FLORIDA INTERNATIONAL UNIVERSITY
Mechanical and Materials Engineering Department

Spring 2017 Advanced Vibration Analysis EML6223

The FIU Civility Initiative is a collaborative effort by students, faculty, and staff to promote civility as a cornerstone of the FIU Community. We believe that civility is an essential component of the core values of our University. We strive to include civility in our daily actions and look to promote the efforts of others that do the same. Show respect to all people, regardless of differences; always act with integrity, even when no one is watching; be a positive contributing member of the FIU community.

— Review:
- o one degree of freedom systems.
- o Free, forced, damped and undamped vibrations.
- o Forced support vibration,

— Newton's law in non-inertial coordinate frame.
— Effective stiffness calculation for combined bar-beam-string-plate systems.
— Systems with multiple degrees of freedom. Some models. General analysis. Frequencies and mode shapes for undamped systems. Principal or normal coordinates. Damping in multidegree systems.
— Continuous systems with infinite number of degrees of freedom. Longitudinal vibrations of prismatic bars. Free and forced vibrations. Prismatic bar with a mass or spring at the end. The problem of bar impact.
— Torsional vibrations of shafts.
— Transverse vibrations of beams.
— Transverse vibrations of membranes and plates.
— Stability analysis. Introduction to Liapunoffs method.
— Non-linear conservative systems. Free and forced vibrations. Piecewise-linear systems. Numerical solution.
— Non-linear non-conservative systems. Self-excited vibrations. Van der Pol's equation.
— Parametric resonance. Mathieu's equation. The Ince-Strutt diagram.
— Inelastic (especially, viscoelastic) material damping and vibration attenuation using VEM

Book to be used S.S. Rao, Mechanical Vibrations, 5th Edition, Pearson-Prentice Hall Publishers.
Also notes will be provided from other books as well.

GRADES

Grades will be determined on the basis of

1 Midterm Exam	40 % each
HW	20 %
Final Exam	40 %

Letter Grades will be based as follows:

(A) 95 & above	(B+) 85-89	(C+) 73-76	(D) 60-64
(A-) 90-94	(B) 80-84	(C) 70-72	(F) below 60
	(B-) 77-79	(C-) 65-69	

Please be on time to class and keep up with the work. There is a lot of work to cover and it will be difficult for you if you do not do the homework assignments. My office hours will be posted during the first week of classes. Please come to see me if you are having problems or have suggestions on how to improve this course.

We will be meeting twice a week T-R from 1230-145pm. Our meeting room will be EC1116, though that may change.

My office hours will be determined during the first week of classes.

THIS IS A PRELIMINARY SYLLABUS. ALL CHANGES WILL BE ANNOUNCED IN CLASS, INCLUDING ANY CHANGES IN CLASSROOM.

- VIBRATIONS EXIST IN ELECTRIC CIRCUITS
  ACOUSTICS
  CAN BE ELECTROMAGNETIC WAVES
  MECHANICAL SYSTEMS

• MECHANICAL VIB CAN CAUSE ACOUSTICAL VIBRATION (VICE VERSA) — *pluck a string* — *Ella Fitzgerald & glass*

• " CAN CAUSE AN ELECTRIC OSCILLATION (")

• BASIC PRINCIPLES ARE THE SAME

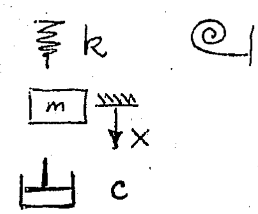
• FOR VIBRATIONS TO OCCUR NEED AT LEAST TWO ENERGY STORAGE ELEMENTS

POSSIBILITIES

INERTIA OR MASS – STORES KINETIC ENERGY

ELASTIC OR SPRING MEMBER – " POTENTIAL ENERGY

DAMPER (DASHPOT) – DISSIPATES ENERGY



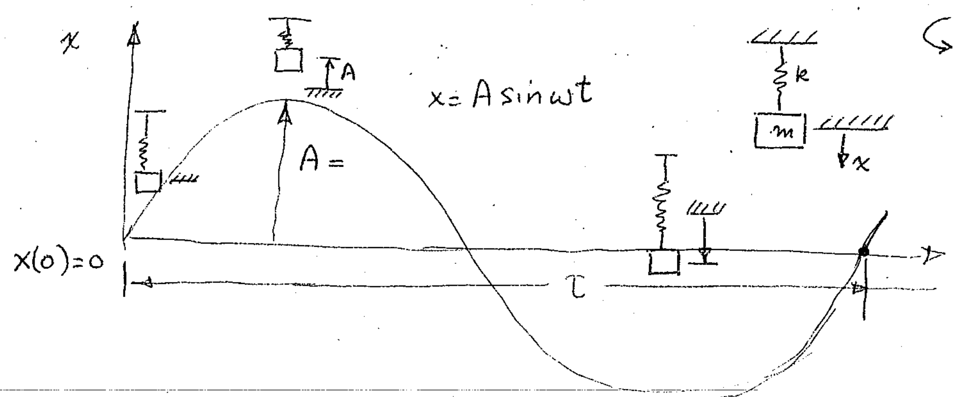
VERTICAL VIBRATIONS OF MASS

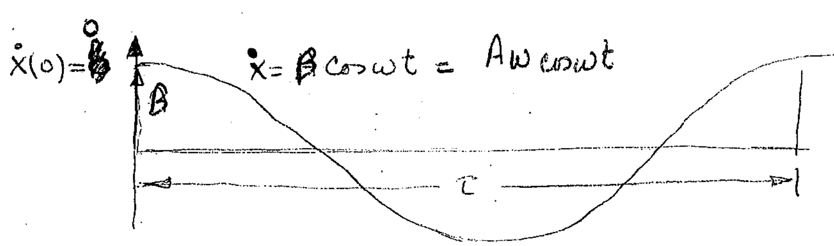
• ASSUMPTION – NO HORIZONTAL MOTION OF MASS

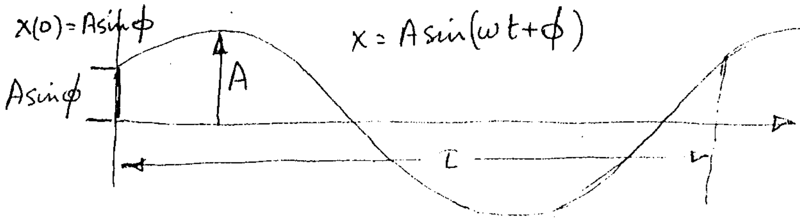
• VIBRATION OR OSCILLATION INVOLVES TRANSFER OF ENERGY FROM KE ⇄ PE

• MOTION OF MASS WILL BE CYCLIC OR PERIODIC

PERIODIC HARMONIC MOTION
$\ddot{x} = -\text{const}\, x$
MOTION OF MASS IF ELASTIC MEMBER ∝ TO DEFORMATION





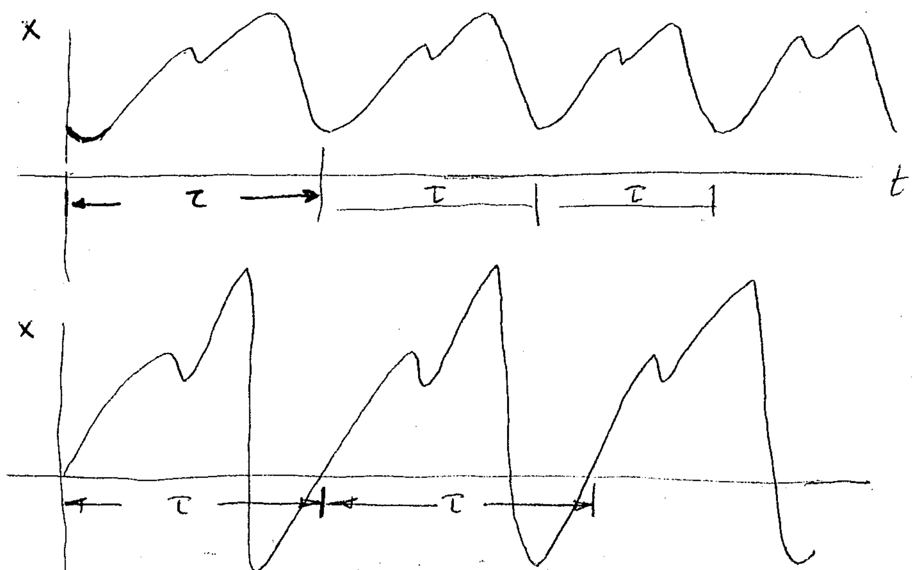


- MOTION IS DEPENDENT ON THE INITIAL CONDITIONS
- A – AMPLITUDE = MAX DISP.

  ϕ – phase ANGLE rad or degrees

  ω – CIRCULAR FREQUENCY rad/sec or degrees/sec

NONHARMONIC PERIODIC MOTION



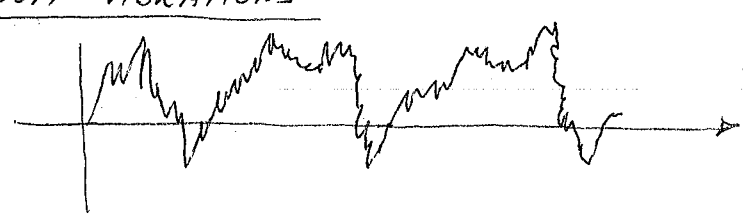
ONE COMPLETE MOVEMENT IS A CYCLE

TIME OF 1 CYCLE IS A PERIOD = τ (seconds)

FREQUENCY (# OF TIMES) OF CYCLE IN A UNIT TIME = f (cycles/second, Hz)

$\tau f = 1$.

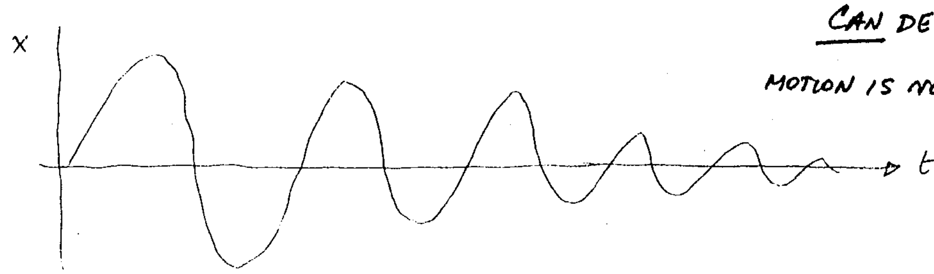
RANDOM VIBRATIONS



PRODUCED BY FORCES THAT ARE IRREGULAR

- CAN OCCUR DURING TRANSPORTATION OF GOODS
- IMPACT CONDITIONS

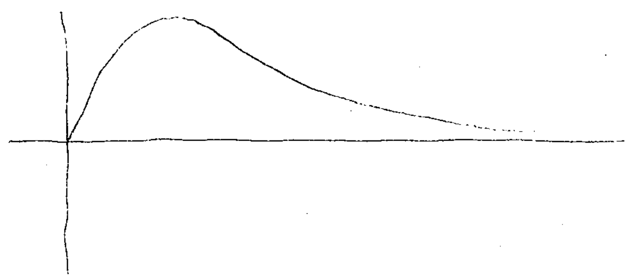
WHEN SYSTEM UNDERGOES RESISTANCE (DAMPING) OSCILLATORY BEHAVIOR DIES OUT



CAN DEFINE A PERIOD

MOTION IS NOT HARMONIC

OR



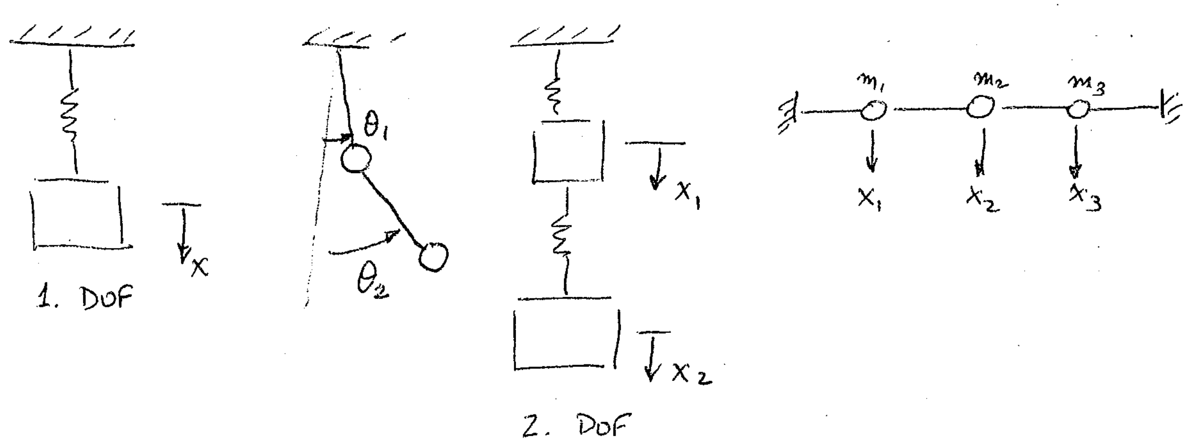
APERIODIC - NONPERIODIC

IF ∃ EXTERNAL FORCES CAUSING OSCILLATION - FORCED VIB

∄ " " - FREE VIBS

NO. OF DEGREES OF FREEDOM

NO. OF INDEP COORDINATES NEEDED TO FULLY DESCRIBE MOTION



FOR EACH DISCRETE POINT MASS 3 DOF

- IN GENERAL SYSTEMS THAT UNDERGO VIBS ARE COMPLICATED
- CAN BE REPLACED BY SIMPLER SYSTEMS INVOLVING masses, springs, dampers
- THE MODEL IS AN EQUIVALENT SYSTEM
- EACH ELEMENT IS CONSIDERED AS <u>LUMPED</u>
  - SPRING HAS NO MASS, DAMPING
  - MASS IS RIGID - DOES NOT DEFORM, NO DAMPING
  - DAMPER HAS NO MASS, ELASTICITY

SEE PG 12 for car
20 for motorcycle 4th

<u>STATEMENT OF VIBRATIONS PROBLEM</u>

USE CAR AS EXAMPLE

1. RECOGNIZE THAT VIBS CAN OCCUR
2. DETERMINE WHAT VIBS ARE SIGNIFICANT
3. FORMULATE A SIMPLE MODEL THAT CAPTURES GYST OF PROBLEM
4. * DERIVE GOVERNING EQS.
5. * SOLVE EQS
6. * INTERPRET RESULTS
7. MAKE APPROPRIATE RECOMENDATIONS

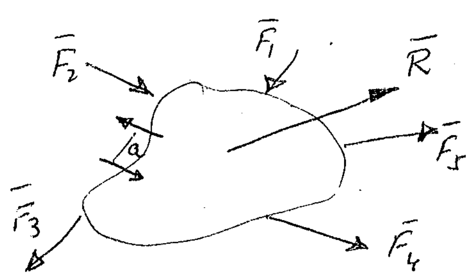
GOVERNING EQS DEPENDENT ON NEWTON'S LAWS

FIRST LAW - BODY REMAINS AT REST OR CONTINUES MOVING AT CONST. VEL. UNLESS ACTED UPON BY AN UNBALANCED EXTERNAL FORCE

(*) SECOND LAW - $\bar{F} = m\bar{a}$ $\quad \bar{a}$ is in direction of $\bar{F}$

THIRD LAW - FOR EVERY ACTION IS AN EQUAL & OPPOSITE REACTION

BODY ACTED UPON BY A SET OF FORCES & COUPLES



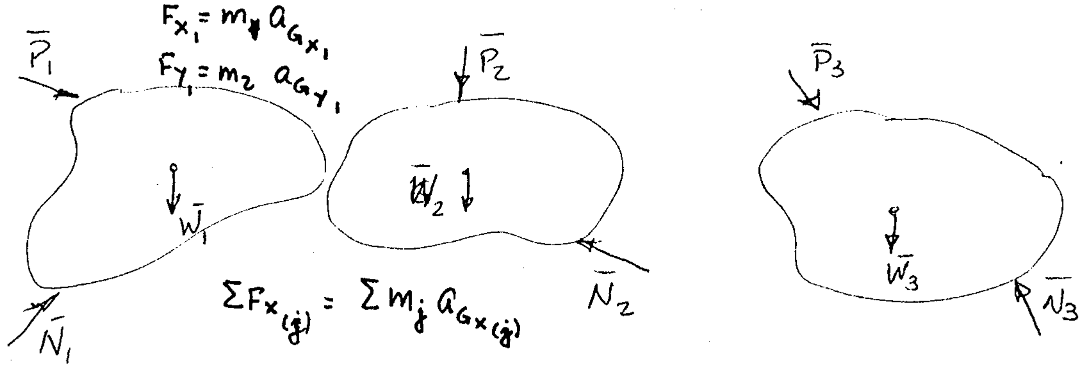
$$\Sigma \bar{F}_i = \bar{R} = m\bar{a}_G$$

where $\bar{a}_G$ is the accel of the mass center

SESSION #4

$\bar{a}_G$ & $\bar{R}$ have same direction <u>but</u> different lines of action

FOR A SYSTEM OF RIGID BODIES UNDER EXTERNAL FORCES



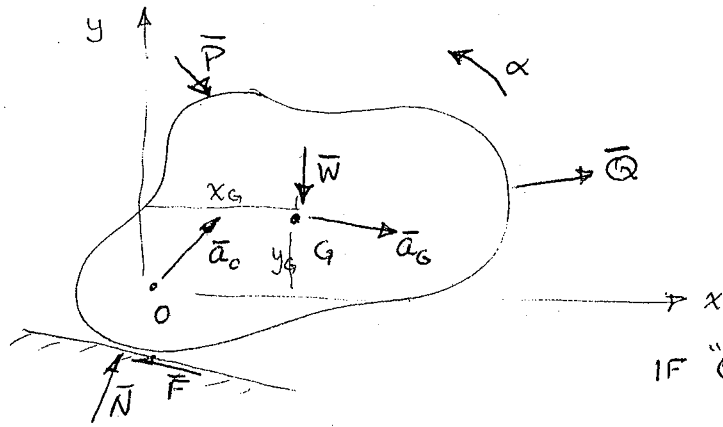
FOR EACH BODY

$$\bar{R} = \Sigma m_i \bar{a}_G^{(i)}$$

$\bar{R}$ resultant of <u>EXTERNAL FORCES</u>

BODIES MAY OR MAY NOT BE IN CONTACT — CONTACT FORCES CANCEL

PRINCIPLE OF ANGULAR MOTION — PICK AXES THROUGH A PT



$$\Sigma T_o = I_o \alpha + m a_{o_y} x_G - m a_{o_x} y_G$$

$\bar{r}_{G/o} \times m\bar{a}_o$

$$\Sigma T_G = I_G \alpha \quad \text{ABOUT } G$$

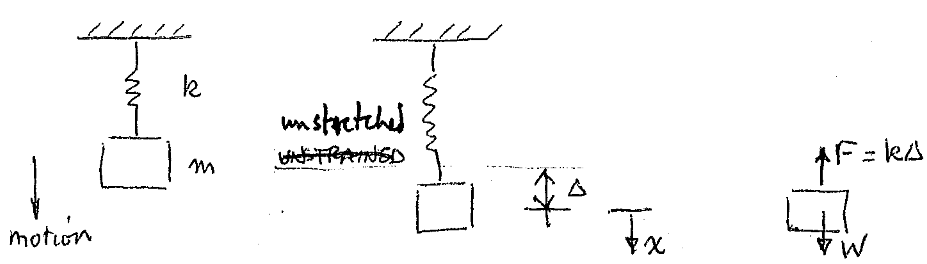
IF "O" IS FIXED $\bar{a}_o = 0$

$$\Sigma T_o = I_o \alpha$$

DO EQUIV SYSTEMS HERE FIRST (LESSON #5).

UNDAMPED FREE VIBS FOR SDOF SYS

SIMPLEST IS AN ELASTIC MEMBER + MASS

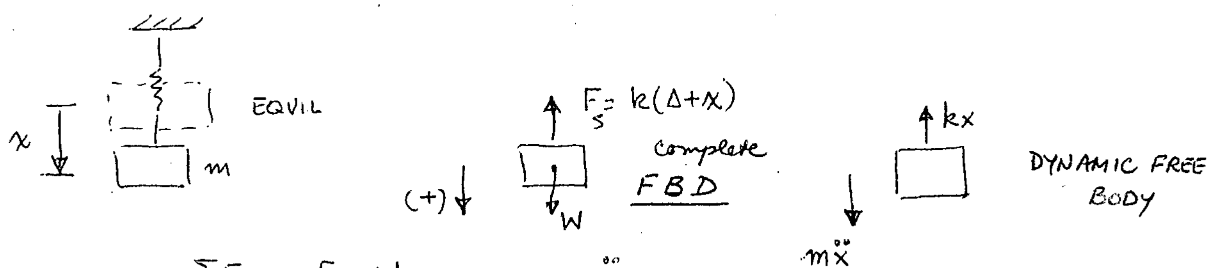


NO EXTERNAL FORCES - FREE VIBS

SDOF - SINCE X IS ONLY VAR. TO DEFINE POSITION OF MASS

BY STATIC EQUILIB $F = k\Delta = W$ $\quad \Delta = \frac{W}{k}$ STATIC DISP. δ_{ST} in RAO

- ASSUME NO HORIZONTAL MOTION OF MASS
-
- SUPPOSE WE DISPLACE MASS & LET GO
- MASS MOVES VERTICALLY ABOUT EQUILIB POSITION



$$\Sigma F = -F_s + W = ma = m\ddot{x}$$

$$= -(\Delta + x)k + W = -kx = m\ddot{x}$$

THIS SHOWS VIB TAKES PLACE ABOUT THE EQUIL POSITION

LET $\omega^2 = \frac{k}{m}$ $\qquad \ddot{x} + \omega^2 x = 0$ $\qquad \omega_n^2$ in RAO

$x(t) = Ce^{st} \Rightarrow Ce^{st}(s^2 + \omega^2) = 0 \quad s = \pm i\omega$

$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$

SOLUTION: $x = A\sin\omega t + B\cos\omega t$ $\quad$ Euler $e^{i\theta} = \cos\theta + i\sin\theta$

$= C\sin(\omega t + \phi)$

LET $A = C\cos\phi$ $\quad B = C\sin\phi$ $\quad \therefore A = iC_2 + iC_1 \quad B = C_1 + C_2$

$$x = C[\cos\phi\sin\omega t + \sin\phi\cos\omega t] = C\sin(\omega t + \phi)$$

$$C = \sqrt{A^2 + B^2} \qquad \phi = \tan^{-1} B/A$$

ϕ is the phase angle $\qquad C$: amplitude

ω - CIRCULAR FREQ. $\qquad \omega t = 2\pi$ 1 cycle $\quad \tau = \frac{2\pi}{\omega}$

TO SOLVE COMPLETELY NEED IC

$$x(0) = x_0 \Rightarrow B = x_0$$

$$\dot{x}(0) = \dot{x}_0 \Rightarrow A = \dot{x}_0/\omega$$

$$x = \frac{\dot{x}_0}{\omega}\sin\omega t + x_0\cos\omega t = X\sin(\omega t + \phi)$$

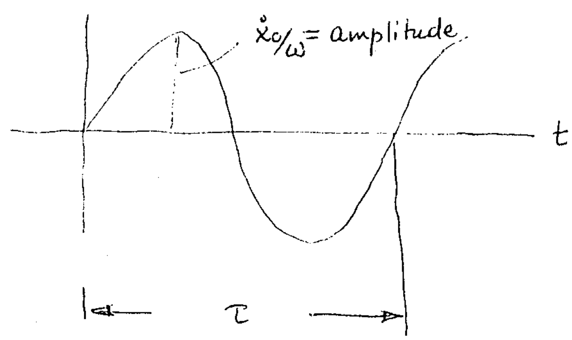
$$X = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega}\right)^2} \qquad \phi = \tan^{-1}\frac{x_0}{\dot{x}_0/\omega}$$

MOTION IS SINUSOIDAL OR HARMONIC

TIME FOR 1 CYCLE $\omega t = 2\pi$ $\quad \frac{2\pi}{\omega} = \tau$ PERIOD

$\tau f = 1 \quad \therefore \quad f = \frac{1}{\tau} = \frac{\omega}{2\pi}$ FREQUENCY

ω = CIRCULAR FREQ $= 2\pi f = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{mg/k}} = \sqrt{\frac{g}{W/k}} = \sqrt{\frac{g}{\Delta_{st}}}$



FOR $x_0 = 0 \quad \dot{x}_0 \neq 0$

- amplitude - max. travel about equil position
- ω^2 is prop. to g & inversely prop. to static disp

units of ω is radians/sec

τ is seconds

f is cycles/sec or Hertz (Hz)

FOR $x = \frac{\dot{x}_0}{\omega} \sin\omega t + x_0 \cos\omega t = X \sin(\omega t + \phi) \qquad X = \sqrt{\left(\frac{\dot{x}_0}{\omega}\right)^2 + x_0^2} \qquad \phi = \tan^{-1}\frac{x_0\omega}{\dot{x}_0}$

$\dot{x} = \dot{x}_0 \cos\omega t - x_0\omega \sin\omega t = X\omega\cos(\omega t + \phi) = X\omega \sin(\omega t + \phi + \frac{\pi}{2})$

$= \dot{X}\cos(\omega t + \phi) \qquad \dot{X} = X\omega$ amplitude of veloc.

- velocity is ω times the displacement
- this shows that the velocity is out of phase with the displ by $\pi/2$

$\ddot{x} = -\dot{x}_0\omega \sin\omega t - x_0\omega^2 \cos\omega t = -X\omega^2 \sin(\omega t + \phi)$

this is governing DE $\quad = -\omega^2 x$

$= -\ddot{X}\sin(\omega t + \phi) \qquad \ddot{X} = X\omega^2 = \dot{X}\omega$ Amplitude of accel

- accel is negative of displacement curve mult. by ω^2

$$\tau = \frac{1}{f} = \frac{2\pi}{\omega_n} = 2\pi \Big/ \sqrt{\frac{G\pi}{32 I l_1 l_2}\left(l_1 D_2^4 + l_2 D_1^4\right)}$$

$$= 2\pi \sqrt{\frac{32 I l_1 l_2}{G\pi} \frac{1}{\left(l_1 D_2^4 + l_2 D_1^4\right)}} = 8\sqrt{\frac{2\pi I l_1 l_2}{G\left(l_1 D_2^4 + l_2 D_1^4\right)}}$$

PROBLEM 2-11 Given: mass decreased by 0.4 kg & freq ↑ by 25%

suppose you are told a system represented by a spring-mass had a freq of 2 Hz by ~~adding~~ subtracting 0.4 kg freq increases by 25%. what is k, m

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 2\,Hz \qquad f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m-.4}} = 1.25 f$$

$$(2\pi f)^2 = [2\pi(2)]^2 = \frac{k}{m} \Rightarrow \frac{m}{k} = \frac{1}{16\pi^2}$$

$$(2\pi f')^2 = [2\pi(2.5)]^2 = \frac{k}{m-.4} \Rightarrow \frac{m-.4}{k} = \frac{1}{25\pi^2}$$

$$-\frac{.4}{k} = \frac{1}{25\pi^2} - \frac{m}{k} = \frac{1}{25\pi^2} - \frac{1}{16\pi^2} = \frac{1}{\pi^2}\left[\frac{-9}{16\cdot 25}\right]$$

$$k = \frac{\pi^2(16\cdot 25)}{9}(.4) = 175.46\ N/m.$$

$$m = \frac{k}{16\pi^2} = 1.11\ kg$$

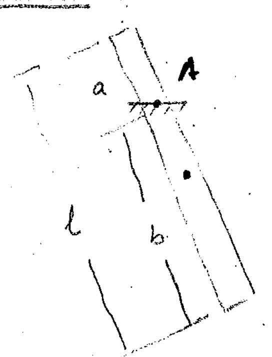
PROBLEM 2-16 A vibrating SDOF system shows an amplitude of 2cm and a period of 3 sec. Determine the max. velocity & accel

given $X = 2cm$ $\tau = 3\,sec = \frac{2\pi}{\omega}$ $\omega = \frac{2}{3}\pi = 2.094$ rad/sec

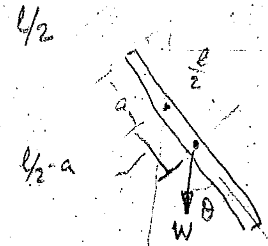
max veloc $\dot{X} = X\omega = 4.19$ cm/sec

$\ddot{X} = -X\omega^2 = 8.77\ cm/sec^2$

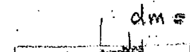
PROBLEM 2-21 A slender uniform bar pivots ~~about~~ in a vertical plane about A for small vibs obtain the DE & freq



center of gravity is at $l/2$



$$I_o = I_G + m\left(\tfrac{l}{2} - a\right)^2$$



$$I_G = \int_{-l/2}^{l/2} x^2\,dm = \int_{-l/2}^{l/2} x^2 \rho\,dx = \rho\frac{l^3}{12} = \frac{ml^2}{12}$$

$$I_0 = \frac{m\ell^2}{12} + m\left(\frac{\ell^2}{4} - 2\frac{\ell}{2}a + a^2\right) = m\left[\frac{\ell^2}{3} - \ell a + a^2\right]$$

TORQUE OF weight about O $= -W\sin\theta\,(\ell/2 - a)$

$$I_0\ddot\theta = -mg\sin\theta(\ell/2 - a) \quad \text{or}$$

$$m\left[\ell^2/3 - \ell a + a^2\right]\ddot\theta + mg(\ell/2 - a)\theta = 0 \qquad \text{for small } \theta$$

$$\omega = \sqrt{\frac{mg(\ell/2 - a)}{(\ell^2/3 - \ell a + a^2)m}} = \sqrt{\frac{g(\ell/2 - a)}{\ell^2/3 - \ell a + a^2}}$$

$$f = \frac{1}{2\pi}\omega$$

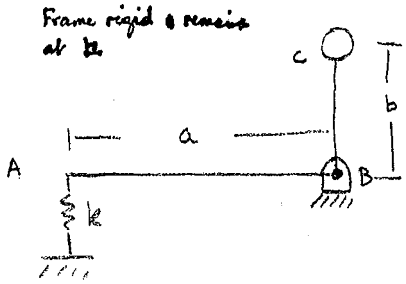
now $\omega_n = \sqrt{g/\ell'}$ for simple pendulum $\ell' = \dfrac{\ell^3/3 - \ell a + a^2}{\ell/2 - a}$

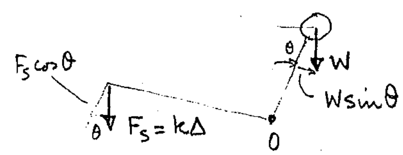
center of percussion

PROBLEM 2-24

WRITE EQ wrt B

Frame rigid & remains at B





Torque due to Weight $W\sin\theta \cdot b$

$\Delta = a\sin\theta$

$F_s = ka\sin\theta$ Torque due to $F_s = (-ka\sin\theta)\cos\theta\, a$

$I_o = mb^2$ if I_G of ball is small

$$I_0\ddot\theta = -ka^2\sin\theta\cos\theta + Wb\sin\theta$$

$$\text{or} \quad mb^2\ddot\theta + (ka^2 - Wb)\theta = 0 \qquad \text{for small } \theta$$

$$\omega = \sqrt{\frac{ka^2 - Wb}{mb^2}} \qquad f = \frac{1}{2\pi}\omega.$$

energy $\frac{1}{2}k_{eq}\Delta_{eq}^2 = \frac{1}{2}k\Delta^2 + mgb(\cos\theta - 1)$

$\frac{1}{2}k_{eq}\Delta_{eq}^2 = \frac{1}{2}ka^2\theta^2 - \frac{mgb}{2}\theta^2$

$(a\theta)^2$

$\therefore k_{eq} = k - \dfrac{mgb}{a^2}$

energy $\frac{1}{2}m_{eq}\dot\Delta_{eq}^2 = \frac{1}{2}m(b\dot\theta)^2$

$m_{eq} = m\left(\dfrac{b}{a}\right)^2$

$\therefore \omega_n = \sqrt{\dfrac{k_{eq}}{m_{eq}}} = \sqrt{\dfrac{k - mgb/a^2}{mb^2/a^2}}$

DO PROBLEMS 2-30, 2-32, 2-33

~~$\tau = \frac{1}{f} = \frac{2\pi}{\omega_n} = 2\pi \Big/ \sqrt{\frac{G\pi}{32 I l_1 l_2}\left(l_1 D_2^4 + l_2 D_1^4\right)}$~~

~~$= 2\pi \sqrt{\frac{32 I l_1 l_2}{G\pi} \frac{1}{\left(l_1 D_2^4 + l_2 D_1^4\right)}}$~~ $= 8\sqrt{\frac{2\pi I l_1 l_2}{G\left(l_1 D_2^4 + l_2 D_1^4\right)}}$

DAMPED FREE VIBS FOR SDOF

- SO FAR VIBS HAVE BEEN SELF-SUSTAINING
- REAL LIFE VIBS DIE AWAY (VIBS ARE DAMPED)
- THREE TYPES OF DAMPING

1 VISCOUS DAMPING $F_d = -c\dot{x}$ $c = \text{CONST.}$ F_d - DAMPING FORCE

F_d - lbs, N kg/sec, $\frac{N\,sec}{m}$, $\frac{lb\text{-}sec}{in}$

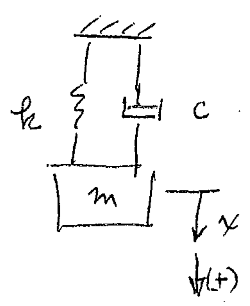
- EXAMPLE - BODY MOVING THRU FLUIDS AT LOW VELOC.
- FORCE OPPOSES MOTION
  - BODIES SLIDE OVER LUBRICATED SURFACE
  - BODIES THROUGH AIR, OIL

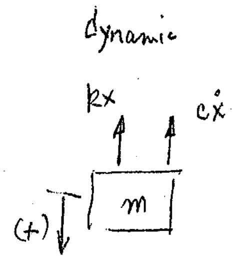
SHOCK ABSORBERS, DASHPOTS

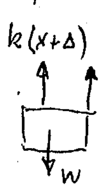
2. COULOMB FRICTION $F = \mu N$ SLIDING OVER DRY SURFACE

3. HYSTERESIS DAMPING DUE TO INTERNAL FRICTION IN MATERIAL

CONSIDER MASS-SPRING-DASHPOT SYSTEM







$c\frac{d}{dt}(x+\Delta) = c\dot{x}$ measure every thing from unstretched length of spring

$\frac{d^2}{dt^2}(x+\Delta) = \ddot{x}$

$m\ddot{x} = -kx - c\dot{x}$

$\ddot{x} + \frac{k}{m}x + \frac{c}{m}\dot{x} = 0$

$x = Ce^{st} \Rightarrow \quad s^2 + \frac{k}{m} + \frac{cs}{m} = 0$

$s = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}}{2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$

SESSION #7

if $\frac{c}{2m} = \sqrt{\frac{k}{m}} = \omega_n$ then $c = c_c$ critical damping cond.

$$(C_1 + C_2 t)e^{-\frac{c_c}{2m}t} = (C_1 + C_2 t)e^{-\omega t} \qquad \underline{c_c = 2\sqrt{km} = 2m\omega}$$

CAN DEFINE DAMPING FACTOR

$$\frac{c}{c_c} = \zeta \qquad \therefore \quad \frac{c}{2m} = \frac{c}{c_c}\cdot\frac{c_c}{2m} = \zeta(\omega) \qquad c = 2m\omega_n\zeta$$

now $\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = \sqrt{\zeta^2\omega_n^2 - \omega_n^2} = \sqrt{\zeta^2 - 1}\,\omega_n$

General Solution $= C_1 e^{(-\zeta + \sqrt{\zeta^2-1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2-1})\omega_n t}$ for $\zeta \neq 1$

$C_1 e^{s_1 t} + C_2 e^{s_2 t}$

- if $\zeta > 1$ DAMPING CONST > CRIT. DAMP. CONST.

  $\zeta > 1$; $\sqrt{\zeta^2 - 1} < \zeta$ $\quad \therefore \quad -\zeta + \sqrt{\zeta^2 + 1} < 0$

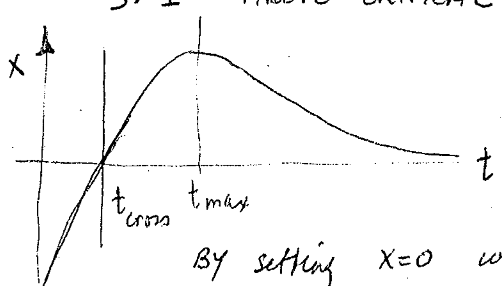
  $-\zeta - \sqrt{\quad} < 0$

  $x(t) =$ sum of 2 decreasing exponential $= C_1 e^{-\alpha t} + C_2 e^{-\beta t}$

  $\alpha = (\zeta - \sqrt{\zeta^2-1})\omega_n \quad \beta = (\zeta + \sqrt{\zeta^2-1})\omega_n$

  $x(t)$ is not periodic or aperiodic

- $\zeta > 1$ ABOVE CRITICAL DAMPING <u>overdamped</u>



$x_0 = C_1 + C_2$
$\dot{x}_0 = -\alpha C_1 - \beta C_2$ @ $t = 0$

$\frac{\alpha x_0 + \dot{x}_0}{\alpha - \beta} = C_2 \qquad \frac{\beta x_0 + \dot{x}_0}{\beta - \alpha} = C_1$

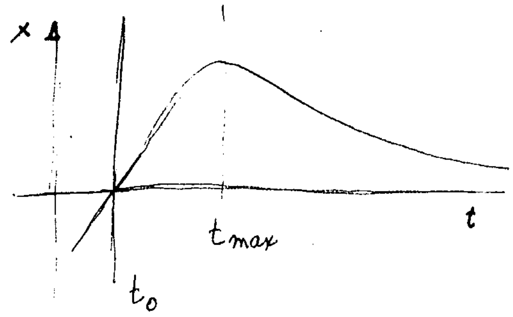
BY setting $x = 0$ we get first crossing pt. $t_{crossing} = \frac{\ln(-C_2/C_1)}{\beta - \alpha}$

$$t_{max} = \frac{\ln(-\beta C_2/\alpha C_1)}{\beta - \alpha} \qquad \beta - \alpha = 2\omega\sqrt{\zeta^2 - 1}$$

$\frac{dx}{dt} = 0 \Rightarrow -C_1\alpha e^{-\alpha t} - C_2\beta e^{-\beta t}$

- IF $\zeta = 1$ $\quad x(t) = (A + Bt)e^{-\omega t}$ $\quad$ APERIODIC FN $\quad x_0 = A$, $\dot{x}_0 = B - A\omega$

FOR THE CRITICALLY DAMPED CASE $\zeta = 1$



$x = 0 \Rightarrow A + Bt_0 = 0 \qquad t_0 = -\frac{A}{B}$

t_0 is first crossing

$\dot{x} = \left[(B - A\omega) - B\omega t\right]e^{-\omega t} \qquad \dot{x} \Rightarrow 0 \Rightarrow$

$t_{max} = \frac{1}{\omega} - \frac{A}{B}$

- FOR THE SUBCRITICAL DAMPING CASE $\zeta < 1$ underdamped.

$$x(t) = c_1 e^{(-\zeta + i\sqrt{1-\zeta^2})\omega_n t} + c_2 e^{(-\zeta - i\sqrt{1-\zeta^2})\omega_n t}$$

$$= C e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ $\quad$ damped circular freq

- FOR $x = 0$

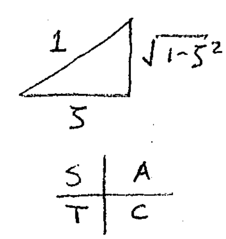
Crossing points $\quad \omega_d t + \phi = n\pi \qquad t = (n\pi - \phi)/\omega_d$

- TO FIND THE MAX OR MIN

$$\dot{x}(t^*) = 0 = C e^{-\zeta\omega t^*}\left[-\zeta\omega \sin(\omega_d t^* + \phi) + \omega_d \cos(\omega_d t^* + \phi)\right]$$

IF $t^* \neq \infty$ $\quad e^{-\zeta\omega t^*}$ and C are not zero

$$\Rightarrow \tan(\omega_d t^* + \phi) = \frac{\omega_d}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$



since $\zeta < 1$ $\quad$ tan is (+) in 1^{ST} & 3^{rd} QUAD.

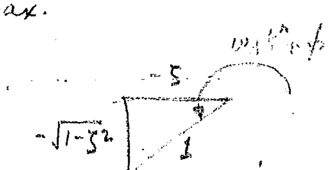
in 1^{ST} QUAD $\quad \sin(\omega_d t_1^* + \phi) = \sqrt{1-\zeta^2} < 1$ $\qquad$ MAXS

$$\sin(\omega_d t_1^* + \phi) = \sin(\omega_d t_1^* + \phi + 2\pi) = \ldots$$

$$t_1^* = \left[\sin^{-1}\sqrt{1-\zeta^2} - \phi - 2n\pi\right] \tfrac{1}{\omega_d} \qquad \text{max.}$$

First max is at $\quad t_1^* = \left[(\sin^{-1}\sqrt{1-\zeta^2}) - \phi\right]/\omega_d$

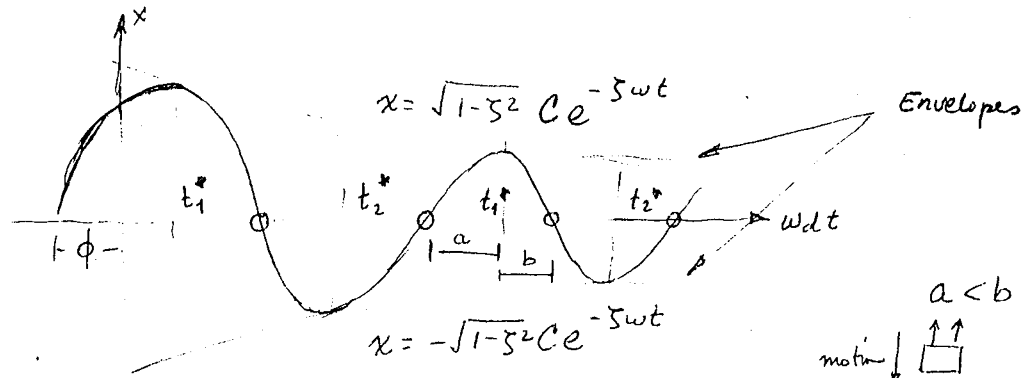
in 3^{rd} QUAD $\quad \sin(\omega_d t_2^* + \phi) = -\sqrt{1-\zeta^2} > -1$ $\qquad$ MINS

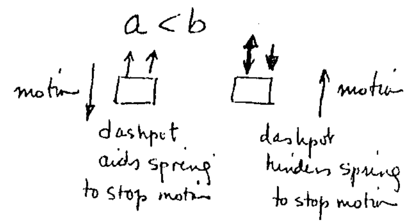


$$\sin(\omega_d t_2^* + \phi) = \sin(\omega_d t_2^* + \phi + 2\pi) = \ldots$$

$$t_2^* = \left[\sin^{-1}(-\sqrt{1-\zeta^2}) - \phi - 2n\pi\right]/\omega_d = \left[\sin^{-1}(\sqrt{1-\zeta^2}) - \phi - (2n-1)\pi\right]/\omega_d$$

$$\omega_d t_2^* = \omega_d t_1^* + \pi$$





- period between maximums. $\omega_d \tau = 2\pi$

$$\tau = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega}.$$

ENVELOPES

- since at max. $\sin(\omega_d t_1^* + \phi) = \sqrt{1-\zeta^2}$

$$x_{ENV.} = C e^{-\zeta\omega t} \sin(\omega_d t_1^* + \phi) = C e^{-\zeta \omega t} \sqrt{1-\zeta^2}$$

- at min $\sin(\omega_d t_2^* + \phi) = -\sqrt{1-\zeta^2}$

$$x_{ENV.} = C e^{-\zeta\omega t} \sin(\omega_d t_2^* + \phi) = -\sqrt{1-\zeta^2}\, C e^{-\zeta\omega t}$$

EFFECTS OF DAMPING. $\zeta > 0$

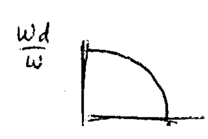
- REDUCES THE AMPLITUDE OF THE MOTION
- REDUCES TIME TO FIRST PEAK $\omega t_1^* = \dfrac{\sin^{-1}\sqrt{1-\zeta^2} - \phi}{\sqrt{1-\zeta^2}} \cong 1 + \dfrac{(1-\zeta^2)}{6} + \dfrac{3}{40}(1-\zeta^2)^2 + \ldots$

  $\omega t_1^* = \pi/2 - \phi$ vs. $\frac{\pi}{2} - \zeta - \phi$
- INCREASES TIME OF MOTION TO RETURN TO NEUTRAL POSITION

  $\omega t_{cross} = \pi - \phi$ vs. $\omega t_{cross} = \dfrac{\pi - \phi}{\sqrt{1-\zeta^2}}$ FIRST crossing

  $\sin(\omega_n t + \phi)$ $\qquad$ $\sin(\omega_d t + \phi)$
- PERIOD OF DAMPED MOTION > PERIOD OF UNDAMPED MOTION

  $\omega_n \tau_n = 2\pi$ VS $\omega_n \tau_d = \dfrac{2\pi}{\sqrt{1-\zeta^2}}$ $\qquad T_n = \dfrac{2\pi}{\omega_n}$ & $T_d = \dfrac{2\pi}{\omega_d}$ but $\omega_d < \omega_n$
- PEAKS STARTED TO THE LEFT OF THE UNDAMPED PEAKS
  - SINCE PERIOD INCREASES $\Rightarrow$ PEAKS MOVE TO RIGHT OF UNDAMPED
- DAMPING DUE TO CRITICAL DAMPING CAUSES AMPLITUDE TO DROP TO ZERO MOST QUICKLY FOR SAME I.C.
- FREQ DECREASES $\omega_d = \sqrt{1-\zeta^2}\,\omega$



LOGARITHMIC DECREMENT

- WANT TO MEASURE THE RATE OF DECAY OF SYSTEM
- TAKE 2 POINTS IN PHASE (i.e. amplitudes)

$$x_j = C\sqrt{1-\zeta^2}\, e^{-\zeta\omega t_j} \quad \text{or} \quad Ce^{-\zeta\omega t_j}\sin(\omega_d t_j + \phi)$$

$$x_{j+1} = C\sqrt{1-\zeta^2}\, e^{-\zeta\omega(t_j+\tau)} \quad \text{or} \quad Ce^{-\zeta\omega(t_j+\tau)}\sin[\omega_d(t_j+\tau)+\phi]$$

$$\sin[\omega_d(t_j+\tau)+\phi] = \sin(\omega_d t_j + \phi + 2\pi)$$

$$\frac{x_j}{x_{j+1}} = e^{\zeta\omega_n\tau_d} = \text{constant}; \quad \ln\frac{x_j}{x_{j+1}} = \zeta\omega_n\tau_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\tau_d = 2\pi/\omega_d$$

LOGARITHMIC DECREMENT

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \ln\left(\frac{x_j}{x_{j+1}}\right)$$

SAME FOR ANY SUCCESSIVE AMPLITUDES

- FOR THE DECAY AFTER n cycles

$$\frac{x_0}{x_n} = \frac{x_0}{x_1}\,\frac{x_1}{x_2}\,\frac{x_2}{x_3}\cdots\frac{x_{n-1}}{x_n} = \left(\frac{x_j}{x_{j+1}}\right)^n$$

$$\ln\left(\frac{x_0}{x_n}\right) = n\ln\left(\frac{x_j}{x_{j+1}}\right) = n\delta$$

$$\delta = \frac{1}{n}\ln\left(\frac{x_0}{x_n}\right)$$

or

- DETERMINE NO. OF CYCLES TO CAUSE DECAY FROM x_0 TO x_n

$$n = \frac{1}{\delta}\ln\left(\frac{x_0}{x_n}\right) = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta}\ln\left(\frac{x_0}{x_n}\right)$$

NOT INTEGER

- TIME FOR THIS TO OCCUR IS

$$n\tau = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta}\ln\left(\frac{x_0}{x_n}\right)\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$$

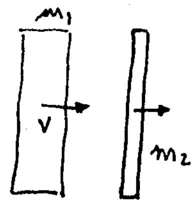
(τ: period between cycles)

$$\Delta T = n\tau = \frac{1}{\zeta\omega_n}\ln\left(\frac{x_0}{x_n}\right)$$

if from exper we can get $\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \zeta = \frac{\delta}{\sqrt{(2\pi)^2+\delta^2}}$

if k & m are known $\omega = \sqrt{\frac{k}{m}}$

$$\frac{c}{2m} = \zeta\omega \Rightarrow c = 2m\zeta\omega$$

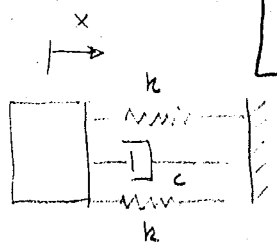


$m_1 V_1 = (m_1 + m_2) V_2$

$V_2 = \frac{m_1}{m_1+m_2} V_1$

RAO

2-52



$k_{eq} = 2k =$

c

Given $c = .2 \frac{N\text{-}s}{mm}$, $k = 40 N/mm$

$m\ddot{x} + c\dot{x} + k_{eq}x = 0$ w/IC that $x = 0$ @ $t = 0$, $\dot{x} = 80 m/s$ @ $t = 0$

Now $\omega = \frac{c_c}{2m} = \sqrt{\frac{k_{eq}}{m}} \Rightarrow c_c = 2\sqrt{m k_{eq}} = 2\sqrt{(1)(2 \cdot 40 \cdot 1000)} = 565.686 \; N\text{-}s/m = .566 \; \frac{N\text{-}s}{mm}$

$\zeta\omega = \frac{c}{2m} = 100$

$c = .2 \; N\text{-}s/mm$ $\qquad \zeta = \frac{c}{c_c} = .3536 \Rightarrow$ underdamped

$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{80000}{1}} = 282.84 \text{ rad/s}$

initially $x = 0$, $\dot{x} = 80 m/s$

$x = X e^{-\zeta\omega t} \sin(\omega_d t + \phi)$ $\qquad$ when $t = 0$ $\qquad 0 = X e^0 \sin\phi$

either $\phi = 0$ or $X = 0$ $\qquad$ if $X = 0 \Rightarrow x = 0 \; \forall t$ trivial soln

$\dot{x} = -\zeta\omega X e^{-\zeta\omega t} \sin(\omega_d t + \phi) + X\omega_d e^{-\zeta\omega t} \cos(\omega_d t + \phi)$

$\dot{x} = 80 = -\zeta\omega X e^0 \sin\phi + X\omega_d e^0 \cos\phi$ $\qquad$ if $X = 0$ cannot satisfy $\Rightarrow \phi = 0$

$80 = X\omega_d e^0 \cdot 1 = X\omega\sqrt{1-\zeta^2} \Rightarrow X = .3023 \; m$ (~~.3266~~)

$x = .3023 \, e^{-(.3536)(282.84)t} \sin\left(\frac{80}{.3023} t\right)$

$\dot{x} = -\zeta\omega_n X e^{-\zeta\omega_n t} \sin\omega_d t + X\omega_n\sqrt{1-\zeta^2} \cos(\omega_d t) = 0$ when $t = t^*$

$\Rightarrow -\zeta \sin\omega_d t + \sqrt{1-\zeta^2} \cos\omega_d t$ or $\tan\omega_d t^* = \frac{\sqrt{1-\zeta^2}}{\zeta}$

$t^* = \frac{\sin^{-1}\sqrt{1-\zeta^2}}{\omega\sqrt{1-\zeta^2}} = \frac{69.29 \pi / 180}{264.57} = .00457 \text{ sec}$

$x = .3023 \, e^{-\zeta\omega t} \sin(\omega_d t)$ w/ $\zeta\omega = 100$ $\quad \omega_d = 264.57$ $\quad t = .00457$ sec

$x = .1790 \; m$

Given that 2 successive mass are differ a factor of 12

2-49

$\left(\frac{x_j}{x_{j+1}}\right) = \frac{12}{1} \Rightarrow \ln(12) = \delta = 2.4849 \Rightarrow \zeta = \frac{2.4849}{\sqrt{(2\pi)^2 + \delta^2}} = .36777$

$\zeta_{new} = .73554$

$\delta = \ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\zeta_{new}}{\sqrt{1-\zeta_{new}^2}} = 6.822$

$\frac{x_j}{x_{j+1}}$ $\quad 917.49$

HW 3-26, 3-27, 3-30, 3-35

a system is composed of mass & spring subjected to a coulomb force. If initial amplitude is find amplitude after eight cycles & frequency

Problem 3-36 Given [mass-spring sketch] FIND X AFTER 8 CYCLES

$m = 57.61\ kg \quad k = 9100\ N/m \quad F_c = 6.825N \quad X_0 = 4cm \quad$ find X_8

$$\Delta X = \frac{4F_c}{k} = \frac{4(6.825N)}{9100\ N/m} = .003\,m = .3\,cm$$

at end of 8 cycles decreases by $8(.3\,cm) = 2.4\,cm \quad \therefore \quad X_8 = X_0 - 8(\Delta X) = 1.6cm$

- $f = \dfrac{\omega}{2\pi} \qquad \omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{9100\ N/m}{57.61\ kg}} = 12.5682\ rad/sec$

$$f = 2\ Hz = \frac{\omega}{2\pi}$$

CESSATION OF MOTION

- $X_{max} = \dfrac{F_c}{k} = .075cm \qquad \dfrac{X_0 - .075}{n} = .3$ or $\dfrac{4 - .075}{.3} = \dfrac{3.925}{.3} \cong 13.1$ cycles TO CESSATION

TOPIC IV Harmonically Forced Vibs – UNDAMPED CASE

- LOOK AT CASE OF MASS-SPRING SYSTEM INFLUENCED BY A TIME DEPENDENT FORCING FUNCTION

- FORCE MAY BE HARMONIC, OR NONHARMONIC BUT PERIODIC, OR NONPERIODIC OR RANDOM
- WANT DYNAMIC RESPONSE OF SYSTEM CAUSED BY FORCE

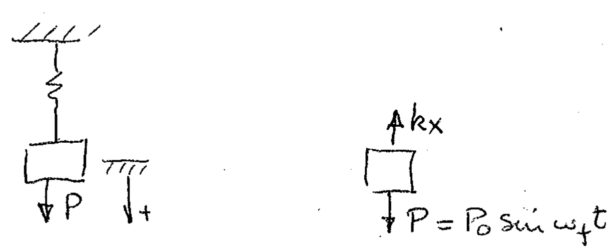
- LOOK AT SDOF W/ HARMONIC FORCING CONDITION OF FORM $P(t) = P_0 \sin(\omega_f t + \psi)$ **RAD use $F_o \cos \omega t$**

P_0 is amplitude of $P(t)$

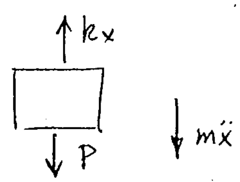
ω_f is the circular freq of FORCING FUNC **ω in RAO**

ψ is the phase angle $\qquad \psi$ is DEPENDENT ON I.C. of $P(t)$

LOOK AT CAUSE WHERE $\psi = 0$



FOR DYNAMIC FREE-BODY DIAG.



$$m\ddot{x} = -kx + P$$

$$m\ddot{x} + kx = P_o \sin \omega_f t$$

Solution is $X_0 + X_p = X$

FREE VIBS SOLN : X_0 satisfies $m\ddot{x} + kx = 0$ $\qquad X_0 = A \sin \omega_n t + B \cos \omega_n t \quad \omega_n = \sqrt{\frac{k}{m}}$

FORCED VIBS : X_p " $\quad m\ddot{x} + kx = P_o \sin \omega_f t$

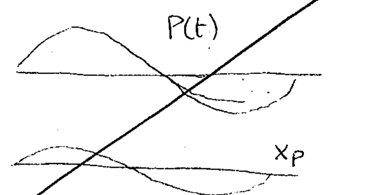
$\qquad P_o \cos \omega_f t$

- if we choose $X_p = \begin{cases} C \sin \omega_f t \\ C \cos \omega_f t \end{cases} \qquad C = \dfrac{P_o}{k - m\omega_f^2} = \dfrac{P_o}{m} \dfrac{1}{\omega_n^2 - \omega_f^2}$

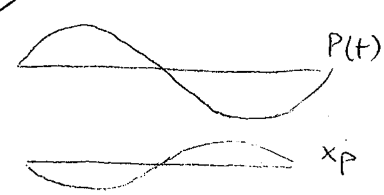
$$\frac{P_o}{m} \frac{1}{\omega_n^2 \left[1 - \left(\omega_f/\omega_n\right)^2\right]} = \frac{P_o}{m \cdot k/m} \frac{1}{\left[1 - r^2\right]} = \frac{P_o/k}{1 - r^2} = X = \frac{X_o}{1 - r^2}$$

- $X_o = P_o/k$ STATIC DISPL OF SPRING DUE TO P_o $\qquad \delta_{ST}$ in RAO
- $r = \omega_f/\omega_n$ IS THE FREQ ratio $\qquad \dfrac{\omega}{\omega_n}$ in RAO

- if $r < 1 \quad 0 < 1 - r^2 < 1 \quad X_p > 0$
  and X_p is in phase w/ $P(t)$



- if $r > 1 \quad 1 - r^2 < 0 \quad X_p < 0$
  and X_p is opposite to $P(t)$



- if $r = 1 \quad 1 - r^2 = 0$ and we have <u>resonance</u>

if $x(0) =$ [redacted] $= 0$ $\Rightarrow$ $\dot{x}(0) = -\dfrac{X_o \omega_n}{2}$

$$X_p = \frac{P_o t}{2m\omega_f} \begin{bmatrix} -\cos \omega_f t \\ +\sin \omega_f t \end{bmatrix} = \frac{P_o/k}{2\omega_f \left(\frac{1}{\omega_f}\right)^2} \begin{bmatrix} -\cos \omega_f t \\ +\sin \omega_f t \end{bmatrix} = \frac{X_o \omega_f t}{2} \begin{bmatrix} -\cos \omega_f t \\ +\sin \omega_f t \end{bmatrix}$$

Note 1: amplitude of $X_p = \dfrac{X_o \omega_f t}{2}$ increases linearly w/ time

2: since $\sin(\omega_f t + \pi/2) = -\cos \omega_f t$ $\qquad$ <u>displ</u> lags <u>force</u> by $\pi/2$

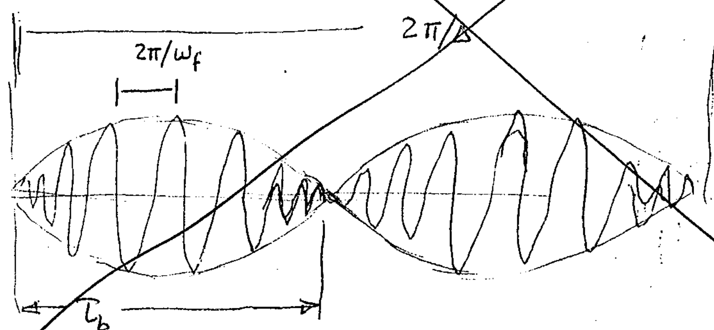
$\cos(\omega_f t - \pi/2) = \sin \omega_f t$

- IF $\omega_f \approx \omega$ but $\neq$ BEATING OCCURS
  - AMPLITUDE ↑ THEN ↓

$r \cong 1$

$$x = -\frac{X_0 \omega}{2\Delta r} \sin \Delta t \cos \omega_f t = \left(-\frac{X_0 \omega}{2\Delta} \cos \omega_f t\right) \sin \Delta t = \left(-\frac{X_0 \omega}{2\Delta} \sin \Delta t\right) \cos \omega_f t$$

- since $\omega_f >> \Delta$ cosine goes through many cycles before sine goes through one
  $\tau_f << \tau$
- $-\frac{X_0 \omega}{2\Delta r} \cos \omega_f t$ is a cyclic varying amplitude
- $\sin \Delta t$ represents envelope of the amplitude

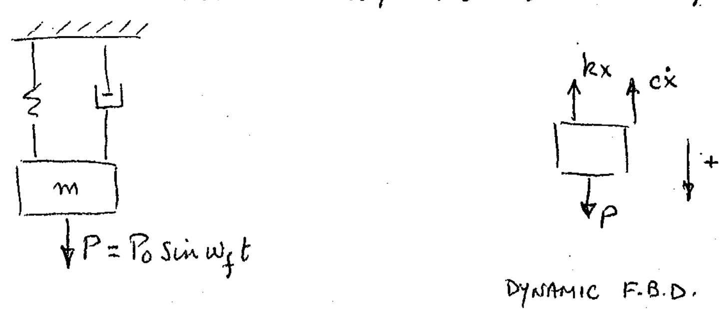


- BEAT PERIOD $\tau_b = \frac{2\pi}{\Delta}$ TIME BETWEEN POINTS OF ZERO MOTION
  $= \frac{2\pi}{\omega_n - \omega_f}$ OR BETWEEN MAXIMUM MOTIONS

→ HW 4-2, 6, 13

SESSION #11 + PROBLEMS

FORCED VIBRATIONS w/ VISCOUS DAMPING



$m\ddot{x} = -kx - c\dot{x} + P$

$$m\ddot{x} + c\dot{x} + kx = P_0 \sin \omega_f t$$

HAS SOLUTION $x = x_p + x_c$

x_c solves $m\ddot{x} + c\dot{x} + kx = 0$

x_p " $= P_0 \sin \omega_f t$

FLORIDA INTERNATIONAL UNIVERSITY
Mechanical and Materials Engineering Department

Spring 2011 Advanced Vibration Analysis EML6223

— Review:
- one degree of freedom systems.
- Free, forced, damped and undamped vibrations.
- Forced support vibration,

— Newton's law in non-inertial coordinate frame.
— Effective stiffness calculation for combined bar-beam-string-plate systems.
— Non-linear conservative systems. Free and forced vibrations. Piecewise-linear systems. Numerical solution.
— Non-linear non-conservative systems. Self-excited vibrations. Van der Pol's equation.
— Parametric resonance. Mathieu's equation. The Ince-Strutt diagram.
— Inelastic (especially, viscoelastic) material damping.
— Systems with multiple degrees of freedom. Some models. General analysis. Frequencies and mode shapes for undamped systems. Principal or normal coordinates. Damping in multidegree systems.
— Continuous systems with infinite number of degrees of freedom. Longitudinal vibrations of prismatic bars. Free and forced vibrations. Prismatic bar with a mass or spring at the end. The problem of bar impact.
— Torsional vibrations of shafts.
— Transverse vibrations of beams.
— Transverse vibrations of membranes and plates.
— 3D waves in continua.
— Stability analysis. Introduction to Liapunoffs method.

Book to be used S.S. Rao, Mechanical Vibrations, 4th Edition, Pearson-Prentice Hall Publishers.
Also notes will be provided from other books as well.

GRADES

Grades will be determined on the basis of

1 Midterm Exam	40 % each
HW	20 %
Final Exam	40 %

Letter Grades will be based as follows:

(A) 95 & above	(B+) 85-89	(C+) 73-76	(D) 60-64
(A-) 90-94	(B) 80-84	(C) 70-72	(F) below 60
	(B-) 77-79	(C-) 65-69	

Please be on time to class and keep up with the work. There is a lot of work to cover and it will be difficult for you if you do not do the homework assignments. My office hours will be posted during the first week of classes. Please come to see me if you are having problems or have suggestions on how to improve this course.

We will be meeting 2 times a week M and W 500-615pm. Our meeting room will be EC3327, though that may change.

My office hours tentatively are set for T-W 330-5pm and also by appointment.

THIS IS A PRELIMINARY SYLLABUS. ALL CHANGES WILL BE ANNOUNCED IN CLASS, INCLUDING ANY CHANGES IN CLASSROOM.

$$X = \sqrt{A^2+B^2} = \frac{P_o}{\sqrt{denom}} = \frac{P_o/k}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} = \frac{X_o}{\sqrt{(1-r^2)^2+(2r\zeta)^2}}$$

$$\tan\psi = -\frac{B}{A} = \frac{c\omega_f}{k-m\omega_f^2} = \frac{2\zeta r}{1-r^2}$$

BOTH X & ψ are fns of ζ & r

$x_p = X \sin(\omega_f t - \psi)$ $\longleftrightarrow$ $P_o \sin\omega_f t = P$ TIME LAG IN RESPONSE $r<1$ LEAD $r>1$

SESSION #12

- SAME MOTION BUT x_p lags $P(t)$ by ψ
- AMPLITUDE is X
- $\forall\ \zeta>0$ REDUCES X and for

NOTE IF $\zeta = 0$ (NO DAMPING) $\psi = 0$ & $X = \frac{X_o}{|1-r^2|}$ AS BEFORE

- THE LAG TIME t' IS TIME x_p LAGS $P(t)$
  - IE when $t=0$ $\sin\omega_f t = 0$ but $\sin(\omega_f t-\psi) \neq 0$
  - $\sin(\omega_f t'-\psi) = 0$ when $t' = \frac{\psi}{\omega_f}$

IF UNDERDAMPED

TOTAL SOL'N is $x = Ce^{-\zeta\omega t}\sin(\omega_d t+\phi) + X\sin(\omega_f t-\psi)$

GIVE HANDOUT

2nd ed. RAO 129

$x = (A+Bt)e^{-\omega_n t} + X\sin(\omega_f t - \psi)$ for critical

- LOOK AT PAGE 107 FOR TRANSIENT $\longrightarrow$ STEADY STATE CONVERSION

ASSUMED : $\omega_f < \omega_d$ $=$ $\omega_f < \sqrt{1-\zeta^2}\,\omega_n \Rightarrow r < \sqrt{1-\zeta^2} < 1$

$\tau_f > \tau_d$

- NOTE THAT $X_{DAMPED} < X_{UNDAMPED}$ FOR ALL ω_f
- NOTE WHEN $r = 1$ $X_{RES} = \frac{X_o}{2\zeta} \neq \infty$ EXCEPT FOR UNDAMPED

EXTREMUMS OF MF

- $MF = \frac{X}{X_o} = \frac{1}{\sqrt{(1-r^2)^2+(2r\zeta)^2}}$ $MF = MF(r,\zeta)$

take $\frac{d(MF)}{dr} = -\frac{1}{2}\frac{[2(1-r^2)(-2r)+4\zeta(2r\zeta)]}{[\quad]^{3/2}} = \frac{4r[-1+r^2+2\zeta^2]}{2[\quad]^{3/2}} = 0$

$$dMF/dr = 0$$

when $r=0$

when $r\to\infty$ $\quad d(MF)/dr \to -\frac{(-2r^3)}{r^6} = \frac{2}{r^3} \to 0 \quad r\to\infty$ ✓S

when $1-r^2 - 2\zeta^2=0 \quad r=\sqrt{1-2\zeta^2}<1$ for $0\le\zeta\le\frac{1}{\sqrt{2}} = .707$

for $0\le\zeta\le .707$ there is a relative max. $\Rightarrow \omega=\omega_n\sqrt{1-2\zeta^2}<\omega_d$.

- THIS DEFINES MAX POINT IN THE RESONANT REGION & since $r<1$

Pg 125 OCCURS TO THE LEFT OF THE RESONANT VALUE $r=1$

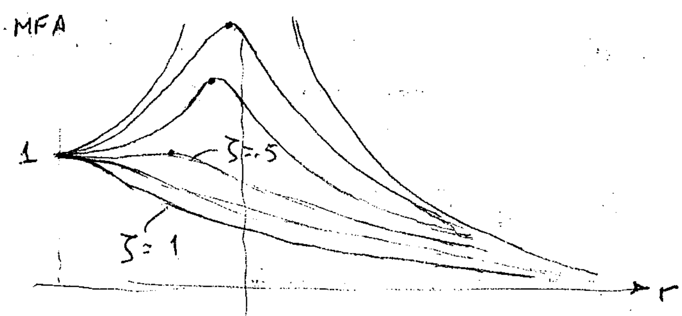
PUT $r=\sqrt{1-2\zeta^2}$ in $MF = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

$$(MF)_{max} = \frac{X_{max}}{X_o} = \frac{1}{\sqrt{(2\zeta^2)^2+(2\zeta)^2[1-2\zeta^2]}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \text{for } \zeta<0.707$$

SOLN TO 3.9 $\quad \Rightarrow \frac{X_{max}}{X_o} = \frac{X_{RES}}{X_o}\frac{1}{\sqrt{1-\zeta^2}} \Rightarrow \frac{X_{max}}{X_{res}} = \frac{1}{\sqrt{1-\zeta^2}}$

- FOR $\zeta\geq 0.707$ $\quad (MF)_{max}$ is at $r=0$

3 extremums for $\zeta<.707$
2 extrem for $\zeta>.707$



- $(MF)_{max}$ occurs when $r=\sqrt{1-2\zeta^2}$ or when $\omega_f=\omega_n\sqrt{1-2\zeta^2}$

$$\omega_f=\omega_n\sqrt{1-2\zeta^2}<\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}<\omega_n$$

- MF will decrease if r is large
- look at this as ω_f change only when r changes

*** GO TO MASS & VISCOSITY VARIATIONS NEXT - THEN RETURN**

- PHASE ANGLE VARIATION with r

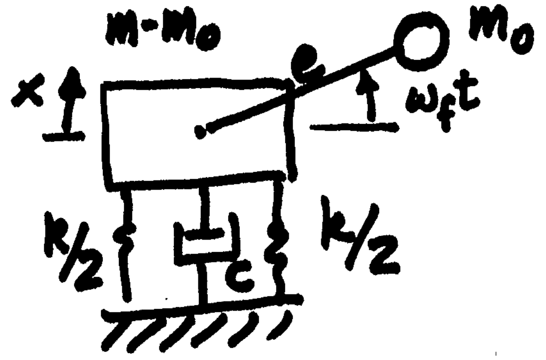
$\tan\psi = \frac{2\zeta r}{1-r^2} = \frac{c\omega_f}{k-m\omega_f^2}$; for ANY const ζ as $r\uparrow$ $\tan\psi\uparrow$ $r<1$

when $r=1$ $\tan\psi=\infty \Rightarrow \psi=\pi/2$

when $r>1$ when $r\uparrow$ $\tan\psi\to -0$

EML 4220 10/16/97

ROTATING UNBALANCE

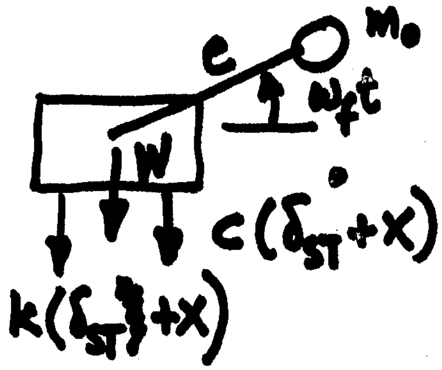


TOTAL SYSTEM MASS = m_0

$$\delta_{st} = \frac{mg}{k}$$

x: IS THE DISPL MEASURED FROM EQUIL OF ~~MAIN~~ TOTAL MASS

x: DEFINE FORCED MOTION OF $m - m_0$



$$(m-m_0)\frac{d^2x}{dt^2} + m_0 \frac{d^2}{dt^2}(x + e\sin\omega_f t)$$

$$= -k(\cancel{\delta_{st}} + x) - c(\dot{\delta}_{st} + \dot{x}) - \cancel{W}$$

$$m\ddot{x} + c\dot{x} + kx = \underbrace{m_0 e \omega_f^2}_{P_0} \sin\omega_f t$$

$$m\ddot{x} + c\dot{x} + kx = P_0 \sin\omega_f t$$

$$x_{ss} = x_p = \frac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \sin(\omega_f t - \psi)$$

$$\tan\psi = \frac{c\omega_f}{k - m\omega_f^2} = \frac{2\zeta r}{1 - r^2} \qquad \zeta = \frac{c}{c_c} \qquad r = \frac{\omega_f}{\omega_n}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

FOR ROTATING UNBALANCE

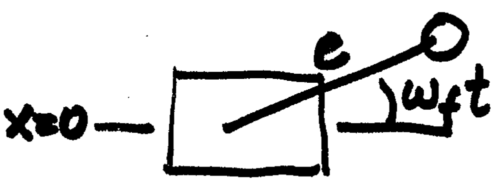
$$x_p = \frac{m_o e \omega_f^2}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \sin(\omega_f t - \psi)$$

$$\tan\psi = \frac{c\omega_f}{k - m\omega_f^2}$$

TOTAL DISPL = $X_{TRANSIENT} + X_p$

OVERDAMPED C.D. UNDERDAMPED

FREE VIBS FOR A DAMPED SYSTEM



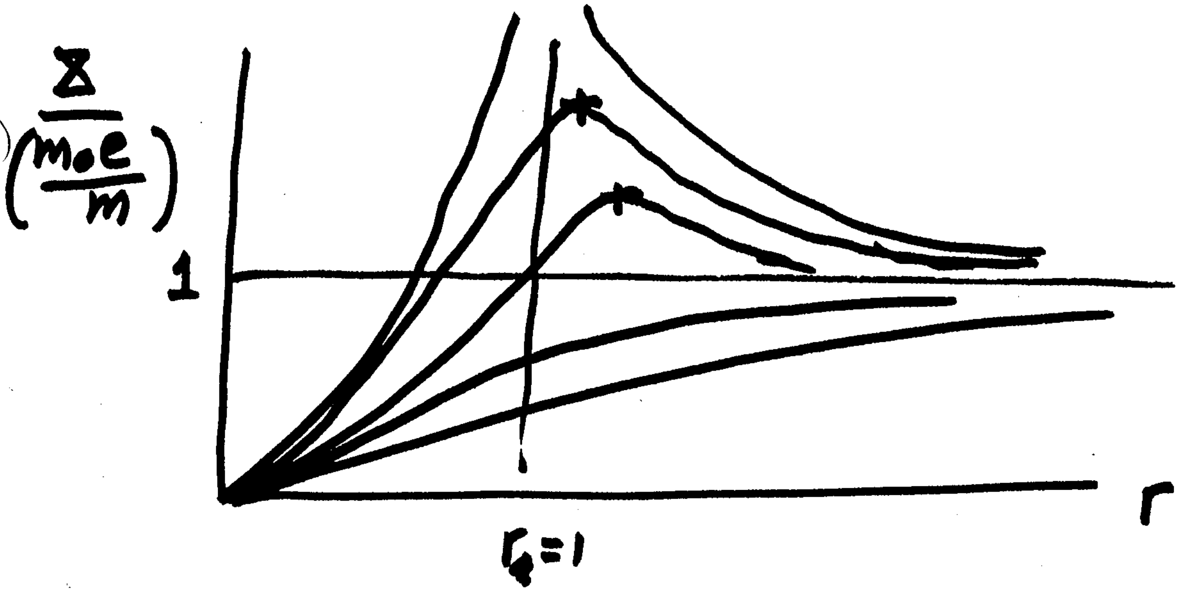
WHEN MAIN MASS PASSES THROUGH $x_p = 0 \Rightarrow \psi = \omega_f t$

- THE SAME GRAPHS THAT DEFINE HOW X VARIES WITH m & k FOR

$F(t) = P_o \sin \omega_f t$

ALSO APPLIES FOR ROTATING UNBALANCE

- VARIATION OF X DUE TO VARIATIONS IN ω_f IS NOT THE SAME



- FOR VERY LARGE ω_f $(r \to \infty)$ ALL CURVES TEND TO $\dfrac{X_{RU}}{(m_o e/m)} = 1$

- $\dfrac{d}{dr}\left(\dfrac{X_{RU}}{m_o e/m}\right) = 0 \Rightarrow$ max. OCCURS

  WHEN $r = \dfrac{1}{\sqrt{1-2\zeta^2}} > 1$

  ONLY TRUE UNTIL $\zeta = \dfrac{1}{\sqrt{2}}$

  r HAS LOCAL MAX WHEN $\zeta \le \dfrac{1}{\sqrt{2}}$

  FOR $\zeta > \dfrac{1}{\sqrt{2}}$ NO <u>LOCAL</u> MAX

$$X_{RU_{max}} = \frac{m_o e}{m} \cdot \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$ THIS IS FOR LOCAL MAXES $\zeta \le \dfrac{1}{\sqrt{2}}$

$$X_{RU}\Big|_{r=1} = \frac{m_o e}{m} \cdot \frac{1}{2\zeta} \; ; \; X_{RU_{max}} = X_{RU}\Big|_{r=1} \cdot \frac{1}{\sqrt{1-\zeta^2}}$$

$$x_p = \frac{m_o e \omega_f^2}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \sin(\omega_f t - \psi)$$

$$= \frac{m_o e \omega_f^2/k \cdot \frac{m}{m}}{\sqrt{\left(1 - \frac{m}{k}\omega_f^2\right)^2 + \left(\frac{c\omega_f}{k}\right)^2}}$$

$$= \underbrace{\left[\frac{m_o e}{m} \cdot \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\right]}_{X_{RU}} \sin(\omega_f t - \psi)$$

$$\frac{X_{RU}}{(m_o e/m)} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\frac{X}{X_o = P_o/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

FLORIDA INTERNATIONAL UNIVERSITY
Mechanical and Materials Engineering Department

Summer 2015 | Advanced Vibration Analysis | EML6223

— Review:
- one degree of freedom systems.
- Free, forced, damped and undamped vibrations.
- Forced support vibration,

— Newton's law in non-inertial coordinate frame.
— Effective stiffness calculation for combined bar-beam-string-plate systems.
— Systems with multiple degrees of freedom. Some models. General analysis. Frequencies and mode shapes for undamped systems. Principal or normal coordinates. Damping in multidegree systems.
— Continuous systems with infinite number of degrees of freedom. Longitudinal vibrations of prismatic bars. Free and forced vibrations. Prismatic bar with a mass or spring at the end. The problem of bar impact.
— Torsional vibrations of shafts.
— Transverse vibrations of beams.
— Transverse vibrations of membranes and plates.
— 3D waves in continua.
— Stability analysis. Introduction to Liapunoffs method.
— Non-linear conservative systems. Free and forced vibrations. Piecewise-linear systems. Numerical solution.
— Non-linear non-conservative systems. Self-excited vibrations. Van der Pol's equation.
— Parametric resonance. Mathieu's equation. The Ince-Strutt diagram.
— Inelastic (especially, viscoelastic) material damping.

Book to be used S.S. Rao, Mechanical Vibrations, 4th Edition, Pearson-Prentice Hall Publishers.
Also notes will be provided from other books as well.

GRADES

Grades will be determined on the basis of

1 Midterm Exam	40 % each
HW	20 %
Final Exam	40 %

Letter Grades will be based as follows:

(A) 95 & above	(B+) 85-89	(C+) 73-76	(D) 60-64
(A-) 90-94	(B) 80-84	(C) 70-72	(F) below 60
	(B-) 77-79	(C-) 65-69	

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Figure 3.3.6 shows a model simulating a long rotor with sensors at the two bearings. The two end disks may be initially unbalanced by adding weights at any location. By adding a trial weight at one of the disks and recording the amplitude and phase and then removing the first trial weight and placing a second trial weight to the other disk and making similar measurements, the initial unbalance of the simulated rotor can be determined.

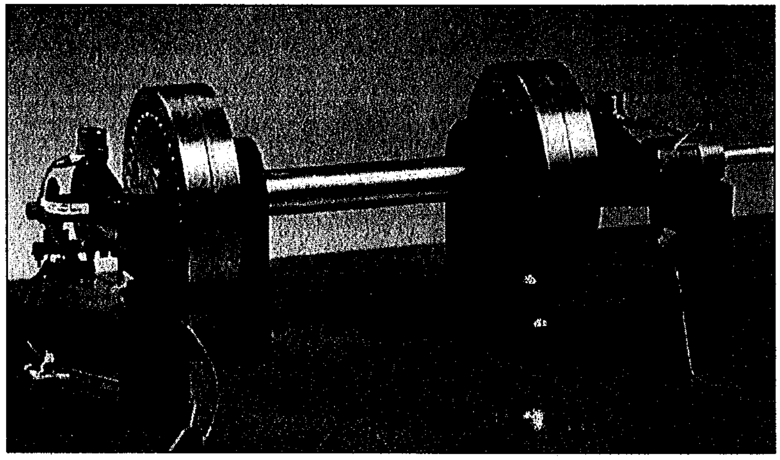


FIGURE 3.3.6. The plane-balancing experiment. (*Courtesy of UCSB Mechanical Engineering Undergraduate Laboratory.*)

3.4 WHIRLING OF ROTATING SHAFTS

Rotating shafts tend to bow out at certain speeds and whirl in a complicated manner. *Whirling* is defined as the rotation of the plane made by the bent shaft and the line of centers of the bearings. The phenomenon results from such various causes as mass unbalance, hysteresis damping in the shaft, gyroscopic forces, fluid friction in bearings, and so on. The whirling of the shaft can take place in the same or opposite direction as that of the rotation of the shaft and the whirling speed may or may not be equal to the rotation speed.

We will consider here a single disk of mass m symmetrically located on a shaft supported by two bearings, as shown in Fig. 3.4.1. The center of mass G of the disk is at a distance e (eccentricity) from the geometric center S of the disk. The center line of the bearings intersects the plane of the disk at O, and the shaft center is deflected by $r = OS$.

We will always assume the shaft (i.e., the line $e = SG$) to be rotating at a constant speed ω, and in the general case, the line $r = OS$ to be whirling at speed $\dot{\theta}$ that is not equal to ω. For the equation of motion, we can develop the acceleration of the mass center as follows:

$$\mathbf{a}_G = \mathbf{a}_S + \mathbf{a}_{G/S} \tag{3.4.1}$$

Thomson & Dahleh

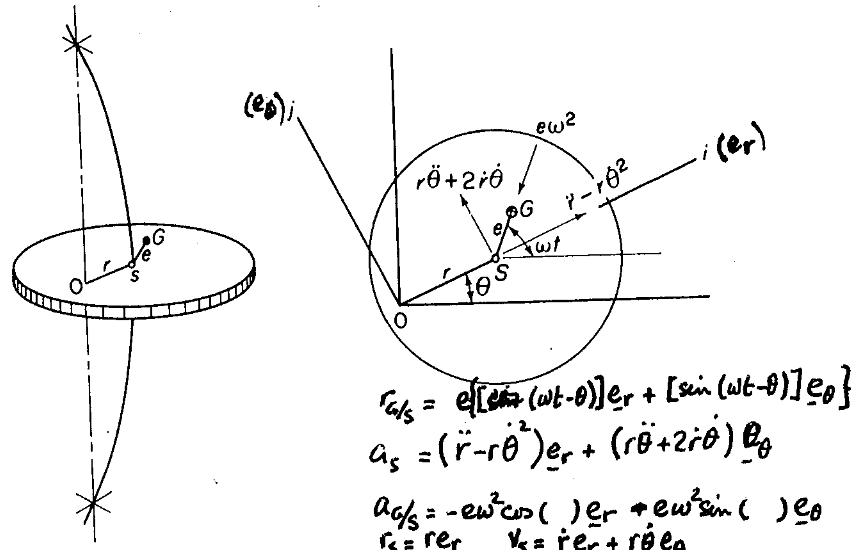


FIGURE 3.4.1. Whirling of shaft.

where $\mathbf{a}_S$ is the acceleration of S and $\mathbf{a}_{G/S}$ is the acceleration of G with respect to S. The latter term is directed from G to S, because ω is constant. Resolving $\mathbf{a}_G$ in the radial and tangential directions, we have

$$\mathbf{a}_G = [(\ddot{r} - r\dot{\theta}^2) - e\omega^2 \cos(\omega t - \theta)]\,\mathbf{i} + [(r\ddot{\theta} + 2\dot{r}\dot{\theta}) - e\omega^2 \sin(\omega t - \theta)]\,\mathbf{j} \qquad (3.4.2)$$

Aside from the restoring force of the shaft, we will assume a viscous damping force to be acting at S. The equations of motion resolved in the radial and tangential directions then become

$$-kr - c\dot{r} = m[\ddot{r} - r\dot{\theta}^2 - e\omega^2 \cos(\omega t - \theta)]$$

$$-cr\dot{\theta} = m[r\ddot{\theta} + 2\dot{r}\dot{\theta} - e\omega^2 \sin(\omega t - \theta)]$$

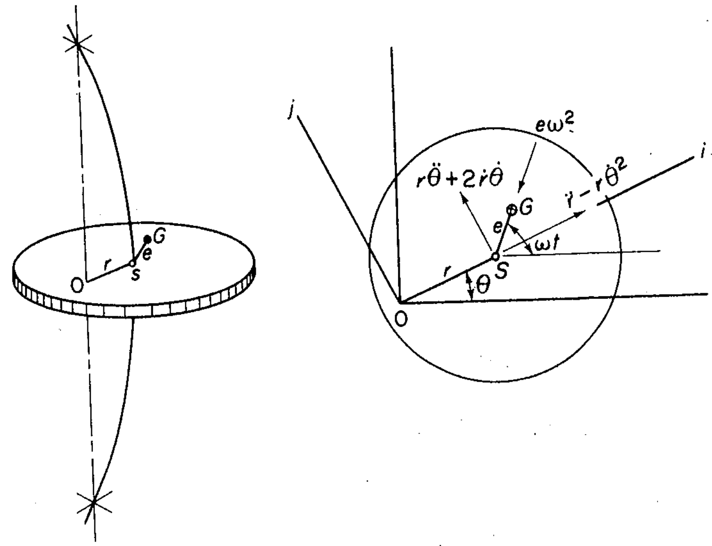
which can be rearranged to

$$\ddot{r} + \frac{c}{m}\dot{r} + \left(\frac{k}{m} - \dot{\theta}^2\right)r = e\omega^2 \cos(\omega t - \theta) \qquad (3.4.3)$$

$$r\ddot{\theta} + \left(\frac{c}{m}r + 2\dot{r}\right)\dot{\theta} = e\omega^2 \sin(\omega t - \theta) \qquad (3.4.4)$$

The general case of whirl as described by the foregoing equations comes under the classification of self-excited motion, where the exciting forces inducing the motion are controlled by the motion itself. Because the variables in these equations are r and θ, the problem is that of 2 DOF. However, in the steady-state synchronous whirl, where $\dot{\theta} = \omega$ and $\ddot{\theta} = \ddot{r} = \dot{r} = 0$, the problem reduces to that of 1 DOF.

Synchronous whirl. For the synchronous whirl, the whirling speed $\dot{\theta}$ is equal to the rotation speed ω, which we have assumed to be constant. Thus, we have



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which can be rearranged to

$$\ddot{r} + \frac{c}{m}\dot{r} + \left(\frac{k}{m} - \dot{\theta}^2\right)r = e\omega^2 \cos(\omega t - \theta) \quad (3.4.3)$$

$$r\ddot{\theta} + \left(\frac{c}{m}r + 2\dot{r}\right)\dot{\theta} = e\omega^2 \sin(\omega t - \theta) \quad (3.4.4)$$

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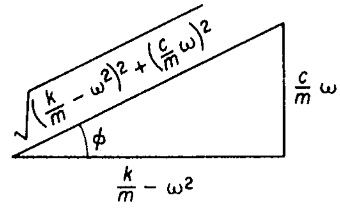

FIGURE 3.4.2.

$$\dot{\theta} = \omega$$

and on integrating we obtain

$$\theta = \omega t - \phi$$

where ϕ is the phase angle between e and r, which is now a constant, as shown in Fig. 3.4.1. With $\ddot{\theta} = \ddot{r} = \dot{r} = 0$, Eqs. (3.4.3) and (3.4.4) reduce to

$$\left(\frac{k}{m} - \omega^2\right)R = e\omega^2 \cos\phi$$

$$\frac{c}{m}\omega R = e\omega^2 \sin\phi \tag{3.4.5}$$

Dividing, we obtain the following equation for the phase angle:

$$\tan\phi = \frac{\frac{c}{m}\omega}{\frac{k}{m} - \omega^2} = \frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{3.4.6}$$

where $\omega_n = \sqrt{k/m}$ is the critical speed, and $\zeta = c/c_c$. Noting from the vector triangle of Fig. 3.4.2 that

$$\cos\phi = \frac{\frac{k}{m} - \omega^2}{\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{c}{m}\omega\right)^2}}$$

and substituting into the first of Eq. (3.4.5) gives the amplitude equation

$$R = \frac{me\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{e\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}} \tag{3.4.7}$$

These equations indicate that the eccentricity line $e = SG$ leads the displacement line $R = OS$ by the phase angle ϕ, which depends on the amount of damping and

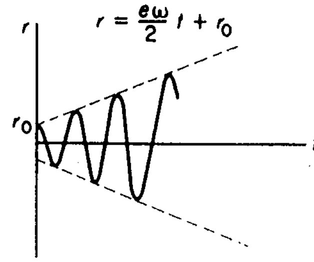

FIGURE 3.4.4. Amplitude and phase relationship of synchronous whirl with viscous damping.

Because the right side of this equation is constant, it is satisfied only if the coefficient of t is zero:

$$\left(\frac{k}{m} - \omega^2\right)\sin\phi = 0 \tag{d}$$

which leaves the remaining terms:

$$\left(\frac{k}{m} - \omega^2\right)r_0 = e\omega^2\cos\phi \tag{e}$$

With $\omega = \sqrt{k/m}$, the first equation is satisfied, but the second equation is satisfied only if $\cos\phi = 0$ or $\phi = \pi/2$. Thus, we have shown that at $\omega = \sqrt{k/m}$, or at resonance, the phase angle is $\pi/2$ as before for the damped case, and the amplitude builds up linearly according to the equation shown in Fig. 3.4.4.

■

3.5 SUPPORT MOTION

In many cases, the dynamical system is excited by the motion of the support point, as shown in Fig. 3.5.1. We let y be the harmonic displacement of the support point and measure the displacement x of the mass m from an inertial reference.

In the displaced position, the unbalanced forces are due to the damper and the springs, and the differential equation of motion becomes

$$m\ddot{x} = -k(x - y) - c(\dot{x} - \dot{y}) \tag{3.5.1}$$

By making the substitution

$$z = x - y \tag{3.5.2}$$

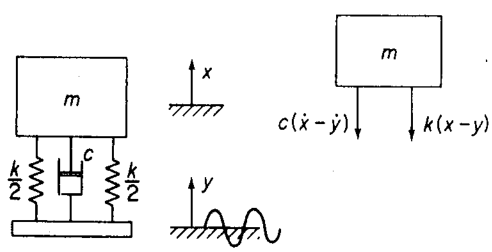

FIGURE 3.5.1. System excited by motion of support point.

vector ab is rotated 107° ccw, it will be opposite the vector oa. To cancel oa it must be shortened by $oa/ab = 3.2/5.4 = 0.593$. Thus, the trial weight $W_t = 2.5$ oz must be rotated 107° ccw and reduced in size to $2.5 \times 0.593 = 1.48$ oz. Of course, the graphical solution for ab and ϕ can be found mathematically by the law of cosines.

Figure 3.3.6 shows a model simulating a long rotor with sensors at the two bearings. The two end disks may be initially unbalanced by adding weights at any location. By adding a trial weight at one of the disks and recording the amplitude and phase and then removing the first trial weight and placing a second trial weight to the other disk and making similar measurements, the initial unbalance of the simulated rotor can be determined.

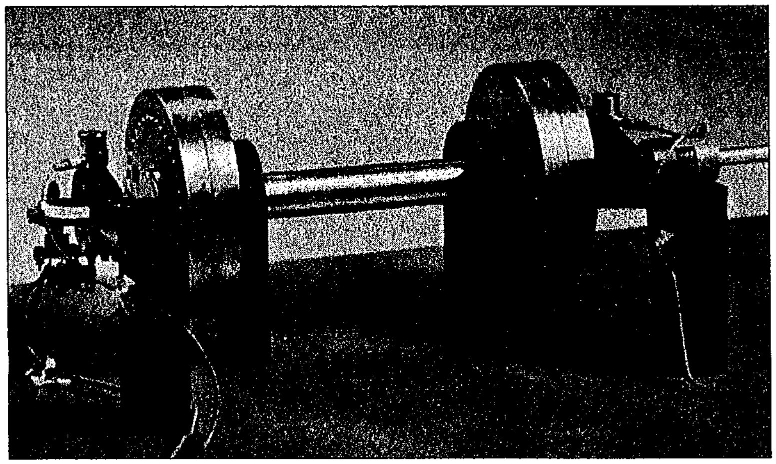

FIGURE 3.3.6. The plane-balancing experiment. (*Courtesy of UCSB Mechanical Engineering Undergraduate Laboratory.*)

3.4 WHIRLING OF ROTATING SHAFTS

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We will consider here a single disk of mass m symmetrically located on a shaft supported by two bearings, as shown in Fig. 3.4.1. The center of mass G of the disk is at a distance e (eccentricity) from the geometric center S of the disk. The center line of the bearings intersects the plane of the disk at O, and the shaft center is deflected by $r = OS$.

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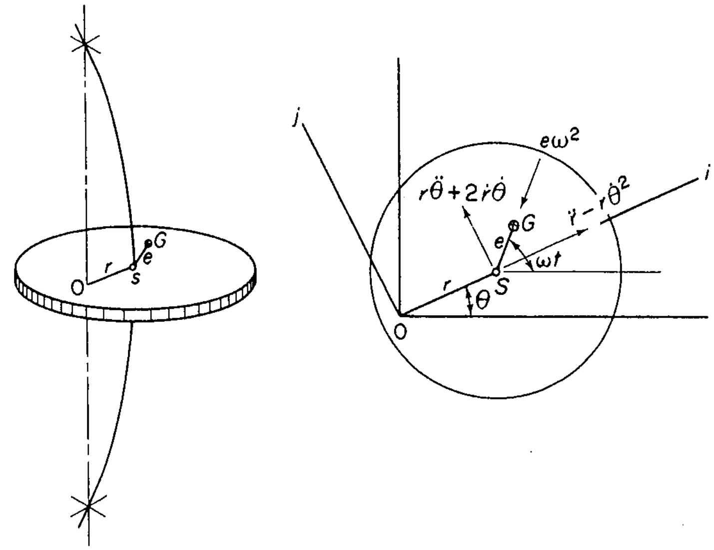


FIGURE 3.4.1. Whirling of shaft.

where $\mathbf{a}_S$ is the acceleration of S and $\mathbf{a}_{G/S}$ is the acceleration of G with respect to S. The latter term is directed from G to S, because ω is constant. Resolving $\mathbf{a}_G$ in the radial and tangential directions, we have

$$\mathbf{a}_G = [(\ddot{r} - r\dot{\theta}^2) - e\omega^2\cos(\omega t - \theta)]\,\mathbf{i} + [(r\ddot{\theta} + 2\dot{r}\dot{\theta}) - e\omega^2\sin(\omega t - \theta)]\,\mathbf{j} \tag{3.4.2}$$

Aside from the restoring force of the shaft, we will assume a viscous damping force to be acting at S. The equations of motion resolved in the radial and tangential directions then become

$$-kr - c\dot{r} = m[\ddot{r} - r\dot{\theta}^2 - e\omega^2\cos(\omega t - \theta)]$$
$$-cr\dot{\theta} = m[r\ddot{\theta} + 2\dot{r}\dot{\theta} - e\omega^2\sin(\omega t - \theta)]$$

which can be rearranged to

$$\ddot{r} + \frac{c}{m}\dot{r} + \left(\frac{k}{m} - \dot{\theta}^2\right)r = e\omega^2\cos(\omega t - \theta) \tag{3.4.3}$$

$$r\ddot{\theta} + \left(\frac{c}{m}r + 2\dot{r}\right)\dot{\theta} = e\omega^2\sin(\omega t - \theta) \tag{3.4.4}$$

The general case of whirl as described by the foregoing equations comes under the classification of self-excited motion, where the exciting forces inducing the motion are controlled by the motion itself. Because the variables in these equations are r and θ, the problem is that of 2 DOF. However, in the steady-state synchronous whirl, where $\dot{\theta} = \omega$ and $\ddot{\theta} = \ddot{r} = \dot{r} = 0$, the problem reduces to that of 1 DOF.

Synchronous whirl. For the synchronous whirl, the whirling speed $\dot{\theta}$ is equal to the rotation speed ω, which we have assumed to be constant. Thus, we have

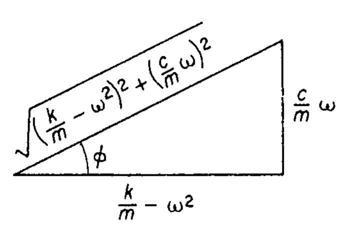


FIGURE 3.4.2.

$$\dot{\theta} = \omega$$

and on integrating we obtain

$$\theta = \omega t - \phi$$

where ϕ is the phase angle between e and r, which is now a constant, as shown in Fig. 3.4.1. With $\ddot{\theta} = \ddot{r} = \dot{r} = 0$, Eqs. (3.4.3) and (3.4.4) reduce to

$$\left(\frac{k}{m} - \omega^2\right) r = e\omega^2 \cos\phi$$

$$\frac{c}{m}\omega r = e\omega^2 \sin\phi \tag{3.4.5}$$

Dividing, we obtain the following equation for the phase angle:

$$\tan\phi = \frac{\dfrac{c}{m}\omega}{\dfrac{k}{m} - \omega^2} = \frac{2\zeta\dfrac{\omega}{\omega_n}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} \tag{3.4.6}$$

where $\omega_n = \sqrt{k/m}$ is the critical speed, and $\zeta = c/c_c$. Noting from the vector triangle of Fig. 3.4.2 that

$$\cos\phi = \frac{\dfrac{k}{m} - \omega^2}{\sqrt{\left(\dfrac{k}{m} - \omega^2\right)^2 + \left(\dfrac{c}{m}\omega\right)^2}}$$

and substituting into the first of Eq. (3.4.5) gives the amplitude equation

$$r = \frac{me\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{e\left(\dfrac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\dfrac{\omega}{\omega_n}\right)\right]^2}} \tag{3.4.7}$$

These equations indicate that the eccentricity line $e = SG$ leads the displacement line $r = OS$ by the phase angle ϕ, which depends on the amount of damping and

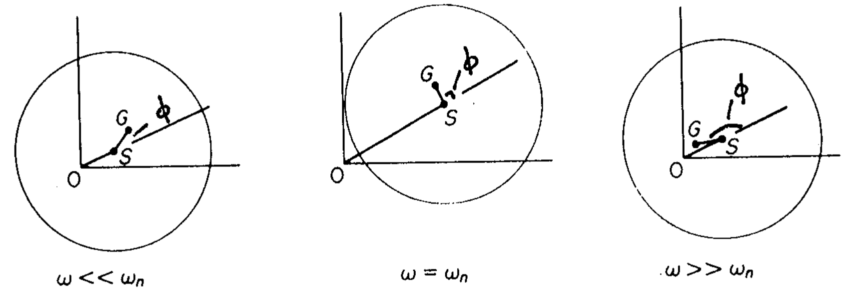


FIGURE 3.4.3. Phase of different rotation speeds.

the rotation speed ratio ω/ω_n. When the rotation speed coincides with the critical speed $\omega_n = \sqrt{k/m}$, or the natural frequency of the shaft in lateral vibration, a condition of resonance is encountered in which the amplitude is restrained only by the damping. Figure 3.4.3 shows the disk-shaft system under three different speed conditions. At very high speeds, $\omega \gg \omega_n$, the center of mass G tends to approach the fixed point O, and the shaft center S rotates about it in a circle of radius e.

$R = \dfrac{me\omega^2/K}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

It should be noted that the equations for synchronous whirl appear to be the same as those of Sec. 3.2. This is not surprising, because in both cases the exciting force is rotating and equal to $me\omega^2$. However, in Sec. 3.2 the unbalance was in terms of the small unbalanced mass m, whereas in this section, the unbalance is defined in terms of the total mass m with eccentricity e. Thus, Fig. 3.2.2 is applicable to this problem with the ordinate equal to R/e instead of MX/me.

EXAMPLE 3.4.1

Turbines operating above the critical speed must run through dangerous speed at resonance each time they are started or stopped. Assuming the critical speed ω_n to be reached with amplitude r_0, determine the equation for the amplitude buildup with time. Assume zero damping.

Solution We will assume synchronous whirl as before, which makes $\dot{\theta} = \omega =$ constant and $\ddot{\theta} = 0$. However, $\ddot{r}$ and $\dot{r}$ terms must be retained unless shown to be zero. With $c = 0$ for the undamped case, the general equations of motion reduce to

$$\ddot{r} + \left(\frac{k}{m} - \omega^2\right)r = e\omega^2 \cos\phi$$

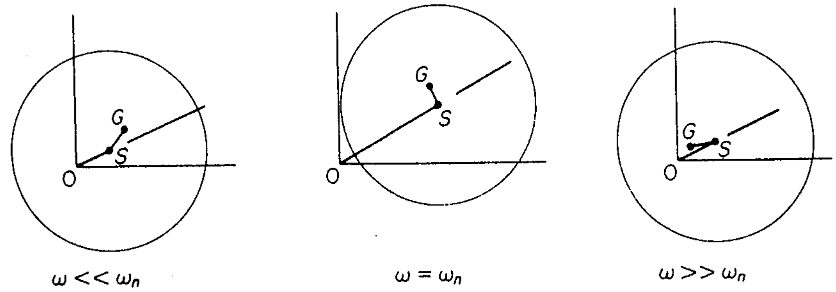
$$2\dot{r}\omega = e\omega^2 \sin\phi \qquad \text{(a)}$$

The solution of the second equation with initial deflection equal to r_0 is

$$r = \frac{e\omega}{2} t \sin\phi + r_0 \qquad \text{(b)}$$

Differentiating this equation twice, we find that $\ddot{r} = 0$; so the first equation with the above solution for r becomes

$$\left(\frac{k}{m} - \omega^2\right)\left(\frac{e\omega}{2} t \sin\phi + r_0\right) = e\omega^2 \cos\phi \qquad \text{(c)}$$



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The solution of the second equation with initial deflection equal to r_0 is

$$r = \frac{e\omega}{2}t\sin\phi + r_0 \tag{b}$$

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$$\left(\frac{k}{m} - \omega^2\right)\left(\frac{e\omega}{2}t\sin\phi + r_0\right) = e\omega^2\cos\phi \tag{c}$$

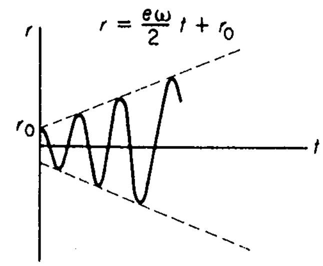

FIGURE 3.4.4. Amplitude and phase relationship of synchronous whirl with viscous damping.

Because the right side of this equation is constant, it is satisfied only if the coefficient of t is zero:

$$\left(\frac{k}{m} - \omega^2\right)\sin\phi = 0 \tag{d}$$

which leaves the remaining terms:

$$\left(\frac{k}{m} - \omega^2\right)r_0 = e\omega^2\cos\phi \tag{e}$$

With $\omega = \sqrt{k/m}$, the first equation is satisfied, but the second equation is satisfied only if $\cos\phi = 0$ or $\phi = \pi/2$. Thus, we have shown that at $\omega = \sqrt{k/m}$, or at resonance, the phase angle is $\pi/2$ as before for the damped case, and the amplitude builds up linearly according to the equation shown in Fig. 3.4.4.

■

3.5 SUPPORT MOTION

In many cases, the dynamical system is excited by the motion of the support point, as shown in Fig. 3.5.1. We let y be the harmonic displacement of the support point and measure the displacement x of the mass m from an inertial reference.

In the displaced position, the unbalanced forces are due to the damper and the springs, and the differential equation of motion becomes

$$m\ddot{x} = -k(x - y) - c(\dot{x} - \dot{y}) \tag{3.5.1}$$

By making the substitution

$$z = x - y \tag{3.5.2}$$

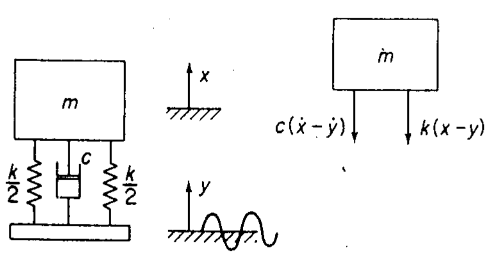

FIGURE 3.5.1. System excited by motion of support point.

3.16. A solid disk weighing 10 lb is keyed to the center of a $\frac{1}{2}$-in. steel shaft 2 ft between bearings. Determine the lowest critical speed. (Assume the shaft to be simply supported at the bearings.)

3.17. Convert all units in Prob. 3.16 to the SI system and recalculate the lowest critical speed.

3.18. The rotor of a turbine 13.6 kg in mass is supported at the midspan of a shaft with bearings 0.4064 m apart, as shown in Fig. P3.18. The rotor is known to have an unbalance of 0.2879 kg · cm. Determine the forces exerted on the bearings at a speed of 6000 rpm if the diameter of the steel shaft is 2.54 cm. Compare this result with that of the same rotor mounted on a steel shaft of diameter 1.905 cm. (Assume the shaft to be simply supported at the bearings.)

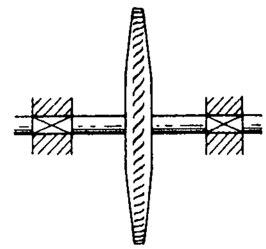

FIGURE P3.18.

3.19. For turbines operating above the critical speed, stops are provided to limit the amplitude as they run through the critical speed. In the turbine of Prob. 3.18, if the clearance between the 2.54-cm shaft and the stops is 0.0508 cm, and if the eccentricity is 0.0212 cm, determine the time required for the shaft to hit the stops. Assume that the critical speed is reached with zero amplitude.

3.20. Figure P3.20 represents a simplified diagram of a spring-supported vehicle traveling over a rough road. Determine the equation for the amplitude of W as a function of the speed, and determine the most unfavorable speed.

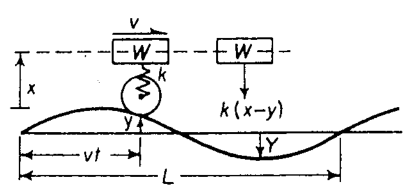

FIGURE P3.20.

3.21. The springs of an automobile trailer are compressed 10.16 cm under its weight. Find the critical speed when the trailer is traveling over a road with a profile approximated by a sine wave of amplitude 7.62 cm and wavelength of 14.63 m. What will be the amplitude of vibration at 64.4 km/h? (Neglect damping.)

3.22. The point of suspension of a simple pendulum is given by a harmonic motion $x_0 = X_0 \sin \omega t$ along a horizontal line, as shown in Fig. P3.22. Write the differential equation of motion for a small amplitude of oscillation using the coordinates shown. Determine the solution for x/x_0, and show that when $\omega = \sqrt{2}\omega_n$, the node is found at the midpoint of l. Show that in general the distance h from the mass to the node is given by the relation $h = l(\omega_n/\omega)^2$, where $\omega_n = \sqrt{g/l}$.

3-17

$W = 10 \text{ lb} = 44.48 \text{ N}$ $\qquad m = \frac{44.48}{9.81} = 4.534 \text{ kg}$

$g = 386 \frac{in}{sec^2} = 9.81 \ m/s^2$

$$k = \frac{48EI}{\ell^3} = \begin{cases} E = 200\times 10^9 \ N/m^2 \\ \ell = 2\times .3048 = .6096 \ m \\ d = .5\times 2.54\times 10^{-2} = 1.270\times 10^{-2} \ m \\ I = \frac{\pi d^4}{64} = .1277\times 10^{-8} \end{cases}$$

$$k = \frac{200\times 10^9 \times 48 \times .1277\times 10^{-8}}{(.6096)^3} = 54116 \ N/m$$

$$.486 \ m_{shaft} = .486 \times \frac{\pi}{4}(1.270\times 10^{-2})^2(.6096)\times \rho = .2938$$

$\rho = 7830 \frac{kg}{m^3}$ = density of steel

$$f = \frac{1}{2\pi}\sqrt{\frac{54116}{4.534+.2938}} = 16.86 \text{ Hz}$$

3-18

dia $= 2.54$ cm $\qquad I = \frac{\pi d^4}{64} = \frac{\pi\times 41.62}{64\times 100^4} = 2.043\times 10^{-8} \ m^4$

$$k = \frac{48EI}{\ell^3} = \frac{48(200\times 10^9)\,2.043\times 10^{-8}}{.4064^2} = 2.922\times 10^6 \ N/m$$

$m_{shaft} = \rho V = \rho \cdot A \cdot \ell$

$$m = 13.6 + .486\left(\underbrace{\frac{\pi}{4}\times .0254^2}_{A} \times \underset{\ell}{.4064}\right)(\underset{\rho}{7830}) = 13.6 + .784 = 14.38 \text{ kg}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{2.922\times 10^6}{14.38}} = 71.74 \text{ Hz} = 4304 \text{ rpm}$$

$$\frac{\omega}{\omega_n} = \frac{6000}{4304} = 1.394 \qquad r = \frac{e\left(\frac{\omega}{\omega_n}\right)^2}{1-\left(\frac{\omega}{\omega_n}\right)^2} = -2.060\,e$$

$me = .2879$ kg cm (given) $\quad \therefore \quad e = \frac{.2879}{13.6} = .02117$ cm

$r = -2.060(.02117) = -.04316$ cm

$$F = m(r+e)\omega^2 = 14.38\left(\frac{-.04316+.02117}{100}\right)(2\pi\times 100)^2 = 1273 \text{ N}$$

(/100: in m; 100: cps)

3-18 Cont.

for diam = 1.905 cm

$$.486\, m_{shaft} = .486\left(\frac{\pi}{4}\times .01905^2 \times .4064\right)(7830) = .4408$$

$$m = m_{disk} + .483\, m_{shaft} = 14.04 \text{ kg.}$$

$$I = \frac{\pi}{64}\ 1.905^4 \times 100^{-4} = .6464 \times 10^{-8}$$

$$k = \frac{48\,(200\times 10^9)\,.6464\times 10^{-8}}{.4064^3} = 0.9441 \times 10^6$$

$$f_n = 41.27 \text{ Hz} = 2476 \text{ rpm} \qquad \frac{\omega}{\omega_n} = \frac{6000}{2476} = 2.423$$

$$r = \frac{.02117\,(2.423)^2}{1-(2.423)^2} = -.02552 \qquad r+e = .00435$$

$$F = 14.042\left(\frac{.00435}{100}\right)\left(2\pi\times\frac{6000}{60}\right)^2 = 241.1 \text{ N}$$

3-19

$$r = r_0 + \frac{e\omega t}{2} \qquad (\text{see Ex. } 3.4-1)$$

$$.0508 = 0 + .0212\,(2\pi\times 100)\frac{t}{2}$$

$$t = \frac{.0508}{6.6602} = .0075 \text{ sec}$$

3-20

$$m\ddot{x} = -k(x-y)$$

where x = displ. of m measured from static equilib. position of m with $y = 0$

Let $y = Y \sin \frac{2\pi Vt}{L}$

then $m\ddot{x} + kx = kY \sin\frac{2\pi Vt}{L} = kY\sin\omega t$

where $\omega = \frac{2\pi V}{L}$ Sol is $x = X \sin\omega t$

$$X = \frac{Y}{1-(\omega/\omega_n)^2} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

The most unfavorable speed corresponds to $\frac{\omega}{\omega_n} = 1$

$$\therefore\ V = \frac{L}{2\pi}\sqrt{\frac{k}{m}}$$

vector ab is rotated 107° ccw, it will be opposite the vector oa. To cancel oa it must be shortened by $oa/ab = 3.2/5.4 = 0.593$. Thus, the trial weight $W_t = 2.5$ oz must be rotated 107° ccw and reduced in size to $2.5 \times 0.593 = 1.48$ oz. Of course, the graphical solution for ab and ϕ can be found mathematically by the law of cosines.

Figure 3.3.6 shows a model simulating a long rotor with sensors at the two bearings. The two end disks may be initially unbalanced by adding weights at any location. By adding a trial weight at one of the disks and recording the amplitude and phase and then removing the first trial weight and placing a second trial weight to the other disk and making similar measurements, the initial unbalance of the simulated rotor can be determined.

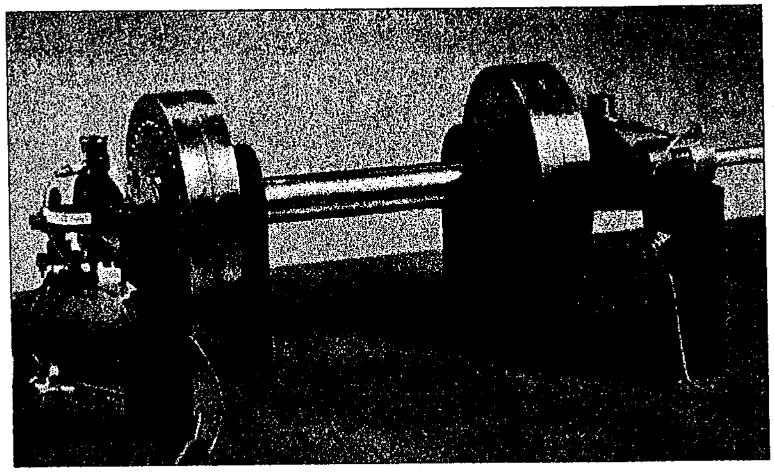

FIGURE 3.3.6. The plane-balancing experiment. (*Courtesy of UCSB Mechanical Engineering Undergraduate Laboratory.*)

3.4 WHIRLING OF ROTATING SHAFTS

Rotating shafts tend to bow out at certain speeds and whirl in a complicated manner. *Whirling* is defined as the rotation of the plane made by the bent shaft and the line of centers of the bearings. The phenomenon results from such various causes as mass unbalance, hysteresis damping in the shaft, gyroscopic forces, fluid friction in bearings, and so on. The whirling of the shaft can take place in the same or opposite direction as that of the rotation of the shaft and the whirling speed may or may not be equal to the rotation speed.

We will consider here a single disk of mass m symmetrically located on a shaft supported by two bearings, as shown in Fig. 3.4.1. The center of mass G of the disk is at a distance e (eccentricity) from the geometric center S of the disk. The center line of the bearings intersects the plane of the disk at O, and the shaft center is deflected by $r = OS$.

We will always assume the shaft (i.e., the line $e = SG$) to be rotating at a constant speed ω, and in the general case, the line $r = OS$ to be whirling at speed $\dot{\theta}$ that is not equal to ω. For the equation of motion, we can develop the acceleration of the mass center as follows:

$$\mathbf{a}_G = \mathbf{a}_S + \mathbf{a}_{G/S} \tag{3.4.1}$$

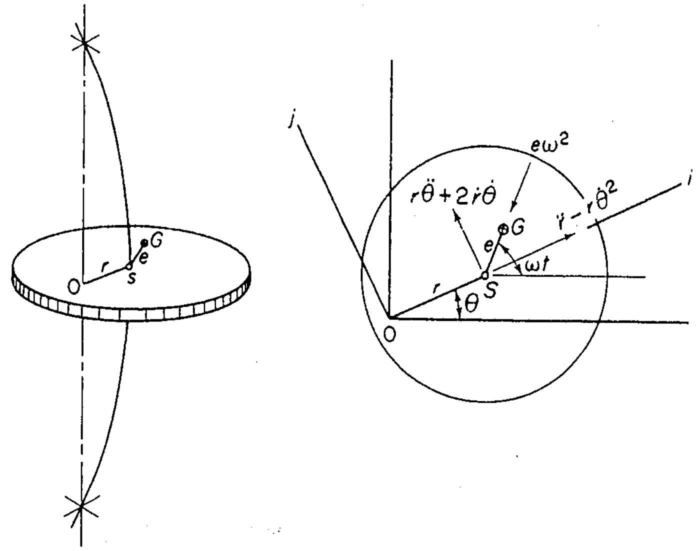

FIGURE 3.4.1. Whirling of shaft.

where $\mathbf{a}_S$ is the acceleration of S and $\mathbf{a}_{G/S}$ is the acceleration of G with respect to S. The latter term is directed from G to S, because ω is constant. Resolving $\mathbf{a}_G$ in the radial and tangential directions, we have

$$\mathbf{a}_G = [(\ddot{r} - r\dot{\theta}^2) - e\omega^2 \cos(\omega t - \theta)]\mathbf{i} + [(r\ddot{\theta} + 2\dot{r}\dot{\theta}) - e\omega^2 \sin(\omega t - \theta)]\mathbf{j} \qquad (3.4.2)$$

Aside from the restoring force of the shaft, we will assume a viscous damping force to be acting at S. The equations of motion resolved in the radial and tangential directions then become

$$-kr - c\dot{r} = m[\ddot{r} - r\dot{\theta}^2 - e\omega^2 \cos(\omega t - \theta)]$$
$$-cr\dot{\theta} = m[r\ddot{\theta} + 2\dot{r}\dot{\theta} - e\omega^2 \sin(\omega t - \theta)]$$

which can be rearranged to

$$\ddot{r} + \frac{c}{m}\dot{r} + \left(\frac{k}{m} - \dot{\theta}^2\right)r = e\omega^2 \cos(\omega t - \theta) \qquad (3.4.3)$$

$$r\ddot{\theta} + \left(\frac{c}{m}r + 2\dot{r}\right)\dot{\theta} = e\omega^2 \sin(\omega t - \theta) \qquad (3.4.4)$$

The general case of whirl as described by the foregoing equations comes under the classification of self-excited motion, where the exciting forces inducing the motion are controlled by the motion itself. Because the variables in these equations are r and θ, the problem is that of 2 DOF. However, in the steady-state synchronous whirl, where $\dot{\theta} = \omega$ and $\ddot{\theta} = \ddot{r} = \dot{r} = 0$, the problem reduces to that of 1 DOF.

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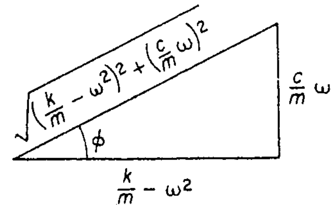

FIGURE 3.4.2.

$$\dot{\theta} = \omega$$

and on integrating we obtain

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where ϕ is the phase angle between e and r, which is now a constant, as shown in Fig. 3.4.1. With $\ddot{\theta} = \ddot{r} = \dot{r} = 0$, Eqs. (3.4.3) and (3.4.4) reduce to

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where $\omega_n = \sqrt{k/m}$ is the critical speed, and $\zeta = c/c_c$. Noting from the vector triangle of Fig. 3.4.2 that

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and substituting into the first of Eq. (3.4.5) gives the amplitude equation

$$r = \frac{me\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{e\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}} \tag{3.4.7}$$

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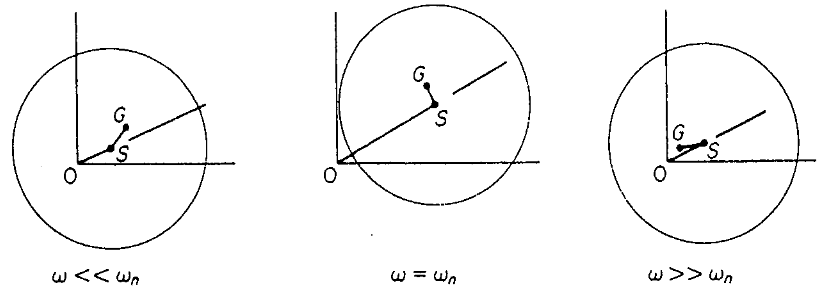

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The solution of the second equation with initial deflection equal to r_0 is

$$r = \frac{e\omega}{2} t \sin \phi + r_0 \qquad \text{(b)}$$

Differentiating this equation twice, we find that $\ddot{r} = 0$; so the first equation with the above solution for r becomes

$$\left(\frac{k}{m} - \omega^2\right)\left(\frac{e\omega}{2} t \sin \phi + r_0\right) = e\omega^2 \cos \phi \qquad \text{(c)}$$

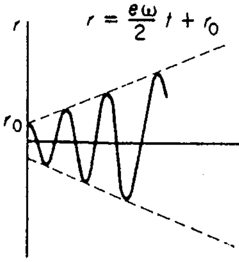


FIGURE 3.4.4. Amplitude and phase relationship of synchronous whirl with viscous damping.

Because the right side of this equation is constant, it is satisfied only if the coefficient of t is zero:

$$\left(\frac{k}{m} - \omega^2\right)\sin\phi = 0 \tag{d}$$

which leaves the remaining terms:

$$\left(\frac{k}{m} - \omega^2\right) r_0 = e\omega^2 \cos\phi \tag{e}$$

With $\omega = \sqrt{k/m}$, the first equation is satisfied, but the second equation is satisfied only if $\cos\phi = 0$ or $\phi = \pi/2$. Thus, we have shown that at $\omega = \sqrt{k/m}$, or at resonance, the phase angle is $\pi/2$ as before for the damped case, and the amplitude builds up linearly according to the equation shown in Fig. 3.4.4.

■

3.5 SUPPORT MOTION

In many cases, the dynamical system is excited by the motion of the support point, as shown in Fig. 3.5.1. We let y be the harmonic displacement of the support point and measure the displacement x of the mass m from an inertial reference.

In the displaced position, the unbalanced forces are due to the damper and the springs, and the differential equation of motion becomes

$$m\ddot{x} = -k(x - y) - c(\dot{x} - \dot{y}) \tag{3.5.1}$$

By making the substitution

$$z = x - y \tag{3.5.2}$$

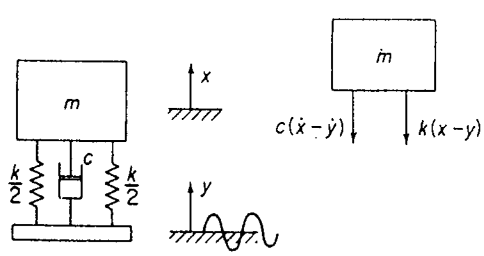


FIGURE 3.5.1. System excited by motion of support point.

LOOK AT P. 125 $\frac{d}{dr}\left(F_T / \frac{m_0 e k}{m}\right) = 0$

- $r = 0$ is a relative minimum
- $0 < \zeta < \sqrt{2}/4$ FOR $0 < r < \sqrt{2}$ RELATIVE MAX
  $r > \sqrt{2}$ " MIN
- $\zeta > \sqrt{2}/4$ NO RELATIVE MAX ONLY AN ABSOLUTE MAX AT $r = \infty$

- FIGURE ON P. 124 IS PLOT OF $\left(F_T / \frac{m_0 e k}{m}\right)$ VS. r

  NOTE IN PRACTICE $r \geq 10$ $\zeta \neq 0$ WE HAVE LARGE TRANSMITTED FORCE

- IF WE VARY k or m. LOOK AT $\frac{F_T}{m_0 e \omega_f^2} = \frac{\sqrt{k^2 + (c\omega_f)^2}}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} = \frac{F_T}{P_0}$

  THIS IS SAME AS WHAT WE DID EARLIER FOR TRANSMITTED FORCE P122 / 123

SESSION #17

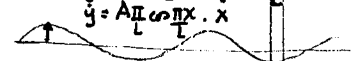
OSCILLATING SUPPORT

- COMMON SOURCE OF VIBS

EXAMPLES VEHICLES NO STREET IS FLAT

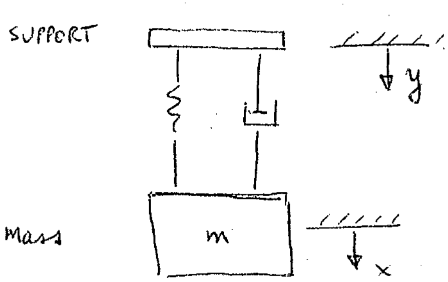
$y = A\sin\frac{\pi x}{L} = A\sin\frac{\pi v}{L}t$

$\dot{y} = A\frac{\pi}{L}\cos\frac{\pi x}{L}\cdot\dot{x}$

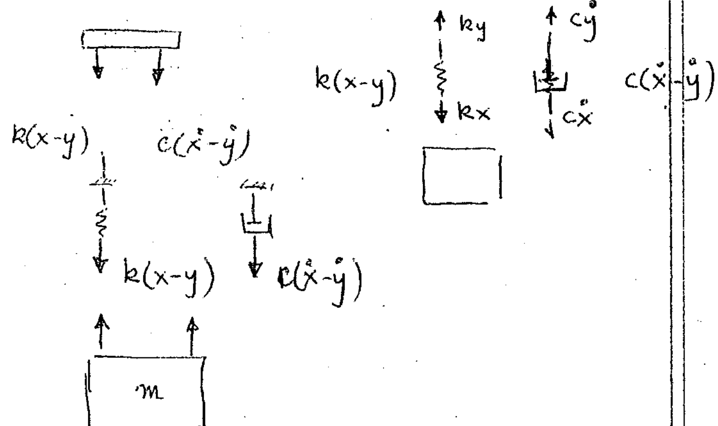


SHIPS OCEAN DUE TO TROUGHS & CRESTS

AIRCRAFT



ABSOLUTE DISP. OF SUPPORT $y = Y\sin\omega_f t$



x-y is RELATIVE MOTION

$m\ddot{x} = \sum forces = -k(x-y) - c(\dot{x} - \dot{y})$ or $m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$

$= kY\sin\omega_f t + c\omega_f Y\cos\omega_f t$

$= Y\left[k\sin\omega_f t + c\omega_f\cos\omega_f t\right]$

let $k = C\cos\beta$ let $-c\omega_f = C\sin\beta$ thus $\tan\beta = -\frac{c\omega_f}{k} = -2\zeta r$

$C = \sqrt{k^2+(c\omega_f)^2} = k\sqrt{1+(2\zeta r)^2}$

and $m\ddot{x} + c\dot{x} + kx = Y\sqrt{k^2+(c\omega_f)^2}\,\sin(\omega_f t - \beta)$

as before if we let $P_o = Y\sqrt{k^2+(c\omega_f)^2}$ then

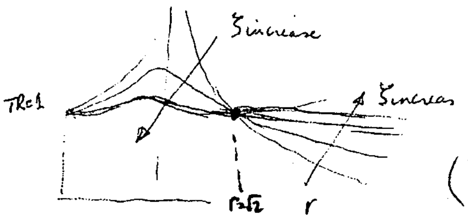
$$x_p = X\sin(\omega_f t - \gamma) = \frac{P_o}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}\sin(\omega_f t-\gamma)$$

LIKE THE FORCE TRANSMISSION

$$X = \frac{Y\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}$$

$\gamma = \beta + \psi \quad \tan\psi = \frac{2\zeta r}{1-r^2}$

$\tan\beta = -2\zeta r$



$$= \frac{Y\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

for $r=\sqrt{2}$, $r=0$: $\frac{X}{Y}=1 \;\forall \zeta$

as $\zeta\uparrow$ $X\uparrow$

$(X/Y) = F_T/P_o = TR$ transmissibility $\frac{\text{motion of mass}}{\text{motion of base}}$

$(X/Y)_{max}$ occurs @ $r = \frac{\sqrt{(1+8\zeta^2)^{1/2}-1}}{2\zeta}$

Force on support is

$$F = k(x-y) + c(\dot{x}-\dot{y}) = -m\ddot{x}$$

$$= \frac{m\,\omega_f^2 Y\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}\sin(\omega_f t-\gamma)$$

$$= m\omega_f^2 X\sin(\omega_f t-\gamma) = F_{TRANS}\sin(\omega_f t-\gamma)$$

$$= Yk\,\frac{r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega_f t-\gamma)$$

NOTE THAT

F is in phase w/ mass

PLOTTING $\frac{F_{MAX}}{Yk}$ vs. r same graph as $F_T\Big/\frac{m_o e k}{m}$ vs. r

AS $r\to\infty$ $X\to 0$ but $F\to\infty$ since velocity $\to\infty$ damper transmits F

IF we let $z = x-y$ RELATIVE DISP.

$m\ddot{x} = -kz - c\dot{z}$

$m(\ddot{z}+\ddot{y}) = -kz - c\dot{z}$ or $m\ddot{z}+c\dot{z}+kz = -m\ddot{y}$

$= m\omega_f^2 Y\sin\omega_f t$

let $m\omega_f^2 Y = P_o$ $z = Z\sin(\omega_f t-\psi)$ $Z = \frac{P_o}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}} = \frac{mY\omega_f^2}{\sqrt{\quad}} = \frac{Yr^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

VELOMETERS – measure velocity of a body

$y = Y\sin\omega t \qquad \dot{y} = Y\omega\cos\omega t$

$z_p = Z\sin(\omega t - \psi) \qquad \dot{z}_p = Z\omega\cos(\omega t - \psi) = \frac{Y\omega r^2}{\sqrt{\ }}\cos(\omega t - \psi)$

if $\frac{r^2}{\sqrt{\ }} \sim 1 \Rightarrow \dot{z}_p \sim \dot{y}$ outside of the phase difference

$\Rightarrow r$ must be large

$\Rightarrow$ same problems as the vibrometer.

(**) FOR vibrometer & velocity; phase shift causes $z = -y$, $\dot{z} = -\dot{y}$,

READ 3.11 & 3.12 FOR errors < 30% $\zeta \in [.6 - 1]$ and $-z\omega_n^2 = \ddot{y}(t)$ (τ)

$\tau = t - \frac{\text{const}}{\omega_n}$

3.54 given $x(t) = 20\sin 6.5\pi t + 5\sin 19.5\pi t$ (f_1, $3f_1$)

$$\ddot{x} = -\left[20(6.5\pi)^2\sin 6.5\pi t + 5(19.5\pi)^2\sin(19.5\pi)t\right] \quad mm/s^2$$

$$= -8339.8547\sin(6.5\pi t) - 18764.673\sin 19.5\pi t$$

Given $\zeta = 0.6 \qquad \omega_n = 45$ rad/s.

for each one $r_1 = \frac{\omega_1}{\omega_n} = \frac{6.5\pi}{45} = .4538 \qquad Y_1 = 20$

$r_2 = \frac{\omega_2}{\omega_n} = \frac{19.5\pi}{45} = 1.3614 = 3r_1 \qquad Y_2 = 5$

$$\ddot{z}_p = \frac{-Y_1\omega_1^2\sin(\omega_1 t - \psi_1)}{\sqrt{(1-r_1^2)^2 + (2\zeta r_1)^2}} - \frac{Y_2\omega_2^2\sin(\omega_2 t - \psi_1)}{\sqrt{(1-r_2^2)^2 + (2\zeta r_2)^2}}$$

$$= -8661.7957\sin(6.5t - \psi_1) - 10180.750\sin(19.5t - \psi_2)$$

$\psi_1 = \tan^{-1}\left(\frac{2\zeta r_1}{1-r_1^2}\right) = 34.4418^\circ \qquad \psi_2 = \tan^{-1}\left(\frac{2\zeta r_2}{1-r_2^2}\right) = \tan^{-1}(-1.9143)$

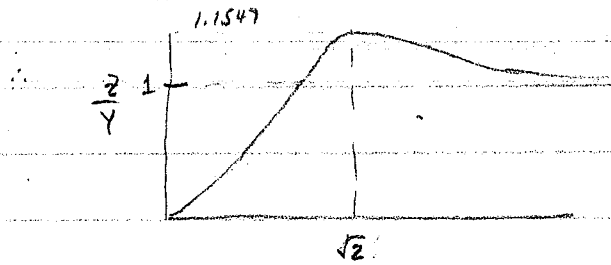
$= 117.5818^\circ$

10.9 Vibrometer

~~3.58~~ $f_n = 5$ Hz $\qquad \zeta = 0.5 \qquad$ FOR A VIBR $\frac{Z}{Y} \cong 1$

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Note $\left(\frac{Z}{Y}\right)_{max}$ occurs at $r = \frac{1}{\sqrt{1-2\zeta^2}}$ and $\left(\frac{Z}{Y}\right)_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.1547$



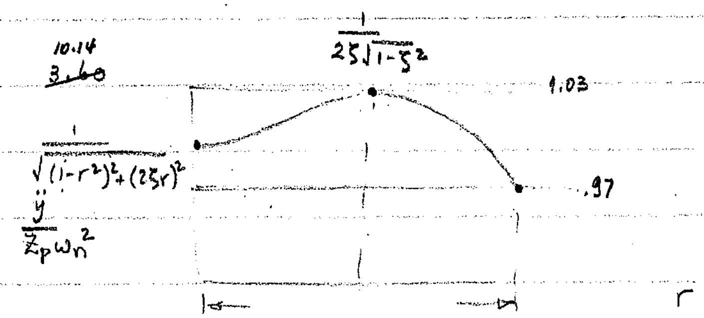
$\therefore \frac{Z}{Y} \rightarrow 1+$

for a 1% error $\frac{Z}{Y} = 1.01$

$\therefore \quad 1.01 = \frac{r^2}{\left[(1-r^2)^2 + r^2\right]^{1/2}} \Rightarrow r = 1.0101 , \quad 7.0527 = \frac{\omega}{\omega_n} = \frac{f}{f_n}$

$5 f_n = f = 5.0505\ Hz , \quad \underline{f = 35.2635\ Hz}$

remember $r \geq 3$



$r^4 = \sqrt{1-2\zeta^2}$
$= .49$

$\frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.03$ gives $\zeta = .6184$

for this ζ

$\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = .97$ gives r_{opt}.

$r_{opt} = .7662$

TO FIND OPTIMAL r & ζ FOR ACCELEROMETER GIVEN required error.

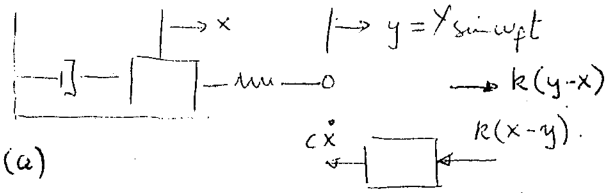
$0 \leq r \leq .7662 \quad \therefore \quad \omega_n = \omega/r \Rightarrow \frac{100\ Hz}{.7662} = f_n \cong 130.5\ Hz ; \omega_n =$

$2\pi f_n = \omega_n$ & $2m\omega_n = C_{crit} = 205.\ \frac{N\text{-}s}{m} \quad \therefore \quad c = 2m\omega_n \zeta = 2(.05)(820)(.6164) = 126.4\ \frac{N\text{-}s}{m}$ & $k = \omega_n^2 m = 33625\ N/m$

An accelerometer is to be designed for optimum range of freq ratio for max error of 3%. Determine ζ and range.

$$Z/Y = \frac{X}{m_o e/m} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad \tan\psi = \frac{2\zeta r}{1-r^2}$$

Problem 4-16



(a)

$$F = k(x-y)$$

Find DE for m (b) $m\ddot{x} = -c\dot{x} - k(x-y)$ or $m\ddot{x} + kx + c\dot{x} = ky = kY\sin\omega_f t$

what is displ of mass due to $y = Y\sin\omega_f t$ (c)

$$x = X\sin(\omega_f t - \psi) \qquad X = \frac{P_o}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} = \frac{kY}{\sqrt{\quad}} = \frac{Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\tan\psi = \frac{2\zeta r}{1-r^2}$$

what is F on mass due to spring (d)

$$F = k(x-y) = -[m\ddot{x} + c\dot{x}] = \omega_f^2 X m \sin(\omega_f t - \psi) - c\omega_f X\cos(\omega_f t - \psi)$$

$$= \omega_f X\{m\omega_f \sin(\omega_f t - \psi) - c\cos(\omega_f t - \psi)\}$$

$$= \omega_f X\, \mathcal{C}\sin(\omega_f t - \psi - \beta)$$

$$\mathcal{C}\cos\beta = \omega_f m \qquad \mathcal{C}\sin\beta = c$$

$$\mathcal{C} = \sqrt{m^2\omega_f^2 + c^2} \qquad \tan\beta = \frac{c}{\omega_f m} = \frac{2\zeta r}{r^2} = \frac{2\zeta}{r}$$

$$= \omega_f\sqrt{m^2\omega_f^2 + c^2}\; X\sin(\omega_f t - \psi - \beta)$$

$$= m\omega_f^2\sqrt{1 + \left(\frac{2\zeta}{r}\right)^2}\; X\sin(\omega_f t - \gamma) \qquad \gamma = \psi + \beta$$

what's F at support (e) $F = c\dot{x} = c\omega_f X\cos(\omega_f t - \psi) = c\omega_f X\sin(\omega_f t - \psi + \pi/2)$

READ SECTIONS 4-16, 4-17

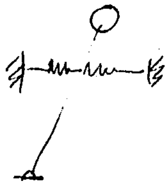
SELF-EXCITED VIBRATION & INSTABILITY

- CONSIDERED FORCING FUNC. EXTERNAL TO SYSTEM & INDEPENDENT OF MOTION

  i.e IF SYSTEM IS CLAMPED, FORCING FUNC. WILL CONTINUE

- CONSIDER CASE WHEN $P = P(x)$, $P = P(\dot{x})$ or $P(\ddot{x})$

  → SELF EXCITED MOTION IF SYSTEM IS CLAMPED ⇒ $P = 0$.



NON RESTORING ~~FORCE~~ MOMENT $Wl\theta$ was a function of θ

- WILL CONCENTRATE ON CASE WHERE FORCE IS FUNCTION OF VELOC.
  - EXAMPLE IS WING FLUTTER, NOSE-GEAR SHIMMY
  - AUTO WHEEL SHIMMY
  - AERODYNAMIC INDUCED MOTION OF BRIDGES

- SUPPOSE $m\ddot{x} + c\dot{x} + kx = P_0\dot{x} = P$

$\Rightarrow \quad \ddot{x} + (c - P_0)\frac{\dot{x}}{m} + \frac{kx}{m} = 0 \qquad$ let $x = Ce^{st}$

- CHAR EQ IS $\left[s^2 + (c-P_0)\frac{s}{m} + k/m\right] Ce^{st} = 0 \Rightarrow \left[s^2 + (c-P_0)\frac{s}{m} + k/m\right] = 0$

$$s_{1,2} = \frac{P_0 - c}{2m} \pm \sqrt{\left(\frac{P_0 - c}{2m}\right)^2 - \frac{k}{m}}$$

- SUPPOSE $P_0 > c \Rightarrow$ negative damping (makes situation worse)
- IF $\left(\frac{P_0 - c}{2m}\right)^2 > \frac{k}{m} \Rightarrow s_1, s_2$ are > 0 & real

$\Rightarrow x \uparrow$ as $t \uparrow \qquad x = C_1 e^{s_1 t} + C_2 e^{s_2 t} \qquad$ DIVERGENT APERIODIC

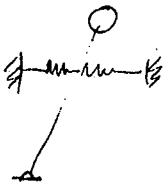
- IF $\left(\frac{P_0 - c}{2m}\right)^2 < \frac{k}{m} \Rightarrow$ conjugate pairs $\quad s_{1,2} = \frac{P_0 - c}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{P_0 - c}{2m}\right)^2}$

$\Rightarrow x = \bar{X} e^{\frac{P_0 - c}{2m}t} \sin\left[\sqrt{\frac{k}{m} - \left(\frac{P_0 - c}{2m}\right)^2}\, t + \phi\right] \qquad$ DIVERGENT OSCILLATORY since power of e is > 0

- SUPPOSE $P_0 < c \Rightarrow$ positive damping: same as damped free vibs

- $P_0 = c \Rightarrow \ddot{x} + \frac{kx}{m} = 0 \Rightarrow$ undamped free vibs

<u>HW</u> 4-20, 4-42, 4-43 Due Monday 17 March



NON RESTORING ~~FORCE~~ MOMENT $Wl\theta$ was a function of θ

- WILL CONCENTRATE ON CASE WHERE FORCE IS FUNCTION OF VELOC.
  - EXAMPLE IS WING FLUTTER, NOSE-GEAR SHIMMY
  - AUTO WHEEL SHIMMY
  - AERODYNAMIC INDUCED MOTION OF BRIDGES

- SUPPOSE $m\ddot{x} + c\dot{x} + kx = P_0\dot{x} = P$

$\Rightarrow \quad \ddot{x} + (c - P_0)\frac{\dot{x}}{m} + \frac{kx}{m} = 0 \qquad$ let $\quad x = Ce^{st}$

- CHAR EQ IS $\left[s^2 + (c - P_0)\frac{s}{m} + k/m\right]Ce^{st} = 0 \Rightarrow \left[s^2 + (c-P_0)\frac{s}{m} + k/m\right] = 0$

$$s_{1,2} = \frac{P_0 - c}{2m} \pm \sqrt{\left(\frac{P_0 - c}{2m}\right)^2 - \frac{k}{m}}$$

- SUPPOSE $P_0 > c \Rightarrow$ negative damping (makes situation worse)
- IF $\left(\frac{P_0 - c}{2m}\right)^2 > \frac{k}{m} \Rightarrow s_1, s_2$ are >0 & real

$\Rightarrow x \uparrow$ as $t \uparrow \qquad x = C_1e^{s_1t} + C_2e^{s_2t} \qquad$ DIVERGENT APERIODIC

- IF $\left(\frac{P_0 - c}{2m}\right)^2 < \frac{k}{m} \Rightarrow$ conjugate pairs $\quad s_{1,2} = \frac{P_0 - c}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{P_0 - c}{2m}\right)^2}$

$\Rightarrow \quad x = \overline{X}e^{\frac{P_0 - c}{2m}t} \sin\left[\sqrt{\frac{k}{m} - \left(\frac{P_0 - c}{2m}\right)^2}\,t + \phi\right] \qquad$ DIVERGENT OSCILLATORY since power of e is >0

- SUPPOSE $P_0 < c \Rightarrow$ positive damping: same as damped free vibs

- $P_0 = c \Rightarrow \ddot{x} + \frac{kx}{m} = 0 \Rightarrow$ undamped free vibs

<u>HW</u> 4-20, 4-42, 4-43 Due Monday 17 March

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THUS $A = 0 \quad B = \frac{1}{m\omega_d}$

- Thus $x = \frac{1}{m\omega_d} e^{-\zeta\omega(t-0)} \sin\omega_d(t-0)$ response to unit impulse at $t=0$
- FOR AN IMPULSE $= P d\tau \Rightarrow dx = \frac{P d\tau}{m\omega_d} e^{-\zeta\omega(t-\tau)} \sin\omega_d(t-\tau)$

$\therefore$ TOTAL RESPONSE TO ANY FORCE P IS

- $$x = \frac{1}{m\omega_d}\int_0^t P e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$$

if $x_0, \dot{x}_0 \neq 0 \quad x = x_h + \frac{1}{m\omega_d}\int_0^t P e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$

HW Prob 5-6, 5-28 Due Monday 17 March.

$$x_h = \left(x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \sin\omega_d t\right) e^{-\zeta\omega_n t}$$

SESSION #19

Chapter 5 in Rao

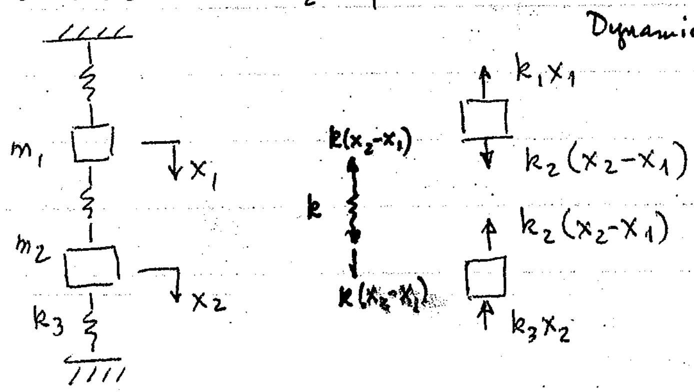
READ SECTIONS 4-16 THRU 4-18 IF NOT ALREADY DONE SO

READ CH. 6

- SO FAR HAVE LOOKED AT SDOF systems
- FREE UNDAMPED VIBS $\longrightarrow$ HARMONIC OSCILLATIONS $\omega_n = \sqrt{\frac{k}{m}}$
- ONE FREQ

- WANT TO LOOK AT 2 DOF SYSTEMS
- REQUIRES 2 INDEP COORDINATES
- LOOK AT UNDAMPED CASE FIRST
- FOR EACH DEGREE OF FREEDOM $\Rightarrow$ AN ω EXISTS

CONSIDER $x_2 > x_1$



Dynamic FBD - measured from static equilb of system

$$m_1\ddot{x}_1 = k_2(x_2 - x_1) - k_1 x_1$$

$$m_2\ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1)$$

$$\ddot{x}_1 + \left(\frac{k_1+k_2}{m_1}\right)x_1 - \frac{k_2}{m_1}x_2 = 0$$

HOMOG EQS – CONST COEFF

$$\ddot{x}_2 + \frac{(k_2+k_3)}{m_2}x_2 - \frac{k_2}{m_2}x_1 = 0$$

COUPLED EQNS — STATIC COUPLING

METHODS OF SOLUTION

A. ① TAKE 2 deriv of eq 1 TO FIND EQ OF $\ddddot{x}_1$ & $\ddot{x}_1$ IN TERMS OF $\ddot{x}_2$ } ELIMINATE x_2

② EQ. (1) GIVES x_2 IN TERMS OF $\ddot{x}_1$, x_1

B. LET $x_1 = A_1 \sin(\omega_1 t + \varphi_1)$
$x_2 = A_2 \sin(\omega_2 t + \varphi_2)$ } most general

$$\left[-A_1\omega_1^2 + \left(\frac{k_1+k_2}{m_1}\right)A_1\right]\sin(\omega_1 t+\varphi_1) - \frac{k_2}{m_1}A_2\sin(\omega_2 t+\varphi_2) = 0 \quad (1)$$
$$\left[-A_2\omega_2^2 + \frac{k_2+k_3}{m_2}A_2\right]\sin(\omega_2 t+\varphi_2) - \frac{k_2}{m_2}A_1\sin(\omega_1 t+\varphi_1) = 0$$

- True $\forall t$ ∴ pick $t=0$

$$\left[-A_1\omega_1^2 + \left(\frac{k_1+k_2}{m_1}\right)A_1\right]\sin\varphi_1 - \frac{k_2}{m_1}A_2\sin\varphi_2 = 0 \quad (2)$$
$$\left[-A_2\omega_2^2 + \frac{k_2+k_3}{m_2}A_2\right]\sin\varphi_2 - \frac{k_2}{m_2}A_1\sin\varphi_1 = 0$$

TRIVIAL CASE

- IF φ_1 & φ_2 are independent $\Rightarrow A_1 = A_2 = 0$ if $\omega_1, k_1, k_2, k_3, \omega_2 \neq 0$
- IF $\varphi_1 = \varphi_2 \Rightarrow A_1\left[\left(\frac{k_1+k_2}{m_1}\right) - \omega_1^2\right] = \frac{k_2}{m_1}A_2$ &

$$\frac{A_1}{A_2}\,\frac{[\quad]}{k_2/m_1} = \text{const} = \frac{\sin(\omega_2 t+\varphi_2)}{\sin(\omega_1 t+\varphi_1)} \quad \forall t \text{ from (1)}$$

- $\Rightarrow \omega_2 = \omega_1$ $\quad \sin(\omega_2 t + \varphi_2) = \sin(\omega_1 t + \varphi_1) = \sin(\omega t + \varphi)$

$\Rightarrow$

$x_1 = A_1 \sin(\omega t + \varphi)$
$x_2 = A_2 \sin(\omega t + \varphi)$ } x_1, x_2 <u>only vary in amplitude</u>

BACK TO (1) $\Rightarrow$ — Amplitude eqns.

$$\left\{A_1\left[\left(\frac{k_1+k_2}{m_1}\right) - \omega^2\right] - A_2\left[\frac{k_2}{m_1}\right]\right\}\sin(\omega t+\varphi) = 0$$
$$\left\{A_1\left[-\frac{k_2}{m_2}\right] + A_2\left[\left(\frac{k_2+k_3}{m_2}\right) - \omega^2\right]\right\}\sin(\omega t+\varphi) = 0$$

this is true $\forall t \Rightarrow$ either $A_1 = A_2 = 0$ TRIVIAL CASE

or

$$\begin{vmatrix} \left(\frac{k_1+k_2}{m_1}\right)-\omega^2 & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \left(\frac{k_2+k_3}{m_2}\right)-\omega^2 \end{vmatrix} = 0$$

$$m_1 m_2 \omega^4 - \left[m_2(k_1+k_2) + (k_2+k_3)m_1\right]\omega^2 + (k_1+k_2)(k_2+k_3) - k_2^2 = 0$$

$$\omega^4 - [\qquad\qquad]\omega^2 + \frac{k_1k_2+k_2k_3+k_1k_3}{m_1m_2} = 0$$

- THIS IN CHARACTERISTIC EQ IN ω_n^2 OR FREQ. EQN

THIS NORMALLY HAS 2 POSITIVE SOLUTIONS SINCE ALL K's, m's > 0

- LOOK AT SPECIAL CASE $k_1=k_2=k_3=k$ $m_1=m_2=m$

$$\omega^4 - \left[\frac{4k}{m}\right]\omega^2 + \frac{3k^2}{m^2} = 0$$

$$\left(\omega^2 - \frac{3k}{m}\right)\left(\omega^2-\frac{k}{m}\right) = 0 \quad \text{or} \quad \omega_{n_2} = \sqrt{\frac{3k}{m}} \qquad \omega_{n_1} = \sqrt{\frac{k}{m}}$$

FUNDAMENTAL OR FIRST MODE

LET $\omega = \omega_{n_1} = \sqrt{\frac{k}{m}}$ put into Amplitude Eqn

$$\left[\left(\frac{k_1+k_2}{m_1}-\omega^2\right)A_1 - \frac{k_2}{m_1}A_2 = 0 \right. \Rightarrow \left[\left(\frac{2k}{m}-\frac{k}{m}\right)A_1 - \frac{k}{m}A_2\right] = 0$$

$$\left(-\frac{k_2}{m_2}\right)A_1 + \left[\left(\frac{k_2+k_1}{m_2}\right)-\omega^2\right]A_2 = 0 \qquad \left[-\frac{k}{m}A_1 + \left(\frac{2k}{m}-\frac{k}{m}\right)A_2\right] = 0$$

NOTE THAT 1^{st} eq = 2^{nd} eq ⇒ DETERMINANT = 0 CAN'T ACTUALLY SOLVE FOR A_1 & A_2 BUT ONLY GET RELATION OF A_2 TO A_1

$$\frac{k}{m}A_1 - \frac{k}{m}A_2 = 0 \Rightarrow A_1 = A_2 \Rightarrow A_2/A_1 = 1 = \eta_1$$

$$\Rightarrow x_1 = x_2 = A_1 \sin(\omega_{n_1} t + \varphi_1)$$ both masses move in unison

A_1 & ϕ_1 come from IC

LET $\omega = \omega_{n_2} = \sqrt{\frac{3k}{m}}$ put into amplitude eqn ω_{n_2} is freq for 2^{nd} mode of vib.

$$\left[\left(\frac{2k}{m}-\frac{3k}{m}\right)A_1 - \frac{k}{m}A_2\right] = 0$$
$$\left[-\frac{k}{m}A_1 + \left(\frac{2k}{m}-\frac{3k}{m}\right)A_2\right] = 0$$
$$\Rightarrow -\frac{k}{m}A_1 - \frac{k}{m}A_2 = 0 \quad \text{or} \quad A_1 = -A_2$$
$$\Rightarrow A_1/A_2 = -1 = \eta_2$$

$$\Rightarrow \text{IF} \quad x_1 = A_1 \sin(\omega_{n_2} t + \varphi_2)$$
$$x_2 = -A_1 \sin(\omega_{n_2} t + \varphi_2)$$

masses move in opposite

A_1 & ϕ_2 come from IC

if $k_1 = k_2 = k_3 = k \qquad m_1 = m_2 = m$

$$\ddot{x}_1 + \left(\frac{2k}{m}\right)x_1 - \frac{k}{m}x_2 = 0$$

$$\ddot{x}_2 + \left(\frac{2k}{m}\right)x_2 - \frac{k}{m}x_1 = 0$$

add eq

$$(\ddot{x_1 + x_2}) + \frac{2k}{m}(x_1 + x_2) - \frac{k}{m}(x_1 + x_2) = 0$$

$$(\ddot{x_1 + x_2}) + \frac{k}{m}(x_1 + x_2) = 0$$

let $q_1 = x_1 + x_2$ then $\ddot{q}_1 + \frac{k}{m}q_1 = 0$ & $\omega_{n_1} = \sqrt{\frac{k}{m}}$

and $q_1 = A\sin(\omega_{n_1}t + \phi)$

subtract eq

$$(\ddot{x_1 - x_2}) + \frac{2k}{m}(x_1 - x_2) - \frac{k}{m}(x_2 - x_1) = 0$$

$$(\ddot{x_1 - x_2}) + \frac{3k}{m}(x_1 - x_2) = 0$$

let $q_2 = x_1 - x_2$

$$\ddot{q}_2 + \frac{3k}{m}q_2 = 0 \qquad \omega_{n_2} = \sqrt{\frac{3k}{m}} \qquad q_2 = A\sin(\omega_{n_2}t + \phi)$$

$\therefore$

$$\frac{q_1 + q_2}{2} = x_1$$

$$\frac{q_1 - q_2}{2} = x_2$$

now $\tau_1 = 2\pi/\omega_1$ $\tau_2 = 2\pi/\omega_2$ since $\omega_2 > \omega_1$ $\tau_2 < \tau_1$

LOOK AT p. 207/208 wave forms P 230 in RAO

LOOK AT p. 208/209 TO DEFINE GENERAL CASE

IF $2\alpha = \frac{k_1+k_2}{m_1}$ $\gamma = \frac{k_2}{m_1}$ $\Rightarrow \ddot{x}_1 + 2\alpha x_1 - \gamma x_2 = 0$

$2\beta = \frac{k_2+k_3}{m_2}$ $\varepsilon = \frac{k_2}{m_2}$ $\Rightarrow \ddot{x}_2 + 2\beta x_2 - \varepsilon x_1 = 0$

the amplitude eqn.

$$(2\alpha - \omega^2)A_1 - \gamma A_2 = 0$$
$$(-\varepsilon A_1 + (2\beta - \omega^2)A_2 = 0$$

FREQ EQ.

$$\omega^4 - 2(\alpha+\beta)\omega^2 + 4\alpha\beta - \varepsilon\gamma = 0$$

and

$$\omega_1 = \sqrt{(\alpha+\beta) - \sqrt{(\alpha-\beta)^2 + \varepsilon\gamma}}$$
$$\omega_2 = \sqrt{(\alpha+\beta) + \sqrt{(\alpha-\beta)^2 + \gamma\varepsilon}}$$

when $\omega = \omega_1 \Rightarrow \frac{A_2}{A_1} = \frac{(\alpha-\beta) + \sqrt{(\alpha-\beta)^2 + \gamma\varepsilon}}{\gamma}$ $= \eta_1$ $\eta_1 > 0$

$$\left. \begin{aligned} x_1 &= A_1 \sin(\omega_1 t + \varphi_1) \\ x_2 &= A_2 \sin(\omega_1 t + \varphi_1) = \eta_1 A_1 \sin(\ ) \end{aligned} \right\}$$

when $\omega = \omega_2 \Rightarrow \frac{A_2}{A_1} = \frac{(\alpha-\beta) - \sqrt{(\alpha-\beta)^2 + \gamma\varepsilon}}{\gamma}$ $= -\eta_2$ $\eta_2 > 0$

$$\left. \begin{aligned} x_1 &= \tilde{A}_1 \sin(\omega_2 t + \varphi_2) \\ x_2 &= -\eta_2 \tilde{A}_1 \sin(\omega_2 t + \varphi_2) \end{aligned} \right\}$$

- BY LINEAR SUPERPOSITION

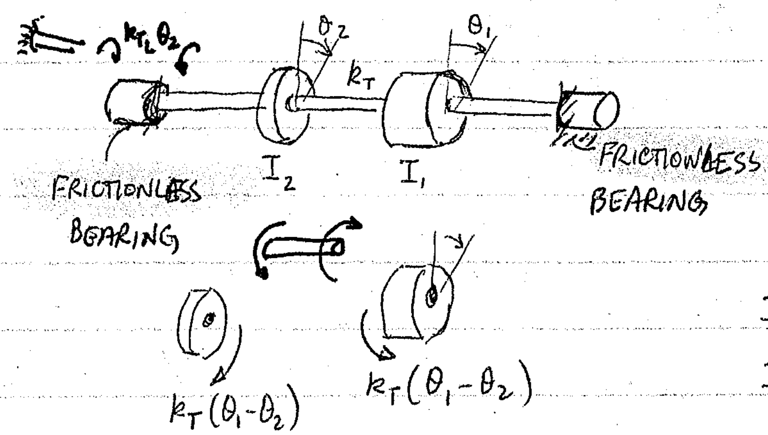
$$x_1 = A_1 \sin(\omega_1 t + \varphi_1) + \tilde{A}_1 \sin(\omega_2 t + \varphi_2)$$
$$x_2 = \eta_1 A_1 \sin(\omega_1 t + \varphi_1) - \eta_2 \tilde{A}_1 \sin(\omega_2 t + \varphi_2)$$

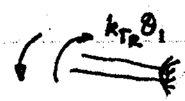
NOTE η_1, η_2 depend on k_1, k_2, k_3, m_1, m_2

- THERE ARE 4 unknowns $A_1, \tilde{A}_1, \varphi_1, \varphi_2$ $\Rightarrow$ need IC on x_1 & x_2

SESSION # 20

FOR THE TORSIONAL SHAFT





if not frictionless

$$I_1\ddot{\theta}_1 = -k_T(\theta_1-\theta_2) - k_{T_R}\theta_1$$
$$I_2\ddot{\theta}_2 = k_T(\theta_1-\theta_2) - k_{T_L}\theta_2$$

thus $\ddot{\theta}_1 + \frac{k_T}{I_1}\theta_1 - \frac{k_T}{I_1}\theta_2 = 0$

$$\ddot{\theta}_2 + \frac{k_T}{I_2}\theta_2 - \frac{k_T}{I_2}\theta_1 = 0$$

CHOOSE $\theta_1 = A_1\sin(\omega t+\varphi)$, $\theta_2 = A_2\sin(\omega t+\varphi)$ $\Rightarrow$

$$\begin{bmatrix}\left(-\omega^2+\frac{k_T}{I_1}\right)A_1 - \frac{k_T}{I_1}A_2 \\ -\frac{k_T}{I_2}A_1 + \left(-\omega^2+\frac{k_T}{I_2}\right)A_2\end{bmatrix} = 0$$

FREQ EQN.

$$\omega^4 - \left(\frac{k_T}{I_1}+\frac{k_T}{I_2}\right)\omega^2 + \frac{k_T^2}{I_1I_2} - \frac{k_T^2}{I_1I_2} = 0$$

$\omega = 0$ or $\omega = \sqrt{k_T\left(\frac{1}{I_1}+\frac{1}{I_2}\right)}$ SEMI DEFINITE SYSTEM

if $\omega = 0 \Rightarrow A_1 = A_2$ (from amplitude eqs) $\Rightarrow \theta_1 = \theta_2 \Rightarrow$ (FROM DE.) $I_1\ddot{\theta}_1 = 0$ or $\theta_1 = Ct + D$, $\theta_2 = Ct + D$.

NON OSCILLATORY MOTION HAVING CONST VEL { SAME VELOC & DISP. THIS IS DEGENERATE MODE

if $\omega = \sqrt{k_T\left(\frac{1}{I_1}+\frac{1}{I_2}\right)} \Rightarrow$ (from amplitude eqs) $\frac{A_1}{I_2} = -\frac{A_2}{I_1}$ or $\frac{A_2}{A_1} = -\frac{I_1}{I_2}$

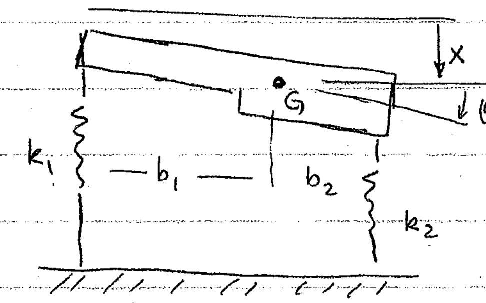
thus $\theta_1 = A_1\sin(\omega t+\varphi)$

$$\theta_2 = A_2\sin(\omega t+\varphi) = -\frac{I_1A_1}{I_2}\sin(\omega t+\varphi).$$

HW. DO 7-1, 7-4, 7-7

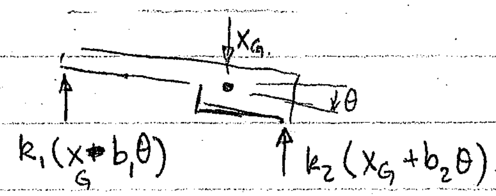
SESSION # 23

MIXED COORDINATES - CONSIDER UNSYMMETRICAL BODY.



MODEL OF VEHICLE SUSPENSION

- NEED COORDINATE FOR LINEAR POSITION
- " " FOR ANGULAR POSITION
- BOTH MEASURED FROM NEUTRAL POSITION equilibrium



$$m\ddot{x}_G = -k_1(x_G - b_1\theta) - k_2(x_G + b_2\theta)$$

$$J_G\ddot{\theta} = -k_2(x_G + b_2\theta)b_2 + k_1(x_G - b_1\theta)b_1$$

STATIC COUPLING

$$m\ddot{x}_G + (k_2b_2 - k_1b_1)\theta + x_G(k_1 + k_2) = 0$$

$$J_G\ddot{\theta} + (k_2b_2 - k_1b_1)x_G + (k_1b_1^2 + k_2b_2^2)\theta = 0$$

IF $(k_2b_2 - k_1b_1) = 0$, NO COUPLING

COUPLING COEFF

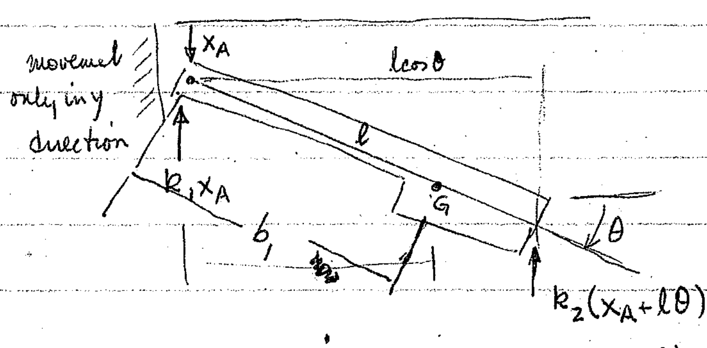
$$\Rightarrow \quad m\ddot{x}_G + (k_1 + k_2)x_G = 0$$

$$J_G\ddot{\theta} + (k_1b_1^2 + k_2b_2^2)\theta = 0$$

$\theta = \Theta \sin(\omega t + \varphi)$

$x = A\sin(\omega t + \varphi)$

IF YOU CHOOSE x_A, θ



GO TO PG 12 $\quad + ma_y x_G - ma_x y_G$ o@A

$$\Sigma T_A = J_A\ddot{\theta} + m\ddot{x}_A(b_1\cos\theta) \quad \text{for small } \theta,\ \cos\theta \sim 1$$

$$\Sigma T_A = -k_2(x_A + l\theta)\cdot l\cos\theta \cong -k_2(x_A + l\theta)l$$

$$m(\ddot{x}_G) = m(\ddot{x}_A + b_1 \ddot{\tan\theta}) = \Sigma F$$

$$= m(\ddot{x}_A + b_1\ddot{\theta})$$

$$\Sigma F = -k_1x_A - k_2(x_A + l\theta) = m\ddot{x}_G$$

Gives rise to coupling in accel & displ

$$\frac{d}{dt}\tan\theta = \sec^2\theta\,\dot{\theta} \qquad \frac{d^2}{dt^2}\tan\theta = \underbrace{2\sec^2\theta\tan\theta\,\dot{\theta}}_{\text{nonlin}} + \sec^2\theta\,\ddot{\theta}$$

$2\theta\dot{\theta}$

STATIC, DYNAMIC OR COMBINATION ARISE FROM COORDINATES CHOSEN

$$\frac{1}{\sqrt{mk}} \int_{\frac{4\pi}{\omega}}^{t} \cancel{P}^{\,0} \sin(\omega[t-\tau])\,d\tau$$

since $t > \frac{4\pi}{\omega}$ and $P = \text{const} \Rightarrow x = \text{const}$

We can see this since $m\ddot{x} + kx = P$ and $P = \text{const} \Rightarrow$ particular solution $x_p = P/k$

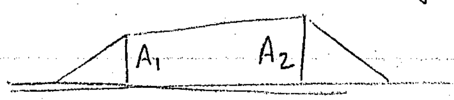
—— o —— o —— o ——

SESSION # 21

RETURNING TO p. 208 TALK ABOUT NODE

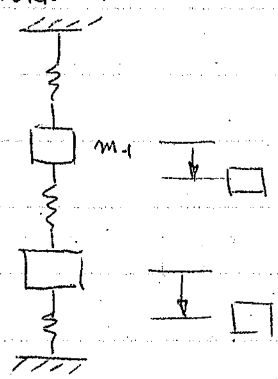
$x_1 = A_1 \sin(\omega_1 t + \varphi_1)$ $\quad x_2 = A_2(\omega_1 t + \varphi_1)$ is defined as the ~~mode~~ displacement equations for the masses

A_1 & A_2 found for ω_1 define the mode pattern



DISPLACEMENT METHOD - ANOTHER METHOD

RETURN TO



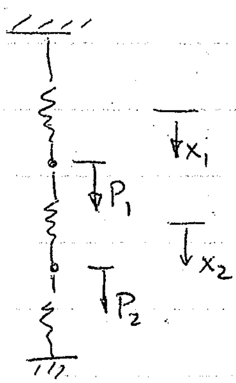
LOOK AT FBD OF mass m_1

TOTAL FORCE ON MASS M_1 ~~ON~~ DUE TO SPRINGS

$m_1\ddot{x}_1 = -P_1$ (1)

TOTAL FORCE ON MASS M_2 DUE TO SPRINGS

$m_2\ddot{x}_2 = -P_2$ (2)



LOOK AT SPRINGS ALONE

P_1 - FORCE EXERTED BY MASS 1 ON SPRINGS

P_2 - " " " " 2 ON SPRINGS

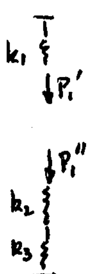
- DEFORMATION IS PROPORTIONAL TO LOADS APPLIED

FOR SMALL DEFORMATIONS $\quad x_1 = a_{11}P_1 + a_{12}P_2$ (3)

$x_2 = a_{21}P_1 + a_{22}P_2$ (4)

where a_{ij} is a flexibility coeff

DISPL AT POINT i DUE TO FORCE AT POINT j



$P_1 = P_1' + P_1''$ also

$P_1' = k_1 x_1$ $\quad P_1'' = k_1' x_1$ $\quad$ since k_2 & k_3 in series $\frac{1}{k_1'} = \frac{1}{k_2} + \frac{1}{k_3} \Rightarrow k_1' = \frac{k_2 k_3}{k_2 + k_3}$

but P_1'' is the load on k_3 that causes x_2 since load for springs in series are equal

$\therefore$ $P_1'' = k_3 x_2$ $\quad$ hence $k_1' x_1 = k_3 x_2$ $\quad$ or $x_2 = k_1'/k_3 \, x_1 = \frac{k_2}{k_2 + k_3} x_1$

~~but $P_1''/k_1 = x_1 = \frac{k_2 + k_3}{k_2} x_2$~~

also $P_1 = k'' x_1 = \left[k_1 + \frac{k_2 k_3}{k_2 + k_3}\right] x_1$ $\quad$ or $\left[\frac{k_1 k_2 + k_1 k_3 + k_2 k_3}{k_2 + k_3}\right] x_1 = P_1$

but we want the relationship $x_2 = a_{21} P_1$ $\quad$ disp @ 2 due to force at 1

$$\therefore \quad x_1 = P_1 \left[\frac{k_2 + k_3}{\sum_{i \neq j} k_i k_j}\right] = \frac{k_2 + k_3}{k_2} \cdot x_2$$

$$\Rightarrow \quad \frac{k_2}{\sum k_i k_j} \cdot P_1 = x_2$$

substitute (1) & (2) into (3) & (4) FOR P_1 and P_2

$\Rightarrow$

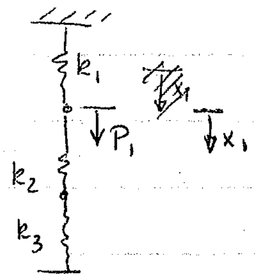
$$x_1 = a_{11}(-m_1\ddot{x}_1) + a_{12}(-m_2\ddot{x}_2) \Rightarrow a_{11}m_1\ddot{x}_1 + a_{12}m_2\ddot{x}_2 + x_1 = 0$$

$$x_2 = a_{21}(-m_1\ddot{x}_1) + a_{22}(-m_2\ddot{x}_2) \Rightarrow a_{21}m_1\ddot{x}_1 + a_{22}m_2\ddot{x}_2 + x_2 = 0$$

DYNAMIC COUPLING

TO FIND THE FLEXIBILITY COEFF a_{11} assume $P_2 = 0$

FORCE P_1 applied at point 1



k_2 & k_3 are in series $\frac{1}{k'} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{k_3+k_2}{k_2k_3}$

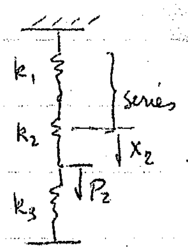
and k_1 & k' are in parallel $k_1 + \frac{k_1k_3}{k_3+k_2} = \frac{k_1k_3+k_1k_2+k_1k_3}{k_3+k_2} = k''$

$x_1 = a_{11}P_1 + a_{12}P_2$ ($P_2 = 0$)

now $P_1 = k''x_1 \Rightarrow k'' = \frac{1}{a_{11}}$ or $\boxed{a_{11} = \frac{1}{k''}}$

$x_1 = a_{11}P_1$ & $P_1 = k''x_1$

TO FIND a_{22} assume $P_1 = 0$ $\quad P_2 = k''x_2$ DISPL IS $\propto$ LOAD



series $k''' = \frac{k_2k_1}{k_1+k_2}$ } Parallel $k''' + k_3 = k^{iv} = \frac{k_1k_2 + k_1k_3 + k_2k_3}{k_1+k_2}$

and $a_{22} = \frac{k_1+k_2}{k_1k_2+k_2k_3+k_3k_1}$

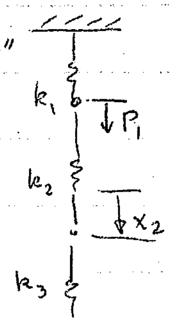
$x_2 = a_{21}P_1 + a_{22}P_2$ ($P_1 = 0$) $\Rightarrow k^{iv} = \frac{1}{a_{22}}$ $\quad a_{22} = \frac{1}{k^{iv}}$

disp @ 2 due to load @ 1

$P_1 = P_1' + P_1''$

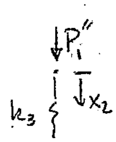
$P_1' = k_1x_1$

$P_1'' = k'x_1$



TO FIND a_{21} assume $P_2 = 0 \Rightarrow x_2 = a_{21}P_1$

since k_2 & k_3 in series each feels the force P_1''

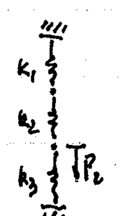


$\therefore P_1 = \underbrace{k_1x_1}_{P_1'} + \underbrace{k'x_1}_{P_1''} = k_1x_1 + \frac{k_2k_3}{k_3+k_2}x_1 = \underbrace{k_3x_2}_{P_1''} + \underbrace{k_1x_1}_{P_1'}$; leads to $x_2 = \frac{k_2}{k_3+k_2}x_1$

but $P_1 = k''x_1$ or $x_1 = \frac{P_1}{k''} = P_1\left[\frac{k_3+k_2}{k_1k_3+k_1k_2+k_1k_3}\right]$

$$k'' = k_1 + \frac{1}{1/k_2 + 1/k_3} = k_1 + \frac{k_2k_3}{k_2+k_3} = \frac{k_1k_2+k_1k_3+k_2k_3}{k_2+k_3}$$

$$x_2 = \frac{k_2}{k_2+k_3}x_1 = P_1\left[\frac{k_2+k_3}{k_1k_2+k_1k_3+k_2k_3}\right]\cdot\frac{k_2}{k_2+k_3} = a_{21}P_1 = P_1\cdot\frac{k_2}{k_1k_2+k_1k_3+k_2k_3}$$



TO FIND a_{12} assume $P_1 = 0 \Rightarrow x_1 = a_{12}P_2$

GOING THROUGH SAME THING WE CAN SHOW THAT

$$x_1 = a_{12}P_2 = P_2\left[\frac{k_2}{k_1k_2+k_1k_3+k_2k_3}\right]$$

$x_1 = \frac{k_2}{k_1+k_2}x_2 = \frac{k_2}{k_1+k_2}P_2a_{22}$

$= \frac{k_2}{k_1k_2+k_2k_3+k_3k_1}$

$P_2 = P_2' + P_2''$

$P_2' = k_3x_2$

$P_2'' = k'''x_2$

$P_2'' = k_1x_1 = \frac{k_2k_1}{k_1+k_2}x_2$

or $x_1 = \frac{k_2}{k_1+k_2}x_2$

$P_2 = \underbrace{k_3x_2}_{P_2'} + \underbrace{k'''x_2}_{P_2''} = k_3x_2 + \frac{k_2k_1}{k_1+k_2}x_2$

but $P_2 = \frac{1}{a_{22}}x_2 = \frac{k_1+k_2}{[k_1k_2+k_1k_3+k_3k_2]}$ [illegible] $x_2 = \frac{k_1+k_2}{[\ ]}$ [illegible]

• THIS RESULT SHOWS $a_{12} = a_{21}$ WHICH COMES FROM MAXWELL'S RECIPROCAL LAW

FOR $M_1 = M_2 = M$ & $k_1 = k_2 = k_3 = k$

$$a_{11} = \frac{2}{3}\frac{1}{k} \qquad a_{12} = \frac{1}{3k} = a_{21} \qquad a_{22} = \frac{2}{3k}$$

THUS

$$\frac{2}{3}\frac{m}{k}\ddot{x}_1 + \frac{1}{3}\frac{m}{k}\ddot{x}_2 + x_1 = 0$$

$$\frac{1}{3}\frac{m}{k}\ddot{x}_1 + \frac{2}{3}\frac{m}{k}\ddot{x}_2 + x_2 = 0$$

IF $x_1 = A_1 \sin(\omega t + \phi)$ $\quad x_2 = A_2 \sin(\omega t + \phi)$

$$\begin{bmatrix} (1 - \frac{2}{3}\frac{m}{k}\omega^2) A_1 & - \frac{1}{3}\frac{m}{k}\omega^2 A_2 \\ -\frac{1}{3}\frac{m}{k}\omega^2 A_1 & + (1 - \frac{2}{3}\frac{m}{k}\omega^2) A_2 \end{bmatrix} \begin{matrix} = 0 \\ = 0 \end{matrix}$$

$\omega^2 \neq \frac{k}{m}$ in general.

let

$\frac{k}{m\omega^2} = \lambda$

INVERSE FREQ. FACTOR

$$(\lambda - 2/3)A_1 - 1/3\, A_2 = 0$$

$$-1/3\, A_1 + (\lambda - 2/3)A_2 = 0$$

FOR $A_1 = A_2 \neq 0$ determinant $= 0$

$$\begin{vmatrix} \lambda - 2/3 & -1/3 \\ -1/3 & (\lambda - 2/3) \end{vmatrix} = 0$$

or ~~$1 - \frac{4}{3}\lambda + \frac{1}{3}\lambda^2 = 0 \Rightarrow \lambda^2 - 4\lambda + 3 = 0$~~ ~~$1 - \frac{1}{3}$~~

$$\lambda^2 - 4/3\,\lambda + 1/3 = 0 \qquad (\lambda - 1/3)(\lambda - 1) = 0 \qquad \lambda = 1 \quad \lambda = 1/3$$

$\lambda = 1 \Rightarrow \omega^2 = k/m \quad \omega = \sqrt{k/m}$

$\lambda = 1/3 \Rightarrow \omega^2 = 3k/m \quad \omega = \sqrt{\frac{3k}{m}}$

if $\lambda = 1 \Rightarrow (1 - 2/3)A_1 - 1/3\, A_2 = 0 \qquad A_1 = A_2 \qquad \frac{A_2}{A_1} = 1$

IF $\lambda = 1/3 \qquad (\frac{1}{3} - 2/3)A_1 - 1/3\, A_2 = 0 \qquad A_1 = -A_2 \qquad A_2/A_1 = -1$

THESE ARE THE SAME AS WHAT WERE OBTAINED FOR THE DIRECT APPROACH

CAN DEFINE FOR SIMPLE PROBLEMS ~~mode shapes~~ coordinate that will uncouple the equations

ORIGINALLY

$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 = 0$$
$$m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2 x_1 = 0$$

} STATIC OR ELASTIC COUPLING

COUPLING IS IN THE DISPL COORD.

INVERSE APPROACH

$$\frac{k_2+k_3}{k_1k_2+k_1k_3+k_2k_3} m_1\ddot{x}_1 + \frac{k_2}{k_1k_2+k_1k_3+k_2k_3} m_2\ddot{x}_2 + x_1 = 0$$

$$\frac{k_2}{k_1k_2+k_1k_3+k_2k_3} m_1\ddot{x}_1 + \frac{k_1+k_2}{k_1k_2+k_1k_3+k_2k_3} m_2\ddot{x}_2 + x_2 = 0$$

DYNAMIC OR INERTIA COUPLING - COUPLING IS IN THE ACCELERATION VARIABLE

TYPE OF COUPLING IS IN METHOD USED TO DERIVE EQS.

LET $m_1 = m_2 = m \qquad k_1 = k_2 = k_3 = k$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 + 2kx_2 - kx_1 = 0$$

ADD — $m(\ddot{x}_1+\ddot{x}_2) + k(x_1+x_2) = 0$

SUBTRACT $m(\ddot{x}_1-\ddot{x}_2) + 3k(x_1-x_2) = 0$

LET $q_1 = x_1+x_2 \qquad q_2 = x_1 - x_2$ GENERALIZED COORD

$$m\ddot{q}_1 + kq_1 = 0 \quad \text{or} \quad \omega_1 = \sqrt{k/m} \qquad q_1 = A_1\sin(\omega_1 t+\phi_1)$$
$$m\ddot{q}_2 + 3kq_2 = 0 \quad \text{or} \quad \omega_2 = \sqrt{\frac{3k}{m}} \qquad q_2 = A_2\sin(\omega_2 t+\phi_2)$$

$q_1+q_2 = 2x_1 \qquad q_1 - q_2 = 2x_2$

if

$$x_1 = [A_1\sin(\omega_1 t+\phi_1) + A_2\sin(\omega_2 t+\phi_2)]/2$$
$$x_2 = [A_1\sin(\omega_1 t+\phi_1) - A_2\sin(\omega_2 t+\phi_2)]/2$$

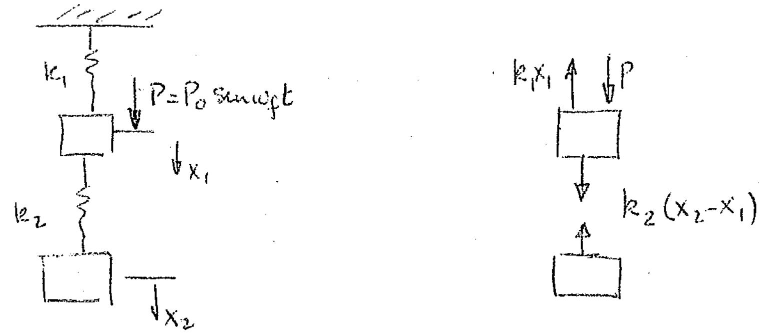
SESSION #22

- ANOTHER METHOD TO FIND THE EQUATIONS - LAGRANGE'S METHOD
  - MAKES USE OF ENERGY OF SYSTEM
  - POTENTIAL ENERGY V IS A FN OF GENERALIZED COORDINATES q_k
  - KINETIC ENERGY T IS A FN OF GENERALIZED VELOCITY $\dot{q}_k$ and q_k
  - FOR A CONSERVATIVE SYSTEM <u>READ APPENDIX B</u>

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = 0$$

- GENERALIZED COORDINATE IS SUCH THAT A CHANGE IN ONE COORDINATE DOESN'T CAUSE OR DOESN'T REQUIRE A CHANGE IN ANY OTHER COORD.

TWO DEGREE OF FREEDOM W/O DAMPING W/ FORCED VIBS.



$$m_1\ddot{x}_1 = k_2(x_2 - x_1) - k_1 x_1 + P$$

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = P_0 \sin\omega_f t$$

$$m_2\ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$

$$x_i = x_{i_h} + x_{i_p} \qquad i = 1, 2$$

solution to the homogeneous equations $x_{1_h} = A_1 \sin(\omega t + \phi)$ $\quad x_{2_h} = A_2 \sin(\omega t + \phi)$

solution to the forcing function choose

$$x_{1p} = \bar{X}_1 \sin\omega_f t \qquad x_{2p} = \bar{X}_2 \sin\omega_f t$$

$\div k_1$ $\qquad -m_1 \bar{X}_1 \omega_f^2 + (k_1 + k_2)\bar{X}_1 - k_2 \bar{X}_2 = P_0$

$\div$ by k_2 $\qquad -m_2 \bar{X}_2 \omega_f^2 + k_2 \bar{X}_2 - k_2 \bar{X}_1 = 0 \qquad$ if $\sin\omega_f t \neq 0$

$$\bar{X}_1\left[1 - r_1^2 + k_2/k_1\right] - k_2/k_1\, \bar{X}_2 = P_0/k_1 = \bar{X}_0 \qquad r_1 = \frac{\omega_f}{\omega_1} \qquad \omega_1 = \sqrt{\frac{k_1}{m_1}}$$

$$-1\,\bar{X}_1 + (1 - r_2^2)\bar{X}_2 = 0 \qquad r_2 = \omega_f/\omega_2 \qquad \omega_2 = \sqrt{\frac{k_2}{m_2}}$$

$\Rightarrow$

$$\bar{X}_1/\bar{X}_0 = \frac{(1 - r_2^2)}{(1 - r_2^2)(1 - r_1^2 + k_2/k_1) - k_2/k_1} \Rightarrow \frac{P_0(k_2 - m_2\omega_f^2)}{(k_2 - m_2\omega_f^2)(k_1 - m_1\omega_f^2 + k_2) - k_2^2} = \bar{X}_1$$

$$\bar{X}_2/\bar{X}_0 = \frac{1}{(1 - r_2^2)(1 - r_1^2 + k_2/k_1) - k_2/k_1} = \frac{P_0 k_2}{(k_2 - m_2\omega_f^2)(k_1 - m_1\omega_f^2 + k_2) - k_2^2} = \bar{X}_2$$

DETERMINANT $\quad (1 - r_2^2)(1 - r_1^2 + k_2/k_1) - k_2/k_1 = 0 \quad$ we have resonance or $m_1 m_2 \omega_f^4 - (k_1 m_2 + k_2 m_1 + k_2 m_2)\omega_f^2 + k_2 k_1 = 0$

if $\quad r_2 = 1 \quad$ ie $\quad \omega_f = \omega_2 \quad \Rightarrow \quad \bar{X}_1 = 0 \qquad \bar{X}_2/\bar{X}_0 = -\frac{k_1}{k_2} \Rightarrow \bar{X}_2 = -\frac{k_1}{k_2}\bar{X}_0 = -\frac{P_0}{k_2}$

if $\quad r_1 = 1 \quad$ ie $\quad \omega_f = \omega_1 \quad \Rightarrow \quad \bar{X}_1 = -\frac{(1 - r_2^2)}{r_2^2}\frac{k_1}{k_2}\bar{X}_0 = -\frac{(1 - r_2^2)}{r_2^2}\frac{P_0}{k_2}$

$$\bar{X}_2 = -\frac{1}{r_2^2}\frac{k_1}{k_2}\bar{X}_0 = -\frac{1}{r_2^2}\frac{P_0}{k_2}$$

if $\quad \omega_f = 0 \qquad r_1 = r_2 = 0 \qquad \bar{X}_1 = \bar{X}_2 = \bar{X}_0$

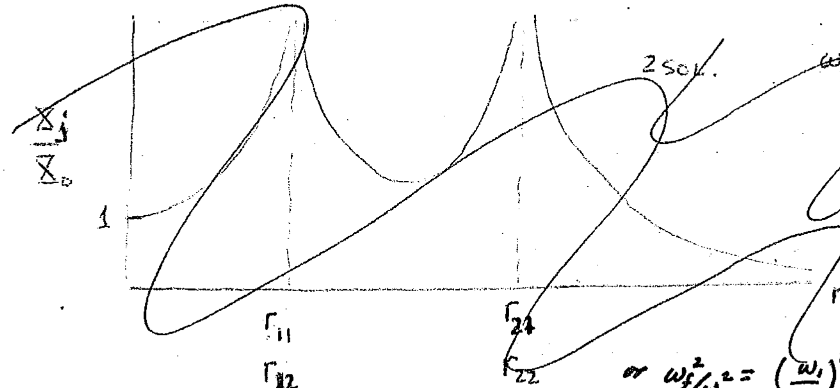
if $\quad \omega_f \to \infty \qquad \bar{X}_1/\bar{X}_0 \to 0 \qquad \bar{X}_2/\bar{X}_0 \to 0$

FROM RESONANCE EQ:

$$\omega_f^4 - \omega_f^2\left[\frac{k_2}{k_1}\omega_1^2 + \omega_1^2 + \omega_2^2\right] + \omega_1^2\omega_2^2 = 0$$

this has 2 roots for ω_f^2

$$\omega_f^2 = \frac{\omega_1^2(1+k_2/k_1) + \omega_2^2 \pm \sqrt{\left[\omega_1^2(1+k_2/k_1)+\omega_2^2\right]^2 - 4\omega_1^2\omega_2^2}}{2}$$



2 sol. ω_{f_1}, ω_{f_2}

$r_{ij} = \frac{\omega_{fi}}{\omega_j}$ $i = 1,2$ $j = 1,2$

r_i

or $$\omega_f^2/\omega_2^2 = \frac{\left(\frac{\omega_1}{\omega_2}\right)^2\left(1+\frac{k_2}{k_1}\right) + 1 \pm \sqrt{\left\{\left(\frac{\omega_1}{\omega_2}\right)^2\left[1+\frac{k_2}{k_1}\right]+1\right\}^2 - 4\left(\frac{\omega_1}{\omega_2}\right)^2}}{2}$$

FOR EQUAL MASSES & EQUAL SPRINGS. $k_2 = k_1 = k$ $m_1 = m_2 = m$ $\omega_1 = \omega_2 = \omega = \sqrt{\frac{k}{m}}$

$$r^4 - 3r^2 + 1 = 0 \qquad r^2 = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} = .382,\ 2.618$$

or $$\omega_f^2/\omega_2^2 = \frac{\left[1+(1+\mu)\left(\frac{\omega_2}{\omega_1}\right)^2\right] \pm \sqrt{\left[1+(1+\mu)\left(\frac{\omega_2}{\omega_1}\right)^2\right]^2 - 4\left(\frac{\omega_2}{\omega_1}\right)^2}}{2\left(\frac{\omega_2}{\omega_1}\right)^2}$$

$r_1 = .618$ $r_2 = 1.618$

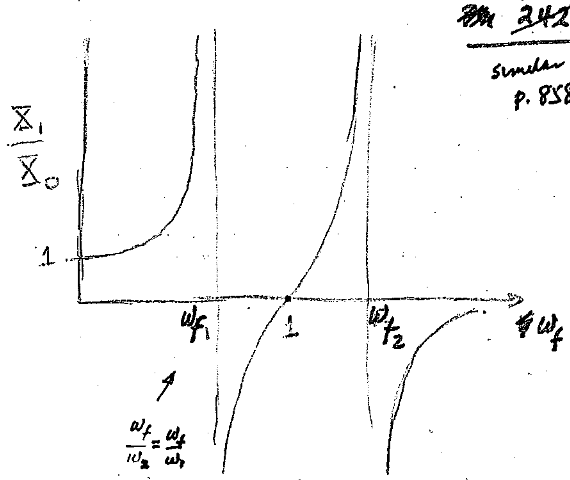
if $\omega_1 = \omega_2$ (tuned)

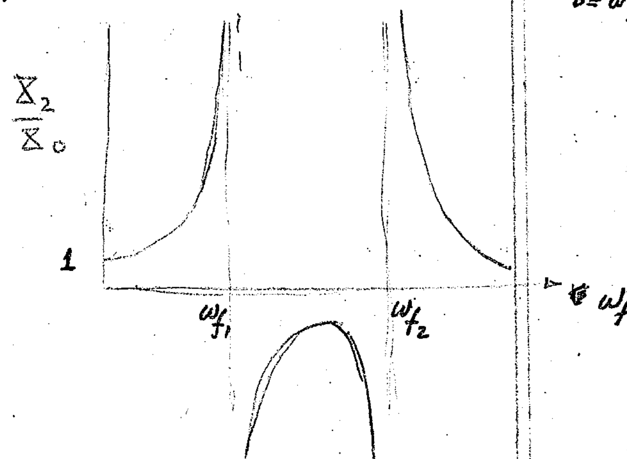
$$\omega_f^2/\omega_2^2 = \left[(1+\mu/2) \pm \sqrt{(1+\mu/2)^2 - 1}\right]$$

$\mu = m_2/m_1$

$b = \omega_2/\omega_1$

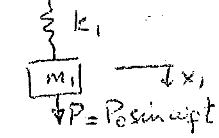
Sect 9.11
~~242~~ Rao. 6th ed.
similar to this
p. 858





SESSION # 25

- UNDAMPED VIBRATION ABSORBER
  - FOR A MASS SPRING SYSTEM



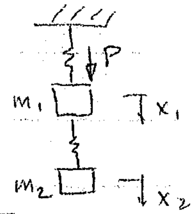
WHEN $r = 1$ RESONANCE

- $m_1\ddot{x}_1 + k_1 x_1 = P \Rightarrow x = X \sin\omega_f t$

$$X = \frac{X_0}{1-r^2} \qquad r = \frac{\omega_f}{\omega} \qquad X_0 = \frac{P_0}{k_1}$$

- IF $r \approx 1$ $X \uparrow \infty$

- SUPPOSE WE LOOK AT



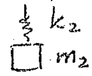
Assume

$x_1 = X_1 \sin \omega_f t$

$x_2 = X_2 \sin \omega_f t$

- WE FOUND
  - IF $r_2 = \frac{\omega_f}{\omega_2} = 1 \left(\omega_2 = \sqrt{\frac{k_2}{m_2}} \right)$ then $X_1 = 0$ $\quad X_2 = -X_0 \frac{k_1}{k_2} = -\frac{P_0}{k_2}$

- THIS ALLOWS FOR DESIGNING OF ABSORBER



  - PICK k_2, m_2 : knowing ω_f, P_0 and X_2 (perhaps allowable travel of mass 2)
  - since

$$k_2 = \frac{P_0}{X_2} \quad \text{put into} \quad \omega_2 = \omega_f = \sqrt{\frac{k_2}{m_2}} \ (r_2 = 1) \implies m_2$$

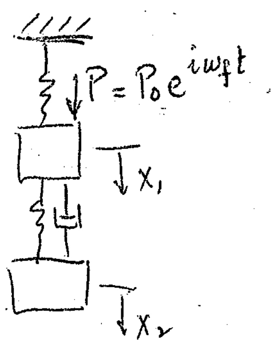
- NORMALLY PICK $k_2 << k_1$ $\quad m_2 << m_1$ $\implies X_2$ will be large since $X_2 = -P_0/k_2$

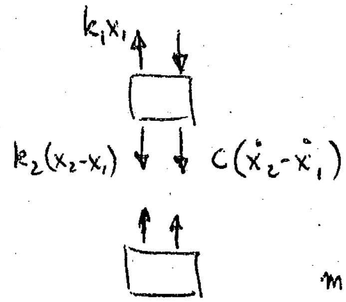
- DISADV. EFFECTIVE ONLY AT $\omega_f = \omega_2$
  - IF ω_f VARIES A LOT $\implies$ WILL CAUSE TROUBLE
  - since $\omega_2 - \omega_1$ small change in ω_f can cause you to approach second resonant condition

- GOOD FOR FIXED SPEED MACHINES BUT NOT AT STARTUP
  - MUST GO THROUGH LOWER RESONANT FREQ TO REACH OPERATING SPD

- CAN DO THE SAME FOR TORSIONAL ABSORBER

- VIBS OF FORCED DAMPED SYSTEM





$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_1) + P$$

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - c(\dot{x}_2 - \dot{x}_1)$$

$m_1\ddot{x}_1 + (k_2+k_1)x_1 + c\dot{x}_1 - k_2x_2 - c\dot{x}_2 = P$

$m_2\ddot{x}_2 + k_2x_2 + c\dot{x}_2 - k_2x_1 - c\dot{x}_1 = 0$

- solution to homog. gives transient ; solution to nonzero rhs leads to steady state

- choose $x_1 = X_1 e^{i(\omega_f t - \psi)} = \bar{X}_1 e^{i\omega_f t}$, $x_2 = X_2 e^{i(\omega_f t - \psi)} = \bar{X}_2 e^{i\omega_f t}$ } like SDOF

$P_0 \cos\omega_f t = Re(\bar{X} e^{i\omega_f t}) = X_1 \cos(\omega_f t - \psi)$

$$\left[-m_1\omega_f^2 + (k_2+k_1) + ic\omega_f\right]\bar{X}_1 - (k_2 + ic\omega_f)\bar{X}_2 = P_0$$
$$\left[-m_2\omega_f^2 + ic\omega_f + k_2\right]\bar{X}_2 - (k_2 + ic\omega_f)\bar{X}_1 = 0$$

Now
$$\bar{X}_1 = \frac{\begin{vmatrix} P_0 & -k_2 + ic\omega_f \\ 0 & -m_2\omega_f^2 + ic\omega_f + k_2 \end{vmatrix}}{\begin{vmatrix} -m_1\omega_f^2 + (k_2+k_1) + ic\omega_f & -(k_2+ic\omega_f) \\ -(k_2+ic\omega_f) & (-m_2\omega_f^2 + ic\omega_f + k_2) \end{vmatrix}} = \frac{A+iB}{C+iD}$$

$$\bar{X}_2 = \frac{\begin{vmatrix} -m_1\omega_f^2 + (k_2+k_1) + ic\omega_f & P_0 \\ -(k_2+ic\omega_f) & 0 \end{vmatrix}}{\text{denom.}}$$

$\left\{ A = -P_0 k_2 \quad B = -P_0 c\omega_f \quad C = \quad D = \right\}$

$A = P_0(k_2 - m_2\omega_f^2) \quad B = P_0 c\omega_f$

$C = (k_1 - m_1\omega_f^2)(k_2 - m_2\omega_f^2) - m_2 k_2 \omega_f^2$

$D = c\omega_f(-m_1\omega_f^2 + k_1 - m_2\omega_f^2)$

$$\tan\psi = \frac{AD - BC}{AC + BD}$$

$a + bi = re^{i\theta}$, $r = \sqrt{a^2+b^2}$, $\tan\theta = b/a$

$$\frac{A+iB}{C+iD} = \frac{\sqrt{A^2+B^2}}{\sqrt{C^2+D^2}} e^{i(\tan^{-1}B/A - \tan^{-1}D/C)}$$

$\tan\eta = \dfrac{-(BC - AD)}{AC+BD}$

$$X_1 = \frac{P_0\sqrt{(k_2 - m_2\omega_f^2)^2 + (c\omega_f)^2}}{\sqrt{\left[(k_1 - m_1\omega_f^2)(k_2 - m_2\omega_f^2) - m_2k_2\omega_f^2\right]^2 + (c\omega_f)^2(m_1\omega_f^2 - k_1 + m_2\omega_f^2)^2}}$$

if $c_{cr} = 2m_2\omega_1$, $\zeta = c/c_{cr}$, $X_0 = P_0/k_1$, $r_1 = \omega_f/\omega_{n_1}$, $r_2 = \omega_f/\omega_{n_2}$, $\mu = \frac{m_2}{m_1}$, $b = \frac{\omega_{n_2}}{\omega_{n_1}}$

$$\frac{X_1}{X_0} = \frac{\sqrt{(r_1^2 - b^2)^2 + (2\zeta r_1)^2}}{\sqrt{(2\zeta r_1)^2(r_1^2 - 1 + \mu r_1^2)^2 + [\mu b^2 r_1^2 - (r_1^2-1)(r_1^2-b^2)]^2}}$$

$$X_2 = \frac{P_0\sqrt{k_2^2 + (c\omega_f)^2}}{\sqrt{}}$$

and $X_2/X_0 = \dfrac{\sqrt{b^4 + (2\zeta r_1)^2}}{\text{denom}}$

if $P = P_0\cos\omega_f t \Rightarrow x_1 = X_1\cos(\omega_f t - \psi_1)$, $x_2 = X_2\cos(\omega_f t - \psi_2)$ }

ψ_1, ψ_2 dependent of A

If $b = 1$ $\mu = 1$ & $r_2 = r_1 = r$

$$\frac{\delta_1}{\delta_0} = \frac{\sqrt{(r^2-1)^2+(2\zeta r)^2}}{\sqrt{(2\zeta r)^2(2r^2-1)^2+(r^4-3r^2+1)^2}}$$

& when $\zeta = 0$ $\quad r^4 - 3r^2 + 1 = 0$ gives r for resonance

$$r^2 = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} = \frac{3 \pm 2.236}{2}$$

$$= \frac{5.236}{2}, \ \frac{.764}{2}$$

$$r^2 = 2.618, \ .382$$

$$r = 1.618 \quad .618$$

$$\frac{\delta_2}{\delta_1} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(2\zeta r)^2(2r^2-1)^2+(r^4-3r^2+1)^2}}$$

when $\zeta \to \infty$ denom $\to (2\zeta r)(2r^2-1)$

num of $\delta_1/\delta_0 \to (2\zeta r)$ num of $\frac{\delta_2}{\delta_0} \to 2\zeta r$

so both $\frac{\delta_1}{\delta_0}, \frac{\delta_2}{\delta_0} \to \frac{1}{|1-2r^2|}$ and this $\to \infty$ if $r = \frac{1}{\sqrt{2}}$

masses locked & act as one

Vibration Analysis by Vierck 2nd ed

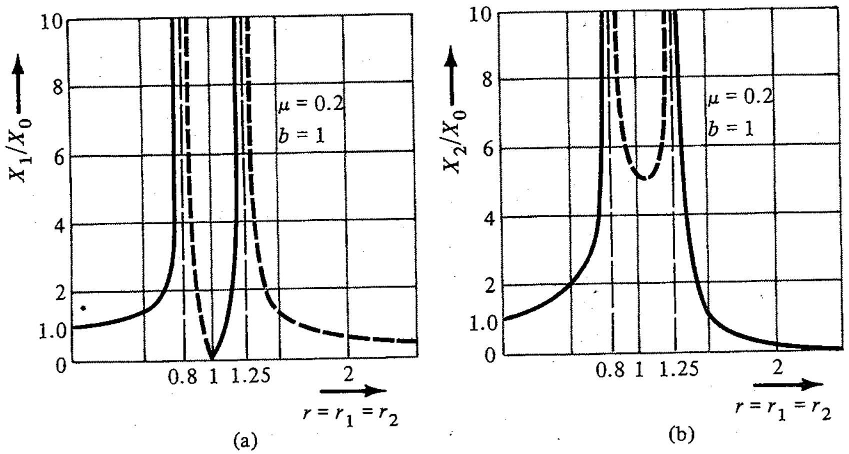


Figure 7-19

frequency in order to reach the operating frequency, and this is an undesirable condition. Also, if loss of load or some other factor results in a gain in speed, difficulty can be encountered at the upper resonant frequency.

For a torsional system composed of a flywheel or disk on a shaft, an absorber consisting of an additional disk and shaft can be employed. The foregoing analysis can then be applied simply by substituting the mass moment of inertia I of the disks for m of the mass elements and using the torsional spring constant K_T for the shaft parts in place of k for the rectilinear spring constants. Such a torsional absorber works satisfactorily, but it is subject to the same limitations as noted above for the rectilinear-system absorber.

EXAMPLE 7-6 A machine weighing 50 lb is supported by a set of four springs each of which has a spring modulus of 15 lb/in. It is subjected to a harmonic force having an amplitude of 3 lb and a frequency of 180 cycles/min. Design a vibration absorber for this system, determining the absorber spring constant and the mass. Clearance limits the absorber amplitude to 0.8 in., and standard springs are available with constants in 1 lb/in. multiples.

SOLUTION The preliminary calculation for k_2 is

$$k_2 = \frac{P_0}{X_2} = \frac{3 \text{ lb}}{0.8 \text{ in.}} = 3.75 \text{ lb/in.}$$

Accordingly, select a spring so that

$$k_2 = 4 \text{ lb/in.}$$

Then

$$X_2 = \frac{P_0}{k_2} = \frac{3 \text{ lb}}{4 \text{ lb/in.}} = 0.75 \text{ in.} < 0.8 \text{ in.}$$

$$m_2 = \frac{k_2}{\omega_f^2} = \frac{4}{(\frac{180}{60} \times 2\pi)^2} = 0.011\,2579 \text{ lb sec/in.}$$

$$W_2 = m_2 g = 0.011\,2579 \times 386 = 4.3455 \text{ lb}$$

EXAMPLE 7-7 Consider an undamped vibration absorber for which $\omega_2 = \omega_1$ and $\mu = 0.2$. Determine the operating range of the r_2 frequency for which $|X_1/X_0| \leq 0.5$.

SOLUTION Equation 7-102 becomes

$$\pm 0.5 = \frac{1 - r_2^2}{r_2^4 - 2.2r_2^2 + 1}$$

As indicated by Fig. 7-19(a), the positive sign will give the upper limiting value for r_2, and the negative sign will yield the lower limit of r_2, for the range extending from somewhat below $r_2 = 1$ to slightly above this value.

For plus 0.5, the above expression can be rearranged as

$$r_2^4 - 0.2r_2^2 - 1 = 0$$

whence

$$r_2^2 = 1.105, \qquad r_2 = 1.051$$

For minus 0.5, the relation becomes

$$r_2^4 - 4.2r_2^2 + 3 = 0$$

whence

$$r_2^2 = 0.9126, \qquad r_2 = 0.9553$$

(There is another root here, for which $r_2 = 1.813$. This defines $X_1/X_0 = -0.5$ at r_2 above the upper resonance point.) Thus the range is less than 10% of the frequency value for which the absorber was designed ($\omega_f = \omega_2$ or $r_2 = 1$).

7-14. VIBRATION OF FORCED DAMPED SYSTEM

Forcing conditions act on damped systems of more than one degree of freedom, for example, as represented in Fig. 7-20. Although a damping element is shown only between the two mass elements, this does not greatly reduce the generality of the conditions or problem. The force P is harmonic

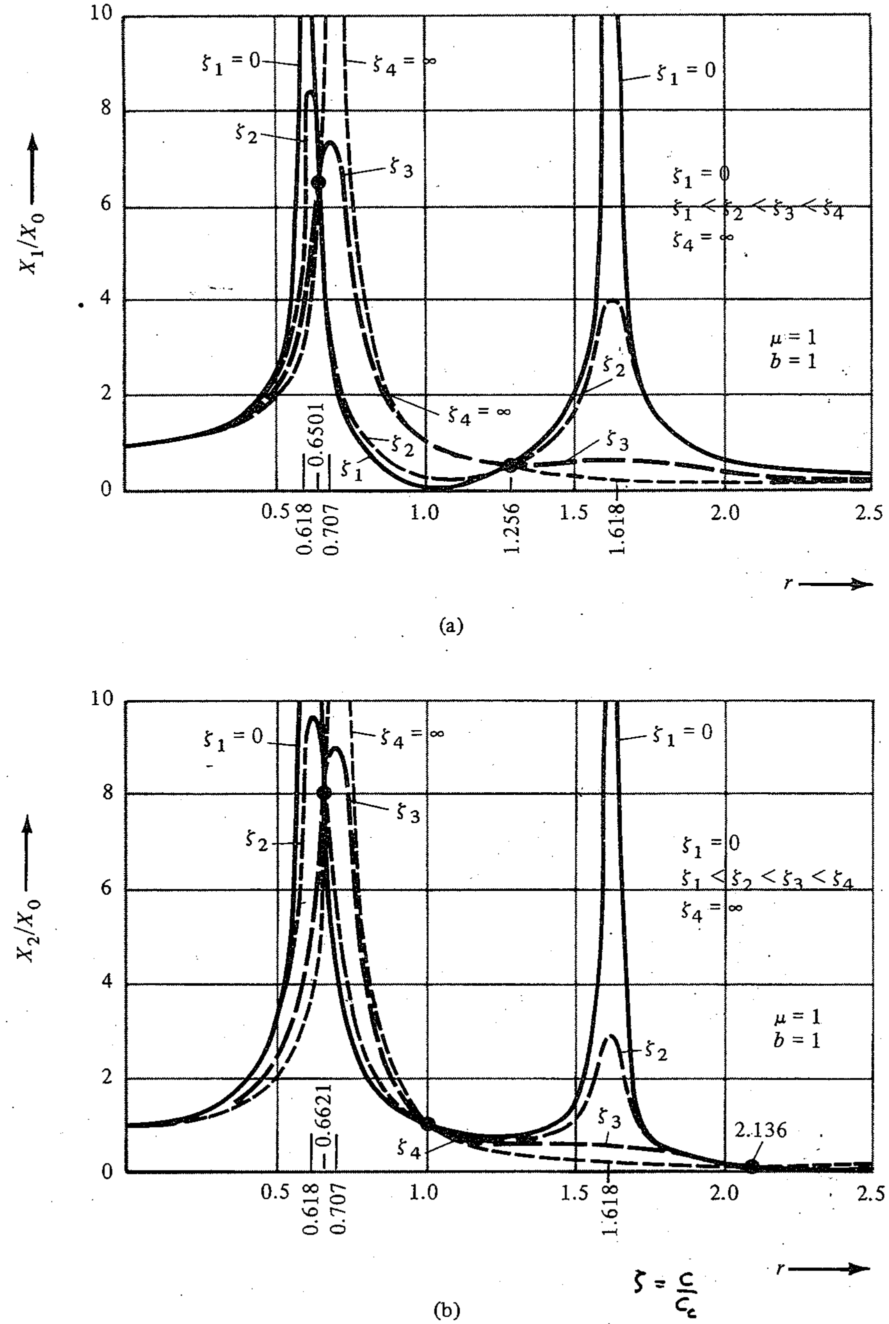


Figure 7-21

$\zeta = \frac{c}{c_c}$

$C_{cr} = 2m_2\omega_1$

$\omega_1 = \sqrt{\frac{k_1}{m_1}}$

have been repeated in Fig. 7-21. For $\zeta = \infty$, Eqs. 7-128 and 7-129 become

$$\frac{X_1}{X_0} = \frac{\pm 1}{1 - 2r^2} = \frac{X_2}{X_0} \tag{7-130}$$

In effect, the two masses are locked, but with a single spring acting, so that $\omega = \sqrt{k/2m}$ and $r = \sqrt{2r_1}$, and Eq. 7-130 thus checks Eqs. 4-16 and 4-17. Resonance then occurs at $r = 0.707$. Curves for this case are also shown in Fig. 7-21. For small damping, the curve should be similar to that for $\zeta_1 = 0$, but with both resonant amplitudes reduced to finite values. This is shown by the curve for ζ_2. Similarly, for moderately large damping, the curve should be similar to that for $\zeta_4 = \infty$, but with the resonant amplitude reduced to a finite amount. A curve of this type is represented by that for ζ_3. Curves such as those for ζ_2 and ζ_3 can be determined from Eqs. 7-128 and 7-129. All the curves have common crossover points, as shown by the circled points in the figures.

The preceding discussions and relations can be used as the basis for the damped vibration absorber since the arrangement contains the same system elements. A thorough analysis of such an absorber is lengthy and detailed and has not been included here.†

7-15. EXCITATION AND STABILITY OF SYSTEMS

For single-degree-of-freedom systems subjected to some form of self-excitation, it is relatively simple to determine stability from the physical constants of the system, as explained in Section 4-19. However, the problem of determining stability becomes more involved with increase in the number of degrees of freedom, and accordingly, it is desirable to ascertain the criteria for stability of such systems.

Consider a damped two-degree-of-freedom system to be subjected to self-excitation which is a linear function of the velocities, that is, for forcing functions of the form $P_1\dot{x}_1$, $P_{12}(\dot{x}_1 - \dot{x}_2)$ and $P_2\dot{x}_2$. Also assume that no other forcing conditions act on the system. The self-exciting functions may be combined with the damping terms, resulting in differential equations of the form

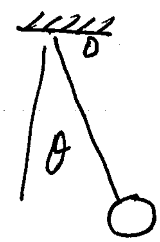
$$\ddot{x}_1 + \alpha_{11}\dot{x}_1 + \alpha_{12}\dot{x}_2 + \beta_{11}x_1 + \beta_{12}x_2 = 0$$
$$\ddot{x}_2 + \alpha_{21}\dot{x}_1 + \alpha_{22}\dot{x}_2 + \beta_{21}x_1 + \beta_{22}x_2 = 0 \tag{7-131}$$

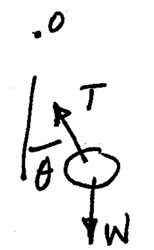
Note that the signs of the α coefficients are not defined here. Substituting in the solutions $x_1 = C_1 e^{st}$ and $x_2 = C_2 e^{st}$ and expanding the determinant of

† See J. P. Den Hartog, *Mechanical Vibrations*, 4th ed. (New York: McGraw-Hill, 1956), pp. 97–105. See also S. P. Timoshenko, D. H. Young, and W. Weaver, Jr., *Vibration Problems in Engineering*, 4th ed. (New York: Wiley, 1974), Example, pp. 273–278.

Example of nonlinear problem with solution

look at the pendulum case





$$\Sigma T_o = I_o \ddot{\theta}$$

$$-Wl\sin\theta = I_o \ddot{\theta} \qquad I_o = ml^2$$

Governing eqn $\ddot{\theta} + g/l \sin\theta = 0$ Nonlinear

if we mult by $\dot{\theta}$ & integrate

$$\frac{\dot{\theta}^2}{2} - g/l \cos\theta = \text{const}$$

if when $\theta = \theta_o$ $\dot{\theta} = \Omega$ @ $t = 0$

$$\text{then } \dot{\theta}^2 = 2g/l \cos\theta + \underbrace{2\left[\frac{\Omega^2}{2} - g/l \cos\theta_o\right]}_{C}$$

$$\dot{\theta} = \sqrt{C + 2g/l \cos\theta}$$

$$\frac{d\theta}{\sqrt{C + \frac{2g}{l}\cos\theta}} = dt \quad \& \quad t = \int_{\theta_o}^{\theta} \frac{d\bar{\theta}}{\sqrt{C + 2g/l \cos\bar{\theta}}}$$

since $\cos\theta = 1 - 2\sin^2\theta/2$

$$t = \int_{\theta_o}^{\theta} \frac{d\bar{\theta}/2}{\sqrt{a^2 - b^2 \sin^2 \bar{\theta}/2}} \quad \text{when } a^2 = \frac{1}{4}\left(C + \frac{2g}{l}\right) \quad b^2 = g/l$$

if $b/a = s < 1$ $\bar{\theta}/2 = X$ $\frac{d\bar{\theta}}{2} = d\bar{x}$

$$t = \frac{1}{a}\int_{x_o}^{x} \frac{d\bar{x}}{\sqrt{1 - s^2 \sin^2 \bar{x}}} \quad \text{or} \quad \frac{1}{a}\left[\int_0^{x} \cdot - \int_0^{x_o} \cdot \right]$$

where $\int_0^{x} \frac{d\bar{x}}{\sqrt{1 - s^2 \sin^2 \bar{x}}}$ elliptic integral of first kind $F(x, s)$

Euler's method (Taylor's Algorithm of order 1)

$$y' = f(x,y).$$

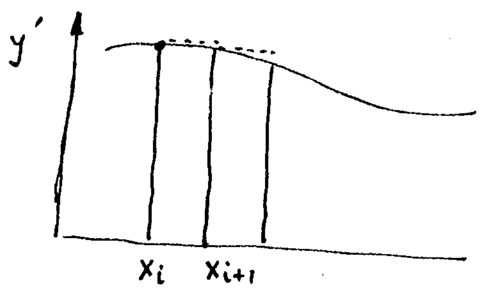
$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} y'\,dx = \int_{x_0}^{x_1} f(x,y)\,dx$$

ASSUME $f(x,y) \cong$ const.

RECTANGULAR AREA UNDER y' vs. X CURVE

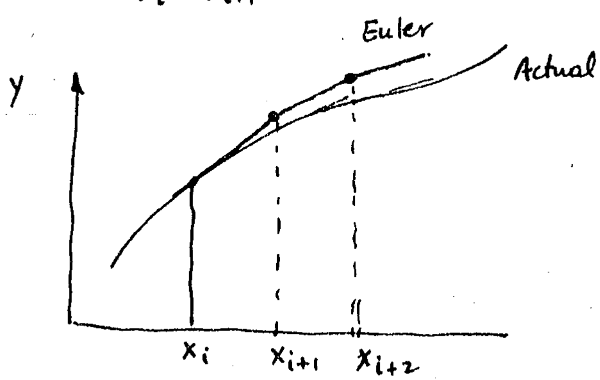
$$y_1 = y_0 + f(x_0,y_0)\,\Delta x = y_0 + y'\big|_{x=x_0} \Delta x$$

$$\therefore \quad y_{i+1} = y_i + y_i'\,\Delta x \qquad y_i' = f(x_i,y_i).$$



SELF STARTING.

ASSUMES $y_i' =$ const.



Error exists since

$$y_1 = y_0 + y_0'\,\Delta x + y''\frac{\Delta x^2}{2} + \dots$$

local error is $O(\Delta x)^2$

global error is $O(\Delta x)$

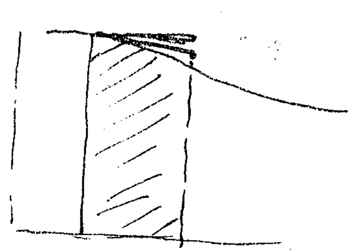
Modified Euler method. (Taylor's Algorithm of order 2)

let $\tilde{y}_{i+1} = y_i + y_i'\,\Delta x$ — let $\tilde{y}_{i+1}$ be the predicted y_{i+1}.

to find the corrected y_{i+1} define $\tilde{y}'_{i+1} = f(x_{i+1}, \tilde{y}_{i+1}) = f(x_i,y_i) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y} y_i' \Delta x$

and then take

$$\frac{\tilde{y}'_{i+1} + y_i'}{2}\,\Delta x + y_i = y_{i+1}.$$



this has an local error of $O(\Delta x^3)$

global error of $O(\Delta x^2)$

Modified Euler is self starting

General Taylor Algorithm of order k.

$$T_k(x,y) = f(x,y) + \frac{\Delta x}{2!} f'(x,y) + \frac{\Delta x^2}{3!} f''(x,y) + \dots + \frac{\Delta x^{k-1}}{k!} f^{(k-1)}(x,y)$$

if we now choose a step size $h = (b-a)/N$

- let $x_0 = a$ $x_i = a + ih$ $y_i \doteq y(x_i)$ $x_N = b$
- Then $y_{i+1} = y_i + hT_k(x_i, y_i)$ $i = 0, 1, 2, \dots, N$

$\therefore$ for $k=1$ $T_1 = f(x,y)$

$k=2$ $T_2 = f(x,y) + \frac{\Delta x}{2!} \underbrace{f'(x,y)}_{df/dx}$

where $f' = f_x + f_y \cdot f = \frac{\partial f}{\partial x} + f \cdot \frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$

THE REASON WHY NONE ARE USED ABOVE $k=2$ IS THE REQUIREMENT OF DEFINING THE DERIVATIVES OF f

error for k^{th} order is $\frac{\Delta x^{k+1}}{(k+1)!} f^{(k)}(x)$ or $\frac{\Delta x^{k+1}}{(k+1)!} y^{k+1}$

WHAT IF you had $y'' = f(x, y, y')$

HOW WOULD THE MODIFIED EULER METHOD LOOK.

$y' = p$

$p' = f(x,y,p)$

$y' = g(x,y,p) = p$

let $\tilde{p} = p_0 + f\,\Delta x$

$\tilde{y} = y_0 + g(\Delta x)$

$\tilde{x} = x_0 + \Delta x$

$\rightarrow$ find $f(\tilde{x}, \tilde{y}, \tilde{p}) = \tilde{p}'$

$g(\tilde{x}, \tilde{y}, \tilde{p}) = \tilde{y}'$

then

$$p_{i+1} = p_i + \frac{p_i' + \tilde{p}_{i+1}'}{2}\Delta x$$

$$y_{i+1} = y_i + \frac{y_i' + \tilde{y}_{i+1}'}{2}\Delta x$$

Non-self starting modified Euler — Multistep

$$\tilde{y}_{i+1} = P(y_{i+1}) = y_{i-1} + 2y_i' \Delta x \qquad \text{where} \quad \frac{dy}{dt} \cong \frac{y_{i+1} - y_{i-1}}{2\Delta} \quad O(\Delta x^2)$$

Then $P(y_{i+1}') = f(x_{i+1}, \tilde{y}_{i+1}) = \tilde{y}_{i+1}'$

then $y_{i+1} = y_i + \frac{\Delta}{2}\left[y_i' + \tilde{y}_{i+1}'\right]$

To start: use self starting modified Euler since it has same error per step

---

For Runge Kutta

$$y_{i+1} = y_i + a_1 k_1 + a_2 k_2$$

where

$$k_1 = h f(x,y)$$

$$k_2 = h f(x + p_1 h, y + q_1 k_1)$$

$$y_{i+1} = y_i + p_1 h \frac{\partial f}{\partial x} + q_1 k_1 \frac{\partial f}{\partial y} + \frac{1}{2}\left[p_1 q_1 h k \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + (p_1 h)^2 \frac{\partial^2 f}{\partial x^2} + (q_1 k)^2 \frac{\partial^2 f}{\partial y^2}\right]$$

(hf written above k_1)

$$a_2 k_2 = a_2 h\left\{f(x,y) + p_1 h \frac{\partial f}{\partial x} + q_1 k_1 \frac{\partial f}{\partial y} + \dots\right\}$$

$$a_1 k_1 = a_1 h f$$

$\therefore a_2 = \frac{1}{2} \quad p_1 = 1 \quad q_1 = 1 \qquad \therefore (a_1 + a_2) h f$

but

$$y_{i+1} = y_i + \frac{dy}{dx} \cdot \Delta x + \frac{d^2 y}{dx^2} \cdot \frac{\Delta x^2}{2} + \dots \quad \text{by taylor series}$$

$$\frac{dy}{dx} = f \qquad \frac{d^2y}{dx^2} = \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right]\frac{\Delta x^2}{2} \qquad \frac{d}{dx} f(x,y)$$

$$\therefore (a_1 + a_2) h f = f \Delta x \quad \text{or} \quad a_1 + a_2 = 1$$

$$\frac{\Delta x^2}{2} f_x = p_1 h$$

Runge-Kutta

$$y_{n+1} = y_n + a_1 k_1 + a_2 k_2$$

where $k_1 = hf(x_n, y_n) = hf_n$

$$k_2 = h f(x_n + \alpha h, y_n + \beta k_1)$$

$f(x+\Delta x, y+\Delta y) = f(x,y) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \text{hot}$

$\Delta x = \alpha h \quad \Delta y = \beta k_1$

$$y_{n+1} = y_n + a_1[hf_n] + a_2 h \left\{ f_n + \frac{\partial f}{\partial x}\Big|_n \alpha h + \frac{\partial f}{\partial y}\Big|_n \beta k_1 \right\}$$

error is O(stepsize³)

$$= y_n + h(a_1 + a_2) f_n + a_2 \alpha h^2 f_x + a_2 \beta h^2 f_n \frac{\partial f}{\partial y} + \cdots$$

$$y_{n+1} = y_n + \frac{dy}{dx} h + \frac{d}{dx}\left(\frac{dy}{dx}\right) \frac{h^2}{2} + \cdots$$

$\frac{dy}{dx} = f$, $\frac{d}{dx} f(x,y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$, with $\frac{dy}{dx} = f$

now $\frac{d^3 y}{dx^3} = \frac{d}{dx}\{f_x + f_y f\} = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)($

$$= f_{xx} + 2f_{xy} f + f_{yy} f^2 + f_y f_x + f$$

$\therefore \quad a_1 + a_2 = 1 \qquad a_2 \alpha = \tfrac{1}{2} \qquad a_2 \beta = \tfrac{1}{2}$

if $\quad a_1 = a_2 = \tfrac{1}{2} \quad \alpha = \beta = 1$

could have picked $a_1 = 3/4 \quad a_2 = 1/4 \quad \alpha = \beta = 2$; $\beta = 3/2 \quad \alpha = 3/2 \quad a_2 = \frac{1}{3} \quad a_1 = 2/3$; $a_2 = 2 \quad a_1 = -1 \quad \beta = \frac{1}{4} \quad \alpha = 1/4$

get modified euler

$$y_{n+1} = y_n + \tfrac{1}{2}(k_1 + k_2)$$

$$k_1 = hf_n = h y_n'$$

$$k_2 = h f(x_n + h, y_n + k_1)$$

$$= h f(\underbrace{x_n + h}_{x_{n+1}}, \underbrace{y_n + hf_n}_{\tilde{y}_{n+1}}) = h \tilde{y}_{n+1}'$$

in general $\quad y_{n+1} = y_n + \sum_{i=1}^{m} a_i k_i$

$$k_i = h f\left(x_n + \alpha_{i-1} h, y_n + \sum_{j=1}^{i-1} \beta_{i-1,j} k_j\right)$$

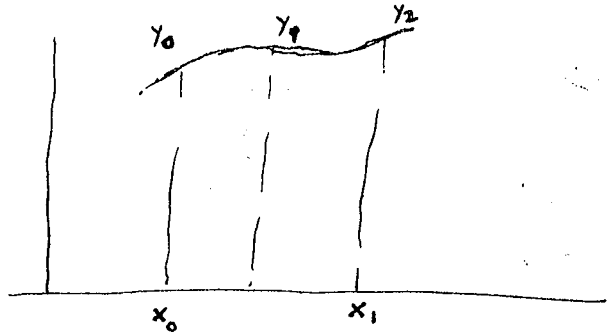
Pick a_i's α_i's & β_{ij}'s in such a manner to meet some $O(h^p)$

The runge kutta technique

assume $y_{i+1} = y_i + \sum a_j k_j$ such that it matches

~~such that~~ $y_{i+1} = y_i + y_i' \Delta x + y_i'' \frac{\Delta x^2}{2}$

$y' = f(x,y)$ $y_{i+1} = y_i + \int f(x,y)\,dx$



Using $\int f(x,y)\,dx$ as being simpson's rule * $\frac{1}{6}\left[k_0 + 4k_1 + k_2\right]$

† * $k_0 = hf(x_0, y_0)$

† * $k_1 = hf(x_0 + \frac{\Delta x}{2}, y_0 + \frac{k_0}{2})$ $\Delta y = k_0 \Delta x$

$k_3 = hf(x_0 + \Delta x, y_0 + k_0)$

$k_2 = hf(x_0 + \Delta x \quad y_0 + k_3)$

this is of local error $O(\Delta x^4)$ global $O(\Delta x^3)$

if however we defined * $k_2 = hf\{x_0 + \Delta x, y_0 + (k_0 - 2k_1)\}$ Wolford original edition

then the results are of local error $O(\Delta x^4)$.   global $O(\Delta x^3)$

Using † $\frac{1}{6}\left[k_0 + 2k_1 + 2k_2 + k_3\right]$

† $k_2 = hf(x_0 + \frac{\Delta x}{2}, y_0 + \frac{k_1}{2})$

† $k_3 = hf(x_0 + \Delta x, y_0 + k_2)$ local error $O(\Delta x^5)$

global error $O(\Delta x^4)$

To determine accuracy, we may divide stepsize in half & redo calculations to the same x_n

If results are negligible continue to march; otherwise reduce stepsize in half & redo calculations

1) expensive in computational effort
2) not a good idea to constantly change stepsize

Runge-Kutta-Fehlberg (RKF) provides w/ same computations $O(h^5)$ & $O(h^6)$ methods & also the error estimate as well.

$$k_1 = h f(x_n, y_n)$$
$$k_2 = h f(x_n + h/4, y_n + k_1/4)$$
$$k_3 = h f(x_n + 3h/8, y_n + \tfrac{3}{32}(k_1 + 3k_2))$$
$$k_4 = h f(x_n + 12h/13, y_n + \tfrac{1}{2197}(1932k_1 - 7200k_2 + 7296k_3))$$
$$k_5 = h f(x_n + h, y_n + \tfrac{439k_1}{216} - 8k_2 + 3680k_3/513 - 845k_4/4104)$$
$$k_6 = h f(x_n + h/2, y_n - 8k_1/27 + 2k_2 - 3544k_3/2565 + 1859k_4/4104 - 11k_5/40)$$

$$\hat{y}_{n+1} = y_n + (25k_1/216 + 1408k_3/2565 + 2197k_4/4104 - k_5/5) \qquad \text{local err } O(h^5)$$
$$y_{n+1} = y_n + (16k_1/135 + 6656k_3/12825 + 28561k_4/56430 - 9k_5/50 + 2k_6/55) \qquad \text{local err } O(h^6)$$
$$E = k_1/360 - 128k_3/4275 - 2197k_4/75240 + k_5/50 + 2k_6/55$$

We compare y_{n+1} using two different orders RK formulas instead of halving stepsize that use same k's

Also we get error estimate at same time

$$y'' = f(x, y, y')$$

$$y' = p \longrightarrow y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} p\,dx$$
$$y'' = p' = f(x, y, y') \longrightarrow p_{i+1} = p_i + \int_{x_i}^{x_{i+1}} f(\bar{x}, y, p)\,d\bar{x}$$

$$\int p\,dx = \tfrac{1}{6}(q_0 + 2q_1 + 2q_2 + q_3) \qquad \begin{aligned} q_0 &= \Delta x(p_i) \\ q_1 &= \Delta x(p_i + q_0/2) \\ q_2 &= \Delta x(p_i + q_1/2) \\ q_3 &= \Delta x(p_i + q_2) \end{aligned}$$

over

$$y_{i+1} = y_i + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$$

$$k_0 = \Delta x\, f(x_i, y_i, p_i)$$

$$k_1 = \Delta x\, f(x_i + \frac{\Delta x}{2}, y_i + \frac{q_0}{2}, p_i + \frac{k_0}{2})$$

$$k_2 = \Delta x\, f(x_i + \frac{\Delta x}{2}, y_i + \frac{q_1}{2}, p_i + \frac{k_1}{2})$$

$$k_3 = \Delta x\, f(x_i + \Delta x, y_i + q_2, p_i + k_2)$$

↑ $q's$ ↑ $k's$

$$M\dot{d}_n + Kd_n = F_n \quad \Rightarrow \quad Mv_{n+1} + Kd_{n+1} = F_{n+1}$$

if $\alpha = 0$ $\quad v_{n+\alpha} = (1-\alpha)v_n + \alpha v_{n+1} \quad \Rightarrow \quad v_n = v_n$

$$d_{n+1} = d_n + \Delta t\, v_n \quad \text{or} \quad v_n = \frac{d_{n+1} - d_n}{\Delta t} \cong \dot{d}_n$$

$$M[v_{n+1}] + K[d_n + \Delta t\, v_{n+\alpha}]$$

$$M[v_{n+1}] + Kd_n + K\Delta t[(1-\alpha)v_n + \alpha v_{n+1}] = F_{n+1}$$

$$[M + \alpha\Delta t K]v_{n+1} + K[d_n + \Delta t(1-\alpha)]v_n = F_{n+1}$$

REMARKS:

1.) $\underset{\sim}{M}$ (AND $\underset{\sim}{\bar{M}}$) IS SYMMETRIC POSITIVE DEFINITE AND HAS THE SAME BAND PROFILE AS $\underset{\sim}{K}$

2.) IT IS COMMON PRACTICE TO DIRECTLY SPECIFY $\underset{\sim}{d}_0$ THUS AVOIDING AN ADDITIONAL MATRIX SOLUTION.

3.) RECALL $p = N_{ed}(a-1) + i$, FOR HEAT $N_{ed} = 1$ THUS $i = 1$

$\therefore p = (a-1) + 1 = a$

ONE-STEP ALGORITHMS FOR THE SEMI-DISCRETE HEAT EQUATION: GENERALIZED TRAPAZOIDAL METHOD

GIVEN: (THE FOLLOWING EQ.'S DERIVED IN (M))

F.D.

$$\underset{\sim}{M}\dot{\underset{\sim}{d}} + \underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F} \qquad \text{if } \dot{\underset{\sim}{d}} \Rightarrow d_{n+1} = d_n + \Delta t\, \dot{d}_n \quad (1)$$

$$\underset{\sim}{d}(0) = \underset{\sim}{D}_0 \quad (2)$$

WHERE:

$\underset{\sim}{M}$ ... CAPACITY MATRIX; SYMM. POS. DEF.

$\underset{\sim}{K}$ ... CONDUCTIVITY MATRIX, SYMM. POS. SEMI-DEF.

$\underset{\sim}{F} = \underset{\sim}{F}(t) \quad t \in [0,T]$ ... HEAT SUPPLY VECTOR (PRESC.)

$\underset{\sim}{d}$ ... TEMPERATURE VECTOR.

$\dot{\underset{\sim}{d}}$ ... TIME (t) DERIVATIVE OF $\underset{\sim}{d}$.

$\underset{\sim}{D}_0$ ... GIVEN INITIAL TEMPERATURE.

APPLYING THE GENERALIZED TRAPAZOIDAL METHOD FOR FIRST ORDER SYSTEMS TO (1), WE GET

$$\underset{\sim}{M}\underset{\sim}{v}_{n+1} + \underset{\sim}{K}\underset{\sim}{d}_{n+1} = \underset{\sim}{F}_{n+1} \quad (3)$$

$$\underset{\sim}{d}_{n+1} = \underset{\sim}{d}_n + \Delta t\, \underset{\sim}{v}_{n+\alpha} \quad \leftarrow \dot{d} \text{ eqn} \quad (4)$$

$$\underset{\sim}{v}_{n+\alpha} = (1-\alpha)v_n + \alpha\, \underset{\sim}{v}_{n+1} \quad (5)$$

$$\underset{\sim}{d}_{n+1} = \underset{\sim}{d}_n + \Delta t\,[1-\alpha]\dot{d}_n + \Delta t\, \alpha\, \dot{d}_{n+1}$$

NOTATION:

n_{ts} ... NUMBER OF TIME STEPS

$\Delta t = T/n_{ts}$ ... TIME STEP (ASSUMED CONSTANT)

$t_n = n\Delta t$ ... TIME n

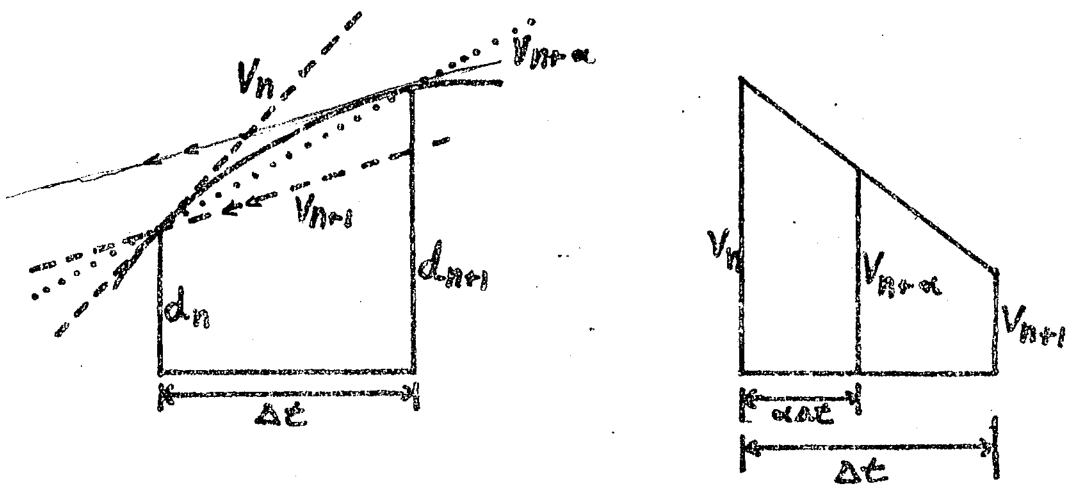
n ... INTEGER STEP NUMBER ($0 \le n \le n_{ts}$)

$d_n \approx d(t_n)$ ... APPROX. TEMP.

$v_n \approx \dot{d}(t_n)$ ... APPROX. TIME DERIVATIVE OF TEMP.

$\alpha \in [0, 1]$ ... PARAMETER

ILLUSTRATION OF $v_{n+\alpha}$ (1 dof)



SOME WELL KNOWN MEMBERS OF THE GENERALIZED TRAPAZOIDAL FAMILY,

α	METHOD
0	FORWARD DIFFERENCES; FORWARD EULER
1/2	TRAPAZOIDAL RULE; MIDPOINT RULE; CRANK-NICOLSON
2/3	GALERKIN
1	BACKWARD DIFFERENCES; BACKWARD EULER

REMARKS:

1.) FOR $\alpha = 0$ THE METHOD IS SAID TO BE EXPLICIT. IF $\alpha = 0$ AND $\underset{\sim}{M}$ IS "LUMPED" (DIAGONAL) THEN IT CAN BE SEEN (3)-(5) THAT THE SOLUTION CAN BE ADVANCED WITHOUT THE NECESSITY OF SOLVING MATRIX EQUATIONS.

2) FOR $\alpha \neq 0$ THE METHOD IS SAID TO BE IMPLICIT. IF $\alpha \neq 0$ A SYSTEM OF EQUATIONS WITH COEFFICIENT MATRIX $\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K}$ MUST BE SOLVED AT EACH STEP k for $\underset{\sim}{v}_k$ for $\underset{\sim}{v}_k$ then put into two formulas of pg 9 to get $\underset{\sim}{d}_k$

DEFINITION: CONVERGENCE

WE CALL AN ALGORITHM CONVERGENT IF FOR EACH FIXED t_n, $\Delta t = t_n / n$

$$\underset{\sim}{d}_n \rightarrow \underset{\sim}{d}(t_n) \text{ AS } \Delta t \rightarrow 0 \quad (n \rightarrow \infty)$$

TO BE SHOWN: STABILITY + CONSISTENCY $\Rightarrow$ CONVERGENCE

IN ORDER TO STUDY THE CHARACTERISTICS OF A TIME INTEGRATION ALGORITHM (i.e ITS STABILITY, CONSISTENCY, ACCURACY AND CONVERGENCE) WE WILL EMPLOY THE MODAL APPROACH (ALSO CALLED SPECTRAL OR FOURIER ANALYSIS)

MODAL ANALYSIS: REDUCTION TO A SINGLE DEGREE-OF-FREEDOM (SDOF) MODEL EQUATION

WE WILL NOW DECOMPOSE (1) INTO n_{eq} UNCOUPLED SCALAR EQUATIONS. AND THUS THE ANALYSIS OF THE ALGORITHM WILL REDUCE TO THE ANALYSIS OF A SIMPLE SINGE-DEGREE-OF FREEDOM CASE.

let $\dot{d} = v$

$$M v + K d = F$$

then $M v_{n+1} + K d_{n+1} = F_{n+1}$

for $d_{n+1} = d_n + \Delta t\, v_n$ forward diff $\alpha = 0$ $\quad \frac{d_{n+1} - d_n}{\Delta t} = v_n$

$v_{n+\alpha} = (1-\alpha) v_n + \alpha v_{n+1} \Rightarrow v_n = 1 \cdot v_n + 0 \cdot v_{n+1}$

for $d_n = d_{n+1} - \Delta t\, v_{n+1}$ backward diff $\alpha = 1 \Rightarrow d_{n+1} = \Delta t\, v_{n+1} + d_n$; $\frac{d_{n+1} - d_n}{\Delta t}$

$v_{n+\alpha} = (1-\alpha) v_n + \alpha v_{n+1} \Rightarrow v_{n+1} = 0 \cdot v_n + 1 \cdot v_{n+1}$

for $v_{n+1/2}$ $\quad \frac{v_{n+1} + v_n}{2} = \alpha v_{n+1} + (1-\alpha) v_{n+1}$ centered diff

$d_{n+1} = d_n + \Delta t\, v_{n+1/2} \longleftarrow v_{n+1/2} = \frac{d_{n+1} - d_n}{2 \cdot \Delta t/2}$

proof $d_{n+1} = d_{n+1/2} + \frac{\Delta t}{2} v_{n+1/2}$

$d_n = d_{n+1/2} - \frac{\Delta t}{2} v_{n+1/2}$; $\quad d_{n+1} - d_n = \Delta t\, v_{n+1/2}$

thus $d_{n+1} = d_n + \Delta t\, v_{n+\alpha} = d_n + \Delta t \{ (1-\alpha) v_n + \alpha v_{n+1} \}$

if $\alpha = 0$ explicit & M is diag (lumped) then from the above $d_{n+1} = d_n + \Delta t\, v_n$ ∴

$M v_{n+1} = F_{n+1} - K (d_n + \Delta t\, v_n)$ = known rhs ∴ $(v_i)_{n+1} = (F_{n+1} - K d_{n+1})_i / [M_{ii}]$

if $\alpha \neq 0$ then since K is not diag then $\{ M + \alpha \Delta t K \} v_{n+1} = F_{n+1} - K \{ d_n + \Delta t (1-\alpha) v_n \}$

this is implicit (not diag) (known rhs)

solve for v_{n+1}; put into $v_{n+\alpha} = (1-\alpha) v_n + \alpha v_{n+1}$ to get $v_{n+\alpha}$; put into $d_{n+1} = d_n + \Delta t\, v_{n+\alpha}$ to get d_{n+1}

Now $\alpha M v_{n+1} + \alpha K d_{n+1} = \alpha F_{n+1}$

$(1-\alpha) M v_n + K (1-\alpha) d_n = (1-\alpha) F_n$

add $M \left\{ \frac{d_{n+1} - d_n}{\Delta t} \right\} + K \{ (1-\alpha) d_n + \alpha d_{n+1} \} = \alpha F_{n+1} + (1-\alpha) F_n$ (with $v_{n+\alpha}$ over the first bracket)

∴ $(M + \alpha K \Delta t)\, d_{n+1} = \{ M - \Delta t K (1-\alpha) \} d_n + \Delta t\, F_{n+\alpha}$

$F_{n+\alpha} = \alpha F_{n+1} + (1-\alpha) F_n$ $\quad$ **let $d_n = \sum d_{\ell_n} \cdot \psi_\ell$ & $d_{n+1} = \sum d_{\ell_{n+1}} \psi_\ell$**

Now let ψ_ℓ^T be applied to both sides of eqn w/ $d = d_m^T \psi_m$ & $\psi_\ell^T M \psi_m = \delta_{\ell m}$ $\psi_\ell^T K \psi_m = \lambda$

then $[1 + \alpha \lambda_\ell \Delta t]\, d_{\ell_{n+1}} = [1 - \Delta t \lambda_\ell (1-\alpha)]\, d_{\ell_n} + \Delta t\, \psi_\ell^T F_{n+\alpha}$

w/ $d_\ell(0) = \psi_\ell^T M d(0)$

The matrix formulation then becomes:

Find: $\underset{\sim}{d} : [0,T] \rightarrow \mathbb{R}^{n_{eq}}$ $\ni$.

$$\underset{\sim}{M}\ddot{\underset{\sim}{d}} + \underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F}(t)$$

where $\underset{\sim}{K}$ is same as before for elastostatic

where $\underset{\sim}{M} = \overset{n_{el}}{\underset{e=1}{A}}(\underset{\sim}{m}^e)$ $\quad \underset{\sim}{m}^e = [m^e_{pq}]$ $\quad p = n_{ed}(a-1)+i$ $\quad q = n_{ed}(b-1)+j$ $\quad n_{ed} = n_{sd}$

$$[m^e_{pq}] = \int_{\Omega^e} \rho N_a N_b \delta_{ij}\, d\Omega$$

now $\underset{\sim}{f}^e = \{f^e_p\}$; $f^e_p = \int_{\Omega^e} N_a f_i(t)\, d\Omega + \int_{\Gamma^e_h} N_a h_i\, d\Gamma$

$$- \sum_{q=1}^{n_{ee}} \left(k^e_{pq} g^e_q + m^e_{pq} \ddot{g}^e_q \right)$$

Now the IC's are that

$$\underset{\sim}{d}(0) = \underset{\sim}{d}_0 \; ; \quad \dot{\underset{\sim}{d}}(0) = \dot{\underset{\sim}{d}}_0$$

the discreting is for the IC's

$$\underset{\sim}{d}_0 = \bar{\underset{\sim}{M}}^{-1}\left\{ \overset{n_{el}}{\underset{e=1}{A}} (\bar{\underset{\sim}{d}}^e) \right\} \quad \bar{\underset{\sim}{M}} = A(\bar{\underset{\sim}{m}}^e) \quad \bar{m}^e_{pq} = \int_{\Omega^e} N_a N_b \delta_{ij}\, d\Omega$$

$$\bar{\underset{\sim}{d}}^e = \{\bar{d}^e_p\} \quad \bar{d}^e_p = \int_{\Omega^e} N_a u_{0i}\, d\Omega - \sum_q \bar{m}^e_{pq} g^e_q(0)$$

$$\dot{\underset{\sim}{d}}_0 = \bar{\underset{\sim}{M}}^{-1}\left\{ \overset{n_{el}}{\underset{e=1}{A}} (\dot{\bar{\underset{\sim}{d}}}^e) \right\}$$

where $\dot{\bar{d}}^e_p = \int_{\Omega^e} N_a \dot{u}_{0i}\, d\Omega - \sum \bar{m}^e_{pq} \dot{g}^e_q(0)$

before we had

$$\underset{\sim}{M}\ddot{\underset{\sim}{d}} + \underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F}(t)$$

and IC

now let's add a viscous damping matrix

(*) $\underset{\sim}{M}\ddot{\underset{\sim}{d}} + \underset{\sim}{C}\dot{\underset{\sim}{d}} + \underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F}(t)$ where $\underset{\sim}{C}$ is a viscous damping matrix

if $\underset{\sim}{K} = 0$ then we really have a 1st order system on $\dot{\underset{\sim}{d}}$ but this also leads to nonlinear analysis also. We will now look at integrators of the "NEWMARK Family" ie "one-step" algorithm.

We can write (*) as 3 eqns

$$\underset{\sim}{M}\underset{\sim}{a}_{n+1} + \underset{\sim}{C}\underset{\sim}{v}_{n+1} + \underset{\sim}{K}\underset{\sim}{d}_{n+1} = \underset{\sim}{F}_{n+1} = \underset{\sim}{F}(t_{n+1})$$

where $\underset{\sim}{a}_n \approx \underset{\sim}{a}(t_n)$ $\quad \underset{\sim}{v}_n \approx \underset{\sim}{v}(t_n)$ $\quad \underset{\sim}{d}_n \approx \underset{\sim}{d}(t_n)$

$\approx \ddot{\underset{\sim}{d}}(t_n)$ $\quad \approx \dot{\underset{\sim}{d}}(t_n)$

if linear accel

$a_n + (a_{n+1} - a_n)\frac{(t-t_n)}{\Delta t}$

if $\dot v = a$ & $a = a_0 + a_1 t$

$\frac{V_{n+1} - V_n}{\Delta t} = a_0 + a_1 t$

$V_{n+1} = V_n + \int_{t_n}^{t_{n+1}} a_0 + a_1 t \, dt$

$= V_n + \left(a_0 t + a_1 \frac{t^2}{2}\right)\Big|_{t_n}^{t_{n+1}}$

$a_0 \Delta t + \frac{a_1}{2}(\Delta t)(t_{n+1}+t_n)$

$\int a_n + \frac{\Delta a}{\Delta t}(t-t_n)$

$a_n t + \frac{\Delta a}{\Delta t}\frac{(t-t_n)^2}{2}\Big|_{t_n}^{t_{n+1}}$

$a_n \Delta t + \frac{\Delta a}{2\Delta t}[\Delta t^2] = a_n \Delta t + \Delta a \frac{\Delta t}{2}$

$v_{n+1} - v_n = a_n + \frac{a_{n+1} - a_n}{2} = \frac{a_{n+1} + a_n}{2}\Delta t$

$\dot d = v_n + \Delta t\, a_n + \dots$

$d_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2} a_n$

$\beta = 0 \quad \gamma = 1/2$

$v_{n+1} = v_n + \Delta t \frac{(a_n + a_{n+1})}{2}$

$d_{n+2} = d_{n+1} + \Delta t\, v_{n+1} + \frac{\Delta t^2}{2} a_{n+1}$

$d_{n+2} - d_{n+1} = d_{n+1} - d_n + \Delta t(v_{n+1} - v_n) + \frac{\Delta t^2}{2}(a_{n+1} - a_n)$

$\uparrow \; \Delta t\left(\frac{a_n + a_{n+1}}{2}\right)$

$d_{n+2} - 2d_{n+1} + d_n = \Delta t^2 a_{n+1}$ ✓

$\therefore \quad \frac{d_{n+1} - 2d_n + d_{n-1}}{2} = \frac{\Delta t^2}{2} a_n$

since $d_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2} a_n$

$= d_n + \Delta t\, v_n + \frac{d_{n+1}}{2} - d_n + \frac{d_{n-1}}{2}$

$\frac{d_{n+1} - d_{n-1}}{2} = \Delta t\, v_n$ ✓

$v_n = v_{n+1} - \left[\frac{a_{n+1} + a_n}{2}\right]\Delta t$

$d_{n+1} = d_n + \dots$

$a_n \frac{t^2}{2} + \frac{\Delta a}{\Delta t}\frac{(t-t_n)^3}{6}\Big|_{t_n}^{t_{n+1}}$

$a_n[\Delta t]\frac{t_{n+1}+t_n}{2} + \frac{\Delta a}{\Delta t}\cdot\frac{\Delta t^3}{6}$

$a_n\left[\frac{\Delta t^2}{2}\right] + a_n t_n + \Delta a \cdot \frac{\Delta t^2}{6}$

These are Taylor series approx.

$$\underset{\sim}{d}_{n+1} = \underset{\sim}{d}_n + \Delta t\, \underset{\sim}{v}_n + \frac{\Delta t^2}{2}\left\{(1-2\beta)\underset{\sim}{a}_n + 2\beta \underset{\sim}{a}_{n+1}\right\}$$

$$\underset{\sim}{v}_{n+1} = \underset{\sim}{v}_n + \Delta t\left\{(1-\gamma)\underset{\sim}{a}_n + \gamma \underset{\sim}{a}_{n+1}\right\}$$

where β is a parameter, " γ " a " } they control accuracy & stability of algorithm

For the algorithmic eqns suppose we are given everything at $\underset{\sim}{d}_n, \underset{\sim}{v}_n, \underset{\sim}{a}_n$ and we want to find $\underset{\sim}{a}_{n+1}, \underset{\sim}{d}_{n+1}, \underset{\sim}{v}_{n+1}$

for the initial step.

$$\underset{\sim}{M}\underset{\sim}{a}_0 = \underset{\sim}{F}_0 - \underset{\sim}{C}\underset{\sim}{v}_0 - \underset{\sim}{K}\underset{\sim}{d}_0 \quad \text{now this is known.}$$

Now look at members of newmark family

Ex. 1. suppose we look at the (constant) average accel. implicit method $\beta = 1/4$ $\gamma = 1/2$

this is equiv to applying trapez rule to first order form of $M\ddot{d} + C\dot{d} + Kd$

what is first order form.

let $\underset{\sim}{y} = (\underset{\sim}{d}\ \dot{\underset{\sim}{d}})^T$ let $\underset{\sim}{f} = \underset{\sim}{f}(\underset{\sim}{y}, t) = \begin{Bmatrix} \dot{d} \\ M^{-1}(F(t) - C\dot{d} - Kd) \end{Bmatrix}$

Thus $\dot{\underset{\sim}{y}} = \underset{\sim}{f}(\underset{\sim}{y}, t)$ the first order form. Now apply trapez form.

now $\underset{\sim}{z}_n \approx \dot{\underset{\sim}{y}}(t_n) \implies \underset{\sim}{y}_{n+1} = \underset{\sim}{y}_n + \frac{\Delta t}{2}(\underset{\sim}{z}_n + \underset{\sim}{z}_{n+1})$ trapez. rule $\quad \underset{\sim}{y}_n \approx \underset{\sim}{y}(t_n)$

$\underset{\sim}{z}_{n+1} = \underset{\sim}{f}(\underset{\sim}{y}_{n+1}, t_{n+1})$ trap is uncond. stable & 2nd order accurate.

exercise show that the above integrator is the newmark member w/ $\beta = 1/4$ $\gamma = 1/2$.

Ex. 2. if we use linear accel method $\beta = 1/6$, $\gamma = 1/2$ conditionally stable, an 2nd order

this is an implicit method.

what is implicit or explicit

implicit is a matrix problem needs to be solved to advance sol. from step to step.

explicit – no matrix problem needs to be solved.

the conditionally stable system is dependent on the highest e.value.

Ex. 3. Central difference method $\beta = 0$ $\gamma = 1/2$

ie $\underset{\sim}{v}_n = (\underset{\sim}{d}_{n+1} - \underset{\sim}{d}_{n-1})/2\Delta t$

$\underset{\sim}{a}_n = (\underset{\sim}{d}_{n+1} - 2\underset{\sim}{d}_n + \underset{\sim}{d}_{n-1})/\Delta t^2$

2nd order accurate
explicit; conditionally stable
attractive due to 2nd order & explicit

$$\underset{\sim}{M}\underset{\sim}{a}_{n+1} + C\underset{\sim}{v}_n + C\Delta t(1-\gamma)\underset{\sim}{a}_n + C\gamma\Delta t\, \underset{\sim}{a}_{n+1} + K\underset{\sim}{d}_n + K\Delta t\,\underset{\sim}{v}_n + K\frac{\Delta t^2}{2}\left[(1-2\beta)\underset{\sim}{a}_n + 2\beta\, a_{n+1}\right]$$
$$= \underset{\sim}{F}_{n+1}$$

$$\left(\underset{\sim}{M} + \underset{\sim}{C}\gamma\Delta t + 2\underset{\sim}{K}\beta\frac{\Delta t^2}{2}\right)\underset{\sim}{a}_{n+1} + \underset{\sim}{a}_n\left(K\frac{\Delta t^2}{2}(1-2\beta) + C\Delta t(1-\gamma)\right) + K\underset{\sim}{d}_n + K\Delta t\,\underset{\sim}{v}_n$$

$$M\dot{\underset{\sim}{v}} + C\underset{\sim}{v} + K\underset{\sim}{d} =$$

$$\overset{\underset{\sim}{y}_{n+1}}{\begin{pmatrix} d_{n+1} \\ \dot{d}_{n+1}\end{pmatrix}} = \overset{\underset{\sim}{y}_n}{\begin{pmatrix} d_n \\ \dot{d}_n\end{pmatrix}} + \frac{\Delta t}{2}\begin{bmatrix} \dot{d}_n + \dot{d}_{n+1} \\ M^{-1}\left[F_n - C\dot{d}_n - Kd_n\right] + M^{-1}\left[F_{n+1} - C\dot{d}_{n+1} - Kd_{n+1}\right]\end{bmatrix}$$

(where $\underset{\sim}{z}_n = \underset{\sim}{f}(\underset{\sim}{y}_n, t_n)$, $\underset{\sim}{z}_{n+1} = \underset{\sim}{f}(\underset{\sim}{y}_{n+1}, t_{n+1})$)

1st eqn $\quad d_{n+1} = d_n + \frac{\Delta t}{2}(\dot{d}_n + \dot{d}_{n+1})$

2nd eq $\quad \dot{d}_{n+1} = \dot{d}_n + \frac{\Delta t}{2}M^{-1}\left\{(F_n + F_{n+1}) - \underset{\sim}{C}(\dot{d}_n + \dot{d}_{n+1}) - \underset{\sim}{K}(d_n + d_{n+1})\right\}$

use original eqn. $\quad \rightarrow = M(\underset{\sim}{a}_{n+1} + \underset{\sim}{a}_n) + \underset{\sim}{C}(\dot{d}_{n+1} + \dot{d}_n) + \underset{\sim}{K}(d_{n+1} + d_n)$

to lead to $\quad \dot{d}_{n+1} = \dot{d}_n + \frac{\Delta t}{2}\{\underset{\sim}{a}_{n+1} + \underset{\sim}{a}_n)$ ie $\gamma = 1/2$

take this & subst in 1st eq. for $\dot{d}_{n+1}$

$$d_{n+1} = d_n + \frac{\Delta t}{2}\left\{\dot{d}_n + \dot{d}_n + \frac{\Delta t}{2}(\underset{\sim}{a}_{n+1} + \underset{\sim}{a}_n)\right) \text{ to get}$$
$$= d_n + \Delta t\,\dot{d}_n + \frac{\Delta t^2}{4}(\underset{\sim}{a}_{n+1} + \underset{\sim}{a}_n) \quad \text{if } \beta = 1/4 \quad 1-2\beta = 1/2 \quad 2\beta = 1/2$$

---

if $\quad \underset{\sim}{y}_{n+1} = \underset{\sim}{y}_n + \Delta t\,\underset{\sim}{z}_{n+1/2}$

$$\begin{pmatrix} d_{n+1} \\ \dot{d}_{n+1}\end{pmatrix} = \begin{pmatrix} d_n \\ \dot{d}_n\end{pmatrix} + \Delta t\begin{pmatrix} \dot{d}_{n+1/2} \\ M^{-1}\left[F_{n+1/2} - C\dot{d}_{n+1/2} - Kd_{n+1/2}\right]\end{pmatrix}$$

$$d_{n+1} = d_n + \Delta t\,\dot{d}_{n+1/2}$$

~~$\dot{d}_{n+1} = \dot{d}_n + \Delta t\{M^{-1}[F_{n+1/2} - C\dot{d}_{n+1/2} - Kd_{n+1/2}]\} = \dot{d}_n + \Delta t\, a_{n+1/2}$ $\gamma = 0$~~

~~$d_{n+1} = d_n + \Delta t[\dot{d}_n + \Delta t\, a_{n+1}] = d_n + \dot{d}_n\Delta t + \Delta t^2 a_{n+1}$ $\beta =$~~

~~$\underset{\sim}{y}_{n+1} = \underset{\sim}{y}_n + \Delta t$~~

$$\dot{d}_{n+1} = \dot{d}_n + \Delta t\, a_{n+1/2} \qquad \gamma = 1/2$$
$$= \dot{d}_n + \Delta t\left(\tfrac{1}{2}\underset{\sim}{a}_n + \tfrac{1}{2}a_{n+1}\right)$$
$$\dot{d}_{n+1/2} = \tfrac{1}{2}\dot{d}_n + \tfrac{1}{2}\dot{d}_{n+1}$$

if $\underset{\sim}{C} = 0$ and $\underset{\sim}{M}$ is diag: $\underset{\sim}{M}\, \underset{\sim}{a}_{n+1} + \underset{\sim}{K}\, d_{n+1} = \underset{\sim}{F}_{n+1}$
by some method.

if $\beta = 0$ $\quad d_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2} a_n$; if d_n, v_n, a_n are known; d_{n+1} is found

then $\underset{\sim}{K}\, d_{n+1}$ is known then $a_{n+1} = (F_{n+1} - \underset{\sim}{K} d_{n+1}) / [M_{ii}]$ explicit in nature
using the 2nd of the newmark eqns we can get v_{n+1}

if $\beta \neq 0$ then a_{n+1} requires implicitness
for example look back at

$$\underset{\sim}{M}\, a_{n+1} + \underset{\sim}{K}\, d_{n+1} = \underset{\sim}{F}_{n+1}$$
$$d_{n+1} = \tilde{d}_{n+1} + \beta \Delta t^2 a_{n+1}$$
$$v_{n+1} = \tilde{v}_{n+1} + \gamma \Delta t\, a_{n+1}$$

} newmark forms; i.e. $\tilde{d}_{n+1} = d_n + \Delta t\, v_n + \frac{(1-2\beta)}{2} \Delta t^2 a_n$, $\tilde{v}_{n+1} = v_n + (1-\gamma)\Delta t\, a_n$ — are predictors

here a_{n+1} & d_{n+1} are coupled. If we eliminate we can get

$$(\underset{\sim}{M} + \beta \Delta t^2 \underset{\sim}{K})\, a_{n+1} = F_{n+1} - \underset{\sim}{K}\, \tilde{d}_{n+1} = \text{known}.$$

we may be able to diagonalize $\underset{\sim}{M}$ but we can't diag K hence we have a matrix problem for a_{n+1} since $\underset{\sim}{K}$ is banded. $(\underset{\sim}{M} + \beta \Delta t^2 \underset{\sim}{K})$ is implicit

if $\underset{\sim}{C}$ terms were in ie $\underset{\sim}{C}\, v_{n+1}$ by substituting for v_{n+1}

thus $\quad (\underset{\sim}{M} + \beta \Delta t^2 \underset{\sim}{K} + \gamma \Delta t\, \underset{\sim}{C})\, a_{n+1} = F_{n+1} - \underset{\sim}{C}\, \tilde{v}_{n+1} - \underset{\sim}{K}\, \tilde{d}_{n+1}$

if $\beta = 0$ $\underset{\sim}{C}$ may be diag & we have explicit if $\underset{\sim}{M}$ is diag
but if $\beta \neq 0$ even if $\underset{\sim}{C}$ is diag we have an implicit system.

5/20/81

$$\underset{\sim}{M}\ddot{\underset{\sim}{d}} + \underset{\sim}{C}\dot{\underset{\sim}{d}} + \underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F}, \quad \underset{\sim}{d}(0) = \underset{\sim}{d}_0, \quad \dot{\underset{\sim}{d}}(0) = \dot{\underset{\sim}{d}}_0$$

temporal discretiz

$$\underset{\sim}{M}\, a_{n+1} + \underset{\sim}{C}\, v_{n+1} + \underset{\sim}{K}\, d_{n+1} = \underset{\sim}{F}_{n+1}$$
$$d_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2}\{(1-2\beta) a_n + 2\beta a_{n+1}\}$$
$$v_{n+1} = v_n + \Delta t\{(1-\gamma) a_n + \gamma a_{n+1}\}$$
d_0 given, $v_0 = \dot{d}_0$ given

homework : quad hourg & incomp good $\quad$ hourg is faster than incomp

bisect – degen toward .499
cross diag – doesn't deg as much

ORIGINALLY

$$m_1 \ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 = 0$$
$$m_2 \ddot{x}_2 + (k_2+k_3)x_2 - k_2 x_1 = 0$$

} STATIC OR ELASTIC COUPLING

COUPLING IS IN THE DISPL COORD.

INVERSE APPROACH

$$\frac{k_2+k_3}{k_1k_2+k_1k_3+k_2k_3} m_1\ddot{x}_1 + \frac{k_2}{k_1k_2+k_1k_3+k_2k_3} m_2\ddot{x}_2 + x_1 = 0$$

$$\frac{k_2}{k_1k_2+k_1k_3+k_2k_3} m_1\ddot{x}_1 + \frac{k_1+k_2}{k_1k_2+k_1k_3+k_2k_3} m_2\ddot{x}_2 + x_2 = 0$$

DYNAMIC OR INERTIA COUPLING - COUPLING IS IN THE ACCELERATION VARIABLE

TYPE OF COUPLING IS IN METHOD USED TO DERIVE EQS.

LET $m_1 = m_2 = m \quad k_1 = k_2 = k_3 = k$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 + 2kx_2 - kx_1 = 0$$

ADD $m(\ddot{x}_1+\ddot{x}_2) + k(x_1+x_2) = 0$

SUBTRACT $m(\ddot{x}_1-\ddot{x}_2) + 3k(x_1-x_2) = 0$

LET $q_1 = x_1+x_2 \quad q_2 = x_1 - x_2$ GENERALIZED COORD

$m\ddot{q}_1 + kq_1 = 0$ or $\omega_1 = \sqrt{k/m}$ $\quad q_1 = A_1 \sin(\omega_1 t+\phi_1)$

$m\ddot{q}_2 + 3kq_2 = 0$ or $\omega_2 = \sqrt{\frac{3k}{m}}$ $\quad q_2 = A_2 \sin(\omega_2 t+\phi_2)$

$q_1+q_2 = 2x_1 \quad q_1 - q_2 = 2x_2$

if

$$x_1 = [A_1 \sin(\omega_1 t+\phi_1) + A_2 \sin(\omega_2 t+\phi_2)]/2$$
$$x_2 = [A_1 \sin(\omega_1 t+\phi_1) - A_2 \sin(\omega_2 t+\phi_2)]/2$$

~~SESSION #22~~ Lecture 14

- ANOTHER METHOD TO FIND THE EQUATIONS - LAGRANGE'S METHOD
  - MAKES USE OF ENERGY OF SYSTEM
  - POTENTIAL ENERGY V IS A FN OF GENERALIZED COORDINATES q_k
  - KINETIC ENERGY T IS A FN OF GENERALIZED VELOCITY $\dot{q}_k$ and q_k
  - FOR A CONSERVATIVE SYSTEM <u>READ APPENDIX B</u>

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = 0$$

- GENERALIZED COORDINATE IS SUCH THAT A CHANGE IN ONE COORDINATE DOESN'T CAUSE OR DOESN'T REQUIRE A CHANGE IN ANY OTHER COORD.

here if we use x_1, y_1, & x_2, y_2 we would appear to have 4 indep coord

but $x_1^2 + y_1^2 = l_1^2$

$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$ } 2 constraints $\Rightarrow$ only 2 indep coords

if we take θ_1 & θ_2 to specify location of masses then there are no constraints

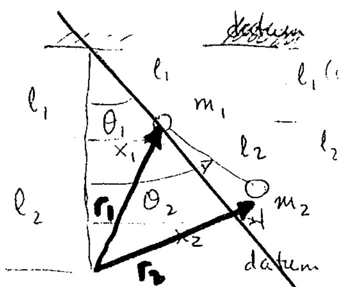
q

- MAY BE LINEAR, ANGULAR OR COMBO $\exists$ 1 for each DOF
- REQUIRES KNOWLEDGE OF GENERALIZED COORD.

CAN DEFINE A LAGRANGIAN $L = T - V$ $\quad \frac{\partial L}{\partial \dot q} = \frac{\partial T}{\partial \dot q} \quad \frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q} + \frac{\partial T}{\partial q}$

IF $T = T(q_k, \dot q_k)$ and $V = V(q_k)$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_k}\right) + \frac{\partial V}{\partial q_k} - \frac{\partial T}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad q = 1, 2, 3, \ldots$$



$l_1(1-\cos\theta_1)$

$l_2(1-\cos\theta_2)$

$(l_1 + l_2)$

$$V = m_1 g\, l_1(1-\cos\theta_1) + m_2 g l_2 + m_2 g\left[l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)\right]$$

$$T = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 (l_1\dot\theta_1)^2 + \tfrac{1}{2} m_2 (l_2\dot\theta_2 + l_1\dot\theta_1)^2$$

relative velocity

~~$x_1 = l_1 \sin\theta_1;\ \dot x_1 = l_1\cos\theta_1 \dot\theta_1 \cong l_1\dot\theta_1$~~

~~$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2;\ \dot x_2 = l_1\cos\theta_1\dot\theta_1 + l_2\cos\theta_2\dot\theta_2 \cong l_1\dot\theta_1 + l_2\dot\theta_2$~~

$q_1 = \theta_1 \quad q_2 = \theta_2$

$$\frac{\partial L}{\partial \theta_1} = \frac{\partial (T-V)}{\partial \theta_1} = -\frac{\partial V}{\partial \theta_1} = -\left[m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1\right]$$

(with $\sin\theta_1$ crossed out and replaced by θ_1)

$$\frac{\partial L}{\partial \theta_2} = \frac{\partial (T-V)}{\partial \theta_2} = -\frac{\partial V}{\partial \theta_2} = -\left[m_2 g l_2 \sin\theta_2\right]$$

(with $\sin\theta_2$ crossed out and replaced by θ_2)

$$\frac{\partial L}{\partial \dot\theta_1} = \frac{\partial (T-V)}{\partial \dot\theta_1} = \frac{\partial T}{\partial \dot\theta_1} = m_1 l_1^2\dot\theta_1 + m_2 l_1^2\dot\theta_1 + m_2 l_1 l_2 \dot\theta_2$$

$$\frac{\partial L}{\partial \dot\theta_2} = \frac{\partial (T-V)}{\partial \dot\theta_2} = \frac{\partial T}{\partial \dot\theta_2} = m_2(l_2\dot\theta_2 + l_1\dot\theta_1) l_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) - \frac{\partial L}{\partial \theta_1} = m_1 l_1^2\ddot\theta_1 + m_2 l_1^2\ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 + m_1 g l_1\theta_1 + m_2 g l_1 \theta_1 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) - \frac{\partial L}{\partial \theta_2} = m_2 l_2^2\ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 + m_2 g l_2\theta_2 = 0$$

$$\bar r_1 = l_1\sin\theta_1\,\bar\imath + \left[l_2 + l_1(1-\cos\theta_1)\right]\bar\jmath$$

$$\dot{\bar r}_1 = \left[l_1\cos\theta_1\,\bar\imath + l_1\sin\theta_1\,\bar\jmath\right]\dot\theta_1$$

$$|\dot{\bar r}_1|^2 = l_1^2\dot\theta_1^2$$

$$\bar r_2 = (l_1\sin\theta_1 + l_2\sin\theta_2)\bar\imath + \left[l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)\right]\bar\jmath$$

$$\dot{\bar r}_2 = (l_1\cos\theta_1\dot\theta_1 + l_2\cos\theta_2\dot\theta_2)\bar\imath + (l_1\sin\theta_1\dot\theta_1 + l_2\sin\theta_2\dot\theta_2)\bar\jmath$$

$$|\dot{\bar r}_2|^2 = l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1\dot\theta_2\cos(\theta_1 - \theta_2)$$

$$\cong (l_1\dot\theta_1 + l_2\dot\theta_2)^2$$

matrix has at least one off-diagonal term nonzero, the system is said to be dynamically coupled. Further, if both the stiffness and mass matrices have nonzero off-diagonal terms, the system is said to be coupled both statically and dynamically.

EXAMPLE 6.2 Equations of Motion of a Trailer–Compound Pendulum System

Derive the equations of motion of the trailer–compound pendulum system shown in Fig. 6.4(a).

Solution

Approach: Draw the free-body diagrams and apply Newton's second law of motion.

The coordinates $x(t)$ and $\theta(t)$ are used to describe, respectively, the linear displacement of the trailer and the angular displacement of the compound pendulum from their respective static equilibrium

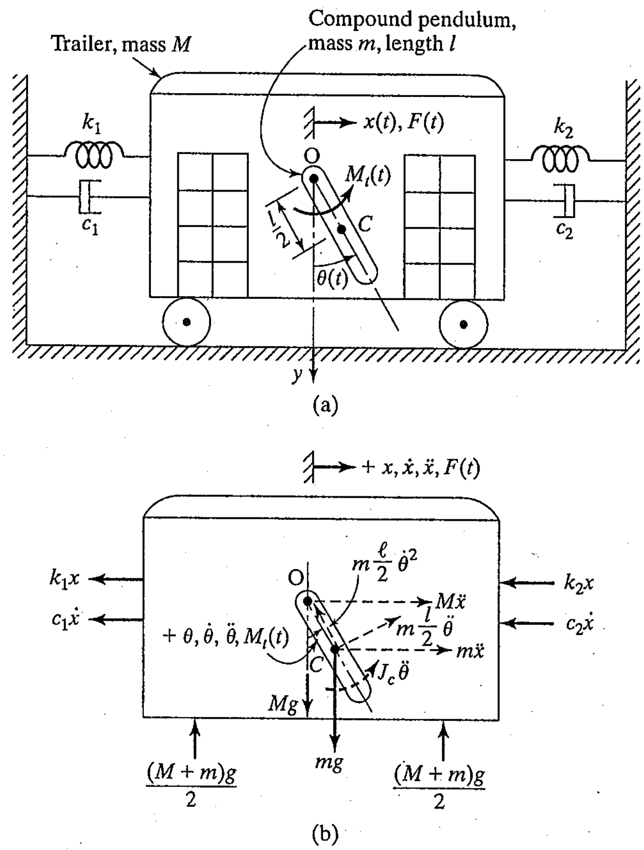


FIGURE 6.4 Compound pendulum and trailer system.

There are external moments applied to the components, so Eq. (6.42) gives

$$Q_j^{(n)} = \sum_{k=1}^{3} M_{tk}\frac{\partial\theta_k}{\partial q_j} = \sum_{k=1}^{3} M_{tk}\frac{\partial\theta_k}{\partial\theta_j} \tag{E.4}$$

from which we can obtain

$$\begin{aligned}
Q_1^{(n)} &= M_{t1}\frac{\partial\theta_1}{\partial\theta_1} + M_{t2}\frac{\partial\theta_2}{\partial\theta_1} + M_{t3}\frac{\partial\theta_3}{\partial\theta_1} = M_{t1}\\
Q_2^{(n)} &= M_{t1}\frac{\partial\theta_1}{\partial\theta_2} + M_{t2}\frac{\partial\theta_2}{\partial\theta_2} + M_{t3}\frac{\partial\theta_3}{\partial\theta_2} = M_{t2}\\
Q_3^{(n)} &= M_{t1}\frac{\partial\theta_1}{\partial\theta_3} + M_{t2}\frac{\partial\theta_2}{\partial\theta_3} + M_{t3}\frac{\partial\theta_3}{\partial\theta_3} = M_{t3}
\end{aligned} \tag{E.5}$$

Substituting Eqs. (E.1), (E.3), and (E.5) in Lagrange's equations, Eq. (6.41), we obtain for $j = 1, 2, 3$ the equations of motion

$$\begin{aligned}
J_1\ddot{\theta}_1 + (k_{t1} + k_{t2})\theta_1 - k_{t2}\theta_2 &= M_{t1}\\
J_2\ddot{\theta}_2 + (k_{t2} + k_{t3})\theta_2 - k_{t2}\theta_1 - k_{t3}\theta_3 &= M_{t2}\\
J_3\ddot{\theta}_3 + k_{t3}\theta_3 - k_{t3}\theta_2 &= M_{t3}
\end{aligned} \tag{E.6}$$

which can be expressed in matrix form as

$$\begin{bmatrix} J_1 & 0 & 0\\ 0 & J_2 & 0\\ 0 & 0 & J_3 \end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\end{Bmatrix} + \begin{bmatrix} (k_{t1}+k_{t2}) & -k_{t2} & 0\\ -k_{t2} & (k_{t2}+k_{t3}) & -k_{t3}\\ 0 & -k_{t3} & k_{t3}\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = \begin{Bmatrix} M_{t1}\\ M_{t2}\\ M_{t3}\end{Bmatrix} \tag{E.7}$$

■

EXAMPLE 6.9

Lagrange's Equations

Derive the equations of motion of the trailer–compound pendulum system shown in Fig. 6.4(a).

Solution: The coordinates $x(t)$ and $\theta(t)$ can be used as generalized coordinates to describe, respectively, the linear displacement of the trailer and the angular displacement of the compound pendulum. If a y-coordinate is introduced, for convenience, as shown in Fig. 6.4(a), the displacement components of point C can be expressed as

$$x_C = x + \frac{l}{2}\sin\theta \tag{E.1}$$

$$y_C = \frac{l}{2} \cos \theta \tag{E.2}$$

Differentiation of Eqs. (E.1) and (E.2) with respect to time gives the velocities of point C as

$$\dot{x}_C = \dot{x} + \frac{l}{2}\dot{\theta} \cos \theta \tag{E.3}$$

$$\dot{y}_C = -\frac{l}{2}\dot{\theta} \sin \theta \tag{E.4}$$

The kinetic energy of the system, T, can be expressed as

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_C^2 + \dot{y}_C^2) + J_C\dot{\theta}^2 \tag{E.5}$$

where $J_C = \frac{1}{12}ml^2$. Using Eqs. (E.3) and (E.4), Eq. (E.5) can be rewritten as

$$\begin{aligned} T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \frac{l^2\dot{\theta}^2}{4} + \dot{x}\dot{\theta}\, l \cos \theta\right) + \frac{1}{2}\left(\frac{ml^2}{12}\right)\dot{\theta}^2 \\ &= \frac{1}{2}(M + m)\dot{x}^2 + \frac{1}{2}\left(\frac{ml^2}{3}\right)\dot{\theta}^2 + \frac{1}{2}(ml \cos \theta)\dot{x}\dot{\theta} \end{aligned} \tag{E.6}$$

The potential energy of the system, V, due to the strain energy of the springs and the gravitational potential, can be expressed as

$$V = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 x^2 + mg\frac{l}{2}(1 - \cos \theta) \tag{E.7}$$

where the lowest position of point C is taken as the datum. Since there are nonconservative forces acting on the system, the generalized forces corresponding to $x(t)$ and $\theta(t)$ are to be computed. The force, $X(t)$, acting in the direction of $x(t)$ can be found from Eq. (6.42) as

$$X(t) = Q_1^{(n)} = F(t) - c_1\dot{x}(t) - c_2\dot{x}(t) \tag{E.8}$$

where the negative sign for the terms $c_1\dot{x}$ and $c_2\dot{x}$ indicates that the damping forces oppose the motion. Similarly, the force $\Theta(t)$ acting in the direction of $\theta(t)$ can be determined as

$$\Theta(t) = Q_2^{(n)} = M_t(t) \tag{E.9}$$

where $q_1 = x$ and $q_2 = \theta$. By differentiating the expressions of T and V as required by Eqs. (6.41) and substituting the resulting expressions, along with Eqs. (E.8) and (E.9), we obtain the equations of motion of the system as

$$\begin{aligned} (M + m)\ddot{x} + \frac{1}{2}(ml \cos \theta)\ddot{\theta} - \frac{1}{2}ml \sin \theta\dot{\theta}^2 + k_1 x + k_2 x \\ = F(t) - c_1\dot{x} - c_2\dot{x} \end{aligned} \tag{E.10}$$

$$\left(\frac{1}{3}ml^2\right)\ddot{\theta} + \frac{1}{2}(ml\cos\theta)\ddot{x} - \frac{1}{2}ml\sin\theta\,\dot{\theta}\,\dot{x} + \frac{1}{2}ml\sin\theta\,\dot{\theta}\,\dot{x}$$
$$+ \frac{1}{2}mgl\sin\theta = M_t(t) \qquad \text{(E.11)}$$

Equations (E.10) and (E.11) can be seen to be identical to those obtained using Newton's second law of motion (Eqs. E.1 and E.2 in Example 6.2).

■

6.8 Equations of Motion of Undamped Systems in Matrix Form

We can derive the equations of motion of a multidegree of freedom system in matrix form from Lagrange's equations.[2]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i} = F_i, \quad i = 1, 2, \ldots n \qquad (6.44)$$

where F_i is the nonconservative generalized force corresponding to the ith generalized coordinate x_i and $\dot{x}_i$ is the time derivative of x_i (generalized velocity). The kinetic and potential energies of a multidegree of freedom system can be expressed in matrix form as indicated in Section 6.5

$$T = \frac{1}{2}\dot{\vec{x}}^T[m]\dot{\vec{x}} \qquad (6.45)$$

$$V = \frac{1}{2}\vec{x}^T[k]\vec{x} \qquad (6.46)$$

where $\vec{x}$ is the column vector of the generalized coordinates

$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{Bmatrix} \qquad (6.47)$$

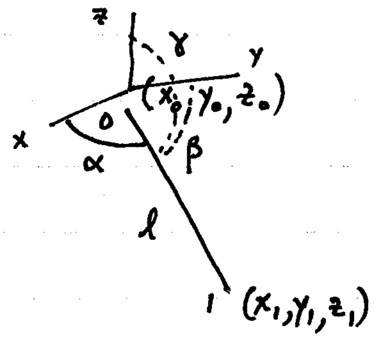
From the theory of matrices, we obtain, by taking note of the symmetry of $[m]$,

$$\frac{\partial T}{\partial \dot{x}_i} = \frac{1}{2}\vec{\delta}^T[m]\dot{\vec{x}} + \frac{1}{2}\dot{\vec{x}}^T[m]\vec{\delta} = \vec{\delta}^T[m]\dot{\vec{x}}$$
$$= \vec{m}_i^T\,\dot{\vec{x}}, \quad i = 1, 2, \ldots, n \qquad (6.48)$$

[2]The generalized coordinates are denoted as x_i instead of q_i and the generalized forces as F_i instead of $Q_i^{(n)}$ in Eq. (6.44).

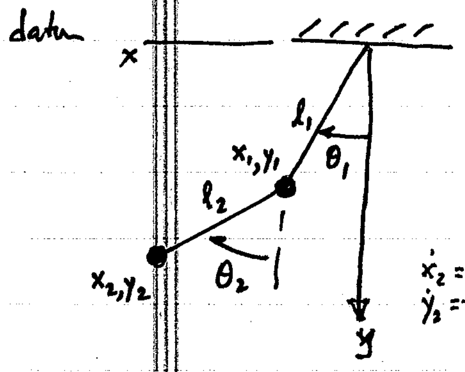
Generalized Coordinates - ANY SET OF INDEPENDENT COORD. = TO # OF DOFs of system

More complex systems may be described in terms of coordinates some of which may not be independent. Some coordinates are related to each other by constraint equations



$$(x_1, y_1, z_1) = (x_0, y_0, z_0) + \ell(\cos\alpha, \cos\beta, \cos\gamma)$$

constraint equation



$$\ell_1^2 = x_1^2 + y_1^2$$

$$\ell_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$x_1 = \ell_1 \sin\theta_1 \qquad y_1 = \ell_1 \cos\theta_1$$

$$x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 \qquad y_2 = \ell_1 \cos\theta_1 + \ell_2 \cos\theta_2$$

$$\dot{x}_2 = \ell_1 \cos\theta_1 \dot{\theta}_1 + \ell_2 \cos\theta_2 \dot{\theta}_2 \qquad (\ell_1^2 c^2\theta_1 \dot{\theta}_1^2 + 2\ell_1 \ell_2 c\theta_1 c\theta_2 \dot{\theta}_1 \dot{\theta}_2 + \ell_2^2 c^2\theta_2 \dot{\theta}_2^2) = \dot{x}_2^2$$

$$\dot{y}_2 = -\ell_1 \sin\theta_1 \dot{\theta}_1 - \ell_2 s\theta_2 \dot{\theta}_2 \qquad (\ell_1^2 s^2\theta_1 \dot{\theta}_1^2 + 2\ell_1 \ell_2 s\theta_1 s\theta_2 \dot{\theta}_1 \dot{\theta}_2 + \ell_2^2 s^2\theta_2 \dot{\theta}_2^2) = \dot{y}_2^2$$

$$V_1^2 = (\ell_1 \dot{\theta}_1)^2 = \dot{x}_1^2 + \dot{y}_1^2 = (\ell_1 \cos\theta_1 \dot{\theta}_1)^2 + (-\ell_1 \sin\theta_1 \dot{\theta}_1)^2$$

$$V_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = \text{[illegible]}$$

~~$= [\ell_1 \dot{\theta}_1 + \ell_2 \dot{\theta}_2 \cos(\theta_2 - \theta_1)]^2 + [\ell_2 \dot{\theta}_2 \sin(\theta_2 - \theta_1)]^2$~~

$$T = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = T(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = T(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \qquad \ell_1^2 \dot{\theta}_1^2 + \ell_2^2 \dot{\theta}_2^2 + 2\ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$q = \theta$ Generalized coordinate

$$T = \tfrac{1}{2} m_1 (\ell_1 \dot{\theta}_1)^2 + \tfrac{1}{2} m_2 (\ell_1 \dot{\theta}_1 + \ell_2 \dot{\theta}_2)^2 \qquad T = T(q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots)$$

$$U = +m_1 (\ell_1(1 - \cos\theta_1)) + m_2 (\ell_1(1 - \cos\theta_1) + \ell_2(1 - \cos\theta_2))$$

Note $U = U(q_1, q_2, \ldots)$

Theory of Vibrations by Thomson & Dahleh

Lagrange formulated a scalar procedure starting from KE, PE & work in terms of generalized coordinates q_i

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \qquad i = 1, 2, \ldots$$

Q_i is related to work done by nonpotential forces & is called the generalized force

This comes from

Conservation of energy $\frac{d}{dt}(PE + KE) = 0$ for conservative system

since $PE + KE = \text{constant}$ and $KE = T(q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2 \ldots)$

$PE = U(q_1, q_2, \ldots)$ as measured from static equil

then $\frac{d}{dt}(PE + KE) = \frac{d}{dt}\left[T(q_1, q_2, \ldots \dot{q}_i) + U(q_1, q_2, \ldots)\right] = 0$

$$\frac{d}{dt}T = \sum \frac{\partial T}{\partial q_i}\frac{dq_i}{dt} + \sum \frac{\partial T}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} \qquad \text{chain rule} \quad (1)$$

$$\frac{d}{dt}U = \sum \frac{\partial U}{\partial q_i}\frac{dq_i}{dt}$$

since T is also $\frac{1}{2}\sum m_i \dot{q}_i^2$ then

$$\sum_i \left(\frac{d}{d\dot{q}_i}T\right)\cdot \dot{q}_i = \left(\sum \frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i = \sum m_i \dot{q}_i^2 = 2T \qquad (1')$$

Now take $\frac{d}{dt}(1')$: $\quad 2\frac{dT}{dt} = \sum \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i + \sum \frac{\partial T}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} \qquad (2)$

subtract (1) from (2) to get $\frac{dT}{dt} = \sum \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i - \sum \frac{\partial T}{\partial q_i}\dot{q}_i$

Now $\frac{d}{dt}\left[T + U\right] = \sum \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i}\right\}\dot{q}_i = 0$ for any $\dot{q}_i$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0 \qquad \forall i = 1, 2, \ldots$$

[illegible]

Show conservation of energy [illegible] from Lagrange's equations [illegible]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot q_i} = Q_i$$

[illegible] the same form.

Conservation of energy $\frac{d}{dt}(PE + KE) = 0$ [illegible] conservative system

Since $PE + KE = $ constant and $KE = T(q_i, \dot q_i, \ldots)$

$PE = U(q_1, q_2, \ldots)$ [illegible]

Now $\frac{d}{dt}(PE + KE) = \frac{d}{dt}\left[T(q_i, \dot q_i) + U(q_i)\right] = 0$

(1) chain rule $\frac{dT}{dt} = \sum \frac{\partial T}{\partial q_i}\dot q_i + \sum \frac{\partial T}{\partial \dot q_i}\ddot q_i$

$$\frac{dU}{dt} = \sum \frac{\partial U}{\partial q_i}\dot q_i$$

Since T is [illegible] $\frac{1}{2}\sum m \dot q_i^2$ [illegible]

(2) $\sum \frac{\partial T}{\partial \dot q_i}\dot q_i = \sum m \dot q_i^2 = 2T$

Now take $\frac{d}{dt}$ (2): $2\frac{dT}{dt} = \sum \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right)\dot q_i + \sum \frac{\partial T}{\partial \dot q_i}\ddot q_i$ (3)

Subtract (1) from (3): $\frac{dT}{dt} = \sum \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right)\dot q_i - \sum \frac{\partial T}{\partial q_i}\dot q_i$

Now $\frac{d}{dt}[T + U] = \sum\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i}\right]\dot q_i = 0$ for any $\dot q_i$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0$$ [illegible]

if we define $L = T - U = T + \text{Work}$

& since $U \neq U(\dot{q}_1, \dot{q}_2, \ldots) \implies \frac{\partial U}{\partial \dot{q}_i} = 0$ then

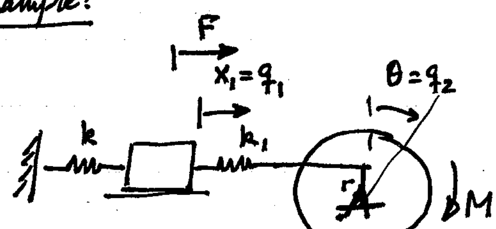
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

work = conservative work + NC work

conserv work = − potential energy

Example:

assume external forces M & F are applied



$$T = \tfrac{1}{2} m\dot{x}_1^2 + \tfrac{1}{2} J\Omega^2 = \tfrac{1}{2} m\dot{q}_1^2 + \tfrac{1}{2} J\dot{q}_2^2$$

$$U = \tfrac{1}{2} k x_1^2 + \tfrac{1}{2} k_1 (r\theta - x_1)^2 = \tfrac{1}{2} k q_1^2 + \tfrac{1}{2} k_1 (r q_2 - q_1)^2$$

Non conservative: $\delta W = M\delta\theta + F\delta x = M\delta q_2 + F\delta q_1$ $\qquad \frac{\delta W}{\delta q_1} = F \qquad \frac{\delta W}{\delta q_2} = M$

$$L = T - U = \tfrac{1}{2} m\dot{q}_1^2 + \tfrac{1}{2} J\dot{q}_2^2 - \tfrac{1}{2} k q_1^2 - \tfrac{1}{2} k_1 (r q_2 - q_1)^2$$

$$\frac{\partial L}{\partial \dot{q}_1} = m\dot{q}_1 \qquad \frac{\partial L}{\partial \dot{q}_2} = J\dot{q}_2$$

$$\frac{\partial L}{\partial q_1} = -k q_1 - k_1 (r q_2 - q_1)(-1) = +k_1 (r q_2) - (k + k_1) q_1$$

$$\frac{\partial L}{\partial q_2} = -k_1 (r q_2 - q_1) \cdot r = +k_1 r q_1 - k_1 r^2 q_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = m_1 \ddot{q}_1 + (k + k_1) q_1 - k_1 r q_2 = 0 \quad \checkmark \quad F$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = J\ddot{q}_2 + k_1 r^2 q_2 - k_1 r q_1 = 0 \quad \checkmark \quad M$$

Give 6.34/6.38 in 5th

6.39/6.43 in 6th

If we define $L = T - U$ [illegible]

[illegible]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

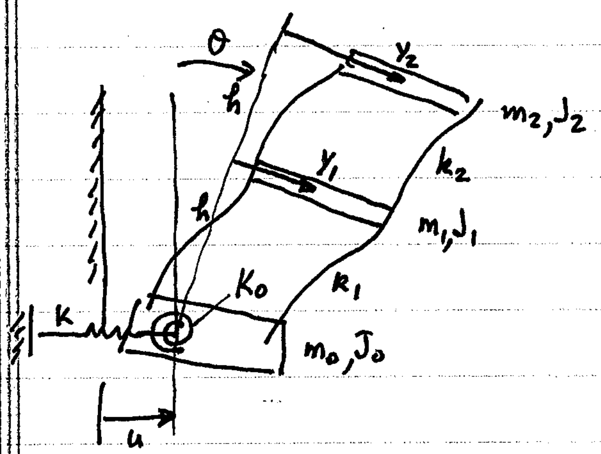
[illegible]

[illegible]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = $$ [illegible]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = $$ [illegible]

[illegible]



model of 2-story building whose foundation is subject to translation & rotation

let y_1 & y_2 be the elastic displ of the floors
u, θ " " transl & rot of the foundation

$$T = \tfrac{1}{2} m_0 \dot{u}^2 + \tfrac{1}{2} J_0 \dot{\theta}^2 + \tfrac{1}{2} m_1 (\dot{u} + h\dot{\theta} + \dot{y}_1)^2 + \tfrac{1}{2} J_1 \dot{\theta}^2 + \tfrac{1}{2} m_2 (\dot{u} + 2h\dot{\theta} + \dot{y}_2)^2 + \tfrac{1}{2} J_2 \dot{\theta}^2$$

$$U = \tfrac{1}{2} k u^2 + \tfrac{1}{2} K_0 \theta^2 + \tfrac{1}{2} k_1 y_1^2 + \tfrac{1}{2} k_2 (y_2 - y_1)^2$$

if we let u, θ, y_1, y_2 be the generalized coordinates then

$$\frac{\partial T}{\partial \dot{\theta}} = J_0 \dot{\theta} + m_1(\dot{u} + h\dot{\theta} + \dot{y}_1)h + J_1 \dot{\theta} + m_2(\dot{u} + 2h\dot{\theta} + \dot{y}_2)\cdot 2h + J_2 \dot{\theta}$$

$$\frac{\partial T}{\partial \theta} = 0 \qquad \frac{\partial U}{\partial \dot{\theta}} = 0 \qquad \frac{\partial U}{\partial \theta} = K_0 \theta$$

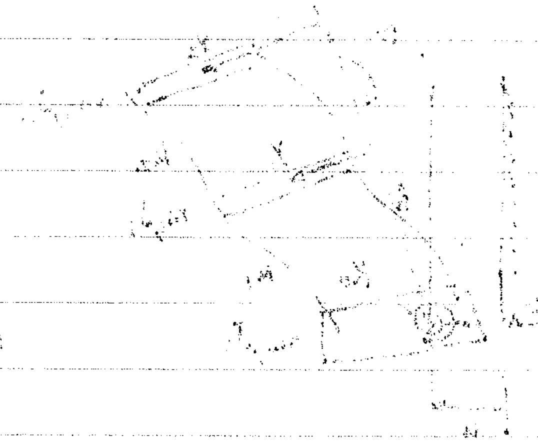
for example $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = (J_0 + J_1 + J_2)\ddot{\theta} + (m_1 + 4m_2)h^2\ddot{\theta} + m_1 \ddot{y}_1 h + 2m_2 \ddot{y}_2 h + (m_1 h + 2m_2 h)\ddot{u} + K_0 \theta = 0$$

$$\frac{\partial T}{\partial \dot{u}} = m_1(\dot{u} + h\dot{\theta} + \dot{y}_1) + m_2(\dot{u} + 2h\dot{\theta} + \dot{y}_2) + m_0 \dot{u} \qquad \frac{\partial T}{\partial u} = 0 \qquad \frac{\partial U}{\partial \dot{u}} = 0 \qquad \frac{\partial U}{\partial u} = ku$$

for example $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} + \frac{\partial U}{\partial u} = (m_0 + m_1 + m_2)\ddot{u} + (m_1 h + 2m_2 h)\ddot{\theta} + m_1 \ddot{y}_1 + m_2 \ddot{y}_2 + ku = 0$$

$$\frac{\partial T}{\partial \dot{y}_1} = m_1(\dot{u} + h\dot{\theta} + \dot{y}_1) \qquad \frac{\partial T}{\partial y_1} = 0 \qquad \frac{\partial U}{\partial y_1} = k_1 y_1 + k_2(y_2 - y_1)(-1) \qquad \frac{\partial U}{\partial \dot{y}_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_1}\right) - \frac{\partial T}{\partial y_1} + \frac{\partial U}{\partial y_1} = m_1(\ddot{u}_1 + h\ddot{\theta} + \ddot{y}_1) + (k_1 + k_2)y_1 - k_2 y_2 = 0$$



[illegible]

$T = $ [illegible]

$U = $ [illegible]

[illegible]

Nonconservative systems. The right side of Lagrange's equation (7.3.1) results from dividing the work term in the dynamical relationship $dT = dW$ into the work done by the potential and nonpotential forces as follows.

$$dT = dW_p + dW_{np} \tag{7.3.12}$$

The work of the potential forces was shown earlier to be equal to $dW_p = -dU$, which is included in the left side of Lagrange's equation. The nonpotential work is equal to the work done by the nonpotential forces in a virtual displacement expressed in terms of the generalized coordinates. Thus, Lagrange's equation, Eq. (7.3.1) is the q_i component of the energy equation

$$d(T + U) = \delta W_{np} \tag{7.3.13}$$

We can write the right side of this equation as

$$\delta W = \sum_{i=1}^{N} Q_i \delta q_i = Q_1 \delta q_i + Q_2 \delta_2 + \cdots \tag{7.3.14}$$

The quantity Q_i is called the *generalized force.* In spite of its name Q_i can have units other than that of force; i.e., if δq_i is an angle, Q_i has the units of moment. The only requirement is that the product $Q_i \delta q_i$ be in the units of work. We now demonstrate the use of Lagrange's equation as applied to some simple examples.

Example 7.3.1

Using Lagrange's method, determine the equation of motion for the 3-DOF system shown in Fig. 7.3.1.

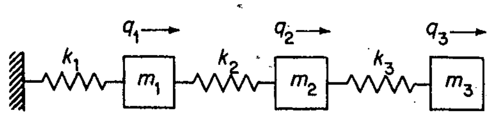


FIGURE 7.3.1.

Solution The kinetic energy here is not a function of q_i so that the term $\partial T/\partial q_i$ is zero. We have the following for the kinetic and potential energies:

$$T = \tfrac{1}{2}m_1\dot{q}_1^2 + \tfrac{1}{2}m_2\dot{q}_2^2 + \tfrac{1}{2}m_3\dot{q}_3^2$$
$$U = \tfrac{1}{2}k_1q_1^2 + \tfrac{1}{2}k_2(q_2 - q_1)^2 + \tfrac{1}{2}k_3(q_3 - q_2)^2$$

and T for this problem is a function of only $\dot{q}_1$ and not of q_i.

By substituting into Lagrange's equation for $i = 1$,

$$\frac{\partial T}{\partial q_1} = m_1\dot{q}_1, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = m_1\ddot{q}_1$$
$$\frac{\partial U}{\partial q_1} = k_1q_1 - k_2(q_2 - q_1)$$

and the first equation is

$$m_1\ddot{q}_1 + (k_1 + k_2)q_1 - k_2q_2 = 0$$

For $i = 2$, we have

$$\frac{\partial T}{\partial \dot{q}_2} = m_2\dot{q}_2 \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = m_2\ddot{q}_2$$

$$\frac{\partial U}{\partial q_2} = k_2(q_2 - q_1) - k_3(q_3 - q_2)$$

and the second equation becomes

$$m_2\ddot{q}_2 - k_2q_1 + (k_2 + k_3)q_2 - k_3q_3 = 0$$

Similarly for $i = 3$,

$$\frac{\partial T}{\partial \dot{q}_3} = m_3\dot{q}_3 \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) = m_3\ddot{q}_3$$

$$\frac{\partial U}{\partial q_3} = k_3(q_3 - q_2)$$

with the third equation

$$m_3\ddot{q}_3 - k_3q_2 + k_3q_3 = 0$$

These three equations can now be assembled into matrix form:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

We note from this example that the mass matrix results from the terms $(d/dt)(\partial T/\partial \dot{q}_i) - \partial T/\partial q_i$ and the stiffness matrix is obtained from $\partial U/\partial q_i$.

Example 7.3.2

Using Lagrange's method, set up the equations of motion for the system shown in Fig. 7.3.2.

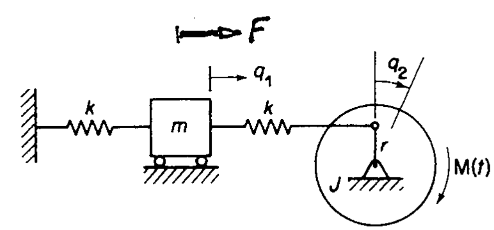


FIGURE 7.3.2.

Solution The kinetic and potential energies are

$$T = \tfrac{1}{2}m\dot{q}_1^2 + \tfrac{1}{2}J\dot{q}_2^2$$

$$U = \tfrac{1}{2}kq_1^2 + \tfrac{1}{2}K(rq_2 - q_1)^2$$

Substituting into Lagrange's equation, the equations of motion are

$$m\ddot{q}_1 + 2kq_1 - krq_2 = 0$$
$$J\ddot{q}_2 - krq_1 + kr^2q_2 = \mathcal{M}(t)$$

which can be rewritten as

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -kr \\ -kr & kr^2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \mathcal{M}(t) \end{Bmatrix}$$

■

Example 7.3.3

Figure 7.3.3 shows a simplified model of a two-story building whose foundation is subject to translation and rotation. Determine T and U and the equations of motion.

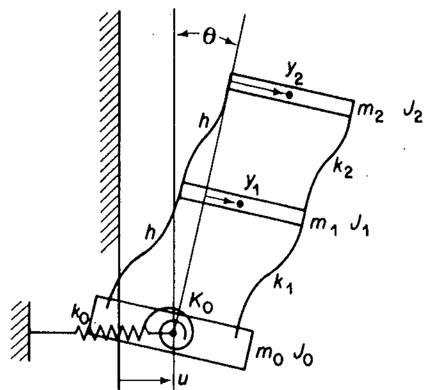


FIGURE 7.3.3.

Solution We choose u and θ for the translation and rotation of the foundation and y for the elastic displacement of the floors. The equations for T and U become

$$T = \tfrac{1}{2}m_0\dot{u}^2 + \tfrac{1}{2}J_0\dot{\theta}^2 + \tfrac{1}{2}m_1(\dot{u} + h\dot{\theta} + \dot{y}_1)^2 + \tfrac{1}{2}J_1\dot{\theta}^2$$
$$+ \tfrac{1}{2}m_2(\dot{u} + 2h\dot{\theta} + \dot{y}_2)^2 + \tfrac{1}{2}J_2\dot{\theta}^2$$
$$U = \tfrac{1}{2}k_0u^2 + \tfrac{1}{2}K_0\theta^2 + \tfrac{1}{2}k_1y_1^2 + \tfrac{1}{2}k_2(y_2 - y_1)^2$$

where u, θ, y_1, and y_2 are the generalized coordinates. Substituting into Lagrange's equation, we obtain, for example,

$$\frac{\partial T}{\partial \dot{\theta}} = (J_0 + J_1 + J_2)\dot{\theta} + m_1h(\dot{u} + h\dot{\theta} + y_1) + m_22h(\dot{u} + 2h\dot{\theta} + \dot{y}_2)$$

$$\frac{\partial U}{\partial \theta} = K_0$$

The four equations in matrix form become

$$\left[\begin{array}{cc:cc} (m_0+m_1+m_2) & (m_1+2m_2)h & m_1 & m_2 \\ (m_1+2m_2)h & (\sum J+m_1h^2+4m_2h^2) & m_1h & 2m_2h \\ \hdashline m_1 & m_1h & m_1 & 0 \\ m_2 & 2m_2h & 0 & m_2 \end{array}\right]\begin{Bmatrix} \ddot{u} \\ \ddot{\theta} \\ \hdashline \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix}$$

$$+\left[\begin{array}{cc:cc} k_0 & 0 & 0 & 0 \\ 0 & K_0 & 0 & 0 \\ \hdashline 0 & 0 & (k_1+k_2) & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{array}\right]\begin{Bmatrix} u \\ \theta \\ \hdashline y_1 \\ y_2 \end{Bmatrix} = \{0\}$$

It should be noted that the equation represented by the upper left corner of the matrices is that of rigid-body translation and rotation.

■

Example 7.3.4

Determine the generalized coordinates for the system shown in Fig. 7.3.4(a) and evaluate the stiffness and the mass matrices for the equations of motion.

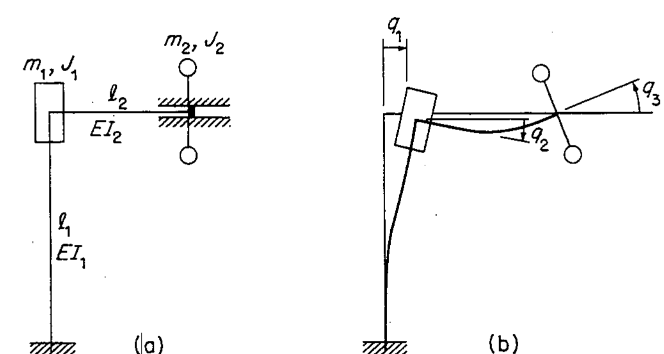


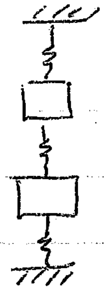
FIGURE 7.3.4. (a) and (b).

Solution Figure 7.3.4(b) shows three generalized coordinates for which the stiffness matrix can be written as

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

The elements of each column of this matrix are the forces and moments required when the corresponding coordinate is given a value with all other coordinates equal to zero. The configu-...ns for this determination are shown in Fig. 7.3.4(c), and the forces and m...ts necessary ...maintain these deflections are obtained from the free-body diagrams of Fi...4(d) with the

1. LETS ASSUME WE HAD



$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$$
$$m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2x_1 = 0$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$[M]\ddot{\underset{\sim}{x}} + [K]\underset{\sim}{x} = \underset{\sim}{0}$$

M is the mass matrix — Notice coupling (static) produces
K is the stiffness matrix — a diagonal matrix [M]

remember when $x_1 = A_1 \sin(\omega t + \varphi)$ $\quad x_2 = A_2 \sin(\omega t+\varphi)$

$$\therefore \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \sin(\omega t + \varphi) = -\omega^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\therefore \quad \left\{-\omega^2[M] + [K]\right\}(\underset{\sim}{x}) = (\underset{\sim}{0})$$

BY SETTING

$$\det\left\{-\omega^2[M] + [K]\right\} = \begin{bmatrix} -\omega^2 m_1 + (k_1+k_2) & -k_2 \\ -k_2 & -\omega^2 m_2 + (k_2+k_3) \end{bmatrix} = 0$$

we get frequency equation

$$\det\left(-\omega^2[M] + [K]\right) = 0$$

Rayleigh's Method for a conservative system $\quad T_{max} = V_{max}$

FIND: $T = \frac{1}{2} m_1 \dot{x}_1^{\,2} + \frac{1}{2} m_2 \dot{x}_2^{\,2} = \frac{1}{2}\dot{\underset{\sim}{x}}^T [M] \dot{\underset{\sim}{x}} \qquad \dot{\underset{\sim}{x}}^T = [\dot{x}_1 \quad \dot{x}_2]$

$V = \frac{1}{2} k_1 (x_1)^2 + \frac{1}{2} k_2 (x_1 - x_2)^2 + \frac{1}{2} k_3 x_2^2 = \frac{1}{2} \underset{\sim}{x}^T [K] \underset{\sim}{x}$

if $\left.\begin{aligned} x_1 &= A_1 \sin(\omega t + \varphi) \\ x_2 &= A_2 \sin(\omega t + \varphi)\end{aligned}\right\} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \sin(\omega_n t + \phi)$

$$\therefore T = \left(\frac{1}{2} m_1 \omega^2 A_1^2 + \frac{1}{2} m_2 \omega^2 A_2^2\right)\cos^2(\omega t + \varphi) = \frac{1}{2}[A_1 \ A_2]\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\omega^2 \cos^2(\omega t$$

$$V = \left[\frac{1}{2} k_1 A_1^2 + \frac{1}{2} k_2 (A_1 - A_2)^2 + \frac{1}{2} k_3 A_2^2\right] \sin^2(\omega t + \varphi)$$
$$= \frac{1}{2}[A_1 \ A_2]\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_3+k_2 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\sin^2($$

Rayleigh $\Rightarrow T_{max} = V_{max}$, $\therefore$ $\omega^2 \underset{\sim}{A}^T [M] \underset{\sim}{A} = \underset{\sim}{A}^T [K] \underset{\sim}{A}$

$$\therefore \quad \omega^2 = \frac{\underset{\sim}{A}^T [K] \underset{\sim}{A}}{\underset{\sim}{A}^T [M] \underset{\sim}{A}}$$

if $\underset{\sim}{A}$ is close to the mode shapes then ω_{RAYLEIGH} will be close to the value of ω obtained from the characteristic eqn. Actual $\underset{\sim}{A}$ minimizes value of ω^2

EXAMPLE if $m_1 = m_2 = m$ $\quad k_1 = k_2 = k = k_3$ $\quad [K] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} k$

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} m$$

If you are given $[A_1 \ A_2]^T = [1 \ \ 2]$

$$\omega^2 = \frac{[1 \ \ 2] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} k}{[1 \ \ 2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} m} = \frac{[1 \ \ 2] \begin{bmatrix} 0 \\ 3 \end{bmatrix} k = 6k}{[1 \ \ 2] \begin{bmatrix} 1 \\ 2 \end{bmatrix} m = 5m}$$

$$= 1.2 \frac{k}{m} \quad \text{or} \quad \omega_{\text{rayleigh}} \cong 1.1 \sqrt{\frac{k}{m}}$$

FOR ACTUAL $[A_1 \ \ A_2]^T = [1 \ \ 1] \Rightarrow \omega = \sqrt{\frac{k}{m}}$ } error 10%

$\omega_{rayleigh} \geq \omega_{n_1}$

DUNKERLEY'S FORMULA

remember $\{-\omega^2 [M] + [K]\} \underset{\sim}{X} = \underset{\sim}{0}$

if $\det [K] \neq 0$ then $\left\{ [K]^{-1} [M] - \frac{1}{\omega^2} [I] \right\} \underset{\sim}{X} = 0$

where $[K]^{-1} = [a] = \begin{bmatrix} \frac{k_2 + k_3}{k_1 k_3 + k_2 k_3 + k_1 k_2} & \frac{k_2}{k_1 k_3 + k_2 k_3 + k_1 k_2} \\ \frac{k_2}{k_1 k_3 + k_2 k_3 + k_1 k_2} & \frac{k_1 + k_2}{k_1 k_3 + k_1 k_2 + k_2 k_3} \end{bmatrix}$

where $[I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad [a] = \begin{bmatrix} \frac{2}{3k} & \frac{1}{3k} \\ \frac{1}{3k} & \frac{2}{3k} \end{bmatrix}$ if $k_1 = k_2 = k_3$

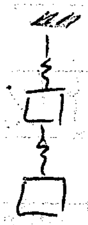
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

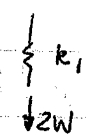
$$A^{-1} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{\det A}$$

$$\det A = a_{11}a_{22} - a_{12} \cdot a_{21}$$

signed minor

$$[a_{ij}]^{-1} = \frac{1}{\det A} \text{ [redacted] } (-1)^{i+j}$$





$$\delta_{st} = \frac{W_1+W_2}{k_1} = \delta_1$$

$$\delta_{st_2} = \delta_1 + \frac{W_2}{k_2} = \delta_2$$

if $m_1 = m_2 = m$ & $k_1 = k_2 = k$

$$\delta_1 = \frac{2W}{k}$$

$$\delta_2 = \frac{2W}{k} + \frac{W}{k} = \frac{3W}{k}$$

$$\frac{W}{k}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \quad \text{choose } \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

then
$$\omega^2 = \frac{[2 \quad 3]\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}}{[2 \quad 3]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}} = \frac{14}{13}\frac{k}{m}$$

$$\det\left\{[a][M]-\frac{1}{\omega^2}[I]\right\}=0 = \begin{bmatrix} a_{11}m_1-\frac{1}{\omega^2} & a_{12}m_2 \\ a_{21}m_1 & a_{22}m_2-\frac{1}{\omega^2}\end{bmatrix}$$

$$\frac{1}{\omega^4}-\underbrace{\left(a_{11}m_1+a_{22}m_2\right)}\frac{1}{\omega^2}+a_{11}m_1a_{22}m_2-a_{12}^2m_1m_2=0$$

$a_{11}m_1+a_{12}m_2$ represent the sum of the terms of the factors (ROOTS)

$$=\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}$$

$(x-s_1)(x-s_2)=0$

$x^2-(s_1+s_2)x+s_1s_2=0$

IN GENERAL IF $\omega_1 \ll \omega_2 \Rightarrow \omega_1^2 \ll \omega_2^2$ and $\frac{1}{\omega_1^2} \gg \frac{1}{\omega_2^2}$

$$\Rightarrow \frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}\cong\frac{1}{\omega_1^2} \quad \text{IF } \omega_2,\omega_3,\omega_4,\ldots \gg \omega_1$$

FOR OUR PROBLEM IF $m_1=m_2=m$ & $k_1=k_2=k$

$$\underset{a_{11}m_1}{\frac{2}{3k}m} + \underset{a_{22}m_2}{\frac{2}{3k}m} \cong \frac{1}{\omega_1^2}=\frac{4m}{3k} \quad \therefore\ \omega_1\cong .866\sqrt{\frac{k}{m}}$$

NOTE THAT FOR $\omega_1 < \omega_{1_{REAL}}$

WORKS BEST FOR VERY LARGE SYSTEMS - MULTIDEGREE SYSTEMS

~~6.34~~ 6.61

$$\omega^2=\frac{[.275\ \ .4\ \ .45]\begin{bmatrix}6&-4&0\\-4&10&0\\0&0&6\end{bmatrix}\begin{bmatrix}.275\\.4\\.45\end{bmatrix}=[.275\ .4\ .45]\begin{bmatrix}.05\\2.9\\2.7\end{bmatrix}=2.39}{[.275\ \ .4\ \ .45]\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}.275\\.4\\.45\end{bmatrix}=[\quad]\begin{bmatrix}.275\\.8\\1.35\end{bmatrix}=1.003}=2.39$$

$\omega=1.544$

$$\overset{[k]}{\left[\begin{array}{ccc|ccc}6&-4&0&1&0&0\\-4&10&0&0&1&0\\0&0&6&0&0&1\end{array}\right]}\rightarrow\left[\begin{array}{ccc|ccc}6&-4&0&1&0&0\\0&44&0&4&6&0\\0&0&1&0&0&\frac{1}{6}\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}66&0&0&\frac{5}{22}&\frac{1}{11}&0\\0&44&0&\frac{1}{11}&\frac{3}{22}&0\\0&0&1&0&0&\frac{1}{6}\end{array}\right]^{[a]}$$

$$\frac{5}{22}\cdot 1+\frac{3}{22}\cdot 2+\frac{1}{6}\cdot 3=1=\frac{1}{\omega_n^2} \qquad \omega_n=1$$

$$a_{11}m_1+a_{22}m_2+a_{33}m_3=\frac{11}{22}+2=2.7$$

2.1 + .06 + .67 = 2.83

$$\omega_n^2 = \frac{[.7 \;\; .3 \;\; -1/3]\begin{bmatrix}6 & -4 & 0\\ -4 & 10 & 0\\ 0 & 0 & 6\end{bmatrix}\begin{bmatrix}.7\\ .3\\ -1/3\end{bmatrix}}{[.7 \;\; .3 \;\; -1/3]\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}.7\\ .3\\ -1/3\end{bmatrix}} = \frac{[.7 \;\; .3 \;\; -1/3]\begin{bmatrix}3\\ .2\\ -2\end{bmatrix}}{[.7 \;\; .3 \;\; -1/3]\begin{bmatrix}.7\\ .6\\ -1\end{bmatrix}} = \frac{2.83}{1} \cong 2.83$$

.49 + .18 + .33

$\omega_n = \sqrt{2.83}$

DO 6.6 but only for 2 masses find: [k], [m]; find ω and modes

For $[A_1 \; A_2] = [1 \;\; 1]$ find ω using Rayleigh

$[\quad] = [1 \;\; -1]$ " " " "

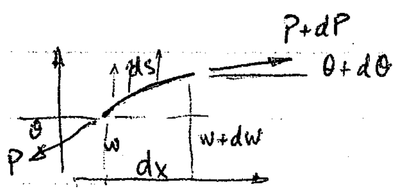
Find ω_1 using dunkerley

- DEALT WITH DISCRETE SYSTEMS ie mass, elasticity, damping at discrete pts
- SYSTEMS WHERE M, K, C occur at ∞ no. of pts = CONTINUOUS SYSTEMS
  - INFINITE DEGREES OF FREEDOM

- DISCRETE SYSTEMS ARE O.D.E's (LUMPED)
- CONTINUOUS SYSTEMS ARE P.D.E's
- CONTINUOUS SYSTEM IS MOST ACCURATE DESCRIPTION
- CHOICE OF LUMPED OR CONTINUOUS DEPENDENT ON WHY THE ANALYSIS

TRANSVERSE VIBS OF STRING

- CONSIDER A TAUT CABLE LENGTH l SUBJECTED TO A force $f(x,t)$ per unit length
- TRANSVERSE DISPLACEMENT IS $w(x,t)$ w is small wrt l



$$(P+dP)\sin(\theta+d\theta) - P\sin\theta + f\,ds = \rho\,ds\,\frac{\partial^2 w}{\partial t^2}$$

$ds = dx\sqrt{1+w'^2}$ $w' \ll 1 \Rightarrow dx = ds$

$$P\sin(\theta + \Delta\theta) - P\sin\theta = P\left[\frac{\overset{\cos\theta}{\sin(\theta + d\theta) - \sin\theta}}{d\theta}\right] d\theta$$

$$dP \sin(\theta + d\theta)$$

$$\sin(\theta + d\theta) = \sin\theta\cos d\theta + \cos\theta\sin d\theta$$

$$\frac{dw}{dx}\cos d\theta + \frac{dx}{ds}\cdot\sin d\theta$$

$2.1 + .06 + .67 = 2.91$

$$\omega_n^2 \; \frac{[.7 \;\; .3 \;\; -1/3] \begin{bmatrix} 6 & -4 & 0 \\ -4 & 10 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} .7 \\ .3 \\ -1/3 \end{bmatrix} = [.7 \;\; .3 \;\; -1/3] \begin{bmatrix} 3 \\ .2 \\ -2 \end{bmatrix} = 2.83}{[.7 \;\; .3 \;\; -1/3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} .7 \\ .3 \\ -1/3 \end{bmatrix} = [.7 \;\; .3 \;\; -1/3] \begin{bmatrix} .7 \\ .6 \\ -1 \end{bmatrix} = 1} \cong 2.83$$

$.49 + .18 + .33$

$\omega_n = \sqrt{2.83}$

DO 6.6 but only for 2 masses find: [k], [m]; find ω and modes

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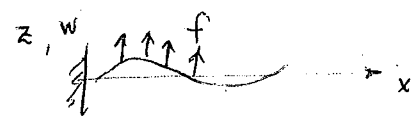
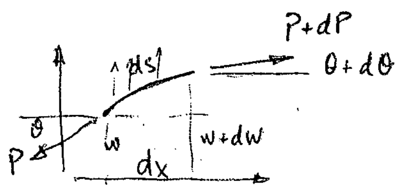


- CONSIDER A TAUT CABLE LENGTH l SUBJECTED TO A force $f(x,t)$ per unit length
- TRANSVERSE DISPLACEMENT IS $W(x,t)$ w is small wrt l



$$(P+dP)\sin(\theta+d\theta) - P\sin\theta + f\,ds = \rho\, ds\, \frac{\partial^2 w}{\partial t^2}$$

$ds = dx\sqrt{1+w'^2}$ $w' << 1 \Rightarrow dx = ds$

$$dP = \frac{\partial P}{\partial x} dx$$

$$\sin\theta = \frac{dw}{ds} = \frac{\partial w}{\partial x} \cong \tan\theta$$

$$P\sin(\theta + d\theta) - P\sin\theta = P\left[\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}dx\right] - P\frac{\partial w}{\partial x} = P\frac{\partial^2 w}{\partial x^2}dx$$

$$dP\sin(\theta + d\theta) = \frac{\partial P}{\partial x}dx \cdot \left[\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}dx\right] = \frac{\partial P}{\partial x}\frac{\partial w}{\partial x}dx + \ldots$$

$$\therefore (P+dP)\sin(\theta+d\theta) - P\sin\theta = \frac{\partial}{\partial x}\left(P\frac{\partial w}{\partial x}\right)dx$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\sin(\theta + d\theta) \cong \tan(\theta + d\theta) = \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}dx$$

put into Diff Eqn.

$$\frac{\partial}{\partial x}\left[P\frac{\partial w}{\partial x}\right] + f = \rho\frac{\partial^2 w}{\partial t^2}$$

if $P = \text{const}$ & $f = 0$ $\quad P+dP-P=0$ $\quad P\frac{\partial^2 w}{\partial x^2} = \rho\frac{\partial^2 w}{\partial t^2} \Rightarrow \boxed{c^2 W_{xx} = W_{tt}}$ $\quad c = \sqrt{P/\rho}$ (wave eqn.)

B.C. $\quad w(x= \ ,t)$ — displacement of pt.

$\quad P\frac{\partial w}{\partial x} = 0$ free end — angular

$\quad P\frac{\partial w}{\partial x} = -kW$ elastic constraint

I.C. $\quad w(x, t=0) = 0$ no displ.

$\quad \dot{w}(x, t=0) = 0$ no velocity

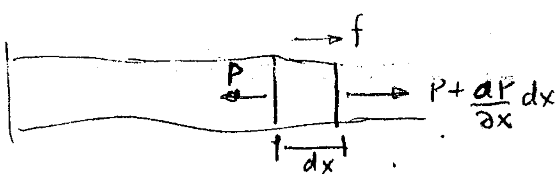
Solution Separation of Variables Let $w(x,t) = X(x)T(t)$

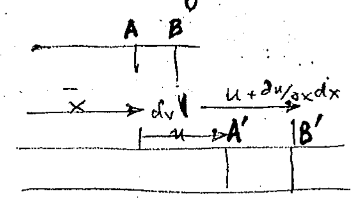
$$c^2 X''T = X\ddot{T} \qquad \text{or} \qquad \frac{c^2 X''}{X} = \frac{\ddot{T}}{T} = -k^2$$

$$X_k = A\sin kx + B\cos kx$$
$$T_k = C\sin kct + D\cos kct$$

then $w = \sum_k X_k T_k$

Longitudinal Vibs of a rod. $\qquad u(x,t)$ longitudinal displ.





elongation Δl

$$\varepsilon = \frac{\Delta l}{l = dx} = \frac{A'B' - AB}{AB} = \frac{\{[x + \Delta x + u + \Delta u] - [u + x]\} - ([x+\Delta x] - x)}{AB = \Delta x} = \frac{\Delta x + \Delta u - \Delta x}{\Delta x}$$

$$\sigma = \text{stress} = E\varepsilon = E\frac{\partial u}{\partial x} \qquad P = \sigma A = EA\frac{\partial u}{\partial x}$$

$$P + dP - P + f\,dx = \underbrace{\rho A\,dx}_{\text{mass}}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right]dx + f\,dx = \rho A\,dx\, u_{tt} \Rightarrow EA\,u_{xx} = \rho A\,u_{tt} \text{ if } f = 0 \text{ \& } EA = \text{const}$$

$$\frac{E}{\rho}u_{xx} = u_{tt} \quad \text{where} \quad c^2 = E/\rho \quad \text{bar or rod velocity}$$

TYPICAL BC are found on pg 372   512 in 3rd ed
599 4th ed
711 5th ed.
729 6th ed.

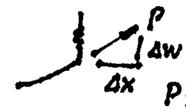
IC. $u(x, t=0) = u_0$

$\dot{u}(x, t=0) = \dot{u}_0$

$EA \frac{\partial u}{\partial x}(x,t) = 0$ no force at end if string/cable connected to a pin that can move $\perp$ to x, end cannot support a transverse force

if end is constrained elastically $P\frac{\partial u}{\partial x} = -ku$ at rhs

— — — — — — — — — — $= ku$ at lhs



$P\frac{\partial w}{\partial x}$ = Force comp vertical

look at string or rod fixed so that $u(x,t)=0$ at $x=0$ & $x=l$

$\Rightarrow$ $u = X \cdot T = 0$ when $x=0$ & $x=l$ $\forall t$ $\Rightarrow$ $X=0$ @ $x=0$
$X=0$ @ $x=l$

$\therefore$ $0 = A\cos 0 + B\sin 0 \Rightarrow A=0$

$0 = \cancel{A\cos l} + B\sin kl \Rightarrow kl = n\pi \qquad k = n\pi/l$

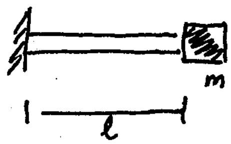
$\therefore$ $T = \sum C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}$

if $u(x,t=0) = u_0 \Rightarrow u_0 = \left(\sum C_n \cos\frac{n\pi c}{L}t + \sum D_n \sin\frac{n\pi c}{L}t\right)\sin\frac{n\pi x}{L}$ (cos term → 1, sin term → 0)

$\therefore$ $C_n = \frac{2}{L}\int_0^L u_0 \sin\frac{n\pi x}{L}\,dx$ since fourier series gives $\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \end{cases}$

$D_n = \frac{2}{nc\pi}\int_0^L \dot{u}_0 \sin\frac{n\pi x}{L}dx$

$\dot{u}(x,t=0) = \dot{u}_0 \Rightarrow \dot{u}_0 = \left(\sum C_n\left(-\frac{n\pi c}{L}\right)\sin\frac{n\pi c}{L}t + \sum D_n \cdot \frac{n\pi c}{L}\cos\frac{n\pi c}{L}t\right)\sin\frac{n\pi x}{L}$ (sin term → 0, cos term → 1)

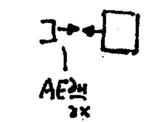


Frequencies of vibrations of bar & mass at end.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad c = \sqrt{E/\rho} \qquad (1)$$

& $u(x=0,t) = 0$ (2)

@ $x = L$ force $= AE\frac{\partial u}{\partial x}(l,t) = -m\frac{\partial^2 u}{\partial t^2}(x=l,t)$ (3)



since solution of (1) is $u = X(x)T(t)$ $\qquad X''T = \frac{1}{c^2}X\ddot{T} \qquad c^2\frac{X''}{X} = \frac{\ddot{T}}{T} = -k^2$

and $X(x) = \tilde{A}\cos\frac{k}{c}x + \tilde{B}\sin\frac{k}{c}x$

$T(t) = C\cos kt + D\sin kt$

the condition (2) $\Rightarrow X(0) = 0 \quad \forall t \Rightarrow \tilde{A} = 0$

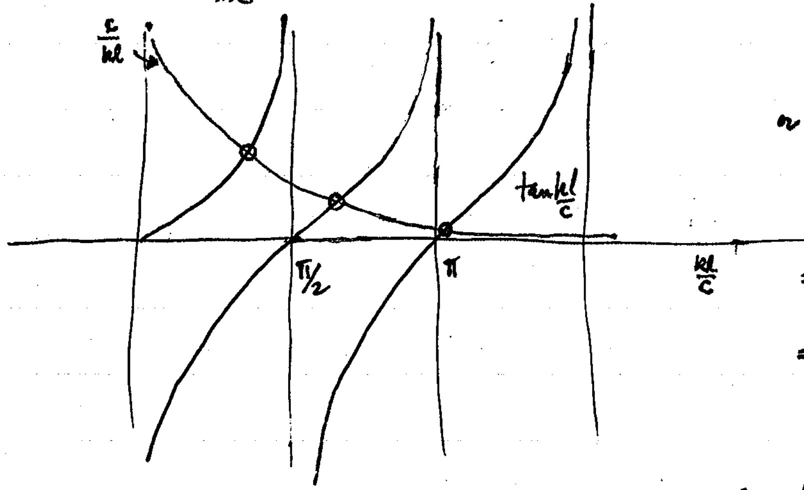
$\therefore \quad u(x,t) = \sum \sin\frac{k}{c}x\left[\bar{C}\cos kt + \bar{D}\sin kt\right] \qquad \bar{C} = C\tilde{B} \quad \bar{D} = D\tilde{B}$

the condition (3)

$$AE\frac{\partial u}{\partial x} = AE\sum\frac{k}{c}\cos\frac{k}{c}l\left[C\cos kt + D\sin kt\right] = +m\sum k^2\sin\frac{k}{c}l\left[C\cos kt + D\sin kt\right]$$

or $AE\frac{k}{c}\cos\frac{k}{c}l = mk^2\sin\frac{k}{c}l \qquad \frac{AE}{mck} = \tan\frac{kl}{c}$ or $\frac{AE}{mc^2}\cdot\frac{c}{k} = \tan\frac{kl}{c}$ or $\frac{AE}{mc^2} = \frac{k}{c}\tan\frac{kl}{c}$

so $\frac{AEl}{mc^2} = \frac{kl}{c}\tan\frac{kl}{c}$ but $c^2 = E/\rho$ or $\frac{A\rho l}{m} = \frac{kl}{c}\tan\frac{kl}{c}$



where $A\rho l = m_{bar}$

or $\frac{c}{kl} = \frac{m}{m_{bar}}\tan\frac{kl}{c}$

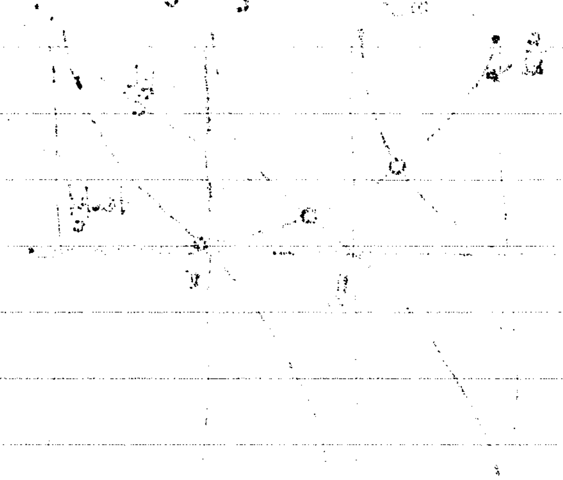
if $m_{bar} \ll m$

$\Rightarrow \rho \ll 1 \Rightarrow c \to \infty \quad c = \sqrt{E/\rho}$

$\Rightarrow \frac{AEl}{mc^2} \to 0$ & $\frac{kl}{c}\tan\frac{kl}{c} \to \frac{k^2l^2}{c^2}$

$\therefore \frac{k^2l^2}{c^2} \cong \frac{m_{bar}}{m}$

[illegible]

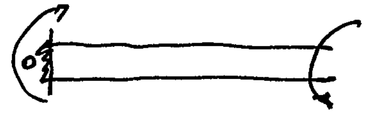


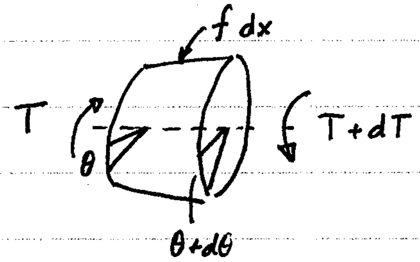
$$\frac{TL}{JG} = \Delta\theta$$

$$\frac{T\Delta x}{JG} = \Delta\theta$$

$$\frac{T}{JG} = \frac{\Delta\theta}{\Delta x} \Rightarrow T = JG\frac{\Delta\theta}{\Delta x}$$

$$\Delta T = \Delta\left(JG\frac{\Delta\theta}{\Delta x}\right)$$





Σ Torques $= I_0 \ddot{\theta}\, dx$ where I_0 is mass moment of inertia per length

$T + dT - T + f\,dx$ where f = torque/length

$$\Delta T = \Delta\left(\frac{JG\,\Delta\theta}{\Delta x}\right)$$

& [illegible]

$I_0 = \int r^2 dm = \rho \int r^2 dx$

$\rho \int r^2 dA = \rho J \Delta x$

if ρ = const along length

Now $dT + f\,dx = I_0 \ddot{\theta}\, dx$ or $\dfrac{\partial T}{\partial x} = \dfrac{\partial}{\partial x}\left(JG \dfrac{\partial \theta}{\partial x}\right)$

& $\dfrac{dT}{dx} + f = I_0 \ddot{\theta}$ or $\dfrac{\partial}{\partial x}\left(JG\dfrac{\partial\theta}{\partial x}\right) + f = I_0\ddot{\theta}$

if JG = const (Torsional Stiffness) $JG\dfrac{\partial^2\theta}{\partial x^2} + f = I_0 \ddot{\theta}$; if $f = 0$ $\dfrac{\partial^2\theta}{\partial x^2} = \dfrac{1}{c_r^2}\dfrac{\partial^2\theta}{\partial t^2}$

where $c_r = \sqrt{\dfrac{JG}{I_0}}$

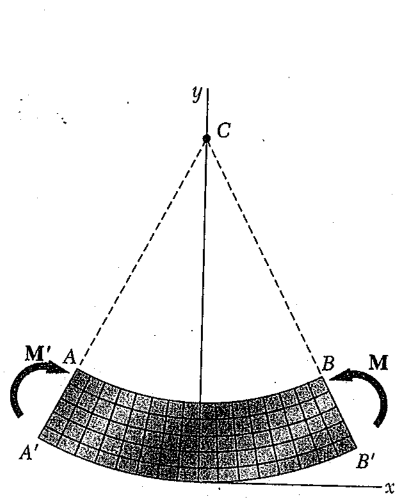
for a circular shaft of uniform density $J\rho = I_0$ $\therefore$ $c_r = \sqrt{\dfrac{G}{\rho}}$

$$\sqrt{\frac{N/m^2}{kg/m^3}} = \sqrt{\frac{N\text{-}m}{kg}} = \sqrt{\frac{kg\,m^2/s^2}{kg}} = \sqrt{m^2/s^2} = m/s$$

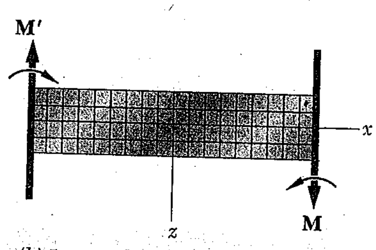
this is the rotational equivalent or the longitudinal vibrations of a bar

$G = \dfrac{E}{2(1+\nu)}$ $\therefore$ $c_r = c \cdot \dfrac{1}{\sqrt{2(1+\nu)}}$ or $c_r < c = \sqrt{\dfrac{E}{\rho}}$

BC's given on pg 738 6th ed



(*a*) Longitudinal, vertical section (plane of symmetry)

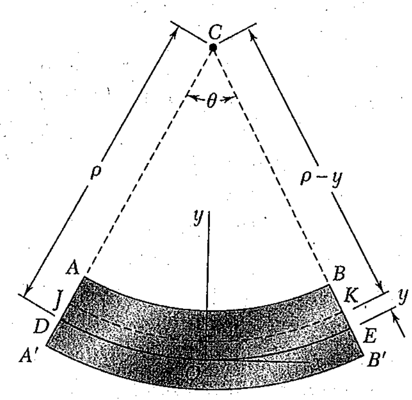


(*b*) Longitudinal, horizontal section

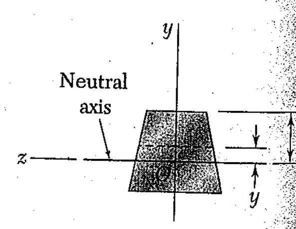
Fig. 4.11

Suppose that the member is divided into a large number of small cubic elements with faces respectively parallel to the three coordinate planes. The property we have established requires that these elements be transformed as shown in Fig. 4.11 when the member is subjected to the couples **M** and **M'**. Since all the faces represented in the two projections of Fig. 4.11 are at 90° to each other, we conclude that $\gamma_{xy} = \gamma_{zx} = 0$ and, thus, that $\tau_{xy} = \tau_{xz} = 0$. Regarding the three stress components that we have not yet discussed, namely, σ_y, σ_z, and τ_{yz}, we note that they must be zero on the surface of the member. Since, on the other hand, the deformations involved do not require any interaction between the elements of a given transverse cross section, we can assume that these three stress components are equal to zero throughout the member. This assumption is verified, both from experimental evidence and from the theory of elasticity, for slender members undergoing small deformations.† We conclude that the only nonzero stress component exerted on any of the small cubic elements considered here is the normal component σ_x. Thus, at any point of a slender member in pure bending, we have a state of *uniaxial stress*. Recalling that, for $M > 0$, lines AB and $A'B'$ are observed, respectively, to decrease and increase in length, we note that the strain ϵ_x and the stress σ_x are negative in the upper portion of the member (*compression*) and positive in the lower portion (*tension*).

It follows from the above that there must exist a surface parallel to the upper and lower faces of the member, where ϵ_x and σ_x are zero. This surface is called the *neutral surface*. The neutral surface intersects the plane of symmetry along an arc of circle DE (Fig. 4.12*a*), and it intersects a transverse section along a straight line called the *neutral axis* of the section (Fig. 4.12*b*). The origin of coordinates will now be selected on the neutral surface, rather than on the lower face of the member as done earlier, so that the distance from any point to the neutral surface will be measured by its coordinate y.



(*a*) Longitudinal, vertical section (plane of symmetry)



(*b*) Transverse section

Fig. 4.12

†Also see Prob. 4.38.

Denoting by ρ the radius of arc DE (Fig. 4.12*a*), by θ the central angle corresponding to DE, and observing that the length of DE is equal to the length L of the undeformed member, we write

$$L = \rho\theta \tag{4.4}$$

Considering now the arc JK located at a distance y above the neutral surface, we note that its length L' is

$$L' = (\rho - y)\theta \tag{4.5}$$

Since the original length of arc JK was equal to L, the deformation of JK is

$$\delta = L' - L \tag{4.6}$$

or, if we substitute from (4.4) and (4.5) into (4.6),

$$\delta = (\rho - y)\theta - \rho\theta = -y\theta \tag{4.7}$$

The longitudinal strain ϵ_x in the elements of JK is obtained by dividing δ by the original length L of JK. We write

$$\epsilon_x = \frac{\delta}{L} = \frac{-y\theta}{\rho\theta}$$

or

$$\epsilon_x = -\frac{y}{\rho} \tag{4.8}$$

The minus sign is due to the fact that we have assumed the bending moment to be positive and, thus, the beam to be concave upward.

Because of the requirement that transverse sections remain plane, identical deformations will occur in all planes parallel to the plane of symmetry. Thus the value of the strain given by Eq. (4.8) is valid anywhere, and we conclude that the *longitudinal normal strain ϵ_x varies linearly with the distance y from the neutral surface.*

The strain ϵ_x reaches its maximum absolute value when y itself is largest. Denoting by c the largest distance from the neutral surface (which corresponds to either the upper or the lower surface of the member), and by ϵ_m the *maximum absolute value* of the strain, we have

$$\epsilon_m = \frac{c}{\rho} \tag{4.9}$$

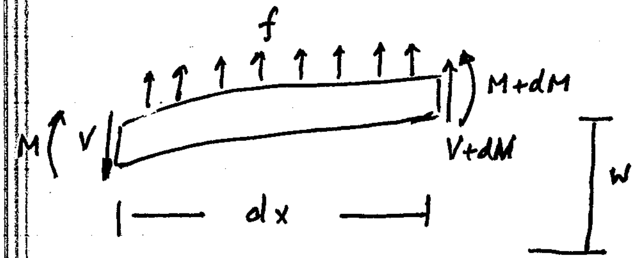
Solving (4.9) for ρ and substituting the value obtained into (4.8), we can also write

$$\epsilon_x = -\frac{y}{c}\epsilon_m \tag{4.10}$$

We conclude our analysis of the deformations of a member in pure bending by observing that we are still unable to compute the strain or stress at a given point of the member, since we have not yet located the neutral surface in the member. In order to locate this surface, we must first specify the stress-strain relation of the material used.†

†Let us note, however, that if the member possesses both a vertical and a horizontal plane of symmetry (e.g., a member with a rectangular cross section), and if the stress-strain curve is the same in tension and compression, the neutral surface will coincide with the plane of symmetry (cf. Sec. 4.8).

for transverse vibs of beam.



$$m \cdot accel = \rho A\,dx \cdot \frac{\partial^2 w}{\partial t^2}$$

Now $\Sigma F_y = V + dV - V + f\,dx = \rho A\,dx \frac{\partial^2 w}{\partial t^2}$

$\Sigma M = M + dM - M + V\,dx = 0$

$\therefore \frac{dM}{dx} = -V$

so $\frac{\partial V}{\partial x} + f = \rho A \frac{\partial^2 w}{\partial t^2}$ or $-\frac{\partial^2 M}{\partial x^2} + f = \rho A \frac{\partial^2 w}{\partial t^2}$

Now from mechanics of materials

$M = EI\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{\partial M}{\partial x}\right) = -\frac{\partial^2 M}{\partial x^2}$

$$\therefore \quad \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t)$$

if EI is constant then $EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t)$

if $f=0$ (free vibs) $\quad EI\,w^{IV} + \rho A \ddot{w} = 0$

let $w(x,t) = X(x)\,T(t)$ $\quad EI\,X^{IV}T + \rho A X \ddot{T} = 0$

& $\frac{X^{IV}}{X} + \frac{\rho A}{EI}\frac{\ddot{T}}{T} = 0$ $\quad$ let $\sqrt{\frac{EI}{\rho A}} = c$

$c^2 \frac{X^{IV}}{X} = -\frac{\ddot{T}}{T} = k^4$ (crossed-out terms in original)

$\therefore \quad T = A\cos k^2 t + B \sin k^2 t$

$X = \check{C} e^{\frac{k}{\sqrt{c}}x} + \check{D} e^{-\frac{k}{\sqrt{c}}x} + E \sin\frac{k}{\sqrt{c}}x + F\cos\frac{k}{\sqrt{c}}x$

$= C\sinh\frac{k}{\sqrt{c}}x + D\cosh\frac{k}{\sqrt{c}}x + E\sin\frac{k}{\sqrt{c}}x + F\cos\frac{k}{\sqrt{c}}x$

$\check{C} = \frac{C}{2} + \frac{D}{2}$

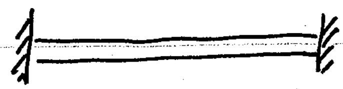
$\check{D} = -\frac{C}{2} + \frac{D}{2}$

BC. S.S. $\Rightarrow w = 0$ & $M = M_{prescribed} = EIw''$

Free $\Rightarrow$ shear or $-(EIw'')' = V_{prescribed}$ & $M = M_{prescr} = EIw''$

Fixed $\Rightarrow w = 0$ & $\frac{\partial w}{\partial x} = 0$

let us look at



$w(x=0,t)=0$ ⓐ $\qquad$ $w(x=l,t)=0$ ⓒ

$\frac{\partial w}{\partial x}(x=0,t)=0$ ⓑ $\qquad$ $\frac{\partial w}{\partial x}(x=l,t)=0$ ⓓ

Fixed Fixed

$$w = XT$$

ⓐ $0 = X(0)T(t) \Rightarrow X(0)=0 \ \forall T \qquad X(0) = C\cdot 0 + D\cdot 1 + E\cdot 0 + F\cdot 1 = 0$

ⓑ $0 = X'(0)T(t) \Rightarrow X'(0)=0 \qquad X'(0) = C\cdot\frac{k}{\sqrt{c}}\cdot 1 + D\cdot\frac{k}{\sqrt{c}}\cdot 0 + E\frac{k}{\sqrt{c}}\cdot 1 + F\frac{k}{\sqrt{c}}\cdot 0 = 0$

ⓒ $0 = X(l)T(t) \Rightarrow X(l)=0 \qquad X(l) = C\sinh\frac{kl}{\sqrt{c}} + D\cosh\frac{kl}{\sqrt{c}} + E\sin\frac{kl}{\sqrt{c}} + F\cos\frac{kl}{\sqrt{c}} = 0$

ⓓ $0 = X'(l)T(t) \Rightarrow X'(l)=0 \qquad X'(0) = C\frac{k}{\sqrt{c}}\cosh\frac{kl}{\sqrt{c}} + D\frac{k}{\sqrt{c}}\sinh\frac{kl}{\sqrt{c}} + E\frac{k}{\sqrt{c}}\cos\frac{kl}{\sqrt{c}} - F\frac{k}{\sqrt{c}}\sin\frac{kl}{\sqrt{c}} = 0$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \frac{k}{\sqrt{c}} & 0 & \frac{k}{\sqrt{c}} & 0 \\ \mathrm{sh}\frac{kl}{\sqrt{c}} & \mathrm{ch}\frac{kl}{\sqrt{c}} & \mathrm{s}\frac{kl}{\sqrt{c}} & \mathrm{c}\frac{kl}{\sqrt{c}} \\ \frac{k}{\sqrt{c}}\mathrm{ch}\frac{kl}{\sqrt{c}} & \frac{k}{\sqrt{c}}\mathrm{sh}\frac{kl}{\sqrt{c}} & \frac{k}{\sqrt{c}}\mathrm{c}\frac{kl}{\sqrt{c}} & -\frac{k}{\sqrt{c}}\mathrm{s}\frac{kl}{\sqrt{c}} \end{bmatrix} \begin{pmatrix} C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$$

$$\Rightarrow \det(\ ) = 0 = 0\cdot\left| \quad \right| - 1\begin{vmatrix} k/\sqrt{c} & k/\sqrt{c} & 0 \\ \mathrm{sh}\cdot & \mathrm{s}\cdot & \mathrm{c}\cdot \\ \frac{k}{\sqrt{c}}\mathrm{ch}\cdot & \frac{k}{\sqrt{c}}\mathrm{c}\cdot & -\frac{k}{\sqrt{c}}\mathrm{s}\cdot \end{vmatrix} + 0\left| \quad \right|$$

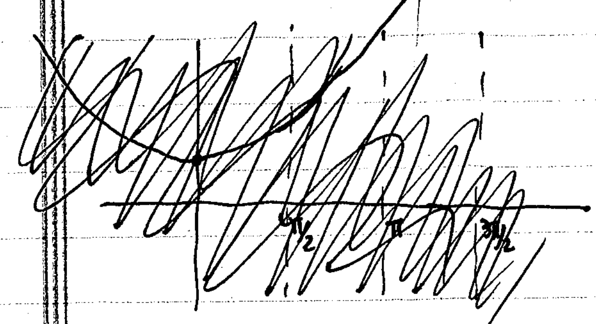
$$+1\begin{vmatrix} k/\sqrt{c} & 0 & \frac{k}{\sqrt{c}} \\ \mathrm{sh}\cdot & \mathrm{ch}\cdot & \mathrm{s}\cdot \\ \frac{k}{\sqrt{c}}\mathrm{ch}\cdot & \frac{k}{\sqrt{c}}\mathrm{sh}\cdot & \frac{k}{\sqrt{c}}\mathrm{c}\cdot \end{vmatrix}$$

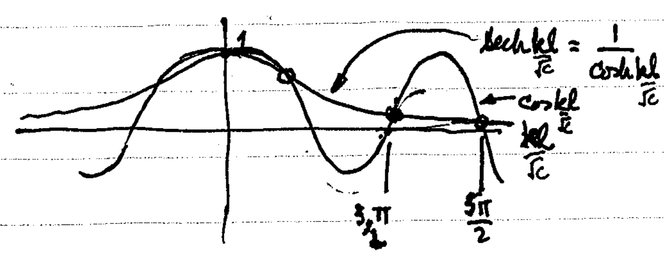
$$-1\left(-\frac{k^2}{c}\sin^2\frac{kl}{\sqrt{c}} + \frac{k^2}{c}\cos\frac{kl}{\sqrt{c}}\cosh\frac{kl}{\sqrt{c}} + \frac{k^2}{c}\sin\frac{kl}{\sqrt{c}}\sinh\frac{kl}{\sqrt{c}} - \frac{k^2}{c}\cos^2\frac{kl}{\sqrt{c}}\right) + 1\left(\frac{k^2}{c}\cos\frac{kl}{\sqrt{c}}\cosh\frac{kl}{\sqrt{c}} + \frac{k^2}{c}\sinh^2\frac{kl}{\sqrt{c}} - \frac{k^2}{c}\cosh^2\frac{kl}{\sqrt{c}} - \frac{k^2}{c}\sin kl\sinh kl\right)$$

$$\frac{k^2}{c} - \frac{k^2}{c}(\mathrm{c}\cdot\mathrm{ch} + \mathrm{s}\cdot\mathrm{sh}) + \frac{k^2}{c} + \frac{k^2}{c}(\mathrm{c}\cdot\mathrm{ch} - \mathrm{s}\cdot\mathrm{sh}) =$$

$$2\frac{k^2}{c} - 2\frac{k^2}{c}\cos\frac{kl}{\sqrt{c}}\cosh\frac{kl}{\sqrt{c}} - \frac{k^2}{c}(\mathrm{s}\cdot\mathrm{sh} - \mathrm{s}\cdot\mathrm{sh}) = 0$$

or $\cos\frac{kl}{\sqrt{c}}\cosh\frac{kl}{\sqrt{c}} = 1$ $\qquad$ characteristic eqn if $\frac{kl}{\sqrt{c}} \neq 0$





Thus the response of the beam is given by Eq. (8.110):

$$w(x, t) = \frac{2 f_0}{\rho A l} \sum_{n=1}^{\infty} \frac{1}{\omega_n^2 - \omega^2} \sin \frac{n\pi a}{l} \sin \frac{n\pi x}{l} \sin \omega t \qquad \text{(E.7)}$$

■

8.5.7 Effect of Axial Force

The problem of vibrations of a beam under the action of axial force finds application in the study of vibrations of cables and guy wires. For example, although the vibrations of a cable can be found by treating it as an equivalent string, many cables have failed due to fatigue caused by alternating flexure. The alternating flexure is produced by the regular shedding of vortices from the cable in a light wind. We must therefore consider the effects of axial force and bending stiffness on lateral vibrations in the study of fatigue failure of cables.

To find the effect of an axial force $P(x, t)$ on the bending vibrations of a beam, consider the equation of motion of an element of the beam, as shown in Fig. 8.18. For the vertical motion, we have

$$-(V + dV) + f\,dx + V + (P + dP)\sin(\theta + d\theta) - P \sin\theta = \rho A\, dx \frac{\partial^2 w}{\partial t^2} \qquad (8.118)$$

$dV + f dx + \frac{d(P\sin\theta)}{dx} dx$

and for the rotational motion about 0,

$$(M + dM) - (V + dV)\, dx + f\, dx \frac{dx}{2} - M = 0 \qquad (8.119)$$

$dM + V dx = 0$

$\rightarrow \frac{dV}{dx} dx + f dx + \frac{d}{dx}(P\sin\theta) dx$

$- \frac{d^2M}{dx^2} dx + f dx + \frac{d}{dx}\left(P \frac{\partial w}{\partial x}\right) dx$

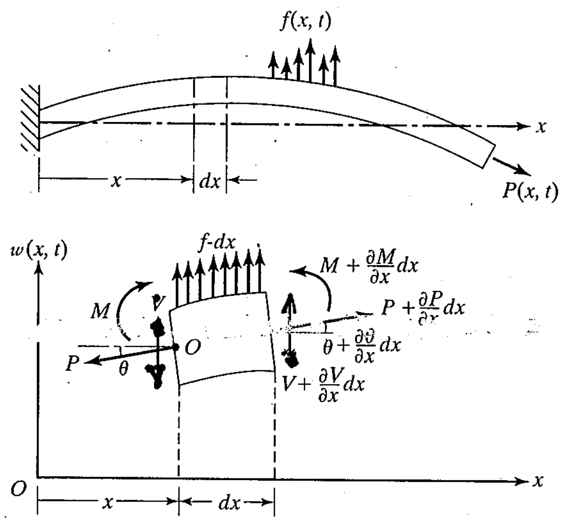


FIGURE 8.18 An element of a beam under axial load.

For small deflections,

$$\sin(\theta + d\theta) \simeq \theta + d\theta = \theta + \frac{\partial \theta}{\partial x} dx = \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx$$

With this, Eqs. (8.118), (8.119), and (8.75) can be combined to obtain a single differential equation of motion:

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right] + \rho A \frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} = f \tag{8.120}$$

For the free vibration of a uniform beam, Eq. (8.120) reduces to

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} = 0 \tag{8.121}$$

The solution of Eq. (8.121) can be obtained using the method of separation of variables as

$$w(x, t) = W(x)(A \cos \omega t + B \sin \omega t) \tag{8.122}$$

Substitution of Eq. (8.122) into Eq. (8.121) gives

$$EI\frac{d^4 W}{dx^4} - P\frac{d^2 W}{dx^2} - \rho A \omega^2 W = 0 \tag{8.123}$$

By assuming the solution $W(x)$ to be

$$W(x) = Ce^{sx} \tag{8.124}$$

in Eq. (8.123), the auxiliary equation can be obtained:

$$s^4 - \frac{P}{EI}s^2 - \frac{\rho A \omega^2}{EI} = 0 \tag{8.125}$$

The roots of Eq. (8.125) are

$$s_1^2, s_2^2 = \frac{P}{2EI} \pm \left(\frac{P^2}{4E^2 I^2} + \frac{\rho A \omega^2}{EI}\right)^{1/2} \tag{8.126}$$

and so the solution can be expressed as (with absolute value of s_2)

$$W(x) = C_1 \cosh s_1 x + C_2 \sinh s_1 x + C_3 \cos s_2 x + C_4 \sin s_2 x \tag{8.127}$$

W'' $s_1^2 C_1 \cosh + s_1^2 C_2 \sinh - s_2^2 C_3 \cos - s_2^2 C_4 \sin$

where the constants C_1 to C_4 are to be determined from the boundary conditions.

EXAMPLE 8.9 Beam Subjected to an Axial Compressive Force

Find the natural frequencies of a simply supported beam subjected to an axial compressive force.

Solution: The boundary conditions are

$$W(0) = 0 \tag{E.1}$$

$$\frac{d^2W}{dx^2}(0) = 0 \tag{E.2}$$

$$W(l) = 0 \tag{E.3}$$

$$\frac{d^2W}{dx^2}(l) = 0 \tag{E.4}$$

Equations (E.1) and (E.2) require that $C_1 = C_3 = 0$ in Eq. (8.127), and so

$$W(x) = C_2 \sinh s_1 x + C_4 \sin s_2 x \tag{E.5}$$

The application of Eqs. (E.3) and (E.4) to Eq. (E.5) leads to

$$\sinh s_1 l \cdot \sin s_2 l = 0 \tag{E.6}$$

Since $\sinh s_1 l > 0$ for all values of $s_1 l \neq 0$, the only roots to this equation are

$$s_2 l = n\pi, \qquad n = 0, 1, 2, \ldots \tag{E.7}$$

Thus Eqs. (E.7) and (8.126) give the natural frequencies of vibration:

$$\omega_n = \frac{\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}}\left(n^4 + \frac{n^2 P l^2}{\pi^2 EI}\right)^{1/2} \tag{E.8}$$

Since the axial force P is compressive, P is negative. Further, from strength of materials, the smallest Euler buckling load for a simply supported beam is given by [8.9]

$$P_{\text{cri}} = \frac{\pi^2 EI}{l^2} \tag{E.9}$$

Thus Eq. (E.8) can be written as

$$\omega_n = \frac{\pi^2}{l^2}\left(\frac{EI}{\rho A}\right)^{1/2}\left(n^4 - n^2\frac{P}{P_{\text{cri}}}\right)^{1/2} \tag{E.10}$$

The following observations can be made from the present example:

1. If $P = 0$, the natural frequency will be same as that of a simply supported beam given in Fig. 8.15.
2. If $EI = 0$, the natural frequency (see Eq. E.8) reduces to that of a taut string.
3. If $P > 0$, the natural frequency increases as the tensile force stiffens the beam.
4. As $P \rightarrow P_{\text{cri}}$, the natural frequency approaches zero for $n = 1$.

■

$\epsilon = \frac{\partial u}{\partial x}$

$\sigma = E\epsilon = E\frac{\partial u}{\partial x}$

$U = \frac{1}{2}\int\left(EA\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)dx$

$T = \frac{1}{2}\int\rho\left(\frac{\partial u}{\partial t}\right)^2\cdot A\,dx$

$W = P_l u_l - P_0 u_0 - \int_0^l p(x,t)\cdot u(x,t)\,dx$

A

$$\delta\int_{t_0}^{t_1}\left\{\frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial x}\right)^2 dx - \frac{1}{2}\int_0^l \rho\left(\frac{\partial u}{\partial t}\right)^2\cdot A\,dx\right.$$

$$\iint\left(m\frac{\partial y}{\partial t}\cdot\frac{\partial \delta y}{\partial t}\right)dx\,dt$$

u dv

$u = m\frac{\partial y}{\partial t}$ $dv = \frac{\partial}{\partial t}\delta y\,dt$

$du = \frac{\partial}{\partial t}\left(m\frac{\partial y}{\partial t}\right)dt$ $v = \delta y$

$$\iint m\frac{\partial y}{\partial t}\frac{\partial \delta y}{\partial t}\,dt\,dx = \underbrace{\int m\frac{\partial y}{\partial t}\delta y\Big|_{t_0}^{t_1}}_{=0} - \int\left(\int\frac{\partial}{\partial t}\left(m\frac{\partial y}{\partial t}\right)dt\cdot\delta y\right)dx$$

$\sigma = \frac{My}{I} = EI\frac{d^2y}{dx^2}\cdot\frac{y}{I}$

$\epsilon = \sigma/E$

$$\frac{1}{2}\int\frac{\sigma^2}{E}dV = \int\frac{M^2y^2}{I^2E}\cdot dx\,dA = \frac{M^2 I}{I^2E}dx = \frac{\left(EI\frac{d^2y}{dx^2}\right)^2}{IE} = EI(y'')^2$$

u dv

$EIy''\cdot\delta(y'')dx = EIy''(\delta y)''dx$

$\frac{\partial}{\partial x}(EIy'')dx$ $v = \frac{\partial}{\partial x}\delta y$

du

$$-\int\frac{\partial}{\partial x}(EIy'')\frac{\partial\delta y}{\partial x} + EIy''\cdot\frac{\partial}{\partial x}\delta y\Big|_{t_0}^{t_1}$$

dx

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The term

(12) $$L \equiv U - K + A$$

(or sometimes $-L$) is called the *Lagrangian function* and the equation (11) represents *Hamilton's principle*, which states that:

The time integral of the Lagrangian function over a time interval t_0 to t_1 is an extremum for the "actual" motion with respect to all admissable virtual displacements which vanish, first, at instants of time t_0 and t_1 at all points of the body, and, second, over S_u, where the displacements are prescribed, throughout the entire time interval.

To formulate this principle in another way, let us call $u_i(x_1, x_2, x_3; t)$ a dynamic path. Then Hamilton's principle states that *among all dynamic paths that satisfy the boundary conditions over S_u at all times and that start and end with the actual values at two arbitrary instants of time t_0 and t_1 at every point of the body, the "actual" dynamic path is distinguished by making the Lagrangian function an extremum.*

In rigid body dynamics the term U drops out, and we obtain Hamilton's principle in the familiar form. The symbol A replaces the usual symbol V in books on dynamics because we have used V for something else.

Note that the potential energy $-A$ of the external loads exists and is a linear function of the displacements if the loads are independent of the elastic displacements, as is commonly the case. In aeroelastic problems, however, the aerodynamic loading is sensitive to the small surface displacements u_i; moreover, it depends on the time history of the displacements and cannot be derived from a potential. Hence, in aeroelasticity we are generally forced to use the variational form (9) of Hamilton's principle.

In some applications of the direct method of calculation, it is even desirable to liberalize the variations δu_i at the instants t_0 and t_1 and use Hamilton's principle in the variational form (4) which cannot be expressed elegantly as the minimum of a well-defined functional. On the other hand, such a formulation will be accessible to the direct methods of solution. On introducing (5), (7), and (10), we may rewrite Eq. (4) in the following form:

(13) $$\int_{t_0}^{t_1} \delta(U - K + A)\, dt = \int_{t_0}^{t_1}\int_V F_i\, \delta u_i\, dv\, dt + \int_{t_0}^{t_1}\int_S \overset{\nu}{T}_i\, \delta u_i\, dS - \int_V \rho \frac{\partial u_i}{\partial t}\, \delta u_i\, dv \Big|_{t_0}^{t_1}.$$

Here U is the total strain energy, K is the total kinetic energy, A is the potential energy for the conservative external forces, F_i and $\overset{\nu}{T}_i$ are, respectively, those external body and surface forces that are not included in A, and δu_i are the virtual displacements.

Problem 11.1. Prove the converse theorem that, for a conservative system, the variational Eq. (11) leads to the equation of motion

$$\rho \frac{\partial^2 u_i}{\partial t^2} = F_i + \frac{\partial}{\partial x_j}\frac{\partial W}{\partial e_{ij}}$$

and the boundary conditions

$$\text{either} \quad \delta u_i = 0 \quad \text{or} \quad \frac{\partial W}{\partial e_{ij}} \nu_j = \overset{\nu}{T}_i.$$

11.2. EXAMPLE OF APPLICATION—EQUATION OF VIBRATION OF A BEAM

As an example of the application of Hamilton's principle in the formulation of approximate theories in elasticity, let us consider the free, lateral vibration of a straight simple beam. We assume that the beam possesses principal planes and that the vibration takes place in one of the principal planes, and let y denote the small deflection of the neutral axis of the beam from its initial, straight configuration. In Sec. 10.8 it is shown that the strain energy of the beam is, for small deflections,

(1) $$U = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx,$$

where E is the Young's modulus of the beam material, I is the cross-sectional moment of interia, and l is the length of the beam.

The kinetic energy of the beam is derived partly from the translation, parallel to y, of the elements composing it, and partly from the rotation of the same elements about an axis perpendicular to the neutral axis and the plane of vibration. The former part is

$$\frac{1}{2}\int_0^l m\left(\frac{\partial y}{\partial t}\right)^2 dx,$$

where m is the mass per unit length of the beam. The latter part is, for each element dx, the product of moment of inertia times one-half of the square of the angular velocity. Let I_ρ denote the mass moment of inertia about the neutral axis per unit length of the beam. The angular velocity being $\partial^2 y/\partial t\, \partial x$, the kinetic energy of the beam is

(2) $$K = \frac{1}{2}\int_0^l m\left(\frac{\partial y}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^l I_\rho\left(\frac{\partial^2 y}{\partial x\, \partial t}\right)^2 dx.$$

If the beam is loaded by a distributed lateral load of intensity $p(x, t)$ per unit length and moment and shear M and Q, respectively, at the ends as shown in Fig. 11.2:1, then the potential energy of the external loading is

(3) $$A = -\int_0^l p(x, t)\,y(x)\, dx - M_l\left(\frac{\partial y}{\partial x}\right)_l + M_0\left(\frac{\partial y}{\partial x}\right)_0 + Q_l y_l - Q_0 y_0.$$

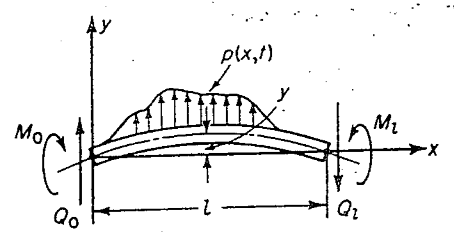


Fig. 11.2:1. Applications to a beam.

The equation of motion is given by Hamilton's principle:

(4) $$\delta \int_{t_0}^{t_1} (U - K + A)\, dt = 0;$$

i.e.,

(5) $$\delta \int_{t_0}^{t_1} \left\{ \int_0^l \left[\frac{1}{2} EI \left(\frac{\partial^2 y}{\partial x^2} \right)^2 - \frac{1}{2} m \left(\frac{\partial y}{\partial t} \right)^2 - \frac{1}{2} I_\rho \left(\frac{\partial^2 y}{\partial x\, \partial t} \right)^2 - py \right] dx - M_l \left(\frac{\partial y}{\partial x} \right)_l + M_0 \left(\frac{\partial y}{\partial x} \right)_0 + Q_l y_l - Q_0 y_0 \right\} dt = 0.$$

Following the usual procedure of the calculus of variations, noting that the virtual displacement must be so specified that $\delta y \equiv 0$ at t_0 and t_1, and, hence, $\partial(\delta y)/\partial x = \delta(\partial y/\partial x) \equiv 0$ at t_0 and t_1, we obtain

$$\int_{t_0}^{t_1} \left[\int_0^l \left(EI \frac{\partial^2 y}{\partial x^2} \frac{\partial^2 \delta y}{\partial x^2} - m \frac{\partial y}{\partial t} \frac{\partial \delta y}{\partial t} - I_\rho \frac{\partial^2 y}{\partial x\, \partial t} \frac{\partial^2 \delta y}{\partial x\, \partial t} - p\, \delta y \right) dx - M_l\, \delta \left(\frac{\partial y}{\partial x} \right)_l + M_0\, \delta \left(\frac{\partial y}{\partial x} \right)_0 + Q_l\, \delta y_l - Q_0\, \delta y_0 \right] dt = 0.$$

Integrating by parts, we obtain

(6) $$\int_{t_0}^{t_1} \int_0^l \left[\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y}{\partial x^2} \right) + m \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2}{\partial x\, \partial t} \left(I_\rho \frac{\partial^2 y}{\partial x\, \partial t} \right) - p(x, t) \right] \delta y\, dx\, dt - \int_{t_0}^{t} \left[EI \frac{\partial^2 y}{\partial x^2} - M \right] \delta \left(\frac{\partial y}{\partial x} \right) \Big|_0^l \, dt - \int_{t_0}^{t} \left[\frac{\partial}{\partial x} \left(EI \frac{\partial^2 y}{\partial x^2} \right) - \frac{\partial}{\partial t} \left(I_\rho \frac{\partial^2 y}{\partial x\, \partial t} \right) - Q \right] \delta y \Big|_0^l \, dt = 0.$$

Hence, the Euler equation of motion is

(7) $$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y}{\partial x^2} \right) + m \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2}{\partial x\, \partial t} \left(I_\rho \frac{\partial^2 y}{\partial x\, \partial t} \right) = p(x, t),$$

and a proper set of boundary conditions at each end is

(8a) $$\text{either} \quad EI \frac{\partial^2 y}{\partial x^2} = M \quad \text{or} \quad \delta \left(\frac{\partial y}{\partial x} \right) = 0$$

and

(8b) $$\text{either} \quad \frac{\partial}{\partial x} \left(EI \frac{\partial^2 y}{\partial x^2} \right) - \frac{\partial}{\partial t} \left(I_\rho \frac{\partial^2 y}{\partial x\, \partial t} \right) = Q \quad \text{or} \quad \delta y = 0.$$

*These are equations governing the motion of a beam including the effect of the rotary inertia, due to Lord Rayleigh, and known as **Rayleigh's equations.*** If the rotary inertia is neglected and if the beam were uniform, then the governing equation is simplified into:

(9) $$\frac{\partial^2 y}{\partial t^2} + c_0^2 R^2 \frac{\partial^4 y}{\partial x^4} = \frac{1}{EI} p,$$

where

(10) $$c_0^2 = \frac{E}{\rho}, \qquad R^2 = \frac{I}{A}.$$

The constant c_0 has the dimension of speed and can be identified as the phase velocity of longitudinal waves in a uniform bar.† R is the radius of gyration of the cross section. A is the cross-sectional area, so that $m = \rho A$.

In the special case of a uniform beam of infinite length free from lateral loading, $p = 0$, Eq. (9) becomes

(11) $$\frac{\partial^2 y}{\partial t^2} + c_0^2 R^2 \frac{\partial^4 y}{\partial x^4} = 0.$$

It admits a solution in the form

(12) $$y = a \sin \frac{2\pi}{\lambda}(x - ct),$$

which represents a progressive wave of phase velocity c and wave length λ. On substituting (12) into (11), we obtain the relation

(13) $$c = \pm c_0 R \frac{2\pi}{\lambda},$$

which states that the phase velocity depends on the wave length and that it tends to infinity for very short wave lengths. Somewhat disconcerting is the fact that, according to Eq. (13), the group velocity (see Sec. 11.3) also tends to infinity as the wave length tends to zero. Since group velocity is the velocity at which energy is transmitted, this result is physically unreasonable. If Eq. (13) were correct, then the effect of a suddenly applied concentrated load will be felt at once everywhere in the beam, as the Fourier representation for a concentrated load contains harmonic components with infinitesimal wave length, and hence infinite wave speed. Thus, Eq. (11) cannot be very accurate in describing the effect of impact loads on a beam.

† See Prob. 11.2, p. 325.

This difficulty of infinite wave speed is removed by the inclusion of the rotary inertia. However, the speed versus wave length relationship obtained from Rayleigh's Eq. (7) for a uniform beam of circular cross section with radius a, as is shown in Fig. 11.2:2, still deviates appreciably from Pochhammer and Chree's results, which were derived from the exact three-dimensional linear elasticity theory. A much better approximation is obtained by including the shear deflection of the beam, as was first shown by Timoshenko.

To incorporate the shear deformation, we note that the slope of the deflection curve depends not only on the rotation of cross sections of the beam but also on the shear. Let ψ denote the slope of the deflection curve when the shearing force is neglected and β the angle of shear at the neutral axis in the same cross section. Then the total slope is

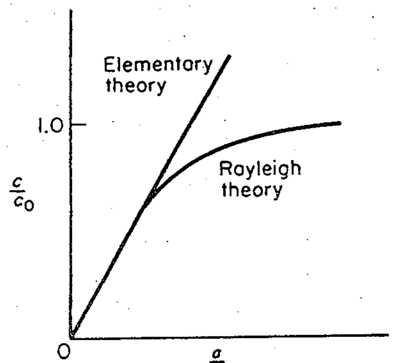


Fig. 11.2:2. Phase velocity curves for flexural elastic waves in a circular cylinder of radius a.

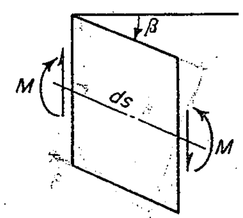


Fig. 11.2:3. A Timoshenko beam element.

$$\frac{\partial y}{\partial x} = \psi + \beta. \tag{14}$$

The strain energy due to bending, Eq. (1), must be replaced by

$$\frac{1}{2}\int_0^l EI\left(\frac{\partial \psi}{\partial x}\right)^2 dx, \tag{15}$$

because the internal bending moment does no work when shear deformation takes place (see Fig. 11.2:3). The strain energy due to shearing strain β must be a quadratic function of β if linear elasticity is assumed. We shall write

$$\frac{1}{2}\int_0^l k\beta^2\,dx = \frac{1}{2}\int_0^l k\left(\frac{\partial y}{\partial x} - \psi\right)^2 dx \tag{16}$$

for the strain energy for shear. The kinetic energy is

$$K = \frac{1}{2}\int_0^l m\left(\frac{\partial y}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^l I_\rho\left(\frac{\partial \psi}{\partial t}\right)^2 dx, \tag{17}$$

because the translational velocity is $\partial y/\partial t$, but the angular velocity is $\partial\psi/\partial t$. Hence, Hamilton's principle states that

$$\delta\int_{t_0}^{t_1}\int_0^l \frac{1}{2}\left[EI\left(\frac{\partial \psi}{\partial x}\right)^2 + k\left(\frac{\partial y}{\partial x} - \psi\right)^2 - m\left(\frac{\partial y}{\partial t}\right)^2 - I_\rho\left(\frac{\partial \psi}{\partial t}\right)^2\right] dx\,dt + \delta A = 0, \tag{18}$$

where A is given by (3) except that $\partial y/\partial x$ at the ends is to be replaced by ψ. The virtual displacements now consist of δy and $\delta\psi$, which must vanish at t_0 and t_1 and also where displacements are prescribed. On carrying out the calculations, the following two Euler equations are obtained:

$$\frac{\partial}{\partial x}\left(EI\frac{\partial \psi}{\partial x}\right) + k\left(\frac{\partial y}{\partial x} - \psi\right) - I_\rho\frac{\partial^2 \psi}{\partial t^2} = 0, \tag{19a}$$

$$m\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left[k\left(\frac{\partial y}{\partial x} - \psi\right)\right] - p = 0. \tag{19b}$$

The appropriate boundary conditions are, at each end of the beam,

$$\text{Either } -EI\frac{\partial \psi}{\partial x} = M \quad \text{or} \quad \delta\psi = 0, \tag{20a}$$

and

$$\text{either } k\left(\frac{\partial y}{\partial x} - \psi\right) = Q \quad \text{or} \quad \delta y = 0. \tag{20b}$$

These are the differential equation and boundary conditions of the so-called *Timoshenko beam theory*.

For a uniform beam, EI, k, m, etc., are constants, and the function ψ can be eliminated from the equations above to obtain the well-known *Timoshenko equation for lateral vibration of prismatic beams*,

$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} - \left(I_\rho + \frac{EIm}{k}\right)\frac{\partial^4 y}{\partial x^2\,\partial t^2} + I_\rho\frac{m}{k}\frac{\partial^4 y}{\partial t^4} = p + \frac{I_\rho}{k}\frac{\partial^2 p}{\partial t^2} - \frac{EI}{k}\frac{\partial^2 p}{\partial x^2}. \tag{21}$$

So far we have not discussed the constants m, I_ρ, and k. For a beam of uniform material, $m = \rho A$, $I_\rho = \rho AR^2$, where ρ is the mass density of the beam material, A is the cross-sectional area, and R is the radius of gyration

of the cross section about an axis perpendicular to the plane of motion and through the neutral axis. But k depends on the distribution of shearing stress in the beam cross section. Timoshenko writes

$$k = k'AG, \tag{22}$$

where G is the shear modulus of elasticity and k' is a numerical factor depending on the shape of the cross section, and ascertains that according to the elementary beam theory, $k' = \frac{2}{3}$ for a rectangular cross section. The use of such a value of k is, however, a subject of controversy in the literature. Mindlin[11.1] suggests that the value of k can be so selected that the solution of Eq. (21) be made to agree with certain solution of the exact three-dimensional equations of Pochhammer (1876) and Chree (1889) (see Love,[1.2] *Elasticity*, 4th ed., pp. 287–92). Indeed, I_ρ, which arises in the assumption of plane sections remain plane in bending, may also be regarded, when such an assumption is relaxed, as an empirical factor to be determined by comparison with exact solutions.

For a uniform beam free from lateral loadings, Eq. (21) can be written as

$$\frac{\partial^4 y}{\partial x^4} - \left(\frac{1}{c_0^2} + \frac{1}{c_Q^2}\right)\frac{\partial^4 y}{\partial x^2\,\partial t^2} + \frac{1}{c_0^2 c_Q^2}\frac{\partial^4 y}{\partial t^4} + \frac{1}{c_0^2 R^2}\frac{\partial^2 y}{\partial t^2} = 0, \tag{23}$$

where

$$c_0^2 = \frac{E}{\rho}, \qquad c_Q^2 = \frac{k'G}{\rho}, \qquad R^2 = \frac{I}{A}. \tag{24}$$

If the beam is of infinite length, a solution of the form (12) may be substituted into (23), and we see that the wave speed c must satisfy the equation

$$1 - \left(\frac{c^2}{c_0^2} + \frac{c^2}{c_Q^2}\right) + \frac{c^4}{c_0^2 c_Q^2} - \frac{c^2}{c_0^2 R^2}\left(\frac{\lambda}{2\pi}\right)^2 = 0. \tag{25}$$

The solution of this equation for c/c_0 versus λ yields two branches, corresponding to two "modes" of motion (two different shear-to-bending deflection ratios for the same wavelength). They are plotted in Fig. 11.2:4 for the special case of a beam of circular cross section with radius a. The results of the exact solution of Pochhammer and Chree for Poisson's ratio $\nu = 0.29$ are also plotted there for comparison. It is seen that the Timoshenko theory agrees reasonably well with the exact theory in the first mode, but wide discrepancy occurs in the second mode. The approximate theory gives no information about higher modes: an infinite number of which exist in the exact theory.

The equations derived above are, of course, appropriate for the determination of the free-vibration modes and frequencies of a beam. The effects of rotary inertia and shear are unimportant if the wavelength of the vibration mode is large compared with the cross-sectional dimensions of the beam; but these effects become more and more important with a decrease of wavelength, i.e., with an increase in the frequency of vibration. In the example of a uniform beam with rectangular cross section and simply supported at both ends, with $E = \frac{8}{3}G$ and $k' = \frac{2}{3}$, we find that the shear deflection and rotary inertia reduce the natural frequencies. If the wavelength

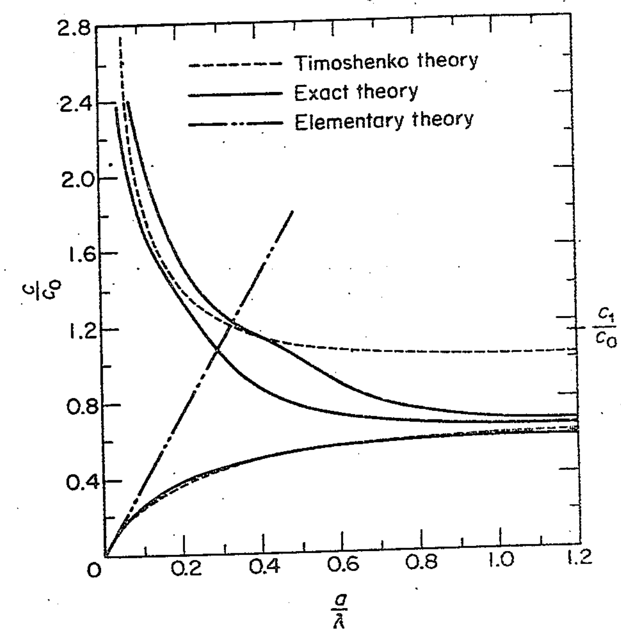


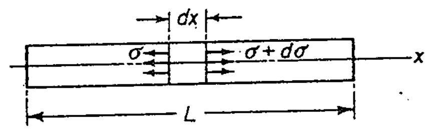
Fig. 11.2:4. Phase velocity curves for flexural elastic waves in a solid circular cylinder of radius a. (From Abramson,[11.1] *J. Acoust. Soc. Am.*, 1957.)

is ten times larger than the depth of the beam, the correction on the frequency due to rotary inertia alone is about 0.4 per cent, and the correction due to rotary inertia and shear together will be about 2 per cent.

The Timoshenko beam theory has attracted much attention in recent years. For a survey of literature, see Abramson, Plass, and Ripperger.[11.1]

PROBLEMS

11.2. Consider the free longitudinal vibration of a rod of uniform cross section and length L, as shown in Fig. P11.2. Let us assume that plane cross sections remain plane, that only axial stresses are present, being uniformly distributed over the cross section, and that radial displacements are negligible (i.e., the displacements consist of only one nonvanishing component u in the x-direction). Derive



P11.2. Longitudinal vibration of a rod.

expressions for the potential and kinetic energy and show that the equation of motion is

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} = 0, \qquad c_0^2 = \frac{E}{\rho}. \tag{26}$$

Show that the general solution is of the form

$$u = f(x - c_0 t) + F(x + c_0 t), \tag{27}$$

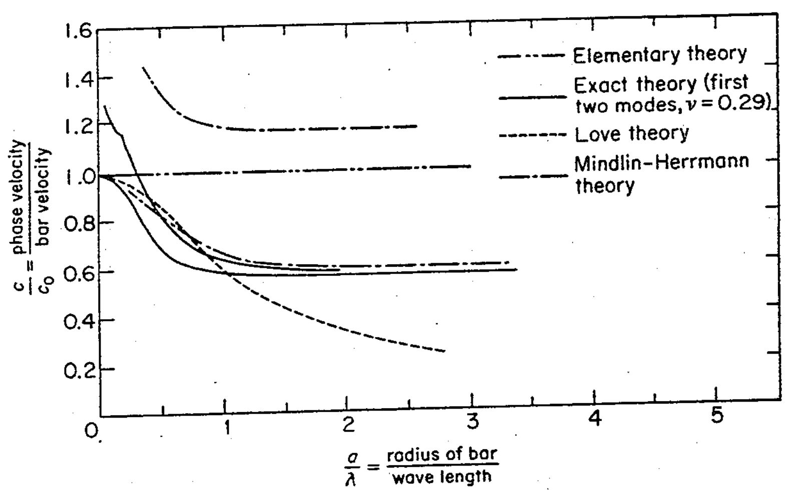
where f and F are two arbitrary functions.

11.3. Consider the same problem as above, but now incorporate approximately the transverse inertia associated with the lateral expansion or contraction connected with axial compression and extension, respectively. Let the (Love's) assumption be made that the displacement in the radial direction v is proportional to the radial coordinate r, measured from a centroidal axis, and to the axial strain $\partial u/\partial x$; i.e.,

$$v = -\nu r \frac{\partial u}{\partial x}, \tag{28}$$

where ν is Poisson's ratio. Derive expressions of the kinetic and potential energy and obtain the equation of motion according to Hamilton's principle,

$$\rho\left[\frac{\partial^2 u}{\partial t^2} - (\nu R)^2 \frac{\partial^4 u}{\partial x^2\,\partial t^2}\right] - E\frac{\partial^2 u}{\partial x^2} = 0, \tag{29}$$



P11.3. Phase velocity curves for longitudinal elastic waves in a solid circular cylinder of radius a. (After Abramson et al., *Adv. Applied Mech.*, 5, 1958.)

where R is the polar radius of gyration of the cross section. The natural boundary condition at the end $x = 0$, if that end is subject to a stress $\sigma_0(t)$, is

$$\rho\nu^2 R^2 \frac{\partial^3 u}{\partial x\,\partial t^2} + E\frac{\partial u}{\partial x} = \sigma_0(t) \qquad \text{at } x = 0. \tag{30}$$

Note. It is important to note that, according to the last equation, the familiar proportionality between axial stress σ and axial strain $\partial u/\partial x$ does not exist in this theory.

Comparison of the dispersion curves obtained from the elementary theory (Prob. 11.2), the Love theory, the Pochhammer-Chree "exact" theory, and another approximate theory due to Mindlin and Herrmann,[11.1] are shown in Fig. P11.3 The last-mentioned theory accounts for the strain energy associated with the transverse displacement v, of which the most important contribution comes from the shearing strain caused by the lateral expansion of the cross section near a wave front.

11.4. The method of derivation of the various forms of equations of motion of beams as presented above has the advantage of being straightforward, but it does not convey the physical concepts as clearly as in an elementary derivation. Hence, rederive the basic equations by considering the forces that act on an element of length dx, as shown in Fig. P11.2 and Fig. P11.4. Obtain the following equations, and then derive the wave equations by proper reductions.

Longitudinal waves, elementary theory (Fig. P11.2):

$$\frac{\partial \sigma_{xx}}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \quad \text{(equation of motion)}, \tag{31}$$

$$\frac{\partial^2 u}{\partial x\,\partial t} = \frac{\partial \epsilon_{xx}}{\partial t} \quad \text{(equation of strain)}, \tag{32}$$

$$\sigma_{xx} = E\epsilon_{xx} \quad \text{(equation of material behavior)}, \tag{33}$$

where σ_{xx} = axial stress, ϵ_{xx} = axial strain, $\partial u/\partial t$ = axial particle velocity, x = axial coordinate, t = time, E = modulus of elasticity, and ρ = mass density.

Flexural waves, Timoshenko theory (Fig. P11.4, p. 328):

$$\frac{\partial M}{\partial x} - Q = \rho I\frac{\partial \omega}{\partial t} \quad \text{(rotational)} \tag{34}$$

$$\frac{\partial Q}{\partial x} = \rho A\frac{\partial v}{\partial t} \quad \text{(transverse)} \tag{35}$$

(equations of motion),

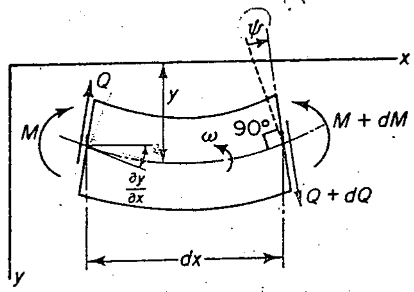
$$\frac{\partial K}{\partial t} = \frac{\partial \omega}{\partial x} \quad \text{(bending)} \tag{36}$$

$$\frac{\partial \beta}{\partial t} = \frac{\partial v}{\partial x} + \omega \quad \text{(shear)} \tag{37}$$

$$M = EIK \quad \text{(bending)} \tag{38}$$

$$Q = A_s\beta \quad \text{(shear)} \tag{39}$$

(equations of material behavior),



P11.4. Element of a beam in bending.

where M = moment, Q = shear force, K = axial rate of change of section angle = $-\partial\psi/\partial x$, β = shear strain = $\partial y/\partial x - \psi$, ω = angular velocity of section = $-\partial\psi/\partial t$, v = transverse velocity = $\partial y/\partial t$, I = section moment of inertia, A = section area, and A_s = area parameter defined by $\iint \gamma(z)\, dA = \beta A_s$ where $\gamma(z)$ is the shear strain at a point z in the cross section.

11.3. GROUP VELOCITY

Since we have been concerned in the preceding sections about wave propagations in beams, it seems appropriate to make a digression to explain the concept of *group velocity* as distinguished from the *phase velocity*. We have seen that for certain equations a solution of the following form exists:

(1)
$$u = a \sin(\mu x - \nu t).$$

If x is increased by $2\pi/\mu$, or t by $2\pi/\nu$, the sine takes the same value as before, so that $\lambda = 2\pi/\mu$ is the wavelength and $T = 2\pi/\nu$ is the period of oscillation. If $\mu x - \nu t$ = constant, i.e. x = const. + $\nu t/\mu$, the argument of the sine function remains constant in time; which means that the whole waveform is displaced towards the right with a velocity $c = \nu/\mu$. The quantity c is called the phase velocity, in terms of which Eq. (1) may be exhibited as

(2)
$$u = a \sin \frac{2\pi}{\lambda}(x - ct).$$

If the phase velocity c depends on the wavelength λ, the wave is said to exhibit *dispersion*. Our examples in the previous section show that dispersion exists in both longitudinal and flexural waves in rods and beams.

What happens when two sine waves of the same amplitude but slightly different wavelengths and frequencies are superposed? Let these two waves be characterized by two sets of slightly different values μ, ν and μ', ν'. The resultant of the superposed waves is

$$u + u' = A[\sin(\mu x - \nu t) + \sin(\mu' x - \nu' t)].$$

Using the well-known formula

$$\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta),$$

we have

(3)
$$u + u' = 2A \sin[\tfrac{1}{2}(\mu+\mu')x - \tfrac{1}{2}(\nu+\nu')t] \cos[\tfrac{1}{2}(\mu-\mu')x - \tfrac{1}{2}(\nu-\nu')t].$$

This expression represents the well-known phenomenon of "beats." The sine factor represents a wave whose wave number and frequency are equal to the mean of μ, μ' and ν, ν', respectively. The cosine factor, which varies very slowly when $\mu - \mu'$, $\nu - \nu'$ are small, may be regarded as a varying amplitude,

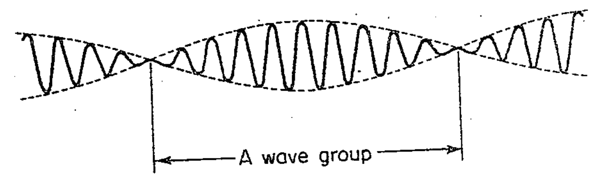


Fig. 11.3:1. An illustration of a wave group.

as shown in Fig. 11.3:1. The "wave group" ends wherever the cosine becomes zero. The velocity of advance of these points is called the *group velocity*; its value U is equal to $(\nu - \nu')/(\mu - \mu')$. For long groups (or slow beats), the group velocity may be written with sufficient accuracy as

(4)
$$U = \frac{d\nu}{d\mu}.$$

In terms of the wavelength λ $(= 2\pi/\mu)$, we have

(5)
$$U = \frac{d(\mu c)}{d\mu} = c - \lambda\frac{dc}{d\lambda},$$

where c is the phase velocity.

From the fact that no energy can travel past the nodes, one can infer that the rate of transfer of energy is identical with the group velocity. This fact is capable of rigorous proof for single trains of waves.

The most familiar examples of propagation of wave groups are perhaps the water waves. It has often been noticed that when an isolated group of waves, of sensibly the same length, advancing over relatively deep water, the velocity of the group as a whole is less than that of the individual waves composing it. If attention is fixed on a particular wave, it is seen to advance through the group, gradually dying out as it approaches the front, while its former place in the group is occupied in succession by other waves which have come forward from the rear. Another familiar example is the wave train set up by ships. The explanation as presented above seems to have been first given by Stokes (1876). Other derivations and interpretations of

8.5.8 Effects of Rotary Inertia and Shear Deformation

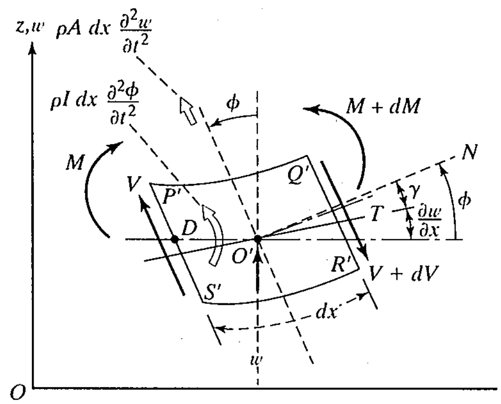
If the cross-sectional dimensions are not small compared to the length of the beam, we need to consider the effects of rotary inertia and shear deformation. The procedure, presented by Timoshenko [8.10], is known as the *thick beam theory* or *Timoshenko beam theory*. Consider the element of the beam shown in Fig. 8.19. If the effect of shear deformation is disregarded, the tangent to the deflected center line $O'T$ coincides with the normal to the face $Q'R'$ (since cross sections normal to the center line remain normal even after deformation). Due to shear deformation, the tangent to the deformed center line $O'T$ will not be perpendicular to the face $Q'R'$. The angle γ between the tangent to the deformed center line ($O'T$) and the normal to the face ($O'N$) denotes the shear deformation of the element. Since positive shear on the right face $Q'R'$ acts downward, we have, from Fig. 8.19,

$$\gamma = \phi - \frac{\partial w}{\partial x} \tag{8.128}$$

where ϕ denotes the slope of the deflection curve due to bending deformation alone. Note that because of shear alone, the element undergoes distortion but no rotation.

The bending moment M and the shear force V are related to ϕ and w by the formulas[2]

$$M = EI\frac{\partial \phi}{\partial x} \tag{8.129}$$



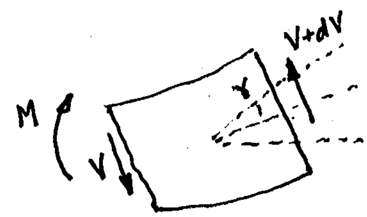


FIGURE 8.19 An element of Timoshenko beam.

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[2]Equation (8.129) is similar to Eq. (8.75). Eq. (8.130) can be obtained as follows:

Shear force = Shear stress × Area = Shear strain × Shear modulus × Area

$\tau = V/A = \gamma G$

or

$$V = \gamma GA$$

This equation is modified as $V = kAG\gamma$ by introducing a factor k on the right-hand side to take care of the shape of the cross section.

and

$$V = kAG\gamma = kAG\left(\phi - \frac{\partial w}{\partial x}\right) \tag{8.130}$$

where G denotes the modulus of rigidity of the material of the beam and k is a constant, also known as *Timoshenko's shear coefficient*, which depends on the shape of the cross section. For a rectangular section the value of k is 5/6; for a circular section it is 9/10 [8.11].

The equations of motion for the element shown in Fig. 8.19 can be derived as follows:

1. For translation in the z direction:

$$-[V(x,t) + dV(x,t)] + f(x,t)\,dx + V(x,t)$$
$$= \rho A(x)\,dx\,\frac{\partial^2 w}{\partial t^2}(x,t)$$
$$\equiv \text{Translational inertia of the element} \tag{8.131}$$

2. For rotation about a line passing through point D and parallel to the y axis:

$$[M(x,t) + dM(x,t)] + [V(x,t) + dV(x,t)]\,dx$$
$$+ f(x,t)\,dx\,\frac{dx}{2} - M(x,t)$$
$$= \rho I(x)\,dx\,\frac{\partial^2 \phi}{\partial t^2} \equiv \text{Rotary inertia of the element} \tag{8.132}$$

Using the relations

$$dV = \frac{\partial V}{\partial x}\,dx \qquad \text{and} \qquad dM = \frac{\partial M}{\partial x}\,dx$$

along with Eqs. (8.129) and (8.130) and disregarding terms involving second powers in dx, Eqs. (8.131) and (8.132) can be expressed as

$$-kAG\left(\frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) + f(x,t) = \rho A\,\frac{\partial^2 w}{\partial t^2} \tag{8.133}$$

$$EI\,\frac{\partial^2 \phi}{\partial x^2} - kAG\left(\phi - \frac{\partial w}{\partial x}\right) = \rho I\,\frac{\partial^2 \phi}{\partial t^2} \tag{8.134}$$

By solving Eq. (8.133) for $\partial\phi/\partial x$ and substituting the result in Eq. (8.134), we obtain the desired equation of motion for the forced vibration of a uniform beam:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4}$$

$$+ \frac{EI}{kAG}\frac{\partial^2 f}{\partial x^2} - \frac{\rho I}{kAG}\frac{\partial^2 f}{\partial t^2} - f = 0 \tag{8.135}$$

For free vibration, $f = 0$, and Eq. (8.135) reduces to

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4} = 0 \tag{8.136}$$

The following boundary conditions are to be applied in the solution of Eq. (8.135) or (8.136):

1. Fixed end:

$$\phi = w = 0$$

2. Simply supported end: M & W=0

$$EI\frac{\partial \phi}{\partial x} = w = 0$$

3. Free end: V & M=0

$$kAG\left(\frac{\partial w}{\partial x} - \phi\right) = EI\frac{\partial \phi}{\partial x} = 0$$

EXAMPLE 8.10 Natural Frequencies of a Simply Supported Beam

Determine the effects of rotary inertia and shear deformation on the natural frequencies of a simply supported uniform beam.

Solution: By defining

$$\alpha^2 = \frac{EI}{\rho A} \quad \text{and} \quad r^2 = \frac{I}{A} \tag{E.1}$$

Eq. (8.136) can be written as

$$\alpha^2\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - r^2\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho r^2}{kG}\frac{\partial^4 w}{\partial t^4} = 0 \tag{E.2}$$

We can express the solution of Eq. (E.2) as

$$w(x, t) = C \sin \frac{n\pi x}{l} \cos \omega_n t \tag{E.3}$$

which satisfies the necessary boundary conditions at $x = 0$ and $x = l$. Here, C is a constant and ω_n is the nth natural frequency. By substituting Eq. (E.3) into Eq. (E.2), we obtain the frequency equation:

$$\omega_n^4\left(\frac{\rho r^2}{kG}\right) - \omega_n^2\left(1 + \frac{n^2\pi^2r^2}{l^2} + \frac{n^2\pi^2r^2}{l^2}\frac{E}{kG}\right) + \left(\frac{\alpha^2 n^4\pi^4}{l^4}\right) = 0 \tag{E.4}$$

It can be seen that Eq. (E.4) is a quadratic equation in ω_n^2 and for any given n; there are two values of ω_n that satisfy Eq. (E.4). The smaller value corresponds to the bending deformation mode, while the larger one corresponds to the shear deformation mode.

The values of the ratio of ω_n given by Eq. (E.4) to the natural frequency given by the classical theory (in Fig. 8.15) are plotted for three values of E/kG in Fig. 8.20 [8.22].[3]

Note the following aspects of rotary inertia and shear deformation:

1. If the effect of rotary inertia alone is considered, the resulting equation of motion does not contain any term involving the shear coefficient k. Hence we obtain (from Eq. 8.136):

k=∞ since V=kAGγ & if no shear deformation => γ=0 but k=∞ for V≠0

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \tag{E.5}$$

Timoshenko

Euler

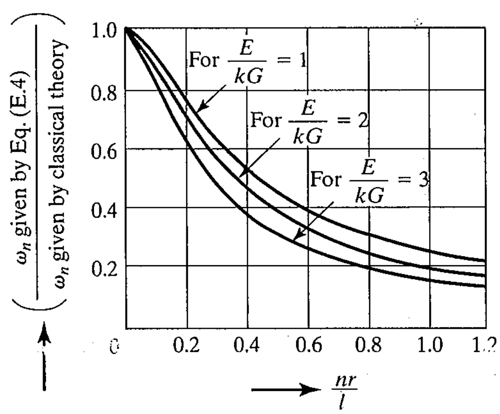


FIGURE 8.20 Variation of frequency.

[3]The theory used for the derivation of the equation of motion (8.76), which disregards the effects of rotary inertia and shear deformation, is called the *classical* or *Euler-Bernoulli* or *thin beam theory*.

In this case the frequency equation (E.4) reduces to

$$\omega_n^2 = \frac{\alpha^2 n^4 \pi^4}{l^4\left(1 + \dfrac{n^2\pi^2 r^2}{l^2}\right)} \tag{E.6}$$

2. If the effect of shear deformation alone is considered, the resulting equation of motion does not contain the terms originating from $\rho I(\partial^2\phi/\partial t^2)$ in Eq. (8.134). Thus we obtain the equation of motion

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \frac{EI\rho}{kG}\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \tag{E.7}$$

and the corresponding frequency equation

$$\omega_n^2 = \frac{\alpha^2 n^4 \pi^4}{l^4\left(1 + \dfrac{n^2\pi^2 r^2}{l^2}\dfrac{E}{kG}\right)} \tag{E.8}$$

3. If both the effects of rotary inertia and shear deformation are disregarded, Eq. (8.136) reduces to the classical equation of motion, Eq. (8.78)

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \tag{E.9}$$

and Eq. (E.4) to

$$\omega_n^2 = \frac{\alpha^2 n^4 \pi^4}{l^4} \tag{E.10}$$

■

8.5.9 Other Effects

The transverse vibration of tapered beams is presented in Refs. [8.12, 8.14]. The natural frequencies of continuous beams are discussed by Wang [8.15]. The dynamic response of beams resting on elastic foundation is considered in Ref. [8.16]. The effect of support flexibility on the natural frequencies of beams is presented in [8.18, 8.19]. A treatment of the problem of natural vibrations of a system of elastically connected Timoshenko beams is given in Ref. [8.20]. A comparison of the exact and approximate solutions of vibrating beams is made by Hutchinson [8.30]. The steady-state vibration of damped beams is considered in Ref. [8.21].

8.6 Vibration of Membranes

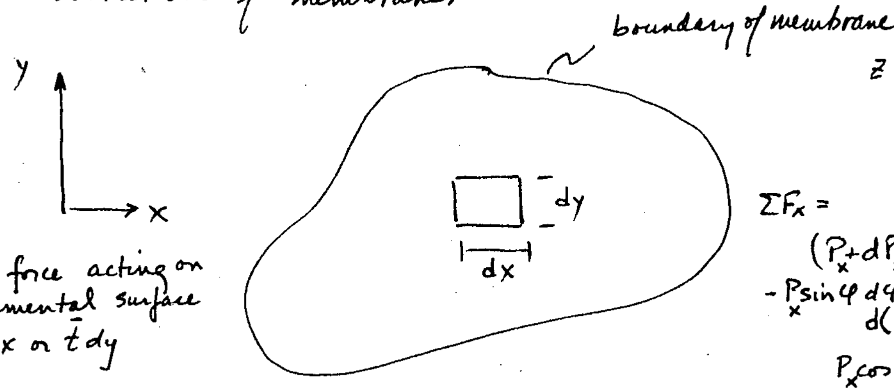
A membrane is a plate that is subjected to tension and has negligible bending resistance. Thus a membrane bears the same relationship to a plate as a string bears to a beam. A drumhead is an example of a membrane.

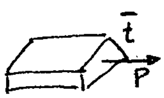
~~if E and A are not functions of x and $f(x,t) = 0$~~

~~$u_{xx} - \frac{1}{c^2} u_{tt} = 0$ $\qquad c = \sqrt{E/\rho}$ is the bar velocity~~

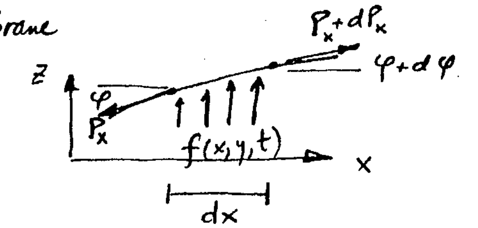
~~if $f(x,t) \neq 0$ we have an inhomogeneous problem: example vertical rod where weight cannot be neglected~~

- vibration of membranes





P force acting on elemental surface $\bar{t}dx$ or $\bar{t}dy$



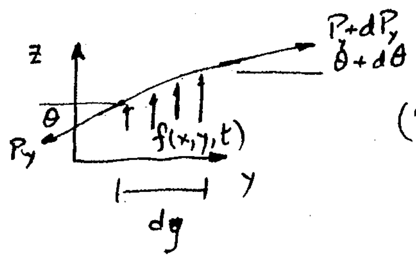
$\Sigma F_x =$

$(P_x + dP_x)\cos(\varphi + d\varphi) - P_x \cos\varphi = 0$

$-P_x \sin\varphi \, d\varphi + dP_x \cos\varphi = 0$ TO FIRST ORDER

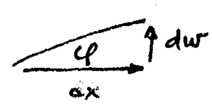
$d(P_x \cos\varphi) = 0$

$P_x \cos\varphi = \text{const. or fn of } y$

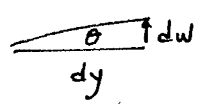


$(P_y + dP_y)\cos(\theta + d\theta) - P_y \cos\theta = 0 \quad = \Sigma F_y$

$d(P_y \cos\theta) = 0 \implies P_y \cos\theta = \text{const or fn of } x$



$\cos\varphi = \frac{dx}{\sqrt{dx^2 + dw^2}} \cong 1 \quad \text{if } \left|\frac{\partial w}{\partial x}\right| \ll 1$



$\cos\theta = \frac{dy}{\sqrt{dy^2 + dw^2}} \cong 1 \quad \text{if } \left|\frac{\partial w}{\partial y}\right| \ll 1$

$\implies$ P_x is essentially constant everywhere
P_y " " " " $\Big\}$ $P_x = P_y = P$

$\Sigma F_z = m \cdot \text{accel} = \rho \frac{\partial^2 w}{\partial t^2} dy dx \cdot \bar{t}$

$f(x,y,t)dxdy + (P_y + dP_y)\sin(\theta + d\theta) - P_y \sin\theta + (P_x + dP_x)\sin(\varphi + d\varphi) - P_x \sin\varphi = \Sigma F_z$

$d(P_y \sin\theta) + d(P_x \sin\varphi) + f(x,y,t)dxdy = \Sigma F_z$

$\frac{d}{dy}(\underbrace{P_y}_{\bar{P}\cdot\bar{t}dx} \sin\theta)dy + \frac{d}{dx}(\underbrace{P_x}_{\bar{P}\cdot\bar{t}dy} \sin\varphi)dx + f(x,y,t)dxdy = \Sigma F_z$

$\sin\theta \sim \frac{\partial w}{\partial y} \qquad \sin\varphi \sim \frac{\partial w}{\partial x}$

$$\frac{\partial}{\partial y}\left(\bar{P}\bar{t}\,\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(\bar{P}\bar{t}\,\frac{\partial w}{\partial x}\right) + f(x,y,t) = \rho \frac{\partial^2 w}{\partial t^2}\,\bar{t}$$

IF $\bar{t}$ = const $\quad \frac{\partial}{\partial y}\left(\bar{P}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(\bar{P}\frac{\partial w}{\partial x}\right) + f/\bar{t} = \rho\, w_{tt}$

if $\bar{P}$ is constant $\Longrightarrow \quad w_{xx} + w_{yy} + f/\bar{t}\bar{P} = \rho/\bar{P}\; w_{tt}$

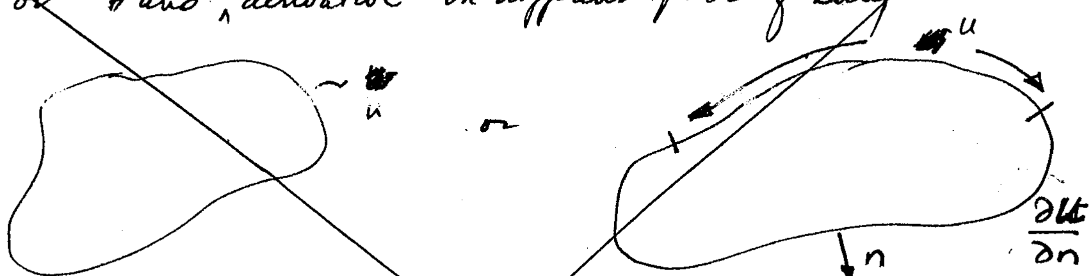
if $f = 0 \quad \Longrightarrow \quad w_{xx} + w_{yy} = \frac{1}{c^2} w_{tt}$ and $\nabla^2 w = \frac{1}{c^2} w_{tt} \qquad c = \sqrt{\frac{\bar{P}}{\rho}}$

- IF steady state $\frac{\partial}{\partial t} = 0 \Longrightarrow \nabla^2 w = 0$ Elliptic eqn.

- Boundary Conditions + initial conditions IN GENERAL
  - for each time derivative you need 1 initial condition
  - for each space derivative $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ you need 1 boundary condi

must also satisfy consistency of Neumann $\int \frac{\partial u}{\partial n} ds = \iint \nabla^2 u\, dA$ using Divergence theorem.

- FOR ELLIPTIC TYPE EQNS need to specify value on bdry, or deriv on bdry or value and normal derivative on different parts of bdry



- FOR PARABOLIC TYPE – heat eqn $u_{xx} = a u_t$
  - need 1 initial condition $u(x, t=t_0) = u_0(x)$
  - need value or value + normal deriv on boundary : $\alpha u + \beta u_n$

- FOR HYPERBOLIC – wave eqn.
  - need 2 initial conditions $u(x, t=t_0) = u_0(x)$, $\frac{\partial}{\partial t} u(x, t=t_0) = u_1(x)$
  - need value or value + normal deriv on boundary

vibration of circular membrane clamped at edge

$$\nabla^2 W - \frac{1}{a^2} W_{tt} = 0 \qquad a = \sqrt{P/\rho}$$

let $W(r,\theta,t) = w(r,\theta)T(t)$

$$\Rightarrow \quad T \cdot \nabla^2 w - \frac{1}{a^2} w \cdot T'' = 0 \quad \text{or}$$

$$a^2 \frac{\nabla^2 w}{w} - \frac{T''}{T} = 0$$

$$\Rightarrow \quad a^2 \frac{\nabla^2 w}{w} = \frac{T''}{T} = -\omega^2 \qquad \text{frequency of vibration}$$

$$\Rightarrow \quad T'' + \omega^2 T = 0 \quad \sim \text{temporal mode of vib}$$

$$\nabla^2 w + \frac{\omega^2}{a^2} w = 0 \quad \Rightarrow \quad \nabla^2 w + \lambda^2 w = 0 \quad \text{— spatial mode of vibration}$$

$\lambda = \frac{\omega}{c}$ eigenvalue

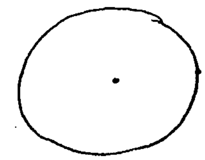
$w(r,\theta)$ eigenmode

EIGENVALUE PROBLEM (POISSON'S EQN IN A CIRCULAR REGION)

- OBTAINED BY SEPARATING VARIABLES – ONLY LOOKS AT SPATIAL MODE SHAPES
- $\nabla^2 W - \frac{1}{c^2} W_{tt} = 0 \Rightarrow W = w(r,\theta)T(t) \Rightarrow \nabla^2 w + \lambda^2 w = 0$ & $T'' + \omega^2 T = 0$ $\quad \lambda^2 = \omega^2/c^2$
- let us solve $\nabla^2 w + \lambda^2 w = 0$ in a circular region under the condition that $w(r,\theta) = 0$ when $r = a$

$$\nabla^2 w = \frac{\partial^2}{\partial r^2} w + \frac{1}{r}\frac{\partial}{\partial r} w + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} w \quad \text{in cylindrical coordinates}$$

$$\therefore \quad \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} w + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \lambda^2 w = 0$$



method of S.O.V. yields $w(r,\theta) = R(r)\Theta(\theta)$

$$\therefore \quad \nabla^2 w = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''$$

and

$$\nabla^2 w + \lambda^2 w = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' + \lambda^2 R\Theta = 0$$

- divide by $R\Theta$ if $w(r,\theta) \neq 0$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \lambda^2 = 0$$

- $\Rightarrow \underbrace{\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \lambda^2\right) r^2}_{\text{fn of } r} = \underbrace{-\frac{\Theta''}{\Theta}}_{\text{fn of } \theta} = \text{constant} = k^2$

$$\Rightarrow \quad \Theta'' + k^2\Theta = 0 \quad \text{or} \quad \Theta = A\cos k\theta + B\sin k\theta$$

$$\Rightarrow \quad r^2 R'' + rR' + (\lambda^2 r^2 - k^2)R = 0$$

$$(\lambda r)^2 \frac{d^2}{d(\lambda r)^2} R(\lambda r) + (\lambda r)\frac{d}{d(\lambda r)} R(\lambda r) + (\lambda^2 r^2 - k^2)R = 0$$

look at the Bessel's equation $x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$

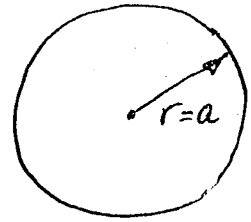
if $x = \lambda r$, $p = k$ then the solutions to $x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$

are the solutions to $r^2 R'' + rR' + (\lambda^2 r^2 - k^2)R = 0$

for example

ON THURSDAY WE SOLVED

$$\nabla^2 W - \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = 0 \quad \text{IN A CIRCULAR REGION}$$



with $W(r,\theta,t) = W$

AND $W(r=a,\theta,t) = 0$

By writing $W = w(r,\theta)\cdot T(t)$ we can separate spatial & temporal functions

$$\nabla^2 W = \frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}$$

THUS $$\nabla^2 W - \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = T\nabla^2 w - \frac{1}{c^2} w \ddot{T} = 0$$

or $$\frac{c^2 \nabla^2 w}{w} = \frac{\ddot{T}}{T} = -\omega^2 \quad \Longrightarrow \quad T = C_1 \cos\omega t + C_2 \sin\omega t$$

ω = frequency of vibration

$$\Longrightarrow \nabla^2 w + \left(\frac{\omega}{c}\right)^2 w = 0 \qquad \text{let } \frac{\omega}{c} = \lambda$$

or $\nabla^2 w + \lambda^2 w = 0$ this is an eigenvalue problem

TO SOLVE EIGENVALUE PROBLEM, LET $w(r,\theta) = R(r)\,\Theta(\theta)$

$$\therefore \nabla^2 w + \lambda^2 w = \left(R'' + \frac{1}{r}R'\right)\Theta + \frac{1}{r^2}\Theta'' + \lambda^2 R\Theta = 0$$

$$\Longrightarrow \frac{r^2\left(R'' + \frac{1}{r}R'\right) + \lambda^2 r^2}{R} = -\frac{\Theta''}{\Theta} = +k^2 \quad \text{SINCE } \Theta \text{ FUNCTION MUST BE PERIODIC}$$

$$\Longrightarrow \Theta = A\cos k\theta + B\sin k\theta$$

$$\Longrightarrow r^2 R'' + rR' + (\lambda^2 r^2 - k^2)R = 0$$

BESSEL EQN OF FORM $x^2 y'' + xy' + (x^2 - p^2)y = 0$

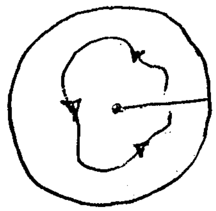
IF $x = \lambda r$
$R = y$
$p = k$

SOLUTIONS ARE

$$y = C_1 J_p(x) + C_2 J_{-p}(x) \qquad \text{IF } p \neq 0 \text{ or an integer}$$

$$y = C_1 J_n(x) + C_2 Y_n(x) \qquad \text{IF } p \text{ is zero or an integer}$$

- TO DETERMINE IF k IS integer $W(r,\theta,t) = W(r,\theta+2\pi,t) \Rightarrow w(r,\theta) = w(r,\theta+2\pi) \Rightarrow$



$$\Theta \Rightarrow \Theta(\theta+2k\pi) = \Theta(\theta) \Rightarrow \boxed{k=n}$$

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,\Gamma(i+p+1)}\left(\frac{x}{2}\right)^{2i}$$

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,(i+n)!}\left(\frac{x}{2}\right)^{2i}$$

$$\therefore \; R(\lambda r) = C_1 J_n(\lambda r) + C_2 Y_n(\lambda r)$$

- NOW $J_n(\lambda r)$ IS BOUNDED AT $r=0$

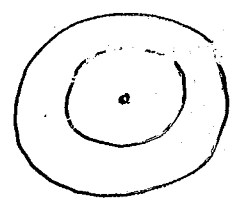
  $Y_n(\lambda r)$ IS NOT BOUNDED AT $r=0$

PHYSICAL PROBLEM DICTATES THAT $W(r,\theta,t)$ IS BOUNDED AT $r=0$

$\Rightarrow$ MUST TAKE $\boxed{C_2 = 0}$ SINCE $Y_n(\lambda r)$ CONTAINS $\log(\lambda r)$ TERM

$\Rightarrow$ NOTE $J_{-p}(x)$ IS NOT BOUNDED AT $x=0$ EITHER

$\Rightarrow$ FOR AN ANNULAR MEMBRANE ORIGIN NOT INCLUDED THUS $Y_n(\lambda r)$ IS kept



- LET'S LOOK AT HOW TO HANDLE $W(r=a,\theta,t)=0$

$$W(r=a,\theta,t) = w(r=a,\theta,t) = R(\lambda r=\lambda a)\,\Theta(\theta)\,T(t) = 0 \qquad \text{IRRESPECTIVE OF } t \,\&\, \theta$$

$\Rightarrow R(\lambda r = \lambda a) = 0$ THIS IS THE WAY TO FIND THE λ's

SO FAR $w(r,\theta) = J_n(\lambda r)\left[\bar{A}\cos n\theta + \bar{B}\sin n\theta\right]$

SINCE TRUE FOR ANY n & EQ $(\nabla^2 w + \lambda^2 w = 0)$ IS LINEAR

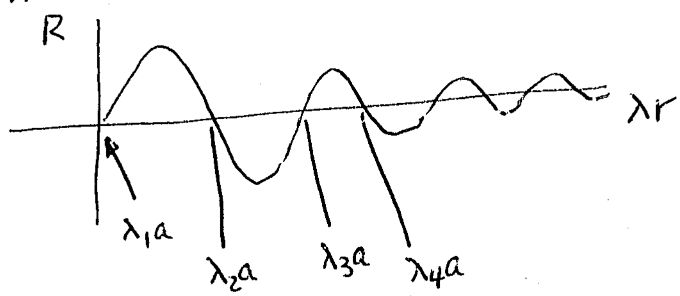
SO $w(r,\theta)$ DEPENDS ON $n \Rightarrow w_n(r,\theta)$

AND

$$w(r,\theta) = \sum_n w_n(r,\theta) = \sum_n J_n(\lambda r)\left[\bar{A}_n\cos n\theta + \bar{B}_n\sin n\theta\right]$$

$\Rightarrow$ SINCE $R(\lambda a)=0 \Rightarrow J_n(\lambda a) = 0$

FOR any n



$\Rightarrow$ THERE ARE AN INFINITE NO. OF VALUES FOR $J_0(\lambda r)$ & $J_1(\lambda r)$, & $J_2(\lambda r)$ ...

$\therefore$ WE MUST NUMBER THE ZEROES OF J_0, J_1, J_2 etc. AND PUT IN ORDER OF INCREASING MAGNITUDE ...

THUS

$$w(r,\theta) = \sum_m\sum_n J_n(\lambda_{nm} r)\left[\bar{A}_n\cos n\theta + \bar{B}_n\sin n\theta\right]$$

AND

$$W(r,\theta,t) = \sum_m\sum_n J_n(\lambda_{nm} r)\left[\bar{A}_n\cos n\theta + \bar{B}_n\sin n\theta\right]\left[\mathcal{C}_{1_{mn}}\cos\omega_{mn}t + \mathcal{C}_{2_{mn}}\sin\omega_{mn}t\right]$$

$$= \sum_m\sum_n J_n(\lambda_{nm} r)\left[\bar{C}_n\cos(n\theta + \psi_n)\right]\left[D_{mn}\cos(\omega_{mn}t + \phi_{mn})\right]$$

$\Rightarrow$ ω_{mn} has DOUBLE SUBSCRIPT SINCE $\frac{\omega}{c} = \lambda$ & λ DEPENDS ON m & n

$\Rightarrow$ $\bar{C}_n$, ψ_n, D_{mn} & ϕ_{mn} CANNOT BE FOUND WITHOUT IC's FOR $T(t)$ & BC's ON Θ

Table 3.5.1 gives the first few values of these frequencies in dimensionless form. The solution for the membrane displacement u_{nm} in the vibration eigenmode n,m is then

$$u_{nm}(r,\theta,t) = A_{nm} J_n(\lambda_{nm} r) \cos(n\theta - \psi) \cos(\omega_{nm} t - \phi)$$

$$\lambda_{nm} = j_{n,m}/r_o \qquad \omega_{nm} = \lambda_{nm} a \qquad (3.5.14)$$

The phase angles ϕ and ψ, and the amplitude A remain undetermined.

The lowest frequency occurs for the 0,1 mode. Note that for n = 0 the motion is axisymmetric, and has no nodes. The next higher frequency occurs for the 1,1 mode. This mode has one diametral node along which the

TABLE 3.5.1

DIMENSIONLESS MEMBRANE FREQUENCIES

n	m	$j_{n,m} = \omega_{nm} r_o/a$
0	1	2.40483
1	1	3.83171
2	1	5.13562
0	2	5.52008
3	1	6.38016
1	2	7.01559
4	1	7.58834

membrane does not move. (The phase angle of this node cannot be determined without initial conditions). The third mode is the 2,1 mode, which has <u>two</u> diametral nodes, and the fourth is the 0,2 mode, with one circular node at the point where $J_0(\lambda_{02} r) = 0$, i.e. at $\lambda_{02} r = j_{0,1} = 2.40483$. Figure 3.5.3 shows the nodal lines for the first several modes.

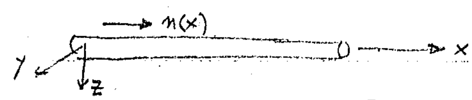
What is a plate:

first 3-D Structures are usually not efficient usage of structure

1-D wire } beam } one dim is >> than other 2 dimensions $EIw^{iv} = p(x)$

2-D plate one dim is small wrt the other 2 $EI\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = p(x,$

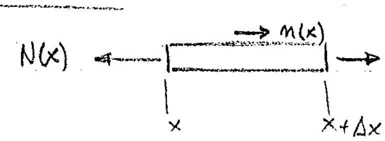
Stretching of a rod



$n(x)$ = distributed load / unit length

This load produces a 3-D stress. Since the length >> the radius we can define a stress resultant $N(x) = \iint \sigma_x \, dA$ to get rid of 3 dimensionality

looking at equilibrium



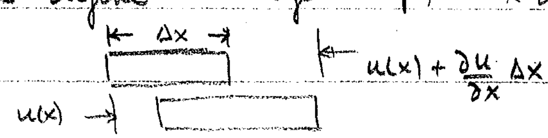
$N(x+\Delta x) = N(x) + \left(\frac{\partial N}{\partial x}\right)_x \Delta x + O(\Delta x)$

thus $\left(\frac{\partial N}{\partial x}\right)_x \Delta x + \dots + n(x)\Delta x = 0$; taking limit as $\Delta x \to 0$ is $\boxed{\frac{dN}{dx} = -n(x)}$

if $n(x)$ is known then $\boxed{N(x) = -\int_{x_0}^{x} n(x)\,dx + N(x_0)}$

Normally we don't know $N(x_0)$ but we do know what displacements are

thus define average disp. in x direction $u(x) = \frac{1}{A}\int_A u \, dA$

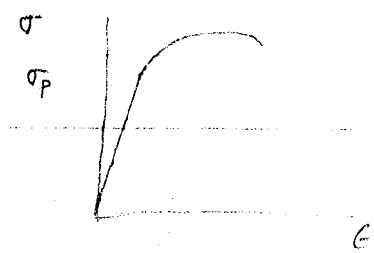


we can thus define the engineering strain $\epsilon = \dfrac{\text{final length} - \text{init length}}{\text{initial length}} = \dfrac{\frac{\partial u}{\partial x}\Delta x}{\Delta x} = \dfrac{\partial u}{\partial x}$

$$\boxed{\epsilon = \frac{\partial u}{\partial x}}$$

This is relation that defines the displacement in terms of the strain

Now we use the stress strain relation to relate the stresses to the strain



Generally $|\sigma| < |\sigma_p|$ in design – we thus use

Hooke's law $\sigma = E\epsilon$ E is Young's modulus

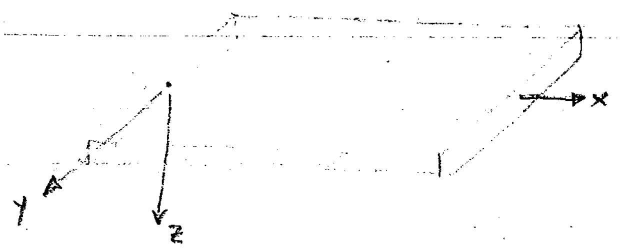
where σ is the 3-D stress

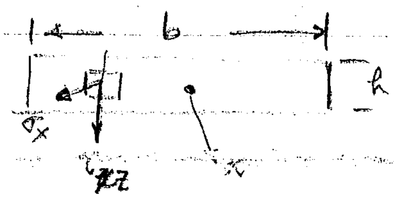
now $N(x) = \int \sigma\, dA = EA\epsilon$ if σ is a constant

now $\dfrac{dN}{dx} = -n(x) \Rightarrow \boxed{\dfrac{\partial}{\partial x}\left(EA\dfrac{\partial u}{\partial x}\right) = -n(x)}$

this is the relation between the distributed load and the displacement

9/28 Beams in bending



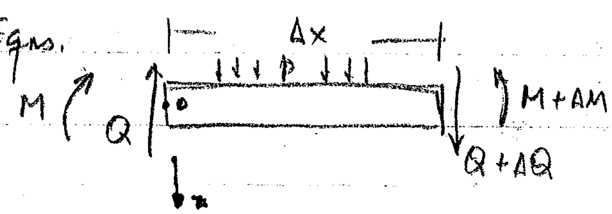


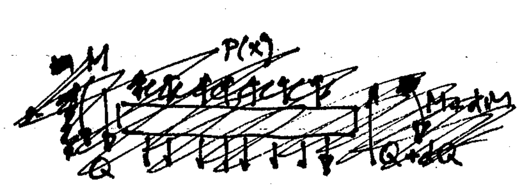
Define stress Resultants

$$Q = \int_{-h/2}^{h/2} \tau_{zx}\, dz \qquad \text{force per unit width}$$

$$M = \int_{-h/2}^{h/2} \sigma_x z\, dz \qquad \text{moment / unit width}$$

Equilib Eqns.





Force: $\Delta Q + p\,\Delta x \sim 0$ p = load/unit width

$$\lim_{\Delta x \to 0} \frac{\Delta Q}{\Delta x} = -p \quad \text{or} \quad \boxed{\frac{dQ}{dx} = -p} \qquad (1)$$

ΣM_o Moment $p\Delta x \cdot \frac{\Delta x}{2} + (Q+\Delta Q)\Delta x - M - \Delta M + M \sim 0$ or $\Delta M \sim Q\Delta x$

$$\Rightarrow \boxed{\frac{dM}{dx} = Q} \qquad (2)$$

(1) & (2) $\Longrightarrow$ $\boxed{\frac{d^2M}{dx^2} = -p}$ (3)

Displacement of midsurface

w – in z-dir

0 in x-dir

For the displacement of a point a distance z from the midsurface we must make a kinematic hypothesis

For Beams – Bernoulli or Euler

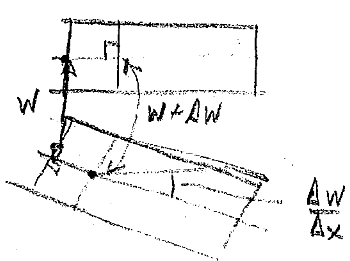
For plate – Kirchhoff

For shell – Kirchhoff, Love

Assume:

(1) A normal to the neutral surface remains straight & unstretched (plane surfaces remain plane)

(2) Normals remain ⊥ to the neutral surface



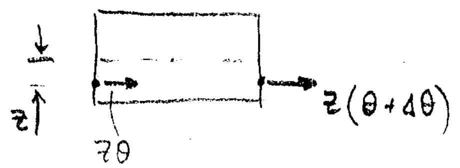
Displacement of pt a distance z from the midsurface is:

w – in z direction

$z\theta$ – in x direction

Thus rotation of the midsurface $\boxed{\theta = -\frac{dw}{dx}}$ . . . (4)

Strains



$$\epsilon_x \cong z \frac{\Delta\theta}{\Delta x}$$

or
$$\boxed{\epsilon_x = z\frac{d\theta}{dx}} \qquad (5)$$

Using the stress strain law

$$\sigma_x = E\epsilon_x$$

$$M = \int_{-h/2}^{h/2} \sigma_x z\,dz = \int E\frac{d\theta}{dx} z^2 dz = E\frac{d\theta}{dx}\frac{h^3}{12} \qquad \text{using (5)}$$

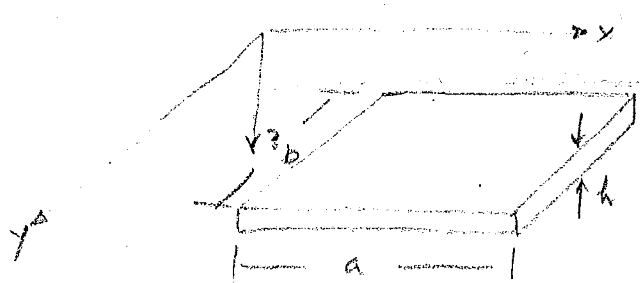
$$Mb = E\frac{d\theta}{dx}\frac{bh^3}{12} = EI\frac{d\theta}{dx} \qquad \text{for a rectangular beam.}$$

or
$$\boxed{Mb = EI\left(-\frac{d^2w}{dx^2}\right)} \qquad \text{using (4)} \qquad (6)$$

but
$$\frac{d^2M}{dx^2} = -p \quad (3) \qquad \therefore \quad \boxed{pb = \frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)} \qquad (7)$$

beams bending in terms of displacement

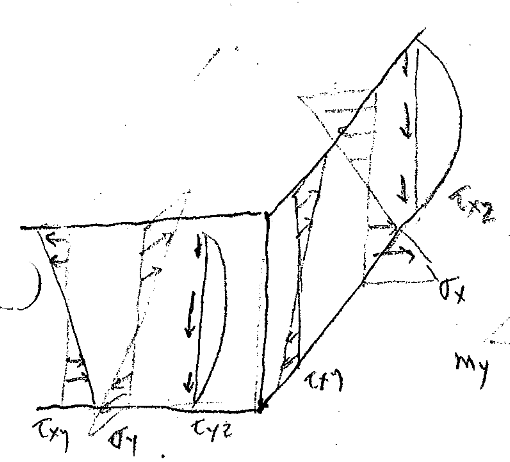
10/1/79 Definition of a plate

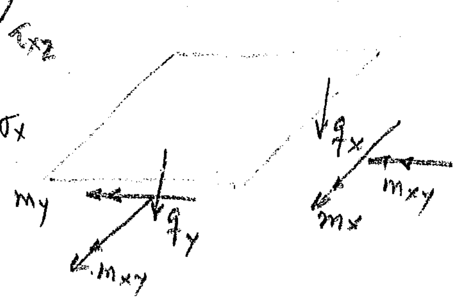


Assume: $h << a, b$

(2) plate is elastic, homog, isotropic
(3) small deflections $w \le \frac{h}{5}$
(4) $\frac{\partial w}{\partial x}$ & $\frac{\partial w}{\partial y} << 1$
(5) Kirchhoff Hypothesis
- normals remain straight / unstretched and normal
- $w = w(x,y)$ only

(6) Inextensional theory – no stretching
(7) no initial imperfections
(8) stresses normal to midsurface are negligible $\sigma_z << \sigma_x, \sigma_y$



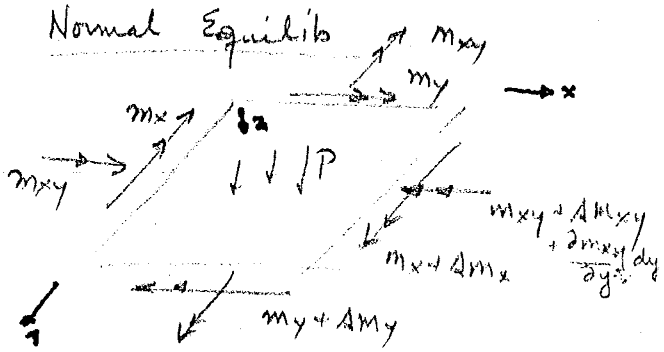


Define our stress resultants

$$q_x = \int_{-h/2}^{h/2} \tau_{xz}\, dz \qquad q_y = \int_{-h/2}^{h/2} \tau_{yz}\, dz \qquad m_x = \int_{-h/2}^{h/2} \sigma_x z\, dz \qquad m_y = \int_{-h/2}^{h/2} \sigma_y z\, dz$$

$$m_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\, dz \qquad m_{yx} = \int_{-h/2}^{h/2} \tau_{yx} z\, dz$$

Normal Equilib



$m_{xy} + \Delta m_{xy} = m_{xy} + \frac{\partial m_{xy}}{\partial x} dx$

$$\frac{\partial q_y}{\partial y} \Delta y \Delta x + \frac{\partial q_x}{\partial x} \Delta x \Delta y + p \Delta x \Delta y \cong 0$$

or

$$\boxed{\frac{\partial q_y}{\partial y} + \frac{\partial q_x}{\partial x} + p = 0} \qquad (1)$$

Moment Equil in x-direction

$$\frac{\partial m_y}{\partial y} \Delta x \Delta y + \frac{\partial m_{xy}}{\partial x} \Delta x \Delta y - q_y \Delta x \Delta y \cong 0$$

or

$$\boxed{\frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} - q_y = 0} \qquad (2)$$

Moment Equil in y dir:

$$\boxed{\frac{\partial m_{yx}}{\partial y} + \frac{\partial m_x}{\partial x} - q_x = 0} \qquad (3)$$

take $\frac{\partial}{\partial y}(2)$ $\frac{\partial}{\partial x}(3)$ add & substitute (1) to give

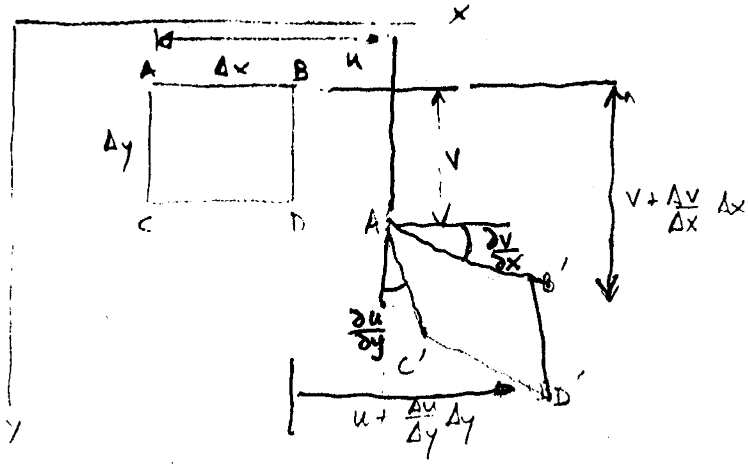
$$\boxed{\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} + P = 0} \qquad (4)$$

Displacements $\quad w = w(x, y)$

$$u = z\,\theta_x = -z\frac{\partial w}{\partial x} \qquad v = z\,\theta_y = -z\frac{\partial w}{\partial y}$$

Strains

$$\epsilon_x = z\frac{\partial \theta_x}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \qquad \epsilon_y = z\frac{\partial \theta_y}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}$$



$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

Stress - Strain relations

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

neglect $\sigma_z << \sigma_x, \sigma_y$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y)$$

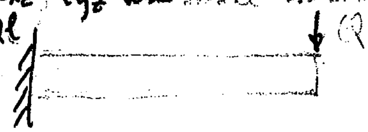
$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x)$$

obtained by inverting ϵ_x, ϵ_y for σ_x & σ_y

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

10/6/99

o show τ_{xz}, τ_{yz} are small in comparison to σ_x



$$\sigma_x = \frac{Mc}{I} = \frac{6M}{h^2} = \frac{6Mh}{2 \cdot h^3/12} = \frac{6Q}{h}\left(\frac{l}{h}\right)$$

$$\text{average } \tau_{xz} = \frac{Q}{h}$$

$\Rightarrow \sigma_x >> \tau_{xz}, \tau_{yz}$ thus neglect τ_{xz}, τ_{yz}

however we do not neglect q_x, q_y

$$m_x = \int_{-h/2}^{h/2} \sigma_x z\, dz = \frac{E}{1-\nu^2}\left(\frac{\partial\theta_x}{\partial x} + \nu\frac{\partial\theta_y}{\partial y}\right)\int_{-h/2}^{h/2} z^2 dz = \boxed{-\frac{EI}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = m_x}$$

in doing this σ_x is assumed constant across cross section $\quad D = \frac{EI}{1-\nu^2} = \frac{Eh^3}{12(1-\nu^2)}$ Bending Rigidity

Constitutive eqs

Thus $\boxed{m_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)} \quad \boxed{m_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]}$

$$m_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\,dz = \frac{E}{2(1+\nu)} \int_{-h/2}^{h/2} -2 \frac{\partial^2 w}{\partial x \partial y} z^2 dz = -\frac{EI}{1+\nu} \frac{\partial^2 w}{\partial x \partial y} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = m_{xy}$$

now $q_x = m_{x,x} + m_{xy,y}$ from (8)

$$= -D\left\{\frac{\partial}{\partial x}\left[w_{,xx} + \nu w_{,yy}\right] + (1-\nu)\frac{\partial}{\partial y}(w_{,xy})\right\} = -D\frac{\partial}{\partial x}(\Delta w) = -D\frac{\partial}{\partial x}(\nabla^2 w) = q_x$$

$$\therefore \quad q_y = -D\frac{\partial}{\partial y}(\nabla^2 w)$$

by similarity $x \to y$, $y \to x$; also from (2)

Normal equil eqn.

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + p = 0 \quad \Rightarrow \quad -D\left(\nabla^2\left(\nabla^2 w\right)\right) + p = 0$$

or $\Delta^2 w = \nabla^4 w = p/D$ beam bending eq in terms of displacement

Nonhomogeneous Biharmonic Eqn.

$\nabla^2 w = 0$ Laplace $\quad \nabla^2 w \neq 0$ Poisson $\quad \nabla^4 w = 0$ Biharmonic

Define Curvature strain measures

$$K_x = -\frac{\partial^2 w}{\partial x^2} \qquad K_y = -\frac{\partial^2 w}{\partial y^2} \qquad K_{xy} = -\frac{\partial^2 w}{\partial x \partial y}$$

Stress Resultants in terms of strain measures (Kinematic relations)

$$m_x = D(K_x + \nu K_y) \qquad m_y = D(K_y + \nu K_x) \qquad m_{xy} = D(1-\nu)K_{xy}$$

<u>Summary</u> of how we got governing eqns.

- Eqns of Equil.
- Strain Displ relations
- Stress - Strain Relations
  - * stress resultants & strain measures
- kinematic relations

Boundary Conditions

Since 4th order ODE need 4 bc. (2 on each end)

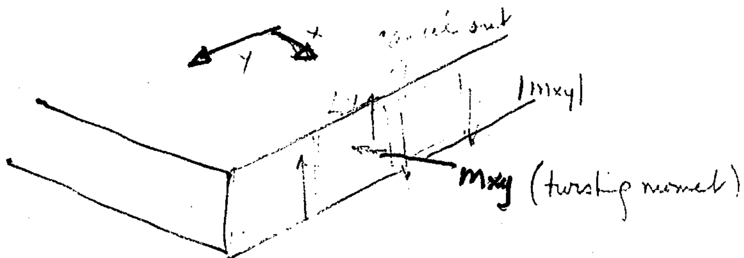
Geometric BC

$w, \frac{\partial w}{\partial x}$ on $x = const$

$w, \frac{\partial w}{\partial y}$ on $y = const$

- fixed edge $w = 0$ $\frac{\partial w}{\partial x} = 0$ at $x = ...$

Stress BC $\Rightarrow$ m_x q_x & m_{xy} on $x = const.$



can be replaced by a force couple of magnitude $|m_{xy}|$. This adds to the vertical shear

Net shear resultant on edge is $V_x = q_x + \frac{\partial m_{xy}}{\partial y}$

$$V_y = q_y + \frac{\partial m_{xy}}{\partial x}$$

thus our BC is V_x & m_x on $x = const$

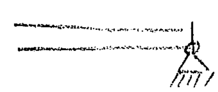
on a free edge $m_x = V_x = 0$

$$V_x = f(w) = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]$$

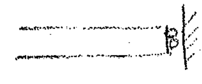
$$V_y = g(w) = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]$$

Mixed boundary conditions

Simply - Supported Edge $w = 0$, $m_x = 0$ at $x = const.$



$V_x = 0$ $\frac{\partial w}{\partial x} = 0$



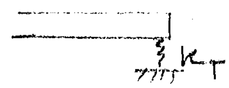
Summary

fixed $w = \frac{\partial w}{\partial x} = 0$ $A\left(1-\cos\left(\frac{2n\pi x}{L}\right)\right)$ n integer

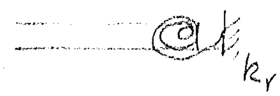
free $m_x = v_x = 0$

Simply Supported $w = 0$, $m_x = 0$

elastic support $m_x = 0$ $w = \frac{v_x}{k_t}$



$v_x = 0$ $\frac{\partial w}{\partial x} = \frac{m_x}{k_r}$



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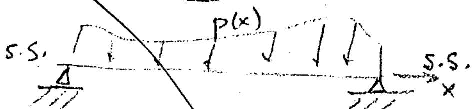
Fourier Series method of soln.

Beam $EI\, w^{iv} = p(x)$



Simple Support $w=0$ $M=0$ or $w=0$ & $\frac{\partial^2 w}{\partial x^2}=0$

v_z

Look at sol. $w = \sum_{n=1}^{\infty} w_n \sin\frac{n\pi x}{L}$ this satisfies B.C. : any sufficiently smooth fn. can be represented by this series

put w into ODE each term satisfies BC identically & doesn't necessarily solve de ident.

$\therefore\ EI \sum_{n=1}^{\infty} w_n \left(\frac{n\pi}{L}\right)^4 \sin\frac{n\pi x}{L} = p(x) \Rightarrow \frac{2EI}{L}\int_0^L \sum w_n \left(\frac{n\pi}{L}\right)^4 \sin\frac{n\pi x}{L}\sin\frac{k\pi x}{L}\,dx = \frac{2}{L}\int_0^L \sin\frac{k\pi x}{L}\,p(x$

$\Rightarrow\ EI\, w_k \left(\frac{k\pi}{L}\right)^4 = \frac{2}{L}\int_0^L p(x)\sin\frac{k\pi x}{L}\,dx$ $\therefore$ $w_k = \frac{2}{EIL}\left(\frac{L}{k\pi}\right)^4 \int_0^L p(x)\sin\frac{k\pi}{L}$

$\therefore$

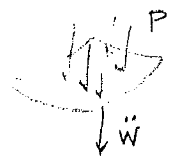
$$w_k = \frac{2}{EIL}\left(\frac{L}{k\pi}\right)^4 \int_0^L p(x)\sin\frac{k\pi x}{L}\,dx$$

alternate write $p(x) = \sum p_n \sin\frac{n\pi x}{L}$ $p_n = \frac{2}{L}\int_0^L p(x)\sin\frac{n\pi x}{L}\,dx$

$\Rightarrow\ EI\left(\frac{n\pi}{L}\right)^4 w_n = p_n \Rightarrow$ $w_n = \frac{p_n}{EI}\left(\frac{L}{n\pi}\right)^4$ very good convergence in comparison to p series

Consider Plate vibrations problem.

$D\Delta^2 w = p$



Look at $D\Delta^2 w = \frac{\text{mass}}{\text{Area}} \cdot \text{accel.}$

inertia force $= \rho h \frac{\partial^2 w}{\partial t^2}$

$\therefore \quad D\Delta^2 w = p - \rho h \frac{\partial^2 w}{\partial t^2}$; consider free vibes

set $P = 0$ & look at harmonics $w(x,y,t) = W(x,y)\sin\omega t$

$\longrightarrow \quad D\nabla^2\nabla^2 W = +\rho h \omega^2 W$ like foundation w/ negative spring const

Look at SS plates   Let $W(x,y) = A \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

This satisfies bc on w & moment. put into DE

$\therefore \quad D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 - \rho h \omega^2 = 0$

$\Longrightarrow \quad \omega = \sqrt{\frac{D}{\rho h}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$ for a square plate (a=b) $\omega_{mn} = \sqrt{\frac{D}{\rho h}}\left(\frac{\pi}{a}\right)^2(m^2+n^2)$

Forced Vibration $p(t)$

$$D\nabla^2\nabla^2 w = p - \rho h \ddot{w} \qquad \text{let } p(t,x,y) = \sum\sum p_{mn}(t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$w(x,y,t) = \sum\sum w_{mn}(t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

put into DE

look at m,n term: $D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 w_{mn}(t) + \rho h \ddot{w}_{mn}(t) = p_{mn}(t)$

let $\omega_{mn}^2 = D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 / \rho h \quad \therefore \quad \ddot{W}_{mn}(t) + \omega_{mn}^2 W_{mn} = \frac{p_{mn}(t)}{\rho h}$

natural freq of m,n mode

let $p_{mn}(t) = A_{mn} \sin\Omega t \qquad A_{mn} = \frac{2}{\tau}\int_0^\tau p_{mn}(t)\sin\Omega t\, dt \qquad \Omega\tau = 2\pi$

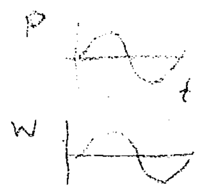
Let $W_{mn}(t) = B_{mn} \sin\Omega t$

if $\Omega = \omega_{mn}$ then we have resonance.

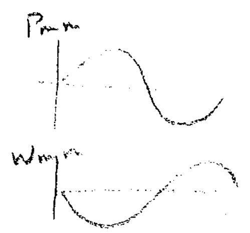
put into $\ddot{W}_{mn} + \omega_{mn}^2 W = p_{mn}/\rho h$

$$-B_{mn}\Omega^2 + \omega_{mn}^2 B_{mn} = A_{mn}/\rho h$$

$$B_{mn} = \frac{A_{mn}}{\rho h(\omega_{mn}^2 - \Omega^2)} \qquad \text{if } \omega_{mn} > \Omega$$



if $\omega_{mn} < \Omega$



when we're at resonance PDE is $\ddot{W} + \omega^2 W = f(t) = e^{i\omega t}$ let $f(t) = e^{i\omega t}$

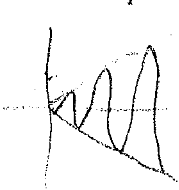
Guess $W = \hat{f}(t) e^{i\omega t}$

$$\dot{W} = \dot{\hat{f}} e^{i\omega t} + i\omega \hat{f} e^{i\omega t}$$
$$\ddot{W} = \ddot{\hat{f}} e^{i\omega t} + 2\dot{\hat{f}} i\omega e^{i\omega t} + (i\omega)^2 \hat{f} e^{i\omega t}$$

put into DE

$$\ddot{f} + 2i\omega\dot{f} = 1 \qquad \text{Let } f = t/2i\omega \quad \dot{f} = 1/2i\omega \quad \ddot{f} = 0$$

$$w = \frac{t}{2i\omega} e^{i\omega t} + W_h$$



linear theory is good for small time.

since $\sigma_z, \gamma_{xz}, \gamma_{yz}$ are negligible in plate theory

then $U(w) = \frac{1}{2}\int(\sigma_x\varepsilon_x + \varepsilon_y\sigma_y + \tau_{xy}\gamma_{xy})\,dV$

$\varepsilon_x = z\cdot K_x$ etc where K_x is the curvature

$$U(w) = \frac{1}{2}D\iint(\nabla^2 w)^2\,dx\,dy + (1-\nu)D\iint[(w_{,xy})^2 - (w_{,xx})(w_{,yy})]\,dx\,dy$$

Now for our simply supported plate uniformly load $\hat{w} = C\sin\frac{\pi x}{l}\sin\frac{\pi y}{l}$

$$\alpha_{in} = \iint D(\nabla^2\varphi_i)\nabla^2\varphi_n\,dx\,dy + (1-\nu)D\iint\ldots$$

$$\beta_i = \iint p\,\varphi_i\,dx\,dy$$

since only ∇^2 we get less differentiation - Major advantage can relax bc. (natural ones).

geometric bc w, w_n

natural bc Moment, Shear

Finite Element is a composition of Ritz method + automatic procedures to generate test fn.

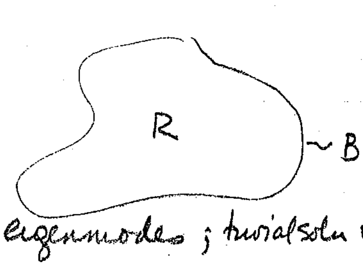
5/29/79

Plate Vibrations - General Problem.

inertial loading $p(\underline{r},t) = -\rho h \frac{\partial^2 w}{\partial t^2}$

PDE $\Rightarrow$

$$D\nabla^4 w + \rho h\, w_{,tt} = 0 \quad \text{in } R$$



w/BC ie $w = \frac{\partial w}{\partial n} = 0$ on $\partial R = B$

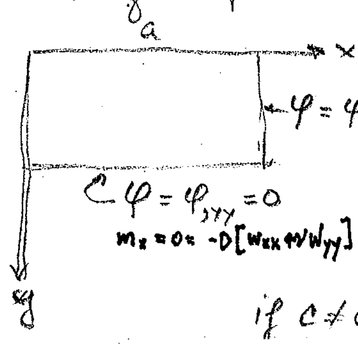
Not well specified since IC not specified - look for eigenmodes; trivial soln $w=0$

Look for soln $w = \varphi(\underline{r})e^{i\omega t}$

$$\Rightarrow D\nabla^4\varphi - \rho h\omega^2\varphi = 0 \text{ in } R \text{ w/ } \varphi = \frac{\partial\varphi}{\partial n} = 0 \text{ on } \partial R \qquad \varphi = 0 \text{ is triv sol.}$$

this is an eigenvalue problem for ω

Rectangular problem - SS BC.



$M_y = 0 = -D[w_{,yy} + \nu w_{,xx}]$

$M_x = 0 = -D[w_{xx} + \nu w_{yy}]$

$$\nabla^4\varphi - \frac{\rho h}{D}\omega^2\varphi = 0$$

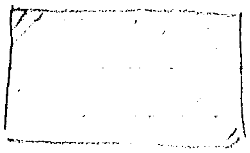
using Navier soln method: then $\varphi = C\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

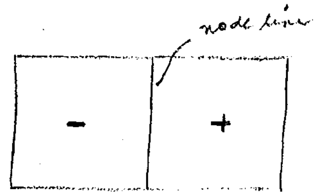
φ satisfies bc then

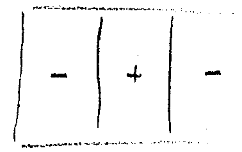
$$\left\{C\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)\right]^2 - \frac{\rho h}{D}\omega^2 C\right\}\sin(\ )\sin(\ ) = 0$$

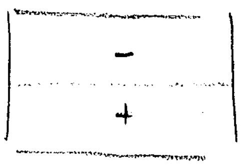
if $C \neq 0 \Rightarrow \omega_{mn}^2 = \frac{\pi^4 D}{\rho h}\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^2 \qquad \varphi_{mn} = C\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

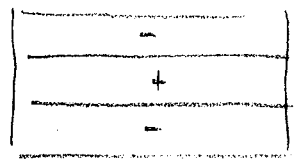
this doesn't take into account damping
C can only be found if IC are specified : we really don't care about it

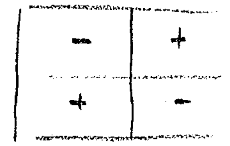
φ_{11} = 

φ_{21} = 

φ_{31} = 

φ_{12} = 

φ_{13} = 

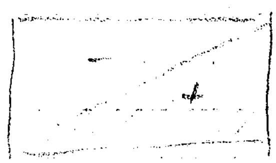
φ_{22} = 

for a square plate $\varphi_{12} = \varphi_{21}$ $\varphi_{13} = \varphi_{31}$ this leads to degenerate modes (where 2 modes collapse into 1) $\omega_{21} = \frac{5\pi^2}{a^2}\sqrt{D/\rho h}$

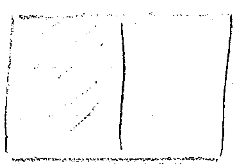
Look at

$$w = C(\varphi_{12}\cos\omega_{12}t + \varphi_{21}\cos\omega_{21}t) = C(\varphi_{12} + \varphi_{21})\cos\omega_{12}t$$ this is an eigenmode

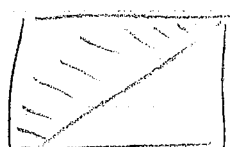
$$w = C(\varphi_{12}\cos\omega_{12}t - \varphi_{21}\cos\omega_{21}t) = C(\varphi_{12} - \varphi_{21})\cos\omega_{12}t$$ thus this is also an eigenmode



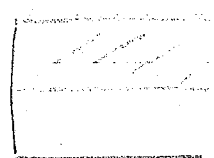
if $w = C(\varphi_{12}\cos\omega_{12}t \pm \varphi_{12}\sin\omega_{12}t)$ then



t=0



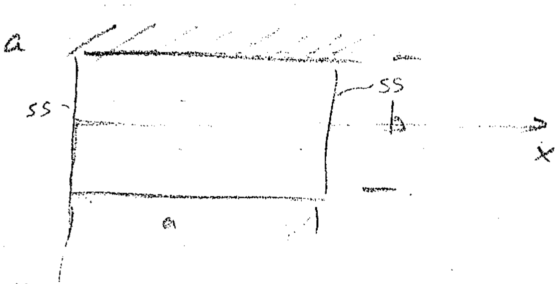
t = T/8



t = T/4

thus the nodal line is rotating

HW #8 Look at plate with bxa



find eigensoln for symmetric modes about centerline

General Case $\nabla^2(\nabla^2\varphi) - \frac{\rho h}{D}\omega^2\varphi = 0$

$\Rightarrow$

$$\left(\nabla^2 - \sqrt{\tfrac{\rho h}{D}}\,\omega\right)\left(\nabla^2 + \sqrt{\tfrac{\rho h}{D}}\,\omega\right)\varphi = 0$$ not separable in most cases

if $\left(\nabla^2 - \sqrt{\frac{\rho h}{D}}\,\omega\right)\varphi = 0$ or $\left(\nabla^2 + \sqrt{\frac{\rho h}{D}}\,\omega\right)\varphi = 0$ then original eqn is satisfied. This is the helmholtz eqn. which is separable in many case.

Circular plate $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$; with $k^2 = \sqrt{\frac{\rho h}{D}}\,\omega$

$\therefore\ (\nabla^2 \pm k^2)\varphi = 0$ using $\varphi = R(r)T(\theta) \Rightarrow \frac{r^2R'' + rR' \pm k^2r^2R}{R} = -\frac{T''}{T} = \lambda$ const

R = bessel fns ; T are trig fn.

for a complete plate causality requires at $r=0$ R must be finite and single valuedness requires periodicity of T. $\Rightarrow$ λ must be integer

$\therefore\ T = \sin n\theta, \cos n\theta$ and $r^2R'' + rR' \dot{-} (n^2 \dot{\mp} k^2r^2)R = 0$ • lead to other lead to I_n

$\therefore\ R = A_nJ_n(kr) + B_nI_n(kr)$ I_n is modified Bessel fns. bounded at $r=0$

for a clamped plate $R(r=a) = 0$ $\frac{dR}{dr}(r=a) = 0$ since $W(a,\theta,t) = 0 = \left.\frac{\partial W}{\partial r}\right|_{r=a} = 0$

for nontrivial solns requires Wronskian of $\left(R, \frac{dR}{dr}\right) = 0$

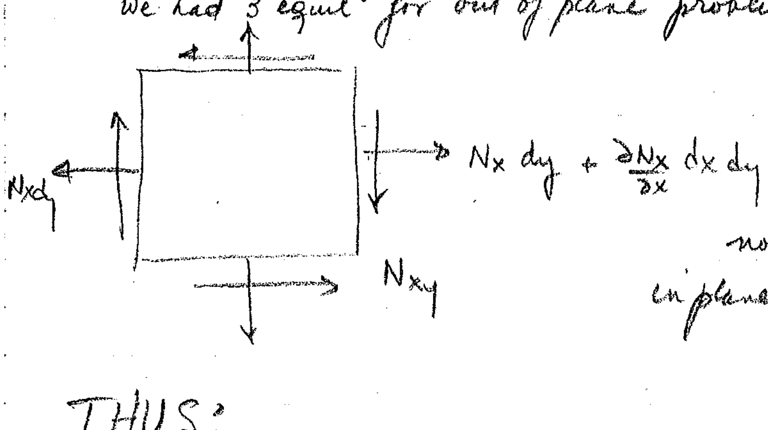
Wronskian gives $\boxed{\frac{J_{n-1}(ka)}{J_n(ka)} = \frac{I_{n-1}(ka)}{I_n(ka)}}$ since R_n' can be written in terms of R_{n-1}

This is our characteristic eqn that gives the value of ω through k

Effects of midplane forces — ie Buckling

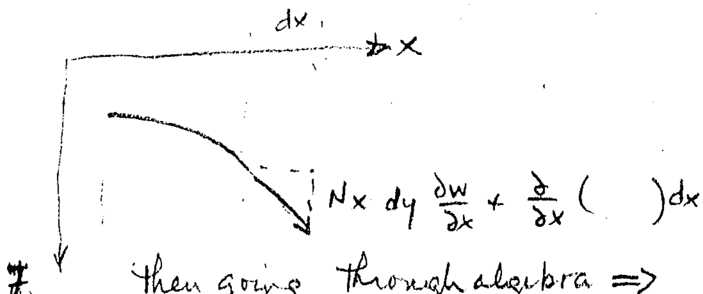
we had 3 equil for out of plane problem $\Sigma F_x, \Sigma F_y, \Sigma M_z = 0$

ie $\Sigma F_x = 0 \Rightarrow \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$



now we want to find how these in plane affect out of plane deformation

THUS:

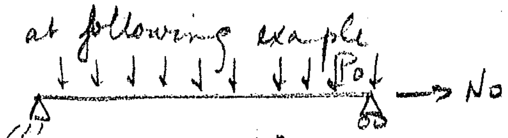


then going through algebra $\Rightarrow$ $D\nabla^4 w = p(x,y) + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}\left(N_{xy}\frac{\partial w}{\partial y}\right)$

$$+ \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y}\right)$$

Normally since the inplane & out of plane problem are decoupled solve inplane problem to get N_x, N_{xy}, N_y put here then solve for w the out of plane deformation

Look at following example



SS plate $\Rightarrow$ Navier soln is OK here

PDE $\quad D\nabla^4 w - N_0 \frac{\partial^2 w}{\partial x^2} = p(x,y)$

Navier soln $\quad w = \sum\sum C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

$$p(x,y) = \frac{16 p_0}{\pi^2} \sum_{mn\ \text{odd}} \frac{1}{mn} \sin(\quad)\sin(\quad)$$

Put w & p into PDE to get

$$\sum\sum\left[C_{mn} D\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + N_0\left(\frac{m\pi}{a}\right)^2 - \frac{16p_0}{\pi^2 mn}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{b} = 0$$

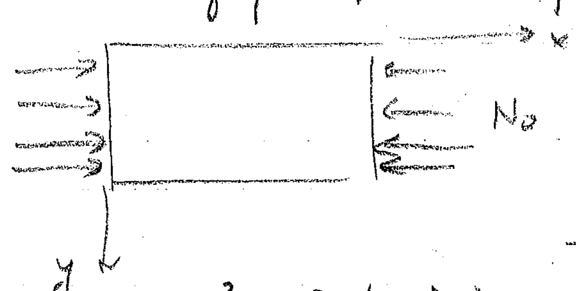
$\Rightarrow$ C_{mn} can be obtained $\quad C_{mn} =$ ________

$$\therefore \quad w = \sum_{n,m\ \text{odd}}\sum \frac{16p_0}{\pi^6 D} \frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{nm\left\{\left[\left(\frac{m^2}{a^2}\right)+\frac{n^2}{b^2}\right]^2 + \frac{N_0 m^2}{D\pi^2 a^2}\right\}}$$

as $N_0 \uparrow$ $\quad w \downarrow$ since denom $\uparrow$

for buckling for $N_0 < 0$ $\Rightarrow$ denom can be made $= 0$.

Buckling of SS plate w $p = 0$



Continuation $\quad$ 5/31/79

$$D\nabla^4 w - \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) = 0$$

BC $\quad w = w_{xx} = 0 \quad x = 0, a \qquad w = w_{yy} = 0 \quad y = 0, b$

EML 6223

FOR 4/19 & 4/20

ADVANCED VIB. ANALYSIS

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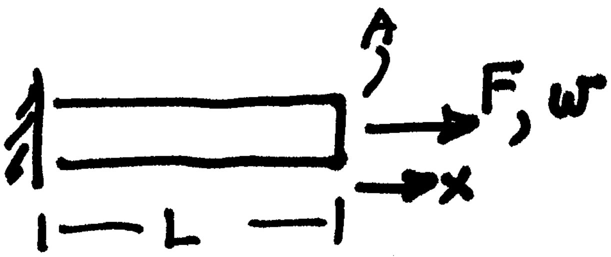
ADVANCED VIB. ANALYSIS

STRAIN ENERGY METHOD

FOR PROBS WHERE EXACT SOLN'S CAN'T BE OBTAINED



$$\mathcal{W} = \int_0^w F\, dw'$$

IF F is linear in w $\Rightarrow \mathcal{W} = \frac{1}{2} F w$

FOR UNIAXIAL CASE

$$\sigma = F/A = E\epsilon$$

$$\epsilon = w/L$$

$$\mathcal{W} = \tfrac{1}{2} A \sigma \epsilon L = \frac{1}{2} \underbrace{(AL)}_{\text{Vol}} E \epsilon^2$$

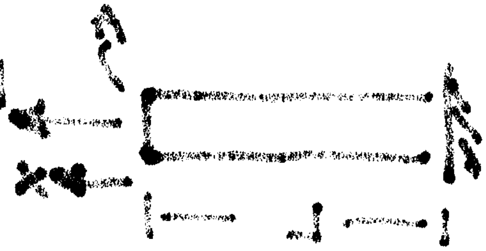
For beam $\epsilon = -\kappa z$ $\quad \sigma = -\frac{Mz}{I}$ $\quad \mathcal{W} = \frac{1}{2} \int \kappa z \frac{Mz}{I}\, dx\, dA$

$$\mathcal{W} = \int \frac{1}{2} \kappa M \int \frac{z^2 dA}{I}\, dx \qquad M = EI\kappa$$

$$= \int \frac{1}{2} EI \kappa^2 dx = \int \frac{1}{2} EI (w'')^2 dx$$

STRAIN ENERGY METHOD

FOR PROBS WHERE EXACT SOL'NS CAN'T BE OBTAINED



$W = \int_0^{w} F\,dw'$

IF F is linear in w $\Rightarrow$ $W = \frac{1}{2} F w$

FOR UNIAXIAL CASE

$\sigma = F/A = E\epsilon$

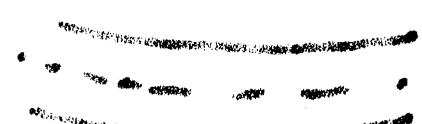
$\epsilon = w/L$

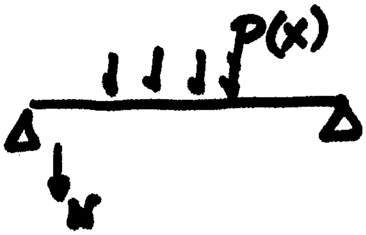
$W = \frac{1}{2} A \sigma \epsilon L = \frac{1}{2}\underbrace{(AL)}_{vol} E\epsilon^2$

For beam $\epsilon = -\kappa z$ $\quad \sigma = -\frac{Mz}{I}$ $\quad W = \frac{1}{2}\int \kappa z \frac{Mz}{I}\,dA\,dx$

$W = \frac{1}{2}\int \kappa M \int \frac{z^2 dA}{I}\,dx \qquad M = EI\kappa$

$= \frac{1}{2}\int EI\kappa^2\,dx = \frac{1}{2}\int EI(w'')^2\,dx$

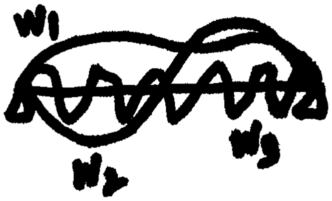




$$\bar{U} = \int_0^L \tfrac{1}{2} EI\kappa^2 dx - \int_0^L p(x)w(x)dx$$

$$\bar{U} = \int_0^L \left\{\tfrac{1}{2} EI\left(w''(x)\right)^2 - pw\right\}dx \qquad 0 \le x \le L$$

$$\bar{U}(w)$$



Out of all possible w's that satisfy the BC's, the solution of the equilibrium eqn. makes $\bar{U}$ a minimum

$w_i = w + \epsilon f$ — if w minimizes & satisfies BC's the f must satisfy the BC's, since w_i satisfies the BC's

$$\bar{U}[w+\epsilon f] = g(\epsilon) \Rightarrow \bar{U} \text{ min when } \frac{dg}{d\epsilon} = 0 \text{ @ } \epsilon = 0$$

$$\bar{U}[w+\epsilon f] = \int_0^L \left[\tfrac{1}{2} EI\left(w'' + \epsilon f''\right)^2 - P(w+\epsilon f)\right]dx$$

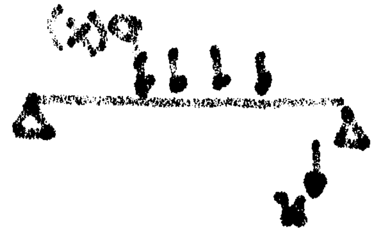
$$\frac{\partial \bar{U}}{\partial \epsilon} = \int_0^L \left[\tfrac{1}{2} EI\, 2\left(w'' + \epsilon f''\right)f'' - Pf\right]dx$$

$$\lim_{\epsilon \to 0} \frac{\partial \bar{U}}{\partial \epsilon} = \int_0^L \left[\tfrac{1}{2} \cdot 2EI\left(w''\right)f'' - Pf\right]dx$$

$$= \int_0^L \left[(EIw'')'' - p\right]f\,dx + \underbrace{EIw''}_{\text{mom}}\underbrace{f'}_{\text{slope}}\Big|_0^L - \underbrace{(EIw'')'}_{\text{shear force}}\underbrace{f}_{\text{disp}}\Big|_0^L = 0$$

Either M=0 or slope=0 @ x=0 or L

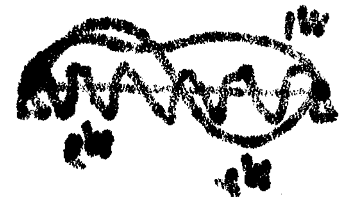
Either shear force or displ =0 @ x=0 or L



$$U = \int_0^L \tfrac{1}{2} EI \kappa^2 \, dx - \int_0^L p(x) w(x) \, dx$$

$$U = \int_0^L \left[\tfrac{1}{2} EI \left(w''(x)\right)^2 - pw\right] dx \qquad 0 \le x \le L$$

$U(w)$



Out of all possible w's that satisfy the BC's, the solution of the equilibrium eqn. makes U a minimum

$w_1 = w + \epsilon f$ if w minimizes & satisfies BC's the f must satisfy the BC's, since w_1 satisfies the BC's

$U[w+\epsilon f] = g(\epsilon) \Rightarrow U$ min when $\frac{dg}{d\epsilon} = 0$ @ $\epsilon = 0$

$$U[w+\epsilon f] = \int_0^L \left[\tfrac{1}{2} EI \left(w'' + \epsilon f''\right)^2 - P(w+\epsilon f)\right] dx$$

$$\frac{\partial U}{\partial \epsilon} = \int_0^L \left[\tfrac{1}{2} EI \, 2 (w'' + \epsilon f'') f'' - Pf\right] dx$$

at $\epsilon \to 0$

$$\frac{\partial U}{\partial \epsilon} = \int_0^L \left[\tfrac{1}{2} \cdot 2 EI (w'') f'' - Pf\right] dx$$

$$= \int_0^L \left[(EI w'')'' - p\right] f \, dx + \underbrace{EI w''}_{\text{mom}} \underbrace{f'}_{\text{slope}} \Big|_0^L - \underbrace{(EI w'')'}_{\text{shear force}} \underbrace{f}_{\text{disp}} \Big|_0^L = 0$$

Either $M = 0$ or slope $= 0$ @ $x = 0$ or L

Either shear force or displ $= 0$ @ $x = 0$ or L.

$$\int_0^L [(EIw'')'' - p]\, f\, dx = 0$$

for general $f \Rightarrow (EIw'')'' - p = 0$ for $\forall x$

$$U = U\,[w, w', w'', w''', w^{iv}, x]\,dx$$

The necessary condition for $w(x)$ to give a stationary value for U is the Euler Eqn.

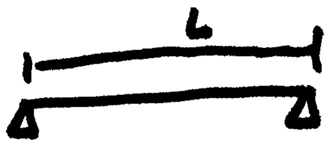
$$\Rightarrow \frac{d^4}{dx^4}\left(\frac{\partial \mathcal{L}}{\partial w^{iv}}\right) - \frac{d^3}{dx^3}\left(\frac{\partial \mathcal{L}}{\partial w'''}\right) + \frac{d^2}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial w''}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial w'}\right) + \frac{\partial \mathcal{L}}{\partial w} = 0$$

For example: $\mathcal{L} = \mathcal{L}\{w, w'', x\} = \frac{1}{2}EI(w'')^2 - pw$

$$\frac{\partial \mathcal{L}}{\partial w''} = EIw'' \qquad \frac{\partial \mathcal{L}}{\partial w} = -p$$

$$\frac{d^2}{dx^2}[EIw''] - p = 0$$

EXAMPLE:



$w = 0$ $\quad$ $w = 0$

$w'' = 0$ $\quad$ $w'' = 0$

Choose $w(x) = A \sin \frac{\pi x}{L}$

$w = 0$ $\quad$ $w = 0$

$w' = 0$ $\quad$ $w' = 0$

Choose $w(x) = A\left(1 - \cos\frac{2\pi x}{L}\right)$

$$w' = A \cdot \frac{2\pi}{L} \sin \frac{2\pi x}{L}$$

$$w'' = A \cdot \frac{4\pi^2}{L^2} \cos \frac{2\pi x}{L}$$

$$\int_0^l \left[(EIu'')'' - p\right]\delta u\, dx = 0$$

for general $\delta u \Rightarrow (EIu'')'' - p = 0$ for $\forall x$

$$U = \int U\left[u, u', u'', u''', x\right] dx$$

The necessary condition for $u(x)$ to give a stationary value for U is the Euler Eqn.

$$\Rightarrow -\frac{d^3}{dx^3}\left(\frac{\partial F}{\partial u'''}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)$$

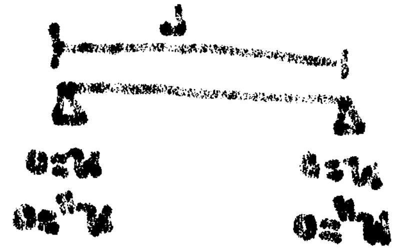
$$+ \frac{\partial F}{\partial u} = 0$$

For example: $F = F\{x, u, u''\} = \frac{1}{2}EI(u'')^2 - pu$

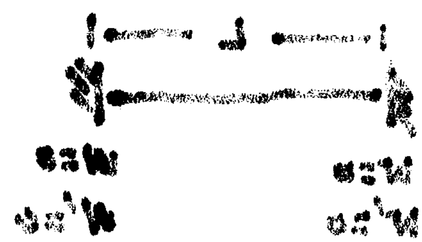
$$\frac{\partial F}{\partial u''} = EIu'' \qquad \frac{\partial F}{\partial u} = -p$$

$$\frac{d^2}{dx^2}\left[EIu''\right] - p = 0$$

EXAMPLE:



choose $u(x) = A \sin\frac{\pi x}{L}$



choose $u(x) = A\left(\cos\frac{2\pi x}{L} - 1\right)$

$u' = -A\cdot\frac{2\pi}{L}\sin\frac{2\pi x}{L}$

$u'' = -A\cdot\frac{4\pi^2}{L^2}\cos\frac{2\pi x}{L}$

look at fixed fixed beam choose $w(x) = A\left(1 - \cos\frac{2\pi x}{L}\right)$

$$\mathcal{L} = \frac{1}{2}\left\{EI(w'')^2 - pw\right\}$$

$$= \frac{1}{2} EI\, A^2 \cdot \frac{16\pi^4}{L^4} \cos^2 \frac{2\pi x}{L} - pA\left(1 - \cos\frac{2\pi x}{L}\right)$$

$$U = \frac{A^2}{2}\int_0^L EI \left(\frac{2\pi}{L}\right)^4 \cos^2\frac{2\pi x}{L}\,dx - A\int_0^L p\left(1 - \cos\frac{2\pi x}{L}\right)dx$$

$$U = U(A)$$

$$\frac{\partial U}{\partial A} = 0 \Rightarrow A = \frac{\int_0^L p(x)\left(1 - \cos\frac{2\pi x}{L}\right)dx}{EI\int_0^L \left(\frac{2\pi}{L}\right)^4 \cos^2\frac{2\pi x}{L}\,dx}$$

if $p = \text{const}$

$$A = \frac{pL}{EI\left(\frac{2\pi}{L}\right)^4 \cdot \frac{L}{2}} = \frac{pL^4}{8EI\pi^4}$$

[redacted] $w(x) = A\left(1 - \cos\frac{2\pi x}{L}\right) = \frac{pL^4}{8\pi^4 EI}\left(1 - \cos\frac{2\pi x}{L}\right)$

$$w(x = L/2) = \frac{pL^4}{4\pi^4 EI}$$

Exact $w(x = L/2) = \frac{pL^4}{384EI}$ $\qquad 4\pi^4 = 389.64$

Bending Stress $\sigma = \frac{Mz}{I} \Rightarrow$

$$M \sim w'' \cong A \cdot \frac{4\pi^2}{L^2}\cos\frac{2\pi x}{L}$$

$$M = EIw'' = \frac{pL^4}{8\pi^4} \cdot \frac{4\pi^2}{L^2}\cos\frac{2\pi x}{L}$$

$$M_{max} = \frac{pL^2}{2\pi^2} \cong \frac{pL^2}{19.7}$$

$$M_{max}\ \text{exact} = \frac{pL^2}{12}$$

$U_0 = \frac{1}{2}EI(w'')^2 - pw$

$U = \frac{1}{2}\int_0^l EI\left(\frac{\pi}{2l}\right)^4 A^2 \cos^2\frac{\pi x}{2l}\,dx - A\int_0^l p\left(1-\cos\frac{\pi x}{2l}\right)dx$

$U = U(A)$

$\frac{\partial U}{\partial A} = 0 \Rightarrow A = \frac{\int_0^l p(x)\left(1-\cos\frac{\pi x}{2l}\right)dx}{\int_0^l EI\left(\frac{\pi}{2l}\right)^4\cos^2\frac{\pi x}{2l}\,dx}$

if $p = \text{const}$: $A = \frac{pl\left(1-\frac{2}{\pi}\right)}{\frac{l}{2}\left(\frac{\pi}{2l}\right)^4 EI} = \frac{[illegible]\,pl^4}{\pi^4 EI}$

approx. $w(x) = A\left(1-\cos\frac{\pi x}{2l}\right) = \frac{[illegible]\,pl^4}{\pi^4 EI}\left(1-\cos\frac{\pi x}{2l}\right)$

$w(x=l) = \frac{pl^4}{[illegible]\,EI}$

exact $w(x=l) = \frac{pl^4}{8EI}$ $\qquad \pi^4 = 97.4$

Bending moment $M = EIw'' = A\left(\frac{\pi}{2l}\right)^2\cos\frac{\pi x}{2l}$

$M_{max} = \frac{pl^2}{[illegible]}$

M_{max} exact $= \frac{pl^2}{2}$

$w(x) = w_0(x) + w_1(x)$ $\qquad w_1(x) = A_1\left(1 - \cos\frac{4\pi x}{L}\right)$

$w_0(x) = A\left(1 - \cos\frac{2\pi x}{L}\right)$

$U(A, A_1) = \min$

$$\left.\begin{array}{l}\frac{\partial U}{\partial A} = 0 \\ \frac{\partial U}{\partial A_1} = 0\end{array}\right] \rightarrow \text{2 eqs for 2 unknowns}$$

For a beam $\quad U[w] = \int_0^L \left[\frac{1}{2} EI (w'')^2 - pw\right] dx$

$\frac{1}{2} EI w'' \cdot w'' = \frac{1}{2} M K$

For a plate $\quad \frac{1}{2}(m_x K_x + 2 m_{xy} K_{xy} + m_y K_y)$

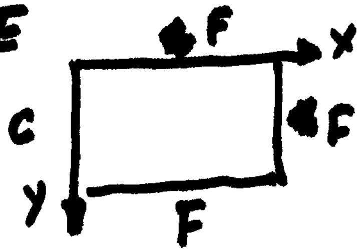
$m_x = D(K_x + \nu K_y)$, $\quad m_{xy} = D(1-\nu) K_{xy}$

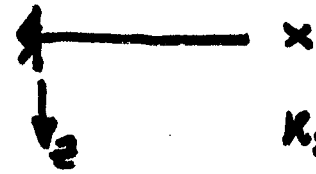
$m_y = D(K_y + \nu K_x)$

$$\frac{D}{2}\left[(K_x + K_y)^2 + 2(1-\nu)\{K_{xy}^2 - K_x K_y\}\right]$$

$$U[w(x,y)] = \iint \left(\frac{D}{2}\left[\qquad \right] - pw\right) dx\,dy$$

EXAMPLE





$w_1(x,y) = C_1 x^2 + C_2 x^2 y^2$

$K_x = -w_{xx} = -\frac{\partial^2 w}{\partial x^2} = -2C_1 - 2C_2 y^2$

$K_y = -\frac{\partial^2 w}{\partial y^2} = 2C_2 x^2$

$$w(x) = w_0(x) + \alpha_1 \phi(x)$$

$$w_0(x) = A_1\left(1-\cos\frac{\pi x}{L}\right)$$
$$w_1(x) = A_2\left(1-\cos\frac{3\pi x}{L}\right)$$

$$U(A_1, A_2) = \min$$

$$\left.\begin{array}{l}\frac{\partial U}{\partial A_1} = 0 \\ \frac{\partial U}{\partial A_2} = 0\end{array}\right] \rightarrow \text{2 eqs for 2 unknowns}$$

For a beam $U[w] = \int_0^L \left[\frac{1}{2} EI (w'')^2 - pw\right] dx$

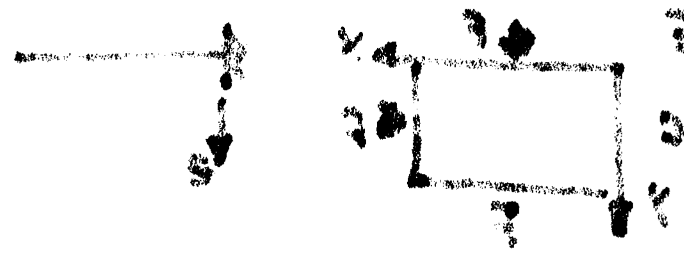
$$\frac{1}{2} EI w'' \cdot w'' = \frac{1}{2} M\kappa$$

For a plate $\frac{1}{2}(M_x \kappa_x + 2 M_{xy}\kappa_{xy} + M_y \kappa_y)$

$$M_x = D(\kappa_x + \nu\kappa_y), \quad M_{xy} = D(1-\nu)\kappa_{xy}$$
$$M_y = D(\kappa_y + \nu\kappa_x)$$

$$U[w(x,y)] = \iint \left(\frac{D}{2}\left[(\kappa_x + \kappa_y)^2 + 2(1-\nu)\{\kappa_{xy}^2 - \kappa_x\kappa_y\} \right] - pw \right) dx\,dy$$

EXAMPLE

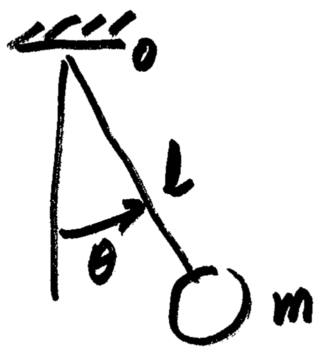


$$w(x,y) = c_1 x^2 + c_2 x^2 y^2$$
$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} = -2c_1 - 2c_2 y^2$$
$$\kappa_y = -\frac{\partial^2 w}{\partial y^2} = -2c_2 x^2$$

Take $\kappa_x, \kappa_y, \kappa_{xy}$ & put into $\bar{U} \Rightarrow \bar{U}(c_1, c_2)$

$$\frac{\partial \bar{U}}{\partial c_1} = 0$$
$$\frac{\partial \bar{U}}{\partial c_2} = 0$$

] 2 eqs in 2 unknowns & solve for c_1 & c_2



$$\sum M_o = I_o \ddot{\theta}$$

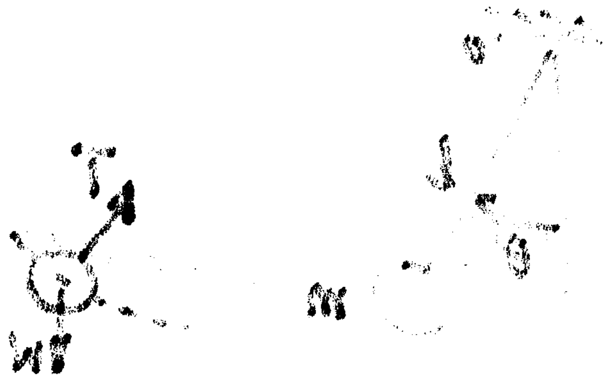
$$I_o \ddot{\theta} + mgl \sin\theta = 0 \qquad I_o = ml^2$$

$$\ddot{\theta} + g/l \underbrace{\sin\theta}_{\theta - \theta^3/3!} = 0$$

Take K_x, K_y, K_{xy} & put into $U \rightarrow U(\varepsilon_1, \varepsilon_2)$

$$\left.\begin{aligned}\frac{\partial U}{\partial \varepsilon_1} &= 0 \\ \frac{\partial U}{\partial \varepsilon_2} &= 0\end{aligned}\right] \text{ 2 eqs in 2 unknowns \& solve for } \varepsilon_1 \ \& \ \varepsilon_2$$

$$\Sigma M_o = I_o \ddot{\theta}$$



$$I_o = ml^2$$

$$I\ddot{\theta} + mgl \sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

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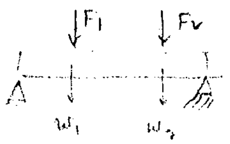
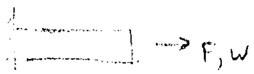
Maxwell–Betti Reciprocity Laws.



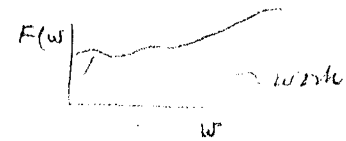
$F_1 = F_1(w_1, w_2)$

$F_2 = F_2(w_1, w_2)$



$\delta \text{work} = F\,dw$

$\text{work} = \int_0^w F(w')\,dw'$



For F_1 & F_2, $\text{work} = \int_0^P F_1\,dw_1 + F_2\,dw_2$

In a conservative system the work done is indep. of path. Thus $F_1\,dw_1 + F_2\,dw_2$ must be an exact derivative ∴

$$d\mathcal{W} = F_1\,dw_1 + F_2\,dw_2 \quad \text{where} \quad \mathcal{W} = \mathcal{W}(F_1, F_2)$$

Thus $F_1 = \dfrac{\partial \mathcal{W}}{\partial w_1}$ $\quad F_2 = \dfrac{\partial \mathcal{W}}{\partial w_2}$

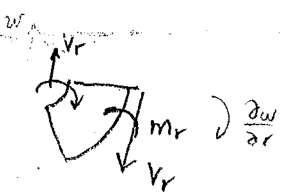
$$\frac{\partial F_1}{\partial w_2} = \frac{\partial F_2}{\partial w_1} = \frac{\partial^2 \mathcal{W}}{\partial w_1 \partial w_2}$$

If the forces are linear fns of displ they can be written as a matrix

$$\begin{pmatrix} F \\ \sim \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} \underset{\sim}{w} \end{pmatrix}$$

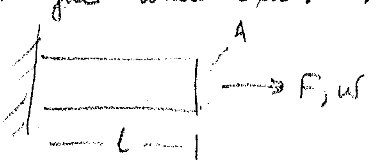
since $\dfrac{\partial F_1}{\partial w_2} = k_{12}$ $\quad \dfrac{\partial F_2}{\partial w_1} = k_{21}$, the matrix (= stiffness matrix) must be sym.

$$\delta w = M_r\,\delta\left(\frac{\partial w}{\partial r}\right) + V_r\,\delta(-w)$$



<u>Strain Energy Method</u>

Useful when exact solution cannot be obtained



$$\mathcal{W} = \int_0^w F\,dw' \quad \text{and} \quad \mathcal{W} = \int M\,d\theta = \int (EI\,w'')\frac{d\theta}{dx}\,dx$$

but $\theta = \dfrac{dw}{dx}$

If F is linear in w then $\mathcal{W} = \frac{1}{2} F w$

For uniaxial case

$$\mathcal{W} = \int \sigma A\,d(l\epsilon) = \int AE\epsilon \cdot l\,d\epsilon = \tfrac{1}{2} AEl\epsilon^2 \text{ vol}$$

$\sigma = F/A = E\epsilon$

$\epsilon = w/l$

$$\therefore \quad \mathcal{W} = \tfrac{1}{2} A\sigma\epsilon l = \frac{1}{2}(Al)E\epsilon^2$$

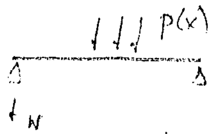
(Volume) this is strain energy

For beam $\epsilon = Kz$ $\quad \sigma = \dfrac{Mz}{I}$ $\quad \mathcal{W} = \dfrac{1}{2}\displaystyle\int Kz\,\frac{Mz}{I}\,dA\,dx$

$W = \frac{1}{2} \kappa M$ strain energy density now $W = \frac{1}{2}\int \kappa z \cdot E\kappa z \, dA\, dx = \frac{1}{2}\int E\left(\int z^2 dA\right)\kappa^2 dx = \frac{1}{2}\int EI\kappa^2 dx$

Thus for entire beam Strain energy $= \int_0^L \frac{EI}{2} \kappa^2 dx$

The potential energy U is the strain energy in beam + potential of external loads.



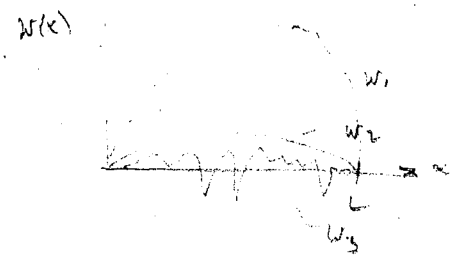
$$U = \int_0^L \tfrac{1}{2} EI K^2 dx - \int_0^L p(x)\, w(x)\, dx$$

$$U = \int_0^L \left\{ \tfrac{1}{2} EI \left[w''(x) \right]^2 - p w \right\} dx$$

U is a scalar that depends on a fn $w(x)$ on the interval $0 \le x \le L$. Thus U is a fnl and we denote

$$U(w(x)) = \int_0^L \{ \quad \} dx$$

thus



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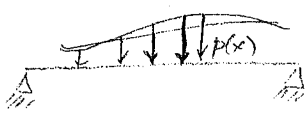
Approx. Methods

Continuation of last time



$$U[w(x)] = \int_0^L \Big\{ \underbrace{\tfrac{1}{2} EI [w'']^2}_{\text{internal pot}} - \underbrace{p w}_{\text{external pot}} \Big\} dx$$

out of all possible w that satisfy B.C.'s, the soln of equil eqn makes U a min.

assume that w is the fn. that minimizes U. Then $U[w(x) + \varepsilon f(x)] = g(\varepsilon)$ is a fn. of ε with w & f fixed. $\Rightarrow \left.\frac{\partial g}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$

$$U(w+\varepsilon f) = \int_0^L \left[\tfrac{1}{2} EI \left(w'' + \varepsilon f''\right)^2 - P(w + \varepsilon f) \right] dx$$

$$\frac{\partial U}{\partial \varepsilon} = \int_0^L \left[\tfrac{1}{2} EI\, 2 (w'' + \varepsilon f'') f'' - P f \right] dx$$

$$\lim_{\varepsilon \to 0} \frac{\partial U}{\partial \varepsilon} = \int_0^L \left(EI\, w'' f'' - p f \right) dx = -\int_0^L (EI\, w'')' f' dx - \int_0^L p f\, dx + EI w'' f' \Big|_0^L$$

$$= \int_0^L \left[(EI w'')'' - p \right] f dx + EI w'' f' \Big|_0^L - (EI w'')' f \Big|_0^L$$

f & w must satisfy BC

$$\therefore \quad EIw''f'\Big|_0^L - (EIw'')'f\Big|_0^L$$

(mom. rot) (shear displ)

either mom or rot = 0 either shear or displ = 0 at Boundary.

BC. 1) either (EIw'') or $w' = 0$ @ x=0 or L

2) " $(EIw'')'$ or $w = 0$ @ x=0 or L ∴ $\lim_{\epsilon \to 0} \frac{dU}{d\epsilon} = \int_0^L \{(EIw'')'' - p\} f\, dx = 0$

$\forall f$ arbitrary $\Rightarrow$ $\boxed{(EIw'')'' - p = 0}$ $\forall x$ — Euler Beam Eqn.

we've used stress-strain rel. to obtain result.

this result is a necessary condition for minim not a sufficient condition since $\frac{dU}{d\epsilon} = 0$ can also produce a max or even a pt of inflection.

suppose $U = \int \mathcal{L}[w, w', w'', w''', w^{IV}, x]\,dx$. The necessary condition for $w(x)$ to give a stationary value of U is the Euler eqn.

$$\frac{d^4}{dx^4}\left(\frac{\partial \mathcal{L}}{\partial w^{IV}}\right) - \frac{d^3}{dx^3}\left(\frac{\partial \mathcal{L}}{\partial w'''}\right) + \frac{d^2}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial w''}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial w'}\right)$$

$$+ \frac{\partial \mathcal{L}}{\partial w} = 0$$

For example $\mathcal{L} = \mathcal{L}(w, w'', x) = \frac{1}{2}EI(w'')^2 - pw$

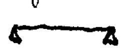
$$\frac{\partial \mathcal{L}}{\partial w''} = EIw''$$

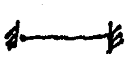
$$\frac{\partial \mathcal{L}}{\partial w} = -p$$

$$\therefore \quad \frac{d^2}{dx^2}(EIw'') - p = 0$$

Ex

Given SS beam with loads.

 chose as 1st approx $w(x) = A \sin \frac{\pi x}{L}$

 for clamped ends $w(x) = A\left(1 - \cos \frac{2\pi x}{L}\right)$ 1st app.

$w' = A \frac{2\pi}{L} \sin \frac{2\pi x}{L}$ $\quad w'' = A\left(\frac{2\pi}{L}\right)^2 \cos \frac{2\pi x}{L}$

$$\mathcal{L} = \frac{1}{2}EI(w'')^2 - p = \frac{1}{2}EI\left\{A^2\left(\frac{2\pi}{L}\right)^4 \cos^2 \frac{2\pi x}{L}\right\} - pA\left[1 - \cos \frac{2\pi x}{L}\right]$$

$$U = \frac{A^2}{2}\int_0^L EI\left(\frac{2\pi}{L}\right)^4 \cos^2 \frac{2\pi x}{L}\,dx - A\int_0^L p(x)\left(1 - \cos \frac{2\pi x}{L}\right)dx$$

but $U(A) = U$ only

$$\therefore \frac{\partial U}{\partial A} = 0$$

$$\frac{\partial U}{\partial A} = \frac{2A}{2}\int_0^L EI\left(\frac{2\pi}{L}\right)^4 \cos^2\frac{2\pi x}{L}\,dx - \int_0^L p\left(1-\cos\frac{2\pi x}{L}\right)dx = 0$$

$$\therefore A = \frac{\int_0^L p(1-\cos\ )\,dx}{\int_0^L EI\left(\frac{2\pi}{L}\right)^2\cos^2} = \frac{pL}{EI\left(\frac{2\pi}{L}\right)^4\cdot\frac{1}{2}L}$$ for const p & EI

$$\therefore A = \frac{pL^4}{EI\,8\pi^4}$$

what is $w(x=L/2) = \frac{2pL^4}{EI\,8\pi^4} = \frac{pL^4}{4EI\pi^4}$

exact soln:

$w(L/2) = \frac{pL^4}{EI\cdot 384}$ $\qquad 4\pi^4 = 389.64$

<u>Calculate bending stress</u>

moment$|_{x=0}$ $\quad w'' = A\cdot\left(\frac{2\pi}{L}\right)^2 \Rightarrow M = -EIw'' = -EI\cdot\frac{pL^4}{EI\,8\pi^4}\cdot\frac{4\pi^2}{L^2}$

$$M = -\frac{pL^2}{2\pi^2}$$

<u>exact sol.</u>

Mom $= \frac{pL^2}{12}$ $\qquad 2\pi^2 \simeq 19.7$

to get better mom. resolution take $w = w_0 + w_1$

$w_1 = A_1\left(1-\cos\frac{4\pi x}{L}\right)$

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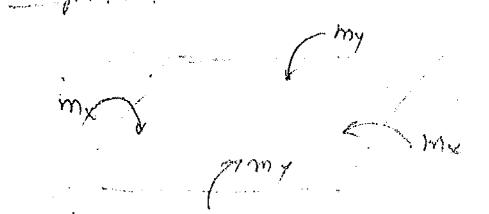
HW problem



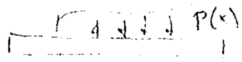
$W_{TOT} = W_I + W_{II}$

BC $v_r = m_r = 0$

W_I: soln w/o hole — known soln

W_{II}: soln due to the hole
1. localized soln
2. BC on edge of hole $(m_r, v_r)_{II} = (-m_r, -v_r)_I$

Continuation of Energy Methods



$$U[w(x)] = \int_0^L \left\{ \tfrac{1}{2} EI (w'')^2 - pw \right\} dx$$

Plate from beam: $\frac{1}{2} EI(w'')^2 = \frac{1}{2} M\kappa = \frac{1}{2} EI\kappa^2$

thus for plate $\frac{1}{2}(M_x \kappa_x + 2M_{xy}\kappa_{xy} + M_y \kappa_y)$

$$M_x = D(\kappa_x + \nu\kappa_y) \qquad M_y = D(\kappa_y + \nu\kappa_x) \qquad M_{xy} = D(1-\nu)\kappa_{xy}$$

Invariant condition

$$\therefore \quad \frac{D}{2}\left[(\kappa_x + \nu\kappa_y)\kappa_x + 2(1-\nu)\kappa_{xy}^2 + (\kappa_y + \nu\kappa_x)\kappa_y \right] = \frac{D}{2}\left[(\kappa_x + \kappa_y)^2 + 2(1-\nu)\left\{ \kappa_{xy}^2 - \kappa_x\kappa_y \right\} \right]$$

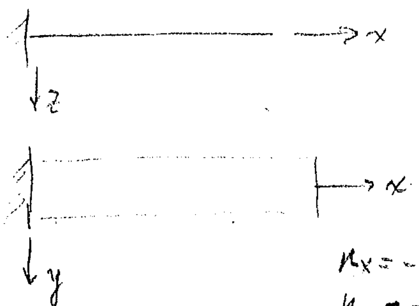
second invariant

for Plate $$U[w(x,y)] = \iint \left(\frac{D}{2}[\quad] - pw \right) dx\,dy.$$

For plate

Example

BC clamped at $x=0$ free everywhere else.



$w(x,y) = c_1 x^2$ only gives bending approx ~~conditions~~ in x direction

$+ c_2 x^2 y^2$ (gives bending approx in y direction)

$$\kappa_x = -w_{xx} = -2c_1 + 2c_2 y^2$$
$$\kappa_y = -w_{yy} = 2c_2 x^2$$

Since $U = fn(\kappa_x^2, \kappa_{xy}, \kappa_x\kappa_y, \kappa_y^2) = fn(c_1^2, c_1 c_2, c_2^2, c_1, c_2)$

$$= \frac{1}{2}\left(c_1^2 k_{11} + 2c_1c_2 k_{12} + c_2^2 k_{22} - L_1 c_1 - L_2 c_2 \right)$$

to get min $\Rightarrow$ $$\frac{\partial U}{\partial c_1} = 0 = c_1 k_{11} + c_2 k_{12} - L_1 = 0$$

$$\frac{\partial U}{\partial c_2} = 0 = c_1 k_{12} + c_2 k_{22} - L_2 = 0$$

thus we get a linear system of eqns.

$$\therefore \quad c_1 = \frac{\begin{pmatrix} L_1 & k_{12} \\ L_2 & k_{22} \end{pmatrix}}{\begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}} \qquad c_2 = \frac{\begin{pmatrix} k_{11} & L_1 \\ k_{12} & L_2 \end{pmatrix}}{\begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}}$$

if $\underset{\sim}{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ $\quad \underset{\sim}{k} = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$ $\quad \underset{\sim}{L} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$

$$U = \frac{1}{2} \underset{\sim}{c}^T \underset{\sim}{k}\, \underset{\sim}{c} - \underset{\sim}{c}^T \underset{\sim}{L} \qquad \frac{\partial U}{\partial \underset{\sim}{c}} = \underset{\sim}{k}\,\underset{\sim}{c} - \underset{\sim}{L} = 0$$

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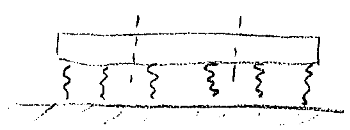
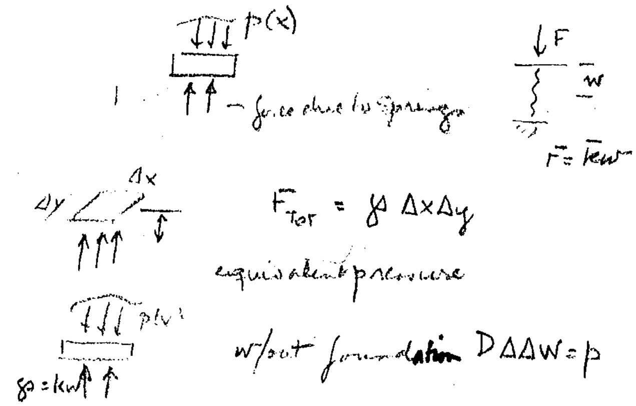
for brittle material look for max tensile stress to define SCF
for material that fails in shear look for max shearing stress to define SCF
for material that flows plastically look for value that will bring onset of plasticity

Consider Plate w/ Elastic Foundation (Winkler foundation for soils)



Take a Δx out of plate



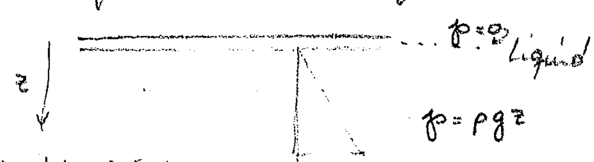
w/out foundation $D\Delta\Delta w = p$

$k \; [=] \; \frac{\text{force}}{\text{unit area} \times (\text{unit disp})}$

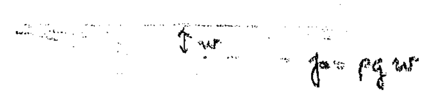
to include found: replace $p \rightarrow p - kw$ $\quad \therefore \quad D\Delta\Delta w + kw = p$

foundation doesn't affect defn. of m_x, m_y, m_{yx} & also doesn't affect b.c.

Example — Case of an exact elastic foundation — plate floating on liquid (ice over water



for case of deflected plate



equivalent $k = \rho g$.

For soils, use elastic foundation (Winkler) for a first approx. Good only when fndtn is softer than beam (plate).

Solution of $D\Delta\Delta w + kw = p$

Foundations of Solid Mechanics by Y.C. Fung

The term

(12) $$L \equiv U - K + A$$

(or sometimes $-L$) is called the *Lagrangian function* and the equation (11) represents *Hamilton's principle*, which states that:

The time integral of the Lagrangian function over a time interval t_0 to t_1 is an extremum for the "actual" motion with respect to all admissable virtual displacements which vanish, first, at instants of time t_0 and t_1 at all points of the body, and, second, over S_u, where the displacements are prescribed, throughout the entire time interval.

To formulate this principle in another way, let us call $u_i(x_1, x_2, x_3; t)$ a dynamic path. Then Hamilton's principle states that *among all dynamic paths that satisfy the boundary conditions over S_u at all times and that start and end with the actual values at two arbitrary instants of time t_0 and t_1 at every point of the body, the "actual" dynamic path is distinguished by making the Lagrangian function an extremum.*

In rigid body dynamics the term U drops out, and we obtain Hamilton's principle in the familiar form. The symbol A replaces the usual symbol V in books on dynamics because we have used V for something else.

Note that the potential energy $-A$ of the external loads exists and is a linear function of the displacements if the loads are independent of the elastic displacements, as is commonly the case. In aeroelastic problems, however, the aerodynamic loading is sensitive to the small surface displacements u_i; moreover, it depends on the time history of the displacements and cannot be derived from a potential. Hence, in aeroelasticity we are generally forced to use the variational form (9) of Hamilton's principle.

In some applications of the direct method of calculation, it is even desirable to liberalize the variations δu_i at the instants t_0 and t_1 and use Hamilton's principle in the variational form (4) which cannot be expressed elegantly as the minimum of a well-defined functional. On the other hand, such a formulation will be accessible to the direct methods of solution. On introducing (5), (7), and (10), we may rewrite Eq. (4) in the following form:

(13) $$\int_{t_0}^{t_1} \delta(U - K + A)\, dt = \int_{t_0}^{t_1}\int_V F_i\, \delta u_i\, dv\, dt + \int_{t_0}^{t_1}\int_S \overset{\nu}{T}_i\, \delta u_i\, dS - \int_V \rho \frac{\partial u_i}{\partial t}\, \delta u_i\, dv \Big|_{t_0}^{t_1}.$$

Here U is the total strain energy, K is the total kinetic energy, A is the potential energy for the conservative external forces, F_i and $\overset{\nu}{T}_i$ are, respectively, those external body and surface forces that are not included in A, and δu_i are the virtual displacements.

Problem 11.1. Prove the converse theorem that, for a conservative system, the variational Eq. (11) leads to the equation of motion

$$\rho \frac{\partial^2 u_i}{\partial t^2} = F_i + \frac{\partial}{\partial x_j}\frac{\partial W}{\partial e_{ij}}$$

and the boundary conditions

$$\text{either} \quad \delta u_i = 0 \quad \text{or} \quad \frac{\partial W}{\partial e_{ij}} \nu_j = \overset{\nu}{T}_i.$$

11.2. EXAMPLE OF APPLICATION—EQUATION OF VIBRATION OF A BEAM

As an example of the application of Hamilton's principle in the formulation of approximate theories in elasticity, let us consider the free, lateral vibration of a straight simple beam. We assume that the beam possesses principal planes and that the vibration takes place in one of the principal planes, and let y denote the small deflection of the neutral axis of the beam from its initial, straight configuration. In Sec. 10.8 it is shown that the strain energy of the beam is, for small deflections,

(1) $$U = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx,$$

where E is the Young's modulus of the beam material, I is the cross-sectional moment of interia, and l is the length of the beam.

The kinetic energy of the beam is derived partly from the translation, parallel to y, of the elements composing it, and partly from the rotation of the same elements about an axis perpendicular to the neutral axis and the plane of vibration. The former part is

$$\frac{1}{2}\int_0^l m\left(\frac{\partial y}{\partial t}\right)^2 dx,$$

where m is the mass per unit length of the beam. The latter part is, for each element dx, the product of moment of inertia times one-half of the square of the angular velocity. Let I_ρ denote the mass moment of inertia about the neutral axis per unit length of the beam. The angular velocity being $\partial^2 y/\partial t\, \partial x$, the kinetic energy of the beam is

(2) $$K = \frac{1}{2}\int_0^l m\left(\frac{\partial y}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^l I_\rho\left(\frac{\partial^2 y}{\partial x\, \partial t}\right)^2 dx.$$

If the beam is loaded by a distributed lateral load of intensity $p(x, t)$ per unit length and moment and shear M and Q, respectively, at the ends as shown in Fig. 11.2:1, then the potential energy of the external loading is

(3) $$A = -\int_0^l p(x, t)y(x)\, dx - M_l\left(\frac{\partial y}{\partial x}\right)_l + M_0\left(\frac{\partial y}{\partial x}\right)_0 + Q_l y_l - Q_0 y_0.$$

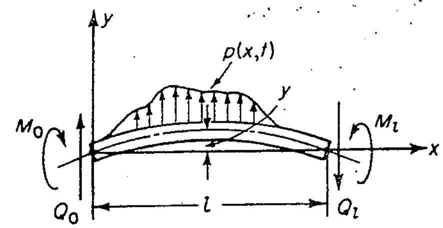


Fig. 11.2:1. Applications to a beam.

The equation of motion is given by Hamilton's principle:

(4) $$\delta \int_{t_0}^{t_1} (U - K + A)\, dt = 0;$$

i.e.,

(5) $$\delta \int_{t_0}^{t_1} \left\{ \int_0^l \left[\frac{1}{2} EI \left(\frac{\partial^2 y}{\partial x^2}\right)^2 - \frac{1}{2} m \left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2} I_\rho \left(\frac{\partial^2 y}{\partial x\, \partial t}\right)^2 - py \right] dx - M_l \left(\frac{\partial y}{\partial x}\right)_l + M_0 \left(\frac{\partial y}{\partial x}\right)_0 + Q_l y_l - Q_0 y_0 \right\} dt = 0.$$

Following the usual procedure of the calculus of variations, noting that the virtual displacement must be so specified that $\delta y \equiv 0$ at t_0 and t_1, and, hence, $\partial(\delta y)/\partial x = \delta(\partial y/\partial x) \equiv 0$ at t_0 and t_1, we obtain

$$\int_{t_0}^{t_1} \left[\int_0^l \left(EI \frac{\partial^2 y}{\partial x^2} \frac{\partial^2\, \delta y}{\partial x^2} - m \frac{\partial y}{\partial t} \frac{\partial\, \delta y}{\partial t} - I_\rho \frac{\partial^2 y}{\partial x\, \partial t} \frac{\partial^2\, \delta y}{\partial x\, \partial t} - p\, \delta y \right) dx - M_l\, \delta\left(\frac{\partial y}{\partial x}\right)_l + M_0\, \delta\left(\frac{\partial y}{\partial x}\right)_0 + Q_l\, \delta y_l - Q_0\, \delta y_0 \right] dt = 0.$$

Integrating by parts, we obtain

(6) $$\int_{t_0}^{t_1} \int_0^l \left[\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) + m \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2}{\partial x\, \partial t}\left(I_\rho \frac{\partial^2 y}{\partial x\, \partial t}\right) - p(x, t) \right] \delta y\, dx\, dt - \int_{t_0}^{t} \left[EI \frac{\partial^2 y}{\partial x^2} - M \right] \delta \left(\frac{\partial y}{\partial x}\right)\Big|_0^l\, dt - \int_{t_0}^{t} \left[\frac{\partial}{\partial x}\left(EI \frac{\partial^2 y}{\partial x^2}\right) - \frac{\partial}{\partial t}\left(I_\rho \frac{\partial^2 y}{\partial x\, \partial t}\right) - Q \right] \delta y \Big|_0^l\, dt = 0.$$

Hence, the Euler equation of motion is

(7) $$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) + m \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2}{\partial x\, \partial t}\left(I_\rho \frac{\partial^2 y}{\partial x\, \partial t}\right) = p(x, t),$$

and a proper set of boundary conditions at each end is

(8a) $$\text{either} \quad EI \frac{\partial^2 y}{\partial x^2} = M \quad \text{or} \quad \delta\left(\frac{\partial y}{\partial x}\right) = 0$$

and

(8b) $$\text{either} \quad \frac{\partial}{\partial x}\left(EI \frac{\partial^2 y}{\partial x^2}\right) - \frac{\partial}{\partial t}\left(I_\rho \frac{\partial^2 y}{\partial x\, \partial t}\right) = Q \quad \text{or} \quad \delta y = 0.$$

These are equations governing the motion of a beam including the effect of the rotary inertia, due to Lord Rayleigh, and known as ***Rayleigh's equations.*** If the rotary inertia is neglected and if the beam were uniform, then the governing equation is simplified into:

(9) $$\frac{\partial^2 y}{\partial t^2} + c_0^2 R^2 \frac{\partial^4 y}{\partial x^4} = \frac{1}{EI} p,$$

where

(10) $$c_0^2 = \frac{E}{\rho}, \qquad R^2 = \frac{I}{A}.$$

The constant c_0 has the dimension of speed and can be identified as the phase velocity of longitudinal waves in a uniform bar.† R is the radius of gyration of the cross section. A is the cross-sectional area, so that $m = \rho A$.

In the special case of a uniform beam of infinite length free from lateral loading, $p = 0$, Eq. (9) becomes

(11) $$\frac{\partial^2 y}{\partial t^2} + c_0^2 R^2 \frac{\partial^4 y}{\partial x^4} = 0.$$

It admits a solution in the form

(12) $$y = a \sin \frac{2\pi}{\lambda}(x - ct),$$

which represents a progressive wave of phase velocity c and wave length λ. On substituting (12) into (11), we obtain the relation

(13) $$c = \pm c_0 R \frac{2\pi}{\lambda},$$

which states that the phase velocity depends on the wave length and that it tends to infinity for very short wave lengths. Somewhat disconcerting is the fact that, according to Eq. (13), the group velocity (see Sec. 11.3) also tends to infinity as the wave length tends to zero. Since group velocity is the velocity at which energy is transmitted, this result is physically unreasonable. If Eq. (13) were correct, then the effect of a suddenly applied concentrated load will be felt at once everywhere in the beam, as the Fourier representation for a concentrated load contains harmonic components with infinitesimal wave length, and hence infinite wave speed. Thus, Eq. (11) cannot be very accurate in describing the effect of impact loads on a beam.

† See Prob. 11.2, p. 325.

This difficulty of infinite wave speed is removed by the inclusion of the rotary inertia. However, the speed versus wave length relationship obtained from Rayleigh's Eq. (7) for a uniform beam of circular cross section with radius a, as is shown in Fig. 11.2:2, still deviates appreciably from Pochhammer and Chree's results, which were derived from the exact three-dimensional linear elasticity theory. A much better approximation is obtained by including the shear deflection of the beam, as was first shown by Timoshenko.

To incorporate the shear deformation, we note that the slope of the deflection curve depends not only on the rotation of cross sections of the

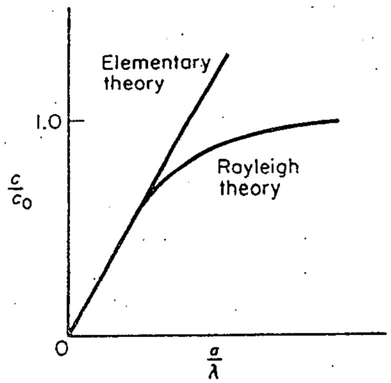


Fig. 11.2:2. Phase velocity curves for flexural elastic waves in a circular cylinder of radius a.

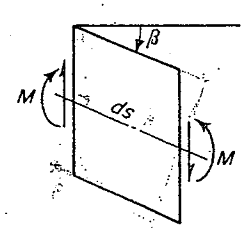


Fig. 11.2:3. A Timoshenko beam element.

beam but also on the shear. Let ψ denote the slope of the deflection curve when the shearing force is neglected and β the angle of shear at the neutral axis in the same cross section. Then the total slope is

(14) $$\frac{\partial y}{\partial x} = \psi + \beta.$$

The strain energy due to bending, Eq. (1), must be replaced by

(15) $$\frac{1}{2}\int_0^l EI\left(\frac{\partial \psi}{\partial x}\right)^2 dx,$$

because the internal bending moment does no work when shear deformation takes place (see Fig. 11.2:3). The strain energy due to shearing strain β must be a quadratic function of β if linear elasticity is assumed. We shall write

(16) $$\frac{1}{2}\int_0^l k\beta^2\, dx = \frac{1}{2}\int_0^l k\left(\frac{\partial y}{\partial x} - \psi\right)^2 dx$$

for the strain energy for shear. The kinetic energy is

(17) $$K = \frac{1}{2}\int_0^l m\left(\frac{\partial y}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^l I_\rho\left(\frac{\partial \psi}{\partial t}\right)^2 dx,$$

because the translational velocity is $\partial y/\partial t$, but the angular velocity is $\partial\psi/\partial t$. Hence, Hamilton's principle states that

(18) $$\delta\int_{t_0}^{t_1}\int_0^l \frac{1}{2}\left[EI\left(\frac{\partial \psi}{\partial x}\right)^2 + k\left(\frac{\partial y}{\partial x} - \psi\right)^2 - m\left(\frac{\partial y}{\partial t}\right)^2 - I_\rho\left(\frac{\partial \psi}{\partial t}\right)^2\right] dx\, dt + \delta A = 0,$$

where A is given by (3) except that $\partial y/\partial x$ at the ends is to be replaced by ψ. The virtual displacements now consist of δy and $\delta\psi$, which must vanish at t_0 and t_1 and also where displacements are prescribed. On carrying out the calculations, the following two Euler equations are obtained:

(19a) $$\frac{\partial}{\partial x}\left(EI\frac{\partial \psi}{\partial x}\right) + k\left(\frac{\partial y}{\partial x} - \psi\right) - I_\rho\frac{\partial^2 \psi}{\partial t^2} = 0,$$

(19b) $$m\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left[k\left(\frac{\partial y}{\partial x} - \psi\right)\right] - p = 0.$$

The appropriate boundary conditions are, at each end of the beam,

(20a) $$\text{Either } -EI\frac{\partial \psi}{\partial x} = M \quad \text{or} \quad \delta\psi = 0,$$

and

(20b) $$\text{either } \quad k\left(\frac{\partial y}{\partial x} - \psi\right) = Q \quad \text{or} \quad \delta y = 0.$$

These are the differential equation and boundary conditions of the so-called *Timoshenko beam theory*.

For a uniform beam, EI, k, m, etc., are constants, and the function ψ can be eliminated from the equations above to obtain the well-known *Timoshenko equation for lateral vibration of prismatic beams*,

(21) $$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} - \left(I_\rho + \frac{EIm}{k}\right)\frac{\partial^4 y}{\partial x^2\,\partial t^2} + I_\rho\frac{m}{k}\frac{\partial^4 y}{\partial t^4} = p + \frac{I_\rho}{k}\frac{\partial^2 p}{\partial t^2} - \frac{EI}{k}\frac{\partial^2 p}{\partial x^2}.$$

So far we have not discussed the constants m, I_ρ, and k. For a beam of uniform material, $m = \rho A$, $I_\rho = \rho A R^2$, where ρ is the mass density of the beam material, A is the cross-sectional area, and R is the radius of gyration

of the cross section about an axis perpendicular to the plane of motion and through the neutral axis. But k depends on the distribution of shearing stress in the beam cross section. Timoshenko writes

$$k = k'AG, \tag{22}$$

where G is the shear modulus of elasticity and k' is a numerical factor depending on the shape of the cross section, and ascertains that according to the elementary beam theory, $k' = \frac{2}{3}$ for a rectangular cross section. The use of such a value of k is, however, a subject of controversy in the literature. Mindlin[11.1] suggests that the value of k can be so selected that the solution of Eq. (21) be made to agree with certain solution of the exact three-dimensional equations of Pochhammer (1876) and Chree (1889) (see Love,[1.2] *Elasticity*, 4th ed., pp. 287–92). Indeed, I_ρ, which arises in the assumption of plane sections remain plane in bending, may also be regarded, when such an assumption is relaxed, as an empirical factor to be determined by comparison with exact solutions.

For a uniform beam free from lateral loadings, Eq. (21) can be written as

$$\frac{\partial^4 y}{\partial x^4} - \left(\frac{1}{c_0^2} + \frac{1}{c_Q^2}\right)\frac{\partial^4 y}{\partial x^2\,\partial t^2} + \frac{1}{c_0^2 c_Q^2}\frac{\partial^4 y}{\partial t^4} + \frac{1}{c_0^2 R^2}\frac{\partial^2 y}{\partial t^2} = 0, \tag{23}$$

where

$$c_0^2 = \frac{E}{\rho}, \qquad c_Q^2 = \frac{k'G}{\rho}; \qquad R^2 = \frac{I}{A}. \tag{24}$$

If the beam is of infinite length, a solution of the form (12) may be substituted into (23), and we see that the wave speed c must satisfy the equation

$$1 - \left(\frac{c^2}{c_0^2} + \frac{c^2}{c_Q^2}\right) + \frac{c^4}{c_0^2 c_Q^2} - \frac{c^2}{c_0^2 R^2}\left(\frac{\lambda}{2\pi}\right)^2 = 0. \tag{25}$$

The solution of this equation for c/c_0 versus λ yields two branches, corresponding to two "modes" of motion (two different shear-to-bending deflection ratios for the same wavelength). They are plotted in Fig. 11.2:4 for the special case of a beam of circular cross section with radius a. The results of the exact solution of Pochhammer and Chree for Poisson's ratio $\nu = 0.29$ are also plotted there for comparison. It is seen that the Timoshenko theory agrees reasonably well with the exact theory in the first mode, but wide discrepancy occurs in the second mode. The approximate theory gives no information about higher modes: an infinite number of which exist in the exact theory.

The equations derived above are, of course, appropriate for the determination of the free-vibration modes and frequencies of a beam. The effects of rotary inertia and shear are unimportant if the wavelength of the vibration mode is large compared with the cross-sectional dimensions of the beam; but these effects become more and more important with a decrease of wavelength, i.e., with an increase in the frequency of vibration. In the example of a uniform beam with rectangular cross section and simply supported at both ends, with $E = \frac{8}{3}G$ and $k' = \frac{2}{3}$, we find that the shear deflection and rotary inertia reduce the natural frequencies. If the wavelength

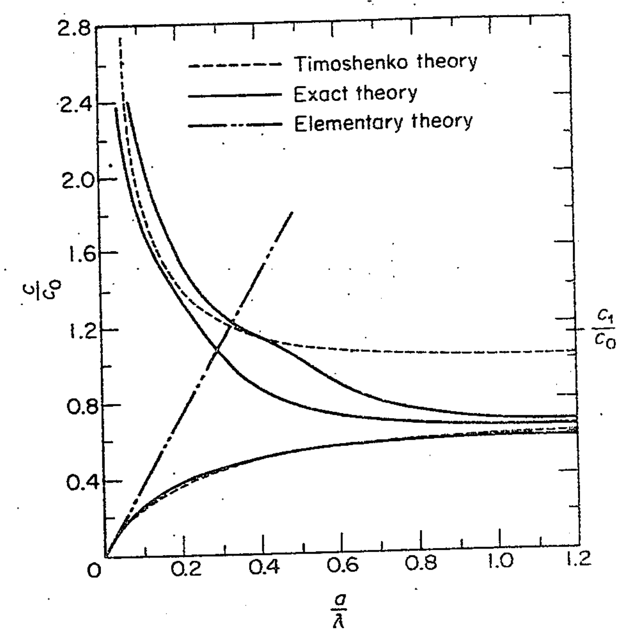


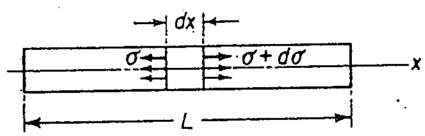
Fig. 11.2:4. Phase velocity curves for flexural elastic waves in a solid circular cylinder of radius a. (From Abramson,[11.1] *J. Acoust. Soc. Am.*, 1957.)

is ten times larger than the depth of the beam, the correction on the frequency due to rotary inertia alone is about 0.4 per cent, and the correction due to rotary inertia and shear together will be about 2 per cent.

The Timoshenko beam theory has attracted much attention in recent years. For a survey of literature, see Abramson, Plass, and Ripperger.[11.1]

PROBLEMS

11.2. Consider the free longitudinal vibration of a rod of uniform cross section and length L, as shown in Fig. P11.2. Let us assume that plane cross sections remain plane, that only axial stresses are present, being uniformly distributed over the cross section, and that radial displacements are negligible (i.e., the displacements consist of only one nonvanishing component u in the x-direction). Derive



P11.2. Longitudinal vibration of a rod.

expressions for the potential and kinetic energy and show that the equation of motion is

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} = 0, \qquad c_0^2 = \frac{E}{\rho}. \tag{26}$$

Show that the general solution is of the form

$$u = f(x - c_0 t) + F(x + c_0 t), \tag{27}$$

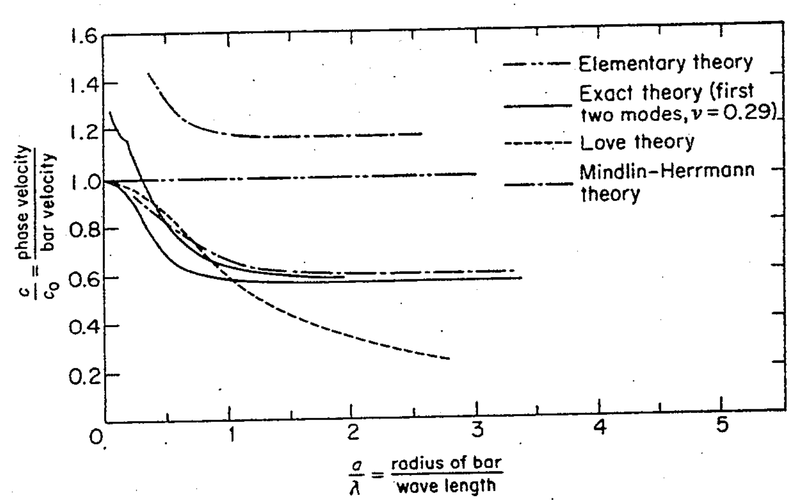
where f and F are two arbitrary functions.

11.3. Consider the same problem as above, but now incorporate approximately the transverse inertia associated with the lateral expansion or contraction connected with axial compression and extension, respectively. Let the (Love's) assumption be made that the displacement in the radial direction v is proportional to the radial coordinate r, measured from a centroidal axis, and to the axial strain $\partial u/\partial x$; i.e.,

$$v = -\nu r \frac{\partial u}{\partial x}, \tag{28}$$

where ν is Poisson's ratio. Derive expressions of the kinetic and potential energy and obtain the equation of motion according to Hamilton's principle,

$$\rho\left[\frac{\partial^2 u}{\partial t^2} - (\nu R)^2 \frac{\partial^4 u}{\partial x^2\, \partial t^2}\right] - E\frac{\partial^2 u}{\partial x^2} = 0, \tag{29}$$



P11.3. Phase velocity curves for longitudinal elastic waves in a solid circular cylinder of radius a. (After Abramson et al., *Adv. Applied Mech.*, 5, 1958.)

where R is the polar radius of gyration of the cross section. The natural boundary condition at the end $x = 0$, if that end is subject to a stress $\sigma_0(t)$, is

$$\rho\nu^2 R^2 \frac{\partial^3 u}{\partial x\, \partial t^2} + E\frac{\partial u}{\partial x} = \sigma_0(t) \qquad \text{at } x = 0. \tag{30}$$

Note. It is important to note that, according to the last equation, the familiar proportionality between axial stress σ and axial strain $\partial u/\partial x$ does not exist in this theory.

Comparison of the dispersion curves obtained from the elementary theory (Prob. 11.2), the Love theory, the Pochhammer-Chree "exact" theory, and another approximate theory due to Mindlin and Herrmann,[11.1] are shown in Fig. P11.3 The last-mentioned theory accounts for the strain energy associated with the transverse displacement v, of which the most important contribution comes from the shearing strain caused by the lateral expansion of the cross section near a wave front.

11.4. The method of derivation of the various forms of equations of motion of beams as presented above has the advantage of being straightforward, but it does not convey the physical concepts as clearly as in an elementary derivation. Hence, rederive the basic equations by considering the forces that act on an element of length dx, as shown in Fig. P11.2 and Fig. P11.4. Obtain the following equations, and then derive the wave equations by proper reductions.

Longitudinal waves, elementary theory (Fig. P11.2):

$$\frac{\partial \sigma_{xx}}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \quad \text{(equation of motion)}, \tag{31}$$

$$\frac{\partial^2 u}{\partial x\, \partial t} = \frac{\partial \epsilon_{xx}}{\partial t} \quad \text{(equation of strain)}, \tag{32}$$

$$\sigma_{xx} = E\epsilon_{xx} \quad \text{(equation of material behavior)}, \tag{33}$$

where σ_{xx} = axial stress, ϵ_{xx} = axial strain, $\partial u/\partial t$ = axial particle velocity, x = axial coordinate, t = time, E = modulus of elasticity, and ρ = mass density.

Flexural waves, Timoshenko theory (Fig. P11.4, p. 328):

$$\frac{\partial M}{\partial x} - Q = \rho I \frac{\partial \omega}{\partial t} \quad \text{(rotational)} \tag{34}$$

$$\frac{\partial Q}{\partial x} = \rho A \frac{\partial v}{\partial t} \quad \text{(transverse)} \tag{35}$$

(equations of motion),

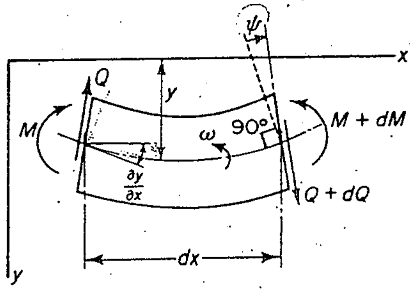
$$\frac{\partial K}{\partial t} = \frac{\partial \omega}{\partial x} \quad \text{(bending)} \tag{36}$$

$$\frac{\partial \beta}{\partial t} = \frac{\partial v}{\partial x} + \omega \quad \text{(shear)} \tag{37}$$

$$M = EIK \quad \text{(bending)} \tag{38}$$

$$Q = A_s\beta \quad \text{(shear)} \tag{39}$$

(equations of material behavior),



P11.4. Element of a beam in bending.

where M = moment, Q = shear force, K = axial rate of change of section angle = $-\partial\psi/\partial x$, β = shear strain = $\partial y/\partial x - \psi$, ω = angular velocity of section = $-\partial\psi/\partial t$, v = transverse velocity = $\partial y/\partial t$, I = section moment of inertia, A = section area, and A_s = area parameter defined by $\iint \gamma(z)\,dA = \beta A_s$ where $\gamma(z)$ is the shear strain at a point z in the cross section.

11.3. GROUP VELOCITY

Since we have been concerned in the preceding sections about wave propagations in beams, it seems appropriate to make a digression to explain the concept of *group velocity* as distinguished from the *phase velocity*. We have seen that for certain equations a solution of the following form exists:

(1)
$$u = a \sin(\mu x - \nu t).$$

If x is increased by $2\pi/\mu$, or t by $2\pi/\nu$, the sine takes the same value as before, so that $\lambda = 2\pi/\mu$ is the wavelength and $T = 2\pi/\nu$ is the period of oscillation. If $\mu x - \nu t$ = constant, i.e. x = const. + $\nu t/\mu$, the argument of the sine function remains constant in time; which means that the whole waveform is displaced towards the right with a velocity $c = \nu/\mu$. The quantity c is called the phase velocity, in terms of which Eq. (1) may be exhibited as

(2)
$$u = a \sin \frac{2\pi}{\lambda}(x - ct).$$

If the phase velocity c depends on the wavelength λ, the wave is said to exhibit *dispersion*. Our examples in the previous section show that dispersion exists in both longitudinal and flexural waves in rods and beams.

What happens when two sine waves of the same amplitude but slightly different wavelengths and frequencies are superposed? Let these two waves be characterized by two sets of slightly different values μ, ν and μ', ν'. The resultant of the superposed waves is

$$u + u' = A[\sin(\mu x - \nu t) + \sin(\mu' x - \nu' t)].$$

Using the well-known formula

$$\sin\alpha + \sin\beta = 2 \sin \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta),$$

we have

(3)
$$u + u' = 2A \sin[\tfrac{1}{2}(\mu + \mu')x - \tfrac{1}{2}(\nu + \nu')t] \cos[\tfrac{1}{2}(\mu - \mu')x - \tfrac{1}{2}(\nu - \nu')t].$$

This expression represents the well-known phenomenon of "beats." The sine factor represents a wave whose wave number and frequency are equal to the mean of μ, μ' and ν, ν', respectively. The cosine factor, which varies very slowly when $\mu - \mu'$, $\nu - \nu'$ are small, may be regarded as a varying amplitude,

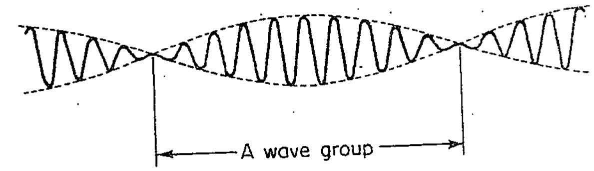


Fig. 11.3:1. An illustration of a wave group.

as shown in Fig. 11.3:1. The "wave group" ends wherever the cosine becomes zero. The velocity of advance of these points is called the *group velocity*; its value U is equal to $(\nu - \nu')/(\mu - \mu')$. For long groups (or slow beats), the group velocity may be written with sufficient accuracy as

(4)
$$U = \frac{d\nu}{d\mu}.$$

In terms of the wavelength λ ($= 2\pi/\mu$), we have

(5)
$$U = \frac{d(\mu c)}{d\mu} = c - \lambda \frac{dc}{d\lambda},$$

where c is the phase velocity.

From the fact that no energy can travel past the nodes, one can infer that the rate of transfer of energy is identical with the group velocity. This fact is capable of rigorous proof for single trains of waves.

The most familiar examples of propagation of wave groups are perhaps the water waves. It has often been noticed that when an isolated group of waves, of sensibly the same length, advancing over relatively deep water, the velocity of the group as a whole is less than that of the individual waves composing it. If attention is fixed on a particular wave, it is seen to advance through the group, gradually dying out as it approaches the front, while its former place in the group is occupied in succession by other waves which have come forward from the rear. Another familiar example is the wave train set up by ships. The explanation as presented above seems to have been first given by Stokes (1876). Other derivations and interpretations of

J. J. S[illegible], "Non Linear Vibrations" – Interscience, a div of J. Wiley
Pure & Applied Math. Vol II, 1950

response curve" associated with a value L_1^* of the external torque such that the vertical tangent drawn on it from the point P intersects the upper branch of *the same response curve* at Q on the line $|A| = |A|_{\max}$. Points to the left of PQ and above T_1P lie on "unstable" branches of response curves whose upper stable branches correspond to values of $|A|$ larger than $|A|_{\max}$, while points to the right of PQ lead to values of $|A|$ less than $|A|_{\max}$. The critical value L_1^* of the external torque thus has the following property: If the external torque L_1 is kept below this value the amplitude $|A|$ of the motion will be less than $|A|_{\max}$ for all frequencies, but this will not hold good for all frequencies if $L_1 > L_1^*$.

One sees from Figure 5.1 that *the range of frequencies a little below the frequency $\sqrt{\alpha}$ of the linear free oscillation is much more critical for the operation of the synchronous motor than the higher range of frequencies*, and this results because of the fact that the nonlinear restoring torque is soft in the present case. Figure 5.1 was constructed assuming the presence of damping. One sees readily, however, that a non-vanishing critical external torque L_1^* would exist without the presence of damping. In other words, even without damping it would be possible to vary the frequency of the external torque from values well below $\sqrt{\alpha}$ to values well above it without causing amplitudes higher than any given value, provided that the external torque amplitude is kept below the critical value.† This is, of course, not possible with a linear restoring force.

If the constant part L_0 of the external torque is not zero, the problem of hunting of the synchronous motor can be attacked in much the same way as for $L_0 = 0$. A special case of this kind has been treated by Duffing.

6. *The perturbation method*

One of the commonest methods for treating nonlinear problems in mechanics is the perturbation method, which consists in developing the desired quantities in powers of some parameter which can be considered small, and determining the coefficients of the developments stepwise, usually by solving a sequence of linear problems. The

† Naturally, this statement (as well as other earlier statements) is made only on the assumption that it is the harmonic oscillation studied here which is actually excited.

method has the advantage that it is relatively foolproof, in the sense that it can often be applied fairly safely even in the absence of foreknowledge regarding the general character of the solution.* The method of perturbations is also very useful in settling important theoretical questions of a purely mathematical character; for example, the existence of the various types of periodic motions discussed in this chapter can probably be established most readily by proving that the appropriate perturbation series converge. In Appendix I to this book we take up such questions in some detail. However, the perturbation method has the disadvantage that it is often rather cumbersome for actual computations, particularly if more than one or two terms in the perturbation series are desired. Consequently, it is often advantageous to begin the attack on a new problem by the perturbation method in order to gain a first insight into the character of its solution, but to abandon this attack eventually in favor of other approaches once sufficient knowledge about the behavior of the solutions has been gained. For this reason, as well as for reasons indicated at the beginning of this chapter, we treat the Duffing problem in detail once more by the perturbation method as applied directly to the differential equation.

In this section we illustrate the use of the perturbation method (in a form first used by Lindstedt [26], apparently, in connection with problems in astronomy) to obtain the harmonic solutions of

$$\ddot{x} + (\alpha x + \beta x^3) = F \cos \omega t. \tag{6.1}$$

As we have stated, the method consists in developing the desired solution $x(t)$ in a power series with respect to a small parameter ϵ, the coefficients in the series being functions of t. We write, therefore,

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots, \tag{6.2}$$

the x_i being functions of t. A periodic solution $x(t)$ is desired which has the same frequency as $F \cos \omega t$. It would be natural to regard the amplitude of $x(t)$ as a quantity to be determined for any given frequency ω. However, our previous experience has taught us that the amplitude of the vibration rather than its frequency should be prescribed. In order to avoid working with functions of unknown

* The problem under consideration here, however, are not altogether of this character. One must gain some advance insight about the character of the solutions in order to fix the details of the procedure.

$\ddot{\theta} + g/l \sin\theta = F\cos\omega t$

$+ g/l[\theta - \theta^3/3!]$ $\therefore\ \alpha = g/l \quad \beta = -g/l\cdot 6$

period it is of advantage to introduce a new independent variable θ replacing t through the relation $\theta = \omega t$. The equation (6.1) becomes

(6.3) $$\omega^2 \frac{d^2 x}{d\theta^2} + (\alpha x + \beta x^3) = F \cos \theta$$

We now require that $x(\theta)$ satisfy the following conditions:

(6.4)
$$\begin{aligned} &\text{a)} \quad x(\theta + 2\pi) = x(\theta) \\ &\text{b)} \quad x(0) = A \\ &\text{c)} \quad x'(0) = 0. \end{aligned}$$

The prime on a quantity means differentiation with respect to θ here and in what follows. The condition a) states that $x(\theta)$ is to be of period 2π, while *b*) and *c*) fix, roughly speaking, the amplitude and phase of the vibration. It should be noted that A is the maximum of $x(t)$ here rather than the first Fourier coefficient, and hence is not quite the same as the quantity A in the preceding sections. The value of ω will depend upon A, as we know.

The parameter ϵ is arbitrary to a certain degree in any perturbation procedure. It is, however, natural in this case to choose $\epsilon = \beta$, so that the perturbation series (6.2) may be considered as a development in the neighborhood of the solution of the linearized vibration problem. In addition to x it is also necessary to develop the quantity ω with respect to β. If the series

(6.5) $$x(\theta) = x_0(\theta) + \beta x_1(\theta) + \beta^2 x_2(\theta) + \cdots,$$

(6.6) $$\omega = \omega_0 + \beta\omega_1 + \beta^2\omega_2 + \cdots$$

are inserted in (6.3) we obtain a power series in β which must vanish identically in β; hence the coefficients of the successive powers of β must vanish. The coefficients are second order linear differential equations in the $x_i(\theta)$, which involve also the constants ω_i. To determine the x_i and the ω_i we have the conditions (6.4) which lead to the new conditions

(6.7)
$$\begin{aligned} &\text{a)} \quad x_i(\theta + 2\pi) = x_i(\theta) \\ &\text{b)} \quad x_0(0) = A, \qquad x_i(0) = 0 \\ &\text{c)} \quad x_0'(0) = 0, \qquad x_i'(0) = 0. \end{aligned}$$

The condition a) will serve to determine the constants ω_i in (6.6).

It is more convenient, though not strictly necessary, to assume that the amplitude F of the applied force is also small with β; we assume, therefore:

(6.8) $$F = \beta F_0 .$$

This means that our development is one in the neighborhood of the linear *free* vibration, just as it was in the treatment of the same problem by the iteration method.

The result of inserting (6.5), (6.6), and (6.8) in (6.3) is

(6.9)
$$\begin{aligned} (\omega_0^2 + 2\beta\omega_0\omega_1 + \cdots)(x_0'' + \beta x_1'' + \cdots) \\ + \alpha(x_0 + \beta x_1 + \beta^2 x_2 + \cdots) \\ + \beta(x_0^3 + 3x_0^2 x_1 \beta + \cdots) = \beta F_0 \cos\theta. \end{aligned}$$

By x_i'' is meant, of course, $d^2x_i/d\theta^2$.

The term of zero order in β yields

(6.10) $$\omega_0^2 x_0'' + \alpha x_0 = 0,$$

the general solution of which is

(6.11) $$x_0 = A_0 \cos \frac{\sqrt{\alpha}}{\omega_0}\theta + B_0 \sin \frac{\sqrt{\alpha}}{\omega_0}\theta.$$

The conditions (6.7) lead at once to

(6.12)
$$\begin{aligned} &\text{a)} \quad \omega_0 = \sqrt{\alpha} \\ &\text{b)} \quad A_0 = A \\ &\text{c)} \quad B_0 = 0. \end{aligned}$$

Hence we have determined

(6.13) $$x_0 = A \cos\theta$$

and

(6.14) $$\omega_0 = \sqrt{\alpha},$$

i.e., the zero ord terms in the perturbation series (6.5) and (6.6).

We continue the process by taking the first order term in (6.9). This leads to a differential equation for $x_1(\theta)$:

(6.15) $$\omega_0^2 x_1'' + \alpha x_1 = -2\omega_0\omega_1 x_0'' - x_0^3 + F_0 \cos\theta.$$

Insertion of x_0 from (6.13) yields

(6.16) $$\omega_0^2 x_1'' + \alpha x_1 = (2\omega_0\omega_1 A - \tfrac{3}{4}A^3 + F_0)\cos\theta - \tfrac{1}{4}A^3\cos 3\theta.$$

The periodicity condition for x_1 requires that the coefficient of $\cos\theta$ *be zero, since otherwise a term $\theta\sin\theta$ would arise in the general solution of* (6.16), which presents the exceptional resonance case. Hence we set $2\omega_0\omega_1 A - 3A^3/4 + F_0 = 0$, or

(6.17) $$\omega_1 = \frac{1}{2\sqrt{\alpha}}\left(\frac{3}{4}A^2 - \frac{F_0}{A}\right),$$

which fixes the quantity ω_1 in (6.6). We note that the first two terms in (6.6) yield, in view of (6.12a) and (6.17), the same relation (within terms of order less than β^2) between ω and A as was found previously (cf. equation (2.8)). From $\omega^2 = \omega_0^2 + 2\omega_0\omega_1\beta + \cdots$, we obtain in fact

$$\omega^2 = \alpha + \beta\left(\frac{3}{4}A^2 - \frac{F_0}{A}\right) + \cdots$$

$$= \alpha + \tfrac{3}{4}\beta A^2 - \frac{\beta F_0}{A} + \cdots$$

where the dots refer to terms of order β^2 and higher. The general solution of (6.16) may now be written

(6.18) $$x_1 = A_1\cos\frac{\sqrt{\alpha}}{\omega_0}\theta + B_1\sin\frac{\sqrt{\alpha}}{\omega_0}\theta - \frac{A^3}{4(\alpha - 9\omega_0^2)}\cos 3\theta$$

$$= A_1\cos\theta + B_1\sin\theta + \frac{A^3}{32\alpha}\cos 3\theta$$

upon setting $\omega_0 = \sqrt{\alpha}$. The conditions (6.7b, c) require $A_1 = -A^3/32\alpha$ and $B_1 = 0$. Hence we have finally

(6.19) $$x_1 = \frac{A^3}{32\alpha}(-\cos\theta + \cos 3\theta).$$

The approximation $x = x_0 + \beta x_1$ can now be seen to coincide with the second approximation (2.10) furnished by the iteration procedure, again within terms of order β^2. Our solution, up to terms of first order in β, is

(6.20) $$x = A\cos\theta + \beta\frac{A^3}{32\alpha}(-\cos\theta + \cos 3\theta) + \cdots,$$

and

(6.21) $$\omega = \sqrt{\alpha} + \beta\frac{1}{2\sqrt{\alpha}}\left(\frac{3}{4}A^2 - \frac{F_0}{A}\right) + \cdots.$$

The method of procedure from this point on should be clear.

7. *Subharmonic response*

Up to now we have considered only the harmonic solutions of the Duffing equation, that is, solutions for which the frequency is the same as that of the external force $F\cos\omega t$. Permanent oscillations whose frequency is a fraction $\frac{1}{2}, \frac{1}{3}, \cdots, 1/n$, of that of the applied force can, however, occur in nonlinear systems, in particular in our case of the Duffing equation. To this phenomenon the term subharmonic response is usually applied, though the term frequency demultiplication is also used and is perhaps a better one. (For literature on this subject see the papers by Baker [3], Krylov and Bogoliuboff [21], v. Kármán [20], and Friedrichs and Stoker [13].)

The fact that subharmonic oscillations occur in systems with nonlinear restoring forces can hardly be denied since they have been often observed (cf. the paper of Ludeke [27], for example). But it is not an entirely simple matter to give a plausible physical explanation for their occurrence. Let us recall the behavior of linear systems. If the frequency of the free oscillation of a linear system is ω/n (n an integer, say) then a periodic external force of frequency ω can excite the free oscillation in addition to the forced oscillation of frequency ω. But since some damping is always present in a physical system, the free oscillation is damped out so that the eventual "steady state" consists solely of the oscillation of frequency ω. Why should the situation be different in a nonlinear system? The explanation usually offered is as follows: Any free oscillation of a nonlinear system contains the higher harmonics in profusion, and hence it is possible that an external force with a frequency the same as one of these might be able to excite and sustain the harmonic of lowest frequency. Of course that this actually should occur probably requires that the damping be not too great and that proper precautions of various kinds be taken.

We shall not attempt to present a solution of the problem of subharmonic response for the Duffing equation in all generality. Rather,

Florida International University
Department of Mechanical and Materials Engineering

EML 6223 FINAL EXAMINATION Dec. 13, 2004

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DO YOUR OWN WORK -- SHOW ALL WORK

Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of this will result in failure of the examination and the course.

_________________________ PRINT NAME

_________________________ SIGN NAME

This examination consists of 3 problems. Several problems have multiple parts. Do all problems and show all your work and any assumptions you make.

PROBLEM #1	25 POINTS
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	100 POINTS

Problem #1. For the following two degree of freedom system, using Lagrange's equation derive the equations of motion.

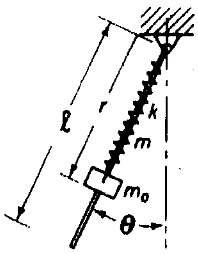


FIGURE P7.15.

Problem #1b. Read the following example, then answer the question below.

Example 7.1.1

Consider the plane mechanism shown in Fig. 7.1.3, where the members are assumed to be rigid. Describe all possible motions in terms of generalized coordinates.

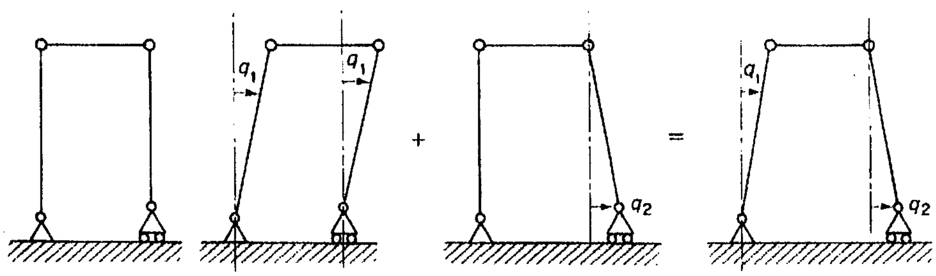


FIGURE 7.1.3.

Solution As shown in Fig. 7.1.3, the displacements can be obtained by the superposition of two displacements q_1 and q_2. Because q_1 and q_2 are independent, they are generalized coordinates, and the system has 2 DOF.

■

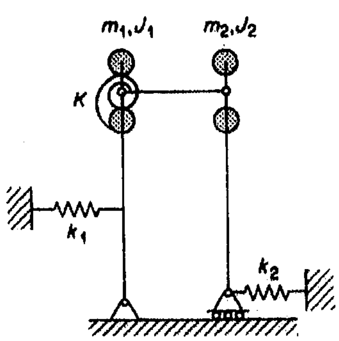


FIGURE P7.18.

The rigid bar linkages of Example 7.1.1 are loaded by spring and masses as show in the figure to the left. Using Lagrange's Equation, derive the equations of motion. Assume that the springs are unstretched when the upright bar linkages are vertical and the longitudinal bar linkage is horizontal. The vertical bar linkages are of length l and the horizontal bar linkage is of length $l/2$.

Problem #2.

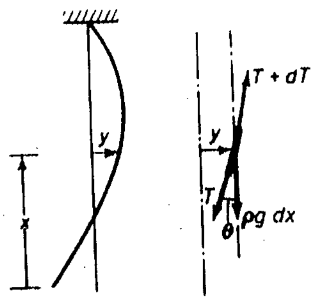


FIGURE P9.6.

1) Shown is a flexible cable supported at the upper end and free to oscillate under the influence of gravity. Show that the equation of lateral motion, i.e., y=y(x,t), is

$$\frac{\partial^2 y}{\partial t^2} = g\left(x\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x}\right)$$

2) If a solution of the form y=Y(x)•cosωt is assumed, show that Y(x) can be reduced to a Bessel's differential equation, viz.,

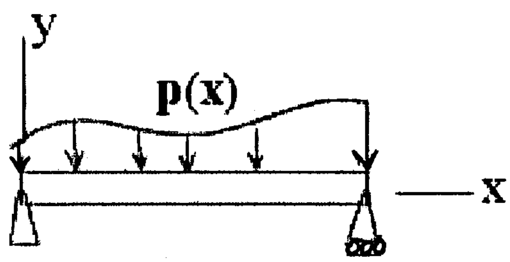
$$\frac{d^2Y(z)}{dz^2} + \frac{1}{z}\frac{dY(z)}{dz} + Y(z) = 0$$

with solution $Y(z)=J_0(z)$ or $Y(x)=J_0(2\omega\sqrt{\frac{x}{g}})$ by a change in variable $z^2=4\omega^2 x/g$.

3) What are the boundary conditions for this problem?

Problem #3

Given the functional $U[w(x)] = \int_0^L \{ \frac{1}{2} EI[w'']^2 - pw \} dx$ for the beam shown below, for



which we then replaced w(x) by w(x)+εf(x) and obtained the governing equation of equilibrium and boundary conditions required for this problem, you are to do the following:

Modify the Lagrangian L to include the kinetic energy of the beam, namely, $\frac{1}{2}\rho A \dot{w}^2\, dx$ and repeat the process we followed to derive the governing equation of motion and the required boundary conditions

Problem #4.

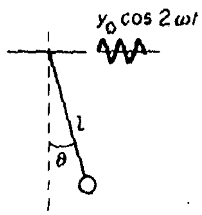


FIGURE P14.38.

1) Derive the governing equation of motion for this oscillating support problem.

2) IF y_o were zero, then this would become the simple pendulum problem. Apply the *perturbation method* to the simple pendulum problem with

$$\sin\theta \text{ being replaced by } \theta - (1/6)\theta^3$$

Use only the first three terms of the series for x and ω.

3) Show that if y_0 is not zero, then the θ_1 or θ_2 differential equation must satisfy the *Mathieu Equation*, namely, an equation of the form

$$\frac{d^2y}{dz^2} + (a - 2b\cos 2z)y = 0 \quad \text{where } z=\omega t$$

FIU
FLORIDA INTERNATIONAL UNIVERSITY
Miami's public research university

December 3, 2004

TO ALL TENURED FACULTY

Dear Faculty Colleague:

Academic Affairs is pleased to announce the sabbatical application and review procedure for the **2005-06** academic year. In order to save on duplicating costs, we are making the FIU Sabbatical Policy and Procedures statement, and sabbatical application available online at **http://academic.fiu.edu/docs/budget_personnel_sabbatical.htm**. Please take particular note that the Collective Bargaining Agreement(CBA) has expired, nevertheless, FIU intends to maintain the existing policies and procedures for sabbaticals.

Full-time tenured employees at the time the proposed sabbatical is to be taken, with at least six years of full-time service within the State University System, are eligible for one semester, full pay sabbaticals or two semester, half pay sabbaticals. Faculty are not normally eligible for a second sabbatical until six years of continuous service are completed following a previous sabbatical. The six years are measured from the end of the first sabbatical to the beginning of the second sabbatical.

Sabbaticals for two semesters at half pay will be made available to all eligible faculty whose applications are deemed to be complete according to the FIU Sabbatical Policy and Procedures. Sabbaticals for one semester at full pay are fewer in number and highly competitive. We are authorized to award 11 one-semester, full pay sabbaticals for the **2005-06** academic year.

Sabbaticals for two semesters at two-thirds pay are available on a competitive basis to tenured faculty who have at least nine years of continuous service at FIU without having had a sabbatical. These sabbaticals are usually equal in number to the required number of one semester, full pay sabbaticals awarded each year. As with the two semester, half pay sabbaticals, deans have the prerogative to postpone a two-thirds pay sabbatical for one year if a faculty member's absence would create a hardship for a department with respect to staffing academic programs.

Insofar as departments and colleges are able to do so, faculty performance while on sabbatical leave will be included in the annual review process. A report on activities pursued while on sabbatical is required upon completion of the leave.

The deadline for receipt of completed sabbatical applications by Geri Plummer in the Office of the Provost (PC 526) is **5:00 p.m. Monday, January 17, 2005.** The Sabbatical Committee will be unable to consider applications submitted after the deadline.

Sincerely,

Arthur Herriott
Executive Vice Provost
Academic Budget & Personnel

c: Provost Staff
Academic Affairs Deans

Office of the Provost
University Park · Miami, FL 33199 · Tel 305-348-2151 · Fax 305-348-2994 · www.fiu.edu

OUR POST-PRODUCTION SYSTEM UPDATE

PantherSoft News
Volume II, Issue 3
December 2004

PantherSoft News

INSIDE THIS ISSUE:

University Community Transitions to New PantherSoft System

Four months ago, on July 1, the PantherSoft Financials System went live with the General Ledger, Purchasing, Accounts Payable and Asset Management modules. A few days later, on July 6, the Student Administration System became available to the University Community as well. Since then, many students, faculty and staff have been taking advantage of the new features PantherSoft offers its users. This includes access to real-time information, faster processing, direct deposit of financial aid funds, online routing and approval, and access from anywhere through an internet connection.

However, as with every system implementation, the transition period has brought many new benefits, as well as several challenges. As the University Community transitions to the new functionality PantherSoft provides, the Project Team has been monitoring and tweaking the System based on valuable feedback received from users; such as the class rosters, and the class search page. As we continue to move forward with improving the System, the PantherSoft Project Team appreciates your cooperation and patience. Below are just a few facts based on transactions in PantherSoft since its full implementation in July.

FINANCIALS SYSTEM	
Online requisitions processed	9,240
Purchased orders processed	8,751
Total online vouchers (invoices) processed	17,123
Checks / payments processed	10,277
TARs processed	2,427
Cash advances processed	132
Expense / Pro-card reports processed	12,782
Journals processed	19,353
Phone calls handled by FAST (Financials Application Support Team)	3,535
Faculty / staff that have attended training for Financials System	1,500

STUDENT ADMINISTRATION SYSTEM	
Total students enrolled for Fall 2004	35,046
Highest amount of students using the System at the same time	2,000
Total financial aid disbursed through PantherSoft	$42,000,000
Number of times students added a class	287,752
Number of times students dropped a class	48,661
Number of times students swapped a class	68,086
Total number of times a permission number was used to add a class	13,853
Total number of transactions in PantherSoft... and counting	404,499
Phone calls handled by the UTS Support Center pertaining to PantherSoft	13,487
Number of seats filled in training	1,588

CHECK THE FACTS:

8,751 PURCHASE ORDERS PROCESSED...

68,086 TIMES A STUDENT 'SWAPPED' A CLASS ON PANTHERSOFT

IN OUR NEXT ISSUE:

Additional information on **Phase II**

And much more!

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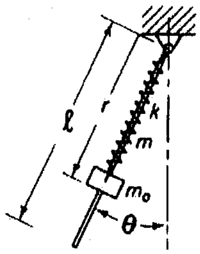

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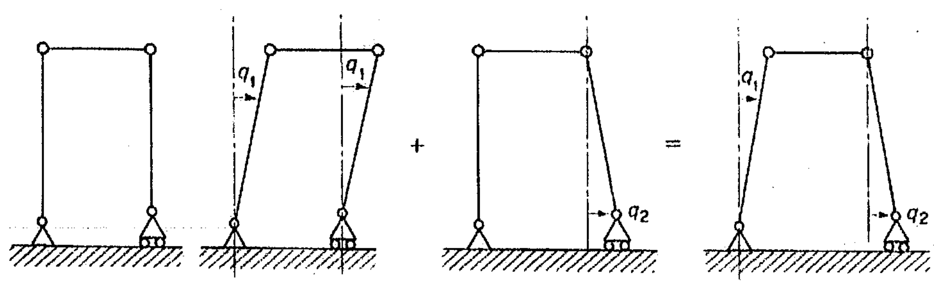

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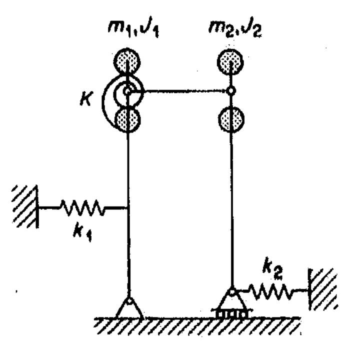

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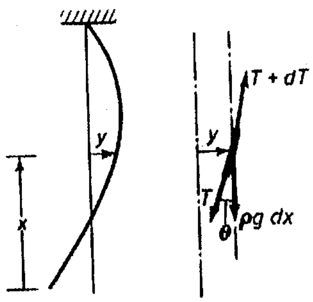

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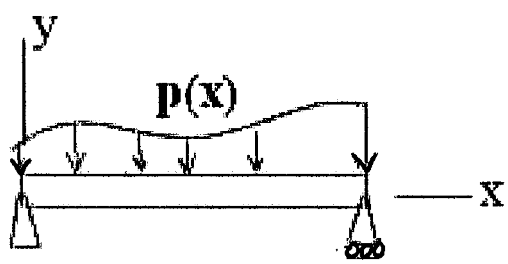
Problem #3

Given the functional $U[w(x)] = \int_0^L \{ \frac{1}{2} EI[w'']^2 - pw \} dx$ for the beam shown below, for which we then replaced w(x) by w(x)+εf(x) and obtained the governing equation of equilibrium and boundary conditions required for this problem, you are to do the following:



Modify the Lagrangian L to include the kinetic energy of the beam, namely, $\frac{1}{2}\rho A \dot{w}^2 dx$ and repeat the process we followed to derive the governing equation of motion and the required boundary conditions

Problem #4.

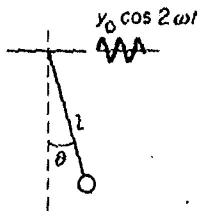


FIGURE P14.38.

1) Derive the governing equation of motion for this oscillating support problem.

2) IF y_o were zero, then this would become the simple pendulum problem. Apply the *perturbation method* to the simple pendulum problem with

$$\sin\theta \text{ being replaced by } \theta - (1/6)\theta^3$$

Use only the first three terms of the series for x and ω.

3) Show that if y_0 is not zero, then the θ_1 or θ_2 differential equation must satisfy the *Mathieu Equation*, namely, an equation of the form

$$\frac{d^2y}{dz^2} + (a - 2b\cos 2z)y = 0 \quad \text{where } z = \omega t$$

Using the relation $\int \sin \alpha t \, \sin \beta t \, dt = \frac{1}{2}\left\{ \frac{\sin(\alpha-\beta)t}{\alpha-\beta} - \frac{\sin(\alpha+\beta)t}{\alpha+\beta} \right\}; \alpha \neq \beta$

Eq. (E_5) can be simplified as

$$A_{mn} = \frac{4 w_0}{ab} \left\{ \frac{1}{2} \left[\frac{\sin \frac{(m-1)\pi x}{a}}{\frac{(m-1)\pi}{a}} - \frac{\sin \frac{(m+1)\pi x}{a}}{\frac{(m+1)\pi}{a}} \right]_0^a \right\} \left\{ \frac{1}{2} \left[\frac{\sin \frac{(n-1)\pi y}{b}}{\frac{(n-1)\pi}{b}} - \frac{\sin \frac{(n+1)\pi y}{b}}{\frac{(n+1)\pi}{b}} \right]_0^b \right\}$$

$$= 0 \quad \text{for } m > 1 \text{ and/or } n > 1 \qquad (E_7)$$

For $m = 1$ and $n = 1$, Eq. (E_5) can be simplified as

$$A_{11} = \frac{4 w_0}{ab} \int_0^a \sin^2 \frac{\pi x}{a} \, dx \int_0^b \sin^2 \frac{\pi y}{b} \, dy = w_0 \qquad (E_8)$$

$$\therefore \quad w(x,y,t) = w_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \cos \omega_{11} t$$

with $\omega_{11} = \left\{ c^2 \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \right\}^{1/2}$

(from the solution of problem 8.52)

8.54 General solution:

$$w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left(A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t \right)$$

If $w(x,y,0) = w_0(x,y)$ and $\frac{\partial w}{\partial t}(x,y,0) = \dot{w}_0(x,y)$; $\quad 0 \le x \le a, \quad 0 \le y \le b$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = w_0(x,y)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \dot{w}_0(x,y)$$

These are double Fourier sine series expansions of $w_0(x,y)$ and $\dot{w}_0(x,y)$.

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b w_0(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$

$$= 0 \text{ for given data}$$

$$B_{mn} = \frac{4}{ab\,\omega_{mn}} \int_0^a \int_0^b \dot{w}_0(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$

$$= \frac{4 \dot{w}_0}{ab\,\omega_{mn}} \int_0^a \sin \frac{m\pi x}{a} \sin \frac{\pi x}{a} \, dx \int_0^b \sin \frac{n\pi y}{b} \sin \frac{2\pi y}{b} \, dy \quad ---(E_1)$$

using the relation

$$\int \sin \alpha t \cdot \sin \beta t \cdot dt = \frac{1}{2} \left\{ \frac{\sin(\alpha-\beta)t}{\alpha-\beta} - \frac{\sin(\alpha+\beta)t}{\alpha+\beta} \right\} \quad \text{for } \alpha \neq \beta,$$

$$w(y) = e^{\alpha y} \tag{E_{11}}$$

Substitution of (E_{11}) into (E_{10}) gives

$$EI\,\alpha^4 + \rho A v_0^2 + k = 0 \tag{E_{12}}$$

whose roots can be expressed as

$$\alpha_{1,2} = \pm\,(a + i\,b)\,, \qquad \alpha_{3,4} = \pm\,(a - i\,b) \tag{E_{13}}$$

with

$$\left.\begin{aligned} & a = \sqrt{1-c}\;d\,, \quad b = \sqrt{1+c}\;d\,, \quad c = \frac{v_0^2}{\sqrt{\dfrac{4EIk}{\rho^2 A^2}}} \\ & \text{and} \quad d = \left(\frac{k}{4EI}\right)^{1/4} \end{aligned}\right\} \tag{E_{14}}$$

Thus the solution of (E_{10}) becomes

$$w(y) = A_1 e^{\alpha_1 y} + A_2 e^{\alpha_2 y} + A_3 e^{\alpha_3 y} + A_4 e^{\alpha_4 y} \tag{E_{15}}$$

where A_i ($i = 1, 2, 3, 4$) are constants which can be determined from the following conditions:

$$\left.\begin{aligned} & w = 0 \quad \text{at} \quad y = \infty \\ & \frac{d^2 w}{dy^2} = 0 \quad \text{at} \quad y = \infty \end{aligned}\right\} \text{deflection \& bending moment are zero at } y = \infty$$

$$\left.\frac{dw}{dy} = 0 \quad \text{at} \quad y = 0\right\} \text{slope is zero under the load}$$

$$\left.\begin{aligned} & EI \left.\frac{d^3 w}{dy^3}\right|_{y=0^+} - EI \left.\frac{d^3 w}{dy^3}\right|_{y=0^-} = P \\ & \text{i.e.,} \quad EI\,\frac{d^2 w}{dy^2} = \frac{P}{2} \end{aligned}\right\} \text{shear force has discontinuity under the load}$$

8.57 $W(x) = \dfrac{c_0\, x^2}{24\,EI}\,(l - x)^2 = \dfrac{c_0}{24\,EI}\,(x^2 l^2 + x^4 - 2\,l\,x^3)$

$$\frac{dW}{dx} = \frac{c_0}{24\,EI}\,(2x\,l^2 + 4x^3 - 6\,l\,x^2)$$

$$\frac{d^2 W}{dx^2} = \frac{c_0}{24\,EI}\,(2\,l^2 + 12\,x^2 - 12\,l\,x)$$

$$N = EI \int_0^l \left(\frac{d^2 W}{dx^2}\right)^2 dx = 4\,EI \left(\frac{c_0}{24\,EI}\right)^2 \int_0^l (l^2 + 6x^2 - 6\,l\,x)^2\,dx$$

$$= \frac{c_0^2\, l^5}{720\,EI} \tag{E_1}$$

$$D = \rho A \int_0^l \big(W(x)\big)^2 dx = \rho A \left(\frac{c_0}{24\,EI}\right)^2 \int_0^l (x^2 l^2 + x^4 - 2\,l\,x^3)^2\,dx$$

$$= \frac{\rho A\, l^9\, c_0^2}{362\,880\; E^2 I^2} \tag{E_2}$$

$EIw^{IV} + \rho A\ddot{w} = 0 \qquad w = XT$

$EI\dfrac{X^{IV}}{X} + \rho A\dfrac{\ddot{T}}{T} = 0 \qquad \dfrac{EI}{\rho A}\dfrac{X^{IV}}{X} = -\dfrac{\ddot{T}}{T} = \omega^2$

$T = \tilde{A}\sin\omega t + \hat{B}\cos\omega t$

let $\beta^4 = \dfrac{EI}{\rho A} \qquad \beta^4 \dfrac{X^{IV}}{X} = \omega^2$

$$X^{IV} - \frac{\omega^2}{\beta^4} X = 0$$

let $X = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + C_3 e^{\lambda_3 x} + C_4 e^{\lambda_4 x}$

$$\lambda^4 - \frac{\omega^2}{\beta^4} = 0$$

$\lambda^2 = \dfrac{\omega}{\beta^2} \qquad \lambda^2 = \dfrac{-\omega}{\beta^2}$

$\lambda = \pm\dfrac{\sqrt{\omega}}{\beta} \qquad \lambda = \pm i\dfrac{\sqrt{\omega}}{\beta}$

$X = C_1 e^{\frac{\sqrt{\omega}}{\beta}x} + C_2 e^{-\frac{\sqrt{\omega}}{\beta}x} + C_3 \sin\dfrac{\sqrt{\omega}}{\beta}x + C_4 \cos\dfrac{\sqrt{\omega}}{\beta}x$

$= C_1 \sinh\dfrac{\sqrt{\omega}}{\beta}x + C_2 \cosh\dfrac{\sqrt{\omega}}{\beta}x$

free-free $\quad M = 0 = EIw''(0) \neq EIw''(l)$

$V = 0 \quad EIw'''(0) \neq EIw'''(l)$

$X'' = C_1\dfrac{\omega}{\beta^2}\sinh(\,)x + C_2\dfrac{\omega}{\beta^2}\cosh(\,) - C_3\dfrac{\omega}{\beta^2}\sin(\,)x - C_4\dfrac{\omega}{\beta^2}\cos(\,)x$

$X''(0) = 0 = C_2\dfrac{\omega}{\beta^2} - C_4\dfrac{\omega}{\beta^2}$

$X''(l) = C_1\dfrac{\omega}{\beta^2}\sinh\left(\dfrac{\sqrt{\omega}}{\beta}l\right) + C_2\dfrac{\omega}{\beta^2}\cosh\left(\dfrac{\sqrt{\omega}l}{\beta}\right) - C_3\dfrac{\omega}{\beta^2}\sin\left(\dfrac{\sqrt{\omega}l}{\beta}\right) - C_4\dfrac{\omega}{\beta^2}\cos\left(\dfrac{\sqrt{\omega}l}{\beta}\right)$

$X'''(0) = C_1\dfrac{\omega\sqrt{\omega}}{\beta^3} - C_3\dfrac{\omega\sqrt{\omega}}{\beta^2} = 0$

$X'''(l) = C_1\dfrac{\omega\sqrt{\omega}}{\beta^3}\cosh\left(\dfrac{\sqrt{\omega}}{\beta}l\right) + C_2\dfrac{\omega\sqrt{\omega}}{\beta^3}\sinh\left(\dfrac{\sqrt{\omega}}{\beta}\right) - C_3\dfrac{\omega\sqrt{\omega}}{\beta^3}\cos\left(\dfrac{\sqrt{\omega}}{\beta}l\right) + C_4\dfrac{\omega\sqrt{\omega}}{\beta^3}\sin\left(\dfrac{\sqrt{\omega}}{\beta}\right.$

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$0 \qquad \frac{\omega}{\beta^2} \qquad 0 \qquad -\frac{\omega}{\beta^2}$

$\frac{\omega}{\beta^2}\sinh\frac{\sqrt{\omega}}{\beta}l \qquad \frac{\omega}{\beta^2}\cosh\frac{\sqrt{\omega}}{\beta}l \qquad -\frac{\omega}{\beta^2}\sin\frac{\sqrt{\omega}l}{\beta} \qquad -\frac{\omega}{\beta^2}\cos\frac{\sqrt{\omega}}{\beta}l$

$\frac{\omega\sqrt{\omega}}{\beta^2} \qquad 0 \qquad -\frac{\omega\sqrt{\omega}}{\beta^2} \qquad 0$

$\frac{\omega\sqrt{\omega}}{\beta^3}\cosh\frac{\sqrt{\omega}}{\beta}l \qquad \frac{\omega\sqrt{\omega}}{\beta^3}\sinh\frac{\sqrt{\omega}}{\beta}l \qquad -\frac{\omega\sqrt{\omega}}{\beta^3}\cos\frac{\sqrt{\omega}}{\beta}l \qquad \frac{\omega\sqrt{\omega}}{\beta^3}\sin\frac{\sqrt{\omega}}{\beta}l$

$\frac{\omega}{\beta^2}\cdot\frac{\omega}{\beta^2}\cdot\frac{\omega\sqrt{\omega}}{\beta^3}\cdot\frac{\omega\sqrt{\omega}}{\beta^3}\Big[$

0	1	0	1	
sh	ch	-s	-c	sh
1	0	-1	0	1
ch	sh	-c	s	ch

sh	-s	-c
1	-1	0
ch	-c	s

$-1\ (-sh\cdot s + \cancel{c^2} + \cancel{0\cdot ch\cdot s} + ch\cdot c + \cancel{s^2} + sh\cdot c)\ -1\left(\cancel{0\cdot sh\cdot -c} - ch^2 - s\cdot sh + c\cdot ch + sh^2\right)$

$\frac{\omega^5}{\beta^{10}}\left(2s\cdot sh + 1 + 1 - 2c\cdot ch + c\cdot sh\right) = 0$ $\quad$ <u>$\omega = 0$ here</u> or ( )

$\beta^4 X^{IV} = 0$

$a + bx + cx^2 + dx^3 \ ; \quad b + 2cx + 3dx^2 ; \quad 2c + 6dx \ ; \quad 6d$

$2c + 6dx$ $\qquad$ <u>$c = 0 \quad d = 0$</u>

Now $X''' = 0$ @ $x = 0, l \Rightarrow d = 0$

$X'' = 0$ @ $x = 0, l$

$\Rightarrow c = 0$ the other automatically satisfied $\therefore\ a + bx = X$ when <u>$\omega = 0$</u>

$C_1 + D_1 t = T$

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$\mathfrak{X} = 0 + \mathscr{C}_2 + 0 + \mathscr{C}_4$

$' = \mathscr{C}_1 sh + \mathscr{C}_2 ch + \mathscr{C}_3 s + \mathscr{C}_4 c$

$\mathfrak{X}'' = \frac{\omega}{\beta^2}\left[\cancel{sh} + \mathscr{C}_2 \quad - \mathscr{C}_4 \right]$

$= \frac{\omega}{\beta^2}\left[\mathscr{C}_1 sh + \mathscr{C}_2 ch - \mathscr{C}_3 s - \mathscr{C}_4 c \right]$

$$\frac{\omega^2}{\beta^4} \begin{pmatrix} 0 & 1 & 0 & 1 \\ sh & ch & s & c \\ 0 & 1 & 0 & -1 \\ sh & ch & -s & -c \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ sh & s & c \\ 0 & 0 & -1 \\ sh & -s & -c \end{matrix}$$

$+1(-s \cdot sh - sh \cdot s$

$$\begin{matrix} sh & ch & s \\ 0 & 1 & 0 \\ sh & ch & -s \end{matrix}$$

$-s \cdot sh - s \cdot sh$

$\frac{\omega^2}{\beta^4}\left(-4 \sin \frac{\sqrt{\omega}}{\beta} l \sinh \frac{\sqrt{\omega}}{\beta} l \right) \qquad \omega = 0$ or $(\quad) = 0$

$\mathfrak{X} = a + bx + cx^2 + dx^3$; $b + 2cx + 3dx^2$; $2c + 6dx$

$\mathfrak{X}(0) = a = 0 \qquad \mathfrak{X}(l) = bl = 0 \Rightarrow b = 0 \quad \Rightarrow \mathfrak{X} = 0 \ \forall x$ trivial

$\mathfrak{X}''(0) = c = 0 \qquad \mathfrak{X}''(l) = 0 \Rightarrow d = 0$

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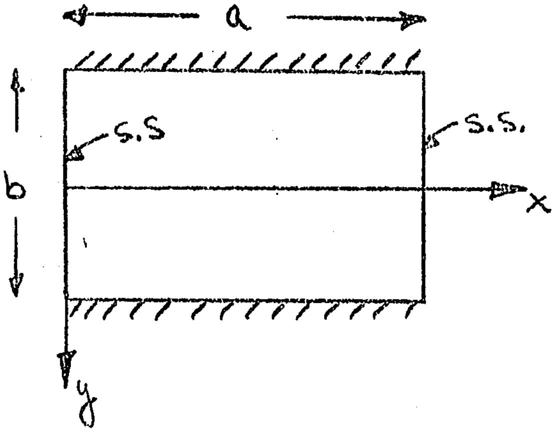
SOLUTION : PROB. [redacted]



Determine natural vibration frequencies

$$w(x,y,t) = \varphi(x,y)e^{i\omega t}$$

PDE $\rightarrow$ $\nabla^4 \varphi = \frac{\rho h \omega^2}{D}\varphi$

BC $\rightarrow$ $\begin{cases} \varphi = \varphi_{,xx} = 0 & \text{at } x = 0, a \\ \varphi = \varphi_{,y} = 0 & \text{at } y = \pm b/2 \end{cases}$

Use Lévy idea to reduce PDE to ODE : let n = integer

$$\varphi = F(y)\sin\frac{n\pi x}{a} \rightarrow F'''' - 2\left(\frac{n\pi}{a}\right)^2 F'' + \left[\left(\frac{n\pi}{a}\right)^4 - \frac{\rho h\omega^2}{D}\right]F = 0$$

$$F = e^{\beta y} \rightarrow \beta^4 - 2\left(\frac{n\pi}{a}\right)^2\beta^2 + \left[\left(\frac{n\pi}{a}\right)^4 - \frac{\rho h \omega^2}{D}\right] = 0$$

$$\rightarrow \beta = \pm\sqrt{\sqrt{\frac{\rho h\omega^2}{D}} + \left(\frac{n\pi}{a}\right)^2}, \quad \pm i\sqrt{\sqrt{\frac{\rho h\omega^2}{D}} - \left(\frac{n\pi}{a}\right)^2}$$

Note it may be assumed that $\frac{\rho h\omega^2}{D} > \left(\frac{n\pi}{a}\right)^4$ because we know that the mode shapes must be oscillatory in y.

let $\lambda^2 = \sqrt{\frac{\rho h\omega^2}{D}} + \left(\frac{n\pi}{a}\right)^2$ $\quad \mu^2 = \sqrt{\frac{\rho h\omega^2}{D}} - \left(\frac{n\pi}{a}\right)^2$

Modes symmetric about the x-axis

$$F = A\cosh\lambda y + B\cos\mu y$$

BC $\rightarrow$ $F(\tfrac{b}{2}) = 0 \rightarrow A\cosh\lambda b/2 + B\cos\mu b/2 = 0$

$F'(\tfrac{b}{2}) = 0 \rightarrow A\lambda\sinh\lambda b/2 - B\mu\sin\mu b/2 = 0$

nontrivial solution $\rightarrow$ $\mu\cosh\lambda b/2\,\sin\mu b/2 + \lambda\sinh\lambda b/2\,\cos\mu b/2 =$

let $\alpha_n = \frac{n\pi b}{2a}$, $\frac{\mu b}{2} = \theta \rightarrow \frac{\lambda b}{2} = \sqrt{\theta^2 + 2\alpha_n^2}$

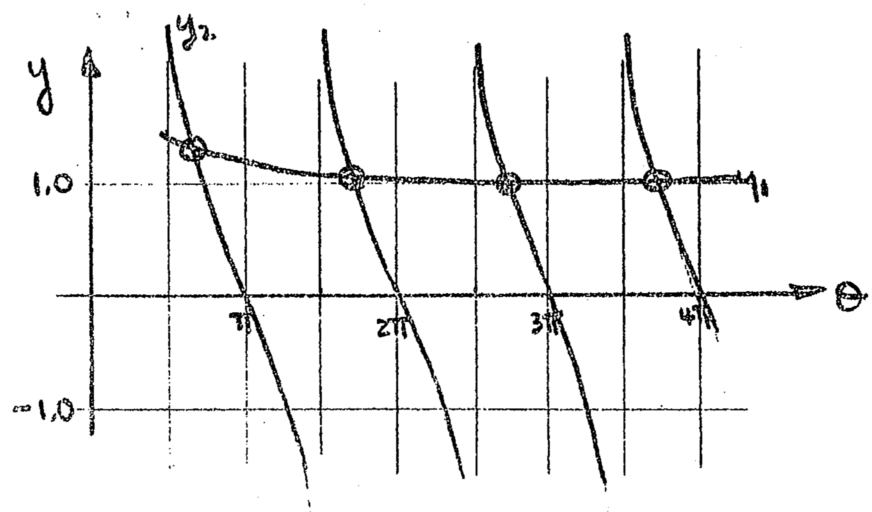
Frequency eq. $\rightarrow$ $\frac{\sqrt{\theta^2+\alpha_n^2}}{\theta}\tanh\sqrt{\theta^2+\alpha_n^2} = -\tan\theta$

For a given value of b/a, choose n (and thus $\alpha_n = \frac{n\pi b}{2a}$)

Plot $y_1 = \frac{\sqrt{\theta^2+\alpha_n^2}}{\theta} \tanh\sqrt{\theta^2+\alpha_n^2}$ and $y_2 = -\tan\theta$

The intersections of these two curves give the natural frequencies associated with the chosen value of n : $\theta_{n1}, \theta_{n2}, \theta_{n3}, \ldots$

$$\omega_{nm} = \sqrt{\frac{D}{\rho h (b/2)^4}\left(\theta_{nm} + \alpha_n^2\right)} \quad (m=1,2,3,\ldots)$$



This is the plot for $n=1$ and $b/a=1$ (a square plate)

For larger values of n the y_1 curve simply becomes flatter

Note that $\theta_{nm} = (m-\frac{1}{4})\pi$ is very accurate for $m \geq 3$

A rapidly converging iteration formula is easily developed :

$$\theta_{nm}^{(k+1)} = m\pi - \tan^{-1}\left[\frac{\sqrt{\theta_{nm}^{(k)2}+\alpha_n^2}}{\theta_{nm}^{(k)}} \tanh\sqrt{\theta_{nm}^{(k)2}+\alpha_n^2}\right]$$

which should be used with the initial guess: $\theta_{nm}^{(0)} = (m-\frac{1}{4})\pi$

<u>Modes antisymmetric about the x axis</u>

$$F = A\sinh\lambda y + B\sin\mu y \rightarrow A\sinh\frac{\lambda b}{2} + B\sin\frac{\mu b}{2} = 0$$

$$A\lambda\cosh\frac{\lambda b}{2} + B\mu\cos\frac{\mu b}{2} = 0$$

freq eq : $\mu\sinh\frac{\lambda b}{2}\cos\frac{\mu b}{2} - \lambda\cosh\frac{\lambda b}{2}\sin\frac{\mu b}{2} = 0$

$$\rightarrow \frac{\theta}{\sqrt{\theta^2+\alpha_n^2}} \tanh\sqrt{\theta^2+\alpha_n^2} = \tan\theta$$

8.5 At $x = 0$: $\quad P \dfrac{\partial w}{\partial x} = m \dfrac{\partial^2 w}{\partial t^2}$ (E

At $x = \ell$: $\quad P \dfrac{\partial w}{\partial x} = -k\, w$ (E

General solution is

$$w(x,t) = W(x) \cdot T(t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$$ (

Equations (E_1) and (E_3) give:

$$\Big\{P\left(-A \frac{\omega}{c} \sin \frac{\omega x}{c} + B \frac{\omega}{c} \cos \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$$
$$= -m\,\omega^2 \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)\Big\}_{x=0}$$

i.e., $\quad A\,(m\,\omega^2) + B\left(\dfrac{P\,\omega}{c}\right) = 0$ (E

Eqs. (E_2) and (E_3) yield

$$P\left(-\frac{\omega}{c} A \sin \frac{\omega \ell}{c} + B \frac{\omega}{c} \cos \frac{\omega \ell}{c}\right) = -k\left(A \cos \frac{\omega \ell}{c} + B \sin \frac{\omega \ell}{c}\right)$$

i.e.,

$$A\left(-\frac{P\omega}{c} \sin \frac{\omega \ell}{c} + k \cos \frac{\omega \ell}{c}\right) + B\left(\frac{P\omega}{c} \cos \frac{\omega \ell}{c} + k \sin \frac{\omega \ell}{c}\right) = 0 \qquad (E_5)$$

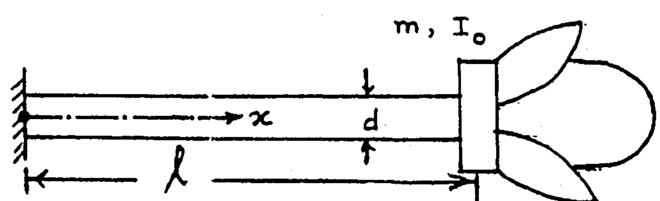
Eqs. (E_4) and (E_5) give the frequency equation:

$$\begin{vmatrix} (m\,\omega^2) & (P\omega/c) \\ \left(-\frac{P\omega}{c} \sin \frac{\omega \ell}{c} + k \cos \frac{\omega \ell}{c}\right) & \left(\frac{P\omega}{c} \cos \frac{\omega \ell}{c} + k \sin \frac{\omega \ell}{c}\right) \end{vmatrix} = 0$$

which, upon simplification, becomes

$$\tan \alpha = \left\{ \frac{P k - \left(\frac{P m c^2}{\ell^2}\right) \alpha^2}{\left(\frac{c^2 m k}{\ell}\right) \alpha + \left(\frac{P^2 c}{\ell}\right) \alpha} \right\} \qquad (E_6)$$

8.19 (a) Axial vibration:



$$\beta = \frac{m_0}{m} = \frac{\text{mass of rod}}{\text{end mass}}$$

Using $\rho = 76.5\ \text{kN/m}^3$ for steel, we find

$$m_0 = \frac{\pi\, d^2\, \ell\, \rho}{4} = \frac{\pi}{4}\left(\frac{5}{100}\right)^2 (1) \left(\frac{76.5\,(10^3)}{9.81}\right) = 15.3117 \text{ kg}$$

$$\frac{m_0}{m} = \frac{15.3117}{100} = 0.1531$$

From Table 8.1, the value of α_1 for $\beta = 0.1531$ (using linear interpolation between values of $\beta = 0.1$ and $\beta = 1.0$) is:

$$\alpha_1 = 0.6099\,(0.1531) + 0.2503 = 0.3437$$

$$\omega_1 = \frac{\alpha_1\, c}{\ell} = \frac{\alpha_1}{\ell}\sqrt{\frac{E}{\rho}} = \frac{0.3437}{1}\sqrt{\frac{207\,(10^9)\,(9.81)}{76500}} = 1770.7958 \text{ rad/sec}$$

(b) Torsional vibration:

In this case, we use the result of Example 8.6.

$$\beta = \frac{\tilde{J}_{\text{rod}}}{I_0} = \frac{J\,\rho\,\ell}{I_0}$$

where J = polar moment of inertia of the shaft:

.10^{-3} kg/m

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$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32} (0.05)^4 = 2.4544 (10^{-8})\ m^4 \quad [\text{handwritten correction: } 10^{-6}]$$

$$\rho = 76500/9.81\ kg/m^3 \ ; \ \ell = 1\ m$$

$$\tilde{J}_{rod} = (2.4544 (10^{-8})) \frac{76500}{9.81} (1) = 1.9140 (10^{-4})\ kg{-}m^2 \quad [\text{handwritten: } 6.14 \times 10^{-7};\ 4.785 (10^{-3})]$$

$$\beta = \frac{1.9140 (10^{-4})}{10} = 1.9140 (10^{-5}) \quad [\text{handwritten: } 4.785 (10^{-4})]$$

Since β is close to zero, we have from Example 8.6,

$$\omega_1 \approx \frac{c}{\ell} \sqrt{\beta} = \sqrt{\frac{G\,\beta}{\rho}} = \sqrt{\frac{(80\,(10^9))\,(9.81)}{76500}} = 14.0126\ \text{rad/sec} \quad [\text{handwritten: } 4.785 \times 10^{-4};\ 70.063]$$

8.28 $I = \frac{1}{12}(0.1)(0.3)^3 = 2.25 \times 10^{-4}\ m^4$

$A = 0.03\ m^2$, $\ell = 2\ m$, $E = 20.5 \times 10^{10}\ N/m^2$, $\rho = 7.83 \times 10^3\ kg/m^3$

Fig. 8.15 gives the values of $\beta_n \ell$:

$$\omega_n = (\beta_n \ell)^2 \sqrt{\frac{EI}{\rho A \ell^4}}$$

Here $\sqrt{\frac{EI}{\rho A \ell^4}} = \left\{ \frac{(20.5 \times 10^{10})(2.25 \times 10^{-4})}{7830 \times 0.03 \times 16} \right\}^{\frac{1}{2}} = 110.7814$

(a) For pinned-pinned beam :

$\beta_1 \ell = \pi$, $\omega_1 = \pi^2 (110.7814) = 1093.3737$ rad/sec

$\beta_2 \ell = 2\pi$, $\omega_2 = 4\,\omega_1 = 4373.4948$ rad/sec

$\beta_3 \ell = 3\pi$, $\omega_3 = 9\,\omega_1 = 9840.3634$ rad/sec

(b) For fixed-fixed beam :

$\beta_1 \ell = 4.730841$, $\omega_1 = 2479.3826$ rad/sec

$\beta_2 \ell = 7.853205$, $\omega_2 = 6832.2023$ rad/sec

$\beta_3 \ell = 10.995608$, $\omega_3 = 13393.8474$ rad/sec

(c) For fixed-free beam :

$\beta_1 \ell = 1.875104$, $\omega_1 = 389.5091$ rad/sec

$\beta_2 \ell = 4.694091$, $\omega_2 = 2441.0117$ rad/sec

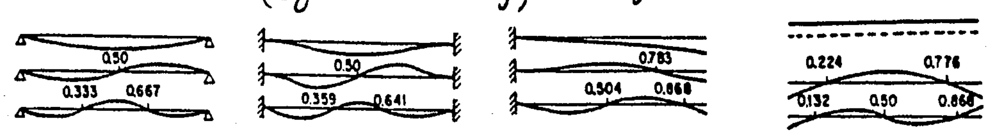
$\beta_3 \ell = 7.854757$, $\omega_3 = 6834.9030$ rad/sec

(d) For free-free beam :

$\beta_1 \ell = 0$, $\omega_1 = 0$; $\beta_2 \ell = 4.730841$, $\omega_2 = 2479.3826$ rad/sec

$\beta_3 \ell = 7.853205$, $\omega_3 = 6832.2023$ rad/sec.

Mode shapes: The mode shapes are given in Fig. 8.15 (equations only). They result in the following.



Pinned-pinned   Fixed-fixed   Fixed-free   Free-free

8.33 For a simply supported beam, the first three natural frequencies are given by

$$\omega_1 = (\beta_1 l)^2 \left(EI/\rho A l^4\right)^{\frac{1}{2}} = \pi^2 \left(EI/\rho A l^4\right)^{1/2}$$

$$\omega_2 = (\beta_2 l)^2 \left(EI/\rho A l^4\right)^{1/2} = 4\pi^2 \left(EI/\rho A l^4\right)^{\frac{1}{2}}$$

$$\omega_3 = (\beta_3 l)^2 \left(EI/\rho A l^4\right)^{\frac{1}{2}} = 9\pi^2 \left(EI/\rho A l^4\right)^{1/2}$$

For steel, $E = 2.07 \times 10^{11}$ N/m^2, $\rho = 7880$ kg/m^3, $l = 1.0$ m.

Setting $\omega_1 \geq 1500$ Hz $= 9424.8$ rad/sec $= \pi^2 \left\{ \frac{(2.07 \times 10^{11}) I}{(7880) A (1)} \right\}^{1/2}$

we get

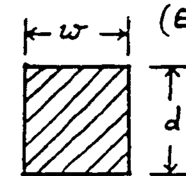
$$(9424.8)^2 \geq \pi^4 \left(\frac{2.07 \times 10^{11}}{7880} \right) \left(\frac{I}{A} \right)$$

or $\quad \frac{I}{A} \leq 0.03471 \qquad (E_1)$

Setting $\omega_3 \leq 5000$ Hz $= 31416.0$ rad/sec $= 9\pi^2 \left\{ \frac{(2.07 \times 10^{11}) I}{7880 \, A \, (1)} \right\}^{\frac{1}{2}}$

we get

$$\frac{I}{A} \geq 0.0047617 \qquad (E_2)$$



Let the beam have a rectangular section

$A = wd$, $\quad I = \frac{1}{12} w d^3$, $\quad \frac{I}{A} = \frac{d^2}{12}$

Let $\frac{I}{A} = 0.005$ to satisfy inequalities (E_1) and (E_2).

Then $d^2 = 0.06$ or $d = 0.2449$ m

Taking $w = 0.1$ m (w can have any value), we get

$A = wd = 0.02449$ m^2 and $I = 1.2240 \times 10^{-4}$ m^4.

Department of Mechanical and Materials Engineering
Florida International University

EML 6223 | **Fall 2004** | **DR. C. LEVY**

Midterm EXAMINATION

3 November 2004

General Instructions -- This examination is 75 minutes long. Put away all your books and other material except for:

a) your 4 8.5" x 11" pages of notes
b) your calculator and straight edge.

Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of this will result in failure of the examination.

PRINT NAME ____________ SIGN NAME ____________

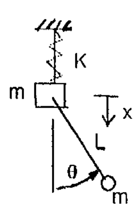
Problem 1.

The rotor of a turbine 13.6 kg in mass is supported at the midspan of the shaft with bearings 0.4064 m apart. The rotor is known to have an unbalance of 0.2879 kg-cm. Determine the forces exerted on the bearings at a speed of 6000 rpm if the diameter of the steel shaft is 2.54 cm. Compare this result with that of the same rotor mounted on a shaft of diameter 1.905 cm (assume the shaft is simply supported at the bearings). Also determine the critical speed of the shaft.

Assume the shaft has an $E=207 \times 10^9$ GPa and that $I_{zz}=\pi r^4/4$ where r is the radius of the shaft and z is the axis perpendicular to the page and passing through the centroid of the shaft.

Problem 2.

Derive the equations of motion of the system shown in the figure using Lagrange's equations with x and θ as generalized coordinates.



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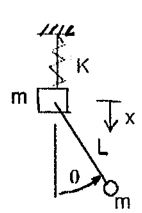


$(y+\delta_{st})^2 \quad (y+\delta_{st}) \quad (y+\delta_{st}+L)$

$$U = \frac{1}{2}k\cancel{y^2} - \left(mg\cancel{y} + mg\,\cancel{L}(\cos\theta - 1)\right)$$

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\left[(\dot{y} + \dot{\overline{l\cos\theta}})^2 + (\dot{\overline{l\sin\theta}})^2\right] = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{y}^2 - m\dot{y}\,l\sin\theta\,\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2$$

$$\dot{y}^2 + 2\dot{y}\,l\sin\theta\,\dot{\theta} + l^2\sin^2\theta\,\dot{\theta}^2 + l^2\cos^2\theta\,\dot{\theta}^2$$

$$\frac{\partial T}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{y}} = \frac{1}{2}m\cdot 2\dot{y} + \frac{1}{2}m\,2(\dot{y} + \dot{\overline{l\cos\theta}})\cdot 1 = m\dot{y} + m(\dot{y} - l\sin\theta\,\dot{\theta})$$

$$\frac{\partial U}{\partial q_1} = \frac{\partial U}{\partial y} = \frac{1}{2}k\cdot 2(y+\delta_{st}) - 2mg = ky \cancel{-2mg} \cancel{+k\delta_{st}}$$

$$\frac{\partial T}{\partial q_1} = \frac{\partial T}{\partial y} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial U}{\partial q_1} = m\ddot{y} + m\ddot{y} - ml\cos\theta\,\dot{\theta}^2 - ml\sin\theta\,\ddot{\theta} + ky + \cancel{mg} = 0$$

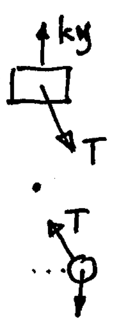
$$\frac{\partial T}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{\theta}} = \frac{1}{\cancel{2}}m\left[-\cancel{2}\dot{y}\,l\sin\theta + \cancel{2}l^2\dot{\theta}\right]$$

$$\frac{\partial U}{\partial q_2} = \frac{\partial U}{\partial \theta} = +mgL\sin\theta$$

$$\frac{\partial T}{\partial q_2} = \frac{\partial T}{\partial \theta} = \frac{1}{\cancel{2}}m\left[-\cancel{2}\dot{y}\,l\cos\theta\,\dot{\theta}\right]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial U}{\partial q_2} = \cancel{-ml\cos\theta\,\dot{\theta}\,\dot{y}} - m\ddot{y}\,l\sin\theta + ml^2\ddot{\theta} + \cancel{\tfrac{1}{2}m\dot{y}\,l\cos\theta\,\dot{\theta}} + mgL\sin\theta = 0$$

$$= ml^2\ddot{\theta} - ml\sin\theta\,\ddot{y} + mgL\sin\theta = 0$$



$m\ddot{x} = T\sin\theta$ [redacted]

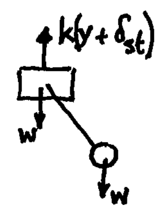
$m\ddot{y} = -ky + T\cos\theta$

$\Sigma T_o = I_o\ddot{\theta} + \cancel{m a_{o_x} y_G} - m a_{o_y} x_G$

$-Wl\sin\theta = ml^2\ddot{\theta} - m\ddot{y}\,l\sin\theta$ ✓

$-T\cos\theta + W = m\ddot{y} + m\ddot{y} = m(a_G = a_o + a_{G/o})$

$-T\sin\theta = m\ddot{x} = m(\ddot{\overline{l\sin\theta}})$



$k\delta_{st} = 2W$

$m\ddot{y} + m(\ddot{\overline{y + l\cos\theta}}) = -ky$

$2m\ddot{y} + m(\dot{\overline{-l\sin\theta\,\dot{\theta}}})$

$+ m(-l\cos\theta\,\dot{\theta}^2 - l\sin\theta\,\ddot{\theta})$

$$U = \frac{1}{2}k(y+\delta_{st})^2 - mg(y+\delta_{st}) - mg(y+\delta_{st}+L(\cos\theta - 1))$$

$\dot{y} = f(x,y)$

$$y_{n+1} = y_n + \frac{1}{2}(k_0 + k_1)$$

$$k_0 = \Delta x f(x_0, y_0)$$

$$k_1 = \Delta x f(x_1, y_0 + \Delta x k_0)$$

$$d_{n+1} = d_n + V_{n+\alpha} \Delta t$$

$$y_{n+1} = y_n + \frac{\Delta x}{2}\left(f(x_0,y_0) + f(x_1, y_0 + \Delta x k_0)\right)$$

$$V_{n+\alpha} = V_n(1-\alpha) + V_{n+1}\alpha$$

$$V_{n+\alpha} = \frac{1}{2}\ f(x_0,y_0) + f(x_1, y_0 + \Delta x k_0)$$

$\downarrow$ V_n $\quad$ $\downarrow$ $\sim V_{n+1}$

$\alpha = \frac{1}{2}$ is the equivalent to modified eulers method.

$$R = m_0 e \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

here $\zeta = 0$ $\quad \frac{m_0 e}{m} \frac{r^2}{|1-r^2|}$

$I = \frac{\pi d^4}{64} = 2.04 \times 10$ m^4

$$k = \frac{48EI}{l^3} = \frac{48\left(207\times10^9 \frac{N}{m}\right)\left(\pi d^4/64\right)}{(.4064)^3} = \frac{48\left(207\times10^9\frac{N}{m}\right)\left(\frac{\pi(2.54\times10^{-2})^4}{64}\right)}{(.4064)^3} = 3024512\ \frac{N}{m}$$

$$\omega_n = \sqrt{\frac{k}{13.6}} = \sqrt{222391} = 471.6\ \frac{rad}{s}$$

$$r = \frac{\omega_f}{\omega_n} = \frac{6000\frac{rev}{min}\cdot\frac{2\pi rad}{rev}\cdot\frac{1 min}{60}}{\omega_n} = \frac{628.3\ rad/s}{\omega_n} = 1.333$$

when $r = 1$ crit speed. $\frac{\omega_n}{2\pi} = f_{crit}$

$$R = \frac{m_0 e}{m}\frac{r^2}{|1-r^2|} = \frac{.2879\times10^{-2}}{13.6}\frac{(1.333)^2}{|1-(1.333)^2|} = -4.842\times10^{-4}\ m$$

$F_{trans} = kR =$

$$F = kR = 3024512 \times 4.842\times10^{-4} = 1464.4\ N$$

---

when I changes k change $\quad k = 956974.5\ N/m \quad \omega_n = 265.265\ rad/s$

$I = .6455\times10^{-8}\ m^4$

$r = 2.369$

$$R = \frac{m_0 e}{m}\frac{r^2}{(1-r^2)} = \frac{.2879\times10^{-2}}{13.6}\frac{(2.3686)^2}{(1-2.3686^2)} = \frac{-.0035}{13.6}\ m = -2.576\times10^{-4}\ m$$

$$F = kR = 246.52\ N$$

$$\omega_f = \omega_n = 265.265$$

Theory of Vibrations by Thomson & Dalch

Lagrange formulated a scalar procedure starting from KE, PE & work in terms of generalized coordinates q_i

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \qquad i=1,2,\ldots$$

Q_i is (related to) work done by nonpotential forces & is called the generalized force

This comes from

Conservation of energy $\frac{d}{dt}(PE+KE)=0$ for conservative system

since $PE+KE=$ constant and $KE = T(q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2 \ldots)$

$PE = U(q_1, q_2, \ldots)$ as measured from static equil

then $\frac{d}{dt}(PE+KE) = \frac{d}{dt}\left[T(q_1,q_2,\ldots,\dot{q}_i) + U(q_1,q_2,\ldots)\right] = 0$

$$\frac{d}{dt}T = \sum \frac{\partial T}{\partial q_i}\frac{dq_i}{dt} + \sum \frac{\partial T}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} \qquad \text{chain rule} \quad (1)$$

$$\frac{d}{dt}U = \sum \frac{\partial U}{\partial q_i}\frac{dq_i}{dt}$$

since T is also $\frac{1}{2}\sum m_i \dot{q}_i^2$ then

$$\sum_i\left(\frac{d}{d\dot{q}_i}T\right)\cdot\dot{q}_i = \sum \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i = \sum m_i \dot{q}_i^2 = 2T \qquad (1')$$

Now take $\frac{d}{dt}(1')$: $2\frac{dT}{dt} = \sum \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i + \sum \frac{\partial T}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} \qquad (2)$

subtract (1) from (2) to get $\frac{dT}{dt} = \sum \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i - \sum \frac{\partial T}{\partial q_i}\dot{q}_i$

Now $\frac{d}{dt}[T+U] = \sum\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i}\right\}\dot{q}_i = 0$ for any $\dot{q}_i$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0 \qquad \forall i=1,2,\ldots$$

if we define $L = T - U = T + \text{Work}$

& since $U \neq U(\dot{q}_1, \dot{q}_2, \ldots) \Rightarrow \frac{\partial U}{\partial \dot{q}_i} = 0$ then

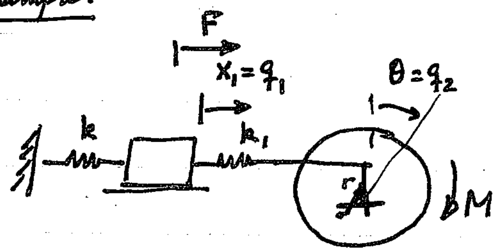
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

work = conservative work + NC work

conserv work = − potential energy

Example:

assume external forces M & F are applied



$$T = \tfrac{1}{2} m \dot{x}_1^2 + \tfrac{1}{2} J \Omega^2 = \tfrac{1}{2} m \dot{q}_1^2 + \tfrac{1}{2} J \dot{q}_2^2$$

$$U = \tfrac{1}{2} k x_1^2 + \tfrac{1}{2} k_1 (r\theta - x_1)^2 = \tfrac{1}{2} k q_1^2 + \tfrac{1}{2} k_1 (r q_2 - q_1)^2$$

Non conservative: $\delta W = M\delta\theta + F\delta x = M\delta q_2 + F\delta q_1$ $\qquad \frac{\delta W}{\delta q_1} = F \qquad \frac{\delta W}{\delta q_2} = M$

$$L = T - U = \tfrac{1}{2} m \dot{q}_1^2 + \tfrac{1}{2} J \dot{q}_2^2 - \tfrac{1}{2} k q_1^2 - \tfrac{1}{2} k_1 (r q_2 - q_1)^2$$

$$\frac{\partial L}{\partial \dot{q}_1} = m \dot{q}_1 \qquad \frac{\partial L}{\partial \dot{q}_2} = J \dot{q}_2$$

$$\frac{\partial L}{\partial q_1} = -k q_1 - k_1 (r q_2 - q_1)(-1) = +k_1 (r q_2) - (k + k_1) q_1$$

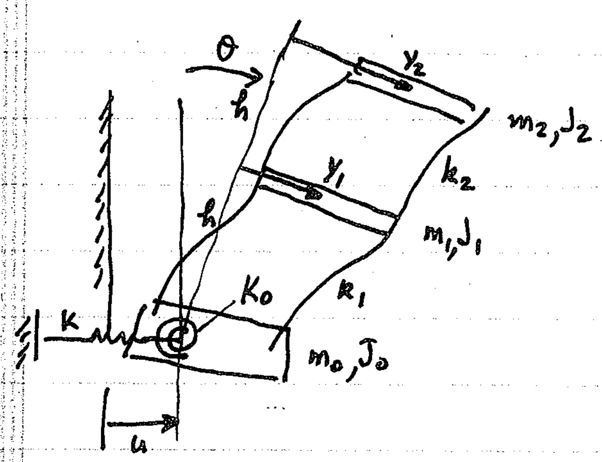
$$\frac{\partial L}{\partial q_2} = -k_1 (r q_2 - q_1) \cdot r = +k_1 r q_1 - k_1 r^2 q_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = m_1 \ddot{q}_1 + (k + k_1) q_1 - k_1 r q_2 = 0 \quad ✓ \quad F$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = J \ddot{q}_2 + k_1 r^2 q_2 - k_1 r q_1 = 0 \quad ✓ \quad M$$

Give 6.38/6.28 in 5th

6.39/6.43 in 6th



model of 2-story building whose foundation is subject to translation & rotation

let y_1 & y_2 be the elastic displ of the floors

u, θ " " transl & rot of the foundation

$$T = \tfrac{1}{2} m_0 \dot{u}^2 + \tfrac{1}{2} J_0 \dot{\theta}^2 + \tfrac{1}{2} m_1 (\dot{u} + h\dot{\theta} + \dot{y}_1)^2 + \tfrac{1}{2} J_1 \dot{\theta}^2$$
$$+ \tfrac{1}{2} m_2 (\dot{u} + 2h\dot{\theta} + \dot{y}_2)^2 + \tfrac{1}{2} J_2 \dot{\theta}^2$$

$$U = \tfrac{1}{2} k u^2 + \tfrac{1}{2} K_0 \theta^2 + \tfrac{1}{2} k_1 y_1^2 + \tfrac{1}{2} k_2 (y_2 - y_1)^2$$

if we let u, θ, y_1, y_2 be the generalized coordinates then

$$\frac{\partial T}{\partial \dot{\theta}} = J_0 \dot{\theta} + m_1 (\dot{u} + h\dot{\theta} + \dot{y}_1) h + J_1 \dot{\theta} + m_2 (\dot{u} + 2h\dot{\theta} + \dot{y}_2) \cdot 2h + J_2 \dot{\theta}$$

$$\frac{\partial T}{\partial \theta} = 0 \qquad \frac{\partial U}{\partial \dot{\theta}} = 0 \qquad \frac{\partial U}{\partial \theta} = K_0 \theta$$

for example $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = (J_0 + J_1 + J_2)\ddot{\theta} + (m_1 + 4m_2) h^2 \ddot{\theta} + m_1 \ddot{y}_1 h + 2m_2 \ddot{y}_2 h + (m_1 h + 2m_2 h)\ddot{u} + K_0 \theta = 0$$

$$\frac{\partial T}{\partial \dot{u}} = m_1 (\dot{u} + h\dot{\theta} + \dot{y}_1) + m_2 (\dot{u} + 2h\dot{\theta} + \dot{y}_2) + m_0 \dot{u} \quad \frac{\partial T}{\partial u} = 0 \quad \frac{\partial U}{\partial \dot{u}} = 0 \quad \frac{\partial U}{\partial u} = ku$$

for example $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} + \frac{\partial U}{\partial u} = (m_0 + m_1 + m_2)\ddot{u} + (m_1 h + 2m_2 h)\ddot{\theta} + m_1 \ddot{y}_1 + m_2 \ddot{y}_2 + ku = 0$$

$$\frac{\partial T}{\partial \dot{y}_1} = m_1 (\dot{u} + h\dot{\theta} + \dot{y}_1) \qquad \frac{\partial T}{\partial y_1} = 0 \quad \frac{\partial U}{\partial y_1} = k_1 y_1 + k_2 (y_2 - y_1)(-1) \qquad \frac{\partial U}{\partial \dot{y}_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_1}\right) - \frac{\partial T}{\partial y_1} + \frac{\partial U}{\partial y_1} = m_1 (\ddot{u} + h\ddot{\theta} + \ddot{y}_1) + (k_1 + k_2) y_1 - k_2 y_2 = 0$$

vector ab is rotated 107° ccw, it will be opposite the vector oa. To cancel oa it must be shortened by $oa/ab = 3.2/5.4 = 0.593$. Thus, the trial weight $W_t = 2.5$ oz must be rotated 107° ccw and reduced in size to $2.5 \times 0.593 = 1.48$ oz. Of course, the graphical solution for ab and ϕ can be found mathematically by the law of cosines.

Figure 3.3.6 shows a model simulating a long rotor with sensors at the two bearings. The two end disks may be initially unbalanced by adding weights at any location. By adding a trial weight at one of the disks and recording the amplitude and phase and then removing the first trial weight and placing a second trial weight to the other disk and making similar measurements, the initial unbalance of the simulated rotor can be determined.

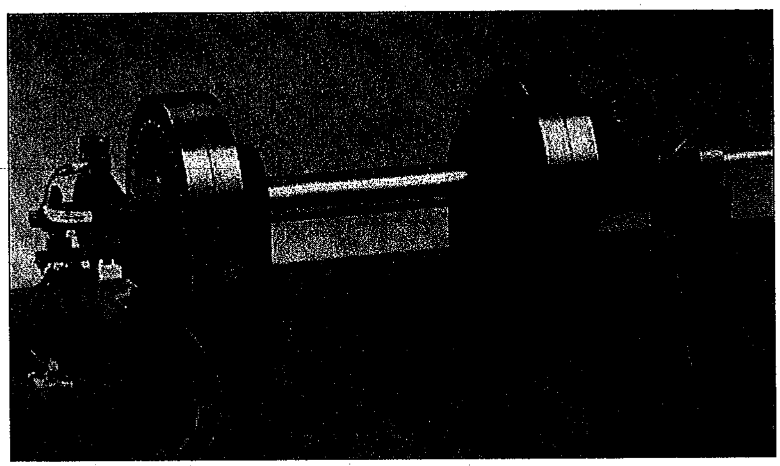


FIGURE 3.3.6. The plane-balancing experiment. (*Courtesy of UCSB Mechanical Engineering Undergraduate Laboratory.*)

3.4 WHIRLING OF ROTATING SHAFTS

Rotating shafts tend to bow out at certain speeds and whirl in a complicated manner. *Whirling* is defined as the rotation of the plane made by the bent shaft and the line of centers of the bearings. The phenomenon results from such various causes as mass unbalance, hysteresis damping in the shaft, gyroscopic forces, fluid friction in bearings, and so on. The whirling of the shaft can take place in the same or opposite direction as that of the rotation of the shaft and the whirling speed may or may not be equal to the rotation speed.

We will consider here a single disk of mass m symmetrically located on a shaft supported by two bearings, as shown in Fig. 3.4.1. The center of mass G of the disk is at a distance e (eccentricity) from the geometric center S of the disk. The center line of the bearings intersects the plane of the disk at O, and the shaft center is deflected by $r = OS$.

We will always assume the shaft (i.e., the line $e = SG$) to be rotating at a constant speed ω, and in the general case, the line $r = OS$ to be whirling at speed $\dot{\theta}$ that is not equal to ω. For the equation of motion, we can develop the acceleration of the mass center as follows:

$$\mathbf{a}_G = \mathbf{a}_S + \mathbf{a}_{G/S} \tag{3.4.1}$$

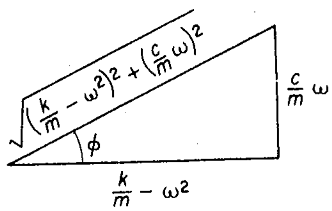


FIGURE 3.4.2.

$$\dot{\theta} = \omega$$

and on integrating we obtain

$$\theta = \omega t - \phi$$

where ϕ is the phase angle between e and r, which is now a constant, as shown in Fig. 3.4.1. With $\ddot{\theta} = \ddot{r} = \dot{r} = 0$, Eqs. (3.4.3) and (3.4.4) reduce to

$$\left(\frac{k}{m} - \omega^2\right)r = e\omega^2 \cos\phi$$

$$\frac{c}{m}\omega r = e\omega^2 \sin\phi \tag{3.4.5}$$

Dividing, we obtain the following equation for the phase angle:

$$\tan\phi = \frac{\frac{c}{m}\omega}{\frac{k}{m} - \omega^2} = \frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{3.4.6}$$

where $\omega_n = \sqrt{k/m}$ is the critical speed, and $\zeta = c/c_c$. Noting from the vector triangle of Fig. 3.4.2 that

$$\cos\phi = \frac{\frac{k}{m} - \omega^2}{\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{c}{m}\omega\right)^2}}$$

and substituting into the first of Eq. (3.4.5) gives the amplitude equation

$$r = \frac{me\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{e\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}} \tag{3.4.7}$$

These equations indicate that the eccentricity line $e = SG$ leads the displacement line $r = OS$ by the phase angle ϕ, which depends on the amount of damping and

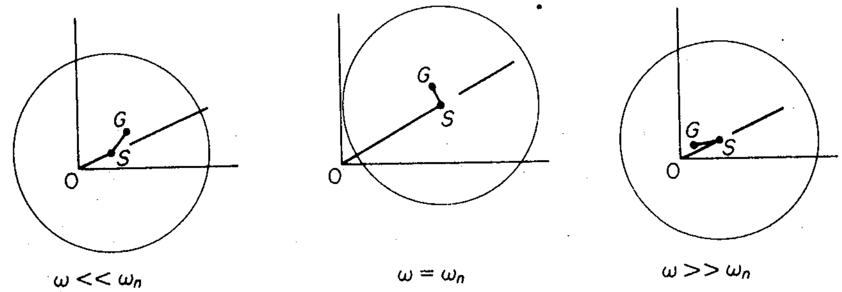


FIGURE 3.4.3. Phase of different rotation speeds.

the rotation speed ratio ω/ω_n. When the rotation speed coincides with the critical speed $\omega_n = \sqrt{k/m}$, or the natural frequency of the shaft in lateral vibration, a condition of resonance is encountered in which the amplitude is restrained only by the damping. Figure 3.4.3 shows the disk-shaft system under three different speed conditions. At very high speeds, $\omega \gg \omega_n$, the center of mass G tends to approach the fixed point O, and the shaft center S rotates about it in a circle of radius e.

It should be noted that the equations for synchronous whirl appear to be the same as those of Sec. 3.2. This is not surprising, because in both cases the exciting force is rotating and equal to $me\omega^2$. However, in Sec. 3.2 the unbalance was in terms of the small unbalanced mass m, whereas in this section, the unbalance is defined in terms of the total mass m with eccentricity e. Thus, Fig. 3.2.2 is applicable to this problem with the ordinate equal to r/e instead of MX/me.

EXAMPLE 3.4.1

Turbines operating above the critical speed must run through dangerous speed at resonance each time they are started or stopped. Assuming the critical speed ω_n to be reached with amplitude r_0, determine the equation for the amplitude buildup with time. Assume zero damping.

Solution We will assume synchronous whirl as before, which makes $\dot{\theta} = \omega =$ constant and $\ddot{\theta} = 0$. However, $\ddot{r}$ and $\dot{r}$ terms must be retained unless shown to be zero. With $c = 0$ for the undamped case, the general equations of motion reduce to

$$\ddot{r} + \left(\frac{k}{m} - \omega^2\right)r = e\omega^2 \cos\phi$$

$$2\dot{r}\omega = e\omega^2 \sin\phi \tag{a}$$

The solution of the second equation with initial deflection equal to r_0 is

$$r = \frac{e\omega}{2} t \sin\phi + r_0 \tag{b}$$

Differentiating this equation twice, we find that $\ddot{r} = 0$; so the first equation with the above solution for r becomes

$$\left(\frac{k}{m} - \omega^2\right)\left(\frac{e\omega}{2} t \sin\phi + r_0\right) = e\omega^2 \cos\phi \tag{c}$$

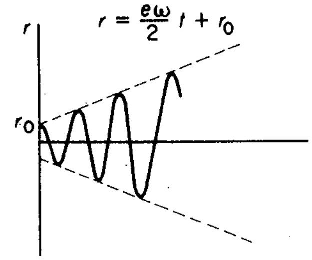


FIGURE 3.4.4. Amplitude and phase relationship of synchronous whirl with viscous damping.

Because the right side of this equation is constant, it is satisfied only if the coefficient of t is zero:

$$\left(\frac{k}{m} - \omega^2\right) \sin \phi = 0 \tag{d}$$

which leaves the remaining terms:

$$\left(\frac{k}{m} - \omega^2\right) r_0 = e\omega^2 \cos \phi \tag{e}$$

With $\omega = \sqrt{k/m}$, the first equation is satisfied, but the second equation is satisfied only if $\cos \phi = 0$ or $\phi = \pi/2$. Thus, we have shown that at $\omega = \sqrt{k/m}$, or at resonance, the phase angle is $\pi/2$ as before for the damped case, and the amplitude builds up linearly according to the equation shown in Fig. 3.4.4.

3.5 SUPPORT MOTION

In many cases, the dynamical system is excited by the motion of the support point, as shown in Fig. 3.5.1. We let y be the harmonic displacement of the support point and measure the displacement x of the mass m from an inertial reference.

In the displaced position, the unbalanced forces are due to the damper and the springs, and the differential equation of motion becomes

$$m\ddot{x} = -k(x - y) - c(\dot{x} - \dot{y}) \tag{3.5.1}$$

By making the substitution

$$z = x - y \tag{3.5.2}$$

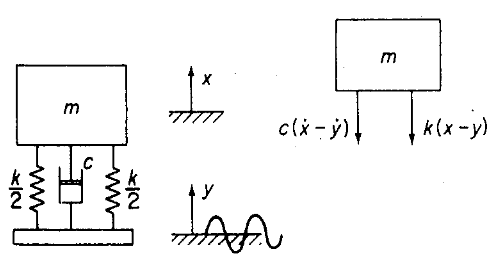


FIGURE 3.5.1. System excited by motion of support point.

3.16. A solid disk weighing 10 lb is keyed to the center of a $\frac{1}{2}$-in. steel shaft 2 ft between bear ings. Determine the lowest critical speed. (Assume the shaft to be simply supported a the bearings.)

3.17. Convert all units in Prob. 3.16 to the SI system and recalculate the lowest critical speed.

3.18. The rotor of a turbine 13.6 kg in mass is supported at the midspan of a shaft with bearing 0.4064 m apart, as shown in Fig. P3.18. The rotor is known to have an unbalance of 0.287 kg · cm. Determine the forces exerted on the bearings at a speed of 6000 rpm if the diam eter of the steel shaft is 2.54 cm. Compare this result with that of the same rotor mounte on a steel shaft of diameter 1.905 cm. (Assume the shaft to be simply supported at th bearings.)

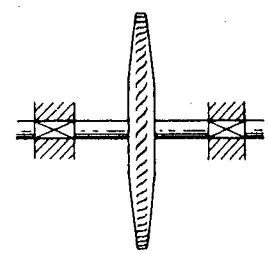


FIGURE P3.18.

3.19. For turbines operating above the critical speed, stops are provided to limit the amplitud as they run through the critical speed. In the turbine of Prob. 3.18, if the clearanc between the 2.54-cm shaft and the stops is 0.0508 cm, and if the eccentricity is 0.0212 cm determine the time required for the shaft to hit the stops. Assume that the critical spee is reached with zero amplitude.

3.20. Figure P3.20 represents a simplified diagram of a spring-supported vehicle traveling ove a rough road. Determine the equation for the amplitude of W as a function of the speed and determine the most unfavorable speed.

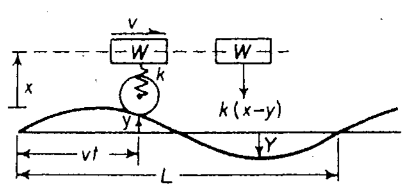


FIGURE P3.20.

3.21. The springs of an automobile trailer are compressed 10.16 cm under its weight. Find th critical speed when the trailer is traveling over a road with a profile approximated by sine wave of amplitude 7.62 cm and wavelength of 14.63 m. What will be the amplitude o vibration at 64.4 km/h? (Neglect damping.)

3.22. The point of suspension of a simple pendulum is given by a harmonic motio $x_0 = X_0 \sin \omega t$ along a horizontal line, as shown in Fig. P3.22. Write the differential equa tion of motion for a small amplitude of oscillation using the coordinates shown Determine the solution for x/x_0, and show that when $\omega = \sqrt{2}\omega_n$, the node is found a the midpoint of l. Show that in general the distance h from the mass to the node is give by the relation $h = l(\omega_n/\omega)^2$, where $\omega_n = \sqrt{g/l}$.

since $\sigma_z, \gamma_{xz}, \gamma_{yz}$ are negligible in plate theory

then $u(w) = \frac{1}{2}\int(\sigma_x \varepsilon_x + \varepsilon_y \sigma_y + \tau_{xy}\gamma_{xy})\,dV$
$\varepsilon_x = z \cdot K_x$ etc where K_x is the curvature

$$u(w) = \frac{1}{2} D \iint (\nabla^2 w)^2 dxdy + (1-\nu) D \iint [(w_{,xy})^2 - (w_{,xx})(w_{,yy})] dxdy$$

Now for our simply supported plate uniformly load $\hat{w} = C \sin\frac{\pi x}{\ell} \sin\frac{\pi y}{\ell}$

$$\alpha_{in} = \iint D(\nabla^2 \varphi_i)\nabla^2 \varphi_n \, dxdy + (1-\nu) D \iint \cdots$$

$$\beta_i = \iint p\, \varphi_i \, dxdy$$

since only ∇^2 we get less differentiation - Major advantage can relax bc. (natural ones).
geometric bc w, w_n
natural bc Moment, Shear

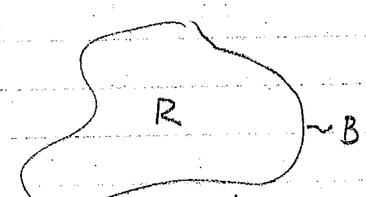
Finite Element is a composition of Ritz method + automatic procedures to generate test fn.

5/29/79

Plate Vibrations - General Problem.
inertial loading $p(\underline{r}, t) = -\rho h \frac{\partial^2 w}{\partial t^2}$
PDE ⇒

$$D\nabla^4 w + \rho h\, w_{,tt} = 0 \quad \text{in } R$$

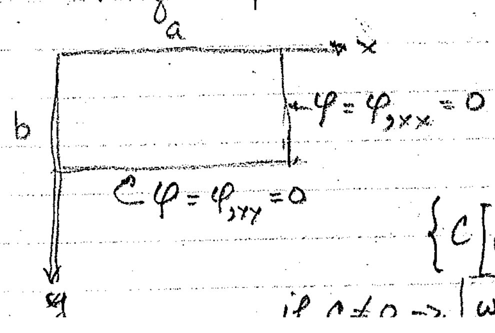
w/ BC ie $w = \frac{\partial w}{\partial n} = 0$ on $\partial R = B$



Not well specified since IC not specified - look for eigenmodes; trivial soln w=0
Look for soln $w = \varphi(\underline{r}) e^{i\omega t}$

⇒ $D\nabla^4\varphi - \rho h \omega^2 \varphi = 0$ in R w/ $\varphi = \frac{\partial \varphi}{\partial n} = 0$ on ∂R $\quad$ $\varphi = 0$ is triv sol
this is an eigenvalue problem for ω

Rectangular problem - SS BC.



$$\nabla^4 \varphi - \frac{\rho h}{D}\omega^2 \varphi = 0$$

using Navier soln method: then $\varphi = C \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$
φ satisfies bc then

$$\left\{ C\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 - \frac{\rho h}{D}\omega^2 C \right\} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = 0$$

if $C \neq 0$ ⇒ $\omega_{mn}^2 = \frac{\pi^4 D}{\ldots}\left[m^2 + \frac{n^2}{\ldots}\right]^2$ $\quad$ $\varphi_{mn} = C \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

$$A_n J_n(ka) + B_n I_n(ka) = 0$$

$$k A_n J_n'(ka) + B_n k I_n'(ka) = 0$$

$$\begin{pmatrix} J_n & I_n \\ kJ_n' & kI_n' \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} = 0 \Rightarrow k\left(J_n I_n' - J_n' I_n\right) = 0$$

or $$\frac{I_n'}{I_n} = \frac{J_n'}{J_n}$$

$$J_n' = -\frac{\nu}{x} J_\nu(x) + J_{\nu-1}(x)$$

$$Y_n' = -\frac{\nu}{x} Y_\nu(x) + Y_{\nu-1}(x)$$

$$I_n J_n' = -\frac{n}{x} J_n(x) I_n + J_{n-1}(x) I_n$$

$$J_n Y_n' = -\frac{n}{x} Y_n(x) J_n + Y_{n-1}(x) J_n$$

$$I_n J_n' - J_n Y_n' = -\frac{n}{x} JI + \frac{n}{x} IJ + J_{n-1} I_n - I_{n-1} J_n = 0$$

$$\therefore \quad \frac{I_{n-1}}{I_n} = \frac{J_{n-1}}{J_n}$$

$$(D\nabla^4 W)T + \rho h W\ddot{T} = 0$$

$$\frac{D\nabla^4 W}{W} + \rho h\frac{\ddot{T}}{T} = 0$$

$$\frac{D}{\rho h W}\nabla^4 W = -\frac{\ddot{T}}{T} = \omega^2 \quad \therefore$$

$$D\nabla^4 W - \rho h W\omega^2 = 0$$

$$\ddot{T} + \omega^2 T = 0$$

let $W(x,y) = X(x)Y(y)$

$$D\left[X^{IV}Y + 2X''Y'' + XY^{IV}\right] - \rho h XY\omega^2 = 0$$

let

$$D\left[\frac{X^{IV}}{X} + 2\frac{X''Y''}{XY} + \frac{Y^{IV}}{Y}\right] - \rho h\omega^2 = 0$$

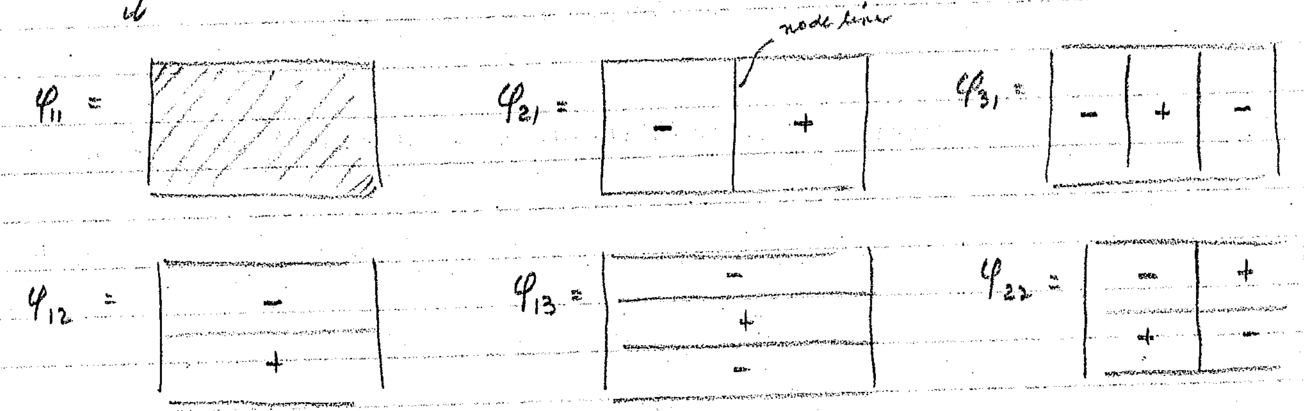
let $W(x,y) = C\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$

satisfies $W(0,y)=0 \quad W(a,y)=0 \quad W(x,0)=0 \quad W(x,b)=0$

also $W_{,xx}(0,y)=0 \implies -\left(\frac{n\pi}{a}\right)^2 CW(0,y)=0$

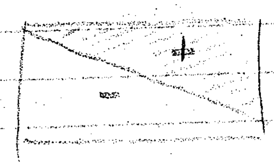
$W_{,xx}(a,y)=0$

this doesn't take into account damping
C can only be found if IC are specified! we really don't care about it

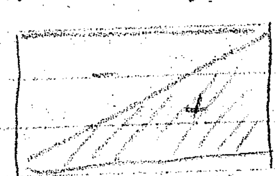


for a square plate $\varphi_{12} = \varphi_{21}$ $\quad \varphi_{13} = \varphi_{31}$ this leads to degenerate modes (where 2 modes collapse into 1) $\omega_{21} = \frac{5\pi^2}{a^2}\sqrt{D/\rho h}$

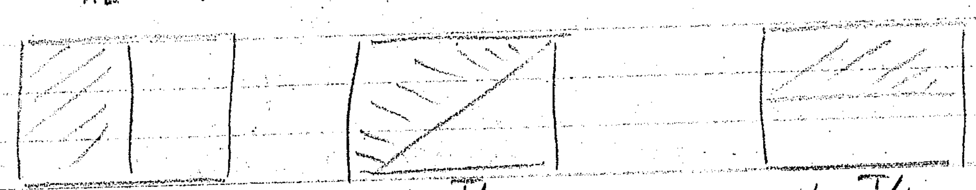
Look at
$$w = C(\varphi_{12}\cos\omega_{12}t + \varphi_{21}\cos\omega_{21}t) = C(\varphi_{12} + \varphi_{21})\cos\omega_{12}t$$ this is an eigenmode



$$w = C(\varphi_{12}\cos\omega_{12}t - \varphi_{21}\cos\omega_{21}t) = C(\varphi_{12} - \varphi_{21})\cos\omega_{12}t$$ thus this is also an eigenmode



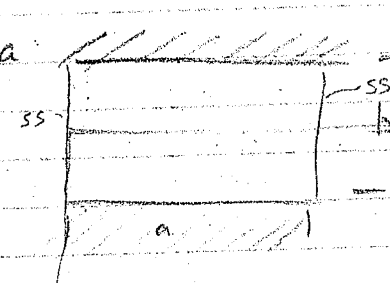
if $w = C(\varphi_{12}\cos\omega_{12}t \pm \varphi_{12}\sin\omega_{12}t)$ then



t=0 $\quad$ t = 1/8 $\quad$ t = T/4

thus the nodal line is rotating

HW #8 Look at plate with b×a



find eigensolns for symmetric modes about centerline

General Case $\nabla^2(\nabla^2\varphi) - \frac{\rho h}{D}\omega^2\varphi = 0$

$$\Rightarrow \left(\nabla^2 - \sqrt{\tfrac{\rho h}{D}}\,\omega\right)\left(\nabla^2 + \sqrt{\tfrac{\rho h}{D}}\,\omega\right)\varphi = 0 \quad \text{not separable in most cases}$$

if $\left(\nabla^2 - \sqrt{\frac{\rho h}{D}}\,\omega\right)\varphi = 0$ or $\left(\nabla^2 + \sqrt{\frac{\rho h}{D}}\,\omega\right)\varphi = 0$ then original eqn is satisfied. This is the Helmholtz eqn. which is separable in many case.

Circular plate $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$; with $k^2 = \sqrt{\frac{\rho h}{D}}\,\omega$

$$\therefore (\nabla^2 \pm k^2)\varphi = 0 \quad \text{using } \varphi = R(r)T(\theta) \Rightarrow \frac{r^2R'' + rR' \pm k^2r^2R}{R} = -\frac{T''}{T} = \lambda^2$$

R = Bessel fns, T are trig fn.
for a complete plate causality requires at $r=0$ R must be finite and single valuedness requires periodicity of T. $\Rightarrow \lambda$ must be integer

$\therefore T = \sin n\theta, \cos n\theta$ and $r^2R'' + rR' - (n^2 \mp k^2r^2)R = 0$

$\therefore R = A_n J_n(kr) + B_n I_n(kr)$ I_n is modified Bessel fns.

for a clamped plate $R(r=a) = 0$ $\frac{dR}{dr}(r=a) = 0$ since $w(a,\theta,t) = 0 = \left.\frac{\partial w}{\partial r}\right|_{r=a}$

for nontrivial solns requires Wronskian of $\left(R, \frac{dR}{dr}\right) = 0$

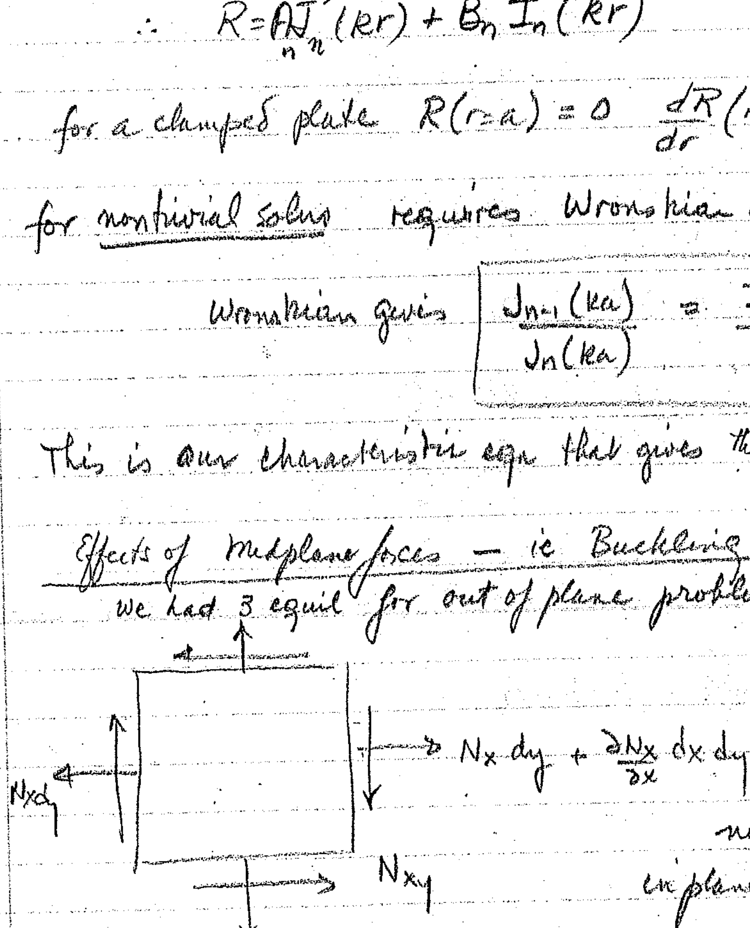
Wronskian gives $\boxed{\frac{J_{n-1}(ka)}{J_n(ka)} = \frac{I_{n-1}(ka)}{I_n(ka)}}$ since R'_n can be written in terms of R_{n-1}

This is our characteristic eqn that gives the value of ω through k

Effects of midplane forces — ie Buckling

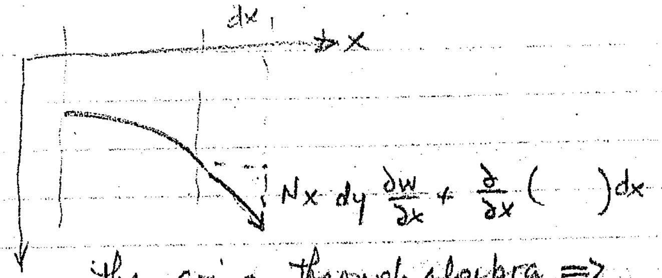
we had 3 equil for out of plane problem $\Sigma F_x, \Sigma F_y ; \Sigma M_z = 0$

ie $\Sigma F_x = 0 \Rightarrow \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$



now we want to find how these inplane affect out of plane deformation

THUS:

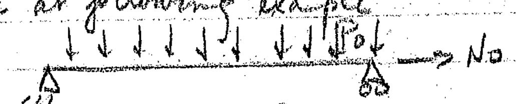


then going through algebra $\Rightarrow$ $D\nabla^4 w = p(x,y) + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}\left(N_{xy}\frac{\partial w}{\partial y}\right)$

$$+ \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y}\right)$$

Normally since the inplane & out of plane problems are decoupled solve inplane problem to get N_x, N_{xy}, N_y put here then solve for w the out of plane deformation

Look at following example



SS plate $\Rightarrow$ Navier soln is OK here

PDE $\quad D\nabla^4 w - N_0 \frac{\partial^2 w}{\partial x^2} = p(x,y)$

Navier sol $\quad w = \sum\sum C_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

$$p(x,y) = \frac{16 p_0}{\pi^2}\sum_{\substack{m,n\\odd}} \frac{1}{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

Put w & p into PDE to get

$$\sum_m\sum_n \left[C_{mn}\left\{D\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + N_0\left(\frac{m\pi}{a}\right)^2\right\} - \frac{16p_0}{\pi^2 mn}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0$$

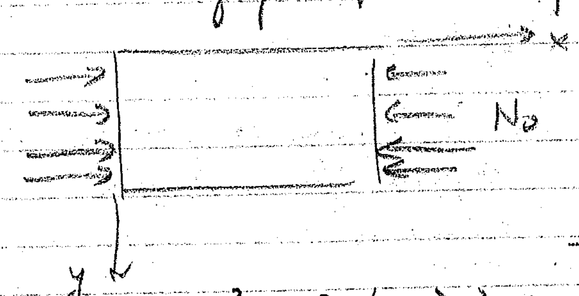
$\Rightarrow C_{mn}$ can be obtained $\quad C_{mn} = \dfrac{16p_0/\pi^2 mn}{D\pi^4\left(\frac{m^2}{a^2}+n^2/b^2\right)^2 + N_0\left(\frac{m\pi}{a}\right)^2}$

$$\therefore\quad w = \sum_{n,m\ odd}\sum \frac{16p_0}{\pi^6 D}\,\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{nm\left\{\left[\left(\frac{m^2}{a^2}\right)+\frac{n^2}{b^2}\right]^2 + \frac{N_0 m^2}{D\pi^2 a^2}\right\}}$$

as $N_0 \uparrow$ $\quad w \downarrow$ since denom $\uparrow$

for buckling for $N_0 < 0$ $\Rightarrow$ denom can be made $= 0$.

Buckling of SS plate w $p = 0$



Continuation $\quad$ 5/31/79

$$D\nabla^4 w - \frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x}\right) = 0$$

$w = w_{,yy} = 0 \qquad y = 0, b$

for $D w_{xx} - n w = 0$ $\quad w = A e^{\sqrt{n/D}\,x} + B e^{-\sqrt{n/D}\,x}$ since we note that

On the boundary we don't feel effect of P if b is large

$\therefore$

$w = C_1 \ln b/r + A\left(e^{\sqrt{n/D}\,(r-b)} - 1\right)$ for large r note w/ r=b $\quad w = 0$ at bdy

now $\frac{dw}{dr} = -C_1/r + A\sqrt{n/D}\; e^{\sqrt{n/D}(r-b)}$ $\quad$ from last time $\delta = \frac{\pi}{\sqrt{n/D}} = \frac{\pi}{\lambda}$

at $r=b$ $\quad \theta = 0$ $\quad \therefore \quad 0 = -\frac{C_1}{b} + A\sqrt{n/D}$ $\quad A = \frac{C_1}{\sqrt{n/D}\; b} = \frac{C_1\delta}{\pi b}$

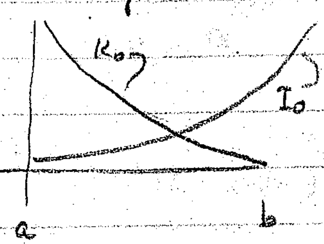
$$\left.\frac{d^2w}{dr^2}\right|_{r=b} = C_1\left[\frac{1}{r^2} + \frac{A(n/D)}{C_1}\, e^{\sqrt{n/D}(r-b)}\right] = \frac{C_1}{b^2}\left[1 + \frac{\pi b}{\delta}\right]$$

$\frac{b}{\delta} >> 1$ we have large curvature

at r=a we can do the same if $a >> \delta$

if a is not $>> \delta$ the solns are bessel fns.

if $n>0$ $\quad D\nabla^2 w_{II} - n\, w_{II} = 0$ $\quad w_{II} = A\, I_0\left(\sqrt{n/D}\, r\right) + B\, K_0\left(\sqrt{n/D}\, r\right)$

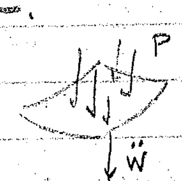


$K_0 \sim \ln r$ for real

if $n<0$ $\quad$ soln $\quad w_{II} = A\, J_0\left(\sqrt{n/D}\, r\right) + B\, Y_0\left(\sqrt{n/D}\, r\right)$ $\quad Y_0 \sim \ln r \quad r<<1$; $\quad J_0 = 1 \quad r=0$

Consider. Plate vibrations problem.

$D\Delta^2 w = p$



Look at $D\Delta^2 w = \frac{\text{mass}}{\text{Area}} \cdot \text{accel.}$

inertia force $= \rho h \frac{\partial^2 w}{\partial t^2}$

$\therefore \quad D\Delta^2 w = p - \rho h \frac{\partial^2 w}{\partial t^2}$ $\quad$; consider free vibes

set $P = 0$ & look at harmonics $\quad w(x,y,t) = W(x,y) \sin\omega t$

$\longrightarrow \quad D\nabla^2\nabla^2 W = +\rho h \omega^2 W$ $\quad$ like foundation w/ negative spring const

Look at SS plates $\quad$ Let $W(x,y) = A \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

This satisfies bc on w & moment. put into DE

$$\therefore \quad D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 - \rho h \omega^2 = 0$$

$\longrightarrow \quad \omega = \sqrt{\frac{D}{\rho h}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{+1/2}$ $\quad$ for a square plate $\omega_{mn} = \sqrt{\frac{D}{\rho h}}\left(\frac{\pi}{a}\right)^2 (m^2$

Forced Vibration $p(t)$

$$D\nabla^2\nabla^2 w = p - \rho h \ddot{w} \qquad \text{Let } p(t,x,y) = \sum\sum p_{mn}(t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$w(x,y,t) = \sum\sum w_{mn}(t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

put into DE

look at m,n term: $D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 w_{mn}(t) + \rho h \ddot{w}_{mn}(t) = p_{mn}(t)$

let $\omega_{mn}^2 = D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 / \rho h \quad \therefore \quad \ddot{W}_{mn}(t) + \omega_{mn}^2 W_{mn} = \frac{p_{mn}(t)}{\rho h}$

natural freq of m,n mode

let $p_{mn}(t) = A_{mn} \sin \Omega t$

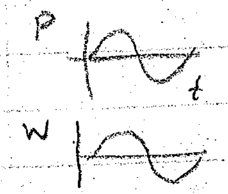
Let $W_{mn}(t) = B_{mn} \sin \Omega t$

if $\Omega = \omega_{mn}$ then we have resonance.

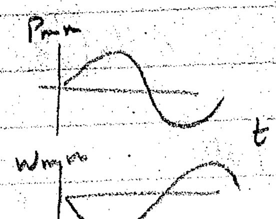
put into $\ddot{W}_{mn} + \omega_{mn}^2 W = p_{mn}/\rho h$

$$-B_{mn}\Omega^2 + \omega_{mn}^2 B_{mn} = A_{mn}/\rho h$$

$$B_{mn} = \frac{A_{mn}}{\rho h\,(\omega_{mn}^2 - \Omega^2)} \qquad \text{if } \omega_{mn} > \Omega$$



if $\omega_{mn} < \Omega$



when we're at resonance PDE is $\ddot{W} + \omega^2 W = f(t)$ let $f(t) = e^{i\omega t}$

$= e^{i\omega t}$

Guess $W = \hat{f}(t) e^{i\omega t}$

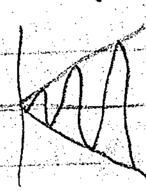
$$\dot{W} = \dot{\hat{f}} e^{i\omega t} + i\omega \hat{f} e^{i\omega t}$$

$$\ddot{W} = \ddot{\hat{f}} e^{i\omega t} + 2\dot{\hat{f}} i\omega e^{i\omega t} + (i\omega)^2 \hat{f} e^{i\omega t}$$

put into DE

$$\ddot{f} + 2i\omega \dot{f} = 1 \qquad \text{Let } f = t/2i\omega \quad \dot{f} = 1/2i\omega \quad \ddot{f} = 0$$

$$w = \frac{t}{2i\omega} e^{i\omega t} + W_h$$



linear theory is good for small time!

ME 241 Steele, C. Theory of Plates

Prereqs: CE 114 (Strength of Materials)

Math: ODE; PDE; $\Delta\varphi = 0$ $\Delta^2\varphi = p$

Approach will be mathematical

Objective → fundamental understanding & Computational capability

Method : solve problems weekly assignments

Grade : 50% HW 50% in-class Final (not open book)

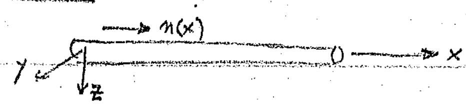
What is a plate :

first 3-D Structures are usually not efficient usage of structure

1-D wire / beam } one dim is >> than other 2 dimensions $EIw''' = p(x)$

2-D plate one dim is small wrt the other 2 $EI\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = p($

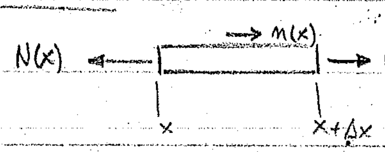
Stretching of a rod



$n(x)$ = distributed load / unit length

This load produces a 3-D stress. Since the length >> the radius we can define a stress resultant $N(x) = \iint \sigma_x \, dA$ to get rid of 3 dimensionality

Looking at equilibrium



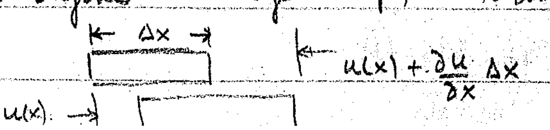
$\rightarrow N(x+\Delta x) = N(x) + \left(\frac{\partial N}{\partial x}\right)_x \Delta x + O(x)$

thus $\left(\frac{\partial N}{\partial x}\right)_x \Delta x + \ldots + n(x)\Delta x = 0$; taking limit as $\Delta x \to 0$ is $\boxed{\frac{dN}{dx} = -n(x)}$

if $n(x)$ is known then $\boxed{N(x) = -\int_{x_0}^{x} n(x)\,dx + N(x_0)}$

Normally we don't know $N(x_0)$ but we do know what displacements are

thus define average disp, in x direction $u(x) = \frac{1}{A}\int_A u \, dA$

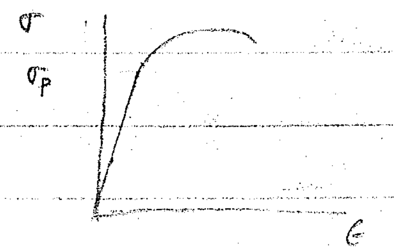


we can thus define the engineering strain $\epsilon = \dfrac{\text{final length} - \text{init length}}{\text{initial length}} = \dfrac{\frac{\partial u}{\partial x}\Delta x}{\Delta x} = \dfrac{\partial u}{\partial x}$

$$\boxed{\epsilon = \frac{\partial u}{\partial x}}$$

This is relation that defines the displacement in terms of the strain

Now we use the stress strain relation to relate the stresses to the strain



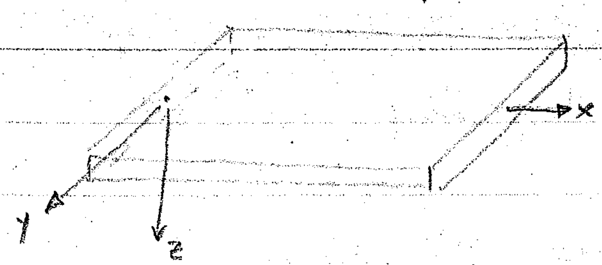
Generally $|\sigma| < |\sigma_p|$ in design - we thus use Hooke's law $\sigma = E\epsilon$ E is Young's modulus where σ is the 3-D stress

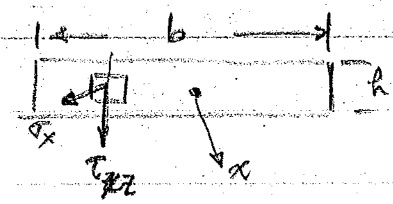
now $N(x) = \int \sigma\, dA = EA\epsilon$ if σ is a constant

now $\dfrac{dN}{dx} = -n(x) \Rightarrow$ $$\boxed{\frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = -n(x)}$$

this is the relation between the distributed load and the displacement

9/28 Beams in bending



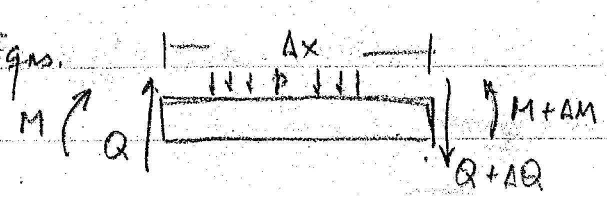


Define stress Resultants

$$Q = \int_{-h/2}^{h/2} \tau_{zx}\, dz \quad \text{force per unit width}$$

$$M = \int_{-h/2}^{h/2} \sigma_x z\, dz \quad \text{moment / unit width}$$

Equilib Eqns.



Force : $\Delta Q + p\,\Delta x \sim 0$ $\qquad p$ = load / unit width

$$\lim_{\Delta x \to 0} \frac{\Delta Q}{\Delta x} = -p \quad \text{or} \quad \boxed{\frac{dQ}{dx} = -p} \qquad (1)$$

Moment $p\Delta x \cdot \frac{\Delta x}{2} + (Q+\Delta Q)\Delta x - M - \Delta M + M \sim 0$ or $\Delta M \sim Q\Delta x$

$$\Rightarrow \boxed{\frac{dM}{dx} = Q} \qquad (2)$$

(1) & (2) $\Longrightarrow$ $\boxed{\frac{d^2M}{dx^2} = -p}$ (3)

<u>Displacement of midsurface</u>

w - in z-dir.

0 in x-dir

For the displacement of a point a distance z from the midsurface we must make a kinematic hypothesis

For Beams - Bernoulli or Euler

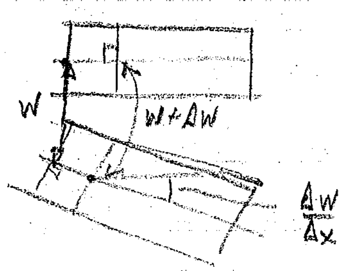
For plate - Kirchhoff

For shells - Kirchhoff, Love

<u>Assume:</u>

(1) A normal to the neutral surface remains straight & unstretched (plane surfaces remain plane)

(2) Normals remain ⊥ to the neutral surface



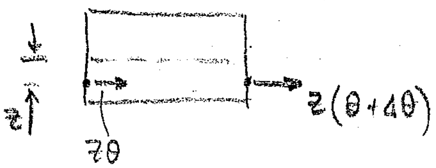
Displacement of pt a distance z from the midsurface is:

w - in z direction

$z\theta$ - in x direction

thus rotation of the midsurface $\boxed{\theta = -\frac{dw}{dx}}$ (4)

Strains



$$\epsilon_x \cong z\frac{\Delta\theta}{\Delta x}$$

or
$$\boxed{\epsilon_x = z\frac{d\theta}{dx}} \qquad (5)$$

Using the stress strain law

$$\sigma_x = E\epsilon_x$$

$$M = \int_{-h/2}^{h/2} \sigma_x z\,dz = \int E\frac{d\theta}{dx}z^2 dz = E\frac{d\theta}{dx}\frac{h^3}{12} \qquad \text{using (5)}$$

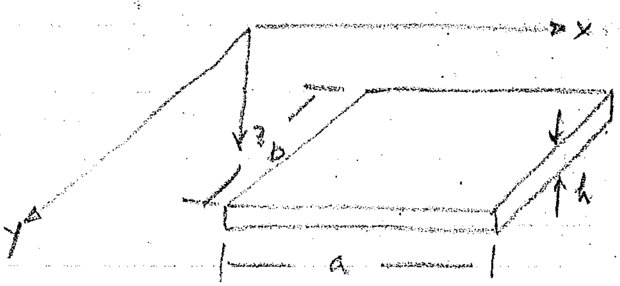
$$Mb = E\frac{d\theta}{dx}\frac{bh^3}{12} = EI\frac{d\theta}{dx} \qquad \text{for a rectangular beam.}$$

or
$$\boxed{Mb = EI\left(-\frac{d^2w}{dx^2}\right)} \qquad \text{using (4)} \qquad (6)$$

but
$$\frac{d^2M}{dx^2} = -p \quad (3) \qquad \therefore \quad \boxed{pb = \frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)} \qquad (7)$$

beam bending in terms of displacement

10/1/79 **Definition of a plate**



Assume: $h \ll a, b$

(2) plate is elastic, homog, isotropic

(3) small deflections $w \leq \frac{h}{5}$

(4) $\frac{\partial w}{\partial x}$ & $\frac{\partial w}{\partial y} \ll 1$

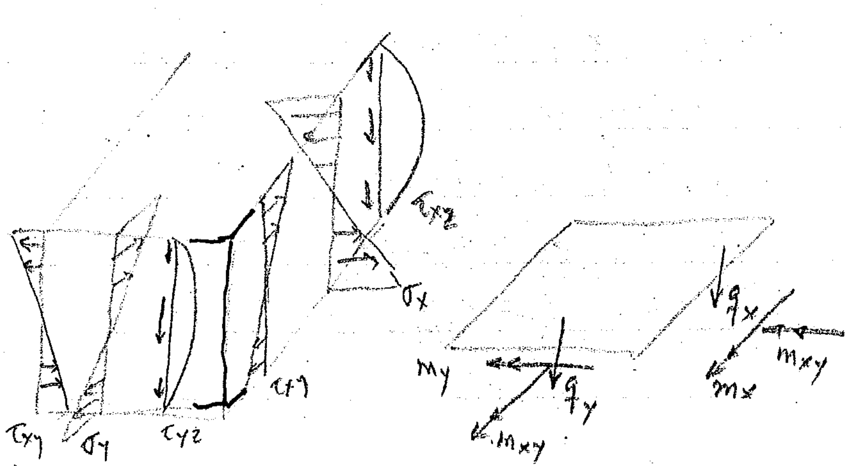
(5) Kirchhoff Hypothesis

- normals remain straight & unstre[tched] and normal
- $w = w(x,y)$ only

(6) Inextensional theory - no stretching

(7) no initial imperfections

(8) stresses normal to midsurface are negligible $\sigma_z \ll \sigma_x, \sigma_y$

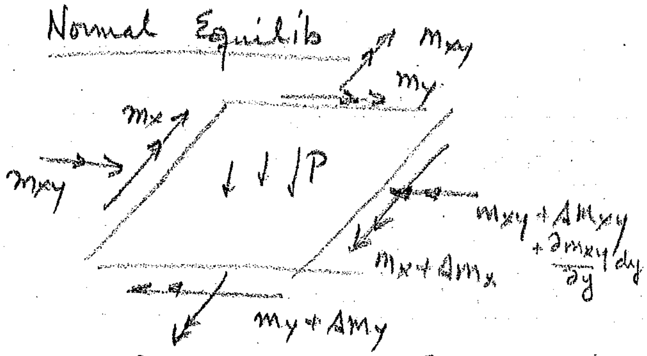


our stress resultants

$$q_x = \int_{-h/2}^{h/2} \tau_{xz}\, dz \qquad q_y = \int_{-h/2}^{h/2} \tau_{yz}\, dz \qquad m_x = \int_{-h/2}^{h/2} \sigma_x z\, dz \qquad m_y = \int_{-h/2}^{h/2} \sigma_y z\, dz$$

$$m_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\, dz \qquad m_{yx} = \int_{-h/2}^{h/2} \tau_{yx} z\, dz$$

Normal Equilib



$$m_{xy} + \Delta m_{xy} = m_{xy} + \frac{\partial m_{xy}}{\partial x} dx$$

$$\frac{\partial q_y}{\partial y} \Delta y \Delta x + \frac{\partial q_x}{\partial x} \Delta x \Delta y + p \Delta x \Delta y \cong 0$$

or
$$\boxed{\frac{\partial q_y}{\partial y} + \frac{\partial q_x}{\partial x} + p = 0} \qquad (1)$$

Moment Equil in x-direction

$$\frac{\partial m_y}{\partial y} \Delta x \Delta y + \frac{\partial m_{xy}}{\partial x} \Delta x \Delta y - q_y \Delta x \Delta y \cong 0$$

or
$$\boxed{\frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} - q_y = 0} \qquad (2)$$

Moment Equil in y dir:

$$\boxed{\frac{\partial m_{yx}}{\partial y} + \frac{\partial m_x}{\partial x} - q_x = 0} \qquad (3)$$

take $\frac{\partial}{\partial y}(2)$ $\frac{\partial}{\partial x}(3)$ add & substitute (1) to give

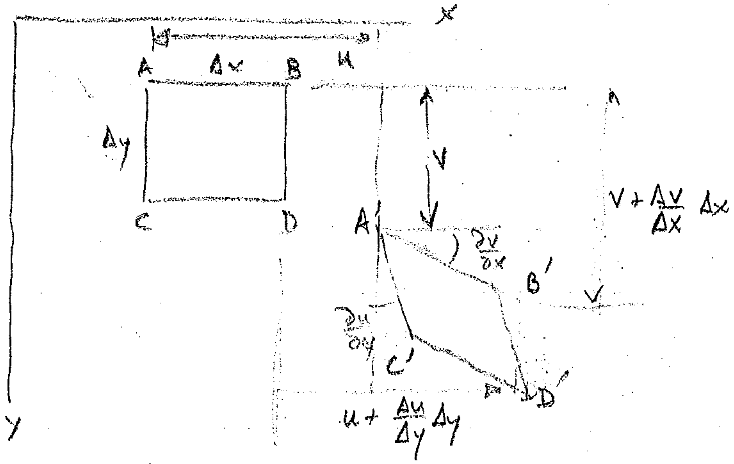
$$\boxed{\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} + P = 0} \qquad (4)$$

Displacements $\quad w = w(x, y)$

$$u = z\,\theta_x = -z\frac{\partial w}{\partial x} \qquad v = z\,\theta_y = -z\frac{\partial w}{\partial y}$$

Strains

$$\epsilon_x = z\frac{\partial \theta_x}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \qquad \epsilon_y = z\frac{\partial \theta_y}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}$$



$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

Stress - Strain relations

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

neglect $\sigma_z << \sigma_x, \sigma_y$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\epsilon_x + \nu\epsilon_y\right)$$

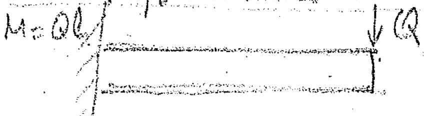
$$\sigma_y = \frac{E}{1-\nu^2}\left(\epsilon_y + \nu\epsilon_x\right)$$

obtained by inverting ϵ_x, ϵ_y for σ_x, σ_y

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

10/3/79

To show τ_{xz}, τ_{yz} are small in comparison to σ_x



$$\sigma_x = \frac{Mc}{I} = \frac{6M}{h^2} = \frac{M h}{2 \cdot h^3/12} = \frac{6Q}{h}\left(\frac{l}{h}\right)$$

aver $\tau_{xz} = \dfrac{Q}{h}$ $\Rightarrow$ $\sigma_x >> \tau_{xz}, \tau_{yz}$ thus neglect τ_{xz}, τ_{yz}

however we do not neglect q_x, q_y

$$m_x = \int_{-h/2}^{h/2} \sigma_x z\, dz = \frac{E}{1-\nu^2}\left(\frac{\partial\theta_x}{\partial x} + \nu\frac{\partial\theta_y}{\partial y}\right)\int_{-h/2}^{h/2} z^2 dz = \boxed{\frac{-EI}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = m_x}$$

in doing this σ_x is assumed constant across cross section $D = \dfrac{EI}{1-\nu^2} = \dfrac{Eh^3}{12(1-\nu^2)}$ Bending Rigidity

Constitutive Eqs

Thus $\boxed{m_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)}$ $\boxed{m_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]}$

$$m_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\, dz = \frac{E}{2(1+\nu)} \int_{-h/2}^{h/2} -2 \frac{\partial^2 w}{\partial x \partial y} z^2 dz = -\frac{EI}{1+\nu} \frac{\partial^2 w}{\partial x \partial y} = \boxed{-D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = m_{xy}}$$

now $q_x = m_{x,x} + m_{xy,y}$

$$= -D\left\{\frac{\partial}{\partial x}\left[w_{,xx} + \nu w_{,yy}\right] + (1-\nu)\frac{\partial}{\partial y}(w_{,xy})\right\} = -D\frac{\partial}{\partial x}(\Delta w) = \boxed{-D\frac{\partial}{\partial x}(\nabla^2 w) = q_x}$$

$$\therefore \boxed{q_y = -D\frac{\partial}{\partial y}(\nabla^2 w)}$$

Normal equil eqn.

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + p = 0 \Rightarrow -D\left(\nabla^2(\nabla^2 w)\right) + p = 0$$

or $\boxed{\Delta^2 w = \nabla^4 w = p/D}$ beam bending eq in terms of displacement

Nonhomogeneous Biharmonic Eqn.

$\nabla^2 w = 0$ Laplace $\nabla^2 w \neq 0$ Poisson $\nabla^4 w = 0$ Biharmonic

Define Curvature strain measures

$$K_x = -\frac{\partial^2 w}{\partial x^2} \qquad K_y = -\frac{\partial^2 w}{\partial y^2} \qquad K_{xy} = -\frac{\partial^2 w}{\partial x \partial y}$$

Stress Resultants in terms of strain measures (Kinematic relations)

$$m_x = D(K_x + \nu K_y) \qquad m_y = D(K_y + \nu K_x) \qquad m_{xy} = D(1-\nu)K_{xy}$$

<u>Summary</u> of how we got governing eqns.

- Eqns of Equil.
- Strain Displ relations
- Stress - Strain Relations
  - * stress resultants & strain measures
- Kinematic relations

Boundary Conditions

Since 4^{th} order ODE need 4 bc. (2 on each end)

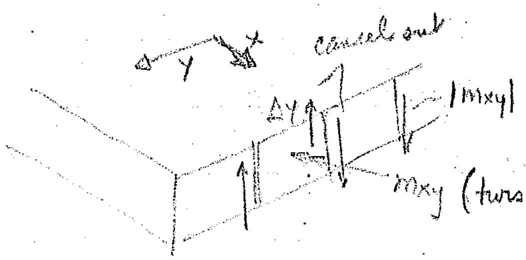
Geometric BC

$w, \frac{\partial w}{\partial x}$ on $x = const$

$w, \frac{\partial w}{\partial y}$ on $y = const$

- fixed edge $w = 0 \quad \frac{\partial w}{\partial x} = 0$ at $x = a$

Stress BC $\Longrightarrow$ m_x q_x & m_{xy} on $x = const.$



can be replaced by a force couple of magnitude $|m_{xy}|$. This adds to the vertical shear

Net Shear Resultant on edge is $V_x = q_x + \frac{\partial m_{xy}}{\partial y}$

$$V_y = q_y + \frac{\partial m_{xy}}{\partial x}$$

Thus our BC is V_x & m_x on $x = const.$

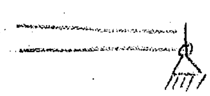
on a free edge $m_x = V_x = 0$

$$V_x = f(w) = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]$$

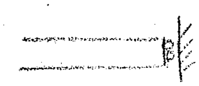
$$V_y = g(w) = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]$$

Mixed boundary conditions

Simply-Supported Edge $w = 0$, $m_x = 0$ at $x = const.$



$V_x = 0 \quad \frac{\partial w}{\partial x} = 0$



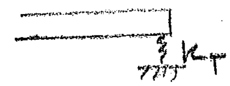
Summary

fixed $w = \frac{\partial w}{\partial x} = 0$

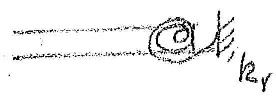
free $m_x = v_x = 0$

Simply Supported $w = 0,\ m_x = 0$

elastic support $m_x = 0$ $w = \frac{v_x}{k_t}$



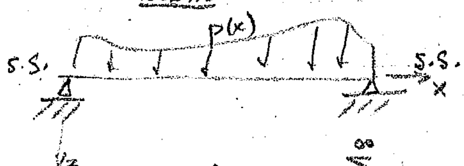
$v_x = 0$ $\frac{\partial w}{\partial x} = \frac{m_x}{k_r}$



10/15/79

Fourier Series method of soln.

Beam $EI\, w^{iv} = p(x)$



simple support $w = 0$ $M = 0$ or $w = 0$ & $\frac{\partial^2 w}{\partial x^2} = 0$

look at soln. $w = \sum_{n=1}^{\infty} w_n \sin\frac{n\pi x}{L}$ this satisfies B.C. : any sufficiently smooth fn. can be represented by this series

put w into ODE each term satisfies BC identically & doesn't necessarily solve the ident.

$$\therefore\ EI \sum_{n=1}^{\infty} w_n \left(\frac{n\pi}{L}\right)^4 \sin\frac{n\pi x}{L} = p(x) \Rightarrow \frac{2EI}{L}\int_0^L \sum w_n \left(\frac{n\pi}{L}\right)^4 \sin\frac{n\pi x}{L}\sin\frac{k\pi x}{L}\,dx = \frac{2}{L}\int_0^L \sin\frac{k\pi x}{L}\,p($$

$$\Rightarrow EI\, w_k \left(\frac{k\pi}{L}\right)^4 = \frac{2}{L}\int_0^L p(x)\sin\frac{k\pi x}{L}\,dx \qquad \therefore \boxed{w_k = \frac{2}{EIL}\left(\frac{L}{k\pi}\right)^4 \int_0^L p(x)\sin\frac{k}{l}}$$

$$\therefore \boxed{w_k = \frac{2}{EIL}\left(\frac{L}{k\pi}\right)^4 \int_0^L p(x)\sin\frac{k\pi x}{L}\,dx}$$

alternate write $p(x) = \sum p_n \sin\frac{n\pi x}{L}$ $\qquad p_n = \frac{2}{L}\int_0^L p(x)\sin\frac{n\pi x}{L}\,dx$

$$\Rightarrow EI\left(\frac{n\pi}{L}\right)^4 w_n = p_n \Rightarrow \boxed{w_n = \frac{p_n}{EI}\left(\frac{L}{n\pi}\right)^4}$$

very good convergence in comparison to p se

ME 241 Steele, C. Theory of Plates

Prereqs: CE 114 (Strength of Materials)

Math: ODE; PDE; $\Delta\varphi = 0$ $\Delta^2\varphi = p$

Approach will be mathematical

Objective → fundamental understanding & Computational capability

Method: solve problems weekly assignments

Grade: 50% HW 50% in-class Final (__not__ open book)

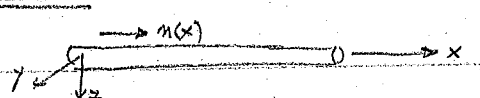
__What is a plate:__

first 3-D Structures are usually not efficient usage of structure

1-D wire/beam } one dim is >> than other 2 dimensions $EIw''' = p(x)$

2-D plate one dim is small wrt the other 2 $EI\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = p(x,y)$

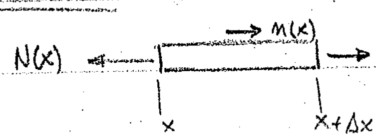
__Stretching of a rod__



$n(x)$ = distributed load / unit length

This load produces a 3-D stress. Since the length >> the radius we can define a stress resultant $N(x) = \iint \sigma_x \, dA$ to get rid of 3 dimensionality

__looking at equilibrium__



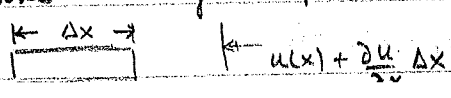
$N(x+\Delta x) = N(x) + \left(\frac{\partial N}{\partial x}\right)_x \Delta x + O(x)$

thus $\left(\frac{\partial N}{\partial x}\right)_x \Delta x + \ldots + n(x)\Delta x = 0$; taking limit as $\Delta x \to 0$ is $\boxed{\frac{dN}{dx} = -n(x)}$

if $n(x)$ is known then $\boxed{N(x) = -\int_{x_0}^{x} n(x)\,dx + N(x_0)}$

Normally we don't know $N(x_0)$ but we do know what displacements are

thus define average disp. in x direction $u(x) = \frac{1}{A}\int_A u \, dA$

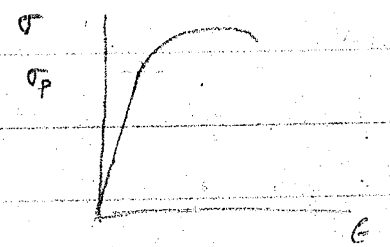


we can thus define the engineering strain $\epsilon = \dfrac{\text{final length - init length}}{\text{initial length}} = \dfrac{\frac{\partial u}{\partial x}\Delta x}{\Delta x} = \dfrac{\partial u}{\partial x}$

$$\boxed{\epsilon = \frac{\partial u}{\partial x}}$$

This is relation that defines the displacement in terms of the strain

Now we use the stress strain relation to relate the stresses to the strain



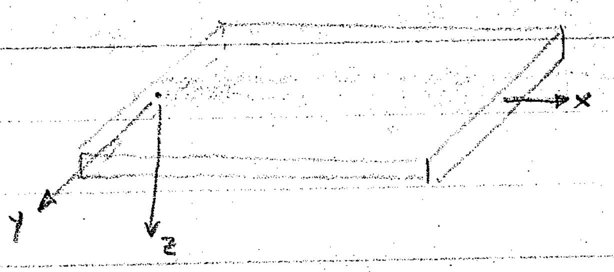
Generally $|\sigma| < |\sigma_p|$ in design - we thus use Hooke's law $\sigma = E\epsilon$ E is Young's modulus where σ is the 3-D stress

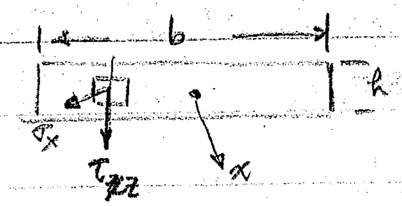
now $N(x) = \int \sigma dA = EA\epsilon$ if σ is a constant

now $\dfrac{dN}{dx} = -n(x) \Rightarrow \boxed{\dfrac{\partial}{\partial x}\left(EA\dfrac{\partial u}{\partial x}\right) = -n(x)}$

this is the relation between the distributed load and the displacement

9/28 Beams in bending



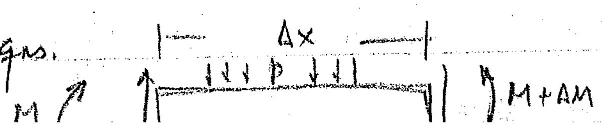


Define stress Resultants

$$Q = \int_{-h/2}^{h/2} \tau_{zx}\, dz \quad \text{force per unit width}$$

$$M = \int_{-h/2}^{h/2} \sigma_x z\, dz \quad \text{moment / unit width}$$

Equilib Eqns.



Force: $\Delta Q + p\,\Delta x \sim 0$ $\qquad p$ = load / unit width

$$\lim_{\Delta x \to 0} \frac{\Delta Q}{\Delta x} = -p \quad \text{or} \quad \boxed{\frac{dQ}{dx} = -p} \qquad (1)$$

Moment $p\Delta x \cdot \frac{\Delta x}{2} + (Q+\Delta Q)\Delta x - M - \Delta M + M \sim 0$ or $\Delta M \sim Q\Delta x$

$$\Rightarrow \boxed{\frac{dM}{dx} = Q} \qquad (2)$$

(1) & (2) $\Longrightarrow$ $\boxed{\frac{d^2M}{dx^2} = -p}$ (3)

Displacement of midsurface

w – in z-dir

0 in x-dir

For the displacement of a point a distance z from the midsurface we must make a kinematic hypothesis

For Beams – Bernoulli or Euler

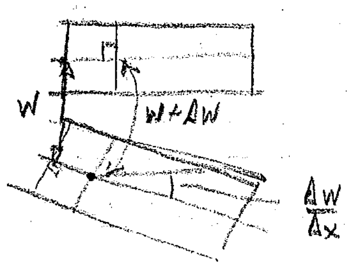
For plate – Kirchhoff

For shell – Kirchhoff, Love

Assume:

(1) A normal to the neutral surface remains straight & unstretched (plane surfaces remain plane)

(2) normals remain ⊥ to the neutral surface



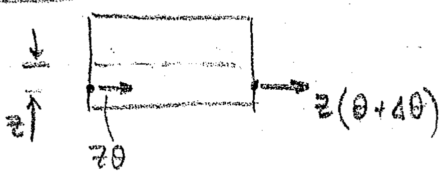
Displacement of pt a distance z from the midsurface is:

w – in z direction

zθ – in x direction

Thus rotation of the midsurface $\boxed{\theta = -\frac{dw}{dx}}$ (4)

Strains



$$\epsilon_x \cong z\frac{\Delta\theta}{\Delta x}$$

or
$$\boxed{\epsilon_x = z\frac{d\theta}{dx}} \qquad (5)$$

Using the stress strain law

$$\sigma_x = E\epsilon_x$$

$$M = \int_{-h/2}^{h/2} \sigma_x z\,dz = \int E\frac{d\theta}{dx}z^2 dz = E\frac{d\theta}{dx}\frac{h^3}{12} \qquad \text{using (5)}$$

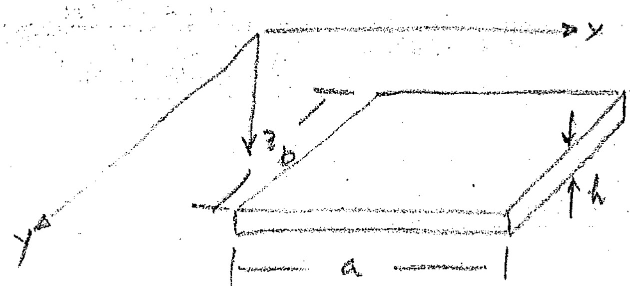
$$Mb = E\frac{d\theta}{dx}\frac{bh^3}{12} = EI\frac{d\theta}{dx} \qquad \text{for a rectangular beam}$$

or
$$\boxed{Mb = EI\left(-\frac{d^2w}{dx^2}\right)} \qquad \text{using (4)} \qquad (6)$$

but
$$\frac{d^2M}{dx^2} = -p \quad (3) \qquad \therefore \quad \boxed{pb = \frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)} \qquad (7)$$

beam bending in terms of displacement

10/1/79 Definition of a plate



Assume $h \ll a, b$

(2) plate is elastic, homog, isotropic

(3) small deflections $w \leq \frac{h}{5}$

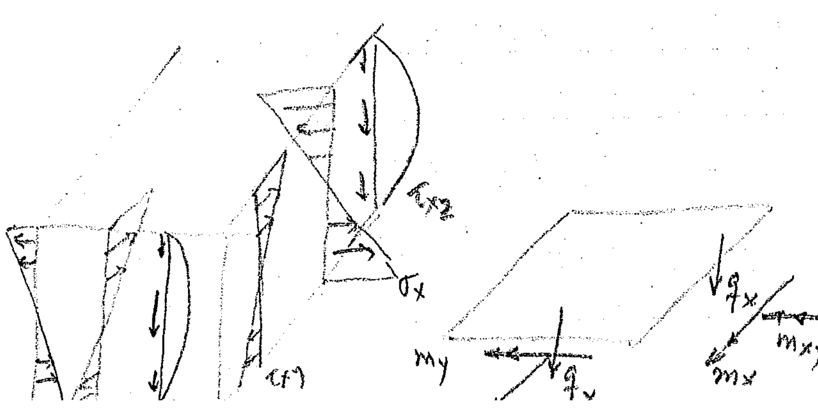
(4) $\frac{\partial w}{\partial x}$ & $\frac{\partial w}{\partial y} \ll 1$

(5) Kirchhoff Hypothesis
- normals remain straight & unstretched and normal
- $w = w(x,y)$ only

(6) Inextensional theory – no stretching

(7) no initial imperfections

(8) stresses normal to midsurface are negligible $\sigma_z \ll \sigma_x, \sigma_y$

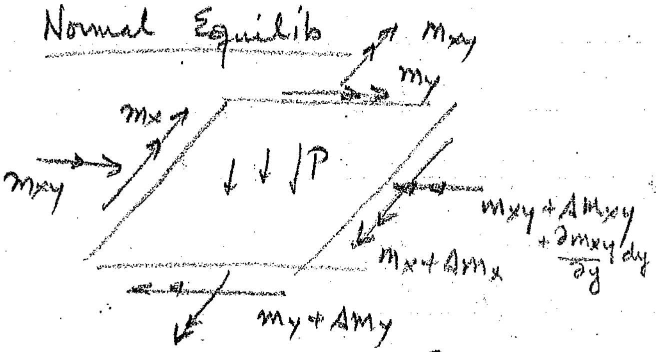


Define our stress resultants

$$q_x = \int_{-h/2}^{h/2} \tau_{xz}\,dz \qquad q_y = \int_{-h/2}^{h/2} \tau_{yz}\,dz \qquad m_x = \int_{-h/2}^{h/2} \sigma_x z\,dz \qquad m_y = \int_{-h/2}^{h/2} \sigma_y z\,dz$$

$$m_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\,dz \qquad m_{yx} = \int_{-h/2}^{h/2} \tau_{yx} z\,dz$$

Normal Equilib



$m_{xy} + \Delta m_{xy} = m_{xy} + \frac{\partial m_{xy}}{\partial x}\,dx$

$$\frac{\partial q_y}{\partial y}\Delta y \Delta x + \frac{\partial q_x}{\partial x}\Delta x \Delta y + p\Delta x \Delta y \cong 0$$

or

$$\boxed{\frac{\partial q_y}{\partial y} + \frac{\partial q_x}{\partial x} + p = 0} \qquad (1)$$

Moment Equil in x-direction

$$\frac{\partial m_y}{\partial y}\Delta x \Delta y + \frac{\partial m_{xy}}{\partial x}\Delta x \Delta y - q_y \Delta x \Delta y \cong 0$$

or

$$\boxed{\frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} - q_y = 0} \qquad (2)$$

Moment Equil in y dir:

$$\boxed{\frac{\partial m_{yx}}{\partial y} + \frac{\partial m_x}{\partial x} - q_x = 0} \qquad (3)$$

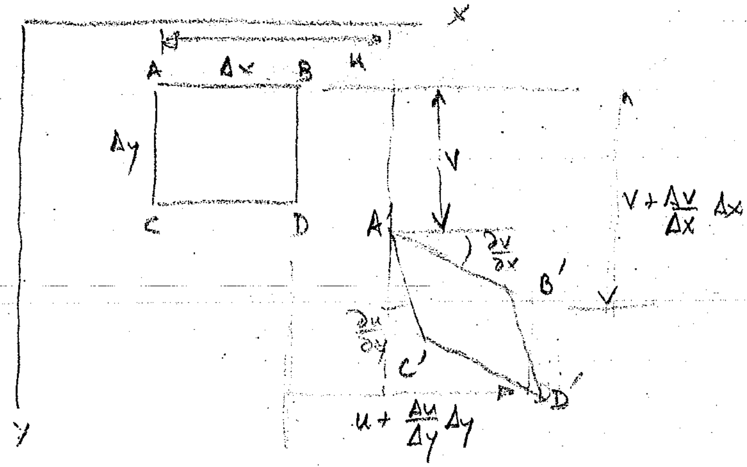
take $\frac{\partial}{\partial y}(2)$ $\frac{\partial}{\partial x}(3)$ add & substitute (1) to give

$$\boxed{\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} + P = 0} \qquad (4)$$

Displacements $\quad w = w(x, y)$

$$u = z\,\theta_x = -z\frac{\partial w}{\partial x} \qquad v = z\,\theta_y = -z\frac{\partial w}{\partial y}$$

Strains



$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

Stress - Strain relations

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

neglect $\sigma_z \ll \sigma_x, \sigma_y$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\epsilon_x + \nu\epsilon_y\right)$$

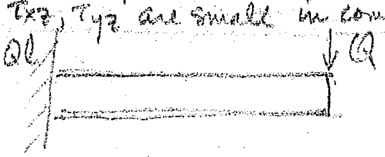
$$\sigma_y = \frac{E}{1-\nu^2}\left(\epsilon_y + \nu\epsilon_x\right)$$

obtained by inverting ϵ_x, ϵ_y for σ_x, σ_y

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

10/3/99

Show τ_{xz}, τ_{yz} are small in comparison to σ_x



$$\sigma_x = \frac{Mc}{I} = \frac{6M}{h^2} = \frac{6Mh}{2 \cdot h^3/12} = \frac{6Q}{h}\left(\frac{l}{h}\right)$$

aver $\tau_{xz} = \frac{Q}{h}$ $\Rightarrow$ $\sigma_x \gg \tau_{xz}, \tau_{yz}$ thus neglect τ_{xz}, τ_{yz}

however we do not neglect q_x, q_y

$$m_x = \int_{-h/2}^{h/2} \sigma_x z\, dz = \frac{E}{1-\nu^2}\left(\frac{\partial \theta_x}{\partial x} + \nu\frac{\partial \theta_y}{\partial y}\right)\int_{-h/2}^{h/2} z^2 dz = \boxed{\frac{-EI}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = m_x}$$

in doing this σ_x is assumed constant across cross section $D = \frac{EI}{1-\nu^2} = \frac{Eh^3}{12(1-\nu^2)}$ Bending Rigidity

Constitutive eqs

$$m_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\,dz = \frac{E}{2(1+\nu)} \int_{-h/2}^{h/2} -2\frac{\partial^2 w}{\partial x \partial y} z^2 dz = -\frac{EI}{1+\nu}\frac{\partial^2 w}{\partial x \partial y} = \boxed{-D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = m_{xy}}$$

now $q_x = m_{x,x} + m_{xy,y}$

$$= -D\left\{\frac{\partial}{\partial x}\left[w_{,xx} + \nu w_{,yy}\right] + (1-\nu)\frac{\partial}{\partial y}(w_{,xy})\right\} = -D\frac{\partial}{\partial x}(\Delta w) = \boxed{-D\frac{\partial}{\partial x}(\nabla^2 w) = q_x}$$

$\therefore$ $\boxed{q_y = -D\frac{\partial}{\partial y}(\nabla^2 w)}$

Normal equil eqn.

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + p = 0 \quad \Rightarrow \quad -D\left(\nabla^2(\nabla^2 w)\right) + p = 0$$

or $\boxed{\Delta^2 w = \nabla^4 w = p/D}$ beam bending eq in terms of displacement

Nonhomogeneous Biharmonic Eqn.

$\nabla^2 w = 0$ Laplace $\quad$ $\nabla^2 w \neq 0$ Poisson $\quad$ $\nabla^4 w = 0$ Biharmonic

Define Curvature strain measures

$$K_x = -\frac{\partial^2 w}{\partial x^2} \qquad K_y = -\frac{\partial^2 w}{\partial y^2} \qquad K_{xy} = -\frac{\partial^2 w}{\partial x \partial y}$$

Stress Resultants in terms of strain measures (Kinematic relations)

$$m_x = D(K_x + \nu K_y) \qquad m_y = D(K_y + \nu K_x) \qquad m_{xy} = D(1-\nu)K_{xy}$$

<u>Summary</u> of how we got governing eqns.

- Eqns of Equil.
- Strain Displ relations
- Stress - Strain Relations
  - * stress resultants & strain measures
- Kinematic relations

Boundary Conditions

Since 4^{th} order ODE need 4 bc. (2 on each end)

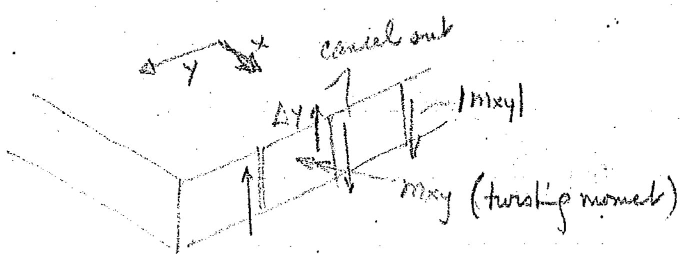
<u>Geometric BC</u>

$w, \frac{\partial w}{\partial x}$ on $x = const$

$w, \frac{\partial w}{\partial y}$ on $y = const$

- fixed edge $\quad w = 0 \quad \frac{\partial w}{\partial x} = 0 \quad$ at $x = a$

<u>Stress BC</u> $\Rightarrow$ m_x q_x & m_{xy} on $x = const.$



can be replaced by a force couple of magnitude $|m_{xy}|$. This adds to the vertical shear

Net Shear resultant on edge is $\quad v_x = q_x + \frac{\partial m_{xy}}{\partial y}$

$$v_y = q_y + \frac{\partial m_{xy}}{\partial x}$$

Thus our BC is v_x & m_x on $x = const.$

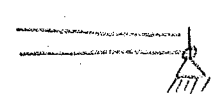
on a free edge $\quad m_x = v_x = 0$

$$v_x = f(w) = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]$$

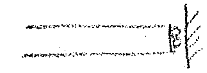
$$v_y = g(w) = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]$$

<u>Mixed boundary conditions</u>

Simply - Supported Edge $\quad w = 0, \quad m_x = 0 \quad$ at $x = const.$



$$v_x = 0 \quad \frac{\partial w}{\partial x} = 0$$



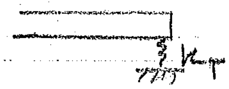
Summary

fixed $w = \frac{\partial w}{\partial x} = 0$

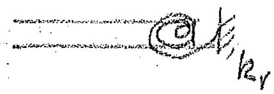
free $m_x = v_x = 0$

Simply Supported $w = 0, \ m_x = 0$

elastic support $m_x = 0 \quad w = \frac{v_x}{k_t}$



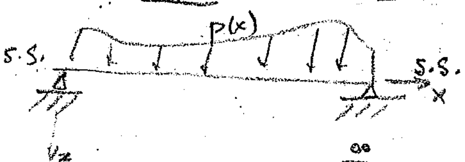
$v_x = 0 \quad \frac{\partial w}{\partial x} = \frac{m_x}{k_r}$



10/5/79

Fourier Series method of soln.

Beam $EI w^{iv} = p(x)$



simple support $w=0 \quad M=0$ or $w=0$ & $\frac{\partial^2 w}{\partial x^2} = 0$

look at sol. $w = \sum_{n=1}^{\infty} w_n \sin \frac{n\pi x}{L}$ this satisfies B.C. : any sufficiently smooth fn. can be represented by this series

put w into ODE each term satisfies BC identically & doesn't necessarily solve the ident.

$$\therefore EI \sum_{n=1}^{\infty} w_n \left(\frac{n\pi}{L}\right)^4 \sin \frac{n\pi x}{L} = p(x) \Rightarrow \frac{2EI}{L} \int_0^L \sum w_n \left(\frac{n\pi}{L}\right)^4 \sin \frac{n\pi x}{L} \sin \frac{k\pi x}{L} dx = \frac{2}{L} \int_0^L \sin \frac{k\pi x}{L} p(x) dx$$

$$\Rightarrow EI \ w_k \left(\frac{k\pi}{L}\right)^4 = \frac{2}{L} \int_0^L p(x) \sin \frac{k\pi x}{L} dx \qquad \therefore \boxed{w_k = \frac{2}{EIL} \left(\frac{L}{k\pi}\right)^4 \int_0^L p(x) \sin \frac{k\pi x}{L}}$$

$$\therefore \boxed{w_k = \frac{2}{EIL} \left(\frac{L}{k\pi}\right)^4 \int_0^L p(x) \sin \frac{k\pi x}{L} dx}$$

alternate write $p(x) = \sum p_n \sin \frac{n\pi x}{L}$ $\quad p_n = \frac{2}{L} \int_0^L p(x) \sin \frac{n\pi x}{L} dx$

$$\Rightarrow EI \left(\frac{n\pi}{L}\right)^4 w_n = p_n \Rightarrow \boxed{w_n = \frac{p_n}{EI} \left(\frac{L}{n\pi}\right)^4}$$

very good convergence in comparison to p series

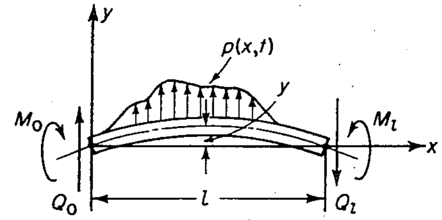


Fig. 11.2:1. Applications to a beam.

The equation of motion is given by Hamilton's principle:

$$\delta \int_{t_0}^{t_1} (U - K + A)\, dt = 0; \tag{4}$$

i.e.,

$$\delta \int_{t_0}^{t_1} \left\{ \int_0^l \left[\frac{1}{2} EI \left(\frac{\partial^2 y}{\partial x^2}\right)^2 - \frac{1}{2} m \left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2} I_\rho \left(\frac{\partial^2 y}{\partial x\, \partial t}\right)^2 - py \right] dx - M_l \left(\frac{\partial y}{\partial x}\right)_l + M_0 \left(\frac{\partial y}{\partial x}\right)_0 + Q_l y_l - Q_0 y_0 \right\} dt = 0. \tag{5}$$

Following the usual procedure of the calculus of variations, noting that the virtual displacement must be so specified that $\delta y \equiv 0$ at t_0 and t_1, and, hence, $\partial(\delta y)/\partial x = \delta(\partial y/\partial x) \equiv 0$ at t_0 and t_1, we obtain

$$\int_{t_0}^{t_1} \left[\int_0^l \left(EI \frac{\partial^2 y}{\partial x^2} \frac{\partial^2\, \delta y}{\partial x^2} - m \frac{\partial y}{\partial t} \frac{\partial\, \delta y}{\partial t} - I_\rho \frac{\partial^2 y}{\partial x\, \partial t} \frac{\partial^2\, \delta y}{\partial x\, \partial t} - p\, \delta y \right) dx - M_l\, \delta\left(\frac{\partial y}{\partial x}\right)_l + M_0\, \delta\left(\frac{\partial y}{\partial x}\right)_0 + Q_l\, \delta y_l - Q_0\, \delta y_0 \right] dt = 0.$$

Integrating by parts, we obtain

$$\int_{t_0}^{t_1} \int_0^l \left[\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) + m \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2}{\partial x\, \partial t}\left(I_\rho \frac{\partial^2 y}{\partial x\, \partial t}\right) - p(x, t) \right] \delta y\, dx\, dt - \int_{t_0}^{t} \left[EI \frac{\partial^2 y}{\partial x^2} - M \right] \delta\left(\frac{\partial y}{\partial x}\right)\Big|_0^l\, dt - \int_{t_0}^{t} \left[\frac{\partial}{\partial x}\left(EI \frac{\partial^2 y}{\partial x^2}\right) - \frac{\partial}{\partial t}\left(I_\rho \frac{\partial^2 y}{\partial x\, \partial t}\right) - Q \right] \delta y \Big|_0^l\, dt = 0. \tag{6}$$

Hence, the Euler equation of motion is

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) + m \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2}{\partial x\, \partial t}\left(I_\rho \frac{\partial^2 y}{\partial x\, \partial t}\right) = p(x, t), \tag{7}$$

and a proper set of boundary conditions at each end is

$$\text{either} \quad EI \frac{\partial^2 y}{\partial x^2} = M \quad \text{or} \quad \delta\left(\frac{\partial y}{\partial x}\right) = 0 \tag{8a}$$

and

$$\text{either} \quad \frac{\partial}{\partial x}\left(EI \frac{\partial^2 y}{\partial x^2}\right) - \frac{\partial}{\partial t}\left(I_\rho \frac{\partial^2 y}{\partial x\, \partial t}\right) = Q \quad \text{or} \quad \delta y = 0. \tag{8b}$$

These are equations governing the motion of a beam including the effect of the rotary inertia, due to Lord Rayleigh, and known as ***Rayleigh's equations.*** If the rotary inertia is neglected and if the beam were uniform, then the governing equation is simplified into:

$$\frac{\partial^2 y}{\partial t^2} + c_0^2 R^2 \frac{\partial^4 y}{\partial x^4} = \frac{1}{EI} p, \tag{9}$$

where

$$c_0^2 = \frac{E}{\rho}, \qquad R^2 = \frac{I}{A}. \tag{10}$$

The constant c_0 has the dimension of speed and can be identified as the phase velocity of longitudinal waves in a uniform bar.† R is the radius of gyration of the cross section. A is the cross-sectional area, so that $m = \rho A$.

In the special case of a uniform beam of infinite length free from lateral loading, $p = 0$, Eq. (9) becomes

$$\frac{\partial^2 y}{\partial t^2} + c_0^2 R^2 \frac{\partial^4 y}{\partial x^4} = 0. \tag{11}$$

It admits a solution in the form

$$y = a \sin \frac{2\pi}{\lambda}(x - ct), \tag{12}$$

which represents a progressive wave of phase velocity c and wave length λ. On substituting (12) into (11), we obtain the relation

$$c = \pm c_0 R \frac{2\pi}{\lambda}, \tag{13}$$

which states that the phase velocity depends on the wave length and that it tends to infinity for very short wave lengths. Somewhat disconcerting is the fact that, according to Eq. (13), the group velocity (see Sec. 11.3) also tends to infinity as the wave length tends to zero. Since group velocity is the velocity at which energy is transmitted, this result is physically unreasonable. If Eq. (13) were correct, then the effect of a suddenly applied concentrated load will be felt at once everywhere in the beam, as the Fourier representation for a concentrated load contains harmonic components with infinitesimal wave length, and hence infinite wave speed. Thus, Eq. (11) cannot be very accurate in describing the effect of impact loads on a beam.

† See Prob. 11.2, p. 325.

This difficulty of infinite wave speed is removed by the inclusion of the rotary inertia. However, the speed versus wave length relationship obtained from Rayleigh's Eq. (7) for a uniform beam of circular cross section with radius a, as is shown in Fig. 11.2:2, still deviates appreciably from Pochhammer and Chree's results, which were derived from the exact three-dimensional linear elasticity theory. A much better approximation is obtained by including the shear deflection of the beam, as was first shown by Timoshenko.

To incorporate the shear deformation, we note that the slope of the deflection curve depends not only on the rotation of cross sections of the

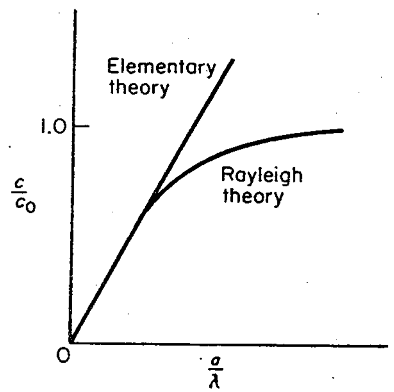


Fig. 11.2:2. Phase velocity curves for flexural elastic waves in a circular cylinder of radius a.

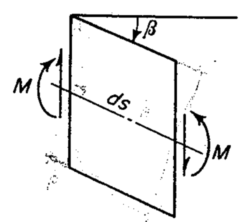


Fig. 11.2:3. A Timoshenko beam element.

beam but also on the shear. Let ψ denote the slope of the deflection curve when the shearing force is neglected and β the angle of shear at the neutral axis in the same cross section. Then the total slope is

$$\frac{\partial y}{\partial x} = \psi + \beta. \tag{14}$$

The strain energy due to bending, Eq. (1), must be replaced by

$$\frac{1}{2}\int_0^l EI\left(\frac{\partial \psi}{\partial x}\right)^2 dx, \tag{15}$$

because the internal bending moment does no work when shear deformation takes place (see Fig. 11.2:3). The strain energy due to shearing strain β must be a quadratic function of β if linear elasticity is assumed. We shall write

$$\frac{1}{2}\int_0^l k\beta^2\, dx = \frac{1}{2}\int_0^l k\left(\frac{\partial y}{\partial x} - \psi\right)^2 dx \tag{16}$$

for the strain energy for shear. The kinetic energy is

$$K = \frac{1}{2}\int_0^l m\left(\frac{\partial y}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^l I_\rho\left(\frac{\partial \psi}{\partial t}\right)^2 dx, \tag{17}$$

because the translational velocity is $\partial y/\partial t$, but the angular velocity is $\partial \psi/\partial t$. Hence, Hamilton's principle states that

$$\delta\int_{t_0}^{t_1}\int_0^l \frac{1}{2}\left[EI\left(\frac{\partial \psi}{\partial x}\right)^2 + k\left(\frac{\partial y}{\partial x} - \psi\right)^2 - m\left(\frac{\partial y}{\partial t}\right)^2 - I_\rho\left(\frac{\partial \psi}{\partial t}\right)^2\right] dx\, dt + \delta A = 0, \tag{18}$$

where A is given by (3) except that $\partial y/\partial x$ at the ends is to be replaced by ψ. The virtual displacements now consist of δy and $\delta \psi$, which must vanish at t_0 and t_1 and also where displacements are prescribed. On carrying out the calculations, the following two Euler equations are obtained:

$$\frac{\partial}{\partial x}\left(EI\frac{\partial \psi}{\partial x}\right) + k\left(\frac{\partial y}{\partial x} - \psi\right) - I_\rho\frac{\partial^2 \psi}{\partial t^2} = 0, \tag{19a}$$

$$m\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left[k\left(\frac{\partial y}{\partial x} - \psi\right)\right] - p = 0. \tag{19b}$$

The appropriate boundary conditions are, at each end of the beam,

$$\text{Either } -EI\frac{\partial \psi}{\partial x} = M \quad \text{or} \quad \delta\psi = 0, \tag{20a}$$

and

$$\text{either } k\left(\frac{\partial y}{\partial x} - \psi\right) = Q \quad \text{or} \quad \delta y = 0. \tag{20b}$$

These are the differential equation and boundary conditions of the so-called *Timoshenko beam theory*.

For a uniform beam, EI, k, m, etc., are constants, and the function ψ can be eliminated from the equations above to obtain the well-known *Timoshenko equation for lateral vibration of prismatic beams*,

$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} - \left(I_\rho + \frac{EIm}{k}\right)\frac{\partial^4 y}{\partial x^2\,\partial t^2} + I_\rho\frac{m}{k}\frac{\partial^4 y}{\partial t^4} = p + \frac{I_\rho}{k}\frac{\partial^2 p}{\partial t^2} - \frac{EI}{k}\frac{\partial^2 p}{\partial x^2}. \tag{21}$$

So far we have not discussed the constants m, I_ρ, and k. For a beam of uniform material, $m = \rho A$, $I_\rho = \rho A R^2$, where ρ is the mass density of the beam material, A is the cross-sectional area, and R is the radius of gyration

of the cross section about an axis perpendicular to the plane of motion and through the neutral axis. But k depends on the distribution of shearing stress in the beam cross section. Timoshenko writes

$$k = k'AG, \tag{22}$$

where G is the shear modulus of elasticity and k' is a numerical factor depending on the shape of the cross section, and ascertains that according to the elementary beam theory, $k' = \frac{2}{3}$ for a rectangular cross section. The use of such a value of k is, however, a subject of controversy in the literature. Mindlin[11.1] suggests that the value of k can be so selected that the solution of Eq. (21) be made to agree with certain solution of the exact three-dimensional equations of Pochhammer (1876) and Chree (1889) (see Love,[1.2] *Elasticity*, 4th ed., pp. 287–92). Indeed, I_ρ, which arises in the assumption of plane sections remain plane in bending, may also be regarded, when such an assumption is relaxed, as an empirical factor to be determined by comparison with exact solutions.

For a uniform beam free from lateral loadings, Eq. (21) can be written as

$$\frac{\partial^4 y}{\partial x^4} - \left(\frac{1}{c_0^2} + \frac{1}{c_Q^2}\right)\frac{\partial^4 y}{\partial x^2\,\partial t^2} + \frac{1}{c_0^2 c_Q^2}\frac{\partial^4 y}{\partial t^4} + \frac{1}{c_0^2 R^2}\frac{\partial^2 y}{\partial t^2} = 0, \tag{23}$$

where

$$c_0^2 = \frac{E}{\rho}, \qquad c_Q^2 = \frac{k'G}{\rho}, \qquad R^2 = \frac{I}{A}. \tag{24}$$

If the beam is of infinite length, a solution of the form (12) may be substituted into (23), and we see that the wave speed c must satisfy the equation

$$1 - \left(\frac{c^2}{c_0^2} + \frac{c^2}{c_Q^2}\right) + \frac{c^4}{c_0^2 c_Q^2} - \frac{c^2}{c_0^2 R^2}\left(\frac{\lambda}{2\pi}\right)^2 = 0. \tag{25}$$

The solution of this equation for c/c_0 versus λ yields two branches, corresponding to two "modes" of motion (two different shear-to-bending deflection ratios for the same wavelength). They are plotted in Fig. 11.2:4 for the special case of a beam of circular cross section with radius a. The results of the exact solution of Pochhammer and Chree for Poisson's ratio $\nu = 0.29$ are also plotted there for comparison. It is seen that the Timoshenko theory agrees reasonably well with the exact theory in the first mode, but wide discrepancy occurs in the second mode. The approximate theory gives no information about higher modes: an infinite number of which exist in the exact theory.

The equations derived above are, of course, appropriate for the determination of the free-vibration modes and frequencies of a beam. The effects of rotary inertia and shear are unimportant if the wavelength of the vibration mode is large compared with the cross-sectional dimensions of the beam; but these effects become more and more important with a decrease of wavelength, i.e., with an increase in the frequency of vibration. In the example of a uniform beam with rectangular cross section and simply supported at both ends, with $E = \frac{8}{3}G$ and $k' = \frac{2}{3}$, we find that the shear deflection and rotary inertia reduce the natural frequencies. If the wavelength

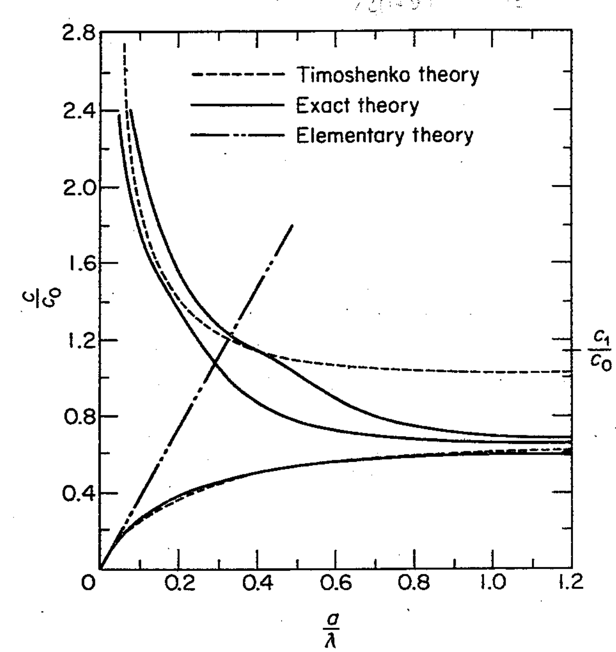


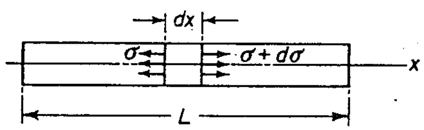
Fig. 11.2:4. Phase velocity curves for flexural elastic waves in a solid circular cylinder of radius a. (From Abramson,[11.1] *J. Acoust. Soc. Am.*, 1957.)

is ten times larger than the depth of the beam, the correction on the frequency due to rotary inertia alone is about 0.4 per cent, and the correction due to rotary inertia and shear together will be about 2 per cent.

The Timoshenko beam theory has attracted much attention in recent years. For a survey of literature, see Abramson, Plass, and Ripperger.[11.1]

PROBLEMS

11.2. Consider the free longitudinal vibration of a rod of uniform cross section and length L, as shown in Fig. P11.2. Let us assume that plane cross sections remain plane, that only axial stresses are present, being uniformly distributed over the cross section, and that radial displacements are negligible (i.e., the displacements consist of only one nonvanishing component u in the x-direction). Derive



P11.2. Longitudinal vibration of a rod.

expressions for the potential and kinetic energy and show that the equation of motion is

(26) $$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} = 0, \qquad c_0^2 = \frac{E}{\rho}.$$

Show that the general solution is of the form

(27) $$u = f(x - c_0 t) + F(x + c_0 t),$$

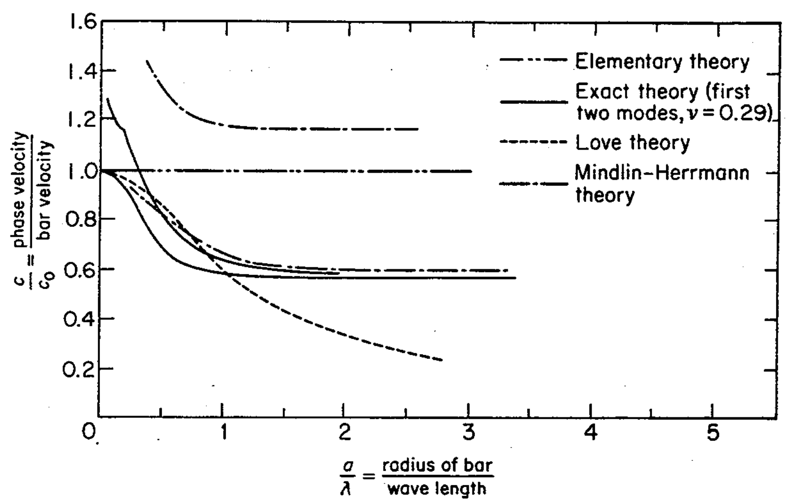
where f and F are two arbitrary functions.

11.3. Consider the same problem as above, but now incorporate approximately the transverse inertia associated with the lateral expansion or contraction connected with axial compression and extension, respectively. Let the (Love's) assumption be made that the displacement in the radial direction v is proportional to the radial coordinate r, measured from a centroidal axis, and to the axial strain $\partial u/\partial x$; i.e.,

(28) $$v = -\nu r \frac{\partial u}{\partial x},$$

where ν is Poisson's ratio. Derive expressions of the kinetic and potential energy and obtain the equation of motion according to Hamilton's principle,

(29) $$\rho\left[\frac{\partial^2 u}{\partial t^2} - (\nu R)^2 \frac{\partial^4 u}{\partial x^2\,\partial t^2}\right] - E\frac{\partial^2 u}{\partial x^2} = 0,$$



P11.3. Phase velocity curves for longitudinal elastic waves in a solid circular cylinder of radius a. (After Abramson et al., *Adv. Applied Mech.*, 5, 1958.)

where R is the polar radius of gyration of the cross section. The natural boundary condition at the end $x = 0$, if that end is subject to a stress $\sigma_0(t)$, is

(30) $$\rho\nu^2 R^2 \frac{\partial^3 u}{\partial x\,\partial t^2} + E\frac{\partial u}{\partial x} = \sigma_0(t) \qquad \text{at } x = 0.$$

Note. It is important to note that, according to the last equation, the familiar proportionality between axial stress σ and axial strain $\partial u/\partial x$ does not exist in this theory.

Comparison of the dispersion curves obtained from the elementary theory (Prob. 11.2), the Love theory, the Pochhammer-Chree "exact" theory, and another approximate theory due to Mindlin and Herrmann,[11.1] are shown in Fig. P11.3 The last-mentioned theory accounts for the strain energy associated with the transverse displacement v, of which the most important contribution comes from the shearing strain caused by the lateral expansion of the cross section near a wave front.

11.4. The method of derivation of the various forms of equations of motion of beams as presented above has the advantage of being straightforward, but it does not convey the physical concepts as clearly as in an elementary derivation. Hence, rederive the basic equations by considering the forces that act on an element of length dx, as shown in Fig. P11.2 and Fig. P11.4. Obtain the following equations, and then derive the wave equations by proper reductions.

Longitudinal waves, elementary theory (Fig. P11.2):

(31) $$\frac{\partial \sigma_{xx}}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \quad \text{(equation of motion)},$$

(32) $$\frac{\partial^2 u}{\partial x\,\partial t} = \frac{\partial \epsilon_{xx}}{\partial t} \quad \text{(equation of strain)},$$

(33) $$\sigma_{xx} = E\epsilon_{xx} \quad \text{(equation of material behavior)},$$

where σ_{xx} = axial stress, ϵ_{xx} = axial strain, $\partial u/\partial t$ = axial particle velocity, x = axial coordinate, t = time, E = modulus of elasticity, and ρ = mass density.

Flexural waves, Timoshenko theory (Fig. P11.4, p. 328):

(34) $$\frac{\partial M}{\partial x} - Q = \rho I \frac{\partial \omega}{\partial t} \quad \text{(rotational)}$$

(35) $$\frac{\partial Q}{\partial x} = \rho A \frac{\partial v}{\partial t} \quad \text{(transverse)}$$

(equations of motion),

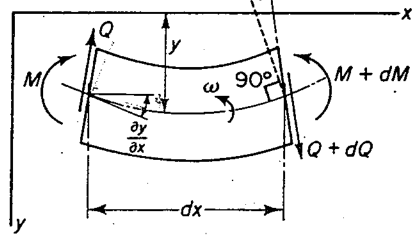
(36) $$\frac{\partial K}{\partial t} = \frac{\partial \omega}{\partial x} \quad \text{(bending)}$$

(37) $$\frac{\partial \beta}{\partial t} = \frac{\partial v}{\partial x} + \omega \quad \text{(shear)}$$

(38) $$M = EIK \quad \text{(bending)}$$

(39) $$Q = A_s\beta \quad \text{(shear)}$$

(equations of material behavior),



P11.4. Element of a beam in bending.

where M = moment, Q = shear force, K = axial rate of change of section angle = $-\partial\psi/\partial x$, β = shear strain = $\partial y/\partial x - \psi$, ω = angular velocity of section = $-\partial\psi/\partial t$, v = transverse velocity = $\partial y/\partial t$, I = section moment of inertia, A = section area, and A_s = area parameter defined by $\iint \gamma(z)\, dA = \beta A_s$ where $\gamma(z)$ is the shear strain at a point z in the cross section.

11.3. GROUP VELOCITY

Since we have been concerned in the preceding sections about wave propagations in beams, it seems appropriate to make a digression to explain the concept of *group velocity* as distinguished from the *phase velocity*. We have seen that for certain equations a solution of the following form exists:

(1) $$u = a \sin(\mu x - \nu t).$$

If x is increased by $2\pi/\mu$, or t by $2\pi/\nu$, the sine takes the same value as before, so that $\lambda = 2\pi/\mu$ is the wavelength and $T = 2\pi/\nu$ is the period of oscillation. If $\mu x - \nu t$ = constant, i.e. x = const. + $\nu t/\mu$, the argument of the sine function remains constant in time; which means that the whole waveform is displaced towards the right with a velocity $c = \nu/\mu$. The quantity c is called the phase velocity, in terms of which Eq. (1) may be exhibited as

(2) $$u = a \sin \frac{2\pi}{\lambda}(x - ct).$$

If the phase velocity c depends on the wavelength λ, the wave is said to exhibit *dispersion*. Our examples in the previous section show that dispersion exists in both longitudinal and flexural waves in rods and beams.

What happens when two sine waves of the same amplitude but slightly different wavelengths and frequencies are superposed? Let these two waves be characterized by two sets of slightly different values μ, ν and μ', ν'. The resultant of the superposed waves is

$$u + u' = A[\sin(\mu x - \nu t) + \sin(\mu' x - \nu' t)].$$

Using the well-known formula

$$\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta),$$

we have

(3) $$u + u' = 2A \sin[\tfrac{1}{2}(\mu+\mu')x - \tfrac{1}{2}(\nu+\nu')t]\cos[\tfrac{1}{2}(\mu-\mu')x - \tfrac{1}{2}(\nu-\nu')t].$$

This expression represents the well-known phenomenon of "beats." The sine factor represents a wave whose wave number and frequency are equal to the mean of μ, μ' and ν, ν', respectively. The cosine factor, which varies very slowly when $\mu - \mu'$, $\nu - \nu'$ are small, may be regarded as a varying amplitude,

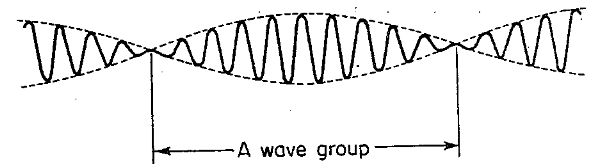


Fig. 11.3:1. An illustration of a wave group.

as shown in Fig. 11.3:1. The "wave group" ends wherever the cosine becomes zero. The velocity of advance of these points is called the *group velocity*; its value U is equal to $(\nu - \nu')/(\mu - \mu')$. For long groups (or slow beats), the group velocity may be written with sufficient accuracy as

(4) $$U = \frac{d\nu}{d\mu}.$$

In terms of the wavelength $\lambda\ (= 2\pi/\mu)$, we have

(5) $$U = \frac{d(\mu c)}{d\mu} = c - \lambda\frac{dc}{d\lambda},$$

where c is the phase velocity.

From the fact that no energy can travel past the nodes, one can infer that the rate of transfer of energy is identical with the group velocity. This fact is capable of rigorous proof for single trains of waves.

The most familiar examples of propagation of wave groups are perhaps the water waves. It has often been noticed that when an isolated group of waves, of sensibly the same length, advancing over relatively deep water, the velocity of the group as a whole is less than that of the individual waves composing it. If attention is fixed on a particular wave, it is seen to advance through the group, gradually dying out as it approaches the front, while its former place in the group is occupied in succession by other waves which have come forward from the rear. Another familiar example is the wave train set up by ships. The explanation as presented above seems to have been first given by Stokes (1876). Other derivations and interpretations of

$$m_R = \sqrt{(-0.005017)^2 + (-0.001742)^2} = 0.005311 \text{ kg} = 5.311 \text{ g}$$

$$\theta_R = \tan^{-1}\left(-0.001742 / -0.005017\right) = 19.1480^\circ + 180^\circ = 199.1480^\circ$$

$$\vec{B}_L = -\vec{R}_A = 28.4021\ \vec{j} + 3.5436\ \vec{k} = m_L\, r\, \omega^2 (\cos\theta_L\ \vec{j} + \sin\theta_L\ \vec{k})$$

i.e.

$$m_L \cos\theta_L = \frac{28.4021}{(0.25)(104.72)^2} = 0.01036$$

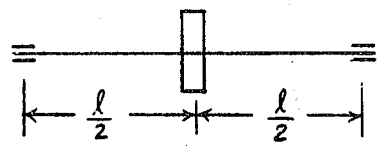
$$m_L \sin\theta_L = \frac{3.5436}{(0.25)(104.72)^2} = 0.001293$$

$$m_L = \sqrt{(0.01036)^2 + (0.001293)^2} = 0.01044 \text{ kg} = 10.44 \text{ g}$$

$$\theta_L = \tan^{-1}\left(0.001293 / 0.01036\right) = 7.1141^\circ$$

Note: Angles are measured clockwise from z-axis while looking from A towards B.

9.13 Stiffness of steel shaft between bearings $= k = \left\{\frac{48EI}{l^3}\right\}$



$$k = \frac{48(30\times10^6)}{(30)^3}\left(\frac{\pi}{64}(1)^4\right) = 2618 \text{ lb/in.}$$

(a) Critical speed $= \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2618\ (386.4)}{100}} = 100.5781$ rad/sec

(b) Vibration amplitude of the rotor (steady state value):

Eq. (9.32) gives the amplitude in x-direction as

$$X = D\, m\omega^2 e \quad \text{when damping is zero} = \frac{m\omega^2 e}{|k - m\omega^2|}$$

Here $\omega = 1200$ rpm $= 1200\,(2\pi)/60 = 125.664$ rad/sec

$$e = 0.5'', \quad m\omega^2 = \frac{100}{386.4}(125.664)^2 = 4086.8118$$

$$X = \left(\frac{100}{386.4}\right)(125.664)^2(0.5)\frac{1}{|2618 - 4086.8118|} = 1.3912''$$

Similarly the amplitude in y-direction is given by

$$Y = X = 1.3912''$$

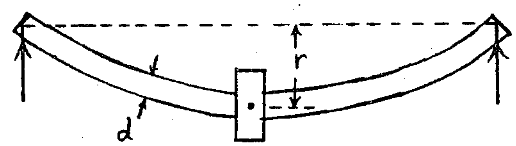
Resultant amplitude of the flywheel $= R = \sqrt{X^2 + Y^2} = 1.9675''$

(c) Force transmitted to the bearing supports

$$= k\,R = 2618\ (1.9675) = 5150.915 \text{ lb}$$

9.14 Considering bearings as simple supports, the spring constant of the beam is $k = \frac{48EI}{\ell^3}$ where ℓ = distance between bearings.

Let r = variable position of center of mass, and
δ_{st} = static radial displacement of center of mass.



Then equation of motion is

$$m r \omega^2 = k (r - \delta_{st}) \quad \text{or} \quad \frac{k}{k - m\omega^2} = \frac{r}{\delta_{st}} \quad \text{or} \quad r = \frac{k \cdot \delta_{st}}{k - m\omega^2}$$

$$\text{Dynamic force} = F = m r \omega^2 = \frac{m \omega^2 k \delta_{st}}{k - m\omega^2}$$

Since F acts at the middle of the beam, $\sigma_{max} = \frac{M \cdot y_{max}}{I} = \frac{F\ell}{4} \cdot \frac{d}{2I}$

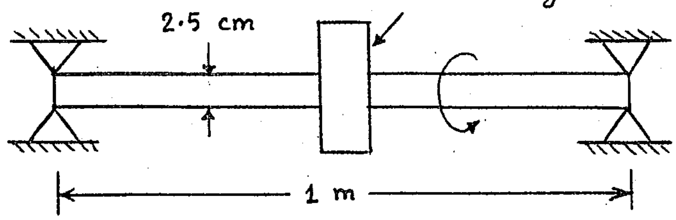
where d = diameter of shaft.

$$\sigma_{max} = \frac{F \ell d}{8I} = \frac{m \omega^2 k \, \delta_{st} \, \ell \, d}{8I (k - m\omega^2)}$$

Substituting the expression for k,

$$\sigma_{max} = \frac{m\omega^2 \delta_{st} \, \ell \, d}{8I} \left(\frac{48EI}{\ell^3} \right) \left\{ \frac{1}{\left(\frac{48EI}{\ell^3} - m\omega^2 \right)} \right\}$$

9.15



Stiffness of a simply supported beam:

$$k = \frac{48 \, EI}{\ell^3} = \frac{48 \, (207 \, (10^9)) \left| \frac{\pi}{64} \, (0.025^4) \right|}{1^3} = 19.0521 \, (10^4) \text{ N/m}$$

Natural frequency of the system:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{19.0521 \, (10^4)}{20}} = 97.6014 \text{ rad/sec}$$

$$\text{Frequency of rotor (speed of shsft):} \quad \omega = \frac{6000}{60} \, (2\,\pi) = 628.32 \text{ rad/sec}$$

$$\text{Whirl amplitude of the disc:} \quad A = \frac{a \, r^2}{\sqrt{(1 - r^2)^2 + (2 \, \varsigma \, r)^2}}$$

(a) At operating speed:

$$r = \frac{\omega}{\omega_n} = \frac{628.32}{97.6014} = 6.4376$$

$$A = \frac{(0.005)\,(6.4376^2)}{\sqrt{(1-6.4376^2)^2 + (2\,(0.01)\,(6.4376))^2}} = 0.005124 \text{ m}$$

(b) At critical speed (Eq. 9.34):

Critical speed:

$$\omega = \omega_{cri} = \frac{\omega_n}{\left\{1 - \frac{1}{2}\,(\frac{c}{\omega_n})^2\right\}^{\frac{1}{2}}} = \frac{97.6014}{\left\{1 - \frac{1}{2}\,(\frac{39.0406}{97.6014})^2\right\}^{\frac{1}{2}}} = 101.7565 \text{ rad/sec}$$

where $c = 2\sqrt{k\,m}\,\varsigma = 2\sqrt{(19.0521\,(10^4))\,(20)}\,(0.01) = 39.0406$ N–s/m.

$$r = \frac{\omega}{\omega_n} = \frac{101.7565}{97.6014} = 1.0426$$

$$A = \frac{(0.005)\,(1.0426^2)}{\sqrt{(1-1.0426^2)^2 + (2\,(0.01)\,(1.0426))^2}} = 0.06074 \text{ m}$$

(c) At 1.5 times critical speed:

$$r = \frac{1.5\,\omega_{cri}}{\omega_n} = \frac{152.6347}{97.6014} = 1.5638$$

$$A = \frac{(0.005)\,(1.5638^2)}{\sqrt{(1-1.5638^2)^2 + (2\,(0.01)\,(1.5638))^2}} = 0.008457 \text{ m}$$

9.16 (a) At operating speed:

$r = 6.4376$; $\omega = 628.32$ rad/sec. Deflection of mass center:

$$R = a\left\{\frac{1 + (2\,\varsigma\,r)^2}{(1-r^2)^2 + (2\,\varsigma\,r)^2}\right\}^{\frac{1}{2}}$$

$$= (0.005)\left\{\frac{1 + (2\,(0.01)\,(6.4376))^2}{(1-6.4376^2)^2 + (2\,(0.01)\,(6.4376))^2}\right\}^{\frac{1}{2}} = 1.2465\,(10^{-4}) \text{ m}$$

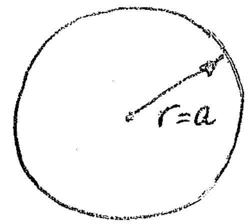
Centrifugal force: $m\,\omega^2\,R = (20)\,(628.32^2)\,(1.2465\,(10^{-4})) = 984.2015$ N

Bearing reactions: $R_1 = R_2 = \frac{m\,\omega^2\,R}{2} = 492.1007$ N

$$I = \frac{\pi}{64}\,d^4 = \frac{\pi}{64}\,(0.025^4) = 1.9175\,(10^{-8}) \text{ m}^4$$

ON THURSDAY WE SOLVED

$$\nabla^2 W - \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = 0$$ IN A CIRCULAR REGION



with $W(r,\theta,t) = W$

AND $W(r=a,\theta,t) = 0$

By writing $W = w(r,\theta)\cdot T(t)$ we can separate spatial & temporal functions

$$\nabla^2 W = \frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}$$

THUS $$\nabla^2 W - \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = T\nabla^2 w - \frac{1}{c^2} w \ddot{T} = 0$$

or $$\frac{c^2 \nabla^2 w}{w} = \frac{\ddot{T}}{T} = -\omega^2 \quad \Rightarrow \quad T = \mathcal{E}_1 \cos\omega t + \mathcal{E}_2 \sin\omega t$$

ω = frequency of vibration

$$\Rightarrow \nabla^2 w + \left(\frac{\omega}{c}\right)^2 w = 0 \qquad \text{let } \frac{\omega}{c} = \lambda$$

or $\nabla^2 w + \lambda^2 w = 0$ this is an eigenvalue problem

TO SOLVE EIGENVALUE PROBLEM, LET $w(r,\theta) = R(r)\,\Theta(\theta)$

$$\therefore \nabla^2 w + \lambda^2 w = (R'' + \tfrac{1}{r}R')\Theta + \frac{1}{r^2}\Theta'' + \lambda^2 R\Theta = 0$$

$$\Rightarrow \frac{r^2(R'' + \frac{1}{r}R') + \lambda^2 r^2}{R} = -\frac{\Theta''}{\Theta} = +k^2$$ SINCE Θ FUNCTION MUST BE PERIODIC

$$\Rightarrow \Theta = A\cos k\theta + B\sin k\theta$$

$$\Rightarrow r^2 R'' + rR' + (\lambda^2 r^2 - k^2)R = 0$$

BESSEL EQN OF FORM $x^2 y'' + xy' + (x^2 - p^2)y = 0$

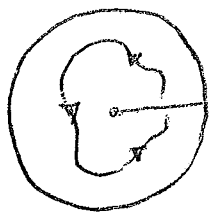
IF $x = \lambda r$
$R = y$
$p = k$

SOLUTIONS ARE

$y = C_1 J_p(x) + C_2 J_{-p}(x)$ IF $p \neq 0$ or an integer

$y = C_1 J_n(x) + C_2 Y_n(x)$ IF p is zero or an integer

- TO DETERMINE IF k IS integer $W(r,\theta,t) = W(r,\theta+2\pi,t) \Rightarrow w(r,\theta) = w(r,\theta+2\pi) \Rightarrow$



$\Theta \Rightarrow \Theta(\theta + 2k\pi) = \Theta(\theta) \Rightarrow \boxed{k=n}$

$\therefore R(\lambda r) = C_1 J_n(\lambda r) + C_2 Y_n(\lambda r)$

- NOW $J_n(\lambda r)$ IS BOUNDED AT $r=0$

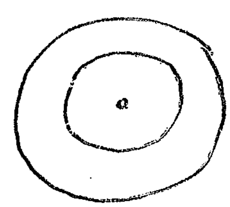
  $Y_n(\lambda r)$ IS NOT BOUNDED AT $r=0$

  PHYSICAL PROBLEM DICTATES THAT $W(r,\theta,t)$ IS BOUNDED AT $r=0$

  $\Longrightarrow$ MUST TAKE $\boxed{C_2 = 0}$ SINCE $Y_n(\lambda r)$ CONTAINS $\log(\lambda r)$ TERM

  $\Longrightarrow$ NOTE $J_{-p}(x)$ IS NOT BOUNDED AT $x=0$ EITHER

  $\Longrightarrow$ FOR AN ANNULAR MEMBRANE ORIGIN NOT INCLUDED THUS $Y_n(\lambda r)$ IS kept



- LET'S LOOK AT HOW TO HANDLE $W(r=a,\theta,t)=0$ IRRESPECTIVE OF t & θ

  $W(r=a,\theta,t) = w(r=a,\theta,t) = R(\lambda r=\lambda a)\Theta(\theta)T(t) = 0$

  $\Longrightarrow R(\lambda r=\lambda a)=0$ THIS IS THE WAY TO FIND THE λ's

SO FAR $w(r,\theta) = J_n(\lambda r)\left[\bar{A}\cos n\theta + \bar{B}\sin n\theta\right]$

SINCE TRUE FOR ANY n & EQ $(\nabla^2 w + \lambda^2 w = 0)$ IS LINEAR

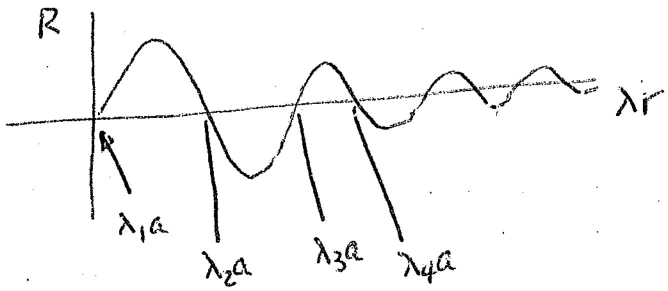
SO $w(r,\theta)$ DEPENDS ON n $\Rightarrow w_n(r,\theta)$

AND

$$w(r,\theta) = \sum_n w_n(r,\theta) = \sum_n J_n(\lambda r)\left[\bar{A}_n \cos n\theta + \bar{B}_n \sin n\theta\right]$$

$\Rightarrow$ SINCE $R(\lambda a) = 0 \Rightarrow J_n(\lambda a) = 0$

FOR any n



FOR $J_0(\lambda r)$ & $J_1(\lambda r)$, & $J_2(\lambda r)$ ...

$\Rightarrow$ THERE ARE AN INFINITE NO. OF VALUES

$\therefore$ WE MUST NUMBER THE ZEROES OF J_0, J_1, J_2 etc. AND PUT IN ORDER OF INCREASING MAGNITUDE

THUS

$$w(r,\theta) = \sum_m \sum_n J_n(\lambda_{nm} r)\left[\bar{A}_n \cos n\theta + \bar{B}_n \sin n\theta\right]$$

AND

$$W(r,\theta,t) = \sum_m \sum_n J_n(\lambda_{nm} r)\left[\bar{A}_n \cos n\theta + \bar{B}_n \sin n\theta\right]\left[\mathcal{C}_{1mn} \cos \omega_{mn} t + \mathcal{C}_{2mn} \sin \omega_{mn} t\right]$$
$$= \sum_m \sum_n J_n(\lambda_{nm} r)\left[\bar{C}_n \cos(n\theta + \psi_n)\right]\left[D_{mn} \cos(\omega_{mn} t + \phi_{mn})\right]$$

$\Rightarrow$ ω_{mn} has DOUBLE SUBSCRIPT SINCE $\frac{\omega}{c} = \lambda$ & λ DEPENDS ON m & n

$\Rightarrow$ $\bar{C}_n$, ψ_n, D_{mn} & ϕ_{mn} CANNOT BE FOUND WITHOUT IC'S FOR $T(t)$ & BC'S ON θ

IF p is not zero or a positive integer

$$y = C_1 J_p(x) + C_2 J_{-p}(x)$$

Bessel fn of 1st kind

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+p}}{k!\,(k+p)!}$$ for $J_{-p}(x)$ replace p by $-p$

if p is zero or a positive integer $p=n$

$$y = C_1 J_n(x) + C_2 Y_n(x)$$ Bessel fn of 2nd Kind

J_n is same as J_p but replace p by n

- $$Y_n(x) = \frac{2}{\pi}\left[\left(\log \frac{x}{2} + \gamma\right) J_n(x) - \frac{1}{2}\sum_{k=0}^{n-1} \frac{(n-k-1)!\left(\frac{x}{2}\right)^{2k-n}}{k!} + \frac{1}{2}\sum_{k=0}^{\infty}(-1)^{k+1}\left[\varphi(k) + \varphi(k+n)\right]\frac{\left(\frac{x}{2}\right)^{2k+n}}{k!\,(k+n)!}\right]$$
- $$\varphi(k) = \sum_{m=1}^{k} \frac{1}{m} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}$$
- γ is Euler's constant $= \lim_{k\to\infty}\left[\varphi(k) - \log k\right] = 0.5772157\ldots$

(**)
- note that at $x=0$ $J_n(x) =$ bdd & $Y_n(x)$ is ∞ ; $J_{-p}(x)$ is ∞ at x:
- FOR OUR PROBLEM since ω is unknown we don't know which form to use
- Secondly our problem must satisfy the condition that $w(r,\theta)=0$ at $r=a$

(**)
- IF our problem involves annular membrane - not see $C_2 \neq 0$
- $\Rightarrow$ for circular problems to have unique solutions $\Rightarrow w(r,\theta) = w(r,\theta+2\pi$

  $\Rightarrow$ <u>k must be integer</u> $\omega = n$

- $\Rightarrow R_n(\lambda r) = C_1 J_n(\lambda r) + C_2 Y_n(\lambda r)$
- also since $w(r=a,\theta) = R(\lambda a)\,\Theta(\theta) = 0$ for all Θ $\Rightarrow R(\lambda a) = 0$

Table 3.5.1 gives the first few values of these frequencies in dimensionless form. The solution for the membrane displacement u_{nm} in the vibration eigenmode n,m is then

$$u_{nm}(r,\theta,t) = A_{nm}J_n(\lambda_{nm}r)\cos(n\theta-\psi)\cos(\omega_{nm}t-\phi)$$

$$\lambda_{nm} = j_{n,m}/r_o \qquad \omega_{nm} = \lambda_{nm}a \tag{3.5.14}$$

The phase angles ϕ and ψ, and the amplitude A remain undetermined.

The lowest frequency occurs for the 0,1 mode. Note that for n = 0 the motion is axisymmetric, and has no nodes. The next higher frequency occurs for the 1,1 mode. This mode has one diametral node along which the

TABLE 3.5.1

DIMENSIONLESS MEMBRANE FREQUENCIES

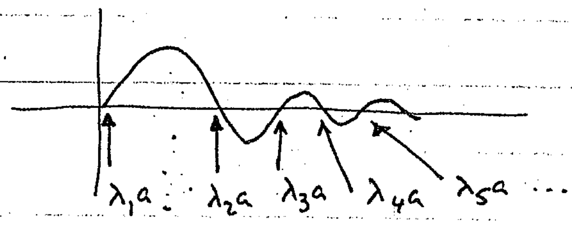
n	m	$j_{n,m} = \omega_{nm}r_o/a$
0	1	2.40483
1	1	3.83171
2	1	5.13562
0	2	5.52008
3	1	6.38016
1	2	7.01559
4	1	7.58834

membrane does not move. (The phase angle of this node cannot be determined without initial conditions). The third mode is the 2,1 mode, which has two diametral nodes, and the fourth is the 0,2 mode, with one circular node at the point where $J_0(\lambda_{02}r) = 0$, i.e. at $\lambda_{02}r = j_{0,1} = 2.40483$. Figure 3.5.3 shows the nodal lines for the first several modes.

- since the origin is included and our solution must be bounded at origin $\Rightarrow C_2 \equiv 0$ since $Y_n(0) = \infty$

- $\Rightarrow \quad R_n(\lambda r) = C_1 J_n(\lambda r) \quad \Rightarrow \quad R_n(\lambda a) = 0 \Rightarrow J_n(\lambda a) = 0$

- THESE POINTS AT WHICH $R_n = 0$ are nodal pts of the mode shape R_n
- Since $J_n(\lambda r)$ is a series made up of terms alternating in sign and each term decreases $\Rightarrow$ there will be a set of values λa for which $R_n(\lambda a) = 0$



thus we can show that $\lambda_1 < \lambda_2 < \dots < \lambda_m$ for each fn J_n and this is true for each n

- The solution then is $w_{n,m}(r,\theta) = J_n(\lambda_{nm} r)\left[C_n \cos n\theta + D_n \sin n\theta\right]$
- see table 3.5.1 Reynolds

$n=0$	$m=1$	$\lambda_{01} \cdot a = 2.40483$
1	1	$\lambda_{11}\, a = 3.83171$
2	1	$\lambda_{21}\, a = 5.13562$
⋮		

we can set in order λ_{nm}

- $w_{nm}(r,\theta)$ represents the mode shapes of vibration
- λ_{nm} represents ~~frequency~~ eigenvalues of vibration

- note: EIGENVALUE PROBLEMS are time independent

DO 3.3 & 3.5 & 3.2 also derive the indicial equation & the series solutions for the bessel fn.

$$x^2 y'' + x y' + (x^2 - p^2) y = 0$$

$$\int_0^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} L/2 & n=m \\ 0 & n\neq m \end{cases}$$

what if we wanted to find the solution to

$\nabla^2 W - \frac{1}{c^2}W_{tt} = 0$ in a circular region including the origin and $W(r=a,\theta,t)=0$

w/ I.C. $W(r,\theta,t=0) = f(r)$

$W_t(r,\theta,t=0) = g(r)$

if we choose $W(r,\theta,t) = F(r)G(\theta)H(t)$

then $\nabla^2 W - \frac{1}{c^2}W_{tt} = W_{rr} + \frac{1}{r}W_r + \frac{1}{r^2}W_{\theta\theta} - \frac{1}{c^2}W_{tt} = 0$

$= F''GH + \frac{1}{r}F'GH + \frac{1}{r^2}FG''H - \frac{1}{c^2}FGH'' = 0$

Divide By FGH & multiply by c^2: $\underbrace{c^2\frac{F''}{F} + \frac{c^2}{r}\frac{F'}{F} + \frac{c^2}{r^2}\frac{G''}{G}}_{\text{spatial}} = \underbrace{\frac{H''}{H}}_{\text{time}} = -\omega^2$ ↖ frequency

$\Rightarrow H'' + \omega^2 H = 0$ or $H(t) = B\cos\omega t + A\sin\omega t$.

and $c^2\frac{F''}{F} + \frac{c^2}{r}\frac{F'}{F} + \frac{c^2}{r^2}\frac{G''}{G} = -\omega^2$ Now multiply both sides by r^2/c^2 & separate

$$r^2\frac{F''}{F} + \frac{r^2}{r}\frac{F'}{F} + \frac{\omega^2 r^2}{c^2} = -\frac{G''}{G} = \alpha^2$$

$\Rightarrow G'' + \alpha^2 G = 0$ or $\underline{G(\theta) = C\cos\alpha\theta + D\sin\alpha\theta}$

$\Rightarrow r^2F'' + rF' + \left(\left[\frac{\omega r}{c}\right]^2 - \alpha^2\right)F = 0$ or $\underline{F\left(\frac{\omega r}{c}\right) = M J_\alpha\left(\frac{\omega r}{c}\right) + N Y_\alpha\left(\frac{\omega r}{c}\right)}$

- since we employ a circular region $G(\theta) = G(\theta + 2n\pi) \Rightarrow \alpha = \text{integer} =$
- since we include the origin $\Rightarrow N \equiv 0$

$\therefore$

- Solution is $W(r,\theta,t) = F\left(\frac{\omega r}{c}\right)G(\theta)H(t)$
- since $W(r=a,\theta,t) = 0 \Rightarrow F\left(\frac{\omega a}{c}\right) = 0 = M J_n\left(\frac{\omega a}{c}\right) = 0$ or $J_n\left(\frac{\omega a}{c}\right) = 0$

- $J_n\left(\frac{\omega a}{c}\right)=0$ defines ω since $\frac{\omega_1 a}{c} = r_1$ so that $J_n(r_1)=0$
  $\frac{\omega_2 a}{c} = r_2$ " " $J_n(r_2)=0$ etc

- also since <u>initial conditions</u> are independent of θ
  reasonable to assume that $W(r,\theta,t)$ is independent of θ

  $\Rightarrow \alpha = 0 \Rightarrow G(\theta) = \text{constant}$ why? $G''+\alpha^2 G = G'' = 0 \Rightarrow G = C_1\theta + C_2$
  FOR $G(\theta+2n\pi) = G(\theta) \Rightarrow C_1 = 0 \Rightarrow G(\theta) = \text{constant}$. $\therefore J_n\left(\frac{\omega r}{c}\right) = J_0\left(\frac{\omega r}{c}\right)$

$\therefore \quad W(r,\theta,t) \equiv W(r,t) = F\left(\frac{\omega r}{c}\right) H(t)$ where $F\left(\frac{\omega r}{c}\right) = J_0\left(\frac{\omega r}{c}\right)$

and $W_m(r,t) = F\left(\frac{\omega_m r}{c}\right) H(t)$ if $B\cdot C\cdot M = \tilde{B}$, $A\cdot C\cdot M = \tilde{A}$

$= (\tilde{B}\cos\omega_m t + \tilde{A}\sin\omega_m t)\, J_0\left(\frac{\omega_m r}{c}\right)$

$$W(r,t) = \sum_m W_m(r,t) = \sum \left(\tilde{B}_m \cos\omega_m t + \tilde{A}_m \sin\omega_m t\right) J_0\left(\frac{\omega_m r}{c}\right)$$

at $t=0$ $\quad W(r,t=0) = f(r) = \sum \tilde{B}_m J_0\left(\frac{\omega_m r}{c}\right) = \sum \tilde{B}_m R_m(r)$

$t=0$ $\quad W_t(r,t=0) = g(r) = \sum \tilde{A}_m \omega_m J_0\left(\frac{\omega_m r}{c}\right) = \sum \tilde{A}_m \omega_m R_m(r)$

suppose we can write $f(r) \equiv E_1 J_0\left(\frac{\omega_1 r}{c}\right) + E_2 J_0\left(\frac{\omega_2 r}{c}\right) + \dots + E_n J_0\left(\frac{\omega_n r}{c}\right) + \dots$

$= \sum_{i=1}^{\infty} E_i J_0\left(\frac{\omega_i r}{c}\right) = W(r,t=0) = \sum_{m=1}^{\infty} \tilde{B}_m J_0\left(\frac{\omega_m r}{c}\right)$

$\Rightarrow \quad E's = \tilde{B}'s \quad \forall\ i=m$

suppose we can write $g(r) = L_1 J_0\left(\frac{\omega_1 r}{c}\right) + L_2 J_0\left(\frac{\omega_2 r}{c}\right) + \dots + L_n J_0\left(\frac{\omega_n r}{c}\right) + \dots$

$= \sum L_i J_0\left(\frac{\omega_i r}{c}\right) = \left.\frac{\partial W}{\partial t}\right|_{t=0} = \sum \tilde{A}_m \omega_m J_0\left(\frac{\omega_m r}{c}\right)$

$\Rightarrow \quad L_i = \tilde{A}_i \cdot \omega_i$ or $\tilde{A}_i = \frac{L_i}{\omega_i} \quad \forall\ i=m$

- so how do we get the $E's$ & $L's$. See Chapter 8 §8.4

- Sturm-Liouville gives the method for solving for the $\tilde{A}_m$ & $\tilde{B}_m$

Given $\frac{d}{dx}\left(S(x)\frac{dy}{dx}\right) + \left[Q(x) + \lambda^2 P(x)\right] y = 0$

$S(x)y'' + S'(x)y' + \left[Q(x) + \lambda^2 P(x)\right] y = 0 \qquad r^2 R'' + rR' + \left[\lambda^2 r^2 - \nu^2\right] R = 0$

if we chose $\Rightarrow S(x) = r \quad Q(x) = -\nu^2/r$
$S'(x) = 1 \quad P(x) = r$

if not ie $S(x) = r^2 \quad S'(x) = 2r \rightarrow$ doesn't fit

subjected to boundary conditions $\alpha y + \beta y' = 0$ at $x = a$

$\alpha y + \beta y' = 0$ at $x = b$

- Homogeneous ODE & B.C.

- assume for $\left.\begin{array}{l}\lambda = \lambda_m \\ y = y_m\end{array}\right\}$ & $\left.\begin{array}{l}\lambda = \lambda_n \\ y = y_n\end{array}\right\}$ both ODE & BC are satisfied $y_m \neq y_n$ $\lambda_m \neq \lambda_n$

$$\Rightarrow \quad [S y_m']' + [Q + \lambda_m^2 P] y_m = 0 \qquad (1)$$

$$[S y_n']' + [Q + \lambda_n^2 P] y_n = 0 \qquad (2)$$

$$\Rightarrow \int_a^b \left\langle \left\{ [S y_m']' + [Q + \lambda_m^2 P] y_m \right\} y_n - \left\{ [S y_n']' + [Q + \lambda_n^2 P] y_n \right\} y_m \right\rangle dx$$

$$\int_a^b \left\{ \underset{①}{[S y_m']' y_n} - \underset{②}{[S y_n']' y_m} \right\} dx + (\lambda_m^2 - \lambda_n^2) \int_a^b P y_m y_n \, dx = 0$$

$$① = \int_a^b [S y_m']' y_n \, dx = S y_m' y_n \Big|_a^b - \int_a^b S y_m' y_n' \, dx \qquad \text{integration by parts}$$

$$② = \int_a^b [S y_n']' y_m \, dx = S y_n' y_m \Big|_a^b - \int_a^b S y_n' y_m' \, dx$$

at $x = a$ and at $x = b$ $\left.\begin{array}{l}\alpha y_n + \beta y_n' = 0 \\ \alpha y_m + \beta y_m' = 0\end{array}\right\}$ either $\alpha = 0$ & $\beta = 0$ or $\begin{bmatrix} y_n & y_n' \\ y_m & y_m' \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

det wronskian of $y_n, y_m = 0$.

$$\Rightarrow \quad y_n y_m' - y_m y_n' = 0 \ \text{at both } x = a, b \qquad S[y_m' y_n - y_n' y_m] \Big|_a^b + (\lambda_m^2 - \lambda_n^2) \int_a^b P y_n y_m \, dx = 0$$

$$\Rightarrow \text{if } \lambda_m \neq \lambda_n \quad \boxed{\int_a^b P y_n y_m \, dx = 0}$$

orthogonality condition but wrt weight func $P(x)$

example $T'' + \omega^2 T = 0 \Rightarrow S(x) = 1 \quad Q(x) = 0 \quad \lambda^2 = \omega^2 \quad P(x) = 1$

$$\Rightarrow \int P y_n y_m \, dx = 0 \quad \Rightarrow \quad \int_a^b P \sin \omega_n x \cdot \sin \omega_m x \, dx = 0$$

bc. suppose $T(x) = 0$ @ $x = 0$ $T(x) = 0$ @ $x = L$ $\Rightarrow T_n(x) = \sin \frac{n\pi x}{L}$

now suppose we want to write $f(x)$ as a fn of the eigenfunctions $y_n(x)$

Let: $$f(x) = \sum A_n y_n(x)$$

if we mult. by $P y_m$ & integrate

$$\int_a^b P(x) f(x) y_m(x)\,dx = \int_a^b P \sum A_n y_n(x) y_m(x)\,dx$$

$$= \sum A_n \int_a^b P y_n y_m\,dx = \begin{cases} A_m \int_a^b P y_m^2\,dx & n=m \\ 0 & n \neq m \end{cases}$$

$$\therefore \quad \boxed{A_m = \frac{\int_a^b P(x) f(x) y_m\,dx}{\int_a^b P y_m^2\,dx}}$$

FOR $T'' + \omega^2 T = 0$ $\quad y_m = \sin\frac{m\pi x}{L}$ $\quad P = 1$ $\quad a=0,\ b=L$

$$A_m = \int_0^L f(x) \sin\frac{m\pi x}{L}\,dx \Big/ \int_0^L \sin^2\frac{m\pi x}{L}\,dx$$

$$\int_0^L \sin^2\frac{m\pi x}{L}\,dx = L/2 \quad \Rightarrow \quad \int_0^L \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2m\pi x}{L}\right)dx = L/2$$

$$A_m = \frac{2}{L}\int_0^L f(x) \sin\frac{m\pi x}{L}\,dx$$ normal fourier coefficient

- Returning to our problem $\quad f(r) = \sum E_i J_0\left(\frac{\omega_i r}{c}\right)$

for the bessel fn. $P(r) = +r$

$$E_i = \frac{\int_0^a r f(r) J_0\left(\frac{\omega_i r}{c}\right) dr}{\int_0^a r\, J_0^2\left(\frac{\omega_i r}{c}\right) dr} = \tilde{B}_i$$

since $E_i = \tilde{B}_i$

also $\quad g(r) = \sum L_i J_0\left(\frac{\omega_i r}{c}\right)$

$$L_i = \frac{\int_0^a r\, g(r) J_0\,dr}{\int_0^a r\, J_0^2\,dr} = \tilde{A}_i \omega_i$$

$\tilde{A}_i \omega_i \ldots \Rightarrow \tilde{A}_i = L_i / \omega_i$

$\int_0^a r\, J_0^2(r\frac{\omega_i}{c})\,dr$ can be evaluated as follows

using the ODE: if you remember $J_0\left(\frac{\omega_i}{c} r\right)$ satisfies

$$r R'' + R' + \left(\left[\frac{\omega_i}{c}\right]^2 r - 0^2\right) R = 0 \qquad R_i = J_0(\lambda_i r)$$

(with $\lambda_i^2 r$ written above the bracketed term)

let $R(r,\lambda)$ be a solution to $r R'' + R' + (\lambda^2 r) R = 0$
and satisfy that $R(r=0)$ is not ∞

$$R(r) = J_0(\lambda r)$$

now take $\frac{\partial}{\partial \lambda}$ and multiply by R_i

$$R_i\, r \frac{\partial^3 R}{\partial r^2 \partial \lambda} + R_i \frac{\partial^2 R}{\partial \lambda \partial r} + 2\lambda r\, R R_i + \lambda^2 r \frac{\partial R}{\partial \lambda} R_i = 0$$

integrate over $0 \le r \le a$

$$\int_0^a R_i \left\{ \frac{\partial}{\partial r}\left(r \frac{\partial^2 R}{\partial r \partial \lambda}\right) + \lambda^2 r \frac{\partial R}{\partial \lambda} + 2\lambda r R \right\} dr = 0$$

integrate by parts the first term. $\int u\,dv = uv - \int v\,du \qquad u = R_i \quad v = r\frac{\partial^2 R}{\partial r \partial \lambda}$

$$r R_i \frac{\partial^2 R}{\partial r \partial \lambda}\Big|_0^a - \overset{(*)}{\int_0^a r \frac{\partial^2 R}{\partial r \partial \lambda} R_i'\, dr} + \int_0^a \Big(\overset{(**)}{\lambda^2 r \frac{\partial R}{\partial \lambda} R_i} + \overset{(***)}{2\lambda r\, R R_i} \Big) dr = 0$$

R_i by definition is $=0$ @a: $(J_0(\lambda_i a) = 0)$. For $r=0$ at lower limit $R_i = 0$

integrate by parts

$(*)$ $$-\int_0^a r R_i' \frac{\partial^2 R}{\partial r \partial \lambda} dr = -r R_i' \frac{\partial R}{\partial \lambda}\Big|_0^a + \int_0^a \frac{\partial R}{\partial \lambda} (r R_i')' \quad \text{with } u = r R_i' \quad v = \frac{\partial R}{\partial \lambda}$$

$$\Rightarrow \underbrace{-r R_i' \frac{\partial R}{\partial \lambda}\Big|_0^a}_{-a R_i' \frac{\partial R}{\partial \lambda}} + \int_0^a \frac{\partial R}{\partial \lambda} \Big\{ \underbrace{(r R_i')' + \overset{(**)}{\lambda^2 r R_i}}_{=0} \Big\} dr + \int_0^a \overset{(***)}{2\lambda r\, R R_i}\, dr = 0$$

if $R(r,\lambda)$ is $R_i \Rightarrow \lambda = \lambda_i$; thus the middle term is zero

and $$2\lambda_i \int_0^a \overset{(***)}{r R R_i}\, dr \Rightarrow \int_0^a r R_i^2\, dr = \frac{a R_i' \frac{\partial R}{\partial \lambda}}{2\lambda_i}\Bigg|_{\substack{r=a \\ \lambda=\lambda_i}}$$

THUS $\int_0^a r J_0^2(\frac{\omega_i}{c}r)\,dr = \frac{a}{2(\frac{\omega_i}{c})}\left[J_0'(\frac{\omega_i}{c}a)\right]^2 \cdot a\cdot\frac{\omega_i}{c} = \frac{a^2}{2}\left[J_0'(\frac{\omega_i}{c}a)\right]^2$

~~since $R_i'(r) = \frac{d}{dr} J_0(\frac{\omega_i}{c}r) = \frac{\omega_i}{c} J_0'(\frac{\omega_i}{c}r)$~~

~~$\frac{\partial R}{\partial\lambda} = r J_0'(\lambda r)$~~

~~now $R_i'\frac{\partial R}{\partial\lambda}\Big|_{\substack{r=a\\ \lambda=\lambda_i}} = \frac{\omega_i}{c}\cdot r\, J_0'(\frac{\omega_i}{c}r)\cdot J_0'(\lambda r) = \frac{\omega_i a}{c}\left[J_0'(\frac{\omega_i}{c}a)\right]^2$~~

THUS $\tilde{B}_i = \frac{2}{a^2}\frac{\int_0^a r f(r) J_0(\frac{\omega_i}{c}r)\,dr}{\left[J_0'(\frac{\omega_i}{c}a)\right]^2}$

$$\tilde{A}_i = \frac{2}{a^2\omega_i}\frac{\int_0^a r g(r) J_0(\frac{\omega_i}{c}r)\,dr}{\left[J_0'(\frac{\omega_i}{c}a)\right]^2}$$

Bessel Relations: $J_0'(\lambda x) = -\lambda J_1(\lambda x) \quad \therefore \quad J_0'(\frac{\omega_i}{c}a) = -\frac{\omega_i}{c} J_1(\frac{\omega_i}{c}a)$

- note Problem in Reynolds pg 4.10 involves — SPHERICAL BESSEL FNS.

produces a bessel fn that solves

$$r^2R'' + \underline{\underline{2}}rR' + \lambda^2r^2R = 0$$ and this has an

$$[r^2R']' + \lambda^2r^2R = 0$$

orthogonality condition $\int_0^r \underline{\underline{r^2}} R_n R_m\,dr = 0$

- in general if $[Sy_n']' + [Q+\lambda_n^2P]y_n = 0$ Sturm-Liouville

under the conditions $\alpha y_n + \beta y_n' = 0$ at $x=a$, $x=b$

y_n is an eigenfunction λ_n eigenvalue

- then if we want to construct $f(x) = \sum A_n y_n(x)$

$$A_n = \frac{\int_a^b f(x)P(x)y_n(x)\,dx}{\int_a^b P(x)\,y_n^2(x)\,dx}$$

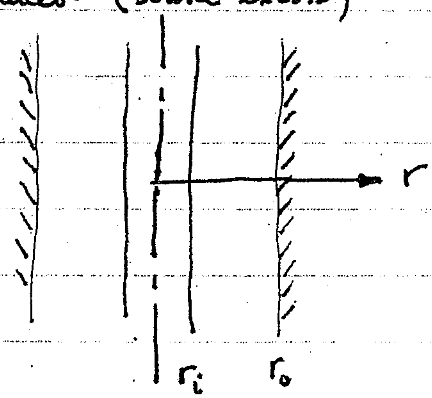
The denominator $\int_a^b P(x)\, y_n^2(x)\,dx = \frac{1}{2\lambda_n}\left\{ y_n' \, S \frac{\partial y}{\partial \lambda}\Big|_a^b - y_n \, S \frac{\partial^2 y}{\partial x \partial \lambda}\Big|_a^b \right\}_{\lambda=\lambda}$

$y_n'(z) = \frac{d}{dz} y_n$

$y = y(x;\lambda)$ satisfies ODE & is bounded

$y = y(x;\lambda_n) = y_n$ satisfies ODE & B.C.

- Removing inhomogeneities in the PDE & BC's
  - IN PDE & IN BC.
  - Time history of diffusion of a contaminant $c(r,t)$ in an annular region in which the contaminant in continually produced. (source exists)



$$\frac{\partial}{\partial r}\left(r \frac{\partial c}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial c}{\partial t} - rS \qquad (3)$$

α diffusivity

S source term

- assume at $r = r_o$ barrier blocks outer diffusion $\ni$ $\frac{\partial c}{\partial r} = 0$ (1)

- Also assume contaminant is convectively removed at $r = r_i$

$$h(c - c_\infty) = D \frac{\partial c}{\partial r} \qquad (2)$$

h convective transport coeff; c_∞ is the fixed concentration in fluid passing through annular hole

D - diffusion coeff for the contaminant in the solid

- initially $c = c_{?}$ at $t = 0$ (... IC)

ran 3 + 3

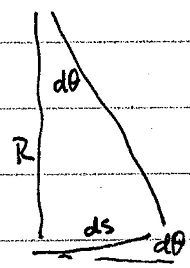
work = F · d $F dx = \sigma . A dx$ energy

$= \sigma \epsilon$ $F dv$ $F \frac{dx}{dt} dt$

$\sigma . A . \epsilon L$

σ $\sigma A dx$

work = $\epsilon E A dx$



$M d\theta$ = work $d\theta = \frac{ds}{R}$

$\epsilon = \frac{-y}{R}$

$\sigma = E\epsilon = \frac{-Ey}{R}$ $\frac{-My}{I_z} = \sigma = \frac{-Ey}{R}$

$\therefore \quad M = +\frac{EI_z}{R}$

$\therefore \quad M d\theta = \frac{EI_z}{R^2} ds$

but $\frac{1}{R} \sim \frac{d^2w}{dx^2}$

$\therefore \quad M d\theta = EI_z \cdot \left(\frac{d^2w}{dx^2}\right)^2 ds$

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1 Midterm Exam	40 % each
HW	20 %
Final Exam	40 %

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(A-) 90-94	(B) 80-84	(C) 70-72	(F) below 60
	(B-) 77-79	(C-) 65-69	

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We will be meeting 2 times a week M and W 1140-1255. Our meeting room will be EAS 1104, though that may change.

My office hours are M and W 1-4pm and also by appointment.

THIS IS AN EXTREMELY PRELIMINARY EXAM DUE TO THE PROBLEMS IN SETTING THE CLASS. A MORE ACCURATE ONE WILL BE PROVIDED WITHIN THE NEXT WEEK.

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Found of Solid Mech
Y.C. Fung

The term

(12) $$L \equiv U - K + A$$

(or sometimes $-L$) is called the *Lagrangian function* and the equation (11) represents *Hamilton's principle*, which states that:

The time integral of the Lagrangian function over a time interval t_0 to t_1 is an extremum for the "actual" motion with respect to all admissable virtual displacements which vanish, first, at instants of time t_0 and t_1 at all points of the body, and, second, over S_u, where the displacements are prescribed, throughout the entire time interval.

To formulate this principle in another way, let us call $u_i(x_1, x_2, x_3; t)$ a dynamic path. Then Hamilton's principle states that *among all dynamic paths that satisfy the boundary conditions over S_u at all times and that start and end with the actual values at two arbitrary instants of time t_0 and t_1 at every point of the body, the "actual" dynamic path is distinguished by making the Lagrangian function an extremum.*

In rigid body dynamics the term U drops out, and we obtain Hamilton's principle in the familiar form. The symbol A replaces the usual symbol V in books on dynamics because we have used V for something else.

Note that the potential energy $-A$ of the external loads exists and is a linear function of the displacements if the loads are independent of the elastic displacements, as is commonly the case. In aeroelastic problems, however, the aerodynamic loading is sensitive to the small surface displacements u_i; moreover, it depends on the time history of the displacements and cannot be derived from a potential. Hence, in aeroelasticity we are generally forced to use the variational form (9) of Hamilton's principle.

In some applications of the direct method of calculation, it is even desirable to liberalize the variations δu_i at the instants t_0 and t_1 and use Hamilton's principle in the variational form (4) which cannot be expressed elegantly as the minimum of a well-defined functional. On the other hand, such a formulation will be accessible to the direct methods of solution. On introducing (5), (7), and (10), we may rewrite Eq. (4) in the following form:

(13) $$\int_{t_0}^{t_1} \delta(U - K + A)\, dt = \int_{t_0}^{t_1}\int_V F_i\, \delta u_i\, dv\, dt + \int_{t_0}^{t_1}\int_S \overset{\nu}{T}_i\, \delta u_i\, dS - \int_V \rho \frac{\partial u_i}{\partial t}\, \delta u_i\, dv \Big|_{t_0}^{t_1}.$$

Here U is the total strain energy, K is the total kinetic energy, A is the potential energy for the conservative external forces, F_i and $\overset{\nu}{T}_i$ are, respectively, those external body and surface forces that are not included in A, and δu_i are the virtual displacements.

Problem 11.1. Prove the converse theorem that, for a conservative system, the variational Eq. (11) leads to the equation of motion

$$\rho \frac{\partial^2 u_i}{\partial t^2} = F_i + \frac{\partial}{\partial x_j}\frac{\partial W}{\partial e_{ij}}$$

and the boundary conditions

$$\text{either} \quad \delta u_i = 0 \quad \text{or} \quad \frac{\partial W}{\partial e_{ij}} \nu_j = \overset{\nu}{T}_i.$$

11.2. EXAMPLE OF APPLICATION—EQUATION OF VIBRATION OF A BEAM

As an example of the application of Hamilton's principle in the formulation of approximate theories in elasticity, let us consider the free, lateral vibration of a straight simple beam. We assume that the beam possesses principal planes and that the vibration takes place in one of the principal planes, and let y denote the small deflection of the neutral axis of the beam from its initial, straight configuration. In Sec. 10.8 it is shown that the strain energy of the beam is, for small deflections,

(1) $$U = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx,$$

where E is the Young's modulus of the beam material, I is the cross-sectional moment of interia, and l is the length of the beam.

The kinetic energy of the beam is derived partly from the translation, parallel to y, of the elements composing it, and partly from the rotation of the same elements about an axis perpendicular to the neutral axis and the plane of vibration. The former part is

$$\frac{1}{2}\int_0^l m\left(\frac{\partial y}{\partial t}\right)^2 dx,$$

where m is the mass per unit length of the beam. The latter part is, for each element dx, the product of moment of inertia times one-half of the square of the angular velocity. Let I_ρ denote the mass moment of inertia about the neutral axis per unit length of the beam. The angular velocity being $\partial^2 y/\partial t\, \partial x$, the kinetic energy of the beam is

(2) $$K = \frac{1}{2}\int_0^l m\left(\frac{\partial y}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^l I_\rho\left(\frac{\partial^2 y}{\partial x\, \partial t}\right)^2 dx.$$

If the beam is loaded by a distributed lateral load of intensity $p(x, t)$ per unit length and moment and shear M and Q, respectively, at the ends as shown in Fig. 11.2:1, then the potential energy of the external loading is

(3) $$A = -\int_0^l p(x, t)y(x)\, dx - M_l\left(\frac{\partial y}{\partial x}\right)_l + M_0\left(\frac{\partial y}{\partial x}\right)_0 + Q_l y_l - Q_0 y_0.$$

Thwin Siss
M&M

20/42

H.W 4

2.5, 17, 27, 33, 41

2.5 a) $\frac{\delta}{L} = \frac{0.025}{10.0} = 0.00025$

$\sigma = \frac{E\delta}{L} = (69\times10^9)(.00025) = 17.25\times10^6$ pa

b) $\sigma = P/A$, $A = P/\sigma = \frac{7.2\times10^3 N}{17.25\times10^6 pa} = 0.000417\ m^2$

$A = \frac{\pi}{4}(d_o^2 - d_i^2)$

$d_i^2 = d_o^2 - \frac{4A}{\pi} = .050^2 m - \frac{(4)(4.17\times10^{-4} m^2)}{\pi} = 0.001969\ m^2$

$d_i = .04437 m$

$t = \frac{1}{2}(d_o - d_i) = \frac{1}{2}(0.05 - 0.04437)$

$t = 0.00563 m$

2.17 (10)

$A_{AB} = \pi/4\,(.030)^2 = 70.68\times10^{-5}\ m^2$

$A_{BC} = \pi/4\,(.050)^2 = 19.63\times10^{-4}\ m^2$

$P_{AB} = -30 KN = -30\times10^3 N$

$P_{BC} = -30 KN - 40 KN = -70\times10^3 N$

$L_{AB} = 0.25 m$, $L_{BC} = 0.3 m$, $E_{AB} = 200 GPA$, $E_{BC} = 105 GPa$

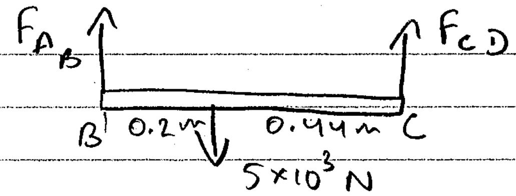
$\delta_{AB} = \frac{P_{AB} L_{AB}}{A_{AB} E_{AB}} = \frac{-(30\times10^3 N)(0.25 m)}{(70.68\times10^{-5} m^2)(200\times10^9 Pa)} = -5.30\times10^{-5} m$

$\delta_{BC} = \frac{P_{BC} L_{BC}}{A_{BC} E_{BC}} = \frac{-(70\times10^3 N)(0.3 m)}{(19.63\times10^{-4} m^2)(105\times10^9 Pa)} = -10.19\times10^{-5} m$

a) $\delta_{ABC} = \delta_{AB} + \delta_{BC} = -15.488\times10^{-5}\ m$

b) $\delta_B = \delta_{BC} = -10.19\times10^{-5} m$

27



$+\circlearrowleft \Sigma M_C = 0 \qquad -(0.64)F_{AB} + (0.44)(5\times10^3) = 0$

$F_{AB} = 3.4375\times10^3\ N$

$+\circlearrowleft \Sigma M_B = 0 \qquad (0.64)F_{CD} - (0.2)(5\times10^3) = 0$

$F_{CD} = 1.5625\times10^3\ N$

$A_{AB} = A_{BC} = 125\ mm^2 = 125\times10^{-6}\ m^2$

$\delta_{AB} = \frac{F_{AB}L_{AB}}{A_{AB}E_{AB}} = \frac{(3.4375\times10^3)(0.36)}{(75\times10^9)(125\times10^{-6})} = 132.00\times10^{-6} m$

$\delta_{\substack{BC\\CD}} = \frac{F_{CD}L_{CD}}{A_{CD}E_{CD}} = \frac{(1.5625)\times10^3)(0.36)}{(75\times10^9)(125\times10^{-6})} = 60.00\times10^{-6} m$

slope $\theta = \frac{\delta_B - \delta_C}{L_{BC}} \quad \frac{72\times10^{-6}}{0.64} = 112.5\times10^{-6}$ rad

$\delta_E = \delta_C + l_{EC}\,\theta$

$\delta_E = 60.00\times10^{-6} + (0.44)(112.5\times10^{-6})$

$= 109.5\times10^{-6} m$

(10)

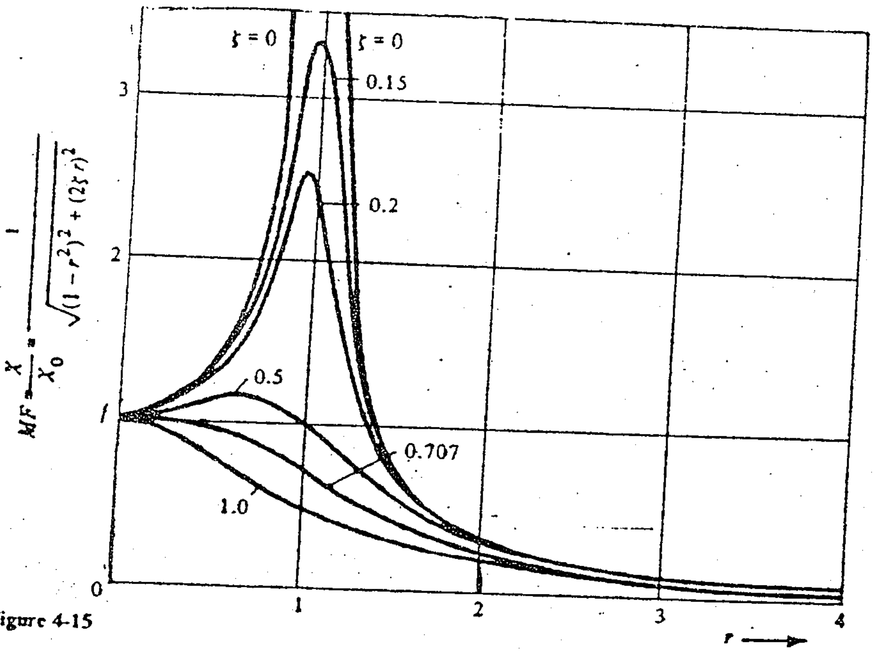


Figure 4-15

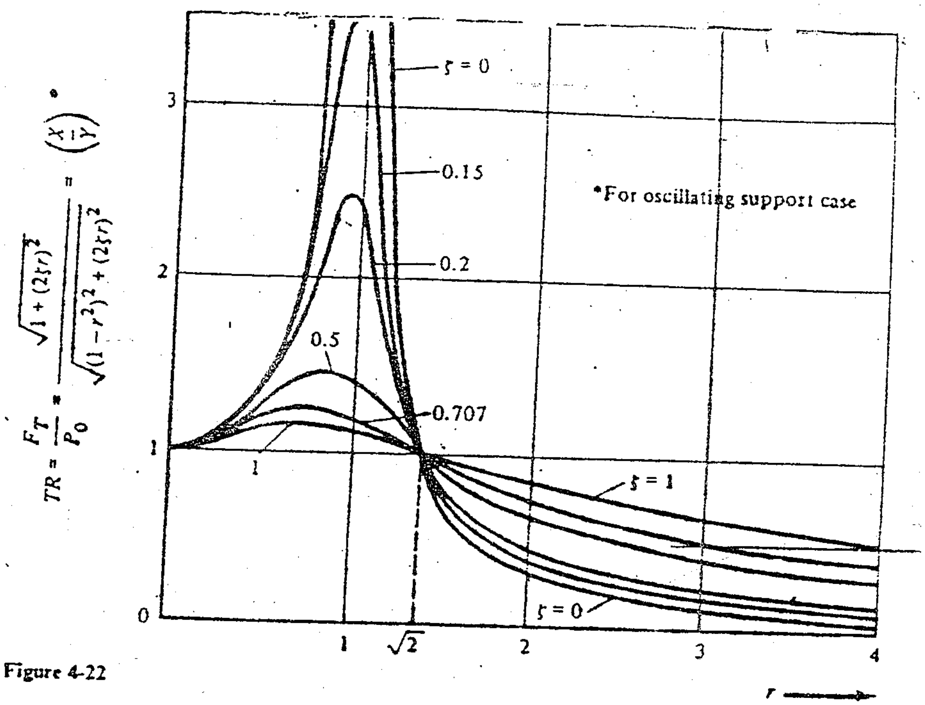


Figure 4-22

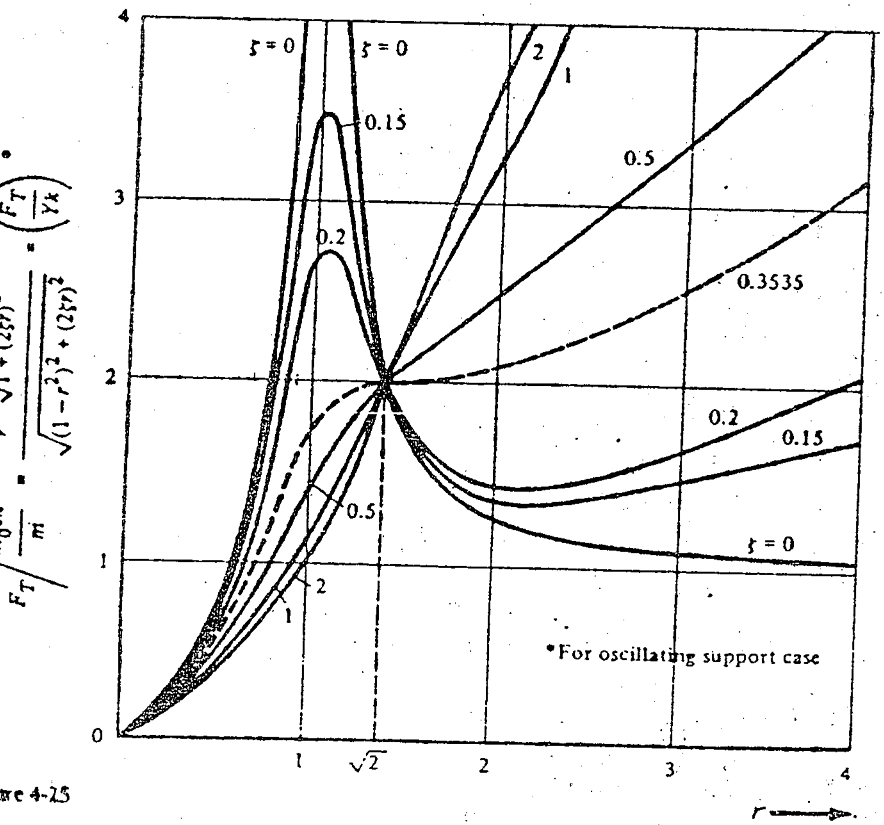


Figure 4-25

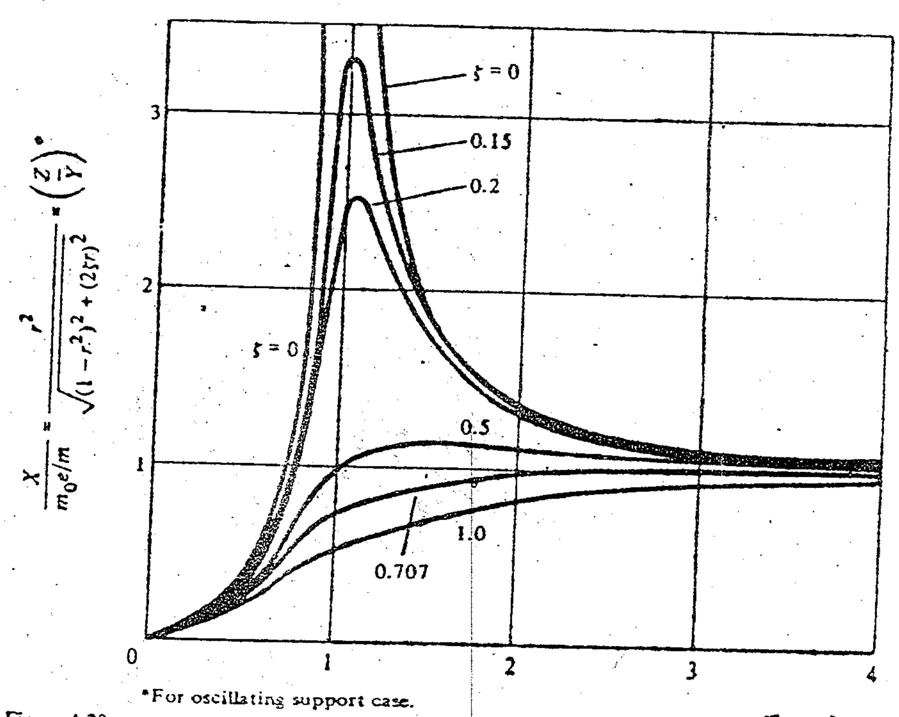


Figure 4-20

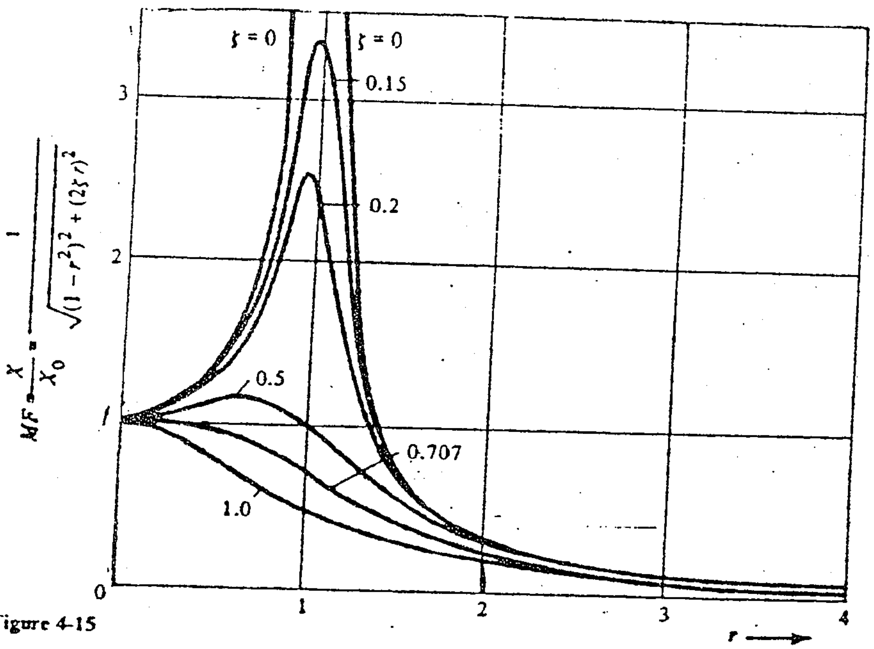


Figure 4-15

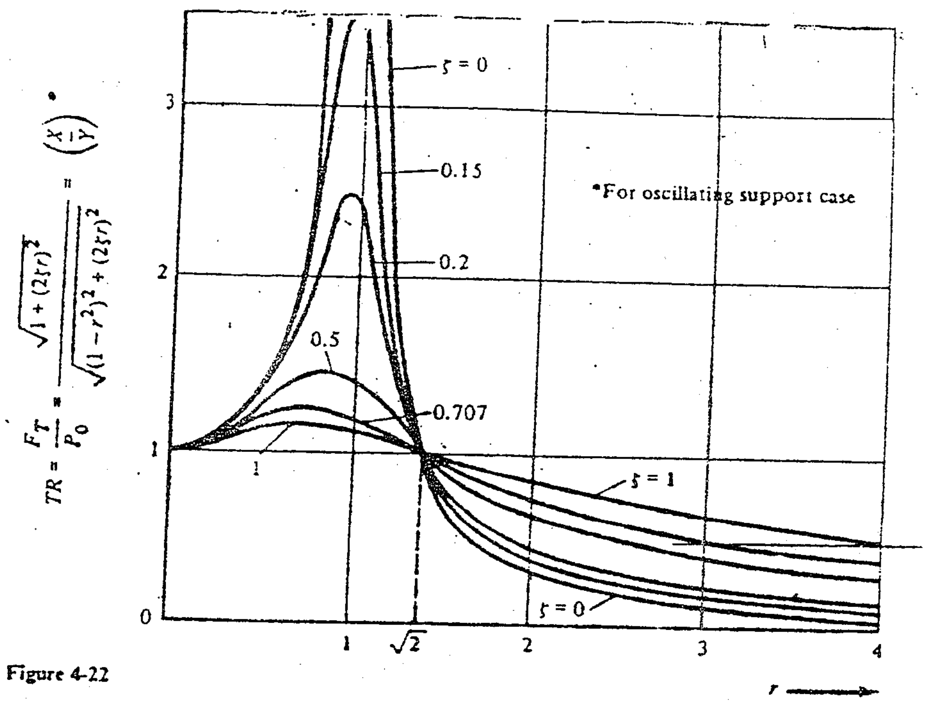


Figure 4-22

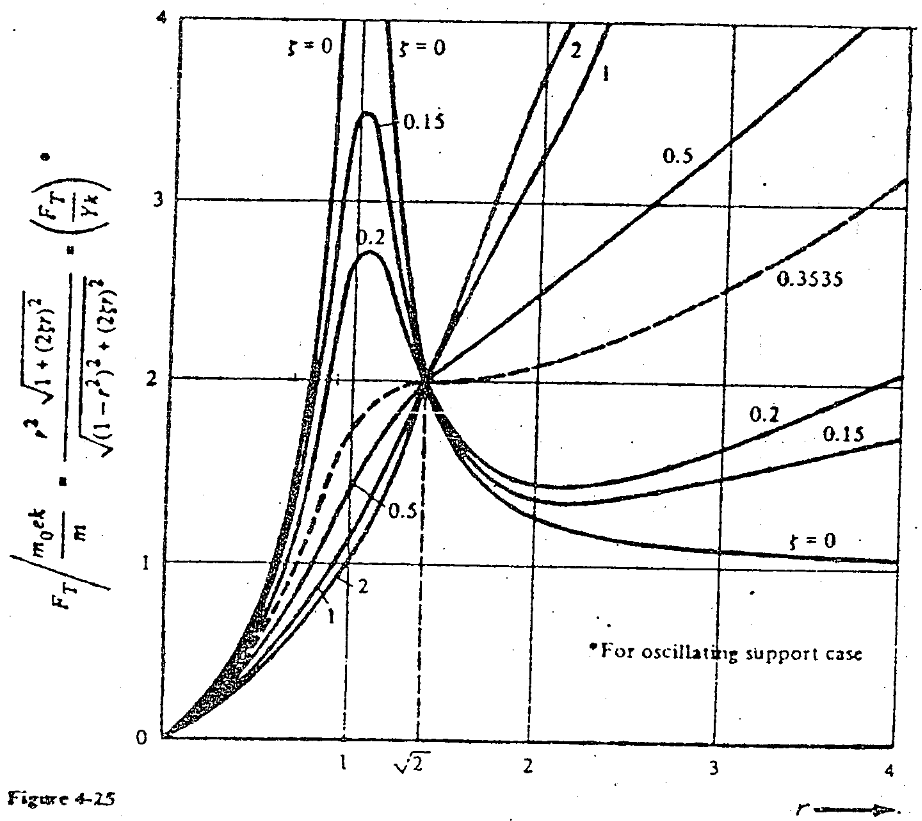
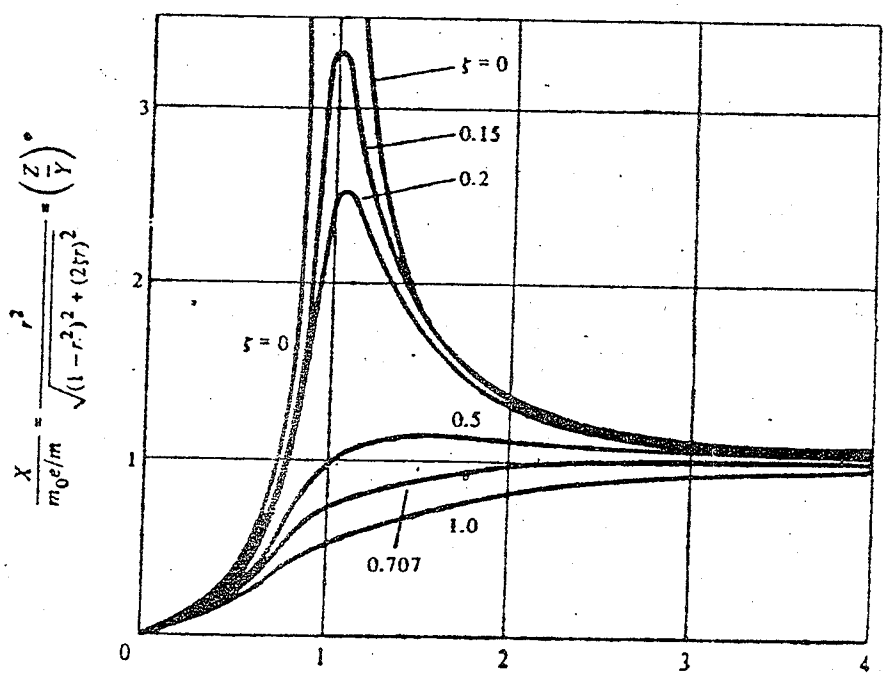


Figure 4-25



*For oscillating support case.

Figure 4-20

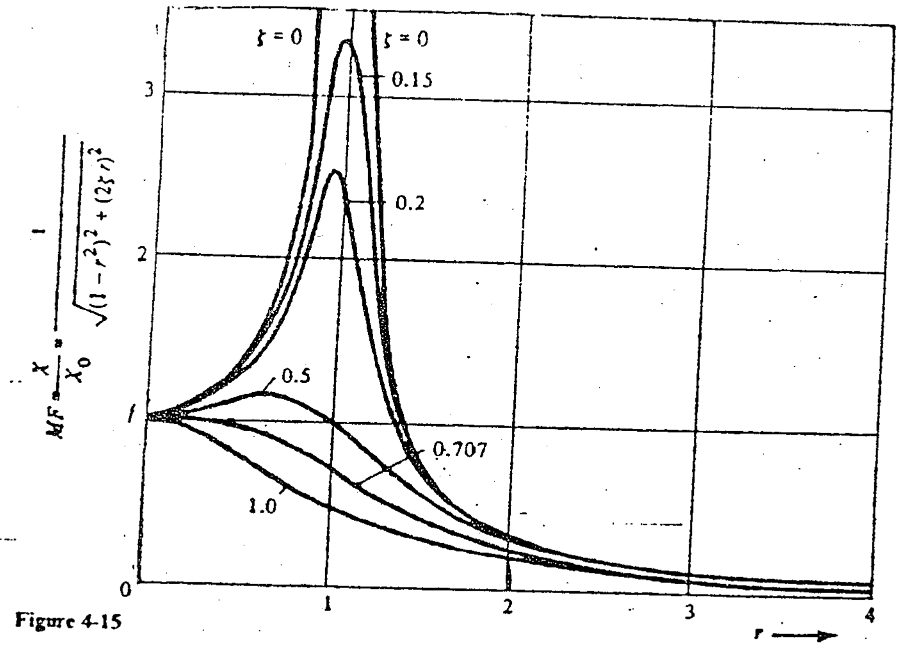


Figure 4-15

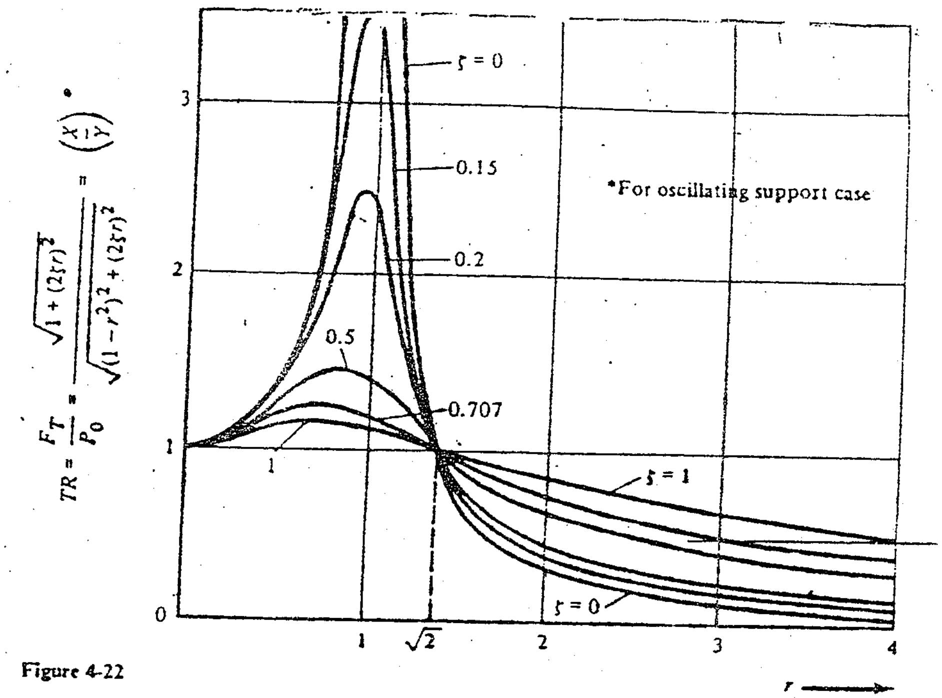


Figure 4-22

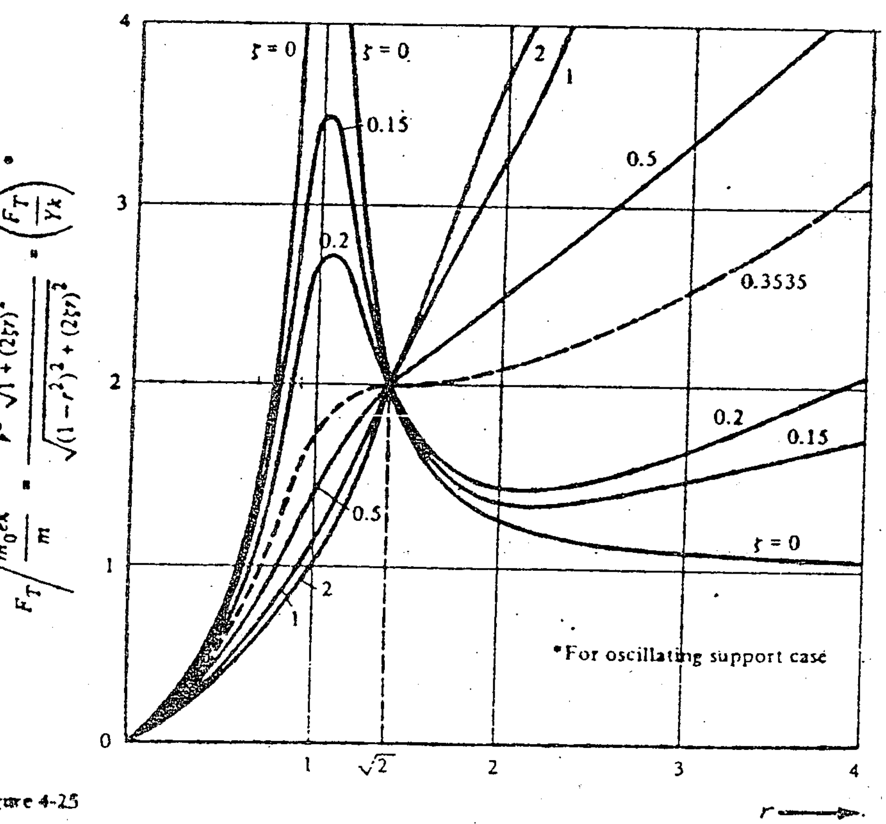


Figure 4-25

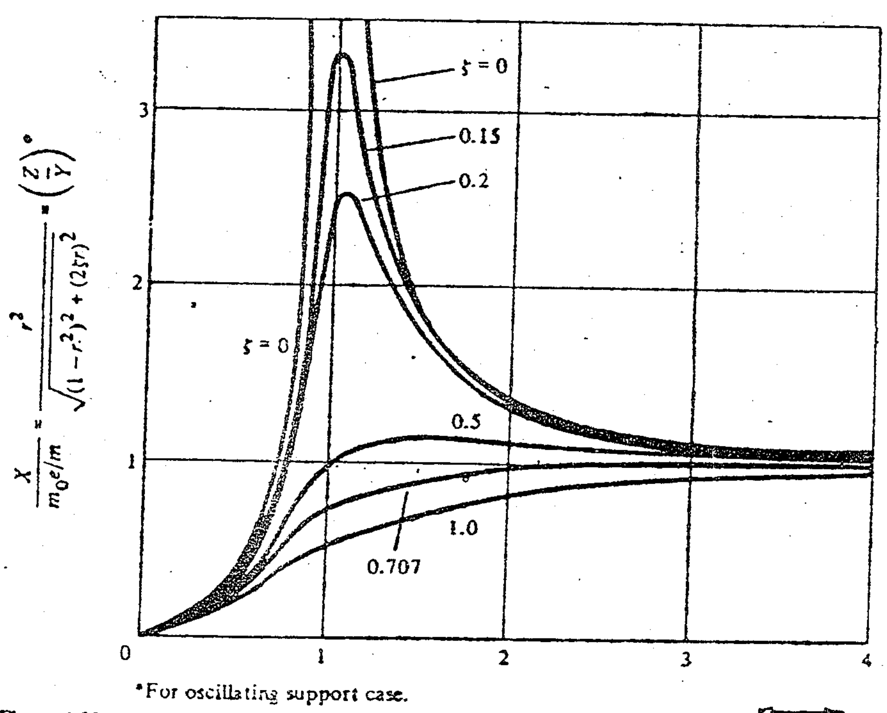


Figure 4-20

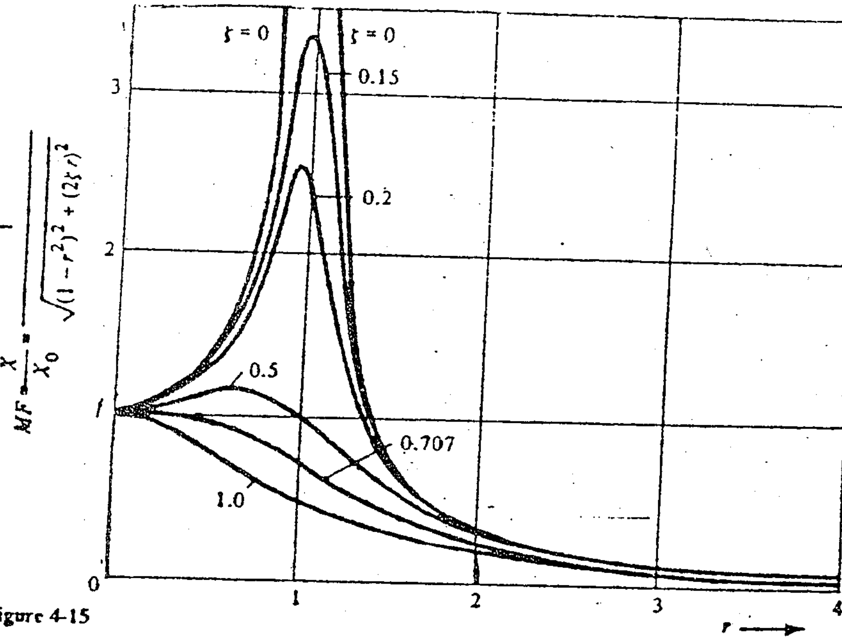


Figure 4-15

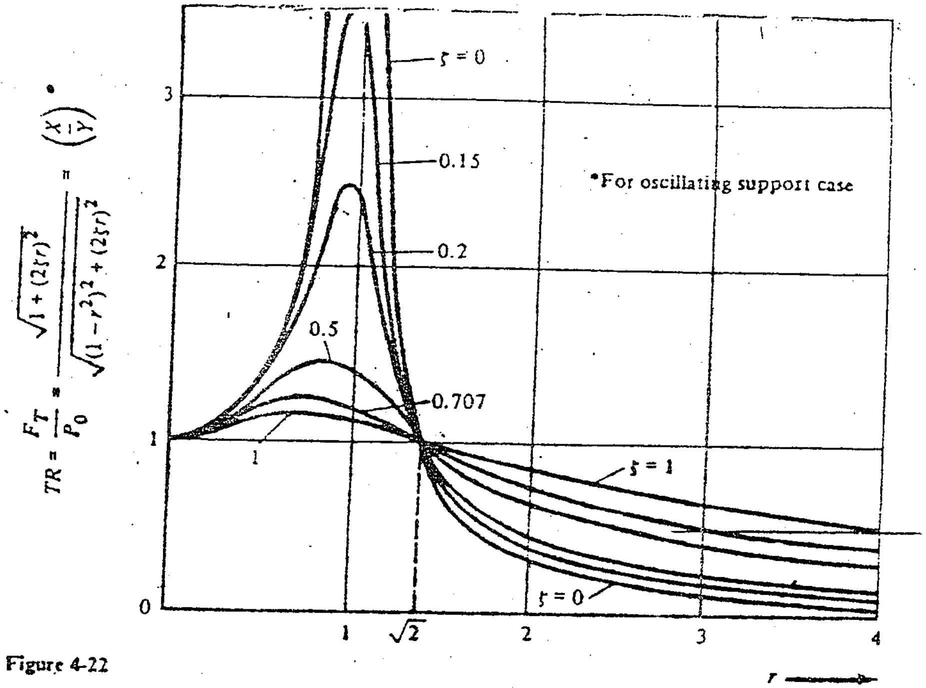


Figure 4-22

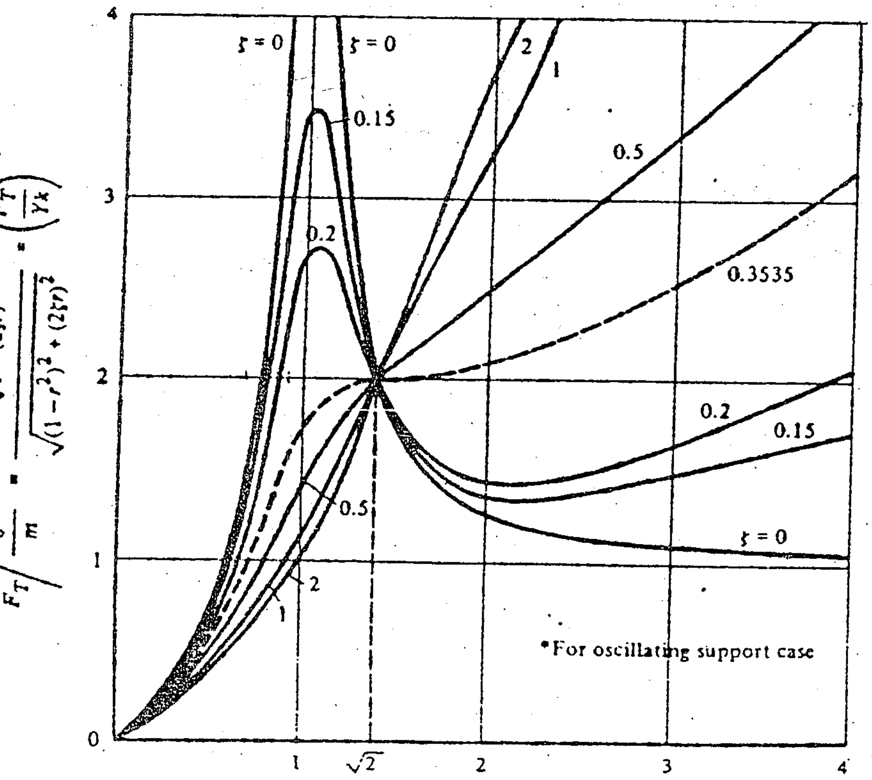
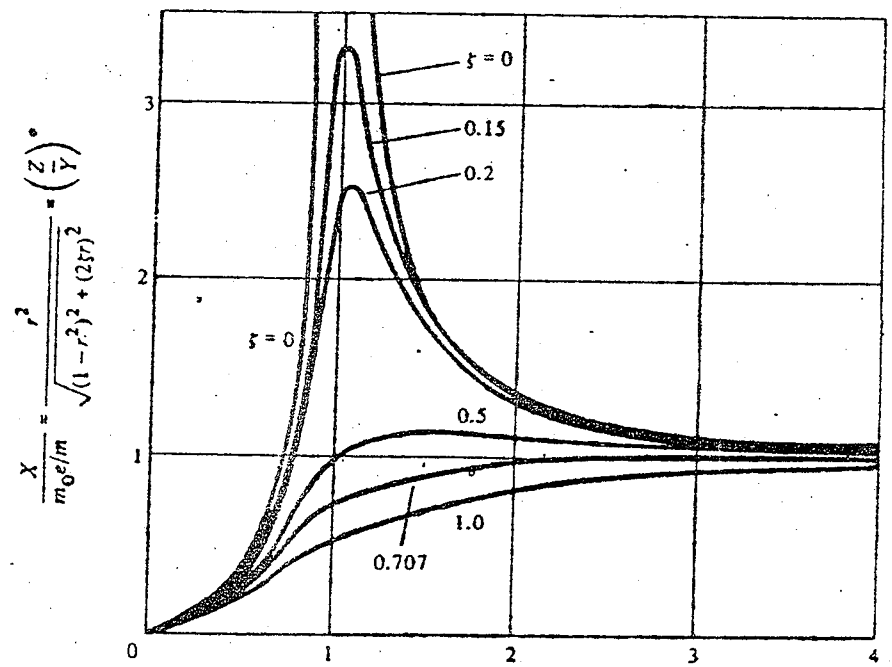


Figure 4-25

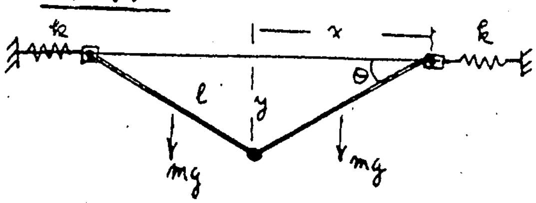


*For oscillating support case.

r

Figure 4-20

7-14



Let θ_0 = equilib angle

Spring force = F_{s0} at $\theta = \theta_0$

$x = l\cos\theta \qquad \delta x = -l\sin\theta_0\,\delta\theta$

$y = l\sin\theta \qquad \delta y = l\cos\theta_0\,\delta\theta$

$$\delta W = 2mg\frac{\delta y}{2} + 2F_{s0}\,\delta x = 0$$

$$(mg\cos\theta_0 - 2F_{s0}\sin\theta_0)\,l\,\delta\theta = 0$$

If $F_s = 0$ at $\theta = 0$ then $F_{s0} = k(l-x) = kl(1-\cos\theta_0)$

$$\tan\theta_0 = \frac{mg}{2kl(1-\cos\theta_0)}$$

solve by trial for given value of $\frac{mg}{2kl}$

7-15

$$T = \frac{1}{2}m_0\left[(r\dot\theta)^2 + \dot r^2\right] + \frac{1}{2}\left(m\frac{l^2}{3}\right)\dot\theta^2$$

$$U = \frac{1}{2}k(r-r_0)^2 - m_0 g r\cos\theta - mg\frac{l}{2}\cos\theta$$

$$\frac{\partial T}{\partial\dot\theta} = m_0 r^2\dot\theta + m\frac{l^2}{3}\dot\theta \qquad \frac{\partial T}{\partial\theta} = 0$$

$$\frac{\partial U}{\partial\theta} = m_0 g r\sin\theta + mg\frac{l}{2}\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial U}{\partial\theta} = 0$$

$$m_0[r\ddot\theta + 2\dot r\dot\theta]r + \frac{ml^2}{3}\ddot\theta + \left(m_0 g r + mg\frac{l}{2}\right)\sin\theta = 0$$

$$\frac{\partial T}{\partial\dot r} = m_0\dot r \qquad \frac{\partial T}{\partial r} = m_0 r\dot\theta^2$$

$$\frac{\partial U}{\partial r} = -m_0 g\cos\theta + k(r-r_0)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot r}\right) - \frac{\partial T}{\partial r} + \frac{\partial U}{\partial r} = 0$$

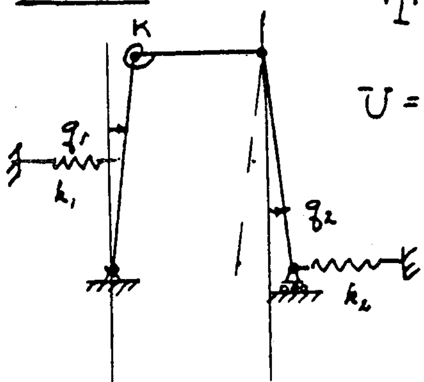
$$m_0\ddot r - m_0 r\dot\theta^2 + k(r-r_0) - m_0 g\cos\theta = 0$$

$n=4$

$\bar{x}=64.5$

$\sigma: 4.77$

7-18



$$T = \tfrac{1}{2}(m_1+m_2)(2\ell\dot{q}_1)^2 + \tfrac{1}{2}J_1\dot{q}_1^{\,2} + \tfrac{1}{2}J_2\dot{q}_2^{\,2}$$

$$U = \tfrac{1}{2}k_1(\ell q_1)^2 + \tfrac{1}{2}Kq_1^2 + \tfrac{1}{2}k_2(2\ell)^2(q_1+q_2)^2$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = (m_1+m_2)(4\ell^2)\ddot{q}_1 + J_1\ddot{q}_1$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} = J_2\ddot{q}_2$$

$$\frac{\partial U}{\partial q_1} = \ell^2 k_1 q_1 + Kq_1 + k_2(4\ell^2)(q_1+q_2)$$

$$\frac{\partial U}{\partial q_2} = 4\ell^2 k_2(q_1+q_2)$$

$L = 2\ell$

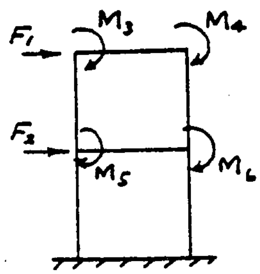
(above terms: L^2, $L^2/4$, L^2, L^2)

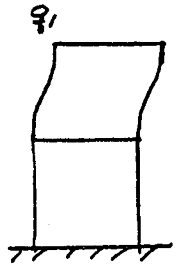
$$\left[(m_1+m_2)4\ell^2 + J_1\right]\ddot{q}_1 + \left[\ell^2 k_1 + K + 4\ell^2 k_2\right]q_1 + 4\ell^2 k_2 q_2 = 0$$

$$J_2\ddot{q}_2 \quad + 4\ell^2 k_2(q_1+q_2) = 0$$

7-19

Refer to FIG 7.1-4. Assume $\ell_2 = \ell_1 = \ell$

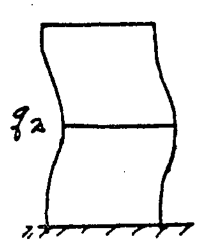


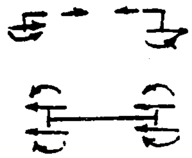


$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 24 & 0 & 0 & 0 & 0 & 0 \\ -24 & \vdots & \vdots & \vdots & \vdots & \vdots \\ -6\ell & & & & & \\ -6\ell & & & & & \\ -6\ell & & & & & \\ -6\ell & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}\begin{Bmatrix} q_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Examine FBD of each of the four corners for above

Refer to FIG 6.4-2





$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{\ell^3}\begin{bmatrix} 0 & -24 & 0 & 0 & 0 & 0 \\ \vdots & 48 & \vdots & \vdots & \vdots & \vdots \\ & 6\ell & & & & \\ & 6\ell & & & & \\ & 0 & & & & \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots \end{bmatrix}\begin{Bmatrix} 0 \\ q_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

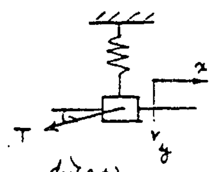
9-1

$$c = \sqrt{\frac{T}{\rho}} = \sqrt{\frac{444}{.372}} = 34.55 \ m/s$$

9-2

$$f_n = \frac{n}{2\ell}\sqrt{\frac{T}{\rho}}, \quad n = 1, 2, 3 \ ---$$

9-3



gen. sol. $y(x,t) = \left(A \sin\frac{\omega x}{c} + B\cos\frac{\omega x}{c}\right)\sin\omega t$

At $x=0$ $\quad y = 0 \quad \therefore B = 0$

At $x=\ell \quad y = y(\ell,t)$ of spring mass

$$\text{Vertical force} = -T\frac{dy}{dx}(\ell,t)$$

$$= \left(-TA\frac{\omega}{c}\cos\frac{\omega\ell}{c}\right)\sin\omega t$$

$$= m\ddot{y} + ky$$

$$\therefore y(\ell) = \frac{-TA\frac{\omega}{c}\cos\frac{\omega\ell}{c}}{k - m\omega^2} = \frac{-TA\frac{\omega}{c}\cos\frac{\omega\ell}{c}}{m\omega_n^2\left(1-\frac{\omega^2}{\omega_n^2}\right)}, \quad \omega_n = \sqrt{\frac{k}{m}}$$

$$\text{or} \quad \tan\frac{\omega\ell}{c} = -\left(\frac{T}{k\ell}\right)\frac{\left(\frac{\omega\ell}{c}\right)}{\left[1-\left(\frac{\omega\ell}{c}\right)^2\left(\frac{mc^2}{k\ell^2}\right)\right]}$$

9-4

$$y_1 = a\cos kx\sin\omega t, \qquad y_2 = a\cos\left(\frac{k}{x}+\frac{\pi}{2}\right)\sin\left(\omega t+\frac{\pi}{2}\right)$$

$$= -a\sin kx\cos\omega t$$

$$y = y_1 + y_2 = a\left[\cos kx\sin\omega t - \sin kx\cos\omega t\right]$$

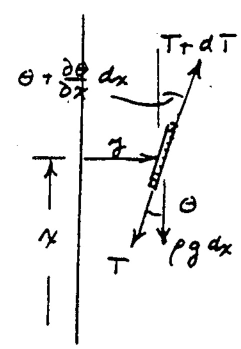
$$= a\sin(\omega t - kx) = a\sin k\left(\frac{\omega}{k}t - x\right) \quad \therefore c = \frac{\omega}{k}$$

9-5

$$c = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{200\times10^9}{7810}} = 5060 \ m/s = 16,600 \ ft/s$$

9-6

$dT \cong \rho g\,dx$



$$(T+dT)\left(\theta+\frac{\partial\theta}{\partial x}dx\right) - T\theta = \rho\,dx\,\ddot{y}$$

$$\therefore T\frac{\partial\theta}{\partial x} + \rho g\,\theta = \rho\ddot{y} \qquad \theta = \frac{\partial y}{\partial x}, \quad T = \rho g x$$

$$\rho g x\frac{\partial^2 y}{\partial x^2} + \rho g\frac{\partial y}{\partial x} = \rho\ddot{y}$$

$$\ddot{y} = g\left(x\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x}\right)$$

9-7

$$y = Y(x)\cos\omega t$$
$$\ddot{y} = -\omega^2 Y(x)\cos\omega t$$

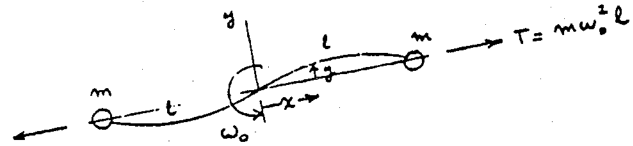
$$-\omega^2 Y(x) = g\left(x\frac{d^2Y(x)}{dx^2} + \frac{dY(x)}{dx}\right)$$

Let $z^2 = \frac{4\omega^2}{g}x \qquad dx = \frac{g}{4\omega^2}2z\,dz, \quad (dx)^2 = \left(\frac{g2z}{4\omega^2}\right)^2 dz^2$

$$-\omega^2 Y = g\left\{\frac{g}{4\omega^2}z^2\frac{d^2Y}{\frac{gz^2}{4\omega^2}dz^2} + \frac{2\omega^2}{gz}\frac{dY}{dz}\right\}$$

$$\therefore \quad \frac{d^2Y}{dz^2} + \frac{1}{z}\frac{dY}{dz} + Y = 0 \qquad \text{Bessel's D.E}$$

9-8



Assume mode shape as shown. Accel. at y is $\ddot{y} - y\omega_0^2$ in lateral direction

$T\frac{\partial^2 y}{\partial x^2} = \rho(\ddot{y} - y\omega_0^2) \qquad$ Let $y = Y(x)e^{i\omega t}$

$\frac{d^2Y}{dx^2} + \left[\left(\frac{\omega}{c}\right)^2 + \left(\frac{\omega_0}{c}\right)^2\right]Y(x) = 0 \qquad c = \sqrt{\frac{T}{\rho}}$

$Y(0) = 0 \quad \therefore \quad B = 0$ and $Y(x) = A\sin\Omega x + 0$

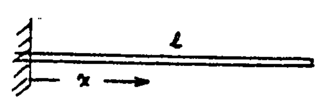
where $\Omega = \sqrt{\left(\frac{\omega}{c}\right)^2 + \left(\frac{\omega_0}{c}\right)^2}$

$Y(l) = 0 \quad \therefore \quad \sin\Omega l = 0$

$\sqrt{\left(\frac{\omega}{c}\right)^2 + \left(\frac{\omega_0}{c}\right)^2}\; l = \pi \qquad \omega^2 = \left(\frac{\pi c}{l}\right)^2 - \omega_0^2$

$$\therefore \quad \omega^2 = \left(\frac{\pi}{l}\right)^2\left(\frac{m\omega_0^2 l}{\rho}\right) - \omega_0^2$$

9-9



$u = \left(A\sin\frac{\omega x}{c} + B\cos\frac{\omega x}{c}\right)\sin\omega t$

$u(0) = 0 \quad \therefore B = 0$

$\sigma = E\left(\frac{du}{dx}\right)_{x=l} = 0 \quad \therefore \quad \frac{\omega}{c}\cos\frac{\omega l}{c} = 0$

$\frac{\omega l}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \cdots = \left(n + \frac{1}{2}\right)\pi \qquad n = 0, 1, 2 \cdots$

$$f_n = \frac{\omega_n}{2\pi} = \left(n + \frac{1}{2}\right)\frac{c}{2l}, \quad n = 0, 1, 2, \cdots$$

14-37 Cont:

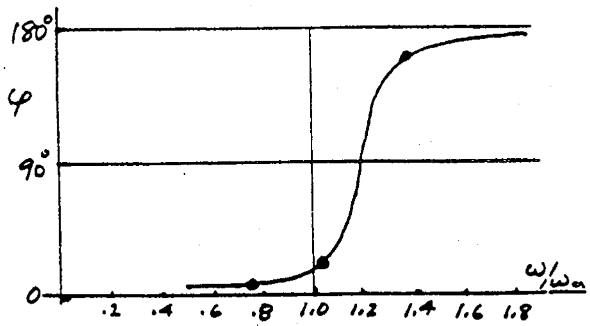
$$(\tan\phi)_{A=1} = \frac{.15\sqrt{15.78}}{(10-15.78)+.75} = \frac{.60}{-5.03} = -.119 \qquad \phi = 173^\circ 12'$$

$$(\tan\phi)_{A=2} = \frac{.15\sqrt{10.69}}{(10-10.69)+3} = \frac{.490}{2.31} = .213 \qquad \phi = 12^\circ$$

$$= \frac{.15\sqrt{15.29}}{(10-15.29)+3} = \frac{.590}{-2.29} = -.257 \qquad \phi = 185^\circ 35'$$

$$(\tan\phi)_{A=0} = \frac{\infty}{-\infty^2} = -\frac{1}{\infty} = -0 \qquad \phi = 180^\circ -$$

$$(\tan\phi)_{A=.5} = \frac{0}{(10-0)+.375} = 0 \qquad \phi = 0^\circ$$



ω/ω_n	φ_1	φ_2
0	0	
.755	4° 4'	
1.036	12°	
1.25		173° 12'
1.24		185° 35'
∞		180°

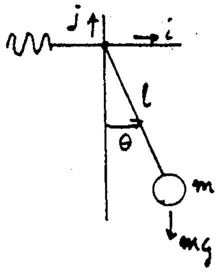
14-38 Moment eq. about an accelerating point A is

$$\vec{M}_A = I_A\,\dot{\vec{\omega}} + \vec{\rho}_{AC} \times m\,\vec{a}_A$$ (see Dynamics by Pestel & Thomson p 213 McGraw Hill.)

where c is the center of mass and $\vec{\rho}_{AC}$ is a vector from A to c. For this problem $y_A = y_0 \cos 2\omega t$

$I_A = m\ell^2 \qquad |\rho_{AC}| = \ell \qquad |a_A| = -4y_0\omega^2\cos 2\omega t$

$$\vec{\rho}_{AC} \times m\vec{a}_A = \ell(\sin\theta\,\vec{i} - \cos\theta\,\vec{j}) \times m(-4y_0\omega^2\cos 2\omega t)\,\vec{j}$$



$$= -4y_0\omega^2\ell\, m\cos 2\omega t \cdot \vec{k}$$

$$\dot{\vec{\omega}} = \ddot{\theta}\,\vec{k} \qquad \vec{M}_A = -mg\ell\sin\theta\,\vec{k}$$

$$\therefore -mg\ell\sin\theta = m\ell^2\ddot{\theta} - 4y_0\omega^2\ell m\cos 2\omega t$$

$$\ddot{\theta} + \left(\frac{g}{\ell} - \frac{4y_0\omega^2\cos 2\omega t}{\ell}\right)\sin\theta = 0$$

Florida International University
Department of Mechanical and Materials Engineering

EML 6223 FINAL EXAMINATION April 25, 2011

This is a takehome examination that is due at 1200 hrs. April 27, 2011. You have access to your notes and your book and nothing else.

DO YOUR OWN WORK -- SHOW ALL WORK

Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of this will result in failure of the examination and the course.

________________________ ________________________
PRINT NAME SIGN NAME

This examination consists of 4 problems. You must do Problems 1 and 2; you have a choice between Problem 3 and Problem 4. Several problems have multiple parts. Do all problems and show all your work and any assumptions you make.

PROBLEM #1	40 POINTS
PROBLEM #2	25 POINTS
Choose either of these two problems	
PROBLEM #3	35 POINTS
PROBLEM #4	35 POINTS
	------------
	100 POINTS

Problem #1. For the following two degree of freedom system in r and θ, using Lagrange's equation derive the equations of motion.

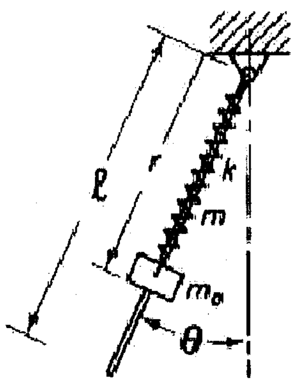


FIGURE 1

Problem #1b. Read the following example, then answer the question below.

Consider the plane mechanism shown in Fig. 7 where the members are assumed to be rigid. Describe all possible motions in terms of generalized coordinates.

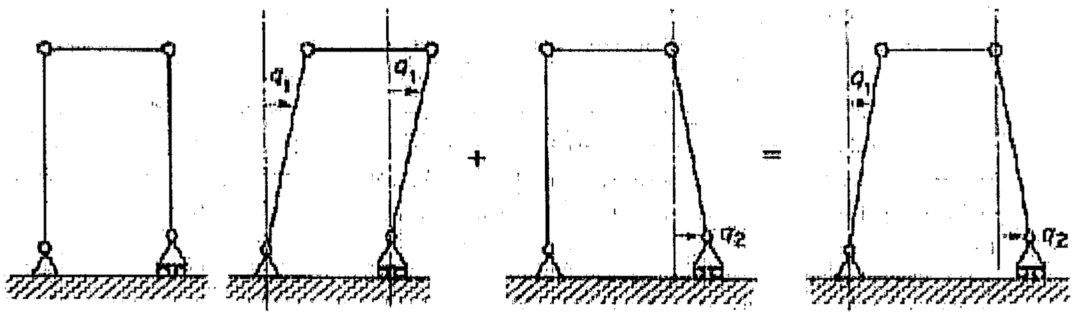


FIGURE 7

Solution As shown in Fig. 7 the displacements can be obtained by the superposition of two displacements q_1 and q_2. Because q_1 and q_2 are independent, they are generalized coordinates, and the system has 2 DOF.

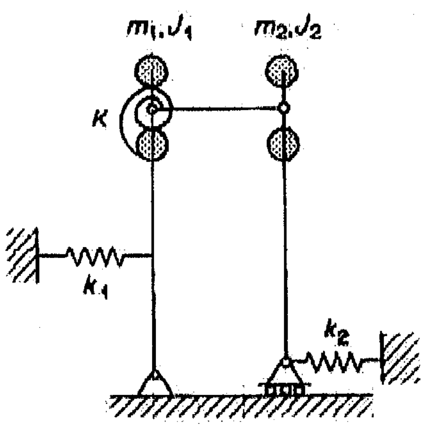


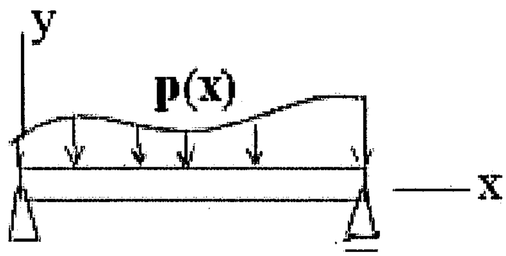
FIGURE 1b

The rigid bar linkages are loaded by spring and masses as shown in the figure to the left. Using Lagrange's Equation, derive the equations of motion. Assume that the springs are unstretched when the upright bar linkages are vertical and the longitudinal bar linkage is horizontal.

The vertical bar linkages are of length *l* and the horizontal bar linkage is of length *l/2*. Spring k_4 is connected halfway up the vertical linkage. There is a torsional spring K at the juncture of the left vertical linkage and the horizontal linkage. The masses and moments of inertia are concentrated at the joints.

Problem #2

Given the functional $U[w(x)] = \int_0^L \{ \frac{1}{2} EI[w'']^2 - pw \} dx$ for the beam shown below, for



which we then replaced w(x) by w(x)+εf(x) and obtained the governing equation of equilibrium and boundary conditions required for this problem, you are to do the following:
Modify the Lagrangian L to include the kinetic energy of the beam, namely, $\frac{1}{2}\rho A \dot{w}^2 dx$ and repeat the process we followed to derive the governing equation of motion and the required boundary conditions

Choose This Problem:

Problem #3.

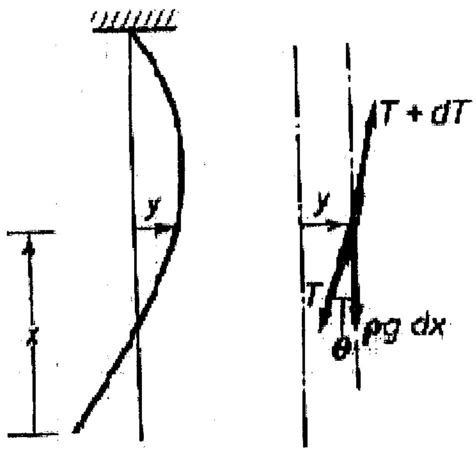


FIGURE 3

1) Shown is a flexible cable supported at the upper end and free to oscillate under the influence of gravity. Show that the equation of lateral motion, i.e., y=y(x,t), is

$$\frac{\partial^2 y}{\partial t^2} = g\left(x\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x}\right)$$

2) If a solution of the form $y=Y(x)\bullet\cos\omega t$ is assumed, show that Y(x) can be reduced to a Bessel's differential equation, viz.,

$$\frac{d^2Y(z)}{dz^2} + \frac{1}{z}\frac{dY(z)}{dz} + Y(z) = 0$$

with solution $Y(z)=J_0(z)$ or $Y(x)=J_0\left(2\omega\sqrt{\frac{x}{g}}\right)$ by a change in variable $z^2=4\omega^2 x/g$.

3) What are the boundary conditions for this problem?

Or Choose This Problem:

Problem #4.

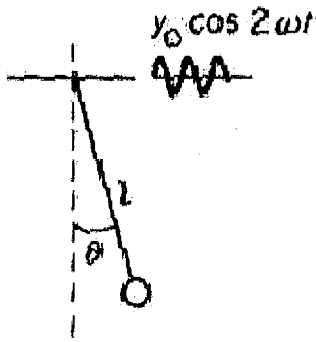


FIGURE 4

1) Derive the governing equation of motion for this oscillating support problem. DO NOT MAKE SMALL ANGLE APPROXIMATION.

2) IF y_o were zero, then this would become the simple pendulum problem. Apply the *perturbation method* to the simple pendulum problem with

$$\sin\theta \text{ being replaced by } \theta - (1/6)\theta^3$$

Use only the first three terms of the series for x and ω.

3) Show that if y_0 is not zero, then the θ_1 or θ_2 differential equation must satisfy the *Mathieu Equation*, namely, an equation of the form

$$\frac{d^2y}{dz^2} + (a - 2b\cos 2z)y = 0 \quad \text{where } z = \omega t$$

Hint, see the handout given to you with the last videotape lecture.

Florida International University
Department of Mechanical and Materials Engineering

EML 6223 MIDTERM EXAMINATION July 2, 2015

This examination is a take-home examination and you have access to your book and your notes only. This examination is due July 6 at 12 noon in my office, EC3443.

DO YOUR OWN WORK -- SHOW ALL WORK

Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of this will result in failure of the examination and the course.

____________________________ ____________________________
PRINT NAME SIGN NAME

This examination consists of 4 problems. You are to do ALL FOUR problems. ALL problems have multiple parts. Do all problems and show all your work and any assumptions you make.

		SCORE
PROBLEM #1	25 POINTS	__________
PROBLEM #2	25 POINTS	__________
PROBLEM #3	25 POINTS	__________
PROBLEM #3	25 POINTS	__________
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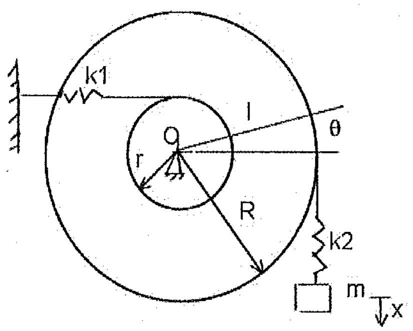
	100 POINTS	__________

PROBLEM #1.

(a) Derive the equations of motion for this two degree of freedom system by deriving the equations using sum of forces=mass x acceleration and sum of moments=mass moment of inertia x angular acceleration. The compound flywheel is pinned at O so that it rotates by an angle θ. The mass, m, is constrained to move vertically only. The compound flywheel is composed of a double disk, one with radius, R, and one with radius r. The compound flywheel has a moment of inertia I about point O.

Assume that motion for angle θ and for vertical displacement x is measured from when the springs k1 and k2 are unstretched.



(b) Now, suppose k1 and k2 had values of 7500 N/m, the radius r=1 m and the radius R=3 m, the moment of inertia of the compound flywheel is 500 kg-m^2 and the mass m is 20 kg. What are the natural frequencies of the system and what are the η values for this system?

PROBLEM #2.

A weight of 77.2 lbf is suspended by a spring having a modulus of 80 lb/in. A harmonic force having a frequency of 152.8 cycles/minute acts on the mass, resulting in a steady motion with an amplitude of forced motion of 1.111 inches.

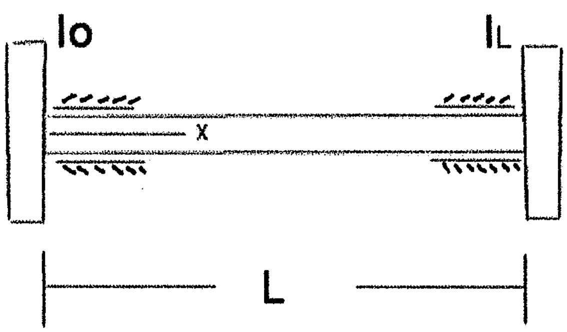
a) What is the system's natural frequency?
b) What is the frequency ratio, $r = \omega/\omega_n$

If a damper were added to the system whose value was 7.5 lbf-sec/in,
c) What would the amplitude of forced motion be reduced to?
d) What would be the force transmitted to the support?

PROBLEM #3.

In the case of a beam of length L undergoing torsion, mass moments of inertia are attached at both ends of the beam.
Assume the beam has its own mass moment of inertia per unit length $I_b = \rho I_p$ and, at each end, the mass moment of inertia is I_o at x=0 and I_L at x=L, which are not necessarily equal to each other. Assume that the beam is supported by frictionless bearings. Let the beam have shear modulus equal to G and a density equal to ρ.

(a) *First determine the governing equation for the beam in torsion.*
(b) *State the boundary conditions you would use at the ends.*
(c) *Solve the governing equation and boundary conditions so that you can find the eigenfunction equation.*
(d) *Find an expression for the angular displacement, $\Theta(x, t)$=...*
(e) *How would you find the frequencies ω for such a problem*



PROBLEM #4.

Starting from the equation of transverse motion of a beam, derive the eigenfunction equation for tranverse motion for the case of a beam that is fixed at x=L and free at x=0. Also determine the first 3 frequencies of vibration.