FLORIDA INTERNATIONAL UNIVERSITY
Mechanical Engineering Department

ummer 1991 Intermediate Gas Dynamics EML 5997

PREREQUISITES: EGN 3343, EGN 3353 or permission of the instructor

COURSE CONTENT

1. Overview
2. Review of basic thermodynamic and fluid mechanics principles
3. Review of Navier-Stokes Equations
4. Isentropic Flow of a Perfect Gas
5. Normal Shock Waves
6. Oblique Shock Waves
7. Prandtl-Meyer Flow
8. Flow with Friction
9. Flow with Heat Addition/Heat Loss
10. Linearized Flows
11. Method of Characteristics

Textbook:
- Compressible Fluid Flow, M. A. Saad, Prentice-Hall, 1985
- Handouts

References: Compressible Fluid Flow, A. Shapiro, Wiley Publishers Vol II
Gas Dynamics, J.E.A. John, Allyn and Bacon Publ.

Grade will be determined on the basis of

Quizzes	40 %
HW/Project	30 %
Final Exam	30 %

Tentative Grading Scheme:	90 and above	A	86 - 89	A-	83 - 86	B+
	80 - 82	B	76 - 79	B-	73 - 76	C+
	70 - 72	C	60 - 69	D	below 60	F

Note: The grading scheme and syllabus is tentative and may be changed. All changes will be announced in class.

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Office Hours: Monday and Wednesday 1600-1815

GAS DYNAMICS - BRANCH OF FLUIDS WHICH DEALS WITH COMPRESSIBLE FLUID

- FLOW IS SUCH THAT FLUID DENSITY IS VARIABLE

FLOW IS GOVERNED BY FIRST LAW OF THERMO (ENERGY BALANCE)
SECOND LAW (HEAT INTERACTION / ENTROPY)
NEWTON'S LAWS OF MOTION
CONSERVATION OF MASS

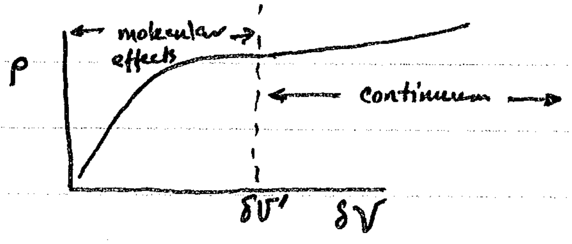
TO DEFINE FLUID PROPERTIES

CONTINUUM - NEED NOT INVESTIGATE A MOLECULE BUT A CONGLOMERATE OF MOLECULES & THEN DEFINE PROPERTIES FROM THAT CONGLOMERATE. ASSUME CONGLOMERATE HAS A SMALL VOLUME CONTAINING A LARGE NO. OF MOLECULES FROM WHICH THE AVERAGE PROPERTIES OF THE MOLECULES.

FLUID PARTICLE - A CONGLOMERATION OF MOLECULES AT A GIVEN POINT IN SPACE

FLUID - IS A SUBSTANCE THAT CONTINUOUSLY DEFORMS UNDER THE ACTION OF SHEARING FORCES

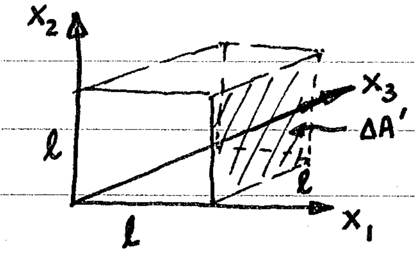
DENSITY $\rho = \lim_{\Delta V \to \delta U'} \frac{\Delta m}{\Delta V}$ $\frac{\text{mass CONTAINED IN THE FLUID PARTICLE}}{\text{VOLUME OF THE PARTICLE}}$



For $\Delta U'$ a cube exists that have area $\Delta A'$

PRESSURE

$$p = \lim_{\Delta A \to \Delta A'} \frac{\Delta F_\perp}{\Delta A}$$



FLUID PRESSURE ACTS NORMAL TO A SURFACE

HOW IS PRESSURE RELATED TO K.E. OF MOLECULAR MOTION?

$\bar{c}_1$ = AVE. MOLECULAR VELOC. IN x_1 DIRECTION

- ONE MOLECULE TRAVERSES ℓ DIRECTION, BOUNCES OFF WALL & BACK IN $2\ell/\bar{c}_1$
- NO INTERMOLECULAR COLLISIONS ARE ASSUMED; ELASTIC COLLISION WITH WALL
  - CHANGE IN MOMENTUM BY COLLISION IS $2m\bar{c}_1$
- m is mass OF MOLECULE

$\therefore$ rate of change of momentum/molecule $= \dfrac{2m\bar{c}_1}{2\ell/\bar{c}_1} = \dfrac{m\bar{c}_1^{\,2}}{\ell}$ (FORCE)

- IF n molecules $\quad \dfrac{nm\bar{c}_1^{\,2}}{\ell}$
- in 3 DIRECTIONS $\quad \displaystyle\sum_{i=1}^{k} \frac{n_i m_i}{\ell}\left(\bar{c}_{i_1}^{\,2} + \bar{c}_{i_2}^{\,2} + \bar{c}_{i_3}^{\,2}\right)$
- in any one direction $\quad \displaystyle\frac{1}{3}\sum_{i=1}^{k} \frac{n_i m_i}{\ell}\left(\qquad\qquad \right)$

$$p = \frac{\text{force}}{\ell^2} = \frac{1}{3}\sum_{i=1}^{k}\frac{n_i m_i}{\ell^3}\left(\bar{c}_{i_1}^{\,2} + \bar{c}_{i_2}^{\,2} + \bar{c}_{i_3}^{\,2}\right)$$

$\dfrac{n_i m_i}{\ell^3} = \rho_i$ & let $\left(\bar{c}_{i_1}^{\,2} + \bar{c}_{i_2}^{\,2} + \bar{c}_{i_3}^{\,2}\right) = \tilde{c}_i^{\,2}$

$$p = \frac{2}{3}\sum_{i=1}^{k}\frac{1}{2}\rho_i \tilde{c}_i^{\,2} = \frac{2}{3}\left(\frac{1}{2}\rho_{ave}\, c_{ave}^{2}\right)$$

$\frac{1}{2}\rho_{ave} c_{ave}^{2}$ = KE/unit volume of molecule

now the ideal gas law gives $p = \rho RT$

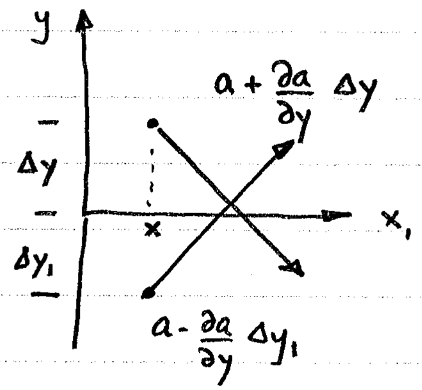
$\Rightarrow \quad T \sim c_{ave}^2$. WE CAN USE IDEAL GAS LAW IF

MEAN FREE PATH - DISTANCE A MOLECULE TRAVELS BEFORE INTERMOLECULAR COLLISION OCCURS IS THE FREE PATH (λ). MEAN FREE PATH ($\bar{\lambda}$) IS THE AVERAGE DISTANCE THAT ALL MOLECULES TRAVEL

FOR CONTINUUM : $\bar{\lambda}/l << 1$ WHERE l IS THE CHARACTERISTIC LENGTH OF THE PROBLEM.

FLUID TRANSPORT PROPERTIES - AS FLUID MOVES THERE IS A TRANSFER OF MASS, MOMENTUM & ENERGY OF A MOLECULE

LOOK AT A PROPERTY "a" AT A FIXED X : $a = a(x_1, y)$



- HERE "a" CHANGES DUE TO A CHANGE IN y
- IF "a" IS THE x_1 momentum (mu) THEN $\frac{\partial u}{\partial y}$ CAN BE RELATED TO A SHEARING STRAIN ON THE FLUID PARTICLE (WHERE m IS particle mass)

THE SHEAR STRESS CAN BE RELATED TO $\frac{\partial u}{\partial y}$ BY $\tau = \mu \frac{\partial u}{\partial y}$

$\tau = \lim_{\Delta A \to \Delta A'} \frac{\Delta F_{//\Delta A}}{\Delta A}$: $\Delta F_{//\Delta A} =$ CHANGE in TANGENTIAL MOMENTUM

$$\frac{a + \Delta a - \Delta a}{\Delta t} = \frac{\frac{\partial a}{\partial y} \Delta y}{\Delta t} = \frac{m \frac{\partial u}{\partial y} \Delta y}{\Delta t}$$

$\Delta t = \Delta y / \bar{c}$ & $\Delta A = \Delta y$

THUS FOR ONE moleculer $\tau = \lim_{\Delta A \to \Delta A'} \left[\frac{m \frac{\partial y}{\partial y} \Delta y}{\Delta y / \bar{c}} \right] / \Delta y \cdot 1 = \bar{c}\, m \frac{\partial u}{\partial y} \Delta y$

for n such molecules $\tau = n\bar{\bar{c}} m \frac{\partial u}{\partial y} \Delta y = n\bar{\bar{c}} m \bar{\lambda} \frac{\partial u}{\partial y} = \rho \bar{\bar{c}} \bar{\lambda} \frac{\partial u}{\partial y}$

WHERE $\bar{\bar{c}}, \bar{\lambda}$ are averages for all the molecules ; $\bar{\lambda}$ is mean free path

- $\rho\bar{\lambda}$ is $\approx$ constant (LOWER $\rho \Rightarrow$ higher mean free path)

$$\therefore \quad \tau = \rho\bar{\lambda}\bar{\bar{c}} \frac{\partial u}{\partial y} = \mu \frac{\partial u}{\partial y} \implies \mu \propto \bar{\bar{c}} \propto \sqrt{T}$$

- since $\mu \propto \rho\bar{\lambda}\bar{\bar{c}}$ & $\bar{\lambda}/_{l} << 1$ THEN

$$\frac{\bar{u}}{\bar{u}} \frac{\mu}{\rho\bar{\bar{c}} l} = \frac{\mu}{\rho\bar{\bar{c}} l} \propto \bar{\lambda}/_{l} << 1$$

$$\frac{\bar{u}/\bar{\bar{c}}}{\rho \bar{u} l/\mu} = \frac{M}{R_N} = K_N \propto \frac{\bar{\lambda}}{l} << 1$$

HERE $\bar{\bar{c}}$ is proportional to speed of sound

FOR A CONTINUUM : $K_N << 1$

THUS WHEN "a" is mu $\tau = \mu \frac{\partial u}{\partial y}$

"a" is m $J = D \frac{\partial n}{\partial y}$

"a" is energy $q = k \frac{\partial T}{\partial y}$

COMPRESSIBILITY - IS DEFINED AS THE CHANGE IN PRESSURE DUE TO A CHANGE IN SPECIFIC VOLUME

$$-\frac{dp}{dV} / V = \rho \frac{dp}{d\rho} = \text{BULK MODULUS OF COMPRESSION} = E$$

IF INCOMPRESSIBLE, $E = \infty$; IF COMPRESSIBLE, $E < \infty$.

PRESSURE - A SCALAR, SHOWN IN FLUIDS COURSE TO BE EQUAL IN ALL DIRECTIONS (STATIC PRESSURE)

FOR AN IDEAL (INVISCID) FLUID (WHETHER STATIC OR DYNAMIC), $p \neq$ DIRECTION DEPENDENT

THREE TYPES OF FLUIDS

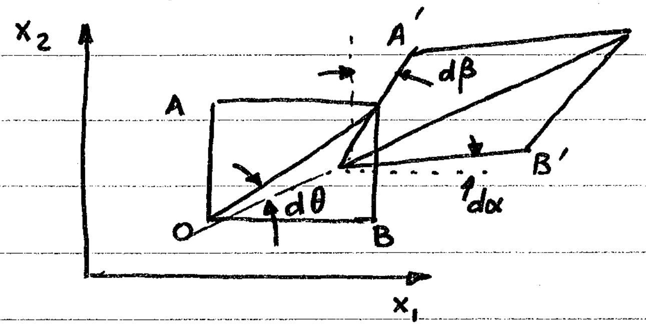
- IDEAL OR PERFECT - HOMOGENEOUS, INVISCID, INCOMPRESSIBLE;

  GOOD FOR REGIONS OUTSIDE BDRY LAYER OR WAKES, $M < .2$, $h < 10^5 ft$
- IDEAL COMPRESSIBLE - INVISCID, $M > .2$, REGIONS OUTSIDE BDRY LAYER & WAKES, $h < 10^5 ft$, HOMOGENEOUS
- VISCOUS - VISCOSITY EFFECTS INCLUDED; BDRY LAYER & WAKES

  FOR ALL M DEPENDING ON COMPRESSIBILITY; $h < 10^5 ft$
- ABOVE 200,000 ft $\bar{\lambda}$ is $O(\ell)$

KINEMATICS OF A FLUID FIELD

VORTICITY & SHEARING STRAIN RATE (LOOK AT COMPONENTS ABOUT x_3 AXIS)



O (u_1, u_2)

A $\left(u_1 + \frac{\partial u_1}{\partial x_2} dx_2,\ u_2 + \frac{\partial u_2}{\partial x_2} dx_2\right)$

B $\left(u_1 + \frac{\partial u_1}{\partial x_1} dx_1,\ u_2 + \frac{\partial u_2}{\partial x_1} dx_1\right)$

$$d\alpha \sim \tan d\alpha = \frac{B'B}{O'B'} = \left[\left(u_2 + \frac{\partial u_2}{\partial x_1} dx_1\right) dt - u_2\, dt\right] \Big/ \left[\left(u_1 + \frac{\partial u_1}{\partial x_1} dx_1\right) dt - u_1\, dt + dx_1\right]$$

$$= \frac{\partial u_2}{\partial x_1} dt \Big/ \left(1 + \frac{\partial u_1}{\partial x_1} dt\right) \sim \frac{\partial u_2}{\partial x_1} dt \quad \text{since} \quad \frac{\partial u_1}{\partial x_1} dt \ll 1$$

$$d\beta \sim \tan d\beta = \frac{A'A}{A'O'} = \left[\left(u_1 + \frac{\partial u_1}{\partial x_2} dx_2\right) dt - u_1\, dt\right] \Big/ \left[\left(u_2 + \frac{\partial u_2}{\partial x_2} dx_2\right) dt - u_2\, dt + dx_2\right]$$

$$= \frac{\partial u_1}{\partial x_2} dt \Big/ \left(1 + \frac{\partial u_2}{\partial x_2} dt\right) \sim \frac{\partial u_1}{\partial x_2} dt$$

RATE OF SHEAR STRAIN IS $\frac{d\alpha + d\beta}{2\, dt} = \gamma_z = \frac{1}{2} \left[\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right] = -\frac{1}{2}\frac{d}{dt}\left(90^\circ - \alpha - \beta\right)$

$\circlearrowleft +$) VORTICITY is $\frac{d\theta}{dt} = \frac{1}{2}\left[-\frac{d\beta}{dt} + \frac{d\alpha}{dt}\right] = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right] = \frac{\Omega_z}{2} = \omega_z$

THE AMOUNT THE DIAGONAL ROTATES IN TIME Δt & IT IS 2 TIMES ROTATION ω_z

DEFINING $\underset{\sim}{q} = u\underset{\sim}{i} + v\underset{\sim}{j} + w\underset{\sim}{k}$ & $\nabla = \frac{\partial}{\partial x}\underset{\sim}{i} + \frac{\partial}{\partial y}\underset{\sim}{j} + \frac{\partial}{\partial z}\underset{\sim}{k}$

$$\nabla \times \underset{\sim}{q} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \underset{\sim}{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \underset{\sim}{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \underset{\sim}{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= \underset{\sim}{\Omega} = \Omega_x \underset{\sim}{i} + \Omega_y \underset{\sim}{j} + \Omega_z \underset{\sim}{k}$$

FOR IRROTATIONAL FLOW $\underset{\sim}{\Omega} = \underset{\sim}{0}$; WE CAN DEFINE $\underset{\sim}{q} = \nabla\varphi$

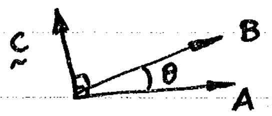
REVIEW OF VECTOR ANALYSIS

1. DOT PRODUCT

- GIVEN: $\underset{\sim}{A} = A_x \underset{\sim}{i} + A_y \underset{\sim}{j} + A_z \underset{\sim}{k}$ $\quad |\underset{\sim}{A}| = (A_x^2 + A_y^2 + A_z^2)^{1/2} = \sqrt{\underset{\sim}{A}\cdot\underset{\sim}{A}}$
- $\underset{\sim}{A}\cdot\underset{\sim}{B} = \underset{\sim}{B}\cdot\underset{\sim}{A} = |\underset{\sim}{A}||\underset{\sim}{B}| \cos\theta_{AB} = A_x B_x + A_y B_y + A_z B_z$
- $\underset{\sim}{A}\cdot\underset{\sim}{B} = 0$ IF $\underset{\sim}{A} \perp \underset{\sim}{B}$

2. CROSS PRODUCTS

$\underset{\sim}{C} = \underset{\sim}{A}\times\underset{\sim}{B} = |\underset{\sim}{A}||\underset{\sim}{B}| \sin\theta_{AB}\, \underset{\sim}{\epsilon}$ $\quad \underset{\sim}{\epsilon} \perp \underset{\sim}{A}$ & $\underset{\sim}{B}$ $\quad |\underset{\sim}{\epsilon}| = 1$



$$\underset{\sim}{A}\times\underset{\sim}{B} = -\underset{\sim}{B}\times\underset{\sim}{A} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \underset{\sim}{i}(A_y B_z - A_z B_y) - \underset{\sim}{j}(A_x B_z - A_z B_x)$$
$$+ \underset{\sim}{k}(A_x B_y - A_y B_x)$$

3. TRIPLE PRODUCTS

$$\underset{\sim}{C}\cdot(\underset{\sim}{A}\times\underset{\sim}{B}) = (\underset{\sim}{A}\times\underset{\sim}{B})\cdot\underset{\sim}{C} = \det\begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

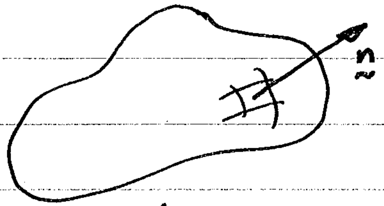
$$\underset{\sim}{C}\times(\underset{\sim}{A}\times\underset{\sim}{B}) = (\underset{\sim}{C}\cdot\underset{\sim}{B})\underset{\sim}{A} - (\underset{\sim}{C}\cdot\underset{\sim}{A})\underset{\sim}{B}$$

4. VECTOR DERIVATIVES

(i) $$\nabla\varphi = \frac{\partial\varphi}{\partial x}\underset{\sim}{i} + \frac{\partial\varphi}{\partial y}\underset{\sim}{j} + \frac{\partial\varphi}{\partial z}\underset{\sim}{k} = \mathrm{grad}\,\varphi$$

$$\nabla\varphi = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\int_S \varphi\,\underset{\sim}{n}\,dS$$

$\underset{\sim}{n}\cdot\underset{\sim}{n}=1$ $\underset{\sim}{n}$ is + out from surface, $\underset{\sim}{n} \perp$ TO SURFACE



$$\int_V \nabla\varphi\,dV = \int_S \varphi\,\underset{\sim}{n}\,dS$$

(ii) Divergence Theorem

$$\nabla\cdot\underset{\sim}{A} = \mathrm{div}\,A = \left[\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right]\cdot\left[A_x\underset{\sim}{i} + A_y\underset{\sim}{j} + A_z\underset{\sim}{k}\right]$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\cdot\underset{\sim}{A} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\int_S \underset{\sim}{A}\cdot\underset{\sim}{n}\,dS$$

$$\int_V \nabla\cdot\underset{\sim}{A}\,dV = \int_S \underset{\sim}{A}\cdot\underset{\sim}{n}\,dS$$ Green's (Divergence Theorem)

(iii) Curl Theorem

$$\nabla\times\underset{\sim}{A} = \mathrm{Curl}\,\underset{\sim}{A} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\int_S \underset{\sim}{n}\times\underset{\sim}{A}\,dS$$

$$\int_V \nabla\times\underset{\sim}{A}\,dV = \int_S \underset{\sim}{n}\times\underset{\sim}{A}\,dS$$

(iv) Stokes' Theorem

$$\int_S \underset{\sim}{n}\cdot\nabla\times\underset{\sim}{A}\,dS = \oint_C \underset{\sim}{A}\cdot d\underset{\sim}{l} \Rightarrow \text{if } \underset{\sim}{A} = \underset{\sim}{q} \text{ THEN } \oint = \Gamma$$

CIRCULATION

DESCRIPTION OF FLUID MOTION

LAGRANGIAN : GIVEN A FLUID PARTICLE AT (x_o, y_o, z_o) & AT TIME t_o WE FOLLOW THAT FLUID PARTICLE AS IT MOVES THROUGH SPACE

$$x = x(x_o, y_o, z_o, t-t_o)$$
$$y = y(x_o, y_o, z_o, t-t_o)$$
$$z = z(x_o, y_o, z_o, t-t_o)$$

x_o, y_o, z_o & t_o are independent variables

WHAT IS VELOCITY AT x_o, y_o, z_o

$$u(x_o, y_o, z_o, t-t_o) = \lim_{\Delta x,\, \Delta t \to 0} \Delta x / \Delta t = \left.\frac{\partial x}{\partial t}\right|_{x_o, y_o, z_o}$$

similarly $v = \left.\dfrac{\partial y}{\partial t}\right|_{x_o, y_o, z_o}$ $\quad w = \left.\dfrac{\partial z}{\partial t}\right|_{x_o, y_o, z_o}$

EULERIAN : LOOK AT A POINT IN SPACE AND WATCH AS PARTICLES PASS THROUGH THAT POINT

$$u = u(x, y, z, t) = \frac{dx}{dt}$$
$$v = v(x, y, z, t) = dy/dt$$
$$w = w(x, y, z, t) = dz/dt$$

x, y, z, t are independent variables but

WHAT IS RATE OF CHANGE OF A PROPERTY AS IT MOVES THROUGH THE POINT AFTER A SHORT TIME dt ?

$$\frac{da}{dt} = \frac{\partial a}{\partial x}\frac{dx}{dt} + \frac{\partial a}{\partial y}\frac{dy}{dt} + \frac{\partial a}{\partial z}\frac{dz}{dt} + \frac{\partial a}{\partial t}$$

SINCE dx, dy, dz depend on trajectory of particle & they in turn depend on t only

$$= u\frac{\partial a}{\partial x} + v\frac{\partial a}{\partial y} + w\frac{\partial a}{\partial z} + \frac{\partial a}{\partial t}$$

CALLED MATERIAL DERIVATIVE , EULERIAN SUBSTANTIAL DERIVATIVE : $\dfrac{Da}{Dt}$

$\dfrac{\partial a}{\partial t}$: local derivative $\qquad \underset{\sim}{q} \cdot \nabla a$: convective derivative

$$\frac{Da}{Dt} = \frac{\partial a}{\partial t} + \underset{\sim}{q} \cdot \nabla a$$

LAGRANGIAN RATE = EULERIAN FORM $\left(\frac{\partial a}{\partial t}, \frac{\partial a}{\partial x}, \frac{\partial a}{\partial y}, \frac{\partial a}{\partial z} \right)$

EULERIAN FORM SAYS A PROPERTY CAN CHANGE DUE TO CHANGE IN POSITION $(\underset{\sim}{q} \cdot \nabla a)$ OR CHANGES IN TIME $\left(\frac{\partial a}{\partial t}\right)$ OR BOTH

EXAMPLE : SUPPOSE AT $t=0$ A PARTICLE IS AT x_0

AND $u = kx_0$

USE LANGRAGIAN APPROACH FIND $x = x(t)$ and THE PARTICLE ACCELERATION

$$u = kx_0 = \left.\frac{\partial x}{\partial t}\right|_{x_0}$$

$$\therefore \int_{x=x_0}^{x} d\bar{x} = \int_{t=0}^{t} u\, d\bar{t} = kx_0 t + C \qquad @\ t=0 \quad x=x_0 \implies C = x_0$$

$$x = (kt+1)\,x_0$$

$$\text{ACCEL} = \left.\frac{\partial u}{\partial t}\right|_{x_0} = \left.\frac{\partial (kx_0)}{\partial t}\right|_{x_0} = 0 \qquad \therefore \ u \text{ is constant}$$

EULERIAN

$$x = x_0(1+kt) \quad \therefore \quad x_0 = \frac{x}{1+kt} \quad \& \quad u = kx_0 = \frac{kx}{1+kt}$$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{-k^2x}{(1+kt)^2} + \frac{kx}{(1+kt)} \cdot \frac{k}{(1+kt)} = 0$$

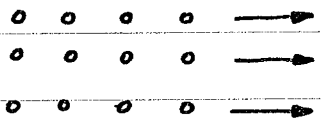
since $\frac{Du}{Dt} = 0 \implies u$ is constant

THERE ARE MANY WAYS ONE CAN DESCRIBE THE MOTION OF A PARTICLE

PATHLINE - LOCUS OF ALL POINTS OF THE PATH OF A PARTICLE, USED TO DESCRIBE MOTION OF A SINGLE PARTICLE & GIVES LONG-TERM PICTURE

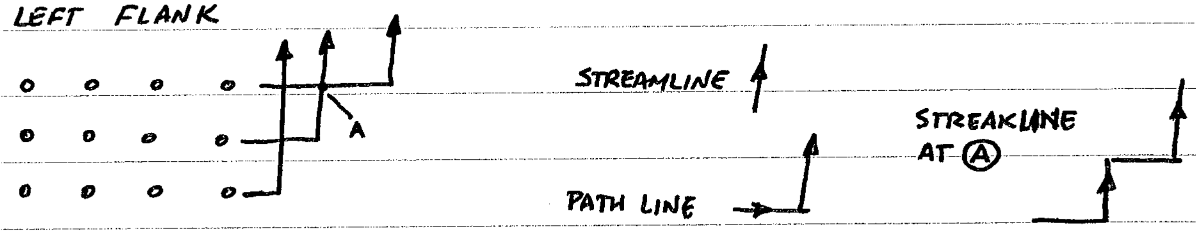
STREAMLINE - LOCUS OF TANGENTS TO THE INSTANTANEOUS VELOCITY VECTOR GIVES AN INSTANTANEOUS PICTURE

STREAKLINE - LOCUS OF ALL PARTICLES PASSING THROUGH A PARTICULAR POINT GIVES A LONG TERM PICTURE

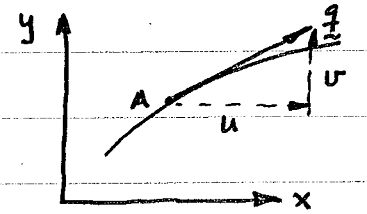


PATHLINE, STEAMLINE, STREAKLINE ARE EQUAL WHEN FLOW IS STEADY (AT SOME POINT IN SPACE THE VELOCITY IS NOT A FUNCTION OF TIME)

IF LEFT FLANK



BY DEFINITION FOR 2-D FLOWS



$$\left.\frac{dy}{dx}\right|_A = \frac{v}{u}$$ FOR A STREAMLINE

FROM THIS WE CAN DEFINE FLOW

EXAMPLE

$$u = \frac{2Ay}{x^2+y^2} \qquad v = -\frac{2Ax}{x^2+y^2} \qquad \frac{dy}{dx} = \frac{v}{u} = -\frac{x}{y} \quad \text{or} \quad x^2+y^2 = \mathcal{C}$$

VORTEX FLOW

WE HAVE SAID THAT IF FLOW WAS IRROTATIONAL

$$u = \frac{\partial\phi}{\partial x} \qquad v = \frac{\partial\phi}{\partial y} \qquad \boxed{w = \frac{\partial\phi}{\partial z}} \quad \phi \text{ is a potential function}$$

$$\underset{\sim}{q} = \nabla\phi = \underset{\sim}{i}\frac{\partial\phi}{\partial x} + \underset{\sim}{j}\frac{\partial\phi}{\partial y} \boxed{+ \underset{\sim}{k}\frac{\partial\phi}{\partial z}}$$

IF $\nabla\cdot\underset{\sim}{q} = 0$ (FLOW IS DIVERGENCE FREE [STEADY STATE, INCOMPRESSIBLE]) $\Rightarrow \nabla^2\phi = 0$

IF FLOW IS IRROTATIONAL $\nabla\times\underset{\sim}{q} = 0$ or $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

2-D: IF WE DEFINE $\rho u = \frac{\partial\psi}{\partial y}$ $\rho v = -\frac{\partial\psi}{\partial x}$ $\Rightarrow \nabla\cdot\rho\underset{\sim}{q} \equiv 0$ & $\nabla\times\underset{\sim}{q} = \nabla^2\psi = 0$

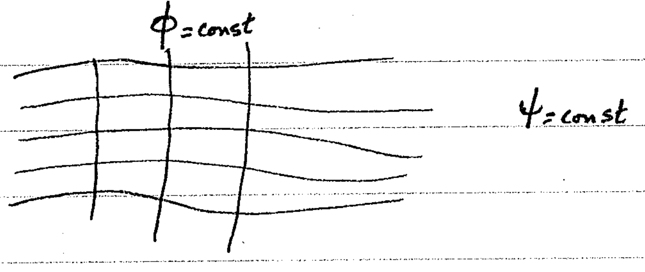
NOW $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy \boxed{+ \frac{\partial\phi}{\partial z}dz} = u\,dx + v\,dy$; ALONG $\phi = const$ $d\phi = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{u}{v}$$

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = -\rho v\,dx + \rho u\,dy \; ; \quad \text{along } \psi = const \quad d\psi = 0$$

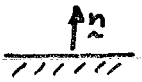
$\Rightarrow \frac{dy}{dx} = \frac{v}{u}$ streamline equation : $\Rightarrow \psi$ stream fn

since $\left.\frac{dy}{dx}\right|_{\psi=const} \cdot \left.\frac{d\psi}{dx}\right|_{\phi=const} = -1$ ψ & ϕ are $\perp$ TO EACH OTHER



WHAT DO WE KNOW AT A SOLID SURFACE

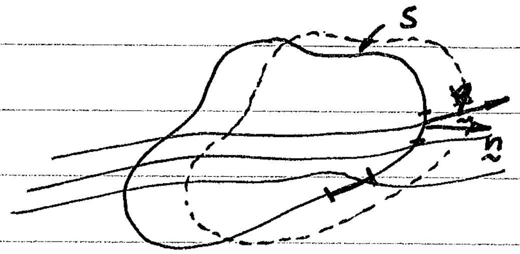
1) NO FLOW THROUGH SURFACE $q_n = \underset{\sim}{q}\cdot\underset{\sim}{n}$



1. CONSERVATION OF MASS

MASS IS NEITHER CREATED OR DESTROYED

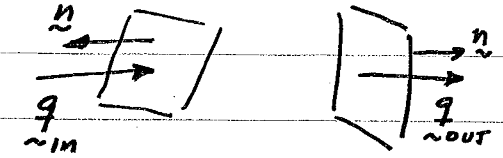
FIX A <u>CONTROL</u> <u>VOLUME</u> IN A FLUID



THE CHANGE IN MASS OF FLUID IN THE VOLUME PER UNIT TIME $= \dot{m}_{IN} - \dot{m}_{OUT}$

$$\dot{m}_{IN} = \rho \underset{\sim}{q}_{in} \cdot \underset{\sim}{n}\, dS \qquad \dot{m}_{OUT} = \rho \underset{\sim}{q}_{out} \cdot \underset{\sim}{n}\, dS$$

$$\dot{m}_{in} - \dot{m}_{OUT} = \int_S \rho \underset{\sim}{q} \cdot \underset{\sim}{n}\, dS \qquad \text{but}$$



$$\therefore \quad \dot{m}_{in} - \dot{m}_{OUT} = -\int_S \rho \underset{\sim}{q} \cdot \underset{\sim}{n}\, dS$$

now

$$\dot{m}_{in} - \dot{m}_{OUT} = \frac{\partial}{\partial t}\int_V \rho\, dV + \boxed{\begin{matrix}\text{Source}\\ \text{term}\end{matrix}}$$

$$\frac{\partial}{\partial t}\int_V \rho dV + \int_S \rho \underset{\sim}{q} \cdot \underset{\sim}{n}\, dS = \frac{\partial}{\partial t}\int_V \rho\, dV + \int_V \nabla \cdot (\rho \underset{\sim}{q})\, dV = 0$$

<u>IF NO</u> SOURCE TERM

SINCE VOLUME IS FIXED $\frac{\partial}{\partial t}\int = \int \frac{\partial}{\partial t} \Rightarrow 0 = \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{q})\right] dV$

and $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{q}) = 0 \Rightarrow 0 = \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \underset{\sim}{q} + \underset{\sim}{q} \cdot \nabla \rho = \frac{D\rho}{Dt} + \rho \nabla \cdot \underset{\sim}{q}$

if flow is steady $\frac{\partial}{\partial t} = 0 \Rightarrow \nabla \cdot (\rho \underset{\sim}{q}) = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$

if flow is incompressible $\rho \neq \rho(x,y,z,t) \Rightarrow \nabla \cdot \underset{\sim}{q} = 0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

$\hookrightarrow \frac{D\rho}{\partial t} = 0$

LAGRANGE DESCRIPTION OF CONTINUITY USING FIXED MASS

fixed mass : $\int_V \rho dV = \int_{V_o} \rho_o dV_o$

ρ, dV = values at time $t+\Delta t$ ρ_o, dV_o = values at time t (FIXED VOLUME)

$$\underset{\sim\sim\sim}{dV = J dV_o} \qquad J = \frac{\partial(x,y,z)}{\partial(x_o,y_o,z_o)} = \begin{vmatrix} \partial x/\partial x_o & \partial y/\partial x_o & \partial z/\partial x_o \\ \partial x/\partial y_o & \partial y/\partial y_o & \partial z/\partial y_o \\ \partial x/\partial z_o & \partial y/\partial z_o & \partial z/\partial z_o \end{vmatrix}$$

J relates the CHANGE IN COORDINATES x,y,z wrt COORDINATED x_o, y_o, z_o

$$\int_V \rho dV = \int_{V_o} \rho J dV_o = \int_{V_o} \rho_o dV_o \quad \Rightarrow \int (\rho J - \rho_o) dV_o = 0$$

THUS THE CONTINUITY EQUATION STATE $\rho/\rho_o = \frac{1}{J}$ IN LAGRANGIAN TERMS

EULARIAN DESCRIPTION USING FIXED MASS (CONTROL VOLUME ARBITRARY)

$$\frac{D}{Dt}\int_V \rho dV = 0 \quad \text{IF} \quad dV = J dV_o \quad \Rightarrow \quad \frac{D}{Dt}\int_{V_o} \rho J dV_o = 0$$

SINCE dV_o IS FIXED VOLUME COMPARED TO NEW VOLUME dV

$$\frac{D}{Dt}\int_{V_o} \rho J \, dV_o = \int_{V_o} \frac{D}{Dt}(\rho J) dV_o = \int_{V_o} \left(J \frac{D\rho}{Dt} + \rho \frac{DJ}{Dt}\right) dV_o = 0$$

$$\Rightarrow \int_V \frac{D\rho}{Dt} dV + \int_{V_o} \rho \frac{DJ}{Dt} dV_o = 0$$

$$\text{if} \quad J = \frac{\partial(x,y,z)}{\partial(x_o,y_o,z_o)} \quad \Rightarrow \quad \frac{DJ}{Dt} = J \nabla \cdot \underset{\sim}{q}$$

$$\Rightarrow \int_{V_o} \rho \frac{DJ}{Dt} dV_o = \int_{V_o} \rho J \nabla \cdot \underset{\sim}{q} \, dV_o = \int_V \rho \nabla \cdot \underset{\sim}{q} \, dV$$

$$\text{or} \quad \frac{D}{Dt}\int_V \rho dV = 0 \quad \Longrightarrow \quad \int_V \left(\frac{D\rho}{Dt} + \rho \nabla \cdot \underset{\sim}{q}\right) dV = 0$$

$$\frac{DJ}{Dt} = \frac{\partial}{\partial t}J + (\underset{\sim}{q}\cdot\nabla)J \qquad J = \begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial y}{\partial x_0} \\ \frac{\partial x}{\partial y_0} & \frac{\partial y}{\partial y_0} \end{vmatrix}$$

$$\frac{\partial}{\partial t}J = \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial x_0}\right)\frac{\partial y}{\partial y_0} + \frac{\partial x}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial y_0}\right) - \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial y_0}\right)\frac{\partial y}{\partial x_0} - \frac{\partial x}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x_0}\right)$$

$$J = \frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} - \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}$$

Now $\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial x_0}\right) = \frac{\partial}{\partial x_0}\left(\frac{\partial x}{\partial t}\right) = \frac{\partial}{\partial x_0}u$

$$\therefore \frac{\partial J}{\partial t} = \frac{\partial u}{\partial x_0}\frac{\partial y}{\partial y_0} + \frac{\partial x}{\partial x_0}\frac{\partial v}{\partial y_0} - \frac{\partial u}{\partial y_0}\frac{\partial y}{\partial x_0} - \frac{\partial x}{\partial y_0}\frac{\partial v}{\partial x_0}$$

$$\frac{\partial J}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} + \frac{\partial x}{\partial x_0}\frac{\partial v}{\partial y}\frac{\partial y}{\partial y_0} - \frac{\partial u}{\partial x}\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0} - \frac{\partial x}{\partial y_0}\frac{\partial v}{\partial y}\frac{\partial y}{\partial x_0}$$

$$= \frac{\partial u}{\partial x}\left[\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} - \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\right] + \frac{\partial v}{\partial y}\left[\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} - \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\right]$$

$$\frac{\partial J}{\partial t} = \frac{\partial u}{\partial x}J + \frac{\partial v}{\partial y}J = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)J = J(\nabla\cdot\underset{\sim}{q})$$

$$\boxed{\partial J/\partial t = J(\nabla\cdot\underset{\sim}{q})}$$

$$(\underset{\sim}{q}\cdot\nabla)J = u\frac{\partial}{\partial x}J + v\frac{\partial}{\partial y}J$$

$$u\frac{\partial}{\partial x}J = u\left[\frac{\partial}{\partial x}\left[\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} - \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\right]\right] = u\left[\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial x_0}\right)\frac{\partial y}{\partial y_0} + \frac{\partial x}{\partial x_0}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial y_0}\right) + \dots\right]$$

$$\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial x_0}\right) = \frac{\partial}{\partial x_0}\left(\frac{\partial x}{\partial x}\right) = \frac{\partial 1}{\partial x_0} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial y_0}\right) = \frac{\partial}{\partial y_0}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial y_0}(0) = 0 \text{ since } y, x \text{ are independent}$$

$$\therefore u\frac{\partial J}{\partial x} = 0 \ \ \& \ \ v\frac{\partial J}{\partial y} = 0 \implies \boxed{(\underset{\sim}{q}\cdot\nabla)J \equiv 0}$$

$$\therefore \boxed{DJ/Dt = J(\nabla\cdot\underset{\sim}{q})}$$

2. MOMENTUM EQUATION OR EQUATION OF MOTION (NEWTON'S SECOND LAW)

- VISCOSITY IS NOT CONSIDERED

EULERIAN FIXED MASS : $\frac{d}{dt}$(momentum) = ΣForces (pressure, body, friction

$$\frac{D}{Dt}\int_V \rho \underset{\sim}{q}\, dV = -\int_S p\underset{\sim}{n}\, dS + \int_V \rho \underset{\sim}{F}\, dV + \int_S \cancelto{0}{\text{friction force}}\, dS$$

$\underset{\sim}{F}$ is body force (force/mass)

friction term must be retained : if working w/ boundary layers, shock waves, areas with high gradients

$$\frac{D}{Dt}\int_V \rho\underset{\sim}{q}\, dV = \frac{D}{Dt}\int_{V_0}\rho\underset{\sim}{q}\, J dV_0 = \int_{V_0}\frac{D}{Dt}(\rho\underset{\sim}{q}J) dV_0 = \int_{V_0} J\frac{D}{Dt}(\rho\underset{\sim}{q}) dV_0 + \int_{V_0}\rho\underset{\sim}{q}\frac{DJ}{Dt} dV_0$$

$$= \int_{V_0}\left[J\frac{D(\rho\underset{\sim}{q})}{Dt} + J\rho\underset{\sim}{q}(\nabla\cdot\underset{\sim}{q})\right] dV_0 = \int_{V_0}\left[\frac{D}{Dt}(\rho\underset{\sim}{q}) + \rho\underset{\sim}{q}(\nabla\cdot\underset{\sim}{q})\right] dV$$

$$= \int_V\left\{\left[\frac{\partial}{\partial t}(\rho\underset{\sim}{q}) + \underbrace{\underset{\sim}{q}\cdot\nabla(\rho\underset{\sim}{q})\right] + \rho\underset{\sim}{q}(\nabla\cdot\underset{\sim}{q})}\right\} dV$$

$$\frac{D}{Dt}\int_V \rho\underset{\sim}{q}\, dV = \int_V\left[\frac{\partial}{\partial t}(\rho\underset{\sim}{q}) + \nabla\cdot(\rho\underset{\sim}{q}\underset{\sim}{q})\right] dV$$

$$\Rightarrow \boxed{\int_V \frac{\partial}{\partial t}\rho\underset{\sim}{q}\, dV + \int_S \underset{\sim}{q}\underbrace{(\rho\underset{\sim}{q}\cdot\underset{\sim}{n})}_{q_n} dS = -\int_S p\underset{\sim}{n}\, dS + \int_V \rho\underset{\sim}{F}\, dV}$$

from $\int_V\left[\frac{D}{Dt}(\rho\underset{\sim}{q}) + \rho\underset{\sim}{q}(\nabla\cdot\underset{\sim}{q})\right] dV$

$$\searrow \qquad \searrow$$

$$\left[\rho\frac{D}{Dt}\underset{\sim}{q} + \underbrace{\underset{\sim}{q}\frac{D\rho}{Dt} + \rho\underset{\sim}{q}(\nabla\cdot\underset{\sim}{q})}\right]$$

$$\rho\frac{D\underset{\sim}{q}}{Dt} + \underset{\sim}{q}\left[\frac{D\rho}{Dt} + \rho(\nabla\cdot\underset{\sim}{q})\right]$$

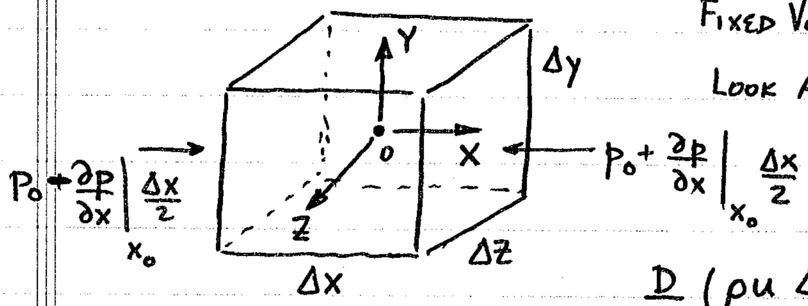
$$\therefore \int_V \rho \frac{D\underset{\sim}{q}}{Dt}\, dV = -\int_S p\underset{\sim}{n}\, dS + \int_V \rho \underset{\sim}{F}\, dV$$

$$\downarrow$$

$$-\int_V \nabla p\, dV$$

$$\therefore \int_V \left[\rho \frac{D\underset{\sim}{q}}{Dt} + \nabla p - \rho \underset{\sim}{F}\right] dV = 0 \quad \text{or} \quad \boxed{\rho \frac{D\underset{\sim}{q}}{Dt} = -\nabla p + \rho \underset{\sim}{F}}$$

Momentum Equation for a fluid particle using Eulerian fixed mass



FIXED VOLUME APPROACH OF Euler

LOOK AT $\underline{\underline{X}}$ momentum:

$$\frac{D}{Dt}\left(\rho u\, \Delta x \Delta y \Delta z\right) = \text{CHANGE IN X MOMENTUM}$$

$$\left[P_o - \frac{\partial p}{\partial x}\Big|_{x_o} \frac{\Delta x}{2}\right]\Delta z \Delta y - \left[P_o + \frac{\partial p}{\partial x}\Big|_{x_o} \frac{\Delta x}{2}\right]\Delta z \Delta y + \rho F_x \Delta x \Delta y \Delta z = \sum \text{forces}_x$$

$$\frac{D}{Dt}(\rho u) = -\frac{\partial p}{\partial x}\Big|_{x_o} + \rho F_x$$

HOW MANY UNKNOWNS DO WE HAVE ? u, v, w, ρ, P

EQUATIONS ? CONTINUITY (ρ, u, v, w) 1 eqn

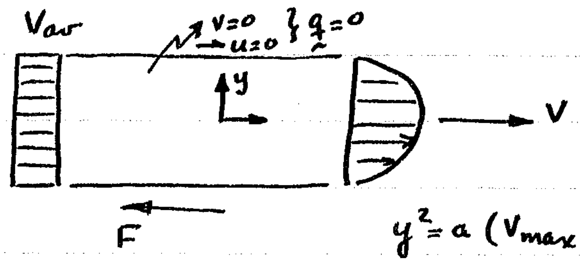
momentum (ρ, u, v, w, P) 3 eqs

need one more : equation of state $p = \rho R T$ adds T as unknown

need one more : energy

NOW WE LOOK AT THERMODYNAMICS

Problem 2.3



$y^2 = a\,(V_{max} - V)$

@ $y=0 \quad V=V_{max}$

$y=\pm h/2 \quad V=0 \Rightarrow \frac{h^2}{4} = a\,V_{max}$

$\therefore \quad a = h^2/4V_{max}$

& $$\frac{V}{V_{max}} = \left(1 - 4y^2/h^2\right)$$

continuity eqn: $\int \frac{D}{Dt}\rho\, dV + \int \rho \underset{\sim}{q}\cdot\hat{n}\, dS = 0 \qquad \frac{D}{Dt}=0$ steady & incomp

$\therefore \int \rho \underset{\sim}{q}\cdot\hat{n}\, dS = 0$ or $\dot{m}_{in} = \dot{m}_{out}$

$\dot{m}_{in} = \rho\, V_{av}\cdot h\cdot 1 \qquad \dot{m}_{out} = \int_{-h/2}^{h/2} \rho\, V\, dy\cdot 1 = 2\rho\int_0^{h/2} V_{max}\left(1 - \frac{4y^2}{h^2}\right)d$

$= 2\rho V_{max}\cdot\frac{1}{3}h\cdot 1$

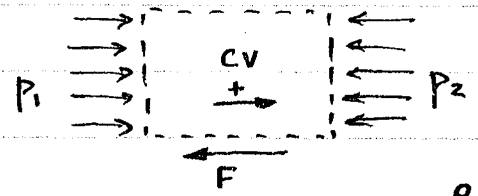
$\therefore \quad V_{av} = 2\rho V_{max}\cdot\frac{1}{3}h\cdot 1 \,/\, \rho\cdot h\cdot 1 = \frac{2}{3}V_{max}$

Momentum Eq:

$$\int \rho \frac{D}{Dt}\underset{\sim}{q}\, dV + \int \nabla p\, dV - \int \rho \underset{\sim}{F}\, dV = 0 \quad \text{or}$$

$$\int_V \frac{\partial}{\partial t}\rho \underset{\sim}{q}\, dV + \int_S \rho \underset{\sim}{q}\left(\underset{\sim}{q}\cdot\underset{\sim}{n}\right)dS = -\int_S p\underset{\sim}{n}\, dS + \int_V \rho \underset{\sim}{F}\, dV + \text{friction force}$$

steady state $\frac{\partial}{\partial t}=0$; also no body force $\therefore \int_S \rho \underset{\sim}{q}(\underset{\sim}{q}\cdot\underset{\sim}{n})dS = -\int p\hat{n}\, dS + \text{fri}$



$p_1\cdot h\cdot 1 - p_2\cdot h\cdot 1 - F = -\int p\underset{\sim}{n}\, dS + \text{friction}$

$\rho V_{av}(-V_{av})\cdot h\cdot 1 + \int_{-h/2}^{h/2} \rho V^2 dy\cdot 1 = \int \rho \underset{\sim}{q}(\underset{\sim}{q}\cdot\underset{\sim}{n})dS$

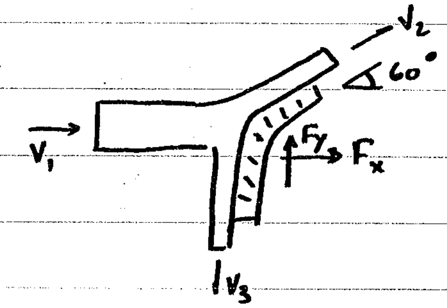
$\therefore \; -\rho V_{av}^2 h\cdot 1 + 2\rho\int_0^{h/2} V_{max}^2\left[1 - 4y^2/h^2\right]^2 dy\cdot 1 = -\rho\cdot\frac{4}{9}V_{max}^2 h\cdot 1 + 2\rho V_{max}^2\left[\frac{1}{2}h - \frac{1}{3}h + \frac{1}{10}h\right.$

$$\therefore\ p_1 - p_2 = \left[-\int p\,\underset{\sim}{n}\,dS + F\right]/h = \left[\int \rho\underset{\sim}{q}(\underset{\sim}{q}\cdot\underset{\sim}{n})\,dS + F\right]/h$$

$$= \left\{\rho h\left[-\tfrac{4}{9}V_{max}^2 + 2V_{max}^2\,\frac{8}{30}\right] + F\right\}/h$$

$$= \rho\cdot\frac{4}{45}V_{max}^2 + F/h$$

2.5



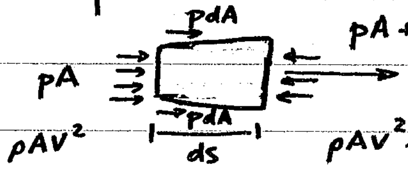
$$\int_V \frac{D}{Dt}\rho\,dV + \int_S \rho\underset{\sim}{q}\cdot\underset{\sim}{n}\,dS = 0 \qquad \text{incompressible, steady}$$

$$\int_S \rho\underset{\sim}{q}\cdot\hat{n}\,dS = -\rho V_1A_1 + \rho V_2A_2 + \rho V_3A_3 = 0$$

$$\text{or}\quad V_1A_1 = V_2A_2 + V_3A_3$$

now $\dot{m}_2 = \frac{2}{3}\dot{m}_1 \qquad \dot{m}_3 = \frac{1}{3}\dot{m}_1$

Along a streamtube : look at steady state flow ; w/o friction or body force



$pA + \frac{\partial}{\partial s}(pA)dS$: pressure force variation

$\rho AV^2 + \frac{\partial}{\partial s}(\rho AV^2)ds$: momentum flux variation

$$F_s = pA - \left[pA + \frac{\partial}{\partial s}(pA)\,dS\right] + p\,dA = -\frac{\partial p}{\partial s}\cdot A\,ds + p\frac{\partial A}{\partial s}ds + p\,\partial A = -A\frac{\partial p}{\partial s}ds$$

Momentum flux is $-\rho AV^2 + \rho AV^2 + \frac{\partial}{\partial s}(\rho AV^2)dS = \frac{\partial}{\partial s}(\rho AV^2)\,dS$

since stream tube is a collection of streamlines V is $\underset{\sim}{q}$ $\quad V^2 = (\underset{\sim}{q}\cdot\underset{\sim}{q})$

From momentum equation $\underbrace{\int_V \frac{\partial\rho\underset{\sim}{q}}{\partial t}dV}_{0} + \underbrace{\int_S \rho\underset{\sim}{q}(\underset{\sim}{q}\cdot\hat{n})\,dS}_{} = \underbrace{-\int_S p\hat{n}\,dS}_{} + \int_V \rho\hat{F}\,dV$

$$v\frac{\partial}{\partial s}\underbrace{(\rho \cancel{Av})}_{=0}dS + \rho Av\frac{\partial v}{\partial s}dS = \frac{\partial}{\partial s}(\rho AV^2)dS = -A\frac{\partial p}{\partial s}ds$$

Continuity

or $$\rho Av\frac{\partial v}{\partial s}ds = -A\frac{\partial p}{\partial s}ds \quad \text{or} \quad \rho v\,\partial v + \partial p = 0$$

Thus $\rho \frac{V^2}{2} + p$ = constant along streamline

But if there is a change in altitude

$$\rho \frac{V^2}{2} + p + \rho g z = \text{constant}$$

for out case between 1 & 3 where z changes are negligible

$$\rho \frac{V_1^2}{2} + p_1 = \rho \frac{V_3^2}{2} + p_3 \qquad \text{since } p_1 = p_3 \Rightarrow V_1 = V_3$$

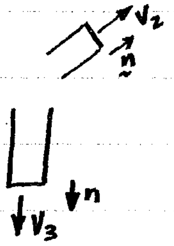
between 1 & 2 : $\rho \frac{V_2^2}{2} + p_2 = \rho \frac{V_1^2}{2} + p_1$ or $V_1 = V_2$ since $p_1 = p_2$

since $V_1 = V_2 = V_3$ & since $\dot{m}_2 = 2/3\, \dot{m}_1 \Rightarrow A_2 = 2/3\, A_1$

$\dot{m}_3 = 1/3\, \dot{m}_1 \Rightarrow A_3 = 1/3\, A_1$

$$\Sigma F_x = F_x = \left[\iint_S \rho \underset{\sim}{q} (\underset{\sim}{q} \cdot \underset{\sim}{n}) dS\right]_{x \text{ component}} = +\rho V_1 A_1 (-V_1) + (\rho V_2 A_2) V_2 \cos 60^\circ$$

$$\Sigma F_y = F_y = \left[\iint_S \rho \underset{\sim}{q} (\underset{\sim}{q} \cdot \underset{\sim}{n}) dS\right]_{y \text{comp}} = (\rho V_2 A_2) V_2 \sin 60^\circ - (\rho V_3 A_3) V_3$$



$\underset{\sim 3}{q} \cdot \underset{\sim}{n} = V_3 \qquad \underset{\sim 3}{q} = -V_3 \underset{\sim}{j} \qquad \underset{\sim 2}{q} \cdot \underset{\sim}{n} = V_2 \qquad \underset{\sim 2}{q} = V_2 \cos 60^\circ \underset{\sim}{i} + V_2 \sin 60^\circ \underset{\sim}{j}$

$$F_x = -\dot{m}_1 V_1 + \dot{m}_2 V_2 \cdot \frac{1}{2} = -\dot{m}_1 V_1 + 2/3\, \dot{m}_1 V_1 \cdot \frac{1}{2} = \dot{m}_1 V_1 \left[-1 + \frac{1}{3}\right] = -\frac{2}{3} \dot{m}_1 V_1$$

$$F_y = \dot{m}_2 V_2 \cdot \frac{\sqrt{3}}{2} - \dot{m}_3 V_3 = \frac{2}{3} \dot{m}_1 V_1 \frac{\sqrt{3}}{2} - \frac{1}{3} \dot{m}_1 V_1 = \dot{m}_1 V_1 \left[\frac{\sqrt{3}}{3} - \frac{1}{3}\right] = .244\, \dot{m}_1 V_1$$

2.13 $\phi = y^3/3 - x^2 y$

$$\frac{\partial\phi}{\partial x} = u = -2xy \qquad \frac{\partial\phi}{\partial y} = v = y^2 - x^2$$

For irrot flow $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -2x - [-2x] = 0$ ✓

$$\nabla\cdot\underset{\sim}{q} = \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 2y - 2y = 0 \Rightarrow \nabla^2\phi = 0$$

To find ψ: $u = -2xy = \frac{1}{\rho}\frac{\partial\psi}{\partial y} \Rightarrow \psi = -xy^2\rho + \cancel{\rho}f(x)$

$$\frac{1}{\rho}\frac{\partial\psi}{\partial x} = -\cancel{\rho}y^2 + \cancel{\rho}f'(x)/\rho = -\cancel{v} = x^2 - y^2 \Rightarrow f'(x)/\rho = x^2 \text{ or } f(x) = \rho x^3/3 + \mathcal{C}$$

$$\therefore \psi = \rho[-xy^2 + x^3/3 + \mathcal{C}]$$

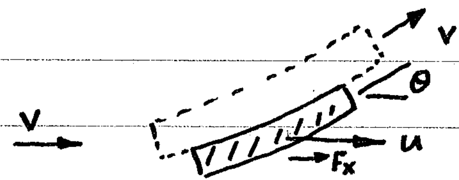
$$\underset{\sim}{q} = u\underset{\sim}{i} + v\underset{\sim}{j} = -2xy\underset{\sim}{i} + (y^2 - x^2)\underset{\sim}{j}$$

Notes: The equations are given in an inertial frame ($\underset{\sim}{q} = 0$ or $\underset{\sim}{q} = \text{const}$) velocities must be expressed <u>relative</u> to the coordinate system of control volume

in $\oint_S \underset{\sim}{q}\,(\rho\underset{\sim}{q}\cdot\underset{\sim}{n})\,dS$

$\underset{\sim}{q}$ (second) ← relative to control volume

$\underset{\sim}{q}$ (first) ← relative to an inertial frame



$V' \begin{cases} (V-u) & \text{relative to C.V.} \\ [u + (V-u)\cos\theta]\underset{\sim}{i} & \text{relative to inertial fram} \\ \quad + (V-u)\sin\theta\,\underset{\sim}{j} & \end{cases}$

$\begin{cases} V-u & \text{relative to C.V.} \\ V & \text{relative to inertial frame} \end{cases}$

$$F_x = -V\{\rho(V-u)A\} + \rho(V-u)A\,[u + (V-u)\cos\theta]$$

$$\underset{\sim}{q}_{\substack{\text{absolute}\\ \text{inertia}}} = \underset{\sim}{q}_{\text{vehicle}} + \underset{\sim}{q}_{\text{relative}}$$

WE HAVE SHOWN THAT FROM BERNOULLI EQUATION $p + \frac{1}{2}\rho V^2 = \text{Const}$

ALONG STREAMLINE FOR INCOMPRESSIBLE FLOW

$\frac{1}{2}\rho V^2$: K.E. /unit volume

p : related to energy attributed to particle random motion/unit vol

$\rho g z$: P.E. /unit volume

$\Rightarrow$ energy conserved ; $p \cdot \text{volume} = \text{work}$ ($p \cdot A = \text{force}$; length [streamline]

$A \cdot \text{length} = \text{volume}$ $\therefore$ $p = \text{work}/\text{volume}$

- here mechanical energy is conserved for incompressible flow
- from compressible fluids – mechanical energy (KE + PE + work) $\neq$ const
- MUST INCLUDE [redacted] ENERGY EFFECTS AS WELL & LOOK AT ENERGY TRANSFER DUE TO COMPRESSIBILITY
-

DFNS - THERMODYNAMICS

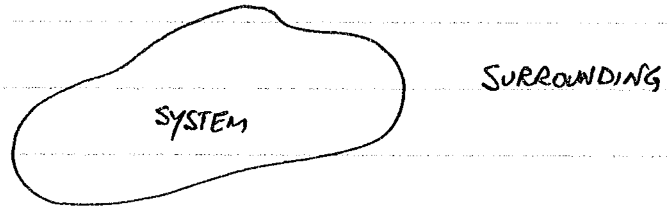
STATE - WHEN OBSERVABLE CHARACTERISTICS OF A FLUID CAN BE FIXED

PROPERTY - OBSERVABLE CHARACTERISTICS , PHYSICAL QUANTITIES $(P, \rho, \underset{\sim}{V}, \underset{\sim}{X})$

PROCESS - CHANGE OF STATE

CYCLE - PROCESS WHERE BEGINNING & FINAL STATES ARE SAME

ENVIRONMENT OR SURROUNDINGS -



TEMPERATURE - A PROPERTY RELATED TO THE ENERGY OF THE SYSTEM (ABSOLUTE)

- ZEROTH LAW OF THERMO : 2 BODIES IN CONTACT, & ARE IN THERMAL EQUIL., ARE AT THE SAME TEMP. TEMP IS IN̲TENSIVE (I̲N̲DEPENDENT OF MASS)

- LAWS OF THERMO DEAL W/ INTERACTIONS BETWEEN SYSTEM & SURROUNDINGS AS THE PASS THROUGH EQUILIB. STATES THROUGH WORK & HEAT

  WORK DONE BY SYSTEM ON ITS SURROUNDINGS IS +

  WORK DONE ON SYSTEM BY SURROUNDINGS IS −

- FIRST LAW STATES :

  WORK DONE BY A SYSTEM IN GOING FROM ONE STATE TO ANOTHER DOESN'T DEPEND ON THE INTERMEDIATE STATES ACHIEVED NOR ON MANNER OF WORK INTERACTION, SO LONG AS SYSTEM & SURROUNDING ARE EQUAL IN TEMP AT EACH STEP OF PROCESS

WE CAN DEFINE THE ENERGY E

$$E_1 - E_2 = W_{\text{equal temp.}}$$

- IF TEMPERATURE EQUALITY DOESN'T PREVAIL, HEAT INTERATION DUE TO THIS INEQUALITY IS GIVEN BY

$$\boxed{Q = \underset{\text{actual process}}{W} - \underset{\text{process}}{W_{e.t.}} = W - [E_1 - E_2]}$$

Normal Statement: $dE = \delta Q - \delta W$ where δW is + when transferred out & δQ is + when added to system

- in book δQ same δW is + when transfered in
- $\delta \Rightarrow$ path dependent & differentials inexact : $\delta Q, \delta W$
- $d \Rightarrow$ path independent & differential is exact : dE

FOR A CYCLE $\oint dE = 0 \Longrightarrow \oint \delta Q = \oint dW$

ADIABATIC PROCESS : Q is ZERO (ADIABTIC = EQUAL TEMP PROCESS)

$$dE = dU + dKE + dPE + dU'$$

U : relates to molecular internal energy

KE : relates to global energies KE $\left(m V^2/2\right)$

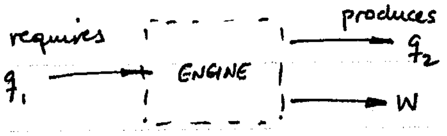
PE : " " " " PE (mgz)

U' : relates electrical, capillary, magnetic

} properties

SECOND LAW OF THERMODYNAMICS

STATES - NO SYSTEM CAN PASS THROUGH A COMPLETE CYCLE OF STATES & DELIVER + WORK TO SURROUNDINGS WHILE EXCHANGING HEAT WITH ONLY A SINGLE SOURCE OF HEAT AT UNIFORM TEMP.



$W = q_1 - q_2$ &

$\eta = \dfrac{W}{q_1} < 1$

- NATURAL EVENTS PROCEED IN ONE DIRECTION DUE TO IRREVERSIBILITY OF PROCESS

  DUE TO VISCOSITY, HEAT CONDUCTION, MASS DIFFUSION (MOLECULAR ACTIONS)

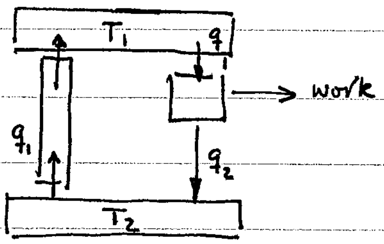
- 2nd LAW $\Longleftrightarrow$ IRREVERSIBILITY
- USE REVERSIBLE PROCESSES AS A GAUGE FOR REAL PROCESSES

2nd LAW IMPLIES
- REVERSIBLE PROCESS ENGINE IS MOST EFFICIENT ENGINE
- TEMPERATURE SCALE BASED ON REVERSIBLE ENGINE EFFICIENCY

  DEPENDS ON TEMPERATURES OF HEAT RECEIVED & REJECTED

  NOT ON TYPE OF ENGINE OR WORKING FLUID $q_1/q_2 = T_1/T_2$

- IF AN IRREVERSIBLE PROCESS CAN BE REVERSED WE COULD CONSTRUCT A PERPETUAL MOTION MACHINE OF THE SECOND KIND



SYSTEM PASSING THROUGH A COMPLETE CYCLE CAN PRODUCE PERPETUAL MOTION OF SECOND KIND UNLESS $\oint \frac{\delta Q}{T} \leq 0$

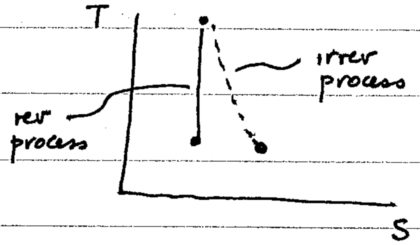
- FOR A REVERSIBLE PROCESS

$$\oint \left(\frac{\delta Q}{T}\right)_{rev} = 0 \quad \& \quad \int_1^2 \left(\frac{\delta Q}{T}\right)_{rev} \text{ depends on end states}$$

i.e. $\left(\frac{\delta Q}{T}\right)_{rev} = dS$ entropy (deduced from 2^{nd} LAW)

- WE CAN STILL DEFINE entropy for irreversible processes

$$Q_{rev} = \int T dS$$



- for irreversible processes $dS \geq \left(\frac{\delta Q}{T}\right)_{irreversible}$

- WHEN SYSTEM IS ISOLATED FROM ALL HEAT EXCHANGE w/ SURROUNDING

$$dS_{adiabatic} \geq 0 \quad \text{PRINCIPLE OF INCREASE IN ENTROPY}$$

- WE CAN DEFINE FLOW WORK : $p\mathcal{V}$ $\qquad pv \qquad v = \mathcal{V}/m$

ENTHALPY : $H = U + p\mathcal{V}$ $\qquad h = u + pv \qquad u = U/m$

$h = u + p/\rho$

- FIRST LAW & SECOND LAW ARE RELATED BY

$$dS = \frac{dU + pd\mathcal{V}}{T}$$

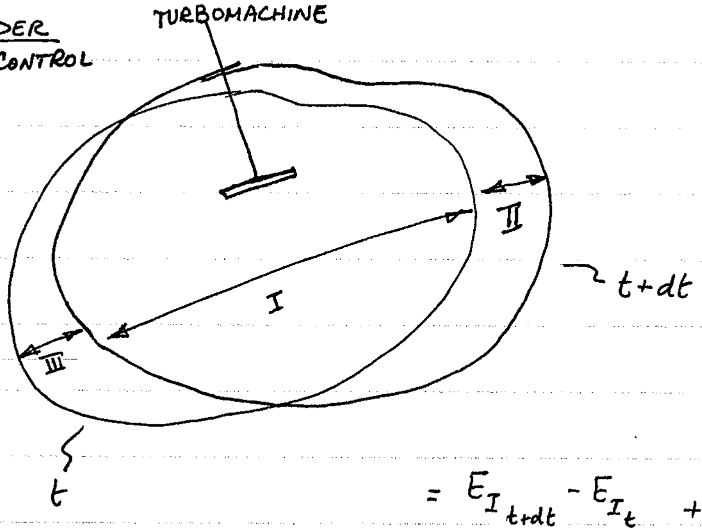
PURE SUBSTANCE, EXCLUDING ELECTRICAL, MAGNETIC & CAPILLARY FORCES, INERTIAL FRAME OF REFERENCE, SURFACE TENSION, ANISOTROPY

$$E = U + m\left(\frac{V^2}{2} + gz\right) \qquad \text{or} \qquad e = u + \frac{V^2}{2} + gz$$

is the internal energy for the system

FOR THE CONTROL VOLUME: $\delta Q + \delta W + \sum m\left(h + \frac{V^2}{2} + gz\right) = dE$

CONSIDER A FIXED CONTROL VOLUME



$$\frac{\delta Q}{dt} = \frac{dE}{dt} + \frac{\delta W}{dt}$$

$$\frac{dE}{dt} = \left[\left(E_{I_{t+dt}} + E_{II_{t+dt}}\right) - \left(E_{I_t} + E_{III_t}\right)\right] / dt$$

$$= \frac{E_{I_{t+dt}} - E_{I_t}}{dt} + \frac{E_{II_{t+dt}}}{dt} - \frac{E_{III_t}}{dt}$$

$$\overset{\uparrow}{\left(\frac{\partial E}{\partial t}\right)_{cv}} + \frac{\int e\,\delta m_{II_{t+dt}}}{dt} - \frac{\int e\,\delta m_{III_t}}{dt}$$

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\int_{\mho} e\rho\, d\mho + \int e\,\delta \dot{m}_{out} - \int e\,\delta \dot{m}_{in}$$

$\frac{\delta W}{dt} = \frac{\delta\left(W_{shear} + W_{flow} + W_{shaft}\right)}{dt}$ $\quad W_{flow}$ is done by hydrostatic pressure p

$$\frac{\delta W_{flow}}{dt} = \int p \frac{d\mho}{dt}\Big|_{II_{t+dt}} - \int p\frac{d\mho}{dt}\Big|_{III_t}$$

note I does not contribut

$$= \int p/\rho \;\frac{dm}{dt}\Big|_{out} - \int p/\rho \;\frac{dm}{dt}\Big|_{in}$$

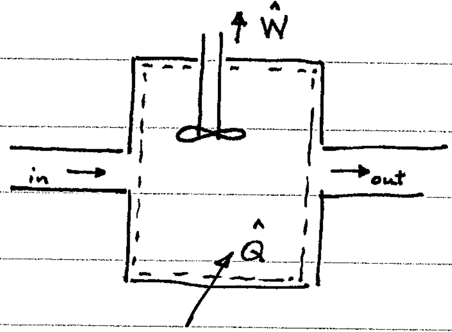
now let $\frac{\delta W_{shear}}{dt} = P_{shear}$ $\quad \frac{\delta W_{shaft}}{dt} = P_{shaft}$

$$\therefore \quad \frac{\delta Q}{dt} = \frac{\partial}{\partial t}\int e\rho\, dU + \int e\,(\rho \underset{\sim}{q}\cdot\underset{\sim}{n})\,dA + P_{shear} + P_{shaft} + \int P/\rho\,(\rho\underset{\sim}{q}\cdot\underset{\sim}{n})\,dA$$

$$= \frac{\partial}{\partial t}\int e\rho\, dU + \underbrace{\int (e + P/\rho)\,(\rho\underset{\sim}{q}\cdot\underset{\sim}{n})\,dA}_{\int\left[u + \frac{V^2}{2} + gz + P/\rho\right](\rho\underset{\sim}{q}\cdot\underset{\sim}{n})\,dA} + P_{shear} + P_{shaft}$$

$$= \frac{\delta Q}{dt} = \frac{\partial}{\partial t}\int_U e\rho\, dU + \int_A \left[h + \frac{V^2}{2} + gz\right](\rho\underset{\sim}{q}\cdot\underset{\sim}{n})\,dA + P_{shear} + P_{shaft}$$

FOR A STEADY ONE-DIMENSIONAL PROBLEM C.V. IS INNER CASING OF ENGINE



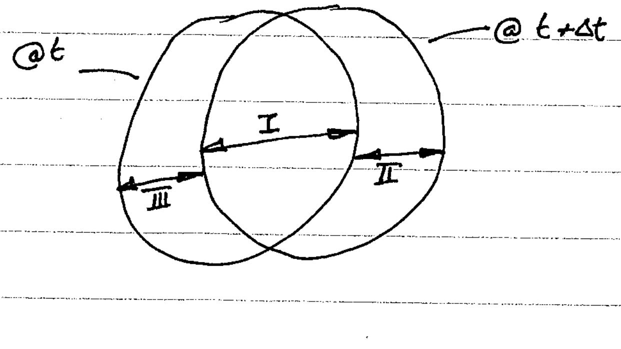
$P_{shear} = 0 \qquad \dot{m} = \text{constant}$

$\hat{Q} = \frac{1}{\dot{m}}\frac{\delta Q}{dt} \qquad \hat{W} = \frac{P_{shaft}}{\dot{m}}$

$$\hat{Q} = \hat{W} + \left(h + \frac{V^2}{2} + gz\right)_{out} - \left(h + \frac{V^2}{2} + gz\right)_{in}$$

second law for a control volume $\quad s = S/m$

$$\oint \frac{\delta Q}{T} \le \oint dS = \frac{\partial}{\partial t}\int_U (s\rho)\,dU + \int s(\rho\underset{\sim}{q}\cdot\underset{\sim}{n})\,dA$$



entropy flux is $s\frac{dm}{dt} = \frac{dS}{dt}$

Equation of State : gives pressure - density relationship

$$pv = p/\rho = RT \qquad \text{or} \qquad R = 287.04 \text{ J/kg °K}$$

$$R = 1715 \text{ ft-lb/slug °R}$$

A perfect gas satisfies $p/\rho = RT$

$+ \ (\partial u/\partial v)_T = 0$ i.e. internal energy is independent of density

(Joule-Thomson Experiments)

THUS $u = u(v,T)$ for a pure substance

Ideal or perfect gas must satisfy

1) small intermolecular forces
2) Molecular dimensions small compared to mean free path & to particle volume
3) No chemical reactions to alter state of gass

since $u = u(v,T)$

$$du = \underset{0}{\underbrace{\cancel{\left(\frac{\partial u}{\partial v}\right)_T}}} dv + \left(\frac{\partial u}{\partial T}\right)_v dT \qquad \text{we define } \left(\frac{\partial u}{\partial T}\right)_v = c_v$$

and

$$u_2 - u_1 = \underset{\text{IF constant}}{c_v}(T_2 - T_1) \quad \text{or} = \int_{T_1}^{T_2} c_v(T)\, dT$$

since we can define $h = h(p,T) = u + pv$ enthalpy

$$dh = \left(\frac{\partial h}{\partial T}\right)_p dT + \left(\frac{\partial h}{\partial p}\right)_T dp$$

by definition $\left(\frac{\partial h}{\partial T}\right)_p \equiv c_p$ and $c_p = c_p(T)$

since $pv = RT$ & $u = u(T)$ only $\Rightarrow h = h(T)$ only

$$\therefore \ h_2 - h_1 = \int_{T_1}^{T_2} c_p(T)\, dT = c_p(T_2 - T_1) \quad \text{IF } c_p = \text{constant}$$

$$\underset{= c_pT}{h} = u + pv = c_vT + RT = (c_v + R)T \ \Big] \Longrightarrow \ c_p = c_v + R \ \text{ or } \ \boxed{R = c_p - c_v}$$

SPECIAL RELATIONS FOR AN IDEAL GAS

① $$T\,dS = du + p\,dv = dh - v\,dp$$

② $$p = \rho R T \Rightarrow p = RT/v$$

ASSUME

if $s = s(T, v)$ & $u = u(T, v)$ ← NON PERFECT GAS

$$T\,ds = du + p\,dv$$

$$T\left[\frac{\partial s}{\partial T}dT + \frac{\partial s}{\partial v}dv\right] = \left[\frac{\partial u}{\partial T}dT + \frac{\partial u}{\partial v}dv\right] + p\,dv$$

$$\left(\frac{\partial s}{\partial T}\right)_v = \frac{1}{T}\left(\frac{\partial u}{\partial T}\right)_v \quad \& \quad \left(\frac{\partial s}{\partial v}\right)_T = \frac{1}{T}\left(\frac{\partial u}{\partial v}\right)_T + p/T$$

now $$\frac{\partial}{\partial v}\left(\frac{\partial s}{\partial T}\right) = \frac{\partial}{\partial T}\left(\frac{\partial s}{\partial v}\right)$$

$$\frac{1}{T}\frac{\partial^2 u}{\partial T \partial v} = -\frac{1}{T^2}\left(\frac{\partial u}{\partial v}\right)_T + \frac{1}{T}\frac{\partial^2 u}{\partial T \partial v} + \frac{\partial}{\partial T}\left(\cancelto{0}{\frac{R}{v}}\right)$$

$$\Rightarrow \left(\frac{\partial u}{\partial v}\right)_T = 0$$ ← FOR PERFECT GAS

now: $T\,dS = du + p\,dv$ $\qquad v = 1/\rho \qquad dv = -\frac{1}{\rho^2}d\rho$

& $$ds = \frac{du}{T} + p/T\,dv = \frac{du}{T} - \frac{p}{\rho^2 T}d\rho$$

$$ds = C_v\frac{dT}{T} - R\frac{d\rho}{\rho}$$

FROM $p = \rho R T \qquad \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$

$$ds = C_v\frac{dp}{p} - (R + C_v)\frac{d\rho}{\rho}$$

FOR CALORICALLY PERFECT GASES $\quad C_p, C_v = \text{constants}$

$$\therefore \frac{ds}{C_v} = \frac{dp}{p} - \frac{R + C_v}{C_v}\frac{d\rho}{\rho} \Rightarrow \frac{ds}{C_v} = \frac{dp}{p} - \frac{C_p}{C_v}\frac{d\rho}{\rho}$$

$$\frac{s - s_r}{C_v} = \ln p/p_r - \ln \left(\rho/\rho_r\right)^{\gamma} \qquad \gamma = C_p/C_v$$

[illegible]

[illegible] (4)

[illegible] (5)

[illegible]

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$$p/\rho^{\gamma} = e^{\frac{s-s_r}{c_v}} \cdot p_r/\rho_r^{\gamma}$$ r: reference value

$$p/p_r = \left(\rho/\rho_r\right)^{\gamma} e^{\frac{s-s_r}{c_v}}$$ for a non isentropic process

- IF FLOW IS FRICTIONLESS ADIABATIC (REVERSIBLE ADIABATIC) $\Rightarrow ds = 0$
  - S IS CONSTANT ALONG A STREAMLINE & FLOW IS ISENTROPIC
  - IF s = constant everywhere FLOW IS HOMENTROPIC (HOMOGENEOUS ENTROPY)

ASSUME: FOR AIR $c_p = 1.0035$ kJ/kg °K $\quad \gamma = 1.4$

$c_v = .7165$ kJ/kg °K

FOR AN ISENTROPIC PROCESS $\boxed{p/p_r = \left(\rho/\rho_r\right)^{\gamma}}$ but $p = \rho R T$

$$\therefore \quad \rho/\rho_r \cdot T/T_r = \left(\rho/\rho_r\right)^{\gamma} \quad \text{or} \quad \boxed{T/T_r = \left(\rho/\rho_r\right)^{\gamma-1}}$$

also $\boxed{\left(T/T_r\right)^{\gamma/\gamma-1} = p/p_r}$

IF $\rho = \rho(p)$ only THEN FLUID IS BAROTROPIC

RETURNING TO FLUIDS

$$(\underset{\sim}{q}\cdot\nabla)\underset{\sim}{q} = \nabla \frac{q^2}{2} - \underset{\sim}{q}\times\nabla\times\underset{\sim}{q} \quad \Rightarrow (\underset{\sim}{q}\cdot\nabla)\underset{\sim}{q} = \nabla q^2/2 \text{ if } \nabla\times\underset{\sim}{q} = 0$$

RETURNING TO ENERGY

$$\frac{\delta Q}{dt} = \frac{\partial}{\partial t}\int e\rho\, d\mathcal{V} + \int \left(e + p/\rho\right)(\rho \underset{\sim}{q}\cdot\underset{\sim}{n})\, dA + P_{shear} + P_{shaft}$$

$$e = u + \frac{V^2}{2} + gz$$

$$\frac{\delta Q}{dt} = \frac{\partial}{\partial t}\int e\rho\, d\mathcal{V} + \int \nabla\cdot\left(e\underset{\sim}{q}\rho + p\underset{\sim}{q}\right) d\mathcal{V} + P_{shear} + P_{shaft}$$

THIS WAS DERIVED FOR FIXED CONTROL VOLUME $\therefore \frac{\partial}{\partial t}\int e\rho\, d\mathcal{V} = \int \frac{\partial}{\partial t}(e\rho)\, d\mathcal{V}$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{q}\cdot\nabla$$

∴ FOR NO SHEAR OR SHAFT WORK

$$\int \dot{Q}\, d\upsilon = \frac{\delta Q}{dt} = \int \frac{\partial}{\partial t}(\rho e)\, d\upsilon + \int_{\upsilon} [\nabla\cdot(e\rho \underset{\sim}{q} + p\underset{\sim}{q})]\, d\upsilon$$

$$= \underbrace{\downarrow \quad + \int_{\upsilon} [\underset{\sim}{q}\cdot\nabla(\rho e)}\, + \rho e\, \nabla\cdot\underset{\sim}{q} + \nabla\cdot(p\underset{\sim}{q})]\, d\upsilon$$

$$= \overbrace{\int \frac{D}{Dt}(\rho e)\, d\upsilon} + \int_{\upsilon} \rho e\, \nabla\cdot\underset{\sim}{q}\, d\upsilon + \int_{\upsilon} \nabla\cdot(p\underset{\sim}{q})\, d\upsilon$$

$$\int \left[\rho \frac{D}{Dt} e\, d\upsilon\right] + \int \left[e \frac{D}{Dt}\rho\, d\upsilon + e\,\rho\nabla\cdot\underset{\sim}{q}\, d\upsilon\right] + \int_{\upsilon} \nabla\cdot(p\underset{\sim}{q})\, d\upsilon$$

$$\searrow\ e\cdot \text{continuity} = 0$$

$$\therefore \quad \int \dot{Q}\, d\upsilon = \int \rho \frac{D}{Dt}(e)\, d\upsilon + \int_{\upsilon} \nabla\cdot(p\underset{\sim}{q})\, d\upsilon$$

$$\searrow \int [\cancel{\rho}\underset{\sim}{q}\cdot\nabla p/\rho + p\cancel{/\rho}\,\nabla\cdot\underset{\sim}{q}\cancel{\rho}]\, d\upsilon$$

ok

$$\cancel{\rho}\frac{D}{Dt}p\cancel{/\rho} - \cancel{\rho}\frac{\partial}{\partial t}(p\cancel{/\rho}) + p\cancel{/\rho}\,\nabla\cdot\cancel{\rho}\underset{\sim}{q}$$

$$-\rho\left[\frac{1}{\rho}\frac{\partial p}{\partial t} - p/\rho^2 \frac{\partial\rho}{\partial t}\right] + p/\rho\left[\rho\nabla\cdot\underset{\sim}{q} + \underset{\sim}{q}\cdot\nabla\right.$$

$$p/\rho\,[\text{contintinuity}]$$

$$= \int \rho \frac{D}{Dt}(e + p/\rho)\, d\upsilon - \int \frac{\partial}{\partial t} p\, d\upsilon$$

if adiabatic $\Rightarrow$ $\rho \frac{D}{Dt}(e + p/\rho) = \frac{\partial p}{\partial t}$

IF not $\Rightarrow$ $\rho \frac{D}{Dt}(e + p/\rho) = \partial p/\partial t + \dot{Q}$

if steady adiabatic $\Rightarrow$ $e + p/\rho = \text{constant}$ or $\cancel{u} + p/\rho + V^2/2 + gz = \text{constant}$

$h + V^2/2 + gz = \text{constant}$

if steady, adiabatic & no altitude changes $h + V^2/2 = \text{Const} = H_0$ stagnation enthalpy

- Enthalpy that flow would have if is brought to rest isentropically $= H_0$

$$[p-(p+\Delta p)]A = -\rho u^2 A + \overset{\rho u A(u+\Delta u)}{(\rho+\Delta\rho)(u+\Delta u)^2 A}$$

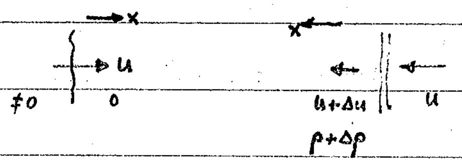
$$\rho u A = (\rho+\Delta\rho)(u+\Delta u)A \qquad \Rightarrow \quad u\Delta\rho + \rho\Delta u = 0 \quad \text{or}$$

$$② \rightarrow ① \qquad -\Delta p A = -\rho u^2 A + \rho u A(u+\Delta u) = \rho u \Delta u A$$

$$\therefore \quad -\Delta p = \rho u \Delta u \qquad -\frac{\Delta p}{p} = \frac{\rho u \Delta u}{p} = \frac{u\Delta u}{RT}$$

$$\text{FROM} \quad p = k\rho^\gamma \qquad \frac{\Delta p}{p} = \gamma\frac{\Delta\rho}{\rho} \Rightarrow a^2\Delta\rho = \Delta p \Rightarrow \quad -a^2\Delta\rho = \rho u \Delta u \quad \text{or}$$

$$\Delta\rho = -\frac{\rho u}{a^2}\Delta u$$



$$\text{if } \Delta p \nless 0 \Rightarrow \Delta u \ngtr 0$$

FROM MOMENTUM $\rho \frac{D\underset{\sim}{q}}{Dt} + \nabla p = \rho \bar{F} = \rho \nabla \tilde{U}$

$\underset{\sim}{F}$ is body force $= \nabla \tilde{U}$ is $\underset{\sim}{F}$ is conservative

if steady $\rho \underset{\sim}{q} \cdot \nabla \underset{\sim}{q} + \nabla p = \rho \nabla \tilde{U}$

IRROTATIONAL

$$\rho \nabla \frac{V^2}{2} + \nabla p = \rho \nabla \tilde{U}$$

For constant density flow : $\nabla \left[\frac{V^2}{2} + p/\rho - \tilde{U} \right] = 0$ or $\frac{V^2}{2} + p/\rho - \tilde{U} = \text{const}$ along streamline

For compressible flow fluid is baratropic $\rho = \rho(p)$

$$\therefore \int \frac{dp}{\rho} + \frac{V^2}{2} - \tilde{U} = \text{constant}$$

if homentropic flow $p = K\rho^{\gamma}$ $\gamma = \frac{C_p}{C_v} = 1.4$ for air $c_p = 1.0035$ kJ/kg $c_v = .7165$ kJ/kg K

& $\frac{\gamma}{\gamma - 1} p/\rho + \frac{V^2}{2} - \tilde{U} = \text{Constant}$

$$\frac{C_p}{C_p - C_v} p/\rho + \qquad = \frac{C_p}{R} \cdot RT + \ldots = C_pT + \ldots = h + \ldots$$

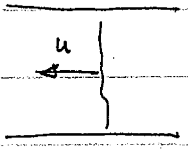
if calorically perfect

$\therefore h + \frac{V^2}{2} - \tilde{U} = \text{constant}$

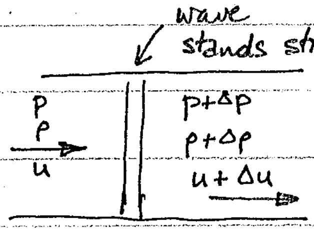
if $\tilde{U} = 0$ (no change is altitude) $\Rightarrow H_0 = \text{constant}$ (isoenergetic flow)

For compressible flow

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$$



unsteady



steady

- Propagation of an infinitesimal disturb
- infinitesimal disturbance produces compression & expansion of fluid but it is reversible

$$[p - (p + \Delta p)]A = -\rho u^2 A + (\rho + \Delta\rho)(u + \Delta u)^2 A$$

$$-\rho u A + (\rho + \Delta\rho)(u + \Delta u)A = 0$$

$$\frac{\Delta u}{u} + \frac{\Delta \rho}{\rho} + \frac{\Delta\rho\Delta u}{\rho u} = 0$$

$$u\,\Delta u + \frac{\Delta p}{\rho} = 0 \qquad \Rightarrow \qquad u^2 = \frac{-\Delta p}{-\Delta\rho - \Delta\rho\,(\Delta u/u)}$$

small discontinuities $\lim_{\Delta u, \Delta p, \Delta\rho \to 0} u \to a \qquad \therefore \; u^2 = \frac{\partial p}{\partial \rho} = a^2$

$\therefore \; a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$ since $p = k\rho^\gamma \qquad \frac{\partial p}{\partial \rho} = k\gamma\rho^{\gamma-1} = \frac{\cancel{k}\gamma p}{\rho}$

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT}$$

FOR NON ISENTROPIC $a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$ DOES NOT APPLY SINCE $p \neq p(\rho)$ ONLY

For incompressible flow ρ = constant $d\rho = 0 \Rightarrow a = \infty$

what does mach no. represent

$M = \frac{u}{a}$; $M^2 = \frac{u^2}{a^2} = \frac{\rho u^2}{\gamma p} =$ ~~kinet~~ $\frac{\text{dynamic pressure}}{\text{static pressure}}$

$$= \frac{u^2}{\gamma RT} = \frac{\text{kinetic energy}}{\text{thermal energy}}$$

STAGNATION CONDITIONS - WHEN BROUGHT ISENTROPICALLY TO REST

$H_0 = h + V^2/2 = C_pT + V^2/2$ ALSO APPLIES FOR ADIABATIC w/ VISCOUS DISSIP.

$C_pT_0 =$

$$\Rightarrow \frac{T_0}{T} = 1 + \frac{V^2}{2C_pT} = 1 + \frac{\gamma - 1}{2\gamma R}\frac{V^2}{T}$$

$$\boxed{\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2}$$

since $\frac{T}{T_r}$ & p/p_r & ρ/ρ_r are related through $p = k\rho^\gamma$

$$p/p_0 = \left(\rho/\rho_0\right)^\gamma = \left(\frac{T}{T_0}\right)^{-\gamma/\gamma-1} \quad \text{or} \quad p_0/p = \left(\frac{T_0}{T}\right)^{\gamma/\gamma-1} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/\gamma-1}$$

FOR INVISCID STEADY FLOW p_0 = constant along a streamline

for steady flow, adiabatic T_0 = constant along a streamline

All states with same entropy & same stagnation temperature have the same isentropic stagnation state & crit state. When a stream with a given pressure, temperature & velocity is decelerated to zero velocity, the final pressure will be less than the isentropic stagnation pressure if the deceleration is irreversible, but the final temperature will be equal to the adiabatic stagnation temperature irrespective of the reversibility/irreversibility of the deceleration

A MOVING DISTURBANCE CAN CREATE CHANGES IN PRESSURE FIELD

IN ACOUSTICS – FLUID VELOC ARE SMALL BUT CAUSE $\Delta p, \Delta T, \Delta \rho$

IN INCOMPRESSIBLE FLOWS – FLUID VELOCS ARE SMALL & SO IS $\Delta \rho$

BUT $\Delta p, \Delta T$

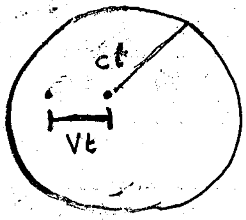
IN COMPRESSIBLE FLOWS – VELOCITY IS ON ORDER OF a CAUSING

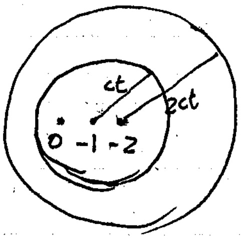
Δp, ΔT & $\Delta \rho$

IN INCOMPRESSIBLE [redacted] VELOC < a

DISTURBANCE MOVES at speed Vt

- PRESSURE WAVE MOVES at speed ct

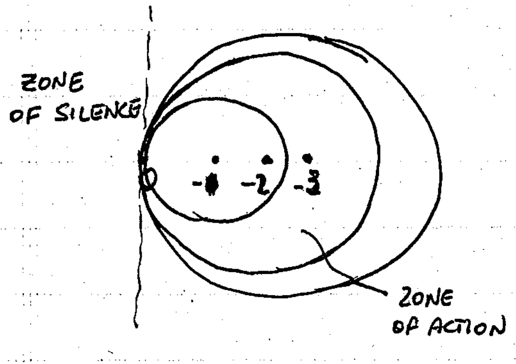




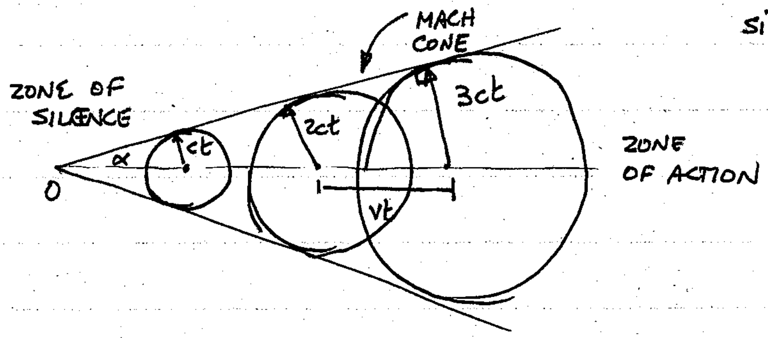
PRESSURE DISTURBANCE IS PROPAGATED IN ALL DIRECTIONS

FOR VELOC = a

NO PROPAGATION BEFORE WAVE



FOR VELOC > a



$$\sin\alpha = \frac{3ct}{3vt} = \frac{1}{M}$$

VON KARMAN'S RULES FOR SUPERSONIC FLOW

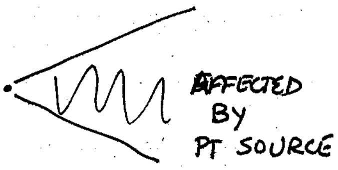
- EXACT FOR SMALL DISTURBANCES
- QUALITATIVE FOR LARGE DISTURBANCES

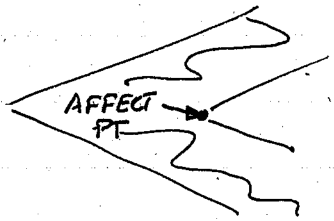
- RULE OF FORBIDDEN SIGNALS - IF V > a PRESSURE CHANGES CANNOT REACH POINTS AHEAD OF BODY

- ZONE OF ACTION / ZONE OF SILENCE

  PRESSURE & VELOCITY AT SOME POINT IN STREAM CAN BE INFLUENCED ONLY BY DISTURBANCES ACTING AT POINTS LYING ON OR INSIDE CONE EXTENDING UPSTREAM FROM POINT CONSIDERED WITH α SAME AS MACH CONE. A STATIONARY POINT SOURCE AFFECTS POINTS THAT LIE ON OR

IN THE MACH CONE EXTENDING DOWNSTREAM FROM POINT SOURCE





- RULE OF CONCENTRATED ACTION

CLOSENESS OF CIRCLES REPRESENT PRESSURE INTENSITY DUE TO DISTURBANCE. SUBSONIC (INTENSITY ISASYMMETRIC) STATIONARY (SYMMETRIC). SUPERSONIC – PRESSURE DISTURBANCE IS LARGELY CONCENTRATED IN THE NEIGHBORHOOD OF THE MACH CANE

Problem 1.14

$V = 1500 \text{ km/hr} = \dfrac{1500{,}000}{3600 \text{ sec/hr}} \cdot \text{m/hr} = 416.67 \text{ m/s}$; $T = 273.15 - 60 = 213.15$

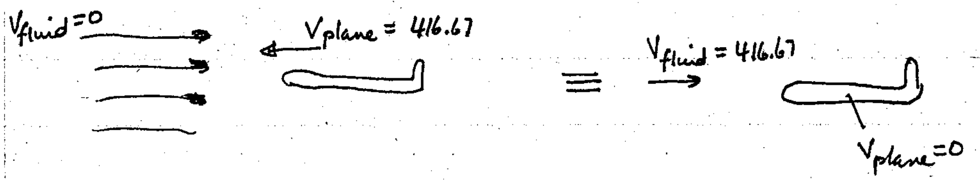
$M = \dfrac{V}{a}$ $\quad a = \sqrt{\gamma R T} = \sqrt{(1.4)(287.04)(213.15)} = 292.67 \text{ m/s}$

$= \dfrac{416.67}{292.67} = 1.4237$

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2 = 1 + .2(1.4237)^2 = 1.4054$$

Given T $\quad T_{surface} = T_o$ $\quad \therefore \quad T_o = 213.15\,[1.4054] = 299.56\ ^\circ K$

$= 26.41^\circ C$



$\therefore T_{surface} = T_o$

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10. Schardin, H. Theorie und Anwendung des Mach-Zehnderschen Interferenz-Refraktometers, *Zeitschrift fur Instrumentenkunde*, Vol. 53 (1933), pp. 396, 424. Also see R.A.E. Translation, No. 79.
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12. Pankhurst, R. C., and Holder, D. W. *Wind-Tunnel Technique.* London: Sir Isaac Pitman & Sons, 1952.

PROBLEMS

3.1. The compressibility of a liquid is usually expressed in terms of the bulk modulus of compression,

$$\beta = \rho \frac{dp}{d\rho}$$

Show that

$$c = \sqrt{\beta/\rho}$$

3.2. Calculate the velocity of sound at 70°F in the following media:

(a) Air, (b) hydrogen, (c) uranium hexafluoride, (d) mercury vapor, (e) water vapor, (f) liquid water at 14.7 psia.

3.3. To what pressure must liquid water be compressed in order that it leave a nozzle at atmospheric pressure with a jet velocity equal to the sonic velocity? Assume a constant bulk modulus of compression of 300,000 psi.

3.4. Plot and compare curves of the sonic velocity in air versus absolute temperature (a semi-log plot is suggested) using

(a) a value of γ of 1.4 for all temperatures, and (b) the actual value of γ for each temperature.

3.5. A projectile in flight carries with it a more or less conical-shaped shock front. From physical reasoning it appears that at great distances from the projectile this shock wave becomes truly conical and changes in velocity and density across the shock become vanishingly small.

Photographs of a bullet in flight show that at a great distance from the bullet the total included angle of the cone is 50.3°. The pressure and temperature of the undisturbed air are 14.62 psia and 73°F, respectively.

Calculate the velocity of the bullet, in ft/sec, and the Mach Number of the bullet relative to the undisturbed air.

3.6. Show that, for a perfect gas, the fractional change in pressure across a small pressure pulse is given by

$$\frac{dp}{p} = \gamma \frac{dV}{c}$$

and that the fractional change in absolute temperature is given by

$$\frac{dT}{T} = (\gamma - 1) \frac{dV}{c}$$

3.7. (a) Demonstrate that a compression wave (i.e., a pressure pulse which increases the density of the fluid over which it passes) which moves rightward imparts a rightward velocity to the fluid.

(b) Derive similar rules for a rightward-moving rarefaction wave and for leftward-moving compression and rarefaction waves.

(c) The rightward-moving compression wave of Fig. 3.1a strikes a stationary wall closing the right-hand end of the duct. Demonstrate that it is necessary for a reflected wave to travel leftward, and determine whether the reflected wave is a compression or rarefaction. Compare the pressure change across the incident wave with that across the reflected wave.

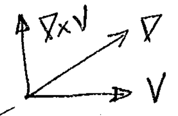
3.8. A compression pulse changes the velocity of the fluid over which it passes by 10 ft/sec. Calculate the pressure rise (psi) across the pulse for (a) water, and (b) air at 14.7 psia and 70°F.

ALSO

DO PROBLEMS 1.3, 1.4, 2.8 IN SAAD

Do problem 1.4, 1.3, 2.8

DO 3.5, 3.7, 3.8



$$\boxed{\frac{\partial \rho_\infty}{\partial t}} + \frac{\partial \rho'}{\partial t} + (\rho_\infty+\rho')\,\nabla\cdot(q_\infty+q') + (q_\infty+q')\cdot\nabla(\rho_\infty+\rho')$$

$$\cancel{\frac{\partial \rho_\infty}{\partial t}} \quad \frac{\partial \rho'}{\partial t} + \rho_\infty\nabla\cdot q' + \cancel{\rho'\nabla\cdot q_\infty} + \underbrace{\rho'\nabla\cdot q'}_{\text{2nd order}} + \cancel{\rho_\infty\nabla\cdot q_\infty} + \cancel{q_\infty\cdot\nabla\rho_\infty} + q_\infty\cdot\nabla\rho'$$

$$+ \cancel{q'\cdot\nabla\rho_\infty} + \underbrace{q'\cdot\nabla\rho'}_{\text{2nd order}}$$

$$\nabla\frac{V^2}{2} + \frac{1}{\rho}\nabla p$$

Let's look at what the governing equation is for a small perturbation in a fluid

- Suppose fluid initially at rest & has p_∞ & ρ_∞
- Suppose a disturbance occurs so that $p = p_\infty + p'$; $\rho_\infty + \rho' = \rho$; $\underset{\sim}{q} = \underset{\sim}{q}'$
  here p', ρ' & q' are small compared to p_∞, ρ_∞

Governing equation will be $\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \underset{\sim}{q} + \underset{\sim}{q} \cdot \nabla \rho = 0$ (Continuity)

IF FLOW INVISCID, no body forces (no gravitational) $\frac{\partial q}{\partial t} + \underset{\sim}{q} \cdot \nabla \underset{\sim}{q} + \frac{1}{\rho} \nabla p = 0$ (Momentum)

we assume process is reversible adiabatic (homentropic s = constant)

$$p = k\rho^\gamma \qquad \text{(Equation of State)}$$

This is an <u>unsteady flow</u>

SPEED OF SOUND $a^2 = \left.\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho} = \frac{\gamma(p_\infty + p')}{\rho_\infty + \rho'} = \frac{\gamma p_\infty}{\rho_\infty}\frac{(1 + p'/p_\infty)}{(1 + \rho'/\rho_\infty)}$

$$= a_\infty^2 \left[1 + p'/p_\infty\right]\left(1 - \rho'/\rho_\infty + (\rho'/\rho_\infty)^2 - \dots\right]$$

$$= a_\infty^2 \left[1 + p'/p_\infty - \rho'/\rho_\infty + \text{second order terms}\right]$$

$$a^2 = a_\infty^2 + a_\infty^2 \left(p'/p_\infty - \rho'/\rho_\infty\right) + \dots = a_\infty^2 + a'^2 \quad \text{when linearized}$$

<u>Continuity</u>

$$\frac{\partial}{\partial t}(\rho_\infty + \rho') + (\rho_\infty + \rho')\nabla \cdot \underset{\sim}{q}' + \underset{\sim}{q}' \cdot \nabla(\rho_\infty + \rho') = 0$$

since ρ_∞, p_∞ is not changing $\Rightarrow \frac{\partial \rho'}{\partial t} + \rho_\infty \nabla \cdot \underset{\sim}{q}' + \underbrace{\rho' \cancel{\nabla \cdot \underset{\sim}{q}'} + \underset{\sim}{q}' \cdot \cancel{\nabla \rho'}}_{2^{nd}\text{ order terms}} = 0$

now LINEARIZED CONTINUITY is $\boxed{\frac{\partial \rho'}{\partial t} + \rho_\infty \nabla \cdot \underset{\sim}{q}' = 0}$

Momentum

$$\frac{\partial \underset{\sim}{q}'}{\partial t} + \underbrace{\cancel{(\underset{\sim}{q}' \cdot \nabla)\underset{\sim}{q}'}}_{2^{\text{nd}} \text{ order small}} + \frac{1}{(\rho_\infty + \rho')} \nabla (p_\infty + p') = 0$$

$$\Big\downarrow \qquad + \frac{1}{\rho_\infty} \frac{1}{(1 + \rho'/\rho_\infty)} \Big[\underbrace{\cancel{\nabla p_\infty}}_{=0} + \nabla p' \Big] = 0 \quad \text{since } p_\infty \text{ not changing}$$

now $$\frac{\partial \underset{\sim}{q}'}{\partial t} + \frac{1}{\rho_\infty} \nabla p' \left(1 - \rho'/\rho_\infty + \left(\rho'/\rho_\infty\right)^2 + \cdots \right) = 0$$

$$\boxed{\frac{\partial \underset{\sim}{q}'}{\partial t} + \frac{1}{\rho_\infty} \nabla p' = 0} \quad \text{keeping first order terms only}$$

Linearized Momentum   note: $\underset{\sim}{q}', p', \rho'$ but 2 eqns.

Use equation of state to relate p & ρ

since $$p = k\rho^\gamma \qquad dp = k\gamma\rho^{\gamma-1} d\rho = k\gamma\rho^\gamma \frac{d\rho}{\rho} = \frac{\gamma p}{\rho} d\rho$$

or $$dp = a^2 d\rho \qquad \text{now} \quad dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz = \nabla p \cdot d\underset{\sim}{r}$$

$$\therefore \quad dp \left(= \nabla p \cdot d\underset{\sim}{r} \right) = a^2 d\rho = a^2 \left(\nabla\rho \cdot d\underset{\sim}{r} \right) \; ; \; \text{or } \left(\nabla p - a^2 \nabla\rho \right) \cdot d\underset{\sim}{r} = 0$$

- since p, ρ are point functions without preferred direction $\Rightarrow \nabla p - a^2 \nabla\rho = 0$

$$\nabla p = a^2 \nabla \rho$$

$$\nabla (p_\infty + p') = \left[a_\infty^2 + a'^2 \right] \nabla (\rho_\infty + \rho')$$

∇p_∞, $\nabla \rho_\infty$ are zero

$$\therefore \quad \nabla p' = a_\infty^2 \nabla \rho' + \underbrace{a'^2 \cancel{\nabla \rho'}}_{3^{\text{rd}} \text{ order small}}$$

$$\therefore \quad \boxed{\nabla p' = a_\infty^2 \nabla \rho'}$$

substitute

$$\frac{\partial \underset{\sim}{q}'}{\partial t} + \frac{1}{\rho_\infty} \nabla p' = \frac{\partial \underset{\sim}{q}'}{\partial t} + \frac{a_\infty^2}{\rho_\infty} \nabla \rho' = 0 \qquad \text{Linearized Momentum}$$

now take $\frac{\partial}{\partial t}$ of Linearized Continuity $-\rho_\infty\nabla\cdot$ (Linearized Momentum)

$$\left[\frac{\partial^2}{\partial t^2}\rho' + \rho_\infty\frac{\partial}{\partial t}\nabla\cdot\underset{\sim}{q}'\right] - \rho_\infty\nabla\cdot\left[\frac{\partial \underset{\sim}{q}'}{\partial t} + \frac{a_\infty^2}{\rho_\infty}\nabla\rho'\right] = \frac{\partial}{\partial t}(0) - \rho_\infty\nabla\cdot(0) = 0$$

$$\frac{\partial^2\rho'}{\partial t^2} + \rho_\infty\frac{\partial}{\partial t}\nabla\cdot\underset{\sim}{q}' - \rho_\infty\frac{\partial}{\partial t}\nabla\cdot\underset{\sim}{q}' - a_\infty^2\nabla^2\rho' = 0$$

since x, y, z are indep of t

or $\underline{\frac{\partial^2\rho'}{\partial t^2} = a_\infty^2\nabla^2\rho'}$

wave equation
Governing Equation for perturbation density ρ'

similarly we can show that $\frac{\partial^2\underset{\sim}{q}'}{\partial t^2} = a_\infty^2\nabla(\nabla\cdot\underset{\sim}{q}')$

TAKE $\frac{a_\infty^2}{\rho_\infty}\nabla$ (Linearized Cont) $-\frac{\partial}{\partial t}$ (Linearized Momentum) TO GET IT.

since $p = k\rho^\gamma$ $\Rightarrow$

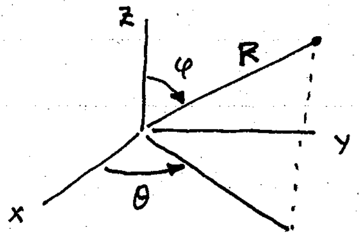
$$\begin{aligned}(p_\infty + p') &= k(\rho_\infty + \rho')^\gamma = k\rho_\infty^\gamma(1 + \rho'/\rho_\infty)^\gamma \\ &= k\rho_\infty^\gamma\left(1 + \frac{\gamma\rho'}{\rho_\infty} + h.o.t\right) \\ \Big\downarrow\quad &= k\rho_\infty^\gamma + k\rho_\infty^\gamma\cdot\frac{\gamma\rho'}{\rho_\infty} \\ p_\infty + p' &= p_\infty + p_\infty\frac{\gamma}{\rho_\infty}\rho' = p_\infty + a_\infty^2\rho'\end{aligned}$$

$\therefore$ $\underline{p' = a_\infty^2\rho'}$ Governing relation between p' & ρ'

<u>What does this mean?</u> physically

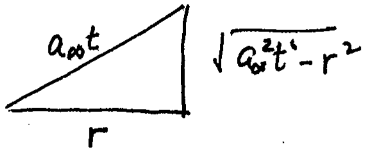
a) spherical symmetry (no dependence on θ, φ)

$$\nabla^2(\ ) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\ \right) + \frac{1}{R^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{R^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}$$

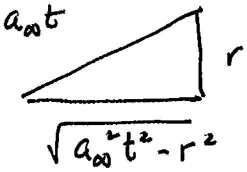


$$\nabla^2(\ ) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) = \frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R}$$

$$\rho' = \frac{1}{R}\left[f(R - a_\infty t) + g(R + a_\infty t)\right]$$



or



$\therefore \quad \int \frac{r}{(a_\infty^2 t^2 - r^2)^{3/2}}\, dt \qquad \tan\theta = \frac{r}{\sqrt{}}$

~~$\sin\theta\cos\theta$~~

$$\frac{1}{\sin\theta} = \csc\theta = \frac{a_\infty t}{r}$$

$$-\frac{1}{\sin^2\theta}\cos\theta\, d\theta = \quad = \frac{a_\infty}{r}\, dt$$

$$-\csc\theta\cot\theta\, d\theta = \frac{a_\infty}{r}\, dt$$

$$\tan^2\theta \cdot \cancel{\tan}\theta \left(-\csc\theta\, \cancel{\cot}\theta\right) d\theta = \frac{a_\infty\, dt}{\cancel{r}} \cdot \frac{\cancel{r}}{\sqrt{}} \cdot \frac{r^2}{(\quad)}$$

$$\therefore \quad -\csc\theta\tan^2\theta\, d\theta = \frac{a_\infty r^2 dt}{(\quad)^{3/2}}$$

or $$\frac{-\csc\theta\tan^2\theta\, d\theta}{a_\infty r} = \frac{r\, dt}{(\quad)^{3/2}}$$

$$\int -\frac{1}{\cancel{\sin}} \cdot \frac{s^{\cancel{2}}}{c^2} \cdot \frac{1}{a_\infty r} = \int$$

$$\int \frac{1}{a_\infty r} \frac{-\sin\theta\, d\theta}{c^2} = \frac{1}{a_\infty r}\frac{dc}{c^2}$$

$$\frac{1}{a_\infty r}\int \frac{dc}{c^2} = \frac{-1}{c}\frac{1}{a_\infty r}$$

$$-\frac{a_\infty t}{a_\infty r} \cdot \frac{1}{\sqrt{}} = \frac{-(\cos\theta)^{-1}}{a_\infty r} = \int$$

$$\therefore \quad u' = \frac{a_\infty^2 t}{\rho_\infty r} \frac{1}{(a_\infty^2 t^2 - r^2)^{1/2}}$$

BB PRINCIPAL INVESTIGATOR/PROGRAM DIRECTOR: Levy/Gordon

DESCRIPTION: State the application's broad, long-term objectives and specific aims, making reference to the health relatedness of the proj Describe concisely the experimental design and methods for achieving these goals. Avoid summaries of past accomplishments and the of the first person. This abstract is meant to serve as a succinct and accurate description of the proposed work when separated from application. **DO NOT EXCEED THE SPACE PROVIDED.**

The objectives of this project are to raise mice under varying conditions of exercise and correlate th changes in the architecture of trabecular bone in the proximal one-half of the femur with the changes i breaking strength, and the associated stresses and strains in the surrounding cortical bone. The specific goal are to expose juvenile mice to normal voluntary exercise (NE), burrowing exercise in a high-litter cage (HL) and four times normal gravity for six, ten minute periods per day (4g). Mice will be sacrificed after 30 day of treatment; the breaking strength, mineral content, mass, cortical wall thickness, moments of inertia in A- and M-L directions, and changes in trabecular architecture will be determined and compared to the NE controls These data will provide the input for a failure analysis and a finite element model that will determine th changes in stresses and strains of the cortical bone of the shaft of the femur (based on our load to failur experiments) as a result of changes in the architecture of the associated trabecular bone. The significance o this project is 1) to demonstrate that significant changes are possible in trabecular architecture and in stresse and strains of associated cortical bone of femora in groups raised under different exercise regimens; 2) to sho the correlation of change in trabecular architecture and cortical bone with the method of failure of the femur and 3) to use the finite element model to predict the locations of the shaft where trabecular bone would b placed to reduce stress concentrations. These predictions will be compared with demonstrated changes i trabecular architecture. As a result, within our experimental treatment groups it will be possible to predict th likely mode of failure and change in breaking strength associated with demonstrated changes in trabecula architecture and the structure of cortical bone.

KEY PERSONNEL ENGAGED ON PROJECT

NAME, DEGREE(S), SSN	POSITION TITLE AND ROLE IN PROJECT	DEPARTMENT AND ORGANIZATION
Cesar Levy B.S., M.S., Ph.D. 097-42-1895	Associate Professor Co-Principal Investigator	Mechanical Engineering Florida International Univ. Miami, FL 33199
Kenneth R. Gordon B.S., M.A., Ph.D. 563-62-8465	Associate Professor Co-Principal Investigator	Biological Sciences Florida International Univ. Miami, FL 33199

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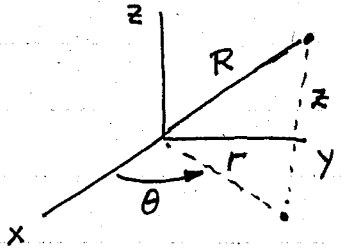
Example : an explosion at $r=0$ propagates radially at speed $\frac{dr}{dt} = a_\infty$

b) Cylindrical Symmetry (no z or θ variation)



$$\nabla^2(\;) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta^2} + \frac{\partial}{\partial z^2}$$

$$\rho' = (a_\infty^2 t^2 - r^2)^{-1/2}$$

$R = \sqrt{r^2+z^2}$

c) One dimensional flow $\quad \frac{\partial}{\partial y}, \frac{\partial}{\partial z} = 0 \quad \Rightarrow \frac{\partial^2 \rho'}{\partial t^2} = a_\infty^2 \frac{\partial^2 \rho'}{\partial x^2}$

u' = x component of $\underset{\sim}{q}'$ $\qquad \Longrightarrow \frac{\partial^2 u'}{\partial t^2} = a_\infty^2 \frac{\partial^2 u'}{\partial x^2}$

$$\rho' = f(x - a_\infty t) + g(x + a_\infty t)$$

To determine u' in all cases use solution for ρ' then

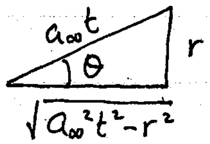
$$\frac{\partial \underset{\sim}{q}'}{\partial t} + \frac{a_\infty^2}{\rho_\infty}\nabla\rho' = 0$$

for example in cylindrical $\rightarrow \frac{\partial u'}{\partial t} + \frac{a_\infty^2}{\rho_\infty}\frac{\partial \rho'}{\partial r} = 0$

$$\frac{\partial \rho'}{\partial r} = -\frac{1}{2}(a_\infty^2 t^2 - r^2)^{-3/2}(-2r) = \frac{r}{(a_\infty^2 t^2 - r^2)^{3/2}}$$

$$\therefore \quad u' = -\frac{a_\infty^2}{\rho_\infty}\int \frac{r}{(a_\infty^2 t^2 - r^2)^{3/2}}\,dt + f(r)$$

using



$$\int \frac{r}{(a_\infty^2 t^2 - r^2)^{3/2}}\,dt = -\frac{a_\infty t}{a_\infty r}\frac{1}{(a_\infty^2 t^2 - r^2)^{1/2}}$$

$$\int -\frac{\csc\theta \tan^2\theta\, d\theta}{a_\infty r} = -\frac{1}{a_\infty r}\cdot\frac{1}{\cos\theta}$$

$$\therefore \quad u' = \frac{a_\infty^2}{\rho_\infty}\frac{t}{r}\frac{1}{(a_\infty^2 t^2 - r^2)^{1/2}} + f(r)$$

here u' = velocity in radial direction.

ated in φ. It should be noticed that all the preceding relations involve only the in[illegible]ant operator ∇, insofar as space differentiation is concerned, and hence [illegible] be readily expressed in terms of any convenient coordinate system.

We now suppose that the uniform flow is parallel to the x axis, and so write

$$\mathbf{U} = U\,\mathbf{i}, \tag{191}$$

in which case Equation (190) becomes

$$\delta = -\frac{\rho_0}{V_{S0}^2}\left(U\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial t}\right). \tag{192}$$

If Equations (189), (191), and (192) are introduced into (187), we obtain the equation

$$\nabla^2\varphi - \frac{1}{V_{S0}^2}\left(U\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^2\varphi = 0, \tag{193}$$

which must be satisfied by the potential function φ. For an *incompressible* fluid $V_S = \infty$, and the equation reduces to Laplace's equation, in accordance with the results of Section 6.15.

In the special case of *steady flow in two dimensions*, where the flow is parallel to the xy plane and the velocity and density do not vary with time, Equation (193) reduces to the form

$$(1 - M^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0, \tag{194}$$

where M is the so-called local *Mach number*,

$$M = \frac{U}{V_{S0}}, \tag{195}$$

and is the ratio of the speed of the uniform flow to the sonic velocity corresponding to this flow. A flow is said to be *subsonic* when $M < 1$ and *supersonic* when $M > 1$. It is important to notice that the coefficient of $\dfrac{\partial^2\varphi}{\partial x^2}$ in Equation (194) changes sign when the sonic velocity is passed.

In the special case of *nonsteady one-dimensional flow*, when the velocities in the y and z directions are assumed to be negligible, Equation (193) becomes

$$(U^2 - V_{S0}^2)\frac{\partial^2\varphi}{\partial x^2} + 2U\frac{\partial^2\varphi}{\partial x\,\partial t} + \frac{\partial^2\varphi}{\partial t^2} = 0. \tag{196}$$

If the above developments are still assumed to be satisfactorily approximate when *small deviations* are measured from a *state of rest*, so that $U = 0$, we may

$$\delta = \rho - \rho_0 = -\frac{\rho_0}{V_{S0}^2}\frac{\partial\varphi}{\partial t} \tag{197}$$

and

$$\nabla^2\varphi = \frac{1}{V_{S0}^2}\frac{\partial^2\varphi}{\partial t^2}, \tag{198}$$

where

$$\mathbf{V} = \nabla\varphi. \tag{199}$$

Here V_{S0}^2 is essentially an effective "mean value" of $\dfrac{dp}{d\rho}$. Thus we have also, approximately,

$$p - p_0 = V_{S0}^2(\rho - \rho_0),$$

so that the pressure change due to the disturbing flow is given approximately by

$$p - p_0 = -\rho_0\frac{\partial\varphi}{\partial t}. \tag{200}$$

To conclude this section, we obtain an explicit exact integral of Euler's equation, when external forces are absent, in the case of irrotational flow ($\nabla \times \mathbf{V} = \mathbf{0}$). From the identity

$$\nabla(\mathbf{V}\cdot\mathbf{V}) = 2\mathbf{V}\cdot\nabla\mathbf{V} + 2\mathbf{V}\times(\nabla\times\mathbf{V})$$

[see Equation (74e), Section 6.9], we here obtain the result

$$\mathbf{V}\cdot\nabla\mathbf{V} = \nabla(\tfrac{1}{2}V^2),$$

where $V = |\mathbf{V}| = \sqrt{\mathbf{V}\cdot\mathbf{V}}$. Thus Euler's equation (182) can be written in the form

$$\nabla\left(\frac{1}{2}V^2\right) + \frac{\partial\mathbf{V}}{\partial t} + \frac{\nabla p}{\rho} = \mathbf{0}. \tag{201}$$

If we now write

$$\mathbf{V} = \nabla\varphi \tag{202}$$

and *define* a function P by the relation

$$\nabla P = \frac{1}{\rho}\nabla p \quad\text{or}\quad P = \int\frac{dp}{\rho}, \tag{203}$$

Equation (201) takes the form

$$\nabla\left(\frac{1}{2}V^2\right) + \frac{\partial}{\partial t}\nabla\varphi + \nabla P = \mathbf{0}$$

or

$$\nabla\left(\frac{1}{2}V^2 + \frac{\partial\varphi}{\partial t} + P\right) = \mathbf{0}. \tag{204}$$

given particle in its motion. For this reason this derivative is frequently termed the "particle," "substantial," or "material" derivative of **V**, and is often indicated by the special notation $\dfrac{D\mathbf{V}}{Dt}$.

For fixed initial coordinates, **V** depends directly on x,y,z and t, and x, y, and z themselves depend upon the time t, in such a way that $\dfrac{dx}{dt} = V_x$, and so on. Hence we may write the substantial derivative of **V** in the expanded form (see Section 7.1),

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{V}}{\partial t}$$

$$= V_x\frac{\partial \mathbf{V}}{\partial x} + V_y\frac{\partial \mathbf{V}}{\partial y} + V_z\frac{\partial \mathbf{V}}{\partial z} + \frac{\partial \mathbf{V}}{\partial t}$$

or

$$\frac{d\mathbf{V}}{dt} = (\mathbf{V}\cdot\nabla)\mathbf{V} + \frac{\partial \mathbf{V}}{\partial t}. \tag{180}$$

In terms of differential operators we may write also

$$\frac{d}{dt} = \mathbf{V}\cdot\nabla + \frac{\partial}{\partial t} = V_x\frac{\partial}{\partial x} + V_y\frac{\partial}{\partial y} + V_z\frac{\partial}{\partial z} + \frac{\partial}{\partial t}. \tag{181}$$

Thus Equation (179) can be written in the form

$$\rho\left(\mathbf{V}\cdot\nabla\mathbf{V} + \frac{\partial \mathbf{V}}{\partial t}\right) + \nabla p = \mathbf{0} \tag{182}$$

and is equivalent to three scalar equations, the first of which is of the form

$$\rho\left(V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z} + \frac{\partial V_x}{\partial t}\right) + \frac{\partial p}{\partial x} = 0.$$

These equations of motion are known as *Euler's equations.*

In addition to the vector equation (182), the equation of conservation of mass [Equation (112)],

$$\rho\nabla\cdot\mathbf{V} = -\left(\mathbf{V}\cdot\nabla\rho + \frac{\partial\rho}{\partial t}\right) = -\frac{d\rho}{dt}, \tag{183}$$

must be satisfied. This equation is known as the *equation of continuity.*

Finally, to complete the system of basic equations, we may take the effect of compressibility into account by assuming a suitable relationship between pressure p and density ρ. (For an incompressible fluid, ρ is constant.) Assuming such a relationship, we may then write

$$\nabla p = \mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y} + \mathbf{k}\frac{\partial p}{\partial z} = \frac{dp}{d\rho}\left(\mathbf{i}\frac{\partial \rho}{\partial x} + \mathbf{j}\frac{\partial \rho}{\partial y} + \mathbf{k}\frac{\partial \rho}{\partial z}\right) = \frac{dp}{d\rho}\nabla\rho.$$

The quantity

$$\sqrt{\frac{dp}{d\rho}} = V_S \tag{184}$$

is known as the *local sonic velocity*, and is seen to vary with ρ from point to point in the fluid, and with time.

With these results Euler's equations in the vector form (182) become

$$\rho\mathbf{V}\cdot\nabla\mathbf{V} + \rho\frac{\partial\mathbf{V}}{\partial t} + V_S^2\,\nabla\rho = \mathbf{0}. \tag{185}$$

Equations (183) and (185) are sufficient to determine ρ and **V** if suitable boundary conditions and/or initial conditions are prescribed.

In many problems the flow is nearly uniform. That is, the vector **V** and the scalar functions ρ and p are nearly constant. In such cases, Equations (183) and (185) can be *linearized* by first writing

$$\mathbf{V} = \mathbf{U} + \mathbf{u}, \qquad \rho = \rho_0 + \delta, \qquad V_S = V_{S0} + v_S \tag{186}$$

where **U**, ρ_0, and V_{S0} are the *constant values* corresponding to the uniform flow and the quantities **u**, δ, and v_S are considered as small deviations. With this notation Equations (183) and (185) become

$$(\rho_0 + \delta)\nabla\cdot\mathbf{u} + (\mathbf{U} + \mathbf{u})\cdot\nabla\delta + \frac{\partial\delta}{\partial t} = 0,$$

$$(\rho_0 + \delta)(\mathbf{U} + \mathbf{u})\cdot\nabla\mathbf{u} + (\rho_0 + \delta)\frac{\partial\mathbf{u}}{\partial t} + (V_{S0} + v_S)^2\,\nabla\delta = \mathbf{0}.$$

If products of small deviations are assumed to be relatively negligible, these equations are reduced to the linear forms

$$\rho_0\nabla\cdot\mathbf{u} + \left(\mathbf{U}\cdot\nabla + \frac{\partial}{\partial t}\right)\delta = 0, \tag{187}$$

$$\left(\mathbf{U}\cdot\nabla + \frac{\partial}{\partial t}\right)\mathbf{u} + \frac{V_{S0}^2}{\rho_0}\,\nabla\delta = \mathbf{0}. \tag{188}$$

For an *irrotational* flow (see Section 6.15), the velocity **V**, and hence also **u**, can be expressed as the gradient of a scalar function. Hence, if we write

$$\mathbf{u} = \nabla\varphi, \tag{189}$$

Equation (188) becomes

$$\nabla\left[\delta + \frac{\rho_0}{V_{S0}^2}\left(\mathbf{U}\cdot\nabla + \frac{\partial}{\partial t}\right)\varphi\right] = \mathbf{0},$$

from which the density deviation δ is expressed in terms of the *velocity deviation potential* φ in the form

$$\delta = -\frac{\rho_0}{V_{S0}^2}\left(\mathbf{U}\cdot\nabla + \frac{\partial}{\partial t}\right)\varphi, \tag{190}$$

To find $f(r)$ remember we must also satisfy continuity

$$\frac{\partial \rho'}{\partial t} + \rho_\infty \nabla\cdot \underset{\sim}{q}' = 0 \quad \text{where for cylindrical coordinates} \quad \nabla\cdot\underset{\sim}{q}' = \frac{1}{r}\frac{\partial}{\partial r}(ru')$$

$$\frac{\partial \rho'}{\partial t} = \frac{-a_\infty^2 t}{(a_\infty^2 t^2 - r^2)^{3/2}} \; ; \; \nabla\cdot\underset{\sim}{q}' = \frac{1}{r}\frac{\partial}{\partial r}(ru') = \frac{a_\infty^2}{\rho_\infty}\frac{t}{(a_\infty^2 t^2 - r^2)^{3/2}} + \frac{1}{r}\frac{\partial}{\partial r}(rf)$$

$$\Rightarrow \quad \frac{\rho_\infty}{r}\frac{\partial}{\partial r}(rf) = 0 \quad \text{or} \quad \cancel{f(r) = c_1 \ln r + c_2} \quad f(r) = \frac{c_1}{r}$$

if for example at $t=0$ & $r=0$, $u' = 0 \Rightarrow c_1 = 0$ since origin is includes & $\cancel{c_2 = 0}$ $\therefore f(r) = 0$

FOR INFORMATION :

CART: $\nabla\cdot\underset{\sim}{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$; $\nabla\varphi = \underset{\sim}{i}\frac{\partial\varphi}{\partial x} + \underset{\sim}{j}\frac{\partial\varphi}{\partial y} + \underset{\sim}{k}\frac{\partial\varphi}{\partial z}$

$\underset{\sim}{q} = u\underset{\sim}{i} + v\underset{\sim}{j} + w\underset{\sim}{k}$

CYL: $\nabla\cdot\underset{\sim}{q} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial w}{\partial z}$; $\nabla\varphi = \underset{\sim}{i}_r\frac{\partial\varphi}{\partial r} + \underset{\sim}{i}_\theta\frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \underset{\sim}{k}\frac{\partial\varphi}{\partial z}$

$\underset{\sim}{q} = u_r\underset{\sim}{i}_r + u_\theta\underset{\sim}{i}_\theta + u_z\underset{\sim}{k}$

SPH: $\nabla\cdot\underset{\sim}{q} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\varphi}\frac{\partial}{\partial\varphi}(u_\varphi\sin\varphi) + \frac{1}{r\sin\varphi}\frac{\partial u_\theta}{\partial\theta}$; $\nabla f = \underset{\sim}{i}_r\frac{\partial f}{\partial r} + \underset{\sim}{i}_\varphi\frac{1}{r}\frac{\partial f}{\partial\varphi} + \underset{\sim}{i}_\theta\frac{1}{r\sin\varphi}\frac{\partial f}{\partial\theta}$

$\underset{\sim}{q} = u_r\underset{\sim}{i}_r + u_\varphi\underset{\sim}{i}_\varphi + u_\theta\underset{\sim}{i}_\theta$

CYL: $0 \le r < \infty \qquad 0 \le \theta \le 2\pi \qquad |z| < \infty$

SPH: $0 \le r < \infty \qquad 0 \le \theta \le 2\pi \qquad 0 \le \varphi \le \pi$

IN PROBLEMS WHERE FLOW IS NEARLY UNIFORM $\Rightarrow$ WHERE $\underset{\sim}{q}, p, \rho$ ARE ALMOST CONSTANT

assume $\underset{\sim}{q} = \underset{\sim}{q}_\infty + \underset{\sim}{q}'$; $p = p_\infty + p'$; $\rho = \rho_\infty + \rho'$ & homentropic flow

Continuity $\frac{\partial\rho}{\partial t} + \underset{\sim}{q}\cdot\nabla\rho + \rho\nabla\cdot\underset{\sim}{q} = 0$

$$\frac{\partial}{\partial t}(\rho_\infty + \rho') + (\underset{\sim}{q}_\infty + \underset{\sim}{q}')\cdot\nabla(\rho_\infty + \rho') + (\rho_\infty + \rho')\nabla\cdot(\underset{\sim}{q}_\infty^* + \underset{\sim}{q}') = 0$$

$$\Rightarrow \frac{\partial \rho'}{\partial t} + \underset{\sim}{q}_\infty \cdot \nabla \rho' + \overbrace{\cancel{\underset{\sim}{q}' \cdot \nabla \rho'} + \rho_\infty \nabla \cdot \underset{\sim}{q}' + \cancel{\rho' \nabla \cdot \underset{\sim}{q}'}}^{\text{2nd order}} = 0$$

note: $\frac{\partial \rho_\infty}{\partial t}$, $\nabla \cdot \underset{\sim}{q}_\infty$ & $\nabla \rho_\infty = 0$ since $\underset{\sim}{q}_\infty, p_\infty, \rho_\infty$ are constant

Also <u>Momentum</u>

$$\frac{\partial \underset{\sim}{q}}{\partial t} + (\underset{\sim}{q} \cdot \nabla)\underset{\sim}{q} + \frac{1}{\rho}\nabla p = 0$$

$$\frac{\partial}{\partial t}(\underset{\sim}{q}_\infty + \underset{\sim}{q}') + \left[(\underset{\sim}{q}_\infty + \underset{\sim}{q}') \cdot \nabla\right](\underset{\sim}{q}_\infty + \underset{\sim}{q}') + \frac{1}{\rho_\infty + \rho'} \nabla(p_\infty + p') = 0$$

$$\Rightarrow \frac{\partial \underset{\sim}{q}'}{\partial t} + (\underset{\sim}{q}_\infty \cdot \nabla)\underset{\sim}{q}' + (\underset{\sim}{q}' \cdot \nabla)\underset{\sim}{q}' + \frac{1}{\rho_\infty} \cdot \frac{1}{(1 + \rho'/\rho_\infty)} \nabla p' = 0$$

$$\frac{1}{\rho_\infty}\left[1 - \rho'/\rho_\infty + \left(\frac{\rho'}{\rho_\infty}\right)^2 + \dots\right] \nabla p' = 0$$

$$\frac{\partial \underset{\sim}{q}'}{\partial t} + (\underset{\sim}{q}_\infty \cdot \nabla)\underset{\sim}{q}' + \underbrace{\cancel{(\underset{\sim}{q}' \cdot \nabla)\underset{\sim}{q}'} + \frac{1}{\rho_\infty}\nabla p' - \cancel{\frac{1}{\rho_\infty^2}\rho' \nabla p'}}_{\text{2nd order}} + \dots = 0$$

$$\therefore \frac{\partial \underset{\sim}{q}'}{\partial t} + (\underset{\sim}{q}_\infty \cdot \nabla)\underset{\sim}{q}' + \frac{1}{\rho_\infty}\nabla p' = 0$$

now $\nabla p' = a_\infty^2 \nabla \rho'$ from before for homentropic flow $(p = k\rho^\gamma)$

$$\therefore \frac{\partial \underset{\sim}{q}'}{\partial t} + (\underset{\sim}{q}_\infty \cdot \nabla)\underset{\sim}{q}' + \frac{a_\infty^2}{\rho_\infty}\nabla \rho' = 0 \qquad \text{Linearized Momentum}$$

$$\frac{\partial \rho'}{\partial t} + (\underset{\sim}{q}_\infty \cdot \nabla)\rho' + \rho_\infty \nabla \cdot \underset{\sim}{q}' = 0 \qquad \text{Linearized Continuity}$$

1) Look at a specific case of steady state $\left(\frac{\partial}{\partial t} = 0\right)$
2) where original flow was only in x direction ie $\underset{\sim}{q}_\infty = u_\infty \underset{\sim}{i}$
3) disturbance $\underset{\sim}{q}' = u'\underset{\sim}{i} + v'\underset{\sim}{j}$

Thus Linearized Momentum is $$u_\infty \frac{\partial}{\partial x} u' + \frac{a_\infty^2}{\rho_\infty} \frac{\partial \rho'}{\partial x} = 0 \qquad x\text{-momentum}$$

$$u_\infty \frac{\partial}{\partial x} v' + \frac{a_\infty^2}{\rho_\infty} \frac{\partial}{\partial y} \rho' = 0 \qquad y\text{-momentum}$$

Linearized Continuity $$u_\infty \frac{\partial}{\partial x} \rho' + \rho_\infty \left[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right] = 0$$

x momentum + continuity eliminate $\frac{\partial \rho'}{\partial x}$:

$$\frac{u_\infty^2}{a_\infty^2} \rho_\infty \frac{\partial u'}{\partial x} - \rho_\infty \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) = 0$$

or $$\left(1 - M_\infty^2\right) \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

if flow is irrotational $\underset{\sim}{q}' = \nabla \varphi \quad \therefore \quad \frac{\partial \varphi}{\partial x} = u' \quad \frac{\partial \varphi}{\partial y} = v'$

$$\Rightarrow \quad \left(1 - M_\infty^2\right) \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

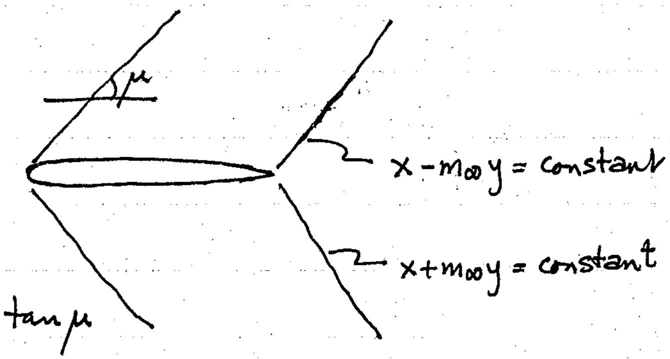
if $1 - M_\infty^2 = -m_\infty^2 < 0 \quad \Rightarrow \quad \frac{\partial^2 \varphi}{\partial y^2} = m_\infty^2 \frac{\partial^2 \varphi}{\partial x^2} \qquad M_\infty > 1$

$1 - M_\infty^2 = 0 \quad \Rightarrow \quad \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad M_\infty = 1$

$1 - M_\infty^2 = +\beta_\infty^2 > 0 \quad \Rightarrow \quad \beta_\infty^2 \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad M_\infty < 1$

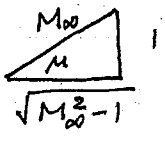
For supersonic flow

$$\frac{\partial^2 \varphi}{\partial y^2} = m_\infty^2 \frac{\partial^2 \varphi}{\partial x^2}$$



$$\therefore \quad \frac{dy}{dx} = \frac{1}{m_\infty} = \frac{1}{\sqrt{M_\infty^2 - 1}} = \tan \mu$$

$$\varphi = f(x - m_\infty y) + g(x + m_\infty y)$$



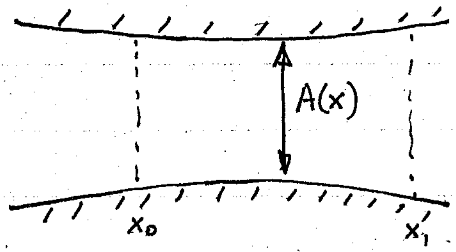
$$\sin \mu = \frac{1}{M_\infty}$$

- This forms a mach wave (weakest form of shock wave)
- Mach cone is formed by the characteristics for the equation

Let's now look at 2-D flows that act like <u>Quasi 1-D Flows</u>

assume flow is inviscid, steady, homentropic ($s = \text{const}$)

let's assume
- $T_o = \text{constant}$ (frictionless adiabatic flow)
- $p_o = \text{constant}$
- flow across ↑ channel doesn't change too much
- channel is long & slender
- entrance & exit flow do not dominate
- flow changes occur in in y-direction but much smaller than veloc in x direction
- flow properties change due to changes in area but due to isentropy T_o, p_o constant
- area variations are small $\Rightarrow A'(x)/A(x) \ll 1$
- viscous effects small



1.) Mass Conservation $\quad \frac{\partial}{\partial t}\int \cancel{\rho \, d\mathcal{V}} + \int \rho \underset{\sim}{q}\cdot\underset{\sim}{n}\, dS = 0$

Steady

$\underset{\sim}{q}\cdot\underset{\sim}{n} = 0$ at wall $\quad \Rightarrow \quad \int \rho \underset{\sim}{q}\cdot n\, dS = 0 \quad$ or $\; -\rho_{av}(x_o)\, \underset{av}{u}(x_o) A(x_o) + \rho_{av}(x_1)\, \underset{av}{u}(x_1) A(x_1) = 0$

or $\rho_{av}(x)\, u_{av}(x)\, A(x) = \text{constant}$

Momentum

2) $$\frac{\partial}{\partial t}\int \underset{\text{steady}}{\cancel{\rho \underset{\sim}{q}}}\, d\mho + \int \rho(\underset{\sim}{q}\cdot\underset{\sim}{n})\, \underset{\sim}{q}\, dS + \int p\,\underset{\sim}{n}\, dS = 0$$ if no height variations

x - component : $$-\rho(x_0)u^2(x_0)A(x_0) + \rho(x_1)u^2(x_1)A(x_1) - p(x_0)A(x_0) + p(x_1)A(x_1) - 2\int_{x_0}^{x_1} p(\bar{x})\,\frac{A'(\bar{x})}{2}\,d\bar{x} = 0$$

differentiate wrt x_1 :

$$(\rho u A)\,u' + \underbrace{u\frac{d}{dx_1}\cancel{(\rho u A)}}_{0 \text{ by continuity}} + p'A + \underbrace{pA'}_{\text{cancel}} - \underbrace{2\frac{\partial}{\partial x_1}\int_{x_0}^{x_1} p\,\frac{A'}{2}\,d\bar{x}}_{-2\,p\frac{A'}{2}\,\frac{\cancel{dx}}{\cancel{dx_1}}} = 0$$

$\therefore \quad (\rho u A)u' + p'A = 0$

or $\boxed{\rho u u' + p' = 0}$ Momentum

For shock less isentropic flow

Eqn of state: $$p' = \frac{dp}{dx} = \frac{d}{dx}\left(k\rho^{\gamma}\right) = k\gamma\rho^{\gamma-1}\frac{d\rho}{dx} = \frac{\gamma p}{\rho}\rho' = a^2\rho'$$

also momentum : $\rho u u' + p' = \rho u u' + a^2\rho' = 0$ or

$$\rho'/\rho = -\frac{uu'}{a^2} = -M^2\frac{u'}{u}$$

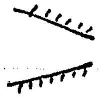
From continuity $\frac{d}{dx}(\rho u A) = 0 \implies \rho u A = \text{const}$

$$\rho'/\rho + \frac{u'}{u} + \frac{A'}{A} = 0$$

$$\therefore \quad (1-M^2)\frac{u'}{u} + \frac{A'}{A} = 0$$

Thus $$\frac{A'}{A} = (M^2-1)\frac{u'}{u} = -\frac{(M^2-1)}{M^2}\,\rho'/\rho = -\frac{(M^2-1)}{M^2}\frac{1}{\rho}\left[p'/a^2\right]$$

$$= -\frac{(M^2-1)}{\gamma M^2}\,p'/p$$

Look at  Converging Nozzle

$A' < 0$ $\quad M > 1 \quad u' < 0 \quad \rho' > 0 \quad p' > 0$

$\quad M < 1 \quad u' > 0 \quad \rho' < 0 \quad p' < 0$

$A' > 0$  diverging nozzle

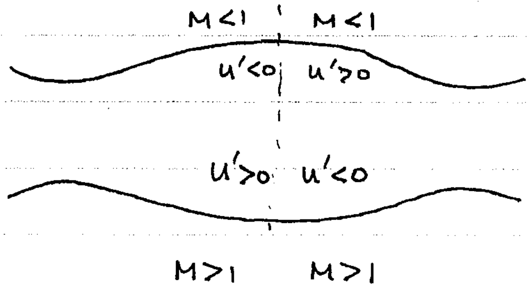
$M > 1 \quad u' > 0 \quad \rho' < 0 \quad p' < 0$

$M < 1 \quad u' < 0 \quad \rho' > 0 \quad p' > 0$

what about $M = 1 \Rightarrow A' = 0$ or $A = A^*$ CRITICAL CONDITIONS

$\Rightarrow u' = \infty$ (PHYSICALLY IMPOSSIBLE)

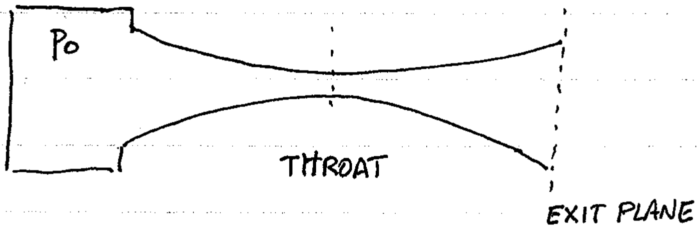
- FLOW BEHAVIOR OPPOSITE FOR SUPERSONIC & SUBSONIC FLOWS
- SONIC FLOW CAN ONLY BE ACHIEVED AT A <u>CONSTRICTION</u>



<u>NOT</u> AT THE LARGE SECTION OF CHANNEL

- IN CONVERGING-DIVERGING NOZZLES CONSTRICTION CAN MEAN THROAT

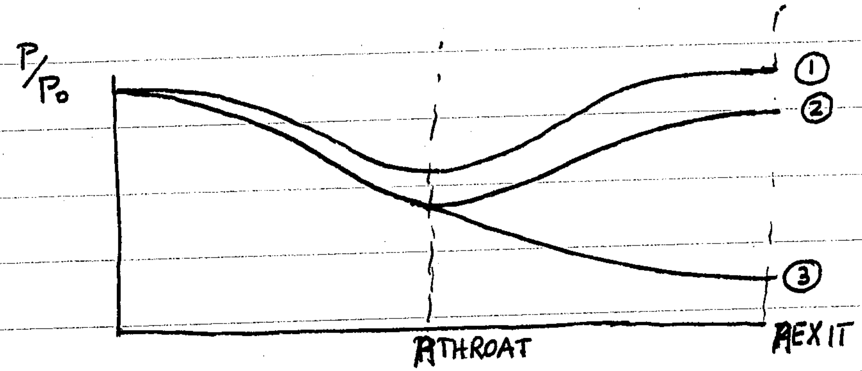
- LOOK AT CONVERGING-DIVERGING NOZZLE



IF FLOW SUBSONIC p ↓ UNTIL THROAT ; IF $M < 1$ AT THROAT IN DIVERGING SECTION $M < 1$ & p ↑ UNTIL EXIT PLANE

IF FLOW IS SUCH THAT $p\downarrow$ & $M=1$ AT THROAT; THEN

FLOW CAN EXIT SUBSONIC OR SUPERSONIC DEPENDING ON P_{EXIT}



IF $M_{THROAT} < 1$ $\quad P_E/P_o > P_E/P_o$ FOR $MACH_{THROAT} = 1$ $(M^* = 1)$ ①

IF $M_{THROAT} = 1$ 2 CHOICES ② or ③ : WHICH ONE DEPENDS ON P_E/P_o

MASS FLOW EQUATION – $\dot{m}$

$$\rho u A = \rho^* u^* A^* \qquad * \text{ CONDITIONS ARE A } M=1$$

$$\frac{A}{A^*} = \frac{\rho^* u^*}{\rho u} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{a^*}{a}\cdot\frac{a}{u} \qquad \text{since } @\ M=1\ \ u^* = a^*$$

WHAT IS ρ^*/ρ_0? $\quad \underline{\dfrac{\rho_0}{\rho} = \left(1 + \dfrac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}}$

when $M=1$ $\rho = \rho^*$ $\quad \therefore \underline{\rho_0/\rho_*} = \left(1 + \frac{\gamma-1}{2}\right)^{\frac{1}{\gamma-1}} = \underline{\left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}}}$

$$a = \sqrt{\gamma R T} \quad \& \quad a^* = \sqrt{\gamma R T^*} \qquad \therefore \frac{a^*}{a} = \sqrt{\frac{T^*}{T}} \quad \& \quad \underline{\frac{a}{u} = \frac{1}{M}}$$

$$\therefore \frac{A}{A^*} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \cdot \left[1 + \frac{\gamma-1}{2}M^2\right]^{\frac{1}{\gamma-1}} \sqrt{\frac{T^*}{T}}\ \frac{1}{M}$$

$$\sqrt{\frac{T^*}{T}} = \sqrt{\frac{T^*}{T_0}}\cdot\sqrt{\frac{T_0}{T}} \quad ; \quad \frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2 \quad \& \quad \frac{T_0}{T^*} = \frac{\gamma+1}{2}$$

$$\therefore \frac{A}{A^*} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\left[1 + \frac{\gamma-1}{2}M^2\right]^{\frac{1}{\gamma-1}}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}\left[1 + \frac{\gamma-1}{2}M^2\right]^{\frac{1}{2}}\cdot\frac{1}{M}$$

$$= \left[\left(\frac{2}{\gamma+1}\right)\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\cdot\frac{1}{M}$$

Now since $\dot{m} = \rho u A = \rho^* u^* A^* = \underset{\downarrow}{\rho^* u^*} \frac{A^*}{A} \cdot A$

$$= \rho^* a^* \cdot \frac{A^*}{A} \cdot A \;;\; \rho^* = \frac{p^*}{RT^*} \;\&\; a^* = \sqrt{\gamma R T^*}$$

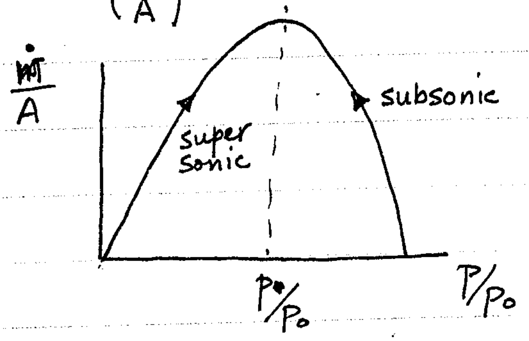
$$\frac{\dot{m}}{A} = \frac{p^*}{\sqrt{T^*}} \cdot \sqrt{\frac{\gamma}{R}} \frac{A^*}{A} = \frac{p_0}{\sqrt{T_0}} \cdot \frac{p^*}{p_0} \cdot \sqrt{\frac{T_0}{T^*}} \frac{A^*}{A} \cdot \sqrt{\frac{\gamma}{R}}$$

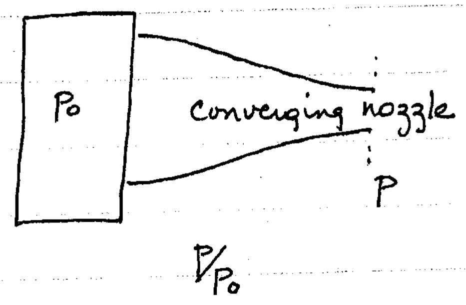
$$= \frac{p_0}{\sqrt{T_0}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}} \sqrt{\frac{\gamma}{R}} \frac{A^*}{A}$$

$$= \frac{p_0}{\sqrt{T_0}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \sqrt{\frac{\gamma}{R}} \frac{A^*}{A}$$

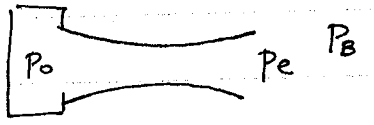
$\therefore \frac{\dot{m}}{A}$ = function of p_0, T_0, γ & Mach number

The max $\left(\frac{\dot{m}}{A}\right)$ occurs when $M=1$; for [redacted]





How do we do problems



1) Assume $A_{THROAT} = A^*$
2) Given A_{exit}
3) Find $A_{exit}/A^* \Longrightarrow M|_{exit}$ (2 values ① $M>1$ ② $M<1$)
4) find p_e/p_0 for ① & ②
5) match p_e/p_0 TO p_B/p_0
6) IF $p_B/p_0 \neq p_e/p_0 \Longrightarrow$ SHOCK INSIDE NOZZLE
SHOCK AT EXIT
NOT SUPERSONIC AT THROAT

for air $\quad T^*/T_0 = \frac{2}{\gamma+1} = .8333 \quad$ } $\gamma = 1.4$

$p^*/p_0 = .5283$

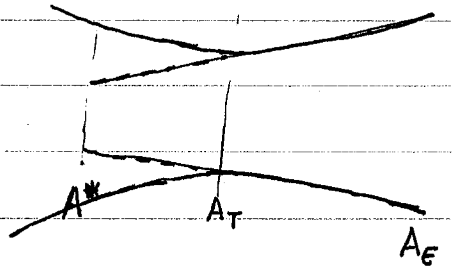
$\rho^*/\rho_0 = .6339$

also $\frac{\dot{m}}{A}\frac{\sqrt{T_0}}{p_0}$ = function of γ, R only = .04042 in T in °K p in N/m²

.532 if T in °R p in psi

FOR CASE WHERE NOT SUPERSONIC AT THROAT & $p_B/p_0 > p_E/p_0$

1) set $p_B/p_0 = p_E/p_0$ & get M_E subsonic & A_E/A_*



2) $A^* = \frac{A^*}{A_E} \cdot A_E$ to get A^* (non existent physically

3) A_T/A^* is found & get $M_{subsonic}$
THIS is M at THROAT

EXAMPLE 3.2 ON page 97

$\dot{m} = 1$ kg/s $\quad T_0 = 310°K \quad p_0 = 810$ kPA $\quad p_E = 101.3$ kPA

(a) IF FLOW IS ISENTROPIC THROUGHOUT SO THAT $p_E = p_B$

$p_E/p_0 = .125 \quad p^*/p_0 = .5283 \quad \therefore$ since $p_E/p_0 < p^*/p_0$ SUPERSONIC EXIT FROM CONVERGING-DIVERGING NOZZLE

AND $p_E/p_0 = .125 \Rightarrow M_E \sim 2.014$ & $A_E/A_* = 1.7074 \quad T_E/T_0 = .5154$

- now $A_* = A_{THROAT}$
- $\frac{\dot{m}}{A_*}\frac{\sqrt{T_0}}{p_0} = .04042 = \frac{\dot{m}\sqrt{T_0}}{p_0}$ [illegible] $\cdot \frac{1}{A_*} = 1\frac{kg}{s}\frac{\sqrt{310}}{810000} \cdot$ [illegible] $\frac{1}{A_*}$

$1/A_* = .04042\left(\frac{810,000}{\sqrt{310}}\right)$ [illegible] $\Rightarrow A_* = 5.378 \times 10^{-4}\,m^2 = 5.378\,cm^2$

Now to find $V_E \Rightarrow$ find T_E then $a_E = \sqrt{\gamma R T_E}$ & $V_E = M_E a_E$

$T_E/T_0 = .55154$ $T_E = 310 \times .55154 = 170.98$ °K

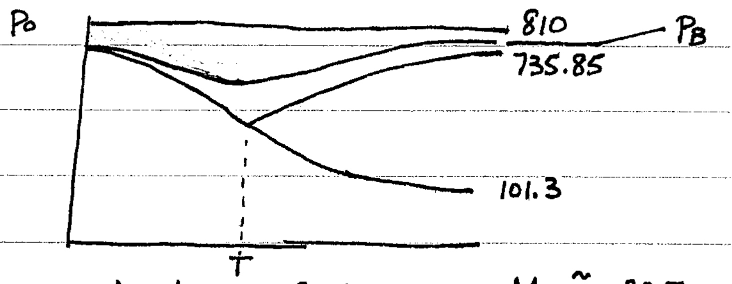
$a_E = \sqrt{1.4(287.04)(170.98)} = 262.98$ m/s

$V_E = 2.014\,(262.98) = 529.64$ m/s

since $A_* = 5.378\ \text{cm}^2$ $A_E = A_E/A_* \cdot A_* = 1.7074(5.378) = 9.183\ \text{cm}^2$

(b) for an isentropic subsonic exit $A_E/A^* = 1.7074 \Rightarrow M_E = .3675$ $p_E/p_0 = .91092$
(pg 519) & $p_E = p_E/p_0 \cdot p_0 = 737.85$ kPA

(c) WHAT IF $p_B = 750$ kPA



$\therefore\ p_B/p_0 = .9259 \Rightarrow M_E \cong .335$ $\frac{A_E}{A_*} = 1.8468$

$\Rightarrow A^* = \frac{A_*}{A_E} \cdot A_E = \frac{9.183}{1.8468} = 4.9724\ \text{cm}^2$ (smaller than A_T: not physical)

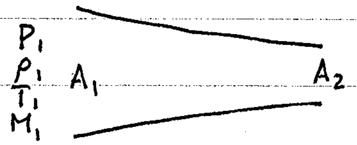
but M=1 at A*

now at A_T : $A_T/A^* = \frac{5.378}{4.9724} = 1.0815 \Rightarrow M_T \cong .7187$ &

$p_T/p_0 \cong .7089$ $p_T = .7089\,(810) = 574.21$ kPA

what about mass flow rate

Assume we have a converging nozzle w/ section p, ρ, T, M @ 1



for $\frac{A_2}{A_1} = \frac{A_2/A_*}{A_1/A_*}$ we can get

if $A_2 \geq A_*$ we can get p_2, ρ_2, M_2, T_2

& also $\frac{\dot{m}}{A}$

Working Tables. For accurate or extensive calculations, Table B.2 lists the various isentropic flow functions for $k = 1.4$, with Mach Number as independent argument.

Illustrative Example. The use of the compressible flow functions is best explained by an example.

PROBLEM. A supersonic wind-tunnel nozzle is to be designed for $M = 2$, with a throat section 1 sq ft in area. The supply pressure and temperature at the nozzle inlet, where the velocity is negligible, are 10 psia and 100°F, respectively. The preliminary design is to be based on the assumptions that the flow is isentropic, with $k = 1.4$, and that the flow is one-dimensional at the throat and test section. It is desired to compute the mass flow, the test-section area, and the fluid properties at the throat and test section.

SOLUTION. Table B.2 is to be used. First we work out the reference stagnation properties. We are given $p_0 = 10$ psia and $T_0 = 459.7 + 100 = 559.7$°R. From these we compute

$$\rho_0 = \frac{p_0}{RT_0} = \frac{10(144)}{53.3(559.7)} = 0.0483 \text{ lbm/ft}^3$$

$$c_0 = 49.1\sqrt{T_0} = 49.1\sqrt{559.7} = 1162 \text{ ft/sec}$$

Next, we find the properties at the throat by selecting values from Table B.2 at $M = 1$:

$$p^*/p_0 = 0.528; \qquad \therefore\ p^* = 10(0.528) = 5.28 \text{ psia}$$

$$T^*/T_0 = 0.833; \qquad \therefore\ T^* = 0.833(559.7) = 466°\text{R}$$

$$\rho^*/\rho_0 = 0.634; \qquad \therefore\ \rho^* = 0.634(0.0483) = 0.0306 \text{ lbm/ft}^3$$

$$c^*/c_0 = \sqrt{T^*/T_0} = 0.913; \quad \therefore\ c^* = V^* = 0.913(1162) = 1060 \text{ ft/sec}$$

Entering Table B.2 at $M = 2$, we now calculate properties in the test section:

$$M^* = V/c^* = 1.6330; \quad \therefore\ V = 1.633(1060) = 1731 \text{ ft/sec}$$

$$p/p_0 = 0.1278; \qquad \therefore\ p = 0.1278(10) = 1.278 \text{ psia}$$

$$\rho/\rho_0 = 0.2300; \qquad \therefore\ \rho = 0.2300(0.0483) = 0.01110 \text{ lbm/ft}^3$$

$$T/T_0 = 0.556; \qquad \therefore\ T = 0.556(559.7) = 311°\text{R}$$

$$A/A^* = 1.6875; \qquad \therefore\ A = 1.6875(1) = 1.6875 \text{ ft}^2$$

Finally, we compute the mass flow from Eq. 4.18:

$$w = \frac{0.532 p_0 A^*}{\sqrt{T_0}} = \frac{0.532(10)(144)(1)}{\sqrt{559.7}} = 32.4 \text{ lbm/sec}$$

Alternatively, the mass flow may be computed from the continuity equation at the throat or test section. For example,

$$w = \rho^* A^* V^* = 0.0306(1)(1060) = 32.4 \text{ lbm/sec}$$

4.6. Choking in Isentropic Flow

The fact that the curve of mass flow per unit area has a maximum is connected with an interesting and important effect called choking.

Let us consider two sections of a stream tube having a ratio of areas A_2/A_1, and let us specify all flow properties at section 1, such as p_1, T_1, M_1, etc. From the tables or graphs we can then solve for the properties at section 2, except as discussed later. For example, corresponding to M_1 we may find in the tables $(p/p_0)_1$, $(T/T_0)_1$, and $(A/A^*)_1$. Then, since A^* is constant during the process, we may write

$$\frac{A_2}{A_1} = \frac{(A/A^*)_2}{(A/A^*)_1}$$

and so we may compute $(A/A^*)_2$. Returning to the tables, we then obtain at this value of $(A/A^*)_2$ the corresponding values of M_2, $(p/p_0)_2$, $(T/T_0)_2$, etc. Since p_0 and T_0 are constant, this allows us to compute p_2 and T_2 from

$$\frac{p_2}{p_1} = \frac{(p/p_0)_2}{(p/p_0)_1}; \quad \frac{T_2}{T_1} = \frac{(T/T_0)_2}{(T/T_0)_1}$$

Now, let us consider a passage with a given area ratio A_2/A_1 and compute in the manner outlined above the values of M_2 corresponding to several values of M_1. The results may then be plotted as in Fig. 4.9. Examination of this chart indicates two peculiarities:

(i) For a given initial Mach Number M_1 and a given area ratio A_2/A_1, there are either two solutions for the final state M_2 or none at all. When there are two solutions, one of the two is subsonic and the other is supersonic. Which one of the two occurs depends in part on whether a throat intervenes between sections 1 and 2, for we have demonstrated that in order to go from supersonic speed to subsonic speed, or vice versa, it is necessary for the flow to pass through a throat at $M = 1$. For example, if M_1 is subsonic and the passage is converging, then M_2 must also be subsonic. On the other hand, if M_1 is subsonic and the passage is converging-diverging (i.e., has a minimum area between sections 1 and 2) the flow at section 2

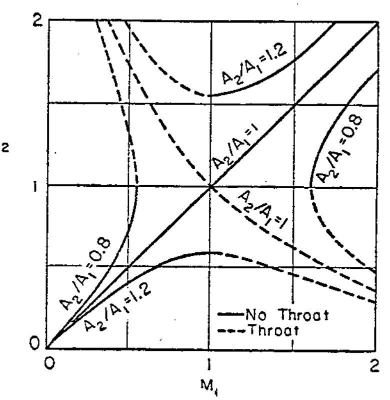


FIG. 4.9. Typical curves of M_2 versus M_1 for fixed values of area ratio A_2/A_1.

may be either subsonic, as in a conventional venturi, or supersonic, as in a supersonic nozzle; which of these two situations prevails depends on the pressures imposed at the inlet and exit of the passage, as discussed more fully in Art. 4.7.

(ii) When, for selected values of M_1 and A_2/A_1, there is no solution in Fig. 4.9, the solution is imaginary in the mathematical sense. This occurs only when A_2 is smaller than A_1. Physically, this result signifies that for given conditions at section 1, there is a maximum contraction which is possible; this maximum contraction corresponds to sonic velocity at section 2. Or, put quite simply, if conditions at section 1 are specified, the mass flow is accordingly determined, and there is then a minimum cross-sectional area required to pass this flow. This phenomenon is called choking, and may be summarized by saying that for a given area reduction, there is in subsonic flow a maximum initial Mach Number which can be maintained steadily; and in supersonic flow a minimum initial Mach Number which can be maintained steadily. At either of these limiting conditions, the flow at section 2 is sonic, and is said to be choked.

choked p_2 71 p_0 $M_2 = 1$

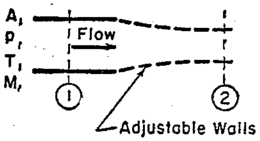


FIG. 4.10. Illustrates choking of flow.

To illustrate further the phenomenon of choking, let us suppose that at a section 1 in a duct there is a subsonic flow with certain values of M_1, p_1, T_1, and A_1. These parameters fix the flow rate, w. Let us imagine further that, at a section 2 downstream, the walls are flexible; thus the area A_2 is adjustable, as shown in Fig. 4.10. If A_2 is equal to A_1, all conditions at section 2 will be identical with the corresponding conditions at section 1. A slight reduction in A_2 will produce certain effects at section 2, which, according to Fig. 4.8, will comprise an increase in M_2, a decrease in p_2, and a decrease in T_2. This slight reduction in A_2 without a change in conditions at section 1 must, therefore, be accompanied by a reduction in the back pressure, p_2, according to the requirements of Fig. 4.8. Further reductions in A_2 may be made in the same way until the value of M_2 reaches unity.

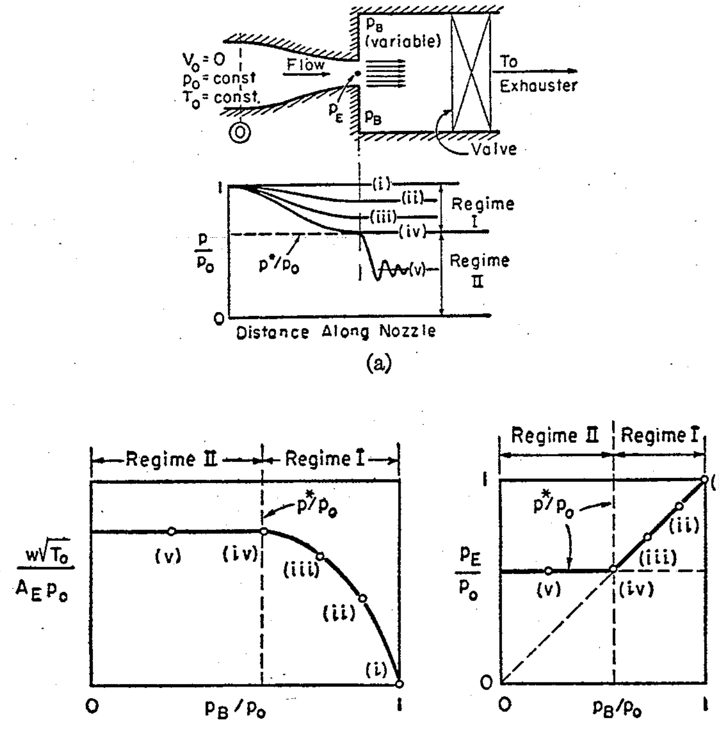
After this point has been reached, there is no way of reducing the area further without a simultaneous change in the steady-state conditions at section 1. If, for example, the pressure and temperature at section 1 are maintained constant, a reduction in A_2/A_1 beyond its limiting value will, after a transient period of wave propagation, result in a reduced steady-state M_1, which in turn means that the flow rate will be decreased. The maximum possible value of M_1 (which will correspond to the maximum possible flow rate) is obtained when $M_2 \doteq 1$. To obtain this limiting flow, the back pressure, p_2, must of course be adjusted accordingly. Fig. 4.8 shows that any area reduction whatsoever may be made if the initial Mach Number is sufficiently low or sufficiently high.

reduce ($\frac{A_2}{A_1} = \frac{A^*}{A_1}$ reduction M_1 to adjust (but $M_2 = 1$) and $\left(\frac{A_2}{A_1}\right)_{max}$ to decrease

4.7. Operation of Nozzles Under Varying Pressure Ratios

The phenomenon of choking discussed above may be manifested in several different ways. To illustrate still another aspect of choking, let us discuss the practical problem of nozzles operating under varying pressure ratios.

Converging Nozzles. Suppose, for the sake of concreteness, that a converging passage (Fig. 4.11a) with a large entrance area at section 0 discharges into a region where the back pressure, p_B, is controllable by means of a valve. The values of p_0 and T_0 will be maintained constant, and the experiment will involve variations in p_B. If p_E denotes the pressure in the exit plane of the nozzle, we inquire as to the effects of variations in back pressure on the distribution of pressure in the passage, on the flow rate, and on the exit-plane pressure. These effects are portrayed graphically in Figs. 4.11a, b, and c, respectively.



(a)

(b)

(c)

FIG. 4.11. Operation of converging nozzle at various back pressures.

To begin with, suppose that $p_B/p_0 = 1$, shown as condition (i) in Fig. 4.11a. The pressure is then constant through the nozzle, and there is no flow.

If p_B is now reduced to a value slightly less than p_0, as shown by condition (ii), there will be flow with a constantly decreasing pressure through the nozzle. Because the exit flow is subsonic, the exit-plane pressure p_E must be the same as the back pressure p_B, except for minor secondary circulation effects in the exhaust space. That this must be so can be seen by supposing for a moment that p_E is substantially larger than p_B. If this were so the stream would expand laterally upon leaving the nozzle; however, such an area increase at subsonic speeds causes the stream pressure to rise even further. Since the back pressure is, by definition, the pressure which the stream ultimately achieves in the exhaust space, it follows that p_E cannot be larger than p_B. A similar argument leads to the conclusion that p_E cannot be substantially less than p_B.

A further reduction in p_B to condition (iii) acts to increase the flow rate and to change the pressure distribution, but there is no qualitative change in performance.

Similar considerations apply until condition (iv) is reached, at which point p_B/p_0 equals the critical pressure ratio and the value of M_E equals unity.

Further reductions in p_B/p_0, say to condition (v), cannot produce further changes in conditions within the nozzle, for the value of p_E/p_0 cannot be made less than the critical pressure ratio unless there is a throat upstream of the exit section (it is assumed here that the stream fills the passage). Consequently, at condition (v), the pressure distribution within the nozzle, the value of p_E/p_0, and the flow rate, are all identical with the corresponding quantities for condition (iv). The pressure distribution outside the nozzle cannot be predicted on one-dimensional grounds, and for the present is shown in Fig. 4.11a as a wavy curve.

To summarize the preceding discussion, the two different types of flow will be denoted as regime I and regime II. These regimes may be compared as follows:

Regime I	Regime II
$p_B/p_0 > p^*/p_0$	$p_B/p_0 < p^*/p_0$
$p_E/p_0 \cong p_B/p_0$	$p_E/p_0 = p^*/p_0$
$M_E < 1$	$M_E \doteq 1$
$\dfrac{w\sqrt{T_0}}{A_E p_0}$ dependent on p_B/p_0	$\dfrac{w\sqrt{T_0}}{A_E p_0}$ independent of p_B/p_0

In regime I the values of p_E and p_B are virtually identical. Hence, except for a constant multiplier, the flow curve in regime I of Fig. 4.11b is identical with the curve of w/A in the subsonic part of Fig. 4.3.

A simple converging nozzle of the type discussed often serves as a flow nozzle. It is particularly useful when p_B/p_0 is less than the critical pressure ratio, for then the flow rate is given by Eq. 4.18, and measurements of only p_0, T_0, and A_E are necessary for computation of the flow rate.

For accurate measurements, the effects of boundary layer and of departures from one-dimensionality require that the nozzle be calibrated. Discharge coefficients for rounded-entrance nozzles are usually of the order 0.98 to 0.99, except for very low Reynolds Numbers where they may be considerably less.

The converging nozzle may occasionally be used to advantage as a simple flow regulator because of the fact that the flow rate is independent of back pressure when the latter is less than about half the supply pressure.

Converging-Diverging Nozzles. Consider an experiment similar to the one described above, except that a converging-diverging nozzle is to be used (Fig. 4.12).

With p_B less than p_0 by only a small amount, the flow is similar to that through a venturi passage, and it may be treated approximately as incompressible. The corresponding pressure distribution is shown by curves (i) and (ii) in Fig. 4.12.

When p_B/p_0 is reduced to the value corresponding to curve (iii), the Mach Number at the throat is unity, and no further reductions in p_T/p_0 are possible if the stream fills the passage.

We consider next the operation when the flow is entirely supersonic, corresponding to curve (iv). The value of p_B/p_0 for curve (iv) corresponds exactly to the area ratio of the nozzle, A_E/A_T, as given by the isentropic tables (in this case $A_T = A^*$, since $M_T = 1$). This is often called the "design" pressure ratio of the nozzle.

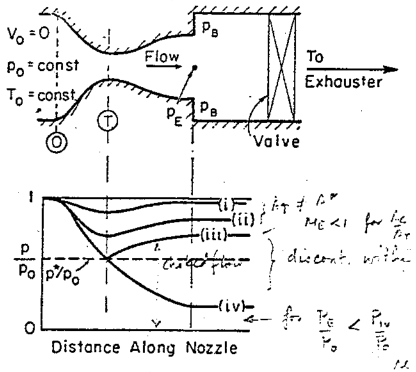


FIG. 4.12. Operation of converging-diverging nozzle at various back pressures.

No flow pattern fulfilling the conditions of isentropic and one-dimensional flow can be found which will correspond to values of p_B/p_0 between those of curves (iii) and (iv) in Fig. 4.12. One method of finding solutions for these boundary conditions is to suppose that irreversible

nches diameter at a rate of 2.20
At one section of the duct the

:agnation pressure at this section.

'lbm°R) is supplied to a converg-
l 540°F. The nozzle discharges
frictionless adiabatic flow, and a

ne, in square feet

:t gas, the pressure, temperature,
)ect to the corresponding critical

of the isentropic flow relations
:ompared with unity.

ure of 5.20 psig and a difference
19.42 inches of mercury. The
;nation temperature of the air
ssuming the air incompressible,

pressure of 20 psia, has a Mach
The cross-sectional area of the

e stream in degrees F.

n in area which could be intro-
.m?

', find the velocity and pressure
io heat transfer.

of one square inch is supplied
erature of 100 psia and

'ersus back pressure, assuming

4.12. A rocket combustion chamber is supplied with 24 lb/sec of hydrogen and 76 lb/sec of oxygen. Before entering the nozzle all the oxygen is consumed, the pressure is 23 atmospheres, and the temperature is 4960°F. Neglecting dissociation and friction, find the throat area of the nozzle required. Assume $k = 1.25$.

inviscid isentropic adiab.

4.13. At a certain point in a stream tube, air flows with a velocity of 500 ft/sec and has a pressure and temperature of 10 psia and 40°F, respectively.

(a) Calculate the following quantities at a point further downstream in the stream tube where the cross-sectional area is 15 per cent smaller than at the upstream section: the stagnation pressure and temperature, the stream pressure and temperature, the velocity, the Mach Number, and the value of M*.

(b) Compute the maximum possible reduction in area of the stream tube. For the section with the minimum area, compute the quantities listed in part (a).

4.14. When a body is placed in a stream which at infinite distance upstream is in uniform flow with free-stream conditions V_∞, p_∞, M_∞, etc., the local pressures in the neighborhood of the body are usually reported in terms of the dimensionless pressure coefficient, C_p:

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{P/p_\infty - 1}{\frac{1}{2}\rho_\infty \frac{V_\infty^2}{p_\infty}\frac{K}{K}} = \frac{P/p_\infty - 1}{\frac{1}{2} K M_\infty^2} = \frac{\left(T/T_\infty\right)^{\frac{K}{K-1}} - 1}{\frac{1}{2} K M_\infty^2}$$

(a) Show that the value of the pressure coefficient corresponding to the appearance of the critical velocity is given by

$$C_p{}^* = \frac{\left[\dfrac{2 + (k-1)M_\infty{}^2}{k+1}\right]^{\frac{k}{k-1}} - 1}{\dfrac{k}{2}M_\infty{}^2}$$

now for $\frac{T}{T_\infty} = \frac{T^*}{T_\infty} = \frac{T^*}{T_0} \cdot \frac{T_0}{T_\infty}$

$\frac{T_0}{T_\infty} = 1 + \frac{k-1}{2} M_\infty^2$, $\frac{T^*}{T_0} = \frac{2}{k+1}$

hence result

(b) Plot log ($-C_p{}^*$) versus log M_∞ for $k = 1.4$ and for values of M_∞ between 0.1 and 1.0.

(c) Suppose that an airplane is flying at sea level with a velocity of 500 mph. What is the maximum pressure coefficient which may be attained on the wings without the speed becoming anywhere supersonic?

4.15. Pressure coefficients, lift coefficients, drag coefficients, etc., of airfoils which are in a free stream with conditions p_∞, M_∞, etc., are usually expressed in terms of the dynamic head of the free stream. Thus

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty{}^2}; \quad C_L \equiv \frac{L/A}{\frac{1}{2}\rho_\infty V_\infty{}^2}; \quad \text{etc.}$$

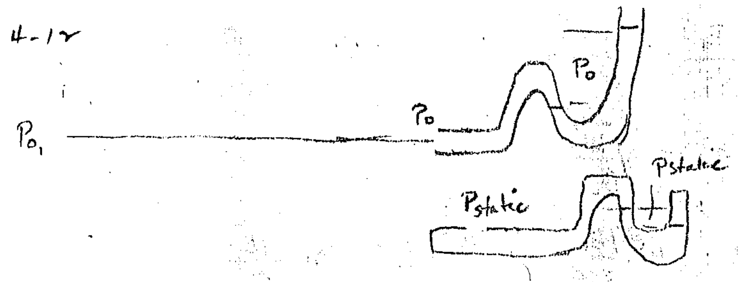
Alternate definitions for compressible flow, not usually employed, are as follows:

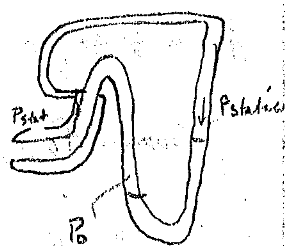
$$C_p' \equiv \frac{p - p_\infty}{p_0 - p_\infty}; \quad C_L' \equiv \frac{L/A}{p_0 - p_\infty}; \quad \text{etc.}$$

where p_0 is the isentropic stagnation pressure corresponding to p_∞ and M_∞.

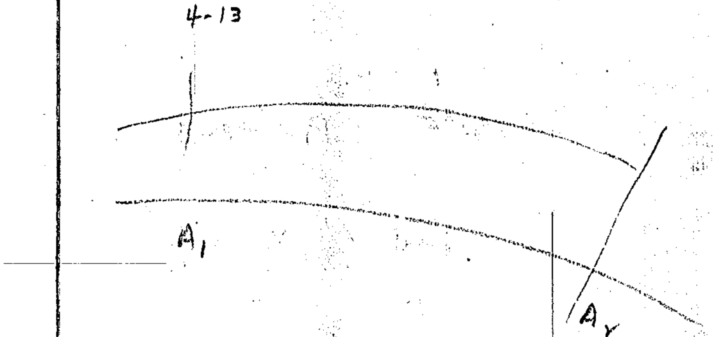
Derive an expression for C_p'/C_p in terms of M_∞ and k. Plot C_p'/C_p versus M_∞ for $k = 1.4$ and for values of M_∞ between 0 and ∞.

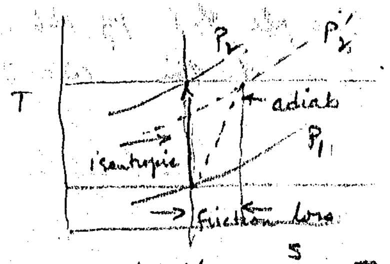
4-12





4-13





$\frac{A_2}{A_1} = .85$

adiabatic $\Longleftrightarrow$ $T_{o_1} = T_{o_2}$

Isentropic $\Longleftrightarrow$ $P_{o_1} = P_{o_2}$

not isentropic $\Longleftrightarrow$ $P_{o_2}' < P_{o_2}$

$S_2' > S_2$

$V_1 = 500$ ft/sec

$P_1 = 10$ psia

$T_1 = 500°R$

find P_{o_2}, T_{o_2}, P_2, T_2, V_2, M_2, M_2^*

since we are adiab & isentropic

$$V_1 \, \& (T_1 \rightarrow C_1) \rightarrow M_1 \xrightarrow{\text{Table B2}} \frac{P_{o_1}}{P_1} \rightarrow P_{o_1} = P_{o_2}$$

$$\frac{T_{o_1}}{T_1} \rightarrow T_{o_1} = T_{o_2}$$

$$M_1 = \frac{V_1}{C_1} = \frac{500}{49.02\sqrt{500}} = .452$$

Table B2 given $\rightarrow \frac{P_1}{P_{o_1}} = .869 \Rightarrow P_{o_1} = 11.5 \text{psia} = P_{o_2}$

$\frac{T_1}{T_{o_1}} = .96 \Rightarrow T_{o_1} = 520.8°R$

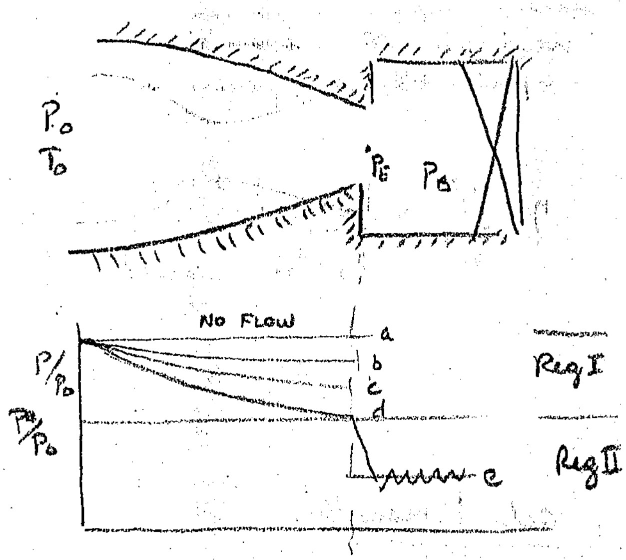
from $\frac{A_2}{A_1} = \frac{A_2/A^*}{A_1/A^*} = .85$ $\quad A_1/A^*$ (for $M = .452$) $= 1.436$

$$\therefore \; A_2/A^* = \frac{A_2}{A_1} \cdot \frac{A_1}{A^*} = .85(1.436) = 1.22$$

for $\frac{A_2}{A^*} = 1.22$ go to B2 & get M_2 (for $\frac{A_2}{A^*} = 1.22$) $\Big\langle$ $M_2 = .575$; $M_2 = 1.56$

using $M_2 \xrightarrow{B2} \frac{P_2}{P_{o_2}} \rightarrow P_2$; $\frac{T_2}{T_{o_2}} \rightarrow T_2 \rightarrow C_2 \xrightarrow{\& M_2} V_2$ & M_2^* is found from M_2

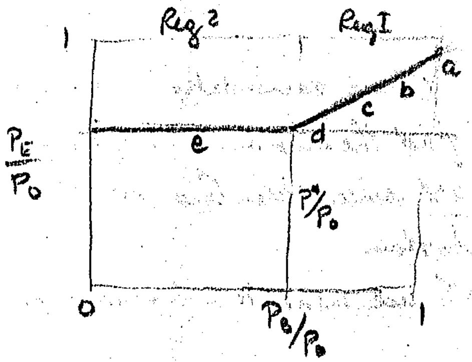
Part 2 $A_{min} = A^*$ $\frac{A_1}{A^*} = 1.436$ $\Rightarrow A^* = \frac{A_1}{1.436}$ or 30.4% reduction in area $= \frac{1-1.43}{1.436}$

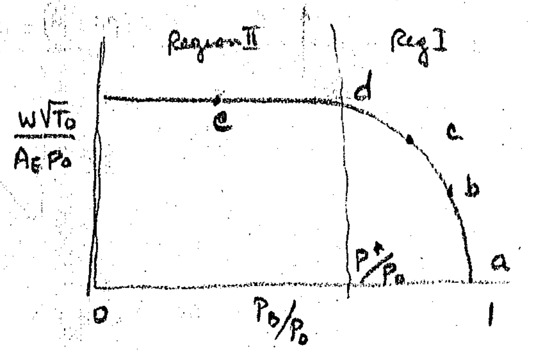


Region I
for $M \leq 1$ $P_B = P_e$, $P_B > P^*$, $M_E \leq 1$, $\frac{\omega}{A} \propto \frac{1}{P_B}$
if $P_B \neq P_E$ look at stream tube

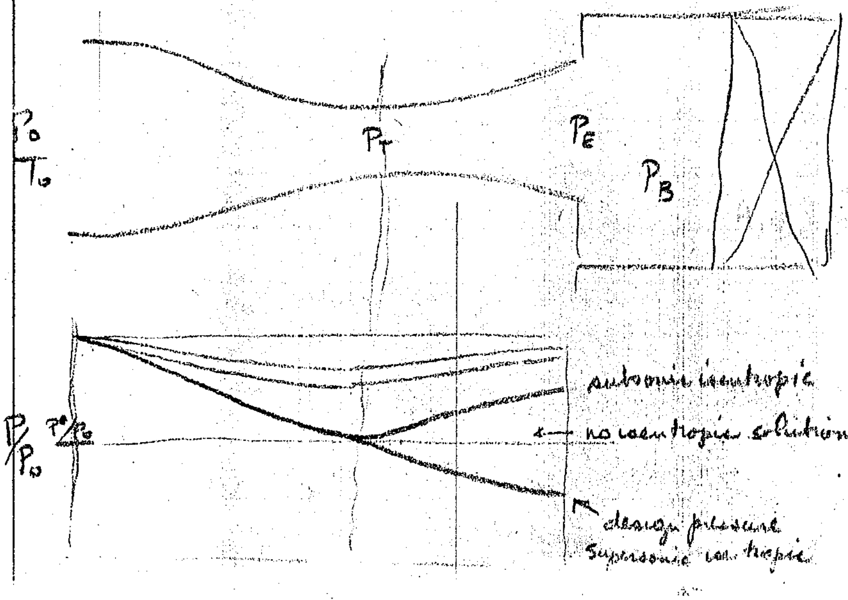
since pressure is a point function $\Rightarrow \Delta p$ is set up on $\downarrow P_B$ $\uparrow P_e$ causing the stream tube to expand or contract depending on whether $p_e - p_B$

for Reg II $P_E = P^*$ $M_E = 1$, choked flow
$P_B < P^*$, ω is indep of P_B





Index pressure $= P^*$ controlling parameter.



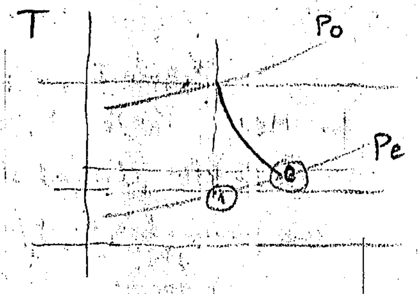
Reg I $M_E < 1$

$M_G > 1$

performance of real nozzles

Nozzle Efficiency $\eta = \dfrac{\text{actual KE / unit mass at exit}}{\text{isentropic KE / unit mass at exit}}$

expanded to the same pressure in some hypothetical nozzle



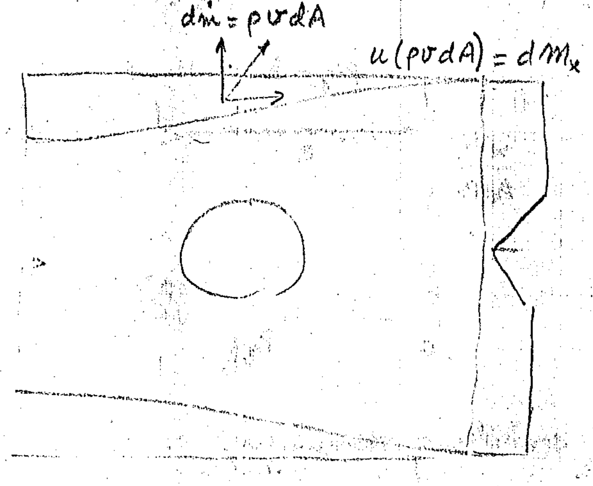
$$\eta = \frac{Ve^2}{V'^2_{isent}} = \frac{h_0 - h_e}{h_0 - h_i}$$

straight nozzles .94
.99

curved .90
.95

discharge coeff $c_w = \dfrac{\text{actual mass flow}}{\text{isentropic mass flow}}$

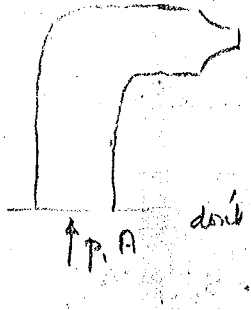
10/23/78



1st use conservation of mass to find the displacement of the streamline

2nd since streamline no ṁ through tops bottom

3rd use momentum to find F on cylinder



don't forget p_1A

SHOCKWAVES

for pure substance 2 ways for irreversibility
1) friction
2) heat loss over finite temp drop

1) shock is produced by a set of pressure pulses emitted such that at each instant veloc. of pressure pulse is higher that previous

$$-\Delta p = \rho u \Delta u$$

$$\Delta \rho = -\frac{\rho u}{a^2}\,\Delta u$$

$$\begin{array}{l|l} \rho_1 & \rho_2 \\ P_1 & P_2 \end{array} \rightarrow$$

$\rho_2 < \rho_1$

$P_2 < P_1$

if Δu is + $\Rightarrow \Delta p -$

$\Delta \rho \rightarrow$

$u_2 > u_1$

$$c_1 = \frac{\gamma p_1}{\rho_1} \qquad c_2 = \frac{\gamma(p_2)}{\rho_2} = \frac{\gamma(p_1 - \Delta p)}{\rho_1\left(1 - \frac{\Delta\rho}{\rho_1}\right)}$$

$$= c_1 \frac{[1-\zeta]}{1-\theta} = c_1\,[1+\theta-\zeta+\dots]$$

$$= c_1\left[\frac{\Delta\rho}{\rho} - \frac{\Delta p}{p_1} + 1\right]$$

$$= c_1\left[1 + \frac{u\Delta u}{c_1^2} + \gamma\frac{u\Delta u}{c_1^2}\right.$$

$$\left[1 + (\gamma-1)\frac{u\Delta u}{c_1^2}\right]$$

$c_2 > c_1$

- waves move forward until the total Δp is balanced by frictional & heat losses

- control volume about shock
  - energy balance & momentum balance don't give direction of changes
  - 2nd law does

Equation of state outside the shock is same as for isentropic

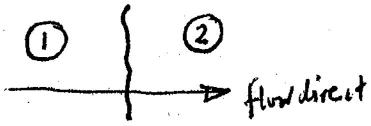
2nd law of thermo states

$$\frac{\partial}{\partial t}\int s\,d\mathcal{V} \int s\,\rho \underset{\sim}{q}\cdot\underset{\sim}{n}\,dA = \text{entropy flow from surroundings}$$

if steady state, across shock entropy flow from surrounds ≥ 0

$$\therefore \int s\,\rho \underset{\sim}{q}\cdot\underset{\sim}{n}\,dA \geq 0 \Rightarrow s_2(\rho v A)_2 \geq s_1(\rho v A)_1$$

$s_2 > s_1$



3. $\frac{p_{exit}}{p_0} \longleftarrow M_{exit}$

If $\frac{p_{exit}}{p_0} >$ given p/p_0 then

calculate $M_e \rightarrow \frac{A_e}{A^*}$ & we obtain A^* since $A^* = A(x)$

11/16/70

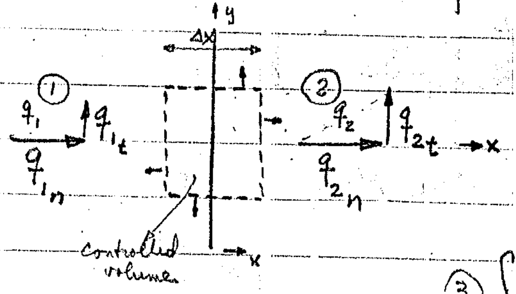
Exam: Know assumptions and conditions for equations

Exam is long

<u>Investigation of other flows</u> (Discontinuity in flows, s≠c)

Q: Can a discontinuity exist in a steady inviscid flow?

fixed discontinuity assumed.



① $\int_S \rho \bar{q} \cdot \bar{n} \, ds = 0$

② $\int_S \rho \bar{q} \, \bar{q} \cdot \bar{n} \, ds + \int p \bar{n} \, ds = 0$

③ $\int_S \rho \left(u + \frac{q^2}{2}\right)(\bar{q} \cdot \bar{n}) \, ds + \int_S p \, \bar{q} \cdot \bar{n} \, ds = 0$

Steady flow, $\bar{F} = 0$, $\dot{Q} = 0$

Viscosity is unimportant outside of the discontinuity

In ①, ②, ③ the contributions in (x_0, y_0) and $(x_0, -y_0)$ are opposite or $(x_0, -y_0) \equiv -(x_0, y_0)$

① Continuity Eq $\quad -\rho_1 q_{1n} \Delta y + \rho_2 q_{2n} \Delta y = 0$

$$\boxed{\rho_1 q_{1_n} = \rho_2 q_{2_n}}$$

② Momentum eq : x comp

$$-\rho_1 q_{1n} q_{1n} \Delta y + p_1(-1)\Delta y + \rho_2 q_{2n} q_{2n} \Delta y + p_2 \Delta y = 0$$

$$\boxed{p_1 + \rho_1 q_{1_n}^2 = p_2 + \rho_2 q_{2_n}^2}$$

③ Energy Eq

$$-\rho_1\left(e_1 + \frac{q_1^2}{2}\right)\left(q_{1_n}\Delta y\right) - p_1 q_{1n}\Delta y + \rho_2\left(e_2 + \frac{q_2^2}{2}\right)\left(q_{2_n}\Delta y\right) + p_2 q_{2_n}\Delta$$

$$\left(e_1 + \frac{p_1}{\rho_1} + \frac{q_1^2}{2}\right)\rho_1 q_{1n} = \left(e_2 + \frac{p_2}{\rho_2} + \frac{q_2^2}{2}\right)\rho_2 q_{2n}$$

if $\rho_1 q_{1n} = \rho_2 q_{2n}$ then $\boxed{H_{o_1} = H_{o_2}}$

② Mom: y-comp

$$-\rho_1 q_{1_t} q_{1_n} + \rho_2 q_{2_t} q_{2_n} = 0$$

$$\boxed{\rho_1 q_{1_t} q_{1_n} = \rho_2 q_{2_t} q_{2_n}}$$

if $q_n \neq 0$ $\quad q_n \rho = C$ across discontinuity

$p + \rho q_n^2 = C$ across discontinuity

$q_t = C$ " "

$H_o = C$ " "

Normal Shocks: $q_t = 0$ let $q_n = u$

1. $\rho_2 q_{2_n} = \rho_1 q_{1_n}$ 3a. Calorically perfect gas $h = C_p T$

2. $p_2 + \rho_2 q_{2_n}^2 = p_1 + \rho_1 q_{1_n}^2$ 4. $P = \rho RT$

3. $h_2 + \dfrac{q_{2n}^2}{2} = h_1 + \dfrac{q_{1n}^2}{2}$ adiabatic

$$h + \frac{q^2}{2} = h_0$$

$$\left.\begin{array}{l} c_p T + q^2/2 = c_p T_0 \\ T = \frac{a^2}{\gamma R} \quad \& \quad \frac{c_p}{\gamma R} = \frac{1}{\gamma - 1} \end{array}\right) \quad \frac{a^2}{\gamma-1} + \frac{V^2}{2} = \frac{a_0^2}{\gamma-1}$$

@ $M = 1 \quad a = a^*, \; V = a^* \quad \Rightarrow \quad \frac{a_0^2}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)} a_*^2$

since $h_{o_1} = h_{o_2} \quad \Rightarrow \quad a_{o_1} = a_{o_2} \quad \Rightarrow \quad a_{*_1} = a_{*_2}$

- using $a^2 = \frac{\gamma p}{\rho}$

adiabatic, calorically perfect $\quad c_p, c_v = \text{const}$
thermally perfect

$$\frac{\gamma}{\gamma-1} p/\rho + \frac{V^2}{2} = \frac{a_0^2}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)} a_*^2 \text{ on both sides}$$

also $\quad p_1 + \rho_1 q_{1n}^2 = p_2 + \rho_2 q_{2n}^2 = p_2 + \overset{\text{cont.}}{\overbrace{\rho_1 q_{1n}}} q_{2n}$

$$\Rightarrow \quad p_2/\rho_2 q_{2n} - p_1/\rho_1 q_{1n} = q_{1n} - q_{2n}$$

$$\frac{1}{q_{2n}}\left[\frac{\gamma+1}{2(\gamma-1)} a_*^2 - \frac{q_{2n}^2}{2}\right]\frac{\gamma-1}{\gamma} - \frac{1}{q_{1n}}\left[\qquad\qquad\right] = q_{1n} - q_{2n}$$

$$\Rightarrow \quad \frac{\gamma+1}{2\gamma}\left[\frac{a_*^2}{q_{2n} q_{1n}} - 1\right](q_{1n} - q_{2n}) = 0$$

$\Rightarrow \quad q_{1n} = q_{2n} \quad$ or $\quad \boxed{a_*^2 / q_{2n} q_{1n} = 1} \qquad$ also $\quad q_{1n} q_{2n} = a_*^2 = \frac{p_2 - p_1}{\rho_2 - \rho_1}$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1}{a_*} \cdot \frac{a_*}{u_2} = \frac{u_1}{a_*} \cdot \frac{u_1}{a_*} = \frac{u_1^2}{a_*^2} = \frac{u_1^2}{a_1^2} \cdot \frac{a_1^2}{a_*^2}$$

$$= M_1^2 \left(\frac{T_1}{T_*}\right)$$

$$= M_1^2 \left(T_1/T_0\right)\left(T_0/T_*\right)$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} = \frac{\frac{\gamma+1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_1^2} = \frac{T_0/T_*}{T_0/T_1}M_1^2$$

$$M_1 = \frac{u_1}{a_1} \;;\quad a_1^2 = \frac{\gamma p_1}{\rho_1}$$

$$p_2/p_1 = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)$$

$$T_2/T_1 = 1 + \frac{2(\gamma-1)}{(\gamma+1)^2}\left[\frac{\gamma M_1^2+1}{M_1^2}\left(M_1^2-1\right)\right]$$

$$M_2^2 = \frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}}$$

between stagnation values 1st & 2nd law
$\Rightarrow T ds_0 = dh_0 - v_0 dp_0$
for no work & no heat $dh_0 = 0$
also for perfect gas $v_0 = RT_0/p_0$
$\Rightarrow ds_0/R = -dp_0/p_0$

adiabatic equation but work is done due

$$\frac{S_2-S_1}{R} = \ln\frac{P_{0_1}}{P_{0_2}} = \ln\left\{\left[1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)\right]^{\gamma/\gamma-1}\left[\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}\right]^{-\gamma/\gamma-1}\right\}$$

since $S_2 > S_1 \Rightarrow P_{0_1}/P_{0_2} > 1$ or $\underline{P_{0_2} < P_{0_1}}$

11/23/70

Normal Shock

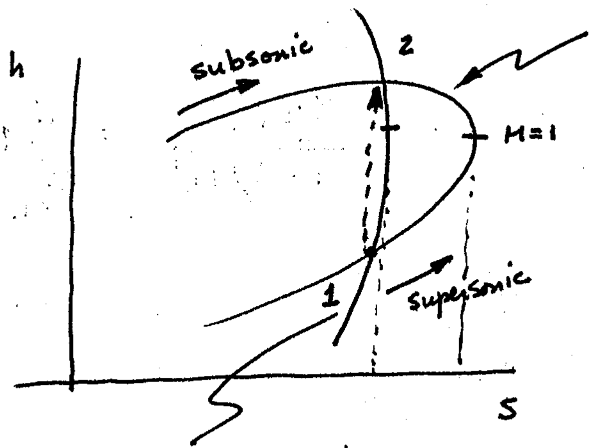
$q_n = u \neq 0$
$q_t = 0$

$\rho u = \text{const}$
$p + \rho u^2 = \text{const}$
$h + \frac{u^2}{2} = H_0 = \text{const.} = C_p T_0$
$h = C_p T$
$p = \rho R T$

mach behind the shock with respect to mach in front of shock

$$M_2^2 = \frac{1+\frac{(\gamma-1)}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}}$$

$$\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} \;;\quad \frac{T_2}{T_1} = 1 + \frac{2(\gamma-1)}{(\gamma+1)^2}\left[\frac{\gamma M_1^2+1}{M_1^2}\left(M_1^2-1\right)\right]$$



rayleigh (satisfies momentum + cont but not energy)

Fanno line (satisfies cont. + energy but not moment)

	quantities that are constant
"Four simple processes 1-D steady isentropic, adiabatic, area change	energy impulse $p+\rho v^2$ no shock mass flow
shock	area, impulse energy mass flow
Energy Exchange (Rayleigh)	area, impulse no shock
Simple friction (Fanno)	area, energy no shock

shock process goes from 1 → 2
process is dotted since h, s represents only equilib states
& shock doesn't go from 1→2 via equilib

$$\Delta s = c_p \ln\left(\frac{T_y}{T_x}\right) - R \ln\left(\frac{P_y}{P_x}\right)$$

$$= \frac{\gamma}{\gamma-1} R \ln\left(T_2/T_1\right) - R \ln\left(P_2/P_1\right)$$

$$= c_v \ln\left(P_2/P_1\right) - c_p \left(\rho_2/\rho_1\right)$$

$$= \frac{1}{\gamma-1} R \ln\left(\quad\right) - \frac{\gamma}{\gamma-1} R \left(\quad\right)$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2-1) \quad ; \quad \frac{S_2-S_1}{R} = \ln \frac{p_{o_1}}{p_{o_2}}$$

this is obtained from

$$p/p_r = \exp\left(\frac{s-s_r}{c_v}\right)(\rho/\rho_r)^\gamma$$

$$p/p_0 = (\rho/\rho_0)^\gamma = (T/T_0)^{\gamma/\gamma-1}$$

$$\frac{S_2-S_1}{R} = \ln\left\{\left[1+\frac{2\gamma}{\gamma+1}(M_1^2-1)\right]^{1/\gamma-1}\left[\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}\right]^{-\gamma/\gamma-1}\right\}$$

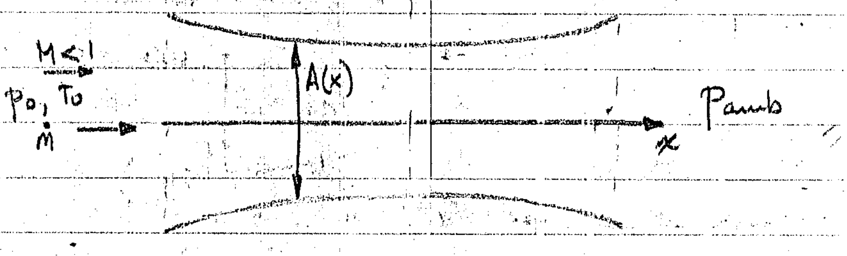
$S_2 \geqslant S_1$ only if $M_1 \geqslant 1$

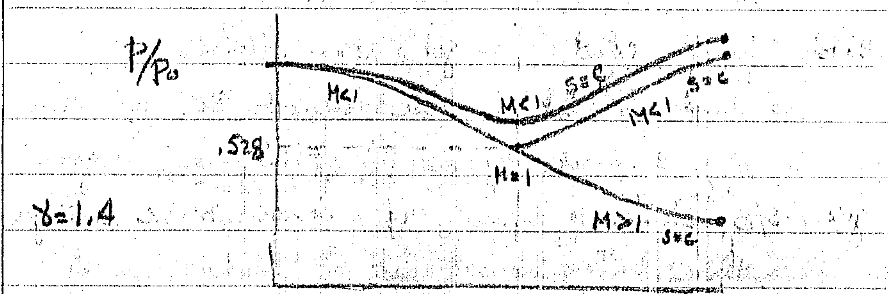
$M_1 = 1+\epsilon \quad \epsilon << 1$ weak discontinuity or shock

$M_1 >> 1$, $\frac{\rho_2}{\rho_1} \to \frac{\gamma+1}{\gamma-1}$ strong wave (shock)

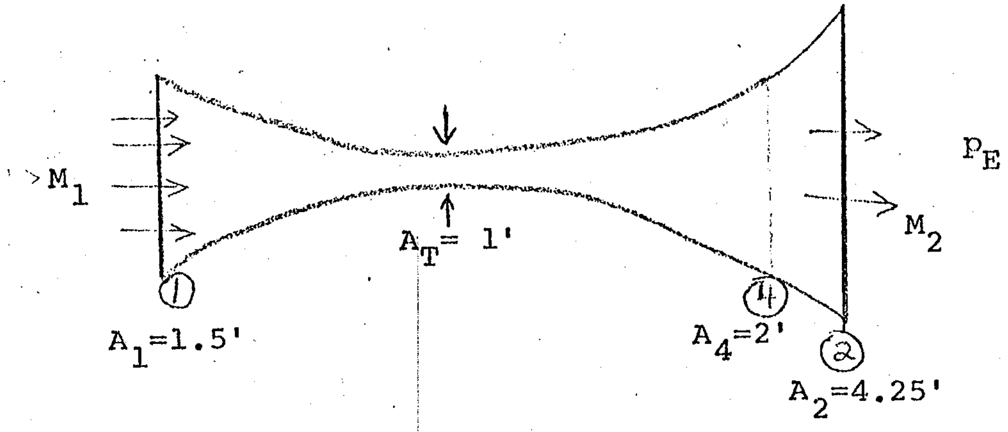
$$\left.\begin{array}{ll} \rho_2/\rho_1 \geqslant 1 & S_2 - S_1 \geqslant 0 \\ u_2/u_1 \leqslant 1 & p_{o_2}/p_{o_1} \leqslant 1 \\ T_2/T_1 \geqslant 1 & \\ p_2/p_1 \geqslant 1 & \end{array}\right\}$$

if $M_1 \geqslant 1$ this is a <u>compressio</u> <u>discontinuity</u>





1. Consider a "quasi"-one - dimensional flow in a converging-diverging channel with a subsonic entrance (Steady with zero viscosity)



Given: At the entrance the stagnation pressure is $p_o=150$ psi

(i) Determine p_2 for a subsonic isentropic exit with sonic conditions (M=1) at the throat.

(ii) Determine p_2 for a supersonic isentropic exit.

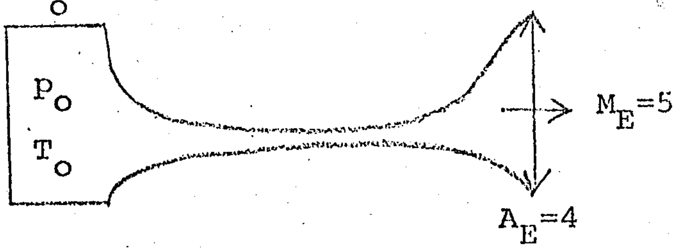
(iii) Determine p_2 (behind the shock) for a normal shock that is standing at the exit.

(iv) If $p_E= 90$ psi, what will the ultimate flow look like?

(v) Determine the final $\overset{\text{Pexit}}{p_2}$ for a shock located at $A_4=2'$

(vi) Determine M_{THROAT} if $p_E=149$ psi.

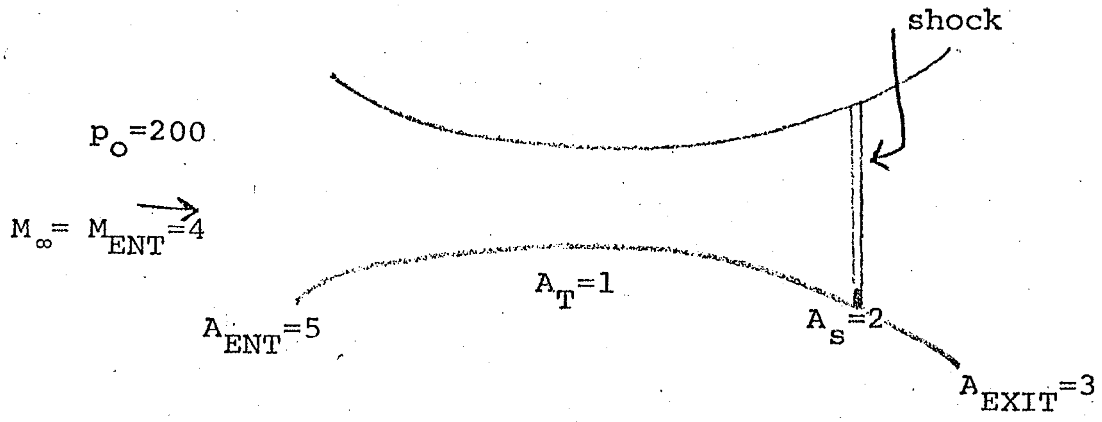
2. Given a channel, with exit area $A_E=4$ and exit Mach number $M_E=5$, $p_o=200$, $T_o=500$.



1. Determine the exit pressure.
2. Determine the exit Mach number if T_o is raised to 700.
3. Determine the exit Mach number if A_t^o is reduced by ½.

(p_o and T_o are the original values.)

3. Consider a jet engine



1. Determine the Mach No. upstream of the shock.
2. Determine the Mach exit pressure.

Cesar Levy
AE 331

i) $A^* = 1$ $A = 4.25$ $A^*/A = .2355$

M	A*/A	p/po
.13	.2224	.9883
M_x	.2355	p_x/p_o
.14	.2391	.9864

$M_x = .13785$
$p_x/p_o = .98681$

$p_x = p_o(.98681) = 148.021$ psia ✓

ii)

M	A*/A	p/po
3.00	.2362	.02722
M_1	.2355	p_1/p_o
3.01	.2339	.02682

$M_1 = 3.00304$
$p_1/p_o = .0270984 \approx .0271$

$p_1 = p_o(.02741) = 4.06$ psia ✓

iii) $p_2 = p_1 K$ K from tables → 10.33

$p_2 = (4.06)(10.33) = 42$ psia ✓

(iv) take $A = 2.03$ $A_1^*/A_1 = .4933$ $M_1 = 2.2125$

$p_1/p_o = .09171$

$p_1 = (150)(.09171) = 13.757$ psia

$p_{o_2}/p_{o_1} = .6225$

$p_{o_2} = (150)(.6225) = 93.38$ psia

$p_2 = p_1(5.544) = 76.2$ psia $p_2/p_{o_2} = \frac{76.2}{93.38} = .816$

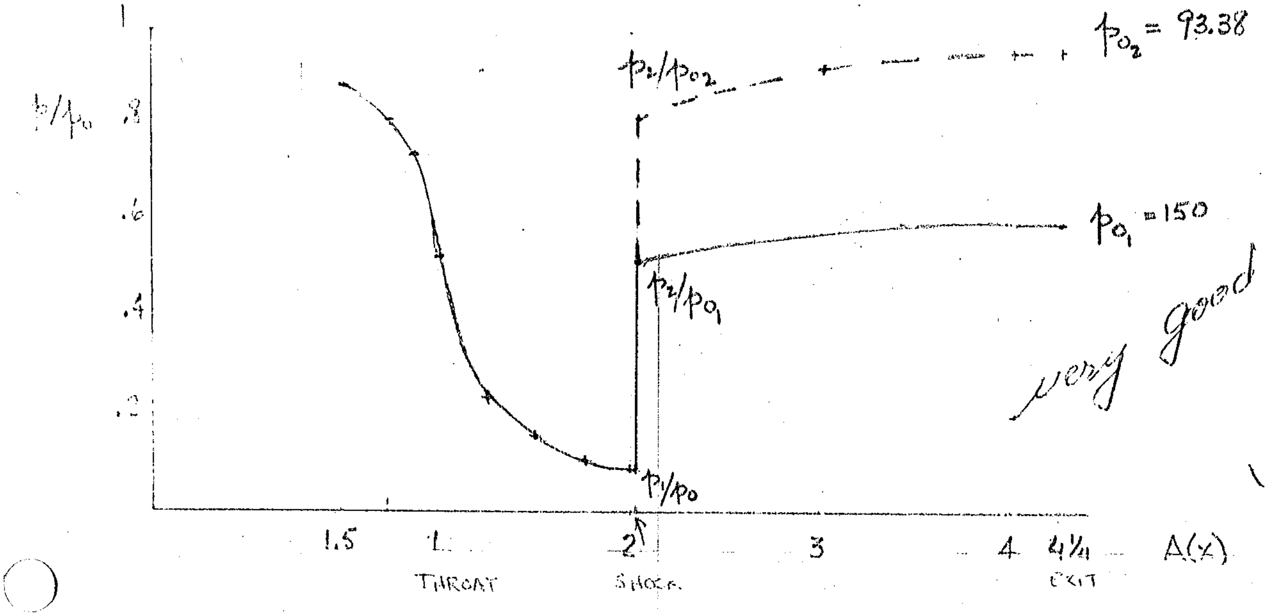
$A_2^*/A_1 = .7939$ $A_2^* = (2.03)(.7939) = 1.61$

$A_2^*/A_E = \frac{1.61}{4.25} = .379$ $p_E/p_{o_2} = .965$ $p_E = .965(93.38) = 90$ psia

A	A*/A	p/po
1.5	.667	.881
1.25	.800	.8122
1.125	.890	.7444
1.00	1.00	.528

A	A*/A	p/po
1.25	.8	.2353
1.50	.667	.1602
1.75	.572	.1197
2.00	.5	.0939

A	A^*/A	p/p_0	p/p_{0_2}
2.03	.4933	.09171	
2.03	.794	.507	.816
3.00	.537	.9248	
4.00	.403	.9602	
4.25	.379	.6	.965



very good ✓

(v) $A_1^*/A_1 = .5 \qquad M_1 = 2.1975 \qquad p/p_0 = .0939$

$p_1 = 150\,(.0939) = 14.08$ psia

$p_2 = (14.08)(5.467) = 77$ psia $\qquad p_{0_2} = 150\,(.629) = 94.35$

$p_2/p_{0_2} = \dfrac{77}{94.35} = .817 \rightarrow M_s = .545 \qquad A_2^*/A_1 = .792$

$A_2^* = 2\,(.792) = 1.584$

$A_2^*/A_2 = \dfrac{1.584}{4.25} = .373 \qquad M_2 = .2223$

$p_E/p_{0_2} = .966 \qquad p_E = (94.35)(.966) = 91$ psia ✓

(vi) $p_E/p_0 = \dfrac{149}{150} = .993 \qquad A^*/A_E = .1718 \qquad M_E = .1$

$A^* = 4.25\,(.1718) = .73$

$A^*/A_t = .73/1 = .73 \qquad M_t = .48491$

2.

(i) $p_E/p_0 = K = 1.89\times10^{-3}$

$p_E = 200(1.89\times10^{-3}) = .378$ psia ✓

(ii) since $A^*/A_E = f(M)$ only if both A^* & A_E remain the same and T_0 increases then M_E must remain the same as from the equation above. This means $T_{E_2}/T_{0_2} = T_{E_1}/T_{0_1}$ this means that if p_0 remains constant from the isentropic equations p_E must remain constant from the mass flow equation:

before T_0 increased at a particular A_1:

$\rho_{1_b}u_{1_b}A_{1_b} = C_1$ since $A_{1_b} = A_{1_a}$ (b - before, a - after)

$\rho_{1_b}u_{1_b}A_{1_a} = C_1$ since $u_{1_a} > u_{1_b}$ (since $u_{1_b} = f(T_{1_b})$, $u_{1_a} = f(T_{1_a})$ and $T_{1_b} < T_{1a}$ ∴ $u_{1a} > u_{1b}$)

∴ $\rho_{1_b}u_{1_a}A_{1_a} > C_1$

now if $p_0 = $ const & p_E remains constant $p_E/p_0 = (\rho_E/\rho_0)^\gamma$

∴ if $p_0 = \rho_0 R T_0$ if $p_0 = C$, T_0 increases, ρ_0 must decrease if ρ_0 decreases ρ_E must decrease, at A_1 $\rho_{1_a} < \rho_{1_b}$

we can transform all the above saying $c_1 = \rho_{1_a}u_{1a}A_1\frac{\sqrt{T_{0a}}}{\sqrt{T_{0b}}}$.

and since $T_{0b} < T_{0a}$ $\rho_{1_a}u_{1_a}A_1 < \rho_{1_b}u_{1_b}A_1$ ✓

or mass flow decreases

(iii) $A^*/A_E = .04$ $A^* = .16$ let $A^* = .08$

$A^*/A_E = .08/4 = .02$ $M_E = 5.943$ ✓

3. (i) $M_{ENT} = 4$ $A^*/A = .09329$ $A^* = .46645$

$A^*/A_S = .23323$ $M_1 =$ ~~3.06~~ 3.013 $M_2 =$ ~~.472~~ .4746

p/p_0 = ~~.02489~~ .02669 p_1 = ~~4.998~~ 5.338

p_2/p_1 = ~~10.76~~ 10.42 p_2 = ~~53.56~~ 55.622

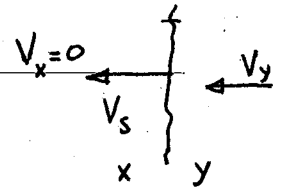
p_{o_2}/p_{o_1} = ~~.312~~ .3249 p_{o_2} = ~~62.4~~ 64.98

$p_2/p_{o_2} = \frac{53.56}{62.4}$ = ~~.8583~~ .8570 A^*/A = ~~.7157~~ .7189

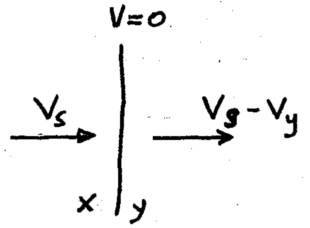
A^* = ~~1.4314~~ 1.4378 $A^*/A_E = \frac{1.4314}{3.0}$ = ~~.4771~~ .4793 ~~$M_E = .29$~~ .2918

p_E/p_{o_2} = ~~.9433~~ .9426 p_E = ~~58.86 psia~~ ≈ ~~61.0~~ 61.25 psia

WHAT IF WE HAD A WAVE MOVING AT CONSTANT SPEED



must use



SINCE STATIC CONDITIONS NOT VELOCITY DEPENDENT P_x, P_y are unchanged
T_x, T_y
a_x, a_y

stagnation conditions are

$$T_{o_x} = T_x \left(1 + \frac{\gamma-1}{2} M_x^2\right)$$

$$T_{o_y}' = T_y \left(1 + \frac{\gamma-1}{2} M_y'^2\right) \qquad M_y' = \frac{V_s - V_y}{a_y} \qquad a_y = \sqrt{\gamma R T_y}$$

$$T_{o_y} = T_y \left(1 + \frac{\gamma-1}{2} M_y^2\right) \qquad M_y = \frac{V_y}{a_y}$$

$$P_{o_y}' = P_y \left(1 + \frac{\gamma-1}{2} M_y'^2\right)^{\gamma/\gamma-1} \qquad M_x = 0$$

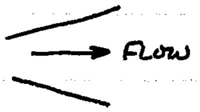
$$M_x' = V_s/a_x$$

1) DO TRANSFORMATION ; THEN DETERMINE CONDITIONS ; THEN TRANSFORM BACK

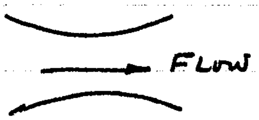
WE use the actual values to determine results See examples on p. 176 - 180

- SUPERSONIC DIFFUSERS
  - CAUSES STATIC PRESSURE TO RISE WHILE GAS DECELERATES

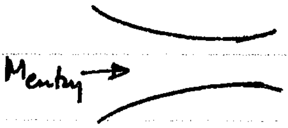
IF $M < 1$



IF $M > 1$

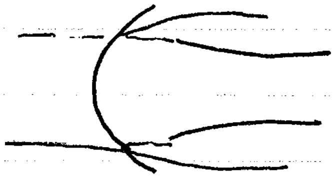


IF A_{inlet}/A^* FOR SUPERSONIC is > A_{in}/A^* FOR M_{ENTRY}



THEN THROAT TOO SMALL $\Rightarrow \dot{m}$ reduced

& FLOW SPILLS OUTSIDE



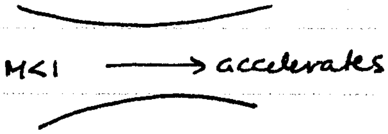
BOW SHOCK STANDS OUTSIDE

since $F = \dot{m}V_e + (p - p_e)A_e$

& $\dot{m}$ is reduced $\Rightarrow F \downarrow$

TO CIRCUMVENT: increase $M > M_{design}$ CAUSING SHOCK TO MOVE TO ENTRY

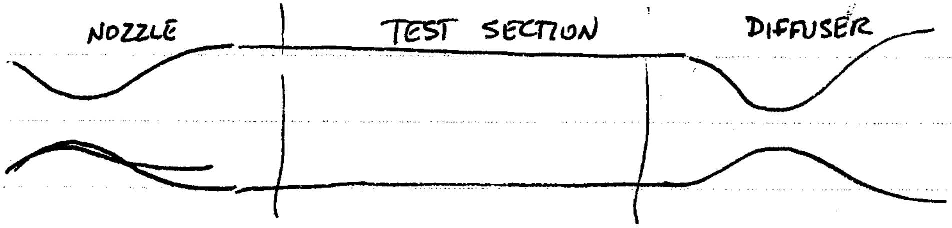
M AFTER SHOCK is SUBSONIC



AS M↑ SHOCK IS SWALLOWED PAST ~~THROUGHT~~ THROAT; FINALLY M is SET EQUAL TO M_{design}; SHOCK is AT THROAT NOW & IS A WEAK SHOCK HAVE ZERO STRENGTH. (STRENGTH IS MEASURED) BY $(M_x^2 - 1)$

TO CIRCUMVENT USE VARIABLE AREA DIFFUSER: INCREASE THROAT AREA $\Rightarrow \dot{m}$ WILL INCREASE. SHOCK MOVES TO INLET THEN IS SWALLOWED; NOW DECREASE A_t TO MOVE SHOCK TO THROAT

- WIND TUNNELS



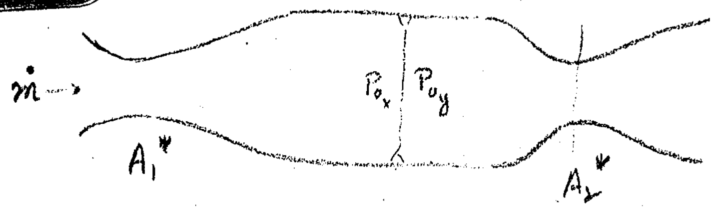
- WANT TEST SECTION TO BE CLEAR OF SHOCKS & HAVE $M > 1$
- AS $M \uparrow$ (or back pressure drops) SHOCK DEVELOPS IN NOZZLE DUE TO CHOKING IN DIFFUSER & MOVES TO TEST SECTION CAUSING M IN TEST SECTION TO BE $M < 1$
- WE WANT $M = 1$ AT NOZZLE & DIFFUSER
- IF $A_{DIFF_t} > A_{NOZZLE_t}$ SHOCK CAN EXIST AT DIFFUSER THROAT

SMALLEST LOSS IN STAGNATION pressure IS WHERE SHOCK IS AT THROAT OF DIFFUSER

$$A \text{ diffuser throat} = \left(\frac{P_{0x}}{P_{0y}}\right)\Bigg|_{M_{design}} A \text{ nozzle throat}$$

See example 4.9

11/11/78



ṁ = const throughout this double nozzle

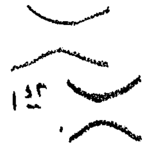
$$\text{for } \dot{m} = f(K,R)\frac{P_0 A_1^*}{\sqrt{T_0}} \quad \text{since } T_0 = \text{const and } P_{0x} > P_{0y}$$

$$\text{then } \frac{P_{0x} A_1^*}{\sqrt{T_0}} = \frac{P_{0y} A_2^*}{\sqrt{T_0}} \Rightarrow A_2^* > A_1^*$$

$s_y - s_x = -R \ln \frac{P_{0y}}{P_{0x}}$ for noenergetic perfect gas flow.

if $A_2^* > A_2^*$ flow will be subsonic in next

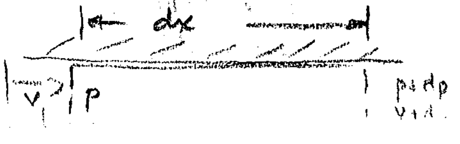
if $A_2 < A_2^*$ will drive shock back down 1st



- Fanno line ≡ simple friction : Steady 1-D flow without energy (i.e. energy exchange), or area exchange

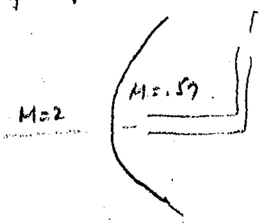
- Shocks are allowable as discontinuities.

Look at CV

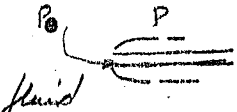


1-D velocity profile
all viscosity is at the wall (no body layer
effect on V until be

for pitot tubes



for P_{o_x} = const & $M_x \uparrow$ $\quad \frac{P_{o_y} - P_{o_x}}{P_{o_x}} \uparrow$



impact pressure = stag press = total pressure.
static pressure = thermodynamic press w/observer or fluid
dynamic = impact - static

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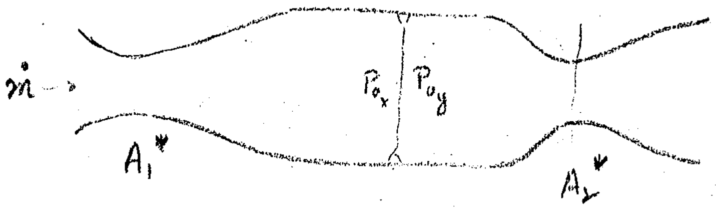


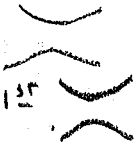
$\dot{m}$ = const throughout this double nozzle

for $\dot{m} = f(\gamma, R) \frac{P_0 A_1^*}{\sqrt{T_0}}$ since T_0 = const and $P_{o_x} > P_{o_y}$

$$\text{then } \frac{P_{o_x} A_1^*}{\sqrt{T_0}} = \frac{P_{o_y} A_2^*}{\sqrt{T_0}} \Rightarrow A_2^* > A_1^*$$

$s_y - s_x = -R \ln \frac{P_{o_y}}{P_{o_x}}$ for isoenergetic perfect gas flow.

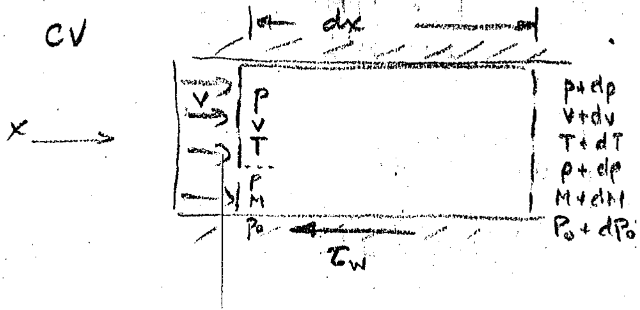
if $A_2 > A_2^*$ flow will be subsonic in next
if $A_2 < A_2^*$ will drive shock back down 1st



- Fanno line ≡ simple friction : Steady 1-D flow without energy (isoenergetic) exchange, or area exchange

- Shocks are allowable as discontinuities.

Look at CV



1-D velocity profile
all viscosity is at the wall (no bdy layer effect on V will be allowed)
we will neglect body forces

Governing Eqs $p=\rho RT$ equations of State (1)

$$\frac{dp}{p}=\frac{d\rho}{\rho}+\frac{dT}{T}$$

$$\frac{dM^2}{M^2}=\frac{dV^2}{V^2}-\frac{dc^2}{c^2}=\frac{dV^2}{V^2}-\frac{dT}{T}$$ def of mach no. (2)

$$\frac{dT}{T}+\frac{\gamma-1}{2}M^2\frac{dV^2}{V^2}=0$$ conserv of energy (3)

$\dot{m}=\rho AV=\text{const}$ $$\frac{d\dot{m}}{\dot{m}}=\frac{d\rho}{\rho}+\frac{dV}{V}=0$$ $$=\frac{d\rho}{\rho}+\frac{1}{2}\frac{dV^2}{V^2}=0$$ cont (4) since A=const

$(p_{er}\cdot dx)$

$$-A\,dp-\tau_w\,dA_W=\dot{m}\,dV$$ (5) no body forces steady flow since cv is fixed

var $p;\rho;T;V,\tau_w,M^2$ only 5 eqs $\therefore$ need 1 more eq must correlate τ_w w/something

HYDRAULIC DIAM

let $$D=\frac{4A}{P}=\frac{4A}{dA_W/dx}=4A\frac{dx}{dA_W}$$ $$f\triangleq\frac{\tau_W}{\frac{1}{2}\rho V^2}$$ FANNING FRICTION FACTO FOR DUCTS OR C_D FOR FLAT PLATE

$$p_o/p=\left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/\gamma-1}$$ can be written (even though process is not isentropic) because we can imagine that if the flow were brought to rest isentropically this would define p_o

$$\frac{dp_0}{p_0}=\frac{dp}{p}+\frac{\frac{1}{2}\gamma M^2}{T_0/T}\frac{dM^2}{M^2}$$

and now since isoenergetic $T_0=\text{const}\Longrightarrow -\frac{ds}{R}=\frac{dp_o}{p_o}$ (see shock relation)

thus using all the above

$$\frac{dp}{p}=-\frac{\gamma M^2}{2}\frac{[1+(\gamma-1)M^2]}{(1-M^2)}4f\frac{dx}{D}$$

$$\frac{dM^2}{M^2}=\frac{\gamma M^2}{1-M^2}\left(1+\frac{\gamma-1}{2}M^2\right)4f\frac{dx}{D}$$

$$\frac{dT}{T}=\frac{1}{2}\frac{dc}{c}=-\frac{\gamma(\gamma-1)M^4}{2(1-M^2)}4f\frac{dx}{D}$$

$$\frac{d\rho}{\rho}=-\frac{\gamma M^2}{2(1-M^2)}4f\frac{dx}{D}$$

$$\frac{ds}{R}=-\frac{dp_0}{p_0}=\frac{\gamma M^2}{2}4f\frac{dx}{D}$$ as $x\uparrow$ $s\uparrow$ $p_o\downarrow$

$$\frac{dV}{V}=\frac{\gamma M^2}{1-M^2}4f\frac{dx}{D}$$

$$\frac{d\dot{m}}{\dot{m}}=\left[-\frac{\gamma M^2}{2(1-M^2)}+\frac{\gamma M^2}{1-M^2}\right]\frac{4f\,dx}{D}$$
$$=\frac{\gamma M^2}{2[1-M^2]}\frac{4f\,dx}{D}$$

from $ds = c_v \frac{dT}{T} - R\frac{d\rho}{\rho}$

$+R\frac{dV}{V}$ Continuity

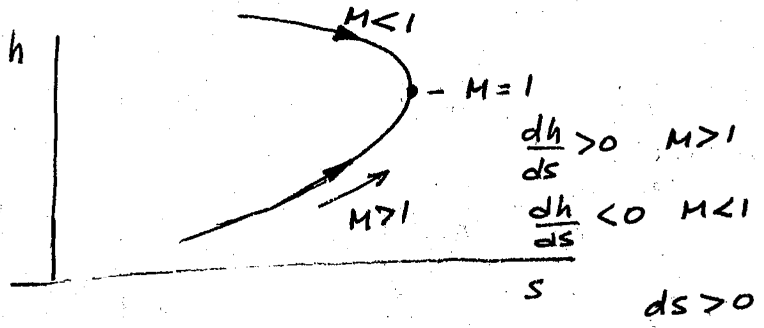
also $V = \sqrt{2c_p(T_0-T)}$

$$\frac{dV}{V} = \frac{d(T_0-T)}{2(T_0-T)}$$

or $\frac{ds}{c_v} = \frac{dT}{T} + \frac{\gamma-1}{2}\frac{d(T_0-T)}{T_0-T}$

from $T\,ds = dh - v\,dp$

$$\frac{dh}{ds} = T + v\frac{dp}{ds} = \frac{\gamma T M^2}{M^2-1}$$

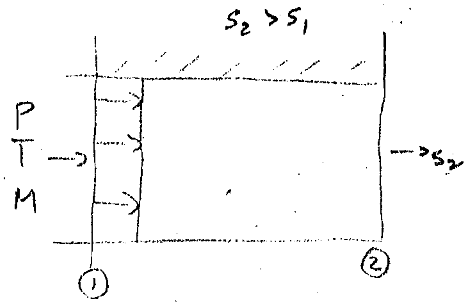


always drive towards M=1

M > 1	dx>0	dp ↑	dv ↓	dT ↑	dρ ↑	ds ↑	dp_0 ↓	dM ↓
M < 1	dx>0	dp ↓	dv ↑	dT ↓	dρ ↓	ds ↑	dp_0 ↓	dM ↑

as M goes through 1 inequality signs flows over.

11/13/78



$$\frac{dp}{p} = -\frac{\cancel{\gamma}M^2\left[1+(k-1)M^2\right]}{2(1-M^2)}\,4f\frac{dx}{D}$$

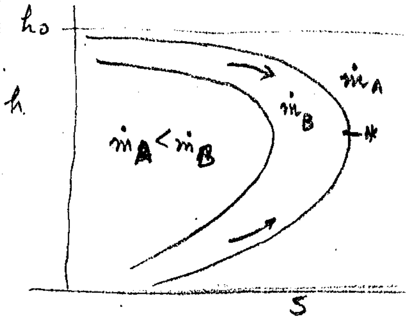
$$\frac{dV}{V} = \frac{\cancel{\gamma}M^2}{2(1-M^2)}\,4f\frac{dx}{D}$$

$$\frac{dT}{T} = \frac{1}{2}\frac{dc}{c} = -\frac{\cancel{\gamma}(k-1)\,M^4}{2(1-M^2)}4f\frac{dx}{D}$$

$$\frac{d\rho}{\rho} = -\frac{\cancel{\gamma}M^2}{2(1-M^2)}\,4f\frac{dx}{D}$$

$$\frac{ds}{R} = -\frac{dp_0}{p_0} = \frac{\cancel{\gamma}M^2}{2}\,4f\frac{dx}{D}$$

$$D = \frac{4 \times \text{cross sectional area}}{\text{perimeter}}$$



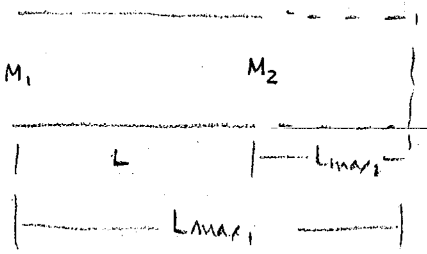
Variables: M^2, V, P, T, ρ, p_0

$$\int_0^{L_{max}} 4f\frac{dx}{D} = \int_{M_1^2}^{1} \frac{1-M^2}{\cancel{\gamma}M^4\left(1+\frac{\cancel{\gamma}-1}{2}M^2\right)}\,dM^2 \quad \text{using } \frac{dM^2}{M^2} = g(k,M)$$

$$= 4\bar{f}\frac{L_{max}}{D} = \frac{1-M^2}{\cancel{\gamma}M^2} + \frac{\cancel{\gamma}+1}{2}\ln\frac{(\cancel{\gamma}+1)M^2}{2\left(1+\frac{\cancel{\gamma}-1}{2}M^2\right)}$$

- NOW WE USE MACH NO. AS INDEPENDENT VARIABLES & INTEGRATE ALL VARIABLES

$$\bar{f} = \frac{1}{L_{max}}\int_0^{L_{max}} f\,dx$$



$$L = L_{max_1} - \bar{L}_{max_2}$$

$$\frac{4fL}{D} = 4\bar{f}\frac{L_{max_1}}{D} - 4\bar{f}\frac{L_{max_2}}{D}$$

p^* is not p^* for conv-div nozzle

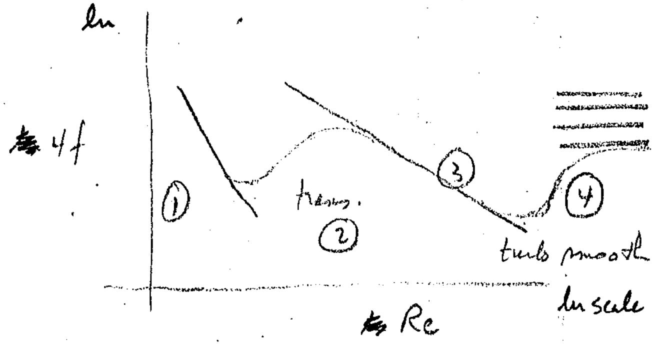
∃ an L_{max} ∋. $M_{L_{max}} = 1$ $p_{L_{max}} = p^*$ etc. for a given $\dot{m}_1$ if $L > L_{max}$ flow will choke and $\dot{m}_1$ will decrease (you will jump the fanno lines). for $L < L_{max}$ compute for a given M_1 the L_{max_1} then get $L_{max_2} = L_{max_1} - L$ for this length go to tables to find M_2

DUE TO LARGE LOSSES IN STAGNATION PRESSURE, WE AVOID LONG DUCTS OF SUPERSONIC FLOW

we define 4f from $\Delta p \triangleq 4f \frac{L}{D} \frac{\rho V^2}{2}$

Assume fully-established flow <u>ie</u> velocity profile is the same at any cross section.

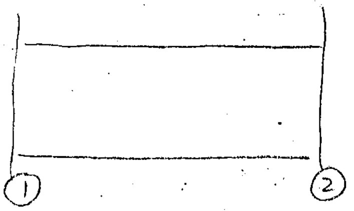
Assume square profile.



⑤ turbulent rough
independent of R_N - bumps on wall disturb sublayer

bumps on wall are not big enough to disturb sublayer.

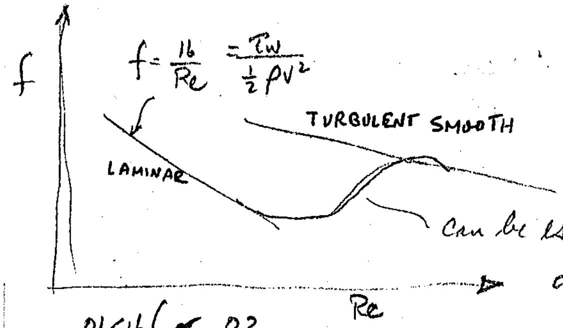
11/15/78



$$\int_1^* 4f \frac{dx}{D} = \int_1^2 4f \frac{dx}{D} + \int_2^* 4f \frac{dx}{D}$$

$$\overline{4f} \frac{L_{max_1}}{D} = \overline{4f} \frac{L_{12}}{D} + \overline{4f} \frac{L_{max_2}}{D}$$

$$\therefore \ 4f \frac{L_{12}}{D} = \overline{4f} \frac{L_{max_1}}{D} - \overline{4f} \frac{L_{max_2}}{D}$$



can be estimated by erf fn. the flow here is gauss distribution of laminar & turbulent flow.

$.01 < 4f < .02$

A NORMAL SHOCK CAN EXIST (EQUIV. TO $4fL/D = 0$)

SINCE $4fL/D$ is the governing variable like p_B

subsonic there is a given Length for ~~same~~ given initial conditions $M_1 \ni M_2 = 1$

as $L \uparrow$ flow readjusts $\ni M_2 = 1 \Rightarrow M_1$ readjusts $\downarrow \therefore \dot{m} \downarrow$

<u>supersonic</u>

what if $L > L_{max}$ a shock appears at exit

as $L \uparrow$ shock moves into duct towards inlet

always $M_2 = 1$ @ exit

as L goes past a certain value shock would lie outside inlet (ie if C-D nozzle connected to duct the shock is in Diverging section)

$$\frac{d\dot{m}}{\dot{m}} = \frac{\cancel{\gamma M^2}}{2(\cancel{1-M^2}) \cdot \frac{\cancel{\gamma M^2}}{\cancel{(1-M^2)}} (1+\frac{\gamma-1}{2}M^2)} \frac{dM^2}{M^2}$$

$$= \frac{du}{2u\,(1+\frac{\gamma-1}{2}u)} = du\left[\frac{c_1}{u} + \frac{c_2}{A+Bu}\right]$$

$$(A+Bu)C_1 + C_2 u = \frac{1}{2}$$

$$C_1 A = \frac{1}{2} \qquad C_2 = -BC_1$$

$$A = 1 \qquad C_1 = 1/2 \qquad C_2 = -\frac{\gamma-1}{4}$$

$$du\left[\frac{1}{2u} - \frac{\gamma-1}{4}\frac{1}{1+\frac{\gamma-1}{2}u}\right]$$

$$\ln\dot{m}\Big|_{\dot{m}_{in}}^{\dot{m}^*} = \frac{1}{2}\ln u - \frac{1}{2}\ln\left(1+\frac{\gamma-1}{2}u\right)\Big|_{M_{in}}^{M}$$

$$\frac{\dot{m}}{\dot{m}_{in}} = \left[\frac{M^2}{1+\frac{\gamma-1}{2}M^2}\right]^{1/2} \Big/ \left(\frac{2}{\gamma+1}\right)^{1/2}$$

$$= \rho VA =$$

$$\dot{m} = \dot{m}_{in}\left[\qquad\qquad\right]^{1/2} \Big/ \left(\frac{2}{\gamma+1}\right)^{1/2}$$

Task IV:

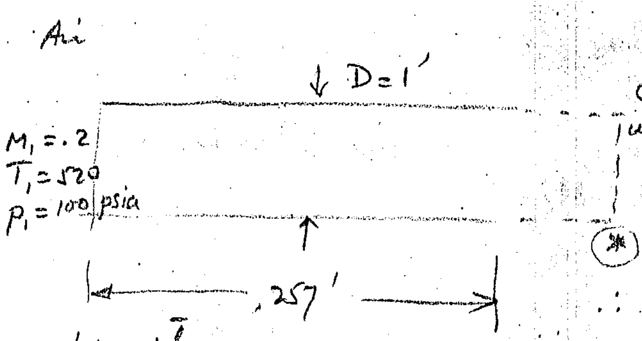
Undergraduate students will be conducting heat transfer analysis of the MVST off-gas ducts above the top of the vault to determine heat losses and the rate of condensation in the system as a function of off-gas temperature and ambient air temperature.

The off-gas line located near the vault ceiling is a 6-inch pipe. This line has about 25 feet of horizontal runs, and 9 feet of vertical runs from the tank exit point to the top surface of the 3 foot thick vault roof. There is an additional 3 feet of vertical and 8 feet of horizontal run at the top of the vault prior to entering the 2 inch reducer of the flow measuring device. All condensation and entrained liquid which collects within the pipe between the tank and this reducer flows back into the tank. Condensation or entrained liquid collected by the pipe system after this reducer flows into the demister.

Presently, the refluxing of liquid prior to this reducer, plus returning the demister liquid back into a tank, effectively minimizes the net evaporation of water from the tanks. However, the flow of 100 cfm of ventilation air through a tank will remove water by evaporation and entrainment. A fraction of this water will be refluxed back into the tank; another fraction returns to a tank from the demister and the remaining water vapor vents to the stacks.

The sparge system was operated on a continuous basis during the period of time when waste disposal via hydrofracture was operating. However, due to atmospheric condensation combined with instrumentation sensitivity, the evaporated water exiting to the stack with this tank ventilation and sparge air did not result in a detectable reduction in tank volume. The spargers were operated at about 1/3 of the design capacity in this time interval.

To affect a significant net increase in the removal of liquid from MVSTs by evaporation, the demister liquid must be intercepted and diverted to other treating



Go to table using $M_1=.2 \Rightarrow \left.\frac{4\bar{f}L_{max}}{D}\right|_{M_1=.2} = 14.533$

$$\frac{4\bar{f}L_{12}}{D} = .020 \times 257 = 5.143$$

$\therefore$ Case I $\quad \frac{4\bar{f}L_{12}}{D} < \frac{4\bar{f}L_{max}}{D}$

take $4\bar{f}=.02$

$$\frac{4\bar{f}L_{max_1}}{D} - \frac{4\bar{f}L_{12}}{D} = \therefore \frac{4\bar{f}L_{max_2}}{D} = 14.533 - 5.143 \doteq 9.39$$

find $p_2, T_2, M_2, \dot{m}$

$$p^* = \frac{P^*}{P_1} P_1 = \frac{1}{5.4555} \times 100 = 18.33 \text{ psia}$$

$$T^* = \frac{T^*}{T_1} \cdot T_1 = \frac{1}{1.1905} \cdot 520 = 436\ °R$$

$M_2 = .24$

$$P_2 = P_2/p^* \cdot p^* = 4.5383 \times 18.33 = 83.187 \text{ psia}$$

$$T_2 = T_2/T^* \cdot T^* = 436 \times 1.1863 = 517\ °R$$

$$\dot{m} = \rho A V = \frac{P_2}{RT_2} A M_2 \sqrt{KRT_2}$$

$$= .4347\ lb_m/ft^3 \cdot \frac{\pi}{4} (0.24)\left(\sqrt{(1.4 \times 1716)(517°R)}\right) = 91.328$$

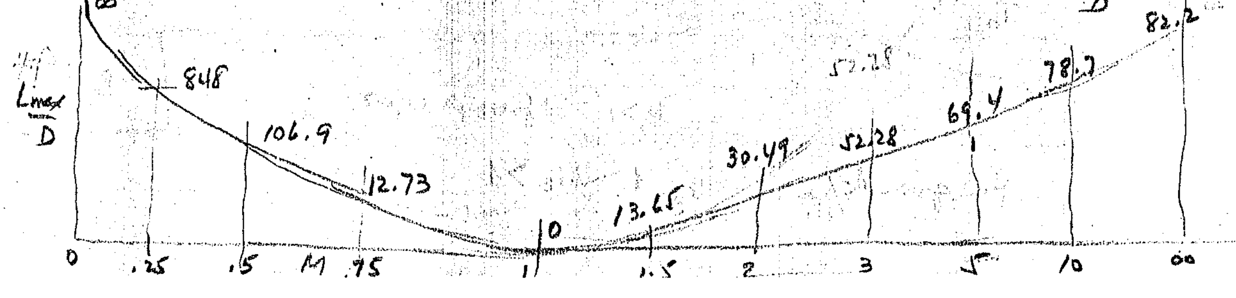
$$P_{0_1} = \left(\frac{P_0}{P_1}\right)_{M_1=.2} P_1 = \frac{1}{.9725} \times 100 = 102.88 \text{ psia}$$

$$P_{0_2} = \left(\frac{P_0}{P_2}\right)_{M_2=.24} P_2 \quad \frac{1}{.9607} \times 83.187 = 86.468 \text{ psia}$$

$\Delta P_0 = -16.412$ psi

from isentropic tables

let $4f = .01$ $(f = .0025)$ and look at $k=1.4$ M vs. $\frac{L_{max}}{D}$



Case 1 $\quad 4\frac{fL_{12}}{D} < \frac{4fL_{max}}{D}$

a. $\quad p_B > p^*$ unchoked $\qquad$ this p^* is not same as isentropic p^*

b $\quad p_B \leq p^*$ choked $\quad M_e = 1$

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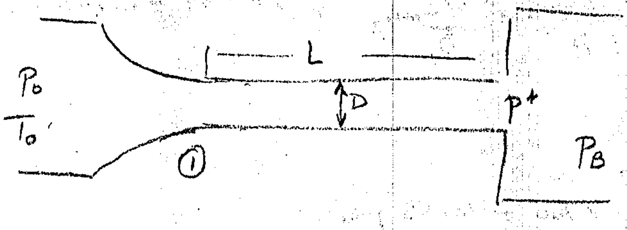
$p^* = fn\,(f, L, D, P_0)$



$L = 2270$ ft $\qquad D = 1'$

$4\bar{f} = .02$

$$4\bar{f}\frac{L}{D} = 45.40$$

$\left.\begin{array}{l}P_0\\ T_0\end{array}\right\}$ given as $\begin{array}{l}100\text{ psi}\\ 520\,°R\end{array}$

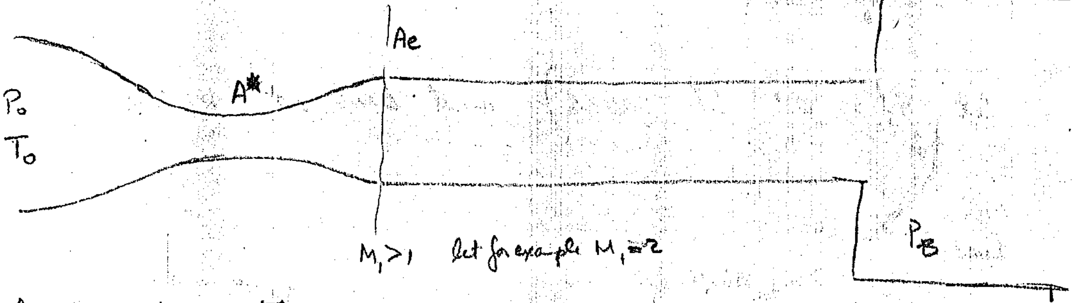
if we ask, will $M_1 = .2$? $\Rightarrow \left.\frac{4fL_{max_1}}{D}\right|_{M_1=.2} = 14.533 \quad \Rightarrow$ pipe is too long for $M_1 = .2$

$\Rightarrow$ somewhere the pipe will reach $M = 1 \Rightarrow M_e = M^* = 1 \Rightarrow$ flow will choke

thus $\dot{m} < \dot{m}_{M=.2}$. To find $\dot{m}$ use $\frac{4fL}{D} = \frac{4fL_{max}}{D}$

$\frac{4fL}{D} = \frac{4fL_{max_2}}{D} \Rightarrow M_1 = .12$ ∴ we can then get $\dot{m}_{M=.12}$

Thus no matter how much you drop $P_B < p^*$ you cannot get $M_1 > .12$

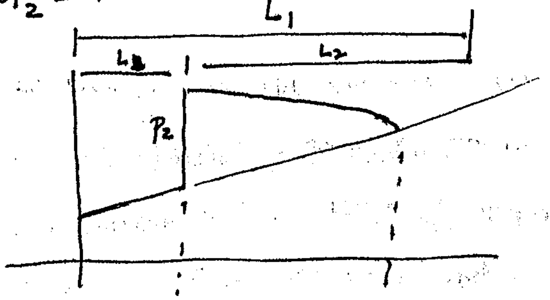


for a given A_e/A^* $\exists$ only 1 $M_e > 1$

5.15

$M_1 = 2.5 \quad T_1 = 310\ ^\circ K \quad p_1 = 70\ kPa. \qquad D = D_H = .02 m \qquad f = .005$

$M_2 = 2$



$$\frac{4fL_1}{D} = .43197 \qquad \frac{4fL_2}{D} = .30499$$

$$\frac{4fL}{D} = .127 = \frac{4fL_1}{D} - \frac{4fL_2}{D}$$

$$L_s = \frac{.127(2)cm}{4(.005)} = 12.7\ cm$$

$$\frac{.127\cancel{(2)}^{100}}{\cancel{.02}}$$

$$\left.p/p^*\right|_{M_1} = .29212 \qquad p^* = \left.p^*/p_1\right|_{M_1} \cdot p_1 = \frac{70}{.29212} \sim 235 kPa$$

$$\left.p/p^*\right|_{M_2} = .40825 \qquad p_2 = p_2/p_* \cdot p_* = p_2/p_* \cdot p^*/p_1 \cdot p_1$$

$$= \boxed{\frac{.40825}{.29212}}_{\sim 1.4} \cdot 70 = 98$$

$$M_{2x} = 2 \quad M_{2y} = .5774 \qquad p_y/p_x = \cancel{.45}\ 4.5 \quad \therefore \quad p_y = p_2 \times 4.5 = 441\ kPa$$

$$1.8384 = p/p_* = p_x/p_* \qquad \frac{4fL}{D} = .5896$$

$$p_* = p_*/p \cdot p_y = \frac{441}{1.838} \qquad .5173 = \frac{4fL_s}{D}$$

$$M = .8 \quad p/p_* = 1.2892 \quad \frac{4fL}{D} = .072\overset{3}{\cancel{29}} \qquad \frac{.5173 \cdot \cancel{2}^{100}}{\cancel{.005} \cdot 4} = 51.73$$

Total is 64.43 cm

$p = 1.2892\ (235\ kPa) = 300\ kPa$

Engineering Department will be initiated and data gathered by a half-time statistician.

9) Video cassette modules covering those topics defined in (7) would be developed and kept in the library, for use by the students in preparation for the Engineering Fundamentals examination. A more advanced aspect of this would be implementation of a self-paced schedule for courses listed in (7). Along with the self-paced modules, the student would be provided with instructor support to answer questions.

10) Passing the Engineering Fundamentals (EF) examination would not affect the normal class grade as class work is considered to be at a higher level than that tested by the EF examination.

By accomplishing these ten steps, we can increase the pass rate of our students in the Fundamentals of Engineering examination, increase the number of qualified graduating minorities and innovatively provide instruction to both normal and, in the future, "fast track" students.

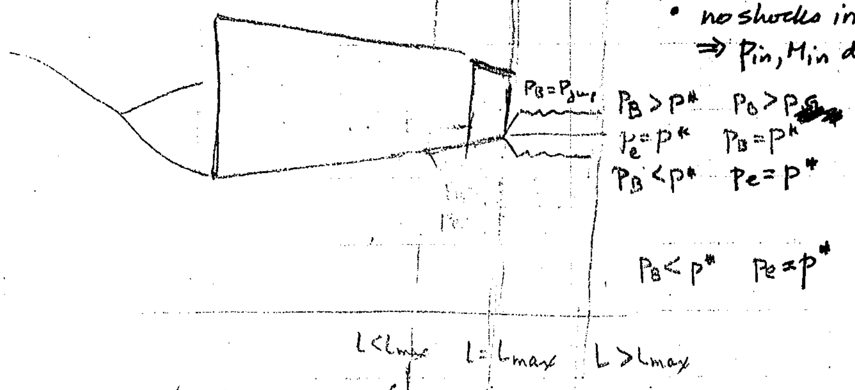
move before 7)

6) To further increase the number of minorities entering the university, a basic level synthesis type course will be developed and inserted into Dade County Public Schools/FIU's Florida Action for Minorities in Engineering (FLAME) program. ~~The program is a three-year program that emphasizes academic excellence in the field of engineering. It takes students from the tenth to twelfth grade of high school and teaches them what engineering is all about. In the senior year of high school, the student is dual enrolled with FIU, taking undergraduate courses in Engineering Drawing and Applied Engineering Power Applications.~~ By designing a "synthesis course" similar to those described above but at a more basic level, the FLAME program can be a pipeline providing FIU with minorities (predominantly Black at this time) that are interested in engineering. This aspect of the plan is scheduled for completion by the end of year 1. It continues to be on schedule

ACTION: ←

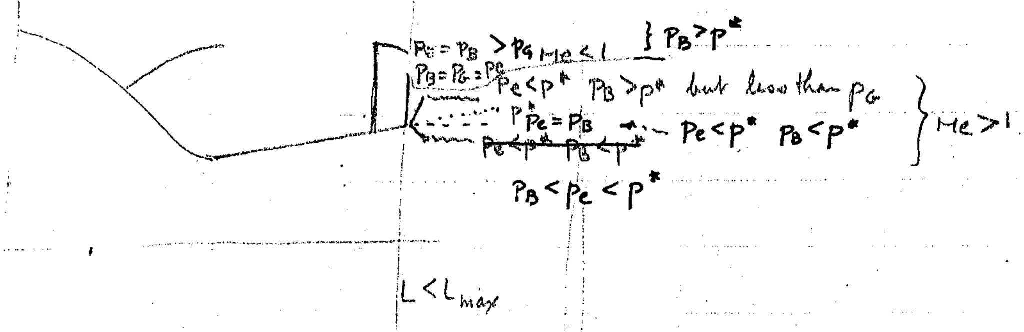
- Confined ourselves to case where no shocks in nozzle & flow in duct is <u>not</u> subsonic
- no shocks in nozzle
  $\Rightarrow p_{in}, M_{in}$ defined by A_E/A_t

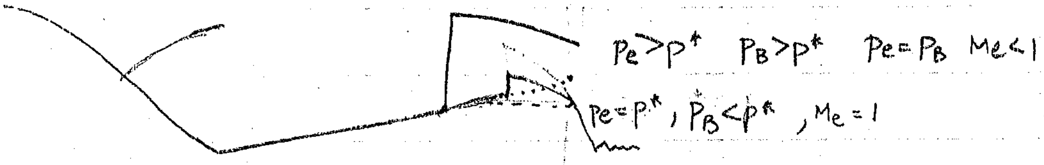
$$\boxed{\begin{array}{l} p^* \text{ is a fn of } M_{in}, p_{in} - \\ \text{but } p^*_{e_{si}} \text{ is a fn of } M_{in}, p_{in}, \frac{4fL}{D_H} \end{array}}$$



$\frac{dQ}{T} < ds$

$\therefore Tds - dQ > 0 \implies$





~~$Tds - dQ = dh$~~

$$dQ = c_p dT + d\left(\frac{V^2}{2}\right) = c_p dT_0$$

$$Tds = dh - dp/\rho$$

$$\therefore Tds - dQ = \cancel{dh} - dp/\rho - c_p \cancel{dT} - d\left(\frac{V^2}{2}\right)$$

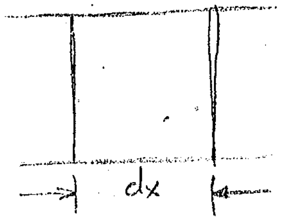
$$= - dp/\rho - d\left(\frac{V^2}{2}\right)$$

$$= - \frac{dp}{kp} \cdot kp/\rho$$

$$=$$

Isothermal flow

long ducts are isothermal rather than adiabatic



Momentum remains same

Cont. remains same

energy changes $dQ = C_p \cancelto{0}{dT} + d\left(V^2/2\right) = C_p dT_0$

for a perfect gas. isothermal cond

$$T_0 = T\left(1 + \frac{k-1}{2} M^2\right) \qquad \frac{dT_0}{T_0} = \frac{(k-1)M^2}{2\left(1 + \frac{k-1}{2}M^2\right)} \frac{dM^2}{M^2}$$

from perfect gas eq of state $p = \rho R T \Rightarrow \frac{dp}{p} \overset{\text{EQ of state}}{=} \frac{d\rho}{\rho} \overset{\text{cont}}{=} -\frac{dV}{V}$

and from def of Mach no. $\frac{dM^2}{M^2} = 2\frac{dV}{V}$ (since $dc^2 = 0$)

Cont. can be rewritten $\frac{d\rho}{\rho} = -\frac{1}{2}\frac{dV^2}{V^2}$

Mom. " " " $\frac{dp}{p} + \frac{kM^2}{2} 4f\frac{dx}{D} + \frac{kM^2}{2}\frac{dV^2}{V^2} = 0$

we can get a relation from $\frac{P_0}{P} = \left(1 + \frac{k-1}{2}M^2\right)^{k/k-1}$ for dp_0, dp, dM^2

hence we finally show $\underbrace{dp/p = d\rho/\rho}_{\text{STATE}} \underbrace{= -\frac{dV}{V}}_{\text{Continuity}} \underbrace{= -\frac{dM^2}{2M^2}}_{\text{MACH}} \underbrace{= \frac{-kM^2}{2} \cdot \frac{1}{(1-kM^2)} 4f\frac{dx}{D}}_{\text{MOM}}$

$$\frac{dT_0}{T_0} = \frac{k(k-1)M^4}{2(1-kM^2)\left(1 + \frac{k-1}{2}M^2\right)} 4f\frac{dx}{D}$$

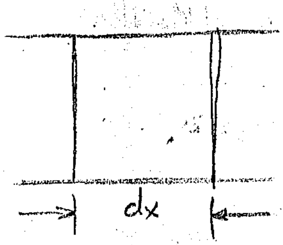
$$\frac{dP_0}{P_0} = \frac{kM^2\left(1 - \frac{k+1}{2}M^2\right)}{2(1-kM^2)\left(1 + \frac{k-1}{2}M^2\right)} 4f\frac{dx}{D}$$

Singular @ $M = \frac{1}{\sqrt{k}}$ or choking occurs when $M = \frac{1}{\sqrt{k}}$

$$\boxed{4f\frac{L_{max}}{D} = \frac{1-kM^2}{kM^2} + \ln kM^2}$$

Isothermal flow

long ducts are isothermal rather than adiabatic : in temp equilibrium w/ surround



Momentum remains same

Cont. remains same

energy <u>change</u> $dQ = c_p \overset{\to 0}{\cancel{dT}} + d\left(\frac{V^2}{2}\right) = c_p dT_0$

for a perfect gas, isothermal cond

$$T_0 = T\left(1 + \frac{\gamma-1}{2}M^2\right) \qquad \frac{dT_0}{T_0} = \frac{(\gamma-1)M^2}{2\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM^2}{M^2}$$

from perfect gas eq of state $p = \rho RT \Rightarrow \frac{dp}{p} \overset{\text{EQ of state}}{=} \frac{d\rho}{\rho} \overset{\text{cont}}{=} -\frac{dV}{V}$

and from def of Mach no. $\frac{dM^2}{M^2} = 2\frac{dV}{V}$ (since $dc^2 = 0$)
since $T =$ constant

Cont. can be rewritten $\frac{d\rho}{\rho} = -\frac{1}{2}\frac{dV^2}{V^2}$

Mom. " " " $\frac{dp}{p} + \frac{\gamma M^2}{2} 4f\frac{dx}{D} + \frac{\gamma M^2}{2}\frac{dV^2}{V^2} = 0$

we can get a relation from $\frac{P_0}{P} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/\gamma-1}$ for dp_0, dp, dM^2

hence we finally show $\underbrace{dp/p = d\rho/\rho}_{\text{state}} \underbrace{= -\frac{dV}{V}}_{\text{continuity}} \underbrace{= -\frac{dM^2}{2M^2}}_{\text{mach \#}} \underbrace{= -\frac{\gamma M^2}{2}\cdot\frac{1}{(1-\gamma M^2)}4f\frac{dx}{D}}_{\text{momentum}}$

$$\frac{dT_0}{T_0} = \frac{\gamma(\gamma-1)M^4}{2(1-\gamma M^2)\left(1+\frac{\gamma-1}{2}M^2\right)} 4f\frac{dx}{D}$$

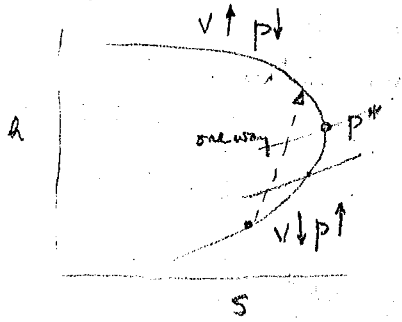
$$\frac{dP_0}{P_0} = \frac{\gamma M^2\left(1 - \frac{\gamma+1}{2}M^2\right)}{2(1-\gamma M^2)\left(1+\frac{\gamma-1}{2}M^2\right)} 4f\frac{dx}{D}$$

Singular @ $M = \frac{1}{\sqrt{\gamma}}$ or choking occurs when $M = \frac{1}{\sqrt{\gamma}}$

$$\boxed{4f\frac{L_{max}}{D} = \frac{1-\gamma M^2}{\gamma M^2} + \ln \gamma M^2}$$

$P_E' \triangleq$ exit pressure for shock free flow in nozzle & duct
$P_G \triangleq$ Pressure behind normal shock at exit plane.
P^* flow ∋. $M_{exit} = 1$
} 3 under pressures

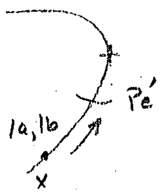
5 flow regimes exist and we have possibilities of the shock.



for $\frac{4fL}{D}$ and $M_1 = 2$

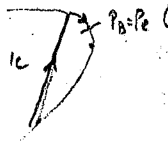
Class 1a. $p_B \le p_e' < p^*$ Supersonic throughout $L < L_{max}$
since $p_e' < p^*$ $L < L_{max}$

Class 1 $P_e' < P^*$



(a) $P_B \le P_e' < P^*$ Supersonic to end of duct w/ Expansion waves outside nozzle

(b) $P_G \ge P_B > P_E'$ Supersonic to end of duct w/ compression waves outside nozzle



(c) $P_B > P_G$; $P_E' < P^*$ shock in duct. $4\bar{f}\frac{L_1}{D} < 4\bar{f}\frac{L_{max}}{D}$

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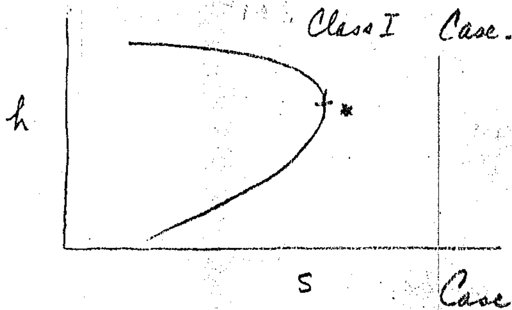
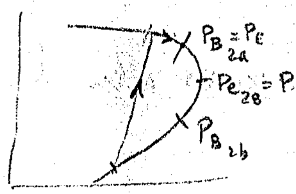
Supersonic nozzle feeding a const. area duct w/ friction but w/o energy exch.
SSF

Class I Case I $P_E' < P^* \Rightarrow L < L_{max}$
P_G



Case IV Class II $P_e' > p^*$
$L > L_{max}$



2a $P_B > p^*$ shock w/ $P_E = P_B$

2b $P_e' > p^*$ and $P_B < P^*$ shock w/ $P_e = P^*$ expansion outside

Rayleigh - difficult to achieve in practice; also known as T_0 change Process

$\frac{d\rho}{\rho}+\frac{dV}{V}=0$ Continuity $(\rho VA)=\text{const}$

$dp+\rho V dV=0$ momentum , $p+\rho V^2=\text{const}$

$\delta q=\frac{Q}{\dot{m}}=dh+V dV=dh_0$ energy

$\frac{dp}{p}=\frac{d\rho}{\rho}+\frac{dT}{T}$ state $p=\rho RT$

now since $\rho V^2=\gamma p M^2 \Rightarrow p_2-p_1=\rho V_1^2-\rho V_2^2=\gamma p_1 M_1^2-\gamma p_2 M_2^2$

or $p_2(1+\gamma M_2^2)=p_1(1+\gamma M_1^2) \Rightarrow p_2/p_1=\frac{(1+\gamma M_1^2)}{(1+\gamma M_2^2)}$

Now since Δs change p_{o_2} & p_{o_1} are not same

Limitations

normally heat transfer & friction are connected

are changes in stagnation temperature due to combustion $\Rightarrow$ chem composition changes

Good model when heating/cooling is through external heat exchange

if chemical composition remains same $\gamma_1=\gamma_2$

note $dq=\dot{m}c_p\Delta T_0 \Rightarrow$ change in Q ~ change in T_0

∴ independent variable is T_0 (Q)

Second Law of Thermodynamics. Application of the Second Law of Thermodynamics is simplified through use of the entropy. For a semi-perfect gas and no changes in chemical composition, the entropy change of the main gas stream alone is $T ds = c_p dT - \frac{1}{\rho} dp$

$\frac{dT}{T} - \frac{dp}{\rho T} \rightarrow$

$$\frac{ds}{c_p} = \frac{dT}{T} - \frac{k-1}{k}\frac{dp}{p} \tag{8.25}$$

More important, however, is the total entropy change of the main stream plus injected gas and evaporated liquid. It may be shown that for unchanged chemical composition

$$d\bar{s} = ds + (s - s_g)\frac{dw_g}{w} + (s - s_L)\frac{dw_L}{w} \tag{8.26}$$

where $d\bar{s}$ is the total entropy change per unit mass of main gas stream.

8.3. Working Equations and Tables of Influence Coefficients

Eight independent relations between the differential parameters have been set forth, namely, Eqs. 8.2, 8.4, 8.6, 8.8, 8.16, 8.21, 8.24, and 8.25. As there are fourteen differential variables, six may be chosen as independent variables and eight as dependent variables. For the independent variables we choose those most easily controlled in practice, as indicated below.

Independent	Dependent
dA/A	$d\text{M}^2/\text{M}^2$
$(dQ - dW_x + dH)/c_pT$	dV/V
$4f\frac{dx}{D} + \frac{dX}{\frac{1}{2}kpA\text{M}^2} - 2y\frac{dw}{w}$	dc/c $d\rho/\rho$
dw/w	dp/p
dW/W	dF/F
dk/k	$d\bar{s}/c_p$
	dT/T

The usual methods for solving a system of simultaneous, linear, algebraic equations may be employed for obtaining each dependent variable in terms of the six independent parameters. For example, by elimination we obtain for $d\text{M}^2/\text{M}^2$ the equation

$$\begin{aligned} d\text{M}^2/\text{M}^2 = &- \frac{2\left(1 + \frac{k-1}{2}\text{M}^2\right)}{1 - \text{M}^2}\frac{dA}{A} + \frac{1 + k\text{M}^2}{1 - \text{M}^2}\frac{dQ - dW_x + dH}{c_pT} \\ &+ \frac{k\text{M}^2\left(1 + \frac{k-1}{2}\text{M}^2\right)}{1 - \text{M}^2}\left[4f\frac{dx}{D} + \frac{dX}{\frac{1}{2}kpA\text{M}^2} - 2y\frac{dw}{w}\right] \\ &+ \frac{2(1 + k\text{M}^2)\left(1 + \frac{k-1}{2}\text{M}^2\right)}{1 - \text{M}^2}\frac{dw}{w} - \frac{1 + k\text{M}^2}{1 - \text{M}^2}\frac{dW}{W} - \frac{dk}{k} \end{aligned} \tag{8.27}$$

Influence Coefficients. We shall call the coefficients of the independent variables *influence coefficients,* since they indicate the influence of each independent variable on each of the dependent parameters. Table 8.1 lists the influence coefficients found by solution of the simultaneous equations. The examples presented later illustrate the usefulness of these influence coefficients in treating a variety of practical problems.

Useful Integral Relations. In addition to the differential relations summarized by Table 8.1, a number of useful integral relations are obtainable from Eqs. 8.1, 8.5, 8.7, 8.16, and 8.23. They may be summarized as follows:

$$\begin{aligned} Q - W_x + \Delta H \cong c_{p2}T_2\left(1 + \frac{k_2 - 1}{2}\text{M}_2^2\right) - c_{p1}T_1\left(1 + \frac{k_1 - 1}{2}\text{M}_1^2\right) \\ - (c_{p2} - c_{p1})\frac{T_1 + T_2}{2} \end{aligned} \tag{8.28}$$

from cont.

$$\frac{p_2}{p_1} = \frac{w_2}{w_1}\frac{A_1}{A_2}\frac{\text{M}_1}{\text{M}_2}\sqrt{\frac{T_2}{T_1}}\sqrt{\frac{k_1W_1}{k_2W_2}} \tag{8.29}$$

defn of mach no

$$\frac{V_2}{V_1} = \frac{\text{M}_2}{\text{M}_1}\sqrt{\frac{k_2}{k_1}\frac{T_2}{T_1}\frac{W_1}{W_2}} \tag{8.30}$$

cont — eq of state

$$\frac{\rho_2}{\rho_1} = \frac{w_2}{w_1}\frac{A_1}{A_2}\frac{V_1}{V_2} = \frac{p_2}{p_1}\frac{T_1}{T_2}\frac{W_2}{W_1} \tag{8.31}$$

$$\frac{F_2}{F_1} = \frac{p_2}{p_1}\frac{A_2}{A_1}\frac{1 + k_2\text{M}_2^2}{1 + k_1\text{M}_1^2} \tag{8.32}$$

These formulas are derived by writing the energy equation and continuity equation for two sections (1 and 2) of the flow, and by introducing

TABLE 8.1

Influence Coefficients for Variable Specific Heat and Molecular Weight

INDEP → DEP ↓	$\frac{dA}{A}$	$\frac{dQ - dW_x + dH}{c_pT}$	$4f\frac{dx}{D} + \frac{-dX}{\frac{1}{2}kpA\mathsf{M}^2} - 2y\frac{dw}{w}$	$\frac{dw}{w}$	$\frac{dW}{W}$	$\frac{dk}{k}$
$\frac{d\mathsf{M}^2}{\mathsf{M}^2}$	$-\frac{2\left(1 + \frac{k-1}{2}\mathsf{M}^2\right)}{1 - \mathsf{M}^2}$	$\frac{1 + k\mathsf{M}^2}{1 - \mathsf{M}^2}$	$\frac{k\mathsf{M}^2\left(1 + \frac{k-1}{2}\mathsf{M}^2\right)}{1 - \mathsf{M}^2}$	$\frac{2(1 + k\mathsf{M}^2)\left(1 + \frac{k-1}{2}\mathsf{M}^2\right)}{1 - \mathsf{M}^2}$	$-\frac{1 + k\mathsf{M}^2}{1 - \mathsf{M}^2}$	-1
$\frac{dV}{V}$	$-\frac{1}{1 - \mathsf{M}^2}$	$\frac{1}{1 - \mathsf{M}^2}$	$\frac{k\mathsf{M}^2}{2(1 - \mathsf{M}^2)}$	$\frac{1 + k\mathsf{M}^2}{1 - \mathsf{M}^2}$	$-\frac{1}{1 - \mathsf{M}^2}$	0
$\frac{dc}{c}$	$\frac{\frac{k-1}{2}\mathsf{M}^2}{1 - \mathsf{M}^2}$	$\frac{1 - k\mathsf{M}^2}{2(1 - \mathsf{M}^2)}$	$-\frac{k(k-1)\mathsf{M}^4}{4(1 - \mathsf{M}^2)}$	$-\frac{\frac{k-1}{2}\mathsf{M}^2(1 + k\mathsf{M}^2)}{1 - \mathsf{M}^2}$	$\frac{k\mathsf{M}^2 - 1}{2(1 - \mathsf{M}^2)}$	$\frac{1}{2}$
$\frac{dT}{T}$	$\frac{(k-1)\mathsf{M}^2}{1 - \mathsf{M}^2}$	$\frac{1 - k\mathsf{M}^2}{1 - \mathsf{M}^2}$	$-\frac{k(k-1)\mathsf{M}^4}{2(1 - \mathsf{M}^2)}$	$-\frac{(k-1)\mathsf{M}^2(1 + k\mathsf{M}^2)}{1 - \mathsf{M}^2}$	$\frac{(k-1)\mathsf{M}^2}{1 - \mathsf{M}^2}$	0
$\frac{d\rho}{\rho}$	$\frac{\mathsf{M}^2}{1 - \mathsf{M}^2}$	$-\frac{1}{1 - \mathsf{M}^2}$	$-\frac{k\mathsf{M}^2}{2(1 - \mathsf{M}^2)}$	$-\frac{(k+1)\mathsf{M}^2}{1 - \mathsf{M}^2}$	$\frac{1}{1 - \mathsf{M}^2}$	0
$\frac{dp}{p}$	$\frac{k\mathsf{M}^2}{1 - \mathsf{M}^2}$	$-\frac{k\mathsf{M}^2}{1 - \mathsf{M}^2}$	$-\frac{k\mathsf{M}^2[1 + (k-1)\mathsf{M}^2]}{2(1 - \mathsf{M}^2)}$	$-\frac{2k\mathsf{M}^2\left(1 + \frac{k-1}{2}\mathsf{M}^2\right)}{1 - \mathsf{M}^2}$	$\frac{k\mathsf{M}^2}{1 - \mathsf{M}^2}$	0
$\frac{dF}{F}$	$\frac{1}{1 + k\mathsf{M}^2}$	0	$-\frac{k\mathsf{M}^2}{2(1 + k\mathsf{M}^2)}$	0	0	0
ds/c_p (Note 2)	0	1	$\frac{(k-1)\mathsf{M}^2}{2}$	$(k-1)\mathsf{M}^2$	0	0

Notes: (1) Each influence coefficient represents the logarithmic partial derivative of the variable in the left-hand column with respect to the variable in the top row; for example

$$\frac{d\mathsf{M}^2}{\mathsf{M}^2} = -\frac{2\left(1 + \frac{k-1}{2}\mathsf{M}^2\right)}{1 - \mathsf{M}^2}\frac{dA}{A} + \frac{1 + k\mathsf{M}^2}{1 - \mathsf{M}^2}\frac{dQ - dW_x + dH}{c_pT} + \dots\dots - \frac{dk}{k}$$

(2) For unaltered chemical composition only, and referring to entropy change of main stream. See Eq. 8.26 for total entropy change.

(3) gas withdrawal works only with $y = 1$

the definitions of M and F together with the relations $p = \rho RT$ and $c = \sqrt{k\mathcal{R}T/W}$.

In deriving Eq. 8.28 it was assumed that for a moderate temperature interval it is valid to write

$$\int_{T_1}^{T_2} c_p\,dT \cong \frac{c_{p1} + c_{p2}}{2}(T_2 - T_1)$$

mean value theor

trapezoidal rule

8.4. Flow with Constant Specific Heat and Molecular Weight

There are many instances where the changes in molecular weight and specific heat are of secondary importance and may, therefore, be neglected for the sake of simplicity.

The equations of Arts. 8.2 and 8.3 remain valid, of course, but are simplified by noting that

$$dW = dc_p = dk = 0$$

and

$$W_1 = W_2; \quad c_{p1} = c_{p2}; \quad k_1 = k_2$$

Stagnation Temperature and Pressure. The stagnation temperature and stagnation pressure are, as we have seen, useful parameters for problems of one-dimensional flow.

Although these stagnation properties are always significant, they have not been brought explicitly into the equations of Arts. 8.2 and 8.3 because there are no simple algebraic relations connecting the local properties with the stagnation properties in the general case of variable specific heat and molecular weight.

Stagnation Temperature. With the assumptions of constant c_p and W, however, we may express the stagnation temperature as

$$T_0 = T + \frac{V^2}{2c_p} = T\left(1 + \frac{k-1}{2}\mathsf{M}^2\right) \tag{8.33}$$

or, in differential form, as

$$\frac{dT_0}{T_0} = \frac{dT}{T} + \frac{\frac{k-1}{2}\mathsf{M}^2}{1 + \frac{k-1}{2}\mathsf{M}^2}\frac{d\mathsf{M}^2}{\mathsf{M}^2} \tag{8.34}$$

Eq. 8.14, when combined with Eq. 8.33, yields

$$dQ - dW_x + dH = c_p\,dT + d(V^2/2) = c_p\,dT_0 \tag{8.35}$$

or

$$\frac{dQ - dW_x + dH}{c_pT} = \left(1 + \frac{k-1}{2}\mathsf{M}^2\right)\frac{dT_0}{T_0} \tag{8.36}$$

Eq. 8.35 shows that the change in T_0 is a measure of the total energy effects produced by heat transfer, shear and shaft work, chemical reaction, change of phase, injection of gases, etc. We shall, therefore, employ Eq. 8.34 rather than Eq. 8.16 as a representation of the energy equation. The parameter $(dQ - dW_x + dH)/c_pT$ is then replaced by dT_0/T_0 in our system of equations. These parameters are related through Eq. 8.36.

STAGNATION PRESSURE. The isentropic stagnation pressure is related to the local pressure and Mach Number through the familiar formula

$$p_0 = p\left(1 + \frac{k-1}{2}\mathrm{M}^2\right)^{\frac{k}{k-1}} \tag{8.37}$$

or, in differential form,

$$\frac{dp_0}{p_0} = \frac{dp}{p} + \frac{k\mathrm{M}^2/2}{1 + \frac{k-1}{2}\mathrm{M}^2}\frac{d\mathrm{M}^2}{\mathrm{M}^2} \tag{8.38}$$

The change of entropy, in terms of p_0 and T_0, is then

$$\frac{ds}{c_p} = \frac{dT_0}{T_0} - \frac{k-1}{k}\frac{dp_0}{p_0} \tag{8.39}$$

Working Equations and Table of Influence Coefficients. As in Art. 8.3, the differential working equations are arranged in terms of independent parameters, influence coefficients, and dependent parameters.

For example, Eq. 8.27 becomes

$$\frac{d\mathrm{M}^2}{\mathrm{M}^2} = -\frac{2\left(1 + \frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}\frac{dA}{A} + \frac{(1+k\mathrm{M}^2)\left(1 + \frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}\frac{dT_0}{T_0}$$
$$+ \frac{k\mathrm{M}^2\left(1 + \frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}\left(4f\frac{dx}{D} + \frac{dX}{\frac{1}{2}kpA\mathrm{M}^2} - 2y\frac{dw}{w}\right)$$
$$+ \frac{2(1+k\mathrm{M}^2)\left(1 + \frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}\frac{dw}{w} \tag{8.40}$$

The complete array of influence coefficients is given in Table 8.2.

TABLE 8.2

INFLUENCE COEFFICIENTS FOR CONSTANT SPECIFIC HEAT AND MOLECULAR WEIGHT

INDEP → DEP ↓	$\frac{dA}{A}$	$\frac{dT_0}{T_0}$	$4f\frac{dx}{D} + \frac{dX}{\frac{1}{2}kpA\mathrm{M}^2} - 2y\frac{dw}{w}$	$\frac{dw}{w}$
$\frac{d\mathrm{M}^2}{\mathrm{M}^2}$	$-\frac{2\left(1+\frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}$	$\frac{(1+k\mathrm{M}^2)\left(1+\frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}$	$\frac{k\mathrm{M}^2\left(1+\frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}$	$\frac{2(1+k\mathrm{M}^2)\left(1+\frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}$
$\frac{dV}{V}$	$-\frac{1}{1-\mathrm{M}^2}$	$\frac{1+\frac{k-1}{2}\mathrm{M}^2}{1-\mathrm{M}^2}$	$\frac{k\mathrm{M}^2}{2(1-\mathrm{M}^2)}$	$\frac{1+k\mathrm{M}^2}{1-\mathrm{M}^2}$
$\frac{dc}{c}$	$\frac{\frac{k-1}{2}\mathrm{M}^2}{1-\mathrm{M}^2}$	$\frac{\frac{1-k\mathrm{M}^2}{2}\left(1+\frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}$	$-\frac{k(k-1)\mathrm{M}^4}{4(1-\mathrm{M}^2)}$	$-\frac{\frac{k-1}{2}\mathrm{M}^2(1+k\mathrm{M}^2)}{1-\mathrm{M}^2}$
$\frac{dT}{T}$	$\frac{(k-1)\mathrm{M}^2}{1-\mathrm{M}^2}$	$\frac{(1-k\mathrm{M}^2)\left(1+\frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}$	$-\frac{k(k-1)\mathrm{M}^4}{2(1-\mathrm{M}^2)}$	$-\frac{(k-1)\mathrm{M}^2(1+k\mathrm{M}^2)}{1-\mathrm{M}^2}$
$\frac{d\rho}{\rho}$	$\frac{\mathrm{M}^2}{1-\mathrm{M}^2}$	$-\frac{1+\frac{k-1}{2}\mathrm{M}^2}{1-\mathrm{M}^2}$	$-\frac{k\mathrm{M}^2}{2(1-\mathrm{M}^2)}$	$-\frac{(k+1)\mathrm{M}^2}{1-\mathrm{M}^2}$
$\frac{dp}{p}$	$\frac{k\mathrm{M}^2}{1-\mathrm{M}^2}$	$-\frac{k\mathrm{M}^2\left(1+\frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}$	$-\frac{k\mathrm{M}^2[1+(k-1)\mathrm{M}^2]}{2(1-\mathrm{M}^2)}$	$-\frac{2k\mathrm{M}^2\left(1+\frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}$
$\frac{dp_0}{p_0}$	0	$-\frac{k\mathrm{M}^2}{2}$	$-\frac{k\mathrm{M}^2}{2}$	$-k\mathrm{M}^2$
$\frac{dF}{F}$	$\frac{1}{1+k\mathrm{M}^2}$	0	$-\frac{k\mathrm{M}^2}{2(1+k\mathrm{M}^2)}$	0
$\frac{ds}{c_p}$	0	$1+\frac{k-1}{2}\mathrm{M}^2$	$\frac{(k-1)\mathrm{M}^2}{2}$	$(k-1)\mathrm{M}^2$

NOTE: Each influence coefficient represents the partial logarithmic derivative of the variable in the left-hand column with respect to the variable in the top row; for example

$$\frac{d\mathrm{M}^2}{\mathrm{M}^2} = -\frac{2\left(1+\frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}\frac{dA}{A} + \frac{(1+k\mathrm{M}^2)\left(1+\frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}\frac{dT_0}{T_0} + \frac{k\mathrm{M}^2\left(1+\frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}\left(4f\frac{dx}{D} + \frac{dX}{\frac{1}{2}kpA\mathrm{M}^2} - 2y\frac{dw}{w}\right)$$
$$+ \frac{2(1+k\mathrm{M}^2)\left(1+\frac{k-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}\frac{dw}{w}$$

gas withdrawal works only with $y=1$

Same as table 6.5 in Saad

USEFUL INTEGRAL RELATIONS. The several useful integral relations between the properties at two cross-sections may be summarized as follows:

$$Q - W_x + \Delta H = c_p(T_{02} - T_{01}) \tag{8.41}$$

since C_p = const
W = const
k = const

$$\frac{p_2}{p_1} = \frac{w_2}{w_1}\frac{A_1}{A_2}\frac{\mathsf{M}_1}{\mathsf{M}_2}\sqrt{\frac{1 + \frac{k-1}{2}\mathsf{M}_1{}^2}{1 + \frac{k-1}{2}\mathsf{M}_2{}^2}}\sqrt{\frac{T_{02}}{T_{01}}} \tag{8.42}$$

$$\frac{V_2}{V_1} = \frac{\mathsf{M}_2}{\mathsf{M}_1}\sqrt{\frac{T_2}{T_1}} \tag{8.43}$$

$$\frac{\rho_2}{\rho_1} = \frac{w_2}{w_1}\frac{A_1}{A_2}\frac{V_1}{V_2} = \frac{p_2}{p_1}\frac{T_1}{T_2} \tag{8.44}$$

$$\frac{T_2}{T_1} = \frac{T_{02}}{T_{01}}\,\frac{1 + \frac{k-1}{2}\mathsf{M}_1{}^2}{1 + \frac{k-1}{2}\mathsf{M}_2{}^2} \tag{8.45}$$

$$\frac{p_{02}}{p_{01}} = \frac{p_2}{p_1}\left[\frac{1 + \frac{k-1}{2}\mathsf{M}_2{}^2}{1 + \frac{k-1}{2}\mathsf{M}_1{}^2}\right]^{\frac{k}{k-1}} \tag{8.46}$$

8.5. General Features of Flow Patterns

The influence coefficients are indicative of some interesting aspects of compressible flow patterns.

w/ variable k, W, Cp

Effects of Independent Parameters. The influence coefficients for M, V, T, ρ, and p have, with the exception of those pertaining to the independent parameter dk/k, the term $(1 - \mathsf{M}^2)$ in the denominator. From this it is evident that the Mach Number, velocity, temperature, pressure, and density, all change at an infinite rate in the neighborhood of Mach Number unity. Moreover, wherever the signs of the influence coefficients change at $\mathsf{M} = 1$, the effects of the independent variables are of opposite sign for subsonic and supersonic flow. An exception concerns the effect of heating on the temperature, for which it will be noted that the influence coefficient changes sign at $\mathsf{M} = 1/\sqrt{k}$ as well as at $\mathsf{M} = 1$.

The individual influence of each independent variable is readily found from Table 8.1. From Eqs. 8.27 and 8.40 these influences are seen to be independent of each other, i.e., the incremental effects of small changes in the independent variables are linearly additive.

EFFECTS ON M. An increase in area tends to reduce M in subsonic flow and to increase it in supersonic flow.

Heating or combustion tends to increase M at subsonic speeds, and to decrease M at supersonic speeds.

The effect of friction, or drag of internal bodies, acts to increase M at subsonic speeds, and to decrease M at supersonic speeds. Note that the friction term, $f\,dx$, and internal-drag term, dX, are always positive, while the other independent variables may be either positive or negative.

An increase in mass flow (with $y < 1$ for the added flow) tends to increase M at subsonic speeds and to decrease M at supersonic speeds.

When the molecular weight is increased, the Mach Number tends to decrease for subsonic flow and to increase for supersonic flow.

An increase in the isentropic exponent k always tends to reduce M, for either subsonic or supersonic speeds.

EFFECTS ON STAGNATION PRESSURE. Changes in area have no effect on stagnation pressure, at least when c_p and W are constant.

none when C_p, W con.
ie look at isentropic flow

An increase in stagnation temperature always tends toward a reduction in p_0. Thus the stagnation-pressure loss in combustion chambers can never be entirely eliminated.

Friction and drag likewise always act to reduce p_0.

Gas injection tends to decrease p_0 if $y < 1$, i.e., if the gas is injected with a forward velocity less than that of the main stream; while the stagnation pressure tends to be increased when gas is injected with a forward velocity greater than that of the main stream (as in a jet pump).

The Second Law of Thermodynamics. Considering the simple case where there is no combustion, no W_x, and no injection, we find from Table 8.1 that

[illegible] = 0, W_x = 0, [illegible] = 0

$$ds = \frac{dQ}{T} + c_p\frac{k-1}{2}\mathsf{M}^2\left[4f\frac{dx}{D} + \frac{dX}{\frac{1}{2}kpA\mathsf{M}^2}\right]$$

Application of the Second Law of Thermodynamics to the flow through the control surface yields

$$ds - \frac{dQ}{T} \geq 0$$

from which it may be shown that both $f\,dx$ and dX are always positive, i.e., negative friction or negative drag would constitute violations of the Second Law.

The term $4f\,dx/D$ is always positive. Therefore, if the area continually decreases and the stagnation temperature continually increases it is clear that $(1 - M^2)\,dM^2$ will always be positive and that choking effects must ensue. On the other hand, if the area were continually increased and the stagnation temperature continually decreased, these effects might outweigh frictional effects and act to maintain $(1 - M^2)\,dM^2$ always negative, in which case there would be no difficulty in continuing the flow pattern indefinitely.

CONTINUOUS TRANSITION THROUGH SPEED OF SOUND. In order to obtain a continuous transition from subsonic to supersonic flow, or vice versa, it is evident that the rates of change of area, stagnation temperature, and frictional length must be such as to produce a change of sign in $(1 - M^2)\,dM^2$ at Mach Number unity. That is, $(1 - M^2)\,dM^2$ must be generally positive until Mach Number unity is reached, at which point $(1 - M^2)\,dM^2$ must pass through zero and become negative, after which the Mach Number will change in the direction away from unity. This is a delicate process near $M = 1$, since dA/A, $4f\,dx/D$ and dT_0/T_0 must be so controlled relative to each other as to make $(1 - M^2)\,dM^2$ pass from positive values through zero to negative values exactly at the moment when $M = 1$. When the flow is from subsonic to supersonic speeds, this continuous transition is easy to carry out in practice. The continuous transition from supersonic to subsonic speeds, on the other hand, is hardly ever realized in practice, and is probably unstable under most conditions.

8.6. General Method of Solution

Integration of Differential Equations. A typical problem involving simultaneous effects will usually involve known initial conditions (M_1, p_1, T_1, etc.) at some point in the duct, together with certain prescribed variations in area, stagnation temperature, and in the remaining independent parameters of Tables 8.1 and 8.2. Integration of Eq. 8.27 then yields the value of the Mach Number at each section of the duct downstream of the initial section. Sometimes this integration may be carried out analytically, but more often an approximate step-wise integration is necessary.

In order to find the corresponding values of p, T, V, etc. at each section of the duct, two procedures are open. First, the differential equations of Table 8.1 for dp/p, dT/T, dV/V, etc. may be integrated analytically or numerically. Or, secondly, since the value of M is known at each section from the previous integration, Eqs. 8.28 to 8.32 and 8.41 to 8.46 may be used for finding the remaining stream properties.

Suppose that the values of M_1, p_1, T_1, A_1, etc. are known at a section 1, and that the values of A_2, $(Q - W_x + \Delta H)_{1-2}$, $x_2 - x_1$, w_2, W_2, k_2, etc. are known for a section 2, a small distance downstream of section 1. Then, from Eq. 8.27, the value of M_2 may be found by numerical or graphical integration. Next, Eq. 8.28 is solved for T_2, Eq. 8.29 for p_2, and Eqs. 8.30 and 8.31 for V_2 and ρ_2, respectively. The procedure is then repeated for a step between sections 2 and 3, and so on.

Integration Near Mach Number Unity. Special precautions in numerical integration must be taken near $M = 1$ because many of the influence coefficients there approach infinity. It is usually possible to avoid difficulty by obtaining an approximate analytical formula for a step near $M = 1$, making use of the approximation that $1 - M^2$ is negligible compared with unity. For example, consider two of the terms of Eq. 8.40:

$$\frac{dT_0}{T_0} = \frac{1 - M^2}{M^2(1 + kM^2)\left(1 + \dfrac{k-1}{2}M^2\right)}\,dM^2$$

Define, for compactness, $N \equiv 1 - M^2$. Then, when $M \cong 1$, $N \ll 1$, and it may be shown by means of binomial expansions that

$$\frac{dT_0}{T_0} \cong \frac{2}{(k+1)^2}N(1 + N + \cdots)\left(1 + \frac{2k-1}{k+1}N + \cdots\right)dN$$

Retaining terms only up to the first order in N, and integrating, we finally obtain

$$(1 - M^2)^2 \cong (k+1)^2\,\frac{T_0{}^* - T_0}{T_0}$$

where $T_0{}^*$ is the stagnation temperature at Mach Number unity.

Tables of Influence Coefficients. For convenience in performing numerical calculations the influence coefficients (multiplied by M^2) are tabulated in Table B.6 for $k = 1.4$, with M as the independent argument. In order to explain the notation of these tables, we rewrite Eqs. 8.27 and 8.40 in the respective forms

$$\begin{aligned} dM^2 = F_A\frac{dA}{A} &+ F_Q\frac{dQ - dW_x + dH}{c_pT} \\ &+ F_f\left(4f\frac{dx}{D} + \frac{dX}{\frac{1}{2}kpAM^2} - 2y\frac{dw}{w}\right) \\ &+ F_w\frac{dw}{w} + F_W\frac{dW}{W} - \frac{dk}{k} \end{aligned} \qquad (8.47)$$

$$dM^2 = F_A\frac{dA}{A} + F_{T_0}\frac{dT_0}{T_0} + F_f\left(4f\frac{dx}{D} + \frac{dX}{\frac{1}{2}kpAM^2} - 2y\frac{dw}{w}\right) + F_w\frac{dw}{w} \qquad (8.48)$$

where

$$F_A \equiv -\frac{2M^2\left(1+\frac{k-1}{2}M^2\right)}{1-M^2} \tag{8.49a}$$

$$F_Q = -F_W \equiv \frac{M^2(1+kM^2)}{1-M^2} \tag{8.49b}$$

$$F_f \equiv \frac{kM^4\left(1+\frac{k-1}{2}M^2\right)}{1-M^2} \tag{8.49c}$$

$$F_{T_0} = \frac{F_w}{2} \equiv \frac{M^2(1+kM^2)\left(1+\frac{k-1}{2}M^2\right)}{1-M^2} \tag{8.49d}$$

The influence coefficients F_A, F_Q, etc., are seen to be greater than those in Tables 8.1 and 8.2 by the factor M^2.

Specific illustrations of both analytical and numerical procedures are given in the examples which follow.

8.7. Simple Types of Flow

A *simple type of flow* is defined here as one in which all but one of the independent differentials in Table 8.2 are zero. Thus Simple Area-Change refers to a process in which the area changes without frictional, stagnation-temperature, or gas-injection effects. Similarly, Simple T_0-Change refers to a process in which the stagnation temperature changes (due to external heat exchange) without area change, friction, or gas injection. In each of the simple types of flow the specific heat and molecular weight are constant.

Three of the simple types of flow have, of course, been discussed in Chapters 4, 6, and 7.

The integration of Eq. 8.40 may be carried out analytically for each of the simple types of flow. For example, consider Simple Area-Change. Using the first influence coefficient in Table 8.2, we obtain

$$\int_{A^*}^{A}\frac{dA}{A} = -\int_1^{M^2}\frac{(1-M^2)\,dM^2}{2M^2\left(1+\frac{k-1}{2}M^2\right)}$$

Integration yields

$$\frac{A}{A^*} = \frac{1}{M}\sqrt{\left[\frac{2\left(1+\frac{k-1}{2}M^2\right)}{k+1}\right]^{\frac{k+1}{k-1}}}.$$

TABLE 8.3

FORMULAS FOR SIMPLE TYPES OF ONE-DIMENSIONAL FLOW WITH CONSTANT SPECIFIC HEAT AND MOLECULAR WEIGHT

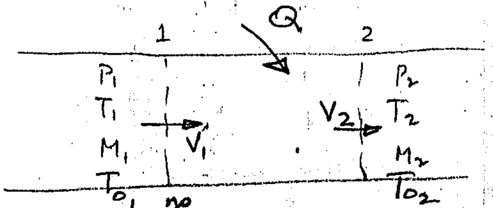
	Area change only $dT_0 = 0$ $f = 0$ $dw = 0$	Change in T_0 only $dA = 0$ $f = 0$ $dw = 0$	Friction only $dA = 0$ $dT_0 = 0$ $dw = 0$	Gas injection only $y = 0$ $dA = 0$ $dT_0 = 0$ $f = 0$	Gas injection only $y = 1$ $dA = 0$ $dT_0 = 0$ $f = 0$
$\frac{A}{A^*}$	$\frac{1}{M}\sqrt{\left[\frac{2\left(1+\frac{k-1}{2}M^2\right)}{k+1}\right]^{\frac{k+1}{k-1}}}$	1	1	1	1
$\frac{T_0}{T_0^*}$	1	$\frac{2(k+1)M^2\left(1+\frac{k-1}{2}M^2\right)}{(1+kM^2)^2}$	1	1	1
$4f\frac{L_{\max}}{D}$	0	0	$\frac{1-M^2}{kM^2}+\frac{k+1}{2k}\log_e\frac{(k+1)M^2}{2\left(1+\frac{k-1}{2}M^2\right)}$	0	0
$\frac{w}{w^*}$	1	1	1	$\frac{M\sqrt{2(k+1)\left(1+\frac{k-1}{2}M^2\right)}}{1+kM^2}$	$M\sqrt{\left[\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}\right.}$
$\frac{V}{V^*}$	$M\sqrt{\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}}$	$\frac{(k+1)M^2}{1+kM^2}$	$M\sqrt{\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}}$	$M\sqrt{\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}}$	$M\sqrt{\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right.}}$
$\frac{T}{T^*} = \left(\frac{c}{c^*}\right)^2$	$\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}$	$\frac{(k+1)^2M^2}{(1+kM^2)^2}$	$\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}$	$\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}$	$\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}$
$\frac{\rho}{\rho^*}$	$\left[\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}\right]^{\frac{1}{k-1}}$	$\frac{1+kM^2}{(k+1)M^2}$	$\frac{1}{M}\sqrt{\frac{2\left(1+\frac{k-1}{2}M^2\right)}{k+1}}$	$\frac{2\left(1+\frac{k-1}{2}M^2\right)}{1+kM^2}$	$\left[\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}\right]^{k}$
$\frac{p}{p^*} = \frac{p}{p_0}\cdot\frac{p_0}{p^*}$	$\left[\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}\right]^{\frac{k}{k-1}}$	$\frac{k+1}{1+kM^2}$	$\frac{1}{M}\sqrt{\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}}$	$\frac{k+1}{1+kM^2}$	$\left[\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}\right]^{k}$
$\frac{p_0}{p_0^*}$	1	$\frac{k+1}{1+kM^2}\left[\frac{2\left(1+\frac{k-1}{2}M^2\right)}{k+1}\right]^{\frac{k}{k-1}}$	$\frac{1}{M}\sqrt{\left[\frac{2\left(1+\frac{k-1}{2}M^2\right)}{k+1}\right]^{\frac{k+1}{k-1}}}$	$\frac{k+1}{1+kM^2}\left[\frac{2\left(1+\frac{k-1}{2}M^2\right)}{k+1}\right]^{\frac{k}{k-1}}$	1
$\frac{F}{F^*}$	$\frac{1+kM^2}{M\sqrt{2(k+1)\left(1+\frac{k-1}{2}M^2\right)}}$	1	$\frac{1+kM^2}{M\sqrt{2(k+1)\left(1+\frac{k-1}{2}M^2\right)}}$	1	$\frac{1+kM^2}{k+1}\left[\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right.}\right.$
$\frac{s-s^*}{c_p}$	0	$\log_e M^2\left(\frac{k+1}{1+kM^2}\right)^{\frac{k+1}{k}}$	$\log_e M^2\sqrt{\left[\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)M^2}\right]^{\frac{k+1}{k}}}$	$\log_e\frac{k+1}{2\left(1+\frac{k-1}{2}M^2\right)}\left(\frac{1+kM^2}{k+1}\right)^{\frac{k-1}{k}}$	0

Note: Each quantity in the table represents the value of the variable in the left-hand column under the conditions specified in the top row.

11/22/78

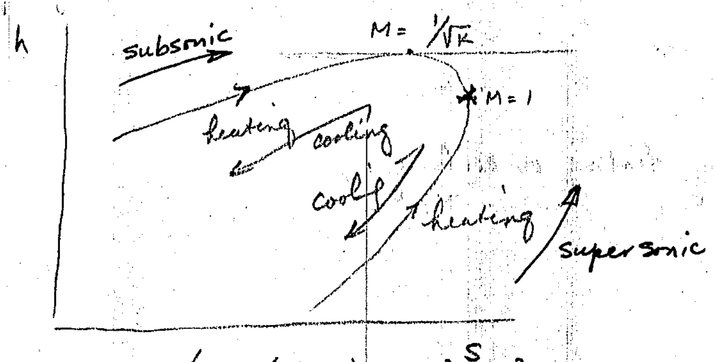
Simple Energy Exch – Rayleigh Line <u>Reversible</u>

Assume : SSF 1-D flow, const area, $\tau_W = 0$ Shocks allowed



Since ~~now~~ no wall shear mom becomes $p + \rho V^2 = \text{const.} = F/A$

cont. becomes $\rho_1 V_1 = \rho_2 V_2 \triangleq G$



as you add heat you move to the right so that entropy ↑

Energy $\frac{Q - \underset{0}{\cancel{W}}}{\dot m} = C_p(T_2 - T_1) + \frac{V_2^2}{2} - \frac{V_1^2}{2} = c_p(T_{0_2} - T_{0_1})$ $\qquad C_p T_0 \equiv C_p T + \frac{V^2}{2}$

Using perfect gas flow $\rho V^2 = \gamma p M^2$ $\qquad P_2/P_1 = \frac{\rho_2 T_2}{\rho_1 T_1}$ and $\frac{M_2}{M_1} = \frac{V_2}{V_1}\frac{C_1}{C_2} = \frac{V_2}{V_1}\sqrt{\frac{T_1}{T_2}}$

$$\frac{P_{0_2}}{P_{0_1}} = \frac{P_2}{P_1}\left[\frac{1 + \frac{\gamma-1}{2}M_2^2}{1 + \frac{\gamma-1}{2}M_1^2}\right]^{\gamma/\gamma-1} \qquad \frac{T_{0_2}}{T_{0_1}} = \frac{T_2}{T_1}\left[\frac{1 + \frac{\gamma-1}{2}M_2^2}{1 + \frac{\gamma-1}{2}M_1^2}\right] \qquad \frac{S_2 - S_1}{R} = \ln\frac{(T_2/T_1)}{(P_2/P_1)^{\frac{\gamma-1}{\gamma}}}$$

Tabulate everything wrt * conditions $(M = 1)$

$$\frac{T}{T^*} = \frac{(\gamma+1)^2 M^2}{(1+\gamma M^2)^2} \qquad \frac{V}{V^*} = \frac{(\gamma+1)M^2}{1+\gamma M^2} = \frac{\rho^*}{\rho}$$

$$\frac{T_0}{T_0^*} = \frac{2(\gamma+1)M^2\left(1 + \frac{\gamma-1}{2}M^2\right)}{(1+\gamma M^2)^2} \qquad \frac{S - S^*}{C_p} = \ln\left[M^2\left(\frac{\gamma+1}{1+\gamma M^2}\right)^{\frac{\gamma+1}{\gamma}}\right]$$

$$\frac{P}{P^*} = \frac{\gamma+1}{1+\gamma M^2}$$

if you add more heat than what is necessary to bring it to star you choke flow. If you do $\dot m$ decreases and you jumps to another Rayleigh line

heating reduces stagnation pressure

If you cool the flow you will never choke it.

	Add energy		Extract energy	
	$M^2<1$	$M^2>1$	$M^2<1$	$M^2>1$
To	↑	↑	↓	↓
M	↑	↓	↓	↑
T	↑ to $M=\frac{1}{\sqrt{K}}$ then ↓	↑	↓ for $M<\frac{1}{\sqrt{K}}$ then ↑	↓ $M=\frac{1}{\sqrt{K}}$ then ↓
ρ	↓	↑	↑	↓
P_0	↓	↓	↑	↑
$\frac{1}{\rho}$ & V	↑	↓	↓	↑

I can pump a gas by heating or cooling.

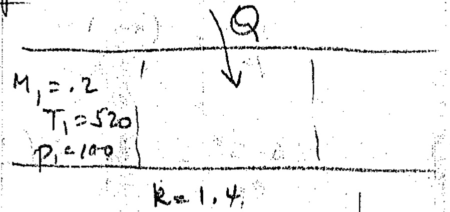
11/27/78

How to raise To

External energy add
- a. add Q
- b. and work

Change in form of internal energy
- c. Combustion (Chem. Rearrangement)
- d. Phase Change

Example



$T_{o_1} = T_1\left(1+\frac{k-1}{2}M^2\right) = 524°R$

T = 535K
P = 101.3kPa
V = 130 m/s
a = 463.64
M = .28

$\frac{T_o}{T_o^*} = .17355$

$\frac{T_o}{\;*} = .31035$

$T_o^* = \frac{T_o^*}{T_{o_1}} \cdot T_{o_1} = 3019°R$ can be unrealistic since tube will melt.

$T_o = 543.39°K$
$T_o^* = 1750.89°K$
$\Delta T_o = 1207.5$

$Q = \dot{m}\, C_p\,(\Delta T_o)$

$\frac{\dot{Q}}{\dot{m}} = C_p \Delta T_o = \frac{\gamma R}{\gamma-1}\Delta T_o = \frac{1.4}{.4} 287\,(1207.5°) = 1212933.75\ KJ/kg$

(b) When M_1 is subsonic, $\Delta H/c_pT_{01} = 0$, and $\alpha = \infty$, show that

$$\frac{p_{01}}{p_{02}} = \left(1 + \frac{k-1}{2} M_1^2\right)^{\frac{k}{k-1}}$$

and comment on the significance of this.

(c) Derive appropriate relations for the case where M_2 is exactly unity.

7.15. Air with a stagnation temperature of 70°F is supplied to an insulated pipe at Mach Number 3 by means of a converging-diverging nozzle. Assuming a recovery factor of 0.85, estimate the wall temperature of the pipe.

7.16. A ram jet flies at sea level with a speed of 1500 miles per hour. At one point in the inlet duct the Mach Number is 0.5. Estimate the adiabatic wall temperature at that section.

Chapter 8

GENERALIZED ONE-DIMENSIONAL CONTINUOUS FLOW

8.1. Introductory Remarks

Analyses were presented in the previous chapters for simple types of flow, that is, for flow processes in which only a single independent parameter was allowed to change. For example, in Chapter 4 the effects of area change alone were considered; in Chapter 6 the effects of friction alone were considered; and in Chapter 7 the effects of changes in stagnation temperature alone were considered.

In most practical problems these effects occur simultaneously, and, in addition, there may be present such other phenomena as chemical reaction, change of phase, injection or withdrawal of gases, and changes in molecular weight and specific heat. As examples in which simultaneous effects are present, we may cite rocket nozzles, ram jets, combustion chambers for gas turbines and turbojet engines, injectors and ejectors, moisture condensation shocks, moving flame fronts, detonation waves, heat exchangers, and gas-cooled nuclear reactors.

The phenomena which we shall take into account in our analysis include:

(i) Area change
(ii) Wall friction
(iii) Drag of internal bodies
(iv) External heat exchange
(v) Chemical reaction
(vi) Change of phase, e.g., evaporation or condensation of water or fuel
(vii) Mixing of gases which are injected into the main stream
(viii) Changes in molecular weight and specific heat occasioned by combustion, evaporation, gas injection, etc.

The assumptions made are as follows:

(i) The flow is one-dimensional and steady.
(ii) Changes in stream properties are continuous.
(iii) The gas is semiperfect, i.e., it obeys Boyle's and Charles' laws and has a specific heat which varies only with temperature and composition.

The method of analysis follows closely that of Shapiro and Hawthorne.(1)

Additional material relevant to the subject matter of the present chapter may be found in Chapters 4, 5, 6, 7, and in Volume II, Chapters 26, 27, and 28.

NOMENCLATURE

A	area
A_w	wetted area of pipe wall
c = a	speed of sound
c_p	specific heat at constant pressure
D	mean hydraulic diameter
e	base of natural logarithms, 2.718
f	coefficient of friction
F = I	impulse function
F_{T_0}, F_f, etc.	influence coefficient defined by Eqs. 8.49
$\bar{F}_{T_0}$	mean influence coefficient evaluated at $(M_1 + M_2)/2$
h	enthalpy per unit mass
h_{pr}	constant-pressure heat of reaction per unit mass, positive when exothermic
H	energy term defined by Eq. 8.15
$\mathcal{H}$	heat transfer coefficient
k = γ	ratio of specific heats
L	length
M	Mach Number
p	pressure
Q = q	net heat per unit mass of gas
R	gas constant
$\mathcal{R}$	universal gas constant; recovery factor

s	entropy per unit mass (single phase)
$\bar{s}$	total entropy per unit mass of gas stream
T	absolute temperature
T_{aw}	adiabatic wall temperature
T_w	temperature of pipe wall
V	velocity
V'	forward component of velocity of injected fluid $V_i \cos\alpha_i$
w	mass rate of flow of gas stream $= \dot{m}$
W	molecular weight
W_x	work
X	drag force
x	Cartesian coordinate
y	V'/V; also see Eq. 8.22c $= r_i = \frac{V_i \cos\alpha_i}{V}$
ρ	density
τ_w	wall shearing stress
ξ	x/L
$(\ )_0$	signifies stagnation state
$(\ )^*$	signifies state at which M = 1
$(\ )_g$	signifies injected gas dw_g
$(\ )_L$	signifies evaporated liquid dw_L entering control volume

8.2. Physical Equations and Definitions

Consider the flow in a duct between two sections an infinitesimal distance dx apart (Fig. 8.1). In this element of duct length gas is injected into the stream at the mass rate of flow dw_g, liquid evaporates into the stream at the mass rate of flow dw_L, heat in the amount dQ is added to the stream from external sources, and shaft work, shearing work, or electrical work is delivered by the stream to external bodies in the amount dW_x.

The various physical equations and definitions will be expressed in logarithmic differential form. It will be seen that this procedure allows easy separation of the physical variables.

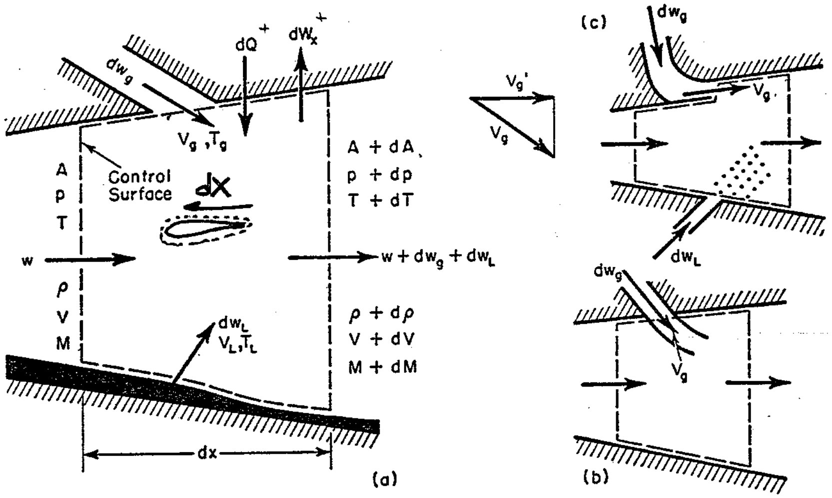


FIG. 8.1. Control surfaces defined for several methods of gas injection and liquid evaporation.

Equation of State. The pressure–density–temperature relation is

$$p = \rho \mathcal{R} T/W \tag{8.1}$$

Taking logarithms, we get

$$\log p = \log \rho + \log \mathcal{R} + \log T - \log W$$

Then, taking the differential of each side, we have

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} - \frac{dW}{W} \tag{8.2}$$

Sound Velocity. The expression for the sound velocity in a semi-perfect gas is

$$c^2 = k\mathcal{R}T/W \tag{8.3}$$

or

$$\frac{dc}{c} = \frac{1}{2}\left(\frac{dk}{k} + \frac{dT}{T} - \frac{dW}{W}\right) \tag{8.4}$$

Definition of Mach Number. From the definition of the Mach Number and Eq. 8.3, we find that

$$M^2 = V^2/c^2 = V^2W/k\mathcal{R}T \tag{8.5}$$

or

$$\frac{dM^2}{M^2} = \frac{dV^2}{V^2} + \frac{dW}{W} - \frac{dk}{k} - \frac{dT}{T} \tag{8.6}$$

Equation of Continuity. Fig. 8.1 shows several methods by which the injected gas and liquid may be brought into the main stream. Also shown are the corresponding control surfaces employed for purposes of analysis. Any liquid travelling through the duct, whether it be in the form of droplets, filaments, or sheets, is considered to lie outside the control volume. Accordingly, liquid which travels through the length dx without evaporating neither enters nor leaves the control volume. The infinitesimal flow dw_L which evaporates in the length is considered to be liquid which enters the control volume at the liquid–gas interface and which leaves as a gas with the main gas stream.

The injected gas flow dw_g is assumed to be injected continuously along the length of the duct, i.e., dw_g/dx is assumed to be finite or zero. This may be thought of as a simple model of real injection processes.

It is assumed further that the injected streams dw_L and dw_g are mixed perfectly with the main stream at the downstream boundary of the control volume.

The mass flow of the main gas stream may be expressed as

$$w = \rho AV \tag{8.7}$$

or

$$\frac{dw}{w} = \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} \tag{8.8}$$

In this expression dw denotes the total increase of mass flow in the main stream and includes both injected gas and evaporated gas. The equation of continuity is, therefore,

$$dw = dw_L + dw_g$$

If vapor is condensed, dw_L is negative. It is important to note also that dA refers to the change in cross-sectional area occupied by the main gas stream and does not include the cross-sectional area of the injected streams before mixing. Nor does it include that part of the cross section of the pipe occupied by liquid, although usually the volume of liquid may in fact be neglected in comparison with the volume of gas.

Energy Equation. Only the liquid crossing the control-surface boundary and evaporating within the control surface is taken into account in evaluating the flux of enthalpy. Changes in temperature of the liquid traveling along with the stream are taken to be the result of external heat exchange to or from the main stream. As the flows dw_L and dw_g, thoroughly mixed with the main stream, pass out of the control surface, they are assumed to be at the temperature of the latter.

The energy equation for the flow through the control surface of Fig. 8.1 may be written, assuming gravity effects to be negligible, as

$$\begin{aligned} w(dQ - dW_x) &= [w(h \overset{\text{flow work in}}{+} dh) + h_{gT}\,dw_g + h_V\,dw_L] \\ &\quad - [wh \overset{\text{flow work out}}{+} h_g\,dw_g + h_L\,dw_L] \\ &\quad \overset{\text{KE out}}{+ (w + dw_g + dw_L)}\left(\frac{V^2}{2} + d\frac{V^2}{2}\right) \\ &\quad \overset{\text{KE in}}{-} \left[w\frac{V^2}{2} + \frac{V_g{}^2}{2}dw_g + \frac{V_L{}^2}{2}dw_L\right] \end{aligned} \tag{8.9}$$

external work

$P_{shaft} + P_{shear} + P_{elect}$

In this expression dQ is the net heat added to the stream by conduction or radiation from sources external to the main gaseous stream, per unit mass of gas entering the control surface. Likewise, dW_x is the net external work delivered to outside bodies per unit mass of gas entering the control boundary. The external work includes shaft work, electrical work, and shear work on moving bodies adjacent to the control-surface boundaries. The symbols h_{gT} and h_g denote, respectively, the enthalpy of the injected gas dw_g at the temperature T, and the enthalpy at the temperature T_g with which dw_g enters the control volume. The symbol h_V denotes the enthalpy of the evaporated liquid dw_L at the temperature T, and h_L denotes the enthalpy of the liquid about to evaporate as it enters the control volume.

? shouldn't it be at T+dT?

Eq. 8.9 may be rearranged in the form

$$dQ - dW_x = dh + d\frac{V^2}{2} + \left[h_{gT} - h_g + \frac{V^2 - V_g{}^2}{2}\right]\frac{dw_g}{w} + \left[h_V - h_L + \frac{V^2 - V_L{}^2}{2}\right]\frac{dw_L}{w} \tag{8.10}$$

The enthalpy change of the main gas stream, dh, is the sum of the changes due to chemical reaction and to temperature change. Thus

$$dh = -dh_{pr} + c_p\,dT \tag{8.11}$$

where dh_{pr}, the enthalpy increase at the temperature T and pressure p for a change from products to reactants, is positive for exothermic reactions. It is usually called the constant-pressure heat of reaction and, in special cases, the constant-pressure heat of combustion. In evaluating dh_{pr}, one considers of course only the chemical changes which actually occur, but computes dh_{pr} per unit mass of the main gas stream.

We may also write

$$h_{gT} - \left[h_g + \frac{V_g^2}{2}\right] = h_{gT} - h_{0g} = \bar{c}_{pg}(T - T_{0g}), \qquad (8.12)$$

where

$$\bar{c}_{pg} = \frac{1}{T - T_{0g}} \int_{T_{0g}}^{T} c_{pg}\, dT \qquad (8.13)$$

and h_{0g} and T_{0g} are, respectively, the stagnation enthalpy and stagnation temperature of the injected gas stream.

Inserting Eqs. 8.11 and 8.12 into 8.10, and rearranging, the energy equation is finally put into the convenient form

$$dQ - dW_x + dH = c_p\, dT + d(V^2/2) \qquad (8.14)$$

where dH is an energy term defined by

$$dH = dh_{pr} - [\bar{c}_{pg}(T - T_{0g}) + V^2/2]\frac{dw_g}{w}$$

$$- \left[h_L - h_V + \frac{V^2 - V_L^2}{2}\right]\frac{dw_L}{w} \qquad (8.15)$$

Dividing Eq. 8.14 by c_pT, we obtain

$$\frac{dQ - dW_x + dH}{c_pT} = \frac{dT}{T} + \frac{k-1}{2}\mathrm{M}^2\frac{dV^2}{V^2} \qquad (8.16)$$

Momentum Equation. The net force acting on the material within the control surface of Fig. 8.1 is equal to the increase of momentum flux of the streams flowing through the control surface.

Consider forces in the direction of flow acting on the control surface in Fig. 8.1a. Assuming that the injected gas and liquid streams are at the control-surface pressure as they cross the boundary and that the angle of divergence of the walls is small, the momentum equation may be written *

$$pA + p\,dA - (p + dp)(A + dA) - \tau_w\, dA_w - dX$$

$$= (w + dw_g + dw_L)(V + dV) - V_g'\, dw_g - V_L'\, dw_L - wV \qquad (8.17)$$

In the foregoing expression τ_w represents the frictional shearing stress acting on the pipe wall area dA_w; dX is the sum of (i) the drag of stationary bodies immersed in the stream within the control-surface boundaries, (ii) the drag of liquid droplets and filaments traveling more slowly than the main gas stream, and (iii) the component of body or gravity forces acting on the material within the control surface in the direction opposite to that of the velocity vector; V_g' is the forward component of the velocity V_g with which the injected gas dw_g crosses the control surface, and similarly for V_L'.

The wall shearing stress is related to the coefficient of friction, f, though the definition of the latter:

$$\tau_w \equiv f\rho V^2/2 \qquad (8.18)$$

It is convenient to define the quantities

$$y_g = V_g'/V \quad \text{and} \quad y_L = V_L'/V \qquad (8.19)$$

From the definition of the mean hydraulic diameter D, we obtain

$$dA_w/A = 4dx/D \qquad (8.20)$$

Substituting Eqs. 8.18, 8.19, and 8.20 into Eq. 8.17, and noting that $\rho V^2 = kp\mathrm{M}^2$, we obtain, following rearrangement,

$$\frac{dp}{p} + \frac{k\mathrm{M}^2}{2}\frac{dV^2}{V^2} + \frac{k\mathrm{M}^2}{2}\left(4f\frac{dx}{D} + \frac{dX}{\frac{1}{2}kpA\mathrm{M}^2}\right) + k\mathrm{M}^2(1 - y)\frac{dw}{w} = 0 \qquad (8.21)$$

where

$$(1 - y)\frac{dw}{w} \equiv (1 - y_g)\frac{dw_g}{w} + (1 - y_L)\frac{dw_L}{w} \qquad (8.22a)$$

$$\frac{dw}{w} = \frac{dw_g}{w} + \frac{dw_L}{w} \qquad (8.22b)$$

$$y\frac{dw}{w} \equiv y_g\frac{dw_g}{w} + y_L\frac{dw_L}{w} = \sum y_i \frac{dw_i}{\omega} \qquad (8.22c)$$

These equations apply also for the methods of injection shown in Figs. 8.1b and 8.1c, provided that A is always taken as the cross-sectional area of the main gas stream.

Definition of Impulse Function. The impulse function, which is defined by

$$F = pA + \rho AV^2 = pA(1 + k\mathrm{M}^2) \qquad (8.23)$$

is useful for evaluating the thrust of propulsion systems. The increase in this function represents the total force exerted by the stream on the internal walls of the duct and acting on the duct walls in the direction opposite to the flow. In differential form,

$$\frac{dF}{F} = \frac{dA}{A} + \frac{dp}{p} + \frac{k\mathrm{M}^2}{1 + k\mathrm{M}^2}\frac{d\mathrm{M}^2}{\mathrm{M}^2} + \frac{k\mathrm{M}^2}{1 + k\mathrm{M}^2}\frac{dk}{k} \qquad (8.24)$$

---

* If the injected gas stream is not at the control-surface pressure as it crosses the boundary, the term $(p_g - p)dA_g'$ should be added to the left-hand side of Eq. 8.17, where p_g is the pressure of the injected stream at the boundary and dA_g' is the projected area of the part of the control surface occupied by the injected gas stream.

197 200
150 ~~148~~ 175] 350
173 175
119 110
120 ~~142~~ 1
760.

178
~~126~~ 168
168
~~492~~ 510
160 126 636.

1. Discuss in a few paragraphs:

   (i) How the coefficient of viscosity might be measured.
   (ii) The temperature dependence of the viscosity coefficient for a liquid as opposed to a gas.
   (iii) At an altitude of 250,000 ft., if a vehicle having a typical dimension L=50ft, were travelling at 1000 fps, would continuum theory be applicable?

2. Consider a flow in which the velocity components in the x and y directions are respectively,

$$u=Ax, \qquad v= -Ay$$

   (i) If at t=0, $x=x_o$ and $y=y_o$, determine the Lagrangian formulas for $x=x(x_o,y_o,t)$ and $y=y(x_o,y_o,t)$. Determine the rate of change of u for a particle passing through a point (x_1,y_1) at $t=t_1$ by using both the Eulerian and Lagrangian approaches.

   (ii) Determine the shape of the streamlines associated with the above flow. Draw a sketch of this flow field.

3. Consider a flow in which the velocity components in the radial and tangential directions (r,θ) are respectively

$$u_r= U \cos\theta \left(\frac{R^2}{r^2} - 1\right) \qquad , r \leq R$$

$$u_\theta= U \sin\theta \left(\frac{R^2}{r^2} + 1\right)$$

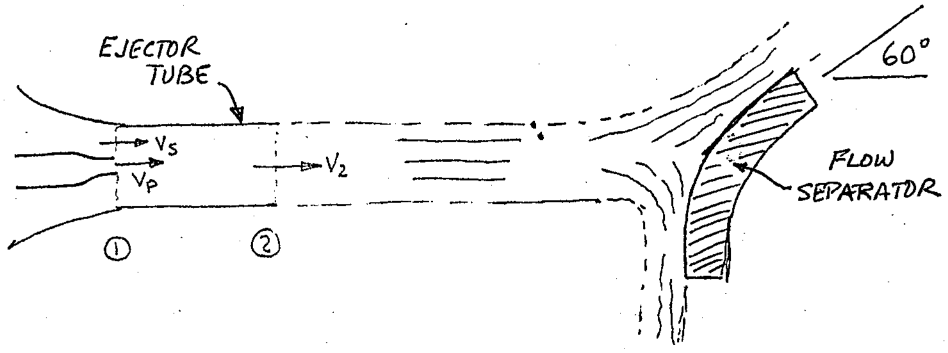
   Show that the streamline equation is given as

$$\frac{1}{r}\,\frac{dr}{d\theta} = \frac{u_r}{u_\theta}$$

   and prove that the circle r=R is in fact a streamline. Draw a qualitative sketch of the flow field.

4. Particles are dropping from rest off the edge of a table continuously. Express the velocity distribution by the Eulerian representation.

EML 5997 QUIZ # 1 NAME:____________________



A primary flow of water is mixed with a secondary flow of water at the same temperature of 20 degrees C in an ejector tube as shown. The primary flow speed is 160 m/s and the secondary flow speed is 35 m/s. The primary flow has a cross-sectional area of 0.05 square meters and the area at section 2 is 0.60 square meters. The mixed jet is the directed at the flow separator, which deflects the flow so that one half is directed upward.

1) Determine the pressure rise between sections 1 and 2 due to mixing and the velocity at section 2. Assume the initial pressure at section 1 is atmospheric. Neglect shear stresses at the wall of the tube.

2) Find the magnitude and direction of the forces acting on the separator. Assume that the pressure that surrounds the stream outside the ejector tube is the same as the pressure at section 2. Assume gravitational and frictional effects are negligible.

$R = 287.04\ m^2 / s^2 - K$

one atmosphere = 14.696 psi = 101,325 N / m^2

CURRICULUM VITAE

Cesar Levy, Ph.D.

~~February~~ 1991
April

PERSONAL

Name: Cesar Levy

Date of Birth: 22 November 1950

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MILITARY SERVICE

U.S. Army 1974-1978 (Active Service),
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Rank - Major, U.S. Army Reserves
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ACADEMIC DEGREES

a. Stanford University, Stanford, California
Ph.D. in Mechanical Engineering 1983

b. Courant Institute of Mathematical Sciences,
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M.S. in Applied Mathematics 1974

c. Polytechnic Institute of Brooklyn, New York
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PRESENT POSITION

Tenured Associate Professor, Department of Mechanical Engineering, Florida International University, Miami, FL 33199
Since: 1 August 1990

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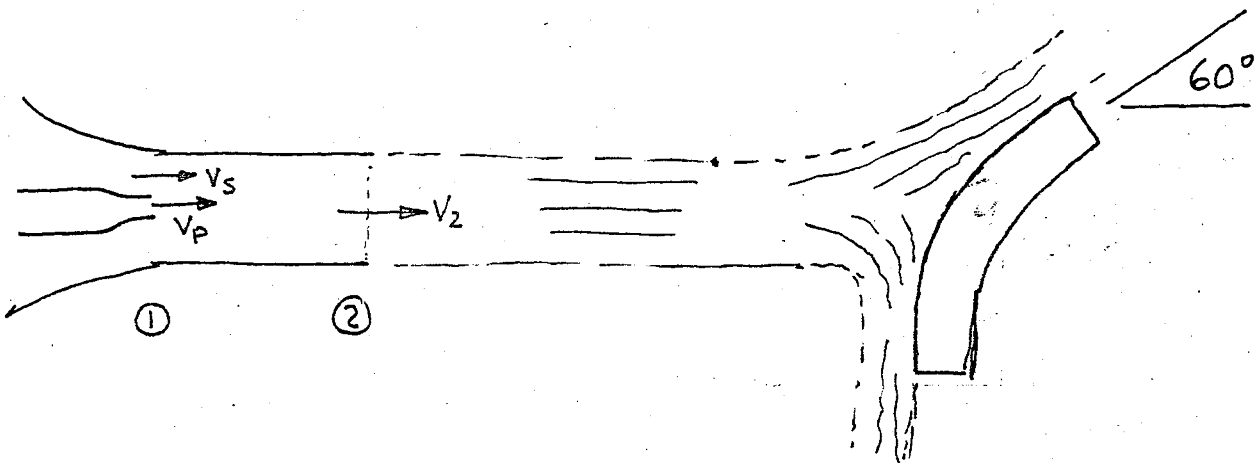
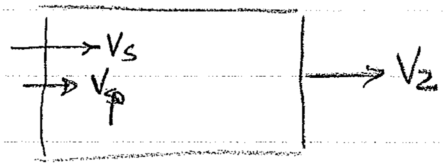

V_s
V_P
V_2
①
②
60°



③

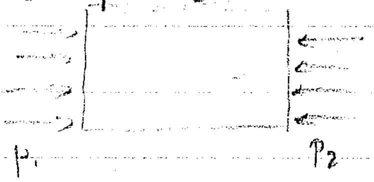
1) $-\rho V_s A_s - \rho V_p A_p + \rho V_2 A_2 = 0$

$V_s A_s + V_p A_p = V_2 A_2$

$V_2 = \dfrac{V_s A_s + V_p A_p}{A_2} = \dfrac{35(.55) + 160(.05)}{.60}$

$= \underline{45.41\bar{6} \ m/s}$ ①

$\dot{m}_s + \dot{m}_p = \dot{m}_2$



② ③

$$p_1 A_2 - p_2 A_2 = -(\rho A_s V_s) V_s - (\rho A_p V_p) V_p + (\rho A_2 V_2) V_2$$

$$(p_2 - p_1) A_2 = \rho \left[A_s V_s^2 + A_p V_p^2 - A_2 V_2^2 \right]$$

$$= \rho \left[A_s V_s^2 + A_p V_p^2 - V_2 (V_s A_s + V_p A_p) \right]$$

$$= \rho \left[A_s V_s (V_s - V_2) + A_p V_p (V_p - V_2) \right]$$

$$= \rho \left[(.55) 35 (-10.41\bar{6}) + .05(160)(114.58\bar{3}) \right]$$

$$(p_2 - p_1) .60 = \rho \left[716.1458332 \right] \frac{m^4}{s^2}$$

① ①

$$\rho = \frac{p}{RT} = \frac{101.325 \times 10^3 N/m^2}{287.04 \ m^2/s^2\text{-}K \ \ 293.15 K} = \underline{1.2042 \frac{N s^2}{m^4}} = 1.2042 \frac{kg}{m^3}$$

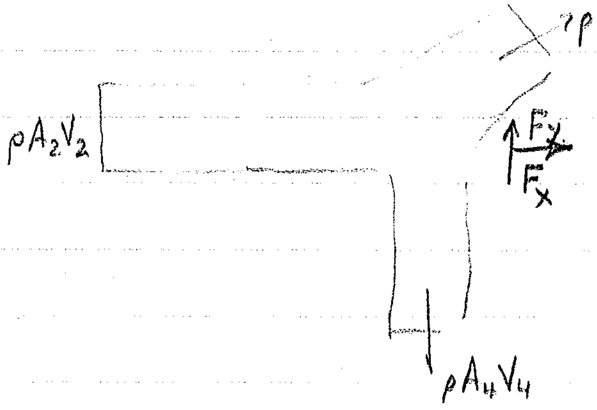
$$p_2 - p_1 = \frac{\rho}{A_2} \left[716.1458332 \right] = \underline{1437.257 \frac{N}{m^2}}$$ ①

①

$$p_2 = p_1 + 1.427 \times 10^3 \ N/m^2 = \left[101.325 + 1.427 \right] \times 10^3 N/m^2 = 102.752 \frac{kN}{m^2}$$

$= \underline{102.752 \ kPa}$

2)



③

$$-\rho A_2 V_2 + \rho A_3 V_3 + \rho A_4 V_4 = 0$$

$$\rho A_2 V_2 = \rho A_3 V_3 + \rho A_4 V_4$$

$$\therefore \ \underline{\rho A_3 V_3 = \rho A_4 V_4 = \tfrac{1}{2} \rho A_2 V_2}$$

② $p_2 + \rho \dfrac{V_2^2}{2} = p_3 + \rho \dfrac{V_3^2}{2} \Rightarrow \underline{V_2 = V_3}$ ①

② $p_2 + \rho V_2^2/2 = p_4 + \rho V_4^2/2 \Rightarrow \underline{V_2 = V_4}$ ①

and F_x + ~~pressure forces~~ $^{=0}$ $= -(\rho A_2 V_2)V_2 + (\rho A_3 V_3)\cos 60^\circ V_3$ ③

F_y + pressure forces $= (\rho A_2 V_3 \sin 60^\circ)V_3 + \rho A_4 V_4(-V_4)$ ③

$$F_x = \rho A_2 V_2\left[-V_2 + \tfrac{1}{2}V_3\cos 60^\circ\right] = (\rho A_2 V_2)V_2\left[-1+\tfrac{1}{4}\right] = \dot{m}_2 V_2\left[-\tfrac{3}{4}\right]$$

$$F_y = \rho A_3 V_3\left[V_3 \sin 60^\circ - V_4\right] = (\rho A_3 V_3)V_3\left[\tfrac{\sqrt{3}}{2} - 1\right]$$

$$\dot{m}_2 = \rho A_2 V_2 = (1.2042)(.60)(45.41\overline{6}) = 32.81336\ \text{kg/s} = 2\dot{m}_3$$

$F_x = \dot{m}_2(45.41\overline{6})(-3/4) = -1117.7$ N on fluid ① $\underline{F_x = 1117.7\text{ N}}$ on sep

$F_y = \dfrac{\dot{m}_2}{2}V_3[-.134] = \dfrac{32.81336}{2}(45.41\overline{6})(-.134) = -99.85$ N on fluid ①

$\underline{\text{or } 99.85\text{ N on sep.}}$

1) 3 + 1 V_2

5 + 2(ρ) + 1 $p_2 - p_1$

1 p_2

2) 3 + 3 + 3 + 3 + 3 + 1 + 1 F_x, F_y

3 + 1	V_2	4
5 + 1	$p_2 - p_1$	6
1 + 1	ρ	2
1	p_2	1 / 13
3 + 2 + 1	$V_2 = V_3$	6
2 + 1	$V_2 = V_4$	3
3 + 1	F_x	4
3 + 1	F_y	4
		17
		30

EML 5997 QUIZ # 2 NAME:____________________

Determine the solution for ρ' , p′, u′, the perturbation values of the density, pressure and displacement, given that the continuity, momentum and state equations are:

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{q} \cdot \nabla \rho + \rho \nabla \cdot \underset{\sim}{q} = 0$$

$$\frac{\partial \underset{\sim}{q}}{\partial t} + \underset{\sim}{q} \cdot \nabla \underset{\sim}{q} + \frac{1}{\rho} \frac{\partial p}{\partial t} = 0$$

$$p = \rho R T$$

1) Assume that the pressure, density and velocity of the undisturbed medium are p_∞, ρ_∞, and zero respectively.

2) Derive the continuity, momentum and state equations governing the density, pressure and velocity perturbations

3) Show that the density is governed by the wave equation

4) Give the solution for the density perturbation ρ' for a one-dimensional cartesian flow

5) Give the solution for p′ and u′, knowing the solution for ρ'.

Assume homentropic flow $p = k\rho^\gamma$

$$\rho' = f(x - a_\infty t) + g(x + a_\infty t)$$

$$p' = a_\infty^2 \rho' = a_\infty^2 [f + g]$$

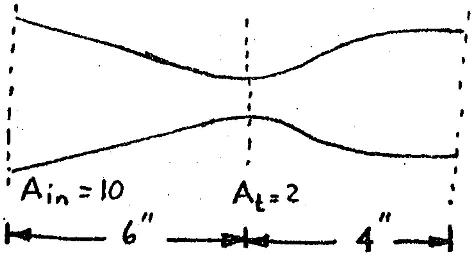
$$u' = \frac{a_\infty^2}{\rho_\infty} \int \frac{\partial \rho'}{\partial x} dt = -\frac{a_\infty}{\rho_\infty} [f - g]$$

EML 5997 QUIZ # 3 NAME: ______________________

A converging-diverging nozzle is designed such that the inlet area is 10 inches squared and the throat area is 2 inches squared.

The nozzle length is 10 inches. The nozzle is designed to produce a linear increase in Mach number from inlet to exit.

The inlet conditions are ~~helium~~ air with a γ value of ~~1.66~~ 1.4 at 100 psia and a temperature of 450 degrees Rankine.



1) Compute the following exit pressures (also known as index pressures):
  a) Subsonic isentropic exit pressure
  b) Supersonic isentropic exit pressure
  c) the exit pressure for which a shock exists at the exit plane

2) What is the design mach number (mach number at supersonic isentropic exit?)

3) For the design mach number what is the mass flow rate and exit mach number

For all the throat is crit $\Rightarrow$ $A_{in}/A_* = 5$ since M Must always increase

M_{in} is subsonic $\sim .115$ $\Rightarrow$ $\frac{\Delta M}{\Delta x} = \frac{1-.115}{6} = \frac{.885}{6}$ $\therefore$ $M = \frac{.885}{6}(x \cancel{-}) + .115$

$\therefore$ $M\Big|_{x=10} = \frac{.885}{6}(10) + .115 = 1.475 + .115 = 1.59$ design

$A/A_* = 1.2422$ $A_e = 2.484 \text{ in}^2$ $P/P_o = .2388$

$p_{in}/p_o = .9910$ $P_o = \frac{100}{.9910} \sim 100.9$ psia ; since M_{in} small $p_o \approx p_{in}$

1 b) $p_{exit} = p_{exit}/p_o \cdot p_o = .2388 \times \frac{100}{.9910} = 24.1$psia super isentropic

for $\frac{A_{ex}}{A_*}\Big|_{sub} = 1.2422 \Rightarrow$ $p_e/p_o = .8080$ $M_e = .559$

1 a) $p_{exit} = .8080 \times \frac{100}{.9910} \cong 81.53$psia sub isentropic

1c) for shock at exit

$M_x = 1.59$ $M_y = .6715$ $p_y/p_x = 2.783$ $\therefore\ p_y = 2.783\ (24.1\ \text{psia})$

$= 67.07\,\text{psia}$

for shock at exit $\underline{p_y = 67.07\ \text{psia}}$

2) design mach number is $\underline{1.59}$; $\frac{T_o}{T_{in}} = 1 + \frac{\gamma-1}{2} M_{in}^2 \Rightarrow T_o = 451.2\,°R$ $T_{ex} = T_o\left(1 + \frac{\gamma-1}{2} Me^2\right)^{-1}$

$= 299.7\,°R$

3) $$\dot{m} = \rho_e V_e A_e = \frac{p_e}{RT_e} \cdot Me\, a_e\, A_e = \frac{p_e}{RT_e} \cdot Me \sqrt{g_c \gamma R T_e}\ A_e$$

2.48

$$= \frac{24.1\ \text{psia} \times 144\ \text{sq in/sq ft}}{(53.30)(299.7\,°R)} (1.59) \sqrt{1.4\,(53.30)(299.7°)\,g_c} \cdot \frac{}{14}$$

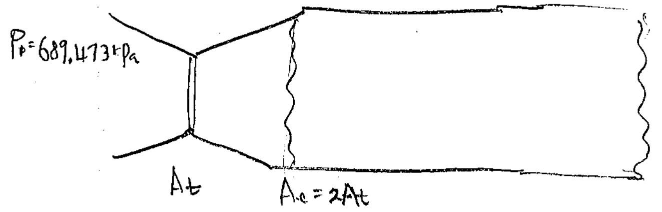
$$= \underline{5.056\ \text{lbm/sec}} = \underline{.1570\ \text{slugs/sec}}$$

also $\dot{m} = \frac{A_t p_o}{\sqrt{T_o}} \frac{.532}{32.2}$

$R_{air} = 53.30\ \text{ft-lb/lbm}\,°R$

$g_c = \frac{32.2}{1} \frac{\text{lbm}}{\text{slug}} = 32.2\ \text{ft/sec}^2$

Seongho Kang.



D:L = 1:10

$\bar{f}$ = 0.02.

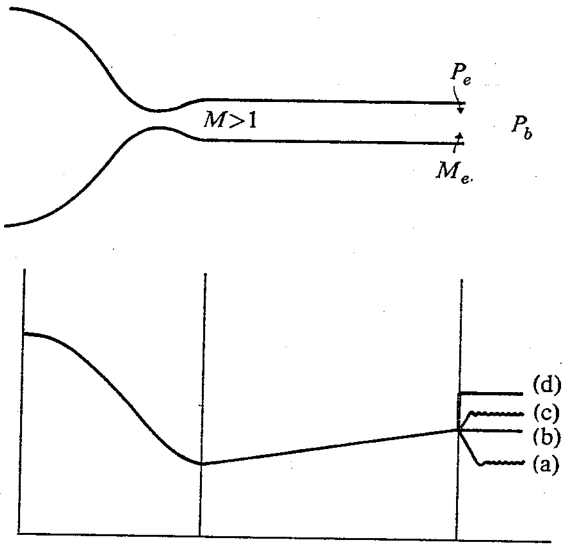
L/D = 10

I did not know how to use Ratio informations.

Ju

AA'

with no further increase in mass flow possible. Unlike the case previously discussed, in which a converging nozzle exhausted into the duct, and mass flow was affected by duct length, here, once the throat velocity reaches the velocity of sound, the mass flow rate is unaffected by duct length. Now, the system is choked by the nozzle, not the duct. Let us consider the flow pattern obtained with supersonic flow at the duct inlet. First, suppose the duct length L is less than L_{max} corresponding to the given inlet supersonic Mach number M_1. The change in flow pattern is to be described as the back pressure is increased from 0 psia. A back pressure of 0 psia, or a very low back pressure implies the existence of expansion waves at the duct exit. This means that the exit Mach number must be either supersonic or one. Since L is less than L_{max}, supersonic flow occurs at the duct exit, with P_{exit} greater than P_b. (*See* curve (a), Figure 9.14.) When P_b is raised to a value corresponding to curve (b), the exit plane pressure is equal to the back pressure. A further increase in back pressure yields oblique shock waves at the duct exit [curve (c)], until eventually, for a back pressure equal to that of curve (d), a normal shock stands at the duct exit. It can be seen that the flow described is exactly the same as that obtained at the exit of a converging–diverging nozzle (see Chapter 5, Section 5.2). Increases in back pressure over that of curve (d) cause the shock to move into the duct. For a high enough back



$L < L_{max}$

Figure 9.14

pressure, the shock moves into the nozzle, eliminating supersonic flow in the duct.

Example 9.3

A converging-diverging nozzle, with area ratio 2 to 1, is supplied by a reservoir containing air at 100 psia. The nozzle exhausts into a constant-area duct of length-to-diameter ratio 10 and friction coefficient $4f$ of .02. Determine the range of system back pressures over which a normal shock appears in the duct. Assume isentropic flow in the nozzle, Fanno flow in the duct.

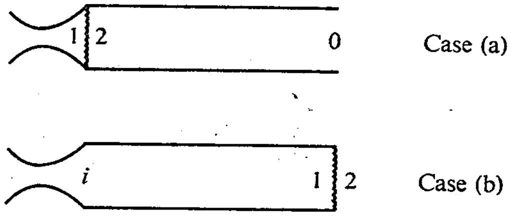


Figure 9.15

Solution:

From Appendix A, at $A/A^* = 2.0$, $M = 2.20$. From Appendix E, $4fL_{max}/D = .361$. For the duct under consideration, $4fL/D = .20$ so that $L < L_{max}$. Calculations must be made for two limiting cases, one with shock at the duct inlet (Figure 9–15), the other with shock at the duct outlet. For case (a), $M_1 = 2.20$. From Appendix B, for a normal shock, $M_2 = .547$. From Appendix E, $(4fL_{max}/D)_2 = .746$. Thus,

$$\left(\frac{4fL_{max}}{D}\right)_0 = \overset{\left(\frac{4fL_{max}}{D}\right)_2}{.746} - \frac{4fL}{D} = .546, \text{ so that } M_0 = .587.$$

For this case,

Diff L of duct ↓ ↓ shock jump ← isentropic

$$P_b = P_0 = \frac{P_0}{P^*}\frac{P^*}{P_2}\frac{P_2}{P_1}\frac{P_1}{P_r}100 = (1.80)\left(\frac{1}{1.94}\right)(5.48)(.0935)100$$

or

$$P_b = 47.5 \text{ psia}$$

For case (b), $M_i = 2.20$ so that $(4fL_{max}/D)_1 = .361 - .20 = .161$, or, from Appendix E, $M_1 = 1.57$. From Appendix B, for a normal shock, $P_2/P_1 = 2.71$. For this case,

jump duct conv

$$P_b = P_2 = \frac{P_2}{P_1}\frac{P_1}{P^*}\frac{P^*}{P_i}\frac{P_i}{P_r}100 = (2.71)\frac{(.571)}{.355}.0935(100) = 40.8 \text{ psia}$$

Thus, a shock will appear in the duct over the back pressure range 40.8 to 47.5 psia.

281,305 kPa to 327.5 kPa

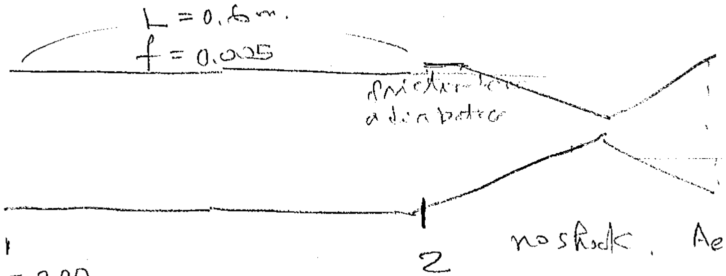
For L greater than L_{max}, the duct length is greater than that required to reach Mach 1 for supersonic duct flow. For very low back pressures, however, there must be sonic or supersonic flow at the duct exit to provide

Seongho Hong

$\frac{10}{30}$

3.76) An air stream enters an adiabatic, c-area duct

$D_H = D = 5\times10^{-2}$ m. ID.



no shock. $A_e = 3\times A_t$.

$M < 1$

$M_1 = 2.00$
$D_H = 5,\times10^{-2}$ m.
$P_1 = 80\times10^3$ pa
$T_1 = 290$ K.

P_e, M_e ?

$M = 2,00$

$$\frac{4\bar{f}\,L_{1\to max}}{D} = .30499 \quad ✓$$

$$L_{1\to max} = \frac{.30499 * 5*10^{-2}}{4*0,005} = 0,7625.$$

$$\frac{4\bar{f}\,L_{2\to max}}{D} = \frac{4\bar{f}\,L_{1\to max}}{D} - \frac{4\bar{f}\,L_{1\to 2}}{D} = .30499 - \frac{4*0.005*0.6}{0.05} = 0,06499 \quad ✓$$

TA-4. $M_2 \doteq 1.30$ ✓
$P_2/p^* \doteq 0.72848$. ✓

$M_1 = 2,00$.
$P_1/p^* = 0,40825$

$$P_2 = P_1\frac{P_2/p^*}{P_1/p^*} = 80\times10^3 \cdot \frac{0.72848}{0,40825} \quad ✓$$
$$= 142751 \text{ pa}.$$

at $M = 1.30$
$A/A^* = 1.06631$,

$$A^* = \frac{\frac{\pi}{4}(0.05)^2}{1.06631} \doteq 1,8414\times10^{-3}$$

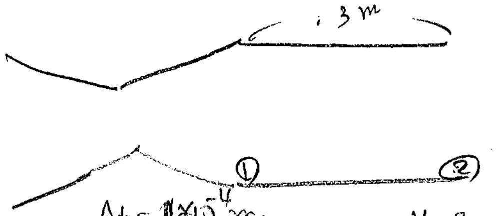
$A^* \doteq A_t$

since flow supersonic at inlet & subsonic at exit $\Rightarrow$ $M=1$ @ throat

$\therefore \frac{A_e}{A_t} = \frac{A_e}{A^*} = 3 \Rightarrow M_{exit} = .198$

etc

5,12). $p_0 = 600$ kPa.
$T_0 = 600$ K.
isentropically in a supersonic C-D nozzle



$A_t = 1\times10^{-4}$ m. $A_D = 3\times10^{-4} m^2 \rightarrow D_H \equiv D = \sqrt{\frac{4}{\pi} 3\times10^{-4}} = 1.9544\times10^{-2}$ m ✓

$\bar{f} = 0,004$.

$\dot{m}_{max}$. ⇾

p_e

p_b if a normal shock after the nozzle?

For max. flow rate $M_2 = 1$. X max flow controlled by nozzle throat

$$\frac{4\bar{f}L_{12}}{D_H} = \frac{4 * 0.004 * 0.3}{1.9544\times10^{-2}} = 0,246.$$

TA-4 $M_1 \doteq 1,813$ ok

TA-2 $\frac{p_1}{p_{01}} = 0.170$ $\frac{T_1}{T_{01}} = 0,603$

$p_1 = 0.170 * 600k$ $T_1 = 0.603 * 600.0 =$
$p_1 = 102$ kPa $T_1 = 361.8$ K.

$$\dot{m} = \rho_1 A_1 V_1 = \left(\frac{p_1}{RT_1}\right) A_1 M_1 C_1$$

$$= \left(\frac{102000}{287 * 361,8}\right) 3\times10^{-4} * 1.813 * \left(20.1\sqrt{361.8}\right) = 0.204 \text{ kg/s.}$$ ok

for this case need to find $L_{12} = L^*$
if $M_{exit} = 1$

at $M_1 = 1.813$

$$\frac{p_1}{p^*} = 0,469$$

$$p^* = \frac{102k}{0,469} = 217 \text{ kPa}$$

$p_2 = p^* = 217$ kPa.

P_b if a normal shock is after the nozzle.

P_0

$M_x = 1.813$.

I think ~~there~~ 's an effect of the nozzle.
I mean the method above for P_2 is not right.
correct

find M_y & P_y/P_x

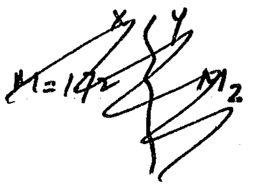
$\dot{m}_{max}$ is at $M_{throat} = 1$ $\dot{m} = \rho_e A_e V_e = .099$ kg/s

$L_{12} < L_1^* = 56$ cm

$M_1 = 2.637$ based on $A_e/A_t = 3$

$\frac{4fL_1^*}{D} = .46$ $\frac{4fL_{12}}{D} = .246$ $\frac{4fL^*_2}{D} = .214 \Rightarrow M_2 = 1.72$

find p_2 based on M_2

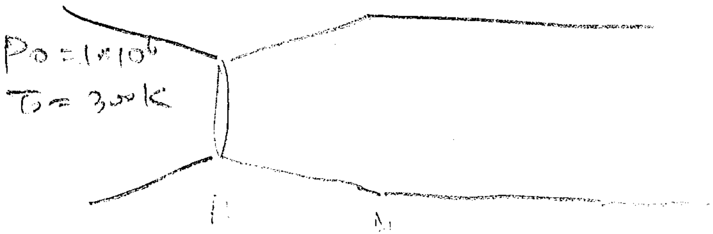


b) $M_x = M_1 = 2.64$ $M_y = .5$ $P_y/P_x = 7.965$

$P_x = P_1$ $P_y = 226.05$ kPa

5-14. $P_o = 1 \times 10^6$ pa.
$T_o = 300$ K. isenthropically through C-D → adiabatic Duct

Area ratio of the nozzle 3.0
L-D ratio of the duct 15.
$f = 0.005$.



$A_{in} : A_{out} = 1 : 3$.
$D : L = 15$.

$P_b = ?$ a) a normal shock is the throat.
b) the duct exit

Ratio

please see me

0

EML 5997 FINAL EXAMINATION 6 August 1991

This is a take home examination. It is due by Friday, 9 August 1991 at 1630 hours (4:30 PM) **at my office ECS 463.**

Please do your own work!!!

Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of this will result in failure of the exam.

PRINT NAME ________________ SIGN NAME ________________

This examination consists of **six problems. Do all problems.** Read each question carefully. Show all work!!!!!

PROBLEM NO.	POINTS	RAW SCORE
1	50	
2	50	
3	40	
4	30	
5	25	
6	30	
TOTAL	225	

(d) Failure of the offeror to provide the certification required by paragraphs (b) or (c) of this provision, renders the offeror unqualified and ineligible for award. (See FAR 9.104-1(g) and 19.602-1(a)(2)(i).)

(e) In addition to other remedies available to the Government, the certification in paragraphs (b) and (c) of this provision concerns a matter within the jurisdiction of an agency of the United States and the making of a false, fictitious, or fraudulent certification may render the maker subject to prosecution under Title 18, United States Code, Section 1001.

K-22 NOTICE OF EVALUATION PREFERENCE FOR SMALL DISADVANTAGED BUSINESS (SDB) CONCERNS (UNRESTRICTED) (JUL 1989) (DFARS 252.219-7007)

(a) Definition. The term "small disadvantaged business (SDB) concern", as used in this clause, has the meaning set forth in the clause entitled Small Disadvantaged Business Concern Representation (DoD FAR Supplement).

(b) Evaluation.

(1) After all other evaluation factors described in this solicitation are applied, and except as set forth in paragraph (b)(2) below, offers will be evaluated by adding a factor of ten percent (10%) to offers from concerns that are not SDB concerns and to offers from those SDB concerns which elect to waive the SDB evaluation preference (see paragraph (c) below) by checking the box below. However, in no event may award be made to an SDB concern at a price which exceeds fair market price (as determined under FAR under FAR 19.806-2) by more than ten percent (10%).

[__] The SDB Offeror requests that the evaluation preference in paragraph (b)(1) above not be given to this offer.

(2) The evaluation factor described in paragraph (b)(1) above shall not be applied to (i) otherwise low offers of (A) eligible products under the Trade Agreements Act as defined in DFARS 225.401 when the acquisition equals or exceeds the dollar threshold stated in FAR 25.402, or (B) qualifying country end products as defined in DFARS 225.001; or (ii) where the application would be inconsistent with a Memorandum of Understanding or any other international agreement with a foreign government (see Appendix T to the DoD FAR Supplement).

(c) Agreement.

(1) By submission of an offer and execution of a contract, the SDB Offeror/Contractor (except a regular dealer) who did not waive the evaluation preference by checking the box in paragraph (b) above agrees that in performance of the contract in the case of a contract for --

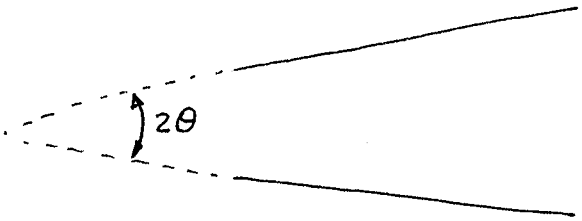
(i) Services (except construction). At least fifty percent

1. Air flows adiabatically in a conical duct of circular cross-section, the included angle between the walls being 2θ.

) If f is the coefficient of friction, find a relation between θ, D/D^*, M and f, where D is the diameter at Mach Number M and D^* is the diameter at M=1.

b) Plot the angle θ needed to keep the mach number M constant versus the mach number M for $0 < M < 5$ with f as a parameter for values

i) $4f = 0.010$
ii) $4f = 0.015$
iii) $4f= 0.020$



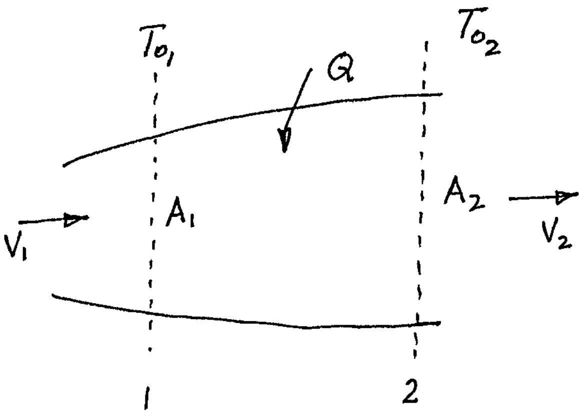
2. Consider the compressible flow of steam in an electrically heated tube. It is desired to keep the flow at constant mach number of 0.5, and changes in cross-sectional area will therefore be necessary.

To simplify the analysis, wall friction will be neglected and constant specific heats will be assumed.

a) Express the area ratio (A_2/A_1) as a function of the heat flux per unit mass, Q and the stagnation temperature T_{0_1}.

b) Compute the area ratio of heat flux of 1884 kJ/kg and $T_{0_1} = 500\ ^\circ K$.

c) Compute the velocities V_1 and V_2 for the conditions in 2(b).



Labor; and (3) employs in the production of commodities and in the provision of services, handicapped individuals for not less than 75 percent of the direct labor required for the production or provision of the commodities or services.

(End of Clause)

K-25 REPRESENTATION OF EXTENT OF TRANSPORTATION OF SUPPLIES BY SEA (JAN 1990) (DFARS 252.247-7202)

The Offeror is required to indicate whether transportation by sea is anticipate under the resultant contract by checking the appropriate blank as follows:

[_] The Offeror represents that it does anticipate that supplies, as defined in the clause at 252.247-7203, Transportation of Supplies by Sea, will be transported by sea in the performance of any contract or subcontract resulting from this solicitation.

[_] The Offeror represents that it does not anticipate that supplies, as defined in the clause at 252.247-7203, Transportation of Supplies by Sea, will be transported by sea in the performance of any contract or subcontract resulting from this solicitation. If the Offeror represents that it will not use ocean transportation, the clause at 252.247-7204 will also be inserted in the resulting contract.

K-26 CERTIFICATION AND DISCLOSURE REGARDING PAYMENTS TO INFLUENCE CERTAIN FEDERAL TRANSACTIONS (JAN 1990) (FAR 52.203-11)

(a) The definitions and prohibitions contained in the clause, at FAR 52.203-12, Limitation on Payments to Influence Certain Federal Transactions, included in this solicitation, are hereby incorporated by reference in paragraph (b) of this certification.

(b) The offeror, by signing its offer, hereby certifies to the best of his or her knowledge and belief as of December 23, 1989, that --

(1) No Federal appropriated funds have been paid or will be paid to any person for influencing or attempting to influence an officer or employee of any agency, a Member of Congress, an officer or employee of Congress, or an employee of a Member of Congress on his or her behalf in connection with the awarding of any Federal contract, the making of any Federal grant, the making of any Federal loan, the entering into of any cooperative agreement, and the extension, continuation, renewal, amendment or modification of any Federal

3. Suppose a blast wave, which might have been initiated by an explosion, is traveling through still air at standard atmospheric conditions (59° F, and 1 atmosphere) with a speed of 200,000 feet per second.

Estimate the changes in pressure (in atmospheres), temperature (in °F), stagnation pressure (in atm), stagnation temperature (in °F), and velocity (in ft/sec) produced by the wave with respect to an observer who is stationary with respect to the undisturbed air. (Assume air acts as a perfect gas with constant specific heats for simplicity).

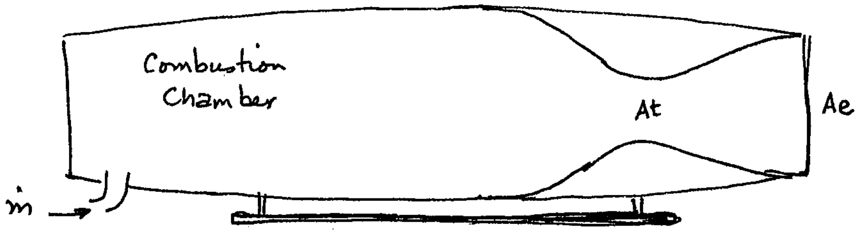
4. A rocket motor is to be tested on a horizontal thrust stand. The design machnumber at the nozzle exit is 5. The test is to be run at sea level (p_a = 14.7 psia). The area of the nozzle throat is 2 square feet. The stagnation pressure and temperature in the combustion chamber just ahead of the nozzle are P_o = 2000 psia and T_o = 3000 degrees Rankine. You can assume the combustion gases act like a perfect gas with a = 1.40 and a mol weight of 100. Find:

a) The mass flow rate

b) The pressure, temperature and velocity in the exit plane of the nozzle.

c) The force the nozzle exerts on the test stand. Define the direction of the force by a sketch. You can neglect forces of connector hoses and inflows which are at low velocity.

d) What would the force be if a shock stood at the exit plane.



5. An impact tube in a wind tunnel with air at 21.11 degrees <u>Centigrade</u> reads 137.895 kPa. If a wall static tap at the same section, where the flow streamlines are straight, reads 27.579 kPa, what is the mach number in the tunnel at this section? What is the change in mach number if the temperature rises to 35 degrees <u>Centigrade</u> and all else is unchanged?

HINT: see pitot tubes

K-23 SMALL BUSINESS SIZE REPRESENTATION FOR TARGETED INDUSTRY CATEGORIES UNDER THE SMALL BUSINESS COMPETITIVENESS DEMONSTRATION PROGRAM (JAN 1989) (DFARS 252.219-7014)

Complete only if Offeror has certified itself under the clause FAR 52.219-01 as a small business concern under the size standards of this solicitation.

The Offeror represents and certifies as follows:

Offeror's number of employees for the past twelve (12) months or Offeror's average annual gross revenue for the last three (3) fiscal years. (Check one of the following.)

	No. of Employees		Average Annual Gross Revenues
	50 or fewer		$1 million or less
_____	51 - 100	_____	$1,000,001 - $2 million
_____	101 - 250	_____	$2,000,001 - $3.5 million
_____	251 - 500	_____	$3,500,001 - $5 million
_____	501 - 750	_____	$5,000,001 - $10 million
_____	751 - 1,000	_____	$10,000,001 - $17 million
_____	Over 1,000	_____	Over $17 million

(End of provision)

K-24 NOTICE OF PARTICIPATION BY ORGANIZATIONS FOR THE HANDICAPPED (JUN 1989) (FAR 52.219-15)

(a) Definitions. "Handicapped individual" means a person who has a physical, mental, or emotional impairment, defect, ailment, disease, or disability of a permanent nature which in any way limits the selection of any type of employment for which the person would otherwise be qualified or qualifiable.

(b) The Offeror certifies that it is [_] is not [_] a public or private organization for the handicapped. An offeror certifying in the affirmative is eligible to participate in any resultant contract as if it were a small business concern.

(c) An Offeror certifying as a public or private organization for the handicapped agrees that at least 75 percent of the direct labor required in the performance of the contract will be performed by handicapped individuals.

"Public or private organization for the handicapped" means one which (1) is organized under the laws of the United States or of any State, operated in the interest of handicapped individuals, the net income of which does not inure in whole or in part to the benefit of any shareholder or other individual; (2) complies with any applicable occupational health and safety standard prescribed by the Secretary of

6. (a) Describe at least four different physical effects that can cause choking. Please be brief

(b) Give a general word description of the choking phenomenon (this can include more than one kind of qualitative view if you desire).

(c). Apply your description to the examples of 6(a) to indicate (physically) how choking arises in each case. Please keep answers brief.

absence thereof, by the Contracting Officer's best estimate of the actual transportation costs. If the item shipping costs, based on the actual shipping characteristics, exceed the item shipping costs used for evaluation purposes, the Contractor agrees that the contract price shall be reduced by an amount equal to the difference between the transportation costs actually incurred, and the costs which would have be incurred if the evaluated shipping characteristics had been accurate.

(1) To be completed by the offeror:

(i) Type of container: Wood Box___, Fiber Box___,
Barrel___, Reel___, Drum___, Other (Specify)_______;
(ii) Shipping configuration: Knocked-down___, Set-up ___,
Nested ____, Other (specify) ____________;
(iii) Size of container:_____" (Length), x ____"
(Width),x____" (Height) = ____ Cubic FT;
(iv) Number of items per container _________Each;
(v) Gross wight of container and contents ____LBS
(vi) Palletized/skidded _________Yes __________ No;
(vii) Number of containers per pallet/skid _____;
(viii) Weight of empty pallet bottom/skid and sides
______LBS;
(ix) Sizes of pallet/skid and contents ________LBS Cube
_____;
(x) Number of containers or pallets/skids per
railcar _______ *--
Size of railcar _________
Type of railcar _________
(xi) Number of containers or pallets/skids per trailer
______ *--
Size of trailer ___________FT
Type of trailer ___________

* Number of complete units (contract line item) to be shipped in carrier's equipment.

(2) To be completed by the Government after evaluation but before contract award:

c(77i) Rate used in evaluation ___________;
(ii) Tender/Tariff ___________;
77 (iii) Item ____________;
77

(b) The guaranteed shipping characteristics requested in subparagraph (a)(1) of this clause do not establish actual transportation requirements, which are specified elsewhere in this solicitation. The guaranteed shipping characteristics will be used only for the purpose of evaluating offers and establishing any liability of the successful offeror for increased transportation costs resulting from actual shipping characteristics which differ form those

EML 5997 QUIZ # 4 NAME:______________________

A converging-diverging nozzle, with an exit to throat area ratio of 2, is supplied by a reservoir containing air at 689.473 kPa. The nozzle exhausts into a constant area duct of length-to-diameter ratio 10 and friction coefficient f of 0.02. Determine the range of system back pressures over which a normal shock appears in the duct. Assume isentropic flow in the nozzle and fanno flow in the duct.

Another constant area duct is connected to the end of the fanno flow duct. This duct is frictionless but heat is added to it. If the inlet conditions (to this new duct) are the exit conditions to the fanno flow duct with a shock standing at the exit, what is the maximum amount of heat that can be transferred to the flow per unit mass.

$$\frac{DJ}{Dt} = \frac{\partial}{\partial t}J + (\underset{\sim}{q}\cdot\nabla)J \qquad J = \begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial y}{\partial x_0} \\ \frac{\partial x}{\partial y_0} & \frac{\partial y}{\partial y_0} \end{vmatrix}$$

$$\frac{\partial}{\partial t}J = \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial x_0}\right)\frac{\partial y}{\partial y_0} + \frac{\partial x}{\partial x_0}\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial y_0}\right) \qquad J = \frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} - \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}$$

$$-\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial y_0}\right)\frac{\partial y}{\partial x_0} - \frac{\partial x}{\partial y_0}\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x_0}\right)$$

Now $\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial x_0}\right) = \frac{\partial}{\partial x_0}\left(\frac{\partial x}{\partial t}\right) = \frac{\partial}{\partial x_0}u$

$$\therefore \frac{\partial J}{\partial t} = \frac{\partial u}{\partial x_0}\frac{\partial y}{\partial y_0} + \frac{\partial x}{\partial x_0}\frac{\partial v}{\partial y_0} - \frac{\partial u}{\partial y_0}\frac{\partial y}{\partial x_0} - \frac{\partial x}{\partial y_0}\frac{\partial v}{\partial x_0}$$

$$\frac{\partial J}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} + \frac{\partial x}{\partial x_0}\frac{\partial v}{\partial y}\frac{\partial y}{\partial y_0} - \frac{\partial u}{\partial x}\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0} - \frac{\partial x}{\partial y_0}\frac{\partial v}{\partial y}\frac{\partial y}{\partial x_0}$$

$$= \frac{\partial u}{\partial x}\left[\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} - \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\right] + \frac{\partial v}{\partial y}\left[\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} - \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\right]$$

$$\frac{\partial J}{\partial t} = \frac{\partial u}{\partial x}J + \frac{\partial v}{\partial y}J = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)J = J(\nabla\cdot\underset{\sim}{q})$$

$$\boxed{\partial J/\partial t = J(\nabla\cdot\underset{\sim}{q})}$$

$$(\underset{\sim}{q}\cdot\nabla)J = u\frac{\partial}{\partial x}J + v\frac{\partial}{\partial y}J$$

$$u\frac{\partial}{\partial x}J = u\left[\frac{\partial}{\partial x}\left[\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} - \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\right]\right] = u\left[\boxed{\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial x_0}\right)}\frac{\partial y}{\partial y_0} + \frac{\partial x}{\partial x_0}\boxed{\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial y_0}\right)} + \ldots\right]$$

$$\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial x_0}\right) = \frac{\partial}{\partial x_0}\left(\frac{\partial x}{\partial x}\right) = \frac{\partial 1}{\partial x_0} = 0$$

$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial y_0}\right) = \frac{\partial}{\partial y_0}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial y_0}(0) = 0$ since y, x are independent

$\therefore u\frac{\partial J}{\partial x} = 0$ & $v\frac{\partial J}{\partial y} = 0 \Rightarrow \boxed{(\underset{\sim}{q}\cdot\nabla)J \equiv 0}$

$$\therefore \boxed{DJ/Dt = J(\nabla\cdot\underset{\sim}{q})}$$

1. Discuss in a few paragraphs:

   (i) How the coefficient of viscosity might be measured.

   (ii) The temperature dependence of the viscosity coefficient for a liquid as opposed to a gas.

   (iii) At an altitude of 250,000 ft., if a vehicle having a typical dimension L=50ft, were travelling at 1000 fps, would continuum theory be applicable?

2. Consider a flow in which the velocity components in the x and y directions are respectively,

$$u=Ax, \qquad v=-Ay$$

   (i) If at t=0, $x=x_o$ and $y=y_o$, determine the Lagrangian formulas for $x=x(x_o,y_o,t)$ and $y=y(x_o,y_o,t)$. Determine the rate of change of u for a particle passing through a point (x_1,y_1) at $t=t_1$ by using both the Eulerian and Lagrangian approaches.

   (ii) Determine the shape of the streamlines associated with the above flow. Draw a sketch of this flow field.

3. Consider a flow in which the velocity components in the radial and tangential directions (r,θ) are respectively

$$u_r = U\cos\theta\left(\frac{R^2}{r^2}-1\right) \qquad , r \leq R$$

$$u_\theta = U\sin\theta\left(\frac{R^2}{r^2}+1\right)$$

   Show that the streamline equation is given as

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{u_r}{u_\theta}$$

   and prove that the circle r=R is in fact a streamline. Draw a qualitative sketch of the flow field.

4. Particles are dropping from rest off the edge of a table continuously. Express the velocity distribution by the Eulerian representation.

FOR INVISCID STEADY FLOW $p_0 =$ constant along a streamline

for steady flow, adiabatic $T_0 =$ constant along a streamline

All states with same entropy & same stagnation temperature have the same isentropic stagnation state & crit state. When a stream with a given pressure, temperature & velocity is decelerated to zero velocity, the final pressure will be less than the isentropic stagnation pressure if the deceleration is irreversible, but the final temperature will be equal to the adiabatic stagnation temperature irrespective of the reversibility/irreversibility of the deceleration

A MOVING DISTURBANCE CAN CREATE CHANGES IN PRESSURE FIELD

IN ACOUSTICS – FLUID VELOC ARE SMALL BUT CAUSE $\Delta p, \Delta T, \Delta \rho$

IN INCOMPRESSIBLE FLOWS – FLUID VELOCS ARE SMALL & SO IS $\Delta \rho$

BUT $\Delta p, \Delta T$

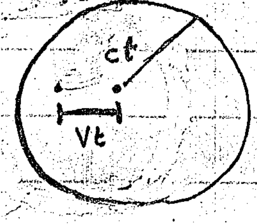
IN COMPRESSIBLE FLOWS – VELOCITY IS ON ORDER OF a CAUSING

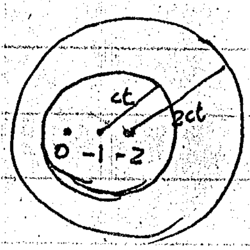
$\Delta p, \Delta T$ & $\Delta \rho$

IN INCOMPRESSIBLE VELOC $< a$

DISTURBANCE MOVES at speed Vt

PRESSURE WAVE MOVES at speed ct

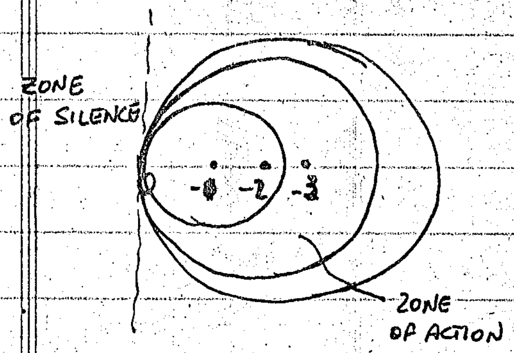




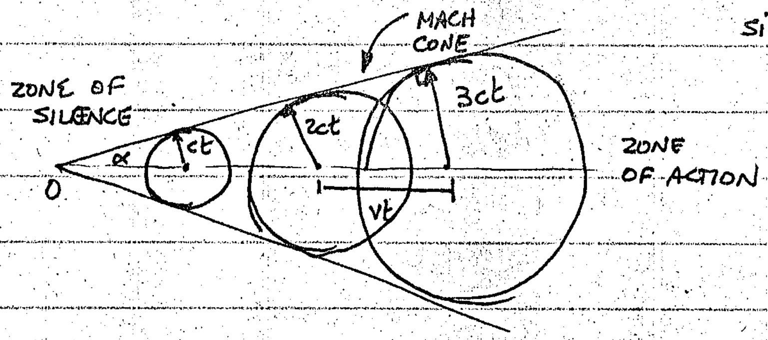
PRESSURE DISTURBANCE IS PROPAGATED IN ALL DIRECTIONS

FOR VELOC = a

NO PROPAGATION BEFORE WAVE



FOR VELOC > a



$$\sin\alpha = \frac{3ct}{3vt} = \frac{1}{M}$$

VON KARMAN'S RULES FOR SUPERSONIC FLOW

- EXACT FOR SMALL DISTURBANCES
- QUALITATIVE FOR LARGE DISTURBANCES

- RULE OF FORBIDDEN SIGNALS - IF $V > a$ PRESSURE CHANGES CANNOT REACH POINTS AHEAD OF BODY

- ZONE OF ACTION / ZONE OF SILENCE

  PRESSURE & VELOCITY AT SOME POINT IN STREAM CAN BE INFLUENCED ONLY BY DISTURBANCES ACTING AT POINTS LYING ON OR INSIDE CONE EXTENDING UPSTREAM FROM POINT CONSIDERED WITH α SAME AS MACH CONE. A STATIONARY POINT SOURCE AFFECTS POINTS THAT LIE ON OR

9. DeFrate, L. A., Barry, F. W., and Bailey, D. Z. A Portable Mach-Zehnder Interferometer, *Meteor Report*, No. 51, M. I. T., Cambridge, Mass. (1950).

10. Schardin, H. Theorie und Anwendung des Mach-Zehnderschen Interferenz-Refraktometers, *Zeitschrift fur Instrumentenkunde*, Vol. 53 (1933), pp. 396, 424. Also see R.A.E. Translation, No. 79.

11. Liepmann, H. W., and Puckett, A. E. *Introduction to Aerodynamics of a Compressible Fluid.* New York: John Wiley & Sons, Inc., 1947.

12. Pankhurst, R. C., and Holder, D. W. *Wind-Tunnel Technique.* London: Sir Isaac Pitman & Sons, 1952.

PROBLEMS

3.1. The compressibility of a liquid is usually expressed in terms of the bulk modulus of compression,

$$\beta = \rho \frac{dp}{d\rho}$$

Show that

$$c = \sqrt{\beta/\rho}$$

3.2. Calculate the velocity of sound at 70°F in the following media:

(a) Air, (b) hydrogen, (c) uranium hexafluoride, (d) mercury vapor, (e) water vapor, (f) liquid water at 14.7 psia.

3.3. To what pressure must liquid water be compressed in order that it leave a nozzle at atmospheric pressure with a jet velocity equal to the sonic velocity? Assume a constant bulk modulus of compression of 300,000 psi.

3.4. Plot and compare curves of the sonic velocity in air versus absolute temperature (a semi-log plot is suggested) using

(a) a value of γ of 1.4 for all temperatures, and (b) the actual value of γ for each temperature.

3.5. A projectile in flight carries with it a more or less conical-shaped shock front. From physical reasoning it appears that at great distances from the projectile this shock wave becomes truly conical and changes in velocity and density across the shock become vanishingly small.

Photographs of a bullet in flight show that at a great distance from the bullet the total included angle of the cone is 50.3°. The pressure and temperature of the undisturbed air are 14.62 psia and 73°F, respectively.

Calculate the velocity of the bullet, in ft/sec, and the Mach Number of the bullet relative to the undisturbed air.

3.6. Show that, for a perfect gas, the fractional change in pressure across a small pressure pulse is given by

$$\frac{dp}{p} = \gamma \frac{dV}{c}$$

and that the fractional change in absolute temperature is given by

$$\frac{dT}{T} = (\gamma - 1) \frac{dV}{c}$$

3.7. (a) Demonstrate that a compression wave (i.e., a pressure pulse which increases the density of the fluid over which it passes) which moves rightward imparts a rightward velocity to the fluid.

(b) Derive similar rules for a rightward-moving rarefaction wave and for leftward-moving compression and rarefaction waves.

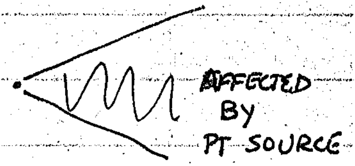
(c) The rightward-moving compression wave of Fig. 3.1a strikes a stationary wall closing the right-hand end of the duct. Demonstrate that it is necessary for a reflected wave to travel leftward, and determine whether the reflected wave is a compression or rarefaction. Compare the pressure change across the incident wave with that across the reflected wave.

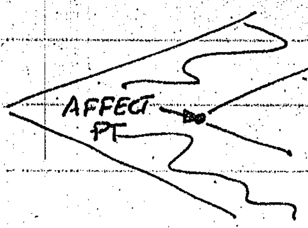
3.8. A compression pulse changes the velocity of the fluid over which it passes by 10 ft/sec. Calculate the pressure rise (psi) across the pulse for (a) water, and (b) air at 14.7 psia and 70°F.

ALSO

DO PROBLEMS 1.3, 1.4, 2.8 IN SAAD

IN THE MACH CONE EXTENDING DOWNSTREAM FROM POINT SOURCE





- RULE OF CONCENTRATED ACTION

  CLOSENESS OF CIRCLES REPRESENT PRESSURE INTENSITY DUE TO DISTURBANCE. SUBSONIC (INTENSITY ISASYMMETRIC) STATIONARY (SYMMETRIC). SUPERSONIC – PRESSURE DISTURBANCE IS LARGELY CONCENTRATED IN THE NEIGHBORHOOD OF THE MACH CONE

Problem 1.14

$V = 1500 \text{ km/hr} = \frac{1500{,}000}{3600 \text{ sec/hr}} \cdot \text{m/hr} = 416.67 \text{ m/s}$; $T = 273.15 - 60 =$

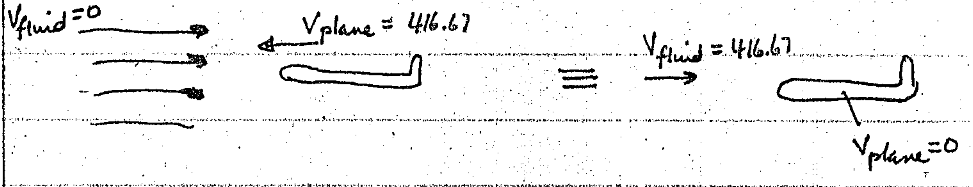
$M = \frac{V}{a}$ $\qquad a = \sqrt{\gamma R T} = \sqrt{(1.4)(287.04)(213.15)} = 292.67 \text{ m/s}$

$= \frac{416.67}{292.67} = 1.4237$

$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2 = 1 + .2(1.4237)^2 = 1.4054$

Given T $\qquad T_{surface} = T_0$ $\qquad \therefore \qquad T_0 = 213.15\,[1.4054] = 299.56\ ^\circ K$

$= 26.41\ ^\circ C$



$\therefore T_{surface} = T_0$

Let's look at what the goveining equation is for a small perturbation in a fluid

- Suppose fluid initially at rest & has p_∞ & ρ_∞
- Suppose a disturbance occurs so that $p = p_\infty + p'$; $\rho_\infty + \rho' = \rho$; $\underset{\sim}{q} = \underset{\sim}{q}'$
  here p', ρ' & $\underset{\sim}{q}'$ are small compared to p_∞, ρ_∞

Governing equation will be $\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \underset{\sim}{q} + \underset{\sim}{q} \cdot \nabla \rho = 0$ (Continuity)

IF FLOW INVISCID, no body forces (no gravitational) $\frac{\partial \underset{\sim}{q}}{\partial t} + \underset{\sim}{q} \cdot \nabla \underset{\sim}{q} + \frac{1}{\rho} \nabla p = 0$ (Momentum)

we assume process is reversible adiabatic (homentropic s = constant)

$p = k\rho^\gamma$ (Equation of State

This is an <u>unsteady flow</u>

SPEED OF SOUND $a^2 = \left.\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho} = \frac{\gamma(p_\infty + p')}{\rho_\infty + \rho'} = \frac{\gamma p_\infty}{\rho_\infty} \frac{(1 + p'/p_\infty)}{(1 + \rho'/\rho_\infty)}$

$$= a_\infty^2 \left[1 + p'/p_\infty\right] \left(1 - \rho'/\rho_\infty + (\rho'/\rho_\infty)^2 - \dots\right]$$

$$= a_\infty^2 \left[1 + p'/p_\infty - \rho'/\rho_\infty + \text{second order terms}\right]$$

$$a^2 = a_\infty^2 + a_\infty^2 \left(p'/p_\infty - \rho'/\rho_\infty\right) + \dots = a_\infty^2 + a'^2 \quad \text{when linear}$$

<u>Continuity</u>

$$\frac{\partial}{\partial t}(\rho_\infty + \rho') + (\rho_\infty + \rho') \nabla \cdot \underset{\sim}{q}' + \underset{\sim}{q}' \cdot \nabla(\rho_\infty + \rho') = 0$$

since ρ_∞, p_∞ is not changing $\Rightarrow \frac{\partial \rho'}{\partial t} + \rho_\infty \nabla \cdot \underset{\sim}{q}' + \underbrace{\rho' \cancel{\nabla \cdot \underset{\sim}{q}'} + \underset{\sim}{q}' \cdot \cancel{\nabla \rho'}}_{2^{nd}\text{ order terms}} = 0$

now LINEARIZED CONTINUITY IS $\boxed{\frac{\partial \rho'}{\partial t} + \rho_\infty \nabla \cdot \underset{\sim}{q}' = 0}$

Momentum

$$\frac{\partial \underset{\sim}{q}'}{\partial t} + \underbrace{\cancel{(\underset{\sim}{q}'\cdot\nabla)\underset{\sim}{q}'}}_{2^{\text{nd}}\text{ order small}} + \frac{1}{(\rho_\infty+\rho')}\nabla(p_\infty+p') = 0$$

$$+\frac{1}{\rho_\infty}\frac{1}{(1+\rho'/\rho_\infty)}\left[\underbrace{\cancel{\nabla p_\infty}}_{=0\text{ since } p_\infty \text{ not changing}} + \nabla p'\right] = 0$$

now $$\frac{\partial q'}{\partial t} + \frac{1}{\rho_\infty}\nabla p'\left(1-\rho'/\rho_\infty + (\rho'/\rho_\infty)^2 + \cdots\right) = 0$$

$$\boxed{\frac{\partial \underset{\sim}{q}'}{\partial t} + \frac{1}{\rho_\infty}\nabla p' = 0}$$ keeping first order terms only

Linearized Momentum   note: $\underset{\sim}{q}', p', \rho'$ but 2 eqns.

Use equation of state to relate p & ρ

since $p = k\rho^\gamma$ $dp = k\gamma\rho^{\gamma-1}d\rho = k\gamma\rho^\gamma\frac{d\rho}{\rho} = \frac{\gamma p}{\rho}d$

or $dp = a^2 d\rho$ now $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = \nabla p\cdot d\underset{\sim}{r}$

∴ $dp\ (=\nabla p\cdot d\underset{\sim}{r}) = a^2 d\rho = a^2(\nabla\rho\cdot d\underset{\sim}{r})$; or $(\nabla p - a^2\nabla\rho)\cdot d$

- since p, ρ are point functions without preferred direction ⇒ $\nabla p - a^2\nabla\rho =$

$$\nabla p = a^2\nabla\rho$$

$$\nabla(p_\infty+p') = [a_\infty^2 + a'^2]\nabla(\rho_\infty+\rho')$$

∇p_∞, $\nabla\rho_\infty$ are zero

$$\nabla p' = a_\infty^2\nabla\rho' + \underbrace{a'^2\cancel{\nabla\rho'}}_{3^{\text{rd}}\text{ order small}}$$

$$\therefore\ \boxed{\nabla p' = a_\infty^2\nabla\rho'}$$

$$\frac{\partial \underset{\sim}{q}'}{\partial t} + \frac{1}{\rho_\infty}\nabla p' = \frac{\partial \underset{\sim}{q}'}{\partial t} + \frac{a_\infty^2}{\rho_\infty}\nabla\rho' = 0$$ Linearized Momentum

now take $\frac{\partial}{\partial t}$ of Linearized Continuity $-\rho_\infty \nabla \cdot$ (Linearized Momentum)

$$\left[\frac{\partial^2}{\partial t^2}\rho' + \rho_\infty \frac{\partial}{\partial t}\nabla\cdot \underset{\sim}{q}'\right] - \rho_\infty \nabla\cdot\left[\frac{\partial \underset{\sim}{q}'}{\partial t} + \frac{a_\infty^2}{\rho_\infty}\nabla\rho'\right] = \frac{\partial}{\partial t}(0) - \rho_\infty\nabla\cdot(0) = 0$$

$$\frac{\partial^2\rho'}{\partial t^2} + \rho_\infty\frac{\partial}{\partial t}\nabla\cdot\underset{\sim}{q}' - \rho_\infty \frac{\partial}{\partial t}\nabla\cdot\underset{\sim}{q}' - a_\infty^2\nabla^2\rho' = 0$$

since x, y, z are inde of t

or $$\underline{\frac{\partial^2\rho'}{\partial t^2} = a_\infty^2\nabla^2\rho'}$$

wave equation
Governing Equation for perturbation density ρ'

similarly we can show that $\frac{\partial^2 \underset{\sim}{q}'}{\partial t^2} = a_\infty^2 \nabla(\nabla\cdot\underset{\sim}{q}')$

TAKE $\frac{a_\infty^2}{\rho_\infty}\nabla$ (Linearized Cont) $-\frac{\partial}{\partial t}$ (Linearized Momentum) TO GET IT.

since $p = k\rho^\gamma \Rightarrow (p_\infty + p') = k(\rho_\infty+\rho')^\gamma = k\rho_\infty^\gamma(1+\rho'/\rho_\infty)$

$$= k\rho_\infty^\gamma\left(1+\frac{\gamma\rho'}{\rho_\infty} + h.o.t\right)$$

$$= k\rho_\infty^\gamma + k\rho_\infty^\gamma\cdot\frac{\gamma\rho'}{\rho_\infty}$$

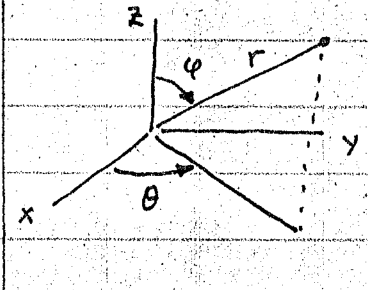
$$= p_\infty + p_\infty\frac{\gamma}{\rho_\infty}\rho' = p_\infty + a_\infty^2\rho$$

$\therefore \underline{p' = a_\infty^2\rho'}$ Governing relation between p' & ρ'

What does this mean

a) spherical symmetry (no dependence on θ, φ)

$$\nabla^2() = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}$$

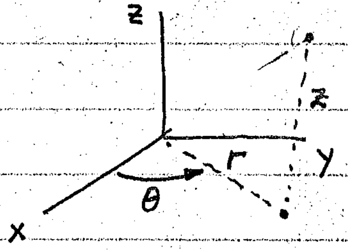


$$\nabla^2(\ ) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

$$\rho' = \frac{1}{r}\left[f(r-a_\infty t) + g(r+a_\infty t)\right]$$

Example : an explosion at $r=0$ propagates radially at speed $\frac{dr}{dt} = a$

b) Cylindrical Symmetry (no z or θ variation)



$$\nabla^2(\ ) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta^2} + \frac{\partial}{\partial z^2}$$

$$\rho' = (a_\infty^2 t^2 - r^2)^{-1/2}$$

c) One dimensional flow $\frac{\partial}{\partial y}, \frac{\partial}{\partial z} = 0$ $\Longrightarrow$ $\frac{\partial^2\rho'}{\partial t^2} = a_\infty^2 \frac{\partial^2\rho'}{\partial x^2}$

$\Longrightarrow$ $\frac{\partial^2 u'}{\partial t^2} = a_\infty^2 \frac{\partial^2 u'}{\partial x^2}$

u' = x component of $\underset{\sim}{q}'$

$$\rho' = f(x - a_\infty t) + g(x + a_\infty t)$$

To determine u' in all cases use solution for ρ' then

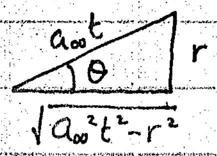
$$\frac{\partial \underset{\sim}{q}'}{\partial t} + \frac{a_\infty^2}{\rho_\infty}\nabla\rho' = 0$$

for example in cylindrical $\rightarrow$ $\frac{\partial u'}{\partial t} + \frac{a_\infty^2}{\rho_\infty}\frac{\partial\rho'}{\partial r} = 0$

$$\frac{\partial\rho'}{\partial r} = -\frac{1}{2}(a_\infty^2 t^2 - r^2)^{-3/2}(-2r) = \frac{r}{(a_\infty^2 t^2 - r^2)^{3/2}}$$

$$\therefore \quad u' = -\frac{a_\infty^2}{\rho_\infty}\int \frac{r}{(a_\infty^2 t^2 - r^2)^{3/2}}\,dt + f(r)$$

using



$$\int \frac{r}{(a_\infty^2 t^2 - r^2)^{3/2}}\,dt = -\frac{a_\infty t}{a_\infty r}\,\frac{1}{(a_\infty^2 t^2 - r^2)^{1/2}}$$

$$\therefore \quad u' = \frac{a_\infty^2}{\rho_\infty}\,\frac{t}{r}\,\frac{1}{(a_\infty^2 t^2 - r^2)^{1/2}} + f(r)$$

here u' = velocity in radial direction

To find $f(r)$ remember we must also satisfy continuity

$$\frac{\partial \rho'}{\partial t} + \rho_\infty \nabla \cdot \underset{\sim}{q}' = 0 \quad \text{where for cylindrical coordinates} \quad \nabla \cdot \underset{\sim}{q}' = \frac{1}{r}\frac{\partial}{\partial r}(r$$

$$\frac{\partial \rho'}{\partial t} = \frac{-a_\infty^2 t}{(a_\infty^2 t^2 - r^2)^{3/2}} \; ; \quad \nabla \cdot \underset{\sim}{q}' = \frac{1}{r}\frac{\partial}{\partial r}(ru') = \frac{a_\infty^2}{\rho_\infty}\frac{t}{(a_\infty^2 t^2 - r^2)^{3/2}} + \frac{1}{r}\frac{\partial}{\partial r}(rf')$$

$$\Rightarrow \quad \frac{\rho_\infty}{r}\frac{\partial}{\partial r}(rf') = 0 \quad \text{or} \quad f(r) = C_1 \ln r + C_2$$

if for example at $t=0$ & $r=0$, $u'=0 \Rightarrow C_1 = 0$ since origin is included & $C_2 = 0$ $\therefore f(r) = 0$

FOR INFORMATION :

CART: $\nabla \cdot \underset{\sim}{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$; $\nabla \varphi = \underset{\sim}{i}\frac{\partial \varphi}{\partial x} + \underset{\sim}{j}\frac{\partial \varphi}{\partial y} + \underset{\sim}{k}\frac{\partial \varphi}{\partial z}$

CYL: $\nabla \cdot \underset{\sim}{q} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial w}{\partial z}$; $\nabla \varphi = \underset{\sim}{i}_r \frac{\partial \varphi}{\partial r} + \underset{\sim}{i}_\theta \frac{1}{r}\frac{\partial \varphi}{\partial \theta} + \underset{\sim}{k}\frac{\partial \varphi}{\partial z}$

SPH: $\nabla \cdot \underset{\sim}{q} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\varphi}\frac{\partial}{\partial \varphi}(u_\varphi \sin\varphi) + \frac{1}{r\sin\varphi}\frac{\partial u_\theta}{\partial \theta}$; $\nabla f = \underset{\sim}{i}_r \frac{\partial f}{\partial r} + \underset{\sim}{i}_\varphi \frac{1}{r}\frac{\partial f}{\partial \varphi} + \underset{\sim}{i}_\theta \frac{1}{r\sin\varphi}\frac{\partial f}{\partial \theta}$

CYL: $0 \le r < \infty \quad 0 \le \theta \le 2\pi \quad |z| < \infty$

SPH: $0 \le r < \infty \quad 0 \le \theta \le 2\pi \quad 0 \le \varphi \le \pi$

IN PROBLEMS WHERE FLOW IS NEARLY UNIFORM $\Rightarrow$ WHERE $\underset{\sim}{q}$, p, ρ ARE ALMOST CONSTANT

$$\Rightarrow \frac{\partial \rho'}{\partial t} + \underset{\sim}{q}_\infty \cdot \nabla\rho' + \overbrace{\cancel{\underset{\sim}{q}'\cdot\nabla\rho'} + \rho_\infty \nabla\cdot\underset{\sim}{q}' + \cancel{\rho'\nabla\cdot\underset{\sim}{q}'}}^{2^{nd}\text{ order}} = 0$$

note: $\frac{\partial \rho_\infty}{\partial t}$, $\nabla\cdot\underset{\sim}{q}_\infty$ & $\nabla\rho_\infty = 0$ since $\underset{\sim}{q}_\infty, p_\infty, \rho_\infty$ are constant

Also <u>Momentum</u>

$$\frac{\partial \underset{\sim}{q}}{\partial t} + (\underset{\sim}{q}\cdot\nabla)\underset{\sim}{q} + \frac{1}{\rho}\nabla p = 0$$

$$\frac{\partial}{\partial t}(\underset{\sim}{q}_\infty + \underset{\sim}{q}') + \left[(\underset{\sim}{q}_\infty + \underset{\sim}{q}')\cdot\nabla\right](\underset{\sim}{q}_\infty + \underset{\sim}{q}') + \frac{1}{\rho_\infty + \rho'}\nabla(p_\infty + p') = 0$$

$$\Rightarrow \frac{\partial \underset{\sim}{q}'}{\partial t} + (\underset{\sim}{q}_\infty\cdot\nabla)\underset{\sim}{q}' + (\underset{\sim}{q}'\cdot\nabla)\underset{\sim}{q}' + \frac{1}{\rho_\infty}\cdot\frac{1}{(1+\rho'/\rho_\infty)}\nabla p' = 0$$

$$\frac{1}{\rho_\infty}\left[1 - \rho'/\rho_\infty + \left(\frac{\rho'}{\rho_\infty}\right)^2 + \dots\right]\nabla p' = 0$$

$$\frac{\partial \underset{\sim}{q}'}{\partial t} + (\underset{\sim}{q}_\infty\cdot\nabla)\underset{\sim}{q}' + \underbrace{\cancel{(\underset{\sim}{q}'\cdot\nabla)\underset{\sim}{q}'} + \frac{1}{\rho_\infty}\nabla p' - \cancel{\frac{1}{\rho_\infty^2}\rho'\nabla p'}}_{2^{nd}\text{ order}} + \dots = 0$$

$$\therefore \frac{\partial \underset{\sim}{q}'}{\partial t} + (\underset{\sim}{q}_\infty\cdot\nabla)\underset{\sim}{q}' + \frac{1}{\rho_\infty}\nabla p' = 0$$

now $\nabla p' = a_\infty^2 \nabla\rho'$ from before for homentropic flow $(p = k\rho^\gamma)$

$$\therefore \frac{\partial \underset{\sim}{q}'}{\partial t} + (\underset{\sim}{q}_\infty\cdot\nabla)\underset{\sim}{q}' + \frac{a_\infty^2}{\rho_\infty}\nabla\rho' = 0 \qquad \text{Linearized Momentum}$$

$$\frac{\partial \rho'}{\partial t} + (\underset{\sim}{q}_\infty\cdot\nabla)\rho' + \rho_\infty\nabla\cdot\underset{\sim}{q}' = 0 \qquad \text{Linearized Continuity}$$

1) Look at a specific case of steady state $\left(\frac{\partial}{\partial t} = 0\right)$
2) where original flow was only in x direction ie $\underset{\sim}{q}_\infty = u_\infty \underset{\sim}{i}$
3) disturbance $\underset{\sim}{q}' = u'\underset{\sim}{i} + v'\underset{\sim}{j}$

Thus Linearized Momentum is $u_\infty \frac{\partial}{\partial x} u' + \frac{a_\infty^2}{\rho_\infty} \frac{\partial}{\partial x} \rho' = 0$ x-momentum

$$u_\infty \frac{\partial}{\partial x} v' + \frac{a_\infty^2}{\rho_\infty} \frac{\partial}{\partial y} \rho' = 0 \qquad \text{y-momentum}$$

Linearized Continuity $u_\infty \frac{\partial}{\partial x} \rho' + \rho_\infty \left[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right] = 0$

x momentum + continuity eliminate $\frac{\partial \rho'}{\partial x}$:

$$\frac{u_\infty^2}{a_\infty^2} \rho_\infty \frac{\partial u'}{\partial x} - \rho_\infty \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) = 0$$

or $(1 - M_\infty^2) \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$

if flow is irrotational $\underset{\sim}{q}' = \nabla \varphi \quad \therefore \quad \frac{\partial \varphi}{\partial x} = u' \quad \frac{\partial \varphi}{\partial y} = v'$

$$\Rightarrow \quad (1 - M_\infty^2) \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

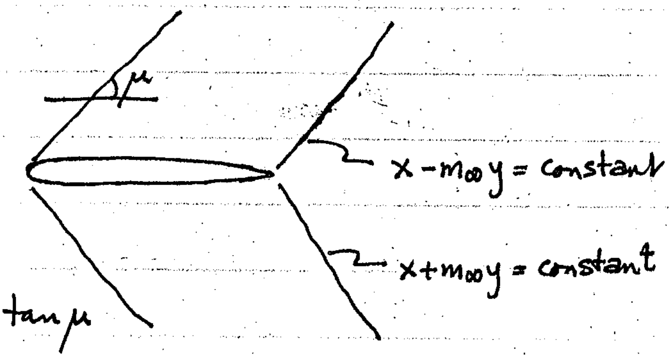
if $1 - M_\infty^2 = -m_\infty^2 < 0 \quad \Rightarrow \quad \frac{\partial^2 \varphi}{\partial y^2} = m_\infty^2 \frac{\partial^2 \varphi}{\partial x^2} \qquad M_\infty > 1$

$1 - M_\infty^2 = 0 \quad \Rightarrow \quad \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad M_\infty = 1$

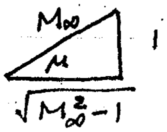
$1 - M_\infty^2 = +\beta_\infty^2 > 0 \quad \Rightarrow \quad \beta_\infty^2 \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad M_\infty < 1$

For supersonic flow

$$\frac{\partial^2 \varphi}{\partial y^2} = m_\infty^2 \frac{\partial^2 \varphi}{\partial x^2}$$



$$\therefore \quad \frac{dy}{dx} = \frac{1}{m_\infty} = \frac{1}{\sqrt{M_\infty^2 - 1}} = \tan \mu$$



$\sin \mu = \frac{1}{M_\infty}$

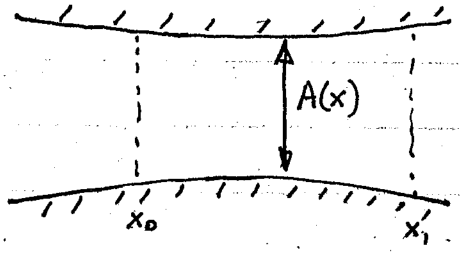
$$\varphi = f(x - m_\infty y) + g(x + m_\infty y)$$

- This forms a mach wave (weakest form of shock wave)
- Mach cone is formed by the characteristics for the equation

Let's now look at 2-D flows that act like <u>Quasi 1-D Flows</u>

assume flow is inviscid, steady, homentropic ($s = \text{const}$)

let's assume
- $T_0 = \text{constant}$ (frictionless adiabatic flow)
- $p_0 = \text{constant}$
- flow across ↑ channel doesn't change too much
- channel is long & slender
- entrance & exit flow do not dominate
- flow changes occur in in y-direction but much smaller than veloc in x direction
- flow properties change due to changes in area but due to isentropy T_0, p_0 constant
- area variations are small $\Rightarrow A'(x)/A(x) \ll 1$
- viscous effects small



1.) Mass Conservation $\frac{\partial}{\partial t}\int \cancel{\rho}\, d\mathcal{V} + \int \rho \underset{\sim}{q}\cdot\underset{\sim}{n}\, dS = 0$

Steady

$\underset{\sim}{q}\cdot\underset{\sim}{n} = 0$ at wall $\Rightarrow \int \rho \underset{\sim}{q}\cdot n\, dS = 0$ or $-\rho_{av}(x_0)\, u_{av}(x_0)A(x_0) + \rho_{av}(x_1)\, u_{av}(x_1)A(x_1) = 0$

or $\rho_{av}(x)\, u_{av}(x)\, A(x) = \text{constant}$

Momentum

2) $$\frac{\partial}{\partial t}\int \underbrace{\cancel{\rho \underset{\sim}{q}}}_{\text{steady}}\, dV + \int \rho(\underset{\sim}{q}\cdot\underset{\sim}{n})\,\underset{\sim}{q}\, dS + \int p\,\underset{\sim}{n}\, dS = 0$$ if no height variations

x - component : $$-\rho(x_0)u^2(x_0)A(x_0) + \rho(x_1)u^2(x_1)A(x_1) - p(x_0)A(x_0) + p(x_1)A(x_1) - 2\int_{x_0}^{x_1} p(\bar{x})\,\frac{A'(\bar{x})}{2}\,d\bar{x} = 0$$

differentiate wrt x_1 :

$$(\rho u A)\,u' + \underbrace{u\,\frac{d}{dx_1}(\rho u A)}_{0 \text{ by continuity}} + p'A + \underbrace{pA'}_{\text{cancel}} - \underbrace{2\frac{\partial}{\partial x_1}\int_{x_0}^{x_1} p\,\frac{A'}{2}\,d\bar{x}}_{-2\,p\frac{A'}{2}\,\cancel{\frac{dx}{dx_1}}} = 0$$

$$\therefore\ (\rho u A)u' + p'A = 0$$

or $\boxed{\rho u u' + p' = 0}$ Momentum

For shock less isentropic flow

Eqn of state: $$p' = \frac{dp}{dx} = \frac{d}{dx}\left(k\rho^{\gamma}\right) = k\gamma\rho^{\gamma-1}\frac{d\rho}{dx} = \frac{\gamma p}{\rho}\,\rho' = a^2\rho'$$

also momentum : $\rho u u' + p' = \rho u u' + a^2\rho' = 0$ or

$$\rho'/\rho = -\frac{u u'}{a^2} = -M^2\frac{u'}{u}$$

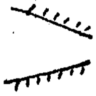
From continuity $\frac{d}{dx}(\rho u A) = 0 \implies \rho u A = \text{const}$

$$\rho'/\rho + \frac{u'}{u} + \frac{A'}{A} = 0$$

$$\therefore \qquad (1-M^2)\frac{u'}{u} + \frac{A'}{A} = 0$$

Thus $$\frac{A'}{A} = (M^2-1)\frac{u'}{u} = -\frac{(M^2-1)}{M^2}\,\rho'/\rho = -\frac{(M^2-1)}{M^2}\,\frac{1}{\rho}\left[p'/a^2\right]$$

$$= -\frac{(M^2-1)}{\gamma M^2}\,p'/p$$

Look at  Converging Nozzle

$A' < 0$ $\quad M > 1 \quad u' < 0 \quad \rho' > 0 \quad p' > 0$

$M < 1 \quad u' > 0 \quad \rho' < 0 \quad p' < 0$

$A' > 0$  diverging nozzle

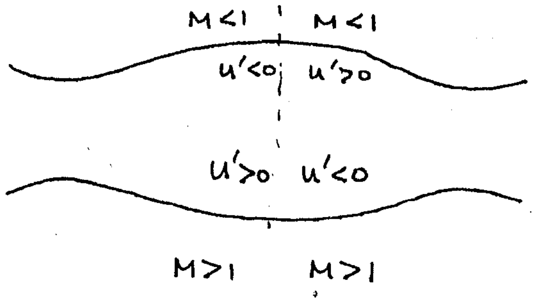
$M > 1 \quad u' > 0 \quad \rho' < 0 \quad p' < 0$

$M < 1 \quad u' < 0 \quad \rho' > 0 \quad p' > 0$

what about $M = 1 \Rightarrow A' = 0$ or $A = A^*$ CRITICAL CONDITIONS

$\Rightarrow u' = \infty$ (PHYSICALLY IMPOSSIBLE)

- FLOW BEHAVIOR OPPOSITE FOR SUPERSONIC & SUBSONIC FLOWS
- SONIC FLOW CAN ONLY BE ACHIEVED AT A <u>CONSTRICTION</u>



NOT AT THE LARGE SECTION OF CHANNEL