g h_0 v_0^2$

This is the adiabatic eq of state which doesn't involve entropy & adiabatic exponent is 2.

From these eqs the shock velocity U & particle velocity u behind the shock can be found in terms of c behind the shock.

Since the rarefaction wave moves with sound speed c_1 into medium whose velocity is u_1. The wave front is given by $\frac{dx}{dt} = u_1 + c_1$ [this is the characteristic] & catches up @ $x = l$, $t = k$. We assume that when the shock is at $x = 1$, $t=0$ the wall is placed into the flow.

Hence @ distance $x^* = Uk + 1 = l$; from the wall we get a centered simple wave $\frac{x}{t} = u_1 + c_1$ (rarefaction) $\therefore$ $x^* = l = k(u_1 + c_1)$

$$\Rightarrow \quad k = \frac{1}{u_1 + c_1 - U}$$

assume that $\bar{x}, \bar{t}$ are new coord from (l, k) i.e. $\bar{x} = x - l$ $\bar{t} = t - k$

We assume that reflection of the rarefaction wave from the shock is ignored since effect enters in terms of higher than second order.

Since centered simple wave $\exists$ a unique characteristic intersection emanating from the origin $(x,t) = (0,0)$, with the negative $\bar{x}$ axis hence from $\frac{x}{t} = u + c$ then if intersection is at $\bar{x}, \bar{t} = (\xi, 0)$ then

$$\frac{\xi + l}{K} = u(\xi) + c(\xi)$$

using the Riemann invariant across the rarefaction wave

$$\frac{u}{2} - \frac{c}{\gamma - 1} = \frac{s}{2} \quad \text{and since } \gamma = 2 \quad \text{we get} \quad u - 2c = s$$

@ the front of the rarefaction $s = u_1 - 2c_1$

$$\therefore \quad u(\xi) - 2c(\xi) = u_1 - 2c_1$$

$$\Rightarrow \quad u(\xi) = \frac{1}{3}\left[2\,\frac{\xi + l}{K} + (u_1 - 2c_1)\right]$$

$$c(\xi) = \frac{1}{3}\left[\frac{\xi + l}{K} - (u_1 - 2c_1)\right]$$

We assume that the Riemann invariant s is a const across the shock which is consistent with the approx theory since changes in s appear in third or higher order terms. $\therefore \quad u - 2c = -2$

To find the shock curve $U = f[\bar{x}(\xi), \bar{t}(\xi)]$ a diff eq. for $\bar{t}(\xi)$ is set up and solved. Behind the shock

$$\frac{\bar{x} - \xi}{\bar{t}} = u(\xi) + c(\xi) \quad \text{or} \quad \bar{x}_\xi = (u + c)\,\bar{t}_\xi + \bar{t}\,(u'(\xi) + c'(\xi)) + 1$$

$$\bar{x}_{\bar{t}}\,\bar{t}_\xi = U\,\bar{t}_\xi$$

or $$\bar{t}_\xi\left[U - (c(\xi) + u(\xi))\right] = \bar{t}\left[u'(\xi) + c'(\xi)\right] + 1$$

which has H.OM sol $$\frac{\bar{t}_\xi}{\bar{t}} = \frac{u' + c'}{U - (c + u)} \qquad \ln \bar{t} = \int_0^\xi \frac{u' + c'}{U - (c + u)}\,d\xi = I(\xi)$$

$$\therefore \quad \bar{t}_p(\xi) = e^{I(\xi)} \int_0^\xi \frac{e^{-I(\xi)}}{U - (u + c)}\,d\xi \qquad \bar{t}_h = e^{I(\xi)}$$

& $$\bar{x}(\xi) = \xi + (u + c)\,\bar{t}$$

to get a simple expression for $t(\xi)$ we write u & U in terms of $\sigma(\xi) = u+c-1$ to get the eqs; why pick $\sigma(\xi)$ to be this, see below.

Find that only a part of the depression wave interacts with the shock; this part is that part of the wave whose height is $> h_0$ ie. that region in x, t space whose characteristics satisfy $x/t = u+c \geq 1$

this is so if we looks at

$$t(\xi) = k\left[\frac{(4/_3 - \frac{\xi+l}{k})(s-1)}{(4/_3 - s)(\frac{\xi+l}{k} - 1)}\right]$$

when $\frac{\xi+l}{k} = 1$ x & t are ∞ these are the coord of intersection between the shock & characteristics $\Rightarrow u = 0$ & $c = 1$ $\Rightarrow$ forward moving part decays shock for an infinitely long time.

If the wall moves with a given velocity $u_1 > w > u_1 - 2c_1$. The rarefaction wave is narrowed & a const state appears to left of variable zone. $u = w$

If $w \leq 0$ rarefaction wave is still wide enough to include that part which decays the shock. If $0 < w < u_1$, there's a finite time interaction in which shock decays from one constant state to another. from the results indicated the height decreases rapidly at first & then approaches its asymptotic value slowly

Finite Diff eqs.

Govern D.E. $u_t + uu_x + 2cc_x = 0$

$$2c_t + cu_x + 2uc_x = 0$$

with chara eq

$$\left.\begin{aligned} dx/dt &= u+c \\ u+2c &= \text{const} \end{aligned}\right\} \text{ along } C^+ \text{ char}$$

$dx/dt = u-c$ $\}$ along C^- char

the reason for the formulae as given is we look at

$$\frac{x_{ik}-x_{i-1,k}}{t_{ik}-t_{i-1,k}} = \left.\frac{\Delta x}{\Delta t}\right|_{i-1/2,k} = \frac{1}{2}\left[S_{ik}+S_{i-1,k}\right]$$ Centered difference on forward characteristic

$$\frac{x_{ik}-x_{i,k-1}}{t_{ik}-t_{i,k-1}} = \left.\frac{\Delta x}{\Delta t}\right|_{i,k-1/2} = \frac{1}{2}\left[T_{ik}+T_{i,k-1}\right]$$ centered diff on backward characteristic

these can be solved for $u_{ik}+c_{ik}$, $u_{ik}-c_{ik}$, t_{ik}, x_{ik}

then from the jump conditions V_{ij} can be found.

Curves in x,t space are given for shock strength $h/h_0 = 1.8, 3.0$ and ~~we show~~ it is shown that even for 3.0 results are in good agreement between Finite Diff + Approx method. For $x<7$ both methods give coinciding results. For large x,t they no longer coincide, probably due to the large mesh size of the F.D. Scheme in that region. Also in this region the Approx method ~~is better~~ should become more accurate since the shock becomes weaker.

Conclusion even higher than second order terms in the expansion of quantities across the shock differ little from what they would be in the case of a continuous wave. It therefore seems reasonable to apply the method of Friedrichs in cases where the pressure ratios are high.