2-7-74

Complex Variables – Lax

$$\sin \pi z = \pi z \prod_{n=1} \left(1 - \frac{z^2}{n^2}\right)$$

$$1/\Gamma(z) = e^{\gamma z} \prod_{n=1} (1 + z/n) e^{-z/n}$$

any fn of certain type can be factored into a product: from Fundam theorem of algebra

$p(z) = a \prod_{j=1}^{n} (z - z_j)$ a is coeff of z^n this doesn't converge when n is ∞

we can rewrite $p(z) = p(0) \prod_{j=1} (1 - z/z_j)$

$|f(z)| \le \text{const}\ e^{\text{const}\,|z|^p}$; $p < 1$ Entire fns of order p.

Examples $e^{\sqrt{z}} + e^{-\sqrt{z}}$; $\dfrac{e^{\sqrt{z}} - e^{-\sqrt{z}}}{\sqrt{z}}$

if quotient of two fns is entire then it is of order p

Thm: Every entire fn of order $p < 1$ can be factored.

To estimate growth: Maximum Principle + Gauss average mod thm

$$|g(0)| \le \max_{|z|=R} |g(z)|$$

$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z-a}$ let $z - a = Re^{i\theta}$

$f(a) = \frac{1}{2\pi i}\oint_{|z-a|=R} \frac{Ri\, f(a+Re^{i\theta})\, e^{i\theta}\, d\theta}{Re^{i\theta}}$

$f(a) = \frac{1}{2\pi}\oint f(a+Re^{i\theta})\, d\theta$

$|f(a)| \le \frac{1}{2\pi} \max_{|z-a|=R} |f(a+Re^{i\theta})|\, 2\pi$

Construct $|B_R(z)| = 1$ on $|z| = R$

$B_R(z_j) = 0$ $|z_j| < R$

$R\dfrac{z - z_j}{R^2 - z\bar{z}_j}$ maps circle of rad $R \Rightarrow \odot_{R=1}$ & maps $z_j \to$ origin

$$B_R(z) = \prod R \frac{z - z_j}{R^2 - z\bar{z}_j}$$

let $g(z) = f(z)/B_R(z)$ entire since zeroes of f = zeroes of $B_R(z)$

$$\left|\frac{f(0)}{B_R(0)}\right| \le \max_{|z|=R} \left|\frac{f(z)}{B_R(z)}\right| = \max_{|z|=R} |f(z)| \quad \text{since } |B_R(z)| = 1$$

$= M(R)$

$$B_R(0) = \prod \frac{R\,(-z_j)}{R^2} = \prod \frac{-z_j}{R} \qquad |B_R(0)| = \prod \frac{z_j}{R} = \frac{\prod z_j}{R^N}$$

or $|f(0)| \left(\prod \frac{z_j}{R^N}\right)^{-1} = \left|\frac{f(0)}{B_R(0)}\right| \le \max_{|z|=R} |f(z)| = M(R)$

take log assume $|f(0)|=1$

$$N \log R - \sum \log|z_j| \le \log M(R)$$

defines $N(R)$ = no of zeroes of f $|z|<R$

then $\cancel{N(R) \log R} - \sum \log|z_j| \le \log M(R)$; write $\sum \log|z_j|$ as Stieltje Integ

$$- \int_0^R \log r \; dN(r)$$

$$\int_a^b \overset{df}{f_x g\,dx} + \overset{dg}{f g_x dx} = fg\Big|_a^b \quad \text{Integ by parts}$$

$$= \cancel{\log r\; N(r)}\Big|_0^R + \int_0^R N(r)\frac{dr}{r}$$

$$\boxed{\int_0^R N(r)\frac{dr}{r} \le \log M(r)}$$

$$N(R/2)\int_{R/2}^R \frac{dr}{r}; \quad N(R/2) \le \frac{\log M(R)}{\log 2}$$

$$N(R) \le \text{const} \log M(2R)$$

$$M(r) = e^{r^p}$$

$$N(R) \le \text{const } R^p$$

take $R = |z_N|$ $\quad N \le \text{const } |z_N|^p$

$$|z_N| \ge \text{const } N^{1/p}$$

$$\left|\frac{1}{z_N}\right| \le \text{const } N^{1/p}$$

this is what is needed for conv criterion of $P(z) = \prod\left(1 - \frac{z}{z_N}\right)$

conv Criterion $\prod(1+a_n) < \infty$ if $\sum |a_n| < \infty$

Consider $\frac{f(z)}{p(z)}$ which has no zeros $\Rightarrow f/p = \text{const}$.

$\frac{f(z)}{p(z)} = h(z)$ if we can show $|h(z)| \leq \text{const}\, e^{\text{const}|z|^p}$ $p<1$
then h has zeros.

<u>Pf:</u> Suppose not, then $\log h(z)$ is entire $\log h(z) = l(z)$

$$h = e^{l} \qquad |h| = e^{\operatorname{Re} l}$$

$$e^{|z|^p} \geq |h| = e^{\operatorname{Re} l} \qquad \operatorname{Re} l(z) \leq |z|^p$$

$\therefore$ $l = \text{const}$ something like Liouville's thm.

suppose $|l(z)| \leq |z|^p$ we use proof for Liouville's th.

$$l(z_1) - l(z_2) = \int l(w) \left[\frac{1}{w-z_1} - \frac{1}{w-z_2}\right] dw$$

for large W

$$= \int l(w) \frac{z_1 - z_2}{(w-z_1)(w-z_2)} dw$$

$$\leq |w|^p \frac{1}{|w|^2} |W| = |W|^{p-1} < 0$$

hence $|l(z_1) - l(z_2)| \to 0$ for large W hence $l(z_1) = l(z_2)$

since we have an upper bound on h we need lower bound on p

$|p(z)| \geq e^{-c|z|^p}$ on a sequence of circles which avoid the zeros.

$$p = \prod(\quad) = \prod_{|z_j|<2R}\left(1-\frac{z}{z_n}\right) \cdot \prod_{|z_j|>2R}\left(1 - z/z_n\right) \qquad \text{pick} \quad R/2 < |z| < R$$

second factor $\left(1-\frac{z}{z_n}\right)$ $z/z_n < \frac{1}{2}$; if $|a| < \frac{1}{2}$ $\quad 1 - a > e^{-2a}$

$$\Rightarrow \quad |1 - z/z_n| \geq 1 - \left|\frac{z}{z_n}\right| > e^{-2|z/z_n|}$$

$$\left|\prod\left(1 - z/z_n\right)\right| \geq \prod\left(1 - |z/z_n|\right) > e^{-2|z|\sum \frac{1}{|z_n|}}$$

$$\sum_{|z_n|>2R} \frac{1}{|z_n|} = \int_{2R}^{\infty} \frac{1}{r} dN(r) = \frac{1}{R} N(R) + \int N(r)\frac{dr}{r^2} = O(R^{p-1})$$

$\therefore$ this part of product is bdd from below by $\geq e^{-\text{const}\, R^{\rho}} \geq e^{-\text{const}\,|z|^{\rho}}$

For first product

$$\prod_{|z_n|<2R} \left(1 - z/z_n\right) = P(z) \qquad \text{degree } N(2R) \leq R^{\rho}; \text{ also } P(0) = 1$$

let $|z| = r$ $\qquad |P(z)| \geq \prod \left(1 - \frac{r}{|z_n|}\right) = Q(r)$, $Q(0) = 1$

$\exists$ r in $R/2 < r < R$ $\ni$. $Q(r) \geq e^{-cR^{\rho}}$

Must show $\max Q(r) \geq e^{-\text{const}\, R^{\rho}}$

$\text{Max} = R/2<r<R$

Let $\max = \max\limits_{R/2<r<R} Q(r)$

Use of Lagrange Interpolation formula

$$Q(r) = \sum_j Q(r_j)\, L_j(r) \qquad L_j(r_k) = \delta_{kj} \quad . \ L_j \text{ of degree } N$$

$$L_j(r) = \prod_{k \neq j} \frac{(r - r_k)}{r_j - r_k}$$

Choose r_j $\qquad$ N+1 pt equid. $\qquad 0 \quad R/2 \quad R/2N \quad R \quad r$; take $r = 0$

distance from mid pt is $\frac{kR}{2N}$ $\qquad k = 1, \ldots, N/2$

$$Q(0) = \sum_j Q(r_j)\, L_j(0) \leq \text{Max} \cdot N \cdot (4e)^N \qquad r_j - r_k \geq \left[\left(\frac{R}{2N}\right)^{N/2} \left(\frac{N}{2}\right)!\right]^2$$

$$= \left(\frac{N}{2}\right)!^2 \left(\frac{R}{2N}\right)^N$$

$$L_j(0) = \frac{\prod r_k}{\prod r_j - r_k} \leq \frac{R^N}{e^{-N}(R/4)^N} = (4e)^N$$

Stirling Formula $\quad N! = e^{-N} N^N$

$(N/2)! = e^{-N/2} (N/2)^{N/2}$

$[(N/2)!]^2 = e^{-N} (N/2)^N$

$1 \leq \text{Max} \cdot N \cdot (4e)^N$

$\text{Max} \geq \frac{1}{N(4e)^N}$ or $\text{Max} \geq e^{-(1+\log 4)N}$ $\qquad \therefore\ e^{-N}\left(\frac{N}{2}\right)^N \left(\frac{R}{2N}\right)^N$

$= e^{-N}\left(\frac{R}{4}\right)^N$

midpt $\quad \frac{R}{2N} \quad \frac{2R}{2N} \quad \frac{3R}{2N} \qquad \frac{N}{2}\frac{R}{2N}$

$$\prod (r_j - r_k) \geq \prod \left(\frac{kR}{2N}\right)^2$$

H.W. - write up $\prod$ ( ) argument for homework.

Prove $\ell(z)$ analytic fn

$$\operatorname{Re} \ell(z) \le \text{const } |z|^p, \quad p<1 \quad \text{on } \infty \text{ many arbitrarily large circles}$$

$\therefore \quad \ell(z) = \text{const}$

<u>Poisson integral Formula</u> $\quad |z| < R \qquad \tilde{z} = R^2/z$

$$\ell(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{\ell(w)}{w-z}\, dw$$

$$0 = \frac{1}{2\pi i} \int \frac{\ell(w)}{w - R^2/\bar{z}}\, dw$$

$$\ell(z) = \frac{1}{2\pi i} \int \ell(w) \left[\frac{(\bar{z}w - R^2) - (w-z)\bar{z}}{(w-z)(\bar{z}w - R^2)} \right] dw = \frac{1}{2\pi i} \int \ell(w) \cdot \frac{z\bar{z} - R^2}{(w-z)(\bar{z}w - R^2)}\, dw$$

let $\quad w = Re^{i\varphi} \qquad z = re^{i\theta}$ then

$$\ell(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \ell(Re^{i\varphi}) \frac{r^2 - R^2}{r^2 - 2rR\cos(\varphi-\theta) + R^2}\, d\varphi$$

$$\operatorname{Re} \ell(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \ell(w) \quad P(w,z)\, d\varphi \qquad P(w,z) \text{ is Poisson Kernel.}$$

Work it out

2-14-74

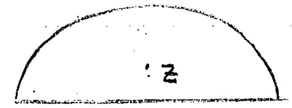
Thm to be proved: If $\operatorname{Re} f(z)$ grows slower than linearly as radius $\to \infty$ then the function $f(z)$ = constant

$$f(0) = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{w}\, dw = \frac{1}{2\pi} \int f(Re^{i\theta})\, d\theta \qquad \text{where } w = Re^{i\theta}$$

$$\operatorname{Re} f(0) = \frac{1}{2\pi} \int \operatorname{Re} f(Re^{i\theta})\, d\theta$$

Suppose $f(w)$ analytic in upper half plane & tends to zero as $|w| \to \infty$

$$f(z) = \frac{1}{2\pi i}\int \frac{f(w)\,dw}{w-z}$$



Contribution from half circle $\to 0$ since $f(w) \to 0$ as $|w| \to \infty$

$$f(x+iy) = \frac{1}{2\pi i}\int f(u)\,\frac{1}{u-x-iy}\,du$$

since integration contrib is from real axis only $\to w \to u$

for inverse pt $\quad 0 = \frac{1}{2\pi i}\int \frac{f(w)\,dw}{w-\bar{z}}$

$$\therefore \quad 0 = \frac{1}{2\pi i}\int \frac{f(u)\,du}{u-x+iy}$$

Subtract the two

$$f(x+iy) - 0 = \frac{1}{2\pi i}\int \frac{f(u)\cdot 2iy\,du}{(u-x)^2+y^2} = \frac{1}{\pi}\int \frac{f(u)\,y}{(x-u)^2+y^2}\,du$$

$$\operatorname{Re} f(x+iy) = \frac{1}{\pi}\int \operatorname{Re} f(u)\,\frac{y\,du}{(x-u)^2+y^2}$$

: $\operatorname{Re} f(z)$ is harmonic when $f(z)$ analyt
converse true conditionally

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given $f(x+iy) = u(x,y) + i\,v(x,y)$ use C.R. eq if $u(x,y)$ is harmonic $\to v(x,y)$ is obtained within constant. But between any two pts P, Q

$$v(P) = v(q) + \int_p^q \underbrace{(v_x \dot{x} + v_y \dot{y})\,ds}_{dv}$$

s is along path of integration

use Green's thm to show that if integral around any closed curve $= 0$ then v can be found. This is true only for simply connected domain.

Another method: since u is harmonic $\Rightarrow \Delta u = 0$ but $\Delta = \partial_z \partial_{\bar{z}}$

$\partial_{\bar{z}}(\partial_z u) = 0 \qquad \partial_z u$ is an analytic fn

$\exists$ an analytic fn $f \ni\ \partial_z f = (\partial_z u)$

H.W. Derive formula for $\operatorname{Im} f(z)$ in terms of real part of f on bdy.

Returning to original problem $w = Re^{i\theta}$ $z = re^{i\varphi}$ $r < R$ (Deriving Poisson Integral)

$$f(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{w-z}\, dw \qquad dw = i w\, d\theta$$

For reflected point $z = R^2/\bar{z}$ (a moment's reflection will show that this is correct ha!)

$$0 = \frac{1}{2\pi i} \int \frac{f(w)}{w - R^2/\bar{z}}\, dw = \frac{1}{2\pi} \int \frac{\bar{z} f(w)\, w\, d\theta}{\bar{z} w - R^2}$$

$$\frac{w}{w - \frac{R^2}{\bar{z}}} \cdot \frac{\bar{z}}{\bar{z}} = \frac{w\bar{z}}{w\bar{z} - R^2} \frac{\bar{w}}{\bar{w}} = \frac{|w|^2 \bar{z}}{|w|^2 \bar{z} - R^2 \bar{w}} = \frac{\bar{z}}{\bar{z} - \bar{w}}$$

$$0 = \frac{1}{2\pi} \int f(w) \frac{\bar{z}}{\bar{z} - \bar{w}}\, d\theta = \frac{1}{2\pi} \int \overline{f(w)} \frac{z}{z - w}\, d\theta$$

Let $\frac{w}{w-z} = \frac{1}{2} \frac{w+z}{w-z} + \frac{1}{2}$; $\frac{z}{w-z} = \frac{1}{2} \frac{w+z}{w-z} - \frac{1}{2}$

$$\therefore f(z) = \frac{1}{2\pi} \int f(w) \frac{w}{w-z}\, d\theta = \frac{1}{2\pi} \int f(w) \left[\frac{1}{2} \frac{w+z}{w-z} + \frac{1}{2} \right] d\theta$$

$$0 = \frac{1}{2\pi} \int \overline{f(w)} \frac{z}{z-w}\, d\theta = \frac{1}{2\pi} \int \overline{f(w)} \left[\frac{1}{2} \frac{w+z}{w-z} - \frac{1}{2} \right] d\theta$$

add

$$f(z) = \frac{1}{2\pi} \int \left\{ \left(\frac{w+z}{w-z} \right) \underbrace{\tfrac{1}{2}\left[f(w) + \overline{f(w)} \right]}_{\mathrm{Re}\, f(w)} + \underbrace{\tfrac{1}{2}\left[f(w) - \overline{f(w)} \right]}_{i\,\mathrm{Im}\, f(w)} \right\} d\theta$$

$$f(z) = \frac{1}{2\pi} \int \mathrm{Re}\, f(w) \left(\frac{w+z}{w-z} \right) d\theta + \underbrace{\frac{i}{2\pi} \int \mathrm{Im}\, f(w)\, d\theta}$$

$i f(0)$ by mean value th.

$$f(z) = \frac{1}{2\pi} \int \mathrm{Re}\, f(w) \left(\frac{w+z}{w-z} \right) d\theta + i f(0)$$

$$\therefore \quad \mathrm{Re}\, f(z) = \frac{1}{2\pi} \int \mathrm{Re}\, f(w)\, P\, d\theta \quad ; \quad \mathrm{Im}\, f(z) = \frac{1}{2\pi} \int \mathrm{Re}\, f(w)\, Q\, d\theta + \mathrm{Im}\, f(0)$$

$$P + iQ = \frac{w+z}{w-z}$$

$$\frac{w+z}{w-z}\;\frac{\bar w-\bar z}{\bar w-\bar z} = \frac{|w|^2-|z|^2}{|w-z|^2} + \frac{(\bar w z-\bar z w)}{|w-z|^2}$$

$$P = \frac{|w|^2-|z|^2}{|w-z|^2} = \frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\varphi)}$$

H.W. Show $Q = \dfrac{\bar w z-\bar z w}{|w-z|^2} = \dfrac{Rr\left(e^{-i\theta+i\varphi}-e^{-i\varphi+i\theta}\right)}{R^2+r^2-2Rr\cos(\theta-\varphi)} = \dfrac{2iRr\sin(\varphi-\theta)}{R^2+r^2-2Rr\cos(\theta-\varphi)}$

H.W. Poisson formula for unit circle

Take circle map onto half plane apply Poisson formula to half plane

now take it back to the circle & check if the two formulas are O.K.

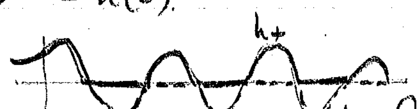
want to Prove Given $l(z)$ $\log l(z) \le |z|^p$ $p<1$ on a sequence of circles $|z|=R\to\infty$

Conclusion : $l(z) = \text{constant}$

Denote by $h(z) = \operatorname{Re}\log l(z)$ Lemma two in notes

$$\frac{1}{2\pi}\int h(Re^{i\theta})\,d\theta = h(0)$$

let $h = h_+ + h_-$



$h_+(\theta) = \max(h(\theta),0)$

$h_-(\theta) = \min(h(\theta),0)$ $h(\theta)$ is no longer differentiable but is cont & Lipschitz

$|h| = h_+ - h_-$

Assumption $h(w) \le R^p \Rightarrow h_+(w) \le R^p$

$\frac{1}{2\pi}\int h_+(w)\,d\theta \le R^p$ use $\frac{1}{2\pi}\int h(Re^{i\theta})\,d\theta = h(0)$

$-\frac{1}{2\pi}\int h_-(w)\,d\theta = \int h_+(w)\,d\theta - h(0)$ to get $\int h_+ + \int h_- = h(0)$

$\le R^p - h(0)$

add two: $\frac{1}{2\pi}\int |h|\,d\theta \le 2R^p - h(0)$

now take Poisson formula for $h(z)$ let $\bar{d}\theta = \frac{d\theta}{2\pi}$

$$h(z) = \int h(w)\, P \,\bar{d}\theta \qquad P \text{ is poisson kernel}$$

$$\frac{\partial h}{\partial r} = \int h(w) \frac{\partial P}{\partial r} \bar{d}\theta$$

$$\frac{\partial h}{\partial \varphi} = \int h(w) \frac{\partial P}{\partial \varphi} \bar{d}\theta$$

as z stays fixed but $R \to \infty$
$P \to 1$
derivatives are $O\left(\frac{1}{R}\right)$

as $R \to \infty$ $\quad \frac{\partial h}{\partial r}, \frac{\partial h}{\partial \varphi} \to 0 \quad \Rightarrow \quad h = \text{constant}.$

From last wk. Every entire fn of order < 1 can be factored & if it's not a constant has infinitely many zeroes

EIGENVALUE PROBLEM.

Sine & Cosines are Efunctions of a self adjoint set of eq and such Efns are orthogonal

we now look at some differential operator $Ly = \lambda y$ $\ni$. $y(0) = y(\pi) = 0$

$L = \partial^2$: $\quad y = \sin\sqrt{-\lambda}\, x + \cos\sqrt{-\lambda}\, x \qquad y(0) = 0 \Rightarrow \cos\sqrt{-\lambda}\, x$ is dropped

$y = \sin\sqrt{-\lambda}\,\pi = 0$ gives cond on λ $\quad \sqrt{-\lambda} = n \quad \lambda = -n^2$

Linear Algebra analogy

$$Ly = \lambda y \qquad L = \begin{pmatrix} L_{11} & \cdots & L_{1n} \\ \vdots & & \vdots \\ L_{n1} & & L_{nn} \end{pmatrix} \qquad \therefore \det(L - \lambda I) = 0$$

can row reduce or row eliminate L $\ni$. we satisfy $n-1$ eq ; the last eq is reduced to $y_1 P(\lambda) = 0$

$\Rightarrow P(\lambda) = 0$ Characteristic Eq

if $\quad L = \partial^2 + q(x)$

$y'' + qy = \lambda y \qquad y'' = (\lambda - q)y$ to solve this solve initial value proble

$y(0) = 0 \quad y'(0) = 1 \quad ; \; y = y(x, \lambda) \quad \Rightarrow$ EV eq $\; y = y(\pi, \lambda) = 0$ is the near solution

This work on entire fns will be connected to the above.

to show y is analytic in λ use Picard iteration let $y' = p \qquad p' = (\lambda - q)\, y$

$$y^{(n+1)}(x) = \int_0^x p^{(n)}(u)\, du \qquad\qquad y(0) = 0$$

$$p^{(n+1)}(x) = 1 + \int_0^x (\lambda - q(u))\, y^{(n)}(u)\, du \qquad p(0) = 1$$

to show analytic by induction. if $y^{(n)}$ $p^{(n)}$ are analytic $\int \lambda y^{(n)}$ is again analytic hence $p^{(n+1)}$ & hence $y^{(n+1)}$ are again analytic since the iteration converges in some domain then the iterates converge uniformly to some limit $\therefore$ the limit is a seq of analytic fns $\therefore$ limit must be analytic.

Rate of growth $\quad |y(x,\lambda)| \leq e^{\text{const}\sqrt{|\lambda|}}$

let $y' = \sqrt{\lambda}\, r \qquad r = p/\sqrt{\lambda}$

$$r' = p'/\sqrt{\lambda} = \left(\frac{\lambda - q}{\sqrt{\lambda}}\right) y \qquad \text{assume } q \text{ is negligible wrt } \lambda$$

Form quantity $\quad |y|^2 + |r|^2 = A \qquad A' = y'\bar{y} + y\bar{y}' + r'\bar{r} + r\bar{r}'$

$$\leq 4\sqrt{|\lambda|}\, yr \leq 2\sqrt{|\lambda|}\left(|y|^2 + |r|^2\right)$$

$$\leq 2\sqrt{|\lambda|}\, A$$

Use Gronwall eq $\quad A \leq e^{\text{const}\sqrt{|\lambda|}} \qquad$ let $\left(A e^{-2\sqrt{|\lambda|}\,\text{const}}\right)' \leq 0$

A is decreasing fn $\Rightarrow \quad A(\pi)\, e^{-2\sqrt{|\lambda|}\,\pi} \leq A(0) = \dfrac{1}{|\lambda|} \quad : \; y(0) = 0,\; r(0) = \dfrac{1}{\sqrt{\lambda}}$

$$A(\pi) \leq \frac{e^{2\sqrt{|\lambda|}\,\pi}}{|\lambda|}$$

$$|y(\pi)|^2 + |r(\pi)|^2 = A(\pi) \leq \frac{e^{2\sqrt{|\lambda|}\,\pi}}{|\lambda|}$$

$$|y(\pi,\lambda)|^2 \leq \frac{e^{2\sqrt{|\lambda|}\,\pi}}{|\lambda|} - |r(\pi)|^2 \leq \frac{e^{2\sqrt{|\lambda|}\,\pi}}{|\lambda|}$$

hence $\quad |y(\pi,\lambda)| \leq \dfrac{e^{\sqrt{|\lambda|}\,\pi}}{|\lambda|^{1/2}} \leq \dfrac{1}{|\lambda|^{1/2}} + \text{const} + |\lambda|^{1/2} + \ldots$

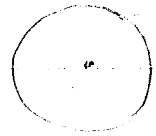
$\therefore \; |y(\pi,\lambda)|$ is $O(|\lambda|^{-1/2})$

2-21-74

Poisson integral formula

#1 Suppose $f(z)$ is analytic

$$g(z) = f\left(\frac{z+w}{1+z\bar{w}}\right)$$



$$g(0) = f(w)$$

Gauss mean value $\quad g(0) = \frac{1}{\pi}\int g(e^{i\theta})\,d\theta \quad$ express $g(e^{i\theta})$ in terms of f.

In terms of fourier series: $\varphi(\theta)$ periodic in 2π

$$\varphi(\theta) = \sum_{-\infty}^{\infty} a_n e^{in\theta}$$

$$a_n = \frac{1}{2\pi}\int \varphi(\theta) e^{-in\theta} d\theta$$

$$\varphi'(\theta) = \sum_{-\infty}^{\infty} in\,a_n e^{in\theta} = \sum c_n e^{in\theta} \qquad c_n = \frac{1}{2\pi}\int \varphi'(\theta) e^{-in\theta} d\theta$$

$$c_n = -\frac{1}{2\pi}\int \varphi(-in) e^{in\theta} d\theta = in\,a_n$$

$$\varphi''(\theta) = \sum_{-\infty}^{\infty} -n^2 a_n e^{in\theta}$$

$$\varphi^{(k)}(\theta) = \sum_{-\infty}^{\infty} (in)^k a_n e^{in\theta}$$

Assume $\exists$ a fn h $\ni$. $\Delta h = 0 \qquad h(r,\theta)$ in unit circle

$$\Delta h = \left(\frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\theta^2\right) h = 0$$

$$h_{rr} + \frac{1}{r} h_r + \frac{1}{r^2} h_{\theta\theta} = 0$$

Let $h(r,\theta) = \sum a_n(r) e^{in\theta}$

$$a_n(r) = \frac{1}{2\pi}\int h(r,\theta) e^{-in\theta} d\theta$$

$$\left[a_n''(r) + \frac{1}{r} a_n'(r) - \frac{n^2}{r^2} a_n(r)\right] e^{in\theta} = 0$$

r^k: $k = \pm n \Rightarrow r^n, r^{-n}$ are solutions $\qquad n=0 \Rightarrow 1 \;\; \log r$

dump r^{-n} at origin since sol blows. $\quad a_n(r) = r^{|n|} a_n(1)$

$$h(r,\theta) = \sum a_n(r)\, e^{in\theta} = \sum a_n(1)\, r^{|n|} e^{in\theta} = \sum \frac{1}{2\pi}\int h(1,\chi)\, e^{-in\chi}\, d\chi\, r^{|n|} e^{in\theta}$$

convergence is uniform interchange $\int, \Sigma$

$$\frac{1}{2\pi}\int h(1,\chi) \sum r^{|n|} e^{in(\theta-\chi)}\, d\chi$$

#2 Prove $\sum r^{|n|} e^{in(\theta-\chi)}$ is poisson Kernel.

Derivation of fourier integral

$h(x)$ periodic in $2\pi R$

$$h(x) = \sum a_n e^{\frac{in}{R}x}$$

$$a_n = \frac{1}{2\pi R}\int_{-\pi R}^{\pi R} h(x)\, e^{-i\frac{n}{R}x}\, dx$$

$$R a_n = \frac{1}{2\pi}\int_{-\pi R}^{\pi R} h(x)\, e^{-i\frac{n}{R}x}\, dx$$ depends on R through exp & lim

define $$b(\xi, R) = \frac{1}{2\pi}\int_{-\pi R}^{\pi R} h(x)\, e^{-i\xi x}\, dx$$

$$a_n = \frac{1}{R}\, b\left(\frac{n}{R}, R\right)$$

$$h(x) = \sum \frac{1}{R}\, b\left(\frac{n}{R}, R\right) e^{\frac{in}{R}x} \cong \int_{-\infty}^{\infty} b(\xi, R)\, e^{i\xi x}\, d\xi$$

look at sum at integer pts $1/R$ apart $\lim_{R\to\infty} \frac{1}{R}\sum f\left(\frac{n}{R}\right) = \int f(x)\, dx$ etc

$b(\xi, R) \to b(\xi)$ as $R \to \infty$

$$b(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x)\, e^{-i\xi x}\, dx$$

$$h(x) = \int_{-\infty}^{\infty} b(\xi)\, e^{i\xi x}\, d\xi$$

fourier integral

$b(\xi) = \mathcal{F}(h(x))$

$\mathcal{F} h_x = i\xi \mathcal{F} h \qquad \mathcal{F} h_{xx} = -\xi^2 \mathcal{F} h \qquad \mathcal{F} h^{(n)}(x) = (i\xi)^n \mathcal{F} h$

Fourier Series of two variables $\varphi(\theta, \chi) = \sum a_n(\chi)\, e^{in\theta} = \sum_{m,n} a_{nm} e^{im\chi} e^{in\theta}$

$$a_n(\chi) = \frac{1}{2\pi} \int \varphi(\theta,\chi) e^{-in\theta} d\theta$$
$$a_{nm} = \frac{1}{2\pi} \int a_n(\chi) e^{-im\chi} d\chi$$

$$a_{nm} = \frac{1}{4\pi^2} \iint \varphi(\theta,\chi) e^{-im\chi} d\chi \, e^{-in\theta} d\theta$$

Work out results for fourier integral. $b(\xi,\eta) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) e^{-i\xi x - i\eta y} dx dy$

$$h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b(\xi,\eta)\, e^{i\xi x + i\eta y} d\xi d\eta$$

To obtain result for half plane. h is harmonic

$$\tilde{h}(\xi, y) = \frac{1}{2\pi} \int h(x,y) e^{-i\xi x} dx$$

$\mathcal{F} h_{xx} = -\xi^2 \mathcal{F} h \qquad : \ \Delta h = -\xi^2 \tilde{h} + \tilde{h}_{yy} = 0$

$$\tilde{h} = a(\xi) e^{\xi y} + b(\xi) e^{-\xi y}$$

$Lh = 0$ L is diff oper with const coeff : if $h(y)$ is a sol so is $\frac{d}{dy} h$ & so is $h(y+a)$

let N: null space of L $\quad \partial: N \longrightarrow N$ (differentiation) linear operator is ∂

N is spanned by the EV of the operator $\partial \implies N$ is spanned by exponential fns. Constancy of coeff $\implies L\partial = \partial L$ Commutation if ∂ & L commute ∂ maps the null space N of L into itself.

$$Lh = 0 \Rightarrow \quad \partial L h = 0 \Rightarrow \quad L \partial h = 0 \quad \text{QED}$$

commutation

if L is not const coeff $\quad \partial L \neq L \partial$

Back to problem

want $\tilde{h} \to 0$ as $y \to \infty$ $\quad \therefore$ take $\tilde{h}(\xi, y) = \text{const}\; e^{-|\xi| y}$ $\quad$ const = const(ξ)

$$\tilde{h}(\xi, y) = \tilde{h}(\xi, 0)\, e^{-|\xi| y}$$

$$h(x,y) = \int \tilde{h}(\xi, y)\, e^{i\xi x}\, d\xi$$

$$= \int \tilde{h}(\xi, 0)\, e^{-|\xi| y + i\xi x}\, d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(p,0)\, \underbrace{e^{-ip\xi - |\xi| y + i\xi x}}_{e^{i\xi(x-p) - |\xi| y}}\, d\xi\, dp$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} h(p,0) \quad P(x-p, y)\; dp$$

\# 3 $\quad P(x-p, y) = \int_{-\infty}^{\infty} e^{i\xi(x-p) - |\xi| y}\, d\xi$

$$h(\omega) = \int h(z)\, P(z, \omega)\, dz \qquad P(z,\omega) \text{ poisson kernel}$$

in upper half plane

* $h(x,y) = \frac{1}{\pi} \int \frac{h(p,0)\, y}{(x-p)^2 + y^2}\, dp$

as $x, y \to x, 0$ does $h(x,y) \Longrightarrow h(x,0)$ $\quad$ yes

If (x,y) is near bdy i.e. y is small then $\quad h(x,y) = \frac{1}{\pi}\int h(x,0)\, P(x-p,y)\,dp$

$h(x,y)$ as defined by $(*)$ is near $h(x,0)$

$$\int P(x-p,y)\,dp = 1$$

$$\int_{x-\delta}^{x+\delta} + \int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty} \quad \leq \text{Max } h(p,0) \cdot \text{total weight outside } |x|<\delta$$
$\to 0$ as $y \to 0$.

$$\arctan \frac{x-p}{y}\Big|_{x-\delta}^{x+\delta} = \frac{2}{\pi}\arctan \delta/y$$ for fixed δ & y small $\arctan \delta/y \to \frac{\pi}{2}$ as $y \to 0$

This shows that $h(x,y) \to h(x,0)$ as $y \to 0$.

$h(x,y)$ must be continuous and bdd. The integral exists since

$$\left|\int \quad\right| \leq \max h(p,0) \left|\int P(x-p,y)\,dy\right| \leq \max h(p,0)\cdot 1$$

h is harmonic show that it is harmonic by differentiat'g inside integral sign since x,y are inside Region not on bdy

$$\Delta h(x,y) \text{ depends on } \Delta P(x-p,y) = \Delta \,\mathrm{Re}\, \frac{1}{\pi i}\frac{1}{p-z} = 0$$

2-28-74

if $h(x,y)$ harmonic in upper half plane $\quad h(x,0)$ are bdy values $\overset{\text{def}}{=} h(x)$

$$h(x,y) = \int P(x-u,y)\, h(u)\,du = \frac{1}{\pi}\int \frac{y}{(x-u)^2+y^2}\, h(u)\,du$$

Conjugate harmonic fn

$$k(x,y) = \int Q(x-u,y)\,h(u)\,du = \frac{1}{\pi}\int \frac{(x-u)\,h(u)\,du}{(x-u)^2+y^2}$$

$$P+iQ = \frac{1}{\pi}\frac{i}{(x-u)+iy}; \text{ derivs satisfy } \begin{bmatrix} P_x = Q_y \\ = -P_u \end{bmatrix}, \begin{bmatrix} P_y = -Q_x \\ = +Q_u \end{bmatrix} \quad \text{C.R. eq}$$

Thm: If $h(u)$ is bdd & continuous, then $\lim\limits_{(x,y)\to(x_0,0)} h(x,y) = h(x)$

look at conj harmonic fn at bdy let $y=0$

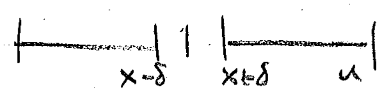
does $k(x,y) \to \frac{1}{\pi}\int \frac{h(u)}{x-u}\,du$? no since integral is divergent.

look at $\int \frac{x-u}{(x-u)^2+y^2}\,[h(u)-h(x)]\,du = k(x,y)$ since $h(x)=\text{const}$ & $\int \frac{(x-u)\,du}{(x-u)^2+y^2} =$

take $y=0$ $\Rightarrow$ does $k(x,y) \to \int \frac{h(u)-h(x)}{x-u}\,du$

if $|h(u)-h(x)| < M\,|u-x|^{\alpha}$ if $\alpha=1$ Lipschitz condition
$\alpha<1$ hölder condition

for both cases we have conv. integrals

Proof of $k(x,y)$ break up interval 

within 2δ $\left|\int\right| \le \int_{x-\delta}^{x+\delta} \left|\frac{h(u)-h(x)}{|x-u|}\right| dx$

between $(x+\delta\,;\,M)$ $(x-\delta,\,-M)$ we have u.c. to $\int \frac{h(u)-h(x)}{x-u}\,dx$
at ∞'s then since if we take $|h(u)| \le \frac{1}{|u|^{\alpha}}$ then $\int \to 0$
then we can get $k(x,y) \to \int \frac{h(u)-h(x)}{x-u}\,du$.

"Cauchy Principal Value Th."

Thm: If $h(u)$ is bdd & hölder cont, $\lim\limits_{(x,y)\to(x_0,0)} k(x,y) = \frac{1}{\pi}\,\overset{P.V.}{\int} \frac{h(u)\,du}{x_0-u}$

$$\text{P.V.}\int \frac{h(u)}{x-u}\,du = \lim_{\delta\to 0} \int_{-\infty}^{x-\delta} + \int_{x-\delta}^{\infty}$$

$$\int \frac{h(u)-h(x)}{x-u}\,du = \int_{-\infty}^{x-\delta} \frac{h(u)-h(x)}{x-u}\,du + \int_{x+\delta}^{\infty} \frac{h(u)-h(x)}{x-u}\,du$$

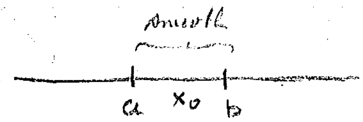
cancel by symmetry.

If bdy values are cont & diff the harmonic fns are diff up to the bdy.

Thm if $h(x)$ is bdd & differentiable (cont 1ST deriv), $h(x,y)$ has 1ST derivs bdd which are cont up to bdy.

$$\frac{\partial h}{\partial x} = \int P_x\, h(u)\,du = \int -P_u\, h(u)\,du = \int P h_u\,du$$

tho' this is local not global in result; suppose



then $\frac{\partial h}{\partial x}(x_0,0) = \int P h'(u)\,du + P h\Big|_a^b$ for interval containing x_0

$$\frac{\partial h}{\partial y} = \int P_y\, h(u)\,du = \int Q_u\, h(u)\,du = -\int Q\, h'(u)\,du + Q h\Big|_a^b$$

for h_y to exist the above th must be modified to read hölder cont 1ST deriv

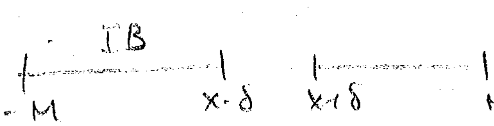
Privaloff Thm. (Schauder) if $h(x,y)$ is hölder cont of exp α, $k(x,y)$ is hölder cont of exp α.

$$k(x) = \frac{PV}{\pi}\int_{-\infty}^{\infty} \frac{h(u)}{x-u}\,du \qquad k = Hh$$

H is a singular integral operator. H - is Hilbert transform

Hilbert 1860-1943

Question of convergence of $k(x,y) \to \int_{-M \leftrightarrow x+\delta}^{x-\delta \leftrightarrow M} \frac{h(u)-h(x)}{x-u}\,du$

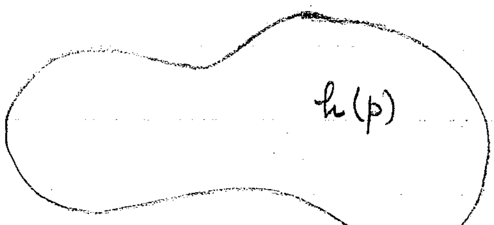


$k(x,y) = k_M + k_{IB} + k_\delta$ claim k_M, k_δ are uniformly small pick $\int \frac{x-u}{(x-u)^2+y^2} [h(u)-h(x)] \, du$ for k_{IB}, $k\delta$

$$\int \frac{x-u}{(x-u)^2+y^2} h(u) \, du \text{ for } k_M; \text{ if } |h(u)| \leq \frac{1}{|u|^\alpha} \text{ then}$$

$$\int \frac{1}{|x-u||u^\alpha|} du \to 0 \quad \text{as } M \to \infty$$

Smooth simple curve $z(t)$, $\frac{dz}{dt} \neq 0$ if h is harmonic



$$= \int h(z) \, P(p,z) \, dt$$

uniqueness proven by max principle on 2 fns (h_1, h_2) use max on $h_1 - h_2$ which has zero b.c. $\Rightarrow h_1 - h_2 = 0$ inside $\therefore h_1 = h_2$

Existence

$h(p) = \ell_p(h)$ $\quad$ h is on bdy

ℓ is a linear fnl ; if h_1 & h_2 are harmonic $a_1 h_1 + a_2 h_2$ is also harmonic $\therefore$

$$\ell(a_1 h_1 + a_2 h_2) = a_1 \ell(h_1) + a_2 \ell(h_2)$$

Let C: space of all continuous fns on bd.

H: Bdy values of harmonic fns.

$H \subset C$ $\quad$ ℓ: linear fnl defined on H

$|\ell(h)| \leq \|h\|$ $\quad$ $\|h\| = \max |h(z)|$ on bdy.

Hahn-Banach Thm: Linear fnl defined on subspace of linear space & bdd as above

We can always extend it to whole space without increasing the norm.

Proof: suppose $g \notin H$ extend l $\ni$ $l(g)$ is defined

look at $h + cg$ $\quad l(h+cg) = l(h) + c\,l(g) \le \|h + cg\|$

let $c > 0$ $\quad l(h/c) + l(g) \le \|h/c + g\|$ $\quad h_1 = h/c \in H$

$c < 0$ $\quad l(h/{-c}) - l(g) \le \|\frac{h}{-c} - g\|$ $\quad h_2 = -h/c \in H$

$$l(h_2) - \|h_2 - g\| \le l(g) \le \|h_1 + g\| - l(h_1)$$

$l(g)$ exists if lim sup (left hand ineq) $\le$ elements of right hand inequality.

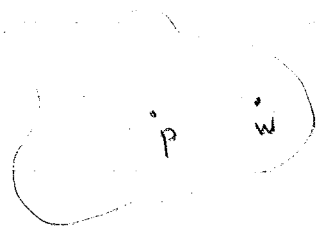
$$l(h_2) + l(h_1) \le \|h_1 + g\| + \|h_2 - g\|$$

$$l(h_1 + h_2) \le \|h_1 + h_2\| = \|h_1 + g + h_2 - g\| \le \|h_1 + g\| + \|h_2 - g\|$$

Let $h = \log|z - w|$

$l(h) = g(w)$ $\quad$ claim $g(w)$ is harmonic fn.

h is harmonic since log is harmonic



l & Laplacian commute hence $\Delta l(h) = l(\Delta h) = \Delta g$

$\Delta h = 0 \quad l(0) = 0 \quad \Rightarrow \Delta g = 0$.

$$\frac{l(h(x+\delta)) - l(h(x))}{\delta} = l\left(\frac{h(x+\delta) - h(x)}{\delta}\right) \to l(h_x) \text{ in normed sense}$$

$$g_x = \partial_x g = \lim \frac{g(x+\delta, y) - g(x,y)}{\delta}$$

3-7-74

Linear fnl

V Linear Space $f, g \in V$ $\quad af + gb \in V$ $\quad a, b$ Scalars

Space of continuous fns $\quad$ Linear fnl space $\quad l(af + bg) = a\,l(f) + b\,l(g)$

defined on whole- or sub-space

Bddness on Space of cont fns $\quad l(f) \le \|f\|_{max}$

if $\ell(f) \le \|f\|_{max}$ then $|\ell(f)| \le \|f\|_{max}$

Extension Hahn Banach Th. ℓ is a linear fnl defined on a subspace H of C (linear space) which is bdd: can be so extended ie expand H to C

ℓ - linear fnl of $f \in C$ $\quad \ell(f) =$ number $\quad$ if $f = f(x)$ then $\ell(f) = \ell(f(x))$

if $f(x)$ is a cont fn of x then $\ell(f)$ is a cont fnl of x

ie $f(x+h)$ differs little from $f(x)$ if h is small

$$\|f(x+h) - f(x)\|_{max} \to 0 \text{ as } h \to 0$$

$$\ell(f(x+h)) - \ell(f(x)) = \ell(f(x+h) - f(x)) \; ; \quad |\ell(\quad)| \le \| \quad \|_m$$

but $\|f(x+h) - f(x)\|_{max} \to 0$ as $h \to 0$

if $f(x)$ "depends differentiably on x" so does $\ell(f(x))$

$$\equiv \frac{f(x+h) - f(x)}{h} \to f'(x)$$

$$\text{means } \left\| \frac{f(x+h) - f(x)}{h} - f'(x) \right\|_{max} \to 0$$

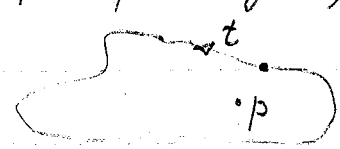
fn of 2 variables $\frac{f(t,x+h) - f(t,x)}{h} \to f'(x,t)$ uniformly in t for t fixed

we can show that if $f(x,y)$ depends cont on x,y ℓf does also

& all other prop of f are transferred to $\ell(f)$.

$$\frac{d}{dx}\ell(f) = \ell\left(\frac{d}{dx}f\right) \text{ etc.}$$

now let C: be the space of cont fns $f(t)$ on bdy



z pts on bdy

H : bdy values of harmonic fns.

Want to prove $H \subset C$ $C \subset H$. for any pt p in region

$$\ell(h) = h(p) \qquad \ell = \ell_p \text{ depends on pt chosen}$$

now we can use extension thm to extend ℓ from H to C

suppose $\exists$ a ω outside the region; fix it.

define $h(t) = \log |z(t) - w|$ claim for w outside ω belongs to H

$$\ell_p(h(w)) = \log|p - w| \quad \text{if } w \text{ outside}$$

$-g(p, w)$ inside; will study how g varies with w

g harmonic fn of w since h is harmonic fn of w $\quad h = \log|z - w|$

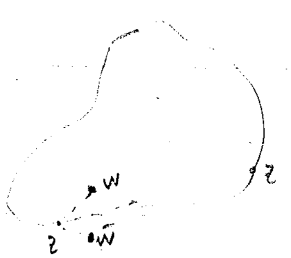
as w approaches bdy, h is large we don't know what happens to g

from outside $\ell_p(h(w))$ approaches some fixed limit

claim: $g(p, w)$ is cont in w across the bdy let $\bar{w}$ be reflected pt of w

$$g(p,w) - g(p,\bar{w}) = g(p,w) - \log|p - \bar{w}| = \ell(h(w)) - \ell(h(\bar{w}))$$

$$= \ell(h(w) - h(\bar{w})) = \ell\left(\log \frac{|z-w|}{|z-\bar{w}|}\right)$$



$$\left|\frac{z-w}{z-\bar{w}}\right| \sim 1 \quad \forall z$$

$$\left|\left|\frac{z-w}{z-\bar{w}}\right| - 1\right| < \epsilon \quad \forall z \quad \text{if } w \text{ is near } z$$

$$\log\left|\frac{z-w}{z-\bar{w}}\right| < 2\varepsilon \quad \forall z$$

using bddness of ℓ then $\ell\left(\log\left|\frac{z-w}{z-\bar{w}}\right|\right) < 2\varepsilon$

we have constructed a g which is harmonic & has value on bdy $\log|p - w|$

$\log(p - w)$ as fn of $w \in H$

$\log(r - w)$ as f. of $w \in H$

$g(r,w)$ is a harmonic fn whose bdy value is $\log|r-w|$

$\therefore \quad \ell_p(\log|r-w|) = g(r,p)$

bdy pt / param

but $\quad \ell_p(\log|z-r|) = g(p,r)$

$\therefore \quad g$ is a symmetric fn.

p is fixed $\quad g$ harmonic $\xRightarrow{\text{define by C.R. equa.}}$ k conj fn $\quad g+ik$ – anal fn.

$$\log(w-p) - (g+ik) = n(w)$$

Green's fn. $=$ $\mathrm{Re}\, n(w) = \log(w-p) - g(p,w)$ is a harmonic fn which is zero on bdy

inside $\quad \log|w-p| - g(p,w) \leq 0 \quad$ since for pts inside $\quad$ w near p $\quad |w-p|<1$

$\log \varepsilon$ is large $- \quad \therefore \quad \log|\;\;| - g < 0 \quad$ w

$$(w-p)\,e^{-(g+ik)} = f(w)$$

$f(w) = e^{n(w)} \qquad \mathrm{Re}\, n = 0$ on bdy & ≤ 0 inside

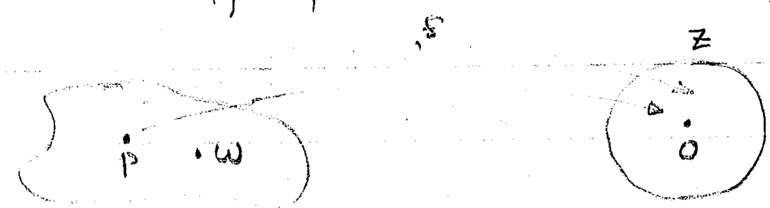
$|f(w)| = e^{\mathrm{Re}\, n} \qquad [\,|f(w)| = 1$ on bdy$\,]$ *

$|f(w)| < 1$ inside

$f(p) = 0 \quad f(w) \neq 0$ for $w \neq p$

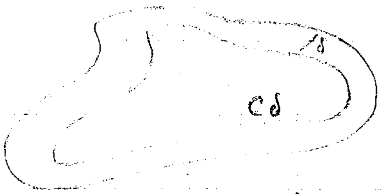
* $f(w)$ is defined inside since $\log|w-p|$ is def everywhere $\quad g$ defined inside & on bdy but k is only defined inside $\therefore$ $n(w)$ is defined in min region of definition $\therefore$ $n(w)$ is defined inside only $\Rightarrow f(w)$ is defined inside $\therefore \quad |f(w)| \longrightarrow 1$ as $w \longrightarrow$ bdy

this maps

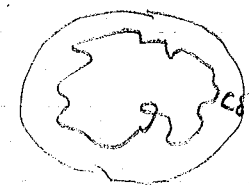


take $|z| < 1$ we want to count the no of times f takes on value of z

Draw curve $\parallel$ to bdy in w plane. ϕ inside C_δ



$$\text{look at } \int_{C_\delta} \frac{f'(w)}{f(w)-z}\, dw = \text{integer}$$



show $\displaystyle\int \frac{f'(w)}{f(w)-z}\, dw = 1$ δ small enough

choose δ small enough $\ni$ $|f(w)| > |z| + \text{small}$ for w on C_δ.

we know that for $z=0$ $\int = 1$; deform origin to z

since this can be done continuously & f is cont & $\int$ takes on integer values

$\Rightarrow \int = 1$ independent of z $\therefore$ f is 1-1 onto.

This is Riemann Mapping Thm.

to show that k is cont up to bdy

if we show that g_x & g_y are bdd up to bdy by CR we can show that k_x, k_y are bdd up to bdy hence with

$K = \int k_x dx + k_y dy$ we know K is defined up to bdy

to show g has bdd deriv : if h harmonic in disk radius R & $|h(x,y)| \le M$ Then $|h_x(0)|; |h(0)| \le \frac{M}{R}$

e^{trick} of

$$|f(z)| \le e^M$$

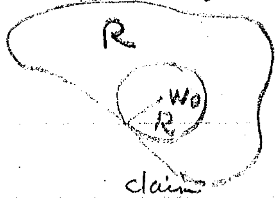
$$|f'(0)| \le \frac{e^M}{R} \quad \text{by Schwarz inequality}$$

$$|h'(0)\, e^{h(0)}| \le \frac{e^M}{R} \quad \text{use lower bd } e^{h(0)} \ge e^{-M}, \; |h'(0)| \le \frac{e^{2M}}{R}$$

H.W.

expres h_x, h_y in terms of bdy values of h by differentiating Poisson formula

$$h(w) = \log|w-p| - g(p,w)$$



In disk $|w - w_0| < R$

claim $|h(w)| \le \text{const} \cdot R$ if curve is twice differentiable

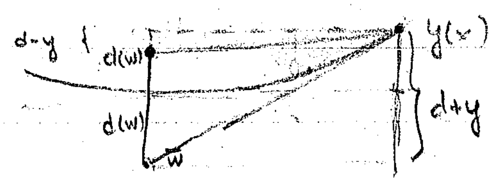
any pt w in disk is at most $2R$ from bdy construct reflected pt

$$h(w) = \log|w-p| - \log|\hat{w}-p| + \log|\hat{w}-p| - g(p,w)$$

$$= \log\frac{|w-p|}{|\hat{w}-p|} + \underbrace{\ell_p\left(\log\frac{|\hat{w}-z|}{|w-z|}\right)}$$

$$= \text{const} \times R \quad \Big\} \le \left\|\log\frac{|\hat{w}-z|}{|w-z|}\right\|_{max} \le \text{const } d(w)$$

$d(w)$ distance to bdy.



$$\frac{(d-y(x))^2 + x^2}{(d+y(x))^2+x^2} = \frac{d^2 - 2dy + y^2 + x^2}{d^2+2dy+y^2+x^2}$$

Since y is twice diff $y(x) \le kx^2$

$$= 1 - \frac{4dy}{d^2+2dy+y^2+x^2}$$

$$\le 1 - \underbrace{\frac{4d\kappa x^2}{x^2}}_{\kappa d = \text{const}}$$

HW. Suppose y is once differentiable & $y'(x)$ is Hölder cont.

$$|y'(x_1) - y'(x_2)| \le \kappa|x_1 - x_2|^\alpha \qquad \alpha < 1$$

Show 1) $y(x) \le \text{Const } x^{1+\alpha}$

2) $h(w) \le d^\alpha(w) \text{const}$

3) $|g_x|, |g_y| \le \text{const } d^{\alpha-1}$

4) g, k (conj g) are hölder cont in sense that $|g(w_1) - g(w_2)| < A|w_1 - w_2|^\alpha$

$|k(w_1) - k(w_2)| < B|w_1 - w_2|^\alpha$

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This was homework from last week $y = y(x)$

$|y'(x_1) - y'(x_2)| \le \text{const}\, |x_1 - x_2|^{\alpha}, \quad 0<\alpha<1$

$h(w)$ harmonic, $=0$ on bdy

$|h(w)| \le \text{const}\, d(w)^{\alpha}$

$|h_{x_y}| \le \text{const}\, d(w)^{\alpha-1}$

Review $\quad h(w) = \log|p-w| - g(p,w)$

$n(w) = h + ik = \log(p-w) - \cdots$

$f(w) = e^{n(w)} = (p-w)\, e^{(\ \ )}$

this maps domain onto unit disk $\ni$ $p \to$ origin

these methods of mapping is indirect, i.e. since g is constructed we can stick to w plane and continue to solve problem. What we don't have to do is map into unit circle. Another objection is conformal maps only holds for 2-D and this is a severe restriction.

Green's fn $\quad G = \log|p-w| - g(p,w)$

$$\iint (u\,\Delta v - v\,\Delta u)\, d\tau = \int \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) ds$$

Problem: wish to find harmonic fn h inside which takes on the given nice bdy values on C.

{ Tietze extension th. – fn continuous on a closed subset of a compact space can be extended.

suppose h_0 takes on bdy values but is not harmonic. Let $h_0 + u = h$

u has to be $=0$ on bdy and $\Delta h = \Delta u + \Delta h_0 = 0$

$f = \Delta u = -\Delta h_0$ inside

Whitney Extension Thm – extension with differentiability

To do the extension from bdy into interior, we do it locally by mapping some part of C into a straight line. if values on line is $b(x)$

then $h_0(x,y) = b(x) - \frac{y^2}{2} b_{xx}(x)$

$\Delta h_0(x,0) = h_{0xx} + h_{0yy} = \cancel{b_{xx}} - \frac{y^2}{2} b_{xxxx} - \cancel{b_{xx}} = 0$ since $y=0$

no we have $\Delta u = f$, $u = 0$ choose $v = G$

$$\iint (u \underbrace{\Delta v}_{c\,\delta(w-p)} - v \Delta u)\, d\tau = \iint (u \Delta G - G \Delta u)\, d\tau = \iint (u \cdot \text{const}\, \delta(w-p) - G f)\, d\tau \qquad c = \text{const} = 2\pi$$

$$= 2\pi u(p) - \iint f(w)\, G(w)\, d\tau = \underbrace{\int \left(u \frac{\partial G}{\partial n} - \frac{\partial u}{\partial n} G\right) ds}_{=0}$$

but $G = 0$ on bdy & $u = 0$ on bdy

$$\therefore \quad 2\pi u(p) = \iint f(w)\, G(w,p)\, d\tau$$

to show that as $p \to$ bdy $u(p) \to 0$. We have show $G(w,p) = G(p,w)$

and also for fixed w & p near bdy $G(w,p) \to 0$

but we integrate over all w in τ $\therefore$ if both w, p are near bdy

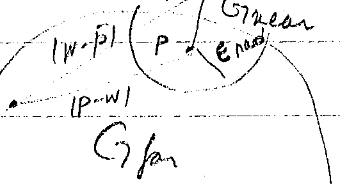
$G(w,p) \to 0$ not uniformly. Break up w into 2 sets

w near p & w away from p. draw ϵ rad about p

for $|w-p| > \epsilon$ go into G_{far}

for $|w-p| < \epsilon$ " " G_{near}

$\log \frac{|p-w|}{|\bar{p}-w|} \sim \log 1 \sim 0$



$$\underbrace{\iint_{G_{far}}}_{\substack{\text{for } w \text{ away from } p \\ G(\text{as } p \to \text{bdy}) \to 0}} + \iint_{G_{near}}$$

$G_{near} \to 0$ because we can make ϵ as small as we wish.

$$G = \log|p-w| - g(p-w) \leq 0 \quad \text{inside region}$$

$$\log|p-w| \leq g(p,w) \leq ? \max_{z \text{ on } C} \log|p-z| = K(\epsilon)$$

since $g(p,w) = \log|p-w|_{\text{on bdy}}$ by max principle

$$|G(p,q)| \leq 2\big|\log|p-w|\big| + K$$

let $|f| \leq M$

$$\iint_{G_{near}} f\,G\,d\tau \leq M \iint_{C_\epsilon} \{2\log|p-w| + K\}\,d\tau = \text{small depending on } \epsilon$$

this shows that as $p \to$ bdy $u(p) \to 0$.

$$\Delta u(p) = \iint f(w)\,\Delta_p G(p,w)\,d\tau = \iint f(w)\,\delta(p-w)\,d\tau = f(p) \quad ?$$

$$u(p) = \iint f(w)\,\big[\log|p-w| - g(p,w)\big]\,d\tau$$

$\Delta_p g(p,w) = 0$ { since g is smooth and as long as p is not near bdy.
(& harmonic in p

$$u^{(\text{singular})} = \iint f(w)\log|p-w|\,d\tau$$

$$u_x^{(s)} = \iint f(w)\,\frac{w_x - p_x}{|p-w|^2}\,d\tau = \iint f(w)\,\underbrace{\partial_x \log|p-w|}_{\text{integrable in 2-D.}}\,dw$$

H.W. prove that the x derivative can be obtained by diff under integral sign.

we assume $f_x = -w_x$ $\therefore$

$$\iint f(w)\, \partial_x \log|p-w|\, d\tau = \iint -f(w)\, \partial_{w_1} \log|p-w|\, d\tau$$

$$= \iint f_{w_1}(w) \log|p-w_1|\, d\tau$$

$$u_{xx}^{(sing)} = -\iint f_{w_1}(w)\, \log_{w_1}|p-w|\, d\tau \; ; \; u_{yy}^{(sing)} = -\iint f_{w_2}(w) \log_{w_2}|p-w|\, d\tau$$

$$\Delta u^{(sing)} = -\iint_G f_{w_1} \partial_{w_1} \log|p-w| + f_{w_2} \partial_{w_2}\log|p-w|\, d\tau + \int f \partial_n \log|p-w|\, ds$$

Let $G_\varepsilon = G - (p)_\varepsilon$

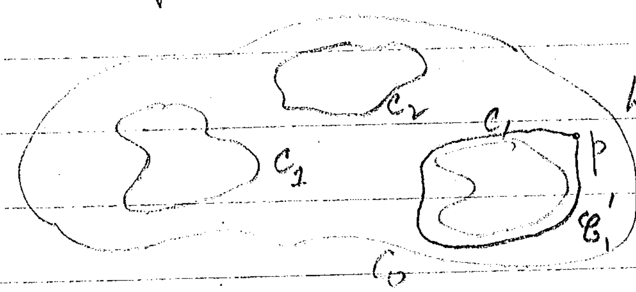
$$= \lim_{\varepsilon \to 0} \iint_{G_\varepsilon} [f_{w_1} \partial_{w_1} \log|\;| + f_{w_2} \partial_{w_2} \log|\;|]\, d\tau + \cdots$$

$$= -\iint_{G_\varepsilon} f \underbrace{\Delta_w \log|p-w|}_{0 \text{ since } p \neq w} - \int_\varepsilon f \partial_n \log|p-w|\, d\tau - \int_C f \partial_n \log|p-w|\, ds$$

$$+ \iint f \frac{1}{r} r\, d\theta$$

$$\to f(p) \quad \text{as } \varepsilon \to 0$$

Multiply Connected Region Mappings



take arbitrary pt p & send to ∞ by.

Let f map rest onto exterior of $n+1$ vertical strips

f maps G onto exterior of vertical slits ; $p \to \infty$

$f = \frac{1}{w-p} + j(w)$ for mapping onto slits $\Rightarrow$ $Re f = const$ on each bdy component $\therefore Re f = c_j$ on $\mathcal{C}_j$, $c_0 = 0$

$\therefore Re\, j(w) = c_j - Re \frac{1}{w-p}$

Let $j = h + ik$ if h is found then $k(w) = \int_p^w k_x dx + k_y dy = \int_p^w -h_y dx + h_x dy$

Condition that k be single valued should be path independent. $\Rightarrow$ n conditions

$$\oint_{\mathcal{C}_j'} -h_y dx + h_x dy = 0 \quad \& \text{ n parameters } c_j$$

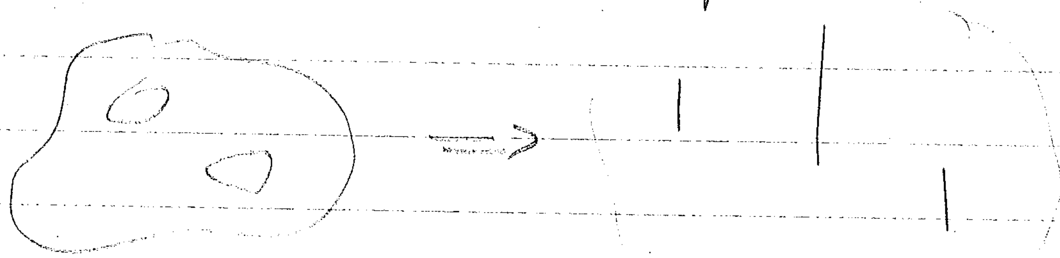
$Ac = b$ can be solved if $A\underset{\sim}{c} = 0 \Rightarrow \underset{\sim}{c} = 0 \Rightarrow \det A \neq 0$

h depends linearly on the c_j & the integral is linear

let $h = h_1 + h_2$ $h_1 = c_j$ $h_2 = Re \frac{1}{w-p}$

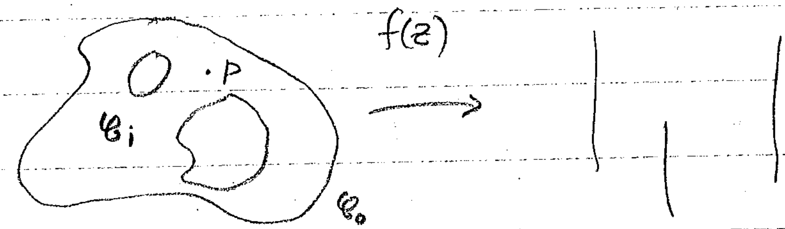
$$\int -h_{1y} dx + h_{1x} dy = 0 = \underbrace{\int h_2}_{\text{right hand side } b}$$

if there was a soln $\overset{c_j \neq 0}{\wedge}$ that $\Rightarrow$ we can map



into the open set with bdy as the slits but slits don't bound open set $\Rightarrow$ $j = const$

3-21-74



use of $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps the circle into a slit in

$f = h + ik$

h harmonic

$h(z) = C_i$ on $\mathscr{C}_i$

has singularity at p 0 on $\mathscr{C}_0$

$$h(z) = \operatorname{Re} \frac{1}{z-p} + j(z)$$

$$j(z) = C_i - \operatorname{Re} \frac{1}{z-p} \quad \text{on } \mathscr{C}_i$$

$$j(z) = j_0 + w(z)$$

where $j_0 = -\operatorname{Re} \dfrac{1}{z-p}$ on $\mathscr{C}_i$

$$w(z) = C_i \text{ on } \mathscr{C}_i$$

$$w(z) = \sum C_i w_i(z) \qquad w_i \text{ harmonic}$$

$$w_i(z) = \begin{matrix} 1 & \text{on } \mathscr{C}_i \\ 0 & \text{on } \mathscr{C}_k \quad k \neq i \end{matrix}$$

Conjugate of $j(z)$ should be single valued

$$0 = \int_{\mathscr{C}} \frac{dj^{\text{conj}}}{ds} \, ds = \int_{\mathscr{C}} \frac{dj}{dn} \, ds \qquad \int_{\mathscr{C}_k} \frac{dj}{dn} = 0 \text{ for } k = 1, \dots, n$$

$$\int_{\mathscr{C}_k} \frac{dj_0}{dn} + \sum_{i=1}^{n} C_i \int_{\mathscr{C}_k} \frac{dw_i}{dn} \, ds = 0 \qquad (*)$$

Should note $w(z; C_1, \dots, C_n)$ is linear in the C_i

Claim homogeneous system has trivial sol

if not w would be single valued analytic mapping of n connected domain onto

[illegible] bounded domain $\Rightarrow$ no pt is mapped into infinity $\therefore$ We can connect an interior & exterior pt without passing through body $\rightarrow\leftarrow$

$\therefore$ solution must be trivial

Let $p_{ik} = \int_{\mathscr{C}_k} \frac{dw_i}{dn} ds$

$\therefore \int_{\mathscr{C}_n} \frac{dj_0}{dn} + \sum_{i=1}^{n} c_i p_{ik} = 0$ to be (*)

Another proof

Let $w = \sum \xi_i w_i$ $\quad z = x+iy$ $\quad$ since w is harmonic $\iint_G w\Delta w \, dxdy = 0$

K.E. of system $\iint_G (w_x^2 + w_y^2) dxdy$ Dirichlet Problem.

Use Greens second formula $\iint f \cdot \nabla w = -\iint w \nabla \cdot f + \int \frac{\partial f}{\partial n} w \, ds$

Let F be a vector $\iint_G \nabla \cdot \underline{F} = \int \frac{\partial F}{\partial n} ds$ $\quad$ let $\underline{F} = w \underline{f}$

$\iint \nabla \cdot \underline{F} = \iint (w \nabla \cdot \underline{f} + f \cdot \nabla w) = \int w \frac{\partial \underline{f}}{\partial n} ds$

Let $f = (w_x, w_y)$ $\quad \nabla w = (w_x, w_y)$

$\Rightarrow \iint_G w_x^2 + w_y^2 \, dx dy = -\iint w \Delta w + \int w \underset{\underline{n}\cdot\nabla w}{\frac{\partial w}{\partial n}} ds$

$= \int \sum \xi_i w_i \sum \xi_j \frac{\partial w_j}{\partial n} ds$

$= \sum \xi_i \xi_j \int_{\mathscr{C}} w_i \frac{\partial w_j}{\partial n} ds$

$w_i = \begin{cases} 1 & \mathscr{C}_i \\ 0 & \mathscr{C}_k \quad k \neq i \end{cases}$ $\quad \int_{\mathscr{C}} = \int_{\mathscr{C}_i} = \int_{\mathscr{C}_i} \frac{\partial w_j}{\partial n} ds = p_{ji}$

w Prove that this matrix is symmetric $p_{ij} = p_{ji}$

$\Rightarrow \sum \xi_i \xi_j p_{ij} > 0$ $\quad \therefore$ matrix is non singular it is not zero since $\iint w_x^2 + w_y^2 \, dx\, dy > 0$

if $=0 \Rightarrow w = \text{const} \Rightarrow w$

HW here's how to start $p_{ik} = \int_{C_k} \frac{dw_i}{dn} ds = \int_C \frac{dw_i}{dn} w_k \, ds$ integrate by parts.

if P were not symmetric write $P = S + A$ S = sym A = antisym

$$p_{ik} = \underbrace{\frac{p_{ik}+p_{ki}}{2}}_{= s_{ik}} + \underbrace{\frac{p_{ik}-p_{ki}}{2}}_{a_{ik}}$$

$$\sum p_{ik}\xi_i\xi_k = \sum s_{ik}\xi_i\xi_k + \underbrace{\sum a_{ik}\xi_i\xi_k}_{0},$$

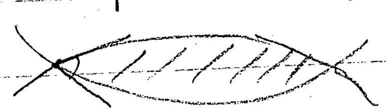
$$\sum p_{ik}\xi_i\xi_k = \sum s_{ik}\xi_i\xi_k$$

Thm if p has its symmetric part positive definite then p_{ik} is invertible

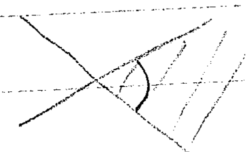
HW Prove the above

Mapping: half plane into unit circle

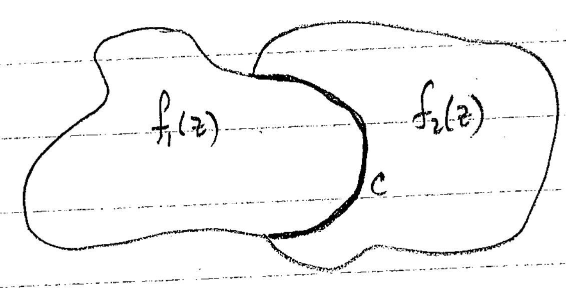
quarter plane $\rightarrow$ half plane



perform inversion to



Analytic continuation & Schwarz Reflection principle



$f_1(z) = f_2(z)$ on C

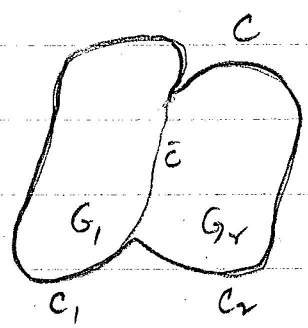
if $f(z)$ is defined as $f_1(z)$
$f_2(z)$

then $f(z)$ is defined in union

pick a pt w

$$f_1(w) = \int_{\mathscr{C}_1} \frac{f_1(z)}{z-w}\,dz$$

$$0 = \qquad w \notin G_1$$



$$\int_{\mathscr{C}_2} \frac{f_2(z)}{z-w}\,dz = \begin{cases} 0 & \text{if } w \in G_1 \\ f_2(w) & w \in G_2 \end{cases}$$

$$\int_{C_1} \frac{f_1}{z-w}\,dz + \int_{C_2} \frac{f_2}{z-w}\,dz = f(w)$$

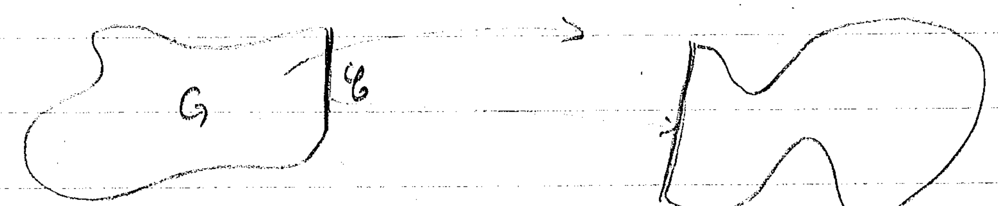
$$\int_{C_1-\bar{C}} + \cancel{\int_{\bar{c}} \frac{f_1}{z-w}\,dz} + \int_{C_2-\bar{C}} - \cancel{\int \frac{f_2}{z-w}\,dz} = \int_{\mathscr{C}} \frac{f(z)}{z-w}\,dz = \begin{matrix} f_1(w) & w \text{ in } G_1 \\ f_2(w) & w \text{ in } G_2 \end{matrix}$$

Analytic cont holds if $f(z)$ not continuous up to bdy in this integral sense

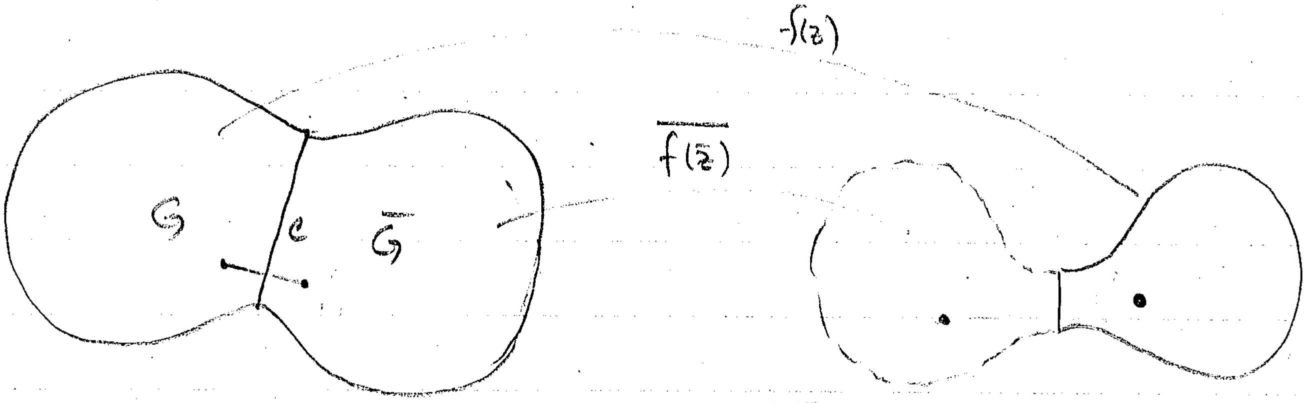
Schwarz reflection principle

$f(z)$ in G, part of whose bdy C is a straight line or circl

$f(z)$ cont up to bdy & maps C into part of straight line or circle



$f(z)$ can be continued analytically into $\overline{G}$, reflection of G across C



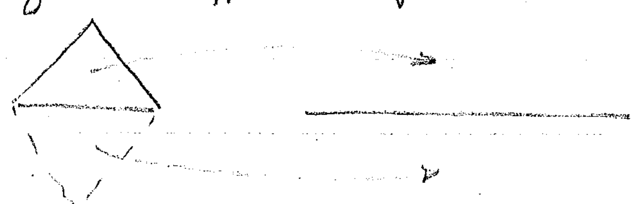
define for z in $\bar{G}$ $\quad f_2(z) = \overline{f(\bar{z})}$ analytic fn.

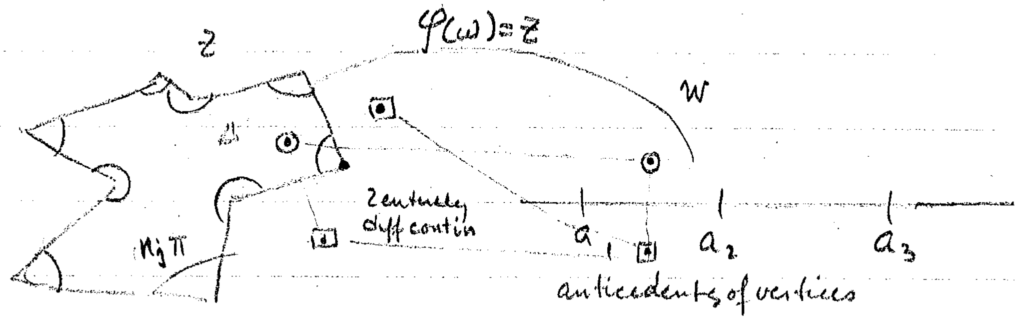
$f(z) = \overline{f(\bar{z})}$ on bdy where $^{-}$ refers to reflection across c.

This works only if $\bar{G}$ doesn't intersect G : $\bar{G} \cap G = \phi$

Reflection across circle is inversion

Schwarz - Cristoffel Transformation





let $\varphi_1(w)$ $\quad \varphi_2(w)$ be continuations

$$\varphi_1(w) = \alpha\, \varphi_2(w) + \beta$$

$$\frac{\varphi_1''}{\varphi_1'} = \frac{\varphi_2''}{\varphi_2'} \quad \therefore \quad \frac{\varphi''}{\varphi'} \text{ can be continued}$$

we see that at the vertices $z \sim (w-a)^{\kappa}$

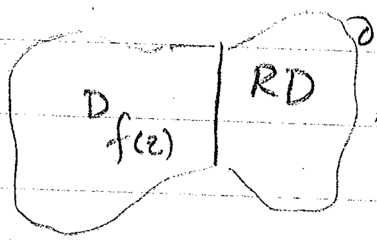
$$\therefore \quad \frac{\varphi''}{\varphi'} = \frac{\kappa - 1}{w-a} \quad \text{near } a$$

$$\therefore \quad \frac{\varphi''}{\varphi'} = \sum \frac{\kappa-1}{w-a} \qquad \log \varphi' = \sum_i \kappa_i - 1 \ \log(w-a_i)$$

$$\varphi' = \prod_i (w-a_i)^{\kappa_i - 1}$$

4-4-74

Schwarz Reflection Principle



∂D contains an interval C or an arc of a circle

f maps C onto C': interval or arc

$\therefore$ f can be cont analytically across C

R: reflection across C reflection is anticonformal

R': " " C'

$$f(Rz) = R' f(z)$$

z in RD $\qquad f(z) = R' f(Rz)$

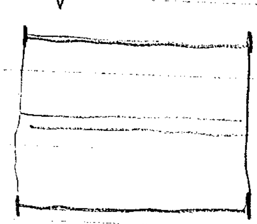
we must stipulate that $RD \cap D = C$ so that we don't get contradiction

Given two different curves with same endpts: analytic cont should not depend on curve

Monodromy Principle map each curve into unit length form unit square by analytic cont

in on plane

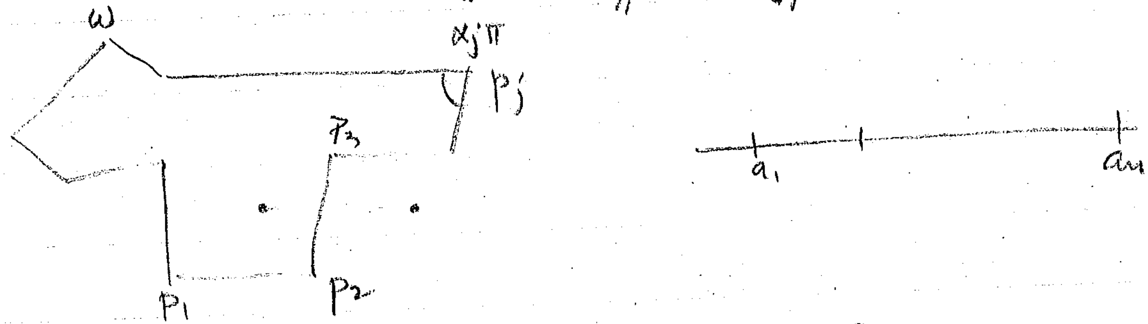
2 curves close lie in a strip. Cover square



by finite no. of curves (unit square is [illegible]

Schwarz - Cristoffel Transformation

$f(z)$



Let $f_j(z)$ be the fn after reflection across $P_2 P_3$

Let $f_k(z)$ " " " " " " $P_1 P_2$

$$f_j(z) \neq f_k(z)$$

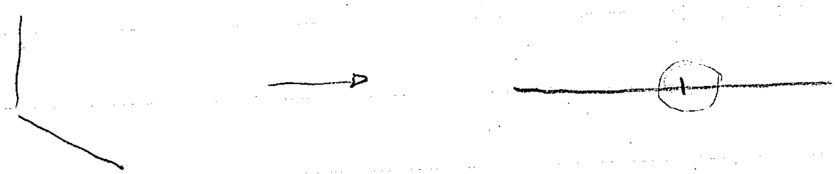
we can define $\quad f_j(z) = \rho f_k(z) + s \qquad \rho$ - rotation $|\rho| = 1$

s - shift

$$\frac{f_j''}{f_j'} = \frac{f_k''}{f_k'} = \frac{f''}{f'} = g(z)$$ defined in whole plane

regular fn except at vertices

$$f(a_j) = p_j \qquad (f(z) - p_j)^{\frac{1}{\alpha_j}}$$ regular simple zero at $z = a_j$



by analytic continuation we can define things in nbd of bdy curve

$$(f(z) - p)^{\frac{1}{\alpha}} = c[z-a]\{1 + \dots \quad \}$$

$$(f(z) - p) = c^\alpha [z-a]^\alpha \{1 + \dots \quad \}^\alpha$$

$$f' = c^\alpha \alpha (z-a)^{\alpha-1} \{ \quad \}^\alpha + c^\alpha (z-a)^\alpha \{ \quad \}'$$

$$f'' = c^\alpha \alpha(\alpha-1)(z-a)^{\alpha-2} \{ \quad \}^\alpha + c^\alpha \alpha (z-a)^{\alpha-1} \{ \quad \}' + c^\alpha (z-a)^\alpha \{ \quad \}''$$

$$f' = c^\alpha \alpha [z-a]^{\alpha-1} \{ 1 + \dots \}$$

$$f'' = c^\alpha \alpha(\alpha-1)(z-a)^{\alpha-2} \{ 1 + \dots \}$$

$$\frac{f''}{f'} = \frac{\alpha-1}{z-a}\left[1+\dots\right] \qquad \textit{local analysis}$$

$g(z)$ is regular at ∞ and has a finite no. of simple poles.

$$\frac{f''}{f'} - \sum \frac{\alpha_j-1}{z-a_j} = \text{everywhere bounded fn} = \text{const by Liouville's thm.}$$

as $z \to \infty$ $\quad f''/f' \to 0 \quad \therefore$ const $= 0$

$$\log f' - \sum(\alpha_j-1)\log(z-a_j) = c$$

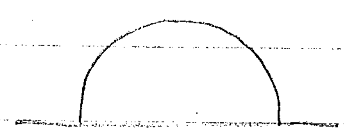
$$f' = \prod(z-a_j)^{\alpha_j-1}\cdot K \qquad \text{or} \qquad f(z) = K\int_0^z \prod(z-a_j)^{\alpha_j-1}dz$$

we can always find polygons with same α_j but not same sides $\therefore$ a_j are parameters

There are n parameters here but we note that upper half plane into upper half plane is $w = \frac{az+b}{cz+d}$ which is a 3 parameter family.

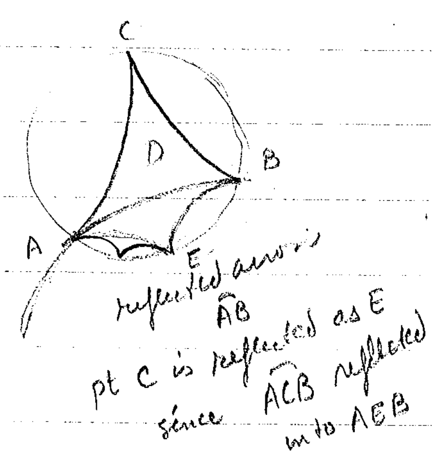
$\therefore$ we have $n-3$ parameters left over to play with.

Polygons with circular sides



$$\frac{az+b}{cz+d} \qquad \text{rigid motion}$$

non Euclidean - parallel postulate doesn't hold, nor angles of triangle $= 180°$



$f(z)$ is multivalued but always lies inside unit disk. Elliptic Modular fn.

Ell. Function

Multiple valued fn between 0 – 1 maps entire plane into circle

Picard Thm there is not analytic entire fn of z $\ni$. 2 pts are omitted

Let $g(z)$ entirely analytic fn

$g(z) \neq 0 \quad g(z) \neq 1$ for any z

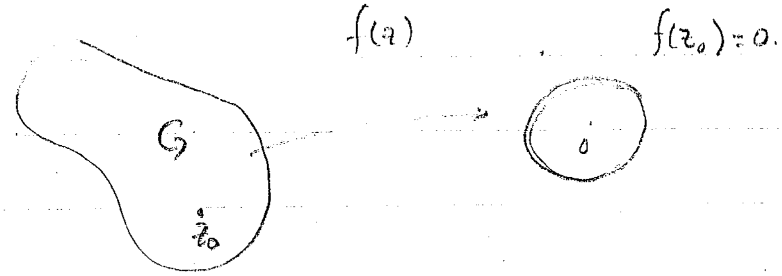
$f(g(z))$ analytic $\forall z$

at some $z = z_0$ $g(z)$ is analytic so is $f(g(z))$

we can analytic continue to any pt in domain. By Monodromy Thm.

we can map domain in the disk $\rightarrow\leftarrow$ by Liouville's Thm. Hence no such fn exists.

$\therefore$ entire fn can omit at most one pt.



take $g(z)$ analytic in G $g(z_0) = 0$ $|g(z)| < 1$ ⊛

Look at $h(w) = g(f^{-1}(w))$: disk $\rightarrow$ disk

$h(0) = 0$

Lemma: then $|h'(0)| \leq 1$ = only if $h(w) = cw$ $|c| = 1$

$$h(w) = \sum_{1}^{\infty} a_n w^n$$

$$h(e^{i\theta}) = \sum a_n e^{in\theta}$$

$2\pi \geq \int |h(e^{i\theta})|^2 = 2\pi \sum |a_n|^2 \Rightarrow \sum |a_n|^2 \leq 1 \quad \therefore \quad 1 \geq |a_1| \quad = \text{ if } a_i = 0 \; i > 1$

Schwarz Lemma

If $|h(w)| \le 1$, $h(0)=0$

$|h(w)| \le |w|$; since $h(0)=0$ then $\frac{h(w)}{w}$ is analytic in disk

use max prin $\frac{h(w)}{w}$. max is on bdy $\frac{|h(w)|}{|w|} \le 1$ $\therefore$ $|h(w)| \le |w|$

$h'(0) = \frac{d}{dz} g(z_0) \left[f^{-1}(0)\right]'$ $\qquad \frac{d}{dw} f^{-1} = \frac{1}{\frac{d}{dw} f}$

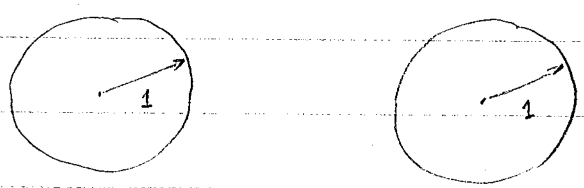
$= \frac{d}{dz} g(z_0) \Big/ \frac{d}{dz} f(z_0)$

$$|h'(0)| = \left| \frac{\frac{dg(z_0)}{dz}}{\frac{df(z_0)}{dz}} \right| \le 1$$

$\therefore$ $\left|\frac{d}{dz} g(z_0)\right| \le \left|\frac{d}{dz} f(z_0)\right|$ $\quad$ = if $g = \text{Const} \times f$ $\quad$ $|\text{const}| = 1$

Among all g satisfying (*) max $\left|\frac{d}{dz} g(z_0)\right|$ is reached for $g = cf$

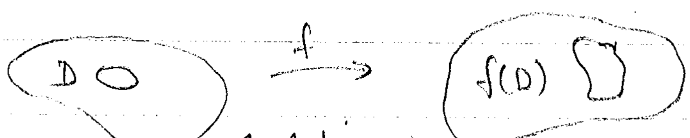
4-11-74



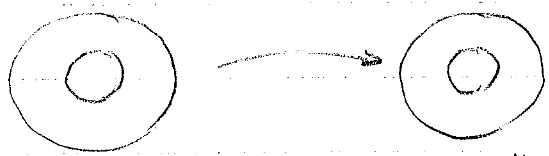
z plane $\qquad$ w-plane

$w = e^{i\theta} \frac{z-b}{1-\bar{b}z}$ $\qquad \theta$ real, $|b|<1$

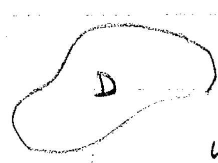
Thm : If $w = f(z)$ is a univalent analytic map of $\{|z|<1\}$ onto $\{|w|<1\}$ then $w = e^{i\theta} \frac{(z-b)}{1-\bar{b}z}$ where θ real and $|b|<1$



analytic
& univalent



only if ratio of radii are the same, if not then $\nexists$ an analytic mapping.

Riemann Mapping thm:  $\exists$ an analytic univalent map onto unit circle.

D is a domain, simply connected, not the whole plane.

Riemann 1851 – H.A. Schwarz – Kobe

Hilbert gives proof using Harmonic fns & Dirichlet Principle (Variational Principle)

Kobe's Proof.: Boundary is not important in this proof

Thms needed for proof   D need not be simply connected in these thms.

I. If f_n analytic in D, and if $f_n \to f$ uniformly on every compact subset of D then f is analytic in D. Proof By Morera's Thm. (see Dawson's Final)

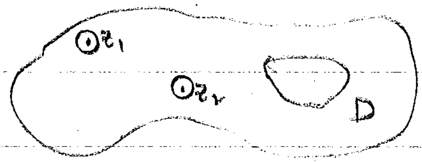
II Same hypothesis as I $\Rightarrow$ $f_n'(z) \to f'$ uniformly on every compact set of D Use Cauchy Integral formula for deriv & pass to limit

III Same hypothesis as I and if $f_n \neq 0$ in D, then either $f \equiv 0$ in D or $f \neq 0$ in D. Hurwitz Thm proof by Rouché's Thm.

IV Same hypothesis as I and if each f_n is univalent then either $f \equiv$ const or limit f_n is univalent. Proof almost identical to Hurwitz Thm.

Proof of IV:

Suppose $f(z_1) = f(z_2) = c$, and $f(z) \not\equiv c$ where $z_1, z_2 \in D$ and $z_1 \neq z_2$ (We are assuming nonunivalence but $f(z) \not\equiv c$)



Draw δ nbd about z_1, z_2 $\ni$ δ circles $\subset D$ nonoverlapping
also on $|z-z_1| = \delta$ $\quad |z-z_2| = \delta$ $\quad f(z) \neq c$.

we know $0 < m = \min\limits_{\substack{|z-z_1|=\delta \\ |z-z_2|=\delta}} |f(z)-c|$ since $f(z)$ is a cont fn.

now look at $\quad f_n(z) - c = [f_n(z) - f(z)] + [f(z) - c]$; $\quad |f(z)-c| > m$

choose n so large $\ni$ $|f_n(z) - f(z)| < |f(z) - c|$ on $\begin{matrix}|z-z_1| = \delta \\ |z-z_2|=\delta\end{matrix}$

Using Rouché's thm $\quad f_n(z) - c$ has same no. of zeros as $f(z) - c$ in
$\begin{matrix}|z-z_1| < \delta \\ |z-z_2| < \delta\end{matrix}$ $\quad \Rightarrow f_n(z) - c$ has 2 zeros since $f(z)-c$ has 2 zeros
at $z = z_1, z_2$. But $f_n(z)$ is univalent $\Rightarrow\Leftarrow$
Contradiction

$\therefore$ f is univalent otherwise $f \equiv$ const.

V $\quad$ If f_n analytic fns in D and if $|f_n| \le M$ in D then $\exists$ a subsequence $f_{n'}$ that converges uniformly on every compact subset of D.

Proof: will reduce this thm to real variables counterpart: Arzela's Th. or Ascoli's Th.

Consequence of hypothesis Claim: If K is a compact subset of D then $\exists$ $\mu > 0$ $\ni$ $|f_n(z_1) - f_n(z_2)| \le \mu |z_1 - z_2|$ $\quad \forall z_1, z_2 \in K$ $\quad$ Equicontinuity?

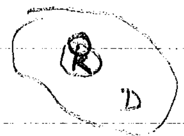
f_n are equicontinuous ; μ is a Lipschitz constant. Arzela's Th needed boundedness of f_n & equicontinuity.

Will prove $\quad |f_n'(z)| \le \mu \quad \rightarrow$ equicontinuity

Let K be a compact subset of D. Let $r > 0$ $\ni$ $0 < r <$ distance$(K, \partial D)$

since K is compact set $\subset D$
such a no. exists

If $z \in K$ then $|z - z_1| = r \in D$



$$f'(z) = \frac{1}{2\pi i} \oint\limits_{|\zeta - z| = r} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^2} \qquad |f'(z)| \le \frac{1}{2\pi} \frac{M}{r^2} \cdot 2\pi r = \frac{M}{r}$$

$\therefore \quad |f_n'(z)| \le \frac{M}{r} = \mu$

take z_1 and $z_2 \in K$ & straight line segment $\overline{z_1 z_2}$ is in D

$$f(z_1) - f(z_2) = \int_{z_2}^{z_1} f'(\zeta)\,d\zeta \implies |f(z_1) - f(z_2)| \le \mu\, |z_1 - z_2|$$

This method here could have been used for one of the homeworks.

We thus have equicontinuity & boundedness Hence Arzela's th

$\therefore$ f_n contains a subsequence f_n' that converges uniformly on K.

Let $r_p = \frac{1}{p} \to 0$ as $p \to \infty$

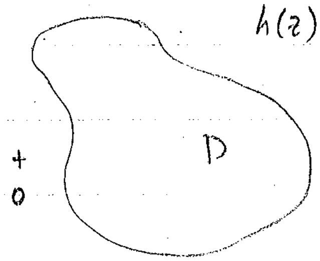
Define $K_p = \left\{ z \in D \;\middle|\; \text{distance}(z, \partial D) \ge \frac{1}{p} \text{ and } |z| \le p \right\}$ K_p is a compact subset

$K_p \subseteq K_{p+1}$ and $\bigcup_{p=1}^{\infty} K_p = D$

Start with K_1: with the above argument we can find a subsequence which works on K_1 f_{1n}. From this pick f_{2n} which works on K_2 etc

and now look at f_{nn} & show it is uniformly convergent $\forall$ K_p

Suppose D is a simply connected domain & $0 \notin D$. Then you can define $h(z) = \log z$ in D as an analytic fn.



$e^{h(z)} = z$ $\qquad \log z = \int^{z} \frac{d\zeta}{\zeta}$ & since $\frac{1}{\zeta}$ is analytic in D $\log z$ is single valued

Riemann Mapping Thm.

Suppose D is a simply connected domain, not the whole plane. Given z_0 in D $\exists$ a unique univalent analytic fn $f(z)$ $\ni$. $f: D \xrightarrow{\text{onto}}$ unit circle $|w| <$

$f(z_0) = 0$ and $f'(z_0) =$ positive real number

Uniqueness is simple since if not we can map one unit disk onto another by only certain unique univalent maps etc

Step i Reduction to case where D is bounded.

Proof: suppose $a \notin D$. Define a fn $s(z) = \sqrt{z-a}$ analytic in D.

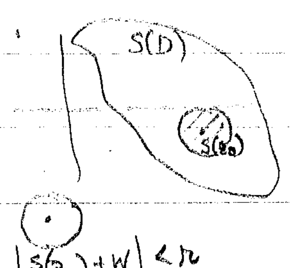
$s(z)$ is univalent on D. If $s(z_1) = s(z_2) \Rightarrow s^2(z_1) = s^2(z_2) \Rightarrow z_1 = z_2$

If $z_1 \in D$ then $-\sqrt{z_1 - a} \notin s(D)$. Suppose $-\sqrt{z_1-a} = \sqrt{z-a}$, $z \in D \Rightarrow z_1 = z$

$-s(D)$ does not intersect $s(D)$ | or $-\sqrt{z_1-a} = \sqrt{z_1-a}$ or $z_1 = a \notin D$

$s(D)$ is a domain $s(z_0) = \sqrt{z_0 - a} \in s(D)$ Take a disk $\{|W - s(z_0)| < r\} \in s(D)$

look at $-W$ this maps into $-s(D)$ which doesn't intersect $s(D)$ or even the circle $\{|W - s(z_0)| < r\}$



hence have found circle which doesn't intersect $s(D)$ and we can use an inversion wrt this circle

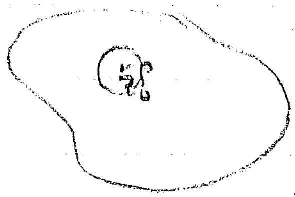
$\therefore \dfrac{r^2}{s(z_0)+W}$ maps $s(D)$ into the circle $|s(z_0)+W| < r$

use a translation to move to center of coordinate system as magnification factor to get to unit circle. Hence with finite no of steps we have mapped D into a unit circle

Step ii Φ: class of fns
1) g analytic univalent on D
2) $|g(z)| \le 1$ for z in D
3) $g(z_0) = 0$, $g'(z_0)$ real positive

$g(z) = \epsilon z$ ϵ is small $\Phi \neq \{\phi\}$ not null set

Step iii Construct $f \in \Phi \ni f'(z_0) = \max$



$|g'(z_0)| \le \frac{1}{\rho}$ Cauchy Inequality

$\sup_{g \in \Phi} g'(z_0) = B \le \frac{1}{\rho}$

choose $g_n \in \Phi$ such that $g_n'(z_0) \to B$

$|g_n| \le 1$ in D

Using V select subseq $g_{n'}$ which converges uniformly on every compact subset of D. Let $f = \lim g_{n'}$: f is analytic in D

Since all $g(z_0) = 0 \Rightarrow f(z_0) = 0$. also $f'(z_0) = \lim g_{n'}'(z_0) = B$

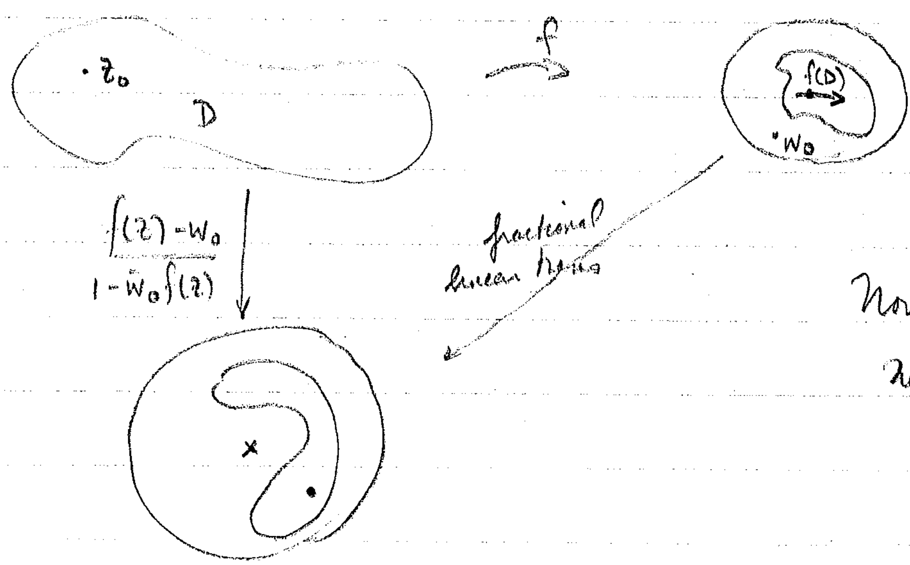
$|f| = \lim |g_{n'}| \le 1$.

By IV $f \in \Phi$

Step IV f maps D onto $\{|w| < 1\}$

Proof: By contradiction; Suppose not so

then $\exists\, w_0 \ni |w_0| < 1$ and $w_0 \ne f(z)\ \forall z \in D$.



Now let z_0 be origin

no loss in Generality

$\dfrac{f(z) - w_0}{1 - \bar{w}_0 f(z)}$ analytic $|\bar{w}_0| < 1$ $|f(z)| \leq 1$ ∴ denom never vanishes

hence analytic univalent on D. $f(z) \neq w_0$

Let $F^2(z) = \dfrac{f(z) - w_0}{1 - \bar{w}_0 f(z)}$ is analytic univalent on $D \Rightarrow F(z)$ is analytic univ on D

also $|F(z)| < 1$

let $G(z) = \dfrac{|F'(0)|}{F'(0)} \dfrac{F(z) - F(0)}{1 - \overline{F(0)} F(z)}$ fractional linear trans which maps disk into disk

$|G(z)| < 1$ for $z \in D$

$G(0) = 0$

$G'(0) = \dfrac{|F'(0)|}{1 - |F(0)|^2}$ $\quad 2F'(0)F(0) = \dfrac{f'(0)}{1 - \bar{w}_0 f(0)} - \dfrac{f(0) - w_0}{(1 - \bar{w}_0 f(0))^2} (-\bar{w}_0 f'(0))$

$f(0) = 0$ $\quad 2F'(0)F(0) = B - B|w_0|^2$

$$G'(0) = \frac{(B - B|w_0|^2)/2|F(0)|}{(1 - |F(0)|^2)} = B \frac{1 - |F(0)|^4}{2|F(0)|(1 - |F(0)|^2)}$$

$$= B \frac{1 + |F(0)|^2}{2|F(0)|} > B \quad \rightarrow \leftarrow$$

∴ f maps D onto $\{|w| < 1\}$

For the case when ∂D is piecewise cont. mapping

4-18-74

Lemma: Extension of Hahn-Banach Thm to Complex Field

Normed Linear Space B

a) Linear $\Rightarrow$ $f, g \in B$ $\quad af + bg \in B \quad \forall a, b \in$ Complex

b) norm $\Rightarrow$ $\|f\|$

1) $\|f+g\| \leq \|f\| + \|g\|$ Triangle

2) $\|af\| = |a| \, \|f\|$ homogeneity

3) $\|f\| \geq 0 \qquad \|f\| = 0$ for $f = 0$

Thm deals with bdd linear fnls $l(f)$ fn defined on B $\ni$. $l(f): B \to \begin{matrix}\mathbb{R}\\ \mathbb{C}\end{matrix}$

linear $\Rightarrow$ $l(af + bg) = a\,l(f) + b\,l(g)$

bddness $\Rightarrow$ $\|l(f)\| \leq$ const $\|f\|$ bounded by constant

Hahn Banach concerns bdd linear fnls defined on a subspace B' of B

Thm: space can be extended keeping the properties of linear bdd fnls.

Extension: Given by Bohnenblust - Sobczyk - Sukhomlinov

$l(f)$ is complex & if l is linear wrt complex coeff

$$l(f) = l_1(f) + i\, l_2(f) \qquad l_1, l_2 \text{ linear wrt real coeff}$$

$$l(if) = i\,l(f) = i\left[l_1(f) + i\,l_2(f)\right] = i\,l_1(f) - l_2(f)$$

$$= l_1(if) + i\, l_2(if)$$

$$\Rightarrow \boxed{l_1(if) = -l_2(f) \qquad l_2(if) = l_1(f)}$$

decompose $l(f)$ & extend $l_1(f)$ since $\|l(f)\| \leq \|f\| \Rightarrow \|l_1(f)\| \leq \|f$

using $l_1(f)$ define $l(f) = l_1(f) - i\,l_1(if)$

Choose $\alpha \in \mathbb{C} \quad |\alpha| = 1 \quad \alpha\, l_1(f) > 0$

since $l(\alpha f)$ is real

$$\|l(f)\| = \|\alpha\, l(f)\| = \|l(\alpha f)\| = \|l_1(\alpha f)\| \leq \|\alpha f\|$$

$$= |\alpha|\,\|f\| = \|f\|$$

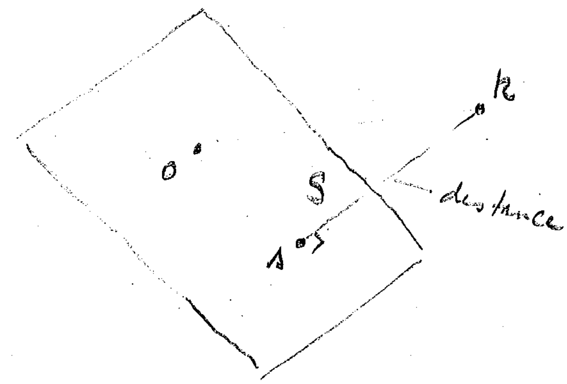
B normed linear space

S subspace, k: element of B

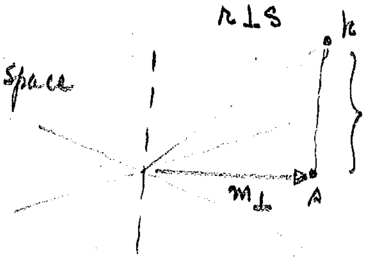
find: distance of k from S

define as $\inf_{s \in S} \|k - s\| = m$

$\inf_{r \perp S} \|k - r\| = m_\perp$



In hilbert space



$$|k|^2 = m^2 + m_\perp^2$$

$$m^2 = |k|^2 - m_\perp^2$$

$$m^2 = \sup_{r \perp S} \left[k^2 - \|k - r\|^2 \right]$$

In Banach Space

Given $S \subset B$ k in B

$$m = \inf_{s \text{ in } S} \|k - s\|$$

$$M = \max_{\substack{l \text{ linear fnl} \\ \text{bdd by } 1 \\ l(S) = 0}} |l(k)|$$

Thm: $m = M$

by 2 parts Prove $M \le m$ & $M \ge m$

Part I $M \overset{?}{\le} m$

by definition of $m \Rightarrow \exists\, s \ni \|k - s\| < m + \epsilon$

$$l(k) = l(k) \underset{l(s)=0}{-} l(s) \underset{\text{linearity}}{=} l(k - s)$$

$$\|l(k)\| \le \|l(k-s)\| \le \|k - s\| < m + \epsilon$$

but $\max |l(k)| = M < m + \epsilon$

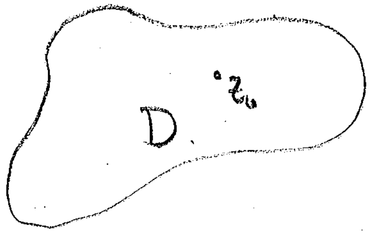
Part II $M \ge m$ exhibit some $l_0(k) = m$ then since $M = \max |l(k)| > l_0(k)$

$l_0(k) = m$, $l_0(s) = 0$ $\therefore$ by linearity $l_0(ak + s) = a\, l_0(k) + l_0(s) = am$

$m = l_0(k+s) \le \|k + s\|$ since $m \overset{\text{def}}{\le} \|k - s\|$ now by Hahn Banach

can be to the whole space & hence $\exists$ an $M = \max |l(k)| \ge m$ etc

$\therefore$ we have shown $M \le m$ $M \ge m$ ...



Max $|f'(z_0)|$ among all f which

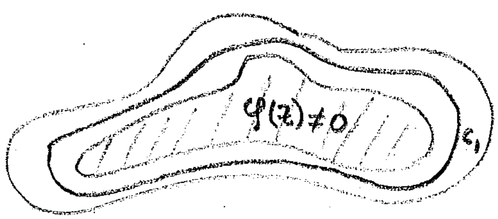
i) $|f(z)| \le 1$

ii) $f(z_0) = 0$

iii) f is 1 to 1

$f'(z) = \max |\mathcal{L}(f)|$ $\quad$ $\mathcal{L}(f) = \iint_D f(z)\,\varphi(z)\,dx\,dy$, $\quad$ $\varphi(z) = 0$ outside a compact subset

Cauchy Integral $\quad f'(z_0) = \int_{c_1} \dfrac{f(z)\,dz}{(z-z_0)^2}$



$$f(z) = \int \frac{dw\, f(w)}{w-z}$$

$$\therefore\ \mathcal{L}(f) = \iiint_{c_1} \frac{f(w)\,\varphi(z)}{w-z}\,dw\,dx\,dy$$

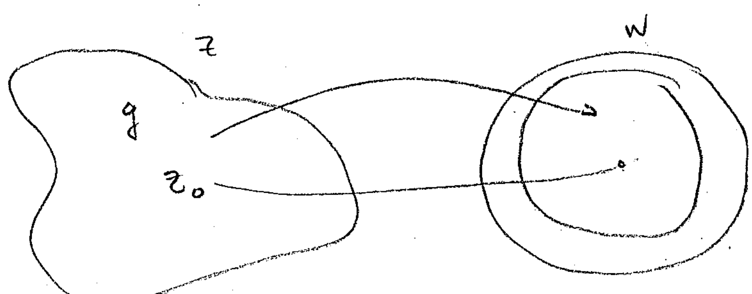
$$= \int_{c_1} f(w) \iint \frac{1}{w-z}\,\varphi(z)\,dx\,dy\,dw$$

$$= \int_{c_1} f(w)\,k(w)\,dw$$

must now max $|\mathcal{L}(f)|$

$\ni$. i) $|f(z)| \le 1$

ii) f has zero of order n_j at z_j



$f(g^{-1}(w)) = h(w)$ $\quad$ $|h(w)| \le 1$

$|h'(0)| \le 1$

$$\int h(w)\,\frac{dw}{w^2} \le \max \frac{|h|}{r^2} \cdot 2\pi r \le \frac{1}{r}$$

Banach Space

B : Cont fns on C, $\|f\| = \int_C |f(z(\sigma))| \, d\sigma$

S : Bd values of analytic fn in D with poles of order n_j at z_j

S is a linear space defined this way

$$M' = \operatorname{Max} |\mathcal{L}(f)| \overset{?}{=} M = m$$

$$M \overset{?}{\leq} M' \overset{?}{\leq} m \qquad \text{we will prove two sided inequality}$$

$$\int_{C_1} |k(z) - s(z)| \, |dz| < m + \epsilon$$

if C_1 is close enough & $\|$ enough to C.

$$|\mathcal{L}(f)| = \int f(w) \, k(w) \, dw = \int f(w)\big(k(w) - s(w)\big) \, dw$$

Claim $\int f(w) \, s(w) \, dw = 0$ true by Cauchy integral th.

$$\leq \operatorname{Max} |f| \int |k(w) - s(w)| \, dw \leq m + \epsilon$$

$$\therefore \quad M' \leq m$$

$$f_0(z) : \quad \mathcal{L}(f_0) = M$$

$$u(z) \in B : \frac{1}{2\pi} \frac{1}{w - z} \qquad z \text{ parameter}$$

Let l_0 be maximizing fnl.

Construct $f_0(z)$ as $\quad f_0(z) = l_0(u(z))$

$$\mathcal{L}(f_0) = \iint f_0(z) \varphi(z) \, dx\,dy = \frac{1}{2\pi} \iint l_0\left(\frac{1}{w - z}\right) \varphi(z) \, dx\,dy$$

$$= l_0 \left\{ \frac{1}{2\pi} \left[\iint \frac{1}{w - z} \varphi(z) \, dx\,dy \right] \right\}$$

need only linearity & bddness to pull l_0 outside but $\frac{1}{2\pi}\iint \frac{1}{w-z}\varphi(z)\,dx\,dy = k$

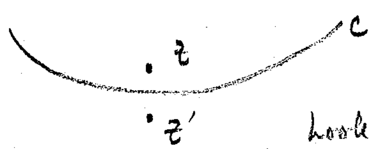
$$= l_0(k) = M$$

$f_0(z)$ analytic since u dependent on z analytically
can take difference quotients & pass to limit under functional

$u(z_1) \in S$

$f_0(z_1) = l_0(u(z_1)) = 0$

$$f_0'(z) = l_0\left[\frac{1}{(w-z)^2}\right]$$



$|f_0(z)| \le 1+\epsilon$ if distance $(z, C) < \delta$

Look at $f_0(z) - f_0(z')$
for z' outside D

Sufficient to show this true near bdy only since by max principle it follows that $f_0(z) \le 1+\epsilon$ throughout

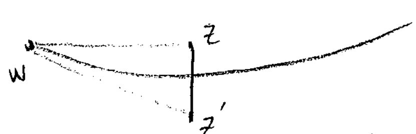
$f_0(z') = 0$

$$f_0(z) - f_0(z') = l_0(u(z'))$$

$u(z')$ belong to S & claim $u(z')$ has no poles.

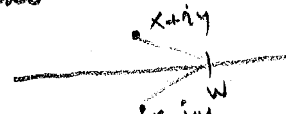
$$|f_0(z)| = |f_0(z) - f_0(z')| = |l_0(u(z) - u(z'))|$$
$$\le \|u(z) - u(z')\| < 1+\epsilon$$

$$\frac{1}{w-z} - \frac{1}{w-z'} = \frac{-(z'-z)}{(w-z)(w-z')}$$



for w far away, integrand

$$\frac{1}{2\pi}\int \frac{|z'-z|}{|w-z||w-z'|}|dw| < 1+\epsilon$$



Poisson kernel

$\frac{1}{\pi}\int \frac{y}{[illegible]}dw = 1$ for straight li

Homework get this result for curved line assume bdy smoothly turning tangent

s_0 sol of min problem
f_0 sol of max problem

$$\int |k - s_0| \, |dw| = m$$

$$m = \mathcal{L}(f_0) = M' = M = m$$

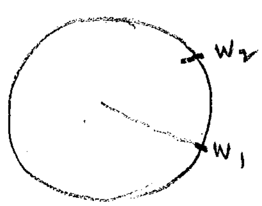
$$\int f(w)\, k(w)\, dw = \int f(w) \left[k(w) - s_0(w) \right] dw$$

$$\leq \operatorname{Max} |f_0| \int |k - s_0| \, |dw| = m$$

$$\Rightarrow |f_0(w)| = 1 \quad \text{on } C$$

$$\& \quad f_0 (k - s_0) \frac{dw}{d\sigma} > 0$$

denote $\quad g(w) = \int f_0 (k - s_0)\, dw$



$$g(w_2) = g(w_1) + \int_{w_1}^{w_2} (\quad)$$

$$\operatorname{Im} g(w_2) = \operatorname{Im} g(w_1)$$

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Find $\quad \operatorname{Max} |\mathcal{L}(f)| = M$

i) $f(z)$ analytic $\quad$ zero of order n_j at z_j in D

ii) $|f(z)| \leq 1$

$$\mathcal{L}(f) = \iint f(z)\, \phi(z)\, dx\, dy \qquad \phi \in \text{compact support in } D$$

$$= \int_{C'} f(w)\, k(w)\, dw \qquad C' \supset \text{support } \phi$$

where $\quad k(w) = \frac{1}{2\pi} \iint \frac{\varphi(x,y)}{z - w}\, dx\, dy$

Find $\quad \text{Min} \int_C |k-s| \, d|z| = m \qquad C = \partial D$

3. i) s analytic in D except for poles of order $\leq n_j$ at z_j

<u>Thm</u> $M = m$

and if f_M = sol of max problem

$g_m = k - s$ soln of min problem

i) $|f_M| = 1$ on C

ii) $f_M \, g_M \dfrac{dz}{d\sigma} > 0$ on C

$$Mm = \mathcal{L}(f_M) = \int f_M(w) \overbrace{\left(k(w) - s(w)\right)}^{g_m} \frac{dw}{d\sigma} d\sigma \leq 1 \int |g_m| \, d\sigma = m$$

since $\int f_M(w) s(w) \frac{dw}{d\sigma} d\sigma = 0$ by Cauchy

$\dfrac{dw}{d\sigma} = 1$ since if we break up curve into segments & σ is arc length $\Delta w = |w_1 - w_2| \cong \Delta\sigma$

Now apply results to these problems.

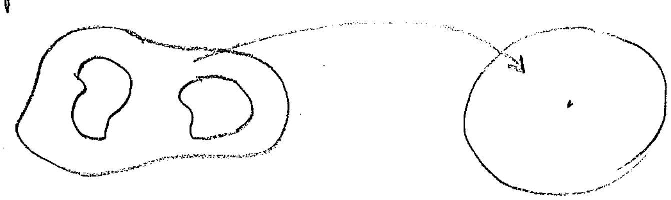
1) Max $|f'(z_0)| \iff k(w) = \dfrac{1}{2\pi}\left(\dfrac{1}{w - z_0}\right)^2$

2) Maximize $|f(z_0)|$, with restriction $f(z_1) = 0 \iff k(w) = \dfrac{1}{2\pi} \dfrac{1}{w - z_0}$, "

3) Maximize $|f(z_0) - f(z_{00})| \iff k(w) = \dfrac{1}{2\pi}\left(\dfrac{1}{w - z_0} - \dfrac{1}{w - z_{00}}\right)$

Solution maps domain D into unit disk in n to 1 manner

Proof:



maps 3 to 1

will show origin is triply covered and hence any other pt is also triply covered this can be shown by winding number which changes continuously as long as we don't jump the boundary

from (3) $\arg f_M + \arg g_M + \arg \frac{dz}{d\sigma} = 0$

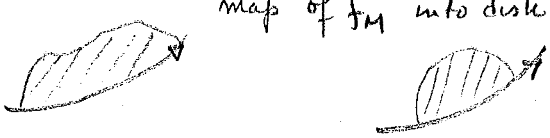
let C_j be the boundary of jth curve

$F_j = \frac{1}{2\pi}$ Change in arg f_M around C_j

G_j = " " " g_m " "

$\therefore$ $F_j + G_j + \begin{matrix} 1 \\ -1 \end{matrix} = 0$ $\begin{matrix} \text{for } j=1 & \text{outer boundary} \\ \text{for } j=2,\dots,n & \text{inner boundary} \end{matrix}$

map of f_M into disk



analytic mappings are sense preserving & boundary $\to$ boundary

$\therefore \Rightarrow F_j \geq 1$

$$\sum F_j + \sum G_j - n + 2 = 0$$

$\sum F_j \geq n$ $\sum G_j$ = no. of zeros − no. of poles of g_m by argument principle

$g_m = K - S$ what are poles & zeros.

in the cases considered g_m must have at most 2 poles

$\therefore$ $\sum G_j \geq -2$

$0 = n - 2 - n + 2 \leq \sum F_j + \sum G_j + 2 - n = 0$ $\Rightarrow$ equality signs must hold

$\Rightarrow$ $F_j = 1$ & g_m has no zeros. $\Rightarrow \sum F_j = n$ $\Rightarrow$ mapping at origin must be covered n times and hence we have an n to 1 mapping.

Garabedian's Result concerning problem 1 g has no zeros but two poles

take $g(z) = h^2(z)$ $h(z)$ has pole at z_0 with residue $\frac{1}{\sqrt{2\pi}}$

takes min $\int |h(z)|^2 dz$ take $h(z) = \frac{1}{\sqrt{2\pi}} \frac{1}{z - z_0} + a_0 + a_1 z + \dots + a_{15} z^{15} + \dots$

$$\int |h(z)|^2 |dz| = \int h(z)\cdot\overline{h(z)}\,|dz| = \int\left(\frac{1}{\sqrt{2\pi}}\,\frac{1}{z-z_0} + a_0 + \sum a_j z^j\right)\left(\frac{1}{\sqrt{2\pi}}\,\frac{1}{\bar z-\bar z_0} + \sum \bar a_k \bar z^k\right)|dz|$$

$$= \frac{1}{2\pi}\int \frac{|dz|}{|z-z_0|^2} + \sum_j c_j a_j + \sum_j \bar c_j \bar a_j + \sum_{kj}\sum b_{jk} a_j \bar a_k$$

$$c_j = \frac{1}{2\pi}\int \frac{1}{\bar z - \bar z_0}\, z^j |dz| \qquad b_{jk} = \int z^j \bar z^k |dz|$$

this is quadratic problem which is easier to solve.

HW

Show $B = \{b_{jk}\}$ is positive definite & minimize

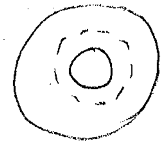
& show that B is diag if C is unit circle

B is hilbert matrix if C is unit interval.

Thm: Every fn which is analytic (defined in D) can be approx by polynomials (on compact subsets K)

$$\max_{z\in K} |f(z) - p(z)| < \epsilon$$

to show a counter example $f(z) = \frac{1}{z}$

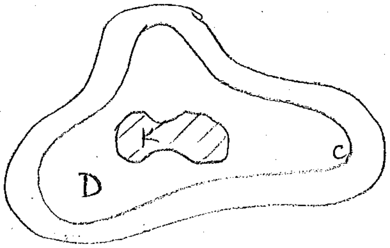


$\int f(z)dz = 2\pi i$

$\int p(z)dz = 0$

∴ thm is false unless D is simply connected.

Runge's Thm



$$f(z) = \int \frac{f(w)}{w-z}\, dw$$

$$= \sum \frac{f(w_j)}{w_j - z} \Delta_j$$

Let P be linear subspace of a normed linear space B.

Which f can be approx arbitrarily closely by elements of P.

ie $\|f-p\| < \epsilon \quad p \in P$.

Possible iff $\ell(f) = 0$ $\forall$ bounded linear fnls. ℓ which are equal to 0 on P

Show if $\ell(p) = 0 \quad \forall p \Rightarrow \ell(f) = 0$

when f can be approx. : $\Rightarrow$ for every ϵ $\exists p$

$$\ell(f) = \ell(f) - \ell(p) \underset{\text{linearity}}{=} \ell(f-p) \underset{\text{bounded}}{\leq} K\|f-p\| < K\epsilon$$

Converse : f cannot be approximated $\Rightarrow \quad \|f-p\| \geq \delta > 0$.

then $\exists$ an ℓ $\ni$. $\ell(p) = 0 \;\; \forall\, p \in P$ then $\ell(f) \neq 0$ assume $\ell(f) = 1$

$\ell(af+p) = a \qquad \|af+p\| = |a|\, \|f + p/a\| \geq |a|\delta$

$$|\ell(af+p)| = |a| \leq \frac{1}{\delta}\|af+p\|$$

ℓ is bounded; using Hahn - Banach & extend to whole space

B : continuous functions on K

P : Polynomials

$f : \dfrac{1}{z-w} \quad w \in D$

if it can be done $\ell(f) = \ell\left(\dfrac{1}{z-w}\right) = a(w)$ analytic fn of w

for large W $\qquad \frac{1}{z-w} = \frac{1}{w} \cdot \frac{1}{\frac{z}{w}-1} = \frac{-1}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n$

$\Rightarrow$ W large f can be approx by elements of p

& $a(w)=0$ for W large

Hörmander defined on connected domain which vanishes on portion of domain

then must vanish everywhere by analytic contin, hence $\Rightarrow \frac{1}{z-w}$ can be approx by elem of P

Weierstrass approx thm. thm in Real

$f(x) - \sum a_n x^{\lambda_n}$ approximation $\Rightarrow \sum \frac{1}{\lambda_n} = \infty$ is condition

Bernstein - Müntz - Szász Thm

5/2/74

X : normed linear space

P : linear subspace

Th. f in closure of P iff $l(f)=0$

for all l which $=0$ on P $\quad l$ bdd linear fnl on X

See page 29 ***

Thm Müntz

X: cont fns on $\mathbb{R}_+$.э. $f=0$ at ∞

P : $\{e^{-\lambda_j s}\}$ all linear comb. norm : max

take $f = e^{-ns}$ $\quad n \in I$

e^{-zs} $\quad Re z > 0$

we will show elements of P approx f. Then know that $f \in C$ by Weierstrass $p(x^n)$ can approx $f \Rightarrow$ if $p(x^\lambda) \sim p(x^n)$ then $f \sim p(x^\lambda)$

then $g(z) = l(e^{-zs})$ i) $g(z)$ analytic in rt half plane

since max e^{-zs} is at $z=0$

$|g(z)| = |l(e^{-zs})| \le K \, \|e^{-zs}\| = K$

$f(x)$ is approx on $0 \le x \le 1$ by x^{λ} $0 < \lambda$

*** $x = e^{-s}$ $\quad 0 \le s < \infty$

$f(e^{-s}) = g(s) \qquad x^{\lambda} \to e^{-\lambda s}$

$f(0) = 0 \Rightarrow g(\infty) = 0$

$l = 0$ on P $\Leftrightarrow l(e^{-\lambda_j s}) = 0$ but using hypothesis

$\Leftrightarrow g(\lambda_j) = 0$

$l(f) = l(e^{-ns}) = g(\tfrac{n}{z}) \overset{?}{=} 0$

we know g is analytic & bdd
to show: g doesn't have too many zeros.

if a fn has to many zeroes $\Rightarrow$ fn $\equiv 0$

Consider $\dfrac{z-\lambda_j}{z+\lambda_j}$ $\quad z = \lambda_j$ fn $= 0$ $\quad$ we will look in right have plane

$\Rightarrow$ fn has abs value 1 on bdy y axis

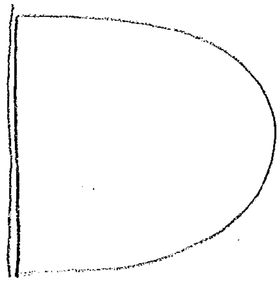
$$\prod_{j=1}^{N} \frac{z-\lambda_j}{z+\lambda_j} = B_N(z)$$ has simple zeroes at λ_j $j = 1, \dots, N$

& $|B_N(z)| = 1$ for $\operatorname{Re} z = 0$

Let $q_N = \dfrac{g(z)}{B_N(z)}$

i) q_N analytic in right have plane

ii) q_N is bdd by K



for large contour $|B_N(z)| \sim 1$ & on im axis

$\therefore$ by max principle $|B_N(z)| \sim 1$

$\therefore |q_N| = |g(z)| \le K + \varepsilon$

$g(z) = q_N(z) B_N(z)$ $\qquad |B_N(z)| = \prod \left| \dfrac{\lambda_j - z}{\lambda_j + z} \right|$

$$= \prod \left| 1 - \frac{2z}{\lambda_j + z} \right| \qquad \text{let } z \text{ be real i.e. } z = x$$

$$\prod \left| 1 - \frac{2x}{\lambda_j + x} \right| \qquad 1 - a < e^{-a} \text{ for small } a \quad a > 0$$

for $a < .1$ $\quad 1 - a < 1$

$$< \prod_{a_j < .1} e^{-a_j} = e^{-\sum a_j} \qquad a_j \text{ here} = \frac{2x}{\lambda_j + x}$$

for large λ_j $\qquad a_j < \frac{2x}{\lambda_j}$

$$|B_N(x)| \le e^{-2x \sum_j \frac{1}{\lambda_j}} \qquad \text{for } \frac{2x}{\lambda_j} < \frac{1}{10} \text{ or } \lambda_j > 20x$$

we had taken i) $\lambda_j \to \infty$

& ii) $\sum \frac{1}{\lambda_j} = \infty$

$$\Rightarrow \quad |B_N(x)| \le e^{-2x \sum_j^N \frac{1}{\lambda_j}} < \varepsilon \quad \text{for } N \text{ large enough.}$$

$$|g(x)| \le |g_N(x)|\,|B_N(x)| \le K\varepsilon$$

$$\Rightarrow \quad g(x) = 0$$

M. Riesz Convexity thm $\qquad$ Proof By Thorin

Let T be a matrix.

& $Tf = k \qquad f = \{f_1, \dots, f_N\} \qquad k = \{k_1, \dots, k_m\}$

bounded operator $\|Tf\| \le K\|f\| \quad \forall f.$ the smallest K = norm of T

$$K = \|T\|$$

$$\|T\| = \sup_f \frac{\|Tf\|}{\|f\|}$$

$\|f\|_2 = \left(\sum |f_j|^2\right)^{1/2}$

$\|f\|_\infty = \max |f_j|$

$\|f\|_1 = \sum |f_j|$

$\|f\|_p = \left(\sum |f_j|^p\right)^{1/p}$

$p \geq 1$ necessary for triangle ineq

$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$

$$\|T\|_{p,q} = \sup_f \frac{\|Tf\|_q}{\|f\|_q}$$

$f(x)$ fn on $\mathbb{R}$ $\quad \|f\|_p = \left(\int |f(x)|^p dx\right)^{1/p}$

Hölder's Inequality

$$(f, g) = \sum f_j g_j$$

$|(f,g)| \leq \|f\|_p \|g\|_{p'} \qquad \frac{1}{p} + \frac{1}{p'} = 1$ Rudin Real Analysis

given f $\exists g \ni$. sign of equality holds.

Suppose $\|g\|_{p'} = 1$

$\Rightarrow \quad (f,g) \leq \|f\|_p$

$$\|f\|_p = \max_{\|g\|_{p'}} (f,g)$$

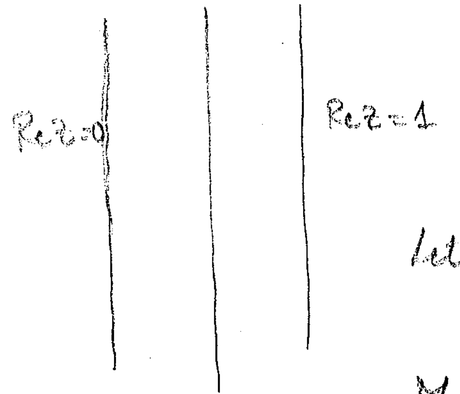
We had Corollary to Hahn Banach Th.

$$\|f\| = \max_{\|l\|=1} l(f) \qquad \text{with } l(f) = (f,g)$$

$\|l\| = \|g\|_{p'}$

Take Home exam given last day of classes & two weeks to do it.

given a vertical strip



3 lines theorem

$s(z)$ analytic & bdd in strip

$0 \le \operatorname{Re} z \le 1$

let $M_x = \sup_y |s(x+iy)|$ — thms b'ds the max in middle wrt the two end lines

$$M_x \le M_0^{(1-x)} M_1^{x} \qquad 0 \le x \le 1$$

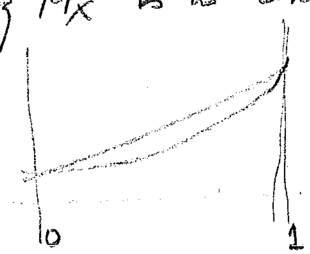
First portion $s(z)$ is bdd but $\to 0$ as $y \to \infty$

$$\log M_x \le (\log M_0)(1-x) + x \log M_1$$

$$m(x) \le (1-x)\, m(0) + x\, m(1)$$

$\Rightarrow \log M_x$ is a convex fn. of x.

Proof:



let $r(z) = s(z) e^{\lambda z}$

$|r(z)| = |s(z)|\, e^{\lambda x}$ — adjust λ $\ni$. $M_0 = M_1 e^{\lambda}$

$$\sup_{\operatorname{Re} z = x} |r(z)| = \sup |s(z)|\, e^{\lambda x} = M_x e^{\lambda x}$$

$\therefore$ max $r(z)$ at $\operatorname{Re} z = 0$ $=$ max $r(z)$ at $\operatorname{Re} z = 1$

we use fact that $s(z) \to 0$ and max principle to show fn is max on bd'y

$$M_x e^{\lambda x} \le M_0 = M_1 e^{\lambda}$$

$$\log M_x + \lambda x \le \log M_0 = \log M_1 + \lambda$$

linear fn. we had chosen which has same values at end pts

$$e^{\lambda} = \frac{M_0}{M_1} \qquad e^{-\lambda} = \frac{M_1}{M_0} \qquad (e^{-\lambda})^x = \frac{M_1^x}{M_0^x}$$

$$M_x \leq e^{-\lambda x} M_0 = \frac{M_1^x}{M_0^x} M_0 = M_1^x M_0^{1-x}$$

If $s(z) \not\to 0$ as $y \to \infty$

let $s(z)\frac{n}{n+z} = s_n(z)$ ~ 1 for z small

$\to 0$ $z \to \infty$

Then use $s_n(z)$ has same properties as $s(z)$

$$\sup_{Re\, z = x} |s_n(z)| \leq \sup_{Re\, z = x} |s(z)|$$

for very large n $\sup_{Re\, z = x} |s_n(z)| \to \sup_{Re\, z = x} |s(z)|$

$$\|T\|_{p_0, q_0} \qquad \|T\|_{p_1, q_1}$$

Convexity then gives information for every p, q between p_0, q_0 & p_1, q_1

$$\|T\|_{p,q} \leq \|T\|_{p_0,q_0}^{1-t} \; \|T\|_{p_1,q_1}^{t} \qquad 0 \leq t \leq 1$$

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

Let $p_0, q_0 = 1, \infty$ $\qquad \frac{1}{p} = 1-t + \frac{t}{2}, \quad \frac{1}{q} = \frac{t}{2}$

$p_1, q_1 = 2, 2$

$$\frac{1}{p} = 1 - t/2 \qquad \frac{1}{q} = t/2$$

$$\frac{1}{p} = 1 - \frac{1}{q}$$ p, q are conjugate

Suppose T is fourier transform

$$Tf = \int f(x)\, e^{ixy}\, dy$$

$$\|T\|_{2,2} = \sqrt{2\pi}$$

$$\|T\|_{1,\infty} \leq 1$$

$$\|T\|_{p,q} \leq \left(\sqrt{2\pi}\right)^{t} \qquad q(t) = \frac{2}{t} \qquad p(t) = \frac{2}{2-t}$$

Hausdorff-Young inequality

Proof of $\|Tf\|_q \leq \underbrace{\|T\|_{p_0,q_0}^{1-t}\, \|T\|_{p_1,q_1}^{t}}\, \|f\|_p$

$$(Tf, g) \leq \qquad \|f\|_p\, \|g\|_{q'}$$

let $f(z)$ z in strip $\qquad f = (f_1, \ldots, f_n) \quad |f_j|\, e^{iu_j} = f_j$

$g(z)$ z in strip $0 \leq \operatorname{Re} z \leq 1 \qquad g = (g_1, \ldots, g_n) \quad |g_j|\, e^{iv_j} = g_j$

$$f_j(z) = |f_j|^{\frac{p(t)}{p(z)}}\, e^{iu_j}$$

$$g_j(z) = |g_j|^{\frac{q'(t)}{q'(z)}}\, e^{iv_j}$$

5-9-74

3 Lines Thm.

$A(z) \quad 0 \le \operatorname{Re} z \le 1, \qquad M_x = \sup_y |s(x+iy)|$

$\Rightarrow \qquad M_t \le M_0^{1-t} M_1^t$

vector $f = (f_1, \dots, f_n)$

weighted L_p : $\|f\|_p = \left(\sum |f_j|^p w_j\right)^{1/p} \quad w_j > 0$

Hölder inequality : given $(f, g) = \sum f_j g_j w_j$

$|(f,g)| \le \|f\|_p \|g\|_{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1$

Reverse Hölder : $\|f\|_p = \max_{\|g\|_{p'}=1} |(f,g)|$

Mappings: $Tf = k \qquad T$ linear

$$\|T\|_{p,q} = \sup \frac{\|Tf\|_q}{\|f\|_p}$$

Given (p_0, q_0) & (p_1, q_1) define interpolated pair $[p(t), q(t)]$ $\quad |t| \le 1$

$$\frac{1}{p(t)} = \frac{1}{p_0}(1-t) + \frac{1}{p_1} t, \quad \frac{1}{q(t)} =$$

Riesz-Thorin Thm.

$$\|T\|_{p(t),q(t)} \le \|T\|_{p_0,q_0}^{1-t} \|T\|_{p_1,q_1}^{t} \equiv B$$

$\|Tf\|_{q(t)} \le \mathcal{B} \qquad$ for $\|f\|_{p(t)} = 1 \qquad (*)$

use reverse holder

if $|(Tf, g)| \le \mathcal{B} \qquad$ for $\|f\|_p = 1$ & $\|g\|_{q'(t)} = 1$ this implies $(*)$

Let $f = f(t) \quad g = g(t) \qquad f_j(z) = |f_j|^{\frac{p(t)}{p(z)}} e^{i\mu_j}$

$$g_j(z) = |g_j|^{\frac{q'(t)}{q'(z)}} e^{i\nu_j}$$

Let $s(z) = [Tf(z), g(z)]$ must show $M_0 \le \|T\|_{p_0 q_0}$

but

$|s(t)| = |(Tf(t), g(t))| = |(Tf, g)|$ to show the above apply 3 lines thm to $s(z)$

let $z = iy$ $\quad \frac{1}{p(iy)} = \frac{1}{p_0}(1-iy) + \frac{1}{p_1}(iy) = \frac{1}{p_0} + i\{ \quad \}$

$$\frac{p(t)}{p(iy)} = \frac{p(t)}{p_0} + i\{\cdots\}$$

uniphölder $|s(iy)| \le \|Tf(iy)\|_{q_0} \|g(iy)\|_{q_0'} \le \overbrace{\|T\|_{p_0,q_0} \|f(iy)\|_{p_0}}\ \|g(iy)\|_{q_0'}$

we will show $\begin{cases} \|f(iy)\|_{p_0} = \|f\|_{p(t)}^{\boxtimes} = 1 \\ \|g(iy)\|_{q_0'} = \|g\|_{q'(t)}^{\boxtimes} = 1 \end{cases}$ $\quad \boxtimes$ powers to be determined

$$\Rightarrow \quad |s(iy)| \le \|T\|_{p_0 q_0} \quad \text{or} \quad M_0 \le \|T\|_{p_0,q_0}$$

$$\left(\sum |f_j(iy)|^{p_0} w_j\right)^{1/p_0} = \left(\sum \left||f_j|^{\frac{p(t)}{p(iy)} p_0}\right| w_j\right)^{1/p_0} = \|f\|_{p(t)}^{\bigcirc}$$

$$\left||f_j|^{\frac{p(t)}{p(iy)} p_0}\right| = \left||f_j|^{p(t) + i\{\ \}}\right| = |f_j|^{p(t)}$$

$\therefore$ the power $\longrightarrow$ $\left(\sum |f_j|^{p(t)} w_j\right)^{1/p_0} = \|f\|_{p(t)}^{\frac{p(t)}{p_0}}$

Prediction Theory : Wiener - Kolmogoroff - Szegö

$\ldots, X_{-2}, X_{-1}, x_0, x_1, \ldots$ real valued random variables

Expected value $\cong$ average

Correlations: $E\,(x_j x_k)$

a) Linearity $E(x+y) = E(x) + E(y)$

$E(ax) = a\,E(x)$

b) Positivity $x \geq 0$, $E(x) \geq 0$

Positive Corr. $E(x_{-1}, x_0) > 0$ denote $E(x_j x_k) = e_{jk}$ a matrix

1) $\{e_{jk}\}$ is symmetric

2) nonnegativity $\sum e_{jk} \xi_j \xi_k \geq 0$

$$\sum E(x_j x_k)\xi_j\xi_k = \sum E(x_j \xi_j x_k \xi_k) = E\left(\sum |x_j \xi_j|^2\right) \geq 0$$

Stationary $e_{jk} = e_{j-k}$

Form $\sum e_n e^{in\theta} = m(\theta)$ $\quad e_n = \int e^{-in\theta} m(\theta)\, d(\theta)$

$\Big\{$ to show that the e_{jk} never increase look at $x_j x_k \leq \frac{1}{2}(x_j^2 + x_k^2)$

then $e_n = e_{jk} \leq \frac{1}{2}(e_{jj} + e_{kk}) = e_0 = \frac{1}{2}(e_{j-j} + e_{k-k})$ $\Big\}$

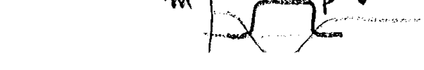
Thus: $m(\theta) \geq 0$

Form $\sum \xi_j e^{ij\theta} = \varphi(\theta)$ & look at $\int |\varphi(\theta)|^2 m(\theta)\, d\theta = \int \varphi \bar{\varphi}\, m\, d\theta$

$\bar{\varphi}(\theta) = \sum \bar{\xi}_k e^{-ik\theta}$

$$\int \sum_{jk,n} \xi_j \bar{\xi}_k e_n\, e^{i(j-k+n)\theta}\, d\theta = \sum \xi_j \bar{\xi}_k e_{k-j}$$
$$= \sum \xi_j \bar{\xi}_k e_{kj} > 0$$

if $m < 0$ in some region we can find a fn $p > 0$ ∋. $\int p\, m\, d\theta < 0$



then if we take $p = \sqrt{10}$ we get a contradiction hence $m \geq 0$.

We want to predict x_0 given $x_{-1}, x_{-2}, \ldots$

i.e. $x_0 = \mathcal{F}(x_{-1}, x_{-2}, \ldots)$

We pick a special class of fns. $\sum_{j=1}^{\infty} a_j x_{-j}$

Fn is optimal if it minimizes $E(|x_0 - \mathcal{F}(x_{-1}, \ldots)|^2)$

$$E((x_0 - \sum a_j x_{-j})^2) = E(x_0^2 - 2\sum a_j x_{-j} x_0 + \sum a_j a_k x_{-j} x_{-k})$$
$$= \sum_{j,k\,0}^{\infty} a_j a_k e_{j-k} = \int |\varphi(\theta)|^2 m(\theta)\, d\theta$$

$$\sum_0^{\infty} a_j e^{ij\theta} = \varphi(\theta) \quad \Rightarrow \quad f(z) = \sum a_j z^j \qquad \varphi(\theta) = f(e^{i\theta})$$

$$\int |f(z)|^2 m(\theta)\, d\theta, \quad z = e^{i\theta} \qquad f(0) = -1 \quad \text{linear constraint}$$

let $m(\theta) = |k(z)|^2 \qquad z = e^{i\theta}$

$k(z)$ analytic in $|z| \leq 1 \qquad k(z) \neq 0$ there

then $$\int |f(z)\, k(z)|^2 d\theta = \int |g(z)|^2 d\theta \quad \Rightarrow \quad f = g/k \quad k \neq 0 \text{ anywh}$$

with condition $g(0) = f(0)\,k(0) = -k(0)$

$$\int |g(z)|^2 d\theta \quad \& \quad g(z) = \sum b_j z^j$$

$$\int |g|^2 d\theta = \sum_0^{\infty} |b_j|^2 \quad \text{to min choose } b_j = 0 \;\forall j \neq 0$$

then we must choose $g(z) = -k(0)$

$$\therefore \; f(z) = -\frac{k(0)}{k(z)} \qquad \therefore \; \sum |b_j|^2 = |b_0|^2 = |k(0)|^2$$

Must prove $m(\theta) = |k(z)|^2$

$$\log m(\theta) = 2 \log|k(z)| = 2\, Re\,(\underbrace{\log k(z)}_{\text{harmonic}})$$

harmonic fn. whose bdy values are given by Poisson integral.

$\Rightarrow$ conjugate harmonic fn can be found. hence $k(z)$ can be obtained

at $z=0$

$$\int \log m(\theta)d\theta = 2 \log |k(0)| \; ; \quad |k(0)|^2 = \exp \int \log (m(\theta))\, d\theta$$

$= $ geometric mean of m.

arithmetic mean given K pts equid $\quad A.M. = \frac{\Sigma m_j}{N} \quad \lim_{N\to\infty} \int m(\theta)d\theta$

geom mean $\quad [\Pi(m_j)]^{1/N} \quad \log GM = \frac{1}{N}\Sigma \log m_j \quad \lim_{N\to\infty} \int \log m(\theta)d\theta$

$$\therefore \quad GM = \exp \int \log m(\theta)d\theta$$

Dirichlet's Principle. Riemann, Hilbert, Courant, Douglas, Tonelli

Application to harmonic fns, conformal maps, minimal surfaces.

1. HARMONIC FUNCTIONS

Dirichlet Integral

$f(x,y)$ real valued in G

$$D(f) = \frac{1}{2} \iint_G (f_x^2 + f_y^2)dxdy$$

Consider among all fns f in G whose bdy value $= \varphi$ on ∂G then harmonic fn has the smallest dirichlet integral.

Greens formula $\quad \iint_G k\,\Delta h\, dxdy = -\iint_G \nabla k \cdot \nabla h\, dxdy + \int_{\partial G} k \frac{\partial h}{\partial n} ds$

with h harmonic $\quad k=0$ on bdy $\Longrightarrow \iint_G \nabla k \cdot \nabla h\, dxdy = 0$

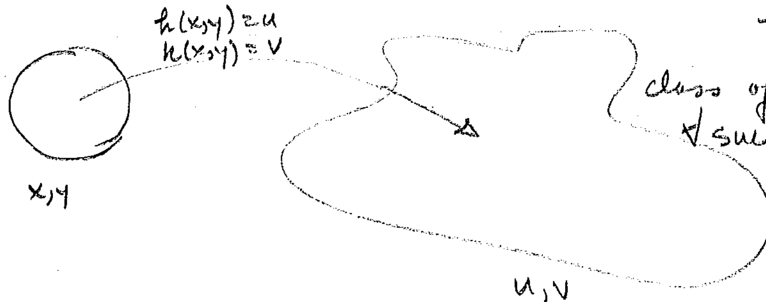
$\therefore D(k,h)$ dirichlet scalar product

$f = h + k$ $\quad$ h harmonic $\quad$ $h = \varphi$ on bdy

k has bdy value zero

$$D(f) = D(h+k) = D(h) + 2\underbrace{D(\cancel{h,k})}_{0 \text{ as shown before.}} + D(k) \Rightarrow D(h) + D(k) \geq D(h)$$

CONFORMAL MAPS

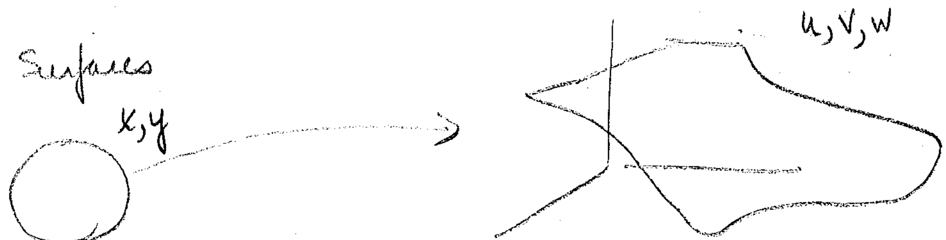


$D(\text{mapping}) = D(h) + D(k)$

class of mappings takes $\partial \to \partial$

$\forall$ such mappings with smallest dirichlet integral is the conformal map.

Minimal Surfaces



map unit circle into 3-D surface with bdy into bdy

define 3 fns $\quad h(x,y) = u$

$k(x,y) = v$

$j(x,y) = w$

then such mappings with smallest D. integ is that map

$D(h) + D(k) + D(j)$

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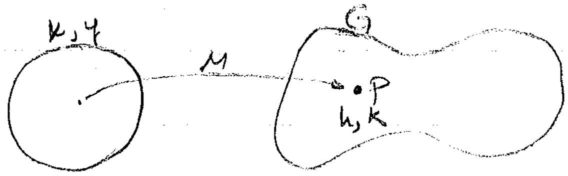
Dirichlet's Principle for Conformal mapping

Map of domain G into unit circle



mapping of domain G onto unit circle with pt P into origin has max deriv at P.

<u>Exam</u> - variations on 4 themes.

mapping $M(x,y) \to h(x,y),\ k(x,y)$ $\qquad D(h) = \frac{1}{2}\iint h_x^2 + h_y^2\, dx\,dy$

$D(M) = D(h) + D(k)$: minimizing fns h, k must satisfy Dirichlet integ seperately

M maps bdy into 1-1 onto] but such fns must be harmonic

h & k are seperately harmonic must prove h, k are conjugate & mapping conformal.

Imbed mapping M into one param family ∋ $M(0) = M$

$$\frac{d}{d\varepsilon} D(M(\epsilon))\Big|_{\varepsilon} = 0$$

How do we imbed consider $V(\varepsilon)$ mapping unit disk in u v plane to unit disk in (x, y) plane $\quad V(\epsilon) : (u,v) \to (x,y) \quad$ bdy is mapped into unit disk.

$\qquad\qquad x, y\ (u, v, \epsilon)$

define $M(\epsilon) = M \circ V(\epsilon)$ (composition) $\quad\therefore\quad V(0) = I$ since $M(0) = M$ our old mapping fn

hence $\quad x(u,v,0) = u \qquad y(u,v,0) = v$

$$\Rightarrow \text{Jacobian } \frac{\partial(x,y)}{\partial(u,v)}\Big|_{\epsilon=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Calculate $M(\epsilon) = M \circ V(\epsilon) \Longrightarrow \begin{matrix} H \\ K \end{matrix}(u,v,\epsilon) = \begin{matrix} h \\ k \end{matrix}\big(x(u,v,\epsilon), y(u,v,\epsilon)\big)$

$D(\epsilon) = \frac{1}{2}\iint (H_u^2 + H_v^2 + K_u^2 + K_v^2)\,du\,dv$; using the above

$D(\epsilon) = \frac{1}{2}\iint [(h_x x_u + h_y y_u)^2 + \ldots \quad]\,du\,dv$; but $du\,dv = \frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy = J^{-1}\,dx\,dy$

Notation

$\frac{d}{d\epsilon}$ differentiation with x, y fixed; ε subscript used when differentiating keeping u, v fixed now $(Hu)^2 = (h_x^2 x_u^2 + 2h_xh_y x_u y_u + h_y^2 y_u^2)$

For any fn $a(u,v,\epsilon)$ $\quad \frac{d}{d\epsilon}a = a_\epsilon + a_u \frac{du}{d\epsilon} + a_v \frac{dv}{d\epsilon} \quad a = x_u, y_u$

Let $a = x_u^2 \quad \frac{d}{d\epsilon} x_u^2 = 2x_u \frac{dx_u}{d\epsilon}$

but $\left.\frac{d}{d\epsilon} x_u\right|_{\epsilon=0} = \left.\left(x_{u\epsilon} + x_{uu}\frac{du}{d\epsilon} + x_{uv}\frac{dv}{d\epsilon}\right)\right|_{\epsilon=0}$; @ $\epsilon=0$ $(x,y)=(u,v)$

$x_u=1 \quad x_{uu}=0 \quad x_{uv}=0$
$y_u=0 \quad y_{uu}=0 \quad y_{uv}=0$
$x_v=0 \quad x_{vv}=0 \quad x_{vu}=0$
$y_v=1 \quad y_{vv}=0 \quad y_{vu}=0$

$= x_{u\epsilon}|_{\epsilon=0}$

Let $a = J \quad \left.\frac{d}{d\epsilon}J\right|_{\epsilon=0} = \left.\left(\frac{dx_u}{d\epsilon}y_v + x_u\frac{dy_v}{d\epsilon} - \frac{dx_v}{d\epsilon}y_u - \frac{dy_u}{d\epsilon}x_v\right)\right|_{\epsilon=0}$

$= x_{u\epsilon}\cdot 1 + 1\cdot y_{v\epsilon} - 0$ since $y_u, x_v = 0$ at $\epsilon=0$

$\left.\frac{d}{d\epsilon}J^{-1}\right|_{\epsilon=0} = -\left.J^{-2}\frac{dJ}{d\epsilon}\right|_{\epsilon=0} = -x_{u\epsilon} - y_{u\epsilon}$ since $J|_{\epsilon=0} = 1$

$$\therefore \left.\frac{d}{d\epsilon}D(\epsilon)\right|_{\epsilon=0} = \frac{1}{2}\iint h_x^2\cdot 2\underset{1}{x_u}\, x_{u\epsilon} + 2h_xh_y\left(x_{u\epsilon}\overset{0}{\cancel{y_u}} + \overset{1}{\cancel{x_u}}y_{u\epsilon}\right) + h_y^2\cdot 2\overset{0}{\cancel{y_u}}\, y_{u\epsilon}$$
$$+ \left(h_x^2x_u^2 + \dots\right)\frac{dJ^{-1}}{d\epsilon}\,dx\,dy$$

$$= \iint\left(h_x^2\, x_{u\epsilon} + h_xh_y\, y_{u\epsilon}\right) - \underbrace{\left(x_{u\epsilon}+y_{u\epsilon}\right)}_{\text{combine}}\left(h_x^2\right) \quad + 4 \text{ other terms}$$

$\therefore$ by symmetry

$$= \frac{1}{2}\iint\left(h_x^2 - h_y^2 + k_x^2 - k_y^2\right) \cdot \left(x_{u\epsilon} - y_{u\epsilon}\right)dx\,dy$$
$$+ \left(h_xh_y + k_xk_y\right)\cdot\left(y_{u\epsilon} + x_{v\epsilon}\right)$$

$= \frac{1}{2}\iint (Aa + Bb)\,dx\,dy$ where $h_x^2 + k_x^2 - h_y^2 - k_y^2 = A$

$a = x_{u\epsilon} - y_{u\epsilon}$ etc.

Now we must construct the map $V(\epsilon)$

Consider $\quad x_\epsilon = f(x,y) \quad y_\epsilon = g(x,y) \quad (**)$

$x(0) = u \quad y(0) = v$

to make sure bdy gets mapped into bdy must have on unit circle (f,g)

tangent to unit circle

$$\Rightarrow (f,g)\cdot(x,y) = xf + gy = 0 \quad \text{for } x^2+y^2=1$$

diff wrt ε the d.e. (**) $\quad x_{\varepsilon u} = f_x x_u + f_y y_u$

$$\Rightarrow x_{\varepsilon u}\Big|_{\varepsilon=0} = f_x \qquad x_{\varepsilon v}\Big|_{\varepsilon=0} = f_y$$

$$y_{\varepsilon u}\Big|_{\varepsilon=0} = g_x \qquad y_{\varepsilon v}\Big|_{\varepsilon=0} = g_y$$

$\therefore \quad a = f_x - g_y \qquad\qquad f, g$ are still under my control

$\quad\;\; b = g_x + f_y \qquad\qquad$ can pick f, g to satisfy a, b

but $\frac{d}{d\varepsilon} D(\varepsilon)\Big|_{\varepsilon=0} = 0$ for min; since a, b are arbitrary

$$\Rightarrow A, B = 0$$

$$\Rightarrow h_x^2 + k_x^2 = h_y^2 + k_y^2$$

from $A=0$ $\quad \|(h_x,k_x)\|^2 = \|(h_y,k_y)\|^2 \qquad$ Let $(h_x,k_x) = V \quad (h_y,k_y) = W$

$$\|V\|^2 = \|W\|^2$$

$B=0 \qquad V\cdot W = 0 \qquad\qquad$ V describes changes in x direction

$\qquad\qquad\qquad\qquad\qquad\qquad\;\; W$ " " " y direction

$\Rightarrow$ changes in mapping are orthog & since mag is same $\Rightarrow$ magnification & rotation

$(x+\Delta x, y+\Delta y) = (x,y) + (\Delta x, \Delta y)$ maps into

$(h,k) + \Delta x\, V + \Delta y\, W$

but mapping hence is conformal.

$a = f_x - g_y$
$b = g_x + f_y$ $\qquad$ when C.R. eq. $\Rightarrow$ $xf + yg = 0$

to get sol take a_x & b_y & add

$$a_x + b_y = f_{xx} + f_{yy} = \Delta f \qquad \text{Poisson equation}$$

Let f_0 be a particular solution

$$g_x = b - f_{0y} \qquad g_y = f_{0x} - a$$

$$g_0(x,y) = \int_0^{(x,y)} g_x\,dx + g_y\,dy$$

$$g_{xy} = b_y - f_{0yy} \qquad g_{yx} = f_{0xx} - a_x$$

Given a particular solution $\left.\begin{matrix} f = f_0 + f_1 \\ g = g_0 + g_1 \end{matrix}\right.$ where $\Delta f_1 = 0 \quad \Delta g_1 = 0$

the general sol. is

let $f_1 + i g_1 = F$ analytic. Now look at $xf + gy = 0$ @ $x^2+y^2=1$

bdy cond. $x f_1 + y g_1 = -x f_0 - y g_0 = \varphi_0 \qquad x + iy = z$ on bdy $\therefore\ z\bar{z} = 1 \ \therefore \bar{z} = 1/z$

$$\underbrace{Re\left(\tfrac{1}{z} F\right)} = \varphi_0 \quad \text{on bdy}$$

let ( ) = H now use poisson formula to get H since bdy values are given

$\therefore$ set $F = zH$ & we are done

We have existence but not uniqueness.

Matrix Valued Analytic fn.

$f(z) \in X$ normed linear space

$$\frac{f(z+h) - f(z)}{h} \to f'(z)$$

Examination in Function Theory

January 17, 1974 10-12:00 Rm. 613 Prof. P. Lax

1) Prove that the points represented by the complex numbers z_1, z_2, z_3 are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z ,$$

[10 pts.]

2) a) Which of the following functions are analytic in the indicated region? Give very briefly your reason:

i) $f(z) = z$ in $\mathbb{C}$

ii) $f(z) = \frac{1}{z^2+5}$ in $|z| < 2$

iii) $f(z) = x^2 - y^2 + 2ixy$ in $\mathbb{C}$, $z = x + iy$.

b) Show that if $f(z)$ is analytic in D,

$$g(z) = \overline{f(\bar{z})}$$

is analytic in $\bar{D}$.

[10 pts.]

3) Suppose that the power series for f:

$$f(z) = \Sigma\, a_n z^n$$

converges for $|z| < R$, and that the power series for g:

$$g(z) = \Sigma\, b_n z^n$$

converges for $|z| < S$.

a) Prove that the power series

$$h(z) = \Sigma\, a_n b_n z^n$$

converges for $|z| < RS$.

b) Prove that for $|z| < RS$,

$$h(z) = \int f(w)\, g\left(\frac{z}{w}\right) \frac{dw}{w} ,$$

the integration being over some circle, around the origin, on which

$$\frac{|z|}{S} < |w| < R .$$

[20 pts.]

4) a) Suppose that f and g are analytic, and that $g(z_o) = 0$, $g'(z_o) \neq 0$. Calculate the residue of

$$\frac{f(z)}{g^2(z)}$$

b) Find the value of the integral

$$\int_o^{2\pi} \frac{d\Theta}{(a+\cos\,\Theta)^2}$$

[30 pts.]

5) The function

$$F(w) = e^{\frac{a}{2}(w-\frac{1}{w})}$$

is analytic for $w \neq 0$ for all complex values of the parameter a. Therefore F has a Laurent expansion

$$F(w) = \sum_{-\infty}^{\infty} J_n(a) w^n .$$

a) Show that

$$J_n(a) = \frac{1}{\pi} \int_0^{\pi} \cos(n\Theta - a \sin \Theta) d\Theta$$

b) Show that

$$|J_n(a)| \leq 1$$

for a real.

c) Show that $J_n(a)$ is an analytic function of a in the whole complex plane

d) Show that for complex a

$$|J_n(a)| \leq e^{|\mathrm{Im}\, a|}$$

e) Show that

$$J_o(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k}(k!)^2}$$

[30 pts.] (Hint: go back to definition of F(w))

The factorization of entire functions.

<u>Theorem</u>: Let $f(z)$ be an entire analytic function of order $p < 1$, i.e. satisfying

1) $$|f(z)| \leq \text{const}\, e^{c|z|^p}, \quad c = \text{const}$$

Denote the zeros of $f(z)$ by z_n, and assume that $f(0) \neq 0$. Then f can be factored as

2) $$f(z) = f(0) \prod \left(1 - \frac{z}{z_n}\right).$$

<u>Proof</u>: Define $N(R)$ to be the <u>counting function</u> of the zeros:

3) $$N(R) = \text{Number of } z_j \text{ with } |z_j| \leq R$$

Next we construct a function $B_R(z)$ with the following properties:

i) $B_R(z_n) = 0$ for $|z_n| < R$

ii) $|B_R(z)| = 1$ for $|z| = R$

$B_R(z)$ can be constructed as the product

α

of functions which have property i) for a single z_n and has property ii). Such a function maps the circle of radius R into the unit circle, and z_n into the origin; thus it is a fractional linear transformation:

$$R\,\frac{z - z_0}{R^2 - \bar{z}_0 z};$$

Then

$$(4) \qquad B_{R}(z) = \prod_{1}^{N(R)} R\,\frac{z - z_1}{R^2 - \bar{z}_1 z}.$$

Now define the quotient q_R by

$$(5) \qquad \frac{f(z)}{B_R(z)} = q_R(z).$$

By property i) of B_R, we conclude that q_R is regular analytic; applying the maximum principle we get

$$(6) \qquad |q_R(0)| \leq \max_{|z|=R} |q_R(z)|$$

Using the definition (5) of q_R and (4) of B_R as well as property ii) of B_R

we deduce from (6) that

$$\frac{|f(0)|\, R^N}{\prod |z_j|} \le \max_{|z|=R} |f(z)|$$

Denote the maximum of $|f|$ on $|z|=R$ by M:

7) $$M(R) = \max_{|z|=R} |f(z)|$$

Taking the log of the above inequality we get

$$N \log R - \sum \log |z_j| = \log M(R) - \log |f(0)|.$$

We can rewrite the left side as an integral:

$$N \log R - \int^R \log r \, dN(r);$$

integration by parts turns this into

$$\int^R N(r) \frac{dr}{r} \le \log M(R) - \log |f(0)|.$$

Since $N(r)$ is an increasing function, the left side above can be estimated from below:

$$N(R/2) \log 2 \le \log M(R) - \log |f(0)|$$

So

8) $$N(R) \le \log [M(2R)/2|f(0)|].$$

For f satisfying (1) we get

9) $$N(R) \le \text{const } R^p$$

Suppose the zeros are indexed according to increasing absolute value; then $N(|z_n|) = n$; so 9) implies

$$n \le \text{const } |z_n|^p.$$

This implies

10) $$\text{const } n^{1/p} \le |z_n|,$$

from which we deduce the convergence of the series

$$\sum \frac{1}{|z_n|}$$

This implies the convergence of the infinite product

11) $$g(z) = \prod_1^\infty \left(1 - \frac{z}{z_n}\right)$$

<u>Lemma</u>: There exists a sequence of radii r tending to ∞ such that for $|z| = r$,

12) $$|g(z)| \ge e^{-c|z|^p}$$

<u>Proof</u>: Let R be any positive no, break up the product g as follows:

13) $$g = \prod_1^\infty \left(1 - \frac{z}{z_n}\right) = \prod_{|z_n| < 2R} \prod_{|z_n| > 2R} = g_R \, g^R.$$

<u>assertions</u> a)

14) $$|g^R(z)| \ge e^{-cR^p} \text{ for } |z| < R$$

b) There exists an r_R between $R/2$ and R such that

15) $$|g_R(z)| > e^{-c|z|^p} \quad \text{for } |z| = r_R.$$

Proof of a) It is an elementary fact that

$$1 - a > e^{-2a} \quad \text{for } 0 < a < 1/2$$

Using this we conclude that

$$\left|1 - \frac{z}{z_n}\right| \geq 1 - \left|\frac{z}{z_n}\right| > e^{-2\left|\frac{z}{z_n}\right|}$$

for $|z| < R$, $|z_n| > 2R$. Therefore for $|z| < R$

16) $$|g^R(z)| = \left|\prod_{|z_n| > 2R} \left(1 - \frac{z}{z_n}\right)\right| \geq e^{-2|z| \sum \frac{1}{|z_n|}}$$

Now integration by parts gives

17) $$\sum_{|z_n| > 2R} \frac{1}{|z_n|} = \int_{2R}^{\infty} \frac{1}{r}\, dN(r) = \frac{N(2R)}{2R} + \int \frac{N(r)}{r^2}\, dr,$$

where $N(r)$ is the counting function for the sequence $\{z_n\}$, i.e.

$$N(r) = \text{no. of } z_n \text{ with } |z_n| < r.$$

It follows from 1) that

18) $$N(r) \leq \text{const } r^p.$$

Substituting this into 17) shows that

$$\sum_{2R<|z_n|} \frac{1}{|z_n|} \le \text{const } R^{p-1},$$

Substituting this into 16) gives

$$|g^R(z)| \ge e^{-c|z|R^{p-1}} \ge e^{-\text{const} R^p} \quad \text{for } |z|<R$$

as asserted in 14) of part a).

To prove part b), we write $|z| = r$, and estimate $|g_R|$ as follows:

19) $$|g_R(z)| = \prod_{|z_n|<2R} \left|1 - \frac{z}{z_n}\right| \ge \prod\left(1 - \frac{r}{|z_n|}\right) = P(r)$$

The function $P(r)$ defined above has these properties

i) $P(r)$ is a polynomial of degree $N(2R)$

20) ii) $P(0) = 1$.

Assertion:

21) $$\max_{R/2 \le r \le R} P(r) \ge e^{-\text{const } R^p}.$$

Proof: We represent $P(r)$ by Lagrange

interpolation at $N+1$ equidistant points r_j in $[R/2, R]$:

22) $$r_j = R/2 + \frac{jR}{2N}, \; j = 0, \ldots, N.$$

The interpolation formula is

23) $$P(r) = \sum P(r_j) L_j(r)$$

where

24) $$L_j(r) = \prod_{k \neq j} \frac{r - r_k}{r_j - r_k}$$

The following estimate holds:

25) $$|L_j(0)| \leq \frac{(4e)^N}{\pi N}.$$

Proof: By definition 24)

26) $$|L_j(0)| = \frac{\prod r_k}{\prod |r_j - r_k|}$$

Since each r_k is $\leq R$,

27) $$\prod r_k \leq R^N$$

By definition 22)

28) $$\prod |r_j - r_k| = \prod \frac{|j-k| R}{2N} = j!(N-j)! \left(\frac{R}{2N}\right)^N$$

It is easy to show that among all j between 0 and N, $j!(N-j)!$ is smallest when $j = N/2$. So it follows from 22) that

$$\prod_{k \neq j} |r_j - r_k| \geq (N/2)!^2 \left(\frac{R}{2N}\right)^N$$

Using Stirling's formula we can write the right side as

$$\left(\sqrt{\pi N}\, e^{-N/2} \left(\frac{N}{2}\right)^{N/2}\right)^2 \left(\frac{R}{2N}\right)^N = \pi N \left(\frac{R}{4e}\right)^N$$

Combining this lower bound for $\prod |r_j - r_k|$ with the upper bound 27) for $\prod r_k$ we get inequality 25) for $L_j(0)$.

Setting $r = 0$ in 23) and using inequality 25) we get

$$P(0) \leq \frac{(4e)^N}{\pi N} \sum P(r_j) \leq \frac{N+1}{\pi N} (4e)^N \max_{R/2 \leq r \leq R} P(r).$$

Since according to 20), $P(0) = 1$, we conclude that

$$\text{const}\,(4e)^{-N} \le \max_{R/2 \le r \le R} P(r);$$

Using estimate 8) for $N = N(2R)$ we get the inequality asserted in 11).

Denote by r_R the value where $\max P(r)$ is achieved; then by 11)

$$e^{-\text{const.}\,R^p} \le P(r_R).$$

Substituting this into 19) for z with $|z| = r_R$ we get

$$|g_R(z)| \ge e^{-\text{const}\,R^p} = e^{-\text{const}\,|z|^p}$$

for all z with $|z| = r_R$. This is inequality 15) of assertion b).

Combining inequalities 14) and 15) with the definition 11):

$$g(z) = g_R(z)\,g^R(z)$$

we get inequality 12) of lemma 1.

Consider now the quotient

29) $$l(z) = f(z)/g(z);$$

l is entire because the zeros of the

denominator are matched by those of the numerator. Furthermore, by construction of g in 11), every zero of f is matched by a zero of the denominator, so that $l(z)$ is nonzero.

It follows from the upper bound 1) for $|f|$ and the lower bound 12) for $|g|$ that

30) $$|l(z)| \leq \text{const}\, e^{c|z|^p}$$

Since l is nonzero, its log is analytic; it follows from 30) that for a sequence of r tending to ∞

31) $$\operatorname{Re} \log l(z) \leq \text{const}\, |z|^p,$$

for $|z| = r$.

<u>Lemma 2</u>: A harmonic function which satisfies ~~an~~ inequality 31) with

$p<1$ on a sequence of circles whose radii tend to ∞ is constant.
Abbreviate $\mathrm{Re}\log l(z)$ as $h(z)$.

<u>Proof</u>: By the mean value property

32) $$\int h(Re^{i\theta})\,d\theta = h(0)$$

Denote by h_- and h_+ the positive and negative parts of h, i.e.

$$h_+(z) = \mathrm{Max}(h(z),0), \quad h_-(z) = \mathrm{Min}(h(z),0),$$

Clearly, $|h(z)| = h_+ - h_-$.
It follows from 32) that

33) $$\int h_+(Re^{i\theta})\,d\theta + \int h_-(Re^{i\theta})\,d\theta = h(0)$$

From 31) it follows that

$$\int h_+(Re^{i\theta})\,d\theta \leq \text{const } r^p;$$

therefore it follows from 33) that also

34) $$\int |h(Re^{i\theta})|\,d\theta \leq \text{const } r^p$$

for a sequence of R tending to ∞.

According to Poisson's formula

$$h(re^{i\varphi}) = \int h(Re^{i\theta})\, P(\theta-\varphi, R, r)\, d\theta$$

~~where~~ for $r < R$, where

$$P(\chi, R, r) = \frac{1}{2\pi}\,\frac{R^2 - r^2}{R^2 - 2Rr\cos\chi + r^2}$$

Differentiating with respect to r and φ respectively we get

$$\frac{\partial}{\partial r} h = \int h(Re^{i\theta})\, \frac{\partial P}{\partial r}\, d\theta$$

$$\frac{\partial}{\partial \varphi} h = \int h(Re^{i\theta})\, \frac{\partial P}{\partial \varphi}\, d\theta$$

It is easy to verify that both $\partial P/\partial r$ and $\partial P/\partial\varphi$ are $O(1/R)$ for R large.

Letting $R \to \infty$ on a sequence of radii for which 34) holds we conclude that $h_r = h_\varphi = 0$. So $h = \text{const}$, as asserted in lemma 2.

$h = \text{const}$ implies that $l = \text{const}$. By

definition
29) of $l(z)$ we conclude that $f(z)$ is a constant multiple of the function g defined by 11). This proves ~~the~~ assertion 2) of the main theorem.

Existence and smoothness of Green's function.

$\mathcal{C}$: Smooth, simple curve
G : Domain enclosed by $\mathcal{C}$.

C : linear space of all continuous functions on $\mathcal{C}$.

H : subspace of C consisting of boundary values of harmonic functions.

Let h be a harmonic function in G, continuous up to the boundary. Then the boundary values of h belong to H. Let p be any point of G. We define the linear functional l_p as follows:

1) $$l_p(h) = h(p)$$

l_p is defined on H, and has these properties:

a) l_p is linear
b) $|l_p(h)| \leq |h|_{max}$

Part a) follows from the linearity of the Laplace equation $\Delta h = 0$ for harmonic functions

part b) follows from the maximum principle.
By the Hahn-Banach theorem, l_p can be extended to C, so that it retains both properties a) and b).

Let w denote any point not on $\mathcal{C}$; denote by h the following element of C

2) $$h = \log|z-w|, \quad z \text{ on } \mathcal{C}.$$

h depends on the parameter w: $h = h(w)$. When w lies outside G, $h(w)$ belongs obviously to H. By definition 1) of l_p and 2) of h we have

3) $$l_p(h(w)) = \log|p-w|, \quad w \notin G.$$

For any w not on $\mathcal{C}$, we define the value of the left side of 3) to be g; g depends of course on both parameters p and w:

4) $$l_p(h(w)) = g(p,w)$$

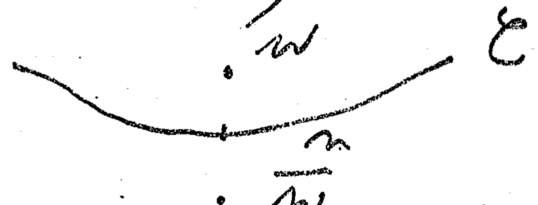
Assertion: For fixed p, $g(p,w)$ depends continuously on w even as w crosses the boundary of G.

Formula 3) shows clearly that g depends continuously on w as w approaches C from the outside. To consider the behavior of g as w approaches ~~the~~ C from G we consider the difference

$$g(p, w) - g(p, \bar{w}) \tag{5}$$

where $\bar{w}$ is the reflection of w across C, constructed as follows:

Take the ~~nearest~~ point n on C which is nearest to w, and reflect w across n.



If C has a continuous turning tangent, it is easy to show that

$$\lim \frac{|z-w|}{|z-\bar{w}|} \to 1$$

as w tends to n, uniformly for all z. Later on we shall estimate the rate at which

$$\max_{z \text{ on } C} \frac{|z-w|}{|z-\bar{w}|} \tag{6}$$

tends to 1.

Using the definition 4) of g we have

7) $$g(p,w) - g(p,\bar{w}) = l_p(h(w)) - l_p(h(\bar{w})) =$$
$$= l_p(h(w) - h(\bar{w})) = l_p\left(\log \frac{|z-w|}{|z-\bar{w}|}\right)$$

Since the quantity 6) tends to zero as w approaches the boundary, it follows that

$$\left| \log \frac{|z-w|}{|z-\bar{w}|} \right|_{Max} \longrightarrow 0$$

From the boundedness 6) of l_p it follows that the right side of 7) tends to zero. This shows that as w tends to C,

$$g(p,w) - g(p,\bar{w}) \longrightarrow 0$$

If w lies in G and is close enough to the boundary, $\bar{w}$ lies outside G. For such values g is given explicitly by 3) as $\log|z-\bar{w}|$. So we have shown that $g(p,w)$ as function of w is continuous across C.

From the definition 2) of h we deduce that $h(w)$, as element of C,

is a harmonic function of w. That means that $h(w)$ satisfies the differential equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)h(w)=0,$$

where $w=x+iy$ and where the limits which define the derivatives occurring above exist in the ~~top~~ topology of C. It follows from this that if l is any continuous linear functional on C,

$$l(h(w))$$

is a harmonic function of w. In particular

$$l_p(h(w))=g(p,w)$$

is a harmonic function of w.

We have shown above that $g(p,w)$ is a continuous function of w as w approaches the C, and that the value of $g(p,w)$ as w approaches C from the exterior is $\log|p-w|$. This shows that

<u>Theorem 1</u>: $g(p,w)$ is a <u>regular harmonic function of</u> w <u>in</u> G, <u>whose boundary value is</u> $\log|p-w|$.

It follows from this result that the function

$$\log|z-r| \quad , \; z \text{ in } C$$

belongs to H, for any r in G, being the boundary value of $g(r, z)$. By definition of l_p,

$$l_p(\log|z-r|) = g(r,p)$$

On the other hand, by definition 2) and 4) of g we have that

$$l_p(\log|z-r|) = g(p,r)$$

Comparing these two expressions we conclude that

8) $$g(r,p) = g(p,r),$$

i.e. that <u>g is a symmetric function of its two arguments.</u>

For theorem 1 we had to assume that C has a continuously turning tangent. If we assume more, we get a sharper result.

<u>Theorem 2:</u> a) Suppose C is of class C^2 i.e. has a differentially turning tangent. Then the first derivatives of $g(p,w)$ with resp. to w are bounded.

b) Suppose that $\mathcal{C}$ is of class $C^{1+\alpha}$, i.e. that the tangent of $\mathcal{C}$ turns Hölder continuously. Then the first derivatives of $g(p,w)$ satisfy the inequality

$$9) \qquad |\partial_w g| + |\partial_{\bar w} g| \le K(p)\, d(w)^{\alpha-1},$$

where $d(w)$ denotes the distance of w to $\mathcal{C}$ and $K(p)$ is a function of p alone, bounded on subsets of p away from the boundary.

Remark 1: Part a) is a special case of part b)

Remark 2: Denote by h the harmonic function conjugate to g. From the Cauchy–Riemann equations we conclude that the first derivatives of h also satisfy inequality 9), from which we conclude that h is Hölder continuous, up to the boundary $\mathcal{C}$ of G.

In the proof of theorem 2 we make use of the following estimate for harmonic functions:

<u>Lemma</u>: Suppose that g is harmonic in a circular disk of radius R, and is bounded there by some constant M in absolute value. Then the first derivatives of g at the center w_0 of the disk are bounded,

$$10) \qquad |\partial_w g(w_0)| + |\partial_{\bar w} g(w_0)| \le \text{const} \frac{M}{R};$$

This is an immediate consequence of the Poisson formula representing harmonic functions in a disk.

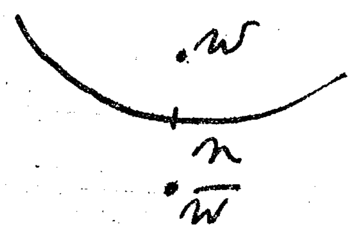
The proof ~~=~~ of theorem 2, part a) relies on the following quantitative ~~version~~ ~~of~~ estimate of 6):

Suppose the curve $\mathscr{C}$ belongs to class C^2; then for all z on $\mathscr{C}$

$$11) \qquad \left| \cancel{\mathrm{Max}} \left|\frac{z-w}{z-\bar w}\right| - 1 \right| \le K(p)\, d(w),$$

<u>Proof</u>: We introduce such coordinates that the point n on $\mathscr{C}$ nearest to w is the origin and that the x axis is tangent to $\mathscr{C}$. We write the equation of $\mathscr{C}$ near n as

$$y = y(x)$$



The assumption that $\mathcal{C}$ is of class C^2 ~~[illegible]~~ implies that

12) $$|y(x)| \le \text{const}\, x^2$$

By definition of the distance $d(w) = d$,

$$w = (0, d), \quad \bar{w} = (0, -d),$$

Then for $z = (x, y)$

13) $$\frac{|z-w|^2}{|z-\bar{w}|^2} = \frac{x^2+(y-d)^2}{x^2+(y+d)^2} = \frac{x^2+y^2+d^2-2yd}{x^2+y^2+d^2+2yd} =$$

$$= 1 - \frac{4yd}{x^2+y^2+d^2+2yd},$$

Using 12) we get that for all z on $\mathcal{C}$

$$\left|\left|\frac{z-w}{z-\bar{w}}\right|^2 - 1\right| \le \frac{4yd}{x^2} \le 4\,\text{const}\, d,$$

as asserted in 11). From this we deduce that for all z on $\mathcal{C}$

[illegible] $$\log\left|\frac{z-w}{z-\bar{w}}\right| \le \text{const.}\, d(w)$$

Using this estimate in 7) and the boundedness 6) of l_p we deduce that

$$14) \qquad |g(p,w) - g(p,\bar{w})| \le \text{const } d(w).$$

according to 7) $g(p,\bar{w}) = \log|p-\bar{w}|$; since $|w-\bar{w}| = 2d(w)$, $\log|p-\bar{w}|$ differs from $\log|p-w|$ by const $d(w)$ and we get from 14) the

$$15) \qquad |g(p,w) - \log|p-w|| \le \text{const } d(w).$$

Let w_0 be another point in G, n_0 the nearest boundary point to w_0. Since n was the nearest boundary point to w, we have

$$16) \quad d(w) = |w-n| \le |w-n_0| \le |w-w_0| + |w_0-n_0| = |w-w_0| + d(w_0)$$

Furthermore

$$\big|\log|p-w| - \log|p-w_0|\big| \le \text{const}\,|w-w_0|$$

Combining the above with 15) and using estimate 16) for $d(w)$ we get

~~Using 16) in 15) gives this estimate:~~

$$|g(p,w) - \log|p-w_0|| \leq \text{const}\ d(w_0) + \text{const}\ |w-w_0|.$$

Suppose w lies within the disk of radius $d(w_0)$ around w_0, i.e.

$$|w - w_0| \leq d(w_0);$$

then the above inequality asserts that

17) $$|g(p,w) - \log|p-w_0|| \leq \text{const}\ d(w_0).$$

We apply now inequality 10) to the function

$$g = g(p,w) - \log|p-w_0|$$

and the disk of radius $R = d(w_0)$ around w_0. Inequality 17) says that

$$M \leq \text{const}\ d(w_0);$$

so we deduce from 10) that

$$|\partial_w g(p,w)| + |\partial_{\bar{w}} g(p,w)| \leq \text{const} \frac{\text{const}\ d(w_0)}{d(w_0)} = \text{const};$$

this is precisely assertion a) of theorem 2. Assertion b) of theorem 2 can be proved similarly, except that inequality 12) is replaced by

$$12)_\alpha \qquad |y(x)| \leq \text{const}\; x^{1+\alpha}.$$

Assuming $y(x) \geq 0$ we deduce from 13) ~~that~~ and $12/\alpha$ that

$$\left|\left(\frac{z-w}{z-\bar{w}}\right)^2 - 1\right| \leq \frac{4yd}{x^2+d^2} \leq 4\text{const}\,\frac{x^{1+\alpha}d}{x^2+d^2}$$

$$= 4\,\text{const}\,\frac{x^{1+\alpha}d^{1-\alpha}}{x^2+d^2}\,d^\alpha \leq 4\,\text{const}\,d^\alpha,$$

since $x^{1+\alpha}d^{1-\alpha} \leq (x^2+d^2)$. The rest of the proof proceeds as before.

The difference

$$18) \qquad G(p,w) = \log|p-w| - g(p,w)$$

is called Green's function of the domain G. It is a harmonic function with a logarithmic singularity at p, zero on the boundary ~~of~~ C!

$$z = \frac{1}{i}\,\frac{w+i}{w-i}\,\frac{\bar w + i}{\bar w + i} = \frac{1}{i}\,\frac{w\bar w + i(w+\bar w) - 1}{w\bar w + i(w-\bar w)+1} = \frac{1}{i}\,\frac{r^2-1+2ir(\cos\theta)}{r^2+1+2r\sin\theta} = \frac{1}{i}\frac{r^2-1}{(\quad)} + \frac{2r\cos\theta}{(\quad)}$$

$re^{i\theta} - re^{-i\theta}$

$x+iy \qquad y=0 \Rightarrow r=1 \qquad x = \dfrac{\cos\theta}{1 - \sin\theta} \qquad \theta = \frac{\pi}{2},\ \frac{5\pi}{2}$

$\theta \to \frac{\pi}{2}+ \quad x \to -\infty$

$\theta \to \frac{5\pi}{2}- \quad x \to +\infty$

then

$$\frac{1}{\pi}\int \frac{f(x)\,y_0\,dx}{(x-x_0)^2+y_0^2}$$

$$\frac{i(1-r^2)}{r^2-2r\sin\theta+1} + \frac{2r\cos\theta}{r^2-2r\sin\theta+1} = x+iy$$

$$\frac{1}{\pi}\int_{\pi/2}^{5\pi/2} \frac{f\left(\frac{\cos\theta}{1-\sin\theta}\right)\frac{(1-r_0^2)}{r_0^2-2r_0\sin\theta_0+1}\,\frac{d\theta}{(1-\sin\theta)}}{\left(\frac{\cos\theta}{1-\sin\theta} - \frac{2r_0\cos\theta_0}{r_0^2-2r_0\sin\theta_0+1}\right)^2 + \left(\frac{1-r_0^2}{r_0^2-2r_0\sin\theta_0+1}\right)^2}$$

$i\,\frac{z-i}{z+i} = w \qquad i\,\frac{x-i}{x+i}\frac{(x-i)}{x-i} = w$

$x^2-2xi-1 / x^2+1$

$w = i\,\frac{x^2-1}{x^2+1} + \frac{2x}{x^2+1}$

$(x^2-1)^2 + 4x^2$

$$\left[(\cos\theta)(r_0^2-2r_0\sin\theta_0+1) - 2r_0\cos\theta_0 + 2r_0\cos\theta_0\sin\theta\right]^2 + \left[(1-r_0^2)(1-\sin\theta)\right]^2$$

$$\left(1-\sin\theta - r_0^2 + r_0^2\sin\theta\right)^2$$

$$\frac{1}{\pi}\int_{\pi/2}^{\frac{5\pi}{2}}$$

$(1-\sin\theta)dx - \cos\theta\, x\, d\theta = -\sin\theta\, d\theta$

$(1-\sin\theta)dx = -\sin\theta\,d\theta + \frac{\cos^2\theta}{1-\sin\theta}$

$= \frac{-\sin\theta\,d\theta + d\theta}{1-\sin\theta}$

$= \frac{\sin d\theta}{1-\sin\theta}$

RECOMMENDED FOR USE IN UNIT IX - ELECTRICITY

THE FOLLOWING EXPERIMENTS AND WORKSHEETS ARE

$$\left[r_0^2\cos\theta - 2r_0\sin(\theta-\theta_0) + \cos\theta - 2r_0\cos\theta_0\right]^2 =$$

$$r_0^4\cos^2\theta - 4r_0^3\cos\theta\sin(\theta-\theta_0) + 2r_0^2\cos^2\theta - 4r_0^3\cos\theta\cos\theta_0 + 4r_0^2\sin^2(\theta-\theta_0)$$
$$- 4r_0\cos\theta\sin(\theta-\theta_0) + 8r_0^2\cos\theta_0\sin(\theta-\theta_0) + \cos^2\theta - 4r_0\cos\theta_0\cos\theta$$
$$+ 4r_0^2\cos^2\theta_0;\ +1 - 2\sin\theta - 2r_0^2 + 2r_0^2\sin\theta + \sin^2\theta + 2r_0^2\sin\theta$$
$$- 2r_0^2\sin^2\theta + r_0^4 - 2r_0^4\sin\theta + r_0^4\sin^2\theta$$

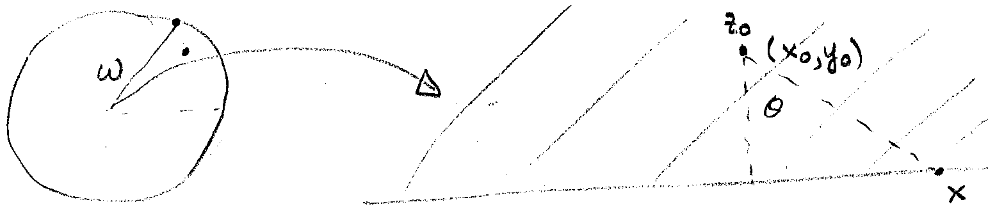
$2r_0^4 + 2r_0^2 + 2 - 2r_0^4\sin\theta - 2r_0^2\sin\theta$

$-4r_0^2\sin\theta$

$$\frac{1}{\pi}\int_{\pi/2}^{5\pi/2} f\left(\frac{\cos\theta}{1-\sin\theta}\right)\frac{(1-r_0^2)\,d\theta\,(r_0^2-2r_0\sin\theta_0+1)(1-\sin\theta)}{2(1-\sin\theta)(r_0^2-2r_0\sin\theta_0+1)\left(r_0^2+2r_0\sin(\theta-\theta_0)+\ldots\right)}$$

$$\frac{1}{\pi}\int^{2\pi} f(\ldots)(1-r_0^2)\ d\theta$$

$$w = i\left(\frac{z-i}{z+i}\right) \qquad z = \frac{1}{i}\left(\frac{w+i}{w-i}\right) \qquad Z$$



$$f(z_0) = \frac{1}{2\pi i}\int \frac{f(z)}{z - z_0}\,dz$$

$$0 = \frac{1}{2\pi i}\int \frac{f(z)}{z-\bar{z}_0}\,dz$$

$x_0 + iy_0 - x_0 + iy_0$

$$f(z_0) = \frac{1}{2\pi i}\int \frac{f(z)\,[z - \bar{z}_0 - z + z_0]}{(z-z_0)(z-\bar{z}_0)}\,dz$$

$(x_0 - iy_0)(x_0 + iy_0)$

$x_0^2 + y_0^2$

$$= \frac{1}{2\pi i}\int \frac{f(z)\;2iy_0}{z^2 - z\bar{z}_0 - z_0 z + z_0\bar{z}_0}\,dz$$

$$f(z_0) = \frac{1}{\pi}\int \frac{f(x)\,y_0\,dx}{[x-(x_0+iy_0)][x-(x_0-iy_0)]}$$

$\theta - \frac{\pi}{2}$

in the halfplane
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x)\,y_0\,dx}{(x-x_0)^2 + y_0^2}$$

$$w = \frac{i\,(x-i)}{x+i}\,\frac{(x-i)}{x-i} \qquad z = \frac{1}{i}\left(\frac{w+i}{w-i}\right) \qquad w = re^{i\theta} \qquad r<1$$

$w = i$

$i = e^{i\pi/2}$

$$i\,\frac{x^2 - 2ix + 1}{x^2+1} \qquad w = \frac{2x}{x^2+1} + i \qquad \frac{w+i}{w-i} \quad \frac{\bar{w}+i}{\bar{w}+i} \qquad \frac{re^{i\theta} + e^{-i\pi/2}}{re^{i\theta} - e^{i\pi/2}}$$

$$w = \frac{2x}{x^2+1} + i\,\frac{(x^2+1)}{x^2+1}$$

$w = re^{i\theta} \qquad r_0 e^{i\theta_0}$

$$\frac{w\bar{w} + i(w+\bar{w}) - 1}{w\bar{w} + iw - i\bar{w} + 1}$$

$$z = x+iy = \frac{1}{i}\,\frac{re^{i\theta}+i}{re^{i\theta}-i}$$

$r^2 \quad -2r(\cancel{e^{i\theta}} - \cancel{e^{i\theta}})$
$\sin\theta$

$$z_0 \longrightarrow w_0 = r_0 e^{i\theta_0}$$

$(re^{i\theta} + i)(re^{-i\theta}$

$$x + iy = \frac{1}{i}\left\{\frac{r^2-1}{r^2 - 2r\cos(\theta - \frac{\pi}{2})} + \frac{2r\sin(\frac{\pi}{2}-\theta)\,i}{r^2 - 2r\cos(\theta-\frac{\pi}{2})} + \right.$$

$x = \infty$ if $r\cos\theta > 0$

$r^2 - 2r\sin\theta + 1 = 0$

$\sin\theta = \frac{1+r^2}{2r}$

$$x = \frac{2r\cos\theta}{r^2 + 1 - 2r\sin\theta}$$

$$f(x+iy) + 0 = \frac{1}{2\pi i}\int \frac{f(u)\,du}{u-x-iy} + \frac{1}{2\pi i}\int \frac{f(u)\,du}{u-x+iy}$$

$x+iy=z$

$x-iy=\bar z$

$[(u-x)-iy][(u-x)+iy]$

$$= \frac{1}{2\pi i}\int \frac{f(u)\,du}{(u-z)(u-\bar z)}\left[(u-z)+(u-\bar z)\right]$$

$$= \frac{1}{2\pi i}\int \frac{f(u)\,du}{(u-x)^2+y^2}\left[u-x-\cancel{iy}+u-x+\cancel{iy}\right]$$

$$f(x+iy) = \frac{1}{2\pi i}\int \frac{2f(u)\,[u-x]\,du}{(u-x)^2+y^2} = -\frac{i}{\pi}\int \frac{f(u)\,[u-x]}{(u-x)^2+y^2}\,du$$

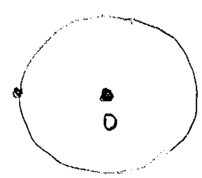
$$= \frac{i}{\pi}\int \frac{f(u)\,(x-u)\,du}{(u-x)^2+y^2}$$

$$\operatorname{Im} f(x+iy) = \frac{1}{\pi}\int \frac{\operatorname{Re} f(u)\,(x-u)\,du}{(u-x)^2+y^2}$$

$$Q = \frac{\bar w z - \bar z w}{|w-z|^2} \qquad w = Re^{i\theta} \quad z = re^{i\varphi}$$

$$\bar w = Re^{-i\theta} \quad \bar z = re^{-i\varphi}$$

$$= \frac{Rre^{-i(\theta-\varphi)} - Rre^{i(\theta-\varphi)}}{R^2+r^2-2Rr\cos(\theta-\varphi)} = -\frac{Rr\cdot 2i\sin(\theta-\varphi)}{R^2+r^2-2Rr\cos(\theta-\varphi)}$$



$$w = \frac{z}{z+1}$$

$\cdot\, i$

$\cdot\, -i$

$$w = \frac{z-i}{z+i}$$

Name ______________________ Date ____________ Class ____________

ELECTRIC CURRENTS FROM CELLS

Purpose of this experiment: To make a simple electric cell and also to learn how to connect dry cells in a battery.

Apparatus you'll need: Zinc strip; copper strip; carbon rod; glass tumbler; battery stand; dilute sulfuric acid, 1 pard acid to 20 parts water; insulated wire, No. 14 for connections; voltmeter, range 0 to 10; enameled pan; 2 dry cells; electric bell.

What this experiment is about: When you connect two unlike strips of metal with a wire and dip the other ends of the strips in a suitable liquid, an electric current flows through the wire. This makes an electric cell. We use electric cells for operating flashlights and for ringing doorbells. They're practical and convenient for generating small amounts of current. When we need larger amounts of current, however, it's cheaper to produce the electric current in other ways.

What to do:

Part 1. The simple cell. Dip one end of a strip of zinc into dilute sulfuric acid that fills a glass tumbler. Note the chemical action that occurs as the zinc and acid react. Remove the zinc and put it in an enameled pan so the acid won't get on the table. Now put a strip of copper in the acid. Then try a carbon rod in the acid. CAUTION: be careful not to get any acid on your clothing or spilled on the table. Does the acid react noticeably with the copper or carbon?

Using a battery stand, clamp the zinc and copper strips in position as shown in Fig. 1. Connect a wire from the zinc binding post to the negative terminal of a voltmeter. Similarly, connect the copper strip to the positive terminal of the voltmeter. When all connections are made, and not until then, dip the strips in a tumbler that is half-full of dilute sulfuric acid. Does the pointer of the voltmeter show that a current is flowing? Allow the strips to remain in the acid for several minutes. Meanwhile, watch to see if the pointer of the voltmeter shows a change in the current. As the action continues, where do you now see bubbles of gas collecting? These bubbles of gas interfere with the working of the cell. This defect of a simple cell is called polarization.

Repeat this test, using a carbon rod instead of the copper strip. Where do you find the bubbles collecting this time?

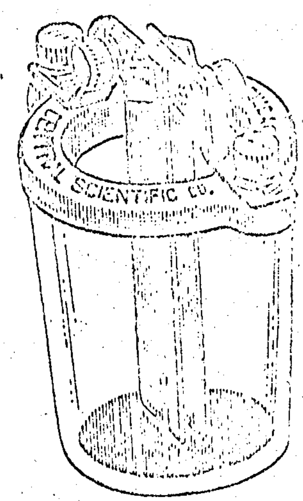


Fig. 1.

Homework. Due March 14

1) Let $f(z)$ be an analytic function in $|z| \leq 1$, Let w be some point, $|w| < 1$. The function

$$g(z) = f\left(\frac{z + w}{1 + z\bar{w}}\right)$$

is analytic in $|z| \leq 1$. From the mean value theorem

$$g(0) = \int g(z) \frac{dz}{z}$$

deduce, by changing the variable of integration, the Poisson formula for $f(w)$

2) Prove that

$$\sum_{-\infty}^{\infty} e^{in\theta} r^{|n|}$$

is the Poisson kernel for the unit circle

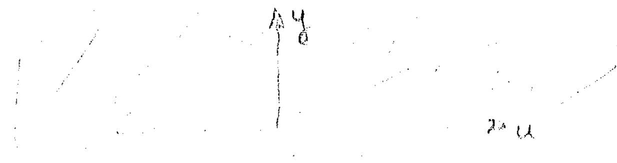
3) Prove that

$$\int_{-\infty}^{\infty} e^{i\xi x - |\xi| y} d\xi$$

is the Poisson kernel for the halfplane.

4) Let $f(z)$ be analytic in $|z| \leq 1$. Express ~~the~~ $\operatorname{Im} f(e^{i\theta})$ as an integral involving $\operatorname{Re} f(e^{i\theta})$.

5) $\Delta h(u,v,y)=0$ in $y \geq 0$ harmonic in half space



use fourier integral method to obtain poisson integral.

$$g(z) = f\left(\frac{z+w}{1+z\bar{w}}\right)$$

$$f(w) = g(0) = \int f\left(\frac{z+w}{1+z\bar{w}}\right)\frac{dz}{z}$$

let $\gamma = \dfrac{z+w}{1+z\bar{w}}$ then $z = \dfrac{\gamma - w}{1-\gamma\bar{w}}$ and $dz = \dfrac{1-w\bar{w}}{(1-\gamma\bar{w})^2}\,d\gamma$

put into formula

$$f(w) = \int f(\gamma)\,\frac{1-w\bar{w}}{(\gamma-w)(1-\gamma\bar{w})}\,d\gamma$$

$$\frac{1-w\bar{w}}{(\gamma-w)(1-\gamma\bar{w})} = \text{poisson kernel}$$

2 From class $\quad h(r,\theta) = \dfrac{1}{2\pi}\displaystyle\int h(1,\chi)\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-\chi)}\,d\chi$

$$\sum_{n=\infty}^{\infty} r^{|n|}e^{in(\theta-\chi)} = \sum_{n=1}^{\infty} r^n[\cos n(\theta-\chi) + i\sin n(\theta-\chi)]$$

$$1 + \sum_{n=-1}^{-\infty} r^{|n|}\left[\cos n(\theta-\chi) + i\sin n(\theta-\chi)\right]$$

$$\sum_{n=-1}^{-\infty} = \sum_{n=1}^{\infty} r^n\left[\cos n(\theta-\chi) - i\sin n(\theta-\chi)\right]$$

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} &= 1 + 2\sum r^n\cos n(\theta-\chi) \\
&= 2\left\{\tfrac{1}{2} + \sum r^n\cos n(\theta-\chi)\right\} \\
&= 2\left\{\tfrac{1}{2} + \tfrac{1}{2}\sum_{n=1}^{\infty} r^n\left[e^{in(\theta-\chi)} + e^{-in(\theta-\chi)}\right]\right\} \\
&= \left\{1 + \frac{re^{i(\theta-\chi)}}{1-re^{i(\theta-\chi)}} + \frac{re^{-i(\theta-\chi)}}{1-re^{-i(\theta-\chi)}}\right\} \\
&= \left\{\frac{1-r^2}{1-2r\cos(\theta-\chi)+r^2}\right\} \quad |r|<1
\end{aligned}$$

3. $\int_{-\infty}^{\infty} e^{i\xi(x-p)} |\xi| y \, d\xi = \int_0^{\infty} e^{\xi(i(x-p)-y)} d\xi + \int_{-\infty}^{0} e^{\xi(i(x-p)+y)} d\xi$

$$= \frac{1}{i(x-p)-y} e^{\xi(i(x-p)-y)} \Big|_0^{\infty} + \frac{1}{i(x-p)+y} e^{\xi(i(x-p)+y)} \Big|_{-\infty}^{0}$$

$$= \underbrace{\lim_{\xi \to \infty} \frac{e^{\xi(i(x-p)-y)}}{i(x-p)-y}}_{0} - \frac{1}{i(x-p)-y} + \frac{1}{i(x-p)+y} - \underbrace{\lim_{\xi \to -\infty} \frac{e^{\xi(i(x-p)+y)}}{i(x-p)+y}}_{0}$$

since $y > 0$ $\xi y > 0$ $e^{-\xi y} \to 0$ as $\xi \to \infty$ $|e^{i\xi(x-p)}| = 1$

$y > 0$ $\xi y < 0$ $e^{\xi y} \to 0$ as $\xi \to -\infty$ $|e^{i\xi(x-p)}| = 1$

$$\frac{1}{i(x-p)+y} - \frac{1}{i(x-p)-y} = \frac{i(x-p)-y - i(x-p)-y}{(i(x-p)+y)(i(x-p)-y)}$$

$$\frac{-2y}{-(x-p)^2 - y^2} = \frac{2y}{(x-p)^2 + y^2}$$ which is the poisson kernel for half plane

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4. $f(re^{i\varphi}) = \int \frac{f(e^{i\theta})}{z - \zeta} dz$ $\quad \zeta = re^{i\varphi}$ $\quad \bar{\zeta} = e^{-i\varphi}$

$\zeta\bar{\zeta} = 1$

$\bar{\zeta} = 1/\zeta =$

$0 = \int \frac{f(e^{i\theta})}{z - 1/\bar{\zeta}} dz$ $\quad \bar{\zeta}z - 1 + \bar{\zeta}z - \bar{\zeta}\zeta$

$$f(re^{i\varphi}) = \frac{1}{2\pi i} \int f(e^{i\theta}) \left[\frac{1}{z-\zeta} + \frac{\bar{\zeta}}{\bar{\zeta}z - 1} \right] dz$$

$$= \frac{1}{2\pi i} \int f(e^{i\theta}) \left[\frac{2(\bar{\zeta}z - 1)}{(z-\zeta)(\bar{\zeta}z - 1)} \right] dz$$

$$= -\frac{i}{2\pi} \int f(e^{i\theta}) \left[\frac{-2\left(\frac{1}{r} e^{i(\theta-\varphi)} - 1\right)}{(z-\zeta)(1 - \bar{\zeta}z)} \right] dz$$

$$= \frac{i}{\ } \int f(e^{i\theta}) \left[\frac{2(\bar{\zeta}z - 1)}{\ } \right.$$

$$f(re^{i\varphi}) = \frac{1}{2\pi i}\int \frac{f(e^{i\theta})\, ie^{i\theta}\,d\theta}{e^{i\theta} - re^{i\varphi}}$$

$$0 = \frac{1}{2\pi i}\int \frac{f(e^{i\theta})\, ie^{i\theta}\,d\theta}{e^{i\theta} - \frac{1}{r}e^{i\varphi}}$$

$$f(re^{i\varphi}) = \frac{1}{2\pi i}\int \frac{f(e^{i\theta})\, ie^{i\theta}\,d\theta\,\left[e^{i\theta} - re^{i\varphi} + e^{i\theta} - \frac{1}{r}e^{-i\varphi}\right]}{(e^{i\theta} - re^{i\varphi})(e^{i\theta} - \frac{1}{r}e^{-i\varphi})}$$

$$= \frac{1}{2\pi}\int \frac{f(e^{i\theta})\, e^{i\theta}\,d\theta\,\left[re^{i\theta} - re^{i\varphi} + re^{i\theta} - e^{-i\varphi}\right]}{(e^{i\theta} - re^{i\varphi})(re^{i\theta} - e^{-i\varphi})}$$

$z = e^{i\theta}$ $\quad \zeta = re^{i\varphi}$

$$f(\zeta) = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{z-\zeta}$$

$$0 = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{z - 1/\bar{\zeta}} = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{\bar{\zeta}z - 1}\,\bar{\zeta}$$

$$f(\zeta) = \frac{1}{2\pi i}\int \frac{f(z)\,dz\,\left[(\bar{\zeta}z - 1) + \bar{\zeta}(z-\zeta)\right]}{(z-\zeta)(\bar{\zeta}z - 1)}$$; since $\zeta\bar{\zeta} = 1$ then

$$f(\zeta) = \frac{1}{2\pi i}\int \frac{f(z)\,dz\;2\left[\bar{\zeta}z - 1\right]}{(z-\zeta)\,(\bar{\zeta}z - 1)}$$

$$f(\zeta) = \frac{1}{2\pi i}\int \frac{2f(z)\,dz}{(z-\zeta)} = \frac{1}{\pi i}\int \frac{f(z)\,dz}{(z-\zeta)}$$

$$f(e^{i\varphi}) = \frac{1}{\pi}\int \frac{f(e^{i\theta})\,e^{i\theta}\,d\theta}{e^{i\theta} - e^{i\varphi}} = \frac{1}{\pi}\int \frac{f(e^{i\theta})\,e^{i\theta}\,d\theta\,(e^{-i\theta} - e^{-i\varphi})}{(e^{i\theta} - e^{i\varphi})(e^{-i\theta} - e^{-i\varphi})}$$

$$= \frac{1}{2\pi}\int \frac{f(e^{i\theta})\left[1 - e^{i(\theta-\varphi)}\right]d\theta}{1 - \cos(\theta-\varphi)} \qquad \frac{2 - e^{i(\theta-\varphi)} - e^{-i(\theta-\varphi)}}{2 - 2\cos(\theta-\varphi)}$$

$$f(e^{i\varphi}) = \frac{1}{2\pi}\int \frac{f(e^{i\theta})\,[1-\cos(\theta-\varphi) - i\sin(\theta-\varphi)]}{1-\cos(\theta-\varphi)}\,d\theta$$

$$\underbrace{\frac{1}{2\pi}\int f(e^{i\theta})\,d\theta}_{\text{Gauss mean value}} - \frac{i}{2\pi}\int \frac{f(e^{i\theta})\sin(\theta-\varphi)}{1-\cos(\theta-\varphi)}\,d\theta$$

$$f(0) - \frac{i}{2\pi}\int \frac{f(e^{i\theta})\sin(\theta-\varphi)}{1-\cos(\theta-\varphi)}\,d\theta$$

$$\operatorname{Im} f(e^{i\varphi}) = \operatorname{Im} f(0) - \frac{1}{2\pi}\int \frac{\operatorname{Re} f(e^{i\theta})\sin(\theta-\varphi)}{1-\cos(\theta-\varphi)}\,d\theta$$

$$= \operatorname{Im} f(0) + \frac{1}{2\pi}\int \frac{\operatorname{Re} f(e^{i\theta})\sin(\varphi-\theta)}{1-\cos(\varphi-\theta)}\,d\theta$$

C. LEVY

2-21-74

 # 1 Get formula for $\operatorname{Im} f(z)$ in terms of $\operatorname{Re} f(w)$ on bdy

In class it was shown that

$$f(x+iy) = \frac{1}{2\pi i}\int_{|u|<\infty} \frac{f(u)\,du}{u-(x+iy)} \qquad 0 = \frac{1}{2\pi i}\int_{|u|<\infty} \frac{f(u)\,du}{u-(x-iy)}$$

where $w = u+iv$ and the contribution from the semi circular arc goes to zero as the radius of the arc goes to infinity. If we add the two and get a common denominator

$$f(z) + 0 = \frac{1}{2\pi i}\int \frac{f(u)\,du}{(u-z)(u-\bar{z})}\left[(u-z)+(u-\bar{z})\right]$$

$$f(z) = \frac{1}{2\pi i}\int \frac{2f(u)\,(x-u)\,du}{(u-x)^2+y^2} = \frac{i}{\pi}\int \frac{f(u)\,(x-u)\,du}{(u-x)^2+y^2}$$

$$\therefore \quad \operatorname{Im} f(x+iy) = \frac{1}{\pi}\int \frac{(\operatorname{Re} f(u))\,(x-u)\,du}{(u-x)^2+y^2} \qquad |u|<\infty$$

✓

 # 2 If $P+iQ = \dfrac{w+z}{w-z}$ find Q

$$\frac{w+z}{w-z}\left(\frac{\bar{w}-\bar{z}}{\bar{w}-\bar{z}}\right) = \frac{|w|^2-|z|^2}{|w-z|^2} + \frac{(\bar{w}z-\bar{z}w)}{|w-z|^2}$$

$$\therefore \quad iQ = \frac{\bar{w}z-\bar{z}w}{|w-z|^2}\ ; \quad \text{if} \quad w = Re^{i\theta} \quad z = re^{i\varphi} \quad \text{then}$$

$$iQ = \frac{Rr\,e^{i\varphi-i\theta} - Rr\,e^{i\theta-i\varphi}}{R^2+r^2-2Rr\cos(\theta-\varphi)} = \frac{2iRr\left\{\dfrac{e^{i(\varphi-\theta)}-e^{-i(\varphi-\theta)}}{2i}\right\}}{R^2+r^2-2Rr\cos(\theta-\varphi)}$$

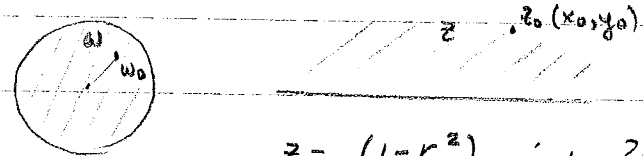
$$= \frac{2iRr\sin(\varphi-\theta)}{R^2+r^2-2Rr\cos(\theta-\varphi)} \qquad \therefore$$

$$Q = \frac{2Rr\sin(\varphi-\theta)}{R^2+r^2-2Rr\cos(\theta-\varphi)}$$

✓

#3 map unit circle into half plane; apply poisson equation to half plane & transform back to unit circle to check poisson integral for unit circle.

mapping the unit circle in half plane is $z = \frac{1}{i}\frac{(w+i)}{(w-i)}$ $\qquad w = i\frac{(z-i)}{z+i}$



$$z = \frac{(1-r^2)}{r^2-2r\sin\theta}\,i + \frac{2r\cos\theta}{r^2-2r\sin\theta+1}$$

if $w = re^{i\theta}$ $\quad r<1$

For the half plane $f(z_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)\,y_0\,dx}{(x-x_0)^2+y_0^2}$

$x = \frac{2\cos\theta}{2(1-\sin\theta)}$ since x is a pt on bdy $\Rightarrow$ x is mapped into pt on bdy of unit circle. Let $z_0 \to w_0 = r_0 e^{i\theta_0}$ then $dx = d\theta/(1-\sin\theta)$

$$\therefore\ f(z_0) = \frac{1}{\pi}\int_{\pi/2}^{\frac{5\pi}{2}}\frac{f\left(\frac{\cos\theta}{1-\sin\theta}\right)\frac{(1-r_0^2)}{r_0^2-2r_0\sin\theta_0+1}\,\frac{d\theta}{1-\sin\theta}}{\left(\frac{\cos\theta}{1-\sin\theta}-\frac{2r_0\cos\theta_0}{r_0^2-2r_0\sin\theta_0+1}\right)^2+\left(\frac{1-r_0^2}{r_0^2-2r_0\sin\theta_0+1}\right)^2}$$

let $\theta' = \theta - \pi/2$

$$f(z_0) = \frac{1}{\pi}\int_0^{2\pi}\frac{f\left(\frac{-\sin\theta'}{1-\cos\theta'}\right)\left[\frac{1-r_0^2}{r_0^2-2r_0\sin\theta_0+1}\right]\frac{d\theta'}{1-\cos\theta'}}{\left(\frac{-\sin\theta'}{1-\cos\theta'}-\frac{2r_0\cos\theta_0}{r_0^2-2r_0\sin\theta_0+1}\right)^2+\left(\frac{1-r_0^2}{r_0^2-2r_0\sin\theta_0+1}\right)^2}$$

which reduces to

$$\therefore\ f(w_0) = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{i\theta'})\,(1-r_0^2)\,d\theta'}{1-2r_0\cos(\theta'-\theta_0)+r_0^2}$$ does it ?

Prof. Lax - I do not know if you assigned these problems outright two weeks ago. If you did, they were ready to be handed in last week but there was some discussion as to whether they were assigned and that is why they were not handed in.

C. LEVY G63.2460

Prof. Lax.

1. Suppose $p(z) = \prod_{j=1} \left(1 - \frac{z}{z_j}\right) = \prod_{|z_j|<2R} \left(1 - z/z_j\right) \cdot \prod_{|z_j|>2R} \left(1 - z/z_j\right)$

pick $\frac{R}{2} < |z| < R$

then for $|z_j| > 2R$ then $|z/z_j| < \frac{1}{2}$. We note that $|1 - z/z_j| \geq 1 - |z/z_j|$ by triangle inequality and that for $|z/z_j| < \frac{1}{2}$ $\quad 1 - |z/z_j| > e^{-2|z/z_j|}$.

$\therefore \left|1 - \frac{z}{z_j}\right| > e^{-2|z/z_j|}$,

But $\left|\prod_{|z_j|>2R} (1 - z/z_j)\right| \geq \prod (1 - |z/z_j|) > e^{-2|z| \sum \frac{1}{|z_j|}}$

We now use $\sum_{|z_j|>2R} \frac{1}{|z_j|} = \int_{2R}^{\infty} \frac{1}{r}\, dN(r)$ where $N(R)$ = no. of zeros of $p(z)$ for $2R < |z_j| < \infty$

Integrate by parts to obtain $\frac{1}{2R} N(2R) + \int_{2R}^{\infty} N(r) \frac{dr}{r^2}$; it had been shown that

$N(R) \leq \text{const} \cdot \log\left(e^{2R^p}\right) = \text{const } R^p$ hence the above is $O(R^{p-1})$

using the fact that $\frac{R}{2} < |z| < R$ then $e^{-2|z| \sum \frac{1}{|z_j|}} > e^{-2|z| \text{const } R^{p-1}} > e^{-2R \text{const } R^{p-1}}$

$\therefore \prod_{|z_j|>2R} (1 - z/z_j)$ is bounded below by $e^{-\text{const } R^p} \geq e^{-\text{const } |z|^p}$ ✓

2. Prove Poisson's Integral formula $|z| < R$

$$f(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{w - z}\, dw$$

the inverse pt to the circle is $z = R^2/\bar{z}$ but this gives no contribution, or

$$0 = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{w - R^2/\bar{z}}\, dw$$

Subtract the two

$$f(z) = \frac{1}{2\pi i} \int \frac{\bar{z}\left[w - R^2/\bar{z} - w + z\right]}{(w - z)(w\bar{z} - R^2)} f(w)\, dw$$

$$f(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{[w\bar{z} - R^2 - w\bar{z} + z\bar{z}]}{(w-z)(w\bar{z} - R^2)} f(w)\,dw$$

Let $z = re^{i\theta}$ $\bar{z} = re^{-i\theta}$
$w = Re^{i\phi}$

$$f(re^{i\theta}) = \frac{1}{2\pi i} \int_{|w|=R} \frac{(r^2 - R^2)\, f(Re^{i\phi})}{w^2\bar{z} - wR^2 - wz\bar{z} + zR^2} Rie^{i\phi} d\phi$$

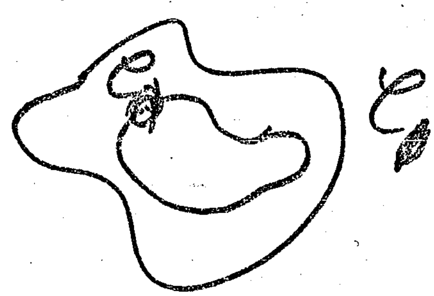
$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{(r^2 - R^2)\, f(Re^{i\phi})\, d\phi}{w\bar{z} - R^2 - z\bar{z} + z/w\, R^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{(r^2 - R^2)\, f(Re^{i\phi}) d\phi}{Rre^{i(\phi-\theta)} - R^2 - r^2 + rRe^{i(\theta-\phi)}}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2)\quad f(Re^{i\phi})\, d\phi}{r^2 - 2Rr\cos(\phi - \theta) + R^2}$$

$$\therefore \quad \operatorname{Re} f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2)}{r^2 - 2Rr\cos(\phi-\theta) + R^2} \operatorname{Re} f(Re^{i\phi})\, d\phi$$

Prof Lax Homework due April 4

1) a) Let G be a doubly connected domain, bounded by C_1 and C_0: Let h be a harmonic function in G, $= 0$ on C_0, 1 on C_1



Denote by k the conjugate harmonic function.

a) Show that k is not single valued

b) Show that for appropriate choice of a, a real

$$e^{(h+ik)a}$$

is single valued and maps G onto the annulus bounded by two concentric circles

c) Prove that if $r_1 \neq r_2$, there is no analytic function mapping the annulus $1 \leq |z| \leq r_1$ onto $1 \leq |w| \leq r_2$.

2) Show that the mapping $w \to z$ given by

$$z = \cancel{\int_0^w \frac{dt}{(1-t^n)^{2}}} \int_0^w \frac{dt}{(1-t^n)^{\frac{2}{n}}}$$

maps a regular n-gon with center at $w=0$ to the unit disk

G63.2460

C. LEVY

H.W. Express h_x & h_y in terms of bdy values of h by differentiation of poisson formula

For the circle

$$h(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\, h(R,\phi)\, d\phi}{[R^2-2Rr\cos(\theta-\phi)+r^2]}$$

$$\frac{h(r+\delta r,\theta)-h(r,\theta)}{\delta r} = \frac{1}{2\pi}\int h(R,\phi)\,d\phi \left[\frac{2R^3\cos(\theta-\phi)+2r^2R\cos(\theta-\phi)-4rR^2-\delta r[2R^2-2rR\cos(\theta-\phi)]}{(R^2-2Rr\cos(\theta-\phi)+r^2)(R^2-2R\hat{r}\cos(\theta-\phi)+\hat{r}^2)}\right.$$

$\hat{r} = r+\delta r$

$$\lim_{\delta r\to 0} \frac{h(r+\delta r,\theta)-h(r,\theta)}{\delta r} = \frac{\partial h}{\partial r}(r,\theta) = \frac{1}{2\pi}\int h(R,\phi)\,d\phi \frac{[2R^3\cos(\theta-\phi)+2r^2R\cos(\theta-\phi)-4rR^2]}{[R^2-2Rr\cos(\theta-\phi)+r^2]^2}$$

since $h(r,\theta)$ is continuous

In same manner

$$\lim_{\delta\theta\to 0} \frac{1}{r}\frac{h(r,\theta+\delta\theta)-h(r,\theta)}{\delta\theta} = \frac{1}{r}\frac{\partial h}{\partial\theta}(r,\theta) = \frac{1}{2\pi}\int h(R,\phi)\,d\phi \frac{(r^2-R^2)2R\sin(\theta-\phi)}{[R^2-2Rr\cos(\theta-\phi)+r^2]^2}$$

now $h_x = h_r\cos\theta - \frac{1}{r}h_\theta\sin\theta$

$\therefore$

$$h_x = \frac{1}{2\pi}\int_0^{2\pi} \frac{h(R,\phi)\,d\phi \quad \{(R^2-r^2)\cos\phi + 2r\cos\theta\,[r\cos(\theta-\phi)-R]\}\cdot 2R}{[R^2-2Rr\cos(\theta-\phi)+r^2]^2}$$

with $x_B = R\cos\phi$ $\quad y_B = R\sin\phi$

$x = r\cos\theta \quad y = r\sin\theta$

& $\dfrac{y_B dy_B - x_B dx_B}{x_B y_B} = d\phi$

$$h_x(0,0) = \frac{1}{2\pi}\int \frac{h(R,\phi)\,d\phi\cdot 2R\,\{R^2\cos\phi\}}{R^4}$$

if $|h(R,\phi)| \le M$ on bdy

$$|h_x(0,0)| \le \frac{1}{2\pi}\cdot\frac{2}{R}\,M\int_0^{2\pi} d\phi = \frac{2M}{R}$$

Better bound : $\dfrac{4M}{\pi R}$
(using $\int|\cos| = 4$)

also $h_y = h_r\sin\theta + \frac{1}{r}h_\theta\cos\theta$

$$h_y = \frac{1}{2\pi}\int_0^{2\pi} \frac{h(R,\phi)\,d\phi \quad \{(R^2-r^2)\sin\phi + 2R\sin\theta[R\cos(\theta-\phi)-r]\}2R}{[R^2-2rR\cos(\theta-\phi)+r^2]^2}$$

$$|h_y(0,0)| \le \frac{1}{2\pi}\cdot\frac{2M}{R}\int d\phi = \frac{2M}{R}$$

3-14-74

C. LEVY

G63.2460

1. with $g(z) = f\left(\frac{z+w}{1+z\bar{w}}\right)$

$$f(w) = g(0) = \int f\left(\frac{z+w}{1+z\bar{w}}\right)\frac{dz}{z} \quad ; \text{ let } \gamma = \frac{z+w}{1+z\bar{w}} \quad \text{then } z = \frac{\gamma - w}{1-\gamma\bar{w}}$$

and $dz = \dfrac{1-w\bar{w}}{(1+\gamma\bar{w})^2}\,d\gamma$; now put into formula

$$f(w) = \int f(\gamma)\,\frac{\frac{1-w\bar{w}}{(1-\gamma\bar{w})^2}\,d\gamma}{\frac{\gamma-w}{1-\gamma\bar{w}}} = \int f(\gamma)\,\frac{1-w\bar{w}}{(\gamma-w)(1-\gamma\bar{w})}\,\frac{d\gamma}{\gamma}$$

but $\dfrac{1-w\bar{w}}{(\gamma-w)(1-\gamma\bar{w})}$ = poisson kernel $P(w,\gamma)$ but note: $\frac{1}{2\pi}P(w,\gamma)\gamma$ = the Poisson kernel. ✓

2. From class we had shown $h(r,\theta) = \dfrac{1}{2\pi}\displaystyle\int h(1,\varphi)\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-\varphi)}\,d\varphi$

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-\varphi)} = \sum_{n=-\infty}^{-1} r^{|n|}e^{in(\theta-\varphi)} + 1 + \sum_{n=1}^{\infty} r^{|n|}e^{in(\theta-\varphi)}$$

$$= 1 + \sum_{n=1}^{\infty} r^n\left\{e^{in(\theta-\varphi)} + e^{-in(\theta-\varphi)}\right\}$$

$$= 1 + re^{i(\theta-\varphi)}\sum_{n=1}^{\infty} r^n e^{in(\theta-\varphi)} + \sum_{n=1}^{\infty} r^n e^{-in(\theta-\varphi)}$$

$$= 1 + \frac{re^{i(\theta-\varphi)}}{1-re^{i(\theta-\varphi)}} + \frac{re^{-i(\theta-\varphi)}}{1-re^{-i(\theta-\varphi)}}$$

$$= \frac{1+r^2-\left(re^{i(\theta-\varphi)}+re^{-i(\theta-\varphi)}\right)+re^{i(\theta-\varphi)}-r^2+re^{-i(\theta-\varphi)}-r^2}{\left[1-re^{i(\theta-\varphi)}\right]\left[1-re^{-i(\theta-\varphi)}\right]}$$

$$= \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} \qquad r < 1$$

$$= P(r, \theta, \varphi) \qquad \text{poisson kernel for the circle}$$

3. $$\int_{-\infty}^{\infty} e^{i\xi(x-p) - |\xi| y}\, d\xi = \int_0^{\infty} e^{\xi[i(x-p) - y]}\, d\xi + \int_{-\infty}^{0} e^{\xi[i(x-p)+y]}\, d\xi$$

$$= \frac{1}{i(x-p) - y} e^{\xi[i(x-p)-y]}\Big|_0^{\infty} + \frac{1}{i(x-p)+y} e^{\xi[i(x-p)+y]}\Big|_{-\infty}^{0}$$

$$= \lim_{\substack{\xi \to \infty \\ y > 0}} \frac{e^{\xi[i(x-p)-y]}}{i(x-p)-y} - \frac{1}{i(x-p)-y} + \frac{1}{i(x-p)+y} - \lim_{\substack{\xi \to -\infty \\ y > 0}} \frac{e^{\xi[i(x-p)+y]}}{i(x-p)+y}$$

The first term $y > 0$, $\xi y > 0$, $e^{-\xi y} \to 0$ as $\xi \to \infty$ & $|e^{\xi[i(x-p)]}| = 1$

The last term $y > 0$, $\xi y < 0$, $e^{\xi y} \to 0$ as $\xi \to -\infty$ & $|e^{\xi[i(x-p)]}| = 1$

Thus the integral equals

$$\frac{1}{i(x-p)+y} - \frac{1}{i(x-p)-y} = \frac{2y}{(x-p)^2 + y^2}$$ which is the poisson kernel for the half plane

$$\frac{2y}{(x-p)^2 + y^2} = P(x-p, y)$$

4. for the circle we have

$$f(\zeta) = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{z - \zeta} \qquad z = e^{i\theta} \qquad \zeta = re^{i\varphi} \qquad r < 1$$

$$0 = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{z - 1/\bar{\zeta}} = \frac{1}{2\pi i}\int \frac{f(z)\,\bar{\zeta}\,dz}{\bar{\zeta} z - 1}$$

?

$$f(\zeta) + 0 = \frac{1}{2\pi i}\int \frac{f(z)\,dz \quad 2[\bar{\zeta} z - 1]}{(z - \zeta)(\bar{\zeta} z - 1)}$$

since $\bar{\zeta}\zeta = 1$ $\quad$ no! $|\zeta| < 1$

this is impossible unless $\phi = 0$

$\frac{1}{2}\phi \neq \phi$

$$\therefore \quad f(\zeta) = \frac{1}{\pi i}\int \frac{f(z)\,dz}{z-\zeta} \qquad \text{with} \quad \bar{\zeta} z - 1 \neq 0$$

now let $|\zeta| \to 1$ then

$$f(e^{i\varphi}) = \frac{1}{\pi}\int \frac{f(e^{i\theta})\,e^{i\theta}\,d\theta}{e^{i\theta}-e^{i\varphi}} = \frac{1}{\pi}\int \frac{f(e^{i\theta})\left[1-e^{i(\theta-\varphi)}\right]}{2(1-\cos(\theta-\varphi))}\,d\theta$$

$$= \frac{1}{2\pi}\int f(e^{i\theta})\,d\theta - \frac{i}{2\pi}\int \frac{f(e^{i\theta})\sin(\theta-\varphi)}{1-\cos(\theta-\varphi)}\,d\theta$$

by Gauss Mean Value thm the first integral $= f(0)$

$\therefore$

$$f(e^{i\varphi}) = f(0) + \frac{i}{2\pi}\int \frac{f(e^{i\theta})\sin(\varphi-\theta)}{1-\cos(\varphi-\theta)}\,d\theta$$

and $\operatorname{Im} f(e^{i\varphi}) = \operatorname{Im} f(0) + \frac{1}{2\pi}\int \frac{\operatorname{Re} f(e^{i\theta})\sin(\varphi-\theta)\,d\theta}{1-\cos(\varphi-\theta)}$ in principal value sense

since $\frac{\sin(\varphi-\theta)}{1-\cos(\varphi-\theta)}$ is unbdd as $\theta \to \varphi$ as shown by applying l'hôpital's rule.

5. For $\Delta h(u,v,y) = 0$ in $y > 0$ we assume h, h_u, h_v to be regular at infinity and $h(u,v,0) = f(u,v)$

let $h^*(\xi,\eta,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u,v,y)\,e^{-i\xi u - i\eta v}\,du\,dv = \mathcal{F}h$

and $h(u,v,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(\xi,\eta,y)\,e^{i\xi u + i\eta v}\,d\xi\,d\eta = \mathcal{F}^{-1}h^*$

Then $\mathcal{F}(h_{uu} + h_{vv} + h_{yy}) = 0 \implies \left[-\xi^2 - \eta^2\right]h^* + h^*_{yy} = 0$

$$\therefore \quad h^*(\xi,\eta,y) = a(\xi,\eta)\,e^{\sqrt{\xi^2+\eta^2}\,y} + b(\xi,\eta)\,e^{-\sqrt{\xi^2+\eta^2}\,y}$$

since we assume that as $y \to +\infty$ $h^* \to 0$ then $a(\xi,\eta) = 0$

$$\therefore \quad h^*(\xi,\eta,y) = b(\xi,\eta)\, e^{-\sqrt{\xi^2+\eta^2}\,y}$$

B.C. : $$h^*(\xi,\eta,0) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\, e^{-i\xi u - i\eta v}\, du\, dv = f^*(\xi,\eta)$$

$$\therefore \quad h^*(\xi,\eta,y) = f^*(\xi,\eta)\, e^{-\sqrt{\xi^2+\eta^2}\,y}$$

Now we invert the transform

$$h(u,v,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(\xi,\eta)\, e^{-\sqrt{\xi^2+\eta^2}\,y}\, e^{i\xi u + i\eta v}\, d\xi\, d\eta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u',v')\, e^{-i\xi u' - i\eta v'}\, du'\, dv'\, e^{-\sqrt{\xi^2+\eta^2}\,y}\, e^{i\xi u + i\eta v}\, d\xi\, d\eta$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u',v')\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\xi(u'-u) - i\eta(v'-v) - \sqrt{\xi^2+\eta^2}\,y}\, d\xi\, d\eta\right] du'\, dv'$$

define $$P(u'-u, v'-v, y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\xi(u'-u) - i\eta(v'-v) - \sqrt{\xi^2+\eta^2}\,y}\, d\xi\, d\eta$$

$$\therefore \quad P(u,v,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\xi u + i\eta v - \sqrt{\xi^2+\eta^2}\,y}\, d\xi\, d\eta$$

Let $\xi = \rho\cos\varphi$, $\eta = \rho\sin\varphi$, $d\xi\, d\eta = \rho\, d\rho\, d\varphi$, $\sqrt{\xi^2+\eta^2} = \rho$, $u = r\cos\theta$, $v = r\sin\theta$

$\therefore$ $\xi u + \eta v = \rho r\cos(\varphi - \theta)$

$$P(u,v,y) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{\infty} e^{-\rho y}\, e^{i\rho r\cos(\varphi-\theta)}\, \rho\, d\rho\, d\theta = P(r,\theta,y)$$

if $$F(r,\theta,y) = \int_0^{2\pi}\int_0^{\infty} e^{-\rho y}\, e^{i\rho r\cos(\varphi-\theta)}\, d\rho\, d\theta$$

then $$P(r,\theta,y) = -\frac{\partial F}{\partial y}\left(\frac{1}{2\pi}\right)^2$$

$$\therefore \quad F(r,\theta,y) = \int_0^{2\pi} \left. \frac{e^{-\rho[y - ir\cos(\varphi-\theta)]}}{-[y - ir\cos(\varphi-\theta)]} \right|_{\rho=0}^{\rho=\infty} d\varphi$$

as $\rho \to \infty$ & $y > 0$ $\quad e^{-\rho y} \to 0$ $\quad \therefore$ only contribution is at $\rho = 0$

$$\therefore \quad F(r,\theta,y) = \int_0^{2\pi} \frac{d\varphi}{y - ir\cos(\varphi-\theta)}$$

integrate in complex plane; let $z = e^{i(\varphi-\theta)}$, $\quad dz = i e^{i(\varphi-\theta)} d\varphi$

$$\therefore \quad F(r,\theta,y) = \oint_c \frac{-i \frac{dz}{z}}{y - ir\left(\frac{z^2+1}{2z}\right)} = -\oint_c \frac{i\, dz}{yz - \frac{ir}{2}(z^2+1)}$$

$$= \frac{2}{i}\oint_c \frac{dz}{2yz - ir(z^2+1)} = \frac{2}{r}\oint \frac{dz}{(z^2 + \frac{2yiz}{r} + 1)}$$

denom has roots $\quad z_\pm = \dfrac{-\frac{2yi}{r} \pm \sqrt{-\frac{4y^2}{r^2} - 4}}{2} = -\dfrac{2yi}{2r} \pm i\sqrt{y^2/r^2 + 1}$

$$= i\left[\frac{-y}{r} \pm \sqrt{y^2/r^2 + 1}\right]$$

the curve C is the circle $|z| = 1$

$$|z_+| < 1 \quad \text{and} \quad |z_-| > 1 \quad \Rightarrow \quad \text{a residue @ } z_+$$

if $(y/r) < 1$ then $|z_+| < 1$

$(y/r) > 1$ then $|z_+| < 1$

$(y/r) = 1$ then $|z_+| < 1$

$$\therefore \quad F(r,\theta,y) = \frac{2}{r}\oint \frac{dz}{[z - z_+][z - z_-]} = \frac{2}{r}\cdot 2\pi i \frac{1}{z_+ - z_-} = \frac{2}{r}\cdot 2\pi i \frac{r}{2i\sqrt{y^2+r^2}}$$

$$= \frac{2\pi}{\sqrt{y^2+r^2}} = F(r,y)$$

also $P(u,v,y) = P(r,\theta,y) = -\left(\frac{1}{2\pi}\right)^2 \frac{\partial F(r,\theta,y)}{\partial y} = -\left(\frac{1}{2\pi}\right)^2 \frac{2\pi}{(y^2+r^2)^{3/2}}\left(-\frac{1}{2}\right)(2y)$

$$= \frac{1}{2\pi} \frac{y}{(y^2+r^2)^{3/2}}$$

but $r^2 = u^2+v^2$ $\therefore$ $P(u,v,y) = \frac{1}{2\pi} \frac{y}{(u^2+v^2+y^2)^{3/2}}$

or $P(u-u',v-v',y) = \frac{1}{2\pi} \frac{y}{[(u-u')^2+(v-v')^2+y^2]^{3/2}}$

This is the poisson integral for the half space $y \geq 0$

$$\therefore\ h(u,v,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{y}{[(u-u')^2+(v-v')^2+y^2]^{3/2}} f(u',v')\,du'dv'$$

G63.2460

C. LEVY

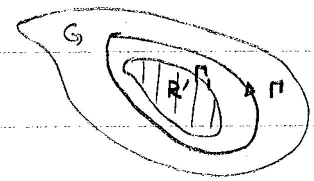
1. Let G be doubly connected domain, bounded by C_1 and C_0:

Let h be a harmonic function in G, $\begin{matrix} 0 & \text{on } C_0 \\ 1 & \text{on } C_1 \end{matrix}$

Denote by k the conjugate harmonic fn.

a) Show that k is not single valued

For some Γ curve in G



we can define a period

$$\rho = -\int_{\Gamma} h_y dx - h_x dy = -\iint_{R_\Gamma} \Delta h \, dx dy$$

where R_Γ is region enclosed by Γ

$$\iint_{R_\Gamma} = \iint_{R_\Gamma - R'} + \iint_{R'} \Delta h \, dx dy$$; the first integral is zero since $\Delta h = 0$ in $R_\Gamma - R'$ but the second is not since Δh is not zero in R'

$$\therefore \quad \rho = -\int_{\Gamma_1} h_y dx - h_x dy = \int_{\Gamma_1} \frac{\partial h}{\partial n} ds$$

b) $e^{(h+ik)a} = \omega$ is single valued & maps G onto the annulus bounded by 2 concentric circles

pick $a = \frac{2\pi}{\rho}$; $e^{(h+ik)a}$ is single valued since $\frac{dw}{dz} \neq 0$ anywhere

Since a full sweep about Γ_1 increases k by ρ

then $e^{(h+ik)a}$ on C_1 gives $e^{\frac{2\pi}{\rho} + 2\pi i} = e^{\frac{2\pi}{\rho}}$

$$\therefore \quad |w| = e^{\frac{2\pi}{\rho}}$$

on C_0 $e^{(h+ik)a}$ gives $e^{(0+i\rho)\frac{2\pi}{\rho}} = e^{2\pi i} = 1$

$$\therefore \quad |w| = 1$$

hence we have the boundaries C_0, C_1 mapped onto 2 concentric circles and by the theorem proved in class, any closed curve between the two

boundaries are such that $\nabla R \neq 0$ in G, and any pt z in the domain G is covered only once.

c) Prove that if $r_1 \neq r_2$, there is no analytic fn mapping the annulus $1 \leq |z| \leq r_1$: G onto $1 \leq |w| \leq r_2$: D

This is equivalent to Thm 1.1 Markushevich Vol III Theory of Functions of a complex variable (P. 4-8) (The pf in Rudin is much shorter)

Mapping of annulus onto annulus ($0 < r_1 < |z| < r_2 < \infty \longrightarrow 0 < R_1 < |w| < R_2 < \infty$) iff $\frac{1}{r_1} = \frac{1}{r_2}$. This condition is satisfied if it is a möbius transformation

Sufficiency: since if $\frac{1}{r_1} = \frac{1}{r_2}$ holds the entire linear transformation $w = f(z) = z = \frac{r_1}{r_2} z$ maps one annulus onto the other.

Necessity; suppose $w = f(z)$ with inverse $z = \varphi(w)$, maps $G \to D$

let Γ_ρ be image of circle $\gamma_\rho : z = \rho e^{i\theta}$ ($0 \leq \theta \leq 2\pi$ $1 < \rho < r_1$) under the mapping w. Then Γ_ρ is a closed Jordan curve given by

$$w = f(\rho e^{i\theta}) \quad 0 \leq \theta \leq 2\pi$$

Clearly

$$1 = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{dz}{z} = \pm \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\varphi'(w)}{\varphi(w)} dw$$

γ_ρ & Γ_ρ traversed in positive direction, $\pm$ denote whether $w = f(z)$ traverses in same or opposite sense to each other. Hence $w = 0$ in inside Γ_ρ and moreover $\frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\varphi'(w)}{\varphi(w)} dw$ continuous fn of ρ takes on integral values and the sign chosen is independent of ρ. Hence the direction of traverse of the pt $w = f(z)$ follows the same direction $\forall \rho \in (1, r_1)$.

Assume traversing in same direction calculate area S_ρ whose bdy is Γ_ρ

$S_\rho = \frac{1}{2}\int_{\Gamma_\rho} (u\,dv - v\,du) > 0 \qquad w = u + iv$ & Γ_ρ traversed in positive sense.

$$S_\rho = \frac{1}{2}\int_{\Gamma_\rho} \mathrm{Im}(\bar{w}\,dw) = \frac{1}{2}\int_0^{2\pi} \mathrm{Im}\left\{ \overline{f(\rho e^{i\theta})}\ \frac{\partial f(\rho e^{i\theta})}{\partial \theta} \right\} d\theta$$

Expand in Laurent series

$$f = \sum_{n=-\infty}^{\infty} a_n \rho^n e^{in\theta} \qquad \frac{\partial f}{\partial \theta} = \sum i n a_n \rho^n e^{in\theta}$$

$$S_\rho = \frac{1}{2}\int_0^{2\pi} \mathrm{Im}\left\{ \sum_{m=-\infty}^{\infty} \bar{a}_m \rho^m e^{-im\theta} \sum_{n=-\infty}^{\infty} i n a_n \rho^n e^{in\theta} \right\} d\theta$$

$$= \mathrm{Im}\left\{ \frac{1}{2} \sum_{n=-\infty}^{\infty} i n |a_n|^2 \rho^{2n} \cdot 2\pi \right\} = \pi \sum_{n=-\infty}^{\infty} n |a_n|^2 \rho^{2n} > 0.$$

since Γ_ρ lies inside $|w| = r_2$ & $|w| = 1$ is inside Γ_ρ, $\forall \rho \in (1, r_1)$ then

$$\pi < \pi \sum n |a_n|^2 \rho^{2n} < \pi r_2^2 \qquad (1 < \rho < r_1) \qquad (1)$$

It follows that the series

$$\sum_{n=-\infty}^{\infty} n |a_n|^2 \quad \& \quad \sum_{-\infty}^{\infty} n |a_n|^2 r_1^{2n} \qquad (2)$$

Converge. Proof for second series $\rightarrow$

$$\sum_{-\infty}^{\infty} n |a_n|^2 \rho^{2n} = \sum_{n=1} n |a_n|^2 \rho^{2n} - \sum_{n=1} n |a_{-n}|^2 \rho^{-2n} \qquad (1 < \rho < r_1)$$

using (1) into the above

$$\sum_{n=1}^{\infty} n |a_n|^2 \rho^{2n} \le r_2^2 + \sum_{n=1} n |a_{-n}|^2 \rho^{-2n} \qquad (1 < \rho < r_1).$$

Therefore

$$\sum_{n=1}^{N} n |a_n|^2 \rho^{2n} < r_2^2 + \sum_{n=1}^{\infty} n |a_{-n}|^2 \rho^{-2n} \qquad (1 < \rho < r_1)$$

for all $N > 0$ take limit as $\rho \to r_1 -$, then

$$\sum n |a_n|^2 r_1^{2n} \le r_2^2 + \sum n |a_{-n}|^2 r_1^{-2n}$$

or

$$\sum_{-\infty}^{\infty} n |a_n|^2 r_1^{2n} \le r_2^2$$

Similarly, we can show the first series of (2)

$$\sum_{n=-\infty}^{\infty} n |a_n|^2 \ge 1$$

$$\therefore \quad \lambda^2 = \frac{\sum_{-\infty}^{\infty} |a_n|^2 n \, r_1^{2}}{\sum_{-\infty}^{\infty} |a_n|^2 n} \leq r_2^2$$

but

$$\lambda^2 - r_1^2 = \frac{r_1^2 \sum_{-\infty}^{\infty} n|a_n|^2 \left(r_1^{2n-2} - 1\right)}{\sum_{-\infty}^{\infty} n|a_n|^2} \geq 0 \qquad (3)$$

since $n(r_1^{2n-2} - 1) \geq 0$

then we conclude $r_1^2 \leq \lambda^2 \leq r_2^2$

or $r_1 \leq r_2$

However this argument holds if we interchange the roles of the two planes this implies that $r_1 \equiv r_2$.

Then (3) must be

$$\lambda^2 - r_1^2 = \frac{r_1^2 \sum_{-\infty}^{\infty} n|a_n|^2 \left(r_1^{2n-2} - 1\right)}{\sum_{-\infty}^{\infty} n|a_n|^2} = 0$$

or $a_n = 0$ unless $n = 0$ or $n = 1$ $\quad \therefore f(z) = a_0 + a_1 z$

to conclude proof we consider case when the two traverse in opposite directions; using $w' = 1/w$ carries $D: \frac{1}{r_2} < |w'| < 1$

$\ni. \quad w' = \frac{1}{w} = \frac{1}{f(z)}$

$\Rightarrow \arg w' = -\arg f(z) = -\arg w$ hence sense of w & $f(z)$ is same

by same argument as before

$$\frac{1}{1/r_2} = r_2 = r_1$$

& $f(z) = \dfrac{1}{a_0 + a_1 z}$ which is again a mobius transformation.

✓ (half-hearted)

2. Show that the mapping $w \to z$ given by
$$z = \int_0^w \frac{dt}{(1-t^n)^{2/n}}$$
maps a regular n-gon with center at $w=0$ into unit disk

By Schwartz-Christoffel transformation the interior angles change by $\pi(1-2/n)$ hence $\frac{\alpha_i}{\pi} - 1 = -2/n$. Since for mapping into upper half plane
$$\frac{dz}{dw'} = (w'-w_0')^{-2/n} \cdots (w'-w_{n-1}')^{-2/n}$$
then
$$z = \int_0^{w'} \frac{dw'}{\left[\prod_{i=0}^{n-1}(w'-w_i')\right]^{2/n}} \qquad w_i' \text{ on } 1$$

and the mapping of half plane into a unit circle is
$$w = \frac{i-w'}{i+w'}$$
then $w' = i\left(\frac{1-w}{1+w}\right)$

let $w'-w_j' = i\left(\frac{1-w}{1+w}\right) - i\left(\frac{1-w_j}{1+w_j}\right) = \frac{-2i(w-w_j)}{(1+w)(1+w_j)}$

$\therefore \quad dw' = \frac{-2i\,dw}{(1+w)^2}$ then

$$z = A\int_0^w \frac{-2i\,dw/(1+w)^2}{\left[\prod_{j=0}^{n-1}\frac{(-2i)(w-w_j)}{(1+w)(1+w_j)}\right]^{2/n}} = \frac{A(-2i)}{\left[\prod_{j=0}^{n-1}\frac{(-2i)}{1+w_j}\right]^{2/n}}\int_0^w \frac{dw\,\prod(1+w)^{2/2}}{(1+w)^2\left[\prod(w-w_j)\right]^{2/n}}$$

where w_j can be defined by $e^{\frac{2\pi i}{n}j}$. This should reduce to the form given It should!

Prediction Theory.

$\ldots, x_{-1}, x_0, x_1, \ldots$ a doubly infinite sequence of random variables, real valued.

<u>Expected value</u> $E(x)$

Properties: a) linearity: $E(x+y) = E(x) + E(y)$
$E(ax) = aE(x)$, a const

b) Positivity: If the random variable x takes on positive values only, $E(x) \geq 0$.

<u>~~Autocorrelation~~ Correlation</u>

1) $$E(x_j x_k) = c_{j,k}.$$

Suppose x_j have mean zero; a positive correlation of x_j and x_k, $c_{jk} > 0$ means that if x_j is pos., x_k is more likely to be positive than negative.

<u>Thm</u>: The ~~collection~~ correlation matrix is symmetric & nonneg.

Pf: Symmetry obvious from def. 1).

$$\sum c_{jk} \xi_j \xi_k = \sum E(x_j x_k) \xi_j \xi_k =$$
$$= E(\sum \xi_j \xi_k x_j x_k) = E(|\sum \xi_j x_j|^2) \geq 0.$$

<u>Stationary sequence:</u>

2) $$c_{j,k} = c_{j-k}.$$

<u>Spectral representation</u> of stationary correlation.

Suppose that $c_n \to 0$ rapidly enough as $n \to \infty$,

Form the Fourier series

3) $$\sum e_n e^{in\theta} = m(\theta)$$

Theorem: m defined by 3) is nonnegative

Pf: Set, for any decent coefficient ξ_j,

4) $$\sum \xi_j e^{ij\theta} = \varphi(\theta)$$

and form

5) $$\int |\varphi(\theta)|^2 m(\theta) d\theta$$

Writing

$\overline{4)}$ $$\overline{\varphi}(\theta) = \sum \overline{\xi_k} e^{-ik\theta}$$

we get for 5)

$$\int \sum \xi_j \overline{\xi_k} e_n e^{i(j-k+n)\theta} d\theta$$

$$= \sum \xi_j \overline{\xi_k} e_{k-j}$$

which by previous thm is ≥ 0, so 5) is nonneg for all decent φ. Since any decent nonneg function $p(\theta)$ can be written as

$$p(\theta) = |\varphi(\theta)|^2$$

it follows that

$$\int p(\theta) m(\theta) d\theta \geq 0$$

for nonneg. $p(\theta)$. But then $m(\theta) \geq 0$.

Assumption that $e_n \to 0$ suff. rapidly not necessary; we skip proof of this.

The values e_n are obtained by long-term observation.

Prediction formula (linear)

6) $$x_0 = \sum_{-\infty}' a_j x_j$$

Best prediction, in root mean square sense: minimize expected deviation between predicted and observed value of x_0:

7) $$E(|x_0 - \sum a_j x_j|^2)$$

which equals

8) $$\sum a_j a_k e_{j-k}$$

where we set $a_0 = -1$, Now set

9) $$\varphi(\theta) = \sum_0^\infty a_{-j}\, e^{i\theta}$$

8) can be expressed by 5) as

10) $$\int |\varphi(\theta)|^2 m$$

Now $\varphi(\theta)$ can be regarded as the boundary value of the function analytic in the unit disk

11) $$f(z) = \sum_0^\infty a_j z^j$$

10) can be written as

12) $$\int |f(z)|^2 m(\theta)\, d\theta, \quad z = e^{i\theta}.$$

$a_0 = -1$ means

13) $$f(0) = -1$$

To minimize 12) among all anlyt. functions $f(z)$ subject to 13) we factor m as follows:

14) $$m(\theta) = |k(z)|^2, \quad z = e^{i\theta},$$

where $k(z)$ is analytic in the unit disk, and doesn't vanish there. Set

15) $$f(z)\,k(z) = g(z)$$

Using 14), 15), 12) can be rewritten as

16) $$\int |g(z)|^2 d\theta$$

and condition 13) becomes

17) $$|g(0)| = |k(0)|$$

As g ranges over all anlyt. fctns, $f = g/k$ ranges over all analytic fctns (because $k \neq 0$ in the unit disk). So the minimum problem 12) for f and 16) for g are the same. 16) has the obvious ~~solution~~ minimizer satisfying (6):

18) $$g(z) \equiv k(0)$$

The value of the minimum 16) is

~~16~~) $$|k(0)|^2$$

<u>The factorization of m</u>: Take the logarithm of 14):

19) $$\log m(\theta) = \log k + \log \bar{k}$$

$k(z) \neq 0$, $$= 2 \operatorname{Re} \log k = 2 \log |k|$$

Since Re log k is harmonic, it satisfies the mean value property

$$2 \log |k(0)| = \int \log m(\theta)\, d\theta$$

So

20) $$\boxed{|k(0)|^2 = \exp \int \log m(\theta)\, d\theta}$$

The expression on the right is the geometric mean of m. 19) gives the <u>prediction error</u>, i.e. the expected value of the square of the difference of the observed and predicted value, when the best linear predictor is used.

Note that the prediction error can be zero, when log m is not integrable.

Using 19) we can determine first the real, then the imag. part of log k, from which k is obtained by exp. The optimizing f is gotten from 18) and 15) as

$$f(z) = \frac{-k(0)}{k(z)},$$

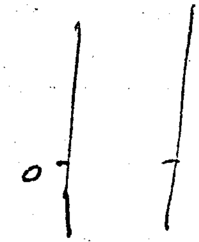
Marcel Riesz Convexity Theorem.

I) Three lines thm:

1) $$M_x \le M_0^{x} M_1^{1-x},$$

where

2) $$M_x = \sup_y |f(x+iy)|$$



II) L_p norms: $f = (f_1, \dots, f_n)$ $\quad 0 \le p \le \infty$

Weighted

3) $$\|f\|_p = \left(\sum |f_j|^p w_j \right)^{1/p}, \quad w_j > 0.$$

Hölder:

4) $$(f, g) = \sum f_j g_j w_j$$

5) $$|(f, g)| \le \|f\|_p \|g\|_{p'},$$

where

6) $$\frac{1}{p} + \frac{1}{p'} = 1.$$

Converse Holders:

7) $$\|f\|_p = \max_{\|g\|_{p'} = 1} |(f, g)|.$$

IIB) $Tf = k$, a linear mapping $f \to k$, f and k vectors of not nec. the same dimension.

8) $$\|T\|_{p,q} = \sup \frac{\|Tf\|_q}{\|f\|_p}.$$

Let p_0, q_0 and p_1, q_1 be two pairs, def

Set $p(t)$, $q(t)$, $0 \le t \le 1$ equal to:

9) $$\frac{1}{p(t)} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q(t)} = \frac{1-t}{q_0} + \frac{t}{q_1},$$

<u>M. Riesz–Thorin's thm:</u>

10) $$\|T\|_{p(t),q(t)} \le \|T\|_{p_0,q_0}^{1-t} \, \|T\|_{p_1,q_1}^{t}.$$

<u>Pf:</u> We show that for $\|f\|_{p(t)} = 1$,

$\|Tf\|_{p(t)} \le$

By 7), this is implied by

11) $|(Tf, g)| \le$

for $\|f\|_{p(t)} = 1$, $\|g\|_{q'(t)} = 1$.

To see this, set

12) $f(z) = \sum |f_j|^{\frac{p(t)}{p(z)}} e^{i\mu_j}$, where $e^{i\mu_j} = \frac{f_j}{|f_j|}$

and $g(z) = |g_j|^{\frac{q'(t)}{q'(z)}} e^{i\nu_j}$, where $e^{i\nu_j} = \frac{g_j}{|g_j|}$,

Note that $f(t) = f$, $g(t) = g$.

Define

13) $s(z) = (T f(z), g(z))$

f and g depend analytically on z, so does $s(z)$.

<u>Assertion:</u> 14) $M_0 \le \|T\|_{p_0,q_0}$, $M_1 \le \|T\|_{p_1,q_1}$

<u>Proof:</u> Suppose z is purely imaginary: $z = iy$

Then it follows from def 9) that

$$\frac{p(t)}{p(z)} = \frac{p(t)}{p_0} + \text{imag}.$$

So it follows from this and def. 12 that

$$\| f(iy) \|_{p_0} = \| f \|_{p(t)}^{\frac{p(t)}{p_0}}$$

Since f has been chosen so that $\|f\|_{p(t)} = 1$

$$\| f(iy) \|_{p_0} = 1.$$

Similarly

$$\| g(iy) \|_{q_0'} = 1$$

Estimating $s(z)$ by Hölder we get for $z = iy$, using the above values, that

$$|s(z)| \leq \| T f(z) \|_{q_0} \| g(z) \|_{q_0'}$$

$$\leq \| T \|_{p_0, q_0} \| f(z) \|_{p_0} \| g(z) \|_{q_0'} = \|T\|_{p_0, q_0}.$$

This proves the first part of the assertion, the rest follows the same way.

By the three lines thm, ~~$M_0 \leq$~~

$$M(t) \leq M_0^{1-t} M_1^{t}, \text{ Since } f(t) = f,\ g(t) = g,$$

$$|s(t)| = |(Tf, g)| \leq M(t)$$

Inequality 10) follows from this and 14) and 1).

Dirichlet's Principle

1) $D(f) = \iint (f_x^2 + f_y^2)\,dx\,dy$ Dirichlet's integral

For harmonic functions:

Among all functions f defined in a domain G with bd. values φ the harmonic function h has the smallest Dirichlet integral.

Pf: By Green's formula

2) $$\iint k\,\Delta h = -\iint k_x h_x + k_y h_y + \int k \frac{\partial h}{\partial n}\,ds$$

Therefore

3) $$\iint k_x h_x + k_y h_y = D(k,h) = 0$$

for all harmonic h, and k which are zero on the boundary. for all

Let h be the harmonic function $=\varphi$ on bd., let f be any fctn $=\varphi$ on bd. Set

$$f = h + k\,;$$

$k = 0$ on the boundary. So by 3)

$$D(f) = D(h+k) = D(h) + 2D(h,k) + D(k)$$
$$= D(h) + D(k) \geq D(h)$$

This proves the minimum property of h.

7

Dirichlet's Principle for Conformal Mapping.

Let

4) $$M : (x,y) \longrightarrow h(x,y), k(x,y)$$

be a mapping of the unit disk $x^2+y^2 \leq 1$ into the Plane. The Dirichlet integral of this mapping is defined as the sum of DI-s of its components.

5) $$D(M) = D(h) + D(k).$$

Given a simply connected domain G, consider all mappings M of $x^2+y^2 \leq 1$ ~~which~~ whose restriction to the boundary $x^2+y^2=1$ describes the curve bounding G. <u>A mapping which minimizes $D(M)$ is conformal or anticonformal</u>

<u>Pf</u>: We shall imbed the map 4) which minimizes M into a one-parameter family of maps $M(\varepsilon)$, so that $M(0)=M$. We start with a one-parameter family of maps

6) $$V(\varepsilon) : (u,v) \rightarrow x(u,v,\varepsilon), y(u,v,\varepsilon)$$

of $u^2+v^2 \leq 1$ onto $x^2+y^2 \leq 1$, such that $V(0)=I$:

7) $$x(u,v,0)=u, \quad y(u,v,0)=y.$$

From 7) we get

8) $$x_u(u,v,0)=1, \; x_v(u,v,0)=0, \; y_u(u,v,0)=0, \; y_v(u,v,0)=$$

3

We define $\mathcal{M}(\varepsilon)$ as the composite

$$\mathcal{M}(\varepsilon) = \mathcal{M}\circ V(\varepsilon) :$$

$$9)\quad \begin{aligned} H(u,v,\varepsilon) &= h(x(u,v,\varepsilon), y(u,v,\varepsilon)) \\ K(u,v,\varepsilon) &= k(\quad ,, \quad) \end{aligned}$$

~~The Dirichlet of H, K~~ $D(\mathcal{M}(\varepsilon)) = D(\varepsilon)$ is then

$$D(\varepsilon) = \frac{1}{2}\iint (H_u^2 + H_v^2 + K_u^2 + K_v^2)\,du\,dv$$

Using 9) we can write this as

$$D(\varepsilon) = \frac{1}{2}\iint [(h_x x_u + h_y y_u)^2 + \dots \quad]\,du\,dv$$

Introducing x, y as new variables of integration we get

$$10)\quad D(\varepsilon) = \frac{1}{2}\iint (\{h_x^2 x_u^2 + 2 h_x h_y x_u y_u + h_y^2 y_u^2 + \dots) J^{-1}\,dx\,dy$$

where J is the Jacobian

$$11)\qquad J = x_u y_v - x_v y_u$$

We denote by $\frac{d}{d\varepsilon}$ differentiation with respect to ε with x, y kept fixed, and by subscript ε differentiation with respect to ε with u, v being fixed. The relation of the two is

$$\frac{d}{d\varepsilon} x_u = x_{u\varepsilon} + x_{uu}\frac{du}{d\varepsilon} + x_{uv}\frac{dv}{d\varepsilon}$$

according to 8), at $\varepsilon = 0$ we have

$$x_u = 1, \dots$$

Therefore at $\varepsilon = 0$, $x_{uu} = x_{uv} = \dots = 0$, so at $\varepsilon = 0$,

12) $$\frac{d}{d\varepsilon} x_u = x_{u\varepsilon}, \text{ etc}$$

Using this relations and 11) we get at $\varepsilon = 0$:

13) $$\frac{d}{d\varepsilon} J^{-1} = -J^{-2}\frac{dJ}{d\varepsilon} = -x_{u\varepsilon} - y_{v\varepsilon}.$$

We differentiate 10) at $\varepsilon = 0$; using 8), 12), 13) we get

$$\frac{d}{d\varepsilon} D(\varepsilon)\Big|_{\varepsilon=0} = \iint h_x^2 x_{u\varepsilon} + h_x h_y y_{u\varepsilon} + \dots - \frac{1}{2}(h_x^2 + h_y^2 + \dots)(x_{u\varepsilon} + y_{v\varepsilon})\, dx\, dy$$

(14) $$= \iint (h_x^2 + k_x^2 - h_y^2 - k_y^2)(x_{u\varepsilon} - y_{v\varepsilon}) + (h_x h_y + k_x k_y)(y_{u\varepsilon} + x_{v\varepsilon})\, dx\, dy$$

We turn now to the task of constructing the one-parameter family 6), as solution of the system of differential equations (nonlinear)

15) $$x_\varepsilon = f(x,y), \quad y_\varepsilon = g(x,y),$$
$$x(0) = u, \quad y(0) = v;$$

Subscript ε denotes differentiation with respect to ε. The condition that ~~that~~

$x(u,v;\varepsilon)$, $y(u,v;\varepsilon)$ map the unit disk onto the unit disk is satisfied if the the vectorfield (f, g) is tangential to the unit circle on the unit circle:

16). $$x f(x,y) + y g(x,y) = 0 \quad \text{for } x^2+y^2=1$$

The solution of the diff. equ. 15) depends differentiably on the initial values. Differentiating 15) with respect to u and v we get

$$x_{\varepsilon u} = f_x x_u + f_y y_u, \quad x_{\varepsilon v} = f_x x_v + f_y y_v$$

$$y_{\varepsilon u} = g_x x_u + g_y y_u, \quad y_{\varepsilon v} = g_x x_v + g_y y_v$$

Using relation 8) we conclude that at $\varepsilon = 0$

17) $$x_{\varepsilon u} = f_x, \quad x_{\varepsilon v} = f_y$$
$$y_{\varepsilon u} = g_x, \quad y_{\varepsilon v} = g_y$$

If the original mapping minimizes the D.I, then the derivative $\frac{d}{d\varepsilon} D(\varepsilon)\big|_{\varepsilon=0} = 0$ in 14) Introducing the abbreviations

18) $$h_x^2 + k_x^2 - h_y^2 - k_y^2 = A$$
$$h_x h_y + k_x k_y = B$$

and

$$x_{u\varepsilon} - y_{v\varepsilon} = a$$
(19)
$$y_{u\varepsilon} + x_{v\varepsilon} = b$$

we can rewrite 14) = 0 as

20) $$\iint (A\, a + B\, b)\, dx\, dy = 0 .$$

Substituting 17) into (19) we get

21) $$a = f_x - g_y, \quad b = f_y + g_x ,$$

<u>Assertion</u>: a and b can be prescribed as arbitrary smooth functions.

Proof: 21) is the inhomogeneous Cauchy–Riemann equation, subject to the boundary condition 16), We solve it as follows:

First we find a solution f_0, g_0 of 21) which does not satisfy the boundary condition. We differentiate the first equation in 21) with respect to x, the second with respect to y and add; we get

22) $$f_{xx} + f_{yy} = a_x + b_y$$

Denote by f_0 any solution of 22); g has to satisfy

23) $$g_x = a - f_{0y}$$
$$g_y = f_{0y} - b$$

We define the solution g_0 of 23) by the line integral

$$g_0(x,y) = \int^{(x,y)} g_x dx + g_y dy,$$

where g_x, g_y are taken from 23).

The general solution of 21) can then be written as

$$f = f_0 + f_1, \quad g = g_0 + g_1,$$

where f_1, g_1 satisfy the homogeneous equation 21). That means that

24) $$f_1 + i g_1 = F$$

is an analytic function.

The boundary condition 16) can be written as

$$x f_1 + y g_1 = -x f_0 - y g_0 \text{ for } x^2+y^2=1$$

Denoting the right side by φ_0 this can be rewritten as follows

$$\operatorname{Im}(x-iy)(f_1+ig_1) = \varphi_0 \text{ for } x^2+y^2=1$$

In terms of the function F defined in 24) this can be rewritten as follows:

and $z = x+iy$

25) $$\operatorname{Im} \frac{F(z)}{z} = \varphi_0 \quad \text{on } |z| = 1.$$

We solve 25) as follows:

Construct a harmonic function ~~whose imaginary po~~ whose value on $x^2+y^2=1$ equals φ_0; then construct the analytic function $H(z)$ whose imaginary part is the harmonic function above ~~imaginary portion~~. Set

$$F(z) = zH(z).$$

Obviously, 25) is satisfied. This completes the proof of our assertion.

We return now to 20); having shown that a and b are arbitrary smooth functions, it follows that

$$A = 0, \quad B = 0.$$

To interpret these relations we introduce the vector

26) $$(h_x, k_x) = V, \quad (h_y, k_y) = W$$

Setting $A=0$, $B=0$ in (18) we can interpret these as follows:

$$27) \quad \|V\|^2 = \|W\|^2 \quad , \quad V \cdot W = 0 \, ,$$

where $\| \ \|$ denotes Euclidean ~~[illegible]~~ length and the dot Euclidean scalar product. Since

$$\begin{pmatrix} h \\ k \end{pmatrix}(x+\Delta x, y+\Delta y) - \begin{pmatrix} h \\ k \end{pmatrix}(x, y) = \Delta x V + \Delta y W,$$

conformality of the mapping follows from 27).

To extend these ideas to conformal mapping of multiply connected domains one must replace the unit disk by a suitable family of standard domains. E.g. for doubly connected domains we take as standard domain the annulus

$$r \leq x^2 + y^2 \leq 1, \quad r \text{ arbitrary}.$$

In deforming the standard domain via solutions differential equations of form 15), the arbitrariness of r ~~impose~~ relaxes the boundary condition 16) to

$$x f(x, y) + y g(x, y) = c$$

c an arbitrary constant. c appears in the boundary condition 25), and can be used to make $H(z)$ single valued.

Dirichlet's Principle for minimal surfaces

Let

28) $M : (x, y) \longrightarrow h(x,y),\ g(x,y),\ k(x,y)$

be a mapping of the disk $x^2+y^2 \leq 1$ into 3-space. The Dirichlet integral $D(M)$ of this mapping is

$$D(M) = D(h) + D(g) + D(k).$$

Let C be a curve in 3-space; we consider all mappings of $x^2+y^2 \leq 1$ into 3-space, whose restriction to $x^2+y^2 = 1$ is C. A solution mapping which minimizes $D(M)$ is conformal.

Pf: Arguing exactly as in 2 dim., we obtain relation 27), where now

29) $V = (h_x, g_x, k_x)$, $W = (h_y, g_y, k_y)$.

As before, 27) implies conformality of the mapping.

We denote by $A(M)$ the area of the surface into which M carries the unit disk $x^2+y^2 \leq 1$.

~~in the x, y plane~~).

Assertion: For all mappings

30) $$D(\mathcal{M}) \geq A(\mathcal{M})$$

Proof: Introduce the notation

31) $$\|V\|^2 = e, \quad V \cdot W = f, \quad \|W\|^2 = g$$

where V, W are given by 29). The formulas for $D(\mathcal{M})$ and $A(\mathcal{M})$ are

32) $$D(\mathcal{M}) = \tfrac{1}{2}\iint (e+g)\, dx\, dy$$
$$A(\mathcal{M}) = \iint \sqrt{eg - f^2}\, dx\, dy$$

By the arithmetic-geometric mean inequality

33) $$\frac{e+g}{2} \geq \sqrt{eg - f^2};$$

inequality 30) follows from this. Equality holds in 33) iff

$$e = g, \quad f = 0$$

This is precisely relation 27) and is equivalent to conformality. So we conclude this

Corollary: Equality holds in 30) iff $\mathcal{M}$ is conformal. (smooth)

Imbed $\mathcal{M}$ in a one-parameter family of mappings $\mathcal{M}(\varepsilon)$, $\mathcal{M}(0) = \mathcal{M}$, so that the boundary of all $\mathcal{M}(\varepsilon)$ is C.

If $A(\mathcal{M}(\varepsilon))$ has a stationary value at $\varepsilon = 0$, i.e. if

34) $$\frac{d}{d\varepsilon} A(\mathcal{M}(\varepsilon))\Big|_{\varepsilon=0} = 0$$

for all (such) smooth imbeddings of $\mathcal{M}$, then $\mathcal{M}$ is called a minimal surface.

Note that if $\mathcal{M}$ minimizes the surface area among all mappings whose boundary is C, then $\mathcal{M}$ is a minimal surface.

Assertion: If $\mathcal{M}$ ~~is conformal and~~ minimizes the Dirichlet integral among all mappings whose bd. is C, then $\mathcal{M}$ is a minimal surface.

Pf: Let $\mathcal{M}(\varepsilon)$ be an imbedding of $\mathcal{M}$ in a one-parameter family of mappings as above. Since $D(\mathcal{M}(\varepsilon))$ has a minimum at $\varepsilon = 0$, it follows that

35) $$\frac{d}{d\varepsilon} D(\mathcal{M}(\varepsilon))\Big|_{\varepsilon=0} = 0$$

By 30), for all ε

36) $$D(\mathcal{M}(\varepsilon)) \geq A(\mathcal{M}(\varepsilon))$$

and since $\mathcal{M}$ is conformal

37) $$D(\mathcal{M}(0)) = A(\mathcal{M}(0))$$

Therefore $D(M(\varepsilon)) - A(M(\varepsilon))$ has a minimum at $\varepsilon = 0$, so

$$\frac{d}{d\varepsilon}\Big[D(M(\varepsilon)) - A(M(\varepsilon))\Big]\Big|_{\varepsilon=0} = 0$$

Subtracting this from 35) gives 34), as asserted.

Matrix valued analytic functions.

Vector valued analytic functions.
Matrix valued functions; sums, products, reciprocals,
A any matrix; define

1) $$F(z) = (zI - A)^{-1}$$

Defined for z large, ~~then~~

2) $$F(z) = \sum_0^\infty \frac{A^n}{z^{n+1}}, \quad |z| \geq \|A\|$$

For $R > \|A\|$,

3) $$\int_{|z|=R} F(z)\, dz = I,$$

4) $$\int_{|z|=R} z F(z)\, dz = A.$$

The singularities of F(z) are the eigenvalues of A, denote these by $z_1, \ldots, z_N$,
By ~~the~~ Cauchy's theorem

5) $$\int_{|z|=R} F(z)\, dz = \sum_{j=1}^{N} \int_{|z-z_j|=\varepsilon} F(z)\, dz = \sum P_j$$

<u>Assertion</u>: i) Each P_j commutes with A

ii) Each P_j is an idempotent:

$$P_j^2 = P_j$$

iii) $\sum P_j = I$

<u>Pf</u>: Part i) is trivial since the integrand $F(z)$ commutes with A. Part iii) follows if one expresses the left side of 5) from 3. To prove ii) we write

$$P_j^2 = P_j P_j = \int_{|z-z_j|=\varepsilon} F(z)\,dz \int_{|w-z_j|=\delta} F(w)\,dw =$$

6) $$= \iint_{\substack{|z-z_j|=\varepsilon \\ |w-z_j|=\delta}} F(z)F(w)\,dz\,dw$$

~~Let's take $\delta \to$~~ Using partial fraction expansion we have

$$F(z)F(w) = \frac{1}{z-A}\,\frac{1}{w-A} = \frac{1}{w-z}\left(\frac{1}{z-A} - \frac{1}{w-A}\right)$$
$$= \frac{1}{w-z}\left(F(z) - F(w)\right)$$

Substitute this into the right side of 6) and take $\delta > \varepsilon$; ~~then in the dz integration $\frac{1}{w-z}$ is analytic~~ then $\frac{1}{w-z}$

is analytic inside $|z-z_j| \le \varepsilon$, so we have

$$\int_{|z-z_j|=\varepsilon} F(z)F(w)\,dz = \int \frac{1}{w-z} F(z)\,dz - \int \frac{F(w)}{w-z}\,dz$$

$$= \int_{|z-z_j|=\varepsilon} \frac{1}{w-z} F(z)\,dz .$$

Next we carry out the dw integration; since for $|z-z_j| = \varepsilon < \delta$

$$\int_{|w-z_j|=\delta} \frac{1}{w-z}\,dw = 1 ,$$

Integrating (7) with respect to dw we have, by (6):

$$P_j^2 = \int_{|w-z_j|=\delta} \left(\int_{|z-z_j|=\varepsilon} \frac{1}{w-z} F(z)\,dz \right) dw =$$

$$= \int_{|z-z_j|=\varepsilon} F(z)\,dz = P_j ,$$

as asserted.

Exercise: Show that if $F(z)$ has a simple pole at z_j, then

$$AP_j = z_j P .$$

Deduce from this and (iii) that

$$\sum z_j P_j = A .$$

The Paley–Wiener theorem.

Let $f(\xi)$ be a square integrable function on $\mathbb{R}$; denote its Fourier transform by F:

1) $$F(x) = \int_{\mathbb{R}} f(\xi)\, e^{ix\xi}\, d\xi$$

According to the Parseval formula

2) $$\int_{\mathbb{R}} |F(x)|^2\, dx = \int_{\mathbb{R}} |f(\xi)|^2\, d\xi$$

Suppose $f(\xi)$ is zero outside the interval (a, b); then 1) defines $F(x)$ as an entire analytic function:

3) $$F(z) = \int_{\mathbb{R}} f(\xi)\, e^{iz\xi}\, d\xi .$$

Setting $z = x + iy$:

$$F(x+iy) = \int_{\mathbb{R}} f(\xi)\, e^{-\xi y}\, e^{ix\xi}\, d\xi$$

This shows that for y fixed, $F(x+iy)$ is the Fourier transform of $f(\xi)\, e^{-\xi y}$

So by the Parseval formula

4) $$\int |F(x+iy)|^2 dx = \int |f(\xi)|^2 e^{-2\xi y} d\xi.$$

Since $f(\xi)$ is zero outside (a,b),

$$\int |f(\xi)|^2 e^{-2\xi y} d\xi \leq \begin{array}{ll} e^{-2ay} M & \text{for } y>0 \\ e^{-2by} M & \text{for } y<0, \end{array}$$

where M abbreviates $\int |f(\xi)|^2 d\xi$. Substituting this into 4) we get

5) $$\int |F(x+iy)|^2 dx \leq \begin{array}{ll} e^{-2ay} M, & y>0 \\ e^{-2by} M, & y<0 \end{array}$$

The converse of this result is the celebrated <u>Paley-Wiener theorem</u>: Suppose $F(z)$ is an entire analytic function which satisfies 5). Then F is the Fourier transform of a function $f(\xi)$ which is zero outside (a, b).

Pf: 5) shows that $F(x)$ is square integrable. Therefore it has a Fourier inverse f which is square integrable.

To show that $f(\xi)$ vanishes outside (a, b) it suffices to show that

6) $$\int f(\xi) \bar{g}(\xi) d\xi = 0$$

for all square integrable functions of compact support which vanish inside an interval of the form

$$(a-\varepsilon,\ b+\varepsilon)$$

Such g can be decomposed as

$$g = g_a + g_b$$

where

7) $$g_a(\xi) = 0 \quad \text{for } \xi > a-\varepsilon$$
$$g_b(\xi) = 0 \quad \text{for } \xi < b+\varepsilon$$

~~By Parseval~~ We shall verify 6) by showing that separately

$$(f, \overline{g_a}) = 0, \quad (f, \overline{g_b}) = 0$$

By Parseval's formula

8) $$\int f_\xi \overline{g_a} d\xi = \int F(x)\, \overline{G_a}(x)\, dx$$

where G_a is the Fourier transform of g_a:

$$G_a(x) = \int g_a(\xi) e^{ix\xi} d\xi$$

Taking the complex conj. we get

9) $$\overline{G_a}(x) = \int \overline{g_a}(\xi)\, e^{-ix\xi} d\xi$$

Since $\overline{g_a}$ has compact support $\overline{G_a}(x)$ is the restriction to the real axis of the entire analytic function

10) $$H(z) = \int \bar{g}_a(\xi)\, e^{-iz\xi}\, d\xi;$$

we can rewrite 8) as

11) $$\int f \bar{g}_a \, d\xi = \int F(x) H(x)\, dx ,$$

Using the same reasoning which led to inequality 5) we deduce from 7) that

12) $$\int |H(x+iy)|^2 \, dx \le M e^{2(a-\varepsilon)y} \quad \text{for } y > 0.$$

We write now the integral on the right in 11) as

$$\lim_{R \to \infty} \int_{-R}^{R} F(x) H(x)\, dx$$

Since both F and H are entire analytic functions, the integral above can be written, using the Cauchy integral theorem, as

13) $$\int_{-R}^{R} F(x+iy) H(x+iy)\, dx + E(R)$$

where

14) $$E(R) = \int_0^y \left(F(-R+iy) H(-R+iy) - F(R+iy) H(R+iy) \right) dy$$

We shall show below that $E(R) \to 0$ as $R \to \infty$ through a suitable sequence, for by the Schwarz inequality

$$E(R) \leq \left(\int_0^Y |F(R+iy)|^2 \int_0^Y |H(R+iy)|^2 \right)^{1/2} + \cdots$$

$$\leq \frac{1}{2} \int_0^Y |F(R+iy)|^2 dy + \frac{1}{2} \int_0^Y |H(R+iy)|^2 dy + $$

It follows from 5) and 12) resp. that ~~the~~ double integrals

$$\int_0^Y \int_{-\infty}^{\infty} |F(x+iy)|^2 dx\, dy, \quad \int_0^Y \int_{-\infty}^{\infty} |H(x+iy)|^2 dx\, dy$$

are both bounded. Therefore

$$\int_0^Y |F(x+iy)|^2 + |H(x+iy)|^2 dy$$

is an integrable function of x; this shows that there is a sequence of R-s for which the above function tends to zero. This shows that over the same sequence $E(R) \to 0$.

We conclude from 13) that

$$\int_{-\infty}^{\infty} F(x+iY)\, H(x+iY)\, dx$$

is independent of Y. By the Schwarz inequality and by 5) and 12)

$$\left|\int F(x+iY)\,H(x+iY)\,dx\right| \leq \int |F^2(x+iY)| \int |H^2(x+iY)|$$

$$\leq M e^{-2aY}\, N e^{2(a-\varepsilon)Y} = MN e^{-2aY}$$

which tends to zero as $Y \to \infty$. This shows that the integral 11) is zero. Quite analogously,

$$\int f\,\bar{\theta}_b\,d\xi$$

is zero. This completes the proof of the theorem.

FINAL EXAM IN COMPLEX VARIABLES

PROF P.D. LAX

DUE MAY 30

$\leq 1 - 2\,\mathrm{Re}\frac{z}{\lambda_j} + \frac{|z|^2}{|\lambda_j|^2} \leq$ $\left|1 + z/\lambda_j\right|^2 = 1 + 2\,\mathrm{Re}\, z/\lambda_j + \frac{|z|^2}{|\lambda_j|^2} \leq 1 + 2\,\mathrm{Re}\frac{z}{\lambda_j} + |z|$

$\left|1 - (-z/\lambda_j)\right|^2 \leq 1 - 2\,\mathrm{Re}\, z/\lambda_j$

1) a) $f(z)$ is an analytic function in $\mathrm{Re}\, z > 0$ and bounded there. It has zeros at the points $\{z_m\}$, all of which lie outside the unit circle. Prove that if the sum $\sum \mathrm{Re}\frac{1}{z_m}$ diverges, $f(z) \equiv 0$

page 29, 30 in my note, Bernstein-Müntz-Szasz Thm. on approx.

b) Let $\{\lambda_n\}$ be an infinite sequence of complex numbers satisfying:

$$\mathrm{Re}\,\lambda_n > 0 \quad , \quad |\lambda_n| > 1 \quad , \quad \sum \mathrm{Re}\frac{1}{\lambda_n} = \infty$$

Show that every continuous function, $f(x)$, on $[0,1]$ ~~on~~ which vanishes at $x=0$ can be approximated arbitrarily closely in the maximum norm by linear combinations of the functions $\{x^{\lambda_n}\}$.

variation of Müntz Th. pg 29-30 in my notes

2) ✓ a) Let $h(\rho)$ be a positive function of the real argument ρ. Show that $\log h(\rho)$ is a convex function of ρ if

check 3 times

then $e^{\log h(\rho)} > 0$ $\quad h''h \geq h'^2 \qquad \left({}' = \frac{d}{d\rho} \right)$ ✓

$\frac{h''}{h'} \geq \frac{h'}{h} \quad \log h' \geq \log h \qquad h' \geq h \Rightarrow h'/h \geq 1$

$\log h \geq \rho$

✓ b) Let w_n, λ_n be positive constants. Show that for

$$h(\rho) = \sum w_n e^{\lambda_n \rho}$$

$\log h(\rho)$ is a convex function of ρ for $\rho < \beta$, provided that $\sum w_n e^{\lambda_n \beta} < \infty$.

✓ c) Let $f(z) = \sum a_n z^n$ be an analytic function in some annulus $a < |z| < b$

Define $M_2(r)$ ~~m~~ by

$$M_2(r) = \left(\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right)^{\frac{1}{2}}$$

Show that $\log M_2(r)$ is a convex function of $\rho = \log r$.

✓ d) Define $M_{2k}(r)$ by:

$$M_{2k}(r) = \left(\int_0^{2\pi} |f(re^{i\theta})|^{2k} \, d\theta \right)^{\frac{1}{2k}}$$

Show that $\log M_{2k}(r)$ is a convex function of $\rho = \log r$.

e) Define $M_\infty(r)$ by

$$M_\infty(r) = \max_{|z|=r} |f(z)|$$

Show that $\log M_\infty(r)$ is a convex function of $\rho = \log r$.

✓✓

3) Let $f(z)$ be analytic in $|z|<1$ and continuous up to the boundary. Suppose that $f(z) \neq 0$ on $|z|=1$. Show that

$$|f(0)| \leq \exp \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta$$

(Hint: Prove it first for the case where $f(z) \neq 0$ in $|z| \leq 1$) Jensen's inequality

Fragmen - Lindelhöf TC.

✓✓ 4) Let $f(z)$ be an analytic function defined in a wedge of opening $< \pi$, say the wedge:

i) $-\alpha < \arg z < \alpha$, $0 < 2\alpha < \pi$

Suppose $f(z)$ satisfies the following inequalities:

ii) $|f(z)| < A e^{B|z|}$ for all z in the wedge i) & some $A, B > 0$

iii) $|f(z)| \leq M$ for $\arg z = \pm \alpha$

Let λ be any number satisfying: $1 < \lambda < \frac{\pi}{2\alpha}$

Define the function $f_\varepsilon(z)$ ~~to~~ by:

$$f_\varepsilon(z) = f(z)e^{-\varepsilon z^\lambda}$$

✓ a) Show that for any $\varepsilon > 0$, $f_\varepsilon(z)$ tends to 0 as z tends to ∞ in the wedge, and that

$$|f_\varepsilon(z)| \leq M$$

for all z in the wedge.

✓ b) Show that $|f(z)| \leq M$ for all z in the wedge

— x —

$$\int_a^b |u\ v|\,dt \leq \left\{\int_a^b |u|^2 dt\right\}^{\frac{1}{2}} \left\{\int_a^b |v(t)|^2 dt\right\}^{\frac{1}{2}}$$

since $|p_K| \geq 1$ then $\left(1+\frac{q+1}{p_K}\right)\left(1+\frac{\bar{q}+1}{\bar{p}_K}\right) \leq (1+q+1)(1+\bar{q}+1)$

$$\left|1+\frac{q+1}{p_K}\right|^2 \leq |2+q|^2$$

$$\geq \frac{4\,\mathrm{Re}\,q\,\mathrm{Re}\,p_K + 2\,\mathrm{Re}\,p_K}{|p_K|^2\,\left|1+\frac{q+1}{p_K}\right|^2}$$

$$\geq \frac{(4\,\mathrm{Re}\,q+2)\,\mathrm{Re}\,p_K}{|2+q|^2\,|p_K|^2} = \frac{(4\,\mathrm{Re}\,q+2)}{|2+q|^2}\,\mathrm{Re}\,\frac{1}{p_K}$$

$1-\alpha \leq e^{-\alpha}$ for small α

$$1-\alpha \leq 1 - \frac{4\,\mathrm{Re}\,q+2}{|2+q|^2}\,\mathrm{Re}\,\frac{1}{p_K} \leq e^{-\frac{4\,\mathrm{Re}\,q+2}{|2+q|^2}\,\mathrm{Re}\,\frac{1}{p_K}}$$

$$\therefore \prod_{K=1}^{N} |\quad|^2 \leq e^{-\frac{4\,\mathrm{Re}\,q+2}{|2+q|^2}\sum^{N}\mathrm{Re}\,\frac{1}{p_K}} < \epsilon^2$$

& since $\sum \mathrm{Re}\,\frac{1}{p_K} = \infty$

$\exists$ an $N \ni.\ e^{-(\quad)} < \epsilon^2$

hence $|t^q - \sum \lambda_i t^{p_i}| \leq |q|\epsilon$

hence $\max\limits_{0\leq t\leq 1} |\quad| \leq \max\limits_{0\leq t\leq 1} |q|\epsilon$ & can be made as small as we please

let $\max_x \left| f(x) - \sum a_j x^{\lambda_j} \right| \le \epsilon$

let

If we assume that q is a number $\operatorname{Re} q > 0$ for $0 \le x \le 1$
we know that for p_i $i = 1, 2, \dots$ $|p_i| > 1$

$$\left| t^q - \sum_{i=1}^m \lambda_i t^{p_i} \right| = |q| \left| \int_0^t \{\tau^{q-1} - \sum^m \mu_i \tau^{p_i - 1}\} dt \right|$$

$$\le |q| \int_0^1 \left| \tau^{q-1} - \sum_{i=1}^m \mu_i \tau^{p_i - 1} \right| dt \le |q| \sqrt{\int_0^1 \left| \tau^{q-1} - \sum^m \mu_i \tau^{p_i - 1} \right|^2 dt}$$

by section 13, 14 Akhiezer P. 15-20

~~we must~~ we had seen that in order to minimize the $\int_0^1 \left| t^q - \sum A_i t^{p_i} \right|^2 dt = \delta^2$
then $\delta^2 = \dfrac{1}{q + \bar{q} + 1} \lim_{n \to \infty} \prod_{k=1}^{n} \left| \dfrac{\bar{q} - p_k}{q + p_k + 1} \right|^2$

& for minimizing i.e. $\exists$ an $N \Rightarrow \prod_{k=1}^{N} \left| \dfrac{\bar{q} - p_k}{q + p_k + 1} \right|^2 < \epsilon^2$

but looking at any one term $\left(\dfrac{\bar{q} - p_k}{q + p_k + 1} \right) \left(\dfrac{q - \bar{p}_k}{\bar{q} + \bar{p}_k + 1} \right) =$

or $$\frac{|q|^2 + |p_k|^2 - 2 \operatorname{Re}(p_k q)}{|q|^2 + |p_k|^2 + 2 \operatorname{Re}(p_k \bar{q}) + 2 \operatorname{Re}(p_k + q) + 1}$$

$$- 2 \operatorname{Re}(p_k q) - 2 \operatorname{Re}(p_k \bar{q}) - 2 \operatorname{Re}(p_k + q) - 1$$

$$= 1 - \frac{4 \operatorname{Re} q \operatorname{Re} p_k + 2 \operatorname{Re} p_k + 2 \operatorname{Re} q + 1}{|q|^2 + |p_k|^2 + 2 \operatorname{Re}(p_k \bar{q}) + 2 \operatorname{Re}(p_k + q) + 1}$$

$$\alpha = \frac{4 \operatorname{Re} q \operatorname{Re} p_k + 2 \operatorname{Re} p_k + 2 \operatorname{Re} q + 1}{(\text{deno})} \geqslant \frac{4 \operatorname{Re} q \operatorname{Re} p_k + 2 \operatorname{Re} p_k}{|p_k|^2 \left(1 + \frac{q+1}{p_k}\right)\left(1 + \frac{\bar{q}+1}{\bar{p}_k}\right)}$$

Consider $\prod_{j=1}^{N} \frac{z-\lambda_j}{z+\bar{\lambda}_j}$ has simple zeros at λ_j

Analytic in RH plane since $z=-\bar{\lambda}_j$ is in LHP

$$\left|\frac{z-\lambda_j}{z+\bar{\lambda}_j}\right|^2 \Rightarrow \left(\frac{z-\lambda_j}{z+\bar{\lambda}_j}\right)\left(\frac{\bar{z}-\bar{\lambda}_j}{\bar{z}+\lambda_j}\right) = \left(1-\frac{\lambda_j+\bar{\lambda}_j}{z+\bar{\lambda}_j}\right)\left(1-\frac{\lambda_j+\bar{\lambda}_j}{\bar{z}+\lambda_j}\right)$$

$$\frac{|z|^2 - z\bar{\lambda}_j - \bar{z}\lambda_j + |\lambda_j|^2}{|z|^2 + z\lambda_j + \bar{z}\bar{\lambda}_j + |\lambda_j|^2} = \frac{|z|^2+|\lambda_j|^2 - 2\,\mathrm{Re}(z\bar{\lambda}_j)}{|z|^2+|\lambda_j|^2 + 2\,\mathrm{Re}(z\lambda_j)}$$

$$1 - \frac{2\,\mathrm{Re}(z\bar{\lambda}_j + z\lambda_j)}{|z|^2+|\lambda_j|^2+2\,\mathrm{Re}(z\lambda_j)} = \frac{1 - 2\,\mathrm{Re}(z\cdot 2\,\mathrm{Re}\,\lambda_j)}{|z|^2+|\lambda_j|^2+2\,\mathrm{Re}(z\lambda_j)}$$

$$\frac{1 - 2\,\mathrm{Re}\,z \cdot \cancel{\ }\mathrm{Re}\,\lambda_j}{|z|^2+|\lambda_j|^2+2\,\mathrm{Re}(z\lambda_j)}$$

For large λ_j then

$$(z+\bar{\lambda}_j)(\bar{z}+\lambda_j) = |\bar{\lambda}_j|\left(1+\frac{z}{\bar{\lambda}_j}\right)\left(1+\frac{\bar{z}}{\lambda_j}\right)$$

since $|\lambda_j| > 1$

$$\leq |\lambda_j|^2 (1+z)(1+\bar{z}) \qquad \therefore\ 1 - \frac{4\,\mathrm{Re}\,z\,\mathrm{Re}\,\lambda_j}{|z+\bar{\lambda}_j|^2} \leq 1 - \frac{4\,\mathrm{Re}\,z\,\mathrm{Re}\,\lambda_j}{|\lambda_j|^2|1+z|^2}$$

$$\therefore\quad 1 - \frac{2\,\mathrm{Re}\,z}{|1+z|^2}\cdot \cancel{4}\,\mathrm{Re}\,\frac{1}{\lambda_j}$$

let use define $[M_2(r)]^2 = \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$

$f(z) = \sum a_n z^n$ where $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$ $n=0, \pm1, \pm2, \ldots$

$$\log M_2(r) = \frac{1}{2} \log \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \geq \frac{1}{2}\int_0^{2\pi} \log |f(re^{i\theta})|^2 d\theta$$
$$= \int_0^{2\pi} \log |f(re^{i\theta})| d\theta$$

if $\underline{h(\rho)} = \sum w_n e^{\lambda_n \rho}$ let $\rho = \rho_1$ $h(\rho_1) = x$ $\rho = \rho_2$ $h(\rho_2) = y$

$\log h(\rho)$ is a convex fn. $\varphi = \log h$ $\varphi(\_\_) = \log(\_\_)$

$\varphi(\alpha x + (1-\alpha)y) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y)$

$\alpha x + (1-\alpha)y \leq x^\alpha (y)^{1-\alpha}$ $0 < \alpha < 1$

then $h(\rho_1)^\lambda h(\rho_2)^{1-\lambda} \leq \lambda h(\rho_1) + (1-\lambda) h(\rho_2)$

$\lambda \log h(\rho_1) + (1-\lambda) h(\rho_2) \leq \log [\qquad\qquad]$

$\alpha \log h(\rho_1) + (1-\alpha)$

$\int_0^{2\pi}$ $\sum w_j e^{\lambda_n \rho_1}$ $\sum w_j e^{\lambda_n \rho_2}$

$h(\rho_1)$

Look at $\log h[\alpha\rho_1 + (1-\alpha)\rho_2] \leq \alpha \log h(\rho_1) + (1-\alpha)\log h(\rho_2)$

$\alpha\rho_1 + (1-\alpha)\rho_2 \leq h(\rho_1)^\alpha h(\rho_2)^{1-\alpha}$

$h(\rho) = \sum w_n e^{\lambda_n \rho}$

$h'(\rho) = \sum \lambda_n w_n e^{\lambda_n \rho}$

$h''(\rho) = \sum \lambda_n^2 w_n e^{\lambda_n \rho}$

$h'' h = \sum w_k e^{\lambda_k \rho} \sum \lambda_k^2 w_k e^{\lambda_n \rho} = \sum \lambda_k^2 w_k^2 e^{2\lambda_k \rho} + \sum_{j \neq k} w_j w_k e^{(\lambda_k + \lambda_j)\rho}$

$h'' h =$

$$h(\rho) = w_1 e^{\lambda_1\rho} + w_2 e^{\lambda_2\rho} + w_3 e^{\lambda_3\rho}$$

$$h'(\rho) = w_1\lambda_1 e^{\lambda_1\rho} + w_2\lambda_2 e^{\lambda_2\rho} + w_3\lambda_3 e^{\lambda_3\rho}$$

$$h''(\rho) = w_1\lambda_1^2 e^{\lambda_1\rho} + w_2\lambda_2^2 e^{\lambda_2\rho} + w_3\lambda_3^2 e^{\lambda_3\rho}$$

$$h''h = w_1^2\lambda_1^2 e^{2\lambda_1\rho} + 2w_1w_2\lambda_1^2 e^{(\lambda_2+\lambda_1)\rho} + 2w_1w_3\lambda_1^2 + w_2^2\lambda_2^2 + 2w_2w_3\lambda_3^2 + w_3^2\lambda_3^2$$

$$= \sum w_i^2\lambda_i^2 + 2\sum_{i>j} w_iw_j\lambda_i^2 e^{(\lambda_i+\lambda_j)\rho}$$

$$h_1^2 = w_1^2\lambda_1^2 + 2w_1w_2\lambda_1\lambda_2 + 2w_1w_3\lambda_1\lambda_3 + w_2^2\lambda_2^2 + 2w_2\lambda_2w_3\lambda_3 + w_3^2\lambda_3^2$$

$$= \sum w_i^2\lambda_i^2 + 2\sum_{i>j} w_iw_j\lambda_i\lambda_j$$

must show $\sum_{i>j} w_i \quad e^{(\ )\rho} > \sum_{i>j} w_iw_j\lambda_j\lambda_i e^{(\ )\rho}$

$$h''h = \sum w_i^2\lambda_i^2 e^{2\lambda_i\rho} + \sum w_i\lambda_i^2 w_j + w_j\lambda_j^2 w_i$$

$$+ \sum w_iw_j(\lambda_i^2+\lambda_j^2)e^{(\lambda_i+\lambda_j)\rho}$$

$$\sum w_i^2\lambda_i^2 e^{2\lambda_i\rho} + \sum_{i>j} w_iw_j(2\lambda_i\lambda_j)e^{(\lambda_i+\lambda_j)\rho} + \sum_{i>j} w_iw_j(\lambda_i-\lambda_j)^2 e$$

but $h''h = h'^2 + \sum_{i>j} w_iw_j(\lambda_i-\lambda_j)^2 e^{(\lambda_i+\lambda_j)\rho} \quad \therefore \quad h''h \geq h'^2$

Scrap of #3

Let $f(z)$ be analytic in $|z|<1$ and continuous up to the boundary.

Suppose that $f(z) \neq 0$ on on $|z| = 1$. Show that

$$|f(0)| \leq \exp \int_0^{2\pi} \log |f(e^{i\theta})|\, d\theta$$

Prove it first for the case where $f(z) \neq 0$ in $|z| \leq 1$

look at Jensen's th. if $f(z)$ is analytic inside & on the circle $|z| = R$ except for zeros at $a_1, a_2, \ldots, a_m$ of multiplicity $p_1, p_2, \ldots, p_m$ & poles @ $b_1, b_2, \ldots, b_n$ of multiplicities $q_1, q_2, \ldots, q_n$ respectively & if $f(0)$ is finite & not zero then

$$\frac{1}{2\pi}\int_0^{2\pi} \ln |f(Re^{i\theta})|\, d\theta = \ln|f(0)| - \sum_{k=1}^{m} p_k \ln\left(\frac{R}{|a_k|}\right) + \sum_{k=1}^{n} q_k \ln\left(\frac{R}{|b_k|}\right)$$

proof in markushevich

look at $\int_{|z|=R} \ln z \{f'(z)/f(z)\} dz$

assume $\exists$ an analytic fn $F(z)$ s.

~~let~~ $f(z) = \dfrac{F(z)}{(z-a_j)^{p_j}}$ where a_j is a zero of multiplicity p_j

then $\dfrac{f'(z)}{f(z)} = \dfrac{F'(z)}{F(z)} - \dfrac{p_j}{z-a_j}$ & if $f(z) = G(z)\,(z-b_k)^{q_k}$ then $f'/f = G'/G + \dfrac{q_k}{z-b_k}$

if $f(z) \neq 0$ in $|z| = R \Rightarrow \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \ln |f(Re^{i\theta})|\, d\theta = \ln|f(0)| + \sum_{k=1}^{n} q_k \ln\left(\frac{R}{|b_k|}\right)$

since b_k are in the circle $\dfrac{R}{|b_k|} > 1$ $\therefore$ the sum $\displaystyle\sum_{k=1}^{n} q_n \ln\left(\frac{R}{|b_k|}\right) \geq 0$

$\therefore \ln|f(0)| + \sum \geq \ln|f(0)|$

$0 \leq +\sum$

$\ln|f(0)| \leq \ln|f(0)| + \sum$

$\therefore \displaystyle\int_0^{2\pi} \ln |f(Re^{i\theta})|\, d\theta \geq \ln|f(0)|$

Dvořák – Carnwar Overture   I have at home.

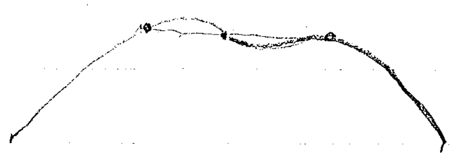
$$\exp \int_0^{2\pi} \ln |f(Re^{i\theta})| \, d\theta = |f(0)| \exp\left[+ \sum q_k \ln\left(\frac{R}{|b_k|}\right)\right] \geq |f(0)|$$

$\therefore$ we have shown that if $f(z) \neq 0$ in interior the result holds

if $f(z) \neq 0$ in interior then we have

$$\exp \int_0^{2\pi} \ln |f(Re^{i\theta})| \, d\theta = |f(0)| \exp\left\{ \sum_{k=1}^{n} q_k \ln\left(\frac{R}{|b_k|}\right) - \sum_{k=1}^{m} p_k \ln\left(\frac{R}{|a_k|}\right) \right\}$$

Since $f(z)$ is analytic in the circle we can write a Taylor expansion
$f(z) = f(0) + f'(0) z + \ldots$



$g(\lambda) \geq 0$ $g(\lambda)$ is concave

If $\exists$ also a min then $\exists$ a pt of inflection

ie $g'' = 0 \Rightarrow \varphi''(\lambda x + (1-\lambda) y) = 0$

$\neq$ $g' = 0 \Rightarrow \dfrac{\varphi(x) - \varphi(y)}{x - y} = \varphi'(\lambda x + (1 - \lambda y))$

$\Rightarrow$ we can find two pts $\ni$ concavity does not hold i.e. secant
~~if min~~ $g'' \geq 0$? $\varphi'' \leq 0$ line lies above part of the curve

$g(\lambda)$ must satisfy $\quad \kappa g(\lambda_1) + (1-\kappa) g(\lambda_2) \leq g(\kappa \lambda_1 + (1-\kappa)\lambda_2)$

$g(\lambda_1) = \lambda_1 \varphi(x) + (1-\lambda_1)\varphi(y) - \varphi(\lambda_1 x + (1-\lambda_1) y)$

$g(\lambda_2) = \lambda_2 \varphi(x) + (1-\lambda_2)\varphi(y) - \varphi(\lambda_2 x + (1-\lambda_2) y)$

$[\kappa\lambda_1 + (1-\kappa)\lambda_2]\varphi(x) + [\kappa(1-\lambda_1) + (1-\kappa)(1-\lambda_2)]\varphi(y)$
$- \kappa \varphi(\lambda_1 x + (1-\lambda_1) y) - (1-\kappa)\varphi(\lambda_2 x + (1-\lambda_2) y)$

since e^ρ is an increasing fn of ρ for real ρ

Let $e^\rho = x$. e^ρ is a continuous fn of ρ & hence x must be continuous

then

$$h(\rho) = \sum \omega_n e^{\lambda_n \rho} = \sum \omega_n x^{\lambda_n} = h(x)$$

$$M_2(r)^2 = \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta$$

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n \qquad f(re^{i\theta}) = \sum a_n r^n e^{in\theta} \qquad a < r < b$$

$$|f(re^{i\theta})|^2 = f(re^{i\theta})\overline{f(re^{i\theta})} = \sum a_n r^n e^{in\theta} \sum \bar{a}_n r^n e^{-in\theta}$$

$$= \sum a_n \bar{a}_m r^{n+m} e^{i\theta(n-m)}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sum a_n \bar{a}_m r^{n+m} e^{i\theta(n-m)} \, d\theta = \begin{cases} 0 & n \neq m \\ \sum |a_n|^2 r^{2n} \end{cases} = M_2^2(r)$$

Show that $\log M_2(r)$ is convex fn of $\rho = \log r$ $\qquad r = e^\rho$

$$\frac{d}{d\rho} \log M_2 = \frac{M'(r)e^\rho}{M_2(r)} \qquad \log M_2(e^\rho) \qquad \frac{M'(e^\rho)\cdot e^\rho}{M_2(e^\rho)}$$

$$\frac{d}{d\rho} h(r) = \frac{d}{dr}h(r)\frac{dr}{d\rho} = \frac{M'(r)}{M(r)} e^\rho \qquad \frac{MM'' - M'^2}{M_2^2} e^{2\rho} + \frac{M'}{M} e^\rho$$

$$\frac{d}{d\rho}\frac{d}{d\rho} h(r) = \frac{d}{dr}\frac{dr}{d\rho}\left\{\frac{d}{dr}h(r)\frac{dr}{d\rho}\right\} = h''\left(\frac{dr}{d\rho}\right)^2 + h'\frac{dr}{d\rho}\frac{d^2r}{d\rho^2}$$

$$\frac{dr}{d\rho}\left\{h''r' + h'r''\right\}$$

if h is convex so is h^2

look at $h^2(\alpha x + (1-\alpha)y) \leq \alpha h^2(x) + (1-\alpha)h^2(y)$

let $g(\alpha) = \alpha h^2(x) + (1-\alpha)h^2(y) - h^2(\alpha x + (1-\alpha)y)$

$g'(\alpha) = h^2(x) - h^2(y) - 2h(\quad)h'(\quad)(x-y)$

$g''(\alpha) = -2\{h'(\quad)\}^2(x-y)^2 - 2h(\quad)h''(\quad)(x-y)^2$

$\Rightarrow g''(\alpha)$